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Equivalent Martingale Measures and Option Pricing in Jump-Diffusion Markets

Yury Dranev

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in partial fulfillment of the requirements for the degree of
Doctorate of Philosophy in Mathematics

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Abstract

One of the key questions in financial mathematics is the choice of an appropriate model for the financial market. There are a number of models available, such as Geometrical Brownian motion and different types of Levy processes, that are not general enough to reflect all the characteristics of fluctuations in stock price but for which the parameters can be estimated with relative ease. There are more general semimartingale models for which parameter estimation and numerical calculation become very difficult questions. The goal of this thesis is to present a tractable model for which we can carry out computations, and it seems that by varying the parameters this model can be related to real market data. We will use the equivalent measure approach to obtain estimates of the price of European call options for our model. Since our market is incomplete, a consequence of the inclusion of jump processes in the model, we will choose the “best” equivalent martingale measure by applying various techniques and compare the results for different choices. We will also illustrate how this theory works on particular examples. We consider applications not only to the cases of continuous and Levy process markets but also to cases that reflect the main advantages of our jump diffusion model. Finally we numerically illustrate option pricing in our setting.

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Dedication

to my wife Marina

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Chapter 1

Introduction

1.1 History of jump-diffusion models in derivative pricing.

From the beginning of the last century market models have evolved in different ways, and consequently so has option pricing theory. We will briefly describe this evolution and will talk about advantages and disadvantages of these results.

The first and simplest model, where a stock was modelled as a Brownian motion, was introduced by Bachelier in 1900. A subsequent step expanding stock price modelling was that of Samuelson (1965), where he assumed that Geometrical Brownian motion is a better fit to real financial data. Based on a geometrical Brownian motion model, Black and Scholes (1973) obtained an explicit formula for pricing a European call option. There are numerous results extending the Black and Scholes formula to more general models involving predictable coefficients. Other generalizations include stochastic volatility models where the volatility of the market (the coefficient of the Brownian motion) is a stochastic process itself. The main disadvantage of a Geometrical Brownian motion pricing model is that jumps apparent in the real market cannot be described by continuous processes.

This disadvantage is the reason why a number of researchers started to work on models where point processes were involved. The first pure jump models were

considered by Mandelbrot in 1963. Press (1967) was one of the first who put Brownian motion and a Poisson process together to model the logarithm of stock prices. Sums of a Brownian motion and an independent Poisson (or compound Poisson process) were studied by many authors (Merton (1976), Aase (1988) and others) as a relatively good model because it reflected the diffusion and the jump part in a real market. But again a Poisson process is intuitively not the best fit to the jumps in a real market since real jumps may have a variable size.

A further generalization of the model leads us to Levy processes and their particular cases. Different forms for the jump part of Levy processes were of a special interest. Variance-gamma processes (pure jump) introduced by Madan and Seneta (1990) have many advantages (such as existence of the moment generating function) and are currently used by many major investors in stock pricing models. Hyperbolic Levy processes were used in another approach for modelling the market which was developed by Eberlein and Keller (1995) based on the work of Barndorff-Nielsen (1977). Later on generalized hyperbolic models were developed. Mittnik and Rachev (1999) studied the special case of stable asset returns. Some of the works mentioned above provided us with a generalization known as subordinated processes, where the time parameter is a stochastic process (Brownian motion, etc.) itself. In a real market the intensity of jumps should depend on the history of the price process. That is why the processes with independent increments (e.g. Levy processes) are not always good enough to describe such market behavior. This observation now leads us to an even more general price model, the semimartingale structure.

Semimartingales are one of the most general choices available for modeling the stock price. There are also some popular non-semimartingale models, such as fractional Brownian motion, but these models are not of interest in this work. The first and most famous steps in developing the semimartingale approach in option pricing were taken by Harrison and Pliska (1981). In these papers the authors considered the price of the option in a market without arbitrage opportunities (the frictionless market) as an expectation with respect to an equivalent martingale measure (the measure under which the price process, a semimartingale, becomes a local martingale). This measure is also called the risk-free measure. The conditions for the existence

and uniqueness of the equivalent martingale measures (the so-called fundamental theorem of option pricing) in the semimartingale market were developed in Delbaen and Schachermayer (1994). They showed that, under certain restrictions, in a market without arbitrage at least one equivalent martingale measure always exists. In a complete market (where every contingent claim is attainable) there is only one equivalent martingale measure (again some restrictions applied). However, the construction of the equivalent martingale measure was a serious problem.

Jacod and Shiryaev (1987) introduced semimartingale characteristics and described the density of the probability measure transformation in terms of these characteristics. Shiryaev (1999), Kallsen (2000) and others obtained equations for the density process parameters such that the transformed measure is the desired equivalent martingale measure. In a general semimartingale market the solution of these equations is quite difficult.

Another major problem is that the semimartingale market is incomplete in general. Hence there is more than one equivalent martingale measure. There are number of ways to pick up the “optimal” measure among the possible choices. We will mention only the most popular methods here. Follmer and Schweizer (1991) introduced the minimal martingale measure for option pricing in incomplete markets. Later on Schweizer (1994, 1996) introduced the variance optimal measure (minimizes the variance of the density process) which under certain conditions coincides with the minimal martingale measure. Both of these measures are generally signed and this is one of the disadvantages of these methods. Gerber and Shiu (1994) introduced the so-called Esscher transform but for a market with Levy processes. Later Shiryaev and Kallsen (2002) generalized this transform to the case of semimartingales. Another choice for the optimal martingale measure is the minimal relative entropy measure introduced by a number of authors and completely studied by Fritelly (2000). A more general approach, utility maximization, was developed in Kallsen (1999) and Kallsen (2002). The optimal density process in this case is determined by a utility function. We can choose different utilities and obtain different optimal measures. There are also other ways to price options in incomplete markets such as super-hedging, and quantile hedging which we do not discuss here.

1.2 Guide and contribution of the thesis.

Chapter 2 of this thesis will give a short introduction to option pricing using the equivalent martingale measure approach. First we will give basic notions of market and option pricing. Then we will consider the case of geometrical Brownian motion to illustrate these definitions. Then we will consider the equivalent martingale measure approach and general results concerning the existence and uniqueness of equivalent martingale measures. We will illustrate this approach on Levy processes using results from Chan (1999) and Shiryaev (1999). We will also discuss equivalent martingale measures in examples of markets with jumps such as a pure jump market (Pringent (2000)) and a jump-diffusion market (Bardhan, Chao (1995,1996)).

In Chapter 3 we will discuss our market model and obtain a class of equivalent martingale measures for that model. As we mentioned before, in general semimartingale models the construction of the equivalent martingale measure (and the choice of the optimal one in the incomplete market) is a difficult question. From the other side if we assume that the market has a Levy structure we remove from consideration a significant part of interesting cases with dependent increments. That is why we consider in this thesis a particular class of semimartingale market models which is more general than the class of Levy processes, but preserves most of its advantages. Our stock market model consists of m risky assets, where every asset price in this market is represented as an exponential of a semimartingale. We consider k , $k > m$ driving “uncertainties”: d Brownian motions and $k - d$ marked point processes determined by their jump intensities and jump size distributions. We will express the price processes in terms of these Brownian motions and marked point processes. In other words the price processes will be a sum of integrals with respect to Brownian motions and with respect to marked point processes. We also will require that some conditions be satisfied. The advantage of this model is that the semimartingale characteristics of the logarithm of the price process are continuous processes and moreover can be represented as integrals with respect to time. This makes these semimartingales look almost like Levy processes, but they still may have dependent increments like more general semimartingale models. Another advantage of this model is that when we try

to fit it to real market data we do not have to exactly estimate parameters of the marked point processes to obtain expressions for the price. We can approximately guess the structure of the jump part of the price and later adjust it to the real data by changing parameters of the model. In Section 3.2 we will represent the price processes as stochastic exponents of semimartingales and obtain conditions for when these semimartingales are special. The main property of a stochastic exponent is that it preserves the martingale structure, i.e. the semimartingale and its stochastic exponent are (local) martingales at the same time. Hence in order to obtain the equivalent martingale measure for the price process it is enough to obtain the equivalent martingale measure for its stochastic logarithm. The class of equivalent martingale measures will be obtained in Section 3.3 following the method of Bardhan and Chao (1996) and keeping in mind that the jump part of our model is different from theirs. We will obtain the equations for the parameters of the density process of the equivalent martingale measure. Since our market is incomplete these equations will not provide us with a unique equivalent martingale measure. Hence to price options in this market we have to pick the optimal martingale measure.

Chapter 4 is completely devoted to the choice of the optimal martingale measure in our market model. First of all, in Section 4.1 we consider the general case of maximum utility. Based on work of Kallsen (2002) we obtain the equivalent measure parameters that correspond to the utility function. We also obtain the system of equations for the utility optimal strategy in our market. Section 4.2 describes another optimal choice of the measure transform. It is the General Esscher transform introduced by Shiryaev and Kallsen (2002). We again obtain the measure parameters for the Esscher transform. It is not possible to determine the utility function in the multidimensional case, which is why we introduce a slightly different transform for which we can determine the utility function. Through Sections 4.3, 4.4, 4.5 and 4.6 we obtain the measure parameters for the variance-optimal, minimal relative entropy, reverse relative entropy and the minimal martingale measure. In the case of the minimal martingale and the variance optimal measures we may generally obtain signed measures. That is why the optimization procedure in this case is restricted to probability measures. In the case of minimal relative entropy we also obtained

formulas not only for the minimal relative entropy measure (minimizes expectation with respect to the new measure of the logarithm of the density process) but also for the measure that minimize the expectation with respect to the new measure of the *stochastic* logarithm of the density process. In Section 4.6 we classify all optimal measures by corresponding utility functions. In Section 4.7 we obtain the classification of optimal measures based on Goll and Ruchendorf (2001). In this thesis we obtained similar equations for different optimal measures and classify them by utility functions. Previously this had been done for Levy processes, but our market model is not Levy.

In Chapter 5 we apply our results for some simple situations. First we consider a continuous market and show that, by using formulas from Chapter 4, all optimal measures considered here are the same in this market. The next example is that of Levy processes. We consider a very simple case of the market where all parameters are constants and the jump part consists of a sum of standard Poisson processes with coefficients. Since generally this market is incomplete, we need to find an optimal measure in this market. We choose the minimal martingale measure and obtain explicitly the optimal strategy and the price of the option in this case. We also consider an example where the marked point processes have exponential and normal jump size distributions. In this case we can simplify the equation for the measure parameters in cases of the minimal martingale measure and the general Esscher transform. In Section 5.4 we consider one more application. We can calculate (Section 3.3) the parameters of the point processes in the stock price expression after the measure transformation. Hence we can specify the class of point processes after transformation (the target class) and this will specify the class of corresponding equivalent martingale measures (the class of “convenient measures”). We will describe some examples of convenient measures where we pick different target classes for which we can simplify the calculation of the option price (as the expectation with respect to the new measure).

Finally in Chapter 6 we illustrate option pricing for our model numerically. Most of the model parameters of this numerical example are deterministic functions but the intensities of the point processes are predictable processes that depend on the previous

values of the stock prices. We obtain the price of the option using expectation with respect to the original and minimal martingale measure. We will present pictures of the paths of the processes before and after the measure transformation to illustrate the changes. We also compare histograms for option prices before and after measure transform.

Chapter 2

EMM approach in option pricing.

2.1 Basic definitions.

We start by presenting the main facts of a general theory of option pricing in financial markets. This section is based on works of Harrison, Pliska (1981), Protter (2001), Delbaen, Schachermayer (1994), Kallianpur, Karandikar (2001) and Shiryaev (1999).

Consider a financial market $S = (S_0, \dots, S_m)$, where S_0 is a bond (riskless asset) and there are m stocks $S_1, \dots, S_m$. We assume here that S_i for $i = 0, \dots, m$ are positive cadlag semimartingales on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$. We will consider a finite time horizon T , thus $t \in [0, T]$.

Definition 1 *The stochastic process $\pi = (\pi_0, \dots, \pi_m)$ is said to be a **trading strategy** if $\pi_i(t)$ are $\mathbf{F}$ -predictable and the and the stochastic integral $\int_0^T \pi_i(s) dS_i(s)$ exists for $i = 0, \dots, m$.*

Definition 2 *The accumulated gain or loss up to the instant t is called the **gains process generated by** π and is given by*

$$G_\pi(t) := \sum_{i=0}^m \int_0^t \pi_i(s) dS_i(s).$$

Definition 3 *We call our holdings in the assets with values $S_0, \dots, S_m$ our **portfolio**. We represent the portfolio by $\pi = (\pi_0, \dots, \pi_m)$ and the associated **value (wealth)***

process is defined to be

$$V_\pi(t) := \sum_{i=0}^m \pi_i(t) S_i(t).$$

Definition 4 Let $L(S)$ denote the space of predictable processes which are integrable with respect to S . On the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ we call $(S, L(S), P)$ a **pricing model**.

Definition 5 A trading strategy π is said to be **self-financing** ($\pi \in SF(S)$) if there is no investment or consumption at any time $t > 0$. That is

$$V_\pi(t) = V_\pi(0) + G_\pi(t) \text{ a.s. } \forall t \in [0, T].$$

Remark 1 . In Protter (2001) $\pi_0(t)$ is optional (not necessary predictable).

If we assume that S_0 is strictly positive we can define a *discounted* process $\tilde{S} = (1, \frac{S_1}{S_0}, \dots, \frac{S_m}{S_0})$. S_0 is called a **numeraire**.

Proposition 1 (Numeraire invariance Protter (2001)).

Let π be a strategy for S . If the process $\frac{1}{S_0}$ is predictable and of finite variation then $\tilde{S}$ is a semimartingale and $\pi \in SF(S)$ if and only if $\pi \in SF(\tilde{S})$.

If we assume that S_0 is (strictly) positive we can define a *discounted* processes $\tilde{S} = (1, \frac{S_1}{S_0}, \dots, \frac{S_d}{S_0})$.

Hence from now on we can assume without loss of generality that $S_0 \equiv 1$ and $S \equiv \tilde{S}$.

Let us now introduce some classes of **admissible** strategies.

Definition 6 (i) $\Pi_a(S) = \{\pi \in SF(S) : V_\pi(t) \geq -a, t \in [0, T]\}$. This class restricts the maximum loss with strategy π over the entire period $[0, T]$ to a , the so-called class of **a -admissible strategies**.

(ii) $\Pi_+ = \cup_{a \geq 0} \Pi_a(S)$, the so-called class of **admissible strategies**.

(iii) $\Pi_a(S) = \{\pi \in SF(S) : V_\pi(t) \geq -\sum_{i=0}^m \mathbf{a}_i S_i(t), t \in [0, T]\}$, where $\mathbf{a} = (\mathbf{a}_0, \dots, \mathbf{a}_m)$ is a m -dimensional vector with non-negative components. This class of strategies restricts the maximum loss of an investor to his original capital $\mathbf{a}_0$ and $\mathbf{a}_i$ shares of each stock S_i .

We can consider more restrictive classes of admissible strategies but the above classes are the most natural and useful for no-arbitrage markets (Shiryaev 1999).

Definition 7 $S = (S_0, \dots, S_m)$ has the **no-arbitrage (NA) property** if for any strategy $\pi \in SF(S)$ with $V_\pi(0) = 0$ we have:

$$P(V_\pi(T) \geq 0) = 1 \implies P(V_\pi(T) = 0) = 1.$$

Now we introduce $L_\infty^+(\Omega, \mathcal{F}_T, P)$ as the subspace of non-negative random variables of a Banach space $L_\infty(\Omega, \mathcal{F}_T, P)$ equipped with norm

$$\|x\|_\infty = \inf\{0 \leq c < \infty : P(|x| > c) = 0\}.$$

Further $\bar{\Psi}_+(S)$ is the closure with respect to norm $\|\cdot\|_\infty$ of the set

$$\Psi_+ = \{x \in L_\infty(\Omega, \mathcal{F}_T, P) : x \leq G_\pi(T), \text{ where } \pi \in \Pi_+(S)\}.$$

Remark 2 The above definition of no-arbitrage can be also rewritten in the following form:

$$\Psi_+(S) \cap L_\infty^+(\Omega, \mathcal{F}_T, P) = \{0\}.$$

Definition 8 $S = (S_0, \dots, S_m)$ has the $\widetilde{NA}_+$ **property** (also called **no free lunch with vanishing risk**) if

$$\bar{\Psi}_+(S) \cap L_\infty^+(\Omega, \mathcal{F}_T, P) = \{0\}.$$

Clearly $\widetilde{NA}_+$ implies NA.

Definition 9 A **contingent claim** is an $\mathcal{F}_T$ -measurable nonnegative bounded random variable f_T .

Definition 10 A contingent claim is said to be **attainable** if there exists a hedging strategy $\pi \in SF(S)$ such that $V_\pi(T) = f_T$ (P -a.s.).

Definition 11 A market S is said to be **complete** if every contingent claim is attainable.

Definition 12 *If a contingent claim is attainable then its (**hedging**) **price** is the value*

$$\mathcal{C}_0(f_T) := \inf\{x \geq 0 : \exists \pi \in \Pi_+(S) \text{ with } V_\pi(0) = x, V_\pi(T) = f_T \text{ (P - a.s.)}\}.$$

The main problem of interest in option pricing is to find $\mathcal{C}(f_T)$ for a European call option, i.e. for $f_T = (S_T - K)^+$.

2.2 Black-Scholes approach and its disadvantages.

In this section we consider a market (B, S) where B (previously S_0) is a bond and $S(t)$ is a one-dimensional stock price. Again we consider a discounted process and assume that $B(t) \equiv 1$.

In their original paper Black and Scholes assumed that a hedging strategy π exists and that the value of this strategy $V_\pi(t)$ depends only on the present value of stock $S(t)$, i.e. does not depend on the “history” $(S(s), s < t)$ (a sort of Markov property). Their next a priori assumption was that $V_\pi(t) = V_\pi(t, S(t)) \in C^{1,2}$ (with continuously differentiable derivative). Hence we can apply the Ito formula and follow Kallianpur-Karandikar (2000):

$$V_\pi(t, S(t)) = V_\pi^1(t) + V_\pi^2(t) + V_\pi^3(t). \quad (2.2.1)$$

Here

$$V_\pi^1(t) = V_\pi(0, S_0) + \int_0^t \frac{\partial V_\pi}{\partial t}(s, S(s-))ds + \int_0^t \frac{\partial V_\pi}{\partial S}(s, S(s-))dS(s)$$

is the continuous part not involving any unforeseen changes;

$$V_\pi^2(t) = \frac{1}{2} \int_0^t \frac{\partial^2 V_\pi}{\partial S^2}(s, S(s-))d[S, S]$$

is the absolutely continuous part involving unforeseen terms; and

$$V_\pi^3(t) = \sum_{0 \leq s \leq t} (V_\pi(t, S(t)) - V_\pi(s, S(s-))) - \sum_{0 \leq s \leq t} \left(\frac{\partial V_\pi}{\partial S}(s, S(s-)) \Delta S(s) + \frac{1}{2} \frac{\partial^2 V_\pi}{\partial S^2}(s, S(s-)) (\Delta S(s-))^2 \right)$$

is the singular part from the chain rule.

On the other hand since $\pi \in SF(S)$ we have:

$$V_\pi(t, S(t)) = \pi_0(t) + \pi_1(t)S(t) = V_\pi(0, S(0)) + \int_0^t \pi_1(s) dS(s). \quad (2.2.2)$$

Our next goal is a comparison of expressions (2.2.1) and (2.2.2). If we consider continuous $S(t)$ with known quadratic variation $[S, S]$, such as geometric Brownian motion, then

$$dS(t) = S(t)(bdt + \sigma dW(t)),$$

and we have that (2.2.1) and (2.2.2) become

$$dV_\pi = \left(\frac{\partial V_\pi}{\partial t} + bS \frac{\partial V_\pi}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_\pi}{\partial S^2} \right) dt + \sigma S \frac{\partial V_\pi}{\partial S} dW; \quad (2.2.3)$$

$$dV_\pi = \pi_1(t)S(t)(bdt + \sigma dW(t)). \quad (2.2.4)$$

From (2.2.3) and (2.2.4) we have two Doob decompositions of the special semimartingale $V_\pi(t, S(t))$. Because of the uniqueness of this decomposition the coefficients of dt and $dW(t)$, respectively, must be equal. Recalling $S(t) > 0$, we get (P-a.s.) from the $dW(t)$ terms that:

$$\pi_1(t) = \frac{\partial V_\pi}{\partial S}(t, S(t)). \quad (2.2.5)$$

Now comparing coefficients on dt and substituting (2.2.5) we get the fundamental (Black-Scholes) PDE:

$$\frac{\partial V_\pi}{\partial t}(t, S(t)) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_\pi}{\partial S^2}(t, S(t)) = 0, \quad (2.2.6)$$

with boundary condition

$$V_\pi(T, S(T)) = f_T, \quad S > 0.$$

Here f_T is our contingent claim and $\mathcal{C}_0(f_T) = V_\pi(0, S(0))$ is its price. Since $S(t)$ is geometrical Brownian motion we can apply the Feynman-Kac formula (Karatzas Shreve (1999, 5.7.B)) to (2.2.6) and get the Black-Scholes formula for the price of European call option (i.e. for $f_T = (S_T - K)^+$ for the discounted market).

$$\begin{aligned}\mathcal{C}_0(f_T) &= S(0)\Phi(\beta) - K\Phi(\beta - \sigma\sqrt{T}); \\ \beta &= \frac{1}{\sigma\sqrt{T}}(\log S(0) - \log K) + \frac{\sigma}{2}\sqrt{T}\end{aligned}\tag{2.2.7}$$

For a general cadlag semimartingale we can obtain a stochastic PDE similar to (2.2.6) but it is difficult to solve it since the Feynman-Kac solution depends strongly on the structure of Brownian motion. Moreover we do not even know if a solution exists and is unique. The additional assumptions that Y has a Markov property and belong to class $C^{1,2}$ also seem too restrictive.

These limitations of the Black-Scholes approach lead us to the so-called martingale approach.

2.3 Equivalent martingale measure approach.

Again we consider a discounted market $S = (1, S_1, \dots, S_m)$ where S_i are nonnegative cadlag semimartingales on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$.

Definition 13 *If there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\tilde{P}$ is equivalent to $(\sim) P$, and S is a martingale (local martingale) with respect to this measure $\tilde{P}$, we say that an **Equivalent Martingale Measure (Equivalent Local Martingale Measure)** exists and the **EMM (ELMM)** property holds.*

Definition 14 *S is said to be a σ -martingale if there exists an m -dimensional martingale $M^S = (M^S(t))_{t \leq T}$ and an M^S -integrable predictable positive one-dimensional process $\gamma^S = (\gamma^S(t))_{t \leq T}$ such that: $S(t) = S(0) + \int_0^t \gamma^S(s) dM^S$. We call such γ^S a **density of S with respect to M^S** . If there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\tilde{P} \sim P$ and the semimartingale S is a σ -martingale with respect to this measure $\tilde{P}$, we say that an **Equivalent σ -Martingale Measure** exists and the **E σ MM** property holds.*

At this point we recall some necessary and sufficient conditions for semimartingales to become (local) martingales after a measure transformation.

We assume that S is a semimartingale with decomposition:

$$S(t) = S_0 + S^M(t) + S^A(t),$$

where S^M is a martingale and S^A is a predictable process of finite variation. Let $Z = \frac{d\tilde{P}}{dP}$ and $Z(t-) = E_P(Z|\mathcal{F}_t)$. The Meyer-Girsanov theorem gives a decomposition of S as

$$S = S_0 + (S^M(t) - \int_0^t \frac{1}{Z(s-)} d[Z, S^M]) + (S^A(t) + \int_0^t \frac{1}{Z(s-)} d[Z, S^M]).$$

Proposition 2 .

1. S is a $\tilde{P}$ -local martingale if $S^A(t) + \int_0^t \frac{1}{Z(s-)} d[Z, S^M] = 0$.
2. If S is a $\tilde{P}$ -local martingale then $S^A(t) = S^A(0) + \int_0^t \gamma^A(s) d[S^M, S^M]$ for a predictable $\gamma^A(t)$.
3. S is a $\tilde{P}$ -martingale if for any bounded stopping time τ with $E(S_\tau) < \infty$ we have that $E(S_\tau) = E(S(0))$.

The following theorem gives us the relationship between different NA and EMM properties.

Proposition 3 (Shiryaev (1999.VII.2)) *For a nonnegative cadlag semimartingale $S = (1, S_1, \dots, S_m)$:*

$$E\sigma MM \iff \widetilde{NA}_+.$$

If S has bounded (locally bounded) components then:

$$E(L)MM \iff \widetilde{NA}_+.$$

On $(\Omega, \mathcal{F}_T)$ let us denote the class of EMMs by $\mathcal{M} = \mathcal{M}(P)$. If there exists a unique EMM we write $|\mathcal{M}(P)| = 1$.

Definition 15 *Fix $\tilde{P}$ an EMM for S , so S is a $\tilde{P}$ -martingale. If whenever M is a $\tilde{P}$ -martingale we can write*

$$M(t) = M(0) + \sum_{i=1}^m \int_0^t \pi_i(s) dS_i(s), \quad t \leq T$$

*for a strategy π then we say that a **S -representation** with respect to $\tilde{P}$ holds for M .*

Proposition 4 (Shiryaev (1999,VII.2d), Kallianpur,Karandikar (2000)). *Let the class of EMM consist of only one measure $\tilde{P}$. Then $\exists \pi \in SF(S)$ such that $V_\pi(T) = f_T$ (P-a.s.) for a contingent claim f_T with $E_{\tilde{P}}|f_T| < \infty$. Moreover*

$$|\mathcal{M}(P)| = 1 \iff S - \text{representation} \iff \text{completeness}$$

Remark 3 *From now on when we talk about a contingent claim f_T in a complete NA market we assume that $E_{\tilde{P}}|f_T| < \infty$.*

Now we can give another definition of the contingent claim price in a complete NA market:

Definition 16 *Let $\tilde{P}$ be a unique EMM. Then the (**hedging**) price is*

$$\mathcal{C}(f_T) = E_{\tilde{P}}f_T.$$

Now it is reasonable to give an interpretation of the martingale approach in option pricing theory from the point of view of economics. We will follow Harrison, Pliska (1981) and Kallianpur, Karandikar (2000). First, we are looking only at self-financing strategies in the market, which means that no funds are added or withdrawn from the value of the portfolio at any time $t \in [0, T]$. We also assume that the market has the NA property i.e. there does not exist such a strategy that represents a riskless plan for making profit without any investment. A security market containing arbitrage opportunities cannot be one in which an economic equilibrium exists. From the above we can see that the NA property holds if and only if the EMM $\tilde{P}$ exists. Moreover Harrison and Pliska showed that for any self-financing admissible strategy π the value process V_π becomes a martingale with respect to $\tilde{P}$ and

$$E_{\tilde{P}}(V_\pi(T)) = V_\pi(0).$$

In other words the expectation of the future value of the self-financing portfolio with respect to a EMM is equal to the value of the original investment. Now if we consider an attainable contingent claim f_T , i.e. for an admissible strategy π we have $V_\pi(T) = f_T$ then the price of the contingent claim is equal to the corresponding value of the original investment:

$$\mathcal{C}(f_T) = V_\pi(0).$$

If the market is complete there is the unique EMM and hence

$$\mathcal{C}(f_T) = E_{\tilde{P}}(V_\pi(T)) = E_{\tilde{P}}(f_T).$$

In general NA markets the equivalence of the two definitions (12 and 17) of hedging price is not clear. However this was proven for geometrical Brownian motion (type) models.

If we use the second definition (Definition 16) of a hedging price, our problem is to construct an equivalent martingale measure and to calculate the expectation of a contingent claim with respect to this measure.

In following three sections we will review some papers that illustrate how the equivalent martingale measure approach will work for different models.

2.4 Construction of EMM for Levy processes.

One of the most common approaches to the construction of an EMM in case of Levy processes is obtaining an EMM through the Esscher transformation. We will illustrate this approach using results of Shiryaev(1999).

Let R be a Levy process. We can use the Levy-Khintchin formula for characteristic functions to obtain

$$E(e^{\varrho R(t)}) = e^{t\kappa_\varrho}$$

where

$$\kappa_\varrho = b\varrho - \frac{1}{2}c\varrho^2 + \int_{\mathbb{R}} (e^{e^z} - 1 - \varrho z I_{\{|z| \leq 1\}}) \nu(dz).$$

For each $\vartheta \in \mathbb{R}$ let $\tilde{P}^{(\vartheta)}$ be a probability measure defined by

$$d\tilde{P}^{(\vartheta)} = Z^\vartheta(T) dP \quad (\text{Esscher transformation})$$

where P is a probability measure on $(\Omega, \mathcal{F}_T)$ with respect to which $R = (R(t))_{t \leq T}$ is a Levy process with triplet (b, c, ν) and $Z^\vartheta(t) = e^{\vartheta R(t) - t\kappa_\vartheta}$ is a martingale.

In the following proposition we get the triplet for R after the Esscher transformation, i.e. with respect to $\tilde{P}^{(\vartheta)}$.

Proposition 5 (Shiryaev (1999,VII.3c)) *Let $\varrho \in \mathbb{R}$. With respect to measure $\tilde{P}^{(\vartheta)}$, the process $R = (R(t))_{t \leq T}$ is a Levy process with Laplace transform*

$$E_{\tilde{P}^{(\vartheta)}}(e^{\varrho R(t)}) = e^{t\kappa_{\varrho}^{(\vartheta)}}, \text{ where } \kappa_{\varrho}^{(\vartheta)} = \kappa_{\vartheta+\varrho} - \kappa_{\vartheta},$$

and with triplet $(b^{(\vartheta)}, c^{(\vartheta)}, \nu^{(\vartheta)})$:

$$b^{(\vartheta)} = b + \vartheta c + \int_{\mathbb{R}} z I_{\{|z| \leq 1\}}(e^{\vartheta z} - 1) \nu(dz), \quad c^{(\vartheta)} = c, \quad \nu^{(\vartheta)}(dz) = e^{\vartheta z} \nu(dz).$$

Now we are looking for conditions on the triplet so that the Levy process R becomes a local martingale. Following Shiryaev (1999, VII.3c), R is a special semi-martingale with respect to P if and only if

$$\int (z^2 \wedge |z|) \nu(dz) < \infty,$$

and R is a local martingale if and only if

$$b + \int_{|z| > 1} z \nu(dz) = 0.$$

Similarly a Levy process is a local martingale with respect to $\tilde{P}^{(\vartheta)}$ if and only if:

$$\int (z^2 \wedge |z|) e^{\vartheta z} \nu(dz) < \infty$$

and

$$b + \vartheta c + \int_{|z| > 1} z \nu(dz) + \int_{\mathbb{R}} z(e^{\vartheta z} - 1) \nu(dz) = 0.$$

Now we consider a stock $S = (S(t))_{t \leq T}$ generated from the Levy process $R = (R(t))_{t \leq T}$ by $S(t) = e^{R(t)}$.

Proposition 6 (Shiryaev (1999,VII.3c)). *$S = e^R$ is a martingale with respect to measure P if and only if*

$$\int_{|z| \leq 1} e^z \nu(dz) < \infty$$

and

$$b + \frac{c}{2} + \int (e^z - 1 - z I_{\{|z| \leq 1\}}) \nu(dz) = 0.$$

If S is not a martingale with respect to P (but still a semimartingale), again we must find a measure $\tilde{P}^{(\vartheta)}$ so that S becomes a martingale with respect to this measure. From Proposition 5 we have

$$E_{\tilde{P}^{(\vartheta)}}(e^{R(t)-R(s)}|\mathcal{F}_s) = e^{(t-s)(\kappa_{\vartheta+1}-\kappa_{\vartheta})}.$$

Now it is clear that if ϑ satisfies the equation $\kappa_{\vartheta+1} - \kappa_{\vartheta} = 0$, then S becomes a martingale with respect to the measure $\tilde{P}^{(\vartheta)}$.

Proposition 7 (Shiryaev (1999, VII.3c)). *If there exist $\vartheta \in \mathbb{R}$ such that:*

$$\int |e^{\vartheta z}(e^z - 1) - zI_{\{|z| \leq 1\}}| \nu(dz) < \infty$$

and

$$b + (\vartheta + \frac{1}{2})c + \int (e^{\vartheta z}(e^z - 1) - zI_{\{|z| \leq 1\}}) \nu(dz) = 0$$

then $S = (S(t))_{t \leq T}$ is a martingale with respect to $P_T^{(\vartheta)}$.

If we find such a ϑ then the Radon-Nykodim derivative is:

$$Z_T^{(\vartheta)} = \exp \left(\vartheta R_T - b\vartheta - \frac{1}{2}c\vartheta^2 + \int_{\mathbb{R}} (e^{\vartheta z} - 1 - \vartheta zI_{\{|z| \leq 1\}}) \nu(dz) \right).$$

The above construction was introduced by Gerber, Shiu (1996) and generalized to the case where the stock price is modelled by a process of the form $\exp(bt + \sigma R(t))$ where b and σ are constants and R is a Levy process. They also obtained results for multidimensional stocks. But since the market in this model is incomplete i.e. there are many equivalent martingale measures in this market, it is not clear why the EM measure obtained by Essher transform should be optimal in any sense.

Another similar model was introduced by Chan (1999). It is more restrictive than the Gerber-Shiu model but, under some assumptions, this model allows us to talk about optimality of the measure. The stock price in this model is driven by a geometric Levy process

$$dS(t) = b(t)S(t-)dt + \sigma S(t-)dR(t)$$

where R is a general Levy process with $E(\exp(-aR(1))) < \infty$ for all $a_1 < a < a_2$ $0 < a_1, a_2 \leq \infty$ (so R has finite moments of all orders). Following Chan for this

Levy process we have $R(t) = \sigma^W W(t) + N^R(t)$ where $W(t)$ is a Brownian motion and $N^R(t)$ is a quadratic pure jump Levy process with Levy measure ν supported on a subset of $[-a_1, a_2]$, where at least one of $a_1 \geq 0$, $a_2 \geq 0$ is finite. The Doob-Meyer decomposition of N^R is given by

$$N^R(t) = Q^R(t) + E(N^R(1))t,$$

where Q^R is a quadratic pure jump martingale with $Q^R(0) = 0$. If $\mu^P(dt, dz)$ is the Poisson measure associated with N^R and $q^R(dt, dz) = \mu^P(dt, dz) - \nu(dz)dt$ denotes the compensated measure, then the martingale part of N^R can be written as $Q^R(t) = \int_0^t \int_{\mathbb{R}} z q^R(dt, dz)$.

Proposition 8 (Chan (1999)) *Consider a predictable process $v(t)$ and a predictable function $\hbar(t, z)$ with respect to measure P . Suppose that $E(\int_0^t v^2(s)ds) < \infty$ and $\hbar > 0$, $\hbar(t, 0) = 1$ for all $t \geq 0$. Let $h(t, z)$ be another predictable function (a sort of truncation function) such that*

$$\int_{\mathbb{R}} |\hbar(t, z) - 1 - h(t, z)| \nu(dz) < \infty.$$

Define a process $Z(t)$ by

$$\begin{aligned} Z(t) &= \exp \left(- \int_0^t v(s) dW(s) - \frac{1}{2} \int_0^t v^2(s) ds - \int_0^t \int_{\mathbb{R}} h(s, z) q^R(ds, dz) \right) \\ &\quad \times \exp \left(- \int_{[0, t) \times \mathbb{R}} [\hbar(s, z) - 1 - h(s, z)] \nu(dz) ds \right) \\ &\quad \times \Pi_{0 < s \leq t} \hbar(s, \Delta N^R(s)) \exp(-h(s, \Delta N^R(s))). \end{aligned}$$

Then Z is a positive local martingale with respect to P with $Z(0) = 1$. Let $\tilde{P}$ be a measure which is absolutely continuous with respect to P on $\mathcal{F}_T$ with density Z . There exist $\tilde{v}$, $\tilde{\hbar}$ and $\tilde{h}$ for which $E(Z(T)) = 1$, i.e. Z is a density process. Moreover under $\tilde{P}$, the process $\tilde{W} = W(t) + \int_0^t v(s)ds$ is a Brownian motion and the process N^R is a quadratic pure jump process with compensator measure given by $\tilde{\nu}(dt, dz) = \tilde{\nu}(dz)dt$, where $\tilde{\nu} = \hbar \nu$, and the predictable part is given by

$$E_{\tilde{P}}(N^R(t)) = E(N^R(1))t - \int_0^t \int_{\mathbb{R}} z(\hbar(z, s) - 1) \nu(dz) ds.$$

Moreover $\tilde{P}$ becomes an EMM in this model if and only if

$$b(t) + \sigma(t)E(N^R(1)) - \sigma(t) \left(\sigma^W v(t) + \int_{\mathbb{R}} z(\tilde{h}(t, z) - 1)\nu(dz) \right) = 0.$$

Remark 4 The above result does not uniquely specify v and $\tilde{h}$ and hence there are numerous EMMs for this model.

Follmer, Schweizer (1991) introduced a minimal martingale measure which restricted the class of EMM's and allows to obtain some additional conditions for the density process in Chan's model.

Definition 17 1. A strategy (π_0, π_1) is called **optimal** if for a contingent claim f_T we have $V_\pi(T) = f_T$ and $V_\pi(t) - G_\pi(t)$ is a square-integrable martingale orthogonal to the martingale part of S with respect to P .

2. An EMM $\tilde{P}$ is called **minimal martingale** if any square-integrable P -martingale which is orthogonal to the martingale part of S with respect to P remains a martingale with respect to $\tilde{P}$.

Chan did not provide us with an optimal strategy in his paper, but we do have a minimal martingale measure.

Proposition 9 (Chan (1999)). Let Z be a density process such that

$$dZ(t) = \gamma^Z(t)Z_{t-}(\sigma^W dW(t) + dQ^R(t))$$

where

$$\gamma^Z(t) = -\frac{b(t) + \sigma(t)E(N^R(1))}{\sigma(t)((\sigma^W)^2 + \int_{\mathbb{R}} z^2 \nu(dz))},$$

$W(t)$ is a Brownian motion, and $Q^R(t)$ is a quadratic pure jump martingale from the above model. Then $\tilde{P}$ is a minimal martingale measure.

Moreover comparing the coefficient in dW and dQ^R to those of Proposition 8

$$v(t) = \sigma^W \gamma^Z(t) = \sigma^W \frac{b(t) + \sigma(t)E(N^R(1))}{\sigma(t)((\sigma^W)^2 + \int_{\mathbb{R}} z^2 \nu(dz))}$$

$$\tilde{h}(t, z) = 1 - \gamma^Z(t)z = 1 - \frac{b(t) + \sigma(t)E(N^R(1))}{\sigma(t)((\sigma^W)^2 + \int_{\mathbb{R}} z^2 \nu(dz))}z.$$

Remark 5 1. In this Proposition the Brownian motion in the classical Black-Scholes formula for Z has been replaced by the martingale part of R .

2. In the continuous case an important property of the minimal measure is that it gives S the law of its martingale part under the Doob decomposition; in fact the minimal measure can be uniquely characterized by this property (Follmer, Schweizer (1991)). This property is no longer the case in Chan's model, because generally the process R is not continuous. Moreover if the Levy process R has a Brownian motion component, we can construct a measure which will differ from the minimal martingale measure and still have this minimality property (Chan (1999)).

3. Since a positive semimartingale S can be written uniquely as $S = S_0 \mathcal{E}(\hat{R}^A) \mathcal{E}(\hat{R}^M)$, where $\hat{R}^A$ is predictable and $\hat{R}^M$ is a martingale, Elliot, Hunter, Kopp, Madan (1995) proposed pricing contingent claims using the martingale measure under which S has the law of its martingale part $\hat{R}^M$ under this multiplicative representation. For Chan's model this construction gives the same result as in the previous remark.

An additional optimality criterion considered in Chan's paper is minimizing the **relative entropy**, $I_P(\tilde{P}) = E_{\tilde{P}} \log \frac{d\tilde{P}}{dP}$. Chan (1999) showed that such a measure for his model can be obtained by the **Esscher-Chan transformation** with the density process

$$Z_\theta(t) = \exp \left(- \int_0^t \theta(s) dR(s) - \int_0^t \log(E e^{-\theta(s)R(1)}) ds \right)$$

and

Proposition 10 (Chan(1999)) *The density for the Esscher-Chan transformation is the density of the martingale measure if and only if the following condition on θ holds:*

$$b(t) + \sigma(t)E(N^R(1)) - \sigma(t) \left((\sigma^W)^2 \theta(t) + \int_{\mathbb{R}} z(\exp(-\theta(t)z) - 1) \nu(dz) \right) = 0.$$

However for the Gerber-Shiu model (R is a Levy process) the measure obtained by the usual Esscher transform does not provide us with minimum relative entropy.

The Black-Scholes approach described earlier could be carried over to Chan's model since we have the Markov property in that case. That is, if $\mathcal{C}(t) = E_{\tilde{P}}(f_T | \mathcal{F}_t)$

is the price of the contingent claim $f_T = (S(T) - K)^+$ of a European call then the valuation process $\mathcal{C}$ admits a Feynman-Kac type representation $\mathcal{C}(t) = v_f(t, S(t))$ where $v_f(t, x)$ is the solution of the following Cauchy problem:

$$\frac{dv}{dt} + \rho(t)v = 0, \quad v(T, x) = (x - K)^+,$$

where $\rho(t)$ is the following integro-differential operator:

$$\rho(t)g(x) = 1/2(\sigma^W)^2\sigma^2(t)x^2g''(x) + \int_{\mathbb{R}} [g(x + \sigma(t)xy) - g(x) - \sigma(t)xyg'(x)]\tilde{\nu}(dy).$$

2.5 Construction of EMM for non Levy pure jump market.

The construction of an EMM for the case of Levy processes was based on the properties of processes with independent increments such as the Levy-Khintchin formula. Now we turn our attention to the case of processes with dependent increments, in particular general marked point processes. Aase (1988) introduced a model with R given by the sum of an Ito process and a jump component, but there were a number of mistakes in his paper and finally it holds only for the case of a Poisson process. Since that time other authors have studied this problem.

Prigent (2001) considers the general model where the stock price S is defined by $S(t) = S(0)\Pi_{t_n \leq t}(z_n)$, for $z_n > 0$ a.s. and t_n denoting the sequence of price variation times. Hence S is purely a jump process which can be also expressed as $S(t) = S(0) \exp(N(t))$ with $N(t) = \sum_{t_n \leq t} \ln z_n$. The sequence $(t_n, \ln z_n)$ is a marked point process. Define $\mu = \mu(\omega; ds; dz)$ to be the random measure of jumps of N . That is $\forall A \in \mathcal{B}(\mathbb{R})$ we have $\mu(\omega; (0, t] \times A) = \sum_{t_n \leq t} I_{\{N(t_n, z_n) \in A\}}$. Let ν be the compensator of μ . Suppose N is a special semimartingale and $\int \int (e^z - 1 - z)d\nu$ has locally integrable variation. Then S can be rewritten as $S = \mathcal{E}(\hat{R})$, where $\hat{R}(t) = N(t) + \int_0^t \int_{\mathbb{R}} (e^z - 1 - z)d\mu$.

Remark 6 1. *Generally financial markets modeled by marked point processes are incomplete. We have completeness only for the case where jump sizes belong to a*

finite set of values and where there are sufficient number of hedging instruments.

2. We continue to assume that the bond in our market is identically one and work with discounted processes.

Proposition 11 (Prigent(2001)) *Each EMM is characterized by a density process Z given by:*

$$Z(t) = \mathcal{E}(\tau_0 + \int_0^t \int (\hbar(\omega, s, z) - 1) d(\mu - \nu)),$$

where $\hbar$ is a predictable function. Moreover to guarantee the positivity of Z we need $\hbar > 0$ a.s. and in order to have $E(Z(t)) = 1$, $\hbar$ must satisfy:

$$E\Pi_{t_n \leq T} \left(\hbar(\omega, t_n, \ln z_n) \exp \frac{1 - \hbar(\omega, t_n, \ln z_n)}{\hbar(\omega, t_n, \ln z_n)} \right) < \infty.$$

The condition under which the measure with the density Z is a EMM (the **martingale condition**) here is:

$$0 = \int_0^t \int_{\mathbb{R}} \hbar(e^z - 1) \nu(ds, dz) - \sum_{s \leq t} \left(\int (\hbar - 1) \nu(\{s\}, dz) \times \int (e^z - 1) \nu(\{s\}, dz) \right).$$

Denote by $\hat{F}_n(dt, dz)$ a regular version of the conditional distribution of $(t_{n+1}, \ln z_{n+1})$ with respect to $\mathcal{F}_{t_n}$. Then from Jacod, Shiryaev (1987, II.1d) we have

$$\nu(dt, dz) = \sum_n I_{\{t_n < t < t_{n+1}\}} \frac{\hat{F}_n(dt, dz)}{\hat{F}_n([t, \infty) \times \mathbb{R})}.$$

Denote by F_n a regular version of the conditional distribution of t_{n+1} with respect to $\mathcal{F}_{t_n}$, i.e. $F_n(dt) = \hat{F}_n(dt \times \mathbb{R})$. Finally introduce the kernel Kr such that $Kr(t_{n+1}, dz)$ is the conditional distribution of $\Delta N(t_{n+1})$ with respect to $\mathcal{F}_{t_n}$ and t_{n+1} . The predictable function $\hbar$ can be decomposed as: $\hbar(\omega, t, z) = 1 + \sum_{t_n \leq t} I_{(t_n, t_{n+1}]}(t) \hat{\hbar}_n(z)$ for some $\hat{\hbar}_n$. If $F_n(dt) \ll dt$ (no fixed time of discontinuity) the Girsanov equation becomes:

$$\forall n, \int_{\mathbb{R}} \hat{\hbar}_n(z) (e^z - 1) Kr(t_{n+1}, dz) = 0.$$

Having established a form for an EMM, Prigent (2001) also obtained characteristics of the stock price under a risk-neutral probability (EMM). He followed the

Girsanov theorem for semimartingales. Denote by $\hat{Y}$ a predictable process such that the process $\tilde{h}$ has the following decomposition:

$$\tilde{h} = Y + \frac{(\hat{Y} - a)}{1 - a} I_{a < 1}$$

with $a(t) = \nu(\omega; t \times \mathbb{R})$ and $\hat{Y} = \int Y(\omega; t, z) \nu(\omega; t, z) dz$. The characteristics $(\tilde{B}(h), \tilde{C}, \tilde{\nu})$ of N under the martingale measure associated to $\tilde{h}$ are:

$$\tilde{B} = B + \int \int z(Y - 1) d\nu,$$

$$\tilde{C} = C = 0,$$

$$d\tilde{\nu} = Y d\nu.$$

Since under $\tilde{P}$, the process N has the following canonical decomposition:

$$N = N(0) + \int \int z d(\mu - \tilde{\nu}) + B,$$

and from the relation $S(t) = S_0 \exp(N(t))$, the dynamic of the discount stock price S under $\tilde{P}$ is deduced.

Prigent (2001) applied these results to the cases of semi-Markov processes and some trivial cases where we have some kind of Markov property. Also Prigent found an exact formula for the risk premium process $\tilde{h}$ (and hence for the density process) for the case of a minimal martingale measure (which is similar to Chan's formula in the case of Levy processes). An open question is to consider a filtration generated by a marked point process and a dependent process with diffusion component. From this paper it is still not clear how to apply the above results to the case of a process without any sort of Markov property.

2.6 Construction of EMM for the example of non Levy complete jump-diffusion market.

Bardhan, Chao(1992,1995) considered the market with a single bond and m stock prices determined by d independent Brownian motions and $m - d$ point processes with

predictable stochastic intensities (number of stocks is equal to number of uncertainties in each formula for stock price). For simplicity we illustrate this paper for the case when we have only two discounted stock prices $S_i(t)$ ($i = 1, 2$) each of which is the weighted sum of the Brownian motion $W(t)$ and a point process $N(t)$ with stochastic intensity λ and a sequence of jump points $(t_1, t_2, \dots)$. In other words each stock price was governed by the following stochastic differential equation

$$dS_i(t) = S_i(t)(\hat{b}_i(t) + \sigma_i^W(t)dW(t) + \sigma_i^N dQ),$$

where $\hat{b}_i = b_i + \sigma_i^N \lambda$, $i = 1, 2$ and $Q = N - \int_0^t \lambda(s)ds$. We assume that processes λ , σ_i^N , $\hat{b}_i$ and σ_i^W are predictable and bounded uniformly on $[0, T] \times \Omega$. In addition $\sigma_i^N > -1$ and the covariance matrix given by $\tilde{\sigma}\tilde{\sigma}^T$ is positive definite, where $\tilde{\sigma} = (\sigma, \sigma^N)$ is the 2×2 volatility matrix process.

Now consider the $\mathbb{R}^2$ -valued process, $\theta(t) = (\tilde{\sigma})^{-1}b(t) = [\theta^W, \theta^N]^T$. This process is bounded, measurable and predictable. We also assume that $\theta^N(t) \leq \lambda(t)$. We define the processes

$$\tilde{W}(t) = W(t) + \int_0^t \theta^W(s)ds,$$

$$\tilde{Q}(t) = Q(t) + \int_0^t \theta^N(s)ds,$$

$$Z^W(t) = \exp\left(-\int_0^t \theta^W(s)dW(s) - \frac{1}{2}\int_0^t (\theta^W(s))^2 ds\right)$$

and

$$Z^N(t) = \prod_{n \geq 1} \left(\left(\frac{-\theta^N(t_n)}{\lambda(t_n)} + 1 \right) I_{t_n \leq t} + I_{t_n > t} \right) \exp\left(\int_0^t \theta^N(s)ds\right).$$

Proposition 12 (Bardhan, Chao (1995))

1. The process $Z(t) = Z^W(t)Z^N(t)$ is a P -martingale with $E(Z(T)) = 1$.
2. Let Z be a density process with respect to measures $\tilde{P}$ and P . The point process N admits $(\tilde{P}, \mathcal{F})$ stochastic intensity $\lambda(t) - \theta^N$.
3. The processes $\tilde{W}$ and $\tilde{Q}$ are martingales with respect to $\tilde{P}$.
4. $\tilde{P}$ is an EMM (risk-neutral measure).

In their next paper Bardhan, Chao (1996) considered a similar market but the point processes were allowed to have unpredictable jump sizes. They proved that

such a market was always incomplete and they described (implicitly) the family of EMM for this market, but it is still not clear which EMM is preferable. Conversely if the number of stocks is equal to the number of uncertainty parameters (Brownian motions and point processes) and jump sizes are predictable then a unique EMM exists and the market is complete.

Chapter 3

Class of EMMs in jump-diffusion markets.

First we describe a model of a more general jump-diffusion market than those considered in the papers reviewed in the previous chapter.

3.1 Class of jump-diffusion semimartingales

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered probability space. In this space let $W = (W_1, \dots, W_d)$ be a $\mathbb{R}^d$ -valued Brownian motion and let $N = (N_1(t, z), \dots, N_{k-d}(t, z))$, $j = 1, \dots, k-d$ be a vector of independent marked point processes with characteristics $\lambda_j(t)$ and $\phi_j(t, z)$. Here the λ_j are the stochastic intensities of the jumps (intensities of the corresponding counting processes) and the $\phi_j(t, z)$ are the probability density functions of the jump size (absolutely continuous distributions) under the condition that this jump occurs at time t (see Bremaud (1981 p.241) for details). Each $N_j(t, z)$ is a sequence of pairs $(t_n^{(j)}, z_n^{(j)}; n \geq 1)$, where $t_n^{(j)}$ is the time of n -th jump, and $z_n^{(j)}$, the size of this jump, is a random variable with the density function $\phi_j(t, z)$. We assume that each $\lambda_j(t)$ is strictly positive and bounded.

Remark 7 *Technically, the dependence on $k-d$ point processes could be expressed as a single more complex expression in terms of a single underlying point process. The parametric form of the above model is simpler both to work with and to interpret.*

At this point we want to characterize a general class of semimartingales that can be represented as stochastic integrals with respect to the processes W and N . In the next section we will define a market model in terms of this class of semimartingales.

Proposition 13 *We fix truncation functions $h_i(z)$ (bounded with compact support and $h_i(z) = z$ in a neighborhood of zero). Let $b_i(t), \sigma_{ij}^W(t)$ be predictable processes.*

Let $\sigma_{ij}^N(t, z)$ be a predictable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}$ (measurable with respect to the predictable σ -algebra), and so that $\sigma_{ij}^N(t, z)$ as a function of z for any fixed t and ω is strictly monotone and differentiable in z . Hence it has the inverse function $(\sigma_{ij}^N)^{-1}(t, \hat{z})$ (i.e. $\hat{z} = \sigma_{ij}^N(t, z)$) with derivative $(\sigma_{ij}^N)^{-1}_z(t, \hat{z})$.

Let $\mathcal{A}_{loc}^+$ be the class of adapted processes with locally integrable variation. We assume that for every i :

$$\begin{aligned} \int_0^t |b_i(s)| ds &\in \mathcal{A}_{loc}^+, \\ \sum_{j=1}^d \int_0^t (\sigma_{ij}^W(s))^2 ds &\in \mathcal{A}_{loc}^+, \\ \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \min(|\sigma_{ij}^N(s, z)|^2, 1) \phi_j(s, z) dz ds &\in \mathcal{A}_{loc}^+. \end{aligned} \quad (3.1.1)$$

Let

$$\begin{aligned} R_i &= \int_0^t b_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s, z)) dN_j(s, z) + \\ &\quad \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\sigma_{ij}^N(s, z) - h_{ij}(\sigma_{ij}^N(s, z))) dN_j(s, z), i = 1, \dots, m. \end{aligned} \quad (3.1.2)$$

Then R is a d -dimensional semimartingale with characteristics:

$$\begin{aligned} B_i^R &= \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s, z)) \phi_j(s, z) dz \right) ds \\ C_{i_1 i_2}^R &= \sum_{j=1}^d \int_0^t \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ \nu_i^R &= \sum_{j=1}^{k-d} \frac{\lambda_j(s)}{|(\sigma_{ij}^N)_z(s, z)|} \phi_j(s, \sigma_{ij}^N(s, z)). \end{aligned} \quad (3.1.3)$$

Moreover if for any i :

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \min(|\sigma_{ij}^N(s, z)|^2, |\sigma_{ij}^N(s, z)|) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+ \quad (3.1.4)$$

then R is a special semimartingale and we can rewrite (3.1.2) as

$$R_i = \int_0^t b_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \sigma_{ij}^N(s, z) dN_j(s, z). \quad (3.1.5)$$

Additionally if for every i :

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} |\hat{\sigma}_{ij}^N(s, z)|^2 \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+ \quad (3.1.6)$$

then R is a square-integrable semimartingale.

Proof. We are looking for conditions on the predictable parameters of the processes R_i in (3.1.2) such that R_i are semimartingales. Following Jacod, Shiryaev (1987, II.2) every semimartingale has a canonical representation:

$$R_i = R_i(0) + B_i^R + R_i^c + h_i(z) \star (\mu_i^R - \nu_i^R) + (z - h_i(z)) \star \mu_i^R. \quad (3.1.7)$$

where R_i^c is the continuous part of the martingale part R_i^M , μ_i^R is a jump random measure with compensator ν_i^R and $h_i(z)$ is a truncation function. Parameters of the representation (3.1.7) are subject to several conditions, which we will impose shortly.

We want to rewrite (3.1.2) in the form (3.1.7) so we can determine the characteristics of the process R_i and the appropriate conditions.

Let $N_j^c(t)$ be the counting process with intensity $\lambda_j(t)$ corresponding to the marked point process $N_j(t, z)$. Let also $\hat{z} = \hat{z}_t(z) = \sigma_{ij}^N(t, z)$ be a random function of z for every fixed time t and

$$\hat{z}_n^{(j)} = \sigma_{ij}^N(t_n^{(j)}, z_n^{(j)}).$$

Now we introduce a marked point process

$$\hat{N}_{ij}(t, \hat{z}) = \sum_{n=1}^{N_j^c(t)} \sigma_{ij}^N(t_n^{(j)}, z_n^{(j)}) = \sum_{n=1}^{N_j^c(t)} \hat{z}_n^{(j)} \quad (3.1.8)$$

It is clear that the characteristics (the intensity and the jump size distribution density) of $\hat{N}_{ij}(t, \hat{z})$ are going to be

$$\lambda_j(s) \quad \text{and} \quad |(\sigma_{ij}^N)^{-1}(t, \hat{z})| \phi_j(t, \hat{z}).$$

In the expression (3.1.2) we can rewrite the jump part of the process R_i as a sum of integrals with respect to the marked point processes $\hat{N}_{ij}(t, \hat{z})$;

$$\begin{aligned} & \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s, z)) dN_j(s, z) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\sigma_{ij}^N(s, z) - h_{ij}(\sigma_{ij}^N(s, z))) dN_j(s, z) \\ &= \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\hat{z}) d\hat{N}_j(s, \hat{z}) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\hat{z} - h_{ij}(\hat{z})) d\hat{N}_j(s, \hat{z}). \end{aligned} \quad (3.1.9)$$

From (3.1.2), using (3.1.9) we obtain

$$\begin{aligned} R_i &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) \\ &+ \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} h_{ij}(\hat{z}) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} \right) ds \\ &+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\hat{z}) \left(d\hat{N}_{ij}(s, \hat{z}) - \lambda_j(s) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \right) \\ &+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\hat{z} - h_{ij}(\hat{z})) d\hat{N}_{ij}(s, \hat{z}). \end{aligned} \quad (3.1.10)$$

Hence the Jacod-Shiryaev characteristics (B, C, ν) of the process R are

$$\begin{aligned} B_i^R(h) &= \int_0^t \left(b_i(s) + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} h_{ij}(\hat{z}) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} \right) ds \\ C_{i_1 i_2}^R &= \sum_{j=1}^d \int_0^t \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ \nu_i^R &= \sum_{j=1}^{k-d} \lambda_j(s) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}), \end{aligned} \quad (3.1.11)$$

which is equivalent to (3.1.3).

Now we recall the conditions required for “good” characteristics B^R, C^R, ν^R so that R is a semimartingale.

1. B^R is a $\mathbf{F}$ -predictable process of finite variation over finite intervals and $B_i(0) = 0$.
2. C^R is a $\mathbf{F}$ -predictable, continuous, $C^R(0) = 0$, and $C^R(t) - C^R(s)$ is a nonnegative symmetric matrix for $s \leq t$.
3. ν^R is a $\mathbf{F}$ -predictable random measure with $\nu^R(0, z) = \nu^R(t, 0) = 0$ and such that

$$\int_0^t \int_{\mathbb{R}} \min(|x|^2, 1) d\nu^R \in \mathcal{A}_{loc}^+$$

From (3.1.11) we can see that in our case these three conditions are equivalent to (3.1.1). Conditions (3.1.4) and (3.1.6) are straightforward from the definitions and (3.1.1). $\square$

Remark 8 1. *In this form R has no predictable jumps (it is quasi-left continuous) since the drift function is continuous. This means that this model cannot be used to describe markets where there are jumps at predictable times even if the jump sizes at those times are unpredictable – e.g. weekly federal decisions. It may be possible to add a predictable jump component to the continuous drift term in the model to compensate.*

2. *By Jacod, Shiryaev (1987, II.2) one can write the characteristics of the semimartingale R in the following form*

$$\begin{aligned} B^R(t) &= \int_0^t b^R(s) dA^R(s) + \int_0^t \int_{\mathbb{R}} (h(z) - z) d\nu^R; \\ C^R(t) &= \int_0^t c^R(s) dA^R(s); \\ d\nu^R(t, z) &= F^R(t, dz) dA^R(t), \end{aligned} \tag{3.1.12}$$

where $A^R \in \mathcal{A}_{loc}^+$, b^R is a predictable processes, c^R is a predictable non-negative symmetric matrix and F^R is a transition kernel. The above proposition also shows that in our case, (3.1.3), we have the typical choice of $A^R(t) = t$ (e.g. for Levy processes, diffusions, Ito processes and etc.). For R_i in the form (3.1.2) one can also interpret $b_i(t)$ as a drift rate, $\sigma_{ij}^W(t)$ for every j as diffusion coefficients and $\sigma_{ij}^N(t, z) \lambda_j(t) \phi(t, z)$ for every j as local jump coefficients.

Definition 18 *Under conditions (3.1.1) we call R in (3.1.2) a **jump-diffusion semimartingale**. We also call the conditions (3.1.1) **jump-diffusion semimartingale conditions**.*

At this point we say a few words about the class of jump-diffusion semimartingales.

We can compare our definition of jump-diffusion semimartingales to the one given by Jacod-Shiryaev (1987, III.2c). Our definition is more general: we do not require the corresponding counting process to be a one-dimensional Poisson process.

By the uniqueness of the Brownian motion W and of the multivariate marked point process N with given compensator ν , we can apply the fundamental representation theorem (Jacod-Shiryaev 1987 III.4d). We have that every local martingale M adapted to the joint filtration of W, N has the $(W, N - \nu)$ -representation property:

$$M = M_0 + \int f^W dW + \int \int f^N dN - \int \int f^N d\nu,$$

where f^W and f^N are predictable such that

$$\begin{aligned} \int (f^W)^2 &\in \mathcal{A}_{loc}^+, \\ \int \int |f^N| d\nu &\in \mathcal{A}_{loc}^+. \end{aligned} \tag{3.1.13}$$

This means that under (3.1.13) every semimartingale R with decomposition $R = R_0 + A + M$ where A is a process of finite variation and M is a local martingale can be represented in the following form:

$$R = R_0 + \hat{A} + \int f^W dW + \int \int f^N dN,$$

where $\hat{A} = A - \int \int f^N d\nu$ is obviously a process of finite variation. Hence almost every semimartingale adapted to the joint filtration of W, N can be represented in the form given by (3.1.2). Jump-diffusion semimartingale conditions are more restrictive than (3.1.13) (mainly because we require f^N to be monotone and differentiable) but it is clear that the class of jump-diffusion semimartingales is a very wide subclass of semimartingales.

3.2 Market with jump-diffusion semimartingales.

We consider the discounted market consisting of the bond $B \equiv 1$ and m stocks S_i . We assume $m \leq k$. We consider a jump-diffusion semimartingale R given by (3.1.2) as being the driving random component in the market.

There are two generally accepted models for stock prices:

$$S_i = \mathcal{E}(R_i),$$

and

$$S_i = \exp(R_i).$$

The models are similar to each other, but each has its separate advantages. The first model was studied for example in Bardhan-Chao(1996) and requires the jump sizes to be greater than -1 so the Dolean-Dade exponent is well defined and can be represented in terms of the usual exponent. In this model if R_i is a local martingale then so is S_i . In the second model (see, for example Shiryaev (1999)) the stock price is always positive. But in this case if R_i is a local martingale S_i may not be one.

We will concentrate on the second model. In that case the price of the i -th stock S_i $i = 1, \dots, m$ is governed by:

$$\begin{aligned} S_i = & \exp \left(\int_0^t b_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\sigma_{ij}^N(s, z)) dN_j(s, z) \right) \\ & \times \exp \left(\sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (\sigma_{ij}^N(s, z) - h_{ij}(\sigma_{ij}^N(s, z))) dN_j(s, z) \right), i = 1, \dots, m. \end{aligned} \quad (3.2.1)$$

where $b_i(t), \sigma_{ij}^W(t), \sigma_{ij}^N(t, z)$ satisfy the jump-diffusion conditions for fixed truncation functions h_{ij} .

The main idea of the EMM approach is to find an EMM such that S_i is a local martingale. If for some $\hat{R}_i$ we have a representation $S_i = \mathcal{E}(\hat{R}_i)$ then after the EMM transformation we will have that $\hat{R}_i$ is a local martingale, as is S_i . So our next step is to use (3.2.1) to obtain a formula for $\hat{R}_i$.

Proposition 14 *Assume that for every i, j and t :*

$$\int_0^t \int_{\mathbb{R}} \left(|\sigma_{ij}^N(s, z)| I_{|\sigma_{ij}^N(s, z)| \leq 1} + e^{\sigma_{ij}^N(s, z)} I_{|\sigma_{ij}^N(s, z)| > 1} \right) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+. \quad (3.2.2)$$

If we write the stock price in the form $S_i = \mathcal{E}(\hat{R}_i)$ then

$$S_i = \mathcal{E} \left(\int_0^t \hat{b}_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dq_j(s, z) \right), \quad (3.2.3)$$

where

$$\hat{b}_i(t) = b_i(t) + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W(t))^2 + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(t, z) - h_{ij}(\sigma_{ij}^N(s, z))) \phi_j(t, z) dz$$

and

$$dq_j(t, z) = dN_j(t, z) - \lambda_j(t) \phi_j(t, z) dz dt.$$

where $\hat{\sigma}_{ij}^N(t, z) = e^{\sigma_{ij}^N(t, z)} - 1$. Under the condition (3.2.2) $\hat{R}$ is a special semimartingale with characteristics

$$\begin{aligned} B_i^{\hat{R}} &= \int_0^t \left(b_i(s) + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W(s))^2 \right) ds \\ &\quad + \int_0^t \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(s, z) - h_{ij}(\sigma_{ij}^N(s, z))) \phi_j(s, z) dz ds, \\ C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^d \int_0^t \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ \nu_i^{\hat{R}} &= \sum_{j=1}^{k-d} \lambda_j(s) \frac{e^{-\sigma_{ij}^N(s, z)}}{|(\sigma_{ij}^N)_z(s, z)|} \phi_j(s, e^{\sigma_{ij}^N(t, z)} - 1). \end{aligned} \quad (3.2.4)$$

Proof. Recalling the characteristics of the semimartingale R_i given in (3.1.11) and the definition of the Dolean-Dade exponent we obtain (Shiryaev (1999, VII.3d)):

$$\begin{aligned} \hat{R}_i &= B_i^R + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \frac{1}{2} C_{ii}^R \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} h_{ij}(\hat{z}) \left(d\hat{N}_{ij}(s, \hat{z}) - \lambda_j(s) |(\sigma_{ij}^N)_{\hat{z}}^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \right) \\ &\quad + \sum_{j=1}^{k-d} \left(\int_0^t \int_{\mathbb{R}} (\hat{z} - h_{ij}(\hat{z})) d\hat{N}_{ij}(s, \hat{z}) + \int_0^t \int_{\mathbb{R}} (e^{\hat{z}} - 1 - \hat{z}) d\hat{N}_{ij}(s, \hat{z}) \right) \end{aligned} \quad (3.2.5)$$

To simplify the above expression we can use the condition (3.2.2) or equivalently :

$$\int_0^t \int_{\mathbb{R}} (|\hat{z}| I_{|\hat{z}| \leq 1} + e^{\hat{z}} I_{|\hat{z}| > 1}) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \in \mathcal{A}_{loc}^+$$

for every i, j and t . Under this condition we can rewrite (3.2.5) as:

$$\begin{aligned} \hat{R}_i &= B_i^{\hat{R}} + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (e^{\hat{z}} - 1) \left(d\hat{N}_{ij}(s, \hat{z}) - \lambda_j(s) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \right), \end{aligned} \quad (3.2.6)$$

where

$$\begin{aligned} B_i^{\hat{R}} &= B_i^R + \frac{1}{2} C_{ii}^R + \int_0^t (b_i(s) + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W(s))^2) \\ &= \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} (e^{\hat{z}} - 1 - h_{ij}(\hat{z})) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \\ &\quad + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (e^{\hat{z}} - 1 - h_{ij}(\hat{z})) |(\sigma_{ij}^N)^{-1}(s, \hat{z})| \phi_j(s, \hat{z}) d\hat{z} ds \\ &= \int_0^t \left(b_i(s) + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W(s))^2 \right) ds \\ &\quad + \int_0^t \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (e^{\sigma_{ij}^N(s, z)} - 1 - h_{ij}(\sigma_{ij}^N(s, z))) \phi_j(s, z) dz ds. \end{aligned} \quad (3.2.7)$$

We can see that (3.2.6) is equivalent to

$$\begin{aligned} \hat{R}_i &= B_i^{\hat{R}} + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) + \\ &\quad \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) (dN_{ij}(s, z) - \lambda_j(s) \phi_j(s, z) dz ds), \end{aligned} \quad (3.2.8)$$

Thus under condition (3.2.2) $\hat{R}_i$ become a special semimartingale.

The expression (3.2.6) is not a canonical representation of a special semimartingale $\hat{R}_i$. To obtain the characteristics of the process $\hat{R}$ we rewrite it in canonical form. It

is easy to see that if $\hat{z} = \hat{\sigma}_{ij}^N(t, z)$ then for $\hat{z} > -1$

$$z = (\sigma_{ij}^N)^{-1}(t, \log(\hat{z} + 1)).$$

Now if we substitute σ_{ij}^N with $\hat{\sigma}_{ij}^N$ we can apply Proposition 13 for a special semimartingale $\hat{R}_i$ (condition (3.2.2) is an analog of (3.1.4)). We obtain the following characteristics of $\hat{R}$:

$$\begin{aligned} B_i^{\hat{R}} &= \int_0^t \left(b_i(s) + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W(s))^2 \right) ds \\ &\quad + \int_0^t \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (e^{\sigma_{ij}^N(s, z)} - 1 - h_{ij}(\sigma_{ij}^N(s, z))) \phi_j(s, z) dz ds, \\ C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^d \int_0^t \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ \nu_i^{\hat{R}} &= \sum_{j=1}^{k-d} \lambda_j(s) \frac{|(\sigma_{ij}^N)^{-1}(s, \log(\hat{z} + 1))|}{\hat{z} + 1} \phi_j(s, \hat{z}), \end{aligned}$$

which is equivalent to (3.2.4). Hence using (3.2.8) we can rewrite (3.2.1) as (3.2.3).

□

Remark 9 1. We can note here that under condition (3.2.2) R remains a semimartingale, although not necessary a special semimartingale.

2. Since $\hat{R}_i$ is a special semimartingale then by the property of the stochastic exponent, $S = \mathcal{E}(\hat{R})$ is also a special semimartingale. We can also call R **exponentially special** (Kallsen, Shiryaev (2002)).

3. We can avoid the condition (3.2.2) and continue to work in the more general settings of a semimartingale. However the technical difficulties associated with that approach seem unwarranted.

Now define

$$Q_{ij}(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dq_j(s, z).$$

We can rewrite (3.2.3) as the stochastic differential equation:

$$\frac{dS_i}{S_i(t-)} = \hat{b}_i(t) dt + \sum_{j=1}^d \sigma_{ij}^W(t) dW_j + \sum_{j=1}^{k-d} dQ_{ij}(t).$$

Since most methods of optimizing the equivalent martingale measure deal directly with the process S itself and not with its (stochastic) logarithm we want to obtain the characteristics of S using the characteristics of its (stochastic) logarithm. First we recall the stochastic differential equation for S in its integral form:

$$\begin{aligned} S_i(t) &= S_i(0) + \int_0^t \hat{b}_i(s) S_i(s-) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) S_i(s-) dW_j \\ &\quad + \sum_{j=1}^{k-d} \int_0^t S_i(s-) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dq_j(s, z). \end{aligned} \quad (3.2.9)$$

Knowing that $S_i(s-)$ is just a predictable process we can obtain the following.

Corollary 1 *Under the conditions of Proposition 14 the characteristics of S are:*

$$\begin{aligned} B_i^{\hat{R}} &= \int_0^t \hat{b}_i(s) S_i(s-) ds, \\ C_{i_1 i_2}^{\hat{R}} &= \sum_{j=1}^d \int_0^t S_{i_1}(s-) S_{i_2}(s-) \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ \nu_i^{\hat{R}} &= \sum_{j=1}^{k-d} \lambda_j(s) S_i(s-) \frac{e^{-\sigma_{ij}^N(s, z)}}{|(\sigma_{ij}^N)_z(s, z)|} \phi_j(s, e^{\sigma_{ij}^N(t, z)} - 1). \end{aligned} \quad (3.2.10)$$

We want to mention here a very important property of our model which we will use later on. We can rewrite (3.2.9) as

$$S_i = S_i^0 + S_i^M + S_i^A,$$

where

$$\begin{aligned} S_i^A &= \int_0^t \hat{b}_i(s) S_i(s-) ds; \\ S_i^M &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) S_i(s-) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d} \int_0^t S_i(s-) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dq_j(s, z). \end{aligned} \quad (3.2.11)$$

Under the jump-diffusion conditions and (3.2.2) S^M is a square-integrable local martingale. We introduce the predictable processes

$$\begin{aligned}\varsigma_{i_1 i_2}(t) &= \frac{d\langle S_{i_1}^M, S_{i_2}^M \rangle}{S_{i_1}(t-)S_{i_2}(t-)dt} \\ &= \sum_{j=1}^d \sigma_{i_1 j}^W(t)\sigma_{i_2 j}^W(t) + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{i_1 j}^N(t, z) \hat{\sigma}_{i_2 j}^N(t, z) \phi_j(t, z) dz; \\ \hat{\varsigma}_{i_1 i_2}(t) &= \varsigma_{i_1 i_2}(t) S_{i_1}(t-) S_{i_2}(t-).\end{aligned}$$

Proposition 15 *Let S be given by (3.2.9). Then*

$$S_i = S_i^0 + S_i^M + \int_0^t \frac{\hat{b}_i(s)}{S_i(s-)\varsigma_{ii}(s)} d\langle S^M, S^M \rangle. \quad (3.2.12)$$

Proof. Since S^M is a square-integrable local martingale we obtain by (3.2.11):

$$\begin{aligned}\langle S_i^M, S_i^M \rangle &= \sum_{j=1}^d \int_0^t (\sigma_{ij}^W(s) S_i(s-))^2 ds \\ &\quad + \sum_{j=1}^{k-d} \int_0^t (S_i(s-))^2 \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(s, z))^2 \phi_j(s, z) dz ds \\ &= \int_0^t (S_i(s-))^2 \left(\sum_{j=1}^d (\sigma_{ij}^W(s))^2 + \sum_{j=1}^{k-d} \lambda_j(s) \int_{\mathbb{R}} (\hat{\sigma}_{ij}^N(s, z))^2 \phi_j(s, z) dz \right) ds; \\ \langle S_{i_1}^M, S_{i_2}^M \rangle &= \int_0^t S_{i_1}(s-) S_{i_2}(s-) \sum_{j=1}^d \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ &\quad + \int_0^t \sum_{j=1}^{k-d} S_{i_1}(s-) S_{i_2}(s-) \lambda_j(s) \int_{\mathbb{R}} \hat{\sigma}_{i_1 j}^N(s, z) \hat{\sigma}_{i_2 j}^N(s, z) \phi_j(s, z) dz ds\end{aligned} \quad (3.2.13)$$

Hence they have the following expression for S^A ,

$$S_i^A = \int_0^t \frac{\hat{b}_i(s)}{S_i(s-)\varsigma_{ii}(s)} d\langle S_i^M, S_i^M \rangle = \int_0^t \frac{\hat{b}_i(s) S_i(s-)}{\hat{\varsigma}_{ii}(s)} d\langle S_i^M, S_i^M \rangle, \quad (3.2.14)$$

and hence we obtain (3.2.12). $\square$

We recall that by the property of Dolean-Dade exponent, if $\hat{R}_i$ is a local martingale under some measure $\tilde{P}$, then S_i also is a local martingale. Hence if under $\tilde{P}$, $B_i^{\hat{R}} = 0$, then S_i becomes a local martingale under this measure (since $\hat{R}_i - \hat{B}_i^{\hat{R}}$ is a local martingale). In the next section we are looking for this equivalent martingale measure $\tilde{P}$. We will see that in general we may have more than one equivalent martingale measure. We will describe the family of equivalent martingale measures $\tilde{P}$ for this market and in the next chapter find the "optimal" measure under additional criteria.

3.3 The general class of equivalent martingale measures.

If not otherwise specified all (local) martingales from now on will be considered to be (local) martingales with respect to the original measure P .

We will consider the model from the previous section on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$. Stock prices are given by (3.2.3) under jump-diffusion conditions and (3.2.2).

There are number of ways to obtain the class of equivalent martingale measures for a special semimartingale. The most general way is described in Jacod-Shiryaev (1987). They considered characteristics $(\tilde{B}, \tilde{C}, \tilde{\nu})$ under transformed measure $\tilde{P}$ and if $\tilde{B} = 0$ this special semimartingale becomes a local martingale and the measure $\tilde{P}$ become an EMM. Since we know the characteristics of the special semimartingale $\hat{R}$ we can directly apply the Jacod-Shiryaev result and using $\tilde{B} = 0$ obtain the class of EMMs. But this approach seems very technical and general, and does not use the benefits of the specific structure of the jump-diffusion model. So we will instead follow Bardhan-Chao (1996) keeping in mind that our market model is different from theirs.

The best way to find the equivalent martingale measure $\tilde{P}$ is through its density process, $Z(t)$. In other words we are looking for a Z such that $\hat{R}_i Z$ (and therefore $S_i Z$) are local martingales for each i . As a density process $Z(t)$ is a local martingale itself and hence in our case there exists a local martingale $M(t)$ such that $Z = \mathcal{E}(M)$

(see for example Shiryaev, VII.3). Consequently any equivalent measure $\tilde{P}$ can be uniquely specified by M . From now on we are looking only for equivalent measures such that M is a square-integrable local martingale. We need this restriction to use the quadratic variation of M .

Let $g_j = g_j(t, z)$ be a predictable function. For each i and j we introduce the predictable processes

$$\alpha_{ij}^g(t) = \lambda_j(t) \int_{\mathbb{R}} g_j(t, z) \hat{\sigma}_{ij}^N(t, z) \phi_j(t, z) dz.$$

If we assume that:

$$\int_0^t \int_{\mathbb{R}} |g_j(s, z)|^2 \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+, \quad (3.3.1)$$

then from condition (3.2.2) and the Cauchy inequality, $\alpha_{ij}^g(t)$ is well defined and

$$\int_0^t \alpha_{ij}^g(s) ds \in \mathcal{A}_{loc}^+.$$

Now let Σ be the m by k matrix where the first d rows are defined by the elements (σ_{ij}^W) and the remaining $k - d$ rows by the elements (α_{ij}^g) .

The following proposition will provide us with the representation for M corresponding to an EMM. It is an analog of a result of Schweizer(1995), but in our case we have the exponential form of stock prices and we are looking for M in a slightly different form.

Proposition 16 *We assume that the jump-diffusion semimartingale conditions along with (3.2.2) are fulfilled. Further we call these conditions the **jump-diffusion special semimartingale conditions**. We also require that $\det(\Sigma \Sigma^{tr}) \neq 0$. Let $(\theta_j^W(t))_{j=1}^d$ and $(\theta_j^N(t))_{j=1}^{k-d}$ be predictable processes such that:*

$$\hat{b}_i(t) = \sum_{j=1}^d \sigma_{ij}^W(t) \theta_j^W(t) + \sum_{j=1}^{k-d} \alpha_{ij}^g(t) \theta_j^N(t). \quad (3.3.2)$$

If $g_j(t, z)$ is predictable such that (3.3.1) holds then $\tilde{P}$ is an equivalent martingale measure if

$$M(t) = - \sum_{j=1}^d \int_0^t \theta_j^W(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} g_j(s, z) \theta_j^N(s) dq_j(s, z) + H(t), \quad (3.3.3)$$

where H is a local martingale orthogonal for every i to the martingale part $\hat{R}_i^M$ of the process $\hat{R}_i$.

Proof. We know that $d\langle W_j, W_j \rangle = dt$ and $\langle q_j(t, z), q_j(t, z) \rangle = \lambda_j(t)\phi_j(t, z)$. From the properties of “ $\langle \rangle$ ” (see Appendix A) it is easy to see that:

$$\begin{aligned} & \left\langle \sum_{j=1}^d \int_0^t \theta_j^W(s) dW_j + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} g_j(s, z) \theta_j^N(s) dq_j, \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(s) \right\rangle \\ &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) \theta_j^W(s) ds + \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \theta_j^N(s) \int_{\mathbb{R}} g_j(s, z) \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz ds, \end{aligned}$$

which is equal to $\int_0^t \hat{b}_i(s) ds$.

Shiryaev(1999, VII.3) (corollary of the Girsanov theorem) showed that R_i is a local martingale if

$$\int_0^t \hat{b}_i(s) ds + \left\langle M, \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(t) \right\rangle = 0. \quad (3.3.4)$$

This is equivalent to

$$\begin{aligned} & \left\langle M + \sum_{j=1}^d \int_0^t \theta_j^W(s) dW_j + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} g_j(s, z) \theta_j^N(s) dq_j, \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij} \right\rangle \\ &= 0. \end{aligned}$$

Define the local martingale H by

$$H(t) = M + \sum_{j=1}^d \int_0^t \theta_j^W(s) dW_j + \sum_{j=1}^{k-d} \int_0^t \theta_j^N(s) \int_{\mathbb{R}} g_j(s, z) dq_j(s, z). \quad (3.3.5)$$

This proves the proposition. $\square$

Again consider $Z = \mathcal{E}(M)$. Since M is a square integrable martingale it has a representation (Jacod-Shiryaev (1987, III.4)):

$$M(t) = - \sum_{j=1}^d \int_0^t v_j(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \eta_j(s, z) dq_j(s, z) \quad (3.3.6)$$

for some predictable $v_j(t)$ and $\eta_j(t, z)$ such that:

$$\begin{aligned} \int_0^t (v_j(s))^2 ds &\in \mathcal{A}_{loc}^+, \\ \int_0^t \lambda_j(s) \int_{\mathbb{R}} |\eta_j(s, z)|^2 \phi_j(s, z) dz ds &\in \mathcal{A}_{loc}^+. \end{aligned} \quad (3.3.7)$$

Since $\mathcal{E}(M)$ is well defined we know that $\Delta M > -1$ and hence $-\eta_j(s, t) > -1$. From the above conditions we also have that:

$$\int_0^t \lambda_j(s) \int_{\mathbb{R}} |\eta_j(s, z) \hat{\sigma}_{ij}^N(s, z)| \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+.$$

The next proposition uses Proposition 16 to establish when parameters v_j and η_j leads to a martingale density.

Proposition 17 *We assume that all the conditions of Proposition 16 along with (3.3.7) hold. The process*

$$Z = \mathcal{E} \left(- \sum_{j=1}^d \int_0^t v_j(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \eta_j(s, z) dq_j(s, z) \right) \quad (3.3.8)$$

is the density process corresponding to an equivalent martingale measure $\tilde{P}$ if the following conditions holds for every i, j and t :

$$\begin{aligned} \hat{b}_i(t) - \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) - \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \eta_j(t, z) \phi_j(t, z) dz &= 0; \\ \eta_j(t, z) &< 1 \end{aligned} \quad (3.3.9)$$

Proof.

From (3.3.5) and (3.3.6) we have:

$$\begin{aligned} H(t) &= \sum_{j=1}^d \int_0^t (\theta_j^W(s) - v_j(s)) dW_j \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (g_j(s, z) \theta_j^N(s) - \eta_j(s, z)) dq_j(s, z). \end{aligned} \quad (3.3.10)$$

Now from Proposition 16 $\tilde{P}$ is a martingale measure if

$$\left\langle H(t), \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(t) \right\rangle = 0 \quad (3.3.11)$$

Hence we have

$$\begin{aligned} 0 &= \left\langle H(t), \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(t) \right\rangle \\ &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) (\theta_j^W(s) - v_j(s)) ds \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) (g_j(s, z) \theta_j^N(s) - \eta_j(s, z)) \lambda_j(t) \phi_j(t, z) dz ds \\ &= \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) (\theta_j^W(s) - v_j(s)) ds + \sum_{j=1}^{k-d} \int_0^t \alpha_{ij}^g(s) \theta_j^N(s) ds \\ &\quad - \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \eta_j(s, z) \phi_j(t, z) dz ds \\ &= \int_0^t \hat{b}_i(s) ds - \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) v_j(s) ds \\ &\quad - \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \eta_j(s, z) \phi_j(t, z) dz ds. \end{aligned} \quad (3.3.12)$$

Since the above equation holds for every $t \in [0, T]$ we can now remove the integration and obtain the condition (3.3.9). $\square$

We introduce two vectors of positive predictable processes $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t, z))_{j=1}^{k-d}$ such that $\eta_j(s, t) = 1 - r_j(t) \psi_j(t, z)$. We rewrite (3.3.9) as:

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) (1 - r_j(t) \psi_j(t, z)) \phi_j(t, z) dz. \\ r_j(t) \psi_j(t, z) &> 0, \end{aligned} \quad (3.3.13)$$

for every i, j and t .

The next result shows that after the measure transformation (3.3.8) the processes W_j and Q_{ij} become local martingales. We will also obtain the formula for S_i with respect to the transformed processes.

Proposition 18 *Under conditions of Proposition 16, (3.3.7) and (3.3.13):*

$$\tilde{W}_j(t) = W_j(t) + \int_0^t v_j(s) ds \quad (3.3.14)$$

are $\tilde{P}$ -local martingales for each $j = 1, \dots, d$. Moreover $\tilde{W}_j(t)$ is a Brownian motion under $\tilde{P}$. The processes

$$\tilde{Q}_{ij}(t) = Q_{ij}(t) + \int_0^t \lambda_j(s) \int_{\mathbb{R}} (1 - r_j(s) \psi_j(s, z)) \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz dt \quad (3.3.15)$$

are $\tilde{P}$ -local martingales for $i = 1, \dots, m$ and $j = 1, \dots, k - d$. Moreover $\tilde{\lambda}_j(t) = r_j(t) \lambda_j(t)$, $\tilde{\phi}_j(t, z) = \psi_j(t, z) \phi_j(t, z) dz$ are the new characteristics (the intensity and the density function of the jump distribution) for point processes N_j ($j = 1, \dots, k - d$) under the new measure $\tilde{P}$. The stock prices follow the equations:

$$S_i(t) = \mathcal{E} \left(\sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) d\tilde{W}_j + \sum_{j=1}^{k-d} \tilde{Q}_{ij}(t) \right) \quad (3.3.16)$$

and hence they are also $\tilde{P}$ -local martingales.

Proof. To prove that $\tilde{W}_j$ is a $\tilde{P}$ -local martingale it is enough to show that $\mathcal{E}(\tilde{W}_j)Z$ is a martingale. Indeed since Q_{ij} and W_j are orthogonal, and using $Z = \mathcal{E}(M)$,

$$\begin{aligned} \mathcal{E}(\tilde{W}_j)Z &= \mathcal{E} \left(W_j(t) + \int_0^t v_j(s) ds \right) \mathcal{E} \left(- \sum_{j=1}^d \int_0^t v_j(s) dW_j \right) \\ &= \mathcal{E} \left(W_j + \int_0^t v_j ds - \sum_{j=1}^d \int_0^t v_j dW_j + [W_j + \int_0^t v_j ds, - \sum_{j=1}^d \int_0^t v_j dW_j] \right) \\ &= \mathcal{E} \left(W_j(t) - \sum_{j=1}^d \int_0^t v_j(s) dW_j \right), \end{aligned} \quad (3.3.17)$$

which is a local martingale. Since $[\tilde{W}_j, \tilde{W}_j] = [W_j, W_j] = t$ we have that $\tilde{W}_j$ is Brownian motion. Similarly we can prove that $\tilde{Q}_{ij}(t)$ is a $\tilde{P}$ -local martingale.

Now recalling the definition of $Q_{ij}(t)$ and $q(t, dz)$ we have that

$$\tilde{Q}_{ij}(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dN_j - \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \lambda_j(s) \phi_j(s, z) dz ds$$

$$\begin{aligned}
& - \int_0^t \lambda_j(s) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz ds \\
& = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dN_j - \int_0^t \lambda_j(s) r_j(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \psi_j(s, z) \phi_j(s, z) dz ds.
\end{aligned}$$

Since $\tilde{Q}_{ij}(t)$ is a $\tilde{P}$ -local martingale the $\tilde{P}$ -compensator of $\int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dN_j$ is

$$\int_0^t \lambda_j(s) r_j(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \psi_j(s, z) \phi_j(s, z) dz ds. \quad (3.3.18)$$

Now using uniqueness of the compensator we can find versions of $r_j(t)$ and $\psi_j(t, z)$ such that $\tilde{\lambda}_j(t) = r_j(t) \lambda_j(t)$ and $\tilde{\phi}_j = \psi_j(t, z) \phi_j(t, z) dz$ are the new characteristics of the point processes N_j under the new measure $\tilde{P}$.

Using (3.3.14) and (3.3.15) we have

$$\begin{aligned}
& \mathcal{E} \left(\sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) d\tilde{W}_j + \sum_{j=1}^{k-d} \tilde{Q}_{ij}(t) \right) \\
& = \mathcal{E} \left(\sum_{j=1}^d \left(\int_0^t \sigma_{ij}^W dW_j + \int_0^t v_j ds \right) + \sum_{j=1}^{k-d} (Q_{ij} + \int_0^t \lambda_j \int_{\mathbb{R}} (1 - r_j \psi_j) \hat{\sigma}_{ij}^N \phi_j(s, z) dz ds) \right).
\end{aligned}$$

By condition (3.3.13) this is equal to:

$$\mathcal{E} \left(\int_0^t b_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j + \sum_{j=1}^{k-d} Q_{ij}(t) \right) = S_i. \quad \square$$

Remark 10 Since $\tilde{\phi}_j$ is a probability density function of jumps we obtain the following condition for $j = 1, \dots, k-d$:

$$\int_{\mathbb{R}} \psi_j(t, z) \phi_j(t, z) dz = 1. \quad (3.3.19)$$

We call (3.3.7), (3.3.13) and (3.3.19) the **general EMM conditions**. We can now conclude that density process Z and hence the corresponding EMM can be determined by parameters $(v_j(t))_{j=1}^d$, $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t, z))_{j=1}^{k-d}$. Under the general

EMM conditions the density process, $Z(t)$, is given by

$$\mathcal{E} \left(- \sum_{j=1}^d \int_0^t v_j(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (1 - r_j(t) \psi_j(t, z)) dq_j(s, z) \right) \quad (3.3.20)$$

$$\begin{aligned} &= \exp \left(- \sum_{j=1}^d \left(\int_0^t v_j(s) dW_j - \frac{1}{2} \int_0^t v_j^2(s) ds \right) \right) \\ &\times \exp \left(\sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \log((r_j(s) \psi_j(s, z)) dN_j(s, z) \right) \\ &\times \exp \left(\int_0^t \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) \lambda_j(s) \phi_j(s, z) dz ds \right). \end{aligned} \quad (3.3.21)$$

whenever the following is true

$$\int_0^t \int_{\mathbb{R}} \log((r_j(s) \psi_j(s, z)) dN_j(s, z) \in \mathcal{A}_{loc}^+.$$

Since the equations on the parameters v_j, r_j, ψ_j , (3.3.13), may have more than one solution, the general EMM conditions do not provide us always with a unique EMM. Hence for option pricing we have to find an EMM optimal in some sense so we can find the value of the contingent claim with respect to this measure.

Chapter 4

Optimal equivalent martingale measures.

The market described in the previous chapter is incomplete since no unique EMM exists. As we mentioned in the conclusion of the previous section, we are going to somehow optimize the choice of the characteristics $(v_j(t))_{j=1}^d$, $(r_j(t))_{j=1}^{k-d}$ and $(\psi_j(t, z))_{j=1}^{k-d}$ to specify a "best" EMM. There are a number of ways to do this. We assume in this chapter that the stocks are discounted semimartingales given by (3.2.3) and the jump-diffusion special semimartingale conditions and the general EMM conditions hold.

4.1 EMM based on utility maximization.

Recently, Kallsen (1999, 2002) introduced and developed portfolio optimization by maximizing expected local utility. Now this is one of the most common approaches for optimization over a class of equivalent martingale measures.

We will consider the market of discounted stocks S with characteristics (3.2.10). Let $\pi = (\pi_1, \dots, \pi_m)$ be a trading strategy and G_π be the gain process for the strategy π . To apply Kallsen's results (which employ characteristics of S instead of $\hat{R}$) we consider strategies of the form:

$$\pi_i = \frac{\hat{\pi}_i}{S_i(s-)}.$$

We call $\hat{\pi}$ an **exponential strategy** since it specifies holdings as a proportion of past stock values.

We assume that

$$\begin{aligned} & \sum_{i=1}^m \int_0^t |\hat{\pi}_i(s) \hat{b}_i(s)| ds + \sum_{j=1}^d \sum_{i_1, i_2}^m \int_0^t \hat{\pi}_{i_1}(s) \hat{\pi}_{i_2}(s) \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ & + \sum_{i=1}^m \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \min((\hat{\pi}_i(s) \hat{\sigma}_{ij}^N(s, z))^2, |\hat{\pi}_i(s) \hat{\sigma}_{ij}^N(s, z)|) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+ \end{aligned} \quad (4.1.1)$$

Definition 19 (Kallsen (2002))

1. We call a function $u : \mathbb{R} \rightarrow \mathbb{R}$ a **utility function** if
 - (i) u is twice continuously differentiable;
 - (ii) the derivatives u', u'' are bounded and $\lim_{x \rightarrow \infty} u'(x) = 0$;
 - (iii) $u(0) = 0, u'(0) = 1$;
 - (iv) $u'(x) > 0$ for any $x \in \mathbb{R}$;
 - (v) $u''(x) < 0$ for any $x \in \mathbb{R}$.
2. For any exponential strategy $\hat{\pi}$, the random variable

$$\begin{aligned} \xi_{\hat{\pi}}(t) &= \sum_{i=1}^m \hat{\pi}_i(t) \hat{b}_i(t) + \frac{u''(0)}{2} \sum_{j=1}^d \sum_{i_1, i_2} \hat{\pi}_{i_1}(t) \hat{\pi}_{i_2}(t) \sigma_{i_1 j}^W(t) \sigma_{i_2 j}^W(t) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right) - \sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right) \phi_j(t, z) dz \end{aligned} \quad (4.1.2)$$

is termed the **local utility** of the exponential strategy $\hat{\pi}$ at t .

3. We call an exponential strategy $\hat{\pi}$ **u -optimal** if

$$E \left(\int_0^T \xi_{\hat{\pi}}(s) ds \right) \geq E \left(\int_0^T \xi_{\tilde{\pi}}(s) ds \right) \quad (4.1.3)$$

for any other exponential strategy $\tilde{\pi}$.

Remark 11 Kallsen (2002) considered a market with characteristics of the form (3.1.12) and defined u -optimal strategies for this market. There is a correspondence between a u -optimal exponential strategy and a u -optimal strategy. For the second one the expression in (4.1.2) will involve an $S(t-)$ term.

The next proposition follows from the results of Kallsen(2002) applied to our market model.

Proposition 19 *An exponential trading strategy $\hat{\pi}$ is u -optimal if and only if*

$$\begin{aligned} & \hat{b}_i(t) + u''(0) \sum_{j=1}^d \sigma_{ij}^W(t) \left(\sum_{i_1=1}^m \hat{\pi}_{i_1}(t) \sigma_{i_1 j}^W(t) \right) \\ & + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(u' \left(\sum_{i_1=1}^m \hat{\pi}_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z) \right) - 1 \right) \phi_j(t, z) dz = 0 \end{aligned} \quad (4.1.4)$$

almost everywhere for $i = 1, \dots, m$.

Moreover assume that the u -optimal exponential trading strategy $\hat{\pi}$ exists and define the equivalent measure $\tilde{P}$ by the density process $Z^{\hat{\pi}}(t) = \mathcal{E}(M^{\hat{\pi}})$ where:

$$\begin{aligned} M^{\hat{\pi}}(t) &= u''(0) \sum_{j=1}^d \int_0^t \left(\sum_{i=1}^m \hat{\pi}_i(s) \sigma_{ij}^W(s) \right) dW_j(s) \\ &+ \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \left(u' \left(\sum_{i=1}^m \hat{\pi}_i(s) \hat{\sigma}_{ij}^N(s, z) \right) - 1 \right) dq_j(s, z). \end{aligned} \quad (4.1.5)$$

Then $\tilde{P}$ is an EMM with parameters

$$\begin{aligned} v_j(t) &= -u''(0) \sum_{i=1}^m \hat{\pi}_i(t) \sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}} u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right) \phi_j(t, z) dz, \\ \psi_j(t, z) &= \frac{u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right)}{\int_{\mathbb{R}} u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, y) \right) \phi_j(t, dy)}. \end{aligned} \quad (4.1.6)$$

We call $\tilde{P}$ with parameters (4.1.6) the **maximum utility measure**.

Proof. The equation (4.1.4) follows from Kallsen (2002) specialized to our model as does (4.1.5). Since the density given by (3.3.20) describes the general class of EMMs for our market model we can compare it to the density given by (4.1.5). Now we can conclude that the EMM that maximizes the local utility in terms of Definition 19 is given by the parameters (4.1.6), assuming that $\hat{\pi}$ satisfying (4.1.4) exists. Note that for the parameters given by (4.1.6), the conditions (3.3.13) and (4.1.4) are equivalent. And $\tilde{P}$ is indeed a EMM. $\square$

Remark 12 *Kallsen made an assumption that $Z^{\hat{\pi}}$ is a martingale. In our case W is a martingale and if we assume that q is also a martingale (previously we defined it as a local martingale), then $M^{\hat{\pi}}$ is also a martingale and*

$$Z^{\hat{\pi}}(t) = 1 + \int_0^t Z^{\hat{\pi}}(s-) dM^{\hat{\pi}}(s)$$

is a martingale.

There are number of established choices for the utility function u . In the next several sections we will apply different methods to obtain the optimal EMM and in each case we will try to determine the corresponding utility function.

4.2 General Esscher transformation.

In Section 2.4 we described an Esscher transformation for Levy processes. Kallsen and Shiryaev (2002) generalized an Esscher transformation for exponentially special semimartingales. We will apply their results to our model to obtain a general Esscher transformation for exponential jump-diffusion semimartingales.

Let us consider our market in the form (3.2.1) i.e. $S = \exp(R)$ where R is a semimartingale given by (3.1.2). Kallsen and Shiryaev (2002) considered a market in the same form, $S = \exp(R)$, but the semimartingale characteristics of R there have a more general structure, (3.1.12), then the jump diffusion model considered here. Our model is easier to apply since $A^R(t) = t$ and hence $\nu^R(t, z) = 0$ (the process R is quasi-left continuous). This will allow us to obtain more tractable expressions.

Definition 20 (Kallsen, Shiryaev (2002)) *Let $\vartheta \in L(R)$ be such that*

$$S^{\vartheta} = \exp \left(\int_0^t \vartheta_i(s) dR_i \right)$$

*is a special semimartingale (i.e. $\int_0^t \vartheta_i(s) dR_i$ is exponentially special). The **Laplace cumulant process** is $\tilde{K}_{\vartheta}^R$ is defined as the compensator of the special semimartingale $\text{Log}(S^{\vartheta})$. The **modified Laplace cumulant process** $K_{\vartheta}^R := \log(\mathcal{E}(\tilde{K}_{\vartheta}^R))$*

The next proposition follows from Lemma 2.13 and Theorem 2.18 both of Kallsen, Shiryaev (2002).

Proposition 20 1. $\int_0^t \vartheta_i(s) dR_i$ is exponentially special if and only if

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^N \right) - 1 - \sum_{i=1}^m \vartheta_i(s) h_{ij}(\sigma_{ij}^N) \phi_j(s, z) dz \right) ds \in \mathcal{A}_{loc}^+. \quad (4.2.1)$$

2. Let $\int_0^t \vartheta_i(s) dR_i$ be exponentially special then for our market model, the Laplace cumulant process is

$$\begin{aligned} \tilde{K}_{\vartheta}^R &= K_{\vartheta}^R \\ &= \int_0^t \sum_{i=1}^m \vartheta_i(s) b_i(s) ds + \frac{1}{2} \int_0^t \sum_{j=1}^d \sum_{i_1, i_2} \vartheta_{i_1}(s) \vartheta_{i_2}(s) \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^N \right) - 1 - \sum_{i=1}^m \vartheta_i(s) h_{ij}(\sigma_{ij}^N) \right) \phi_j(s, z) dz ds. \end{aligned} \quad (4.2.2)$$

This is a continuous predictable process.

We assume that (4.2.1) holds and construct the density process

$$\begin{aligned} Z^{\vartheta}(t) &= \exp \left(\sum_{i=1}^m \int_0^t \vartheta_i(s) dR_i - K_{\vartheta}^R \right) \\ &= \mathcal{E} \sum_{j=1}^d \int_0^t \sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^W(s) dW_j(s) \\ &\quad \times \mathcal{E} \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^N(s, z) \right) - 1 \right) dq_j(s, z). \end{aligned} \quad (4.2.3)$$

The second equality follows from Theorem 2.19 of Kallsen, Shiryaev (2002). We can also obtain it using the definition of the stochastic exponent and the fact $\nu^R(t, z) = 0$.

In our case Z^{ϑ} is again (as in the previous section) a uniformly integrable martingale if we assume that q is a martingale. Now the following proposition is an analog of Theorem 4.1 Kallsen, Shiryaev (2002)

Proposition 21 (Kallsen, Shiryaev (2002)) *Let ϑ be such that (4.2.1) holds. We set $\vartheta^{(i_0)} = (\vartheta^1, \dots, \vartheta^{i_0-1}, \vartheta^{i_0} + 1, \vartheta^{i_0+1}, \dots, \vartheta^m)$. Let $\tilde{P}$ be a measure given by the transformation density Z^{ϑ} . Then $\tilde{P}$ is an EMM if and only if*

$$K_{\vartheta^{(i_0)}}^R - K_{\vartheta}^R = 0, \quad i_0 = 1, \dots, m. \quad (4.2.4)$$

In this case we call $\tilde{P}$ an **Esscher martingale transform for exponential processes**.

Again, we will obtain expressions specific to our jump-diffusion model. Let us use (4.2.2) to rewrite (4.2.4). First of all

$$\begin{aligned}
K_{\vartheta^{(i_0)}}^R &= \int_0^t \sum_{i=1}^m \vartheta_i(s) b_i(s) ds + \int_0^t b_{i_0}(s) ds \\
&\quad + \frac{1}{2} \int_0^t \sum_{j=1}^d \sum_{i_1, i_2} \vartheta_{i_1}(s) \vartheta_{i_2}(s) \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\
&\quad + \frac{1}{2} \int_0^t \sum_{j=1}^d 2\sigma_{i_0 j}^W(s) \sum_{i_1} \vartheta_{i_1}(s) \sigma_{i_1 j}^W(s) ds + \frac{1}{2} \int_0^t \sum_{j=1}^d (\sigma_{i_0 j}^W(s))^2 ds \\
&\quad + \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^N(s, z) \right) \exp(\sigma_{i_0 j}^N(s, z)) - 1 \right) \phi_j(s, z) dz ds \\
&\quad - \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \sum_{i=1}^m \vartheta_i(s) h_{ij}(\sigma_{ij}^N(s, z)) \phi_j(s, z) dz ds
\end{aligned}$$

Now recalling the expression for $\hat{b}$ from Proposition 14 we have

$$\begin{aligned}
K_{\vartheta^{(i_0)}}^R - K_{\vartheta}^R &= \int_0^t \hat{b}_{i_0}(s) ds \\
&\quad + \int_0^t \sum_{j=1}^d \sigma_{i_0 j}^W(s) \left(\sum_{i_1} \vartheta_{i_1}(s) \sigma_{i_1 j}^W(s) \right) ds \\
&\quad + \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \exp \left(\sum_{i=1}^m \vartheta_i(s) \sigma_{ij}^N(s, z) \right) \left(e^{\sigma_{i_0 j}^N(s, z)} - 1 \right) \phi_j(s, z) dz ds \\
&\quad - \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} (e^{\sigma_{i_0 j}^N(s, z)} - 1) \phi_j(s, z) dz ds
\end{aligned} \tag{4.2.5}$$

Hence the condition (4.2.4) is equivalent to

$$\begin{aligned}
&\hat{b}_{i_0}(t) + \sum_{j=1}^d \sigma_{i_0 j}^W(t) \left(\sum_{i_1} \vartheta_{i_1}(t) \sigma_{i_1 j}^W(t) \right) \\
&+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \vartheta_i(t) \sigma_{ij}^N(t, z) \right) - 1 \right) \hat{\sigma}_{i_0 j}^N(t, z) \phi_j(t, z) dz = 0
\end{aligned} \tag{4.2.6}$$

We again compare formulas (3.3.20) and (3.3.13) for the characteristics under the transformed measure with those of the Esscher transformation, (4.2.3) and (4.2.6). We obtain the following formulas for parameters of the EMM of the general Esscher transformation for exponential processes.

$$\begin{aligned} v_j(t) &= - \sum_{i=1}^m \vartheta_i(t) \sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}} \exp \left(\sum_{i=1}^m \vartheta_i(t) \sigma_{ij}^N(t, z) \right) \phi_j(t, z) dz, \\ \psi_j(t, z) &= \frac{\exp \left(\sum_{i=1}^m \vartheta_i(t) \sigma_{ij}^N(t, z) \right)}{\int_{\mathbb{R}} \exp \left(\sum_{i=1}^m \vartheta_i(t) \sigma_{ij}^N(t, y) \right) \phi_j(t, dy)}, \end{aligned} \quad (4.2.7)$$

where ϑ is the solution of (4.2.6) if it exists.

It is difficult to obtain a utility function in this case because we were working with the price process in the form $S = \exp(R)$ and the utility maximization approach was considered when the price process is given by $S = \mathcal{E}(\hat{R})$. Comparing (4.2.7) and (4.1.6) we conclude that the possible utility function here should satisfy

$$u'(\sum_i \vartheta_i x_i) = \prod_i (x_i + 1)^{\vartheta_i}.$$

In the one-dimensional case this becomes

$$u(x) = \frac{\vartheta}{\vartheta + 1} \left(\left(\frac{x}{\vartheta} + 1 \right)^{\vartheta} - 1 \right),$$

where $\vartheta < 0$ and $x > -\vartheta$. In both cases the utility function depends on the strategy. Another disadvantage of this approach is that in our model the Esscher transformation for exponential processes is not unique. It is unique only in the one-dimensional case (Kallsen, Shiryaev 2002).

These difficulties leads us to a second construction of the Esscher transform for jump-diffusion semimartingales. In this case we consider prices of our market in the stochastic exponential form (3.2.3) i.e. $S = \mathcal{E}(\hat{R})$. We apply here Theorem 2.18 and Theorem 4.4. of Kallsen and Shiryaev (2002).

Proposition 22 (Kallsen, Shiryaev (2002))

1. Let $\hat{\vartheta}_i \in L(S_i)$. $\int_0^t \hat{\vartheta}_i(s) d\hat{R}_i$ is exponentially special if and only if

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N \right) - 1 - \sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N \right) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+. \quad (4.2.8)$$

2. Let $\int_0^t \vartheta_i(s) d\hat{R}_i$ be exponentially special then in our market model Laplace cumulant process for $\hat{R}$:

$$\begin{aligned} \tilde{K}_{\hat{\vartheta}}^{\hat{R}} &= K_{\hat{\vartheta}}^{\hat{R}} = \int_0^t \sum_{i=1}^m \hat{\vartheta}_i(s) \hat{b}_i(s) ds + \frac{1}{2} \int_0^t \sum_{j=1}^d \sum_{i_1, i_2} \hat{\vartheta}_{i_1}(s) \hat{\vartheta}_{i_2}(s) \sigma_{i_1 j}^W(s) \sigma_{i_2 j}^W(s) ds \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \left(e^{\sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N} - 1 - \sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N \right) \phi_j(s, z) dz ds \end{aligned} \quad (4.2.9)$$

is a continuous predictable process.

3. Moreover the density process

$$\begin{aligned} Z^{\hat{\vartheta}}(t) &= \exp \left(\sum_{i=1}^m \int_0^t \hat{\vartheta}_i(s) d\hat{R}_i - K_{\hat{\vartheta}}^{\hat{R}} \right) \\ &= \mathcal{E} \left(\sum_{j=1}^d \int_0^t \left(\sum_{i=1}^m \hat{\vartheta}_i(s) \sigma_{ij}^W(s) \right) dW_j(s) \right. \\ &\quad \left. + \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} \left(\exp \left(\sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N(s, z) \right) - 1 \right) dq_j(s, z) \right) \end{aligned} \quad (4.2.10)$$

provides us with the EMM $\tilde{P}$ if and only if

$$\sum_{j=1}^{k-d} \int_0^t \lambda_j \int_{\mathbb{R}} \hat{\sigma}_{ij}^N \exp \left(\sum_{i=1}^m \hat{\vartheta}_i(s) \hat{\sigma}_{ij}^N \right) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+. \quad (4.2.11)$$

and

$$\begin{aligned} &\hat{b}_i(t) ds + \sum_{j=1}^d \sigma_{ij}^W(t) \sum_{i_0=1}^m \hat{\vartheta}_{i_0}(t) \sigma_{i_0 j}^W(t) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\exp \left(\sum_{i_0=1}^m \hat{\vartheta}_{i_0}(t) \hat{\sigma}_{i_0 j}^N(t, z) \right) - 1 \right) \hat{\sigma}_{ij}^N(t, z) \phi_j(t, z) dz \\ &= 0. \end{aligned} \quad (4.2.12)$$

Remark 13 In the above proposition we used the fact that $\hat{R}$ is a special semimartingale itself and hence the truncation function is the identity i.e. $\hat{h}_{ij}(\hat{\sigma}_{ij}^N) = \hat{\sigma}_{ij}^N$.

Definition 21 The EMM provided by the density process (4.2.10) under conditions (4.2.11) and (4.2.12) we call a **general Esscher transformation for jump-diffusion processes**.

If the solution of (4.2.12) exists for every i then the parameters of the general Esscher transformation are

$$\begin{aligned} v_j(t) &= - \sum_{i=1}^m \hat{\vartheta}_i(t) \sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}} \exp \left(\sum_{i=1}^m \hat{\vartheta}_i(t) \hat{\sigma}_{ij}^N(t, z) \right) \phi_j(t, z) dz, \\ \psi_j(t, z) &= \frac{\exp \left(\sum_{i=1}^m \hat{\vartheta}_i(t) \hat{\sigma}_{ij}^N(t, z) \right)}{\int_{\mathbb{R}} \exp \left(\sum_{i=1}^m \hat{\vartheta}_i(t) \hat{\sigma}_{ij}^N(t, y) \right) \phi_j(t, dy)}, \end{aligned} \quad (4.2.13)$$

It is clear that the corresponding utility function will be $u(x) = 1 - e^{-x}$. In the one-dimensional case the general Esscher transform is unique.

4.3 Variance-optimal EMM.

There is another way to find an optimal EMM. Schweizer(1996) introduced the variance optimal martingale measure. Again we are going to use his approach to obtain this measure for our market model.

Let f_T be a contingent claim. We assume that $E(H)^2 < \infty$ (i.e. $f_T \in L^2(P)$). In this section we will consider strategies π such that the gain process $G_\pi(T) \in L^2(P)$. We will call this strategy **square-integrable**.

We want to determine an initial capital v and a trading strategy π such that the achieved terminal wealth $v + G_\pi(T)$ approximates f_T with respect to the distance in $L^2(P)$. Hence we consider the following optimization problem (Schweizer(1996)):

$$\text{minimize } E \left((f_T - v - G_\pi(T))^2 \right), \quad (4.3.1)$$

over all pairs (v, π) such that π is a square-integrable strategy. If (v_f, π_f) is the solution of this problem then we call v_f the **approximation price** of f_T .

If a contingent claim f_T is attainable then $f_T = v_0 + G_T(\pi_0)$ for some (v_0, π_0) and it obviously solves (4.3.1) and $(v_f, \pi_f) = (v_0, \pi_0)$. Hence this approach is consistent with complete markets.

Definition 22 (Schweizer (1996)) 1. A signed measure Q is called a **signed martingale measure** if $Q(\Omega) = 1$, $Q \ll P$ with $\frac{dQ}{dP} \in L^2(P)$ and

$$E_Q(G_\pi(T)) = 0$$

for all admissible strategies π . We denote by Z_Q its density.

2. A signed martingale measure Q_V is called a **signed variance optimal** if Q_V minimizes:

$$\text{Var} \left(\frac{dQ}{dP} \right) = E \left(\left(\frac{dQ}{dP} - 1 \right)^2 \right) = E \left(\left(\frac{dQ}{dP} \right)^2 \right) - 1 = E_Q(Z_Q) - 1$$

We mention here the main properties of a signed variance optimal measure (see Schweizer(1996) for details).

Proposition 23 (Schweizer (1996)) 1. The signed variance optimal measure is unique and exists whenever there is at least one signed martingale measure.

2. Q_V is signed variance optimal if and only if

$$E \left(\frac{dQ}{dP} \frac{dQ_V}{dP} \right) = E_{Q_V}(Z_Q) = \text{const} \quad (4.3.2)$$

for all signed martingale measures Q .

3. Let Q_V be the signed variance optimal measure. Let (v_f, π_f) be the solution of the problem (4.3.1) and suppose

$$E(f_T - v_f - G_\pi(T)) = 0,$$

then

$$v_f = E_{Q_V}(f_T). \quad (4.3.3)$$

The main disadvantage of the signed variance optimal measure is that it is a signed (not a probability) measure. To avoid this problem we can give another definition of variance optimality for our market model.

Definition 23 *Let $\tilde{P}$ be an EMM given by the density process given by (3.3.20). Then we call $\tilde{P}$ a variance optimal measure if $\tilde{P}$ minimizes:*

$$\text{Var}(Z) = E_{\tilde{P}}(Z) - 1$$

over all v, r, ψ satisfying (3.3.13) and (3.3.19).

To compare this definition to Schweizer's Definition 22 we recall that under (3.3.13) the price S is given by (3.3.16).

$$\begin{aligned} G_{\pi}(t) &= \sum_{i=1}^m \int_0^t \pi_i(s) d\tilde{S}_i(s) \\ &= \sum_{i=1}^m \left(\sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) \pi_i(s) S_i(s-) d\tilde{W}_j + \sum_{j=1}^{k-d} \int_0^t \pi_i(s) S_i(s-) d\tilde{Q}_{ij}(s) \right), \end{aligned} \quad (4.3.4)$$

where $\tilde{W}$ is a $\tilde{P}$ -Brownian motion and $\tilde{Q}$ is a $\tilde{P}$ -compensated point process (martingale). Hence $E_{\tilde{P}}(G_{\pi}(t)) = 0$ and therefore an EMM is a signed martingale measure.

Unfortunately Proposition 23 does not work any longer since densities for signed martingale measures compose a linear space, but this is not true for densities of positive (probability) martingale measures. However, the variance optimal measure seems still to be interesting from the economic point of view because it provides us with the probability EMM whose difference (the density process) from the original measure P has the smallest variation.

The next proposition gives us explicitly the parameters of the variance optimal measure as in Definition 3.3.5. We use the optimization procedure introduced by Chan (1999).

Proposition 24 *Let $l(t) = (l_1(t), \dots, l_m(t))$, if it exists, be nonnegative and determined by the linear system of equations and inequalities:*

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^d \sigma_{ij}^W(t) \left(\sum_{n=1}^m l_n(t) \sigma_{nj}^W(t) \right) \\ &\quad + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(\sum_{n=1}^m l_n(t) \hat{\sigma}_{nj}^N(t, z) \right) \phi_j(t, z) dz, \\ &\quad \sum_{i=1}^m l_i(t) \hat{\sigma}_{ij}^N(t, z) < 1 \end{aligned} \quad (4.3.5)$$

for each $i = 1, \dots, m$.

Then a version of the variance optimal measure is characterized by:

$$\begin{aligned} v_j &= \sum_{i=1}^m l_i(s) \sigma_{ij}^W(s), \\ r_j(s) &= 1 - \sum_{i=1}^m l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz, \\ \psi_j(s, z) &= \frac{1 - \sum_{i=1}^m l_i(s) \hat{\sigma}_{ij}^N(s, z)}{1 - \sum_{i=1}^m l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, y) \phi_j(s, dy)}. \end{aligned} \quad (4.3.6)$$

If such l does not exist then the minimum of density variance cannot be achieved and variance optimal measure does not exist.

Proof. From (3.3.20) and the definition of the Dolean-Dade exponent we can see that:

$$\begin{aligned} E_{\tilde{P}}(Z) &= 1 - E_{\tilde{P}} \left(\sum_{j=1}^d \int_0^t v_j(s) Z(s-) dW_j \right) \\ &\quad + E_{\tilde{P}} \int_0^t Z(s-) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) dq_j(s, z) \\ &= 1 - E_{\tilde{P}} \left(\sum_{j=1}^d \int_0^t v_j(s) Z(s-) d \left(W_j + \int_0^t v_j(s) ds \right) - \sum_{j=1}^d \int_0^t (v_j(s))^2 Z(s-) ds \right) \\ &\quad + E_{\tilde{P}} \int_0^t Z(s-) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) d(N_j(s, z) - \lambda_j(s) \phi_j(s, z) dz ds) \end{aligned} \quad (4.3.7)$$

Now we can use the fact that $W(s) + \int_0^s v_j(u)du = \tilde{W}(s)$ is a martingale under $\tilde{P}$. We also know that if $\tilde{\Lambda}_j$ is a compensator of the marked point process N_j under $\tilde{P}$, then from Proposition 18:

$$\tilde{\Lambda}_j(t, z) = \int_0^t r_j(s) \lambda_j(s) \psi_j(s, z) \phi_j(s, z) dz ds.$$

Hence

$$\begin{aligned} E_{\tilde{P}}(Z) &= 1 + E_{\tilde{P}} \sum_{j=1}^d \int_0^t (v_j(s))^2 Z(s-) ds \\ &\quad + E_{\tilde{P}} \int_0^t Z(s-) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) d\tilde{\Lambda}_j(t, dz) \\ &\quad - E_{\tilde{P}} \int_0^t \lambda_j(s) Z(s-) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1) \phi_j(s, z) dz ds \quad (4.3.8) \\ &= 1 + E_{\tilde{P}} \sum_{j=1}^d \int_0^t (v_j(s))^2 Z(s-) ds \\ &\quad + E_{\tilde{P}} \int_0^t \lambda_j(s) Z(s-) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1)^2 \phi_j(s, z) dz ds \end{aligned}$$

Our goal is to minimize (4.3.8) with respect to v_j, r_j and ψ_j subject to constraints (3.3.13). Let $l(s) = (l_1(s), \dots, l_m(s))$ be Lagrange multipliers for (3.3.13). Following Chan(1999) we fix ω and w.l.o.g. remove $E_{\tilde{P}}$. Since $Z(s-) > 0$ we can remove $Z(s-)$ and the integration over s and fix s w.l.o.g. Hence we want to minimize the following:

$$\sum_{j=1}^d (v_j(s))^2 ds + \lambda_j(s) \int_{\mathbb{R}} (r_j(s) \psi_j(s, z) - 1)^2 \phi_j(s, z) dz ds. \quad (4.3.9)$$

First we fix η_j for $j = 1, \dots, k-d$ and v_j for $j = 1, \dots, j_0-1, j_0+1, \dots, d$ and minimize (4.3.9) with respect to v_{j_0} subject to the constraints (3.3.9). Hence we want to minimize the Lagrange function:

$$L_W(l, v_{j_0}) = v_{j_0}^2(s) - 2 \sum_{i=1}^m l_i(s) \sigma_{ij_0}^W(s) v_{j_0}(s) \quad (4.3.10)$$

and the minimizer is:

$$v_{j_0} = \sum_{i=1}^m l_i(s) \sigma_{ij_0}^W(s), \quad j_0 = 1, \dots, d. \quad (4.3.11)$$

Since j_0 is arbitrary between 1 and d , this is true for each j .

Now we fix v_j for $j = 1, \dots, d$ and η_j for $j = 1, \dots, j_1 - 1, j_1 + 1, \dots, k - d$ and minimize (4.3.9) with respect to η_{j_1} with constraint (3.3.9). Again we can remove the integration over s and fix s w.l.o.g. Similar we want to minimize the corresponding Lagrange function:

$$\begin{aligned} L_N(l, \eta_{j_1}) &= \int_{\mathbb{R}} (\eta_{j_1}(s, z))^2 \lambda_{j_1}(s) \phi_{j_1}(s, dz) \\ &\quad - 2 \sum_{i=1}^m l_i(s) \lambda_{j_1}(s) \int_{\mathbb{R}} (e^{\sigma_{ij_1}^N(s, z)} - 1) \eta_{j_1}(s, z) \phi_{j_1}(s, dz). \end{aligned} \quad (4.3.12)$$

Again following Chan(1999) we require:

$$\frac{d}{dt} L_N(l, \eta_{j_1} + tF) |_{t=0} = 0$$

for all F , which gives:

$$\eta_{j_1}(s, z) = \sum_{i=1}^m l_i(s) (e^{\sigma_{ij_1}^N(s, z)} - 1). \quad (4.3.13)$$

Again j_1 is not fixed and hence this is true for any j between 1 and $k - d$. Obviously $L_N(\eta_{j_1})$ is a convex function of η and hence we have a minimum.

Now we plug (4.3.11) and (4.3.13) into condition (3.3.9) to obtain (4.4.12).

By (4.3.11), (4.3.13) and (3.3.19) we obtain (4.3.6). $\square$

We can also find the corresponding utility for the variance optimal measure. It will be $u(x) = x - \frac{x^2}{2}$. Generally this measure will be a signed measure. If (3.3.13) holds (i.e. u is applied only to $x < 1$), the variance optimal measure is a probability measure.

4.4 Minimal relative entropy.

We will consider two types of minimal relative entropy, one based on the usual logarithmic function and one based on the stochastic logarithm ($\mathcal{L}og$).

We consider our model $S = \exp(R)$. Let $\tilde{P}$ be a EMM with the density process Z .

Definition 24 *The EMM that minimizes the **relative entropy** $(E_{\tilde{P}}(\log Z))$ is the **minimal relative entropy measure**.*

Fritelly (2000) investigated the properties of the minimal relative entropy measure (MRE) and obtained conditions for existence and uniqueness. His main result is stated below.

Proposition 25 (Fritelly (2000)) *If there exists a EMM $\tilde{P}$ such that $(E_{\tilde{P}}(\log Z)) < \infty$ and $\hat{R}$ is bounded then the minimal relative entropy measure exists and it is unique.*

In our case the MRE measure is characterized by its parameters v, r, ψ . We will find these parameters of the MRE measure using the optimization procedure of Chan (1999).

Proposition 26 *Let $l(t) = (l_1(t), \dots, l_m(t))$, whenever it exists, be determined by the following system:*

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^d \sigma_{ij}^W(t) \left(\sum_{n=1}^m l_n(t) \sigma_{nj}^W(t) \right) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(1 - \exp \left(\sum_{n=1}^m l_n(t) \hat{\sigma}_{nj}^N(t, z) \right) \right) \phi_j(t, z) dz, \end{aligned} \quad (4.4.1)$$

for each $i = 1, \dots, m$.

Then a version of the minimal relative entropy measure is characterized by:

$$\begin{aligned} v_j &= \sum_{i=1}^m l_i(s) \sigma_{ij}^W(s), \\ r_j(s) &= \int_{\mathbb{R}} \exp \left(\sum_{i=1}^m l_i(s) \hat{\sigma}_{ij}^N(s, z) \right) \phi_j(s, z) dz, \\ \psi_j(s, z) &= \frac{\exp \left(\sum_{i=1}^m l_i(s) \hat{\sigma}_{ij}^N(s, z) \right)}{\int_{\mathbb{R}} \exp \left(\sum_{i=1}^m l_i(s) \hat{\sigma}_{ij}^N(s, y) \right) \phi_j(s, dy)}. \end{aligned} \quad (4.4.2)$$

If such l does not exist then the minimum of relative entropy cannot be achieved and the minimal relative entropy measure does not exist.

Proof. From (3.3.21) we can see that:

$$\begin{aligned}
E_{\tilde{P}}(\log Z) &= E_{\tilde{P}} \left(- \sum_{j=1}^d \left(\int_0^t v_j(s) dW_j - \frac{1}{2} \int_0^t v_j^2(s) ds \right) \right) \\
&\quad + E_{\tilde{P}} \sum_{j=1}^{k-d} \left(\int_0^t \int_{\mathbb{R}} \log(r_j(s) \psi_j(s, z)) dN_j(s, z) \right) \\
&\quad + E_{\tilde{P}} \int_0^t \int_{\mathbb{R}} (1 - r_j(s) \psi_j(s, z)) \lambda_j(s) \phi_j(s, z) dz ds. \tag{4.4.3}
\end{aligned}$$

Now the first term is the sum over j of the following terms:

$$\begin{aligned}
&E_{\tilde{P}} \left(- \int_0^t v_j(s) dW(s) - \frac{1}{2} \int_0^t v_j^2(s) ds \right) \\
&= E_{\tilde{P}} \left(- \int_0^t v_j(s) d \left(W(s) + \int_0^s v_j(u) du \right) + \frac{1}{2} \int_0^t v_j^2(s) ds \right) \\
&= E_{\tilde{P}} \left(- \int_0^t v_j(s) d\tilde{W}(s) + \frac{1}{2} \int_0^t v_j^2(s) ds \right) \\
&= E_{\tilde{P}} \left(\frac{1}{2} \int_0^t v_j^2(s) ds \right). \tag{4.4.4}
\end{aligned}$$

Here we used the fact that $W(s) + \int_0^s v_j(u) du = \tilde{W}(s)$ is a local martingale under $\tilde{P}$.

The second term is the sum of expectations:

$$\begin{aligned}
&\sum_{j=1}^{k-d} E_{\tilde{P}} \left(\int_0^t \int_{\mathbb{R}} \log(r_j(s) \psi_j(s, z)) dN_j(s, z) \right) \\
&= \sum_{j=1}^{k-d} E_{\tilde{P}} \left(\int_0^t \int_{\mathbb{R}} \log(r_j(s) \psi_j(s, z)) d\tilde{\Lambda}_j(s, dz) \right), \tag{4.4.5}
\end{aligned}$$

where $\tilde{\Lambda}_j$ is a compensator of the marked point process N_j under $\tilde{P}$. We know from the Proposition 18 that:

$$\tilde{\Lambda}_j(t, dz) = \int_0^t r_j(s) \lambda_j(s) \psi_j(s, z) \phi_j(s, z) dz ds.$$

Hence the second term can be written as:

$$\sum_{j=1}^{k-d} E_{\tilde{P}} \left(\int_0^t \int_{\mathbb{R}} \log(r_j(s) \psi_j(s, z)) r_j(s) \lambda_j(s) \psi_j(s, z) \phi_j(s, z) dz ds \right). \tag{4.4.6}$$

Combining (4.4.3), (4.4.4) and (4.4.6) we obtain the relative entropy in terms of the parameters:

$$\begin{aligned} E_{\bar{P}}(\log Z(t)) &= E_{\bar{P}} \left(\frac{1}{2} \sum_{j=1}^d \int_0^t v_j^2(s) ds + \sum_{j=1}^{k-d} \int_{\mathbb{R}} \Lambda_j(t, dz) \right) \\ &+ E_{\bar{P}} \sum_{j=1}^{k-d} \left(\int_0^t \int_{\mathbb{R}} (\log(r_j(s)\psi_j(s, z)) - 1) r_j(s) \lambda_j(s) \psi_j(s, z) \phi_j(s, z) dz ds \right), \end{aligned} \quad (4.4.7)$$

where Λ_j is the compensator of the process N_j under the original measure P .

Our goal is to minimize (4.4.7) with respect to v_j, r_j and ψ_j subject to the constraints (3.3.13). We can notice that the existence of $\log(r_j(s)\psi_j(s, z))$ already requires the positivity of $r_j(s)\psi_j(s, z)$. Hence we do not need Lagrange multipliers to account for this condition. Let $l(s) = (l_1(s), \dots, l_m(s))$ be Lagrange multipliers for (3.3.13). Following Chan(1999) we fix ω and w.l.o.g. remove $E_{\bar{P}}$. We also can remove integration over s and fix s w.l.o.g.

Let $\tilde{h}_j(s, z) = r_j(s)\psi_j(s, z)$. First we fix $\tilde{h}_j$ for $j = 1, \dots, k-d$ and v_j for $j = 1, \dots, j_0-1, j_0+1, \dots, d$ and minimize (4.4.7) with respect to v_{j_0} subject to constraint (3.3.13). Hence we want to minimize the Lagrange function:

$$L_W(l, v_{j_0}) = \frac{1}{2} v_{j_0}^2(s) - \left(\sum_{i=1}^m l_i(s) \sigma_{ij_0}^W(s) \right) v_{j_0}(s), \quad (4.4.8)$$

and the minimizer is:

$$v_{j_0} = \sum_{i=1}^m l_i(s) \sigma_{ij_0}^W(s), \quad j_0 = 1, \dots, d. \quad (4.4.9)$$

Since j_0 is arbitrary between 1 and d , (4.4.9) holds for each j .

Now we consider the point process contribution and fix v_j for $j = 1, \dots, d$ and $\tilde{h}_j$ for $j = 1, \dots, j_1-1, j_1+1, \dots, k-d$. We minimize (4.4.7) with respect to $\tilde{h}_{j_1}$ subject to the constraint (3.3.13). Similarly we want to minimize the corresponding Lagrange function:

$$\begin{aligned} L_N(l, \tilde{h}_{j_1}) &= \int_{\mathbb{R}} \tilde{h}_{j_1}(s, z) (\log(\tilde{h}_{j_1}(s, z)) - 1) \lambda_{j_1}(s) \phi_{j_1}(s, dz) \\ &+ \sum_{i=1}^m l_i(s) \lambda_{j_1}(s) \int_{\mathbb{R}} (e^{\sigma_{ij_1}^N(s, z)} - 1) (1 - \tilde{h}_{j_1}(s, z)) \phi_j(s, z) dz \end{aligned} \quad (4.4.10)$$

Following Chan(1999) we require

$$\frac{d}{dt} L_N(l, \hbar_{j_1} + tF) |_{t=0} = 0$$

for all F , which gives

$$\hbar_{j_1}(s, z) = \exp \left(\sum_{i=1}^m l_i(s) (e^{\sigma_{ij_1}^N(s, z)} - 1) \right). \quad (4.4.11)$$

Again j_1 is arbitrary and hence (4.4.11) holds for any j between 1 and $k - d$. Since $\hbar_{j_1}(s, z) > 0$, the function $L_N(\hbar_{j_1})$ is convex and hence we have a minimum here (“second derivative” is positive).

Now we plug (4.4.9) and (4.4.11) into condition (3.3.13) to obtain (4.4.1). We have m equations for m unknown parameters $l_1, \dots, l_m$.

By (3.3.19), (4.4.9) and the definition of $\hbar_j$ we obtain (4.4.2). $\square$

It can be seen that the relative entropy corresponds to the utility $u(x) = 1 - e^{-x}$. Hence we can conclude that the minimal relative entropy measure is equal to the measure obtained by the general Esscher transform.

Grandits and Rheinlander(2002) gave another characterization of the MRE measure: $\tilde{P}$ is the MRE measure if and only if

- (i) $Z(t) = Ce^{G_\pi(t)}$ for a constant C and a strategy π ;
- (ii) $E_{\tilde{P}} e^{G_\pi(t)} = 0$.

As we checked in the previous section (ii) is fulfilled for all EMMs in our model. We can also see from the above Proposition that (i) is also true in our case.

We also consider a second distinct approach to minimization of relative entropy.

Definition 25 *The EMM that minimizes the expectation of the stochastic logarithm of the density process, $E_{\tilde{P}}(\text{Log} Z)$, we call an **exponential minimal relative entropy measure (exponential MRE)**.*

Proposition 27 *Let $l(t) = (l_1(t), \dots, l_m(t))$, if it exists, be nonnegative and determined by the linear system:*

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^d \sigma_{ij}^W(t) \left(\sum_{n=1}^m l_n(t) \sigma_{nj}^W(t) \right) \\ &+ \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(\sum_{n=1}^m l_n(t) \hat{\sigma}_{nj}^N(t, z) \right) \phi_j(t, z) dz, \\ \sum_{n=1}^m l_n(t) \hat{\sigma}_{nj}^N(t, z) &< 1 \end{aligned} \quad (4.4.12)$$

for each $i = 1, \dots, m$.

Then a version of the exponential MRE measure is characterized by:

$$\begin{aligned} v_j &= \sum_{i=1}^m l_i(s) \sigma_{ij}^W(s), \\ r_j(s) &= 1 - \sum_{i=1}^m l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz, \\ \psi_j(s, z) &= \frac{1 - \sum_{i=1}^m l_i(s) \hat{\sigma}_{ij}^N(s, z)}{1 - \sum_{i=1}^m l_i(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, y) \phi_j(s, y) dy}. \end{aligned} \quad (4.4.13)$$

If such l does not exist then the minimum of the stochastic logarithm of the density cannot be achieved and the exponential MRE measure does not exist.

Proof. In this case the condition that $r_j \psi_j > 0$ is essential. From (3.3.8) we can see that:

$$\begin{aligned} E_{\hat{P}}(\mathcal{L}og Z) &= \\ &-E_{\hat{P}} \sum_{j=1}^d \int_0^t v_j(s) dW_j - E_{\hat{P}} \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \eta_j(s, z) dq_j(s, z) \end{aligned} \quad (4.4.14)$$

Now we can compare this to (4.3.7) and notice that we can remove $Z(s-)$ from the optimization process w.l.o.g. Hence the proof of Proposition 24 gives us the same result. $\square$

Here we can notice that the exponential MRE measure is the variance optimal measure and its utility function is $u(x) = x - \frac{x^2}{2}$, $x < 1$. Again we can notice that in general the exponential MRE can be a signed measure.

4.5 Reverse relative entropy.

We will mention here another interesting optimal measure.

Definition 26 *The EMM is the minimal reverse relative entropy measure if it minimizes $E(-\log Z)$.*

We will follow the Goll and Kallsen (2000) results here. They worked with a slightly different definition of utility. In our definition of the utility function we assume that its domain is the whole real line and Goll and Kallsen (2000) assume that its domain can be bounded from below. We will call $\hat{u}$ a **restricted utility function** if it is defined on $(\bar{x}, \infty)$ with

$$\hat{u}'(\bar{x}) = \lim_{x \downarrow \bar{x}} \hat{u}'(x) = \infty$$

and the rest of the properties of usual utility function are fulfilled on the domain $(\bar{x}, \infty)$. In terms of our definition of utility Goll and Kallsen (2000) showed that the corresponding utility function for the reverse relative entropy is $u(x) = \log(1+x)$, $x > -1$ and obtained the optimal strategy. We will demonstrate the Goll and Kallsen result applied to our model in the following proposition.

Proposition 28 (Goll, Kallsen (2000)) *Let $\varpi = (\varpi_1, \dots, \varpi_m)$ be a predictable process such that*

$$\begin{aligned} & 1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z) > 0; \\ & \int_0^t \lambda_j(s) \int_{\mathbb{R}} \frac{\hat{\sigma}_{ij}^N(s, z)}{1 + \sum_{i_1=1}^m \varpi_{i_1}(s) \hat{\sigma}_{i_1 j}^N(s, z)} \phi_j(s, z) dz \in \mathcal{A}_{loc}^+ \\ & \hat{b}_i(t) - \sum_{j=1}^d \sigma_{ij}^W(t) \left(\sum_{i_1=1}^m \varpi_{i_1}(t) \sigma_{i_1 j}^W(t) \right) \\ & + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(\frac{1}{1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z)} - 1 \right) \phi_j(t, z) dz \\ & = 0 \end{aligned} \tag{4.5.1}$$

almost everywhere for $i = 1, \dots, m$.

Moreover assume that the ϖ exists and define the equivalent measure $\tilde{P}$ by the density

process $Z^\varpi(t) = \mathcal{E}(M^\varpi)$ where:

$$\begin{aligned} M^\varpi(t) = & - \sum_{j=1}^d \int_0^t \left(\sum_{i=1}^m \varpi_i(s) \sigma_{ij}^W(s) \right) dW_j(s) \\ & + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \left(\frac{1}{1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z)} - 1 \right) dq_j(s, z). \end{aligned} \quad (4.5.2)$$

Then $\tilde{P}$ is an EMM minimizing the reverse relative entropy and its parameters are

$$\begin{aligned} v_j(t) &= \sum_{i=1}^m \varpi_i(t) \sigma_{ij}^W(t), \\ r_j(t) &= \int_{\mathbb{R}} \frac{1}{1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z)} \phi_j(t, z) dz, \\ \psi_j(t, z) &= \frac{1}{\int_{\mathbb{R}} \frac{1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, z)}{1 + \sum_{i_1=1}^m \varpi_{i_1}(t) \hat{\sigma}_{i_1 j}^N(t, y)} \phi_j(t, dy)}, \end{aligned} \quad (4.5.3)$$

4.6 Minimal martingale measure.

According to Follmer and Schweizer (1991) and Schweizer (1995) there are three characterizations of the minimal martingale measure. One of these characterizations of this measure is as a solution of a minimization problem involving the relative entropy, considered only in the case of continuous market prices S . Since our model is more general and discontinuous we are not interested in this characterization.

The main disadvantage of the minimal martingale measure is that it is in general a signed martingale measure. We will try to construct only probability minimal martingale measures.

We assume also that there exists predictable processes $\ell = (\ell_1, \dots, \ell_m)$, $\ell \in L^2(S^M)$

such that for every $i = 1, \dots, m$

$$\begin{aligned} \hat{b}_i(t)S_i(t-) &= \sum_{i_1=1}^m \hat{\varsigma}_{i_1 i}(t)\ell_{i_1}(t) \\ &= S_i(t-) \sum_{j=1}^d \left(\sum_{i_1=1}^m S_{i_1}(s-) \sigma_{i_1 j}^W(t) \ell_{i_1}(t) \right) \sigma_{ij}^W(t) \\ &\quad + S_i(t-) \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\sum_{i_1=1}^m S_{i_1}(t-) \hat{\sigma}_{i_1 j}^N(t, z) \ell_{i_1}(t) \right) \hat{\sigma}_{ij}^N(t, z) \phi_j(t, z) dz. \end{aligned}$$

Let $\hat{\ell}_i(t) = S_i(t-) \ell_i(t)$ then we obtain

$$\begin{aligned} \hat{b}_i(t) &= \sum_{j=1}^d \left(\sum_{i_1=1}^m \sigma_{i_1 j}^W(t) \hat{\ell}_{i_1}(t) \right) \sigma_{ij}^W(t) \\ &\quad + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} \left(\sum_{i_1=1}^m \hat{\sigma}_{i_1 j}^N(t, z) \hat{\ell}_{i_1}(t) \right) \hat{\sigma}_{ij}^N(t, z) \phi_j(t, z) dz \end{aligned} \quad (4.6.1)$$

If S satisfies (3.2.12) and (4.6.1) we say that S satisfies the **structure conditions**.

Proposition 29 (Schweizer (1995))

1. Suppose S has a square-integrable martingale part S^M . Let Z , a density of a EMM $\tilde{P}$, be a square-integrable martingale. Then for $\hat{\ell}_i(t)$ defined by (4.6.1) we have

$$Z(t) = \mathcal{E} \left(- \sum_{i=1}^m \int_0^t \frac{\hat{\ell}_i(s)}{S_i(s-)} dS_i^M + \hat{H} \right), \quad (4.6.2)$$

where $\hat{H}$ is a square-integrable local martingale orthogonal to S_i^M for each i . Moreover if

$$\sum_{i=1}^m \frac{\hat{\ell}_i(s)}{S_i(s-)} \Delta S^M < 1 \quad P - a.s., \quad (4.6.3)$$

then

$$Z^{MM}(t) = \mathcal{E} \left(- \sum_{i=1}^m \int_0^t \frac{\hat{\ell}_i(s)}{S_i(s-)} dS_i^M \right)$$

is the density of an EMM.

Definition 27 *The EMM with the density Z^{MM} is the **minimal martingale measure***

Generally the minimal martingale measure (MMM) is a signed measure, but under condition (4.6.3) it is a probability measure.

We can rewrite (4.6.3) as

$$\sum_{i=1}^m \hat{\ell}_i(s) \hat{\sigma}_{ij}^N(s, z) < 1 \quad P - a.s., \quad (4.6.4)$$

We can also see that

$$\begin{aligned} Z^{MM}(t) = & \quad (4.6.5) \\ \mathcal{E} \left(- \sum_{j=1}^d \int_0^t \left(\sum_{i=1}^m \hat{\ell}_i(s) \sigma_{ij}^W(s) \right) dW_j(s) - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \left(\sum_{i=1}^m \hat{\ell}_i(s) \hat{\sigma}_{ij}^N(s, z) \right) dq_j(s, z) \right). \end{aligned}$$

Now we can determine parameters v, r, ψ that correspond to the MMM.

Proposition 30 *Let $\hat{\ell}_i(s)$ satisfy the equations (4.6.1) and (4.6.4). Then a version of the minimal martingale measure is characterized by:*

$$\begin{aligned} v_j &= \sum_{i=1}^m \hat{\ell}_i(s) \sigma_{ij}^W(s), \\ r_j(s) &= 1 - \int_{\mathbb{R}} \sum_{i=1}^m \hat{\ell}_i(s) \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz, \\ \psi_j(s, z) &= \frac{1 - \sum_{i=1}^m \hat{\ell}_i(s) \hat{\sigma}_{ij}^N(s, z)}{1 - \int_{\mathbb{R}} \sum_{i=1}^m \hat{\ell}_i(s) \hat{\sigma}_{ij}^N(s, z) \phi_j(s, z) dz}. \end{aligned} \quad (4.6.6)$$

If such $\hat{\ell}$ does not exist then the minimum martingale measure does not exist.

We can notice now that the minimal martingale measure corresponds to the utility $u(x) = x - \frac{x^2}{2}$. Since it is in general a signed measure we in addition require $x < 1$. We can also conclude that in our model the MMM is the variance optimal measure and also the exponential MRE.

4.7 Classification of optimal EMMs by its utility function.

The first and the most common current way to classify optimal EMMs is by the utility function. As you can see from Sections 4.2-4.6 the density and the parameters of all optimal EMMs satisfy (4.5.2) and (4.2.7) for different utility functions. Here we want to summarize the results of the previous sections in a table

EMM	Utility function
Minimal martingale	$x - \frac{x^2}{2}, x < 1$
Variance optimal	$x - \frac{x^2}{2}, x < 1$
Exponential MRE	$x - \frac{x^2}{2}, x < 1$
general Esscher transform	$1 - e^{-x}$
MRE	$1 - e^{-x}$
Reverse Relative Entropy	$\log(1 + x), x > -1$

We could not classify the EMM obtained by the Esscher transform for exponential processes since we know its utility function only in the one dimensional case

$$u(x) = \frac{\vartheta}{\vartheta + 1} \left(\left(\frac{x}{\vartheta} + 1 \right)^{\vartheta} - 1 \right),$$

where $\vartheta < 0$ and $x > -\vartheta$. As we mentioned before the main problem with this last optimal EMM is that the optimization process was applied when the stock price was given in the form $S = \exp(R)$ and for the other six optimal measures and the general case, utility maximization was considered when the price is in the form $S = \mathcal{E}(\hat{R})$. Despite the fact that both forms of the price are equivalent we obtain different results after optimization.

In our market model the Minimal martingale, Variance optimal and exponential MRE measures are the same. The EMM obtained by the general Esscher transform and the MRE are also the same. We will try now to find the relationship between these two measures. We determine this relationship using the fact that the first measure can

be obtained by minimizing $E_{\hat{P}} \mathcal{L}og(Z)$ and the second one by minimizing $E_{\hat{P}} \log Z$. Let v, η satisfy (3.3.9) and let

$$\int_0^t \lambda_j(s) \int_{\mathbb{R}} (e^{-\eta_j(s,z)} - 1) \phi_j(s, z) dz ds \in \mathcal{A}_{loc}^+.$$

Under this condition we can define $\hat{Z}$ as $\hat{Z} = \mathcal{E}(\hat{M})$, where

$$\begin{aligned} \hat{M} = & - \sum_{j=1}^d \int_0^t v_j(s) dW_j - \frac{1}{2} \sum_{j=1}^d \int_0^t (v_j(s))^2 ds \\ & + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (e^{-\eta_j(s,z)} - 1) dq_j(s, z) \\ & + \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} (e^{-\eta_j(s,z)} - 1 - \eta_j(s, z)) \phi_j(s, z) dz ds. \end{aligned}$$

Hence

$$\begin{aligned} \hat{Z} &= \exp \left(- \sum_{j=1}^d \int_0^t v_j(s) dW_j - \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \eta_j(s, z) dq_j(s, z) \right) \\ &= \exp(\mathcal{L}og Z). \end{aligned} \tag{4.7.1}$$

Hence if we minimize $\log \hat{Z}$ we minimize $\mathcal{L}og Z$. If Z is the density of an EMM we have to check whether $\hat{Z}$ is also a density of the same EMM. This is only possible when $\hat{Z} = Z$ i.e. the following must be true

$$\frac{1}{2} \sum_{j=1}^d \int_0^t (v_j(s))^2 ds - \sum_{j=1}^{k-d} \int_0^t \lambda_j(s) \int_{\mathbb{R}} (e^{-\eta_j(s,z)} - 1 - \eta_j(s, z)) \phi_j(s, z) dz ds = 0 \tag{4.7.2}$$

If the above equation has the solution satisfying (3.3.9) then the exponential MRE can be obtained as the MRE measure for the density $\hat{Z}$ and since $\hat{Z} = Z$ under (4.7.2) both measures are the same under (4.7.2). We can see that if the price is a pure continuous or a pure jump process there is no solution of (4.7.2). Hence the case when (4.7.2) holds is very restrictive.

4.8 Minimal distance martingale measures.

Besides the classification of an optimal EMM by utility function there is another classification introduced by Goll, Ruchendorf (2001). They combined all known optimal EMMs in the class of minimal distance martingale measures.

Definition 28 (Goll, Ruchendorf (2001)) *Let $Q \ll P$ and let $f : (0, \infty) \rightarrow \mathbb{R}$ be a continuous, strictly convex and differentiable function. We assume also that $f(0) = \lim_{x \downarrow 0} f(x)$. Then f -divergence between Q and P is defined as*

$$f(Q||P) = Ef \left(\frac{dQ}{dP} \right),$$

if $Ef \left(\frac{dQ}{dP} \right)$ exists or $f(Q||P) = \infty$ if does not.

Let $\mathcal{K}$ denote a convex set of probability measures dominated by P . A measure $\tilde{P} \in \mathcal{K}$ is called the f -projection of P on $\mathcal{K}$ if $f(Q||P) = \inf_{Q \in \mathcal{K}} f(Q||P) = f(\mathcal{K}||P)$.

There are a number of properties of f -projection. We will not mention those here, except the following important one

Proposition 31 (Goll, Ruchendorf (2001)) *Let $f'(0) = -\infty$. Assume the existence of a measure $Q \in \mathcal{K}$ such that $Q \sim P$ and $f(Q||P) < \infty$. If $\tilde{P}$ is the f -projection of P , then $\tilde{P} \sim P$.*

Let $\mathcal{M}^s$ be the set of signed EMMs. As we checked before for the EMM $\tilde{P}$ in our case we have that $G_\pi(t)$ is a $\tilde{P}$ -martingale and by the definition $E_{\tilde{P}}(G_\pi(t)) = 0$. The next proposition provides the characterization of minimal distance martingale measures (the necessary condition).

Proposition 32 (Goll, Ruchendorf (2001)) *Let $\tilde{P} \in \mathcal{M}^s$ satisfy $f(\tilde{P}||P) < \infty$ and $f'(\frac{d\tilde{P}}{dP}) \in L(\tilde{P})$. If $\tilde{P}$ is the f -projection of P on $\mathcal{M}^s$, then*

$$f'(\frac{d\tilde{P}}{dP}) = v + G_\pi(T) \tag{4.8.1}$$

for the strategy π .

We call the f -projection of P on $\mathcal{M}^s$ in the above proposition a **minimal distance martingale measure**. Generally the minimal distance martingale measure is a signed EMM. The next proposition provides us with sufficient condition

Proposition 33 (Goll, Ruchendorf (2001)) 1. Let $\tilde{P} \in \mathcal{M}^s$ satisfy $f(\tilde{P}||P) < \infty$. Let also S be a square integrable $\tilde{P}$ -martingale and for a bounded strategy π (4.8.1) fulfilled. Then $\tilde{P}$ is the f -projection of P on $\mathcal{M}^s$.

2. Let $\tilde{P} \in \mathcal{M}^s$ satisfy $f(\tilde{P}||P) < \infty$ and $f'(\frac{d\tilde{P}}{dP}) \in L(\tilde{P})$. Let also S be locally bounded and for a bounded strategy π (4.8.1) fulfilled. Then $\tilde{P}$ is the f -projection of P on $\mathcal{M}^s$.

In our case the optimal measures we considered can be represented as minimal distance martingale measures for different functions f

EMM	function $f(y)$
Variance optimal	y^2
Exponential MRE	$y \mathcal{L}og(y)$
MRE	$y \log y$
Reverse Relative Entropy	$-\log y$

It is not clear how to classify the minimal martingale measure and the general Esscher transformation by the function f . Goll and Ruchendorf (2001) obtained a form of f for the Esscher transformation only in case of Levy processes.

Now we want to establish the connection between the minimal distance martingale measures and the measures obtained by the utility maximization. Since the utility maximization measure was defined as a probability measure and the minimal distance measures are generally signed measures, in order to compare them we have to change the definition of the utility function and measure. We recall the definition of restricted utility function (Section 4.5) and slightly change the definition of the maximum utility measure.

Definition 29 Let $\hat{u}$ be the restricted utility function with domain $(\bar{x}, \infty)$. A signed EMM $\tilde{P}(x)$ is called the **minimax measure** for $x > \bar{x}$ if it minimizes over all $Q \in \mathcal{M}^s$ the quantity

$$U_Q(x) = \sup\{Eu(\xi(t)) : E_Q\xi(t) \leq x, E\hat{u}(\xi)(t-) < \infty\}$$

In general the minimax measure depends on choice of x but for the utility functions considered in previous sections the minimax measure is independent of x (Goll, Ruchendorf (2001)). Now we can talk about connection between minimax and minimal distance martingale measures.

Proposition 34 (Goll, Ruchendorf (2001)) *We assume there is such an x that the minimax measure corresponding to this x exists. Let for any $Q \in \mathcal{M}^s$ and any $a > 0$*

$$E_Q \frac{1}{\hat{u}'(a \frac{dQ}{dP})} < \infty.$$

Let $\tilde{P}$ be an EMM (the probability measure here) such that

$$a \frac{d\tilde{P}}{dP} - a \frac{d\tilde{P}}{dP} \frac{1}{\hat{u}'(a \frac{d\tilde{P}}{dP})} < \infty.$$

Then if $\tilde{P}$ is a minimal distance martingale measure then it is a minimax measure. Moreover

$$U_{\tilde{P}}(x) = \sup_{\pi} E \hat{u}(x + G_{\pi}(T));$$

$$\frac{1}{\hat{u}'(a \frac{d\tilde{P}}{dP})} = x + G_{\pi^{\hat{u}}}(T),$$

where $\pi^{\hat{u}}$ is a strategy. And if $Q \in \mathcal{M}^s$ with

$$a \frac{dQ}{dP} - a \frac{dQ}{dP} \frac{1}{\hat{u}'(a \frac{d\tilde{P}}{dP})} < \infty$$

then

$$E_Q \frac{1}{\hat{u}'(a \frac{dQ}{dP})} < E_{\tilde{P}} \frac{1}{\hat{u}'(a \frac{d\tilde{P}}{dP})}.$$

In the above proposition $f = \frac{1}{\hat{u}'}$ for the minimax measure. Using the above proposition for different choices of $\hat{u}$ (or different choices of f) we can construct the densities that correspond to the minimax (or the minimal distance martingale) measures. We also can obtain the optimal strategy for the market. It will be $\pi^{\hat{u}}$. We can conclude that the above proposition shows the connection between maximum utility approach and the minimal distance approach.

Chapter 5

Option pricing for simple models.

5.1 Continuous market model.

It is worth mentioning here how our results apply to a continuous market, i.e. the market where the price of the stock is determined as a continuous semimartingale. We know that every (multidimensional) continuous martingale can be represented as an integral with respect to a (multidimensional) Brownian motion (e.g. Shiryaev (1987)). We discussed in the Section 3.1 that

$$\hat{R} = \int_0^t \hat{b}_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s)$$

with square-integrable σ^W is a general form of continuous semimartingale. Hence in our setting the price is given by:

$$S_i(t) = \mathcal{E} \left(\int_0^t \hat{b}_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) \right) \quad (5.1.1)$$

is a general form of a positive continuous semimartingale.

The equation (3.3.9) can be written as

$$\hat{b}_i(t) = \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) \quad (5.1.2)$$

for $i = 1, \dots, m$, $m < d$. Hence the solution of the above linear system of m equations, if it exists, will provide us with the EMM with density of the following form

$$\begin{aligned} Z(t) &= \mathcal{E} \left(- \sum_{i=1}^d \int_0^t v_j(s) dW_j(s) \right) \\ &= \exp \left(- \sum_{i=1}^d \left(\int_0^t v_j(s) dW_j(s) - \frac{1}{2} \int_0^t (v_j(s))^2 ds \right) \right). \end{aligned}$$

We can conclude that an EMM is characterized by d parameters $v_j(t)$. If the equation (5.1.2) has a unique solution then we have only one EMM in our market and our market is complete. If we have more than one solution we have more than one EMM and we need an optimization procedure to obtain the optimal EMM. As can be seen from Chapter 4 all optimal measures considered here are the same when the jump part disappears. And the EMM parameter in this case is given by

$$v_j(t) = -u''(0) \sum_{i=1}^m \hat{\pi}_i(t) \sigma_{ij}^W(t),$$

where $\hat{\pi}_i \in L(S_i)$ is a predictable process and $\frac{\hat{\pi}_i(t)}{S_i(t)}$ is an optimal strategy for the utility function $u(x)$. From (5.1.2) we obtain

$$\hat{b}_i(t) = -u''(0) \sum_{i_0=1}^m \sum_{j=1}^d \hat{\pi}_{i_0}(t) \sigma_{i_0j}^W(t) \sigma_{ij}^W(t).$$

Since in all the cases described in Chapter 4 we have a utility function with $u''(0) = -1$, we can rewrite the above equation as

$$\hat{b} = (\Sigma \Sigma^{tr}) \hat{\pi}.$$

Since we require $\det(\Sigma \Sigma^{tr}) \neq 0$ (see Proposition 16) we always have a unique solution for $\hat{\pi}$ here. Hence we have a unique optimal EMM and strategy. Option pricing in this case becomes fairly easy since we know the law of W_j under the new measure:

$$\tilde{W}_j = W_j + \int_0^t v_j(s) ds$$

is a Brownian motion under $\tilde{P}$. We will discuss details of option pricing in more general setting.

5.2 Jump-diffusion with constant intensities.

Now we apply our results in the case when the prices are geometric Levy processes. In Section 2.4 we already reviewed different models with prices determined by Levy processes. Chan(1999), Shiryaev (1999) and Fujiwara, Miyahara (2003) described optimal EMMs for the case of Levy processes. Our model is obviously more general since it does not require independent increments of the process. In order to make the process R in our model a Levy process we require some additional conditions. For example if we take

$$\begin{aligned}\sigma_{ij}^N(t, z) &= \sigma_{ij}^N(z); \\ \sigma_{ij}^W(t) &= \sigma_{ij}^W = \text{const}; \\ b_i(t) &= b_i; \\ \lambda_j(t) &= \lambda_j = \text{const}; \\ \phi_j(t, z)dz &= \phi_j(dz).\end{aligned}$$

We then obtain the Levy process

$$\hat{R}_i = \hat{b}_i t + \sum_{j=1}^d \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} (e^{\sigma_{ij}^N(z)} - 1) dq_j(s, z)$$

with

$$\hat{b}_i = b_i + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W)^2 + \sum_{j=1}^{k-d} \lambda_j \int_{\mathbb{R}} (e^{\sigma_{ij}^N(z)} - 1 - h_{ij}(\sigma_{ij}^N(z))) \phi_j(dz).$$

In other words the price process is determined by a Brownian motion and a compound Poisson process.

For the utility function u the corresponding maximum utility EMM $\tilde{P}$ is going to be determined by the density process $Z = \mathcal{E}(M^{\hat{\pi}})$ where:

$$\begin{aligned}M^{\hat{\pi}}(t) &= u''(0) \sum_{j=1}^d \sum_{i=1}^m \sigma_{ij}^W \int_0^t \hat{\pi}_i(s) dW_j(s) \\ &\quad + \sum_{j=1}^{k-d} \int_0^t \int_{\mathbb{R}} \left(u' \left(\sum_{i=1}^m \hat{\pi}_i(s) \hat{\sigma}_{ij}^N(z) \right) - 1 \right) dq_j(s, z)\end{aligned}$$

and

$$\begin{aligned} & \hat{b}_i + u''(0) \sum_{j=1}^d \sum_{i_1=1}^m \sigma_{ij}^W \sigma_{i_1 j}^W \hat{\pi}_{i_1}(t) \\ & + \sum_{j=1}^{k-d} \lambda_j \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(z) \left(u' \left(\sum_{i_1=1}^m \hat{\pi}_{i_1}(t) \hat{\sigma}_{i_1 j}^N(z) \right) - 1 \right) \phi_j(dz) = 0 \end{aligned} \quad (5.2.1)$$

From the above condition on $\hat{\pi}$ we can conclude that $\hat{\pi}$ does not depend on t in this case. Thus $\tilde{P}$ has parameters that do not depend on time:

$$\begin{aligned} v_j &= -u''(0) \sum_{i=1}^m \hat{\pi}_i \sigma_{ij}^W, \\ r_j &= \int_{\mathbb{R}} u' \left(\sum_{i=1}^m \hat{\pi}_i \hat{\sigma}_{ij}^N(z) \right) \phi_j(dz), \\ \psi_j(z) &= \frac{u' \left(\sum_{i=1}^m \hat{\pi}_i \hat{\sigma}_{ij}^N(z) \right)}{\int_{\mathbb{R}} u' \left(\sum_{i=1}^m \hat{\pi}_i \hat{\sigma}_{ij}^N(y) \right) \phi_j(dy)}. \end{aligned} \quad (5.2.2)$$

In the case of a Minimal martingale (or Variance Optimal, Exponential MRE) EMM, the utility function is $u(x) = x - \frac{x^2}{2}, x < 1$. Then the equation for the optimal exponential strategy will be a solution of

$$\hat{b}_i - \sum_{j=1}^d \sum_{i_1=1}^m \sigma_{ij}^W \sigma_{i_1 j}^W \hat{\pi}_{i_1} - \sum_{j=1}^{k-d} \lambda_j \sum_{i_1=1}^m \hat{\pi}_{i_1} \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(z) \hat{\sigma}_{i_1 j}^N(z) \phi_j(dz) = 0 \quad (5.2.3)$$

under the condition

$$\sum_{i_1=1}^m \hat{\pi}_{i_1} \hat{\sigma}_{i_1 j}^N(z) < 1. \quad (5.2.4)$$

This is a linear system of m equations with m unknowns $\pi_1, \dots, \pi_m$. This system has a unique solution if $\det(\Sigma \Sigma^{tr}) \neq 0$, where Σ is defined similarly as in the section 3.3 for the function $g \equiv 1$. If this condition satisfies (5.2.4) then we have an optimal EMM, otherwise we have a signed optimal EMM.

The above example also shows that sometimes the price after transformation remains in the same “class”. Indeed since the EMM parameters are constants the price after transformation will again be determined by the sum of a Brownian motions and compound Poisson processes with coefficients.

Now let us further simplify the above situation. We assume that $N_j, j = 1, \dots, k-d$ are Poisson processes without any marks and with constant intensities λ_j . We also assume that $\sigma_{ij}^N(z) = \sigma_{ij}^N$. In other words the price process S_i for this model will be given by

$$S_i(t) = \exp \left(b_i t + \sum_{j=1}^d \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} \sigma_{ij}^N N_j(t) \right). \quad (5.2.5)$$

We can rewrite this expression as

$$S_i(t) = \exp \left(\int_0^t b_i ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W dW_j(t) + \sum_{j=1}^{k-d} \int_0^t \sigma_{ij}^N dN_j(t) \right).$$

Now we apply Proposition 14 and obtain

$$\begin{aligned} S_i(t) &= \mathcal{E} \left(\int_0^t \hat{b}_i ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W dW_j + \sum_{j=1}^{k-d} \int_0^t (e^{\sigma_{ij}^N} - 1) dq_j(s) \right) \\ &= \mathcal{E} \left(\hat{b}_i t + \sum_{j=1}^d \sigma_{ij}^W W_j(t) + \sum_{j=1}^{k-d} (e^{\sigma_{ij}^N} - 1) q_j(t) \right), \end{aligned} \quad (5.2.6)$$

where

$$\hat{b}_i = b_i + \frac{1}{2} \sum_{j=1}^d (\sigma_{ij}^W)^2 + \sum_{j=1}^{k-d} \lambda_j (e^{\sigma_{ij}^N} - 1 - \sigma_{ij}^N)$$

and

$$dq_j(t) = dN_j(t) - \lambda_j dt.$$

The parameters of the minimal martingale (the variance optimal, exponential MRE) measure $\tilde{P}$ here will be:

$$\begin{aligned} v_j &= \sum_{i=1}^m \hat{\pi}_i \sigma_{ij}^W, \\ r_j &= 1 - \sum_{i=1}^m \hat{\pi}_i (e^{\sigma_{ij}^N} - 1), \end{aligned} \quad (5.2.7)$$

where $\hat{\pi}_i$ satisfy

$$\begin{aligned} \hat{b}_i - \sum_{j=1}^d \sum_{i_1=1}^m \sigma_{ij}^W \sigma_{i_1 j}^W \hat{\pi}_{i_1} - \sum_{j=1}^{k-d} \lambda_j \sum_{i_1=1}^m \hat{\pi}_{i_1} (e^{\sigma_{i_1 j}^N} - 1) (e^{\sigma_{ij}^N} - 1) &= 0 \quad (5.2.8) \\ \sum_{i_1=1}^m \hat{\pi}_{i_1} (e^{\sigma_{i_1 j}^N} - 1) &< 1. \end{aligned}$$

Here we can notice that since there is no jump-size distribution involved the parameter $\psi_j \equiv 1$.

Now let Σ be a $m \times k$ matrix with rows

$$\left(\sigma_{i1}^W, \dots, \sigma_{id}^W, \sqrt{\lambda_j} \left(e^{\sigma_{i1}^N} - 1 \right), \dots, \sqrt{\lambda_j} \left(e^{\sigma_{i(k-d)}^N} - 1 \right) \right),$$

$i = 1, \dots, m$. If $\det(\Sigma \Sigma^{tr}) \neq 0$ then there is a unique solution of (5.2.8)

$$\hat{\pi} = (\Sigma \Sigma^{tr})^{-1} \hat{b}.$$

Now we can calculate the parameters v_j, r_j and hence apply Proposition 18 to obtain

$$\tilde{W}_j(t) = W_j(t) + \sum_{i=1}^m \hat{\pi}_i \sigma_{ij}^W t \quad (5.2.9)$$

which is a Brownian motion under $\tilde{P}$. For the processes

$$\tilde{Q}_{ij}(t) = \left(e^{\sigma_{ij}^N} - 1 \right) \left(N_j(t) + \lambda_j \left(\sum_{i=1}^m \hat{\pi}_i \left(e^{\sigma_{i1j}^N} - 1 \right) - 1 \right) t \right) \quad (5.2.10)$$

we know that $N_j(t)$ under $\tilde{P}$ is again a Poisson process with the constant intensity

$$\tilde{\lambda}_j = \left(1 - \sum_{i=1}^m \hat{\pi}_i \left(e^{\sigma_{ij}^N} - 1 \right) \right) \lambda_j.$$

Now we can rewrite (5.2.5) as

$$S_i(t) = \exp \left(\tilde{b}_i t + \sum_{j=1}^d \sigma_{ij}^W \tilde{W}_j(t) + \sum_{i=1}^{k-d} \sigma_{ij}^N N_j(t) \right). \quad (5.2.11)$$

where

$$\tilde{b}_i = b_i - \sum_{i=1}^m \sum_{j=1}^d \hat{\pi}_i \sigma_{i1j}^W \sigma_{ij}^W t$$

Hence the price of European call option $(S_i(T) - K_i)^+$ is:

$$\begin{aligned} \mathcal{C}_i &= E_{\tilde{P}}(S_i(T) - K_i)^+ \\ &= E_{\tilde{P}} \left(\exp \left(\tilde{b}_i T + \sum_{j=1}^d \sigma_{ij}^W \sqrt{T} \frac{\tilde{W}_j(T)}{\sqrt{T}} + \sum_{j=1}^{k-d} \sigma_{ij}^N N_j(T) \right) - K_i \right)^+. \end{aligned} \quad (5.2.12)$$

Since W and N are independent processes we have:

$$C_i = \int_{A(x_1, \dots, x_k)} c_i(x_1, \dots, x_k) \prod_{j=1}^d f_W(x_j) \prod_{j=1}^{k-d} f_{N_j}(x_j) \prod_{j=1}^k dx_j.$$

where the set

$$A_i(x_1, \dots, x_k) = \left\{ x \in \mathbb{R}^k \mid \tilde{b}_i T + \sum_{j=1}^d \sigma_{ij}^W \sqrt{T} x_j + \sum_{j=1}^{k-d} \sigma_{ij}^N x_j > \log K_i \right\},$$

$$c_i(x_1, \dots, x_k) = \exp \left(\tilde{b}_i T + \sum_{j=1}^d \sigma_{ij}^W \sqrt{T} x_j + \sum_{j=1}^{k-d} \sigma_{ij}^N x_j \right) - K_i$$

and f_X is the probability density function of X . In our case $f_W = f_{(W(T))/\sqrt{T}}$ is a probability density function of a standard normal variable, and f_{N_j} is a Poisson probability density function with parameter $\tilde{\lambda}_j$. To continue here we have to simplify the problem without significant loss of generality by taking $d = 1$ and $k = 3$, i.e. we have one Brownian motion and two Poisson processes.

$$\begin{aligned} C_i &= \sum_{j=1}^2 \sum_{n_j=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{A_i(x, n_1, n_2)} (c_i(x, n_1, n_2) - K) e^{-\frac{x^2}{2}} dx \prod_{j=1}^2 \frac{e^{-\tilde{\lambda}_j T} (\tilde{\lambda}_j T)^{n_j}}{n_j!} \\ &= \sum_{j=1}^2 \sum_{n_j=1}^{\infty} e^{(\tilde{b}_i - \sum_{j=1}^2 \tilde{\lambda}_j) T + \sum_{j=1}^2 \sigma_{ij}^N n_j} (1 - \Phi(B_i^1(n_1, n_2))) \prod_{j=1}^2 \frac{(\tilde{\lambda}_j T)^{n_j}}{n_j!} \\ &\quad - K \sum_{j=1}^2 \sum_{n_j=1}^{\infty} (1 - \Phi(B_i^2(n_1, n_2))) \prod_{j=1}^2 \frac{e^{-\tilde{\lambda}_j T} (\tilde{\lambda}_j T)^{n_j}}{n_j!}, \end{aligned} \quad (5.2.13)$$

where Φ is the distribution function of the standard normal variable, and

$$\begin{aligned} B_i^1(n_1, n_2) &= \frac{1}{\sigma_i^W \sqrt{T}} (\log K_i - (\tilde{b}_i T + \sum_{j=1}^2 \sigma_{ij}^N n_j) + (\sigma_i^W)^2 \frac{T}{2}), \\ B_i^2(n_1, n_2) &= \frac{1}{\sigma_i^W \sqrt{T}} (\log K_i - (\tilde{b}_i T + \sum_{j=1}^2 \sigma_{ij}^N n_j) - (\sigma_i^W)^2 \frac{T}{2}). \end{aligned} \quad (5.2.14)$$

Now we can obtain the final result quite rapidly by simulation. As a conclusion we can notice that (5.2.13) is an explicit expression for the price of the European option

(since we know the exact expressions for $\tilde{\lambda}_j$). This result illustrates the application of this thesis to option pricing in a simple case of Poisson processes. Several authors have obtained similar results in different ways.

Remark 14 *This example provides us with a unique EMM if $m = k$ and all the general EMM conditions are fulfilled. This follows from Bardhan and Chao(1995), but also can be easily checked directly.*

5.3 Example of the general jump-diffusion market.

We consider our model (Chapter 3) with two assets $S_i, i = 1, 2$. These assets are driven by one Brownian motion W and two marked point processes N_1 and N_2 . We assume that

$$\begin{aligned}
 b_i(t) &= b_i, \quad i = 1, 2; \\
 \sigma_{11}^N(t, z) &= \sigma_{11}^N(s)z^2; \\
 \sigma_{12}^N(t, z) &= \sigma_{12}^N(s)z; \\
 \sigma_{21}^N(t, z) &= \sigma_{21}^N(s)z; \\
 \sigma_{22}^N(t, z) &= \sigma_{22}^N(s)z^2; \\
 \phi_1(t, dz) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - z_1)^2}{2}\right), \quad z \in (0, mx); \\
 \phi_2(t, dz) &= \frac{1}{z_2} \exp\left(-\frac{z}{z_2}\right), \quad z \in (0, mx).
 \end{aligned} \tag{5.3.1}$$

In other words the sizes of jumps of process N_1 are normally distributed with mean z_1 and the sizes of jumps of process N_2 are exponentially distributed with parameter z_2 . We choose z_1 and $mx > \max(z_1, z_2)$ large and z_2 is small enough such that jump sizes exponentially and normally distributed are most likely in the interval $(0, mx)$, for some $mx \in \mathbb{R}_+$: i.e. the integrals

$$\begin{aligned}
 MX_1 &= \int_0^{mx} \phi_1(z)dz \approx 1 \\
 MX_2 &= \int_0^{mx} \phi_2(z)dz \approx 1.
 \end{aligned}$$

We also assume that the moment generating functions of distributions ϕ_1 and ϕ_2 are very close (say in uniform norm) to the moment generating function of the normal and exponential distributions respectively. We can see that $\sigma_{ij}^N(t, z)$ is an invertable strictly monotone function of z on the interval $(0, mx)$. The other jump-diffusion conditions are also fulfilled.

Hence the logarithms of prices are given by

$$\begin{aligned}\log(S_1(t)) &= b_1 t + \int_0^t \sigma_1^W(s) dW(s) \\ &\quad + \int_0^t \sigma_{11}^N(s) \int_0^{mx} z^2 dN_1(s, z) + \int_0^t \sigma_{12}^N(s) \int_0^{mx} z dN_2(s, z); \\ \log(S_2(t)) &= b_2 t + \int_0^t \sigma_2^W(s) dW(s) \\ &\quad + \int_0^t \sigma_{21}^N(s) \int_0^{mx} z dN_1(s, z) + \int_0^t \sigma_{22}^N(s) \int_0^{mx} z^2 dN_2(s, z) \quad (5.3.2)\end{aligned}$$

Now we calculate $\hat{b}_i$ using Proposition 3.2.6 and keeping in mind that sizes of jumps are bounded.

$$\begin{aligned}\hat{b}_1(t) &= b_1 t + \frac{1}{2}(\sigma_1^W(t))^2 + \lambda_1(t) \int_0^{mx} \left(e^{\sigma_{11}^N(t)z^2} - 1 - \sigma_{11}^N(t)z^2 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{12}^N(t)z} - 1 - \sigma_{12}^N(t)z \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz; \\ \hat{b}_2(t) &= b_2 t + \frac{1}{2}(\sigma_2^W(t))^2 + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{21}^N(t)z} - 1 - \sigma_{21}^N(t)z \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{22}^N(t)z^2} - 1 - \sigma_{22}^N(t)z^2 \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz.\end{aligned}$$

Recalling the first and the second moments for the exponential and normal distributions and assuming that z is mostly concentrated on the interval $(0, mx)$ we can rewrite $\hat{b}$ as

$$\begin{aligned}\hat{b}_1(t) &\approx b_1 t + \frac{1}{2}(\sigma_1^W(t))^2 - (\sigma_{11}^N(t)(1 + z_1^2) + \sigma_{12}^N(t)z_2) \\ &\quad + \lambda_1(t) \int_0^{mx} \left(e^{\sigma_{11}^N(t)z^2} - 1 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{12}^N(t)z} - 1 \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz; \\ \hat{b}_2(t) &\approx b_2 t + \frac{1}{2}(\sigma_2^W(t))^2 - (\sigma_{21}^N(t)z_1 + \sigma_{22}^N(t)2z_2^2) \\ &\quad + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{21}^N(t)z} - 1 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz + \lambda_2(t) \int_0^{mx} \left(e^{\sigma_{22}^N(t)z^2} - 1 \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz.\end{aligned} \quad (5.3.3)$$

Let also

$$\begin{aligned} bb_1(t) &= b_1 t + \frac{1}{2}(\sigma_1^W(t))^2 - (\sigma_{11}^N(t)(1 + z_1^2) + \sigma_{12}^N(t)z_2); \\ bb_2(t) &= b_2 t + \frac{1}{2}(\sigma_2^W(t))^2 - (\sigma_{21}^N(t)z_1 + \sigma_{22}^N(t)2z_2^2). \end{aligned}$$

Obviously the above market (5.3.2) is not complete in general. Hence we want to find the optimal EMM for this market.

First we construct the Esscher martingale measure for exponential processes (ESEP-measure). We rewrite the condition (4.2.6) for the parameter ϑ replacing ϑ by $-\hat{\pi}$ (however $\hat{\pi}$ here will not be the optimal exponential strategy).

$$\begin{aligned} bb_1(t) &\approx \sigma_1^W(t) (\hat{\pi}_1(t)\sigma_1^W(t) + \hat{\pi}_2(t)\sigma_2^W(t)) \\ &\quad + \lambda_1(t) \int_0^{mx} e^{-\hat{\pi}_1(t)\sigma_{11}^N(t)z^2 - \hat{\pi}_2(t)\sigma_{21}^N(t)z} \left(e^{\sigma_{11}^N(t)z^2} - 1 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^{mx} e^{-\hat{\pi}_1(t)\sigma_{12}^N(t)z - \hat{\pi}_2(t)\sigma_{22}^N(t)z^2} \left(e^{\sigma_{12}^N(t)z} - 1 \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz; \\ bb_2(t) &\approx \sigma_2^W(t) (\hat{\pi}_1(t)\sigma_1^W(t) + \hat{\pi}_2(t)\sigma_2^W(t)) \\ &\quad + \lambda_1(t) \int_0^{mx} e^{-\hat{\pi}_1(t)\sigma_{11}^N(t)z^2 - \hat{\pi}_2(t)\sigma_{21}^N(t)z} \left(e^{\sigma_{21}^N(t)z} - 1 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-z_1)^2}{2}} dz \\ &\quad + \lambda_2(t) \int_0^{mx} e^{-\hat{\pi}_1(t)\sigma_{12}^N(t)z - \hat{\pi}_2(t)\sigma_{22}^N(t)z^2} \left(e^{\sigma_{22}^N(t)z^2} - 1 \right) \frac{1}{z_2} e^{-\frac{z}{z_2}} dz \quad (5.3.4) \end{aligned}$$

The integrals in the above expression are of the following form

$$\int_0^{mx} e^{-a_1 z^2 + a_2 z + a_3} dz.$$

The coefficients a_1, a_2, a_3 depend on z_1 and z_2 . Now we can rewrite these integrals as

$$\exp \left(a_3 + \frac{a_2^2}{4a_1} \right) \int_0^{mx} \exp \left(-\frac{\left(z - \frac{a_2}{2a_1} \right)^2}{2 \left(\frac{1}{2a_1} \right)} \right) dz. \quad (5.3.5)$$

Again if we require that z_1, z_2 and mx satisfy the conditions mentioned earlier in this section so this is close to

$$\text{const} \int_{\mathbb{R}} \exp \left(-\frac{\left(z - \frac{a_2}{2a_1} \right)^2}{2 \left(\frac{1}{2a_1} \right)} \right) dz.$$

If we also require in all these integrals $a_1 > 0$, the above integral can be calculated using the normal distribution.

After calculation of the integrals we can solve the equations for $\hat{\pi}$. Then using (4.3.6) we obtain approximate formulas for the ESEP measure parameters. Thus we can approximately obtain the ESEP density process Z using (3.3.21) and finally the European option price will be given by

$$C_i = EZ(S_1 - K_1)^+$$

This example can be used for more precise calculations of the price of the option then the general case that we will consider in Chapter 6. The main disadvantage here is that we should require the means of distributions ϕ_1 and ϕ_2 to meet the certain conditions so they behave almost as a normal and exponential distribution respectively.

5.4 Convenient EMMs and EMMs invariant within a certain class.

The main idea of option pricing is to find a EMM and obtain the price of the, say, European call option as $E_{\tilde{P}}(S - K)^+$. In our case we know that the price process can be represented as (3.3.16) in terms of processes $\tilde{W}$ and $\tilde{Q}$. We know that after the measure transformation $\tilde{W}$ and $\tilde{Q}$ become martingales and we also know that $\tilde{W}$ is a $\tilde{P}$ -Brownian motion and $\tilde{Q}$ with respect to $\tilde{P}$ is a compensated point process. We can suggest further that after the measure transformation the process $\tilde{Q}$ belongs to a certain class, say Poisson processes. Then we can relatively easily calculate the price of the option. We call the measures with respect to which the process $\tilde{Q}$ belong to the certain class (call it the **target class**) a **convenient measure**. Of course for convenient measures it is hard to talk about optimality. But if we allow the target class be wide enough we can obtain the optimal convenient measure within this class.

We will consider a market model as in chapter 3. Now let

$$N_{ij}^Q(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dN_j.$$

We can also notice that the point process parameters of N^Q are

$$\begin{aligned}\lambda_{ij}^Q(t) &= \lambda_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \phi_j(t, z) dz, \\ \phi_{ij}^Q(t, dz) &= \frac{\hat{\sigma}_{ij}^N(t, z)}{\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, y) \phi_j(t, dy)} \phi_j(t, z) dz\end{aligned}\quad (5.4.1)$$

We use Proposition 18 to obtain the point process $\tilde{P}$ characteristics (the intensity and jump size distribution) of $\tilde{P}$ -martingale $\tilde{Q}$. We recall that

$$\tilde{Q}_{ij}(t) = \int_0^t \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) dN_j - \int_0^t \lambda_j(s) r_j(s) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(s, z) \psi_j(s, z) \phi_j(s, z) dz ds. \quad (5.4.2)$$

Hence

$$\tilde{Q}_{ij}(t) = N_j^Q(t) - \int_0^t \lambda_{ij}^Q(s) r_j(s) \int_{\mathbb{R}} \psi_j(s, z) \phi_{ij}^Q(s, dz) ds. \quad (5.4.3)$$

We conclude that the new point process parameters of $\tilde{Q}$ are

$$\begin{aligned}\lambda_{ij}^{\tilde{Q}}(t) &= r_j(t) \lambda_{ij}^Q(t) \int_{\mathbb{R}} \psi_j(t, z) \phi_{ij}^Q(t, dz) \\ &= \lambda_j(t) r_j(t) \int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, z) \psi_j(t, z) \phi_j(t, z) dz, \\ \phi_{ij}^{\tilde{Q}}(t, dz) &= \frac{\psi_j(t, z) \phi_{ij}^Q(t, dz)}{\int_{\mathbb{R}} \psi_j(t, z) \phi_{ij}^Q(t, dz)} \\ &= \frac{\hat{\sigma}_{ij}^N(t, z) \psi_j(t, z) \phi_j(t, z) dz}{\int_{\mathbb{R}} \hat{\sigma}_{ij}^N(t, y) \psi_j(t, y) \phi_j(t, dy)}.\end{aligned}\quad (5.4.4)$$

Hence

$$r_j(t) \psi_j(t, z) = \frac{\lambda_{ij}^{\tilde{Q}}(t) \phi_{ij}^{\tilde{Q}}(t, dz)}{\hat{\sigma}_{ij}^N(t, z) \lambda_j(t) \phi_j(t, z) dz}. \quad (5.4.5)$$

Now we fix the target distribution law of the point process N^Q , i.e. we fix $\lambda_{ij}^{\tilde{Q}}(t)$ and $\phi_{ij}^{\tilde{Q}}(t, dz)$. Then (3.3.13) can be written as

$$\hat{b}_i(t) + \sum_{j=1}^{k-d} \left(\lambda_{ij}^{\tilde{Q}}(t) - \lambda_{ij}^Q(t) \right) = \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t), \quad (5.4.6)$$

$i = 1, \dots, m$. At this point we assume that $d > m$. Then the above equation has a solution for v_j under the jump-diffusion conditions (including $\det(\Sigma\Sigma^{tr}) \neq 0$). Generally there can be more than one solution. Then we have more than one EMM and we need optimization. Again we recall formulas for the parameters of the maximum utility measure for utility function $u(x)$.

$$r_j(t)\psi_j(t, z) = u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right).$$

Now using (5.4.5) we obtain

$$u' \left(\sum_{i=1}^m \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) \right) = \frac{\lambda_{ij}^{\tilde{Q}}(t) \phi_{ij}^{\tilde{Q}}(t, dz)}{\hat{\sigma}_{ij}^N(t, z) \lambda_j(t) \phi_j(t, z) dz}. \quad (5.4.7)$$

and from (5.4.6)

$$-u''(0) \sum_{j=1}^d \sum_{i_1=1}^m \sigma_{ij}^W(t) \sigma_{i_1 j}^W(t) \hat{\pi}_i(t) = \hat{b}_i(t) + \sum_{j=1}^{k-d} \left(\lambda_{ij}^{\tilde{Q}}(t) - \lambda_{ij}^Q(t) \right). \quad (5.4.8)$$

The last system of m equations and m unknowns has a unique solution π as long as $\det(\Sigma\Sigma^{tr}) \neq 0$. If this solution π satisfies (5.4.7) then we have that $\tilde{P}$, obtained through the density of the form (4.1.5), is the maximum utility measure with the target point process distribution with parameters $\lambda_{ij}^{\tilde{Q}}(t)$ and $\phi_{ij}^{\tilde{Q}}(t, dz)$. If we assume that the target point process distribution belongs to a certain class, we can obtain that the maximum utility measure belongs to the class of corresponding convenient measures. Obviously if we pick a wider class there is a better chance that the solution of (5.4.8) satisfies (5.4.7) and hence the maximum utility measure is a convenient measure for this class.

Remark 15 *Another way to obtain an optimal convenient measure is to maximize the utility but only when the parameters of the EMM belong to a certain class, i.e. the constraints (5.4.5) and (5.4.6) are fulfilled for parameters v_j, r_j, ψ_j and we obtain the maximum under these constraints.*

Another interesting question: when does the process N^Q after the EMM transformation remain in the given target class. This kind of EMM we will call **invariant**.

There are several examples of invariant measures. First of all, if the target distribution is equal to the original then $r_j \psi_j \equiv 1$ and we have to require the original measure to be the EMM. Let us now assume that we fix $\lambda_{ij}^{\tilde{Q}}(t) = \lambda_{ij}^Q(t)$ and allow $\phi_{ij}^{\tilde{Q}}(t, dz) \neq \phi_{ij}^Q(t, dz)$. Then (5.4.6) can be rewritten as

$$\hat{b}_i(t) = \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t), \quad (5.4.9)$$

$i = 1, \dots, m$. If this system has a solution v_j then we obtain the EMM invariant for a class of point processes with fixed intensity $\lambda_{ij}^Q(t)$.

Remark 16 *The examples in the previous section are invariant for the class of Poisson processes.*

To illustrate convenient and invariant measures we consider the following example. We consider the market with the stock price given by:

$$S_i = \exp \left(\int_0^t b_i(s) ds + \sum_{j=1}^d \int_0^t \sigma_{ij}^W(s) dW_j(s) + \sum_{j=1}^{k-d} \int_0^t \sigma_{ij}^N(s) \int_{\mathbb{R}} z dN_j(s) \right) \quad (5.4.10)$$

and the sizes of jumps of the point process N_j do not depend on time, i.e the jump-size distribution is $\phi_j(z)$. The EMM for this market will be characterized by the parameters v_j, r_j, ψ_j satisfying

$$b_i(t) = \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) + \sum_{j=1}^{k-d} \lambda_j(t) \int_{\mathbb{R}} (e^{\sigma_{ij}^N(t)z} - 1)(1 - r_j(t) \psi_j(t)) \phi_j(z) dz. \quad (5.4.11)$$

Let

$$MG_f(u) = \int_{\mathbb{R}} e^{uz} f(z) dz$$

be a moment generating function corresponding to a probability density function $f(z)$. Then (5.4.11) can be rewritten as:

$$\begin{aligned} b_i(t) &+ \sum_{j=1}^{k-d} \lambda_j(t) (1 - MG_{\phi_j}(\sigma_{ij}^N(t))) \\ &= \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) + \sum_{j=1}^{k-d} r_j(t) \lambda_j(t) (1 - MG_{\psi_j \phi_j}(\sigma_{ij}^N(t))). \end{aligned} \quad (5.4.12)$$

As can be seen the left hand side does not depend on the measure parameters. Let the jump distribution of the point process N_j be exponential of the form $\phi_j(z) = \frac{1}{\zeta_j} e^{-\frac{z}{\zeta_j}}$. We further assume that $\zeta_j \sigma_{ij}^N(t) < 1$. The EMM given by (5.4.12) generally is not unique. We choose the measure under which the jump distribution $\psi_j \phi_j$ of the marked point process N_j after transformation is an exponential with parameter $\tilde{\zeta}_j$

$$\psi_j(z) \phi_j(z) = \frac{1}{\tilde{\zeta}_j} e^{-\frac{z}{\tilde{\zeta}_j}}$$

and assume that $\tilde{\zeta}_j \sigma_{ij}^N(t) < 1$. Then by (5.4.12) we obtain:

$$\begin{aligned} b_i(t) - \sum_{j=1}^{k-d} \lambda_j(t) \frac{\zeta_j \sigma_{ij}^N(t)}{1 - \zeta_j \sigma_{ij}^N(t)} \\ = \sum_{j=1}^d \sigma_{ij}^W(t) v_j(t) - \sum_{j=1}^{k-d} \lambda_j(t) \frac{\tilde{\zeta}_j \sigma_{ij}^N(t)}{1 - \tilde{\zeta}_j \sigma_{ij}^N(t)} r_j(t). \end{aligned} \quad (5.4.13)$$

This is a linear system of m equations and k unknowns v_j and r_j . If this system has a solution and all the other jump-diffusion conditions fulfilled we have a EMM such that the sizes of jumps of the point process N_j after transformation are exponentially distributed.

Similarly, we can obtain convenient equivalent measures for different kinds of jump distributions as long as we know formulas for the moment generating functions before and after the measure transformation. We can also optimize the choice of measures by changing the jump distribution function parameters.

Chapter 6

Numerical results.

6.1 Model and market simulation

In this chapter we will consider another example of a jump-diffusion market. We will numerically obtain an option price for this market to illustrate our methods.

We start with the model description. We assume that our market is given by (3.2.1). We also assume that in this market we have two stocks driven by one Brownian motion W and two marked point processes $N_j, j = 1, 2$. N_j has predictable intensity of jumps $\lambda_j(t)$. The jump sizes distribution of N_j is exponential with parameter β_j on the interval $(0, mx)$, i.e. the sizes of jumps is bounded by mx . Hence $\log(S_i) = R_i$ are special semimartingales for $i = 1, 2$ and we can rewrite (3.2.1) as

$$S_i = \exp \left(\int_0^t b_i(s) ds + \int_0^t \sigma_i^W(s) dW(s) + \sum_{j=1}^2 \int_0^t \int_0^{mx} \sigma_{ij}^N(s, z) dN_j(s, z) \right),$$

with parameters:

$$\begin{aligned} b_i(t) &= b_i, \quad i = 1, 2; \\ \sigma_i^W(t) &= \sigma_i^W(1) |\sin(a_i^W t)|, \quad i = 1, 2; \\ \sigma_{ij}^N(t, z) &= \sigma_{ij}^N(1, 1) |\sin(a_{ij}^N t)| z, \quad i = 1, 2, j = 1, 2; \\ \phi_j(t, z) &= \frac{1}{\beta_j} \exp \left(-\frac{z}{\beta_j} \right), \quad z \in (0, mx). \end{aligned}$$

(6.1.1)

We will choose $b_i, \sigma_i^W(1), \sigma_{i1}^N(1, 1)$ to be positive numbers and $\sigma_{i2}^N(1, 1)$ to be negative. Hence the second point process will provide us with negative jumps in the price process. a_i^W and a_{ij}^N are arbitrary real numbers. The parameters of the model are chosen to be periodic functions so they may illustrate seasonal changes in stock prices.

Now we specify the predictable intensity

$$\lambda_j(t) = \lambda_j^{pr}(t) + \lambda_j^0 |\sin(a_j^\lambda t)|,$$

where a_j^λ, λ_j^0 are arbitrary real numbers and $\lambda_j^{pr}(t)$ is a predictable component of the intensity. If the increase in the stock price S_i during the last few time intervals becomes larger than a certain number cgr_i the intensity of downward jumps $\lambda_2(t)$ becomes larger, i.e. $\lambda_2^{pr}(t) = adj_1$. If the decrease in the stock price S_i during the last few time intervals becomes larger than a certain number $cdec_i$ the intensity of upward jumps $\lambda_1(t)$ becomes larger, i.e. $\lambda_1^{pr}(t) = adj_2$. If there are no significant changes in stock prices the predictable components $\lambda_j^{pr}(t)$ of both intensities remain equal to zero. This choice of the intensity allows us to control the big jumps in our market.

Main model parameters are given in the following table.

Parameter	Value
time period T	1
number of time intervals n	200
jumps upper bound mx	20
strike for the first stock K_1	0.2
strike for the second stock K_2	0.2
drift for the first stock $b1$	0.05
drift for the second stock $b2$	0.05

For the rest of this model parameters, see Appendix A (program code).

We can notice here that all the jump-diffusion conditions for this market are fulfilled. Finally to simulate the price S_i we divide the time interval $[0, T]$ into n intervals. We will also assume that the jump part is equal to zero on the first time interval. In Figure 1 (Appendix B) you can see the paths of the stock price process S_1 after 50 simulations.

6.2 MMM transform.

Now we are going to obtain the minimal martingale measure for this market. Following Section 4.6 (4.4.2) the parameters of the density process are given by

$$\begin{aligned} v_j(t) &= \hat{\pi}_1(t)\sigma_1^W(t) + \hat{\pi}_2(t)\sigma_2^W(t), \\ r_j(t) &= 1 - \int_0^{mx} \sum_{i=1}^2 \hat{\pi}_i(s) \hat{\sigma}_{ij}^N(t, z) \phi_j(z) dz, \\ \psi_j(t, z) &= \frac{1 - \sum_{i=1}^2 \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z)}{r_j(t)}. \end{aligned} \quad (6.2.1)$$

where $\hat{\sigma}_{ij}^N(t, z) = e^{\sigma_{ij}^N(t, z)} - 1$ and $\hat{\pi}_i$, $i = 1, 2$ are given by the system of equations (4.6.1). We keep in mind that we have to choose the parameters of the model such that

$$\begin{aligned} r_j(t) &> 0 \\ \sum_{i=1}^2 \hat{\pi}_i(t) \hat{\sigma}_{ij}^N(t, z) &< 1. \end{aligned} \quad (6.2.2)$$

If the above conditions are not fulfilled the measure with the parameters (6.2.1) will be a signed measure.

In our case we can rewrite (4.6.1) as a system of two equations with two unknowns $\hat{\pi}_1$ and $\hat{\pi}_2$

$$\begin{aligned} \hat{b}_1(t) &= \hat{\pi}_1(t)((\sigma_1^W(t))^2 + \lambda_1(t) \int_0^{mx} (\hat{\sigma}_{11}^N(t, z))^2 \phi_1(z) dz \\ &\quad + \lambda_2(t) \int_0^{mx} (\hat{\sigma}_{12}^N(t, z))^2 \phi_2(z) dz) \\ &\quad + \hat{\pi}_2(t)(\sigma_1^W(t)\sigma_2^W(t) + \lambda_1(t) \int_0^{mx} \hat{\sigma}_{11}^N(t, z) \hat{\sigma}_{21}^N(t, z) \phi_1(z) dz \\ &\quad + \lambda_2(t) \int_0^{mx} \hat{\sigma}_{12}^N(t, z) \hat{\sigma}_{22}^N(t, z) \phi_2(z) dz); \end{aligned}$$

$$\begin{aligned}
\hat{b}_2(t) = & \hat{\pi}_1(t)(\sigma_1^W(t)\sigma_2^W(t) + \lambda_1(t) \int_0^{mx} \hat{\sigma}_{11}^N(t, z)\hat{\sigma}_{21}^N(t, z)\phi_1(z)dz \\
& + \lambda_2(t) \int_0^{mx} \hat{\sigma}_{12}^N(t, z)\hat{\sigma}_{22}^N(t, z)\phi_2(z)dz); \\
& + \hat{\pi}_2(t)((\sigma_2^W(t))^2 + \lambda_1(t) \int_0^{mx} (\hat{\sigma}_{21}^N(t, z))^2\phi_1(z)dz \\
& + \lambda_2(t) \int_0^{mx} (\hat{\sigma}_{22}^N(t, z))^2\phi_2(z)dz)
\end{aligned}$$

Again if we choose the appropriate parameters this system will have a unique solution for $\hat{\pi}_1$ and $\hat{\pi}_2$. Hence we can find the minimal martingale measure parameters from (6.2.1). Now we use (3.3.21) to obtain the density process for the minimal martingale measure in this case

$$\begin{aligned}
Z = & \exp \left(- \left(\int_0^t v(s) dW - \frac{1}{2} \int_0^t v^2(s) ds \right) \right) \\
& \times \exp \left(\sum_{j=1}^2 \int_0^t \int_0^{mx} \log((r_j(s)\psi_j(s, z)) dN_j(s, z) \right) \\
& \times \exp \left(\sum_{j=1}^2 \int_0^t \lambda_j(s)(r_j(t) - 1) ds \right). \tag{6.2.3}
\end{aligned}$$

Finally to simulate the paths of the process S_1 after the minimal martingale measure transformation we can simply simulate the paths of $S_1 Z$ (see Figure 2 Appendix B).

6.3 Results interpretation and option pricing.

Comparing the paths of the stock price S_1 before and after transformation we can see that the variance decreased significantly. We can also notice that big jumps in some of the paths disappeared under the minimal martingale measure.

Now we can calculate the price of the European call option. We use a Monte-Carlo method to estimate $E(S_i(T) - K_i)^+$. Similarly we estimate the expectation with respect to the minimal martingale measure i.e.

$$E_{\bar{P}}(S_i(T) - K_i)^+ = E(Z(T)(S_i(T) - K_i)^+),$$

where Z is the density process of the minimal martingale measure.

We obtain the following results after using a Monte-Carlo estimation for 1000 simulations.

	Original measure	Minimal martingale measure
European call price S_1	0.9536	0.8483
Variance for S_1	0.2178	0.0555
European call price S_2	0.9263	0.8348
Variance for S_2	0.1461	0.0307

So if you are offered 0.90 for this European call option, you would naively purchase it based upon the estimate under the original measure (since $0.90 < 0.9536$). However, this implicitly assumes a certain increase in the market (speculation) that is absent from the MMM estimate, 0.8483. There is no risk of overpaying for the option at this price. For this model the minimal martingale, variance optimal and exponential minimal relative entropy measures all agree. The other optimality criteria were not successfully applied in this case because of numeric instability in solving the optimization equations, e.g. (4.4.1).

We can conclude that the minimal martingale measure transform significantly decrease the variance of option prices. We also illustrate this fact comparing histograms of the option price (for the first stock S_1) before and after the measure transformation (see Figure 3 and Figure 4). We can also observe that the stock price after transform behaves as a martingale (see Figure 2).

Chapter 7

Conclusion

In conclusion we would like to mention what else can be done in this framework.

Our first suggestion concerns the structure of the model. In Section 3.1 we require the coefficients $\sigma_{ij}^N(t, z)$ to be continuously differentiable functions invertible in z . Probably this restriction can be eased. This is interesting not only from the mathematical point of view. Say, we observe the market with a certain structure of jumps of the stock price processes. We usually cannot guess or estimate exactly the marked point process that will be the best model for these jumps. Instead we can pick a marked point process from the certain class, say with constant intensities and exponential jump-size distributions, and adjust it to the real data by changing the parameters $\sigma_{ij}^N(t, z)$. If we put significant restrictions on these parameters the adjustment procedure will be difficult. But we also have to keep in mind that the arbitrary choice of $\sigma_{ij}^N(t, z)$ may violate the semimartingale structure of the price processes.

There are a number of techniques for obtaining an optimal martingale measure that are not covered in this thesis. We did not consider the super-hedging, quantile hedging, Hellinger process minimization and other procedures for derivative pricing in incomplete markets. It may be interesting to compare the results obtained by these techniques for our market model to the ones in this thesis.

From our point of view it is very interesting to develop the convenient measure approach further. Using convenient measures we can calculate the prices of the options explicitly or at least without big numeric computations. We can mention some

problems here: what target class do we choose, how far are the results of optimization among convenient measures from the optimization results among all possible equivalent martingale measures, can the convenient measure coincide with a certain choice of optimality for martingale measures and other questions.

There is still a big open question as to the estimation of parameters for our model from the real data sources. It seems that it is a relatively easy problem since we have adjustment parameters for Brownian motions and marked point processes. This will require fitting a model to the data using a particular utility function, and numeric optimization.

We considered only the most common type of options - the European call. It is also interesting to determine how we can price the American, Asian and may be even some exotic options using the techniques described in this thesis.

Appendix A

Program code

The simulations were done using MATLAB R13.

```
function c
% number of simulations
l=50;
%simulation #
a=0;
% time
T=1;
% number of time intervals
n=200;
% number of jump size intervals
nz=50;
% M=100 approximatly cutoff 0.10 of pdf
M=20;
% we have 2 stock model with stock price
%  $\log(S)=b+\sigma W.W+\sigma N1.N1+\sigma N2.N2$ 
% log price simulation
lnS1=zeros(n,l);
S1=ones(n,l);
lnS2=zeros(n,l);
S2=ones(n,l);
```

```

% strike of the option
K1=.2;
K2=.2;
% sum of option prices for Monte-Carlo method for option pricing
histop1=zeros(1,1);
histop2=zeros(1,1);
sumS1=0;
sumS2=0;
varS1=0;
varS2=0;
k1=0;
k2=0;
optprice1=0;
optprice2=0;
% EMM log density process
EMMlnZ=zeros(n,1);
% log price after EMM transformation
EMMlnS1=zeros(n,1);
EMMlnS2=zeros(n,1);
% sum of EMM option prices for Monte-Carlo method for option pricing
EMMhistop1=zeros(1,1);
EMMhistop2=zeros(1,1);
EMMk1=0;
EMMsumS1=0;
EMMvarS1=0;
EMMk2=0;
EMMsumS2=0;
EMMvarS2=0;
EMMoptprice1=0;
EMMvaropt1=0;
EMMoptprice2=0;
EMMvaropt2=0;
while (a<1)

```

```

a=a+1;
% ORIGINAL step-----
% ORIGINAL parameters
% Original conditions fulfilled
par=0;
par1=ones(1,n);
par1(1)=0;
% Original drift
b1=0.05;
b2=0.05;
lambda10=0.75;
lambda20=0.57;
betta1=1.9;
betta2=1.8;
% too fast growth parameter
cgr1=2.1;
cgr2=2.2;
% too fast decrease parameter
cdec1=-1.1;
cdec2=-1.2;
%adjustment
adj1=0.2;
adj2=0.3;
%ORIGINAL modified drift part
hatb1=zeros(1,n);
hatb2=zeros(1,n);
%ORIGINAL drift part
lnStr1=zeros(1,n);
lnStr2=zeros(1,n);
% ORIGINAL continuous part
sigmaW1=zeros(1,n);
sigmaW2=zeros(1,n);
lnSc1=zeros(1,n);

```

```

lnSc2=zeros(1,n);
% ORIGINAL jump part parameters
lnSjmp1=zeros(1,n);
lnSjmp2=zeros(1,n);
sigmaN11=zeros(nz,n);
sigmaN12=zeros(nz,n);
sigmaN21=zeros(nz,n);
sigmaN22=zeros(nz,n);
hatsigmaN11=zeros(nz,n);
hatsigmaN12=zeros(nz,n);
hatsigmaN21=zeros(nz,n);
hatsigmaN22=zeros(nz,n);
% ORIGINAL different integrals w.r.t. exp distribution
intsigmaN11=zeros(1,n);
intsigmaN12=zeros(1,n);
intsigmaN21=zeros(1,n);
intsigmaN22=zeros(1,n);
inthatsigmaN11=zeros(1,n);
inthatsigmaN12=zeros(1,n);
inthatsigmaN21=zeros(1,n);
inthatsigmaN22=zeros(1,n);
% ORIGINAL intesity
lambda1=zeros(1,n);
lambda2=zeros(1,n);
% ORIGINAL compensator
Lambdatj1=zeros(1,n);
Lambdatj2=zeros(1,n);
% ORIGINAL time of the jump
tj1=zeros(1,n);
tj2=zeros(1,n);
% jump size distribution (exponential with parameter betta)
% jump size are exponential between 0 and M
zjs1=zeros(1,n);

```

```

zjs2=zeros(1,n);
zzjs1=M+1;
zzjs2=M+1;
tc0=1;
while (tc0<n+1)
while zzjs1>M
zzjs1=exprnd(betta1);
end
zjs1(tc0)=zzjs1;
while zzjs2>M
zzjs2=exprnd(betta2);
end
zjs2(tc0)=zzjs2;
tc0=tc0+1;
end
tc=1;    % time count
% THE FIRST STEP-----
%parameters
sigmaW1(1)=.88;
sigmaW2(1)=.84;
sigmaN11(1,1)=.59;
sigmaN12(1,1)=-0.48;
sigmaN21(1,1)=.57;
sigmaN22(1,1)=-0.44;
% stoch exp coefficients
hatsigmaN11(1,1)=exp(sigmaN11(1,1))-1;
hatsigmaN12(1,1)=exp(sigmaN12(1,1))-1;
hatsigmaN21(1,1)=exp(sigmaN21(1,1))-1;
hatsigmaN22(1,1)=exp(sigmaN22(1,1))-1;

% trend
lnStr1(1)=b1*T/n;
lnStr2(1)=b2*T/n;

```

```

% continuous part
WT=normrnd(0,sqrt(T/n),1,n);
lnSc1(1)=0;
lnSc2(1)=0;
% jump part
% intensity
lambda1(1)=lambda10;
lambda2(1)=lambda20;
% compensator
Lambdatj1(1)=0;
Lambdatj2(1)=0;
% jump time
PTY1=0;
PTY2=0;
PTY1=PTY1+exprnd(1)*T/n;
if PTY1<Lambdatj1(1)
tj1(1)=1;
end
PTY2=PTY2+exprnd(1)*T/n; if PTY2<Lambdatj2(1)
tj2(1)=1;
end

% point process part
lnSjmp1(1)=tj1(1)*sigmaN11(1,1)+tj2(1)*sigmaN12(1,1);
lnSjmp2(1)=tj1(1)*sigmaN21(1,1)+tj2(1)*sigmaN22(1,1);
% general price
lnS1(1,a)=lnStr1(1)+lnSc1(1)+lnSjmp1(1);
lnS2(1,a)=lnStr2(1)+lnSc2(1)+lnSjmp2(1);
% expectation of sigmaN w.r.t. phi
intsigmaN11(1)=sigmaN11(1,1);
intsigmaN12(1)=sigmaN12(1,1);
intsigmaN21(1)=sigmaN21(1,1);
intsigmaN22(1)=sigmaN22(1,1);

```

```

inthatsigmaN11(1)=hatsigmaN11(1,1);
inthatsigmaN12(1)=hatsigmaN11(1,1);
inthatsigmaN21(1)=hatsigmaN11(1,1);
inthatsigmaN22(1)=hatsigmaN11(1,1);

% modified trend
hatb1(1)=lnStr1(1)+1/2*sigmaW1(1)*sigmaW1(1);
hatb1(1)=hatb1(1)-lambda1(1)*(intsigmaN11(1)-inthatsigmaN11(1));
hatb1(1)=hatb1(1)-lambda2(1)*(intsigmaN12(1)-inthatsigmaN12(1));
hatb2(1)=lnStr2(1)+1/2*sigmaW2(1)*sigmaW2(1);
hatb2(1)=hatb2(1)-lambda1(1)*(intsigmaN21(1)-inthatsigmaN21(1));
hatb2(1)=hatb2(1)-lambda2(1)*(intsigmaN22(1)-inthatsigmaN22(1));

% MMM
% original MMM integrals
d1MMMinsigmaN11=zeros(1,n);
d1MMMinsigmaN12=zeros(1,n);
d2MMMinsigmaN11=zeros(1,n);
d2MMMinsigmaN12=zeros(1,n);
d1MMMinsigmaN21=zeros(1,n);
d1MMMinsigmaN22=zeros(1,n);
d2MMMinsigmaN21=zeros(1,n);
d2MMMinsigmaN22=zeros(1,n);
% optimal strategies
MMMpi1=zeros(1,n);
MMMpi2=zeros(1,n);
% original MMM measure parameters
MMMupsilon=zeros(1,n);
MMMr1=zeros(1,n);
MMMr2=zeros(1,n);
MMMpsi1=zeros(nz,n);
MMMpsi2=zeros(nz,n);

```

```

d1MMIntsigmaN11(1)=hatsigmaN11(1,1)*hatsigmaN11(1,1);
d1MMIntsigmaN12(1)=hatsigmaN12(1,1)*hatsigmaN12(1,1);
d2MMIntsigmaN11(1)=hatsigmaN21(1,1)*hatsigmaN11(1,1);
d2MMIntsigmaN12(1)=hatsigmaN22(1,1)*hatsigmaN12(1,1);
d1MMIntsigmaN21(1)=hatsigmaN11(1,1)*hatsigmaN21(1,1);
d1MMIntsigmaN22(1)=hatsigmaN12(1,1)*hatsigmaN22(1,1);
d2MMIntsigmaN21(1)=hatsigmaN21(1,1)*hatsigmaN21(1,1);
d2MMIntsigmaN22(1)=hatsigmaN22(1,1)*hatsigmaN22(1,1);
EMMlnZ(1,a)=0;
% log price after EMM transformation
EMMlnS1(1,a)=lnS1(1,a);
EMMlnS2(1,a)=lnS2(1,a);
% OTHER STEPS-----
while (tc<n)
tc=tc+1;
% drift
lnStr1(tc)=lnStr1(tc-1)+(b1*T/n);
lnStr2(tc)=lnStr2(tc-1)+(b2*T/n);
% continuous part
sigmaW1(tc)=sigmaW1(1)*abs(sin(1*tc*T/n));
sigmaW2(tc)=sigmaW2(1)*abs(sin(2*tc*T/n));
lnSc1(tc)=lnSc1(tc-1)+(sigmaW1(tc)*WT(tc));
lnSc2(tc)=lnSc2(tc-1)+(sigmaW2(tc)*WT(tc));
% jump part
lambda1(tc)=lambda10*abs(sin(1.3*tc*T/n));
Lambdatj1(tc)=Lambdatj1(tc-1)+lambda1(tc)*T/n;
lambda2(tc)=lambda20*abs(sin(1.2*tc*T/n));
Lambdatj2(tc)=Lambdatj2(tc-1)+lambda2(tc)*T/n;
if tc>3
% too fast growth
if lnS1(tc-1)-lnS1(tc-2)>cgr1
lambda2(tc)=lambda2(tc)+adj1;
Lambdatj2(tc)=Lambdatj2(tc)+adj1*T/n;

```

```

end
if lnS2(tc-1)-lnS2(tc-2)>cgr2
lambda2(tc)=lambda2(tc)+adj1;
Lambdatj2=Lambdatj1+adj1*T/n;
end
% too fast decrease
if lnS1(tc-1)-lnS1(tc-3)<cdec1
lambda1(tc)=lambda1(tc)+adj2;
Lambdatj1(tc)=Lambdatj1(tc)+adj2*T/n;
end
if lnS2(tc-1)-lnS2(tc-3)<cdec2
lambda1(tc)=lambda1(tc)+adj2;
Lambdatj1(tc)=Lambdatj1(tc)+adj2*T/n;
end
end
% jump time
PTY1=PTY1+exprnd(1)*T/n;
if PTY1<Lambdatj1(tc)
tj1(tc)=1;
end
PTY2=PTY2+exprnd(1)*T/n;
if PTY2<Lambdatj2(tc)
tj2(tc)=1;
end

z=0;
zjj1=0;
zjj2=0;

while (z<nz) z=z+1;
% integral of sigmaN and hatsigma w.r.t. phi- exponential on 0 to M
if abs(z-zjs1(tc)*nz/M)<1
    zjj1=z;

```

```

end
if abs(z-zjs2(tc)*nz/M)<1
    zjj2=z;
end
sigmaN11(z,tc)=sigmaN11(1,1)*abs(sin(2*tc*T/n))*log(z)*M/nz;
hatsigmaN11(z,tc)=(exp(sigmaN11(z,tc))-1);
intsigmaN11(tc)=intsigmaN11(tc)+1/betta1*sigmaN11(z,tc)
*exp(-z*M/nz/betta1)*M/nz;
inthatsigmaN11(tc)=inthatsigmaN11(tc)+1/betta1*hatsigmaN11(z,tc)
*exp(-z*M/nz/betta1)*M/nz;
sigmaN12(z,tc)=sigmaN12(1,1)*abs(sin(4*tc*T/n))*log(z)*M/nz;
hatsigmaN12(z,tc)=(exp(sigmaN12(z,tc))-1);
intsigmaN12(tc)=intsigmaN12(tc)+1/betta2*sigmaN12(z,tc)
*exp(-z*M/nz/betta2)*M/nz;
inthatsigmaN12(tc)=inthatsigmaN12(tc)+1/betta2*hatsigmaN12(z,tc)
*exp(-z*M/nz/betta2)*M/nz;
sigmaN21(z,tc)=sigmaN21(1,1)*abs(sin(3*tc*T/n))*log(z)*M/nz;
hatsigmaN21(z,tc)=(exp(sigmaN21(z,tc))-1);
intsigmaN21(tc)=intsigmaN21(tc)+1/betta1*sigmaN21(z,tc)
*exp(-z*M/nz/betta1)*M/nz;
inthatsigmaN21(tc)=inthatsigmaN21(tc)+1/betta1*hatsigmaN21(z,tc)
*exp(-z*M/nz/betta1)*M/nz;
sigmaN22(z,tc)=sigmaN22(1,1)*abs(sin(tc*T/n))*log(z)*M/nz;
hatsigmaN22(z,tc)=(exp(sigmaN22(z,tc))-1);
intsigmaN22(tc)=intsigmaN22(tc)+1/betta2*sigmaN22(z,tc)
*exp(-z*M/nz/betta2)*M/nz;
inthatsigmaN22(tc)=inthatsigmaN22(tc)+1/betta2*hatsigmaN22(z,tc)
*exp(-z*M/nz/betta2)*M/nz;
end
% modified trend
hatb1(tc)=lnStr1(tc)+1/2*sigmaW1(tc)*sigmaW1(tc);
hatb1(tc)=hatb1(tc)-lambda1(tc)*(intsigmaN11(tc)-inthatsigmaN11(tc));
hatb1(tc)=hatb1(tc)-lambda2(tc)*(intsigmaN12(tc)-inthatsigmaN12(tc));

```

```

hatb2(tc)=lnStr2(tc)+1/2*sigmaW2(tc)*sigmaW2(tc);
hatb2(tc)=hatb2(tc)-lambda1(tc)*(intsigmaN21(tc)-inthatsigmaN21(tc));
hatb2(tc)=hatb2(tc)-lambda2(tc)*(intsigmaN22(tc)-inthatsigmaN22(tc));
% point process part
lnSjmp1(tc)=lnSjmp1(tc-1)+tj1(tc)*sigmaN11(zjj1,tc)+tj2(tc)*sigmaN12(zjj2,tc);
lnSjmp2(tc)=lnSjmp2(tc-1)+tj1(tc)*sigmaN21(zjj1,tc)+tj2(tc)*sigmaN22(zjj2,tc);
% general price
lnS1(tc,a)=lnStr1(tc)+lnSc1(tc)+lnSjmp1(tc);
lnS2(tc,a)=lnStr2(tc)+lnSc2(tc)+lnSjmp2(tc);
S1(tc,a)=exp(lnS1(tc,a));
S2(tc,a)=exp(lnS2(tc,a));

% solution for exponential optimal strategy for optimal EMMs.
% Minimal Martingale measure (MMM).
% optimal strategies MMM
MMMhatb=zeros(2,1);
MMMFprime=zeros(2,2);
MMMFprimeinv=zeros(2,2);
MMMstep=zeros(2,1);
% solution of martingale condition
z=0;
while (z<nz)
z=z+1;
d1MMMintsigmaN11(tc)=d1MMMintsigmaN11(tc)+(1/betta1)*hatsigmaN11(z,tc)
*(exp(sigmaN11(z,tc))-1)*exp(-(z*M/nz)/betta1)*M/nz;
d1MMMintsigmaN12(tc)=d1MMMintsigmaN12(tc)+(1/betta2)*hatsigmaN12(z,tc)
*(exp(sigmaN12(z,tc))-1)*exp(-(z*M/nz)/betta2)*M/nz;
d2MMMintsigmaN11(tc)=d2MMMintsigmaN11(tc)+(1/betta1)*hatsigmaN21(z,tc)
*(exp(sigmaN11(z,tc))-1)*exp(-(z*M/nz)/betta1)*M/nz;
d2MMMintsigmaN12(tc)=d2MMMintsigmaN12(tc)+(1/betta2)*hatsigmaN22(z,tc)
*(exp(sigmaN12(z,tc))-1)*exp(-(z*M/nz)/betta2)*M/nz;
d1MMMintsigmaN21(tc)=d1MMMintsigmaN21(tc)+(1/betta1)*hatsigmaN11(z,tc)
*(exp(sigmaN21(z,tc))-1)*exp(-(z*M/nz)/betta1)*M/nz;

```

```

d1MMMinsigmaN22(tc)=d1MMMinsigmaN22(tc)+(1/betta2)*hatsigmaN12(z,tc)
*(exp(sigmaN22(z,tc))-1)*exp(-(z*M/nz)/betta2)*M/nz;
d2MMMinsigmaN21(tc)=d2MMMinsigmaN21(tc)+(1/betta1)*hatsigmaN21(z,tc)
*(exp(sigmaN21(z,tc))-1)*exp(-(z*M/nz)/betta1)*M/nz;
d2MMMinsigmaN22(tc)=d2MMMinsigmaN22(tc)+(1/betta2)*hatsigmaN22(z,tc)
*(exp(sigmaN22(z,tc))-1)*exp(-(z*M/nz)/betta2)*M/nz;
end
MMMFprime(1,1)=sigmaW1(tc)*sigmaW1(tc)+lambda1(tc)*d1MMMinsigmaN11(tc);
MMMFprime(1,1)=MMMFprime(1,1)+lambda2(tc)*d1MMMinsigmaN12(tc);
MMMFprime(1,2)=sigmaW1(tc)*sigmaW2(tc)+lambda1(tc)*d2MMMinsigmaN11(tc);
MMMFprime(1,2)=MMMFprime(1,2)+lambda2(tc)*d2MMMinsigmaN12(tc);
MMMFprime(2,1)=sigmaW2(tc)*sigmaW1(tc)+lambda1(tc)*d1MMMinsigmaN21(tc);
MMMFprime(2,1)=MMMFprime(2,1)+lambda2(tc)*d1MMMinsigmaN22(tc);
MMMFprime(2,2)=sigmaW2(tc)*sigmaW2(tc)+lambda1(tc)*d2MMMinsigmaN21(tc);
MMMFprime(2,2)=MMMFprime(2,2)+lambda2(tc)*d2MMMinsigmaN22(tc);
MMMFprimeinv=inv(MMMFprime);
MMMhatb(1)=hatb1(tc);
MMMhatb(2)=hatb2(tc);
MMMstep=MMMFprimeinv*MMMhatb;
MMMpi1(tc)=MMMstep(1);
MMMpi2(tc)=MMMstep(2);
% MMM measure parameters
MMMupsilon(tc)=MMMpi1(tc)*sigmaW1(tc)+MMMpi2(tc)*sigmaW2(tc);
z=0;
MMMr1(tc)=0; MMMr2(tc)=0;
MMMr1(tc)=MMMpi1(tc)*inthatsigmaN11(tc)+MMMpi2(tc)*inthatsigmaN21(tc);
MMMr2(tc)=MMMpi1(tc)*inthatsigmaN12(tc)+MMMpi2(tc)*inthatsigmaN22(tc);
MMMr1(tc)=1-MMMr1(tc); MMMr2(tc)=1-MMMr2(tc);
MMMpsi1(tc)=(1-MMMpi1(tc)*hatsigmaN11(tc)-MMMpi2(tc)*hatsigmaN21(tc))/MMMr1(tc);
MMMpsi2(tc)=(1-MMMpi1(tc)*hatsigmaN12(tc)-MMMpi2(tc)*hatsigmaN22(tc))/MMMr2(tc);
% MMM log density process
EMMlnZ(tc,a)=EMMlnZ(tc-1,a)-MMMupsilon(tc)*WT(tc)
EMMlnZ(tc,a)=EMMlnZ(tc,a)+(MMMupsilon(tc)*MMMupsilon(tc))/2*T/n;

```

```

if MMMr1(tc)*MMMpsi1(tc)>0
if MMMr2(tc)*MMMpsi2(tc)>0
EMMlnZ(tc,a)=EMMlnZ(tc,a)+tj1(tc)*log(MMMr1(tc)*MMMpsi1(tc))
EMMlnZ(tc,a)=EMMlnZ(tc,a)+tj2(tc)*log(MMMr2(tc)*MMMpsi2(tc));
EMMlnZ(tc,a)=EMMlnZ(tc,a)+(MMMr1(tc)-1)*lambda1(tc)*T/n
EMMlnZ(tc,a)=EMMlnZ(tc,a)+(MMMr2(tc)-1)*lambda2(tc)*T/n;
par1(tc)=0;
end
end
par=par+par1(tc);
% log price after EMM transformation
EMMlnS1(tc,a)=lnS1(tc,a)+EMMlnZ(tc,a);
EMMlnS2(tc,a)=lnS2(tc,a)+EMMlnZ(tc,a);
end
% sum ans sum of squares of otpion prices for each simulation
if par==0
if (S1(n,a)-K1)>0
k1=k1+1;
histop1(a)=S1(n,a)-K1;
sumS1=sumS1+S1(n,a)-K1;
varS1=varS1+(S1(n,a)-K1)*(S1(n,a)-K1);
end
if (S2(n,a)-K2)>0
k2=k2+1;
histop2(a)=S2(n,a)-K2;
sumS2=sumS2+S2(n,a)-K2;
varS2=varS2+(S2(n,a)-K2)*(S2(n,a)-K2);
end
if (exp(EMMlnS1(n,a))-K1)>0
EMMk1=EMMk1+1;
EMMhistop1(a)=exp(EMMlnS1(n,a))-K1;
EMMsumS1=EMMsumS1+exp(EMMlnS1(n,a))-K1;
EMMvarS1=EMMvarS1+(exp(EMMlnS1(n,a))-K1)*(exp(EMMlnS1(n,a))-K1);

```

```

end
if (exp(EMMlnS2(n,a))-K2)>0
    EMMk2=EMMk2+1;
    EMMhistop2(a)=exp(EMMlnS2(n,a))-K2;
    EMMsumS2=EMMsumS2+exp(EMMlnS2(n,a))-K2;
    EMMvarS2=EMMvarS2+(exp(EMMlnS2(n,a))-K2)*(exp(EMMlnS2(n,a))-K2);
end
hold on
TT=ones(1,n);
tt=0;
while tt<n
    tt=tt+1;
    TT(tt)=tt;
end
plot(TT,exp(lnS1(1:n,a)));
plot(TT,lnS1(1:n,a));
plot(TT,exp(EMMlnS1(1:n,a)),'r');
plot(TT,EMMlnS1(1:n,a),'r'); end end hold off
% histograms
hist(histop1(1:a),30);
hist(EMMhistop1(1:a),30);
% price of the European call option w.r.t. the original measure
if k1>0
    optprice1=sumS1/k1
    varopt1=varS1/k1-optprice1*optprice1
end
if k2>0
    optprice2=sumS2/k2
    varopt2=varS2/k2-optprice2*optprice2
end
% price of the European call option w.r.t. the MMM
if EMMk1>0
    EMMoptprice1=EMMsumS1/EMMk1

```

```
EMMvaropt1=EMMvarS1/EMMk1-EMMoptprice1*EMMoptprice1
else
EMMoptprice1=EMMsumS1 EMMvaropt1=0
end
if EMMk2>0
EMMoptprice2=EMMsumS2/EMMk2
EMMvaropt2=EMMvarS2/EMMk2-EMMoptprice2*EMMoptprice2
else
EMMoptprice2=EMMsumS2 EMMvaropt2=0
end
```

Appendix B

Graphs and histograms

Here we present graphs and histograms for the numeric example from Chapter 6. All pictures were obtained using MATLAB R13.

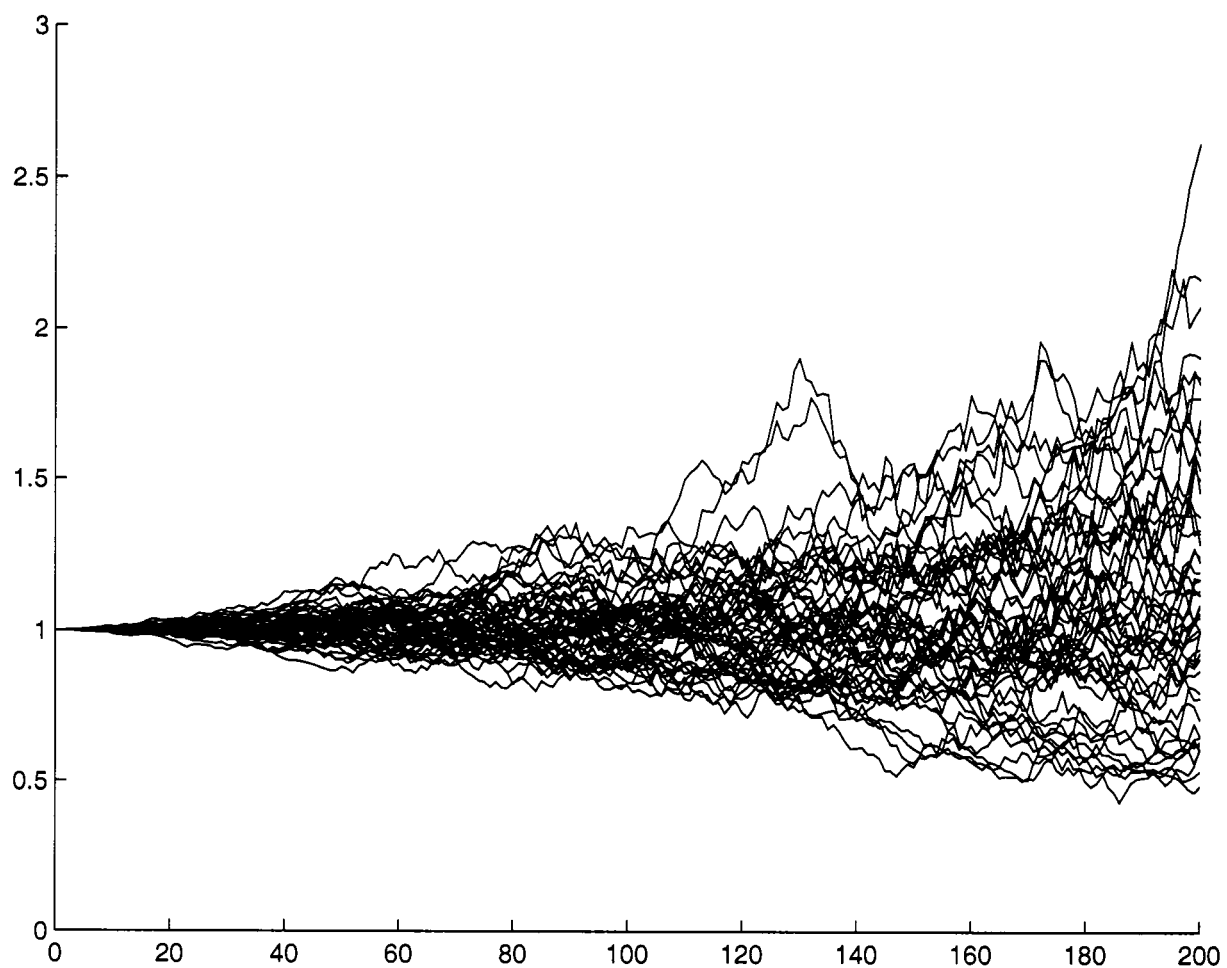


Figure 1: 50 simulations of paths of the stock price S_1 with respect to the original measure

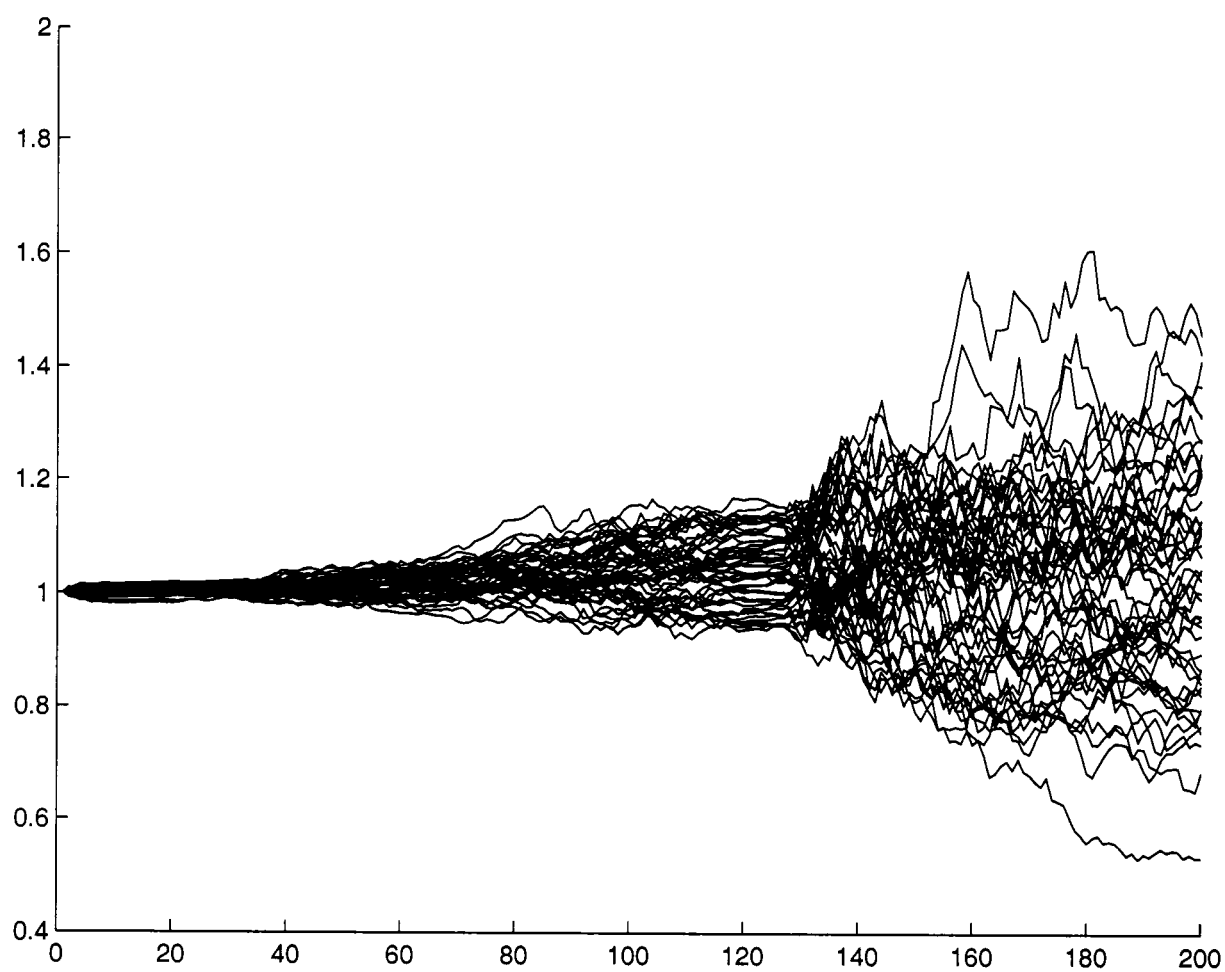


Figure 2: 50 simulations of paths of the stock price S_1 after MMM transform

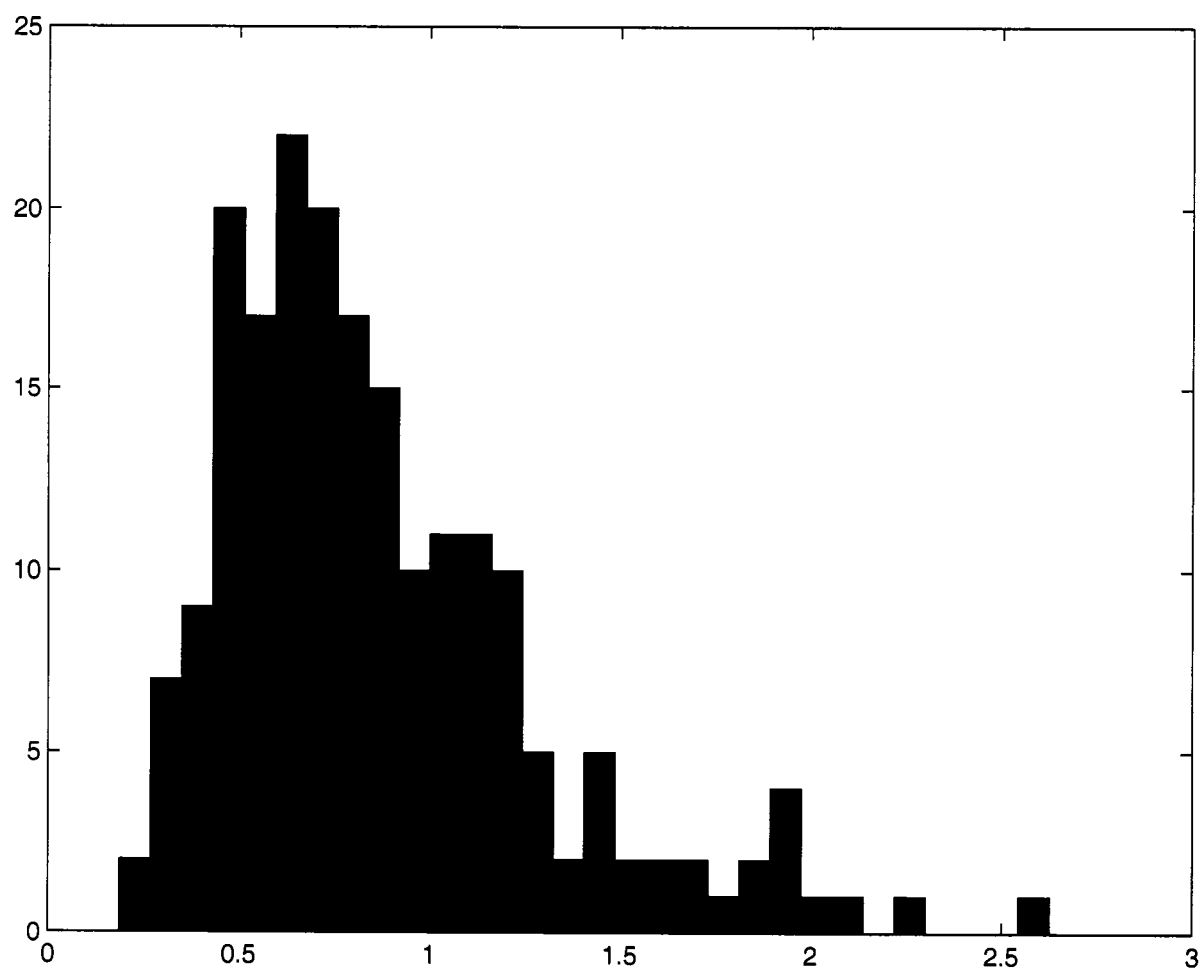


Figure 3: Histogram of the option price $(S(T) - K)^+$ (200 simulations)

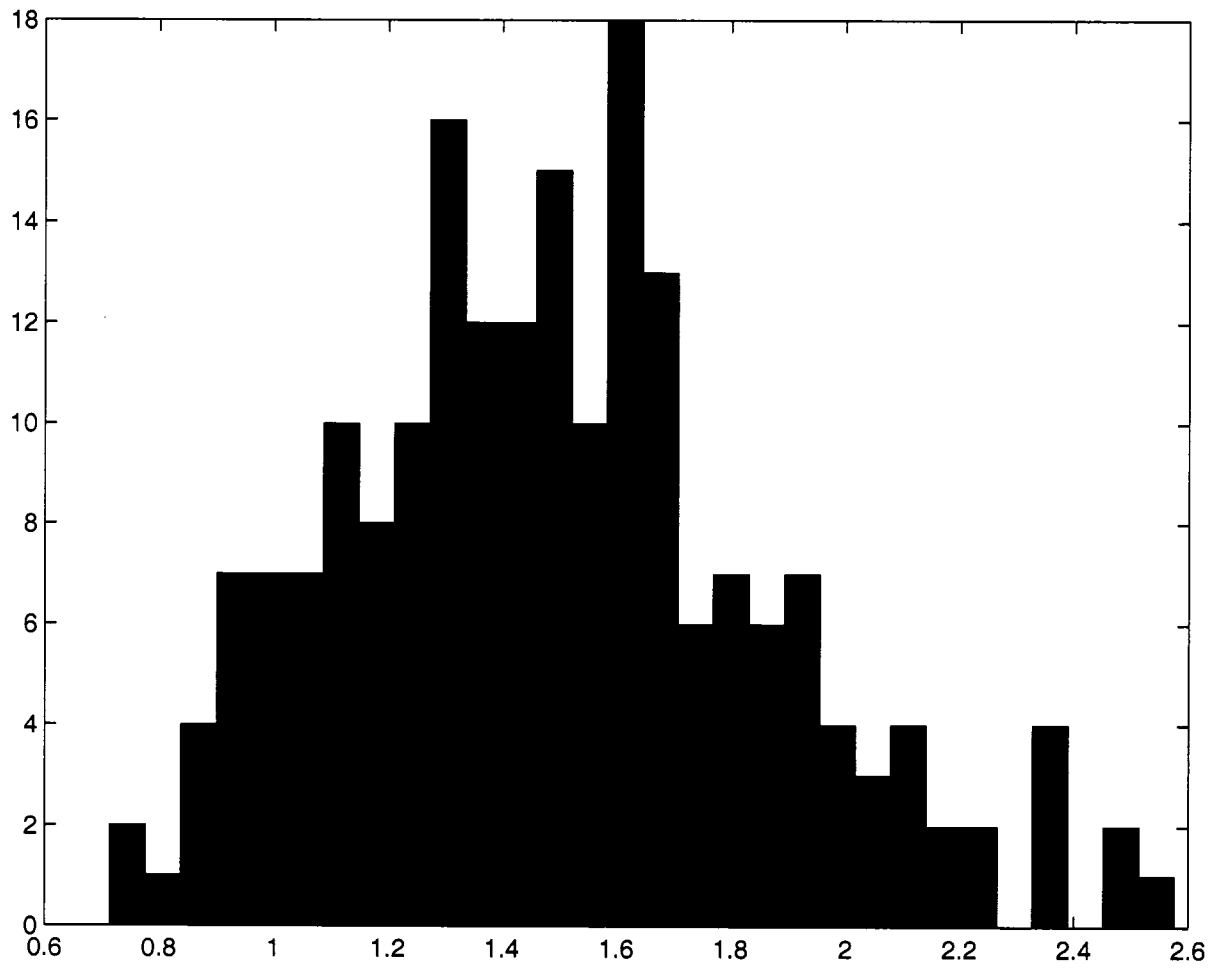


Figure 4: Histogram of the option price $Z(T)(S(T) - K)^+$ after MMM transform (200 simulations)

Appendix C

Stochastic calculus review

C.1 Some probability concepts.

C.1.1 Semimartingales

The goal of this appendix is not to provide a complete introduction to stochastic calculus, but to collect a number of definitions and results in that area in the form and notation used in this thesis. This material is very well known. The results of greatest interest to us here are the Doleans-Dade exponent, the Girsanov theorem, and the characteristics of a semimartingale. We will extensively use Jacod, Shiryaev (1987) (call it JS), Bremaud (1981) (call it B), Karatzas, Shreve (1991) (call it K-S1) and Protter (1991) (call it P).

Definition 30 $(\Omega, \mathcal{F}, P)$ will denote a complete probability space. We equip $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$. By a filtration we mean an increasing family $\mathcal{F}_t$ of sub- σ -algebras of $\mathcal{F}$: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. We set $\mathcal{F}_\infty = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$.

Definition 31 (i) A complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$ is said to satisfy the usual conditions if $\mathcal{F}_0$ contains all the P -null sets of $\mathcal{F}$ and $\mathcal{F}_t = \sigma(\cap_{s \geq t} \mathcal{F}_s)$ for all t , $0 \leq t \leq \infty$ (that is the filtration $\mathbf{F}$ is right continuous).

(ii) A complete probability space with a filtration $(\Omega, \mathcal{F}, \mathbf{F}, P)$ satisfying the usual conditions is also called a filtered probability space.

Definition 32 A stochastic process X on $(\Omega, \mathcal{F}, P)$ is a collection of random variables $(X_t)_{0 \leq t < \infty}$. For all fixed $\omega \in \Omega$ the functions $t \mapsto X_t(\omega)$ mapping $[0, \infty)$ into $\mathbb{R}$ are called the sample paths of the stochastic process X . Two stochastic processes X and Y are modifications if $P(X_t = Y_t) = 1, \forall t \in [0, \infty)$. Two stochastic processes X and Y are indistinguishable if $P(X_t = Y_t, \forall t \in [0, \infty)) = 1$.

Remark 17 We assume that the stochastic process X in the above definition takes values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (in general we can take any “good” measurable space, e.g. $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$).

Definition 33 (i) A random variable $T : \Omega \rightarrow [0, \infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$, for every $t, 0 \leq t \leq \infty$.
(ii) The σ -algebra $\mathcal{F}_T = \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$ is said to be a stopping time σ -algebra.
(iii) Let X be a stochastic process, then X_T is said to be the process stopped at T if $X_t^T = X_{t \wedge T}$.
(iv) A stopping time T is predictable if there exists a sequence of stopping times $(T_n)_{n \geq 1}$ such that $T_n \nearrow T$ a.s. on $\{T > 0\}$. Such a sequence is said to announce T .
(v) A stopping time T is accessible if there exists a sequence $(T_k)_{k \geq 1}$ of predictable times such that

$$P(\cup_{k=1}^{\infty} \{\omega : T_k(\omega) = T(\omega) < +\infty\}) = P(\{T < \infty\}).$$

Such a sequence $(T_k)_{k \geq 1}$ is said to envelop T .

(vi) A stopping time T is totally inaccessible if for every predictable stopping time S , $P(\{\omega : T(\omega) = S(\omega) < \infty\}) = 0$.
(vii) Let τ denote a finite sequence of stopping times: $0 = T_0 \leq T_1 \leq \dots \leq T_k < \infty$. The sequence τ is called a random partition. A sequence of random partitions τ_n is said to tend to identity if $\lim_n \sup_k T_k^n = \infty$ a.s. and $\|\tau_n\| = \sup_k |T_{k+1}^n - T_k^n|$ converges to 0 a.s.

Here are some properties of stopping times.

Proposition 35 1)(P) Let S, T be stopping times. Then $S \wedge T = \min(S, T)$, $S \vee T = \max(S, T)$, $S + T$ and αS for $\alpha > 1$ are stopping times.

2)(J-S) Let T_n be a sequence of stopping times. Then $S = \wedge T_n$ and $T = \vee T_n$ are stopping times and $\mathcal{F}_S = \cap \mathcal{F}_{T_n}$.

3)(P) Let Y be an a random variable and let S, T be stopping times. Then $E(E(Y|\mathcal{F}_S)|\mathcal{F}_T) = E(E(Y|\mathcal{F}_T)|\mathcal{F}_S) = E(Y|\mathcal{F}_{S \wedge T})$.

4)(P) Let T be a stopping time and $\Lambda \in \mathcal{F}_T$. We define $T_\Lambda(\omega) = T(\omega)$ if $\omega \in \Lambda$ and $T_\Lambda(\omega) = \infty$ if $\omega \notin \Lambda$. Then T_Λ is a stopping time and $T = T_\Lambda \wedge T_{\Lambda^c}$.

5)(J-S) T is a predictable stopping time if and only if the random set $[0, T)$ is in predictable σ -algebra.

6)(J-S) Let T_n be a sequence of predictable stopping times then $T = \vee T_n$ is predictable and if $\cup_n \{S = \wedge T_n\} = \Omega$ then S is a predictable stopping time.

7)(J-S) Let T be a predictable stopping time then T_Λ is a predictable stopping time if $\Lambda \in \mathcal{F}_{T-}$.

8)(P) Let T be a stopping time. There exist disjoint events A, B such that $A \cup B = \{T < \infty\}$ a.s., such that T_A is accessible and T_B is totally inaccessible; and $T = T_A \wedge T_B$ a.s. Such a decomposition is a.s. unique.

Definition 34 Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered probability space and X is a stochastic process on $(\Omega, \mathcal{F}, P)$.

(i) X is said to be adapted if for each $t \in [0, \infty)$, X_t is an $\mathcal{F}$ -measurable random variable. For a given stochastic process X , $\mathcal{F}_t^X := \sigma(X_s; 0 \leq s \leq t)$ is the smallest σ -algebra with respect to which X_t is measurable for every $s \in [0, t]$. We say that the filtration $(\mathcal{F}_t^X)_{0 \leq t \leq \infty}$ is generated by X if the usual conditions are fulfilled.

(ii) X is said to be cadlag (r.c.l.l.) if it a.s. has sample paths which are right continuous with left limits. $\mathbb{D}$ denote the space of adapted cadlag processes. Let $\Delta X_t = X_t - X_{t-}$, where $X_{t-} = \lim_{s \nearrow t} X_s$; $X_t^* = \sup_{s \leq t} |X_s|$ and $X^* = \sup_s |X_s|$. Similarly X is said to be caglad (l.c.r.l.) if it a.s. has sample paths which are left continuous with right limits. $\mathbb{L}$ denote the space of adapted caglad processes. Also $b\mathbb{L}$ denote the space of all adapted bounded caglad processes. X is said to be continuous if it a.s. has sample paths which are continuous.

(iii) $\mathcal{Q}$ is said to be the progressive σ -algebra on $(\mathbb{R}_+ \times \Omega)$ if

$$\mathcal{Q} = \sigma((s, t] \times U | 0 \leq s \leq t, U \in \mathcal{F}_t).$$

X is said to be progressively measurable if X is measurable w.r.t. the progressive σ -algebra $\mathcal{Q}$; in other words if, for each $t \geq 0$ the mapping $(s, \omega) \mapsto X_s(\omega) : ([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_t) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable (where $\mathcal{B}(A)$ is Borel σ -algebra on $A \subseteq \mathbb{R}$).

(iv) $\mathcal{O}$ is said to be the optional σ -algebra on $(\mathbb{R}_+ \times \Omega)$ if it is generated by all cadlag adapted processes. X is said to be the optional if X is measurable w.r.t. the optional σ -algebra $\mathcal{O}$.

(v) $\mathcal{P}$ is said to be the predictable σ -algebra on $(\mathbb{R}_+ \times \Omega)$ if

$$\mathcal{P} = \sigma((s, t] \times U | 0 \leq s \leq t, U \in \mathcal{F}_s).$$

X is said to be predictable (measurable) if X is measurable w.r.t. predictable σ -algebra $\mathcal{P}$; in other words if for each $t \geq 0$ the mapping $(s, \omega) \mapsto X_s(\omega) : ([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_s) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. $\mathbb{P}$ denote the space of all predictable processes. Also $b\mathbb{P}$ denote the space of all bounded predictable processes.

(vi) Let X be a bounded, measurable process. The predictable projection of X is the predictable process $\hat{X}$ such that $\hat{X}_T = E(X_T | \mathcal{F}_{T-})$ a.s. on $\{T < \infty\}$ for all predictable stopping times T .

(vii) X is said to be increasing if X is cadlag and a.s. has sample paths which are nondecreasing. An increasing process X is called integrable if $E(X_\infty) < \infty$, where $X_\infty = \lim_{t \rightarrow \infty} X_t$. X is said to be of finite variation (FV) if X is cadlag and a.s. has sample paths which are of finite variation on each compact interval of $\mathbb{R}_+$. For X an FV process, the total variation process, $|A| = (|A|_t)_{t \geq 0}$ is an increasing process, such that

$$|A|_t = \sup_{n \geq 1} \sum_{k=1}^{2^n} |A_{\frac{tk}{2^n}} - A_{\frac{t(k-1)}{2^n}}|.$$

An FV process A is of integrable variation if $E(|A|_\infty) < \infty$.

(viii) A sequence of processes $(H^n)_{n \geq 1}$ converges to H uniformly on compacts in probability (ucp) if, for each $t > 0$ $\sup_{0 \leq s \leq t} |H_s^n - H_s|$ converges to 0 in probability. Let $\mathbb{L}^0$ denote the space of finite-valued random variables topologized by convergence in probability; $\mathbb{D}$ topologized by ucp denotes convergence by $\mathbb{D}_{ucp}$; $\mathbb{L}$ topologized by ucp convergence by $\mathbb{L}_{ucp}$.

(viii) A martingale X is said to be closed by a random variable Y if $E|Y| < \infty$ and $X_t = E(Y|\mathcal{F}_t)$, $0 \leq t < \infty$.

(ix) A real-valued, adapted process X is called a martingale (resp. supermartingale, submartingale) w.r.t. a filtration $\mathbf{F}$ if $E|X_t| < \infty$ for all t and $E(X_t|\mathcal{F}_s) = X_s$ a.s. for all $s \leq t$ (resp. $E(X_t|\mathcal{F}_s) \leq X_s$, resp. $E(X_t|\mathcal{F}_s) \geq X_s$).

(x) X is said to be uniformly integrable if

$$\lim_{n \rightarrow \infty} \sup_t \int_{\{|X_t| \geq n\}} |X_t| dP = 0.$$

(xi) A cadlag martingale X is said to be square-integrable if $E(X_t^2) < \infty$ for every $t \geq 0$.

(xii) An adapted, cadlag process X is said to be a local martingale if there exists a sequence of increasing stopping times $(T_n)_{n \geq 1}$, with $\lim_{n \rightarrow \infty} T_n = \infty$ a.s. such that $X_{t \wedge T_n} I_{\{T_n > 0\}}$ is a martingale for each n .

(xiii) Let X be a stochastic process. A property is said to hold locally if there exists a sequence of stopping times $(T_n)_{n \geq 1}$, with $\lim_{n \rightarrow \infty} T_n = \infty$ a.s. such that $X_{t \wedge T_n} I_{\{T_n > 0\}}$ has this property, for each $n \geq 1$ (e.g. locally bounded and locally square-integrable).

(xiv) A stopping time T reduces a process X if X_T is a uniformly integrable martingale.

(xv) An adapted cadlag process X is a semimartingale if there exists a local martingale M and an FV process A with $M_0 = A_0 = 0$ such that

$$X_t = X_0 + M_t + A_t.$$

(xvi) Let X be a semimartingale. If there exists a local martingale M and an FV predictable process A with $M_0 = A_0 = 0$ such that $X_t = X_0 + M_t + A_t$, then X is said to be a special semimartingale. And such a decomposition is called the canonical (Doob) decomposition.

(xvii) Let X be an FV process with locally integrable variation and $A_0 = 0$. An FV predictable process $\langle X \rangle$ such that $X - \langle X \rangle$ is a local martingale and $\langle X \rangle_0 = 0$ is called a compensator of X .

Now we summarize some of the most important properties of processes introduced in the above definition.

Proposition 36 Basic properties of stochastic processes:

1)(P) Let X and Y be two cadlag processes and X be a modification of Y . Then X and Y are indistinguishable.

2)(P) Let X be an adapted cadlag process and let Λ be a closed set. Then a random variable $T(\omega) = \inf\{t : X_t(\omega) \in \Lambda \text{ or } X_{t-}(\omega) \in \Lambda\}$ is a stopping time.

3)(P) Let T be a finite stopping time then

$$\mathcal{F}_T = \sigma\{X_T; X \text{ all adapted cadlag processes}\}.$$

4)(K-S1) Let X be an adapted cadlag process then there exists a sequence of stopping times $(T_n)_{n \geq 1}$ such that

$$\{(t, \omega) \in (0, \infty) \times \Omega; X_t(\omega) \neq X_{t-}(\omega)\} \subseteq \bigcup_{n=1}^{\infty} \{(t, \omega) \in [0, \infty) \times \Omega; T_n(\omega) = t\}.$$

5)(K-S1) An adapted process X with right-continuous or left-continuous sample paths is progressively measurable. A progressively measurable process is adapted.

6)(J-S) If X is cadlag adapted process, the processes X_- , ΔX are optional. If X is optional and T is a stopping time then X_T is also optional.

7)(P) The space $\mathbb{D}_{ucp}$ is a complete metric space w.r.t. the metric

$$d(X, Y) = \sum_{n=1}^{\infty} 2^{-n} E(1 \wedge (X - Y)_n^*)$$

where $X, Y \in \mathbb{D}$.

Predictable processes and predictable projection:

8)(J-S) An adapted process with left-continuous sample paths is predictable. Actually the predictable σ -algebra is generated by all adapted left-continuous processes. Even more is true: the predictable σ -algebra is generated by all adapted continuous processes.

9)(J-S) If X is a predictable process, then the stopped process X_T and a process ΔX

are also predictable. If X is an adapted cadlag process, then X_- is a predictable process.

10)(J-S) If S, T are stopping times and if Y is $\mathcal{F}_S$ -measurable random variable, the process $YI_{(S,T]}$ is predictable.

11)(B) A predictable process is optional. An optional process is progressively measurable.

12)(P) Let A be an adapted cadlag nondecreasing predictable process and let T be a stopping time. If $P(A_T \neq A_{T-}) > 0$ then T is not totally inaccessible.

13)(J-S) A predictable projection $\hat{X}_T$ of a bounded measurable process X bounded by $\sup |X|$ exists and is unique (for some more properties of predictable projection see (J-S)).

Basic properties of martingales:

14)(P) If X is a martingale then there exists a unique modification Y of X which is cadlag. Every right-continuous martingale is cadlag.

15)(J-S) Let X and Y be two predictable processes that satisfy $X_T = Y_T$ a.s. on $\{T < \infty\}$ for all predictable stopping times, then X and Y are indistinguishable.

Convergence and uniform integrability of martingales:

16)(P)(Martingale Convergence theorem). Let X be a right continuous supermartingale and $\sup_{0 \leq t \leq \infty} E|X_t| < \infty$. Then the random variable $X_\infty = \lim_{t \rightarrow \infty} X_t$ a.s. exists and $E|X_\infty| < \infty$. Moreover (J-S) if X is a supermartingale such that there exists an integrable random variable Y with $X_t \geq E(Y | \mathcal{F}_s)$ for all $t \in \mathbb{R}_+$, then the random variable $X_\infty = \lim_{t \rightarrow \infty} X_t$ a.s. exists and $E|X_\infty| < \infty$.

17) The following conditions are equivalent for a right continuous martingale X_t :

- (a) X is uniformly integrable;
- (b) X converges in L_1 , as $t \rightarrow \infty$;
- (c) $X_\infty = \lim_{t \rightarrow \infty} X_t$ a.s. exists, $E|X_\infty| < \infty$, and $(X_t)_{0 \leq t \leq \infty}$ is a martingale;
- (d) there exists an integrable random variable Y , such that $X_t = E(Y | \mathcal{F}_t)$ a.s., for every $t \geq 0$.

Optional sampling:

18)(P) (Doob's Optional Sampling Theorem) Let X be a right-continuous martingale, which is closed by X_∞ . Let S and T be two stopping times such that $S \leq T$ a.s. Then

X_S and X_T are integrable and $X_S = E(X_T | \mathcal{F}_S)$.

19)(P) Let X be an adapted cadlag process. Suppose $E|X_T| < \infty$ and $EX_T = 0$ for any stopping time T , finite or not. Then X is a uniformly integrable martingale.

20)(P) Let X be a uniformly integrable right-continuous martingale and let T be a stopping time. Then $X_T = (X_{t \wedge T})_{0 \leq t \leq \infty}$ is also a uniformly integrable right continuous martingale.

Martingale inequalities:

21)(P) Let X be a martingale and let ϕ be convex such that $\phi(X_t)$ is integrable, $0 \leq t < \infty$. Then $\phi(X)$ is a submartingale.

22)(P) (Doob's maximal inequality) Let X be a positive submartingale. For all $p > 1$ with q conjugate to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$) we have

$$\left\| \sup_t |X_t| \right\|_{L_p} \leq q \sup_t \|X_t\|_{L_q}.$$

23)(K-S1) Let X be a right-continuous submartingale, let $[t_1, t_2] \subset [0, \infty]$, $\lambda > 0$. Then

$$\lambda P\left[\sup_{t_1 \leq t \leq t_2} X_t \geq \lambda\right] \leq E(X_{t_2}^+) \text{ and } \lambda P\left[\inf_{t_1 \leq t \leq t_2} X_t \leq -\lambda\right] \leq E(X_{t_2}^+) - E(X_{t_1}).$$

Local martingales:

24)(P) Each cadlag martingale is a local martingale. Each locally square integrable martingale is also a local martingale. Each continuous local martingale is locally square integrable.

25)(P) Let X be a local martingale. If T reduces X and $S \leq T$ a.s. then S reduces X . If S, T both reduce X then $S \vee T$ also reduces X .

26)(P) Let X be a local martingale. Then X_T and $X_T I_{T>0}$ are local martingales. Moreover if T is predictable then X_- is also a local martingale.

27)(P) Let X be a local martingale then there exists a sequence of increasing stopping times $(T_n)_{n \geq 1}$, with $\lim_{n \rightarrow \infty} T_n = \infty$ a.s. such that $X_{t \wedge T} I_{\{T>0\}}$ is a uniformly integrable martingale for each n .

28)(J-S) A local martingale X is a uniformly integrable martingale if and only if the set of random variables $\{X_T : T \text{ is a finite-valued stopping time}\}$ is uniformly integrable.

29)(P) Let X be a local martingale such that $EX_t^* < \infty$ for every $t \geq 0$. Then X is a martingale. If $EX^* < \infty$, then X is a uniformly integrable martingale.

30)(P) Local martingales form a vector space.

31)(P) (Fundamental theorem of local martingales) Let X be a local martingale and $\beta > 0$. Then there exists local martingales M, A such that A is FV process, the jumps of M are bounded by 2β , and $X = M + A$.

32)(P) Let X be a continuous local martingale and $S \leq T \leq \infty$ be stopping times. If X has paths of finite variation on the stochastic interval (S, T) , then X is constant on $[S, T]$. In particular if X is a continuous FV local martingale then X is constant (proof uses quadratic variation properties, see below).

33)(J-S) Let X be a local martingale then $\hat{X} = X_-$.

FV processes

34)(P) Let T be a totally inaccessible stopping time. Then there exists a FV martingale with exactly one jump, of size one, occurring at time T .

35)(P) Let X be FV predictable local martingale with locally integrable variation and $X_0 = 0$ then X is identically zero.

36)(J-S) Let X be a FV process. If X is a local martingale then X has a locally integrable variation. Moreover if X is predictable then X has a locally integrable variation.

37)(J-S) Let X be a FV process with locally integrable variation then its compensator $\langle X \rangle$ exists and is unique. If moreover X is predictable and $X_0 = 0$ then $\langle X \rangle = X$. Moreover if X is increasing and integrable then its compensator is also increasing and integrable.

38)(J-S) Let X be a FV process with locally integrable variation and $X_0 = 0$ then $\langle X \rangle$ is its compensator if and only if $E(\langle X \rangle_T) = E(X_T)$ for all stopping times T .

39)(J-S) Let X be a FV process with locally integrable variation and T is a stopping time then $\langle X_T \rangle = \langle X \rangle_T$, $\Delta \hat{X} = \Delta \langle X \rangle$.

40)(J-S) Let X be a FV process with locally integrable variation then $\langle X \rangle$ is a local martingale if and only if $\langle X \rangle = 0$.

Stability properties of semimartingales:

41)(P) The set of semimartingales is a vector space and an algebra.

42)(P) If Q is a probability measure and Q is absolutely continuous w.r.t. P , then every P -semimartingale is a Q -semimartingale.

43)(P) Let $(P_k)_{k \geq 1}$ be a sequence of probability measures such that X is a P_k -semimartingale for each k . Let $R = \sum_{k=1}^{\infty} \lambda_k P_k$, where $\lambda_k \geq 0$, for each k and $\sum_{k=1}^{\infty} \lambda_k = 1$. Then X is an R -semimartingale.

44)(P) (Stricker's theorem) Let X be a semimartingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $(\mathcal{G}_t)_{t \geq 0}$ be a subfiltration of $(\mathcal{F}_t)_{t \geq 0}$ such that X adapted to $(\mathcal{G}_t)_{t \geq 0}$. Then X is a semimartingale w.r.t. $(\mathcal{G}_t)_{t \geq 0}$.

45)(P) Let $\mathcal{A}$ be an infinite collection of disjoint events (finite collection of events) in $\mathcal{F}$. Let $(\mathcal{G}_t)_{t \geq 0}$ be a filtration generated by $\mathcal{F}_t$ and $\mathcal{A}$. Then every $((\mathcal{F}_t)_{t \geq 0}, P)$ -semimartingale is an $((\mathcal{G}_t)_{t \geq 0}, P)$ -semimartingale.

46)(P) Every local semimartingale is a semimartingale.

47)(P) Let X be a semimartingale. Then $|X|$, X^+ , X^- are all semimartingales.

48)(P) Let X, Y be semimartingales. Then $X \vee Y$ and $X \wedge Y$ are semimartingales.

49)(P) The space of semimartingales is stable under C^2 transformation and under convex transformation (proof from Ito formula see below).

50)(P) The following processes are trivial examples of semimartingales: an adapted FV process; a local martingale; a supermartingale; a submartingale.

Decompositions:

51)(P) (Rao's theorem) Let X be a cadlag adapted process defined on $[0, \infty]$. Let $\tau = (t_0, \dots, t_{n+1})$ with $0 = t_0 < \dots < t_{n+1} = \infty$ be a partition of $[0, \infty]$ such that X_{t_i} is integrable for each $t_i \in \tau$. Then for each t $E|X_t| < \infty$ and $\text{Var}(X) = \sup_{\tau} E(\sum_{i=0}^n |E(X_{t_i} - X_{t_{i+1}} | \mathcal{F}_{t_i})|) < \infty$ if and only if X has a decomposition $X = Y - Z$ where Y and Z are each positive right continuous supermartingales. Moreover such a process (called quasimartingale) is a semimartingale.

52)(P) An adapted cadlag process X is semimartingale if and only if there exists a local square integrable locally martingale M and an FV process A with $M_0 = A_0 = 0$ such that $X_t = X_0 + M_t + A_t$.

53)(P) (Riesz decomposition) Every right-continuous, uniformly integrable supermartingale X admits the decomposition $X_t = M_t + Z_t$, a.s., as a sum of a right-continuous, uniformly integrable martingale M_t and a right-continuous nonnegative

supermartingale Z_t with $\lim_{t \rightarrow \infty} E(Z_t) = 0$ (such Z_t is called a potential).

54)(P) (Doob-Meyer Decomposition) Let X be a positive supermartingale and suppose $\mathcal{H} = \{X_T; T \text{ is a stopping time}\}$ is uniformly integrable. Then X has a unique decomposition $X = M - A$ where M is a martingale and A is a right continuous, increasing, predictable process with $A_0 = 0$. Moreover if X is a supermartingale (need not be positive) then X has a unique decomposition $X = X_0 + M - A$ where M is a local martingale and A is an increasing predictable process with $M_0 = A_0 = 0$.

54)(P) The canonical decomposition of a special semimartingale is unique.

56)(P) Let X be a semimartingale with bounded jumps then X is a special semimartingale.

57)(P) Let X be a semimartingale. Then X is a special semimartingale if and only if the process $Y_t = \sup_{s \leq t} |X_s - X_0|$ is an increasing process with locally integrable variation.

C.1.2 Stochastic Integration.

We will first define the stochastic integral of predictable processes w.r.t. FV processes. Then we define the stochastic integral of simple predictable processes w.r.t. semimartingales. Then we define the stochastic integral of bounded caglad processes w.r.t. semimartingales. Then we define the stochastic integral of bounded predictable processes w.r.t. special semimartingales using quadratic variation. Then we define the stochastic integral of predictable processes w.r.t. special semimartingales. And finally we define the stochastic integral of predictable processes w.r.t. semimartingales, i.e. the general case.

C.1.3 Stochastic Integrals w.r.t. FV processes.

First we consider stochastic integrals w.r.t. increasing processes. In this case the stochastic integration becomes the usual Stieltjes integration with all the corresponding properties.

Let A be an increasing process. Fix an ω such that $t \rightarrow A_t(\omega)$ is right-continuous

and nondecreasing. This function induces a measure $\mu_A(\omega, ds)$ on $\mathbb{R}_+$. If f is a bounded Borel function on $\mathbb{R}_+$, then $\int_0^t f(s)\mu_A(\omega, ds)$ is well defined for each $t > 0$. We denote this integral by $\int_0^t f(s)dA_s(\omega)$. If $F_s = F(s, \omega)$ is bounded and jointly measurable (in t and ω), we can define ω -wise the integral $\mathbb{I}(t, \omega) = \int_0^t F(s, \omega)\mu_A(\omega, ds)$. Proceeding analogously for A an FV process (except that the induced measure $\mu_A(\omega, ds)$ is a signed measure), we can define the integral

$$\mathbb{I}(t, \omega) = \int_0^t F(s, \omega)\mu_A(\omega, ds)$$

for F bounded and jointly measurable. We will write $F \cdot A$ to denote the process $(F \cdot A_t)_{t \geq 0} = (\mathbb{I}_t)_{t \geq 0}$. We will also write $\int_0^t F_s |dA_s|$ for $F \cdot |A|_t$.

The integral w.r.t. FV processes has same properties as usual Stieltjes integrals.

Proposition 37 1)(P)(Absolute continuity). *Let A, C be adapted, increasing processes such that $C - A$ is also an increasing process. Then there exists a jointly measurable, adapted process H such that $0 \leq H \leq 1$ and $A = H \cdot C$. If moreover A, C are predictable then one may choose H to be predictable. Moreover if A is FV process then there exists a jointly measurable, adapted process H , $-1 \leq H \leq 1$ such that $|A| = H \cdot A$ and $A = H \cdot |A|$.*

2)(P)(Riemann-Stieltjes integral). *Let A be a FV-process and let H be a continuous jointly measurable process. Let a_n be a sequence of finite random partitions of $[0, t]$ with $\lim_{n \rightarrow \infty} \text{mesh}(a_n) = 0$. Then for $T_k \leq S_k \leq T_{k+1}$,*

$$\lim_{n \rightarrow \infty} \sum_{T_k, T_{k+1} \in a_n} H_{S_k} (A_{T_{k+1}} - A_{T_k}) = \int_0^t H_s dA_s \text{ a.s.}$$

3)(P)(Change of variables). *Let A be an FV process with right-continuous paths. If $f \in C^1(\mathbb{R})$ then $(f(A_t))_{t \geq 0}$ is a FV process and*

$$f(A_t) - f(A_0) = \int_{0+}^t f'(A_{s-}) dA_s + \sum_{0 < s \leq t} (f(A_s) - f(A_{s-}) - f'(A_{s-}) \Delta A_s).$$

4)(P) *Let A be a FV predictable process of integrable variation and let H be bounded, measurable. Then, for $\hat{H}$ the predictable projection of H ,*

$$E\left(\int_0^\infty H_s dA_s\right) = E\left(\int_0^\infty \hat{H}_s dA_s\right).$$

- 5)(J-S) Let A be a predictable FV process, and let H be a predictable process such that $E(\int_0^\infty |H_s| |dA_s|) < \infty$. Then the FV process $(\int_0^t H_s dA_s)_{t \geq 0}$ is also predictable.
- 6)(J-S) Let A be a FV process with $A_0 = 0$ and integrable variation, T be a stopping time and H be a bounded martingale. Then $E(H_T A_T) = E(H \cdot A_T)$ and $HA - H \cdot A$ is a local martingale. If moreover A is predictable then $E(H_T A_T) = E(H_- \cdot A_T)$ and $HA - H_- \cdot A$ is a local martingale.
- 7)(J-S) Let A be a FV process with $A_0 = 0$ and integrable variation then $\langle A \rangle$ is its compensator if and only if $E(H \cdot \langle A \rangle)_\infty = E(H \cdot A)_\infty$ for all nonnegative predictable processes H .
- 8)(J-S) Let A be a FV local martingale, H is a predictable process and $H \cdot A$ is a FV process with integrable variation then $H \cdot A$ is a local martingale.

C.1.4 Stochastic integration of adapted cadlag processes w.r.t. semimartingales.

Definition 35 (i) A process H is said to be simple predictable if H has a representation

$$H_t = H_0 I_0(t) + \sum_{i=1}^n H_i I_{(T_i, T_{i+1}]}(t),$$

where $0 = T_0 \leq \dots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$.

(ii) We denote the space of simple predictable processes by $\mathbb{S}$, $\mathbb{S}$ topologized by uniform convergence (in (t, ω)) by $\mathbb{S}_u$ and $\mathbb{S}$ topologized by ucp convergence by $\mathbb{S}_{ucp}$.

(iii) For X a cadlag adapted process we define the linear mapping $\mathbb{I}_X : \mathbb{S}_u \rightarrow \mathbb{L}^0$ such that

$$\mathbb{I}_X(H) = H_0 X_0 + \sum_{i=1}^n H_i (X_{T_{i+1}} - X_{T_i}),$$

where $H \in \mathbb{S}_u$ is simple predictable.

(iv) For X a cadlag adapted process we define the linear mapping $\mathbb{J}_X : \mathbb{S}_{ucp} \rightarrow \mathbb{D}_{ucp}$ such that

$$\mathbb{J}_X(H) = H_0 X_0 + \sum_{i=1}^n H_i (X_{T_{i+1}} - X_{T_i}),$$

where $H \in \mathbb{S}$ is simple predictable.

- Proposition 38** 1)(P) The space $\mathbb{S}$ is dense in $\mathbb{L}$ under the ucp topology.
 2)(P) (Bichteler-Dellacherie theorem) A cadlag adapted process X is semimartingale if and only if $\mathbb{I}_{X_T}$ is continuous for each $T \in [0, \infty)$.
 3)(P) $\mathbb{J}_X(H)_T = \mathbb{I}_{X_T}(H)$.
 4)(P) Let X be a semimartingale. Then the mapping $\mathbb{J}_X$ is continuous.

Definition 36 Let X be a semimartingale. The continuous linear mapping $\mathbb{J}_X : L_{ucp} \rightarrow \mathbb{D}_{ucp}$ obtained as the extension of $\mathbb{J}_X : S_{ucp} \rightarrow \mathbb{D}_{ucp}$ is called a stochastic integral. We use notations here: $\mathbb{J}_X(H) = \int H_s dX_s = H \cdot X$, $\int_0^t H_s dX_s = H \cdot X_t$ and $\int_0^\infty H_s dX_s = \lim_{t \rightarrow \infty} \int_0^t H_s dX_s$.

- Remark 18** (i) $\mathbb{J}_X(H)$ is a stochastic process and $\mathbb{I}_X(H)$ is a random variable.
 (ii) $\mathbb{I}_X$ plays the role of a definite integral i.e. $\mathbb{I}_X(H) = \int_0^\infty H_s dX_s$.
 (iii) We will give properties of a stochastic integral in the next section where we consider predictable integrands.

C.1.5 Quadratic variation of semimartingale.

Definition 37 (i) Let X be a semimartingale. The quadratic variation process of X , denoted $[X, X] = ([X, X]_t)_{t \geq 0}$, is defined by:

$$[X, X] = X^2 - 2 \int X_- dX.$$

(ii) Let X, Y be semimartingales. The quadratic covariation process of X, Y is defined by:

$$[X, Y] = XY - \int X_- dY - \int Y_- dX.$$

- (iii) For a semimartingale X , the process $[X, X]^c$ denotes the path by path continuous part of $[X, X]$.
 (iv) A semimartingale X is said to be a quadratic pure jump process if $[X, X]^c = 0$.

Properties of quadratic variation.

Proposition 39 1)(P) The quadratic variation process of a semimartingale X is a cadlag increasing adapted process with $[X, X]_0 = X_0$ and $\Delta[X, X] = (\Delta X)^2$.

2)(J-S) The quadratic covariation process $[X, Y]$ is an FV process and $\Delta[X, Y] = \Delta X \Delta Y$.

3)(P) If $\tau_n = (T_0^n, \dots, T_{k_n}^n)$ is a sequence of random partitions tending to identity, then

$$X_0^2 + \sum_i (X_{T_{i+1}^n} - X_{T_i^n})^2 \rightarrow [X, X].$$

4)(J-S) Let X be a semimartingale and Y be a FV process, then $[X, Y] = \Delta X \cdot Y$ and $XY = Y_- \cdot X + X \cdot Y$.

5)(J-S) Let X be a local martingale and Y be a predictable FV process, then $[X, Y]$ is a local martingale.

6)(J-S) Let X be a continuous semimartingale and Y be a FV process then $[X, Y] = 0$.

7)(P) If T is a stopping time, then $[X_T, X] = [X, X_T] = [X_T, X_T] = [X, X]^T$.

8)(P) (Kunita-Watanabe inequality) Let X and Y be two semimartingales, and let H and K be two measurable processes. Then

$$\left| \int_0^\infty H_s |K_s| d[X, Y]_s \right| \leq \left(\int_0^\infty H_s^2 d[X, X]_s \right)^{\frac{1}{2}} \left(\int_0^\infty K_s^2 d[Y, Y]_s \right)^{\frac{1}{2}}.$$

9)(P) $[X, X]_0^c = 0$, $[X^c, X^c] = [X, X]^c$ and we can write

$$[X, X]_t = [X, X]_t^c + X_0^2 + \sum_{0 < s \leq t} (\Delta X_s)^2 = [X, X]_t^c + \sum_{0 \leq s \leq t} (\Delta X_s)^2.$$

10)(P) If X is a quadratic pure jump process, then $[X, X]_t = \sum_{0 \leq s \leq t} (\Delta X_s)^2$.

11)(P) If X is adapted cadlag FV process then X is quadratic pure jump semimartingale. In particular if X is an adapted continuous FV process then $[X, X]_t = X_0^2$.

12)(P) Let X be a continuous local martingale. If its paths are not everywhere constant X then $[X, X]$ is not the constant process X_0^2 , and $X^2 - [X, X]$ is a continuous local martingale. Moreover if $[X, X]_t = 0$ for all t then $X_t = 0$ for all t . Moreover if $S \leq T \leq \infty$ are stopping times and $[X, X]$ is constant on $[S, T] \cap [0, \infty)$, then X is constant there too. Moreover X and $[X, X]$ have the same intervals of constancy a.s.

13)(P) Let X and Y be two locally square integrable local martingales (in $\mathbb{M}^2$). Then $[X, Y]$ is the unique adapted cadlag FV process A such that $XY - A$ is a local martingale, $\Delta A = \Delta X \Delta Y$ and $A_0 = X_0 Y_0$. In particular if XY is a martingale and X

is continuous then $[X, Y] \equiv X_0 Y_0$. Moreover if X is a continuous square integrable martingale and Y is FV square integrable martingale then $[X, Y] = X_0 Y_0$ and hence XY is a martingale.

14)(P) Let X be a local martingale, then its quadratic variation $[X, X]_t$ always exist and is finite a.s. for every $t \geq 0$.

15)(P) Let X be a local martingale. Then X is a square integrable martingale if and only if $E([X, X]_t) < \infty$, all $t \geq 0$. If $E([X, X]_t) < \infty$ then $EX_t^2 = E([X, X]_t)$. Moreover if $E([X, X]_\infty) < \infty$ then $\sup_t EX_t^2 = EX_\infty^2 = E([X, X]_\infty) < \infty$.

16)(P) Let X be a quadratic pure jump semimartingale. Then for any semimartingale Y we have

$$[X, Y]_t = X_0 Y_0 + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s.$$

17)(P) Let H be a cadlag adapted process and X, Y be two semimartingales. Let $\tau_n = (T_0^n, \dots, T_{k_n}^n)$ be a sequence of random partitions tending to the identity. Then

$$\sum H_{T_i^n} (X_{T_{i+1}^n} - X_{T_i^n}) (Y_{T_{i+1}^n} - Y_{T_i^n}) \xrightarrow{ucp} \int H_{s-} d[X, Y]_s.$$

18)(P) Let X be a continuous semimartingale with canonical decomposition $X = M + A$. Then $[X, X] = [M, M]$.

19)(P) Let X be a predictable FV process of locally integrable variation and $X_0 = 0$. Then for all bounded martingales M : $E[M, X]_\infty = 0$ or equivalently

$$E\left(\int_0^\infty M_{s-} dX_s\right) = E(M_\infty X_\infty).$$

16)(P) (Burkholder's inequality) Let X be a continuous local martingale with $X_0 = 0$, $2 \leq p < \infty$, and T a finite stopping time. Then

$$E(X_T^*)^p \leq C(p) E[X, X]_T^{p/2}$$

where $C(p)$ is a constant depends only on p .

C.1.6 Stochastic integration of predictable processes w.r.t. semimartingales - general stochastic integral.

W.l.o.g we assume here that $X_0 = 0$.

Now we define a stochastic integral w.r.t. a special semimartingale.

Definition 38 Let X be a special semimartingale with canonical decomposition $X = M + A$. We define the space $\mathcal{H}^2$ of all special semimartingales with $\mathcal{H}^2$ -norm:

$$\|X\|_{\mathcal{H}}^2 = \|[M, M]_{\infty}^{1/2}\|_{L^2} + \left\| \int_0^{\infty} |dA_s| \right\|_{L^2}.$$

Remark 19 (i)(P) $\mathcal{H}^2$ is a Banach space.

(ii)(P) If H is an adapted caglad, bounded process and $X \in \mathcal{H}^2$ then the stochastic integral (as defined above) $H \cdot X \in \mathcal{H}^2$. Moreover if X is a special semimartingale with canonical decomposition $X = M + A$ then $H \cdot M + H \cdot A$ is a canonical decomposition of $H \cdot X$ and

$$\|H \cdot X\|_{\mathcal{H}^2} = \left\| \left(\int_0^{\infty} H_s^2 d[M, M]_s \right)^{1/2} \right\|_{L^2} + \left\| \int_0^{\infty} |H_s| |dA_s| \right\|_{L^2}.$$

(iii)(P) $[M, M]$, A are FV processes and therefore ω -by- ω the integrals $\int_0^t H_s^2(\omega) d[M, M]_s(\omega)$ and $\int_0^t |H_s| |dA_s|$ make sense for any H bounded predictable.

Definition 39 Let $X \in \mathcal{H}^2$ with $X = M + A$ its canonical decomposition and H, H' are in $b\mathbb{P}$. We define $d_X(H, H')$ by:

$$d_X(H, H') = \left\| \left(\int_0^{\infty} (H_s - H'_s)^2 d[M, M]_s \right)^{1/2} \right\|_{L^2} + \left\| \int_0^{\infty} |H_s - H'_s| |dA_s| \right\|_{L^2}.$$

The following proposition describes some properties of $\mathcal{H}^2$ which are necessary to construct the stochastic integral of bounded predictable processes w.r.t. the special semimartingale.

Proposition 40 1)(P) For $X \in \mathcal{H}^2$ the space $b\mathbb{L}$ is dense in the space $b\mathbb{P}$ under $d_X(\cdot, \cdot)$.

2)(P) Let $X \in \mathcal{H}^2$ and $H^n \in b\mathbb{L}$ such that H^n is Cauchy under d_X . Then $H^n \cdot X$ is Cauchy in $\mathcal{H}^2$.

3)(P) Let $X \in \mathcal{H}^2$ and $H^n \in b\mathbb{P}$. Suppose $H^n \in b\mathbb{L}$ and $H^m \in b\mathbb{L}$ are two sequences such that $\lim_n d_X(H^n, H) = \lim_m d_X(H^m, H) = 0$. Then $H^n \cdot X$ and $H^m \cdot X$ tend to the same limit in $\mathcal{H}^2$.

Definition 40 (i) Let $X \in \mathcal{H}^2$ and $H^n \in b\mathbb{P}$. Suppose $H^n \in b\mathbb{L}$ is a sequence such that $\lim_n d_X(H^n, H) = 0$. The stochastic integral $H \cdot X$ is the unique semimartingale in $\mathcal{H}^2$ such that $\lim_{n \rightarrow \infty} H^n \cdot X = H \cdot X$ in $\mathcal{H}^2$. We write $H \cdot X = (\int_0^t H_s dX_s)_{t \geq 0}$.
(ii) Let $X \in \mathcal{H}^2$ with canonical decomposition $X = M + A$. We say a predictable process H is $(\mathcal{H}^2, X)$ -integrable if

$$E\left(\int_0^\infty H_s^2 d[M, M]_s\right) + E\left(\int_0^\infty |H_s| |dA_s|\right)^2 < \infty.$$

The following proposition describes some properties of $\mathcal{H}^2$ which are necessary to construct the stochastic integral of predictable processes w.r.t. any semimartingale.

Proposition 41 1)(P) Let $X \in \mathcal{H}^2$ with canonical decomposition $X = M + A$. Then $E([M, M]_\infty) \leq E([X, X]_\infty)$.

2)(P) Let $X \in \mathcal{H}^2$. Then $E(\sup_t |X_t|)^2 \leq 8\|X\|_{\mathcal{H}^2}^2$.

3)(P) Let (X^n) be a sequence of semimartingales converging to X in $\mathcal{H}^2$. Then there exists a subsequence (n_k) such that $\lim_{n_k \rightarrow \infty} (X^{n_k} - X)^* = 0$ a.s.

4)(P) Let A be an FV process with $A_0 = 0$ and $\int_0^\infty |dA_s| \in L^2$. Then $A \in \mathcal{H}^2$ and $\|A\|_{\mathcal{H}^2} \leq 6\|\int_0^\infty |dA_s|\|_{L^2}$.

5)(P) Let X be a semimartingale, $X_0 = 0$. Then there exist a nondecreasing sequence of stopping times $T^n \nearrow \infty$ a.s., such that $X^{T^n-} \in \mathcal{H}^2$ for each n .

6)(P) Let X be a semimartingale and let a predictable process H be $(\mathcal{H}^2, X)$ -integrable. Let $H^n = HI_{\{|H| \leq n\}} \in b\mathbb{P}$. Then $H^n \cdot X$ is a Cauchy sequence in $\mathcal{H}^2$.

Definition 41 (i) Let $X \in \mathcal{H}^2$ and let a predictable process H be $(\mathcal{H}^2, X)$ -integrable. The stochastic integral $H \cdot X$ is defined to be $\lim_{n \rightarrow \infty} H^n \cdot X$, with convergence in $\mathcal{H}^2$, where $H^n = HI_{\{|H| \leq n\}}$.

(ii) Let X be a semimartingale and let H be a predictable process. The stochastic integral $H \cdot X$ is said to exist if there exists a sequence of stopping times $T^n \nearrow \infty$ a.s. such that $X^{T^n-} \in \mathcal{H}^2$ for each $n \geq 1$ and such that H is $(\mathcal{H}^2, X)$ -integrable for each n . In this case we say H is X -integrable, written $H \in L(X)$, and we define the stochastic integral by $H \cdot X = H \cdot (X^{T^n-})$ on $[0, T^n)$ for each n .

Proposition 42 Basic properties of the stochastic integral:

(1)(P) Let X be a semimartingale and $H \in L(X)$ then $H \cdot X$ is a semimartingale.

(2)(P) $L(X)$ is a linear space.

(3)(P) Let X, Y be semimartingales and $H \in L(X) \cap L(Y)$. Then $H \in L(H + Y)$ and $H \cdot (X + Y) = H \cdot X + H \cdot Y$.

(4)(P) Let X be a semimartingale and $H \in L(X)$ then the jump process $(\Delta(H \cdot X))_{s \geq 0}$ is indistinguishable from $(H_s(\Delta X_s))_{s \geq 0}$.

(5)(P) Let T be a stopping time, X be a semimartingale and $H \in L(X)$. Then $(H \cdot X)^T = HI_{[0, T]} \cdot X = H \cdot (X_T)$. Also $(H \cdot X)^{T-} = H \cdot (X_{T-})$.

(6)(P) Let X be a FV semimartingale and $H \in L(X)$ such that the Stieltjes integral $\int |H_s| |dX_s|$ exists a.s. each $t \geq 0$. Then the stochastic integral $H \cdot X$ agrees with a path-by-path Stieltjes integral.

(7)(P) Let X be a semimartingale with $K \in L(X)$. Then $H \in L(K \cdot X)$ if and only if $HK \in L(X)$, in which case $H \cdot (K \cdot X) = (HK) \cdot X$.

(8)(P) Let X, Y be semimartingales and let $H \in L(X), K \in L(Y)$ then for $t \geq 0$

$$[H \cdot X, K \cdot Y]_t = \int_0^t H_s K_s d[X, Y].$$

9)(P) For a semimartingale X with $X_0 = 0$ we have

$$\|X\|_{\mathcal{H}^2} \leq 3 \sup_{|H| \leq 1} \|(H \cdot X)_\infty^*\|_{L^2} \leq 9\|X\|_{\mathcal{H}^2}.$$

Cases where the stochastic integral exists:

10)(P) Let X be a semimartingale and H be a locally bounded predictable process then $H \in L(X)$ i.e. the stochastic integral exists.

11)(P) Let X be a continuous local martingale and let H be a predictable process such that $\int_0^t H_s^2 d[X, X]_s < \infty$ a.s. $t \geq 0$. Then the stochastic integral $H \cdot X$ exists and it is a continuous local martingale.

12)(P) Let X be a continuous semimartingale with canonical decomposition $X = M + A$. Let H be a predictable process such that for each $t \geq 0$.

$$\int_0^t H_s^2 d[M, M]_s + \int_0^t |H_s| |dA_s| < \infty \text{ a.s.}$$

Then the stochastic integral $(H \cdot X)_t = \int_0^t H_s dX_s$ exists and it is continuous.

13)(P) Let X be a local martingale with jumps bounded by a constant. Let H be a

predictable process such that $\int_0^t H_s^2 d[X, X]_s < \infty$ a.s. $t \geq 0$ and $EH_T^2 < \infty$ for any bounded stopping time T . Then the stochastic integral $(H \cdot X)_t = \int_0^t H_s dX_s$ exists and it is a local martingale.

Convergence of stochastic integrals:

14)(P) Let X be a semimartingale and H in $\mathbb{L}$ or in $\mathbb{D}$. Let $\tau_n = (T_0^n, \dots, T_{k_n}^n)$ be a sequence of random partitions tending to identity and $H^{\tau_n} = H_0 I_0 + \sum_j H_{T_j^n} I_{(T_j^n, T_{j+1}^n]}$. Then $\int H_s^{\tau_n} dX_s = H_0 X_0 + \sum_j H_{T_j^n} (X_{T_{j+1}^n} - X_{T_j^n})$ and

$$\sum_j H_{T_j^n} (X_{T_{j+1}^n} - X_{T_j^n}) \xrightarrow{ucp} (H_-) \cdot X.$$

15)(P) (Dominated convergence) Let X be a semimartingale and H^m be a sequence of predictable processes converging a.s. to a limit H . If there exists process $G \in L(X)$ such that $|H^m| \leq G$, all n , then H^m, H are in $L(X)$ and $H^m \cdot X$ converges to $H \cdot X$ in ucp.

Stability properties of the stochastic integral:

16)(P) Let X be a semimartingale and $H \in L(X)$. Then the stochastic integral $H \cdot X$ is a semimartingale.

17)(P) Let X be a semimartingale and $H \in L(X)$. If $Q \ll P$ then $H \in L(X)$ under Q as well and $H_Q \cdot X = H_P \cdot X$ a.s.

18)(P) Let X, Y be two semimartingale, $H \in L(X)$, $K \in L(Y)$ and S, T be two stopping times with $S < T$. Let $A = \{\omega : H_t(\omega) = K_t(\omega) \text{ and } X_t(\omega) = Y_t(\omega); S(\omega) < t \leq T(\omega)\}$ and $B = \{\omega : t \rightarrow X_t(\omega) \text{ is of finite variation on } S(\omega) < t < T(\omega)\}$. Then $H \cdot X_T - H \cdot X^S = K \cdot Y^T - K \cdot Y^S$ on A and $H \cdot X_T - H \cdot X^S$ equals a path-by-path Lebesgue-Stieltjes integral on B .

19)(P) Let $(P_k)_{k \geq 1}$ be a sequence of probability measures such that X is a P_k -semimartingale for each k . Let $R = \sum_{k=1}^{\infty} \lambda_k P_k$, where $\lambda_k \geq 0$, for each k and $\sum_{k=1}^{\infty} \lambda_k = 1$. Let $H \in L(X)$ under R . Then $H \in L(X)$ under P_k and $H_R \cdot X = H_{P_k} \cdot X$, P_k -a.s., for all k such that $\lambda_k > 0$.

20)(P) Let X be a square integrable martingale and let H be a predictable process with $E(\int_0^\infty H_s^2 d[X, X]_s) < \infty$. Then $H \cdot X$ is a square integrable martingale. Moreover if X is a local martingale and H is a locally bounded predictable process then the stochastic integral $H \cdot X$ is a local martingale.

21)(P) Let X be a semimartingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $(\mathcal{G}_t)_{t \geq 0}$ be a sub-filtration of $(\mathcal{F}_t)_{t \geq 0}$ such that X adapted to $(\mathcal{G}_t)_{t \geq 0}$. Let H be locally bounded and predictable for $(\mathcal{F}_t)_{t \geq 0}$. Then the stochastic integrals $H_{\mathcal{F}} \cdot X$ and $H_{\mathcal{G}} \cdot X$ both exists and they are equal.

Main theorems of the stochastic integration:

22)(P) (Ito's Formula). Let X be a semimartingale and let $f \in C^2$ be a real-valued function. Then $f(X)$ is again a semimartingale and

$$f(X_t) - f(X_0) = \int_{0+}^t f'(X_{s-}) dX_s + \frac{1}{2} \int_{0+}^t f''(X_{s-}) d[X, X]_s^c + \sum_{0 < s \leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s\}.$$

23)(P) Let f be convex and let X be a semimartingale. Then $f(X)$ is a semimartingale and

$$f(X_t) - f(X_0) = \int_{0+}^t f^l(X_{s-}) dX_s + A_t$$

where f^l is the left derivative of f and A is an adapted, right continuous increasing process with $\Delta A_t = f(X_t) - f(X_{t-}) - f^l(X_{t-}) \Delta X_t$ (for more general cases of Ito's formula see (P)).

24)(P) (Doleans-Dade exponential) Let X be a semimartingale, $X_0 = 0$. Then there exists a unique semimartingale $\mathcal{E} = \mathcal{E}(X)$ that satisfies the equation: $\mathcal{E}_t = 1 + \int_0^t \mathcal{E}_{s-} dX_s$ and $\mathcal{E}$ is given by

$$\mathcal{E}_t = \exp(X_t - \frac{1}{2}[X, X]_t) \prod_{0 < s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2)$$

where the infinite product converges. Such a semimartingale $\mathcal{E}(X)$ is called the stochastic(Doleans-Dade) exponential. And moreover if X and Y are two semimartingales with $X_0 = Y_0 = 0$ then $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y])$.

25)(P) (Fubini's theorem) Let $(A, \mathcal{A}, \mu)$ denote a measurable space and μ a positive finite measure. Let X be a semimartingale, $H_t^a = H(a, t, \omega)$ be a bounded $\mathcal{A} \otimes \mathcal{P}$ measurable function. Then there exists $Z_t^a = Z(a, t, \omega)$ $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ -measurable such that for each a , Z^a is a cadlag version of $H^a \cdot X$. Moreover $Y_t = \int_A Z_t^a \mu(da)$ (if exists) is a cadlag version of $H \cdot X$, where $H_t = \int_A H_t^a \mu(da)$. Moreover if $(\int_A (H_t^a)^2 \mu(da))^{1/2} \in L(X)$ then such Y_t exists.

C.1.7 Square-integrable martingales and martingale representation.

Definition 42 (i) A cadlag martingale X is said to be square-integrable if $E(X_t^2) < \infty$ for every $0 \leq t \leq \infty$. The space of square integrable martingales with $X_0 = 0$ we denote by $\mathbb{M}^2$. Since for $X \in \mathbb{M}^2$ we have $\lim_{t \rightarrow \infty} EX_t^2 = EX_\infty^2 < \infty$ and $X_t = E(X_\infty | \mathcal{F}_t)$. Thus each $X \in \mathbb{M}^2$ can be identified with its terminal value X_∞ . We can endow $\mathbb{M}^2$ with a norm: $\|X\| = (EX_\infty^2)^{1/2}$ and with an inner product: $(X, Y) = EX_\infty Y_\infty$ for $X, Y \in \mathbb{M}^2$. It is evident that $\mathbb{M}^2$ is a Hilbert space and its dual space is also $\mathbb{M}^2$.

(ii) Let X be a locally square-integrable martingale. We define the increasing predictable process $\langle X \rangle = \langle X, X \rangle$ such that $X^2 - \langle X \rangle$ is a uniformly integrable martingale. Another common name for $\langle X \rangle$ is the compensator of X . To each pair X, Y of locally square-integrable martingales we associate a process $\langle X, Y \rangle = \frac{1}{4}(\langle X + Y, X + Y \rangle - \langle X - Y, X - Y \rangle)$.

(iii) A closed subspace $\mathcal{S} \subset \mathbb{M}^2$ is called a stable subspace if it is stable under stopping, that is if $X \in \mathcal{S}$ and if T is a stopping time, then $X_T \in \mathcal{S}$. Let $\mathcal{A} \subset \mathbb{M}^2$. The stable subspace generated by $\mathcal{A}$ denoted $\mathcal{S}(\mathcal{A})$, is the intersection of all closed stable subspaces containing $\mathcal{A}$.

(iv) Two martingales $X, Y \in \mathbb{M}^2$ are said to be orthogonal if $E(X_\infty Y_\infty) = 0$. Two martingales $X, Y \in \mathbb{M}^2$ are said to be strongly orthogonal if $Z = XY$ is a martingale. For $\mathcal{A} \subset \mathbb{M}^2$ we let $\mathcal{A}^\perp$ (respectively $\mathcal{A}^\times$) denote the set of all elements of $\mathbb{M}^2$ orthogonal (respectively strongly orthogonal) to each element of $\mathcal{A}$.

(v) Let $\mathcal{A} \subset \mathbb{M}^2$ be a finite set of martingales. We say that $\mathcal{A}$ has the predictable representation property if

$$\mathbb{M}^2 = \{X : X = \sum_{i=1}^n H^i M^i, M^i \in \mathcal{A}, H^i \in \mathbb{P}, E(\int_0^\infty (H_s^i)^2 d[M^i, M^i]_s) < \infty\}.$$

Proposition 43 1)(S) Let X be a locally square-integrable martingale. Then $\langle X \rangle$ exists and is unique, and $\langle X \rangle$ is a compensator of $[X, X]$ since $[X, X]$ is FV process. Moreover $\langle X^c \rangle = [X, X]^c$. Moreover $\langle X \rangle$ can be defined as a compensator of $[X, X]$ for any semimartingale X .

2)(J-S) Let X, Y be a pair of locally square-integrable martingales. Then $\langle X, Y \rangle$ is a predictable FV process and $XY - \langle X, Y \rangle$ is a local martingale. Moreover $\langle X, Y \rangle$ is a compensator of $[X, Y]$ and $[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \leq t} \Delta X_s \Delta Y_s$.

3)(J-S) $E(X_\infty Y_\infty) = E(\langle X, Y \rangle_\infty)$ if $X_0 Y_0 = 0$.

4)(P) Let $\mathcal{S} \subset \mathbb{M}^2$ is a closed subspace then the following are equivalent

(a) $\mathcal{S}$ is a stable subspace;

(b) For any $\Lambda \in \mathcal{F}_t$, any $X \in \mathcal{S}$ we have $(X - X^t)I_\Lambda \in \mathcal{S}$, any $t \geq 0$.

(c) if $X \in \mathcal{S}$, $H \in \mathcal{bP}$ then $H \cdot X \in \mathcal{S}$.

(d) if $X \in \mathcal{S}$, H is predictable with $E(\int_0^\infty (H_s)^2 d[X, X]_s) < \infty$ then $H \cdot X \in \mathcal{S}$.

5)(P) Strong orthogonality implies orthogonality (the converse is not true).

6)(J-S) A local martingale $X \in \mathbb{M}^2$ is strongly orthogonal to itself if and only if X is indistinguishable from zero.

7)(J-S) Let $X, Y \in \mathbb{M}^2$ be orthogonal then for all stopping times T, S we have X^S and Y^T are strongly orthogonal.

8)(P) If $\mathcal{A} \subset \mathbb{M}^2$, then $\mathcal{A}^\times$ is closed and stable.

9)(P) the following are equivalent

(a) $X, Y \in \mathbb{M}^2$ are strongly orthogonal;

(b) $[X, Y]$ is a uniformly integrable martingale;

(c) $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ are strongly orthogonal;

(d) $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ are orthogonal;

(e) $\langle X, Y \rangle = 0$;

(f) for all stopping times T , X_T and Y are strongly orthogonal.

10) Any local martingale X admits a unique decomposition $X = X_0 + X^c + X^d$, where $X_0^c = X_0^d = 0$ and X^c is continuous local martingale and X^d is a local martingale which is strongly orthogonal to all continuous local martingales.

11)(P) Let $X^1, \dots, X^n \in \mathbb{M}^2$ and suppose X^i, X^j are strongly orthogonal for $i \neq j$. Then

$$\mathcal{S}(X^1, \dots, X^n) = \{X : X = \sum_{i=1}^n H^i M^i, H^i \in \mathcal{bP}, E(\int_0^\infty (H_s^i)^2 d[M^i, M^i]_s) < \infty\}.$$

12)(P) Let $\mathcal{A} \subset \mathbb{M}^2$ is stable then $\mathcal{A}^\times = \mathcal{A}^\perp$ and $\mathcal{S}(\mathcal{A}) = \mathcal{A}^{\perp\perp} = \mathcal{A}^{\perp\times} = \mathcal{A}^{\times\times}$.

13)(P) Let $\mathcal{A} \subset \mathbb{M}^2$ is stable then each $X \in \mathbb{M}^2$ has a decomposition $X = A + B$ with

$A \in \mathcal{A}$ (projection of X) and $B \in \mathcal{A}^\times$.

14)(P) Let $X, Y \in \mathbb{M}^2$ and let Z be the projection of Y onto $\mathcal{S}(X)$. Then there exists a predictable process H such that $Z = H \cdot X$.

15)(P) Let $\mathcal{A} = \{X^1, \dots, X^n\} \in \mathbb{M}^2$ and suppose X^i, X^j are strongly orthogonal for $i \neq j$. Suppose that if $Y \in \mathbb{M}^2$, Y strongly orthogonal to $\mathcal{A}$ implies $Y = 0$. Then $\mathcal{A}$ has the predictable representation property.

Definition 43 (i) Let $\mathcal{A} \subset \mathbb{M}^2$. The set of $\mathbb{M}^2$ martingale measures for $\mathcal{A}$, denoted $\mathcal{M}^2(\mathcal{A})$, is the set of all probability measures Q defined on $\cup_{0 \leq t < \infty} \mathcal{F}_t$ such that $Q \ll P$, $Q = P$ on $\mathcal{F}_0$ and if $X \in \mathcal{A}$ then X is a square integrable martingale w.r.t. Q .

(ii) A measure $Q \in \mathcal{M}^2(\mathcal{A})$ is an extreme point of $\mathcal{M}^2(\mathcal{A})$ if whenever $Q = \lambda Q_1 + (1 - \lambda)Q_2$ with $Q_1, Q_2 \in \mathcal{M}^2(\mathcal{A})$, $Q_1 \neq Q_2$, $0 \leq \lambda \leq 1$ then $\lambda = 0$ or 1 .

Proposition 44 1)(P) $\mathcal{M}^2(\mathcal{A})$ is a convex set.

2)(P) Let $\mathcal{A} \subset \mathbb{M}^2$. If $\mathcal{S}(\mathcal{A}) = \mathbb{M}^2$ then P is an extreme point of $\mathcal{M}^2(\mathcal{A})$.

3)(P) Let $\mathcal{A} \subset \mathbb{M}^2$. If P is an extreme point of $\mathcal{M}^2(\mathcal{A})$ then every bounded P -martingale orthogonal to $\mathcal{A}$ is null.

4)(P) Let $\mathcal{A} = \{X^1, \dots, X^n\} \in \mathbb{M}^2$ with X^i continuous and X^i, X^j are strongly orthogonal for $i \neq j$. Suppose P is an extreme point of $\mathcal{M}^2(\mathcal{A})$ then every stopping time is accessible; every bounded martingale is continuous; every uniformly integrable martingale is continuous; $\mathcal{A}$ has the predictable representation property.

5)(P) Let $X \in \mathbb{M}^2$, $Y^n \in \mathbb{M}^2$, $n \geq 1$ and suppose Y_∞^n converges to Y_∞ in L^2 and that exists a sequence $H^n \in L(X)$ such that $Y_t^n = (H^n \cdot X)_t$, $n \geq 1$. Then there exists a predictable process $H \in L(X)$ such that $Y_t = (H \cdot X)_t$ (more general case see (P)).

6)(P) Let $W = (W^1, \dots, W^n)$ be an n -dimensional Brownian motion and $\mathbb{F}$ denote its completed natural filtration. Then every $\mathbb{F}$ -local martingale X has a representation

$$X_t = X_0 + \sum_{i=1}^n \int_0^t H_s^i dX_s^i$$

where H^i are predictable.

C.1.8 The characteristic of semimartingales and Girsanov theorem.

Definition 44 (i) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(D, \mathcal{D})$ be a measurable space. A random measure on $\mathbb{R}_+ \times D$ is a family $\mu = (\mu(\omega; dt, dx) : \omega \in \Omega)$ of nonnegative measures on $(\mathbb{R}_+ \times D, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{D})$ satisfying $\mu(\omega; \{0\} \times D) = 0$ for any $\omega \in \Omega$.

(ii) We consider $\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times D$ with σ -algebras $\tilde{\mathcal{O}} = \mathcal{O} \otimes \mathcal{D}$ and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{D}$. A function $V = V(\omega, t, x)$ on $\tilde{\Omega}$ that is $\tilde{\mathcal{O}}$ -measurable (respectively $\tilde{\mathcal{P}}$ -measurable) is called optional (respectively predictable) function.

(iii) For an optional V and a random measure μ we denote $V * \mu$ the integral process such that

$$(V * \mu)_t(\omega) = \int_{(0,t] \times D} V(\omega, s, x) \mu(\omega; ds, dx)$$

where the integral is Lebesgue-Stieltjes for any $\omega \in \Omega$ and

$$\int_{(0,t] \times D} |V(\omega, s, x)| \mu(\omega; ds, dx) < \infty$$

for $t > 0$.

(iv) A random measure is called optional (respectively predictable) if the process $V * \mu$ is optional (respectively predictable) for every optional (respectively predictable) function V . An optional measure μ is integrable if the random variable $1 * \mu_\infty = \mu(\cdot, \mathbb{R}_+ \times D)$ is integrable (or equivalently $1 * \mu$ is increasing integrable process). An optional measure μ is called $\tilde{\mathcal{P}}$ - σ -finite if there exists a strictly positive predictable function V on $\tilde{\Omega}$ such that the random variable $V * \mu_\infty$ is integrable (or equivalently, $V * \mu$ is increasing integrable); this property is equivalent to the existence of a $\tilde{\mathcal{P}}$ -measurable partition (A_n) of $\tilde{\Omega}$ such that $(I_{A_n} * \mu)_\infty$ is integrable.

(v) Let μ be an optional $\tilde{\mathcal{P}}$ - σ -finite random measure. $\langle \mu \rangle$ is called a compensator of measure μ if $\langle \mu \rangle$ is predictable and for every nonnegative $\tilde{\mathcal{P}}$ -measurable function V on $\tilde{\Omega}$ we have $E(V * \langle \mu \rangle_\infty) = E(V * \mu_\infty)$.

(vi) An optional $\tilde{\mathcal{P}}$ - σ -finite measure μ with $\mu(\omega; \{t\} \times E) \leq 1$ is called an integer-valued random measure if for each $A \in \mathbb{R}_+ \otimes \mathcal{E}$, $\mu(\cdot, A)$ takes values in $\mathbb{N} \cup \infty$.

Proposition 45 1) For a $\tilde{\mathcal{P}}$ - σ -finite random measure μ the unique compensator $\langle \mu \rangle$ exists and for every $\tilde{\mathcal{P}}$ -measurable function V on $\tilde{\Omega}$ such that $|V| * \mu$ is a locally integrable increasing process then $|V| * \langle \mu \rangle$ is a locally integrable increasing process and $V * \langle \mu \rangle$ is a compensator of the process $V * \mu$ (converse is also true). If μ is predictable $\tilde{\mathcal{P}}$ - σ -finite random measure then $\langle \mu \rangle = \mu$.

2) Let X be an adapted cadlag $\mathbb{R}^d$ -valued process. Then

$$\mu^X(\omega; dt, dx) = \sum_s I_{(\Delta X_s(\omega) \neq 0)} \delta_{(s, \Delta X_s(\omega))}(dt, dx)$$

defines an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

Let μ be an optional $\tilde{\mathcal{P}}$ - σ -finite random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ and ν is its compensator. Let us define a process $V * (\mu - \nu) = V * \mu - V * \nu$ if $|V| * \mu$ is locally integrable.

Proposition 46 1) $V * (\mu - \nu)$ is a purely discontinuous local martingale.

2) For its jumps we have:

$$\tilde{V}_t = \Delta(V * (\mu - \nu))_t = \int V(\omega, t, x) \mu(\omega; \{t\} \times dx) - \int V(\omega, t, x) \nu(\omega; \{t\} \times dx).$$

Definition 45 For each $\tilde{\mathcal{P}}$ -measurable real-valued functions V a purely discontinuous local martingale $X = (X_t)_{t \geq 0}$ is called a stochastic integral of V w.r.t. $\mu - \nu$ if ΔX and $\tilde{V}$ are indistinguishable.

Proposition 47 .(existence) Let $a_t(\omega) = \nu(\omega; \{t\} \times E)$ and for each $\tilde{\mathcal{P}}$ -measurable real-valued functions V define $\hat{V}_t(\omega) = \int_E V(t, \omega, x) \nu(\omega; \{t\} \times dx)$. Let us assume for any stopping time T

$$\int_E |V(T, \omega, x)| \nu(\omega; T \times dx) < \infty \quad P - a.s.$$

Let also

$$\frac{(V - \hat{V})^2}{1 + |V - \hat{V}|} * \nu + \frac{\hat{V}^2}{1 + |\hat{V}|} * \left(\sum_{s \in B} I(a_s(\omega) > 0)(1 - a_s(\omega)) \right), \quad B \in \mathcal{B}(\mathbb{R}_+),$$

be a locally integrable increasing process. Then there exists a unique purely discontinuous local martingale $V * (\mu - \nu)$, such that $\Delta(V * (\mu - \nu)) = \tilde{V}$.

Definition 46 We call $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ a truncation function if it is bounded with compact support and satisfies $h(x) = x$ in a neighborhood of zero.

Let X be a semimartingale. Let $X(h) = X - \sum_{s \leq t} (\Delta X_s - h(\Delta X_s))$. Then $X(h)$ is a special semimartingale with canonical decomposition $X(h) = X_0 + M(h) + B(h)$, where $B(h)$ is a predictable FV process and $M(h)$ is a local martingale. (see (J-S))

Definition 47 Let a truncation function h be fixed. We call characteristics of X the triplet (B, C, ν) where

- (a) $B = B(h)$ is a predictable FV process in the canonical decomposition of $X(h)$.
- (b) $C = \langle X^c, X^c \rangle$ is a continuous FV process.
- (c) ν is the compensator of the random measure μ^X .

Proposition 48 Properties of characteristics of semimartingale.

1)(J-S) Let X be a semimartingale with characteristics (B, C, ν) relative to a truncation function h , and with measure of jumps μ^X . Then X has a canonical representation

$$X = X_0 + X^c + h * (\mu^X - \nu) + (x - h(x)) * \mu^X + B.$$

2)(S) Let X be a semimartingale then $(x^2 \wedge 1) * \nu$ is a process with locally integrable variation.

3)(S) X is a special semimartingale if and only if $(x^2 \wedge |x|) * \nu$ is a process with locally integrable variation. Moreover if $B = 0$ then the special semimartingale X is a local martingale.

4)(S) X is a locally square integrable semimartingale if and only if $x^2 * \nu$ is a process with locally integrable variation.

5)(S) $X = X_0 + X^c + x(\mu - \nu) + A$ is a canonical representation of special semimartingale, where A is a predictable increasing process.

6)(J-S) Let X be a semimartingale with characteristics (B, C, ν) relative to a truncation function h , and with measure of jumps μ^X . For each $u \in \mathbb{R}$ we have a predictable FV process

$$\Psi(u)_t = iuB_t - \frac{1}{2}u^2C_t + (e^{ixu} - 1 - iuh) * \nu_t.$$

Under assumption that $\Delta\Psi(u) \neq -1$ identically for all $u \in \mathbb{R}$, for each $u \in \mathbb{R}$ we have X is a semimartingale with characteristics (B, C, ν) if and only if $\frac{e^{iuX}}{\mathcal{E}(\Psi(u))}$ is a local martingale.

7)(S) If X is a process with independent increments then its characteristics are non-random. In particular for Levy processes $B_t = bt$, $C_t = ct$ and $\nu(dt, dx) = dt\nu(dx)$ where $\nu(dx)$ is a Levy measure (see below).

Density process and Girsanov theorem.

8)(J-S) Let $\tilde{P} \ll P$. There is a unique P -martingale Z , such that $Z_t = \frac{d\tilde{P}}{dP}$ (the Radon-Nykodym derivative) for all $t \in \mathbb{R}_+$. The P -martingale Z is called a density process. We have $E(Z_t) = 1$ and $\tilde{P}(\inf_t Z_t > 0) = 1$. Moreover one may take $Z \geq 0$ for all $t > 0$ and all $\omega \in \Omega$.

9)(J-S) If T is a stopping time, restricted to the set $\{T < \infty\}$ we have $\tilde{P}_T \ll P_T$ and $Z_T = \frac{d\tilde{P}_T}{dP_T}$. If T is a predictable stopping time then $Z_{T-} = \frac{d\tilde{P}_{T-}}{dP_{T-}}$ on $\mathcal{F}_{T-}$.

10)(J-S) Let Z be a density process then Z is a P -uniformly integrable martingale if and only if $\tilde{P}(\sup_t Z_t < \infty) = 1$.

11)(J-S) Let $\tilde{P} \ll P$ and let Z be the density process. Let X be an adapted cadlag process. Then XZ is a P -(local) martingale if and only if X is a $\tilde{P}$ -(local) martingale.

12)(J-S)(Girsanov theorem) Let $\tilde{P} \ll P$ and let Z be the density process. Let X be a P -local martingale such that $X_0 = 0$ and the P -quadratic covariation $[X, Z]$ has P -locally integrable variation, and denote by $\langle X, Z \rangle$ its compensator. Then the process

$$\tilde{X} = X - \frac{1}{Z_-} \cdot \langle X, Z \rangle$$

is a $\tilde{P}$ -local martingale. Moreover the P -quadratic variation $\langle X^c, X^c \rangle$ is also a version of the $\tilde{P}$ -quadratic variation of the continuous part of $\tilde{X}$.

13)(S)(representation of a local martingale) Let X be a semimartingale with characteristics (B, C, ν) . Let P be unique in the following sense: if $\tilde{P} \ll P$ and X has the same characteristics w.r.t. $\tilde{P}$ as w.r.t. P then $\tilde{P} = P$. Then each local martingale M has a representation:

$$M = M_0 + f \cdot X^c + V * (\mu - \nu),$$

where f is a predictable process, $f^2 \cdot \langle X^c \rangle$ is a locally integrable increasing process and V is such that $V * (\mu - \nu)$ exists.

14)(S) Let X be a semimartingale with characteristics (B, C, ν) . Let $\tilde{P} \ll P$ and let Z be the density process. Let

$$\beta = \frac{d\langle Z^c, X^c \rangle}{d\langle X^c, X^c \rangle} \cdot \frac{I(Z_- > 0)}{Z_-}$$

and let Y be a measurable function which can be specified in terms of the density process Z (see J-S). Then the $(\tilde{B}, \tilde{C}, \tilde{\nu})$ characteristics of X w.r.t. $\tilde{P}$ are

$$\tilde{B} = B + \beta \cdot C + h(x)(Y - 1) * \nu; \quad \tilde{C} = C; \quad \tilde{\nu} = Y \cdot \nu.$$

15)(S) Let X be a semimartingale with characteristics (B, C, ν) . Let $\tilde{P} \ll P$ and let Z be the density process. Let $\nu(\omega; \{t\} \times E) = 0$, $t > 0$, β and Y be defined as above then

$$Z = Z_0 + (Z_- \beta) \cdot X^c + Z_- (Y - 1) * (\mu - \nu).$$

If for any $t > 0$

$$\beta^2 \cdot \langle X^c \rangle_t + (1 - \sqrt{Y})^2 * \nu_t < \infty,$$

then the process

$$M_t = \beta \cdot X_t^c + (Y - 1) * (\mu - \nu)_t$$

is a P -local martingale. The process Z can be represented as $Z_t = Z_0 \mathcal{E}(M_t)$.

C.2 Stochastic processes in financial mathematics.

C.2.1 Levy processes.

Definition 48 An adapted cadlag process X with $X_0 = 0$ a.s. is a Levy process if

(i) X has independent increments, that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$;

(ii) X has stationary increments, that is, $X_t - X_s$ has the same distribution as X_{t-s} , $0 \leq s < t < \infty$;

(iii) X_t is continuous in probability, that is, for any $t \geq 0$ and $\epsilon > 0$ we have $\lim_{s \rightarrow t} P(|X_s - X_t| > \epsilon) = 0$.

Now we summarize some properties of Levy processes.

Proposition 49 .Let X be a Levy process.

- 1)(P) There exists a unique modification Y of X which is cadlag and also a Levy process. The proof is based on the Esscher transform, which will be discussed later.
- 2)(P) For each $t > 0$, X_t has an infinitely divisible distribution. Conversely it can be shown that for each infinitely divisible distribution μ there exists a Levy process X such that μ is the distribution of X_1 .
- 3)(P) Let X be cadlag and let T be a stopping time. On the set $\{T < \infty\}$ the process Y defined as $Y_t = X_{T+t} - X_T$ is a Levy process adapted to filtration $(\mathcal{F}_{T+t})$, Y is independent of $\mathcal{F}_T$ and Y has the same distribution as X .
- 4)(P) Let X be cadlag and $\sup_t |\Delta X_t| \leq C < \infty$ a.s., where C is a non-random constant (X has *bounded jumps*). Then $E|X_t|^n < \infty$ for all $n \in \mathbb{N}$.
- 5)(P) A Levy process is a semimartingale.
- 6)(P) Any jump time of a Levy process is not accessible.

Remark 20 . Since every Levy process has a cadlag modification we consider from now on only cadlag Levy processes.

We next turn our attention to an analysis of the jumps of a Levy process. Let Λ be a Borel set in $\mathbb{R}$ bounded away from 0 (that is, $0 \notin \bar{\Lambda}$, where $\bar{\Lambda}$ is the closure of Λ). For a Levy process X define $T_\Lambda^1 = \inf\{t > 0 : \Delta X_t \in \Lambda\}$ and $T_\Lambda^n = \inf\{t > T_\Lambda^{n-1} : \Delta X_t \in \Lambda\}$ for $n \geq 2$. We consider the counting process of jumps

$$N_t^\Lambda = \sum_{n=1}^{\infty} I_{\{T_\Lambda^n \leq t\}}.$$

Proposition 50 (P). N_t^Λ is a Poisson process with parameter $\nu(\Lambda) = EN_1^\Lambda$. The set function $\Lambda \rightarrow N_t^\Lambda(\omega)$ defines a σ -finite (counting) measure on $\mathbb{R} \setminus 0$ for each fixed (t, ω) . We denote this measure $\mu_t(dx)$. The set function

$$\nu(\Lambda) = EN_1^\Lambda = E\left(\sum_{0 < s \leq 1} I_\Lambda(\Delta X_s)\right)$$

also defines a σ -finite measure on $\mathbb{R} \setminus 0$, which is called the *Levy measure* of X .

Proposition 51 (P). Let Λ_1, Λ_2 be two disjoint Borel sets with $0 \notin \tilde{\Lambda}_1$, $0 \notin \tilde{\Lambda}_2$. Then the two processes

$$J_t^i = \sum_{0 < s \leq t} \Delta X_s I_{\Lambda_i}(\Delta X_s), \quad i = 1, 2$$

are independent Poisson processes.

Proposition 52 (P).

- 1) Let X be a Levy process. Then X_t is a semimartingale and $X_t = Y_t + Z_t$, where Y and Z are Levy processes, Y is a martingale with bounded jumps, $Y_t \in L_p$ for all $p \geq 1$ and Z is FV process.
- 2) Let X be a Levy process with jumps bounded by a : $\sup_s |\Delta X_s| \leq a$ a.s. Let $Z_t = X_t - EX_t$. Then Z is martingale and $Z_t = Z_t^c + Z_t^d$, where Z_t^c is a continuous martingale, $Z_t^d = \int_{\{|x| \leq a\}} x(\mu_t(dx) - t\nu(dx))$, and Z^c, Z^d are independent Levy processes. Moreover $Z_t^2 - E \langle Z^c, Z^c \rangle_t$ is also martingale.

The preceding results lead us to the following important theorems.

Proposition 53 (J-S)(*Levy decomposition*). Let X be a Levy process. Then X has a decomposition

$$X_t = W_t + \int_{\{|x| < 1\}} x(\mu - t\nu)(dx) + bt + \sum_{0 < s \leq t} \Delta X_s I_{\{|\Delta X_s| \geq 1\}}$$

where W_t is a Brownian motion; for any set Λ , $0 \notin \tilde{\Lambda}$, $N_t^\Lambda = \int_\Lambda \mu(dx)$ is a Poisson process independent of W ; N_t^Λ is independent of N_t^Γ if Λ and Γ are disjoint; N_t^Λ has parameter $\nu(\Lambda)$; and $\nu(dx)$ is a measure on $\mathbb{R} \setminus \{0\}$ such that $\int \min(1, x^2)\nu(dx) < \infty$. In other words a Levy process can be decomposed into the sum of Brownian motion and a quadratic pure jump semimartingale.

Proposition 54 (J-S)(*Levy-Khintchine formula*). Let X be a Levy process with Levy measure ν . Then

$$E(e^{iuX_t}) = e^{t\psi(u)}$$

where

$$\psi(u) = ibu - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuxI_{\{|x| < 1\}})\nu(dx).$$

Moreover given a triplet (b, c, ν) the corresponding Levy process is unique in distribution. (For the multidimensional case of the above formula see (S)).

C.2.2 Point processes.

Definition 49 1) Let $\mathcal{M} = \mathcal{M}(E)$ be the space of Radon (locally finite) measures on E a locally compact second countable Hausdorff space. The space $\mathcal{M}$ can be endowed with the vague topology, for which the class of all finite intersections of sets of the form $\{\mu \in \mathcal{M} : s < \int_E f d\mu\}$ for $s, t \in \mathbb{R}$ and f is nonnegative continuous function with compact support, may serve a base. The space $\mathcal{M}$ with the vague topology is metrizable as a Polish space. We define a point process N on the space E as a $\mathcal{M}$ -valued random element confined with probability one to the subset $\mathcal{N} = \mathcal{N}(E)$ of $\mathcal{M}$ consisting of all integer Radon measures on the space E . When $E = \mathbb{R}_+$ the simplest description of a point process is by a sequence of random variables $T_0, T_1, \dots$ corresponding to the jump points of N .

2) A point process is called an *inexplosive* if $T_n \rightarrow \infty$ as $n \rightarrow \infty$.

3) A *inexplosive* point process is called *simple* (without multiple points) if $\{T_n\}$ is strictly increasing. In this case there exist a representation $N = \sum_{i=0}^{\infty} I_{T_i}$.

4) Let N be a simple point process with the representation $N = \sum_{i=1}^{\infty} I_{T_i}$ and let E' be a locally compact second countable Hausdorff space. A *marked point process* with underlying process N is any point process $\bar{N} = \sum_{i=0}^{\infty} I_{(T_i, Y_i)}$ on $\mathbb{R}_+ \times E'$. The random variable Y_i of E' is called the *mark* associated to T_i .

Proposition 55 1)(K-S) Let $\mathcal{N}_\infty(\mathbb{R}_+) = \{\mu \in \mathcal{N}(\mathbb{R}_+) : \mu(\mathbb{R}_+) = \infty\}$ and $\Xi_+ = \{\bar{t} = (t_1, t_2, \dots) \in \mathbb{R}_+^\infty : t_1 \leq t_2 \leq \dots \leq t_n \rightarrow +\infty\}$. Consider Ξ_+ to be endowed with the usual product topology. For $\mu \in \mathcal{N}(\mathbb{R}_+)$ let $\tau_0(\mu) = 0$ and for $n \geq 1$ let $\tau_n = \sup\{u > 0 : \mu(0, u] \leq n\}$. Define a mapping $\bar{\tau} : \mathcal{N}_\infty(\mathbb{R}_+) \rightarrow \Xi_+$ as $\bar{\tau}(\mu) = (\tau_1(\mu), \tau_2(\mu), \dots)$. Then $\bar{\tau}$ is invertable and $\bar{\tau}^{-1}(t)$ is defined as a measure μ such that $\mu(C) = \sum_{n=1}^{\infty} I_C t_n$ for all measurable sets $C \subset \mathbb{R}_+$. Moreover the mappings $\bar{\tau}$ and $\bar{\tau}^{-1}$ are measurable.

2) Let N be a simple point process. Then N is a *cadlag increasing process* and hence its *compensator* is well defined.

3)(K-S,B,J-S) Let N be a simple point process and $0 = T_0 < T_1 < \dots$ are the jump points of N on $\mathbb{R}_+$. That is, $T_n = \tau_n(N)$. For $n \leq 0$ let

$$F_{n+1}(x; t_1, \dots, t_n) = P(T_{n+1} - T_n \leq x | T_1 = t_1, \dots, T_n = t_n),$$

$$R_{n+1}(x; t_1, \dots, t_n) = -\log(1 - F_{n+1}(x; t_1, \dots, t_n))$$

be regular versions of conditional probability distributions of the interpoint distances of N and their cumulative hazard functions, respectively. Fix $n \geq 1$ and $0 = t_0 < \dots < t_n < t_{n+1} = \infty$. Define $a_1(t) = R_1(t)$ and

$$a_{n+1}(t; t_1, \dots, t_n) = \sum_{k=1}^n I_{(t_k, t_{k+1}]}(t) \left(\sum_{i=1}^k R_i(t_i - t_{i-1}; t_1, \dots, t_{i-1}) + R_{k+1}(t - t_k; t_1, \dots, t_n) \right).$$

The family of functions $\{a_n(\cdot)\}$ is called the compensator function family associated with the point process N . Then the compensator of the process N is a process $\Lambda(t, N)$ such that, for $t \in \mathbb{R}_+$ and $\mu \in \mathcal{N}(\mathbb{R}_+)$,

$$\Lambda(t, \mu) = \sum_{n=0}^{\infty} a_{n+1}(t; \tau_1(\mu), \dots, \tau_n(\mu)) I_{(\tau_n(\mu), \tau_{n+1}(\mu)]}(t).$$

Definition 50 Let the conditional distributions F_{n+1} (from the above proposition) be absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}_+$. Denote the corresponding probability density functions by $f_{n+1}(x; t_1, \dots, t_n)$. Let

$$r_n(x; t_1, \dots, t_n) = \frac{f_{n+1}(x; t_1, \dots, t_n)}{1 - F_{n+1}(x; t_1, \dots, t_n)}$$

be the conditional failure rate of $(T_{n+1} - T_n)$. The stochastic intensity of the process N is a process $\lambda(t, N)$ such that

$$\lambda(t, \mu) = \sum_{n=0}^{\infty} r_{n+1}(t - \tau_n(\mu); \tau_1(\mu), \dots, \tau_n(\mu)) I_{(\tau_n(\mu), \tau_{n+1}(\mu)]}(t),$$

where $t \in \mathbb{R}_+$ and $\mu \in \mathcal{N}(\mathbb{R}_+)$. That is $\lambda(t, N)$ equals $r_{n+1}(t - T_n; T_1, \dots, T_n)$ for $t \in (T_n, T_{n+1}]$.

Remark 21 . Since $R_{n+1}(x; t_1, \dots, t_n) = \int_0^x r_{n+1}(u; t_1, \dots, t_n) du$, then $\Lambda(t, N) = \int_0^t \lambda(u, N) du$. Moreover $\lambda(t, N)$ is $\mathcal{F}_t^N$ predictable.

Definition 51 1) For $\mu, \nu \in \mathcal{M}(E)$ let $\mu \prec_{\mathcal{M}} \nu$ if $\mu(B) \leq \nu(B)$ for all bounded sets $B \in \mathcal{B}(E)$. This order is closed in $\mathcal{M}$ (see details in K-S and Kallenberg(1983)). The ordering $\prec_{\mathcal{M}}$ restricted to $\mathcal{N}$ will be denoted by $\mu \prec_{\mathcal{N}} \nu$ for $\mu, \nu \in \mathcal{N}$. This order is closed in $\mathcal{N}$.

2) Let N be a simple point process with a stochastic intensity λ . We say that N is positively self-exciting w.r.t. $\prec_{\mathcal{N}}$ if, for all $\mu, \nu \in \mathcal{N}(\mathbb{R}_+)$, $n \geq 1$,

$$\mu \prec_{\mathcal{N}} \nu \Rightarrow (\forall t \in \mathbb{R}_+ \lambda(\mu) \leq \lambda(\nu)).$$

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