CAHIER DE RECHERCHE #2402E
Département de science économique
Faculté des sciences sociales
Université d'Ottawa

WORKING PAPER #2402E
Department of Economics
Faculty of Social Sciences
University of Ottawa

Charitable Giving and NPOs Investment Decision in a Stochastic Dynamic Economy[*]

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April 18, 2024

[*] *We are grateful to Victoria Barham, Anthony Heyes, Jean-Francois Tremblay, Louis-Philippe Morin and Frances Woolley for comments and suggestions.
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Abstract

We study the dynamic interaction between donors' contributions and the investment strategies of a non-profit organization (NPO) amid fluctuating donor income and varying financial market conditions. The analysis reveals a consistent pattern in the NPO's allocation strategy, directing a fixed proportion of its endowment toward higher-risk assets. Notably, increased donor support often correlates with the NPO's heightened activity in financial markets, which can occasionally reduce the provision of charitable goods. A significant finding is the NPO's preference for environments with lower returns on risk-free assets. Additionally, the study delineates the contrasting impacts of financial market uncertainties and donor income variations on the decision-making processes of both donors and the NPO; while market volatility significantly shapes strategies for both groups, fluctuations in donor income have minimal impact on their strategic decisions.

Keywords: *Non-Profit Investments, Donor Dynamics, Financial Volatility, Risk Management, Stochastic Differential Game.*

JEL Classification: D64 , G11, G23, L31, C73.

1 Introduction

Philanthropic giving in the U.S. has experienced significant growth, doubling since the 1990s and reaching \$427.71 billion in 2018. The trend continued, with contributions totaling approximately \$484.85 billion in 2021, reflecting a steady increase even amidst economic uncertainties.[1] The majority of these donations come from individual donors, who contributed \$326 billion, or 67% of the total, underscoring the significant role of personal philanthropy in the U.S. Additionally, the rise of donor-advised funds has also played a crucial role, with contributions to such funds increasing by over 80% since 2015.[2]

While donations form a substantial portion of non-profit organizations (NPOs) revenue, these are often complemented by financial market investments, which form part of a broader strategy to enhance revenue in support of their missions. NPOs engage in financial markets with two main objectives: to generate investment income for operational funds and to hedge against the volatility of contributions. Over 35,000 nonprofits engage in investment activities, collectively managing funds that total approximately \$800 billion in 2018.[3] This strategic approach to investment not only augments their fundraising endeavors but also supports nonprofits in financing specific projects and securing long-term financial stability.[4]

This paper examines the investment behaviors of non-profit organizations. To capture the interaction between an NPO and its donors in the context of income shocks and financial market volatility, we utilize a dynamic, non-cooperative stochastic differential game framework. Our model accounts for fluctuations in the return rates of risky assets and donor incomes, employing Wiener processes. The NPO strategically allocates its resources between charitable activities and investments in risk-free assets, such as Treasury bills, as well as riskier alternatives. Simultaneously, donors decide on their consumption and contributions.

Our analysis yields several key insights into the investment strategies and decisions of non-profit organizations. A principal finding is the NPOs' consistent strategy of allocating a fixed proportion of their endowments to risky assets, influenced by the

[1] See, for example, List (2011) and `https://givingusa.org/`.
[2] See `https://www.nptrust.org/reports/daf-report/`.
[3] See Dahiya and Yermack (2018).
[4] According to Lo et al. (2019), these funds yield an average net return of 5.3 percent.

returns on risk-free assets. Specifically, our findings indicate:

1. High returns on risk-free assets lead NPOs to avoid financial market risks and allocate their entire disposable endowments to these safer assets.
2. With moderate returns on risk-free assets, NPOs diversify their portfolios by investing significantly in risky assets, demonstrating a non-monotonic relationship between risk-free asset returns and charitable good provision.
3. Low returns on risk-free securities prompt NPOs to invest all their disposable endowments in risky assets to potentially increase their expected endowments.

Regarding donor contributions, our study highlights several patterns:

1. Donor contributions intensify when the NPOs' endowments dip below a specific threshold.
2. An abundance of endowment may discourage additional donor contributions.
3. Donation patterns exhibit discontinuities, with a notable surge in the NPOs' provision of charitable goods once endowments surpass a pivotal level.

Our findings reveal that heightened donor contributions may prompt NPOs to engage more in financial markets, potentially at the expense of their charitable activities. This observation suggests that policy interventions, such as setting minimum charitable provision requirements and limiting risky investment exposure, could be beneficial.

The analysis also shows that while investment shocks significantly influence the strategies of NPOs and donors, variations in donor income are less impactful. Moreover, our data underscore the risk-averse stance of NPOs, indicating that more diverse portfolios may necessitate higher risk premiums to offset possible investment setbacks.

Organization of the paper: The structure of the paper is as follows: Section 2 provides a literature review. In Section 3, we set-up the model. Section 4 examines the equilibrium. The impact of government regulations is presented in Section 5. The paper concludes with Section 6, offering a summary and final remarks. For technical details, all proofs are included in Appendix 7.

2 Literature Review

The economic behaviors of Non-Profit Organizations (NPOs), particularly in the context of charitable giving, have been the subject of extensive research. This literature can be broadly categorized into two main strands:

Individual Economic Decisions in Charitable Giving: This perspective views charitable giving as an economic decision made by individuals. Contributions are regarded as activities aimed at maximizing utility, constrained by the donor's budget. NPOs play a facilitative role, channeling these funds towards public initiatives and ensuring the efficient provision of charitable goods. Key references in this area include Varian (1994), Andreoni (1988), and Bergstrom et al. (1986).

Strategic Interactions in Charitable Giving: This approach perceives charitable giving as the outcome of strategic interactions between donors and NPOs. In this framework, NPOs take on a pivotal role, crafting fundraising strategies and mechanisms. This aspect is elaborated by Andreoni (2006).

Financial markets: Despite the depth of these areas, a common omission is the involvement of NPOs in financial markets. Recent literature, such as Jegers and Verschueren (2006), Bowman (2002), and Wedig (1994), has started to explore the role of NPOs in capital markets and its implications for charitable giving, with a focus on how capital structure influences the provision of charitable goods. The investigation of the investment role of NPOs has started with empirical studies that shed light on NPOs' investment patterns. Lo et al. (2019) examine the asset allocation choices and investment returns of nonprofit endowment funds, observing a tendency towards conservative asset allocation. This approach results in an average annual return of 5.3%. They also pointed out on considerable heterogeneity in investment returns across different funds. Dahiya and Yermack (2018) analyze investment returns and distribution rates of U.S. non-profit endowments over a decade, finding underperformance against market benchmarks. Interestingly, they observe an elasticity in donor contributions relative to endowment returns.

Investment theories with risk: Foundational theories on investments with inherent risks are provided by Markowitz (1952) and Tobin (1958), with a focus on optimal portfolio construction and risk diversification. Following Tobin (1974), a rich body of literature, including Brown et al. (2014), has delved into endowment spending policies, echoing Tobin's permanent income hypothesis and the practice of dividend smoothing. In contrast, Merton (1993) argues for strategic investment of endowment funds, advocating for a countercyclical distribution approach aligned with other income sources of an institution.

Handling of uncertainty: Recent research has also illuminated the influence of uncertainty on the efficiency of charitable good provision. In the context of discrete charitable goods, McBride (2006) argues that increased uncertainty about provision thresholds can actually enhance contributions for valuable charitable goods. Wang and Ewald (2010a) introduces a stochastic differential game model, accounting for volatility in the level or provision rate of charitable goods. Moreover, Lohse et al. (2012) highlights that donors with higher risk aversion prefer more efficient charitable good provision, signifying a strategic balance between market insurance and effective charitable good provision.

Our paper builds on strategic interactions in charitable giving focusing on the investment role of NPOs in the risky environment. We emphasize the evolving interactions between NPO strategies and donor decisions, highlighting aspects of the free-rider issue *between NPOs and donors*. Our findings indicate that NPOs consistently provide charitable goods in the absence of donor contributions, a scenario that may alter with increased donor engagement.[5]

3 The Model

Consider the following dynamic model of charitable good in continuous time $t \in [0, \infty)$. There are two players in the model (a NPO and a cohort of homogeneous donors) and two types of assets: a risky asset and a risk-free asset (e.g. T-bills). The NPO's strategy consists of a path (G_t, β_t), where G_t is the provision of the charitable good and β_t refers to the proportion of funds invested in a risky asset for each t. Donors select consumption and contribution level path, (C_t, D_t), for each t.

[5]The literature delves into the complexities of the free-rider problem in charitable giving. Works by Yildrim (2006), Marx and Matthews (2000), and Fershtman and Nitzan (1991) discuss the challenges of coordinating collective action in dynamic settings, suggesting that accumulated contributions, coupled with uncertainty, can intensify the free-rider dilemma.

3.1 NPO's Payoff and Program

To mitigate the risk associated with fluctuations in returns on risky assets, the NPO determines the optimal allocation of its disposable endowment between a risky asset and a risk-free asset.[6] The NPO does not dedicate its entire endowment to the provision of charitable goods; instead, it invests a portion of the disposable endowment in the financial market. The remaining portion is held as an additional fund balance to hedge against potential financial downturns and to ensure adequate liquidity for future operations. Thus, the investment strategy at time t consists of allocating a $\beta_t \in [0, 1]$ proportion of its disposable endowment fund to the risky asset and $1 - \beta_t$ proportion to the risk-free asset.

Formally, the NPO chooses a pair (G_t, β_t) at each time t to maximize its present value of expected lifetime utility, $U^n = \int_0^\infty e^{-\rho_c t} u^n(G_t) dt$, where ρ_c represents the NPO's discount rate and u_t^n is an instantaneous utility function that is increasing and strictly concave.

The rate of return on T-bills is a constant r, while the rate of return on the risky asset, r_t, evolves over the time interval $[\, t,\ t + \Delta t\,]$ according to the Wiener process W_{tr}:

$$r_t \Delta t = \theta\left(\bar{r} - r_t\right)\Delta t + \sigma_r \varepsilon_t \sqrt{\Delta t}, \tag{2}$$

where $\bar{r}$ is the long-term mean of the rate of return on the risky asset, σ_r is the volatility of the risky asset, and $\theta > 0$ is a parameter that captures the system's response to perturbations in the rate of return. The error term is $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Denote by s_t – the state variable, representing NPO's total financial endowment at time t and by w_t donors' wealth. Donors' contribution function, $D_t(w_t, s_t) : (w_t, s_t) \to D_t$, and $D_t(w_t, s_t)$ maps onto $[\, 0, +\infty)$. We have the following:

Proposition 1. *Given donors' consumption and contribution strategy* (C_t, D_t)*, NPOs'*

[6] The disposable endowment refers to the NPO's endowment fund after the provision of the charitable good.

best-response program is the following:

$$\max_{G_t,\beta_t} \mathbb{E}(U^n) = \mathbb{E}\left\{\int_0^\infty e^{-\rho_c t} u^n(G_t) dt\right\}, \tag{3}$$

s.t.

$$ds_t = \left(\left(\beta_t\left(\frac{\theta}{1+\theta}\bar{r} - r\right) + r\right) s_t - \big(G_t - D_t(w_t, s_t)\big)\right))dt + \beta_t s_t \sigma_r dW_{tr}. \tag{4}$$

Note that given donors' choice (C_t, D_t), the NPO's strategy only affects its endowment. Hence, the NPO's program is constrained only by its endowment equation (4). Also, note that the variance of the NPO's endowment, $\beta_t^2 s_t^2 \sigma_r^2$, depends on not only the variance of the risky asset, but also on the portfolio selection and the endowment level. For instance, a higher level of endowment and a larger proportion invested in the risky asset generate more fluctuations than smaller ones.

3.2 Donors' Payoff and Program

Consider a cohort of homogeneous donors with mass 1. An instantaneous utility of a representative donor is quasi-linear in the private consumption C_t and the level of charitable good G_t is given by

$$U_t^d = C_t + u^d(G_t), \tag{5}$$

where $u^d(G_t)$ is an increasing and concave function.

Assume that the consumption C_t of donors at any given time t is constrained within the interval $[0, \eta_c w_t]$, where η_c is parameter within the range $[0, 1)$, representing a fixed proportion of the state variable w_t – donor's total wealth at time t. The contribution D_t from donors is confined within the range $[0, \eta_d s_t]$, where parameter $\eta_d \in [0, 1)$.

The donors' income per unit time, I_t, evolves over the time interval $[\,t,\, t+\Delta t\,]$ according to a Wiener process W_{tI}:

$$I_t \Delta t = \bar{I}\Delta t + \sigma_I \varepsilon_t \sqrt{\Delta t}, \tag{6}$$

where $\bar{I}$ represents the donors' average income (drift), σ_I is the volatility of the income I_t around this average, and $\varepsilon_t \sim \mathcal{N}(0, 1)$. Upon realization of the donors' decisions

(C_t, D_t), the remaining endowment receives a risk-free rate of return, r.

For simplicity, assume that both the NPO and donors have the same discount rate, which is equal to the rate of return on T-bills: $\rho_c = \rho_d = r$. Suppose further that Wiener processes W_{tI} and W_{tr} are not correlated. We have the following:

Proposition 2. *Given the NPO's strategy (G_t, β_t), the donor's best-response program is the following:*

$$\max_{\substack{C_t \in (0, \eta_c w_t] \\ D_t \in [0, \eta_d s_t]}} \mathbb{E}(U^d) = \mathbb{E}\left\{\int_0^\infty e^{-\rho_d t}\big(u^d(G_t) + C_t\big)dt\right\}, \tag{7}$$

s.t.

$$ds_t = \left(\left(\beta_t\left(\frac{\theta}{1+\theta}\bar{r} - r\right) + r\right)s_t - \big(G_t - D_t(s_t)\big)\right)dt + \beta_t s_t \sigma_r dW_{tr}, \tag{8}$$

and

$$dw_t = \big(rw_t + \bar{I} - D_t - C_t\big)\,dt + \sigma_I dW_{tI}. \tag{9}$$

Note that the donors' program is constrained by two differential equations: (8) and (9). These constraints arise because, given the NPO's strategy (G_t, β_t), the choices made by donors (C_t, D_t) influence both the evolution of the NPO's endowment and the progression of the donors' own wealth. Additionally, it is important to recognize that the fluctuations in the donors' wealth are solely dependent on the volatility of their income. This is reflected in the differential equations, which model how changes in income, impacted by its inherent volatility, affect the overall financial dynamics of the donors.

4 Equilibrium Analysis

4.1 Solving Model

In addressing the model's resolution, we employ the Feedback Nash equilibrium framework. The choice between equilibrium notions, such as open-loop Nash and Feedback Nash, depends on underlying assumptions about the pre-commitment of actions and the structure of information available to the players. In scenarios lacking a central enforcing authority, where commitments are inherently non-binding, the Feedback Nash equilibrium is more appropriate. This equilibrium framework allows agents to adjust their strategies based on the current state of the system, adhering to the principles of a stationary Markovian Nash equilibrium.[7]

In this equilibrium framework, the strategies of players are contingent to the current state of the game, denoted as S_t. The solution is governed by the Hamilton-Jacobi-Bellman (HJB) equations, which encapsulate the conditional behavior of players within the Nash equilibrium context.

Denote by $\Gamma(S_\tau, \infty - \tau)$ the non-cooperative stochastic differential game with equations (3), (4), (7), (8), (9), and initial state $S_\tau = \{s_\tau, w_\tau\} \in S$ and $\tau \in [0, \infty)$. The NPO and donors' discounted value functions of any subgame from $t \in [t, \infty)$ satisfy,

$$\Phi^{(\tau)}(t, s_t) = \mathbb{E}\bigg\{\int_t^\infty e^{-r(y-\tau)} u^n(G_y) dy\bigg\}, \tag{10}$$

and

$$\Psi^{(\tau)}(t, s_t, \omega_t) = \mathbb{E}\bigg\{\int_t^\infty e^{-r(y-\tau)} \big(u^d(G_y) + C_y\big) dy\bigg\}, \tag{11}$$

for $\tau \in [0, \infty)$, and $t \in [\tau, \infty)$ respectively.

Using equations $\Phi^{(\tau)}(t, s_t) = e^{-r(t-\tau)}\Phi^{(t)}(t, s_t)$, and $\Psi^{(\tau)}(t, s_t, w_t) = e^{-r(t-\tau)}\Psi^{(t)}(t, s_t, w_t)$, it can be concluded that the game, $\Gamma(S_\tau, \infty - \tau)$, has the property that the discounted value of any subgame value function from $t \in [t, \infty)$ is equal to the present value of the value function at time $t \in [t, \infty)$. For simplicity, denote $\Phi^{(t)}(t, s_t)$ and $\Psi^{(t)}(t, s_t, w_t)$ as $\Phi(s_t)$, $\Psi(s_t, w_t)$, respectively.

Definition 1. (Shadow prices of the NPO and donors) *Donors' shadow prices refer to the partial derivatives, Ψ_{s_t} and Ψ_{w_t}, of the value function, $\Psi(s_t, w_t)$, with respect to their wealth level and the NPO's endowment correspondingly. The NPO's shadow price, Φ_{s_t}, is the derivative of its value function, $\Phi(s_t)$, with respect to endowment s_t.*

Note that value functions $\Psi(s_t, w_t)$ and $\Phi(s_t)$ are increasing and strictly concave in s_t and w_t. Therefore, shadow prices are positive and decreasing.

The NPO's best response program is to choose the (G_t^*, β_t^*), given the donors' choice, (C_t^*, D_t^*), at each time t. The same procedure applies to donors' best-response program. To solve for the equilibrium of the game, we need the following:

[7]The Feedback Nash equilibrium specifically fits scenarios where continual strategy adjustments are necessary and feasible due to changing conditions, as referenced in various studies (Wang and Ewald 2010b,a, Kossioris et al. 2008, Yeung and Petrosjan 2006, Wedig 1994, Fershtman and Nitzan 1991).

Proposition 3. *A strategy profile* $\{(G_t^*, \beta_t^*), (C_t^*, D_t^*)\}$ *is the equilibrium of the game* $\Gamma(S_0, \infty - 0)$ *if there exist two times differentiable functions* $\Phi(s_t)$ *and* $\Psi(s_t, w_t)$ *satisfying the following Hamilton-Jacobi-Bellman equations:*

$$r\Phi(s_t) = \max_{G_t,\beta_t} \left\{ u^n(G_t) + \Phi_{s_t} \left(\left(\beta_t \left(\frac{\theta}{1+\theta}\bar{r} - r \right) + r \right) s_t - (G_t - D_t^*) \right) + \frac{1}{2}\Phi_{s_t,s_t}\beta_t^2 s_t^2 \sigma_r^2 \right\}; \tag{12}$$

$$r\Psi(s_t, w_t) = \max_{\substack{C_t \in (0,\, \eta_c w_t] \\ D_t \in [0,\, \eta_d s_t]}} \left\{ u^d(G_t^*) + C_t + \Psi_{s_t} \left(\left(\beta_t^* \left(\frac{\theta}{1+\theta}\bar{r} - r \right) + r \right) s_t - (G_t^* - D_t) \right) + \Psi_{w_t}\left(r w_t + \bar{I} - D_t - C_t \right) + \frac{1}{2}\Psi_{s_t,s_t}\beta_t^{*2} s_t^2 \sigma_r^2 + \frac{1}{2}\Psi_{w_t,w_t}\sigma_I^2 \right\}. \tag{13}$$

Given donors' strategy, (C_t^*, D_t^*), the NPO chooses (G_t^*, β_t^*) such that the right side of the HJB equation is maximized. Hence, we obtain the following first-order conditions for the NPO's program:

$$u^n_{G_t}(G_t^*) - \Phi_{s_t} = 0; \tag{14}$$

$$\Phi_{s_t} \left(\left(\frac{\theta}{1+\theta}\bar{r} - r \right) s_t \right) + \Phi_{s_t,s_t}\beta_t^* s_t^2 \sigma_r^2 = 0. \tag{15}$$

Donors' best response (C_t, D_t) to (G_t^*, β_t^*) is characterized by the following first-order conditions with respect to C_t and D_t respectively:

$$1 - \Psi_{w_t} > 0, \text{or}, < 0, \tag{16}$$

$$\Psi_{s_t} - \Psi_{w_t} > 0, \text{or}, < 0. \tag{17}$$

From inequality (16), we deduce that $\Psi_{w_t} < 1$ is necessary to ensure that donors have private consumption at time t. Consequently, the optimal private consumption, C_t^*, must assume a corner solution. Considering that the value function $\Psi(s_t, w_t)$ is increasing and strictly concave in its arguments, the following properties are satisfied: $\lim_{w_t \to 0} \Psi_{w_t} = 1$ and $\lim_{w_t \to +\infty} \Psi_{w_t} = 0$. For the shadow price Ψ_{s_t}, it must hold that $\lim_{s_t \to 0} \Psi_{s_t} = \infty$ and $\lim_{s_t \to +\infty} \Psi_{s_t} = 0$. There exists a unique threshold, $\bar{s}$, is determined by the condition: $\Psi_{s_t} = 1$. Below this threshold, $\Psi_{s_t} > \Psi_{w_t}$, as depicted in Figure 1. Note that if $s_t > \bar{s}_U$ then Ψ_{s_t} can be larger than Ψ_{w_t} (as shown in the Figure) or smaller.

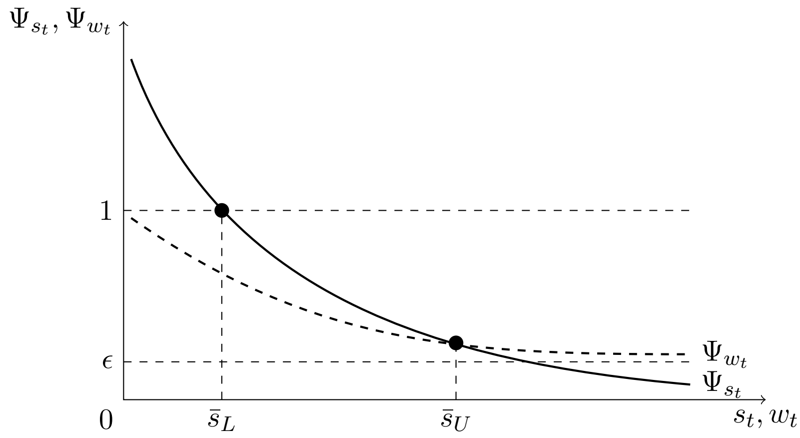


Figure 1: Donors shadow prices Ψ_{s_t} and Ψ_{w_t}

Note: the solid curve represents the donors' shadow price of NPO's endowment, whereas the dashed one represents the donors' shadow price of their wealth level. $\Psi_{s_t} > \Psi_{w_t}$ if $s_t \leq \bar{s}_L$; when $\bar{s}_1 < s_t < \bar{s}_U$, for an arbitrary w_t, $\Psi_{s_t} >< \Psi_{w_t}$; $\Psi_{s_t} < \Psi_{w_t}$ if $s_t > \bar{s}_U$.

To proceed further, let us define the set of rates of return on T-bills.

Definition 2. (Effective space of the rate of return on T-bills) *The effective space, $\mathscr{R}_e \subset \mathbb{R}^+$, is a set of rates of return on T-bills, r, such that the NPO chooses an interior optimal portfolio.*

The next proposition describes the equilibrium of the game. For the sake of tractability, we follow Merton (1971), assuming that the NPO's instantaneous utility function takes the form of CRRA, $u^n(G_t) = G_t^\alpha/\alpha$, where $\alpha \in (0,1)$ and $1-\alpha$ measures NPO's relative risk aversion to the variation of the charitable good provision G_t.

Proposition 4. *There exists a unique threshold, $\bar{s}$, such that*

a) The NPO provides a constant proportion of its current endowment as charitable good, G_t^*, given by

$$G_t^* = \begin{cases} \left[r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right] s_t & \text{if } s_t < \bar{s}, \\ \left[r - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right] s_t & \text{if } s_t > \bar{s}, \text{ and } \Psi_{s_t} < \Psi_{w_t}, \\ \left[r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right] s_t & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \tag{18}$$

b) The NPO exhibits risk-aversion: a higher proportion of the disposable endowment *invested in the risky asset requires a larger risk premium. The proportion, β_t^*, is fixed and given by*

$$\beta_t^* = \begin{cases} \frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2} & \text{if } r \in \mathscr{R}_e, \\ 0, 1 & \text{if } r \notin \mathscr{R}_e. \end{cases} \tag{19}$$

c) Donors' contributions, D_t^*, are given by

$$D_t^* = \begin{cases} \eta_d s_t & \text{if } s_t < \bar{s}, \\ 0 & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} < \Psi_{w_t}, \\ \eta_d s_t & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \tag{20}$$

d) finally, donors consume a fixed proportion of their wealth, $C_t^* = \eta_c w_t$.

Note that equations (18, 19, 20) indicate the optimal strategy $(G_t^*, \beta_t^*), (C_t^*, D_t^*)$ depends only on the present state, satisfying the Markov property.

Proposition 4 establishes that the NPO provides a higher proportion of its present endowment as a charitable good if donors do not contribute. The intuition is that more charitable good provided by NPOs will shrink its endowment, which therefore increases the donors' *shadow price*, Ψ_{s_t},[8] and incentivizes donors to give in the future. Figure 2 illustrates this property. Donors may free ride on the NPO's investment when $s_t > \bar{s}$. In this case, the donors' shadow price from w_t exceeds the shadow price from s_t.

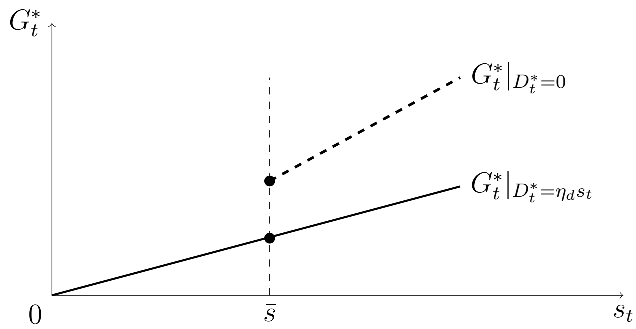


Figure 2: Optimal provision of charitable good conditional on donors' contribution.

The NPO's optimal portfolio selection does not depend on donors' contributions.

[8]Note that the donors' first-order condition with respect to contribution equals the difference between the shadow prices, $\Psi_{s_t} - \Psi_{w_t}$. The properties of decreasing Ψ_{s_t} and Ψ_{w_t} directly lead to the result that a lower NPO endowment encourages donors to contribute.

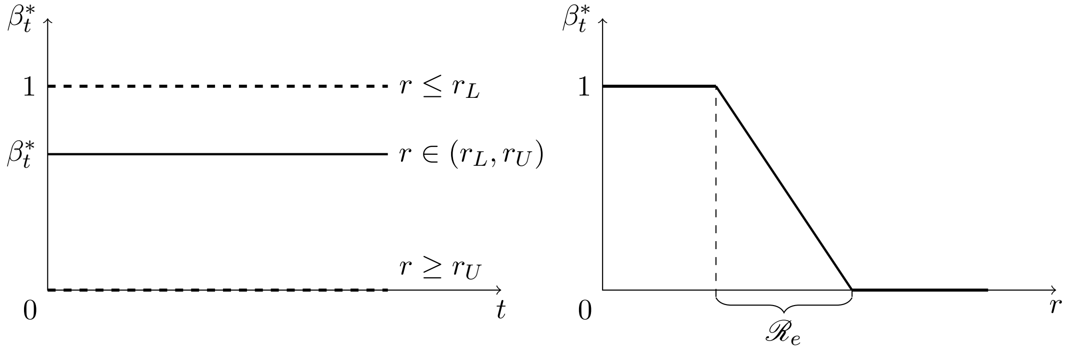


Figure 3 - I

Figure 3 - II

Figure 3: NPO's optimal portfolio β_t^*

Note that $\beta_t^* = \min\left\{\max\left\{\frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2}, 0\right\}, 1\right\}$. The effective space is $\mathscr{R}_e = \{r \in \mathbb{R}^+ \mid r_L < r < r_U\}$, where $r_L = \frac{\theta}{1+\theta}\bar{r} - (1-\alpha)\sigma_r^2$, $r_U = \frac{\theta}{1+\theta}\bar{r}$.

This result aligns with Samuelson (1975) and Merton (1971), who show that for CRRA utilities, the portfolio-selection decision is independent of consumption in both discrete and continuous-time models. Additionally, a sufficiently low rate of return on T-bills will encourage the NPO to allocate all its disposable endowment to the risky portfolio. Conversely, a sufficiently high rate of return on T-bills will deter the NPO from engaging in any risky financial activities. For any given rate of return on T-bills, $r \in \mathscr{R}$, the NPO may opt for an interior solution, $\beta_t^* = (\frac{\theta}{1+\theta}\bar{r} - r)/((1-\alpha)\sigma_r^2)$, or a corner solution: $\beta_t^* = 0$, or $\beta_t^* = 1$, as shown in Figure 3. It is noteworthy that the expected rate of return on the risky asset is an increasing function of β_t^*, indicating the NPO risk-aversion.

Figure 4 illustrates that the decision to contribute by donors depends on whether the shadow price of the NPO's endowment, Ψ_{s_t}, exceeds the shadow price of their own income, Ψ_{w_t}. If it does not, donors tend to prefer free riding on the NPO's efforts rather than making contributions. Consequently, donors will contribute $D_t^* = \eta_d s_t$ if $s < \bar{s}$. When $s > \bar{s}$, the optimal contribution would be either $D_t^* = \eta_d s_t$ or $D_t^* = 0$.

Next, we examine the potential for NPOs to cease distributing charitable goods. Denote by $f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2/(2(1-\alpha)^2\sigma_r^2)$ the gap between the rate of return on T-bills, r, and the adjusted risk premium, $\alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2/(2(1-\alpha)^2\sigma_r^2)$, plus donors' contribution factor, $\frac{\alpha}{1-\alpha}\eta_d$.

Corollary 1. *1. An interior solution for the provision of the charitable good, G_t^*, exists if and only if the subspace of the rate of return on T-bills, $\mathscr{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\}$*

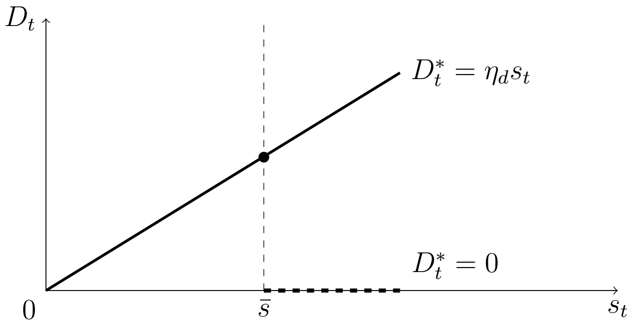


Figure 4: Optimal donors' contributions D_t^*

is non-empty.

2. *The NPO always provides the charitable good when donors do not contribute.*

3. *A higher contribution proportion, η_d, leads the NPO to provide less charitable good*

.

Providing the charitable good adds positive value to the NPO's value function, $\Phi(s_t)$, which requires that $\mathscr{R}_s = r \in \mathbb{R}^+ \mid f(r) > 0$ be non-empty. Corollary 1 establishes that the NPO always provides the charitable good when donors do not contribute; in this case, the subspace $\mathscr{R}_s(\eta_d = 0) \neq \varnothing$. A higher contribution proportion, η_d, decreases $f(r)$ and monotonically shrinks the subspace, $\mathscr{R}_s$. Consequently, the NPO is more inclined to engage in the financial market rather than provide the charitable good. There exists a threshold of the contribution proportion, $\hat{\eta}_d$, beyond which the NPO will allocate all its endowment to the financial market and will not provide the charitable good: $G_t^* = 0$ when $\eta_d \geq \hat{\eta}_d$. Figure 5 illustrates this relationship.

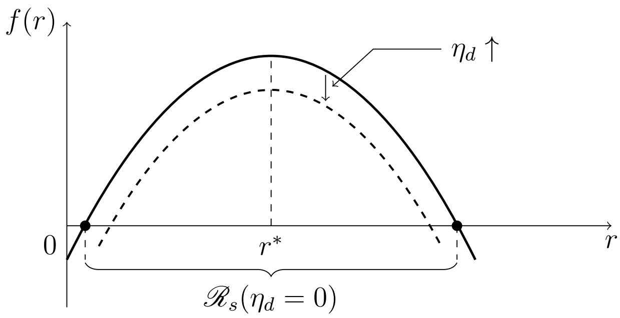


Figure 5: The subspace of rates of return on T-bills, $\mathscr{R}_s(\eta_d)$

The subspace $\mathscr{R}_s(\eta_d = 0)$ is equal to $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d = 0\} \neq \varnothing$, and $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$. The dashed parabola depicts the case when η_d has increased.

Corollary 2. *The subspace of the rate of return on T-bills, $\mathscr{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\}$, increases with parameters σ_r, $\bar{r}$, θ and decreases with NPO's risk parameter α.*

This property is attributed to the fact that the changes in the parameters $\{\alpha, \theta, \bar{r}, \sigma_r\}$ cause the $f(r)$ curve to shift both vertically and horizontally. For example, as the NPO's risk parameter α increases from α_1 to α_2, the critical point $(r^*, f(r^*))$ shifts to the lower left, and the subspace, $\mathscr{R}_s$, shrinks. This impact is different from that of the contribution ratio η_d (See Figures 5 and 6).

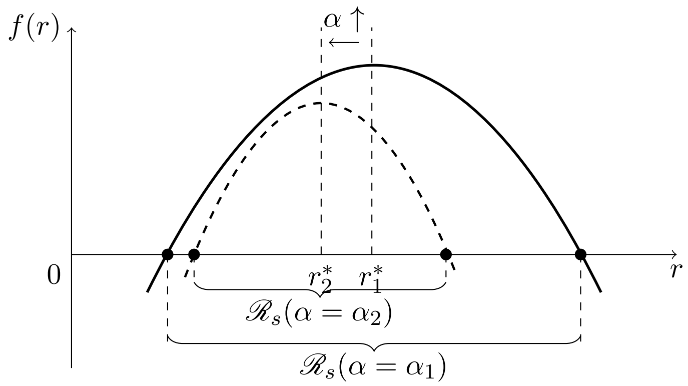


Figure 6: The impact of the risk parameter α

The subspace $\mathscr{R}_s$ is equal to $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d \geq 0\} \neq \varnothing$. The dashed parabola is the case when NPO's risk parameter, α, increases from α_1 to α_2.

4.2 Risk Exposure

It is natural to question whether the impact of risk exposure differs between NPOs and donors. This inquiry requires an examination of how the variance of risky investments and income influences the strategy profile.

4.2.1 Benchmark

In the benchmark scenario, defined in Sections (3.1) and (3.2), NPOs can actively choose their risky portfolio to diversify a certain degree of investment risk. Conversely, donors face income risk and accumulate residual wealth by investing in risk-free assets.

Proposition 5. *Investments risk affects both the NPO and donors' decisions. Increased risk exposure for NPOs encourages charitable goods provision, reduces the proportion of disposable endowment invested in the risky asset, and incentivizes donors to contribute* *less. In contrast, donors' income risk exposure does not influence the decisions of either party.*

Proposition 5 establishes that risk exposure plays distinct roles in shaping the decision-making profiles of NPOs and donors. This proposition complements the experimental evidence presented by Cettolin et al. (2017). A more volatile risky investment compels the NPO to lower its portfolio selection and provide more charitable goods, thus decreasing its endowment level. According to Proposition 4, donors respond by reducing their contributions. Conversely, donors' wealth variation, represented by income variation σ_I^2, does not correlate with the decisions (C_t^*, D_t^*) (see Proposition 2). The negative variance impact $\frac{1}{2}\Psi_{w_t,w_t}\sigma_I^2$ on donors' value function $\Psi(s_t, w_t)$ is irrelevant to donors' first-order conditions (16, 17). Consequently, donors' income risk exposure does not affect the equilibrium of the game.

The underlying rationale for this discrepancy is that donors passively place their residual wealth in risk-free assets, rather than actively diversifying it. As a result, the variance in wealth levels is irrelevant to the decision-making processes of donors.

4.2.2 Extended Risk Exposure

Let us to examine the impact of donors' risk exposure when they allocate their residual wealth to risky investments, rather than risk-free assets. Consider the rate of return on the risky asset, r_t, modeled by the Ornstein-Uhlenbeck process as follows:

$$dr_t = \vartheta(\tilde{r} - r_t)dt + \tilde{\sigma}dW_t. \tag{21}$$

Here, $\tilde{r}$ is the mean rate of return on the risky asset, $\tilde{\sigma}$ is the standard deviation, and W_t is the standard Wiener process.

Corollary 3. *Suppose donors are exposed to income risk and invest their residual wealth in a risky asset. Under these conditions, donors' risk exposures do not influence their optimal decisions.*

Donors investing their entire residual wealth in the risky asset do so without diversifying actively. Consequently, the variance in wealth level does not impact their decisions, and thus, their choices do not affect the negative variance impact on the value function

$\Psi(s_t, w_t)$. The findings from Proposition 5 and Corollary 3 align with the experimental evidence presented by Cettolin et al. (2017), which suggests that while increased risk exposure for donors remains inconsequential, heightened risk exposure for beneficiaries discourages contributions.

4.3 The Expected Stock of Endowment and Its Limitation

Proposition 6. *The NPO's expected stock of endowment is contingent on the rate of return on T-bills. It converges either to infinity or to a level* $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$.[9]

$$\mathbb{E}\{s_t\} = \begin{cases} s_0 e^{\left(\frac{\theta}{1+\theta}\bar{r} - r + \frac{\eta_d}{1-\alpha} + \frac{\alpha}{2}\sigma_r^2\right)t} & \text{if } r \leq r_L, \\ s_0 e^{\left(\frac{\eta_d}{1-\alpha} + \frac{(2-\alpha)\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right)t} & \text{if } r \in \mathscr{R}_e, \\ s_0 e^{\frac{\eta_d}{1-\alpha}t} & \text{if } r \geq r_U. \end{cases}$$

$$\lim_{t\to+\infty} \mathbb{E}\{s_t\} = \begin{cases} \infty & \text{if } r < r_U, \\ \bar{s}_\tau & \text{if } r \geq r_U. \end{cases}$$

Proposition 6 shows that given the NPO's risk parameter α, and donors' contributions proportion η_d, the rate of return on T-bills r plays an important role in determining both $\mathbb{E}\{s_t\}$ and the limit as t approaches infinity, $\lim_{t\to+\infty} \mathbb{E}\{s_t\}$. If r is not sufficiently large, the NPO engages in risky investment activities and allocates part of the investment proceeds to charitable good. This strategy leads to a continuous accumulation in the expected endowment, $\mathbb{E}s_t$, as depicted in Figures 7,-,I and II. Conversely, if r is sufficiently high, the NPO opts to invest its entire disposable endowment in T-bills. As the endowment grows, donor contributions cease, and the NPO provides all net proceeds from the T-bills as charitable good until donors begin contributing again. This cycle continues until the NPO's endowment s_t reaches the level, $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$. This dynamics helps to explain that $\mathbb{E}\{s_t\}$ increases from the initial level, s_0, and converges to $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$ (See Figure 7 - III).

Note that with a relatively lower rate of return on T-bills, the NPO's expected

[9]Thresholds $\bar{s}_L$ and $\bar{s}_U$ are uniquely determined by $\Psi_{s_t} = 1$ and $\Psi_{s_t} = \Psi_{w_t}$ respectively. Donors contribute when $s_t \leq \bar{s}_L$ and free ride the NPO when $s_t \geq \bar{s}_U$. Donors either contribute or free ride when $\bar{s}_L < s_t < \bar{s}_U$ depending on the sign of $\Psi_{s_t} - \Psi_{w_t}$. Note that $\bar{s}_\tau \in (\bar{s}_1, \bar{s}_2]$ is the minimum level of the NPO's endowment such that donors do not contribute $\forall s_t > \bar{s}_\tau$.

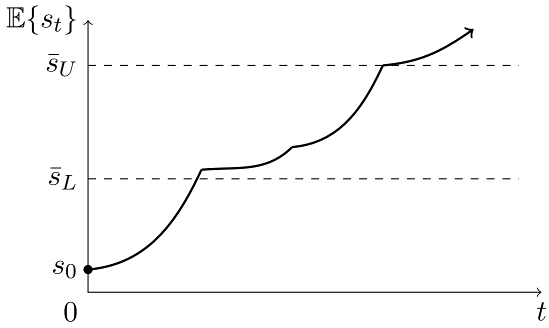


Figure 7 - I

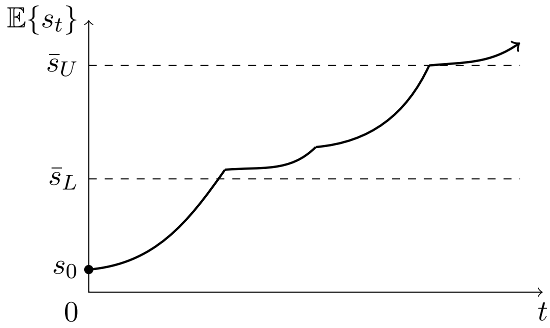


Figure 7 - II

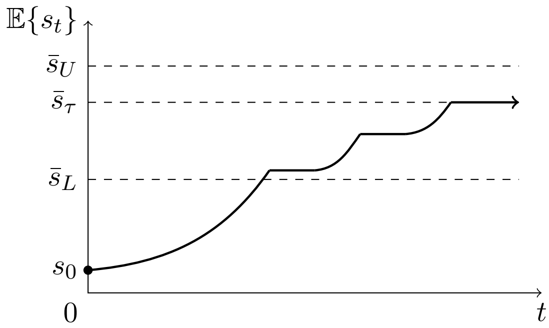
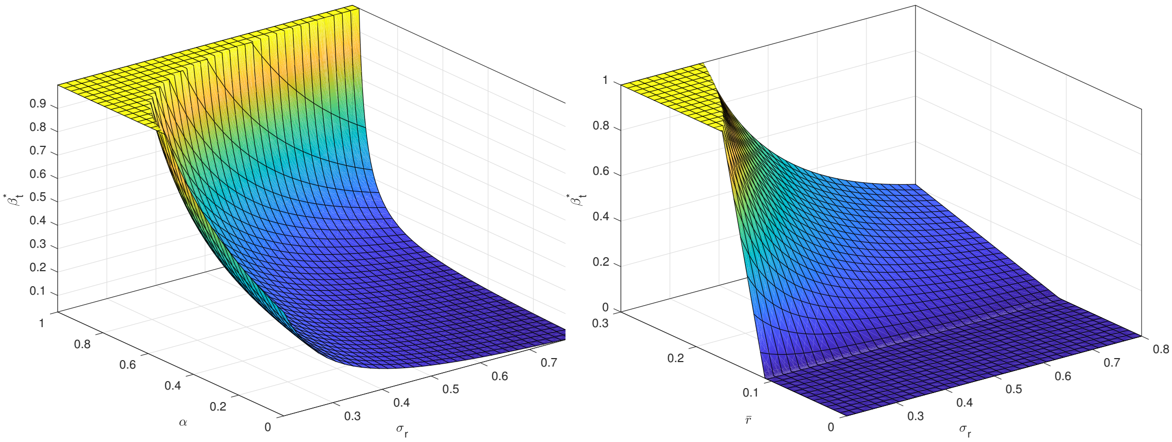


Figure 7 - III

Figure 7: NPO's Expected Endowment $\mathbb{E}\{s_t\}$ and its Limitation

Note: s_0 is the NPO's initial endowment. Figures 8 - I, II, III are cases when $r \leq r_L$, $r \in \mathscr{R}_e$ and $r \geq r_U$, respectively.

endowment, $\mathbb{E}s_t$, reaches the threshold level, $\bar{s}_L$, more quickly because a higher proportion of the disposable endowment is allocated to the risky asset. Subsequently, the $\mathbb{E}s_t$ curve flattens, coinciding with the cessation of donors' contributions. As indicated by Corollary 1, a lower contribution ratio, η_d, increases the NPO's willingness to provide charitable goods. Therefore, when η_d is sufficiently small, the NPO invests less in the risky asset. Consequently, the surplus from investment proceeds, after the provision of charitable goods, accumulates at a slower rate, leading to a flattening of the NPO's expected endowment curve. This behavior is depicted in Figures 7-,I, II, and III.



8-(a) β_t^* vs. α and σ_r 8-(b) β_t^* vs. $\bar{r}$ and σ_r

Figure 8: NPOs' portfolio selection, β_t^*

4.4 Numerical Simulations

Let $\bar{r} = 0.15, r = 0.05, \theta = 0.85, \alpha = 0.35, \sigma_r = \sigma_I = 0.5$, $s_0 = 0.2$, and $\bar{s} = 0.6$. [10] By Proposition 4, the NPO chooses a constant fraction,

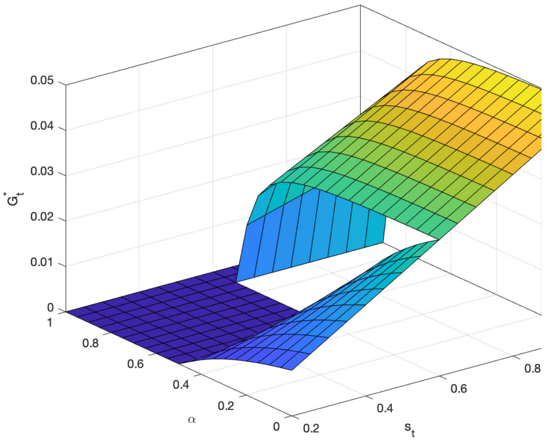
$$\beta_t^* = \min\left\{\max\left\{\frac{\frac{\theta}{1+\theta}\bar{r} - r}{(1-\alpha)\sigma_r^2}, 0\right\}, 1\right\}. \tag{23}$$

Figure 8-(a) demonstrates that, given the parameters $\bar{r}$, r, and θ, the NPO's optimal portfolio selection, β_t^*, increases with its risk parameter, α, but decreases as the standard deviation of the investment project, σ_r, increases. Notably, when the risk parameter, α, is sufficiently large and the standard deviation, σ_r, is small, the NPO is led to invest its entire disposable endowment in the risky asset.

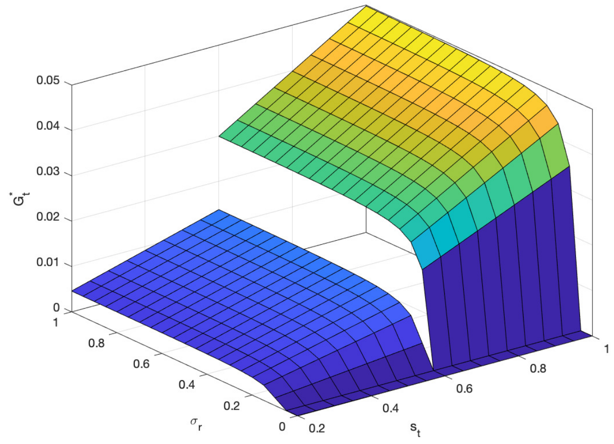
Figure 8-(b) illustrates that with fixed values for the rate of return on T-bills, r, and the NPO's risk parameter, α, the optimal β_t^* increases with the long-term mean of the rate of return on the risky tangency portfolio, $\bar{r}$, and decreases with σ_r. A sufficiently high long-term mean, $\bar{r}$, prompts the NPO to set $\beta_t^* = 1$, thereby allocating all its disposable endowment to the risky asset. Conversely, a sufficiently low $\bar{r}$ results in the NPO opting for $\beta_t^* = 0$, investing entirely in risk-free assets.

Figures 9-(a) and (b) demonstrate that the NPO provides less charitable good as it becomes less risk-averse, but it increases provision when the financial investment is more volatile. The reasoning is that an NPO with a higher risk parameter, α, shows a preference for participating in financial markets and allocating more of its endowment to

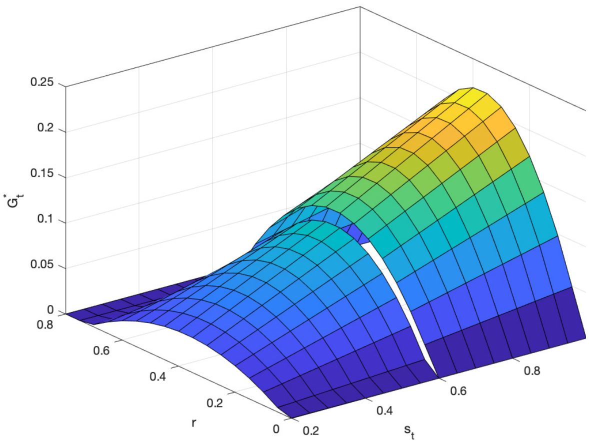
[10]Note that qualitatively similar results are obtained when using other sets of parameters.



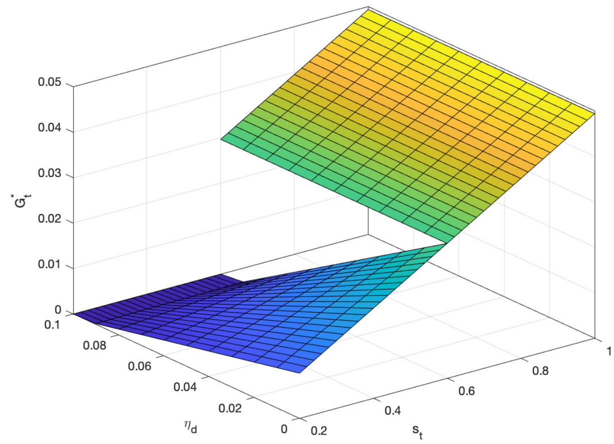
9-(a) G_t^* vs. α and s_t



9-(b) G_t^* vs. σ_r and s_t



9-(c) G_t^* vs. r and s_t



9-(d) G_t^* vs. η_d and s_t

Figure 9: NPO's provision of charitable good.

the risky portfolio, consequently reducing the level of charitable goods, G_t^*. Conversely, investments with higher volatility incentivize the NPO to provide more charitable goods and allocate less endowment to risky assets. Note that sufficiently large values of α, or sufficiently low volatility, σ_r, can lead the NPO to entirely forgo providing the charitable good.

Figure 9-(c) illustrates that the impact of the rate of return on T-bills, r, is not monotonic. Extremely low or high rates of return on T-bills increase the investment risk premium, prompting the NPO to engage more in the financial market and thus provide less charitable good. Figure 9-(d) shows that the NPO decreases the provision of charitable goods when donors increase their contribution proportion, η_d. The results depicted in Figure 9-(c) and (d) align with the insights from Corollary 1 (as seen in

Figure 5).

We present the following comparative statics analysis:

Proposition 7. *The parameters $\alpha, \eta_d, \theta, \bar{r}, \sigma_r$ affect the provision of charitable goods monotonically. However, the impact of the rate of return on T-bills, r, on the provision of charitable goods is non-monotonic.*

The function $f(r)$ takes a quadratic form in the rate of return on T-bills, r (as shown in Figure 5), indicating a non-monotonic relationship. Changes in other parameters shift the $f(r)$ curve (as seen in Figure 6), but the dominant behavior of function $f(r)$ within its domain, $\mathscr{R}_s$, remains consistent. Therefore, except for r, the impact of all other parameters is monotonic, as illustrated in Figure 8.

5 Regulations

To maximize the present value of their expected payoffs, NPOs are likely to allocate a large proportion of their endowments to risky assets. This strategy, however, may result in unexpected investment losses *ex post*. Concurrently, NPOs might provide charitable goods insufficiently, leading to fewer financed philanthropic activities. To address these issues, it is crucial that NPOs operate under meaningful regulations. Specifically, binding constraints such as a portfolio ceiling, $\tilde{\beta}_t$, and a minimum provision floor for charitable goods, $\tilde{G}_t$, are essential.

5.1 Portfolio Ceiling

Proposition 8. *When bound by a portfolio ceiling, $\tilde{\beta}_t$, the NPO adopts $\tilde{\beta}_t$ as its suboptimal portfolio selection. Consequently, the effective investment space $\mathscr{R}_e$ contracts to $\tilde{\mathscr{R}}_e$.*

Proposition 8 demonstrates that, in compliance with regulations, NPOs allocate a ceiling-limited proportion of disposable endowment to the risky asset. It is important to note that the NPO's portfolio selection behaves as a kinked piecewise function of the rate of return on the risk-free asset, as illustrated in Figure 10. This observation underscores that to effectively manage potential investment losses, the setting of the portfolio ceiling must consider the impact of the rate of return on portfolio decisions. For example, an NPO reduces its optimal risky portfolio as the rate of return on the

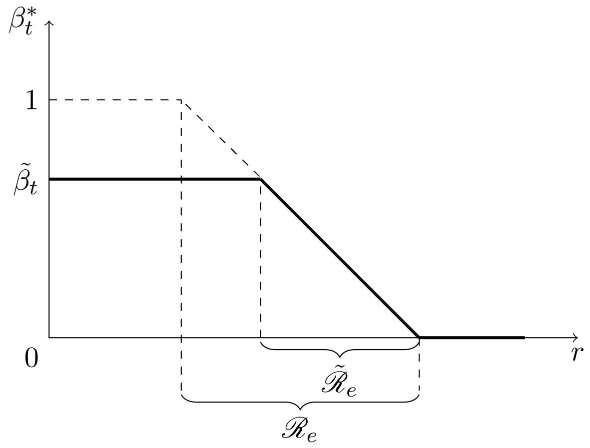


Figure 10: NPO's portfolio selection β_t^* with portfolio ceiling $\tilde{\beta}_t$

Note that $\beta_t^* = \min\left\{\max\{\frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2}, 0\}, 1\right\}$ without portfolio ceiling. The effective space $\mathscr{R}_e = \{r \in \mathbb{R}^+ \mid r_L < r < r_U\}$, where $r_L = \frac{\theta}{1+\theta}\bar{r} - (1-\alpha)\sigma_r^2$, $r_U = \frac{\theta}{1+\theta}\bar{r}$. The solid kinked line represents the optimal portfolio selection with portfolio ceiling $\tilde{\beta}_t$, where $\tilde{\mathscr{R}}_e = \{r \in \mathbb{R}^+ \mid \tilde{r}_L < r < r_U\}$ and $\tilde{r}_L = \frac{\theta}{1+\theta}\bar{r} - (1-\alpha)\sigma_r^2\tilde{\beta}_t^2$, $r_U = \frac{\theta}{1+\theta}\bar{r}$.

risk-free asset increases, necessitating a correspondingly lower portfolio ceiling to remain binding.

5.2 Provision Floor

Recalling from Proposition 7 and discussions in Section 4.1, without a provision floor, NPOs may undersupply charitable goods at equilibrium when certain conditions involving parameters $\alpha, \eta_d, \theta, \bar{r}, \sigma_r$ are met. It is therefore crucial to enforce a regulation that compels NPOs to provide charitable goods at a level higher than the equilibrium level.

Proposition 9. *With a provision floor $\tilde{G}_t$, NPOs are required to deliver charitable goods at a level not lower than $\tilde{G}_t$, diverging from the otherwise optimal level G_t^*. This regulation expands the subspace $\mathscr{R}_s$.*

6 Conclusion

We consider a dynamic charitable market with two types of random shocks: fluctuations of the investment rates of return and income shock. Donors contribute to an NPO's endowment, and the NPO provides the charitable good, which is financed by donors' contributions and investment proceeds. The results show that the NPO chooses a constant share of the risky asset. Donors' contributions are discontinuous; they either

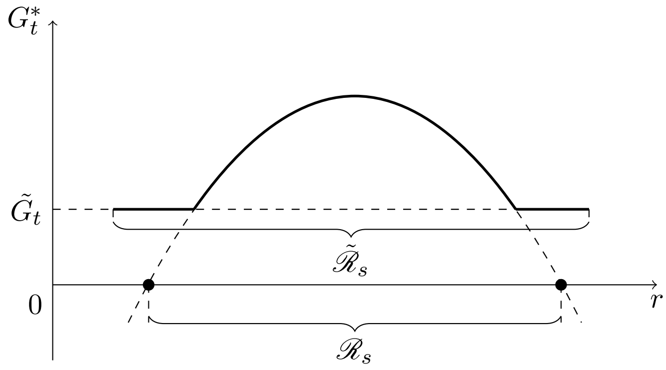


Figure 11: NPOs provision of charitable good with provision floor $\tilde{G}_t$

Note that without the regulation of provision floor, the subspace $\mathscr{R}_s$ equals $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d \geq 0\} \neq \varnothing$, where $G_t^* = (r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2})s_t$. The solid piecewise curve is the case where provision floor $\tilde{G}_t$ is applied.

hit the upper bound of giving or free-ride the NPO, which leads to the jump in the provision of charitable good by NPOs. The NPO always provides charitable good when donors do not contribute, but this does not necessarily apply if donors contribute with a higher contribution ratio.

We also show that the environment with a lower rate of return on T-bills is favourable to the NPO's expected endowment. NPO's investment risk affects both NPO's and donors' decision profiles; however, donors' income and passive investment risk exposures keep silent, which complements the experimental evidence by Cettolin et al. (2017).

7 Appendix

7.1 Proof of Proposition 1

Rewrite equation (2) as:

$$r_t\Delta t = \frac{\theta}{1+\theta}\bar{r}\Delta t + \frac{\sigma_r}{1+\theta}\varepsilon_t\sqrt{\Delta t}. \tag{A1}$$

The total change of NPO's endowment at the time interval $[\,t,\, t+\Delta t\,]$ can be rewritten as

$$s_{t+\Delta t} - s_t = \sum_{j=1}^{m} \beta_{tj}(e^{r_j \Delta t} - 1)\,(s_t + D_t\Delta t - G_t\Delta t) - (G_t - D_t)\,\Delta t \tag{A2}$$
$$= \big(\beta_t(e^{\frac{\theta}{1+\theta}\bar{r}\Delta t + \frac{\sigma_r}{1+\theta}\varepsilon_t\sqrt{\Delta t}} - 1) + (1-\beta_t)\,(e^{r\Delta t} - 1)\big)\big(s_t - (G_t - D_t)\,\Delta t\big)$$
$$- (G_t - D_t)\,\Delta t.$$

Using the Taylor expansion, $e^x = 1 + x$, and the fact that the mean of $\varepsilon_t\sqrt{\Delta t}$ is zero, we have two implications of equation (A2):

$$\mathbb{E}_t\{s_{t+\Delta t} - s_t\} = \left(\left(\beta_t\left(\frac{\theta}{1+\theta}\bar{r} - r\right) + r\right) s_t - (G_t - D_t)\right)\Delta t + O\left((\Delta t)^2\right);$$
$$\mathbb{E}_t\{\,(s_{t+\Delta t} - s_t)^2\,\} = \beta_t^2 s_t^2 \sigma_r^2 \Delta t + O\left((\Delta t)^2\right).$$

The stochastic differential equation of NPO's total financial endowment at time t can be written as

$$ds_t = \left(\big(\beta_t\left(\frac{\theta}{1+\theta}\bar{r} - r\right) + r\big)s_t - \big(G_t - D_t(w_t, s_t)\big)\right)dt + \beta_t s_t \sigma_r dW_{tr}, \tag{A3}$$

where W_t is the standard Wiener process.

7.2 Proof of Proposition 2

Proof. From equation (6), we have for the change of the wealth level, Δw_t:

$$w_{t+\Delta t} - w_t = (e^{r\Delta t} - 1)(w_t + (I_t - D_t - C_t)\Delta t) - (D_t + C_t - I_t)\Delta t. \tag{A4}$$

Inserting equation (6) into equation (A4), the conditional expectations of $w_{t+\Delta t} - w_t$ and $(w_{t+\Delta t} - w_t)^2$ at time t can be expressed as

$$\mathbb{E}_t\{w_{t+\Delta t} - w_t\} = \left(rw_t + \bar{I} - D_t - C_t\right)\Delta t + O((\Delta t)^2),$$
$$\mathbb{E}_t\{\,(w_{t+\Delta t} - w_t)^2\,\} = \sigma_I^2 \Delta t + O((\Delta t)^2).$$

The corresponding stochastic differential equation of the state variable w_t is, therefore,

$$dw_t = \big(rw_t + \bar{I} - D_t - C_t\big)\, dt + \sigma_I dW_{tI}. \tag{A5}$$

■

7.3 Proof of Proposition 3

Proof. The proof directly follows from equations (3, 4, 7, 8, 9) and the Theorem 2.5.1 from Yeung and Petrosjan (2006). ■

7.4 Proof of Proposition 4

Proof. By donors' *Hamilton-Jacobi-Bellman* equation (13), the optimal choice pair (C_t^*, D_t^*) satisfies the following inequalities:

$$1 - \Psi_{w_t} > 0 \ or \ < 0; \tag{A6}$$

$$\Psi_{s_t} - \Psi_{w_t} > 0 \ or \ < 0. \tag{A7}$$

Therefore, donors' private consumption, C_t^*, and contributions, D_t^*, must take the corner solutions.

Assume that donors' value function, $\Psi(s_t, w_t)$, is increasing and strictly concave in both arguments s_t and w_t. Note that the donors' first-order conditions with respect to consumption, C_t, must be positive to ensure private consumptions at time t with any level of wealth, w_t. That is Ψ_{w_t} is monotonically decreasing and set within $(0, 1)$. Besides, Ψ_{w_t} follows the properties: $\lim_{w_t \to 0} \Psi_{w_t} = 1$ and $\lim_{w_t \to +\infty} \Psi_{w_t} = 0$. By equation (A6), the optimal private consumption, C_t^*, would hit the upper bound level, $\eta_c w_t$, where the parameter $\eta_c \in (0, 1)$ represents donors' propensity for consumption.

Donors' value function $\Psi(w_t, s_t)$ is concave in s_t and, therefore, Ψ_{s_t} is a monotonically decreasing function in s_t. It follows that $\lim_{s_t \to 0} \Psi_{s_t} = +\infty$ and $\lim_{s_t \to +\infty} \Psi_{s_t} = 0$, as shown in the Figure above. Consequently, there must exist a unique level of NPO's endowment, $\bar{s}$, such that $\Psi_{s_t} = 1$, which is greater than Ψ_{w_t}. Thus, $\Psi_{s_t} - \Psi_{w_t} > 0$ when $s_t < \bar{s}$. That is the donors' first-order condition with respect to contribution, D_t, is positive, which motivates donors to contribute to the upper bound, $D_t^* = \eta_d s_t$, where parameter $\eta_d \in (0,1)$ represents the donors' desired level of charitable good. On the other hand, $\Psi_{s_t} - \Psi_{w_t} >< 0$ when $s_t > \bar{s}$. Thus, donors do not contribute if $\Psi_{s_t} < \Psi_{w_t}$, but they contribute $\eta_d s_t$ if $\Psi_{s_t} > \Psi_{w_t}$.

$$C_t^* = \eta_c w_t, \tag{A8}$$

$$D_t^* = \begin{cases} \eta_d s_t & \textit{if } s_t < \bar{s} \\ 0 & \textit{if } s_t > \bar{s} \textit{ and } \Psi_{s_t} < \Psi_{w_t} \\ \eta_d s_t & \textit{if } s_t > \bar{s} \textit{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \tag{A9}$$

Next, consider the NPO's optimal program. Following Merton (1971) we assume that the NPO's instantaneous utility function takes the form of CRRA: $u^n(G_t) = {}^{G_t^\alpha}/\alpha$, where $\alpha \in (0,1)$. By the NPO's first-order conditions equations (14, 15), G_t^* and β_t^* can be expressed as,

$$G_t^* = \Phi_{s_t}^{1/(\alpha - 1)}, \tag{A10}$$

$$\beta_t^* = -\frac{\Phi_{s_t}(\frac{\theta}{1+\theta}\bar{r} - r)}{\Phi_{s_t,s_t} s_t \sigma_r^2}. \tag{A11}$$

Inserting equations (A10, A11) into the NPO's *HJB* equation (12), yields,

$$r\Phi(s_t) = \begin{cases} \frac{1-\alpha}{\alpha}\Phi_{s_t}^{\frac{\alpha}{(\alpha-1)}} + \Phi_{s_t} r s_t - \frac{1}{2}\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2 \frac{\Phi_{s_t}^2}{\Phi_{s_t,s_t}\sigma_r^2} & \textit{if } D_t^* = 0, \\ \frac{1-\alpha}{\alpha}\Phi_{s_t}^{\frac{\alpha}{(\alpha-1)}} + \Phi_{s_t}(r + \eta_d) s_t - \frac{1}{2}\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2 \frac{\Phi_{s_t}^2}{\Phi_{s_t,s_t}\sigma_r^2} & \textit{if } D_t^* = \eta_d s_t. \end{cases} \tag{A12}$$

Assume that the NPO's *BJH* equation, equation (A12), has the explicit solution, which takes the form of $\Phi(s_t) = \Lambda s_t^\alpha$ (See Merton (1971)). The function, $\Phi(s_t) = \Lambda s_t^\alpha$, will be the solution of equation (A12) if the parameter Λ satisfies the following conditions

$$\Lambda = \begin{cases} \frac{1}{\alpha}\left(r - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right)^{\alpha-1} & \textit{if } D_t^* = 0, \\ \frac{1}{\alpha}\left(r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right)^{\alpha-1} & \textit{if } D_t^* = \eta_d s_t. \end{cases} \tag{A13}$$

Hence, by equations (A10, A11, A13), we obtain

$$G_t^* = \begin{cases} \left(r - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2} \right) s_t & \textit{if } D_t^* = 0, \\ \left(r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2} \right) s_t & \textit{if } D_t^* = \eta_d s_t, \end{cases} \tag{A14}$$

$$\beta_t^* = \min\left\{ \max\left\{ \frac{\frac{\theta}{1+\theta}\bar{r} - r}{(1-\alpha)\sigma_r^2}, 0 \right\}, 1 \right\} \qquad \textit{if } D_t^* \geq 0. \tag{A15}$$

This proves Proposition 4. ■

7.5 Proof of Corollary 1

Proof. Define the function of the rates of return on T-bills, r,

$$f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2}{2(1-\alpha)^2\sigma_r^2}.$$

Rewrite $f(r)$ as:

$$f(r) = \frac{\alpha}{2(1-\alpha)^2\sigma_r^2}\left(-r^2 + 2\left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}\right) r - \left(\frac{\theta}{1+\theta}\bar{r}\right)^2 - 2(1-\alpha)\sigma_r^2\eta_d \right). \tag{A16}$$

By equation (A14), for any given parameter vector of agents and risky asset, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, function f must map onto $\mathbb{R}^+$. It follows from equation (A16) that we have $B^2 - 4AC > 0$, that is, parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, must satisfy the following inequality

$$\frac{1-\alpha}{\alpha}\left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha}\right) - \eta_d > 0. \tag{A17}$$

This proves that G_t^* exists if and only if the subspace of the rates of return on T-bills, $\mathscr{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\} \neq \varnothing$.

Donors do not contribute when parameter η_d equals zero. Straightforward calculations show that the subspace of rates of return on T-bills $\mathscr{R}_s(\eta_d = 0) = \frac{2(1-\alpha)\sigma_r}{\alpha}\sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2}$ is not empty. This establishes that there exists a subspace $\mathscr{R}_s$ such that the NPO is willing to provide the charitable good when donors do not contribute.

Note that the function $f(r)$ has a critical point, $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$, which does not depend on η_d; and $\mathscr{R}_s = \frac{2(1-\alpha)\sigma_r}{\alpha}\sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2 - \frac{2\alpha^2}{(1-\alpha)}\eta_d}$. One can verify that $\mathscr{R}_s$ has the following property: $\frac{\partial \mathscr{R}_s}{\partial \eta_d} < 0$. As a result, increasing η_d continuously pushes the solid parabola curve downwards, which monotonically shrinks the subspace, $\mathscr{R}_s$ (See Figure below). Therefore, there exists $\hat{\eta}_d$, such that $\mathscr{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\} = \varnothing$. By equation (A17), the threshold $\hat{\eta}_d = \frac{1-\alpha}{\alpha}\left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha}\right)$. ∎

7.6 Proof of Corollary 2

Proof. Note that for any given parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, the function $f(r)$ has the critical point $(r^*, f(r^*)) = \left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}, \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} - \frac{\alpha}{1-\alpha}\eta_d\right)$. The subspace of the rates of return on T-bills is,

$$\mathscr{R}_s = \frac{2(1-\alpha)\sigma_r}{\alpha}\sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2 - \frac{2\alpha^2}{(1-\alpha)}\eta_d}.$$

This subspace $\mathscr{R}_s$ has the property,

$$\begin{cases} \frac{\partial \mathscr{R}_s}{\partial \alpha} < 0, \\ \frac{\partial \mathscr{R}_s}{\partial \sigma_r} > 0,\ \frac{\partial \mathscr{R}_s}{\partial \bar{r}} > 0,\ \frac{\partial \mathscr{R}_s}{\partial \theta} > 0. \end{cases}$$

This establishes that the subspace of rates of return on T-bills, $\mathscr{R}_s$, increases with σ_r, $\bar{r}$, and θ; and decreases with the NPO's risk parameter, α.

Direct calculations show that the critical point $(r^*, f(r^*))$ shifts when parameters $\{\alpha, \theta, \bar{r}, \sigma_r\}$ change:

$$\begin{cases} \frac{\partial r^*}{\partial \alpha} < 0,\ \frac{\partial r^*}{\partial \sigma_r} > 0,\ \frac{\partial r^*}{\partial \bar{r}} > 0,\ \frac{\partial r^*}{\partial \theta} > 0, \\ \frac{\partial f(r^*)}{\partial \alpha} < 0,\ \frac{\partial f(r^*)}{\partial \sigma_r} > 0,\ \frac{\partial f(r^*)}{\partial \bar{r}} > 0,\ \frac{\partial f(r^*)}{\partial \theta} > 0. \end{cases}$$

Hence, the $f(r)$ curve shifts to lower left or upper right. This finishes the proof of Corollary 2. ∎

7.7 Proof of Proposition 5

Proof. By Proposition 3, given the NPO's optimal strategy $\{G_t^*, \beta_t^*\}$, donors' optimal decision $\{C_t^*, D_t^*\}$ is such that value function $\Psi(s_t, w_t)$ must satisfy

$$r\Psi(s_t, w_t) = \max_{\substack{C_t \in (0,\, \eta_c w_t] \\ D_t \in [0,\, \eta_d s_t]}} \Big\{ u^d(G_t^*) + C_t + \Psi_{s_t}\Big(\big(\beta_t^*(\frac{\theta}{1+\theta}\bar{r} - r) + r\big)s_t - (G_t^* - D_t)\Big) \\ + \Psi_{w_t}\left(rw_t + \bar{I} - D_t - C_t\right) + \frac{1}{2}\Psi_{s_t,s_t}\beta_t^{*2}s_t^2\sigma_r^2 + \frac{1}{2}\Psi_{w_t,w_t}\sigma_I^2\Big\}. \quad \text{(A18)}$$

By equation (9), the donors' wealth variation is exactly the same as their income volatility, σ_I^2, which is not related to the donors' decision $\{C_t^*, D_t^*\}$. Thus, the change of the term $\frac{1}{2}\Psi_{w_t,w_t}\sigma_I^2$ in equation (A18) does not affect the donors' first-order conditions (See equations (A6, A7)). That is the change of donors' income risk exposure does not have any effect on the equilibrium of the game.

From Proposition 4, a more volatile investment project leads the NPO to lower its portfolio selection and provide more charitable goods, which decreases its endowment level. Meanwhile, donors' contribution shrinks. ■

7.8 Proof of Corollary 3

Proof. At time interval $[t, t + \Delta t]$, the discrete forms of donors' income and rate of return differential equations are,

$$\begin{cases} I_t\Delta t = \bar{I}\Delta t + \sigma_I\varepsilon_t\sqrt{\Delta t}, \\ r_t\Delta t = \vartheta(\tilde{r} - r_t)\Delta t + \tilde{\sigma}\varepsilon_t\sqrt{\Delta t}. \end{cases} \quad \text{(A19)}$$

It follows that the change of the wealth level, Δw_t, can be written as:

$$w_{t+\Delta t} - w_t = (e^{r_t\Delta t} - 1)(w_t + (I_t - D_t - C_t)\Delta t) - (D_t + C_t - I_t)\Delta t. \quad \text{(A20)}$$

Inserting equation (A19) into equation (A20), the conditional expectations of $w_{t+\Delta t} - w_t$ and $(w_{t+\Delta t} - w_t)^2$ at time t can be written as:

$$\mathbb{E}_t\{w_{t+\Delta t} - w_t\} = \left(\frac{\vartheta}{1+\vartheta}\tilde{r}w_t + \bar{I} - D_t - C_t\right)\Delta t + O((\Delta t)^2),$$
$$\mathbb{E}_t\{\,(w_{t+\Delta t} - w_t)^2\,\} = \left(\frac{1}{(1+\vartheta)^2}w_t^2\tilde{\sigma}^2 + \sigma_I^2\right)\Delta t + O((\Delta t)^2).$$

The corresponding stochastic differential equation of the state variable w_t is, therefore,

$$dw_t = \left(\frac{\vartheta}{1+\vartheta}\tilde{r}w_t + \bar{I} - D_t - C_t\right)dt + \sqrt{\frac{1}{(1+\vartheta)^2}w_t^2\tilde{\sigma}^2 + \sigma_I^2 dW_t}. \tag{A21}$$

By Proposition 3, one can derive that NPOs and donors have the same first-order conditions as in the benchmark case. ■

7.9 Proof of Proposition 6

Proof. Substituting β^*, G_t^*, and D_t^* into equation (4) leads to

$$ds_t = \left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r} - r) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d\right)\right)s_t dt + \beta_t^* s_t \sigma_r dW_{tr},$$

where $\eta_d \geq 0$. Following Kuo (2006), this stochastic differential equation has the following solution:

$$s_t = s_0 e^{\left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r}-r)+r-\left((\Lambda\alpha)^{1/(\alpha-1)}-\eta_d\right)-0.5\beta_t^{*2}\sigma_r^{*2}\right)t+\beta_t^*\sigma_r W_{tr}}$$
$$+ \int_0^t 0 \times e^{\left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r}-r)+r-\left((\Lambda\alpha)^{1/(\alpha-1)}-\eta_d\right)-0.5\beta_t^{*2}\sigma_r^{*2}\right)(t-\tau)+\beta_t^*\sigma_r(W_{tr}-W_{\tau r})}d\tau.$$

We take the expectation of both sides conditional on the information at time t. By the property of geometric Brownian motion, we have

$$\begin{aligned}\mathbb{E}_t\{s_t\} &= \mathbb{E}_t\left\{s_0 e^{\left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r}-r)+r-\left((\Lambda\alpha)^{1/(\alpha-1)}-\eta_d\right)\right)t}\right\}\\ &= s_0 e^{\left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r}-r)+r-\left((\Lambda\alpha)^{1/(\alpha-1)}-\eta_d\right)\right)t},\end{aligned} \tag{A22}$$

where $(\alpha\Lambda)^{1/(\alpha-1)} = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha}{2}\sigma_r^2\beta_t^2$, and β_t^* equals 0, $\frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2}$, 1 for $r > r_U, r \in \mathscr{R}_e$, and $r < r_L$, respectively. By equation (A22), the NPO's expected endowment can be simplified as,

$$\mathbb{E}\{s_t\} = s_0 e^{\left(\beta_t^*(\frac{\theta}{1+\theta}\bar{r}-r)+\frac{1}{1-\alpha}\eta_d+\frac{\alpha}{2}\sigma_r^2\beta_t^2\right)t}. \tag{A23}$$

By Proposition 4, the NPO chooses the optimal share of the risky asset equal to $1, \beta_t^*$, and 0, respectively, contingent on the rate of return on T-bills. It follows,

$$\mathbb{E}\{s_t\} = \begin{cases} s_0 e^{\left(\frac{\theta}{1+\theta}\bar{r}-r+\frac{\eta_d}{1-\alpha}+\frac{\alpha}{2}\sigma_r^2\right)t} & \textit{if } r \leq r_L,\ \beta_t^* = 1, \\ s_0 e^{\left(\frac{\eta_d}{1-\alpha}+\frac{(2-\alpha)(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}\right)t} & \textit{if } r \in \mathscr{R}_e,\ \beta_t^* = \frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2}, \\ s_0 e^{\frac{\eta_d}{1-\alpha}t} & \textit{if } r \geq r_U,\ \beta_t^* = 0. \end{cases}$$

It is innocuous to assume that the donors' shadow price of wealth level, Ψ_{w_t}, follows $\lim_{t\to+\infty}\Psi_{w_t} = \epsilon$, where $0 < \epsilon < 1$ (See Figure above). Thus, we derive the following properties,

$$D_t^* = \begin{cases} \eta_d s_t & \textit{if } s_t \leq \bar{s}_L, \\ \eta_d s_t \textit{ or } 0 & \textit{if } \bar{s}_L < s_t < \bar{s}_U, \\ 0 & \textit{if } s_t \geq \bar{s}_U. \end{cases}$$

Therefore, the limit of $\mathbb{E}\{s_t\}$ is given by

$$\lim_{t\to+\infty}\mathbb{E}\{_t\} = \begin{cases} \infty & \textit{if } r < r_U, \\ \bar{s}_\tau & \textit{if } r \geq r_U. \end{cases}$$

where $\bar{s}_\tau \in (\bar{s}_1, \bar{s}_2]$ is the minimum level of the NPO's endowment such that donors do not contribute $\forall s_t > \bar{s}_\tau$. ■

7.10 Proof of Proposition 7

Proof. According to equation (A17), the function $f(r)$ takes the quadratic form:

$$f(r) = \frac{\alpha}{2(1-\alpha)^2\sigma_r^2}\Big(-r^2 + 2(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha})r - (\frac{\theta}{1+\theta}\bar{r})^2 - 2(1-\alpha)\sigma_r^2\eta_d\Big),$$

with the critical point $(r^*, f(r^*)) = (\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}, \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} - \frac{\alpha}{1-\alpha}\eta_d)$. This implies that the impact of r on the optimal provision of charitable good, G_t^*, is non-monotonic.

However, the change of other parameters shift the function $f(r)$ curve to the lower left, vertically, or to the upper right (see Figure 5), but the dominance of function $f(r)$ in its domain, $\mathscr{R}_s$, is consistent. This shows that the impact of parameters other than r is monotonic, as shown in Figure 8. This completes the proof of Proposition 7. ■

7.11 Proof of Proposition 8

Proof. By applying the NPO's first-order condition with respect to β_t (see equation (15)), it becomes apparent that if the portfolio ceiling is binding ($\tilde{\beta}_t < \beta_t^*$), the NPO opts for $\tilde{\beta}_t$ instead of β_t^*. ■

7.12 Proof of Proposition 9

Proof. Incorporating the NPO's instantaneous utility function and value function into its first-order condition with respect to D_t, equation (14) transforms to:

$$G_t^{\alpha-1} - \Lambda\alpha s_t^{\alpha-1} = 0.$$

Indeed, the NPO is worse off when the provision floor is greater than the optimal level. Therefore, the best choice for the NPO is to set the provision floor. ■

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