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ANALYTICAL AND NUMERICAL EVALUATION OF CONFIGURATION FACTORS IN RADIATION HEAT TRANSFER

By

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CONFIGURATION FACTORS

IN RADIATION HEAT TRANSFER

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Candidate

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ABSTRACT

The object of this thesis is twofold; on the one hand the radiation configuration factors for certain fundamental geometries —— not previously treated in the literature —— have been developed and, on the other hand, an entirely new approach proposed by Feingold* has been investigated, leading to a series of new analytical formulas involving spheres and infinitely long cylinders. Particularly significant are two hitherto unobserved conclusions; namely that the radius of the radiating sphere or of infinitely long cylinder does not affect the factor from that body to an entire class of surfaces described in the thesis, and that the Lambertian postulate of non-preferential distribution of radiation intensity over the surfaces of spheres and infinitely long cylinders is not necessary to the calculation of factors from these bodies.

Several graphs have been plotted which should form a useful addition to the collection of similar data already available in the standard textbooks.

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* Reference 9

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I. INTRODUCTION

There are not many configurations for which the radiant heat-transfer configuration factor can be obtained analytically in a closed form. An all-numerical method can be used to compute the configuration factor in every case, but the degree of accuracy of the result would be difficult to ascertain.

In this thesis an attempt has been made to formulate configuration factors for a variety of geometries, not previously treated in the literature. The final expressions have, in most cases, been arrived at analytically in terms of elementary functions, while in others the solution proceeded in closed form as far as possible and then a numerical method was employed to perform the final integration. A single numerical integration performed on a digital computer is capable of producing results with a desired degree of accuracy, limited only by the capability of the particular machine.

The expression ''radiant interchange configuration factor'' has been chosen here in preference to such other commonly employed terms as ''shape factors'', ''view factors'', etc., because it is more descriptive.

In some of the sections contour integrals have been employed in preference to a double integration over each surface. The advantage of this is apparent when the two methods are carefully compared.

In the existing literature on radiant-interchange configuration factors, the underlying assumption is that the directional distribution of the emitted radiation follows Lambert's cosine law. It is shown here that in the case of radiation from spheres to a certain class of surfaces, the configuration factors are independent of the validity of Lambert's law. A new technique is developed to calculate the configuration factors in radiation from spheres, and formulas are obtained for several important geometries.

Extremely simple formulas resulted from the present investigation in certain cases in which the heretofore used methods lead to multiple integrals of forbidding complexity which often could only be solved by numerical processes.

Reasoning, similar to the one applied to the spheres, is further extended to infinitely long cylinders.

Finally, some errors in the formulas published in well-known texts are pointed out and corrected.

II. CONFIGURATION FACTOR FROM AN INFINITESIMAL AREA TO A TRIANGLE, THEIR PLANES FORMING AN ARBITRARY ANGLE

As a first step, we shall evaluate the configuration factor from an infinitesimal area to a right-angle trapezoid, their planes forming an arbitrary angle. One of the sides of the trapezoid lies on the intersection of the two planes.

Figure 1(a) represents a trapezoid in the y-z plane. The trapezoid is composed of a rectangle and a right-angle triangle with a common base b. The height of the rectangle is a and of the triangle is kb, where $k = \tan\theta$. Figure 1(a) also represents an infinitesimal area in a plane which makes an angle ϕ with the y-z plane. In the figure, the y-coordinate of this infinitesimal area falls arbitrarily anywhere between zero and b. It is to be noted here, that the y-coordinate of the infinitesimal area does not necessarily have to fall in this particular range, it could equally well fall either above b or below zero. These two situations are shown in figures 1(b) and 1(c) respectively. We shall proceed to evaluate the factor for the configuration in figure 1(a). However, the expression will be identical for the other two configurations provided that when the coordinate y_1 is negative it is entered into the formula with its minus sign.

The factor from an infinitesimal area dA_1 to a finite area A_2 is given by

$$F_{dA_1-A_2} = \iint_{A_2} \frac{\cos\beta_1 \cdot \cos\beta_2}{\pi r^2}\, dA_2 \; . \qquad (1)$$

The normals to the elements dA_1 and dA_2 are denoted respectively by unit vectors $\overline{n}_1$ and $\overline{n}_2$ in figure 1(a). β_1 and β_2 are the angles formed by the respective normals and the connecting line between the elements, The length of the connecting line is r.

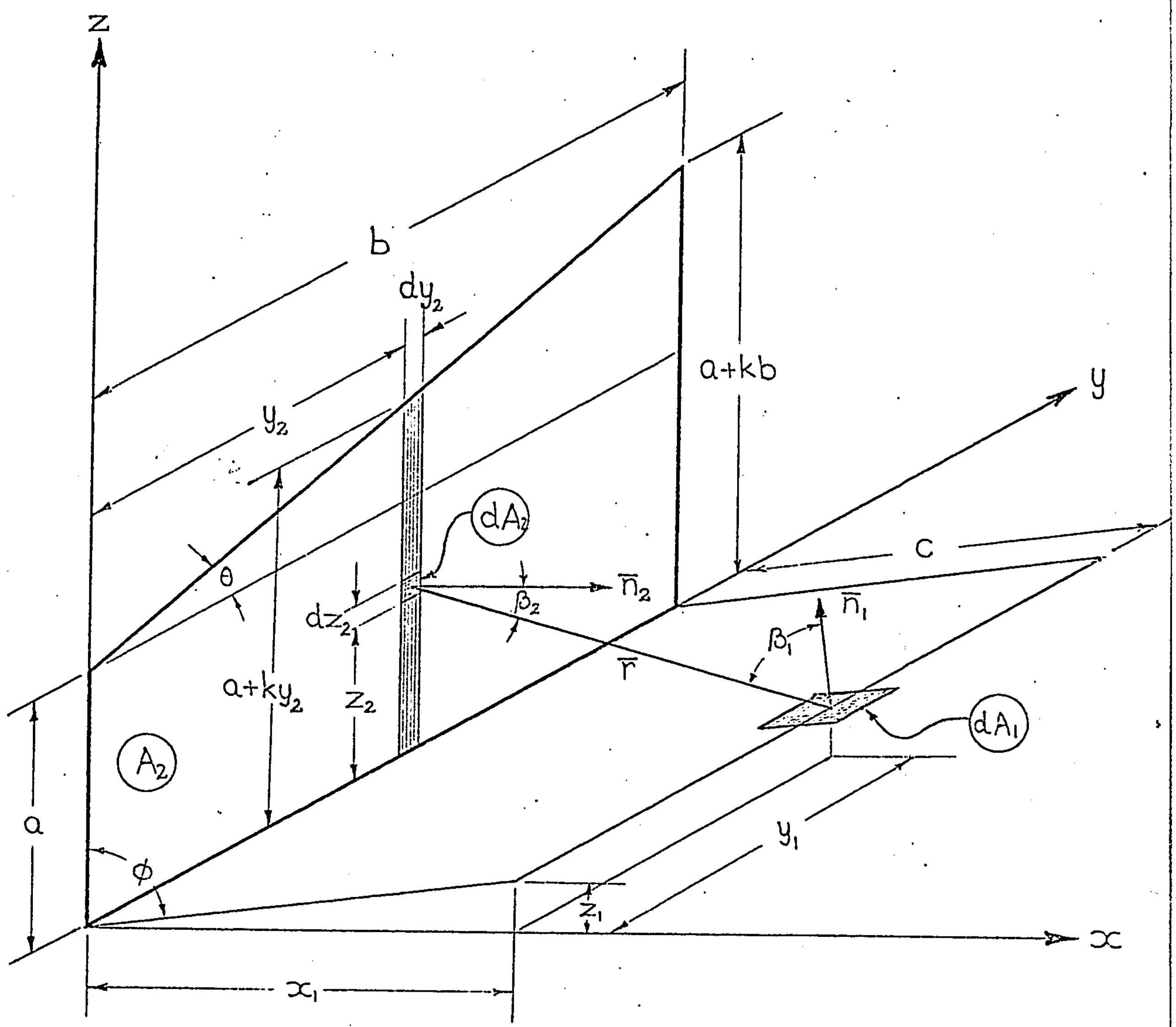

z
b
dy_2
$a+kb$
y
y_2
dA_2
c
θ
$\bar{n}_2$
$\bar{n}_1$
dz_2
β_2
β_1
$\bar{r}$
$a+ky_2$
z_2
dA_1
A_2
a
y_1
φ
z_1
x
x_1

FIGURE 1(a)

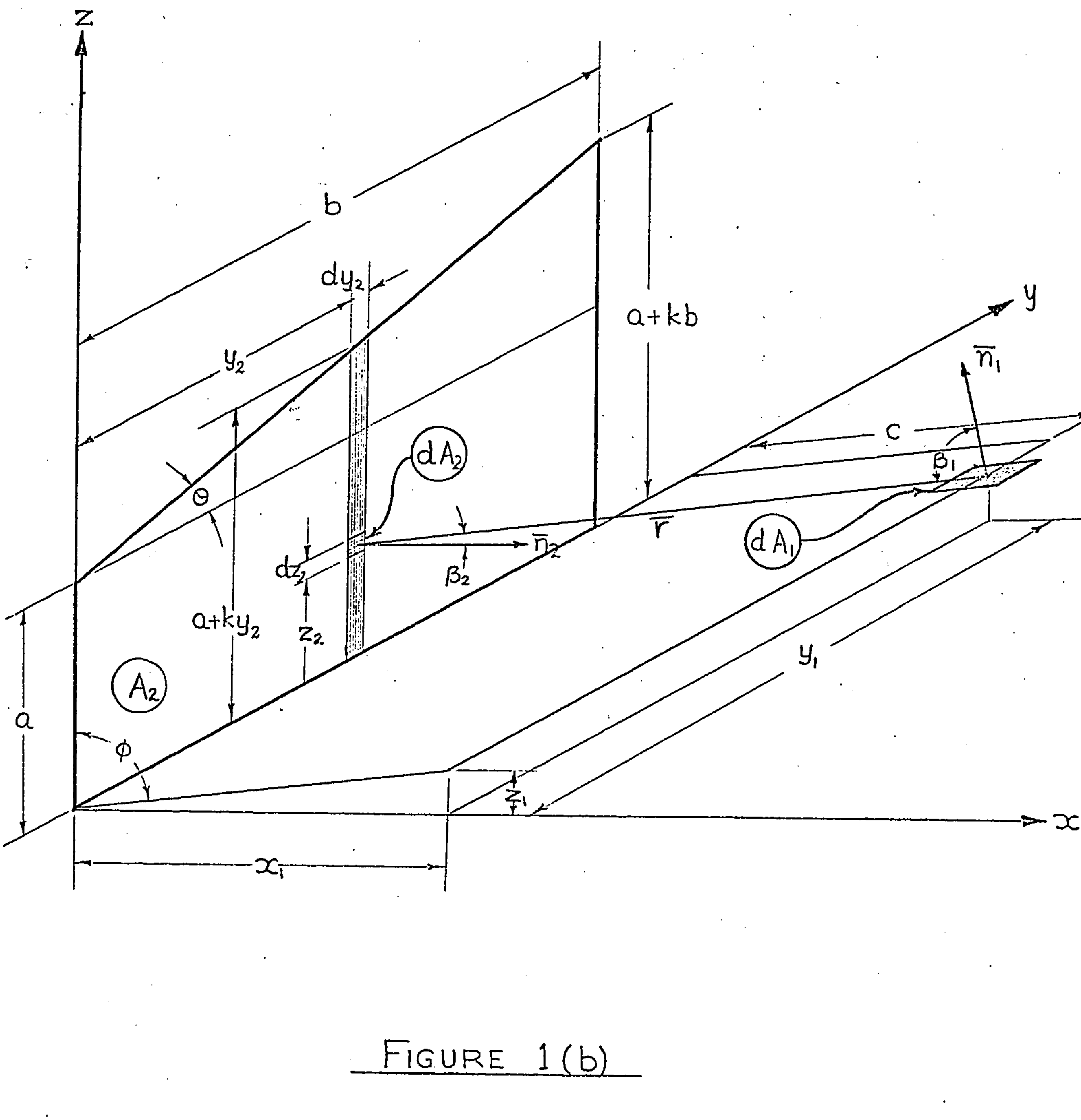


FIGURE 1(b)

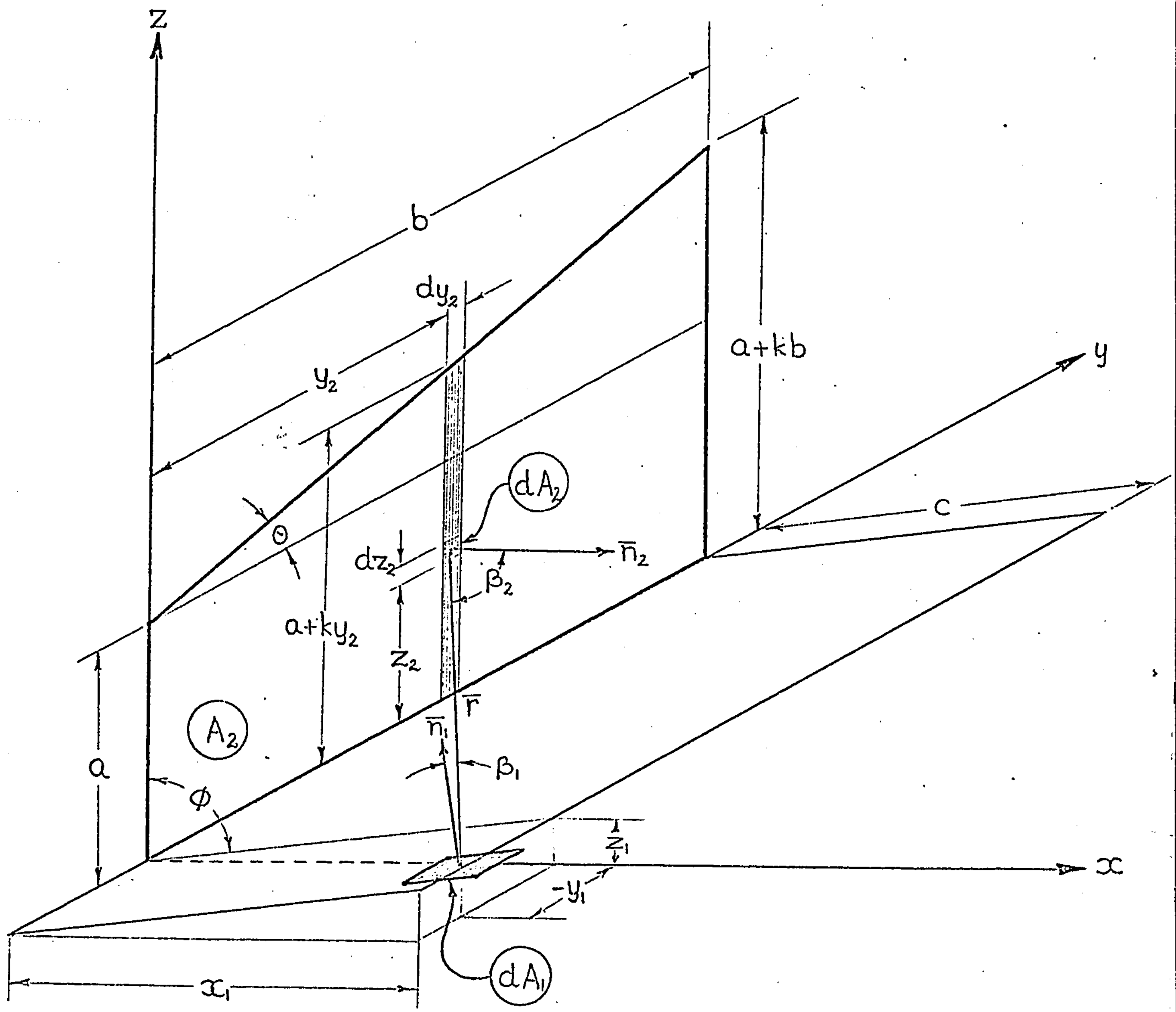


FIGURE 1 (c)

If l_1, m_1, n_1 are the direction cosines of $\overline{n}_1$ and l_2, m_2, n_2 that of $\overline{n}_2$, then we have

$$\overline{n}_1 = l_1\overline{i} + m_1\overline{j} + n_1\overline{k},$$

$$= -\cos\phi\,\overline{i} + 0\overline{j} + \sin\phi\,\overline{k},$$

$$\overline{n}_2 = l_2\overline{i} + m_2\overline{j} + n_2\overline{k},$$

$$= \overline{i} + 0\overline{j} + 0\overline{k},$$

$$\overline{r}_{dA_1-dA_2} = -x_1\overline{i} + (y_2-y_1)\overline{j} + (z_2-z_1)\overline{k},$$

$$= -\overline{r}_{dA_2-dA_1},$$

$$r = \sqrt{x_1^2 + (y_2-y_1)^2 + (z_2-z_1)^2},$$

$$\cos\beta_1 = \frac{\overline{n}_1 \cdot \overline{r}_{dA_1-dA_2}}{r} = \frac{x_1\cos\phi + (z_2-z_1)\sin\phi}{r},$$

$$= \frac{z_2\sin\phi}{r}, \quad \text{since } x_1\cos\phi = z_1\sin\phi.$$

$$\cos\beta_2 = \frac{\overline{n}_2 \cdot \overline{r}_{dA_2-dA_1}}{r} = \frac{x_1}{r},$$

$$dA_2 = dy_2 \cdot dz_2 .$$

It is easier, in this case, to integrate first with respect to z than with respect to y. Substituting the values in equation (1), we get

$$F_{dA_1-A_2} = \int_0^b\int_0^{a+ky_2} \frac{1}{\pi r^2} \cdot \frac{z_2\sin\phi}{r} \cdot \frac{x_1}{r} \cdot dz_2\,dy_2,$$

$$= \frac{x_1\sin\phi}{\pi}\int_0^b dy_2 \int_0^{a+ky_2} \frac{z_2\,dz_2}{[x_1^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^2} .$$

Let

$$y_2 - y_1 = y ,$$

then

$$F_{dA_1-A_2} = \frac{x_1 \sin\phi}{\pi} \int_0^b dy_2 \int_0^{a+ky_2} \frac{z_2\, dz_2}{[z_2^2 - 2z_1 z_2 + x_1^2 + y^2 + z_1^2]^2} .$$

Let

$$x_1^2 + y^2 + z_1^2 = p ,$$

and

$$-2z_1 = q ,$$

then we have

$$F_{dA_1-A_2} = \frac{x_1 \sin\phi}{\pi} \int_0^b dy_2 \int_0^{a+ky_2} \frac{z_2\, dz_2}{[z_2^2 + qz_2 + p]^2} , \quad (1)^*$$

$$= \frac{x_1 \sin\phi}{\pi} \int_0^b dy_2 \left[\frac{-(2p+qz_2)}{(4p-q^2)[z_2^2+qz_2+p]} - \frac{2q}{(4p-q^2)^{3/2}} \tan^{-1} \frac{2z_2+q}{\sqrt{4p-q^2}} \right]_0^{a+ky_2} .$$

Substituting the values of p and q and the limits, we have

$$F_{dA_1-A_2} = \frac{x_1 \sin\phi}{\pi} \int_0^b dy_2 \left[\frac{-2(x_1^2+y^2+z_1^2) + 2z_1(a+ky_2)}{4(x_1^2+y^2+z_1^2-z_1^2)[(a+ky_2)^2 - 2z_1(a+ky_2) + x_1^2+y^2+z_1^2]} \right.$$

$$\left. - \frac{2(x_1^2+y^2+z_1^2)}{4(x_1^2+y^2)(x_1^2+y^2+z_1^2)} - \frac{2(-2z_1)}{8(x_1^2+y^2)^{3/2}} \tan^{-1} \frac{a+ky_2-z_1}{\sqrt{x_1^2+y^2}} + \frac{2(-2z_1)}{8(x_1^2+y^2)^{3/2}} \tan^{-1} \frac{-z_1}{\sqrt{x_1^2+y^2}} \right] ,$$

or

$$\frac{2\pi F_{dA_1-A_2}}{x_1 \sin\phi} = \int_0^b \frac{(a+ky_2)z_1 - (x_1^2+y^2+z_1^2)}{(x_1^2+y^2)[x_1^2+y^2+(a+ky_2-z_1)^2]}\, dy_2 + \int_0^b \frac{dy_2}{x_1^2+y^2}$$

$$+ \int_0^b \frac{z_1}{(x_1^2+y^2)^{3/2}} \tan^{-1} \frac{a+ky_2-z_1}{\sqrt{x_1^2+y^2}}\, dy_2 + \int_0^b \frac{z_1}{(x_1^2+y^2)^{3/2}} \tan^{-1} \frac{z_1}{\sqrt{x_1^2+y^2}}\, dy_2 . \quad \text{--------(2)}$$

* Number in the parenthesis refers to the number in the Appendix

Let us now integrate the third and the fourth terms of equation (2) as illustrated below

$$\int_0^b \frac{z_1}{(x_1^2+y^2)^{3/2}} \tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}}\, dy_2$$

$$= \int_{-y_1}^{b-y_2} \frac{z_1}{(x_1^2+y^2)^{3/2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}}\,dy \quad , \text{ since } \quad y_2-y_1=y$$

$$\text{and} \quad dy_2=dy .$$

$$=\left[\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}}\cdot\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}} - \int \frac{x_1^2+y^2}{x_1^2+y^2+z_1^2}\cdot\frac{z_1(-\frac{1}{2})2y}{(x_1^2+y^2)^{3/2}}\cdot\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\,dy\right]_{-y_1}^{b-y_2},$$

$$=\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}} + \int\frac{z_1^2y^2}{x_1^2(x_1^2+y^2)(x_1^2+y^2+z_1^2)}\,dy.\right]_{-y_1}^{b-y_2},$$

$$=\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}} + \int \frac{x_1^2+z_1^2}{x_1^2(x_1^2+y^2+z_1^2)}\,dy - \int\frac{dy}{x_1^2+y^2}\right]_{-y_1}^{b-y_2},$$

$$=\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}} + \frac{x_1^2+z_1^2}{x_1^2}\,\frac{1}{\sqrt{x_1^2+z_1^2}}\tan^{-1}\frac{y}{\sqrt{x_1^2+z_1^2}} - \int\frac{dy}{x_1^2+y^2}\right]_{-y_1}^{b-y_2} \text{.....(3)}$$

$$\int_0^b \frac{z_1}{(x_1^2+y^2)^{3/2}}\tan^{-1}\frac{a+ky_2-z_1}{\sqrt{x_1^2+y^2}}\,dy_2$$

$$=\int_{-y_1}^{b-y_1}\frac{z_1}{(x_1^2+y^2)^{3/2}}\tan^{-1}\frac{ky+a+ky_1-z_1}{\sqrt{x_1^2+y^2}}\,dy ,$$

$$=\left[\tan^{-1}\frac{ky+a+ky_1-z_1}{\sqrt{x_1^2+y^2}}\cdot\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\right.$$

$$\left. -\int\frac{1}{1+\frac{(ky+a+ky_1-z_1)^2}{x_1^2+y^2}}\cdot\frac{\sqrt{x_1^2+y^2}(k)-\frac{y}{\sqrt{x_1^2+y^2}}(ky+a+ky_1-z_1)}{(x_1^2+y^2)}\,\frac{z_1 y\,dy}{x_1^2\sqrt{x_1^2+y^2}}\right]_{-1}^{b-}$$

$$=\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{ky+a+ky_1-z_1}{\sqrt{x_1^2+y^2}}-\int\frac{z_1 y[x_1^2k+y^2k-(ky+a+ky_1-z_1)y]}{x_1^2(x_1^2+y^2)[x_1^2+y^2+(ky+a+ky_1-z_1)^2]}dy\right]_{-y_1}^{b-y_1} \quad \text{------------(4)}$$

Substituting equations (3) and (4) in (2), we get

$$\frac{2\pi F dA_1-A_2}{x_1\sin\phi}=\int_{-y_1}^{b-y_1}\frac{(a+ky+ky_1)z_1-(x_1^2+y^2+z_1^2)}{(x_1^2+y^2)[x_1^2+y^2+(a+ky+ky_1-z_1)^2]}dy+\int_{-y_1}^{b-y_1}\frac{dy}{x_1^2+y^2}$$

$$+\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{ky+a+ky_1-z_1}{\sqrt{x_1^2+y^2}}-\int\frac{z_1 y[x_1^2k-ay-ky_1y-z_1y]\,dy}{x_1^2(x_1^2+y^2)[x_1^2+y^2+(ky+a+ky_1-z_1)^2]}\right]_{-y_1}^{b-y_1}$$

$$+\left[\frac{z_1 y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}}+\frac{\sqrt{x_1^2+z_1^2}}{x_1^2}\tan^{-1}\frac{y}{\sqrt{x_1^2+z_1^2}}-\int\frac{dy}{x_1^2+y^2}\right]_{-y_1}^{b-y_1} \quad \text{...(5)}$$

The integrals on the right hand side of equation (5) are combined and integrated as follows:

$$\int_{-y_1}^{b-y_1}\frac{(az_1+kyz_1+ky_1z_1-x_1^2-y^2-z_1^2)x_1^2-z_1y(x_1^2k-ay-ky_1y-z_1y)}{x_1^2(x_1^2+y^2)[x_1^2+y^2+(ky+a+ky_1-z_1)^2]}dy$$

$$=\int_{-y_1}^{b-y_1}\frac{x_1^2(az_1+ky_1z_1-x_1^2-z_1^2)+y^2(az_1-z_1^2-x_1^2+ky_1z_1)}{x_1^2(x_1^2+y^2)[x_1^2+y^2+(ky+a+ky_1-z_1)^2]}dy\,,$$

$$=\frac{(az_1+ky_1z_1-z_1^2-x_1^2)}{x_1^2}\int_{-y_1}^{b-y_1}\frac{dy}{[x_1^2+y^2+k^2y^2+2ky(a+ky_1-z_1)+(a+ky_1-z_1)^2]}\,,$$

$$=\frac{(az_1+ky_1z_1-z_1^2-x_1^2)}{x_1^2}\int_{-y_1}^{b-y_1}\frac{dy}{y^2(1+k^2)+y(2ka+2k^2y_1-2kz_1)+[x_1^2+(a+ky_1-z_1)^2]} \quad (2_1)$$

$$= \frac{(az_1+ky_1z_1-z_1^2-x_1^2)}{x_1^2}\left[\frac{2}{2\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\tan^{-1}\frac{2(1+k^2)y+2k(a+ky_1-z_1)}{2\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\right]_{-y_1}^{b-y_1}$$

$$= \frac{az_1+ky_1z_1-z_1^2-x_1^2}{x_1^2\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\left[\tan^{-1}\frac{(1+k^2)y+(a+ky_1-z_1)k}{\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\right]_{-y_1}^{b-y_1}.$$

Substituting the above result in equation (5), it becomes

$$\frac{2\pi F_{dA_1-A_2}}{x_1\sin\phi} = \frac{az_1+ky_1z_1-z_1^2-x_1^2}{x_1^2\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\left[\tan^{-1}\frac{(1+k^2)y+(a+ky_1-z_1)k}{\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\right]_{-y_1}^{b-y_1}$$

$$+\left[\frac{z_1y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{ky+a+ky_1-z_1}{\sqrt{x_1^2+y^2}}\right]_{-y_1}^{b-y_1}$$

$$+\left[\frac{z_1y}{x_1^2\sqrt{x_1^2+y^2}}\tan^{-1}\frac{z_1}{\sqrt{x_1^2+y^2}}+\frac{\sqrt{x_1^2+z_1^2}}{x_1^2}\tan^{-1}\frac{y}{\sqrt{x_1^2+z_1^2}}\right]_{-y_1}^{b-y_1}.$$

Recalling from figure 1(a) that $x_1 = c\sin\phi$ and $z_1 = c\cos\phi$, we have

$$2\pi F_{dA_1-A_2} = \frac{a\cos\phi+ky_1\cos\phi-c}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\left[\tan^{-1}\frac{(1+k^2)y+(a+ky_1-c\cos\phi)k}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\right]_{-y_1}^{b-y_1}$$

$$+\left[\frac{y\cos\phi}{\sqrt{c^2\sin^2\phi+y^2}}\tan^{-1}\frac{ky+a+ky_1-c\cos\phi}{\sqrt{c^2\sin^2\phi+y^2}}\right]_{-y_1}^{b-y_1}$$

$$+\left[\frac{y\cos\phi}{\sqrt{c^2\sin^2\phi+y^2}}\tan^{-1}\frac{c\cos\phi}{\sqrt{c^2\sin^2\phi+y^2}}+\tan^{-1}\frac{y}{c}\right]_{-y_1}^{b-y_1},$$

$$= \frac{a\cos\phi+ky_1\cos\phi-c}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\left[\tan^{-1}\frac{(1+k^2)(b-y_1)+(a+ky_1-c\cos\phi)k}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\right.$$

$$\left.-\tan^{-1}\frac{(1+k^2)(-y_1)+(a+ky_1-c\cos\phi)k}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\right]$$

$$+\frac{(b-y_1)\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\tan^{-1}\frac{k(b-y_1)+a+ky_1-c\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}$$

$$-\frac{-y_1\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\tan^{-1}\frac{k(-y_1)+a+ky_1-c\cos\phi}{\sqrt{c^2\sin^2\phi+(y_1)^2}}$$

$$+\frac{(b-y_1)\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\tan^{-1}\frac{c\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}$$

$$-\frac{-y_1\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\tan^{-1}\frac{c\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}+\tan^{-1}\frac{b-y_1}{c}-\tan^{-1}\frac{-y_1}{c}\ .$$

Simplifying the above expression, we get

$$2\pi F_{dA_1-A_2}=\frac{a\cos\phi+ky_1\cos\phi-c}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\left[\tan^{-1}\frac{b-y_1+k^2+k(a-c\cos\phi)}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\right.$$

$$\left.+\tan^{-1}\frac{y_1-k(a-c\cos\phi)}{\sqrt{(1+k^2)c^2\sin^2\phi+(a+ky_1-c\cos\phi)^2}}\right]$$

$$+\frac{(b-y_1)\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\left[\tan^{-1}\frac{kb+a-c\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}+\tan^{-1}\frac{c\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\right]$$

$$+\frac{y_1\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\left[\tan^{-1}\frac{a-c\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}+\tan^{-1}\frac{c\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\right]$$

$$+\tan^{-1}\frac{b-y_1}{c}+\tan^{-1}\frac{y_1}{c}\ . \qquad \text{-----------------(6)}$$

In order to express the factor in a dimensionless form, let us define the ratios

$$N=\frac{a}{b}\ ,$$

$$L = \frac{c}{b} ,$$

$$M = \frac{y_1}{b} ;$$

and let

$$A = \sqrt{(1+k^2)L^2\sin^2\phi + (N+kM-L\cos\phi)^2} ,$$

$$B = \sqrt{L^2\sin^2\phi + (1-M)^2} ,$$

$$D = \sqrt{L^2\sin^2\phi + M^2} .$$

Utilizing these, we get

$$F_{dA_1-A_2} = \frac{N\cos\phi + kM\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{1-M+k^2+k(N-L\cos\phi)}{A} + \tan^{-1}\frac{M-k(N-L\cos\phi)}{A}\right]$$

$$+ \frac{(1-M)\cos\phi}{2\pi B}\left[\tan^{-1}\frac{k+N-L\cos\phi}{B} + \tan^{-1}\frac{L\cos\phi}{B}\right]$$

$$+ \frac{M\cos\phi}{2\pi D}\left[\tan^{-1}\frac{N-L\cos\phi}{D} + \tan^{-1}\frac{L\cos\phi}{D}\right]$$

$$+ \frac{1}{2\pi}\left[\tan^{-1}\frac{1-M}{L} + \tan^{-1}\frac{M}{L}\right] . \qquad \text{------------}(6')$$

The factor for the same configuration would now be evaluated using contour integral method (reference 4) which simplifies the procedure considerably since only a single integration along the contour of the trapezoid is involved in this case.

The limitation, that each element of either of the objects should "see" each element of the other, as applicable to the quadruple integral method applies to this method as well. This method becomes handier when the surfaces involved are not plane.

The contour integral representation for the configuration factor between an infinitesimal area dA_1 and a finite area A_2 (reference 4) is given by

$$F_{dA_1-A_2} = l_1 \oint_C \frac{(z_2-z_1)dy_2 - (y_2-y_1)dz_2}{2\pi r^2} + m_1 \oint_C \frac{(x_2-x_1)dz_2 - (z_2-z_1)dx_2}{2\pi r^2}$$

$$+ n_1 \oint_C \frac{(y_2-y_1)dx_2 - (x_2-x_1)dy_2}{2\pi r^2} \; . \qquad \text{------------}(7)$$

In our case, with reference to figure 1(a), we have

$$l_1 = -\cos\phi \, ,$$

$$m_1 = 0 \, ,$$

$$n_1 = \sin\phi \, ,$$

$$x_1 = c\sin\phi \, ,$$

$$z_1 = c\cos\phi \, ,$$

$$x_2 = 0 \qquad , \quad dx_2 = 0 \, .$$

Substituting the values for m_1, x_2 and dx_2, equation (7) reduces to

$$2\pi F_{dA_1-A_2} = l_1 \oint_C \frac{(z_2-z_1)dy_2 - (y_2-y_1)dz_2}{r^2} + n_1 \oint_C \frac{x_1 dy_2}{r^2} \quad \text{....}(8)$$

The contour C is composed of four parts as shown in figure 2.

On I :

$$z_2 = 0 \, , \; dz_2 = 0 \, ,$$

$$0 \leq y_2 \leq b \, .$$

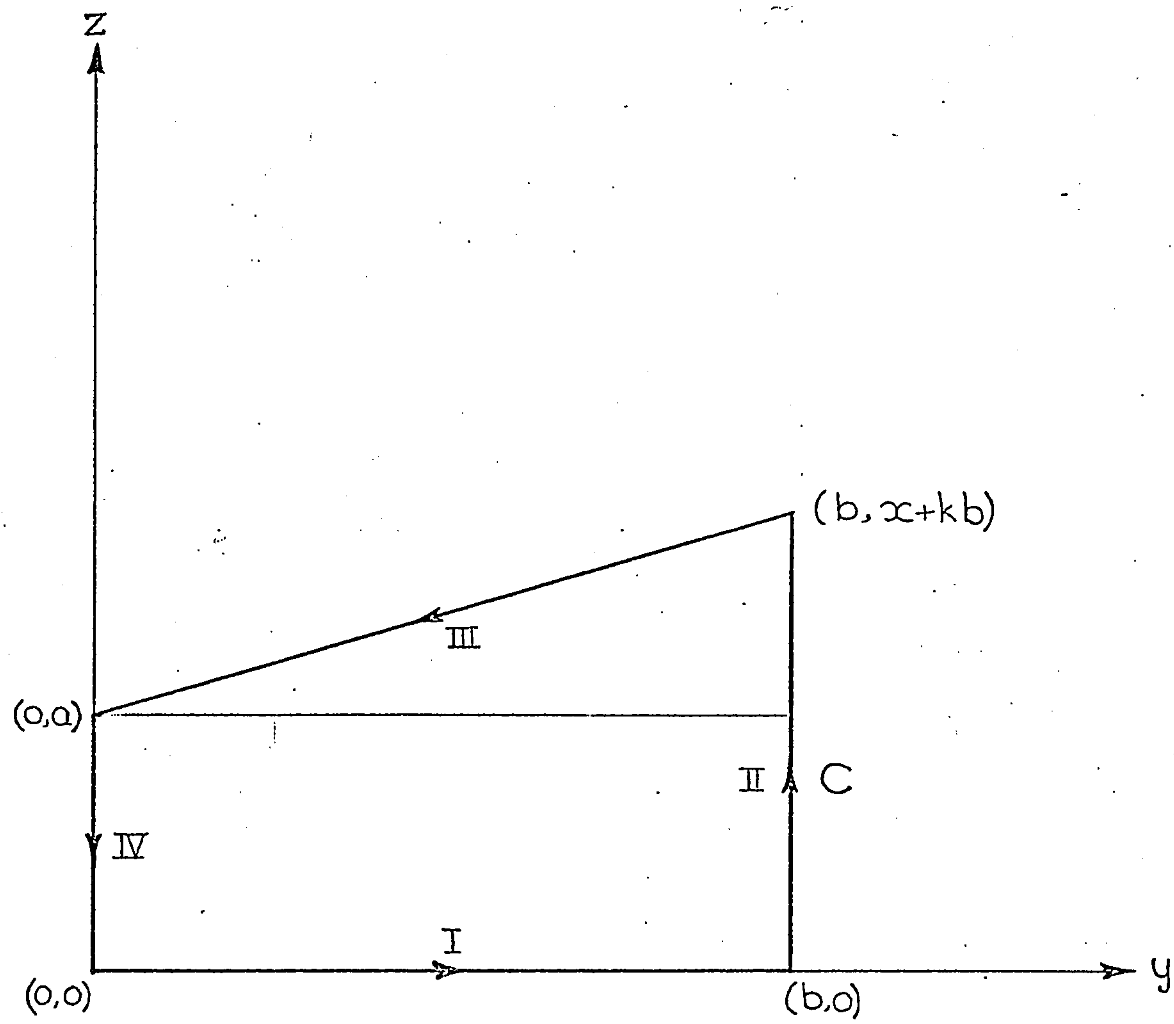


FIGURE 2

On II : $y_2 = b$, $dy_2 = 0$;

$0 \leq z_2 \leq a + kb$.

On III : $z_2 = a + ky_2$, $y_2 = \frac{1}{k}(z_2 - a)$,

$0 \leq y_2 \leq b$, $a \leq z_2 \leq a + kb$.

On IV : $y_2 = 0$, $dy_2 = 0$,

$0 \leq z_2 \leq a$.

Writing proper limits of integration, equation (8) becomes

$$2\pi F_{dA_1-A_2} = l_1 \int_0^b \frac{-z_1\,dy_2}{x_1^2+(y_2-y_1)^2+z_1^2} + n_1 \int_0^b \frac{x_1\,dy_2}{x_1^2+(y_2-y_1)^2+z_1^2}$$

$$+ l_1 \int_0^{a+kb} \frac{-(b-y_1)\,dz_2}{x_1^2+(b-y_1)^2+(z_2-z_1)^2} + l_1 \int_b^0 \frac{(a+ky_2-z_1)\,dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2}$$

$$+ l_1 \int_{a+kb}^{a} \frac{-[\frac{1}{k}(z_2-a)-y_1]\,dz_2}{x_1^2+[\frac{1}{k}(z_2-a)-y_1]^2+(z_2-z_1)^2} + n_1 \int_b^0 \frac{x_1\,dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2}$$

$$+ l_1 \int_a^0 \frac{y_1\,dz_2}{x_1^2+y_1^2+(z_2-z_1)^2},$$

$$= (-l_1 z_1 + n_1 x_1) \int_0^b \frac{dy_2}{x_1^2+(y_2-y_1)^2+z_1^2} - (b-y_1)\, l_1 \int_0^{a+kb} \frac{dz_2}{x_1^2+(b-y_1)^2+(z_2-z_1)^2}$$

$$+ (l_1 a - l_1 z_1 + n_1 x_1) \int_b^0 \frac{dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2} + l_1 k \int_b^0 \frac{y_2\,dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2}$$

$$+ \left(l_1 \frac{a}{k} + l_1 y_1\right) \int_{a+kb}^{a} \frac{dz_2}{x_1^2+[\frac{1}{k}(z_2-a)-y_1]^2+(z_2-z_1)^2} - \frac{l_1}{k} \int_{a+kb}^{a} \frac{z_2\,dz_2}{x_1^2+[\frac{1}{k}(z_2-a)-y_1]^2+(z_2-z_1)^2}$$

$$+L_1 y_1 \int_a^0 \frac{dz_2}{x_1^2+y_1^2+(z_2-z_1)^2} . \qquad \text{---------------(9)}$$

Solving the integrals of equation (9) one by one we get

$$\int_0^b \frac{dy_2}{x_1^2+(y_2-y_1)^2+z_1^2} = \left[\frac{1}{\sqrt{x_1^2+z_1^2}} \tan^{-1}\frac{y_2-y_1}{\sqrt{x_1^2+z_1^2}}\right]_0^b .$$

$$\int_0^{a+kb} \frac{dz_2}{x_1^2+(b-y_1)^2+(z_2-z_1)^2} = \left[\frac{1}{\sqrt{x_1^2+(b-y_1)^2}} \tan^{-1}\frac{z_2-z_1}{\sqrt{x_1^2+(b-y_1)^2}}\right]_0^{a+kb} .$$

$$\int_b^0 \frac{y_2\,dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2} = \int_b^0 \frac{y_2\,dy_2}{y_2^2(1+k^2)+y_2(2y_1+2ka-2kz_1)+(x_1^2+y_1^2+(a-z_1)^2)} ,$$

$$= \left[\frac{1}{2(1+k^2)} \ln\{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2\}\right.$$

$$\left. + \frac{y_1-ka+kz_1}{1+k^2} \int \frac{dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2}\right]_b^0 .$$

$$\int_b^0 \frac{dy_2}{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2} = \int_b^0 \frac{dy_2 \quad (3)}{y_2^2(1+k^2)+y_2(-2y_1+2ka-2kz_1)+\{x_1^2+y_1^2+(a-z_1)^2\}} ,$$

$$= \left[\frac{1}{\sqrt{x_1^2(1+k^2)+(a+ky_2-z_1)^2}} \tan^{-1}\frac{(1+k^2)y_2+(ka-kz_1-y_1)}{\sqrt{x_1^2(1+k^2)+(a+ky_2-z_1)^2}}\right]_b^0 .$$

$$\int_{a+kb}^{a} \frac{z_2\,dz_2}{x_1^2+[\frac{1}{k}(z_2-a)-y_1]^2+(z_2-z_1)^2} = \int_{a+kb}^{a} \frac{k^2 z_2\,dz_2}{z_2^2(1+k^2)-z_2\{2k^2z_1+2(a+ky_1)\}+\{k^2x_1^2+k^2z_1^2+(a+ky_1)^2\}} ,$$

$$= \left[\frac{k^2}{2(1+k^2)} \ln\{k^2x_1^2+(z_2-a-ky_1)^2+k^2(z_2-z_1)^2\}\right.$$

$$\left. + \frac{k^2(k^2z_1+a+ky_1)}{1+k^2} \int \frac{dz_2}{k^2x_1^2+(z_2-a-ky_1)^2+k^2(z_2-z_1)^2}\right]^a .$$

$$\int_{a+kb}^{a} \frac{dz_2}{x_1^2+\left[\frac{1}{k}(z_2-a)-y_1\right]^2+(z_2-z_1)^2} = \int_{a+kb}^{a} \frac{k^2\,dz_2}{z_2^2(1+k^2)-z_2\{2k^2z_1+2(a+ky_1)\}+\{k^2x_1^2+k^2z_1^2+(a+ky_1)^2\}} \quad (4)$$

$$= \left[\frac{k^2}{\sqrt{x_1^2k(1+k^2)+k^2(z_1-a-ky_1)^2}}\tan^{-1}\frac{(1+k^2)z_2-(a+ky_1+k^2z_1)}{\sqrt{x_1^2k^2(1+k^2)+k^2(z_1-a-ky_1)^2}}\right]_{a+kb}^{a}$$

$$\int_{a}^{0} \frac{dz_2}{x_1^2+y_1^2+(z_2-z_1)^2} = \left[\frac{1}{\sqrt{x_1^2+y_1^2}}\tan^{-1}\frac{z_2-z_1}{\sqrt{x_1^2+y_1^2}}\right]_a^0 .$$

and now substituting these results in equation (9) we have

$$2\pi F_{dA_1-A_2} = \frac{(-l_1z_1+n_1x_1)}{\sqrt{x_1^2+z_1^2}}\left[\tan^{-1}\frac{y_2-y_1}{\sqrt{x_1^2+z_1^2}}\right]_0^b - \frac{l_1(b-y_1)}{\sqrt{x_1^2+(b-y_1)^2}}\left[\tan^{-1}\frac{z_2-z_1}{\sqrt{x_1^2+(b-y_1)^2}}\right]_0^{a+kb}$$

$$- \frac{(l_1a-l_1z_1+n_1x_1)+l_1k\frac{y_1-ka+kz_1}{(1+k^2)}}{\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\left[\tan^{-1}\frac{(1+k^2)y_2+(ka-kz_1-y_1)}{\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2}}\right]_0^b$$

$$- \frac{l_1k}{2(1+k^2)}\left[\ln\{x_1^2+(y_2-y_1)^2+(a+ky_2-z_1)^2\}\right]_0^b$$

$$- \frac{k^2(l_1\frac{a}{k}+l_1y_1)-\frac{l_1}{k}k^2\frac{k^2z_1+a+ky_1}{1+k^2}}{\sqrt{x_1^2k^2(1+k^2)+k^2(z_1-a-ky_1)^2}}\left[\tan^{-1}\frac{(1+k^2)z_2-(a+ky_1+k^2z_1)}{\sqrt{x_1^2k^2(1+k^2)+k^2(z_1-a-ky_1)^2}}\right]_a^{a+kb}$$

$$+ \frac{l_1}{k}\frac{k^2}{2(1+k^2)}\left[\ln\{k^2x_1^2+(z_2-a+ky_1)^2+k^2(z_2-z_1)^2\}\right]_a^{a+kb}$$

$$- \frac{l_1y_1}{\sqrt{x_1^2+y_1^2}}\left[\tan^{-1}\frac{z_2-z_1}{\sqrt{x_1^2+y_1^2}}\right]_0^a .$$

Substituting the values of l_1, n_1, x_1 and z_1, we have

$$2\pi F_{dA_1-A_2} = \frac{c\cos^2\phi+c\sin^2\phi}{\sqrt{c^2\sin^2\phi+c^2\cos^2\phi}}\left[\tan^{-1}\frac{y_2-y_1}{\sqrt{c^2\sin^2\phi+c^2\cos^2\phi}}\right]_0^b$$

$$+\frac{\cos\phi\cdot(b-y_1)}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\left[\tan^{-1}\frac{z_2-c\cos\phi}{\sqrt{c^2\sin^2\phi+(b-y_1)^2}}\right]_0^{a+kb}$$

$$-\frac{-a\cos\phi+c\cos^2\phi+c\sin^2\phi+k\cos\phi\,\dfrac{ak-kc\cos\phi-y_1}{1+k^2}}{\sqrt{c^2\sin^2\phi(1+k^2)+(a+ky_1-c\cos\phi)^2}}$$

$$\left[\tan^{-1}\frac{(1+k^2)y_2+(ka-kc\cos\phi-y_1)}{\sqrt{c^2\sin^2\phi(1+k^2)+(a+ky_1-c\cos\phi)^2}}\right]_0^b$$

$$+\frac{k\cos\phi}{2(1+k^2)}\left[\ln\frac{c^2\sin^2\phi+(b-y_1)^2+(a+kb-c\cos\phi)^2}{c^2\sin^2\phi+\;y_1^2+(a-c\cos\phi)^2}\right]$$

$$+\frac{\cos\phi}{1+k^2}\;\frac{ak^2+y_1k^3-ck^2\cos\phi}{\sqrt{c^2\sin^2\phi(1+k^2)+(c\cos\phi-a-ky_1)^2}}$$

$$\left[\tan^{-1}\frac{(1+k^2)z_2-(a+ky_1+k^2c\cos\phi)}{\sqrt{k^2c^2\sin^2\phi(1+k^2)+k^2(c\cos\phi-a-ky_1)^2}}\right]_a^{a+kb}$$

$$-\frac{k\cos\phi}{2(1+k^2)}\left[\ln\frac{k^2c^2\sin^2\phi+(a+kb-a+ky_1)^2+k^2(a+kb-c\cos\phi)^2}{k^2c^2\sin^2\phi+k^2y_1^2+k^2(a-c\cos\phi)^2}\right]$$

$$+\frac{y_1\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\left[\tan^{-1}\frac{z_2-c\cos\phi}{\sqrt{c^2\sin^2\phi+y_1^2}}\right]_0^a.$$

In the foregoing expression the fourth and the sixth members are equal in their absolute values, but have opposite signs, and, therefore, they cancel out.

In order to express the factor in a dimensionless form, let us define the ratios as before, as

$$N=\frac{a}{b},$$

$$L=\frac{c}{b},$$

$$M = \frac{y_1}{b} .$$

Thus,

$$2\pi F_{dA_1-A_2} = \left[\tan^{-1}\frac{\frac{y_2}{b}-M}{L}\right]_0^b + \frac{(1-M)\cos\phi}{\sqrt{L^2\sin^2\phi+(1-M)^2}}\left[\tan^{-1}\frac{\frac{z_2}{b}-L\cos\phi}{\sqrt{L^2\sin^2\phi+(1-M)^2}}\right]_0^{a+kb}$$

$$-\frac{L+Lk^2\sin^2\phi-N\cos\phi-kM\cos\phi}{(1+k^2)\sqrt{L^2\sin^2\phi(1+k^2)+(N+kM-L\cos\phi)^2}}$$

$$\left[\tan^{-1}\frac{(1+k^2)\frac{y_2}{b}+(kN-kL\cos\phi-M)}{\sqrt{L^2\sin^2\phi\,(1+k^2)+(N+kM-L\cos\phi)^2}}\right]_0^b$$

$$+\frac{\cos\phi\cdot k^2}{1+k^2}\,\frac{N+Mk-L\cos\phi}{\sqrt{L^2\sin^2\phi\cdot(1+k^2)+(L\cos\phi-N-kM)^2}}$$

$$\left[\tan^{-1}\frac{(1+k^2)\frac{z_2}{b}-(N+kM+k^2L\cos\phi)}{\sqrt{L^2\sin^2\phi\,(1+k^2)+(L\cos\phi-N-kM)^2}}\right]_a^{a+kb}$$

$$+\frac{M\cos\phi}{\sqrt{L^2\sin^2\phi+M^2}}\left[\tan^{-1}\frac{\frac{z_2}{b}-L\cos\phi}{\sqrt{L^2\sin^2\phi+M^2}}\right]_0^a .$$

Let

$$A = \sqrt{L^2\sin^2\phi\,(1+k^2)+(N+kM-L\cos\phi)^2} ,$$

$$B = \sqrt{L^2\sin^2\phi + (1-M)^2},$$

$$D = \sqrt{L^2\sin^2\phi + M^2} .$$

Substituting for A, B, D and putting the limits, we get

$$2\pi F_{dA_1-A_2} = \left[\tan^{-1}\frac{1-M}{L} + \tan^{-1}\frac{M}{L}\right]$$

$$+\frac{(1-M)\cos\phi}{B}\left[\tan^{-1}\frac{N+k-L\cos\phi}{B}+\tan^{-1}\frac{L\cos\phi}{B}\right]$$

$$- \frac{L + Lk^2\sin^2\phi - N\cos\phi - kM\cos\phi}{(1+k^2)A}\left[\tan^{-1}\frac{1+k^2+kN-kL\cos\phi-M}{A}\right.$$

$$\left. - \tan^{-1}\frac{kN-kL\cos\phi-M}{A}\right]$$

$$+ k^2\cos\phi\,\frac{N+Mk-L\cos\phi}{(1+k^2)A}\left[\tan^{-1}\frac{(1+k^2)(N+k)-(N+kM+k^2L\cos\phi)}{kA}\right.$$

$$\left. - \tan^{-1}\frac{(1+k^2)N-(N+kM+k^2L\cos\phi)}{kA}\right] \qquad (5)$$

$$+ \frac{M\cos\phi}{D}\left[\tan^{-1}\frac{N-L\cos\phi}{D} + \tan^{-1}\frac{L\cos\phi}{D}\right].$$

Simplying, rearranging and adding the third and the fourth members of the above expression, we have

$$F_{dA_1-A_2} = \frac{N\cos\phi + kM\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{1-M+k^2+k(N-L\cos\phi)}{A}\right.$$

$$\left. + \tan^{-1}\frac{M-k(N-L\cos\phi)}{A}\right]$$

$$+ \frac{(1-M)\cos\phi}{2\pi B}\left[\tan^{-1}\frac{k+N-L\cos\phi}{B} + \tan^{-1}\frac{L\cos\phi}{B}\right]$$

$$+ \frac{M\cos\phi}{2\pi D}\left[\tan^{-1}\frac{N-L\cos\phi}{D} + \tan^{-1}\frac{L\cos\phi}{D}\right]$$

$$+ \frac{1}{2\pi}\left[\tan^{-1}\frac{1-M}{L} + \tan^{-1}\frac{M}{L}\right].$$

This is same as equation (6').

The two special cases of this configuration are:

(1) the infinitesimal area in line with the shorter-side-end of the trapezoid,

(2) the infinitesimal area in line with the longer-side-end of the trapezoid.

Case 1:

This is shown in figure 3. For this the factor is obtained by letting $y_1 = 0$ i.e. $M = 0$ in the general expression for $F_{dA_1-A_2}$.

This gives

$$F_{dA_1-A_2} = \frac{N\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{1+k^2+k(N-L\cos\phi)}{A} - \tan^{-1}\frac{k(N-L\cos\phi)}{A}\right]$$

$$+ \frac{\cos\phi}{2\pi B}\left[\tan^{-1}\frac{k+N-L\cos\phi}{B} + \tan^{-1}\frac{L\cos\phi}{B}\right] + \frac{1}{2\pi}\tan^{-1}\frac{1}{L},$$

where

$$A = \sqrt{(1+k^2)L^2\sin^2\phi + (N-L\cos\phi)^2},$$

$$B = \sqrt{L^2\sin^2\phi + 1}.$$

The similar expression is given by Hamiltón and Morgan in reference 1.

Case 2:

This is shown in figure 4, for which the factor is obtained by letting $y_1 = b$ i.e. $M = 1$ in the general expression for $F_{dA_1-A_2}$.

This gives

$$F_{dA_1-A_2} = \frac{N\cos\phi + k\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{k^2+k(N-L\cos\phi)}{A} + \tan^{-1}\frac{1-k(N-L\cos\phi)}{A}\right]$$

$$+ \frac{\cos\phi}{2\pi D}\left[\tan^{-1}\frac{N-L\cos\phi}{D} + \tan^{-1}\frac{L\cos\phi}{D}\right] + \frac{1}{2\pi}\tan^{-1}\frac{1}{L},$$

where

$$A = \sqrt{L^2\sin^2\phi(1+k^2) + (N+k-L\cos\phi)^2},$$

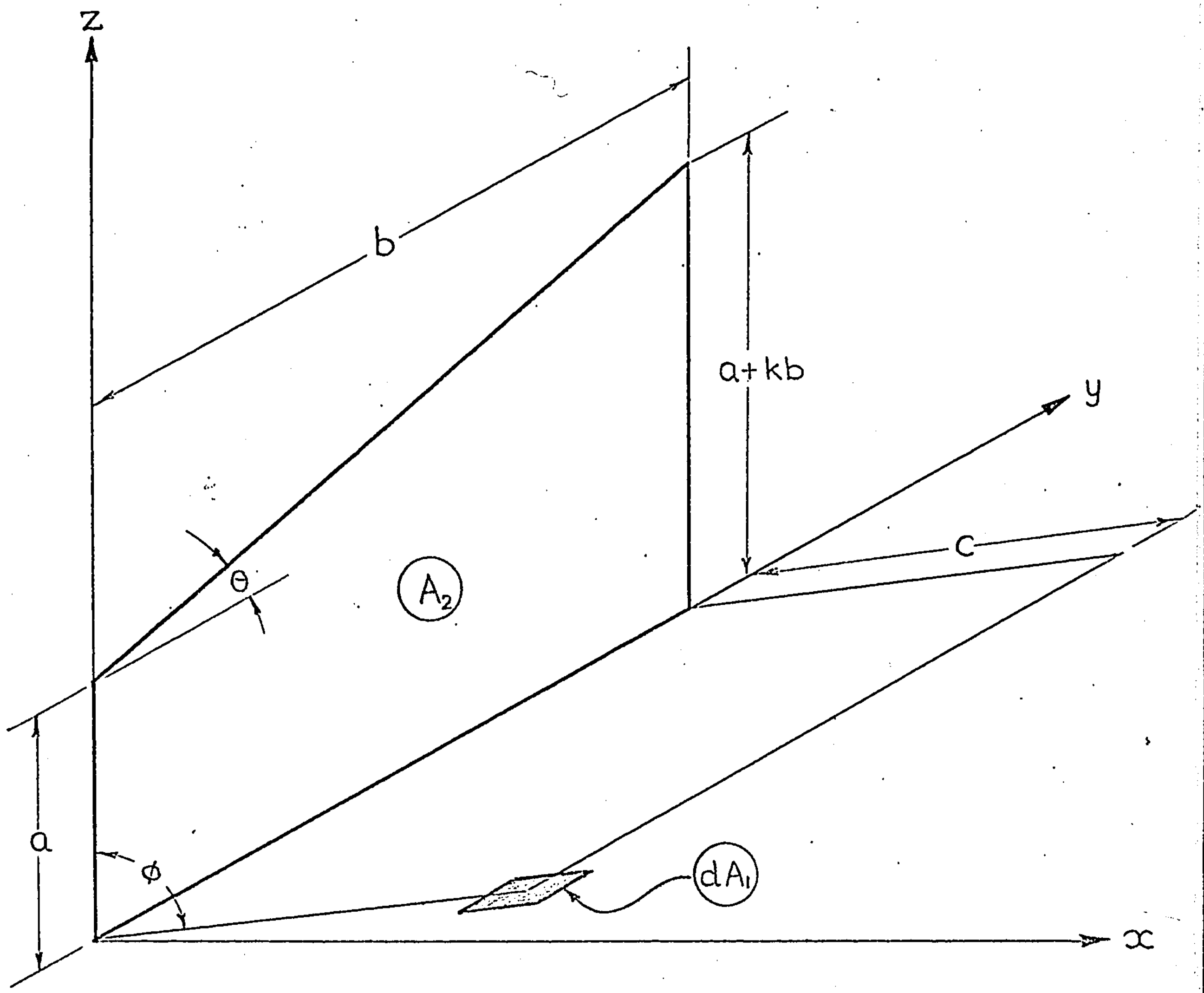


FIGURE 3

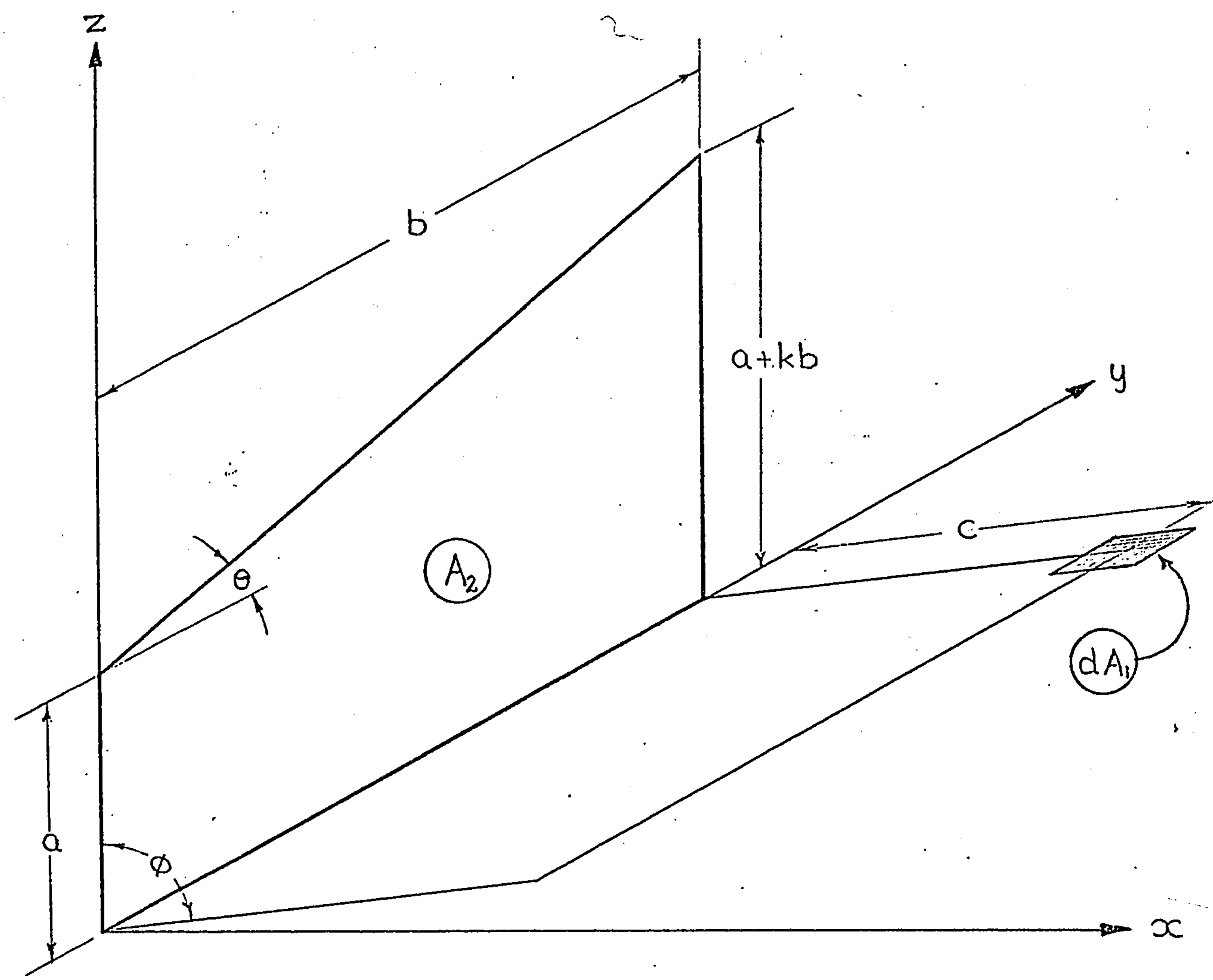


FIGURE 4

$$D = \sqrt{L^2 \sin^2\phi + 1} \quad .$$

Now, if k is negative i.e. $\tan\theta$ is negative, then the configuration shown in figure 5 resembles the configuration P-11 of reference 1. The factor is obtained by letting $y_1 = 0$ i.e. $M_1 = 0$ and replacing k by $-k$ in the general expression for $F_{dA_1-A_2}$.

This gives

$$F_{dA_1-A_2} = \frac{N\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{1+k^2-k(N-L\cos\phi)}{A} + \tan^{-1}\frac{k(N-L\cos\phi)}{A}\right]$$

$$+ \frac{\cos\phi}{2\pi B}\left[\tan^{-1}\frac{N-k-L\cos\phi}{B} + \tan^{-1}\frac{L\cos\phi}{B}\right] + \frac{1}{2\pi}\tan^{-1}\frac{1}{L} ,$$

where

$$A = \sqrt{(1+k^2)L^2\sin^2\phi + (N-L\cos\phi)^2} ,$$

$$B = \sqrt{L^2\sin^2\phi + 1} .$$

The expression for this configuration, given by Hamilton and Morgan is incorrect, which is very obvious from the curves the authors have drawn for $F_{dA_1-A_2}$ in figure 18 of the same reference. The curves indicate increase in $F_{dA_1-A_2}$ with increasing θ, whereas the figure 5 shows that $F_{dA_1-A_2}$ should decrease with increasing θ, because the area of the trapezoid decreases with increasing θ.

This mistake is reproduced in many books, among them is a well-known textbook by Wiebelt (reference 6).

The correct geometry for the expression given in Appendix C of reference 1, configuration P-11, is shown in figure 4.

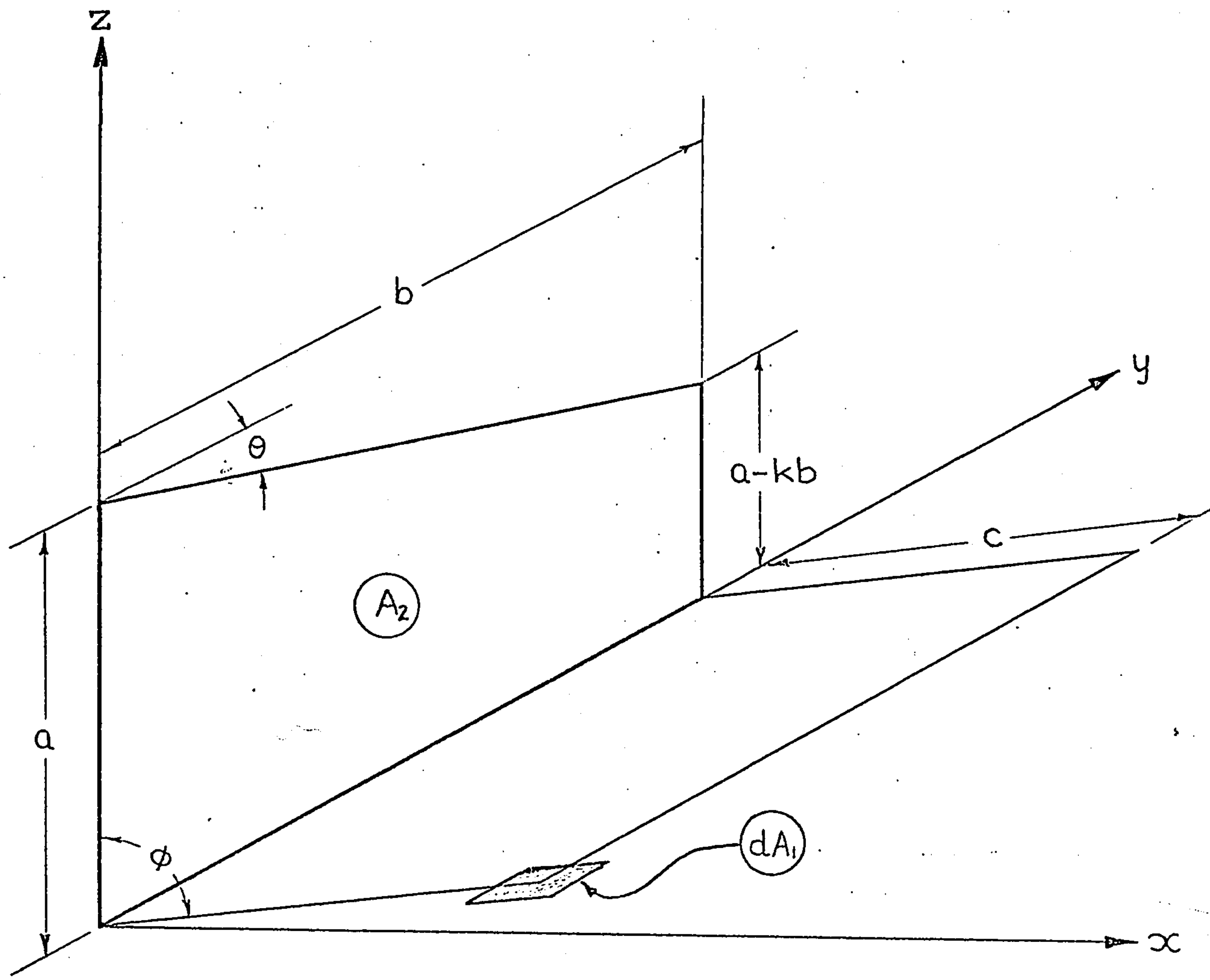


FIGURE 5

Thus, having obtained equation (6) the expression for the factor from an infinitesimal area to a right-angle trapezoid, their planes forming an arbitrary angle, and having discussed two of its specialcases, we shall now proceed to evaluate the factor form an infinitesimal area to any triangle, their planes forming an arbitrary angle.

An infinitesimal area dA_1 and a triangle ABC are shown in figure 6, where, their planes are forming an angle ϕ. The coordinate axes are chosen so that one of the corners (A in this case) of the triangle lies on the z-axis. The coordinates of the points A, B and C and of dA_1 can now be defined with reference to this coordinate system as

$$A\ (0, z_a),\ B\ (y_b, z_b),\ C\ (y_c, z_c),\ dA_1\ (x_1, y_1, z_1).$$

Let us draw a perpendicular CE from point C and BF from point B on the y-axis, intersecting the y-axis at E and F respectively.

Then, the factor from dA_1 to the triangle ABC is obtained by utilising factor algebra as shown below.

$$F_{dA_1-\Delta ABC} = F_{dA_1-ABFO} + F_{dA_1-BCEF} - F_{dA_1-ACEO} \quad \text{.....(10)}$$

The three factors on the right-hand side of equation (10) are the factors from an infinitesimal area dA_1 to the trapezoids, which can be obtained from equation (6') by making appropriate substitutions

F_{dA_1-ABFO} is obtained by comparing the trapezoid ABFO with that shown in figure 1(a) and then substititing as follows.

$$a = z_a$$

$$b = y_b$$

$$c = c$$

$$y_1 = y_1$$

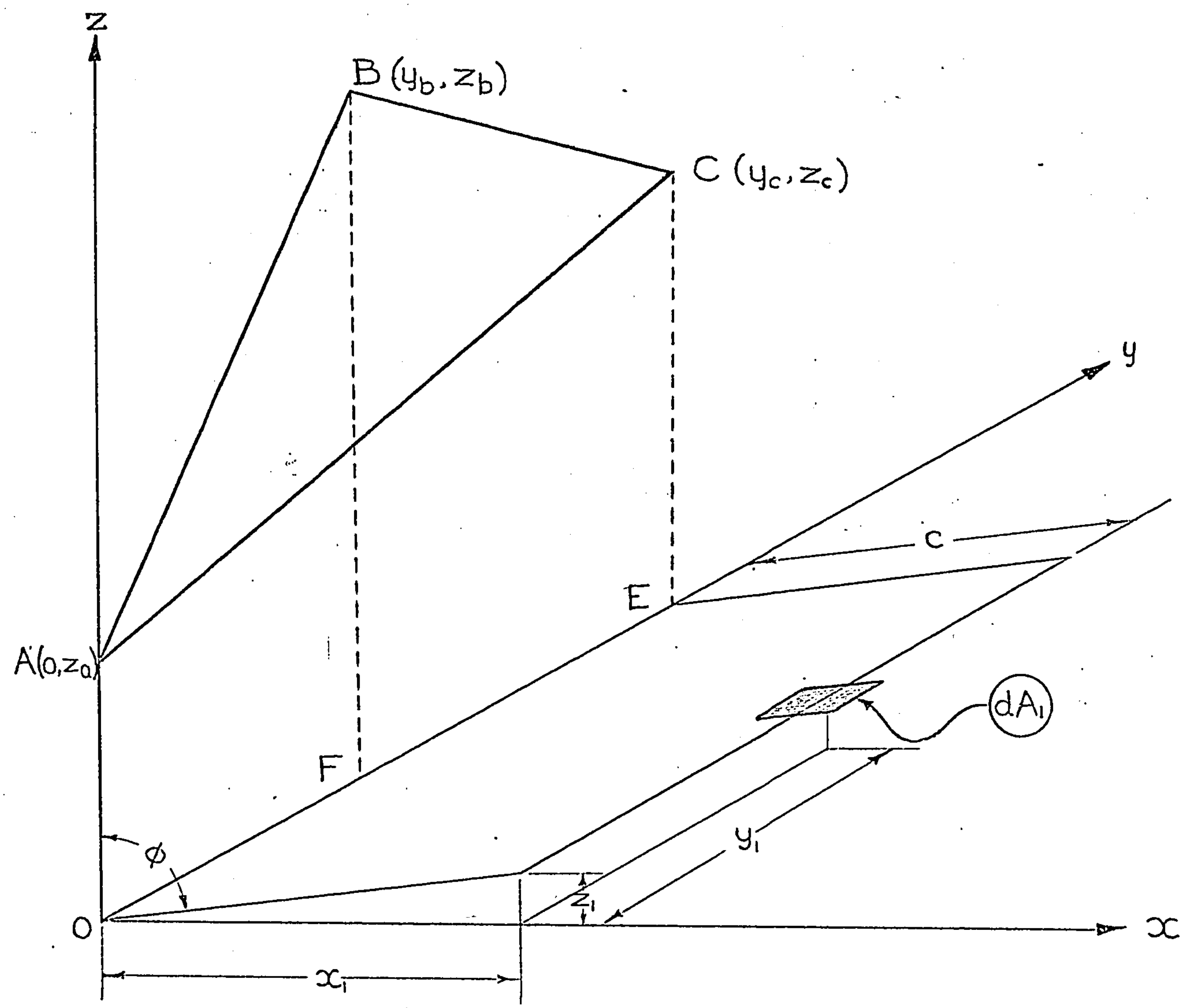


FIGURE 6

$$k \equiv \frac{z_b - z_a}{y_b}$$

$$N \equiv \frac{z_a}{y_b}$$

$$L \equiv \frac{c}{y_b}$$

$$M \equiv \frac{y_1}{y_b} .$$

F_{dA_1-BCEF} is obtained by utilising equation (6') with the following substitutions.

$$a \equiv z_b$$

$$b \equiv y_c - y_b$$

$$c \equiv c$$

$$y_1 \equiv y_1 - y_b$$

$$k \equiv \frac{z_c - z_b}{y_c - y_b}$$

$$N \equiv \frac{z_b}{y_c - y_b}$$

$$L \equiv \frac{c}{y_c - y_b}$$

$$M \equiv \frac{y_1 - y_b}{y_c - y_b} .$$

F_{dA_1-ACEO} is obtained by defining the following equivalent quantities and then using equation (6').

$$a \equiv z_a$$

$$b \equiv y_c$$

$$c \equiv c$$

$$y_1 = y_1$$

$$k = \frac{z_c - z_a}{y_c}$$

$$N = \frac{z_a}{y_c}$$

$$M = \frac{c}{y_c}$$

$$L = \frac{y_1}{y_c} .$$

Now, if the triangle is placed so that B lies below AC, as shown in figure 7, the entire analysis would remain the same except that equation(10) would change to :

$$F_{dA_1-\Delta ABC} = -F_{dA_1-ABFO} - F_{dA_1-BCEF} + F_{dA_1-ACEO} \quad \text{....(11)}$$

Thus, having found $F_{dA_1-\Delta ABC}$, applying the reciprocity rule we can get

$$F_{\Delta ABC-dA_1} = \frac{dA_1}{A_{\Delta ABC}} F_{dA_1-\Delta ABC} \quad \text{---------(12)}$$

Integrating equation (12) with respect to dA_1 over any required plane area A_1, the factor from an arbitrary triangle to that area can be obtained. Using the reciprocity rule, we can now obtain the factor from an arbitrary area A_1 to a triangle. Finally, because any plane area A_2 which is bounded by straight lines can be divided into triangles, we have a method for calculating factors between any such areas A_1 and A_2.

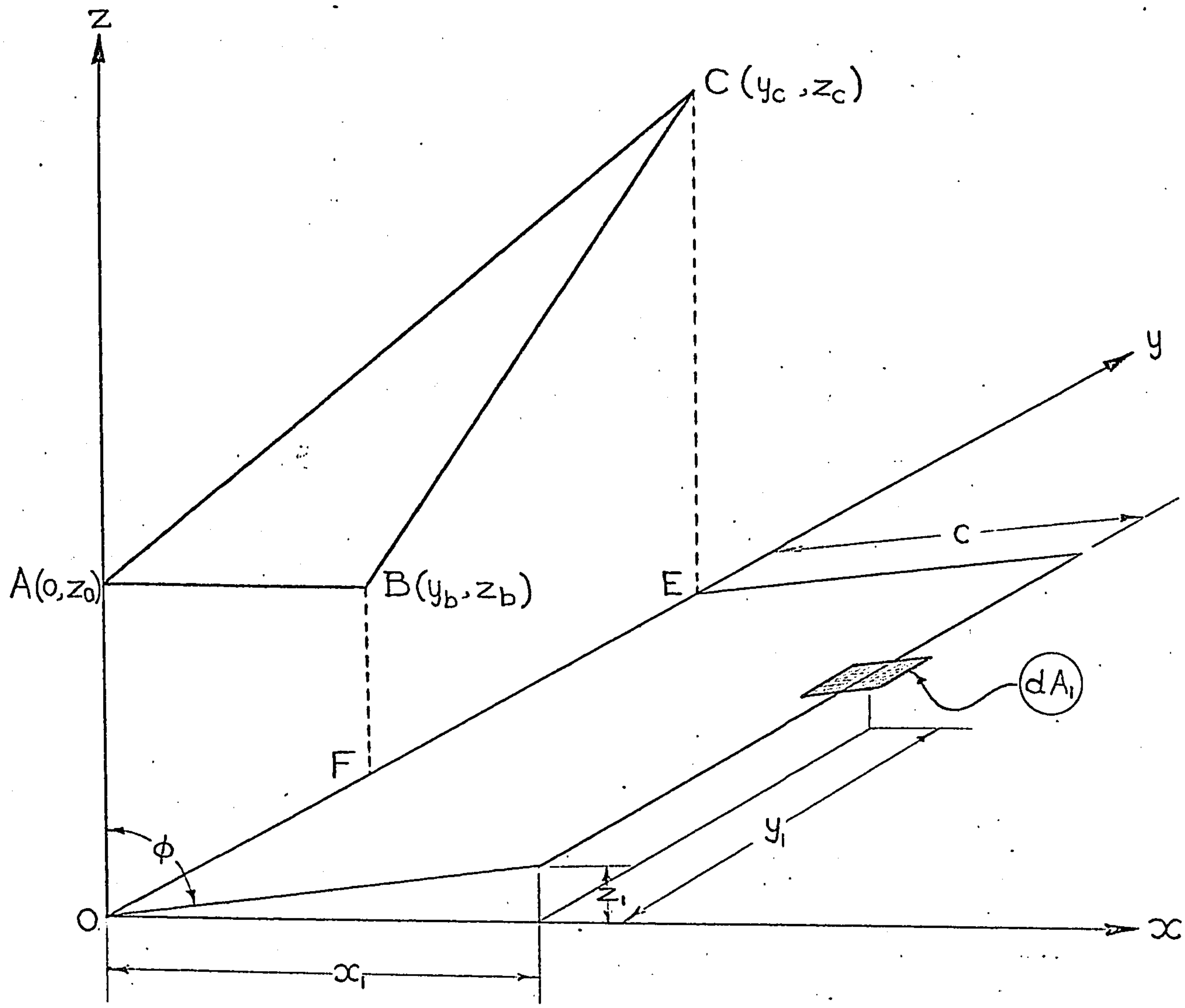


FIGURE 7

III. CONFIGURATION FACTOR FROM A RECTANGLE TO ANOTHER RECTANGLE IN A PARALLEL PLANE, ROTATED THROUGH AN ARBITRARY ANGLE

The closed-form solution for the configuration factor between two identical parallel rectangles facing each other is available in the literature. Factor from a given rectangle to any other rectangle in a parallel plane can be obtained by means of factor algebra, provided, however, that the sides of one rectangle are parallel to the respective sides of the other. If these sides are not parallel, the problem has to be tackled through integration from the first principles.

In this section a particular case is considered. One of the two identical parallel rectangles facing each other is now rotated about one of its corners through an angle θ. There appears to be no possibility of a complete closed-form solution to this problem and the integration in terms of elementary functions leaves us still with one integral to be evaluated numerically.

Figure 8 represents two rectangles measuring $a \times b$, separated by distance c. The rectangle lying in the x-y plane is rotated clockwise through an angle θ.

The factor is independent of the scale. We may, therefore, divide all the dimensions by c, introducing dimensionless quantities $A = a/c$ and $B = b/c$ as has been done in figure 9. The coordinates of the corners of rectangle I, as well as the equations of its sides are also shown in figure 9. The equations of the sides are not valid for $\theta = n\pi/2$, where n is any integer.

The contour-integral representation for the factor from a finite area A_1 to a finite area A_2 is given by equation (19) of reference 4 as follows

$$F_{A_1-A_2} = \frac{1}{2\pi A_1} \oint_{C_1} \oint_{C_2} \left[\ln r \, dx_1 dx_2 + \ln r \, dy_1 dy_2 + \ln r \, dz_1 dz_2 \right] \quad \text{.....(13)}$$

In our case, with reference to figure 9, we have

$$A_1 = A_2 = A \times B,$$

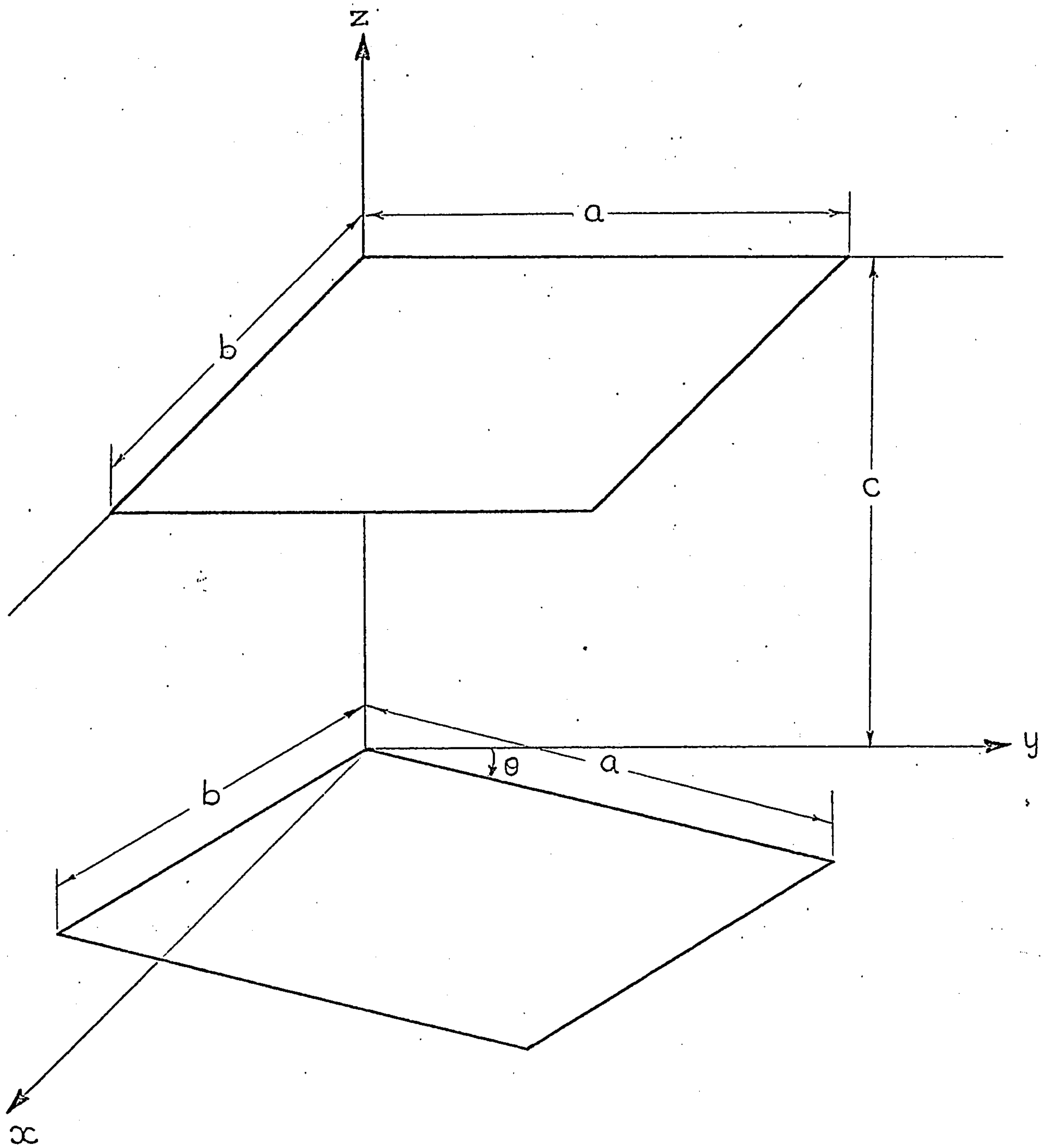


FIGURE 8

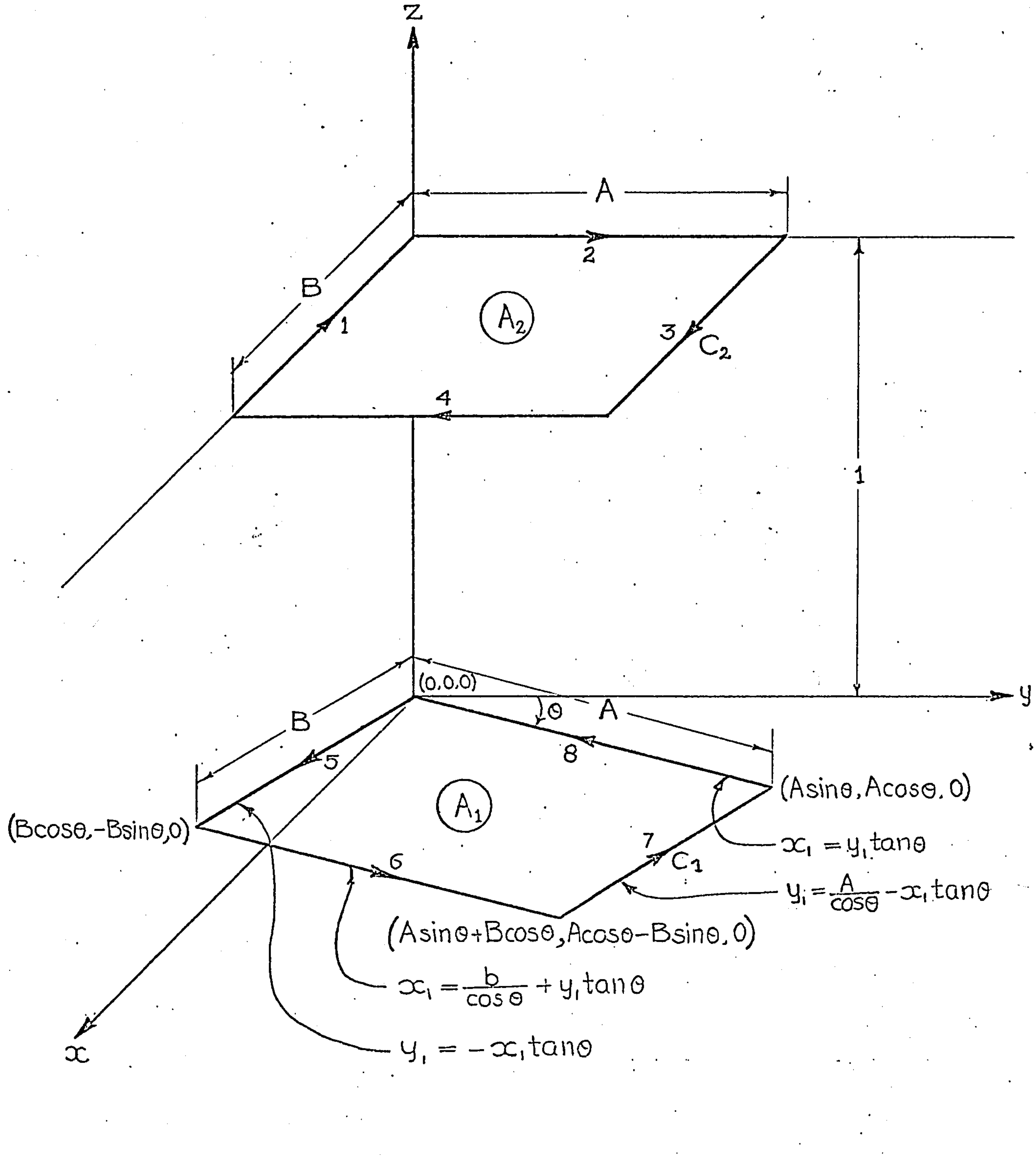

Z
A
B
2
1
3
C2
A2
4
1
(0,0,0)
y
θ
A
B
8
5
(Asinθ, Acosθ, 0)
A1
(Bcosθ,-Bsinθ,0)
7
C1
6
x1 = y1 tanθ
y1 = A/cosθ - x1 tanθ
(Asinθ+Bcosθ, Acosθ-Bsinθ, 0)
x1 = b/cosθ + y1 tanθ
x
y1 = -x1 tanθ

FIGURE 9

r = the distance between the points on the boundaries of A_1 and A_2

$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \;,$$

$$z_2 - z_1 = 1 \;,$$

$$dz_1 = dz_2 = 0 \;.$$

With these, integrating equation (13) over the boundary C_2, we get

$$2\pi A_1 F_{A_1-A_2} = \oint_{C_1} dx_1 \left\{ \int_B^0 \ln\left[(x_2-x_1)^2 + y_1^2 + 1\right]^{1/2} dx_2 + \int_0^B \ln\left[(x_2-x_1)^2 + (A-y_1)^2 + 1\right]^{1/2} dx_2 \right\}$$

$$+ \oint_{C_1} dy_1 \left\{ \int_0^A \ln\left[x_1^2 + (y_2-y_1)^2 + 1\right]^{1/2} dy_2 + \int_A^0 \ln\left[(B-x_1)^2 + (y_2-y_1)^2 + 1\right]^{1/2} dy_2 \right\},$$

---------(14)

where the integrals with respect to x_2 on the boundaries 2 and 4 and with respect to y_2 on the boundaries 1 and 3 are zero, since the respective values of dx_2 and dy_2 are zero.

Now, for simplicity, let us write

$$A\sin\theta = J$$

$$A\cos\theta = K$$

$$A\sin\theta + B\cos\theta = L$$

$$A\cos\theta - B\sin\theta = M$$

$$B\cos\theta = P$$

$$-B\sin\theta = Q \;.$$

Substituting the above in equation(14) and then integrating over the boundary C_1, we get

$$4\pi A_1 F_{A_1-A_2} =$$

$$= \int_0^P dx_1 \left\{ \int_B^0 \ln\left[(x_2-x_1)^2+(-x_1\tan\theta)^2+1\right]dx_2 + \int_0^B \ln\left[(x_2-x_1)^2+(A+x_1\tan\theta)^2+1\right]dx_2 \right\}$$

$$+ \int_P^L dx_1 \left\{ \int_B^0 \ln\left[(x_2-x_1)^2+\left(\frac{x_1}{\tan\theta}-\frac{B}{\sin\theta}\right)^2+1\right]dx_2 + \int_0^B \ln\left[(x_2-x_1)^2+\left(A-\frac{x_1}{\tan\theta}+\frac{B}{\sin\theta}\right)^2+1\right]dx_2 \right\}$$

$$+ \int_L^J dx_1 \left\{ \int_B^0 \ln\left[(x_2-x_1)^2+\left(\frac{A}{\cos\theta}-x_1\tan\theta\right)^2+1\right]dx_2 + \int_0^B \ln\left[(x_2-x_1)^2+\left(A-\frac{A}{\cos\theta}+x_1\tan\theta\right)^2+1\right]dx_2 \right\}$$

$$+ \int_J^0 dx_1 \left\{ \int_B^0 \ln\left[(x_2-x_1)^2+\left(\frac{x_1}{\tan\theta}\right)^2+1\right]dx_2 + \int_0^B \ln\left[(x_2-x_1)^2+\left(A-\frac{x_1}{\tan\theta}\right)^2+1\right]dx_2 \right\}$$

$$+ \int_0^Q dy_1 \left\{ \int_0^A \ln\left[(y_2-y_1)^2+\left(\frac{-y_1}{\tan\theta}\right)^2+1\right]dy_2 + \int_A^0 \ln\left[(y_2-y_1)^2+\left(B+\frac{y_1}{\tan\theta}\right)^2+1\right]dy_2 \right\}$$

$$+ \int_Q^M dy_1 \left\{ \int_0^A \ln\left[(y_2-y_1)^2+\left(\frac{B}{\cos\theta}+y_1\tan\theta\right)^2+1\right]dy_2 + \int_A^0 \ln\left[(y_2-y_1)^2+\left(B-\frac{B}{\cos\theta}-y_1\tan\theta\right)^2+1\right]dy_2 \right\}$$

$$+ \int_M^K dy_1 \left\{ \int_0^A \ln\left[(y_2-y_1)^2+\left(\frac{A}{\sin\theta}-\frac{y_1}{\tan\theta}\right)^2+1\right]dy_2 + \int_A^0 \ln\left[(y_2-y_1)^2+\left(B-\frac{A}{\sin\theta}+\frac{y_1}{\tan\theta}\right)^2+1\right]dy_2 \right\}$$

$$+ \int_K^0 dy_1 \left\{ \int_0^A \ln\left[(y_2-y_1)^2+(y_1\tan\theta)^2+1\right]dy_2 + \int_A^0 \ln\left[(y_2-y_1)^2+(B-y_1\tan\theta)^2+1\right]dy_2 \right\} \quad \text{.....(13)}$$

As all the sixteen double integrals are of the same form, only one general solution is required into which various constants, names of variables, and limits of integration can be appropriately substituted.

$$\int_S^R dz_1 \int_D^P \ln\left[(z_2-z_1)^2+(Gz_1+H)^2+1\right] dz_2$$

$$= \int_S^R dz_1 \left\{ (z_2-z_1)\ln\left[(z_2-z_1)^2+(Gz_1+H)^2+1\right] - 2(z_2-z_1) + 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{z_2-z_1}{\sqrt{(Gz_1+H)^2+1}} \right\}_D^P, \quad (6)$$

$$= \int_S^R dz_1 \left\{ (P-z_1)\ln\left[(P-z_1)^2+(Gz_1+H)^2+1\right] - 2(P-z_1) + 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{P-z_1}{\sqrt{(Gz_1+H)^2+1}} \right.$$

$$\left. -(D-z_1)\ln\left[(D-z_1)^2+(Gz_1+H)^2+1\right] + 2(D-z_1) - 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{D-z_1}{\sqrt{(Gz_1+H)^2+1}} \right\},$$

$$= \int_S^R dz_1 \left\{ (P-z_1)\ln\left[z_1^2(1+G^2)+z_1(2GH-2P)+(P^2+H^2+1)\right] - 2P + 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{P-z_1}{\sqrt{(Gz_1+H)^2+1}} \right.$$

$$\left. -(D-z_1)\ln\left[z_1^2(1+G^2)+z_1(2GH-2D)+(D^2+H^2+1)\right] + 2D - 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{D-z_1}{\sqrt{(Gz_1+H)^2+1}} \right\}, \quad (7)$$

$$= \left[P\left\{ \left(z_1+\frac{GH-P}{1+G^2}\right)\ln\left[(P-z_1)^2+(Gz_1+H)^2+1\right] - 2z_1 + \frac{2\sqrt{(1+G^2)+(GP+H)^2}}{1+G^2}\tan^{-1}\frac{(1+G^2)z_1+GH-P}{\sqrt{(1+G^2)+(GP+H)^2}} \right\} \right.$$

$$- \left\{ \frac{(GH-P)z_1}{1+G^2} - \frac{z_1^2}{2} + \left[\frac{z_1^2}{2} - \frac{(GH-P)^2}{(1+G^2)^2} + \frac{P^2+H^2+1}{2(1+G^2)}\right]\ln\left[(P-z_1)^2+(Gz_1+H)^2+1\right] \right.$$

$$\left. - \frac{2(GH-P)\sqrt{(1+G^2)+(GP+H)^2}}{(1+G^2)^2}\tan^{-1}\frac{(1+G^2)z_1+GH-P}{\sqrt{(1+G^2)+(GP+H)^2}} \right\} - 2Pz_1 + \int 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{P-z_1}{\sqrt{(Gz_1+H)^2+1}}\,d$$

$$- D\left\{ \left(z_1+\frac{GH-D}{1+G^2}\right)\ln\left[(D-z_1)^2+(Gz_1+H)^2+1\right] + 2z_1 - \frac{2\sqrt{(1+G^2)+(GD+H)^2}}{1+G^2}\tan^{-1}\frac{(1+G^2)z_1+GH-D}{\sqrt{(1+G^2)+(GD+H)^2}} \right\}$$

$$+ \left\{ \frac{(GH-D)z_1}{1+G^2} - \frac{z_1^2}{2} + \left[\frac{z_1^2}{2} - \frac{(GH-D)^2}{(1+G^2)^2} + \frac{D^2+H^2+1}{2(1+G^2)}\right]\ln\left[(D-z_1)^2+(Gz_1+H)^2+1\right] \right.$$

$$\left. - \frac{2(GH-D)\sqrt{(1+G^2)+(GD+H)^2}}{(1+G^2)^2}\tan^{-1}\frac{(1+G^2)z_1+GH-D}{\sqrt{(1+G^2)+(GD+H)^2}} \right\} + 2Dz_1 - \int 2\sqrt{(Gz_1+H)^2+1}\,\tan^{-1}\frac{D-z_1}{\sqrt{(Gz_1+H)^2+1}}\,dz$$

$$=\Bigg[\Bigg\{Pz_1+P\frac{GH-P}{1+G^2}-\frac{z_1^2}{2}+\frac{(GH-P)^2}{(1+G^2)^2}-\frac{P^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(P-z_1)^2+(Gz_1+H)^2+1\Big]$$

$$-\Bigg\{Dz_1+D\frac{GH-D}{1+G^2}-\frac{z_1^2}{2}+\frac{(GH-D)^2}{(1+G^2)^2}-\frac{D^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(D-z_1)^2+(Gz_1+H)^2+1\Big]$$

$$+\frac{P-D}{1+G^2}z_1-2(P-D)z_1$$

$$+\frac{2\sqrt{1+G^2+(GP+H)^2}}{(1+G^2)^2}\Big\{P(1+G^2)-(GH-P)\Big\}\tan^{-1}\frac{(1+G^2)z_1+GH-P}{\sqrt{1+G^2+(GP+H)^2}}$$

$$-\frac{2\sqrt{1+G^2+(GD+H)^2}}{(1+G^2)^2}\Big\{D(1+G^2)-(GH-D)\Big\}\tan^{-1}\frac{(1+G^2)z_1+GH-D}{\sqrt{1+G^2+(GD+H)^2}}\Bigg]_S^R$$

$$+2\int_S^R\sqrt{(Gz_1+H)^2+1}\left[\tan^{-1}\frac{P-z_1}{\sqrt{(Gz_1+H)^2+1}}-\tan^{-1}\frac{D-z_1}{\sqrt{(Gz_1+H)^2+1}}\right]dz_1\ ,$$

$$=\Bigg\{PR+P\frac{GH-P}{1+G^2}-\frac{R^2}{2}+\frac{(GH-P)^2}{(1+G^2)^2}-\frac{P^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(P-R)^2+(GR+H)^2+1\Big]$$

$$-\Bigg\{PS+P\frac{GH-P}{1+G^2}-\frac{S^2}{2}+\frac{(GH-P)^2}{(1+G^2)^2}-\frac{P^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(P-S)^2+(GS+H)^2+1\Big]$$

$$-\Bigg\{DR+D\frac{GH-D}{1+G^2}-\frac{R^2}{2}+\frac{(GH-D)^2}{(1+G^2)^2}-\frac{D^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(D-R)^2+(GR+H)^2+1\Big]$$

$$+\Bigg\{DS+D\frac{GH-D}{1+G^2}-\frac{S^2}{2}+\frac{(GH-D)^2}{(1+G^2)^2}-\frac{D^2+H^2+1}{2(1+G^2)}\Bigg\}\ln\Big[(D-S)^2+(GS+H)^2+1\Big]$$

$$+\frac{2G(PG+H)}{(1+G^2)^2}\sqrt{1+G^2+(PG+H)^2}\left[\tan^{-1}\frac{(1+G^2)R+GH-P}{\sqrt{1+G^2+(GP+H)^2}}-\tan^{-1}\frac{(1+G^2)S+GH-P}{\sqrt{1+G^2+(GP+H)^2}}\right]$$

$$-\frac{2G(DG+H)}{(1+G^2)^2}\sqrt{1+G^2+(DG+H)^2}\left[\tan^{-1}\frac{(1+G^2)R+GH-D}{\sqrt{1+G^2+(GD+H)^2}}-\tan^{-1}\frac{(1+G^2)S+GH-D}{\sqrt{1+G^2+(GD+H)^2}}\right]$$

$$+(P-D)(S-R)\frac{1+2G^2}{1+G^2}+2\int_S^R\sqrt{(Gz_1+H)^2+1}\left[\tan^{-1}\frac{P-z_1}{\sqrt{(Gz_1+H)^2+1}}-\tan^{-1}\frac{D-z_1}{\sqrt{(Gz_1+H)^2+1}}\right]dz_1 \quad ---(16)$$

Thus equation (16) gives the general solution of the double integral of the form appearing in equation (15). When the following sixteen sets of values are inserted one by one in equation (16) and the results are added together, we get the solution of equation (15).

	R	S	P	D	G	H
1.	$B\cos\theta$	0	0	B	$\tan\theta$	0
2.	$B\cos\theta$	0	B	0	$\tan\theta$	A
3.	$A\sin\theta+B\cos\theta$	$B\cos\theta$	0	B	$1/\tan\theta$	$-B/\sin\theta$
4.	$A\sin\theta+B\cos\theta$	$B\cos\theta$	B	0	$1/\tan\theta$	$-A-B/\sin\theta$
5.	$A\sin\theta$	$A\sin\theta+B\cos\theta$	0	B	$\tan\theta$	$-A/\cos\theta$
6.	$A\sin\theta$	$A\sin\theta+B\cos\theta$	B	0	$\tan\theta$	$A-A/\cos\theta$
7.	0	$A\sin\theta$	0	B	$1/\tan\theta$	0
8.	0	$A\sin\theta$	B	0	$1/\tan\theta$	$-A$
9.	$-B\sin\theta$	0	A	0	$1/\tan\theta$	0
10.	$-B\sin\theta$	0	0	A	$1/\tan\theta$	B
11.	$A\cos\theta-B\sin\theta$	$-B\sin\theta$	A	0	$\tan\theta$	$B/\cos\theta$
12.	$A\cos\theta-B\sin\theta$	$-B\sin\theta$	0	A	$\tan\theta$	$B/\cos\theta-B$
13.	$A\cos\theta$	$A\cos\theta-B\sin\theta$	A	0	$1/\tan\theta$	$-A/\sin\theta$
14.	$A\cos\theta$	$A\cos\theta-B\sin\theta$	0	A	$1/\tan\theta$	$B-A/\sin\theta$
15.	0	$A\cos\theta$	A	0	$\tan\theta$	0
16.	0	$A\cos\theta$	0	A	$\tan\theta$	$-B$

The term $(P-D)(S-R)(1+2G^2)/(1+G^2)$ cancels out when added sixteen times for the abovementioned set of values. Hence, it can be omitted in the computation.

The numerically computed values of the factor for two equal rectangles are presented in figures 10 to 13, as a function of dimensionless ratios $A=a/c$ for different values of the angle of rotation and of $B=b/c$. The accuracy of the computation is checked by interchanging the values of A and B. The results show an accuracy of twelve significant figures.

The curves are plotted for θ ranging from 0° to 180° only, since, for two equal rectangles, the configuration is symmetrical in the range $0^\circ \leq \theta \leq 180^\circ$ and $360^\circ \geq \theta \geq 180^\circ$. For two unequal rectangles this will no longer be true.

Moreover, these numerically computed values of the factor as θ approaches 0, $\pi/2$, π etc. are found to approach the values of the factor found from factor algebra for θ equal to 0, $\pi/2$, π etc., respectively. This is an additional check of our procedure.

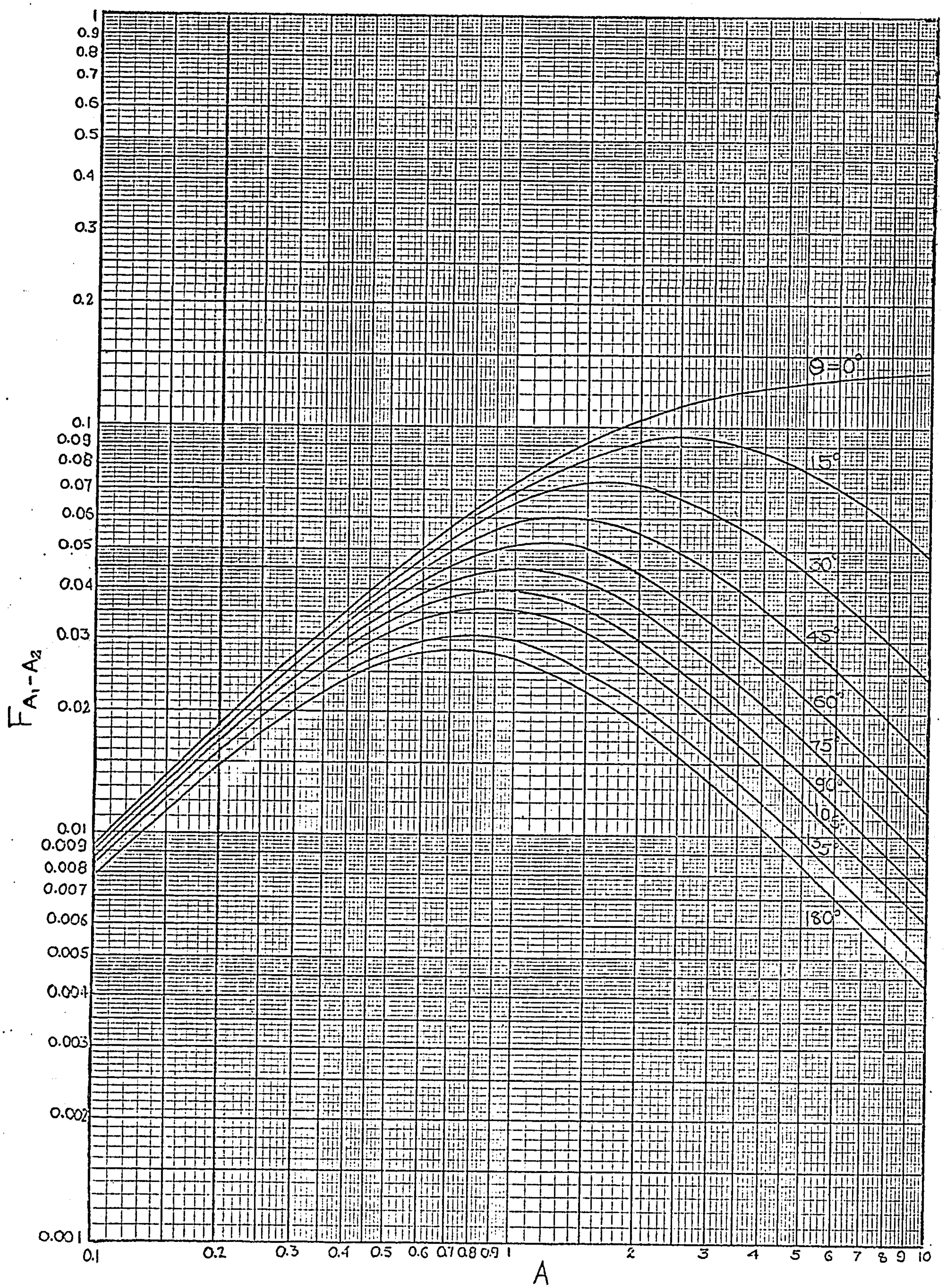
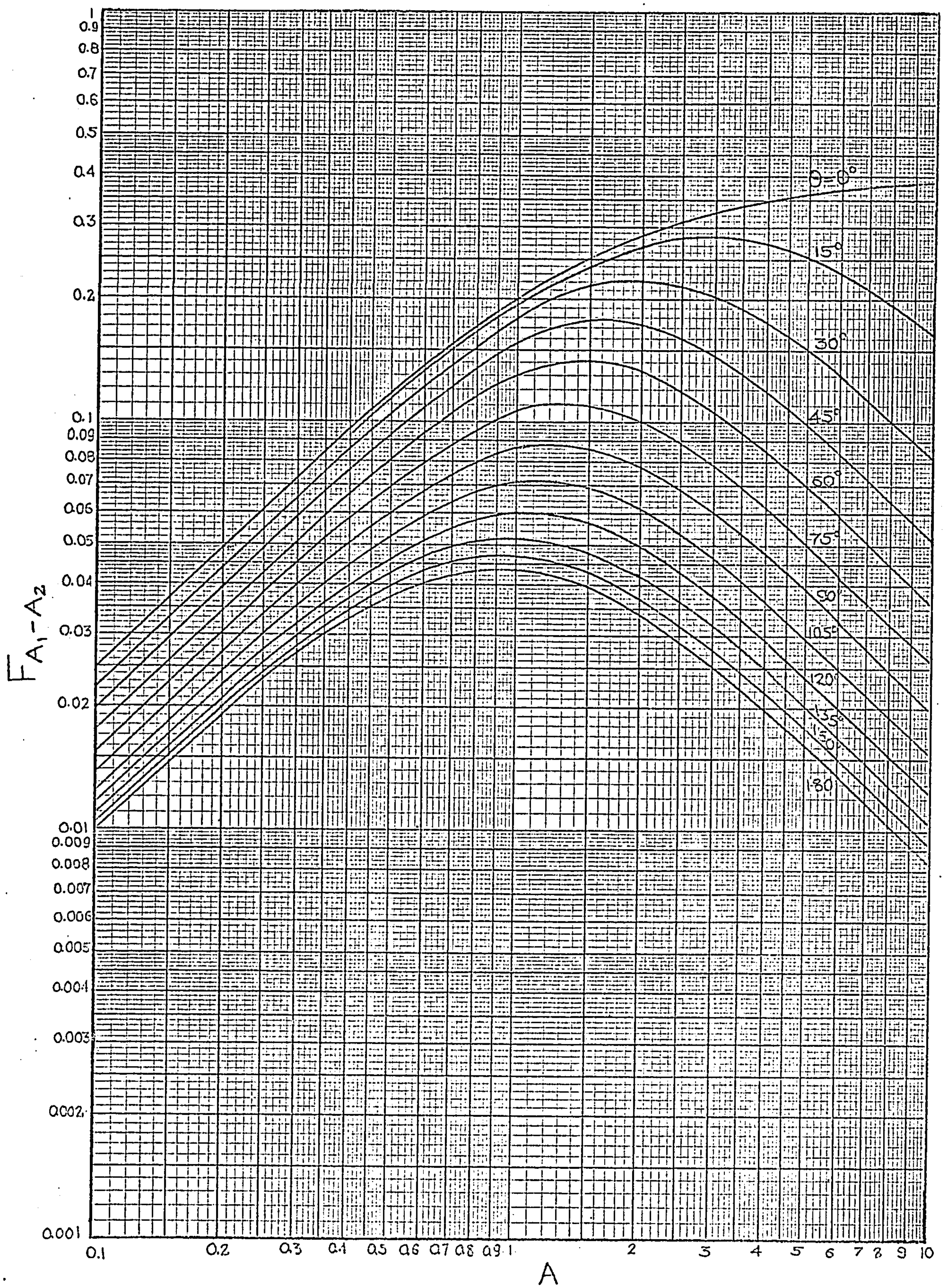
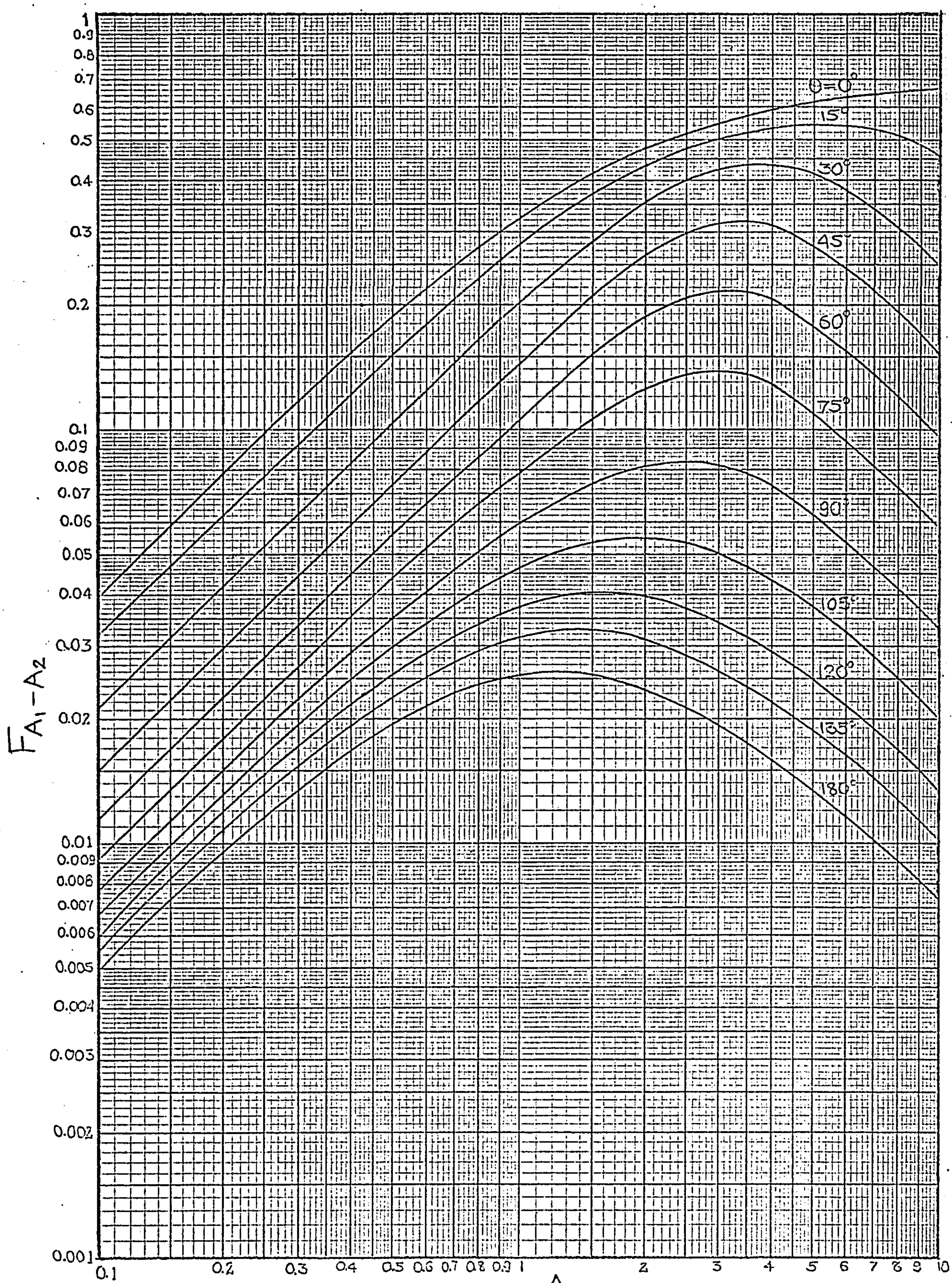


FIGURE 10. B=0.3.



$F_{A_1-A_2}$
A
θ=0°
15°
30°
45°
60°
75°
90
105°
120°
135°
150°
180
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8
9
10

FIGURE 11. B = 1.



$\theta = 0°$
15°
30°
45°
60°
75°
90°
105°
120°
135°
180°
$F_{A_1-A_2}$
A
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8
9
10

FIGURE 12. B=3.

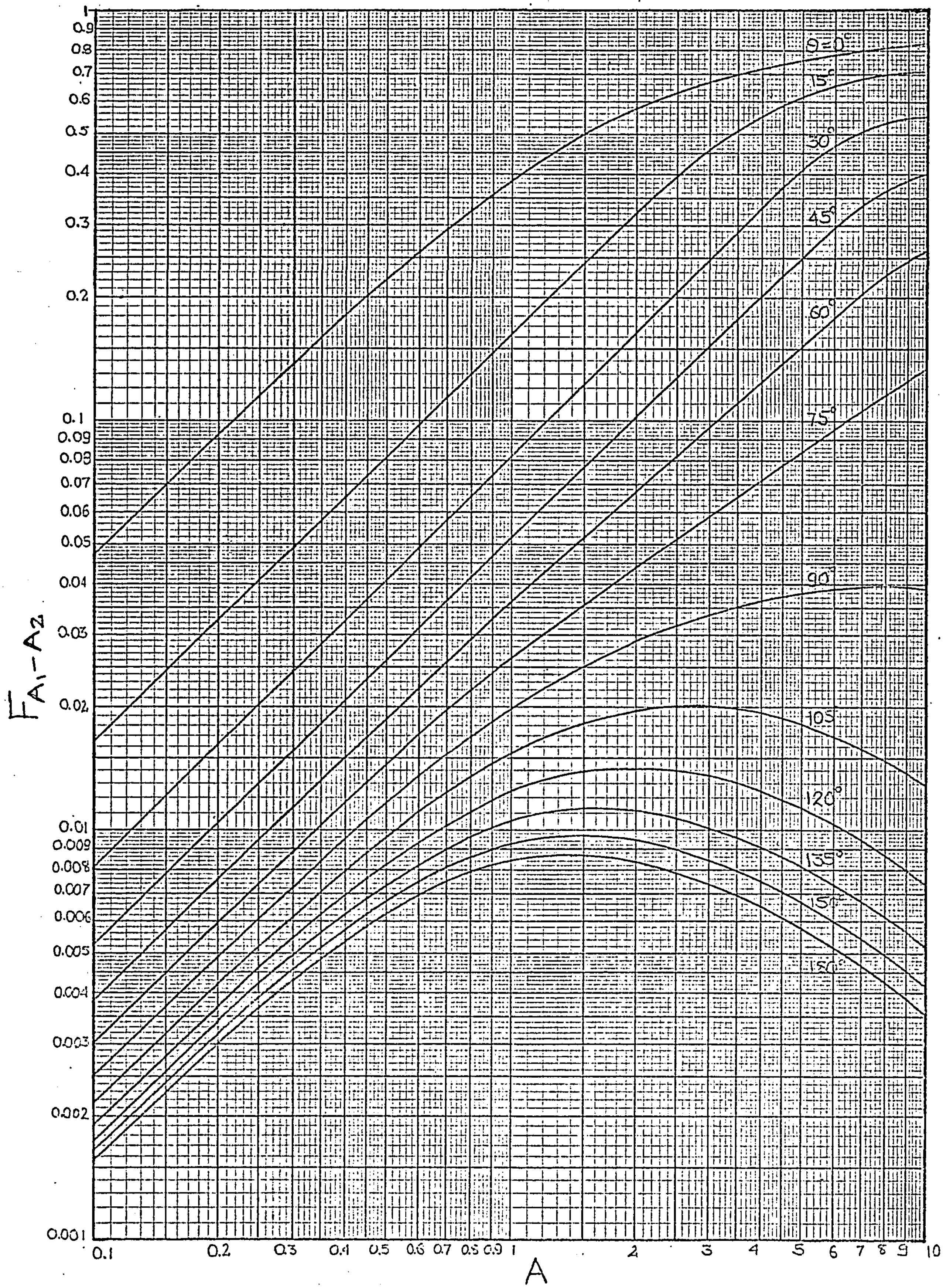


FIGURE 13. B=10.

IV. CONFIGURATION FACTOR FROM A CIRCULAR DISK TO A NON-COAXIAL CIRCULAR DISK LYING IN A PARALLEL PLANE

We have in the literature, the configuration factor between two co-axial circular disks in a parallel plane. Let us now proceed to formulate the factor for the case when the two parallel disks are not coaxial, We shall utilise the factor from an infinitesimal area to a parallel circular disk given in reference 2 and by integrating this factor with appropriate limits we can find the factor from a circular disk to a noncoaxial circular disk.

Figure 14(a) and 14(b) represent two noncoaxial parallel disks having radii b and c and areas A_1 and A_2. Their planes are separated by a distance d and their axes by a distance e. An arbitrary infinitesimal area on A_2 is denoted by dA_2. The factor from dA_2 to A_1 is given (reference 2) as

$$F_{dA_2-A_1} = \frac{1}{2}\left[1 - \frac{r^2+d^2-b^2}{\sqrt{r^4+2(d^2-b^2)r^2+(d^2+b^2)^2}}\right] \cdot \text{------}(17)$$

By the rule of reciprocity

$$dA_2 F_{dA_2-A_1} = A_1 F_{A_1-dA_2}$$

from which we have

$$F_{A_1-dA_2} = \frac{1}{2A_1}\left[1 - \frac{r^2+d^2-b^2}{\sqrt{r^4+2(d^2-b^2)r^2+(d^2+b^2)^2}}\right] dA_2 \text{ ,...}(18)$$

where

$$A_1 = \pi b^2 ,$$

$$dA_2 = r d\theta dr .$$

Letting

$$p = d^2-b^2 ,$$

$$q = d^2+b^2 ,$$

we have

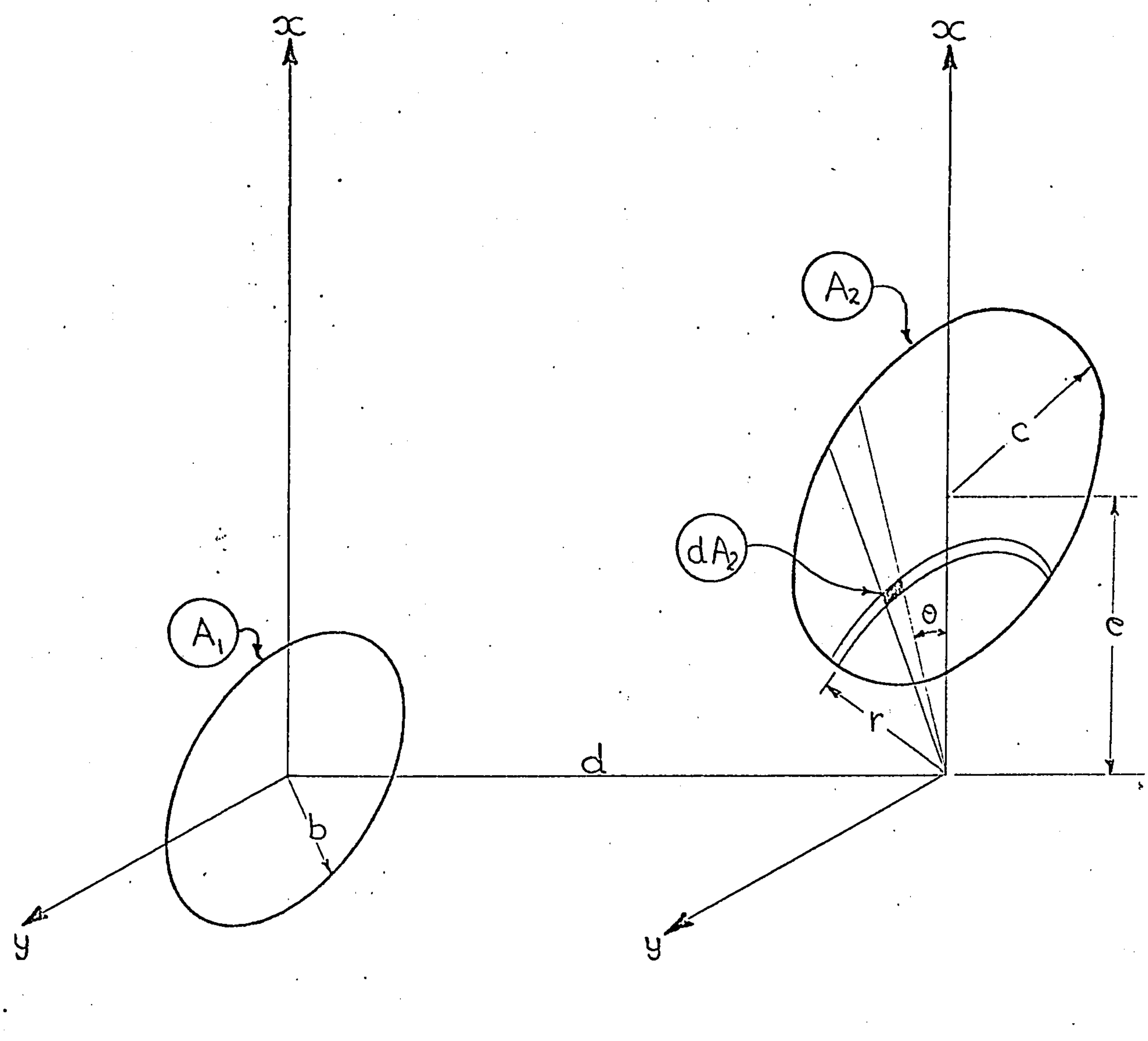


FIGURE 14(a). $e \geq c$.

$$F_{A_1-dA_2} = \frac{1}{2\pi b^2}\left[1-\frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr. \qquad (19)$$

The factor $F_{A_1-A_2}$ is obtained by integrating equation (19) over the area A_2. Thus,

$$F_{A_1-A_2} = \frac{1}{2\pi b^2}\iint_{A_2}\left[1-\frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr. \qquad (20)$$

Now we are faced with two cases. They are:

(1) Axis of A_1 not intersecting A_2, i.e. $e \geqslant c$,

(2) Axis of A_1 intersecting A_2, i.e. $e < c$.

Case 1

In this case (figure 14(a)), area A_2 can be divided into a number of segments of rings of infinitesimal thickness dr. Writing the limits of integration, equation (20) becomes

$$F_{A_1-A_2} = \frac{1}{2\pi b^2}\int_{e-c}^{e+c} 2\int_0^{\cos^{-1}\frac{r^2+e^2-c^2}{2re}}\left[1-\frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr,$$

$$= \frac{1}{\pi b^2}\int_{e-c}^{e+c}\left[1-\frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr,$$

$$= \frac{1}{\pi b^2}\int_{e-c}^{e+c} r\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr - \frac{1}{\pi b^2}\int_{e-c}^{e+c}\frac{r^3+rp}{\sqrt{r^4+2pr^2+q^2}}\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr. \qquad (21)$$

Twice the first integral in equation (21) is obviously the area A_2. The check for this is shown in the Appendix [9].

Thus equation (21) becomes

$$F_{A_1-A_2} = \frac{\pi c^2}{2\pi b^2} - \frac{1}{\pi b^2}\int_{e-c}^{e+c}\frac{r^3+rp}{\sqrt{r^4+2pr^2+q^2}}\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr,$$

$$= \frac{c^2}{2b^2} - \frac{1}{4\pi b^2}\left[2\sqrt{r^4+2pr^2+q^2}\cos^{-1}\frac{r^2+e^2-c^2}{2re}\right]_{e-c}^{e+c} - \frac{1}{4\pi b^2}\int_{e-c}^{e+c}\frac{2\sqrt{r^4+2pr^2+q^2}\,(2re)}{\sqrt{4r^2e^2-(r^2+e^2-c^2)^2}}\,\frac{r^2+c^2-e^2}{2r^2e}\,dr$$

$$= \frac{c^2}{2b^2} - \frac{1}{4\pi b^2}\left[0\right] - \frac{1}{2\pi b^2}\int_{e-c}^{e+c}\frac{\sqrt{r^4+2pr^2+q^2}}{\sqrt{-r^4-e^4-c^4+2(r^2e^2+e^2c^2+c^2r^2)}}\cdot\frac{r^2+c^2-e^2}{r}\,dr,$$

$$= \frac{c^2}{2b^2} - \frac{1}{2\pi b^2}\int_{e-c}^{e+c}\frac{\sqrt{r^4+2pr^2+q^2}}{\sqrt{-r^4+2r^2g-h^2}}\cdot\frac{r^2-h}{r}\,dr, \quad \text{........(22)}$$

where

$$g = e^2+c^2,$$

$$h = e^2-c^2.$$

We find that the integral of equation (22) cannot be solved in terms of elementary functions.

In order to write equation (22) in a dimensionless form, let us introduce the ratios

$$B = b/d$$

$$C = c/d$$

$$E = e/d$$

$$R = r/d$$

$$P = p/d = (d^2-b^2)/d^2 = 1-B^2$$

$$Q = q/d = (d^2+b^2)/d^2 = 1+B^2$$

$$G = (e^2+c^2)/d^2 = E^2+C^2$$

$$H = (e^2-c^2)/d^2 = E^2-C^2.$$

Then

$$F_{A_1-A_2} = \frac{C^2}{2B^2} - \frac{1}{\pi B^2}\int_{E-C}^{E+C} \frac{R^3+RP}{\sqrt{R^4+2PR^2+Q^2}} \cos^{-1}\frac{R^2+E^2-C^2}{2RE}\,dR, \quad \text{---}(22')$$

or

$$F_{A_1-A_2} = \frac{C^2}{2B^2} - \frac{1}{\pi B^2}\int_{E-C}^{E+C} \frac{\sqrt{R^4+2PR^2+Q^2}}{\sqrt{-R^4+2GR^2-H^2}} \cdot \frac{R^2-H}{R}\,dR. \quad \text{---------}(22'')$$

Case 2

The second case is represented by figure 14(b). In this case area A_2 is taken to be composed of a circle and a number of segments of rings. This circle of area A_s is shown shaded in figure 14(b). The limits of integration are different from case 1 and hence equation (20) becomes

$$F_{A_1-A_2} = \frac{1}{2\pi b^2}\int_{-e+c}^{e+c} 2\int_{0}^{\cos^{-1}\frac{r^2+e^2-c^2}{2re}} \left[1 - \frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr + F_{A_1-A_s},$$

$$= \frac{1}{\pi b^2}\int_{-e+c}^{e+c} r\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr - \frac{1}{\pi b^2}\int_{-e+c}^{e+c}\frac{r^3+rp}{\sqrt{r^4+2pr^2+q^2}}\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr + F_{A_1-A_s}$$

$$\text{---------}(23)$$

Twice the first integral in equation (23) is the unshaded area of A_2 i.e. (A_2-A_s) and $F_{A_1-A_s}$ is the factor from A_1 to a parallel coaxial disk A_s, given as

$$F_{A_1-A_s} = \frac{1}{2}\left[Z - \sqrt{Z^2 - \frac{4(c-e)^2}{b^2}}\right],$$

where

$$Z = 1 + \frac{d^2}{b^2} + \frac{(c-e)^2}{b^2}.$$

Thus equation (23) reduces to

$$F_{A_1-A_2} = \frac{\pi c^2 - \pi(c-e)^2}{2\pi b^2} - \frac{1}{2\pi b^2}\int_{-e+c}^{e+c}\frac{r^3+rp}{\sqrt{r^4+2pr^2+q^2}}\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr + F_{A_1-A_s},$$

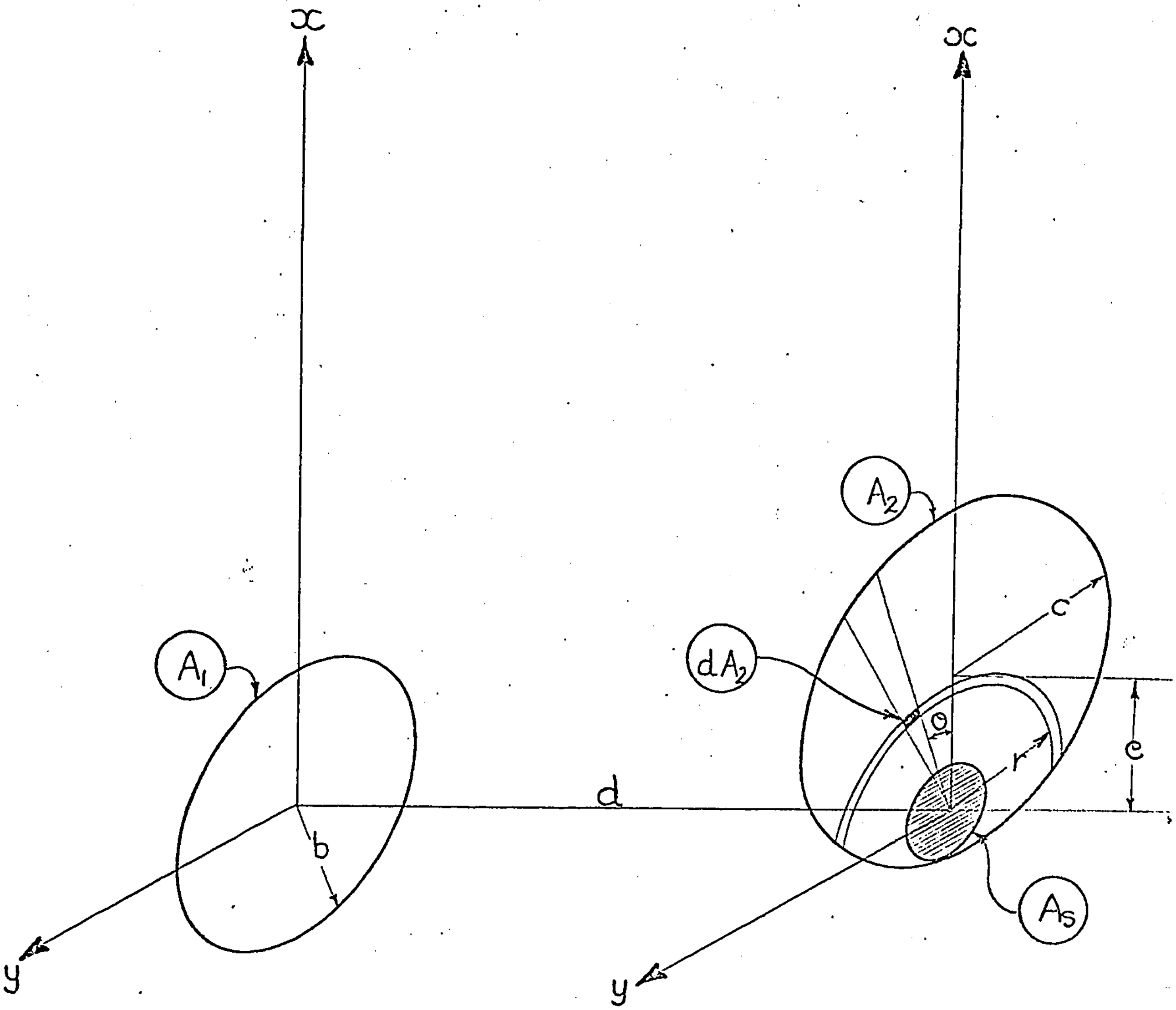


FIGURE 14(b). e < c.

$$= \frac{c^2-(c-e)^2}{2b^2} + \frac{\sqrt{(c-e)^4+2p(c-e)^2+q^2}}{2b^2} - \frac{1}{2\pi b^2}\int_{c-e}^{e+c} \frac{\sqrt{r^4+2pr^2+q^2}}{\sqrt{-r^4+2gr^2-h^2}}\,\frac{r^2-h}{r}\,dr + F_{A_1-A_s}\,; \qquad \text{------------}(24)$$

where

$$g = e^2+c^2\,,$$

$$h = e^2-c^2\,,$$

and the dimensionless form of equation (24) is

$$F_{A_1-A_2} = \frac{C^2-(C-E)^2}{2B^2} - \frac{1}{\pi B^2}\int_{C-E}^{E+C} \frac{R^3+RP}{\sqrt{R^4+2PR^2+Q^2}}\cos^{-1}\frac{R^2+E^2-C^2}{2RE}\,dR + \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{4(C-E)^2}{B^2}}\right], \qquad \text{-----------}(24')$$

or

$$F_{A_1-A_2} = \frac{C^2-(C-E)^2}{2B^2} - \frac{\sqrt{(C-E)^4+2P(C-E)^2+Q^2}}{2B^2} - \frac{1}{2\pi B^2}\int_{C-E}^{E+C} \frac{\sqrt{R^4+2PR^2+Q^2}}{\sqrt{-R^4+2GR^2-H^2}}\,\frac{R^2-H}{R}\,dR$$

$$+\frac{1}{2}\left[Z-\sqrt{Z^2-\frac{4(C-E)^2}{B^2}}\right], \qquad \text{-----------}(24'')$$

where

$$Z = 1+\frac{1}{B^2}+\frac{(C-E)^2}{B^2}\,.$$

Thus the factor from a circular disk to a noncoaxial parallel disk is given either by equation (22) or by equation (24), depending upon the configuration; in terms of a single integral. In figure 15, the numerically computed values of the factor for two equal disks are presented as a function of the dimensionless ratios $Y = E/C = e/c$ and $B = b/d$. Equations (22') and (24') rather than (22'') and (24'') have been used in the numerical integration. It should be noted that the latter equations do not lend themselves to numerical integration because in both cases the integrand becomes infinite at the limits.

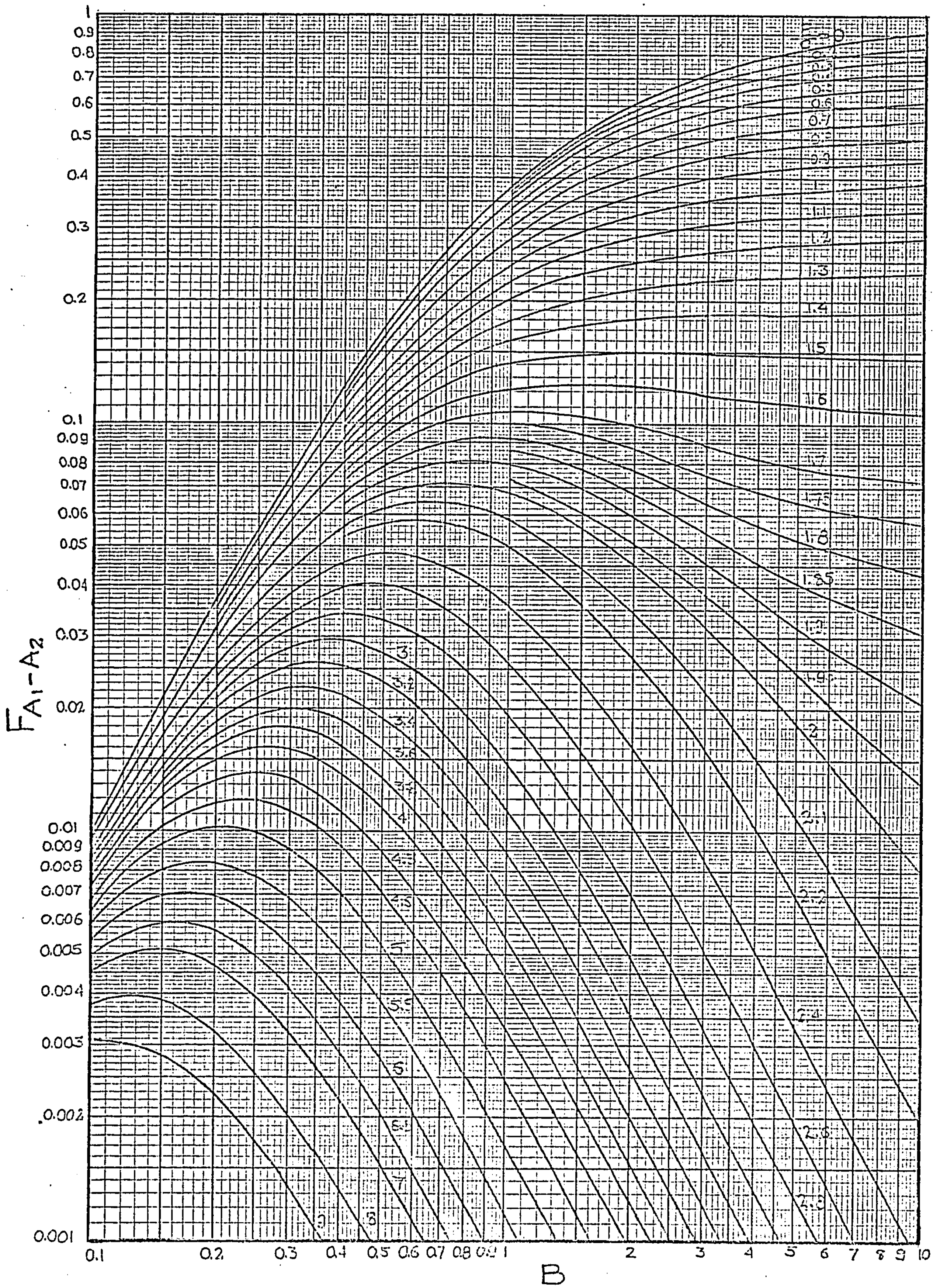


FIGURE 15. A=B.

V. CONFIGURATION FACTOR FROM A CIRCULAR DISK TO A COAXIAL SQUARE LYING IN A PARALLEL PLANE.

Figure 16 represents a circular disk of radius b and area A_1 and a square of side 2c and area A_2. A_1 and A_2 are parallel, coaxial, and distance d apart. Let a circle of radius c be inscribed in the square (shown shaded in the figure) and let its area be A_2'. $A_2'' = A_2 - A_2'$ is the unshaded area of A_2. Then,

$$F_{A_1-A_2} = F_{A_1-A_2'} + F_{A_1-A_2''} \;. \qquad \text{-------------(25)}$$

We know from equation (18) that

$$F_{A_1-dA_2''} = \frac{1}{2A_1}\left[1 - \frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right]dA_2'' \;,$$

where

$$p = d^2-b^2 \;,$$

$$q = d^2+b^2 \;,$$

$$dA_2'' = r\,d\theta\,dr \;,$$

and for coaxial and parallel disks, we have the well-known formula

$$F_{A_1-A_2'} = \frac{1}{2}\left[z - \sqrt{z^2-4c^2/b^2}\right] ,$$

where

$$z = 1 + d^2/b^2 + c^2/b^2 \;.$$

Thus, equation (25) becomes

$$F_{A_1-A_2} = F_{A_1-A_2'} + \frac{1}{2A_1}\int_{A_2''}\left[1 - \frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right]dA_2'' \;,$$

$$= F_{A_1-A_2'} + \frac{8}{2\pi b^2}\int_{c}^{\sqrt{2}c}\int_{\cos^{-1}\frac{c}{r}}^{\pi/4}\left[1 - \frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr \;,$$

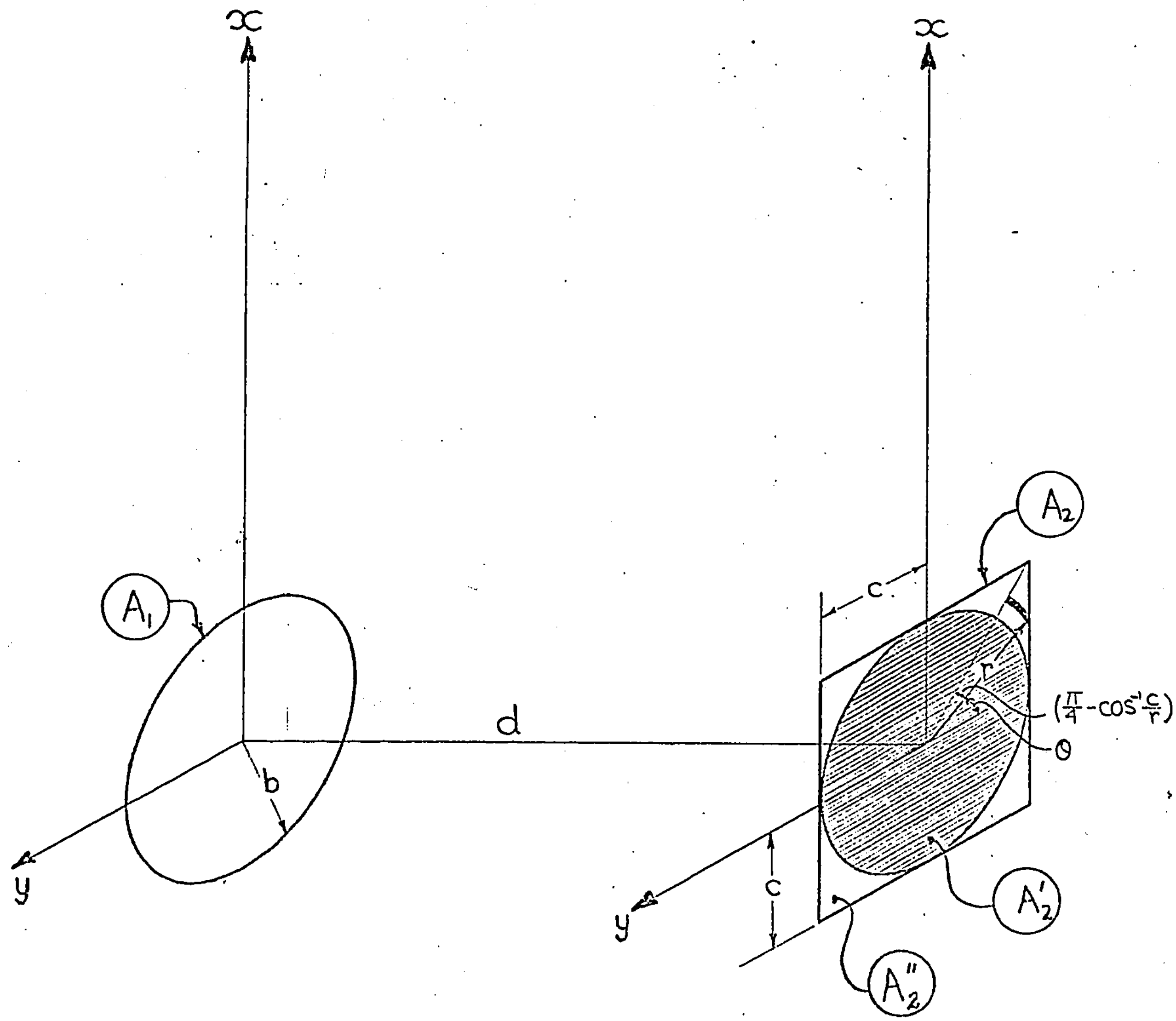


FIGURE 16

$$= F_{A_1 - A_2'} + \frac{4}{\pi b^2}\int_c^{\sqrt{2}c}\int_{\cos^{-1}\frac{c}{r}}^{\pi/4} r\,d\theta\,dr - \frac{4}{\pi b^2}\int_c^{\sqrt{2}c}\int_{\cos^{-1}\frac{c}{r}}^{\pi/4} \frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\,dr\ .$$

The middle term of the above expression is obviously equal to $(4c^2 - \pi c^2)/2\pi b^2$, the check for which is given in the Appendix[10]. And the last integral of the above expression is simplified as follows

$$\int_c^{\sqrt{2}c}\int_{\cos^{-1}\frac{c}{r}}^{\pi/4} \frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\,dr$$

$$= \int_c^{\sqrt{2}c}\left(\frac{\pi}{4} - \cos^{-1}\frac{c}{r}\right)\frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\,dr\ ,$$

$$= \frac{\pi}{4}\int_c^{\sqrt{2}c}\frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\,dr - \int_c^{\sqrt{2}c}\frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\cos^{-1}\frac{c}{r}\,dr\ ,$$

$$= \frac{\pi}{4}\left[\frac{\sqrt{r^4 + 2pr^2 + q^2}}{2}\right]_c^{\sqrt{2}c} - \left[\frac{\sqrt{r^4 + 2pr^2 + q^2}}{2}\cos^{-1}\frac{c}{r}\right]_c^{\sqrt{2}c} - \int_c^{\sqrt{2}c}\frac{\sqrt{r^4 + 2pr^2 + q^2}}{2}\,\frac{\left(-\frac{c}{r^2}\right)dr}{\sqrt{1 - c^2/r^2}}\ ,$$

$$= \frac{\pi}{8}\left[\sqrt{4c^4 + 4pc^2 + q^2} - \sqrt{c^4 + 2pc^2 + q^2}\right] - \frac{\pi}{8}\left[\sqrt{4c^4 + 4pc^2 + q^2}\right] + \frac{c}{2}\int_c^{\sqrt{2}c}\frac{\sqrt{r^4 + 2pr^2 + q^2}}{\sqrt{r^4 - c^2 r^2}}\,dr$$

$$= -\frac{\pi}{8}\sqrt{c^4 + 2pc^2 + q^2} + \frac{c}{2}\int_c^{\sqrt{2}c}\frac{\sqrt{r^4 + 2pr^2 + q^2}}{\sqrt{r^4 - c^2 r^2}}\,dr\ .$$

It is shown in the Appendix[11] that the integral in the above expression can be transformed into a sum of elliptic integrals, which proves that it cannot be integrated in terms of elementary functions.

Equation (26) can now be written as

$$F_{A_1-A_2} = \frac{1}{2}\left[Z - \sqrt{Z^2 - \frac{4c^2}{b^2}}\right] + \frac{c^2}{b^2}\left[\frac{4-\pi}{2\pi}\right] - \frac{4}{\pi b^2}\int_{c}^{\sqrt{2}c}\left(\frac{\pi}{4} - \cos^{-1}\frac{c}{r}\right)\frac{r^3 + rp}{\sqrt{r^4 + 2pr^2 + q^2}}\,dr \;,$$

or

$$F_{A_1-A_2} = \frac{1}{2}\left[Z - \sqrt{Z^2 - \frac{4c^2}{b^2}}\right] + \frac{c^2}{b^2}\left[\frac{4-\pi}{2\pi}\right] + \frac{1}{2b^2}\sqrt{c^4 + 2pc^2 + q^2} - \frac{2c}{\pi b^2}\int_{c}^{\sqrt{2}c}\frac{\sqrt{r^4 + 2pr^2 + q^2}}{r\sqrt{r^2 - c^2}}\,dr \quad \ldots (27)$$

Defining the ratios

$$B = b/d$$
$$C = c/d$$
$$R = r/d$$
$$P = p/d^2 = 1 - B^2$$
$$Q = q/d^2 = 1 + B^2$$

and

$$Z = 1 + 1/B^2 + C^2/B^2 \;,$$

we get the dimensionless form of equation (27) as follows

$$F_{A_1-A_2} = \frac{1}{2}\left[Z - \sqrt{Z^2 - \frac{4C^2}{B^2}}\right] + \frac{C^2}{B^2}\left[\frac{4-\pi}{2\pi}\right] - \frac{4}{\pi B^2}\int_{C}^{\sqrt{2}C}\left(\frac{\pi}{4} - \cos^{-1}\frac{C}{R}\right)\frac{R^3 + RP}{\sqrt{R^4 + 2PR^2 + Q^2}}\,dR \quad \ldots (27')$$

or

$$F_{A_1-A_2} = \frac{1}{2}\left[Z - \sqrt{Z^2 - \frac{4C^2}{B^2}}\right] + \frac{C^2}{B^2}\left[\frac{4-\pi}{2\pi}\right] + \frac{1}{2B^2}\sqrt{C^4 + 2PC^2 + Q^2} - \frac{2C}{\pi B^2}\int_{C}^{\sqrt{2}C}\frac{\sqrt{R^4 + 2PR^2 + Q^2}}{R\sqrt{R^2 - C^2}}\,dR \quad (27'')$$

For the reasons similar to those explained on page 51, the equation used for numerical integration was (27') rather than equation (27'').

We shall now proceed to evaluate the factor for the same configuration in a little different manner. Figure 17 is similar to the figure 16 except that in this the circle is circumscribed on the square instead of being inscribed in it. If $A_{seg.}$ represents the area of the shaded portion i.e. the area of the four segments, as shown in the figure, then

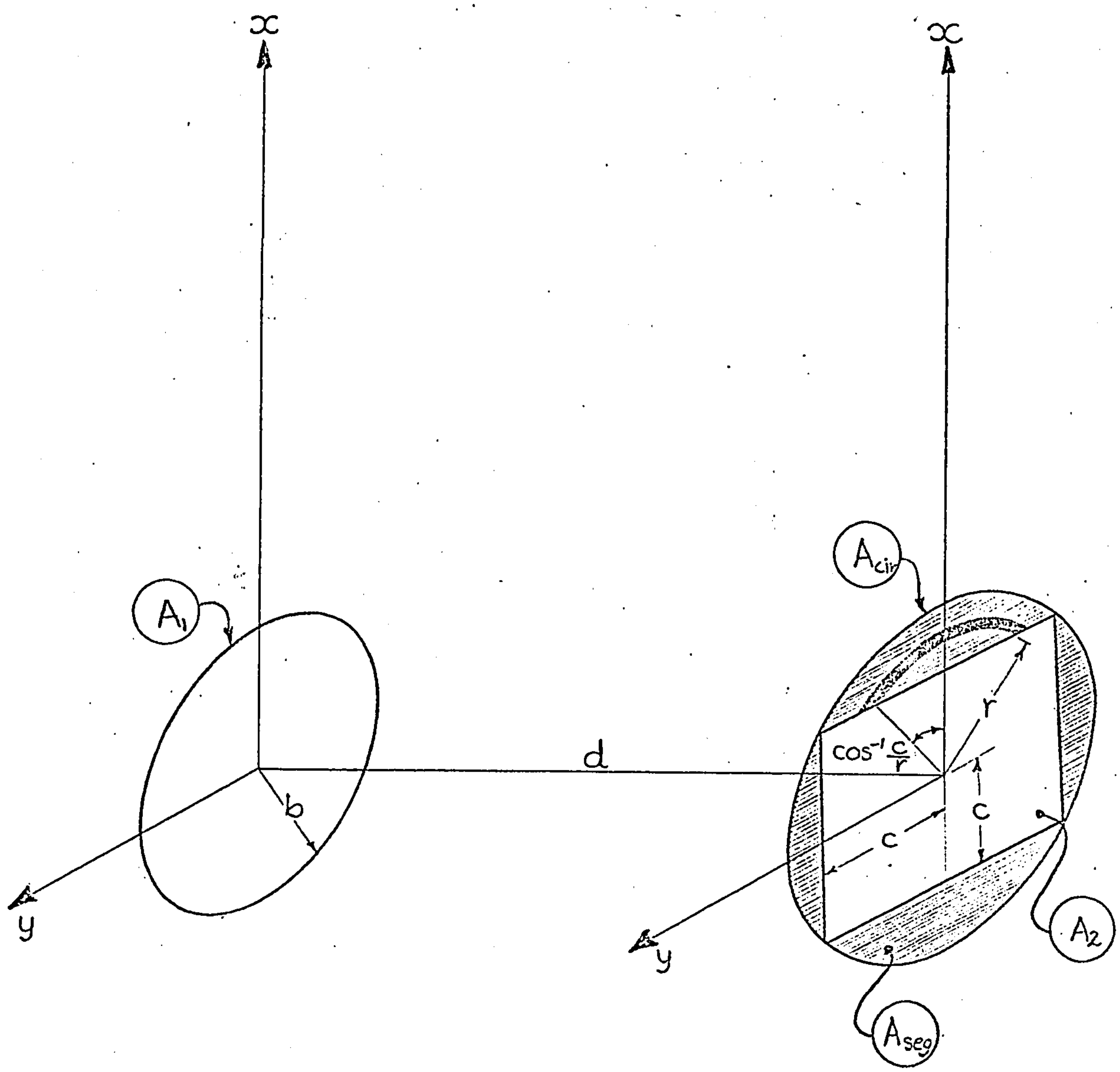


FIGURE 17

$$F_{A_1-A_2} = F_{A_1-A_{cir.}} - F_{A_1-A_{seg.}} , \quad \text{------------(28)}$$

where $A_{cir.}$ is the area of the circumscribed circle. Writing the values of the factors $F_{A_1-A_{cir.}}$ and $F_{A_1-A_{seg.}}$, we have

$$F_{A_1-A_2} =$$

$$= \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{b^2}}\right] - 4\,\frac{1}{2A_1}\int_C^{\sqrt{2}C} 2\int_0^{\cos^{-1}\frac{C}{r}}\left[1-\frac{r^2+p}{\sqrt{r^4+2pr^2+q^2}}\right] r\,d\theta\,dr ,$$

$$= \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{b^2}}\right] - \frac{4}{\pi b^2}\int_C^{\sqrt{2}C} r\cos^{-1}\frac{C}{r}\,dr + \frac{4}{\pi b^2}\int_C^{\sqrt{2}C}\frac{r^3+rp}{\sqrt{r^4+2pr^2+q^2}}\cos^{-1}\frac{C}{r}\,dr ,$$

$$= \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{b^2}}\right] - \frac{4}{\pi b^2}\left[\frac{\pi C^2}{4}-\frac{C^2}{2}\right] + \frac{4}{\pi b^2}\left[\frac{\sqrt{r^4+2pr^2+q^2}}{2}\cos^{-1}\frac{C}{r}\right]_C^{\sqrt{2}C} - \frac{2C}{\pi b^2}\int_C^{\sqrt{2}C}\frac{\sqrt{r^4+2pr^2+q^2}}{\sqrt{r^4-r^2C^2}}\,dr$$

$$= \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{b^2}}\right] - \frac{C^2}{b^2}\left[\frac{\pi-2}{\pi}\right] + \frac{1}{2b^2}\left[\sqrt{4C^4+4pC^2+q^2}\right] - \frac{2C}{\pi b^2}\int_C^{\sqrt{2}C}\frac{\sqrt{r^4+2pr^2+q^2}}{r\sqrt{r^2-C^2}}\,dr .. (29)$$

Using the same ratios that are used for equation (27'), we get the dimensionless form of equation (29) as

$$F_{A_1-A_2} = \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{B^2}}\right] - \frac{C^2}{B^2}\left[\frac{\pi-2}{\pi}\right] + \frac{4}{\pi B^2}\int_C^{\sqrt{2}C}\frac{R^3+RP}{\sqrt{R^4+2PR^2+Q^2}}\cos^{-1}\frac{C}{R}\,dR , \quad \text{.....(29')}$$

or

$$F_{A_1-A_2} = \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{8C^2}{B^2}}\right] - \frac{C^2}{B^2}\left[\frac{\pi-2}{\pi}\right] + \frac{\sqrt{4C^4+4PC^2+Q^2}}{2B^2} - \frac{2C}{\pi B^2}\int_C^{\sqrt{2}C}\frac{\sqrt{R^4+2PR^2+Q^2}}{R\sqrt{R^2-C^2}}\,dR , \quad \text{...(29'')}$$

where

$$Z = 1 + 1/B^2 + 2C^2/B^2 .$$

The numerically computed values of the factor from a circular disk to a parallel, coaxial square are presented in form of graphs in figure 18.

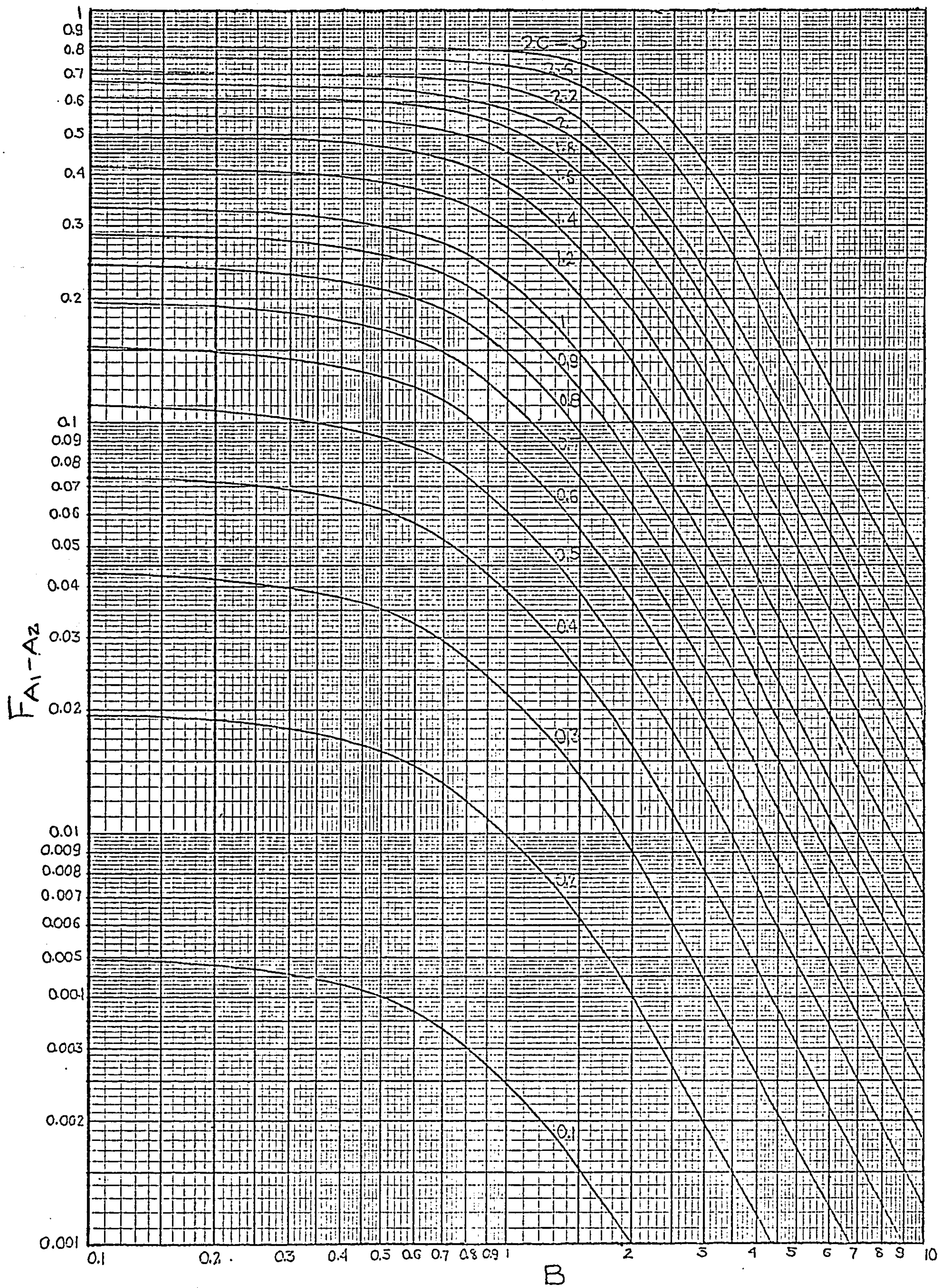


FIGURE 18.

VI. CONFIGURATION FACTOR FROM A SPHERE TO A COAXIAL CIRCULAR DISK

Figure 19 represents a sphere of radius s and area A_1, its center being at O, which is chosen as the origin of the coordinate system. An infinitesimal area dA_2 is centered on the y-axis at the point E distant d from the origin and inclined in such a manner that the normal to dA_2 intersects the z-axis while forming an angle ϕ with the y-axis. Let us draw a cone with its vertex at E, tangent to the sphere, as shown in the figure.

We have

$$k^2 = OP^2 - OQ^2 ,$$

where

$$OP = s = \text{radius of the sphere},$$

$$OQ = OP^2/OE = s^2/d .$$

Hence

$$k^2 = s^2 - \left(\frac{s^2}{d}\right)^2$$

$$= \frac{s^2}{d^2}(d^2 - s^2) . \qquad \text{--------------(30)}$$

Let $dA = dA_2 \cos\phi$ be the projection of dA_2 on a plane parallel to the plane x-z and passing through E. The area dA is shown dotted in figure 19.

Now, the factor from dA to the sphere is the same as the factor from dA to the circular disk of radius k. This is so because any ray of radiant energy emanating from dA and striking the sphere would also strike the disk and vice versa — if it strikes the disk it would also strike the sphere. Thus the factor from dA to the sphere is

$$F_{dA-A_1} = \frac{k^2}{g^2 + k^2} ,$$

which is obtained by letting $r = 0$ in equation (17) and where $g = QE$. But, because dA and dA_2 are infinitesimal,

$$F_{A_1-dA} = F_{A_1-dA_2} ,$$

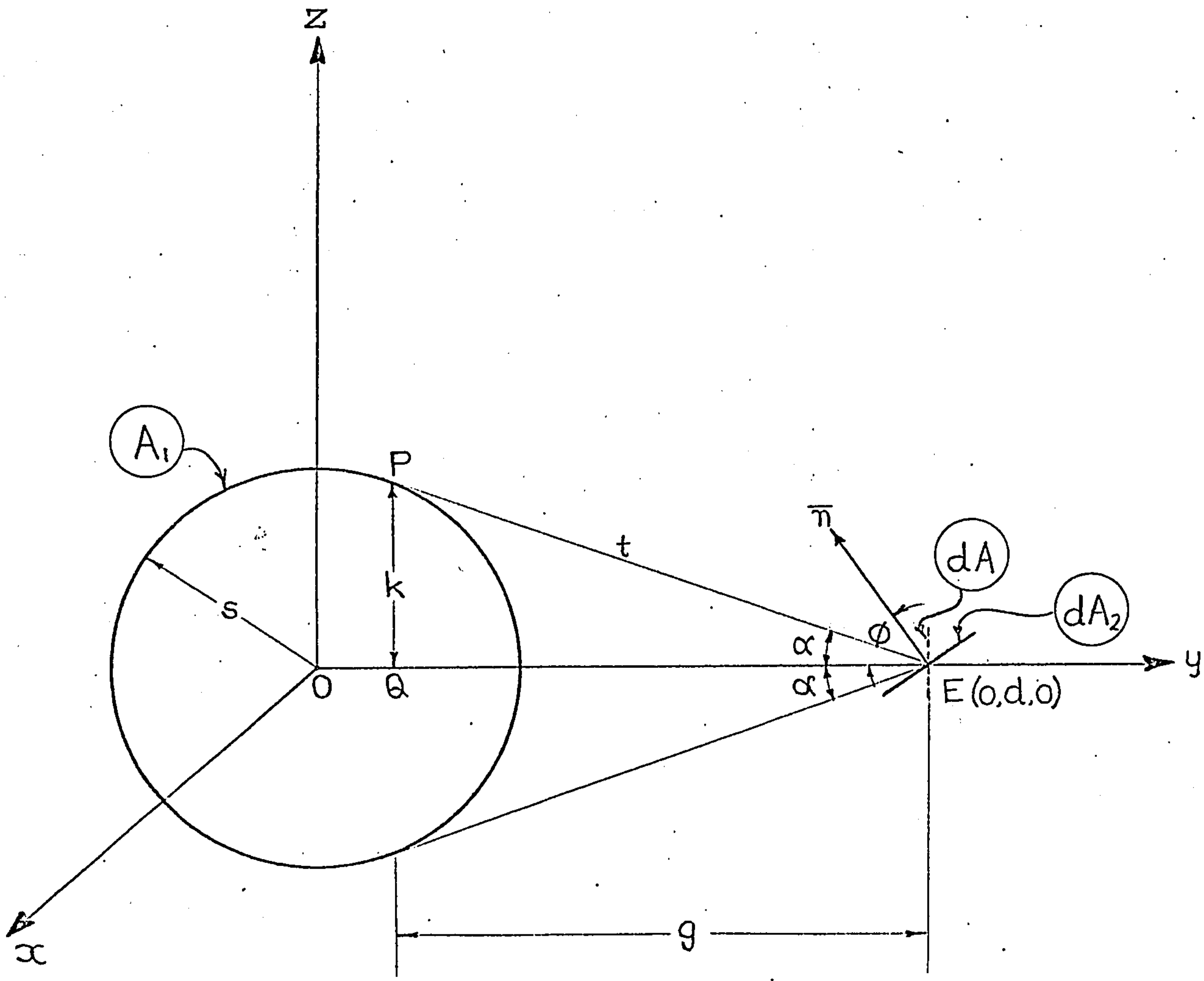


FIGURE 19. $\alpha \leq \phi \leq \frac{\pi}{2} - \alpha$.

(again a ray emanating from any point on the sphere and striking dA, would also strike dA_2 and vice versa)

By reciprocity rule

$$\frac{dA}{A_1} F_{dA-A_1} = \frac{dA_2}{A_1} F_{dA_2-A_1} ,$$

therefore,

$$F_{dA_2-A_1} = \frac{dA}{dA_2} F_{dA-A_1} = \frac{dA_2 \cos\phi}{dA_2} \frac{k^2}{g^2+k^2}$$

$$= \frac{k^2}{g^2+k^2} \cos\phi \; . \qquad \text{------------(31)}$$

This result can also be obtained with a much greater expenditure of energy by the usual double integration or by contour integration as shown in the Appendix[12].

Let us now rotate the y-z plane about x-axis such that dA_2 becomes parallel to the x-z' plane in figure 20. Let the projection of point E on the y'-axis be E', and EE' and OE' be denoted as r and a respectively. Hence

$$d^2 = a^2 + r^2 ,$$

and

$$\cos\phi = \frac{a}{\sqrt{a^2+r^2}} \; .$$

Rewriting equation (31), and noting that

$$t^2 = g^2 + k^2 = d^2 - s^2 ,$$

we have

$$F_{dA_2-A_1} = \frac{k^2}{g^2+k^2} \cos\phi ,$$

$$= \frac{k^2}{d^2-s^2} \frac{a}{\sqrt{a^2+r^2}} \; .$$

Inserting the value of k^2 from equation (30), we obtain

$$F_{dA_2-A_1} = \frac{s^2}{d^2} \frac{a}{\sqrt{a^2+r^2}} ,$$

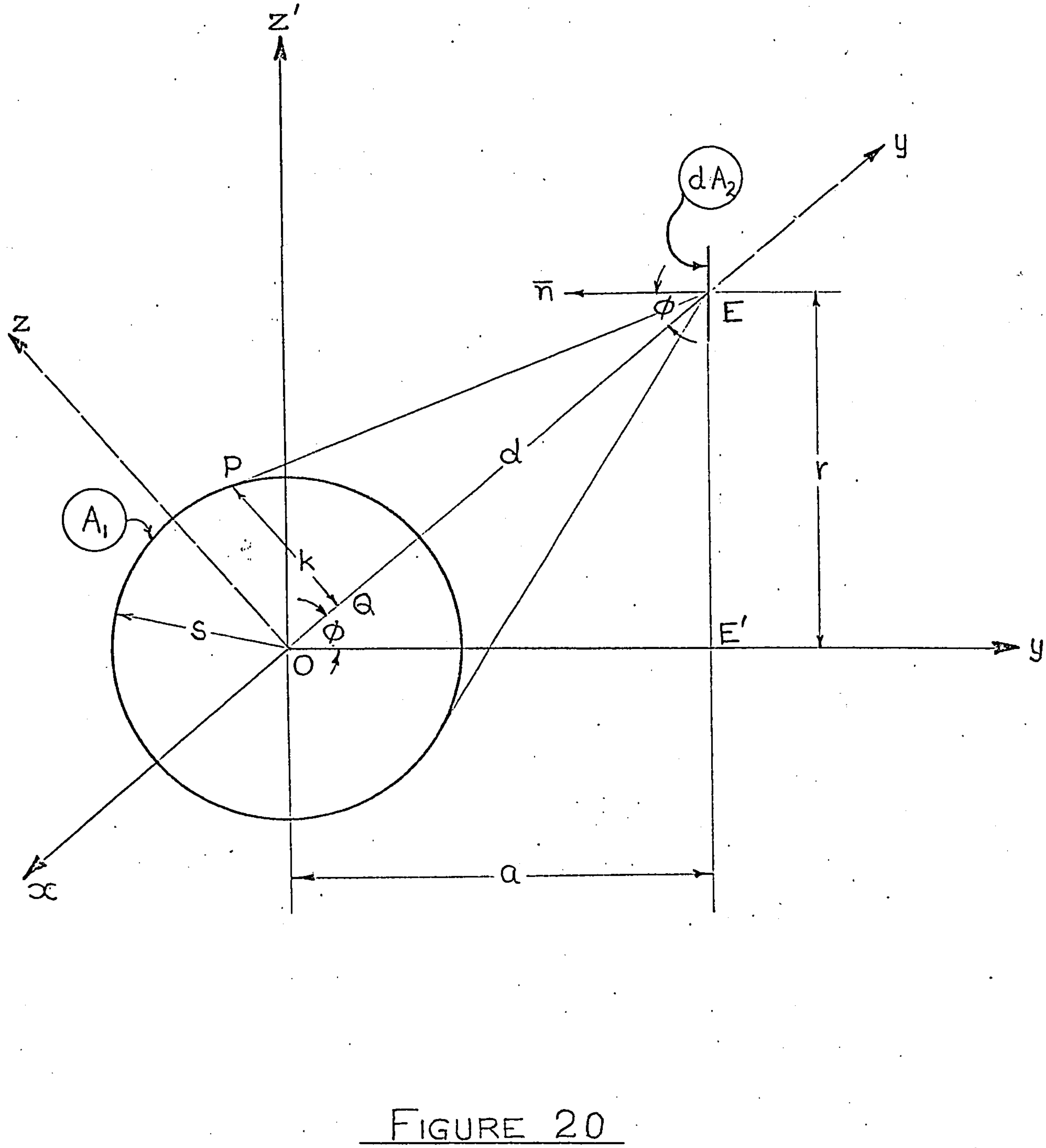


FIGURE 20

or

$$F_{dA_2-A_1} = \frac{a s^2}{(a^2+r^2)^{3/2}} .$$

Using the reciprocity rule,

$$F_{A_1-dA_2} = \frac{dA_2}{A_1} F_{dA_2-A_1} = \frac{dA_2}{A_1} \frac{as^2}{(a^2+r^2)^{3/2}} . \qquad \text{------(32)}$$

Choosing the polar coordinates, $dA_2 = r\ d\theta\ dr$. The factor from the sphere to the ring of thickness dr and radius r is obtained by integrating equation (32) with respect to θ as follows.

$$F_{A_1-\text{inf.ring}} = \int_0^{2\pi} \frac{r\,dr}{4\pi s^2} \frac{a s^2}{(a^2+r^2)^{3/2}} d\theta ,$$

$$= \frac{2\pi r\,dr}{4\pi s^2} \frac{a s^2}{(a^2+r^2)^{3/2}} ,$$

$$= \frac{a}{2(a^2+r^2)^{3/2}} r\,dr. \qquad \text{----------(33)}$$

The factor from the sphere to a coaxial circular disk of radius r is obtained by integrating equation (33) as shown below. (see figure 21).

$$F_{s-d} = \int_0^r F_{A_1-\text{inf.ring}} ,$$

$$= \int_0^r \frac{ar}{2(a^2+r^2)^{3/2}} dr ,$$

$$= \frac{a}{2} \left[\frac{-1}{(a^2+r^2)^{1/2}} \right]_0^r ,$$

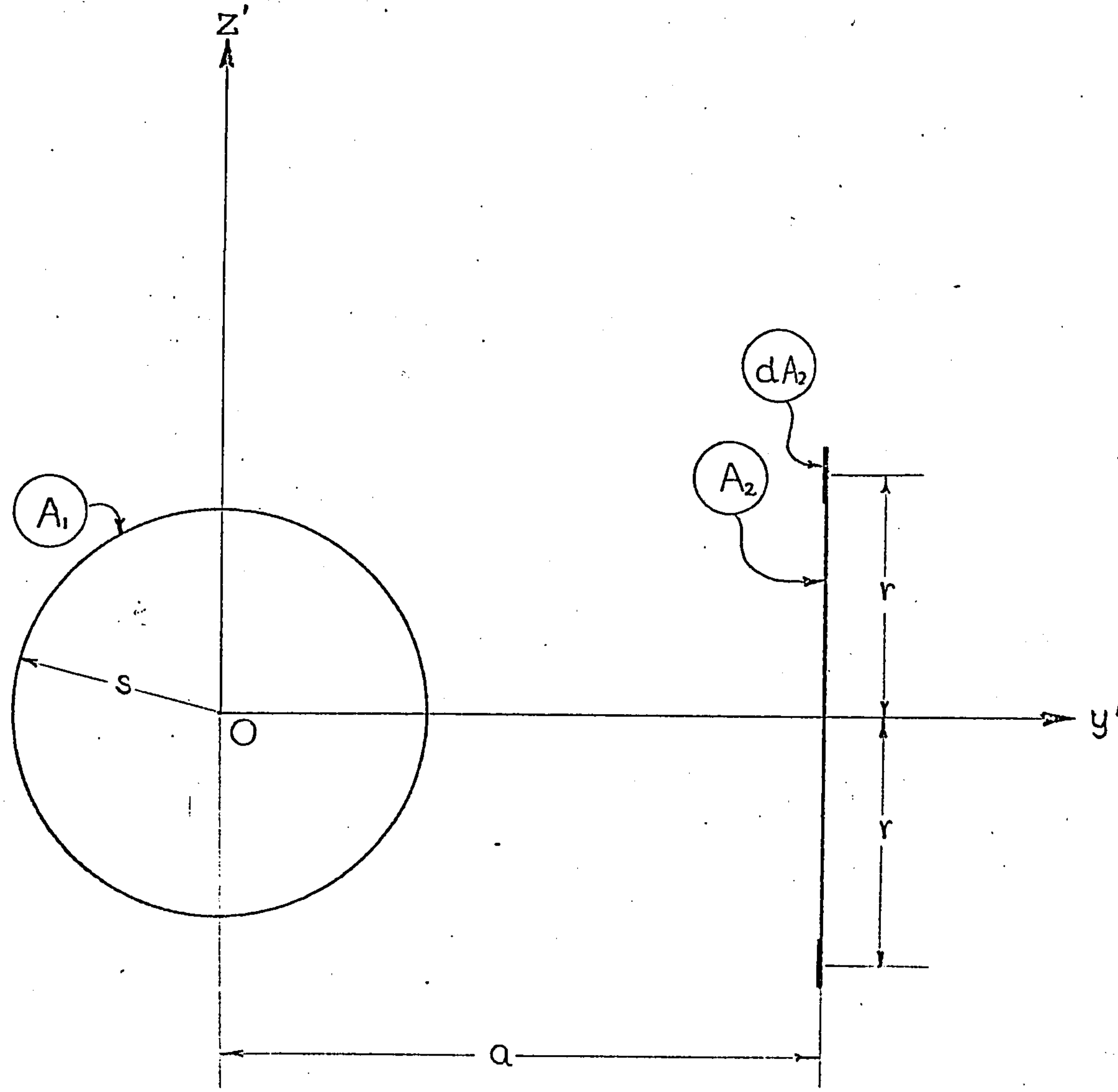


FIGURE 21

$$= \frac{-a}{2}\left[\frac{1}{\sqrt{a^2+r^2}} - \frac{1}{a}\right]$$

$$= \frac{1}{2}\left[1 - \frac{a}{\sqrt{a^2+r^2}}\right]. \qquad \text{--------------(34)}$$

Defining the ratio $R=r/a$, we have the dimensionless form of equation (34) as

$$= \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+R^2}}\right]. \qquad \text{-----------(34')}$$

Thus having found the factor from a sphere to an infinitesimal area, to a ring, and to a circular disk, we can now proceed further to obtain the factors for some other configurations, which are dealt with below.

It should be pointed out, however, that an entirely new approach developed in the next section makes it possible to obtain the results arrived at in this section with almost no intermediate mathematical passages.

VII. NEW METHOD OF EVALUATION OF CONFIGURATION FACTORS FROM A SPHERE TO: 1) A COAXIAL CIRCULAR DISK; 2) AN INFINITESIMAL COAXIAL RING; 3) AN INFINITESIMAL AREA LYING IN A PLANE WHICH DOES NOT INTERSECT THE SPHERE

In the existing literature on radiant-interchange configuration factors, the underlying assumption is that the directional distribution of the emitted radiation follows Lambert's cosine law. A new technique is developed here to calculate the configuration factor from spheres to a certain class of surfaces and it is shown that in these cases the factors are independent of the validity of Lambert's law.

1) Configuration factor from a sphere to a coaxial circular disk

As before, we shall consider a disk coaxial with a given sphere and lying in a plane perpendicular to the line joining their centers.

Figure 22 represents a sphere and the projection of the coaxial disk AB. The factor from the sphere to the disk AB is the same as the factor from the sphere to the hollow spherical surface ACB, provided that AB does not intersect the sphere. This is so because any ray, emanating from any point on the radiating sphere and striking the disk, would necessarily also strike the surface ACB. The opposite is equally true — thus, any ray striking ACB could not come from the radiating sphere without passing through the circle AB. This observation is the keystone of the method developed below.

It must further be observed that the irradiation per unit area on the surface of the outer sphere is the same at every point quite irrespectively of the validity of Lambert's cosine law; provided only that, whatever is the angular distribution of emanating energy, this distribution does not vary throughout the radiating surface. This represents no real restriction, because in most practial cases the surfaces of radiating spheres can be considered homogeneous, while not necessarily behaving in accordance with Lambert's law.

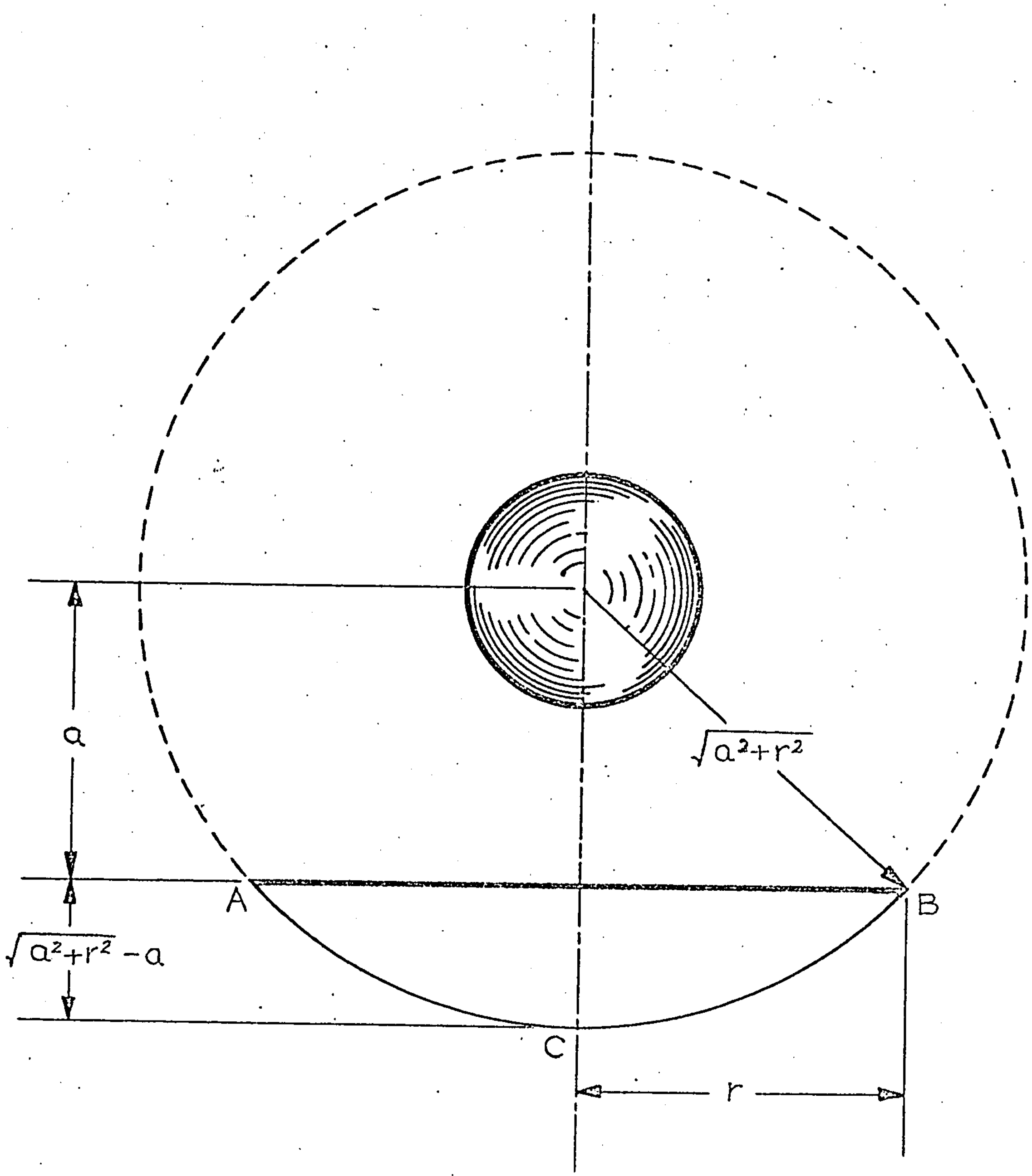


FIGURE 22

The factor from the radiating sphere to the outer sphere is, obviously unity. Thus, bearing in mind our previous remarks, we must conclude that the factor from the sphere to the area ACB (and with it the factor from the sphere to the coaxial disk) is equal to the ratio between the area of the spherical segment ACB and the total area of the outer sphere. Thus, the factor from the sphere to the disk is

$$F_{s-d} = \frac{2\pi\sqrt{a^2+r^2}\left(\sqrt{a^2+r^2}-a\right)}{4\pi(a^2+r^2)},$$

$$= \frac{1}{2} - \frac{a}{2\sqrt{a^2+r^2}} . \qquad \text{------------(35)}$$

This is same as equation (34) obtained before.

In order to obtain this factor as a function of a dimensionless quantity, let us introduce the ratio $R=r/a$ which may conveniently be called "the relative radius." After simplification, we obtain

$$F_{s-d} = \frac{1}{2} - \frac{1}{2\sqrt{1+R^2}} . \qquad \text{------------(35')}$$

For a sector of the disk (figure 23) the factor is

$$F_{s-sector} = \frac{\alpha}{2\pi}\left[\frac{1}{2} - \frac{a}{2\sqrt{a^2+r^2}}\right], \qquad \text{------------(36)}$$

or

$$F_{s-sector} = \frac{\alpha}{2\pi}\left[\frac{1}{2} - \frac{1}{2\sqrt{1+R^2}}\right]. \qquad \text{------------(36')}$$

2) Configuration factor from a sphere to an infinitesimal coaxial ring

We shall now consider the differential increase in the factor F_{s-d} as r grows from a value x to x+dx. This increase represents the factor from a sphere to an infinitesimal ring of radius x and width dx.

$$dF_{s-inf.r} = \frac{ax}{2}(a^2+x^2)^{-\frac{3}{2}}\,dx . \qquad \text{------------(37)}$$

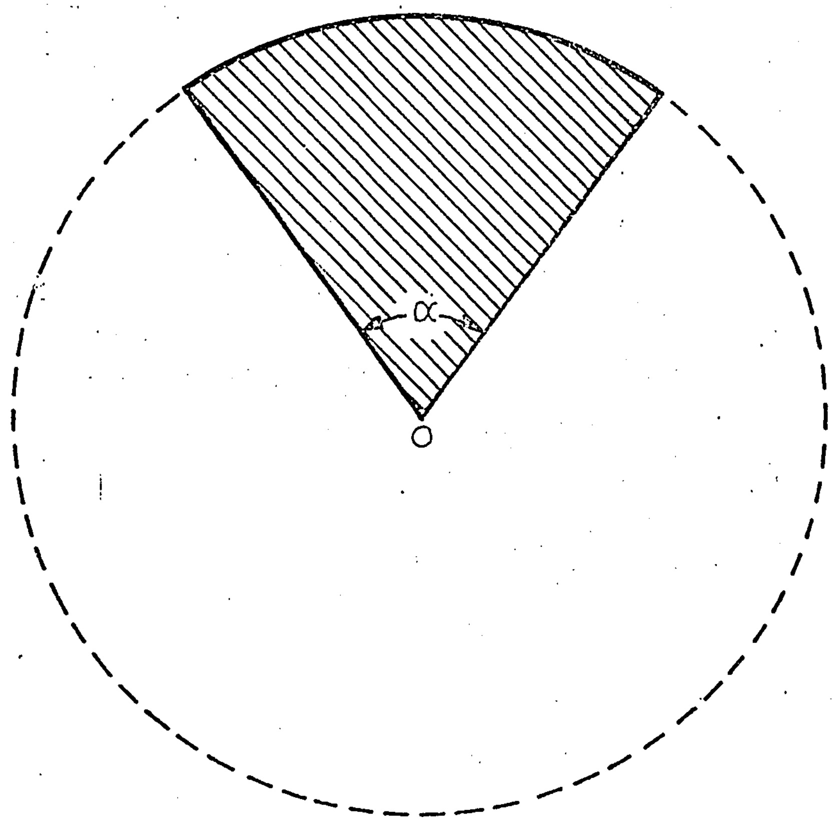


FIGURE 23

For a portion of such a ring subtending an angle α at the center (figure 24), the factor is

$$dF_{s-seg.inf.r} = \frac{\alpha}{2\pi} \frac{ax}{2} (a^2+x^2)^{-\frac{3}{2}} dx. \quad \text{..........(38)}$$

The dimensionless form of these equations is obtained by defining the ratio $X = x/a$ and substituting aX for x and $a\,dx$ for dx. After simplification, we get

$$dF_{s-inf.r} = \frac{X}{2} (1+X^3)^{-\frac{3}{2}} dX. \quad \text{..............(37')}$$

$$dF_{s-seg.inf.r} = \frac{\alpha}{2\pi} \frac{X}{2} (1+X^2)^{-\frac{3}{2}} dX. \quad \text{..........(38')}$$

<u>3) Configuration factor from a sphere to an infinitesimal area lying in a plane which does not intersect the sphere</u>

It will be useful to consider at this point the factor from a sphere to a unit area on a coaxial ring. This factor per unit area is

$$dF_{s-unit\,area} = \frac{dF_{s-inf.r}}{2\pi x\, dx},$$

$$= \frac{a}{4\pi} (a^2+x^2)^{-\frac{3}{2}}.$$

Therefore, the factor from sphere to a differential element of area lying in a plane which does not intersect the sphere is

$$dF_{s-dA_R} = \frac{a}{4\pi} (a^2+x^2)^{-\frac{3}{2}} dA_R, \quad \text{..........(39)}$$

where x is the distance between this element and the perpendicular drawn from the center of the sphere to the plane of this element.

Putting $X = x/a$ and $dA = dA_R / a^2$, we get

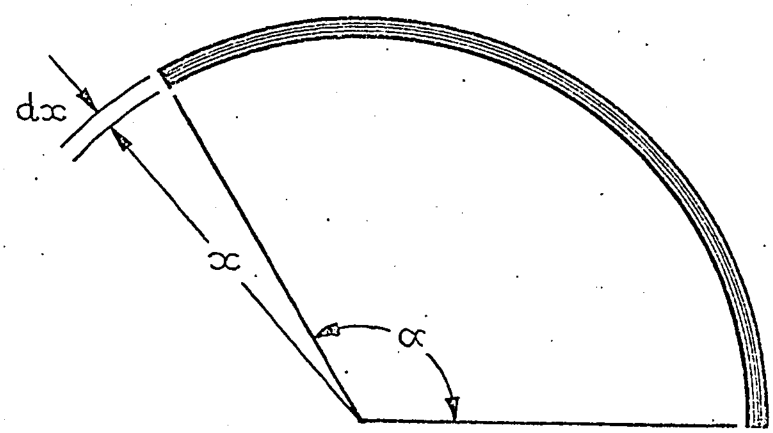


FIGURE 24

$$dF_{s-dA} = \frac{1}{4\pi}(1+X^2)^{-\frac{3}{2}}\, dA. \qquad (39')$$

It should now be noted that equation (39') has been developed from equation (35) through a process which required no reference to Lambert's law and is, therefore, just as independent from this law as equation (35), itself. Further, this independence applies equally to any finite area lying in a plane which does not intersect the sphere.

A digression may be in order at this point in relation to the reciprocity law. It is well-known that this law is in general not valid for non-Lambertian surfaces. Since the factors obtained here are valid also for Lambertian surfaces; the reciprocity law will apply, provided only that the surface paired with the sphere follows Lambert's law.

VIII. CONFIGURATION FACTOR FROM A SPHERE TO A SEGMENT OF A COAXIAL DISK

Figure 25 represents the disk AB in figure 22, viewed from above. Point O is the projection of the center of the sphere. This projection coincides with the center of the disk.

The angle subtended at O by the shaded ring is $2\cos^{-1}\frac{h}{x}$. Thus, utilising equation (38), we obtain

$$F_{s\text{-}seg.} = \int_h^r \frac{2\cos^{-1}\frac{h}{x}}{2\pi}\,\frac{ax}{2}(a^2+x^2)^{-\frac{3}{2}}\,dx\,,$$

$$= \frac{a}{2\pi}\int_h^r \cos^{-1}\frac{h}{x}\,\frac{x}{(a^2+x^2)^{3/2}}\,dx\,,$$

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{x}}{\sqrt{a^2+x^2}} + \int \frac{x}{\sqrt{x^2-h^2}}\,\frac{h}{x^2}\,\frac{1}{\sqrt{a^2+x^2}}\,dx\right]_h^r,$$

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{x}}{\sqrt{a^2+x^2}} + \frac{h}{2}\int \frac{2x\,dx}{x^2\sqrt{x^4+x^2(a^2-h^2)-a^2h^2}}\right]_h^r \quad (13),$$

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{x}}{\sqrt{a^2+x^2}} + \frac{h}{2}\,\frac{1}{ah}\sin^{-1}\frac{(a^2-h^2)x^2-2a^2h^2}{x^2\sqrt{(a^2-h^2)^2+4a^2h^2}}\right]_h^r,$$

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{x}}{\sqrt{a^2+x^2}} + \frac{1}{2a}\sin^{-1}\frac{(a^2-h^2)x^2-2a^2h^2}{x^2(a^2+h^2)}\right]_h^r,$$

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{r}}{\sqrt{a^2+r^2}} + \frac{\cos^{-1}1}{\sqrt{a^2+h^2}} + \frac{1}{2a}\sin^{-1}\frac{(a^2-h^2)r^2-2a^2h^2}{r^2(a^2+h^2)} - \frac{1}{2a}\sin^{-1}\frac{(a^2-h^2)h^2-2a^2h^2}{h^2(a^2+h^2)}\right]$$

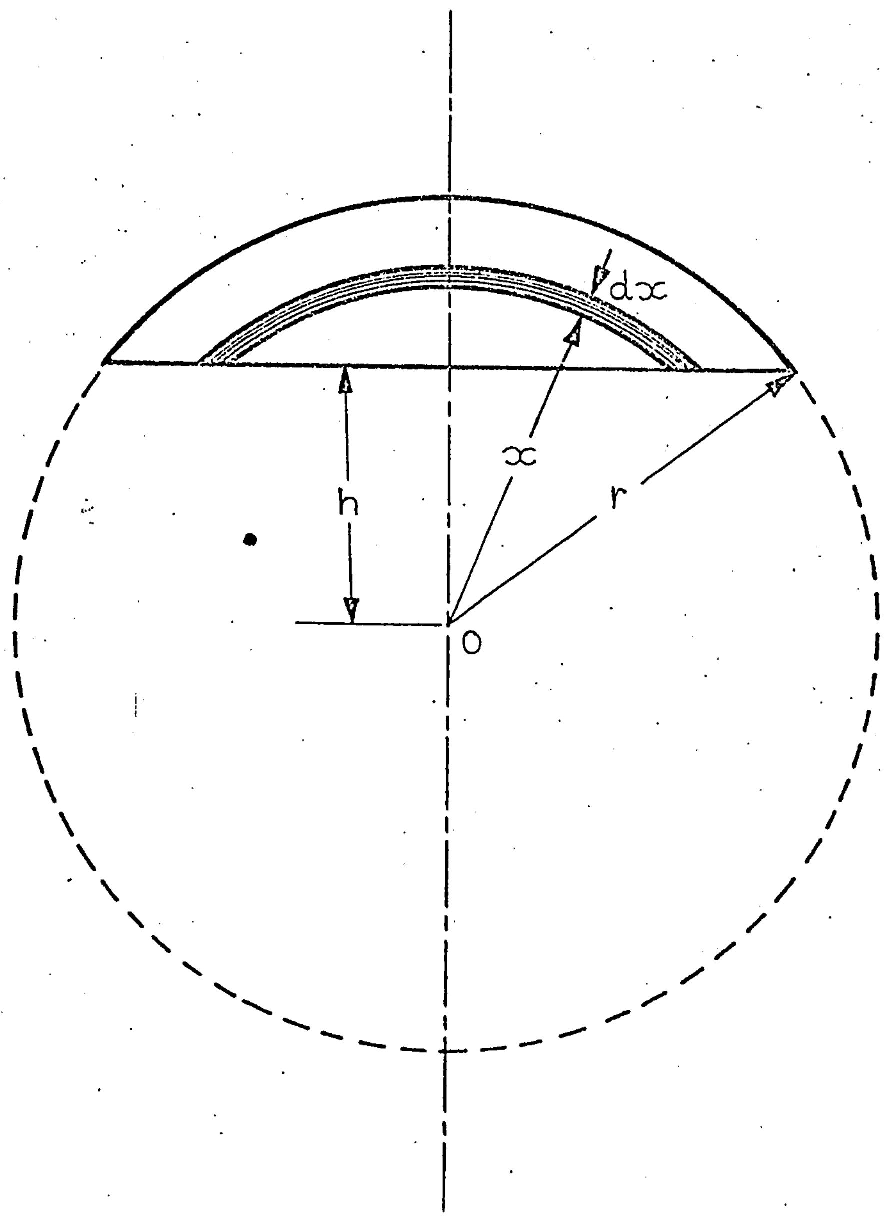


FIGURE 25

$$= \frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{h}{r}}{\sqrt{a^2+r^2}} + 0 + \frac{1}{2a}\sin^{-1}\frac{(a^2-h^2)r^2-2a^2h^2}{r^2(a^2+h^2)} - \frac{1}{2a}\sin^{-1}(-1)\right],$$

$$= -\frac{a\cos^{-1}\frac{h}{r}}{2\pi\sqrt{a^2+r^2}} + \frac{1}{4\pi}\sin^{-1}\frac{(a^2-h^2)r^2-2a^2h^2}{r^2(a^2+h^2)} - \frac{1}{4\pi}\left(-\frac{\pi}{2}\right),$$

$$= \frac{1}{8} - \frac{a\cos^{-1}\frac{h}{r}}{2\pi\sqrt{a^2+r^2}} + \frac{1}{4\pi}\sin^{-1}\frac{(a^2-h^2)r^2-2a^2h^2}{r^2(a^2+h^2)}. \quad \text{------------} (40)$$

Equation (40) is transformed to a dimensionless form by the substitutions $R = r/a$ and $H = h/a$.

$$F_{s-seg.} = \frac{1}{8} - \frac{\cos^{-1}\frac{H}{R}}{2\pi\sqrt{1+R^2}} + \frac{1}{4\pi}\sin^{-1}\frac{(1-H^2)R^2-2H^2}{R^2(1+H^2)}. \text{------} (40')$$

It may perhaps be useful to underline the method employed in this section which can be used to an advantage in calculating factors from spheres to other plane areas. The multiple integration, commonly resorted to in the existing literature, has here been replaced by a single integral.

The graphic representation of the dependence of $F_{s-seg.}$ on R and on a ratio $Z = H/R$ is shown in figure 26.

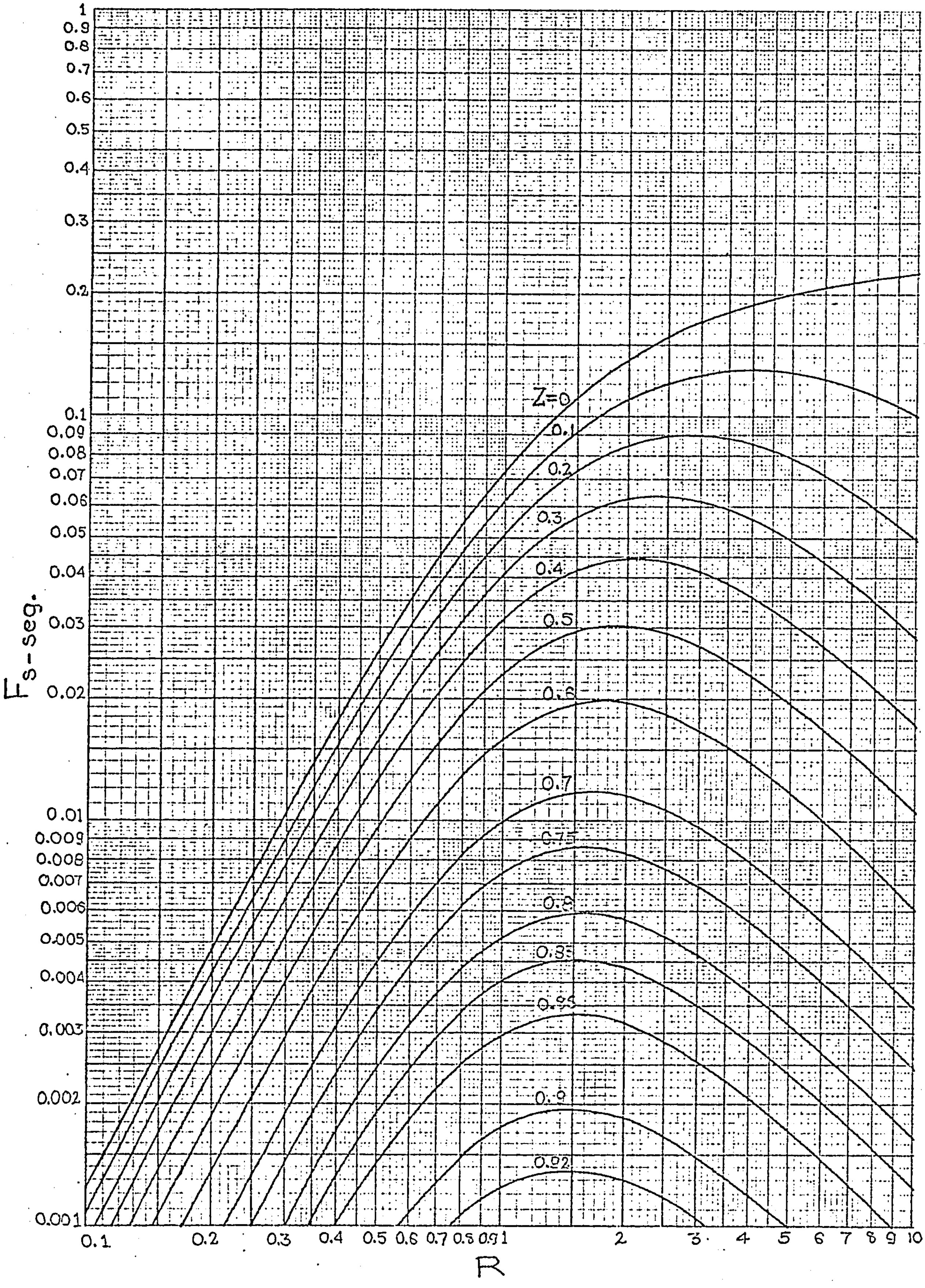


FIGURE 26. Configuration factors from a sphere to a segment of a coaxial disk.

IX. CONFIGURATION FACTOR FROM A SPHERE TO A COAXIAL RECTANGLE

We shall say that a rectangle is coaxial with a given sphere when it lies in a plane perpendicular to the line joining its center with the center of the sphere. Figure 27 represents the view from above of a rectangle measuring $2b_1 \times 2b_2$, which is circumscribed by a circle of radius $\sqrt{b_1^2 + b_2^2}$. The center of the rectangle, 0, coincides with the projection of the center of the sphere.

The factor from the sphere to this rectangle can be obtained by subtracting the sum of the factors from the sphere to the four segments from the factor from the sphere to the circle; it can, therefore, be easily calculated with the help of equation (35) and (40). Thus,

$$F_{s-rect.} = F_{s-d} - 2\int_{b_1}^{\sqrt{b_1^2+b_2^2}} dF_{s-seg.inf.r} - 2\int_{b_2}^{\sqrt{b_1^2+b_2^2}} dF_{s-seg.inf.r} ,$$

$$= \frac{1}{2} - \frac{a}{2\sqrt{a^2+b_1^2+b_2^2}} - 2\int_{b_1}^{\sqrt{b_1^2+b_2^2}} \frac{2\cos^{-1}\frac{b_1}{x}}{2\pi} \frac{ax}{2}(a^2+x^2)^{-\frac{3}{2}} dx - 2\int_{b_2}^{\sqrt{b_1^2+b_2^2}} \frac{2\cos^{-1}\frac{b_2}{x}}{2\pi} \frac{ax}{2}(a^2+x^2)^{-\frac{3}{2}} dx ,$$

$$= \frac{1}{2} - \frac{a}{2\sqrt{a^2+b_1^2+b_2^2}}$$

$$- 2\frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{b_1}{x}}{\sqrt{a^2+x^2}} + \frac{1}{2a}\sin^{-1}\frac{(a^2-b_1^2)x^2-2a^2b_1^2}{x^2(a^2+b_1^2)} \right]_{b_1}^{\sqrt{b_1^2+b_2^2}}$$

$$- 2\frac{a}{2\pi}\left[-\frac{\cos^{-1}\frac{b_2}{x}}{\sqrt{a^2+x^2}} + \frac{1}{2a}\sin^{-1}\frac{(a^2-b_2^2)x^2-2a^2b_2^2}{x^2(a^2+b_2^2)} \right]_{b_2}^{\sqrt{b_1^2+b_2^2}} ,$$

$$= \frac{1}{2} - \frac{a}{2\sqrt{a^2+b_1^2+b_2^2}} - \frac{a}{\pi}\left[-\frac{\cos^{-1}\frac{b_1}{\sqrt{b_1^2+b_2^2}}}{\sqrt{a^2+b_1^2+b_2^2}} + \frac{\cos^{-1}(1)}{\sqrt{a^2+b^2+b_2^2}} \right.$$

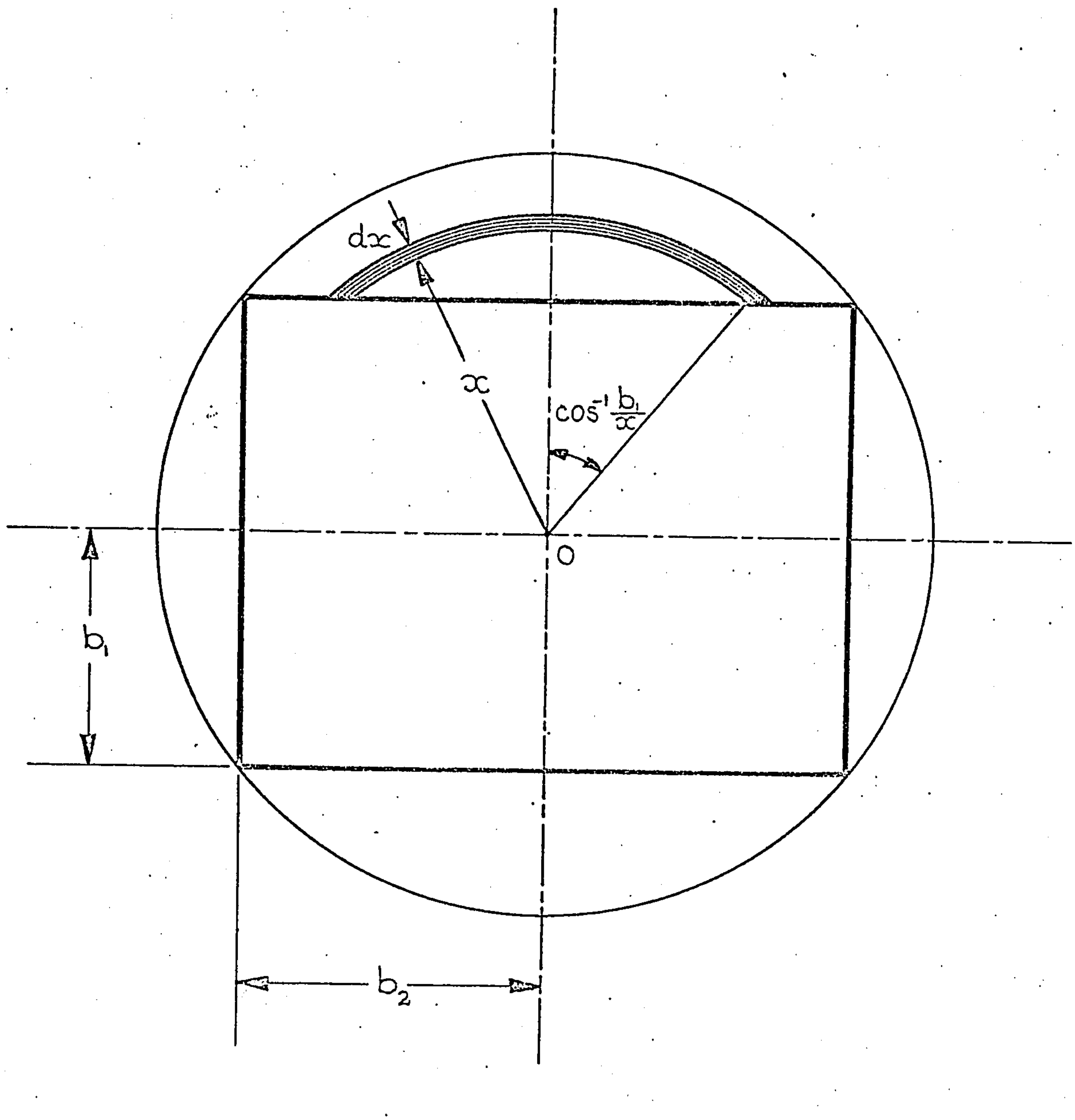


FIGURE 27

$$+\frac{1}{2a}\sin^{-1}\frac{(a^2-b_1^2)(b_1^2+b_2^2)+2a^2b_1^2}{(b_1^2+b_2^2)(a^2+b_1^2)}-\frac{1}{2a}\sin^{-1}\frac{(a^2-b_1^2)b_1^2-2a^2b_1^2}{b_1^2(a^2+b_1^2)}\Bigg]-\frac{a}{\pi}\Bigg[-\frac{\cos^{-1}\frac{b_2}{\sqrt{b_1^2+b_2^2}}}{\sqrt{a^2+b_1^2+b_2^2}}$$

$$+\frac{\cos^{-1}(1)}{\sqrt{a^2+b_1^2+b_2^2}}+\frac{1}{2a}\sin^{-1}\frac{(a^2-b_2^2)(b_1^2+b_2^2)-2a^2b_2^2}{(b_1^2+b_2^2)(a^2+b_2^2)}-\frac{1}{2a}\sin^{-1}\frac{(a^2-b_2^2)b_2^2-2a^2b_2^2}{(b_2^2)(a^2+b_2^2)}\Bigg]$$

$$=\frac{1}{2}-\frac{a}{2\sqrt{a^2+b_1^2+b_2^2}}+\frac{a\cos^{-1}\frac{b_1}{\sqrt{b_1^2+b_2^2}}}{\pi\sqrt{a^2+b_1^2+b_2^2}}-\frac{a}{\pi}(0)$$

$$+\frac{1}{2\pi}\sin^{-1}\frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)}+\frac{1}{2\pi}\sin^{-1}(-1)+\frac{a\cos^{-1}\frac{b_2}{\sqrt{b_1^2+b_2^2}}}{\pi\sqrt{a^2+b_1^2+b_2^2}}$$

$$-\frac{a}{\pi}(0)+\frac{1}{2\pi}\sin^{-1}\frac{2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)}+\frac{1}{2\pi}\sin^{-1}(-1)\ ,$$

$$=\frac{1}{2}-\frac{a}{\pi}\frac{1}{\sqrt{a^2+b_1^2+b_2^2}}\left[\frac{\pi}{2}-\cos^{-1}\frac{b_1}{\sqrt{b_1^2+b_2^2}}-\cos^{-1}\frac{b_2}{\sqrt{b_1^2+b_2^2}}\right]\ ,$$

$$+\frac{1}{2\pi}\sin^{-1}\frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)}-\frac{1}{4}$$

$$+\frac{1}{2\pi}\sin^{-1}\frac{2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)}-\frac{1}{4}\ ,$$

$$=\frac{1}{2\pi}\left[\sin^{-1}\frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)}+\sin^{-1}\frac{2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)}\right]\quad\ldots(41)$$

After introducing the ratios $B_1=b_1/a$ and $B_2=b_2/a$, the nondimensional form of equation (41) becomes

$$F_{s\text{-rect.}} = \frac{1}{2\pi}\left[\sin^{-1}\frac{2B_1^2-(1-B_1^2)(B_1^2+B_2^2)}{(1+B_1^2)(B_1^2+B_2^2)} + \sin^{-1}\frac{2B_2^2-(1-B_2^2)(B_1^2+B_2^2)}{(1+B_2^2)(B_1^2+B_2^2)}\right] \quad (41')$$

Equation (41) is equivalent to the expression obtained by Mackey, Wright, Clark and Gay (reference 3), for the factor from sphere to a coaxial rectangle. This equivalence is shown in the Appendix [(14)]. These authors assumed that the size of the sphere is very small as compared to the dimensions of the rectangle. The same factor has later been obtained independently by Wilson, Hwang and Crank (reference 7) by means of the usual method of quadruple integration, involving in this case some rather complicated three-dimensional drawings. Their contribution lies in the fact that they have shown that the radius of the sphere does not influence the factor, provided that the plane of the rectangle does not cut the sphere. They have, thus, found that the restriction imposed by Mackey et al. (reference 3) was unnecessary. However, the conventional approach used by these authors precluded them from discovering that the assumption of validity of Lambert's law is also superfluous.

X. CONFIGURATION FACTOR FROM A SPHERE TO A COAXIAL RIGHT CIRCULAR CYLINDER

Surprisingly simple expression is obtained for the factor from a sphere to a right circular cylinder of length 2a and radius r, when the sphere is placed at the center of the cylinder as shown in figure 28.

Because the factor from the sphere to an enclosure containing this sphere is unity, the factor from the sphere to the cylinder can be obtained by subtracting from unity the factors from sphere to the two bases. Using equation (35), we have

$$F_{s\text{-}cyl.} = 1 - 2\left[\frac{1}{2} - \frac{a}{2\sqrt{a^2+r^2}}\right],$$

$$= \frac{a}{\sqrt{a^2+r^2}} .$$

After the transformation $R = r/a$, the factor becomes

$$F_{s\text{-}cyl.} = \frac{1}{\sqrt{1+R^2}} .$$

If the sphere and the cylinder are coaxial but not concentric as shown in figure 29, the relevant self-explanatory equation is

$$F_{s\text{-}cyl.} = \frac{1}{2}\left[\frac{a_2}{\sqrt{a_2^2+r^2}} - \frac{a_1}{\sqrt{a_1^2+r^2}}\right] .$$

Putting $R_1 = r/a_1$ and $R_2 = r/a_2$, we obtain

$$F_{s\text{-}cyl.} = \frac{1}{2}\left[\frac{1}{\sqrt{1+R_2^2}} - \frac{1}{\sqrt{1+R_1^2}}\right] .$$

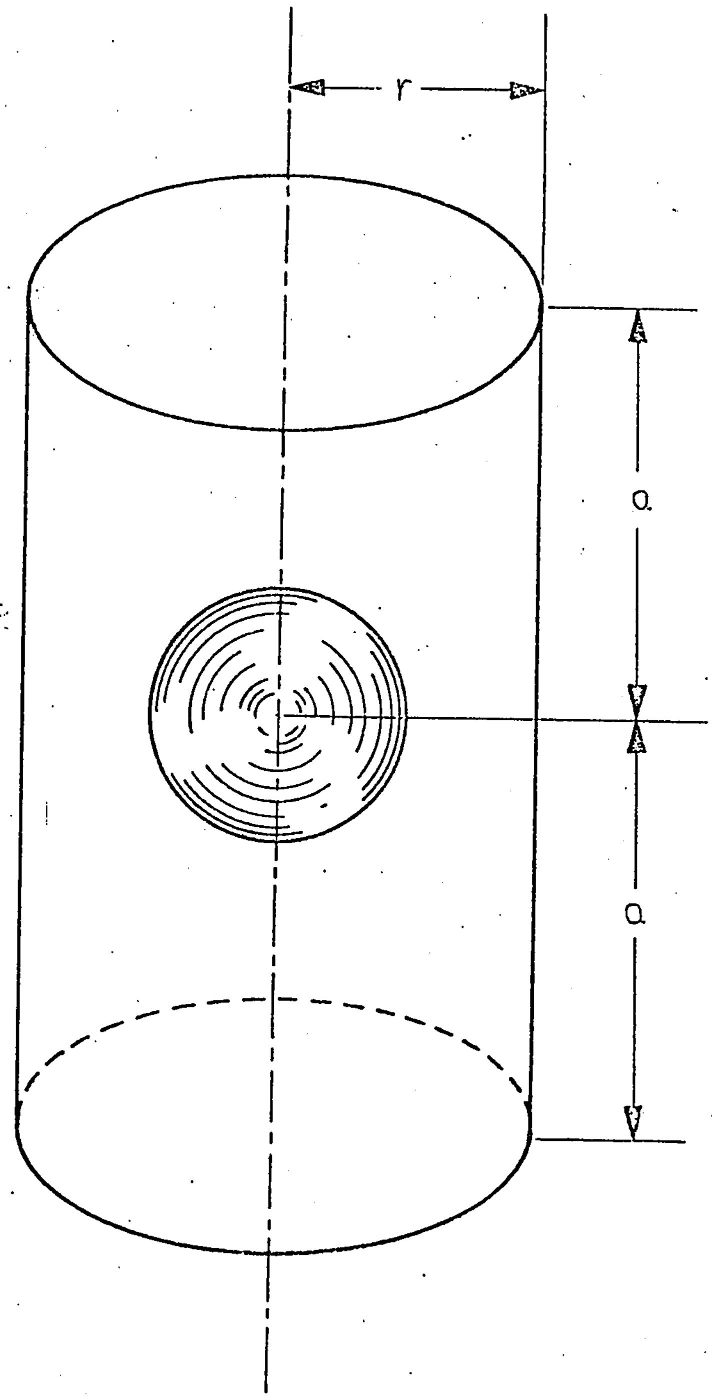


FIGURE 28

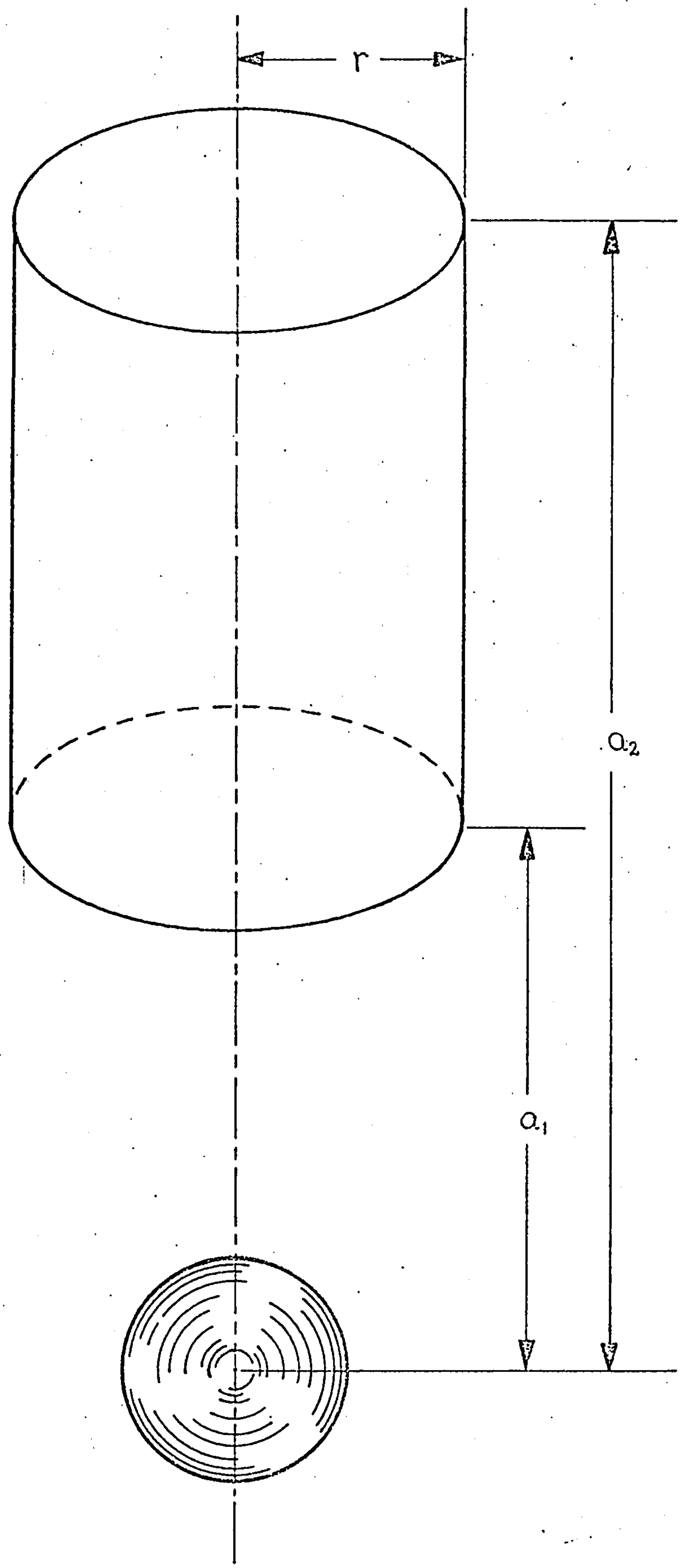


FIGURE 29

XI. CONFIGURATION FACTOR FROM A SPHERE TO A POLYGON

Let us consider an arbitrary n-sided polygon lying in a plane which does not intersect the radiating sphere, e.g. the polygon ABCDEA in figure 30. Let point O to be the projection of the center of the sphere on that plane. This projection can fall either within the polygon, or without. It will be sufficient to examine the latter case, because it is more comprehensive than the former, as will readily be understood from the following analysis.

By joining the corners of the polygon to the point O, we obtain n triangles with a common vertex at that point. The problem is now reduced to the evaluation of the factor from a sphere to an arbitrary triangle with a vertex at O. The factor from the sphere to the polygon will be a simple algebraic sum of n such sphere-to-triangle factors, provided that the triangles which lie completely outside the polygon are considered negative.

Let the triangle OCD in figure 31 be one of the above-mentioned triangles. We shall now draw two circles with the common center at O and with the respective radii OC and OD. The factor $F_{s-\Delta OCF}$ (from the sphere to the triangle OCF) can be obtained by subtracting a factor to a segment (equation (40)) from a factor to a sector (equation (36)). The same applies to the factor $F_{s-\Delta ODE}$ (from the sphere to the triangle ODE).

Clearly,

$$F_{s-\Delta OCD} = \frac{1}{2}\left[F_{s-\Delta OCF} - F_{s-\Delta ODE}\right],$$

where

$$F_{s-\Delta OCF} = \left[\frac{1}{2} - \frac{a}{2\sqrt{a^2+OC^2}}\right]\frac{\cos^{-1}\frac{OP}{OC}}{\pi} - \left[\frac{1}{8} - \frac{a\cos^{-1}\frac{OP}{OC}}{2\pi\sqrt{a^2+OC^2}} + \frac{1}{4\pi}\sin^{-1}\frac{(a^2-OP^2)OC^2-2a^2OP^2}{(a^2+OP^2)OC^2}\right],$$

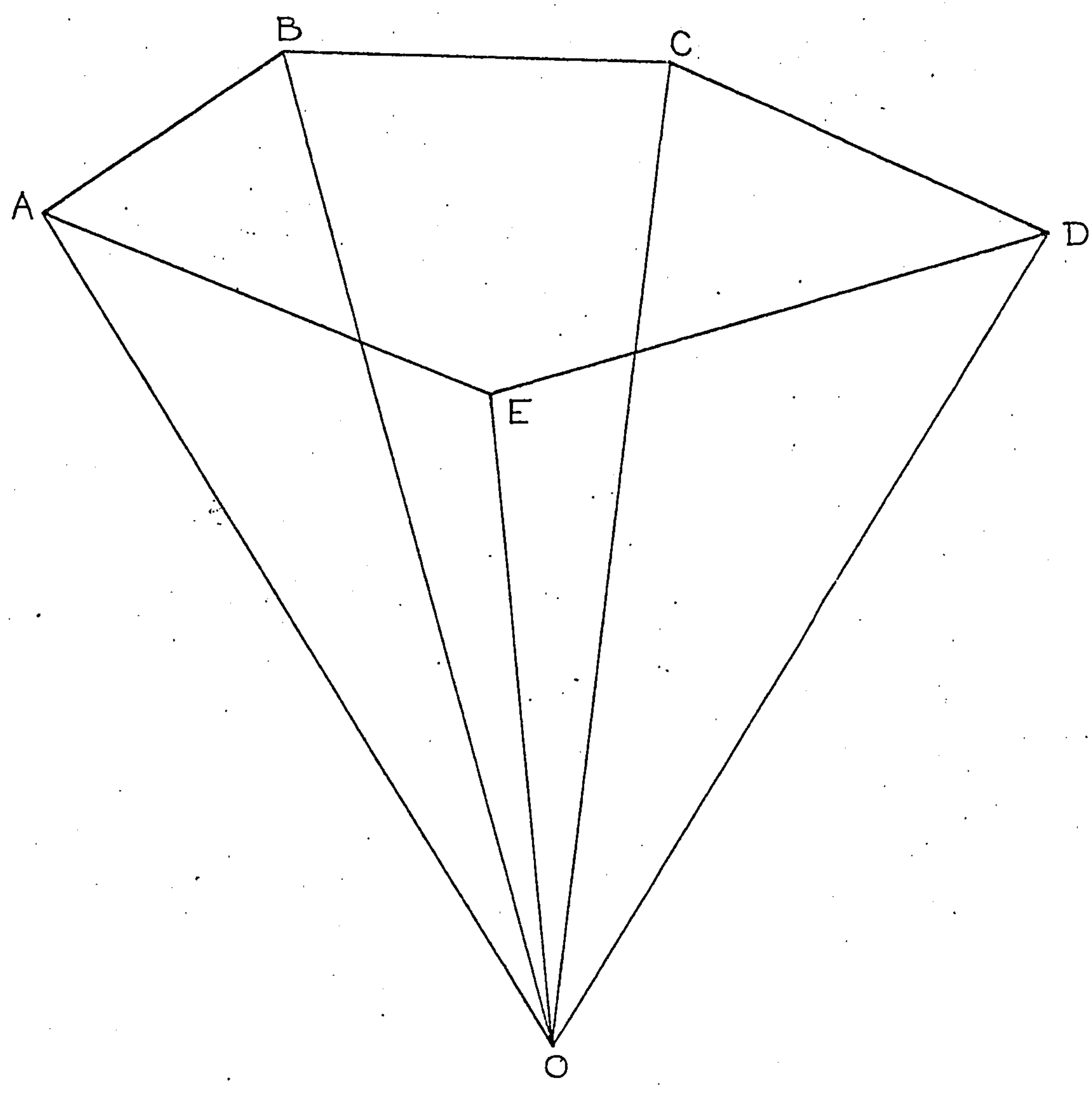


FIGURE 30

$$= \frac{\cos^{-1}\frac{OP}{OC}}{2\pi} - \frac{1}{4\pi}\sin^{-1}\frac{(a^2-OP^2)OC^2-2a^2OP^2}{(a^2+OP^2)OC^2} - \frac{1}{8},$$

and

$$F_{s-\Delta ODE} = \left[\frac{1}{2} - \frac{a}{2\sqrt{a^2+OD^2}}\right]\frac{\cos^{-1}\frac{OP}{OD}}{\pi} - \left[\frac{1}{8} - \frac{a\cos^{-1}\frac{OP}{OD}}{2\pi\sqrt{a^2+OD^2}}\right.$$

$$\left. + \frac{1}{4\pi}\sin^{-1}\frac{(a^2-OP^2)OD^2-2a^2OP^2}{(a^2+OP^2)OD^2}\right]$$

$$= \frac{\cos^{-1}\frac{OP}{OD}}{2\pi} - \frac{1}{4\pi}\sin^{-1}\frac{(a^2-OP^2)OD^2-2a^2OP^2}{(a^2+OP^2)OD^2} - \frac{1}{8}.$$

Therefore,

$$F_{s-\Delta OCD} = \frac{1}{4\pi}\left[\cos^{-1}\frac{OP}{OC} - \cos^{-1}\frac{OP}{OD}\right]$$

$$- \frac{1}{8\pi}\left[\sin^{-1}\frac{(a^2-OP^2)OC^2-2a^2OP^2}{(a^2+OP^2)OC^2} - \sin^{-1}\frac{(a^2-OP^2)OD^2-2a^2OP^2}{(a^2+OP^2)OD^2}\right].$$

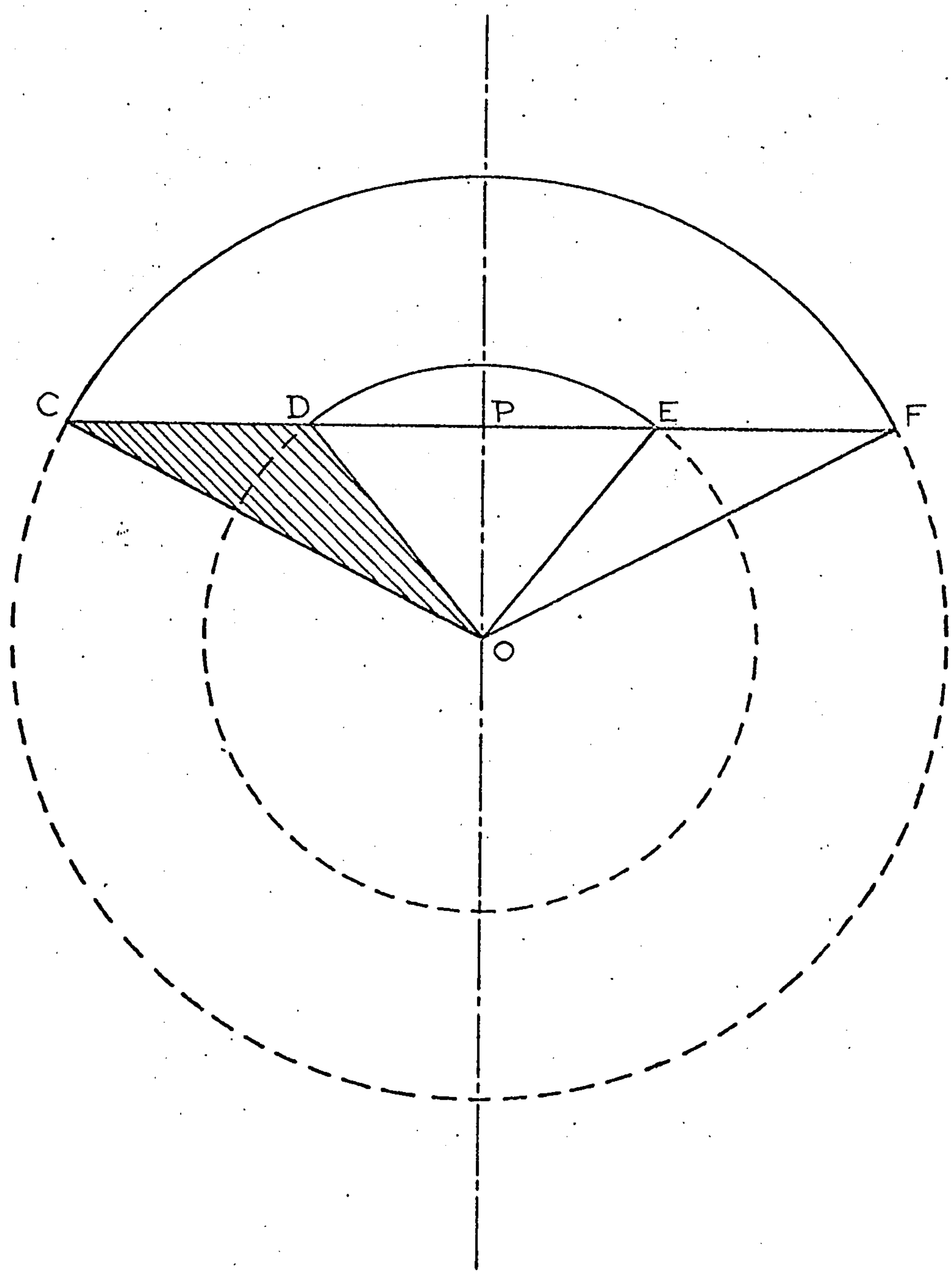


FIGURE 31

XII. CONFIGURATION FACTOR FROM A SPHERE TO A NONCOAXIAL DISK

Point O in figures 32(a) and 32(b) is the projection of the center of the radiating sphere and the shaded arcs are drawn with O as their center. Evidently, the whole surface of the disk can be covered by such arcs in figure 32(a), but not so in figure 32(b). Therefore, the integral of equation (38) with appropriate limits will give us the factor from the sphere to the disk in the former case; while in the latter, another factor (from the sphere to the coaxial disk, shaded in the figure) will have to be added to the integral.

In both figures we have

$$\alpha = \cos^{-1}\frac{x^2+b^2-c^2}{2xb} .$$

Thus, for $b \geqslant c$

$$F_{s\text{-noncoax.d}} = \int_{b-c}^{b+c} \frac{\cos^{-1}\frac{x^2+b^2-c^2}{2xb}}{\pi}\,\frac{ax}{2}\left(a^2+x^2\right)^{-\frac{3}{2}}dx \;, \quad \text{....(42)}$$

and, for $b < c$

$$F_{s\text{-noncoax.d}} = \int_{c-b}^{b+c} \frac{\cos^{-1}\frac{x^2+b^2-c^2}{2xb}}{\pi}\,\frac{ax}{2}\left(a^2+x^2\right)^{-\frac{3}{2}}dx + \frac{1}{2} - \frac{a}{2\sqrt{a^2-(c-b)^2}} \cdot \quad \text{....(43)}$$

It can be shown that the equations (42) and (43) lead to elliptic integrals and cannot, therefore, be integrated in terms of elementary functions. This is shown in the Appendix [15]. It was thought, however, that it may be useful to present in figure 33 the numerically computed values of the factor as a function of the dimensionless ratios $Z=b/c$ and $R=c/a$.

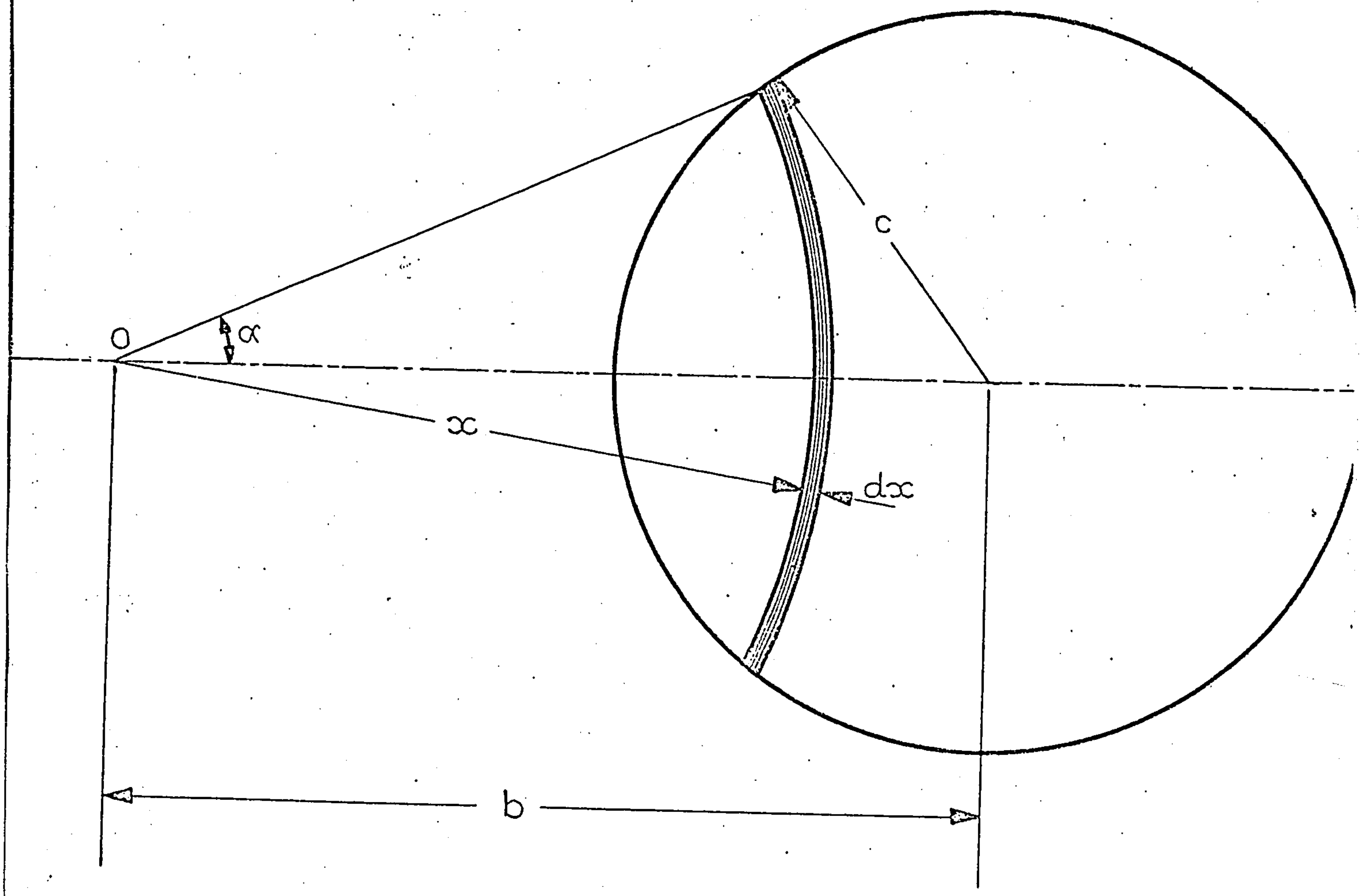


FIGURE 32(a). $b \geq c$.

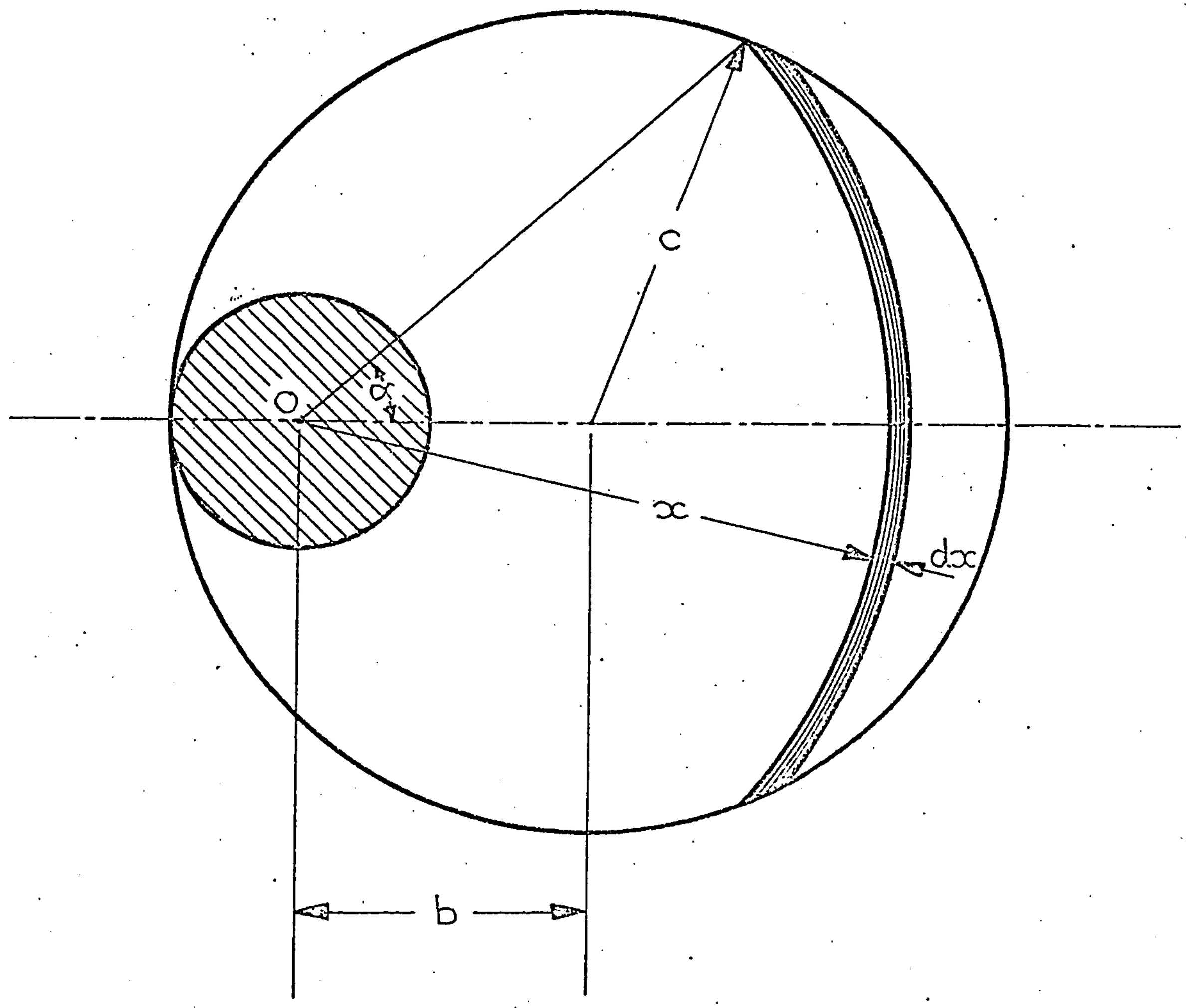


FIGURE 32(b). b<c.

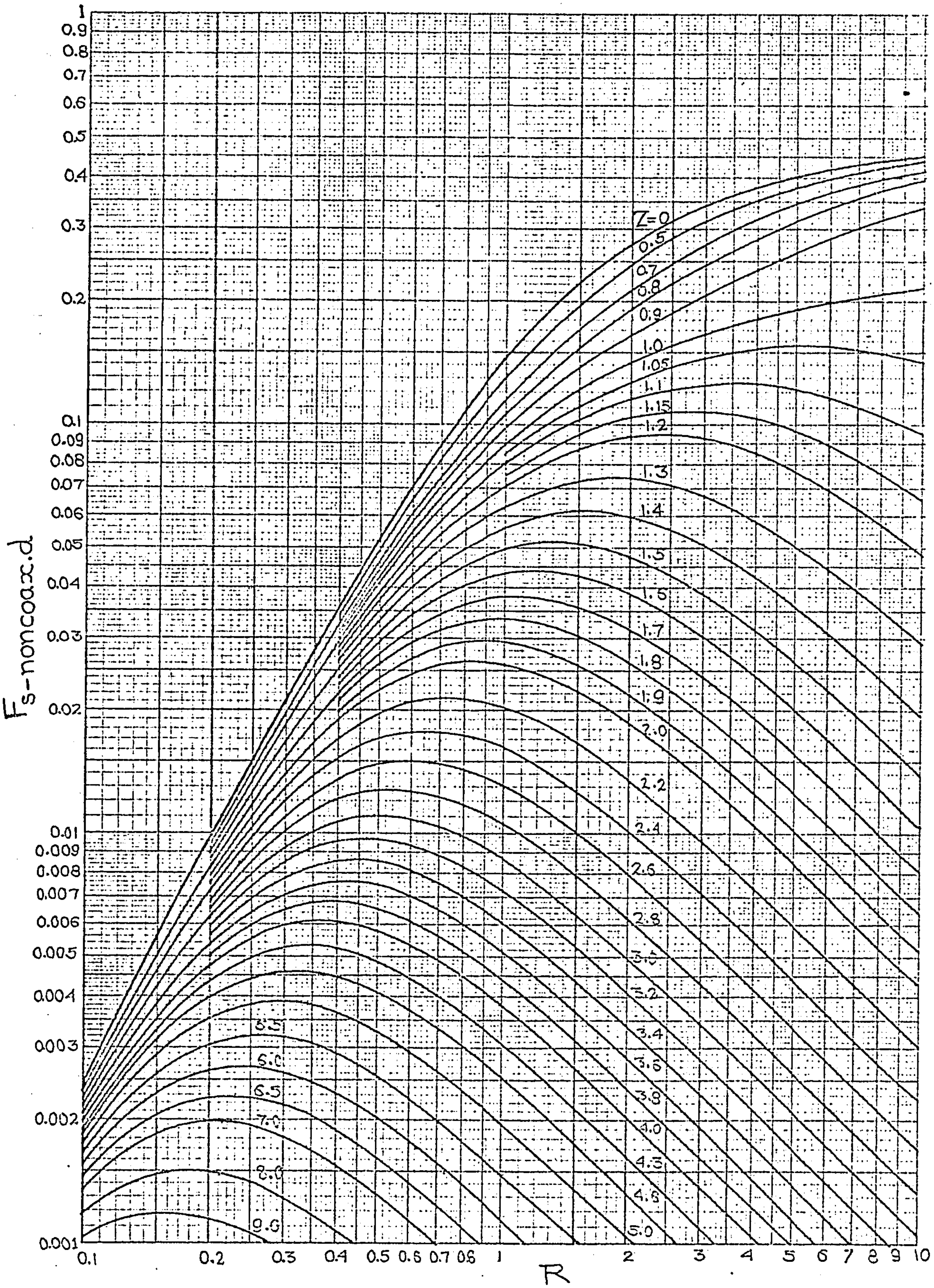


FIGURE 33. Configuration factors from a sphere to a noncoaxial disk.

XIII. CONFIGURATION FACTORS FROM A SPHERE TO SOME SELECTED PLANE AREAS

The radiation from a sphere to a surrounding concentric cube is equally divided among its six sides irrespectively of the validity of Lambert's law. Thus, the factor from a sphere to a coaxial square (figure 34), the side of which is twice the distance between their centers, must be equal to 1/6. This is in agreement with the result obtained from equation (41') by putting $B_1 = B_2 = 1$.

$$F_{s\text{-square}} = \frac{1}{2\pi}\left[\sin^{-1}\frac{2}{4} + \sin^{-1}\frac{2}{4}\right] = \frac{1}{6}\,.$$

The factor from a sphere with its center at the centroid of a regular tetrahedron to one of the four sides of the regular tetrahedron is 1/4, irrespective of the validity of Lambert's law. This can also be obtained by the difference of the factors from a sphere to the disk, circumscribing one of the triangles, and to the three segments of the same disk; as shown in figure 35.

$$F_{s\text{-tri.}} = F_{s\text{-d}} - 3F_{s\text{-seg.}}\,.$$

If b is the distance of the point of projection of the center of the sphere on the triangle, to one of its edges, then each side of this equilateral triangle measures $2\sqrt{3}\,b$. The radius of the circumscribing circle is $2b$ and the height a of the center of the sphere above this triangle is

$$a = \frac{\text{Height of the tetrahedron}}{4}$$

$$= \frac{\sqrt{2/3}\ \text{edge of the triangle}}{4}$$

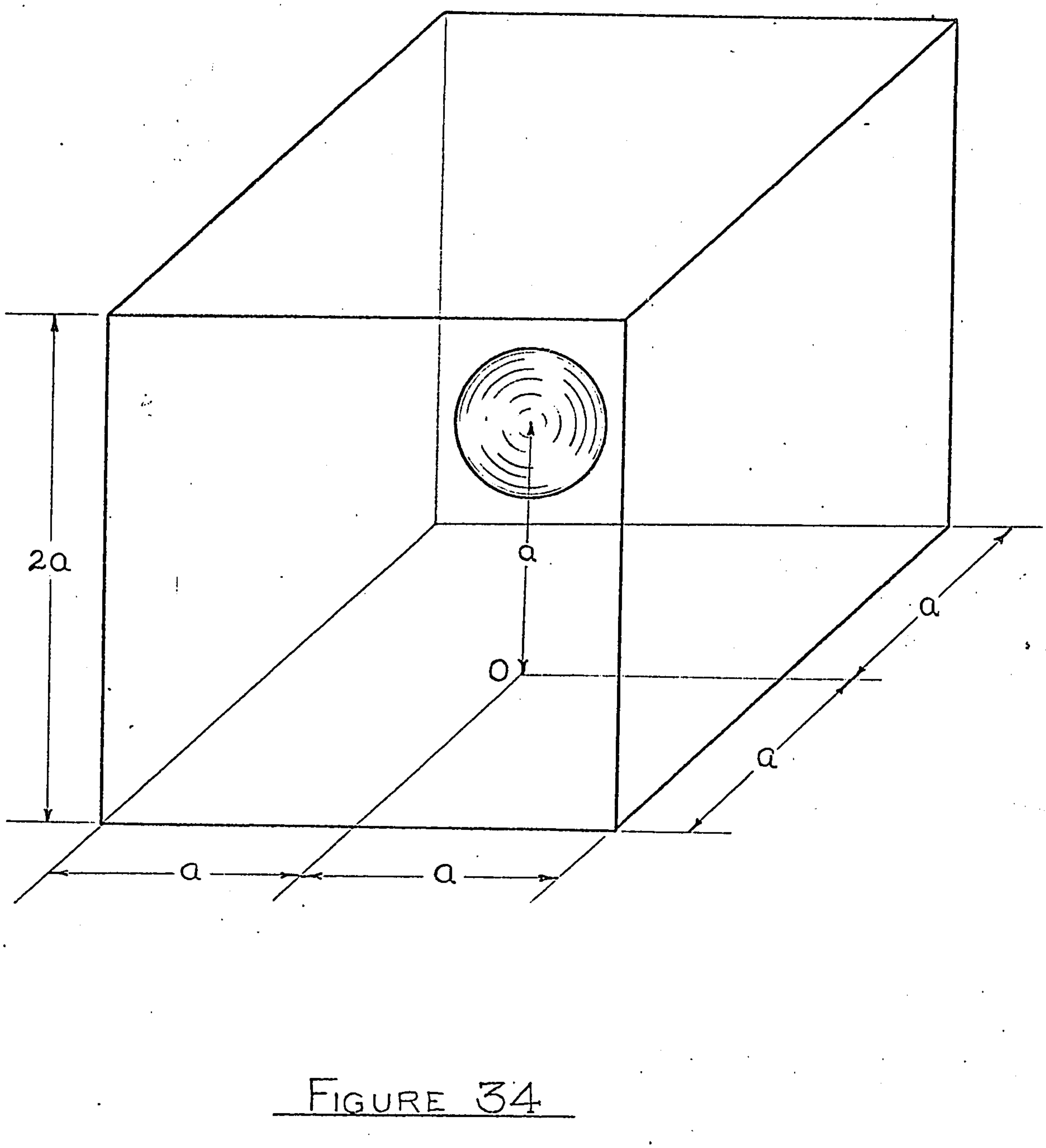


FIGURE 34

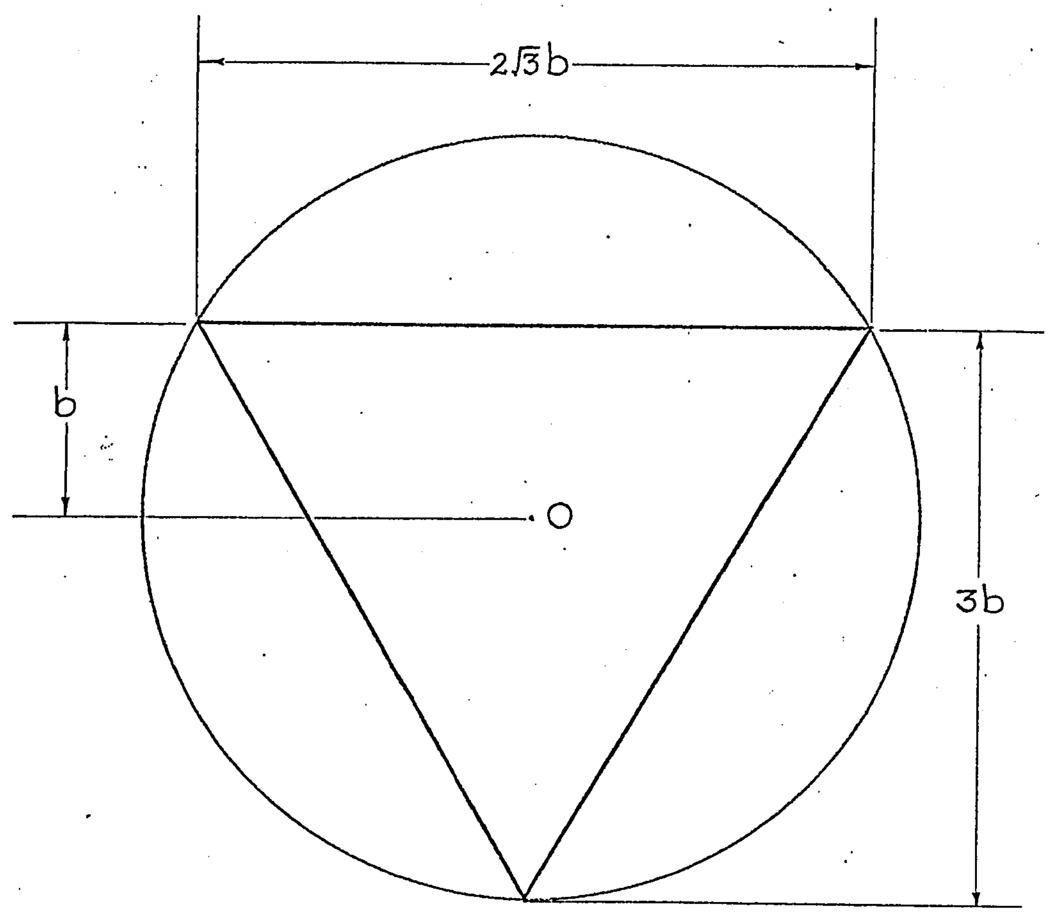


FIGURE 35

$$a = \frac{\sqrt{2/3} \times 2\sqrt{3}\ b}{4},$$

$$= \frac{b}{\sqrt{2}}.$$

Hence,

$$F_{s-tri.} = \left[\frac{1}{2} - \frac{b/\sqrt{2}}{2\sqrt{b^2/2+4b^2}}\right] - 3\left[\frac{1}{4\pi}\sin^{-1}\frac{(b^2/2-b^2)4b^2-b^4}{4b^2(b^2/2+b^2)}\right.$$

$$\left. - \frac{b/\sqrt{2}\ \cos^{-1}\frac{1}{2}}{2\pi\sqrt{b^2/2+4b^2}} + \frac{1}{8}\right],$$

$$= \left[\frac{1}{2} - \frac{1}{6}\right] - 3\left[\frac{1}{4\pi}\sin^{-1}\left(-\frac{1}{2}\right) - \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{2}\right) + \frac{1}{8}\right],$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{3}{4\pi}\cdot\frac{\pi}{6} + \frac{3}{\pi}\cdot\frac{1}{6}\cdot\frac{\pi}{3} - \frac{3}{8},$$

$$= \frac{1}{4}.$$

Let us now consider the geometry shown in figure 36. We are interested in obtaining the factor from the sphere to one of the triangular sides of the hollow pyramid.

The factor from the sphere to one of the sides of the cube is 1/6. Therefore, the factor from this sphere to one of the four triangles of the pyramid is (1 - 1/6) / 4, that is, 5/24.

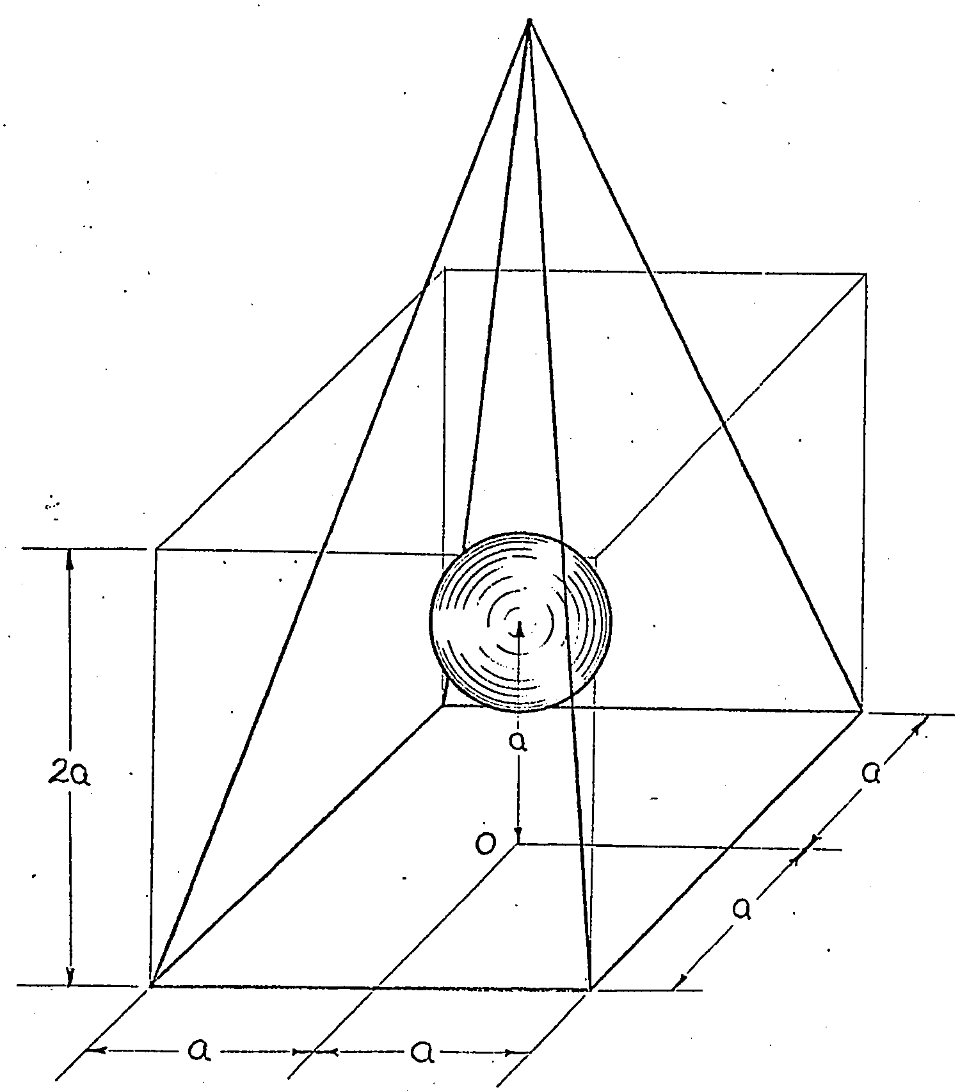


FIGURE 36

XIV. CONFIGURATION FACTOR FROM AN INFINITELY LONG CYLINDER TO AN INFINITELY LONG PARALLEL RECTANGLE OF EITHER FINITE OR INFINITESIMAL WIDTH

The same reasoning employed in the development of equation (35) can be applied to the radiation from an infinitely long cylinder to a parallel, symmetrically placed, infinite rectangle appearing in cross section in figure 37.

AB represents the width of the rectangle and ACB is an arc of a circumscribed circle concentric with the cylinder. Provided that AB does not intersect the cylinder, the factor from the radiating cylinder to the rectangle is the same as the factor from this cylinder to the portion ACB of the cylinder. Irrespectively of the validity of Lambert's law this factor equals α/π and does not depend on the radius of the radiating cylinder. Because $\alpha = \tan^{-1}(b/a)$, we have

$$F_{cyl.-symm.rect.} = \frac{1}{\pi}\tan^{-1}\frac{b}{a}\,. \qquad \text{(44)}$$

Putting $B = b/a$,

$$F_{cyl.-symm.rect.} = \frac{1}{\pi}\tan^{-1}B\,. \qquad \text{(44')}$$

If the rectangle is not placed symmetrically, shown in figure 38, we obtain through simple subtraction

$$F_{cyl.-nonsymm.rect} = \frac{1}{2\pi}\left[\tan^{-1}\frac{b_1}{a} - \tan^{-1}\frac{b_2}{a}\right], \qquad \text{(45)}$$

or

$$F_{cyl.-nonsymm.rect.} = \frac{1}{2\pi}\left[\tan^{-1}B_1 - \tan^{-1}B_2\right] \qquad \text{(45')}$$

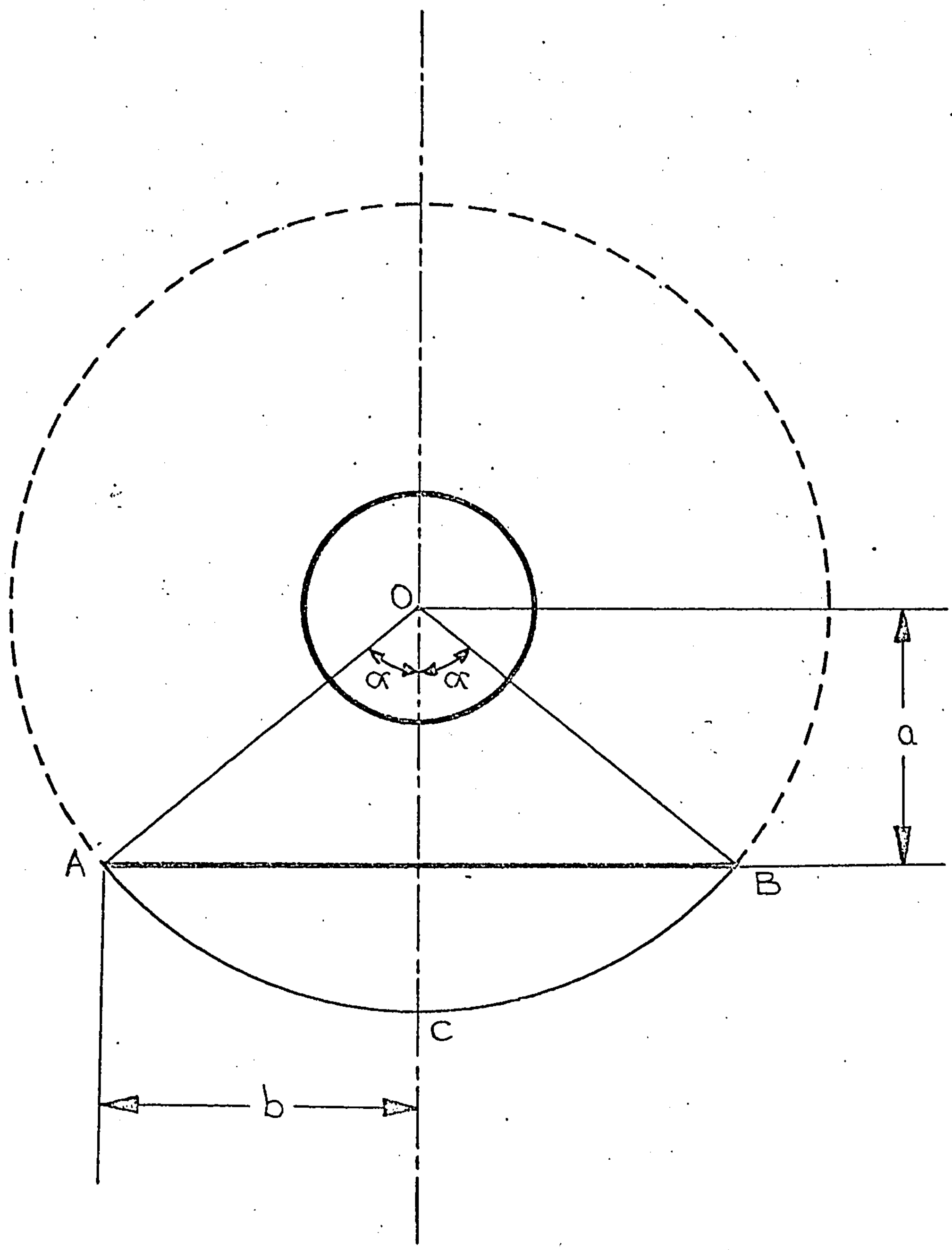


FIGURE 37

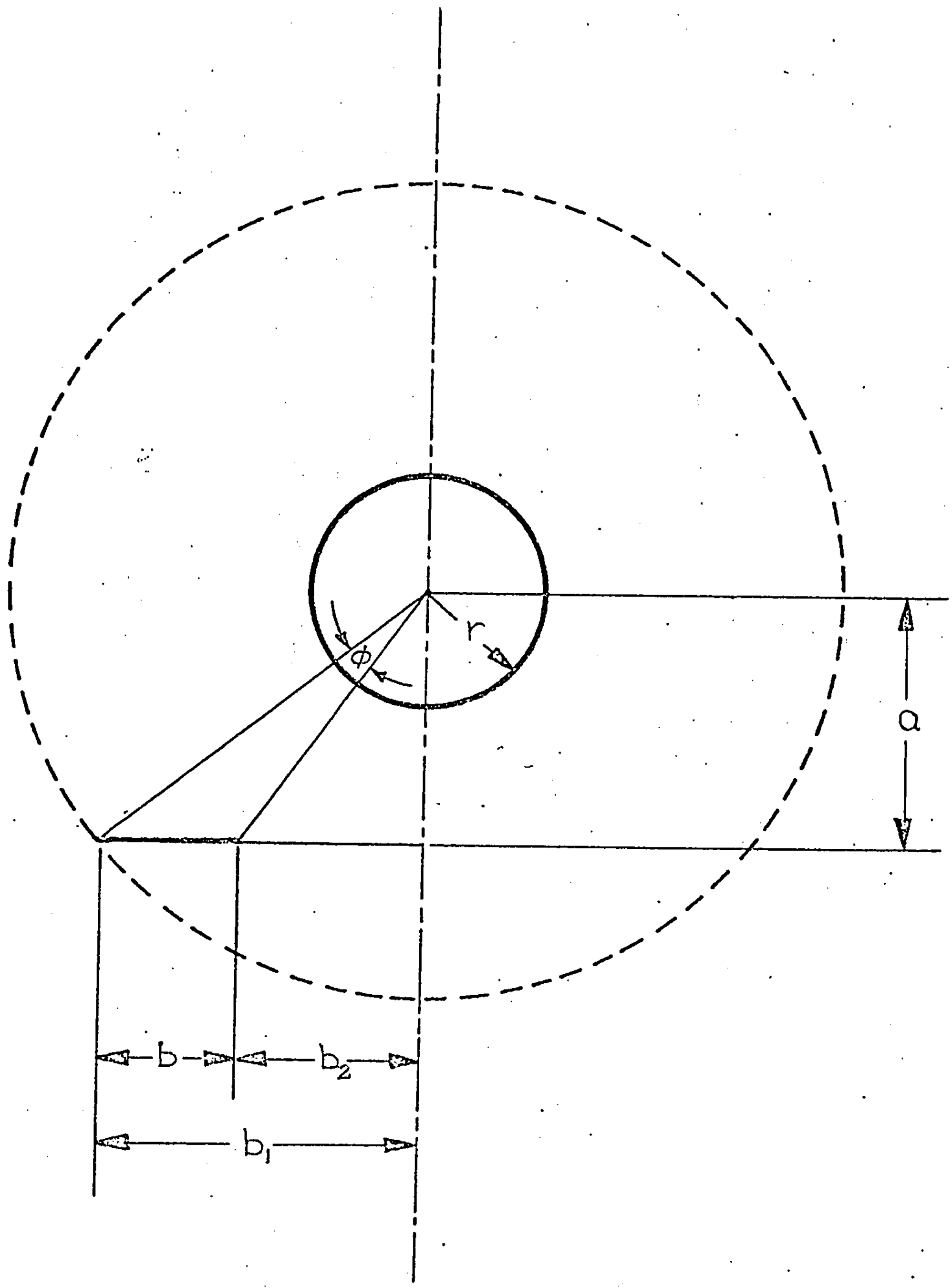


FIGURE 38

It can be seen from figure 38 that this factor is simply $\phi/2\pi$. This could have been guessed intuitively, but it is not easy to prove directly.

Assuming now that the rectangle radiates in accordance with Lambert's law, we can apply the reciprocity theorem, and obtain

$$F_{rect.-cyl.} = \frac{r}{b}\left[\tan^{-1}\frac{b_1}{a} - \tan^{-1}\frac{b_2}{a}\right].$$

Putting $b = b_1 - b_2$, $R = r/a$, $B_1 = b_1/a$ and $B_2 = b_2/a$, we obtain

$$F_{rect.-cyl.} = \frac{R}{B_1 - B_2}\left[\tan^{-1}B_1 - \tan^{-1}B_2\right].$$

The parantheses in this formula have been erroneously omitted by Hamilton and Morgan (reference 1) and that mistake has been subsequently reproduced in many books, among them the widely used text by Sparrow and Cess (reference 5). Apart from the error involved, it should be pointed out that in these references the reciprocal factor, from the cylinder to the rectangle, has not been evaluated. Thus, its independence from the radius of the radiating cylinder has not been noted.

In this connection it ought to be mentioned that another formula in Hamilton and Morgan (reference 1) is mistaken. Figure 39 is a reproduction from this work. An infinitely long strip of infinitesimal width is denoted by P_2. The factor from this strip to the cylinder is given as

$$F_{P_2-A_1} = \frac{N^2}{N^2 - M^2}.$$

This is manifestly wrong, as can be seen by putting $N = M$ which would lead to an infinitely large factor. The correct result can be obtained from equation (44).

Let the width of P_2 be dm. Then

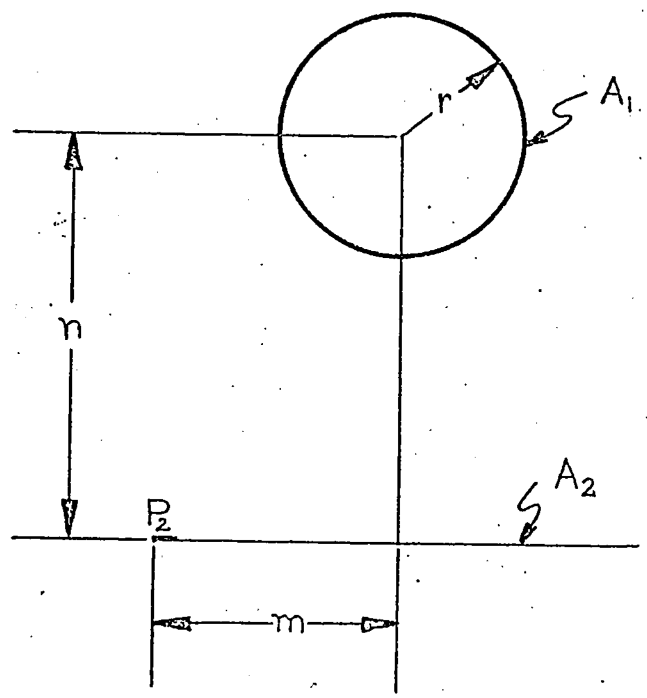


FIGURE 39. $M = m/r$, $N = n/r$.

$$F_{cyl.-rect. of width\ m} = \frac{1}{2\pi} \tan^{-1} \frac{m}{n} ,$$

$$\frac{dF_{cyl.-rect. of width\ m}}{dm} = \frac{1}{2\pi\left(1+\frac{m^2}{n^2}\right)n} .$$

But,

$$dF_{cyl.-rect. of width\ m} = F_{A_1-P_2} .$$

Thus,

$$F_{A_1-P_2} = \frac{n\ dm}{2\pi(n^2+m^2)} , \qquad \text{-----------(46)}$$

and, therefore,

$$F_{P_2-A_1} = \frac{2\pi r}{dm} F_{A_1-P_2} ,$$

$$= \frac{2\pi r}{dm} \frac{n\ dm}{2\pi(n^2+m^2)} ,$$

$$= \frac{rn}{(n^2+m^2)} ,$$

$$= \frac{N}{N^2+M^2} .$$

Again the erroneous result received wide currency; being reproduced, among others, in a book by Wiebelt (reference 6).

XV. CONFIGURATION FACTOR FROM AN INFINITELY LONG CYLINDER TO AN INFINITELY LONG NONCONCENTRIC CYLINDRICAL ENCLOSURE

We shall now use equation (45) as a starting point in our quest for an analytical expression for a factor from the infinitely long inner cylinder shown in figure 40, to the portion PQ of the outer cylinder.

Equation (45) represents the factor to an arbitrary infinitely long strip of width dm identified by the variables m and n shown in figure 40. But m, dm and n are functions of the angle α. Specifically:

$$m = e \sin\alpha ,$$

$$dm = r_2 \, d\alpha ,$$

$$n = r_2 - e \cos\alpha ,$$

where e is the distance between the centers of the two cylinders.

Let dA_2 denote the area of the strip dm. Equation (45) becomes now

$$F_{A_1-dA_2} = \frac{n\,dm}{2\pi(n^2+m^2)} = \frac{(r_2 - e\cos\alpha)\, r_2 \, d\alpha}{2\pi(r_2^2 - 2r_2 e\cos\alpha + e^2)} .$$

Integrating from α_1 to α_2 we obtain the required factor $F_{A_1-A_2}$, from the inner cylinder to the portion PQ of the outer cylinder.

$$F_{A_1-A_2} = \frac{r_2}{2\pi}\int_{\alpha_1}^{\alpha_2} \frac{r_2 - e\cos\alpha}{r_2^2 - 2r_2 e\cos\alpha + e^2}\, d\alpha$$

$$= \frac{r_2}{2\pi}\left[\frac{e\alpha}{2r_2 e} + \frac{-2r_2^2 e + e(r_2^2+e^2)}{-2r_2 e}\int \frac{d\alpha}{r_2^2 + e^2 - 2r_2 e\cos\alpha}\right]_{\alpha_1}^{\alpha_2},$$

since

$$\int \frac{A + B\cos\theta}{a + b\cos\theta}\, d\theta = \frac{B\theta}{b} + \frac{bA - aB}{b}\int \frac{d\theta}{a + b\cos\theta} , \quad b \neq 0 .$$

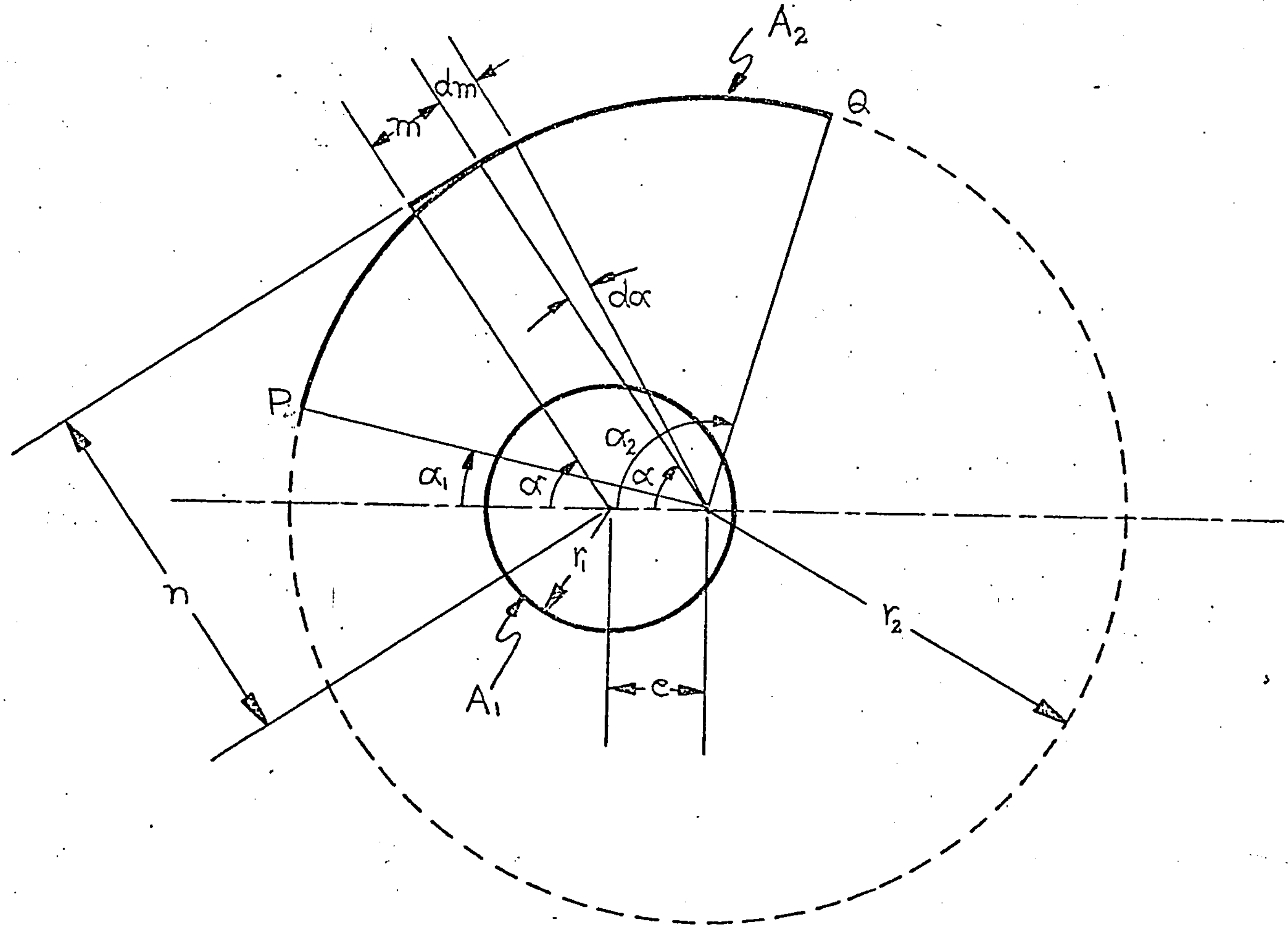


FIGURE 40

And, since

$$\int \frac{d\theta}{a+b\cos\theta} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a^2-b^2}\,\tan\frac{\theta}{2}}{a+b} \quad , \quad b^2<a^2 ,$$

we have

$$F_{A_1-A_2} = \frac{1}{2\pi}\left[\frac{\alpha}{2} + \frac{2r_2^2-r_2^2-e^2}{2}\,\frac{2}{\sqrt{(r_2^2+e^2)^2-(2r_2e)^2}}\tan^{-1}\frac{\sqrt{(r_2^2+e^2)^2-(2r_2e)^2}\,\tan\frac{\alpha}{2}}{r_2^2+e^2-2r_2e}\right]_{-\alpha_1}^{\alpha_2} ,$$

$$= \frac{1}{2\pi}\left[\frac{\alpha}{2} + \tan^{-1}\left(\frac{r_2+e}{r_2-e}\tan\frac{\alpha}{2}\right)\right] ,$$

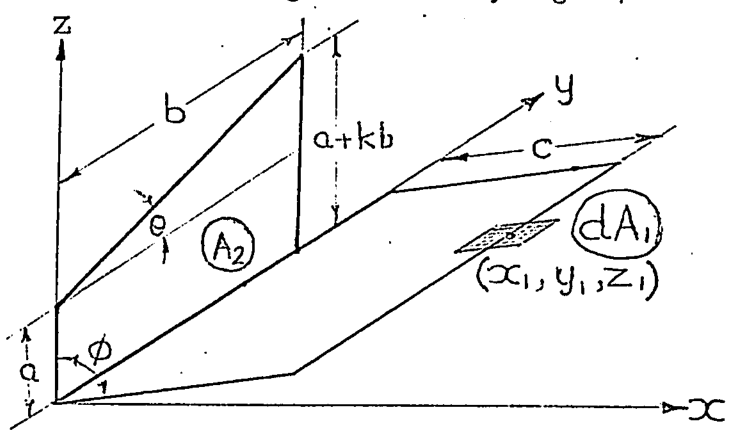
$$= \frac{1}{2\pi}\left[\frac{\alpha_2-\alpha_1}{2} + \tan^{-1}\left(\frac{r_2+e}{r_2-e}\tan\frac{\alpha_2}{2}\right) - \tan^{-1}\left(\frac{r_2+e}{r_2-e}\tan\frac{\alpha_1}{2}\right)\right] .$$

As was the case with the factors given by equations (44) and (45), $F_{A_1-A_2}$ is independent of the radius of the inner cylinder and of the applicability of Lambert's law.

XVI. SUMMARY

The configuration factors, discussed in this thesis are presented here in the form of a catalogue. In the list, the configuration factors from 6 to 15 are all independent of the validity of Lambert's cosine law and of the radius of the radiating bodies (spheres or infinitely long cylinders). None of the factors, with exception of numbers 6 and 14, has previously appeared in the literature. Factor 14 has appeared in Hamilton and Morgan (reference 1) with a printing error which was subsequently reproduced by others. Factor 6 was first calculated by Mackey et al. (reference 3) who, however, failed to recognize its greater generality.

1. Configuration factor from an infinitesimal area to a trapezoid, their planes forming an arbitrary angle ϕ.



$$k = \tan\theta$$

$$N = \frac{a}{b}$$

$$L = \frac{c}{b}$$

$$M = \frac{y_1}{b}$$

$$A = \sqrt{(1+k^2)L^2\sin^2\phi + (N + kM - L\cos\phi)^2}$$

$$B = \sqrt{L^2\sin^2\phi + (1-M)^2}$$

$$D = \sqrt{L^2\sin^2\phi + M^2}$$

$$F_{dA_1-A_2} = \frac{N\cos\phi + kM\cos\phi - L}{2\pi A}\left[\tan^{-1}\frac{1-M+k^2+k(N-L\cos\phi)}{A}\right.$$

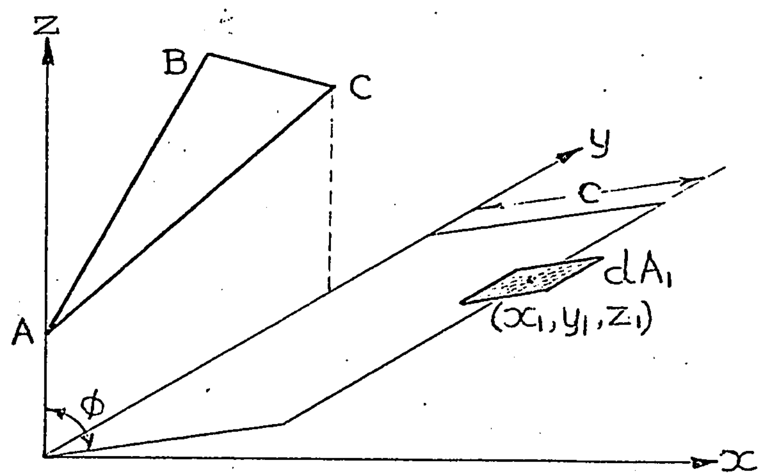
$$+\tan^{-1}\frac{M-k(N-L\cos\phi)}{A}\Bigg]$$

$$+\frac{(1-M)\cos\phi}{2\pi B}\left[\tan^{-1}\frac{k+N-L\cos\phi}{B}+\tan^{-1}\frac{L\cos\phi}{B}\right]$$

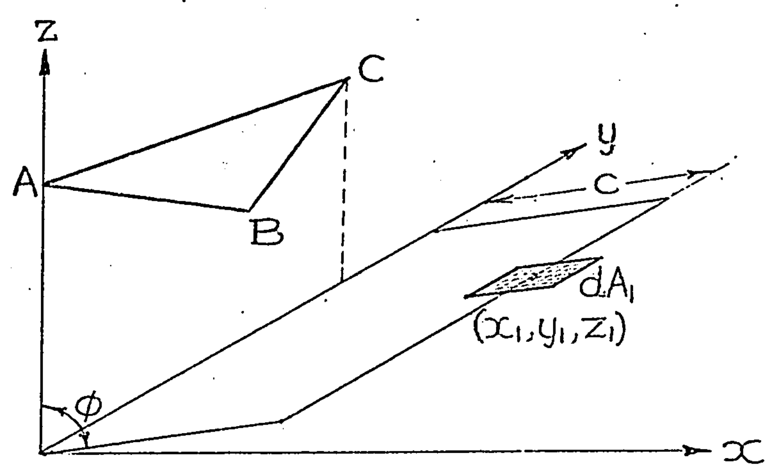
$$+\frac{M\cos\phi}{2\pi D}\left[\tan^{-1}\frac{N-L\cos\phi}{D}+\tan^{-1}\frac{L\cos\phi}{D}\right]$$

$$+\frac{1}{2\pi}\left[\tan^{-1}\frac{1-M}{L}+\tan^{-1}\frac{M}{L}\right].$$

2. Configuration factor from an infinitesimal area to a triangle, their planes forming an arbitrary angle ϕ.

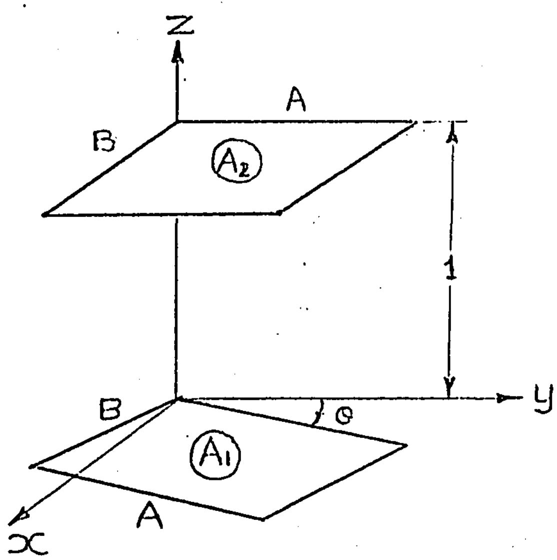


$F_{dA_1-\Delta ABC}$ ⟶ equation (10).



$F_{dA_1-\Delta ABC}$ ⟶ equation (11).

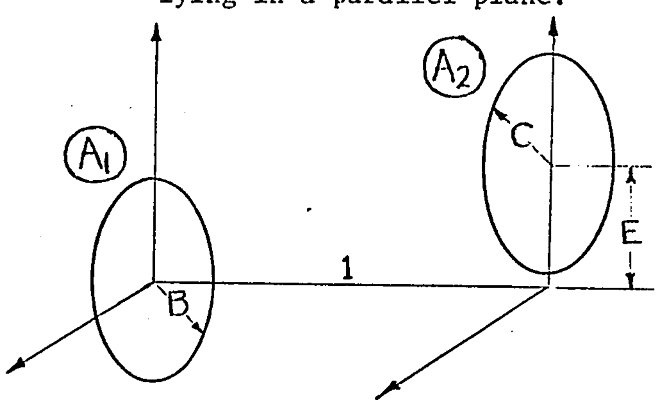
3. Configuration factor from a rectangle to another rectangle in a parallel plane, rotated through an arbitrary angle θ.



$F_{A_1-A_2} \longrightarrow$ equation (13)

Numerical values in figures 10, 11, 12 and 13.

4. Configuration factor from a circular disk to a noncoaxial circular disk lying in a parallel plane.



$$P = 1 - B^2$$

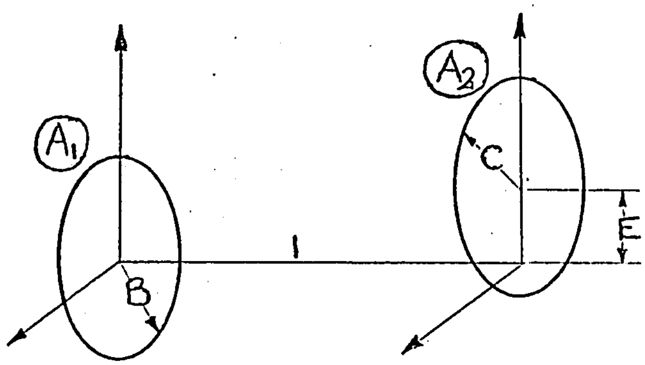
$$Q = 1 + B^2$$

$$G = E^2 + C^2$$

$$H = E^2 - C^2$$

a) $E \geqslant C$

$$F_{A_1-A_2} = \frac{C^2}{2B^2} - \frac{1}{2\pi B^2} \int_{E-C}^{E+C} \sqrt{\frac{R^4 + 2PR^2 + Q^2}{-R^4 + 2GR^2 - H^2}} \cdot \frac{R^2 - H}{R} \, dR ,$$



$$Z = 1 + \frac{1}{B^2} + \frac{(C-E)^2}{B^2}$$

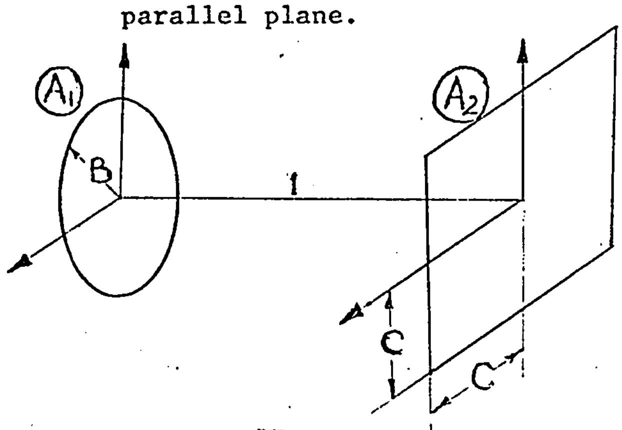
b) $E < C$

$$F_{A_1-A_2} = \frac{C^2-(C-E)^2}{2B^2} - \frac{\sqrt{(C-E)^4+2P(C-E)^2+Q^2}}{2B^2}$$

$$+\frac{1}{2}\left[Z-\sqrt{Z^2-\frac{4(C-E)^2}{B^2}}\right] - \frac{1}{2\pi B^2}\int_{C-E}^{C+E}\sqrt{\frac{R^4+2PR^2+Q^2}{-R^4+2GR^2-H^2}}\,\frac{R^2-H}{R}\,dR.$$

Numerical values in figure 15.

5. Configuration factor from a circular disk to a coaxial square lying in a parallel plane.



$$P = 1 - B^2$$

$$Q = 1 + B^2$$

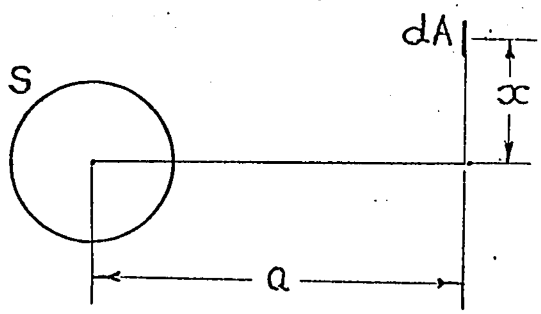
$$Z = 1 + \frac{1}{B^2} + \frac{C^2}{B^2}$$

$$F_{A_1-A_2} = \frac{1}{2}\left[Z-\sqrt{Z^2-\frac{4C^2}{B^2}}\right] + \frac{C^2}{B^2}\left[\frac{4-\pi}{2\pi}\right] + \frac{1}{2B^2}\sqrt{C^4+2PC^2+Q^2}$$

$$-\frac{2C}{\pi B^2}\int_{C}^{\sqrt{2}C}\sqrt{\frac{R^4+2PR^2+Q^2}{R^4-R^2C^2}}\,dR$$

Numerical values in figure 18.

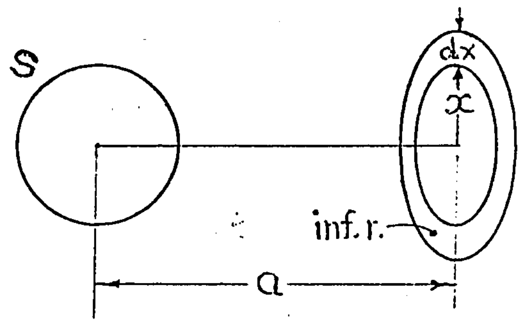
6. Configuration factor from a sphere to an infinitesimal area lying in a plane which does not intersect the sphere.



$$X = \frac{x}{a}$$

$$dF_{S-dA} = \frac{1}{4\pi}(1+X^2)^{-\frac{3}{2}}\,dA.$$

7. Configuration factor from a sphere to an infinitesimal coaxial ring.

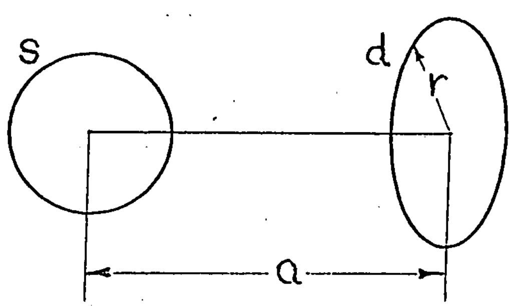


$$X = \frac{x}{a}$$

$$dX = \frac{dx}{a}$$

$$dF_{S-\text{inf.r.}} = \frac{X}{2}(1+X^2)^{-\frac{3}{2}}\,dX.$$

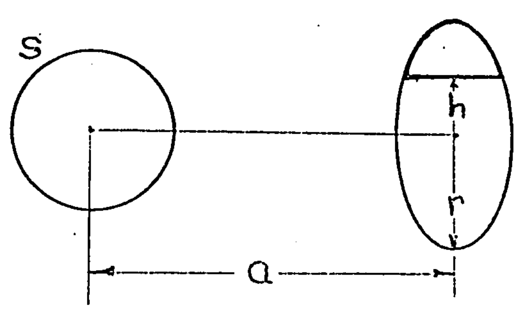
8. Configuration factor from a sphere to a coaxial circular disk.



$$R = \frac{r}{a}$$

$$F_{S-d} = \frac{1}{2} - \frac{1}{2\sqrt{1+R^2}}.$$

9. Configuration factor from a sphere to a segment of a coaxial disk.



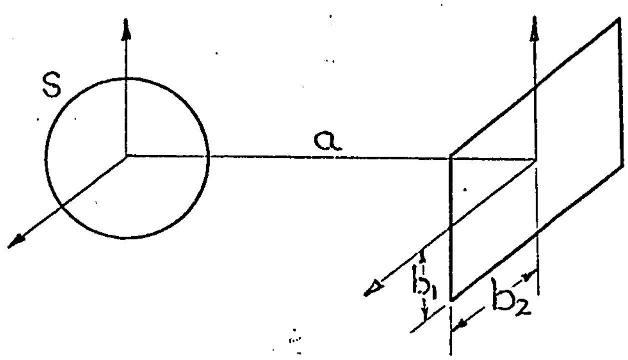
$$H = \frac{h}{a}$$

$$R = \frac{r}{a}$$

$$F_{s-seg.} = \frac{1}{8} - \frac{\cos^{-1}\frac{H}{R}}{2\pi\sqrt{1+R^2}} + \frac{1}{4\pi}\sin^{-1}\frac{(1-H^2)R^2-2H^2}{R^2(1+R^2)} ,$$

Numerical values in figure 26.

10. Configuration factor from a sphere to a coaxial rectangle.

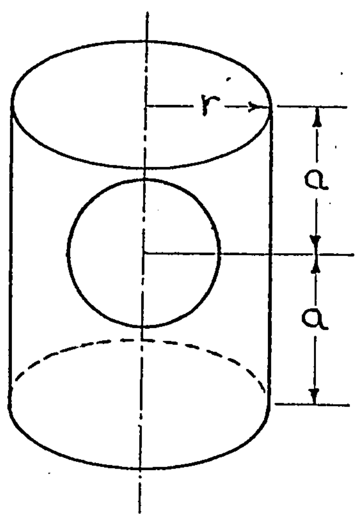


$$B_1 = \frac{b_1}{a}$$

$$B_2 = \frac{b_2}{a}$$

$$F_{s-rect.} = \frac{1}{2\pi}\left[\sin^{-1}\frac{2B_1^2-(1-B_1^2)(B_1^2+B_2^2)}{(1+B_1^2)(B_1^2+B_2^2)} + \sin^{-1}\frac{2B_2^2-(1-B_2^2)(B_1^2+B_2^2)}{(1+B_1^2)(B_1^2+B_2^2)}\right] .$$

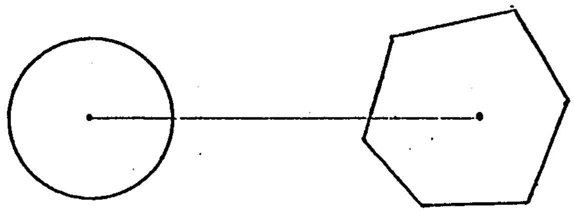
11. Configuration factor from a sphere to a coaxial right circular cylinder.



$$R = \frac{r}{a}$$

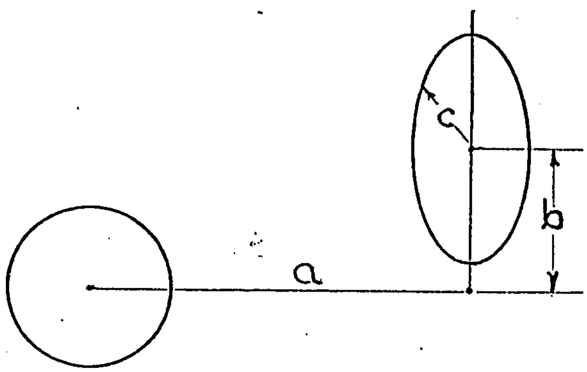
$$F_{s-cyl.} = \frac{1}{\sqrt{1+R^2}} .$$

12. Configuration factor from a sphere to a polygon lying in a plane which does not intersect the sphere.



see section XI .

13. Configuration factor from a sphere to a noncoaxial disk.



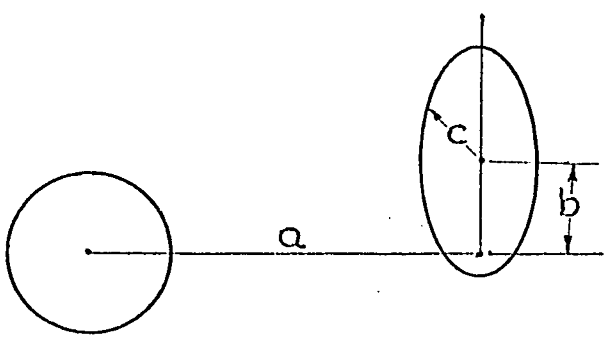
$$B = \frac{b}{a}$$

$$C = \frac{c}{a}$$

$$X = \frac{x}{a}$$

a) $B \geq C$

$$F_{s\text{-noncoax.d}} = \int_{B-C}^{B+C} \cos^{-1}\frac{X^2+B^2-C^2}{2XB}\cdot\frac{X\,dX}{2\pi(1+X^2)^{3/2}},$$

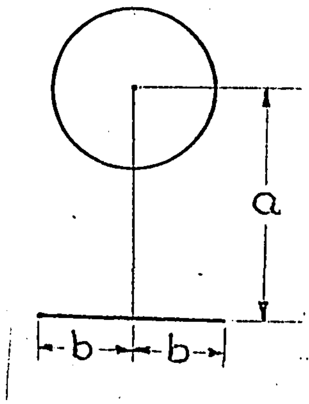


b) $B < C$

$$F_{s\text{-noncoax.d}} = \frac{1}{2} - \frac{1}{2\sqrt{1-(C-B)^2}} + \int_{C-B}^{B+C} \cos^{-1}\frac{X^2+B^2-C^2}{2XB}\cdot\frac{X\,dX}{2\pi(1+X^2)^{3/2}},$$

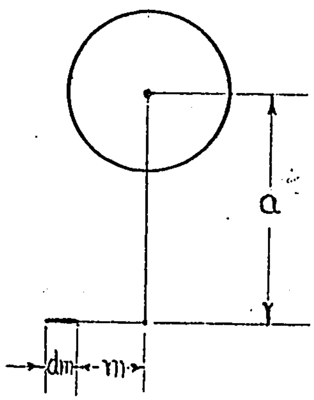
Numerical values in figure 33.

14. Configuration factor from an infinitely long cylinder to an infinitely long parallel rectangle of either finite or infinitesimal width.



$$B = \frac{b}{a}$$

$$F_{\text{cyl.-symm.rect.}} = \frac{1}{\pi} \tan^{-1} B \ .$$

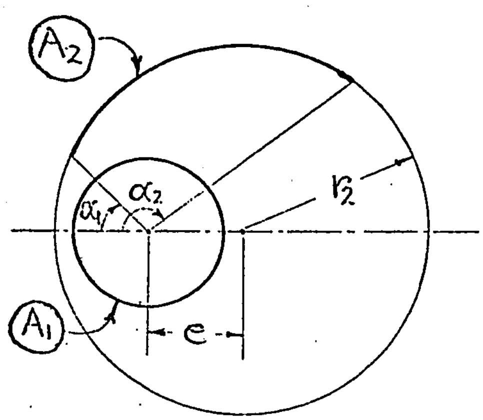


$$M = \frac{m}{a}$$

$$dM = \frac{dm}{a}$$

$$F_{\text{cyl.-rect. of width } dm} = \frac{dM}{2\pi(1+M^2)} \ .$$

15. Configuration factor from an infinitely long cylinder to a portion of an infinitely long nonconcentric cylindrical enclosure.



$$F_{A_1-A_2} = \frac{1}{2\pi}\left[\frac{\alpha_2-\alpha_1}{2} + \tan^{-1}\left(\frac{r_2+e}{r_2-e}\tan\frac{\alpha_2}{2}\right) - \tan^{-1}\left(\frac{r_2+e}{r_2-e}\tan\frac{\alpha_1}{2}\right)\right].$$

XVII. APPENDIX

(1) $$\int \frac{z_2\,dz_2}{z_2^2+qz_2+p} = \frac{-(2p+qz_2)}{(4p-q^2)(z_2^2+qz_2+p)} - \frac{2q}{(4p-q^2)}\int \frac{dz_2}{z_2^2+qz_2+p},$$

since,

$$\int \frac{x\,dx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{n(4ac-b^2)(ax^2+bx+c)^n} - \frac{b(2n-1)}{n(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^n}. \quad (a)$$

and,

$$\int \frac{dz_2}{z_2^2+qz_2+p} = \frac{2}{\sqrt{4p-q^2}}\tan^{-1}\frac{2z_2+q}{\sqrt{4p-q^2}}, \quad q^2<4p$$

i.e. $4z_1^2<4(x_1^2+y^2+z_1^2)$

since,

$$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}, \quad b^2<4ac. \quad (b)$$

---

(2) $$\int \frac{dy}{y^2(1+k^2)+y(2ka+2k^2y_1-2kz_1)+[x_1^2+(a+ky_1-z_1)^2]},$$

Comparing this integral with integral (b), we have

$$a \equiv 1+k^2,$$

$$b \equiv 2k(a+ky_1-z_1),$$

$$c \equiv x_1^2+(a+ky_1-z_1)^2,$$

$$\sqrt{4ac-b^2} \equiv \sqrt{4(1+k^2)[x_1^2+(a+ky_1-z_1)^2]-4k^2(a+ky_1-z_1)^2},$$

$$= 2\sqrt{x_1^2(1+k^2)+(a+ky_1-z_1)^2+k^2(a+ky_1-z_1)^2-k^2(a+ky_1-z_1)^2},$$

$$= 2\sqrt{x_1^2(1+k^2) + (a+ky_1-z_1)^2} \quad ,$$

and

$$4k^2(a+ky_1-z_1)^2 < 4(1+k^2)\left[x_1^2 + (a+ky_1-z_1)^2\right] \quad .$$

$$(3) \int \frac{y_2 dy_2}{y_2^2(1+k^2) + y_2(-2y_1+2ka-2kz_1)+[x_1^2+y_1^2+(a-z_1)^2]} \quad ,$$

Comparing this integral with integral (b), we have

$$a = 1+k^2 ,$$

$$b = 2(ka-kz_1-y_1) ,$$

$$c = x_1^2+y_1^2+(a-z_1)^2 ,$$

$$\sqrt{4ac-b^2} = \sqrt{4(1+k^2)[x_1^2+y_1^2+(a-z_1)^2]-4(ka-kz_1-y_1)^2} \quad ,$$

$$= 2\sqrt{x_1^2(1+k^2)+y_1^2+(a-z_1)^2+k^2y_1^2+k^2(a-z_1)^2-[k^2(a-z_1)^2+y_1^2 - 2y_1k(a-z_1)]} \quad ,$$

$$= 2\sqrt{x_1^2(1+k^2) + (a-z_1)^2+k^2y_1^2+2y_1k(a-z_1)} \quad ,$$

$$= 2\sqrt{(x_1^2)(1+k^2) + (a+ky_1-z_1)^2} \quad .$$

$$(4) \int \frac{dz_2}{z_2^2(1+k^2)-z_2[2k^2z_1+2(a+ky_1)]+[k^2x_1^2+k^2z_1^2+(a+ky_1)^2]} \quad ,$$

Comparing this integral with integral (b), we have

$$a = 1+k^2 ;$$

$$b = -2[k^2z_1+(a+ky_1)] ,$$

$$C \equiv k^2x_1^2 + k^2z_1^2 + (a+ky_1)^2 ,$$

$$\sqrt{4ac-b^2} \equiv \sqrt{4(1+k^2)[k^2x_1^2+k^2z_1^2+(a+ky_1)^2]-4[k^2z_1+(a+ky_1)]^2} ,$$

$$= 2\sqrt{x_1^2k^2(1+k^2)+(1+k^2)[k^2z_1^2+(a+ky_1)^2] \cdots}$$

$$\overline{\cdots -[k^4z_1^2+2k^2z_1(a+ky_1)+(a+ky_1)^2]} ,$$

$$= 2\sqrt{x_1^2k^2(1+k^2)+k^2z_1^2+k^2(a+ky_1)^2-2k^2z_1(a+ky_1)} ,$$

$$= 2\sqrt{x_1^2k^2(1+k^2)+k^2(z_1-a-ky_1)^2} .$$

---

(5) $$\tan^{-1}\frac{(1+k^2)(N+k)-(N+kM+k^2L\cos\phi)}{kA}$$

$$= \tan^{-1}\frac{N+Nk^2+k+k^3-N-kM-k^2L\cos\phi}{kA} ,$$

$$= \tan^{-1}\frac{1+k^2+Nk-M-kL\cos\phi}{A} ,$$

and

$$\tan^{-1}\frac{(1+k^2)N-(N+kM+k^2L\cos\phi)}{kA}$$

$$= \tan^{-1}\frac{Nk-kL\cos\phi-M}{A} .$$

Thus the bracketed terms of the fourth member are equal to the bracketed terms of the third. Hence, adding their coefficients, we have

$$-\frac{L+Lk^2\sin^2\phi-N\cos\phi-kM\cos\phi}{(1+k^2)A}+k^2\cos\phi\frac{N+kM-L\cos\phi}{(1+k^2)A}$$

$$= \frac{-L - Lk^2 + Lk^2\cos^2\phi + N\cos\phi + kM\cos\phi + k^2N\cos\phi + Mk^3\cos\phi - Lk^2\cos^2\phi}{(1+k^2)A},$$

$$= \frac{(1+k^2)(-L) + (1+k^2)(N\cos\phi) + (1+k^2)(kM\cos\phi)}{(1+k^2)A},$$

$$= \frac{N\cos\phi + kM\cos\phi - L}{A}.$$

---

(6) $$\int \ln(x^2+k^2)\,dx = \ln(x^2+k^2)\cdot x - \int \frac{2x}{x^2+k^2}\,x\,dx,$$

$$= x\ln(x^2+k^2) - 2\left[x - k^2\cdot\frac{1}{k}\tan^{-1}\frac{x}{k}\right],$$

$$= x\ln(x^2+k^2) - 2x + 2k\tan^{-1}\frac{x}{k}.$$

---

(7) $$\int \ln(ax^2+bx+c)\,dx = x\ln(ax^2+bx+c) - \int x\,\frac{2ax+b}{ax^2+bx+c}\,dx,$$

$$= x\ln(ax^2+bx+c) - \int \frac{(2ax^2+2bx+2c) - bx - 2c}{ax^2+bx+c}\,dx,$$

$$= x\ln(ax^2+bx+c) - 2x + \int \frac{bx\,dx}{ax^2+bx+c} + \int \frac{2c\,dx}{ax^2+bx+c},$$

$$= x\ln(ax^2+bx+c) - 2x + b\left[\frac{1}{2a}\ln(ax^2+bx+c)\right.$$

$$-\frac{b}{2a}\int\frac{dx}{ax^2+bx+c}+\int\frac{2c\,dx}{ax^2+bx+c},$$

$$=x\ln(ax^2+bx+c)-2x+\frac{b}{2a}\ln(ax^2+bx+c)$$

$$+\left(2c-\frac{b^2}{2a}\right)\int\frac{dx}{ax^2+bx+c},$$

$$=\left(x+\frac{b}{2a}\right)\ln(ax^2+bx+c)-2x+\frac{4ac-b^2}{2a}\,\frac{\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}}{\frac{1}{2}\sqrt{4ac-b^2}}$$

$$=\left(x+\frac{b}{2a}\right)\ln(ax^2+bx+c)-2x+\frac{\sqrt{4ac-b^2}}{a}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

In our case,

$$a = 1+G^2,$$

$$b = 2GH-2P,$$

$$c = P^2+H^2+1,$$

$$\sqrt{4ac-b^2} = \sqrt{4(1+G^2)(P^2+H^2+1)-(2GH-2P)^2},$$

$$=2\sqrt{P^2+H^2+1+G^2P^2+G^2H^2+G^2-G^2H^2-P^2+2GHP},$$

$$=2\sqrt{(1+G^2)+(GP+H)^2},$$

$$x = Z_1,$$

$$\frac{2ax+b}{\sqrt{4ac-b^2}} = \frac{(1+G^2)Z_1+GH-P}{\sqrt{(1+G^2)+(GP+H)^2}}.$$

(8) $\int x \ln(ax^2+bx+c)\,dx =$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \int \frac{x^2}{2}\,\frac{2ax+b}{ax^2+bx+c}\,dx,$$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \frac{1}{2}\int \frac{x(2ax+2bx+2c) - bx^2 - 2cx}{ax^2+bx+c}\,dx,$$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \int x\,dx + \frac{b}{2}\int \frac{x^2\,dx}{ax^2+bx+c} + c\int \frac{x\,dx}{ax^2+bx+c},$$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \frac{x^2}{2} + \frac{b}{2a}\int \frac{(ax^2+bx+c) - bx - c}{ax^2+bx+c}\,dx + c\int \frac{x\,dx}{ax^2+bx+c},$$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \frac{x^2}{2} + \frac{b}{2a}x + \left(c - \frac{b^2}{2a}\right)\int \frac{x\,dx}{ax^2+bx+c} - \frac{bc}{2a}\int \frac{dx}{ax^2+bx+c}$$

$$= \frac{x^2}{2}\ln(ax^2+bx+c) - \frac{x^2}{2} + \frac{bx}{2a} + \left(c - \frac{b^2}{2a}\right)\left[\frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}\right] - \frac{bc}{2a}\int \frac{dx}{ax^2+bx+c},$$

$$= \left(\frac{x^2}{2} + \frac{c}{2a} - \frac{b^2}{4a^2}\right)\ln(ax^2+bx+c) - \frac{x^2}{2} + \frac{bx}{2a} - \left(\frac{bc}{2a} - \frac{b^3}{4a^2} + \frac{bc}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

$$= \frac{bx}{2a} - \frac{x^2}{2} + \left(\frac{x^2}{2} + \frac{c}{2a} - \frac{b^2}{4a^2}\right)\ln(ax^2+bx+c) - \frac{b\sqrt{4ac-b^2}}{2a^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}.$$

The equivalent quantities for a, b and c are the same as in (7).

$$(9)\quad 2\int_{e-c}^{e+c} r\cos^{-1}\frac{r^2+e^2-c^2}{2re}\,dr =$$

$$=2\left[\frac{r^2}{2}\cos^{-1}\frac{r^2+e^2-c^2}{2re}+\int\frac{r^2}{2}\frac{2re}{\sqrt{(2re)^2-(r^2+e^2-c^2)^2}}\;\frac{2re(2r)-2e(r^2+e^2-c^2)}{(2re)^2}\,dr\right]_{e-c}^{e+c},$$

$$=\left[r^2\cos^{-1}\frac{r^2+e^2-c^2}{2re}\right]_{e-c}^{e+c}+\int_{e-c}^{e+c}\frac{r(r^2+c^2-e^2)\;dr}{\sqrt{-r^4-e^4-c^4+2(r^2e^2+e^2c^2+c^2r^2)}},$$

$$=\left[0\right]+\int_{e-c}^{e+c}\frac{r(r^2+c^2-e^2)\,dr}{\sqrt{-r^4+2r^2(e^2+c^2)-(e^2-c^2)^2}},$$

Let $r^2=x$, $2r\,dr=dx$,

$c^2-e^2=g$, $e^2+c^2=h$,

$$=\frac{1}{2}\int_{(e-c)^2}^{(e+c)^2}\frac{x+g}{\sqrt{-x^2+2xh-g^2}}\,dx,$$

$$=\frac{1}{2}\left[\frac{\sqrt{-x^2+2xh-g^2}}{(-1)}-\frac{2h}{2(-1)}\int\frac{dx}{\sqrt{-x^2+2xh-g^2}}+g\sin^{-1}\frac{2x-2h}{\sqrt{4h^2-4g^2}}\right]_{(e-c)^2}^{(e+c)^2},$$

$$=\left[-\frac{1}{2}\sqrt{-x^2+2xh-g^2}+\frac{1}{2}(h+g)\sin^{-1}\frac{x-h}{\sqrt{h^2-g^2}}\right]_{(e-c)^2}^{(e+c)^2},$$

$$=\left[-\frac{1}{2}\sqrt{-x^2+2x(e^2+c^2)-(c^2-e^2)^2}+\frac{1}{2}(2c^2)\sin^{-1}\frac{x-e^2-c^2}{\sqrt{(2c^2)(2e^2)}}\right]_{(e-c)^2}^{(e+c)^2},$$

$$=\left[-0+0+c^2\sin^{-1}(1)-c^2\sin^{-1}(-1)\right],$$

$$=c^2\left(\frac{\pi}{2}\right)-c^2\left(-\frac{\pi}{2}\right),\qquad=\pi c^2.$$

(10) $$\frac{8}{2A_1}\int_c^{\sqrt{2}c}\int_{\cos^{-1}\frac{c}{r}}^{\pi/4} r\,d\theta\,dr =$$

$$= \frac{4}{A_1}\int_c^{\sqrt{2}c}\left(\frac{\pi}{4}-\sec^{-1}\frac{r}{c}\right) r\,dr\ ,$$

$$= \frac{4}{A_1}\left[\int_c^{\sqrt{2}c}\frac{\pi}{4} r\,dr - \int_c^{\sqrt{2}c} r\sec^{-1}\frac{r}{c}\,dr\right],$$

$$= \frac{\pi}{A_1}\left[\frac{r^2}{2}\right]_c^{\sqrt{2}c} - \frac{4}{A_1}\left[\frac{r^2}{2}\sec^{-1}\frac{r}{c} - \int\frac{r^2}{2}\frac{1}{\frac{r}{c}\sqrt{\frac{r^2}{c^2}-1}}\cdot\frac{1}{c}\,dr\right]_c^{\sqrt{2}c},$$

$$= \frac{\pi c^2}{2A_1} - \frac{4}{A_1}\left[\frac{r^2}{2}\sec^{-1}\frac{r}{c}\right]_c^{\sqrt{2}c} + \frac{2c}{A_1}\int_c^{\sqrt{2}c}\frac{r\,dr}{\sqrt{r^2-c^2}}\ ,$$

$$= \frac{\pi c^2}{2A_1} - \frac{4}{A}\left[c^2\frac{\pi}{4} - \frac{c^2}{2}\,0\right] + \frac{c}{A_1}\left[\frac{\sqrt{r^2-c^2}}{1/2}\right]_c^{\sqrt{2}c},$$

$$= \frac{\pi c^2}{2A_1} - \frac{\pi c^2}{A_1} + \frac{2c}{A_1}\left[c-0\right],$$

$$= -\frac{\pi c^2}{2A_1} + \frac{2c^2}{A_1}\ ,$$

$$= \frac{4c^2-\pi c^2}{2A_1}\ ,$$

$$= \frac{4c^2-\pi c^2}{2\pi b^2}\ .$$

---

(11) $$\int \frac{\sqrt{r^4+2pr^2+q^2}}{r\sqrt{r^2-c^2}}\,dr =$$

$$= \int \frac{r^4+2pr^2+q^2}{r\sqrt{r^2-c^2}\sqrt{r^4+2pr^2+q^2}}\,dr\,,$$

Putting $r^2 = x\,,$

$2r\,dr = dx\,,$

$$= \frac{1}{2}\int \frac{x^2+2px+q^2}{x\sqrt{x-c^2}\sqrt{x^2+2px+q^2}}\,dx\,,$$

$$= \frac{1}{2}\int \frac{x^2+2px+q^2}{x\sqrt{x^3+(2p-c^2)x^2+(q^2-2pc^2)x-c^2q^2}}\,dx,$$

$$= \frac{1}{2}\int \frac{x^2+2px+q^2}{x\sqrt{a_0x^3+3a_1x^2+3a_2x+a_3}}\,dx,$$

$$= \frac{1}{2}\int \frac{x\,dx}{\sqrt{a_0x^3+3a_1x^2+3a_2x+a_3}} + p\int \frac{dx}{\sqrt{a_0x^3+3a_1x^2+3a_2x+a_3}}$$

$$+ \frac{q^2}{2}\int \frac{dx}{x\sqrt{a_0x^3+3a_1x^2+3a_2x+a_3}}\,,$$

where $a_0 = 1\,;$

$3a_1 = 2p-c^2,$

$3a_2 = q^2-2pc^2,$

$a_3 = -c^2q^2.$

The solutions of these integrals are given in terms of elliptic integrals in reference 8.

(12) Referring to figure 19, we have the direction cosines of dA_2 as

$$l_2 = 0,$$

$$m_2 = -\cos\phi,$$

$$n_2 = \sin\phi,$$

and the direction cosines of the circular disk of radius k are

$$l_1 = 0,$$

$$m_1 = 1,$$

$$n_1 = 0.$$

The factor from dA_2 to A_1, using contour integration is given by equation (7) as

$$F_{dA_2-A_1} = l_1 \oint_C \frac{(z_1-z_2)dy_1-(y_1-y_2)dz_1}{2\pi r^2} + m_1 \oint_C \frac{(x_1-x_2)dz_1-(z_1-z_2)dx_1}{2\pi r^2}$$

$$+ n_1 \oint_C \frac{(y_1-y_2)dx_1-(x_1-x_2)dy_1}{2\pi r^2},$$

where, in our case

$$r^2 = PE = t^2 \text{ (say)} = d^2-s^2.$$

Therefore,

$$F_{dA_2-A_1} = \frac{-\cos\phi}{2\pi(d^2-s^2)} \oint_C (x_1 dz_1 - z_1 dx_1) + \frac{\sin\phi}{2\pi(d^2-s^2)} \oint g\,dz_1$$

$$= \frac{-\cos\phi}{2\pi(d^2-s^2)} 4\left[\int_K^0 \sqrt{K^2-z_1^2}\,dz_1 - \int_0^k \sqrt{k^2-x_1^2}\,dx_1\right] + \frac{\sin\phi}{2\pi(d^2-s^2)}\left[0\right].$$

The second integral in the above expression reduces to zero, since the line integral of a function around a closed boundary is zero if the function is constant. Therefore,

$$F_{dA_2-A_1} = \frac{\cos\phi}{2\pi(d^2-s^2)} \cdot 4 \cdot 2 \int_0^k \sqrt{k^2-x_1^2}\, dx_1 ,$$

$$= \frac{4\cos\phi}{\pi(d^2-s^2)} \cdot \frac{1}{2} \left[x_1\sqrt{k^2-x_1^2} + k^2\sin^{-1}\frac{x_1}{k} \right]_0^k ,$$

$$= \frac{2\cos\phi}{\pi(d^2-s^2)} \left[\frac{\pi}{2} k^2 \right] ,$$

$$= \frac{k^2\cos\phi}{d^2-s^2} = \frac{k^2\cos\phi}{g^2+k^2} .$$

Let us now obtain the same factor using quadruple integration method. The factor from dA_2 to A_1 is

$$F_{dA_2-A_1} = \iint_{A_1} \frac{\cos\beta_1 \cos\beta_2}{\pi r^2}\, dA_1 ,$$

where

$$r = \sqrt{x_1^2 + g^2 + z_1^2} ,$$

$$\cos\beta_1 = \frac{0 + m_1(y_2-y_1) + 0}{r} ,$$

$$= \frac{g}{r} ,$$

$$\cos\beta_2 = \frac{0 + m_2(y_1-y_2) + n_2(z_1-z_2)}{r} ,$$

$$= \frac{g\cos\phi + z_1\sin\phi}{r} .$$

Hence,

$$F_{dA_2-A_1} = \int_{-k}^{k}\int_{-\sqrt{k^2-z_1^2}}^{\sqrt{k^2-z_1^2}} \frac{1}{\pi r^2}\,\frac{g}{r}\,\frac{g\cos\phi + z_1\sin\phi}{r}\,dx_1 dz_1 ,$$

$$= \frac{g^2\cos\phi}{\pi}\int_{-k}^{k}\int_{-\sqrt{k^2-z_1^2}}^{\sqrt{k^2-z_1^2}} \frac{dx_1 dz_1}{(x_1^2+g^2+z_1^2)^2} + \frac{g\sin\phi}{\pi}\int_{-k}^{k}\int_{-\sqrt{k^2-z_1^2}}^{\sqrt{k^2-z_1^2}} \frac{z_1\,dx_1 dz_1}{(x_1^2+g^2+z_1^2)^2} .$$

Transferring the rectangular coordinates into polar coordinates, we get

$$F_{dA_2-A_1} = \frac{g^2\cos\phi}{\pi}\int_0^k\int_0^{2\pi} \frac{r\,d\theta\,dr}{(r^2+g^2)^2} + \frac{g\sin\phi}{\pi}\int_0^k\int_0^{2\pi} \frac{r^2\sin\theta\,d\theta\,dr}{(r^2+g^2)^2}$$

$$= \frac{g^2\cos\phi}{\pi}\,2\pi\int_0^k \frac{r\,dr}{(r^2+g^2)^2} + \frac{g\sin\phi}{\pi}\int_0^k \frac{r^2}{(r^2+g^2)^2}\Big[-\cos\theta\Big]_0^{2\pi} dr,$$

$$= \frac{2\pi g^2\cos\phi}{2\pi}\left[-\frac{1}{r^2+g^2}\right]_0^k + \frac{g\sin\phi}{\pi}\int_0^k \frac{r^2}{(r^2+g^2)^2}\Big[0\Big] dr,$$

$$= g^2\cos\phi\left[\frac{1}{g^2} - \frac{1}{k^2+g^2}\right] + 0 ,$$

$$= \frac{k^2\cos\phi}{k^2+g^2} ,$$

$$= \frac{k^2\cos\phi}{d^2-s^2} .$$

This result is same as obtained by the contour integration method.

(13) $$\int \frac{2x\,dx}{x^2\sqrt{x^4+x^2(a^2-h^2)-a^2h^2}} =$$

$$= \int \frac{y\,dy}{y\sqrt{y^2+y(a^2-h^2)-a^2h^2}} \; ,$$

where $x^2 = y$

$2x\,dx = dy,$

$$= \frac{1}{ah} \sin^{-1} \frac{(a^2-h^2)y - 2a^2h^2}{y\sqrt{(a^2-h^2)+4a^2h^2}} \; ,$$

$$= \frac{1}{ah} \sin^{-1} \frac{(a^2-h^2)x^2 - 2a^2h^2}{x^2\sqrt{(a^2-h^2)+4a^2h^2}} \; .$$

---

(14) The factor from a sphere to a coaxial rectangle as obtained in equation (41) is

$$F_{s\text{-rect.}} = \frac{1}{2\pi}\left[\sin^{-1}\frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)} + \sin^{-1}\frac{2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)}\right] \quad \text{------(I)}$$

The same factor as given in reference 3 or reference 7 is

$$F_{s\text{-rect.}} = \frac{1}{4\pi}\sin^{-1}\left[\frac{b_1^2b_2^2}{(a^2+b_1^2)(a^2+b_2^2)}\right]^{\frac{1}{2}} . \quad \text{--------(II)}$$

To prove that equation (I) is equivalent to equation (II) , we proceed as follows.

Let,

$$A = \frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)} \; ,$$

$$B = \frac{2a^2b_2^2 - (a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)} ,$$

$$\alpha = \sin^{-1}A ,$$

$$\beta = \sin^{-1}B .$$

Then

$$A = \sin\alpha ,$$

$$\sqrt{1-A^2} = \cos\alpha ,$$

$$B = \sin\beta ,$$

$$\sqrt{1-B^2} = \cos\beta .$$

We have

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta ,$$

or

$$\alpha+\beta = \sin^{-1}(\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta), \quad \alpha+\beta \leq \frac{\pi}{2},$$

$$= \pi - \sin^{-1}(\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta), \quad \alpha+\beta > \frac{\pi}{2}.$$

Substituting for α and β, we have

$$\sin^{-1}A+\sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right),$$

or

$$\sin^{-1}A+\sin^{-1}B = \pi - \sin^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right).$$

A and B being given, let us find the expression for $A\sqrt{1-B^2}+B\sqrt{1-A^2}$.

$$A^2 = \frac{4a^4b_1^4 - 4a^2b_1^2(a^2-b_1^2)(b_1^2+b_2^2) + (a^2-b_1^2)^2(b_1^2+b_2^2)^2}{(a^2+b_1^2)^2(b_1^2+b_2^2)^2} ,$$

$$1-A^2 = \frac{4a^2b_1^2(b_1^2+b_2^2)^2 - 4a^4b_1^4 + 4a^2b_1^2(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)^2(b_1^2+b_2^2)^2},$$

$$= 4a^2b_1^2\,\frac{b_1^4+2b_1^2b_2^2+b_2^4-a^2b_1^2+a^2b_1^2-b_1^4+a^2b_2^2-b_1^2b_2^2}{(a^2+b_1^2)^2(b_1^2+b_2^2)^2},$$

$$= 4a^2b_1^2\,\frac{2b_1^2b_2^2+b_2^4+a^2b_2^2-b_1^2b_2^2}{(a^2+b_1^2)^2(b_1^2+b_2^2)^2},$$

$$= 4a^2b_1^2b_2^2\,\frac{a^2+b_1^2+b_2^2}{(a_1^2+b_1^2)^2(b_1^2+b_2^2)^2},$$

$$\sqrt{1-A^2} = \frac{2ab_1b_2}{(a^2+b_1^2)(b_1^2+b_2^2)}\sqrt{a^2+b_1^2+b_2^2},$$

$$B\sqrt{1-A^2} = \frac{[2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)]\,2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(b_1^2+b_2^2)^2(a^2+b_1^2)(a^2+b_2^2)}.$$

If b_1 and b_2 are interchanged in the above expression, we get

$$A\sqrt{1-B^2} = \frac{[2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)]\,2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(b_1^2+b_2^2)^2(a^2+b_1^2)(a^2+b_2^2)}.$$

Hence,

$$A\sqrt{1-B^2}+B\sqrt{1-A^2} = \frac{[2a^2(b_1^2+b_2^2)-(a^2+a^2-b_1^2-b_2^2)(b_1^2+b_2^2)]\,2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(b_1^2+b_2^2)^2(a^2+b_1^2)(a^2+b_2^2)}$$

$$= \frac{[2a^2-2a^2+b_1^2+b_2^2]\,2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(b_1^2+b_2^2)(a^2+b_1^2)(a^2+b_2^2)},$$

$$= \frac{2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)} .$$

Therefore,

$$\sin^{-1}\left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right) = \sin^{-1}\frac{2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)}$$

$$= \sin^{-1}A + \sin^{-1}B .$$

Now, we are left to prove the following.

$$\sin^{-1}\frac{2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)} = 2\sin^{-1}\frac{b_1b_2}{\sqrt{(a^2+b_1^2)(a^2+b_2^2)}} .$$

Let

$$C = D = \frac{b_1b_2}{\sqrt{(a^2+b_1^2)(a^2+b_2^2)}} .$$

Then

$$2\sin^{-1}\frac{b_1b_2}{\sqrt{(a^2+b_1^2)(a^2+b_2^2)}} = \sin^{-1}C + \sin^{-1}D ,$$

$$= \sin^{-1}\left(C\sqrt{1-D^2}+D\sqrt{1-C^2}\right) ,$$

$$= \sin^{-1}\left(2C\sqrt{1-D^2}\right) ,$$

$$= \sin^{-1}\left[\frac{2b_1b_2}{\sqrt{(a^2+b_1^2)(a^2+b_2^2)}}\sqrt{1-\frac{b_1^2b_2^2}{(a^2+b_1^2)(a^2+b_2^2)}}\right] ,$$

$$= \sin^{-1}\left[\frac{2b_1b_2}{(a^2+b_1^2)(a^2+b_2^2)}\sqrt{(a^2+b_1^2)(a^2+b_2^2)-b_1^2b_2^2}\right] ,$$

$$= \sin^{-1}\left[\frac{2b_1b_2\sqrt{a^4+a^2b_1^2+a^2b_2^2+b_1^2b_2^2-b_1^2b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)}\right] ,$$

$$= \sin^{-1}\left[\frac{2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)}\right].$$

Hence,

$$\sin^{-1}A + \sin^{-1}B = \sin^{-1}\frac{2ab_1b_2\sqrt{a^2+b_1^2+b_2^2}}{(a^2+b_1^2)(a^2+b_2^2)} = 2\sin^{-1}C$$

or

$$\sin^{-1}\frac{2a^2b_1^2-(a^2-b_1^2)(b_1^2+b_2^2)}{(a^2+b_1^2)(b_1^2+b_2^2)} + \sin^{-1}\frac{2a^2b_2^2-(a^2-b_2^2)(b_1^2+b_2^2)}{(a^2+b_2^2)(b_1^2+b_2^2)}$$

$$= 2\sin^{-1}\frac{b_1b_2}{\sqrt{(a^2+b_1^2)(a^2+b_2^2)}}.$$

Thus, equation (I) is equivalent to equation (II).

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(15) $$\int \cos^{-1}\frac{x^2+b^2-c^2}{2xb}\cdot\frac{ax}{2\pi}(a^2+x^2)^{-\frac{3}{2}}dx =$$

$$= \frac{a}{2\pi}\left[\cos^{-1}\frac{x^2+b^2-c^2}{2xb}\left(-\frac{1}{\sqrt{a^2+x^2}}\right) - \int\frac{2xb}{\sqrt{4x^2b^2-(x^2+b^2-c^2)^2}}\cdot\frac{2xb(2x)-2b(x^2+b^2-c^2)}{(2xb)^2\sqrt{a^2+x^2}}dx\right]$$

$$= -\frac{a}{2\pi\sqrt{a^2+x^2}}\cos^{-1}\frac{x^2+b^2-c^2}{2xb} - \frac{a}{2\pi}\int\frac{1}{\sqrt{a^2+x^2}}\,\frac{x^2-b^2+c^2}{x\sqrt{-x^4+2x^2(b^2+c^2)-(b^2-c^2)^2}}dx$$

Let us take the remaining integral of the above expression.

$$\int\frac{x^2-b^2+c^2}{x\sqrt{a^2+x^2}\sqrt{-x^4+2x^2(b^2+c^2)-(b^2-c^2)^2}}\,dx.$$

Putting

$$x^2 = y,$$

$$2x\,dx = dy \,,$$
$$K^2 = b^2 - c^2 \,,$$
$$N^2 = b^2 + c^2 \,.$$

we get

$$\int \frac{x^2 - b^2 + c^2}{x\sqrt{a^2+x^2}\ \sqrt{-x^4 + 2x^2(b^2+c^2) - (b^2-c^2)^2}}\,dx$$

$$= \int \frac{y - K^2}{2y\sqrt{a^2+y}\ \sqrt{-y^2 + 2yN^2 - K^4}}\,dy \,,$$

$$= \frac{1}{2}\int \frac{y - K^2}{y\sqrt{-y^3 + y^2(2N^2 - a^2) + y(2a^2N^2 - K^4) - a^2K^4}}\,dy \,,$$

$$= \frac{1}{2}\int \frac{y - K^2}{y\sqrt{a_0 y^3 + 3a_1 y^2 + 3a_2 y + a_3}}\,dy \,,$$

where
$$a_0 = -1 \,,$$
$$3a_1 = 2N^2 - a^2 \,,$$
$$3a_2 = 2a^2N^2 - K^4 \,,$$
$$a_3 = -a^2K^4 \,.$$

$$= \frac{1}{2}\int \frac{dy}{\sqrt{a_0 y^3 + 3a_1 y^2 + 3a_2 y + a_3}} - \frac{K^2}{2}\int \frac{dy}{y\sqrt{a_0 y^3 + 3a_1 y^2 + 3a_2 y + a_3}}$$

The solutions of these integrals are given in terms of elliptic integrals in reference 8.

XVIII. REFERENCES

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