Rigidity of Pham-Brieskorn Threefolds

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Abstract

Let $\mathbf{k}$ be a field of characteristic zero. A Pham-Brieskorn ring is a $\mathbf{k}$-algebra of the form $B_{a_0,\dots,a_n} = \mathbf{k}[X_0,\dots,X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$, where $n \geq 2$ and $a_0,\dots,a_n$ are positive integers. A ring B is rigid if the only locally nilpotent derivation $D : B \to B$ is the zero derivation. Consider the following conjecture.

Conjecture. *Let $n \geq 2$, and let $B_{a_0,\dots,a_n} = \mathbf{k}[X_0,\dots,X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$ be a Pham-Brieskorn ring. If $\min\{a_0,\dots,a_n\} \geq 2$ and at most one element i of $\{0,\dots,n\}$ satisfies $a_i = 2$, then $B_{a_0,\dots,a_n}$ is rigid.*

The $n = 2$ case of the Conjecture is known to be true. In this thesis, we make progress towards solving the above conjecture. Our main results are:

(1) For any $n \geq 3$, in order to prove the above conjecture, it suffices to prove rigidity of $B_{a_0,\dots,a_n}$ in the cases where $\mathbf{k} = \mathbb{C}$ and $\operatorname{cotype}(a_0,\dots,a_n) = 0$.

(2) For any $n \geq 2$, $X = \operatorname{Proj} B_{a_0,\dots,a_n}$ is a well-formed quasismooth weighted complete intersection if and only if $\operatorname{cotype}(a_0,\dots,a_n) = 0$.

(3) When $n = 3$ and $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$, B_{a_0,a_1,a_2,a_3} is rigid, except possibly in the cases where, up to a permutation of the a_i, $(a_0, a_1, a_2, a_3) \in \{(2, 3, 4, 12), (2, 3, 5, 30)\}$.

(4) We summarize the list of 3-dimensional Pham-Brieskorn rings B_{a_0,a_1,a_2,a_3} for which rigidity is known. It follows in particular that if $B_{2,3,4,12}$ and $B_{2,3,5,30}$ are rigid then the $n = 3$ case of the above conjecture is true.

In addition to the above, we develop techniques for proving rigidity of rings in general; prove rigidity of many Pham-Brieskorn rings whose dimension is greater than 3; give simple examples of rational projective surfaces with quotient singularities that have an ample canonical divisor and prove that the members of a certain family of singular hypersurfaces are not rational.

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Contents

Introduction

In this introduction, and in most parts of this thesis, all rings and fields have characteristic 0. All rings are commutative, associative, and have a multiplicative identity.

Let B be a ring. A derivation $D : B \to B$ is *locally nilpotent* if for every $b \in B$, there exists some $n \in \mathbb{N}$ such that $D^n(b) = 0$. A ring B is *rigid* if the only locally nilpotent derivation $D : B \to B$ is the zero derivation.

Locally nilpotent derivations are an important tool in commutative algebra and affine algebraic geometry. They are closely related to automorphisms of varieties, and are particularly useful when trying to classify rings and varieties. It is also well known that if X is an affine variety with coordinate ring R, then there is a bijection between the set of locally nilpotent derivations $D : R \to R$ and the set of algebraic actions of G_a on X; in particular, R is rigid if and only if the only algebraic action of G_a on X is the trivial action. We refer the reader to [Dai03] for the details.

Whether or not a ring is rigid is a ring invariant; even the seemingly simple task of determining whether or not a ring is rigid is often very difficult.

Let $n \geq 2$, and let $B_{a_0,\dots,a_n} = \mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n} \rangle$ where $\mathbf{k}$ is a field and $a_0, \dots, a_n$ are positive integers. Then $B_{a_0,\dots,a_n}$ is an integral domain, and is called a *Pham-Brieskorn* ring. This thesis is motivated by the following conjecture:

0.0.1 Conjecture. *Let $n \geq 2$, and let $B_{a_0,\dots,a_n} = \mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n} \rangle$ be a Pham-Brieskorn ring. If $\min\{a_0, \dots, a_n\} \geq 2$ and at most one element i of $\{0, \dots, n\}$ satisfies $a_i = 2$, then $B_{a_0,\dots,a_n}$ is rigid.*

The $n = 2$ case of the above conjecture was proved by Kaliman and Zaidenberg in [KZ00]. For $n \geq 3$, only partial results are known. In [FZ03], Flenner and Zaidenberg asked the following question, which belongs to the $n = 3$ case of the conjecture:

Question. Is $\mathbb{C}[X_0, X_1, X_2, X_3]/\langle X_0^3 + X_1^3 + X_2^3 + X_3^3 \rangle$ rigid?

This question remained open for over 10 years before eventually being answered in the affirmative by Cheltsov, Park and Won. (See Corollary 1.9 of [CPW16a].)

The objective of this thesis is to contribute to the solution of Conjecture 0.0.1, particularly in the case where $n = 3$. We give some partial results valid for all n, and we almost completely solve the $n = 3$ case.

Before we began working on the problem, some special cases had already been resolved. In order to properly discuss the state of the problem before we began working on it we require the following definition.

Definition. Let $n \geq 2$ and let $S = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$. For each $j \in \{0, \dots, n\}$, we define $S_j = (a_0, \dots, \widehat{a_j}, \dots, a_n) \in \mathbb{Z}^n$, and define

$$J(S) = \{\, j \in \{0, \dots, n\} \,:\, \operatorname{lcm}(S_j) \neq \operatorname{lcm}(S) \,\} = \{\, j \in \{0, \dots, n\} \,:\, a_j \nmid \operatorname{lcm}(S_j) \,\}.$$

The *cotype* of S is the cardinality of $J(S)$.

Returning to Conjecture 0.0.1 in the $n = 3$ case, if we assume that (a_0, a_1, a_2, a_3) satisfies the hypothesis of the conjecture, it was already known that B_{a_0,a_1,a_2,a_3} is rigid in each of the following cases:

(a) $\gcd(a_0 a_1 a_2, a_3) = 1$;

(b) $\min\{a_0, a_1, a_2, a_3\} \geq 8$;

(c) $(a_0, a_1, a_2, a_3) = (a_0, 3, 3, 3)$ where $a_0 \geq 3$;

(d) $(a_0, a_1, a_2, a_3) = (2, a_1, a_2, a_3)$,
where $a_1, a_2, a_3 \geq 3$, a_1 is even, $\gcd(a_1, a_2) \geq 3$, and $\gcd(a_3, \operatorname{lcm}(a_1, a_2)) = 2$;

(e) $\operatorname{cotype}(a_0, a_1, a_2, a_3) \geq 2$;

(f) $a_0 \not| \operatorname{lcm}(a_1, a_2, a_3)$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} < \frac{1}{2}$;

(g) $(a_0, a_1, a_2, a_3) = (bk_0, bk_1, bk_2, bk_3)$ where $b \geq 3$ and $k_0, k_1, k_2, k_3 \in \mathbb{N}^+$ are pairwise relatively prime.

Items (a) - (d) appear as Theorem 8.1 in [FMJ13]. Item (e) is a reformulation of Theorem 7.2 (b) in [DFMJ17]. Items (f) and (g) appear as Theorem 1.4 (d) and (f) of [CD20]. Additional cases for arbitrary $n \geq 3$ are also known to be rigid. In particular, (assuming $(a_0, \dots, a_n)$ satisfies the hypothesis of Conjecture 0.0.1) $B_{a_0,\dots,a_n}$ is rigid in each of the following cases:

(h) $\operatorname{cotype}(a_0, \dots, a_n) \geq n - 1$;

(i) $\sum_{i=0}^{n} \frac{1}{a_i} \leq \frac{1}{n-1}$.

Item (h) appears as Theorem 1.4 (e) of [CD20] and (i) is Example 2.6 in [FM10]. In [LS22], the authors study rigidity of ring extensions. Remarks 6.3 and 6.4 show that some 4-dimensional Pham-Brieskorn rings are rigid, however, these are already captured by part (h) above.

While the above results cover many cases and the techniques that they demonstrate are interesting, they do not really provide much insight into how to solve Conjecture 0.0.1 in general. Moreover, most cases where the a_i are small and satisfy $\operatorname{cotype}(a_0, \dots, a_n) < n - 1$ are not resolved by previous results. One of the main contributions of this thesis is that Theorem 1.3.12, Proposition 2.4.25 and Theorem 4.1.14 (all discussed later in the Introduction) when combined provide an approach for proving Conjecture 0.0.1 in all dimensions, and especially in dimension 3.

Content of the thesis. In Section 1.1, we give some general results on locally nilpotent derivations, most of which are well known. Sections 1.2 and 1.3 discuss Conjecture 0.0.1. Proposition 1.2.8 recalls the well-known fact that if Conjecture 0.0.1 is true in the special case $\mathbf{k} = \mathbb{C}$ then it is true for any field $\mathbf{k}$ of characteristic 0. This allows us to assume that $\mathbf{k} = \mathbb{C}$ without loss of generality; this assumption is in effect from Section 2.3 to the end of the thesis. The first main result of the thesis is Theorem 1.3.12; here is a simplified version of this result:

> *To prove Conjecture 0.0.1, it suffices to prove rigidity of $B_{a_0,\dots,a_n}$ in the cases where* $\operatorname{cotype}(a_0, \dots, a_n) = 0$.

Chapter 2 provides the geometric background needed in the remainder of the thesis. In particular, we discuss sheaves of modules, divisors and their properties, singularities, and weighted projective varieties. We give the necessary definitions, and state the results needed in the thesis. When unable to find suitable references, we provide proofs.

We use geometry (in two ways) to study $B_{a_0,\dots,a_n} = \mathbb{C}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$:

(1) The solution-set of $X_0^{a_0} + \cdots + X_n^{a_n} = 0$ in $\mathbb{C}^{n+1}$ is an affine variety of dimension n. In the language of schemes (which we adopt in this thesis), it is denoted $\operatorname{Spec}(B_{a_0,\dots,a_n})$.

(2) Let $L = \operatorname{lcm}(a_0, \dots, a_n)$ and (for each $i \in \{0, \dots, n\}$) let $w_i = L/a_i$. Define an $\mathbb{N}$-grading on $\mathbb{C}[X_0, \dots, X_n]$ by declaring that (for each i) X_i is homogeneous of degree w_i. Then $X_0^{a_0} + \cdots + X_n^{a_n}$ is a homogeneous element of $\mathbb{C}[X_0, \dots, X_n]$, so $B_{a_0,\dots,a_n}$ is an $\mathbb{N}$-graded ring. The fact that $X_0^{a_0} + \cdots + X_n^{a_n}$ is homogeneous also allows us to view the solution-set of $X_0^{a_0} + \cdots + X_n^{a_n} = 0$ as a hypersurface of the weighted projective space $\mathbb{P}(w_0, \dots, w_n)$. This hypersurface is a projective variety of dimension $n - 1$. In the language of schemes, it is denoted $\operatorname{Proj}(B_{a_0,\dots,a_n})$.

We will see that both $\operatorname{Spec}(B_{a_0,\dots,a_n})$ and $\operatorname{Proj}(B_{a_0,\dots,a_n})$ are normal varieties; this allows us to consider Weil divisors on these varieties. Studying the hypersurface $\operatorname{Proj}(B_{a_0,\dots,a_n})$ of $\mathbb{P}(w_0,\dots,w_n)$ turns out to be particularly fruitful. In this regard, Proposition 2.4.25 states:

$$\operatorname{Proj}(B_{a_0,\dots,a_n}) \textit{ is a well-formed quasismooth weighted complete intersection in } \mathbb{P}(w_0,\dots,w_n) \iff \operatorname{cotype}(a_0,\dots,a_n)=0,$$

which is very useful in conjunction with the aforementioned Theorem 1.3.12.

Chapter 3 has only two sections. Corollaries 3.1.11, 3.1.15 and 3.1.16 are the main results of the first section. Recall that if $B=\bigoplus_{i\in\mathbb{Z}} B_i$ is a $\mathbb{Z}$-graded ring and d is a positive integer, then the *d-th Veronese subring* of B is $B^{(d)}=\bigoplus_{i\in d\mathbb{Z}} B_i$. Corollary 3.1.11 implies that

> *if* $\operatorname{cotype}(a_0,\dots,a_n)=0$ *and* $B_{a_0,\dots,a_n}$ *is rigid then* $\left(B_{a_0,\dots,a_n}\right)^{(d)}$ *is rigid for every* $d\geq 1$.

Corollary 3.1.15 asserts:

> *if* B_{a_0,a_1,a_2,a_3} *is not rigid then both* $\operatorname{Proj}(B_{a_0,a_1,a_2,a_3})$ *and* $\operatorname{Spec}(B_{a_0,a_1,a_2,a_3})$ *are rational varieties.*

One defines the *amplitude* of $B_{a_0,\dots,a_n}$ (or of $\operatorname{Proj}(B_{a_0,\dots,a_n})$) to be the integer $\alpha=L-\sum_{i=0}^{n}\frac{L}{a_i}$ where $L=\operatorname{lcm}(a_0,\dots,a_n)$. Corollary 3.1.16 asserts:

> *Let* $(a_0,\dots,a_n)\in(\mathbb{N}^+)^{n+1}$ *and let* α *be the amplitude of* $B=B_{a_0,\dots,a_n}=\bigoplus_{i\in\mathbb{N}} B_i$. *If* $\operatorname{cotype}(a_0,\dots,a_n)=0$ *and* $B_\alpha\neq 0$ *then* $\operatorname{Proj}(B)$ *is not birationally ruled and* B *is rigid.*

In Section 3.2 we discuss the notions of unramified morphisms and étale morphisms, in relation with singularities. We then use those ideas to determine the type of the singularities of $\operatorname{Proj} B_{2,3,3,6}$. This information is used in Chapter 4, in order to show that $B_{2,3,3,6}$ is rigid.

The remaining main results of the thesis appear in Chapter 4 where we show that most 3-dimensional Pham-Brieskorn rings are rigid. In Section 4.1, we discuss an important result of Demazure's (Theorem 3.5 in [Dem88]) and apply it to Pham-Brieskorn rings. We require the following definitions, the second of which is used in Theorem 4.1.14, the main result of the section.

Definitions. A scheme U over $\mathbb{C}$ is a *cylinder* if $U\cong C\times_{\mathbb{C}}\mathbb{A}^1_{\mathbb{C}}$ for some affine scheme C. A scheme X over $\mathbb{C}$ *contains a cylinder* if there exists a non-empty open set $U\subseteq X$ such that U is a cylinder.

Let H be an ample $\mathbb{Q}$-divisor on a projective normal variety X over $\mathbb{C}$. Let $U\subseteq X$ be a cylinder of X. Then, the cylinder $U\subseteq X$ is called *H-polar* if $U=X\setminus\operatorname{supp}(D)$ for some effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D\in\operatorname{Div}(X,\mathbb{Q})$ such that $D\sim sH$ where $s\in\mathbb{Q}^+$.

This is Theorem 4.1.14, stated with minor edits for convenience.

Theorem. *Let $n \geq 2$ and consider $(a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ of cotype 0. Let $B = B_{a_0,\dots,a_n}$, let $X = \operatorname{Proj}(B)$, let ω_X and K_X be the canonical sheaf and canonical divisor of X, and let α be the amplitude of X.*

(a) $\omega_X \cong \mathcal{O}_X(\alpha)$

(b) *Let T be a homogeneous element of $\operatorname{Frac} B$ of degree 1 and let H be the unique $\mathbb{Q}$-divisor of X satisfying $B = \bigoplus_{i\in\mathbb{N}} \mathrm{H}^0\big(X, \mathcal{O}_X(nH)\big)T^i$ (see Theorem 4.1.2). Then $H \in \operatorname{Div}(X)$, $K_X \sim \alpha H$ and K_X is $\mathbb{Q}$-Cartier.*

(c) *If $\alpha = 0$ then $B^{(d)}$ is rigid for all $d \geq 1$.*

(d) *Assume that $\alpha \neq 0$ and define $s = \frac{\alpha}{|\alpha|} \in \{1, -1\}$. Then sK_X is ample and the following are equivalent:*

   (i) *B is non rigid;*

   (ii) *$B^{(d)}$ is non rigid for some $d \geq 1$;*

   (iii) *There exists a homogeneous element $h \in B \setminus \{0\}$ of positive degree such that the open subset $D_+(h)$ of $\operatorname{Proj} B$ is a cylinder.*

   (iv) *There exists a (sK_X)-polar cylinder of X.*

So, to show that $B_{a_0,\dots,a_n}$ is rigid, it suffices to prove the non-existence of (sK_X)-polar cylinders in $\operatorname{Proj} B_{a_0,\dots,a_n}$.

In Section 4.2, we reduce the problem (when $n = 3$) to two remaining cases. Define

$$\begin{aligned}
\Gamma_3 &= \big\{\, (a_0, a_1, a_2, a_3) \in (\mathbb{N}^+)^4 \,:\, \min\{a_0, a_1, a_2, a_3\} \geq 2 \text{ and at most one } i \text{ satisfies } a_i = 2 \,\big\},\\
\Gamma_3^+ &= \big\{\, (a_0, a_1, a_2, a_3) \in \Gamma_3 \,:\, \operatorname{cotype}(a_0, a_1, a_2, a_3) = 0 \text{ and } \alpha \geq 0 \,\big\},\\
\Gamma_3^- &= \big\{\, (a_0, a_1, a_2, a_3) \in \Gamma_3 \,:\, \operatorname{cotype}(a_0, a_1, a_2, a_3) = 0 \text{ and } \alpha < 0 \,\big\},
\end{aligned}$$

where α denotes the amplitude of B_{a_0,a_1,a_2,a_3}. Recall (by Theorem 1.3.12) that

> *to prove the $n = 3$ case of Conjecture 0.0.1, it suffices to show that B_{a_0,a_1,a_2,a_3} is rigid for every $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$.*

It turns out that Γ_3^+ is an infinite set but that Γ_3^- is finite. Lemma 4.2.4 shows that, up to permutation, the elements of Γ_3^- are:

$$(2,3,3,6),\ (2,3,4,12),\ (2,3,5,30),\ (2,3,6,6),\ (2,4,4,4),\ (3,3,3,3),\ (3,3,4,4),\ (3,3,5,5).$$

Section 4.3 proves that if $(a_0, a_1, a_2, a_3) \in \Gamma_3^+$ then $(B_{a_0,a_1,a_2,a_3})^{(d)}$ is rigid for every $d \geq 1$. To prove this, we use a result of [CPW16b] that implies that $\operatorname{Proj} B_{a_0,\dots,a_3}$ cannot contain a cylinder.

Another interesting result from Section 4.3 is the following (Corollary 4.3.5):

> *Let $a, c \in \mathbb{N}^+$ be such that $\gcd(a, c) = 1$ and such that $\frac{1}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{c} < 1$. Let $X = \operatorname{Proj} B_{a,a,c,c}$. Then, X is a normal rational surface with quotient singularities and ample canonical divisor.*

Examples of rational surfaces with quotient singularities and ample canonical divisors do not appear very frequently in the literature. Since the above examples are so simple, we consider them to be worthy of mention. (See [HK12] and [Kol08] for some other examples.)

Sections 4.4 to 4.6 prove rigidity of $(B_{a_0,a_1,a_2,a_3})^{(d)}$ for each $(a_0, a_1, a_2, a_3) \in \Gamma_3^-$ which is not a permutation of $(2, 3, 4, 12)$ or $(2, 3, 5, 30)$. In each of these cases $\operatorname{Proj} B_{a_0,a_1,a_2,a_3}$ is a (possibly singular) del Pezzo surface. For each of these surfaces, after determining the singularities and the degree of the del Pezzo surface $\operatorname{Proj} B_{a_0,a_1,a_2,a_3}$, we use results from the literature to prove the absence of $(-K_X)$-polar cylinders. Combining the results from Sections 4.2 to 4.6 we obtain Corollary 4.6.10, which asserts:

> *If $B_{2,3,4,12}$ and $B_{2,3,5,30}$ are rigid then the $n = 3$ case of Conjecture 0.0.1 is true.*

This thesis does not answer the question whether $B_{2,3,4,12}$ and $B_{2,3,5,30}$ are rigid. Remark 4.6.11 explains why these two cases are difficult, and discusses work in progress (with Adrien Dubouloz) which might lead to their solution and hence to completing the proof of the $n = 3$ case of Conjecture 0.0.1.

After considering the cases where $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$, we turn towards some cases where $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 1$. In most cases, if $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 1$, rigidity follows from an easy application of Proposition 1.3.10. However, in some cases, we require other methods to prove rigidity. These cases are discussed in Section 4.7.

In Section 4.8 we summarize our results on the 3-dimensional case of Conjecture 0.0.1. Let Δ be the set of all tuples (a_0, a_1, a_2, a_3) such that (a_0, a_1, a_2, a_3) is a permutation of an element of $\{ (2, 3, 4, 12m) : m \in \mathbb{N}^+ \} \cup \{ (2, 3, 5, 30m) : m \in \mathbb{N}^+ \}$. Observe that $\Delta \subset \Gamma_3$. We obtain the following, which is Theorem 4.8.3:

Theorem. *Let $B = B_{a_0,a_1,a_2,a_3}$ be a Pham-Brieskorn ring and let Δ be as above.*

(a) *If $(a_0, a_1, a_2, a_3) \notin \Gamma_3$ then B is not rigid.*

(b) *If $(a_0, a_1, a_2, a_3) \in \Delta$, then the rigidity of B is not yet known.*

(c) *If $(a_0, a_1, a_2, a_3) \in \Gamma_3 \setminus \Delta$ then B is rigid.*

(d) *If* $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$ *and* B *is rigid, then* $B^{(d)}$ *is rigid for all* $d \geq 1$.

In Section 4.9, we discuss Conjecture 0.0.1 for higher dimensions. We prove two results that are noteworthy, the second of which is already known to be an easy application of existing results.

Assume that $n \geq 4$ and that $\operatorname{cotype}(a_0, \dots, a_n) = 0$. Corollary 4.9.6 and Proposition 4.9.7 show that

$B_{a_0,\dots,a_n}$ is not rigid if and only if $\operatorname{Proj} B$ contains a cylinder,

and

$B_{a_0,\dots,a_n}$ is a unique factorization domain.

Finally, Chapter 5 is optional and does not contribute new information towards answering Conjecture 0.0.1. The results of Chapter 5 were initially developed to prove that many of the Pham-Brieskorn rings B_{a_0,a_1,a_2,a_3} satisfying $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 1$ are rigid. While the methods worked in most cases, the results of Section 4.7 provided simpler proofs. Despite the above, we chose to include the chapter anyway. The techniques are interesting and the main result of the section (Corollary 5.3.6) appears to be new. That result can be stated as follows:

Let $c \geq 2$. *Consider the hypersurface* X_c *of* $\mathbb{P}^3$ *which is the solution-set of* $X_0^2 T^{c-2} + X_1^c + X_2^c + T^c = 0$. *Then* X_c *is rational if and only if* $c \leq 4$.

Notation and Definitions

In this section, we introduce the definitions and notation that we will use throughout the thesis. To the maximum extent possible, we use the definitions and notation in Hartshorne's *Algebraic Geometry* [Har77].

Algebra

- All rings are commutative with a multiplicative identity.
- The symbols $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^+, \mathbb{R}, \mathbb{C}$ denote the natural numbers, the positive natural numbers, the integers, the rational numbers, the positive rational numbers, the real numbers and the complex numbers respectively.
- If $r \in \mathbb{R}$, $\lceil r \rceil$ denotes the smallest integer greater than or equal to r, and $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.
- We use the symbols " $\subseteq$ " and " $\subset$ " to denote inclusion and proper inclusion respectively. We use the symbols " $\supseteq$ " and " $\supset$ " to denote containment and proper containment respectively.
- If S is a set, $|S|$ denotes the cardinality of S.
- If S is a set and $(s_1, \ldots, s_n) \in S^n$ is an n-tuple, then $(s_1, \ldots, s_{i-1}, \widehat{s_i}, s_{i+1}, \ldots, s_n) \in S^{n-1}$ is the $(n-1)$-tuple with the i^{th} coordinate omitted.
- The symbol $\mathbf{k}$ denotes a field and $\bar{\mathbf{k}}$ denotes its algebraic closure.
- Let $K/\mathbf{k}$ be an extension of fields. If K is purely transendental over $\mathbf{k}$ of finite transcendence degree, we say that $K/\mathbf{k}$ is *rational*. If $K/\mathbf{k}$ is rational and has transcendence degree n (where $n \in \mathbb{N}$), we write $K = \mathbf{k}^{(n)}$.
- Throughout the thesis, we will sometimes abbreviate the term "integral domain" by "domain".

- If A and B are integral domains and A is a subring of B, then the *transcendence degree of B over A* is defined to be the transcendence degree of $\operatorname{Frac} B$ over $\operatorname{Frac} A$ and is denoted $\operatorname{trdeg}(B/A)$.
- The polynomial ring in n variables over a ring R is denoted as $R^{[n]}$ or $R[X_1, \dots, X_n]$.
- If R is a ring and I is an ideal of R, we write $I \lhd R$.
- If R is a ring, then R^* is the set of invertible elements of R.
- If R is a ring, the ideal $\langle f_1, \dots, f_r \rangle \lhd R$ is the ideal generated by $f_1, \dots, f_r \in R$.
- If R is a ring, and $\mathfrak{p}$ is a prime ideal, then the *height* of $\mathfrak{p}$ is denoted by $\operatorname{ht}(\mathfrak{p})$. See page 120 of [AM69].
- If R is a subring of S, then we require that $1_R = 1_S$.
- A ring homomorphism $\varphi : R \to S$ is *of finite type* if S is finitely generated as an R-algebra.
- A $\mathbf{k}$-domain is an integral domain containing a field $\mathbf{k}$.
- An affine $\mathbf{k}$-domain is an integral domain that is finitely generated as a $\mathbf{k}$-algebra.
- Suppose $B = \bigoplus_{i \in \mathbb{Z}} B_i$ is a $\mathbb{Z}$-graded ring. Then $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a *graded subring* of B if A itself is a graded ring and $A_i \subseteq B_i$ for all $i \in \mathbb{Z}$.
- If R is a ring, then $\widehat{R}_I$ is the completion of R with respect to the ideal I. If R is a local ring, then $\widehat{R}$ is the completion of R with respect to the maximal ideal $\mathfrak{m} \lhd R$.
- If M is a module, the submodule $\langle m_1, \dots, m_r \rangle \subseteq M$ is the submodule of M generated by $m_1, \dots, m_r \in M$.
- Let R be a ring and let R-**Mod** denote the category of R-modules. If A is an R-module, then $\operatorname{Ext}^i_R(A, -) : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ are the right derived functors of the $\operatorname{Hom}_R(A, -) : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ functor.
- Let R be a ring and let M be an R-module. A sequence $x_1, \dots, x_r$ of elements of R is called a *weak M-regular sequence* if x_1 is not a zero divisor in M and for all $i = 2, \dots, r$, x_i is not a zero divisor in $M/\langle x_1, \dots, x_{i-1} \rangle M$.

  Let $x_1, \dots, x_r$ be a weak M-regular sequence and let $I = \langle x_1, \dots, x_r \rangle \lhd R$. If $M \neq IM$ then $x_1, \dots, x_r$ is called an *M-regular sequence.* If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module, then the *depth* of M is the maximum length of an M-regular sequence $x_1, \dots, x_r$ where $x_i \in \mathfrak{m}$ for all $i = 1, \dots, r$.

- If R is a ring and $f \in R$, then R_f is the localization of R with respect to the multiplicative set $\{1, f, f^2, f^3 \dots\}$. If $\mathfrak{p} \in \operatorname{Spec} R$, then $R_\mathfrak{p}$ is the localization with respect to $R \setminus \mathfrak{p}$. (See Examples on page 38 of [AM69].)

- If R is an $\mathbb{N}$-graded ring and $f \in R$ is a homogeneous element, then $R_{(f)}$ is the degree 0 subring of the $\mathbb{Z}$-graded ring R_f.

- A Noetherian local ring R is *Cohen-Macaulay* if the depth of R (as a module over itself) is equal to the Krull dimension of R. A ring R is *Cohen-Macaulay* if it is Noetherian and if $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec} R$. Note in particular that the zero ring is Cohen-Macaulay.

- A Noetherian local ring R of dimension n is *Gorenstein* if $\operatorname{Ext}^i_R(\mathbf{k}, R) = 0$ for all $i \neq n$ and $\operatorname{Ext}^n_R(\mathbf{k}, R) \cong \mathbf{k}$ where $\mathbf{k}$ is the residue field of R.

- Let R be a ring and let M be an R-module. Let $S \subseteq M$. The *annihilator* of S in R is the set
$$\operatorname{Ann}_R(S) = \left\{ r \in R \,:\, rs = 0 \text{ for all } s \in S \right\}.$$
Note that $\operatorname{Ann}_R(S)$ is an ideal of R.

- If M is a matrix, M^T is the transpose of M.

Geometry

- A *variety* is an integral separated scheme of finite type over an algebraically closed field $\mathbf{k}$. In particular, varieties are irreducible and reduced. (See page 105 of [Har77].)

- A *curve* is a one-dimensional variety. A *surface* is a two-dimensional variety. Since varieties are irreducible by definition, all curves and surfaces are irreducible and reduced.

- *Affine n-space over* $\mathbf{k}$ is denoted by $\mathbb{A}^n_\mathbf{k} = \operatorname{Spec} \mathbf{k}[X_1, \dots, X_n]$.

- *Projective n-space over* $\mathbf{k}$ is denoted by $\mathbb{P}^n_\mathbf{k} = \operatorname{Proj} \mathbf{k}[X_1, \dots, X_n]$.

- If $\mathbf{k}$ is algebraically closed, a *hypersurface of* $\mathbb{A}^n_\mathbf{k}$ *(resp.* $\mathbb{P}^n_\mathbf{k}$*)* is a closed subvariety of $\mathbb{A}^n_\mathbf{k}$ (resp. $\mathbb{P}^n_\mathbf{k}$) of codimension 1.

- If X is a scheme, then $\mathcal{O}_X$ is its structure sheaf.

- We will use the following categories throughout the thesis.
  - **Ab** - abelian groups

  - R-**Mod** - R-modules
  - **Ab**(X) - sheaves of abelian groups on a topological space X
  - **Mod**(X) - sheaves of $\mathcal{O}_X$-modules on a scheme X.

- Let $f : X \to Y$ be a continuous map of topological spaces. The *direct image functor* $f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ is defined on page 65 of [Har77]. The *higher direct image functors*
$$R^i f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$$
are the right derived functors of f_*. (See Section III.8 in [Har77].)

- If R is a ring, $X = \operatorname{Spec} R$ and $f \in R$, then $D(f)$ is the principal open subscheme of X whose underlying set consists of prime ideals in $\operatorname{Spec} R$ that do not contain f. Note that $D(f) \cong \operatorname{Spec} R_f$.

- Let R be an $\mathbb{N}$-graded ring. If $X = \operatorname{Proj} R$ and $f \in R$ is homogeneous with $\deg(f) > 0$, then $D_+(f)$ is the principal open subscheme of X whose underlying set consists of homogeneous prime ideals in $\operatorname{Proj} R$ that do not contain f. Note that $D_+(f) \cong \operatorname{Spec} R_{(f)}$.

- If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a locally ringed space X, then the *dual of* $\mathcal{F}$ is defined to be $\mathcal{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. (The sheaf $\mathscr{H}om$ is defined on page 109 of [Har77].)

- If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a locally ringed space X, then $\bigwedge^n \mathcal{F}$ is the n^{th} *exterior power of* $\mathcal{F}$. See Exercise II.5.16 in [Har77].

- If X is a scheme, a point $x \in X$ is *regular* or *nonsingular* if $\mathcal{O}_{X,x}$ is a regular local ring. Otherwise, the point x is *singular*.

- If X is a scheme, then we write $\operatorname{Sing}(X)$ to denote the set of singular points of X.

- If X is an integral scheme, then $K(X)$ is the function field of X.

- If X is a projective scheme over $\mathbf{k}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F})$ is a finite-dimensional vector space over $\mathbf{k}$ by Theorem II.5.19 of [Har77]. We write $h^i(X, \mathcal{F}) = \dim_{\mathbf{k}} H^i(X, \mathcal{F})$.

- Let X be a projective scheme over $\mathbf{k}$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. The *Euler characteristic* of $\mathcal{F}$ is
$$\chi(\mathcal{F}) = \sum_{i \in \mathbb{N}} (-1)^i \dim_{\mathbf{k}} H^i(X, \mathcal{F}).$$
Assume furthermore that X has dimension r over $\mathbf{k}$. The *arithmetic genus of* X is
$$p_a(X) = (-1)^r (\chi(\mathcal{O}_X) - 1).$$

- A locally Noetherian scheme X is *Cohen–Macaulay* if for each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay.

- Let X be a locally Noetherian scheme. A point $x \in X$ is *Gorenstein* if $\mathcal{O}_{X,x}$ is a Gorenstein local ring. A scheme X is *Gorenstein* if x is Gorenstein for all $x \in X$.

- Let X be an integral, separated, Noetherian scheme. Then X is *locally factorial* if for all $x \in X$, $\mathcal{O}_{X,x}$ is a unique factorization domain.

Chapter 1

Locally Nilpotent Derivations

1.1 Preliminaries on Locally Nilpotent Derivations

1.1.1 Assumption. All rings (and fields $\mathbf{k}$) in this section have characteristic zero.

1.1.2 Definitions. Let B be a ring. A derivation $D : B \to B$ is *locally nilpotent* if for every $b \in B$ there exists some $n \in \mathbb{N}$ such that $D^n(b) = 0$. If B is a ring, the set of locally nilpotent derivations $D : B \to B$ is denoted $\mathrm{LND}(B)$. If $\mathrm{LND}(B) = \{0\}$, we say that B is *rigid*.

1.1.3 Definitions. Let B be a ring and let $D \in \mathrm{LND}(B)$.

- A *local slice* of D is an element $t \in B \setminus \ker(D)$ such that $D(t) \neq 0$ and $D^2(t) = 0$. In particular, if $D(t) = 1$, then t is called a *slice of* D.
- A derivation D is *reducible* if there exists some $b \in B$ such that $D(B) \subseteq \langle b \rangle \neq B$. If no such b exists, then D is *irreducible*.

1.1.4 Definition. Let $A \subseteq B$ be an inclusion of integral domains. Then, A is *factorially closed in* B if for all $x, y \in B \setminus \{0\}$, $xy \in A$ implies that $x, y \in A$.

1.1.5 Theorem. *Let B be an integral domain, let $D : B \to B$ be a derivation, and let $A = \ker(D)$. The following facts are well known. (Refer to [Fre17] for instance.)*

(a) *If D is locally nilpotent, then A is a factorially closed subring of B. Consequently, if D is locally nilpotent and $\mathbf{k}$ is a field included in B, then D is a $\mathbf{k}$-derivation.*

(b) *Assume that $\mathbb{Q} \subseteq B$. If $D \neq 0$ is locally nilpotent then D has a local slice $t \in B$. For any such t, if we define $s = D(t)$ then $B_s = A_s[t]$ is a polynomial ring in one variable over A_s. Consequently, $\mathrm{trdeg}(B/A) = 1$ and $\mathrm{Frac}(B) \supset \mathrm{Frac}(A)$ is a purely transcendental field extension of transcendence degree one.*

(c) *If $b \in B \setminus \{0\}$, then the derivation $bD : B \to B$ is locally nilpotent if and only if D is locally nilpotent and $b \in A$.*

(d) *If $D \neq 0$ is locally nilpotent and B satisfies the ascending chain condition for principal ideals, then there exists an irreducible locally nilpotent derivation $\delta : B \to B$ such that $D = a\delta$ for some $a \in A$.*

(e) *Let $S \subseteq B \setminus \{0\}$ be a multiplicative subset of B containing 1. Then $S^{-1}D : S^{-1}B \to S^{-1}B$ defined by $(S^{-1}D)(\frac{b}{s}) = \frac{sD(b)-bD(s)}{s^2}$ is a derivation and the following hold:*

- *$S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$;*
- *if $S \subseteq A$ then $\ker(S^{-1}D) = S^{-1}A$ and $(S^{-1}A) \cap B = A$.*

1.1.6 Definition. A scheme U over a field $\mathbf{k}$ is a *cylinder* if $U \cong C \times_{\mathbf{k}} \mathbb{A}^1_{\mathbf{k}}$ for some affine scheme C over $\mathbf{k}$. A scheme X over $\mathbf{k}$ *contains a cylinder* if there exists a non-empty open set $U \subseteq X$ such that U is a cylinder. Note that U is affine.

1.1.7 Remark. Suppose B is a $\mathbf{k}$-domain. If B is not rigid, then Theorem 1.1.5 (b) implies that $\operatorname{Spec}(B)$ contains a cylinder. If some $D \in \operatorname{LND}(B)$ has a slice and we define $A = \ker(D)$, then $\operatorname{Spec}(B) \cong \operatorname{Spec} A \times_{\mathbf{k}} \mathbb{A}^1_{\mathbf{k}}$ is a cylinder.

1.1.8 Definition. Let $(G, +)$ be an abelian group. A *G-grading* of a ring B is a family $\{B_g\}_{g \in G}$ of subgroups of $(B, +)$ such that $B = \bigoplus_{g \in G} B_g$ and $B_g B_h \subseteq B_{g+h}$ for all $g, h \in G$. A *G-graded ring* is a ring B together with a G-grading (of B). If B is a G-graded ring, then an element $b \in B$ is *homogeneous* if $b \in B_g$ for some $g \in G$. In the special case where $G = \mathbb{Z}$ and $B_i = 0$ for all $i < 0$, we say that B is $\mathbb{N}$-graded and write $B = \bigoplus_{i \in \mathbb{N}} B_i$.

1.1.9 Definitions. Suppose that G is an abelian group and that $B = \bigoplus_{i \in G} B_i$ is a G-graded ring.

- A derivation $D : B \to B$ is *homogeneous* if there exists an $h \in G$ such that $D(B_g) \subseteq B_{g+h}$ for all $g \in G$. If D is homogeneous and $D \neq 0$, then h is unique and we say that D is homogeneous of degree h. The zero derivation is said to be homogeneous of degree $-\infty$. The set of homogeneous locally nilpotent derivations of B is denoted $\operatorname{HLND}(B)$.
- A *graded subring* of B is a subring A of B satisfying $A = \bigoplus_{g \in G}(A \cap B_g)$. If A is a graded subring of B then A is a G-graded ring ($A = \bigoplus_{g \in G} A_g$ is a G-grading of A, where we define $A_g = A \cap B_g$ for each $g \in G$). Note that if $D \in \operatorname{HLND}(B)$ then $\ker(D)$ is a graded subring of B.

- If A is a graded subring of B, define $G(A)$ to be the subgroup of G generated by the set $\{\, g \in G \, : \, A_g \neq 0 \,\}$. If A is $\mathbb{Z}$-graded, we write $\mathbb{Z}(A)$ instead of $G(A)$.

1.1.10 Definition. A *degree function* on a ring B is a map $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ satisfying:

(a) $\forall x \in B$, $\deg(x) = -\infty \iff x = 0$;

(b) $\forall x, y \in B$, $\deg(xy) = \deg(x) + \deg(y)$;

(c) $\forall x, y \in B$, $\deg(x + y) \leq \max(\deg(x), \deg(y))$.

1.1.11 Definition. Let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be a $\mathbb{Z}$-graded integral domain and let $b \in B$. Define the set map $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ as follows. If $b = 0$, define $\deg(b) = -\infty$. Otherwise, write $b = \sum_{i \in \mathbb{Z}} b_i$ where $b_i \in B_i$, note that $b_i = 0$ for almost all $i \in \mathbb{Z}$ and define $\deg(b) = \max\{i : b_i \neq 0\}$. Since B is an integral domain, it follows that $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ is a degree function on B.

Recall the terms $\mathbf{k}$-*domain* and *affine* $\mathbf{k}$-*domain* from the Notation and Definitions section.

1.1.12 (Homogenization)**.** Let $\mathbf{k}$ be a field of characteristic zero and $B = \bigoplus_{i \in \mathbb{Z}} B_i$ a $\mathbb{Z}$-graded affine $\mathbf{k}$-domain. It can be shown that if $D \in \operatorname{LND}(B)$ then the set

$$S_D \stackrel{\text{def}}{=} \{\, \deg(D(x)) - \deg(x) \, : \, x \in B \setminus \{0\} \,\} \subseteq \mathbb{Z} \cup \{-\infty\}$$

has a greatest element (where $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ is the degree function defined in Definition 1.1.11); one defines $\deg(D) = \max(S_D) \in \mathbb{Z} \cup \{-\infty\}$. Clearly, $\deg(D) = -\infty \Leftrightarrow D = 0$. Also, if D happens to be homogeneous then $\deg(D)$ coincides with the usual degree of a homogeneous derivation (Definition 1.1.9).

If $D \in \operatorname{LND}(B)$ then there is a natural way to define an element $\tilde{D}$ of $\operatorname{HLND}(B)$ satisfying (in particular) $\deg(\tilde{D}) = \deg(D)$. Indeed, if $D = 0$ then set $\tilde{D} = 0$. If $D \in \operatorname{LND}(B) \setminus \{0\}$ then let $d = \deg(D) \in \mathbb{Z}$ and define (for each $i \in \mathbb{Z}$) a map $\tilde{D}_i : B_i \to B_{i+d}$ by $\tilde{D}_i(x) = p_{i+d}(D(x))$, where $p_j : B \to B_j$ is the canonical projection; then if $b = \sum_{i \in \mathbb{Z}} b_i \in B$ ($b_i \in B_i$ for all i, $b_i = 0$ for almost all i), define $\tilde{D}(b) = \sum_i \tilde{D}_i(b_i)$; one can check that $\tilde{D} \in \operatorname{HLND}(B) \setminus \{0\}$ and $\deg(\tilde{D}) = \deg(D)$. Note in particular that $D \neq 0 \implies \tilde{D} \neq 0$. The derivation $\tilde{D}$ is sometimes called the *homogenization* of D. This is in fact a special case of the process of replacing $D : B \to B$ by $\mathbf{Gr}(D) : \mathbf{Gr}(B) \to \mathbf{Gr}(B)$, so one can also call $\tilde{D}$ the *associated homogeneous derivation* of D.

Refer to [Dai12] for proofs of the claims made in 1.1.12, and for an in-depth treatment of this topic.

Note that Lemma 1.1.13 below is a consequence of the above discussion and can be also found in Section 4 of [Dai].

1.1.13 Lemma. *Let B be a $\mathbb{Z}$-graded affine $\mathbf{k}$-domain. If $\operatorname{LND}(B) \neq \{0\}$, then $\operatorname{HLND}(B) \neq \{0\}$.*

We also need the following well-known result.

1.1.14 Proposition. *Let B be a normal $\mathbf{k}$-domain. Then, for all $D \in \operatorname{LND}(B)$, $\ker(D)$ is normal.*

Proof. For all $D \in \operatorname{LND}(B)$, we have that $\ker(D)$ is factorially closed in B and hence is integrally closed in B. Since B is normal, the integral closure of $\ker(D)$ in B is normal (see Lemma 034L of [Sta22]) and since $\ker(D)$ is integrally closed, we conclude that $\ker(D)$ is normal. □

The following appears in [Zar54].

1.1.15 Theorem. *Let B be a normal affine $\mathbf{k}$-domain and suppose that K is a field satisfying $\mathbf{k} \subseteq K \subseteq \operatorname{Frac} B$. If $\operatorname{trdeg}(K/\mathbf{k}) \leq 2$, then $K \cap B$ is a finitely generated $\mathbf{k}$-algebra.*

1.1.16 Corollary. *Let B be a 3-dimensional normal affine $\mathbf{k}$-domain and let $D \in \operatorname{LND}(B)$. Then $\ker(D)$ is a finitely generated $\mathbf{k}$-algebra.*

Proof. Let $A = \ker(D)$ and let $K = \operatorname{Frac}(A)$. If $D = 0$, the claim is trivial. If $D \neq 0$, then $\operatorname{trdeg}(K/\mathbf{k}) = 2$ and $A = \operatorname{Frac}(A) \cap B$ by Theorem 1.1.5 (e). It follows from Theorem 1.1.15 that $A = K \cap B$ is a finitely generated $\mathbf{k}$-algebra. □

1.1.17 Notation. Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be an $\mathbb{N}$-graded ring. We let $B_+ = \bigoplus_{n \in \mathbb{N}^+} B_n$ denote the ideal of positive degree elements of B.

We need the following Generalized Luroth Theorem (see page 29 of [Ohm89]) and an immediate corollary.

1.1.18 Theorem. *Let $\mathbf{k} \subset F \subseteq \mathbf{k}(x_1, \dots x_n)$ be field extensions, with $x_1, \dots, x_n$ algebraically independent over $\mathbf{k}$ such that $\operatorname{trdeg}(F/\mathbf{k}) = 1$. Then there exists $t \in \mathbf{k}(x_1, \dots, x_n)$ such that $F = \mathbf{k}(t)$.*

1.1.19 Corollary. *If $F/\mathbf{k}$ is a field extension of transcendence degree 1, then F is rational over $\mathbf{k}$ if and only if $F^{(1)}$ is rational over $\mathbf{k}$.*

Proof. One direction is obvious, the other follows immediately from Theorem 1.1.18. □

1.1.20 Definition. Let B be a $\mathbb{Z}$-graded domain. Define $\operatorname{HFrac} B$ to be the degree-0 subring of the $\mathbb{Z}$-graded ring $S^{-1}B$, where S is the set of nonzero homogeneous elements of B. Note that $\operatorname{HFrac} B$ is a subfield of $\operatorname{Frac} B$, and that if the grading of B is non-trivial then $\operatorname{Frac} B = (\operatorname{HFrac} B)^{(1)}$. See Lemma 4.7 of [CD22] for a proof.

1.1.21 Remark. Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be an $\mathbb{N}$-graded domain and let $X = \operatorname{Proj} B$. Since $K(X) = \operatorname{HFrac} B$, we have, $K(\operatorname{Spec} B) \cong \operatorname{Frac} B = (\operatorname{HFrac} B)^{(1)} \cong K(X)^{(1)}$.

Proposition 1.1.23 may be known, but we have not previously seen it in the literature. This proof is due to Daigle. We leave Lemma 1.1.22 to the reader.

1.1.22 Lemma. *Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be an $\mathbb{N}$-graded domain and let S be a multiplicatively closed set such that $1 \in S \subseteq \bigcup_{n \in \mathbb{N}} (B_n \setminus \{0\})$. Then $S^{-1}B$ is $\mathbb{Z}$-graded and $\operatorname{HFrac}(S^{-1}B) = \operatorname{HFrac}(B)$.*

1.1.23 Proposition. *Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be an $\mathbb{N}$-graded domain of characteristic 0 such that $B \neq B_0$. If A is the kernel of $D \in \operatorname{HLND}(B) \setminus \{0\}$ then $\operatorname{Spec}(A)$ and $\operatorname{Proj}(B)$ have isomorphic function fields.*

Proof. The fact that A is a graded subring of B immediately implies that $\operatorname{HFrac} A \subseteq \operatorname{HFrac} B$. So we have the following field extensions:

$$\begin{tikzcd}[column sep=small] & \operatorname{Frac} B \arrow[dl, dash] \arrow[dr, dash] & \\ \operatorname{Frac} A \arrow[dr, dash] & & \operatorname{HFrac} B \arrow[dl, dash] \\ & \operatorname{HFrac} A & \end{tikzcd}$$

We must show that $\operatorname{Frac} A \cong \operatorname{HFrac} B$.

Assume first that $A \nsubseteq B_0$, so the $\mathbb{N}$-grading of A is non-trivial and consequently $\operatorname{Frac} A \cong (\operatorname{HFrac} A)^{(1)}$. It now suffices to prove that $\operatorname{HFrac} B \cong (\operatorname{HFrac} A)^{(1)}$. We have $\operatorname{Frac} B \cong (\operatorname{Frac} A)^{(1)} \cong (\operatorname{HFrac} A)^{(2)}$ and $\operatorname{Frac} B \cong (\operatorname{HFrac} B)^{(1)}$, so $\operatorname{trdeg}(\operatorname{HFrac} B / \operatorname{HFrac} A) = 1$. If we write $K = \operatorname{HFrac} A$ and $L = \operatorname{HFrac} B$, then $L \subset \operatorname{Frac} B \cong K^{(2)}$, so $K \subset L \subset K^{(2)}$ and $\operatorname{trdeg}(L/K) = 1$. Theorem 1.1.18 gives $\operatorname{HFrac} B = L \cong K^{(1)} = (\operatorname{HFrac} A)^{(1)}$, as required.

Finally, assume $A \subseteq B_0$. Since $B \neq B_0$ we have $\operatorname{trdeg}(B/B_0) > 0$, so $1 \leq \operatorname{trdeg}(B/B_0) \leq \operatorname{trdeg}(B/A) = 1$, so $\operatorname{trdeg}(B/B_0) = \operatorname{trdeg}(B/A)$, showing that B_0 is algebraic over A. Since A is factorially closed in B, it follows that $A = B_0$. There exists a homogeneous element $\sigma \in B$ such that $D(\sigma) \neq 0$ and $D^2(\sigma) = 0$. Let $d = \deg(\sigma)$; since $\sigma \notin A = B_0$, we have $d > 0$. As is well known, if we set $S = A \setminus \{0\}$, $L = \operatorname{Frac} A$ and $\mathcal{B} = S^{-1}B$, then $\mathcal{B} = L[\sigma] = L^{[1]}$ (by Theorem 1.1.5 (b)). Since $S = B_0 \setminus \{0\}$, we have $\mathcal{B} = \bigoplus_{n \in \mathbb{N}} \mathcal{B}_n$ and $\mathcal{B}_0 = L$. In fact we have $\mathcal{B}_n = L\sigma^{n/d}$ if $n \in d\mathbb{N}$ and $\mathcal{B}_n = 0$ if $n \notin d\mathbb{N}$. If follows that $\operatorname{HFrac}(\mathcal{B}) = L$, so by Lemma 1.1.22 $\operatorname{HFrac}(B) = \operatorname{HFrac}(\mathcal{B}) = L = \operatorname{Frac}(A)$. $\square$

1.1.24 Proposition. *Let B be a $\mathbf{k}$-domain, $D : B \to B$ a nonzero locally nilpotent derivation with $A = \ker(D)$. Let s be a local slice of D and let $f : A \to B/\langle s \rangle$ be the composition of maps $A \hookrightarrow B \to B/\langle s \rangle$. Then f is injective, and if we define $a = D(s) \in A \setminus \{0\}$ then the localization of f with respect to $\{1, a, a^2, \dots\}$ is bijective.*

Proof. Using that A is factorially closed in B, it is easy to verify that f is injective. The localization of f is $f_a : A_a \hookrightarrow A_a[s]/\langle s \rangle$ where $A_a[s] = A_a^{[1]}$, so f_a is bijective. □

1.1.25 Corollary. *Let B be a $\mathbf{k}$-domain, $D : B \to B$ a nonzero locally nilpotent derivation with $A = \ker(D)$ and local slice s. Suppose furthermore that $B/\langle s \rangle$ is also an integral domain. Then* $\operatorname{Frac} A \cong \operatorname{Frac}(B/\langle s \rangle)$.

Proof. This follows immediately from Proposition 1.1.24. □

1.1.26 Remark. Let B be a $\mathbf{k}$-algebra and $D \in \operatorname{LND}(B) \setminus \{0\}$. Given any field K containing $\mathbf{k}$, define $\bar{B} = K \otimes_{\mathbf{k}} B$ and consider the unique K-linear map $\bar{D} : \bar{B} \to \bar{B}$ such that $\bar{D}(k \otimes b) = k \otimes D(b)$ for all $k \in K$ and $b \in B$. Then it is easy to check that $\bar{D} \in \operatorname{LND}(\bar{B}) \setminus \{0\}$.

1.1.27 Definition. Let B be a ring and let $D \in \operatorname{LND}(B)$. For each $b \in B \setminus \{0\}$, define $\deg_D(b) = \max\{n \in \mathbb{N} : D^n(b) \neq 0\}$; define $\deg_D(0) = -\infty$. This defines a set map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$.

The following proposition can be found in [Dai10] as Proposition 4.8.

1.1.28 Proposition. *Let B be an integral domain of characteristic zero and let $D \in \operatorname{LND}(B)$. Then $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.*

1.1.29 Definition. Suppose B is a commutative $\mathbf{k}$-domain where $\mathbf{k}$ is a field of characteristic zero. If B is not rigid, then given $f \in B$, the *absolute degree* of f is defined by

$$|f|_B = \min \left\{ \deg_D(f) \, : \, D \in \operatorname{LND}(B) \setminus \{0\} \right\}.$$

If B is rigid, define $|f|_B = -\infty$ if $f = 0$, and $|f|_B = \infty$ otherwise.

Note that the absolute degree is not a degree function in the sense of Definition 1.1.10. For additional properties of the absolute degree, see Section 2.3 of [FMJ13].

1.2 A Conjecture About Pham-Brieskorn Rings

1.2.1 Assumption. All rings (and fields $\mathbf{k}$) in this section have characteristic zero.

1.2.2 Definition. If $\mathbf{k}$ is a field of characteristic 0 and $a_0, \dots, a_n$ ($n \geq 2$) are positive integers, the $\mathbf{k}$-algebra $B_{a_0,\dots,a_n} = \mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n} \rangle$ is called a *Pham-Brieskorn ring over* $\mathbf{k}$. If $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ we may write B_S instead of $B_{a_0,\dots,a_n}$. We may also omit the words "over $\mathbf{k}$" when the field $\mathbf{k}$ is clear from the context. One can show that $B_{a_0,\dots,a_n}$ is an integral domain.

1.2.3 Notation. If $B = B_{a_0,\dots,a_n}$ is a Pham-Brieskorn ring and $\pi : \mathbf{k}[X_0,\dots,X_n] \to B$ is the canonical quotient homomorphism, then for each $i = 0,\dots,n$, we define $x_i = \pi(X_i) \in B$.

The following proposition is well known. We sketch a proof for the unfamiliar reader.

1.2.4 Proposition. *Let $n \geq 2$ and let $B_{a_0,\dots,a_n} = \mathbf{k}[X_0,\dots,X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$ where $\mathbf{k}$ is a field containing $\mathbb{Q}(i)$. If either of the following conditions hold*

- *$a_i = 1$ for some $i \in \{0,\dots,n\}$*
- *$a_i = a_j = 2$ for some choice of distinct i, j where $i, j \in \{0,\dots,n\}$,*

then $B_{a_0,\dots,a_n}$ is not rigid.

Proof. Let $B = B_{a_0,\dots,a_n}$. If $a_i = 1$ for some index i, then B is a polynomial ring and hence is not rigid. Otherwise, assume without loss of generality that $a_0 = a_1 = 2$. Let $F(\mathbf{Y}) = Y_2^{a_2} + \cdots + Y_n^{a_n} \in \mathbf{k}[Y_0,\dots,Y_n]$. Let $R = \mathbf{k}[Y_0,\dots,Y_n]/\langle Y_0Y_1 + F(\mathbf{Y})\rangle$. Since $\mathbb{Q}(i) \subseteq \mathbf{k}$, there is a well-defined $\mathbf{k}$-isomorphism $\varphi : R \to B_{2,2,a_2,\dots,a_n}$ determined by

- $\varphi(y_0) = x_0 + ix_1$,
- $\varphi(y_1) = x_0 - ix_1$,
- $\varphi(y_j) = x_j$ for all $j \in \{2,\dots,n\}$,

where y_i (respectively x_i) is the image of Y_i (respectively X_i) in R (respectively $B_{2,2,a_2,\dots,a_n}$). To prove the result, it now suffices to show that R is not rigid.

Define $\partial_i(F(\mathbf{Y})) = \frac{\partial F(\mathbf{Y})}{\partial Y_i}$ where $\frac{\partial F(\mathbf{Y})}{\partial Y_i}$ is the formal partial derivative. Consider the $\mathbf{k}$-derivation $D \in \mathrm{Der}(\mathbf{k}[Y_0,\dots,Y_n])$ defined by setting $D(Y_0) = 0, D(Y_1) = -\sum_{i=2}^{n} \partial_i(F(\mathbf{Y}))$ and $D(Y_i) = Y_0$ for all $i = 2,\dots,n$. Then, it is easy to check that D is locally nilpotent and maps the ideal $\langle Y_0Y_1 + F(\mathbf{Y})\rangle$ into itself. So D induces a locally nilpotent derivation $\bar{D} : R \to R$, which is non-zero since $\bar{D}(y_2) = y_0 \neq 0$. We conclude that R is not rigid, and so neither is $B_{2,2,a_2,\dots,a_n}$. □

1.2.5 Remark. Theorem 9.27 of [Fre17] shows that $\mathbb{R}[X,Y,Z]/\langle X^2 + Y^2 + Z^2\rangle$ is rigid. So Proposition 1.2.4 is false if we assume that $\mathbf{k} = \mathbb{R}$.

The following Conjecture is the primary motivation of this thesis. We refer the reader to the Introduction for a list of previously known related results.

1.2.6 Conjecture. *Let $n \geq 2$, and let $B_{a_0,\dots,a_n} = \mathbf{k}[X_0,\dots,X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$ be a Pham-Brieskorn ring. If $\min\{a_0,\dots,a_n\} \geq 2$ and at most one element i of $\{0,\dots,n\}$ satisfies $a_i = 2$, then $B_{a_0,\dots,a_n}$ is rigid.*

1.2.7 Remark. In Lemma 4 of [KZ00], Kaliman and Zaidenberg proved Conjecture 1.2.6 in the special case where $n = 2$ and $\mathbf{k} = \mathbb{C}$.

The following proposition is also well known to the experts in the field. Since we cannot find a published proof, we include one.

1.2.8 Proposition. *If* $\mathbb{C}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$ *is rigid, then* $\mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$ *is rigid for any field* $\mathbf{k}$ *of characteristic zero.*

Proof. Fix positive integers $a_0, \dots, a_n$ where $n \geq 2$. For each field $\mathbf{k}$ of characteristic zero, let $B_{\mathbf{k}} = \mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots X_n^{a_n}\rangle$. Note that if $\mathbf{k}_1 \subseteq \mathbf{k}_2$ then $B_{\mathbf{k}_1} \subseteq B_{\mathbf{k}_2}$.

Let $\mathbf{k}$ be a field of characteristic zero. We prove that if $B_{\mathbf{k}}$ is not rigid then $B_{\mathbb{C}}$ is not rigid. Suppose that $D \in \mathrm{LND}(B_{\mathbf{k}}) \setminus \{0\}$. For each $i = 0, \dots, n$, choose $F_i(X_0, \dots, X_n) \in \mathbf{k}[X_0, \dots, X_n]$ such that $D(x_i) = F_i(x_0, \dots, x_n)$. Let S be a finite subset of $\mathbf{k}$ that contains all coefficients of $F_1(X_0, \dots, X_n), \dots, F_n(X_0, \dots, X_n)$, and define the subfield $\mathbf{k}_0 = \mathbb{Q}(S)$ of $\mathbf{k}$. Then $B_{\mathbf{k}_0}$ is a subring of $B_{\mathbf{k}}$ and D maps $B_{\mathbf{k}_0}$ into itself. It is clear that the restriction $D_0 : B_{\mathbf{k}_0} \to B_{\mathbf{k}_0}$ of D is a locally nilpotent derivation of $B_{\mathbf{k}_0}$. Since $D \neq 0$, there exists $j \in \{0, \dots, n\}$ such that $D(x_j) \neq 0$; then $D_0(x_j) \neq 0$ and hence $B_{\mathbf{k}_0}$ is not rigid. Since $\mathbf{k}_0$ is a countable field of characteristic zero, it is isomorphic to a subfield of $\mathbb{C}$. Remark 1.1.26 then implies that $B_{\mathbb{C}}$ is not rigid. $\square$

1.2.9 Remark. Proposition 1.2.8 immediately implies that the $n = 2$ case of Conjecture 1.2.6 is true for any field $\mathbf{k}$ of characteristic 0. For general n, it also allows us to focus on the $\mathbf{k} = \mathbb{C}$ case of Conjecture 1.2.6. This is important, as the geometric arguments we use may fail or become more complicated when the base field is not $\mathbb{C}$.

Type and Cotype

The following definitions appear in [CD20]. In order to keep the notation as consistent as possible throughout the thesis, our indices run from 0 to n (as opposed to from 1 to n).

1.2.10 Definitions. Let $n \geq 2$ and $S = (b_0, \dots, b_n) \in \mathbb{Z}^{n+1}$.

- Define[1] $\gcd(S) = \gcd(b_0, \dots, b_n)$ and $\mathrm{lcm}(S) = \mathrm{lcm}(b_0, \dots, b_n)$.
- If $\gcd(S) = 1$, we say that S is *normal*. If $S \neq (0, \dots, 0)$ and $d = \gcd(S)$, then the tuple $S' = (\frac{b_0}{d}, \dots, \frac{b_n}{d})$ is normal, and is called the *normalization of* S.
- For each $j \in \{0, \dots, n\}$, define $S_j = (b_0, \dots, \widehat{b_j}, \dots, b_n)$.

[1] By convention, $\gcd(S) \geq 0$ and $\mathrm{lcm}(S) \geq 0$.

- We define the sets

$$J^*(S) = \{\, i \in \{0,\dots,n\} \,:\, \gcd(S_i) \neq \gcd(S) \,\} = \{\, i \in \{0,\dots,n\} \,:\, \gcd(S_i) \nmid b_i \,\}$$
$$J(S) = \{\, i \in \{0,\dots,n\} \,:\, \operatorname{lcm}(S_i) \neq \operatorname{lcm}(S) \,\} = \{\, i \in \{0,\dots,n\} \,:\, b_i \nmid \operatorname{lcm}(S_i) \,\}.$$

  Then we define the *type* and the *cotype* of S by:

$$\operatorname{type}(S) = |J^*(S)| \quad \text{and} \quad \operatorname{cotype}(S) = |J(S)|.$$

  Note that $\operatorname{type}(S), \operatorname{cotype}(S) \in \{0,1,\dots,n+1\}$ and that, if S' is the normalization of S, then $\operatorname{type}(S) = \operatorname{type}(S')$ and $\operatorname{cotype}(S) = \operatorname{cotype}(S')$.

1.2.11 Example. For any integers $a, b \in \mathbb{Z}$, $\operatorname{cotype}(a,a,b,b) = 0$.

1.2.12 Definition. Let $n \geq 2$.

- Given $S = (a_0,\dots,a_n) \in (\mathbb{N}^+)^{n+1}$ and $i \in \{0,\dots,n\}$, define $g_i(S) = \gcd(a_i, \operatorname{lcm}(S_i))$.

- Let $S = (a_0,\dots,a_n)$ and $S' = (a'_0,\dots,a'_n)$ be elements of $(\mathbb{N}^+)^{n+1}$ and let $i \in \{0,\dots,n\}$. We write $S \leq^i S'$ if and only if

$$S_i = S'_i \quad \text{and} \quad g_i(S') \mid a_i \mid a'_i.$$

  We write $S <^i S'$ if and only if $S \leq^i S'$ and $S \neq S'$.

1.2.13 Remark. The relation $\leq^i$ is a partial order on $(\mathbb{N}^+)^{n+1}$. Transitivity follows from the fact that if $S \leq^i S'$ then $g_i(S) = g_i(S')$.

1.3 Simplifying Conjecture 1.2.6

In this section we show that if Conjecture 1.2.6 holds whenever $\operatorname{cotype}(a_0,\dots,a_n) = 0$, then it holds in general. This is stated formally in Theorem 1.3.12 (b). Moreover, Theorem 1.3.12 (a) and Remark 1.2.7 together imply that to prove Conjecture 1.2.6 in the $n = 3$ case, it suffices to prove the result when $\operatorname{cotype}(a_0,a_1,a_2,a_3) = 0$. This is stated in Lemma 4.2.3.

1.3.1 Assumption. All rings (and fields $\mathbf{k}$) in this section have characteristic zero.

The following remark is particularly important.

1.3.2 Remark. Let $n \geq 2$ and let $S = (a_0,\dots,a_n) \in (\mathbb{N}^+)^{n+1}$. It is possible to view the Pham-Brieskorn ring B_S as a graded ring. Let $f = X_0^{a_0} + X_1^{a_1} + \cdots + X_n^{a_n} \in \mathbf{k}[X_0,\dots,X_n]$. Let $L = \operatorname{lcm}(a_0,\dots,a_n)$ and let $\deg(X_i) = w_i = L/a_i$ for each $i \in \{0,1,\dots,n\}$. Then f is homogeneous, $B_S = \mathbf{k}_{w_0,\dots,w_n}[X_0,\dots,X_n]/\langle f \rangle$ is a graded ring, and $\deg(x_i) = w_i$ for every $i = 0,\dots,n$. Moreover, $\mathbb{Z}(B_S) = \mathbb{Z}$ (see Definitions 1.1.9).

The following is the $G = \mathbb{Z}$ case of Definition 5.1 of [DFMJ17].

1.3.3 Definition. Let B be a $\mathbb{Z}$-graded ring. A nonzero homogeneous element x of B is $\mathbb{Z}$-*critical* if there exists a graded subring $A \subset B$ such that $\mathbb{Z}(A) \neq \mathbb{Z}(B)$ and $B = A[x]$.

The following is a special case of Theorem 6.2 of [DFMJ17].

1.3.4 Lemma. *Let B be a $\mathbb{Z}$-graded integral domain, and let $D \in \operatorname{LND}(B)$ be homogeneous. For every $\mathbb{Z}$-critical element $x \in B$, $D^2(x) = 0$.*

The following is Lemma 3.1 of [FMJ13]. See Definition 1.1.29 for the notation $|f|_R$.

1.3.5 Lemma. *Assume R is a $\mathbf{k}$-domain, $f \in R$, $n \geq 2$, and $f + Z^n$ is a prime element of $R[Z]$. If $|f|_R \leq 1$, then $B = R[Z]/\langle Z^n + f\rangle$ is not rigid.*

1.3.6 Lemma. *Let $n \geq 2$, let $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$, and consider*

$$B_S = \mathbf{k}[X_0, \dots, X_n]/\langle X_0^{a_0} + \dots + X_n^{a_n}\rangle = \mathbf{k}[x_0, \dots, x_n].$$

Let $m \in \mathbb{N}^+$, let $S^m = (a_0, \dots, a_{n-1}, ma_n)$ and write

$$B_{S^m} = \mathbf{k}[Y_0, \dots, Y_n]/\langle Y^{a_0} + \dots + Y_{n-1}^{a_{n-1}} + Y_n^{ma_n}\rangle = \mathbf{k}[y_0, \dots, y_n].$$

Then there is a $\mathbf{k}$-isomorphism

$$B_S[Z]/\langle Z^m - x_n\rangle \cong B_{S^m}$$

where Z is an indeterminate over B_S.

Proof. We define a $\mathbf{k}$-homomorphism

$$\psi : \mathbf{k}[Y_0, \dots, Y_n] \to B_S[Z]/\langle Z^m - x_n\rangle$$

by

$$\begin{aligned} &\psi(Y_i) = x_i \text{ for each } i = 0, \dots, n-1, \\ &\psi(Y_n) = z. \end{aligned}$$

Since $\psi(Y_0^{a_0} + \dots + Y_{n-1}^{a_{n-1}} + Y_n^{ma_n}) = x_0^{a_0} + \dots + x_{n-1}^{a_{n-1}} + z^{ma_n} = x_0^{a_0} + \dots + x_{n-1}^{a_{n-1}} + x_n^{a_n} = 0$, we have $\langle Y_0^{a_0} + \dots + Y_{n-1}^{a_{n-1}} + Y_n^{ma_n}\rangle \subseteq \ker\psi$. It follows that ψ induces a $\mathbf{k}$-homomorphism $\bar{\psi} : B_{S^m} \to B_S[Z]/\langle Z^m - x_n\rangle$ satisfying

$$\begin{aligned} &\bar{\psi}(y_i) = x_i \text{ for each } i = 0, \dots, n-1, \\ &\bar{\psi}(y_n) = z. \end{aligned}$$

Conversely, we can define a homomorphism $\varphi : \mathbf{k}[X_0, \dots, X_n, Z] \to B_{S^m}$ by

$$\begin{aligned} \varphi(X_i) &= y_i \text{ for each } i = 0, \dots, n-1, \\ \varphi(X_n) &= y_n^m, \\ \varphi(Z) &= y_n. \end{aligned}$$

Since $\varphi(X_0^{a_0} + \cdots + X_n^{a_n}) = y_0^{a_0} + \cdots + y_{n-1}^{a_{n-1}} + y_n^{ma_n} = 0$ and $\varphi(Z^m - X_n) = 0$, we obtain that $\langle X_0^{a_0} + \cdots + X_n^{a_n}, Z^m - X_n \rangle \subseteq \ker \varphi$. So φ induces a $\mathbf{k}$-homomorphism $\bar{\varphi} : B_S[Z]/\langle Z^m - x_n \rangle \to B_{S^m}$ satisfying

$$\begin{aligned} \bar{\varphi}(x_i) &= y_i \text{ for each } i = 0, \dots, n-1, \\ \bar{\varphi}(x_n) &= y_n^m, \\ \bar{\varphi}(z) &= y_n. \end{aligned}$$

It is a direct verification to see that $\bar{\varphi}$ and $\bar{\psi}$ are mutually inverse. □

1.3.7 Proposition. *Let $n \geq 2$, and suppose that $(a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ satisfies*

- $a_n \nmid \operatorname{lcm}(a_0, \dots, a_{n-1})$,
- *there exists $m \in \mathbb{N}^+$ such that $B_{a_0, \dots, a_{n-1}, ma_n}$ is rigid.*

Then $B_{a_0, \dots, a_{n-1}, a_n}$ is rigid.

Proof. Let $R = B_{a_0, \dots, a_n} = \mathbf{k}[x_0, \dots, x_n]$. Let $A = \mathbf{k}[x_0, \dots, x_{n-1}]$ and observe that $R = A[x_n]$. By Remark 1.3.2, $\mathbb{Z}(R) = \mathbb{Z}$. Let $L' = \operatorname{lcm}(a_0, \dots, a_{n-1})$, and let $L = \operatorname{lcm}(a_n, L')$. Since $a_n \nmid L'$, $\mathbb{Z}(A) = (L/L')\mathbb{Z} \subset \mathbb{Z}$. It follows that x_n is a $\mathbb{Z}$-critical element of R. Let $m \in \mathbb{N}^+$ be such that $B_{a_0, \dots, a_{n-1}, ma_n}$ is rigid, let $R' = R[Z]/\langle Z^m - x_n \rangle$ and note that Lemma 1.3.6 implies that $R' \cong B_{a_0, \dots, a_{n-1}, ma_n}$.

Assume that R is not rigid. By Lemma 1.1.13, there exists $D \in \operatorname{LND}(R)$ which is nonzero and homogeneous; since x_n is a $\mathbb{Z}$-critical element of R, Lemma 1.3.4 implies that $D^2(x_n) = 0$. In particular, $|x_n|_R \leq 1$. By Lemma 1.3.5, R' is not rigid, hence $B_{a_0, \dots, a_{n-1}, ma_n}$ is not rigid. This is a contradiction. So R is rigid. □

1.3.8 Notation. For each $n \geq 2$, define

$$\Gamma_n = \left\{ (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1} \ : \ \min(a_0, \dots, a_n) > 1 \text{ and at most one } i \text{ satisfies } a_i = 2 \right\}.$$

and consider the statements:

$$\begin{aligned} P(n) &: B_{a_0, \dots, a_n} \text{ is rigid for all } (a_0, \dots, a_n) \in \Gamma_n \ . \\ P(n, i) &: B_{a_0, \dots, a_n} \text{ is rigid for all } (a_0, \dots, a_n) \in \Gamma_n \text{ satisfying } \operatorname{cotype}(a_0, \dots, a_n) = i. \end{aligned}$$

1.3.9 Remark. Conjecture 1.2.6 holds if and only if $P(n)$ holds for all $n \geq 2$. Recall that $P(2)$ holds by Remark 1.2.7.

We require the following, which appears as Proposition 4.9 of [CD20] with slightly different notation.

1.3.10 Proposition. *Let $n \geq 2$, let $S, S^*, S' \in (\mathbb{N}^+)^{n+1}$ and let $i \in \{0, \dots, n\}$.*

(a) *If $S' \leq^i S$ and $B_{S'}$ is rigid then B_S is rigid.*

(b) *If $S' <^i S$ and $S' <^i S^*$, then B_S is rigid if and only if B_{S^*} is rigid.*

1.3.11 Notation. Recall from Definitions 1.2.10 that if $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ then we define $S_i = (a_0, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots a_n)$.

1.3.12 Theorem.

(a) *Let $n \geq 3$. If $P(n-1)$ and $P(n, 0)$ hold, then $P(n)$ holds.*

(b) *If $P(n, 0)$ holds for all $n \geq 3$, then $P(n)$ holds for all $n \geq 3$.*

Proof. (a) Suppose $S = (a_0, \dots, a_n) \in \Gamma_n$ and assume that $P(n-1)$ and $P(n, 0)$ hold. We must show that B_S is rigid. If $\operatorname{cotype}(S) = 0$ we're done, since $P(n, 0)$ holds. Assume henceforth that $\operatorname{cotype}(S) \geq 1$.

Suppose $\operatorname{cotype}(S) \geq 2$. By contradiction, assume that there exists $D \in \operatorname{LND}(B_S) \setminus \{0\}$. Without loss of generality, by Lemma 1.1.13 together with Theorem 1.1.5 (d) we may assume D is homogeneous and irreducible. For each $i \in \{0, \dots, n\}$, let $w_i = \deg(x_i)$ be as in Remark 1.3.2. Let $H_i = \langle w_0, \dots, w_{i-1}, \hat{w}_i, w_{i+1}, \dots, w_n \rangle \subseteq \mathbb{Z}$. Since $\operatorname{cotype}(S) \geq 2$, there exist distinct $j, k \in \{0, \dots, n\}$ such that $H_j \subset \mathbb{Z}$ and $H_k \subset \mathbb{Z}$. By Corollary 6.3 (b) of [DFMJ17], either $x_j \in \ker(D)$ or $x_k \in \ker(D)$. Without loss of generality, we may assume $j = n$ or $k = n$, so that $x_n \in \ker(D)$. Then, since D is irreducible, D induces a nonzero locally nilpotent derivation on $B_S / \langle x_n \rangle \cong B_{S_n}$. But since $S \in \Gamma_n$, it follows that $S_n \in \Gamma_{n-1}$. This is a contradiction since $P(n-1)$ holds. So B_S is rigid when $\operatorname{cotype}(S) \geq 2$.

Finally, assume $\operatorname{cotype}(S) = 1$. Then, up to permuting the a_i, we may arrange that $a_n \nmid \operatorname{lcm}(a_0, \dots, a_{n-1})$. Let $L = \operatorname{lcm}(a_0, \dots, a_{n-1})$. By Proposition 1.3.7, it suffices to prove that $B_{a_0, \dots, a_{n-1}, a_n L}$ is rigid. We have $(a_0, \dots, a_{n-1}, L) <^n (a_0, \dots, a_{n-1}, a_n L)$ and $\operatorname{cotype}(a_0, \dots, a_{n-1}, L) = 0$. Since $L > 2$, $(a_0, \dots, a_{n-1}, L) \in \Gamma_n$. By assumption, $P(n, 0)$ holds so $B_{a_0, \dots, a_{n-1}, L}$ is rigid and hence $B_{a_0, \dots, a_n L}$ is rigid by Proposition 1.3.10 (a). This proves (a).

The proof of (b) is by induction on n, and follows immediately from (a) together with the base case that $P(2)$ holds.

$\square$

Chapter 2

Preliminaries on Geometry

The purpose of this chapter is to present the preliminary geometric results that are used in Chapters 3, 4 and 5, gathering definitions and results and providing proofs where we could not find suitable references. With the exception of Proposition 2.4.25, all results are already known. The reader that is familiar with these topics can skip most of this chapter (while still reading Proposition 2.4.25), returning where necessary whilst reading Chapters 3, 4 and 5.

2.1 $\mathcal{O}_X$-modules

2.1.1 Definitions. Let X and Y be schemes.

- Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F} \in \mathbf{Ab}(X)$, then the *direct image sheaf* $f_*\mathcal{F} \in \mathbf{Ab}(Y)$ is defined by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for any open subset $V \subseteq Y$.
- If $\mathcal{G} \in \mathbf{Ab}(Y)$, then we may define the presheaf $\mathcal{P}$ on X by $\mathcal{P}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ for any open set $U \subseteq X$. The *inverse image sheaf* $f^{-1}\mathcal{G} \in \mathbf{Ab}(X)$ is defined to be the sheafification of $\mathcal{P}$.
- An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if $\mathcal{F} \cong \bigoplus_{i \in I} \mathcal{O}_X$ for some index set I.
- An $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if X can be covered by open sets $\{U_i\}_{i \in I}$ such that for each $i \in I$, $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module. The *rank of* $\mathcal{F}$ *on an open set* U in the open cover is the cardinality of the set J such that $\mathcal{F}|_U \cong \bigoplus_{j \in J} \mathcal{O}_X|_U$. If X is connected, the rank of $\mathcal{F}$ on every open set in a cover $\{U_i\}$ is the same everywhere and is called the *rank of* $\mathcal{F}$.
- An $\mathcal{O}_X$-module $\mathcal{F}$ is *invertible* if it is locally free of rank one.
- Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules on Y. Then $f^{-1}\mathcal{G}$ is a sheaf of $f^{-1}\mathcal{O}_Y$-modules. Because of the adjoint property of f^{-1} (see Exercise

II.1.18 in [Har77]), there exists a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. The *inverse image of $\mathcal{G}$ by f* is

$$f^*(\mathcal{G}) = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X .$$

Note that $f^*(\mathcal{G})$ is a sheaf of $\mathcal{O}_X$-modules on X and that $f^*(\mathcal{O}_Y) \cong \mathcal{O}_X$.

2.1.2 Remark. If $f : X \to Y$ is a morphism of schemes, then

$$f_* : \mathbf{Mod}(X) \to \mathbf{Mod}(Y) \quad \text{and} \quad f^* : \mathbf{Mod}(Y) \to \mathbf{Mod}(X)$$

are covariant additive functors, and f^* is left-adjoint to f_*. See pages 109-110 of [Har77].

We list some of the important properties of the f^* functor.

2.1.3 Proposition.

(a) *If $f : X \to Y$ is a morphism of schemes then $f^*(\mathcal{O}_Y) \cong \mathcal{O}_X$. Moreover, f^* commutes with direct limits and with arbitrary direct sums.*

(b) *If $f : X \to Y$ is a morphism of schemes and $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{Mod}(Y)$ then*

$$f^*\big(\mathcal{G}_1 \otimes_{\mathcal{O}_Y} \mathcal{G}_2\big) \cong f^*(\mathcal{G}_1) \otimes_{\mathcal{O}_X} f^*(\mathcal{G}_2).$$

(c) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of schemes and $\mathcal{H} \in \mathbf{Mod}(Z)$ then $(g \circ f)^*(\mathcal{H}) \cong f^*(g^*(\mathcal{H}))$.*

(d) *If U is an open subscheme of a scheme X, $i : U \to X$ is the inclusion morphism and $\mathcal{F} \in \mathbf{Mod}(X)$ then $i^*(\mathcal{F}) \cong \mathcal{F}|_U$.*

(e) *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{G} \in \mathbf{Mod}(Y)$. If $\mathcal{G}$ is invertible (resp. locally free, quasi-coherent) then so is $f^*(\mathcal{G})$. If X, Y are Noetherian and $\mathcal{G}$ is coherent then $f^*(\mathcal{G})$ is coherent.*

Proof. (a) The fact that $f^*(\mathcal{O}_Y) \cong \mathcal{O}_X$ is an obvious consequence of the definition of f^*. By (4.3.2) in Chapter 0 of [Gro60], f^* commutes with direct limits and with arbitrary direct sums. Assertions (b) and (c) are special cases of, respectively, (4.3.3.1) and (4.3.6) in Chapter 0 of [Gro60]. Assertion (d) is an easy exercise.

(e) The "quasi-coherent" and "coherent" cases are Proposition II.5.8 of [Har77]. If $\mathcal{G}$ is locally free and $U \subseteq Y$ is an open set such that $\mathcal{G}|_U \cong \bigoplus_{i\in I} \mathcal{O}_U$ then consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ f^{-1}(U) \arrow[u, "i_0"] \arrow[r, "f_0"'] & U \arrow[u, "i"'] \end{tikzcd}$$

where i and i_0 are the inclusion morphisms and f_0 is the restriction of f. We have

$$(f^*\mathcal{G})|_{f^{-1}(U)} \overset{\text{(d)}}{\cong} i_0^*(f^*\mathcal{G}) \overset{\text{(c)}}{\cong} f_0^*(i^*\mathcal{G}) \overset{\text{(d)}}{\cong} f_0^*(\mathcal{G}|_U) \cong f_0^*(\oplus_{i\in I}\mathcal{O}_U) \overset{\text{(a)}}{\cong} \oplus_{i\in I} f_0^*(\mathcal{O}_U) \overset{\text{(a)}}{\cong} \oplus_{i\in I}\mathcal{O}_{f^{-1}(U)},$$

showing that $f^*\mathcal{G}$ is locally free of the same rank as $\mathcal{G}$. In particular, if $\mathcal{G}$ is invertible then so is $f^*\mathcal{G}$. □

2.1.4 Definition. A morphism of schemes $f : X \to Y$ is an *immersion* if it can be factored as a composition $X \xrightarrow{c} \bullet \xrightarrow{\mu} Y$ for some closed immersion c and some open immersion μ.

2.1.5 Remark. It is known that *a composition of immersions is an immersion*. (See Lemma 02V0 in [Sta22].)

2.1.6 Notation. Recall that if $X = \mathrm{Proj}(S)$ for some $\mathbb{N}$-graded ring S, then one defines $\mathcal{O}_X(n) = \widetilde{S(n)}$, for every $n \in \mathbb{Z}$. (See page 117 of [Har77].)

2.1.7 Definitions. Let X be a variety over $\mathbf{k}$.

- An invertible sheaf $\mathcal{L} \in \mathbf{Mod}(X)$ is *very ample* [1] if there exist some $n \in \mathbb{N}$ and an immersion $i : X \to \mathbb{P}^n_{\mathbf{k}}$ such that $\mathcal{L} \cong i^*(\mathcal{O}_{\mathbb{P}^n_{\mathbf{k}}}(1))$;
- An invertible sheaf $\mathcal{L} \in \mathbf{Mod}(X)$ is *ample* if there exists $m \in \mathbb{N}^+$ such that $\mathcal{L}^{\otimes m}$ is very ample.

2.1.8 Remark. Our definition of "ample" is not the most general one. Page 153 of [Har77] gives a definition of ampleness valid on an arbitrary Noetherian scheme. As we do not need this level of generality, we restrict ourselves to the special case of varieties. The equivalence of the two definitions in the case of varieties is due to Theorem II.7.6 of [Har77].

2.1.9 Lemma. *Let $i : X \to Y$ be an immersion, where X, Y are varieties over* $\mathbf{k}$*, and let $\mathcal{L}$ be an invertible sheaf on Y. Then $i^*(\mathcal{L})$ is an invertible sheaf on X and*

(a) *if $\mathcal{L}$ is very ample then so is $i^*(\mathcal{L})$;*

(b) *if $\mathcal{L}$ is ample then so is $i^*(\mathcal{L})$.*

Proof. The fact that $i^*(\mathcal{L})$ is invertible follows from Proposition 2.1.3(e).

(a) Assume that $\mathcal{L}$ is very ample. Then there exist $N > 0$ and an immersion $j : Y \to \mathbb{P}^N_{\mathbf{k}}$ such that $\mathcal{L} \cong j^*(\mathcal{O}_{\mathbb{P}^N_{\mathbf{k}}}(1))$. Since a composition of immersions is an immersion, $j \circ i : X \to \mathbb{P}^N_{\mathbf{k}}$ is an immersion and consequently $(j \circ i)^*(\mathcal{O}_{\mathbb{P}^N_{\mathbf{k}}}(1))$ is very ample. Since $i^*(\mathcal{L}) \cong i^*(j^*(\mathcal{O}_{\mathbb{P}^N_{\mathbf{k}}}(1))) \cong (j \circ i)^*(\mathcal{O}_{\mathbb{P}^N_{\mathbf{k}}}(1))$ by Proposition 2.1.3(c), we conclude that $i^*(\mathcal{L})$ is very ample.

(b) Assume that $\mathcal{L}$ is ample. Then there exists $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample. By part (a), $i^*(\mathcal{L}^{\otimes m})$ is very ample. We have $i^*(\mathcal{L}^{\otimes m}) \cong (i^*\mathcal{L})^{\otimes m}$ by Proposition 2.1.3(b), so $(i^*\mathcal{L})^{\otimes m}$ is very ample, so $i^*\mathcal{L}$ is ample. □

[1] Technically, we should write "very ample over $\mathbf{k}$". We will omit the mention of $\mathbf{k}$ for the sake of brevity.

2.2 Divisors

Normal and Cohen-Macaulay Rings and Schemes

2.2.1 Definitions.

- A *normal domain* is an integral domain that is integrally closed in its field of fractions.
- A ring A is *normal* if $A_{\mathfrak{p}}$ is a normal domain for every $\mathfrak{p} \in \operatorname{Spec} A$.
- A scheme X is *normal* if $\mathcal{O}_{X,x}$ is a normal domain for all $x \in X$.
- A variety X is *normal* if it is normal as a scheme.

2.2.2 Definition. If A is a ring and $k \in \mathbb{N}$, then A *satisfies condition* S_k if $\operatorname{depth} A_{\mathfrak{p}} \geq \inf\{k, \operatorname{ht} \mathfrak{p}\}$ for every prime ideal $\mathfrak{p} \lhd A$.

2.2.3 Remark. A Noetherian ring A is Cohen-Macaulay if and only if A satisfies S_k for all $k \in \mathbb{N}$. See Example B.77 in [GW10].

2.2.4 Definition. A ring A is *regular in codimension* k if $A_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p}$ of A satisfying $\operatorname{ht} \mathfrak{p} \leq k$.

We need Serre's normality criterion. One reference is Theorem II.8.22A in [Har77].

2.2.5 Theorem. *A Noetherian ring A is normal if and only if the following hold:*

- *A is regular in codimension 1,*
- *A satisfies condition S_2.*

2.2.6 Corollary. *Normal surfaces are Cohen-Macaulay.*

Proof. Let X be a normal surface and let $x \in X$. Since $\dim \mathcal{O}_{X,x} \leq 2$, the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ satisfies $\operatorname{ht} \mathfrak{m}_x = \inf\{2, \operatorname{ht} \mathfrak{m}_x\}$. Since $\mathcal{O}_{X,x}$ is a Noetherian normal domain, Theorem 2.2.5 implies that $\mathcal{O}_{X,x}$ satisfies S_2, which gives the inequality in

$$\dim \mathcal{O}_{X,x} = \operatorname{ht} \mathfrak{m}_x = \inf\{2, \operatorname{ht} \mathfrak{m}_x\} \leq \operatorname{depth} \mathcal{O}_{X,x} .$$

Since $\operatorname{depth} \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{X,x}$ (by the $M = R$ case of Lemma 00LK in [Sta22] for instance), $\operatorname{depth} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}$ and hence $\mathcal{O}_{X,x}$ is Cohen-Macaulay. So X is Cohen-Macaulay. $\square$

The following is a special case of Theorem 2.13 (a) in [BH93].

2.2.7 Proposition. *Let S be a Noetherian Cohen-Macaulay ring and let $I = \langle f_1, \ldots, f_k \rangle$ be such that $(f_1, \ldots, f_k)$ is an S-regular sequence. Then S/I is Cohen-Macaulay.*

2.2.8 Remark. The terminology in [Har77] is different from ours, where we instead follow [Mat80] and [BH93]. Hartshorne's "regular sequence" does not require the condition $IM \neq M$ and is what we (and Matsumura) call a "weak regular sequence".

2.2.9 Corollary. *Consider a polynomial ring $A = \mathbf{k}[X_1, \ldots, X_n]$ over a field $\mathbf{k}$, and let $f \in A$.*

(a) *The ring $A/\langle f \rangle$ is Cohen-Macaulay.*

(b) *If $A/\langle f \rangle$ is regular in codimension 1 then $A/\langle f \rangle$ is normal.*

Proof. By Example 1 on page 112 of [Mat80], A is a Cohen-Macaulay ring, so (a) is true if $f = 0$. If f is a unit of A, then $A/\langle f \rangle = 0$ is Cohen-Macaulay. If f is not zero and not a unit then (f) is an A-regular sequence, so the result follows from Proposition 2.2.7.

We prove (b). Since every Cohen-Macaulay ring satisfies S_k for all $k \geq 1$, $A/\langle f \rangle$ satisfies S_2. Since $A/\langle f \rangle$ is regular in codimension 1, Theorem 2.2.5 implies that $A/\langle f \rangle$ is normal. □

2.2.10 Corollary. *Let X be a hypersurface of $\mathbb{A}^n$ or $\mathbb{P}^n$. Then,*

(a) *X is Cohen-Macaulay;*

(b) *if X is regular in codimension 1, then X is normal.*

Proof. It suffices to prove the affine case, since if X is a hypersurface of $\mathbb{P}^n$ then X is covered by open sets X_i where each X_i is a hypersurface of $\mathbb{A}^n$.

Let X be a hypersurface of $\mathbb{A}^n$. We have $X \cong \operatorname{Spec}(A/\langle f \rangle)$ for some $f \in A = \mathbf{k}[Y_1, \ldots, Y_n]$. Note that if R is a Noetherian ring then

(i) R is a Cohen-Macaulay ring if and only if $\operatorname{Spec} R$ is a Cohen-Macaulay scheme;

(ii) R is a normal ring if and only if $\operatorname{Spec} R$ is a normal scheme.

Both (i) and (ii) are straightforward consequences of the definitions. The claim then follows from Corollary 2.2.9. □

The following appears as Exercise 6.B.26 in [PS10].

2.2.11 Proposition. *Let R be an $\mathbb{N}$-graded Noetherian normal domain. Then $\operatorname{Proj}(R)$ is a normal scheme.*[2]

[2] *The Noetherian hypothesis is superfluous but the result as stated is sufficient for us.*

Basic Properties of Divisors

2.2.12 Assumption. Throughout the section on divisors, we assume that X is a normal variety over $\mathbf{k}$, where $\mathbf{k}$ is assumed to be an algebraically closed field. Consequently, $\operatorname{Sing}(X)$ is a closed subset of X of codimension greater than 1.

2.2.13 Definitions.

- A *prime divisor* on X is an irreducible closed subset of codimension one.
- The group of *divisors* (or *Weil divisors*), denoted $\operatorname{Div}(X)$, is the free abelian group on the set of all prime divisors of X. A *divisor* D is an element of the group of divisors and is written as $D = \sum_{i\in I} a_i Y_i$ where each Y_i is a prime divisor and each $a_i \in \mathbb{Z}$.
- Given $f \in K(X)^*$, we write $\operatorname{div}_X(f)$ (or simply $\operatorname{div}(f)$ when X is understood) for the divisor of f. (See Section II.6 of [Har77] for the definition of the group homomorphism $\operatorname{div}_X : K(X)^* \to \operatorname{Div}(X)$, $f \mapsto \operatorname{div}_X(f)$.)
- A divisor $D \in \operatorname{Div}(X)$ is *principal* if there exists $f \in K(X)^*$ such that $D = \operatorname{div}_X(f)$.
- The set of principal divisors is a subgroup of $\operatorname{Div}(X)$ and is denoted $\operatorname{Princ}(X)$. The *divisor class group of* X is the quotient group $\operatorname{Div}(X)/\operatorname{Princ}(X)$ and is denoted $\operatorname{Cl}(X)$.
- The group of $\mathbb{Q}$-*divisors*, denoted $\operatorname{Div}(X,\mathbb{Q})$, is the group $\operatorname{Div}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. A $\mathbb{Q}$-*divisor* D is written as $D = \sum_{i\in I} a_i Y_i$ where each Y_i is a prime divisor and $a_i \in \mathbb{Q}$.
- Two $\mathbb{Q}$-divisors D and D' are *linearly equivalent* (we write $D \sim D'$) if there exists some $f \in K(X)^*$ such that $D - D' = \operatorname{div}(f)$.
- If $D = \sum_{i\in I} a_i Y_i$ is a $\mathbb{Q}$-divisor, then $\lfloor D\rfloor = \sum_{i\in I}\lfloor a_i\rfloor Y_i$.
- A $\mathbb{Q}$-divisor $D = \sum_{i\in I} a_i Y_i$ is *effective* if $a_i \geq 0$ for all i. It is *reduced* if for all $i \in I$, either $a_i = 1$ or $a_i = 0$.
- The *support* of a $\mathbb{Q}$-divisor $D = \sum\limits_{i\in I} a_i Y_i$ is $\operatorname{supp}(D) = \bigcup\limits_{a_i\neq 0} Y_i$.
- Let U be a non-empty open subset of X. If Y is a prime divisor of X, we define

$$Y|_U = \begin{cases} Y\cap U & \text{if } Y\cap U \neq \varnothing \\ 0 & \text{if } Y \cap U = \varnothing \end{cases} \tag{1}$$

  Extending linearly gives two group homomorphisms $\operatorname{Div}(X) \to \operatorname{Div}(U)$ and $\operatorname{Div}(X,\mathbb{Q}) \to \operatorname{Div}(U,\mathbb{Q})$ where in both cases the map is given by $D \mapsto D|_U$.

2.2.14 Remark. As every divisor is also a $\mathbb{Q}$-divisor, every notion that is well defined for $\mathbb{Q}$-divisors is also well defined for divisors.

2.2.15 Notation. Given a ring R, define $\operatorname{Spec}^1(R) = \{\, \mathfrak{p} \in R \,:\, \operatorname{ht} \mathfrak{p} = 1 \,\}$.

The following lemma is Theorem 38 on page 124 of [Mat80].

2.2.16 Lemma. *If A is a Noetherian normal domain then $A = \bigcap_{\mathfrak{p} \in \operatorname{Spec}^1(A)} A_{\mathfrak{p}}$.*

Lemma 2.2.16 has the following consequence:

2.2.17 Lemma. *Consider the group homomorphism $\operatorname{div}_X : K(X)^* \to \operatorname{Div}(X)$.*

(a) *For any $f \in K(X)^*$, $\operatorname{div}_X(f) \geq 0$ if and only if $f \in \mathcal{O}_X(X)$.*

(b) $\ker(\operatorname{div}_X) = \mathcal{O}_X(X)^*$

2.2.18 Definitions. A Weil divisor D is *Cartier* (or *locally principal*) if there exists an open cover $X = \bigcup_{i \in I} U_i$ of X such that, for each $i \in I$, $D|_{U_i}$ is a principal divisor of U_i. We write $\operatorname{CaDiv}(X) = \{\, D \in \operatorname{Div}(X) \,:\, D \text{ is Cartier} \,\}$ and note that $\operatorname{CaDiv}(X)$ is a subgroup of $\operatorname{Div}(X)$. Since every principal divisor is locally principal, $\operatorname{Princ}(X)$ is a subgroup of $\operatorname{CaDiv}(X)$. The quotient group $\operatorname{CaDiv}(X)/\operatorname{Princ}(X)$ is called the *Cartier class group of X* and is denoted $\operatorname{CaCl}(X)$.

A *$\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor* is a $\mathbb{Q}$-divisor $D \in \operatorname{Div}(X, \mathbb{Q})$ such that nD is a Cartier divisor for some $n \in \mathbb{N}^+$. A *$\mathbb{Q}$-Cartier divisor* is a divisor $D \in \operatorname{Div}(X)$ such that nD is a Cartier divisor for some $n \in \mathbb{N}^+$.

2.2.19 Remark. If $D \sim D'$ are linearly equivalent divisors on X, then D is Cartier if and only if D' is Cartier.

2.2.20 Definition. For any scheme X, the *Picard group of X*, denoted $\operatorname{Pic}(X)$, is the group of isomorphism classes of invertible sheaves on X with group operation $\otimes$.

2.2.21 Remarks. We briefly discuss some generalizations of the above notions.

- If X is an arbitrary scheme then one can define a Cartier divisor of X to be a global section of the quotient sheaf $\mathcal{K}_X^* / \mathcal{O}_X^*$. Using this new viewpoint together with a corresponding definition for what it means for a Cartier divisor to be "principal", it makes sense to define both $\operatorname{CaDiv}(X)$ and $\operatorname{CaCl}(X)$ for an arbitrary scheme X. (See the "Cartier Divisors" subsection of Section II.6 in [Har77] for the relevant definitions.)

- If X is any integral scheme, then $\operatorname{CaCl}(X) \cong \operatorname{Pic}(X)$. (See Proposition II.6.15 in [Har77].)

- Remark II.6.11.2 of [Har77] implies that, in the special case where X is a normal variety (which is the only case considered from this point onward), the general notion of a Cartier divisor is equivalent to the one defined in Definition 2.2.18. As such, we will use the phrase "the Cartier divisor D" to mean "the divisor D that is Cartier" as in Definition 2.2.18.

2.2.22 Proposition. *If $D \in \operatorname{Div}(X)$ is such that $\operatorname{supp}(D) \cap \operatorname{Sing}(X) = \varnothing$, then $D \in \operatorname{CaDiv}(X)$. In particular, if X is nonsingular then $\operatorname{CaDiv}(X) = \operatorname{Div}(X)$.*

Proof. Since $U = X \setminus \operatorname{Sing}(X)$ is a nonsingular variety, Proposition II.6.11 of [Har77] implies that $\operatorname{CaDiv}(U) = \operatorname{Div}(U)$, so $D|_U$ is a Cartier divisor of U, so there is an open cover $\{U_i\}_{i\in I}$ of U such that, for each $i \in I$, $D|_{U_i}$ is a principal divisor of U_i. Let $V = X \setminus \operatorname{supp}(D)$ and note that $D|_V = 0 = \operatorname{div}_V(1)$ is a principal divisor of V. Since $\{U_i\}_{i\in I} \cup \{V\}$ is an open cover of X, it follows that $D \in \operatorname{CaDiv}(X)$. □

2.2.23 Definition. Given $D \in \operatorname{Div}(X)$, we define the sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by stipulating that if U is a non-empty open subset of X then

$$\Gamma(U, \mathcal{O}_X(D)) = \{0\} \cup \big\{ f \in K(X)^* \,:\, \operatorname{div}_U(f) + D|_U \geq 0 \big\}.$$

More generally, if $D \in \operatorname{Div}(X, \mathbb{Q})$, we define the sheaf $\mathcal{O}_X(D)$ to be sheaf $\mathcal{O}_X(\lfloor D \rfloor)$.

2.2.24 Remark. For $D \in \operatorname{Div}(X)$, note that $\mathcal{O}_X(D)$ has a nonzero global section if and only if there exists $f \in K(X)^*$ such that $\operatorname{div}_X(f) + D \geq 0$ if and only if D is linearly equivalent to an effective divisor.

2.2.25 Proposition. *Let $D \in \operatorname{Div}(X)$ and let C be a closed subset of X with* $\mathbf{codim}_X(C) > 1$.

(a) *If D is Cartier then $\mathcal{O}_X(D)$ is equal to the invertible sheaf $\mathcal{L}(D)$ defined in Section II.6 of [Har77].*

(b) *If U is an open subset of X then $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$.*

(c) *Let $U = X \setminus C$ and let $i : U \to X$ be the inclusion morphism. Then $i_*\big(\mathcal{O}_X(D)|_U\big) = \mathcal{O}_X(D)$.*

(d) *$\mathcal{O}_X(D)$ is coherent.*

Proof. (a) Consider a pair (U, f) such that U is an open subset of X and $f \in K(X)^*$ is such that $D|_U = \operatorname{div}_U(f)$. Given $g \in K(X)^*$, we have $g \in \Gamma(U, \mathcal{O}_X(D)) \Leftrightarrow \operatorname{div}_U(g) + \operatorname{div}_U(f) \geq 0 \Leftrightarrow \operatorname{div}_U(gf) \geq 0 \Leftrightarrow gf \in \Gamma(U, \mathcal{O}_X)$, the last equivalence by Lemma 2.2.17(a). So $\Gamma(U, \mathcal{O}_X(D)) = \frac{1}{f}\Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{L}(D))$, the last equality by definition of $\mathcal{L}(D)$. This means that there is a basis $\mathscr{B}$ of open sets for the topology of X such that $\Gamma(U, \mathcal{O}_X(D)) = \Gamma(U, \mathcal{L}(D))$ for every $U \in \mathscr{B}$. Since both $\mathcal{O}_X(D)$ and $\mathcal{L}(D)$ are subsheaves of the constant sheaf $K(X)$, it follows that $\mathcal{O}_X(D) = \mathcal{L}(D)$.

Assertion (b) follows from the definitions. To prove (c), first note that there is a morphism $\varphi : \mathcal{O}_X(D) \to i_*\big(\mathcal{O}_X(D)|_U\big)$ defined by stipulating that if W is an open subset of X then $\varphi_W : \Gamma(W, \mathcal{O}_X(D)) \to \Gamma(W, i_*\big(\mathcal{O}_X(D)|_U\big)) = \Gamma(W \cap U, \mathcal{O}_X(D))$ is the restriction homomorphism of the sheaf $\mathcal{O}_X(D)$. Clearly, $\Gamma(W, \mathcal{O}_X(D)) \subseteq \Gamma(W \cap U, \mathcal{O}_X(D))$ and φ_W is the inclusion map. The assumption $\mathbf{codim}_C(X) > 1$ implies that for any $D' \in \mathrm{Div}(X)$ we have $D'|_W \geq 0 \Leftrightarrow D'|_{W\cap U} \geq 0$; so given any $f \in K(X)^*$ we have:

$$f \in \Gamma(W, \mathcal{O}_X(D)) \Leftrightarrow (\mathrm{div}_X(f) + D)|_W \geq 0 \Leftrightarrow (\mathrm{div}_X(f) + D)|_{W\cap U} \geq 0 \Leftrightarrow f \in \Gamma(W \cap U, \mathcal{O}_X(D)),$$

which shows that $\Gamma(W, \mathcal{O}_X(D)) = \Gamma(W \cap U, \mathcal{O}_X(D))$. So φ_W is the identity map, which proves (c).

For (d), let $U = X \setminus \mathrm{Sing}(X)$ and note that $\mathcal{O}_X(D) = i_*(\mathcal{O}_X(D)|_U)$ by (c) (using the fact that $\mathrm{Sing}(X)$ is closed and of codimension > 1 in X). By Proposition 2.2.22, $D|_U$ is a Cartier divisor of U; so parts (a) and (b) imply that $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$ is an invertible sheaf on U. In particular, $\mathcal{O}_X(D)|_U$ is quasi-coherent. Proposition II.5.8 of [Har77] implies that $i_*(\mathcal{O}_X(D)|_U)$ is quasi-coherent, so $\mathcal{O}_X(D)$ is quasi-coherent. To show that $\mathcal{O}_X(D)$ is coherent, it suffices to show that given any affine open subset $W = \operatorname{Spec} A$ of X, the A-module $M = \Gamma(W, \mathcal{O}_X(D))$ is finitely generated. Let X_1 be the set of height 1 prime ideals of A, and let $(m_\mathfrak{p})_{\mathfrak{p}\in X_1}$ be the family of integers (almost all zero) such that $D|_W = \sum_{\mathfrak{p}\in X_1} m_\mathfrak{p} V(\mathfrak{p})$. Note that $X_1^+ = \big\{\, \mathfrak{p} \in X_1 \,:\, m_\mathfrak{p} > 0 \,\big\}$ is a finite set. For each $\mathfrak{p} \in X_1^+$, choose $f_\mathfrak{p} \in \mathfrak{p} \setminus \{0\}$; then the element $f = \prod_{\mathfrak{p}\in X_1^+} f_\mathfrak{p}^{m_\mathfrak{p}}$ of $A \setminus \{0\}$ satisfies $v_\mathfrak{p}(f) \geq m_\mathfrak{p}$ for all $\mathfrak{p} \in X_1^+$, and $v_\mathfrak{p}(f) \geq 0 \geq m_\mathfrak{p}$ for all $\mathfrak{p} \in X_1 \setminus X_1^+$ (where $v_\mathfrak{p}$ is the valuation of the discrete valuation ring $A_\mathfrak{p}$), so in fact

$$v_\mathfrak{p}(f) \geq m_\mathfrak{p} \text{ for all } \mathfrak{p} \in X_1.$$

Define $\mu : M = \Gamma(W, \mathcal{O}(D)) \to K(X)$ by $\mu(h) = fh$ for all $h \in M$. If $h \in M \setminus \{0\}$ then $v_\mathfrak{p}(h) \geq -m_\mathfrak{p}$ for all $\mathfrak{p} \in X_1$, so $v_\mathfrak{p}(\mu(h)) \geq 0$ for all $\mathfrak{p} \in X_1$, so $\mu(h) \in \bigcap_{\mathfrak{p}\in X_1} A_\mathfrak{p} = A$ (the last equality by Lemma 2.2.16), showing that $\mu(M) \subseteq A$. Since M is isomorphic to the ideal $\mu(M)$ of the Noetherian ring A, M is finitely generated. This proves (d). $\square$

The following is Proposition II.6.13 of [Har77] together with Proposition 2.2.25(a).

2.2.26 Proposition.

(a) *If $D_1, D_2 \in \mathrm{CaDiv}(X)$ then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \iff D_1 \sim D_2$.*

(b) *If $\mathcal{L}$ is an invertible sheaf on X then there exists $D \in \mathrm{CaDiv}(X)$ such that $\mathcal{O}_X(D) \cong \mathcal{L}$.*

2.2.27 Lemma. *Let X be a scheme and let $i : U \hookrightarrow X$ be an open immersion. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on U. Then $\mathcal{F} \cong \mathcal{G}$ if and only if $i_*\mathcal{F} \cong i_*\mathcal{G}$.*

Proof. The ($\Rightarrow$) direction is because i_* is a functor. The converse is because the restriction of $i_*\mathcal{F}$ to U is $\mathcal{F}$ and the restriction of $i_*\mathcal{G}$ to U is $\mathcal{G}$. $\square$

2.2.28 Proposition.

(a) *If* $D_1, D_2 \in \mathrm{Div}(X)$ *then* $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \iff D_1 \sim D_2$.

(b) *If* $D \in \mathrm{Div}(X)$ *then* $\mathcal{O}_X(D)$ *is invertible if and only if* D *is Cartier.*

(c) *If* $D_1, D_2 \in \mathrm{Div}(X)$ *and at least one of* D_1, D_2 *is Cartier then*

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1 + D_2).$$

Proof. Recall that $C = \mathrm{Sing}(X)$ is closed and $\mathbf{codim}_X(C) > 1$. Let $U = X \setminus C$ and let $i : U \to X$ be the inclusion morphism. We have $\mathrm{CaDiv}(U) = \mathrm{Div}(U)$ by Proposition 2.2.22.

(a) If $D \in \mathrm{Div}(X)$ and $f \in K(X)^*$ are such that $D|_U = \mathrm{div}_U(f)$, then $(D - \mathrm{div}_X(f))|_U = 0$, so $D - \mathrm{div}_X(f) = 0$ because $\mathbf{codim}_X(C) > 1$. Thus, $D_1 \sim D_2 \Leftrightarrow D_1|_U \sim D_2|_U$. By Proposition 2.2.25, we have $\mathcal{O}_X(D_j) = i_*\big(\mathcal{O}_U(D_j|_U)\big)$ for $j = 1, 2$. It follows that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \iff i_*\left(\mathcal{O}_U(D_1|_U)\right) \cong i_*\left(\mathcal{O}_U(D_2|_U)\right) \iff \mathcal{O}_U(D_1|_U) \cong \mathcal{O}_U(D_2|_U)$, the last equivalence by Lemma 2.2.27. So

$$D_1 \sim D_2 \iff D_1|_U \sim D_2|_U \iff \mathcal{O}_U(D_1|_U) \cong \mathcal{O}_U(D_2|_U) \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2),$$

where the middle equivalence follows from Proposition 2.2.26 because $D_1|_U, D_2|_U \in \mathrm{CaDiv}(U)$.

(b) If D is Cartier then $\mathcal{O}_X(D)$ is invertible by Proposition 2.2.25(a). Conversely, if $\mathcal{O}_X(D)$ is invertible then by Proposition 2.2.26 there exists $D' \in \mathrm{CaDiv}(X)$ such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$; then $D \sim D'$ by (a), so D is Cartier.

(c) If $D_1, D_2 \in \mathrm{Div}(X)$ then, for each open subset W of X,

$$\Gamma(W, \mathcal{O}_X(D_1)) \times \Gamma(W, \mathcal{O}_X(D_2)) \to \Gamma(W, \mathcal{O}_X(D_1 + D_2)), \quad (f, g) \mapsto fg$$

is a well-defined $\mathcal{O}_X(W)$-bilinear map. So, there is an induced homomorphism of $\mathcal{O}_X(W)$-modules

$$\varphi_W : \Gamma(W, \mathcal{O}_X(D_1)) \otimes_{\mathcal{O}_X(W)} \Gamma(W, \mathcal{O}_X(D_2)) \to \Gamma(W, \mathcal{O}_X(D_1 + D_2)), \quad f \otimes g \mapsto fg. \tag{2}$$

These φ_W define a homomorphism of presheaves $\varphi : \mathcal{P} \to \mathcal{O}_X(D_1 + D_2)$, where $\mathcal{P}$ is the presheaf defined by $\mathcal{P}(W) = \Gamma(W, \mathcal{O}_X(D_1)) \otimes_{\mathcal{O}_X(W)} \Gamma(W, \mathcal{O}_X(D_2))$. Let $\mathcal{P}^+$ denote the sheafification of $\mathcal{P}$. Since $\mathcal{P}^+ = \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$, φ induces a homomorphism $\Phi = \varphi^+$,

$$\Phi : \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \to \mathcal{O}_X(D_1 + D_2).$$

We shall argue that if one of D_1, D_2 is Cartier then Φ is an isomorphism. Suppose that D_2 is Cartier. Consider a pair (W, h) such that $W \cong \operatorname{Spec} A$ is an affine open subset of X, $h \in \operatorname{Frac}(A)^* = K(X)^*$ and $D_2|_W = \operatorname{div}_W(h)$. Note that for each $\xi \in \operatorname{Frac}(A)^*$ we have

$$\xi \in \Gamma(W, \mathcal{O}_X(D_2)) \Leftrightarrow 0 \leq \operatorname{div}_W(\xi) + \operatorname{div}_W(h) = \operatorname{div}_W(\xi h) \Leftrightarrow \xi h \in \mathcal{O}_W(W) = A$$

(the last equivalence by Lemma 2.2.17), so $\Gamma(W, \mathcal{O}_X(D_2)) = \frac{1}{h}A$. Similarly, if we define $M = \Gamma(W, \mathcal{O}_X(D_1))$ then $\Gamma(W, \mathcal{O}_X(D_1 + D_2)) = \frac{1}{h}M$. It follows that the map φ_W of (2) is

$$M \otimes_A \tfrac{1}{h}A \longrightarrow \tfrac{1}{h}M, \quad f \otimes g \mapsto fg,$$

which is an isomorphism of A-modules. This shows that there exists a basis $\mathscr{B}$ of open sets for the topology of X such that φ_W is an isomorphism for all $W \in \mathscr{B}$. It follows that the morphism $\varphi_x : \mathcal{P}_x \to \mathcal{O}_X(D_1 + D_2)_x$ is an isomorphism for all $x \in X$. Consequently, $\Phi_x : \big(\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)\big)_x \to \mathcal{O}_X(D_1 + D_2)_x$ is an isomorphism for each $x \in X$, so Φ is an isomorphism.

□

2.2.29 Remark. Throughout the thesis, we may implicitly use Proposition 2.2.28(b) to show that a divisor D on a normal variety X is Cartier, by instead showing that $\mathcal{O}_X(D)$ is an invertible sheaf.

2.2.30 Proposition. *Let $U = X \setminus \operatorname{Sing}(X)$, and let $i : U \to X$ be the inclusion morphism. For $D \in \operatorname{Div}(X)$ and $L \in \operatorname{Div}(U)$, we have $D|_U \sim L \iff \mathcal{O}_X(D) \cong i_*(\mathcal{O}_U(L))$.*

Proof. Suppose that $D|_U \sim L$. By Propositions 2.2.25(b) and 2.2.28(a), $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U) \cong \mathcal{O}_U(L)$, so by Proposition 2.2.25 (c)

$$\mathcal{O}_X(D) \cong i_*\big(\mathcal{O}_X(D)|_U\big) \cong i_*\big(\mathcal{O}_U(L)\big).$$

Conversely, suppose that $\mathcal{O}_X(D) \cong i_*(\mathcal{O}_U(L))$. Then $\mathcal{O}_X(D)|_U \cong \big(i_*(\mathcal{O}_U(L))\big)|_U$. Since $\big(i_*(\mathcal{O}_U(L))\big)|_U = \mathcal{O}_U(L)$, we have by Proposition 2.2.25(b),

$$\mathcal{O}_U(D|_U) = \mathcal{O}_X(D)|_U \cong \big(i_*(\mathcal{O}_U(L))\big)|_U = \mathcal{O}_U(L),$$

which implies that $D|_U \sim L$ by Proposition 2.2.28 (a). □

2.2.31 Definition. Let $\nu : X \to Y$ be a closed immersion, where X and Y are normal varieties. Define $\operatorname{CaDiv}^\nu(Y) = \big\{\, D \in \operatorname{CaDiv}(Y) \ :\ \nu(X) \nsubseteq \operatorname{supp}(D) \,\big\}$ and observe that $\operatorname{CaDiv}^\nu(Y)$ is a subgroup of $\operatorname{CaDiv}(Y)$. Section 21.4 of [Gro67] defines a group homomorphism

$$\nu^* : \operatorname{CaDiv}^\nu(Y) \to \operatorname{CaDiv}(X)$$

that satisfies the following conditions for all $D \in \operatorname{CaDiv}^\nu(Y)$:

(1) $\mathcal{O}_X\big(\nu^*(D)\big) \cong \nu^*\big(\mathcal{O}_Y(D)\big)$;

(2) if $D \geq 0$ then $\nu^*(D) \geq 0$ and $\operatorname{supp}\big(\nu^*(D)\big) = \nu^{-1}\big(\operatorname{supp}(D)\big)$.

Given $D \in \operatorname{CaDiv}^{\nu}(Y)$, the Cartier divisor $\nu^*(D)$ (also denoted $D|_X$) is computed as follows. Since D is Cartier, there exists an open cover $\{U_i\}_{i\in I}$ of Y such that, for each $i \in I$, $D|_{U_i}$ is a principal divisor of U_i. For each $i \in I$, choose $f_i \in K(Y)^*$ such that $D|_{U_i} = \operatorname{div}_{U_i}(f_i)$ and let $V_i = \nu^{-1}(U_i)$. It can be verified that the hypothesis $\nu(X) \nsubseteq \operatorname{supp}(D)$ implies that each f_i determines a well-defined element $\bar{f}_i$ of $K(X)^*$, and that there exists a unique Cartier divisor C of X such that $C|_{V_i} = \operatorname{div}_{V_i}(\bar{f}_i)$ for all $i \in I$. Moreover, C is independent of all the choices that we made. This Cartier divisor C is $\nu^*(D)$.

2.2.32 Definition. Let $\nu : X \to Y$ be a closed immersion, where X and Y are normal varieties. If D is a $\mathbb{Q}$-Cartier divisor of Y such that $\nu(X) \nsubseteq \operatorname{supp}(D)$ then we define the $\mathbb{Q}$-divisor $\nu^*(D)$ of X by

$$\nu^*(D) = \frac{1}{m}\nu^*(mD) \quad \text{for any } m \in \mathbb{N}^+ \text{ such that } mD \text{ is a Cartier divisor,}$$

where $\nu^*(mD)$ is defined in 2.2.31. Clearly, $\nu^*(D)$ is independent of the choice of m. The $\mathbb{Q}$-divisor $\nu^*(D)$ is also denoted $D|_X$.

We will now define a number of properties of divisors on normal varieties. It should be noted that most of these properties are well defined in more general settings. As we do not make use of the additional generality, it makes sense for us to restrict to the case of normal varieties.

2.2.33 Definitions. Let X be a normal variety over $\mathbf{k}$.

- A Cartier divisor D is *very ample* if $\mathcal{O}_X(D)$ is a very ample invertible sheaf;
- a $\mathbb{Q}$-divisor D is *ample* if there exists some $m \in \mathbb{N}^+$ such that mD is a very ample Cartier divisor;

Assume furthermore that X is projective.

- a Cartier divisor D is *big* if there exists $n \in \mathbb{N}^+$ such that $nD \sim A + E$ for some ample divisor A and some effective divisor E;
- a $\mathbb{Q}$-Cartier divisor D is *big* if there exists $n \in \mathbb{N}^+$ such that nD is a big Cartier divisor;
- a $\mathbb{Q}$-Cartier divisor D is *pseudo-effective* if $D + A$ is big for every ample $\mathbb{Q}$-Cartier divisor A on X.

2.2.34 Remark. The definition of "big" used above is a special case of Definition 2.2.14 in [Laz04]. The definition of "pseudo-effective" above can be found in [CU15].

The following remarks are all well known.

2.2.35 Remarks. Let X be a normal variety.

(a) The sum of two very ample Cartier divisors is very ample.

(b) The sum of two ample $\mathbb{Q}$-divisors is ample.

Assume furthermore that X is projective.

(c) An ample $\mathbb{Q}$-Cartier divisor is big.

(d) The sum of a big $\mathbb{Q}$-Cartier divisor and an effective $\mathbb{Q}$-Cartier divisor is big.

(e) An ample $\mathbb{Q}$-Cartier divisor is pseudo-effective.

Proof. Part (a) is due to Exercise II.5.12 together with Proposition II.6.13 of [Har77].

For part (b), let D_1, D_2 be ample $\mathbb{Q}$-divisors. Then, there exist $m, n \in \mathbb{N}^+$ such that mD_1 and nD_2 are very ample Cartier divisors. Then $mn(D_1 + D_2) = n(mD_1) + m(nD_2)$ is very ample, so $D_1 + D_2$ is ample.

Part (c) is clear.

For part (d), let D be a big $\mathbb{Q}$-Cartier divisor. Then there exists some $n \in \mathbb{N}^+$ such that nD is a big Cartier divisor. So there exists some $m \in \mathbb{N}^+$ such that $mnD \sim A + E$ where A is an ample divisor and E is an effective divisor. Let E' be an effective $\mathbb{Q}$-Cartier divisor, and let $p \in \mathbb{N}^+$ be such that pE' is Cartier. Then $pmn(D + E')$ is Cartier and $pmn(D + E') \sim pA + pE + pmnE'$, where pA is ample and $pE + pmnE'$ is effective. So $D + E'$ is big.

For part (e), assume D is ample. Part (b) then implies that $D + A$ is ample for every ample divisor A on X. So $D + A$ is big by part (c), hence D is pseudo-effective. □

2.2.36 Proposition. *Let $i : X \to Y$ be a closed immersion of normal varieties, let D be a $\mathbb{Q}$-Cartier divisor of Y such that $i(X) \nsubseteq \operatorname{supp}(D)$ and consider the $\mathbb{Q}$-divisor $D|_X$ of X.*

(a) *If D is a very ample Cartier divisor then so is $D|_X$.*

(b) *If D is ample then so is $D|_X$.*

Proof. (a) Suppose that D is a very ample Cartier divisor. Then $\mathcal{O}_Y(D)$ is a very ample invertible sheaf. By Lemma 2.1.9, $i^*(\mathcal{O}_Y(D))$ is a very ample invertible sheaf on X. Since $i(X) \nsubseteq \operatorname{supp}(D)$, it follows from the Definition of $i^*(D)$ and from condition (1) in 2.2.31 that $i^*(D) \in \operatorname{CaDiv}(X)$ and

$i^*(\mathcal{O}_Y(D)) \cong \mathcal{O}_X(i^*(D))$. Since $i^*(\mathcal{O}_Y(D))$ is a very ample invertible sheaf on X, $i^*(D) = D|_X$ is a very ample Cartier divisor.

(b) There exists $m \in \mathbb{N}^+$ such that mD is a very ample Cartier divisor of Y. By (a), $i^*(mD)$ is a very ample Cartier divisor on X. We have $i^*(D) = \frac{1}{m} i^*(mD)$ by definition, so $mi^*(D)$ is a very ample Cartier divisor on X, so the $\mathbb{Q}$-divisor $i^*(D) = D|_X$ is ample. □

The following remark is Theorem I.3.4(a) in [Har77]. This remark and the lemma that follows are used in Section 4.9.

2.2.37 Remark. If $X \subseteq \mathbb{P}^n_{\mathbf{k}}$ is a projective variety, then $\mathcal{O}_X(X) = \mathbf{k}$.

2.2.38 Lemma. *Let X be a projective normal variety of nonzero dimension. Then no principal divisor of X is ample.*

Proof. Let D be a principal divisor of X. If D is ample, then $\mathcal{O}_X(D) \cong \mathcal{O}_X$ is an ample invertible sheaf. By definition there exists some $n \in \mathbb{N}^+$ such that $(\mathcal{O}_X)^{\otimes n}$ is very ample. Since $(\mathcal{O}_X)^{\otimes n} \cong \mathcal{O}_X$ for all $n \in \mathbb{N}^+$, it follows that $\mathcal{O}_X$ is very ample on X. Thus, to prove the lemma, it suffices to show that $\mathcal{O}_X$ is not very ample.

Assume for the sake of a contradiction that $\mathcal{O}_X$ is very ample. Then there exist an $N \in \mathbb{N}^+$ and an immersion $i : X \to \mathbb{P}^N$ such that $i^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_X$. Since X is projective over $\mathbf{k}$, Remark II.5.16.1 in [Har77] implies that i is a closed immersion. Recall that $\mathcal{O}_X(X) \cong \mathbf{k}$ by Remark 2.2.37 and that X has infinitely many closed points (since $\dim X > 0$). Part (1) of Proposition II.7.3 in [Har77] then implies

> $(*)$ for every pair of distinct closed points $P, Q \in X$, there exists some $k \in \mathbf{k}$ such that $k \in \mathfrak{m}_P \mathcal{O}_{X,P}$ and $k \notin \mathfrak{m}_Q \mathcal{O}_{X,Q}$ (where $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$).

Let $k \in \mathbf{k}$. If $k \neq 0$ then $k \notin \mathfrak{m}_P \mathcal{O}_{X,P}$ for any closed point $P \in X$; if $k = 0$, then $k \in \mathfrak{m}_P \mathcal{O}_{X,P}$ for every closed point $P \in X$. Consequently, no $k \in \mathbf{k}$ satisfies $(*)$. We conclude that $\mathcal{O}_X$ is not very ample. This completes the proof. □

Canonical sheaf, canonical divisor, dualizing sheaf

2.2.39 Notation. Let $\varphi : R \to S$ be a ring homomorphism. Then $\Omega_{S/R}$ is the module of differentials of S over R. If X is a scheme over $\mathbf{k}$, then $\Omega_{X/\mathbf{k}}$ is the sheaf of relative differentials of X over $\mathbf{k}$. (See Section II.8 of [Har77].)

2.2.40 Definition. Let X be a nonsingular variety over $\mathbf{k}$. Let $\Omega_{X/\mathbf{k}}$ be the sheaf of differentials of X over $\mathbf{k}$ and let $n = \dim X$. (See page 180 of [Har77].) Any sheaf of $\mathcal{O}_X$-modules isomorphic to $\bigwedge^n \Omega_{X/\mathbf{k}}$ is called a *canonical sheaf* of X. We use the symbol ω_X to denote a canonical sheaf of X. Since $\Omega_{X/\mathbf{k}}$ is locally free of rank n (by Theorem II.8.15 of [Har77]), it follows from Exercise II.5.16 (a) of [Har77] that ω_X is an invertible sheaf on X. Proposition 2.2.26(b) implies that there exists a divisor K_X that satisfies $\mathcal{O}_X(K_X) \cong \omega_X$. Any such K_X is called a *canonical divisor* of X. The set of canonical divisors of X is a linear equivalence class.

2.2.41 Definition. Let X be a normal variety over $\mathbf{k}$ and recall that $\operatorname{Sing}(X)$ has codimension at least 2 in X. The *canonical sheaf* ω_X of X is defined by $\omega_X \cong i_*\omega_U$, where $U = X \setminus \operatorname{Sing}(X)$ is open in X, $i : U \to X$ is the inclusion morphism, and ω_U is a canonical sheaf of U in the sense of Definition 2.2.40. Note that ω_X is not necessarily an invertible sheaf. A *canonical divisor* of X is any divisor K_X of X that satisfies the equivalent conditions:

- $K_X|_U \sim K_U$
- $\mathcal{O}_X(K_X) \cong \omega_X$.

These conditions are equivalent by Proposition 2.2.30. By Definition 2.2.40, we may consider a canonical divisor K_U of U. If $D \in \operatorname{Div}(X)$ is the closure of K_U in X then $D|_U = K_U$, so D is a canonical divisor of X and hence K_X exists. By Proposition 2.2.28(a), the set of canonical divisors of X is a linear equivalence class.

2.2.42 Lemma. *Let V be a non-empty open subset of a normal $\mathbf{k}$-variety X.*

(a) *$\omega_X|_V$ is a canonical sheaf of V.*

(b) *$K_X|_V$ is a canonical divisor of V.*

Proof. If X is nonsingular then $\omega_X|_V = \left(\bigwedge^n \Omega_{X/\mathbf{k}}\right)|_V = \bigwedge^n \left(\Omega_{X/\mathbf{k}}|_V\right) = \bigwedge^n \Omega_{V/\mathbf{k}} = \omega_V$. So (a) is true in this special case. In the general case, let $U = X \setminus \operatorname{Sing}(X)$ and consider the commutative diagram

$$\begin{tikzcd} & X & \\ U \arrow[ur, hook, "i"] & & V \arrow[ul, hook'] \\ & U \cap V \arrow[ul, hook] \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where the four arrows are the inclusion morphisms. It is straightforward to verify that if $\mathcal{F}$ is any sheaf on U then $(i_*\mathcal{F})|_V = j_*(\mathcal{F}|_{U\cap V})$ is an equality of sheaves on V. In particular, if $\mathcal{F} = \omega_U$ is a canonical sheaf of U then $(i_*\omega_U)|_V = j_*(\omega_U|_{U\cap V})$. We have $\omega_X \cong i_*\omega_U$ by definition of ω_X, so $\omega_X|_V \cong j_*(\omega_U|_{U\cap V})$. The fact that (a) is true in the nonsingular case implies that $\omega_U|_{U\cap V} \cong \omega_{U\cap V}$, so $j_*(\omega_U|_{U\cap V}) \cong j_*\omega_{U\cap V}$. Since $U \cap V$ is the nonsingular locus of V, we have $j_*\omega_{U\cap V} \cong \omega_V$ by definition of ω_V, so

$$\omega_X|_V \cong j_*(\omega_U|_{U\cap V}) \cong j_*\omega_{U\cap V} \cong \omega_V,$$

which proves (a).

From $\omega_X \cong \mathcal{O}_X(K_X)$ and part (a), we get $\omega_V \cong \omega_X|_V \cong \mathcal{O}_X(K_X)|_V = \mathcal{O}_V(K_X|_V)$, so $K_X|_V$ is a canonical divisor of V, which proves (b). □

2.2.43. Let X be a projective variety over $\mathbf{k}$. Section III.7 of [Har77] defines the *dualizing sheaf* of X, denoted ω_X°. We need the following properties of ω_X°:

(a) The sheaf ω_X° exists, is unique up to isomorphism, and is coherent.

(b) (Serre Duality) If X is Cohen-Macaulay and of dimension n, and if $\mathcal{F}$ is a locally free sheaf of finite rank on X, then $h^i(X, \mathcal{F}) = h^{n-i}(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X^\circ)$ for all $i \in \{0, \dots, n\}$, where $\mathcal{F}^\vee$ is the dual of $\mathcal{F}$.

(c) If X is normal and Cohen-Macaulay then ω_X° coincides with the canonical sheaf ω_X of X.

Propositions III.7.5 and III.7.2 of [Har77] imply that ω_X° exists and is unique. The fact that ω_X° is coherent is part of its definition (page 241 of [Har77]). Assertion (b) follows from Corollary III.7.7 of [Har77]. Assertion (c) is a special case of Remark 5.2 of [Kov12].

The most important case for us is the following:

2.2.44 Corollary. *Let X be a normal projective surface.*

(a) *The sheaves ω_X and ω_X° exist and are isomorphic.*

(b) *If $\mathcal{F}$ is a locally free sheaf of finite rank on X, then $h^i(X, \mathcal{F}) = h^{2-i}(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)$ for all $i \in \{0, 1, 2\}$.*

Proof. We noted that ω_X exists in Definition 2.2.41, and 2.2.43 implies that ω_X° exists. By Corollary 2.2.6, normal surfaces are Cohen-Macaulay. So, part (a) follows from 2.2.43 (c) and part (b) follows from 2.2.43 (b). □

The following version of the adjunction formula is a special case of Proposition 5.73 of [KM98].

2.2.45 Proposition. *Let X be a projective variety over $\mathbf{k}$ which is normal and Cohen-Macaulay, and let D be a prime divisor of X. If D is Cartier then $\omega_D^\circ \cong i^*\big(\omega_X^\circ \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)\big)$, where $i : D \to X$ is the inclusion morphism.*

2.2.46 Corollary. *Let X be a 3-dimensional projective variety over $\mathbf{k}$ which is normal and Cohen-Macaulay, and let $S \subset X$ be a prime divisor of X which is a normal surface. If S is a Cartier divisor of X then the following hold.*

(a) $\omega_S \cong i^*\big(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)\big)$*, where $i : S \to X$ is the inclusion morphism.*

(b) *If ω_X is invertible then so is ω_S.*

(c) *Suppose $D \in \operatorname{Div}(X)$ is Cartier and linearly equivalent to $K_X + S$. If $S \not\subseteq \operatorname{supp}(D)$ then $D|_S$ is a canonical divisor of S.*

Proof. For part (a), we have $\omega_S^\circ \cong \omega_S$ by Corollary 2.2.44 (a) and $\omega_X^\circ \cong \omega_X$ by 2.2.43 (a). The result then follows from Proposition 2.2.45.

For (b), we have that $\mathcal{O}_X(S)$ is invertible since S is Cartier, so $\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)$ is invertible (as it is a tensor product of invertible sheaves). Then, $i^*\big(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)\big)$ is invertible by Proposition 2.1.3 (e) and ω_S is invertible by part (a).

For (c) we have

$$\omega_S \cong i^*\big(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)\big) \cong i^*\big(\mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)\big) \cong i^*\big(\mathcal{O}_X(D)\big)$$

where the first isomorphism is by part (a) and the third is by Proposition 2.2.28 (c). By condition (1) of Definition 2.2.31, $i^*\big(\mathcal{O}_X(D)\big) \cong \mathcal{O}_S(D|_S)$. It follows that $D|_S$ is a canonical divisor of S.

□

The following lemma is well known and is stated explicitly in Definition 3.1 of [Kov12]. [3]

2.2.47 Lemma. *Let X be a normal variety over $\mathbf{k}$. If X is Gorenstein, then the canonical sheaf ω_X is invertible.*

Intersection Numbers

We require some results on intersection numbers of Cartier divisors. As such, we include a few references as well as some (likely already known) lemmas that we will refer to later in the thesis. For a very nice exposition on intersection numbers, we refer the reader to [Arc93].

2.2.48. Our calculations are done using Snapper-Kleiman intersection numbers, discussed thoroughly in [Kle66]. Snapper-Kleiman intersection numbers are defined for invertible sheaves on a

[3] To correctly interpret Definition 3.1 of [Kov12], one should keep in mind that every Gorenstein scheme is Cohen-Macaulay (see Tag 0C00 of [Sta22]). It then follows from 2.2.43 that if X is a variety that is both normal and Gorenstein then the sheaves ω_X and ω_X° exist and are isomorphic.

complete scheme X over an algebraically closed field $\mathbf{k}$. If X is d-dimensional, and if $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d$ are invertible sheaves on X, we write $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_d)_X$ to denote the Snapper-Kleiman intersection number of $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d$ on X.

2.2.49 Definition. Let X be a complete normal $\mathbf{k}$-variety of dimension d. Given Cartier divisors $D_1, D_2, \dots, D_d$ on X, define $(D_1 \cdot D_2 \cdots D_d)_X = (\mathcal{O}_X(D_1) \cdot \mathcal{O}_X(D_2) \cdots \mathcal{O}_X(D_d))_X$. It is known that the map $\operatorname{CaDiv}(X)^d \to \mathbb{Z}$, $(D_1, \dots, D_d) \mapsto (D_1 \cdots D_d)_X$ is $\mathbb{Z}$-multilinear and symmetric. Moreover, if $D_1, \dots, D_d$ and $D'_1, \dots, D'_d$ are Cartier divisors such that $D_i \sim D'_i$ (linear equivalence) for all $i = 1, \dots, d$, then $(D_1 \cdots D_d)_X = (D'_1 \cdots D'_d)_X$.

Given $\mathbb{Q}$-Cartier divisors $E_1, E_2, \dots, E_d$ on X, define

$$(E_1 \cdot E_2 \cdots E_d)_X = \frac{\big((n_1 E_1) \cdot (n_2 E_2) \cdots (n_d E_d)\big)_X}{\prod_{i=1}^d n_i}$$

for any choice of $n_1, n_2, \dots, n_d \in \mathbb{N}^+$ such that $n_1 E_1, n_2 E_2, \dots, n_d E_d$ are Cartier.

2.2.50 Notation. We are mostly interested in the case where X is a surface. In that case, we abbreviate $(D_1 \cdot D_2)_X$ to $(D_1 \cdot D_2)$ or $D_1 \cdot D_2$, and $(D \cdot D)_X$ to D^2.

The main result we need is the following reformulation of Remark 2.5.10 in [Arc93].

2.2.51 Lemma. *Let D_1, D_2 be Cartier divisors on a complete normal surface over $\mathbf{k}$. Then*

$$(D_1 \cdot D_2) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D_1)) - \chi(\mathcal{O}_X(D_2)) + \chi(\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)).$$

2.2.52. Before we state the following lemma, we recall that an isomorphism of varieties is, in particular, a closed immersion. In view of this, Definition 2.2.31 implies that every isomorphism of normal varieties $\nu : U \to U'$ induces an isomorphism of groups $\nu^* : \operatorname{CaDiv}(U') \to \operatorname{CaDiv}(U)$ (since $\operatorname{CaDiv}^\nu(U') = \operatorname{CaDiv}(U')$).

2.2.53 Lemma. *Let X and X' be complete normal surfaces and let $f : U \to U'$ be an isomorphism where $U \subseteq X$ and $U' \subseteq X'$ are non-empty open subsets. Let $D_1, D_2 \in \operatorname{CaDiv}(X)$ and $D'_1, D'_2 \in \operatorname{CaDiv}(X')$. Assume:*

(i) *D_1 and D'_1 are effective, $\operatorname{supp}(D_1) \subset U$ and $\operatorname{supp}(D'_1) \subset U'$;*

(ii) *$D_i|_U = f^*(D'_i|_{U'})$ for $i = 1, 2$, where $f^* : \operatorname{CaDiv}(U') \to \operatorname{CaDiv}(U)$ is the isomorphism determined by f.*

Then $(D_1 \cdot D_2)_X = (D'_1 \cdot D'_2)_{X'}$.

We could not find Lemma 2.2.53 in the literature. One proof can be obtained by using Proposition 4 of [Kle66] and Exercise III.2.3 (e) and (f) of [Har77]. However, that proof is long and requires the definition of the Snapper-Kleiman intersection number $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_t \cdot \mathcal{F})_X$ which is more general than the one we presented. (Our discussion in 2.2.48 is the special case of Definition 1 in [Kle66] where $t = \dim X$ and $\mathcal{F} = \mathcal{O}_X$). So we shall follow general usage and omit the proof.

2.3 Preliminaries on Singularities

2.3.1 Assumption. We assume throughout this section that all varieties are over $\mathbb{C}$. While most results in this section hold in greater generality, it is cumbersome to state each result in its most general form. In view of Proposition 1.2.8, restricting to the case where the base field is $\mathbb{C}$ is sufficient for our purposes.

2.3.2 Definition. Let X be a variety. A *resolution of singularities of* X is a proper birational morphism $f : \tilde{X} \to X$ where $\tilde{X}$ is a nonsingular variety.

2.3.3 Remark. While we assume X is a variety over $\mathbb{C}$, resolutions of singularities exist even in the more general situation where X is an integral separated scheme of finite type over any field $\mathbf{k}$ of characteristic zero. This result is due to the landmark paper of Hironaka. See [Hir64a] and [Hir64b]. Hironaka has also posted a document with an alleged proof of the characteristic p case online (see [Hir17]). As far as we know, its validity has been neither refuted nor widely accepted by the mathematical community.

2.3.4. Consider an affine variety $X = \operatorname{Spec} R$ over an algebraically closed field $\mathbf{k}$, and a finite group G acting on X. Each $g \in G$ determines a $\mathbf{k}$-automorphism $\bar{g}$ of R, so we may consider the subalgebra $R^G = \{ r \in R : \bar{g}(r) = r \text{ for all } g \in G \}$ of R. It is well known (see [Noe26]) that R^G is finitely generated as a $\mathbf{k}$-algebra. The affine variety $X/G = \operatorname{Spec}(R^G)$ is called *the quotient of* X *by the action of* G, and the morphism of varieties $\pi : X \to X/G$ defined by the inclusion homomorphism $R^G \hookrightarrow R$ is called the *quotient morphism.* It is easy to verify that if R is a normal domain then so is R^G; so if X is a normal variety then so is X/G.

2.3.5 Definition. Let Y be a variety and let $y \in Y$. The point $y \in Y$ is a *quotient singularity* if there exists a nonsingular affine variety X and a finite group G acting on X such that X/G is isomorphic to an open neighborhood of y in Y. The point y is a *cyclic quotient singularity* if G can be chosen to be a finite cyclic group. It can be checked that the conditions

- every point of Y is a quotient singularity (resp. a cyclic quotient singularity),
- every closed point of Y is a quotient singularity (resp. a cyclic quotient singularity)

are equivalent. If they are satisfied, we say that Y has *at most (cyclic) quotient singularities.*

2.3.6 Remark. If a variety Y has at most quotient singularities then Y is normal. (This follows from the last sentence in paragraph 2.3.4.)

2.3.7 Definition. A normal variety Y is $\mathbb{Q}$-*factorial* if every divisor $D \in \operatorname{Div}(Y)$ is $\mathbb{Q}$-Cartier.

The following is Proposition 5.15 in [KM98].

2.3.8 Proposition. *If a variety Y has at most quotient singularities then Y is $\mathbb{Q}$-factorial.*

2.3.9 Definitions. Let X be a normal variety and let $f : \tilde{X} \to X$ be a resolution of singularities. Then $f : \tilde{X} \to X$ *is a rational resolution of* X if the following equivalent conditions hold:

(a) $R^i f_* \mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$;

(b) X is Cohen-Macaulay and $f_* \omega_{\tilde{X}} = \omega_X$.

One says that X has *rational singularities* if every resolution of singularities $f : \tilde{X} \to X$ is a rational resolution. Theorem 5.10 of [KM98] implies that if there exists a rational resolution of X, then every resolution of singularities of X is a rational resolution and hence X has rational singularities. Note that the equivalence of (a) and (b) is explained by Definition 5.8, Theorem 5.10 and Lemma 5.12 in [KM98]. A point $x \in X$ is a *rational singularity* if there exists an open neighborhood U of x such that U has rational singularities.

2.3.10 Remark. If X is nonsingular, then X has rational singularities.

2.3.11 Lemma. *For a normal variety X, the following are equivalent.*

(a) *X has rational singularities;*

(b) *each closed point of X is a rational singularity.*

Proof. It is obvious that (a) implies (b). Conversely, suppose that (b) holds. Consider a resolution of singularities of X, $f : \tilde{X} \to X$. Fix $i > 0$. It has to be shown that $R^i f_* \mathcal{O}_{\tilde{X}} = 0$.

Let $\mathbf{P}$ be the set of closed points of X. For each $P \in \mathbf{P}$, there exists an open neighborhood U_P of P in X such that U_P has rational singularities. Since $\{U_P\}_{P \in \mathbf{P}}$ is an open cover of X, it suffices to show that $\left(R^i f_* \mathcal{O}_{\tilde{X}}\right)|_{U_P} = 0$ for every $P \in \mathbf{P}$. (Note that an open set containing all closed points of X must contain X.)

Let $P \in \mathbf{P}$, let $W_P = f^{-1}(U_P)$ and let $g : W_P \to U_P$ be the restriction of f. Then g is a resolution of singularities of U_P, and hence is a rational resolution of U_P. It follows that $R^i g_* \mathcal{O}_{W_P} = 0$, so $\left(R^i f_* \mathcal{O}_{\tilde{X}}\right)|_{U_P} \cong R^i g_*\left(\mathcal{O}_{\tilde{X}}|_{W_P}\right) \cong R^i g_* \mathcal{O}_{W_P} = 0$, where the first isomorphism is Corollary III.8.2 of Hartshorne. Thus, $\left(R^i f_* \mathcal{O}_{\tilde{X}}\right)|_{U_P} = 0$ for every $P \in \mathbf{P}$. It follows that $R^i f_* \mathcal{O}_{\tilde{X}} = 0$, so X has rational singularities. $\square$

The following is well known.

2.3.12 Remark. A ring R is a discrete valuation ring if and only if R is Noetherian normal local domain with Krull dimension equal to one.

We state a lemma which is implicit in many places in the literature, but for which we could not find a published proof. See page 50 of [KKMSD06] for example.

2.3.13 Lemma. *Let $g : X \to Y$ be a surjective birational morphism between Noetherian integral schemes. If Y is normal, then $g^\sharp : \mathcal{O}_Y \to g_* \mathcal{O}_X$ is an isomorphism of sheaves on Y.*

Proof. Since g is birational, we may assume that the canonical map $K(Y) \to K(X)$ is the identity map $K \to K$, where $K = K(X) = K(Y)$. Then

(1) for each $x \in X$ and $y \in Y$ satisfying $g(x) = y$, $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are subrings of K and satisfy $\mathcal{O}_{Y,y} \subseteq \mathcal{O}_{X,x}$ and $\mathfrak{m}_{X,x} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$;

(2) for every non-empty open subset U of Y, $\mathcal{O}_Y(U)$ and $(g_* \mathcal{O}_X)(U) = \mathcal{O}_X(g^{-1}(U))$ are subrings of K such that $\mathcal{O}_Y(U) \subseteq \mathcal{O}_X(g^{-1}(U))$, and $g_U^\sharp : \mathcal{O}_Y(U) \to \mathcal{O}_X(g^{-1}(U))$ is the inclusion map.

To prove the result, it suffices to show that $\mathcal{O}_Y(U) \supseteq \mathcal{O}_X(g^{-1}(U))$ for every non-empty open subset U of Y. Let U be a non-empty open subset of Y, and let $U^{(1)} = \{ y \in U : \dim \mathcal{O}_{Y,y} = 1 \}$. Since Y is Noetherian and normal, we have $\mathcal{O}_Y(U) = \bigcap_{y \in U^{(1)}} \mathcal{O}_{Y,y}$. Let $y \in U^{(1)}$; then $\mathcal{O}_{Y,y}$ is a discrete valuation ring of K by Remark 2.3.12. Since g is surjective, there exists $x \in g^{-1}(U)$ such that $g(x) = y$; then $\mathcal{O}_{Y,y} \subseteq \mathcal{O}_{X,x}$ and $\mathfrak{m}_{X,x} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$; since $\mathcal{O}_{Y,y}$ is a valuation ring of K, it follows that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$. This shows that

$$\{ \mathcal{O}_{Y,y} : y \in U^{(1)} \} \subseteq \{ \mathcal{O}_{X,x} : x \in g^{-1}(U) \}.$$

Since $\mathcal{O}_Y(U) = \bigcap_{y \in U^{(1)}} \mathcal{O}_{Y,y}$ and $\mathcal{O}_X(g^{-1}(U)) = \bigcap_{x \in g^{-1}(U)} \mathcal{O}_{X,x}$, we get $\mathcal{O}_Y(U) \supseteq \mathcal{O}_X(g^{-1}(U))$, so we are done. □

2.3.14 Corollary. *Let X be a normal variety and let $f : \tilde{X} \to X$ be a resolution of singularities. Then $\mathcal{O}_X \cong f_* \mathcal{O}_{\tilde{X}}$.*

Proof. This follows immediately from Lemma 2.3.13. □

The following lemma appears as Exercise III.8.1 in [Har77].

2.3.15 Lemma. *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\mathcal{F} \in \mathbf{Ab}(Y)$ and assume that $R^i f_* \mathcal{F} = 0$ for all $i > 0$. Then, for each $i \geq 0$, there is a natural isomorphism $H^i(Y, \mathcal{F}) \to H^i(X, f_* \mathcal{F})$.*

2.3.16 Theorem. *Let X be a normal projective variety with rational singularities. Let $f : \tilde{X} \to X$ be a resolution of singularities. Then for all $i \geq 0$, $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^i(X, \mathcal{O}_X)$. In particular, $p_a(\tilde{X}) = p_a(X)$.*

Proof. Applying Lemma 2.3.15 with $Y = \tilde{X}$, $\mathcal{F} = \mathcal{O}_{\tilde{X}}$ and $X = X$ gives $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^i(X, f_* \mathcal{O}_{\tilde{X}})$ for all $i \geq 0$. By Corollary 2.3.14, $f_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$, and so $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^i(X, \mathcal{O}_X)$ for all $i \geq 0$. $\square$

We need the following well-known remark. See Section 5.1 of [GW10] for instance.

2.3.17 Remark. If X is a scheme that is locally of finite type over $\mathbb{C}$, then the closed points are exactly the $\mathbb{C}$-rational points.

The following proposition follows from Proposition 1 in [Vie77] together with Remark 2.3.17.

2.3.18 Proposition. *Let X be a normal variety, and let x be a closed point of X. If x is a quotient singularity, then x is a rational singularity.*

2.3.19 Definitions. Let X be a nonsingular projective variety. The *geometric genus* of X is $p_g(X) = \dim_{\mathbb{C}} H^0(X, \omega_X)$. For each $n > 0$, the n^{th} *plurigenus* is $P_n(X) = \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes n})$. Note that $p_g(X) = P_1(X)$.

2.3.20 Remark. If X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ for all $n \geq 1$. (See Exercise II.8.8 of [Har77].)

2.3.21. Let X be a nonsingular projective variety. Refer to Section 10.5 of [Iit82] for the definition of the *Kodaira dimension* $\kappa(X)$ of X, which is an element of $\{-\infty\} \cup \{0, 1, \ldots, d\}$ where $d = \dim X$. We are only interested in the situation where $\kappa(X) = -\infty$, in which case one says that X has *negative Kodaira dimension.* Unravelling the definitions in Sections 10.5 (a), 10.1 (a) and at the bottom of page 126 of [Iit82], we find that $\kappa(X) = -\infty$ if and only if $\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(mK_X)) = 0$ for all $m \in \mathbb{N}^+$. Since $\mathcal{O}_X(mK_X) \cong \mathcal{O}_X(K_X)^{\otimes m} \cong \omega_X^{\otimes m}$ by Proposition 2.2.28(c), we obtain Lemma 2.3.22 below.

2.3.22 Lemma. *A nonsingular projective variety X has negative Kodaira dimension if and only if $P_m(X) = 0$ for all $m \in \mathbb{N}^+$.*

2.3.23 Definition. A variety X is *birationally ruled* if X is birationally equivalent to $C \times_{\mathbb{C}} \mathbb{P}^1$ for some variety C.

2.3.24 Remark. If X is a nonsingular projective variety that is birationally ruled, then $\kappa(X) = -\infty$ and so (by Lemma 2.3.22) $P_m(X) = 0$ for all $m \in \mathbb{N}^+$. (See Proposition 10.8 of [Iit82] as well as the definition and remark that follow it on page 309.)

2.3.25 Remark. Let C_1, C_2 be (not necessarily distinct) curves on a normal surface U. If C_1 and C_2 are complete and are included in $U \setminus \operatorname{Sing}(U)$ then we can define the intersection number $C_1 \cdot C_2$ even without the assumption that U is complete. To do this, embed U as an open subset of a complete surface X_0. (This is possible by [Nag62]). Let $\pi : X \to X_0$ be the normalization of X_0 (so that X is a normal surface). Since π is a finite morphism (by Exercise II.3.8 of [Har77]) and finite morphisms are proper (by Exercise II.4.1 of [Har77]), π is proper and so X is complete over $\mathbb{C}$, since $X \to X_0 \to \mathbb{C}$ is proper (by Corollary II.4.8(b) in [Har77]). Since the restriction $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is the normalization of U and U is normal, $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism. It follows that X is a complete normal surface with an open subset $\pi^{-1}(U)$ that is isomorphic to U. This shows that we can choose a complete normal surface X that contains U as an open subset.

Note that $C_1, C_2 \subset U \subseteq X$. Since C_1, C_2 are complete, they are closed in X and hence are prime divisors of X. Since $C_1, C_2 \subset U \setminus \mathrm{Sing}(U) \subseteq X \setminus \mathrm{Sing}(X)$, Proposition 2.2.22 implies that $C_1, C_2 \in \mathrm{CaDiv}(X)$. So, the Snapper-Kleiman intersection number $(C_1 \cdot C_2)_X$ is defined. By Lemma 2.2.53, this number is independent of the choice of X, so we may define $C_1 \cdot C_2 = (C_1 \cdot C_2)_X$ for any choice of X.

2.3.26 Definition. Let S be a normal surface. A *strong normal crossings divisor* (or *SNC-divisor*) of S is an element $D = C_1 + \cdots + C_n$ $(n \geq 0)$ of $\mathrm{Div}(S)$ satisfying:

- $C_1, \dots, C_n$ are distinct projective nonsingular curves in $S \setminus \mathrm{Sing}(S)$;
- if $i \neq j$ then $C_i \cdot C_j \leq 1$ (where $C_i \cdot C_j$ is well defined by Remark 2.3.25);
- if i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

2.3.27 Remark. Consider C_i, C_j with $i \neq j$ as in Definition 2.3.26. The condition $C_i \cdot C_j \leq 1$ is equivalent to $C_i \cdot C_j \in \{0, 1\}$, which is equivalent to:

if $C_i \cap C_j \neq \varnothing$ then $C_i \cap C_j$ is a one-point set and C_i, C_j meet normally at that point.

2.3.28 Definitions. Let $D = C_1 + \cdots + C_n$ be an SNC-divisor on a normal surface S.

- The *dual graph* $\mathcal{G}(D, S)$ *of* D *in* S is the graph obtained by stipulating that the vertex-set is $\{C_1, \dots, C_n\}$, distinct vertices C_i, C_j are joined by an edge if and only if $C_i \cap C_j \neq \varnothing$, and each vertex C_i is decorated by the pair $(p_g(C_i), C_i^2)$, where $p_g(C_i)$ is the genus of C_i and C_i^2 is the self-intersection number of C_i in S (which is well defined by Remark 2.3.25). If a vertex C_i satisfies $p_g(C_i) = 0$ then we may omit the genus from the decoration, i.e., that vertex may be decorated simply by the number C_i^2.
- The *intersection matrix* M of D is the $n \times n$ (symmetric) matrix defined by $M_{ij} = C_i \cdot C_j$.

2.3.29 Definition. Given a singular point p of a normal surface S, let $\mathcal{R}(S, p)$ be the set of pairs (U, π) such that U is an open neighborhood of p in S with $U \setminus \{p\}$ nonsingular, and $\pi : \tilde{U} \to U$ is a proper birational morphism such that

- $\tilde{U}$ is nonsingular;
- π restricts to an isomorphism $\pi^{-1}(U \setminus \{p\}) \to U \setminus \{p\}$;
- the set $\pi^{-1}(p)$ is the support of an SNC-divisor of $\tilde{U}$.

Define an equivalence relation on $\mathcal{R}(S,p)$ by declaring that elements (U_i, π_i) $(i = 1, 2)$ of $\mathcal{R}(S,p)$ are *equivalent* if for some open neighborhood U of p in $U_1 \cap U_2$ there exists a commutative diagram

$$\begin{tikzcd} \pi_1^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi_1"'] & & \pi_2^{-1}(U) \arrow[dl, "\pi_2"] \\ & U & \end{tikzcd}.$$

A *resolution of the singularity* (S,p) (or simply a *resolution of* (S,p)) *is* an equivalence class of pairs $(U,\pi) \in \mathcal{R}(S,p)$.

Observe that if $(U_1,\pi_1), (U_2,\pi_2) \in \mathcal{R}(S,p)$ are equivalent, and if D_i is the unique SNC-divisor of $\tilde{U}_i$ such that $\operatorname{supp}(D_i) = \pi_i^{-1}(p)$, then Lemma 2.2.53 implies that the dual graphs $\mathcal{G}(D_1, \tilde{U}_1)$ and $\mathcal{G}(D_2, \tilde{U}_2)$ are the same, and that the intersection matrix of D_1 is the same as that of D_2. So, each resolution of (S,p) determines a dual graph and an intersection matrix.

The following facts are stated on page 6 of [Mum61]:

(i) $\mathcal{R}(S,p) \neq \varnothing$;

(ii) for each $(U,\pi) \in \mathcal{R}(S,p)$, the intersection matrix of $\pi^{-1}(p)$ is negative-definite;[4]

(iii) there exists a unique resolution of (S,p) that minimizes the number of irreducible components of $\pi^{-1}(p)$ (or equivalently, the number of vertices of the associated dual graph).

We introduce the following definitions:

(i) The resolution of (S,p) that minimizes the number of irreducible components of $\pi^{-1}(p)$ is called the *minimal resolution* of (S,p).

(ii) Let $(U,\pi) \in \mathcal{R}(S,p)$ be such that the equivalence class of (U,π) is the minimal resolution of (S,p). We have $\pi : \tilde{U} \to U$. The set $\pi^{-1}(p)$ is called the *exceptional locus of* p, and each curve included in $\pi^{-1}(p)$ is called an *exceptional curve*. If D is the unique SNC-divisor of $\tilde{U}$ such that $\operatorname{supp}(D) = \pi^{-1}(p)$, then the dual graph $\mathcal{G}(D, \tilde{U})$ is called the *resolution graph* of the singularity (S,p).

For the notion of a *complex analytic space*, we refer the reader to Section 2.3 of [Ish18]. This notion is only needed in 2.3.30–2.3.33. Proposition 2.3.33 allows us to use the notion of analytic equivalence (for closed points of $\mathbb{C}$-varieties) without needing to resort to analytic spaces.

[4] We mean the intersection matrix of the unique SNC-divisor whose support is $\pi^{-1}(p)$.

2.3.30 Definition. Consider pairs (X, x) and (Y, y) where X, Y are complex analytic spaces and $x \in X$, $y \in Y$. One says that (X, x) is *analytically equivalent* to (Y, y) if there exist open neighborhoods U of x in X and V of y in Y (in the standard $\mathbb{C}$-topologies) and an isomorphism of analytic spaces $f : U \to V$ such that $f(x) = y$. Each equivalence class, with respect to this equivalence relation, is called a *germ of analytic space.*

2.3.31 Remark. The above definition of *germ of analytic space* is the one given in [Ish18]. By Proposition 3.16 on page 23 of [Dim92], the above definition of *analytic equivalence* is equivalent to the one given in [Dim92]. We point this out because we will use results from both [Ish18] and [Dim92].

If $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-variety then there is a natural way to define a sheaf $\mathcal{O}_X^{\text{hol}}$ on X such that $(X, \mathcal{O}_X^{\text{hol}})$ is an analytic space. If U is any open subset of X in the standard $\mathbb{C}$-topology then U has a structure of analytic space, obtained by setting $\mathcal{O}_U^{\text{hol}} = \mathcal{O}_X^{\text{hol}}|_U$. We can thus introduce the following:

2.3.32 Definition. Consider pairs (X, x) and (Y, y) where X, Y are $\mathbb{C}$-varieties and $x \in X$, $y \in Y$ are closed points. Then, (X, x) is *analytically equivalent* to (Y, y) if there exist open neighborhoods U of x in X and V of y in Y (in the standard $\mathbb{C}$-topologies) and an isomorphism of analytic spaces $f : U \to V$ such that $f(x) = y$.

2.3.33 Proposition. *Given (X, x) and (Y, y) where X, Y are $\mathbb{C}$-varieties and $x \in X$, $y \in Y$ are closed points, the following are equivalent.*

(a) *(X, x) is analytically equivalent to (Y, y);*

(b) *$\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$ as $\mathbb{C}$-algebras.*

Proof. By a result of Hironaka-Rossi and Artin (see Theorem 4.2.3 in [Ish18]), we have the equivalences (a) $\Leftrightarrow \mathcal{O}_{X,x}^{\text{hol}} \cong \mathcal{O}_{Y,y}^{\text{hol}} \Leftrightarrow \widehat{\mathcal{O}}_{X,x}^{\text{hol}} \cong \widehat{\mathcal{O}}_{Y,y}^{\text{hol}}$. By Proposition 3 of [Ser56], we have $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X,x}^{\text{hol}}$ and $\widehat{\mathcal{O}}_{Y,y} \cong \widehat{\mathcal{O}}_{Y,y}^{\text{hol}}$. □

2.3.34 Remark. In the statement of Proposition 2.3.33, $\mathcal{O}_{X,x}$ is the local ring of the variety X at x (not to be confused with $\mathcal{O}_{X,x}^{\text{hol}}$) and $\widehat{\mathcal{O}}_{X,x}$ denotes the completion of $\mathcal{O}_{X,x}$ with respect to its maximal ideal.

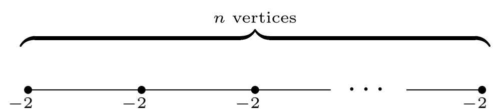
The following is a consequence of Theorem 2 of [Neu81].

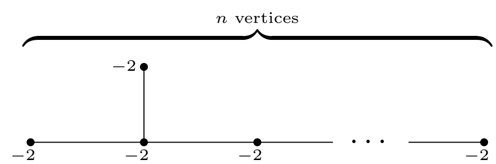
2.3.35 Proposition. *Consider (X, x) and (Y, y) where X, Y are normal surfaces over $\mathbb{C}$ and $x \in X$, $y \in Y$ are closed points. If (X, x) and (Y, y) are analytically equivalent then they have the same resolution graph.*

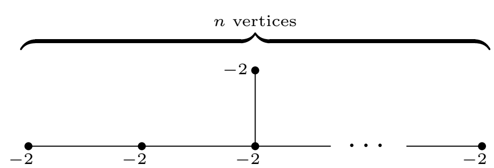
2.3.36 Definition. Let X be a normal surface. A singular point $x \in X$ is *Du Val* if it is rational and Gorenstein.

The following remark follows from Theorem 7.5.1 in [Ish18].

2.3.37 Remark. The exceptional curves in the minimal resolution of Du Val singularities are fully understood, and consist of projective lines with self-intersection -2. If X is a normal surface, then a singular point $x \in X$ is a du Val singularity if and only if its resolution graph is one of the following Dynkin diagrams:

- A_n for $n \geq 1$: 

- D_n for $n \geq 4$: 

- E_n for $6 \leq n \leq 8$: 

2.3.38 Definition. Let T be one of the Dynkin diagrams listed in Remark 2.3.37. We say that a Du Val singularity is of *type* T if its resolution graph is T.

2.3.39 Definition. Given coprime positive integers P, Q, the *cyclic quotient singularity of type* $\frac{1}{P}(1, Q)$ is defined to be the quotient $\mathbb{C}^2/\Gamma$ where Γ is the action of the group of P^{th} roots of unity on $\mathbb{C}^2$ given by $\mu \cdot (x, y) = (\mu x, \mu^Q y)$.

2.3.40 Definition. Let $\frac{P}{Q} \in \mathbb{Q}$ where P, Q are coprime positive integers. The *Hirzebruch-Jung continued fraction of* $\frac{P}{Q}$ is the continued fraction

$$\frac{P}{Q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\cdots - \frac{1}{b_r}}}}$$

such that $b_i \in \mathbb{N}^+$ for all $i = 1, \ldots, r$ and is denoted by $[b_1, b_2, \ldots, b_r]$.

2.3.41 Remark. The method for calculating the exceptional locus of cyclic quotient surface singularities is well known. (See Theorem 7.4.19 of [Ish18] for example.) For a cyclic quotient singularity x of type $\frac{1}{P}(1, Q)$, let $\frac{P}{Q} = [b_1, \ldots, b_r]$ be the Hirzebruch-Jung continued fraction. Then, the resolution graph of x is

$$\underset{-b_1}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{-b_2}{\bullet}\!\!-\!\!-\!\!-\qquad\cdots\qquad-\!\!-\!\!-\!\!\underset{-b_r}{\bullet}$$

where each node represents a projective line with self-intersection number $-b_i$, $1 \leq i \leq r$.

2.3.42 Remark. It is unfortunate that the term *type* is used in two different ways to describe singular points. It will be clear from the context whether we are talking about the type of a cyclic quotient singularity or the type of a Du Val singularity.

2.3.43 Example. Let $n \in \mathbb{N}^+$, and consider the weighted projective space $\mathbb{P}(n, 1, 1)$. (See Definition 2.4.2.) It is known that the point $x = [1 : 0 : 0] \in \mathbb{P}(n, 1, 1)$ is a cyclic quotient singularity of type $\frac{1}{n}(1, 1)$. Since the Hirzebruch-Jung continued fraction of $n/1$ is $[n]$, Remark 2.3.41 implies that the resolution graph of this singularity is

$$\underset{-n}{\bullet}$$

and so the resolution locus is a projective line with self-intersection $-n$.

2.4 Weighted Projective Spaces and Hypersurfaces

2.4.1 Assumption. We assume throughout this section that all varieties are over $\mathbb{C}$.

2.4.2 Definition. Let $n \geq 2$, $(w_0, \ldots, w_n) \in (\mathbb{N}^+)^{n+1}$. Let $\mathbb{C}_{w_0,\ldots,w_n}[X_0, \ldots, X_n]$ denote the $\mathbb{N}$-graded polynomial ring over $\mathbb{C}$ where X_i is homogeneous of degree w_i for each $i = 0, \ldots, n$. Then $\operatorname{Proj} \mathbb{C}_{w_0,\ldots,w_n}[X_0, \ldots, X_n]$ is denoted by $\mathbb{P}(w_0, \ldots, w_n)$ and is called a *weighted projective space over* $\mathbb{C}$.

2.4.3 Definition. A weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ is said to be *well-formed* if $\gcd(w_0, \ldots, w_{i-1}, \hat{w}_i, w_{i+1}, \ldots, w_n) = 1$ for each $i = 0, \ldots, n$.

2.4.4 Remark. It is well known that every weighted projective space is isomorphic to a well-formed weighted projective space.

2.4.5 Remark. Being well-formed is a property of the $n + 1$-tuple $(w_0, \ldots, w_n)$, and is not a geometric invariant. For example, the two weighted projective spaces $\mathbb{P} = \mathbb{P}(1, 1, 1)$ and $\mathbb{P}' = \mathbb{P}(6, 10, 15)$ are isomorphic as projective varieties. The first is a well-formed weighted projective space. The second is not.

2.4.6 Remark. Observe that $\mathbb{P}(w_0, \ldots, w_n)$ is well-formed if and only if $S = (w_0, \ldots, w_n)$ is normal and $\mathrm{type}(S) = 0$. (See Definitions 1.2.10.)

We include a few results on weighted projective spaces before discussing weighted projective varieties in general.

The following is part of Theorem 3.1A(c) in [BR86].

2.4.7 Proposition. *Every weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ is normal and Cohen-Macaulay.*

2.4.8 Definition. Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ be a well-formed weighted projective space. If Y is a prime divisor of $\mathbb{P}$, then $Y = V_+(h)$ for some irreducible homogeneous polynomial $h \in \mathbb{C}_{w_0,\ldots,w_n}[X_0, \ldots, X_n]$. We define $\deg(Y) = \deg(h)$. Extending linearly, we obtain a homomorphism $\deg : \mathrm{Div}(\mathbb{P}) \to \mathbb{Z}$.

2.4.9 Proposition. *If $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ is a weighted projective space, then $\mathrm{Cl}(\mathbb{P}) = \mathbb{Z}$.*

Proof. The case where $\mathbb{P}$ is well-formed is a special case of Exercise 4.1.5 of [CLS11]. The general case follows from Remark 2.4.4. $\square$

The following is Proposition 5.3.7 of [CLS11], applied to $\mathbb{P}(a_0, \ldots, a_n)$.

2.4.10 Proposition. *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ be a well-formed weighted projective space and let $D, E \in \mathrm{Div}(\mathbb{P})$. If $\deg(D) = \deg(E) = d$, then $\mathcal{O}_{\mathbb{P}}(D) \cong \mathcal{O}_{\mathbb{P}}(E) \cong \mathcal{O}_{\mathbb{P}}(d)$.*

The following appears in [Del75]. Part (a) appears as Proposition 2.3(i) and part (b) is in Section 1.8.

2.4.11 Lemma. *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots a_n)$ be a well-formed weighted projective space and let $m = \mathrm{lcm}(a_0, \ldots, a_n)$. Then*

(a) *$\mathcal{O}_{\mathbb{P}}(m)$ is an ample invertible sheaf,*

(b) *$\mathcal{O}_{\mathbb{P}}(jm) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k) \cong \mathcal{O}_{\mathbb{P}}(jm + k)$ for all $j, k \in \mathbb{N}^+$.*

2.4.12 Definition. A *weighted projective variety* is a closed subvariety of a weighted projective space.

2.4.13 Assumption. In the paragraphs below, we assume $n \geq 1$ and $(w_0, \ldots, w_n) \in (\mathbb{N}^+)^{n+1}$. Whenever we write "the variety $X \subseteq \mathbb{P}(w_0, \ldots, w_n)$", we mean that X is a *closed* subvariety of $\mathbb{P}(w_0, \ldots, w_n)$.

2.4.14 Remark. Note that $\mathbb{P}(w_0, \ldots, w_n)$ is a projective variety. (See Proposition 1.3.3 of [Dol82] for instance.) Consequently, every weighted projective variety is also a projective variety.

2.4.15 Definitions. Let I be a homogeneous prime ideal of the graded ring $S = \mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n]$ and consider the closed subvariety $X = V_+(I)$ of $\mathbb{P} = \mathbb{P}(w_0, \dots, w_n) = \text{Proj}(S)$. Note that X is isomorphic to $\text{Proj}(S/I)$. The closed subset $C_X = V(I) \subseteq \mathbb{A}^{n+1}$ is called the *affine cone over* X; note that C_X passes through the origin of $\mathbb{A}^{n+1}$, and that $C_X \cong \text{Spec}(S/I)$ is an integral affine scheme. The variety X is *quasismooth* if C_X is nonsingular away from the origin.

The following is Definition 6.4 in [IF00].

2.4.16 Definitions. Let I be a homogeneous prime ideal of the graded ring $S = \mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n]$ and define $X_I = \text{Proj}(S/I) \subseteq \mathbb{P}(w_0, \dots, w_n)$. If $f_1, \dots, f_k$ are homogeneous elements of S, we abbreviate $X_{\langle f_1,\dots,f_k\rangle}$ by $X_{f_1,\dots,f_k}$. If I is generated by a regular sequence $(f_1, \dots, f_k)$ of homogeneous elements of S of respective degrees $d_i = \deg(f_i)$ where $i = 1, \dots, k$, then X_I is called a *weighted complete intersection* of *multidegree* $(d_1, \dots, d_k)$. In particular, if $k = 1$, $f_1 = f$ and $d_1 = d$, we will say that $X_I = X_f$ is a *weighted hypersurface* of *degree* d.

2.4.17 Remark. In Definitions 2.4.16, I is a prime ideal of $\mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n]$, so every weighted complete intersection is a variety over $\mathbb{C}$.

Example 2.4.18 below is particularly important for us. We recall from Notation 1.2.3 that if $\pi : \mathbf{k}[X_0, \dots, X_n] \to B_{a_0,\dots,a_n}$ is the canonical quotient homomorphism, then for each $i = 0, \dots, n$ we define $x_i = \pi(X_i) \in B$.

2.4.18 Example. Let $f = X_0^{a_0} + X_1^{a_1} + \cdots + X_n^{a_n} \in \mathbb{C}[X_0, \dots, X_n]$ where $n \geq 2$ and $a_i \geq 1$ for each $i \in \{0, \dots, n\}$. Let $L = \text{lcm}(a_0, \dots, a_n)$. By Remark 1.3.2, $\mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n]/\langle f\rangle = B_{a_0,\dots,a_n}$ is a graded ring and for all $i \in \{0, \dots, n\}$, $\deg(x_i) = w_i$ where $w_i = L/a_i$. We have that $\text{Proj}(\mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n])$ is the weighted projective space $\mathbb{P}(w_0, \dots, w_n)$, that f is irreducible and that $X = \text{Proj}(\mathbb{C}[X_0, \dots, X_n]/\langle f\rangle) = \text{Proj}(B_{a_0,\dots,a_n})$ is a weighted projective hypersurface of degree L. The reader can easily verify that $C_X = \text{Spec}(\mathbb{C}[X_0, \dots, X_n]/\langle f\rangle)$ is nonsingular away from the origin, and hence that X is quasismooth.

2.4.19 Remark. Two weighted projective varieties can be isomorphic as projective varieties but can have non-isomorphic affine cones. For instance, let $B_{2,3,5} = \mathbb{C}_{5,3,2}[X_0, X_1, X_2]/\langle f\rangle$ where $f = X_0^2 + X_1^3 + X_2^5$. Let $X = \text{Proj}\, B_{2,3,5}$ and let $Y = \text{Proj}(\mathbb{C}_{1,1}[Y_0, Y_1])$. Then $X \cong \mathbb{P}^1 \cong Y$, however, $C_X = \text{Spec}\, B_{2,3,5} \not\cong \text{Spec}\, \mathbf{k}[Y_0, Y_1] = C_Y$.

The following is Definition 6.9 in [IF00].

2.4.20 Definition. Let $\mathbb{P}$ be a weighted projective space and let $X \subseteq \mathbb{P}$ be a weighted subvariety. Then X is *well-formed* if the following conditions hold:

- $\mathbb{P}$ is well-formed,

- $\mathbf{codim}_X(X \cap \mathrm{Sing}(\mathbb{P})) \geq 2$.

2.4.21 Remark. The above terminology isn't entirely consistent in the literature. Most of the results we will cite below are due to Dimca, and can be found in [Dim86]. Instead of writing that X is well-formed, Dimca writes "a quasismooth complete intersection X is *in general position relative to* $\mathrm{Sing}(\mathbb{P})$ if $\mathbf{codim}_X(X \cap \mathrm{Sing}(\mathbb{P})) \geq 2$". We use the words "well-formed" (as other authors do) simply because they are less cumbersome.

The following definition appears at the bottom of Definition 1 in [Dim86]

2.4.22 Definition. Let $\mathbb{P} = \mathbb{P}(w_0, \dots, w_n)$ be a well-formed weighted projective space and let $X = X_{f_1,\dots,f_k} \subset \mathbb{P}$ be a quasismooth weighted complete intersection that is not contained in a hyperplane $x_i = 0$. Let $d_i = \deg f_i$ for each i. For each prime p, define the following integers:

- $m(p) = |\{i : p|w_i\}|$
- $k(p) = |\{j : p|d_j\}|$
- $q(p) = \dim X + 1 - m(p) + k(p)$

The following is Proposition 2 in [Dim86].

2.4.23 Proposition. *Assume $\mathbb{P} = \mathbb{P}(w_0, \dots, w_n)$ is a well-formed weighted projective space and let $X = X_{f_1,f_2,\dots,f_k} \subset \mathbb{P}$ be a quasismooth weighted complete intersection. Then X is well-formed if and only if $q(p) \geq 2$ for all primes p.*

We need Lemma 4.2 (c) of [CD20].

2.4.24 Lemma. *Let $n \geq 1$ and $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$. Define $\bar{S} = \left(\frac{L}{a_0}, \dots, \frac{L}{a_n}\right)$ where $L = \mathrm{lcm}(S)$. Then,*

$$\mathrm{type}(\bar{S}) = \mathrm{cotype}(S) \text{ and } \mathrm{type}(S) = \mathrm{cotype}(\bar{S}).$$

Proposition 2.4.25 follows from the special case of a hypersurface in the context of Example 5 in [Dim86]. Since the result is not obvious, we provide a proof.

2.4.25 Proposition. *Let $f = X_0^{a_0} + \cdots + X_n^{a_n}$ where $n \geq 2$ and $a_i \geq 1$ for all i. Let $L = \mathrm{lcm}(a_0, \dots, a_n)$ and let $w_i = L/a_i$. Then, the weighted hypersurface $X_f \subset \mathbb{P} = \mathbb{P}(w_0, \dots, w_n)$ is a well-formed quasismooth weighted complete intersection if and only if $\mathrm{cotype}(a_0, \dots, a_n) = 0$.*

Proof. Since X_f is always a quasismooth weighted complete intersection (see Example 2.4.18), it suffices to prove that X_f is well-formed if and and only if $\mathrm{cotype}(a_0, \dots, a_n) = 0$.

If $\operatorname{cotype}(a_0, \ldots, a_n) \neq 0$, then by Lemma 2.4.24, $\operatorname{type}(w_0, \ldots, w_n) = \operatorname{cotype}(a_0, \ldots, a_n) \neq 0$. By Remark 2.4.6, $\mathbb{P}$ is not well-formed, so by Definition 2.4.20, X_f is not well-formed.

Conversely, assume that $\operatorname{cotype}(a_0, \ldots a_n) = 0$. Then, $\mathbb{P}$ is well-formed and X_f is quasismooth. By Proposition 2.4.23, it suffices to prove that $q(p) \geq 2$ for all primes p.

If p does not divide w_i for any i, then $m(p) = 0$, $q(p) = \dim X + 1 + k(p) \geq \dim X + 1 \geq 2$ and we are done. Otherwise, $p \mid w_i$ for some i, so $p \mid \deg(f)$ and so $k(p) = 1$. Since $\mathbb{P}$ is well-formed, $m(p) \leq n - 1$ and so $q(p) \geq (n-1) + 1 - (n-1) + 1 \geq 2$. $\square$

2.5 Generalities on Quasismooth Complete Intersections

2.5.1 Assumption. We assume throughout this section that all varieties are over $\mathbb{C}$.

The following appears on page 185 of [Dim86].

2.5.2 Proposition. *Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ be a well-formed weighted projective space and let $\boldsymbol{x} = [x_0 : \ldots : x_n] \in \mathbb{P}$. Define $I(\boldsymbol{x}) = \{\, j \,:\, x_j \neq 0 \,\}$. For any prime p, let*

$$\mathbb{P}_p = \{\, \boldsymbol{x} \in \mathbb{P} \,:\, p|w_i \text{ for every } i \in I(\boldsymbol{x}) \,\}.$$

Then,

$$\operatorname{Sing} \mathbb{P} = \bigcup_{p \text{ prime}} \mathbb{P}_p.$$

2.5.3 Example. We compute $\operatorname{Sing} \mathbb{P}$ for various weighted projective spaces.

- Let $k \in \mathbb{N}$ and let $\mathbb{P} = \mathbb{P}^k$. We have $\mathbb{P}_p = \varnothing$ for all p and so $\operatorname{Sing} \mathbb{P}^k = \varnothing$, as expected.
- For $\mathbb{P} = \mathbb{P}(2, 1, 1, 1)$, $\operatorname{Sing} \mathbb{P} = \mathbb{P}_2 = \{[1 : 0 : 0 : 0]\}$.
- For $\mathbb{P} = \mathbb{P}(3, 2, 1, 1)$, $\operatorname{Sing} \mathbb{P} = \mathbb{P}_2 \cup \mathbb{P}_3 = \{[0 : 1 : 0 : 0], [1 : 0 : 0 : 0]\}$.
- For $\mathbb{P} = \mathbb{P}(3, 2, 2, 1)$,

$$\operatorname{Sing} \mathbb{P} = \mathbb{P}_2 \cup \mathbb{P}_3 = \{\, [0 : x_1 : x_2 : 0] \,:\, x_1, x_2 \text{ are not both } 0 \,\} \cup \{[1 : 0 : 0 : 0]\}.$$

The following result, due to Dimca, is particularly important for us. It can be found as Proposition 8 of [Dim86].

2.5.4 Proposition. *Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ and let $X \subseteq \mathbb{P}$ be a well-formed quasismooth weighted complete intersection of dimension at least 2. Then $\operatorname{Sing}(X) = X \cap \operatorname{Sing}(\mathbb{P})$.*

2.5.5 Definition. Let $B = \mathbb{C}_{w_0, \ldots, w_n}[X_0, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$ and let $X = X_{f_1, \ldots f_r} = \operatorname{Proj} B \subseteq \mathbb{P}(w_0, \ldots, w_n)$ be a weighted complete intersection where $\deg(f_i) = d_i$. We define

- $\mathbf{w} = (w_0, \dots, w_n)$,
- $|\mathbf{w}| = \sum_{i=0}^{n} w_i$,
- $\mathbf{d} = (d_1, \dots, d_r)$,
- $|\mathbf{d}| = \sum_{i=1}^{r} d_i$,

and will say that X (or B) *is of type* $(\mathbf{w}, \mathbf{d})$. The *amplitude* of X (or B) is $\alpha = |\mathbf{d}| - |\mathbf{w}|$.

2.5.6 Remark. In the special case where $X = \mathbb{P}(w_0, \dots, w_n)$, we obtain that $|\mathbf{d}| = 0$ is the empty sum and the amplitude of $\mathbb{P}(w_0, \dots, w_n)$ is $\alpha = -\sum_{i=0}^{n} w_i$.

2.5.7 Remark. The integer α introduced in Definition 2.5.5 depends on the ambient weighted projective space and on the equations $f_1, \dots f_r$. Thus when we say that "X has amplitude α", the equations defining X will be clear from the context.

2.5.8 Example. Let $n \geq 2$, let $B = B_{a_0,\dots,a_n}$ be a Pham-Brieskorn ring and let $L = \operatorname{lcm}(a_0, \dots, a_n)$. Then, the amplitude α of B is $L - \sum_{i=0}^{n} \frac{L}{a_i}$. In particular,

$$\alpha > 0 \Leftrightarrow \textstyle\sum_{i=0}^{n} \frac{1}{a_i} < 1, \quad \alpha = 0 \Leftrightarrow \sum_{i=0}^{n} \frac{1}{a_i} = 1, \quad \text{and} \quad \alpha < 0 \Leftrightarrow \sum_{i=0}^{n} \frac{1}{a_i} > 1.$$

2.5.9 Proposition. *Let $\mathbb{P} = \mathbb{P}(w_0, \dots, w_n)$ be a well-formed weighted projective space. If $X \subseteq \mathbb{P}$ is a well-formed quasismooth weighted complete intersection then the following hold:*

(a) *X is normal;*

(b) *X has at most cyclic quotient singularities;*

(c) *X has rational singularities;*

(d) *X is Cohen-Macaulay;*

(e) *X is $\mathbb{Q}$-factorial.*

Proof. We prove (a). If $\dim X = 0$ the result is clear so assume $\dim X \geq 1$. Let S and I be as in Definitions 2.4.16 so that $X = \operatorname{Proj}(S/I)$ where S is a graded polynomial ring over $\mathbb{C}$ and I is a homogeneous prime ideal generated by a regular sequence $(f_1, \dots, f_r)$. By Proposition 2.2.11, it suffices to show that S/I is normal. By Proposition 2.2.7, S/I is Cohen-Macaulay and hence is S_2 by Remark 2.2.3. Since X is quasismooth, S/I is regular in codimension one (noting that $\dim(S/I) \geq 2$). It follows from Theorem 2.2.5 that S/I is normal.

Part (b) is well known. It suffices to show that every closed point of X is a cyclic quotient singularity (see Definition 2.3.5), and this is stated explicitly in this form on p.186 of [Dim86].

For part (c), suppose $x \in X$ is a closed point. By parts (a) and (b) together with Proposition 2.3.18, we have that x is a rational singularity. It follows from Lemma 2.3.11 that X has rational singularities. Part (d) follows from part (c), because condition (b) in Definition 2.3.9 asserts that X is Cohen-Macaulay. Part (e) follows from Proposition 2.3.8 together with part (b). □

2.5.10 Corollary. *Let $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ and let $X \subset \mathbb{P}$ be a well-formed quasismooth weighted complete intersection which is a surface. Then $\mathrm{Sing}(X)$ is a finite set of closed points.*

Proof. By Proposition 2.5.4, we have $\mathbf{codim}_X(\mathrm{Sing}(X)) = \mathbf{codim}_X(X \cap \mathrm{Sing}(\mathbb{P})) \geq 2$ and so $\dim(\mathrm{Sing}(X)) = 0$. Since X is a variety, $\mathrm{Sing}(X)$ is a closed set. Since X is Noetherian, every closed set is a finite union of irreducible closed sets, so $\mathrm{Sing}(X)$ is a finite set of closed points. □

2.5.11 Theorem. *Let $X \subseteq \mathbb{P}(w_0, \ldots, w_n)$ be a well-formed quasismooth weighted complete intersection. Then $\omega_X \cong \omega_X^\circ \cong \mathcal{O}_X(\alpha)$ where α is the amplitude of X.*

Proof. By Proposition 2.5.9, X is normal and Cohen-Macaulay. By 2.2.43 (a) and (c), ω_X and ω_X° exist and are isomorphic. Finally, $\omega_X^\circ \cong \mathcal{O}_X(\alpha)$ by Theorem 3.3.4 of [Dol82]. □

We state Lemma 7.1 of [IF00]. (We fix a small typo. The last line should read $A_{-(k-\alpha)}$ instead of $A_{-k-\alpha}$ as in [IF00].)

2.5.12 Lemma. *Let $X_{f_1,\ldots,f_r} \subset \mathbb{P}(w_0, \ldots, w_n)$ be a well-formed quasismooth weighted complete intersection. Let A be the graded ring $\mathbb{C}_{w_0,\ldots,w_n}[X_0, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$. Then,*

$$H^i(X, \mathcal{O}_X(k)) = \begin{cases} A_k & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i < \dim X \\ A_{-(k-\alpha)} & \text{if } i = \dim X \end{cases}$$

for all $k \in \mathbb{Z}$.

2.5.13 Remark. The assumption in Theorem 2.5.11 that X is well-formed is necessary (but is omitted in Dolgachev's statement of the result). See 6.15 in [IF00].

Chapter 3

Properties of Pham-Brieskorn Rings

3.0.1 Assumption. Recall the notation for B_S from Definition 1.2.2. Throughout this chapter, we assume that $\mathbf{k} = \mathbb{C}$, and so for $S = (a_0, \dots a_n) \in (\mathbb{N}^+)^{n+1}$, we have

$$B_S = B_{a_0,\dots,a_n} = \mathbb{C}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n} \rangle.$$

Whenever B_S is regarded as a graded ring, the grading is the one defined in Remark 1.3.2.

3.0.2 Assumption. We assume throughout this chapter that all varieties are over $\mathbb{C}$.

3.0.3 Remark. While many of the results of this chapter are true without assuming the base field is $\mathbb{C}$, this assumption is used in our proof of Theorem 3.1.3, which is then used in Corollary 3.1.16. It is also used in the proof of Corollary 3.2.23.

3.1 Some General Results

3.1.1 Proposition. *Let $n \geq 2$ and let $S \in (\mathbb{N}^+)^{n+1}$. Then B_S is a normal integral domain and* $\operatorname{Proj} B_S$ *is a normal projective variety.*

Proof. We noted in Definition 1.2.2 that B_S is an integral domain. Since $\operatorname{Spec} B_S$ has only one singular point, it is regular in codimension 1 (since $n > 1$). Moreover, $\operatorname{Spec} B_S$ is a hypersurface of $\mathbb{C}^{n+1}$, so by Proposition 2.2.10 (b), it is normal. So B_S is a normal domain. By Proposition 2.2.11, $\operatorname{Proj} B_S$ is normal. □

The following theorem is well known. As we could not find a suitable reference for the statement in its current form, we include a short proof.

3.1.2 Theorem. *Let S be a nonsingular projective surface.*

(a) *If S is birationally ruled then $h^2(S, \mathcal{O}_S) = 0$.*

(b) *If S is birationally ruled and $h^1(S, \mathcal{O}_S) = 0$ then S is rational.*

Proof. For (a), we have $h^2(S, \mathcal{O}_S) = \dim_{\mathbb{C}} H^2(S, \mathcal{O}_S) = p_g(S) = P_1(S) = 0$ by Corollary 2.2.44 and Remark 2.3.24.

For (b), note that S is birationally ruled if and only if S is birational to $C \times_{\mathbb{C}} \mathbb{P}^1$ where C is a smooth curve. By Proposition III.21 of [Bea96], $0 = h^1(S, \mathcal{O}_S) = p_g(C)$. So, C is a smooth projective curve of genus 0. So $C = \mathbb{P}^1$ and hence S is birational to $\mathbb{P}^1 \times_{\mathbb{C}} \mathbb{P}^1$ and hence is rational. $\square$

3.1.3 Theorem. *If $X_{f_1,\dots,f_r} \subseteq \mathbb{P}(w_0, \dots, w_n)$ is a 2-dimensional birationally ruled well-formed quasismooth weighted complete intersection then X is rational.*

Proof. Let $\tilde{X} \to X$ be a resolution of singularities. Since X (by Proposition 2.5.9) is normal and has rational singularities, Theorem 2.3.16 implies that $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^i(X, \mathcal{O}_X)$ for all $i \geq 0$. Since $\tilde{X}$ is birationally ruled and $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^1(X, \mathcal{O}_X) = 0$ by Lemma 2.5.12, Theorem 3.1.2 implies that $\tilde{X}$ is rational. It follows that X is rational. $\square$

3.1.4 Definition. If $R = \bigoplus_{n \in \mathbb{N}} R_n$ is an $\mathbb{N}$-graded ring and $d \in \mathbb{N}^+$, then $R^{(d)} = \bigoplus_{n \in \mathbb{N}} R_{nd}$ is called the d^{th} *Veronese subring* of the graded ring R.

The following proposition is Exercise 9.5 in [Eis95].

3.1.5 Proposition. *If $R = \bigoplus_{n \in \mathbb{N}} R_n$ is an $\mathbb{N}$-graded Noetherian ring and $d \in \mathbb{N}^+$, then* $\operatorname{Proj} R \cong \operatorname{Proj} R^{(d)}$.

3.1.6 Definition. Let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be a $\mathbb{Z}$-graded domain. The *saturation index of B* is defined as $e(B) = \gcd \{ i \in \mathbb{Z} : B_i \neq 0 \}$.

The following definition appears at the start of Section 3 in [CD22].

3.1.7 Definition. Let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be a $\mathbb{Z}$-graded domain. We say that B is *saturated in codimension 1* if $e(B/\mathfrak{p}) = e(B)$ for every homogeneous height 1 prime ideal $\mathfrak{p}$ of B.

The following is a special case of Theorem 1.2 of [CD22].

3.1.8 Theorem. *Let $B = \bigoplus_{i \in \mathbb{N}} B_i$ be an $\mathbb{N}$-graded domain such that the transcendence degree of B over B_0 is at least 2. If B is finitely generated as an algebra over $\mathbb{C}$ then the following are equivalent.*

(a) *There exists $d \geq 1$ such that $B^{(d)}$ is not rigid.*

(b) *There exists a homogeneous element* $h \in B \setminus \{0\}$ *of positive degree such that the open subset* $D_+(h)$ *of* $\operatorname{Proj} B$ *is a cylinder.*

If, moreover, B *is normal and is saturated in codimension 1, then the above conditions are equivalent to*

(c) B *is not rigid.*

3.1.9 Remark. In order to be consistent with our assumption at the start of the chapter that all varieties are over $\mathbb{C}$, Theorem 3.1.8 assumes that B is finitely generated over $\mathbb{C}$. Nonetheless, the result is true as long as B is finitely generated over any field $\mathbf{k}$ of characteristic zero.

The following remark is established in Example 3.16 in [CD22].

3.1.10 Remark. Let $n \geq 2$ and let $B = B_{a_0,\dots,a_n}$. Then the following are equivalent:

- B is saturated in codimension 1;
- $\operatorname{cotype}(a_0, \dots, a_n) = 0$.

3.1.11 Corollary. *Let* $n \geq 2$ *and suppose* $\operatorname{cotype}(a_0, \dots, a_n) = 0$. *The following are equivalent:*

(a) $B_{a_0,\dots,a_n}$ *is rigid;*

(b) $(B_{a_0,\dots,a_n})^{(d)}$ *is rigid for all* $d \in \mathbb{N}^+$.

Proof. It suffices to check that $B_{a_0,\dots,a_n}$ satisfies the assumptions of Theorem 3.1.8. The only non-trivial things to check are normality and saturation in codimension 1. Proposition 3.1.1 implies that $B_{a_0,\dots,a_n}$ is normal and Remark 3.1.10 implies that $B_{a_0,\dots,a_n}$ is saturated in codimension 1. □

3.1.12 Remark. The assumption that $\operatorname{cotype}(a_0, \dots, a_n) = 0$ is necessary for Corollary 3.1.11 to hold. We will see later (in Corollary 4.7.12) that $B_{2,3,3,4}$ is rigid whereas $(B_{2,3,3,4})^{(2)} \cong B_{2,3,3,2}$ is not rigid by Proposition 1.2.4.

We require the following result, which is Proposition 5.2 of [CD20].

3.1.13 Proposition. *Let* $n \geq 2$, $S = (a_0, \dots, a_n)$, $S' = (a'_0, \dots, a'_n) \in (\mathbb{N}^+)^{n+1}$, *and assume that* $i \in \{0, \dots, n\}$ *is such that* $S' \leq^i S$. *Then* $B_{S'} \cong B_S^{(k)}$, *where we define* $k = a_i/a'_i \in \mathbb{N}^+$. *Consequently,* $\operatorname{Proj} B_S \cong \operatorname{Proj} B_{S'}$.

3.1.14 Lemma. *Let* $n \geq 2$ *and let* $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$.

(a) *If* $\operatorname{cotype}(S) = k > 0$, *then there exist* $i \in \{0, \dots, n\}$ *and* $S' \in (\mathbb{N}^+)^{n+1}$ *such that* $S' <^i S$ *and* $\operatorname{cotype}(S') = k - 1$.

(b) *There exists* $\hat{S} \in (\mathbb{N}^+)^{n+1}$ *such that*

- $\text{cotype}(\hat{S}) = 0$,
- $\text{Proj}(B_S) \cong \text{Proj}(B_{\hat{S}})$,
- *if* B_S *is not rigid, then neither is* $B_{\hat{S}}$.

Proof. We prove (a). Suppose $\text{cotype}(S) > 0$. Then, there exists some $i \in \{0, \dots, n\}$ such that $\text{lcm}(S_i) \neq \text{lcm}(S)$. Let $b = \gcd(a_i, \text{lcm}(S_i))$ and let $S' = (a_0, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$. Then $S' <^i S$ and $\text{cotype}(S') = k - 1$.

For (b), let S, S' be as in part (a). By induction, it suffices to prove the following two claims:

- $\text{Proj}(B_S) \cong \text{Proj}(B_{S'})$;
- if B_S is not rigid, then neither is $B_{S'}$.

The first claim follows from Proposition 3.1.13, the second from Proposition 1.3.10 (a). □

3.1.15 Corollary. *Let* $B = B_{a_0,a_1,a_2,a_3}$ *be a Pham-Brieksorn ring. If* B *is not rigid, then the varieties* $\text{Proj}\, B$ *and* $\text{Spec}\, B$ *are rational.*

Proof. Let $S = (a_0, a_1, a_2, a_3)$, and let $X = \text{Proj}\, B_S$. By Lemma 3.1.14 (b), X is isomorphic to $X' = \text{Proj}\, B_{S'}$ where $S' \in (\mathbb{N}^+)^4$, $B_{S'}$ is not rigid and $\text{cotype}(S') = 0$. By Proposition 2.4.25, X' is well-formed. Since $B_{S'}$ is not rigid, Theorem 3.1.8 implies that X' contains a cylinder and hence is birationally ruled. Since X' is quasismooth (see Example 2.4.18), Theorem 3.1.3 implies that X' is rational. Since $X \cong X'$, X too is rational. This proves that $\text{Proj}\, B_S$ is rational. If K denotes the function field of $\text{Proj}\, B_S$, then the function field of $\text{Spec}\, B_S$ is $K^{(1)}$ by Remark 1.1.21. So $\text{Spec}\, B_S$ too is rational. □

3.1.16 Corollary. *Let* $n \geq 2$ *and let* $S \in (\mathbb{N}^+)^{n+1}$ *be such that* $\text{cotype}(S) = 0$. *Let* α *be the amplitude of* $B = B_S$. *If* $B_\alpha \neq \{0\}$, *then*

(a) $\text{Proj}\, B$ *is not birationally ruled over* $\mathbb{C}$*;*

(b) B *is rigid.*

Proof. The weighted projective hypersurface $X = \text{Proj}\, B_S$ is well-formed (by Proposition 2.4.25), quasismooth, and has dimension $n - 1$; hence $h^{n-1}(X, \mathcal{O}_X) = \dim_{\mathbb{C}} B_\alpha \neq 0$ by Lemma 2.5.12. Applying Theorem 2.3.16 and 2.2.43 (b), we have that X is birational to a smooth projective $\mathbb{C}$-variety $\tilde{X}$ such that $0 \neq h^{n-1}(X, \mathcal{O}_X) = h^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^0(\tilde{X}, \omega_{\tilde{X}}) = p_g(\tilde{X})$. It follows that $\tilde{X}$ is smooth and has positive geometric genus and hence by Remark 2.3.24 cannot be birationally ruled. Consequently, X cannot be birationally ruled. This proves (a).

If B is not rigid then Theorem 3.1.8 implies that there exists $h \in B$ homogeneous of positive degree such that $D_+(h)$ is a cylinder in X; in particular X is birationally ruled, contradicting (a). This proves (b).

□

Recall Definition 2.5.5.

3.1.17 Corollary. *Suppose $B = B_{a_0,\dots,a_n}$ is a Pham-Brieskorn ring such that the following conditions hold:*

- *B has amplitude $\alpha \geq 0$;*
- $a_n = \operatorname{lcm}(a_0, \dots, a_{n-1})$.

Then $B_\alpha \neq 0$ and B is rigid.

Proof. Since $a_n = \operatorname{lcm}(a_0, \dots, a_{n-1})$, it follows that $\operatorname{cotype}(a_0, \dots, a_n) = 0$ and $\deg(x_n) = 1$. Since $\alpha \geq 0$, $B_\alpha \neq 0$ so the result follows from Corollary 3.1.16. □

3.1.18 Remark. Let $B = B_{a_0,\dots,a_n}$ and let $d_i = \deg(x_i)$ for each i. Then $B_\alpha \neq 0$ if and only if there exist $c_i \in \mathbb{N}$ such that $\sum_{i=0}^{n} c_i d_i = \alpha$.

3.2 Projections to Weighted Projective Spaces

Preliminaries

3.2.1 Definition. A morphism $f : X \to Y$ is *finite* if for every open affine subscheme $V = \operatorname{Spec} A$ in Y, the inverse image of V in X is affine, of the form $\operatorname{Spec} B$, with B finitely generated as an A-module.

We include the following well-known remark.

3.2.2 Remark. If $f : X \to Y$ is a morphism of projective varieties such that for each closed point $y \in Y$ the set $f^{-1}(y)$ is finite, then f is a finite morphism.

3.2.3 Proposition. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a degree-preserving homomorphism of $\mathbb{Z}$-graded rings. Suppose that Φ makes $\mathcal{B}$ a free $\mathcal{A}$-module and that there exists a basis $(e_i)_{i \in I}$ of $\mathcal{B}$ over $\mathcal{A}$ with the property that $e_i \in \mathcal{B}_0$ for all $i \in I$. Then, $\mathcal{B}_0$ is a free $\mathcal{A}_0$-module, with basis $(e_i)_{i \in I}$.*

Proof. If $\mathcal{B} = 0$, then the result is obvious. If $\mathcal{B} \neq 0$, then the fact that $\mathcal{B}$ is a free $\mathcal{A}$-module implies that Φ is injective and $\mathcal{A}$ is a graded subring of $\mathcal{B}$. Let $b \in \mathcal{B}_0$. Since $\mathcal{B}_0 \subseteq \mathcal{B}$, we may write $b = \sum_{i \in I} a_i e_i$ where $a_i \in \mathcal{A}$ are homogeneous. Since $b \in \mathcal{B}_0$ and $e_i \in \mathcal{B}_0$ for all $i \in I$, it follows that we may choose $a_i \in \mathcal{A}_0$ since all terms of nonzero degree must cancel. This proves that $\mathcal{B}_0$ is spanned by the $\{e_i\}_{i \in I}$ as an $\mathcal{A}_0$-module. It is clear that $\{e_i\}_{i \in I}$ is linearly independent over $\mathcal{A}_0$ since it is linearly independent over $\mathcal{A}$. □

3.2.4 Remark. Suppose $\mathcal{B}$ is a free $\mathcal{A}$-module with basis $\{e_i\}_{i \in I}$. If $\{u_i\}_{i \in I} \subseteq \mathcal{A}^*$, then $\{u_i e_i\}_{i \in I}$ is also a basis for $\mathcal{B}$ as a free $\mathcal{A}$-module.

We need the following, which can be found as Corollary 18 in Section 10.4 of [DF04].

3.2.5 Proposition. *Let $f : R \to S$ be a homomorphism of rings and let M be a free R-module. Then $M \otimes_R S$ is a free S-module, and any basis of M is a basis of $M \otimes_R S$.*

We use the following two definitions which appear as Definitions 00UT and 02G4 of The Stacks Project [Sta22] respectively.

3.2.6 Definition. A ring homomorphism $\varphi : R \to S$ is *unramified* if φ is of finite type and $\Omega_{S/R} = 0$.

3.2.7 Definition. Let X, Y be schemes and let $\pi : X \to Y$ be a morphism of finite type. We say that π is *unramified at $x \in X$* if there exists an affine open neighbourhood $U = \operatorname{Spec}(S) \subseteq X$ of x and an affine open $V = \operatorname{Spec}(R) \subseteq Y$ with $\pi(U) \subseteq V$ such that the induced ring homomorphism $\varphi : R \to S$ is unramified. The morphism π is *unramified* if it is unramified at every point $x \in X$.

3.2.8 Proposition. *A ring homomorphism $\varphi : R \to S$ is unramified if and only if the corresponding morphism $\pi : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is unramified.*

Proof. Suppose φ is unramified. The claim that π is unramified follows from the definitions. The converse is a special case of Lemma 02G5 of [Sta22]. □

3.2.9 Definition. Let X, Y be schemes of finite type over $\mathbf{k}$. A $\mathbf{k}$-morphism $\pi : X \to Y$ is *étale* if it is flat and unramified.

3.2.10 Notation. We use the notation $B\langle dT \rangle$ to denote the free B-module generated by the symbol dT.

The following lemma appears as Example 15.2.8 in [Kob].

3.2.11 Lemma. *Let $C = A[T]$, let $f \in C$ and let $B = C/\langle f \rangle$. Let $f' = \frac{df}{dT}$ be the formal T-derivative. Then $\Omega_{B/A} = \frac{B\langle dT \rangle}{B\langle f' dT \rangle}$ where $B\langle f' dT \rangle$ is the B-submodule of $B\langle dT \rangle$ generated by $f' dT$.*

The following lemma is a slight generalization of a result that appears on the "Unramified morphism" Wikipedia page. However, due to the dynamic nature of Wikipedia and the fact that we could not find a published proof, we provide one here.

3.2.12 Lemma. *Let A be a ring, let $B = A[T]/\langle f \rangle$ for some polynomial f with coefficients in A. Let $f' = \frac{df}{dT}$. Let $A \to B$ be the inclusion and let $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism of affine schemes. Then π is unramified if and only if f and f' generate the unit ideal of $A[T]$.*

Proof. By Proposition 3.2.8, π is unramified if and only if $A \to B$ is unramified. Since $A \to B$ is of finite type, π is unramified if and only if $\Omega_{B/A} = 0$. By Lemma 3.2.11, $\Omega_{B/A} = 0$ if and only if $B\langle dT \rangle = B\langle f' dT \rangle$ if and only if $f' dT$ generates $B\langle dT \rangle$ if and only if there exist $c_1, c_2 \in A[T]$ such that $1 = c_1 f' + c_2 f$ (since $f = 0$ in B) if and only if $\langle f, f' \rangle = \langle 1 \rangle \lhd A[T]$. □

We require the following, which is part of Proposition 17.6.3 in [Gro67].

3.2.13 Proposition. *Let Y be a locally Noetherian scheme. Suppose $f : X \to Y$ is an étale morphism of schemes that is locally of finite type. Let $x \in X$ and let $y = f(x)$. Let $\kappa(y)$ denote the residue field of Y at y. If $\kappa(y)$ is separably closed, then $\widehat{\mathcal{O}_{Y,y}} \cong \widehat{\mathcal{O}_{X,x}}$.*

The General Setup

3.2.14 Notation. Recall (from Section 1.2 of [Dol82] for instance) that a closed point $P \in \mathbb{P}(w_0, \ldots, w_n)$ can be viewed as an equivalence class of points in $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if and only if there exists some $\lambda \in \mathbb{C}^*$ such that $(\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n) = (b_0, \ldots, b_n)$. We use the notation $P = [a_0 : \cdots : a_n]$ to denote the equivalence class of $(a_0, \ldots, a_n)$.

3.2.15 Definition. Let $f : X \to Y$ be a morphism of varieties and let $n \in \mathbb{N}^+$. The morphism $f : X \to Y$ is $n:1$ if there exists a non-empty open set $U \subseteq Y$ such that for each closed point $y \in U$, $f^{-1}(y)$ consists of n distinct closed points of X. The morphism $f : X \to Y$ is said to be *strictly* $n:1$ if we may choose $U = X$.

3.2.16. Suppose $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ is well-formed, and let $\mathbb{P}_i = \mathbb{P}(w_0, \ldots, w_{i-1}, \widehat{w_i}, w_{i+1}, \ldots, w_n)$ be the weighted projective space of dimension $n-1$ where w_i is omitted. Let $F = X_i^{b_i} + G(X_0, \ldots, X_{i-1}, \widehat{X_i}, X_{i+1}, \ldots, X_n) \in \mathbb{C}_{w_0,\ldots,w_n}[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree $d > 0$. Let $X = V_+(F) \subset \mathbb{P}$ be the weighted hypersurface defined by F. Consider the rational map $\varphi_i : X \dashrightarrow \mathbb{P}_i$ given by $[y_0 : \ldots : y_n] \mapsto [y_0 : \ldots : y_{i-1} : \widehat{y_i} : y_{i+1} : \ldots, y_n]$.

Observing that

$$\begin{aligned}\varphi_i(y_0,\dots,y_n) &= (y_0,\dots,y_{i-1},\widehat{y_i},y_{i+1},\dots,y_n)\\ &\sim (\lambda^{w_0}y_0,\dots,\lambda^{w_{i-1}}y_{i-1},\widehat{\lambda^{w_i}y_i},\lambda^{w_{i+1}}y_{i+1},\dots,\lambda^{w_n}y_n)\\ &= \varphi_i(\lambda^{w_0}y_0,\dots,\lambda^{w_{i-1}}y_{i-1},\lambda^{w_i}y_i,\lambda^{w_{i+1}}y_{i+1},\dots,\lambda^{w_n}y_n)\end{aligned} \tag{3}$$

we see that φ_i is indeed well-defined as a rational map of weighted projective varieties. In fact, φ_i is a surjective morphism. Indeed, if φ_i is not defined for some point on X, its indeterminacy locus (when viewed as a subset of $\mathbb{P}$) must contain a closed point of form $[0:\dots:0:\beta:0:\dots:0]$ where $\beta \neq 0$ at the i^{th} coordinate. But, such a point is not on X, so $\varphi_i : X \to \mathbb{P}_i$ is a morphism.

For surjectivity, let $Q = [y_0:\dots:y_{i-1}:\widehat{y_i}:y_{i+1}:\dots:y_n] \in \mathbb{P}_i$. Let $P = [y_0:\dots:y_{i-1}:y_i:y_{i+1}:\dots:y_n]$ for any $y_i \in \mathbb{C}$ satisfying $y_i^{b_i} = -G(y_0,,\dots,y_{i-1},\widehat{y_i},y_{i+1},\dots,y_n)$. Then $\varphi_i(P) = Q$. (Note we used that $\mathbb{C}$ is algebraically closed.)

3.2.17 Notation. Since $G \in \mathbb{C}_{w_0,\dots,w_{i-1},\hat{w}_i,w_{i+1},\dots,w_n}[X_0,\dots,X_{i-1},\widehat{X_i},X_{i+1},\dots,X_n]$ is a homogeneous polynomial of degree d, it makes sense to write both $V_+(G) \subset \mathbb{P}$ and $V_+(G) \subset \mathbb{P}_i$.

3.2.18 Remark. Using the notation of 3.2.16, if $Q \in V_+(G) \subset \mathbb{P}_i$ is a closed point, it follows that $\varphi_i^{-1}(Q)$ is a single closed point $P \in V_+(G) \subseteq \mathbb{P}$.

3.2.19 Remark. When $F = X_0^{b_0} + X_1^{b_1} + X_2^{b_2} + X_3^{b_3}$ and $X = V_+(F) \subset \mathbb{P}$, the construction above gives 4 projection morphisms $\varphi_i : X \to \mathbb{P}_i$ (where $i \in \{0,1,2,3\}$), each to a possibly different weighted projective space. The discussion in 3.2.16 shows that for every closed point Q of $\mathbb{P}_i$, $\varphi_i^{-1}(Q)$ is a finite set. Since φ_i is a morphism of projective varieties, Remark 3.2.2 implies that φ_i is a finite morphism.

A Useful Example

In this subsection, we apply the techniques introduced in this section to determine the singularities of the projective surface $\operatorname{Proj} B_{2,3,3,6}$.

3.2.20 Notation. In this subsection, we set $\mathbb{P} = \mathbb{P}(3,2,2,1)$, $F = X_0^2 + X_1^3 + X_2^3 + X_3^6$, $G = X_0^2 + X_2^3 + X_3^6 \in \mathbb{C}_{3,2,2,1}[X_0,X_1,X_2,X_3]$ and $X = V_+(F) = \operatorname{Proj} B_{2,3,3,6} \subseteq \mathbb{P}$. We also let $x_0,x_1,x_2,x_3,f,g \in B_{2,3,3,6}$ denote the images of X_0,X_1,X_2,X_3,F,G with respect to the quotient homomorphism $\pi : \mathbb{C}[X_0,X_1,X_2,X_3] \to B_{2,3,3,6}$.

3.2.21 Proposition. *Let $U = D_+(gx_2) \subseteq X$ and consider the morphism*

$$\varphi_1|_U : U \longrightarrow D_+(GX_2) \subseteq \mathbb{P}_1 = \mathbb{P}(3,2,1) = \operatorname{Proj} \mathbb{C}_{3,2,1}[X_0,X_2,X_3].$$

Then,

(a) $\varphi_1|_U$ *is a strictly* $3:1$ *surjective morphism of affine schemes,*

(b) $\varphi_1|_U$ *is étale.*

Proof. We will write φ_U instead of $\varphi_1|_U$ to simplify the notation. Let $Q \in D_+(GX_2)$. The discussion in 3.2.16 shows that there are at most 3 points in $\varphi_U^{-1}(Q)$. Suppose that $P, P' \in \varphi^{-1}(Q)$. Without loss of generality, we may assume $P = [y_0 : y_1 : y_2 : y_3]$ and $P' = [y_0 : y_1' : y_2 : y_3]$. We must show that if $y_1 \neq y_1'$, then P and P' are distinct.

We note that (using the equation $f = 0$) either $y_1' = \zeta_3 y_1$ or $y_1' = \zeta_3^2 y_1$ where ζ_3 is a 3^{rd} root of unity. Suppose $[y_0 : y_1 : y_2 : y_3] = [y_0 : \zeta_3 y_1 : y_2 : y_3]$. It then follows that for some $\lambda \in \mathbb{C}^*$ we have,

- $y_0 = \lambda^3 y_0$;
- $y_1 = \lambda^2 \zeta_3 y_1$;
- $y_2 = \lambda^2 y_2$;
- $y_3 = \lambda^1 y_3$.

Since $[y_0 : y_1 : y_2 : y_3] \in U$, we have $y_2 \neq 0$, so $\lambda = \pm 1$. This implies that $\zeta_3 y_1 = y_1$ which is impossible since $y_1 \neq 0$ for all points in U. (If $y_1 = 0$, then $g(y_0, y_2, y_3) = 0$ which is impossible since $U = D_+(gx_2)$.) We conclude that $[y_0 : y_1 : y_2 : y_3] \neq [y_0 : \zeta_3 y_1 : y_2 : y_3]$. Similar arguments show that $[y_0 : y_1 : y_2 : y_3] \neq [y_0 : \zeta_3^2 y_1 : y_2 : y_3]$ and that $[y_0 : \zeta_3 y_1 : y_2 : y_3] \neq [y_0 : \zeta_3^2 y_1 : y_2 : y_3]$.

It is not hard to see that φ_U is surjective, and since we have

$$\mathrm{Spec}((B_{2,3,3,6})_{(gx_2)}) \cong D_+(gx_2) \to D_+(GX_2) \cong \mathrm{Spec}(\mathbb{C}_{3,2,1}[X_0, X_2, X_3]_{(GX_2)}),$$

φ_U is a morphism of affine schemes. This proves (a).

For (b), we have the degree preserving inclusion of $\mathbb{Z}$-graded rings $\psi : A = \mathbb{C}_{3,2,1}[X_0, X_2, X_3] \hookrightarrow \mathbb{C}_{3,2,1,2}[X_0, X_2, X_3, X_1]/\langle F \rangle = B$. Observe that B is a free A-module with basis $\{1, x_1, x_1^2\}$. Let $S = \{1, GX_2, (GX_2)^2, \dots\} \subset A$. We write the localization $S^{-1}\psi : S^{-1}A \to S^{-1}B$ as

$$\psi_{GX_2} : \mathcal{A} = \mathbf{k}[X_0, X_2, X_3]_{GX_2} \to (\mathbf{k}[X_0, X_1, X_2, X_3]/\langle F \rangle)_{gx_2} = \mathcal{B}.$$

Then ψ_{GX_2} is also a degree preserving homomorphism of $\mathbb{Z}$-graded rings and by Proposition 3.2.5, $\mathcal{B}$ is a free $\mathcal{A}$-module with basis $\{1, x_1, x_1^2\}$.

Letting $v = \frac{g}{gx_2} \in \mathcal{B}$, we obtain by Remark 3.2.4 that $\{1, vx_1, v^2x_1^2\} \subset \mathcal{B}_0$ is a basis of $\mathcal{B}$ as an $\mathcal{A}$-module; so by Proposition 3.2.3, $\mathcal{B}_0$ is a free and hence flat $\mathcal{A}_0$-module with basis $\{1, vx_1, v^2x_1^2\}$. Also, the restriction of ψ_{GX_2} to $\mathcal{A}_0$ is exactly the ring homomorphism which corresponds to the affine morphism of schemes φ_U.

If we define $t = vx_1 \in \mathcal{B}_0$ and $a = v^3 g \in \mathcal{A}_0$ then $t^3 + a = v^3(x_1^3 + g) = v^3 f = 0$ and $\mathcal{B}_0 = \mathcal{A}_0[t] \cong \mathcal{A}_0[T]/\langle T^3 + a\rangle$. Then, $a \in \langle t^3 + a, 3t^2\rangle \lhd \mathcal{B}_0$. Since a is a unit of $\mathcal{A}_0$ and also of $\mathcal{B}_0$, φ_U is unramified by Lemma 3.2.12. This proves (b). □

3.2.22 Proposition. *The point* $[0 : 1 : 0] \in \mathbb{P}(3, 2, 1)$ *is of type* A_1.

Proof. Consider the open neighborhood $D_+(X_1)$ of $[0 : 1 : 0]$, which is isomorphic to $\mathbb{A}^2/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action is given by $t \cdot (x, y) \mapsto (t^3x, ty) = (tx, ty)$. (See the paragraph just before 1.20 of [DR01] for instance.) The singular point thus has type $\frac{1}{2}(1, 1)$, so its resolution graph is determined by the Hirzebruch-Jung continued fraction of $\frac{2}{1}$. The exceptional locus is thus a single projective line with self intersection -2. By Remark 2.3.37, this completes the proof. □

3.2.23 Corollary. *Let* $X = \operatorname{Proj} B_{2,3,3,6}$. *Then* X *has exactly three singular points, each of type* A_1.

Proof. Let $\varphi_1 : X \to \mathbb{P}(3, 2, 1)$ be the projection morphism onto the first, third and fourth coordinates. Let $U = D_+(gx_2) \subset X$ where $g = x_0^2 + x_2^3 + x_3^6$. Then $\varphi_1|_U : U \to D_+(GX_2) \subset \mathbb{P}(3, 2, 1)$ is étale by Proposition 3.2.21. Let $y = [0 : 1 : 0] \in \mathbb{P}(3, 2, 1)$. For each of the distinct points $x \in \operatorname{Sing}(X) = \{[0 : -1 : 1 : 0], [0 : \zeta_6 : 1 : 0], [0 : \zeta_6^{-1} : 1 : 0]\}$ we have $\varphi_1(x) = y$, so by Proposition 3.2.13 we have $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{Y,y}}$. For each $x \in \operatorname{Sing}(X)$, Proposition 2.3.33 implies that (X, x) is analytically equivalent to (Y, y) and Proposition 2.3.35 implies that (X, x) and (Y, y) have the same resolution graph. By Proposition 3.2.22, $y \in \mathbb{P}(3, 2, 1)$ is of type A_1, so (X, x) too is of type A_1. □

3.2.24 Remark. Corollary 3.2.23 is used in the proof of Lemma 4.6.3 and the proof of Corollary 4.6.7.

Chapter 4

Rigidity of Pham-Brieskorn Threefolds

4.0.1 Assumption. Throughout this chapter, all varieties are over $\mathbb{C}$.

4.0.2 Assumption. As in Chapter 3, for $S = (a_0, \dots, a_n) \in (\mathbb{N}^+)^{n+1}$ we assume that $B_S = B_{a_0,\dots,a_n} = \mathbb{C}[X_0, \dots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n} \rangle$, with the grading defined in Remark 1.3.2.

4.1 Demazure's Construction and Polar Cylinders

4.1.1 Definition. Let $B = \bigoplus_{i \in \mathbb{N}} B_i$ be an $\mathbb{N}$-graded integral domain. An element $\xi \in \operatorname{Frac} B$ is *homogeneous* if $\xi = \frac{a}{b}$ for some homogeneous elements $a, b \in B$ with $b \neq 0$. If ξ is homogeneous, then its *degree* is $\deg(\xi) = \deg(a) - \deg(b)$ where a and b are as before.

The following is a special case of Theorem 3.5 of [Dem88]. (See Definition 3.1.6 and Notation 1.1.17 for the definitions of $e(B)$ and B_+ respectively.) To be coherent with Assumptions 2.2.12 and 4.0.1, we assume in the statement of Theorem 4.1.2 that B is finitely generated over $\mathbb{C}$. Nonetheless, Theorem 3.5 of [Dem88] is valid as long as the graded ring B is finitely generated over any field $\mathbf{k}$.

4.1.2 Theorem. *Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be an $\mathbb{N}$-graded normal domain that is finitely generated over $\mathbb{C}$ and such that $e(B) = 1$ and $\operatorname{ht}(B_+) > 1$. Let $X = \operatorname{Proj} B$. Then, there exists a homogeneous element T of $\operatorname{Frac} B$ of degree 1, and for each such T, there exists a unique $\mathbb{Q}$-divisor H of X such that $B_n = \mathrm{H}^0(X, \mathcal{O}_X(nH))T^n$ for all $n \in \mathbb{N}$.*

4.1.3 Remark. The $\mathbb{Q}$-divisor H in Theorem 4.1.2 is ample. This is stated in Section 1.10 of [KPZ12], Theorem 3.3.4 of [Dol06], and is implicit in Section 1 of [Wat81].

The following is Lemma 2.1 in [Wat81].

4.1.4 Lemma. *Let the assumptions and notation be as in Theorem 4.1.2. For all $n \in \mathbb{Z}$, there is a canonical isomorphism $\mathcal{O}_X(n) = \widetilde{B(n)} \to \mathcal{O}_X(nH)$.*

4.1.5 Caution. In Lemma 4.1.4, nH is a $\mathbb{Q}$-divisor of X, so $\mathcal{O}_X(nH)$ is an abbreviation for $\mathcal{O}_X(\lfloor nH \rfloor)$ by Definition 2.2.23. Thus, Lemma 4.1.4 asserts that $\mathcal{O}_X(n) \cong \mathcal{O}_X(\lfloor nH \rfloor)$ for all $n \in \mathbb{Z}$. Similarly, Theorem 4.1.2 asserts that $B_n = \mathrm{H}^0(X, \mathcal{O}_X(\lfloor nH \rfloor))T^n$ for all $n \in \mathbb{N}$.

The following appears in Section 5.4 of [CD22], modulo a change of notation.

4.1.6. Let B be an $\mathbb{N}$-graded Noetherian normal domain such that the prime ideal $B_+ = \bigoplus_{i>0} B_i$ has height greater than 1. Let $\Omega = \operatorname{Spec} B$ and $X = \operatorname{Proj} B$. We shall now define a $\mathbb{Q}$-linear map $D \mapsto D^*$ from $\operatorname{Div}(X, \mathbb{Q})$ to $\operatorname{Div}(\Omega, \mathbb{Q})$.

Let $K(\Omega)$ and $K(X)$ be the function fields of Ω and X respectively. Let $X^{(1)}$ be the set of homogeneous prime ideals of B of height 1. Since $\operatorname{ht} B_+ > 1$, $X^{(1)} = \{ x \in X \, : \, \dim \mathcal{O}_{X,x} = 1 \}$. For each $\mathfrak{p} \in X^{(1)}$, $B_\mathfrak{p} \supset B_{(\mathfrak{p})}$ is an extension of discrete valuations rings; let $e_\mathfrak{p}$ denote the ramification index of this extension. Then $e_\mathfrak{p} \in \mathbb{N} \setminus \{0\}$. If $v_\mathfrak{p}^X : K(X)^* \to \mathbb{Z}$ and $v_\mathfrak{p}^\Omega : K(\Omega)^* \to \mathbb{Z}$ denote the normalized[1] valuations of $B_{(\mathfrak{p})}$ and $B_\mathfrak{p}$ respectively, then $v_\mathfrak{p}^\Omega(\xi) = e_\mathfrak{p} v_\mathfrak{p}^X(\xi)$ for all $\xi \in K(X)^*$. Let $C_\mathfrak{p}^X$ (resp. $C_\mathfrak{p}^\Omega$) denote the closure of $\{\mathfrak{p}\}$ in X (resp. in Ω). Then $C_\mathfrak{p}^X$ (resp. $C_\mathfrak{p}^\Omega$) is a prime divisor of X (resp. of Ω), and every prime divisor of X is a $C_\mathfrak{p}^X$ for some $\mathfrak{p} \in X^{(1)}$. We define $(C_\mathfrak{p}^X)^* = e_\mathfrak{p} C_\mathfrak{p}^\Omega$ for each $\mathfrak{p} \in X^{(1)}$, and extend linearly to a $\mathbb{Q}$-linear map $\operatorname{Div}(X, \mathbb{Q}) \to \operatorname{Div}(\Omega, \mathbb{Q})$, $D \mapsto D^*$. It can be verified that the linear map $D \mapsto D^*$ is injective and has the following two properties:

(a) $\big(\operatorname{div}_X(\xi)\big)^* = \operatorname{div}_\Omega(\xi)$ for all $\xi \in K(X)^*$;

(b) if f is a nonzero homogeneous element of B and $D \in \operatorname{Div}(X, \mathbb{Q})$ satisfies $D^* = \operatorname{div}_\Omega(f)$, then $D \geq 0$ and $\operatorname{supp}(D) = V_+(f)$.

4.1.7 Lemma. *Let the assumptions and notation be as in Theorem 4.1.2.*

(a) *If T and H are as in Theorem 4.1.2 then $H^* = \operatorname{div}_\Omega(T)$.*

(b) *Let T_1, T_2 be homogeneous elements of* $\operatorname{Frac} B$ *of degree* 1 *and for $i = 1, 2$ let H_i be the $\mathbb{Q}$-divisor of X that corresponds to T_i as in Theorem 4.1.2. Then $T_1/T_2 \in K(X)^*$ and $\operatorname{div}_X(T_1/T_2) = H_1 - H_2$. In particular, $H_1 \sim H_2$.*

Proof. Part (a) is stated in the first sentence on page 52 of [Dem88], and is also part of the statement of Theorem 5.9 of [CD22]. To prove (b), we use the notation of 4.1.6. It is clear that $T_1/T_2 \in K(X)^*$ and we have

$$(H_1 - H_2)^* = H_1^* - H_2^* = \operatorname{div}_\Omega(T_1) - \operatorname{div}_\Omega(T_2) = \operatorname{div}_\Omega(T_1/T_2) = \big(\operatorname{div}_X(T_1/T_2)\big)^*,$$

[1]The word "normalized" means that the maps $v_\mathfrak{p}^X$ and $v_\mathfrak{p}^\Omega$ are surjective.

where the second equality follows from part (a) and the last one follows from 4.1.6 (a). Since $D \mapsto D^*$ is injective, $\operatorname{div}_X(T_1/T_2) = H_1 - H_2$, proving (b). □

Recall Notation 1.2.3.

4.1.8 Lemma. *With the notation of 4.1.6, let $B = B_{a_0,\dots,a_n}$ where* $\operatorname{cotype}(a_0,\dots,a_n) = 0$ *and let $\mathfrak{p}_i = \langle x_i \rangle \lhd B$. For each $i = 0,\dots,n$, $\mathfrak{p}_i \in X^{(1)}$ and $e_{\mathfrak{p}_i} = 1$.*

Proof. As in Remark 1.3.2, we let $w_i = \deg(x_i)$ for each $i \in \{0,\dots,n\}$. Define $I_i = \{0,\dots,n\}\setminus\{i\}$. It is clear that $\mathfrak{p}_i \in X^{(1)}$. Observe that $\frac{x_i}{1}$ is a generator of the maximal ideal $\mathfrak{p}_i B_{\mathfrak{p}_i}$ of the local ring $B_{\mathfrak{p}_i}$. Since $\operatorname{cotype}(a_0,\dots,a_n) = 0$, there exists a sequence $\{b_j\}_{j\in I_i} \in \mathbb{Z}^n$ such that $\sum_{j\in I_i} b_j w_j = w_i$. Let $\xi = \frac{x_i}{\prod_{j\in I_i} x_j^{b_j}}$. Then ξ is homogeneous of degree zero and is an element of $B_{(\mathfrak{p}_i)} \subset \operatorname{HFrac}(B) = K(X)$. Since $v^{\Omega}_{\mathfrak{p}_i}(\xi) = v^{\Omega}_{\mathfrak{p}_i}(x_i) = 1$ (because $\prod_{j\in I_i} x_j^{b_j}$ is a unit in $B_{\mathfrak{p}_i}$), and since $v^{\Omega}_{\mathfrak{p}_i}(\xi) = e_{\mathfrak{p}_i} v^X_{\mathfrak{p}_i}(\xi)$ (by 4.1.6), it follows that $v^X_{\mathfrak{p}_i}(\xi) = 1$ and $e_{\mathfrak{p}_i} = 1$. □

4.1.9 Definition. Let H be an ample $\mathbb{Q}$-divisor on a projective normal variety X over $\mathbf{k}$. Let $U \subseteq X$ be a cylinder of X. Then, the cylinder $U \subseteq X$ is called *H-polar* if $U = X \setminus \operatorname{supp}(D)$ for some effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D \in \operatorname{Div}(X,\mathbb{Q})$ such that $D \sim sH$ where $s \in \mathbb{Q}^+$.

4.1.10 Remark. Let H and H' be ample $\mathbb{Q}$-divisors on a projective normal variety X over $\mathbf{k}$. Assume that there exist $q, q' \in \mathbb{Q}^+$ such that $qH \sim q'H'$. Then a cylinder U is H-polar if and only if it is H'-polar.

4.1.11 Definition. Let X be a normal variety and let D, D' be $\mathbb{Q}$-divisors on X. We say that D and D' are *$\mathbb{Q}$-linearly equivalent* and write $D \sim_{\mathbb{Q}} D'$ if there exists $n \in \mathbb{N}^+$ such that nD and nD' are linearly equivalent divisors.

4.1.12 Remark. Some authors define an H-polar cylinder of X to be a cylinder $U = X \setminus \operatorname{supp}(D)$ where D is an effective $\mathbb{Q}$-divisor satisfying $D \sim_{\mathbb{Q}} H$. See [KPZ14] and [CPPZ21]. It can be checked that every H-polar cylinder defined in the sense of Definition 4.1.9 is an H-polar cylinder in the sense defined in this remark, and vice versa.

The following is a special case of Lemma 5.20(a) in [CD22].

4.1.13 Lemma. *Let the assumptions and notation be as in Theorem 4.1.2. Fix some choice of homogeneous $T \in \operatorname{Frac} B$ of degree 1 as well as its corresponding $\mathbb{Q}$-divisor H. Then, a cylinder U of X is H-polar if and only if there exist $n \geq 1$ and $h \in B_n \setminus \{0\}$ such that $U = D_+(h)$.*

4.1.14 Theorem. *Let $n \geq 2$ and consider $(a_0,\dots,a_n) \in (\mathbb{N}^+)^{n+1}$ of cotype* 0*. Let $B = B_{a_0,\dots,a_n}$, let $X = \operatorname{Proj}(B)$, and let α be the amplitude of X. Then,*

(a) *$\omega_X \cong \mathcal{O}_X(\alpha)$.*

(b) *Let T be a homogeneous element of $\operatorname{Frac} B$ of degree 1 and let H be the unique $\mathbb{Q}$-divisor of X determined by T as in Theorem 4.1.2. Then $H \in \operatorname{Div}(X)$, $K_X \sim \alpha H$ and K_X is $\mathbb{Q}$-Cartier.*

(c) *If $\alpha = 0$ then $B^{(d)}$ is rigid for all $d \geq 1$.*

(d) *Assume that $\alpha \neq 0$ and define $s = \frac{\alpha}{|\alpha|} \in \{1, -1\}$. Then sK_X is ample and the following are equivalent:*

   (i) *B is not rigid;*

   (ii) *for some $d \geq 1$, $B^{(d)}$ is not rigid;*

   (iii) *there exists a homogeneous element $h \in B \setminus \{0\}$ of positive degree such that the open subset $D_+(h)$ of $\operatorname{Proj} B$ is a cylinder;*

   (iv) *there exists a (sK_X)-polar cylinder of X.*

Proof. First, since $\operatorname{cotype}(a_0, \dots, a_n) = 0$, X is a well-formed quasismooth weighted complete intersection (by Proposition 2.4.25) so (a) follows from Theorem 2.5.11.

We prove (b). By Lemma 4.1.7 (b), if assertion (b) is true for one particular choice of a homogeneous element T in $\operatorname{Frac} B$ of degree 1, then it is true for every choice of such a T. As such, we assume henceforth that $T = \prod_{i=0}^n x_i^{b_i}$ where $b_0, \dots, b_n \in \mathbb{Z}$ are such that $\sum_{i=0}^n b_i \deg(x_i) = 1$. Then, $H^* = \operatorname{div}_\Omega(T)$ by Lemma 4.1.7 (a) and $\operatorname{div}_\Omega(T) = \sum_{i=0}^n b_i C^\Omega_{\mathfrak{p}_i}$. Letting $E = \sum_{i=0}^n b_i C^X_{\mathfrak{p}_i} \in \operatorname{Div}(X)$ we obtain that $E^* = \sum_{i=0}^n b_i C^\Omega_{\mathfrak{p}_i} = \operatorname{div}_\Omega(T) = H^*$ (since by Lemma 4.1.8, $e_{\mathfrak{p}_i} = 1$ for all $i = 0, \dots, n$). Since $D \mapsto D^*$ is injective, we get $E = H$, so $H \in \operatorname{Div}(X)$. By (a) together with Lemma 4.1.4, $\mathcal{O}_X(K_X) \cong \omega_X \cong \mathcal{O}_X(\alpha) \cong \mathcal{O}_X(\alpha H)$ and so $K_X \sim \alpha H$ by Proposition 2.2.28 (a). Finally, K_X is $\mathbb{Q}$-Cartier by Proposition 2.5.9 (e), proving (b).

For (c), assume that $\alpha = 0$. Then, $B_\alpha = B_0 = \mathbb{C} \neq \{0\}$, so Corollary 3.1.16 implies that B is rigid. By Corollary 3.1.11, $B^{(d)}$ is rigid for all $d \geq 1$. This proves (c).

We prove (d). Let H be as in part (b). Since H is ample by Remark 4.1.3, Remark 2.2.35 (b) implies that rH is ample for every integer $r > 0$. Since $K_X \sim \alpha H$, $sK_X \sim |\alpha| H$ is ample. By Remark 3.1.10, B is saturated in codimension 1. By Theorem 3.1.8, (i), (ii) and (iii) are equivalent. Assume (iii) holds, and let $h \in B \setminus \{0\}$ be a homogeneous element of positive degree such that $D_+(h) \subset X$ is a cylinder. By Lemma 4.1.13, $D_+(h)$ is H-polar and since $sK_X \sim |\alpha| H$ it is also (sK_X)-polar by Remark 4.1.10, so (iii) implies (iv). Conversely, assume (iv) holds. Let U be an (sK_X)-polar cylinder of X. Since $sK_X \sim |\alpha| H$, U is H-polar by Remark 4.1.10. By Lemma 4.1.13, (iii) holds and so (iv) implies (iii). This proves (d). $\square$

4.2 Remaining Open Cases in Dimension 3

We recall the special case of Conjecture 1.2.6 where $n = 3, \mathbf{k} = \mathbb{C}$.

4.2.1 Conjecture. *Let $B_{a_0,a_1,a_2,a_3} = \mathbb{C}[X_0, X_1, X_2, X_3]/\langle X_0^{a_0} + X_1^{a_1} + X_2^{a_2} + X_3^{a_3} \rangle$. Assume that $\min\{a_0, a_1, a_2, a_3\} \geq 2$ and at most one element i of $\{0, 1, 2, 3\}$ satisfies $a_i = 2$. Then B_{a_0,a_1,a_2,a_3} is rigid.*

4.2.2 Notation. See Notation 1.3.8 for the definition of Γ_3. Let $L = \operatorname{lcm}(a_0, a_1, a_2, a_3)$ and let $\alpha = L - \sum_{i=0}^{n} \frac{L}{a_i}$. (See Example 2.5.8.) Define the following sets :

$$\Gamma_3^+ = \left\{ (a_0, a_1, a_2, a_3) \in \Gamma_3 \,:\, \operatorname{cotype}(a_0, a_1, a_2, a_3) = 0 \text{ and } \alpha \geq 0 \right\}$$
$$\Gamma_3^- = \left\{ (a_0, a_1, a_2, a_3) \in \Gamma_3 \,:\, \operatorname{cotype}(a_0, a_1, a_2, a_3) = 0 \text{ and } \alpha < 0 \right\}.$$

4.2.3 Lemma. *To prove Conjecture 4.2.1, it suffices to prove that for each $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$, the ring B_{a_0,a_1,a_2,a_3} is rigid.*

Proof. Proving that B_{a_0,a_1,a_2,a_3} is rigid for each $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$ is exactly the same as proving $P(3, 0)$ (see Notation 1.3.8). Since $P(2)$ holds, the result follows from Theorem 1.3.12 (a). $\square$

4.2.4 Lemma. *Up to a permutation of a_0, a_1, a_2, a_3, the set Γ_3^- consists of the following 4-tuples:*

- (2,3,3,6)
- (2,3,4,12)
- (2,3,5,30)
- (2,3,6,6)
- (2,4,4,4)
- (3,3,3,3)
- (3,3,4,4)
- (3,3,5,5)

Proof. We assume without loss of generality that $a_0 \leq a_1 \leq a_2 \leq a_3$. Let $\sigma = \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. By Example 2.5.8, $\alpha < 0$ if and only if $\sigma > 1$. Let $(a_0, a_1, a_2, a_3) \in \Gamma_3^-$. We will show that (a_0, a_1, a_2, a_3) is in the above list. Since $\sigma > 1$, either $a_0 = 2$ or $a_0 = 3$.

Assume first that $a_0 = 2$. Since $\sigma > 1$ we must have $a_1 = 3, 4$ or 5. The $a_1 = 5$ case is impossible. Indeed, if $a_1 = 5$, then $\frac{1}{a_2} + \frac{1}{a_3} > \frac{3}{10}$, and since $a_2 \leq a_3$, the only possibilities are $a_2 = 5, 6$. But since $\operatorname{cotype}(2, 5, a_2, a_3) = 0$, we must have $a_3 \geq 10$ in which case $\frac{1}{2} + \frac{1}{5} + \frac{1}{a_2} + \frac{1}{a_3} \leq 1$ which is impossible. If $a_1 = 4$, we obtain the $(2, 4, 4, 4)$ case above. It is easy to check that no pair (a_2, a_3) other than $(4, 4)$ satifies $a_2, a_3 \geq 4$, $\operatorname{cotype}(2, 4, a_2, a_3) = 0$ and $\sigma > 1$. If $a_1 = 3$, then for each $a_2 = 3, 4, 5, 6$ the requirement that $\operatorname{cotype}(2, 3, a_2, a_3) = 0$ uniquely determines a_3, and each of these four cases is included in the list above. It is again easy to check that no pair (a_2, a_3) satifies $a_2, a_3 \geq 7$, $\operatorname{cotype}(2, 3, a_2, a_3) = 0$ and $\sigma > 1$. This proves the result in the case where $a_0 = 2$.

Assume now that $a_0 = 3$. Then $\sigma > 1$ implies $a_1 = 3, 4$. If $a_1 = 3$, then $\sigma > 1$ implies that $a_2 = 3, 4, 5$. If $a_2 = 3$ then $\operatorname{cotype}(3, 3, 3, a_3) = 0$ implies $a_3 = 3$, and this case is listed above. If $a_2 = 4$, then $\sigma > 1$ implies $a_3 < 12$ and $\operatorname{cotype}(3, 3, 4, a_3) = 0$ then implies $a_3 = 4$, and this case is also listed above. If $a_2 = 5$, the same type of argument implies $a_3 = 5$; this case too is listed above. If $a_1 = 4$, then $\sigma > 1$ implies that $\frac{1}{a_2} + \frac{1}{a_3} > \frac{5}{12}$, so we must have $(a_2, a_3) \in \{(4, 4), (4, 5)\}$. But both of these cases are impossible since $\operatorname{cotype}(3, 4, a_2, a_3) = 0$. □

4.3 B_{a_0,a_1,a_2,a_3} is rigid for all $(a_0, a_1, a_2, a_3) \in \Gamma_3^+$

In this section we show that if $(a_0, a_1, a_2, a_3) \in \Gamma_3^+$ (recall Notation 4.2.2), then B_{a_0,a_1,a_2,a_3} is rigid.

We require the following result which appears just after Remark 1.1.3 in [CPW16b]. Recall that a variety with at most quotient singularities is normal. (See Remark 2.3.6.)

4.3.1 Proposition. *If X is a rational projective surface with quotient singularities and pseudo-effective canonical divisor K_X, then X does not contain a cylinder.*

4.3.2 Corollary. *Let $X = X_{f_1,\dots,f_r} \subseteq \mathbb{P}(w_0, \dots, w_n)$ be a well-formed quasismooth weighted complete intersection and let $B = \mathbb{C}_{w_0,\dots,w_n}[X_0, \dots, X_n]/\langle f_1, \dots, f_r\rangle$. Assume that $\dim X = 2$ and that K_X is pseudo-effective. Then $B^{(d)}$ is rigid for every $d \geq 1$.*

Proof. By Proposition 2.5.9, X is normal and has cyclic quotient singularities. Suppose that there exists $d \geq 1$ such that $B^{(d)}$ is not rigid. By Theorem 3.1.8, there exists a homogeneous element $h \in B \setminus \{0\}$ of positive degree such that the open subset $D_+(h)$ of $\operatorname{Proj} B$ is a cylinder; so $X = \operatorname{Proj} B$ contains a cylinder and hence is birationally ruled. By Theorem 3.1.3, X is rational. Since K_X is pseudo-effective, Proposition 4.3.1 implies that X does not contain a cylinder, a contradiction. □

4.3.3 Corollary. *Let $(a_0, a_1, a_2, a_3) \in \Gamma_3^+$ and let $B = B_{a_0,a_1,a_2,a_3}$. Then $B^{(d)}$ is rigid for every $d \geq 1$.*

Proof. Since $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$, Proposition 2.4.25 implies that $X = \operatorname{Proj}(B)$ is a well-formed quasismooth weighted complete intersection; so by Corollary 4.3.2, it suffices to verify that K_X is pseudo-effective. Theorem 4.1.14 (using the notation of said theorem) implies that $H \in \operatorname{Div}(X)$, H is ample, and $K_X = \alpha H$. By assumption, $\alpha \geq 0$. The divisor 0 is pseudo-effective, so if $\alpha = 0$ then K_X is pseudo-effective. (Note that the $\alpha = 0$ case also follows from Theorem 4.1.14 (c).) If $\alpha > 0$ then $K_X = \alpha H$ is an ample $\mathbb{Q}$-Cartier divisor (by Theorem 4.1.14 (d)), so K_X is pseudo-effective by Remark 2.2.35. □

As an aside, we note that rational varieties with ample canonical divisor do not appear very frequently in the literature. Examples of such varieties can be found in [Kol08] (see Theorem 39) and in [HK12]. With this in mind we include two results. The first, Proposition 4.3.4, is due to Michela Artebani. Since the result is not published, we include a proof which is a slight reformulation of her original proof so that it is consistent with the language and techniques discussed in this thesis. The second, Corollary 4.3.5, follows almost immediately and gives further examples of rational varieties with ample canonical divisor.

4.3.4 Proposition. *Let $a, c \in \mathbb{N}^+$ be such that $\gcd(a, c) = 1$ and let $B = B_{a,a,c,c}$. Then $\operatorname{Proj} B$ is rational.*

Proof. We write $B = \mathbb{C}[x_0, x_1, x_2, x_3]$ where x_i is the canonical image of X_i in B. Consider the elements $t = \frac{x_0}{x_1}$, $u = \frac{x_1^a}{x_3^c}$ and $v = \frac{x_2}{x_3}$ of the function field K of $\operatorname{Proj} B$. We claim that

$$B_{(x_3)} \subseteq \mathbb{C}(t, u, v) \subseteq K, \tag{4}$$

where the second inclusion is obvious. To prove the first inclusion, note that $B_{(x_3)}$ is generated as a $\mathbb{C}$-algebra by elements of the form $\frac{x_0^i x_1^j x_2^k}{x_3^l}$ such that $i, j, k, l \in \mathbb{N}$ and $ci + cj + ak = al$. Since $a \mid c(i+j)$ and $\gcd(a, c) = 1$, we have $a \mid (i + j)$. Then $i + j = ma$ for some $m \in \mathbb{N}$, so $j = ma - i$. As $ci + cj + ak = al$ gives $l = cm + k$, we then have

$$\frac{x_0^i x_1^j x_2^k}{x_3^l} = \frac{x_0^i x_1^{ma-i} x_2^k}{x_3^{cm+k}} = \left(\frac{x_0}{x_1}\right)^i \left(\frac{x_1^a}{x_3^c}\right)^m \left(\frac{x_2}{x_3}\right)^k = t^i u^m v^k \in \mathbb{C}(t, u, v). \tag{5}$$

This proves (4). Since $\operatorname{Frac}(B_{(x_3)}) = K$, it follows that $K = \mathbb{C}(t, u, v)$. Note that $x_0^a + x_1^a + x_2^c + x_3^c = 0$ in B; dividing this by x_3^c gives $t^a u + u + v^c + 1 = 0$, then $u = -\frac{v^c+1}{t^a+1} \in \mathbb{C}(t, v)$, so $K = \mathbb{C}(t, v)$ and $\operatorname{Proj} B$ is rational. □

4.3.5 Corollary. *Let $a, c \in \mathbb{N}^+$ be such that $\gcd(a, c) = 1$ and $\frac{1}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{c} < 1$. Let $X = \operatorname{Proj} B_{a,a,c,c}$. Then, X is a normal rational surface with cyclic quotient singularities and ample canonical divisor.*

Proof. By Proposition 4.3.4, X is rational; by Proposition 2.5.9, X has cyclic quotient singularities; by Remark 2.3.6, X is normal; by Theorem 4.1.14 (d), K_X is ample. □

4.4 Some Remarks on Fano Varieties

Recall that the base field of a variety is assumed to be algebraically closed.

4.4.1 Definition. A *Fano variety* is a normal projective variety over $\mathbf{k}$ such that the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

4.4.2 Remark. Definition 4.4.1 encompasses both the definitions of a *Fano variety* and a *singular Fano variety* in Definition 2.1.1 of [IP99]. Since we are mostly concerned with singular Fano varieties, we use the term "Fano variety" to include both singular and nonsingular normal projective varieties over $\mathbf{k}$ with ample $\mathbb{Q}$-Cartier anticanonical divisor $-K_X$.

4.4.3 Definition. A *del Pezzo surface* X is a normal projective surface over $\mathbf{k}$ with at most quotient singularities such that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

4.4.4 Example. A del Pezzo surface is a Fano variety.

4.4.5 Definition. The *degree* of a del Pezzo surface X is the self-intersection number of the canonical divisor K_X.

4.4.6. We require Definition 4.4.7 below to properly state Theorem 4.4.9 and Proposition 4.5.4. Since the theory of singularities of pairs is used only in these two special cases, we have chosen to omit what it means for a pair (X, D) to be "log canonical" as well as what it means for X to have "Kawamata log terminal" singularities. We refer the reader to Section 2.3 of [KM98] for the background material on the topic.

The following appears as Definition 1.1 in [CPW14], as the $G = \{1\}$ case in Definition 1.1 of [Che08], and under the name of Tian's α-invariant in 1.25 of [CPPZ21].

4.4.7 Definition. Let X be a Fano variety. The *log canonical threshold of X is*

$$\mathbf{lct}(X) = \sup\left\{ \lambda \in \mathbb{Q} \,:\, (X, \lambda D) \textit{ is log canonical for any effective } \mathbb{Q}\textit{-divisor } D \sim_{\mathbb{Q}} -K_X. \right\}$$

4.4.8 Remark. Let X be a Fano variety. The *log canonical threshold* of X, denoted $\mathbf{lct}(X)$, and Tian's *α-invariant* of X, denoted $\alpha(X)$ are used interchangeably in the literature. For example, the definition of the α-invariant in Section 1.25 of [CPPZ21] is exactly the definition of the log canonical threshold in many other sources. See [Che08], [CS08], and [KP15] for instance.

The following is Theorem 1.26 of [CPPZ21].

4.4.9 Theorem. *Let X be a Fano variety that has at most Kawamata log terminal singularities. If $\mathbf{lct}(X) \geq 1$, then X does not contain a $-K_X$-polar cylinder.*

4.5 $B_{3,3,4,4}$ and $B_{3,3,5,5}$ are rigid.

4.5.1 Proposition. *Let $B = B_{a_0,a_1,a_2,a_3}$, let $X = \operatorname{Proj} B$ and let α be the amplitude of X. If $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$ and $\alpha < 0$ then the following hold:*

(a) *X is a del Pezzo surface.*

(b) *If there exists $d \in \mathbb{N}^+$ such that $B^{(d)}$ is not rigid, then X contains a $-K_X$-polar cylinder.*

Proof. By Theorem 4.1.14 (d) and (b), $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. Since X has quotient singularities (by Proposition 2.5.9), X is a del Pezzo surface. This proves (a). Part (b) follows from the equivalence of (ii) and (iv) in Theorem 4.1.14 (d). $\square$

Since the dual graph of the resolution locus of a cyclic quotient singularity is a chain, the following remark follows from Theorem 4.17 in [KM98].

4.5.2 Remark. Cyclic quotient surface singularities are Kawamata log terminal. Consequently, every del Pezzo surface is a Fano variety with Kawamata log terminal singularities.

4.5.3 Remark. If $B = B_{3,3,4,4}$ or $B = B_{3,3,5,5}$, then Proposition 4.5.1 and Remark 4.5.2 imply that $X = \operatorname{Proj} B$ is a Fano variety with at most Kawamata log terminal singularities.

The following combines Lemmas 4.1 and 5.1 in [CPS10].

4.5.4 Proposition. *Let $X = \operatorname{Proj} B_{3,3,5,5}$ and let $Y = \operatorname{Proj} B_{3,3,4,4}$. Then $\mathbf{lct}(X) = 2$ and $\mathbf{lct}(Y) = 1$.*

4.5.5 Proposition. *Let $B = B_{3,3,c,c}$ where $c = 4$ or 5, and let $X = \operatorname{Proj} B$. Then X does not contain a $-K_X$-polar cylinder. Moreover, for every $d \geq 1$, $B^{(d)}$ is rigid.*

Proof. By Remark 4.5.3, X is a Fano variety with Kawamata log terminal singularities. By Proposition 4.5.4, $\mathbf{lct}(X) \geq 1$. By Theorem 4.4.9, X does not contain a $-K_X$-polar cylinder, and so by Proposition 4.5.1 (b), $B^{(d)}$ is rigid for every $d \geq 1$. $\square$

4.6 $B_{3,3,3,3}$, $B_{2,4,4,4}$, $B_{2,3,6,6}$ and $B_{2,3,3,6}$ are rigid.

The following is Theorem 1.5 in [CPW16c].

4.6.1 Theorem. *Let S_d be a del Pezzo surface of degree d with at most Du Val singularities. Then, the surface S_d does not admit a $-K_{S_d}$-polar cylinder if and only if one of the following holds:*

(i) $d = 1$ *and* S_d *allows only singular points of types* A_1, A_2, A_3, D_4 *if any;*

(ii) $d = 2$ *and* S_d *allows only singular points of type* A_1 *if any;*

(iii) $d = 3$ *and* S_d *allows no singular point.*

4.6.2 Lemma. *Let* $S \in \{(3,3,3,3),(2,4,4,4),(2,3,6,6)\}$. *Then* $\operatorname{Proj} B_S$ *is nonsingular.*

Proof. In all three cases the proof is identical. We have that $\operatorname{cotype}(S) = 0$, so Proposition 2.4.25 implies that $\operatorname{Proj} B_S$ is a well-formed quasismooth complete intersection. By Example 2.5.3 and Proposition 2.5.4, we obtain that $\operatorname{Sing}(\operatorname{Proj} B_S) = \varnothing$. □

4.6.3 Lemma. *Let* B *be any of* $B_{3,3,3,3}$, $B_{2,4,4,4}$, $B_{2,3,6,6}$, $B_{2,3,3,6}$. *Let* $X = \operatorname{Proj} B$ *and let* α *be the amplitude of* X. *Then,*

(a) K_X *is Cartier;*

(b) $K_X^2 = \dim_{\mathbb{C}} B_{-\alpha} - 1$

Proof. We prove (a). By Lemma 4.6.2, $\operatorname{Proj} B_{3,3,3,3}$, $\operatorname{Proj} B_{2,4,4,4}$ and $\operatorname{Proj} B_{2,3,6,6}$ are nonsingular and so (a) holds in these three cases since every divisor on a nonsingular variety is Cartier. By Lemma 2.2.47, it suffices to show that $\operatorname{Proj} B_{2,3,3,6}$ is Gorenstein. By Corollary 3.2.23, the singular locus of $\operatorname{Proj} B_{2,3,3,6}$ consists of three singular points of type A_1; since A_1 singularities are Du Val by Remark 2.3.37 and hence are Gorenstein by Definition 2.3.36, $\operatorname{Proj} B_{2,3,3,6}$ is Gorenstein. This proves (a).

For (b), consider a homogeneous $T \in K(X)^*$ of degree 1 and let H be the unique $\mathbb{Q}$-divisor of X determined by T as in Theorem 4.1.2. By Theorem 4.1.14 (b), $H \in \operatorname{Div}(X)$ and $K_X = \alpha H$. By Remark 4.1.3, H is ample. By (a), K_X is Cartier. By Lemma 2.2.51, we have

$$(K_X)^2 = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_X(K_X)) + \chi(\mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X)).$$

We have $\mathcal{O}_X(K_X) = \mathcal{O}_X(\alpha H) \cong \mathcal{O}_X(\alpha)$ by Lemma 4.1.4 and since K_X is Cartier, we obtain $\mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X) \cong \mathcal{O}_X(2K_X) \cong \mathcal{O}_X(2\alpha)$, the first isomorphism by Proposition 2.2.28 (c), the second by Lemma 4.1.4. This gives

$$(K_X)^2 = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_X(\alpha)) + \chi(\mathcal{O}_X(2\alpha)).$$

Define $a_i = \dim_{\mathbb{C}} B_i$ for $i \in \mathbb{Z}$ and note that $a_0 = 1$, and $a_\alpha = a_{2\alpha} = 0$ since $\alpha < 0$. Applying Lemma 2.5.12, we find

$$\begin{aligned} \chi(\mathcal{O}_X) &= a_0 + a_\alpha = 1, \\ \chi(\mathcal{O}_X(\alpha)) &= a_\alpha + a_0 = 1, \\ \chi(\mathcal{O}_X(2\alpha)) &= a_{2\alpha} + a_{-\alpha} = a_{-\alpha}. \end{aligned}$$

It follows that $K_X^2 = a_{-\alpha} - 1$ proving (b). □

4.6.4 Corollary. *Let* $X = \operatorname{Proj} B_{3,3,3,3}$. *Then,*

(a) *X is a smooth degree 3 del Pezzo surface;*

(b) *$(B_{3,3,3,3})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$.*

Proof. The amplitude of X is -1. By Proposition 4.5.1 (a), X is a del Pezzo surface, by Lemma 4.6.2 it is smooth, and by Lemma 4.6.3 (b),

$$K_X^2 = 4 - 1 = 3.$$

This proves (a). It follows from Theorem 4.6.1 that X does not contain a $-K_X$ polar cylinder. By Theorem 4.5.1, $(B_{3,3,3,3})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. This proves (b). □

4.6.5 Corollary. *Let* $X = \operatorname{Proj} B_{2,4,4,4}$. *Then,*

(a) *X is a smooth degree 2 del Pezzo surface;*

(b) *$(B_{2,4,4,4})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$.*

Proof. The amplitude of X is -1. By Proposition 4.5.1 (a), X is a del Pezzo surface, by Lemma 4.6.2 it is smooth, and by Lemma 4.6.3 (b),

$$K_X^2 = 3 - 1 = 2.$$

This proves (a). It follows from Theorem 4.6.1 that X does not contain a $-K_X$ polar cylinder. By Theorem 4.5.1, $(B_{2,4,4,4})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. This proves (b). □

4.6.6 Corollary. *Let* $X = \operatorname{Proj} B_{2,3,6,6}$. *Then,*

(a) *X is a smooth degree 1 del Pezzo surface;*

(b) *$(B_{2,3,6,6})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$.*

Proof. The amplitude of X is -1. By Proposition 4.5.1 (a), X is a del Pezzo surface, by Lemma 4.6.2 it is smooth, and by Lemma 4.6.3 (b),

$$K_X^2 = 2 - 1 = 1.$$

This proves (a). It follows from Theorem 4.6.1 that X does not contain a $-K_X$ polar cylinder. By Theorem 4.5.1, $(B_{2,3,6,6})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. This proves (b). □

4.6.7 Corollary. *Let* $X = \operatorname{Proj} B_{2,3,3,6}$. *Then,*

(a) *X is a degree 2 del Pezzo surface whose three singular points are A_1 singularities;*

(b) $(B_{2,3,3,6})^{(d)}$ *is rigid for all* $d \in \mathbb{N}^+$.

Proof. The amplitude of X is -2. By Proposition 4.5.1 (a), X is a del Pezzo surface, by Corollary 3.2.23, the singular points of X are three A_1 singularities and by Lemma 4.6.3 (b),

$$K_X^2 = 3 - 1 = 2.$$

This proves (a). It follows from Theorem 4.6.1 that X does not contain a $-K_X$ polar cylinder. By Theorem 4.5.1, $(B_{2,3,3,6})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. This proves (b). □

4.6.8 Corollary. *Let* $(a_0, a_1, a_2, a_3) \in \Gamma_3^-$ *and assume that* (a_0, a_1, a_2, a_3) *is not a permutation of* $(2, 3, 4, 12)$ *or* $(2, 3, 5, 30)$. *Then* $(B_{a_0,a_1,a_2,a_3})^{(d)}$ *is rigid for all* $d \in \mathbb{N}^+$.

Proof. This follows from Lemma 4.2.4, Proposition 4.5.5, and Corollaries 4.6.7, 4.6.6, 4.6.4 and 4.6.5. □

4.6.9 Corollary. *Let* $(a_0, a_1, a_2, a_3) \in \Gamma_3$ *be such that* $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$ *and assume that* (a_0, a_1, a_2, a_3) *is not a permutation of* $(2, 3, 4, 12)$ *or* $(2, 3, 5, 30)$. *Then* $(B_{a_0,a_1,a_2,a_3})^{(d)}$ *is rigid for all* $d \in \mathbb{N}^+$.

Proof. Since $(a_0, a_1, a_2, a_3) \in \Gamma_3$ satisfies $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$, it follows that $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$ (see Notation 4.2.2). If $(a_0, a_1, a_2, a_3) \in \Gamma_3^+$, Corollary 4.3.3 gives that $(B_{a_0,a_1,a_2,a_3})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. Otherwise, since (a_0, a_1, a_2, a_3) is not a permutation of $(2, 3, 4, 12)$ or $(2, 3, 5, 30)$, Corollary 4.6.8 gives that $(B_{a_0,a_1,a_2,a_3})^{(d)}$ is rigid for all $d \in \mathbb{N}^+$. □

4.6.10 Corollary. *If* $B_{2,3,4,12}$ *and* $B_{2,3,5,30}$ *are rigid, then Conjecture 4.2.1 is true.*

Proof. By Lemma 4.2.3, to prove Conjecture 4.2.1, it suffices to prove that B_{a_0,a_1,a_2,a_3} is rigid for all $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$. Since Corollaries 4.3.3 and 4.6.8 imply rigidity of B_{a_0,a_1,a_2,a_3} for all $(a_0, a_1, a_2, a_3) \in \Gamma_3^+ \cup \Gamma_3^-$ except for when (a_0, a_1, a_2, a_3) is a permutation of $(2, 3, 4, 12)$ or $(2, 3, 5, 30)$, the result follows. □

4.6.11 Remark. Unfortunately, the methods discussed in the thesis do not appear to work for the $B_{2,3,4,12}$ and $B_{2,3,5,30}$ cases. We can calculate the degrees and the singularity types of the del Pezzo surfaces $\operatorname{Proj}(B_{2,3,4,12})$ and $\operatorname{Proj}(B_{2,3,5,30})$, but we cannot apply Theorem 4.6.1 because $\operatorname{Proj}(B_{2,3,4,12})$ and $\operatorname{Proj}(B_{2,3,5,30})$ have singular points that are not Du Val. Consequently, in order to prove the rigidity of $B_{2,3,4,12}$ and $B_{2,3,5,30}$ using the techniques described in this section, one must generalize Theorem 4.6.1 to include cases where the del Pezzo surface S_d admits singular points that are not Du Val. As far as we know, this approach seems to be a promising one. In unpublished work, Dubouloz has recently managed to extend the results of Theorem 4.6.1 by applying techniques of birational geometry. While the details still need to be verified, it appears as though these techniques may be able to show that $B_{2,3,4,12}$ is rigid. We are hopeful that with some slight adaptations, the $B_{2,3,5,30}$ case can also be resolved, finally resolving Conjecture 4.2.1.

4.7 Some Cotype 1 Cases

4.7.1. Recall Definition 1.2.12 and consider the following sets T, U (noting that the indices run from 0 to 3).

$$T = \{\, (2,2,a_2,a_3) \,:\, a_2, a_3 \geq 3 \text{ and } \operatorname{cotype}(2,2,a_2,a_3) = 0 \,\}$$
$$U = \{\, (2,a_1,a_2,a_3) \,:\, a_1,a_2,a_3 \geq 3, \operatorname{cotype}(2,a_1,a_2,a_3) = 1 \text{ and } t <^1 (2,a_1,a_2,a_3) \text{ for some } t \in T \,\}$$

We show that if $(2,a_1,a_2,a_3) \in U$, then B_{2,a_1,a_2,a_3} is rigid.

4.7.2 Notation. Using that each $(2,2,a_2,a_3) \in T$ satisfies $\operatorname{cotype}(2,2,a_2,a_3) = 0$, we may write $T = T_1 \cup T_2 \cup T_3$ where

$$T_1 = \{\, (2,2,a_2,a_2) \,:\, a_2 \geq 4 \text{ and } a_2 \text{ is even} \,\}$$
$$T_2 = \{\, (2,2,a_2,2a_2) \,:\, a_2 \geq 3 \text{ and } a_2 \text{ is odd} \,\}$$
$$T_3 = \{\, (2,2,a_2,a_2) \,:\, a_2 \geq 3 \text{ and } a_2 \text{ is odd} \,\}.$$

Consequently, we may write $U = U_1 \cup U_2 \cup U_3$ where

$$U_1 = \{\, u \in U \,:\, t <^1 u \text{ for some } t \in T_1 \,\}$$
$$U_2 = \{\, u \in U \,:\, t <^1 u \text{ for some } t \in T_2 \,\}$$
$$U_3 = \{\, u \in U \,:\, t <^1 u \text{ for some } t \in T_3 \,\}.$$

4.7.3 Remark. Let X be a nonsingular projective curve. Using 2.2.43 (b) together with the fact that $(\mathcal{O}_X)^\vee \cong \mathcal{O}_X$ gives $p_g(X) = h^1(X, \mathcal{O}_X)$.

4.7.4 Definition. A $\mathbf{k}$-domain is said to be *rational over* $\mathbf{k}$ if its field of fractions is rational over $\mathbf{k}$.

4.7.5 Lemma. *Let $a,b,c \geq 2$ be such that $c = \operatorname{lcm}(a,b)$ and $(a,b,c) \neq (2,2,2)$. Let $B = B_{a,b,c}$. Then $X = \operatorname{Proj}(B_{a,b,c})$ is not rational, and consequently $B_{a,b,c}$ is not rational. In particular,*

- *if $c \geq 4$ is even, then $B_{2,c,c}$ is not rational;*
- *if $c \geq 3$ is odd, then $B_{2,c,2c}$ is not rational.*

Proof. Let $B = B_{a,b,c}$ be such that $c = \operatorname{lcm}(a,b)$ and $(a,b,c) \neq (2,2,2)$. The condition $c = \operatorname{lcm}(a,b)$ implies that $\operatorname{cotype}(a,b,c) = 0$. Since $(a,b,c) \neq (2,2,2)$ and $c = \operatorname{lcm}(a,b)$ it follows that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$ and so by Example 2.5.8, $\alpha \geq 0$. By Corollary 3.1.17, $B_\alpha \neq 0$, and by Corollary 3.1.16, X is not birationally ruled, and hence is not rational. By Remark 1.1.21 and Corollary 1.1.19, $\operatorname{Spec} B$ is not rational and so neither is B.

∎

The following remark appears in Example IV.1.3.5 in [Har77].

4.7.6 Remark. Any smooth, projective, rational curve over $\mathbb{C}$ is isomorphic to $\mathbb{P}^1$.

4.7.7 Proposition. *Let B be a 3-dimensional, normal, $\mathbb{N}$-graded $\mathbb{C}$-domain. Assume that* $\operatorname{Proj} B$ *is rational. For every $D \in \operatorname{HLND}(B) \setminus \{0\}$, we have*

(a) $\ker D$ *is rational;*

(b) $\operatorname{Proj}(\ker(D)) \cong \mathbb{P}^1$.

Proof. Part (a) follows immediately from Proposition 1.1.23. For (b), let $D \in \operatorname{HLND}(B) \setminus \{0\}$ and let $A = \ker(D)$. By Proposition 1.1.14 and Corollary 1.1.16, A is a normal affine $\mathbb{C}$-domain. Since A is 2-dimensional (by Theorem 1.1.5 and Theorem I.1.8A in [Har77]), Proposition 2.2.11 implies that $\operatorname{Proj} A$ is a normal rational curve. Since a normal curve is nonsingular, the result follows from Remark 4.7.6. □

See Notation 1.3.8 for the definition of Γ_n.

4.7.8 Lemma. *Let $(a_0, a_1, a_2, a_3) \in \Gamma_3$ be such that $a_1 \nmid \operatorname{lcm}(a_0, a_2, a_3)$. If B_{a_0,a_1,a_2,a_3} is not rigid, then*

$$\text{for each } \lambda \in \mathbb{C}, \quad \mathbb{C}[X_0, X_2, X_3]/\langle X_0^{a_0} + X_2^{a_2} + X_3^{a_3} + \lambda\rangle \text{ is rational.}$$

Proof. Since B_{a_0,a_1,a_2,a_3} is not rigid, there exists $D \in \operatorname{HLND}(B_{a_0,a_1,a_2,a_3}) \setminus \{0\}$ by Lemma 1.1.13. Moreover, we may (and shall) choose D to be irreducible. Also note that $\operatorname{Proj} B_{a_0,a_1,a_2,a_3}$ is rational by Corollary 3.1.15.

The condition $a_1 \nmid \operatorname{lcm}(a_0, a_2, a_3)$ implies that x_1 is a $\mathbb{Z}$-critical element of B_{a_0,a_1,a_2,a_3}, so Lemma 1.3.4 implies that $D^2(x_1) = 0$. If $D(x_1) = 0$ then D induces a nonzero locally nilpotent derivation of $B_{a_0,a_1,a_2,a_3}/\langle x_1\rangle \cong B_{a_0,a_2,a_3}$, which is impossible because $(a_0, a_2, a_3) \in \Gamma_2$. So, x_1 is a local slice of D and so is $x_1 - c$ for any $c \in \mathbb{C}$. Since $B_{a_0,a_1,a_2,a_3}/\langle x_1 - c\rangle \cong \mathbb{C}[X_0, X_2, X_3]/\langle X_0^{a_0} + X_2^{a_2} + X_3^{a_3} + c^{a_1}\rangle$ are domains, their prime spectra are birationally equivalent to $\operatorname{Spec}(\ker(D))$ by Corollary 1.1.25. Since $\operatorname{Spec}(\ker(D))$ is birationally equivalent to $\operatorname{Proj} B_{a_0,a_1,a_2,a_3}$ by Proposition 1.1.23, and since $\operatorname{Proj} B_{a_0,a_1,a_2,a_3}$ is rational, we obtain that $\mathbb{C}[X_0, X_2, X_3]/\langle X_0^{a_0} + X_2^{a_2} + X_3^{a_3} + c^{a_1}\rangle$ is rational for any $c \in \mathbb{C}$. For any λ, setting $c = \sqrt[a_1]{\lambda}$ gives the result. □

See Notation 4.7.2 for the definitions of U_1, U_2 and U_3.

4.7.9 Remark. Note that if $(a_0, a_1, a_2, a_3) \in U$ then $a_0 = 2$, so the statements of Propositions 4.7.10, 4.7.11 and Corollary 4.7.12 below capture all 4-tuples (a_0, a_1, a_2, a_3) in $U_1 \cup U_2$, U_3, and U respectively.

4.7.10 Proposition. *If $(2, a_1, a_2, a_3) \in U_1 \cup U_2$, then B_{2,a_1,a_2,a_3} is rigid.*

Proof. Let $(2, a_1, a_2, a_3) \in U_1 \cup U_2$. Observe that $(2, a_1, a_2, a_3) \in \Gamma_3$ and $a_1 \nmid \operatorname{lcm}(2, a_2, a_3)$; so, by Lemma 4.7.8 with $\lambda = 0$, it suffices to show that $\mathbb{C}[X_0, X_2, X_3]/\langle X_0^2 + X_2^{a_2} + X_3^{a_3} \rangle$ is not rational, i.e., that B_{2,a_2,a_3} is not rational. Since $a_3 = \operatorname{lcm}(2, a_2)$ and $(2, a_2, a_3) \neq (2, 2, 2)$, Lemma 4.7.5 implies that B_{2,a_2,a_3} is not rational. □

4.7.11 Proposition. *If $(2, a_1, a_2, a_3) \in U_3$, then B_{2,a_1,a_2,a_3} is rigid.*

Proof. Let $(2, a_1, a_2, a_3) \in U_3$. Observe that $(2, a_1, a_2, a_3) = (2, a_1, a_2, a_2)$ with $a_2 \geq 3$ odd. Since $(2, a_1, a_2, a_2) \,{}^1{>}\, (2, 2, a_2, a_2) <^1 (2, 4, a_2, a_2)$, it suffices (by Proposition 1.3.10 (b)) to show that $B_{2,4,a_2,a_2}$ is rigid. We will show (equivalently) that $B_{2,a_2,a_2,4}$ is rigid. Since a_2 is odd, $4 \nmid \operatorname{lcm}(2, a_2, a_2)$, so by Proposition 1.3.7, it suffices to show that $B_{2,a_2,a_2,4a_2}$ is rigid. Observe that $\operatorname{cotype}(2, a_2, a_2, 4a_2) = 1$ and that $(2, a_2, a_2, 2a_2) <^3 (2, a_2, a_2, 4a_2)$, so by Proposition 1.3.10 (a), it suffices to show that $B_{2,a_2,a_2,2a_2}$ is rigid. Since $(2, a_2, a_2, 2a_2) \in \Gamma_3$ has cotype 0 and is not a permutation of $(2, 3, 4, 12)$ or $(2, 3, 5, 30)$, this follows from Corollary 4.6.9. □

4.7.12 Corollary. *Let U be as in 4.7.1. If $(2, a_1, a_2, a_3) \in U$, then B_{2,a_1,a_2,a_3} is rigid.*

Proof. Since $U = U_1 \cup U_2 \cup U_3$, the result follows from Propositions 4.7.10 and 4.7.11. □

While the main results of this section are now complete, we apply similar techniques to prove rigidity for another small set of Pham-Brieskorn rings. These results will be used in our summary, Theorem 4.8.3.

Let $U_{1,2} = \big\{ (a_0, a_1, a_2, a_3) \,:\, (2, 3, 4, 12) <^i (a_0, a_1, a_2, a_3) \text{ for some } i \in \{0, 1, 2\} \big\}$.
Let $U_{1,3} = \big\{ (a_0, a_1, a_2, a_3) \,:\, (2, 3, 5, 30) <^i (a_0, a_1, a_2, a_3) \text{ for some } i \in \{0, 1, 2\} \big\}$.

4.7.13 Lemma. *Let $(a, b, c) \in \{(2, 3, 30), (2, 5, 30), (3, 5, 30), (2, 3, 12), (2, 4, 12), (3, 4, 12)\}$. Then, $B_{a,b,c}$ is not rational.*

Proof. We must prove that $\operatorname{Spec} B_{a,b,c}$ is not rational. Let $Y = \operatorname{Spec} B_{a,b,c}$ and let $X = \operatorname{Proj} B_{a,b,c}$. By Remark 1.1.21, $K(Y) = K(X)^{(1)}$ and so by Corollary 1.1.19 it suffices to prove that X is not rational. Using Proposition 3.1.13, note that we have the following isomorphisms

- $\operatorname{Proj} B_{2,3,12} \cong \operatorname{Proj} B_{2,3,6}$
- $\operatorname{Proj} B_{2,5,30} \cong \operatorname{Proj} B_{2,5,10}$
- $\operatorname{Proj} B_{3,5,30} \cong \operatorname{Proj} B_{3,5,15}$
- $\operatorname{Proj} B_{2,3,12} \cong \operatorname{Proj} B_{2,3,6}$

- $\operatorname{Proj} B_{2,4,12} \cong \operatorname{Proj} B_{2,4,4}$

and so $X \cong \operatorname{Proj} B_S$ for some $S \in \{(2,3,6),(2,5,10),(3,5,15),(2,4,4)\}$. Since (by Lemma 4.7.5) $\operatorname{Proj} B_S$ is not rational for any such S, X is not rational. $\square$

4.7.14 Proposition. *If $(a_0, a_1, a_2, a_3) \in U_{1,2} \cup U_{1,3}$, then $B = B_{a_0,a_1,a_2,a_3}$ is rigid.*

Proof. Let $S = (a_0, a_1, a_2, a_3) \in U_{1,2} \cup U_{1,3}$ and observe that for some $i \in \{0, 1, 2\}$, $a_i \nmid \operatorname{lcm}(S_i)$. If B is not rigid, then (setting $\lambda = 0$) Lemma 4.7.8 implies that $B_{a,b,c}$ is rational for some $(a, b, c) \in \{(2,3,30),(2,5,30),(3,5,30),(2,3,12),(2,4,12),(3,4,12)\}$. This contradicts Lemma 4.7.13. $\square$

4.8 Summary in Dimension Three

4.8.1. Let Δ be the set of all tuples (a_0, a_1, a_2, a_3) such that (a_0, a_1, a_2, a_3) is a permutation of an element of $\{\, (2,3,4,12m) \,:\, m \in \mathbb{N}^+ \,\} \cup \{\, (2,3,5,30m) \,:\, m \in \mathbb{N}^+ \,\}$. Observe that $\Delta \subset \Gamma_3$.

4.8.2 Remark. A 4-tuple $S = (a_0, a_1, a_2, a_3)$ is an element of Δ if and only if exactly one of the following hold:

(i) S is a permutation of $(2,3,4,12)$;

(ii) S is a permutation of $(2,3,5,30)$;

(iii) S is a permutation of some tuple $\bar{S}$ which satisfies $(2,3,4,12) <^3 \bar{S}$;

(iv) S is a permutation of some tuple $\bar{S}$ which satisfies $(2,3,5,30) <^3 \bar{S}$;

We obtain the following theorem which summarizes the current state of Conjecture 4.2.1.

4.8.3 Theorem. *Let $B = B_{a_0,a_1,a_2,a_3}$ be a Pham-Brieskorn ring and let Δ be as in 4.8.1.*

(a) *If $(a_0, a_1, a_2, a_3) \notin \Gamma_3$ then B is not rigid.*

(b) *If $(a_0, a_1, a_2, a_3) \in \Delta$, then the rigidity of B is not yet known.*

(c) *If $(a_0, a_1, a_2, a_3) \in \Gamma_3 \setminus \Delta$ then B is rigid.*

(d) *If $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$ and B is rigid, then $B^{(d)}$ is rigid for all $d \geq 1$.*

Proof. If $(a_0, a_1, a_2, a_3) \notin \Gamma_3$, then $a_i = 1$ or $a_i = 2 = a_j$ (where $i \neq j$ and $i, j \in \{0, 1, 2, 3\}$) and so Proposition 1.2.4 shows that B_{a_0,a_1,a_2,a_3} is not rigid. This proves (a). For (b), there is nothing to prove.

We prove (c). Let $(a_0, a_1, a_2, a_3) \in \Gamma_3 \setminus \Delta$. If $\operatorname{cotype}(a_0, a_1, a_2, a_3) \geq 2$, then B_{a_0,a_1,a_2,a_3} is rigid by Theorem 7.2(b) of [DFMJ17] (note this is also statement (e) in the Introduction). If $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 0$, then since $(a_0, a_1, a_2, a_3) \notin \Delta$, (a_0, a_1, a_2, a_3) is not a permutation of $(2, 3, 4, 12)$ or of $(2, 3, 5, 30)$. It follows that B_{a_0,a_1,a_2,a_3} is rigid by Corollary 4.6.9. Thus, it suffices to prove (c) in the special case where $\operatorname{cotype}(a_0, a_1, a_2, a_3) = 1$.

Assume henceforth that $S = (a_0, a_1, a_2, a_3) \in \Gamma_3 \setminus \Delta$ is such that $\operatorname{cotype}(S) = 1$. By Lemma 3.1.14, there exists some $S' = (b_0, b_1, b_2, b_3)$ such that $\operatorname{cotype}(S') = 0$ and $S' <^i S$ for some $i \in \{0, 1, 2, 3\}$. If $S' \notin \Gamma_3$, then (since $S \in \Gamma_3$) either

(α) $b_i = 1$, or

(β) $b_i = 2$, $b_j = 2$ for some $j \neq i$, and $b_k \geq 3$ for both $k \in \{0, 1, 2, 3\} \setminus \{i, j\}$.

In case (α), we have $\gcd(a_i, \operatorname{lcm}(S_i))|b_i$ and so $\gcd(a_i, \operatorname{lcm}(S_i)) = 1$. Let $I_i = \{0, 1, 2, 3\} \setminus \{i\}$. Since $\gcd(a_i, \operatorname{lcm}(S_i)) = 1$ if and only if $\gcd(a_i, \prod_{j\in I_i} a_j) = 1$, it follows that $\gcd(a_i, \prod_{j\in I_i} a_j) = 1$. Consequently, B_{a_0,a_1,a_2,a_3} is rigid by Theorem 8.1 (a) in [FMJ13]. (See also statement (a) in the Introduction.) In case (β), some permutation of S' is a 4-tuple contained in the set

$$T = \{\, (2, 2, c_2, c_3) \,:\, c_2, c_3 \geq 3 \text{ and } \operatorname{cotype}(2, 2, c_2, c_3) = 0 \,\},$$

and so some permutation of S is contained in

$$U = \{\, (2, c_1, c_2, c_3) \,:\, c_1, c_2, c_3 \geq 3, \operatorname{cotype}(2, c_1, c_2, c_3) = 1 \text{ and } t <^1 (2, c_1, c_2, c_3) \text{ for some } t \in T \,\}.$$

Since B_{2,c_1,c_2,c_3} is rigid for all $(2, c_1, c_2, c_3) \in U$ (by Corollary 4.7.12), this proves case (β), and completes the proof of the case where $S' \notin \Gamma_3$.

Finally assume that $S' \in \Gamma_3$. If S' is not a permutation of either $(2, 3, 4, 12)$ or of $(2, 3, 5, 30)$, then since $B_{S'}$ is rigid (by Corollary 4.6.9), so is B_S (by Proposition 1.3.10 (a)). It remains to prove the cases where S' is a permutation of $(2, 3, 4, 12)$ or of $(2, 3, 5, 30)$. Assume that S' is a permutation of $(2, 3, 4, 12)$. Then some permutation of S, denoted $\bar{S}$, satisfies $(2, 3, 4, 12) <^i \bar{S}$ for some $i \in \{0, 1, 2, 3\}$. Since $S \notin \Delta$, it follows from Remark 4.8.2 that $i \neq 3$. Since $i \neq 3$, Proposition 4.7.14 implies $B_{\bar{S}}$ is rigid and hence so is B_S (since S is a permutation of $\bar{S}$). The proof in the case where S' is a permutation of $(2, 3, 5, 30)$ is identical. This proves (c).

Part (d) is the $n = 3$ case of Corollary 3.1.11.

□

4.8.4 Remark. While Theorem 4.8.3 assumes that the base field of B_{a_0,a_1,a_2,a_3} is $\mathbb{C}$, the theorem is valid under more general circumstances. Part (a) is valid for every field $\mathbf{k}$ of characteristic zero such that $\sqrt{-1} \in \mathbf{k}$. Parts (b) - (d) are valid for all fields $\mathbf{k}$ of characteristic zero. For parts (b) and (c), this claim follows immediately from Proposition 1.2.8. Part (d) is due to the fact that Theorem 3.1.8 is true as long as B is finitely generated over any field $\mathbf{k}$ of characteristic zero. See Remark 3.1.9.

4.8.5 Remark. In Section 7 of [DFMJ17], the authors ask if $B_{2,3,3,4}$ and $B_{2,3,3,6}$ are rigid. They also ask if there exists some $D \in \text{LND}(B_{2,3,3,2})$ with $\deg_D(x_3) = 1$. Proposition 4.7.12 shows that $B_{2,3,3,4}$ is rigid. Corollary 4.6.7 shows that $B_{2,3,3,6}$ is rigid. Lastly, since $B_{2,3,3,4}$ is rigid, $|x_3|_{B_{2,3,3,2}} > 1$ by Corollary 3.3 (b) of [FMJ13], and so no $D \in \text{LND}(B_{2,3,3,2})$ satisfies $\deg_D(x_3) = 1$.

4.9 Some Remarks for Higher Dimensions

4.9.1. We conclude by offering some remarks on Conjecture 1.2.6 in the higher dimensional cases (where $n > 3$). First, we recall that because of Theorem 1.3.12, it suffices to prove the conjecture for the cases $B_{a_0,\dots,a_n}$ where $\text{cotype}(a_0, \dots, a_n) = 0$. Again, by Proposition 2.4.25, in exactly these cases $\text{Proj}\, B_{a_0,\dots,a_n}$ is a well-formed quasismooth weighted complete intersection. We also note that canonical sheaf and canonical divisor of $X = \text{Proj}\, B_{a_0,\dots,a_n}$ can still be computed using Theorems 2.5.11 and 4.1.14.

4.9.2 Lemma. *Let $n \geq 4$ and let $X = \text{Proj}\, B_{a_0,\dots,a_n}$. Then $\text{CaCl}(X) \cong \mathbb{Z}$.*

Proof. By Lemma 3.1.14, we may assume that $\text{cotype}(a_0, \dots, a_n) = 0$. We have $\text{CaCl}(X) \cong \text{Pic}(X) \cong \mathbb{Z}$, the first isomorphism by Remarks 2.2.21 and the second by Theorem 3.2.4 of [Dol82] (using that $\dim(X) \geq 3$). □

4.9.3 Definition. A morphism of varieties $f : X \to Y$ is *affine* if the inverse image of every affine open subvariety of Y is affine.

4.9.4 Lemma. *Let $i : U \hookrightarrow X$ be an open immersion of varieties where U is affine. Then i is an affine morphism.*

Proof. Let V be an affine open subvariety of X. Then $i^{-1}(V) = V \cap U$. Since V and U are affine, Exercise II.4.3 in [Har77] implies that $V \cap U$ is affine (noting that varieties are separated schemes). □

4.9.5 Proposition. *Let X be a projective normal $\mathbb{Q}$-factorial variety and suppose that* $\text{CaCl}(X) \cong \mathbb{Z}$*. Let H be any ample $\mathbb{Q}$-divisor on X. Then every cylinder U in X is an H-polar cylinder.*

Proof. Let U be a cylinder in X and recall that U is affine (by definition). By Lemma 4.9.4 the inclusion $i : U \to X$ is an affine morphism so we can apply Corollary 21.12.7 of [Gro67] which (since U is dense in X) implies that $U = X \setminus \text{supp}(D)$ for some effective divisor $D \in \text{Div}(X)$.

Since by assumption X is $\mathbb{Q}$-factorial, we may assume that D is Cartier (replace D by rD for large enough $r \in \mathbb{N}^+$). Let H be any ample $\mathbb{Q}$-divisor of X. We must show that U is H-polar.

If $D = 0$, then $\operatorname{supp}(D) = \varnothing$ and so $X = U$ is both affine and projective. Since X is a projective variety over $\mathbf{k}$, $\mathcal{O}_X(X) = \mathbf{k}$ by Remark 2.2.37 and since X is affine, $X = \operatorname{Spec} \mathcal{O}_X(X) = \operatorname{Spec} \mathbf{k}$ is a point. This contradicts the assumption that $\operatorname{CaCl}(X) \cong \mathbb{Z}$; it follows that $D \neq 0$. Since D is a nonzero effective divisor and (by Remarks 2.2.21) $\operatorname{Pic}(X) \cong \operatorname{CaCl}(X) \cong \mathbb{Z}$, Example 1.2.4 of [Laz04] implies that D is ample.

Choose $s \in \mathbb{N}^+$ such that sH is Cartier. Then both sH and D belong to $\operatorname{CaDiv}(X)$. Since $\operatorname{CaCl}(X) \cong \mathbb{Z}$, there exist $m, n \in \mathbb{Z}$ such that $m > 0$ and $m[sH] = n[D]$, or equivalently, such that $m(sH) \sim nD$. Since $m(sH)$ is ample, so is nD, and consequently $n \neq 0$ by Lemma 2.2.38. In fact, $n > 0$. Indeed, assume that $n < 0$; since D is ample, so is $|n|D$ by Remark 2.2.35; so both nD and $|n|D$ are ample and consequently $nD + |n|D = 0$ is ample (again by Remark 2.2.35), contradicting Lemma 2.2.38. So $n > 0$. Since $m(sH) \sim nD$ where $m, s, n \in \mathbb{N}^+$, Remark 4.1.10 implies that every D-polar cylinder is H-polar. Since U is D-polar, it is therefore H-polar. □

4.9.6 Corollary. *Let $n \geq 4$, and let $B = B_{a_0,\dots,a_n}$ be such that* $\operatorname{cotype}(a_0, \dots, a_n) = 0$. *Then B is not rigid if and only if* $\operatorname{Proj} B$ *contains a cylinder.*

Proof. Let α denote the amplitude of B and let $X = \operatorname{Proj} B$. If $\alpha = 0$, then Corollary 3.1.16 implies that B is rigid and that $\operatorname{Proj} B$ does not contain a cylinder so the result holds in this special case.

Assume now that $\alpha > 0$ (resp. $\alpha < 0$). Then K_X is ample (resp. $-K_X$ is ample) by Theorem 4.1.14 (d). Then B is not rigid if and only if X contains a K_X-polar cylinder (resp. $-K_X$ polar cylinder) if and only if X contains a cylinder. (The first equivalence is by Theorem 4.1.14 (d), the second is by Proposition 4.9.5.) □

We also note the following proposition. We would like to thank the examiner Professor Gurjar for simplifying our proof.

4.9.7 Proposition. *Let $n \geq 4$. Then $B_{a_0,\dots,a_n}$ is a unique factorization domain.*

Proof. Let $X = \operatorname{Spec} B$ where $B = B_{a_0,\dots,a_n}$. Since B is normal it suffices to prove that $\operatorname{Cl}(X) = 0$. Let $\mathfrak{m}$ denote the maximal ideal $\langle x_0, \dots, x_n \rangle \lhd B$. For all $\mathfrak{p} \in \operatorname{Spec} B$ such that $\mathfrak{p} \neq \mathfrak{m}$, $B_\mathfrak{p}$ is regular and hence is a unique factorization domain. Since $B_\mathfrak{m}$ is a local Noetherian complete intersection, Corollaire XI.3.14 of [G$^+$68] implies that $B_\mathfrak{m}$ is a unique factorization domain, and consequently X is locally factorial. By Corollary II.6.16 of [Har77], $\operatorname{Pic}(X) \cong \operatorname{Cl}(X)$. Viewing B as a graded ring, Lemma 5.1 of [Mur69] implies $0 = \operatorname{Pic}(X) \cong \operatorname{Cl}(X)$. □

4.9.8 Remark. Proposition 4.9.7 fails for $n = 2, 3$. See Exercise II.6.5 in [Har77] as well as [Sto84].)

4.9.9. Let $n \geq 4$, suppose $\operatorname{cotype}(a_0, \dots, a_n) = 0$ and let $X = \operatorname{Proj}(B_{a_0,\dots,a_n})$. Similar to the case when $n = 3$, the problem splits quite naturally into the cases where the amplitude α is either negative, zero or positive.

If $\alpha < 0$, then Theorem 4.1.14 implies that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor, so X is a Fano variety. By Corollary 4.9.6, we are motivated to investigate the more general question of existence of cylinders in singular Fano varieties, a question which is generally open in dimension 2 or larger.

If $\alpha = 0$, we know that $B_{a_0,\dots,a_n}$ is rigid by Corollary 3.1.16.

If $\alpha > 0$, then the cases satisfying $(B_{a_0,\dots,a_n})_\alpha \neq 0$ are rigid by Corollary 3.1.16. For the cases where $B_\alpha = 0$ (such as $B_{5,5,5,6,6}$ for example), noting that K_X is an ample $\mathbb{Q}$-Cartier divisor, we are led to investigate the question of which varieties with ample canonical divisors can contain cylinders (again by Corollary 4.9.6). In particular, it is worthwhile investigating to what extent Proposition 4.3.1 can be generalized to higher dimensions.

Chapter 5

On the Rationality of $V_+(X_0^2T^{c-2} + X_1^c + X_2^c + T^c) \subset \mathbb{P}^3$

5.0.1 Notation. Let $\mathbb{C}[X_0, X_1, X_2, T]$ be a polynomial ring with the standard grading. Define $S_c = \mathbb{C}[X_0, X_1, X_2, T]/\langle X_0^2T^{c-2} + X_1^c + X_2^c + T^c\rangle$ where $c \geq 2$ and define $X_c = \operatorname{Proj} S_c$ so that X_c is a hypersurface of $\mathbb{P}^3_{\mathbb{C}}$.

5.0.2 Proposition. *If $c \geq 5$ and c is odd, then X_c is not rational.*

We wanted to prove Proposition 5.0.2 because it has the following consequence:

5.0.3 Proposition. *If $b \geq 3$ is even, $c \geq 5$ is odd and $\gcd(b, c) = 1$, then $B_{2,b,c,c}$ is rigid.*

Proof. Note that $b \nmid \operatorname{lcm}(2, c, c)$; so, to prove that $B_{2,b,c,c}$ is rigid, it suffices (by the case $\lambda = 1$ of Lemma 4.7.8) to show that $\mathbb{C}[X_0, X_2, X_3]/\langle X_0^2 + X_2^c + X_3^c + 1\rangle$ is not rational. Observe that $\operatorname{Spec}(\mathbb{C}[X_0, X_2, X_3]/\langle X_0^2 + X_2^c + X_3^c + 1\rangle) \cong D_+(t) \subset X_c$. By Proposition 5.0.2, X_c is not rational, and so neither is $\mathbb{C}[X_0, X_2, X_3]/\langle X_0^2 + X_2^c + X_3^c + 1\rangle$. So $B_{2,b,c,c}$ is rigid. $\square$

5.0.4 Remark. Note that if $c \in \mathbb{N}^+$ is odd, then $\mathbb{C}[X_0, X_2, X_3]/\langle X_0^2 + X_2^c + X_3^c\rangle$ is rational, and so the $\lambda \neq 0$ case of Lemma 4.7.8 is required in the proof of Proposition 5.0.3.

5.0.5 Remark. Note that Proposition 5.0.3 is a special case of Corollary 4.7.12; using the notation of 4.7.1, we have $(2, b, c, c) \overset{1}{>} (2, 2, c, c) \in T$, so $(2, b, c, c) \in U$. Thus, Proposition 5.0.2 is no longer needed to prove Proposition 5.0.3 or Conjecture 4.2.1. We proved Propositions 5.0.2 and 5.0.3 because at the time, we had not yet discovered Corollary 4.7.12.

In the end, we decided to include this chapter because Proposition 5.0.2 appears to be a new result, and the techniques used in its proof are interesting and instructive. Moreover, Proposition 5.0.3 shows another method of proving rigidity, and this method is particulary interesting in view of Remark 5.0.4. The reader interested only in Conjecture 4.2.1 may skip the rest of this chapter. In the end, we managed to generalize Proposition 5.0.2 to obtain Corollary 5.3.6, which states:

$$\text{for } c\geq 2,\ X_c \text{ is rational if and only if } c\leq 4.$$

5.1 Odd c via resolution of singularities

Our first proof of Proposition 5.0.2 was the one we included in the initially submitted version of the thesis. For a shorter proof, the reader can skip this section and read Section 5.2.

5.1.1 Remark. It is easy to check that for $c\geq 2$, S_c is regular in codimension 1, and so by Corollary 2.2.9, S_c is normal.

5.1.2 Remark. Let $c\geq 4$ and let $X=X_c$. The only singular point of X is $[1:0:0:0]$ and so, since X is a hypersurface of $\mathbb{P}^3$, X is normal by Corollary 2.2.10 (b). By Corollary 2.2.44 (a), the canonical sheaf ω_X exists and coincides with the dualizing sheaf ω_X° of X. Moreover, since S_c is generated by its degree 1 elements as a $\mathbb{C}$-algebra, Proposition II.5.12 of [Har77] implies that for all $n,s,t\in\mathbb{Z}$, $\mathcal{O}_X(n)$ is an invertible sheaf, $\mathcal{O}_X(s)\otimes_{\mathcal{O}_X}\mathcal{O}_X(t)\cong\mathcal{O}_X(s+t)$, and the dual $\mathcal{O}_X(n)^\vee$ of $\mathcal{O}_X(n)$ is $\mathcal{O}_X(-n)$.

5.1.3 Lemma. *Let $c\geq 4$ and let $X=X_c$. Then,*

(a) $\omega_X\cong\mathcal{O}_X(c-4)$*;*

(b) *ω_X is invertible and so every canonical divisor K_X of X is Cartier;*

(c) *there exists an effective canonical divisor K_X of X such that* $\operatorname{supp}(K_X)\cap\operatorname{Sing}(X)=\varnothing$*.*

Proof. First, note that ω_X cannot be obtained from Theorem 2.5.11, because X is not quasismooth. Let $i:X\hookrightarrow\mathbb{P}^3$ be the canonical closed immersion. We have $\omega_{\mathbb{P}^3}\cong\mathcal{O}_{\mathbb{P}^3}(-4)$ (by Example II.8.20.1 of [Har77]) and (viewing X as a divisor of $\mathbb{P}^3$) $\mathcal{O}_{\mathbb{P}^3}(X)\cong\mathcal{O}_{\mathbb{P}^3}(c)$ by Proposition 2.4.10. We then obtain that

$$\omega_{\mathbb{P}^3}\otimes_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{O}_{\mathbb{P}^3}(X)\cong\mathcal{O}_{\mathbb{P}^3}(-4)\otimes_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{O}_{\mathbb{P}^3}(c)\overset{5.1.2}{\cong}\mathcal{O}_{\mathbb{P}^3}(c-4).$$

Since X is a Cartier divisor of $\mathbb{P}^3$, Corollary 2.2.46 (a) then gives $\omega_X\cong i^*(\omega_{\mathbb{P}^3}\otimes_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{O}_{\mathbb{P}^3}(X))\cong i^*(\mathcal{O}_{\mathbb{P}^3}(c-4))\cong\mathcal{O}_X(c-4)$, where we used Proposition II.5.12 (c) of [Har77] for the last isomorphism. This proves (a).

Since $\omega_{\mathbb{P}^3}$ is invertible, (b) follows from Corollary 2.2.46 (b).

For (c), choose an effective divisor $D\in\operatorname{Div}(\mathbb{P}^3)$ of degree $c-4$ such that $\operatorname{supp}(D)\cap\operatorname{Sing}(X)=\varnothing$. (Take $D=(c-4)V_+(X_0)$ for example.) Then $\mathcal{O}_{\mathbb{P}^3}(D)\overset{2.4.10}{\cong}\mathcal{O}_{\mathbb{P}^3}(c-4)\cong\omega_{\mathbb{P}^3}\otimes_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{O}_{\mathbb{P}^3}(X)$, so $i^*\mathcal{O}_{\mathbb{P}^3}(D)\cong i^*(\omega_{\mathbb{P}^3}\otimes_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{O}_{\mathbb{P}^3}(X))\cong\omega_X$. Condition (1) of Definition 2.2.31 gives $\mathcal{O}_X(D|_X)\cong i^*(\mathcal{O}_{\mathbb{P}^3}(D))$, so

$$\mathcal{O}_X(D|_X)\cong i^*(\mathcal{O}_{\mathbb{P}^3}(D))\cong\omega_X,$$

showing that $D|_X$ is a canonical divisor of X. Condition (2) of Definition 2.2.31 gives that $D|_X$ is effective and that $\operatorname{supp}(D|_X)=X\cap\operatorname{supp}(D)$. Finally,

$$\operatorname{supp}(D|_X)\cap\operatorname{Sing}X=X\cap\operatorname{supp}(D)\cap\operatorname{Sing}(X)=X\cap\varnothing=\varnothing$$

proving (c). □

5.1.4 Notation. We denote the n^{th} *triangular number* by T_n where $\mathrm{T}_n=\sum_{i=1}^n i$.

We denote the n^{th} *tetrahedral number* by Te_n where $\mathrm{Te}_n=\sum_{k=1}^n\mathrm{T}_k$.

By convention, we have $\mathrm{T}_n=0=\mathrm{Te}_n$ for all $n\le 0$.

5.1.5 Notation. Let $n,r\in\mathbb{N}$ with $n\ge r$. We write $\binom{n}{r}=\frac{n!}{r!(n-r)!}$.

5.1.6 Remark. We have the following equalities for all $n\ge 1$.

(a) $\mathrm{T}_n=|\{\,(i,j,k)\;:\;i,j,k\in\mathbb{N}\text{ and }i+j+k=n-1\,\}|=\frac{(n)(n+1)}{2}=\binom{n+1}{2}$,

(b) $\mathrm{Te}_n=|\{\,(i,j,k,l)\;:\;i,j,k,l\in\mathbb{N}\text{ and }i+j+k+l=n-1\,\}|=\binom{n+2}{3}$.

5.1.7 Lemma. *Let* $c\in\mathbb{Z}_{\ge 4}$*,* $X=X_c$ *and* $n\in\mathbb{Z}$*. Then,*

(a) $h^0(X,\mathcal{O}_X(n))=\mathrm{Te}_{n+1}-\mathrm{Te}_{n+1-c}$;

(b) $h^1(X,\mathcal{O}_X(n))=0$;

(c) $h^2(X,\mathcal{O}_X(n))=h^0(X,\mathcal{O}_X(-n+c-4))=\mathrm{Te}_{-n+c-3}-\mathrm{Te}_{-n-3}$.

Proof. Let $R=\mathbb{C}[X_0,X_1,X_2,T]$ and let $R=\bigoplus_{m\in\mathbb{Z}}R_m$ denote the standard grading of R (noting that $R_m=0$ for all $m<0$). Observe that $\dim_{\mathbb{C}}R_m=\mathrm{Te}_{m+1}$ for all $m\in\mathbb{Z}$.

Let $F=X_0^2T^{c-2}+X_1^c+X_2^c+T^c\in R$. We abbreviate S_c to S; so $S=R/\langle F\rangle=\bigoplus_{m\in\mathbb{Z}}S_m$, with $S_m=R_m/\langle F\rangle_m$ for all $m\in\mathbb{Z}$. Let $n\in\mathbb{Z}$. If $\varphi:R_{n-c}\to R_n$ denotes the multiplication by F then the image of φ is $\langle F\rangle_n$, so there is an exact sequence of $\mathbb{C}$-linear maps $0\to R_{n-c}\xrightarrow{\varphi}R_n\to S_n\to 0$. Consequently,

$$h^0(X,\mathcal{O}_X(n))=\dim_{\mathbb{C}}S_n=\dim_{\mathbb{C}}R_n-\dim_{\mathbb{C}}R_{n-c}=\mathrm{Te}_{n+1}-\mathrm{Te}_{n+1-c},$$

where the first equality follows from the fact that S is normal (by Remark 5.1.1) together with Exercise II.5.14(a) in [Har77]. This proves (a).

Assertion (b) follows from Exercise III.5.5(c) of [Har77].

We prove (c). Since $\mathcal{O}_X(n)^\vee \overset{5.1.2}{\cong} \mathcal{O}_X(-n)$ and $\omega_X \cong \mathcal{O}_X(c-4)$, we get

$$\begin{aligned} h^2(X, \mathcal{O}_X(n)) &\overset{2.2.44(b)}{=} h^0(X, \mathcal{O}_X(n)^\vee \otimes_{\mathcal{O}_X} \omega_X) = h^0(X, \mathcal{O}_X(-n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(c-4)) \\ &\overset{5.1.2}{=} h^0(X, \mathcal{O}_X(-n+c-4)) \overset{(a)}{=} \mathrm{Te}_{-n+c-3} - \mathrm{Te}_{-n-3}\,. \end{aligned}$$

□

5.1.8 Lemma. *Let $c \in \mathbb{Z}_{\geq 5}$ and $X = X_c$. Then $K_X^2 = c(c-4)^2$.*

Proof. First, K_X is a Cartier divisor and $\mathcal{O}_X(K_X) \cong \omega_X \cong \mathcal{O}_X(c-4)$ by Lemma 5.1.3. Moreover, $\mathcal{O}_X(2K_X) \cong \omega_X \otimes_{\mathcal{O}_X} \omega_X \cong \mathcal{O}_X(c-4) \otimes_{\mathcal{O}_X} \mathcal{O}_X(c-4) \cong \mathcal{O}_X(2c-8)$ by Proposition 2.2.28(c) and Remark 5.1.2. Lemma 2.2.51 gives

$$K_X^2 = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_X(c-4)) + \chi(\mathcal{O}_X(2c-8)).$$

For every $n \in \mathbb{Z}$ and every $i \in \mathbb{N}$, we abbreviate $h^i(X, \mathcal{O}_X(n))$ to $h^i(n)$; since $h^1(n) = 0$ and $h^2(n) = h^0(c-4-n)$ (by Lemma 5.1.7 (c)), we have $\chi(\mathcal{O}_X(n)) = h^0(n) + h^0(c-4-n)$ for all $n \in \mathbb{Z}$. So

$$\begin{aligned} K_X^2 &= h^0(0) + h^0(c-4) - 2(h^0(c-4) + h^0(0)) + h^0(2c-8) + h^0(4-c) \\ &= -h^0(0) - h^0(c-4) + h^0(2c-8) + h^0(4-c) \\ &= h^0(2c-8) - h^0(c-4) - 1 \\ &\overset{5.1.7}{=} \mathrm{Te}_{2c-7} - \mathrm{Te}_{c-7} - \mathrm{Te}_{c-3} + \mathrm{Te}_{-3} - 1 = \mathrm{Te}_{2c-7} - \mathrm{Te}_{c-7} - \mathrm{Te}_{c-3} - 1. \end{aligned}$$

Since the formula $\mathrm{Te}_n = \frac{n(n+1)(n+2)}{6}$ is valid for all $n \geq -2$, we conclude that

$$K_X^2 = \tfrac{(2c-7)(2c-6)(2c-5)}{6} - \tfrac{(c-7)(c-6)(c-5)}{6} - \tfrac{(c-3)(c-2)(c-1)}{6} - 1 = c(c-4)^2.$$

□

The following definition appears on page 10 of [Dim92].

5.1.9 Definitions. Define $\mathbb{C}\{X_1, \ldots, X_n\}$ to be the $\mathbb{C}$-algebra of convergent power series in $X_1, \ldots, X_n$. Suppose $g \in \mathbb{C}[X_1, \ldots, X_n]$ is such that $X = V(g) \subset \mathbb{C}^n$ has an isolated singularity x at the origin. The *Jacobian ideal of* g is $J(g) = \langle \frac{\partial g}{\partial X_1}, \ldots, \frac{\partial g}{\partial X_n} \rangle \lhd \mathbb{C}\{X_1, \ldots, X_n\}$, and the *Milnor number* of $x \in X$ is

$$\mu(X, x) = \dim_{\mathbb{C}} \left(\mathbb{C}\{X_1, \ldots, X_n\}/J(g)\right).$$

See Definition 2.3.32 for the definition of analytical equivalence.

5.1.10 Proposition. *Suppose* (X,x) *and* (Y,y) *are analytically equivalent isolated hypersurface singularities. Then* $\mu(X,x)=\mu(Y,y)$.

Proof. This follows from the remark on page 2 of [Dim92] that "analytically equivalent implies topologically equivalent" together with Proposition 1.2.7 in [Dim92]. □

5.1.11 Notation. We denote $\operatorname{Spec}\left(\mathbb{C}[X_1,X_2,T]/\langle X_1^u+X_2^v+T^w\rangle\right)$ by $U_{u,v,w}$.

5.1.12 Lemma. *Let* $u,v,w\geq 2$ *and let* o *be the singular point of* $U_{u,v,w}$.
Then $\mu(U_{u,v,w},o)=(u-1)(v-1)(w-1)$.

Proof. Let $g=X^u+Y^v+Z^w\in\mathbb{C}[X,Y,Z]$. Then $\mu(U_{u,v,w},o)=\dim_{\mathbb{C}}\left(\mathbb{C}\{X,Y,Z\}/J(g)\right)=\dim_{\mathbb{C}}\left(\mathbb{C}\{X,Y,Z\}/\langle X^{u-1},Y^{v-1},Z^{w-1}\rangle\right)$. Observe that

$$\mathcal{B}=\left\{\,x^iy^jz^k\,:\,0\leq i<u-1,0\leq j<v-1,0\leq k<w-1\,\right\}$$

is a basis for the vector space $\mathbb{C}\{X,Y,Z\}/\langle X^{u-1},Y^{v-1},Z^{w-1}\rangle$. Since $|\mathcal{B}|=(u-1)(v-1)(w-1)$, the proof is complete. □

5.1.13 Remark. Lemma 5.1.12 easily generalizes to higher dimensions.

The following is Proposition 10.12 in [AM69].

5.1.14 Lemma. *Let*

$$0\to M'\to M\to M''\to 0$$

be an exact sequence of finitely generated modules over a Noetherian ring R. *Let* $\mathfrak{a}$ *be an ideal of* R. *Then the sequence of* $\mathfrak{a}$*-adic completions*

$$0\to\widehat{M'}\to\widehat{M}\to\widehat{M''}\to 0$$

is an exact sequence of $\hat{R}$*-modules.*

Recall from Remark 5.1.2 that if $c\geq 4$, then $\operatorname{Sing}(X_c)=\{[1:0:0:0]\}$ is a single point.

5.1.15 Proposition. *Let* $c\in\mathbb{Z}_{\geq 4}$ *and let* x *denote the singular point* $[1:0:0:0]$ *of* X_c.

(a) *The singularities* (X_c,x) *and* $(U_{c,c,c-2},o)$ *are analytically equivalent.*

(b) $\mu(X_c,x)=(c-1)^2(c-3)$.

Proof. (a) Let $U=U_{c,c,c-2}$. By Proposition 2.3.33, it suffices to prove that $\widehat{\mathcal{O}}_{X_c,x}\cong\widehat{\mathcal{O}}_{U,o}$.

Let $A=\mathbb{C}[X_1,X_2,T]/\langle g\rangle$ where $g=T^c+T^{c-2}+X_1^c+X_2^c$. Observe that the affine open subset $D_+(x_0)\subset X_c$ is isomorphic to $\operatorname{Spec}A$. The singular point $x=[1:0:0:0]\in X_c$ corresponds to the origin of $\operatorname{Spec}A$, i.e., to the maximal ideal $\mathfrak{m}=\langle x_1,x_2,t\rangle$ of A. Let $\mathfrak{M}$ be the maximal ideal $\langle X_1,X_2,T\rangle$ of $\mathbb{C}[X_1,X_2,T]$. We have the exact sequence $0\to\langle g\rangle\to\mathbb{C}[X_1,X_2,T]\to A\to 0$ of $\mathbb{C}[X_1,X_2,T]$-modules. By exactness of localization at $\mathfrak{M}$, we get the exact sequence

$$0\to\langle g\rangle_{\mathfrak{M}}\xrightarrow{\varphi}\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}}\to A_{\mathfrak{M}}\to 0$$

of $\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}}$-modules. Since $A_{\mathfrak{M}}=A_{\mathfrak{m}}=\mathcal{O}_{X_c,x}$ and the image of φ is the principal ideal $\langle g\rangle$ of $\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}}$, we may rewrite the above sequence as $0\to\langle g\rangle\to\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}}\to\mathcal{O}_{X_c,x}\to 0$. If $\mathfrak{N}$ denotes the maximal ideal of $\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}}$, then by exactness of $\mathfrak{N}$-adic completion (i.e., by Lemma 5.1.14) we obtain the exact sequence

$$0\to\widehat{\langle g\rangle}\xrightarrow{\psi}(\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}})\widehat{\ }\to\widehat{\mathcal{O}}_{X_c,x}\to 0$$

of $(\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}})\widehat{\ }$-modules. Since $(\mathbb{C}[X_1,X_2,T]_{\mathfrak{M}})\widehat{\ }\cong\mathbb{C}[[X_1,X_2,T]]$ (by Theorems I.5.4A(d) and I.5.5A in [Har77] for instance) and the image of ψ is the principal ideal $\langle g\rangle$ of $\mathbb{C}[[X_1,X_2,T]]$, we conclude that

$$\widehat{\mathcal{O}}_{X_c,x}\cong\mathbb{C}[[X_1,X_2,T]]/\langle g\rangle.$$

By the same argument, we have $\widehat{\mathcal{O}}_{U,o}\cong\mathbb{C}[[X_1,X_2,T]]/\langle X_1^c+X_2^c+T^{c-2}\rangle$. So it suffices to show that $\mathbb{C}[[X_1,X_2,T]]/\langle g\rangle\cong\mathbb{C}[[X_1,X_2,T]]/\langle X_1^c+X_2^c+T^{c-2}\rangle$.

Viewing g as an element of $\mathbb{C}[[X_1,X_2,T]]$, observe that $g=X_1^c+X_2^c+T^{c-2}(1+T^2)$ and that $1+T^2$ is a unit in $\mathbb{C}[[X_1,X_2,T]]$ (see Exercise 1.5 in [AM69] for instance). Since 1 admits a $(c-2)$-th root in $\mathbb{C}$, there exists $\zeta\in\mathbb{C}[[X_1,X_2,T]]$ satisfying $\zeta^{c-2}=1+T^2$. The $\mathbb{C}$-automorphism $\theta:\mathbb{C}[[X_1,X_2,T]]\to\mathbb{C}[[X_1,X_2,T]]$ defined by $\theta(X_1)=X_1$, $\theta(X_2)=X_2$ and $\theta(T)=\zeta T$ satisfies

$$\theta(X_1^c+X_2^c+T^{c-2})=X_1^c+X_2^c+T^{c-2}(1+T^2)=g,$$

and hence induces an isomorphism $\bar{\theta}:\mathbb{C}[[X_1,X_2,T]]/\langle X_1^c+X_2^c+T^{c-2}\rangle\to\mathbb{C}[[X_1,X_2,T]]/\langle g\rangle$. So $\widehat{\mathcal{O}}_{U,o}\cong\widehat{\mathcal{O}}_{X_c,x}$, which proves (a). Furthermore, we have $\mu(X_c,x)=\mu(U_{c,c,c-2},o)$ by part (a) together with Proposition 5.1.10. Then, $\mu(U_{c,c,c-2},o)=(c-1)^2(c-3)$ by Lemma 5.1.12, proving (b). $\square$

We need some results about the topological Euler characteristic.

5.1.16 Definition. Let X be a topological space and let $H_i(X)$ $(i\geq 0)$ be the singular homology groups of X. If the condition the abelian group $H_i(X)$ has finite rank for every $i \geq 0$, and has nonzero rank for finitely many i

is satisfied, we say that the *topological Euler characteristic* $\chi_{\mathrm{top}}(X)$ of X is defined, and we define it by $\chi_{\mathrm{top}}(X) = \sum_{i \geq 0}(-1)^i \operatorname{rank} H_i(X)$.

5.1.17 Remark. Let X be a Zariski-closed subset of a complex algebraic variety. The following properties of χ_{top} are well known. See page 95 of [Ful93] for a reference for the first two properties. In all cases below, the terms "open" and "closed" refer to the Zariski topology of X. Note also that Fulton's definition of "variety" (see page 15 of [Ful93]) does not require irreducibility.

- If W is either an open or closed subset of X, then $\chi_{\mathrm{top}}(W)$ is defined.
- If C is a closed subset of X, then $\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(X \setminus C) + \chi_{\mathrm{top}}(C)$.
- If S_1 and S_2 are closed subsets of X, then $\chi_{\mathrm{top}}(S_1 \cup S_2) = \chi_{\mathrm{top}}(S_1) + \chi_{\mathrm{top}}(S_2) - \chi_{\mathrm{top}}(S_1 \cap S_2)$. Indeed, we have

$$\begin{aligned} \chi_{\mathrm{top}}(S_1 \cup S_2) &= \chi_{\mathrm{top}}(S_1) + \chi_{\mathrm{top}}((S_1 \cup S_2) \setminus S_1) \\ &= \chi_{\mathrm{top}}(S_1) + \chi_{\mathrm{top}}(S_2 \setminus (S_1 \cap S_2)) \\ &= \chi_{\mathrm{top}}(S_1) + \chi_{\mathrm{top}}(S_2) - \chi_{\mathrm{top}}(S_1 \cap S_2), \end{aligned}$$

as required.

We need the following proposition, which is Exercise 5.3.7 in [Dim92] applied to surfaces.

5.1.18 Proposition. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^3$. Then*

$$\chi_{\mathrm{top}}(X) = \frac{1}{d}[(1-d)^4 - 1] + 4.$$

The following is a special case of Corollary 5.4.4 of [Dim92].

5.1.19 Proposition. *Let V be a complex hypersurface of $\mathbb{P}^n$ whose singular locus is a finite set of points $\{a_1, \dots, a_p\}$ and let V_0 be a smooth hypersurface of $\mathbb{P}^n$ of the same degree. Then*

$$\chi_{\mathrm{top}}(V) = \chi_{\mathrm{top}}(V_0) + (-1)^n \sum_{i=1}^{p} \mu(V, a_i).$$

5.1.20 Corollary. *If $c \in \mathbb{Z}_{\geq 4}$ then $\chi_{\mathrm{top}}(X_c) = c^2 - c + 3$.*

Proof. We have $\chi_{\mathrm{top}}(X_c) = \frac{1}{c}[(1-c)^4 - 1] + 4 + (-1)^3\mu(X_c, x)$ by Propositions 5.1.18 and 5.1.19, and $\mu(X_c, x) = (c-1)^2(c-3)$ by Proposition 5.1.15 (b). The result follows by algebraic manipulation. $\square$

5.1.21. Let o be the unique singular point of $U_{u,v,w}$ where $u,v,w\geq 2$. Then, as discussed on pages 710–711 of [Orl79], there is something called "the link of the singularity $(U_{u,v,w},o)$", with notation $K(U_{u,v,w},o)=\{b;g;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r)\}$, where $b\in\mathbb{Z}$ and $g,\alpha_i,\beta_i\in\mathbb{N}$. We only need to know two facts about $K(U_{u,v,w},o)$:

(a) There are simple formulas for calculating $K(U_{u,v,w},o)$ in terms of u,v,w. The formulas are given on page 711 of [Orl79].

(b) The resolution graph of $(U_{u,v,w},o)$ is completely determined by $K(U_{u,v,w},o)$. The Theorem on pages 710–711 of [Orl79] describes how to obtain the graph from the data contained in $K(U_{u,v,w},o)$.

The pairs (α_i,β_i) are called "orbits". We write $(\alpha,\beta)^n$ to denote that there are n copies of (α,β) in the link. For example,

$$\{-3;0;(2,1),(2,1),(2,1),(3,1),(3,1)\}=\{-3;0;(2,1)^3,(3,1)^2\}.$$

The $(1,0)$ orbits are called "trivial", and must be deleted from the link before drawing the resolution graph.

Using the methods introduced in [Orl79], we obtain the following proposition.

5.1.22 Proposition. *Consider the singularity* $(U_{c-2,c,c},o)$ *where* $c\geq 5$ *is odd. Then, the link of* $(U_{c-2,c,c},o)$ *(after removing the trivial* $(1,0)$ *orbits) is*

$$K(U_{c-2,c,c},o)=\left\{\frac{-c-1}{2};0;\left(c-2,\frac{c-1}{2}\right)^c\right\}$$

Proof. We follow the Example on page 711 of [Orl79]. We list the values needed to perform the necessary computations.

- $d=c(c-2)$
- $q_0=c,q_1=c-2,q_2=c-2$

The formula for g is given by

$$2g=\frac{d^2}{q_0q_1q_2}-\frac{d\gcd(q_0,q_1)}{q_0q_1}-\frac{d\gcd(q_1,q_2)}{q_1q_2}-\frac{d\gcd(q_0,q_2)}{q_0q_2}+\frac{\gcd(d,q_0)}{q_0}+\frac{\gcd(d,q_1)}{q_1}+\frac{\gcd(d,q_2)}{q_2}-1$$

$$=c-1-c-1+1+1+1-1=0.$$

This shows that $g=0$. Next we must calculate the values of α_i and β_i. There are $\frac{d}{\mathrm{lcm}(q_1,q_2)}=c$ orbits with $\alpha=\gcd(q_1,q_2)=c-2$. For each of these orbits, we solve for β such that $\beta<\alpha=c-2$ and $c\beta \equiv 1 \mod (c-2)$. Since c is odd, $\beta=\frac{c-1}{2}$. This gives us the c required orbits of form $(c-2,\frac{c-1}{2})$. By permuting the indices, we find there are two other $(1,0)$ orbits which (as mentioned in the statement of the proposition) are omitted from the link.

The formula for b is given by

$$b=\frac{d}{q_0q_1q_2}+\sum_{i=1}^{r}\frac{\beta_i}{\alpha_i}=\frac{1}{c-2}-c\left(\frac{c-1}{2(c-2)}\right)=\frac{-c-1}{2}.$$

This completes the proof.

□

In order to calculate the resolution graph of $(U_{c-2,c,c},o)$, we need to compute the Hirzebruch-Jung continued fractions of

$$\frac{\alpha}{\alpha-\beta}=\frac{c-2}{(c-2-(\frac{c-1}{2}))}=\frac{2c-4}{c-3}=3-\frac{c-5}{c-3}.$$

5.1.23 Proposition. *Let $c\geq 5$ be odd. The Hirzebruch-Jung continued fraction of $\frac{2c-4}{c-3}=3-\frac{c-5}{c-3}$ is $[3,\underbrace{2,\dots,2}_{\frac{c-5}{2}}]$.*

Proof. For $c=5$, the result is clear. For $c\geq 7$, it suffices to prove that $\frac{c-5}{c-3}=\frac{1}{[2,\dots,2]}$, where there are $\frac{c-5}{2}$ 2's in the continued fraction. We proceed by induction on c. For the base case, we set $c=7$, and see clearly that $\frac{2}{4}=\frac{1}{2}$ as required. Now assume $c\geq 7$. We assume the result for $\frac{c-5}{c-3}$ and prove it for $\frac{c-3}{c-1}$. We have

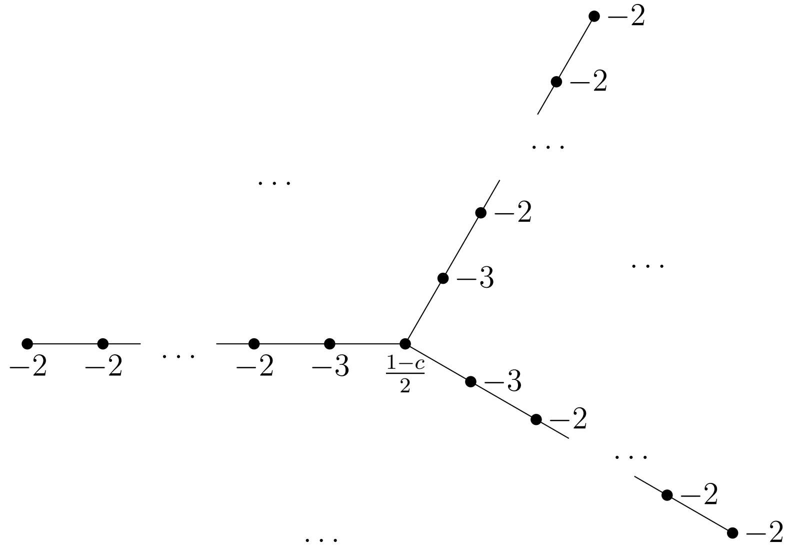
$$\frac{c-5}{c-3}=2-\frac{c-1}{c-3}=2-\frac{1}{\frac{c-3}{c-1}}$$

and so, after rearranging the terms, we find

$$\frac{c-3}{c-1}=\frac{1}{2-\frac{c-5}{c-3}}.$$

This completes the proof. □

5.1.24 Corollary. *The resolution graph of the singularity $o\in U_{c-2,c,c}$ is the following star:*



where

- *all nodes are projective lines,*
- *the central node has self intersection* $\frac{c+1}{2} - c = \frac{1-c}{2}$*,*
- *there are* c *rays made up of* $\frac{c-3}{2}$ *nodes (excluding the central node) with self intersection numbers* $-3, -2, \ldots, -2$ *when considering the rays from the inside-most node outwards.*

Proof. It is easy to check that the $\mathbb{C}^*$-action on $U_{c-2,c,c}$ is "good". (The definition of a *good* $\mathbb{C}^*$ *action* appears on page 708 of [Orl79].) The result then follows from the Theorem on page 710 of [Orl79] combined with Propositions 5.1.22 and 5.1.23. □

5.1.25 Corollary. *Let* $x = [1 : 0 : 0 : 0]$ *be the singular point of* X_c*. The resolution graph of the singularity* (X_c, x) *is given by the star in Corollary 5.1.24.*

Proof. By Corollary 5.1.24, the star is the resolution graph of $(U_{c-2,c,c}, o)$. Since (X_c, x) is analytically equivalent to $(U_{c-2,c,c}, o)$ by Proposition 5.1.15 (a), Proposition 2.3.35 implies that (X_c, x) and $(U_{c-2,c,c}, o)$ have the same resolution graph. □

5.1.26 Remark. In total, the exceptional locus consists of a union of $c(\frac{c-3}{2}) + 1 = \frac{c^2-3c+2}{2}$ projective lines.

5.1.27 Notation. We will order the nodes in the graph which correspond to the projective lines E_i in the following way:

- E_0 corresponds to the root node and $E_0^2 = \frac{1-c}{2}$.

- $E_1, \dots, E_c$ correspond to the innermost nodes of each ray and $E_k^2 = -3$ for each $k = 1, \dots, c$. We say that E_k is in the k^{th} ray.

- E_{pc+k} corresponds to the $(p+1)^{th}$ node from the center on the k^{th} ray, $1 \leq p \leq \frac{c-5}{2}$, $1 \leq k \leq c$ and $E_{pc+k}^2 = -2$.

Note that the self-intersection numbers are due to Corollary 5.1.24.

We need the following fact. (See Exercise V.1.3 from [Har77]).

5.1.28 Lemma. *Let X be an nonsingular projective surface over an algebraically closed field $\mathbf{k}$, and let D be an effective divisor on X. Then $2p_a(D) - 2 = D \cdot (D + K_X)$.*

5.1.29 Definition. Let $E_0, \dots E_n$ be curves on a nonsingular surface. We define the *intersection matrix* M by setting $M_{ij} = E_i \cdot E_j$ where $0 \leq i, j \leq n$.

5.1.30. Fix some $c \geq 5$ odd, and let $X = X_c$. Let $x = [1 : 0 : 0 : 0]$ be the singular point of X. Let $n = \frac{c(c-3)}{2}$. Let $\pi : X' \to X$ be a resolution of singularities with exceptional locus $E = \cup_{i=0}^n E_i$. We now want to calculate $K_{X'}^2$. To do this, we require the following lemma:

5.1.31 Lemma. *With notation as in 5.1.30, there exist $L \in \operatorname{Div}(X')$ and $m_0, \dots, m_n \in \mathbb{Z}$ such that*

$$K_{X'} = L + \sum_{i=0}^{n} m_i E_i, \quad \operatorname{supp}(L) \cap E = \varnothing \quad \textit{and} \quad L^2 = K_X^2.$$

Proof. By Lemma 5.1.3 (c), we can choose K_X so that K_X is effective and $x \notin \operatorname{supp}(K_X)$. Then we can write $K_X = \sum_{j=1}^r a_j C_j$ where $a_1, \dots, a_r \in \mathbb{N}$ and $C_1, \dots, C_r$ are distinct irreducible curves in X such that $x \notin \bigcup_{j=1}^r C_j$. Define $L = \sum_{j=1}^r a_j \pi^{-1}(C_j) \in \operatorname{Div}(X')$ and note that L is effective. Let $U = X \setminus \{x\}$ and $U' = \pi^{-1}(U)$, and note that $\operatorname{supp}(K_X) \subset U$ and $\operatorname{supp}(L) \subset U'$ (so $\operatorname{supp}(L) \cap E = \varnothing$). Since (by Lemma 2.2.42) $K_X = K_X|_U$ is a canonical divisor of U, and since the isomorphism $\pi : U' \to U$ transforms L into K_X, we obtain that L is a canonical divisor of U'. Since K_X and L are effective, Lemma 2.2.53 implies that $L^2 = K_X^2$, where L^2 is computed in X' and K_X^2 in X. Let $D \in \operatorname{Div}(X')$ be any canonical divisor of X'. Then, by Lemma 2.2.42, $D|_{U'}$ is a canonical divisor of U' and hence is linearly equivalent (in U') to L. So there exists $g \in K(U')^* = K(X')^*$ such that $D|_{U'} + \operatorname{div}_{U'}(g) = L$. Then $\big(D + \operatorname{div}_{X'}(g)\big)|_{U'} = L$, which implies that $D + \operatorname{div}_{X'}(g) = L + \sum_{i=0}^n m_i E_i$ for some $m_0, \dots, m_n \in \mathbb{Z}$. Then $L + \sum_{i=0}^n m_i E_i$ is a canonical divisor of X', since it is linearly equivalent to D. $\square$

5.1.32. By Lemma 5.1.31, $K_{X'} = L + \sum_{i=0}^n m_i E_i$ where the E_i are the projective lines that make up the exceptional locus and $m_i \in \mathbb{Z}$. Note also that L is disjoint from $\cup_{i=0}^n E_i$, so $L \cdot E_i = 0$ for all $i=0,\dots,n$. By Lemma 5.1.28, for each $j=0,\dots,n$ we have:

$$2p_a(E_j)-2=E_j(E_j+L+\sum_{i=0}^n m_iE_i)=E_j^2+\sum_{i=0}^n(E_j\cdot E_i)m_i.$$

Since $p_a(E_j)=0$ (E_j is a projective line for all j), we obtain:

$$-2-E_j^2=\sum_{i=0}^n(E_j\cdot E_i)m_i.$$

This gives us a system of $n+1$ equations in $n+1$ unknowns $m_0,\dots,m_n$. Let $\vec{m}=[m_0,\dots,m_n]^T$ and let $\vec{b}=[-2-E_0^2,\dots,-2-E_n^2]^T$. To compute the values of the m_i, we must solve the system $M\vec{m}=\vec{b}$, where $M_{ij}=E_i\cdot E_j$. By Lemma 5.1.31 and the above discussion, we know that this system has at least one solution $\vec{m}\in\mathbb{Z}^{n+1}$. It is well known that the intersection matrix M is negative definite and is hence invertible. (See page 6 of [Mum61] for instance.) So there is exactly one solution $(m_0,\dots,m_n)$ to the system $M\vec{m}=\vec{b}$. Thus, we can determine the m_i and can then compute $K_{X'}^2=L^2+(\sum_{i=0}^n m_iE_i)^2$ (since $L\cdot E_i=0$ for all i).

5.1.33 Corollary. *Let $X_c, E=\cup_{i=0}^n E_i$ be as in 5.1.30. Ordering the E_i as in Notation 5.1.27, the intersection matrix M_c for the projective lines $\{E_i\}_{i=0}^n$ is given by the $(n+1)\times(n+1)$ block matrix*

$$M_c=\left[\begin{array}{c|c|c|c|c|c}
\frac{1-c}{2} & \vec{1} & \vec{0} & \vec{0} & \vec{0}\cdots\vec{0} & \vec{0}\\ \hline
\vec{1} & -3\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0}\cdots\mathbf{0} & \mathbf{0}\\ \hline
\vec{0} & \mathbf{I} & -2\mathbf{I} & \mathbf{I} & \mathbf{0}\cdots\mathbf{0} & \mathbf{0}\\ \hline
\vec{0} & \mathbf{0} & \mathbf{I} & -2\mathbf{I} & \ddots & \vdots\\ \hline
\vec{0} & \mathbf{0} & \mathbf{0} & \ddots & \ddots & \mathbf{I}\\ \hline
\vdots & \vdots & \vdots & \vdots & & \\
\vec{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -2\mathbf{I}
\end{array}\right]$$

where

- *$M_{00}=\frac{1-c}{2}$ is a 1×1 matrix,*
- *$\vec{1},\vec{0}$ in the first row of the block matrix are the $1\times c$ matrices $[1,1,\cdots,1,1]$ and $[0,0,\cdots,0,0]$,*
- *$\vec{1},\vec{0}$ in the first column of the block matrix are the $c\times1$ matrices $[1,1,\cdots,1,1]^T$ and $[0,0,\cdots,0,0]^T$,*
- *the boldface blocks $\mathbf{I},-3\mathbf{I},-2\mathbf{I},\mathbf{0}$ are $c\times c$ matrices.*

Proof. This follows from Notation 5.1.27 together with Corollary 5.1.24. $\square$

5.1.34 Lemma. *Let X_c be as in 5.1.30 and let $\vec{b}$ be as in 5.1.32. Then,* $\vec{b} = [\frac{c-5}{2}, \underbrace{1, \dots, 1}_{c \text{ times}}, 0, \dots, 0]^T$.

Proof. From the discussion in 5.1.32, we have $\vec{b} = [-2-E_0^2, \dots, -2-E_n^2]^T$. For each $i = 0, \dots, n$, the values of E_i^2 are given by Corollary 5.1.24 and the result follows. $\square$

5.1.35 Example. We compute M_c for the cases $c = 5, 7$ and 9 and denote these matrices by M_5, M_7 and M_9 respectively. We have

$$M_5 = \left[\begin{array}{c|ccccc}
-2 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & -3 & 0 & 0 & 0 & 0 \\
1 & 0 & -3 & 0 & 0 & 0 \\
1 & 0 & 0 & -3 & 0 & 0 \\
1 & 0 & 0 & 0 & -3 & 0 \\
1 & 0 & 0 & 0 & 0 & -3
\end{array}\right]$$

$$M_7 = \left[\begin{array}{c|ccccccc|ccccccc}
-3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{array}\right]$$

and

$$M_9 = \left[\begin{array}{c|ccccccccc|ccccccccc|ccccccccc}
-4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{array}\right]$$

5.1.36 Example. Let $X = X_5$. We compute $\vec{b} = [0, 1, 1, 1, 1, 1]^T$. Reducing $[M_5|\vec{b}]$ into reduced row echelon form, we find $m_0 = -5, m_i = -2$ for $i = 1, \dots, c$. It follows that, with notation as in Lemma 5.1.31,

$$K_{X'}^2 = L^2 + (\sum_{i=0}^{n} m_i E_i)^2 \overset{5.1.31}{=} K_X^2 + (\sum_{i=0}^{n} m_i E_i)^2 \overset{5.1.8}{=} 5 + (-10) = -5.$$

5.1.37 Proposition. *Let $M_c, \vec{m}, \vec{b}$ be as in Corollary 5.1.33 and let $p = \frac{c-3}{2}$. The unique solution to the equation $M_c\vec{m} = \vec{b}$ is*

$$\vec{m} = \begin{bmatrix} -c^2+5c-5 \\ \frac{c-3}{2}(3-c) \\ \vdots \\ \frac{c-3}{2}(3-c) \\ \frac{c-5}{2}(3-c) \\ \vdots \\ \frac{c-5}{2}(3-c) \\ \vdots \\ \vdots \\ 2(3-c) \\ \vdots \\ 2(3-c) \\ (3-c) \\ \vdots \\ (3-c) \end{bmatrix}$$

where $m_0 = -c^2+5c-5$, and for each $i = 0, \ldots, c-1$ and each $k = 0, \ldots, p-1$, $m_{n-kc-i} = m_{n-kc} = (k+1)(3-c)$.

Proof. First note that the solution is unique since M_c is negative definite, as discussed in 5.1.32. As such, it suffices to check that $\vec{m}$ satisfies the matrix equation $M_c\vec{m} = \vec{b}$. We leave this verification to the reader. □

5.1.38 Corollary. *With notation as in 5.1.30, $K_{X'}^2 = \frac{25-7c}{2}$.*

Proof. By Lemma 5.1.31, we have $K_{X'} = L + \sum_{i=0}^{n} m_i E_i$ and $\operatorname{supp}(L) \cap E = \varnothing$, so $K_{X'}^2 = L^2 + \left(\sum_{i=0}^{n} m_i E_i\right)^2$ where $L^2 = K_X^2 = c(c-4)^2$ by Lemma 5.1.8.

It remains to compute $\left(\sum_{i=0}^{n} m_i E_i\right)^2$. Let $p = \frac{c-3}{2}$ (so $p \geq 1$). For each $j \in \{1, \ldots, p\}$, define $I_j = \left\{ i \in \mathbb{N} : (j-1)c < i \leq jc \right\}$ and $\Sigma_j = \sum_{i \in I_j} E_i$. Using the values of the m_i given in Proposition 5.1.37, we find

$$\sum_{i=0}^{n} m_i E_i = (-c^2+5c-5)E_0 + (3-c)\left[p\Sigma_1 + (p-1)\Sigma_2 + \cdots + 1\Sigma_p\right].$$

The values of $E_i\cdot E_j$ for all i,j can be deduced from the matrix M_c in Corollary 5.1.33. We have $E_0^2=\frac{1-c}{2}$, $E_0\cdot\Sigma_1=c$, and $E_0\cdot\Sigma_j=0$ for all $j>1$, so

$$\begin{aligned}\left(\sum_{i=0}^n m_iE_i\right)^2 &= (-c^2+5c-5)^2\left(\frac{1-c}{2}\right)+2(-c^2+5c-5)(3-c)pc\\ &\qquad\qquad + (3-c)^2\big[p\Sigma_1+(p-1)\Sigma_2+\cdots+1\Sigma_p\big]^2\\ &= \frac{(c^2-5c+5)(c^3-6c^2+8c+5)}{2}+(3-c)^2\big[p\Sigma_1+(p-1)\Sigma_2+\cdots+1\Sigma_p\big]^2.\end{aligned}$$

We have $E_i^2=-3$ for all $i\in I_1$ and $E_i^2=-2$ for all $i\in I_2\cup\cdots\cup I_p$; for any $j\in\{1,\dots,p\}$, if i_1,i_2 are distinct elements of I_j then $E_{i_1}\cdot E_{i_2}=0$. So $\Sigma_1^2=\sum_{i\in I_1}E_i^2=-3c$ and similarly $\Sigma_j^2=-2c$ for all $j>1$. Also, if j_1,j_2 are distinct elements of $\{1,\dots,p\}$ then $\Sigma_{j_1}\cdot\Sigma_{j_2}$ is equal to c if $|j_1-j_2|=1$ and to 0 otherwise. So

$$\begin{aligned}\big[p\Sigma_1+(p-1)\Sigma_2+\cdots+1\Sigma_p\big]^2 &= \big[p^2\Sigma_1^2+(p-1)^2\Sigma_2^2+\cdots+1^2\Sigma_p^2\big]\\ &\quad +2\big[p(p-1)\Sigma_1\cdot\Sigma_2+(p-1)(p-2)\Sigma_2\cdot\Sigma_3+\cdots+(2\cdot1)\Sigma_{p-1}\cdot\Sigma_p\big]\\ &= \big[-3cp^2-2c(p-1)^2-\cdots-2c\cdot1^2\big]+2c\big[p(p-1)+(p-1)(p-2)+\cdots+(2\cdot1)\big]\\ &= -p^2c+2c\big[(1\cdot2+2\cdot3+\cdots(p-1)p)-(1^2+2^2+\cdots+p^2)\big]\\ &= -p^2c+2c\big[-\tfrac{p(p+1)}{2}\big]=\tfrac{-c^3+5c^2-6c}{2},\end{aligned}$$

and finally $(\sum_{i=0}^n m_iE_i)^2=\frac{(c^2-5c+5)(c^3-6c^2+8c+5)}{2}+(3-c)^2\left(\frac{-c^3+5c^2-6c}{2}\right)=\frac{-2c^3+16c^2-39c+25}{2}$.

Thus, $K_{X'}^2=c(c-4)^2+\left(\frac{-2c^3+16c^2-39c+25}{2}\right)=\frac{25-7c}{2}$. □

5.1.39 Corollary. *Let $\mathcal{E}_c$ be the exceptional locus in the resolution of the singular point of X_c given by the graph in Corollary 5.1.24. Then $\chi_{\mathrm{top}}(\mathcal{E}_c)=\frac{c^2-3c+4}{2}$.*

Proof. Each complex projective line has topological Euler characteristic equal to 2 as it is homeomorphic to a real 2-sphere. Also, the topological Euler characteristic of a point is 1. Using $\chi_{\mathrm{top}}(S_1\cup S_2)=\chi_{\mathrm{top}}(S_1)+\chi_{\mathrm{top}}(S_2)-\chi_{\mathrm{top}}(S_1\cap S_2)$ from Remark 5.1.17, we find by induction that any tree of projectives lines with N irreducible components has topological Euler characteristic equal to $N+1$. Since $\mathcal{E}_c$ has $\frac{c^2-3c+2}{2}$ irreducible components by Remark 5.1.26, the result follows. □

5.1.40 Corollary. *Let $c\geq5$ be odd and let $\pi:X_c'\to X_c$ be as in 5.1.30. Then,*

$$\chi_{\mathrm{top}}(X_c')=\frac{3c^2}{2}-\frac{5c}{2}+4.$$

Proof. Let $U\subset X_c$ be the nonsingular locus of X_c. Using Remark 5.1.17, we have that $\chi_{\mathrm{top}}(U)=\chi_{\mathrm{top}}(X_c)-1$ since χ_{top} of a point equals 1. Let $\mathcal{E}_c$ be as in Corollary 5.1.39. Let $U'=\pi^{-1}(U)$ and note that $\pi|_{U'}: U' \to U$ is an isomorphism. This implies that $\chi_{\text{top}}(U') = \chi_{\text{top}}(U)$ so that

$$\chi_{\text{top}}(X_c') = \chi_{\text{top}}(U') + \chi_{\text{top}}(\mathcal{E}_c) = \chi_{\text{top}}(U) + \chi_{\text{top}}(\mathcal{E}_c) = \chi_{\text{top}}(X_c) - 1 + \chi_{\text{top}}(\mathcal{E}_c)$$

where we used Remark 5.1.17 in the first equality. Since $\chi_{\text{top}}(X_c) = c^2 - c + 3$ by Corollary 5.1.20, and $\chi_{\text{top}}(\mathcal{E}_c) = \frac{c^2-3c+4}{2}$ by Corollary 5.1.39, we obtain

$$\chi_{\text{top}}(X_c') = c^2 - c + 3 - 1 + \frac{c^2-3c+4}{2} = \frac{3c^2}{2} - \frac{5c}{2} + 4.$$

□

The following is Noether's formula. One reference is Section 7.4 of [IF00].

5.1.41 Theorem. *Let S be a nonsingular projective surface over $\mathbb{C}$. Then,*

$$12\chi(S) = K_S^2 + \chi_{\text{top}}(S).$$

5.1.42 Corollary. *Let $c \geq 5$ be odd, and let X_c' be as in 5.1.30. Then,*

(a) $\chi(X_c') = \frac{k(k-1)}{2} + 1$, *where* $k = \frac{c-1}{2} \in \mathbb{N} \setminus \{0,1\}$*;*

(b) X_c *is not rational.*

Proof. We have

$$12\chi(X_c') = K_{X_c'}^2 + \chi_{\text{top}}(X_c') = \frac{25-7c}{2} + \frac{3c^2}{2} - \frac{5c}{2} + 4 = \frac{3c^2-12c+33}{2},$$

where the first equality is due to Theorem 5.1.41 and the second is due to Corollary 5.1.40. This gives $\chi(X_c') = \frac{3c^2-12c+33}{24} = \frac{k(k-1)}{2} + 1$, where $k = \frac{c-1}{2}$. This proves (a).

For (b), note that the Euler characteristic of every smooth projective rational surface is 1 (by Theorem V.6.2 of [Har77] together with the definition of the arithmetic genus). By part (a), $\chi(X_c') > 1$, so X_c' is not rational, and hence neither is X_c. □

5.1.43 Remark. The reader can check that the surface $X_3 \subset \mathbb{P}^3$ is a nonsingular cubic surface, and hence is rational by Example II.8.20.3 in [Har77].

5.2 Odd c via double covers

We preserve the notation for S_c and X_c from the previous section. In this section, we give a second proof of Proposition 5.0.2 by applying results on double covers. The outline of the proof below was provided to us by the examiner Professor Gurjar after reading Section 5.1. Since this proof is shorter and simpler than our initial proof, we include it here.

Recall the following well-known facts:

- If D_1, D_2 are divisors of $\mathbb{P}^2$ then $D_1 \cdot D_2 = \deg(D_1)\deg(D_2)$.
- If D is an effective divisor of $\mathbb{P}^2$ and $d = \deg(D)$ then $p_a(D) = \frac{(d-1)(d-2)}{2}$.
- $\chi(\mathbb{P}^2) = 1$

5.2.1 Proposition. *Let $c \geq 1$ be odd and consider the graded polynomial ring $\mathbf{k}_{\frac{c+1}{2},1,1,1}[W, X, Y, Z]$. Then $B = \mathbf{k}[W, X, Y, Z]/\langle W^2 - (X^c + Y^c + Z^c)Z\rangle = \mathbf{k}[w, x, y, z]$ is an $\mathbb{N}$-graded ring and $B_{(z)} \cong \mathbf{k}[S, T, U]/\langle S^2 - (T^c + U^c + 1)\rangle$.*

Proof. Let $d = \frac{c+1}{2}$. Define $s, t, u \in B_{(z)}$ by $s = w/z^d$, $t = x/z$ and $u = y/z$, and observe that $s^2-(t^c+u^c+1) = 0$. The ring $B_{(z)}$ is spanned (as a $\mathbf{k}$-vector space) by elements $\frac{w^ix^jy^k}{z^{di+j+k}}$ with $i, j, k \in \mathbb{N}$. Since $\frac{w^ix^jy^k}{z^{di+j+k}} = s^it^ju^k$, we have $B_{(z)} = \mathbf{k}[s, t, u]$. So the $\mathbf{k}$-homomorphism $\varphi : \mathbf{k}[S, T, U] \to B_{(z)}$ defined by $\varphi(S) = s$, $\varphi(T) = t$, $\varphi(U) = u$ is surjective. Considering dimensions shows that $\ker\varphi$ is a height 1 prime ideal of $\mathbf{k}[S, T, U]$. Since $S^2-(T^c+U^c+1) \in \ker\varphi$ and is irreducible in $\mathbf{k}[S, T, U]$, we have $\ker\varphi = \langle S^2 - (T^c + U^c + 1)\rangle$ and consequently $B_{(z)} \cong \mathbf{k}[S, T, U]/\langle S^2 - (T^c + U^c + 1)\rangle$. □

5.2.2 Notation. Let $c \geq 1$ be odd and define

$$Y_c = \operatorname{Proj}\left(\mathbb{C}_{\frac{c+1}{2},1,1,1}[X_0, X_1, X_2, X_3]/\langle X_0^2 - (X_1^c + X_2^c + X_3^c)X_3\rangle\right).$$

5.2.3 Lemma. *If $c \geq 3$ is odd, then X_c and Y_c are birationally equivalent.*

Proof. Consider the affine open subsets $V = \operatorname{Spec}(\mathbb{C}[X_0, X_1, X_2]/\langle X_0^2 + X_1^c + X_2^c + 1\rangle \subset X_c$ and (by Proposition 5.2.1) $U = D_+(x_3) = \operatorname{Spec}(\mathbb{C}[X_0, X_1, X_2]/\langle X_0^2 -(X_1^c+X_2^c+1)\rangle \subset Y_c$. It is easy to check that U and V are isomorphic. Consequently, X_c, V, U and Y_c are birationally equivalent. □

We state Proposition 1.2 in [Per81].

5.2.4 Proposition. *Let Y, X be nonsingular projective surfaces and let $\pi : Y \to X$ be a finite morphism of degree 2 with branching curve C. Then*

$$\chi(Y) = 2\chi(X) + \frac{p_a(C) - 1}{2} - \frac{C^2}{8}$$

5.2.5 Proposition. *Let $c \geq 1$ be odd. If Y_c is rational, then $c = 1$ or $c = 3$.*

Proof. Let $Y = Y_c$ and let $C = V_+((X_1^c + X_2^c + X_3^c)X_3) \subset \mathbb{P}^2$ and observe that C is the union of the two curves $C_1 = V_+(X_1^c + X_2^c + X_3^c)$ and $C_2 = V_+(X_3)$. Consider the projection $\varphi_0 : Y_c \to \mathbb{P}^2$, $[x_0 : x_1 : x_2 : x_3] \mapsto [x_1 : x_2 : x_3]$, as defined in 3.2.16. By Remark 3.2.19, φ_0 is a surjective 2:1 finite morphism. Since c is odd, C has even degree in $\mathbb{P}^2$, and so there exists some $B \in \operatorname{Div}(\mathbb{P}^2)$ such $C \in |2B|$. With the terminology of [Per81], C is the "branching curve" of the double covering $\varphi_0 : Y_c \to \mathbb{P}^2$. (Note that Persson's definition of a branching curve does not require irreducibility.)

We have $C_1 \cdot C_2 = \deg(C_1)\deg(C_2) = c$ and $C_1 \cap C_2 = \{ [1 : -\zeta^i : 0] : i = 0, \ldots, c-1 \}$ where ζ is a primitive c^{th} root of unity. Since $C_1 \cap C_2$ consists of c distinct points and $C_1 \cdot C_2 = c$, C has normal crossings. In particular, every singular point of C is a double point and in the terminology of [Per81] is "inessential" (by Proposition 1.8 of [Per81]). Proposition 1.9 of [Per81], then implies that the singular points of Y are rational double points. Let X' be the blowing-up of $\mathbb{P}^2$ at the c singular points of C and let C' be the strict transform of C on X' (so C' is the disjoint union of two irreducible nonsingular curves). In the terminology of Definition 1.5 of [Per81], (X', C') is the *even resolution* of $(\mathbb{P}^2, C)$. Since there exists $B' \in \operatorname{Div}(X')$ such that $C' \in |2B'|$, page 8 of [Per81] implies that there exists a double covering $Y' \to X'$ branched along C'. Since the singular points of C are all double points, Observation 1.16 of [Per81] implies that $\chi(Y') = \chi(Y)$. Also, since the singular points of C are all double points, each blowing-up decreases the value of $p_a(C)$ by 1 and the value of C^2 by 4, so $\frac{p_a(C')-1}{2} - \frac{C'^2}{8} = \frac{p_a(C)-1}{2} - \frac{C^2}{8}$; since $X' \to \mathbb{P}^2$ is a sequence of blowups at nonsingular (hence rational) points of $\mathbb{P}^2$, Theorem 2.3.16 gives $\chi(\mathbb{P}^2) = \chi(X')$. Putting these facts together, we find

$$\chi(Y) = \chi(Y') = 2\chi(X') + \frac{p_a(C')-1}{2} - \frac{C'^2}{8} = 2\chi(\mathbb{P}^2) + \frac{p_a(C)-1}{2} - \frac{C^2}{8},$$

where we used Proposition 5.2.4 for the second equality. Using $\chi(\mathbb{P}^2) = 1$, $p_a(C) = \frac{c(c-1)}{2}$ and $C^2 = (c+1)^2$, it follows that $\chi(Y) = \frac{(c-1)(c-3)}{8} + 1$. Since Y is a surface with rational double points, we have (again by Theorem 2.3.16) $\chi(Y) = \chi(\tilde{Y})$ where $\tilde{Y} \to Y$ is a resolution of singularities. If Y is rational then so is $\tilde{Y}$, so $\chi(Y) = \chi(\tilde{Y}) = 1$ and hence $1 = \chi(Y) = \frac{(c-1)(c-3)}{8} + 1$, which implies that $c = 1$ or $c = 3$. $\square$

5.2.6 Proposition. *Let $S_c = \mathbb{C}[X_0, X_1, X_2, T]/\langle X_0^2T^{c-2} + X_1^c + X_2^c + T^c \rangle$ where $c \geq 3$ is odd and let $X_c = \operatorname{Proj} S_c$. Then X_c is rational if and only if $c = 3$.*

Proof. Suppose X_c is rational. Then Y_c is rational by Lemma 5.2.3 and so $c = 3$ by Proposition 5.2.5. If $c = 3$, then X_c is rational by Remark 5.1.43 $\square$

5.3 Even c

Keeping the notation for S_c and X_c from the previous section, we now turn to the cases where c is even. Some of the proofs of the results that follow are almost identical to those that appear in the cases where c is odd and are often easier. As such, we will generally leave these simpler proofs to the reader.

5.3.1 Proposition. *Let $c \geq 2$ be even and consider the graded polynomial ring $\mathbf{k}_{\frac{c}{2},1,1,1}[W, X, Y, Z]$. Then $B = \mathbf{k}[W, X, Y, Z]/\langle W^2 - (X^c + Y^c + Z^c)\rangle = \mathbf{k}[w, x, y, z]$ is an $\mathbb{N}$-graded ring and $B_{(z)} \cong \mathbf{k}[S, T, U]/\langle S^2 - (T^c + U^c + 1)\rangle$.*

Proof. We leave it to the reader to prove this proposition by following the argument in Proposition 5.2.1. □

5.3.2 Notation. Let $c \geq 2$ be even, let $A_c = \mathbb{C}_{\frac{c}{2},1,1,1}[X_0, X_1, X_2, X_3]/\langle X_0^2 - (X_1^c + X_2^c + X_3^c)\rangle$ and define $Z_c = \text{Proj}(A_c)$.

5.3.3 Remark. It is easy to check using Example 2.5.3 and Proposition 2.5.4 that Z_c is nonsingular and quasismooth.

5.3.4 Lemma. *If $c \geq 2$ is even, then X_c and Z_c are birationally equivalent.*

Proof. We leave it to the reader to prove this lemma by following the argument in Lemma 5.2.3. □

5.3.5 Proposition. *Let $c \geq 2$ be even. Then Z_c is rational if and only if $c \leq 4$.*

Proof. Using Notation 5.3.2, observe that A_c and $B_{2,c,c,c}$ are isomorphic as graded rings so it suffices to prove that $\text{Proj}\, B_{2,c,c,c}$ is rational if and only if $c \leq 4$. Let $P_c = \text{Proj}(B_{2,c,c,c})$ and recall that P_c is a well-formed quasismooth weighted hypersurface of $\mathbb{P}(\frac{c}{2}, 1, 1, 1)$ with amplitude $\alpha = c - \frac{c}{2} - 3 = \frac{c-6}{2}$.

If $c \geq 6$ then $\alpha \geq 0$ and hence $(A_c)_\alpha \neq \{0\}$; so $p_g(P_c) = h^2(P_c, \mathcal{O}_{P_c}) = \dim_{\mathbb{C}}(A_c)_\alpha > 0$ (by Lemma 2.5.12) and so P_c is not rational for $c \geq 6$. If $c \leq 4$ then $\alpha < 0$ and by Proposition 4.5.1 (since $\text{cotype}(2, c, c, c) = 0$ because c is even), P_c is a del Pezzo surface. It is easily verified that P_c is nonsingular. Since a nonsingular del Pezzo surface is rational, the proof is complete. □

In view of Proposition 5.3.5, we obtain the following result.

5.3.6 Corollary. *Let $c \geq 2$. Then X_c is rational if and only if $c \leq 4$.*

Proof. If c is even then (by Lemma 5.3.4) X_c is birationally equivalent to Z_c which (by Proposition 5.3.5) is rational if and only if $c \leq 4$. If c is odd, the result follows from Proposition 5.2.6. □

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