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Abstract

We propose new unitary space-time codes with high diversity products for multiple antenna wireless communications. These Hamiltonian and product constellations have full diversity, and can be used for any number of transmitter antennas and for any data rate. Hamiltonian constellations for two transmitter antennas are based on Slepian's group codes. We construct Hamiltonian constellations for any M transmitter antennas by using a direct sum of 2×2 Hamiltonian matrices for M even, and a direct sum of 2×2 Hamiltonian matrices with the roots of unity for M odd. Product constellations are proposed using a product of a Hamiltonian constellation and a representation of cyclic group. We also present product constellations of two Hamiltonian constellations with diagonal blocks in different order for M odd. Many of our constellations outperform, and have higher diversity products, than the best known space-time constellation designs in the literature.

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Chapter 1

Introduction

The use of multiple antenna systems is a technique for increasing the data rate of wireless communications in a fading environment [8, 9, 32]. Space-time coding was developed for use in multiple antenna wireless communications to achieve high data rate and reliability. It employs a combination of techniques in error control coding and transmission diversity. The space-time encoder encodes data, and splits it into M encoded streams for transmission by M transmitter antennas. The received signal at each receiver antenna is the linear superposition of multiplication of M encoded streams with fading coefficients, adding with noise. The design of a good space-time code with high coding gain and a simple encoding-decoding algorithm is still an open problem. The construction of full diversity space-time codes for any number of transmitter antennas and for any data rate poses a particular challenge.

1.1 Literature Review

Tarokh *et.al.* [31] introduced *space-time trellis codes* for multiple antenna systems in Rayleigh or Rician fading channels. They defined the diversity and coding gain, which are the main design criteria of space-time coding, for both slow and fast fading channels. Although the space-time trellis code performs very well, its disadvantage is its complexity, which grows exponentially with the number of encoder memory states. Alamouti [1] presented a simple transmission scheme for two transmitter antennas. The decoder algorithm of Alamouti's scheme was shown to be very simple using a technique of signal combining and maximum-likelihood (ML) decoding. This idea led to *space-time block codes* [30] for any number of transmitter antennas, which are constructed from the theory of orthogonal designs. Space-time block codes achieve full diversity gain and also have a simple decoder algorithm. The performance of space-time block codes was first shown in [29]. Both space-time trellis codes and space-time block codes were presented for use with known channels, that is, when the transmitter/receiver antennas know the fading coefficient of the channel.

In practical applications and without imposing any training schemes, the fading coefficient of the channel is generally not known to the receiver antenna. Marzetta and Hochwald [24] suggested that a transmitted signal matrix, which is a product of a matrix with orthonormal columns and a real-nonnegative diagonal matrix, can achieve capacity for unknown channels. This suggestion was the motivation for the development of *unitary space-time modulation* techniques [14], in which all transmitted matrices are unitary, for use in those cases of neither the transmitter nor the receiver antennas know the fading coefficient of a channel. These techniques can work well either for high signal-to-noise ratio (SNR) or when the time period of transmitted matrix is much greater than the number of transmitter antennas. The systematic design of unitary space-time codes was presented later in [15]. Hochwald *et.al.* [16] and Hughes [17] proposed *differential unitary space-time modulation* techniques for use in unknown fading channels. These techniques can be considered as an extension of differential phase shift keying (DPSK) of noncoherent modulation for single antenna communications. The pairwise error probabilities in high SNR when using differential modulations were computed for both known and unknown channels. It has been shown that the differential modulations degrade approximately 3 dB coding gain compared to known channel cases. In [28] and [20], the space-time block codes of [1] and [30], respectively, are modified for differential transmissions.

For high SNR, a pairwise error probability is approximated by a fraction whose denominator is the determinant of the difference of two signal matrices in its constellation. The diversity product is defined as the minimum value of one half of the M roots of this determinant. A constellation has full diversity if its diversity product is greater than zero. The design problem of differential unitary space-time coding is to minimize the pairwise error probability, or equivalently to maximize the diversity product. The problem of constructing a constellation with high diversity product has been studied in many prior works. For example, some of the group structures proposed to represent unitary constellations are dicyclic and cyclic groups [16, 17, 18] and fixed-point free groups [26]. Some examples of nongroup unitary constellations include products of fixed-point free groups [26], subset of the compact symplectic groups [34], Cayley codes [12], parametric codes [22] and numerical methods [11]. These designs suffer variously from limitations in performance, the number of transmitters used, and the data rate achieved. Our goal in this thesis is to find a set of $M \times M$ unitary matrices which has a diversity product as large as possible, where M is the arbitrary number of transmitter antennas. We propose two new unitary space-time constellation designs: Hamiltonian and product constellations. These constellations can be used for any number of transmitter antennas and for any data rate. Furthermore, they achieve full diversity and can be used for both known and unknown channels using differential unitary space-time modulation.

We begin with a 2×2 unitary constellation which is obtained from 2×2 Hamiltonian matrices. The diversity product of a 2×2 Hamiltonian constellation equals one half of the Euclidean distance between two points in two-dimensional complex space.

By considering the transformation from four-dimensional real space to two-dimensional complex space, the idea of Slepian's group codes [27] is used to construct constellations for any order L. We present a new 2×2 Hamiltonian constellation which is constructed from an $(L, 4)$ cyclic group code [2]. The $M \times M$ Hamiltonian constellations for any M transmitter antennas can be constructed by using a direct sum of 2×2 Hamiltonian matrices for M even, and a direct sum of 2×2 Hamiltonian matrices with the L^{th} roots of unity for M odd. A product of a Hamiltonian constellation and a representation of a cyclic group is further proposed to increase the data rate and improve the diversity product. We also propose a product of two Hamiltonian constellations with diagonal blocks in different order for use in a case where M is odd. Although our Hamiltonian and product constellations do not form groups, we show that the optimization of their diversity products will not be computationally intensive for large L. It only requires checking $L - 1$ distinct matrices, making it comparable to those that use group constellations. We show that many of our proposed constellations outperform, and have higher diversity products, than the best known space-time constellation designs in the literature. These include orthogonal designs, dicyclic groups, cyclic groups, parametric codes, numerical approaches, nongroup constellation designs, Cayley codes, TAST codes and some constellations obtained from fixed-point free groups.

1.2 Outline of Thesis

The content of this thesis is described as follows. Chapter 2 gives the background material of this thesis. We briefly give an overview of multiple-antenna wireless communications for both known and unknown channel cases. The design criteria of unitary space-time constellations, the upper bound of diversity product and prior works on unitary space-time constellation designs are reviewed. The idea of group codes, and particularly simplex, orthogonal, symmetric and cyclic group codes, are also explained in this chapter. The determination of the best initial vector, which is the main problem of group codes, is discussed for each group structure. Chapter 3 begins with examples of constructing 2×2 Hamiltonian constellations from the basic group codes with different group structures, as discussed in Chapter 2. In particular the new signal matrix form of a 2×2 Hamiltonian constellation for any cardinality L, which is constructed from an $(L, 4)$ cyclic group code, is presented. A direct sum method is used to extend our construction to the general case of $M \times M$ constellations. At the end of the chapter, we give some examples of specific Hamiltonian constellations with their best diversity products. Product constellations are proposed in Chapter 4. We discuss the necessary conditions for constructing full diversity product constellations. Some examples of product constellations are also given. Chapter 5 shows the results and performance of our Hamiltonian and product constellations for arbitrary numbers of transmitter and receiver antennas compared with different known space-time constellation designs. In Chapter 6, we present our conclusions and discuss possible extensions of the work of this thesis. Appendix A contains a review of the group theory and representation theory which are necessary for understanding the group codes in Chapter 2. The four-dimensional irreducible representation of the symmetric group S_5 is also computed in this appendix. Appendix B is the table listing optimal parameter for the $(L, 4)$ cyclic group codes for $L = 3$ to 100. Appendix C contains the proofs for the simplification of the optimizations Hamiltonian and product constellations.

Chapter 2

Preliminaries

2.1 Multiple Antenna Systems

We review some background of multiple antenna systems in wireless communications as given in [14, 16, 17, 31] in this section.

2.1.1 Channel Models

Consider multiple antennas in a Rayleigh flat fading channel with M transmitter antennas and N receiver antennas. At time t, the fading coefficient from transmitter antenna m to receiver antenna n, h_{tmn}, and the additive noise on receiver antenna n, ω_{tn}, are independent complex Gaussian variables with mean zero and variance one, $\mathcal{CN}(0,1)$. Denote by s_{tm} a transmitted signal at time t on transmitter antenna m, $m = 1, 2, \ldots, M$. The received signal at time t on a receiver antenna n, x_{tn} is defined as

$$x_{tn} = \sqrt{\rho} \sum_{m=1}^{M} h_{tmn} s_{tm} + \omega_{tn}, \quad t = 0, 1, \ldots \text{ and } n = 1, \ldots, N \tag{2.1}$$

where ρ is the signal-to-noise ratio (SNR) at each receiver antenna. At each time t, the expected value of the sum of all transmitted signal powers equal to one, that is

$$\mathbf{E}\sum_{m=1}^{M} |s_{tm}|^2 = 1. \tag{2.2}$$

Generally the transmitted signals can be transmitted in a block period T, therefore from (2.1) a received signal X_τ can be written in a matrix form as

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + W_\tau, \qquad \tau = 0, 1, \ldots \tag{2.3}$$

where τ is an index of a block. The sizes of the received signal matrix X_τ and the transmitted signal matrix S_τ are $T \times N$ and $T \times M$ respectively. W_τ is the $T \times N$ additive noise matrix. H_τ is called the $M \times N$ *channel matrix*, and we assume that the fading coefficient is constant within the block. Let R be the data rate in bits/channel use, and L be the size of the alphabet of a message for transmission. We have $L = 2^{RT}$.

Known Channels

We first explain *unitary space-time modulations* as given in [14] for a case of known channel. If receiver antenna knows the channel matrix, H_τ, then the receiver antennas is said to have *the perfect channel state information (Perfect CSI)*. A message is sent to the space-time encoder as a sequence $z_0, z_1, z_2 \ldots$ with $z_\tau \in \{0, 1, \ldots L-1\}$. Then the transmitted signal matrix S_τ is chosen from a signal constellation $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ by index z_τ. Equivalently

$$S_\tau = V_{z_\tau} \tag{2.4}$$

where V_l is a $T \times M$ unitary matrix, which satisfies $V_l V_l^* = I_T$. $(\ )^*$ and I_T denotes conjugate transpose and a $T \times T$ identity matrix respectively. The received signal matrix is defined by (2.3). Using the maximum-likelihood (ML) decoder, the receiver antenna will decode to a message $\hat{z}_\tau$ as

$$\hat{z}_\tau = \arg \min_{l=0,\ldots,L-1} \|X_\tau - V_l H_\tau\| \tag{2.5}$$

where $\|A\|^2 = \mathrm{Tr}(AA^*) = \mathrm{Tr}(A^*A) = \sum_{i,j} |a_{ij}|^2$ (Tr denotes the trace). Suppose the time period in one block equals the number of transmitter antennas, that is, $T = M$. Then the size of V_l is $M \times M$, and the data rate will be

$$R = \frac{\log_2 L}{M}. \tag{2.6}$$

Using the Chernoff bound, the pairwise error probability that the receiver antenna decodes an error from V_l to $V_{l'}$ can be bounded by [14]

$$P_e \leq \frac{1}{2} \prod_{m=1}^{M} [1 + \frac{\rho}{4}\sigma_m^2(V_l - V_{l'})]^{-N} \tag{2.7}$$

where $\sigma_m(V_l - V_{l'})$ is the m^{th} singular value of the matrix $V_l - V_{l'}$. We know that the product of the squares of the singular values equals the norm squared of the determinant. Hence at high SNR, the pairwise error probability P_e from (2.7) can be approximated by

$$P_e \leq \frac{1}{2}(\frac{4}{\rho})^{MN} \frac{1}{|\det(V_l - V_{l'})|^{2N}} \tag{2.8}$$

Unknown Channels

The receiver antennas is said to have *no CSI* if it does not know the channel matrix H_τ. Hughes [17] and Hochwald *et.al.* [16] proposed *differential unitary space-time modulations* for multiple antennas with no knowledge of the channel information. The idea of differential space-time modulation is similar to differential phase shift keying modulation (DPSK) for noncoherent modulation in the single antenna communications. The current transmitted signal S_τ is obtained by multiplication of the previous signal $S_{\tau-1}$ with its signal constellation index V_{z_τ}. Equivalently,

$$S_\tau = V_{z_\tau} S_{\tau-1}, \tag{2.9}$$

with $S_0 = I_M$. In the case of unknown channel, we have to modify the received signal equation of (2.3), since H_τ is unknown. From the differential modulation [16, 17], we assume that the channel matrix H_τ is constant over two consecutive time periods, $H_\tau \approx H_{\tau-1} = H$. From (2.3), we will have

$$X_{\tau-1} = \sqrt{\rho} S_{\tau-1} H + W_{\tau-1}, \tag{2.10}$$

$$X_\tau = \sqrt{\rho} S_\tau H + W_\tau. \tag{2.11}$$

Substitute $S_{\tau-1} = V^{-1} S_\tau$ in (2.10), and then adding the resulting equation to (2.11) gives

$$X_\tau = V_{z_\tau} X_{\tau-1} + W_\tau - V_{z_\tau} W_{\tau-1}. \tag{2.12}$$

Since additive noise is independent and invariant under multiplication with a unitary matrix, the received signal matrix X_τ for unknown channel will be

$$X_\tau = V_{z_\tau} X_{\tau-1} + \sqrt{2} W'_\tau \tag{2.13}$$

where W'_τ is also independent $\mathcal{CN}(0,1)$. The ML decoder is also used for the receiver antenna to decide a message $\hat{z}_\tau$ to be

$$\hat{z}_\tau = \arg \min_{l=0,\dots,L-1} \|X_\tau - V_l X_{\tau-1}\|. \tag{2.14}$$

Now suppose $T = M$. The data rate is also defined as (2.6). The pairwise error probability of receiver antenna decodes an error from V_l to $V_{l'}$ can be computed using Chernoff bound as [14]

$$P_e \leq \frac{1}{2}\prod_{m=1}^{M}[1+\frac{\rho^2}{4(1+2\rho)}\sigma_m^2(V_l - V_{l'})]^{-N}. \tag{2.15}$$

At high SNR, P_e can be also estimated by

$$P_e \leq \frac{1}{2}(\frac{8}{\rho})^{MN}\frac{1}{|\det(V_l - V_{l'})|^{2N}}. \tag{2.16}$$

Clearly from (2.8) and (2.16), at high SNR, the unknown channel has twice the pairwise error probability of the known channel. This implies a 3 dB penalty for differential space-time modulation for use in an unknown channel.

2.1.2 Design Criteria for Unitary Space-Time Codes

Let $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ be a signal constellation, where $|\mathcal{V}| = L$, and V_l is an $M \times M$ unitary matrix. We define the *diversity product*, $\zeta_\mathcal{V}$ as given in [26] which is computed from a constellation $\mathcal{V}$ by

$$\zeta_\mathcal{V} = \frac{1}{2}\min_{0\leq l<l'\leq L-1} |\det(V_l - V_{l'})|^{\frac{1}{M}} \tag{2.17}$$

where M is the number of transmitter antennas. $0 \leq \zeta_\mathcal{V} \leq 1$. A constellation $\mathcal{V}$ which has $\zeta_\mathcal{V} > 0$ is said to have *full diversity*. Clearly we need to minimize P_e of (2.8) and (2.16) for known and unknown channels respectively. Therefore a design criteria for a full diversity constellation $\mathcal{V}$ is to find a set $\mathcal{V}$ of $M \times M$ unitary matrices which has $\zeta_\mathcal{V} \neq 0$ as large as possible. The upper bounds of the largest possible values of diversity products for $M \geq 2$ as given in [22] are

$$\begin{aligned}
\zeta_{\mathcal{V}\text{ upper}} &\leq \sqrt{\frac{L}{2(L-1)}} \quad \text{for } 2 \leq L \leq 2M^2+1 && (2.18)\\
&\leq \frac{1}{\sqrt{2}} \quad \text{for } 2M^2+1 < L \leq 4M^2 && (2.19)\\
&< \frac{1}{\sqrt{2}} \quad \text{for } L > 4M^2 && (2.20)
\end{aligned}$$

We will call a constellation whose diversity product achieves the upper bound as above, *an optimal diversity product constellation.* Furthermore the value $\sqrt{\frac{L}{2(L-1)}}$ is also the exact value of optimal diversity product for $L = 2$ to 5. Table 2.1 shows the largest possible values of diversity product for $L = 2$ to 5.

L	2	3	4	5
$\zeta = \sqrt{\frac{L}{2(L-1)}}$	1.0000	0.8660	0.8165	0.7906

Table 2.1: Optimal diversity product for $L = 2$ to 5

2.1.3 Summary of Prior Work

The problem of constructing a full diversity unitary space-time constellation with high diversity product has been studied in many prior works. We give a brief summary of them in this section. These designs will be compared to our proposed constellations in next three chapters.

Orthogonal Designs

A 2×2 orthogonal design matrix has the form

$$V_l = \frac{1}{\sqrt{2}} \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix}, \tag{2.21}$$

where $|x|^2 = |y|^2 = 1$. The term $\frac{1}{\sqrt{2}}$ makes this matrix unitary, and thus this constellation can be used for differential detection in unknown channel [28]. The values of x, y are chosen from the Q^{th} roots of unity,

$$x, y \in \{1, e^{2\pi j/Q}, e^{2\pi j2/Q}, \ldots, e^{2\pi j(Q-1)/Q}\}.$$

The order of the constellation is given by $L = Q^2$. The diversity product can be computed as

$$\zeta_\mathcal{V} = \frac{\sin(\pi/Q)}{\sqrt{2}}. \tag{2.22}$$

The summary of orthogonal designs with their diversity product for $M = 2$ transmitter antennas [28] is shown in Table 2.2 on page 27. The 4×4 orthogonal square matrix for 4 transmitter antennas as given in [33] is

$$V_l = \frac{1}{\sqrt{3}} \begin{bmatrix} x & y & z & 0 \\ -y^* & x^* & 0 & -z \\ -z^* & 0 & x^* & y \\ 0 & z^* & -y^* & x \end{bmatrix}. \tag{2.23}$$

The value of x, y and z are also chosen from the Q^{th} roots of unity. This constellation has the order $L = Q^3$.

Dicyclic and Cyclic Group Designs

Hughes [17, 18] and Hochwald *et.al* [16] used dicyclic and cyclic groups to represent their constellations. For $M = 2$, the dicyclic group constellation, a quaternion group Q_p, is

$$Q_p = \left\langle \begin{bmatrix} e^{j2\pi/2^p} & 0 \\ 0 & e^{-j2\pi/2^p} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle. \tag{2.24}$$

The diversity product of Q_p is computed as [18]

$$\zeta_{\mathcal{V}} = \min \left\{ \frac{1}{\sqrt{2}}, \min_{l=1,2,\ldots,2^p-1} \left| \sin \frac{2\pi l}{2^p} \right| \right\}. \tag{2.25}$$

And the order of the dicyclic constellation is $L = 2^{p+1}$. On the other hand, for any L, an $M \times M$ cyclic group constellation has the form $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ where

$$V_l = \text{diag}(e^{j2\pi u_1 l/L}, e^{j2\pi u_2 l/L}, \ldots, e^{j2\pi u_M l/L}), \tag{2.26}$$

and $u_i \in \{0, 1, \ldots, L-1\}$. The diversity product is given by

$$\zeta_{\mathcal{V}} = \min_{l=1,2,\ldots,L-1} \left| \prod_{i=1}^{M} \sin \frac{\pi u_i l}{L} \right|^{\frac{1}{M}}. \tag{2.27}$$

The summary of dicyclic [17] and cyclic group [16] designs is shown in Table 2.3 on page 27.

Fixed-point Free Group Designs

A group is called a fixed-point free group if it has a unitary representation which has full diversity, $\zeta_{\mathcal{V}} > 0$ [13]. Hassibi *et.al.* [26] classified all six classes of fixed-point free groups, $G_{m,r}, D_{m,r,l}, E_{m,r}, F_{m,r,l}, J_{m,r}$ and $K_{m,r,l}$, which are all induced from cyclic groups. Some of these resulting constellations have excellent diversity product which are higher than constellations from [16, 17, 18, 28]. But there are still some limitations to these designs. First, the size of possible constellations is limited when M is large and odd. Second, there exist only even order 2×2 constellations. A summary of fixed-point free group designs [26] is shown in Table 2.4 on page 28.

Nongroup Designs

There are three different nongroup constellation designs for any order L and M transmitter antennas which were also proposed in [26]: a Hamiltonian matrix, for only $M = 2$; a nongroup $S_{m,r}$ which is a generalization of the fixed-point free group $G_{m,r}$, and a matrix product of two different representations of fixed-point free groups. It is shown that some constellations from $S_{m,r}$ and the product nongroups have diversity product higher than group constellation designs. A summary of the nongroup constellation designs of [26] is shown in Table 2.5 on page 28.

Symplectic Group Designs

Subsets of the infinite symplectic group $Sp(2)$ are used to construct full diversity constellations for $M = 4$ transmitter antennas in [34]. $Sp(2)$ is not a fixed-point free group. The constellations produced are finite subsets of $Sp(2)$ with nonzero diversity product. Although these $Sp(2)$ constellations underperform the fixed-point free group designs, their significance is in their simple decoding algorithm. The signal matrix is defined as

$$V_{P,Q,\theta} = \frac{1}{\sqrt{2}} \begin{bmatrix} AB & A\bar{B} \\ -\bar{A}B & \bar{A}\bar{B} \end{bmatrix} \tag{2.28}$$

where $(\bar{\ })$ denotes complex conjugate.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\frac{2\pi k}{P}} & e^{j\frac{2\pi l}{P}} \\ -e^{-j\frac{2\pi l}{P}} & e^{-j\frac{2\pi k}{P}} \end{bmatrix} \text{ and } B = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\frac{2\pi m}{Q}+\theta} & e^{j\frac{2\pi n}{Q}+\theta} \\ -e^{-j\frac{2\pi n}{Q}+\theta} & e^{-j\frac{2\pi m}{Q}+\theta} \end{bmatrix}. \tag{2.29}$$

$0 \leq k, l < P, 0 \leq m, n < Q$. We can see that there are P^2 possible choices for A and Q^2 possible choices for B. Hence the data rate is

$$R = \frac{1}{2}(\log_2 P + \log_2 Q). \tag{2.30}$$

The values of P, Q and $\theta \in [0, 2\pi)$ are chosen to maximize the diversity product.

Parametric Code Designs

A parametric code design [22] was proposed for only $M = 2$ transmitter antennas. A parametric code matrix is defined as a product of three 2×2 unitary matrices as:

$$V_l = \begin{bmatrix} e^{j\theta_L} & 0 \\ 0 & e^{jk_1\theta_L} \end{bmatrix}^l \begin{bmatrix} \cos(k_2\theta_L) & \sin(k_2\theta_L) \\ -\sin(k_2\theta_L) & \cos(k_2\theta_L) \end{bmatrix}^l \begin{bmatrix} e^{jk_3\theta_L} & 0 \\ 0 & e^{-jk_3\theta_L} \end{bmatrix}^l \tag{2.31}$$

where $\theta_L = 2\pi/L$ and $k_1, k_2, k_3 \in \{0, 1, \dots, L-1\}$. The value of $k = (k_1, k_2, k_3)$ is found by exhaustive search to maximize $\zeta_{\mathcal{V}}$. This search is computationally intensive when L is large as it needs to consider $L(L-1)/2$ distinct values of $V_l, V_{l'}$ in (2.17). Table 2.6 gives a summary of parametric code designs [22] on page 28.

Numerical Approaches

In [11], numerical approaches are used to construct large diversity product constellations for any dimension M and order L. A simulated annealing and a genetic algorithm are used to search optimized constellations from the algebraic structures, $A^kB^l, AB, A^kB^k, A^kB^lC^m, ABC$ and $A^kB^kC^k$ where A, B, C are $M \times M$ unitary matrices and k, l, m are arbitrary numbers for a given L. A summary of numerical approach designs [11] is also shown in Table 2.7 on page 29.

Cayley Code Designs

Cayley codes [12] were proposed for any number of transmitter antennas and any data rate by considering the transformation of a nonlinear Stiefel manifold of unitary matrices to a linear space of skew-Hermitian matrices. The signal matrix is

$$V = (I_M + jA)^{-1}(I_M - jA). \tag{2.32}$$

$A = \sum_{q=1}^{Q} A_q\alpha_q$, where $\{\alpha_i\}_i^q \in \mathbb{R}$ and the A_q are $M \times M$ complex Hermitian matrices, $A_q = A_q^*$. The values of $Q, \{A_q\}$ and $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ are chosen to optimize the diversity product of a constellation. The data rate is $R = (Q/M)\log_2 r$, where r is the number of distinct real values in $\mathcal{A}$.

TAST Code Designs

In [10], the threaded algebraic space time (TAST) codes, $\mathcal{T}_{M,L,L}$ where M is the number of transmitter antennas and L is the number of threads or layers in space-time layering technique, were presented for both known channel, and unknown channel by combining with Cayley transformation. The differential TAST code for $M = L = 2$ as given in [10] is

$$V = \frac{I_M - jB}{I_M + jB}, \tag{2.33}$$

where B is defined by

$$B = \frac{1}{\sqrt{2(1+\phi)}} \begin{bmatrix} \sqrt{2}s_{21} & j\phi^{1/2}s_{11} + \phi^{1/2}s_{12} \\ \phi^{1/2}s_{12} - j\phi^{1/2}s_{11} & \sqrt{2}s_{22} \end{bmatrix}. \tag{2.34}$$

$(s_{11}, s_{12})^t = M(u_{11}, u_{12})^t$ and $(s_{21}, s_{22})^t = M(u_{21}, u_{22})^t$. The values of $u_{11}, \ldots, u_{22}$ are chosen from pulse code modulation (PAM) constellations, $\phi = n_g = (1 + \sqrt{5})/2$, and

$$M = \frac{1}{\sqrt{1+n_g^2}} \begin{bmatrix} 1 & -n_g \\ n_g & 1 \end{bmatrix}. \tag{2.35}$$

2.2 A 2 × 2 Hamiltonian Matrix

A 2 × 2 Hamiltonian matrix is used to design a full diversity constellation for $M = 2$ transmitter antennas as given in [26]. This matrix is defined by

$$H = \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix} \tag{2.36}$$

where $x, y \in \mathbb{C}$ and $|x|^2 + |y|^2 = 1$. This differs from the 2×2 orthogonal designs in [28] which require $|x|^2 = |y|^2 = 1$. Let $\mathcal{H} = \{H_l\}_{l=1}^{L-1}$ be a 2×2 Hamiltonian constellation, generally $\mathcal{H}$ does not form a group. From (2.17), the diversity product $\zeta_{\mathcal{H}}$ of a 2×2 Hamiltonian constellation for two transmitter antennas can be computed as

$$\begin{aligned} \zeta_{\mathcal{H}} &= \frac{1}{2}|\det(H - H')|^{\frac{1}{2}} & (2.37) \\ &= \frac{1}{2}\left|\det \begin{bmatrix} x - x' & -(y - y')^* \\ y - y' & (x - x')^* \end{bmatrix}\right|^{\frac{1}{2}} & (2.38) \\ &= \frac{1}{2}\sqrt{|x - x'|^2 + |y - y'|^2} & (2.39) \end{aligned}$$

From (2.39), we can easily see that now $\zeta_{\mathcal{H}}$ equals one half of the Euclidean distance between two points (x, y) and (x', y') in $\mathbb{C}^2$. Consider the transformation from $\mathbb{R}^4$ to $\mathbb{C}^2$. If $A(a_1, a_2, a_3, a_4)^{\mathbb{R}^4}$ is a point on the unit sphere in $\mathbb{R}^4$ where $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$, then we can convert this point onto the unit sphere in $\mathbb{C}^2$ using the mapping:

$$A(a_1, a_2, a_3, a_4)^{\mathbb{R}^4} \longmapsto A(a_1 + ja_2, a_3 + ja_4)^{\mathbb{C}^2} = A(x, y)^{\mathbb{C}^2} \tag{2.40}$$

where $x = a_1 + ja_2, y = a_3 + ja_4$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Consequently the problem of constructing a 2×2 Hamiltonian constellation can be reduced to finding L points in $\mathbb{R}^4$ such that the minimum distance between two points as large as possible. A possible solution to this problem is to use *Slepian's group codes* [27] to get these maximum equidistant L points in $\mathbb{R}^4$, that is, $(L, 4)$ group codes when in the group codes these points are considered as a set of L codewords in four-dimensional space. In the next section, we will explain the idea of group codes which will be used for constructing the Hamiltonian constellations in Chapter 3.

2.3 Group Codes

An (L, n) group code is a set of L codewords in the Euclidean space of dimension n. Basically we can think that all L codewords are on the surface of unit sphere in n dimensional space. Let $\{X_l\}_{l=0}^{L-1}$ be codewords and assume that all X_l are equiprobable. The distance from any codeword X_l to all nearest neighbors is the same as from codeword X_k to all nearest neighbors for all $l, k = 0, 1, \ldots, L-1$, or equivalently, all codewords have the same error probability. We first introduce two basic group codes, simplex and biorthogonal codes, which can be obtained by simple structures. Then group codes generated from symmetric and cyclic groups are described.

- **Simplex Codes**

One of the simplest group codes is a simplex code. Simplex codes can be constructed by translating the basis vectors by their average vector. Let us define an orthonormal basis in n dimensional space by

$$X_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0), \quad j = 1, 2, \ldots, n \tag{2.41}$$

where the n-tuple X_j has 1 at the j^{th} position and other entries are zero. The average of all vectors $\{X_j\}_{j=1}^{n}$ is given by

$$X_{\text{avr}} = \frac{1}{n}\sum_{j=1}^{n} X_j = \frac{1}{n}(1,1,\ldots,1). \tag{2.42}$$

Then translating the X_j of (2.41) by $-X_{\text{avr}}$ gives

$$X'_j = \left(-\frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}, 1-\frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}\right). \tag{2.43}$$

To get all these codewords be unit codewords or lie on the unit sphere, we have to divide (2.43) by its modulus, $|X'_j|$, where $|X'_j| = \sqrt{(n-1)/n}$. An n-dimensional simplex code is thus obtained by

$$X''_j = \left(-\frac{1}{\sqrt{n(n-1)}}, \ldots, -\frac{1}{\sqrt{n(n-1)}}, \sqrt{\frac{n-1}{n}}, -\frac{1}{\sqrt{n(n-1)}}, \ldots, -\frac{1}{\sqrt{n(n-1)}}\right) \tag{2.44}$$

We can see that this set of codewords $\{X''_j\}_{j=1}^{n}$ is linearly dependent in n-dimensional space since $\sum_{j=1}^{n} X'' = 0$, and therefore they can not span in $\mathbb{R}_n$ [4]. Consequently this is an $(n, n-1)$ group code with the minimum distance

$$d_{\min} = \sqrt{\frac{2n}{n-1}} \tag{2.45}$$

- **Biorthogonal Codes**

An n-dimensional biorthogonal code can be easily obtained by

$$X_k = (0,0,\ldots,0,\pm 1,0,\ldots,0), \quad k = 1,2,\ldots,n \tag{2.46}$$

where the n-tuple X_k has ± 1 at the k^{th} position and other entries are zero. An $n-$dimensional biorthogonal code has $2n$ codewords. This is a $(2n, n)$ group code with $d_{\min} = \sqrt{2}$.

2.3.1 Definition

Let $\{O_l\}_{l=0}^{L-1}$ be a group of real orthogonal $n \times n$ matrices. The codewords $\{X_l\}_{l=0}^{L-1}$ of an (L, n) group code can be generated by

$$X_l = O_l X \tag{2.47}$$

where X is called *an initial vector.* The initial vector X can be any of the codewords $\{X_l\}_{l=0}^{L-1}$. To guarantee that $O_l X$ will generate exactly L codewords, the transformation between X_l and the orthogonal matrix O_l must be one-to-one correspondence, or *faithful transformation.* If there are only L' distinct codewords, where $L' < L$, then L' must divide L. Intuitively to be able to generate L codewords, an initial vector X can not be an eigenvector of unit eigenvalue of any O_l. The distance, d, between an initial vector X and any codeword X_l is computed using

$$d^2(X, X_l) = \|X - X_l\|^2 = 2 - 2X \cdot O_l X. \tag{2.48}$$

The main problem of group codes is how to choose the best initial vector X in (2.47) to minimize the error probability, or to maximize the minimum distance of nearest codewords in (2.48). This is extremely difficult and a solution has not yet been found, except for special cases such as for a cyclic group [2], for a finite reflection group [25] and for a full homogeneous representation and the $(n-1)$-dimensional representation of the symmetric group S_n [3].

2.3.2 Symmetric Group Codes

The best initial vector, X, and maximum $d_{\min}$ of a symmetric group code have been found in [3] for the $(n-1)$-dimensional representation of S_n by viewing it as a subrepresentation of the natural representation T of S_n (see Appendix A) which is a doubly transitive representation of symmetric group. In other words, there is an orthogonal matrix R such that

$$R^t T R = 1 \oplus \sigma \tag{2.49}$$

where σ is the $(n-1)$-dimensional representation of S_n. $(\ )^t$ and $\oplus$ denotes a transpose of matrix and a direct sum respectively. The orthogonal matrix R has the form as

$$R^t = \begin{bmatrix} \gamma & \gamma & \gamma & \cdots & \gamma & \gamma \\ \beta_1 & -\gamma_1 & -\gamma_1 & \cdots & -\gamma_1 & -\gamma_1 \\ 0 & \beta_2 & -\gamma_2 & \cdots & -\gamma_2 & -\gamma_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \beta_{n-1} & -\beta_{n-1} \end{bmatrix} \tag{2.50}$$

where $n\gamma^2 = 1, (n-j)\gamma_j^2 + \beta_j^2 = 1$ and $\beta_j - (n-j)\gamma_j = 0$. Let $X = (x_1, x_2, \ldots, x_n)$ be an initial vector of the natural representation T; the codewords are $T(g)X$ where $g \in S_n$. Since $T(g)$ permutes the component of X, we may assume that $x_1 \geq x_2 \geq \ldots \geq x_n$. The minimum distance between codewords which we want to maximize is thus given by

$$\min_i 2(x_i - x_{i+1})^2 \quad i = 1, 2, \ldots, n-1. \tag{2.51}$$

From [3], we assume that $\sum_{i=1}^n x_i = 0$ and set $x_i - x_{i+1}$ to have constant value α. We will get a best initial vector for the natural representation T as:

For n odd

$$X = \left(\frac{n-1}{2}\alpha, \frac{n-3}{2}\alpha, \ldots, \alpha, 0, -\alpha, \ldots, -\frac{n-1}{2}\alpha\right) \tag{2.52}$$

For n even

$$X = \left(\frac{n-1}{2}\alpha, \frac{n-3}{2}\alpha, \ldots, \frac{\alpha}{2}, -\frac{\alpha}{2}, \ldots, -\frac{n-1}{2}\alpha\right) \tag{2.53}$$

From (2.52) and (2.53), α can be computed by

$$\alpha = \sqrt{\frac{12}{(n-1)n(n+1)}}. \tag{2.54}$$

The best initial vector for the $(n-1)$-dimensional representation, σ, can be obtained by deleting the first entry of $R^t X$, which is zero. The order of codewords is $n!$. This is an $(n!, n-1)$ group code with the minimum distance

$$d_{\text{min}} = \sqrt{2}\alpha. \tag{2.55}$$

2.3.3 Cyclic Group Codes

The problem to find a best initial vector, X and maximum d_{min} of an (L, n) cyclic group code has been solved in [2] by a linear programming, the simplex method. Every real n-dimensional representation of a cyclic group is generated by a matrix O whose diagonal form is

$$O = \text{diag}(\underbrace{1, 1, \ldots 1}_{\leftarrow n_1 \rightarrow}, \underbrace{-1, -1, \ldots -1}_{\leftarrow n_2 \rightarrow}, A(k_1), \ldots A(k_{n_3})) \tag{2.56}$$

where $A(k_i)$ is defined by

$$A(k_i) = \begin{bmatrix} \cos\frac{2\pi}{L}k_i & \sin\frac{2\pi}{L}k_i \\ -\sin\frac{2\pi}{L}k_i & \cos\frac{2\pi}{L}k_i \end{bmatrix}, \tag{2.57}$$

and

$$n_1 + n_2 + n_3 = n. \tag{2.58}$$

The values of n_1 and n_2 are the number of repetitions of 1 and -1 in O respectively. If L is odd then n_2 must be 0. In group codes, generally we avoid to have a direct sum of identity in the orthogonal matrix O_l of (2.47) because the component of $O_l X$ corresponding to the identity will be constant for all X_l implying that the codewords do not span n-dimensional space. Therefore n_1 in (2.56) must be zero.

Theorem 2.3.1 *If O is a generator matrix of an (L, n) cyclic group code, the codewords which are generated by $\{O_l X\}_{l=0}^{L-1}$ will span a space whose dimensionality equals to the number of distinct eigenvalues of O. [2]*

Since we want the codewords to span n dimensions, from Theorem 2.3.1, O must have n distinct eigenvalues. We can see that the eigenvalues of O when $n_1 = 0$ are $-1, -1, \ldots, -1$ (with n_2 times) and $e^{\pm j \frac{2\pi}{L} k_i}, i = 1, 2, \ldots, n_3$. Consequently we need to choose $n_2 = 1$ for L even and n odd, and different values of k_i, $0 \leq k_i < L$ (not equal to $L/2$ for L even and n odd). Then O_l of an (L, n) cyclic group code will have the form

For L even and n odd, $n = 2\nu + 1$

$$O_l = \operatorname{diag}((-1)^l, A(lk_1), \ldots, A(lk_\nu)) \tag{2.59}$$

For L unrestricted and n even, $n = 2\nu$

$$O_l = \operatorname{diag}(A(lk_1), \ldots, A(lk_\nu)) \tag{2.60}$$

Let $X = (x_1, x_2, \ldots, x_n)$ be an initial vector. We can compute the Euclidean distance between X and $O_l X$ by

$$\begin{aligned}
\|X - O_l X\|^2 &= 2 - 2\sum_{i=1}^{\nu} \mu_i \cos\frac{2\pi}{L} l k_i, \qquad \text{for } n = 2\nu && (2.61)\\
&= 2 - 2(-1)^l \mu_0 - 2\sum_{i=1}^{\nu} \mu_i \cos\frac{2\pi}{L} l k_i, \quad \text{for } n = 2\nu + 1 && (2.62)
\end{aligned}$$

where

$$\begin{aligned}
\mu_0 &= x_1^2 \\
\mu_i &= \begin{cases} x_{2i}^2 + x_{2i+1}^2 & n = 2\nu + 1 \\ x_{2i-1}^2 + x_{2i}^2 & n = 2\nu \end{cases}
\end{aligned} \tag{2.63}$$

From (2.61) and (2.63), we can observe that the distance between X and X_l does not depend on a single entry of the initial vector X, but rather on the sum of squares of pair of entries. Using the property that $\cos\frac{2\pi}{L} l = \cos\frac{2\pi}{L}(L - l)$ in (2.61) or (2.62) we deduce that

$$\|X - O_l X\|^2 = \|X - O_{L-l} X\|^2. \tag{2.64}$$

The problem of computing a best initial vector, X which maximizes the distance in (2.61) or (2.62) is thus equivalent to finding

$$\max_{X} \min_{l=1,2,\ldots,L/2} \|X - O_l X\|. \tag{2.65}$$

First let consider in the case of n even, $n = 2\nu$. By using $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$ in (2.61), the distance from X to $O_l X$ will be

$$d_l^2 = 4\sum_{i=1}^{\nu} \mu_i \sin^2 \frac{\pi}{L} l k_i. \tag{2.66}$$

Let $d^2_{\min}$ be the maximum nearest distance. Clearly $d_l^2 \geq d^2_{\min}$, so

$$\sum_{i=1}^{\nu} \frac{4\mu_i}{d^2_{\min}} \sin^2 \frac{\pi}{L} l k_i \geq 1 \tag{2.67}$$

Let us define

$$y_i = \frac{4\mu_i}{d^2_{\min}}. \tag{2.68}$$

Since $\mu_1 + \mu_2 + \ldots + \mu_\nu = 1$, we have that

$$z = \sum_{i=1}^{\nu} y_i = \frac{4}{d^2_{\min}} \tag{2.69}$$

We want $d^2_{\min}$ as large as possible, therefore we need to minimize

$$z = \sum_{i=1}^{\nu} y_i. \tag{2.70}$$

According to (2.67) and (2.68), μ_i is always positive and so is $d^2_{\min}$, therefore $\mathbf{y} = (y_1, y_2, \ldots, y_\nu)$ must satisfy

$$y_i \geq 0 \quad \text{and} \quad \mathcal{A}\mathbf{y} \geq (1, 1, \ldots, 1) \tag{2.71}$$

where $\mathcal{A}$ is an $(L/2) \times \nu$ matrix which is defined by

$$(\mathcal{A})_{ij} = \sin^2 \frac{\pi}{L} i k_j \tag{2.72}$$

where $i = 1, 2, \ldots, \frac{L}{2}$ and $j = 1, 2, \ldots, \nu$.

Similarly for the case of n odd, $n = 2\nu + 1$ and L is even, we also need to minimize (2.70), and $\mathbf{y} = (y_1, y_2, \ldots, y_{\nu+1})$ must satisfy

$$y_i \geq 0 \quad \text{and} \quad \mathcal{B}\mathbf{y} \geq (1,1,\ldots,1) \tag{2.73}$$

where $\mathcal{B}$ is an $(L/2) \times (\nu+1)$ matrix which is defined by

$$(\mathcal{B})_{ij} = \begin{cases} \frac{1-(-1)^i}{2}, & \text{if } j = 1 \\ \sin^2 \frac{\pi}{L} i k_{j-1}, & \text{if } j > 1 \end{cases} \tag{2.74}$$

where $i = 1, 2, \ldots, L/2$ and $j = 1, 2, \ldots, \nu+1$. Thus we can find the best initial vector, X, and maximum $d_{\min}$ for any given L, n and $k_i, i = 1, 2, \ldots, \nu$ using the simplex programming method. For fixed L, the different values of k_i will give us different values of $d_{\min}$. The next step is to find the value of $k = (k_1, k_2, \ldots, k_\nu)$ which gives the maximum $d_{\min}$ for fixed L and n. We observe that the number of possible choices of k_i is $\binom{L/2}{2}$ since k_i and $L - k_i$ will give us the same distance $d_{\min}$.

L	R	ζ	Q^{th} roots
4	1.00	0.7071	2
16	2.00	0.5000	4
64	3.00	0.2706	8 (Figure 5.2)
121	3.46	0.1922	11
256	4.00	0.1379	16
4096	6.00	0.0347	64 (Figure 5.3)

Table 2.2: Summary of orthogonal designs for $M = 2$

M	L	R	ζ	Constellation designs
2	4	1.00	0.7071	dicyclic group Q_1
2	4	1.00	0.7071	cyclic group $u = (1, 1)$ (Figure 5.1)
2	8	1.50	0.7071	dicyclic group Q_2
2	8	1.50	0.5946	cyclic group $u = (1, 3)$
2	16	2.00	0.3827	dicyclic group Q_3
2	16	2.00	0.3827	cyclic group $u = (1, 7)$
2	32	2.50	0.1951	dicyclic group Q_4
2	32	2.50	0.2494	cyclic group $u = (1, 7)$
2	64	3.00	0.0980	dicyclic group Q_5 (Figure 5.2)
2	64	3.00	0.1985	cyclic group $u = (1, 19)$ (Figure 5.2)
2	120	3.45	0.1353	cyclic group $u = (1, 43)$
2	128	3.50	0.0491	dicyclic group Q_6
2	128	3.50	0.1498	cyclic group $u = (1, 47)$
2	240	3.95	0.1045	cyclic group $u = (1, 151)$
2	256	4.00	0.0245	dicyclic group Q_7
2	256	4.00	0.0988	cyclic group $u = (1, 75)$
3	8	1.00	0.5134	cyclic group $u = (1, 1, 3)$
3	63	1.99	0.3301	cyclic group $u = (1, 17, 26)$
3	64	2.00	0.2765	cyclic group $u = (1, 11, 27)$
4	16	1.00	0.5453	cyclic group $u = (1, 3, 5, 7)$
4	240	1.98	0.2145	cyclic group $u = (1, 31, 133, 197)$
4	256	2.00	0.2208	cyclic group $u = (1, 25, 97, 107)$ (Figure 5.5)
5	32	1.00	0.4095	cyclic group $u = (1, 5, 7, 9, 11)$
6	64	1.00	0.3792	cyclic group $u = (1, 7, 15, 23, 25, 31)$

Table 2.3: Summary of dicyclic and cyclic group designs

M	L	R	ζ	Classes of fixed-point free groups	
2	24	2.29	0.5000	$E_{3,1} = SL_2(\mathbb{F}_3)$	
2	48	2.79	0.3868	$F_{3,1,-1}$	
2	120	3.45	0.3090	$J_{1,1} = SL_2(\mathbb{F}_5)$	
2	240	3.95	0.2257	$F_{15,1,11}$	
3	9	1.06	0.6004	$G_{9,1}$ with $u = (1, 2, 5)$	(Figure 5.4)
3	63	1.99	0.3851	$G_{21,4}$	
3	513	3.00	0.1353	$G_{171,64}(t = 19)$	
4	240	1.98	0.5000	$K_{1,1,-1}$	

Table 2.4: Summary of fixed-point free group designs

M	L	R	ζ	Parameters of nongroups
2	81	3.17	0.2417	$L_A = 9, u = (1, 2)$
2	289	4.09	0.1625	$L_A = 17, u = (1, 12)$
2	1089	5.04	0.0794	$L_A = 33, u = (1, 26)$
2	4225	6.02	0.0436	$L_A = 65, u = (1, 19)$
3	529	3.02	0.1863	$L_A = 23, u = (1, 13, 19)$
4	289	2.04	0.3105	$L_A = 17, u = (1, 3, 4, 11)$
4	4225	3.01	0.1539	$L_A = 65, u = (1, 14, 21, 34)$
5	33	1.01	0.5580	$\mathcal{S}_{11,3}, u = (1, 3, 4, 5, 9)$

Table 2.5: Summary of nongroup designs

M	L	R	ζ	Parameters of parametric codes	
2	16	2.00	0.5946	$k = (3, 4, 2)$	
2	32	2.50	0.3827	$k = (7, 8, 2)$	
2	55	2.89	0.3874	$k = (34, 15, 0)$	
2	64	3.00	0.3070	$k = (7, 10, 0)$	(Figure 5.2)
2	75	3.11	0..3535	$k = (49, 18, 0)$	
2	91	3.25	0.3451	$k = (64, 21, 0)$	
2	105	3.36	0.3116	$k = (34, 42, 0)$	
2	128	3.50	0.2606	$k = (1, 8, 20)$	

Table 2.6: Summary of parametric code designs

M	L	R	ζ	Algebraic structures
2	27	2.38	0.4122	$A^kB^kC^k$
2	36	2.59	0.3860	A^kB^l
2	49	2.81	0.3781	A^kB^l
2	64	3.00	0.3090	A^kB^k
2	120	3.45	0.2377	-
2	256	4.00	0.1651	A^kB^k
3	5	0.77	0.7183	-
4	9	0.79	0.5904	-

Table 2.7: Summary of numerical approach designs

Chapter 3

Hamiltonian Constellations

In this chapter, we begin with the basic examples of constructing 2×2 Hamiltonian constellations from the group codes as discussed in the previous chapter. A new signal matrix form of 2×2 Hamiltonian constellations for any L, which is obtained from an $(L, 4)$ cyclic group code, is then presented. We further extend the construction to general $M \times M$ Hamiltonian constellations for any number of transmitter antennas M using a direct sum method.

3.1 Basic Examples

3.1.1 For $L = 2$

Consider a unit sphere in $\mathbb{R}^4$. For the case of $L = 2$, it is trivial to choose any two points which are opposite each other on the surface of a unit sphere to be the two codewords, X_0 and X_1. Let us choose $X_0 = (1, 0, 0, 0)$ and $X_1 = (-1, 0, 0, 0)$. Then $d_{\min} = 2$, and the data rate is $R = \log_2 2/2 = 0.50$. Transform the codewords X_0 and X_1 to $\mathbb{C}^2$ by (2.40) to obtain $X_0^{\mathbb{C}^2} = (1, 0)$ and $X_1^{\mathbb{C}^2} = (-1, 0)$. Substituting to a 2×2 Hamiltonian matrix in (2.36), we get a 2×2 Hamiltonian constellation, $|\mathcal{H}| = 2$,

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \tag{3.1}$$

The diversity product is $\zeta_{\mathcal{H}} = 2/2 = 1$. This is an optimal constellation.

3.1.2 A Simplex Code, $L = 4$

The Hamiltonian constellation which is obtained from a simplex code has $L = 4$ and $R = (\log_2 4)/2 = 1.00$. From (2.44), with $n = 4$, the 4 four-dimensional codewords, $\{X_l\}_{l=0}^{3}$ are

$$X_0 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}} \right) \tag{3.2}$$

$$X_1 = \left(-\frac{1}{\sqrt{12}}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}} \right) \tag{3.3}$$

$$X_2 = \left(-\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{12}} \right) \tag{3.4}$$

$$X_3 = \left(-\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{\sqrt{3}}{2} \right) \tag{3.5}$$

Then $d_{\min} = \sqrt{8/3} = 1.633$. Transforming these codewords from $\mathbb{R}^4$ to $\mathbb{C}^2$ by (2.40) gives

$$X_0^{\mathbb{C}^2} = \left(\frac{\sqrt{3}}{2} - j\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} \right) \tag{3.6}$$

$$X_1^{\mathbb{C}^2} = \left(-\frac{1}{\sqrt{12}} + j\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} \right) \tag{3.7}$$

$$X_2^{\mathbb{C}^2} = \left(-\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}}, \frac{\sqrt{3}}{2} - j\frac{1}{\sqrt{12}} \right) \tag{3.8}$$

$$X_3^{\mathbb{C}^2} = \left(-\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}} + j\frac{\sqrt{3}}{2}\right) \tag{3.9}$$

Substituting these in the form of a 2×2 Hamiltonian matrix in (2.36), we get a 2×2 Hamiltonian constellation, $|\mathcal{H}| = 4$, as

$$\begin{aligned}\mathcal{H} = \Bigg\{ & \begin{bmatrix} \frac{\sqrt{3}}{2} - j\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} & \frac{\sqrt{3}}{2} + j\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{12}} + j\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} - j\frac{\sqrt{3}}{2} \end{bmatrix}, \\ & \begin{bmatrix} -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} & -\frac{\sqrt{3}}{2} - j\frac{1}{\sqrt{12}} \\ \frac{\sqrt{3}}{2} - j\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} + j\frac{1}{\sqrt{12}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{12}} - j\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} + j\frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{12}} + j\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{12}} + j\frac{1}{\sqrt{12}} \end{bmatrix} \Bigg\}\end{aligned} \tag{3.10}$$

The diversity product is $\zeta_{\mathcal{H}} = 1.633/2 = 0.8165$. Table 3.1 compares the diversity product of this constellation with orthogonal [28], cyclic [16] and dicyclic group [16] designs at the same $L = 4$ and $R = 1.00$. We can see that the Hamiltonian constellation obtained from a simplex code has diversity product higher than the other three designs, and in fact it is optimal (see Table 2.1).

Constellation designs	ζ
cyclic group	0.7071
dicyclic group	0.7071
orthogonal	0.7071
$\mathcal{H}$ by a simplex code	0.8165

Table 3.1: Comparison of different 2×2 constellation designs at $R = 1.00$

3.1.3 A Biorthogonal Code, $L = 8$

The order of a 2×2 Hamiltonian constellation obtained from a biorthogonal code is $L = 8$. The data rate is $R = \log_2 8/2 = 1.50$. From (2.46), the 8 four-dimensional codewords are

$$X = \{(\pm 1,0,0,0), (0,\pm 1,0,0), (0,0,\pm 1,0), (0,0,0,\pm 1)\}, \tag{3.11}$$

hence $d_{\min} = \sqrt{2}$. Transforming to $\mathbb{C}^2$ by (2.40) gives

$$X^{\mathbb{C}^2} = \{(\pm 1,0), (\pm j,0), (0,\pm 1), (0,\pm j)\}. \tag{3.12}$$

We substitute this into a 2×2 Hamiltonian matrix form in (2.36). This 2×2 Hamiltonian constellation, $|\mathcal{H}| = 8$, is

$$\mathcal{H} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \right\}. \tag{3.13}$$

We can see that the Hamiltonian constellation of (3.13) obtained from a biorthogonal code is a special case in which $\mathcal{H}$ forms a group. This $\mathcal{H}$ of (3.13) is equivalent to a dicyclic group [17], denoted Q_2 in Table 2.3. We have a diversity product $\zeta_{\mathcal{H}} = 1.414/2 = 0.7071$ (the diversity product for dicyclic group) which is still higher than a cyclic group design [16], as shown in Table 3.2.

Constellation designs	ζ
cyclic group	0.5946
dicyclic group	0.7071
$\mathcal{H}$ by a biorthogonal code	0.7071

Table 3.2: Comparison of different 2×2 constellation designs at $R = 1.50$

3.1.4 A Symmetric Group Code from S_5, $L = 120$

Since the solution of an initial vector of $(n-1)$-dimensional representation of S_n has been solved as explained in section 2.3.2 and we are searching for four-dimensional group codes, we thus consider the symmetric group S_5 for generating an $(L, 4)$ group code. We begin by computing the best initial vector of the S_5 group code. From (2.54) with $n = 5$, we have

$$\alpha = \sqrt{\frac{12}{4 \times 5 \times 6}} = \frac{1}{\sqrt{10}}. \tag{3.14}$$

In this case $n = 5$ is odd, so from (2.52), the best initial vector in the natural representation is

$$X' = \left(\frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0, -\frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right). \tag{3.15}$$

The orthogonal matrix R will be

$$R^t = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}}) \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{3.16}$$

We can easily see that the matrix R^t is the result from changing to new orthonormal basis vectors $\{\psi_i\}_{i=1}^{5}$ as explained in Appendix A.3. This gives

$$R^t X' = \left(0, \frac{5}{\sqrt{50}}, \frac{3}{\sqrt{30}}, \frac{3}{\sqrt{60}}, \frac{1}{\sqrt{20}}\right). \tag{3.17}$$

By deleting the first entry of $R^t X'$, we obtain the best initial vector for the four-dimensional irreducible representation of S_5 as

$$X = \left(\frac{5}{\sqrt{50}}, \frac{3}{\sqrt{30}}, \frac{3}{\sqrt{60}}, \frac{1}{\sqrt{20}}\right). \tag{3.18}$$

The four-dimensional irreducible representation matrices of S_5 are generated by the following matrices. (see Appendix A.3)

$$\rho(12) = \begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(23) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(34) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \rho(45) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.19}$$

$L = |S_5| = 5! = 120$. The data rate is $R = (\log_2 120)/2 = 3.45$. The 120 four-dimensional codewords can be generated by $\rho(g)X$, where $g \in S_5$. Computing $d_{\min}$ using (2.55) gives $d_{\min} = 0.4472$. Thus diversity product of the corresponding 2×2 Hamiltonian constellation is $\zeta_{\mathcal{H}} = 0.2236$. Our constellation again has diversity product higher than the orthogonal [28], cyclic [16] and dicyclic group [17] designs, as shown in Table 3.3.

Constellation designs	L	R	ζ
dicyclic group	128	3.50	0.0491
cyclic group	120	3.45	0.1353
orthogonal	121	3.46	0.1992
$\mathcal{H}$ by a symmetric group S_5	120	3.45	0.2236

Table 3.3: Comparison of different 2×2 constellation designs at $R \approx 3.45$

3.1.5 A Cyclic Group Code, for any L

We want to generate the codewords of an $(L, 4)$ cyclic group code by the method described in the section 2.3.3. In this case $n = 4$ is even and $\nu = 2$. Let $X = (x_1, x_2, x_3, x_4)$ be the best initial vector. From (2.63), we will have

$$\mu_1 = x_1^2 + x_2^2, \quad \text{and } \mu_2 = x_3^2 + x_4^2. \tag{3.20}$$

For any given k_1 and k_2, we have to minimize $z = y_1 + y_2$ which satisfies $y_1, y_2 \geq 0$, and

$$\begin{aligned}
\sin^2 \frac{\pi}{L} k_1 y_1 + \sin^2 \frac{\pi}{L} k_2 y_2 &\geq 1, \\
\sin^2 \frac{\pi}{L} 2k_1 y_1 + \sin^2 \frac{\pi}{L} 2k_2 y_2 &\geq 1, \\
\vdots \qquad\qquad\qquad \vdots &\quad \vdots \\
\sin^2 \frac{\pi}{L}(\frac{L}{2}) k_1 y_1 + \sin^2 \frac{\pi}{L}(\frac{L}{2}) k_2 y_2 &\geq 1.
\end{aligned} \tag{3.21}$$

After we get y_1 and y_2, the values of d_{min} and μ_1, μ_2 can be computed from (2.69) and (2.68) respectively. The values of k_1 and k_2 , which give the maximum d_{min} of an $(L, 4)$ cyclic group code for $L = 3$ to 100, are found by simplex method. Our results are listed in Table B.1 in Appendix B.

Example 3.1.1 *A* $(10, 4)$ *cyclic group code with* $k_1 = 1$ *and* $k_2 = 3$ *gives the maximum* $d_{\text{min}} = 1.2244$ *(from Table B.1 in Appendix B). The generator matrix,* O_l*, of (2.60), will thus be*

$$O_l = \begin{bmatrix} \cos \frac{\pi l}{5} & \sin \frac{\pi l}{5} & 0 & 0 \\ -\sin \frac{\pi l}{5} & \cos \frac{\pi l}{5} & 0 & 0 \\ 0 & 0 & \cos \frac{3\pi l}{5} & \sin \frac{3\pi l}{5} \\ 0 & 0 & -\sin \frac{3\pi l}{5} & \cos \frac{3\pi l}{5} \end{bmatrix}. \tag{3.22}$$

Let $X = (x_1, x_2, x_3, x_4)$ *be a best initial vector. Also from Table B.1,* $y_1 = 1.3350$ *and* $y_2 = 1.3332$*. We have*

$$x_1^2 + x_2^2 = \frac{d_{\min}^2 y_1}{4} = \frac{1.2244^2 \times 1.3350}{4} = 0.5003 \tag{3.23}$$

$$x_3^2 + x_2^4 = \frac{d_{\min}^2 y_2}{4} = \frac{1.2244^2 \times 1.3332}{4} = 0.4997 \tag{3.24}$$

We can arbitrarily choose the values of x_1 and x_3, and then compute x_2 and x_4 respectively. Let choose $x_1 = 0.4$ and $x_3 = 0.6$. This gives

$$x_2 = \sqrt{0.5003 - 0.4^2} = 0.5834 \tag{3.25}$$

$$x_4 = \sqrt{0.4997 - 0.6^2} = 0.3738 \tag{3.26}$$

Consequently, the initial vector is

$$X = (0.4000,\ 0.5834,\ 0.6000,\ 0.3738). \tag{3.27}$$

The total 10 four-dimensional codewords, $\{X_l\}_{l=0}^{9}$, are generated by $\{O_l X\}_{l=0}^{9}$:

$$X_l = \begin{bmatrix} \cos\frac{\pi l}{5} & \sin\frac{\pi l}{5} & 0 & 0 \\ -\sin\frac{\pi l}{5} & \cos\frac{\pi l}{5} & 0 & 0 \\ 0 & 0 & \cos\frac{3\pi l}{5} & \sin\frac{3\pi l}{5} \\ 0 & 0 & -\sin\frac{3\pi l}{5} & \cos\frac{3\pi l}{5} \end{bmatrix} \begin{bmatrix} 0.4000 \\ 0.5834 \\ 0.6000 \\ 0.3738 \end{bmatrix} \tag{3.28}$$

We then use (2.40) to transform these codewords to $\mathbb{C}^2$, and substitute them in a 2×2 Hamiltonian matrix as defined in (2.36). This gives a 2×2 Hamiltonian constellation, $|\mathcal{H}| = 10, R = \log_2 10/2 = 1.66$ with the diversity product $\zeta_{\mathcal{H}} = 1.2244/2 = 0.6122$. □

We can see that the method for constructing a 2×2 Hamiltonian constellation from a cyclic group code in Example 3.1.1 is still complicated to compute. In the next section, we present a new 2×2 Hamiltonian matrix from an $(L, 4)$ cyclic group code which is very simple to construct for any L codewords.

3.2 The Case $M = 2$

From (2.61), we observe that the distance between an initial vector X and any codeword X_l depends on the sum of squares of pair of entries. Let $X = (\sqrt{x_1}, 0, \sqrt{x_2}, 0)$ be the initial vector with

$$x_1 + x_2 = 1 \quad \text{and} \quad x_1, x_2 \geq 0. \tag{3.29}$$

Generate the $(L, 4)$ cyclic codewords by $\{X_l\}_{l=0}^{L-1} = O_l X$, where O_l for $n = 4$ is defined as

$$O_l = \begin{bmatrix} \cos\frac{2\pi lk_1}{L} & \sin\frac{2\pi lk_1}{L} & 0 & 0 \\ -\sin\frac{2\pi lk_1}{L} & \cos\frac{2\pi lk_1}{L} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi lk_2}{L} & \sin\frac{2\pi lk_2}{L} \\ 0 & 0 & -\sin\frac{2\pi lk_2}{L} & \cos\frac{2\pi lk_2}{L} \end{bmatrix} \tag{3.30}$$

This gives

$$X_l = \left(\sqrt{x_1}\cos\frac{2\pi lk_1}{L}, -\sqrt{x_1}\sin\frac{2\pi lk_1}{L}, \sqrt{x_2}\cos\frac{2\pi lk_2}{L}, -\sqrt{x_2}\sin\frac{2\pi lk_2}{L}\right). \tag{3.31}$$

Transforming these codewords to the new codewords in $\mathbb{C}^2$, $X_{l\mathbb{C}^2}$, by (2.40) gives

$$X_{l\mathbb{C}^2} = \begin{bmatrix} \sqrt{x_1}(\cos\frac{2\pi lk_1}{L} - j\sin\frac{2\pi lk_1}{L}) \\ \sqrt{x_2}(\cos\frac{2\pi lk_2}{L} - j\sin\frac{2\pi lk_2}{L}) \end{bmatrix}. \tag{3.32}$$

Using the property $e^{j\theta} = \cos\theta + j\sin\theta$ for (3.32) gives

$$X_{l\mathbb{C}^2} = \begin{bmatrix} \sqrt{x_1} e^{-j\frac{2\pi l k_1}{L}} \\ \sqrt{x_2} e^{-j\frac{2\pi l k_2}{L}} \end{bmatrix}. \tag{3.33}$$

Substitute in the form of a Hamiltonian matrix in (2.36) to get a 2×2 Hamiltonian constellation $\mathcal{H}_{2\times 2} = \{H_l\}_{l=0}^{L-1}$ with

$$H_l = \begin{bmatrix} \sqrt{x_1} e^{-j\frac{2\pi l k_1}{L}} & -\sqrt{x_2} e^{j\frac{2\pi l k_2}{L}} \\ \sqrt{x_2} e^{-j\frac{2\pi l k_2}{L}} & \sqrt{x_1} e^{j\frac{2\pi l k_1}{L}} \end{bmatrix} \tag{3.34}$$

We can also write H_l of (3.34) in terms of $H_{l=0} = H_0$ as

$$H_l = e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l \tag{3.35}$$

where

$$R_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{-j\frac{2\pi l k_2}{L}} \end{bmatrix}, \; H_0 = \begin{bmatrix} \sqrt{x_1} & -\sqrt{x_2} \\ \sqrt{x_2} & \sqrt{x_1} \end{bmatrix} \text{ and } T_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{j\frac{2\pi l k_2}{L}} \end{bmatrix}.$$

Both $\{R_l\}_{l=0}^{L-1}$ and $\{T_l\}_{l=0}^{L-1}$ form cyclic groups of order L. The diversity product of this 2×2 Hamiltonian constellation is computed by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{0 \le l < l' \le L-1} |\det(H_l - H_{l'})|^{\frac{1}{2}} \tag{3.36}$$

Even though the H_l do not form a group, we may compute $\zeta_{\mathcal{H}}$ just by taking $H_{l=0}$ and $H_{l'=l}$ (see the proof in Appendix C.1), since

$$|\det(H_l - H_l')| = |\det(H_0 - H_{l'-l})|. \tag{3.37}$$

Then the optimization of diversity product in (3.36) will be

$$\begin{aligned}\zeta_{\mathcal{H}} &= \frac{1}{2}\min_{l=1,2,\ldots,L-1} |\det(H_0 - H_l)|^{\frac{1}{2}} \\ &= \frac{1}{2}\min_{l=1,2,\ldots,L-1} \left\{4\left(x_1 \sin^2\frac{\pi l k_1}{L} + x_2 \sin^2\frac{\pi l k_2}{L}\right)\right\}^{\frac{1}{2}}. \end{aligned} \tag{3.38}$$

The values of $x = (x_1, x_2)$ of (3.29) and $k = (k_1, k_2)$ are chosen to maximize $\zeta_{\mathcal{H}}$ of (3.38).

3.3 The Case M Even, $M > 2$

An $M \times M$ constellation for a case where M is even can be constructed by the direct sum of 2×2 Hamiltonian matrices. $\mathcal{H}_{M\times M} = \{J_l\}_{l=0}^{L-1}$ for M even has the block diagonal form

$$J_l = \mathrm{diag}(H_l^{1,2}, H_l^{3,4}, \ldots, H_l^{M-1,M}) \tag{3.39}$$

where $H_l^{m,n}$ is defined as

$$H_l^{m,n} = \begin{bmatrix} \sqrt{x_1}e^{-j\frac{2\pi l k_m}{L}} & -\sqrt{x_2}e^{j\frac{2\pi l k_n}{L}} \\ \sqrt{x_2}e^{-j\frac{2\pi l k_n}{L}} & \sqrt{x_1}e^{j\frac{2\pi l k_m}{L}} \end{bmatrix} \tag{3.40}$$

Using a similar derivation as above, the diversity product can be computed as

$$\zeta_{\mathcal{H}} = \frac{1}{2}\min_{l=1,2,\ldots,L-1} \left| 2^M \prod_{j=1}^{M/2} \sum_{i=1}^{2} x_i \sin^2\frac{\pi l k_{2j-2+i}}{L} \right|^{\frac{1}{M}} \tag{3.41}$$

The values of $x = (x_1, x_2)$ of (3.29) and $k = (k_1, k_2, \ldots, k_M)$ are chosen to maximize $\zeta_{\mathcal{H}}$ of (3.41).

3.4 The Case M Odd, $M \geq 3$

For the case where M is odd, an $M \times M$ constellation is constructed using the direct sum of 2×2 Hamiltonian matrices and the L^{th} roots of unity, $\{1, e^{j2\pi/L}, e^{j2\pi 2/L}, \ldots, e^{j2\pi(L-1)/L}\}$. The matrices of $\mathcal{H}_{M\times M} = \{J_l\}_{l=0}^{L-1}$, for M odd, are of the block diagonal form

$$J_l = \mathrm{diag}(e^{j\frac{2\pi lk_1}{L}}, H_l^{2,3}, \ldots, H_l^{M-1,M}). \tag{3.42}$$

Using a derivation similar to the above, the diversity product can be given by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| 2^M \sin\frac{\pi lk_1}{L} \prod_{j=2}^{(M+1)/2} \sum_{i=1}^{2} x_i \sin^2 \frac{\pi lk_{2j-3+i}}{L} \right|^{\frac{1}{M}}. \tag{3.43}$$

The values of $x = (x_1, x_2)$ of (3.29) and $k = (k_1, k_2, \ldots, k_M)$ are found such that they maximize (3.43).

3.5 Results of Hamiltonian Constellations

Table 3.4 shows some Hamiltonian constellations with their best diversity products. The constellations were found by computer-simulated searching. We use a simplex method only for cases of $M = 2$ and 3. The Hamiltonian constellations for every M even case and $L = 2$ to 5 are optimal constellations whose diversity products achieve the upper bound as given in Table 2.1. These optimal $M \times M$ constellations can be constructed by the direct sum of 2×2 Hamiltonian constellations as listed in Table 3.4. The Hamiltonian $\mathcal{H}_{3\times 3}$ with $L = 3$, which has a diversity product $\zeta = 0.8660$, is also an optimal constellation for $M = 3$.

For $M = 3$, the diversity product of Hamiltonian $\mathcal{H}_{3\times 3}$ with $L = 9$ is 0.6632, which is even higher than, that obtained by the fixed-point free group $G_{9,1}$ [26]. Hamiltonian constellations for $M = 4, 5$ and 6 also have diversity product much higher than cyclic group designs [16]. For example, the diversity product of $\mathcal{H}_{6\times6}$ is 0.5185 while the diversity product of the cyclic group is 0.3792 at the same $L = 64$. Unfortunately Hamiltonian constellations do not give competitive diversity products for the case where the number of transmitter antennas, $M = 2$.

M	L	R	ζ	Parameters of Hamiltonian constellations
2	2	0.50	1.0000	$x_1 = 0.5000,\ k = (1,1)$
2	3	0.79	0.8660	$x_1 = 0.5000,\ k = (1,1)$
2	4	1.00	0.8165	$x_1 = 0.6667,\ k = (1,2)$ (Figure 5.6)
2	5	1.16	0.7906	$x_1 = 0.5000,\ k = (1,2)$
2	8	1.50	0.7071	$x_1 = 0.5000,\ k = (1,3)$
2	16	2.00	0.5098	$x_1 = 0.5198,\ k = (1,4)$
2	32	2.50	0.3827	$x_1 = 0.4953,\ k = (1,7)$
2	64	3.00	0.2816	$x_1 = 0.6281,\ k = (1,27)$
2	121	3.46	0.2106	$x_1 = 0.5590,\ k = (1,22)$
2	128	3.50	0.2031	$x_1 = 0.5142,\ k = (1,12)$
2	240	3.95	0.1511	$x_1 = 0.4173,\ k = (1,85)$
2	256	4.00	0.1477	$x_1 = 0.5526,\ k = (1,119)$
3	3	0.53	0.8660	$x_1 = 0.5000,\ k = (1,1,1)$
3	5	0.77	0.7673	$x_1 = 0.2316,\ k = (1,1,2)$
3	8	1.00	0.6588	$x_1 = 0.8089,\ k = (1,3,4)$ (Figure 5.6)
3	9	1.06	0.6632	$x_1 = 0.4679,\ k = (1,4,3)$ (Figure 5.4)
3	63	1.99	0.3498	$x_1 = 0.3758,\ k = (1,20,27)$
3	64	2.00	0.3478	$x_1 = 0.6994,\ k = (1,23,30)$
4	3	0.40	0.8660	$x_1 = 0.5000,\ k = (1,1,1,1)$
4	4	0.50	0.8165	$x_1 = 0.6667,\ k = (1,2,1,2)$
4	5	0.58	0.7906	$x_1 = 0.5000,\ k = (1,2,1,2)$
4	9	0.79	0.7119	$x_1 = 0.4094,\ k = (1,2,6,5)$
4	16	1.00	0.6377	$x_1 = 0.3680,\ k = (1,3,7,5)$
4	256	2.00	0.3320	$x_1 = 0.4834,\ k = (1,121,79,87)$ (Figure 5.5)
4	289	2.04	0.3287	$x_1 = 0.4646,\ k = (1,126,12,67)$
5	32	1.00	0.5444	$x_1 = 0.4500,\ k = (1,11,13,15,7)$
6	3	0.26	0.8660	$x_1 = 0.5000,\ k = (1,1,1,1,1,1)$
6	4	0.33	0.8165	$x_1 = 0.6667,\ k = (1,2,1,2,1,2)$
6	5	0.39	0.7906	$x_1 = 0.5000,\ k = (1,2,1,2,1,2)$
6	64	1.00	0.5185	$x_1 = 0.4549,\ k = (1,19,3,57,23,31)$

Table 3.4: Summary of some Hamiltonian constellations with their best diversity products

Chapter 4

Product Constellations

In this chapter, we increase the data rate and improve the diversity product of the Hamiltonian constellations constructed in the previous chapter by using a matrix product method. We propose two structures of product constellations: a product $\mathcal{P}$ for any M, and a product $\mathcal{P}_{\mathcal{H}}$ for M odd transmitter antennas.

4.1 Product Constellation $\mathcal{P}$

Let $\mathcal{C}$ be a cyclic group of order L_C. An M-dimensional representation of $\mathcal{C} = \{O_g\}_{g=0}^{L_C-1}$ has the diagonal form

$$O_g = \text{diag}(e^{j2\pi g r_1/L_C}, e^{j2\pi g r_2/L_C} \dots, e^{j2\pi g r_M/L_C}). \tag{4.1}$$

We use the product of Hamiltonian constellation $\mathcal{H}_{M\times M} = \{J_l\}_{l=0}^{L_H-1}$ in (3.39) and (3.42) and an M-dimensional representation of a cyclic group $\mathcal{C}$ in (4.1) to construct a product constellation $\mathcal{P}$ for M even and odd respectively, as follows:

$$\mathcal{P} = \mathcal{H} \times \mathcal{C} = \{J_l O_g\}_{l,g=0}^{L_H-1, L_C-1} \tag{4.2}$$

The product constellation $\mathcal{P}$ is unitary, and $|\mathcal{P}|$ can be at most $L_H L_C$. From (2.17), the diversity product of $\mathcal{P}_{M\times M}$ is

$$\begin{aligned}\zeta_{\mathcal{P}} &= \frac{1}{2}\min_{P,P'\in\mathcal{P}} |\det(P-P')|^{\frac{1}{M}} &(4.3)\\ &= \frac{1}{2}\min_{\substack{0\le l,l'\le L_H-1\\ 0\le g,g'\le L_C-1\\ (l,g)\neq(l',g')}} |\det(J_l O_g - J_{l'} O_{g'})|^{\frac{1}{M}} &(4.4)\end{aligned}$$

It is a straightforward calculation (see the proof in Appendix C.2) to check that

$$|\det(J_l O_g - J_{l'} O_{g'})| = |\det(J_0 - J_{l'-l} O_{g'-g})|. \tag{4.5}$$

Therefore to compute the diversity product, it suffices to consider the case of $l = g = 0$ and l', g' are arbitrary. This will give the optimization of $\zeta_{\mathcal{P}}$ in (4.4) be

$$\zeta_{\mathcal{P}} = \frac{1}{2}\min_{(l,g)\neq(0,0)} |\det(J_0 - J_l O_g)|^{\frac{1}{M}}. \tag{4.6}$$

Thus although the product constellation $\mathcal{P}$ does not form a group, our optimization will require the checking of only $L_H L_C - 1$ distinct values of $P, P' \in \mathcal{P}$. We search for the values of $x = (x_1, x_2)$ of (3.29), $k = (k_1, k_2, \ldots, k_M)$ and $r = (r_1, r_2, \ldots, r_M)$ which maximize $\zeta_{\mathcal{P}}$ in (4.6). A necessary condition for full diversity, $\zeta_{\mathcal{P}} > 0$, is that $\gcd(r_i, L_C) = 1$ for all i. Hence we may restrict our search to choices of r such that $r_1 = 1$ and $\gcd(r_i, L_C) = 1$ for $i = 2, 3, \ldots, M$.

Theorem 4.1.1 *The product constellation $\mathcal{P}$ will reduce to the matrix form of Hamiltonian constellation of (3.39) and (3.42) if $r_{2i-1} + r_{2i} = L_C$ for all $i = 1, 2, \ldots, M/2$ for M even and $r_{2i} + r_{2i+1} = L_C$ for all $i = 1, \ldots, M - 1/2$ for M odd respectively.*

Proof: The proof is shown for the $\mathcal{P}_{2\times 2}$ case, where the proof for $\mathcal{P}_{M\times M}$ follows immediately. From (4.2), we have

$$\{J_l O_g\} = \begin{bmatrix} \sqrt{x_1}e^{-j\frac{2\pi l k_1}{L_H}} & -\sqrt{x_2}e^{j\frac{2\pi l k_2}{L_H}} \\ \sqrt{x_2}e^{-j\frac{2\pi l k_2}{L_H}} & \sqrt{x_1}e^{j\frac{2\pi l k_1}{L_H}} \end{bmatrix} \begin{bmatrix} e^{j\frac{2\pi g r_1}{L_C}} & 0 \\ 0 & e^{j\frac{2\pi g r_2}{L_C}} \end{bmatrix} \tag{4.7}$$

$$= \begin{bmatrix} \sqrt{x_1}e^{-j\frac{2\pi l k_1}{L_H}}e^{j\frac{2\pi g r_1}{L_C}} & -\sqrt{x_2}e^{j\frac{2\pi l k_2}{L_H}}e^{j\frac{2\pi g r_2}{L_C}} \\ \sqrt{x_2}e^{-j\frac{2\pi l k_2}{L_H}}e^{j\frac{2\pi g r_1}{L_C}} & \sqrt{x_1}e^{j\frac{2\pi l k_1}{L_H}}e^{j\frac{2\pi g r_2}{L_C}} \end{bmatrix} \tag{4.8}$$

We can see that $J_l O_g$ in (4.8) will reduce to the matrix form of a 2×2 J_l for all $g = 0, 1, \ldots, L_C - 1$ if $e^{j\frac{2\pi g r_1}{L_C}} = e^{-j\frac{2\pi g r_2}{L_C}}$, or equivalently $e^{j\frac{2\pi g(r_1+r_2)}{L_C}} = 1$. This gives L_C divides $r_1 + r_2$. The values of r_1 and r_2 are chosen from $\{0, 1, \ldots, L_C - 1\}$ thus the only one possible condition which satisfies L_C divides $r_1 + r_2$ is $r_1 + r_2 = L_C$. $\square$

To get a product constellation $\mathcal{P}$ which has a diversity product higher than a Hamiltonian constellation $\mathcal{H}$, we thus omit those choices of r in Theorem 4.1.1.

Theorem 4.1.2 *For M odd, if $\gcd(L_H, L_C) > 1$ then any product constellation $\mathcal{P}$ will have $\zeta_{\mathcal{P}} = 0$. If $\gcd(L_H, L_C) = 1$, then there exist values of $k = (k_1, k_2, \ldots, k_M)$ and $r = (r_1, r_2, \ldots, r_M)$ such that $\zeta_{\mathcal{P}} > 0$, with k_1 relatively prime to L_H, and each $r_i, i = 1, 2, \ldots, M$ relatively prime to L_C.*

Proof: In (4.6), $\zeta_{\mathcal{P}}$ will be 0 if $e^{j2\pi(\frac{k_1 l}{L_H}+\frac{r_1 g}{L_C})} = 1$ for some $(l, g) \neq (0, 0)$. The first part of Theorem 4.1.2 can be proved by showing that if $\gcd(L_H, L_C) > 1$, then for any choices of k_1 and r_1 there exist choices of g and l, not both zero, such that $\frac{lk_1}{L_H} - \frac{gr_1}{L_C}$ is an integer. There are two cases that need to be considered.

Case 1: If $\gcd(k_1, L_H) = t > 1$, then take $l = L_H/t$, so that $k_1 l = k_1 L_H/t = \mathrm{lcm}(k_1, L_H)$, which is an integer; and also take $g = 0$. Since $t > 1$, $l \neq 0$, and $l < L_H$. The case of $\gcd(r_1, L_C) > 1$ can be treated similarly.

Case 2: If $\gcd(k_1, L_H) = 1$ *and* $\gcd(r_1, L_C) = 1$, *then set* $q = \gcd(L_H, L_C)$. *Then the values of* L_H/q *and* L_C/q *are integers strictly less than* L_H *and* L_C *respectively. Moreover, we compute*

$$\frac{L_H/q}{L_H} - \frac{L_C/q}{L_C} = \frac{1}{q} - \frac{1}{q} = 0. \tag{4.9}$$

We require the following lemma.

Lemma 4.1.1 *If* $\gcd(a, A) = 1$, *then the set* $S = \{al \mod A \mid l = 0, 1, \ldots, A-1\}$ *has* A *distinct values.*

Proof of lemma: If not, then some al *must equal another* al' *modulo* A*; say* $l' < l$. *This would mean* $a(l - l') = 0 \mod A$. *Both sides are integers, so this means* A *divides* $a(l - l')$. *But this is impossible unless* $l = l'$*: since* $\gcd(a, A) = 1$, *no part of* A *can divide* a, *and since* $l - l' < A$, A *cannot divide* $l - l'$ *unless* $l - l' = 0$. *Thus all values are distinct, and give representatives of all residue classes. (In particular, the value* $l = 0$ *gives the zero residue class.)* □

Using the Lemma 4.1.1, we may choose l *so that* $lk_1 \equiv L_H/q \mod L_H$ *and also choose* g *so that* $gr_1 \equiv L_C/q \mod L_C$ *for proving the first part of the theorem.*

Now to prove the second part of the theorem, suppose that $\gcd(L_H, L_C) = 1$. *If* $\gcd(k_1, L_H) > 1$ *or* $\gcd(r_1, L_C) > 1$, *then a similar argument as we made in Case 1 above shows that the diversity product will be zero. So suppose we choose* $\gcd(k_1, L_H) = 1$ *and* $\gcd(r_1, L_C) = 1$. *In this case, the lemma shows that* $e^{j2\pi(\frac{lk_1}{L_H} + \frac{gr_1}{L_C})} = 1$ *if and only if for some choice of* $a = 0, 1, \ldots, L_H - 1$ *and* $b = 0, 1, \ldots, L_C - 1$, *we have*

$$\frac{a}{L_H} - \frac{b}{L_C} = 0. \tag{4.10}$$

This is equivalent to saying that $aL_C = bL_H$ *for some* a, b. *But this product is strictly* *less that $L_H L_C$, which is the lcm of L_H and L_C, by the hypothesis that* $\gcd(L_H, L_C) = 1$. *Hence it must be zero, and thus $a = 0$ and $b = 0$. Consequently, the only values of g and l giving $e^{j2\pi(\frac{lk_1}{L_H}+\frac{gr_1}{L_C})} = 1$ in the original product are $g = 0, l = 0$. That the remaining 2×2 blocks will not produce a zero diversity product for all choices in the given set follows from (2.39) and the remarks preceding Theorem 4.1.1.* □

4.2 Product Constellation $\mathcal{P}_{\mathcal{H}}$

Due to a limitation of possible full diversity product constellations $\mathcal{P}$ for a case where M is odd (from Theorem 4.1.2), we propose another unitary product constellation, $\mathcal{P}_{\mathcal{H}}$, which is constructed by a product of two Hamiltonian constellations with the diagonal blocks in different order.

$$\mathcal{P}_{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2^{\dagger} = \{J_l J_g^{\dagger}\}_{l,g=0}^{L_{H_1}-1, L_{H_2}-1} \tag{4.11}$$

where L_{H_1}, L_{H_2} are the order of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. $|\mathcal{P}_{\mathcal{H}}|$ can be at most $L_{H_1} L_{H_2}$. $J_g^{\dagger}$ denotes a block diagonal matrix with different order than J_g in (3.42):

$$J_g^{\dagger} = \mathrm{diag}(H_g^{1,2}, H_g^{3,4}, \ldots, H_g^{M-2,M-1}, e^{j2\pi g r_M / L_{H_2}}) \tag{4.12}$$

where $H_g^{m,n}$ ia also defined in (3.40). To optimize the diversity product it also suffices to consider a case of $l, g = 0$ and arbitrary l', g' since (see the proof in Appendix C.3)

$$|\det(J_l J_g^{\dagger} - J_{l'} J_{g'}^{\dagger})| = |\det(J_0 J_0^{\dagger} - J_{l'-l} J_{g'-g}^{\dagger})|. \tag{4.13}$$

Thus the optimization of a product constellation $\mathcal{P}_{\mathcal{H}}$ is

$$\zeta_{\mathcal{P}_{\mathcal{H}}} = \frac{1}{2} \min_{\substack{0\le l,l'\le L_{H_1}-1 \\ 0\le g,g'\le L_{H_2}-1 \\ (l,g)\neq(l',g')}} |\det(J_l J_g^\dagger - J_{l'} J_{g'}^\dagger)|^{\frac{1}{M}} \tag{4.14}$$

$$= \frac{1}{2} \min_{\substack{0\le l,g\le L_{H_1}-1, L_{H_2}-1 \\ (l,g)\neq(0,0)}} |\det(J_0 J_0^\dagger - J_l J_g^\dagger)|^{\frac{1}{M}} \tag{4.15}$$

Consequently, the optimization of the diversity product $\zeta_{\mathcal{P}_{\mathcal{H}}}$ of (4.15) also requires checking only $L_{H_1}L_{H_2}-1$ pairs of matrices. We assume that both $\mathcal{H}_1$ and $\mathcal{H}_2^\dagger$ have the same value of $x = (x_1, x_2)$. We choose x which satisfies (3.29), $k = (k_1, k_2, \ldots, k_M)$ of $\mathcal{H}_1$ and $r = (r_1, \ldots, r_M)$ of $\mathcal{H}_2^\dagger$ in order to maximize $\zeta_{\mathcal{P}_{\mathcal{H}}}$ in (4.15).

4.3 Results of Product Constellations

Table 4.1 shows some product constellations with their best diversity products. The constellations were also found by computer-simulated searching. Our product constellations have diversity product higher than orthogonal designs [28], dicyclic groups [17], cyclic groups [16], nongroups [26], numerical methods [11], parametric codes [22] and some of those obtained from fixed-point free groups [26]. From Table 4.1, we can see that many product constellation $\mathcal{P}_{2\times2}$ for $M = 2$ transmitter antennas give very good diversity product. For example, $\mathcal{P}_{2\times2}$ with $L = 120$ has $\zeta = 0.3090$, which equals that of the excellent fixed-point free group $SL_2(\mathbb{F}_5)$ constellation. The product constellation $\mathcal{P}_{2\times2}$ with $L = 1089$ also has an excellent diversity product of 0.1142. For $M = 3$, the constellation $\mathcal{P}_{\mathcal{H}_{3\times3}}$ with $L = 513$ has good diversity product = 0.2283, which is higher than that of the fixed-point free group $G_{171,94}$. The $\mathcal{P}_{4\times4}$ with $L = 16$ also has diversity product higher than the cyclic group design.

M	L	R	ζ	Parameters of product constellations
2	16	2.00	0.5412	$\mathcal{P}$ $L_H = 8, x_1 = 0.5858, k = (1,2), r = (1,1)$
2	24	2.29	0.5000	$\mathcal{P}$ $L_H = 8, x_1 = 0.5000, k = (1,3), r = (1,1)$
2	27	2.38	0.4122	$\mathcal{P}$ $L_H = 9, x_1 = 0.7733, k = (1,3), r = (1,1)$
2	32	2.50	0.4082	$\mathcal{P}$ $L_H = 8, x_1 = 0.6667, k = (1,2), r = (1,1)$
2	36	2.59	0.4039	$\mathcal{P}$ $L_H = 9, x_1 = 0.2577, k = (1,2), r = (1,1)$
2	48	2.79	0.3678	$\mathcal{P}$ $L_H = 3, x_1 = 0.2113, k = (1,2), r = (1,7)$
2	49	2.81	0.4118	$\mathcal{P}$ $L_H = 7, x_1 = 0.5000, k = (1,6), r = (1,4)$
2	55	2.89	0.4074	$\mathcal{P}$ $L_H = 11, x_1 = 0.5904, k = (1,2), r = (1,1)$
2	64	3.00	0.3678	$\mathcal{P}$ $L_H = 4, x_1 = 0.6533, k = (1,2), r = (1,9)^{\pounds}$
2	75	3.11	0.3535	$\mathcal{P}$ $L_H = 25, x_1 = 0.5000, k = (1,7), r = (1,1)$
2	81	3.17	0.2974	$\mathcal{P}$ $L_H = 27, x_1 = 0.4024, k = (1,12), r = (1,1)$
2	91	3.25	0.3451	$\mathcal{P}$ $L_H = 13, x_1 = 0.5000, k = (1,5), r = (1,1)$
2	105	3.36	0.3116	$\mathcal{P}$ $L_H = 35, x_1 = 0.5000, k = (1,13), r = (1,1)$
2	120	3.45	0.3090	$\mathcal{P}$ $L_H = 24, x_1 = 0.5000, k = (1,5), r = (1,1)$
2	121	3.46	0.2795	$\mathcal{P}$ $L_H = 11, x_1 = 0.3670, k = (1,6), r = (1,1)$
2	128	3.50	0.2793	$\mathcal{P}$ $L_H = 16, x_1 = 0.6104, k = (1,6), r = (1,3)$
2	240	3.95	0.2381	$\mathcal{P}$ $L_H = 10, x_1 = 0.2960, k = (1,4), r = (1,5)$
2	256	4.00	0.1981	$\mathcal{P}$ $L_H = 8, x_1 = 0.3477, k = (1,4), r = (1,13)$
2	289	4.09	0.1838	$\mathcal{P}$ $L_H = 17, x_1 = 0.6640, k = (1,4), r = (1,1)$
2	1089	5.04	0.1142	$\mathcal{P}$ $L_H = 99, x_1 = 0.7900, k = (1,9), r = (1,1)$
2	4096	6.00	0.0685	$\mathcal{P}$ $L_H = 64, x_1 = 0.3898, k = (1,28), r = (1,33)^{\ddagger}$
2	4225	6.02	0.0671	$\mathcal{P}$ $L_H = 65, x_1 = 0.4026, k = (1,39), r = (1,33)$
3	9	1.06	0.6283	$\mathcal{P}_{\mathcal{H}}$ $L_{H_1} = 3, x_1 = 0.3820, k = (1,1,3), r = (2,2,1)$
3	63	1.99	0.4023	$\mathcal{P}_{\mathcal{H}}$ $L_{H_1} = 7, x_1 = 0.4603, k = (1,1,6), r = (5,3,1)$
3	513	3.00	0.1664	$\mathcal{P}$ $L_H = 27, x_1 = 0.4110, k = (1,3,11), r = (1,18,15)$
3	513	3.00	0.2028	$\mathcal{P}_{\mathcal{H}}$ $L_{H_1} = 9, x_1 = 0.4970, k = (1,1,5), r = (15,20,1)$
3	529	3.02	0.2283	$\mathcal{P}_{\mathcal{H}}$ $L_{H_1} = 23, x_1 = 0.3671, k = (1,19,1), r = (2,20,1)$
4	16	1.00	0.6580	$\mathcal{P}$ $L_H = 4, x_1 = 0.5000, k = (1,2,1,4), r = (1,3,3,1)^{\S}$
4	240	1.98	0.3614	$\mathcal{P}$ $L_H = 16, x_1 = 0.3614, k = (1,2,9,18), r = (1,11,7,2)$

Table 4.1: Summary of some product constellations with their best diversity products (Note: £ for Figure 5.2, ‡ for Figure 5.3 and § for Figure 5.6)

Chapter 5

Performance

We compare the performance of the Hamiltonian and product constellations in Table 3.4 and 4.1 respectively with different space-time code designs as introduced in Section 2.1.3 at the same data rate for arbitrary numbers of transmitter and receiver antennas. The performance is considered by plotting the block error rate, *bler*, against signal-to-noise ratio, *SNR* in dB. All plots are considered in unknown Rayleigh flat fading channels which receiver antenna does not know the information of channels, except the Figure 5.1 which is also considered in the case of known channel. The fading coefficient and additive noise are independent $\mathcal{CN}(0,1)$. For the case of unknown channel, we use the differential modulation to transmit signals. The channel matrix is chosen randomly and is assumed to be constant within two consecutive time periods as explained in Section 2.1.1. We use the ML decoder of (2.5) and (2.14) for decoding for the case of known and unknown channels respectively.

5.1 Hamiltonian vs. Cyclic Group Designs

Figure 5.1 shows the block error rate performance in both known and unknown channels for $M = 2$ transmitter antennas and $N = 1$ receiver antennas at $R = 1.00, L = 4$ of the 2×2 Hamiltonian constellation of (3.10), which is obtained from a simplex group code in Section 3.1.2 with $\zeta = 0.8165$, and the cyclic group $u = (1, 1), \zeta = 0.7071$ [16] as given in Table 2.3. The 2×2 Hamiltonian constellation has gain improvement ≈ 1 dB over the cyclic group. We can also see that the performance of constellations for a known channel has 3 dB gain over the differential modulation of an unknown channel as mentioned in Section 2.1.3.

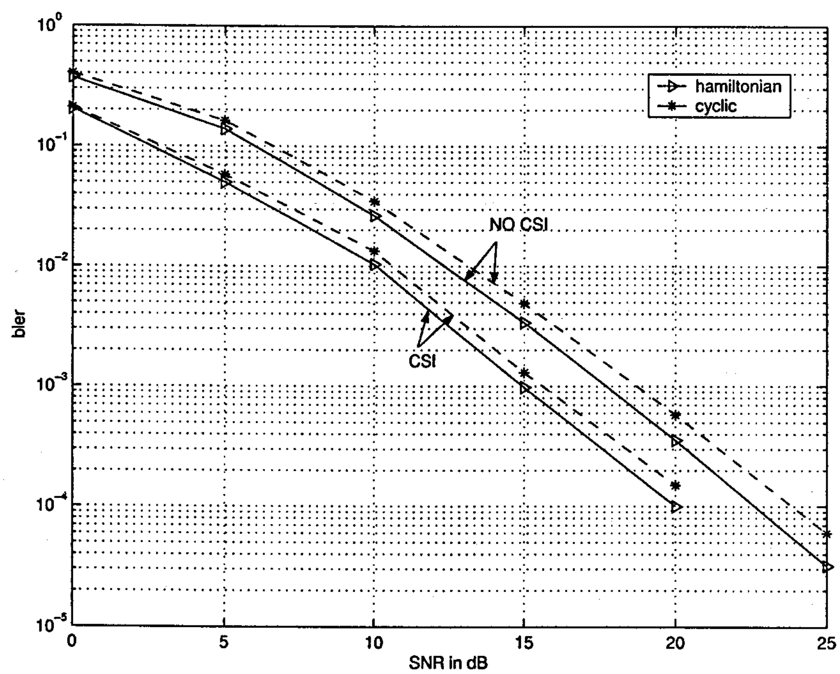

Figure 5.1: Block error rate performance for $M = 2, N = 1$ at $R = 1.00$ of the 2×2 Hamiltonian constellation from a simplex code and cyclic group

5.2 Product vs. Group, Orthogonal and Parametric Code Designs

Figure. 5.2 shows the block error rate performance for $M = 2$ transmitter antennas and $N = 2$ receiver antennas at $R = 3.00, L = 64$ of the product $\mathcal{P}_{2\times2}$ with $\zeta = 0.3678$, dicyclic group Q_5, $\zeta = 0.0980$ [17], cyclic group $u = (1, 19)$, $\zeta = 0.1985$ [16], orthogonal design with 8^{th}-roots of unity $\zeta = 0.2706$ [28] and parametric code $k = (7, 10, 0)$, $\zeta = 0.3070$ [22]. Our product constellation outperforms the other four designs. The gain improvement is 2 and 3 dB over the orthogonal design and cyclic group respectively.

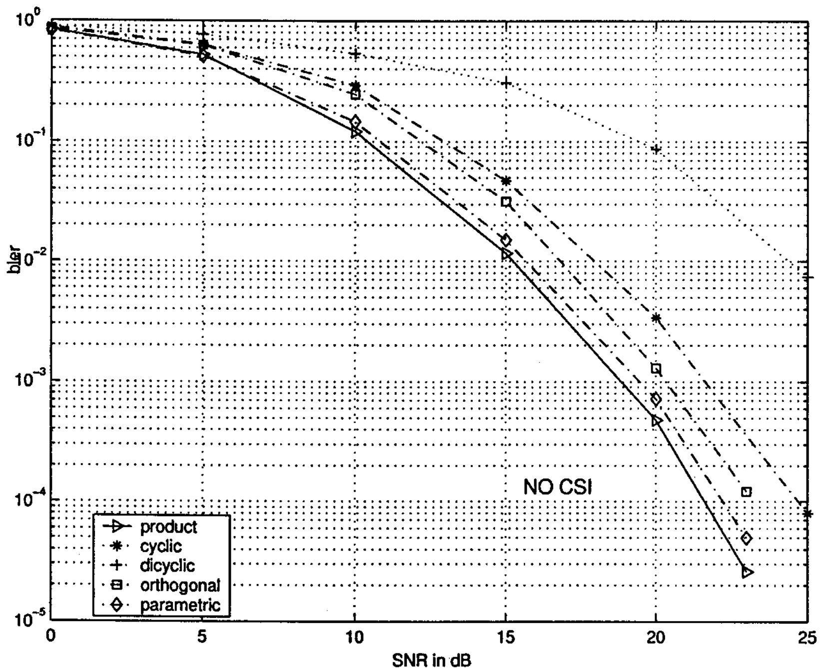

Figure 5.2: Block error rate performance for $M = 2$ and $N = 2$ at $R = 3.00$ of the product $\mathcal{P}_{2\times2}$, dicyclic group, cyclic group, orthogonal design and parametric code

5.3 Product vs. Orthogonal, Cayley and TAST Code Designs

Figure. 5.3 compares the block error rate performance for $M = 2$ transmitter antennas, $N = 1$ receiver antenna at high rate $R = 6.00, L = 4096$ of the product $\mathcal{P}_{2\times2}$ with $\zeta = 0.0685$ and orthogonal design with 64^{th}-roots of unity $\zeta = 0.0347$ [28], Cayley code $Q = 4$ [12] and differential TAST code $\mathcal{T}_{2,2,2}$ [10] with an exhaustive search ML decoder. We can see that our proposed constellation also outperforms these other three designs.

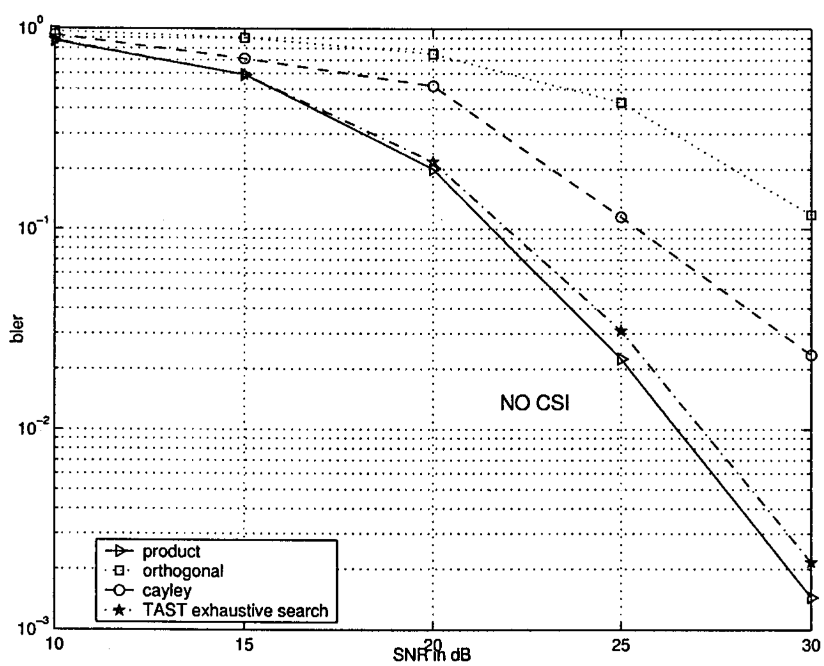

Figure 5.3: Block error rate performance for $M = 2$, $N = 1$ at $R = 6.00$ of the product $\mathcal{P}_{2\times2}$, orthogonal design, Cayley code and TAST code

5.4 Hamiltonian vs. Fixed-point Free Group Designs

Figure. 5.4 compares the block error rate performance for $M = 3$ transmitter antennas, $N = 1$ and 2 receiver antenna at the same $R = 1.06$ and $L = 9$ of the Hamiltonian $\mathcal{H}_{3\times3}$ with $\zeta = 0.6632$ and fixed-point free group $G_{9,1}$, $\zeta = 0.6004$ [26]. We can see that the Hamiltonian constellation outperforms the fixed-point free group. The gain improvement is ≈ 1 dB.

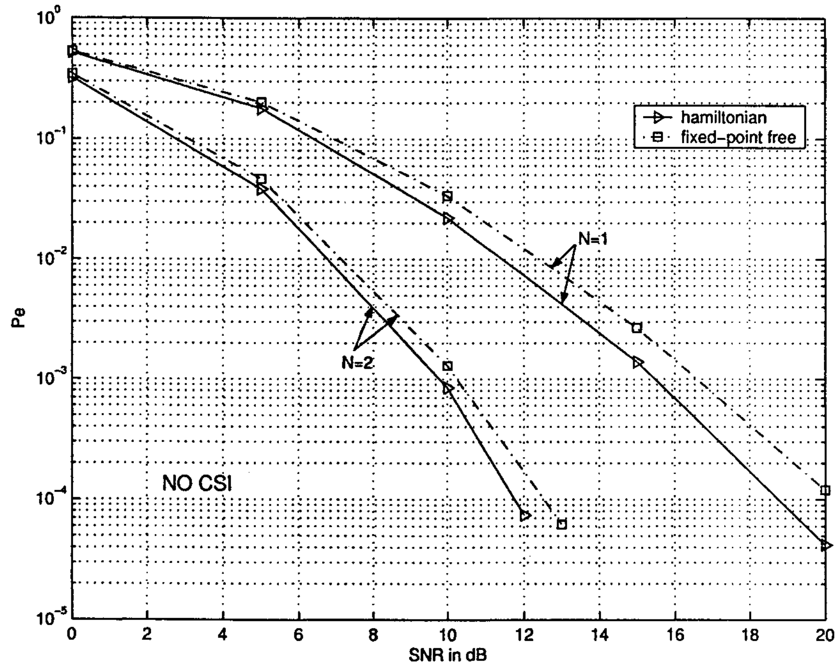

Figure 5.4: Block error rate performance for $M = 3, N = 1$ and 2 at $R = 1.06$ of the Hamiltonian $\mathcal{H}_{3\times3}$ and fixed-point free group

5.5 Hamiltonian vs. Cyclic Group, Cayley and Orthogonal designs

Figure. 5.5 shows the bler performance of different 4×4 constellations for $M = 4$ transmitter antennas and $N = 1$ receiver antenna: Hamiltonian $\mathcal{H}_{4\times4}$ at $R = 2.00$ with $\zeta = 0.3320$, cyclic group $u = (1, 25, 97, 107)$ at $R = 2.00$ with $\zeta = 0.2208$ [16] , cayley code with $Q = 7, r = 2$ at $R = 1.75$ [12] and orthogonal design as (2.23) with x, y, z are chosen from 6-PSK at $R = 1.94$ [33]. We can see that our Hamiltonian constellation outperforms other three designs.

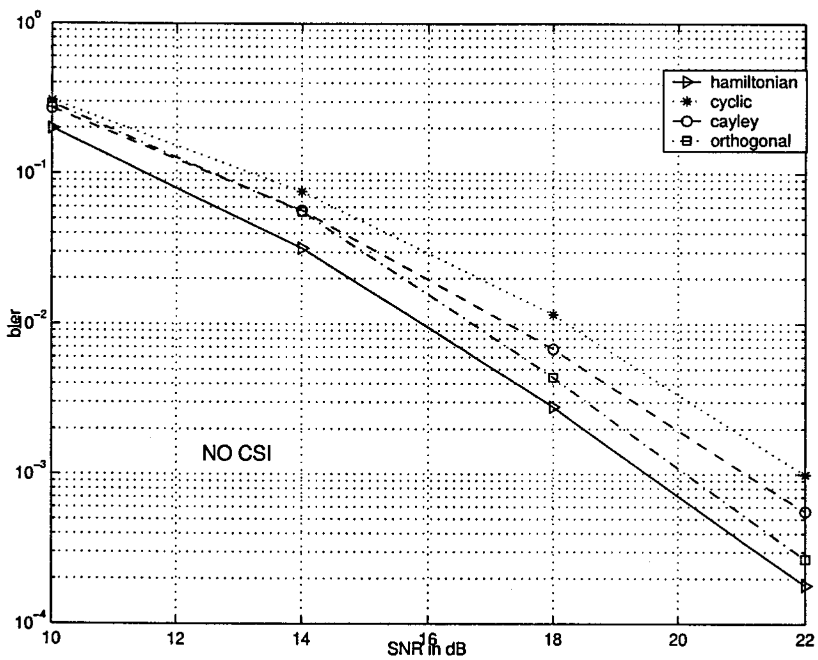

Figure 5.5: Block error rate performance for $M = 4$ and $N = 1$ of the Hamiltonian $\mathcal{H}_{4\times4}$ $R = 2.00$, cyclic group $R = 2.00$, Cayley code $R = 1.75$ and orthogonal design $R = 1.94$

5.6 Our Proposed Constellations at $R = 1.00$ for $M = 2, 3$ and 4

Figure. 5.6 displays the block error rate performance of our proposed constellations of $\mathcal{H}_{2\times2}$ with $\zeta = 0.8165$, $\mathcal{H}_{3\times3}$ with $\zeta = 0.6588$ in Table 3.4 and $\mathcal{P}_{4\times4}$ with $\zeta = 0.6580$ in Table 4.1 at the same $R = 1.00, N = 1$ for $M = 2, 3, 4$ transmitter antennas respectively. We can see that $\mathcal{P}_{4\times4}$ performs very well for high SNR.

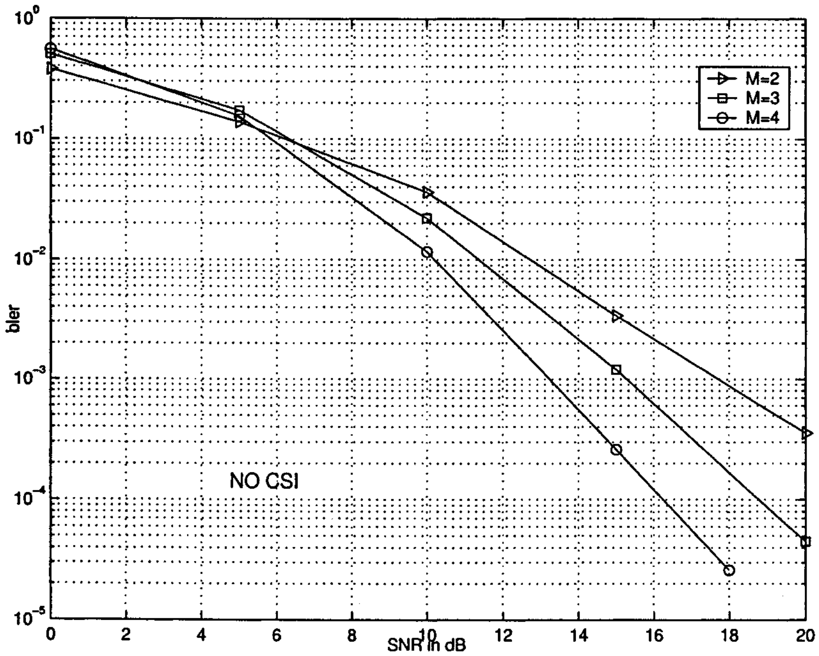

Figure 5.6: Block error rate performance for $M = 2, 3, 4$, and $N = 1$ at $R = 1.00$ of our proposed constellations, $\mathcal{H}_{2\times2}, \mathcal{H}_{3\times3}$ and $\mathcal{P}_{4\times4}$ respectively

Chapter 6

Conclusion

6.1 Summary

We have constructed new unitary space-time constellations with high diversity products, which can be used for any number of transmitter antennas and for any data rate. Our Hamiltonian and product constellations have full diversity and can be used for both known and unknown channels with differential modulations. We use the cyclic group codes for constructing 2×2 Hamiltonian constellation for any cardinality as shown in (3.34). The Hamiltonian constellations for any number of transmitter antennas can be constructed by the direct sum of 2×2 Hamiltonian matrices and the roots of unity. Product constellations, $\mathcal{P}$, are constructed by the matrix product method of Hamiltonian and cyclic group representation. According to the limitation of possible constellations of product $\mathcal{P}$ when the number of transmitter antennas are odd and the gcd of the order of Hamiltonian constellation and cyclic group is greater than one, we have also presented another structure of product constellation, $\mathcal{P}_{\mathcal{H}}$ which is the product of two Hamiltonian constellation with the diagonal blocks in different order. This product method increases the rate of constellations and also gives an excellent value of the diversity product. Although the Hamiltonian constellations and the product constellations do not form groups, the optimization of the diversity product requires checking only $L-1$ distinct matrices in their constellations, which is comparable with group designs. Our Hamiltonian constellations for $L \leq 5$ for the case where M is even achieve the optimal proven bound of the largest possible value of diversity product. In addition, many of our proposed constellations have the best known diversity product in the literature, as listed in Table 3.4 and 4.1, and outperform all other constellation designs as shown in Chapter 5.

6.2 Future Works

In this thesis, for simplicity, we choose cyclic groups to generate group codes. There are still many abstract groups which we have not investigated, e.g., finite reflection groups. It has shown in [7, 25] that several good group codes can be generated by reflection groups. For example, from the Table 8.3, pp.249 in [7], the subgroup of finite reflection group $\mathcal{I}_4$ has $d_{\min} = 0.6181$ for $L = 120$. This gives the 2×2 Hamiltonian constellation, which is constructed from this subgroups, have a diversity product =0.6181/2=0.3090 which equals the excellent fixed-point free group $SL_2(\mathbb{F}_5)$ and our product $\mathcal{P}$ at the same L. While the 2×2 Hamiltonian constellations which are constructed from the symmetric group S_5 with $d_{\min} = 0.4472$ in Chapter 3 and cyclic group with $d_{\min} = 0.4194$ for $L = 120$ have diversity products 0.2236 and 0.2097 respectively.

Taking the tensor product of two different Hamiltonian constellations is also a possible extension of the work of this thesis. Not only does tensor product increase the data rate, it also increases the dimension of the constellation with the small number of optimized variables. This may be found useful when working with the large number of transmitter antennas, which typically requires computationally intensive optimization. For example, for $M = 16$, a cyclic group design in (2.26) has $u = (u_1, u_2, \ldots, u_{12})$ 12 variables for optimization the diversity product. While the tensor product with the same $M = 16$, $\mathcal{H}^1_{4\times4} \otimes \mathcal{H}^2_{4\times4}$, has only 10 variables for optimization (x_1 and $k = (k_1, k_2, k_3, k_4)$ from each $\mathcal{H}_{4\times4}$).

Concatenation of our proposed constellations and forward error correction (FEC) codes is another method for increasing the performance of coding gain. There were many works which have already studied the performance of concatenated space-time codes with different FEC codes, for example, convolutional codes, turbo codes and trellis code modulations were considered in [23]. Also it has been shown that these FEC codes improve the performance of the systems. Most of these works were considered with the case of space-time block codes from orthogonal designs [1, 30]. It is thus interesting to concatenate these FEC codes with our proposed constellations, since the performance of Hamiltonian and product constellations are superior to those orthogonal designs.

When the data rate R and the number of transmitter antennas M are large, the maximum-likelihood decoder will have a high exponential complexity in M and R. Thus fast decoding algorithm is another interesting extension that can be considered for decoding the proposed constellations. These algorithms which are based on lattice decoding have been introduced for nongroup constellations in [26] and for diagonal constellations in [6] with a polynomial complexity in R and M.

Only for the case of Hamiltonian constellations for $M = 2$ and 3 that we use the simplex method for optimization the diversity product. Thus another venue of further research is to use the advanced optimization techniques in computer science or mathematics to improve the results obtained by Hamiltonian constellation designs as given in (3.41) and (3.43) for M even and odd respectively when $M \geq 4$.

Appendix A

Representation Theory of Groups

A.1 Group Theory

• A set G of elements with a binary operation is called a *group* if G satisfies [5]

1. *Closure.* $\forall a, b \in G, ab \in G$

2. *Identity.* There is an element $e \in G$ such that $ae = ea = a$ for every $a \in G$.

3. *Inverse.* For each $a \in G$. There is a unique solution, a^{-1}, such that $a^{-1}a = aa^{-1} = e$.

4. *Associativity.* $a(bc) = (ab)c$ for all $a, b, c \in G$.

A group G is called *an abelian group* if it is also *commutative*, meaning $ab = ba$ for all $a, b \in G$.

• The *order* or *cardinality* of a group G, $|G|$, is the number of elements in G.

• H is a *subgroup* of G, $H \subset G$, if H is also a group itself under the binary operation from G. A subgroup H satisfies $e \in H$ and $hk^{-1} \in H$ for every $h, k \in H$, and $|H|$ must divide $|G|$. A subgroup N of G is a *normal subgroup* of G iff $g^{-1}Ng = N$ for all $g \in G$. We write $N \lhd G$.

• *The symmetric group* S_n is the set of all permutations on n symbols, which forms a group. We call n *the degree* of the symmetric group. $|S_n| = n!$.

Every element in S_n can be written as a product of transpositions, e.g., $(1234) = (123)(34) = (12)(23)(34)$. In fact S_n is generated by $\{(12), (23), \ldots, (n-1\ n)\}$. *Note*: In this thesis, we take the action of a product of cycles from right to left, e.g.,

$$(12)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123).$$

A symmetric group S_n has subgroups which are symmetric groups of smaller degree, e.g.,

$$\begin{aligned}
S_2 &= \{1, (12)\} \\
S_3 &= \{1, (12), (13), (23), (123), (132)\} \\
S_4 &= \{1, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (23)(14), (123), (132), \ldots \\
&\quad (124), (142), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}
\end{aligned}$$

with $S_2 \subset S_3 \subset S_4$.

• *The Alternating group* A_n is a normal subgroup of S_n, $A_n \lhd S_n$ consisting of all $g \in S_n$ which can be expressed as a product of an even number of transpositions.

A.2 Representation Theory

• A representation of a finite group G over the field F is a homomorphism ρ defined in [21] as

$$\rho : G \longmapsto GL(n, F) \tag{A.1}$$

where $GL(n, F)$ is the general linear group consisting of $n \times n$ invertible matrices, and F can be either $\mathbb{R}$ or $\mathbb{C}$. We call ρ *an n-dimensional representation* or *a representation degree n* of G. n is called *the degree or dimension* of ρ. $\rho = \{\rho(g) : \text{for all } g \in G\}$. The representation ρ satisfies:

1. $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.
2. $\rho(1) = I_n$ where I_n is the $n \times n$ identity matrix.
3. $\rho(g^{-1}) = \rho(g)^{-1}$

• The representation ρ is *a unitary representation* if, for all $g \in G$, $\rho(g)$ satisfies $\rho(g)\rho(g)^* = I_n$ where $\rho(g)^*$ is the conjugate transpose of $\rho(g)$.

• The representation ρ is called *a faithful representation* if ρ is a one-to-one mapping between g and $\rho(g)$, for all $g \in G$. If there exist $g \neq h \in G$ such that $\rho(g) = \rho(h)$, then we call ρ *an unfaithful representation.*

• Let ρ and ρ' be two n-dimensional representations of G. These two representations will be *equivalent* if there exists an $n \times n$ nonsingular matrix T such that $\rho(g) = T^{-1}\rho'(g)T$ for all $g \in G$.

• If ρ is a reducible representation of G, then there is a unitary matrix T such that it ρ can be written as the direct sum of irreducible representation matrices ρ^i and ρ^j as follows

$$T^{-1}\rho(g)T = \rho^i(g) \oplus \rho^j(g) = \begin{bmatrix} \rho^i(g) & 0 \\ 0 & \rho^j(g) \end{bmatrix} \tag{A.2}$$

where ρ is an $l_i + l_j$-dimensional reducible representation. l_i and l_j are the dimension of ρ^i and ρ^j respectively.

• Let $l_1, l_2, \ldots l_c$ be the dimensions of all the inequivalent irreducible representations of G. Then we have [27] $\sum_{i=1}^{c} l_i^2 = |G|$, e.g., $|S_3| = 3! = 6 = 2^2 + 1^2 + 1^2$. This means that a symmetric group S_3 has 2 one-dimensional irreducible and 1 two-dimensional irreducible representations all inequivalent.

• *The natural representation of* S_n is by $n \times n$ matrices of zeros and exactly one one in each row and column which correspond to the action of $g \in S_n$, e.g., the natural representation of S_4 satisfies

$$\rho_n(134) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \rho_n(1234) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

• *The character*, χ of ρ is $\chi(g) = \mathrm{Tr}(\rho(g)) = \sum_{i=1}^{n} \rho(g)_{ii}$, where Tr denotes the trace of a matrix.

A.3 Four-Dimensional Irreducible Representation of S_5

Let's first consider the 5 basic basis vectors in $\mathbb{R}^5$:

$$v_1 = (1,0,0,0,0) \tag{A.3}$$
$$v_2 = (0,1,0,0,0) \tag{A.4}$$
$$v_3 = (0,0,1,0,0) \tag{A.5}$$
$$v_4 = (0,0,0,1,0) \tag{A.6}$$
$$v_5 = (0,0,0,0,1) \tag{A.7}$$

We want to express this with S_5 acting by the natural representation in the form $1 \oplus \rho$, where ρ is the 4-dimensional irreducible representation of S_5. The trivial representation 1 will be spanned by a vector u satisfying $gu = u$ for all $g \in G$. Hence u is the sum of five basis vectors as above:

$$u = v_1 + v_2 + v_3 + v_4 + v_5 = (1,1,1,1,1) \tag{A.8}$$

Using the Gram Schmidt method to get new 5 orthonormal basis vectors, $\{\psi_i\}_{i=1}^5$ of u, v_1, v_2, v_3, v_4 gives

$$\psi_1 = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \tag{A.9}$$
$$\psi_2 = \left(\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right) \tag{A.10}$$

$$\psi_3 = \left(0, \frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right) \tag{A.11}$$

$$\psi_4 = \left(0, 0, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \tag{A.12}$$

$$\psi_5 = \left(0, 0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \tag{A.13}$$

Now we can start to find the representation of S_5 with the orthonormal basis vectors $\{\psi_i\}_{i=1}^5$ as above. For $g = (12)$

$$\begin{aligned}
\rho(12)\psi_1 &= \left(\tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}\right) &&= \psi_1 \\
\rho(12)\psi_2 &= \left(-\tfrac{1}{2\sqrt{5}}, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{2\sqrt{5}}, -\tfrac{1}{2\sqrt{5}}, -\tfrac{1}{2\sqrt{5}}\right) &&= -\tfrac{1}{4}\psi_2 + \tfrac{\sqrt{15}}{4}\psi_3 \\
\rho(12)\psi_3(12) &= \left(\tfrac{\sqrt{3}}{2}, 0, -\tfrac{1}{2\sqrt{3}}, -\tfrac{1}{2\sqrt{3}}, -\tfrac{1}{2\sqrt{3}}\right) &&= \tfrac{\sqrt{15}}{4}\psi_2 + \tfrac{1}{4}\psi_3 \\
\rho(12)\psi_4 &= \left(0, 0, \sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right) &&= \psi_4 \\
\rho(12)\psi_5 &= \left(0, 0, 0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) &&= \psi_5
\end{aligned} \tag{A.14}$$

Consequently the representation matrix of $g = (12)$ will be

$$\rho(12) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly we also do for $g = \{(23), (34), (45)\}$.

$$\rho(12) = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad \rho(23) = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ 0 & 0 & \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rho(34) = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{array}\right], \quad \rho(45) = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \tag{A.15}$$

This demonstrates the five-dimensional reducible representation of S_5 as the direct sum of a one-dimensional and a four-dimensional irreducible representation. By deleting the first row and column of these matrices, we will get the four-dimensional irreducible representation for S_5 as shown in (3.19). And from the previous section, these elements $g = \{(12), (23), (34), (45)\}$ generate S_5.

Appendix B

$(L, 4)$ Cyclic Group Codes

Table B.1: The values of k_1, k_2 and $d_{\min}$ of an $(L, 4)$ cyclic group code for $L = 3$ to 100

L	k_1, k_2	y_1	y_2	$d_{\min}$
3	1,1	0.6667	0.6667	1.7321
4	1,2	1.0000	0.5000	1.6330
5	1,2	0.8000	0.8000	1.5811
6	1,2	1.0000	1.0000	1.4142
7	1,2	0.7754	1.3972	1.3569
8	1,3	1.0000	1.0000	1.4142
9	1,3	1.3333	1.1254	1.2755
10	1,3	1.3333	1.3333	1.2247
11	1,3	1.0547	1.6043	1.2265
12	1,4	2.0000	1.1547	1.1260
13	1,5	1.5251	1.5251	1.1452
14	1,4	1.8529	1.4859	1.0946
15	1,4	1.6793	1.6793	1.0913
16	1,4	2.0000	1.8478	1.0196
17	1,4	2.0507	2.0507	0.9876
18	1,5	1.9886	2.0114	1.0000
19	1,4	1.7210	2.5271	0.9704
20	1,8	2.0000	2.3416	0.9598
21	1,4	2.0170	3.0101	0.8920
22	1,6	2.9680	1.6689	0.9288
23	1,5	2.2906	2.4042	0.9230
24	1,5	2.5798	2.5798	0.8805
25	1,10	2.8944	2.3763	0.8712
26	1,10	2.5395	3.1517	0.8384
27	1,6	3.6279	2.3019	0.8213
28	1,6	3.1003	2.4724	0.8472

L	k_1, k_2	y_1	y_2	$d_{\min}$
29	1,6	3.6473	2.6141	0.7993
30	1,12	4.0000	2.3940	0.7909
31	1,13	2.8146	3.7617	0.7799
32	1,7	3.3597	3.4687	0.7654
33	1,6	3.5103	3.3127	0.7657
34	1,6	3.1807	3.5107	0.7732
35	1,10	2.8944	4.2238	0.7496
36	1,15	4.2929	3.4822	0.7173
37	1,14	2.6993	4.8839	0.7263
38	1,16	3.5938	3.9847	0.7265
39	1,7	4.2873	3.4036	0.7212
40	1,7	4.4288	3.5631	0.7075
41	1,9	4.3332	4.3332	0.6794
42	1,12	4.0000	4.2598	0.6959
43	1,12	3.6939	5.4336	0.6620
44	1,8	6.0330	3.3162	0.6541
45	1,19	4.2743	4.2743	0.6840
46	1,7	4.1756	4.6327	0.6739
47	1,20	4.8171	4.4954	0.6554
48	1,18	4.0000	5.7889	0.6392
49	1,14	5.3119	4.2809	0.6457
50	1,22	3.7121	7.0173	0.6281
51	1,9	6.7534	3.5161	0.6241
52	1,22	5.5698	4.2556	0.6380
53	1,8	5.2471	4.7076	0.6339
54	1,12	4.0000	6.7300	0.6106
55	1,8	6.4912	5.0281	0.5893
56	1,16	6.8284	4.2942	0.5997
57	1,24	6.0615	5.2282	0.5952
58	1,22	5.2667	6.1111	0.5929
59	1,26	4.3224	7.1725	0.5899
60	1,9	6.8794	4.7604	0.5862
61	1,8	5.3279	6.1478	0.5904
62	1,8	6.0135	6.3310	0.5692
63	1,14	5.3119	6.7658	0.5755
64	1,27	7.9212	4.6904	0.5632

L	k_1, k_2	y_1	y_2	$d_{\min}$
65	1,10	9.0554	4.5325	0.5426
66	1,10	7.9076	4.6914	0.5635
67	1,10	8.7143	4.8031	0.5440
68	1,9	6.7664	6.0408	0.5589
69	1,26	7.7917	5.3910	0.5509
70	1,8	6.0576	8.0008	0.5334
71	1,26	4.3366	9.9458	0.5292
72	1,16	6.8284	6.7884	0.5420
73	1,11	9.7843	4.7240	0.5251
74	1,22	5.5923	8.1975	0.5386
75	1,33	6.7662	7.0296	0.5385
76	1,10	7.9140	6.1136	0.5340
77	1,9	6.6639	7.6730	0.5282
78	1,9	7.0442	7.8618	0.5180
79	1,35	8.0492	7.7348	0.5034
80	1,22	7.1893	8.6297	0.5029
81	1,18	8.5486	6.8038	0.5104
82	1,30	5.6197	10.1728	0.5033
83	1,15	6.0251	10.1287	0.4976
84	1,37	8.3464	7.1429	0.5082
85	1,10	8.7667	7.5714	0.4948
86	1,10	8.0157	7.7524	0.5037
87	1,9	7.1012	9.7176	0.4877
88	1,16	6.8284	9.8866	0.4892
89	1,20	9.8275	8.2538	0.4703
90	1,20	10.4721	6.8147	0.4810
91	1,25	7.1506	10.3100	0.4786
92	1,12	12.1560	6.2110	0.4667
93	1,41	10.3330	7.2279	0.4773
94	1,28	8.9429	8.4609	0.4794
95	1,11	10.0481	7.8128	0.4732
96	1,10	8.0963	9.5945	0.4755
97	1,29	9.7809	8.9649	0.4619
98	1,36	10.3693	9.1083	0.4532
99	1,18	8.5486	9.9100	0.4655
100	1,44	11.3462	7.4389	0.4614

Appendix C

Simplifications of Optimization Diversity Products

C.1 For Hamiltonian Constellation $\mathcal{H}$

We prove that to compute the diversity product of (3.36), it suffices to choose $l = 0$ for H_l and arbitrary $l' = 1, \ldots, L-1$ for $H_{l'}$. This proof can also be worked for the general $M \times M$ case. We show that $|\det(H_l - H_{l'})| = |\det(H_0 - H_{l'-l})|$ for $0 \leq l < l' \leq L-1$.

Proof:

$$\begin{aligned} |\det(H_l - H_{l'})| &= |\det(e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l - e^{j\frac{2\pi l' k_1}{L}} R_{l'} H_0 T_{l'})| \qquad \text{from (3.35).} \\ &= |e^{j2\pi l k_1}||\det R_l||\det(H_0 - e^{j\frac{2\pi(l'-l)k_1}{L}} R_{l'-l} H_0 T_{l'-l})||\det T_l| \\ &= |\det(H_0 - H_{l'-l})| \end{aligned}$$

Since $|e^{j\frac{2\pi l k_1}{L}}| = |\det R_l| = |\det T_l| = 1$. $\square$

C.2 For Product Constellation $\mathcal{P}$

We prove that to compute the diversity product of $\mathcal{P}$ in (4.4) it suffices to choose $l = g = 0$ for P, and arbitrary $0 \le l' \le L_H - 1,\ 0 \le g' \le L_C - 1$ and $(l', g') \neq (0,0)$ for P'. The proof is shown for the 2×2 case, from which the general $M \times M$ case follows immediately. We will show that $|\det(P - P')| = |\det(H_l O_g - H_{l'} O_{g'})| = |\det(H_0 - H_{l'-l} O_{g''})|$ for $0 \le l < l' \le L_H - 1$ and $0 \le, g, g', g'' \le L_C - 1$.

Proof:

$$\begin{aligned}
|\det(H_l O_g - H_{l'} O_{g'})| &= |\det(e^{j\frac{2\pi l g_1}{L_H}} R_l H_0 T_l O_g - e^{j\frac{2\pi l' g_1}{L_H}} R_{l'} H_0 T_{l'} O_{g'})| \\
&= |\det(e^{j\frac{2\pi l g_1}{L_H}} R_l H_0 T_l - e^{j\frac{2\pi l' g_1}{L_H}} R_{l'} H_0 T_{l'} O_{g''})| \\
&= |e^{-j\frac{2\pi l g_1}{L}}||\det R_l||\det(H_0 - e^{j\frac{2\pi (l'-l) g_1}{L_H}} R_{l'-l} H_0 T_{l'-l} O_{g''})||\det T_l| \\
&= |\det(H_0 - e^{j\frac{2\pi (l'-l) g_1}{L_H}} R_{l'-l} H_0 T_{l'-l} O_{g''})| \\
&= |\det(H_0 - H_{l'-l} O_{g''})|
\end{aligned}$$

The second equality is from $O_{g'} O_{g^{-1}} = O_{g''} \in \mathcal{C}$ □

C.3 For Product Constellation $\mathcal{P}_{\mathcal{H}}$

We prove that to compute the diversity product of $\mathcal{P}_{\mathcal{H}}$ in (4.14), it suffices to choose $l = g = 0$ for $J_l J_g^\dagger$, and arbitrary $0 \le l' \le L_{H_1} - 1,\ 0 \le g' \le L_{H_2} - 1$ and $(l', g') \neq (0,0)$ for $J_{l'} J_{g'}^\dagger$. The proof is shown for a 3×3 case; for a case of general $M \times M$ odd case follows immediately. The 3×3 J_l can be written as

$$J_l = K_l R_l J_0 T_l \tag{C.1}$$

where

$$K_l = \begin{bmatrix} e^{j\frac{2\pi g_1 l}{L_{H_1}}} & 0 & 0 \\ 0 & e^{j\frac{2\pi g_2 l}{L_{H_1}}} & 0 \\ 0 & 0 & e^{j\frac{2\pi g_2 l}{L_{H_1}}} \end{bmatrix}, R_l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j\frac{2\pi g_2 l}{L_{H_1}}} & 0 \\ 0 & 0 & e^{-j\frac{2\pi g_3 l}{L_{H_1}}} \end{bmatrix}$$

$$J_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{y_1} & -\sqrt{y_2} \\ 0 & \sqrt{y_2} & \sqrt{y_1} \end{bmatrix} \text{ and } T_l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j\frac{2\pi g_2 l}{L_{H_1}}} & 0 \\ 0 & 0 & e^{j\frac{2\pi g_3 l}{L_{H_1}}} \end{bmatrix}.$$

Similarly, we can also write the 3×3 $J_g^\dagger$ as

$$J_g^\dagger = K_l^\dagger R_l^\dagger J_0^\dagger T_l^\dagger \tag{C.2}$$

where

$$K_l^\dagger = \begin{bmatrix} e^{j\frac{2\pi r_2 g}{L_{H_2}}} & 0 & 0 \\ 0 & e^{j\frac{2\pi r_2 g}{L_{H_2}}} & 0 \\ 0 & 0 & e^{j\frac{2\pi r_3 g}{L_{H_2}}} \end{bmatrix}, R_l^\dagger = \begin{bmatrix} e^{-j\frac{2\pi r_1 g}{L_{H_2}}} & 0 & 0 \\ 0 & e^{-j\frac{2\pi r_2 g}{L_{H_2}}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_0^\dagger = \begin{bmatrix} \sqrt{y_1} & -\sqrt{y_2} & 0 \\ \sqrt{y_2} & \sqrt{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T_l^\dagger = \begin{bmatrix} e^{j\frac{2\pi r_1 g}{L_{H_2}}} & 0 & 0 \\ 0 & e^{-j\frac{2\pi r_2 g}{L_{H_2}}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \ D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1}c \end{bmatrix}$$

where a, b, c are arbitrary number and $|a| = |b| = |c| = 1$. $A = DE$, and also $A^{-1} = D^{-1}E^{-1}$. We will use this property in our proof:

$$J_0 A J_0^\dagger = D J_0 J_0^\dagger E \tag{C.3}$$

We note that D, D^{-1} and E, E^{-1} can commute with J_0 and $J_0^\dagger$ (as well as any diagonal matrices) respectively.

Proof: $|\det(J_l J_g^\dagger - J_{l'} J_{g'}^\dagger)|$

$$\begin{aligned}
&= |\det(K_l R_l J_0 T_l K_g^\dagger R_g^\dagger J_0^\dagger T_g^\dagger - K_{l'} R_{l'} J_0 T_{l'} K_{g'}^\dagger R_{g'}^\dagger J_0^\dagger T_{g'}^\dagger)| \\
&= |\det(J_0 T_l R_g^\dagger J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \\
&= |\det(J_0 A J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \quad \text{where } A = T_l R_g^\dagger. \\
&= |\det(D J_0 J_0^\dagger E - K_{l'-l} R_{l'-l} J_0 T_{l'} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \quad \text{from the property in (C.3).} \\
&= |\det D||\det(J_0 J_0^\dagger - D^{-1} K_{l'-l} R_{l'-l} J_0 T_{l'} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger E^{-1})||\det E| \\
&= |\det(J_0 J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'} D^{-1} E^{-1} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \\
&= |\det(J_0 J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'} A^{-1} R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \\
&= |\det(J_0 J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'} T_{-l} R_{-g}^\dagger R_{g'}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \\
&\quad \text{where } A^{-1} = (T_l R_g^\dagger)^{-1} = T_{-l} R_{-g}^\dagger. \\
&= |\det(J_0 J_0^\dagger - K_{l'-l} R_{l'-l} J_0 T_{l'-l} R_{g'-g}^\dagger K_{g'-g}^\dagger J_0^\dagger T_{g'-g}^\dagger)| \\
&= |\det(J_0 J_0^\dagger - J_{l'-l} J_{l'-l}^\dagger)|
\end{aligned}$$

□

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