

HOLOGRAMS, SPACEPLATES, AND THE PROPAGATION OF LIGHT

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ABSTRACT

The miniaturization of optical systems has been a longstanding interest for physicists. By facilitating the design of smaller optical systems, we can improve their versatility and cost-effectiveness. This aim applies to macroscopic imaging systems, technologies that implicitly image, and micrometer-scale optics. Parallel to this, quantum optical devices have also seen rapid developments. Notably, the need for new quantum communications and quantum imaging devices has recently risen. The thesis outlines advancements in both of these areas and, in many ways, bridges gaps between them. It discusses the development of optics that compress free space, the design of holographic optical elements, and the generation of entangled photon states in thin-film sources.

First, we describe an optic designed to miniaturize free space, termed the *spaceplate*. Spaceplates achieve the propagation of light for a distance greater than their thickness. Therefore, they compress optical space, reducing the required distance between optical elements in an imaging system. In this thesis, we describe a spaceplate based on conventional optics in a $4f$ arrangement, mimicking the transfer function of free space in a thinner system - we term this device a three-lens spaceplate. It is broadband, polarization-independent, and achieves meter-scale space compression. We experimentally measure compression ratios up to 15.6, replacing up to 4.4 meters of free space, three orders of magnitude greater than previous spaceplates. We demonstrate that three-lens spaceplates reduce the length of a full-color imaging system, albeit with reductions in resolution and contrast. We also present theoretical limits on the numerical aperture and the compression ratio. Our design presents a simple, accessible, cost-effective method for optically compressing large volumes of space.

Second, we discuss the design of holographic optical elements. Holograms are extraordinarily versatile optics. They have many applications, including interferometry, spectrometry, data storage, optical filtration, and sensing. We can design various optical elements such as filters, lenses, beam splitters, and solar concentrators by tailoring the phase response of a hologram. In this thesis, we describe the nature and function of holograms, and we experimentally characterize holography in lithium niobate and photopolymers. Using this characterization, we assess the limitations of different holographic analysis methods. Further, we describe novel holographic optical elements like the holographic spaceplate - a holographic optic element whose phase response mimics free space.

Lastly, this thesis outlines the production of entangled photon pairs, or biphotons, via non-phase-matched spontaneous parametric down-conversion in micrometer- and nanometer-scale devices. By producing biphotons in micrometer-scale crystals rather than in bulk, as is done traditionally, we are

allowed to ignore phase matching. These devices produce broadband emission in both angle and frequency not seen in phase-matched bulk sources. We measure entangled biphotons produced via spontaneous parametric down-conversion in gallium arsenide (GaAs) and lithium niobate - both undoped and iron-doped. Lastly, we outline and present initial experiments towards a holographic spontaneous parametric down-conversion optic that combines photon production and mode sorting - an optic of cascaded miniaturization.

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AUTHOR CONTRIBUTIONS

This thesis is a culmination of the work done during my Master's degree. It compiles contributions by me, Jeff S. Lundeen, Michael T. Weil, Maria V. Chekhova, and Vitaliy Sultanov. All work contained herein is my work including all figures, derivations, and experiments, with exceptions listed below or cited in the text. The following authors contributed to

- **Chapter 2:** Sections 2.1 and 2.2 report background work done by the Lundeen Lab and other groups. Section 2.3 introduces a new spaceplate device originally conceptualized by Jeff Lundeen and explored by Nicholas Sorensen and Michael Weil [1].
- **Chapters 3 and 4:** these chapters presents experimental and theoretcal work performed by Nicholas Sorensen and Michael Weil as guided by Jeff Lundeen [1];
- **Chapter 5:** Sections 5.1 and 5.2 review theoretical work on volume holography. Section 5.3 presents new simulation and analysis based on previous theory work;
- **Chapter 6:** this chapter presents experimental measurements and analysis done Nicholas Sorensen under the guidance of Jeff Lundeen;
- **Chapter 7:** Section 7.1 reviews background work on non-phase-matched spontaneous parametric down-conversion, and Section 7.2 proposes an experiment conceptualized by Maria Chekhova, Jeff Lundeen, and Nicholas Sorensen;
- **Appendix A:** this appendix presents experimental and theoretical work performed by Nicholas Sorensen and Michael Weil as guided by Jeff Lundeen;
- **Appendices B and C:** these appendices present derivations in the method of Kogelnik and Toal [2, 3];
- **Appendix D:** this appendix presents a derivation in the method of Gaylord and Moharam [4];
- **Appendix E:** this chapter presents experimental measurements and analysis done Nicholas Sorensen;
- **Appendix F and G:** these appendices present derivations originally done by Michael Weil;

- **Appendix H:** this appendix presents measurements done by Nicholas Sorensen as guided by Vitaliy Sultanov and Maria Chekhova.
- **Appendix I:** this appendix presents derivations done by Nicholas Sorensen.

All code can be found on Nicholas Sorensen's GitHub page, and is available upon request.

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LIST OF ABBREVIATIONS

- AR** anti-reflective 27
- ASC** Abbe sine condition xvi, 122, 126–129
- BBO** beta-barium borate xix, 188, 211
- BP** bandpass xvii, 93, 184, 187
- BS** beam splitter xiv, xvi, xvii, 79, 111, 184, 189
- CERC** Canada Excellence Research Chair iv
- CW** continuous wave xiv, xvi, xvii, 78, 79, 111, 183, 184, 210
- CWT** coupled wave theory 166
- DAQ** data acquisition device 79, 80
- FS** free-space 9
- FWHM** full width at half maximum 28, 183, 184
- GaAs** gallium arsenide (111) xvii, xviii, 180, 181, 189–194, 196
- GP** Glan-Taylor prism xvi, xvii, 78, 92, 111, 184
- HBT** Hanbury Brown Twiss xvii, xviii, 183, 184, 187
- HOE** holographic optical elements 3, 5, 51, 55, 57, 58, 60, 64, 74, 143
- HOM** Hong-Ou-Mandel xix, 199, 205, 211, 212

HWP half wave plate xiv, xvi, xvii, 78, 79, iii, 184–186

JSI joint spectral intensity 205

LiNbO₃:Fe iron-doped lithium niobate xvii, xviii, 180, 181, 186, 189–195

LiNbO₃:FS lithium niobate mounted on fused silica xvii, xviii, 180–182, 189–194

LiNbO₃:Si lithium niobate mounted on silicon xvii, xix, 180–182, 195, 197

LN lithium niobate xix, 211, 212

LP longpass xvii, xix, 184, 187, 188, 194, 198

MMF multi-mode fiber 189

MPL Max Planck Institute for the Science of Light iv

MTF modulation transfer function xiii, 34, 35

NA numerical aperture xvi, 14, 20, 39, 40, 121–123, 125–128

NQS National Quantum Strategy 3

OAM orbital angular momentum 101–103

PBS polarizing beam splitter 185, 186

PD photodiodes xiv, xvi, 79, iii

PSF point spread function 129, 130

QWP quarter wave plate xvii, 184

SHG second harmonic generation xviii, 181, 188, 189, 194–196

SLM spatial light modulators 49

SMF single-mode fiber 183, 187, 194

SNSPD superconducting nanowire single photon detectors xvii, 183, 184, 194

SPAD single photon avalanche detector xvi, 110, 111

SPDC spontaneous parametric down-conversion xvi, 4, 5, 101–105, 107, 109–111, 180, 181, 187, 188, 191, 194, 198

VR virtual reality 2, 3

I

INTRODUCTION

Technologies tend to become smaller and more compact with time and development. Examples of such devices are plentiful: Moore's law and the shrinking of the transistor come to mind, as well as the miniaturization of the radio. And the telephone. And the camera. One could write an entire dissertation on the history of miniaturized technologies. The convenience and advantages of a smaller form factor impel this miniaturization. Smaller devices often perform better and are less expensive and more manageable. To make smaller smartphones, for example, we need to make thinner smartphone camera

lenses; to make better photonic devices, we need to develop smaller and more reliable photonic chips; and so on. The drive for miniaturization is pervasive across many different fields, but, with particular relevance to this thesis, there is recent impetus to miniaturize optical technologies.

The development of flat optics has recently had much attention. Technologies such as microscopes, cameras, and spectrometers rely on optics like lenses and gratings, which can be made using flat optics. Refractive lenses, for example, manipulate and focus light in optical systems, and they function based on the thickness and curvature of their glass. Their thicknesses are on the scale of centimeters, and, therefore, optical systems composed of refractive lenses are bulky and heavy [5]. To combat this problem, one can instead use flat lenses, which have seen rapid development over the last decade. Flat lenses like diffractive lenses [6, 7] and Fresnel lenses [8] can be designed to replicate the effect of a lens using thinner materials. Flat optics can also be made from sub-wavelength structures known as metaatoms, comprising 2D arrays called metasurfaces; these metasurfaces can be designed, for example, to function as lenses. Such devices are called metalenses [5, 9]. To a similar effect, *photonic crystals* - periodic structures comprised of alternating stacks of differently-indexed dielectrics in one, two, or three dimensions - can create different flat optics [9]. Flat optics are also designed using holographic optics. Holograms can be made into thin lenses, solar concentrators, frequency-band sorters, and virtual reality (VR) technologies [6, 7, 10–16]. The thickness of the listed structures is frequently on the order of nanometers or micrometers, improving greatly upon thick optical systems.

In addition to these systems, miniaturization in quantum optics has seen rapid development. Driven by the need for safer communication and encryption, the desire for more robust, higher data-rate quantum networks and devices has grown increasingly [17–19]. It has led to innovations in on-chip quantum photonics devices [20, 21], new fabrication technologies [22, 23], and new sources of quantum

light [24–26]. These innovations enable exciting applications such as chip-to-chip quantum networking [27], quantum information processing, and high-speed transmission [17]. The Governments of Canada and other nations recognize this so-called *quantum revolution*, as demonstrated by Canada’s 360 million dollar National Quantum Strategy (NQS) and similar investments in the UK, the US, and other nations [28]. To contextualize these numbers, the quantum photonic global market is expected to rise from 0.5 billion CAD in 2023 to 4.4 billion CAD in 2030 [29]. This growth indicates the need to develop better, smaller, and more integrable, quantum technologies, as with their classical counterparts.

This thesis is broken into three, interrelated parts. The first discusses our work on a space-compressive optic called the *spaceplate*, recently developed by the Lundeen and Boyd labs. So far in this introduction, we have described the efforts made to make focusing elements like lenses smaller; however, the spaceplate miniaturizes the space *between* lenses, an often neglected element in optical systems. In this thesis, we describe a new spaceplate implementation, based on conventional optics in a $4f$ arrangement. Using this arrangement of lenses, the device mimicks the transfer function of free-space in a thinner system. It is termed a *three-lens spaceplate*.

The second part of this thesis discusses volume holography and the design of holographic optical elements (HOE). Firstly, the demand is increasing for HOEs of better fidelity and more versatility for applications in new technologies like solar concentrators, frequency band sorters, and VR technologies. We want to understand how to better design, fabricate, and characterize these devices. In this thesis we propose an analytical method to design the phase response of a HOE, which, until now, has not existed. Further, this thesis applies this method in the design of holographic spaceplates.

The final part of this thesis discusses SPDC in thin crystals. Traditionally, SPDC has been used to create entangled photon pairs for fundamental research and quantum applications. There are several limitations to using SPDC in integrated and miniaturized systems: firstly, pair production rate via SPDC scales quadratically with interaction length [30]. Therefore, to produce an adequate amount of entangled photons for, say, quantum communication applications, one traditionally needs to use large crystals on the scale of millimeters or even centimeters. Fortunately, recent work has demonstrated that biphotons produced at a moderate rate with high degrees of entanglement can be produced in materials of nano- or micro-scale thickness. We wish to integrate these devices with holographic optics to create a composite optic - one to create and sort biphotons.

Part I discusses compressive optics, introducing the concept of compressing free-space and the spaceplate more formally before reviewing various experimental implementations. After introducing previous work, we describe the work we have done towards a new spaceplate design made of three lenses.

In **Chapter 2**, we discuss light propagation and review various compressive optical devices. We then formalize the spaceplate and discuss previous implementations. Such implementations include those made of homogeneous materials, photonic crystals, and multilayer stacks, among others. Additionally, we describe a three-lens spaceplate.

Experiments on the three-lens spaceplate are covered in **Chapter 3**. Here, we describe the characterizations made on the three-lens spaceplate, including focus advancement, walkoff, and imaging experiments.

Then, in **Chapter 4**, we discuss the fundamental limitations of the three-lens design, comparing its limits to other spaceplates. We also compare its effect to that of other compressive optics.

In **Part II**, we introduce holography, its physical principles, applications, and design methodologies. Then, we overview characterization experiments done on different holographic media. Using the previously developed theory and our experimental characterizations, we propose a new analytic method of designing HOE.

Chapter 5 contains an introduction to rudimentary holography. It precedes a discussion of volume holography, which extends the principles of gratings and 2D holograms to thick geometries. First- and higher-order methods of holographic analysis are presented, along with a discussion of their limitations.

In **Chapter 6**, we present characterizations of holography in various media, including photopolymers and lithium niobate. Further, we outline the design of several HOEs such as an experimental holographic lens and a holographic spaceplate.

In **Part III**, we describe non-phase-matched SPDC in thin films, and we propose an experiment to sort biphotons produced via holographic SPDC.

Chapter 7 outlines a number of fundamental properties of SPDC, introducing phase matching in SPDC and spectral probability distributions in SPDC. This precedes an experimental proposal to sort biphotons produced via holographic SPDC.

Finally, **Chapter 8** summarizes each part of the thesis and outlines future work.

PART I

OPTICAL COMPRESSION OF FREE-SPACE

2

OPTICAL COMPRESSION AND THE SPACEPLATE

The miniaturization of optical systems, such as microscopes, telescopes, and cameras, has long been a topic of interest for physicists. By designing smaller optical systems, we can increase their versatility and reduce their cost. Most work has aimed at making the lenses thinner, e.g. Fresnel lenses [8], diffractive

lenses [31–36], and metalenses [5, 37–43]. Instead, we aim to miniaturize the space *between* lenses, an often neglected element in optical systems. Since this is usually the largest contributor to the total length of an imaging system, its miniaturization promises the largest impact.

In this chapter, we overview methods to reduce the length of optical systems and introduce the concept of the optical compression of free space. Further, we demonstrate how three lenses can be used to attain free-space compression on a much larger scale than previous spaceplate designs. In section Section 2.3 we present work that we published in 2023 [1]. Work from that publication is also presented in the next chapter. Other than that, this chapter presents no new material.

2.1 THE PROPAGATION OF LIGHT

One motif of this thesis is the propagation of light in different media. Specifically, we study how to compress the propagation of light, how light interacts with holographic gratings as it transmits through them, and how to shape propagating biphotons in a nonlinear material. Therefore, it is necessary to introduce how we model the propagation of electromagnetic radiation.

Light can be parameterized in different ways. One can describe light by its wavelength, polarization, intensity, angular momentum, evolution in time, and distribution in space. How electromagnetic waves interact with matter depends on how a material responds to any of these characteristics. One goal of optics and photonics is to describe this light-matter interaction for different materials and states of light.

One favourite tool of optical physicists is the plane wave. It does not exist in the real world, but it

allows us to study how light and matter interact. A plane wave is a monochromatic electromagnetic wave with a well-defined direction and a planar wavefront in a homogeneous medium. We can describe a plane wave at time t and position $\mathbf{r}$ by its complex amplitude, $\tilde{\mathbf{E}}$, given its angular frequency ω , and wavevector $\mathbf{k}$:

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathbf{E}_0 \exp [i(\mathbf{k} \cdot \mathbf{r} - \omega t)].$$

We use bolded notation and tildes to denote vector quantities and time dependence, respectively.¹ The phase of this plane wave is given by the term $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$. The wavevector $\mathbf{k} = (k_x, k_y, k_z)$ describes the angle of the plane wave and is orthogonal to the wavefront. We define $n = \sqrt{\epsilon}$ to be the refractive index of the propagation material, where ϵ is the relative electric permittivity. If, for instance, a plane wave travels at an angle θ from the z -axis, and it propagates a distance d along the z -axis in free-space (FS). It acquires a phase

$$\phi_{(\text{FS})} = k_z^{(\text{FS})} d = k^{(\text{FS})} d \cos \theta. \quad (2.1)$$

Therefore, the phase response of free-space is angle-dependent; we will generalize the phase response of a system later on. Alternatively, if the plane wave passes through some medium with a position-, polarization- or angle-dependent refractive index [$n = n(x \text{ or } \mathbf{E} \text{ or } \theta)$], it acquires a phase which is also

¹Whenever we directly measure an electromagnetic wave, we measure the time-averaged rate of energy flow per unit area of the wave, known as the optical intensity, $I = \epsilon_0 c E_0^2 / 2$.

position-, polarization- or angle-dependent.²

$$\phi(\mathbf{E} \text{ or } \theta) = n(\mathbf{E} \text{ or } \theta)k_z^{(\text{FS})}z. \quad (2.2)$$

We can therefore design the phase response of an optic by tailoring the refractive index of the optical material. For instance, optics called *phase masks* are used in many optics devices and experiments. They have spatially varying phase patterns, meaning that if an optical wave is transmitted through the phase mask, different parts of the optical wave will experience different phase shifts. The phase response of an optic is used to motivate the design of lenses and spaceplates.

PROPAGATION THROUGH A THIN LENS

A lens is simply a phase transformation. Transmission through a lens adds different amounts of phase to the light, depending on what transverse position the light passes through the lens. The added phase curves the wavefront and focuses the beam. For a positive spherical thin lens of focal length f , the added phase is a function of transverse distance $x^2 + y^2$ from the propagation axis:

$$\phi = -\frac{k(x^2 + y^2)}{2f}. \quad (2.3)$$

If we transmit a plane wave $\mathbf{k} = (k_x, k_y, k_z)$ through a lens, where the lens is aligned to the z -axis, the lens will add phase and curve the wavefront. This causes it to focus to a point (a, b) on the focal plane a distance f away from the thin lens. The position of this focus is well-defined, and we represent it in

²This is a simplification. The phase depends on the wave's path through the material, but the path, in turn, depends on the phase as per Fermat's principle of least time and stationary phase. In general, one must solve Maxwell's equations with appropriate boundary conditions.

the paraxial approximation:

$$(a, b) = \frac{f}{k}(k_x, k_y). \quad (2.4)$$

FOURIER OPTICS

Plane waves are appealing, but sometimes we want to understand how more complex optical fields behave in materials. Fields with multiple wavelengths or fields with arbitrarily shaped wavefronts, for instance, cannot be described by a single plane wave. To this aim, Fourier optics allows us to model all optical waves as superpositions of plane waves. To do this, we represent the field in terms of wavevector instead of in position. This distribution in k-space, F , is given by the Fourier transform ($\mathcal{F}$) of the incident field in Cartesian space:

$$f(x, y) \xrightarrow{\mathcal{F}} F(k_x, k_y),$$

for a beam propagating in the $\mathbf{z}$ -direction.

If the field propagates through some material or optical device, we can use the Fourier transfer function, $H(\mathbf{k})$, of the device to calculate the resulting field. The Fourier transfer function describes how the material affects the distribution of wavevectors in the incident field. It is the Fourier transform of the point spread function $h(\mathbf{r})$, which is the system response to a point source - an input containing

2.2. COMPRESSIVE OPTICAL DEVICES

the full range of wavevectors. Using these tools, we can calculate the spatial form of the resulting field:

$$f'(x, y) = \mathcal{F}^{-1} \{ H(k_x, k_y) F(k_x, k_y) \}. \quad (2.5)$$

AN important property of a lens is that it takes a Fourier transform. We saw earlier [Equation (2.4)] how a thin lens focuses a plane wave to a spot on the focal plane. Therefore, it converts the direction of that field to a position - this is what a Fourier transform does mathematically.

2.2 COMPRESSIVE OPTICAL DEVICES

2.2.1 COMPRESSIVE MULTI-OPTIC SYSTEMS

A simple imaging system consists of a single lens and some distance of free-space. This is how the human eye works. The length of such a system is simply the distance between the surface of the lens and the imaging plane. A conventional way to reduce the size of such a system is to use multiple optics arranged to achieve the same effective focal length in a shorter distance. The simplest example of this is the telephoto lens, where a convex lens and a concave lens are combined in a single optical system [see Figure 2.1(a)]. In practice, the effective focal length of a telephoto lens is limited to about 80% the effective focal length [44]. This limitation results from lens aberrations and optic fabrication limits.

Some other examples of compressive multi-optic systems include reflective telescopes (Cassegrain reflectors) and pancake lenses [see Figure 2.1(b) and (c), respectively][45, 46], among others [47]. Cassegrain reflectors use curved mirrors instead of refractive lenses to focus light. The first mirror reflects light

2.2. COMPRESSIVE OPTICAL DEVICES

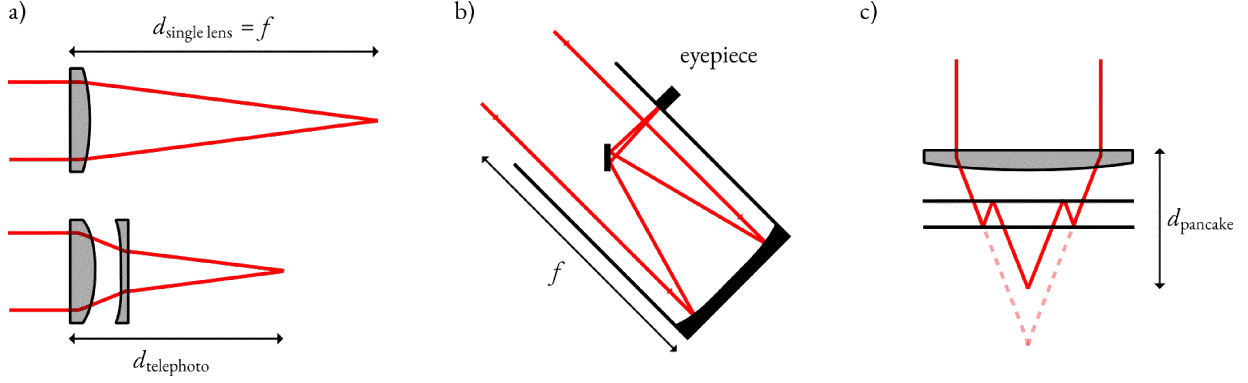


Figure 2.1: Common compressive optical systems: a) a telephoto lens is a composite imaging system whose thickness $d_{\text{telephoto}}$ is less than its effective focal length, f ; b) a Cassegrain reflector with a concave mirror of focal length f folds the propagation of focused light to reduce its size; c) a folding pancake lens has a thickness d_{pancake} less than its effective focal length.

backward to another mirror suspended in the middle of the incident path. The second mirror focuses the light onto an eyepiece or a sensitive array. This backward reflection *folds* light to reduce the size of the optical system. Pancake lenses also achieve this folding of light by combining refractive and reflective elements [48]. Cassegrain reflectors offer several advantages: they avoid the chromatic aberrations present in refractive systems, and their mirrors are larger than conventional lenses, increasing the amount of light they collect; however, they also have drawbacks. In placing the second mirror in the path of the first, Cassegrain reflectors block a portion of the incident light and introduce diffractive distortions [45]. Further, the parabolic mirrors in reflecting telescopes are usually more expensive than their refractive counterparts.

There are other imaging systems whose length is less than their effective focal length, including examples that use cavities and so-called *time-folding* to reduce system length [49]. Different systems use polarization to fold optical propagation [50]. In any of these cases, there are numerous trade-offs

2.2. COMPRESSIVE OPTICAL DEVICES

and limitations³. One drawback of reducing the effective focal length of an imaging system is that the magnification will also decrease. This, in turn, increases the NA and, thus, the aberrations. Beyond the design of multi-optic systems, we can also compress optical systems by reducing the thickness and focal lengths of the components themselves.

2.2.2 THIN LENSES

FOCAL LENGTH

A thin lens of focal length f will focus the light from an object at distance d_o onto a screen at distance d_i . These distances are governed by the thin lens equation:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}.$$

To minimize the length of the imaging system, one can reduce the focal length f of the lens. The ratio of the image and object distances determines the magnification, $\mathcal{M}$, of such a system, and by changing the focal length of an optical system, one changes its magnification:

$$\mathcal{M} := \frac{d_o}{d_i}. \tag{2.6}$$

³A common motif among these examples is that the range of one parameter is limited to achieve the compression of space: in the pancake lens and Cassegrain reflector, we sacrifice spatial and angular modes to fold space; in the telephoto lens, we often lose resolution due to chromatic aberration, etc. This may be related to a conservation of the number of modes in an optical system, but it is beyond the scope of this thesis. Miller touches on this subject in his paper on optical modes [51].

Conventional spherical lenses are made of glass (refractive index n), and their focal lengths are governed by the radii of curvature, $R_{1,2}$, of their sides.

$$\frac{1}{f} = (n - 1) \left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n - 1)t}{nR_1R_2} \right]. \quad (2.7)$$

To reduce the focal length of a lens, one must reduce its radii of curvature. Hence, by Equation (2.7) the lens thickness, t , must increase or the diameter of the lens must decrease. These limitations and those on the refractive index of glasses constrain the achievable focal length of real lenses. Fortunately, designs for thin lenses exist, including Fresnel lenses, diffractive lenses, and, more recently, metalenses.

FRESNEL AND DIFFRACTIVE LENSES

Fresnel lenses resemble plano-convex (or concave) lenses which have been concentrically flattened [see Figure 2.2(b)]. A Fresnel lens consists of a sequence of radial bands whose curvature is the same as a conventional lens but whose thicknesses are periodically stepped [8]. Stepping reduces the thickness of the lens while still producing the required phase response [Equation (2.3)]. The thinner lens reduces system length, material cost, and weight. Fresnel lenses can achieve higher numerical apertures and be much larger than conventional lenses. Unfortunately, they also have several disadvantages. The surface complexity increases manufacturing costs, and the periodic jumps in thickness introduce distortion, diffraction effects, and reduction in inherent image quality [52].

Another example of a flat lens is a diffractive lens [see Figure 2.2(c)]. These lenses can be made of several different materials, but they all function using wave optics and diffraction. A diffractive lens consists of a thin material whose thickness, refractive index, or absorptivity, varies as a function of ra-

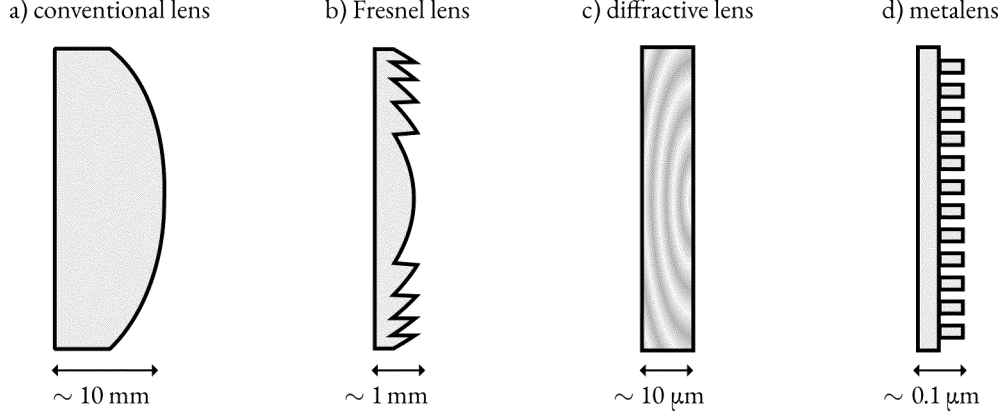


Figure 2.2: Comparison of the geometries of refractive (a), Fresnel (b), holographically diffractive (c), and metalenses (d). We provide approximate thicknesses for comparison.

dius [31–36]. Therefore, as a wavefront passes through the lens, different parts of the wavefront will pick up different phases, and the wavefront will diffract to a point on the focal plane. These lenses can be made inexpensively, very thin, and have high-numerical apertures; however, they are often chromatically dispersive ⁴ and can have high aberrations [33, 37].

METALENSES

Metalenses are a type of metasurface device. They are devices which use arrays of subwavelength structures called meta-atoms to induce a phase response to focus the light [see Figure 2.2]⁵. They can achieve high numerical apertures, small focusing spots, and reduced distortion and aberration [5, 53]. The bandwidth of metalenses is not typically very wide, but they can focus discrete wavelengths to the same point, a characteristic useful for RGB optics. The thickness of a metalens is $\sim 0.1 \text{ }\mu\text{m}$, which is an order of

⁴Work has shown that one can use higher order diffractive modes to operate diffractive lenses over multiple bands, but this also has its limitations [32, 36].

⁵Metalenses usually use a hyperbolic phase response [$\phi = k(\sqrt{r^2 - f^2} + f)$] to focus light without spherical aberration [5, 37–43].

2.2. COMPRESSIVE OPTICAL DEVICES

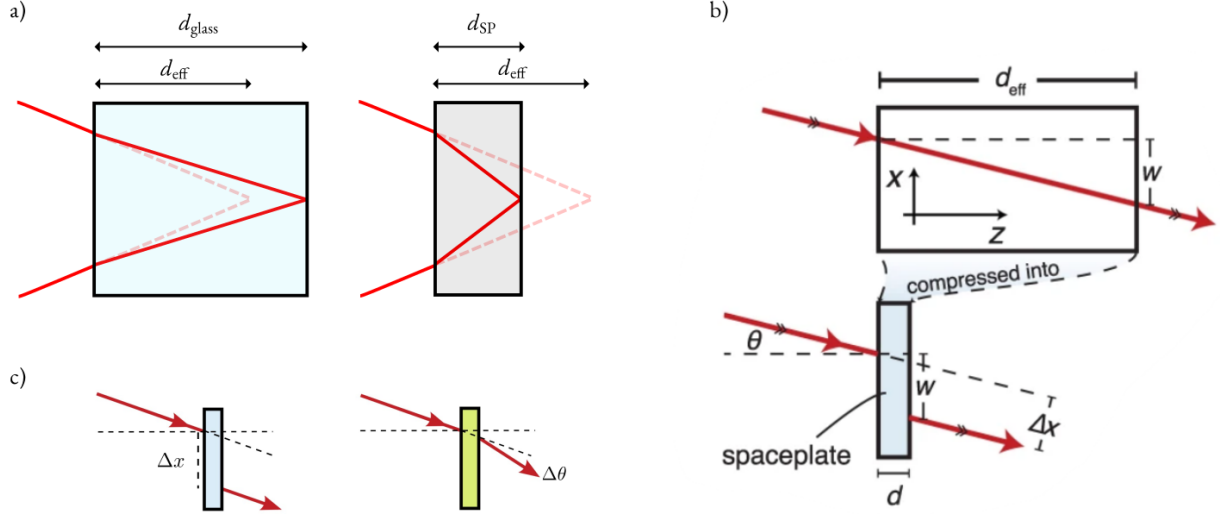


Figure 2.3: The effect of a spaceplate. a) The low-index spaceplate refracts a focusing beam more tightly than a piece of glass, increasing its effective propagation length. b) A spaceplate mimics the effect of free-space propagation in a shorter distance (image taken from [54]). c) A spaceplate changes the transverse position of a propagating ray, while a lens changes its outgoing angle (image taken from [55]).

magnitude improvement over diffractive lenses and many orders of magnitude improvement over conventional lenses.

2.2.3 SPACEPLATES

Thus far, we have discussed ways to make smaller optical systems by using multi-optic systems and smaller lenses; however, we aim to miniaturize the space *between* lenses, an often neglected element in optical systems. Since this is usually the largest contributor to the total length of an imaging system, its miniaturization promises the largest impact. Optics that compress the propagation of light onto a plate are called *spaceplates*, and the best way to introduce them is through an example.

2.2. COMPRESSIVE OPTICAL DEVICES

Referring to Figure 2.3(a), suppose we have two rays focusing to a point in some background medium of index n_0 . Now, we introduce a glass plate whose index is higher than the background index ($n_g > n_0$). Inside the glass, the rays will refract away from the axis, and the resulting focal point will be beyond the original one. Therefore, glass functions like an anti-spaceplate because it introduces additional propagation length compared to free space. We say that its effective thickness d_{eff} is less than its actual thickness d_{glass} . Now, remove the glass and insert a plate whose index is less than that of the background ($n_{\text{SP}} < n_0$). The beams then diffract towards the axis, and the focal spot appears ahead of where it originally appeared. Therefore, its effective thickness is greater than its actual thickness. We call this a low-index spaceplate - first conceptualized by Reshef et al. [54]. A spaceplate mimics the effect of free-space propagation in a shorter distance [see Figure 2.3(b)], and the amount that it compresses space is characterized by its compression ratio:

$$\mathcal{R}_c := \frac{d_{\text{eff}}}{d}, \quad (2.8)$$

The compression ratio of the low-index spaceplate is the ratio of the background and low-index refractive indices: $\mathcal{R}_c = n_0/n_{\text{SP}}$. The higher the compression ratio, the more compressive a spaceplate is, and we often want to maximize this ratio when designing a spaceplate.

Like free-space propagation, a spaceplate changes the transverse position of a ray but leaves its angle unchanged [see Figure 2.3(c)]. Thus, unlike a lens, which changes ray angle, a spaceplate does not alter the magnification of an imaging system⁶. There are two ways to achieve this: increase the transverse group velocity or reduce the longitudinal group velocity [56].

⁶In this way, lenses and spaceplates perform the dual process of one another

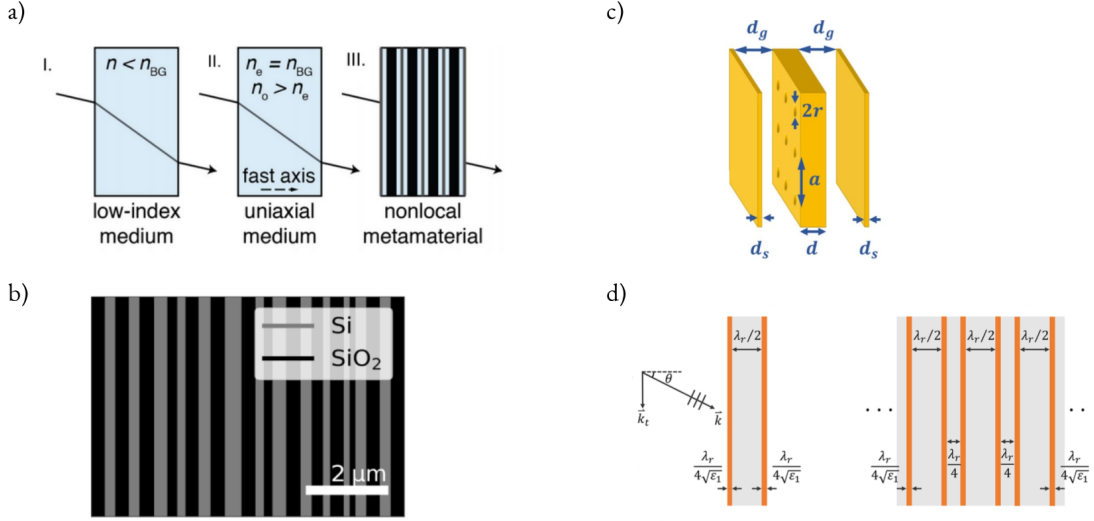


Figure 2.4: Examples of various published spaceplate designs taken from their respective publications. a) The three designs proposed by Reshef et al. [54]. b) Improvements made to the multilayer stack designs [55]. c) Photonic crystal Fano resonance designs [57]. d) Repeated Fabry-Perot cavity design [48].

From the view of physical optics, the spaceplate changes the phase of an incident plane wave without changing its angle. To design a spaceplate, one must mimic the Fourier transfer function [Equation (2.5)] (or phase response) of free space in a material. Specifically, a spaceplate of thickness d must produce the angle-dependent phase response of free space ϕ_{FS} [Equation (2.1)] given some wavevector-invariant global phase ϕ_G and some natural number m :

$$\begin{aligned} \exp [i\phi_{\text{SP}}(\mathbf{k})] &= \exp [i\phi_{\text{FS}}(\mathbf{k}) + i\phi_G] \\ \phi_{\text{SP}}(\mathbf{k}) &= 2\pi m + \phi_{\text{FS}}(\mathbf{k}) + \phi_G \\ &= 2\pi m + \mathcal{R}k_z d + \phi_G. \end{aligned} \quad (2.9)$$

Different optical devices have been proposed to compress space, including the low-index spaceplate.

2.2. COMPRESSIVE OPTICAL DEVICES

The work of Reshef et al. first introduced and experimentally demonstrated the spaceplate [54]. They experimentally tested two designs: a uniaxial crystal and the low-index medium we discussed earlier. The uniaxial crystal spaceplate produces the correct phase response [Equation (2.9)] by introducing a wavevector- or angle-dependent index of refraction. It turns out that a negative birefringent uniaxial crystal (i.e., calcite) produces precisely the correct phase response. The designs were found to have compression ratios [Equation (2.8)] of $\mathcal{R}_c = 1.12$ and 1.48, respectively, which resulted in each spaceplate saving less than 4 mm of space. Remarkably, the uniaxial crystal had a large NA ($\text{NA} = \sin \theta$) and functioned at high incident angles (experimentally $\text{NA} = 0.85$ and theoretically $\text{NA} = 1$) while remaining broadband, i.e., highly transmissive across the visible spectrum. They were able to demonstrate this spaceplate in an imaging system.

In the same paper, the group also proposed and modeled a multi-layer thin-film design, which later work improved upon. A multi-layer stack is a type of *photonic crystal*⁷, comprised of alternating stacks of differently indexed dielectrics; in their work, Reshef et al. propose designs using silica and silicon. The thicknesses of each layer are optimized to reproduce the angle-dependent spaceplate phase response for as wide an angular band as possible, simultaneously maximizing the compression ratio. A deterministic method to design these stacks does not exist, so the optimization used a genetic algorithm. Their simulated design achieved a compression ratio of $\mathcal{R}_c = 5.2$ but was only capable of saving 50 μm of space and was narrowband. The phase response in these multi-layer stacks typically only works within a small frequency band of the design wavelength. In a later publication, Pagé et al. improved upon the design of these multi-layer stacks, optimizing their geometries using a gradient descent technique,

⁷A photonic crystal is a nanostructure whose refractive index changes periodically to manipulate the propagation of light [58, 59]. They can be described in one, two, or three dimensions and have a range of possible geometries.

2.3. DESCRIPTION OF A THREE-LENS SPACEPLATE

increasing the compression ratio to $\mathcal{R}_c = 11.5$.

In another, almost simultaneous work as the first, Guo et al. [57] numerically demonstrated a much higher compression ratio of $\mathcal{R}_c = 144$ using the Fano resonances of a photonic crystal slab. This photonic crystal consisted of two dielectric layers sandwiching another layer containing a square lattice of holes. This geometry replicated the spaceplate phase response but only for specific input beam modes, including linearly polarized cylindrical vector beams and circularly polarized Laguerre-Gaussian beams. Unlike the uniaxial crystal, however, this design was significantly limited in NA ($\text{NA} = 0.01$), and the actual space saved was limited to approximately 360λ [57].

Subsequent designs repeated Fabry-Perot cavities [60, 61], and others [48, 62, 63], made incremental improvements upon compression ratio and NA. None, however, can be scaled up to save space on the order of meters. Additionally, most previously demonstrated designs are limited in either polarization or bandwidth [54, 55, 57, 60]. All of the work presented previously in this section was done previously. However, we present new work in the following section, describing a broadband, polarization-independent three-lens spaceplate capable of meter-scale compression.

2.3 DESCRIPTION OF A THREE-LENS SPACEPLATE

The three-lens spaceplate (Figure 2.5) utilizes the Fourier plane of a $4f$ lens system to impart the same angle-dependent phase that free-space propagation imparts [1, 64]. Our $4f$ lens system consists of two positive lenses of focal length f_{ext} , the exterior lenses. The first lens maps the complex amplitude distribution of an incident field across the front focal plane to the back focal plane via a Fourier transform

2.3. DESCRIPTION OF A THREE-LENS SPACEPLATE

[65]. Thus, there is a one-to-one mapping between the angle of an incoming plane wave and the position in the back focal plane, known as the *Fourier plane*, which is located a distance f_{ext} after the lens. The second lens is a distance $2f_{\text{ext}}$ after the first lens and performs another Fourier transform, undoing the action of the first lens. With a spatially-varying phase element (i.e., a phase mask) in the Fourier plane, the $4f$ system phase shifts each incoming plane wave but otherwise leaves it unchanged. We will show that the angle-dependent phase shift required to emulate free-space propagation happens to correspond to the phase mask created by a thin positive lens. Consequently, such a lens placed in the Fourier plane (i.e., a middle lens with focal length f_{mid}) will cause the field transmitted by the $4f$ system to appear as though it had propagated through a slab of free-space, potentially longer than the $4f$ system

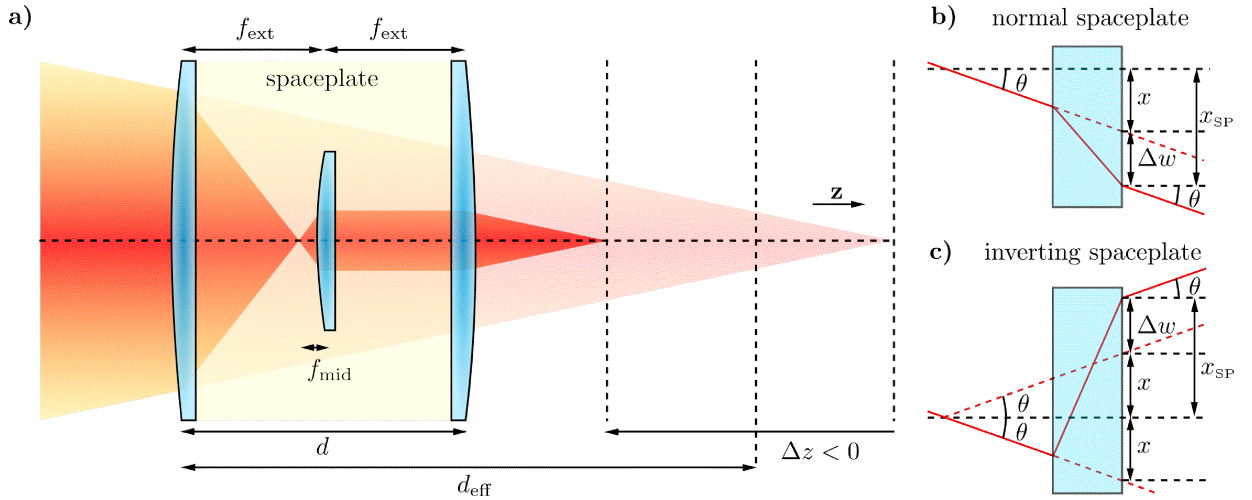


Figure 2.5: Propagation of a light beam in a three-lens spaceplate. a) Light travels through a spaceplate of thickness d with an equivalent propagation through free-space a distance $d_{\text{eff}} > d$. b) Rays passing through a spaceplate will maintain their angle θ , but are displaced by walkoff Δw , just as if they were passing through free space. c) Inverting spaceplates do the same while also inverting the rays about the z -axis. The transverse direction is x , and the longitudinal direction is z . In this figure, f_{ext} is the focal length of the exterior lenses, f_{mid} is the focal length of the middle lens, x and x_{SP} are the transverse displacements of the original and space-compressed rays entering at angle θ , respectively, and Δz is the longitudinal shift of an output light field.

2.3. DESCRIPTION OF A THREE-LENS SPACEPLATE

itself. This three-lens device is similar to a Fourier filtration system which uses a spatial light modulator or a transparency located at the Fourier plane to modify the phase or amplitude of the Fourier decomposition. Note that while it is neither monolithic nor plate-like, we still refer to it as a spaceplate due to its similar effect on light. Note as well that the use of a 4- f system results in inversion of the light about the $\mathbf{z}$ -axis ($x = 0$); while this does not affect the compression or absolute magnification of the device, it does differentiate the device from other spaceplates.

2.3.1 FOURIER OPTICS DESCRIPTION OF A THREE-LENS SPACEPLATE

To motivate the design of the three-lens spaceplate, we follow the Fourier optics description of the spaceplate effect introduced by Reshef et al. [54]. Suppose we have a light field propagating through free-space centered on the $\mathbf{z}$ -axis. We then consider how each transverse spatial Fourier component of this initial field is transformed by this propagation. Each of these components, or plane waves, is described by its momentum vector, $\mathbf{k} = (k_x, k_z) = k(\sin \theta, \cos \theta)$, where θ is the angle between the $\mathbf{z}$ -axis and the direction of the plane wave. Here, we consider a single transverse dimension (x) to reduce notational complexity. This simplification is valid as long as the action of a lens is separable in the x and y directions, which, in the paraxial approximation that we use, is indeed the case. When propagated over a distance z_0 in free-space, the amplitude and direction of each plane wave are conserved, but the phase is not. Between two points on the $\mathbf{z}$ -axis separated by z_0 , each plane wave accumulates phase $\varphi = k_z z_0 = |\mathbf{k}| \cos(\theta) z_0$.

A spaceplate must produce this same phase response but in a shorter distance d than z_0 . An ideal spaceplate, if inserted into the light path, will produce the phase response $\varphi_{\text{SP}} = k_z z_0 = z_0(k^2 - k_x^2)^{1/2}$.

2.3. DESCRIPTION OF A THREE-LENS SPACEPLATE

In the small angle approximation, $\varphi_{\text{SP}} = z_0 k(1 - k_x^2/2k^2)$. Then, by neglecting *global phase* terms, we arrive at our target phase, $\varphi_{\text{SP}} = -z_0 k_x^2/2k$. By global phase, we refer to the spatially invariant phase accumulated by any wave, regardless of the transverse position on the optic through which the wave was transmitted [Equation (2.9)]. An approach for considering larger incident angles is given in Appendix A.1.1. In the Fourier plane of the 4- f system in Figure 2.5(a), the field at position r (i.e., axial distance) is the incoming field component with transverse momentum $k_x = rk/f_{\text{ext}}$. Substitution of this relation into the target phase gives the spatial phase mask required to emulate propagation through space,

$$\varphi_{\text{SP}} = -\frac{z_0 k r^2}{2f_{\text{ext}}^2}. \quad (2.10)$$

The phase added by a spherical, thin lens of focal length f_{mid} in the paraxial approximation is given by $\varphi_{\text{mid}} = -kr^2/2f_{\text{mid}}$, which is of the same form as Equation (2.10). By setting $\varphi_{\text{SP}} = \varphi_{\text{mid}}$ we find that the spaceplate effectively propagates light a distance $z_0 = f_{\text{ext}}^2/f_{\text{mid}}$. Therefore, we can compress space by placing a positive lens at the Fourier plane.

A slight complication is that, as part of the 4- f system, there already is free-space propagation of distance f_{ext} before the first lens and f_{ext} after the last lens. We do not include those regions in the length of space replaced, hence $d_{\text{eff}} = z_0 - 2f_{\text{ext}}$. Similarly, we define the spaceplate thickness to solely be the distance *between* the external lenses, $d = 2f_{\text{ext}}$. Therefore, the compression ratio is

$$\mathcal{R} = \frac{d_{\text{eff}}}{d} = \frac{f_{\text{ext}}}{2f_{\text{mid}}} - 1, \quad (2.11)$$

which only depends on the focal lengths used. If $\mathcal{R} > 1$ or, equivalently, if $f_{\text{mid}} < f_{\text{ext}}/4$, more space is

2.3. DESCRIPTION OF A THREE-LENS SPACEPLATE

replaced than is occupied by the three-lens spaceplate. Further discussion of system length, compression ratio, and Equation (2.11) is given in Appendix A.1.

3

EXPERIMENTALLY COMPRESSING SPACE USING THREE LENSES

In this chapter, we experimentally demonstrate the three-lens spaceplate and characterize its performance for several lens choices: Section 3.1 overviews the experimental design used to characterize the three-lens spaceplate; Section 3.2 describes the experiment used to measure the advance of the focus of

a beam and presents the corresponding results; we show in Section 3.3 how the transverse walkoff of the three-lens spaceplate is characterized; and, finally, we show how it can be straightforwardly inserted into the design of long imaging systems, even if they are broadband.

3.1 DESIGN OF THE FOCUS ADVANCEMENT, WALKOFF, AND IMAGING EXPERIMENTS

The setup used to test the three-lens spaceplate is shown in Figure 3.1. We measure the advance of the focus of a beam, the walk-off of a beam incident at an angle, as well as imaging properties. These three types of measurements require minor setup variations that are detailed in the following three sections.

We start by describing elements in the setup common to all three measurements. The external lenses, $f_{\text{ext}} = 150$ mm, are the same throughout all the tests and have 50.8 mm diameters and 15.0 mm thicknesses. We test the effect of the following series of middle lenses: $f_{\text{mid}} = \{40, 30, 25, 19, 18, 11, 10, 4.51\}$ mm whose respective diameters and thicknesses were $\{25.4, 25.4, 12.7, 12.7, 6.5, 8.0, 5.5\}$ mm and $\{12.5, 14.0, 7.0, 6.0, 2.2, 5.0, 6.5, 2.94\}$ mm. Except for the $f_{\text{mid}} = \{18, 11, 4.51\}$ mm aspheric lenses, all lenses used throughout these tests were achromatic doublets, designed and anti-reflective (AR) coated for the visible range (400-700 nm light). The $f_{\text{mid}} = 18$ mm aspheric lens was designed for 780 nm light, and the $f_{\text{mid}} = \{11, 4.51\}$ mm aspheric lenses were designed for 633 nm light. All aspheric lenses were AR-coated for 600-1100 nm. After the spaceplate, an image sensor ($5.86 \mu\text{m} \times 5.86 \mu\text{m}$ pixel size, 1936×1216 pixels, colour) was mounted on a motorized translation stage capable of moving both in the transverse (x) and longitudinal (z) di-

3.2. FOCUS ADVANCEMENT OF VARIOUS THREE-LENS SPACEPLATES

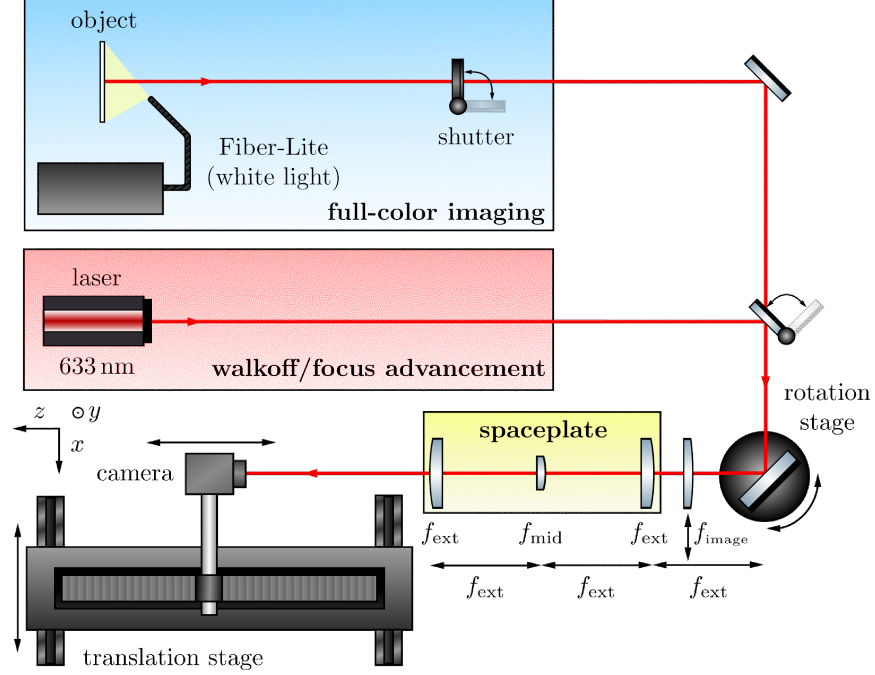


Figure 3.1: Experimental setup used to test the spaceplate focus advancement, the walkoff, as well as the full-colour imaging. The lens labeled f_{image} is removed for walkoff experiments.

rections. With this sensor, we recorded $x \times y$ spatial intensity distributions at chosen z positions. The electronic settings of the image sensor were constant throughout the various tests.

3.2 FOCUS ADVANCEMENT OF VARIOUS THREE-LENS SPACE-PLATES

We measure the focus advancement of a 633 nm wavelength Helium-Neon laser that is collimated so that its beam has an intensity full width at half maximum (FWHM) of 3.5 mm using three different

3.2. FOCUS ADVANCEMENT OF VARIOUS THREE-LENS SPACEPLATES

middle lens focal lengths corresponding to three distinct compression ratios. To focus the beam, the laser travels through a 50.8 mm diameter positive lens with a thickness of 11.3 mm and a focal length of $f_{\text{image}} = 1000$ mm [see Figure 3.1]. The three-lens spaceplate is placed between this positive lens and the nominal focus z -position, then the image sensor records intensity distributions over a range of 500 mm with a 0.25 mm step size. Each x pixel row of the recorded distribution is summed to determine the row with the maximal intensity. This row becomes the 1D intensity profile at each z , which is plotted in Figure 3.2. For each middle lens, the peak intensity along z is taken as the new focal position.

We measure longitudinal focal shifts of $\Delta z = 94$ mm, -49 mm, and -221 mm corresponding to observed compression ratios $\mathcal{R}_{\text{obs}} = 0.69, 1.16$, and 1.74 , respectively. The respective nominal com-

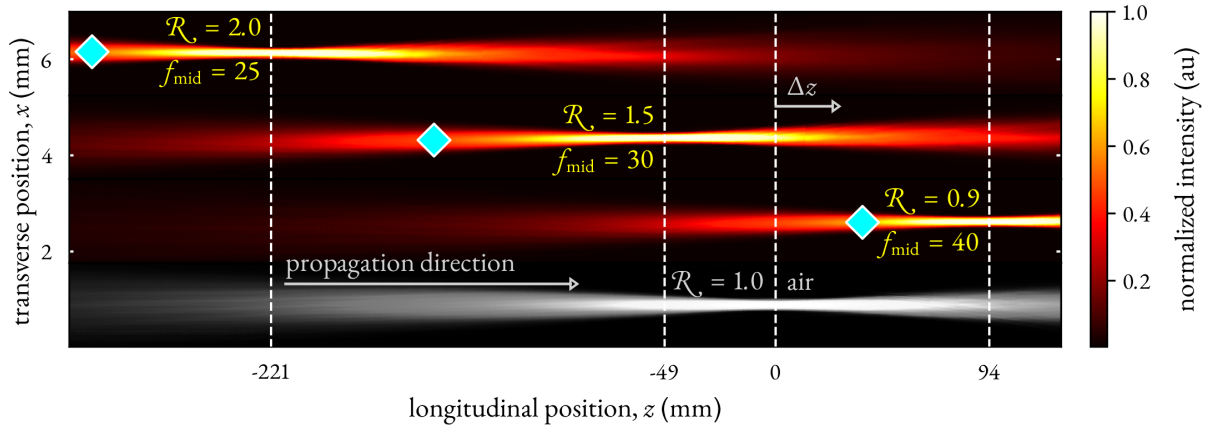


Figure 3.2: Measurement of the focus advance due to the three-lens spaceplate. At each z , we plot the intensity along x (after selecting the x row with the highest intensity) without the spaceplate - just air - (in greyscale, bottom) and for each spaceplate middle lens f_{mid} (in colourscale, associated with nominal compression ratio $\mathcal{R}$). The vertical dotted lines mark the peak intensity along z , i.e. the position of the focus, and the diamond scatter points denote the nominal focus positions. With the air-only focus taken as $z = 0$, the vertical lines indicate the focal advance for each spaceplate. Negative advances correspond to a spaceplate that replaces more space than it occupies, i.e., $\mathcal{R} > 1$. For more experimental details and descriptions see Sections 3.1 and 3.2.

3.3. TRANSVERSE BEAM WALKOFF OF A THREE-LENS SPACEPLATE

pression ratios [Equation (2.11)] are $\mathcal{R}_c = 0.9, 1.5$, and 2.0 . While these are significantly different from the measured values, they demonstrate that the three-lens spaceplate can successfully replace space.

Deviations between $\mathcal{R}_c$ and $\mathcal{R}_{obs}$ result because the three-lens spaceplate relies on a lens located directly at the Fourier plane of the $4f$ system, and any deviation of the thick middle lens from this position results in a reduction of compression and non-unity magnification. Specifically, we subsequently observe that upon transmitting an expanded collimated beam with a FWHM = 3.5 mm through the three-lens spaceplate, the resultant beam width varies (1.7-4.4 mm) depending on small adjustments (0-5 mm) to the longitudinal position of the middle and rear lenses in the spaceplates. Therefore, deviations between the expected and observed longitudinal shifts are due to misalignment of the system. We reduce the effect of these sensitivities in subsequent walkoff and imaging experiments. Additionally, because the compression ratio is derived using geometrical optics not Gaussian optics, the focus position of a Gaussian beam will be slightly different from expected.

Nonetheless, the observed negative longitudinal shift demonstrates that this design can replace free-space on a scale of three orders of magnitude greater than previous spaceplate designs. This experiment is a qualitative demonstration of space compression - the following experiments characterize the three-lens spaceplate with greater accuracy.

3.3 TRANSVERSE BEAM WALKOFF OF A THREE-LENS SPACEPLATE

We next measure the transverse walk-off $\Delta w = |x_{sp}| - |x| = d(\mathcal{R}_c - 1) \sin \theta$ [see Figure 2.5(b) for a diagram] of a beam transmitted through the spaceplate, incident at angle θ to the z -axis, rotating about

3.3. TRANSVERSE BEAM WALKOFF OF A THREE-LENS SPACEPLATE

the y -axis. Recall that this spaceplate functions by increasing the transverse group velocity; this says that the spaceplate will cause a beam to propagate further or *walkoff* more in the transverse direction. For each spaceplate, the walk-off as a function of angle characterizes the NA and aberrations and determines the compression ratio with greater accuracy than in the focal advance measurement. Specifically, before conducting walkoff experiments, we adjust the longitudinal positions of the middle and rear lenses until a beam (FWHM = 3.5 mm) transmitted through the spaceplate becomes collimated and has unity magnification. That is until the exiting beam has a diameter of FWHM = (3.5 ± 0.1) mm both immediately after the rear lens of the spaceplate and one meter after it. This process was instituted after

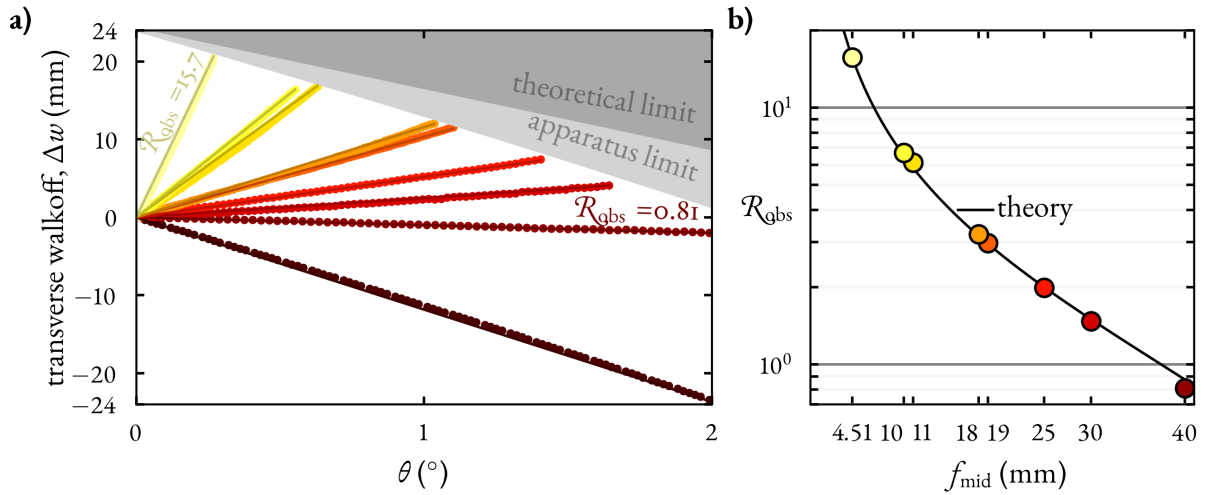


Figure 3.3: Measurement of the transverse walkoff and compression ratio as a function of middle focal length. a) The measured transverse walkoffs (Δw) of each of the tested spaceplates are plotted as a function of angle with fitted linear regressions. The theoretical limit is derived from the aperture sizes of the lenses used and the apparatus limit is produced by the bounds of the translation stage and image sensor. b) Here, the compression ratios measured from the corresponding walkoffs in (a) are plotted against the focal lengths of the middle lens (f_{mid}). The point associated with least compression, $f_{mid} = \infty$, does not appear in b). All points have their uncertainties plotted, though most are obscured by the symbol for the point.

3.3. TRANSVERSE BEAM WALKOFF OF A THREE-LENS SPACEPLATE

observing how sensitive the system was to lens placement during preliminary experiments (Section 3.2).

To test the beam walkoff, the imaging lens and beam expander are absent, leaving the laser beam collimated. The rotation stage in Figure 3.1 varied θ from 0 to 2 degrees in 0.02 degree increments. The stage is located f_{ext} before the first lens, and we measure the transverse displacement $z = 4f_{\text{ext}} = 600$ mm from the rotation axis of the stage. The rotation stage is placed here due to our definition of $\text{NA} = \sin \theta$, where θ is the half-angle of a maximum cone accepted by the system, originating f_{ext} before the first lens. By putting the stage at this position, we can contextualize the experimental angular limitations of the system with an analysis of its numerical aperture. The position at which we measure the transverse displacement, however, does not matter in general. One will measure the same Δw regardless of the position of the imaging plane.

First, we measure the transverse displacement $x(\theta)$ of a beam from the z -axis in free-space and then perform analogous measurements $x_{\text{SP}}(\theta)$ of beams transmitted through each spaceplate. Due to inversion, the beams are displaced in opposite directions, but we can still compute the walkoff as $\Delta w = |x_{\text{SP}}| - |x|$. We measure x and x_{SP} by fitting Gaussians to the measured transverse intensity profiles.

Figure 3.3(a) presents this series of walkoff measurements for each of the eight middle lenses given in Section 3.1, with corresponding observed compression ratios in Figure 3.3(b). Each configuration produced compression ratios within error of their nominal compression ratios over the entire measured range. For the highest experimental compression ratio, $\mathcal{R}_c = 15.6$, $f_{\text{mid}} = 4.51$ mm, the spaceplate replaced 4.39 meters of free-space, which is more than one thousand times greater than that demonstrated by other spaceplate designs.

3.4 BROADBAND IMAGING WITH A THREE-LENS SPACEPLATE

In our final measurement, we investigate the performance of the three-lens spaceplate for imaging with full-visible-spectrum light. We first illuminate a postage stamp with white light and use the f_{image} lens to form a sharp image on the image sensor. The object and image distances are approximately 1.9 m and 2.1 m, respectively, and we measure a magnification of 1.1. We record images with and without a $f_{\text{mid}} = 10$ mm spaceplate [Figure 3.4(c), left and right panes, respectively] in the imaging path. Before imaging, the middle and rear lenses of the spaceplate are calibrated using an expanded collimated beam,

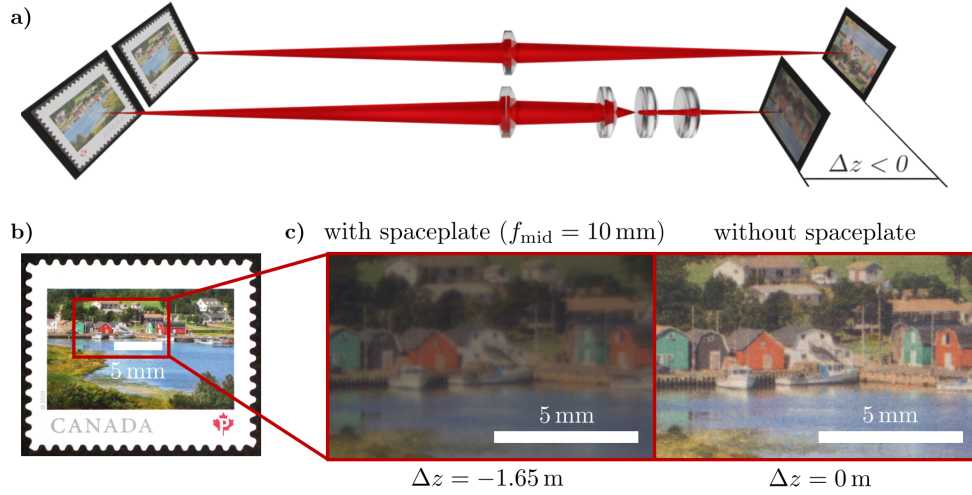


Figure 3.4: Full-colour imaging with a three-lens spaceplate. a) Top: imaging system; Bottom: imaging system with the three-lens spaceplate. b) The imaging object, a stamp; c) An image of the stamp. Right: without the spaceplate; Left: with a $f_{\text{mid}} = 10$ mm spaceplate with nominal compression ratio $\mathcal{R}_c = 6.5$ (image is rotated by 180° to compensate for the three-lens spaceplate inversion). With the spaceplate, a sharp full-colour image is formed 1.65 m closer to the imaging lens, compressing the size of the imaging system. The stamp is a photo of the French River in Prince Edward Island by Barrett and Mackay Photography. Reproduction is permitted under the Canada Post Corporation Act [66].

3.4. BROADBAND IMAGING WITH A THREE-LENS SPACEPLATE

as was done in the walkoff experiment. The camera is placed at the nominal longitudinal position relative to the focus position seen without a spaceplate in the imaging path. Then, the middle and rear lens longitudinal positions are adjusted slightly to achieve the same magnification seen without the spaceplate. To reduce aberration and edge effects, a large iris was positioned immediately before the third lens in the spaceplate and was adjusted until the highest contrast and resolution were seen. This was at an aperture size of (12 ± 2) mm. While there still is a reduction in contrast and brightness, which we discuss later, the images show that the three-lens spaceplate can advance the image plane of a full-visible-spectrum imaging system by 1.65 m.

Next, we image an NBS 1963a resolution target to characterize the reduction in resolution and contrast caused by the addition of four different three-lens spaceplates (see Figure 3.5). The addition of the spaceplates does not change the magnification, as expected. However, Figure 3.5(b) shows that every spaceplate reduces image contrast and resolution in comparison to the image formed without a spaceplate. We quantify this reduction with the MTF of the imaging system following the method of Saiga et al. [67] in Figure 3.5(d). For the case of $f_{\text{mid}} = 10$ mm ($\mathcal{R}_c = 6.5$) and the stamp imaged without an iris, the spaceplate lowers the spatial frequency at half-max of the MTF by 45% (i.e. 10.9 lp/mm is reduced to 6.0 lp/mm). Three potential causes for this reduction are aberrations, scattering from lens edges, and diffraction by the lens apertures. The first two would be mitigated by a reduced aperture, whereas the effect of diffraction would be worsened. Adjusting the diameter of an iris $[(12 \pm 2)$ mm] just before the third lens, we take the images shown in Figure 3.5(c), which are qualitatively improved in contrast and resolution. This improvement is demonstrated by the MTF measured with an iris in place, which increases by 38% (i.e. 6.0 lp/mm to 8.3 lp/mm) suggesting that diffraction is not the main cause of the reduction. Equivalently stated, the addition of the spaceplate and iris causes a 24% reduction in

3.4. BROADBAND IMAGING WITH A THREE-LENS SPACEPLATE

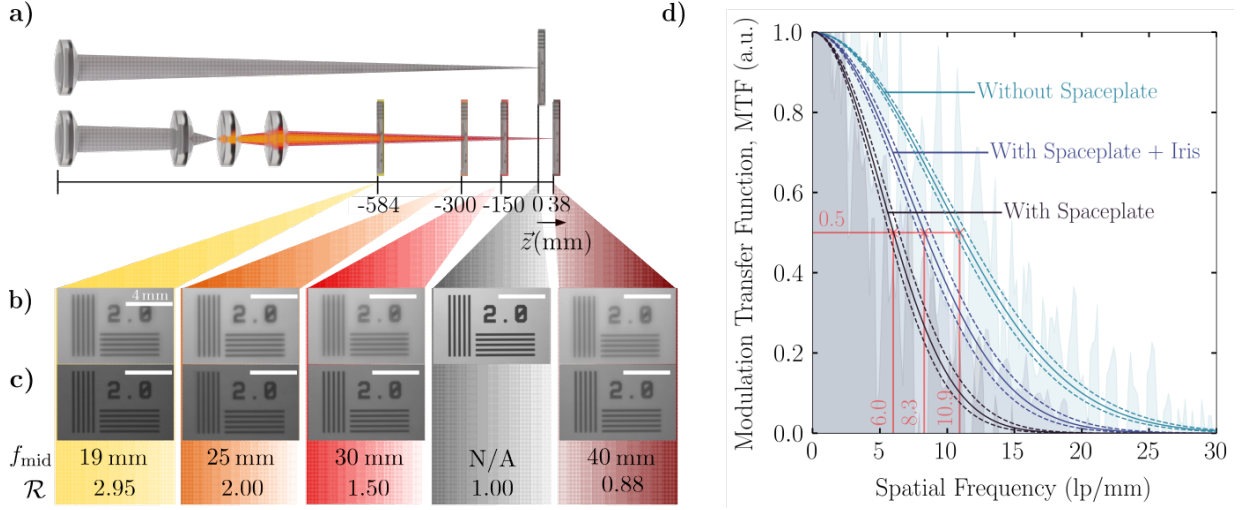


Figure 3.5: The effect of the three-lens spaceplate on resolution and contrast. a) The three-lens spaceplate is used to compress an imaging system consisting of a 1000 mm focal length doublet. b) Images taken using different spaceplates. c) Images taken with a spaceplate and an iris. d) Measured (background, filled plots) and fitted MTF for three full-colour stamp images; one was taken with only the imaging lens, another with the imaging lens and a spaceplate ($\mathcal{R}_s = 6.5, f_{\text{mid}} = 10$ mm), and the third with an iris just before the third spaceplate lens. The dotted curves represent the standard error resulting from the method used to approximate the MTFs. See the text for discussion and comparison of results.

the MTF (i.e. 10.9 lp/mm to 8.3 lp/mm).

In this section, we demonstrated compression of free-space using a device called a “three-lens spaceplate”, a positive lens placed at the Fourier plane of a $4f$ optical setup. We showed that our spaceplate design can be used for meter-scale space compression and the miniaturization of broadband imaging systems. Using the three-lens spaceplate, we achieved experimental compression ratios up to $\mathcal{R}_{\text{obs}} = 15.7$ that replace 4.4 meters of free-space. Until now, replication of the transfer function of free-space over lengths of this scale had not been demonstrated: previously, space compression of only one or two millimeters had been shown [54]. The three-lens spaceplate has not only shown that this is

3.4. BROADBAND IMAGING WITH A THREE-LENS SPACEPLATE

possible, but it has also accomplished this using off-the-shelf optical components. Moreover, the system reduced the length of monochromatic and full-color imaging systems with a 24% loss of resolution as quantified using its modulation transfer function. Finally, we discussed the trade-off between compression ratio and numerical aperture. Specifically, we found that the numerical aperture is highly limited for the three-lens spaceplate and represents its greatest shortcoming. We present more conclusions and future work in Chapter 8.

4

CONTEXTUALIZING THE THREE-LENS SPACEPLATE

This chapter provides context for the three-lens spaceplate, discussing its limitations and contrasting it with existing compressive optics. Section 4.1 outlines some fundamental theoretical limits on the performance of the three-lens spaceplate. We present a more in-depth analysis of the theoretical limits

4.1. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

in Appendix A. In the appendix, we also discuss extensions beyond the small-angle regime, a ray matrix approach to the compression ratio in a three-lens spaceplate, and different approaches to defining system length.

Then, Section 4.2 compares and contrasts the three-lens design to related optical systems, such as the telephoto lens and Cassegrain reflector, and discusses prospective applications.

4.1 FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

In this section, we briefly discuss limits on the NA and compression inherent to the three-lens design of the spaceplate. In most physically realizable setups, the NA of the three-lens spaceplate is limited by the diameter of the third lens, D_3 . For this common case, the NA can be expressed in terms of the f-number of the third lens, $N_3 = f_{\text{ext}}/D_3$, and $\mathcal{R}$:

$$\text{NA} = \left[1 + 4N_3^2 (1 + 2\mathcal{R})^2 \right]^{-\frac{1}{2}}. \quad (4.1)$$

Eq. (4.1) shows that there is a trade-off between compression ratio and NA. The lower the f-number of the third lens, the less strict this trade-off is. The third lens used in our experiment has $N_3 = 3$, which limits the NA of the three-lens spaceplate to be less than 0.06, worsening as $\mathcal{R}$ increases. This represents the greatest shortcoming of the three-lens spaceplate design: even when using a relatively large third lens diameter ($D_3 = 50$ mm), the maximum incident angle is three degrees from the optical axis. For information on the less common cases where the first or second lenses limit the NA, see Appendix A.2.2.

4.2. COMPARISON OF THE THREE-LENS SPACEPLATE TO OTHER COMPRESSIVE OPTICS

We have considered two additional physical mechanisms inherent to the three-lens spaceplate that lead to limitations. However, these are much less restrictive than the above limitation in practical applications with standard lenses. Firstly, the Abbe Sine Condition limits the NA independent of the lens diameters. Secondly, diffraction from the first lens in the spaceplate limits the achievable compression ratio. These two limitations may be relevant for new types of lenses, such as metalenses, and are therefore explored in depth in Appendices A.2.1 and A.2.2, respectively.

4.2 COMPARISON OF THE THREE-LENS SPACEPLATE TO OTHER COMPRESSIVE OPTICS

There are several distinctions between the three-lens spaceplate and other methods to conserve space in imaging systems. We start by comparing the three-lens spaceplate to other spaceplate designs. All of these exhibit some degree of transverse translation invariance [56], whereas the three-lens spaceplate does not since lenses are not translationally invariant. Another difference is that the three-lens spaceplate inverts the transmitted field through the axis of the lenses (i.e., $x \rightarrow -x$ and $y \rightarrow -y$), though this is avoided simply by using a pair of spaceplates.

Many conventional imaging systems are already designed to minimize the space they occupy. Examples include the telephoto lens and the Cassegrain reflector whose goal is to create an imaging system shorter than its effective focal length f_{eff} . Thus the compression of space is intertwined with the main purpose of the imaging system - to provide optical magnification. In practice, telephoto lenses are limited to a length $> 0.8f_{\text{eff}}$ [44–46]. This is equivalent to a non-telephoto lens followed by a three-lens

4.2. COMPARISON OF THE THREE-LENS SPACEPLATE TO OTHER COMPRESSIVE OPTICS

spaceplate with $\mathcal{R}_c \geq 1/0.8 = 1.25$, much lower than the highest value reported here for three-lens spaceplates $\mathcal{R}_c = 15.7$, albeit with a higher NA.

The nature of the components of these three-lens systems means that they are large. In this way, they differentiate themselves from thin spaceplates. Because they use lenses, implementable three-lens spaceplates have limitations on thickness, aperture, and, consequentially, compression ratio. Using off-the-shelf components and manageable system lengths ($2f_{\text{ext}} \lesssim 1 \text{ m}$), compression ratios are limited ($\mathcal{R}_c \lesssim 20$); however, due to the large size of these systems, they can save large absolute quantities of space (on the order of meters). This is in contrast to existing spaceplate designs, which only save space on the order of microns or millimeters [54, 55, 57, 60].

One advantage of using standard optical components in three-lens spaceplates is that the resulting spaceplate is broadband, polarization-independent, and highly configurable in compression ratio. The compression ratio of the three-lens spaceplate can easily be changed by replacing the middle lens, whereas the compression ratio of other spaceplate designs is fixed upon manufacturing.

Similar systems have been used in microscopy to rapidly tune imaging planes [68–71] in transport-of-intensity phase microscopy or to add free space to optical systems [72]. However, such implementations do not compress space and are instead intended to artificially increase the depth of field.

PART II

VOLUME HOLOGRAPHY IN PHOTOPOLYMERS

AND LITHIUM NIOBATE

The initial intention of this thesis was to design a spaceplate using holograms. Recall that a spaceplate has an angle- or wavevector-dependent Fourier transfer function [Equation (2.1)]. Holograms have this wavevector-dependency, so this part overviews our work towards designing a spaceplate using a hologram.

To achieve a holographic spaceplate, we needed to create a framework with which we could design holograms, develop an understanding of holographic materials, and understand the experimental limitations of holograms. This part of the thesis presents investigations into each of these areas. We start with an introduction to holography and its theoretical framework (Chapter 5). The beginning of this chapter is mostly background material, though we demonstrate some novel simulations and analyses using the described framework (Section 5.3.1). This is followed by a description of several holographic materials and our experimental characterizations of them (Chapter 6). Finally, we present new developments in the design of a holographic spaceplate (Section 6.1.1).

5

VOLUME HOLOGRAPHY AND COUPLED WAVE THEORY

This chapter contains an introduction to rudimentary holography, beginning with surface holograms. This precedes a discussion of volume holography, which extends the principles of gratings and two-dimensional holograms to thick geometries. With adequate assumptions and parameters, it is valid to

treat volume simply, and only one or two holographic modes are populated. However, one can have higher-mode mixing and cross-mode population. Such circumstances require more rigorous analysis methods, necessitating a discussion of rigorous coupled wave theory.

5.1 INTRODUCTION TO HOLOGRAPHY

Holograms are sophisticated analogs of photographic images. Instead of imprinting colour or intensity into film, as in photography, holographic films imprint an interference pattern. In photography, the film records an intensity pattern of the image field at the film. It directly records the image in the exposure of the film. Reilluminating this film causes a facsimile of the original light field to be displayed. In holography, an analogous process occurs. When exposing a holographic film to two coherent interfering beams, their interference pattern is imprinted into the film, storing the interference pattern. Upon illumination, the hologram reveals the original light field. Because holograms imprint amplitude and phase, they store three-dimensional images, whereas photographs can store two-dimensional images. We reinforce this concept with a simple example in the next section.

5.1.1 AMPLITUDE HOLOGRAPHIC GRATING

We introduce holography via the surface amplitude hologram, following the method of Toal [3]. Suppose we have two identical, x -polarized plane waves at vacuum wavelength λ and angular frequency ω , traveling at an angle 2θ from one another. They propagate in the yz -plane, offset at an angle θ from the z -axis. This scenario is shown in Figure 5.1(a). Mathematically, these two waves can be described by

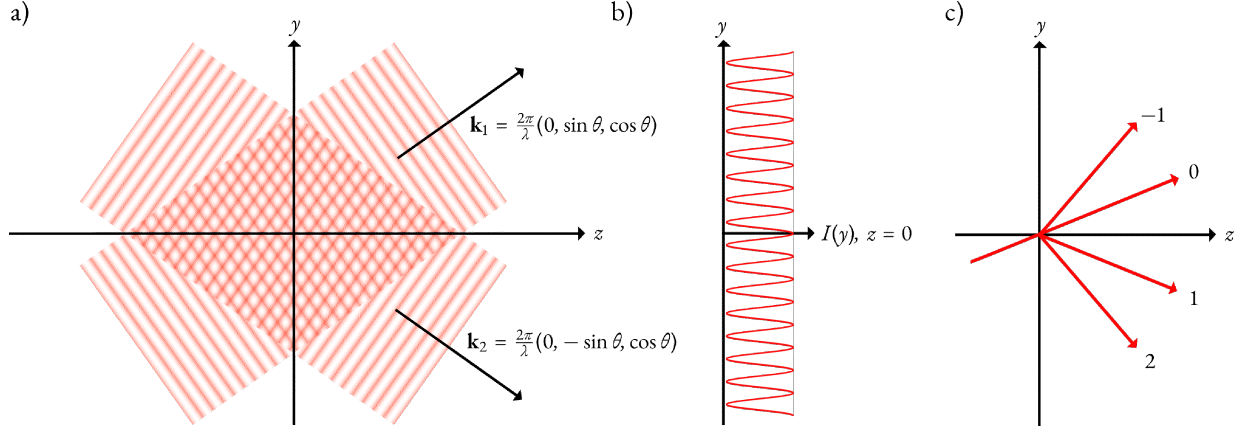


Figure 5.1: The interference of two coherent plane waves. Two coherent plane waves propagating in the z -direction interfere (a), producing an interference pattern at the xy -plane (b). If a 2D hologram is recorded and illuminated at Bragg, it will produce four transmitted modes (c).

their time-varying complex amplitudes at some vector $\mathbf{r}$:

$$\tilde{\mathbf{E}}_1 = \mathbf{E} \exp[i(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)]$$

$$\tilde{\mathbf{E}}_2 = \mathbf{E} \exp[i(\mathbf{k}_2 \cdot \mathbf{r} - \omega t)],$$

with identical wavenumbers $|\mathbf{k}_1| = |\mathbf{k}_2| = k = 2\pi n/\lambda$.

Arbitrarily, we assign $\mathbf{k}_1$ to be in the positive θ direction and $\mathbf{k}_2$ to be in the negative θ direction.

These two plane waves interfere, and the total complex field amplitude is

$$\begin{aligned} \tilde{\mathbf{E}} &= \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2 \\ &= \mathbf{E} \exp[i(ky \sin \theta + kz \cos \theta - \omega t)] + \mathbf{E} \exp[i(-ky \sin \theta + kz \cos \theta - \omega t)], \\ &= 2\mathbf{E} \cos(ky \sin \theta) \exp[i(kz \cos \theta - \omega t)]. \end{aligned}$$

Therefore, the intensity pattern at any position on the xz -plane only depends on y :

$$\begin{aligned} I(y) &= \frac{1}{2} |\tilde{\mathbf{E}}|^2 \\ &= 2E^2 \cos^2(ky \sin \theta). \end{aligned} \tag{5.1}$$

This pattern is pictured in Figure 5.1(b). Placing a flat layer of recording material in the xy -plane will record a holographic diffraction grating. Here, we are assuming they are in phase. In principle, however, it does not matter - a phase offset between the two beams will offset the interference pattern in the y -direction.

In the previous example (Figure 5.1), suppose that the interference pattern imprints an amplitude holographic grating. That is, the transmissivity, t , of the exposed layer is directly proportional to that of the intensity of light [$t(x, y) \propto I(x, y)$]. Constructive interference corresponds to opacity ($I = I_0 = 2E^2 \iff t = 0$), and destructive interference corresponds to transparency ($I = 0 \iff t = 1$). Therefore, from Equation (5.1),

$$t(y) = \frac{I}{I_0} = 1 - \cos^2(ky \sin \theta) = \sin^2(Ky), \tag{5.2}$$

where $K \equiv k \sin \theta$ is the grating frequency. For simplicity's sake, we also assume that the grating is infinite in both x and y ; this allows us to disregard x . Now, let us see what happens if we re-illuminate this amplitude grating with the first beam. Combining Equation (5.1) and Equation (5.2), the near field

complex amplitude of the field immediately after being transmitted through the grating is given by

$$\begin{aligned}\tilde{\mathbf{E}}_{NF} &= t(y)\tilde{\mathbf{E}} \\ &= 2\mathbf{E} \cos(ky \sin \theta) \exp[i(kz \cos \theta - \omega t)][1 - \cos^2(ky \sin \theta)].\end{aligned}\quad (5.3)$$

In the far field, this complex amplitude becomes

$$\begin{aligned}\tilde{\mathbf{E}}_{FF} &= \mathcal{F}_{2D}\{2\mathbf{E} \cos(ky \sin \theta) \exp[i(kz \cos \theta - \omega t)][1 - \cos^2(ky \sin \theta)]\} \\ &\propto \mathbf{E} \left[\underbrace{-\delta(-3k \sin \theta + Y)}_{-1} + \underbrace{\delta(-k \sin \theta + Y)}_0 + \underbrace{\delta(k \sin \theta + Y)}_1 + \underbrace{-\delta(3k \sin \theta + Y)}_2 \right],\end{aligned}\quad (5.4)$$

where $\mathcal{F}_{2D}$ corresponds to the 2D-Fourier transform in x and y ($x, y \rightarrow X, Y$), δ is the Dirac delta function, and we have dropped z and t -dependent terms as well as scaling constants. The terms corresponding to different modes are labeled as such in Equation (5.4). We see that in the far field, with the given assumptions, the incident ray has been diffracted into four different angular modes. We can intuitively understand this in momentum space. Instead of examining the far field, let us instead look to the near field. The real part of Equation (5.3) can be expressed as

$$\begin{aligned}\text{Re}\{\tilde{\mathbf{E}}_{NF}\} &= 2\mathbf{E} \cos(ky \sin \theta) [1 - \cos^2(ky \sin \theta)] \cos(kz \cos \theta - \omega t) \\ &= -\frac{1}{4} \underbrace{\mathbf{E} \cos[(\mathbf{k}_1 + \mathbf{K}) \cdot \mathbf{r} - \omega t]}_{-1} + \frac{1}{4} \underbrace{\mathbf{E} \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t)}_0 \\ &\quad + \frac{1}{4} \underbrace{\mathbf{E} \cos[(\mathbf{k}_1 - \mathbf{K}) \cdot \mathbf{r} - \omega t]}_1 - \frac{1}{4} \underbrace{\mathbf{E} \cos[(\mathbf{k}_1 - 2\mathbf{K}) \cdot \mathbf{r} - \omega t]}_2.\end{aligned}\quad (5.5)$$

Each term in Equation (5.5) represents a different diffracted field mode. We see here that an infinite transmissive sinusoidal grating produces four different, equally intense, Raman-Nath diffracted modes when illuminated by the reference beam $\text{textbf{k}}_1$ [73]: the first term in Equation (5.5) is the first negative mode, created when the grating gives the incident field a momentum kick of amplitude K in the $-\theta$ direction; the second term is the zeroth mode, representing the light which propagates unaffected by the grating; the third term is the first mode, created when the grating gives the incident field a momentum kick of amplitude K in the $+y$ direction; and so on. We visualize this process in Figure 5.1c). Note that the first diffracted mode is identical to the second beam used in the initial interference pattern. This demonstrates that a sinusoidal transmission grating essentially functions to recreate the two writing beams and populate higher-order modes, but what happens if we consider a different type of hologram?

5.1.2 PHASE HOLOGRAMS

There are several ways to record an interference pattern. In the previous section, we recorded a hologram by way of an amplitude grating, but what happens if we use a phase mask instead?

With the same geometry as in the previous example, let us consider the interference pattern created by a more general beam we call the *object beam*. We interfere this with a simple plane wave we call the *reference beam*. These waves can, respectively, be mathematically described as:

$$\tilde{\mathbf{E}}_S = \mathbf{E}_S(x, y) \exp [i(\mathbf{k}_S(x, y) \cdot \mathbf{r} - \omega t)], \quad (5.6)$$

$$\tilde{\mathbf{E}}_R = \mathbf{E}_R \exp [i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)]. \quad (5.7)$$

Here, we consider an object beam with some non-trivial, spatially varying, coherent wavefront. The interference pattern produced by these beams is imprinted into a thin (2D) film, where the amount of phase imparted by the film is proportional to the intensity of the interference pattern. We find that the real amplitude of the transmitted field through this phase mask is

$$\begin{aligned}
 \text{Re}\{\tilde{\mathbf{E}}_{\text{NF}}\} &= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t) - \beta \mathbf{E}_R E_R E_S(x, y) \cos\{[\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r}\} \sin(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \\
 &= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \\
 &\quad - \beta \mathbf{E}_R E_R E_S(x, y) \cos\left\{\mathbf{k}_S(x, y) \cdot \mathbf{r} - \frac{\pi}{2} - \omega t\right\} \\
 &\quad - \beta \mathbf{E}_R E_R E_S(x, y) \cos\left\{[2\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r} - \frac{\pi}{2} - \omega t\right\},
 \end{aligned} \tag{5.8}$$

for some constant β . For a derivation of this result, see Appendix B. The first term in Equation (5.8) is the reference wave [Equation (5.6)], the second term is a $\pi/2$ phase-shifted recreation of the object wave [Equation (5.7)], and the third term is a $\pi/2$ phase-shifted, conjugate recreation of the object wave. This demonstrates that we can imprint the (spatially and temporally coherent) arbitrary wavefront onto a film using a plane reference, and, subsequently, recreate that arbitrary wavefront using the same reference wave. We record an interference pattern in a phase mask and use that phase mask to recall the initial writing beams. This is the basis of holography and lays the groundwork for the analysis to come, as well as the characterization of the various media in later chapters.

GABOR (INLINE) HOLOGRAPHY VS OFF-AXIS HOLOGRAPHY

In this introduction to holography, we have spoken only about *off-axis holography*¹, where the reference

¹a term often used in conjunction with spatial light modulators (SLM) [74, 75].

wave vector is offset from the z -axis and does not share any wavevector component angles with those of the object beam (they do not overlap in angle). It is also possible to write a hologram where the reference and object beams overlap along the z -axis. This is known as *Gabor holography*, and it has several disadvantages. For example, the reconstructed object wave overlaps with the transmitted reference beam, and a conjugate or *twin* image is produced. Off-axis holography has several advantages in that the reconstructed object wave does not overlap with the reference beam and twin images are suppressed.

5.2 VOLUME PHASE HOLOGRAPHY

We extend the ideas of the previous section from two dimensions to three. In a simple sense, we can categorize different diffraction implementations into two regimes: Raman-Nath diffraction, where multiple diffractive orders are non-negligible, and Bragg diffraction, where only one diffracted mode is non-negligible [2, 73, 76]. Diffractive orders, however, are not binary - they are not simply populated or not populated. Diffraction efficiency varies continuously for different modes. Despite this, people have made several attempts to categorize the resulting diffractive regime based on the geometry of the grating itself [77, 78]. In general, however, the thinner a grating is, the more likely it is to exhibit Raman-Nath behaviour, creating many diffracted modes. This suggests, then, that if one wants to design an optic that efficiently couples an incident mode into a different one, a thick grating may do the trick.

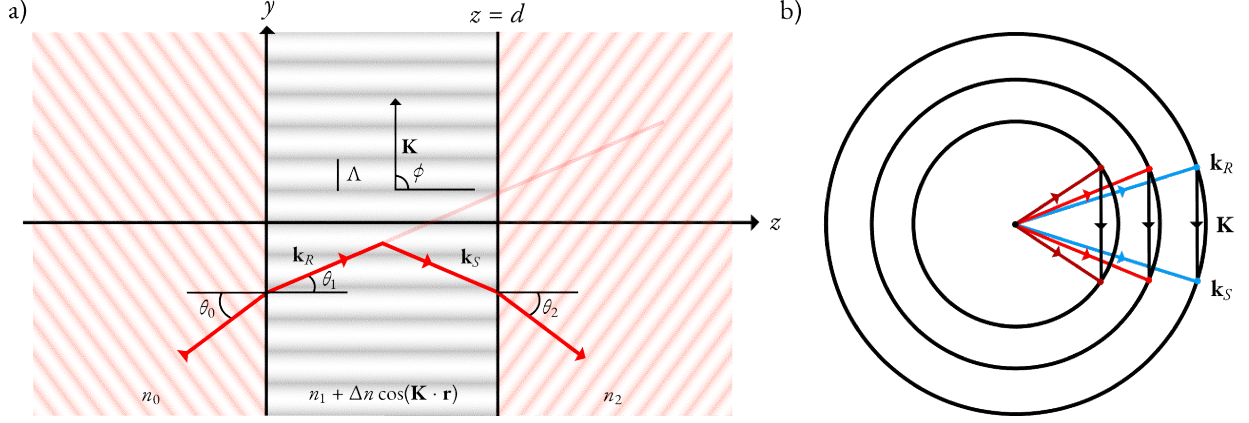


Figure 5.2: Defining the geometry of a dielectric grating (a) and representing the Bragg condition for different incident wavelengths (b). The Bragg angle varies for different wavelengths for a grating vector. In a) we set $\phi = 90^\circ$, but in general the grating can be at any angle.

5.2.1 LIGHT IN A TRANSMISSIVE DIELECTRIC GRATING

First, let us introduce a model from which we can base our development of holography and HOEs. From the model defined below, we can determine how a dielectric grating will affect an incident beam in different geometries. We follow the method of Kogelnik [2].

Suppose we have a dielectric plate of thickness d oriented as shown in Figure 5.2a) with sinusoidally-varying permittivity ε whose grating vector $\mathbf{K}$ makes some angle ϕ with the z axis. The gratings are spaced by distance $\Lambda = 2\pi/|\mathbf{K}|$, and we assume that they are in a holographic medium of base index n_1 . We can write the sinusoidal refractive index of the dielectric as

$$n = n_1 + \Delta n \cos(\mathbf{K} \cdot \mathbf{r}). \quad (5.9)$$

Therefore, $\sqrt{\varepsilon_1} = n_1$, and $\Delta\varepsilon = 2n_1\Delta n$.² Now we ask how the grating might affect an incident optical field.

Generally, suppose a plane wave with vacuum wavelength λ , and wavevector $\mathbf{k}$ is incident on the hologram. This plane wave will satisfy the Bragg condition, visualized in Figure 5.2b), if it transmits through the grating at an angle θ_B :

$$\theta_B = \arcsin\left(\frac{K}{2kn_1}\right) = \arcsin\left(\frac{K\lambda}{4\pi n_1}\right). \quad (5.10)$$

The Bragg angle is the angle at which specular reflection occurs from the planes of the grating and the grating vector K is equal to the change in the wavevector of the light, k . Suppose the plane wave transmits at an angle $\theta_1 = \theta_B - \delta$, where delta is a detuning angle from the first-order Bragg angle. In Appendix C, we show that the plane wave diffracts with first-order diffraction efficiency

$$\eta := \frac{\beta_S}{\beta_R} \mathbf{A}_S \mathbf{A}_S^* = \frac{\sin^2 \sqrt{\xi^2 + \nu^2}}{1 + \xi^2/\nu^2}. \quad (5.11)$$

Where

$$\xi := \frac{\Gamma d}{2\beta_S} = -\frac{kn_1\delta d \sin[2(\phi + \theta_B)]}{2 \cos(2\phi + \theta_B)}, \text{ and} \quad (5.12)$$

$$\nu := \frac{\chi d}{\sqrt{\beta_R \beta_S}} = \frac{\pi \Delta n d}{\lambda \sqrt{-\cos(\theta_B) \cos(2\phi + \theta_B)}}. \quad (5.13)$$

The off-Bragg parameter, ξ , depends on the angle detuning, δ . It measures how far off the incident

²See Appendix C.1 for a derivation of these quantities.

wave is from satisfying the Bragg condition, and for an in-plane grating ($\phi = 0$), $\xi = kn_1\delta d \sin \theta_B$. If δ is too large, negligible energy couples into the first-order diffracted mode. The grating parameter, ν , is grating- and geometry-dependent. It varies with the angle of the grating vector, the index contrast, and the grating thickness. When the Bragg condition is satisfied, $\delta = 0$, and, subsequently, so too does ξ . In such a case, Equation (5.13) allows us to define the maximum efficiency for a given ν :

$$\eta_0 = \sin^2 \nu. \quad (5.14)$$

At Bragg, we can also use Equation (5.11), Equation (5.12) and Equation (5.13) to calculate the index variation parameter of the grating:

$$\Delta n = \frac{\lambda}{\pi d} \sqrt{-\cos \theta_B \cos(2\phi + \theta_B)} \arcsin \sqrt{\eta}. \quad (5.15)$$

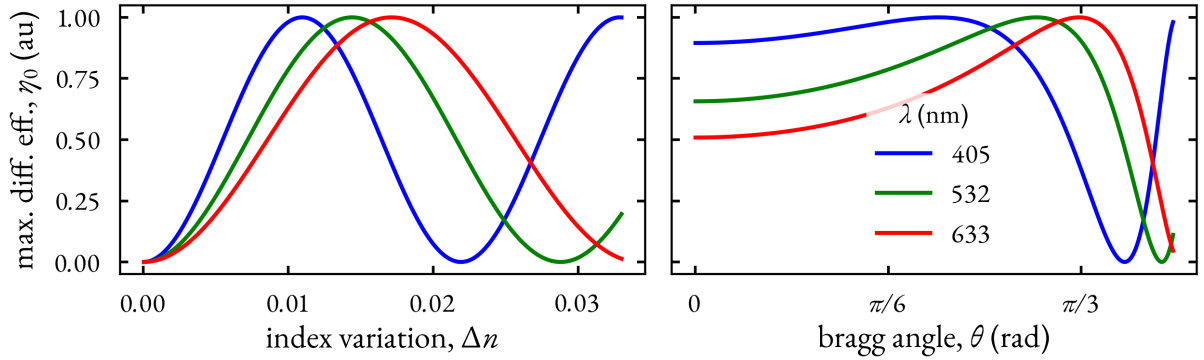


Figure 5.3: Calculated maximal diffraction efficiencies for volume holograms of thickness $d = 16 \mu\text{m}$, base index $n_1 = 1.5$, grating angle $\phi = 90^\circ$, for several wavelengths while varying a) the index contrast (given $\theta_B = \pi/6$) and b) the bragg angle (given $\Delta n = 0.01$).

Now that we have provided a framework to analyze holograms, we can assess their limitations. For simplicity's sake, we assume that the grating vector of the hologram, $\mathbf{K}$, is in the plane of the hologram ($\phi = 0$). To assist in the discussion, we plot several parameters: in Figure 5.3 we plot the maximal diffraction efficiency [Equation (5.11)] given the index variation (a), and, the Bragg angle (b) for several wavelengths. The figure demonstrates the complexity of designing holograms from different perspectives, though it also reveals some potential advantages of holographic materials. Firstly, Figure 5.3a) demonstrates that the diffraction efficiency varies greatly for different index contrasts. It also shows that one can design a grating with any desired diffraction efficiency - considering constraints on other physical parameters like Δn and n_1 . Additionally, Figure 5.3 and Equation (5.11) demonstrate that the diffraction efficiency is periodic in both thickness and index contrast; we see that the diffraction efficiency oscillates from zero unity and back again as index contrast increases. Therefore, one can choose the physical parameters to match different combinations of diffraction efficiencies for different wavelengths. Figure 5.3b) demonstrates how the maximal diffraction efficiency varies with the Bragg angle for different wavelengths. Further, the Bragg angle for a given grating [Equation (5.10)] increases with wavelength - this concept will be useful during the experimental characterization of holograms.

Another important parameter is the angular bandwidth of the hologram. In Figure 5.4, we plot the diffraction efficiency [Equation (5.11)] as a function of angular detuning for two hologram thicknesses, parameterized to both have unity diffraction efficiency at Bragg (we discuss the specific parameters later). These functions are enveloped by a Lorentzian $[1/(1 + \xi^2/\nu^2)]$, whose width is determined by ξ and ν . We define the angular bandwidth by the angular detuning value δ_0 at which the diffraction efficiency reaches its first zero; in Figure 5.4, the bandwidth of the wider, orange curve is denoted by

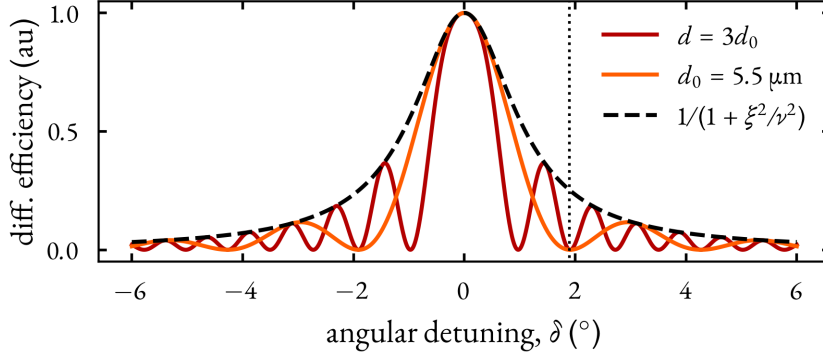


Figure 5.4: Two Bragg curves for a transmission phase grating. We plot the diffraction efficiency as a function of angular detuning, δ given a number of physical parameters: $\lambda = 633$ nm, $\theta = 30^\circ$, $\Delta n = 0.05$, $\phi = 90^\circ$, $n_1 = 1.5$. The vertical dotted line denotes the angular bandwidth, δ_0 [Equation (5.16)] of the d_0 curve.

the dotted black vertical line. The angular bandwidth is

$$\begin{aligned} \delta_0 &= \sqrt{N + \frac{3}{4} \frac{\lambda}{2n_1 d \sin \theta_B}} \\ &= \frac{1}{2N + 1} \sqrt{N + \frac{3}{4} \frac{\Delta n}{n_1 \sin \theta_B \cos \theta_B}}, \end{aligned} \quad (5.16)$$

where $\nu = \pi(2N + 1)/2$ for some natural number N . In Figure 5.4, the orange curve has $N = 0$, and the red curve has $N = 1$.

To reduce the complexity of the angular response, we want to reduce the efficiency of the sidebands and make the envelope narrower, all while maintaining a high Bragg diffraction efficiency (this will be particularly useful when designing the phase response of HOE). To maintain unity efficiency at Bragg,

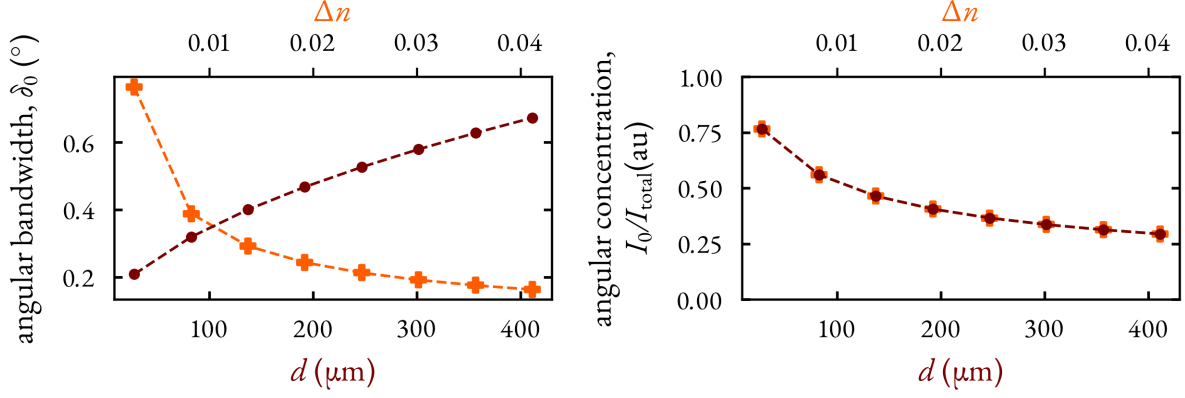


Figure 5.5: Plot of the angular bandwidth (a) and angular concentration (b) as a function of the index contrast and film thickness setting $\Delta nd = \pi/2$, $\lambda = 633$ nm, $\theta = 30^\circ$, and $n_1 = 1.5$. The red scatter points denote varying film thickness, d , and the orange + scatters denote varying index contrast, Δn .

Equation (5.14) tells us that

$$\Delta nd = \frac{2N+1}{2} \lambda \cos \theta. \quad (5.17)$$

Then, to reduce the efficiency of sidebands, we minimize ξ^2/ν^2 or, equivalently, $\Delta n/d$. To demonstrate this, we define an angular concentration parameter, defined as:

$$\begin{aligned} C(\Delta n, d) &:= \frac{I_0}{I} = \frac{\int_{-\delta_0}^{\delta_0} \eta(\delta) d\delta}{\int_{-\pi/2-\theta_B}^{\pi/2-\theta_B} \eta(\delta) d\delta} \\ &= \left(\int_{-\delta_0}^{\delta_0} \frac{\sin^2 \sqrt{\xi(\delta)^2 + \nu^2}}{1 + \xi(\delta)^2/\nu^2} d\delta \right) \left(\int_{-\pi/2-\theta_B}^{\pi/2-\theta_B} \frac{\sin^2 \sqrt{\xi(\delta)^2 + \nu^2}}{1 + \xi(\delta)^2/\nu^2} d\delta \right)^{-1}. \end{aligned} \quad (5.18)$$

We numerically evaluate Equation (5.18) given a set of physical parameters. In Figure 5.5a) the angular bandwidth increases with $\Delta n/d$. Therefore, to reduce the angular bandwidth of a holographic

5.3. RIGOROUS COUPLED WAVE THEORY

film, we need to maximize the film thickness and reduce the index contrast while respecting the unity efficiency condition [Equation (5.17)]. Further, we see in Figure 5.5b) that the angular concentration decreases with both film thickness and index contrast ($C \sim N$). Therefore, to concentrate most power within the first maximum, one should use the minimum index contrast allowable while respecting the efficiency condition for grating of a certain thickness.

This section made several conclusions about the characterization and analysis of volume holograms. We extended the holography to three dimensions, introducing a mathematical framework for volume holograms. It allows us to calculate the diffraction efficiency, η , of a holographic grating, the index contrast, Δn , and the angular bandwidth, δ_0 . It also permits us to characterize the angular diffraction concentration of a hologram given various physical parameters and constraints. These tools will be useful in later sections when we characterize experimental gratings.

5.3 RIGOROUS COUPLED WAVE THEORY

The derivations in Section 5.1 give us a framework from which we can study and design holograms and HOEs; however, they have several assumptions through which certain inaccuracies creep in. First, we assumed that the second-order spatial derivatives of the wave amplitudes are negligible; second, we ignored boundary diffraction; lastly, we considered only first-order diffraction. To determine the effect of these assumptions on any holograms we design, we need another framework to assess them accurately. To do so, we take the approach of Moharam and Gaylord, who produced a more rigorous framework for the study of holograms [4].

5.3. RIGOROUS COUPLED WAVE THEORY

In Appendix D, we introduce a model that more accurately assesses the strengths of high-order diffractive modes in volume holograms. Our model uses the physical setup as the previous section; instead, we consider all potential diffracted modes in all three segments of the hologram (before, inside, and after the dielectric grating). With some modifications, we follow the method of Moharam and Gaylord [4], and the resulting formalism can analyze both reflection and transmission gratings.

These methods are used in the following sections to assess the efficiencies of higher-order diffraction for different holographic geometries and the validity of our previous assumptions. It can also characterize the designs of different HOEs.

5.3.1 THE IMPACT OF HIGHER-ORDER DIFFRACTION

Let us consider a number of volume holograms and calculate the diffraction efficiencies using both first-order (Kogelnik) analysis (Section 5.2.1) and rigorous coupled wave analysis, comparing and contrasting the two. We plot the diffraction efficiencies for in-plane (a, b) and out-of plane gratings (c), and high spatial frequencies (a) and low spatial frequencies (b, c).

Comparing Figure 5.6a) and Figure 5.6b), we see that Kogelnik's assumptions hold water at low grating frequencies (small K , large θ_B) for unslanted gratings. However, as the Bragg angle decreases (or the grating spatial frequency decreases), we get higher-order mode coupling. Comparing Figure 5.6b) and Figure 5.6c) our earlier assumptions in Section 5.2.1 become less accurate for slanted gratings.

We also see again how the diffraction efficiency is pseudo-periodic as a function of film thickness. We initially saw this in Figure 5.3. Heuristically, this can be understood as follows: as the incident mode propagates through the holographic medium, power diffracts into the first-order mode. If the holo-

5.3. RIGOROUS COUPLED WAVE THEORY

gram is thick enough, the first-order mode starts to diffract back into the zeroth-order mode, and this pattern repeats. For gratings that are not in the plane of the hologram ($\phi \neq 90^\circ$) or high spatial frequency gratings, higher-order modes are non-negligible, and the oscillation is not perfectly periodic. In this case, power does not simply oscillate between the zeroth- and first-order modes. This assessment strengthens our earlier conclusion that to design efficient, high angular- and wavelength-resolved holograms, we should use in-plane gratings of moderate to high spatial frequency. To better understand the higher-order behaviour of the angular response of a hologram, we calculate the diffraction efficiency of different volume holographic gratings. We select a static grating d and detune the angle of the incident mode around the first-order Bragg angle ($\theta = \theta_B + \delta$). We use the three gratings introduced earlier (Figure 5.6). We plot the diffraction efficiency as a function of incident angle in Figure 5.7.

Similar to when we varied grating thickness, d , there is increased modal complexity for slanted gratings and gratings with high spatial frequency. Figure 5.7a) demonstrates again that for large incident angles on unslanted gratings, Kogelnik's findings hold quite well. The angular and wavelength band-

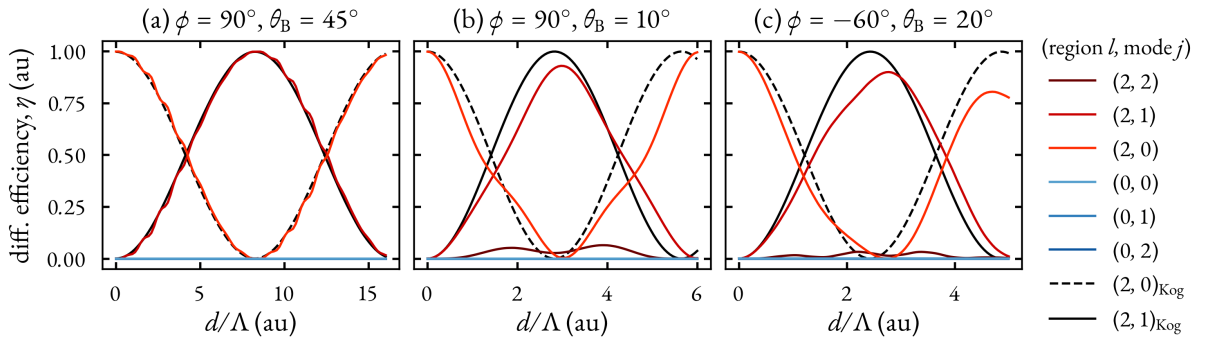


Figure 5.6: Rigorous solutions for regions $\{l\} = \{0, 2\}$ to three physical setups solved for modes $\{j\} = \{4, 3, 2, 1, 0, -1, -2, -3, -4\}$ compared to solutions returned using Kogelnik's first-order assumption (in black). See text for discussion.

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widths are calculated accurately by first-order assumptions. However, Figure 5.7b) demonstrates that for small incident angles (here, 10°) second- and higher-order modes have non-negligible strengths. For slanted gratings [Figure 5.7c)], the situation is even worse. First-order assumptions only approximate the rigorous mode amplitudes near the designed angle, and there is strong second-order mode coupling for a wide range of incident angles.

This analysis allows us to conclude a few things regarding the potential application of holograms to design phase-dependent optical elements. Firstly and most importantly, because Kogelnik's coupled wave theory is the simplest way to deal with HOEs, one should analytically design holograms within bounds where the first-order assumptions introduce negligible error. This is where $\phi \approx 90^\circ$ and $20^\circ \leq \theta_B \leq 70^\circ$.

To visualize the angular response of the hologram further, We plot the diffraction efficiency of each mode (considering modes up to the ninth order) as a function of incident beam angle and resulting

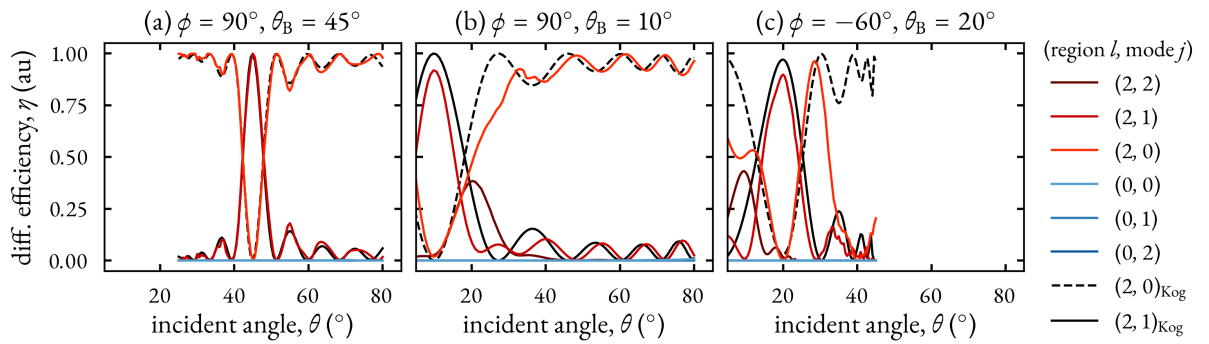


Figure 5.7: Diffraction efficiency as a function of incident angle for several different holographic gratings. We plot rigorous solutions to three physical setups solved for the first nine diffracted reflected and transmitted modes compared to solutions returned using Kogelnik's first-order assumption. We limit the angular range of some of the solutions, as the numerical solver is not stable for certain values. See discussion in text.

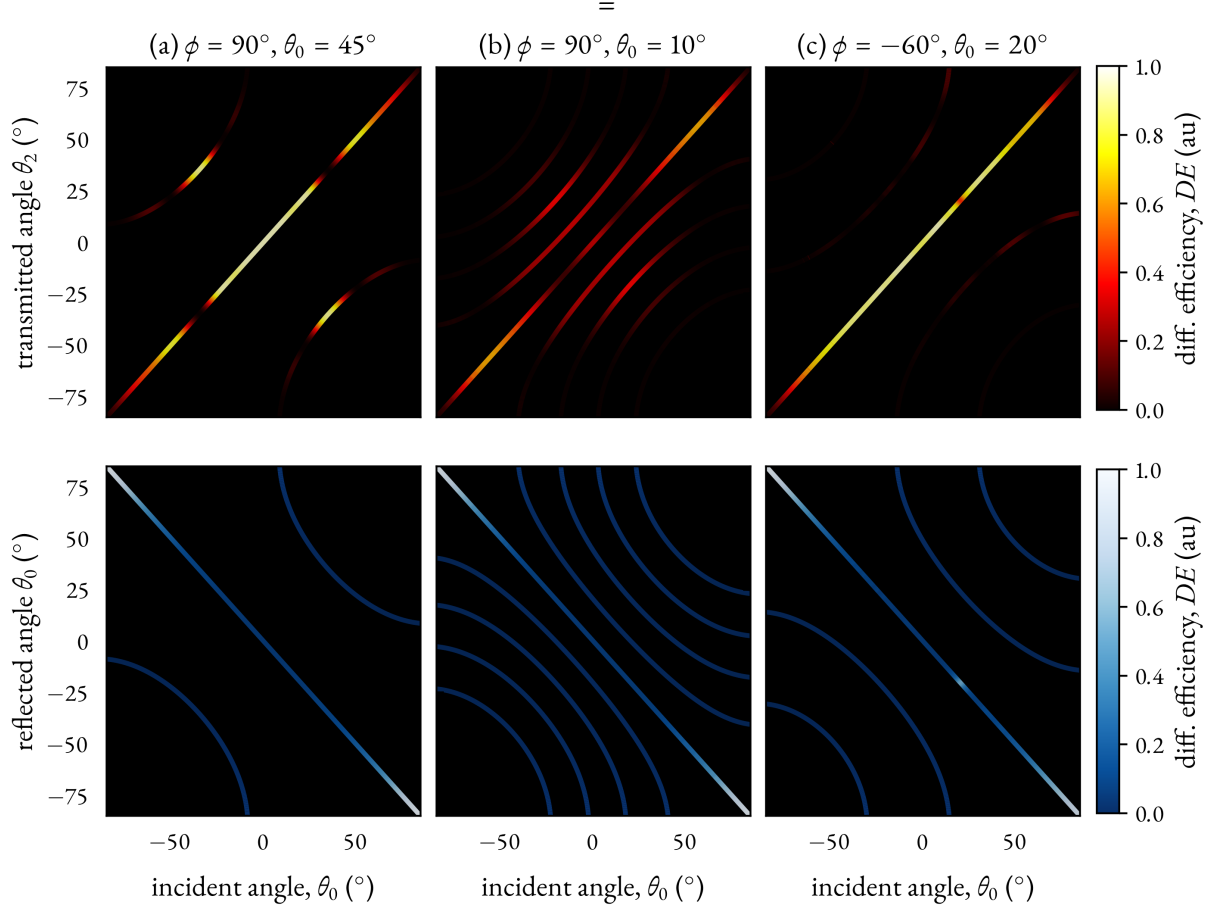


Figure 5.8: Diffraction efficiencies as a function of both incident (θ_0), transmitted mode angle (upper plots, red, θ_2) and reflected mode angle (lower plots, blue, θ_0) for first nine modes for several different gratings.

transmitted or reflected mode angle. We consider the three gratings introduced earlier, though this time we set realistic values of background index: $n_0 = n_2 = 1$ and $n_2 = 1.5$. The color of each labeled mode in Figure 5.8 denotes the efficiency of each mode at each input angle.

Figure 5.8 demonstrates that the coupling of light in holograms is most assuredly not first-order. As we saw before, Figure 5.8a) demonstrates that for moderate incident angles, the hologram behaves

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nicely, producing mostly first-mode coupling with a small off-Bragg region. For shallow incident angles [Figure 5.8b)], however, there is considerable higher-mode coupling, and it turns out that when we use a background index not equal to that of the hologram, there is non-negligible coupling into the reflected modes. In fact, for the case in Figure 5.8b), there are four significant transmitted modes and five non-negligible reflected modes. Therefore, high-order, rigorous analysis is necessary to accurately characterize the efficiency of holographic designs³.

³To demonstrate the usefulness of this analysis, in Appendix D.2 we visualize how the field strength of a Bragg-matched beam diffracts inside a holographic grating.

6

HOLOGRAPHIC DEVICES AND MATERIALS

In this chapter we propose a method to design the refractive index pattern for a holographic spaceplate. Our method can be used to design an angle-dependent phase response for a general input beam. To understand the limitations of our method, we then review the background material on photopolymers and lithium niobate and present new experimental data characterizing samples of different holographic media.

Regarding experimental characterization, to make an angle-dependent HOE we want to be able to write a holographic grating of a specific diffraction efficiency and given a set of physical constraints. Therefore, we need to answer the following questions:

1. What effect does the intensity of the writing laser have on the resulting index contrast and, subsequently, diffraction efficiency?
2. What effect does exposure time have on the index contrast and, subsequently, the diffraction efficiency?
3. How does diffraction efficiency change when we experimentally multiplex holographic gratings?

In this chapter, we answer these questions and produce a method to determine the index contrast of a hologram as it is exposed. These answers are material-dependent, and there are several features to consider when evaluating a holographic material: it should be highly resolving and sensitive to light intensity; a good holographic material should also be stable across a range of temperatures and pressures. The cost of the material, its holographic permanence, and erasability are also important when determining the ideal material for a given application. Some materials are also self-exposing; this can be very useful in different applications. Two materials frequently used in applications are photopolymer holograms and holograms made of lithium niobate. In this chapter, we describe how these materials store holographic information and how we experimentally characterize them.

6.1 HOLOGRAPHIC OPTICAL ELEMENTS

Optical elements change the shape of an incident optical wavefront into a different one. Examples of such devices include lenses, diffraction gratings, wave plates, and mirrors. Holograms work similarly; they take an incident wavefront and convert it into a different one [79]. Therefore, it is easy to imagine designing holograms to perform the transformations made by other optical elements, replacing those elements with their holographic optical element counterparts. Recent work has improved the design of holographic optical elements such as holographic lenses [6, 7], beam splitters [80], and spectrum splitters and combiners [11–14]. Additionally, innovations in device engineering have improved the resolution and efficiency of holographic optical elements [81, 82]. These have enabled applications for holograms in technologies such as heads-up-displays [6, 15, 80], and augmented reality systems [10, 83].

There are several geometrical considerations and limitations to take into account, however, when designing holographic optical elements. Firstly, holograms work based on diffraction and are highly wavelength-dispersive. Therefore, holographic optical elements that replace reflective or refractive conventional optics like lenses or mirrors will be much more dispersive and have lower efficiencies. For instance, the focal length of a holographic lens will depend much more strongly on wavelength than its refractive counterpart. Having said that, holographic optical elements have several advantages over conventional optics. They are inexpensive, lightweight, thin, and can have large aperture sizes. For instance, optics such as refractive lenses and cube beam splitters use bulk material to achieve their optical effects, resulting in heavy, often expensive implementations. Additionally, it is possible to multiplex different optical operations into a single holographic optical element. Even with all these advancements,

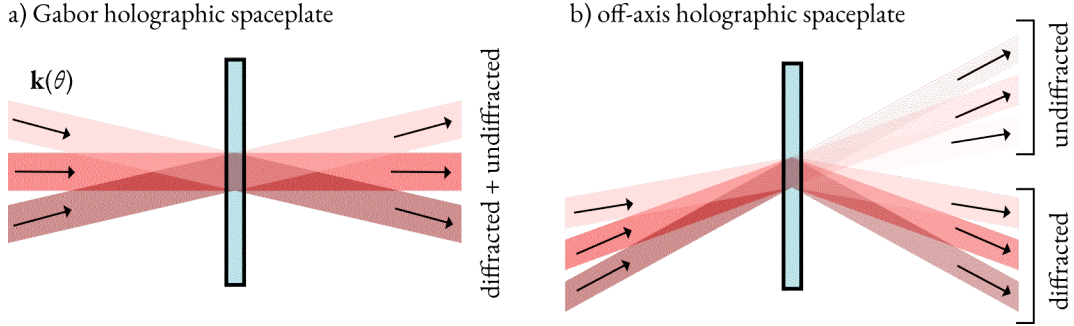


Figure 6.1: Gabor and off-axis implementations of a holographic spaceplate. a) The angular range of the incident beam overlaps with the diffracted, spaceplated beam. Without unity diffraction efficiency, the resulting field will be distorted. b) The zeroth order diffraction is offset from the first order, spaceplated, diffraction. The field is compressed in space and inverted along the propagation and hologram axes.

there is no known method to design a general holographic optical element for a general input waveform [84, 85].

Using the methods described in Section 5.3 and those introduced by Lewis and Solymar [86, 87], we can solve for the transmitted field given a hologram and an incident field, but we cannot do the inverse. Several algorithms in computer-generated holography can solve the index pattern for a hologram, but these require knowledge of the inputted and outputted beams [83, 88]. We cannot yet solve for the grating distribution knowing only the phase difference between the incident and outgoing fields. In the following section, we attempt to do this by designing a holographic spaceplate.

6.1.1 TOWARDS A HOLOGRAPHIC SPACEPLATE

We saw in Section 2.2.3 that the spaceplate must produce a wavevector- or angle-dependent phase response. Transmission holograms are good candidates to achieve this as the diffraction efficiency of a grating highly depends on the wavevector. What would this look like? Firstly, spaceplates do not influ-

ence the resultant momentum of light - the exception being the three-lens spaceplate. This is impossible for holograms, as they function through diffraction, though we can think of a workaround. If we design a Gabor or off-axis hologram using only in-plane gratings, we can use the hologram to add the spaceplate phase to the first-order diffracted beam. This would create an inverting spaceplate, similar to the three-lens spaceplate, which transversely inverts the field. Therefore, we need to tailor the index patterning of the hologram to diffract and phase-shift each wavevector component of the incident light.

In this section, we derive several results. From the spaceplate phase response [Equation (2.10)], we derive the required holographic grating-vector distribution. From that distribution, we then find the holographic index pattern. To do this, we need to make several assumptions, but before we do that, let us reintroduce some notation. Given a holographic film of nominal index n_1 , we wish to solve for the spatially varying index contrast $\Delta n(\mathbf{r})$. Using a Fourier transform this index contrast can be expressed in frequency space as a function of grating vector $N(\mathbf{K})$. Our design must achieve several things: each wavevector component $\mathbf{k}$ of the incident wave must be diffracted in a single direction, each with the same diffraction efficiency. Secondly, each wavevector component must gain the correct relative phase to some global phase offset.

Our derivation also uses several assumptions. Any wavevector component, $\mathbf{k}$, of the incident wave will interact with a single grating component, $\mathbf{K}$. Secondly, we assume that the relative phases of the gratings result in relative phase shifts between the diffracted beams. We discuss the validity of these assumptions at the end of the section.

Suppose we have a holographic grating with grating vector $\mathbf{K}$ defined in the region $0 < z < d$, for which an incident wavevector $\mathbf{k}$ meets the first-order Bragg condition using the same formalism and

geometry as introduced in Section 5.1. For simplicity and efficiency's sake, we assume the grating is in-plane ($\phi = \pi/2$). We introduce the $\mathbf{k}$ -dependent spaceplate phase into the expression for the spatially varying electric permittivity [Equation (5.9)].

$$\epsilon = \epsilon_2 + \Delta\epsilon \cos(\mathbf{K} \cdot \mathbf{r} + \psi(\mathbf{k})).$$

The index inside the hologram can be written, then, as

$$\begin{aligned} n &= \sqrt{\epsilon} \\ &\approx \sqrt{\epsilon_2} + \frac{\Delta\epsilon}{2\sqrt{\epsilon_2}} \cos(K \cdot r + \psi(\mathbf{k})) \\ &= n_1 + \Delta n \cos(\mathbf{K} \cdot \mathbf{r} + \psi(\mathbf{k})), \end{aligned} \tag{6.1}$$

where we explicitly write $\psi = \psi(\mathbf{k}) = \psi(\theta_0)$ to show that the added phase is a function of the incident angle of the plane wave. In the case of the spaceplate this phase is $\psi(\mathbf{k}) = Rdk_z^{(1)}$ [Equation (2.10)] where $k_z^{(1)}$ is the z -component of the k -vector in the background medium. This gives us an expression for the holographic index as a function of the grating vector. It adds the spaceplate phase to the first-order diffracted wave.

To create an optic that appropriately adds phase for all potential input angles, we must create a superimposed grating that includes all grating frequencies.

Now, because we assume $\mathbf{k}$ and $\mathbf{K}$ are one-to-one ($\mathbf{k} \leftrightarrow \mathbf{K}$), we can write, using Equation (2.10)

that

$$\begin{aligned}
 \psi(K) &= \psi(k) \\
 &= \mathcal{R} d k_z^{(0)} \\
 &= \mathcal{R} d \sqrt{(k^{(0)})^2 - (k_y^{(0)})^2} \\
 &= \mathcal{R} d \sqrt{(kn_0)^2 - [kn_0 \sin(\theta_0)]^2} \\
 &= \mathcal{R} d \sqrt{(kn_1)^2 - [kn_1 \sin(\theta_1)]^2},
 \end{aligned} \tag{6.2}$$

where we have dropped the bolded notation for K because it is assumed to be in the y -direction. To satisfy the Bragg condition [originally phrased as Equation (5.10)], $K = 2k_x^{(1)} = 2kn_1 \sin(\theta_1)$, so Equation (6.2) becomes

$$\psi(K) = \mathcal{R} d \sqrt{(kn_0)^2 - \frac{K^2}{4}}. \tag{6.3}$$

Now, we need an expression for the index contrast as a function of the grating wavevector. Equation (5.15) allows us to express the index contrast as a function of efficiency and wavevector angle, but we can rephrase this in terms of the grating wavevector. First, recall our stipulation that all beams must diffract with the same efficiency, η . To first order, using the Bragg condition, we can write Equa-

tion (5.15) as

$$\begin{aligned}
 \Delta n &= \frac{2}{kd} \arcsin \sqrt{\eta} \cos \theta_1 \\
 &= \frac{\pi\eta}{kd} \sqrt{1 - \sin^2 \theta_1} \\
 &= \frac{\pi\eta}{kd} \sqrt{1 - \left(\frac{K}{2kn_1} \right)^2}.
 \end{aligned} \tag{6.4}$$

Thus, using Equations (6.2) and (6.4), we can write the spatially varying index for a certain grating vector as

$$n(K, y) = n_1 + \underbrace{\frac{\pi}{k^2 n_1 d} \sqrt{(kn_1)^2 - \frac{K^2}{4}}}_{\Delta n(\mathbf{K})} \cos \left(Ky + \underbrace{\mathcal{R} d \sqrt{(kn_1)^2 - \frac{K^2}{4}}}_{\psi(\mathbf{K})} \right); \tag{6.5}$$

$$n(\theta_0, y) = n_1 + \frac{\lambda}{2d} \cos(\theta_0) \cos \left(Ky + Rd \sqrt{(kn_1)^2 - (kn_0 \sin(\theta_0))^2} \right). \tag{6.6}$$

Equation (6.6) and Equation (6.5) allow us to express the spatial index pattern [Equation (6.1)] for a specific grating frequency for a certain incident plane wave.

Now, we want to consider what happens if we integrate over all of these grating vectors. Remember, we want to create an index pattern (a volume hologram) that is the sum of all of these plane wave holograms, assuming each hologram component affects only one incoming wavevector. To do so, we

make a notation change, denoting the j -th grating by $K \rightarrow K_j$. We then take a Fourier transform:

$$\begin{aligned}\tilde{n}_j(K, K_j) &:= \mathcal{F} \{n(K, y)\} \\ &= \sqrt{2\pi} \delta(K_j) \delta(K) n_1 + \frac{\pi \Delta n(K_j)}{\sqrt{2}} \left\{ \exp[-i\psi(K_j)] \delta(K - K_j) + \exp[i\psi(K_j)] \delta(K + K_j) \right\}.\end{aligned}$$

Here, δ is the Dirac delta. Now we can either integrate over all grating component frequency components K_j :

$$\tilde{n}(K) = \int_{-\infty}^{\infty} \tilde{n}_j(K, K_j) dK_j; \quad (6.7)$$

$$\tilde{\tilde{n}}(K) = \int_{K_{\min}}^{K_{\max}} \tilde{n}_j(K, K_j) dK_j. \quad (6.8)$$

These equations can assess the effect of different assumptions. Equation (6.7) is only valid in gratings that can achieve unlimited spatial frequencies. This is impossible for any real grating, however, the evaluation of Equation (6.7) could prove elucidating. Equation (6.8) is useful for gratings that have some more physical frequency boundaries, but it still assumes $\mathbf{k} \leftrightarrow \mathbf{K}$, which is strictly not true for gratings.

Here, we only analytically assess Equation (6.7) and find that

$$\begin{aligned}\tilde{n}(K) &= n_1 \delta(K) + \Delta n(K) \cos \psi(K) \\ &= n_1 \delta(K) + \frac{\pi}{k^2 n_1 d} \sqrt{(kn_1)^2 - \frac{K^2}{4}} \cos \left\{ \mathcal{R} d \sqrt{(kn_1)^2 - \frac{K^2}{4}} \right\}.\end{aligned} \quad (6.9)$$

We can now take the inverse Fourier transform to get an expression for the spatially varying index. It is an onerous calculation; therefore, we provide the derivation in Appendix F and simply present the result. The derivation presented is primarily the work of Michael Weil. The form of the spatial index of refraction is given by

$$n(y) = n_1 + \frac{\pi^2 \eta}{k^2 d} (b^2 + y^2)^{-3/2} \left[(y^2 - b^2) J_1 \left(a \sqrt{b^2 + y^2} \right) + a b^2 \sqrt{b^2 + y^2} J_0 \left(a \sqrt{b^2 + y^2} \right) \right], \quad (6.10)$$

where $a = 2kn_1$, $b = \mathcal{R}_\lambda d/2n_1$, and J_0 and J_1 are the zeroth- and first-order Bessel functions of the first kind. This result has been verified by comparing it to the numerically-computed Fourier transform.

There are a few peculiarities about Equation (6.10) we should discuss. We are most interested in how the index oscillates about n_1 , so we define the index variation ($\rho := n - n_1$). Additionally, we plot the index variation in Equation (6.10) in Figure 6.2 assuming $n_1 = 1.5$, $\lambda = 633$ nm, $\mathcal{R}_\lambda = 2$, $d = 16$ μ m, $\eta = 1$.

Firstly, the index modulation is symmetric about the y and is aperiodic. We assumed that the transverse extent of the grating and incident wave are of infinite size. The aperiodicity of this result raises the question of what might happen if, given the grating presented in Figure 6.2, we transmitted a beam of finite size far from the $y = 0$ transverse position. We suspect that it depends on the maximum resolution and fidelity of the grating.

Second, the Bessel functions in Equation (6.10) are scaled by two different functions which depend on the physical parameters. These scaling functions dictate the amplitude of index modulation, so we linearly expand their expressions about $y = 0$ ($|y| \ll b$), where the modulation is highest. We find that

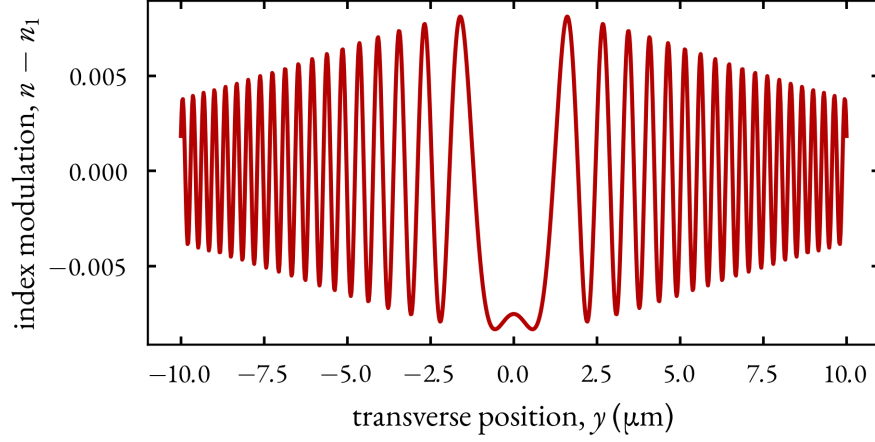


Figure 6.2: The index variation for a perfect holographic spaceplate. This assumes physical parameters $n_1 = 1.5$, $\lambda = 633 \text{ nm}$, $\mathcal{R}_c = 2$, $d = 16 \text{ } \mu\text{m}$, $\eta = 1$. It has a maximum index contrast of 0.07.

the the amplitudes of the first and second bessell functions are approximated by

$$J_0 \text{ scaling : } -\frac{2\pi^2\eta}{\mathcal{R}_c d^2 k^2}; \quad (6.11)$$

$$J_1 \text{ scaling : } \frac{\pi^2 n_1 \eta}{dk}. \quad (6.12)$$

We see from Equation (6.11) and Equation (6.12), therefore, that the maximum index contrast is proportional to efficiency and base refractive index and inversely proportional to the grating thickness d . This is intuitive as our single grating analysis [Equation (5.15)] produced the same results. The stronger the index contrast, the stronger the beam interaction, and the less distance is needed to achieve some beam diffraction efficiency. What is peculiar is that the index variation is *inversely* proportional to the compression ratio. It suggests that to produce a highly compressive spaceplate, we need a very low-index contrast. This does not make immediate sense and needs further thought. It could potentially be

a manifestation of the inherent trade-off between numerical aperture and compression ratio, but this is not clear.

Now, we make several important conclusions about the proposed transmissive holographic spaceplate. Firstly, our analysis has produced a grating with a potentially realizable grating pattern; the necessary index contrast for a 16 μm thick film is only 0.07, and a reduction in this would only result in a lower efficiency. In Section 6.2, we will demonstrate an index contrast of $\Delta n = 0.014$ in a photopolymer, which suggests such an implementation could produce space compression with non-negligible diffraction efficiency. A caveat to this is that in the above derivation we assumed that the hologram could record any spatial frequencies and could, therefore, have an unlimited numerical aperture. Such a feat is impossible, so we have provided frameworks for a more reasonable analysis. Equation (6.8) provides a method by which one can calculate the index pattern for a grating with a finite bandwidth. Further, we assume that the hologram uses only in-plane gratings, with variation in a single transverse direction. A formulation using arbitrary grating vectors might produce better HOEs and may be worth exploring.

A discussion of our initial assumptions is now relevant. We assumed, initially, that the angular bandwidth of each grating component δ_0 was zero. That is, that $\mathbf{k}$ and $\mathbf{K}$ are one-to-one. We saw in our experiments that the bandwidth of the tested films is on the order of degrees - nowhere near zero. With these materials, it is improbable that a holographic spaceplate could work as proposed. The finite bandwidths of real holograms would cause a blurring of the phase response. We still need to determine how the blurring would limit the compression ratio or functionality of the spaceplate. It may render our method completely useless.

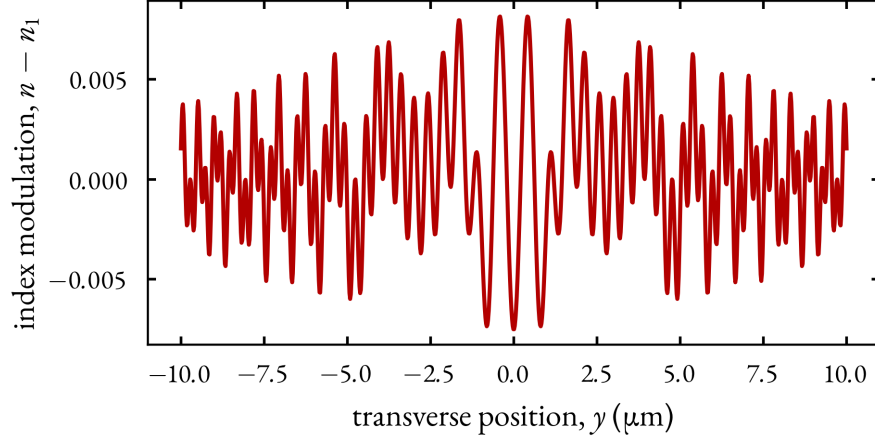


Figure 6.3: The index variation for a perfect holographic spaceplate. This assumes physical parameters $n_1 = 1.5$, $\lambda = 633 \text{ nm}$, $\mathcal{R}_\omega = 2$, $d = 16 \text{ } \mu\text{m}$, $\eta = 1$, $\theta_{\text{OA}} = \pi/6$.

The second assumption was that a phase shift in the phase of a grating component relative to the rest of the grating would result in a relative phase shift of the outgoing beam. We derive this result in Appendix G, and it has been suggested by several theoretical analyses [89–91], but they focused more on the cross-coupling of gratings and the resulting diffraction efficiencies rather than the relative phase of the outgoing wavevectors. Experimental verification of this has not yet been done.

Thirdly, this derivation assumed the Gabor hologram geometry, where the incident and outgoing beams overlap [Figure 6.1(a)]. This introduces several issues including twin image reconstruction, beam overlap, and high-order cross-coupling etc. An off-axis geometry could fix these issues. Mathematically, this would introduce a spatial frequency off-axis offset K_{OA} . In Equation (6.9), $K \rightarrow K + K_{\text{OA}}$ because of the new Bragg condition ($K + K_{\text{OA}} = 2k_x^{(1)} = 2kn_1 \sin[\theta_1 + \theta_{\text{OA}}^{(1)}]$). This would add a phase offset ($\text{Real}\{\exp(-iK_{\text{OA}}x)\}$) to the varying term of the index pattern [Equation (6.10)], but would not influence the index contrast. We plot the resulting index for an offset of $\theta_{\text{OA}} = \pi/6$ in Figure 6.3.

The formalism presented in this section takes several liberties and makes several assumptions, some of which may render a physical implementation highly non-ideal or non-functional. There is considerable work to do to achieve a holographic spaceplate, but we have provided a framework in which further investigations can be made. A summary of these results and future work is given in Chapter 8.

6.2 HOLOGRAPHY IN PHOTOPOLYMERS

6.2.1 THEORY OF PHOTOPOLYMER HOLOGRAMS

Using photopolymers as holographic materials is attractive for many reasons: they are inexpensive, self-exposing, high-resolution, and often require no chemical processing [3, 7]. There are several different kinds of photopolymers used for holography, and the exact behaviours of each photopolymer vary between manufacturers; however, they function generally the same [92, 93].

A self-processing dry photopolymer generally consists of a binder, a cross-linking monomer, a cosensitizer, and a dye sensitizer. Solutions of these materials are crafted into thin sheets and plated on glass substrates. When exposed to intense light beams, phase gratings are imprinted into the monomer solution [92].

When the photopolymer sheet is illuminated by a frequency of light which the dye sensitizers are responsive to, they are excited above their energy ground state. These molecules can fluoresce and drop back down to their ground state by emitting light of a longer wavelength, or they can catalyze polymerization: the excited dye molecule can interact with a cosensitizer molecule, causing the release of

a proton and, subsequently, causing free radical polymerization of the monomers. This polymerization cascades, and the monomers form longer molecules by converting single bonds to double bonds. The creation of double bonds results in changes to the polarizability of the molecule, manifesting in a reduction in the index of refraction in regions of polymerization. As monomers convert to polymers in high intensity regions, monomers diffuse in from dark regions, decreasing the refractive index in areas of low intensity [79, 94, 95]. Therefore, there is an interplay between the rate of polymerization and the diffusion of monomers.

After exposure for long periods at high intensity, many of the monomers will have converted to polymers, and the grating will become permanent. The rate at which this occurs differs for different kinds of photopolymers, and this information is neither widely published nor studied. Therefore, to design holograms with specific refractive index patterns, we must first characterize the effects of different experimental parameters on the photopolymer gratings. We do so in the following sections.

DESCRIPTION OF PHOTOPOLYMER MATERIALS

These experiments use C-RT20 photosensitive film procured from LitiHolo. The sample is comprised of a 1.8 mm thick glass substrate, a 16 μm thick photosensitive film, and a 50 μm thick cover sheet (see Figure 6.4). The film is sensitive across a wide wavelength band (400-690 nm), and LitiHolo reports



Figure 6.4: Dimensions of the litiHolo C-RT20 photosensitive film procured from ©Liti Holographics [96].

diffraction efficiencies above 99% in both the transmission and reflection schemes [96]¹.

The temporal response of these films is not well known, so we aim to characterize them.

6.2.2 CHARACTERIZATION OF PHOTOPOLYMER HOLOGRAMS

We performed several experiments on photopolymer gratings to create a robust method to determine and optimize angular and wavelength efficiencies of holographic gratings in an experimental medium. These experiments include measuring the temporal response of an exposing photopolymer hologram and the angular characterization of a photopolymer hologram. Using methods from Chapter 5 we can determine the index contrast, diffraction efficiency, and angular bandwidth of the gratings. It is possible to use these measurements to characterize the rate of polymerization and diffusion [10, 94], but we leave this as an extension to the project.

We use the setup shown in Figure 6.5 to write and characterize the holograms. A linearly polarized 25 mW CW 633 nm laser diode with a coherence length of ~ 10 m is polarized and power regulated using a HWP and a GP. The vertically polarized beam is sent through a telescope to create a beam with a $1/e^2$ diameter of (10 ± 1) mm, which is then split by a 50:50 beam splitter. The transmitted and reflected beams termed the reference and object beams, respectively, are reflected orthogonally to one another and overlap in the plane of the holographic medium. The optical power in the reference and object beams vary between 1 and 15 mW, and are specified for each set of results. To change the angle of the grating it was necessary to change the angle of one of the beams. Attention was made to keep the path length difference between the beams to a minimum to reduce decoherence.

¹This efficiency does not consider substrate losses - it only considers relative power in the zeroth- and first-order diffracted modes.

6.2. HOLOGRAPHY IN PHOTOPOLYMERS

The holographic medium, a Litiholo CRT20 photopolymer film, is mounted vertically on a motorized rotating and translating stage. The reference and object beams transmit through the hologram, and their transmitted optical powers (P^R , P^O) are measured using two amplified PD (PD1 and PD2) connected to a data acquisition device (DAQ), each photodiode calibrated to the 633 nm laser diode. To measure the diffraction efficiency of the grating, the object beam is blocked, and the telescope is removed, creating a beam with 3 mm diameter. This beam is diffracted by the written hologram, and the powers of the diffracted beams are measured using the same photodiodes as before. In some experiments, we use a probe beam to measure the diffraction efficiency of the exposing hologram. A 532 nm

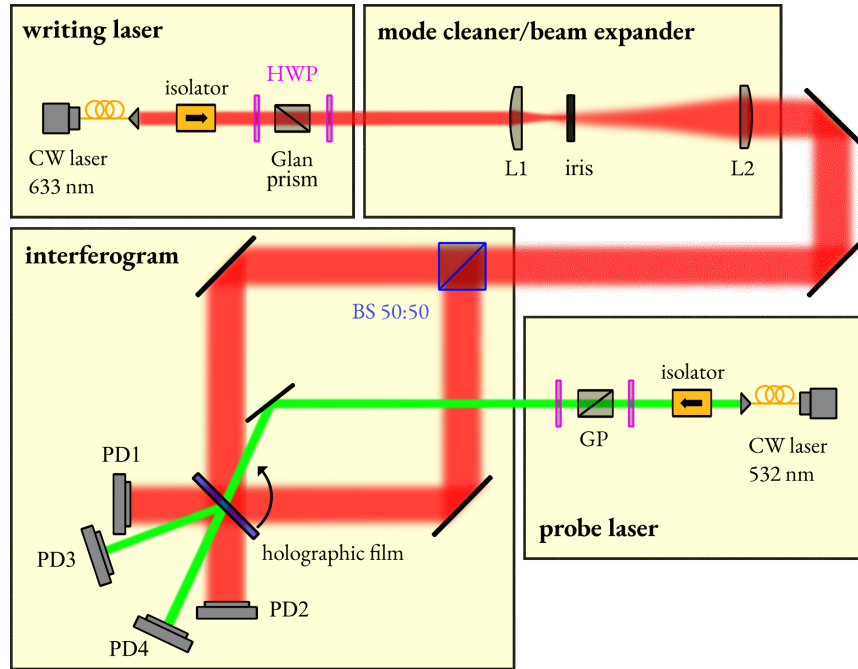


Figure 6.5: The optical setup for writing and characterizing holograms. Lenses L1 and L2 have focal lengths 25 mm and 75 mm and use various contractions: beam splitter, BS; continuous wave, CW; half wave plate, HWP; photodiode, PD.

probe laser diode at low power ($P_i = 15 \text{ uW}$) is transmitted through the center of the 633 nm interference at its Bragg angle. The optical powers of the zeroth and first-order modes P_0 and P_1 , respectively, are measured using two more calibrated photodiodes (PD₃ and PD₄) in conjunction with the DAQ.

We define the first-order diffraction efficiency of the hologram using the writing beams and the probe beam as, respectively

$$\eta := \frac{P_1}{P_i}, \quad (6.13)$$

$$\eta_p := \frac{P^O}{P^R + P^O}. \quad (6.14)$$

TEMPORAL RESPONSE OF EXPOSING PHOTOPOLYMER HOLOGRAMS

Our first experiment measures the optical power of the reference and object beams as they write a hologram into a photopolymer. The two beams are orthogonal to one another, and they produce a hologram with parameters: $\phi = 90^\circ$, $\theta_0 = \theta_2 = 45^\circ$. The writing beams have optical powers $P_i^R = P_i^S = 0.5 \text{ mW}$. We measure the optical powers over the course of a 70-minute exposure. We plot the temporally measured optical power of the beams in Figure 6.6.

It is clear from this figure that during the exposure, the transmitted power of the reference and object beams oscillate, and the period of oscillation changes throughout. When the medium is exposed to light, polymerization occurs in regions of high intensity, and this interference manifests in a changing index contrast. At the beginning of the exposure, the concentration of monomers is still high, so the refractive index contrast changes quickly; however, as the material continues to be exposed, the population of monomers and, subsequently, the rate of polymerization decrease. This causes the rate of

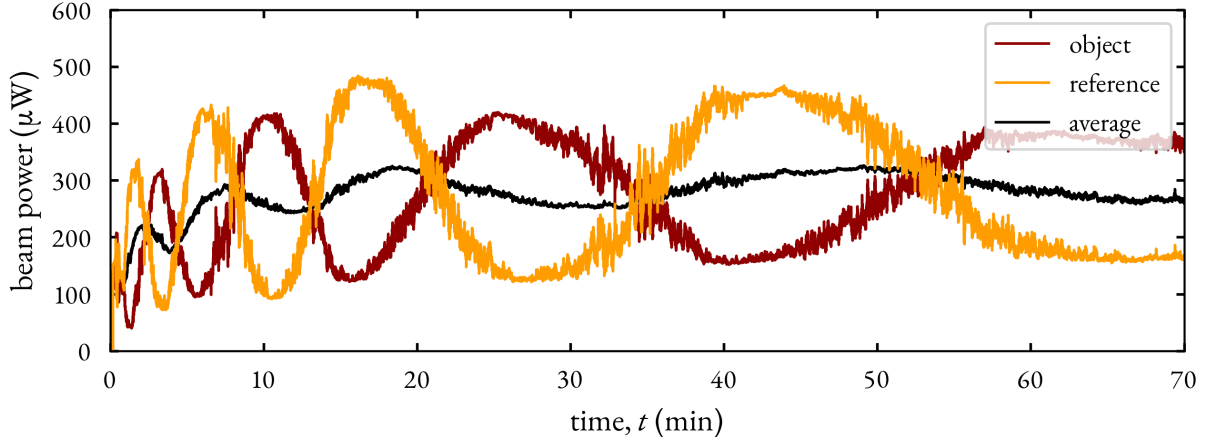


Figure 6.6: Measured optical power of the reference and object beams as they write a hologram into a photopolymer. The powers of the transmitted reference and object beams (P^R and P^O) are plotted as a function of time, as well as the average transmitted power. The reason that the average transmitted power is not constant is that there are also reflected diffraction orders. As the hologram exposes, the efficiency of the reflected orders reduces, and the transmitted beam power increases. We do not present data for the reflected orders as they could not be measured accurately with our experimental scheme.

change of the index contrast to decrease. Eventually all monomers would polymerize, and the hologram would become permanent. By monitoring the rate at which the diffraction efficiency oscillates, it should be possible to estimate the index contrast while the hologram is exposed; however, there are better methods.

PROBED TEMPORAL RESPONSE OF AN EXPOSING PHOTOPOLYMER HOLOGRAM

First, we run the same experiment four more times; however, this time we create four holograms using three different exposure powers (peak intensities): 1 mW (a), 1 mW (b), 5 mW, and 15 mW (0.11 W/cm^2 , 0.27 W/cm^2 , 1.7 W/cm^2)². We measure the diffraction efficiency by way of the probe

²Each power measurement has 5% uncertainty.

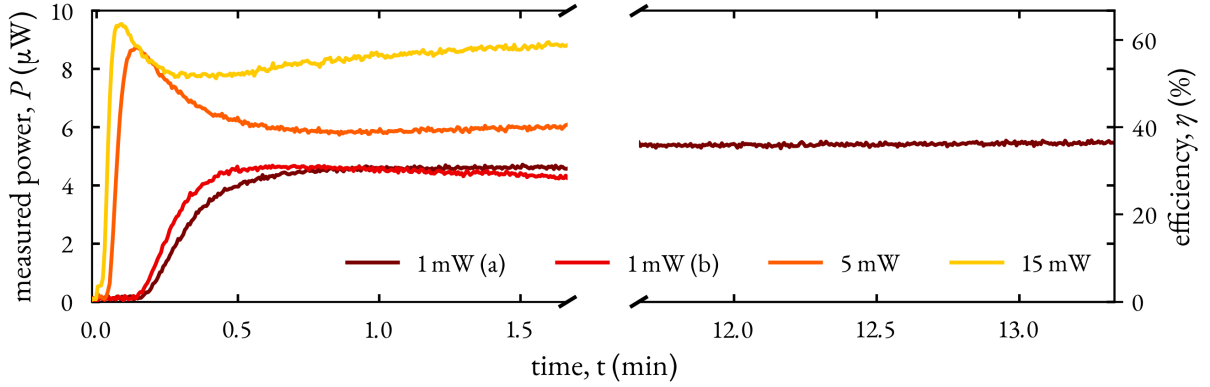


Figure 6.7: Measured optical power and diffraction efficiency of a $15\ \mu\text{W}$ probe beam as a photopolymer hologram is exposed to several different writing beam intensities. The right axis gives the calculated diffraction efficiency of each hologram.

laser, calculating the efficiency using Equation (6.14). Three of the trials are measured over the course of 100 s, and one is measured for 800 s. These measurements are plotted in Figure 6.7.

This figure highlights several important things: first, the more intense the writing beams, the more quickly the hologram is exposed, and the greater the resulting index contrast. Secondly, as demonstrated by the long-exposed 1 mW (b) beam, the index contrast plateaus but continues to increase with exposure - this makes sense when we consider how the photopolymer works. A heuristic explanation of the power curves shown in Figure 6.7 is given in Appendix E.2

Another notable thing about Figure 6.7 is seen upon the comparison of the two 1 mW exposures: they were written one after the other at different positions on the same film, yet they display different behaviours. The diffraction efficiencies rise more rapidly in one compared to the other, and they achieve different steady states (35% and 37%). Therefore, the fidelity of the hologram greatly depends on the specific samples. We recorded over fifty photopolymer holograms during the course of this thesis, and

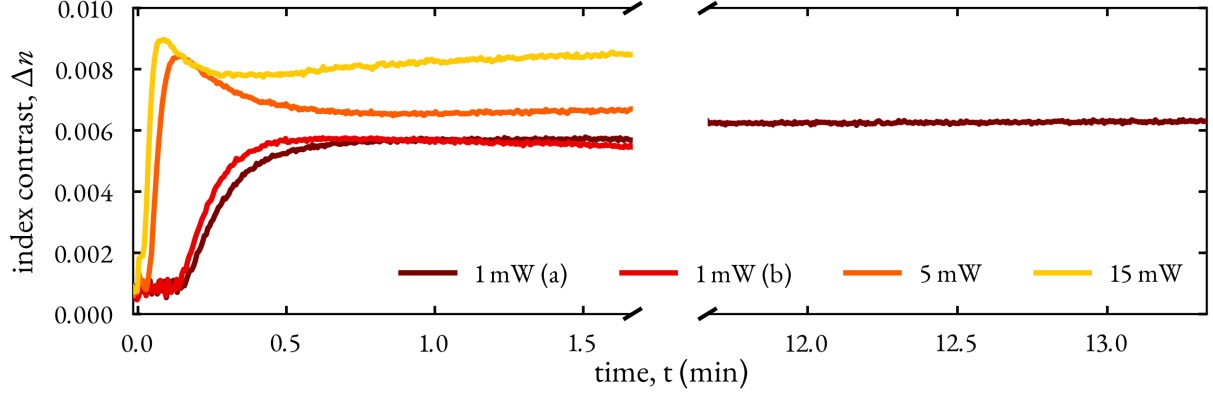


Figure 6.8: Calculated index contrast of an exposing photopolymer hologram. We achieve a maximum index contrast of $\Delta n = 0.0090$.

each responded slightly differently.

Finally, given the diffraction efficiency, we can calculate the change in the diffractive index via Equation (5.15). In Figure 6.8, we plot the index contrast for the holograms introduced in Figure 6.7. The resultant index contrast resembles the diffraction efficiency, and we achieve a maximal index contrast of $\Delta n = 0.0090$. This measured value is probably less than the true index contrast, as we were probably slightly off-Bragg. This value is higher than that reported at 639 nm for a similar photopolymer medium by Wu et al. [10]. In this section, we demonstrated a method to measure the index contrast of a hologram during the writing process.

ANGULAR EFFICIENCY OF PHOTOPOLYMER HOLOGRAMS

Next, we characterized the angular response of different holograms. The following pages present measurements of three different holograms; the parameters of each are similar to those presented in Figure 5.8. All three holograms can be parametrized as follows: $n_0 = 1$, $n_1 = n_2 = 1.5$, $d = 16 \text{ um}$, and $\lambda =$

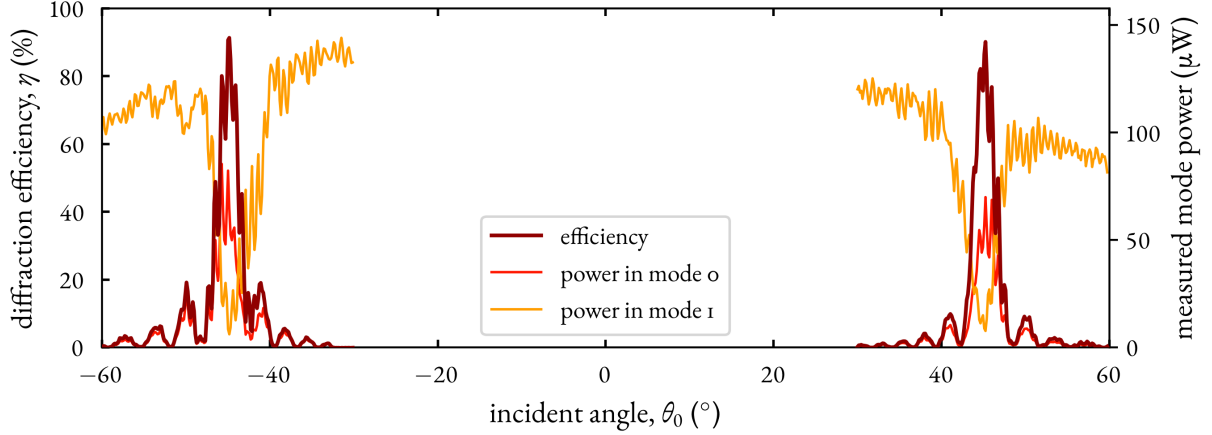


Figure 6.9: Measured angular response of a photopolymer hologram. Relevant physical parameters of the grating are as follows: $n_0 = 1$, $n_1 = n_2 = 1.5$, $d = 16 \mu\text{m}$, $\lambda = 633 \text{ nm}$, $\phi = 90^\circ$, and Bragg angle $\theta = 45^\circ$.

633 nm. The first, second, and third holograms are written such that $\phi = \{90^\circ, 90^\circ, \text{ and } 60^\circ\}$ and the Bragg angle $\theta = \{45^\circ, 18^\circ, \text{ and } 30^\circ\}$, respectively. Each hologram is exposed for ten minutes using the 633 nm laser at 15 mW with a 7 mm $1/e^2$ diameter. The diffraction efficiency is measured using the same laser (no probe) using Equation (6.13). The incident angle is defined in Figure 5.2, and it is swept by controlling the motorized rotation stage atop which the hologram is mounted. We measure the power of the zeroth and first modes by moving the photodiodes along with the setup - this is necessary as the diffraction angles change as a function of the incident angle. The measured powers and resulting diffraction efficiency of each mode are plotted in Figure 6.9.

We measure a maximum diffraction efficiency of $\eta_0 = 91\%$ and calculate an index modulation of $\Delta n = 0.014$.³ Further, we measured an angular bandwidth of $\delta_0 = (2.5 \pm 0.5)^\circ$. This grating was in the hologram with the highest spatial frequency, $K = 14 \text{ rad}/\mu\text{m}$, corresponding to a pitch of less

³The diffraction efficiency gives the efficiency of only the transmitted modes. We do not consider the absorbed or reflected light in these numbers.

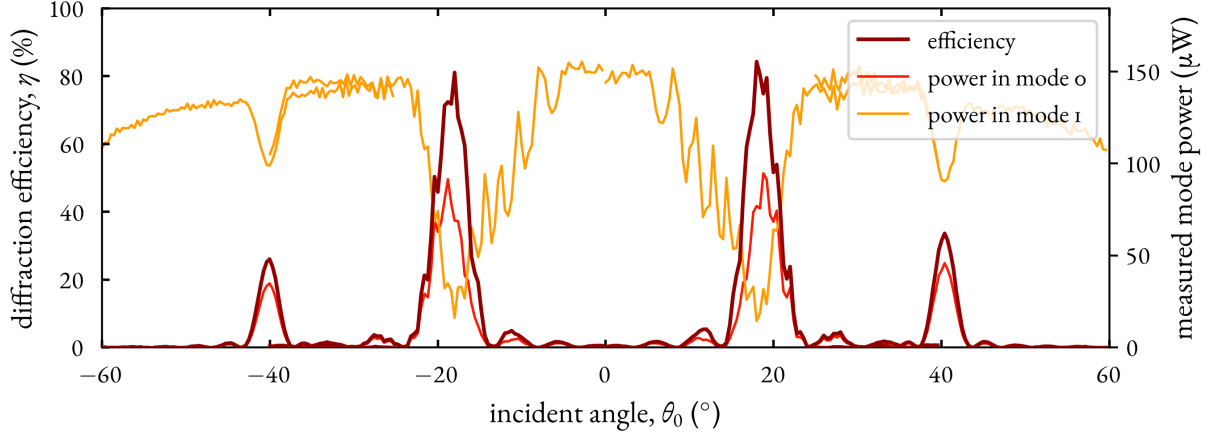


Figure 6.10: Measured angular response of a photopolymer hologram. Relevant physical parameters of the grating are as follows: $n_0 = 1$, $n_1 = n_2 = 1.5$, $d = 16 \text{ } \mu\text{m}$, $\lambda = 633 \text{ nm}$, $\phi = 90^\circ$, and Bragg angle $\theta = 18^\circ$.

than $\Lambda = 0.45 \text{ } \mu\text{m}$. The measured angular response of the hologram is extraordinarily similar to that predicted by the theory in Figure 5.7a), and we did not observe any significant higher order modes. This is because the geometry of this hologram is within the low-order regime (an in-plane grating vector with high spatial grating frequency). One will also notice that at high incident angles, the power measured in both transmitted modes decreases. This is because of Fresnel reflection at the air-hologram interface. Rigorous CWT of Section 5.3 predicts this power loss, too.

In the second hologram (Figure 6.10), we measure a maximum diffraction efficiency of $\eta_0 = 84\%$, and we calculate an index contrast of $\Delta n = 0.014$. Further, we measured an angular bandwidth of $\delta_0 = (5.4 \pm 0.5)^\circ$. In contrast to the first hologram, this one displayed non-negligible coupling into higher orders; we measure the two higher order peaks to be 34% and 26% efficient at their maxima.

Finally, in the third hologram (Figure 6.11), we measure a maximum efficiency of $\eta_0 = 91\%$, an index contrast of $\Delta n = 0.008$. In this case, the angular bandwidth is difficult to measure, but we can be sure

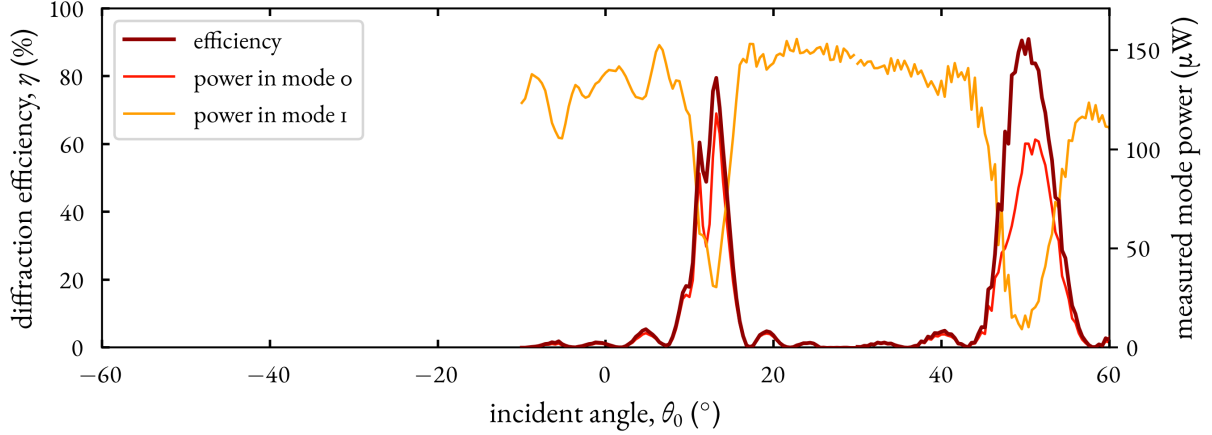


Figure 6.11: Measured angular response of a photopolymer hologram. Relevant physical parameters of the grating are as follows: $n_0 = 1$, $n_1 = n_2 = 1.5$, $d = 16 \text{ } \mu\text{m}$, $\lambda = 633 \text{ nm}$, $\phi = -60^\circ$, and Bragg angle $\theta = 18^\circ$.

that $\delta_0 > 5^\circ$. As predicted, we do not see much higher-order behaviour, but the angular bandwidth of the hologram is very large. We

All measured holograms demonstrated large angular bandwidths δ_0 . We concluded in earlier sections that to reduce the bandwidth of a written grating, we must reduce the ratio $\Delta n/d$; however, we cannot do that in this case without reducing the maximum efficiency as well. To maintain near-unity Bragg efficiency, we need to satisfy Equation (5.17), but we cannot do so for these photopolymer gratings. For example, in the first case where the Bragg angle is $\theta = 45^\circ$, to achieve unity diffraction efficiency, we would need an index contrast of 0.017. However, we only achieved an index contrast of 0.014. Therefore, any reduction in index contrast would cause reductions in diffraction efficiency. To create holograms with high diffraction efficiencies and low angular bandwidths, we need to use thicker gratings.

The investigation in this section has concluded several things. Firstly, there is heuristic agreement

between the measured diffraction efficiencies for all considered modes and the efficiencies predicted by rigorous CMT. We also observed that Kogelnik's first-order assumptions do not hold for many seemingly simple gratings. We have experimentally demonstrated that for first-order analysis to be accurate, we must use gratings with low spatial frequency that lay in the plane of the hologram. Moreover, we have created holograms that exhibit diffraction efficiency of at least 91% with index contrasts of at least $\Delta n = 0.014$. To the author's knowledge, higher index contrasts in these kinds of photopolymers have not been reported.

6.3 HOLOGRAPHY IN LITHIUM NIOBATE

Lithium niobate (LiNbO_3) is an extraordinary optical material that has many applications. It has many interesting optical characteristics, including birefringence, ferroelectricity, piezoelectricity, pyroelectricity, and electrooptical and nonlinear effects [97, 98]. In this thesis, we are specifically interested in two of these: nonlinearity and photorefractivity. We had the idea to combine the two phenomena in a single crystal by mode-shaping photon pairs created using the nonlinearity in lithium niobate. Essentially, we wanted to write a photorefractive hologram into lithium niobate to shape entangled photon pairs - all in a single optic. This has not been done before.

We leave a more in-depth discussion of the nonlinearities and parametric down-conversion to later chapters. This idea motivates the photorefractive measurements in lithium niobate in this section. Here, we introduce the concept of photorefraction in lithium niobate, explaining the relevant physics, the effects of different metal dopants, and existing methods of fixing gratings in lithium niobate. This precedes the characterization of photorefractivity in lithium niobate.

6.3.1 PHOTOREFRACTION IN LITHIUM NIOBATE

Photorefraction is the process by which illumination changes the refractive index of a material; in this way, the intensity-dependent polymerization and diffusion processes explored in Section 6.2 is photorefractive [10, 94, 98]. However, in lithium niobate, we speak of a more specific process known as the electrooptical photorefractive effect, also known as the Pockels effect. Henceforth, this is what we refer to by photorefraction.

The Pockels effect manifests in a change to the refractive index of a material when an electric field is applied. Inhomogeneous illumination of the material excites charge carriers (dopant dependent) into their conduction bands. These charge carriers can migrate from diffusion, drift, or bulk photovoltaic effects [97]. At steady state, a non-negligible spatially-varying charge density builds up, which spatially modulates the index of refraction [98]. The strength of this modulation is characterized by the electrooptical coefficients, which are formally described by a rank-three tensor. Fortunately, due to the molecular geometry of lithium niobate, we only need to consider the coefficients corresponding to ordinarily and extraordinarily polarized light. Respectively, these are r_{13} and r_{33} , given in Table 6.1 along

Material	n_o	n_e	r_{13} (pm/V)	r_{33} (pm/V)	Δn	Λ (μm)
LiNbO ₃	2.29[99]	2.20[99]	11 [100]	34 [100]	0.0007 [101]	-
	-	-	-	-	> 0.000006 *	< 0.16
CRT-20						
Photopolymer	-	-	-	-	0.0085 [10]	< 0.3
	-	-	-	-	0.014 *	< 0.45

Table 6.1: Electrooptic coefficients and refractive index for ordinarily and extraordinarily polarized light in lithium niobate [102], and maximal index contrast, Δn at 633 nm. The index contrast for the CRT-20 photopolymer is also presented for comparison. *this work.

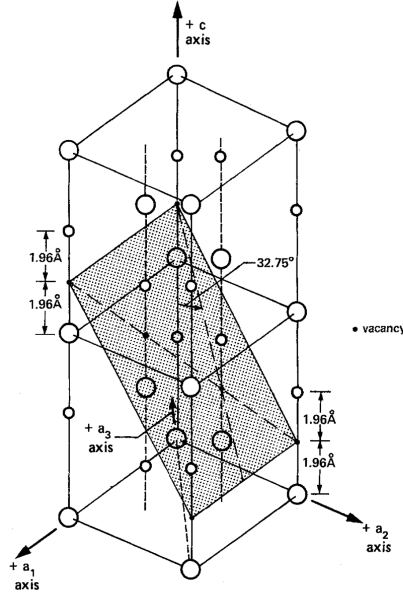


Figure 6.12: Conventional hexagonal unit cell of lithium niobate with hexagonal axes (a_1, a_2, a_3, c) . The large circles denote niobium atoms and the small circles denote lithium atoms. In cartesian coordinates, $\mathbf{z} \equiv \mathbf{c}$, $\mathbf{x} \equiv \mathbf{a}_1$, and $\mathbf{y}$ is chosen to be right handed. Image taken from [97].

with the index of refraction for both ordinarily and extraordinarily polarized light. The effective index contrast for light polarized along one of the optical axes of lithium niobate is given by

$$\Delta n_{o,e} = -\frac{1}{2} n_{o,e}^3 r_{13,33} E_{sc}$$

where $n_{o,e}$ is the index of refraction for ordinarily (extraordinarily) polarized light, and E_{sc} is the magnitude of the photorefractive charge density field created inside the crystal [103, 104]. The ordinary axis is in the xy -direction, and the extraordinary axis is given in the z -direction, as shown in Figure 6.12⁴.

⁴As an aside, we introduce some terminology to define the crystal orientation of a given sample: a crystal described as being i -cut, where $i = x, y$, or z , means that i crystal axis is perpendicular to the large face of the crystal, or equivalently, that the light propagates along the i -axis.

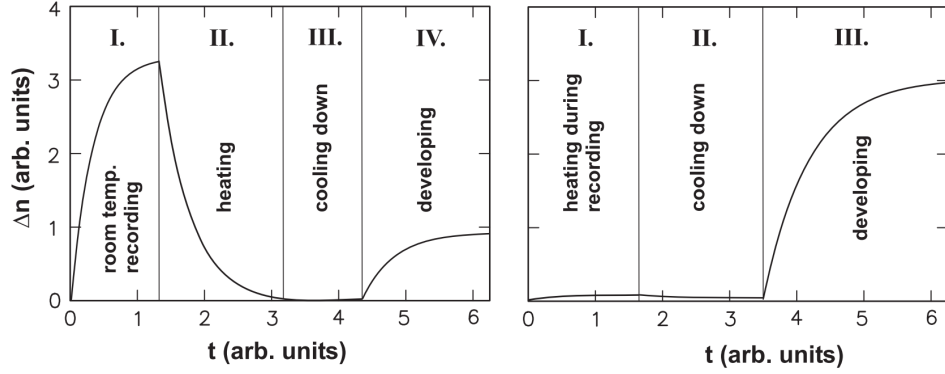


Figure 6.13: Two methods to thermally fix holograms in lithium niobate. See text for discussion. Figure taken from [98].

This charge density, and therefore the index contrast, can be enhanced by doping the lithium niobate with either iron or copper [105–108]. Specifically, the maximum index contrast increases linearly with Fe^{3+} or Cu^{2+} dopant concentration up to a certain point where the index contrast saturates; Weis et al. suggest that a dopant concentration of $20 \times 10^{18} \text{ cm}^{-3}$ produces the highest index contrast, given in Table 6.1 [97]. This index contrast corresponds to a maximal charge density field of approximately $E_{sc} = 4 \text{ kV/mm}$.

Finally, we note that lithium niobate is also highly transmissive for visible, near-infrared, and near-UV (up to 3.8 eV or 326 nm) light.

THERMAL FIXING OF GRATINGS IN LITHIUM NIOBATE

The photorefractivity discussed above is non-permanent; if the writing field is turned off, the inhomogeneous charge distribution will decay, and any attempts to read the hologram out would hasten the decay process [109]. This is not good news if you want to use lithium niobate as holographic data

storage [110, 111]. Fortunately, we can fix the grating by heating it while it is exposed. We briefly summarize two methods to do this as discovered by Amodei and Staebler [112, 113]. Figure 6.13 summarizes these two methods. The first method involved writing a hologram and, subsequently, heating it to $\sim 180^\circ\text{C}$ before cooling it down. Then, the hologram is homogeneously illuminated, and the grating is retrieved. The second method is similar, however, instead of heating the hologram after exposure, it is heated while exposing. Each method produces holograms that do not appreciably decay with time.

DESCRIPTION OF LITHIUM NIOBATE SAMPLES

Writing holograms in lithium niobate is not new; in fact, the methods for storing data in lithium niobate holograms have been studied since the mid-1970s; however, the samples used most often are in thick - on the order of centi- or decimeters. This is because of the low achievable index contrasts in lithium niobate (see 6.1). To achieve non-negligible diffraction efficiencies, it is typical to use thick samples of lithium niobate. Our sample is only 300 μm thick; therefore, its behaviour is not well-known. Several other parameters are unknown in our samples, and these will be characterized in the following section.

Our lithium niobate samples were procured from Xiamen Powerway Advanced Material Company, Ltd. They are x -cut (0.30 ± 0.3) mm thick, double-side polished, and diced into squares approximately 2 cm^2 in area. The samples are iron-doped at a reported concentration of 0.2 mol% = $6 \times 10^{18} \text{cm}^{-3}$.

6.3.2 CHARACTERIZATION OF PHOTOREFRACTIVITY IN LITHIUM NIOBATE

We performed several experiments in lithium niobate to measure the maximum possible saturation index contrast for the given samples and create permanent gratings in lithium niobate. These experiments

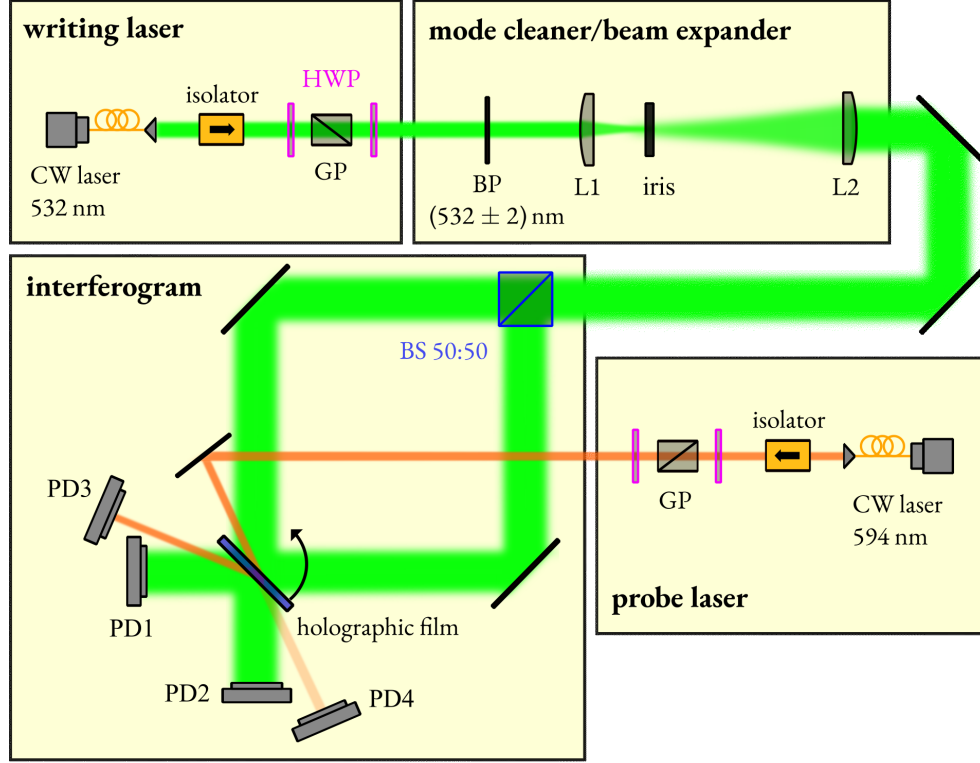


Figure 6.14: Experimental setup of the lithium niobate holography. Lenses L1 and L2 have focal lengths 30 cm and 100 cm, respectively.

include measuring the temporal and thermal responses of an exposing lithium niobate hologram for a set of laser intensities. As was done in Section 6.2, we can determine the diffraction efficiency and index contrast of the writing gratings. The data could also be used to characterize the rates of charge redistribution within the sample, but this is left as an extension to the project.

To write and characterize the holograms, we use the setup shown in Figure 6.14. A 3 W continuous wave 532 nm laser is polarized and power-regulated using a GP with a 1000:1 extinction ratio. The 3 mm $1/e^2$ diameter beam is expanded using a $3.3\times$ telescope. The spatial mode of the laser is cleaned using a

BP filter and a 75 μm pinhole at the Fourier plane of the telescope. As with the previous experiments, this intense beam is split into two, before interfering in the sample. The diffraction efficiency of the process was measured using a 594 nm probe laser, aligned at its Bragg angle. The beam diameter of the probe laser was measured to be 2.5 mm, and the probe power, P_0 , was varied from 1 to 20 mW. We initially tried using a 405 nm probe laser, but we were unable to observe its diffraction, likely due to the low diffraction efficiency at this wavelength. The optical power of the diffracted probe beam was measured using a Hamamatsu S5973 photodiode, and the resulting diffraction efficiency was calculated using Equation (6.14).

TEMPORAL RESPONSE OF EXPOSING LITHIUM NIOBATE HOLOGRAMS

We measured the diffracted power and, subsequently, the diffraction efficiency of a 1 mW probe beam for several different writing laser powers. The holograms are each exposed for 15 s. These measurements are presented in Figure 6.15 where we plot measured power and the diffraction efficiency as a function of time. For the most intense beam, we measure a maximal diffraction efficiency of 0.013%, corresponding to a grating pitch of $\Lambda = 0.16 \mu\text{m}$. In this plot, we can also study the effect of beam intensity on the rate of change of diffraction efficiency both at the beginning and end of the exposure. The more intense the beam, the quicker the index contrast appears, but also the quicker it decays. We measure grating decay times between 0.3 and 0.9 s. We exposed some of the holograms for longer - some for up to two hours - but there was no observed difference in the saturation index contrast or its decay rate. There are some interesting diffusion dynamics happening, but their study is beyond the scope of this work.

Using this data, we can also calculate the index contrast - this is presented in Figure 6.16. Here, we plot the calculated index contrast using the data from Figure 6.15 in conjunction with Equation (5.15).

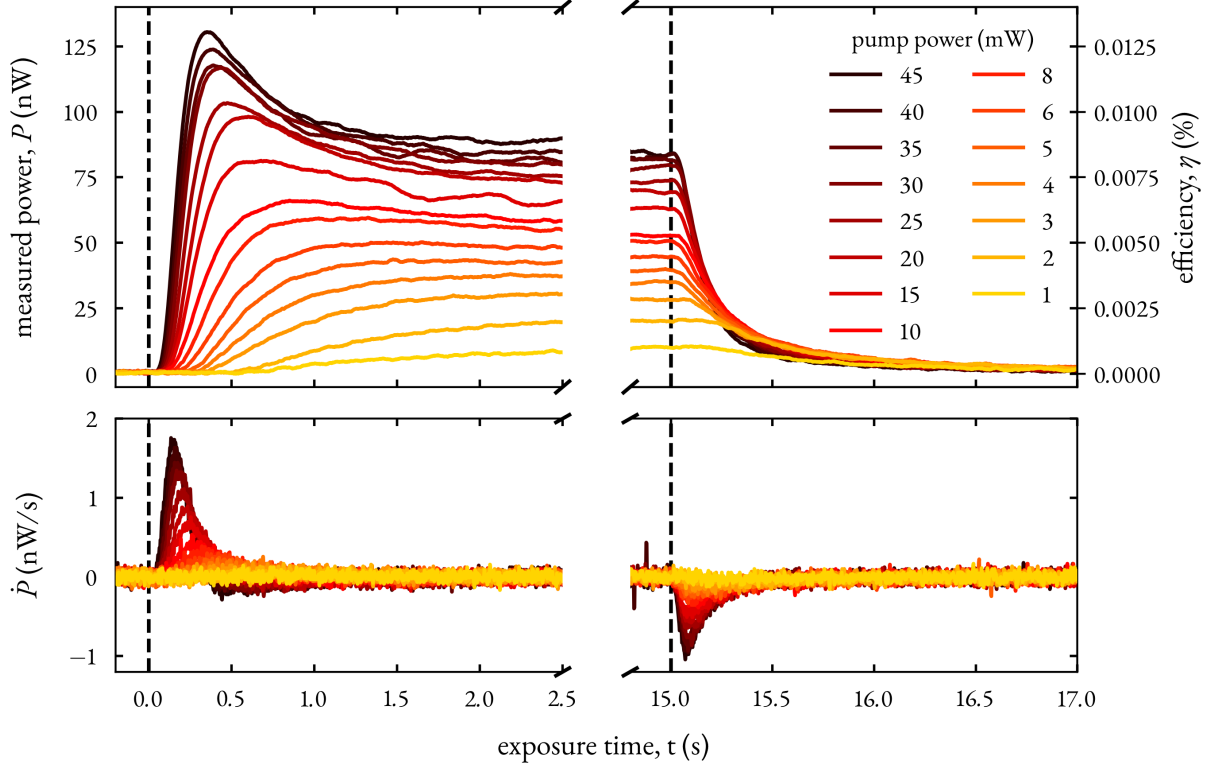


Figure 6.15: Measured optical power and diffraction efficiency of a 1 mW probe beam as a Fe:LiNbO₃ sample is exposed to several different writing beam intensities. The right axis gives the calculated diffraction efficiency of each hologram. The bottom plot presents the rate of change of power over time for each writing intensity.

We average the saturation diffraction efficiency (in this case, from seven to fifteen seconds) and use this value to calculate the index contrast. We plot these values as a function of peak intensity.

We observe a maximal saturation index contrast of 5.6×10^{-6} , and that the index contrast starts to plateau with writing intensity. The observed plateau suggests that our samples have achieved charge fields nearing the maximum allowed given in dopant concentrations. This is peculiar, considering that our measured index contrast is 125 times smaller than the index contrast presented in other works [101].

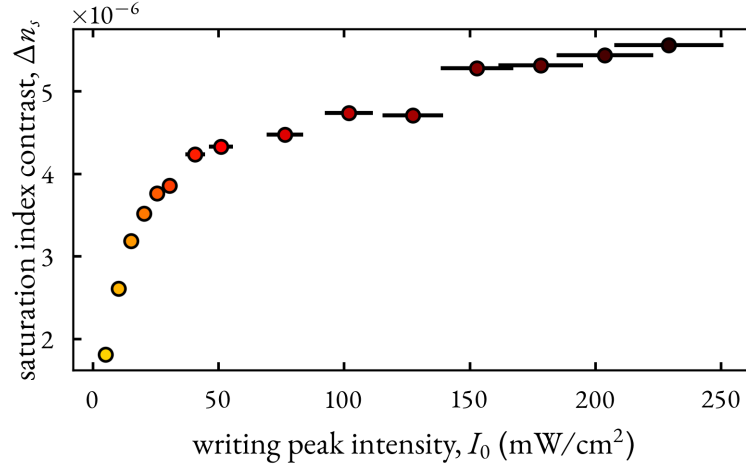


Figure 6.16: The saturation index contrast as a function of the peak intensity of the writing interference of a 532 nm beam. The error bars denote the measurement uncertainty in peak intensity considering the uncertainty in the measured power and beam width. The vertical error bars give the standard error in the averaged saturation index contrast.

We would expect the index contrast to be lower due to the comparatively low Fe dopant concentration (6 mol% compared to 20 mol%), but this would only account for a three-fold reduction. There are several potential culprits which could have resulted in such a reduction in contrast. Firstly, the spatial and temporal coherence of the 532 nm laser could have reduced the interference visibility and, thus, the index contrast; however, we made several photopolymer gratings using the 532 nm setup and achieved diffraction efficiencies of above 60%, so it likely did not significantly contribute. Secondly, the reported dopant concentration could be inaccurate, or the crystal could have high non-stoichiometric defect concentrations; Amodei et al. have suggested that photorefraction is enhanced by reducing the non-stoichiometric defects [112]. This possibility is buttressed by the observation that the holograms decay quickly and without intense homogeneous illumination.

The probe could be partially erasing the grating, but our findings demonstrate that this cannot sufficiently explain the high rate of decay. We wrote several holograms without using a probe, only probing the sample a short time after the writing beam had been turned off, and we observed no diffraction. We also varied the probe intensity during and before an exposure - these are plotted in Appendix E.2. From these plots, we conclude that the probe cannot be entirely responsible for the grating decay.

EXPERIMENTS IN THERMAL FIXING IN LITHIUM NIOBATE

The other goal of this chapter was to thermally fix a photorefractive grating in Fe:LiNbO₃. Two methods are described in Section 6.3.1, but since the rate of decay of these holograms is so fast, the first method shown in Figure 6.13 is not applicable. Therefore, we heated several holograms to $\sim 180^\circ\text{C}$ during their exposures at 225 mW/cm^2 for varying amounts of time (1 - 30 minutes), following the methods of Amodei and Staebler [112, 113]. They were heated using two different apparatus: a Steinel HL1820 S heat gun, as well as a Weller WR 2 hot air station. After heating and exposure, the hologram was allowed to fully cool for 30 min; it was both probed and unprobed, and exposed and unexposed during this period (see Appendix E.2).

This method, however, did not work, regardless of the state of the exposure, the probe, or the length of heating or cooling. The grating would always decay rapidly, and the film would behave as it had before exposure. Upon reexposure, the resultant diffraction efficiency would return to its earlier maxima, demonstrating that any charge patterning had dissipated.

To test this further, we measured the diffraction efficiency as a function of film temperature. We heated the film using the hot air station and varied the temperature between 295 and 473 K in a continuous exposure. The film was allowed to reach steady state for each temperature before increasing

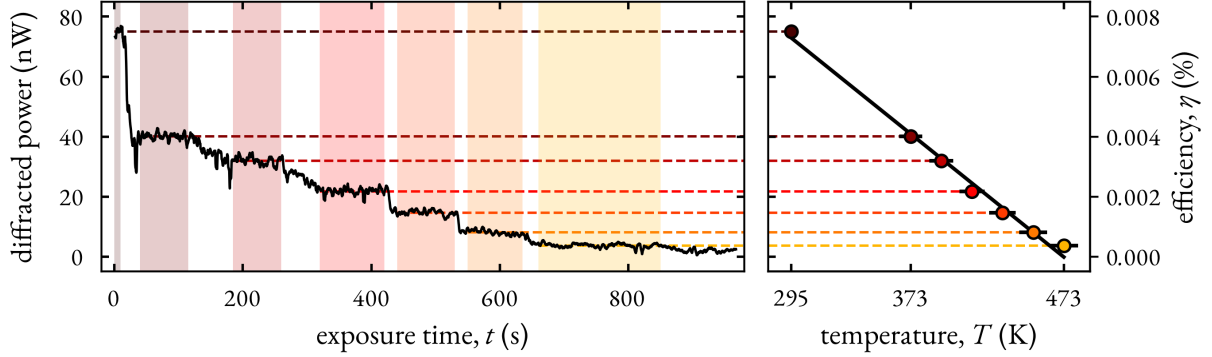


Figure 6.17: Diffraction efficiency of a lithium niobate wafer as a function of temperature. At **left**, the film is continuously exposed using at an intensity of 225 mW/cm^2 and a 1 mW probe. The steady state diffraction efficiencies at each temperature are averaged and plotted at the **right**. The error bars present the uncertainty in film temperature and the standard error in diffraction efficiency.

the temperature. We plot this exposure and the resulting diffracted power as a function of temperature in Figure 6.17. In the right plot, we see that in the tested thermal range, the diffraction efficiency varies linearly with temperature. Beyond the tested temperature range, the diffraction efficiency starts to plateau, as we were able to observe diffraction at temperatures up to 500 K ⁵.

This anti-correlation of temperature and diffraction efficiency is not caused by the thermal decay of the charge field, as thermal excitation of the electrons does not occur below temperatures of 473 K [114]. Instead, it is caused by thermally induced, spatially varying proton and metal ion concentration [115, 116]. In the cited works, they expose 1 mm -thick, copper-doped lithium niobate at 25 W/cm^2 intensities for up to 47 hours to achieve a measurable redistribution in the concentration of ions in the lithium niobate. These are higher exposure times and intensities than in other cited works, but perhaps they are necessary to achieve fixing in gratings of our thicknesses. As well, our method of heating (via

⁵the diffracted power was simply too weak to be accurately measured by our detector.

airflow) could have caused vibrations and instabilities, which could have reduced the interference visibility. However, the linear relationship between temperature and diffraction efficiency suggests that this is not the major cause of our failure to fix a grating in lithium niobate.

PART III

SHAPING SPONTANEOUS PARAMETRIC

DOWN-CONVERSION IN THIN FILMS

7

TOWARDS HOLOGRAPHIC SPONTANEOUS PARAMETRIC DOWN-CONVERSION

This chapter describes work we have done towards combining a linear hologram¹ and a biphoton source into a single optic. This idea was inspired by the holographic work presented earlier in this thesis (Chap-

¹As opposed to nonlinear holograms, which spatially modulate the nonlinear susceptibility. The mode shaping of down-conversion and harmonic generation has been achieved by writing nonlinear holograms into lithium niobate using femtosecond laser pulses [58, 117–119].

ter 6) in concert with the works of Chekhova et al. They use thin (nanometer- to micrometer-thick) crystals to produce biphotons, which are much more broadband in frequency and emission angle than typical SPDC sources [25, 120, 121].

These sources are novel in several aspects. They produce biphotons with a high degree of entanglement in transverse momentum (emission angle) and frequency (energy), and they can be more easily integrated into other optics compared to their bulk counterparts. Their thinness relaxes many phase-matching constraints, allowing for the use of more exotic materials with stronger nonlinearities, and they can be coupled to metasurfaces and other resonant optics to enhance certain emission bands and directions [121–124]

A common material used in this thin SPDC regime is x -cut lithium niobate. Recall from Chapter 6 that lithium niobate also exhibits photorefractivity, which can be used to write holographic gratings into lithium niobate and can be used as a phase-matching mechanism in lithium niobate. Alternatively, it can be thought to diffract the photons once they have been produced in the crystal. We will demonstrate that this can be used to switch or control the emission angle of the entangled photons. This could also be used to shape the mode of the pump, affecting its spatial or angular distributions, as well as its orbital angular momentum (OAM), all of which control the entanglement of the resultant photons.

In this chapter, we provide a brief introduction to SPDC, non-phase-matched SPDC, and we describe work done towards this holographically modulated biphoton source.

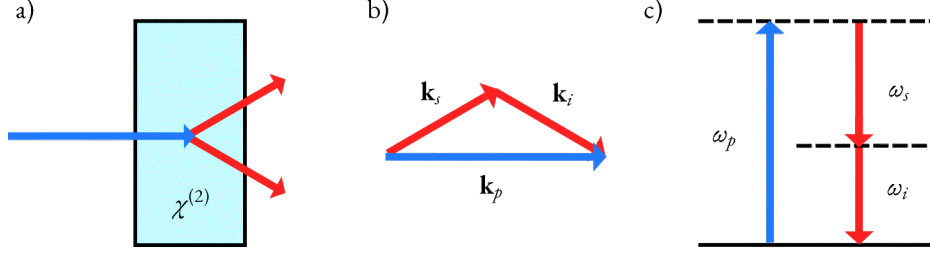


Figure 7.1: Spontaneous parametric down-conversion. a) A pump photon converts into two daughter photons obeying momentum [Equation (7.2)] (b) and energy conservation [Equation (7.1)] (c).

7.1 SPONTANEOUS PARAMETRIC DOWN-CONVERSION

SPDC is the process of converting a high-energy pump photon of angular frequency ω_p into two lower energy daughter photons termed the signal and idler photons with angular frequencies ω_s and ω_i , respectively (Figure 7.1). By convention, the higher energy daughter photon is the signal and the other is the idler. The process occurs in materials with non-zero second-order nonlinearities ($\chi^{(2)} \neq 0$). It is a quantum mechanical phenomenon seeded by vacuum fluctuations [30]. The process must obey certain laws of conservation, including conservation of energy, conservation of momentum, and conservation of OAM [125–129]. The total energy of the daughter photons must equal that of the pump photon, and the transverse and linear momenta of the daughter photons must equal those of the pump:

$$\text{Energy conservation: } \hbar\omega_p = \hbar\omega_s + \hbar\omega_i; \quad (7.1)$$

$$\text{Momentum conservation: } \mathbf{k}^{(p)} = \mathbf{k}^{(s)} + \mathbf{k}^{(i)}. \quad (7.2)$$

The two daughter photons form a quantum state known as a two-photon state or a biphoton state. Because they are generated during a single event, they can be described by a single, non-separable wave-

7.1. SPONTANEOUS PARAMETRIC DOWN-CONVERSION

material	$\chi_{il}^{(2)}$ (pm/V)
LiNbO ₃	$\chi_{32}^{(2)} = 60$ $\chi_{31}^{(2)} = -11.8$
GaAs	$\chi_{36}^{(2)} = 740$
beta-BaB ₂ O ₄ (BBO)	$\chi_{22}^{(2)} = 4.4$

Table 7.1: Second-order susceptibilities for several different materials. Lithium niobate is used extensively due to its attractive qualities as an optical material, including its large susceptibility. GaAs is a semiconductor with a huge susceptibility - we have characterized a sample of GaAs (III), which is presented in Appendix H. BBO is a bulk material used frequently to produce biphotons. We present the susceptibilities in their contracted indexed form, taken from [132, 133].

function and are therefore entangled. Biphotons produced by SPDC can be entangled in several different bases including energy, momentum, polarization, and OAM [128–130], and have been used to violate Bell’s inequalities [131].

7.1.1 NONLINEAR SUSCEPTIBILITY

We describe the interaction of light fields with charge carriers in materials using the material field-dependent polarization:

$$\mathbf{P}(\mathbf{E}) = \epsilon_0 [\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \dots],$$

where $\mathbf{E}$ is the optical field, and $\chi^{(n)}$ is the n -th order susceptibility. In earlier chapters, we assumed that only the linear susceptibility $\chi^{(1)}$ was non-zero, producing a linear material response to an optical field ($\mathbf{P}_L \propto \mathbf{E}$). SPDC is a second-order nonlinear phenomenon, which means that it is facilitated

7.1. SPONTANEOUS PARAMETRIC DOWN-CONVERSION

by the second-order nonlinear coefficient $\chi^{(2)}$. Materials only exhibit a non-zero $\chi^{(2)}$ if they have non-centrosymmetric crystal geometries [132]. In general, this means that isotropic materials do not exhibit SPDC. Further, the susceptibility is enhanced in materials with large densities of weakly bound valence electrons [132]. Intuitively, these conditions lead to greater strength of interaction. Materials with these properties, however, typically have higher absorptivities, making nonlinear crystal selection a game of trade-offs.

The material susceptibility depends on the polarizations of the incident and outgoing fields, and the orientation of the crystal structure. It can therefore be described by a rank-three tensor, and the second-order susceptibility matrix can be reduced to a 3×6 matrix. The form of this matrix depends on the nonlinear material. Lithium niobate exhibits second-order nonlinearity and has two significant second-order susceptibilities depending on the direction of pump propagation and polarization. The lower of the two is relevant for z -cut crystals, and the higher of the two is relevant for x -cut crystals using a z -polarized pump and produces mostly Type o SPDC ².

7.1.2 POLARIZATION

The polarization of the daughter photons depends on the susceptibility tensor and the polarization of the pump field. If the pump field is linearly polarized along the ordinary (o) or extraordinary (e) crystal axes, the polarization of the signal and idler photons are typically characterized into three regularly occurring types (pump polarization $\rightarrow$ signal/polarization):

²The higher susceptibility of x -cut lithium niobate crystals is not used frequently in bulk crystals because it is difficult to phase-match. Instead, z -cut crystals are used in periodically-poled geometries, which are used to quasi-phase-match SPDC.

7.1. SPONTANEOUS PARAMETRIC DOWN-CONVERSION

- Type 0: $o \rightarrow oo$ or $e \rightarrow ee$;
- Type I: $o \rightarrow ee$ or $e \rightarrow oo$;
- Type II: $o \rightarrow eo$ or $e \rightarrow oe$.

Given susceptibility tensor and the polarization of the pump, phase matching determines the type of SPDC which will be most efficient.

7.1.3 PHASE-MATCHING

Phase matching in SPDC comes from conservation of momentum. Explicitly, phase matching requires that the phase of the input fields must maintain a phase relationship (stay in-phase) with the output fields. In SPDC, this can be interpreted as follows: a pump field coherently generates two photons in a nonlinear crystal. If those two daughter photons do not stay in phase with the pump field as they propagate, subsequent down conversions will produce daughter photons out of phase with the first ones, and they will destructively interfere, reducing the field strength of output fields.

This dephasing is caused by wavelength and polarization dispersion in crystals. Because of the vast frequency differences between the daughter and pump photons, they see different refractive indices, and, therefore, they dephase as they propagate. For some materials and SPDC schemes, this dephasing can be corrected by tuning the temperature or relative angle of the crystal, but these phase-matching techniques can only be used to phase match for certain types of SPDC in some birefringent crystals. These techniques can be used to enhance pump conversion to certain wavelengths³.

³This is often used in bulk BBO sources and is used in the characterization of second harmonic generation in GaAs and Fe:LiNbO₃ in Appendix H.

In this work, we can discard phase matching because we are working in the thin-crystal regime. Equation (7.2) can be mathematically simplified by defining the longitudinal and transverse mismatch independently.

$$\Delta k_{\parallel} := k_{\parallel}^{(p)} - k_{\parallel}^{(s)} - k_{\parallel}^{(i)}; \quad (7.3)$$

$$\Delta k_{\perp} := -k_{\perp}^{(p)} + k_{\perp}^{(s)} + k_{\perp}^{(i)}. \quad (7.4)$$

In this work, we define the longitudinal direction to be orthogonal to the plane of the crystal and equal to the pump (z) direction. The transverse direction is in the plane of the crystal (x). We are in the *non-phase-matched regime* when $\Delta k_{\parallel} \neq 0$. The coherence length of the crystal is then

$$L_{\text{coh}} := \frac{\pi}{\Delta k_{\parallel}}. \quad (7.5)$$

Specifically, to calculate the probability of a biphoton emission, we need the phase-matching function

$$F_{\text{pm}}(\Delta k_{\parallel}) = \text{sinc}^2\left(\frac{\Delta k_{\parallel} L}{2}\right), \quad (7.6)$$

and the pump function, for pump waist radius w ,

$$F_{\text{p}}(\Delta k_{\perp}) = e^{-(\Delta k_{\perp} w)^2}, \quad (7.7)$$

which gives the angular correlation width between the signal and idler photons [120, 134]. The proba-

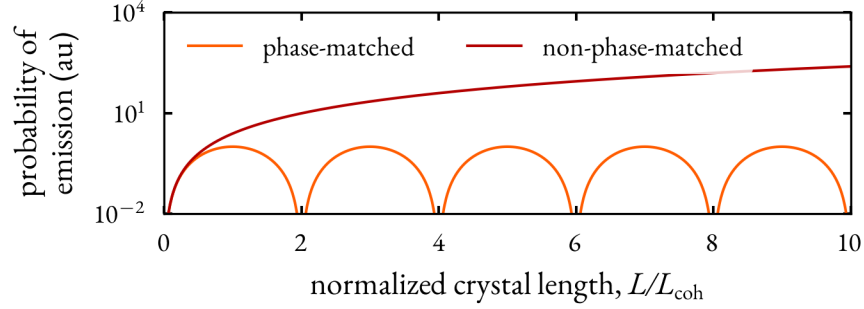


Figure 7.2: The probability of biphoton emission as a function of coherence length-normalized crystal (interaction) length. Comparing the phase matched ($\Delta k_{\parallel} = 0$) to the non-phase-matched case ($\Delta k_{\parallel} \neq 0$).

bility of a biphoton emission is then a function of crystal length L , and Equations 7.6 and 7.7:

$$P \propto [\chi^{(2)}]^2 L^2 F_{pm}(\Delta k_{\parallel}) F_p(\Delta k_{\perp}). \quad (7.8)$$

We plot the emission probability in Figure 7.2 for two different crystals. In the case of non-phase matching, for odd, positive integer multiples of the coherence length, we get constructive interference, and for even, positive integer multiples, we get destructive interference. As long as our crystal is some odd multiple of this coherence length, we will observe down-conversion. Conversely, for a phase-matched crystal, the probability of pair emission increases quadratically with crystal length. This is one downside of non-phase matched SPDC - though we can select materials with large $\chi^{(2)}$, the emission rate is limited by the length of the crystal.

FREQUENCY-ANGULAR SPECTRUM OF THIN LITHIUM NIOBATE

Using Equation (7.8), we can calculate the relative probability that a signal photon at wavelength λ_s ,

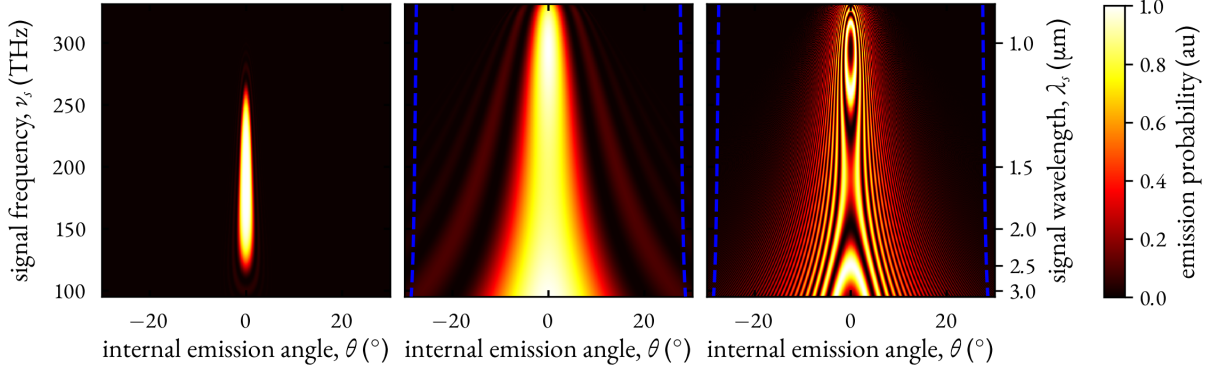


Figure 7.3: Frequency-angular emission probability [Equation (7.8)] as a function of wavelength and emission angle. We calculate the emission probability for a 1 mm-thick BBO crystal, angle-tuned for collinear degenerate SPDC (left). We also calculate the spectrum for two non-phase-matched type o lithium niobate crystal with $L = 10 \mu\text{m} \approx L_{\text{coh}}$ (middle) and $L = 300 \mu\text{m} \approx 30L_{\text{coh}}$ (right). The crystals are pumped at $\lambda_p = 788 \text{ nm}$ with a $100 \mu\text{m}$ waist. The vertical blue, dashed lines denote the angle for total internal reflection inside the crystal. Biphoton production outside those lines will not be emitted outside the crystal.

will be emitted at a given angle. We calculate this probability for a type o process in two different x -cut lithium niobate crystals, pumped using a 788 nm z -polarized pump which is focused to a $100 \mu\text{m}$ beam waist. The coherence length for the collinear, degenerate process ($\lambda_s = \lambda_i$) is approximately $9.8 \mu\text{m}$. The two crystals have thicknesses $L \approx L_{\text{coh}}$ and $L \approx 30L_{\text{coh}}$. The resulting probability as a function of emission angle and signal wavelength is given in Figure 7.3. We see that when L becomes large, the phase matching disallows certain emission frequencies.

In the figure, we see that the probability is relatively across the entire considered signal (or idler) wavelength spectrum, and is very wide in emission angle. This can be intuitively thought of in terms of interaction length. Because the crystal is thin, the longitudinal length of interaction is small. By Heisenberg's uncertainty, this means that the longitudinal wavevector mismatch [Equation (7.3)] is large. A similar intuition can be made for angular bandwidth: if the pump waist is small, the transverse

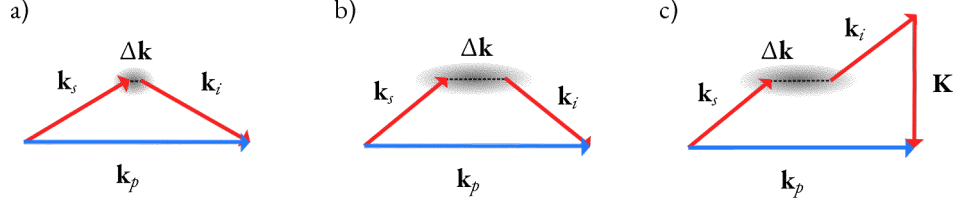


Figure 7.4: Comparison of phase-matching (a), non-phase-matching (b), and holographic non-phase-matching (c) in SPDC.

interaction length is small, so the transverse wavevector mismatch [Equation (7.4)] is large.

Further, the range of possible emission angles are determined by the longitudinal mismatch $\Delta k_{\parallel}$, but the correlation between the daughter photons is given by the transverse mismatch $\Delta k_{\perp}$. The correlations will be tight for a large beam waist. In terms of frequency entanglement, if the pump frequency bandwidth is narrow, so too will the signal-idler frequency correlations. The frequency and angular entanglement in non-phase-matched SPDC can be extremely large compared to phase-matched SPDC (see Figure 7.4).

7.1.4 PHASE MATCHING BY HOLOGRAPHY

If we write an in-plane holographic grating with wavevector $\mathbf{K}$ into a lithium niobate crystal (as was done in Section 6.3, we are essentially adding a diffraction efficiency-dependent term to the transverse phase-matching condition. Equation (7.4) therefore becomes

$$\Delta k_{\perp} := k_{\perp}^{(p)} - k_{\perp}^{(s)} - k_{\perp}^{(i)} + K(t, \mathbf{k}_s, \mathbf{k}_i), \quad (7.9)$$

where we explicitly right the grating vector's dependence on time, and signal and idler angles. Recall that to diffract, the signal or idler photon must approximately satisfy the Bragg condition. When the condition is met by either the signal or the idler, it would diffract from $\theta_{s,i}$ to $-\theta_{s,i}$. And, because, as we saw in Section 6.3, gratings in thin lithium niobate are relatively short-lived. The hologram could therefore be used as a time-dependent switch to modulate angular correlation between the signal and idler photons. The form of this hologram could change, depending on whether we want to shape the angular correlation of the down-conversion, or even the shape of the pump. There is great room for creativity and innovation.

7.2 PROPOSED EXPERIMENT TO MEASURE HOLOGRAPHIC SORTING OF SPDC

We propose an experiment to holographically modulate the frequency angular spectrum of SPDC, and we have made significant progress toward realizing this experiment. An outline of the proposed experiment is shown in Figure 7.5. A linearly polarized and power-regulated laser is used to write an in-plane holographic grating in a sample of thin, iron-doped, x -cut sample of lithium niobate. A blue laser is then focused into the crystal to pump the non-phase-matched SPDC process. Filters are used to remove the pump and writing beams. Angular correlations between signal and idler photons are measured by converting them into spatial correlations using lens L_3 . These spatial correlations are then measured on a SPAD. We expect that, in a given frequency band, with the hologram turned off, the idler and signal will be symmetrically anti-correlated in the spatial domain. With it on, however, we expect

7.2. PROPOSED EXPERIMENT TO MEASURE HOLOGRAPHIC SORTING OF SPDC

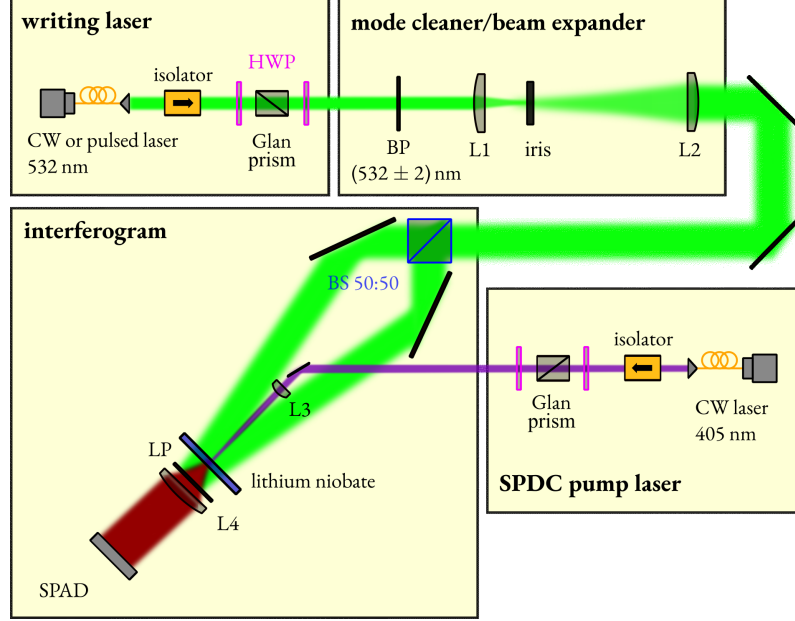


Figure 7.5: Setup to measure the holographic coupling of biphotons in non-phase-matched SPDC in lithium niobate. Lenses L_1 and L_2 magnify the writing beam and use various contractions: beam splitter, BS; continuous wave, CW; Glan-Taylor prism, GP; half wave plate, HWP; photodiode, PD; single photon avalanche detector array, SPAD array.

that the idler and signal photons will be spatially anti-correlated with an offset, albeit inefficiently⁴. We calculate the angular correlation probability for one possible experiment and plot it in Figure 7.6. We see that there is a small chance that one of the signal or idler photons will be diffracted by the grating.

In Figure 7.6 we select a set of parameters that could work for this experiment. These include the wavelengths, intensities, and regimes (CW or pulsed) of the writing and probe lasers, the angle of beam writing beam interference, the focal size of the pump, and the focal length of the coupling lenses. A more advantageous set of these parameters likely exists and is left for future investigation.

⁴For degenerate SPDC, the hologram would cause the angle of the photons to be *correlated* ($\theta_s = \theta_i$) instead of *anti-correlated* ($\theta_s = -\theta_i$).

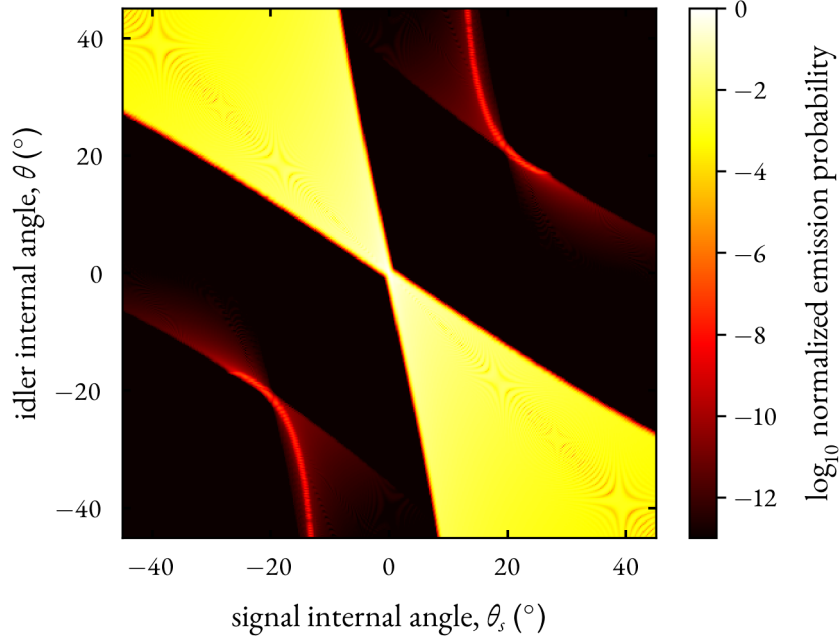


Figure 7.6: Emission probability [Equation (7.8)] of a lithium niobate crystal as a function of signal and idler internal emission angles using holographic phase matching [Equation (7.9)]. The calculation assumes a writing laser of vacuum wavelength 532 nm and external Bragg angle $\pi/6$ ($K = 11.8 \text{ rad}/\mu\text{m}$), a 405 nm pump focused to a $10 \mu\text{m}$ radius, and an index contrast of $\Delta n = 6 \times 10^{-6}$. We integrate over the range of detectable emission wavelengths ($500 \sim 1000 \text{ nm}$).

All of these parameters interplay with one another. Firstly, the writing laser parameters determine the spatial holographic grating frequency. This, in turn, must be physically realizable within the holographic material, producing the highest possible diffraction efficiency. The pump laser and lithium niobate crystal parameters must be chosen so that the frequency-angular spectrum is compatible with the Bragg condition produced in the hologram - this is more complicated than it sounds, as the index modulation in lithium niobate does not innately interact with light in the near-infrared. Further, the detectors we use to correlate the photon pairs must cover the frequency range of biphoton production

- particularly where the Bragg condition is met.

There are also concerns about photoluminescence from the lithium niobate from both the pump and writing laser beams. The biphoton signal could be washed out by the photoluminescence. Moreover, the focused pump laser could wash out the holographic grating, depending on its intensity. The transverse interaction length must also not be so small that the biphotons have a greatly reduced chance of interacting with the grating.

This experiment is intriguing and has great promise, but there is work to do before it is achievable. This thesis has laid a roadmap to achieving holographic coupling in non-phase-matched SPDC, including the characterization of holography in iron-doped lithium niobate, as well as the detection of biphoton production in lithium niobate.

In Appendix H we characterize biphotons in several different thin materials including lithium niobate on silicon, fused silica, and iron-doped lithium niobate, in addition to gallium arsenide (III). The materials are described in Appendix H.1. In Appendix H.2 we present diagrams of all setups used to characterize the samples, and in Appendix H.3 we present measurements of their second-order correlation rates [$g^{(2)}$]. In Appendix H.4 we perform polarization tomography for lithium niobate on fused silica and gallium arsenide. Finally, we perform fiber spectroscopy of all samples and measure the second-order nonlinearities via second harmonic generation in Appendix H.5 and Appendix H.6, respectively.

In Appendix I we describe the quantum state of SPDC, and describe the quantum interference of a biphoton in a beam splitter with variable reflection efficiency.

PART IV

CONCLUSIONS

8

CONCLUSIONS

8.1 OPTICAL COMPRESSION OF FREE SPACE

In Part I, we demonstrated compression of free-space using a device called a “three-lens spaceplate”, a positive lens placed at the Fourier plane of a $4f$ optical setup. We showed that our spaceplate design can be used for meter-scale space compression and the miniaturization of broadband imaging systems.

Using the three-lens spaceplate, we achieved experimental compression ratios up to $\mathcal{R}_{\text{qbs}} = 15.7$ that replace 4.4 meters of free-space. Until now, replication of the transfer function of free-space over lengths of this scale had not been demonstrated: previously, space compression of only one or two millimeters had been shown [54]. The three-lens spaceplate has not only shown that this is possible, but it has also accomplished this using off-the-shelf optical components. Moreover, the system reduced the length of monochromatic and full-color imaging systems with a 24% loss of resolution as quantified using its modulation transfer function. Finally, we discussed the trade-off between compression ratio and numerical aperture. Specifically, we found that the numerical aperture is highly limited for the three-lens spaceplate and represents its greatest shortcoming.

The simplicity and versatility of this spaceplate design motivate its application and future development, though certain limitations differentiate its applications from other spaceplates. The three-lens spaceplate represents a simple, cost-efficient method for reducing the size of axially symmetric, long focal length systems such as telescopes. Because these three-lens spaceplates are so large and have such limited numerical apertures, however, their application in smaller devices such as smartphone cameras or virtual reality headsets would be difficult to implement. That said, the inherent use of free-space within the system itself implies the potential for even higher compression ratios through the use of metalenses and other space-saving devices. Metalenses could also reduce the numerical aperture limitations of the three-lens spaceplate by reducing lens thickness and increasing aperture size. Future work includes using such hybrid designs to reduce overall system length and increase numerical aperture, as this could make a three-lens spaceplate-implementation in small devices more realistic. The characterization of bandwidth and time of flight, in the sense of Shastri et al., also remain open topics of research. [56].

8.2 VOLUME HOLOGRAPHY IN PHOTOPOLYMERS AND LITHIUM NIOBATE

In Part II, we reviewed different analysis methods for determining the diffraction efficiencies of volume holograms, and we proposed a new method to design grating distributions for in-plane holographic optical elements. We also characterized holography in photopolymers and lithium niobate to better understand the validity of our method of designing grating distributions.

We presented a new method of designing the phase response of a holographic optical element, producing a mathematical description for an idealized holographic spaceplate. We also described ways to make a more implementable grating using our formulation. Specifically, we concluded that unless the holographic gratings have large spatial frequencies (large Bragg angles) they will couple non-negligibly into higher-order modes. Secondly, in-plane holograms produce the most efficient first-order coupling, and lastly, most realistic volume holograms have large bandwidths in angle and frequency. To reduce these bandwidths, given some other constraints, one needs to minimize the index contrast to film thickness ratio ($\Delta n/d$).

Finally, we characterized the temporal response of holograms to different exposure conditions. In a photopolymer hologram, we measured a maximum diffraction efficiency of $\eta = 91\%$ and a maximum index contrast of $\Delta n = 0.014$. In lithium niobate, we measured a maximum diffraction efficiency of $\eta = 0.013\%$ and a maximum index contrast of $\Delta n = 0.014$. Our methods can measure the index contrast of a hologram while it is exposed, enabling different holographic optical element applications. We wrote a photopolymer hologram with a minimum pitch $\Lambda < 0.45 \mu\text{m}$. Further, we measured

the angular response of three highly efficient photopolymer holograms. This evidence supported our conclusions about the limitations of the different holographic formalisms.

There are several avenues for further study. Firstly, we have yet to characterize the rates of charge carrier diffusion in thin films of iron-doped lithium niobate. We made several holographic measurements in lithium niobate that showed interesting physics, however, we did not have time to discuss or analyze them. Secondly, we failed to demonstrate thermal fixing in iron-doped lithium niobate holograms. We suggest several reasons why this could be the case, but a more thorough analysis could prove helpful. The search for a better holographic material (i.e., thicker, with a wider range of achievable index contrasts and spatial frequencies) to make holographic optical elements could be fruitful. Our method to design holographic optical elements could also improve. These improvements include removing assumptions, using the method to design other holographic optical elements (i.e., holographic $N \times N$ beam splitters¹ [135–139]), and testing the method in an experiment.

8.3 SPONTANEOUS PARAMETRIC DOWN-CONVERSION IN THIN FILMS

In Part III, we propose an experiment to shape the entanglement between two photons created via non-phase-matched spontaneous parametric down-conversion in lithium niobate. We present a road map to achieve this experiment and contextualize our work towards this aim. This work includes the creation of holograms in iron-doped lithium niobate and the characterization of the polarization, frequency,

¹In the literature, these are also called *star couplers*.

and emission angle of biphotons produced in down-conversion.

There is considerable work left in this area. Firstly, we present a great deal of new experimental material in the appendix of this thesis that we don't discuss at length. This includes measurements of efficient biphoton production in GaAs (III) - a new experimental observation - and fiber spectroscopy of lithium niobate on silicon. The latter is particularly interesting due to the Fabry-Perot resonances observed in the film. Further, there is considerable work left to demonstrate holographic coupling in experimental spontaneous parametric down-conversion.

APPENDICES

A

CONTEXTUALIZING THE THREE-LENS SPACEPLATE

This appendix provides additional context for the implementation of the three-lens spaceplate, discussing its limitations. Appendix A.2 refines some fundamental theoretical limits on the performance of the three-lens spaceplate. Specifically, it focuses on compression ratio, NA, and bandwidth, param-

eters which recent work has identified trade-offs between for other spaceplate designs [56, 60, 61]. We have considered three physical mechanisms inherent to the three-lens spaceplate that lead to limitations. Firstly, it assesses how the lens diameters limit the NA of the optical system. Secondly, the ASC limits the NA independent of the lens diameters. Lastly, diffraction from the first lens in the spaceplate limits the achievable compression ratio.

A.1 DEFINING THE COMPRESSION RATIO AND SYSTEM BOUNDARIES

A.1.1 BEYOND THE SMALL-ANGLE REGIME

In motivating the design of the three-lens spaceplate (see Section 2.3.1), we used the small-angle approximation to draw similarities between the ideal spaceplate phase, φ_{SP} , and the phase of a spherical, thin lens φ_{mid} . In reality, this approximation is more restrictive than necessary for such a comparison, and, thus, the possibility of extending beyond this regime merits further discussion.

The general spaceplate phase is $\varphi_{\text{SP}} = kd_{\text{eff}}(1 - k_x^2/k^2)^{1/2}$, while the spherical lens phase is $\varphi_{\text{mid}} = k(f_{\text{mid}}^2 - x^2)^{1/2}$. Assuming the first lens is capable of performing a perfect Fourier transform of the incident field such that $k_x = rk/f_{\text{ext}}$ holds for large angles, we can directly compare the spaceplate phase to that of a spherical lens outside the small-angle approximation. The maximum value of $k_x/k = \sin \theta$ such that the phase difference is less than some fraction of a period (i.e. $\sim \pi/5$), disregarding any global phase, determines the maximum NA of the three-lens spaceplate. Geometrically, it represents the angle

at which the circular phase profile of a spherical lens deviates a fraction of a cycle from the elliptical phase profile of an ideal spaceplate. This method can be generalized to other middle lens forms as long as the first lens linearly maps each incident transverse wavevector component k_x to a single position x in the back focal plane.

It must be said, however, that conventional lenses produce limits on NA that make this method of little practical use. As we have shown for our setup, this is because the limits on NA will always be more restrictive than those predicted by the method above. There are likely lens designs that are more suited for matching the phase profile of an ideal spaceplate in which case this method may prove useful.

A.1.2 RAY-MATRIX APPROACH TO DERIVING THE COMPRESSION RATIO

One can also assess the three-lens spaceplate using a ray matrix method [65, 140–142]. The ray-transfer matrix for the system is given by

$$M = - \begin{bmatrix} 1 & \frac{f_{\text{ext}}^2}{f_{\text{mid}}} - 2f_{\text{ext}} \\ 0 & 1 \end{bmatrix}$$

which is equivalent to the ray-matrix for inversion and propagation over an effective distance $d_{\text{eff}} = f_{\text{ext}}^2/f_{\text{mid}} - 2f_{\text{ext}}$. The compression ratio can then be expressed as

$$\begin{aligned} \mathcal{R} &= \left(\frac{f_{\text{ext}}^2}{f_{\text{mid}}} - 2f_{\text{ext}} \right) \frac{1}{d} \\ &= \frac{f_{\text{ext}}}{2f_{\text{mid}}} - 1 \end{aligned}$$

which is identical to Equation (2.11).

A.1.3 THE EFFECT OF THE DEFINED SYSTEM LENGTH

In the initial Fourier optics derivation of the spaceplate compression ratio (Section 2.3.1), we consider a $4f$ system without a thickness f_{ext} of free-space before or after the external lenses ($d = 2f_{\text{ext}}$). It is mathematically simpler to consider a system with these preceding and succeeding lengths of free space ($d = 4f_{\text{ext}}$); this is because they are required to perform a Fourier transform without any additional phase factors [65]. In such a system, we can directly equate the spaceplate phase [Equation (2.10)] with the lens phase ($\varphi_{\text{SP}} = \varphi_{\text{mid}}$) and solve for the compression ratio $\mathcal{R}_{4f_{\text{ext}}}$, where we use a subscript to indicate the redefinition of system length:

$$\mathcal{R}_{4f_{\text{ext}}} = \frac{f_{\text{ext}}}{4f_{\text{mid}}}. \quad (\text{A.1})$$

This is similar to the initial definition of compression ratio [Equation (2.11)] but it predicts less compression due to the added free space.

In comparing the two definitions of compression ratios, one may notice that Equation (2.11) can evaluate to be negative, but this is simply a demonstration that the three-lens spaceplate can extend the length of systems, not just compress them. Recall that the compression ratio is defined as $\mathcal{R}_c = d_{\text{eff}}/d$ and that negative compression ratios simply define a system for which $d_{\text{eff}} < 0$. Effectively, such systems produce an output field equivalent to a field seen a distance d_{eff} before the first lens. This is practically irrelevant to our exploration as only compression ratios greater than unity are of interest,

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

though an example can be used to illustrate this discussion as well as demonstrate that Equation (2.11) and Equation (A.1) are physically equivalent.

Suppose we have a three-lens spaceplate without a middle lens ($f_{\text{mid}} \rightarrow \infty$) such that $\mathcal{R}_\infty = -1$ and $\mathcal{R}_{4f_{\text{ext}}} = 0$, as given by Equation (2.11) and Equation (A.1). It is predicted by Equation (2.11) that the light field immediately after the final lens will equal - ignoring inversion - the field a distance $2f_{\text{ext}}$ before the first lens in the spaceplate. This is a separation of $4f_{\text{ext}}$ between equivalent fields. Further, Equation (A.1) predicts that the field at the output (a distance f_{ext} past the final lens in the spaceplate) will equal the field a distance f_{ext} before the first lens. This is also a separation of $4f_{\text{ext}}$ between equivalent fields. This demonstrates that both compression ratio definitions [Equation (2.11) and Equation (A.1)] are contextually equivalent.

A.2 FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

This section presents a discussion of limits on the NA and compression inherent to the three-lens design of the spaceplate.

A.2.1 NUMERICAL APERTURE LIMITATIONS

NUMERICAL APERTURE LIMITATIONS DUE TO LENS DIAMETER

The use of finite-sized lenses results in a number of limitations; the first limitation we discuss is that of NA which depends on the system parameters including compression ratio and the f-numbers of the

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

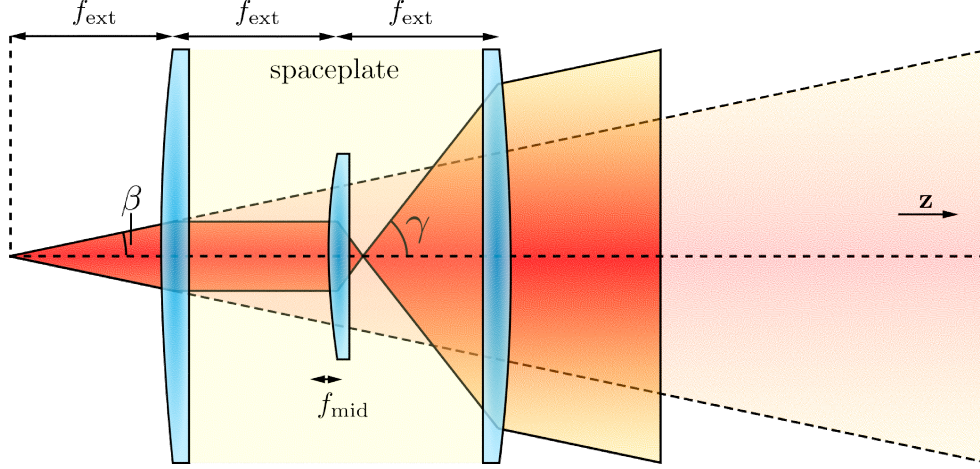


Figure A.1: Maximal light cone used to define the NA of the three-lens spaceplate and contextualize the ASC. We define $\text{NA} = \sin \beta$ for a boundary ray at angle β from the z -axis, crossing the z -axis a distance f_{ext} before the first lens. See text for discussion of NA and the ASC limitations.

three-lenses used, $N_1 = f_{\text{ext}}/D_1$, $N_2 = f_{\text{mid}}/D_1$, and $N_3 = f_{\text{ext}}/D_3$, where D_i is the diameter of the i -th lens. We define $\text{NA} = \sin \beta$ for a boundary ray at angle β from the z -axis, crossing the z -axis a distance f_{ext} before the first lens [see Figure A.1]. The angular range of the spaceplate is defined as the largest angle β at which a ray can successfully be transmitted through the spaceplate. In most cases, the third lens limits the angular range of the spaceplate as evident in Figure A.1, where we assume $f_{\text{ext}} > f_{\text{mid}}$ such that the resulting compression ratio is greater than one. This is because as long as the system compresses space, the third lens must act on a range of angles greater than that incident on the system. However, if the leading or middle lens diameter is reduced enough, the overall limitation on NA will no longer be set by the third lens and instead will be set by the first or second lens. Considering these different cases,

we arrive at the following expressions for the NA:

$$\text{NA} = \begin{cases} (1 + 4N_1^2)^{-\frac{1}{2}}, & D_1 \leq D_2 \leq f_{\text{mid}}D_3/(f_{\text{ext}} - f_{\text{mid}}) \\ \left[1 + 16N_2^2 (1 + \mathcal{R}_\text{c})^2\right]^{-\frac{1}{2}}, & D_2 \leq D_1 \leq f_{\text{mid}}D_3/(f_{\text{ext}} - f_{\text{mid}}) \\ \left[1 + 4N_3^2 (1 + 2\mathcal{R}_\text{c})^2\right]^{-\frac{1}{2}}, & D_1, D_2 \geq f_{\text{mid}}D_3/(f_{\text{ext}} - f_{\text{mid}}). \end{cases} \quad (\text{A.2})$$

Equation (A.2) is plotted in Figure A.2(a) where each piece-wise function is plotted independently. The third case, where the system is N_3 -limited, is most physically realizable. The right-most figure in Figure A.2(a) shows that for compression ratios greater than one, and $N_3 = 3$ as used in our experiment, the achievable NA is less than 0.06.

NUMERICAL APERTURE LIMITATIONS DUE TO THE ABBE SINE CONDITION

In designing high-quality imaging systems using common refractive optics, the minimization of aberrations is of great importance. Aplanatism, or the absence of spherical aberration and coma, may be achieved by satisfying two conditions: the system must be axially stigmatic (fully corrected for on-axis imaging) and the system must satisfy the ASC [143]. For a system to produce sharp images of extended flat objects, which was a central objective of the three-lens spaceplate, the latter is of particular interest and thus merits further discussion.

As we have shown, the three-lens spaceplate does not provide any magnification or lensing effects and hence is not an imaging system, however, the ASC is still relevant. We can see from Figure A.1 that the spaceplate can be divided into two subsystems: a leading imaging system of magnification $\mathcal{M}$ and an auxiliary imaging system of reciprocal magnification $\mathcal{M}^{-1}$. For simplicity, we assume an object

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

placed a distance f_{ext} from the first lens and that each imaging system is free of spherical aberration. In other words, we will only examine limitations on NA due to satisfying the ASC.

We begin by defining the magnification of each imaging system. According to geometrical optics

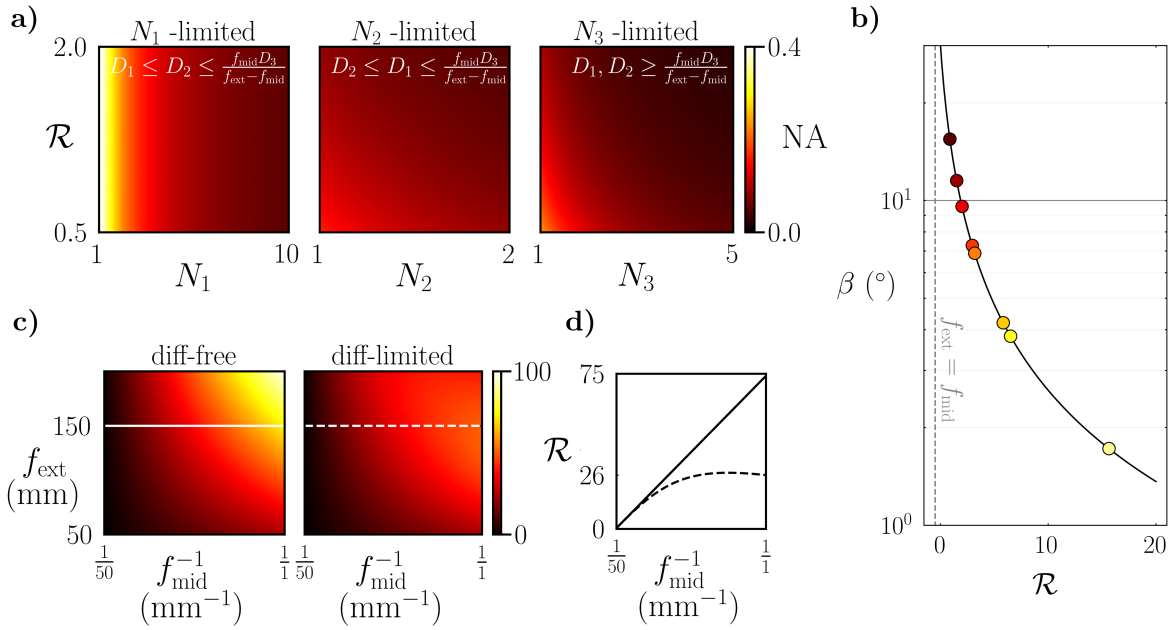


Figure A.2: Various fundamental limitations of three-lens spaceplates. a) These three plots show the three cases in Equation (A.2) labeled by the aperture which limits the system NA. b) A plot of the relationship between $\mathcal{R}$ and the maximum input angle as given by the ASC: each scatter point gives the limit for each experimentally tested spaceplate. c) This plot compares the compression ratio as a function of focal lengths f_{ext} and f_{mid} as predicted by Equation (A.1) to that as predicted by Equation (A.4) which considers diffractive effects. In order to make the effect more apparent, the diameter of the first aperture is made artificially small (~ 10 mm). d) This plot shows a line cut of both plots in (c) for our experimentally tested focal lengths. The solid line denotes the original expression, and the dashed line denotes the diffraction-limited expression.

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

and Equation (2.6), the transverse magnification constant for the leading imaging system is given by:

$$\mathcal{M} = -\frac{f_{\text{mid}}}{f_{\text{ext}}} = -\frac{1}{2(\mathcal{R}_c + 1)}. \quad (\text{A.3})$$

As we are only interested in systems with compression ratios greater than unity, $\mathcal{R}_c > 1$, the magnification constant for the leading imaging system will always be less than $1/4$. In contrast, the magnification of the auxiliary imaging system will always be greater than four. Therefore, the leading imaging system is responsible for creating the largest internal ray angles [set by γ in Figure A.1] and, hence will limit the incident angles according to the ASC: $|\mathcal{M}| = \sin \beta / \sin \gamma$ [144]. This condition states that the ratio of the sine of the angle made by a ray exiting the object of an imaging system to that of the ray at the imaging plane must be equal to the system's magnification constant - where either angle is measured from the optical axis as in Figure A.1. Using Equation (A.3), the ASC gives the maximum incident angle the three-lens spaceplate can act on irrespective of aperture size limitations: $\sin \beta_{\text{max}} = [2(\mathcal{R}_c + 1)]^{-1}$. For practical lenses, this limit is not attainable since the effects of finite lens diameter are more restrictive.

A.2.2 COMPRESSION RATIO LIMITATIONS DUE TO DIFFRACTIVE EFFECTS

Another limitation set by finite aperture size is the diffractive limit on the maximum compression ratio achievable by the spaceplate. For simplicity, we consider diffraction from only the first lens the phase imparted by the middle lens. This assumption is valid for most realistic cases in which the first and third lens are approximately the same size such that the additional diffraction effects attributable to the third lens are negligible. The aperture function of the first lens is Fourier transformed by the first lens, and the resultant point spread function (PSF) is optically convolved with the middle lens phase

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

Equation (2.10) at the Fourier plane. We approximate the PSF of the first aperture by a Gaussian with standard deviation $\sigma = g\lambda f_{\text{ext}}/2D_1$ where $g = 1.22$ is the Airy first node scaling, D_1 is the diameter of the first lens, and λ is the wavelength:

$$H_1 = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

The phase distribution across the range of input k_x is then given by the convolution of the PSF and the phase of the middle lens.

$$\begin{aligned} H * l &= \frac{1}{2\pi\sigma^2} \frac{1}{\sqrt{\frac{2a_1^2}{f_{\text{ext}}^2 g^2 \lambda^2 \pi} + \frac{1}{f_{\text{ext}} \lambda}}} \exp\left(\frac{2a_1^2 \pi x^2}{j2a_1^2 f_{\text{mid}} \lambda - f_{\text{ext}}^2 g^2 \pi \lambda}\right) \\ &= \underbrace{(1 + \alpha^2)^{-1/4} \exp\left(-\frac{\alpha}{1 + \alpha^2} \frac{\pi x^2}{f_{\text{mid}} \lambda}\right)}_{\text{attenuation}} \underbrace{\exp\left(j \frac{\arctan \alpha}{2}\right)}_{\text{global phase}} \underbrace{\exp\left(-j \frac{1}{1 + \alpha^2} \frac{\pi x^2}{f_{\text{mid}} \lambda}\right)}_{\text{phase}} \end{aligned}$$

where $\alpha = 2\pi\sigma^2/f_{\text{mid}}\lambda = g^2\pi f_{\text{ext}}^2\lambda/2f_{\text{mid}}a_1^2$. Here, we can ignore the effects of the global phase and radial attenuation, and extract the compression ratios from this new phase. Recognizing the spaceplate phase as $\varphi_{\text{SP}} = -\pi x^2/f_{\text{mid}}\lambda(1 + \alpha^2)$, we can equate this to Equation (2.10) to solve for the compression ratio:

$$\mathcal{R}_{\mathcal{A}_{\text{ext}}} = \frac{1}{1 + \alpha^2} \frac{f_{\text{ext}}}{4f_{\text{mid}}} \Rightarrow \mathcal{R}_c = \frac{2f_{\text{ext}}f_{\text{mid}}}{4f_{\text{mid}}^2 + \alpha^2 f_{\text{ext}}^4} - 1. \quad (\text{A.4})$$

Note that as the aperture of the first lens gets larger, α goes to zero, and the compression limit approaches our original expression [Equation (2.10)]. Equation (A.4) is plotted in Figure A.2(c) and Figure A.2(d): in c), 2D color plots compare the compression ratios achieved given a pair of focal lengths f_{ext} and f_{mid} ,

A.2. FUNDAMENTAL LIMITS OF THREE-LENS COMPRESSION

and it is clear that diffraction results in a maximally achievable compression ratio. For example, in d) we set $f_{\text{ext}} = 150$ mm to match our experimental parameters and then sweep f_{mid}^{-1} . As expected, Equation (2.10) goes linearly with f_{mid}^{-1} where as the diffractive considerations result in a global compression ratio maximum. Due to the achievable focal lengths and aperture sizes, however, one would be hard-pressed to observe this effect in a real system: Figure A.2(c) assumes an initial lens diameter of 10 mm which is considerably smaller than achievable lens diameters. Even then, one would need middle lenses with focal lengths smaller than two or three millimeters to observe this limitation in compression ratio. This effect is easily designed around.

B

TWO DIMENSIONAL PHASE HOLOGRAMS

There are several ways to record an interference pattern. In the Section 5.1.1, we recorded a hologram by way of an amplitude grating, but what happens if we use a phase mask instead? The derivation below follows the methods of Toal [3].

With the same geometry as in Section 5.1.1, let us instead consider the interference pattern created by a more general beam we call the *object beam*. We interfere this with a simple plane wave we call the

reference beam. These waves can, respectively, be mathematically described as:

$$\begin{aligned}\tilde{\mathbf{E}}_S &= \mathbf{E}_S(x, y) \exp [i(\mathbf{k}_S(x, y) \cdot \mathbf{r} - \omega t)] \\ \tilde{\mathbf{E}}_R &= \mathbf{E}_R \exp [i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)].\end{aligned}$$

Here, we consider an object beam with some non-trivial, spatially varying, coherent wavefront. The summed wave at the interface is then

$$\begin{aligned}\tilde{\mathbf{E}} &= \tilde{\mathbf{E}}_R + \tilde{\mathbf{E}}_S \\ &= \mathbf{E}_R \exp [i(\mathbf{k}_S(x, y) \cdot \mathbf{r} - \omega t)] + \mathbf{E}_S(x, y) \exp [i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)].\end{aligned}$$

The intensity of this wave at the interface is then

$$\begin{aligned}I(x, y) &= \frac{1}{2} |\tilde{\mathbf{E}}|^2 \\ &= \frac{1}{2} [\mathbf{E}_R e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} + \mathbf{E}_S(x, y) e^{i(\mathbf{k}_S(x, y) \cdot \mathbf{r} - \omega t)}] [\mathbf{E}_R e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} + \mathbf{E}_S(x, y) e^{i(\mathbf{k}_S(x, y) \cdot \mathbf{r} - \omega t)}]^* \\ &= \frac{E_R^2 + E_S(x, y)^2}{2} + E_R E_S(x, y) \cos [(\mathbf{k}_R - \mathbf{k}_S(x, y)) \cdot \mathbf{r}]\end{aligned}$$

where $E_R, E_S = |\mathbf{E}_R|, |\mathbf{E}_S|$.

Again, suppose that a thin film is exposed to the interference and that after exposure; however, let the film act as a phase mask, where the amount of phase imparted by the film is proportional to the intensity of the interference. For some scaling invariant β , the spatially-dependent phase can be

written as

$$\phi(x, y) = \beta I(x, y).$$

If the initial reference beam alone is transmitted through the phase grating, the field immediately following the grating can be expressed as

$$\begin{aligned} \tilde{\mathbf{E}}_{\text{NF}} &= \tilde{\mathbf{E}}_R \exp[i\beta I(x, y)] \\ &= \mathbf{E}_R \exp \left[i \left(\mathbf{k}_R \cdot \mathbf{r} - \omega t + \beta \left\{ \frac{E_R^2 + E_S(x, y)^2}{2} + E_R E_S(x, y) \cos [(\mathbf{k}_R - \mathbf{k}_S(x, y)) \cdot \mathbf{r}] \right\} \right) \right]. \end{aligned} \quad (\text{B.1})$$

Assuming that the reference beam more intense than the object beam ($E_R^2 \gg E_S^2$), the real part of the transmitted complex amplitude [Equation (B.1)] can be expressed as

$$\begin{aligned} \text{Re}\{\tilde{\mathbf{E}}_{\text{NF}}\} &= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t + \beta E_R E_S(x, y) \cos\{[\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r}\}) \\ &= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \cos(\beta E_R E_S(x, y) \cos\{[\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r}\}) \\ &\quad - \mathbf{E}_R \sin(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \sin(\beta E_R E_S(x, y) \cos\{[\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r}\}) \end{aligned} \quad (\text{B.2})$$

Assuming the phase added by the film is small, we Maclaurin expand Equation (B.2) and rewrite the

field as

$$\begin{aligned}
\text{Re}\{\tilde{\mathbf{E}}_{\text{NF}}\} &= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t) - \beta \mathbf{E}_R E_R E_S(x, y) \cos\{[\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r}\} \sin(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \\
&= \mathbf{E}_R \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t) \\
&\quad - \beta \mathbf{E}_R E_R E_S(x, y) \cos\left\{\mathbf{k}_S(x, y) \cdot \mathbf{r} - \frac{\pi}{2} - \omega t\right\} \\
&\quad - \beta \mathbf{E}_R E_R E_S(x, y) \cos\left\{[2\mathbf{k}_R - \mathbf{k}_S(x, y)] \cdot \mathbf{r} - \frac{\pi}{2} - \omega t\right\}
\end{aligned} \tag{B.3}$$

The first term in Equation (B.3) is the reference wave, the second term is a $\pi/2$ phase-shifted recreation of the object wave, and the third term is a $\pi/2$ phase-shifted, conjugate recreation of the object wave. This mathematical exploration demonstrates that we can imprint the an arbitrary wavefront onto a film using a coherent plane reference, and, subsequently, recreate that arbitrary wavefront using the same reference wave. This derivation has used several assumptions, but is still quite general. We simply record an interference pattern in a phase mask, and use that phase mask to recall the initial writing beams.

C

VOLUME PHASE HOLOGRAPHY - DERIVATIONS

In the following sections we make more formal derivations regarding light as it propagates through a dielectric grating, relating these findings to a transmission volume hologram.

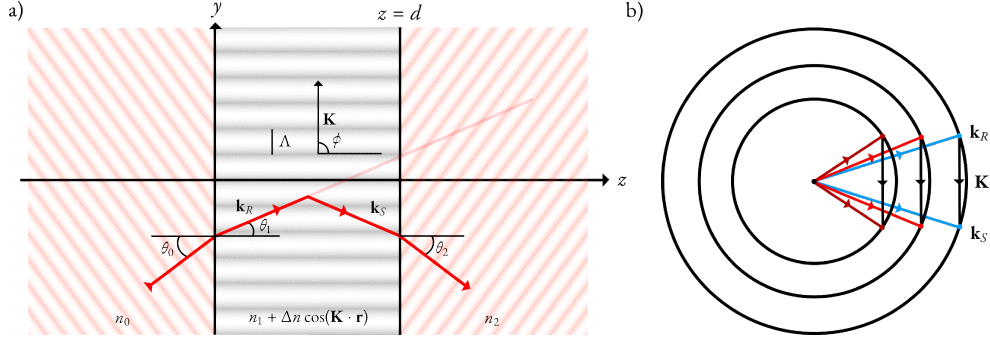


Figure C.I: Defining the geometry of a dielectric grating.

C.I LIGHT IN A DIELECTRIC GRATING

Suppose we have a dielectric plate of thickness d oriented as shown in Figure C.I with sinusoidally-varying permittivity ε whose grating vector $\mathbf{K}$ makes some angle ϕ with the z axis. The gratings are spaced by distance $\Lambda = 2\pi/|\mathbf{K}|$, and we assume that they are in a background medium of index n_1 . We can write the sinusoidal permittivity of the dielectric as

$$\varepsilon = \varepsilon_1 + \Delta\varepsilon \cos(\mathbf{K} \cdot \mathbf{r}), \quad (\text{C.I})$$

where ε_1 is the average dielectric constant, $\Delta\varepsilon$ is the amplitude of the sinusoidal permittivity variation, and $\mathbf{r}$ is the position vector. We can also write the index inside the grating as follows, assuming the index variation is small ($\Delta n \ll n_1$).

$$n^2 = [n_1 + \Delta n \cos(\mathbf{K} \cdot \mathbf{r})]^2 = n_1^2 + 2n_1\Delta n \cos(\mathbf{K} \cdot \mathbf{r}) = \varepsilon = \varepsilon_1 + \Delta\varepsilon \cos(\mathbf{K} \cdot \mathbf{r}).$$

Therefore, $\sqrt{\varepsilon_1} = n_1$, and $\Delta\varepsilon = 2n_1\Delta n$. Now we ask how the grating might affect an incident optical field.

Generally, suppose a reference monochromatic plane wave $\mathbf{E}_R$ with complex amplitude $\mathbf{A}_R(z)$, vacuum wavelength λ , and wavevector $\mathbf{k}_R$ is incident on the hologram.

$$\mathbf{E}_R = \mathbf{A}_R(z) \exp(-i\mathbf{k}_R \cdot \mathbf{r})$$

As shown in Figure C.I, $\mathbf{E}_R$ lies in the yz plane and is x -polarized. The magnitude of the reference wavevector inside the grating is given by $k_R = 2\pi n_1/\lambda$. Suppose that the reference beam is incident on the grating such that it makes an angle θ_1 with the z -axis inside it. The wave on diffracts if it satisfies, or nearly satisfies, the Bragg condition for the grating:

$$\cos(\phi - \theta_1) = \frac{\lambda}{2n_1\Lambda} = \frac{K\lambda}{4\pi n_1}.$$

We assume that there is only single-order diffraction present, whose field can be written as

$$\mathbf{E}_S = \mathbf{A}_S(z) e^{-i\mathbf{k}_S \cdot \mathbf{r}}.$$

Its wavevector, when the reference wave is incident at the Bragg angle, satisfies

$$\mathbf{k}_S = \mathbf{k}_R - \mathbf{K}.$$

This is one way to write the Bragg condition. At some location in the grating described by $\mathbf{r}$, the wave

equation describes propagation through the grating. Assuming that it is lossless, the conductivity goes to zero, and we can write the wave equation for the dielectric as

$$\nabla^2 \mathbf{E} = -\omega^2 \mu_0 \epsilon_1 \epsilon_r \mathbf{E} = -k^2 \epsilon \mathbf{E},$$

where μ_0 is the vacuum permeability, and ω is the frequency of the light wave. Solving the wave equation for $\mathbf{E} = \mathbf{E}_S + \mathbf{E}_R$, assuming that $\mathbf{A}_R(z)$ and $\mathbf{A}_S(z)$ vary slowly. Adding a small Bragg angular perturbation δ such that the incident ray is at angle $\theta_1 + \delta$ inside the grating, we have that

$$\begin{aligned} \frac{\partial \mathbf{A}_R}{\partial z} \beta_R &= -i\chi \mathbf{A}_S \\ -\frac{\partial \mathbf{A}_S}{\partial z} \beta_S - i\Gamma \mathbf{A}_S &= -i\chi \mathbf{A}_R \end{aligned}$$

where $\beta_R = \cos(\theta_1)$ and $\beta_S = -\cos(2\phi + \theta_1)$ are the normalized k_z of the R and S waves, respectively, $\Gamma = kn_1 \delta \sin[2(\phi + \theta_1)]$ describes the effect of the Bragg angular perturbation, and $\chi = \pi \Delta n / \lambda$ describes the coupling between $\mathbf{A}_R$ and $\mathbf{A}_S$. These equations reduce to two second-order differential equations which have general solutions

$$\mathbf{A}_R(z) = \mathbf{A}_{R_1} e^{\zeta_1 z} + \mathbf{A}_{R_2} e^{\zeta_2 z}, \quad \text{and} \quad (\text{C.2})$$

$$\mathbf{A}_S(z) = \mathbf{A}_{S_1} e^{\zeta_1 z} + \mathbf{A}_{S_2} e^{\zeta_2 z}, \quad (\text{C.3})$$

for ζ satisfying the following quadratic equation:

$$\zeta^2 - i\frac{\Gamma}{\beta_S}\zeta + \frac{\chi}{\beta_R\beta_S} = 0 \quad \Rightarrow \quad \zeta_{1,2} = \frac{i\Gamma}{2\beta_S} \pm \frac{1}{2}\sqrt{-\left(\frac{\Gamma}{\beta_S}\right)^2 - \frac{4\chi^2}{\beta_R\beta_S}}. \quad (\text{C.4})$$

In concert with Equation (C.4), geometrically-determined boundary conditions allow us to then solve Equation (C.2) and Equation (C.3), and produce solutions for the transmitted and reflected fields.

C.2 TRANSMISSION PHASE HOLOGRAMS

For transmission gratings, we solve Equations C.2 and C.3 by setting boundary conditions $\mathbf{A}_R(0) = \mathbf{A}_{R_1} + \mathbf{A}_{R_2} = 1$ and $\mathbf{A}_S(0) = \mathbf{A}_{S_1} + \mathbf{A}_{S_2} = 0$. We can then solve for $\mathbf{A}_S(d)$:

$$\begin{aligned} \mathbf{A}_S(d) &= \frac{-i\chi}{\beta_S(\zeta_1 - \zeta_2)} [e^{\zeta_1 d} - e^{\zeta_2 d}] \\ &= \frac{-i\chi d}{i\beta_S \left[\left(\frac{\Gamma d}{2\beta_S} \right)^2 + \frac{\chi^2 d^2}{\beta_R\beta_S} \right]^{1/2}} 2ie^{\frac{i\Gamma d}{2\beta_S}} \sin \left\{ \left[\left(\frac{\Gamma d}{2\beta_S} \right)^2 + \frac{\chi^2 d^2}{\beta_R\beta_S} \right]^{1/2} \right\}. \end{aligned}$$

Let

$$\xi := \frac{\Gamma d}{2\beta_S} = -\frac{kn_1 \delta d \sin[2(\phi + \theta_1)]}{2 \cos(2\phi + \theta_1)}, \text{ and} \quad (\text{C.5})$$

$$\nu := \frac{\chi d}{\sqrt{\beta_R\beta_S}} = \frac{\pi \Delta n d}{\lambda \sqrt{-\cos(\theta_1) \cos(2\phi + \theta_1)}}. \quad (\text{C.6})$$

Then

$$\mathbf{A}_S(d) = -i \sqrt{\frac{\beta_R}{\beta_S}} e^{-i\xi} \frac{\sin \sqrt{\xi^2 + \nu^2}}{\sqrt{1 + \xi^2/\nu^2}}.$$

The off-Bragg parameter, ξ , depends on δ . It measures how far off the incident wave is from satisfying the Bragg condition. If δ is too large, negligible energy couples into $\mathbf{E}_S$. The other parameter, ν , is grating- and geometry-dependent. It varies with the angle of the grating vector, the index variation strength, and the grating thickness.

We can now define the diffraction efficiency, which is the ratio of the flow of energy in the diffracted energy divided by the incident energy, each in the z -direction:

$$\eta := \frac{\beta_S}{\beta_R} \mathbf{A}_S \mathbf{A}_S^* = \frac{\sin^2 \sqrt{\xi^2 + \nu^2}}{1 + \xi^2/\nu^2}. \quad (\text{C.7})$$

When the Bragg condition is satisfied, $\delta = 0$, and, subsequently, so too does ξ . In such a case, Equation (5.13) allows us to define the maximum efficiency for a given ν :

$$\eta_0 = \sin^2 \nu.$$

At Bragg, we can also use Equation (C.7), Equation (C.5) and Equation (C.6) to calculate the index variation parameter of the grating:

$$\Delta n = \frac{\lambda}{\pi d} \sqrt{-\cos \theta_1 \cos(2\phi + \theta_1)} \arcsin \sqrt{\eta}.$$

D

RIGOROUS COUPLED WAVE THEORY: DERIVATIONS AND VISUALIZATIONS

In the following sections we made derivations regarding light transmitting through a dielectric grating, considering high-order diffraction. We also visualize these high-order modes.

D.1 DERIVATION OF A RIGOROUS COUPLED WAVE THEORY

The derivations in Section 5.1 give us a framework from which we can study and design holograms and HOEs; however, they have a number of assumptions through which certain inaccuracies creep in. First, we assumed that the second-order spatial derivatives of the wave amplitudes are negligible; second, we ignored boundary diffraction; and, lastly, we considered only first-order diffraction. To determine the effect of these assumptions on any holograms we design, we need another framework from which can more accurately assess them. To do so, we take the approach of Moharam and Gaylord, who produced a more rigorous framework for the study of holograms [4].

In this section, we introduce a model which allows for a more accurate assessment of the strengths of high-order diffractive modes in volume holograms. Our model uses the same physical setup as is given in the previous section, however we consider all potential modes in all three segments of the hologram (before, inside, and after the dielectric grating). We assume, as before, that the index is unmodulated in the slabs preceding and succeeding the hologram; that is, for regions defined by $z < 0$ and $z > d$, the dielectric constants are ϵ_0 and ϵ_2 , respectively.

Following the method of Moharam and Gaylord [4], we solve the wave equation in each layer and then normalize for each periodic boundary condition at $z = 0, d$. In each region we consider all pre-mitted Bragg modes, both transmitted and reflected. For instance: in region 0, we have the normalized incident plane wave $\{\exp(-i\mathbf{k}_1 \cdot \mathbf{r}) = \exp[-i(k_1 y \sin \theta_1 + k_1 z \cos \theta_1)]\}$ and all reflected modes:

$$E_1 = \exp[-i(\beta_0 y + \xi_{1,0} z)] + \sum_j r_j \exp[-i(\beta_j y - \xi_{1,j} z)]$$

D.1. DERIVATION OF A RIGOROUS COUPLED WAVE THEORY

where β (wave index) and ξ (region index) have been redefined compared to previous sections to follow the derivations of the original authors:

$$\begin{aligned}\beta_j &= k_0 \sin(\theta_0) - jK \sin(\phi); \\ \xi_{0,j} &= \sqrt{k_l^2 - \beta_j^2}\end{aligned}$$

where $l = 0, 2$, r_i is the normalized amplitude of the i -th reflected wave, and the other variables are as before. Similarly, we can define the fields in regions 1 and 2 as

$$E_1 = \sum_j s_j(z) \exp[-i(\beta_j y + \xi_{1,j} z)]; \quad (\text{D.1})$$

$$E_2 = \sum_j t_j \exp\{-i[(\beta_j y + \xi_{2,j}(z - d))]\} \quad (\text{D.2})$$

where $\xi_{2,j} = k_2 \cos(\theta_2) - jK \cos(\phi)$, t_j is the normalized amplitude of the j -th bragg mode, and $s_j(z)$ is the normalized amplitude of the j -th mode in region 2. Due to the variations in index, $s_j(z)$ does not represent a plane wave, but rather a superposition of plane waves.

To solve this problem is equivalent to solving for the various wave amplitudes. To do this, we solve the wave equation first inside of region 1 by using Equation (C.1) and Equation (D.1):

$$\frac{\Delta\epsilon}{8\epsilon_2} \frac{d^2 S_j(u)}{du^2} = (\cos(\theta_2) - j\mu \cos(\phi)) \frac{dS_j(u)}{du} - \rho j(j - B)S_j(u) + S_{j+1}(u) + S_{j-1}(u), \quad (\text{D.3})$$

where

$$\begin{aligned}
 S_j(u) &= s_j(z); \\
 u &= i\pi\Delta\epsilon z/2\lambda n_2 = i\kappa z; \\
 \mu &= \lambda/\Lambda n_2; \\
 \rho &= 2\lambda^2/\Lambda^2\Delta\epsilon = 2\mu^2\epsilon_2/\Delta\epsilon; \quad \text{and} \\
 B &= 2\Lambda n_2 \cos(\phi - \theta_2)/\lambda = 2 \cos(\phi - \theta_2)/\mu.
 \end{aligned}$$

The regime parameter ρ determines the boundary between the Bragg and conjugate diffraction regions, and B represents the Bragg condition such that at the j -th Bragg conjugate, $B = j$, and is a non-integer off Bragg. Finally, κ is the coupling coefficient. Note that this derivation is done without any of the assumptions of previous sections, and in order to avoid any further assumptions we solve it numerically.

Equation D.3 is an eigenvalue equation described by the following matrix state-space equation:

$$\begin{bmatrix} \mathbf{s}' \\ \mathbf{s}'' \end{bmatrix} = [\mathbf{b}] \begin{bmatrix} \mathbf{s} \\ \mathbf{s}' \end{bmatrix};$$

$$\begin{bmatrix} \vdots \\ S'_2 \\ S'_1 \\ S'_0 \\ S'_{-1} \\ S'_{-2} \\ \vdots \\ S''_2 \\ S''_1 \\ S''_0 \\ S''_{-1} \\ S''_{-2} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & & & & \vdots & & & & & \\ & 0 & 0 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 \\ & \vdots & & & & & & \vdots & & & & \\ & b_2 & a & 0 & 0 & 0 & & c_2 & 0 & 0 & 0 & 0 \\ & a & b_1 & a & 0 & 0 & & 0 & c_1 & 0 & 0 & 0 & \dots \\ \dots & 0 & a & b_0 & a & 0 & \dots & 0 & 0 & c_0 & 0 & 0 \\ & 0 & 0 & a & b_{-1} & a & & 0 & 0 & 0 & c_{-1} & 0 \\ & 0 & 0 & 0 & a & b_{-2} & & 0 & 0 & 0 & 0 & c_{-2} \\ & \vdots & & & & & & \vdots & & & & \end{bmatrix} \begin{bmatrix} \vdots \\ S_2 \\ S_1 \\ S_0 \\ S_{-1} \\ S_{-2} \\ \vdots \\ S'_2 \\ S'_1 \\ S'_0 \\ S'_{-1} \\ S'_{-2} \\ \vdots \end{bmatrix}$$

for $a = 8\epsilon_2/\Delta\epsilon$, $b_j = a\rho j(j - B)$, and $c_j = a(\cos(\theta_2) - j\mu \cos(\phi))$. Note that this matrix is an exact representation of the differential equation given in Equation D.3 for n considered modes. As $[\mathbf{b}]$ is a $2n \times 2n$ size matrix, we will get $2n$ eigenvalue/eigenvector sets, given by $\{\nu_j\}$ and $\{\hat{\nu}_j\}$, respectively. Generally, we can write the solutions to this eigenvalue problem as

$$\left[\dots \ S_2 \ S_1 \ S_0 \ S_{-1} \ S_{-2} \ \dots \ S'_2 \ S'_1 \ S'_0 \ S'_{-1} \ S'_{-2} \ \dots \right]^T = \sum_m C_m \hat{\nu}_j \exp(\nu_j u).$$

Here, C_m are the boundary condition determined coefficients. Therefore, we can write the j -th wave as

$$S_j(u) = \sum_m C_m \hat{v}_{m,j} \exp(\nu_j u)$$

where $\hat{v}_{m,j}$ is the m -th value of the j -th wave eigenvector. In the case that the wave modes are solved symmetrically - where we solve for $\{j\} = \{p, p-1, \dots, 2, 1, 0, -1, -2, \dots, -p+1, -p\}$, $S_j(u) = \sum_m C_m \hat{v}_{m,p-j} \exp(\nu_j u)$, where $\hat{v}_{m,p-j}$ is the m -th row and $(p-j)$ -th column of the matrix of eigenvalues. Now we determine C_m (and by doing so, t_j and r_j) by matching the tangential magnetic and electric fields at the boundary according to Maxwell's equations, where δ_{j0} is the Kronecker delta:

$$\begin{aligned} r_j + \delta_{j0} &= \sum_m C_m \hat{v}_{m,j}; \\ \xi_{0j}(R_j - \delta_{j0}) &= \sum_m C_m \hat{v}_{m,j}(\nu_m \kappa - \xi_{1j}); \\ t_j &= \sum_m C_m \hat{v}_{m,j} \exp[i(\nu_m \kappa - \xi_{1j})d]; \\ \xi_{2j} t_j &= \sum_m C_m \hat{v}_{m,j}(\nu_m \kappa - \xi_{1j}) \exp[i(\nu_m \kappa - \xi_{1j})d]. \end{aligned}$$

Substituting the boundary conditions among one another, we remove t_j and r_j :

$$2\delta_{j0}\xi_{0j} = \sum_m C_m \hat{v}_{m,j}(\xi_{1j} - \nu_m \kappa + \xi_{0j}) \text{ and} \quad (\text{D.4})$$

$$0 = \sum_m C_m \hat{v}_{m,j}(\xi_{1j} - \nu_m \kappa - \xi_{2j}) e^{i\nu_m \kappa}. \quad (\text{D.5})$$

We can express the region indices as

$$\begin{aligned}\xi_{1j} &= \frac{2\rho\kappa}{\mu^2}(\cos(\theta_2) - j\mu \cos(\phi)) \\ \xi_{2j} &= \frac{2\rho\kappa}{\mu^2} \sqrt{\frac{\epsilon_l}{\epsilon_2} - (\sin(\theta_2) - j\mu \sin(\phi))^2}\end{aligned}$$

Therefore, if we intend to solve n wave modes, we need to solve for $2n$ C_m coefficients, and we are given $2n$ equations to do so (Equations D.4 and D.5). After solving for all solutions for s_j , t_j , and r_j , we must normalize the complex amplitudes such that they obey power conservation. That is, the sum of efficiencies of all modes must equal 1:

$$\sum_j (\eta_{0j} + \eta_{2j}) = 1$$

where η_{lj} is the diffraction efficiency of the j -th mode in the l -th region. Using the incident and emitted Poynting vectors, these efficiencies can be expressed as

$$\begin{aligned}\eta_{0j} &= \Re \left\{ \frac{\xi_{0j}}{\xi_{10}} \right\} |r_j|^2; \\ \eta_{2j} &= \Re \left\{ \frac{\xi_{2j}}{\xi_{10}} \right\} |t_j|^2.\end{aligned}$$

The diffraction efficiencies can then be used to normalize r_j , s_j , and t_j . Using the above method, we solve for the modal efficiencies of a number of holograms. Following our definitions, we choose values for the following input grating parameters: the vacuum wavelength λ , the dielectric ratio $\Delta\epsilon/\epsilon_1$, the grating angle ϕ , the dielectric constants $\{\epsilon_0, \epsilon_1, \epsilon_2\}$, and the range of modes to solve for $\{j\}$. Additionally, we set

D.2. VISUALIZING HOLOGRAPHIC DIFFRACTION

the grating size Λ by choosing the input angle θ_1 at which the $B = 1$ given the other set parameters. We then vary the grating thickness d and calculate the various diffraction efficiencies for each considered mode. Note that this formalism can be used to analyze reflection gratings in addition to transmission ones.

D.2 VISUALIZING HOLOGRAPHIC DIFFRACTION

In order to visualize what is actually going on inside the hologram, we plot the field strength of a beam as it is transmitted from left to right through a holographic film (Figure D.1). Suppose we have an

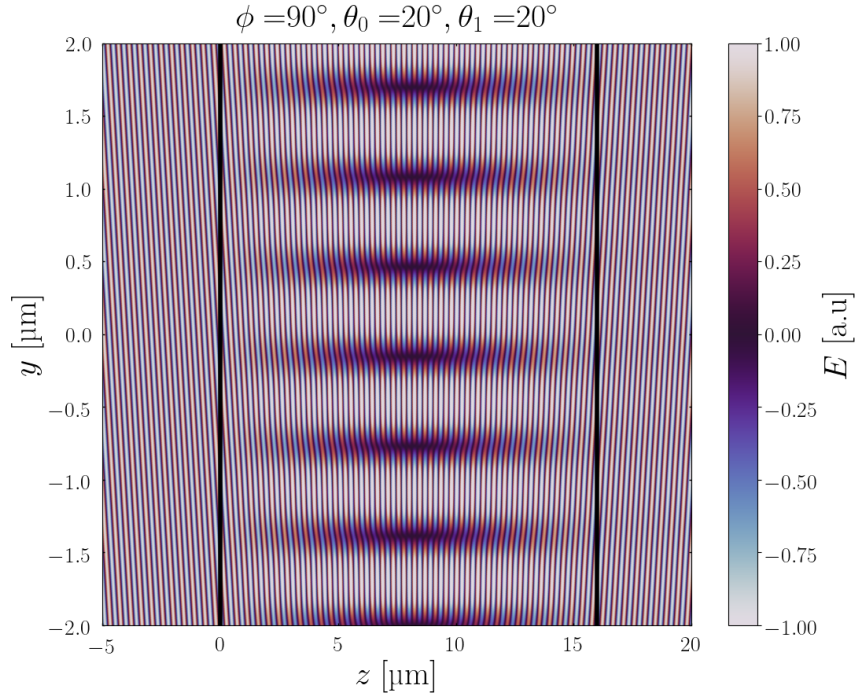


Figure D.1: Visualization of field coupling in a thick hologram.

D.2. VISUALIZING HOLOGRAPHIC DIFFRACTION

unslanted holographic grating designed to diffract a plane wave of vacuum wavelength $\lambda = 633$ nm at $\theta_0 = 20^\circ$. The film thickness is $d = 16$ μm . The indices are: $n_1 = 1.5$, $n_2 = 1.5$, $n_3 = 1.5$. We see that as the beam propagates through the hologram, energy is coupled into the outgoing mode. The black lines in the figure denote the boundary of the hologram as well as the grating period, Λ .

E

SUPPLEMENTARY ANALYSIS ON HOLOGRAPHIC MATERIALS

The following sections provide supplementary analysis on photopolymer and lithium niobate holograms.

E.1 PHOTOPOLYMERS

We propose the following explanation for the power curves in Figure 6.7. If we consider the 15 mW hologram in Figure 6.7, there is an initial rapid increase in the diffraction efficiency before it quickly falls. Slowly, it then increases and saturates. When the hologram is initially exposed, the polymerization rate is high and is proportional to the writing intensity. This polymerization leads to a decrease in refractive index in regions of high intensity, explaining the rapid increase in diffraction efficiency (see Figure E.1). This rapid polymerization then drives a slower monomer diffusion process, where monomers diffuse into regions of high intensity. It initially leads to a decrease in the refractive index, reducing the index contrast. This explains the drop in diffraction efficiency. Then, once the monomers have started to diffuse into regions of high intensity, they polymerize, once again increasing the index contrast and the

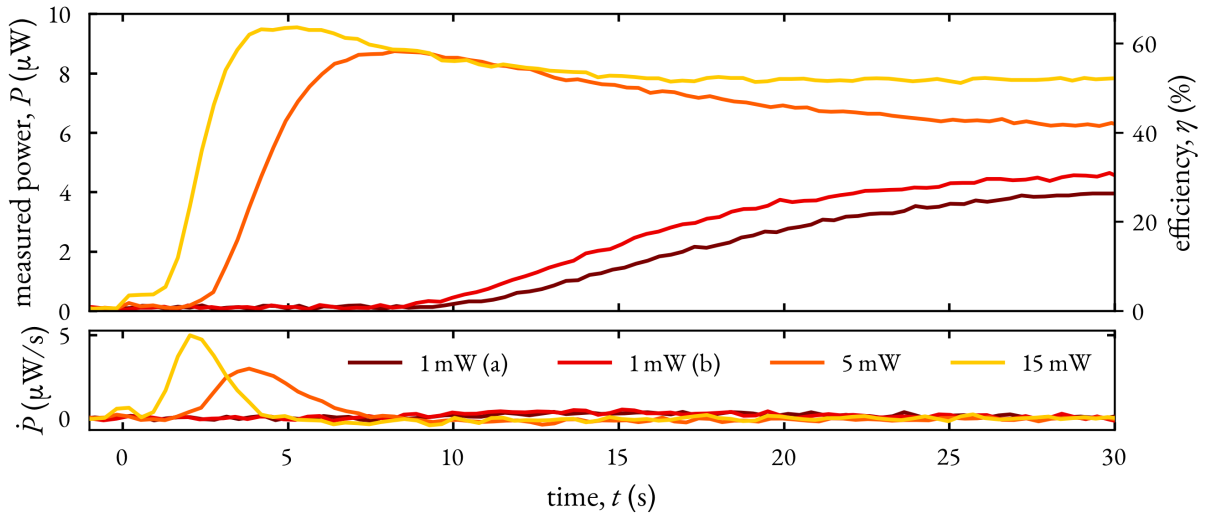


Figure E.1: Measured optical power, diffraction efficiency, and rate of change of optical power (lower) of a probe beam as a photopolymer hologram is exposed to several different writing beam powers.

diffraction efficiency. This process slows as the monomer density approaches zero. We propose that we do not see this phenomenon at lower intensities because of the slower polymerization. Perhaps the rate of diffusion is limited more than the rate of polymerization. To our knowledge, this phenomenon has not been rigorously answered and should be studied further. A rigorous explanation is beyond the intention and scope of this thesis.

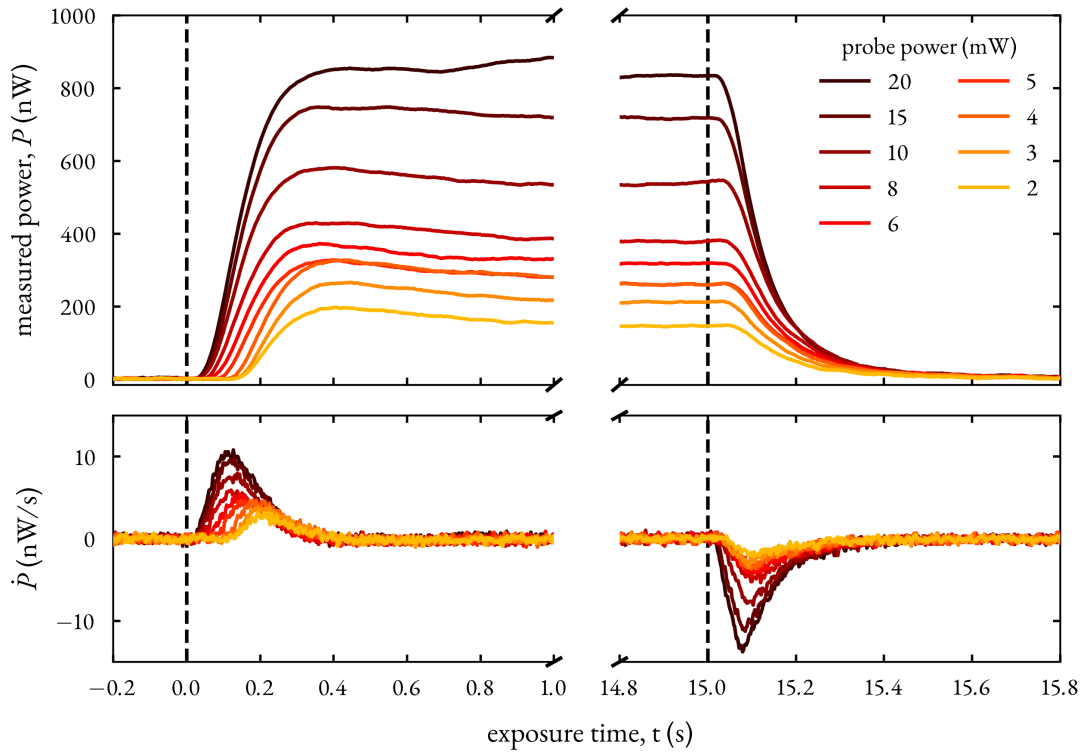


Figure E.2: Measured optical power of different probe beams as a Fe:LiNbO₃ sample is exposed to an interfering writing beam with total power 45 mW and maximal intensity 225 mW/cm². The bottom plot presents the rate of change of power over time for each probe intensity.

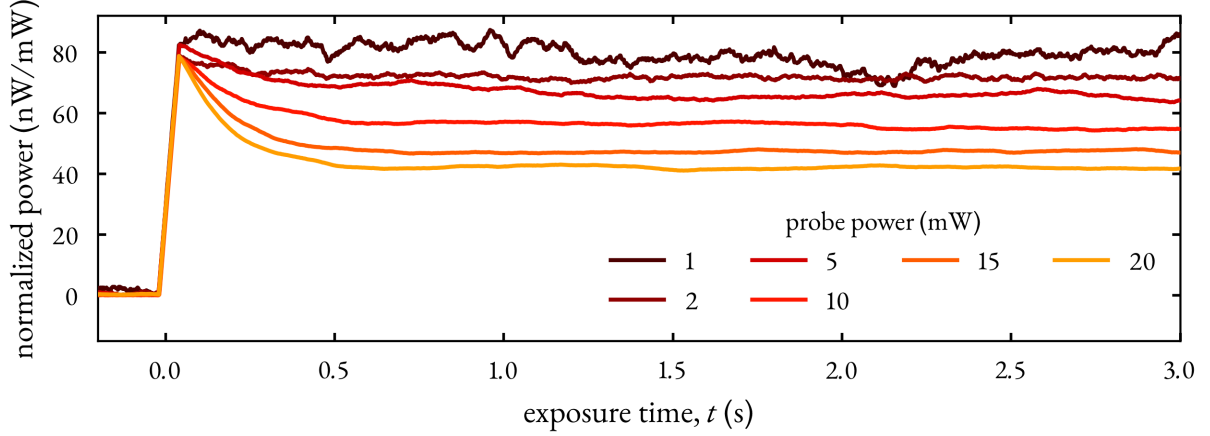


Figure E.3: Normalized measured power of different probe beams as a Fe:LiNbO_3 sample is exposed to an interfering writing beam with total power 45 mW and maximal intensity 225 mW/cm^2 . Normalized power is defined as the measured diffracted probe power divided by the incident probe power. It is proportional to efficiency. The probe is unblocked at time 0 after allowing the the hologram to expose for ten seconds.

E.2 LITHIUM NIOBATE

Using the same experimental setup presented in Figure 6.14 for all measurements presented in this appendix. We measured the diffracted power and, subsequently, the diffraction efficiency of probe beams with several intensities using a writing laser with 45 mW of total power and 225 mW/cm^2 at max intensity. The holograms were exposed for 15 s. These measurements are presented in Figure E.2 where we plot measured probe power as a function of time. We also investigated the effect of probe intensity on the index contrast of the grating by seeing how the diffraction efficiency of a probe beam decays after the crystal is exposed to the probe as a function of probe power. The probe was unblocked at time 0 after allowing the the hologram to expose for ten seconds. Upon exposure, the grating decays exponentially. This is because the higher probe intensities reduce the interference visibility of the writing laser.

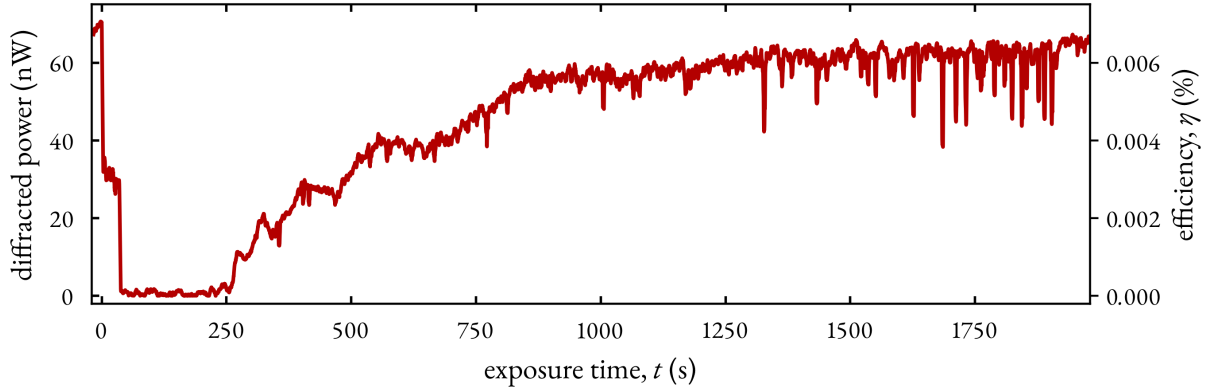


Figure E.4: Diffracted power of a probe of a heated grating over time. The grating starts heating up to 100°C at time 0 s. At time 20 s, the hot air gun is turned up to 180°C . At time 250 s, the heat gun is turned off, the grating cools down, and the writing beam and probe beams are left on. The probe beam has power 1 mW, the writing beam has total intensity of $225\text{ mW}/\text{cm}^2$.

We see that the higher the probe power, the lower the resulting diffraction efficiency.

Finally, we investigated the diffraction efficiency of a probe beam while heating an exposing sample. We started heating the grating using hot air at 100°C at time 0 s. At time 20 s, the Weller WR 2 hot air station is turned up to 180°C . At time 250 s, the heat gun was turned off, the grating cooled down, and the writing beam and probe beams were left on. In Figure E.4, we observe that the grating reappears once it starts to cool down, and it recovers to its original state. After recording this data, we also observed that upon turning the writing beam off, the diffraction efficiency decayed. Therefore, the grating was not thermally fixed by the processing.

F

INVERSE FOURIER TRANSFORM OF THE HOLOGRAPHIC SPACEPLATE INDEX

The goal of this chapter is to derive the expression for $n(\gamma)$ given in Section 6.1.1. This derivation was originally performed by Michael Weil. We perform the derivation by computing the inverse Fourier

transform of $n(K)$ [Equation (6.9)]. For ease of reference, we rewrite it as

$$\begin{aligned}\tilde{n}(K) &= n_1 \delta(K) + \frac{\pi}{k^2 n_1 d} \sqrt{(kn_1)^2 - \frac{K^2}{4}} \cos \left\{ \mathcal{R} d \sqrt{(kn_1)^2 - \frac{K^2}{4}} \right\} \\ &= n_1 \delta(K) + \frac{\pi n_1}{2a^2 d} \sqrt{a^2 - K^2} \cos \left[b \sqrt{a^2 - K^2} \right],\end{aligned}\tag{F.1}$$

where $a := 2kn_1 \gg 1$, $b := \mathcal{R}d/2$ are defined as they were before. We redefine the Equation (F.1) for discrete spatial frequency bounds:

$$n(K) = \begin{cases} n_1 \delta(K) + \frac{\pi n_1}{2a^2 d} \sqrt{a^2 - K^2} \cos \left[b \sqrt{a^2 - K^2} \right], & -a < K < a \\ 0, & \text{otherwise.} \end{cases}\tag{F.2}$$

Hence, we must compute the Fourier transform of Equation (F.2) for the reduced bounds of $K \in [-a, a]$, since values of K outside this range do not contribute.

$$\begin{aligned}n(y) &= \int_{-a}^a \left(n_1 \delta(K) + \frac{\pi n_1}{2a^2 d} \sqrt{a^2 - K^2} \cos \left(b \sqrt{a^2 - K^2} \right) \right) e^{-i(Ky)} dK \\ &= n_1 + \frac{\pi n_1}{2a^2 d} \int_{-a}^a \sqrt{a^2 - K^2} \cos \left(b \sqrt{a^2 - K^2} \right) e^{-i(Ky)} dK.\end{aligned}$$

Note that due to the even nature of Equation (F.2), its inverse Fourier transform reduces to its Fourier cosine counterpart:

$$n(y) = n_1 + \frac{\pi n_1}{a^2 d} \int_0^a \sqrt{a^2 - K^2} \cos \left(b \sqrt{a^2 - K^2} \right) \cos(Ky) dK$$

At this stage, we have a clearly defined integral that needs to be solved in order to determine $n(\gamma)$. We notice that

$$\frac{d}{db} \left[\sin \left(b\sqrt{a^2 - K^2} \right) \right] = \sqrt{a^2 - K^2} \cos \left(b\sqrt{a^2 - K^2} \right),$$

which allows Appendix F to be further reduced via Leibniz integral rule for constant bounds:

$$\begin{aligned} n(\gamma) &= n_1 + \frac{\pi n_1}{a^2 d} \int_0^a \frac{d}{db} \left[\sin \left(b\sqrt{a^2 - K^2} \right) \right] \cos(K\gamma) dK \\ &= n_1 + \frac{\pi n_1}{a^2 d} \frac{d}{db} \left[\int_0^a \sin \left(b\sqrt{a^2 - K^2} \right) \cos(K\gamma) dK \right] \\ &= n_1 + \frac{\pi n_1}{a^2 d} \frac{df_c(\gamma)}{db}, \end{aligned}$$

for

$$f_c(\gamma) := \int_0^a \sin \left(b\sqrt{a^2 - K^2} \right) \cos(K\gamma) dK.$$

We then perform the substitution $u := \arccos(K/a)$:

$$f_c(\gamma) = \int_0^{\frac{\pi}{2}} a \sin(u) \sin[ab \sin(u)] \cos[ay \cos(u)] du.$$

We extend integration bounds by exploiting the integrand's even nature about $u = \frac{\pi}{2}$:

$$f_c(\gamma) = \frac{1}{2} \int_0^{\pi} a \sin(u) \sin[ab \sin(u)] \cos[ay \cos(u)] du.$$

Then, we apply the Leibniz integral rule again by noticing that

$$\frac{d}{db} \{ \cos [ab \sin(u)] \} = a \sin(u) \sin [ab \sin(u)].$$

We find that

$$\begin{aligned} f_c(\gamma) &= \frac{1}{2} \int_0^\pi \frac{d}{db} \{ \cos [ab \sin(u)] \} \cos [ay \cos(u)] du \\ &= \frac{1}{2} \frac{d}{db} \left\{ \int_0^\pi \cos [ab \sin(u)] \cos [ay \cos(u)] du \right\} \\ &= \frac{1}{2} \frac{d}{db} \left\{ \int_0^\pi \frac{1}{2} \{ \cos[ay \cos(u) - ab \sin(u)] + \cos[ay \cos(u) + ab \sin(u)] \} du \right\}. \end{aligned}$$

This expression can be trigonometrically reduced:

$$\begin{aligned} ay \cos(u) - ab \sin(u) &= \sqrt{(ay)^2 + (ab)^2} \cos \left[u + \arctan \left(\frac{ab}{ay} \right) \right] \\ &= a \sqrt{y^2 + b^2} \cos \left[u + \arctan \left(\frac{b}{y} \right) \right]. \end{aligned}$$

Therefore, we find that

$$\Rightarrow f_c(\gamma) = \frac{1}{4} \frac{d}{db} (I_1 + I_2).$$

where I_1 and I_2 are

$$\begin{aligned} I_1 &= \int_0^\pi \cos \left\{ a \sqrt{b^2 + y^2} \cos \left[u + \arctan \left(\frac{b}{y} \right) \right] \right\} du \\ I_2 &= \int_0^\pi \cos \left\{ a \sqrt{b^2 + y^2} \cos \left[u - \arctan \left(\frac{b}{y} \right) \right] \right\} du. \end{aligned}$$

We solve I_1 using the substitution $v := u + \arctan(b/y)$.

$$\begin{aligned} I_1 &= \int_{\arctan(\frac{b}{y})}^{\pi + \arctan(\frac{b}{y})} \cos \left(a \sqrt{b^2 + y^2} \cos v \right) dv \\ &= \pi J_0 \left(a \sqrt{b^2 + y^2} \right). \end{aligned}$$

I_2 is solved using a similar substitution: $w := u - \arctan(b/y)$.

$$I_2 = \int_{-\arctan(\frac{b}{y})}^{\pi - \arctan(\frac{b}{y})} \cos \left(a \sqrt{b^2 + y^2} \cos v \right) dv \quad (\text{F.3})$$

$$= \pi J_0 \left(a \sqrt{b^2 + y^2} \right) \quad (\text{F.4})$$

Here, we use the zeroth-order Bessel function of the first kind. This higher transcendental function has an integral representation of

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \cos \theta) d\theta.$$

By the fact that the integrand is even about $\theta = \frac{\pi}{2}$, shifting the bounds as seen in I_1 and I_2 has no effect on the overall integral and thus the Bessel function may be used. Now, we solve $f_i(y)$ using our

expressions for I_1 and I_2 :

$$\begin{aligned}
f_c(\gamma) &= \frac{1}{4} \frac{d}{db} \left(2\pi J_0 \left(a\sqrt{b^2 + \gamma^2} \right) \right) \\
&= \frac{1}{2} \frac{d}{db} \left(\pi J_0 \left(a\sqrt{b^2 + \gamma^2} \right) \right) \\
&= \frac{1}{2} \frac{d}{db} \left(\int_0^\pi \cos \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) dv \right) \\
&= \frac{1}{2} \int_0^\pi \frac{d}{db} \left(\cos \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) \right) dv \\
&= \frac{1}{2} \int_0^\pi \left(-\sin \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) \right) \frac{d}{db} \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) dv \\
&= \frac{1}{2} \int_0^\pi \left(-a \cos(v) \sin \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) \right) \frac{d}{db} \left(\sqrt{b^2 + \gamma^2} \right) dv \\
&= \frac{1}{2} \int_0^\pi \left(-a \cos(v) \sin \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) \right) \left(\frac{-b}{\sqrt{b^2 + \gamma^2}} \right) dv \\
&= \frac{ab}{2\sqrt{b^2 + \gamma^2}} \int_0^\pi \cos(v) \sin \left(a\sqrt{b^2 + \gamma^2} \cos(v) \right) dv \\
&= \frac{\pi ab}{2\sqrt{b^2 + \gamma^2}} J_1 \left(a\sqrt{b^2 + \gamma^2} \right).
\end{aligned}$$

Here, the first-order Bessel function of the first kind is used. We now express $n(\gamma)$ in terms of the new expression $f_c(\gamma)$:

$$\begin{aligned}
n(\gamma) &= n_1 + \frac{\pi n_1}{a^2 d} \frac{d}{db} f_c(\gamma) \\
&= n_1 + \frac{\pi n_1}{a^2 d} \frac{d}{db} \left[\frac{\pi ab}{2\sqrt{b^2 + \gamma^2}} J_1 \left(a\sqrt{b^2 + \gamma^2} \right) \right] \\
&= n_1 + \frac{\pi n_1}{a^2 d} [D_1 + D_2],
\end{aligned}$$

where

$$D_1 := \frac{d}{db} \left(\frac{\pi ab}{2\sqrt{b^2 + y^2}} \right) J_1 \left(a\sqrt{b^2 + y^2} \right)$$

$$D_2 := \frac{ab}{2\sqrt{b^2 + y^2}} \frac{d}{db} \left(\pi J_1 \left(a\sqrt{b^2 + y^2} \right) \right).$$

Solving D_1 and D_2 , we find that

$$\begin{aligned} D_1 &= \frac{\pi a}{2} \left(\frac{\left(\frac{d}{db}(b) \right) \sqrt{b^2 + y^2} - b \left(\frac{d}{db}(\sqrt{b^2 + y^2}) \right)}{b^2 + y^2} \right) J_1 \left(a\sqrt{b^2 + y^2} \right) \\ &= \frac{\pi a}{2} \left(\frac{\sqrt{b^2 + y^2} - b^2}{b^2 + y^2} \right) J_1 \left(a\sqrt{b^2 + y^2} \right) \\ &= \frac{\pi a}{2} \left(\frac{y^2}{(b^2 + y^2)^{\frac{3}{2}}} \right) J_1 \left(a\sqrt{b^2 + y^2} \right) \end{aligned}$$

,

and

$$\begin{aligned} D_2 &= \frac{ab}{2\sqrt{b^2 + y^2}} \frac{d}{db} \left(\int_0^\pi \cos(v) \sin \left(a\sqrt{b^2 + y^2} \cos(v) \right) dv \right) \\ &= \frac{ab}{2\sqrt{b^2 + y^2}} \int_0^\pi \cos(v) \frac{d}{db} \left(\sin \left(a\sqrt{b^2 + y^2} \cos(v) \right) \right) dv \\ &= \frac{ab}{2\sqrt{b^2 + y^2}} \int_0^\pi \cos(v) \cos \left(a\sqrt{b^2 + y^2} \cos(v) \right) \frac{d}{db} \left(a\sqrt{b^2 + y^2} \cos(v) \right) dv \\ &= \frac{ab}{2\sqrt{b^2 + y^2}} \int_0^\pi \cos^2(v) \cos \left(a\sqrt{b^2 + y^2} \cos(v) \right) \left(\frac{ab}{\sqrt{b^2 + y^2}} \right) dv. \end{aligned}$$

Using the trigonometric identity $\cos^2(v) = (\cos(2v) + 1)/2$,

$$\begin{aligned}
D_2 &= \frac{ab}{2\sqrt{b^2 + y^2}} \int_0^\pi \left(\frac{1}{2} (\cos(2v) + 1) \right) \cos \left(a\sqrt{b^2 + y^2} \cos(v) \right) \left(\frac{ab}{\sqrt{b^2 + y^2}} \right) dv \\
&= \frac{a^2 b^2}{4(b^2 + y^2)} \left(\int_0^\pi \cos(2v) \cos \left(a\sqrt{b^2 + y^2} \cos(v) \right) dv + \int_0^\pi \cos \left(a\sqrt{b^2 + y^2} \cos(v) \right) dv \right) \\
&= \frac{a^2 b^2}{4(b^2 + y^2)} \left(-\pi J_2 \left(a\sqrt{b^2 + y^2} \right) + \pi J_0 \left(a\sqrt{b^2 + y^2} \right) \right).
\end{aligned}$$

We can now express $n(\gamma)$ in terms of D_1 and D_2 :

$$\begin{aligned}
n(\gamma) &= n_1 + \frac{\pi n_1}{a^2 d} \left\{ \frac{\pi a}{2} \frac{y^2}{(b^2 + y^2)^{\frac{3}{2}}} J_1 \left(a\sqrt{b^2 + y^2} \right) \right. \\
&\quad \left. + \frac{\pi a^2 b^2}{4(b^2 + y^2)} \left[J_0 \left(a\sqrt{b^2 + y^2} \right) - J_2 \left(a\sqrt{b^2 + y^2} \right) \right] \right\}.
\end{aligned}$$

Finally, we can simplify $n(\gamma)$ using a recurrence relation for Bessel functions of the first kind:

$$J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z} J_n(z).$$

We can then write that

$$\begin{aligned}
n(y) &= n_1 + \frac{\pi^2 \eta}{k^2 d} (b^2 + y^2)^{-3/2} \left[y^2 J_1 \left(a \sqrt{b^2 + y^2} \right) + \frac{ab^2 \sqrt{b^2 + y^2}}{2} J_0 \left(a \sqrt{b^2 + y^2} \right) \right] \\
&\quad - \frac{\pi^2 \eta}{k^2 d} (b^2 + y^2)^{-3/2} \left[\frac{ab^2 \sqrt{b^2 + y^2}}{2} \left(\frac{2}{a \sqrt{b^2 + y^2}} J_1 \left(a \sqrt{b^2 + y^2} \right) - J_0 \left(a \sqrt{b^2 + y^2} \right) \right) \right] \\
&= n_1 + \frac{\pi^2 \eta}{k^2 d} (b^2 + y^2)^{-3/2} \left[y^2 J_1 \left(a \sqrt{b^2 + y^2} \right) + ab^2 \sqrt{b^2 + y^2} J_0 \left(a \sqrt{b^2 + y^2} \right) - b^2 J_1 \left(a \sqrt{b^2 + y^2} \right) \right] \\
&= n_1 + \frac{\pi^2 \eta}{k^2 d} (b^2 + y^2)^{-3/2} \left[(y^2 - b^2) J_1 \left(a \sqrt{b^2 + y^2} \right) + ab^2 \sqrt{b^2 + y^2} J_0 \left(a \sqrt{b^2 + y^2} \right) \right].
\end{aligned}$$

This is our final expression for the spatial index distribution. Recall that where $a = 2kn_1$ and $b = \mathcal{R}d/2n_1$.

G

THE EFFECT OF RELATIVE GRATING PHASE SHIFTS IN MULTIPLEXED HOLOGRAMS

In this chapter, we derive the effect of relative phase shifts between gratings in multiplexed holograms. This derivation is adapted from work originally done by Michael Weil. We assume that $\mathbf{k}$ and $\mathbf{K}$ are

one-to-one, and we first express the wave equation in the form,

$$\nabla^2 \tilde{\mathbf{E}} = -\omega^2 \mu_0 \epsilon_0 \epsilon \tilde{\mathbf{E}} \quad (\text{G.1})$$

With Equation (G.1) and a form of the spatially varying relative electric permittivity, ϵ , we can derive the coupled wave equations that form the basis of Kogelnik's coupled wave theory (CWT) for thick holographic gratings [2]. The phase grating is modeled as a sinusoid with grating vector, $\mathbf{K}$:

$$\epsilon = \epsilon_1 + \Delta\epsilon \cos(\mathbf{K} \cdot \mathbf{r}),$$

where $\epsilon_1 = n_1^2$ is the average relative permittivity, n_1 is the average refractive index of the grating, and $\Delta\epsilon$ is the modulation amplitude or permittivity contrast of the grating. The index is

$$\begin{aligned} n^2 &= [n_1 + \Delta n \cos(\mathbf{K} \cdot \mathbf{r})]^2 \\ &\simeq n_1^2 + 2n_1 \Delta n \cos(\mathbf{K} \cdot \mathbf{r}), \end{aligned}$$

where we have assumed that $\Delta n \ll n_1$ and $\Delta\epsilon = 2n_1 \Delta n$. However, unlike in Kogelnik's CWT, we assume that the grating takes an arbitrary phase, ϕ :

$$\epsilon = \epsilon_1 + \Delta\epsilon \cos(\mathbf{K} \cdot \mathbf{r} + \phi).$$

Inserting this into Equation (G.1) we find that

$$\begin{aligned}
\nabla^2 \tilde{\mathbf{E}} &= -\omega^2 \mu_0 \epsilon_0 (\epsilon_1 + \Delta \epsilon \cos(\mathbf{K} \cdot \mathbf{r} + \phi)) \tilde{\mathbf{E}} \\
&= -\omega^2 \mu_0 \epsilon_0 \epsilon_1 \tilde{\mathbf{E}} - \omega^2 \mu_0 \epsilon_0 \Delta \epsilon \cos(\mathbf{K} \cdot \mathbf{r} + \phi) \tilde{\mathbf{E}} \\
&= -\omega^2 \mu_0 \epsilon_0 \epsilon_1 \tilde{\mathbf{E}} - \omega^2 \mu_0 \epsilon_0 \Delta \epsilon \left(\frac{e^{i(\mathbf{K} \cdot \mathbf{r} + \phi)} + e^{-i(\mathbf{K} \cdot \mathbf{r} + \phi)}}{2} \right) \tilde{\mathbf{E}} \\
&= -\omega^2 \mu_0 \epsilon_0 \epsilon_1 \tilde{\mathbf{E}} - \frac{1}{2} \omega^2 \mu_0 \epsilon_0 \Delta \epsilon e^{i(\mathbf{K} \cdot \mathbf{r} + \phi)} \tilde{\mathbf{E}} - \frac{1}{2} \omega^2 \mu_0 \epsilon_0 \Delta \epsilon e^{-i(\mathbf{K} \cdot \mathbf{r} + \phi)} \tilde{\mathbf{E}} \\
&= -k^2 \epsilon_1 \tilde{\mathbf{E}} - \frac{1}{2} k^2 \Delta \epsilon e^{i(\mathbf{K} \cdot \mathbf{r} + \phi)} \tilde{\mathbf{E}} - \frac{1}{2} k^2 \Delta \epsilon e^{-i(\mathbf{K} \cdot \mathbf{r} + \phi)} \tilde{\mathbf{E}} \\
&= -k^2 \epsilon_1 \tilde{\mathbf{E}} - 2k \epsilon_1^{\frac{1}{2}} \left(\frac{1}{2} \left(\frac{k \Delta \epsilon e^{i\phi}}{2 \epsilon_1^{\frac{1}{2}}} \right) \right) e^{i(\mathbf{K} \cdot \mathbf{r})} \tilde{\mathbf{E}} - 2k \epsilon_1^{\frac{1}{2}} \left(\frac{1}{2} \left(\frac{k \Delta \epsilon e^{-i\phi}}{2 \epsilon_1^{\frac{1}{2}}} \right) \right) e^{-i(\mathbf{K} \cdot \mathbf{r})} \tilde{\mathbf{E}} \\
&= -\beta^2 \tilde{\mathbf{E}} - 2\beta \chi^+ e^{i(\mathbf{K} \cdot \mathbf{r})} \tilde{\mathbf{E}} - 2\beta \chi^- e^{-i(\mathbf{K} \cdot \mathbf{r})} \tilde{\mathbf{E}},
\end{aligned}$$

where we have made the following substitutions:

$$\begin{aligned}
k^2 &= \omega^2 \mu_0 \epsilon_0 = \frac{\omega^2}{c^2} = \frac{4\pi^2}{\lambda_{vac}^2}; \\
\beta &= k\epsilon_1^{\frac{1}{2}} = kn_1; \\
\chi^+ &= \frac{1}{2} \left(\frac{k\Delta\epsilon e^{i\phi}}{2\epsilon_1^{\frac{1}{2}}} \right) \\
&= \frac{1}{2} \left(\frac{2\pi}{2\lambda_{air}} \frac{\Delta\epsilon}{\epsilon_1^{\frac{1}{2}}} e^{i\phi} \right) \\
&= \frac{1}{2} \left(\frac{\pi}{\lambda_{air}} \frac{2\Delta n n_1}{n_1} e^{i\phi} \right) \\
&= \frac{\pi\Delta n}{\lambda_{air}} e^{i\phi}; \\
\chi^- &= \frac{\pi\Delta n}{\lambda_{air}} e^{-i\phi}.
\end{aligned}$$

It is evident that by adding a relative phase term to the grating, we have transformed Kogelnik's coupling constant into two new coupling constants, χ^+ and χ^- . We will assume the electric field inside the photopolymer is the superposition of two waves with complex amplitudes $R(z)$ and $S(z)$:

$$\tilde{\mathbf{E}} = R(z)e^{-i(\mathbf{k}_r \cdot \mathbf{r})} \hat{i} + S(z)e^{-i(\mathbf{k}_s \cdot \mathbf{r})} \hat{i} = E_R \hat{i} + E_S \hat{i}$$

with

$$\mathbf{k}_r \cdot \mathbf{r} = |\mathbf{k}_r| \sin(\psi) y + |\mathbf{k}_r| \cos(\psi) z = \beta (\sin(\psi) y + \cos(\psi) z), \text{ and } \mathbf{k}_s \cdot \mathbf{r} = k_{s_y} y + k_{s_z} z.$$

Now that we have simplified the right-hand side of Equation (G.1), let us examine the left-hand side.

$$\begin{aligned}\nabla^2 \tilde{\mathbf{E}} &= (\nabla^2 \tilde{E}_x, \nabla^2 \tilde{E}_y, \nabla^2 \tilde{E}_z) = (\nabla^2 \tilde{E}_x, 0, 0) \\ &= \nabla^2 (E_R + E_S)\end{aligned}$$

First, we evaluate the derivative operators on E_R . Firstly,

$$\frac{\partial^2}{\partial x^2} E_R = 0.$$

Then,

$$\begin{aligned}\frac{\partial^2}{\partial y^2} E_R &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} R(z) e^{-i\beta(y \sin(\psi) + z \cos(\psi))} \right) \\ &= \frac{\partial}{\partial y} \left(-i\beta R(z) \sin(\psi) e^{-i\beta(y \sin(\psi) + z \cos(\psi))} \right) \\ &= -\beta^2 R(z) \sin^2(\psi) e^{-i\beta(y \sin(\psi) + z \cos(\psi))}.\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} E_R &= \frac{\partial}{\partial z} \left(\frac{\partial R}{\partial z} e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} + R(z) \frac{\partial}{\partial z} \left(e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} \right) \right) \\
&= \frac{\partial}{\partial z} \left(\frac{\partial R}{\partial z} e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} - \imath\beta R(z) \cos(\psi) e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} \right) \\
&= \frac{\partial^2 R}{\partial z^2} e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} + \left(\frac{\partial R}{\partial z} \right) \frac{\partial}{\partial z} \left(e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} \right) \\
&\quad - \imath\beta \frac{\partial R}{\partial z} \cos(\psi) e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} - \imath\beta R(z) \cos(\psi) \frac{\partial}{\partial z} \left(e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} \right) \\
&= \frac{\partial^2 R}{\partial z^2} e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} - 2\imath\beta \cos(\psi) \left(\frac{\partial R}{\partial z} \right) e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))} \\
&\quad - \beta^2 R(z) \cos^2(\psi) e^{-\imath\beta(y \sin(\psi) + z \cos(\psi))}.
\end{aligned}$$

Similarly, for E_S , we find that

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} E_S &= 0, \\
\frac{\partial^2}{\partial y^2} E_S &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} S(z) e^{-\imath(k_{S_y} y + k_{S_z} z)} \right) \\
&= \frac{\partial}{\partial y} \left(-\imath S(z) k_{S_y} e^{-\imath(k_{S_y} y + k_{S_z} z)} \right) \\
&= -k_{S_y}^2 S(z) e^{-\imath(k_{S_y} y + k_{S_z} z)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} E_S &= \frac{\partial}{\partial z} \left(\frac{\partial S}{\partial z} e^{-i(k_{S_y} y + k_{S_z} z)} + S(z) \frac{\partial}{\partial z} \left(e^{-i(k_{S_y} y + k_{S_z} z)} \right) \right) \\
&= \frac{\partial}{\partial z} \left(\frac{\partial S}{\partial z} e^{-i(k_{S_y} y + k_{S_z} z)} - i S(z) k_{S_z} e^{-i(k_{S_y} y + k_{S_z} z)} \right) \\
&= \frac{\partial^2 S}{\partial z^2} e^{-i(k_{S_y} y + k_{S_z} z)} + \left(\frac{\partial S}{\partial z} \right) \frac{\partial}{\partial z} \left(e^{-i(k_{S_y} y + k_{S_z} z)} \right) \\
&\quad - i k_{S_z} \frac{\partial S}{\partial z} e^{-i(k_{S_y} y + k_{S_z} z)} - i S(z) k_{S_z} \frac{\partial}{\partial z} \left(e^{-i(k_{S_y} y + k_{S_z} z)} \right) \\
&= \frac{\partial^2 S}{\partial z^2} e^{-i(k_{S_y} y + k_{S_z} z)} - 2i k_{S_z} \left(\frac{\partial S}{\partial z} \right) e^{-i(k_{S_y} y + k_{S_z} z)} \\
&\quad - S(z) k_{S_z}^2 e^{-i(k_{S_y} y + k_{S_z} z)}.
\end{aligned}$$

Putting this all together,

$$\begin{aligned}
\nabla^2 (E_R + E_S) &= \left[\frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} - \beta^2 \cos^2(\psi) R(z) - \beta^2 \sin^2(\psi) R(z) \right] e^{-i(\mathbf{k}_R \cdot \mathbf{r})} \\
&\quad + \left[\frac{\partial^2 S}{\partial z^2} - 2i k_{S_z} \frac{\partial S}{\partial z} - k_{S_z}^2 S(z) \right] e^{-i(\mathbf{k}_S \cdot \mathbf{r})}
\end{aligned}$$

Using these results, we find that

$$\begin{aligned}
0 &= \left[\frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} - \beta^2 \cos^2(\psi) R(z) - \beta^2 \sin^2(\psi) R(z) \right] e^{-i(\mathbf{k}_R \cdot \mathbf{r})} \\
&\quad + \left[\frac{\partial^2 S}{\partial z^2} - 2i k_{S_z} \frac{\partial S}{\partial z} - k_{S_z}^2 S(z) \right] e^{-i(\mathbf{k}_S \cdot \mathbf{r})} \\
&\quad + [\beta^2 + 2\beta\chi^+ e^{i(\mathbf{K} \cdot \mathbf{r})} + 2\beta\chi^- e^{-i(\mathbf{K} \cdot \mathbf{r})}] (R(z) e^{-i(\mathbf{k}_R \cdot \mathbf{r})} + S(z) e^{-i(\mathbf{k}_S \cdot \mathbf{r})}) \quad (\text{G.2})
\end{aligned}$$

However, the grating vector, $\mathbf{K}$, and both wave vectors, $\mathbf{k}_R$ and $\mathbf{k}_S$, are all related by the Bragg condition

$$\mathbf{k}_S = \mathbf{k}_R - \mathbf{K}, \quad (\text{G.3})$$

which allows us to reduce Equation (G.2) in terms of the two wave vectors present within the grating

$$\begin{aligned} 0 &= \left[\frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} - \beta^2 \cos^2(\psi) R(z) - \beta^2 \sin^2(\psi) R(z) \right] e^{-i(\mathbf{k}_R \cdot \mathbf{r})} \\ &\quad + \left[\frac{\partial^2 S}{\partial z^2} - 2ik_{S_z} \frac{\partial S}{\partial z} - k_S^2 S(z) \right] e^{-i(\mathbf{k}_S \cdot \mathbf{r})} \\ &\quad + \left[\beta^2 + 2\beta\chi^+ e^{i((\mathbf{k}_R - \mathbf{k}_S) \cdot \mathbf{r})} + 2\beta\chi^- e^{-i((\mathbf{k}_R - \mathbf{k}_S) \cdot \mathbf{r})} \right] (R(z)e^{-i(\mathbf{k}_R \cdot \mathbf{r})} + S(z)e^{-i(\mathbf{k}_S \cdot \mathbf{r})}) \\ &= \left[\frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} \right. \\ &\quad \left. - \beta^2 \cos^2(\psi) R(z) - \beta^2 \sin^2(\psi) R(z) + \beta^2 R(z) + 2\beta\chi^- S(z) \right] e^{-i(\mathbf{k}_R \cdot \mathbf{r})} \\ &\quad + \left[\frac{\partial^2 S}{\partial z^2} - 2ik_{S_z} \frac{\partial S}{\partial z} - k_S^2 S(z) + \beta^2 S(z) + 2\beta\chi^+ R(z) \right] e^{-i(\mathbf{k}_S \cdot \mathbf{r})} \\ &\quad + 2\beta\chi^+ e^{-i((2\mathbf{k}_S - \mathbf{k}_R) \cdot \mathbf{r})} S(z) + 2\beta\chi^- e^{-i((2\mathbf{k}_R - \mathbf{k}_S) \cdot \mathbf{r})} R(z) \\ &= \left[\frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} + 2\beta\chi^- S(z) \right] e^{-i(\mathbf{k}_R \cdot \mathbf{r})} \\ &\quad + \left[\frac{\partial^2 S}{\partial z^2} - 2ik_{S_z} \frac{\partial S}{\partial z} - k_S^2 S(z) + \beta^2 S(z) + 2\beta\chi^+ R(z) \right] e^{-i(\mathbf{k}_S \cdot \mathbf{r})} \\ &\quad + 2\beta\chi^+ e^{-i((2\mathbf{k}_S - \mathbf{k}_R) \cdot \mathbf{r})} S(z) + 2\beta\chi^- e^{-i((2\mathbf{k}_R - \mathbf{k}_S) \cdot \mathbf{r})} R(z) \end{aligned} \quad (\text{G.4})$$

The final two terms in Equation (G.4) are zero as they represent waves which do not satisfy the Bragg condition, (i.e., they do not follow the relation in Equation (G.3)). From this, we arrive at a set of coupled

differential equations,

$$0 = \frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial R}{\partial z} + 2\beta\chi^- S(z) \quad (\text{G.5})$$

$$0 = \frac{\partial^2 S}{\partial z^2} - 2ik_{S_z} \frac{\partial S}{\partial z} - k_S^2 S(z) + \beta^2 S(z) + 2\beta\chi^+ R(z) \quad (\text{G.6})$$

In order to follow the notation commonly used in volume holography, we redefine Equation (G.6).

First we re-express the Bragg condition

$$\begin{aligned} \beta^2 - k_S^2 &= \beta^2 - |\mathbf{k}_R - \mathbf{K}|^2 \\ &= \beta^2 - k_R^2 + 2k_R K \cos\left(\frac{\pi}{2} - \psi'\right) - K^2 \end{aligned} \quad (\text{G.7})$$

$$= 2k_R K \sin(\psi') - K^2 \quad (\text{G.8})$$

where $\psi' = \theta_0 + \delta$, where θ_0 is the Bragg angle of the grating, and δ is a small angular deviation. Assuming δ is sufficiently small, we can approximate $\sin(\psi')$ as:

$$\begin{aligned} \sin(\psi') &= \sin(\theta_0) \cos(\delta) + \cos(\theta_0) \sin(\delta) \\ &\simeq \sin(\theta_0) + \delta \cos(\theta_0), \end{aligned}$$

allowing us to approximate Equation (G.7) as

$$\begin{aligned}
\beta^2 - \mathbf{k}_S^2 &\simeq 2k_R K \left[\frac{K}{2\beta} + \delta \cos(\theta_0) \right] - K^2 \\
&= \frac{k_R K^2}{\beta} + 2k_R K \delta \cos(\theta_0) - K^2 \\
&= 2k_R K \delta \cos(\theta_0) \\
&= 2k_R (2\beta \sin(\theta_0)) \delta \cos(\theta_0) \\
&= 2\beta^2 \delta \sin(2\theta_0) \\
&= 2\beta \Gamma,
\end{aligned}$$

where

$$\Gamma := \beta \delta \sin(2\theta_0).$$

Hence, the coupled differential equations become

$$0 = \frac{\partial^2 R}{\partial z^2} - 2i\beta \cos(\psi) \frac{\partial}{\partial z} R(z) + 2\beta \chi^- S(z); \quad (\text{G.9})$$

$$0 = \frac{\partial^2 S}{\partial z^2} - 2ik_{S_z} \frac{\partial S}{\partial z} + 2\beta \Gamma S(z) + 2\beta \chi^+ R(z). \quad (\text{G.10})$$

We now assume that energy is absorbed slowly by the grating, allowing us to disregard the terms with a second-order derivative in z . A mathematical justification is made shortly, but for now we will follow

this assumption and simplify Equation (G.9) and Equation (G.10):

$$\cos(\psi) \frac{\partial R}{\partial z} = -\imath \chi^- S(z); \quad (\text{G.11})$$

$$\frac{k_{S_z}}{\beta} \frac{\partial S}{\partial z} + \imath \Gamma S(z) = -\imath \chi^+ R(z). \quad (\text{G.12})$$

Solving this set of coupled wave equations is quite straightforward. We begin by differentiating Equation (G.11)

$$\cos(\psi) \frac{\partial^2 R}{\partial z^2} = -\imath \chi^- \frac{\partial S}{\partial z} \Rightarrow \frac{\partial S}{\partial z} = \frac{\imath}{\chi^-} \cos(\psi) \frac{\partial^2 R}{\partial z^2}.$$

We substitute this into and rewrite Equation (G.12) entirely in terms of the reference wave amplitude, $R(z)$:

$$-\imath \chi^+ R(z) = \frac{k_{S_z}}{\beta} \left(\frac{\imath}{\chi^-} \cos(\psi) \frac{\partial^2 R}{\partial z^2} \right) + \imath \Gamma \left(\frac{\imath}{\chi^-} \cos(\psi) \frac{\partial R}{\partial z} \right).$$

Then,

$$\begin{aligned} 0 &= \left(\frac{\imath k_{S_z}}{\beta \chi^-} \cos(\psi) \right) \frac{\partial^2 R}{\partial z^2} - \left(\frac{\Gamma}{\chi^-} \cos(\psi) \right) \frac{\partial R}{\partial z} + (\imath \chi^+) R(z) \\ 0 &= \left(\frac{\imath k_{S_z}}{\beta} \cos(\psi) \right) \frac{\partial^2 R}{\partial z^2} - (\Gamma \cos(\psi)) \frac{\partial R}{\partial z} + (\imath \chi^- \chi^+) R(z) \\ 0 &= \frac{\partial^2 R}{\partial z^2} + \left(\frac{\imath \beta \Gamma}{k_{S_z}} \right) \frac{\partial R}{\partial z} + \left(\frac{\beta}{k_{S_z} \cos(\psi)} \chi^- \chi^+ \right) R(z) \end{aligned} \quad (\text{G.13})$$

This is now in the standard form of a second-order homogeneous equation with constant coefficients.

For this reason, we make the ansatz: $R(z) = \exp(\omega z)$,

$$\begin{aligned}
 0 &= \omega^2 \exp(\omega z) + \left(\frac{i\beta\Gamma}{k_{S_z}} \right) \omega \exp(\omega z) + \left(\frac{\beta}{k_{S_z} \cos(\psi)} \chi^- \chi^+ \right) \exp(\omega z) \\
 \Rightarrow 0 &= \omega^2 + u\omega + v,
 \end{aligned} \tag{G.14}$$

where

$$u = \frac{i\beta\Gamma}{k_{S_z}} \quad \text{and} \quad v = \frac{\beta}{k_{S_z} \cos(\psi)} \chi^- \chi^+.$$

Equation (G.14) is the characteristic equation of Equation (G.13), whose solutions are given by

$$\omega_{1,2} = -\frac{1}{2} \left(\frac{i\beta\Gamma}{k_{S_z}} \right) \pm \frac{1}{2} \sqrt{\left(\frac{i\beta\Gamma}{k_{S_z}} \right)^2 - 4 \left(\frac{\beta \chi^- \chi^+}{k_{S_z} \cos(\psi)} \right)}$$

The general solution of Equation (G.13) is

$$R(z) = R_1 \exp(\omega_1 z) + R_2 \exp(\omega_2 z)$$

where the coefficients, R_1 and R_2 , are determined by the boundary conditions of the holography setup. Similarly, the general solution of $S(z)$ is of the same form:

$$\begin{aligned}
-i\chi^- S(z) &= \cos(\psi) \frac{\partial}{\partial z} (R_1 e^{\omega_1 z} + R_2 e^{\omega_2 z}) \\
-i\chi^- S(z) &= (R_1 \omega_1 e^{\omega_1 z} + R_2 \omega_2 e^{\omega_2 z}) \cos(\psi) \\
S(z) &= \frac{i}{\chi^-} \cos(\psi) (R_1 \omega_1 e^{\omega_1 z} + R_2 \omega_2 e^{\omega_2 z}) \\
S(z) &= S_1 e^{\omega_1 z} + S_2 e^{\omega_2 z}.
\end{aligned}$$

At $z = 0$, the only non-zero contribution to the field is from the reference wave, i.e., $R(0) = R_1 + R_2 = 1$ and $S(0) = S_1 + S_2 = 0$. From the latter condition, we find that $S_1 = -S_2$, allowing us to solve for S_1 and S_2 .

$$\begin{aligned}
\frac{k_{S_z}}{\beta} \frac{\partial}{\partial z} \Big|_{z=0} S(z) + i\Gamma S(0) &= -i\chi^+ R(0) \\
\frac{k_{S_z}}{\beta} \frac{\partial}{\partial z} \Big|_{z=0} (S_1 e^{\omega_1 z} - S_1 e^{\omega_2 z}) + i\Gamma (S_1 e^{\omega_1(0)} - S_1 e^{\omega_2(0)}) &= -i\chi^+ \\
\frac{k_{S_z}}{\beta} (\omega_1 S_1 - \omega_2 S_1) &= -i\chi^+.
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_1 &= \frac{-i\chi^+ \beta}{k_{S_z} (\omega_1 - \omega_2)} \\
S_2 &= \frac{i\chi^+ \beta}{k_{S_z} (\omega_1 - \omega_2)}.
\end{aligned}$$

We now have a general solution for the diffracted signal wave as it propagates through the holographic grating,

$$S(z) = \frac{-\imath\chi^+}{\frac{k_{S_z}}{\beta}(\omega_1 - \omega_2)} (e^{\omega_1 z} - e^{\omega_2 z}).$$

Since we are interested in the field at a distance d , we set $z = d$:

$$\begin{aligned} S(d) &= \frac{-\imath\chi^+ \exp\left(\frac{-\imath\beta\Gamma d}{2k_{S_z}}\right) \left(\exp\left\{\imath\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right\} - \exp\left\{-\imath\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right\} \right)}{\frac{k_{S_z}}{\beta} \left(\sqrt{\left(\frac{\imath\beta\Gamma}{k_{S_z}}\right)^2 - 4\left(\frac{\beta\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)} \right)} \\ &= \frac{-\imath\chi^+ \exp\left(\frac{-\imath\beta\Gamma d}{2k_{S_z}}\right) \left(\exp\left\{\imath\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right\} - \exp\left\{-\imath\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right\} \right)}{\frac{2\imath k_{S_z}}{\beta d} \left(\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)} \right)} \\ &= \frac{-\imath\chi^+ \exp\left(\frac{-\imath\beta\Gamma d}{2k_{S_z}}\right) 2\imath \sin\left(\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right)}{\frac{2\imath k_{S_z}}{\beta d} \left(\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)} \right)} \\ &= -\imath \frac{\chi^+ \beta d}{k_{S_z}} \left(\frac{-\imath\beta\Gamma d}{2k_{S_z}} \right) \frac{\sin\left(\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}\right)}{\sqrt{\left(\frac{\beta\Gamma d}{2k_{S_z}}\right)^2 + \left(\frac{\beta d^2\chi^-\chi^+}{k_{S_z}\cos(\psi)}\right)}} \\ &= -\imath \frac{\chi^+ \beta d}{k_{S_z}} (-\imath\xi) \frac{\sin\left(\sqrt{\xi^2 + \nu^2}\right)}{\sqrt{\xi^2 + \nu^2}} \\ &= -\imath \frac{\chi^+ \beta d}{\nu k_{S_z}} (-\imath\xi) \frac{\sin\left(\sqrt{\xi^2 + \nu^2}\right)}{\sqrt{1 + \frac{\xi^2}{\nu^2}}}, \end{aligned}$$

where

$$\xi = \frac{\beta \Gamma d}{2k_{S_z}};$$

$$\nu = \sqrt{\frac{\beta d^2 \chi^- \chi^+}{k_{S_z} \cos(\psi)}}.$$

To understand the effect of the relative phase term, ϕ , we rewrite the result to find that

$$\begin{aligned} \frac{\chi^+ \beta d}{\nu k_{S_z}} &= \frac{\chi^+ \beta d}{k_{S_z}} \sqrt{\frac{k_{S_z} \cos(\psi)}{\beta d^2 \chi^- \chi^+}} \\ &= \sqrt{\frac{\beta \cos(\psi)}{k_{S_z}}} \frac{\pi \Delta n}{\lambda_{air}} e^{i\phi} \frac{1}{\sqrt{\frac{\pi^2 \Delta n^2}{\lambda_{air}^2} e^{-i\phi} e^{i\phi}}} \\ &= \sqrt{\frac{\beta \cos(\psi)}{k_{S_z}}} e^{i\phi}. \end{aligned}$$

This means the diffracted field at $z = d$ is given by the expression

$$S(d) = -i \sqrt{\frac{\beta \cos(\psi)}{k_{S_z}}} e^{i\phi} e^{-i\xi} \frac{\sin\left(\sqrt{\xi^2 + \nu^2}\right)}{\sqrt{1 + \frac{\xi^2}{\nu^2}}}.$$

Hence, the relative spatial phase shift that was added to the holographic grating appears as a phase shift in the output diffraction field. Hence, we can control the output field via this relative phase term. For an in-plane grating, the diffraction efficiency is not influenced by this relative phase. The idea of adding a relative phase term between an absorption grating and a phase grating had been considered by E. Guibelalde, in 1983, whose results agree with our findings above [91].

H

NON-PHASE MATCHED SPONTANEOUS PARAMETRIC DOWN-CONVERSION

We made several characterizations of SPDC in several different media, including $\text{LiNbO}_3\text{:Fe}$, $\text{LiNbO}_3\text{:FS}$, $\text{LiNbO}_3\text{:Si}$, and GaAs. Detailed descriptions of all samples are given in Appendix H.1, and descriptions of all experiments are given in Appendix H.2. First, we characterized the second-order

temporal correlation (Appendix H.3) and the polarization tomography of each crystal (Appendix H.4). For the silicon-mounted lithium niobate samples, we characterized the effect of the pump polarization. Further, we measured the spectrum of SPDC in the samples using fiber spectroscopy, and, finally, we characterized the strength of the second-order nonlinearities via SHG tomography (Appendix H.6).

H.1 NONLINEAR MATERIAL DESCRIPTIONS

We characterized several samples with different geometries and parameters (see Figure H.1). The x -cut $\text{LiNbO}_3\text{:Fe}$ was $(0.30 \pm 0.03)\text{mm}$ thick and had a 76.2 mm diameter. It was iron-doped at a concentration of 0.2 mol% and double-side polished. The wafer was diced into samples with approximately $1 \times 1 \text{ cm}^2$ surface areas. The x -cut $\text{LiNbO}_3\text{:Si}$ wafer had a thickness of $(10.0 \pm 0.5)\text{mm}$ and a thickness

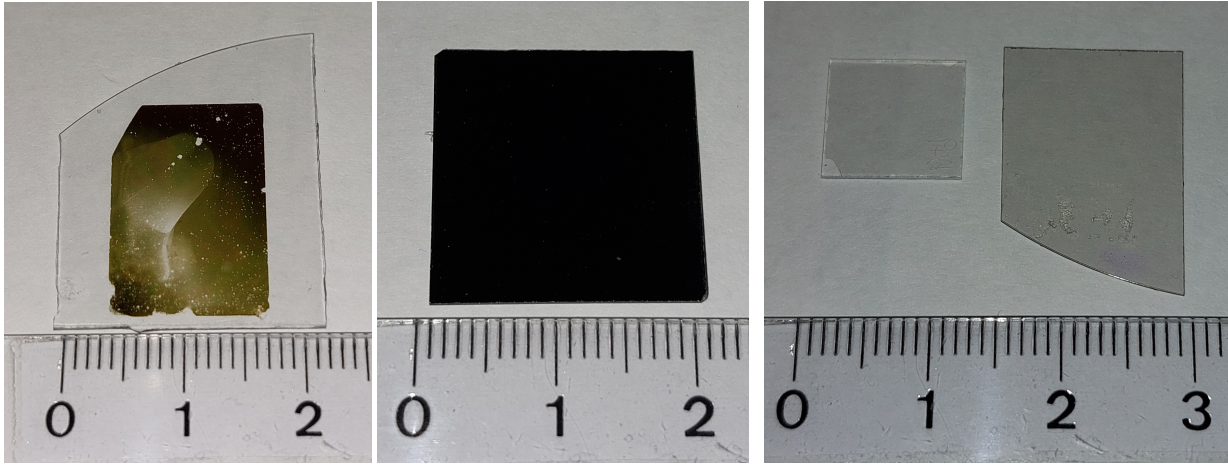


Figure H.1: Images of the GaAs (left), $\text{LiNbO}_3\text{:Si}$ (middle-left), $\text{LiNbO}_3\text{:FS}$ (middle-right), and $\text{LiNbO}_3\text{:Fe}$ (right) samples. The GaAs sample was tested using the upper right hand corner of the sample (the region of highest opacity). The $\text{LiNbO}_3\text{:Si}$ sample is highly reflective at visible wavelengths.

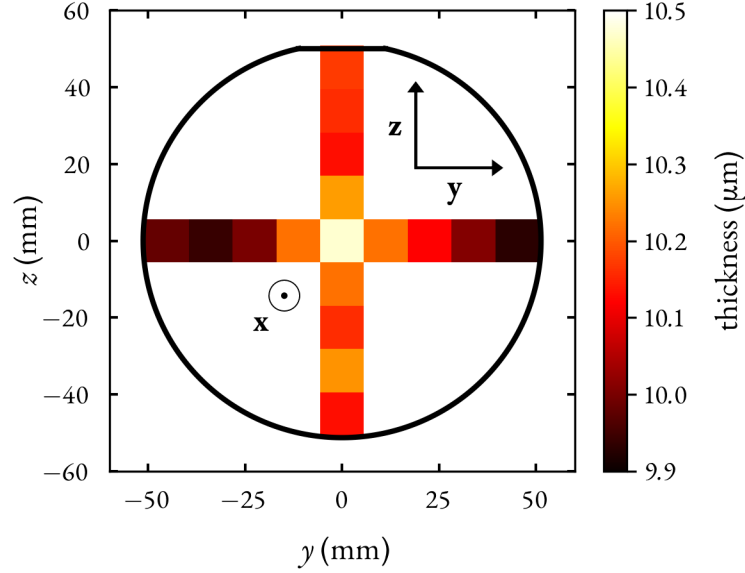


Figure H.2: Measured thickness profile of the $\text{LiNbO}_3\text{:Si}$ sample.

of 101.6 mm. The measured thickness profile of the wafer is given in Figure H.2. The undoped lithium niobate was mounted on a 0.525 mm-thick silicon substrate. It was only polished on the lithium niobate side. The wafer was diced into samples with approximately $1 \times 1 \text{ cm}^2$ surface areas. The final lithium niobate sample - the x -cut $\text{LiNbO}_3\text{:FS}$ sample - had a thickness of $10 \mu\text{m}$ and approximately $1 \times 0.8 \text{ cm}^2$ surface area. The undoped lithium niobate was mounted on a 0.525 mm-thick fused silica substrate. It was polished on both sides of the sample. Normally these samples are doped with magnesium oxide to reduce the effects of photorefractivity, but that was not our aim here. Finally, we tested the a sample of gallium arsenide with a III crystal cut. It had a thickness of 250 nm.

The nonlinear coefficients for each material are given in Section 7.1.1. The coefficient for doped lithium niobate are expected to be approximately the same as those for undoped lithium niobate.

H.2 EXPERIMENTS DONE TO CHARACTERIZE SPDC IN THIN CRYSTALS

We use several setups for different measurements of each sample. The details of each setup are given in the following sections prior to the experimental results, however, most are variations of the HBT experiment. An example of such a setup is given in Figure H.3, where biphotons are produced by focusing the pump into the sample. The biphotons are collimated and filtered before beam splitting and correlating them using two single photon detectors and a correlator. In the figure, we have also added polarization control to permit biphoton polarization tomography. For all correlation measurements presented in this section, we use a CW laser with center wavelength of $\lambda = 788$ nm with a 2 nm FWHM bandwidth, which produces up to 100 mW of power (model LP785-SF100 from Thorlabs). The laser has a $1/e^2$ radius in air of 1.1 mm. It is focused to a spot with a radius of 5 μm using a parabolic mirror with a 15 mm focal length.

The coupling and collimating lenses used depend on the sample, and are selected to optimize mode coupling into the single-mode fiber (SMF). For our experiments however, we found that a collimating lens with a 30 mm focal length worked best. For coupling, we used lenses with either 60 mm or 100 mm focal lengths, depending on the sample. To optimize the setup, we calculated the largest optimal spatial Schmidt mode and optimized the setup to couple that mode to the SMF.

We use two SNSPDs (NW1fC900-1550 detectors from Photon Spot) to detect the biphotons. The detectors are fiber-coupled to SMF which specify detection efficiencies above 80% for 900 $\sim$ 1550 nm light (the actual detection efficiency is narrower and less efficiency because of dispersive loss in the SMF).

H.2. EXPERIMENTS DONE TO CHARACTERIZE SPDC IN THIN CRYSTALS

The detectors have dark counts at $100 \sim 300$ Hz and jitters < 100 ps at FWHM. To correlate the detections, we use a Time Tagger 20 correlator device. Further, to filter out the pump and photoluminescence, we use several 1000 nm LP filters and a 1400 nm LP filter.

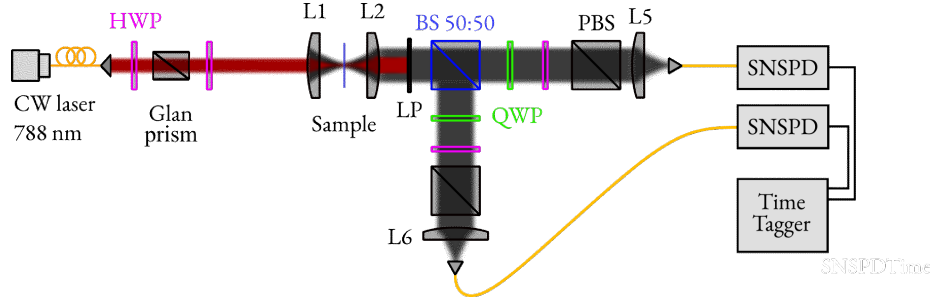


Figure H.3: Polarization tomography HBT setup used to characterize the polarization state of SPDC. A CW ($5 \sim 80$ mW) laser is polarized, power regulated, and focused into the sample using lens one (L1). Biphotons are produced in the sample and collimated using lens two (L2), then a series of LP and BP filters are used to filter out the pump. The photons are split using a 50:50 BS, and the polarization of the photons can be filtered using a GP preceded by a QWP and HWP. The photons are measured using SNSPD and correlated using a Time Tagger. This setup can measure the polarization state of the biphotons. The polarization state is typically represented by a projection onto the Poincaré sphere.

H.2.1 SECOND ORDER TEMPORAL CORRELATION FROM SPDC

For each sample, we measure the second-order temporal correlation, given by

$$g^{(2)}(\tau) = \frac{R_c^{(2)}(\tau)}{R_1 R_2 T}, \quad (\text{H.1})$$

where $R_c^{(2)}$ is the rate of coincidences between the two detectors, R_j is the rate of detections at the j -th detector, and T is the time window. The detected rate of coincidences is a function of both real

H.2. EXPERIMENTS DONE TO CHARACTERIZE SPDC IN THIN CRYSTALS

(biphoton) and accidental (dark or photoluminescent) coincidences:

$$R_c^{(2)} = R^{(2)} + R_A^{(2)}.$$

The rate of accidentals is given by $R_A^{(2)} = R_1 R_2 T$. We can assume that the detection rate R_j will scale linearly with pump power, P_p . Therefore, the second-order correlation will inversely scale with pump power. Measuring this dependence provides good evidence for the generation of a second-order state.

$$g^{(2)}(0) = \frac{1}{P_p} + 1. \quad (\text{H.2})$$

The setup used to perform such a measurement is given in Figure H.4, and can be performed with or without the polarizing beam splitter (PBS) and accompanying HWP. In our case, we do not polarize the biphotons for the $g^{(2)}$ measurements.

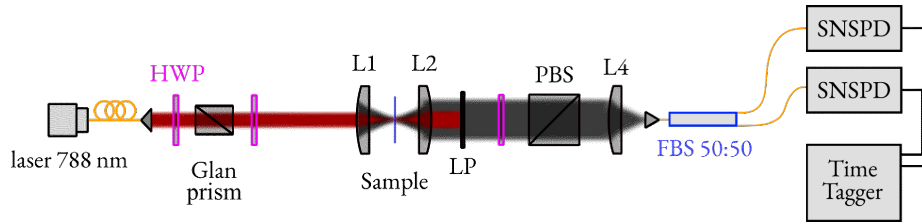


Figure H.4: Reduced polarization tomography setup used to characterize the polarization state of SPDC. Same as Figure H.3, with a simpler form. It is typically only used to characterize linearly polarized photons.

H.2.2 POLARIZATION TOMOGRAPHY OF SPDC

Next, we perform polarization tomography for different samples, and we use the same setup as shown in Figure H.4. The pump is horizontally polarized, and the sample is rotated so that its crystal axis matches that of the strongest second-order nonlinear coefficient. The angle φ of the HWP immediately preceding the PBS is rotated around once entirely [this would be equivalent to rotating a linear polarizer twice ($\theta = 2\varphi$)]. We step the HWP angle, φ in $\pi/8$ intervals, measuring $g^{(2)}(0)$ each time. This characterization allows us to assess the primary crystal process, as the rate of detection (equivalently, $g^{(2)}$) for each process scales like:

$$\text{Type o: } g_0^{(2)}(\theta) \propto \cos^4(\theta); \quad (\text{H.3})$$

$$\text{Type I: } g_I^{(2)}(\theta) \propto \sin^4(\theta); \quad (\text{H.4})$$

$$\text{Type II: } g_{II}^{(2)}(\theta) \propto \cos^2(\theta) \sin^2(\theta). \quad (\text{H.5})$$

H.2.3 PUMP POLARIZATION TOMOGRAPHY OF SPDC

One can investigate the dependence of nonlinear susceptibility on pump polarization through pump polarization tomography. The setup is identical to that in Figure H.4, but instead of varying only the polarizer after the sample, we also vary the pump polarization. In our simple measurements, we perform only four measurements for a single material ($\text{LiNbO}_3\text{:Fe}$), as the $|e\rangle \rightarrow |e\rangle|e\rangle$ is known to be the most efficient process in lithium niobate. We polarize the pump vertically and horizontally, as well as the biphoton polarizer.

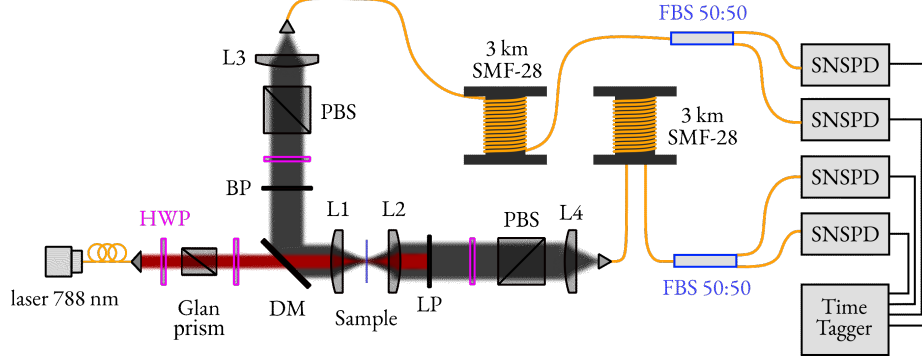


Figure H.5: Modified HBT setup to measure the spectrum of the down-converted photons. This setup adds a second arm to the correlation setup, allowing for the measurement of backwards propagating biphotons.

H.2.4 FIBER SPECTROSCOPY OF SPDC

We also measure the wavelength spectrum of the biphotons produced via SPDC via fiber spectroscopy, which uses the setup shown in Figure H.5. Three kilometers of SMF are added between the fiber coupling and the detectors to temporally delay the photon arrival by dispersion. The effective refractive index of the normally dispersive SMF varies approximately linearly as a function of wavelength. The measured temporal correlations are then linearly related to the wavelength of the radiation. The dispersion of SMF is not widely reported, so we calibrate the system using a broadband source (x -cut lithium niobate) and a series of BP and LP filters. This produces a series of measurements with well-defined edges from which time delay-wavelength correlations are made. From these measurements, a regression is fit (Figure H.6). Then, the rate of biphoton detection can be plotted as a function of wavelength.

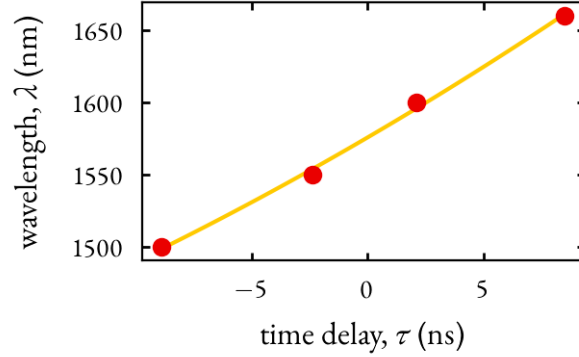


Figure H.6: Calculated regression (solid line) from a series of filtered measurements (scatter points).

H.2.5 CHARACTERIZATION OF SECOND-ORDER NONLINEARITY VIA SECOND HARMONIC GENERATION

We can characterize the strength of the second-order nonlinearity by measuring SHG in the samples. These experiments have been done before, but they are done here to verify several characteristics of the samples. In particular, the nonlinearities in bulk lithium niobate vary greatly across the literature, and the surface effects from thin sources such as ours cause their performances to differ from bulk. SHG can be used to roughly characterize the strength of the second-order nonlinearities across the down-conversion spectrum.

In this measurements, we produce a tunable, broadband, infrared input source via bulk SPDC, then we generate photons via SHG in our samples. A pulsed 532 nm is polarized and power regulated, then is doubly transmitted through a angle-tuned BBO crystal to generate and amplify a beam via SPDC to pump the nonlinear sample. The BBO crystal, depending on the angle radiates light with a wavelength between $1 \sim 2 \mu\text{m}$. This beam is LP-filtered and collimated. This beam is tuned between

H.3. SECOND-ORDER TEMPORAL CORRELATION FROM SPDC

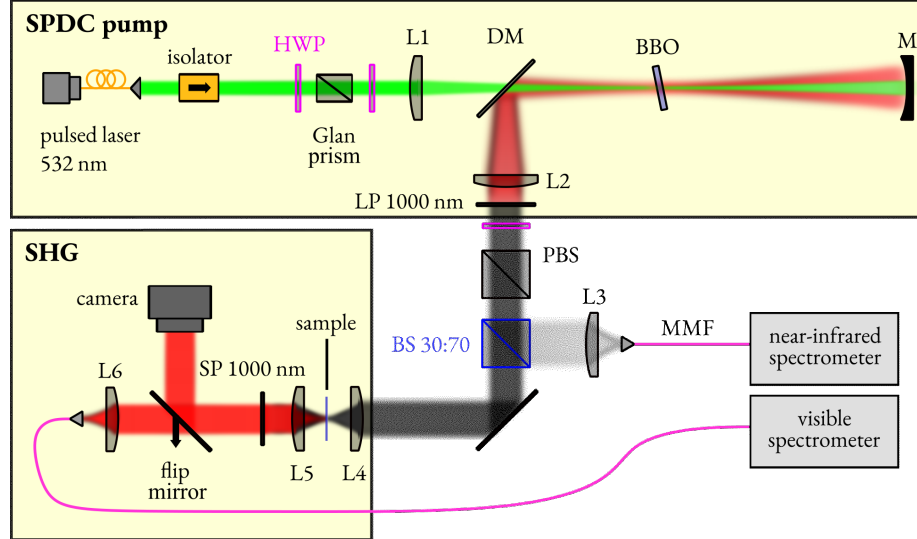


Figure H.7: Setup to perform SHG tomography of crystals with second-order nonlinearities.

$1.2 \sim 1.8 \mu\text{m}$ and has a wavelength bandwidth of $\sim 20 \text{ nm}$. The beam is polarized, tuned, and split into two beams using a 30:70 BS; the less intense beam is sent to a near-infrared spectrometer which measures the spectrum of down-conversion. The more intense beam is focused into the nonlinear sample, which produces visible photons via SPDC. The flip mirror and camera are used to align the setup, and the beam is nominally sent to a visible spectrometer via multi-mode fiber (MMF). The software used to measure each set of spectra for each center pump wavelength is first calibrated by measuring and subtracting dark counts.

H.3 SECOND-ORDER TEMPORAL CORRELATION FROM SPDC

Here, we measure the second-order temporal correlation of biphotons in three different samples $\text{LiNbO}_3\text{:FS}$, $\text{LiNbO}_3\text{:Fe}$, and GaAs . In the following figures, we plot their optimized coincidence rates,

H.3. SECOND-ORDER TEMPORAL CORRELATION FROM SPDC

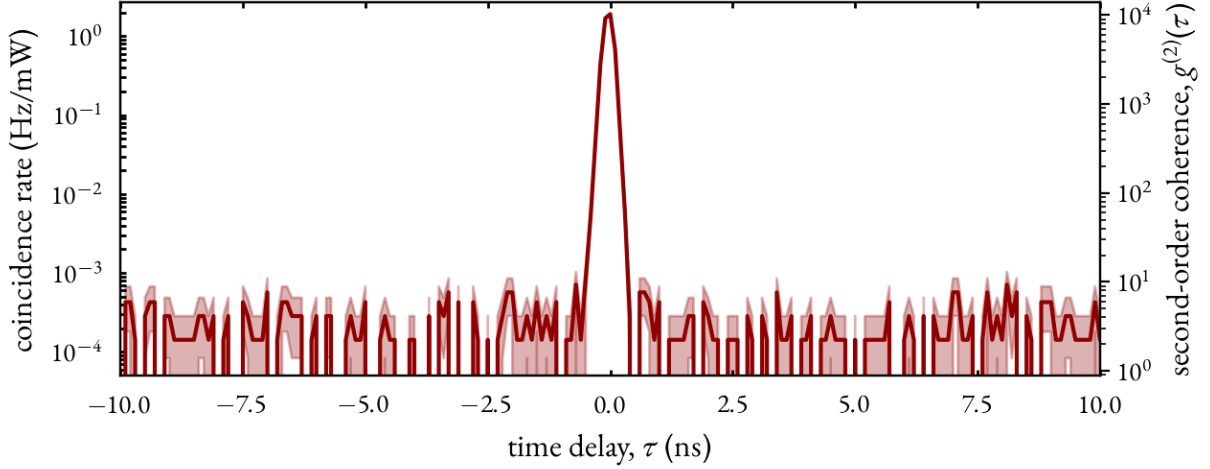


Figure H.8: Coincidence rate and second-order correlation [Equation (H.1)] as a function of time delay for a LiNbO₃:FS sample. The sample was pumped at 70 mW for 100 s. The coincidence rate is defined as the ratio of measured coincidence rate divided by pump power. The translucent bands represent the uncertainty of the measurement - these can be reduced with larger measurement times.

normalized to the pump power. In Figure H.8 we plot the coincidence rate and second-order correlation function as a function of time delay, where, in a 400 ps window, we measure a coincidence rate of 4.7 Hz/mW and $g^{(2)}(0)$ of 10^4 . This coincidence rate is high, but about an order of magnitude less than the highest seen in the Chekhova Lab, though this was for a highly focused pump and a lithium niobate crystal doped with MgO.

We do similar measurements for both LiNbO₃:Fe and GaAs, and plot them in Figure H.9. For the LiNbO₃:Fe and GaAs, respectively, we measure coincidence rates 0.35 Hz/mW and 0.17 Hz/mW, subtracting the rate of accidentals. This rate for the 300 um-thick LiNbO₃:Fe is much lower than that of LiNbO₃:FS but is expected because of the much larger crystal length - the crystal is not an odd integer multiple of the coherence length, and this reduces the rate of biphoton production (see Figure 7.6). For

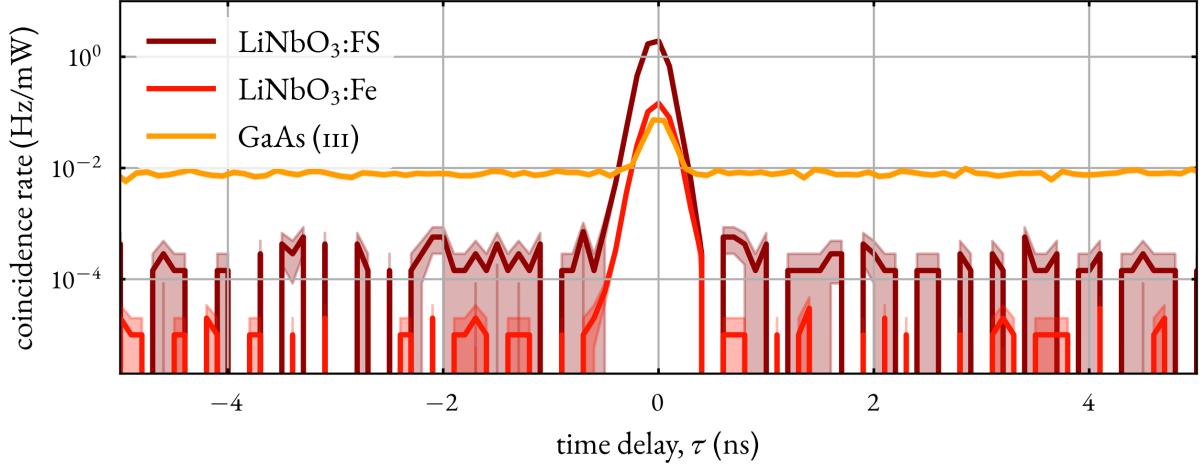


Figure H.9: Comparison of the time delayed detection rate of three different samples: LiNbO₃:FS, LiNbO₃:Fe, and GaAs. The first is pumped at 70 mW for 100 seconds, and the other two are pumped at 30 mW. The LiNbO₃:Fe is measured for 10 minutes and the GaAs for 56 minutes. The translucent bands represent the uncertainty of the measurement - these can be reduced with larger measurement times.

the GaAs however, we have achieved a very high rate of coincidences. Until now, only (100) crystal axis samples had been measured, and no biphotons had been detected [25]. This is also an extremely high rate of biphoton detection, considering that the sample is only 250 nm thick. From the figure, we also see that the rate of accidentals for the GaAs is much higher than the other samples. This demonstrates that we either have poor coupling efficiency or, more likely, that the substrate or bonding of the GaAs is highly photoluminescent. The latter explanation is strengthened by the observation that the sample started to burn at higher pump powers (> 30 mW).

We also measure $g^{(2)}(0)$ as a function of pump power, and plot this for all three samples in the following figure. This pattern evidences the production of biphotons via SPDC.

H.4. POLARIZATION TOMOGRAPHY OF SPDC

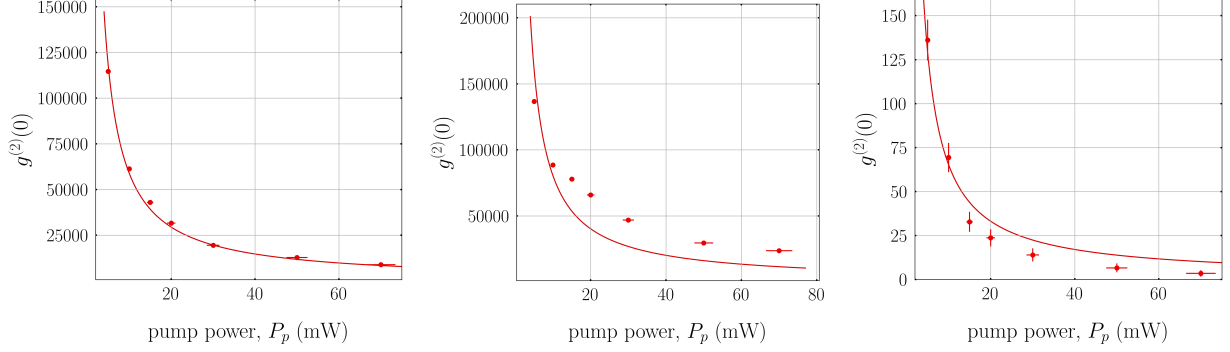


Figure H.10: The second-order correlation function $g^{(2)}(0)$ as a function of pump power for three samples: LiNbO₃:FS (left), LiNbO₃:Fe (center), and GaAs (right). We plot a fitted regression that goes like Equation (H.2).

H.4 POLARIZATION TOMOGRAPHY OF SPDC

We perform simple polarization tomography for the LiNbO₃:FS and GaAs samples. We radially plot the $g^{(2)}(0)$ for each tested polarizer angle, and fit the expected curves for each process. For LiNbO₃:FS, we expect and fit a Type o process [Equation (H.3)], and for GaAs, we expect and fit a Type II process [Equation (H.5)].

H.5 FIBER SPECTROSCOPY OF SPDC

Next, we measure the spectrum of down-version using fiber spectroscopy for three different samples: LiNbO₃:FS, LiNbO₃:Fe, and GaAs. These measurements are plotted in Figure H.12. To more easily compare the measurements, the rates of the LiNbO₃:Fe and GaAs samples are scaled by a factor of ten. We see that the down-conversion is extraordinarily broadband - most bulk sources only emit with

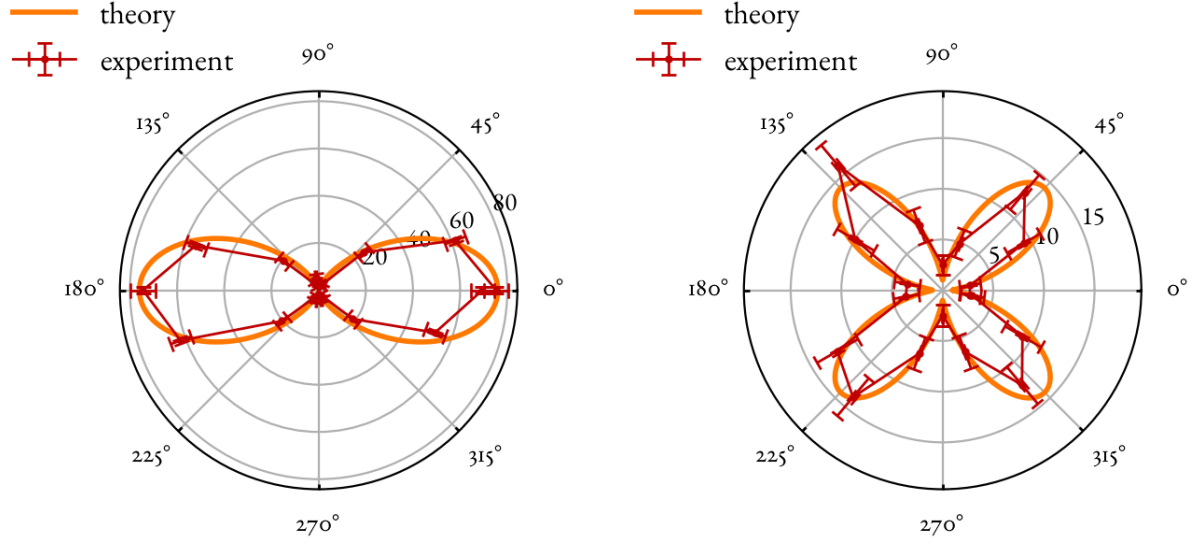


Figure H.11: Measured second-order correlation, $g^{(2)}(0)$, as a function of polarizer angle, θ , for a 10 μm lithium niobate sample (left) and a 250 nm GaAs (III) sample. The z crystal axis is aligned at 0° for the lithium niobate sample. The fitting functions are Equation (H.3) (left) and Equation (H.5) (right).

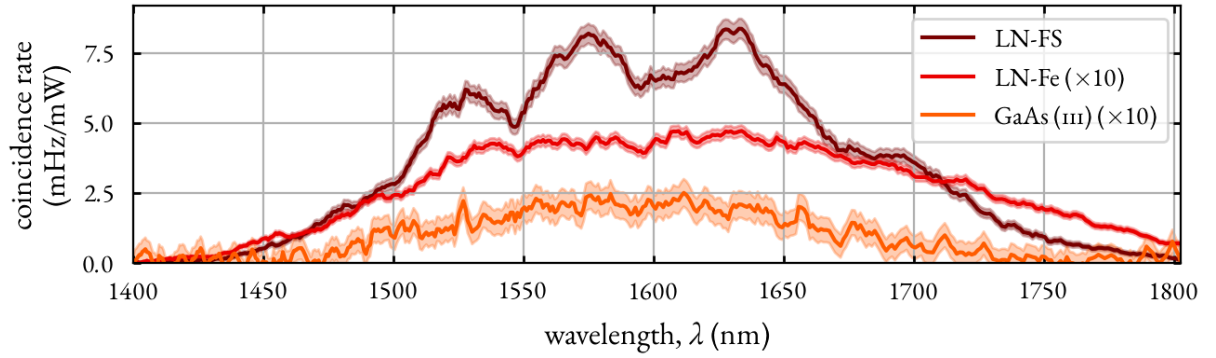


Figure H.12: The biphoton coincidence rate as a function of wavelength for three different samples: LiNbO₃:FS, LiNbO₃:Fe, and GaAs. To more easily compare, the rates of the LiNbO₃:Fe and GaAs samples are scaled by a factor of ten.

H.6. CHARACTERIZATION OF SECOND-ORDER NONLINEARITY VIA SECOND HARMONIC GENERATION

bandwidths less than approximately 10 nm. Note that the bandwidth seen is not the true bandwidth of the source, as the bandwidth of the SMF and SNSPDs are the cause of the drop in detection efficiency, not the inherent bandwidth of the thin film sources. Additionally, recall that we use a 1400 nm LP filter to remove the pump and photoluminescence, so this also limits the observable SPDC spectrum.

In Figure H.12, one will also notice the patterning in the spectrum of the LiNbO₃:FS. This is caused by internal reflections inside the crystal, manifesting in an etalon effect. We have studied this theoretically, but a description of our work is not included in this thesis.

H.6 CHARACTERIZATION OF SECOND-ORDER NONLINEARITY VIA SECOND HARMONIC GENERATION

We also characterize the second-order nonlinearity in LiNbO₃:Fe and GaAs using SHG. We pump each crystal at eleven different central wavelengths between 1300 ~ 1800 nm at 50 nm intervals. We plot the measurements for LiNbO₃:Fe in Figure H.13 where the effect of coherence length dispersion is manifested in a modulation of SHG countrate. This modulation is not observed in GaAs (Figure H.14) because of its thinness.

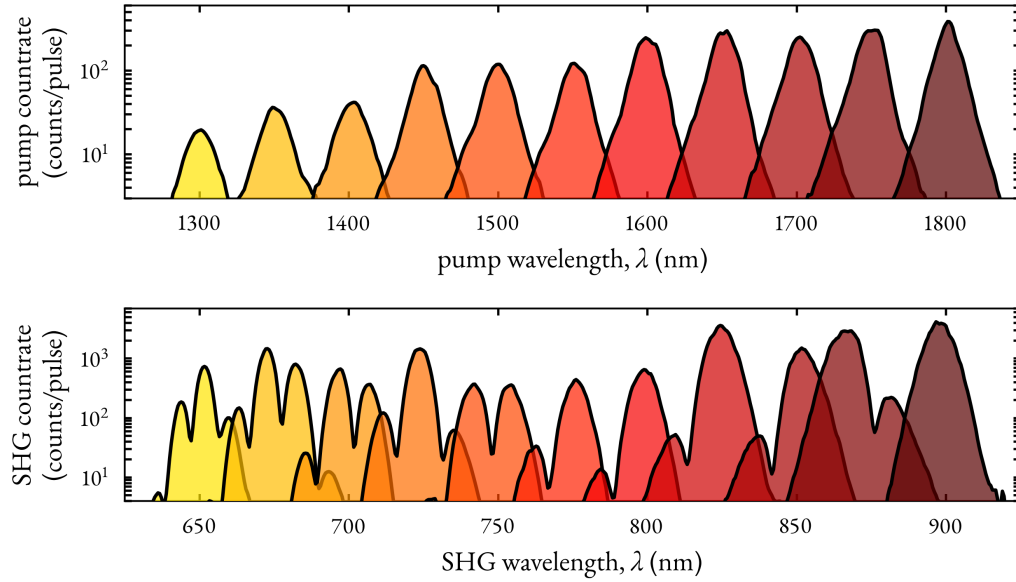


Figure H.13: Measured pump (upper) and SHG (lower) spectra for eleven different central pump wavelengths between 1300 ~ 1800 nm in a 300 μm -thick $\text{LiNbO}_3\text{:Fe}$ crystal.

H.7 THIN FILM SPDC IN LITHIUM NIOBATE MOUNTED ON SILICON

We present all measurements for $\text{LiNbO}_3\text{:Si}$ separately in this section, as the scheme of detection for this sample differed from those presented before. In this scheme, instead of splitting and correlating forward propagating photons, as was done in previous measurements, we also split and correlate photons propagating in the backward direction (see Figure H.5).

The 788 nm pump, when incident upon the $\text{LiNbO}_3\text{:Si}$ propagates through the 10 μm lithium niobate layer before mostly reflecting off of the lithium niobate-silicon interface. The silicon layer is mostly

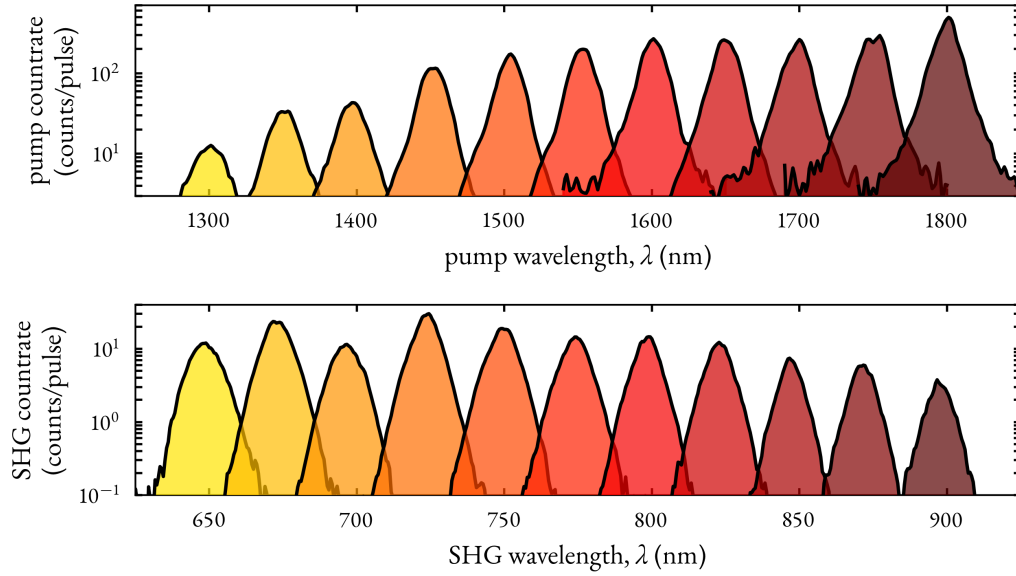


Figure H.14: Measured pump (upper) and SHG (lower) spectra for eleven different central pump wavelengths between 1300 ~ 1800 nm in a 250 nm-thick GaAs crystal.

reflective at the the pump wavelength and mostly transmissive at the biphoton wavelengths. Therefore, most of the forward-generated biphotons are transmitted and detected as normal. However, some are reflected, as well as generated in the backward direction due to the back-propagating pump. These photons are emitted in the backward direction, and are separated from the pump by a dichroic mirror and detected by the same coupling scheme as in the forward scheme. We correlated biphotons which are both transmitted, both reflected, and which have been split in the forward/backward directions. This reflection also manifests in a Fabry-Perot etalon effect, enhancing the production of certain wavelengths and suppressing others.

In Figure H.15, we plot the measured second-order correlation functions for all three correlation schemes as a function of pump power. It demonstrates that biphotons are being produced in all three

schemes.

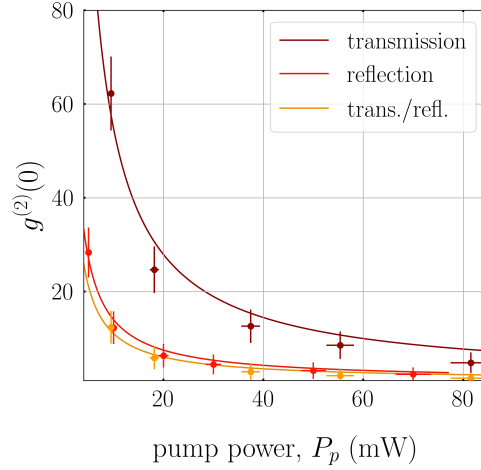


Figure H.15: The second-order correlation function $g^{(2)}(0)$ as a function of pump power in LiNbO₃:Si for three different measurements schemes: transmission, reflection, and between transmission and reflection. We plot a fitted regression that goes like Equation (H.2) for each scheme.

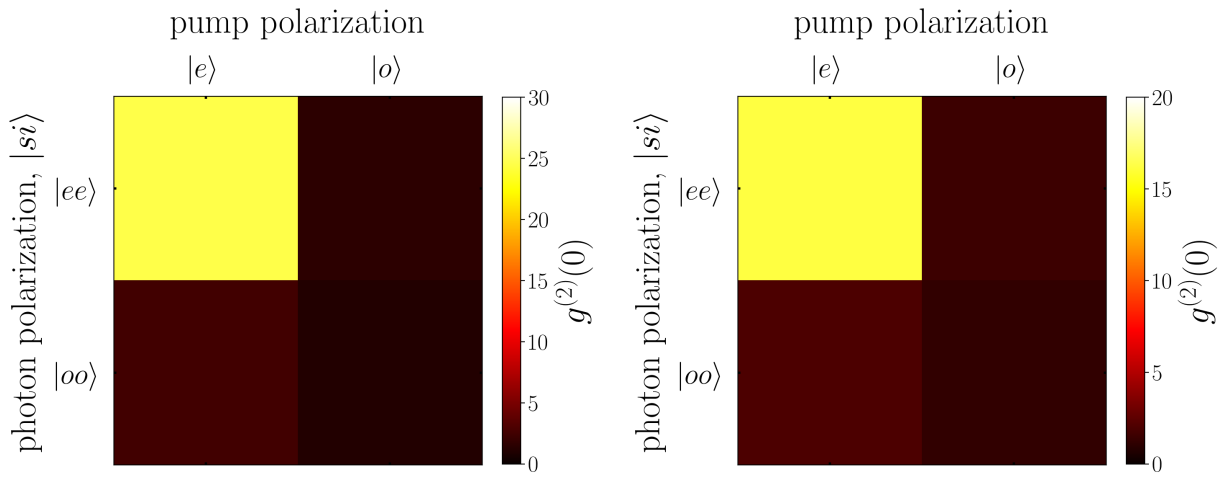


Figure H.16: Measured second-order correlation, $g^{(2)}(0)$, for different pump and photon polarizations in LiNbO₃:Si in both the forward (left) and backward (right) directions.

H.7. THIN FILM SPDC IN LITHIUM NIOBATE MOUNTED ON SILICON

Secondly, we measure the effect of pump polarization, by characterizing the strength of type o and I SPDC in both the transmitted and reflected geometries. These measurements are plotted in Figure H.16. We see that emission in the forward direction is strongest. This is expected, as the silicon is mostly transmissive at those wavelengths. This measurement also demonstrates that, as expected, down-conversion is strongest with a z - (e -) polarized pump in the type o configuration. This is the case for both transmissive and reflective schemes.

Finally, and most interestingly, we measure the pump power-normalized coincidence rate as a function of wavelength using fiber spectroscopy. In this case, we do not discriminate the biphoton polarization. In Figure H.17, we see that the spectrum of down-conversion is highly modulated by the cavity produced by the sample in all three schemes of down-conversion. This necessitates further discussion, but it is out of the scope of this thesis.

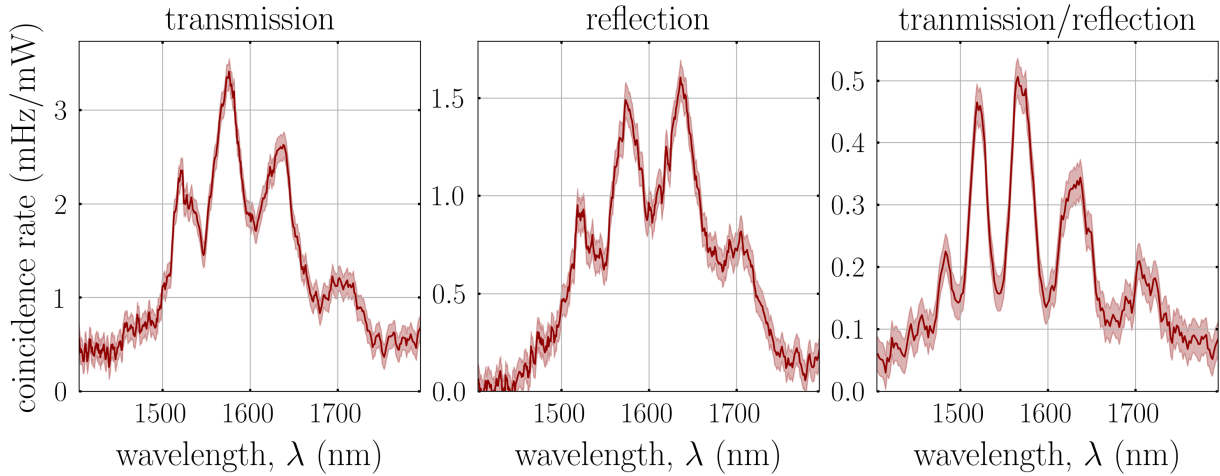


Figure H.17: The biphoton coincidence rate as a function of wavelength for the forward, backward, and forward/backward schemes. We LP filter these spectra using 1400 nm LP filters.

I

THE QUANTUM STATE OF A HOLOGRAPHIC SPDC SOURCE

In this chapter we derive several results on the quantum interference of photon pairs inside a holographic SPDC source. We first review the quantum state produced by SPDC, and then we the effect of a dispersive beam splitter. Two photon interference and the HOM effect is investigated. The theoretical

work in this chapter is based on the work done by Okoth and Branczyk [145, 146].

I.1 THE SPDC QUANTUM STATE

If we consider a beam propagating in the z direction through a nonlinear crystal of length L and nonlinear susceptibility tensor $\chi^{(2)}$, we can write the interaction Hamiltonian as:

$$\hat{H}_I = 2\epsilon_0 \int d^3r \chi^{(2)} E_p(z, t) E_s^{(-)} E_i^{(-)} + \text{h.c.},$$

where h.c. is the hermitian conjugate. This Hamiltonian be considered semi-classically: for a pump with frequency ω_p and spectral amplitude $\alpha(\omega_p)$, we classically represent the pump field and quantize the signal and idler fields:

$$E_p(z, t) = \mathcal{E}_p \int d\omega_p \alpha(\omega_p) \exp[i[k_p(\omega_p)z - \omega_p t]],$$

$$E_j^{(-)} = \int d^3k_j \mathcal{E}_j a_j^\dagger \exp[-i(k_j \cdot r - \omega_j t)].$$

Here, $j \in \{s, i\}$, and $\mathcal{E}_j = i\sqrt{\hbar\omega_j/2\epsilon_0\epsilon_r V_Q}$, for the quantization volume V_Q . The other symbols are as have already been defined in this thesis. As before, we define the longitudinal and transverse phase matching parameters to be

$$\Delta k_{\parallel} := k_{\parallel}^{(p)} - k_{\parallel}^{(s)} - k_{\parallel}^{(i)};$$

$$\Delta \mathbf{k}_{\perp} := \mathbf{k}_{\perp}^{(p)} - \mathbf{k}_{\perp}^{(s)} - \mathbf{k}_{\perp}^{(i)}.$$

Therefore,

$$\hat{H}_I = \epsilon_0 \chi^{(2)} \iiint d\mathbf{r} d\mathbf{k}_s d\mathbf{k}_i d\omega_p \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \alpha(\omega_p) \exp \left[i \left(\Delta \mathbf{k}_\perp \cdot \mathbf{r}_\perp + \Delta k_{\parallel} z - \Delta \omega t \right) \right] a_s^\dagger a_i^\dagger + \text{h.c.},$$

where $\Delta \omega := \omega_p - \omega_s - \omega_i$. We assume that we are in the low gain regime, wherein the coupling between the pump and daughter fields is weak. Therefore, a first-order expansion of the vacuum fluctuation-independent evolution operator gives,

$$\begin{aligned} |\psi\rangle &= \int_{t_0}^{t_1} \exp \left(-\frac{i\hat{H}_I t}{\hbar} \right) |0\rangle \\ &= -\frac{i}{\hbar} \int_{t_0}^{t_1} dt \hat{H}_I |0\rangle \\ &= \frac{2\epsilon_0 \chi^{(2)}}{i\hbar} \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \iiint d\mathbf{r} d\mathbf{k}_s d\mathbf{k}_i d\omega_p \alpha(\omega_p) \exp \left[i \left(\Delta \mathbf{k}_\perp \cdot \mathbf{r}_\perp + \Delta k_{\parallel} z \right) \right] \int_{t_0}^{t_1} dt \exp(-i\Delta \omega t) a_s^\dagger a_i^\dagger |0\rangle. \end{aligned}$$

We can also approximate

$$\int_{\xi_0}^{\xi_1} d\xi \exp(-i\alpha\xi) \approx \exp \left(-\frac{i\alpha(\xi_1 - \xi_0)}{2} \right) \text{sinc} \left(\frac{\alpha(\xi_1 - \xi_0)}{2} \right),$$

where $\text{sinc} \equiv \sin(x)/x$. Therefore, we can write the longitudinal phase matching and the time interaction as

$$\begin{aligned} b(L\Delta k_{\parallel}) &:= \int_0^L dz \exp(i\Delta k_{\parallel} z) \\ &\approx \exp\left(-\frac{iL\Delta k_{\parallel}}{2}\right) \text{sinc}\left(\frac{L\Delta k_{\parallel}}{2}\right); \\ T(\tau\Delta\omega) &:= \int_{t_0}^{t_0+\tau} dt \exp(-it\Delta\omega) \\ &\approx \exp\left(\frac{i\tau\Delta\omega}{2}\right) \text{sinc}\left(\frac{\tau\Delta\omega}{2}\right). \end{aligned}$$

If the interaction time, τ , is much longer than the inverse detection bandwidth ($\tau \gg 1/\Delta\omega_D$), then $T(\tau\Delta\omega) \rightarrow \delta(\Delta\omega)$; this is a representation of the conservation of energy. In the same way, if the interaction length L is much longer than the inverse pump wavevector ($L \gg 1/\Delta k_p$), then $b(L\Delta k_{\parallel}) \rightarrow \delta(\Delta k_{\parallel})$. This is the longitudinal phase matching condition of bulk crystals. We, for accuracy's sake, do not assume this to be true.

If we assume a classical Gaussian pump beam of $1/e^2$ radius w and field intensity $\exp[-r^2/2w^2]/2\pi w^2$,

$$\int d^2\mathbf{r}_{\perp} \exp(-i\Delta\mathbf{k}_{\parallel} \cdot \mathbf{r}_{\parallel}) = \exp(-\Delta\mathbf{k}_{\perp}^2 w^2).$$

Then, we can write the SPDC quantum state as follows, given that $k_s = k_s(\omega_s)$ and $k_i = k_i(\omega_i)$, and

$$\omega_p = \omega_s(\omega_s + \omega_i):$$

$$\begin{aligned} |\psi\rangle &= \frac{2\epsilon_0\chi^{(2)}}{i\hbar} \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \iint d\mathbf{k}_s d\mathbf{k}_i \alpha(\omega_s + \omega_i) \exp\left(-\frac{iL\Delta k_{\parallel}}{2}\right) \text{sinc}\left(\frac{L\Delta k_{\parallel}}{2}\right) \\ &\quad \times \exp(-\Delta\mathbf{k}_{\perp}^2 w^2) \exp\left(\frac{i\tau\Delta\omega}{2}\right) \text{sinc}\left(\frac{\tau\Delta\omega}{2}\right) a_s^{\dagger} a_i^{\dagger} |0\rangle. \\ &= \frac{2\epsilon_0\chi^{(2)}}{i\hbar} \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \iint d\mathbf{k}_s d\mathbf{k}_i f(\Delta\mathbf{k}, \Delta\omega) a_s^{\dagger} a_i^{\dagger} |0\rangle, \end{aligned} \quad (\text{I.1})$$

where

$$\begin{aligned} f(\Delta\mathbf{k}, \Delta\omega) &:= \\ 2\sqrt{2\pi}wL\tau \exp\left(-\frac{iL\Delta k_{\parallel}}{2}\right) \text{sinc}\left(\frac{L\Delta k_{\parallel}}{2}\right) \exp(-\Delta\mathbf{k}_{\perp}^2 w^2) \exp\left(\frac{i\tau\Delta\omega}{2}\right) \text{sinc}\left(\frac{\tau\Delta\omega}{2}\right). \end{aligned} \quad (\text{I.2})$$

This can be represented in its Schmidt decomposition form:

$$f(\omega_s, \omega_s, \mathbf{k}_s, \mathbf{k}_i) = \sum_j k\lambda_{j,k} \rho_j(\mathbf{k}_s) \varrho_j(\mathbf{k}_i) \phi_k(\omega_s) \varphi_k(\omega_s), \quad (\text{I.3})$$

where $\rho_j(\mathbf{k}_s)$ and $\varrho_j(\mathbf{k}_i)$ form a discrete basis of orthonormal functions in angular frequency, and $\phi_k(\omega_s)$ and $\varphi_k(\omega_s)$ form a discrete basis of orthonormal functions in the spatial domain. We define the normalized phase matching function

$$\begin{aligned} \tilde{f}(\Delta\mathbf{k}, \Delta\omega) &:= \frac{f(\Delta\mathbf{k}, \Delta\omega)}{L\tau} \\ 2\sqrt{2\pi}w \exp\left(-\frac{iL\Delta k_{\parallel}}{2}\right) \text{sinc}\left(\frac{L\Delta k_{\parallel}}{2}\right) \exp(-\Delta\mathbf{k}_{\perp}^2 w^2) \exp\left(\frac{i\tau\Delta\omega}{2}\right) \text{sinc}\left(\frac{\tau\Delta\omega}{2}\right). \end{aligned} \quad (\text{I.4})$$

The transition probability of SPDC is then:

$$\begin{aligned}
 P &= \frac{2\epsilon_0 [\chi^{(2)}]^2 P_p}{\hbar c n_p} \mathcal{E}_p^2 \mathcal{E}_s^2 \mathcal{E}_i^2 |f(\Delta \mathbf{k}, \Delta \omega)|^2 \\
 &= \frac{16\pi\epsilon_0 [\chi^{(2)}]^2 P_p w^2 L^2 \tau^2}{\hbar c n_p} \mathcal{E}_p^2 \mathcal{E}_s^2 \mathcal{E}_i^2 \exp(-2\Delta \mathbf{k}_\perp^2 w^2) \text{sinc}^2\left(\frac{L\Delta k_\parallel}{2}\right) \text{sinc}^2\left(\frac{\tau\Delta\omega}{2}\right) \\
 &= \frac{32\pi^2\epsilon_0 [\chi^{(2)}]^2 P_p w^2 L^2 \tau^2}{\hbar c n_p} \mathcal{E}_p^2 \mathcal{E}_s^2 \mathcal{E}_i^2 \exp(-2\Delta \mathbf{k}_\perp^2 w^2) \text{sinc}^2\left(\frac{L\Delta k_\parallel}{2}\right) \delta(\Delta\omega), \tag{I.5}
 \end{aligned}$$

where we have used the Gaussian phase matching function Equation (I.2) and the identity $\lim_{\epsilon \rightarrow \infty} \epsilon \text{sinc}^2(\epsilon\alpha) = \pi\delta(\alpha)$. In the case that the crystal is thin ($L \ll 1/\Delta k_D$) we can use the identity $\lim_{\epsilon \rightarrow 0} \text{sinc}^2(\epsilon\alpha) = 1$ to express the transition probability as

$$P_{\text{thin}} = \frac{32\pi^2\epsilon_0 [\chi^{(2)}]^2 P_p w^2 L^2 \tau^2}{\hbar c n_p} \mathcal{E}_p^2 \mathcal{E}_s^2 \mathcal{E}_i^2 \exp(-2\Delta \mathbf{k}_\perp^2 w^2) \delta(\Delta\omega). \tag{I.6}$$

We can now find the rate of photon production given by the ratio of the total number of biphoton states produced divided by the interaction time. This is given by integration over the density of states:

$$\begin{aligned}
 R &:= \frac{N}{\tau} = \iint d\mathbf{k}_s d\mathbf{k}_i P(\mathbf{k}_s, \mathbf{k}_i) g(\mathbf{k}_s, \mathbf{k}_i) \\
 &= \frac{[\chi^{(2)}]^2 P_p L^2}{128\pi^6 c^7 \epsilon_0 n_p} \int_{\Delta\omega_D} \int_{\Delta\omega_D} \int_0^{\Delta\theta_D} \int_0^{\Delta\theta_D} \int_0^{2\pi} \int_0^{2\pi} d\omega_1 d\omega_i d\theta_s d\theta_i d\varphi_s d\varphi_i \times \\
 &\quad \omega_s^3 \omega_i^3 n_s n_i |\tilde{f}^{(2)}(\Delta \mathbf{k}, \Delta \omega)|^2 \sin(\theta_s) \sin(\theta_i) \tag{I.7}
 \end{aligned}$$

where θ is the axial emission angle, and φ is the longitudinal emission angle, and $\Delta\omega_D$ is the detection angular frequency bandwidth, and $\Delta\theta_D$ is the maximum collection angle of the detector. This expression

comes from the transformation of the volume differential wavevector element:

$$d^3\mathbf{k}_j = \frac{1}{v_j} \left(\frac{\omega_j n_j}{c} \right)^2 \sin(\theta_j) d\omega_j d\theta_j d\varphi_j, \quad (\text{I.8})$$

for the group velocity v_j .

I.2 SPDC INPUT TO A BEAM SPLITTER

Now, suppose we input this broadband SPDC into a beam splitter with transmissivity $T = t^2 = 1 - \eta$ and reflectivity $R = r^2 = \eta$, which depends on the wavevector ($\eta = \eta(\omega_j, \theta_j, \varphi_j)$). We define the input modes $a_s^\dagger$ and $a_i^\dagger$ to be incident on the beam splitter from the left and right sides, respectively. We define the outgoing modes $b_1^\dagger$ and $b_2^\dagger$ to leave the beam splitter to the right and left sides, respectively. The outgoing modes are sent to two detectors. We want to understand how the biphotons interfere, and how the two photon coincidence probability varies with the joint spectral intensity (JSI). We start by deriving the state of the outputted modes. The beam splitter unitary can be treated as it is usually treated for the HOM effect, instead assuming that the reflectivities and transmissivities vary with angle and frequency. The beam splitter has the unitary effect

$$\hat{\mathcal{U}}_{\text{BS}} = \frac{i}{\sqrt{t_{ss}t_{ii} - r_{ss}r_{ii}}} \begin{pmatrix} t_{ss} & r_{ii} \\ r_{ss} & -t_{ii} \end{pmatrix} \rightarrow$$

$$\hat{\mathcal{U}}_{\text{BS}}^{-1} = \frac{-i}{\sqrt{t_{ss}t_{ii} - r_{ss}r_{ii}}} \begin{pmatrix} t_{ii} & r_{ii} \\ r_{ss} & -t_{ss} \end{pmatrix} \text{ and } \hat{\mathcal{U}}_{\text{BS}}^\dagger = \frac{-i}{\sqrt{t_{ss}t_{ii} - r_{ss}r_{ii}}} \begin{pmatrix} t_{ss} & r_{ii} \\ r_{ss} & -t_{ii} \end{pmatrix}$$

Therefore, the beam splitter only has a unitary effect if $t_{sss} \equiv t_{iii}$, which is generally not the case. Here we have used the contracted notation $t_{ijk} \equiv t(\omega_i, \theta_j, \varphi_k)$. We can then write that

$$\begin{pmatrix} a_s^\dagger \\ a_i^\dagger \end{pmatrix} \xrightarrow{\hat{\mathcal{U}}_{\text{BS}}} \begin{pmatrix} t_{sss} & r_{sss} \\ r_{iii} & -t_{iii} \end{pmatrix} \begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix}, \quad (\text{I.9})$$

Therefore, using Equation (I.1) and Equation (I.9) we can write the output state as

$$\begin{aligned} |\psi'\rangle &= \hat{\mathcal{U}}_{\text{BS}} |\psi\rangle \\ &\propto \iint d\mathbf{k}_s d\mathbf{k}_i f(\Delta\mathbf{k}, \Delta\omega) \exp(-i\omega_i \tau) \hat{\mathcal{U}}_{\text{BS}} a_s^\dagger a_i^\dagger |0\rangle, \\ &\propto \iint d\mathbf{k}_s d\mathbf{k}_i f(\Delta\mathbf{k}, \Delta\omega) \exp(-i\omega_i \tau) (t_{sss} b_{1s}^\dagger + r_{sss} b_{2s}^\dagger) (r_{iii} b_{1i}^\dagger - t_{iii} b_{2i}^\dagger) |0\rangle \\ &\propto \iint d\mathbf{k}_s d\mathbf{k}_i f(\Delta\mathbf{k}, \Delta\omega) \exp(-i\omega_i \tau) (t_{sss} r_{iii} b_{1s}^\dagger b_{1i}^\dagger + r_{sss} r_{iii} b_{2s}^\dagger b_{1i}^\dagger - t_{sss} t_{iii} b_{1s}^\dagger b_{2i}^\dagger - r_{sss} t_{iii} b_{2s}^\dagger b_{2i}^\dagger) |0\rangle, \end{aligned}$$

where we have added a time delay, τ , to the incident idler mode. The projector defining detection in mode b_j is given by

$$\hat{P}_j = \int d\omega b_j^\dagger(\omega) |0\rangle_j \langle 0|_j b_j(\omega),$$

so we can write that the probability of detecting a coincidence between the detectors is

$$P = \text{Tr} [|\psi'\rangle_{12} \langle \psi'|_{12} \hat{P}_1 \otimes \hat{P}_2] = \langle \psi'|_{12} \hat{P}_1 \otimes \hat{P}_1 | \psi'\rangle_{12}. \quad (\text{I.10})$$

Inserting the relevant equations into Equation (I.10), we find that

$$\begin{aligned}
 P &= \left[\iint d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha f_\alpha^* \exp(i\omega_i^\alpha \tau) \langle 0|_{12} (t_{sss}^{\alpha*} r_{iii}^{\alpha*} b_{1s}^\alpha b_{1i}^\alpha + r_{sss}^{\alpha*} r_{iii}^{\alpha*} b_{2s}^\alpha b_{1i}^\alpha - t_{sss}^{\alpha*} t_{iii}^{\alpha*} b_{1s}^\alpha b_{2i}^\alpha - r_{sss}^{\alpha*} t_{iii}^{\alpha*} b_{2s}^\alpha b_{2i}^\alpha) \right. \\
 &\quad \times \int d\omega_1 b_{11}^\dagger |0\rangle_1 \langle 0|_1 b_{11} \int d\omega_2 b_{22}^\dagger |0\rangle_2 \langle 0|_2 b_{22} \\
 &\quad \times \left. \left[\iint d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta f_\beta \exp(-i\omega_i^\beta \tau) \left(t_{sss}^\beta r_{iii}^\beta b_{1s}^{\beta\dagger} b_{1i}^{\beta\dagger} + r_{sss}^\beta r_{iii}^\beta b_{2s}^{\beta\dagger} b_{1i}^{\beta\dagger} - t_{sss}^\beta t_{iii}^\beta b_{1s}^{\beta\dagger} b_{2i}^{\beta\dagger} - r_{sss}^\beta t_{iii}^\beta b_{2s}^{\beta\dagger} b_{2i}^{\beta\dagger} \right) |0\rangle_{12} \right] \right. \\
 &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta d\omega_1 d\omega_2 f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \langle 0|_{12} (t_{sss}^{\alpha*} r_{iii}^{\alpha*} b_{1s}^\alpha b_{1i}^\alpha + r_{sss}^{\alpha*} r_{iii}^{\alpha*} b_{2s}^\alpha b_{1i}^\alpha - t_{sss}^{\alpha*} t_{iii}^{\alpha*} b_{1s}^\alpha b_{2i}^\alpha - r_{sss}^{\alpha*} t_{iii}^{\alpha*} b_{2s}^\alpha b_{2i}^\alpha) b_{11}^\dagger b_{22}^\dagger |0\rangle_{12} \\
 &\quad \times \langle 0|_{12} b_{11} b_{22} \left(t_{sss}^\beta r_{iii}^\beta b_{1s}^{\beta\dagger} b_{1i}^{\beta\dagger} + r_{sss}^\beta r_{iii}^\beta b_{2s}^{\beta\dagger} b_{1i}^{\beta\dagger} - t_{sss}^\beta t_{iii}^\beta b_{1s}^{\beta\dagger} b_{2i}^{\beta\dagger} - r_{sss}^\beta t_{iii}^\beta b_{2s}^{\beta\dagger} b_{2i}^{\beta\dagger} \right) |0\rangle_{12}.
 \end{aligned}$$

Now, we discard all terms that have an odd number operators in one mode:

$$\begin{aligned}
 P &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta d\omega_1 d\omega_2 f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \langle 0|_{12} \left(r_{sss}^{\alpha*} r_{iii}^{\alpha*} b_{2s}^\alpha b_{1i}^\alpha b_{11}^\dagger b_{22}^\dagger - t_{sss}^{\alpha*} t_{iii}^{\alpha*} b_{1s}^\alpha b_{2i}^\alpha b_{11}^\dagger b_{22}^\dagger \right) |0\rangle_{12} \\
 &\quad \times \langle 0|_{12} \left(r_{sss}^\beta r_{iii}^\beta b_{11} b_{22} b_{2s}^{\beta\dagger} b_{1i}^{\beta\dagger} - t_{sss}^\beta t_{iii}^\beta b_{11} b_{22} b_{1s}^{\beta\dagger} b_{2i}^{\beta\dagger} \right) |0\rangle_{12}.
 \end{aligned}$$

Now, we replace the terms $\langle 0|_{12} b_{1i} b_{2j} b_{1k}^\dagger b_{2l}^\dagger |0\rangle_{12}$ by delta functions $\delta(\omega_i - \omega_k) \delta(\omega_j - \omega_l)$. Therefore,

$$\begin{aligned}
 P &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta d\omega_1 d\omega_2 f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \left(r_{sss}^{\alpha*} r_{iii}^{\alpha*} \delta(\omega_s^\alpha - \omega_2) \delta(\omega_i^\alpha - \omega_1) - t_{sss}^{\alpha*} t_{iii}^{\alpha*} \delta(\omega_s^\alpha - \omega_1) \delta(\omega_i^\alpha - \omega_2) \right) \\
 &\quad \times \left(r_{sss}^\beta r_{iii}^\beta \delta(\omega_s^\beta - \omega_2) \delta(\omega_i^\beta - \omega_1) - t_{sss}^\beta t_{iii}^\beta \delta(\omega_s^\beta - \omega_1) \delta(\omega_i^\beta - \omega_2) \right).
 \end{aligned}$$

Now, we integrate over ω_1 and ω_2 using the identity $\int d\omega \delta(\omega_2 - \omega) \delta(\omega - \omega_1) = \delta(\omega_2 - \omega_1)$:

$$\begin{aligned}
 P &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta d\omega_1 d\omega_2 f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \left[r_{sss}^{\alpha*} r_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta \delta(\omega_s^\alpha - \omega_2) \delta(\omega_i^\alpha - \omega_1) \delta(\omega_s^\beta - \omega_2) \delta(\omega_i^\beta - \omega_1) \right. \\
 &\quad - t_{sss}^{\alpha*} t_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta \delta(\omega_s^\alpha - \omega_1) \delta(\omega_i^\alpha - \omega_2) \delta(\omega_s^\beta - \omega_2) \delta(\omega_i^\beta - \omega_1) \\
 &\quad - r_{sss}^{\alpha*} r_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \delta(\omega_s^\alpha - \omega_2) \delta(\omega_i^\alpha - \omega_1) \delta(\omega_s^\beta - \omega_1) \delta(\omega_i^\beta - \omega_2) \\
 &\quad \left. + t_{sss}^{\alpha*} t_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \delta(\omega_s^\alpha - \omega_1) \delta(\omega_i^\alpha - \omega_2) \delta(\omega_s^\beta - \omega_1) \delta(\omega_i^\beta - \omega_2) \right]. \\
 &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \left[r_{sss}^{\alpha*} r_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta \delta(\omega_s^\alpha - \omega_s^\beta) \delta(\omega_i^\alpha - \omega_i^\beta) - t_{sss}^{\alpha*} t_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta \delta(\omega_s^\alpha - \omega_i^\beta) \delta(\omega_i^\alpha - \omega_s^\beta) \right. \\
 &\quad \left. - r_{sss}^{\alpha*} r_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \delta(\omega_s^\alpha - \omega_i^\beta) \delta(\omega_i^\alpha - \omega_s^\beta) + t_{sss}^{\alpha*} t_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \delta(\omega_s^\alpha - \omega_s^\beta) \delta(\omega_i^\alpha - \omega_i^\beta) \right]. \\
 &= \int \cdots \int d\mathbf{k}_s^\alpha d\mathbf{k}_i^\alpha d\mathbf{k}_s^\beta d\mathbf{k}_i^\beta f_\alpha^* f_\beta \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \\
 &\quad \times \left[\left(r_{sss}^{\alpha*} r_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta + t_{sss}^{\alpha*} t_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \right) \delta(\omega_s^\alpha - \omega_s^\beta) \delta(\omega_i^\alpha - \omega_i^\beta) \right. \\
 &\quad \left. - \left(t_{sss}^{\alpha*} t_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta + r_{sss}^{\alpha*} r_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \right) \delta(\omega_s^\alpha - \omega_i^\beta) \delta(\omega_i^\alpha - \omega_s^\beta) \right]
 \end{aligned}$$

Now, we change the basis of integration using Equation (I.8) and integrate over ω_s^β and ω_i^β :

$$\begin{aligned}
 P &= \int \cdots \int d\omega_s^\alpha d\omega_i^\alpha d\omega_s^\beta d\omega_i^\beta d\theta_s^\alpha d\theta_i^\alpha d\theta_s^\beta d\theta_i^\beta d\varphi_s^\alpha d\varphi_i^\alpha d\varphi_s^\beta d\varphi_i^\beta (\omega_s^\alpha)^2 (\omega_i^\alpha)^2 (\omega_s^\beta)^2 (\omega_i^\beta)^2 \\
 &\quad \times n_s^\alpha n_i^\alpha n_s^\beta n_i^\beta \sin(\theta_s^\alpha) \sin(\theta_i^\alpha) \sin(\theta_s^\beta) \sin(\theta_i^\beta) \\
 &\quad \times f^*(\omega_s^\alpha, \omega_i^\alpha, \theta_s^\alpha, \theta_i^\alpha, \varphi_s^\alpha, \varphi_i^\alpha) f(\omega_s^\beta, \omega_i^\beta, \theta_s^\beta, \theta_i^\beta, \varphi_s^\beta, \varphi_i^\beta) \\
 &\quad \times \left[\left(r_{sss}^{\alpha*} r_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta + t_{sss}^{\alpha*} t_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \right) \exp \left[i \left(\omega_i^\alpha - \omega_s^\beta \right) \tau \right] \delta(\omega_s^\alpha - \omega_s^\beta) \delta(\omega_i^\alpha - \omega_i^\beta) \right. \\
 &\quad \left. - \left(t_{sss}^{\alpha*} t_{iii}^{\alpha*} r_{sss}^\beta r_{iii}^\beta + r_{sss}^{\alpha*} r_{iii}^{\alpha*} t_{sss}^\beta t_{iii}^\beta \right) \exp \left[i \left(\omega_i^\alpha - \omega_i^\beta \right) \tau \right] \delta(\omega_s^\alpha - \omega_i^\beta) \delta(\omega_i^\alpha - \omega_s^\beta) \right] \\
 &= \int \cdots \int d\omega_s d\omega_i d\theta_s^\alpha d\theta_i^\alpha d\theta_s^\beta d\theta_i^\beta d\varphi_s^\alpha d\varphi_i^\alpha d\varphi_s^\beta d\varphi_i^\beta (\omega_s)^\alpha (\omega_i)^\alpha \quad (I.11) \\
 &\quad \times n_s^\alpha n_i^\alpha n_s^\beta n_i^\beta \sin(\theta_s^\alpha) \sin(\theta_i^\alpha) \sin(\theta_s^\beta) \sin(\theta_i^\beta) \\
 &\quad \times \left[\left(r_{sss}^{\alpha*} r_{iii}^{\alpha*} r_{sss}^{\alpha\beta} r_{iii}^{\alpha\beta} + t_{sss}^{\alpha*} t_{iii}^{\alpha*} t_{sss}^{\alpha\beta} t_{iii}^{\alpha\beta} \right) f^*(\omega_s, \omega_i, \theta_s^\alpha, \theta_i^\alpha, \varphi_s^\alpha, \varphi_i^\alpha) f(\omega_s, \omega_i, \theta_s^\beta, \theta_i^\beta, \varphi_s^\beta, \varphi_i^\beta) \right. \\
 &\quad \left. - \left(t_{sss}^{\alpha*} t_{iii}^{\alpha*} r_{iss}^{\alpha\beta} r_{sii}^{\alpha\beta} + r_{sss}^{\alpha*} r_{iii}^{\alpha*} t_{iss}^{\alpha\beta} t_{sii}^{\alpha\beta} \right) f^*(\omega_s, \omega_i, \theta_s^\alpha, \theta_i^\alpha, \varphi_s^\alpha, \varphi_i^\alpha) f(\omega_i, \omega_s, \theta_s^\beta, \theta_i^\beta, \varphi_s^\beta, \varphi_i^\beta) \exp [i(\omega_s - \omega_i) \tau] \right] \quad (I.12)
 \end{aligned}$$

To solve the coincidence probability exactly, it is necessary to solve this integral, normalizing using the following:

$$1 = \mathcal{A} \int \cdots \int d\omega_s d\omega_i d\theta_s d\theta_i d\varphi_s d\varphi_i (\omega_s)^\alpha (\omega_i)^\alpha n_s n_i \sin(\theta_s) \sin(\theta_i) |f(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i)|^2$$

where $\mathcal{A}$ is a normalization factor.

I.2.1 A DIRECTIONAL SOURCE AND UNITARY BEAM SPLITTER

Suppose we consider emission along only a single pair of emission angles (θ_s, θ_i) along the xz -plane, orthogonal to the plane of reflection ($\varphi_s = \varphi_i = 0$), while also assuming transverse phase matching. Therefore, we can drop all integration of θ and ϕ , and we can drop all α and β terms. If we also assume a CW pump of angular frequency ω_p , we can further reduce the probability by assuming perfect energy conservation ($\omega_i = \omega_p - \omega_s$). The coincidence probability in this case can be approximated by

$$P_{\text{ID}} = \frac{1}{A} \int d\omega_s \times \left[\left(r_{sss}^* r_{iii}^* r_{sss} r_{iii} + t_{sss}^* t_{iii}^* t_{sss} t_{iii} \right) f^*(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) f(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) \right. \\ \left. - \left(t_{sss}^* t_{iii}^* r_{sss} r_{iii} + r_{sss}^* r_{iii}^* t_{sss} t_{iii} \right) f^*(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) f(\omega_i, \omega_s, \theta_s, \theta_i, \varphi_s, \varphi_i) \exp [i(\omega_s - \omega_i) \tau] \right]$$

We can simplify this further if we consider a unitary beam splitter, where the reflection efficiency of the beam splitter does not depend on the frequency or direction of the down-converted light ($\eta \neq \eta(\omega, \theta, \varphi)$).

$$P_{\text{ID}} = \frac{1}{A} \int d\omega_s \left\{ [\eta^2 + (1 - \eta)^2] f^*(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) f(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) \right. \\ \left. - 2\eta(1 - \eta) f^*(\omega_s, \omega_i, \theta_s, \theta_i, \varphi_s, \varphi_i) f(\omega_i, \omega_s, \theta_s, \theta_i, \varphi_s, \varphi_i) \exp [i(\omega_s - \omega_i) \tau] \right\}. \quad (\text{I.13})$$

To facilitate discussion in the following examples, we define the HOM visibility:

$$V := \frac{P_{\text{max}} - P_{\text{min}}}{P_{\text{max}}},$$

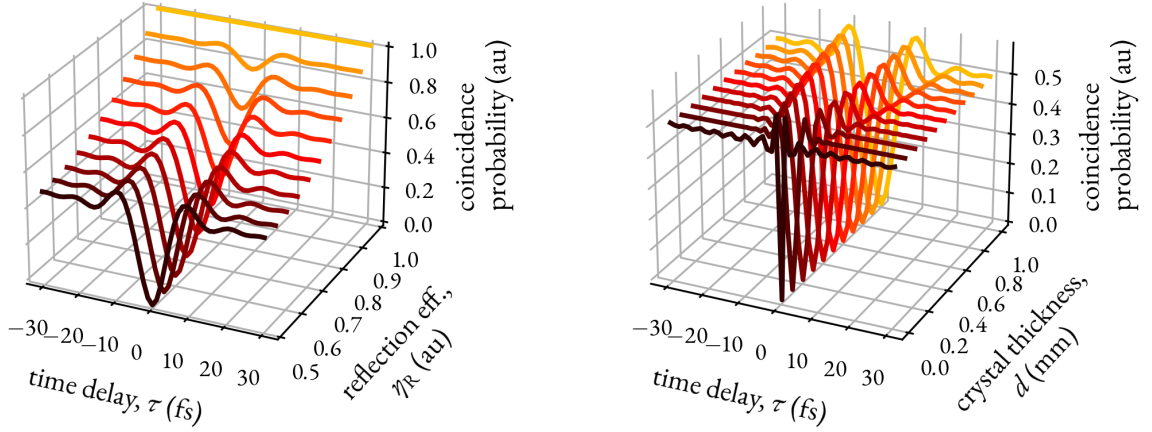


Figure I.1: The HOM dip in the coincidence probability of biphotons produced in a BBO crystal, angle-tuned for collinear, degenerate down-conversion. In the left figure, we use 1 mm-thick crystal and vary the coupling efficiency. In the right figure, we use a 50% coupling efficiency and vary the crystal thickness.

where $P_{\max} := \lim_{\tau \rightarrow \infty} P(\tau)$ and $P_{\min} := \lim_{\tau \rightarrow 0} P(\tau)$. We plot Equation (I.13) for two different crystals and for different coupling efficiencies: first, a crystal made of BBO, and second, a crystal made of x -cut LN. The coincidence probabilities for BBO and LN are plotted in Figure I.1 and Figure I.2, respectively. For both figures, each crystal is pumped with a 405 nm CW laser in the low gain regime. We assume that all light produced to the left of the pump axis is collected by one multi-mode fiber and sent into one arm of a beam splitter with coupling efficiency η . The light produced to the right of the pump axis is separately collected and sent into the other input of the beam splitter, and we plot the coincidence probability of two detectors at the output as a function of idler time delay τ . We integrate over a wide range of angular frequencies ($0.1\omega_p \sim 0.9\omega_p$), and consider different time delays for each crystal. We assume coupling and detection efficiencies of 100%.

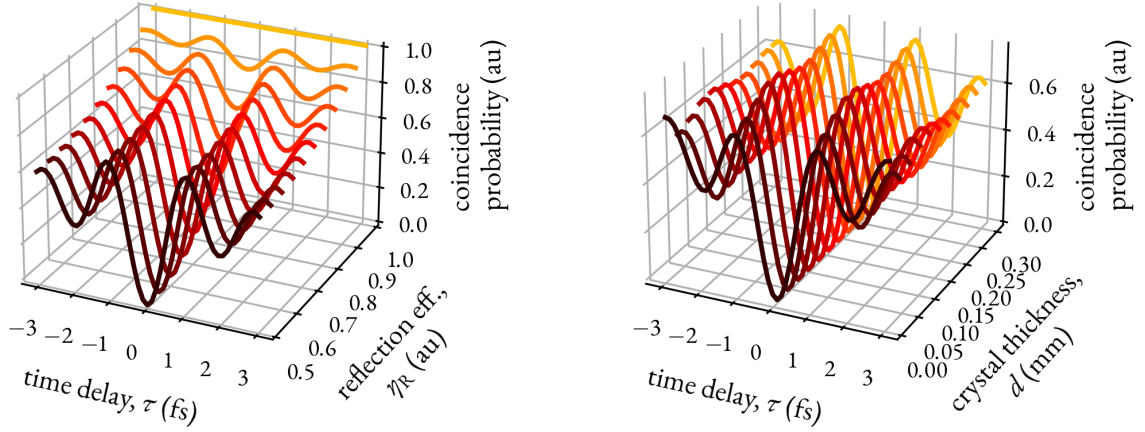


Figure I.2: The HOM dip in the coincidence probability of biphotons produced in a LN crystal, angle-tuned for collinear, degenerate down-conversion. In the left figure, we use 0.3 mm-thick crystal and vary the coupling efficiency. In the right figure, we use a 50% coupling efficiency and vary the crystal thickness.

The left part of Figure I.1 demonstrates several important things. Firstly, as expected, for a 50 : 50 beam splitter at zero time delay, the two daughter photons interfere and exit the beam splitter at the same output port - we see no coincidences [$P(0) = 0$]. In the case that they are distinguished by a time delay, the HOM dip disappears and the beam splitter behaves classically. We also observe that as the reflection efficiency increases (or decreases), the HOM visibility decreases. This is intuitive: if the photons most often pass through the beam splitter without being reflected, there is a very low chance of two photons exiting at the same port. This phenomenon is also seen in LN (see Figure I.2). In the left part of Figure I.2 the HOM visibility is ideal at zero time delay, however, the width of the HOM dip is much smaller in LN than in BBO. This is because the LN crystal produces biphotons with a much broader frequency range than the BBO. The angular frequency range of the incident radiation

is inversely related to the time width of the HOM dip, so we see a much narrower time width in LN as compared to BBO (1 fs in LN compared to 10 fs in BBO).

In the right part of Figure I.1, we study the effect of BBO crystal thickness. As the crystal thickness decreases, so too does the width of the HOM dip. This is because thinner BBO crystals produce biphotons with a wider bandwidth. Therefore as crystal thickness increases, so does the width of the HOM dip. This effect is as clear in LN. In Figure I.2, the time width of the HOM dip is relatively invariant with respect to crystal thickness. This is because the LN is not phase matched, so the LN biphoton bandwidth is relatively invariant of crystal thickness.

Using these results, we make some conclusions on the possibility of observing two-photon interference of biphotons produced in thin crystals. As we saw, thin crystals produce wideband radiation, and this reduces the temporal width of the HOM dip. This limits the experimental viability of such an observation, as the sensing electronics would need to be extremely fast. Secondly, we are primarily interested in splitting the photons using a volume hologram. The efficiency of the hologram would be highly dispersive in frequency and angle, and this dispersion would limit the HOM visibility. The full dispersive calculations have yet to be done [Equation (I.12)]. Further, we want to perform the splitting in the same crystal in which the biphotons are being produced, but in LN, we have experimentally achieved efficiencies less than 1%, which would make the HOM dip extraordinarily difficult to resolve. Finally, we mathematically treated the generation of biphotons and beam splitting as separate operations, but, as they would occur simultaneously in a holographic crystal, it may be worthwhile to create a model which combines the two effects. To summarize, the achievable diffraction efficiencies in holograms made of nonlinear crystals, their limited thickness, and their dispersiveness greatly reduces the viability of observing holographic two photon creation and interference.

J

SYSTEM OF UNITS AND PHYSICAL CONSTANTS

SYSTEM OF UNITS

quantity	unit	symbol	base dimension
length	meter	m	
mass	kilogram	kg	
time	second	s	
temperature	kelvin	K	
current	ampere	A	
frequency	hertz	Hz	
force	nerton	N	kgm/s^2
pressure	pascal	pa	N/m^2
energy	joule	J	Nm
power	watt	W	Nm/s
charge	coulomb	C	As
electric potential	volt	V	Nm/As
conductance	siemens	S	$\text{A}^2\text{s/Nm}$
resistance	ohm	Ω	$\text{Nm/A}^2\text{s}$
capacitance	farad	F	$\text{A}^2\text{s}^2/\text{Nm}$
magnetic flux density	tesla	T	N/Am
inductance	henry	H	Nm/A^2

PHYSICAL CONSTANTS

symbol	quantity	value	units
N_A	Avogadro's number	6.022×10^{23}	mol^{-1}
k_B	Boltzmann's constant	1.381×10^{-23}	J/K
		8.617×10^{-5}	eV/K
e	charge of the electron	-1.602×10^{-19}	C
m_e	mass of electron	9.109×10^{-31}	kg
m_p	mass of proton	1.673×10^{-27}	kg
ϵ_0	permeability of free space (magnetic constant)	$4\pi \times 10^{-7}$	H/m
μ_0	permittivity of free space (electric constant)	8.854×10^{-12}	F/m
h	Planck's constant	6.626×10^{-34}	J s
$\hbar$	Planck's reduced constant	1.055×10^{-34}	J s
n_0	refractive index in vacuum	1	-
c	speed of light in vacuum	2.998×10^8	m/s ²
R	universal gas constant	8.314	J/molK

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