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ON SOLID ANGLES

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ABSTRACT

An ordinary angle between two intersecting curves in Euclidean Space at the point of intersection Zo is defined as the arc length which the tangents at Zo intercept on a unit circle with centre at Zo. We also learn in Complex Analysis that this angle is preserved under a map F iff F is analytic and F'(Zo) does not vanish.

This idea can be extended to a solid angle in $\mathbb{R}^3$ and $\mathbb{R}^n$. This thesis is concerned with what kind of mapping will preserve solid angles. In contrast to the rich variety of conformal maps in two dimensions it turns out that only trivial maps preserve all solid angles.

In Chapter I, the definition and measure of a solid angle are given. Chapter II developes some necessary conditions which a map F must satisfy if it shall preserve all solid angles and shows that these necessary conditions are not sufficient. Chapter III concludes that F is a similarity. Generalizations to higher dimensions will be given in the Appendix.

It is remarkable that a map which preserves all solid angles automatically preserves all ordinary angles, but not conversely. The inversion in spheres preserves ordinary angles, but not solid angles.

CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1. Definition. Let V be a vector space over R, we denote the k-fold product $V \times \ldots\ldots \times V$ by V^k. A function $T : V^k \to R$ is a *multilinear function* (or a *k-tensor* on V) if $\forall i$ with $1 \leq i \leq k$ we have

$$T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k)$$

$$T(v_1, \ldots, av_i, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots, v_k)$$

We denote the set of all k-tensors by $\mathcal{J}^k(V)$

2. Definition. If $S \in \mathcal{J}^k(V)$ and $T \in \mathcal{J}^\ell(V)$, the *tensor product* $S \otimes T \in \mathcal{J}^{k+\ell}(V)$ is defined by

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell})$$

3. Theorem. Let $v_1, \ldots, v_n$ be a basis for V, and let $\phi_1, \ldots, \phi_n$ be the dual basis, $\phi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products

$$\phi_{i_1} \otimes \ldots\ldots \otimes \phi_{i_k} \qquad 1 \leq i_1, \ldots, i_k \leq n$$

is a basis for $\mathcal{J}^k(V)$, which therefore has dimension n^k.

Proof: See bibliography (1) p. 76.

4. Definition. A k-tensor $\omega \in \mathcal{J}^k(V)$ is called *alternating* if

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) \quad \text{for all } v_1, \ldots, v_k \in V .$$

The set of all alternating k-tensors is denoted by $\Lambda^k(V)$.

5. Definition. If $T \in \mathcal{J}^k(V)$, $\mathrm{Alt}(T)$ is defined by

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k is the set of all permutations of the numbers 1 to k .

6. Lemma. If $T \in \mathcal{J}^k(V)$, then $\mathrm{Alt}(T) \in \Lambda^k(V)$.

Proof: See bibliography (1) p. 78.

7. Definition. If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, the *wedge product* $\omega \wedge \eta \in \Lambda^{k+\ell}(V)$ is defined by

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta)$$

8. Theorem. The set of all

$$\phi_{i_1} \wedge \ldots\ldots \wedge \phi_{i_k} \qquad 1 \le i_1 < i_2 < \ldots < i_k \le n$$

is a basis for $\Lambda^k(V)$, which therefore has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof: See bibliography (1) p. 81.

9. Definition. If $p \in \mathbb{R}^n$, the set of all pairs (p,v), for $v \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$, and called the *tangent space* of $\mathbb{R}^n$ at p.

10. Definition. A *vector field* is a function F s.t. $F(p) \in \mathbb{R}^n_p$, $\forall p \in \mathbb{R}^n$.

11. Definition. A function ω with $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ is called a *differential form* on $\mathbb{R}^n$). If $\phi_1(p), \ldots, \phi_n(p)$ is the dual basis to $(e_1)_p, \ldots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1<\ldots<i_k} \omega_{i_1, \ldots, i_k}(p) \cdot [\phi_{i_1}(p) \wedge \ldots \wedge \phi_{i_k}(p)]$$

for certain functions $\omega_{i_1, \ldots, i_k}$; the form ω is continuous, differentiable if these functions are. We shall assume forms are differentiable.

12. Definition. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$. A 1-form df is defined by

$$df(p)(v_p) = Df(p)(v) .$$

13. Consider a particular 1-form dx_i. Since $dx^i(p)(v_p) = Dx^i(p)(v) = v^i$, we see that $dx_1(p), \ldots, dx_n(p)$ is just the dual basis to $(e_1)_p, \ldots, (e_n)_p$. Thus every k-form ω can be written

$$\omega = \sum_{i_1<\ldots<i_k} \omega_{i_1, \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

14. Definition. We define a $(k+1)$-form $d\omega$, the *differential* of ω by

$$d\omega = \sum_{i_1<\dots<i_k} d\omega_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

15. Definition. A form ω is called *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$, for some η.

16. Definition. Let $U, V \subset \mathbb{R}^n$ be open sets, a differentiable function $h : U \to V$ with a differentiable inverse $h^{-1} : V \to U$ is called a *diffeomorphism*. "Differentiable" henceforth means "C^∞".

17. Definition. A subset M of $\mathbb{R}^n$ is called a *k-dimensional manifold* in $\mathbb{R}^n$ if $\forall x \in M$ the following condition is satisfied:

(M) There is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ s.t.

$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = \dots = y^n = 0\}$.

18. Theorem. A subset M of $\mathbb{R}^n$ is a k-dimensional manifold iff $\forall x \in M$ the following "coordinate condition" is satisfied:

There is an open set U containing x, an open set $W \subset \mathbb{R}^k$, and $1-1$ differentiable function $f : W \to \mathbb{R}^n$ s.t.

(1) $f(W) = M \cap U$,

(2) $f'(y)$ has rank k $\forall y \in W$,

(3) $f^{-1} : f(W) \to W$ is continuous .

where f is called a *coordinate system* around x.

Proof: See bibliography (1) p. 111.

19. Definition. A subset M of $\mathbb{R}^n$ is a *k-dimensional manifold-with-boundary* if $\forall x \in M$ either condition (M) or the following condition is satisfied:

(M') There is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ s.t. $h(U \cap M) = V \cap (H^k \times \{0\}) = \{y \in V : y^k \geq 0$ and $y^{k+1} = \dots = y^n = 0\}$ and $h(x)$ has k th component $= 0$, where $H^k = \{x \in \mathbb{R}^k : x^k \geq 0\}$.

20. Definition. The set of all point $x \in M$ satisfied (M') is called the *boundary* of M and denoted ∂M.

21. Definition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, we have a linear transformation $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$. A linear transformation $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)} .$$

22. Definition. Let M be a k-dimensional manifold in $\mathbb{R}^n$ and let $f : W \to \mathbb{R}^n$ be a coordinate system around $x = f(a)$

for $a \in W$. The k-dimensional subspace $f_*(\mathbb{R}^k_a)$ of $\mathbb{R}^n_x$ ($f_* : \mathbb{R}^k_a \to \mathbb{R}^n_x$) is called the *tangent space* of M at x and denoted M_x.

23. Definition. A function ω which assigns $\omega(x) \in \Lambda^p(M_x)$ $\forall x \in M$ is called a *p-form on M* (M is a k-dimensional manifold in $\mathbb{R}^n$). A p-form ω on M can be written as

$$\omega = \sum_{i_1<\dots<i_p} \omega_{i_1, \dots, i_p} \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

24. Theorem. Let M be a two-dimensional manifold (or manifold-with-boundary) with an orientation in $\mathbb{R}^3$ and let N be the unit outward normal. Then

(1) $dS = N_1 \, dx_2 \wedge dx_3 + N_2 \, dx_3 \wedge dx_1 + N_3 \, dx_1 \wedge dx_2$.

Moreover, on M we have

(2) $N^1 dS = dx_2 \wedge dx_3$

(3) $N^2 dS = dx_3 \wedge dx_1$

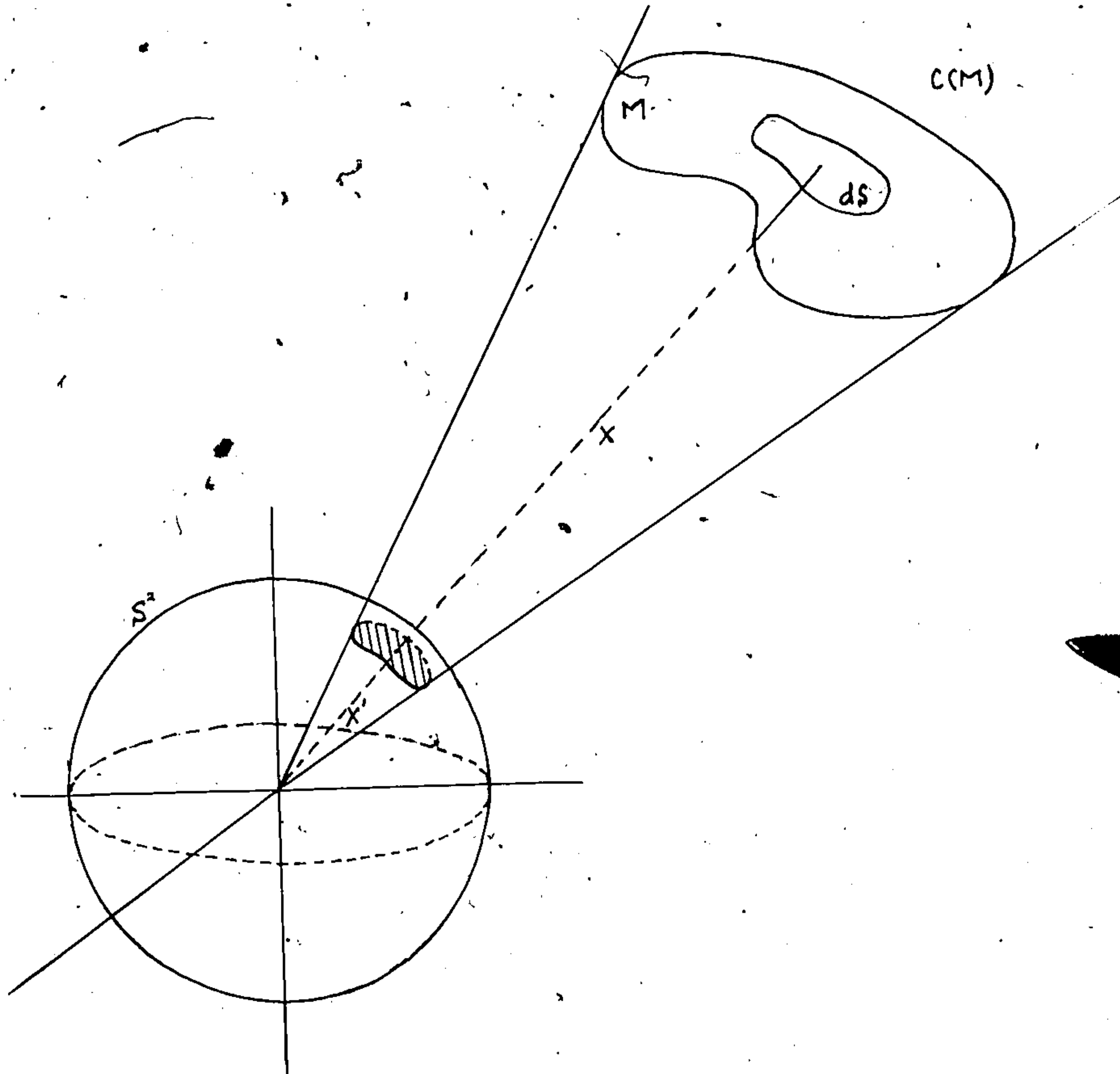
(4) $N^3 dS = dx_1 \wedge dx_2$

where dS denotes the element of volume on M.

Proof: See bibliography (1) p. 128.

25. Definition. Let $M \subset \mathbb{R}^n - \{0\}$ be a compact $(n-1)$-dimensional manifold-with-boundary with an orientation s.t. every ray through 0 intersects M at most once. The union of those rays through 0 which intersect M, is a solid cone $C(M)$. The *solid angle* Ω subtented by M is defined as the

area of $C(M) \cap S^{n-1}$, (here $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$).



Solid angle subtended by a compact 2-dimensional manifold-with-boundary M in $\mathbb{R}^3-\{0\}$.

26. Theorem. Let $M \subset \mathbb{R}^3 - \{0\}$ be a compact 2-dimensional manifold-with-boundary with an orientation. The solid angle Ω subtended by M is given by

$$\Omega = \left| \iint_M \frac{x \cdot N(x)}{r^3} \, dS \right|$$

$$= \left| \iint_M \tau \right|$$

where $\tau = \dfrac{x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2}{r^3}$ is a closed 2-form on $\mathbb{R}^3 - \{0\}$

$x = \{x_1 \; x_2 \; x_3\} \in M, \qquad |x| = r$

$N(x)$ is the unit outward normal at point x

dS is the area element of M.

Moreover, Ω can be locally expressed as

$$\Omega = \left| \int_{\partial M} \frac{x_1}{r} \, d(\text{arc tg} \frac{x_2}{x_3}) \right| = \left| \int_{\partial M} \frac{x_2}{r} \, d(\text{arc tg} \frac{x_3}{x_1}) \right|$$

$$= \left| \int_{\partial M} \frac{x_3}{r} \, d(\text{arc tg} \frac{x_1}{x_2}) \right| .$$

The last expression is remarkable because only ∂M and not M appears.

Proof: (1) Let the surface M be parametrized by the coordinates system u,v, then $x(u,v) = \{x_1(u,v), x_2(u,v), x_3(u,v)\}$ is a point on M. An element of area dS at point

$x(u,v)$ is given by

$$N \cdot (x_u \times x_v)\, du \wedge dv = \frac{(x_u \times x_v)^2}{|x_u \times x_v|}\, du \wedge dv \,, \text{ where } N = \frac{x_u \times x_v}{|x_u \times x_v|}$$

$$= |x_u \times x_v|\, du \wedge dv$$

$$= [(x_u \times x_v)\cdot(x_u \times x_v)]^{\frac{1}{2}}\, du \wedge dv$$

$$= [(x_u \cdot x_u)(x_v \cdot x_v) - (x_u \cdot x_v)(x_u \cdot x_v)]^{\frac{1}{2}}\, du \wedge dv$$

$$= (EG - F^2)^{\frac{1}{2}}\, du \wedge dv$$

where $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_v \cdot x_v$ are the first fundamental coefficients of $x(u,v)$.

Let $x'(u, v)$ be the point on $C(M) \cap S^2$ s.t. $x'(u,v) = \frac{x(u,v)}{|x(u,v)|}$, then the area elements dS' at point $x'(u, v)$ is $(E'G' - F'^2)^{\frac{1}{2}}\, du \wedge dv$ where E', G', F' are the first fundamental coefficients of $x'(u, v)$

$$x' = \frac{x}{\sqrt{x^2}}$$

$$x'_u = \frac{(x \cdot x)x_u - x(x \cdot x_u)}{(x^2)^{3/2}} = \frac{x \times (x_u \times x)}{(x^2)^{3/2}}$$

$$x'_v = \frac{(x \cdot x)x_v - x(x \cdot x_v)}{(x^2)^{3/2}} = \frac{x \times (x_v \times x)}{(x^2)^{3/2}}$$

$$E' = \frac{(x^2)^2\, x_u^2 + (x \cdot x_u)^2\, x^2 - 2[x^2\, x_u \cdot (x \cdot x_u)x]}{(x^2)^3}$$

$$= \frac{(x^2)^2\, E - x^2 (x \cdot x_u)^2}{(x^2)^3}$$

$$G' = \frac{(x^2)^2\, G - x^2(x \cdot x_u)^2}{(x^2)^3}$$

$$E'G' = \frac{1}{(x^2)^6}\,[(x^2)^4\, EG + (x^2)^2\, (x \cdot x_u)^2\, (x \cdot x_v)^2 - (x^2)^3\, E\, (x \cdot x_v)^2 - (x^2)^3\, G\, (x \cdot x_u)^2]$$

$$F' = \frac{1}{(x^2)^3}\,[(x^2)^2\, (x_u \cdot x_v) + x^2(x \cdot x_v)(x \cdot x_u) - x^2\, x_v\, (x \cdot x_u)x - x^2 x_n (x \cdot x_v)x]$$

$$F' = \frac{1}{(x^2)^3}\,[(x^2)^2\, F - x^2(x \cdot x_v)\, (x \cdot x_u)]$$

$$F'^2 = \frac{1}{(x^2)^6}\,[(x^2)^4\, F^2 + (x^2)^2\, (x \cdot x_v)^2\, (x \cdot x_u)^2 - 2(x^2)^3\, F\, (x \cdot x_u)\, (x \cdot x_v)]$$

$$dS'^2 = (E'\, G' - F'^2)\, (du \wedge dv)^2$$

$$= \frac{1}{(x^2)^6}\, \{(x^2)^4\, (EG - F^2) - (x^2)^3\, [E(x \cdot x_v)^2 + G(x \cdot x_u)^2 + 2F\, (x \cdot x_u)\, (x \cdot x_v)]\}\, (du \wedge dv)^2$$

$$= \frac{1}{(x^2)^3}\, \{(x \cdot x)\, (x_u \times x_v)^2 - [(x \cdot x_v)x_u - (x \cdot x_u)\, x_v]^2\}\, (du \wedge dv)^2$$

$$= \frac{1}{(x^2)^3}\, \{(x \cdot x)\, (x_u \times x_v)^2 - [x \times (x_u \times x_v)]^2\}\, (du \wedge dv)^2$$

$$= \frac{1}{(r^2)^3}\, [x \cdot (x_u \times x_v)]^2\, (du \wedge dv)^2$$

$$dS' = \frac{1}{r^3} [x \cdot (x_u \times x_v)] \, du \wedge dv$$

$$= \frac{x \cdot N}{r^3} \, dS$$

$$\therefore \Omega = \int_{C(M) \cap S^2} dS' = \int_M \frac{x \cdot N}{r^3} \, dS .$$

(2) By Theorem (24), we obtain

$$\Omega = \int_M \frac{x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2}{r^3}$$

Let $\tau = \dfrac{x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2}{r^3}$,

$$\omega = x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2$$

and let $\eta = dx_1 \wedge dx_2 \wedge dx_3$,

then $d\omega = 3 \, dx_1 \wedge dx_2 \wedge dx_3 = 3\eta$,

$$d\tau = \frac{1}{r^3} d\omega - \frac{3}{r^{3+2}} (rdr)\omega \qquad \text{where } r^2 = \sum_{i=1} x_i^2$$

$$= \frac{3\eta}{r^3} - \frac{3r^2\eta}{r^{3+2}}$$

$$= 0 .$$

Therefore

$$\Omega = \left| \int_M \tau \right|$$

with τ is a closed 2-form on $\mathbb{R}^3 - \{0\}$

(3) Consider the vector field $A = \frac{x}{r^3}$.

$$\text{div } A = r^{-3}\,\nabla \cdot x + x \cdot \nabla r^{-3}$$

$$= 3r^{-3} + x \cdot (-3r^{-4}\,\nabla r)$$

$$= 3r^{-3} - 3r^{-5}\, x \cdot x$$

$$= 3r^{-3} - 3r^{-3} = 0$$

holds everywhere in $R^3 - \{0\}$, there exists locally a vector-field B s.t.

$$\text{curl } B = A \; .$$

Since $\nabla \times (B + \nabla\phi) = \nabla \times B$ for an arbitrary function ϕ of class C^2 , so B is not uniquely determined.

Let e_1, e_2, e_3 be the basis of R^3 and let $A = \{a(x_1, x_2, x_3),\ b(x_1, x_2, x_3),\ c(x_1, x_2, x_3)\}$ we wish to find $B = \{u(x_1, x_2, x_3),\ v(x_1, x_2, x_3),\ w(x_1, x_2, x_3)\}$ s.t. $A = \nabla \times B$, i.e.

$$a = D_2 w - D_3 v$$
$$b = D_3 u - D_1 w$$
$$c = D_1 v - D_2 u$$

are satisfied in $R^3 - \{0\}$.

Assume $w = 0$, we obtain

$$a = -D_3 v$$
$$b = D_3 u \qquad \text{- - - - - - - - - - - - -} \quad (*)$$
$$c = D_1 v - D_2 u \; .$$

(*) is satisfied if

$$v = -\int_{x_3^0}^{x_3} a(x_1, x_2, \zeta)d\zeta$$

where x_3^0 and subsequently x_2^0 are the x_3 - and x_2-coordinates respectively of an arbitrary fixed point of $R^3 - \{0\}$, and

$$u = \int_{x_3^0}^{x_3} b(x_1, x_2, \zeta)\, d\zeta + \alpha(x_1, x_2) .$$

Since

$$C = D_1 v - D_2 u$$

$$= -\int_{x_3^0}^{x_3} [D_1\, a(x_1, x_2, \zeta) + D_2\, b(x_1, x_2, \zeta)]d\zeta - D_2\, \alpha(x_1, x_2)$$

and $D_1\, a + D_2\, b = - D_3\, c$, we obtain

$$C = C(x_1, x_2, x_3) - C(x_1, x_2, x_3^0) - D_2\, \alpha(x_1, x_2)$$

hence

$$\alpha(x_1, x_2) = -\int_{x_2^0}^{x_2} C(x_1, \eta, x_3^0)\, d\eta .$$

The vector B defined by the functions

$$u = \int_{x_3^0}^{x_3} b(x_1, x_2, \zeta)d\zeta - \int_{x_2^0}^{x_2} C(x, \eta, x_3^0)d\eta$$

$$v = -\int_{x_3^0}^{x_3} a(x_1, x_2, \zeta)d\zeta$$

$$w = 0$$

is a solution of our problem. In general

$$B = u\, e_1 + v\, e_2 + w\, e_3 + \nabla\phi$$

where ϕ is an arbitrary twice continuously differentiable function. Now $A = \frac{x}{r^3}$, we obtain

$$B = \left(\int_0^{x_3} \frac{x_2}{r^3}\, d\zeta\right) e_1 - \left(\int_0^{x_3} \frac{x_1}{r^3}\, d\zeta\right) e_2 + \nabla\phi$$

$$= \int_0^{x_3} \frac{x_2\, e_1 - x_1\, e_2}{(x_1^2 + x_2^2 + \zeta^2)^{3/2}}\, d\zeta + \nabla\phi$$

$$= \frac{x_3(x_2\, e_1 - x_1\, e_2)}{r\,(x_1^2 + x_2^2)} + \nabla\phi \quad .$$

Since

$$\Omega = \int\int_M \nabla \times B \cdot N\, dS \ ,$$

by Stoke's Theorem,

$$\Omega = \int_{\partial M} B \cdot \frac{dx}{ds}\, ds$$ where ds denotes the element of arc length

$$= \int_{\partial M} \frac{x_3(x_2 dx_1 - x_1 dx_2}{r(x_1^2 + x_2^2} + \int_{\partial M} \nabla\phi \cdot dx \ , \quad \text{where}$$

$$\oint \nabla\phi \cdot dx = \oint d\phi = 0$$

$$= \int_{\partial M} \frac{x_3}{r}\, d(\text{arc tg}\, \frac{x_1}{x_2}) \quad .$$

Similarly, we obtain

$$\Omega = \int_{\partial M} \frac{x_2}{r}\, d(\text{arc tg}\, \frac{x_3}{x_1})$$

$$= \int_{\partial M} \frac{x_1}{r}\, d(\text{arc tg}\, \frac{x_2}{x_3}) \quad .$$

CHAPTER II

In analogy to the problem of conformal map in two dimensions we now try to find out under what conditions a map F preserves solid angles. In this chapter we restrict our attention to $\mathbb{R}^3$ space.

1. Definition: Let $f_1, \ldots, f_n$ be n real-valued functions defined on an open set D in $\mathbb{R}^n$, and let $F = (f_1, \ldots, f_n)$. We denote the *Jacobian of* F by $J(F)$,

$$J(F)(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix}$$

at those point $x \in D$ where all $\frac{\partial f_i}{\partial x_j}(x)$ exist.

2. Theorem: Let $D \subset R^3$ be an open set and $F : D \to R^3$, $F = (f_1, f_2, f_3)$ each $f_i(x_1, x_2, x_3)$ a map of class C^1, which preserves all solid angles. Then the fully six partial differential equations hold in D:

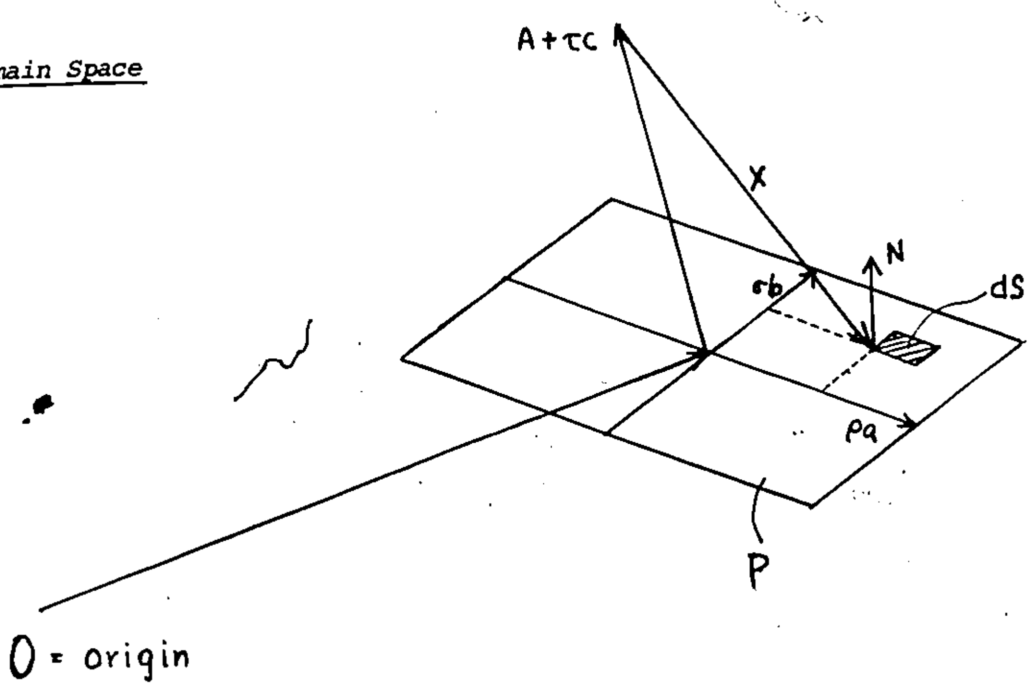
$$\left|\frac{\partial F}{\partial x_i}\right|^3 = |\det J(F)| \neq 0 \qquad \text{for } i = 1,2,3$$

and $\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} = 0$ for $i \neq k$.

Proof: In order to derive necessary conditions which F must satisfy, we consider a special type of solid angle and its image, defined as follow: Let a, b, c be three linearly independent vectors in R^3, and A an arbitrary point of D. (We identify points and their position vectors. A fixed orthonormal Cartesian system of coordinates is used in both spaces).

Consider the parallelogram P with centre A whose sides are parallel to a and b and of length $2\rho|a|$ and $2\sigma|b|$. It is seen from the point $A + \tau c$ under a solid angle Ω. (ρ, σ, τ are positive quantities. Since D is open, they can be chosen so small that P and $A + \tau c$ belong to D)

Domain Space



Let $N = \frac{a \times b}{|a \times b|}$ be the unit normal to the plane of the parallelogram P and $x = A + \alpha a + \beta b - (A + \tau c) = \alpha a + \beta b - \tau c$, a vector from the vertex $A + \tau c$ of the cone to a general point of the parallelogram P, with $|\alpha| \le \rho$ and $|\beta| \le \sigma$ variables. Then the area element on the parallelogram is

$$dS = |d\alpha\ a \times d\beta\ b| = |a \times b|\ d\alpha\ d\beta,$$

hence

$$\Omega = \iint_P d\Omega$$

$$= \iint_P \frac{x \cdot N}{r^3}\, dS \qquad \text{where } r \equiv |x|$$

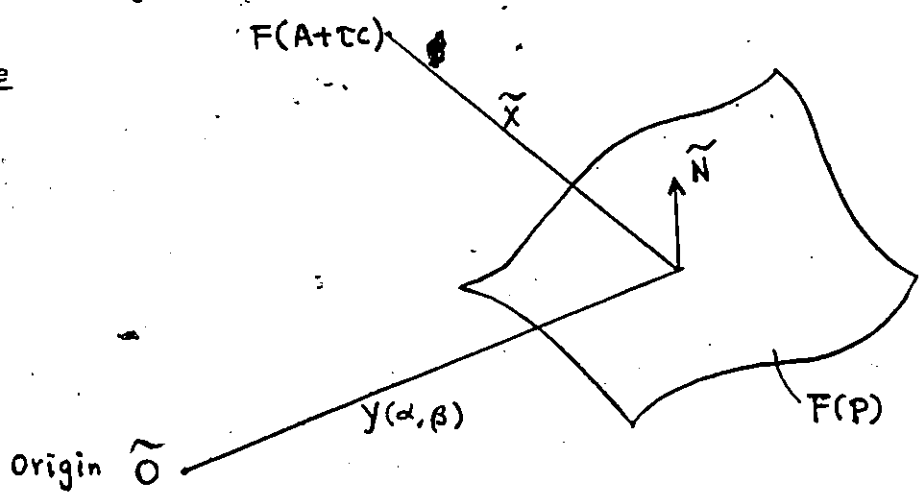
$$= \int_{\alpha=-\beta}^{\alpha=+\beta} \int_{\beta=-\sigma}^{\beta=+\sigma} \frac{(\alpha a + \beta b - \tau c) \cdot (a \times b)}{|\alpha a + \beta b - \tau c|^3\ |a \times b|}\ |a \times b|\ d\beta\ d\alpha$$

By the Mean Value Theorem of the Integral Calculus,

$$\Omega = \left\{ \frac{[\alpha_0\ a + \beta_0\ b - \tau c,\ a,\ b]}{|\alpha_0\ a + \beta_0\ b - \tau c|^3} \right\} 4\rho\,\sigma$$

with certain $|\alpha_0| \le \rho$, $|\beta_0| \le \sigma$.

Image Space



can be substituted to get the transformed solid angle

$$\tilde{\Omega} = \int_{\alpha=-\rho}^{\alpha=+\rho} \int_{\beta=-\sigma}^{\beta=+\sigma} \frac{\tilde{x} \cdot (y_\alpha \times y_\beta)}{\tilde{r}^3 \; |y_\alpha \times y_\beta|} \, |y_\alpha \times y_\beta| \, d\alpha \, d\beta$$

Again using the Mean Value Theorem of the Integral Calculus, we obtain

$$\tilde{\Omega} = \left\{ \frac{\tilde{x} \cdot (y_\alpha \times y_\beta)}{r^3} \right\}_0 4 \rho \sigma$$

which is taken at a certain point $|\alpha_1| \leq \rho$ and $|\beta_1| \leq \sigma$. Since $F \in c^1$, by Taylor's Formula, we obtain the following linear approximation of

$$F(A + \tau c) - F(A)$$

$$= (f_1(A + \tau c) - f_1(A), \quad f_2(A + \tau c) - f_2(A), \quad f_3(A + \tau c) - f_3(A))$$

$$= (\nabla f_1(A) \cdot \tau c + \tau \varepsilon_1, \quad \nabla f_2(A) \cdot \tau c + \tau \varepsilon_2, \quad \nabla f_3(A) \cdot \tau c + \tau \varepsilon_3))$$

$$= \tau (c \cdot \nabla f_1(A) + \varepsilon_1, \quad c \cdot \nabla f_2(A) + \varepsilon_2, \quad c \cdot \nabla f_3(A) + \varepsilon_3)$$

$$= \tau \left(c_1 \frac{\partial F}{\partial x_1} + c_2 \frac{\partial F}{\partial x_2} + c_3 \frac{\partial F}{\partial x_3}\right) + \tau \varepsilon$$

$$= \tau V + \tau \varepsilon$$

where $\varepsilon_i \to 0$ for $\tau \to 0$,

and $c = (c_1, c_2, c_3)$

$\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$V \equiv c_1 \frac{\partial F}{\partial x_1} + c_2 \frac{\partial F}{\partial x_2} + c_3 \frac{\partial F}{\partial x_3} .$$

From $|\Omega| = |\tilde{\Omega}|$ we deduce

$$| = \lim_{\substack{\rho \to 0 \\ \sigma \to 0}} \left| \frac{\tilde{\Omega}}{\Omega} \right|$$

$$= \lim_{\substack{\rho \to 0 \\ \sigma \to 0}} \left| \frac{|\alpha_0 a + \beta_0 b - \tau c|^3}{[\alpha_0 a + \beta_0 b - \tau c, a, b]\, 4\rho\sigma} \frac{F(A + \alpha_1 + \beta_1 b) - F(A + \tau c)}{|F(A + \alpha_1 a + \beta_1 b) - F(A + \tau c)|^3} (y_\alpha(\alpha_1, \beta_1) \times y_\beta(\alpha_1, \beta_1))\, 4\rho\sigma \right|$$

$$= \left| \frac{[F(A + \tau c) - F(A)] \sum_{k,j=1}^{3} a_k b_j \left(\frac{\partial F}{\partial x_k} \times \frac{\partial F}{\partial x_j}\right) \tau^2 |c|^3}{|F(A + \tau c) - F(A)|^3 [a, b, c]} \right| \quad \ldots\ldots (2)$$

where $\frac{\partial F}{\partial x_k}$ are taken at A. $[a, b, c] \neq 0$. Since a, b, c are linearly independent. In (2), we let now $\tau \to 0$ and substitute (1) in (2). Then we have

$$| = \frac{|c|^3 \left| \sum_{k,j=1}^{3} a_k b_j \left[V, \frac{\partial F}{\partial x_k}, \frac{\partial F}{\partial x_j}\right] \right|}{|V|^3 \, |[a, b, c]|} ;$$

hence $\det J(F) \neq 0$.

Hence

$$|[a,b,c]|\,|V|^3 = |c|^3 \left| \sum_{k,j=1}^{3} a_k b_j \left[c_1 \frac{\partial F}{\partial x_1} + c_2 \frac{\partial F}{\partial x_2} + c_3 \frac{\partial F}{\partial x_3}, \frac{\partial F}{\partial x_k}, \frac{\partial F}{\partial x_j}\right] \right|$$

$$= |c|^3 \left| \left[\sum_{i=1}^{3} \frac{\partial F}{\partial x_i} c_i , \sum_{j=1}^{3} \frac{\partial F}{\partial x_j} a_j , \sum_{k=1}^{3} \frac{\partial F}{\partial x_k} b_k \right] \right|$$

$$= |c|^3 \left| [a,b,c] \right| \left| \left[\frac{\partial F}{\partial x_1} , \frac{\partial F}{\partial x_2} , \frac{\partial F}{\partial x_3} \right] \right|$$

$$|V|^3 = |c|^3 \left| \det J(F) \right|$$

with $\det J(F)$ = Jacobian determinant of F at A

i.e. $\det J(F) = \left[\frac{\partial F}{\partial x_1}(A), \frac{\partial F}{\partial x_2}(A), \frac{\partial F}{\partial x_3}(A) \right]$.

So we obtain

$$|V|^2 = |c|^2 \, |\det (J(F)|^{\frac{2}{3}}$$

or $$\left| \sum_{i,k} c_i \, c_k \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} \right| = |c|^2 \, |\det J(F)|^{\frac{2}{3}} .$$

Since this must be an identity in c_1, c_2, c_3, we obtain

$$\left| \frac{\partial F}{\partial x_i} \right|^3 = |\det J(F)| \neq 0 \qquad \text{for } i = 1, 2, 3$$

and $$\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} = 0 \qquad \text{for } i \neq k$$

These relations must hold at all points $A \in D$. They are six partial differential equations for the functions f_i .

(3) Since $$J(F) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_2} \\ \frac{\partial f_1}{\partial x_3} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$$

If $\left|\frac{\partial F}{\partial x_1}\right| = \left|\frac{\partial F}{\partial x_2}\right| = \left|\frac{\partial F}{\partial x_3}\right| = \mu$ and $\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} = 0 \quad \forall i \neq k$.

$\frac{J(F)}{\mu}$ is an orthogonal matrix, so $\det\left[\frac{J(F)}{\mu}\right] = \pm 1$ implies

$\frac{1}{\mu^3} \det J(F) = \pm 1$

then $\mu^3 = |\det J(F)|$

i.e. $\left|\frac{\partial F}{\partial x_i}\right|^3 = |\det J(F)| \qquad$ for $i = 1, 2, 3$.

The above necessary conditions for F in Theorem 2 imply exactly that all $\frac{\partial F}{\partial x_i}$ have the same length and are mutually orthogonal. They are the analogues in 3 dimensions to the Cauchy-Riemann equations: we can write the C - R equations in 2 dimensions in a similar form, i.e.

$$\phi_1 = \pm \psi_2 ; \quad \phi_2 = \mp \psi_1 \quad \text{for } F = (\phi(x_1,x_2), \psi(x_1,x_2))$$

Since $\det J(F) = \begin{vmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{vmatrix} = \phi_1 \psi_2 - \psi_1 \phi_2$

$= \phi_1^2 + \psi_1^2$

or $\det J(F) = \psi_2^2 + \phi_2^2$

i.e. $|\det J(F)| = \left|\frac{\partial F}{\partial x_i}\right|^2 \qquad$ for $i = 1,2$.

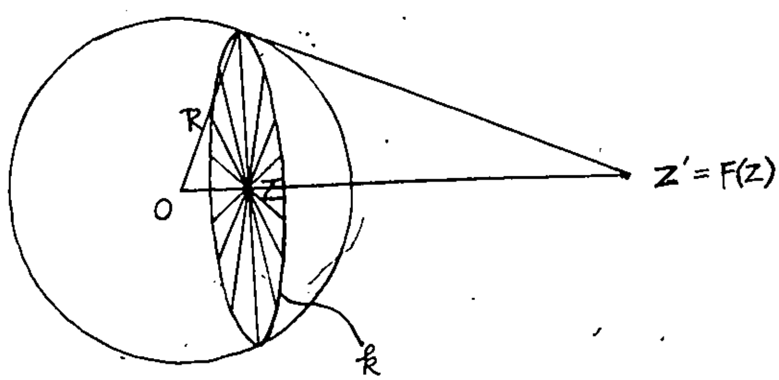
Since $\phi_1 = \pm \psi_2$, $\phi_2 = \mp \psi_1$

$$\phi_1 \phi_2 = - \psi_1 \psi_2$$

$$\phi_1 \phi_2 + \psi_1 \psi_2 = 0$$

i.e. $\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} = 0$ for $i \neq k$.

(4) Since the above necessary conditions found in Theorem 2 have been obtained by considering only special solid angles, we cannot expect them to be sufficient; and indeed they are not. For example, they are satisfied by an inversion in a sphere which does not preserve all solid angles.



Let the sphere under consideration be of center 0 and radius R and let Z be any point ≠ 0 , Z' be its image under the inversion F . Hence Z' lies on the ray OZ and $OZ \cdot OZ' = R^2$.

$$F(x) = \frac{R^2}{x \cdot x} x \qquad \text{where } x = OZ, \; F(x) = OZ', \; (x \neq 0)$$

If we let $g(x) = \frac{R^2}{x \cdot x} x$ be a scalar function of a vector variable $x = (x_1, x_2, x_3) \neq 0$

then
$$\frac{\partial F}{\partial x_1} = g_1 x + g e_1$$
$$\frac{\partial F}{\partial x_2} = g_2 x + g e_2$$
$$\frac{\partial F}{\partial x_3} = g_3 x + g e_3$$

where $\{e_1, e_2, e_3\}$ is the orthonormal basis of the Cartesian coordinate system in both domain space and image space

$$\frac{\partial F}{\partial x_1} \cdot \frac{\partial F}{\partial x_2} = g_1 g_2 x \cdot x + g_1 g x_2 + g_2 g x_1 = 0 .$$

Similarly, $\frac{\partial F}{\partial x_2} \cdot \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial x_3} \cdot \frac{\partial F}{\partial x_1} = 0$.

Since
$$I(x) = \frac{R^2}{x_1^2 + x_2^2 + x_3^2} (x_1, x_2, x_3)$$

$$\frac{\partial f_1}{\partial x_1} = R^2 \cdot \frac{x_2^2 + x_3^2 - x_1^2}{(x_1^2 + x_2^2 + x_3^2)^2}$$

$$\frac{\partial f_1}{\partial x_2} = R^2 \frac{-2x_1 x_2}{(x_1^2 + x_2^2 + x_3^2)^2}$$

$$\frac{\partial f_1}{\partial x_3} = R^2 \frac{-2x_1 x_3}{(x_1^2 + x_2^2 + x_3^2)^2}$$

Similarly we obtain $\frac{\partial f_2}{\partial x_1}$, $\frac{\partial f_2}{\partial x_2}$, $\frac{\partial f_2}{\partial x_3}$, $\frac{\partial f_3}{\partial x_1}$, $\frac{\partial f_3}{\partial x_2}$, $\frac{\partial f_3}{\partial x_3}$.

Then
$$\det J(F) = \left[\frac{R^2}{(x_1^2 + x_2^2 + x_3^2)^2}\right]^3 \begin{vmatrix} x_2^2+x_3^2-x_1^2 & -2x_1x_2 & -2x_1x_3 \\ -2x_1x_2 & x_1^2+x_3^2-x_2^2 & -2x_2x_3 \\ -2x_1x_3 & -2x_2x_3 & x_1^2+x_2^2-x_3^2 \end{vmatrix}$$

$$|\det J(F)| = \frac{R^6}{(x_1^2+x_2^2+x_3^2)^3} \neq 0$$

and $$\left|\frac{\partial F}{\partial x_1}\right|^3 = \left|\frac{\partial F}{\partial x_2}\right|^3 = \left|\frac{\partial F}{\partial x_3}\right|^3 = \left(\sqrt{\left(\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{\partial f_2}{\partial x_1}\right)^2 + \left(\frac{\partial f_3}{\partial x_1}\right)^2}\right)^3$$

$$= \frac{R^6}{(x_1^2 + x_2^2 + x_3^2)} = |\det J(F)| \neq 0$$

Hence F satisfies the necessary conditions for preservation of solid angles found in Theorem 2 above. However, the solid angle determined by the circle k with center Z seen under Z is 2π, but its image is k seen from $F(Z)$ and is obviously less than 2π. Hence the inversion does not preserve solid angles. Therefore, the above conditions for F are not sufficient.

CHAPTER III

In this chapter, we investigate what kind of map preserving solid angles based on what we found in Chapter 2. By means of the Theorem of Liouville on conformal maps of $\mathbb{R}^3$, we shall draw the conclusion that F is a similarity.

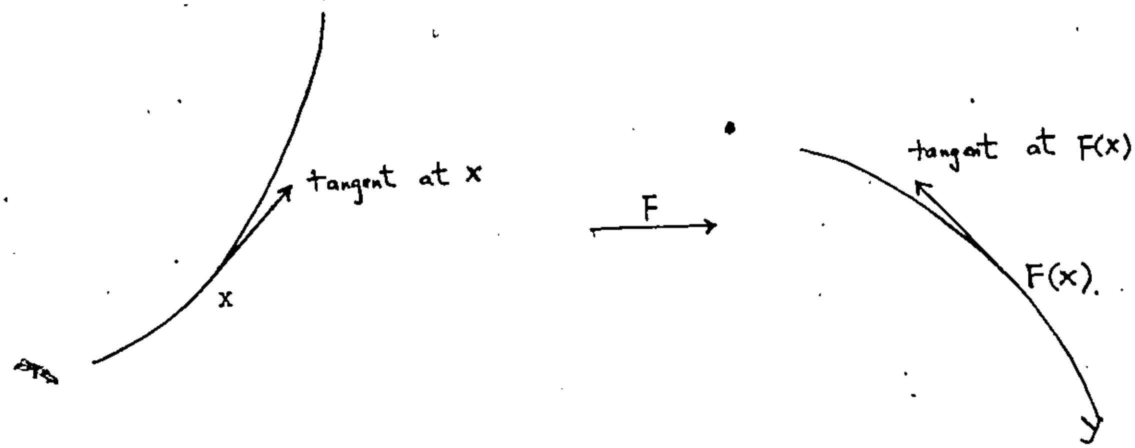
1. Definition: A *homothety* of ratio λ and with its centre the origin is a map : $x \to \lambda x$ (λ, a real number $\neq 0$).

2. Definition: A map of vectors $x \to Ax + b$ is a *euclidean motion* if A is a constant rotation matrix and b a constant vector.

3. Definition: A *similarity* of ratio λ is any map, a composition of homothety of ratio λ and a euclidean motion.

4. Obviously every similarity preserves all solid angles, since motions and homothethies do. We now prove the converse.

From Theorem 2 in Chapter 2, we deduce that the analogues to the C - R Equations mean (as do the C - R equation themselves) that the linear map F_{*p} induced by F from tangents of curves through any $P \in D$ is a similarity, hence F_{*p} preserves all ordinary angles. We are going to show the above statement as follow :



Let x be the curve through P . The tangent of the image curve y at F(x) is given by

$$
\begin{aligned}
y' &= (y_1', y_2', y_3') \\
&= dF(x) \\
&= (df_1(x), df_2(x), df_3(x)) \\
&= \Big(dx_1 \frac{\partial f_1}{\partial x_1} + dx_2 \frac{\partial f_1}{\partial x_2} + dx_3 \frac{\partial f_1}{\partial x_3},\ dx_1 \frac{\partial f_2}{\partial x_1} + dx_2 \frac{\partial f_2}{\partial x_2} + dx_3 \frac{\partial f_2}{\partial x_3}, \\
&\qquad dx_1 \frac{\partial f_3}{\partial x_1} + dx_2 \frac{\partial f_3}{\partial x_2} + dx_3 \frac{\partial f_3}{\partial x_3}\Big)
\end{aligned}
$$

$$
\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}
$$

$$F_{*P}(x) = Ax' \qquad A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$$

i.e. All tangents are transformed by a linear transformation which is determined by the Jacobian matrix A.

Our previous conditions

$$\left|\frac{\partial F}{\partial x_i}\right|^2 = |\det J(F)|^{2/3} \qquad \text{for } i = 1, 2, 3$$

and $$\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_k} = 0 \qquad \text{for } i \neq k$$

can be written in matrix notation as

$$AA^T = \begin{pmatrix} |\det J(F)|^{2/3} & 0 & \\ 0 & |\det J(F)|^{2/3} & 0 \\ 0 & 0 & |\det J(F)|^{2/3} \end{pmatrix} = |\det J(F)|^2 I$$

hence $M = \frac{1}{|\det J(F)|} A$ is an orthogonal matrix and $A = \lambda M$ with $\lambda = |\det J(F)|$ i.e. The linear map F_{*P} induced by F from tangents of curves through P is a similarity of ratio $\lambda = |\det J(F)|$.

In order to draw the same conclusion for F itself instead of F_{*p} , we are going to prove Liouville's Theorem.

5. Theorem (Liouville) Any conformal transformation F in $\mathbb{R}^3$ is either a similarity or an inversion or the composition of one inversion and one similarity.

Before proving this theorem, we now present some definitions, lemmas and theorem which we need.

5.1. Notations: A. Given an immersed surface in $\mathbb{R}^3$ by $x(u^1, u^2)$, we denote the coefficients of the First Fundamental Form by

$$g_{11} = x_1 \cdot x_1 = E \qquad \text{where } \frac{\partial x}{\partial u^i} = x_i$$

$$g_{22} = g_{12} = x_1 \cdot x_2 = F \qquad \text{for } i = 1, 2$$

$$g_{22} = x_2 \cdot x_2 = G$$

and the coefficients of the Second Fundamental Form by

$$L_{ik} = N \cdot x_{ik} = - N_i \, x_k , \qquad L_{ik} = L_{ki}$$

where $$N = \frac{x_1 \times x_2}{|x_1 \times x_2|} \quad \text{and} \quad N_i = \frac{\partial N}{\partial u^i}$$

B. As usual we obtain the derivatives of the vector N, given by the Formulas of Weingarten

$$N_k = - L_k^i x_i \quad \text{with} \quad L_k^i = g^{ij} \, L_{jk} \quad \text{where} \quad g^{i\ell} g_{\ell k} = \delta_k^i, \; g^{i\ell} = g^{\ell i}$$

C. Let the normal curvature of the curve $x(s) = x(u^1(s), u^2(s))$ on the surface $x(u^1, u^2)$ at a given point P_0 be

$$K_n = \frac{L_{ik}\ \xi^i\ \xi^k}{g_{ik}\ \xi^i\ \xi^k} \qquad \text{where} \quad \xi^i = u^{i'} \quad \text{for } i = 1,2. \qquad (1)$$

and let K_1, K_2 be the principal curvatures of the surface at P_0. Then $\lambda = K_1, K_2$ must satisfy

$$(L_{ik} - \lambda\ g_{ik})\xi^k = 0$$

iff

$$\det(L_{ik} - \lambda\ g_{ik}) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

For the corresponding directions of the principal curvature $\xi^i_{(\mu)}\ x_i \quad (\mu = 1,2)$, we have

$$(L_{ik} - K_1\ g_{ik})\xi^k_{(1)} = 0 \qquad g_{ik}\ \xi^i_{(1)}\ \xi^k_{(1)} = 0 \quad \ldots (3)$$

$$(L_{ik} - K_2\ g_{ik})\xi^k_{(2)} = 0 \qquad g_{ik}\ \xi^i_{(2)}\ \xi^k_{(2)} = 0$$

<u>5.2. Lemma</u>: A necessary and sufficient condition for the parametric curves to be lines of curvature is that $g_{12} = L_{12} = 0$.

<u>Proof</u>: (A) Assume the parametric curves (u^1 - curves, u^2 - curves) through point P_0 to be lines of curvature. Then they are mutually perpendicular, and we have $g_{12} = 0$.

Since $x_1 = 1x_1 + 0x_2$ are tangents to u^1-curves where $\xi^1_{(1)} = 1$, $\xi^2_{(1)} = 0$

$x_2 = 0x_1 + 1x_2$ are tangents to u^2-curves where $\xi^1_{(2)} = 0,$
$\xi^2_{(2)} = 1.$

From (3) we have

$$L_{11} = K_1\, g_{11}, \qquad L_{22} = K_2\, g_{22}, \qquad L_{12} = 0$$

(B) If $g_{12} = 0$, then u^1-curves and u^2-curves are mutually orthogonal. From (2), under the assumption $L_{12} = 0$ we have

$$\lambda^2 g_{11} g_{22} - \lambda(g_{11}L_{22} + g_{22}L_{11}) + L_{11}L_{22} = 0$$

$$\lambda = \frac{(g_{11}L_{22} + g_{22}L_{11}) \pm (g_{11}L_{22} - g_{22}L_{11})}{2g_{11}\, g_{22}}$$

then the principal curvatures $K_1 = \dfrac{L_{11}}{g_{11}}$, $K_2 = \dfrac{L_{22}}{g_{22}}$.

On the other hand, the normal curvature K_n of u^1-curves is

$$K_n = \frac{L_{ik}\, \xi^i\, \xi^k}{g_{ik}\, \xi^i\, \xi^k} = \frac{L_{11}}{g_{11}} = K_1$$

and K_n of u^2-curves is

$$K_n = \frac{L_{22}}{g_{22}} = K_2 \ .$$

Hence the parametric curves are lines of curvature.

<u>5.3. Lemma</u>: If a piece of surface consists entirely of umbilics and flat points, then it is a piece of a sphere or of the plane.

Proof: At an umbilic, we have $L_{ik} = \lambda g_{ik}$; at a flat point, $\lambda = 0$. Multiplication by g^{ji} and summation over i yields $L^j_k = \lambda\delta^j_k$, so that Weingartin's equations $N_k = - L^r_k x_r$ here assume the form

$$N_k = -\lambda x_k .$$

Differentiating w.r.t. u^i , we find

$$N_{ki} = -x_k \lambda_i - x_{ki} \lambda,$$

so that $0 = N_{ki} - N_{ik} = -x_k \lambda_i + x_i \lambda_k$, whence, for $i = 1, k = 2$, it follows that $\lambda_i = 0$, hence λ = constant. Thus we can integrate Weingarten's equations explicitly, obtaining $N = -\lambda x + \delta$, where δ is a constant vector. Hence, if $\lambda = 0$, then the normal vector is constant and the surface is plane, while, if $\lambda \neq 0$, we obtain $x - m = \frac{N}{\lambda}$, $(x - m)^2 = \frac{1}{\lambda^2}$, which is the equation of a sphere.

5.4. Definition: Let $H(u^1, u^2, u^3)$ be a vector function of class C^2, assume that $H_i = \frac{\partial H}{\partial u^i}$ are linearly independent, and assume finally that $H_i \cdot H_k = 0$ for $i \neq k$. Then the surfaces u^i = const. constitute three mutually orthogonal families of surface, a so called *triply orthogonal system*.

5.5. Theorem (Dupin): The curves of intersection of the surfaces of a triply orthogonal system are lines of curvature on each of these surfaces.

Proof: Differentiation of $H_i \cdot H_k = 0$ $(i \neq k)$ yields

$$H_{12}\, H_3 + H_1\, H_{23} = H_{23}\, H_1 + H_2\, H_{13} = H_{21}\, H_3 + H_2\, H_{31} = 0,$$

hence

$$H_{12}\, H_3 = H_{23}\, H_1 = H_{31}\, H_2 = 0 \quad .$$

Since the three vectors H_1, H_2, H_{12} all are orthogonal to H_3, they are linearly dependent, i.e.

$$H_1 \times H_2 \cdot H_{12} = 0$$

which implies $L_{12} = 0$, in addition to which we have $H_1 \cdot H_2 = g_{12} = 0$ for the surface $u^3 = \text{const}$. Therefore, the parametric curves are lines of curvature and this applies similarly the surfaces of the other two families.

5.6. Theorem: If the parametric curves of a surface $x = x(u^i)$ are its lines of curvature, then for sufficiently small u^3,

$$H(u^1, u^2, u^3) = x(u^1, u^2) + u^3 N \quad .$$

is a triply orthogonal system.

i.e. Every curvature line on a surface may be otained as an intersection with a surface of a triply orthogonal system.

Proof: Since the parametric curves are lines of curvature, $g_{12} = L_{12} = 0$. Suppose u^1-curves, u^2-curves are in the direction of the principal curvature K_1, K_2 respectively.

From Eq.(3), we have

$$L_{11} - K_1\, g_{11} = 0 \quad , \quad L_{22} - K_2\, g_{22} = 0$$

Since $L_{11} = N_1 x_1$ $\qquad L_{22} = N_2 x_2$

then $N_1 = -K_1 x_1$ $\qquad N_2 = -K_2 x_2$

so N_1 is proportional to x_1. Hence

$$N_1 = x_1 + u^3 N_1$$
$$N_2 = x_2 + u^3 N_2$$
$$N_3 = N$$

imply that

$$\begin{aligned} H_1 \cdot H_2 &= x_1 \cdot x_2 + u^3 x_1 \cdot N_2 + u^3 x_2 \cdot N_1 + N_1 \cdot N_2 \\ &= u^3 [x_1 \cdot (-K_2 x_2)] + u^3 [x_2 \cdot (-K_1 x_1)] + N_1 \cdot N_2 \\ &= 0 . \end{aligned}$$

$$H_2 \cdot H_3 = N \cdot x_2 + u^3 N_2 \cdot N = 0$$
$$H_3 \cdot H_1 = N \cdot x_1 + u^3 N_1 \cdot N = 0$$

which proves the asserted orthogonality. Since $x_1 \times x_2 \neq 0$, we also have, for sufficiently small u^3, that

$$H_1 \times H_2 \neq 0 ,$$

so that we are indeed dealing with surfaces.

5.7. Definition: A map F of class $C^1 : D \to \mathbb{R}^n$ which preserves all ordinary angles in magnitude and sense is called *conformal*.

5.8. Properties of Inversion: Consider an inversion

$$y = R^2 \frac{x}{|x|^2} , \quad x \neq 0 .$$

A. y is the vector of length $R^2 |x|^{-1}$ on the ray of x. Hence the sphere of radius R is invariant in an inversion.

B. Consider $ax^2 + bxR^2 + cR^4 = 0$. It is a sphere for $a \neq 0$, a plane for $a = 0$, $b \neq 0$. Its image in an inversion (at least for the point $x \neq 0$) is

$$a + by + cy^2 = 0$$

which is a sphere for $c \neq 0$, a plane for $c = 0$, $b \neq 0$. Especially, spheres through the origin ($c = 0$) are transformed into planes ($c = 0$, $b \neq 0$) and planes ($a = 0$, $b \neq 0$) are transformed into spheres through origin ($a = 0$).

C. Inversion is conformal.

Let $x(s)$ and $\bar{x}(\sigma)$ be any two intersecting curves, $y(s^*)$, $\bar{y}(\sigma^*)$ respective images. s, σ, s^*, σ^* denote arc lengths, then x_s, $\bar{x}_\sigma$, y_{s^*}, $\bar{y}_{\sigma^*}$ are all unit vectors. Hence the angles of intersection of the curves and their images are equal if

$$x_s \cdot \bar{x}_\sigma = y_{s^*} \cdot \bar{y}_{\sigma^*} .$$

Since
$$y = \frac{x}{x \cdot x}$$

$$y_{s^*} = y_s \frac{ds}{ds^*} = \frac{x_s(x \cdot x) - x(2x \cdot x_s)}{(x \cdot x)^2} \frac{ds}{ds^*}$$

and
$$x_s \cdot x_s = 1, \quad y_{s^*} \cdot y_{s^*} = 1, \quad \text{we have}$$

$$\frac{ds}{ds^*} = x \cdot x$$

Similarly
$$\bar{y}_{\sigma^*} = \frac{\bar{x}_\sigma(\bar{x} \cdot \bar{x}) - \bar{x}(2\bar{x} \cdot \bar{x}_\sigma)}{\bar{x} \cdot \bar{x}}$$

Since $x = \bar{x}$ at the point of intersection, we have

$$x_s \cdot \bar{x}_\sigma = y_{s^*} \cdot \bar{y}_{\sigma^*} .$$

Hence an inversion is conformal.

D. It is an involutive transformation, i.e. two successive applications of it result in the identity transform. Since

$$y = \frac{x}{x \cdot x}, \quad \text{another inversion} \quad z = \frac{y}{y \cdot y}$$

then $$z = \frac{x}{x \cdot x} \frac{(x \cdot x)^2}{x \cdot x} = x$$

Now, we prove Liouville's Theorem as follow:

<u>Proof:</u> A conformal map F transforms a triply orthogonal system of surfaces into a triply orthogonal system of surfaces. By Theorem 5.5, the curves of intersection of the surfaces of a triply orthogonal system are lines of curvature on these surfaces. Then, by Theorem 5.6, the image of curvature lines on the image surfaces may be obtained as an intersection with the image surface of the new orthogonal system. i.e. they are curvature lines. Hence curvature lines are mapped into curvature lines, umbilics go over into umbilics. By Lemma 5.2, planes and spheres are mapped into planes and spheres.

F transforms the planes x_i = const. $(i = 1, 2, 3)$ $(x_1, x_2, x_3$ is a cartesian system of coordinates in the domain space of F) either (A) or (B) :

(A) A triply orthogonal system of planes.

(B) A triply orthogonal system of spehere all passing through $z = F(\infty)$. Here we let I be the one-to-one inversion of center z, then $IF(\infty) = \infty$. Hence the planes x_i = const. are mapped onto some triply orthogonal system of planes, i.e. $F(x_i) = y_i$= const. in a new cartesian system of coordinates in the image space.

A coordinates axis $x_i = x_j = 0$ $(i \neq j)$ is mapped onto the axis $y_i = y_j = 0$. The maps F in (A) and IF in (B) therefore have an analytic representation $y_i = y_i(x_i)$.

On the other hand, F transforms spheres $x \cdot x =$ const. either into (A') or (B'):

(A') A triply orthogonal system of spheres $y \cdot y = 0$.

(B') A triply orthogonal system of planes s.t. $\infty = F(z')$ where z' is the common point through which the spheres $x \cdot x =$ const. all pass. If I be the inversion of center z', then $IF(z') = I(\infty) = z'$. Hence spheres $x \cdot x =$ const. are mapped onto some triply orthogonal system of spheres $y \cdot y =$ const. by IF. The equations $\sum x_i \, dx_i = 0$ and $\sum y_i \frac{dy_i}{dx_i} dx_i = 0$ must hold simultaneously. Since $x \cdot dx = 0$, dx may be any vector onthogonal to x, hence $\left(y_1 \frac{dy_1}{dx_1}, \; y_2 \frac{dy_2}{dx_2}, \; y_3 \frac{dy_3}{dx_3}\right)$ must be linearly dependent on $x = (x_1, x_2, x_3)$. The differential equations

$$y_i \frac{dy_i}{dx_i} = C^2 x_i, \quad y_i(0) = 0 \qquad i = 1, 2, 3$$

imply

$$y = Cx .$$

Therefore either F or IF is a translation bringing the origin of the x_i coordinates on that of the y_i coordinates followed by a rotation of the parallels to the x_i axes onto the y_i axes followed by a homothety of ratio C. Hence either F or IF is a similarity.

By Liouville's Theorem, the map F which preserves all solid angles should be a combination of similarities and inversions. The similarities do preserves solid angles, but the inversions not. Hence we have

6. Conclusion: A map $F : D \to \mathbb{R}^3$ of class C^1, which preserves all solid angles is a similarity

$F(x) = Ax + b$, with $A = \lambda M$, M any orthogonal matrix, $\lambda > 0$ a constant scalar and b a constant vector.

APPENDIX

Generalizations to higher dimensions

1. Let 0 be an arbitrary point of $\mathbb{R}^n$ and denote $D = \mathbb{R}^n \setminus \{0\}$. The scalar product in $\mathbb{R}^n$ with respect ot a fixed orthonormal basis $e_1, e_2, \ldots, e_n$ is denoted by $\langle X, Y\rangle = \sum x_i y_i$. Furthermore let $r(P)$ be the distance of a variable point P with position vector $OP = x$ from 0, $\lambda(r)$ a C^1 function in D, and define the vector field

$$X(P) = \lambda(r)\cdot x \quad .$$

To the vector field X in D belongs a dual differential 1-form defined by

$$\alpha(Y) = \langle X,Y\rangle \quad .$$

If we write

$$\alpha = \sum \alpha_k \, dx_k$$

in the system of coordinates $dx_1, \ldots, dx_n$ dual to $C_1, \ldots, C_n$, we obtain for $Y = \sum y_i e_i$:

$$\begin{aligned}\alpha(Y) &= (\sum \alpha_k \, dx_k)(\sum y_i \, e_i) \\ &= \sum \alpha_k \, y_i \, \delta_{ki} \quad , \\ &= \sum \alpha_i \, y_i \\ &= \sum \lambda x_i \, y_i \quad ,\end{aligned}$$

hence

$$\alpha_i = \lambda\, x_i \qquad\qquad i = 1, 2, \ldots, n,$$

$$\alpha = \lambda \sum x_i\, dx_i = \frac{\lambda}{2}\, d(\sum x_i^2) = \lambda r dr \quad .$$

We conclude

$$d\alpha = \sum d(\lambda x_k) \wedge dx_k$$

$$= \sum_k (\sum_i \frac{\partial(\lambda x_k)}{\partial x_i}\, dx_i) \wedge dx_k$$

$$= \sum_k \{\sum_i [\frac{d\lambda}{dr}(\frac{x_i\, x_k}{r}) + \lambda \delta_{ki}]\, dx_i \wedge dx_k\}$$

$$= \frac{1}{r}\frac{d\lambda}{dr} \sum_{ik} x_i\, x_k\ \ dx_i \wedge dx_k = 0$$

(because for $i = k$ the terms vanish since $dx_i \wedge dx_i = 0$, and for $i \neq k$ the two terms $x_i\, x_k\, dx_i \wedge dx_k + x_k\, x_i\, dx_k \wedge dx_i = 0$) Hence α is always closed in D . It is in general not exact.

2. For an n-dimensional Riemannian manifold R with local coordinates $x_1, \ldots, x_n$ and corresponding local metric components g_{ik} the element of volume is defined by

$$\sigma = \sqrt{G}\quad dx_1 \wedge \ldots \wedge dx_n \quad \text{with} \quad G = \det(g_{ik})$$

Applying this to an $(n - 1)$-dimensional embedded hyper-surface $R = M$ in $\mathbb{R}^n$ we obtain

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1}\, N_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

where $N = (N_1, \ldots, N_n)$ is the unit normal to M and the hat $\wedge$ denotes omission. In particular for the unit sphere in $\mathbb{R}^n$

we obtain

$$\sigma' = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n .$$

If N makes an angle θ with the position vector x of M and if σ' is the central projection of σ from the origin 0 we have

$$\frac{\sigma'}{\sigma \cos \theta} = \frac{1}{r^{n-1}} ;$$

In analogy to the 3-dimensional case we thus obtain the element of solid angle spanned by M and subtended from 0 as

$$\tau = \frac{x \cdot N \sigma}{r^n}$$

$$= \sum (-1)^{i-1} \frac{x_i}{r^n} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n .$$

If M intersects any ray through 0 in at most one point it is seen from 0 under the solid angle

$$\left| \int_M \tau \right|$$

By 1, we find that τ is again a closed, but not exact $(n - 1)$ form.

3. If $\mathbb{R}^n$ is replaced by a connected n-dimensional Riemannian manifold $\mathbb{R}$ in the previous considerations it seems natural to adapt two different definitions of a solid angle Ω :

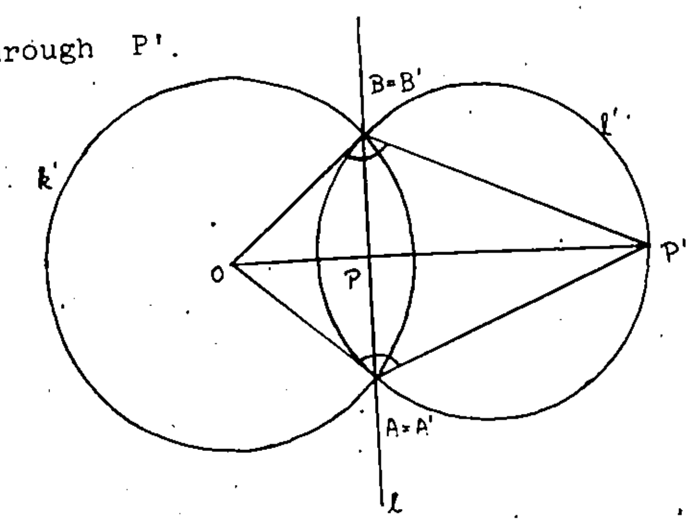
(a) Ω' is formed by a (n - 1)-parameter famuly of line-elements at a point P of R ; i.e. by tangents at P to R and it lies in the tangent space TR_P . It is measured by the volume cut out from the unit sphere in this tangent space.

(b) Ω is formed by shortest geodesic segments within R joining P to a hypersurface M in R.

From our previous results it follows that for the first definition a map $F : R \to R$ preserving all solid angles also preserves all ordinary angles, since every induced map TF_P of the Euclidean tangent spaces $TR_P \to TR_{F(P)}$ must be a homothety. Hence F is then a conformal selfmap of R . It is well-known that these maps form a Lie transformation group of dimension at most $\frac{1}{2}(n + 1)(n + 2)$ for $n \geq 3$.

Note that for $R = \mathbb{R}^3 - \{0\}$ an inversion in a sphere k about 0 preserves the solid angles of the first definition, but not of the second definition:

A straight line ℓ through P goes over into a circle ℓ' through P'.



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