FREE VIBRATIONS OF DEEP SANDWICH SPHERICAL SHELLS

By
Anand Vijoy Singh

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ABSTRACT

With the help of the general equations of motion of a continuum in spherical coordinates, the complete system of differential equations for axi-symmetric vibrations of deep sandwich spherical shell has been derived. The shell consists of a thick core layer, and of two face layers of the same isotropic material and thickness. The core is assumed to be incompressible in the radial direction and the effect of thickness shear deformation has been considered. The stresses in the tangential direction of the core are assumed to be negligible and the faces are taken as membranes. The general solution of the differential equations has been expressed in terms of Legendre functions.

The frequency equations for clamped and free edge hemispherical sandwich shells were developed in terms of Legendre functions. Since the values of Legendre functions with arbitrary complex indices are not available, subroutines have been made to generate these values.

The highly transcendental equations were solved on the digital computer using iterative technique. To examine the behavior of sandwich spherical shells, cellular cellulose acetate as the core material and aluminum, Cu-Zn alloy and mild steel as the face materials were selected. The results, showing the variation of fundamental and higher mode frequencies with the change of elastic and geometric properties of the core and face sheets, are presented in the form of graphs.

CONTENTS

NOMENCLATURE

$2c$	Thickness of the core
$\bar{c}$	Distance between middle surface of core and middle surface of a face sheet
E	Modulus of elasticity of the face sheet
G_c	Modulus of rigidity of the core material
h	Thickness of the face sheet
n_α	Order of the Legendre functions
N_θ , N_ϕ	Normal stress resultants in θ and ϕ directions respectively
$P_{n_\alpha}(\cos\phi)$, $Q_{n_\alpha}(\cos\phi)$	Legendre functions of first and second kind respectively
Q_ϕ	Shear stress resultant
R	Radius of the middle surface of the sandwich shell
R_1, R_2	Radii of the middle surfaces of the face sheet 1 and 2
r, ϕ, θ	Spherical coordinates
t	Instant of time
u, v, w	Displacement components in θ, ϕ and r directions
v_1, v_2	Displacement components in the meridional direction of face sheets 1 and 2
$\gamma_{r\phi}$, $\gamma_{r\theta}$, $\gamma_{\phi\theta}$	Shearing strain-components
ε_r, ε_ϕ, ε_θ	Normal strain-components
ν, ν_c	Poisson's ratios of the face and core materials respectively
ρ, ρ_c	Mass densities of the face and core materials
σ_r, σ_ϕ, σ_θ	Normal stresses in r, ϕ and θ directions

$\tau_{r\phi}$, $\tau_{r\theta}$, $\tau_{\phi\theta}$	Shear stresses
ω	Circular frequency of the shell

In addition, the following quantities have also been used in the thesis.

$$r_h = c/h$$

$$r_\rho = \rho_c/\rho$$

$$r_E = E_c/E$$

$$\Omega^2 = \rho\omega^2R^2/E$$

$$K = Eh/(1 - \nu^2)$$

$$\lambda_\alpha = n_\alpha(n_\alpha + 1)$$

P_1, P_2 and Q' are the quantities defined by Eqs.(3.3).

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Chapter 1

INTRODUCTION

A sandwich shell consists of three or more layers of material bonded together such that they act as a single unit. The outer facings or skins are usually high strength materials, such as steel, aluminum, plastic, or plywood. The function of the facings is to bear most of the outer fiber loads. The thicker central layer (or layers) normally known as core is usually a low strength, low density material such as rigid urethane foam, polystyrene foam, or honeycomb. The functions of the thick core are: (i) to seperate the face layers in a manner similar to the web in an I beam, (ii) to resist shear stresses set up by the external loads, (iii) to stabilize the facings thus preventing buckling and (iv) to provide insulation.

The commercial importance of sandwich construction has long been recognized in the aircraft industries due to its higher strength-to-weight ratios, better stability, high load carrying capacity. Sandwich panels have also increased fatigue life.

In recent years considerable interest has been shown in the dynamic behavior of sandwich shells. With the exception of a few, most of the papers appearing in literature deal with the vibration of homogeneous spherical shells and shallow sandwich shells. The first general dynamic thoery of elastic shells was established by A.E.H.Love who included the effects of both flexural and extensional deformations in the theory. Love also included in his work, as special cases, the previous treatment of inextensional vibrations given by Lord Rayleigh and the extensional vibrations of closed spherical shells discussed by Lamb. Very few papers were published on the vibrations of shells until 1937 when Federhofer reconstructed the basic system of differential equations of motion for the spherical shells in terms of displacements.

Later on in 1946 the problem of axi-symmetric vibrations of a shallow spherical shell was solved by E.Reissner [12] in which the problem was reduced to two simultaneous differential equations for tangential and normal components of displacements. In the above paper the approximate value for the lowest natural frequency was obtained by the use of Rayleigh Ritz procedure. The inextensional vibrations of shallow spherical shells was also investigated in [15] with axial symmetry and in [3] without axial symmetry. In both the cases longitudinal inertia has been neglected in comparison with transverse inertia. The system of equations have been, thus, reduced in terms of the transverse displacement component and a stress function. The exact solution for the torsionless axi-symmetric vibrations of shallow spherical shell segment with various edge conditions has been given by Kalnins and Naghdi in [5]. Van Fo Fy and V.N.Buivol [20] obtained the general solution of the differential equation of motion for the forced vibrations of shallow spherical shell and calculated the numerical values of the natural frequencies for the axi-symmetric case.

In 1961 Naghdi and Kalnins [10] investigated the torsionless axi-symmetric vibrations of non-shallow thin elastic spherical shell and obtained the results for the lowest natural frequency as a function of the thickness of the shell. The effect of bending on vibrations of spherical shells, closed at one pole and open at the other, has been studied by Kalnins [4]. His work also includes the modal shapes for the opening angles ranging from a shallow to a closed spherical shell. The vibration of deep spherical shell under the action of a concentrated force has been reported in detail by Manasyan [8]. Further work on this topic has also been published by Wilkinson and Kalnins [21].

Very little work has been published on the vibrations of sandwich shells. Major portion of the available literature deals with the bending of sandwich plates [14] and shells [18]. In a paper by Y.Y.Yu [23], the dynamic behavior of moderately thin elastic sandwich cylindrical shell has been studied. The problem of free axi-symmetric vibrations of shallow spherical sandwich shell has been solved by Koplik and Yu [6] and numerical values of frequencies upto sixteen natural modes have been calculated for clamped edge condition.

In this thesis an attempt has been made to study the axi-symmetric vibrations of deep sandwich spherical shell, closed at one pole and open at the other. The shell consists of a thick core and of two face sheets of the same isotropic material and of equal thickness. The core is assumed to be incompressible in the radial direction and the effect of thickness shear deformation has been considered. The face parallel stresses in the core are assumed to be negligible and the faces are taken as membranes. The complete system of differential equations, based on linear strain-displacement relationships, has been derived from the general equations of motion in a continuum. The final set of differential equations is linear in nature within the frame work of linear elastic theory of shells.

The highly transcendental equations have been solved by the iteration process on the digital computer. The results obtained, show the variation of frequency with the change of elastic parameters of the core and face sheets. Effect of the change of the ratio of core thickness to face sheet has also been studied. The variation of fundamental and higher mode frequencies are presented in the form of graphs.

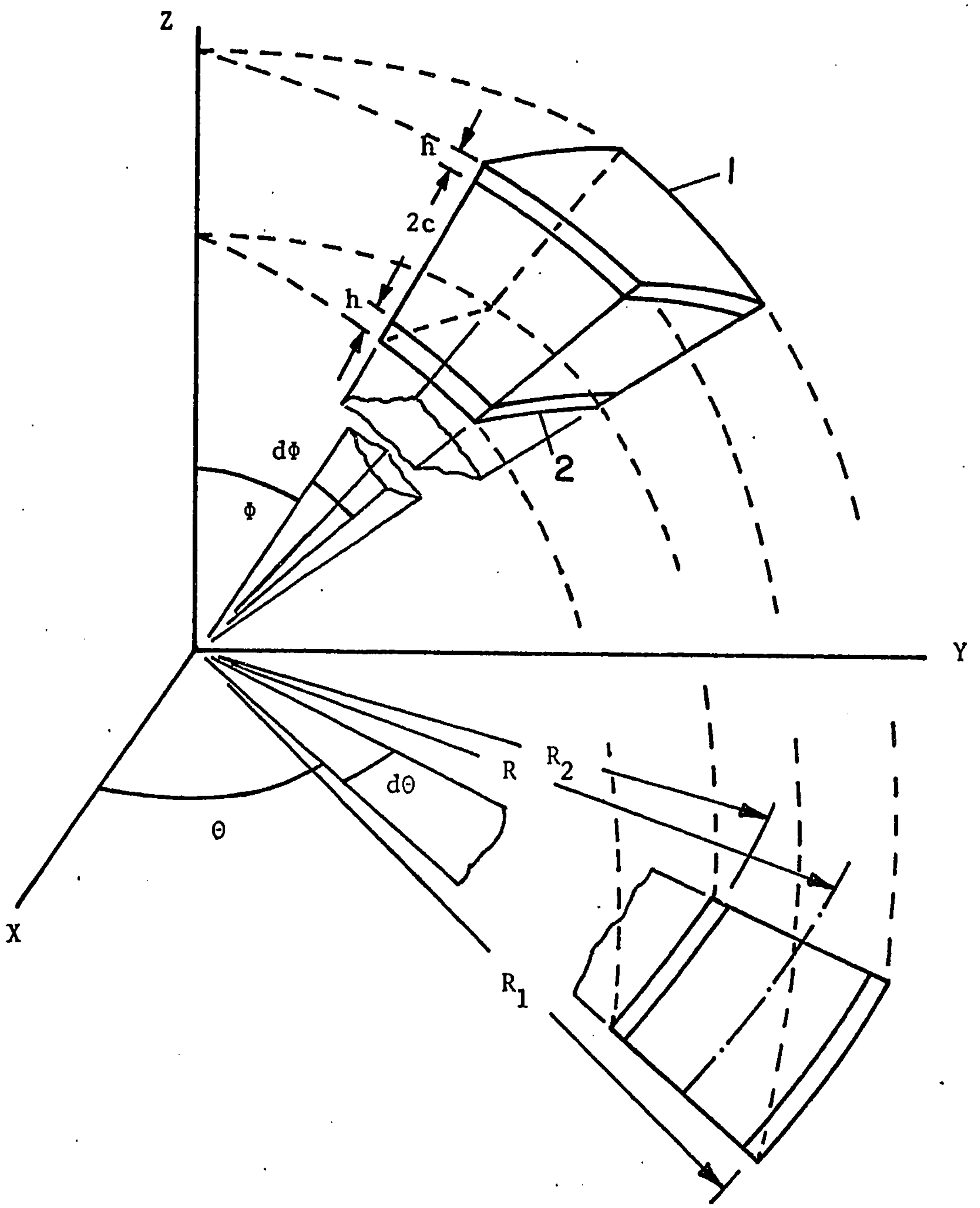
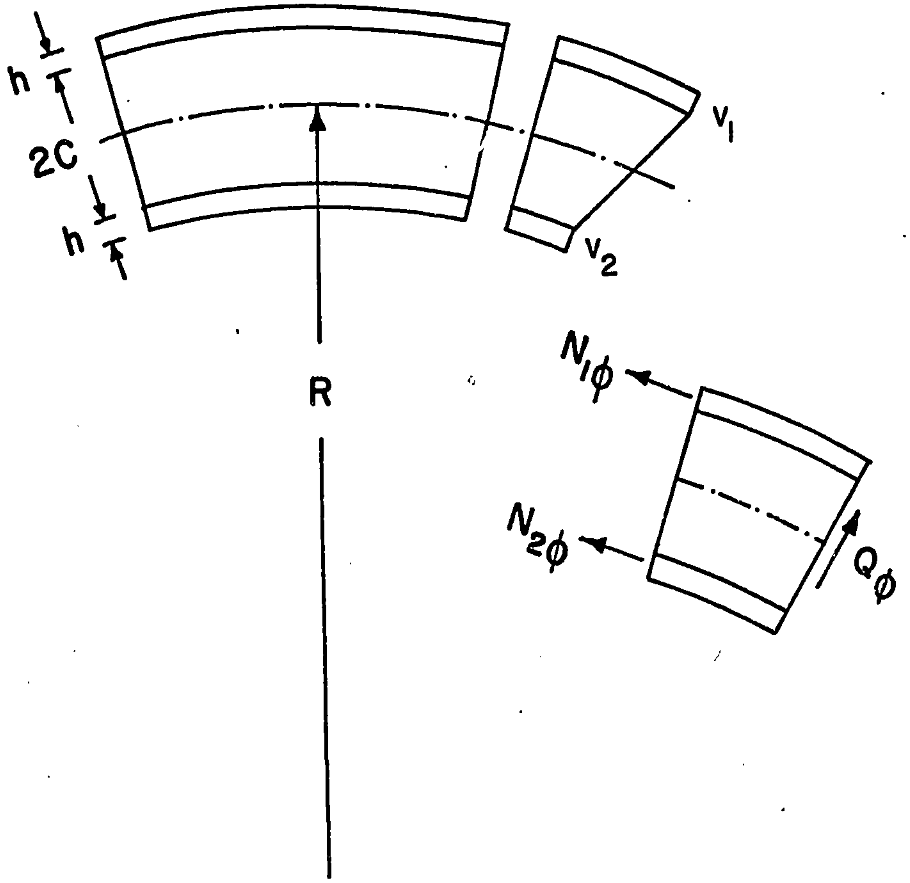


FIG. 1a GEOMETRY OF THE SHELL ELEMENT.



SANDWICH SHELL ELEMENTS SHOWING FORCES AND VARIATION OF DISPLACEMENT

FIG. 1b

Chapter 2

FORMULATION OF THE PROBLEM

General

The present treatment is based on linear strain-displacement relationships. The face sheets and the core are assumed to be elastic and homogeneous. This formulation could be extended to the multi-layered shells. In the following, however, the shell is composed of three layers. The two face sheets are of the same material and the core is made of a different material. The material and their assumptions will be discussed in details later on in this chapter.

Using the generalized equations of motion in a continuum [19], a complete system of differential equations for a sandwich spherical shell has been derived. The geometry of the shell is shown in Fig. 1, in which the thickness of the core is 2c and that of each face layer is h. The radius of the middle surface of the shell is R. The thickness shear deformation is included in the core and the faces are taken as membranes.

Basic Equations of Motion

The equations of motion in a continuum in spherical coordinates, when body forces are neglected, can be written as

$$(r^2\sigma_r)_{,r}/r^2 + \tau_{r\theta,\theta}/(r\sin\phi) + \tau_{r\phi,\phi}/r + (-\sigma_\phi-\sigma_\theta+\tau_{r\phi}\cot\phi)/r = \rho w_{,tt}$$

$$\tau_{r\theta,r} + \sigma_{\theta,\theta}/(r\sin\phi) + \tau_{\theta\phi,\phi}/r + (3\tau_{r\theta}+2\tau_{\theta\phi}\cot\phi)/r = \rho u_{,tt}$$

$$\tau_{r\phi,r} + \tau_{\theta\phi,\theta}/(r\sin\phi) + \sigma_{\phi,\phi}/r + \{3\tau_{r\phi}+(\sigma_\phi-\sigma_\theta)\cot\phi\}/r = \rho v_{,tt}$$

and the strain components are

$$\varepsilon_r = w_{,r}$$

$$\varepsilon_\phi = (v_{,\phi} + w)/r$$

$$\epsilon_\theta = u,_\theta/(r\ \sin\phi) + (w + v\ \cot\phi)/r$$

(2.2)
$$\gamma_{r\phi} = (w,_\phi - v)/r + v,_r$$

$$\gamma_{r\theta} = w,_\theta/(r\ \sin\phi) - u/r + u,_r$$

$$\gamma_{\theta\phi} = u,_\phi/r - u\ \cot\phi/r + v,_\theta/(r\ \sin\phi)$$

where u, v and w are the displacement components in θ,ϕ and r directions respectively. The quqntities σ_r,σ_θ etc. represent normal stresses and $\tau_{r\phi}$, $\tau_{r\theta}$ etc. denote the shear stresses. The derivative $\frac{\partial}{\partial r}(r^2\sigma_r)$ has been written in the symbolic form as $(r^2\sigma_r),_r$ so that

$$\frac{\partial}{\partial r}(r^2\sigma_r) = (r^2\sigma_r),_r$$

$$\frac{\partial}{\partial\theta}(\tau_{\theta\phi}) = \tau_{\theta\phi,\theta}$$

and so on.

Axi-symmetric Case. For axi-symmetric case, all the derivatives with respect to θ must be dropped and the displacement component u in the θ direction should be zero. Thus from Eqs.(2.2)e and (2.2)f, the shear strains and consequently the shear stresses $\tau_{r\theta}$ and $\tau_{\theta\phi}$ vanish.

(2.3)
$$\tau_{r\theta} = 0$$

$$\tau_{\theta\phi} = 0$$

The equations of motion (2.1) now assume the form

(2.4)
$$(r^2\sigma_r),_r/r^2 + \tau_{r\phi,\phi}/r + (-\sigma_\phi - \sigma_\theta + \tau_{r\phi}\cot\phi)/r = \rho w,_{tt}$$

$$\tau_{r\phi,r} + \sigma_{\phi,\phi}/r + \{3\tau_{r\phi} + (\sigma_\phi - \sigma_\theta)\cot\phi\}/r = \rho v,_{tt}$$

and the expressions for the strain components become

$$\begin{aligned} \varepsilon_r &= w,_r \\ \varepsilon_\phi &= (v,_\phi + w)/r \\ \varepsilon_\theta &= (w + v \cot\phi)/r \\ \gamma_{r\phi} &= (w,_\phi - v)/r + v,_r \end{aligned} \tag{2.5}$$

Equations (2.4) and (2.5) have been used seperately for the core and the face layers, in order to obtain a complete system of equations for the composite shell.

Weak Core

Following are the assumptions made for the core material.

(i) Stresses in the tangential directions of the shell surface are small compared to those in the facings.

$$\begin{aligned} \sigma_\phi &= 0 \\ \sigma_\theta &= 0 \end{aligned} \tag{i}$$

From Eq. (2.3), it is seen that the shear stress $\tau_{\theta\phi}$ is zero due to symmetry of the problem. For nonsymmetric dynamic problems of sandwich spherical shells, $\tau_{\theta\phi}$ would be negligible again because the normal stresses σ_ϕ and σ_θ in the core are infinitesimally small.

(ii) The core is incompressible in the radial direction.

$$\begin{aligned} \varepsilon_r &= 0 \\ w_c &= w_1 = w_2 = w \end{aligned} \tag{ii}$$

(iii) The thickness shear deformation is included in the core. Thus the diaplacement component in the meridional direction in the core is given as

$$v = \overset{*}{v} + \bar{v}\, z/c \tag{iii}$$

where,

$$\begin{aligned} \overset{*}{v} &= (v_1 + v_2)/2 \\ \bar{v} &= (v_1 - v_2)/2 \end{aligned} \tag{2.6}$$

and v_1, v_2 are the displacement components in the meridional direction of the middle surface of face 1 and 2 respectively.

With the use of Eqs.(i) and (ii), the equations of motion(2.4) reduce to

$$(2.7)\qquad \begin{aligned} (r^2\sigma_r)_{,r}/r^2 + \tau_{r\phi,\phi}/r + \cot\phi\ \tau_{r\phi}/r &= \rho_c w_{,tt} \\ \tau_{r\phi,r} + 3\ \tau_{r\phi}/r &= \rho_c v_{,tt} \end{aligned}$$

Introducing a new variable z which varies along the thickness of the core such that,

$$(2.8)\qquad r = R + z\ ,$$

then $\quad (\ )_{,r} = (\ )_{,z} \quad$ and $\quad 1/r = (1 - z/R)/R$

Since z/R^3 and $(z/R)\sigma_r$ are higher order terms, as the core has been assumed to be incompressible, the first term in Eq.(2.7)a reduces to $\sigma_{r,r}$.

By making use of Eqs.(2.6) and (2.8) in Eq.(2.7) and integrating with respect to z over the thickness of the core

$$(2.9)\qquad \begin{aligned} &\sigma_r\Big|_{z=c} - \sigma_r\Big|_{z=-c} + \left\{\frac{1}{R}\int_{-c}^{c}\tau_{r\phi}dz - \frac{1}{R^2}\int_{-c}^{c}z\tau_{r\phi}dz\right\}_{,\phi} + \cot\phi\left\{\frac{1}{R}\int_{-c}^{c}\tau_{r\phi}dz - \frac{1}{R^2}\int_{-c}^{c}z\tau_{r\phi}\,dz\right\} = 2c\,\rho_c w_{,tt} \\ &\tau_{r\phi}\Big|_{z=c} - \tau_{r\phi}\Big|_{z=-c} + 3\left\{\frac{1}{R}\int_{-c}^{c}\tau_{r\phi}dz - \frac{1}{R^2}\int_{-c}^{c}z\tau_{r\phi}dz\right\} = 2c\rho_c \overset{*}{v}_{,tt} \end{aligned}$$

From Eqs.(ii),(iii),(2.5)d and (2.8), it can be easily derived that

$$(2.10)\qquad \frac{1}{R}\int_{-c}^{c}\tau_{r\phi}dz - \frac{1}{R^2}\int_{-c}^{c}z\tau_{r\phi}dz = 2cG_c(1+\tfrac{1}{3}c^2/R^2)(w_{,\phi} - \overset{*}{v})/R^2 \ +$$

$$2G_c(1+\tfrac{2}{3}c^2/R^2)\bar{v}/R$$

(For detail derivation refer to appendix A)

With the help of Eq.(2.10), Eqs.(2.9) become

$$\sigma_r\Big|_{z=c} - \sigma_r\Big|_{z=-c} + 2cG_c(1+\tfrac{1}{3}c^2/R^2)(w,_{\phi\phi} + \cot\phi\, w,_\phi - \overset{*}{v},_\phi - \overset{*}{v}\cot\phi)/R^2 +$$

$$2G_c(1+\tfrac{2}{3}c^2/R^2)(\bar{v},_\phi + \bar{v}\cot\phi)/R = 2c\rho_c w,_{tt}$$

(2.11)

$$\tau_{r\phi}\Big|_{z=c} - \tau_{r\phi}\Big|_{z=-c} + 6cG_c(1+\tfrac{1}{3}c^2/R^2)(w,_\phi - \overset{*}{v})/R^2 + 6G_c(1+\tfrac{2}{3}c^2/R^2)$$

$$\bar{v}/R = 2c\rho_c v,_{tt}$$

Face Layers

The following assumptions are made for the face sheets.

(i) The faces are very thin and are considered as membranes.

(ii) They are made of the same isotropic material and are of equal thickness.

The stress resultants in the face sheets are introduced in the following manner.

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi dz$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta dz$$

$$(2.12)\qquad N_{\phi\theta} = \int_{-h/2}^{h/2} \tau_{\phi\theta}\, dz$$

$$Q_{\phi} = \int_{-h/2}^{h/2} \tau_{r\phi}\, dz$$

By using Eqs.(2.5)b, (2.5)c and the Hooke's law

$$(2.13)\qquad \begin{aligned} \sigma_{\phi} &= (\varepsilon_{\phi} + \nu\varepsilon_{\theta})E/(1 - \nu^2) \\ \sigma_{\theta} &= (\varepsilon_{\theta} + \nu\varepsilon_{\phi})E/(1 - \nu^2) \end{aligned}$$

the stress resultants from Eqs.(2.12) are

$$(2.14)\qquad \begin{aligned} N_{1\phi} &= K\{v_{1,\phi} + w + \nu(v_1 \cot\phi + w)\}/R_1 \\ N_{1\theta} &= K\{v_1 \cot\phi + w + \nu(v_{1,\phi} + w)\}/R_2 \\ N_{2\phi} &= K\{v_{2,\phi} + w + \nu(v_2 \cot\phi + w)\}/R_2 \\ N_{2\theta} &= K\{v_2 \cot\phi + w + \nu(v_{2,\phi} + w)\}/R_2 \end{aligned}$$

Where, $K = Eh/(1 - \nu^2)$ and R_1, R_2 are the radii of the middle surface of the facings 1 and 2 respectively. The quantities $N_{1\phi}$ and $N_{1\theta}$ represent the normal stress resultants for face sheet 1 in ϕ and θ directions respectively. Further, since the face sheets are very thin, it can be observed that $1/r \simeq 1/R_{1,2}$.

By substituting the following quantities

$r = R_1 + z'$ and

(2.15) $$(\;)_{,r} = (\;)_{,z'}$$

for face sheet 1 in Eqs.(2.4), it can be easily shown that

$$\sigma_{1r,z'} + \tau_{1r\phi,\phi}/R_1 + (-\sigma_{1\phi} - \sigma_{1\theta} + \tau_{1r\phi}\cot\phi)/R_1 = \rho w_{,tt}$$

$$\tau_{1r\phi,z'} + \sigma_{1\phi,\phi}/R_1 + \{3\tau_{1r\phi} + (\sigma_{1\phi} - \sigma_{1\theta})\cot\phi\}/R_1 = \rho v_{1,tt}$$

Integration of the above equations with respect to z' over the thickness of face sheet 1 gives

(2.16)
$$\sigma_{1r}\Big|_{z'=h/2} - \sigma_{1r}\Big|_{z'=-h/2} + Q_{1\phi,\phi}/R_1 + (-N_{1\phi} - N_{1\theta} + Q_{1\phi}\cot\phi)/R_1 = \rho h w_{,tt}$$

$$\tau_{1r\phi}\Big|_{z'=h/2} - \tau_{1r\phi}\Big|_{z'=-h/2} + N_{1\phi,\phi}/R_1 + \{3Q_{1\phi} + (N_{1\phi} - N_{1\theta})\cot\phi\}/R_1 = \rho h v_{1,tt}$$

Similarly for face sheet 2

(2.17)
$$\sigma_{2r}\Big|_{z'=h/2} - \sigma_{2r}\Big|_{z'=-h/2} + Q_{2\phi,\phi}/R_2 + (-N_{2\phi} - N_{2\theta} + Q_{2\phi}\cot\phi)/R_2 = \rho h w_{,tt}$$

$$\tau_{2r\phi}\Big|_{z'=h/2} - \tau_{2r\phi}\Big|_{z'=-h/2} + N_{2\phi,\phi}/R_2 + \{3Q_{2\phi} + (N_{2\phi} - N_{2\theta})\cot\phi\}/R_2 = \rho h v_{2,tt}$$

For free natural vibrations of the shell, its outer surface is free from external loadings and hence,

$$\sigma_{1r}\Big|_{z'=h/2} = 0\,, \qquad \tau_{1r\phi}\Big|_{z'=h/2} = 0$$

(2.18) $$\sigma_{2r}\Big|_{z'=-h/2} = 0 \qquad \tau_{2r\phi}\Big|_{z'=-h/2} = 0$$

Also, since the faces are membranes, the transverse shearing forces $Q_{1\phi}$ and $Q_{2\phi}$ in the face sheets 1 and 2 respectively, are zreo.

$$Q_{1\phi} = 0$$

$$Q_{2\phi} = 0$$

From Fig.1,

$$R_1 = R + \bar{c}$$

$$R_2 = R - \bar{c}$$

where, $\bar{c} = (2c + h)/2$

Neglecting the higher order terms of $\bar{c}/R$ compared to unity, the following equations can be easily obtained.

(2.19) $$1/R_1 = (1 - \bar{c}/R)/R$$ $$1/R_2 = (1 + \bar{c}/R)/R$$

Substituting the values of $1/R_1$ and $1/R_2$ from Eqs.(2.19), Eqs.(2.16) become

(2.20)
$$-\sigma_{1r}\Big|_{z'=-h/2} - (1 - \bar{c}/R)(N_{1\phi} + N_{1\theta})/R = \rho h w_{,tt}$$

$$-\tau_{1r\phi}\Big|_{z'=-h/2} + (1 - \bar{c}/R)N_{1\phi,\phi}/R + \cot\phi(1 - \bar{c}/R)(N_{1\phi} - N_{1\theta})/R = \rho h v_{1,tt}$$

and Eqs.(2.17)

$$\left.\sigma_{2r}\right|_{z'=h/2} - (1 + \bar{c}/R)(N_{2\phi} + N_{2\theta})/R = \rho h w_{,tt}$$

(2.21)

$$\left.\tau_{2r\phi}\right|_{z'=h/2} + (1 + \bar{c}/R)N_{2\phi,\phi}/R + \cot\phi(1 + \bar{c}/R)(N_{2\phi} - N_{2\theta})/R = \rho h v_{2,tt}$$

Equation for Composite Shell

At this point, it is convenient to introduce the following notations.

(2.22)

$$\overset{*}{N}_\phi = N_{1\phi} + N_{2\phi}$$

$$\bar{N}_\phi = N_{1\phi} - N_{2\phi}$$

$$\overset{*}{N}_\theta = N_{1\theta} + N_{2\theta}$$

$$\bar{N}_\theta = N_{1\theta} - N_{2\theta}$$

Substituting the values of $N_{1\phi}$, $N_{2\phi}$, $N_{1\theta}$, and $N_{2\theta}$ from Eqs. (2.14) and using Eqs. (2.6) and (2.19), the stress resultants $\overset{*}{N}_\phi$, $\overset{*}{N}_\theta$, $\bar{N}_\phi$ and $\bar{N}_\theta$ reduce to

(2.23)

$$\overset{*}{N}_\phi = 2K\{\overset{*}{v}_{,\phi} + w + \nu(\overset{*}{v}\cot\phi + w)\}/R - 2K\bar{c}(\bar{v}_{,\phi} + \nu\bar{v}\cot\phi)/R^2$$

$$\overset{*}{N}_\theta = 2K\{\overset{*}{v}\cot\phi + w + \nu(\overset{*}{v}_{,\phi} + w)\}/R - 2K\bar{c}(\bar{v}\cot\phi + \nu\bar{v}_{,\phi})/R^2$$

$$\bar{N}_\phi = 2K(\bar{v}_{,\phi} + \nu\bar{v}\cot\phi)/R - 2K\bar{c}\{\overset{*}{v}_{,\phi} + w + \nu(\overset{*}{v}\cot\phi + w)\}/R^2$$

$$\bar{N}_\theta = 2K(\bar{v}\cot\phi + \nu\bar{v}_{,\phi})/R - 2K\bar{c}\{\overset{*}{v}\cot\phi + w + \nu(\overset{*}{v}_{,\phi} + w)\}/R^2$$

The equations of motion for the composite shell are obtained by combining the six Eqs.(2.11), (2.20) and (2.21) and simplifying them in a proper manner. The summation of Eqs.(2.11)a, (2.20)a and (2.21)a gives

$$\sigma_r\Big|_{z=c} - \sigma_r\Big|_{z=-c} + 2cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi\phi} + w,_{\phi}\cot\phi - \overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^2 +$$

$$2G_c(1 + \tfrac{2}{3}c^2/R^2)(\bar{v},_{\phi} + \bar{v}\cot\phi)/R - \sigma_{1r}\Big|_{z'=-h/2} + \sigma_{2r}\Big|_{z'=h/2} - (\overset{*}{N}_{\phi} + \overset{*}{N}_{\theta})/R +$$

$$\bar{c}(\bar{N}_{\phi} + \bar{N}_{\theta})/R^2 = (2\rho h + 2\rho_c c)w,_{tt}$$

and from Eqs.(2.11)b, (2.20)b and (2.21)b

$$\tau_{r\phi}\Big|_{z=c} - \tau_{r\phi}\Big|_{z=-c} + 6cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi} - \overset{*}{v})/R^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v}/R -$$

(2.24)

$$\tau_{1r\phi}\Big|_{z'=-h/2} + \tau_{2r\phi}\Big|_{z'=h/2} + \overset{*}{N}_{\phi,\phi}/R - \bar{c}\,\bar{N}_{\phi,\phi}/R^2 + \cot\phi(\overset{*}{N}_{\phi} - \overset{*}{N}_{\theta})/R -$$

$$\bar{c}\cot\phi\,(\bar{N}_{\phi} - \bar{N}_{\theta})/R^2 = (2\rho h + 2\rho_c c)\overset{*}{v},_{tt}$$

Further, subtracting Eq.(2.21)b from (2.20)b

$$-\tau_{1r\phi}\Big|_{z'=-h/2} - \tau_{2r\phi}\Big|_{z'=h/2} + \bar{N}_{\phi,\phi}/R - \bar{c}\,\overset{*}{N}_{\phi,\phi}/R^2 + \cot\phi(\bar{N}_{\phi} - \bar{N}_{\theta})/R -$$

$$\bar{c}\cot\phi\,(\overset{*}{N}_{\phi} - \overset{*}{N}_{\theta})/R^2 = 2\rho h\bar{v},_{tt}$$

The continuity conditions at the interfaces are

$$\sigma_r\Big|_{z=c} = \sigma_{1r}\Big|_{z'=-h/2}\;,\quad \sigma_r\Big|_{z=-c} = \sigma_{2r}\Big|_{z'=h/2}$$

(2.25)

$$\tau_{r\phi}\Big|_{z=c} = \tau_{1r\phi}\Big|_{z'=-h/2}\;,\quad \tau_{r\phi}\Big|_{z=-c} = \tau_{2r\phi}\Big|_{z'=h/2}$$

By making use of Eqs.(2.23) together with the continuity conditions (2.25), the system of Eqs.(2.24) reduces to

$$2cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi\phi} + w,_{\phi}\cot\phi - \overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^2 + 2G_c(1 + \tfrac{2}{3}c^2/R^2)$$

$$(\bar{v},_{\phi} + \bar{v}\cot\phi)/R - 2K(1 + \nu)(\overset{*}{v},_{\phi} + \overset{*}{v}\cot\phi + 2w)/R^2 + 2K\bar{c}(1 + \nu)$$

$$(\bar{v},_{\phi} + \bar{v}\cot\phi)/R^3 + 2K\bar{c}(1 + \nu)(\bar{v},_{\phi} + \bar{v}\cot\phi)/R^3 - 2K\bar{c}^2(1 + \nu)$$

$$(\overset{*}{v},_{\phi} + \overset{*}{v}\cot\phi + 2w)/R^4 = (2\rho h + 2\rho_c c)w,_{tt}$$

$$6cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi} - \overset{*}{v})/R^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v}/R + 2K\{\overset{*}{v},_{\phi} + w +$$

$$\nu(\overset{*}{v}\cot\phi + w)\},_{\phi}/R^2 - 2K\bar{c}(\bar{v},_{\phi} + \nu\bar{v}\cot\phi),_{\phi}/R^3 - 2K\bar{c}(\bar{v},_{\phi} + \nu\bar{v}\cot\phi),_{\phi}/R^3 +$$

$$2K\bar{c}^2\{\overset{*}{v},_{\phi} + w + \nu(\overset{*}{v}\cot\phi + w)\},_{\phi}/R^4 + 2K\cot\phi(1 - \nu)(\overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^2 -$$

$$2K\bar{c}\cot\phi(1-\nu)(\bar{v},_{\phi} - \bar{v}\cot\phi)/R^3 - 2K\bar{c}\cot\phi(1-\nu)(\bar{v},_{\phi} - \bar{v}\cot\phi)/R^3 +$$

$$2K\bar{c}^2\cot\phi(1-\nu)(\overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^4 = (2\rho h + 2\rho_c c)\overset{*}{v},_{tt}$$

and

$$- 2G_c(w,_{\phi} - \overset{*}{v})/R - 2G_c(1 + c^2/R^2)\bar{v}/c + 2K(\bar{v},_{\phi} + \nu\bar{v}\cot\phi),_{\phi}/R^2 -$$

$$2K\bar{c}\{\overset{*}{v},_{\phi} + w + \nu(\overset{*}{v}\cot\phi + w)\},_{\phi}/R^3 - 2K\bar{c}\{\overset{*}{v},_{\phi} + w + \nu(\overset{*}{v}\cot\phi + w)\},_{\phi}/R^3 +$$

$$2K\bar{c}^2(\bar{v},_{\phi} + \nu\bar{v}\cot\phi),_{\phi}/R^4 + 2K\cot\phi(1-\nu)(\bar{v},_{\phi} - \bar{v}\cot\phi)/R^2 - 2K\bar{c}\cot\phi$$

$$(1-\nu)(\overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^3 - 2K\bar{c}\cot\phi(1-\nu)(\overset{*}{v},_{\phi} - \overset{*}{v}\cot\phi)/R^3 + 2K\bar{c}^2$$

$$\cot\phi(1-\nu)(\bar{v},_{\phi} - \bar{v}\cot\phi)/R^4 = 2\rho h\bar{v},_{tt}$$

For detailed explanation of the first two terms in Eq.(2.24)c, refer to appendix B.

Further algebraic simplification of the above three equations, yields the final set of differential equations.

$$2cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi\phi} + w,_{\phi}\cot\phi)/R^2 - 4K(1+\nu)(1+\bar{c}^2/R^2)w/R^2 -$$

$$\{2cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1+\nu)(1+\bar{c}^2/R^2)/R^2\}(\overset{*}{v},_{\phi} + \overset{*}{v}\cot\phi) +$$

$$\{2G_c(1 + \tfrac{2}{3}c^2/R^2)/R + 4K\bar{c}(1 + \nu)/R^3\}(\bar{v},_\phi + \bar{v}\cot\phi) = (2\rho h+2\rho_c c)w,_{tt}$$

$$\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}w,_\phi - 6cG_c(1 + \tfrac{1}{3}c^2/R^2)$$

(2.26) $$\overset{*}{v}/R^2 + 2K(1 + \bar{c}^2/R^2)(\overset{*}{v},_{\phi\phi} + \overset{*}{v},_\phi\cot\phi - \overset{*}{v}\cot^2\phi - \nu\overset{*}{v})/R^2 + 6G_c(1 +$$

$$\tfrac{2}{3}c^2/R^2)\bar{v}/R - 4K\bar{c}(\bar{v},_{\phi\phi} + \bar{v},_\phi\cot\phi - \bar{v}\cot^2\phi - \nu\bar{v})/R^3 = (2\rho h+2\rho_c c)\overset{*}{v},_{tt}$$

and $$-\{2G_c/R + 4K\bar{c}(1 + \nu)/R^3\}w,_\phi + 2G_c\overset{*}{v}/R - 4K\bar{c}(\overset{*}{v},_{\phi\phi} + \overset{*}{v},_\phi\cot\phi - \overset{*}{v}$$

$$\cot^2\phi - \nu\overset{*}{v})/R^3 - 2G_c(1 + c^2/R^2)\bar{v}/c + 2K(1 + \bar{c}^2/R^2)(\bar{v},_{\phi\phi} + \bar{v},_\phi$$

$$\cot\phi - \bar{v}\cot^2\phi - \nu\bar{v})/R^2 = 2\rho h\bar{v},_{tt}$$

For convenience in writing the equations, the following symbols are introduced.

$$\overset{*}{\theta} = (\overset{*}{v},_\phi + \overset{*}{v}\cot\phi)/R$$

(2.27) $$\bar{\theta} = (\bar{v},_\phi + \bar{v}\cot\phi)/R$$

The Laplace operator is given by

$$\nabla^2 = \left(\frac{\partial^2}{\partial\phi^2} + \cot\phi\frac{\partial}{\partial\phi}\right)/R^2$$

Substituting the values of $\overset{*}{\theta}$, $\bar{\theta}$, and ∇^2 from Eq.(2.27), Eq.(2.26)a becomes

$$\{(2cG_c/R^2)(1 + \tfrac{1}{3}c^2/R^2)\; R^2\, \nabla^2 - 4K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}w \; -$$

$$\{\, 2cG_c(1 + \tfrac{1}{3}c^2/R^2)/R \; + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R\,\}\overset{*}{\theta} \; + \; \{2G_c(1 +$$

$$\tfrac{2}{3}c^2/R^2) \; + 4K\bar{c}(1 + \nu)/R^2\,\}\bar{\theta} \; = (2\rho h + 2\rho_c c)w,_{tt}$$

Differentiating Eq.(2.26)b with respect to ϕ and multiplying the same equation by $\cot\phi$ and adding them together gives the following equation.

$$\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}\; R^2\, \nabla^2 w + \{(2K/R)(1 +$$

$$\bar{c}^2/R^2)\; R^2\, \nabla^2 \; + 2K(1 - \nu)(1 + \bar{c}^2/R^2)/R \; - 6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R\}\; \overset{*}{\theta} \; + \tag{2.28}$$

$$\{-(4K\bar{c}/R^2)\; R^2\, \nabla^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2) \; - 4K\bar{c}(1 - \nu)/R^2\}\bar{\theta} \; = 2\rho h R(1 +$$

$$\tfrac{c}{h}\,\rho_c/\rho\;)\; \overset{*}{\theta},_{tt}$$

and similarly Eq.(2.26)c reduces to

$$-\, \{2G_c/R \; + 4K\bar{c}(1 + \nu)/R^3\}R^2\, \nabla^2 w \; + \; \{-(4K\bar{c}/R^2)\; R^2\, \nabla^2 + 2G_c \; -$$

$$4K\bar{c}(1-\nu)/R^2\}\overset{*}{\theta} + \{(2K/R)(1+\bar{c}^2/R^2)\,R^2\,\nabla^2 + 2K(1-\nu)(1+\bar{c}^2/R^2)/R$$

$$-\,2G_cR(1+c^2/R^2)/c\,\}\,\bar{\theta} = 2\rho hR\bar{\theta}_{,tt}$$

For detail derivation of these equations refer to appendix D.

Chapter 3

SOLUTION OF THE DIFFERENTIAL EQUATIONS

The solution of the variables w, $\overset{*}{\theta}$ and $\bar{\theta}$ is now sought in the form

$$
\begin{aligned}
w/h &= e^{i\omega t}\, W \\
\overset{*}{\theta} &= e^{i\omega t}\, \overset{*}{\Theta} \qquad (3.1) \\
\bar{\theta} &= e^{i\omega t}\, \bar{\Theta}
\end{aligned}
$$

where ω is the circular frequency of the shell in rad/sec.

The use of Eqs.(3.1) in Eqs.(2.28) yields

$$
\{(2chG_c/R^2)(1 + \tfrac{1}{3}c^2/R^2)\; R^2\nabla^2 - 4Kh(1 + \nu)(1 + \bar{c}^2/R^2)/R^2 + 2\rho h^2\omega^2
$$

$$
(1 + \tfrac{c}{h}\rho_c/\rho)\}\; W \;-\; \{2K(1 + \nu)(1 + \bar{c}^2/R^2)/R \;+\; 2cG_c(1 + \tfrac{1}{3}c^2/R^2)/R\}\overset{*}{\Theta} \;+
$$

$$
\{2G_c(1 + \tfrac{2}{3}c^2/R^2) + 4K\bar{c}\,(1 + \nu)/R^2\}\bar{\Theta} \;=\; 0
$$

$$
\{6chG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2Kh(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}\; R^2\nabla^2 W \;+
$$

$$
\{(2K/R)(1 + \bar{c}^2/R^2)R^2\nabla^2 + 2K(1 - \nu)(1 + \bar{c}^2/R^2)/R \;-\; 6cG_c(1 + \qquad (3.2)
$$

$$
\tfrac{1}{3}c^2/R^2)/R + 2\rho hR\omega^2(1 + \tfrac{c}{h}\rho_c/\rho)\}\; \overset{*}{\Theta} \;+\; \{6G_c(1 + \tfrac{2}{3}c^2/R^2) \;-\; 4K\bar{c}(1 \;-
$$

$$\nu)/R^2 - (4K\bar{c}/R^2)\ R^2\nabla^2\}\bar{\Theta} = 0$$

and

$$-\{2hG_c/R + 4Kh\bar{c}(1 + \nu)/R^3\}\ R^2\nabla^2 W + \{-(4K\bar{c}/R^2)\ R^2\nabla^2 + 2G_c -$$

$$4K\bar{c}(1 - \nu)/R^2\}\overset{*}{\Theta} + \{(2K/R)(1 + \bar{c}^2/R^2)R^2\nabla^2 + 2K(1 - \nu)(1 + \bar{c}^2/R^2)/R -$$

$$2RG_c(1 + c^2/R^2)/c + 2\rho hR\omega^2\}\bar{\Theta} = 0$$

For simplicity, the following nondimensional quantities have been introduced.

$$r_E = E_c/E$$

$$r_\rho = \rho_c/\rho$$

$$r_h = c/h$$

$$\Omega^2 = \rho\omega^2R^2/E \tag{3.3}$$

$$P = 1 + \bar{c}^2/R^2$$

$$P_1 = 1 + \tfrac{1}{3}c^2/R^2$$

$$P_2 = 1 + \tfrac{2}{3}c^2/R^2$$

$$Q' = (1 + r_\rho\ r_h)\ \Omega^2$$

By using Eqs.(3.3), the system of basic Eqs.(3.2) reduces to

$$(A_1\ R^2\nabla^2 + A_2)\ W + A_3\ \overset{*}{\Theta} + A_4\ \bar{\Theta} = 0$$

$$B_1\ R^2\nabla^2 W + (B_2\ R^2\nabla^2 + B_3)\ \overset{*}{\Theta} + (B_4\ R^2\nabla^2 + B_5)\ \bar{\Theta} = 0 \tag{3.4}$$

$$C_1\, R^2\nabla^2 W + (C_2\, R^2\nabla^2 + C_3)\,\overset{*}{\Theta} + (C_4\, R^2\nabla^2 + C_5)\,\bar{\Theta} = 0$$

In the above equations the constants A_1, A_2; B_1, B_2 etc. are introduced for brevity. These are listed below in details.

$$\begin{aligned}
A_1 &= \{r_E r_h P_1/(1 + \nu_c\ )\}h^2/R^2 \\
A_2 &= \{-4P/(1 - \nu) + 2Q'\}h^2/R^2 \\
A_3 &= -\{r_E r_h P_1/(1 + \nu_c) + 2P/(1 - \nu)\}h/R \\
A_4 &= r_E P_2/(1 + \nu_c) + \{4(r_h + 1/2)/(1 - \nu)\}h^2/R^2 \\
B_1 &= \{3r_E r_h P_1/(1 + \nu_c) + 2P/(1 - \nu)\}h^2/R^2 \\
B_2 &= \{2P/(1 - \nu^2)\}h/R \\
B_3 &= \{-3r_E r_h P_1/(1 + \nu_c) + 2P/(1 + \nu) + 2Q'\}h/R \\
B_4 &= \{-4(r_h + 1/2)/(1 - \nu^2)\}h^2/R^2 \\
B_5 &= 3r_E P_2/(1 + \nu_c) - \{4(r_h + 1/2)/(1 + \nu)\}h^2/R^2 \\
C_1 &= -\{r_E/(1 + \nu_c)\}h/R - \{4(r_h + 1/2)/(1 - \nu)\}h^3/R^3 \\
C_2 &= -\{4(r_h + 1/2)/(1 - \nu^2)\}h^2/R^2 \\
C_3 &= r_E/(1 + \nu_c) - \{4(r_h + 1/2)/(1 + \nu)\}h^2/R^2 \\
C_4 &= \{2P/(1 - \nu^2)\}h/R \\
C_5 &= \{2P/(1 + \nu) + 2\Omega^2\}h/R - r_E(1 + r_h^2 h^2/R^2)/\{(1 + \nu_c) r_h\ h/R\}
\end{aligned} \tag{3.5}$$

The general solution of Eqs.(3.4) can be expressed in terms of Legendre functions of degree n. For a distinct value of $n_\alpha (\alpha=1,2,3)$, the solution takes the form

$$\overset{*}{\Theta} = A_{n_\alpha} Z_{n_\alpha}(\cos\phi)$$

(3.6) $$\bar{\Theta} = B_{n_\alpha} Z_{n_\alpha}(\cos\phi)$$

$$W = C_{n_\alpha} Z_{n_\alpha}(\cos\phi)$$

where $Z_{n_\alpha}(\cos\phi)$ is a linear combination of $P_{n_\alpha}(\cos\phi)$ and $Q_{n_\alpha}(\cos\phi)$ the first and the second kind of Legendre functions respectively.

(3.7) i.e. $$Z_{n_\alpha}(\cos\phi) = D_1 P_{n_\alpha}(\cos\phi) + D_2 Q_{n_\alpha}(\cos\phi)$$

In this case A_{n_α}, B_{n_α}, C_{n_α} are arbitrary constants.

For simplification another parameter λ_α is introduced which is a function of n_α.

$$n_\alpha(n_\alpha + 1) = \lambda_\alpha$$

On simplification, this equation gives

(3.8) $$n_\alpha = (\lambda_\alpha + 1/4)^{1/2} - 1/2$$

Introducing the operator ∇^2 given by Eq.(2.27)c to the general solution for W, $\overset{*}{\Theta}$, and $\bar{\Theta}$ (Eq.3.6) and making use of Eq.(3.8)a, the following three relationships can be established.

$$R^2 \nabla^2 W = -\lambda_\alpha W$$

(3.9) $$R^2 \nabla^2 \overset{*}{\Theta} = -\lambda_\alpha \overset{*}{\Theta}$$

$$R^2 \nabla^2 \bar{\Theta} = -\lambda_\alpha \bar{\Theta}$$

In order to simplify Eq.(3.4) and to check whether the solution of the type mentioned above in Eqs.(3.6) is possible, a routine substitution of Eq.(3.9) in (3.4) yields

$$\begin{aligned}
&(-A_1\lambda_\alpha + A_2)\, C_{n_\alpha} + A_3\, A_{n_\alpha} + A_4\, B_{n_\alpha} = 0 \\
(3.10)\qquad &-B_1\lambda_\alpha\, C_{n_\alpha} + (-B_2\lambda_\alpha + B_3)\, A_{n_\alpha} + (-B_4\lambda_\alpha + B_5) B_{n_\alpha} = 0 \\
&-C_1\lambda_\alpha\, C_{n_\alpha} + (-C_2\lambda_\alpha + C_3)\, A_{n_\alpha} + (-C_4\lambda_\alpha + C_5) B_{n_\alpha} = 0
\end{aligned}$$

It is, therefore, seen that the solution (3.6) is admissible. The quantities n_α, A_{n_α}, B_{n_α} and C_{n_α} will, now, be obtained. In order to have a non-zero solutions of A_{n_α}, B_{n_α} and C_{n_α}, the determinant of the coefficient matrix must vanish.

$$\begin{vmatrix}
-A_1\lambda_\alpha + A_2 & A_3 & A_4 \\
-B_1\lambda_\alpha & -B_2\lambda_\alpha + B_3 & -B_4\lambda_\alpha + B_5 \\
-C_1\lambda_\alpha & -C_2\lambda_\alpha + C_3 & -C_4\lambda_\alpha + C_5
\end{vmatrix} = 0$$

which after expansion gives

$$(3.11)\qquad a_1\lambda_\alpha^3 + a_2\lambda_\alpha^2 + a_3\lambda_\alpha + a_4 = 0$$

The coefficients a_1, a_2, a_3 and a_4 used in Eq.(3.11) are

$$a_1 = A_1(B_4C_2 - B_2C_4)$$

$$a_2 = A_1(B_2C_5 - B_5C_2 + B_3C_4 - B_4C_3) + A_2(B_2C_4 - B_4C_2) +$$

$$A_3(B_4C_1 - B_1C_4) + A_4(B_1C_2 - B_2C_1)$$

$$a_3 = A_1(B_5C_3 - B_3C_5) + A_2(B_5C_2 - B_2C_5 + B_4C_3 - B_3C_4) +$$

$$A_3(B_1C_5 - B_5C_1) + A_4(B_3C_1 - B_1C_3)$$

$$a_4 = A_2(B_3C_5 - B_5C_3)$$

Since the characteristic Eq.(3.11) will have three roots $\lambda_\alpha (\alpha = 1, 2, 3)$, the general solutions of the system of Eqs.(3.4) are

$$\overset{*}{\Theta} = \sum_{\alpha=1}^{3} A_{n_\alpha} P_{n_\alpha}(\cos\phi)$$

$$\bar{\Theta} = \sum_{\alpha=1}^{3} B_{n_\alpha} P_{n_\alpha}(\cos\phi) \tag{3,12}$$

$$W = \sum_{\alpha=1}^{3} C_{n_\alpha} P_{n_\alpha}(\cos\phi)$$

In the above equations, due to the singular character of the function $Q_{n_\alpha}(\cos\phi)$ at $\phi = 0$, the coefficients of $Q_{n_\alpha}(\cos\phi)$ are set to zero.

From Eqs.(2.27), (3,12) and the recurrence relation

$$P_n(\mu) = \{P'_{n+1}(\mu) - P'_{n-1}(\mu)\}/(2n + 1) \tag{3.13}$$

the expression for $\overset{*}{V}$ and $\bar{V}$ are

$$\overset{*}{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}\beta_\alpha C_{n_\alpha}/(2n_\alpha + 1)$$

$$\bar{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}\eta_\alpha C_{n_\alpha}/(2n_\alpha + 1) \tag{3.14}$$

For detailed discussion of Eqs.(3.14) refer to appendix E. The primes in Eq.(3.13) denote the differentiation with respect to μ. While writing Eqs.(3.14) use has been made of the relationships

$$A_{n_\alpha} = \beta_\alpha C_{n_\alpha}$$

(3.15)

$$B_{n_\alpha} = \eta_\alpha C_{n_\alpha}$$

Chapter 4

FREQUENCY EQUATIONS

Equations (3.12) and (3.14) are applied to study the free vibrations of a sandwich spherical shell closed at one pole and open at the other. The frequency equation in general can be written as

$$|D_{i\alpha}| = 0 \qquad (i, \alpha = 1, 2, 3) \tag{4.1}$$

The exact form of the elements $D_{i\alpha}$ is obtained by the use of the boundary conditions prescribed at the open edge $\phi = \phi_0$.

In this chapter, frequency equations for a hemispherical sandwich shell have been derived for two types of boundary conditions such as clamped and free edge conditions.

Clamped edge Condition

The boundary conditions for this case are

$$\begin{aligned} W &= 0 \\ \overset{*}{V} &= 0 \\ \bar{V} &= 0 \end{aligned} \tag{4.2}$$

at $\phi = \pi/2$

Inserting the boundary conditions (4.2) in Eqs.(3.14), the following equations can be obtained.

$$P_{n_1}(0)\, C_{n_1} + P_{n_2}(0)\, C_{n_2} + P_{n_3}(0)\, C_{n_3} = 0$$

(4.3)
$$S_1 C_{n_1} + S_2 C_{n_2} + S_3 C_{n_3} = 0$$

$$H_1 C_{n_1} + H_2 C_{n_2} + H_3 C_{n_3} = 0$$

where,
$$S_\alpha = \beta_\alpha \{ P_{n_\alpha + 1}(0) - P_{n_\alpha - 1}(0) \} / (2n_\alpha + 1)$$ and

(4.4)
$$H_\alpha = \eta_\alpha \{ P_{n_\alpha + 1}(0) - P_{n_\alpha - 1}(0) \} / (2n_\alpha + 1)$$

For a non-trivial solution solution of the constants C_{n_α}, the determinant of the coefficient matrix must be set equal to zero.

(4.5)
$$\begin{vmatrix} P_{n_1}(0) & P_{n_2}(0) & P_{n_3}(0) \\ S_1 & S_2 & S_3 \\ H_1 & H_2 & H_3 \end{vmatrix} = 0$$

Equation(4.5) is the frequency equation of this system.

Free edge Condition

In this case the shell is assumed to be freely supported at the outer edge. Thus the boundary conditions are given as

$$Q_{\phi} = 0$$

(4.6) $$\overset{*}{N}_{\phi} = 0$$

$$\bar{N}_{\phi} = 0$$

at $\phi = \pi/2$.

The shear force Q_{ϕ} can be evaluated by integrating the shear stress $\tau_{r\phi}$ over the thickness of the core.

$$Q_{\phi} = \int_{-c}^{c} \tau_{r\phi} \, dz$$

On simplification (refer to Eq.(C) appendix A), the equation yields.

(4.7)a $$Q_{\phi} = (2cG_{c}/R) \, w,_{\phi} - (2cG_{c}/R) \overset{*}{v} + 2G_{c}(1 + \tfrac{1}{3}c^{2}/R^{2}) \, \bar{v}$$

From Eqs.(2.23), the stress resultants are

$$\overset{*}{N}_{\phi} = 2K\{ \overset{*}{v},_{\phi} + w + \nu(\overset{*}{v} \cot\phi + w) \}/R - 2K\bar{c} \, (\bar{v},_{\phi} + \nu\bar{v} \cot\phi)/R^{2}$$

Before proceeding with further simplification, it will be convenient to write the above equation in a modified form.

$$\overset{*}{N}_{\phi} = 2K\{\overset{*}{v},_{\phi} + \overset{*}{v} \cot\phi + (1 + \nu)w - (1 - \nu)\overset{*}{v} \cot\phi \}/R -$$

$$2K\bar{c} \, \{\bar{v},_{\phi} + \bar{v} \cot\phi - (1 - \nu)\bar{v} \cot\phi \}/R^{2}$$

Using Eqs.(2.27)a and (2.27)b

(4.7)b $$\overset{*}{N}_{\phi} = 2K\{R\overset{*}{\theta} + (1+\nu)w - (1-\nu)\overset{*}{v}\cot\phi\}/R - 2K\bar{c}\{R\bar{\theta} - (1-\nu)\bar{v}\cot\phi\}/R^2$$

The expression for the stress resultant $\bar{N}_{\phi}$ can also be obtained in a similar manner.

(4.7)c $$\bar{N}_{\phi} = 2K\{R\bar{\theta} - (1-\nu)\bar{v}\cot\phi\}/R - 2K\bar{c}\{R\overset{*}{\theta} + (1+\nu)w - (1-\nu)\overset{*}{v}\cot\phi\}/R^2$$

By using the boundary condition (4.6), Eqs.(4.7) reduce to

$$\frac{c}{R}w_{,\phi} - \frac{c}{R}\overset{*}{v} + (1 + \frac{1}{3}c^2/R^2)\bar{v} = 0$$

(4.8) $$\overset{*}{\theta} + (1+\nu)\frac{w}{R} - \frac{\bar{c}}{R}\bar{\theta} = 0$$

$$-\bar{\theta} + \frac{\bar{c}}{R}\overset{*}{\theta} + (1+\nu)\frac{\bar{c}}{R}\frac{w}{R} = 0$$

Substitution of the values of $\overset{*}{\theta}$, $\bar{\theta}$ and w from Eqs.(3.1) and those of $\overset{*}{\Theta}$, $\bar{\Theta}$, W, $\overset{*}{V}$, and $\bar{V}$ from Eqs.(3.12) and (3.14) in Eqs.(4.8) will result in the three Eqs.(4.9)a, b and c.

$$D_{1\alpha} = r_h(h^2/R^2)P'_{n\alpha}(0) + r_h(h/R)\,S_{\alpha} - (1 + \frac{1}{3}c^2/R^2)\,H_{\alpha}$$

(4.9) $$D_{2\alpha} = \{\beta_{\alpha} + (1+\nu)h/R - (r_h + 1/2)\eta_{\alpha}h/R\}\,P_{n_{\alpha}}(0)$$

$$D_{3\alpha} = \{(r_h + 1/2)\beta_{\alpha}h/R + (1+\nu)(r_h + 1/2)h^2/R^2 - \eta_{\alpha}\}P_{n_{\alpha}}(0)$$

In Eqs.(4.9) $D_{1\alpha}$, $D_{2\alpha}$ and $D_{3\alpha}$ are elements of the frequency determinant (4.1) for free edge condition of the sandwich shell.

Chapter 5

NUMERICAL COMPUTATION

The computation have been carried out for the frequency Ω of the sandwich shell. Variation of the frequency parameter has been studied with the geometric and elastic properties of the core and face sheets. Since the analytical closed form solutions of highly transcendental Eqs.(4.5) and (4.9) occuring in frequency computations is not possible, high speed digital computer has been used to solve the equations. The method of iteration has been utilized for the solution. Computer programs, in Fotran IV language, have been developed to compute the frequency of the sandwich shell for various boundary conditions. All the calculations have been done with the aid of IBM(360/65) available at the computer centre of the University of Ottawa.

First, the dimensionless ratios r_E, r_ρ, r_h together with h/R and the Poisson's ratios ν and ν_c were chosen for a particular combination of materials and the geometry of the shell. These values have been presented in the table (1).

The quantities A_1, A_2, A_3,-----C_4, C_5 are functions of non-dimensional parameters given in Eqs.(3.3), h/R, Poisson's ratios of core and face sheets, and the frequency Ω. At this stage, it is seen that there are two unknowns λ_α and Ω and two equations. These two equations, for the clamped hemispherical shell, are the characteristic cubic Eq.(3.11) and the frequency determinant (4.5). By specifying a value for the non-dimensional parameter Ω, the coefficients a_1, a_2, a_3 and a_4 in Eq.(3.11) were evaluated. These calculated values were used as the input quantities to the subroutine CUBIC (page.42). This subroutine, first, calculated one root which is real in all the cases and then reduced the cubic Eq.(3.11) in quadratic form. The subroutine QUAD is called next to calculate the other two roots which may be real or complex conjugate. Thus all the three roots λ_α (α = 1, 2, 3) of the charateristic Eq.(3.11) were obtained.

The frequency determinant (4.5) consists of Legendre functions of different orders $n_\alpha (\alpha = 1, 2, 3)$. These orders of Legendre function were obtained with the help of Eq.(3.8)b. In the case, when all the roots of Eq.(3.11) are real, the order of the function $P_{n_\alpha}(\cos\phi)$ is either real or of the form

$$n_\alpha = -1/2 + ip_\alpha \tag{5.1}$$

where p_α are real quantities. In the event the order n_α is given by Eq.(5.1), the Legendre functions are evaluated as conical functions which yield real values. When the two roots of Eq.(3.11) are complex conjugate, the orders n_α $(\alpha = 2, 3)$ are also complex conjugate. The nature of roots of the characteristic equation will be discussed in detail in the next chapter.

Since the values of Legendre functions $P_n(\cos\phi)$ are not available for arbtrary value of n, which may be real or complex, subroutines MIRLF (page.45) and COMLF (page.47) were developed to generate these values valid in the interval $0 < \phi < \pi$. This evaluation was based on Mehler's integral representation of Legendre functions. The integration was carried out numerically using trapezoidal rule. The derivative of $P_n(\cos\phi)$ with respect to ϕ was obtained from relations between contiguous Legendre functions with the help of these subroutines. Other subroutines LFHCA (page.49), DLF (page.51), and LFUNCT (page.52) were also made to evaluate the Legendre functions of higher orders.

With the aid of these subroutines the frequency determinant given in (4.5) was evaluated. Keeping the parameters r_E, r_h, r_ρ, ν, ν_c and h/R constant, the values of the frequency determinant was calculated for a series of values of Ω. The same procedure was repeated until the value of the determinant changed the sign. For free end conditions of the shell Eqs.(3.11) and (4.9) were used and the same procedure as described for the shell with fixed boundary was followed. This procedure yielded the values of the frequencies for the sandwich shells with free outer edge.

In the case Ia (Table.1), the purpose of numerical computation was to examine the effects of face thickness and its materials on the variation of natural frequencies Ω. The case Ib was investigated to see the variation of the frequency with the thickness of the core layer. Case II yielded the information about the behavior of the spherical sandwich shells at higher modes.

TABLE 1
Geometric and Elastic properties of Sandwich shells.

Description		h/R	r_E	r_ρ	r_h
I. Lowest Natural Frequency	Case a	0.001-0.007 (interval 0.001)	(i) 1/2000 (ii) 1/3000 (iii) 1/6000	1/34.4 1/58.5 1/99.76	7.0
	Case b	0.004	1/2000	1/34.4	1.0 - 8.0 (interval 1.0)
II. Higher Mode Frequencies		0.001-0.007 (interval 0.001)	1/2000	1/34.4	7.0

```
      C      A N A N D   V I J O Y   S I N G H
      C      ****************************************************************
      C      PURPOSE OF THE PROGRAM:
      C            THIS PROGRAM CALCULATES THE FREQUENCY FOR THE CLAMPED-EDGE
      C            DEEP SANDWICH SPHERICAL-SHELL.
      C
      C      METHOD:
      C            THE ROOTS LAMDA1,LAMDA2,LAMDA3 OF THE CUBIC EQUATION ARE
      C            CALCULATED USING SUBROUTINE CUBIC(A,XR,XI)
      C            THE CALCULATION OF THE LOWEST NATURAL FREQUENCY IS CARRIED OUT
      C            BY SPECIFYING THE VALUE OF X AND THE FREQUENCY Y AND THEN THE
      C            DETERMINANT DET IS EVALUATED BY INCREASING THE FREQUENCY.THIS
      C            PROCEDURE IS REPEATED.WHEN THIS DET CHANGES THE SIGN,THE VALUES
      C            OF X AND Y ARE NOTED.THE VALUE OF Y FOR WHICH THE DETERMINANT
      C            IS ZERO IS THE REQUIRED FREQUENCY.THE SAME PROCESS IS REPEATED
      C            BY INCREASING X.
      C
      C      DESCRIPTION OF THE PARAMETERS:
      C            V    =POISSON'S RATIO OF THE FACE MATERIAL
      C            VC   =POISSON'S RATIO OF THE CORE MATERIAL
      C            C    =HALF OF THE THICKNESS OF CORE
      C            R    =RADIUS OF THE MIDDLE SURFACE OF THE SHELL
      C            E    =MODULUS OF ELASTICITY OF THE FACE
      C            EC   =MODULUS OF ELASTICITY OF THE CORE
      C            RE   =EC/E
      C            X    =HSM/R
      C            Y    =OMEGA
      C            RHSM=C/HSM
      C            RRHO=MASS DENSITY RATIO  (CORE/FACE)
      C      ****************************************************************
      C
0001         IMPLICIT REAL*8(A-D,O-R,V-Z)
0002         IMPLICIT COMPLEX*16(F-H,S-U)
0003         DIMENSION A(4),XR(3),U(6),SPV(6),CV(3),PCV(3),AQ(3),AV(3),PVA(3)
0004         X=0.001
0005         DO 100 I=1,4
```

```
FORTRAN IV G LEVEL  19                 MAIN                 DATE = 71224          23/05/03              PAG
 0006                 Y=0.866
 0007                 DO 200 J=1,10
 0008                 RHSM=7.0
 0009                 V=0.3
 0010                 VC=0.091
 0011                 AX1=1.0
 0012                 AX2=24.40
 0013                 AX3=2000.0
 0014                 RRHO=AX1/AX2
 0015                 RE=AX1/AX3
 0016                 P=1.0+((RHSM+0.5)*X)**2.0
 0017                 P1=1.0+((RHSM*X)**2)/3.0
 0018                 P2=P1+((RHSM*X)**2)/3.0
 0019                 Q=(1.0+RRHO*RHSM)*(Y**2.0)
 0020                 ALPHA=(1.0+((RHSM*X)**2)*(2.0*RHSM+3.0)/(3.0*(2.0*RHSM+1.0)))/P1
 0021                 A1=P1*RE*RHSM*(X**2)/(1.0+VC)
 0022                 A2=(-4.0*P/(1.0-V)+2.0*Q)*(X**2)
 0023                 A3=-(RE*P1*RHSM/(1.0+VC)+2.0*P/(1.0-V))*X
 0024                 A4=P2*RE/(1.0+VC)+4.0*(RHSM+0.5)*X*X/(1.0-V)
 0025                 B1=((3.0*RE*RHSM*P1)/(1.0+VC)+(2.0*P)/(1.0-V))*(X**2)
 0026                 B2=2.0*P*X/(1.0-V*V)
 0027                 B3=(-3.0*P1*RE*RHSM/(1.0+VC)+2.0*P/(1.0+V)+2.0*Q)*X
 0028                 B4=-4.0*(0.5+RHSM)*X*X/(1.0-V*V)
 0029                 B5=3.0*P2*RE/(1.0+VC)-4.0*(0.5+RHSM)*X*X/(1.0+V)
 0030                 C1=-(RE*X)/(1.0+VC)-4.0*(0.50+RHSM)*(X**3)/(1.0-V)
 0031                 C2=-(4.0*(0.50+RHSM)*X*X)/(1.0-V*V)
 0032                 C3=RE/(1.0+VC)-(4.0*(0.50+RHSM)*X*X)/(1.0+V)
 0033                 C4=(2.0*P*X)/(1.0-V*V)
 0034                 C5=-RE*(1.0+(RHSM*X)**2)/((1.+VC)*RHSM*X)+2.0*P*X/(1.0+V)+2.0*X*
                     &(Y**2)
 0035                 A(1)=A1*(B4*C2-B2*C4)
 0036                 A(2)=A1*(B2*C5-B5*C2+B3*C4-B4*C3)+A2*(B2*C4-B4*C2)+A3*(B4*C1-B1*
                     &C4)+A4*(B1*C2-B2*C1)
 0037                 A(3)=A1*(B5*C3-B3*C5)+A2*(B5*C2-B2*C5+B4*C3-B3*C4)+A3*(B1*C5-B5*C1
                     &)+A4*(B3*C1-B1*C3)
 0038                 A(4)=A2*(B3*C5-B5*C3)
```

```
0039              IF(A(1).EQ.0.0)GO TO 110
            C
            C     SUBROUTINE CUBIC(A,XR,XI) CALCULATES THE ROOTS OF THE CUBIC
            C     EQUATION:
            C
            C     A(1) * (X**3) + A(2) * (X**2) + A(3) * X + A(4) =0
            C
            C     NOTE:
            C          THE ROOTS MAY BE ALL REAL OR ONE REAL AND TWO COMPLEX CONJUGATE.
            C          IN THE 1ST CASE XR1,XR2,XR3 ARE THE THREE REAL ROOTS.FOR THE
            C          OTHER CASE THE ROOTS ARE XR1,CMPLX(XR2,XI),CMPLX(XR2,-XI).
            C
0040              CALL CUBIC(A,XR,XI)
0041              GO TO 120
            C
0042          110 AQ(1)=A(2)
0043              AQ(2)=A(3)
0044              AQ(3)=A(4)
0045              XR(1)=0.0
0046              CALL QUAD(AQ,XR(2),XR(3),XI)
            C
0047          120 Z1=0.0
0048              Z=-XI
0049              T1=DCMPLX(XR(1),Z1)
0050              T2=DCMPLX(XR(2),XI)
0051              T3=DCMPLX(XR(3),Z)
0052              SV1=CDSQRT(T1+0.25)-0.5
0053              SV2=CDSQRT(T2+0.25)-0.5
0054              SV3=CDSQRT(T3+0.25)-0.5
            C
            C
            C     WRITE(3,11)X,Y,SV1,SV2,SV3
            C  11 FORMAT(10X,2F8.3,5X,6D16.7)
            C     GO TO 21
            C
            C
```

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FORTRAN IV G LEVEL  19                    MAIN                DATE = 71224          23/05/03          PA
 0055            121 U111=-A1*T1+A2
 0056                U121=A3
 0057                U131=A4
 0058                U211=-B1*T1
 0059                U221=-B2*T1+B3
 0060                U231=-B4*T1+B5
 0061                U112=-A1*T2+A2
 0062                U122=A3
 0063                U132=A4
 0064                U212=-B1*T2
 0065                U222=-B2*T2+B3
 0066                U232=-B4*T2+B5
 0067                U113=-A1*T3+A2
 0068                U123=A3
 0069                U133=A4
 0070                U213=-B1*T3
 0071                U223=-B2*T3+B3
 0072                U233=-B4*T3+B5
 0073                UA1=(U131*U211-U111*U231)/(U121*U231-U221*U131)
 0074                UA2=(U132*U212-U112*U232)/(U122*U232-U222*U132)
 0075                UA3=(U133*U213-U113*U233)/(U123*U233-U223*U133)
 0076                UB1=(U111*U221-U121*U211)/(U121*U231-U221*U131)
 0077                UB2=(U112*U222-U122*U212)/(U122*U232-U222*U132)
 0078                UB3=(U113*U223-U123*U213)/(U123*U233-U223*U133)
               C
 0079                IF (XR(1).GT.0.0) GO TO 123
               C
 0080                PR=XR(1)+0.25
 0081                PQ=-PR
 0082                PP=DSQRT(PQ)
 0083                CALL LFHCA(PP,SPV1,SWW1)
 0084                CALL DLF(PP,SPPV1)
 0085                PV1=SPV1
 0086                PPV1=SPPV1
 0087                PP1=SWW1
 0088                GO TO 128
```

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FORTRAN IV G LEVEL  19                  MAIN               DATE = 71224          23/05/03            PAG
 0089             123  V1=DSQRT(XR(1)+0.25)-0.50
 0090                  AV(1)=V1
 0091                  AV(2)=V1+1.0
 0092                  AV(3)=V1-1.0
 0093                  DO 15 M=1,3
 0094                  VV=AV(M)
 0095                  IF (VV.LT.10.0) GO TO 125
 0096                  CALL LFUNCT(VV,APV)
 0097                  GO TO 126
 0098             125  CALL MIRLF(VV,APV)
 0099             126  PVA(M)=APV
 0100              15  CONTINUE
 0101                  PV1=PVA(1)
 0102                  PPV1=AV(2)*PVA(2)
 0103                  PP1=(PVA(2)-PVA(3))/(2.0*V1+1.0)
 0104             128  U(1)=SV2
 0105                  U(2)=SV3
 0106                  U(3)=SV2+1.0
 0107                  U(4)=SV3+1.0
 0108                  U(5)=SV2-1.0
 0109                  U(6)=SV3-1.0
 0110                  DO 10 K=1,6
 0111                  UU=U(K)
                 C
                 C     SUBROUTINE COMLF(V,PV) CALCULATES THE VALUES OF THE LEGENDRE
                 C     FUNCTION OF COMPLEX ORDER 'V' USING ITS MEHLER'S INTEGRAL
                 C     REPRESENTATION.
                 C
 0112                  CALL COMLF(UU,HPV)
                 C
                 C
 0113              30  SPV(K)=HPV
 0114              10  CONTINUE
 0115                  SPV2=SPV(1)
 0116                  SPV3=SPV(2)
 0117                  SPPV2=U(3)*SPV(3)
```

```
 0118                 SPPV3=U(4)*SPV(4)
 0119                 SPP2=(SPV(3)-SPV(5))/(2.0*SV2+1.0)
 0120                 SPP3=(SPV(4)-SPV(6))/(2.0*SV3+1.0)
 0121                 S1=UA1*PP1
 0122                 S2=UA2*SPP2
 0123                 S3=UA3*SPP3
 0124                 H1=UB1*PP1
 0125                 H2=UB2*SPP2
 0126                 H3=UB3*SPP3
 0127                 HA1=RHSM*X*X*PPV1+RHSM*X*S1-H1*P1
 0128                 HA2=RHSM*X*X*SPPV2+RHSM*X*S2-H2*P1
 0129                 HA3=RHSM*X*X*SPPV3+RHSM*X*S3-H3*P1
 0130                 HB1=(UA1+(1.0+V)*X-(RHSM+0.5)*X*UB1)*PV1
 0131                 HB2=(UA2+(1.0+V)*X-(RHSM+0.5)*X*UB2)*SPV2
 0132                 HB3=(UA3+(1.0+V)*X-(RHSM+0.5)*X*UB3)*SPV3
 0133                 HC1=(UB1-(RHSM+0.5)*X*(UA1+(1.0+V)*X))*PV1
 0134                 HC2=(UB2-(RHSM+0.5)*X*(UA2+(1.0+V)*X))*SPV2
 0135                 HC3=(UB3-(RHSM+0.5)*X*(UA3+(1.0+V)*X))*SPV3
 0136                 IF (XI.NE.0.0) GO TO 129
            C
 0137                 PV3  =  SPV3
 0138                 AH3  =  H3
 0139                 AS3  =  S3
 0140                 AHA3  =  HA3
 0141                 AHB3  =  HB3
 0142                 AHC3  =  HC3
 0143                 PV2=SPV2
 0144                 AH2=H2
 0145                 AS2=S2
 0146                 AHA2=HA2
 0147                 AHB2=HB2
 0148                 AHC2=HC2
 0149                 SDET=PV1*(AS2*AH3-AS3*AH2)+PV2*(H1*AS3-S1*AH3)+PV3*(S1*AH2-AS2*H1)
 0150                 SDET1=HA1*(AHB2*AHC3-AHB3*AHC2)+AHA2*(AHB3*HC1-HB1*AHC3)+
                     &AHA3*(HB1*AHC2-AHB2*HC1)
 0151                 GO TO 131
```

```
C
C
  129 Z2=1.0
      SI=DCMPLX(Z1,Z2)
      CA1=PV1
      EA2=(SPV2+SPV3)/2.0
      EA3=SI*(SPV2-SPV3)/2.0
      EB1=S1
      EB2=(S2+S3)/2.0
      EB3=SI*(S2-S3)/2.0
      EC1=H1
      EC2=(H2+H3)/2.0
      EC3=SI*(H2-H3)/2.0
      GA1=HA1
      GA2=(HA2+HA3)/2.0
      GA3=SI*(HA2-HA3)/2.0
      GB1=HB1
      GB2=(HB2+HB3)/2.0
      GB3=SI*(HB2-HB3)/2.0
      GC1=HC1
      GC2=(HC2+HC3)/2.0
      GC3=SI*(HC2-HC3)/2.0
      SDET=CA1*(EB2*EC3-EB3*EC2)+EA2*(EB3*EC1-EB1*EC3)+EA3*(EB1*EC2-
     &EB2*EC1)
      SDET1=GA1*(GB2*GC3-GB3*GC2)+GA2*(GB3*GC1-GB1*GC3)+GA3*(GB1*GC2-
     &GB2*GC1)
C
  131 WRITE(3,17)X,Y,RHSM,SDET,SDET1
   17 FORMAT(5X,3F7.4,5X,4D16.6,/)
      Y=Y+0.002
  200 CONTINUE
      X=X+0.001
  100 CONTINUE
   21 RETURN
      END
```

```
      C     PROGRAM..........(1)
      C     SUBROUTINE CUBIC(A,XR,XI)
      C     PURPOSE:-
      C         THIS SUBROUTINE CALCULATES THE ROOTS OF A CUBIC EQUATION
      C         GIGEN AS:-
      C
      C         A(1)*X**3 + A(2)*X**2 + A(3)*X + A(4) =0
      C
      C         XR(1)=REAL ROOT OF THE CUBIC REAL POLYNOMIAL
      C         XR(2)=REAL PART OF THE FIRST COMPLEX ROOT
      C         XR(3)=REAL PART OF THE  2ND COMPLEX ROOT
      C         XI=IMAGINARY PART OF THE COMPLEX ROOTS
      C
      C       NOTE-
      C         COMPLEX ROOTS ARE COMPLEX CONJUGATE OF EACH OTHER
      C
      C
            SUBROUTINE CUBIC(A,XR,XI)
            IMPLICIT REAL*8(A-H,O-Z)
            DIMENSION A(4),XR(3),AQ(3)
        200 NPATH=2
            AX=1.0
            BX=3.0
            EX=AX/BX
            IF(A(4))1006,1004,1006
       1004 XR(1)=0
            GO TO 1034
       1006 A2=A(1)*A(1)
            Q=(27.*A2*A(4)-9.0*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
            IF(Q)1010,1008,1014
       1008 Z=0.0
            GO TO 1032
       1010 Q=-Q
            NPATH=1
       1014 P=(3.*A(1)*A(3)-A(2)*A(2))/(9.*A2)
            ARG=P*P*P+Q*Q
```

```
0020             IF(ARG)1016,1018,1020
0021       1016  Z=-2.*DSQRT(-P)*DCOS(DATAN(DSQRT(-ARG)/Q)/3.0)
0022             GO TO 1028
0023       1018  Z=-2.*Q**EX
0024             GO TO 1028
0025       1020  SARG=DSQRT(ARG)
0026             IF(P)1022,1024,1026
0027       1022  Z=-(Q+SARG)**EX-(Q-SARG)**EX
0028             GO TO 1028
0029       1024  Z=-(2.*Q)**EX
0030             GO TO 1028
0031       1026  Z=(SARG-Q)**EX-(SARG+Q)**EX
0032       1028  GO TO(1030,1032),NPATH
0033       1030  Z=-Z
0034       1032  XR(1)=(3.*A(1)*Z-A(2))/(3.*A(1))
0035       1034  AQ(1)=A(1)
0036             AQ(2)=A(2)+XR(1)*AQ(1)
0037             AQ(3)=A(3)+XR(1)*AQ(2)
0038             CALL QUAD(AQ,XR(2),XR(3),XI)
0039             RETURN
0040             END
```

```
FORTRAN IV G LEVEL  19                MAIN              DATE = 71224          23/05/03          PA
                  C      PROGRAM .........(2)
                  C      SUBROUTINE QUAD(A,XR1,XR2,XI)
                  C      PURPOSE:-
                  C           THIS SUBROUTINE CALCULATES THE ROOTS OF A QUADRATIC EQUATION
                  C           GIVEN AS:
                  C
                  C           A(1)*X*X+A(2)*X+A(3)=0
                  C
                         SUBROUTINE QUAD(A,XR1,XR2,XI)
                         IMPLICIT REAL*8(A-H,O-Z)
                         DIMENSION A(3)
                     300 X1=-A(2)/(2.*A(1))
                         DISC=X1*X1-A(3)/A(1)
                         IF(DISC)10,20,20
                      10 X2=DSQRT(-DISC)
                         XR1=X1
                         XR2=X1
                         XI=X2
                         GO TO 30
                      20 X2=DSQRT(DISC)
                         XR1=X1+X2
                         XR2=X1-X2
                         XI=0.0
                      30 RETURN
                         END
```

```
FORTRAN IV G LEVEL  19                  MAIN                 DATE = 71224          23/05/03

              C       ***********************************************************************
              C       SUBROUTINE  MIRLF(V,PV)
              C       MIRLF= MEHLER'S  INTEGRAL REPRESENTATION  OF  LEGENDRE  FUNCTION
              C
              C       PURPOSE OF THIS SUBROUTINE:
              C             NUMERICAL COMPUTATION OF LEGENDRE FUNCTION OF FRACTIONAL
              C             ORDERS.
              C
              C       USAGE
              C             CALL MIRLF(V,PV)
              C
              C       METHOD
              C             CALCULATION OF THE VALUES OF LEGENDRE FUNCTION ARE DONE USING
              C             MEHLER'S INTEGRAL  REPRESENTATION (REF.:EQN.27 OF NAGHDI'S
              C             PAPER IN JOURNAL OF APPLIED MECHANICS,PAGE.68,MARCH 1962).
              C             THE INTEGRATION IS DONE NUMERICALLY USING TRAPEZOIDAL RULE.
              C
              C       DESCRIPTION OF THE PARAMETERS:
              C             PI=4.0*ATAN(1.0)
              C             N =NUMBER OF INTERVALS
              C             H =THICKNESS OF THE STRIP
              C             A =LOWER LIMIT OF INTEGRATION
              C             B =UPPER LIMIT OF INTEGRATION
              C             V =ORDER OF THE FUNCTION
              C             PV=LEGENDRE FUNCTION OF ORDER V
              C
              C       REMARKS:
              C             THIS SUBROUTINE GIVES THE VALUES OF LEGENDRE FUNCTION CORRECT
              C             UP TO THREE DESIMAL PLACES.
              C       ***********************************************************************
0001                  SUBROUTINE MIRLF(V,PV)
0002                  IMPLICIT REAL*8(A-H,O-Z)
0003                  C=1.0
0004                  PI=4.0*DATAN(C)
0005                  A=0.0
0006                  B=PI/2.0
```

```
      N=512
      H=(B-A)/512.0
      X=0.0
      P=DCOS((V+1.5)*X)
      Q=DSQRT(DCOS(X))
      R=(V+2.0)*2.0*(2.0**0.5)/PI
      OLDZ=R*Q*P
      SUM=0.0
      DO 25 I=1,N
      X=X+H
      P=DCOS((V+1.5)*X)
      Q=DSQRT(DCOS(X))
      Z=R*Q*P
      SUM=SUM+OLDZ+Z
   25 OLDZ=Z
      PV=SUM*H/2.0
      RETURN
      END
```

```
FORTRAN IV G LEVEL  19                   MAIN                  DATE = 71224           23/05/03
                C       ******************************************************************
                C       SUBROUTINE COMLF(V,PV)
                C
                C       PURPOSE:
                C           NUMERICAL COMPUTATION OF LEGENDRE FUNCTION OF COMPLEX ORDERS.
                C
                C       USAGE:
                C           CALL  COMLF(V,PV)
                C
                C       METHOD:
                C           THE VALUES OF THE LEGENDRE FUNCTIONS ARE CALCULATED USING
                C           MEHLER' INTEGRAL REPRESENTATION.
                C
                C       DESCRIPTION OF THE PARAMETERS:
                C           API=PI,
                C           N   =NUMBER OF INTERVALS
                C           H   =THICKNESS OF THE STRIP
                C           A   =LOWER LIMIT OF INTEGRATION
                C           B   =UPPER LIMIT OF INTEGRATION
                C           V   =ORDER OF THE FUNCTION
                C           PV  =LEGENDRE FUNCTION OF THE COMPLEX ORDER 'V'
                C       ******************************************************************
 0001                   SUBROUTINE COMLF(V,PV)
 0002                   IMPLICIT REAL*8(A-H,T,X-Z)
 0003                   IMPLICIT COMPLEX*16(O-S,U-W)
 0004                   C=1.0
 0005                   API=4.0*DATAN(C)
 0006                   A=0.0
 0007                   B=API/2.0
 0008                   N=256
                C       COMPLEX NUMERICAL INTEGRATION FOLLOWS
 0009                   FN=N
 0010                   H=(B-A)/FN
 0011                   X=A
 0012                   D=2.0*(2.0**0.5)/(API*DSIN(B)*DSIN(B))
 0013                   R1=(V+1.5)*X
```

```
FORTRAN IV G LEVEL  19                    COMLF                DATE = 71224              23/05/03

                   R2=(V+0.5)*X
                   S=(V+2.0)*CDCOS(R1)-V*DCOS(B)*CDCOS(R2)
                   AB=DCOS(X)-DCOS(B)
                   OLDZ=D*S*(AB**0.5)
                   SUM=0.0
                   DO 20 I=1,N
                   X=X+H
                   R1=(V+1.5)*X
                   R2=(V+0.5)*X
                   S=(V+2.0)*CDCOS(R1)-V*DCOS(B)*CDCOS(R2)
                   AB=DCOS(X)-DCOS(B)
                   W=D*S*(AB**0.5)
                   SUM=SUM+OLDZ+W
                20 OLDZ=W
                   PV=SUM*H/2.0
                   RETURN
                   END
```

```
C      ****************************************************************
C      SUBROUTINE  LFHCA(P,DPV,DWW)
C
C      LFHCA= LEGENDRE FUNCTIONS OF HIGHER COMPLEX ARGUMENT
C
C      PURPOSE:
C          TO CALCULATE THE VALUE OF THE CONICAL FUNCTION AND A
C          RECCURRENCE RELATION FOR HIGHER VALUE OF P.THE ORDER OF THE
C          CONICAL FUNCTION IS OF THE FORM  V = - 1/2 + I.P
C
C      USAGE:
C          CALL LFHCA(P,DPV,DWW)
C
C      DESCRIPTION OF THE PARAMETERS
C          PI  = 4.0*ATAN(1.0)
C          P   = COMPLEX PART OF THE ARGUMENT
C          DPV= VAUE OF THE CONICAL FUNCTION
C          DWW= VALUE OF THE RECCURRENCE RELATION
C      ****************************************************************
       SUBROUTINE LFHCA(P,DPV,DWW)
       IMPLICIT REAL*8(A-B,E-H,O-Z)
       IMPLICIT COMPLEX*16(C-D)
   111 Q=1.0
       PI=4.0*DATAN(Q)
       U=PI*P/2.0
       U1=-U
       E1=DEXP(U)/2.0
       E2=DEXP(U1)/2.0
       A=E1+E2
       B=E1-E2
       Z=(1.0+P**2.0)**0.5
       Z1=(1.0+Z)*0.5
       X=DSQRT(Z1)
       Y=P/(2.0*X)
       Z2=(-1.0+Z)*0.5
       X1=DSQRT(Z2)
```

```
FORTRAN IV G LEVEL  19                LFHCA              DATE = 71224            23/05/03               PAG
 0018                 Y1=P/(2.0*X1)
 0019                 AA1=(A+B)/((2.0*P*PI)**0.5)
 0020                 AA2=(B-A)/((2.0*P*PI)**0.5)
 0021                 DPV=DCMPLX(AA1,AA2)
 0022                 P1=(X**2.0+Y**2.0)*(PI**0.5)
 0023                 P2=(X1**2.0+Y1**2.0)*(PI**0.5)
 0024                 BB1=(A*X-B*Y)/P1
 0025                 BB2=-(A*Y+B*X)/P1
 0026                 DPV1=DCMPLX(BB1,BB2)
 0027                 AC1=(-A*X1+B*Y1)/P2
 0028                 AC2=(A*Y1+B*X1)/P2
 0029                 DPV2=DCMPLX(AC1,AC2)
 0030                 W1=0.0
 0031                 W2=1.0
 0032                 DW=DCMPLX(W1,W2)
 0033                 DWW=((DPV2-DPV1)*DW)/(2.0*P)
 0034                 RETURN
 0035                 END
```

```
FORTRAN IV G LEVEL  19                  MAIN                 DATE = 71224          23/05/03                  PAG
               C       ****************************************************************************
               C       CALCULATION OF THE VALUES OF THE LEGENDRE FUNCTION AND ITS
               C       DERIVATIVE HAVING COMPLEX ORDERS OF HIGHER ARGUMENT.
               C       ****************************************************************************
 0001                  SUBROUTINE DLF(P,DPPV)
 0002                  IMPLICIT REAL*8(A-B,E-H,O-Z)
 0003                  IMPLICIT COMPLEX*16(C-D)
 0004                  E=1.0
 0005                  PI=4.0*DATAN(E)
 0006                  X=PI*P/2.0
 0007                  Y=(2.0*PI*P)**0.5
 0008                  Z=(P/(2.0*PI))**0.50
 0009                  U=-X
 0010                  A=DEXP(X)
 0011                  B=DEXP(U)
 0012                  S=-B
 0013                  DP1=DCMPLX(A,B)
 0014                  DP2=DCMPLX(A,S)
 0015                  DPV=DP2/Y
 0016                  DPPV=DP1*Z
 0017                  RETURN
 0018                  END
```

```
FORTRAN IV G LEVEL  19                MAIN                DATE = 71224          23/05/03           PAG
                C       ******************************************************************
                C       SUBROUTINE LFUNCT(V,PV)
                C
                C       USAGE:
                C            CALL LFUNCT(V,PV)
                C
                C       PURPOSE:
                C            THIS SUBROUTINE CALCULATES THE VALUES OF LEGENDRE FUNCTIONS
                C            PV(COS(PI/2)) OF HIGHER ORDERS.
                C
                C       REFERENCE:
                C            EQN.12,PAGE-419.
                C            APPLIED MATHEMATICS FOR ENGINEERS AND SCIENTISTS.
                C            BY:  S. A. SCHELKUNOFF, 1948.
                C
                C       ******************************************************************
 0001                   SUBROUTINE LFUNCT(V,PV)
 0002                   IMPLICIT REAL*8(A-H,O-Z)
 0003                   C=1.0
 0004                   PI=4.0*DATAN(C)
 0005                   R=DSQRT((V+0.5)**2+0.25)
 0006                   B=DSQRT(V*(V+1.0))
 0007                   PV=DCOS((R-0.5)*PI/2.0)/DSQRT(PI*B/2.0)
 0008                   RETURN
 0009                   END
```

Chapter 6

RESULTS AND DISCUSSIONS

Numerical results for the frequency Ω are shown in Figs. 3-7 for sandwich shells with fixed and free boundary conditions. The results for the fixed shell have been generated from the simultaneous solutions of the characteristic cubic Eq.(3.11) and the frequency determinant (4.5). Equations (3.11) and (4.9) were utilized to yield the solutions for the free shell. For the present study, a sandwich spherical shell with cellular cellulose acetate core has been considered. The face sheets were taken to be of three different materials, aluminum, Zn-Cu alloy and mild steel.

It is observed that the solution of the frequency determinant is strongly dependent on the character of the three indices n_1, n_2 and n_3 given by Eq.(3.8). Fig.2 shows a plot of n_α ($\alpha = 1, 2, 3$) vs Ω for a given value of ν and h/R. The indices n_1, n_2, and n_3 are seen to be of different natures within three distinct zones identified in Fig.2. The variation of the indices n_1, n_2 and n_3 has been summarized in Table. 2. In Table 2, p_1, p_2 and p_3 are real quantities. The classification of indices n_1, n_2 and n_3 has been done with the following data about the material and geometry of the shell.

$$h/R = 0.004, \qquad r_h = 7.0$$

$$r_E = 1/2000, \qquad r_\rho = 1/34.4$$

$$\nu = 0.3 \quad \text{and} \quad \nu_c = 0.091$$

The variation of indices n_1 and n_2 vs Ω for free vibrations of spherical shells, based on the membrane theory, have also been plotted in Fig. 2. These results were obtained by substituting $r_E = 0$, $r_h = 0$ and $\nu_c = 0$.

They are identified by the dotted lines. The indices n_1 and n_2 for this type of shell have been found practically independent of the thickness to radius ratio h/R of the face sheet.

Discussion of Results

The lowest natural frequency has been plotted in Fig.3 and Fig.4 with Ω as the ordinate and h/R as the abscissa. Fig.3 shows the variation of Ω with the thickness to radius ratio of the face sheet h/R for clamped edge condition and Fig.4 for free edge condition. Three curves have been drawn for each boundary condition for different values of r_E and r_ρ. It should be noted that frequency Ω increases with the thickness of the face sheets but the slope of the curve decreases. The effect of modulus of elasticity of the face material on the frequency of the shell can also be studied from these curves. It has been found that frequency increases with decrease in r_E. In other words, if the core material is the same and face materials of higher modulus of elasticity are selected, natural frequency of the sandwich shell increases considerably. The curves flatten as r_E decreases for both the boundary conditions.

In Fig.5 and Fig.6, frequencies have been plotted for higher modes. Frequency increases with the thickness and with the mode numbers. For higher values of thickness to radius ratio h/R, the curves are almost flat but for lower nalues, frequencies increase rapidly with the thickness.

The effect of the thickness of the core on the lowest natural frequency Ω of the shell can be studied from Fig.7 which shows the variation of Ω with h/c. For constant thickness of the core, frequency increases with the increase in the thickness h of the face sheet. In Figs. 5, 6 and 7, the characteristic curves have been plotted for the sandwich hemispherical shell with a cellular cellulose acetate core and aluminum face sheets of equal thickness.

Since the amount of literature available on the vibrations of deep sandwich spherical shell is very little, the direct comparison of results could not be made. Naghdi [10] and Kalnins [4], while dealing with the problem of deep spherical shell based on the bending theory of shells, also calculated the natural frequencies for the axi-symmetric vibration of free hemispherical membrane shells. In the present study the faces are considered as membranes and for the sake of comparison of results with those mentioned in [10, 4] the problem was reduced to the vibrations of membrane shells. The calculated values of frequencies have been compared with the results available in literature in the form of graphs. It has been observed that the present results lie within ± 1.0 % of their previous values in [10] and [4]. The membrane modes have been found to be independent of the thickness to radius ratio h/R of the shell.

TABLE 2

Nature of the roots n_α ($\alpha = 1, 2, 3$)

Zone		n_1	n_2	n_3
1	$0 < \Omega < 0.91$	$-1/2 + ip_1$	Real	$-1/2 + ip_3$
2	$0.91 \leq \Omega \leq 1.51$	Real	$-1/2 + ip_2$ for $o.91 \leq \Omega \leq 1.0$ Real for $1.0 < \Omega \leq 1.51$	$-1/2 + ip_3$
3	$1.51 < \Omega$	Real	Real	Real

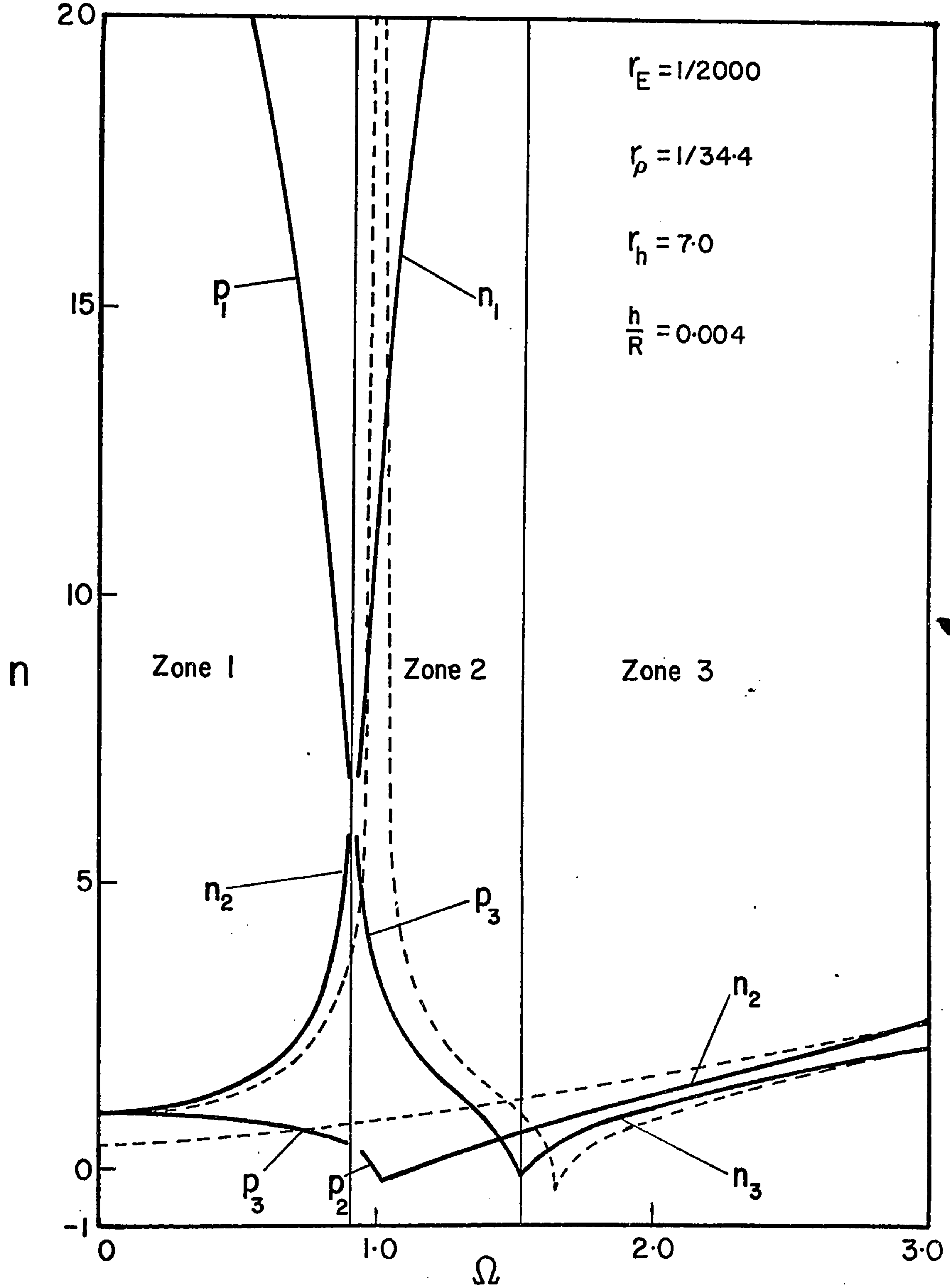

FIG.2. ORDERS OF LEGENDRE FUNCTIONS VS Ω

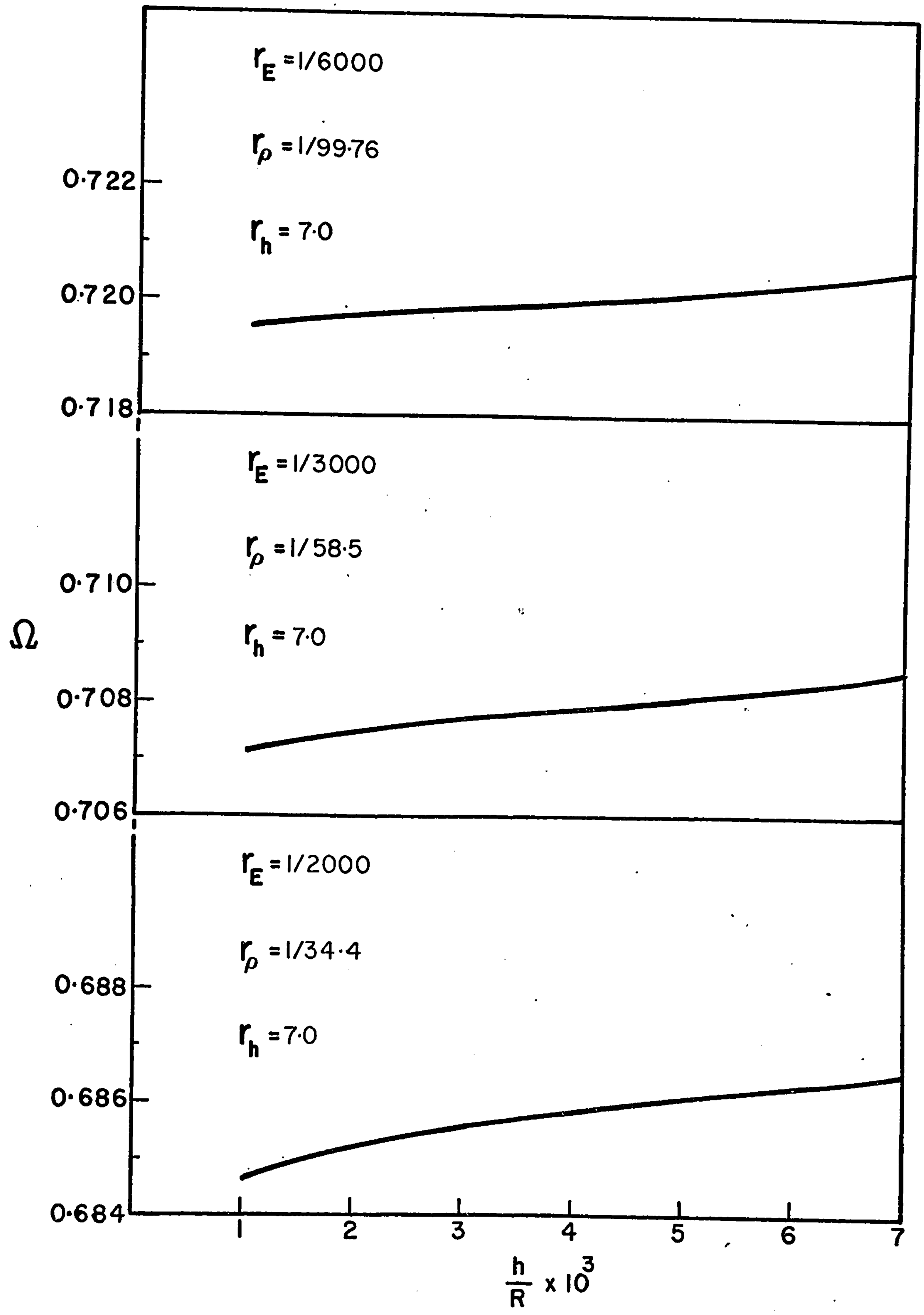

FIG.3 LOWEST NATURAL FREQUENCY Ω VERSUS $\frac{h}{R}$ FOR CLAMPED-EDGE CONDITION

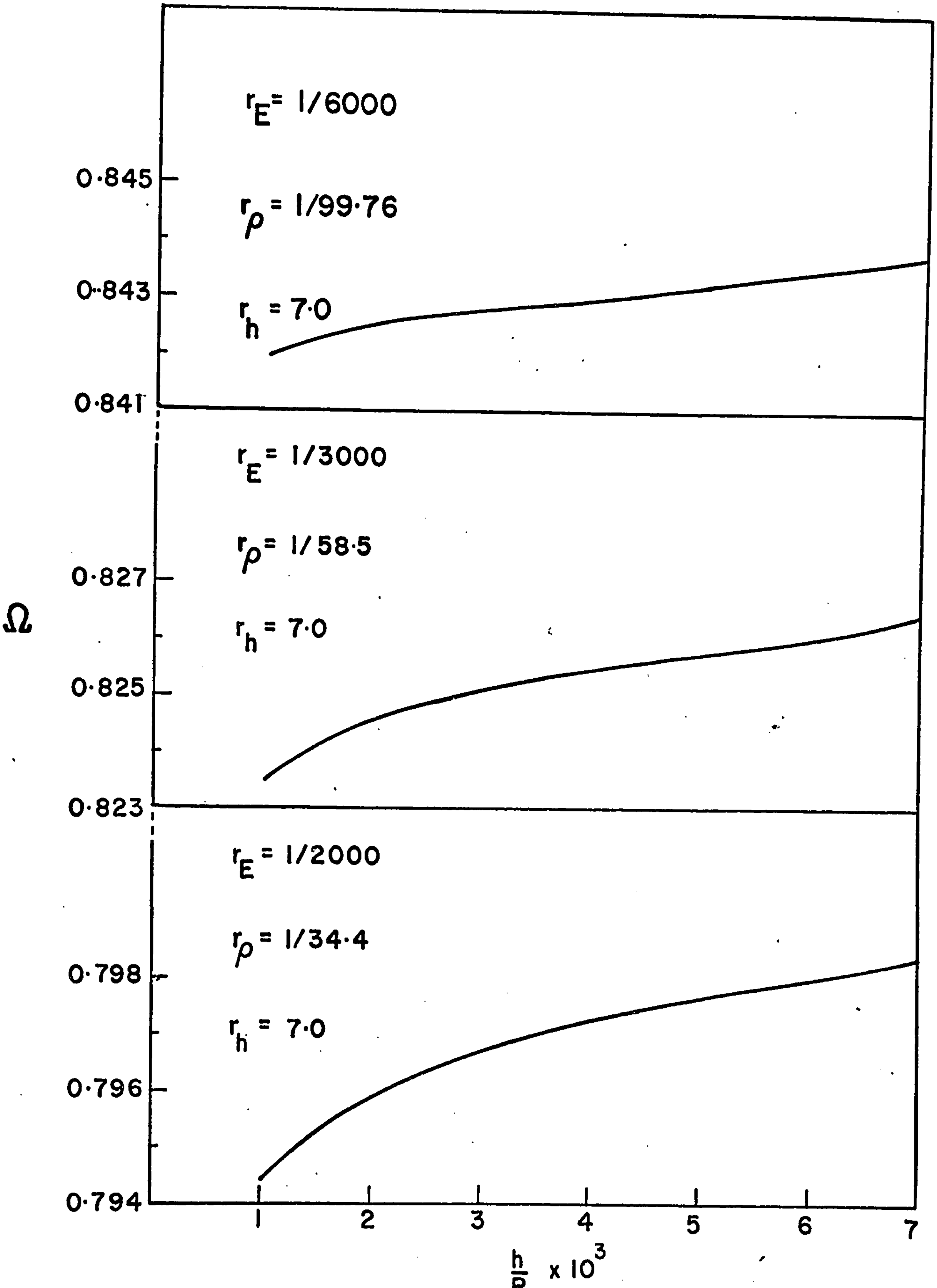

FIG. 4 LOWEST NATURAL FREQUENCY Ω VERSUS h/R FOR FREE-EDGE CONDITION

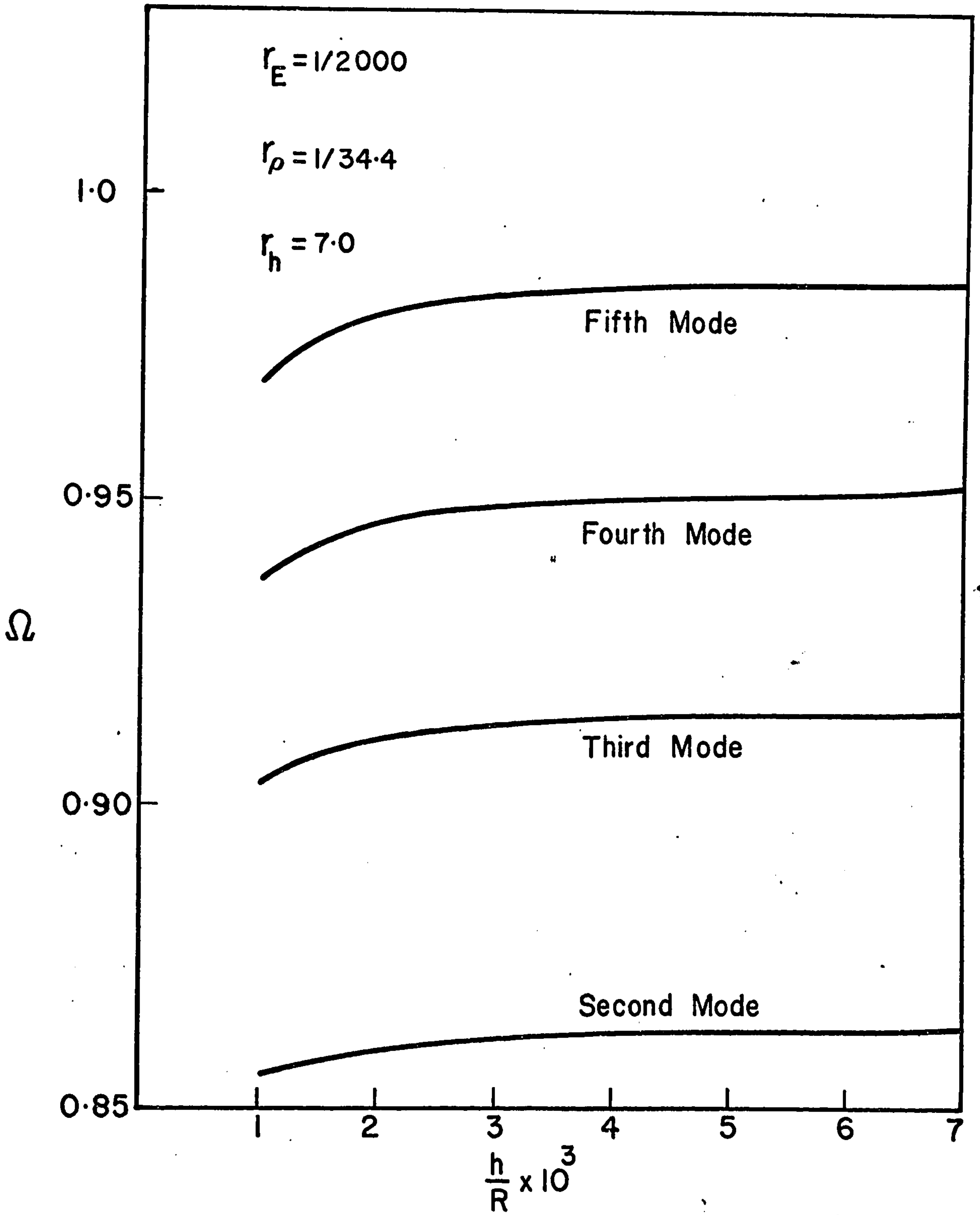

FIG.5 NATURAL FREQUENCY Ω VERSUS $\frac{h}{R}$ FOR CLAMPED-EDGE.

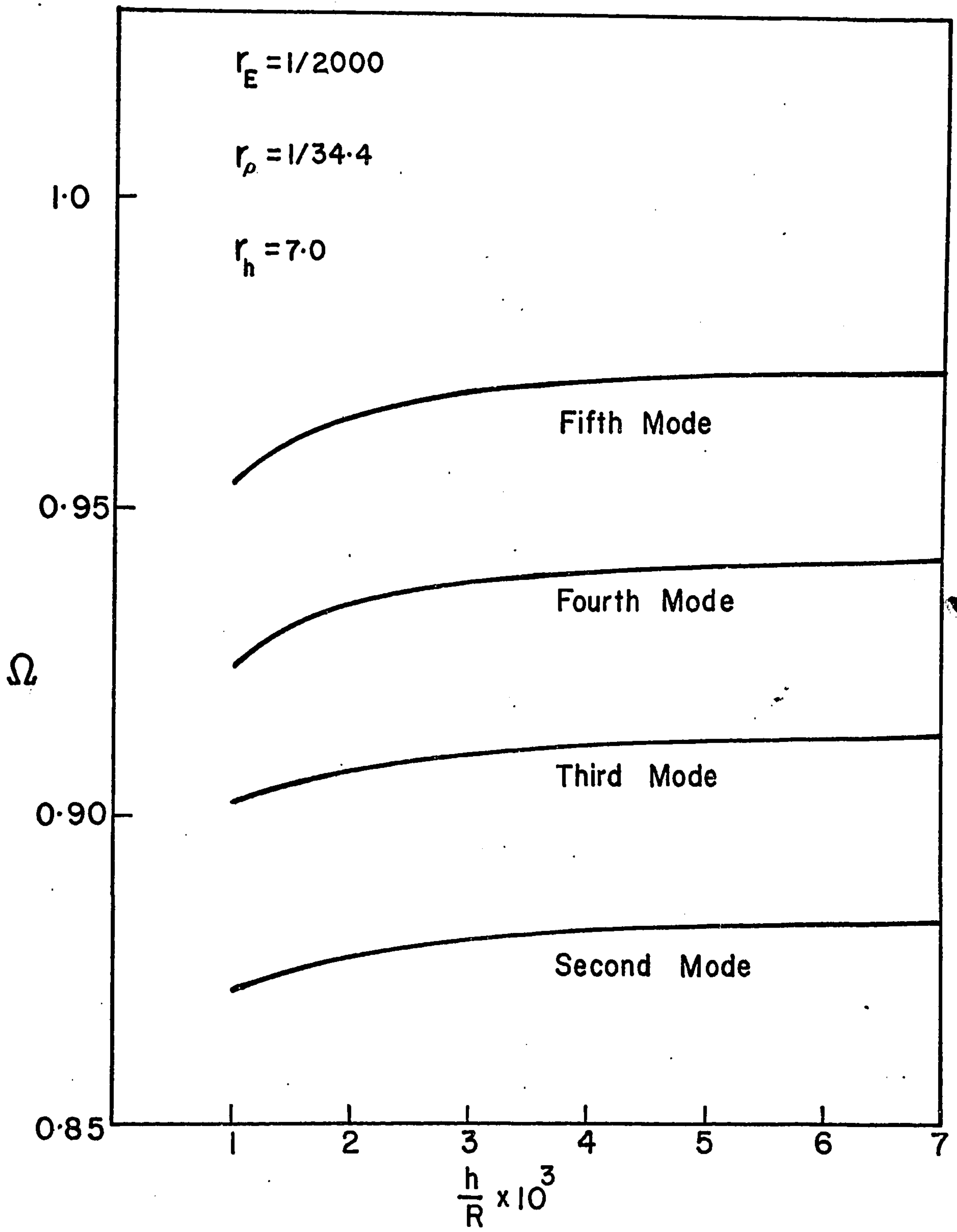

FIG.6 NATURAL FREQUENCY Ω VERSUS $\frac{h}{R}$ FOR FREE-EDGE.

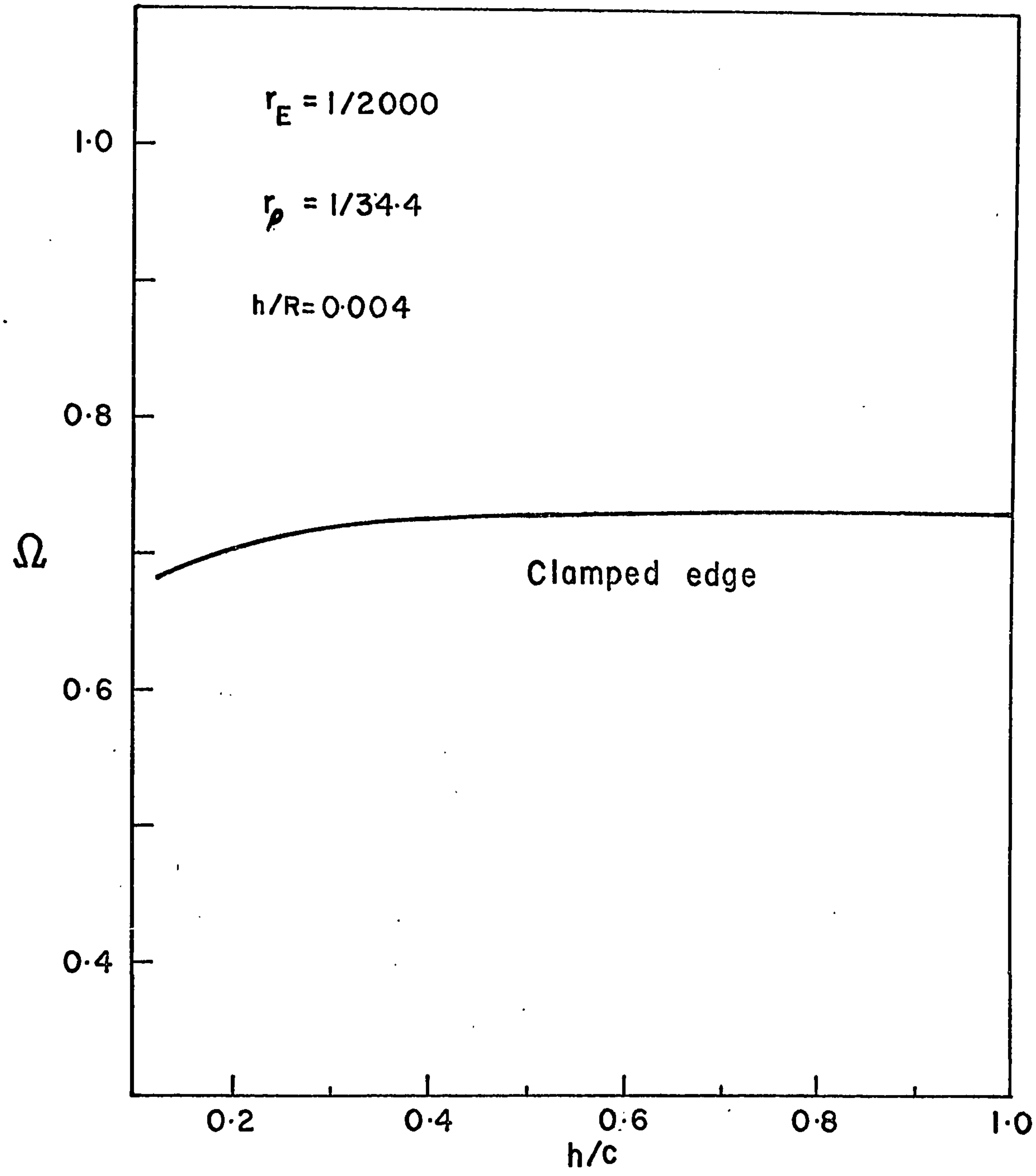

FIG.7 NATURAL FREQUENCY Ω VERSUS h/c.

APPENDICES

APPENDIX A

Using Eqs.(2.5)d, (2.8)c and the stress-strain relationship

$$\tau_{r\phi} = G_c \, \gamma_{r\phi}$$

the expression for $\tau_{r\phi}$ is given by

(a) $$\tau_{r\phi} = G_c(1 - z/R)(w,_\phi - v)/R + G_c \, v,_z$$

Substitution of v from Eq.(iii) gives

$$\tau_{r\phi} = G_c(1 - z/R) \{w,_\phi - (\overset{*}{v} + \bar{v} \, z/c)\}/R + G_c\bar{v}/c$$

Further simplification of the above equation yields

(b) $$\tau_{r\phi} = G_c(1 - z/R)(w,_\phi - \overset{*}{v})/R + G_c(1 - z/R + z^2/R^2) \, \bar{v}/c$$

Multiplying Eq.(b) by $1/R$ and integrating with respect to z over the thickness of the core, following equation has been obtained.

(c) $$\frac{1}{R}\int_{-c}^{c} \tau_{r\phi} dz = 2cG_c(w,_\phi - \overset{*}{v})/R^2 + 2G_c(1 + \tfrac{1}{3}c^2/R^2)\bar{v}/R$$

In a similar manner following equation can also be obtained as

(d) $$\frac{1}{R^2}\int_{-c}^{c} z\tau_{r\phi} \, dz = -\tfrac{2}{3}c^3G_c(w,_\phi - \overset{*}{v})/R^4 - \tfrac{2}{3} \, c^2G_c\bar{v}/R^3$$

Subtracting Eq.(d) from (c)

$$\text{(e)}\qquad \frac{1}{R}\int_{-c}^{c}\tau_{r\phi}dz - \frac{1}{R^2}\int_{-c}^{c}z\tau_{r\phi}dz = 2cG_c(1 + \tfrac{1}{3}c^2/R^2)(w,_{\phi} - \overset{*}{v})/R^2 +$$

$$2G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v}/R$$

APPENDIX B

With the help of Eq.(2.8) , the expression for $\gamma_{r\phi}$ from Eq.(2.5)d reduces to

$$\gamma_{r\phi} = (1 - z/R)(w,_{\phi} - v)/R + v,_{z} \tag{a}$$

Substituting the value of v from Eq.(iii), Eq.(a) becomes

$$\gamma_{r\phi} = (1 - z/R)(w,_{\phi} - \overset{*}{v} - \bar{v}z/c)/R + \bar{v}/c$$

Further simplification of the above equation gives

$$\gamma_{r\phi} = (1 - z/R)(w,_{\phi} - \overset{*}{v})/R + (1 - z/R + z^2/R^2)\bar{v}/c \tag{b}$$

Thus the expression for shear stress $\tau_{r\phi}$ is

$$\tau_{r\phi} = G_c(1 - z/R)(w,_{\phi} - \overset{*}{v})/R + G_c(1 - z/R + z^2/R^2)\bar{v}/c \tag{c}$$

By evaluating $\tau_{r\phi}$ at z = c and z =-c and adding them together, following equation can easily be obtained.

$$\tau_{r\phi}\Big|_{z=c} + \tau_{r\phi}\Big|_{z=-c} = 2G_c(w,_{\phi} - \overset{*}{v})/R + 2G_c(1 + c^2/R^2)\bar{v}/c \tag{d}$$

APPENDIX C

From Eqs.(2.27), the Laplace operator ∇^2 and the expresion for $\overset{*}{\theta}$ are

(a) $$\nabla^2 = \left(\frac{\partial^2}{\partial\phi^2} + \cot\phi \frac{\partial}{\partial\phi} \right)$$

(b) $$\overset{*}{\theta} = (\overset{*}{v}_{,\phi} + \overset{*}{v} \cot\phi)/R$$

Differentiation of Eq.(b) with respect to ϕ, yields

(c) $$\frac{\partial\overset{*}{\theta}}{\partial\phi} = (\overset{*}{v}_{,\phi\phi} + \overset{*}{v}_{,\phi}\cot\phi - \overset{*}{v} \operatorname{cosec}^2\phi)/R$$

Further differentiation reduces Eq.(c) to

(d) $$\frac{\partial^2\overset{*}{\theta}}{\partial\phi^2} = \{\overset{*}{v}_{,\phi\phi\phi} + \overset{*}{v}_{,\phi\phi} \cot\phi - (1 + \cot^2\phi)\overset{*}{v}_{,\phi} - (1 + \cot^2\phi)\overset{*}{v}_{,\phi} + \\ + 2\cot\phi(1 + \cot^2\phi)\overset{*}{v}\}/R$$

With the help of Eqs.(a), (c) and (d), the expression for $\nabla^2\overset{*}{\theta}$ is

(e) $$\nabla^2\overset{*}{\theta} = \{\overset{*}{v}_{,\phi\phi\phi} + 2\overset{*}{v}_{,\phi\phi} \cot\phi - (2 + \cot^2\phi)\overset{*}{v}_{,\phi} + \cot\phi(1 + \cot^2\phi)\overset{*}{v}\}/R^3$$

APPENDIX D

This appendix deals with the simplification of Eq. (2.26)b

$$\begin{aligned}
&\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}w,_{\phi} - 6cG_c(1 + \\
&\tfrac{1}{3}c^2/R^2)\overset{*}{v}/R^2 + 2K(1 + \bar{c}^2/R^2)(\overset{*}{v},_{\phi\phi} + \overset{*}{v},_{\phi}\cot\phi - \overset{*}{v}\cot^2\phi - \nu\overset{*}{v})/R^2 + \\
&6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v}/R - 4K\bar{c}(\bar{v},_{\phi\phi} + \bar{v},_{\phi}\cot\phi - \bar{v}\cot^2\phi - \nu\bar{v})/R^3 = \\
&(2\rho h + 2\rho_c c)\overset{*}{v},_{tt}
\end{aligned} \tag{a}$$

Equation given below is obtained by differentiating Eq.(a) with respect to ϕ.

$$\begin{aligned}
&\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}w,_{\phi\phi} - 6cG_c(1 + \\
&\tfrac{1}{3}c^2/R^2)\overset{*}{v},_{\phi}/R^2 + 2K(1 + \bar{c}^2/R^2)(\overset{*}{v},_{\phi\phi\phi} + \overset{*}{v},_{\phi\phi}\cot\phi - \overset{*}{v},_{\phi}\operatorname{cosec}^2\phi - \overset{*}{v},_{\phi} \\
&\cot^2\phi + 2\overset{*}{v}\cot\phi\operatorname{cosec}^2\phi - \nu\overset{*}{v},_{\phi})/R^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v},_{\phi}/R - \\
&4K\bar{c}(\bar{v},_{\phi\phi\phi} + \bar{v},_{\phi\phi}\cot\phi - \bar{v},_{\phi}\operatorname{cosec}^2\phi - \bar{v},_{\phi}\cot^2\phi + 2\bar{v}\cot\phi\operatorname{cosec}^2\phi - \\
&\nu\bar{v},_{\phi})/R^3 = (2\rho h + 2\rho_c c)\overset{*}{v},_{\phi tt}
\end{aligned} \tag{b}$$

Multiplying Eq.(a) by cotϕ,

$$
\begin{aligned}
&\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}\cot\phi\ w_{,\phi} - 6cG_c(1 + \\
&\tfrac{1}{3}c^2/R^2)\overset{*}{v}\cot\phi/R^2 + 2K(1 + \bar{c}^2/R^2)(\overset{*}{v}_{,\phi\phi}\cot\phi + \overset{*}{v}_{,\phi}\cot^2\phi - \overset{*}{v}\cot^3\phi - \\
&\nu\overset{*}{v}\cot\phi)/R^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{v}\cot\phi/R - 4K\bar{c}(\bar{v}_{,\phi\phi}\cot\phi + \bar{v}_{,\phi}\cot^2\phi - \\
&\bar{v}\cot^3\phi - \nu\bar{v}\cot\phi)/R^3 = (2\rho h + 2\rho_c c)\cot\phi\ \bar{v}_{,tt}
\end{aligned}
\tag{c}
$$

Addition of Eqs.(b) and (c) yields the following equation.

$$
\begin{aligned}
&\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}(w_{,\phi\phi} + \cot\phi\ w_{,\phi}) - \\
&6cG_c(1 + \tfrac{1}{3}c^2/R^2)(\overset{*}{v}_{,\phi} + \overset{*}{v}\cot\phi)/R^2 + 2K(1 + \bar{c}^2/R^2)\{\overset{*}{v}_{,\phi\phi\phi} + 2\cot\phi \\
&\overset{*}{v}_{,\phi\phi} - (1 + \cot^2\phi)\overset{*}{v}_{,\phi} + (2\cot\phi + \cot^3\phi)\overset{*}{v} - \nu(\overset{*}{v}_{,\phi} + \overset{*}{v}\cot\phi)\}/R^2 + \\
&6G_c(1 + \tfrac{2}{3}c^2/R^2)(\bar{v}_{,\phi} + \bar{v}\cot\phi)/R - 4K\bar{c}\{\bar{v}_{,\phi\phi\phi} + 2\bar{v}_{,\phi\phi}\cot\phi - (1 + \\
&\cot^2\phi)\bar{v}_{,\phi} + (2\cot\phi + \cot^3\phi)\bar{v} - \nu(\bar{v}_{,\phi} + \bar{v}\cot\phi)\}/R^3 = (2\rho h + \\
&2\rho_c c)(\overset{*}{v}_{,\phi} + \overset{*}{v}\cot\phi)_{,tt}
\end{aligned}
$$

By making use of Eqs.(2.27) and Eq.(e) of appendix C, the above equation reduces to the following form.

$$\begin{aligned}
&\{6cG_c(1 + \tfrac{1}{3}c^2/R^2)/R^2 + 2K(1 + \nu)(1 + \bar{c}^2/R^2)/R^2\}R^2\nabla^2 w - 6cG_c(1 + \\
&\tfrac{1}{3}c^2/R^2)\overset{*}{\theta}/R + 2K(1 + \bar{c}^2/R^2)\{R^3\nabla^2\overset{*}{\theta} + (1 - \nu)R\overset{*}{\theta}\}/R^2 + 6G_c(1 + \tfrac{2}{3}c^2/R^2)\bar{\theta} - \\
&4K\bar{c}\,\{R^3\nabla^2\bar{\theta} + (1 - \nu)R\bar{\theta}\}/R^3 = (2\rho h + 2\rho_c c)R\,\overset{*}{\theta}_{,tt}
\end{aligned} \tag{d}$$

Equation (2.26)c can also be reduced to the following form in a similar manner.

$$\begin{aligned}
&-\{2G_c/R + 4K\bar{c}(1 + \nu)/R^3\}R^2\,\nabla^2 w + 2G_c\overset{*}{\theta} - 4K\bar{c}\{\,R^3\,\nabla^2\overset{*}{\theta} + \\
&(1 - \nu)R\overset{*}{\theta}\}/R^3 - 2G_cR(1 + c^2/R^2)\bar{\theta}/c + 2K(1 + \bar{c}^2/R^2)\{R^3\nabla^2\bar{\theta} + (1 - \\
&\nu)R\bar{\theta}\}/R^2 = 2\rho hR\,\bar{\theta}_{,tt}
\end{aligned} \tag{e}$$

APPENDIX E

From Eq.(2.27)a, the expression for $\overset{*}{\Theta}$ is

(a) $$\overset{*}{\Theta} = (\overset{*}{V},_{\phi} + \overset{*}{V} \cot\phi)/R$$

where, $$\overset{*}{v} = e^{i\omega t} \overset{*}{V}$$

Substituting the value of $\overset{*}{\Theta}$ from Eq.(3.12)a, Eq.(a) reduces to

(b) $$R \sum_{\alpha=1}^{3} A_{n_\alpha} P_{n_\alpha}(\cos\phi) = (\overset{*}{V} \sin\phi),_{\phi}/\sin\phi$$

A new variable μ is introduced, such that

$$\mu = \cos\phi$$

$$\frac{\partial\mu}{\partial\phi} = -\sin\phi$$

(c) $$(\ ),_{\phi} = -\sin\phi\ (\ ),_{\mu}$$

$$(\ ),_{\mu} = -(\ ),_{\phi}/\sin\phi$$

With the help of Eq.(c), Eq.(b) takes the following form.

$$R \sum_{\alpha=1}^{3} A_{n_\alpha} P_{n_\alpha}(\mu) = -(\overset{*}{V} \sqrt{1-\mu^2}),_{\mu}$$

Integrating the above equation with respect to μ,

(d) $$\overset{*}{V} \sqrt{1-\mu^2} = -R \sum_{\alpha=1}^{3} A_{n_\alpha} \{ \int P_{n_\alpha}(\mu) d\mu \}$$

From reccurence relation given by Eq.(3.13), the expression for $P_{n_\alpha}(\mu)$ is

(e) $$P_{n_\alpha}(\mu) = \{P'_{n_\alpha+1}(\mu) - P'_{n_\alpha-1}(\mu)\}/(2n_\alpha + 1)$$

where, prime denotes the derivative with respect to μ.

Integrating the above equation with respect to μ, following equation is obtained.

$$\text{(f)} \qquad \int P_{n_\alpha}(\mu)\, d\mu = \{P_{n_\alpha+1}(\mu) - P_{n_\alpha-1}(\mu) + CI\}/(2n_\alpha + 1)$$

where, CI is the constant of integration.

By making use of Eq.(f), Eq.(d) reduces to

$$\text{(g)} \qquad \overset{*}{V}\sqrt{1-\mu^2} = -R\sum_{\alpha=1}^{3} A_{n_\alpha}\{P_{n_\alpha+1}(\mu) - P_{n_\alpha-1}(\mu) + CI\}/(2n_\alpha + 1)$$

Due to axi-symmetry of the problem, $\overset{*}{V} = 0$ at $\phi = 0$. Using this condition in Eq.(g), the constant of integration is

$$CI = -\sum_{\alpha=1}^{3}\{P_{n_\alpha+1}(1) - P_{n_\alpha-1}(1)\}$$

Since $P_\upsilon(1) = 1$ for all non-integral value of υ, the value of CI becomes zero and hence Eq.(g) with the help of Eq.(c) assumes the following form,

$$\text{(h)} \qquad \overset{*}{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3} A_{n_\alpha}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}/(2n_\alpha + 1)$$

Similarly, with the help of Eq.(2.27)b, the expression for $\bar{V}$ can also be obtained as follows.

$$\text{(i)} \qquad \bar{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3} B_{n_\alpha}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}/(2n_\alpha + 1)$$

The quantities A_{n_α} and B_{n_α} can be represented in terms of C_{n_α} as follows

(j) $$A_{n_\alpha} = \{(A_{13}A_{21} - A_{11}A_{23})/(A_{12}A_{23} - A_{22}A_{13})\}\, C_{n_\alpha}$$

$$B_{n_\alpha} = \{(A_{11}A_{22} - A_{12}A_{21})/(A_{12}A_{23} - A_{22}A_{13})\}\, C_{n_\alpha}$$

where A_{11}, A_{12}, --- A_{23} are the coefficients of the simultaneous Eqs.(3.10).

Substituting the value of A_{n_α} and B_{n_α} from Eqs.(j) in Eq.(g) and (h), the expressions for $\overset{*}{V}$ and $\bar{V}^\alpha$ are given by the following equations.

$$\overset{*}{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}\beta_\alpha C_{n_\alpha}/(2n_\alpha + 1)$$

$$\bar{V} = -(R/\sin\phi)\sum_{\alpha=1}^{3}\{P_{n_\alpha+1}(\cos\phi) - P_{n_\alpha-1}(\cos\phi)\}\eta_\alpha C_{n_\alpha}/(2n_\alpha+1)$$

where, $$\beta_\alpha = (A_{13}A_{21} - A_{11}A_{23})/(A_{12}A_{23} - A_{22}A_{13})$$

$$\eta_\alpha = (A_{11}A_{22} - A_{12}A_{21})/(A_{12}A_{23} - A_{22}A_{13})$$

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