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State Space Estimation of Wireless Fading Channels

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Abstract

This thesis presents a state space approach to optimal estimation of channel processes, such as inphase, quadrature, square envelope, phase, etc., of wireless fading channels. The models are derived from the Doppler power spectral density using factorization and realization techniques. Several simulations are performed when the channel is Rayleigh and Ricean distributed under both flat and frequency-selective fading assumptions. The simulation studies illustrate the tracking properties of the inphase, quadrature and square envelope estimators for various received signal-to-noise ratios.

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Chapter 1

Introduction

1.1 Mobile Radio Channel

Three fundamental components of a wireless radio communication system are the transmitter, channel and the receiver. The transmitter sends messages to the receiver through a channel. These messages, called the transmitted signals, are some kinds of electromagnetic waves, which obey the well known Maxwell's equations while propagating through the channel. The physical channel is usually the atmosphere (the free space) in wireless radio communications. One of the most important characteristics in wireless radio communications is fading, which causes the received signal power to change with time. The change of the power at some place or on some time interval is so big that the transmitted signal vanishes into the void. A concrete example is signal propagation in an urban area, where natural and man-made objects are concentrated. This kind of signal propagation gives rise to the so called multipath fading, in which the received electromagnetic waves consist of superposition of a number of the transmitted signals, which each one has a differential amplitude and time delay.

Statistical techniques have been introduced to describe this kind of channels. In view of this, the channel statistical properties, like mean, variance, autocorrelation, power spectral density and so on, have been investigated.

Three basic mechanisms which affect signal propagation in wireless radio communications are reflection[1], scattering[2] and diffraction[3]. When the electromagnetic waves propagate through the atmosphere, they undergo reflections, scattering from various surfaces of objects and diffractions. These phenomena can be described by stochastic variables for developing various mathematical fading channel models. After a channel model is constructed, how "good" the model is has to be we evaluated against real data. It is convenient to simulate the channel model in a software simulator at first in stead of implementing the channel in hardware. Doing this way, we will save both time and money.

Channel estimations allow the receiver to approximate the impulse response of the channel and explain the behavior of the channel. The knowledge of the channel's behavior is well-utilized in modern radio communications. It must be stressed here that channel estimation is only a mathematical representation of what is truly happening. A "good" channel estimation is one where some sort of error minimization criteria is satisfied. Kalman filtering theory which belongs to the least-square optimal estimation techniques is used in this paper to estimate the constructed channel inphase, quadrature and square envelope.

Emerging in the area of systems and control theory, the Kalman filter provides a method for generating an optimal estimation of the system state. That is, given a signal model that consists of a linear dynamical system driven by stochastic white noise processes, the Kalman filter provides an optimal way of extracting a signal from noise by exploiting a state space signal model. The prerequisite of using Kalman filter is the existence of a state space channel model. The approximation, factorization and realization techniques will help us for constructing the state space channel model.

In general, the multipath propagation of the transmitted signal can be divided into large-scale fading[4] and small-scale fading[5]. Two typical small-scale fading channels are Rayleigh and Rican, which we will introduce in the following section.

[1]Reflection occurs when a propagating electromagnetic wave strikes a smooth surface with very large dimensions compared to the RF signal wavelength [31].

[2]Scattering occurs when a radio wave impinges on either a large rough surface or any surface whose dimensions are on the order of the signal's wavelength or less, causing the reflected energy to spread out (scatter) in all directions [31].

[3]Diffraction occurs when the radio path between the transmitter and receiver is obstructed by a dense body with large dimensions compared to the wavelength, causing secondary waves to be formed behind the obstructing body. Diffraction is a phenomenon that accounts for RF energy traveling from transmitter to receiver without using a line of site path between these two [31].

[4]Large-scale fading: represents an average received signal power attenuation or path loss due to motion over large areas [31].

[5]Small-scale fading: refers to the dramatic changes in received signal amplitude and phase that can be experienced as a result of small changes in spatial separation between a transmitter and receiver [31].

1.2 Aulin's Channel Model

(i) Representation. In small-scale fading statistics, several multipath fading channel models were reported in the literatures ([1], [2], [3]). Ossanna's model [1] is one of first, and assumes existence of a direct path between the transmitter and receiver. Clarke's model [2], which includes Ossanna's model as a special case, has the property of signal scattered in the vicinity of the receiver. Aulin's model [3], is a generalization of the Clarke's model, because it considers a three dimensional wave propagation model. Our investigation in this thesis, is based on the Aulin three dimensional model which is shown in Figure 1.2.1.

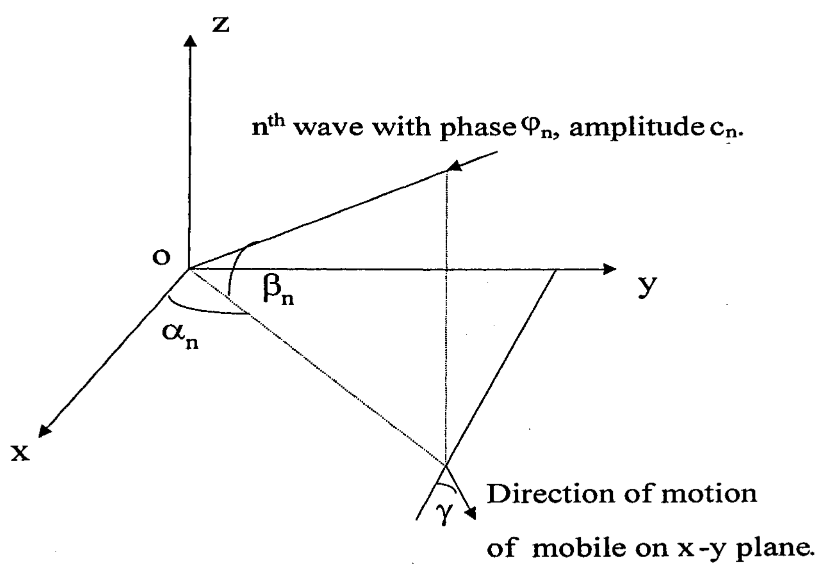

Figure 1.2.1: A wave component in three dimensions

It is important to note that additional models can also be included, such as those in Gans [5]. It assumed that at any point of the received field, the total wave consists of superposition of N plane waves, which each one travels via a different path. Let $O(0,0,0)$ denote the zero reference phase point; let the n^{th} incoming plane wave, traveling through the n^{th} path, be denoted by $E_n(t)$ with its parameters $\{\alpha_n, \beta_n, \phi_n, c_n\}_{n=1}^{N}$. Here c_n and ϕ_n are the amplitude and phase of the n^{th} plane wave. Suppose the transmitted signal is $Re\{e^{j\omega_c t}\}$, then at point

O, the n^{th} incoming wave can be represented by

$$E_n(t) = Re\Big\{c_n e^{j\phi_n} e^{j\omega_c t}\Big\}, \quad 1 \le n \le N,$$

where $Re\{x\}$ denotes the real part of the complex-valued quantity x. Further, at any receiving point $O'(x_0, y_0, z_0)$, the n^{th} plane wave is[6]

$$\begin{aligned} E_n(t) &= Re\Big\{c_n e^{j\phi_n} e^{j\omega_c(t-\tau_n)}\Big\} \\ &= c_n \cos\Big[\omega_c t - \frac{2\pi}{\lambda}(x_0 \cos\alpha_n \cos\beta_n + y_0 \sin\alpha_n \cos\beta_n + z_0 \sin\beta_n) + \phi_n\Big], \end{aligned}$$

where λ is the wavelength. Next, let the point $O'(x_0, y_0, z_0)$ move in the received field with a velocity v on the $x-y$ plane, which has a direction with an angle γ to the $x-z$ plane, then $O'(x_0, y_0, z_0)$ moves to a new position $O''(x_0 + vt\cos\gamma, y_0 + vt\sin\gamma, z_0)$ at time t. Thus,

$$\begin{aligned} E_n(t) &= c_n \cos(\omega_n t + \theta_n)\cos\omega_c t - c_n \sin(\omega_n t + \theta_n)\sin\omega_c t \\ &= I_n(t)\cos\omega_c t - Q_n(t)\sin\omega_c t, \\ &= Re\Big\{c_n e^{j\Phi_n} e^{j\omega_c t}\Big\}, \end{aligned}$$

where

$$\omega_n = -\frac{2\pi v}{\lambda}\cos(\gamma - \alpha_n)\cos\beta_n, \ \theta_n = -\frac{2\pi z_0}{\lambda}\sin\beta_n - \frac{2\pi\sqrt{x_0^2 + y_0^2}}{\lambda}\cos(\alpha_n + \xi)\cos\beta_n + \phi_n.$$

Here ω_n is the Doppler shift, θ_n is the phase associated with the n^{th} wave, $I_n(t)$, $Q_n(t)$ are the inphase and quadrature components, $c_n \triangleq \sqrt{I_n^2(t) + Q_n^2(t)}$, $\Phi_n(t) \triangleq \tan^{-1}(Q_n(t)/I_n(t)) = \omega_n t + \theta_n$ and $\xi \triangleq \tan^{-1}(y_0/x_0)$. Thus, at any receiving point, the superposition of the incoming waves is given by

$$\begin{aligned} E(t) &= \sum_{n=1}^{N} E_n(t) \\ &= \sum_{n=1}^{N} c_n \cos(\omega_n t + \theta_n)\cos\omega_c t - \sum_{n=1}^{N} c_n \sin(\omega_n t + \theta_n)\sin\omega_c t \\ &= I(t)\cos\omega_c t - Q(t)\sin\omega_c t, \end{aligned}$$

where $I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t + \theta_n)$, $Q(t) = \sum_{n=1}^{N} c_n \sin(\omega_n t + \theta_n)$. When N is sufficiently large (typically $N \ge 6$), both $I(t)$ and $Q(t)$ have the statistical property of Gaussian distribution, and so does $E(t)$.

[6] see Appendix A.1.1 for details

(ii) Mean, Variance and Autocorrelation. It is easily verified that[7]

$$E[I(t)] = E[Q(t)] = E[E(t)] = 0,$$

$$E[E(t)E(t+\tau)] = a(\tau)\cos\omega_c\tau - c(\tau)\sin\omega_c\tau,$$

where

$$a(\tau) = E\Big[I(t)I(t+\tau)\Big] = E\Big[Q(t)Q(t+\tau)\Big] = \tfrac{E_0}{2N}\textstyle\sum_{n=1}^{N} E[\cos\omega_n\tau],$$

$$c(\tau) = E\Big[I(t)Q(t+\tau)\Big] = E\Big[I(t+\tau)Q(t)\Big] = \tfrac{E_0}{2N}\textstyle\sum_{n=1}^{N} E[\sin\omega_n\tau].$$

Here E_0 is a positive constant. From above equations, we can easily get the second moment of $I(t)$ or $Q(t)$ by letting $\tau = 0$, that is, $a(0) = E_0/2$. Further, assuming that α_n is uniformly distributed on $[0, 2\pi]$, then we have

$$a(\tau) = \frac{E_0}{2}\int J_0(\frac{2\pi v\tau}{\lambda}\cos\beta)p_\beta(\beta)d\beta, \tag{1.2.1}$$

$$c(\tau) = 0, \tag{1.2.2}$$

where $J_0(\cdot)$ is a Bessel function of the first kind of zero order. Thus, we conclude that $I(t)$ and $Q(t)$ are iid with density $N(0; E_0/2)$.

If there is a direct path between the transmitter and receiver, the means of the inphase and quadrature are finite and not zero, especially, define

$$\mu = E[I_d(t)] \triangleq r_0\cos(\omega_0 t + \theta_0), \quad \nu = E[Q_d(t)] \triangleq r_0\sin(\omega_0 t + \theta_0),$$

where

$$\omega_0 = -\frac{2\pi v}{\lambda}\cos(\gamma - \alpha_0)\cos\beta_0, \quad \theta_0 = -\frac{2\pi z_0}{\lambda}\sin\beta_0 + \phi_0$$

with $\{r_0, \omega_0, \theta_0\}$ are unknown constants, then

$$E(t) = (I(t) + I_d(t))\cos\omega_c t - (Q(t) + Q_d(t))\sin\omega_c t.$$

Obviously, the inphase and quadrature components are still Gaussian with densities $N(\mu; a(0))$ and $N(\nu; a(0))$. Usually, the direct path between the transmitter and receiver is called the specular component or the line of sight (LOS).

(iii) Received Signal Amplitude Distributions. The received signal amplitude which is represented by the inphase and quadrature components has the following distributions.

[7] see Appendix A.1.2 for details

Rayleigh Distribution. Since $I(t),\ Q(t) \sim N(0; a(0))$. It is easily to prove that the signal envelope $r \triangleq \sqrt{I^2(t) + Q^2(t)}$ is Rayleigh distributed with pdf [8]

$$f(r) = \frac{r}{a(0)} e^{-r^2/2a(0)}, \quad r \geq 0,$$

and $E[r] = \sqrt{\frac{a(0)\pi}{2}}$, $Var(r) = (2 - \frac{\pi}{2})a(0)$.

Ricean Distribution. Since $I(t) + I_d(t) \sim N(\mu; a(0))$, $Q(t) + Q_d(t) \sim N(\nu; a(0))$. Obviously, the signal envelope $r \triangleq \sqrt{(I(t) + I_d(t))^2 + (Q(t) + Q_d(t))^2}$ is obeyed Ricean distribution with pdf [9]

$$f(r) = \frac{r}{a(0)} e^{-\frac{r^2 + r_0^2}{2a(0)}} I_0\left(\frac{r r_0}{a(0)}\right), \quad r \geq 0,$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of zero order, and $E[r^2] = r_0^2 + 2a(0)$, $Var(r^2) = 4a^2(0) - 4a(0)r_0^2$.

(iv) Power Spectrum. Suppose that $p_\beta(\beta) = \delta(\beta)$ which means the incoming waves travel only on $x - y$ plane. Integrate (1.2.1), we get $a_0(\tau) = \frac{E_0}{2} J_0(\frac{2\pi v \tau}{\lambda} \cos\beta)$. Thus, the power spectral density function of $I(t)$ or $Q(t)$ is obtained by taking the Fourier transform to $a_0(\tau)$,

$$\begin{aligned} S_0(f) &= \mathcal{F}\{a_0(\tau)\} \\ &= \begin{cases} \frac{E_0}{2} \frac{1}{2\pi} \frac{\lambda}{v} \frac{1}{\sqrt{1 - (\frac{f\lambda}{v})^2}}, & |f| \leq \frac{v}{\lambda}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The processes are strictly band-limited within the maximum Doppler frequency shift f_d (see Figure 1.2.2).

Suppose that

$$p_\beta(\beta) = \begin{cases} \frac{\cos\beta}{2\sin\beta_m}, & |\beta| \leq |\beta_m| \leq \frac{\pi}{2}; \\ 0, & \text{otherwise,} \end{cases}$$

which means the incoming waves travel in the three dimensional space. Take the Fourier transform to (1.2.1), then we obtain the Doppler power spectral density function (DPSD) for small angle β_m[10],

[8]see Appendix A.1.3 for details
[9]see Appendix A.1.4 for details
[10]see Appendix A.1.2 for details

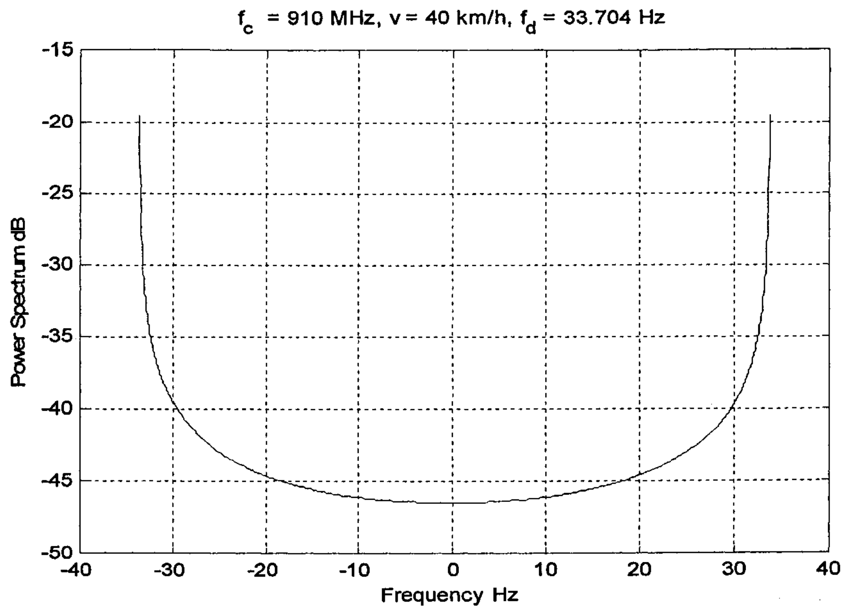

Figure 1.2.2: The Doppler power spectrum of $p_\beta(\beta) = \delta(\beta)$

$$
\begin{aligned}
S_D(f) &= \mathcal{F}\{a(\tau)\} \\
&= \begin{cases}
0, & |f| > \frac{v}{\lambda}; \\
\frac{E_0}{2}\frac{1}{2\sin\beta_m}\frac{\lambda}{v}, & \frac{v}{\lambda}\cos\beta_m \\
& \leq |f| \leq \frac{v}{\lambda}; \\
\frac{E_0}{2}\frac{1}{2\pi\sin\beta_m}\frac{\lambda}{v}\left(\frac{\pi}{2} - \arcsin\frac{2\cos^2\beta_m - 1 - (\frac{f\lambda}{v})^2}{1-(\frac{f\lambda}{v})^2}\right), & |f| < \frac{v}{\lambda} cos\beta_m.
\end{cases}
\end{aligned}
$$

Let the carrier frequency $f_c = 910\ MHz$, v change from 5 km/h to 200 km/h, then we obtain a series of the Doppler power spectrums shown on Figure 1.2.3.

1.3 Concepts about Fading Channel

In this section, we are going to introduce some concepts about fading channel for the purposes of simulation of the channel estimation. Let the transmitter and/or the receiver move, then there is a phase shift, called Doppler shift, on the received signal. The value of Doppler shift depends on the velocity of the moving object, the carrier frequency of the transmitted signal and the angle between the direction of the object's movement and the

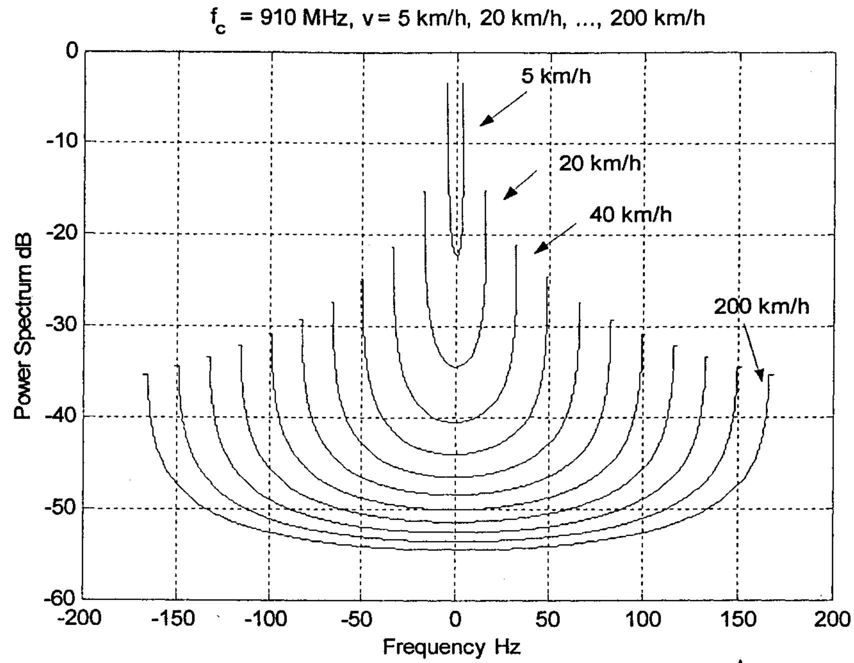

Figure 1.2.3: 3-D power spectrums

direction of the signal arrival. The maximum Doppler shift is defined by $f_d \triangleq v/\lambda$, which is a function of the velocity and the length associate with the carrier wave.

B_d, the Doppler spread, is a measure of the spectral broadening caused by the time rate of change of the mobile radio channel and is defined as the range of frequencies over which the received Doppler spectrum is essentially non-zero. That is, $B_d \triangleq 2f_d$.

The reciprocal of the Doppler spread is defined as the channel coherence time, $T_c \triangleq 1/B_d$, which is the temporal interval over which the channel characteristics remain relatively constant. The time-variant nature of the channel can be viewed in terms of fast fading and slow fading.

A channel is called fast fading when $T_s >> T_c$, where T_s is the time duration of a transmission symbol (signaling interval). It describes a condition where the time duration in which the channel behaves in correlated manner is short compared to the time duration of the symbol. Therefore, it can be expected that the fading character of the channel will change several times while a symbol is propagating. A channel is referred to slow fading when $T_s << T_c$. That is, the time duration which the channel behaves in a correlated manner is long compared to the time duration of a transmission symbol. Thus, one can expect the channel state to remain unchanged during the time in which a symbol is transmitted.

B_c, the channel coherence bandwidth, is a statistical measure of the range of frequencies over which the channel passes all spectral components with approximately equal gain and linear phase. That is, a signal's spectral components in that range are affected by the channel in a similar manner. It is defined as $B_c \triangleq 1/T_m$.

T_m, called the multipath spread or the delay spread of the channel, is the standard deviation (or root-mean-square) value of the delay of reflections, weighted proportional to the energy in the reflected waves. In a fading channel, the relationship between the delay spread T_m and the symbol duration T_s can be viewed in terms of two different categories, frequency-selective fading and frequency-nonselective or flat fading.

A channel is said to exhibit frequency-selective fading if $T_s < T_m$, or when the bandwidth of the transmitted signal is less than the bandwidth of the channel, $B_s < B_c$. A channel is said to exhibit frequency-nonselective fading if $T_s > T_m$, or when the bandwidth of the transmitted signal is greater than the bandwidth of the channel, $B_s > B_c$.

1.4 Outline

A key to our approach is the Doppler power spectral density function which we introduced in section 1.2. Based on it, we will do the following works in this thesis:

1. Approximation of the Doppler power spectral density function for obtaining a system transfer function.

2. Factorization of the system transfer function as in [11].

3. Realization of the system transfer function for getting a state space system model.

4. Estimation of the state space system model.

5. Simulation of the state space system estimations.

The thesis is presented as follows.

Chapter 2, introduces the multipath fading channel model, and some preliminary material from random signal and systems. Next, a state space model is derive from Doppler power spectral density, using approximation, factorization and realization techniques. Then, we simulate some characteristics of Rayleigh and Ricean state space fading channel models.

Chapter 3, introduces the state space channel estimation problem. It presents the Kalman filtering algorithms for estimating the inphase, quadrature and square envelope. In order to use computer to perform these estimates, we discretize the model. In addition, conditional distributions for the channel envelope and phase are derived in closed form.

Chapter 4, presents several case studies of the estimation problem considered in chapter 3, assuming that the state space channel model parameters $\{A, B, G, D\}$ are known. Simulations include inphase, quadrature and square envelope estimates for Rayleigh and Ricean channels. Both flat slow fading and flat fast fading are considered. Finally, a multipath state space fading channel model which involves one Ricean and five Rayleigh's components is simulated.

Chapter 5, presents certain topics of future research which emerge from the finding of the thesis.

Chapter 2

Channel Modeling

In general, in wireless radio communications, radio propagation is mainly by way of scattering from the surfaces of the objects and by diffraction over and/or around them. These natural phenomena produce very complicated signal, which is a summation of the transmitted signal, at the receiver. Each term of the summation stands for a multipath component having an attenuation and a time delay. The number of multipath components, attenuation and time delay are all stochastic processes. Based on the descriptions in the previous chapter, we will do the following tasks in this chapter:

1. Representation of a multipath fading channel model. First giving the received signal model in lowpass and bandpass form, then we introduce the related background about a wide-sense stationary signal which is transmitted through a linear time-invariant system.

2. Factorization of the Doppler power spectral density (DPSD). We approximate the DPSD by a fourth-order rational function which is similar to [6].

3. Realization of DPSD by state space models. We present a fourth-order approximation of the DPSD and its equivalent state space realization as in [8].

4. Simulation of the state space channel models. The inphase, quadrature, envelope and phase of the wireless state space Rayleigh and Ricean fading channel models are simulated by using Simulink.

2.1 Multipath Fading Channel Model

Suppose we transmit a lowpass signal $\{s_l(t)\}_{t\geq 0}$ with a carrier frequency f_c through a wireless multipath fading channel. The received signal in the lowpass representation is

$$y_l(t) = \sum_{i=1}^{N(t)} r_i(t,\tau_i)e^{j\Phi_i(t,\tau_i)}s_l(t-\tau_i) + w_l(t),$$

where $\{w_l(t)\}_{t\geq 0}$ is a complex-valued white Gaussian noise process, $N(t)$ is the total number of multipath components at time t, $r_i(t,\tau_i)$ is the real amplitude of the i^{th} multipath component at time t, $\Phi_i(t,\tau_i)$ is the phase shift due to free space propagation of the i^{th} multipath component plus any additional phase shifts which are encountered in the channel at time t and τ_i is the excess time delay on the i^{th} path. Here $r_i(t,\tau_i)e^{j\Phi_i(t,\tau_i)}$ is called the attenuation factor for the received signal on the i^{th} path, which can be used to describe the wireless multipath fading channel. Another representation is obtained by using the inphase and quadrature components of the i^{th} path. That is,

$$r_i(t,\tau_i)e^{j\Phi_i(t,\tau_i)} = I_i(t,\tau_i) + jQ_i(t,\tau_i),$$

where $I_i(t,\tau)$ and $Q_i(t,\tau)$ are the inphase and quadrature components of the channel corresponding to the i^{th} path. Equivalently, the envelope representation is

$$r_i(t,\tau_i)e^{j\Phi_i(t,\tau_i)} = \sqrt{I_i^2(t,\tau_i) + Q_i^2(t,\tau_i)}e^{j\tan^{-1}\left(\frac{Q_i(t,\tau_i)}{I_i(t,\tau_i)}\right)}.$$

Alternatively, the received signal in the bandpass representation is

$$y(t) = Re\left\{\Big[\sum_{i=1}^{N(t)} r_i(t,\tau_i)e^{j\Phi_i(t,\tau_i)}s_l(t-\tau_i) + w_l(t)\Big]e^{jw_ct}\right\}.$$

Consider the lowpass equivalent representation of the transmitted signal $s_l(t) = S_I(t)+jS_Q(t)$ with the inphase and quadrature components $S_I(t)$ and $S_Q(t)$. Then, the bandpass received signal in the envelope format is given by

$$\begin{aligned} y(t) &= \sum_{i=1}^{N(t)} \Big\{\sqrt{I_i^2(t,\tau_i) + Q_i^2(t,\tau_i)}\sqrt{S_I^2(t-\tau_i) + S_Q^2(t-\tau_i)} \\ &\quad \times \cos\Big(w_ct + \Phi_i(t,\tau_i) + \theta(t,\tau_i) + \sigma(t,\tau_i)\Big)\Big\} \\ &\quad + \sqrt{v_I^2(t) + v_Q^2(t)}\cos\Big(w_ct + \eta(t)\Big), \end{aligned}$$

where

$$\theta(t,\tau_i) = \tan^{-1}\frac{Q_i(t,\tau_i)}{I_i(t,\tau_i)}, \quad \eta(t) = \tan^{-1}\frac{v_Q(t)}{v_I(t)}, \quad \sigma(t,\tau_i) = \tan^{-1}\frac{S_Q(t-\tau_i)}{S_I(t-\tau_i)};$$

The bandpass received signal in the quadratures format is given by

$$\begin{aligned} y(t) &= \sum_{i=1}^{N(t)} \Big\{ \Big[I_i(t,\tau_i)S_I(t-\tau_i) - Q_i(t,\tau_i)S_Q(t-\tau_i)\Big]\cos(\omega_c t) \\ &\quad - \Big[I_i(t,\tau_i)S_Q(t-\tau_i) + Q_i(t,\tau_i)S_I(t-\tau_i)\Big]\sin(\omega_c t)\Big\} \\ &\quad + v_I(t)\cos(\omega_c t) - v_Q(t)\sin(\omega_c t), \end{aligned}$$

where $\{v_I(t)\}_{t\geq 0}$ and $\{v_Q(t)\}_{t\geq 0}$ are two iid white Gaussian noises with density $N(0;\sigma_v^2)$. The simplest bandpass representation of a received signal is

$$y(t) = \sum_{i=1}^{N(t)} r_i(t,\tau_i)\cos\Big(\omega_c t + \Phi_i(t,\tau_i)\Big)s(t-\tau_i) + v(t), \tag{2.1.1}$$

where $\{s(t)\}_{t\geq 0}$ is a real bandpass transmitted signal, $\{v(t)\}_{t\geq 0}$ is a bandpass white Gaussian noise. The quadratures representation of (2.1.1) is

$$\begin{aligned} y(t) &= \textstyle\sum_{i=1}^{N(t)} \Big(I_i(t,\tau_i)\cos(\omega_c t) - Q_i(t,\tau_i)\sin(\omega_c t)\Big)s(t-\tau_i) \\ &\quad + v_I(t)\cos(\omega_c t) - v_Q(t)\sin(\omega_c t) \end{aligned} \tag{2.1.2}$$

2.2 Random Signals and Linear Systems

Consider a wide-sense stationary signal $x(t)$ which passes through a linear time-invariant system having impulse response $h(t)$. Suppose $h \in L_1$[1], then the Fourier transform

$$H(f) = \int_{-\infty}^{+\infty} h(t)e^{-j2\pi ft}dt$$

is an absolutely convergent integral for each $f \in (-\infty,\infty)$. Define $H \in C_0$[2], Then the integral

[1] L_p is the space of all Lebesgue measurable functions satisfying

$$\int_{-\infty}^{+\infty} |f(t)|^p dt < \infty, \quad 1 \leq p < \infty$$

[2] C_0 is the space of all bounded-continuous functions satisfying

$$f(t) \to 0 \ \ as \ \ |t| \to 0$$

$$h(t) = \int_{-\infty}^{+\infty} H(f)e^{j2\pi ft}df$$

is called the inverse Fourier transform of $H(f)$ and is not absolutely convergent unless $H(f)$ is also in L_1. The linear system theory tells us that the convolution product between two time function x and h, called the output of the system, is again a wide-sense stationary random process, and the definition is given by

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau.$$

Thus, $y(t) = h(t) * x(t)$ yields

$$Y(f) = H(f)X(f),$$

for the Fourier transforms of the input $X(f)$, output $Y(f)$ and impulse response $H(f)$. Equivalently, in the Laplace transform domain, the following equation,

$$Y(s) = H(s)X(s), \quad s = j\omega,$$

hold[3] for the Laplace transforms of the input $X(s)$, output $Y(s)$ and impulse response $H(s)$. Further, define the autocorrelations of the input and output signal by

$$R_{xx}(\tau) \triangleq E[x(t+\tau)x^*(t)], \quad R_{yy}(\tau) \triangleq E[y(t+\tau)y^*(t)].$$

Obviously, by using linear system theory, the autocorrelation function of $y(t)$ is given by[4]

$$R_{yy}(\tau) = R_{xx}(\tau) * \Big(h(\tau) * h^*(-\tau)\Big). \tag{2.2.3}$$

Moreover, define the power spectral densities of the input and output signals by

$$S_x(f) \triangleq \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j2\pi f\tau}d\tau, \quad S_y(f) \triangleq \int_{-\infty}^{+\infty} R_{yy}(\tau)e^{-j2\pi f\tau}d\tau, \quad -\infty < f < \infty.$$

[3]The relationship between the Fourier and Laplace transform is

$$\mathcal{F}\{x(t)\} = \mathcal{L}\{x(t)\}|_{s=j\omega}$$

[4]see Appendix A.2 for details

From (2.2.3), the following relationship can be found for the power spectral density functions of the input and output signal[5]

$$S_y(f) = \mid H(f) \mid^2 S_x(f). \tag{2.2.4}$$

If the power spectral density of the input signal satisfies $S_x(f) = 1$, then, from (2.2.4), the power spectral density of the output signal equals the power spectrum $\mid H(f) \mid^2$. For a white noise signal which is a Gaussian process, say $\dot{w}(t)$, with zero-mean variance one, one can easily get $S_{\dot{w}}(f) = 1$[6]. Compared with the discussions in the previous chapter, $\mid H(f) \mid^2$ is actually the DPSD, and $S_y(f)$ is the PSD of the inphase (or quadrature). Thus, one can produce the inphase and quadrature branches based on the equation (2.2.4). In order to generate the inphase and quadrature components, mathematically, one should find an impulse response $h(t)$ which is related to $S_D(f)$. In the next section, we will introduce a rational transfer function $H(s)$ whose square magnitude will approximate to the power spectrum density. In other words, we should find a rational transfer function $H(s)$, which satisfies

$$S_D(f) = \mid H(s) \mid^2, \quad s = j\omega.$$

2.3 Factorization of Doppler Power Spectral Density

Consider factorization of the Doppler power spectral density function $S_D(f)$. If the Paley-Wiener condition $\int_{-\infty}^{\infty} \frac{|\ln S_D(f)|}{1+f^2} df < \infty$ is satisfied, which implies that $S_D(f)$ is factorizable, then we have $|H(s)|^2 = S_D(f), \ s = j\omega$. Unfortunately, here, the Paley-Wiener condition is not satisfied because $S_D(f)$ is band limited. But if $S_D(f)$ is approximated by an even rational transfer function which is factorizable, and the key properties of the DPSD are not affected in this approximation, then this approach will satisfy the Paley-Wiener condition. That is, letting

$$S_D(f) = A^2 \frac{\prod_{i=1}^{n}(s+a_i)(-s+a_i)}{\prod_{i=1}^{n}(s+b_i)(-s+b_i)}\bigg|_{s=j\omega}$$

defining

$$H(s) \triangleq B \frac{\prod_{i=1}^{n}(s+a_i)}{\prod_{i=1}^{n}(s+b_i)}\bigg|_{s=j\omega}$$

[5]see Appendix A.2 for details
[6]see Appendix A.7.1 for details

then

$$S_D(f) = H(s)H(-s) = \mid H(s) \mid^2,$$

where $Re(a_i) > 0$, $Re(b_i) > 0$ and $A^2 = |B|^2$. We choose a second-order rational transfer function

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

to approximate the DPSD. Here ω_n is the undamped natural frequency, ζ is the damping ratio and k is the gain. That is, $S_D(f)$ is approximated by a fourth-order rational polynomial in f through

$$S_D(f) \approx H(j\omega)H(-j\omega).$$

One approximation is to interpolate the points $S_D(0)$ and $S_D(f_{max})$ by determining the parameters $\{\omega_n, \zeta, k\}$. In this case, simple algebra[7] shows that

$$\zeta = \sqrt{\frac{1}{2}\left[1 - \sqrt{1 - \frac{S_D(0)}{S_D(f_{max})}}\right]}, \quad \omega_n = \frac{2\pi f_{max}}{\sqrt{1-2\zeta^2}}, \quad k = \omega_n^2\sqrt{S_D(0)}, \tag{2.3.5}$$

where

$$f_{max} = f_d \cos(\beta_m), \quad S_D(f_{max}) = \frac{E_0}{4 f_d \sin\beta_m}, \tag{2.3.6}$$

$$S_D(0) = \frac{E_0}{4\pi f_d \sin\beta_m}[\frac{\pi}{2} - \arcsin(2\cos^2\beta_m - 1)]. \tag{2.3.7}$$

Figure 2.3.1 shows $S_D(f)$ and its approximation using the above approach for $E_0 = 2$, $f_c = 910\ MHz$, $\beta_m = 10^0$, $v = 5\ km/h$ and $v = 120\ km/h$. Figure 2.3.2 shows $S_D(f)$ and its approximation using the same approach for $E_0 = 2$, $f_c = 910\ MHz$, $\beta_m = 40^0$, $v = 5\ km/h$ and $v = 120\ km/h$. Clearly, one can increase the Doppler spread by increasing the velocity. On the other hand, one can also reduce the Doppler spread by increasing β_m while the velocity is fixed. For a much better approximation, one may increase the degree of the numerator and denominator of $H(s)$. However, for high order approximations it will be difficult to derive explicit equations which are related to the system parameters $\{\omega_n, \zeta, k\}$.

According to the discussions above, one is able to generate the inphase and quadrature components of a bandpass channel model. Figure 2.3.3 shows the channel model is obtained

[7] see Appendix A.3 for details

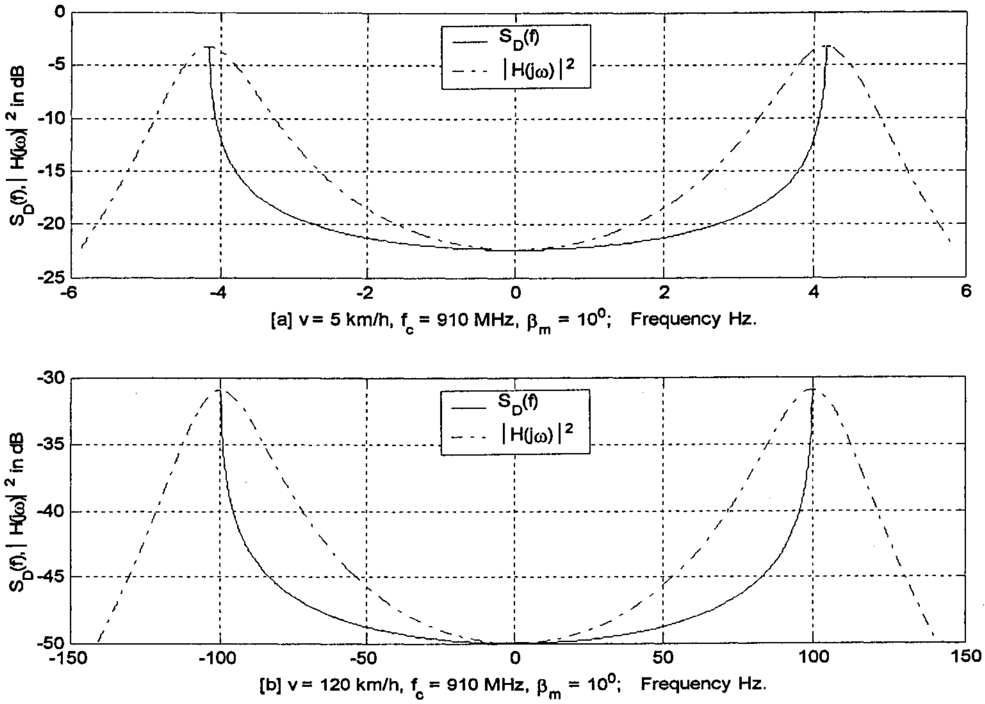

Figure 2.3.1: $S_D(f)$ & $H(j\omega)H(-j\omega)$ at $\beta_m = 10^0$

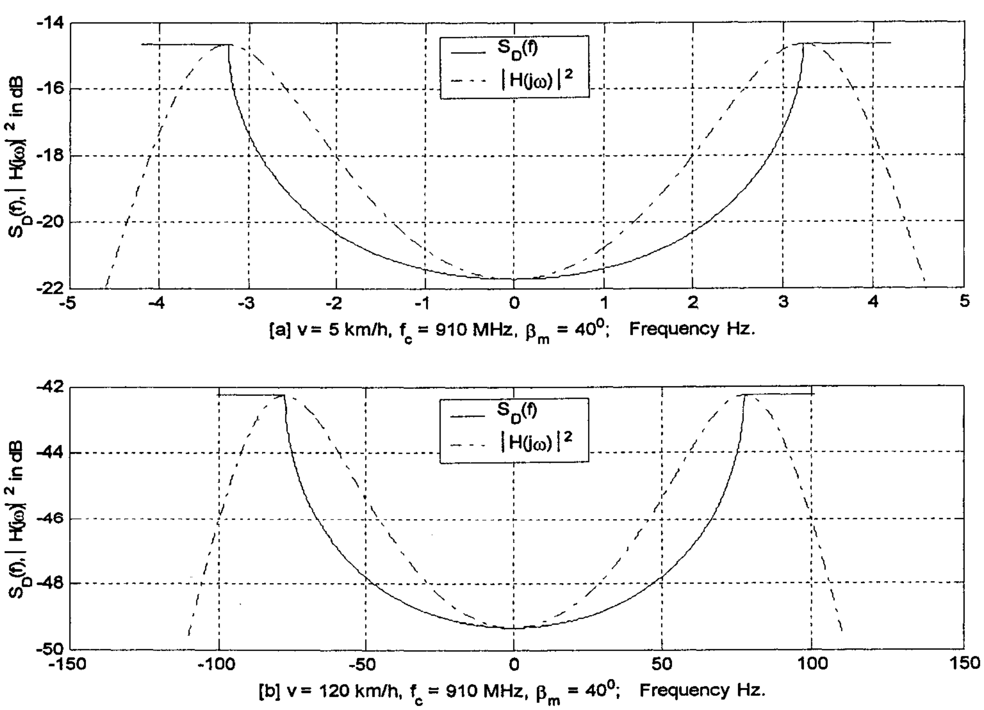

Figure 2.3.2: $S_D(f)$ & $H(j\omega)H(-j\omega)$ at $\beta_m = 40^0$

by passing two white Gaussian noise process, say $\{\dot{w}_I(t)\}_{t\geq 0}$, $\{\dot{w}_Q(t)\}_{t\geq 0}$, through the rational transfer functions, say $H_I(s)$ and $H_Q(s)$ which have the form of $H(s)$. What we have, after filters $H_I(s)$ and $H_Q(s)$, are the inphase and quadrature components of the channel, which we denote as $I(t,\tau)$ and $Q(t,\tau)$. Thus, the bandpass channel model representation is

$$Re\Big\{r(t,\tau)e^{j\Phi(t,\tau)}e^{j\omega_c t}\Big\} = I(t,\tau)\cos(\omega_c t) - Q(t,\tau)\sin(\omega_c t).$$

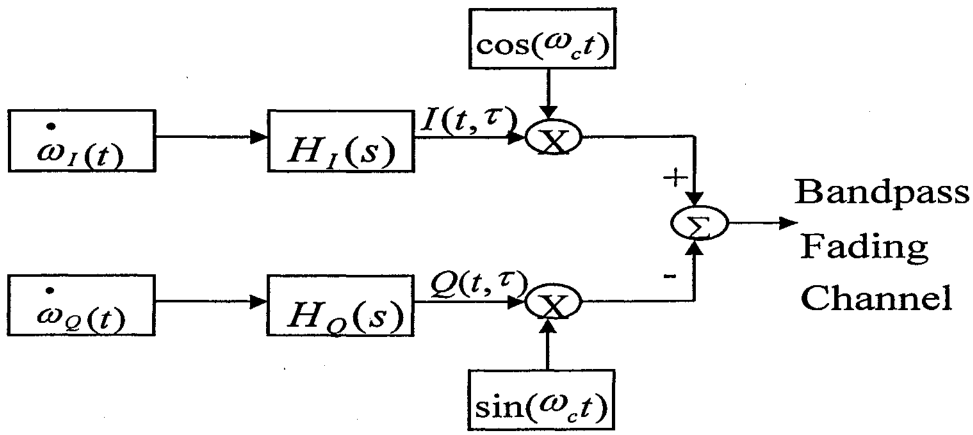

Figure 2.3.3: A Bandpass Channel Model in Frequency Domain

2.4 State Space Realizations

In the previous section, we introduced a rational transfer function $H(s)$ for generating the inphase and quadrature components of a wireless channel, which are obtained from their Doppler power spectral densities. Here a stochastic state space model is used to generate the inphase and quadrature components. The stochastic state space model which can be used to realize any proper channel is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \end{aligned} \tag{2.4.8}$$

where $x \in \Re^n$ is the state of the system, $u \in \Re^m$ is the input signal, $z \in \Re^p$ is the output signal (or say the observation of the system), $A \in \Re^n \times \Re^n$ is the state matrix, $B \in \Re^n \times \Re^m$ is the input matrix, $C \in \Re^p \times \Re^n$ is the output matrix and $D \in \Re^p \times \Re^m$ is the direct feed through matrix. Here $\Re$ denotes the set of real numbers. The block diagram is shown in Figure 2.4.4.

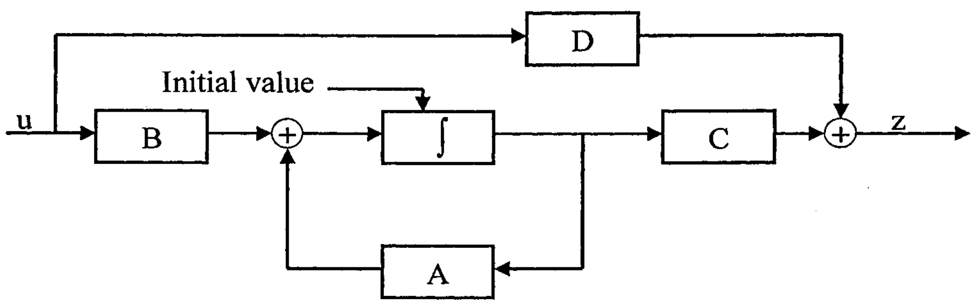

Figure 2.4.4: A Stochastic State Space Model

Although the state space realization for an arbitrary rational transfer function $H(s)$ is not unique, two common ways of the state space realization of a rational transfer function are introduced in this thesis, for generating the inphase and quadrature components of a wireless channel. They are so called the controllable and observable method, respectively. For the rational transfer function

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

a second-order stochastic differential equation is related to the state space realization. This differential equation is given by

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = k\dot{w}(t), \quad \dot{x}(0), x(0) \text{ given.}$$

where $\{w(t)\}_{t\geq 0}$ is a Brownian motion. The state space realization of the Controllable

Canonical Form is[8]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} \dot{w} \\ z &= [1 \quad 0] x \end{aligned},$$

the Observable Canonical Form is[9]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -\omega_n^2 \\ 1 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} k \\ 0 \end{bmatrix} \dot{w} \\ z &= [0 \quad 1] x \end{aligned}.$$

Consider a scattering environment, in which the line of sight (LOS) or specular component is present in the received signal. We assume that there are total N multipath components appearing in the received signal at time t, which one of them is the LOS corresponding to a Ricean channel, while the other $N-1$ multipath components are considered to be Rayleigh channels. For each multipath component, there is a state space representation corresponding to the channel inphase and quadrature components. The state space wireless Rayleigh and Ricean fading channel models will be introduced in the following two subsections.

2.4.1 State Space Realization of Rayleigh Channel

Here we will show how to construct the inphase and quadrature components of the i^{th} path state space Rayleigh channel model. The rational transfer function,

$$H_L(s) = \left. \frac{k(\tau)}{s^2 + 2\zeta(\tau)\omega_n(\tau)s + \omega_n^2(\tau)} \right|_{s=j\omega}, \qquad L \in \{I, Q\},$$

can be related to the following two second order differential equations,

$$\ddot{x}_I(t) + 2\zeta(\tau)\omega_n(\tau)\dot{x}_I(t) + \omega_n^2(\tau)x_I(t) = k(\tau)\dot{w}_I(t), \quad \dot{x}_I(0), x_I(0) \text{ given},$$

$$\ddot{x}_Q(t) + 2\zeta(\tau)\omega_n(\tau)\dot{x}_Q(t) + \omega_n^2(\tau)x_Q(t) = k(\tau)\dot{w}_Q(t), \quad \dot{x}_Q(0), x_Q(0) \text{ given},$$

for constructing the Rayleigh channel inphase and quadrature. Here $\{\dot{w}_I(t)\}_{t\geq 0}$, $\{\dot{w}_Q(t)\}_{t\geq 0}$ are two iid white Gaussian noise processes with density $N(0;1)$. Towards introducing a state space representation, define the phase variables as follows,

$$X_{I_1} \triangleq x_I, \qquad X_{Q_1} \triangleq x_Q,$$

$$X_{I_2} \triangleq \dot{x}_I, \qquad X_{Q_2} \triangleq \dot{x}_Q,$$

$$X_I \triangleq [X_{I_1} \quad X_{I_2}]^{tr}, \quad X_Q \triangleq [X_{Q_1} \quad X_{Q_2}]^{tr}.$$

[8]see Appendix A.4 for details

[9]see Appendix A.4 for details

Then, the state space Controllable Canonical Form of $H_I(s)$, $H_Q(s)$ are

$$\begin{aligned}\dot{X}_I(t) &= A_I(\tau)X_I(t) + B_I(\tau)\dot{w}_I(t), \quad X_I(0) \in \Re^2,\\ \dot{X}_Q(t) &= A_Q(\tau)X_Q(t) + B_Q(\tau)\dot{w}_Q(t), \quad X_Q(0) \in \Re^2,\end{aligned} \tag{2.4.9}$$

where

$$A_I(\tau) = A_Q(\tau) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2(\tau) & -2\zeta(\tau)\omega_n(\tau) \end{bmatrix}, \quad B_I(\tau) = B_Q(\tau) = \begin{bmatrix} 0 \\ k(\tau) \end{bmatrix}.$$

Let $I(t) \triangleq x_I(t)$, $Q(t) \triangleq x_Q(t)$, then

$$I(t) = [1 \quad 0]X_I(t), \quad Q(t) = [1 \quad 0]X_Q(t).$$

Define

$$A_i \triangleq \begin{bmatrix} A_{I_i}(\tau_i) & 0 \\ 0 & A_{Q_i}(\tau_i) \end{bmatrix}, \quad B_i \triangleq \begin{bmatrix} B_{I_i}(\tau_i) & 0 \\ 0 & B_{Q_i}(\tau_i) \end{bmatrix}, \quad X_i \triangleq \begin{bmatrix} X_{I_i}(t) \\ X_{Q_i}(t) \end{bmatrix}, \quad \dot{w}_i \triangleq \begin{bmatrix} \dot{w}_{I_i}(t) \\ \dot{w}_{Q_i}(t) \end{bmatrix}.$$

Then, the i^{th} path Rayleigh channel inphase and quadrature representation of the Controllable Canonical Form is

$$\dot{X}_i = A_i X_i + B_i \dot{w}_i \quad X_i(0) \in \Re^4. \tag{2.4.10}$$

Thus,

$$I_i(t) = [1 \quad 0 \quad 0 \quad 0]X_i(t), \quad Q_i(t) = [0 \quad 0 \quad 1 \quad 0]X_i(t).$$

2.4.2 State Space Realization of Ricean Channel

Let $\{I^R(t)\}_{t\geq 0}$, $\{Q^R(t)\}_{t\geq 0}$ be the inphase and quadrature components of a Ricean channel. Then, the Ricean channel can be generated by assuming $E[I^R(t)] = r_0 \cos(\omega_0 t + \theta_0)$, $E[Q^R(t)] = r_0 \sin(\omega_0 t + \theta_0)$ (see [3]). Thus, the Ricean channel of the Controllable Canonical Form can be represented as follows,

$$X_I^R(t) = X_I(t) + r_0 \cos(\omega_0 t + \theta_0)\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.4.11}$$

$$X_Q^R(t) = X_Q(t) + r_0 \sin(\omega_0 t + \theta_0)\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.4.12}$$

where the parameters $\{r_0, \omega_0, \theta_0\}$ correspond to the specular component or LOS. For the j^{th} multipath component, define

$$A_j \triangleq \begin{bmatrix} A_{I_j}(\tau_j) & 0 \\ 0 & A_{Q_j}(\tau_j) \end{bmatrix}, \quad B_j \triangleq \begin{bmatrix} B_{I_j}(\tau_j) & 0 \\ 0 & B_{Q_j}(\tau_j) \end{bmatrix}.$$

Then, the Ricean channel inphase and quadrature representation of the Controllable Canonical Form is[10]

$$\dot{X}_j^R = A_j X_j^R + f_{los} + B_j \dot{w}_j, \quad X_j^R(0) \in \Re^4, \tag{2.4.13}$$

where

$$f_{los} = \begin{bmatrix} -r_0\omega_0 \sin(\omega_0 t + \theta_0) \\ r_0\omega_n^2 \cos(\omega_0 t + \theta_0) \\ r_0\omega_0 \cos(\omega_0 t + \theta_0) \\ r_0\omega_n^2 \sin(\omega_0 t + \theta_0) \end{bmatrix}.$$

Thus,

$$I_j^R(t) = [1 \quad 0 \quad 0 \quad 0] X_j^R(t), \quad Q_j^R(t) = [0 \quad 0 \quad 1 \quad 0] X_j^R(t).$$

Suppose the received signal over $[0 \;\; T]$ involves N multipath components of which one is the LOS. We assume that the first path is the LOS which corresponds to the Ricean channel. Thus, the state space representation of the output $\{y(t)\}_{t\geq 0}$ corresponding to (2.1.2) is

$$y = \sum_{i=1}^{N} G_i X_i + Dv, \tag{2.4.14}$$

where $G_i \triangleq s(t-\tau_i)C_i(t)$,

$$C_i(t) \triangleq \begin{bmatrix} \cos(\omega_c t) & 0 & -\sin(\omega_c t) & 0 \end{bmatrix}, \quad D \triangleq \begin{bmatrix} \cos(\omega_c t) & -\sin(\omega_c t) \end{bmatrix} \text{ and } v = \begin{bmatrix} v_I(t) \\ v_Q(t) \end{bmatrix}.$$

2.5 State Space Channel Model Simulations

In this section, several simulations of Rayleigh and Ricean state space channels will be performed using the simulink package from Matlab.

2.5.1 Rayleigh Channel Simulations

Here, a fast fading channel is simulated using model (2.4.10) as shown in the block diagram of Figure 2.5.5[11]. There are two possible methods. The first one is based on

[10] see Appendix A.6.1 for details

[11] Here only one of the multipath components is simulated. we choose the i^{th} path state space Rayleigh fading channel model (2.4.10). It is convenient to remove the subscript i in this subsection.

specifying the channel parameters $\{\omega_n, \zeta, k\}$ in prior, the second one is based on extracting the channel parameters from measurement data.

Method 1: Given the values β_m, v, f_c and E_0, one can determine the state space model. For example, when $\beta_m = 10^0$, $v = 120\ km/h$, $f_c = 900\ MHz$ and $E_0 = 2$, we obtain, from (2.3.5) to (2.3.7), $w_n = 637.26\ rad/s$, $\zeta = 0.169\ s^{-1}$, $k = 2.3 \times 10^4$. Figure 2.5.6 shows the inphase $\{I(t)\}_{t\geq 0}$, quadrature $\{Q(t)\}_{t\geq 0}$, phase $\{\Phi(t) \triangleq \tan^{-1}\left(\frac{Q(t)}{I(t)}\right)\}_{t\geq 0}$ (in radian) and envelope $\{r(t) \triangleq \sqrt{I(t)^2 + Q(t)^2}\}_{t\geq 0}$ (in dB).

A typical Rayleigh fading envelope at 900 MHz and 120 km/h is reported in [12], page 173. In order to reach the same signal level, we adjust the channel model parameters $\{\omega_n, \zeta, k\}$[12] for $\beta_m = 10^o$, $v = 120\ km/h$, $f_c = 900\ MHz$ and $E_0 = 2$, which correspond to $w_n = 637.26\ rad/s$, $\zeta = 0.169\ s^{-1}$, $k = 6.11 \times 10^3$ (see Figure 2.5.7). Our further investigations, in this thesis, are based on this modified channel model.

Method 2: Given the steady state mean of a channel envelope, namely, $\lim_{t\to\infty} E[r_{dB}(t)] = \alpha\ dB$, which is extracted from measurement data, one can determine the parameters $\{\zeta, \omega_n, k\}$ of the channel model as follows. Clearly, since $\{I(t)\}_{t\geq 0}$ and $\{Q(t)\}_{t\geq 0}$ are iid, then

$$E[r^2(t)] = E[I^2(t) + Q^2(t)] = E[I^2(t)] + E[Q^2(t)] = 2E[I^2(t)] = 2E[Q^2(t)].$$

The solution of the state space model (2.4.9) is

$$X_L(t) = e^{A_L t} X_L(0) + \int_0^t e^{A_L(t-\tau)} B_L \dot{w}_L(\tau) d\tau.$$

The mean vector is

$$E[X_L(t)] = e^{A_L t} E[X_L(0)].$$

The covariance matrix is

$$\begin{aligned} \Sigma_L(t) &= E[X_L(t) X_L^{tr}(t)] \\ &= e^{A_L t} E[X_L(0) X_L^{tr}(0)] e^{A_L^{tr} t} + \textstyle\int_0^t e^{A_L(t-\tau)} B_L B_L^{tr} e^{A_L^{tr}(t-\tau)} d\tau, \end{aligned} \tag{2.5.15}$$

which satisfies

$$\dot{\Sigma}_L(t) = A_L \Sigma_L(t) + \Sigma_L(t) A_L^{tr} + B_L B_L^{tr}, \quad \Sigma_L(0) = E[X_L(0) X_L^{tr}(0)], \tag{2.5.16}$$

where

[12] Multiply k by $\sqrt{10^{-1.15}}$

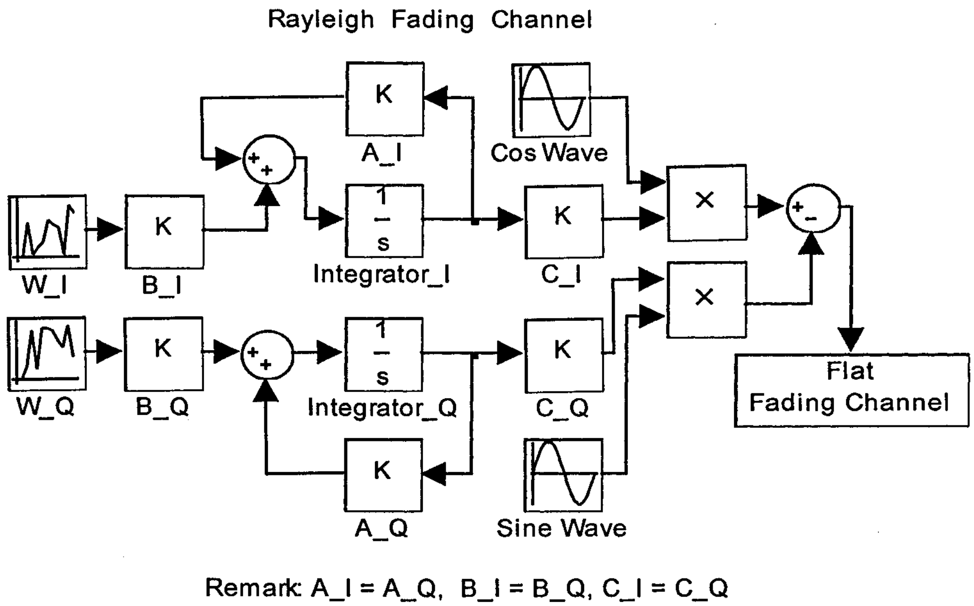

Figure 2.5.5: A Bandpass Rayleigh Fading Channel Model

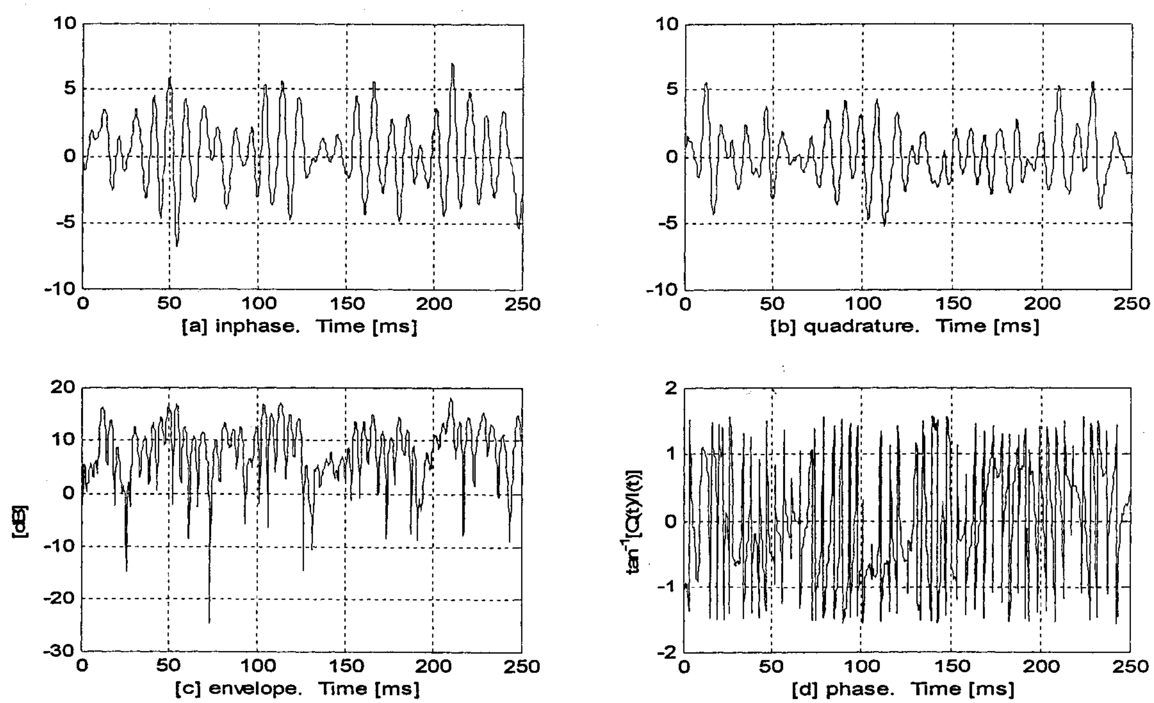

Figure 2.5.6: The Rayleigh Fading Channel; $f_c = 910\ MHz$, $v = 120\ km/h$

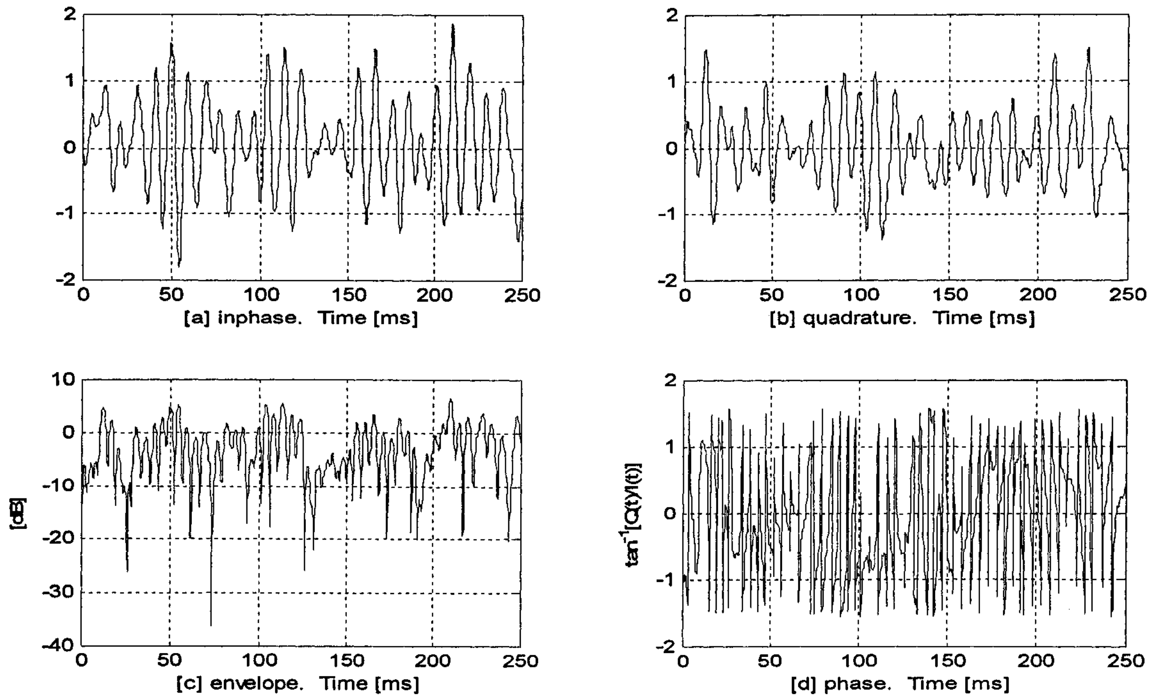

Figure 2.5.7: The Modified Rayleigh Fading Channel; $f_c = 900\ MHz$, $v = 120\ km/h$

$$\Sigma_L(t) = \begin{bmatrix} E[I^2(t)] & E[I(t)\dot{I}(t)] \\ E[I(t)\dot{I}(t)] & E[\dot{I}^2(t)] \end{bmatrix} \triangleq \begin{bmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{12}(t) & \gamma_{22}(t) \end{bmatrix}, \quad L \in \{I, Q\}.$$

The solution $\{\Sigma(t)\}_{t\geq 0}$ can be implemented recursively using (2.5.16), or obtained explicitly through (2.5.15).

Since a Rayleigh R.V. $r(t)$ is generated by adding two iid zero mean Gaussian R.V.'s, then $E[r(t)] = \sqrt{\frac{\pi}{2}}\sigma(t)$. Further, if $\lim_{t\to\infty} E[r(t)] = \sqrt{\frac{\pi}{2}}\sigma$ (e.g., variance converges), then from the data we are given $\lim_{t\to\infty} E[r(t)] = \sqrt{\frac{\pi}{2}}\sigma$. In our case, the inphase and quadrature $I(t)$, $Q(t)$ are generated through the state space model (2.4.9). Therefore, we should find $\{k, \omega_n, \zeta\}$ so that the measured value $\lim_{t\to\infty} E[r(t)] = \sqrt{\frac{\pi}{2}}\sigma = \alpha$ which corresponds to $\sqrt{\frac{\pi}{2}} \lim_{t\to\infty} \sqrt{E[I(t)]}$.

For a Rayleigh channel, we set $E[X_L(0)] = 0$, e.g., X_L is a zero mean Gaussian vector. As $E[I^2(t)] = \gamma_{11}(t) = E[Q^2(t)]$, then $\{k, \omega_n, \zeta\}$ should be chosen so that the steady state $\lim_{t\to\infty} E[I^2(t)] = \lim_{t\to\infty} E[Q^2(t)]$ converges to $\alpha\ dB$. Towards presenting the recipe of determining $\{k, \omega_n, \zeta\}$, we shall present the explicit solution which requires the following preliminary calculations. For A_L corresponding to the Controllable Canonical From, we have

$$\begin{aligned}
e^{A_L t} &= \mathcal{L}^{-1}\Big\{(sI - A_L)^{-1}\Big\} \\
&= \begin{bmatrix} \alpha_1(t)e^{-\zeta\omega_n t} + \alpha_2(t)e^{-\zeta\omega_n t} & \alpha_3(t)e^{-\zeta\omega_n t} \\ \alpha_4(t)e^{-\zeta\omega_n t} & \alpha_1(t)e^{-\zeta\omega_n t} - \alpha_2(t)e^{-\zeta\omega_n t} \end{bmatrix},
\end{aligned}$$

where $s = j\omega$,

$$\alpha_1(t) = \cos\Big(\omega_n\sqrt{1-\zeta^2}t\Big), \qquad \alpha_2(t) = \tfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\Big(\omega_n\sqrt{1-\zeta^2}t\Big),$$

$$\alpha_3(t) = \tfrac{1}{\omega_n\sqrt{1-\zeta^2}}\sin\Big(\omega_n\sqrt{1-\zeta^2}t\Big), \quad \alpha_4(t) = -\tfrac{\omega_n}{\sqrt{1-\zeta^2}}\sin\Big(\omega_n\sqrt{1-\zeta^2}t\Big).$$

Then

$$\begin{aligned}
\gamma_{11}(t) &= \Big(\frac{\gamma_{22}(0)+\gamma_{11}(0)\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} - \frac{k^2}{4\omega_n^3\sqrt{1-\zeta^2}}\Big)\sin\Big(2\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} \\
&+ \frac{\gamma_{11}(0)\zeta^2\omega_n^2 + 2\gamma_{12}(0)\zeta\omega_n + \gamma_{22}(0)}{\omega_n^2(1-\zeta^2)}\sin^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} \\
&+ \gamma_{11}(0)\cos^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} - \frac{k^2}{4\zeta\omega_n^3(1-\zeta^2)}e^{-2\zeta\omega_n t} \\
&+ \frac{\zeta k^2}{4\omega_n^3(1-\zeta^2)}\cos\Big(2\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} + \frac{k^2}{4\zeta\omega_n^3},
\end{aligned}$$

$$\begin{aligned}
\gamma_{12}(t) &= \Big(\frac{\gamma_{22}(0)-\gamma_{11}(0)\omega_n^2}{2\omega_n\sqrt{1-\zeta^2}}\Big)\sin\Big(2\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} + \frac{k^2}{4\omega_n^2(1-\zeta^2)}e^{-2\zeta\omega_n t} \\
&- \frac{\gamma_{11}(0)\omega_n^2\zeta + \gamma_{12}(0)\zeta^2\omega_n + \gamma_{12}(0)\omega_n + \gamma_{22}(0)\zeta}{\omega_n(1-\zeta^2)}\sin^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} \\
&+ \gamma_{12}(0)\cos^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} - \frac{k^2}{4\omega_n^2(1-\zeta^2)}\cos\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t},
\end{aligned}$$

$$\begin{aligned}
\gamma_{22}(t) &= \Big(\frac{k^2\sqrt{1-\zeta^2}}{2\omega_n} - \frac{\gamma_{12}(0)\omega_n + \gamma_{22}(0)\zeta}{\sqrt{1-\zeta^2}}\Big)\sin\Big(2\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} \\
&+ \frac{\gamma_{11}(0)\omega^2 + 2\zeta\omega_n\gamma_{12}(0) + \gamma_{22}(0)\zeta^2}{1-\zeta^2}\sin^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} \\
&+ \gamma_{22}(0)\cos^2\Big(\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} + \frac{(2\zeta^2-\zeta^2-1)k^2}{4\zeta\omega_n(1-\zeta^2)}e^{-2\zeta\omega_n t} \\
&+ \frac{\zeta k^2}{\sqrt{1-\zeta^2}}\cos\Big(2\omega_n\sqrt{1-\zeta^2}t\Big)e^{-2\zeta\omega_n t} + \frac{(1-2\zeta^4+\zeta^2+(2\zeta^3-\zeta)\sqrt{1-\zeta^2})k^2}{4\zeta\omega_n(1-\zeta^2)}.
\end{aligned}$$

As $t \to \infty$, we can compute the steady state covariant matrix which is

$$\Sigma_L \triangleq \lim_{t\to\infty}\Sigma_L(t) = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} \frac{k^2}{4\zeta\omega_n^3} & 0 \\ 0 & \frac{(1-2\zeta^4+\zeta^2+(2\zeta^3-\zeta)\sqrt{1-\zeta^2})k^2}{4\zeta\omega_n(1-\zeta^2)} \end{bmatrix}.$$

Since (see [19])

$$r_{dB} = 20\log_{10} r = c\ln r^2;$$

the mean value of the dB-record is

$$E[r_{dB}] = \int_0^\infty c\ln r^2 f(r)dr,$$

where $c = 10/\ln 10$, $f(r)$ is pdf. Then, for a Rayleigh distributed envelope, we have

$$E[r_{dB}] = 10\log_{10}(2\sigma^2) - 2.51. \tag{2.5.17}$$

In our case, $\alpha = E[r_{dB}]$. That is,

$$\sigma^2 = \frac{10^{\frac{\alpha+2.51}{10}}}{2} = \gamma_{11} = \frac{k^2}{4\zeta\omega_n^3}.$$

Thus, the parameters $\{k, \omega_n, \zeta\}$ satisfy the following equation

$$k = \omega_n\sqrt{2\zeta\omega_n 10^{\frac{\alpha+2.51}{10}}}. \tag{2.5.18}$$

Figure 2.5.8 [a] corresponds to the mean of the channel envelope $\alpha = -10\ dB$, which implies $k = 4.016 \times 10^3$ while $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.8 [b] shows the mean of the channel envelope when $\alpha = 0\ dB$, which gives $k = 1.27 \times 10^4$ for selecting $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.8 [c] displays the mean of the channel envelope at $\alpha = 10\ dB$, which means $k = 4.016 \times 10^4$ for choosing $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.8 [d] shows the mean of the channel envelope at $\alpha = 0\ dB$, which gives $k = 5.34 \times 10^3$ while $\zeta = 0.1\ s^{-1}$, $\omega_n = 250\ rad/s$.

2.5.2 Ricean Channel Simulations

This is very similar to the Rayleigh case, because the Rayleigh case is related to Ricean by (2.4.11) and (2.4.12). A flat fast fading channel is simulated using the model (2.4.13) as shown in the block diagram of Figure 2.5.9[13]. There are two ways of Ricean channel simulation. Both of them are based on a prior knowledge of the LOS parameters $\{r_0, \omega_0, \theta_0\}$.

Method 1. Quite like the *Method 1* of Rayleigh. Given LOS values $\{r_0, \omega_0, \theta_0\}$ and β_m, v, f_c, E_0, one can determine the Ricean state space channel model. For example, when $\beta_m =$

[13]Here again only one multipath component is simulated. We select the j^{th} path ($j = 1$) Ricean channel model (2.4.13). It is convenient to remove the subscript j in this subsection.

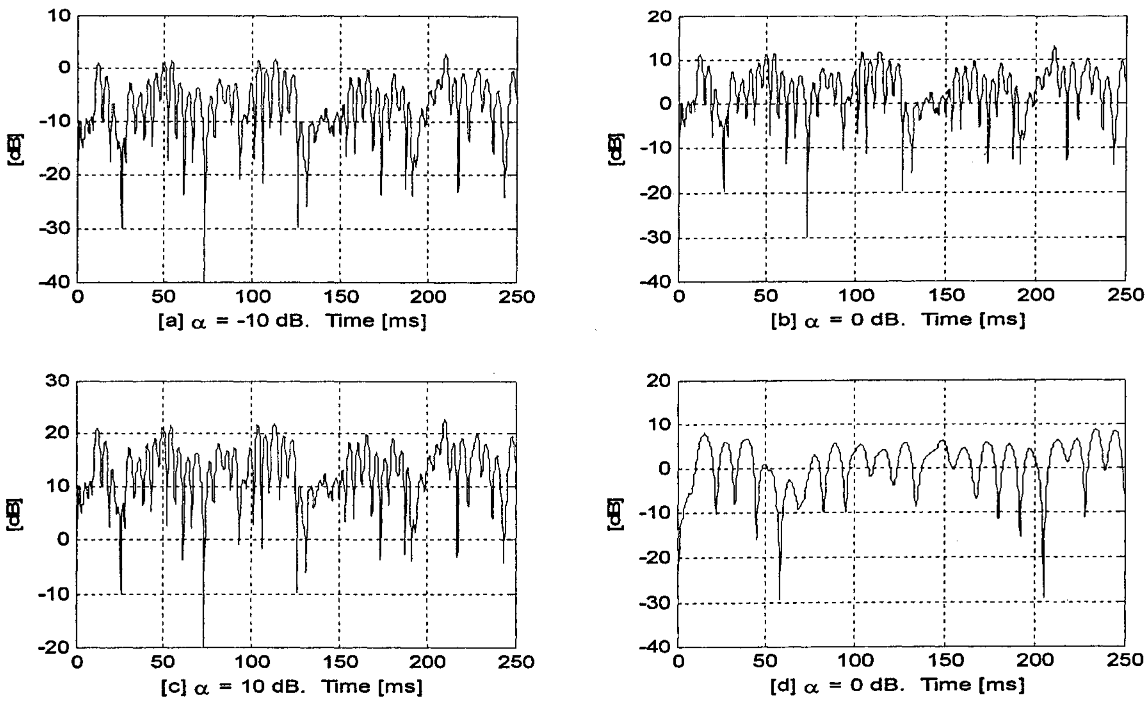

Figure 2.5.8: The Envelopes of Rayleigh Fading Channel; $f_c = 910\ MHz$, $v = 120\ km/h$.

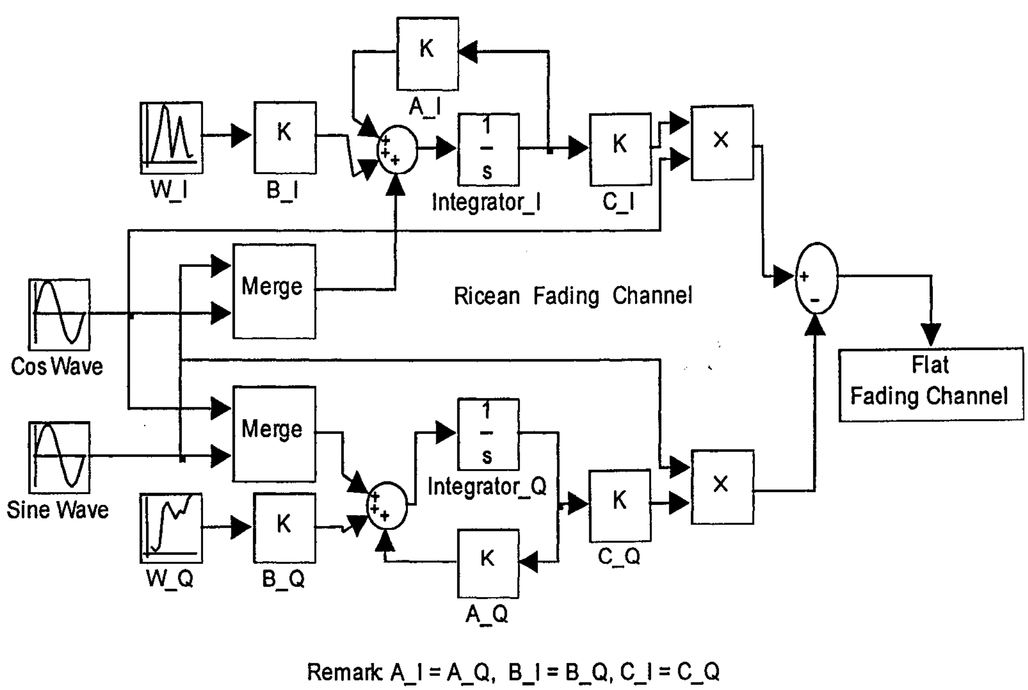

Figure 2.5.9: A Bandpass Ricean Fading Channel Model

10^0, $v = 120\ km/h$, $f_c = 910\ MHz$ and $E_0 = 2$, we obtain $w_n = 644.34\ rad/s$, $\zeta = 0.169\ s^{-1}$, $k = 6.21 \times 10^3$. Figure 2.5.10 shows the inphase $\{I^R(t)\}_{t\geq 0}$, quadrature $\{Q^R(t)\}_{t\geq 0}$, phase $\{\Phi(t) \triangleq \tan^{-1}\left(\frac{Q^R(t)}{I^R(t)}\right)\}_{t\geq 0}$ (in radian) and envelope $\{r(t) \triangleq \sqrt{I^R(t)^2 + Q^R(t)^2}\}_{t\geq 0}$ (in dB) for $r_0 = 1$, $\omega_0 = 6^0$, $\theta_0 = 30^0$.

Method 2. The steady state mean of the square envelope of a Ricean channel is

$$E[r^2] = 2a(0) + r_0^2,$$

which we introduced in the chapter 1. Given the steady state mean of a Rayleigh channel envelope in dB, say

$$\lim_{t\to\infty} E[r_{dB}(t)] = \alpha,$$

which is extracted from measurement data, then by using (2.5.17), we obtain

$$E[r^2] = 10^{\frac{\alpha+2.51}{10}} + r_0^2.$$

We already know that given the steady state mean of a Rayleigh channel envelope, one can determine the state space Rayleigh channel model parameters $\{\zeta, \omega_n, k\}$ from (2.5.18). From the above equation, one can determine the steady state mean of the square envelope of the Ricean channel by given the LOS parameters $\{r_0, \omega_0, \theta_0\}$. For example, while $r_0 = 1$, $\omega_0 = 6^0$, $\theta_0 = 30^0$, Figure 2.5.11 [a] corresponds to the mean of the square envelope is $E[r^2] = 1.42\ dB$ for $\alpha = -10\ dB$, which implies $k = 4.016 \times 10^3$ while $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.11 [b] shows the mean of the square envelope is $E[r^2] = 8.88\ dB$ when $\alpha = 0\ dB$, which gives $k = 1.27 \times 10^4$ for selecting $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.12 [a] displays the mean of the square envelope is $E[r^2] = 25.49\ dB$ at $\alpha = 10\ dB$, which means $k = 4.016 \times 10^4$ for choosing $\zeta = 0.1691\ s^{-1}$, $\omega_n = 644.345\ rad/s$; Figure 2.5.12 [b] shows the mean of the square envelope is $E[r^2] = 8.88\ dB$ when $\alpha = 0\ dB$, which gives $k = 5.34 \times 10^3$ while $\zeta = 0.1\ s^{-1}$, $\omega_n = 250\ rad/s$.

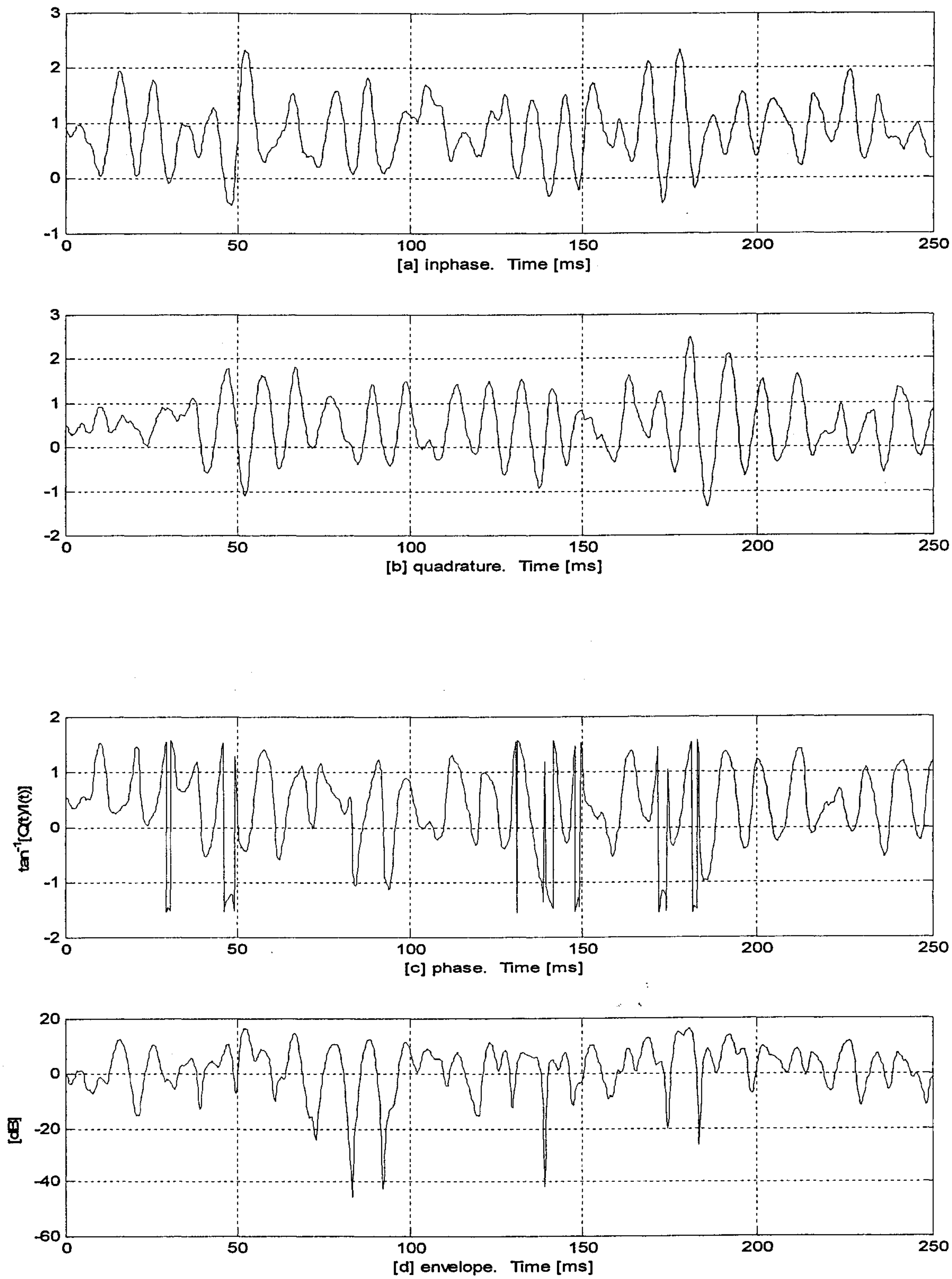

Figure 2.5.10: The Ricean Fading Channel; $f_c = 910\ MHz$, $v = 120\ km/h$

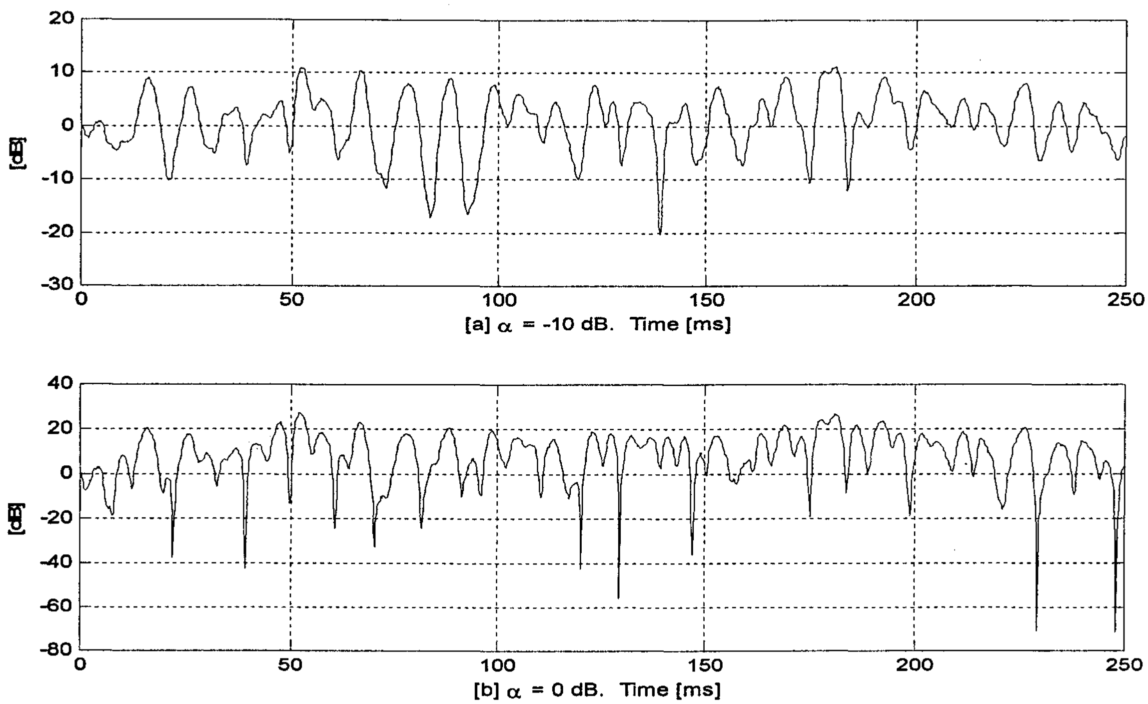

Figure 2.5.11: Part 1: Ricean Channel Square Envelopes; $f_c = 910\ MHz$, $v = 120\ km/h$

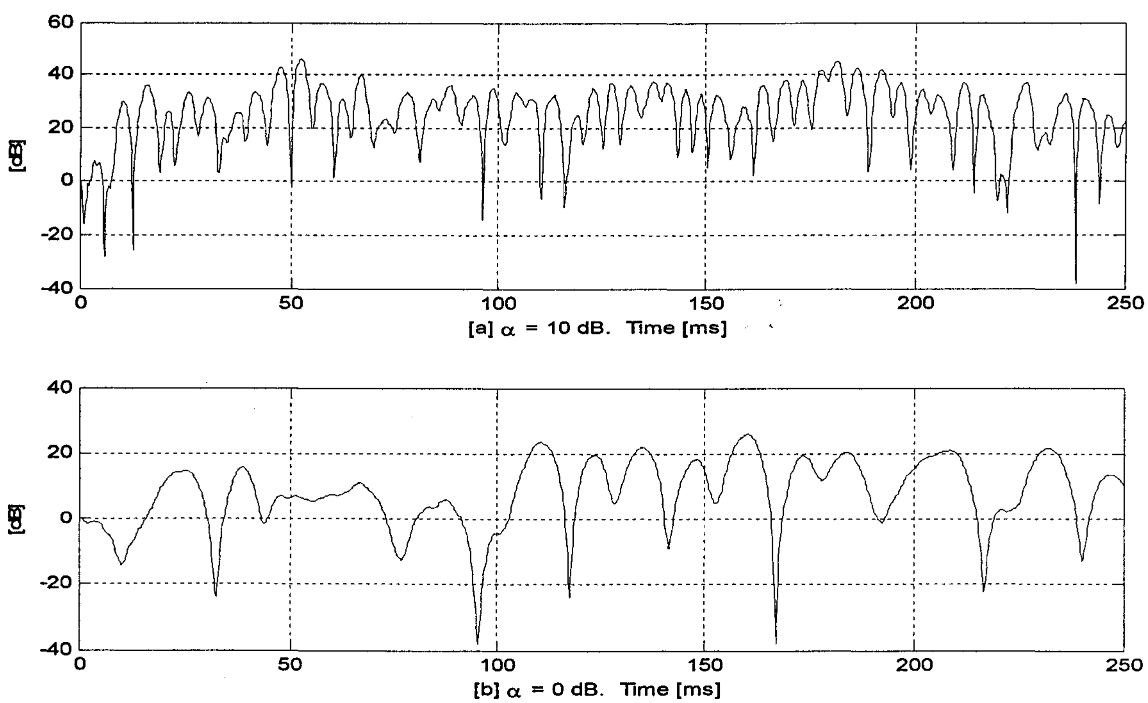

Figure 2.5.12: Part 2: Ricean Channel Square Envelopes; $f_c = 910\ MHz$, $v = 120\ km/h$

Chapter 3

Channel Model Estimation

These results of chapter 2 are first employed to construct wireless state space Rayleigh and Ricean fading channel models. The objective of this chapter is to employ least-square estimation techniques to estimate the channel models from noisy received signals. The focus on is based on Kalman filtering theory, to extract the channel inphase, quadrature and square envelope estimations. The following items will be discussed

1. Continuous time channel state estimation.
2. Discrete time channel state estimation.
3. Computation of the conditional Rayleigh envelope and phase distributions.

All investigations will be done by assuming that the parameters $\{A, B, C, D\}$ of the state space dynamic system (2.4.8) are known. If $\{A, B, C, D\}$ are unknown, one can also employ the state space estimations to estimate the parameters $\{A, B, C, D\}$ as in [7] and [9]. This technique is so called systems identification.

3.1 States Estimation

A general state space dynamic system and measurement models are given by the following equations

$$\begin{aligned} \dot{X} &= AX + B\dot{w} \\ y &= CX + v \end{aligned}$$

Here $X(0) \sim N(\mu, \Sigma)$, $\dot{w} \sim N(0, Q)$, $v \sim N(0, R)$, and $\{\dot{w}(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ are independent white Gaussian processes, which are uncorrelated with $X(0)$. The algorithm corresponding to the Kalman filter is given by the following equations.

Initialization:

$$P(0) = Var(X(0)) = \Sigma(0), \quad X(0) = \widehat{X}(0) = Mean(X(0)) = \mu;$$

Error Covariance (Riccati Equation):

$$\dot{P}(t) = AP(t) + P(t)A^{tr} + BQB^{tr} - P(t)C^{tr}R^{-1}CP(t);$$

Filter Gain:

$$K(t) = P(t)C^{tr}R^{-1};$$

Estimation (Filter):

$$\dot{\widehat{X}} = A\widehat{X} + K(t)(y - C\widehat{X}).$$

3.1.1 Continuous Time

First we apply the Kalman filter algorithm for a flat fading channel and then we introduce the multipath Kalman filter algorithm. The Kalman filter corresponding to (2.4.10), (2.4.13) and (2.4.14) is given below.

Rayleigh Channel Kalman Filter. On the i^{th} path ($i \neq 1$), the state space dynamic system and measurement models are given by (2.4.10), (2.4.14), respectively. For simplifying the notations, we represent the i^{th} path state space Rayleigh channel model as follows,

$$\begin{aligned} \dot{X} &= AX + B\dot{w} \\ y &= GX + Dv \end{aligned} \tag{3.1.1}$$

where

$$X \triangleq X_i, \quad A \triangleq A_i, \quad B \triangleq B_i, \quad \dot{w} \triangleq \dot{w}_i, \quad G \triangleq G_i,$$

and Dv is a white Gaussian noise added on the i^{th} path. Since v has a coefficient D, we should substitute R^{-1} by $DR^{-1}D^{tr}$ in the Kalman filter equations above[1]. Thus, the least-square estimates of $\{I(t)\}_{t\geq 0}$, $\{Q(t)\}_{t\geq 0}$ and $\{r^2(t)\}_{t\geq 0}$ are given by the following equations.
Initialization:

$$P(0) = Var(X(0)), \quad X(0) = \widehat{X}(0) = Mean(X(0));$$

[1]see Appendix A.5.1 for details

Error Covariance (Riccati Equation):

$$\dot{P}(t) = AP(t) + P(t)A^{tr} + BQB^{tr} - P(t)G^{tr}(t)(D(t)RD^{tr}(t))^{-1}G(t)P(t); \qquad (3.1.2)$$

Filter Gain:

$$K(t) = P(t)G^{tr}(t)(D(t)RD^{tr}(t))^{-1},$$

where

$$R \triangleq \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \text{ and } Q \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

Estimate (filter):

$$\dot{\widehat{X}} = A\widehat{X} + K(t)\Big(y(t) - G(t)\widehat{X}\Big); \qquad (3.1.3)$$

Inphase and Quadrature Estimates:

$$\begin{aligned} \widehat{I}(t) &\triangleq E\Big[I(t)|\{y(s);\ 0 \le s \le t\}\Big] &= [1 \quad 0 \quad 0 \quad 0]\widehat{X}(t), \\ \widehat{Q}(t) &\triangleq E\Big[Q(t)|\{y(s);\ 0 \le s \le t\}\Big] &= [0 \quad 0 \quad 1 \quad 0]\widehat{X}(t); \end{aligned}$$

Square-Envelope Estimate:

$$\widehat{r^2}(t) \triangleq \widehat{I^2}(t) + \widehat{Q^2}(t) + e_I^2(t) + e_Q^2(t),$$

where $e_I^2(t) \triangleq E[(I(t) - \widehat{I}(t))^2]$, $e_Q^2(t) \triangleq E[(Q(t) - \widehat{Q}(t))^2]$ are the mean-square errors, which are obtained from (3.1.2), that is, the elements $P_{11}(t)$, $P_{33}(t)$ of $P(t)$, respectively.

Ricean Channel Kalman Filter. On the j^{th} path ($j = 1$), the state space dynamic system and measurement models are given by (2.4.13), (2.4.14), respectively. Similar to the Rayleigh, the j^{th} path state space Ricean channel model is

$$\begin{aligned} \dot{X} &= AX + f_{los} + B\dot{w} \\ y &= GX + Dv \end{aligned} \qquad (3.1.4)$$

where

$$X \triangleq X_j^R, \quad A \triangleq A_j, \quad B \triangleq B_j, \quad \dot{w} \triangleq \dot{w}_j, \quad G \triangleq G_j,$$

and Dv is a white Gaussian noise added on the j^{th} path. The least-square estimates of $\{I^R(t)\}_{t\ge0}$, $\{Q^R(t)\}_{t\ge0}$ and $\{r_R^2(t)\}_{t\ge0}$ are given by the following equations.

Initialization:

$$P(0) = Var(X(0)), \quad X(0) = \widehat{X}(0) = Mean(X(0));$$

Error Covariance (Riccati Equation):

$$\dot{P}(t) \quad = \quad AP(t) + P(t)A^{tr} + BQB^{tr} - P(t)G^{tr}(t)(D(t)RD^{tr}(t))^{-1}G(t)P(t); \tag{3.1.5}$$

Filter Gain:

$$K(t) = P(t)G^{tr}(t)(D(t)RD^{tr}(t))^{-1},$$

where

$$R \triangleq \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \text{ and } Q \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

Estimate (filter)[2]*:*

$$\dot{\widehat{X}} = A\widehat{X} + f_{los} + K(t)\Big(y(t) - G(t)\widehat{X}\Big); \tag{3.1.6}$$

Inphase and Quadrature Estimates:

$$\begin{aligned} \widehat{I}^R(t) &\triangleq E\Big[I^R(t)|\{y(s);\ 0 \le s \le t\}\Big] &= [1 \quad 0 \quad 0 \quad 0]\widehat{X}(t), \\ \widehat{Q}^R(t) &\triangleq E\Big[Q^R(t)|\{y(s);\ 0 \le s \le t\}\Big] &= [0 \quad 0 \quad 1 \quad 0]\widehat{X}(t); \end{aligned}$$

Square-Envelope Estimate:

$$\widehat{r_R^2}(t) \triangleq \widehat{I^{R^2}}(t) + \widehat{Q^{R^2}}(t) + e_{IR}^2(t) + e_{QR}^2(t),$$

where $e_{IR}^2(t) \triangleq E[(I^R(t) - \widehat{I}^R(t))^2]$, $e_{QR}^2(t) \triangleq E[(Q^R(t) - \widehat{Q}^R(t))^2]$ are the mean-square errors, which are obtained from (3.1.5), that is, the elements $P_{11}(t)$, $P_{33}(t)$ of $P(t)$, respectively.

Multipath Channel Kalman Filter. Consider a multipath channel having N multipath components, the first of which is a Ricean channel, while the remaining $N-1$ components are Rayleigh channels. This multipath channel model can be represented by using (2.4.10), (2.4.13) and (2.4.14), as follows.

$$\begin{aligned} \dot{X} &= AX + F_{los} + B\dot{w} \\ y &= GX + Dv \end{aligned} \tag{3.1.7}$$

[2]see Appendix A.6.2 for details

where $X(0) \in \Re^{4N}$,

$$A \triangleq \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_N \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_N \end{bmatrix}, \quad G \triangleq [G_1 \quad \cdots \quad G_N]$$

and

$$F_{los} \triangleq \Big[\overbrace{f_{los}^{tr}}^{1} \quad \overbrace{0 \quad 0 \quad 0 \quad 0}^{2} \quad \cdots \quad \overbrace{0 \quad 0 \quad 0 \quad 0}^{N}\Big]^{tr}.$$

Thus, one can obtain the least-square estimates of the inphases, quadratures and square envelopes by the following equations.

Initialization:

$$P(0) = Var(X(0)), \quad X(0) = \widehat{X}(0) = Mean(X(0));$$

Error Covariance (Riccati Equation):

$$\dot{P}(t) \quad = \quad AP(t) + P(t)A^{tr} + BQB^{tr} - P(t)G^{tr}(DRD^{tr})^{-1}GP(t); \tag{3.1.8}$$

Filter Gain:

$$K(t) = P(t)G^{tr}(DRD^{tr})^{-1},$$

where

$$R = \begin{bmatrix} \sigma_v^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_v^2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \quad R, Q \in \Re^{2N} \times \Re^{2N};$$

Estimate (filter):

$$\dot{\widehat{X}} = A\widehat{X} + F_{los} + K(t)\Big(y(t) - G\widehat{X}\Big), \tag{3.1.9}$$

Inphase and Quadrature Estimates:

$$\begin{aligned} \widehat{I}_i(t) \quad &\triangleq \quad E\Big[I_i(t)|\{y(s)\}_{s=0}^{t}\Big] \\ &= \Big[0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad \overbrace{1 \quad 0 \quad 0 \quad 0}^{i} \quad \cdots \quad 0 \quad 0 \quad 0 \quad 0\Big]\widehat{X}(t), \\ \widehat{Q}_i(t) \quad &\triangleq \quad E\Big[Q_i(t)|\{y(s)\}_{s=0}^{t}\Big] \\ &= \Big[0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad \overbrace{0 \quad 0 \quad 1 \quad 0}^{i} \quad \cdots \quad 0 \quad 0 \quad 0 \quad 0\Big]\widehat{X}(t). \end{aligned}$$

Here $\widehat{I}_1(t)$ denotes the estimate of $I^R(t)$, $\widehat{Q}_1(t)$ denotes the estimate of $Q^R(t)$.
Square-Envelope Estimate:

$$\widehat{r_i^2}(t) \triangleq \widehat{I_i^2}(t) + \widehat{Q_i^2}(t) + e_{Ii}^2(t) + e_{Qi}^2(t),$$

where $e_{Ii}^2(t) \triangleq E[(I_i(t) - \widehat{I}_i(t))^2]$, $e_{Qi}^2(t) \triangleq E[(Q_i(t) - \widehat{Q}_i(t))^2]$ are the mean-square errors, which are obtained from (3.1.8), that is, elements of $P_{4i-3,4i-3}(t)$, $P_{4i-1,4i-1}(t)$ of $P(t)$, $1 \leq i \leq N$, respectively.

3.1.2 Discrete Time

Discrete Time Multipath Channel Kalman Filter. Although estimations can be done by using continuous time model, we prefer the discrete time model in this thesis. In order to use the discrete time algorithm of the Kalman filter, we need to do the discretization for the continuous time multipath state space channel model (3.1.7). Mathematically, we can not prove the existence of a white noise, but we can represent a white noise in terms of the velocity of a Brownian motion, that is, $v(t) = dw(t)/dt = \dot{w}(t)$. Along this way, we can obtain a very good result after the discretization. Sampling (3.1.7) on a time interval $[\,a \quad b\,]$ by using the Ito process method with respect to an equal time subinterval $[\,t_n \quad t_{n+1}\,]$, $\forall t_n \in [\,a \quad b\,]$, then we obtain the discrete time multipath state space channel model as follows[3]

$$\begin{array}{rcl} X_{n+1} & = & \Gamma X_n + \Upsilon + \Lambda \Delta w_n \\ y_n & = & \Omega X_n + \Psi v_n \end{array} \tag{3.1.10}$$

where

$$\Gamma = \delta A + I, \quad \Upsilon = \delta F_{los}, \quad \Lambda = B,$$

$$\Omega = \begin{bmatrix} s(t_n - \tau_1)C_1(t_n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s(t_n - \tau_N)C_N(t_n) \end{bmatrix}$$

$$\Psi = [\,\cos(\omega_c t_n) \quad -\sin(\omega_c t_n)\,], \quad v_n = \begin{bmatrix} v_I(t_n) \\ v_Q(t_n) \end{bmatrix}.$$

Here $\delta = t_{n+1} - t_n$ is called the step size, $I \in \Re^{4N}$ is the unit matrix,

$$C_i(t_n) = [\,\cos(\omega_c t_n) \quad 0 \quad -\sin(\omega_c t_n) \quad 0\,]$$

[3]see Appendix A.5.2 for details

and $\Delta w_n \triangleq w(t_{n+1}) - w(t_n) \in \Re^{2N}$ are iid white Gaussian noises, which each element of $\Delta w_n \sim N(0;\delta)$. Thus, the algorithm of the discrete time Klaman filter produces the least-square estimations of the inphases, quadratures and square envelopes are given by the following equations.

Initialization:

$$P_0 = Var(X_0), \quad X_0 = \widehat{X}_0 = Mean(X_0);$$

Error Covariance (Riccati Equation):

$$P_{n+1} = \Gamma P_n \Gamma^{tr} + \Lambda Q \Lambda^{tr} - \Gamma P_n \Omega^{tr}(\Omega P_n \Omega^{tr} + \Psi R \Psi^{tr})^{-1} \Omega P_n \Gamma^{tr}, \tag{3.1.11}$$

where

$$Q = \begin{bmatrix} \delta & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta \end{bmatrix} \in \Re^{2N};$$

Filter Gain:

$$K_n = \Gamma P_n \Omega^{tr}(\Omega P_n \Omega^{tr} + \Psi R \Psi^{tr})^{-1};$$

Estimate (filter):

$$\widehat{X}_{n+1} = \Gamma \widehat{X}_n + \Upsilon + K_n\Big(y_n - \Omega \widehat{X}_n\Big); \tag{3.1.12}$$

Inphase and Quadrature Estimates:

$$\begin{aligned}
\widehat{I}_{i_n} &\triangleq E\Big[I_{i_n} | \{y_k\}_{k=0}^{t_n}\Big] \\
&= \Big[0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; \overbrace{1 \;\; 0 \;\; 0 \;\; 0}^{i} \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; 0\Big]\widehat{X}_n, \\
\widehat{Q}_{i_n} &\triangleq E\Big[Q_{i_n} | \{y_k\}_{k=0}^{t_n}\Big] \\
&= \Big[0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; \overbrace{0 \;\; 0 \;\; 1 \;\; 0}^{i} \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; 0\Big]\widehat{X}_n.
\end{aligned}$$

Here $\widehat{I}_{1_n}$ denotes the estimate of I_n^R, $\widehat{Q}_{1_n}$ denotes the estimate of Q_n^R.

Square-Envelope Estimate:

$$\widehat{r}_{i_n}^2 \triangleq \widehat{I}_{i_n}^2 + \widehat{Q}_{i_n}^2 + e_{Ii_n}^2 + e_{Qi_n}^2,$$

where $e^2_{Ii_n} \triangleq E[(I_{i_n} - \widehat{I}_{i_n})^2]$, $e^2_{Qi_n} \triangleq E[(Q_{i_n} - \widehat{Q}_{i_n})^2]$ are the mean-square errors, which are obtained from (3.1.11), that is, elements of $P_{(4i-3,4i-3)_n}$, $P_{(4i-1,4i-1)_n}$ of P_n, $1 \le i \le N$, respectively.

Similarly, we can find the discrete time state space Rayleigh channel model of the i^{th} path and Ricean channel model of the j^{th} path using the Ito method.

3.2 Conditional Rayleigh Envelope and Phase Distributions

Channel envelope and phase are two very important parameters in channel estimation. Since the conditional densities of the envelope and phase can be used to compute any conditional estimation of the functions of $r(t)$ and $\theta(t)$, we should obtained their explicit expressions. We are interested in computing the conditional distributions of the channel envelope and phase, given the noisy data. Least-square estimation of Gaussian processes state that the optimal estimates of $X_I(t) \in \Re^2$, $X_Q(t) \in \Re^2$ given $\{y(s);\ 0 \le s \le t\}$ are the conditional means

$$\begin{aligned}
\widehat{X_I}(t) &= E[X_I(t)|Y_0^t], \quad \widehat{X_I}(t) \in \Re^2, \\
\widehat{X_Q}(t) &= E[X_Q(t)|Y_0^t], \quad \widehat{X_Q}(t) \in \Re^2,
\end{aligned}$$

where $Y_0^t \triangleq \{y(s);\ 0 \le s \le t\}$. The conditional joint density of the inphase and quadrature is still Gaussian[4], given by

$$\begin{aligned}
f(x_{I_1}, x_{Q_1}, t|Y_0^t) &\triangleq \frac{\partial^2}{\partial x_{I_1} \partial x_{Q_1}} Prob(X_{I_1}(t) \le x_{I_1}, X_{Q_1}(t) \le x_{Q_1}|Y_0^t) \\
&= \frac{1}{2\pi\sqrt{|M(t)|}} e^{-\frac{1}{2}(x-\widehat{X_{I,Q}}(t))^{tr} M^{-1}(t)(x-\widehat{X_{I,Q}}(t))},
\end{aligned}$$

where

$$x = \begin{bmatrix} x_{I_1} \\ x_{Q_1} \end{bmatrix}, \quad \widehat{X_{I,Q}}(t) = \begin{bmatrix} \widehat{X_{I_1}}(t) \\ \widehat{X_{Q_1}}(t) \end{bmatrix} = \begin{bmatrix} E[I(t)|Y_0^t] \\ E[Q(t)|Y_0^t] \end{bmatrix},$$

$\{M(t)\}_{t\ge 0}$ is the conditional covariant matrix of $\{x(t)\}_{t\ge 0}$, and each element of $M(t)$ is related to the Riccati equation (3.1.2) as follows,

$$M(t) = \begin{bmatrix} P_{11}(t) & P_{13}(t) \\ P_{31}(t) & P_{33}(t) \end{bmatrix}, \quad P_{13}(t) = P_{31}(t).$$

[4]see Appendix A.5.3 for details

3.2.1 Conditional Envelope Distribution

Let $Z = X_{I_1}^2 + X_{Q_1}^2$; then we have, by definition,

$$\begin{aligned}
Prob(Z(t) \le z | Y_0^t) &= Prob(X_{I_1}^2(t) + X_{Q_1}^2(t) \le z | Y_0^t) \\
&= \int_{x_{I_1}^2 + x_{Q_1}^2 \le z} f(x_{I_1}, x_{Q_1}, t | Y_0^t) dx_{I_1} dx_{Q_1} \\
&= \int_{x_{I_1}^2 + x_{Q_1}^2 \le z} \frac{1}{2\pi\sqrt{|M(t)|}} e^{-\frac{1}{2}(x - \widehat{X_{I,Q}}(t))^{tr} M^{-1}(t)(x - \widehat{X_{I,Q}}(t))} dx_{I_1} dx_{Q_1}.
\end{aligned}$$

Define the inverse matrix of $M(t)$

$$\begin{aligned}
M^{-1}(t) &= \frac{1}{P_{11}(t)P_{33}(t) - P_{13}^2(t)} \begin{bmatrix} P_{33}(t) & -P_{13}(t) \\ -P_{13}(t) & P_{11}(t) \end{bmatrix} \\
&\triangleq \begin{bmatrix} g(t) & m(t) \\ m(t) & n(t) \end{bmatrix},
\end{aligned}$$

and introduce to the spherical coordinates

$$\begin{cases} x_{I_1} = \sqrt{\rho}\cos\theta \\ x_{Q_1} = \sqrt{\rho}\sin\theta \end{cases}.$$

Then[5]

$$\begin{aligned}
Prob(Z(t) \le z | Y_0^t) &= \int_0^z \int_0^{2\pi} \frac{1}{2} f(\rho, \theta, t | Y_0^t) d\rho d\theta \\
&= \int_0^z \frac{1}{2\pi\sqrt{|M(t)|}} e^{-\frac{g(t)}{2}\widehat{X_{I_1}}^2(t) - \frac{n(t)}{2}\widehat{X_{Q_1}}^2(t) - m(t)\widehat{X_{I_1}}(t)\widehat{X_{Q_1}}(t) - \frac{g(t)}{2}\rho} \\
&\quad \times \sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!} \left(g(t)\widehat{X_{I_1}}(t) + m(t)\widehat{X_{Q_1}}(t)\right)^h \\
&\quad \times \left(n(t)\widehat{X_{Q_1}}(t) + m(t)\widehat{X_{I_1}}(t)\right)^i \left(g(t) - n(t)\right)^j m^k(t) \\
&\quad \times \rho^{\frac{h+i+2j+2k}{2}} \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})} d\rho.
\end{aligned}$$

Thus, the conditional probability density function of the Rayleigh channel envelope is

$$\begin{aligned}
f_R(r(t)|Y_0^t) &= \frac{1}{\pi\sqrt{|M(t)|}} e^{-\frac{g(t)}{2}\widehat{X_{I_1}}^2(t) - \frac{n(t)}{2}\widehat{X_{Q_1}}^2(t) - m(t)\widehat{X_{I_1}}(t)\widehat{X_{Q_1}}(t) - \frac{g(t)}{2}r^2} \\
&\quad \times \sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!} \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})} \left(g(t)\widehat{X_{I_1}}(t) + m(t)\widehat{X_{Q_1}}(t)\right)^h \\
&\quad \times \left(n(t)\widehat{X_{Q_1}}(t) + m(t)\widehat{X_{I_1}}(t)\right)^i \left(g(t) - n(t)\right)^j m^k(t) r^{h+i+2j+2k+1}.
\end{aligned}$$

[5] see Appendix A.5.3 for details

3.2.2 Conditional Phase Distribution

Here we should first introduce an expression for the conditional density of $\tan^{-1}\left(\frac{Q(t)}{I(t)}\right)$ given $\{y(s);\ 0 \leq s \leq t\}$. Define

$$\begin{cases} x_{I_1} & \triangleq \rho\cos\theta \\ x_{Q_1} & \triangleq \rho\sin\theta \end{cases},$$

and then $\theta = \tan^{-1}\left(\frac{x_{Q_1}}{x_{I_1}}\right)$. Hence[6]

$$\begin{aligned} Prob(\Theta \leq \phi | Y_0^t) &= Prob(\tan^{-1}\frac{X_{Q_1}}{X_{I_1}} \leq \phi | Y_0^t) \\ &= \int_{\tan^{-1}\left(\frac{x_{Q_1}}{x_{I_1}}\right)\leq\phi} f(x_{I_1}, x_{Q_1}, t | Y_0^t) dx_{I_1} dx_{Q_1} \\ &= \int_0^\infty \int_0^\phi \frac{1}{2\pi\sqrt{|M(t)|}} e^{-\frac{1}{2}(x-\widehat{X_{I,Q}}(t))^{tr} M^{-1}(t)(x-\widehat{X_{I,Q}}(t))} \rho\, d\theta\, d\rho \\ &= \int_0^\phi \frac{1}{2\pi\sqrt{|M(t)|}} e^{-\frac{g(t)}{2}\widehat{X_{I_1}}^2(t) - \frac{n(t)}{2}\widehat{X_{Q_1}}^2(t) - m\widehat{X_{I_1}}(t)\widehat{X_{Q_1}}(t)} e^{R^2(\theta)} \\ &\quad \times \Big(\frac{1}{2S^2(\theta)} e^{-R^2(\theta)} + \frac{\sqrt{\pi}R(\theta)}{2S^2(\theta)}\Big) d\theta, \end{aligned}$$

where

$$S(\theta) = \sqrt{\frac{g(t)}{2} - \frac{g(t)-n(t)}{2}\sin^2\theta + m(t)\sin\theta\cos\theta},$$

$$R(\theta) = \frac{\Big(g(t)\widehat{X_{I_1}}(t) + m(t)\widehat{X_{Q_1}}(t)\Big)\cos\theta + \Big(n(t)\widehat{X_{Q_1}}(t) + m(t)\widehat{X_{I_1}}(t)\Big)\sin\theta}{2S(\theta)}.$$

.

Thus, the conditional density of $\theta(t) \triangleq \tan^{-1}\left(\frac{Q(t)}{I(t)}\right)$ given $\{y(s);\ 0 \leq s \leq t\}$ is

$$f(\theta(t)|Y_0^t) = \frac{1 + R(\theta)\sqrt{\pi}e^{R^2(\theta)}}{4\pi S^2(\theta)\sqrt{|M(t)|}} e^{-\frac{g(t)}{2}\widehat{X_{I_1}}^2(t) - \frac{n(t)}{2}\widehat{X_{Q_1}}^2(t) - m(t)\widehat{X_{I_1}}(t)\widehat{X_{Q_1}}(t)}.$$

[6] see Appendix A.5.3 for details

Chapter 4

State Space Channel Estimation: A Case Study

In this chapter, we present a case study of channel state estimation for flat and frequency-selective channels. The estimation approach is based on the Kalman filter presented in the previous chapter. The following items, inphase, quadrature and square envelope, are estimated. For the flat fading channels, we investigate both of slow and fast fading cases. For frequency-selective channels, only the slow fading case is investigated.

4.1 Frequency-Nonselective Fading Channels

A frequency-nonselective fading channel is also called a flat fading channel. In this section, one Ricean channel model given by (3.1.4) and one Rayleigh channel model given by (3.1.1) are performed, respectively. In order to address the performance of the estimator, the received signal-to-noise power is computed using the corresponding state space model.

4.1.1 Rayleigh Channel

Let $X(0) \sim N(0; \Sigma(0))$, then, theoretically, the state space channel model (3.1.1) has the property of Rayleigh fading nature.

(i) Rayleigh Channel Received Signal-to-Noise Ratio. The received SNR is defined by

$$SNR \triangleq \frac{\frac{1}{T_s} E\left[\int_0^{T_s} \mid G(t)X(t) \mid^2 dt \right]}{\frac{1}{T_s} E\left[\int_0^{T_s} \mid D(t)v(t) \mid^2 dt \right]}, \tag{4.1.1}$$

where T_s is the signaling interval. There are two possible ways of using the equation (4.1.1) for the simulation purposes. The first method is specify the desired SNR, and then determine how much the average power of the transmitted signal should be. The second method is to specify the desired SNR, the average noise power, and then determine the power of the receiver. Let E_s be the average noiseless received signal power. Then,

$$
\begin{aligned}
E_s &= \frac{1}{T_s} E\Big[\int_0^{T_s} \mid G(t)X(t) \mid^2 dt\Big] \\
&= \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[X(t)X^{tr}(t)\big]C^{tr}(t)dt.
\end{aligned}
$$

Narrow Band: If $\{s(t)\}_{t\geq 0}$ is a narrow band, and hence without loss of generality is assumed to be $s(t) = 1$, then by performing the above integration, we get

$$
\begin{aligned}
E_s &= \frac{1}{T_s}\int_0^{T_s} C(t)E\big[X(t)X^{tr}(t)\big]C^{tr}(t)dt \\
&= \frac{2\zeta^2\beta_1 - 2\beta_4 - \beta_3 - \beta_2}{4\zeta\omega_n T_s} e^{-2\zeta\omega_n T_s} + \beta_6 \\
&+ \frac{2\zeta\beta_1\sqrt{1-\zeta^2} + 2\beta_4 + 2\zeta^2\beta_5 + \zeta^2\beta_3 + \beta_2 - \zeta^2\beta_2 + \beta_3}{4\zeta\omega_n T_s} \\
&+ \frac{\zeta\beta_2 - \zeta\beta_3 - 2\zeta\beta_5 - 2\beta_1\sqrt{1-\zeta^2}}{4\omega_n T_s}\cos\Big(2\omega_n\sqrt{1-\zeta^2}T_s\Big)e^{-2\zeta\omega_n T_s} \\
&+ \frac{2\sqrt{1-\zeta^2}\beta_5 + \sqrt{1-\zeta^2}\beta_3 + \sqrt{1-\zeta^2}\beta_2}{4\omega_n T_s}\sin\Big(2\omega_n\sqrt{1-\zeta^2}T_s\Big)e^{-2\zeta\omega_n T_s},
\end{aligned}
$$

where $\gamma(0) = E\big[X(0)X^{tr}(0)\big]$ and

$$
\beta_1 = \Big(\frac{\gamma_{22}(0) + \gamma_{11}(0)\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} - \frac{k^2}{4\omega_n^3\sqrt{1-\zeta^2}}\Big), \qquad \beta_2 = \frac{\gamma_{11}(0)\zeta^2\omega_n^2 + 2\gamma_{12}(0)\zeta\omega_n + \gamma_{22}(0)}{\omega_n^2(1-\zeta^2)},
$$

$$
\beta_3 = \gamma_{11}(0), \quad \beta_4 = \frac{k^2}{4\zeta\omega_n^3(1-\zeta^2)}, \qquad \beta_5 = \frac{\zeta k^2}{4\omega_n^3(1-\zeta^2)}, \quad \beta_6 = \frac{k^2}{4\zeta\omega_n^3}.
$$

Then the additive noises are given by[1]

$$
\begin{bmatrix} v_I \\ v_Q \end{bmatrix} = \sqrt{\frac{E_s}{10^{\frac{(SNR)_{dB}}{10}}}}\begin{bmatrix} n_I \\ n_Q \end{bmatrix},
$$

where $\{n_I(t)\}_{t\geq 0} \sim N(0,1)$, $\{n_Q(t)\}_{t\geq 0} \sim N(0,1)$ are independent white Gaussian noise processes, and the covariance matrix is given by

[1] see Appendix A.7 for details

$$R_v = \frac{E_s}{10^{\frac{(SNR)_{dB}}{10}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the average additive noises power is

$$E_n = \frac{1}{T_s} E\left[\int_0^{T_s} \mid D(t)v(t) \mid^2 dt \right] = \frac{1}{T_s} \int_0^{T_s} D(t)E\left[v(t)v^{tr}(t)\right] D^{tr}(t)dt = \frac{E_s}{10^{\frac{(SNR)_{dB}}{10}}}.$$

(ii) Flat Slow Fading Channel Estimation. Let $f_c = 910\ MHz$, $v = 60\ km/h$, which implies a maximum Doppler shift $f_d = 50.556\ Hz$, Doppler spread $B_d = 2f_d = 101.112\ Hz$ and coherence time $T_c = 9.89\ ms$. Select $T_s = 600\ \mu s$, then we have $T_s << T_c$, and therefore the channel is slow fading. The results of the inphase, quadrature, square envelope estimation are shown on Figure 4.1.1 for $(SNR)_{dB} = -3\ dB$.

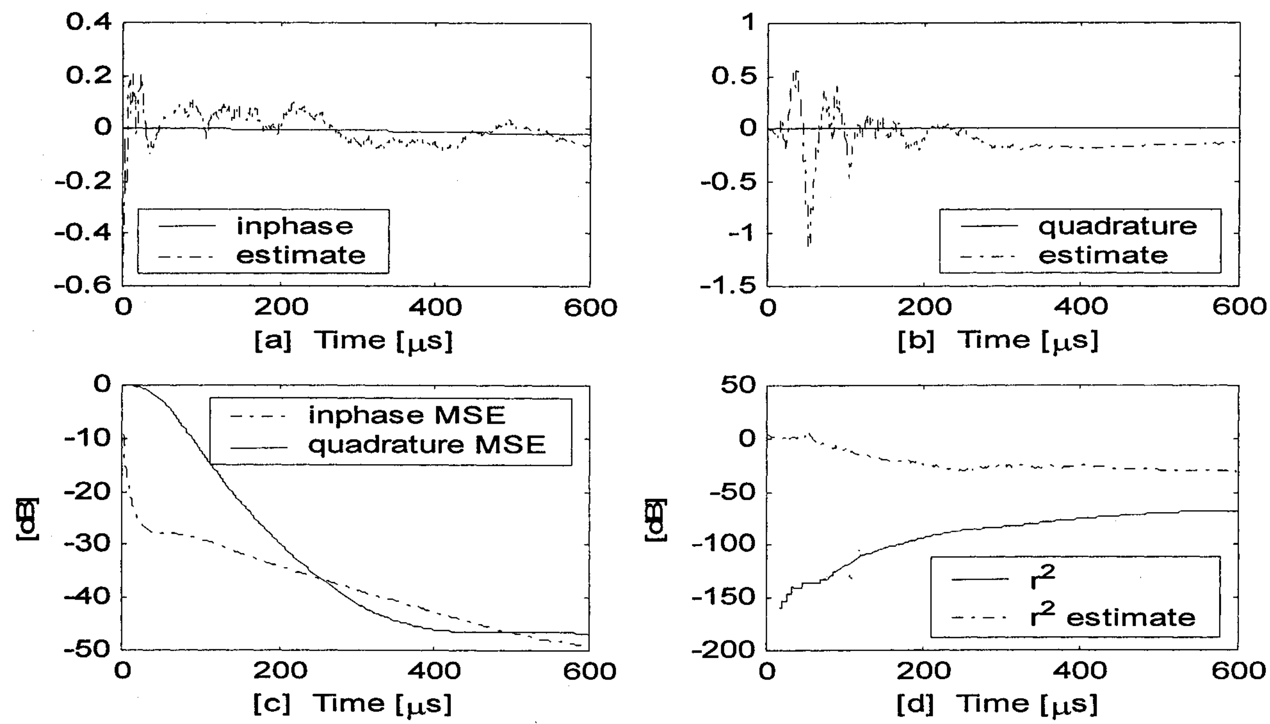

Figure 4.1.1: Flat Slow Rayleigh Fading Channel; $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$

- $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$, $T_s = 600\ \mu s$, $T_c = 9.89\ ms$

  Figure 4.1.1[a] shows the inphase and its least-square estimate. Clearly, tracking is achieved within the first 600 μs; Figure 4.1.1[b] shows the quadrature and its estimate.

Obviously, tracking is also achieved within the first 600 μs; Figure 4.1.1[c] shows the mean square errors of the inphase and quadrature in dB; Figure 4.1.1[d] shows the square envelope $r^2(t)$ and its estimate $\widehat{r^2}(t)$ in dB. Definitely, tracking is achieved within the first 600 μs with error.

(iii) Flat Fast Fading Channel Estimation. Let $f_c = 910\ MHz$, $v \geq 60\ km/h$, then $T_c \leq 9.89\ ms$. Choose $T_s = 250\ ms$, which implies $T_s >> T_c$, and thus the channel is fast fading. The results of the inphase, quadrature, square envelope estimations are shown as follows.

- $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$, $T_s = 250\ ms$, $T_c = 9.89\ ms$
  Figure 4.1.2(a),(b) displays the inphase, quadrature and their least-square estimate, respectively, while Figure 4.1.2(c) displays their mean-square errors. Figure 4.1.2(a),(b) show that tracking of $\{I(t), Q(t)\}_{t\geq 0}$ is possible with mean-square error of 0.049 for the inphase and 0.052 for the quadrature at 4 ms. Figure 4.1.2(d) display the square-envelope and its estimator. Unfortunately, tacking is not possible when the channel experiences deep fades at -3 dB. Next we provide simulation for 10 dB.

- $(SNR)_{dB} = 10\ dB$, $v = 60\ km/h$, $T_s = 250\ ms$, $T_c = 9.89\ ms$
  Figure 4.1.3[a],[b] shows the inphase, quadrature and their least-square estimate, respectively, while Figure 4.1.3[c] shows their mean-square errors. Clearly, these Figures illustrate the perfect tracking properties of the estimator within 3 ms, respectively, with a mean-square error of $-39.1\ dB$ for the inphase, and $-33.3\ dB$ for the quadrature. Figure 4.1.3[d] shows the square envelope and its estimate. It is obvious that tracking is possible at 10 dB even for fast fading channels.

- $(SNR)_{dB} = -3\ dB$, $v = 100\ km/h$, $T_s = 250\ ms$, $T_c = 5.9\ ms$
  Figure 4.1.4(a),(b) show the inphase, quadrature and their least-square estimates. Clearly, tracking is achieved but not well because of the low received signal to noise ratio and the high velocity; Figure 4.1.4(c) shows the mean-square errors; Figure 4.1.4(d) shows the square envelope $r^2(t)$ and its estimate $\widehat{r^2}(t)$. The estimator is not able to track the square envelope because of the low received signal-to-noise ratio and the large velocity.

- $(SNR)_{dB} = 10\ dB$, $v = 100\ km/h$, $T_s = 250\ ms$, $T_c = 5.9\ ms$
  Figure 4.1.5(a),(b) show the inphase, quadrature and their least-square estimates.

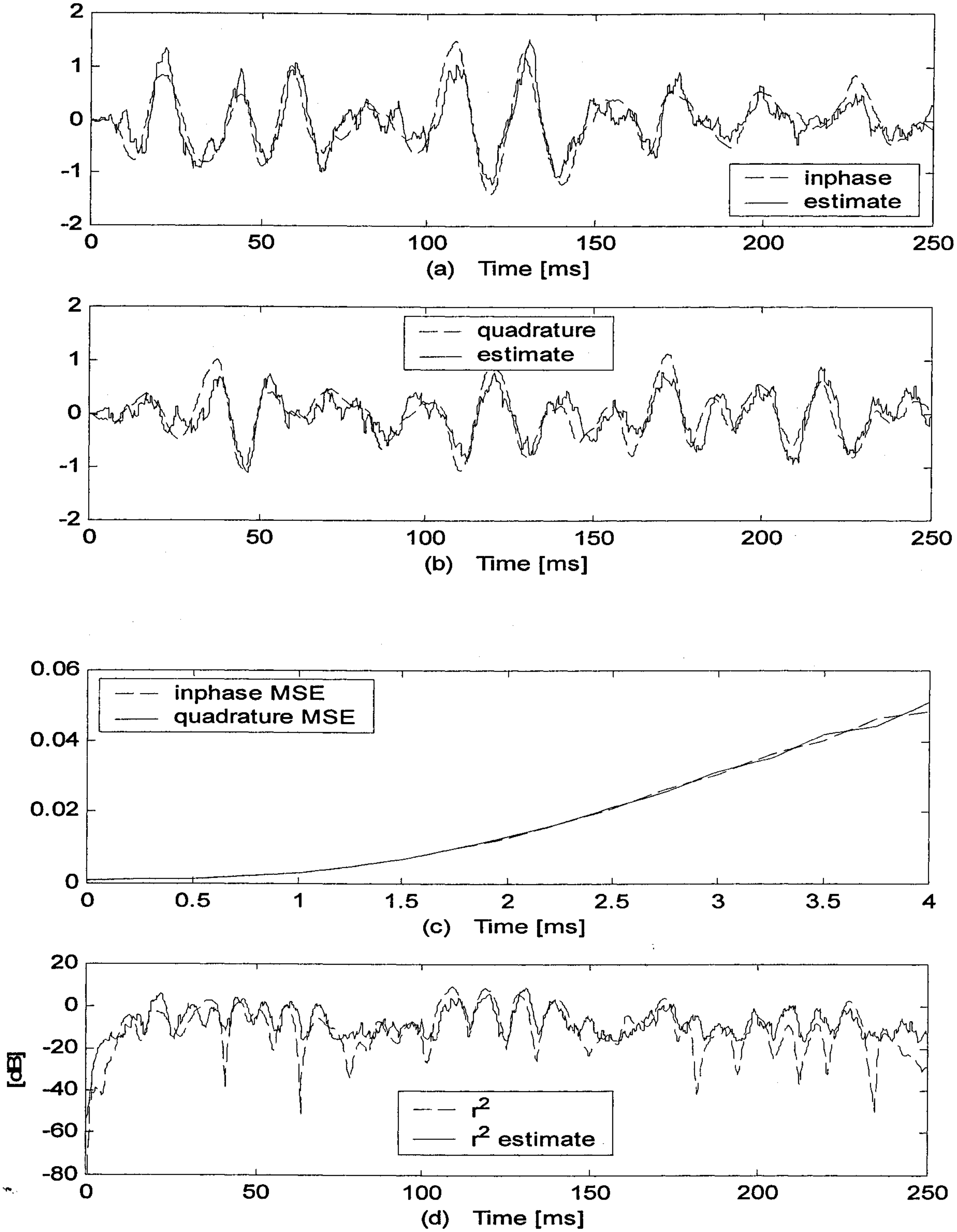

Figure 4.1.2: Flat Fast Rayleigh Fading Channel; $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$

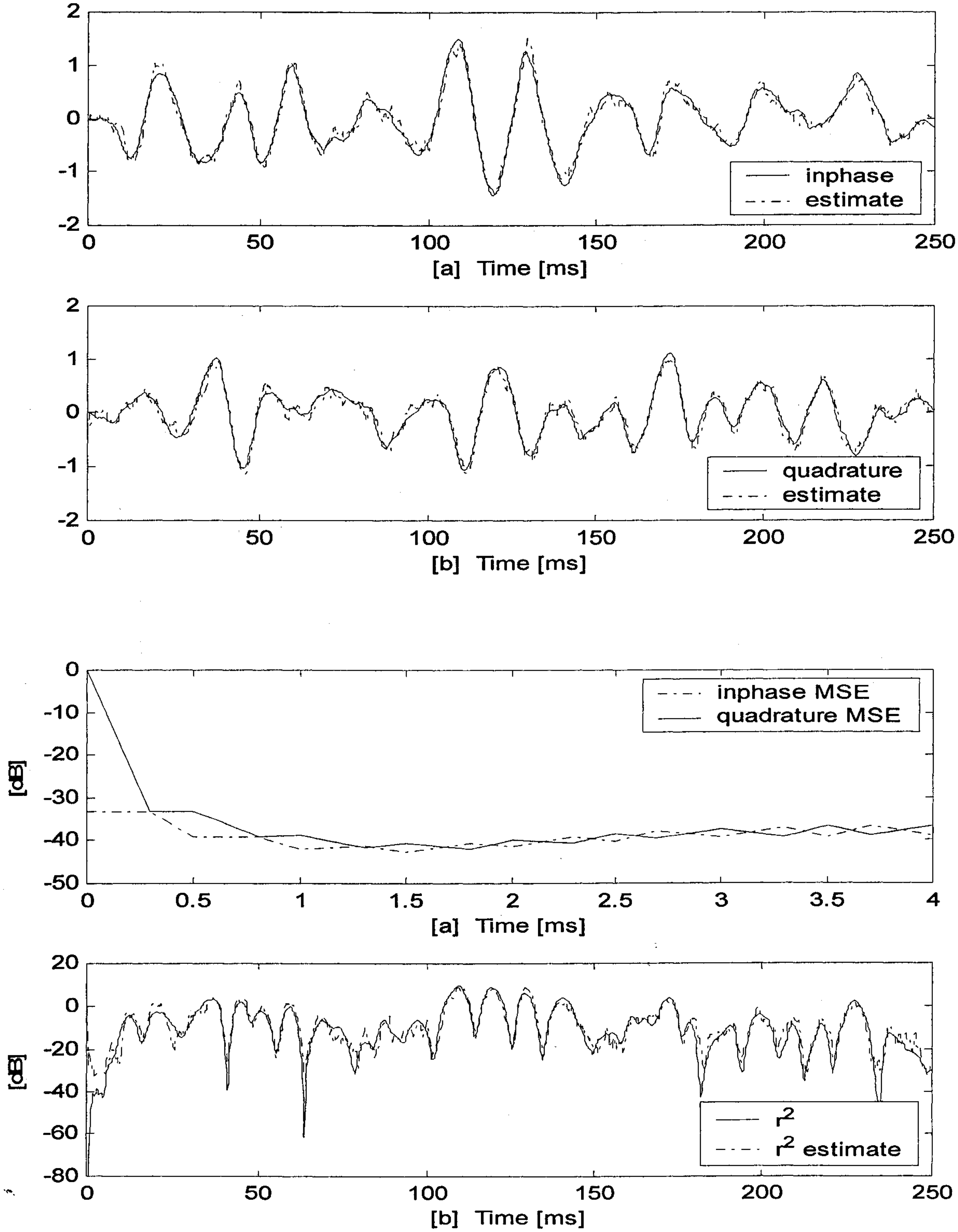

Figure 4.1.3: Flat Fast Rayleigh Fading Channel; $(SNR)_{dB} = 10\ dB$, $v = 60\ km/h$

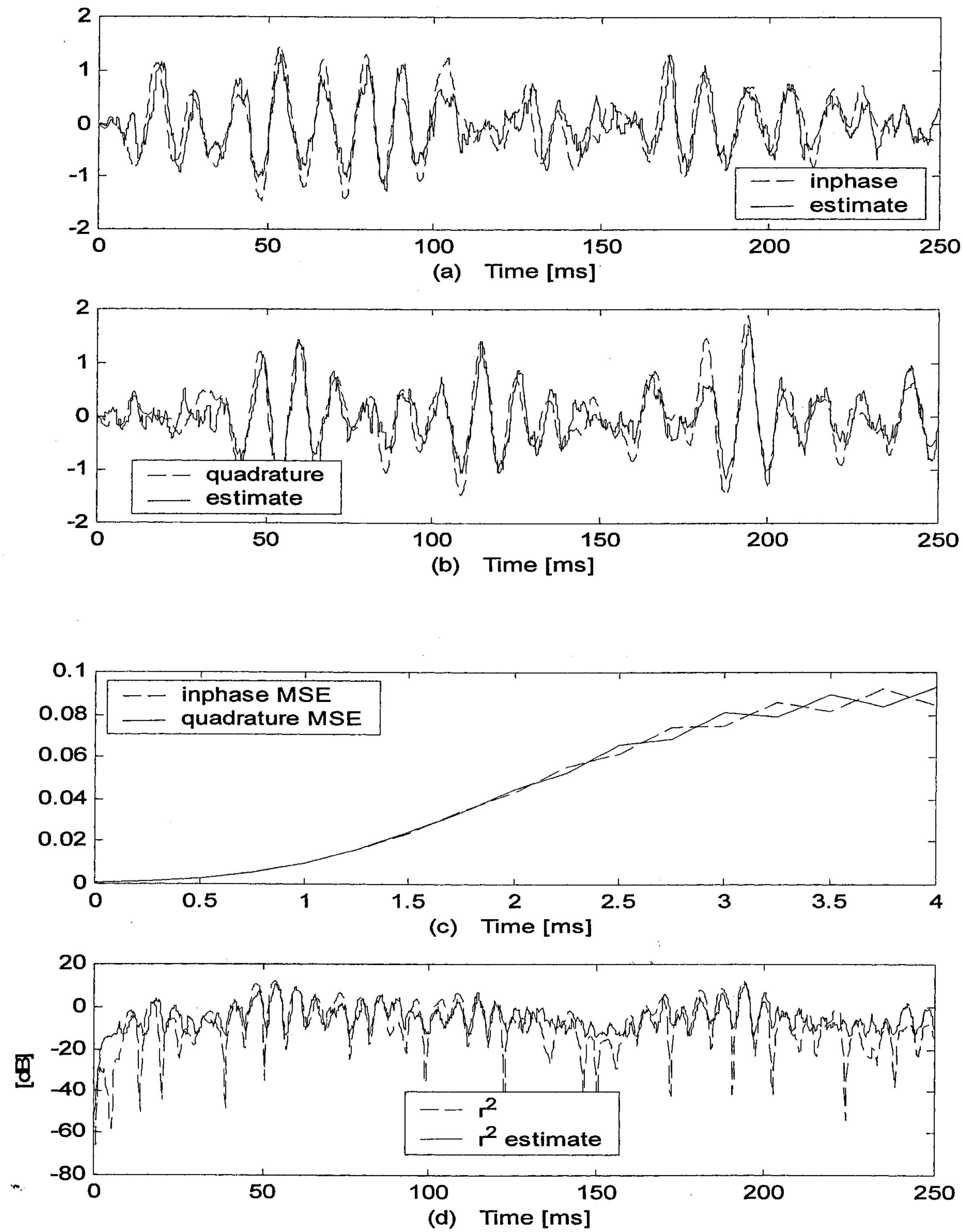

Figure 4.1.4: Flat Fast Rayleigh Fading Channel; $(SNR)_{dB} = -3\ dB$, $v = 100\ km/h$

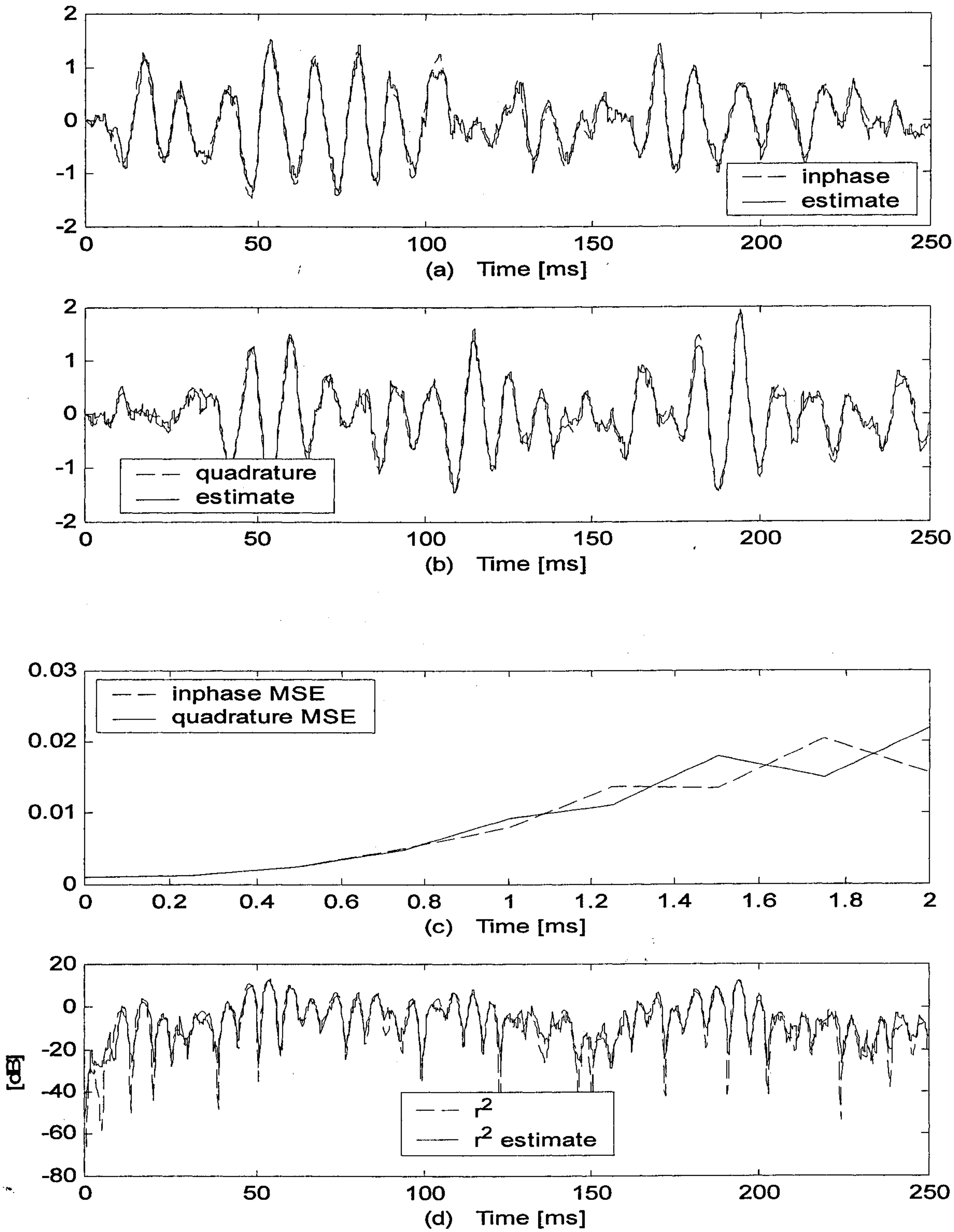

Figure 4.1.5: Flat Fast Rayleigh Fading Channel; $(SNR)_{dB} = 10\ dB$, $v = 100\ km/h$

Clearly, very good tracking is achieved, because of the high received signal-to-noise ratio; Figure 4.1.5(c) shows the mean square errors; Figure 4.1.5(d) shows the square envelope $r^2(t)$ and its estimate $\widehat{r^2}(t)$. The estimator can track the fading even at points which of deep fading because of the high received signal-to-noise ratio.

However, better tracking of the deep fade is expected if one employs $(SNR)_{dB} = 15 - 20\ dB$.

4.1.2 Ricean Channel

If $X(0) \sim N(0; \Sigma(0))$, then, theoretically, the state space channel model (3.1.4) has the property of Ricean fading nature.

(i) Ricean Channel Received Signal-to-Noise Ratio. The received SNR for a Ricean channel can be obtained from that of a Rayleigh channel by using (2.4.11), (2.4.12). Therefore, the received SNR is defined by

$$SNR^R \triangleq \frac{\frac{1}{T_s}E\left[\int_0^{T_s} \mid G(t)X(t) + G(t)h_s(t) \mid^2 dt\right]}{\frac{1}{T_s}E\left[\int_0^{T_s} \mid D(t)v(t) \mid^2 dt\right]},$$

where

$$h_s(t) = \begin{bmatrix} r_0 \cos(\omega_0 t + \theta_0) \\ 0 \\ r_0 \sin(\omega_0 t + \theta_0) \\ 0 \end{bmatrix}.$$

Similar to the Rayleigh channel, the average noiseless received signal power is

$$\begin{aligned} E_s^R &= \frac{1}{T_s}E\left[\int_0^{T_s} \mid G(t)X(t) + G(t)h_s(t) \mid^2 dt\right] \\ &= \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[X(t)X^{tr}(t)\big]C^{tr}(t)dt + \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[h_s(t)h_s^{tr}(t)\big]C^{tr}(t)dt \\ &= part_1(t) + part_2(t), \end{aligned}$$

where

$$part_1(t) = \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[X(t)X^{tr}(t)\big]C^{tr}(t)dt = \frac{1}{T_s}\int_0^{T_s} s^2(t)\gamma_{11}(t)dt = E_s,$$

$$part_2(t) = \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[h_s(t)h_s^{tr}(t)\big]C^{tr}(t)dt = \frac{s^2(t)r_0^2}{2} + \frac{s^2(t)r_0^2 \sin\Big(2(\omega_c + \omega_0)T_s + \theta_0\Big)}{4(\omega_c + \omega_0)T_s}.$$

Thus, the additive noises are given by

$$\begin{bmatrix} v_I \\ v_Q \end{bmatrix} = \sqrt{\frac{E_s^R}{10^{\frac{(SNR^R)_{dB}}{10}}}} \begin{bmatrix} n_I \\ n_Q \end{bmatrix},$$

the covariance matrix is given by

$$R_v^R = \frac{E_s^R}{10^{\frac{(SNR^R)_{dB}}{10}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the average additive noise power is given by

$$E_n^R = \frac{1}{T_s} E\left[\int_0^{T_s} \mid D(t)v(t) \mid^2 dt\right] = \frac{1}{T_s}\int_0^{T_s} D(t)E\left[v(t)v^{tr}(t)\right]D^{tr}(t)dt = \frac{E_s^R}{10^{\frac{(SNR^R)_{dB}}{10}}}.$$

(ii) Flat Slow Fading Channel Estimation. Let $f_c = 910\ MHz$, $v = 60\ km/h$, which implies $f_d = 50.556\ Hz$, $B_d = 2f_d = 101.112\ Hz$ and $T_c = 9.89\ ms$. Select $T_s = 600\ \mu s$, then we have $T_s << T_c$, and therefore the channel is slow fading. The results of the inphase, quadrature, square envelope estimation are shown on Figure 4.1.6 for $r_0 = 1$, $\omega_0 = 5^0$, $\theta_0 = 30^0$ and $(SNR)_{dB} = -3\ dB$.

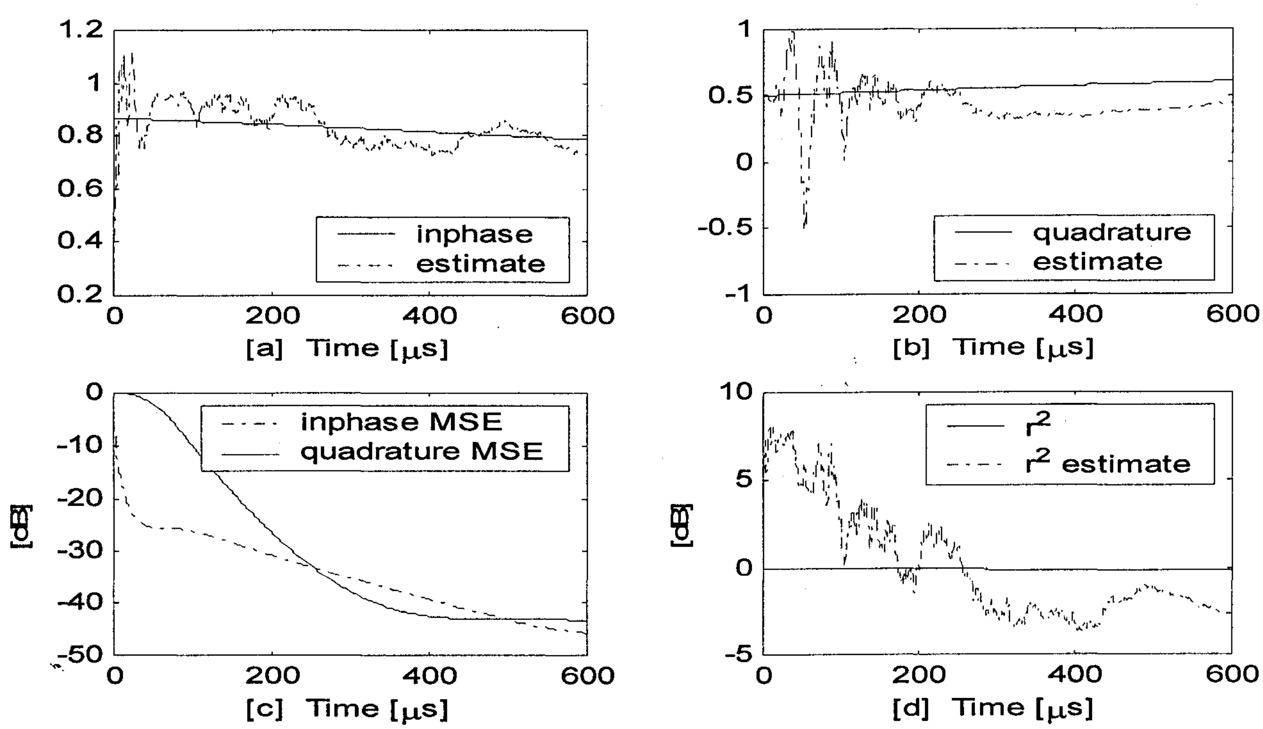

Figure 4.1.6: Flat Slow Ricean Fading Channel; $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$

- $(SNR)_{dB} = -3\ dB$, $v = 60\ km/h$, $T_s = 600\ \mu s$, $T_c = 9.89\ ms$, $r_0 = 1$
  Figure 4.1.6[a] shows the inphase and its least-square estimate. Clearly, tracking is achieved within the first 600 μs; Figure 4.1.6[b] shows the quadrature and its estimate. Obviously, tracking is also achieved within the first 600 μs; Figure 4.1.6[c] shows the mean-square errors of the inphase and quadrature in dB; Figure 4.1.6[d] shows the square envelope $r^2(t)$ and its estimate $\widehat{r^2}(t)$ in dB. Definitely, tracking is achieved within the first 600 μs.

(iii) Flat Fast Fading Channel Estimation. Let $f_c = 910\ MHz$, $v = 60\ km/h$, then $T_c = 9.89\ ms$. Choose $T_s = 250\ ms$, which implies $T_s >> T_c$, and thus the channel is fast fading. The results of the inphase, quadrature, square envelope estimation are shown on Figure 4.1.7 for $r_0 = 1$, $\omega_0 = 5^0$, $\theta_0 = 30^0$ and $(SNR)_{dB} = 10\ dB$.

- $(SNR)_{dB} = 10\ dB$, $v = 60\ km/h$, $T_s = 250\ ms$, $T_c = 9.89\ ms$, $r_0 = 1$
  Figure 4.1.7(a), (b) shows the inphase, quadrature and their least-square estimate, respectively. Clearly, trackings are achieved at $(SNR)_{dB} = 10\ dB$. While Figure 4.1.7(c) shows their mean square errors. Figure 4.1.7(d) shows the square envelope and its estimate. It is obvious that tracking is possible at $(SNR)_{dB} = 10\ dB$ even for fast Ricean fading channels.

4.2 Frequency-Selective Fading Channels

If we transmit a narrow pulse, thus a wideband signal, through a multipath fading channel, then the channel is frequency-selective in this case. The received signal model is given by (2.1.2), which consists of N multipath state space fading channel models given by (3.1.7). In this section, we employ the Kalman filter algorithm to produce the estimates of the inphase, quadrature and square envelope for each multipath component. First, we show how to determine the multipath components N, then we simulate the frequency-selective slow fading channel.

(i) Determination of the Multipath Components N**.** The multipath components N can be obtained by the formula $N = [T_m B_s] + 1$ with the deterministic time delay $\{\tau_i = iT_s\}_{i=1}^{N}$ [2]. The typical values of multipath delay spread T_m are listed in the table of Typical

[2]the formula can be found in [17], page 797

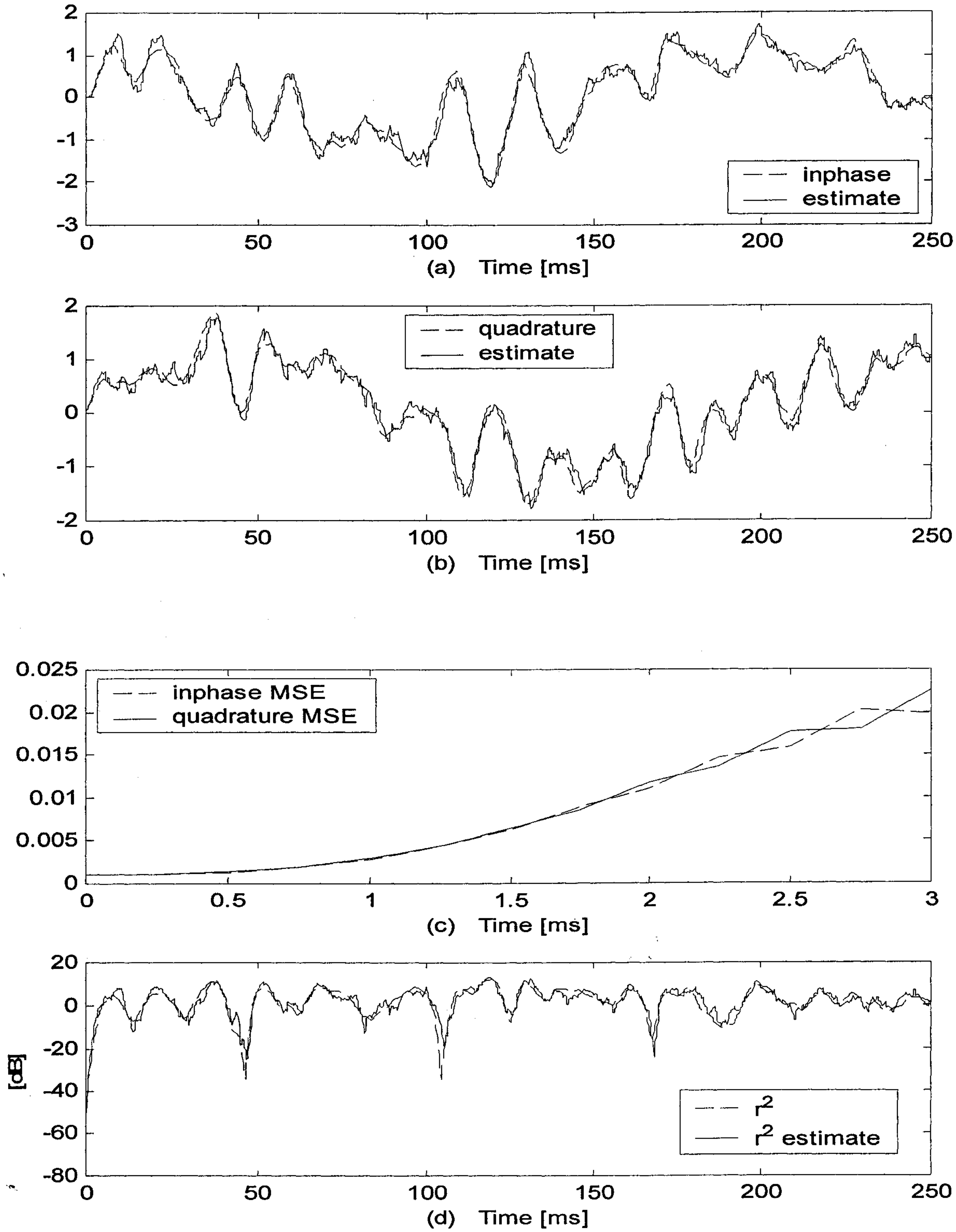

Figure 4.1.7: Flat Fast Ricean Fading Channel; $(SNR)_{dB} = 10\ dB$, $v = 60\ km/h$

Measured Values of RMS Delay Spread, which is reported in [12], page 162. Based on that table, we are able to compute the multipath components N. We choose $T_m = 2\ \mu s$ which corresponds to the New York city environment, carrier frequency is 910 MHz. Let the bandwidth of the transmitted signal be $B_s = 2.5\ MHz$, then the symbol duration (signaling interval) $T_s = 1/B_s = 0.4\ \mu s$. Thus, we get the multipath components $N = 6$, the time delay $\{\tau_i = iT_s\}_{i=1}^{6}$.

(ii) Received Signal-to-Noise Ratio. The received SNR is defined by

$$SNR \triangleq \frac{\frac{1}{T_m}E\left[\int_0^{T_m} \mid \sum_{i=1}^{6} G_i X_i \mid^2 dt\right]}{\frac{1}{T_m}E\left[\int_0^{T_m} \mid D(t)v(t) \mid^2 dt\right]}.$$

By simple computing, we obtain the average noiseless power of the received signal[3]

$$E_s = \frac{1}{T_m}E\left[\int_0^{T_m} \mid \sum_{i=1}^{6} G_i X_i \mid^2 dt\right] = \frac{1}{T_m}\int_0^{T_m} \sum_{i=1}^{6} s^2(t-\tau_i)(\gamma_{11}^i + d_{11}^i)dt,$$

where $\gamma_{11}^i = part_1(t)$ is the steady state variance of the inphase (quadrature) of the i^{th} multipath component, $d_{11}^1 = part_2(t)$ (see [13]), $d_{11}^i = 0$ for $i \neq 1$ and

$$s(t-\tau_i) = \begin{cases} \text{const} & \tau_i \leq t \leq \tau_i + T_s \\ 0 & \text{otherwise} \end{cases};$$

and the average additive noise power

$$E_v = \frac{1}{T_m}E\left[\int_0^{T_m} \mid D(t)v(t) \mid^2 dt\right] = \frac{1}{T_m}\int_0^{T_m} D(t)E[v(t)v^{tr}(t)]D^{tr}(t)dt = \sigma_v^2.$$

(iii) Frequency-Selective Slow Fading Channel Estimation. The estimations are done by using Simulink blocks, which is referred to (3.1.7), as shown in Figure 4.2.8. This multipath fading channel model consists of one Ricean and five Rayleigh's.

Next, we are going to simulate the estimators of the inphase, quadrature and square envelope for each path. Each path at the receiver corresponds to the same transmitted signal with a certain transport delay (T.D.). Let $f_c = 910\ MHz$, $v = 60\ km/h$, then $T_c = 9.89\ ms$. Thus, $T_s < T_m < T_c$. The multipath fading channel shows the property of slow fading nature. Suppose the transmitted signal is a pulse with duty cycle 25, which the pulse width is 0.1 μs. Then the received signal has total 6 pulses separated by $1/B_s$ during T_m. The results of the simulations are shown from Figure 4.2.9 to Figure 4.2.17 for $(SNR)_{dB} = 20\ dB$.

[3]see Appendix A.7.5 for details

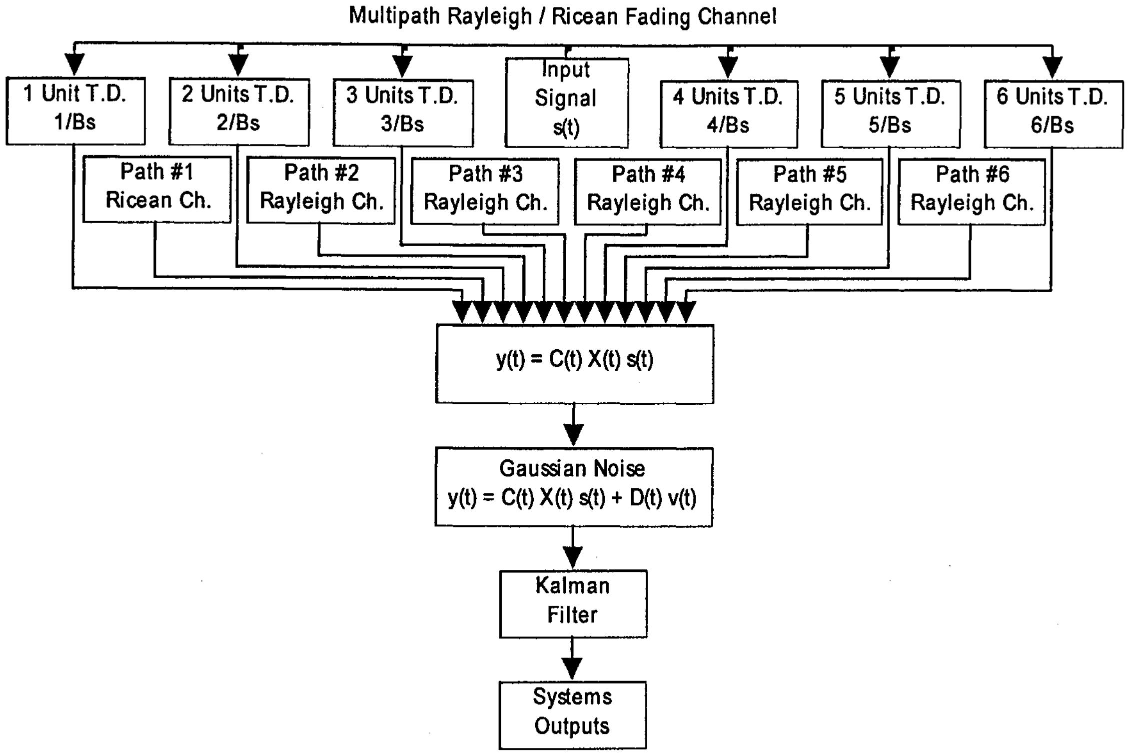

Figure 4.2.8: Multipath Fading Channel Estimator

Figure 4.2.9[a] shows the transmitted signal; Figure 4.2.9[b] displays the absolute value of the received noiseless signal $|y(t)| = |\sum_{i=1}^{6}\big(I_i(t)\cos(\omega_c t) - Q_i(t)\sin(\omega_c t)\big)s(t-\tau_i)|$, the multipath delay profile, which illustrates the amplitude of the received signal of each path goes down because of the longer distance and larger time delay. Path #1 is a Ricean channel, the steady state mean of the channel square envelop is 8.88 dB which corresponds to the channel model parameters $k = 2.197 \times 10^3$, $w_n = 322.1724\ rad/s$, $\zeta = 0.1691\ s^{-1}$ and the LOS component parameters $r_0 = 1$, $\omega_0 = 31.4\ Hz$, $\theta_0 = 30^0$. Path #2 is a Rayleigh channel, the steady state mean of the channel envelop is 10 dB which corresponds to the channel model parameters $k = 6.748 \times 10^3$, $w_n = 322.1724\ rad/s$, $\zeta = 0.1691\ s^{-1}$. The steady state means of the channel envelops from path #3 to path #6 are 15 dB, 20 dB, 25 dB and 30 dB, which correspond to the channel models parameters $(\{k, \omega_n, \zeta\})$ $\{1.2\times10^4, 322.1724, 0.1691\}$, $\{2.1339\times10^4, 322.1724, 0.1691\}$, $\{3.7947\times10^4, 322.1724, 0.1691\}$ and $\{6.748 \times 10^4, 322.1724, 0.1691\}$; Figure 4.2.9[c] shows the absolute value of the received noisy signal; Figure 4.2.9[d] displays $|y(t)|$ and its estimate $|\widehat{y}(t)| = |\sum_{i=1}^{6}\big(\widehat{I}_i(t)\cos(\omega_c t) - \widehat{Q}_i(t)\sin(\omega_c t)\big)s(t-\tau_i)|$.

Figure 4.2.10 shows the received signal of path #1, path #2, path #3 and their estimates.

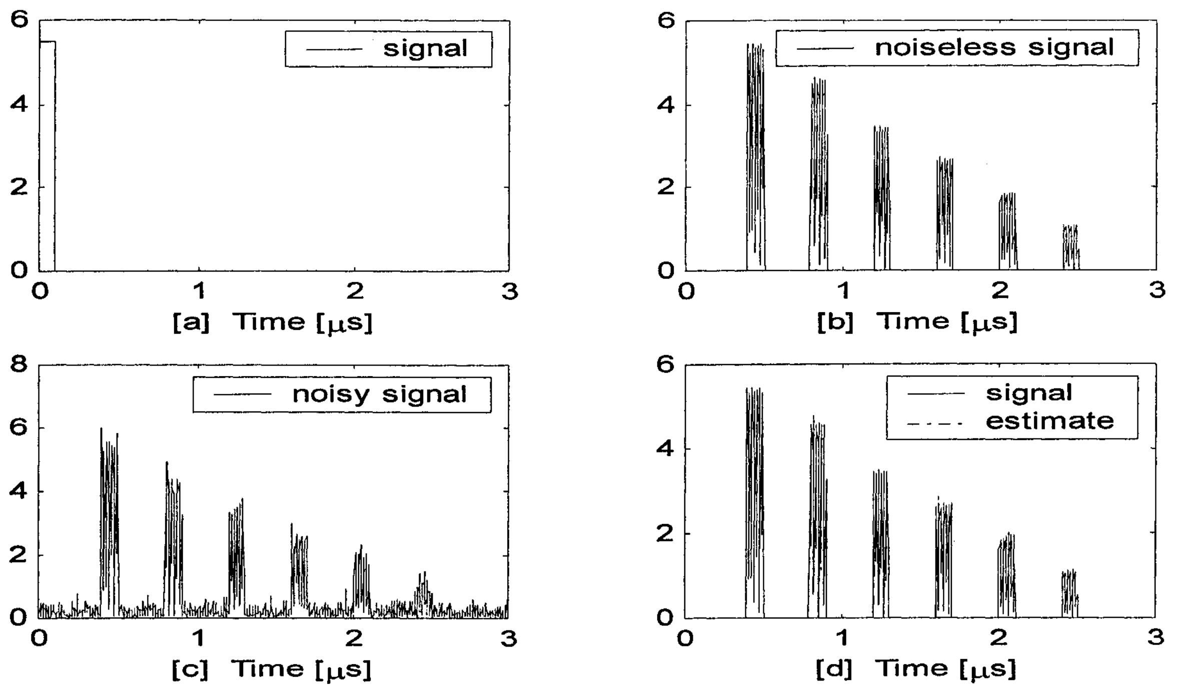

Figure 4.2.9: [a]: the transmitted signal; [b]: the multipath delay profile $\mid y(t) \mid$; [c]: the received signal with noise; [d]: $\mid y(t) \mid$ and its estimate

Figure 4.2.11 shows the received signal of path #4, path #5, path #6 and their estimates. Figure 4.2.10 and Figure 4.2.11 show that tracking is not achieved at the beginning of each path, but it is achieved as the time elapses.

Figure 4.2.12[a] shows the inphase and its least-square estimate. The initial value of the inphase is $r_0 \cos(\theta_0)$. Obviously, tracking is achieved from the time 0.4 μs to 0.5 μs, because there is a transmitted signal on $[\,0.4 \quad 0.5\,]$; Figure 4.2.12[b] displays the quadrature and its least-square estimate. The initial value of the quadrature is $r_0 \sin(\theta_0)$. Clearly, tracking is achieved between the time 0.4 μs and 0.5 μs; Figure 4.2.12[c] shows the mean square errors of the inphase and quadrature; Figure 4.2.12[d] displays the square envelop $r^2(t)$ and its estimate $\widehat{r^2}(t)$ in dB. Clearly, tracking is achieved between 0.4 μs and 0.5 μs, after the time 0.5 μs, the estimator has a predicted value.

Figure 4.2.13[a] shows the inphase and its least-square estimate. Obviously, the initial value of the inphase is 0.6, tracking is achieved between 0.8 μs and 0.9 μs; Figure 4.2.13[b] displays the quadrature and its least-square estimate. Clearly, the initial value of the quadrature is 0.6, tracking is achieved during the time 0.8 μs and 0.9 μs; Figure 4.2.13[c] shows the mean square errors of the inphase and quadrature; Figure 4.2.13[d] displays the square envelop $r^2(t)$ and its estimate $\widehat{r^2}(t)$ in dB. Clearly, tracking is achieved between 0.8 μs and 0.9 μs, after the time 0.9 μs, the estimator has a predicted value.

Path #3, path #4, path #5 and path #6 illustrate the same properties as path #1 and path #2. Tracking is achieved during the different time interval for each path because there is a different time delay signal over it.

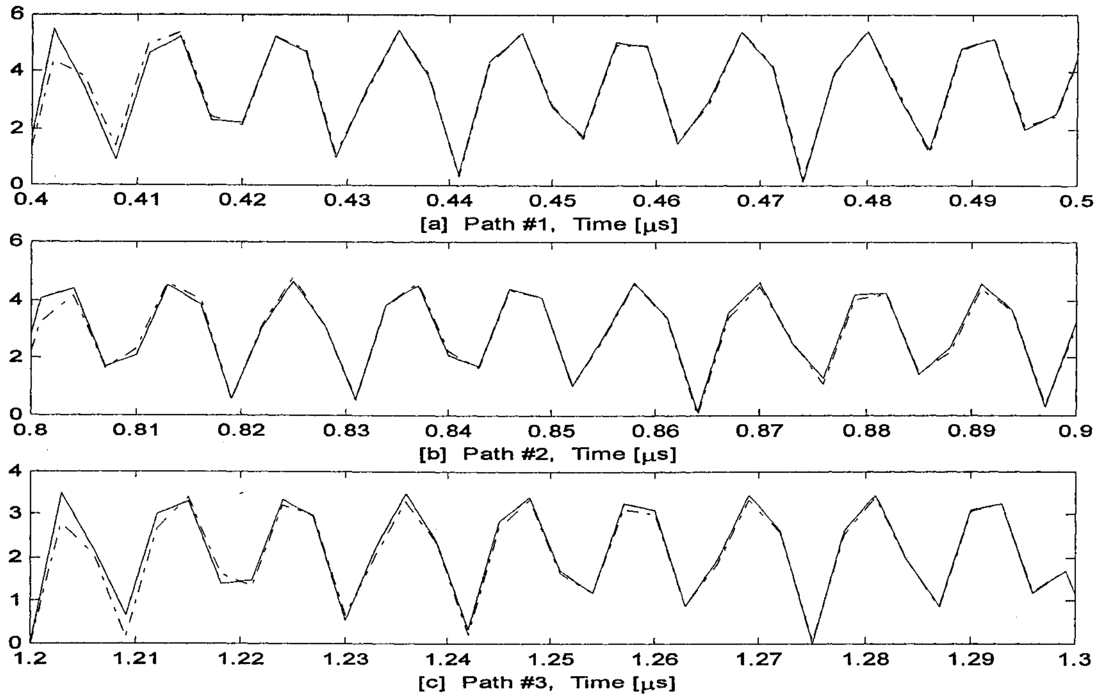

Figure 4.2.10: Zoom in Figure 4.2.9[d]: Path #1 - Path #3

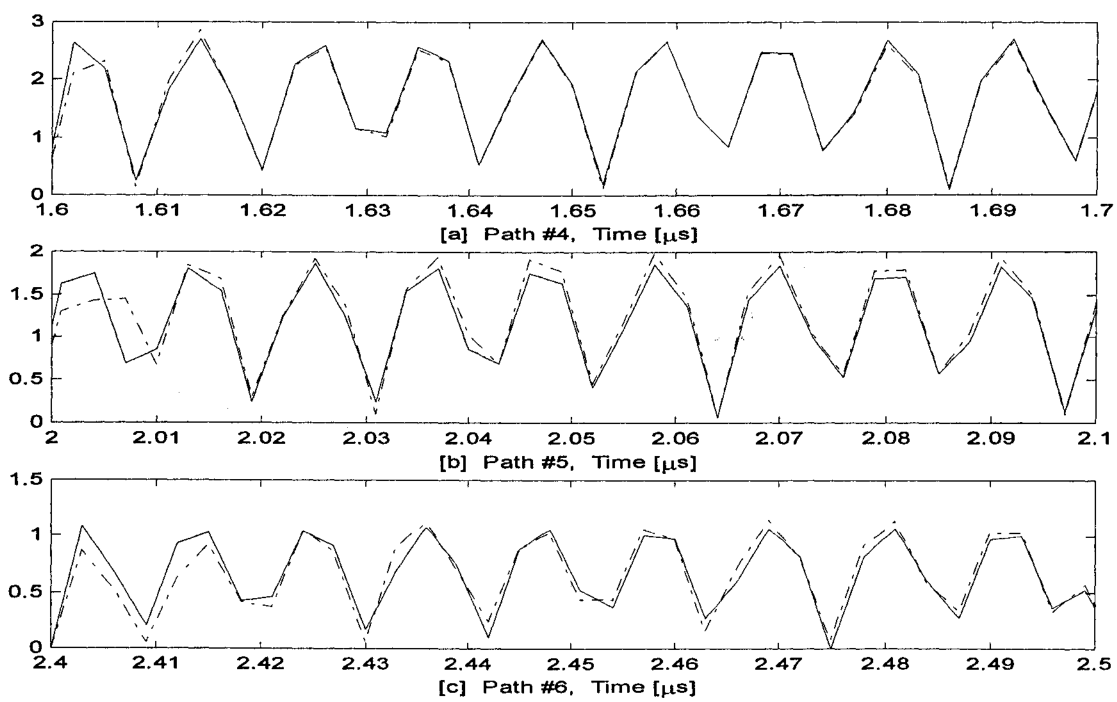

Figure 4.2.11: Zoom in Figure 4.2.9[d]: Path #4 - Path #6

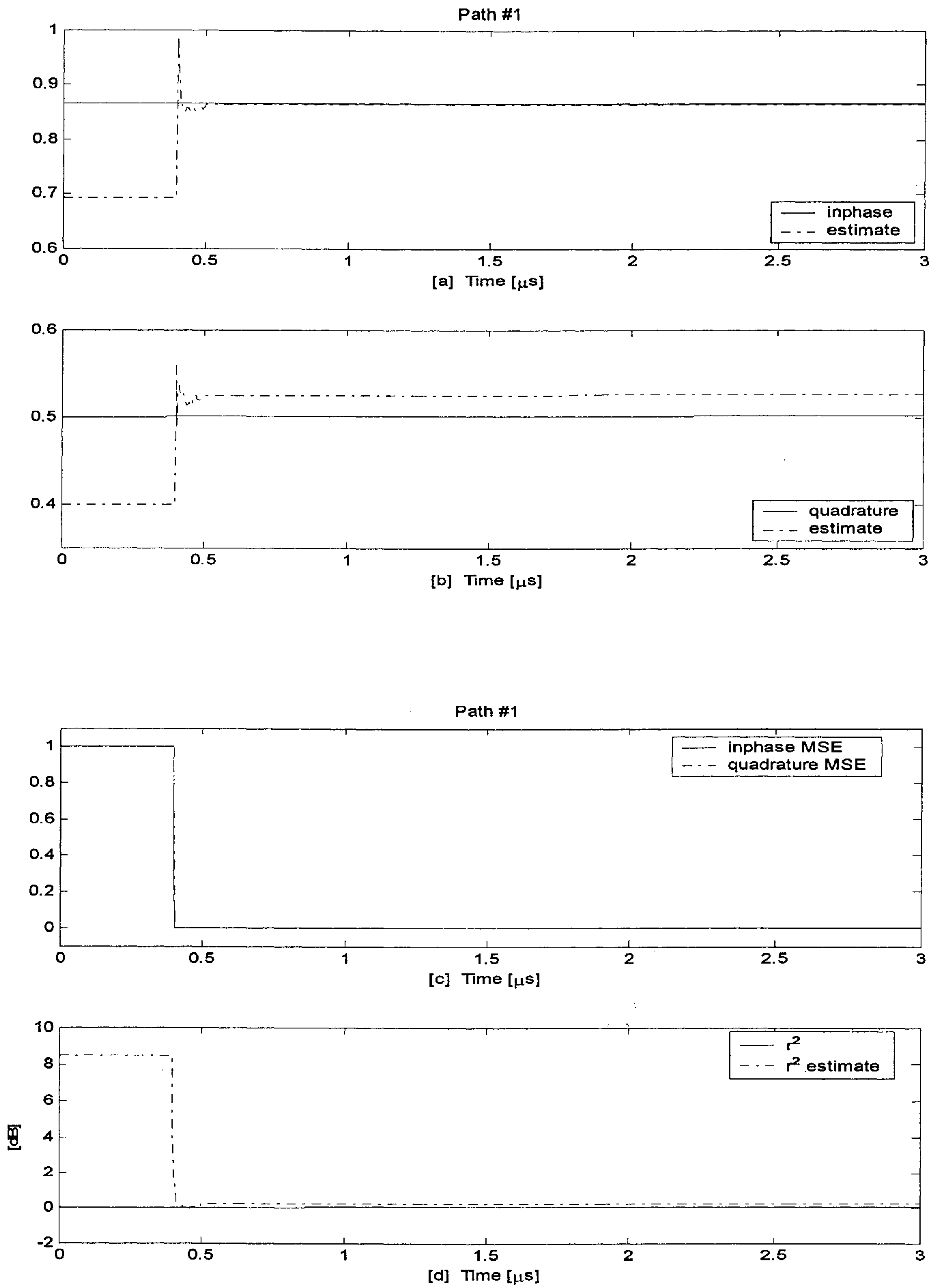

Figure 4.2.12: Multipath Component #1: Ricean Channel

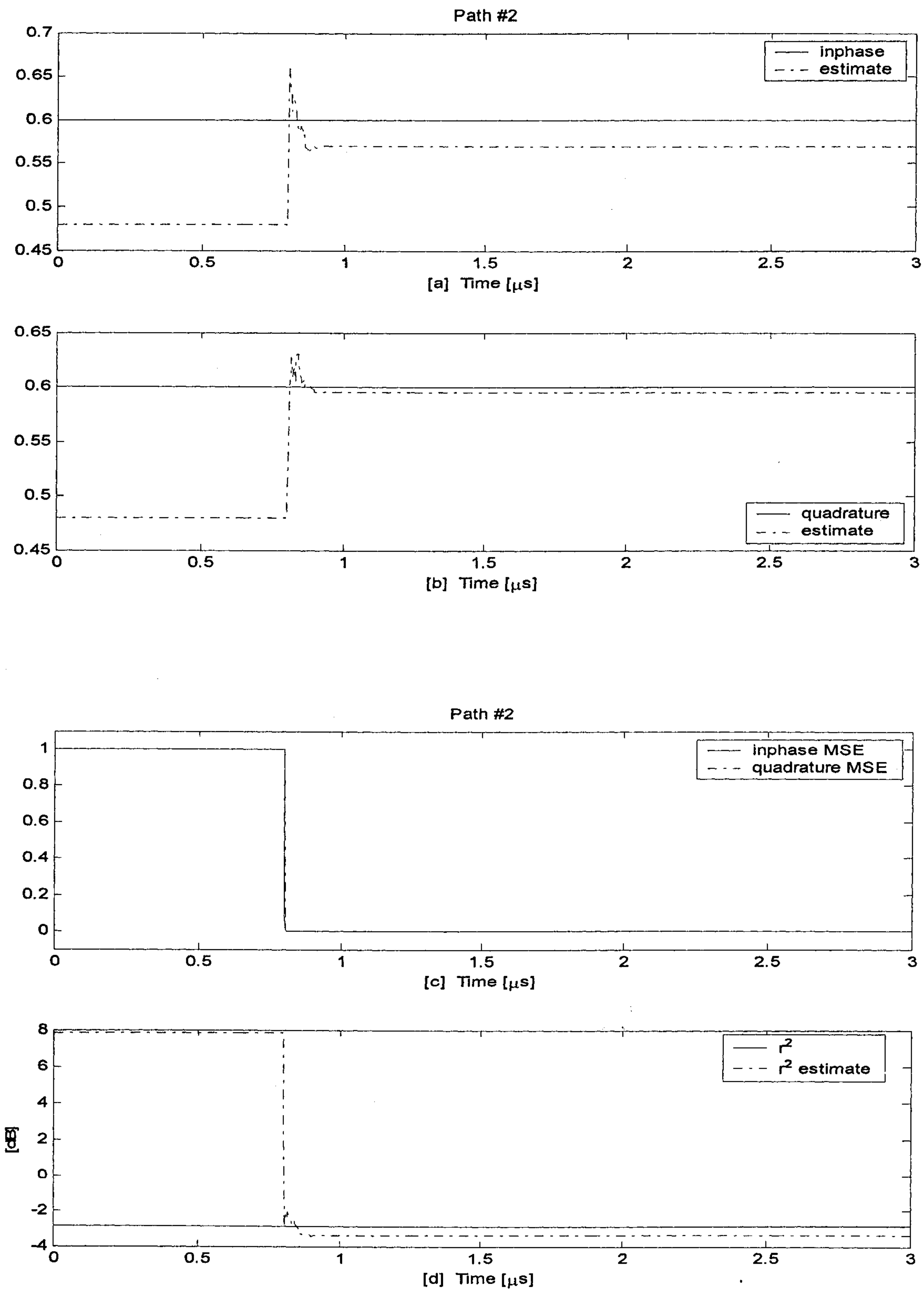

Figure 4.2.13: Multipath Component #2: Rayleigh Channel

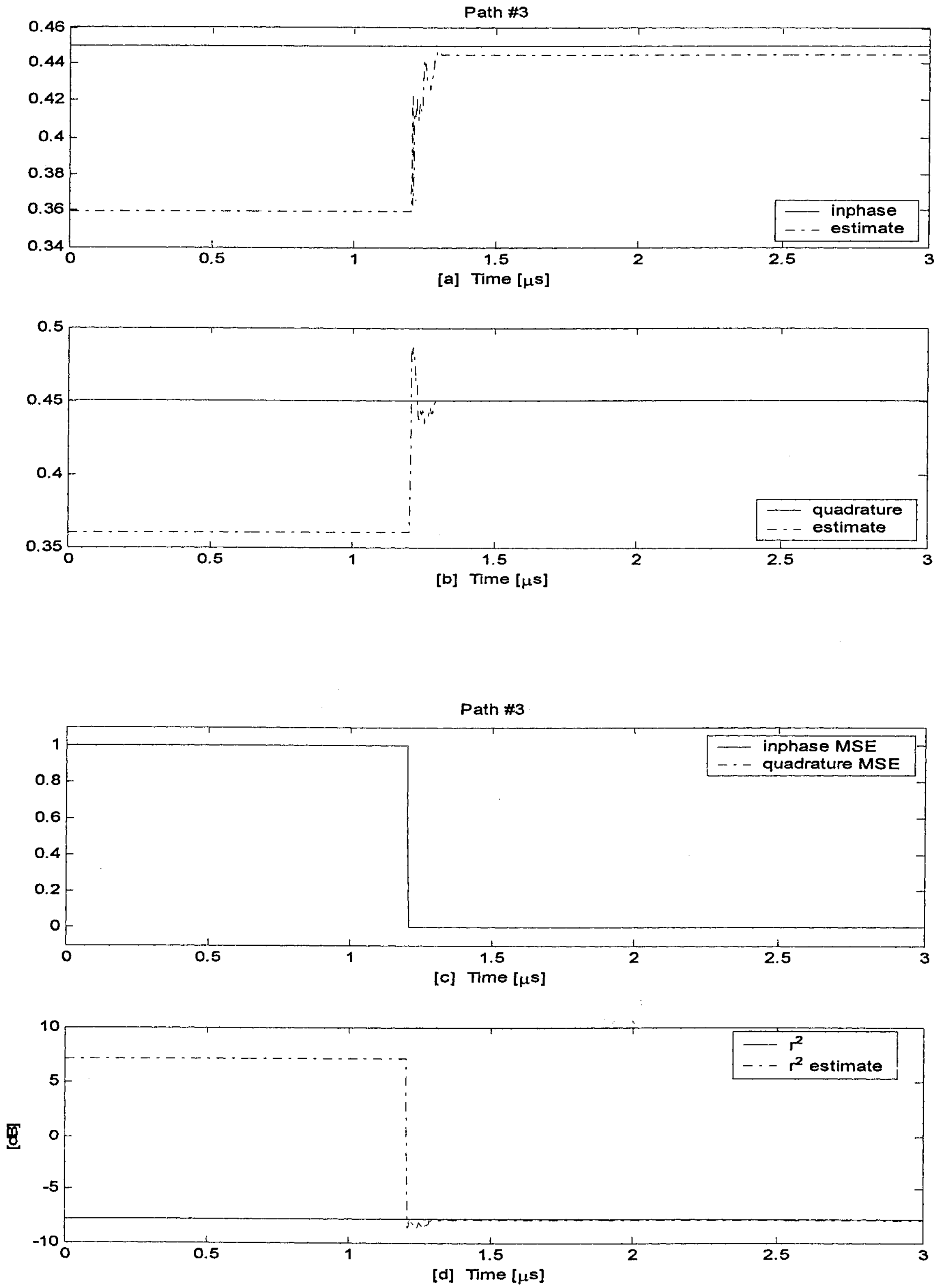

Figure 4.2.14: Multipath Component #3: Rayleigh Channel

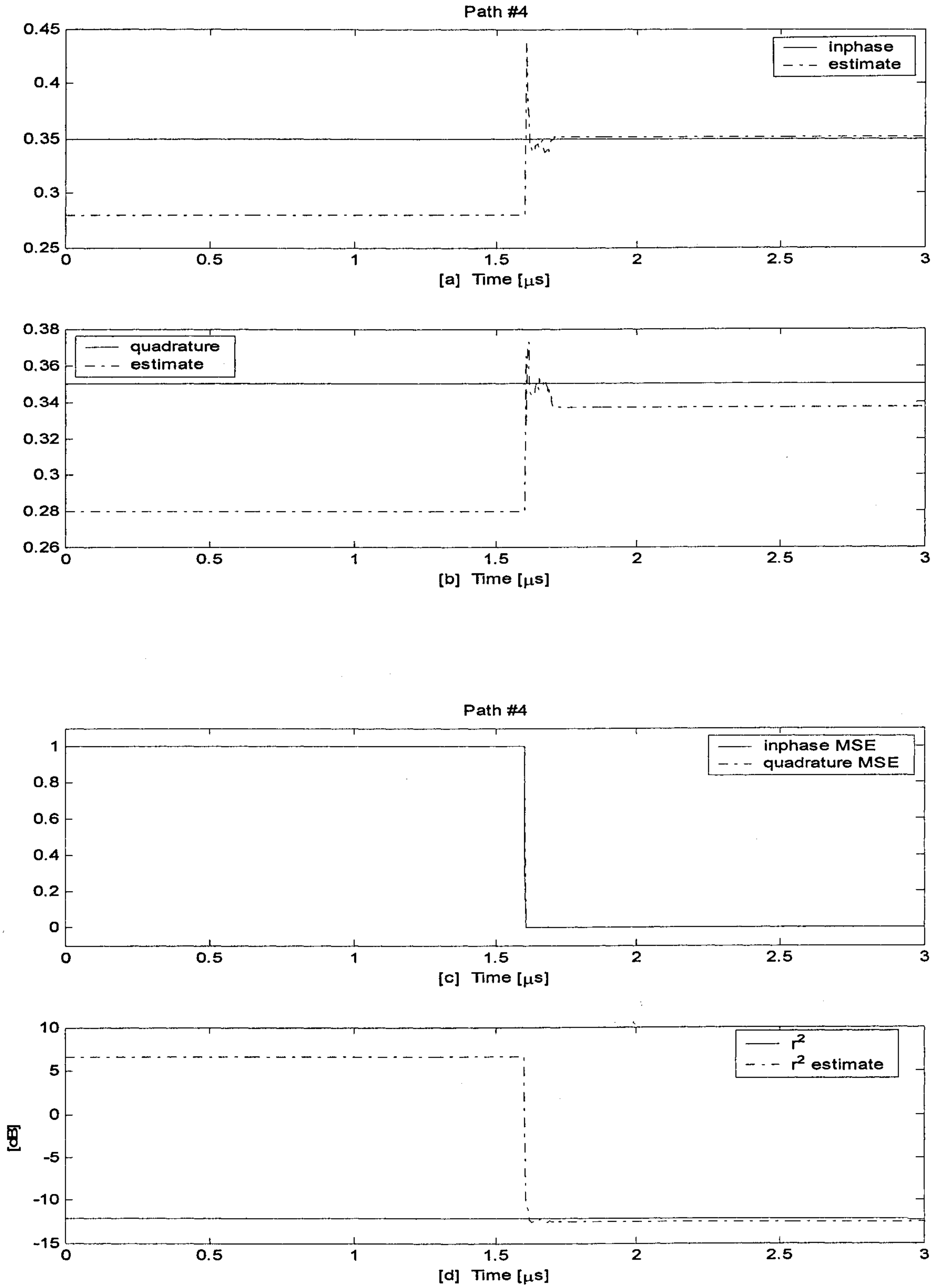

Figure 4.2.15: Multipath Component #4: Rayleigh Channel

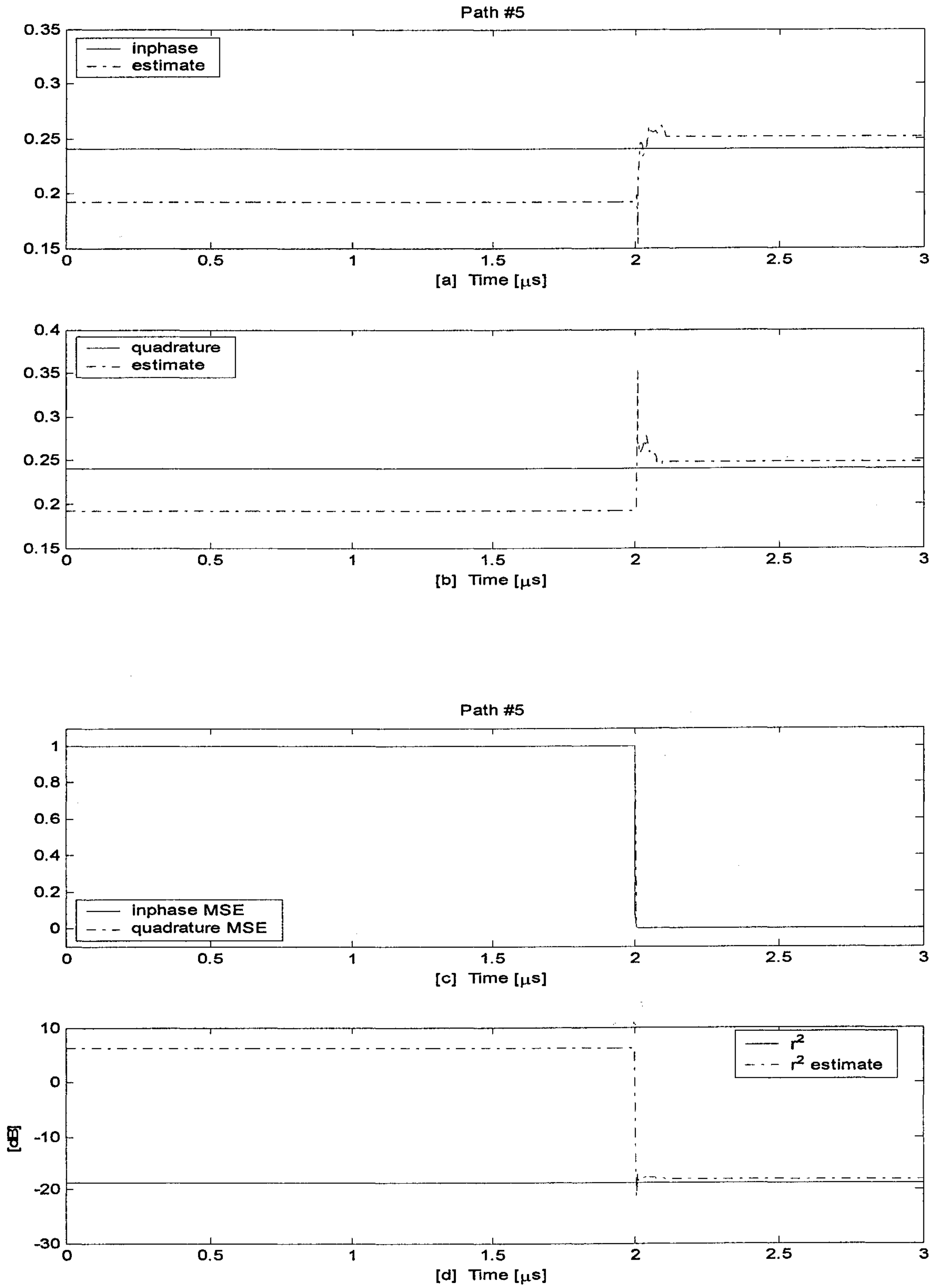

Figure 4.2.16: Multipath Component #5: Rayleigh Channel

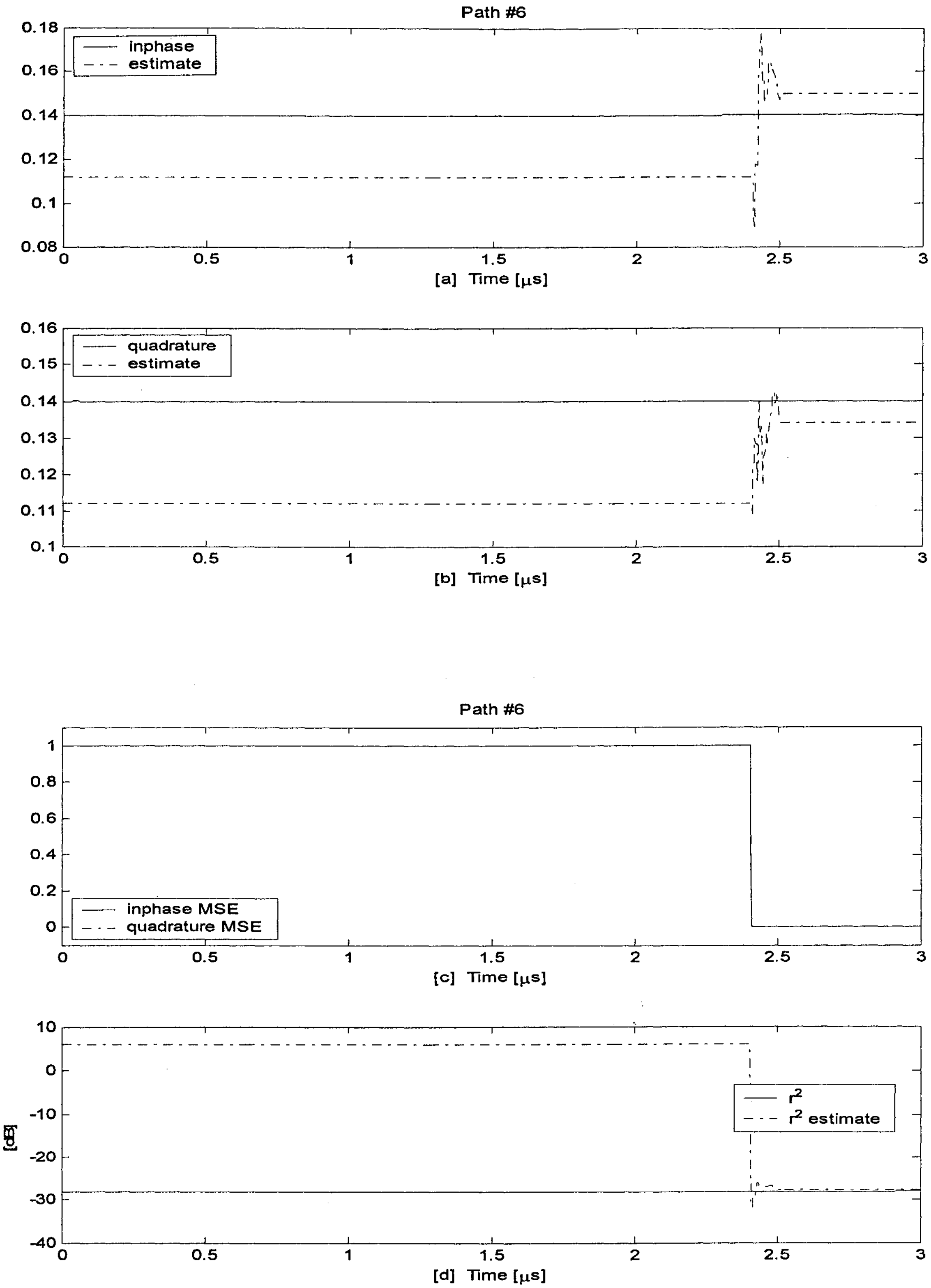

Figure 4.2.17: Multipath Component #6: Rayleigh Channel

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This thesis describes state space fading channel modeling and estimation using the stochastic differential equation. Models and estimates are obtained for inphase, quadrature and square envelope.

1. **Flat Slow Fading Channel, e.g.,** $T_s << T_c$. The state space estimates of $I(t)$, $Q(t)$, $r^2(t)$ show very good tracking at received signal-to-noise ratio as low and as -3 dB.

2. **Flat Fast Fading Channel, e.g.,** $T_s >> T_c$. The state space estimates of $I(t)$, $Q(t)$, $r^2(t)$ also show that tracking is possible, when the received signal-to-noise ratio is 10 dB.

3. **Frequency-Selective Slow Fading Channel.** Similar results are also obtained for frequency-selective slow fading channels, which the state space estimates of the inphase, quadrature, square envelope of each path show very good tracking with respect to mean square errors, when the received signal-to-noise ratio is 20 dB.

5.2 Future Work

The various investigations pursued in this thesis, provide encouraging results, opening the possibility of continuing work in various directions. Some directions for future research are the following,

1. **Systems Identification.** In this paper, all investigations have been done by assuming that the parameters of the state space channel models $\{A, B, C, D\}$ are known. One

can also estimate the channel states by first estimating the parameters $\{A, B, C, D\}$ from measurement data. An algorithm for computing these matrices is given in [7] and [9].

2. **High Order Approach.** One can use high order rational transfer functions to approximate the Doppler power spectral density function, and then employ order reduction techniques to find lower order models which are closer to the true channel power spectral density.

3. **Robustness of the Models.** The dynamic channel model can be described as a nominal one which its parameters are uncertain but having an range. Fixing the velocity v and the carrier frequency f_c, then v/λ is a constant. Thus, $S_D(f)$ is a function of β_m. Given a nominal transfer function $H_n(s)$ by choosing a β_m. For appropriate selecting a weight function $W_2(s)$ and a variable stable transfer function $\Delta(s)$ which satisfies $||\Delta(s)||_\infty \leq 1$, one can get the perturbed transfer function which is defined by

$$H_p(s) \triangleq \Big[1 + \Delta(s) W_2(s)\Big] H_n(s), \quad s = j\omega.$$

   Based on the above perturbed transfer function, one can generate the robust state space channel model and then simulate the multiplicative perturbed inphase, quadrature and square envelope by using the robust Kalman estimation techniques.

4. **Receiver Design.** Another interesting possibility for future work is to design receivers using optimal decision rules which do not assume knowledge of the channels. For the Bayes decision pay-off rule, the optimal decision rule can be computed explicitly by Mrosky to Kalman Filter estimate presented in this thesis.

5. **Optimal Coding Decoding.** One of the most challenging problem is to design encoder and decoder which do not assume knowledge of the channel state information.

Appendix A

Expressions and Derivations

A.1 Notes about Aulin [1]

A.1.1 Part 1: Expressions

Let

$$E(t) = \sum_{n=1}^{N} E_n(t),$$

where $E_n(t) \triangleq Re\{C_n e^{j\phi_n}\}$ at original coordinate, $\{C_n, \phi_n, \alpha_n, \beta_n\}_{n=1}^{N}$ are random variables, $\alpha_n \sim u(0, 2\pi)$ and $\phi_n \sim u(0, 2\pi)$. Here $u(0, 2\pi)$ is denoted the uniform distribution throughout 0 to 2π.

(i) At any receiving point (x_0, y_0, z_0). Suppose that the transmitted signal is $s(t) = Re\{e^{j\omega_c t}\}$, then the received signal is

$$E_n(t) = Re\Big\{C_n e^{j\omega_c(t-\tau_n)+j\phi_n}\Big\},$$

where Re denotes the real part of the expression in its following brace,

$$\omega_c \tau_n = 2\pi f_c \tau_n = 2\pi f_c \frac{d}{c} = 2\pi \frac{d}{\lambda},$$

λ is the wave length associated with the carrier frequency, c is the velocity of light and d is the distance between the transmitter and receiver. That is,

$$E_n(t) = C_n \cos(\omega_c t - \frac{2\pi}{\lambda} d + \phi_n).$$

Let the vector (x_0, y_0, z_0) project to the n^{th} wave path, then $d = x_0 \cos\alpha_n \cos\beta_n + y_0 \sin\alpha_n \cos\beta_n + z_0 \sin\beta_n$. Thus,

$$E_n(t) = C_n \cos[\omega_c t - \frac{2\pi}{\lambda}(x_0 \cos\alpha_n \cos\beta_n + y_0 \sin\alpha_n \cos\beta_n + z_0 \sin\beta_n) + \phi_n].$$

(ii) Let the point (x_0, y_0, z_0) move in the received field. Then, $(x_0, y_0, z_0) \longrightarrow (x_0 + vt\cos\gamma, y_0 + vt\sin\gamma, z_0)$, that is,

$$\begin{aligned}
E_n(t) &= C_n \cos\Big[\omega_c t - \frac{2\pi}{\lambda}\Big((x_0 + vt\cos\gamma)\cos\alpha_n\cos\beta_n + (y_0 + vt\sin\gamma)\sin\alpha_n\cos\beta_n \\
&\quad + z_0\sin\beta_n\Big) + \phi_n\Big] \\
&= C_n \cos\Big[\omega_c t - \frac{2\pi}{\lambda}(vt\cos\gamma\cos\alpha_n\cos\beta_n + vt\sin\gamma\sin\alpha_n\cos\beta_n \\
&\quad + x_0\cos\alpha_n\cos\beta_n + y_0\sin\alpha_n\cos\beta_n + z_0\sin\beta_n) + \phi_n\Big] \\
&= C_n \cos\Big[\omega_c t - \frac{2\pi vt}{\lambda}\cos(\gamma - \alpha_n)\cos\beta_n - \frac{2\pi z_0}{\lambda}\sin\beta_n \\
&\quad - \frac{2\pi\sqrt{x_0^2 + y_0^2}}{\lambda}\cos(\alpha_n + \xi)\cos\beta_n + \phi_n\Big] \\
&= C_n \cos(\omega_c t + \omega_n t + \theta_n) \\
&= C_n \cos(\omega_n t + \theta_n)\cos\omega_c t - C_n \sin(\omega_n t + \theta_n)\sin\omega_n t,
\end{aligned}$$

where

$$\begin{aligned}
\omega_n &= -\frac{2\pi v}{\lambda}\cos(\gamma - \alpha_n)\cos\beta_n, \\
\theta_n &= -\frac{2\pi z_0}{\lambda}\sin\beta_n - \frac{2\pi\sqrt{x_0^2 + y_0^2}}{\lambda}\cos(\alpha_n + \xi)\cos\beta_n + \phi_n, \quad n \in N.
\end{aligned}$$

Here $\xi \triangleq \tan^{-1}(y_0/x_0)$. Let

$$\begin{aligned}
T_c(t) &= \sum_{n=1}^{N} C_n \cos(\omega_n t + \theta_n), \\
T_s(t) &= \sum_{n=1}^{N} C_n \sin(\omega_n t + \theta_n).
\end{aligned}$$

Then we have

$$E(t) = T_c(t)\cos\omega_c t - T_s(t)\sin\omega_c t.$$

A.1.2 Part 2: Mean and Variance

(i) Take the expected value to $T_c(t)$. That is,

$$E[T_c(t)] = E[\sum_{n=1}^{N} C_n \cos(\omega_n t + \theta_n)]$$

$$= \sum_{n=1}^{N} E[C_n]E[\cos(\omega_n t + \theta_n)],$$

since

$$\begin{aligned} E[\cos(\omega_n t + \theta_n)] &= E[E[\cos(\omega_n t + \theta_n)|_{\omega_n = \omega}]] \\ &= E[\frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t + \theta_n) d\theta_n] \\ &= 0, \end{aligned}$$

then $E[T_c(t)] = 0$. In the same way, we get $E[T_s(t)] = 0$. Thus,

$$\begin{aligned} E[E(t)] &= E[T_c(t)\cos\omega_c t - T_s(t)\sin\omega_c t] \\ &= E[T_c(t)]\cos\omega_c t - E[T_s(t)]\sin\omega_c t \\ &= 0. \end{aligned}$$

(ii) Take the autocorrelation to $E(t)$. That is,

$$\begin{aligned} E[E(t)E(t+\tau)] &= E[(T_c(t)\cos\omega_c t - T_s(t)\sin\omega_c t) \\ &\quad \times (T_c(t+\tau)\cos\omega_c(t+\tau) - T_s(t+\tau)\sin\omega_c(t+\tau))] \\ &= E[T_c(t)T_c(t+\tau)]\cos\omega_c t\cos\omega_c(t+\tau) \\ &\quad - E[T_c(t)T_s(t+\tau)]\cos\omega_c t\sin\omega_c(t+\tau) \\ &\quad - E[T_s(t)T_c(t+\tau)]\cos\omega_c(t+\tau)\sin\omega_c t \\ &\quad + E[T_s(t)T_s(t+\tau)]\sin\omega_c t\sin\omega_c(t+\tau), \end{aligned}$$

since

$$\begin{aligned} E[T_c(t)T_c(t+\tau)] &= \sum_{n,m=1}^{N} E[C_n C_m \cos(\omega_m t + \theta_m)\cos(\omega_n(t+\tau) + \theta_n)] \\ &= \frac{1}{2}\sum_{n,m=1}^{N} E[C_n C_m \cos(\omega_m t + \omega_n(t+\tau) + \theta_m + \theta_n)] \\ &= +\frac{1}{2}\sum_{n,m=1}^{N} E[C_n C_m \cos(\omega_m t - \omega_n(t+\tau) + \theta_m - \theta_n)], \end{aligned}$$

where

$$\begin{aligned} (\theta_m + \theta_n) \mod 2\pi &\sim u(0, 2\pi), \\ (\theta_m - \theta_n) \mod 2\pi &\sim u(0, 2\pi). \end{aligned}$$

Thus, we have

$$\begin{aligned} E[C_nC_m\cos(\omega_m t+\omega_n(t+\tau)+\theta_m+\theta_n)] &= E[C_nC_mE(\cos(\omega_m t \\ &\quad + \omega_n(t+\tau)+\theta_m+\theta_n)|_{\omega_n,\omega_m})] \\ &= 0, \end{aligned}$$

and

$$E[C_nC_m\cos(\omega_m t-\omega_n(t+\tau)+\theta_m-\theta_n)] = \begin{cases} \sum_{n=1}^{N} E[C_n^2\cos\omega_n\tau], & m=n \\ 0, & m\neq n \end{cases}.$$

Then,

$$\begin{aligned} E[T_c(t)T_c(t+\tau)] &= \frac{1}{2}\sum_{n,m=1}^{N} E[C_nC_m\cos(\omega_m t-\omega_n(t+\tau)+\theta_m-\theta_n)] \\ &= \frac{1}{2}\sum_{n=1}^{N} E[C_n^2\cos\omega_n\tau] \\ &= \frac{1}{2}\sum_{n=1}^{N} E[C_n^2]E[\cos\omega_n\tau] \\ &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau], \end{aligned}$$

where $E[C_n^2]=E_0/N$. Using the same method, we get

$$\begin{aligned} E[T_s(t)T_s(t+\tau)] &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau] \\ E[T_c(t)T_s(t+\tau)] &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau] \\ E[T_c(t+\tau)T_s(t)] &= -\frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau], \end{aligned}$$

Thus, we have

$$\begin{aligned} E[E(t)E(t+\tau)] &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau]\cos\omega_c t\cos\omega_c(t+\tau) \\ &\quad + \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau]\sin\omega_c t\sin\omega_c(t+\tau) \\ &\quad + \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau]\sin\omega_c t\cos\omega_c(t+\tau) \\ &\quad - \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau]\cos\omega_c t\sin\omega_c(t+\tau) \end{aligned}$$

$$
\begin{aligned}
&= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau]\cos\omega_c\tau - \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau]\sin\omega_c\tau \\
&= E[T_c(t)T_c(t+\tau)]\cos\omega_c\tau - E[T_c(t)T_s(t+\tau)]\sin\omega_c\tau \\
&= a(\tau)\cos\omega_c\tau - c(\tau)\sin\omega_c\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
a(\tau) &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau], \\
c(\tau) &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau].
\end{aligned}
$$

Let $\omega_n = \omega$, then we have

$$
\begin{aligned}
c(\tau) &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\sin\omega_n\tau] \\
&= \frac{E_0}{2N} N E[\sin\omega\tau] \\
&= \frac{E_0}{2} E[\sin\omega\tau] \\
&= \frac{E_0}{2}\int \sin(\omega\tau)p_\omega d\omega \\
&= \frac{E_0}{2}\int\int_{\alpha=0}^{2\pi} \sin(-\frac{2\pi v\tau}{\lambda}\cos(\gamma-\alpha)\cos\beta)p_\alpha(\alpha)p_\beta(\beta)d\alpha d\beta \\
&= \frac{E_0}{2}\int\int_{\theta=0}^{2\pi} \frac{1}{2\pi}\sin(\frac{2\pi v\tau}{\lambda}\sin\theta\cos\beta)p_\beta(\beta)d\theta d\beta \\
&= 0.
\end{aligned}
$$

and

$$
\begin{aligned}
a(\tau) &= \frac{E_0}{2N}\sum_{n=1}^{N} E[\cos\omega_n\tau] \\
&= \frac{E_0}{2N} N E[\cos\omega\tau] \\
&= \frac{E_0}{2} E[\cos\omega\tau] \\
&= \frac{E_0}{2}\int \cos(\omega\tau)p_\omega d\omega \\
&= \frac{E_0}{2}\int\int_{\alpha=0}^{2\pi} \cos(-\frac{2\pi v\tau}{\lambda}\cos(\gamma-\alpha)\cos\beta)p_\alpha(\alpha)p_\beta(\beta)d\alpha d\beta \\
&= \frac{E_0}{2}\int\int_{\alpha=0}^{2\pi} \frac{1}{2\pi}\cos(\frac{2\pi v\tau}{\lambda}\sin(\frac{\pi}{2}-(\gamma-\alpha))\cos\beta)p_\beta(\beta)d\alpha d\beta.
\end{aligned}
$$

Let $\theta = \pi/2 - (\gamma - \alpha)$, then

$$
a(\tau) = \frac{E_0}{2}\int\int_{\theta=\frac{\pi}{2}-\gamma}^{\frac{5\pi}{2}-\gamma} \frac{1}{2\pi}\cos(\frac{2\pi v\tau}{\lambda}\sin\theta\cos\beta)p_\beta(\beta)d\theta d\beta
$$

$$
\begin{aligned}
&= \frac{E_0}{2}\int \frac{1}{2\pi} p_\beta(\beta) \int_{\theta=0}^{2\pi} \cos(\frac{2\pi v\tau}{\lambda}\sin\theta\cos\beta) d\theta d\beta \\
&= \frac{E_0}{2}\int J_0(\frac{2\pi v\tau}{\lambda}\cos\beta) p_\beta(\beta) d\beta,
\end{aligned}
$$

where $J_0(\cdot)$ is a Bessel function of the first kind of zero order. That is,

$$
J_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jz\sin\theta} d\theta.
$$

A.1.3 Part 3: Rayleigh Fading

(i) Suppose $p_\beta(\beta) = \delta(\beta)$, in this case, denote $a(\tau)$ by $a_0(\tau)$, then

$$
\begin{aligned}
a_0(\tau) &= a(\tau) \\
&= \frac{E_0}{2}\int J_0(\frac{2\pi v\tau}{\lambda}\cos\beta)\delta(\beta) d\beta \\
&= \frac{E_0}{2} J_0(\frac{2\pi v\tau}{\lambda}),
\end{aligned}
$$

Take the Fourier transform to $a_0(\tau)$, we get

$$
\begin{aligned}
A_0(f) &= \mathcal{F}[a_0(\tau)] \\
&= \frac{1}{2\pi}\frac{E_0}{2}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{j2\pi v\tau\sin\theta/\lambda} e^{-j2\pi f\tau} d\tau \\
&= \frac{E_0}{4\pi}\int_{\theta=-\pi}^{\pi}\int_{\tau=-\infty}^{\infty} e^{-j2\pi\tau(f - v\sin\theta/\lambda)} d\tau d\theta \\
&= \frac{E_0}{4\pi}\int_{\theta=-\pi}^{\pi}\delta(f - \frac{v}{\lambda}\sin\theta) d\theta.
\end{aligned}
$$

Let $f = \frac{v}{\lambda}\sin\theta$, then $\theta = \arcsin\frac{f\lambda}{v}$, $\mid \frac{f\lambda}{v} \mid = \sin\theta \leq 1$. Thus, we get

$$
d\theta = \frac{1}{\sqrt{1 - (\frac{f\lambda}{v})^2}}\frac{\lambda}{v} df, \quad \mid f \mid \leq \frac{v}{\lambda}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].
$$

Finally, we obtain

$$
\begin{aligned}
A_0(f) &= \frac{E_0}{4\pi}\int_{\theta=-\pi}^{\pi}\delta(f - \frac{v}{\lambda}\sin\theta) d\theta \\
&= \frac{E_0}{4\pi}\int_{f=-\frac{v}{\lambda}}^{\frac{v}{\lambda}}\delta(f - \frac{v}{\lambda}\sin\theta)\frac{1}{\sqrt{1 - (\frac{f\lambda}{v})^2}}\frac{\lambda}{v} df \\
&= \frac{E_0}{4\pi}\frac{\lambda}{v}\frac{1}{\sqrt{1 - (\frac{f\lambda}{v})^2}}.
\end{aligned}
$$

(ii) Suppose

$$
p_\beta(\beta) = \begin{cases} \frac{\cos\beta}{2\sin\beta_m}, & \mid\beta\mid \leq \mid\beta_m\mid \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases},
$$

then

$$a(\tau)=\frac{E_0}{2}\frac{1}{2\sin\beta_m}\int_{\beta=-\beta_m}^{\beta_m}J_0(\frac{2\pi v\tau}{\lambda}\cos\beta)\cos\beta d\beta.$$

Take the Fourier transform to $a(\tau)$, we get[1]

$$\begin{aligned}A(f) &= \mathcal{F}[a(\tau)]\\ &= \frac{E_0}{8\pi\sin\beta_m}\int_{\beta=-\beta_m}^{\beta_m}\int_{\theta=-\pi}^{\pi}\int_{\tau=-\infty}^{\infty}\cos\beta e^{j2\pi v\tau\cos\beta\sin\theta/\lambda}e^{-j2\pi f\tau}d\tau d\theta d\beta\\ &= \frac{E_0}{8\pi\sin\beta_m}\int_{\beta=-\beta_m}^{\beta_m}\int_{\theta=-\pi}^{\pi}\delta(f-\frac{v}{\lambda}\cos\beta\sin\theta)\cos\beta d\theta d\beta\\ &= \frac{E_0}{2\pi\sin\beta_m}\int_{\beta=0}^{\beta_m}\int_{\theta=-\pi/2}^{\pi/2}\delta(f-\frac{v}{\lambda}\cos\beta\sin\theta)\cos\beta d\theta d\beta.\end{aligned}$$

Let

$$\begin{cases} x = \sqrt{\frac{v}{\lambda}}\cos\beta \\ y = \sqrt{\frac{v}{\lambda}}\sin\theta \end{cases},$$

then

$$\begin{cases} \sin\beta = \sqrt{1-\frac{\lambda}{v}x^2} \\ \cos\theta = \sqrt{1-\frac{\lambda}{v}y^2} \end{cases} \Longrightarrow \begin{cases} |x| \le \sqrt{\frac{v}{\lambda}} \\ |y| \le \sqrt{\frac{v}{\lambda}} \end{cases} \Longrightarrow \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

Since Jacobian determinate is

$$\begin{aligned}\frac{d(x,y)}{d(\beta,\theta)} &= \begin{vmatrix} -\sqrt{\frac{v}{\lambda}}\sin\beta & 0 \\ 0 & \sqrt{\frac{v}{\lambda}}\cos\theta \end{vmatrix}\\ &= -\sqrt{\frac{v}{\lambda}-x^2}\sqrt{\frac{v}{\lambda}-y^2}.\end{aligned}$$

Then,

$$\begin{aligned}A(f) &= \frac{E_0}{2\pi\sin\beta_m}\int_{x=\sqrt{\frac{v}{\lambda}}\cos\beta_m}^{\sqrt{\frac{v}{\lambda}}}\int_{y=-\sqrt{\frac{v}{\lambda}}}^{\sqrt{\frac{v}{\lambda}}}\frac{\sqrt{\frac{\lambda}{v}}x\delta(f-xy)}{\sqrt{\frac{v}{\lambda}-x^2}\sqrt{\frac{v}{\lambda}-y^2}}dxdy\\ &= \begin{cases} 0, & |f|>\frac{v}{\lambda}; \\ \frac{E_0}{2}\frac{1}{2\sin\beta_m}\frac{\lambda}{v}, & \frac{v}{\lambda}\cos\beta_m \le |f| \le \frac{v}{\lambda}; \\ \frac{E_0}{2}\frac{1}{2\pi\sin\beta_m}\frac{\lambda}{v}\left(\frac{\pi}{2}-\arcsin\frac{2\cos^2\beta_m-1-(\frac{f\lambda}{v})^2}{1-(\frac{f\lambda}{v})^2}\right), & |f|<\frac{v}{\lambda}\cos\beta_m. \end{cases}\end{aligned}$$

[1]As the integration is symmetry, $\cos\alpha$ is an even function. Thus,

$$\int_{-\gamma}^{\gamma}\cos\alpha d\alpha = 2\int_0^{\gamma}\cos\alpha d\alpha = 2\int_{-\gamma/2}^{\gamma/2}\cos\alpha d\alpha.$$

(iii) The envelope $R(t)$ and the phase $\Phi(t)$ are defined as follows

$$\begin{cases} T_c(t) &= R(t)\cos\Phi(t) \\ T_s(t) &= R(t)\sin\Phi(t) \end{cases},$$

then $R^2(t) = T_c^2(t) + T_s^2(t)$. Since $E[T_c(t)] = E[T_s(t)] = 0$, $E[T_c(t)T_c(t+\tau)] = a(\tau)$, $E[T_s(t)T_s(t+\tau)] = a(\tau)$, thus the second moment (variance) $a(0) = E[T_c^2(t)] = E[T_s^2(t)]$.

Let $y = T_c^2(t) + T_s^2(t)$, then y is chi-square distribution with two degrees of freedom. The pdf is

$$P_Y(y) = \frac{1}{2a(0)} e^{-y/2a(0)}, \quad y \geq 0.$$

Then

$$\int_0^\infty P_Y(y)dy = \int_0^\infty \frac{1}{2a(0)} e^{-y/2a(0)} dy = 1.$$

Let $R(t) = \sqrt{T_c^2(t) + T_s^2(t)} = \sqrt{y}$, then $y = R^2(t)$, $dy = 2R(t)dR(t)$. Integration the pdf with respect to y gives

$$\int_0^\infty P_Y(y)dy = \int_0^\infty \frac{1}{2a(0)} e^{-y/2a(0)} dy = \int_0^\infty \frac{R}{a(0)} e^{-R^2/2a(0)} dR = 1.$$

Thus, the pdf of Rayleigh distribution is

$$P_R(r) = \frac{r}{a(0)} e^{-r^2/2a(0)}, \quad r \geq 0.$$

And the mean of the envelope is

$$\begin{aligned}
E[R(t)] &= \int_0^\infty r P_R(r) dr \\
&= \int_0^\infty \frac{r^2}{a(0)} e^{-r^2/2a(0)} dr \\
&= \int_0^\infty -re^{-r^2/2a(0)} d(-r^2/2a(0)) \\
&= -re^{-r^2/2a(0)}\Big|_0^\infty + \int_0^\infty e^{-r^2/2a(0)} dr \\
&= \int_0^\infty e^{-r^2/2a(0)} dr \\
&= \sqrt{2\pi a(0)} \int_0^\infty \frac{1}{\sqrt{2\pi a(0)}} e^{-r^2/2a(0)} dr \\
&= \frac{1}{2}\sqrt{2\pi a(0)} \\
&= \sqrt{\frac{\pi a(0)}{2}}.
\end{aligned}$$

A.1.4 Part 4: Ricean Fading

For the specular component with the parameters $\{\alpha_0, \beta_0, \phi_0, \sqrt{E_d}\}$, we have the envelope $R_r(t)$, which is Ricean distribution[2],

$$P_{R_r}(r) = \frac{r}{a(0)} e^{-(r^2+E_d)/2a(0)} I_0\Big(r\frac{\sqrt{E_d}}{a(0)}\Big), \quad r \geq 0,$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind.

A.2 Autocorrelation Function and PSD

(i) Consider a wide-sense stationary input signal $x(t)$ which passing through a linear time-invariant system having impulse response $h(t)$. The output signal $y(t)$ is given by

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

in time domain, which has the format

$$Y(f) = H(f)X(f)$$

in frequency domain. Here $Y(f)$, $X(f)$, $H(f)$ are the Fourier transforms of the output signal, input signal, impulse response. The autocorrelation is given by

$$\begin{aligned}
R_{yy}(\tau) &\triangleq E[y(t+\tau)y^*(t)] \\
&= E\Big[\int_{-\infty}^{\infty} h(\beta)x(t+\tau-\beta)d\beta \int_{-\infty}^{\infty} h^*(\gamma)x^*(t-\gamma)d\gamma\Big] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\gamma)h(\beta)E\Big[x^*(t-\gamma)x(t+\tau-\beta)\Big]d\gamma d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\gamma)h(\beta)R_{xx}(\tau+\gamma-\beta)d\gamma d\beta \\
&= \int_{-\infty}^{\infty} \Big(R_{xx}(\tau+\gamma) * h(\tau+\gamma)\Big)h^*(\gamma)d\gamma.
\end{aligned}$$

Let $\gamma = -t$, then

$$\begin{aligned}
R_{yy}(\tau) &= \int_{-\infty}^{\infty} \Big(R_{xx}(\tau+\gamma) * h(\tau+\gamma)\Big)h^*(\gamma)d\gamma \\
&= \int_{-\infty}^{\infty} \Big(R_{xx}(\tau-t) * h(\tau-t)\Big)h^*(-t)dt \\
&= R_{xx}(\tau) * h(\tau) * h^*(-\tau).
\end{aligned}$$

[2]there are two ways to find the pdf, one is on page 43 of [17], the other is like the same way of finding the Rayleigh.

(ii) Take the Fourier transform to $R_{yy}(\tau)$, we obtain

$$\begin{aligned} S_y(f) &\triangleq \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\gamma)h(\beta)R_{xx}(\tau+\gamma-\beta)e^{-j2\pi f\tau}d\gamma d\beta d\tau. \end{aligned}$$

Let $u = \tau+\gamma-\beta$, then

$$\begin{aligned} S_y(f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\gamma)h(\beta)R_{xx}(\tau+\gamma-\beta)e^{-j2\pi f\tau}d\gamma d\beta d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\gamma)h(\beta)R_{xx}(u)e^{-j2\pi f(u+\beta-\gamma)}d\gamma d\beta du \\ &= \int_{-\infty}^{\infty} h^*(-\gamma)e^{-j2\pi f(-\gamma)}d(-\gamma)\int_{-\infty}^{\infty} h(\beta)e^{-j2\pi f\beta}d\beta\int_{-\infty}^{\infty} R_{xx}(u)e^{-j2\pi fu}du \\ &= H^*(-f)H(f)S_x(f) \\ &= \mid H(f)\mid^2 S_x(f). \end{aligned}$$

A.3 Approximation DPSD by $H(s)$

A.3.1 Approximation $H(s)$

Let the second-order rational transfer function be

$$H(s) = \frac{k}{s^2+2\zeta\omega_n s+\omega_n^2},$$

where $s = j\omega$. Then

$$H(j\omega) = \frac{k}{\omega_n^2-\omega^2+2\zeta j\omega_n\omega}.$$

Thus

$$\begin{aligned} \mid H(j\omega)\mid &= \sqrt{H(j\omega)H(-j\omega)} && \text{(A.3.1)} \\ &= \frac{k}{\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}}. && \text{(A.3.2)} \end{aligned}$$

Define

$$f_{max} \triangleq f_d\cos(\beta_m),$$

$$\sqrt{S_D(0)} \triangleq \mid H(j\omega)\mid$$

$$= \frac{k}{\omega_n^2},$$

$$\begin{aligned}\sqrt{S_D(f_{max})} &\triangleq \mid H(j\omega_{max}) \mid \\ &= \frac{k}{\sqrt{(\omega_n^2 - \omega_{max}^2)^2 + 4\zeta^2\omega_n^2\omega_{max}^2}},\end{aligned}$$

where $\omega_{max} = 2\pi f_{max}$. Take the derivative to (A.3.2) with respect to ω, we obtain

$$\frac{d \mid H(j\omega) \mid}{d\omega} = \frac{-k(\omega^3 - \omega_n^2\omega + 2\zeta^2\omega_n^2\omega)}{\left[(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\right]^{3/2}}.$$

Let

$$\frac{d \mid H(j\omega) \mid}{d\omega} = 0,$$

then we get

$$\omega_n = \frac{\omega_{max}}{\sqrt{1 - 2\zeta^2}}.$$

From the previous equations $S_D(0)$ and $S_D(f_{max})$, we get

$$\sqrt{\frac{S_D(f_{max})}{S_D(0)}} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_{max}^2)^2 + 4\zeta^2\omega_n^2\omega_{max}^2}}.$$

Let

$$\begin{aligned}\alpha &= \frac{S_D(f_{max})}{S_D(0)} \\ &= \frac{\omega_n^4}{(\omega_n^2 - \omega_{max}^2)^2 + 4\zeta^2\omega_n^2\omega_{max}^2} \\ &= \frac{\frac{\omega_{max}^4}{(1-2\zeta^2)^2}}{(\frac{\omega_{max}^2}{1-2\zeta^2} - \omega_{max}^2)^2 + \frac{4\zeta^2\omega_{max}^4}{1-2\zeta^2}} \\ &= \frac{1}{4\zeta^2 - 4\zeta^4},\end{aligned}$$

then

$$4\alpha\zeta^4 - 4\alpha\zeta^2 + 1 = 0,$$

that is,

$$\begin{aligned}\zeta^2 &= \frac{4\alpha - \sqrt{16\alpha^2 - 16\alpha}}{8\alpha} \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{\alpha}},\end{aligned}$$

because the property of the damping ratio $\zeta \to 0$ as $S_D(f_{max}) \to \infty$. Finally, we have the parameters

$$\zeta = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{S_D(0)}{S_D(f_{max})}}}, \quad \omega_n = \frac{\omega_{max}}{\sqrt{1-2\zeta^2}} \quad \text{and} \quad k = \omega_n^2\sqrt{S_D(0)}$$

for the transfer function $H(s)$.

A.3.2 Multiplicative Perturbation of $H(s)$

Replace $S_D(f)$ by $A(f)$ in the previous section, we got the parameters $\{\zeta, k, \omega_n\}$ of $H(s)$ for the normalized DPSD, which only depends on β_m.

Let the nominal transfer function be

$$H_n(s) = \frac{k_n}{s^2 + a_n s + b_n}$$

and the perturbed transfer function be

$$H_p(s) = \frac{k_p}{s^2 + a_p s + b_p},$$

then we have, by the definition, the weight function

$$\begin{aligned} W_2(s) &= \frac{H_p(s)}{H_n(s)} - 1 \\ &= \frac{(\frac{k_p}{k_n} - 1)s^2 + (\frac{k_p}{k_n}a_n - a_p)s + (\frac{k_p}{k_n}b_n - b_p)}{s^2 + a_p s + b_p}. \end{aligned}$$

A.4 From $H(s)$ to State Space Representation

Consider the Laplace transform of the impulse response

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

(i) The Controllable Canonical Form. Based on the equation above, we define

$$\begin{aligned} X(s) &= \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}U(s) && \text{— feed back,} \\ Y(s) &= X(s) && \text{— feed forward,} \end{aligned}$$

then we have, by taking the inverse Laplace, the following equations

$$\mathcal{L}^{-1}\Big\{(s^2+2\zeta\omega_n s+\omega_n^2)X(s)\Big\}=k\mathcal{L}^{-1}\Big\{U(s)\Big\}, \tag{A.4.3}$$

$$\mathcal{L}^{-1}\Big\{Y(s)\Big\}=\mathcal{L}^{-1}\Big\{X(s)\Big\}. \tag{A.4.4}$$

From (A.4.3), we get

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=ku.$$

Let $z_1=x$, $z_2=\dot{x}$, that is, $\dot{z}_1=z_2$, $\dot{z}_2=\ddot{x}$, then we have

$$\left\{\begin{array}{lcl}\dot{z}_1 & = & z_2\\ \dot{z}_2 & = & ku-2\zeta\omega_n z_2-\omega_n^2 z_1\end{array}\right.$$

that is,

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}0\\ k\end{bmatrix}u;$$

from (A.4.4), we obtain

$$y=x=z_1$$

that is,

$$y=[1\quad 0]\begin{bmatrix}z_1\\ z_2\end{bmatrix}.$$

(ii) The Observable Canonical Form. From the equation

$$\frac{Y(s)}{U(s)}=\frac{k}{s^2+2\zeta\omega_n s+\omega_n^2},$$

we get

$$\begin{aligned}Y(s) &= -\frac{2\zeta\omega_n}{s}Y(s)-\frac{\omega_n^2}{s^2}Y(s)+\frac{k}{s^2}U(s)\\ &= \frac{1}{s}\Big(-2\zeta\omega_n Y(s)+\frac{1}{s}(-\omega_n^2 Y(s)+kU(s))\Big).\end{aligned}$$

Let

$$\left\{\begin{array}{lcl}Z_1(s) & = & \frac{1}{s}(-\omega_n^2 Y(s)+kU(s))\\ Z_2(s) & = & \frac{1}{s}(-2\zeta\omega_n Y(s)+Z_1(s))\end{array}\right., \tag{A.4.5}$$

then

$$Y(s)=Z_2(s). \tag{A.4.6}$$

By substituting equation (A.4.6) into equation (A.4.5) and multiplying both sides of the equations by s, we obtain

$$\begin{cases} sZ_1(s) &= -\omega_n^2 Z_2(s) + kU(s) \\ sZ_2(s) &= -2\zeta\omega_n Z_2(s) + Z_1(s) \end{cases}.$$

Take the inverse Laplace transforms of the preceding two equations, we get

$$\begin{cases} \dot{z}_1(t) &= -\omega_n^2 z_2(t) + ku(t) \\ \dot{z}_2(t) &= -2\zeta\omega_n z_2(t) + z_1(t) \end{cases},$$

that is,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_n^2 \\ 1 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k \\ 0 \end{bmatrix} u$$

and

$$y = [0 \quad 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

A.5 The Kalman Estimator

A.5.1 The Noise Covariance Matrix

Let the additive bandpass noise is $v(t) = v_I(t)\cos(\omega_c t) - v_Q(t)\sin(\omega_c t)$, $\{v_I(t)\}_{t\geq 0}$ and $\{v_Q(t)\}_{t\geq 0}$ are two iid with density $N(0;\sigma_v^2)$. Then the covariance matrix is

$$\begin{aligned} R &= E[v(t)v^{tr}(t)] \\ &= \begin{bmatrix} E[v_I^2(t)] & E[v_I(t)v_Q(t)] \\ E[v_I(t)v_Q(t)] & E[v_Q^2(t)] \end{bmatrix} \\ &= \begin{bmatrix} E[v_I^2(t)] & 0 \\ 0 & E[v_Q^2(t)] \end{bmatrix} \\ &= \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}, \end{aligned}$$

and so does the Q_i.

We know the state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + v \end{aligned}$$

with error covariance equation and Kalman gain. We want to find the error covariance equation and Kalman gain of the state space model as follows

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Dv \end{aligned}$$

Let the state space model be

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + v' \end{aligned}$$

where $v' = Dv$, then we get

$$\begin{aligned} R' &= E[v'v'^{tr}] \\ &= E[Dvv^{tr}D^{tr}] \\ &= DE[vv^{tr}]D^{tr} \\ &= D\begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}D^{tr} \\ &= DRD^{tr}. \end{aligned}$$

A.5.2 Ito Method

Consider a time interval $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_N = T$, which has step size $\delta_k = t_{k+1} - t_k$. For equation

$$dx = Axdt + Bdw,$$

we have

$$x(t_{k+1}) - x(t_k) = Ax(t_k)(t_{k+1} - t_k) + B(w(t_{k+1}) - w(t_k)),$$

where $x \in \Re^n$. Let $\Delta w_k = w(t_{k+1}) - w(t_k)$, then we get, by denoting $k = t_k$,

$$\begin{aligned} x_{k+1} &= Ax_k\delta_k + x_k + B\Delta w_k \\ &= (A\delta_k + I)x_k + B\Delta w_k, \end{aligned}$$

where $E[\Delta w_k] = 0$, $E[(\Delta w_k)^2] = \delta_k$, $I \in \Re^n \times \Re^n$ is an unit matrix. Let $\delta_k = T/N = \delta$, which means that the step size has an equal length. Denote $\Delta w = \Delta w_k$, then $E[\Delta w] = 0$ and $E[(\Delta w)^2] = \delta$. That is, $\Delta w \sim N(0; \delta)$. Thus,

$$\begin{aligned} x_{k+1} &= Ax_k\delta + x_k + B\Delta w \\ &= (A\delta + I)x_k + B\Delta w \end{aligned}$$

A.5.3 Computation of the Conditional Distributions

We know that if the vector of random variables (X,Y,Z) is joint Gaussian with zero means and covariance matrix

$$M_{XYZ} = \begin{bmatrix} var(X) & cov(X,Y) & cov(X,Z) \\ cov(Y,X) & var(Y) & cov(Y,Z) \\ cov(Z,X) & cov(Z,Y) & var(Z) \end{bmatrix},$$

then the marginal pdf for X and Y is also Gaussian and has the same set of means, variances and covariances[3]. That is, the pair (X,Y) has zero means and covariance matrix

$$M_{XY} = \begin{bmatrix} var(X) & cov(X,Y) \\ cov(Y,X) & var(Y) \end{bmatrix}.$$

(i) Conditional Envelope Distribution. From the least-square estimation of the conditional stochastic process, we have[4]

$$\widehat{I}(t) = E[I(t)|Y_0^t] \text{ and } \widehat{Q}(t) = E[Q(t)|Y_0^t],$$

where $Y_0^t \triangleq \{y(s);\ 0 \leq s \leq t\}$. Then the conditional joint Gaussian pdf of the two stochastic processes, the inphase and quadrature, is

$$f(I,Q,t|Y_0^t) = \frac{1}{2\pi\sqrt{|M|}} e^{-\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x})},$$

where

$$x = \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix}, \quad \widehat{x} = \begin{bmatrix} \widehat{I}(t) \\ \widehat{Q}(t) \end{bmatrix}.$$

Next, we will find the pdf of the envelope given by Y_0^t. Let $Z = I^2 + Q^2$, then we have, by the definition,

$$\begin{aligned} Prob(Z \leq z|Y_0^t) &= Prob(I^2 + Q^2 \leq z|Y_0^t) \\ &= \int_{I^2+Q^2\leq z} f(I,Q|Y_0^t)\,dIdQ \\ &= \int_{I^2+Q^2\leq z} \frac{1}{2\pi\sqrt{|M|}} e^{-\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x})}\,dIdQ. \end{aligned}$$

Let the covariance matrix be

[3] see [15]
[4] see [13]

$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

then the inverse matrix of M is

$$\begin{aligned} M^{-1} &= \frac{1}{ac-b^2}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \\ &= \begin{bmatrix} g & m \\ m & n \end{bmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} -\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x}) &= -\frac{1}{2}\left(\begin{bmatrix} I \\ Q \end{bmatrix} - \begin{bmatrix} \widehat{I} \\ \widehat{Q} \end{bmatrix}\right)^{tr}\begin{bmatrix} g & m \\ m & n \end{bmatrix}\left(\begin{bmatrix} I \\ Q \end{bmatrix} - \begin{bmatrix} \widehat{I} \\ \widehat{Q} \end{bmatrix}\right) \\ &= -\frac{g}{2}(I-\widehat{I})^2 - m(I-\widehat{I})(Q-\widehat{Q}) - \frac{n}{2}(Q-\widehat{Q})^2 \\ &= -\frac{g}{2}\widehat{I}^2 - \frac{n}{2}\widehat{Q}^2 - m\widehat{I}\widehat{Q} - \frac{g}{2}I^2 - \frac{n}{2}Q^2 - mIQ \\ &\quad + (g\widehat{I}+m\widehat{Q})I + (n\widehat{Q}+m\widehat{I})Q. \end{aligned}$$

Let

$$\begin{cases} I = \sqrt{\rho}\cos\theta \\ Q = \sqrt{\rho}\sin\theta \end{cases},$$

then we get $0 \le \rho \le z$, $0 \le \theta \le 2\pi$, $|J| = \frac{1}{2}$, and

$$Prob(Z \le z|Y_0^t) = \int_0^z \int_0^{2\pi} \frac{1}{2} f(\rho,\theta,t|Y_0^t)d\rho d\theta$$

with

$$\begin{aligned} -\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x}) &= -\frac{g}{2}\widehat{I}^2 - \frac{n}{2}\widehat{Q}^2 - m\widehat{I}\widehat{Q} \\ &\quad -\frac{g}{2}\rho + \frac{(g-n)\rho}{2}\sin^2\theta - m\rho\sin\theta\cos\theta \\ &\quad + (g\widehat{I}+m\widehat{Q})\sqrt{\rho}\cos\theta + (n\widehat{Q}+m\widehat{I})\sqrt{\rho}\sin\theta. \end{aligned}$$

Then

$$\begin{aligned} Prob(Z \le z|Y_0^t) &= \int_0^z \int_0^{2\pi} \frac{1}{2} f(\rho,\theta|Y_0^t)d\rho d\theta \\ &= \int_0^z \frac{1}{2}\frac{1}{2\pi\sqrt{|M|}} e^{-\frac{g}{2}\widehat{I}^2 - \frac{n}{2}\widehat{Q}^2 - m\widehat{I}\widehat{Q} - \frac{g}{2}\rho} d\rho \int_0^{2\pi} e^{(g\widehat{I}+m\widehat{Q})\sqrt{\rho}\cos\theta} \\ &\quad \times e^{(n\widehat{Q}+m\widehat{I})\sqrt{\rho}\sin\theta} e^{\frac{(g-n)}{2}\rho\sin^2\theta} e^{-m\rho\sin\theta\cos\theta} d\theta. \end{aligned}$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have[5]

$$
\begin{aligned}
&\int_0^{2\pi} e^{(g\widehat{I}+m\widehat{Q})\sqrt{\rho}\cos\theta} e^{(n\widehat{Q}+m\widehat{I})\sqrt{\rho}\sin\theta} e^{\frac{(g-n)}{2}\rho\sin^2\theta} e^{-m\rho\sin\theta\cos\theta} d\theta \\
&= 4\sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!}(g\widehat{I}+m\widehat{Q})^h(n\widehat{Q}+m\widehat{I})^i(g-n)^j m^k \\
&\quad \times \rho^{\frac{h+i+2j+2k}{2}} \int_0^{\frac{\pi}{2}} \cos^{h+k}\theta \sin^{i+2j+k}\theta d\theta \\
&= 2\sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!}(g\widehat{I}+m\widehat{Q})^h(n\widehat{Q}+m\widehat{I})^i(g-n)^j m^k \\
&\quad \times \rho^{\frac{h+i+2j+2k}{2}} \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
Prob(Z \le z|Y_0^t) &= \int_0^z \frac{1}{2\pi\sqrt{|M|}} e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}-\frac{g}{2}\rho} \\
&\quad \times \sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!}(g\widehat{I}+m\widehat{Q})^h(n\widehat{Q}+m\widehat{I})^i(g-n)^j m^k \\
&\quad \times \rho^{\frac{h+i+2j+2k}{2}} \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})} d\rho.
\end{aligned}
$$

Then the conditional pdf of the channel square envelope is

$$
\begin{aligned}
f_{R^2}(z,t|Y_0^t) &= \frac{1}{2\pi\sqrt{|M|}} e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}-\frac{g}{2}z} \\
&\quad \times \sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!}(g\widehat{I}+m\widehat{Q})^h(n\widehat{Q}+m\widehat{I})^i(g-n)^j m^k \\
&\quad \times \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})} z^{\frac{h+i+2j+2k}{2}}.
\end{aligned}
$$

Let $r = \sqrt{z} = \sqrt{I^2+Q^2}$, then $z = r^2$, $dz = 2rdr$. Thus, we obtain the conditional pdf of the channel envelope

$$
\begin{aligned}
f_R(r,t|Y_0^t) &= \frac{1}{\pi\sqrt{|M|}} e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}-\frac{g}{2}r^2} \sum_{h,i,j,k=0}^{\infty} \frac{(-1)^k 2^{-j}}{h!i!j!k!} \frac{\Gamma(\frac{h+k+1}{2})\Gamma(\frac{i+2j+k+1}{2})}{\Gamma(\frac{h+i+2j+2k+2}{2})} \\
&\quad \times (g\widehat{I}+m\widehat{Q})^h(n\widehat{Q}+m\widehat{I})^i(g-n)^j m^k r^{h+i+2j+2k+1}.
\end{aligned}
$$

(ii) Conditional phase distribution. Let $\theta = \tan^{-1}(\frac{Q}{I})$, then

$$
Prob(\theta \le \phi|Y_0^t) = Prob(\tan^{-1}\left(\frac{Q}{I}\right) \le \phi|Y_0^t)
$$

[5]see [16]

$$
\begin{aligned}
&= \int_{\tan^{-1}\left(\frac{Q}{I}\right)\leq\phi} f(I,Q|Y_0^t)dIdQ \\
&= \int_{\tan^{-1}\left(\frac{Q}{I}\right)\leq\phi} \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x})}dIdQ.
\end{aligned}
$$

Let

$$
\begin{cases} I &= \rho\cos\theta \\ Q &= \rho\sin\theta \end{cases},
$$

then $\theta \leq \phi$, $0 \leq \rho \leq \infty$, $|J| = \rho$. Thus, we have

$$
\begin{aligned}
Prob(\theta \leq \phi|Y_0^t) &= \int_0^\infty \int_0^\phi \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{1}{2}(x-\widehat{x})^{tr}M^{-1}(x-\widehat{x})}\rho d\theta d\rho \\
&= \int_0^\infty \int_0^\phi \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}-\frac{g}{2}\rho^2} \\
&\quad \times e^{(g\widehat{I}+m\widehat{Q})\rho\cos\theta}e^{(n\widehat{Q}+m\widehat{I})\rho\sin\theta}e^{\frac{(g-n)}{2}\rho^2\sin^2\theta}e^{-m\rho^2\sin\theta\cos\theta}\rho d\theta d\rho \\
&= \int_0^\infty \int_0^\phi \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}}e^{-(R\rho-S)^2}e^{S^2}\rho d\theta d\rho,
\end{aligned}
$$

where

$$
R = \sqrt{\frac{g}{2} - \frac{g-n}{2}\sin^2\theta + m\sin\theta\cos\theta},
$$

$$
S = \frac{(g\widehat{I}+m\widehat{Q})\cos\theta + (n\widehat{Q}+m\widehat{I})\sin\theta}{2R}.
$$

Thus[6],

$$
\begin{aligned}
Prob(\theta \leq \phi|Y_0^t) &= \int_0^\phi \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}}e^{S^2}d\theta \\
&\quad \times \int_0^\infty \left[\frac{-1}{2R^2}e^{-(R\rho-S)^2}d\Big(-(R\rho-S)^2\Big) + \frac{S}{R}e^{-(R\rho-S)^2}d\rho\right] \\
&= \int_0^\phi \frac{1}{2\pi\sqrt{|M|}}e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}}e^{S^2}\Big(\frac{1}{2R^2}e^{-S^2} + \frac{S}{R}\frac{1}{2}\sqrt{2\pi}\frac{1}{\sqrt{2}R}\Big)d\theta.
\end{aligned}
$$

Thus, the conditional phase distribution is

$$
f(\phi|Y_0^t) = \frac{1+S\sqrt{\pi}e^{S^2}}{4\pi R^2\sqrt{|M|}}e^{-\frac{g}{2}\widehat{I}^2-\frac{n}{2}\widehat{Q}^2-m\widehat{I}\widehat{Q}}.
$$

[6] $\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}dx = \frac{\sqrt{2\pi}}{2}\sigma$, because $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx = 1$

A.6 Ricean Channel Model

A.6.1 State Space Representation

Let $E[I^R(t)] = r_0\cos(\omega_0 t + \phi_0)$, then the Ricean state space channel model can be represented as follows

$$X_I^R(t) = X_I(t) + r_0\cos(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}, \tag{A.6.7}$$

where $X_I(t)$ is Rayleigh case, the parameters $\{r_0, \omega_0, \phi_0\}$ are given. Take the derivation to (A.6.7), that is,

$$\begin{aligned}
\dot{X}_I^R(t) &= \dot{X}_I(t) - r_0\omega_0\sin(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}\\
&= A_I X_I(t) + B_I\dot{w}_I(t) - r_0\omega_0\sin(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}\\
&= A_I\Big(X_I^R(t) - r_0\cos(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}\Big) + B_I\dot{w}_I(t) - r_0\omega_0\sin(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}\\
&= A_I X_I^R(t) + B_I\dot{w}_I(t) - A_I r_0\cos(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix} - r_0\omega_0\sin(\omega_0 t + \phi_0)\begin{bmatrix}1\\0\end{bmatrix}\\
&= A_I X_I^R(t) + B_I\dot{w}_I(t) + \begin{bmatrix}-r_0\omega_0\sin(\omega_0 t + \phi_0)\\ r_0\omega_n^2\cos(\omega_0 t + \phi_0)\end{bmatrix}.
\end{aligned}$$

Let $E[Q^R(t)] = r_0\sin(\omega_0 t + \phi_0)$, then as the same way of finding $X_I^R(t)$, we get

$$\dot{X}_Q^R(t) = A_Q X_Q^R(t) + B_Q\dot{w}_Q(t) + \begin{bmatrix}r_0\omega_0\cos(\omega_0 t + \phi_0)\\ r_0\omega_n^2\sin(\omega_0 t + \phi_0)\end{bmatrix}.$$

Thus, define $A \triangleq \begin{bmatrix}A_I & 0\\ 0 & A_Q\end{bmatrix}$, $B \triangleq \begin{bmatrix}B_I & 0\\ 0 & B_Q\end{bmatrix}$, the state space representation of $\{y(t)\}_{t\geq 0}$ is given by

$$\begin{aligned}
\dot{X}^R(t) &= AX^R(t) + Los(t) + B\dot{w}(t)\\
y(t) &= G(t)X^R(t) + D(t)v(t)
\end{aligned}$$

where $G(t) = s(t)C(t)$,

$$Los(t) = \begin{bmatrix}-r_0\omega_0\sin(\omega_0 t + \phi_0)\\ r_0\omega_n^2\cos(\omega_0 t + \phi_0)\\ r_0\omega_0\cos(\omega_0 t + \phi_0)\\ r_0\omega_n^2\sin(\omega_0 t + \phi_0)\end{bmatrix}.$$

A.6.2 The Filter for the Ricean Channel Model

Since the Ricean and Rayleigh have the following relationship

$$\widehat{X}^R = \widehat{X} + \begin{bmatrix} r_0\cos(\omega_0 t + \phi_0) \\ 0 \\ r_0\sin(\omega_0 t + \phi_0) \\ 0 \end{bmatrix}$$

Take the derivation to the above equation, we get

$$\begin{aligned}
\dot{\widehat{X}}^R &= \dot{\widehat{X}} + \begin{bmatrix} -r_0\omega_0\sin(\omega_0 t + \phi_0) \\ 0 \\ r_0\omega_0\cos(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} \\
&= A\widehat{X} + K(y - G\widehat{X}) + \begin{bmatrix} -r_0\omega_0\sin(\omega_0 t + \phi_0) \\ 0 \\ r_0\omega_0\cos(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} \\
&= A\widehat{X} + K(GX + Dv - G\widehat{X}) + \begin{bmatrix} -r_0\omega_0\sin(\omega_0 t + \phi_0) \\ 0 \\ r_0\omega_0\cos(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} \\
&= A\Big(\widehat{X}^R - \begin{bmatrix} r_0\cos(\omega_0 t + \phi_0) \\ 0 \\ r_0\sin(\omega_0 t + \phi_0) \\ 0 \end{bmatrix}\Big) + K\Big(G(X^R - \begin{bmatrix} r_0\cos(\omega_0 t + \phi_0) \\ 0 \\ r_0\sin(\omega_0 t + \phi_0) \\ 0 \end{bmatrix})\\
&\quad + Dv - G(\widehat{X}^R - \begin{bmatrix} r_0\cos(\omega_0 t + \phi_0) \\ 0 \\ r_0\sin(\omega_0 t + \phi_0) \\ 0 \end{bmatrix})\Big) + \begin{bmatrix} -r_0\omega_0\sin(\omega_0 t + \phi_0) \\ 0 \\ r_0\omega_0\cos(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} \\
&= A\widehat{X}^R + K\Big(GX^R + Dv - G\widehat{X}^R\Big) \\
&\quad - A\begin{bmatrix} r_0\cos(\omega_0 t + \phi_0) \\ 0 \\ r_0\sin(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} + \begin{bmatrix} -r_0\omega_0\sin(\omega_0 t + \phi_0) \\ 0 \\ r_0\omega_0\cos(\omega_0 t + \phi_0) \\ 0 \end{bmatrix} \\
&= A\widehat{X}^R + Los(t) + K(y - G\widehat{X}^R).
\end{aligned}$$

A.7 Signal-to-Noise Ratio

A.7.1 White Gaussian Noise $X(t) \sim N(0; N_0)$

(i) The definition of the power spectral density function[7] is

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau, \end{aligned}$$

where $X(t)$ is a continuous time WSS random process with mean m_X and autocorrelation function $R_X(\tau)$, which defined as follows

$$\begin{aligned} R_X(\tau) &= E[X(t)X(t+\tau)] \\ &= N_0\delta(\tau). \end{aligned}$$

Especially, $R_X(0) = E[X^2(t)] = N_0$. Thus, we have

$$\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} N_0\delta(\tau) e^{-j2\pi f\tau} d\tau \\ &= N_0, \end{aligned}$$

which means that if $X(t) \sim N(0, \sigma^2)$, then the power spectral density of $X(t)$ is σ^2. Let the bandpass additive noise (substitutes the user's interference, but it is the worst case because the user's interference has band limit) is

$$v(t) = v_I(t)\cos\omega_c t - v_Q(t)\sin\omega_c t,$$

where $\{v_I(t)\}_{t\geq 0}$, $\{v_Q(t)\}_{t\geq 0}$ are two iid white Gaussian noises with zero-mean and variance N_0. Then,

$$\begin{aligned} E[v^2(t)] &= E[(v_I(t)\cos\omega_c t - v_Q(t)\sin\omega_c t)^2] \\ &= E[v_I^2(t)]\cos^2\omega_c t + E[v_Q^2(t)]\sin^2\omega_c t \\ &= E[v_I^2(t)] \\ &= N_0. \end{aligned}$$

(ii) Let $(SNR_t)_{dB}$ be the transmitted signal-to-noise ratio in dB, E_t be the transmitted signal power. Then,

[7]see P404 of [15]

$$E_t = \frac{1}{T_s}\int_0^{T_s} \mid S_{transmitted\ signal}(t) \mid^2 dt,$$

where T_s is the signaling interval; and

$$(SNR_t)_{dB} = 10\log_{10}\frac{E_t}{N_0}.$$

Thus,

$$N_0 = \frac{E_t}{10^{\frac{(SNR_t)_{dB}}{10}}},$$

which means if we have white Gaussian noise $n \sim N(0,1)$, then

$$\sqrt{\frac{E_t}{10^{\frac{(SNR_t)_{dB}}{10}}}}\, n = v_I \sim N(0, N_0), \qquad R = \frac{E_t}{10^{\frac{(SNR_t)_{dB}}{10}}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(iii) Let $(SNR_r)_{dB}$ be the received signal-to-noise ratio in dB, E_r be the noiseless received signal power. Then,

$$E_r = \frac{1}{T_s}\int_0^{T_s} \mid S_{received\ signal}(t) \mid^2 dt.$$

Thus,

$$N_0 = \frac{E_r}{10^{\frac{(SNR_r)_{dB}}{10}}},$$

which means the additive noises are

$$\begin{bmatrix} v_I \\ v_Q \end{bmatrix} = \sqrt{\frac{E_r}{10^{\frac{(SNR_r)_{dB}}{10}}}}\begin{bmatrix} n_I \\ n_Q \end{bmatrix},$$

where $\{n_L(t)\}_{t\geq 0} \sim N(0;1)$, $L = I, Q$. And the covariance matrix is

$$R_v = \frac{E_r}{10^{\frac{(SNR_r)_{dB}}{10}}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A.7.2 The Noise Power

By the definition, the average noise power is

$$\begin{aligned} E_v &= \frac{1}{T_s}E\Big[\int_0^{T_s} \mid D(t)v(t) \mid^2 dt\Big] \\ &= \frac{1}{T_s}\int_0^{T_s} D(t)E[v(t)v^{tr}(t)]D^{tr}(t)dt \\ &= \frac{1}{T_s}\int_0^{T_s} D(t)\begin{bmatrix} E[v_I^2(t)] & 0 \\ 0 & E[v_Q^2(t)] \end{bmatrix}D^{tr}(t)dt \\ &= N_0. \end{aligned}$$

A.7.3 The Rayleigh Channel Received Signal Power

By the definition, we have

$$\begin{aligned}
E_r &= \frac{1}{T_s} E\left[\int_0^{T_s} \mid G(t)X(t) \mid^2 dt\right] \\
&= \frac{1}{T_s} \int_0^{T_s} E\left[\mid G(t)X(t) \mid^2\right] dt \\
&= \frac{1}{T_s} \int_0^{T_s} s^2(t)C(t)E\left[X(t)X^{tr}(t)\right]C^{tr}(t)dt \\
&= \frac{1}{T_s} \int_0^{T_s} s^2(t)\Big(E[X_{I_1}X_{I_1}^{tr}]\cos^2(\omega_c t) + E[X_{Q_1}X_{Q_1}^{tr}]\sin^2(\omega_c t)\Big)dt \\
&= \frac{1}{T_s} \int_0^{T_s} s^2(t)E[X_{I_1}X_{I_1}^{tr}]dt \\
&= \frac{1}{T_s} \int_0^{T_s} s^2(t)\gamma_{11}(t)dt.
\end{aligned}$$

aJJy a1+0If $\{s(t)\}_{t\geq 0}$ is a narrow band signal, and hence without loss of generality assumed to be $s(t) = 1$, by performing the above integration, we get

$$\begin{aligned}
E_r &= \frac{2\zeta^2\beta_1 - 2\beta_4 - \beta_3 - \beta_2}{4\zeta\omega_n T_s} e^{-2\zeta\omega_n T_s} + \beta_6 \\
&+ \frac{2\zeta\beta_1\sqrt{1-\zeta^2} + 2\beta_4 + 2\zeta^2\beta_5 + \zeta^2\beta_3 + \beta_2 - \zeta^2\beta_2 + \beta_3}{4\zeta\omega_n T_s} \\
&+ \frac{\zeta\beta_2 - \zeta\beta_3 - 2\zeta\beta_5 - 2\beta_1\sqrt{1-\zeta^2}}{4\omega_n T_s} \cos\left(2\omega_n\sqrt{1-\zeta^2}T_s\right)e^{-2\zeta\omega_n T_s} \\
&+ \frac{2\sqrt{1-\zeta^2}\beta_5 + \sqrt{1-\zeta^2}\beta_3 + \sqrt{1-\zeta^2}\beta_2}{4\omega_n T_s} \sin\left(2\omega_n\sqrt{1-\zeta^2}T_s\right)e^{-2\zeta\omega_n T_s},
\end{aligned}$$

where $\gamma(0) = E[X(0)X^{tr}(0)]$,

$$\beta_1 = \left(\frac{\gamma_{22}(0)+\gamma_{11}(0)\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} - \frac{k^2}{4\omega_n^3\sqrt{1-\zeta^2}}\right), \quad \beta_2 = \frac{\gamma_{11}(0)\zeta^2\omega_n^2+2\gamma_{12}(0)\zeta\omega_n+\gamma_{22}(0)}{\omega_n^2(1-\zeta^2)},$$

$$\beta_3 = \gamma_{11}(0), \qquad \beta_4 = \frac{k^2}{4\zeta\omega_n^3(1-\zeta^2)},$$

$$\beta_5 = \frac{\zeta k^2}{4\omega_n^3(1-\zeta^2)}, \qquad \beta_6 = \frac{k^2}{4\zeta\omega_n^3}.$$

A.7.4 The Ricean Channel Received Signal Power

Let $s(t) = 1$, then we have, by the definition,

$$E_r^R = \frac{1}{T_s} E\left[\int_0^{T_s} \mid G(t)X(t) + G(t)h_s(t) \mid^2 dt\right]$$

$$
\begin{aligned}
&= \frac{1}{T_s}\int_0^{T_s} E\Big[\mid G(t)X(t) + G(t)h_s(t)\mid^2\Big]dt \\
&= \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[X(t)X^{tr}(t)\big]C^{tr}(t)dt + \frac{1}{T_s}\int_0^{T_s} s^2(t)C(t)E\big[h_s(t)h_s^{tr}(t)\big]C^{tr}(t)dt \\
&= \frac{1}{T_s}\int_0^{T_s}\gamma_{11}(t)dt + \frac{1}{T_s}\int_0^{T_s} C(t)E\big[h_s(t)h_s^{tr}(t)\big]C^{tr}(t)dt \\
&= E_r + los,
\end{aligned}
$$

where

$$
h_s(t) = \begin{bmatrix} r_0\cos(\omega_0 t+\phi_0) \\ 0 \\ r_0\sin(\omega_0 t+\phi_0) \\ 0 \end{bmatrix},
$$

$$
\begin{aligned}
los &= \frac{1}{T_s}\int_0^{T_s} C(t)E\big[H_s(t)H_s^{tr}(t)\big]C^{tr}(t)dt \\
&= \frac{1}{T_s}\int_0^{T_s}\Big(r_0^2\cos^2(\omega_c t)\cos^2(\omega_0 t+\phi_0) - 2r_0^2\sin(\omega_c t)\cos(\omega_c t) \\
&\quad\times\sin(\omega_0 t+\phi_0)\cos(\omega_0 t+\phi_0) + r_0^2\sin^2(\omega_c t)\sin^2(\omega_0 t+\phi_0)\Big)dt \\
&= \frac{r_0^2}{T_s}\int_0^{T_s}\cos^2(\omega_c t+\omega_0 t+\phi_0)dt \\
&= \frac{r_0^2}{2} + \frac{r_0^2\sin\big(2(\omega_c+\omega_0)T_s+\phi_0\big)}{4(\omega_c+\omega_0)T_s}.
\end{aligned}
$$

A.7.5 The Multipath Received Signal Power

$$
\begin{aligned}
E_s &= \frac{1}{T_m}E\Big[\int_0^{T_m}\mid\sum_{i=1}^{6}G_iX_i\mid^2 dt\Big] \\
&= \frac{1}{T_m}\int_0^{T_m} E\Big[\Big(\sum_{i=1}^{6}G_iX_i\Big)\Big(\sum_{j=1}^{6}G_jX_j\Big)\Big]dt \\
&= \frac{1}{T_m}\int_0^{T_m} E\Big[\sum_{i=1}^{6}G_iX_iX_i^{tr}G_i^{tr}\Big]dt + \frac{1}{T_m}\int_0^{T_m} E\Big[\sum_{\substack{i,j=1\\ i\neq j}}^{6}G_iX_iX_j^{tr}G_j^{tr}\Big]dt.
\end{aligned}
$$

Since

$$
s(t-\tau_i)s(t-\tau_j) = \begin{cases} \text{const}^2, & i=j \\ 0, & i\neq j \end{cases}, \quad 1\le i,j\le 6
$$

Then,

$$
E_s = \frac{1}{T_m}E\Big[\int_0^{T_m}\mid\sum_{i=1}^{6}G_iX_i\mid^2 dt\Big]
$$

$$
\begin{aligned}
&= \frac{1}{T_m}\int_0^{T_m} E\Big[\sum_{i=1}^{6} G_i X_i X_i^{tr} G_i^{tr}\Big]dt + \frac{1}{T_m}\int_0^{T_m} E\Big[\sum_{\substack{i,j=1\\ i\neq j}}^{6} G_i X_i X_j^{tr} G_j^{tr}\Big]dt \\
&= \frac{1}{T_m}\int_0^{T_m} \sum_{i=1}^{6} G_i E\Big[X_i X_i^{tr}\Big] G_i^{tr}\, dt \\
&= \frac{1}{T_m}\int_0^{T_m} \sum_{i=1}^{6} s^2(t-\tau_i)(\gamma_{11}^i + d_{11}^i)dt.
\end{aligned}
$$

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