Tensor Maps of Twisted Group Schemes and Cohomological Invariants

Cameron Ruether

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

uOttawa
L'Université canadienne
Canada's university

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Abstract

Working over an arbitrary field $\mathbb{F}$ of characteristic not 2, we consider linear algebraic groups over $\mathbb{F}$. We view these as functors, represented by finitely generated $\mathbb{F}$-Hopf algebras, from the category of commutative, associative, $\mathbb{F}$-algebras $\mathbf{Alg}_{\mathbb{F}}$, to the category of groups. Classical examples of these groups, such as the special linear group $\mathbf{SL}_n$ are split, however there are also linear algebraic groups arising from central simple $\mathbb{F}$-algebras which are non-split. For example, associated to a non-split central simple $\mathbb{F}$-algebra A of degree n is a non-split special linear group $\mathbf{SL}(A)$. It is well known that central simple algebras are twisted forms of matrix algebras. This means that over the separable closure of $\mathbb{F}$, denoted $\mathbb{F}_{\mathrm{sep}}$, we have $A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ and that there is a twisted $\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$-action on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ whose fixed points are A. We show that a similar method of twisted Galois descent can be used to obtain all non-split semisimple linear algebraic groups associated to central simple algebras as fixed points within their split counterparts. In particular, these techniques can be used to construct the spin and half-spin groups $\mathbf{Spin}(A,\tau)$ and $\mathbf{HSpin}(A,\tau)$ associated to a central simple $\mathbb{F}$-algebra of degree $4n$ with orthogonal involution. Furthermore, we develop a theory of twisted Galois descent for Hopf algebras and show how the fixed points obtained this way are the representing Hopf algebras of our non-split groups. Returning to the view of group schemes as functors, we discuss how the group schemes we consider are sheaves on the étale site of $\mathbf{Alg}_{\mathbb{F}}$ whose stalks are Chevalley groups over local, strictly Henselian $\mathbb{F}$-algebras. This allows us to use the generators and relations presentation of Chevalley groups to explicitly describe group scheme morphisms. After showing how the Kronecker tensor product of matrices induces maps between simply connected groups, we give an explicit description of these maps in terms of Chevalley generators. This allows us to compute the kernel of these new maps composed with standard isogenies and thereby construct new tensor product maps between non-simply connected split groups. These new maps are $\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$-morphisms and so we apply our techniques of twisted Galois descent to also obtain new tensor product morphisms between non-split groups schemes. Finally, we use one of our new split tensor product maps to compute the degree three cohomological invariants of $\mathbf{HSpin}_{4n}$ for all n.

Dedication

To online friends and friends online, you made for a much more bearable pandemic.

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Contents

Preface

The motivation for this thesis comes from the theory of cohomological invariants of linear algebraic groups. As defined by Garibaldi, Merkurjev, and Serre in [GMS03], a cohomological invariant of a linear algebraic group $\mathbf{G}$ is a natural transformation $\Delta\colon H^1(-,\mathbf{G}) \to \mathcal{F}$ from the first Galois cohomology of $\mathbf{G}$ to a group valued functor $\mathcal{F}$. For example, the invariants of groups of multiplicative type where $\mathcal{F}\colon \mathbb{E} \mapsto \mathbb{E}^\times \otimes \mathbb{Q}/\mathbb{Z}$ for a field extension $\mathbb{E}/\mathbb{F}$, were studied recently by Wertheim in [Wer21]. A general approach to studying cohomological invariants is to choose a convenient $\mathcal{F}$ and describe the resulting group of invariants, $\operatorname{Inv}(\mathbf{G},\mathcal{F})$. Some common choices are $\mathcal{F} = H^{d+1}(-,\mathbb{Q}/\mathbb{Z}(d))$, which are defined in [GMS03] and recalled in section 5.1.1. The group $\operatorname{Inv}(\mathbf{G}, H^{d+1}(-,\mathbb{Q}/\mathbb{Z}(d))) = \operatorname{Inv}^{d+1}(\mathbf{G})$ is called the group of *degree* $d+1$ *invariants*.

Degree 3 invariants have received the majority of recent attention. The group of degree 3 invariants is known for many, but not all, simple linear algebraic groups. If $\mathbf{G}$ is simply connected then $\operatorname{Inv}^3(\mathbf{G})$ is a cyclic group, generated by the *Rost invariant*, and the order of the Rost invariant is given in [GMS03, pg.130]. The Rost invariant has been studied in further detail for classical groups by Merkurjev, Parimala, and Tignol in [MPT03], and their results were extended to exceptional groups by Garibaldi and Quéguiner-Mathieu in [GQM08]. For split adjoint groups and non-split adjoint groups of inner type, i.e., those associated to a non-split central simple algebra, the groups of degree three invariants were computed in [Mer16]. One technique used in these computations is breaking the group of degree three invariants into a subgroup, $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}} \subseteq \operatorname{Inv}^3(\mathbf{G})$, called the *decomposable invariants*, and into a quotient $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{ind}} = \operatorname{Inv}^3(\mathbf{G}) / \operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}}$, called the *indecomposable invariants*. The indecomposable invariants of some groups which are neither simply connected nor adjoint, in particular $\mathbf{SL}_n / \boldsymbol{\mu}_m$ and $\mathbf{HSpin}_{4n}$, were computed by Bermudez and Ruozzi in [BR14]. Furthermore, Bermudez and Ruozzi also computed the full group of invariants for $\mathbf{HSpin}_{16}$. Another subgroup of invariants, $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{sdec}} \subseteq \operatorname{Inv}^3(\mathbf{G})$, called *semi-decomposable*, were defined by Merkurjev, Neshitov, and Zainoulline in [MNZ15] and studied further by Baek for quotients of semisimple but not simple groups in [Bae17]. Furthermore, degree 3 invariants of another class of non-simple groups, algebraic tori, have been studied by Blinstein and Merkurjev in [BM13].

The group $\operatorname{Inv}^3(\mathbf{HSpin}_{4n})$ for any $n \geq 2$ was computed in [Rue20] by assembling computations from [Mer16] and [BR14] with a generalization of a technique of Garibaldi's found in [Gar09]. This result is outlined in section 5. Garibaldi's technique consists of three steps. First, use Chevalley generators to give a description of the Kronecker tensor product map $\mathbf{Sp}_2 \times \mathbf{Sp}_8 \to \mathbf{SO}_{16}$, then show that this lifts to a map $\mathbf{Sp}_2 \times \mathbf{Sp}_8 \to \mathbf{Spin}_{16}$, and finally, track the images of central elements to compute the kernel and show that there is an induced inclusion $\mathbf{PSp}_2 \times \mathbf{PSp}_8 \hookrightarrow \mathbf{HSpin}_{16}$. This was generalized in [Rue20] to construct various group scheme homomorphisms between split groups induced from the tensor product. In particular, maps $\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$ where n or m are even were used to compute the degree three invariants of half-spin.

If (A, τ) is a non-split central simple $\mathbb{F}$-algebra of degree $4n$ with orthogonal involution of trivial discriminant, then there is a non-split half-spin group denoted $\mathbf{HSpin}(A, \tau)$. Neither the usual degree three invariants nor the indecomposable invariants of this group are known. Hopefully, once the indecomposable invariants are known, then a non-split analogue of the inclusion above, i.e., a map $\mathbf{PSp}(A_1, \psi_1) \times \mathbf{PSp}(A_2, \psi_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$ where ψ_i are symplectic involutions on their respective central simple algebra A_i, could be used to complete the calculation. Hence one may ask the following natural question, "Do analogues of the tensor product maps between split groups exist between their non-split counterparts?" Indeed, this question was asked of me by Dr. Erhard Neher. The answer to this question is yes, and this is shown in chapter 4.

In order to construct these maps, which we call *twisted tensor products*, we extend the known technique of twisted Galois descent for central simple algebras to the setting of linear algebraic groups. Our first main result is the following, where we denote $\Gamma = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$.

Theorem A. *Let (A, τ) be a central simple $\mathbb{F}$-algebra of degree n with an orthogonal involution of trivial discriminant. Let $\alpha\colon \Gamma \to \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$ be a cocycle representing the class $[(A, \tau)] \in H^1(\mathbb{F}, \mathbf{PSO}_n)$. For each $\sigma \in \Gamma$ choose elements $\mathfrak{a}_\sigma \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ such that $\mathfrak{a}_\sigma \mapsto \alpha_\sigma$ under the standard projection. For each $R \in \mathbf{Alg}_{\mathbb{F}}$, let $\mathfrak{a}'_\sigma$ be the image of $\mathfrak{a}_\sigma$ under the map $\mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ and define a twisted Γ-action on $\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ by*

$$\sigma \cdot_\alpha x = \mathfrak{a}'_\sigma \sigma(x) (\mathfrak{a}'_\sigma)^{-1}.$$

Furthermore, if n is divisible by 4 we can also choose elements $\mathsf{a}_\sigma \in \mathbf{HSpin}_n(\mathbb{F}_{\mathrm{sep}})$ such that $\mathfrak{a}_\sigma \mapsto \mathsf{a}_\sigma \mapsto \alpha_\sigma$ and likewise define a twisted action on $\mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ via conjugation by a'_σ. Then the fixed points of these twisted actions are

$$\mathbf{Spin}(A, \tau)(R) \cong \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$$

$$\mathbf{HSpin}(A, \tau)(R) \cong \mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$$

In fact, similar results hold for all simple linear algebraic groups. For example, with the correct setup, $\mathbf{SL}(A)(R) \cong \mathbf{SL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$ for a central simple algebra A of degree n, however these results are already known. Our second main result is actually required to prove the **HSpin** portion of theorem A. While we can prove the twisting result for **Spin** by working within the groups $\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, the lack of a good description of the groups $\mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ required us to pivot our attention to the representing Hopf algebra. We develop a theory of twisted Hopf algebras which is compatible with our twisted groups.

Theorem B. *Let* $\mathbf{G}$ *be a linear algebraic group with representing Hopf algebra* H*. Let* $\mathbf{G}(A)$ *be a twisted form of* $\mathbf{G}$ *with representing Hopf algebra* $H(A)$*. In particular, assume that there is a cocycle* $\alpha\colon \Gamma \to \mathrm{Inn}(\mathbf{G}(\mathbb{F}_{\mathrm{sep}}))$ *of inner automorphisms such that, letting* α'_σ *be the image of* α_σ *in* $\mathrm{Inn}(\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}))$*, the fixed points of the twisted action* $\sigma \cdot_\alpha x = \alpha'_\sigma(\sigma(x))$ *on* $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ *are*

$$\mathbf{G}(A)(R) = \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}.$$

Then there is a cocycle $\mathfrak{A}\colon \Gamma \to \mathrm{Aut}_{\mathbb{F}_{\mathrm{sep}}}(H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ *induced by* α *which defines a twisted action on* $H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$ *given by*

$$\sigma \cdot_\alpha x = \mathfrak{A}_\sigma(\sigma(x))$$

such that $H(A) \cong (H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$*.*

With these new twisting techniques in hand, we are able to produce our final main result which is the existence of various twisted tensor product maps.

Theorem C. *(i) Let* (A_1, ψ_1) *and* (A_2, ψ_2) *be central simple algebras of degree* $2n$ *and* $2m$ *respectively with symplectic involutions. Then there are the following group scheme homomorphisms.*

$$\begin{aligned}
\mathbf{Sp}(A_1,\psi_1) \times \mathbf{Sp}(A_2,\psi_2) &\to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)\\
\mathbf{PSp}(A_1,\psi_1) \times \mathbf{PSp}(A_2,\psi_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \text{ when } n \text{ or } m \text{ even}\\
\mathbf{Sp}(A_1,\psi_1) \times \mathbf{PSp}(A_2,\psi_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \text{ when } n \text{ and } m \text{ odd}\\
\mathbf{PSp}(A_1,\psi_1) \times \mathbf{PSp}(A_2,\psi_2) &\to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)\\
([a],[b]) &\mapsto [a \otimes b]
\end{aligned}$$

All these maps are lying above the fourth map, by which we mean for each of the above maps φ*, there is a commutative diagram*

$$\begin{tikzcd}
\mathbf{G}_1(A_1,\psi_1) \times \mathbf{G}_2(A_2,\psi_2) \arrow[r, "\varphi"] \arrow[d, two heads] & \mathbf{G}_3(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \arrow[d, two heads] \\
\mathbf{PSp}(A_1,\psi_1) \times \mathbf{PSp}(A_2,\psi_2) \arrow[r] & \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \\
([a],[b]) \arrow[r, mapsto] & {[a \otimes b]}
\end{tikzcd}$$

where the vertical maps are the natural surjections.
(ii) Let (A_1, τ_1) *and* (A_2, τ_2) *be central simple algebras of degrees* $2n$ *and* $2m$ *respectively with orthogonal involutions. Then there are group scheme homomorphisms*

$$\begin{aligned}
\mathbf{Spin}(A_1, \tau_1) \times \mathbf{Spin}(A_2, \tau_2) &\to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathbf{PSO}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ or } m \text{ even} \\
\mathbf{SO}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ and } m \text{ odd.}
\end{aligned}$$

(iii) Let (A_1, τ_1) *and* (A_2, τ_2) *be central simple algebras of degrees* $2n$ *and* $2m+1$ *respectively with orthogonal involutions. Then there are group scheme homomorphisms*

$$\begin{aligned}
\mathbf{Spin}(A_1, \tau_1) \times \mathbf{Spin}(A_2, \tau_2) &\to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathbf{HSpin}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ even.}
\end{aligned}$$

(iv) Let (A_1, τ_1) *and* (A_2, τ_2) *be central simple algebras of degrees* $2n+1$ *and* $2m+1$ *respectively with orthogonal involutions. Then there is a group scheme homomorphism*

$$\mathbf{Spin}(A_1, \tau_1) \times \mathbf{Spin}(A_2, \tau_2) \to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2).$$

The homomorphisms in cases (ii), (iii), and (iv) all lie above the map

$$\begin{gathered}
\mathbf{PSO}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) \to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
([a], [b]) \mapsto [a \otimes b]
\end{gathered}$$

We also explain how these maps fit into commutative diagrams with easier to construct examples, such as $\mathbf{SO}(A_1, \tau_1) \times \mathbf{SO}(A_2, \tau_2) \to \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$ which simply comes from restricting the map $A_1 \times A_2 \to A_1 \otimes_{\mathbb{F}} A_2$.

This thesis is organized as follows. Chapter 1 recalls the definitions of, and useful known results about, the objects we will be working with. This begins with a brief overview of central simple algebras including those with involution. Clifford algebras and Clifford bimodules are covered next, both in the split case when associated to a quadratic form and in the non-split case when associated to a central simple algebra. Next, we introduce our most important objects, linear algebraic groups. We give the general framework and emphasize defining the many examples which occur frequently throughout the thesis. The classification of linear algebraic groups follows. Via the Lie algebra and root system of a group, we sort our examples into the classical types A, B, C, and D. We also sort our groups into split versus non-split, and stratify them as simply connected, adjoint, or neither. We then proceed to introduce Chevalley groups via their generators and relations and we show how they sit as subgroups in our split linear algebraic groups. The final section of preliminaries is devoted to Galois cohomology. We recall the definitions of the cohomology sets and then detail how the first cohomology of various linear algebraic groups classifies families of central simple algebras. We finish by recounting the known story of twisting for central simple algebras and some linear algebraic groups.

Chapter 2 contains the proofs of theorem A and theorem B. In order to twist **Spin**, which lives inside a Clifford algebra and is defined in terms of a Clifford bimodule, we prove some twisting results about Clifford objects to translate between their split and non-split versions. The twisting of **Spin** is then accomplished in theorem 2.1.3. As stated above, we then need to develop a theory of twisting for Hopf algebras in order to proceed towards twisting **HSpin**. Section 2.2 begins with a brief overview of Hopf algebras. Then we take the cocycle $\alpha \in \operatorname{Inn}(\mathbf{G}(\mathbb{F}_{\text{sep}}))$ and construct the required $\mathfrak{A}$ from it in theorem 2.2.6. The results of theorem B are then found in theorem 2.2.7. We can then apply these results, along with a bit more twisting of Clifford algebras, to obtain the **HSpin** portion of theorem A in 2.3.4.

Chapter 3 is a setup chapter. We wish to use Garibaldi's technique of working with Chevalley generators in order to define full group scheme homomorphisms. However, there exist $R \in \mathbf{Alg}_{\mathbb{F}}$ where the Chevalley subfunctor, $\mathbf{E}_{\mathbf{G}}$, is a proper subgroup, i.e., $\mathbf{E}_{\mathbf{G}}(R) \subsetneq \mathbf{G}(R)$. We see that this is not a barrier by showing that our linear algebraic groups are sheaves on the étale site of $\mathbf{Alg}_{\mathbb{F}}$. Injectivity and surjectivity of group homomorphisms can then be checked on the stalks of this site, and we show in theorem 3.4.7 and proposition 3.4.8 that the stalks of our linear algebraic groups are completely described by Chevalley generators. Therefore, Chevalley calculations can be used to understand existing maps and to define new maps.

Chapter 4 then proceeds with applying Garibaldi's technique. We recall the Kronecker tensor product map, and in section 4.2 we explicitly describe both the usual and the lifted maps in the split case in terms of Chevalley generators. In section 4.3 we use Chevalley computations to define our various induced maps in the split case. The results of theorem C then appear in section 4.4 where twisting techniques are applied to the maps previously constructed in chapter 4.

Finally, in Chapter 5 the work of this thesis is connected to its motivating subject, cohomological invariants. The chapter begins with an overview of cohomological invariants, in particular of degree 2 and degree 3 invariants including many examples. This setup is then used in section 5.2 to recount the results of [Rue20] as described above.

Chapter 1

Preliminaries

In this chapter we introduce the objects which will be used throughout this thesis. We do so by following the exposition of other sources, highlighting the properties of these objects which are most relevant to us. We work over an arbitrary field $\mathbb{F}$ of characteristic not 2.

1.1 Central Simple Algebras

Central simple algebras and their involutions will underlie many of our later constructions. For example, important linear algebraic groups arise as the automorphisms of an algebra with involution. Additionally, initial notions of *twisted forms* can be explained in the context of central simple algebras. Our exposition in this section follows [KMRT, §1].

Definition 1.1.1. *A unital, associative, finite dimensional algebra A over a field $\mathbb{F}$ is a* central simple algebra *if*

- *The center of A is equal to $\mathbb{F} \cdot 1$ (A is* central*)*
- *The only two-sided ideals in A are $\{0\}$ and A (A is* simple*).*

We may also refer to such A as central simple $\mathbb{F}$-algebras. Matrix algebras $\mathrm{M}_n(\mathbb{F})$ are examples of central simple algebras over $\mathbb{F}$, they are simple and the central elements are exactly scaled copies of the identity matrix. Division algebras (not necessarily commutative algebras where every non-zero element has a multiplicative inverse) are also simple, and since their center is a field, they are central simple algebras over their center.

From a central simple $\mathbb{F}$-algebra A there are two immediate constructions that produce new central simple algebras. First, we can define the *opposite algebra* to have the same elements, addition, and scaling as A but with multiplication reversed. That is, define

$$\begin{aligned} A^{\mathrm{op}} &= \{a^{\mathrm{op}} \mid a \in A\} \\ a^{\mathrm{op}} + b^{\mathrm{op}} &= (a+b)^{\mathrm{op}} \\ c(a^{\mathrm{op}}) &= (ca)^{\mathrm{op}} \\ a^{\mathrm{op}} b^{\mathrm{op}} &= (ba)^{\mathrm{op}} \end{aligned}$$

for $a, b \in A$ and $c \in \mathbb{F}$. Second, if $\mathbb{E}/\mathbb{F}$ is a field extension of $\mathbb{F}$, then we can extend the scalars of A to form $A_{\mathbb{E}} := A \otimes_{\mathbb{F}} \mathbb{E}$ which is a central simple algebra over $\mathbb{E}$. Here we can make precise the notion of twisted forms of a central simple algebra.

Definition 1.1.2. *Let A and B be two central simple $\mathbb{F}$-algebras. We call A a* twisted form *of B (and symmetrically, B a twisted form of A) if there exists a field extension $\mathbb{E}/\mathbb{F}$ such that $A_{\mathbb{E}} \cong B_{\mathbb{E}}$.*

Wedderburn's theorem describes the structure of central simple algebras, and in particular describes how any central simple $\mathbb{F}$-algebra becomes isomorphic to a matrix algebra over a suitable field extension of $\mathbb{F}$.

Theorem 1.1.3 (Wedderburn's Theorem, [KMRT, 1.1])**.** *Let A be a unital, associative, finite dimensional $\mathbb{F}$-algebra. Then the following are equivalent:*

1. *A is a central simple $\mathbb{F}$-algebra.*
2. *The map $A \otimes_{\mathbb{F}} A^{\mathrm{op}} \to \mathrm{End}_{\mathbb{F}}(A)$ sending $a \otimes b$ to the map $x \mapsto axb$ is an isomorphism.*
3. *There is a field extension $\mathbb{E}/\mathbb{F}$ and a natural number n such that $A_{\mathbb{E}} \cong \mathrm{M}_n(\mathbb{E})$.*
4. *There exists a finite dimensional central division algebra D over $\mathbb{F}$ and a natural number m such that $A \cong \mathrm{M}_m(D)$.*

Additionally, if A is a central simple algebra then all simple left A-modules (resp. right A-modules) are isomorphic. The division algebra in point (4) is also unique up to isomorphism and we may take $D = \mathrm{End}_A(L)$ for any simple left A-module L.

If a central simple algebra A is isomorphic to some $M_n(\mathbb{F})$ then A is called *split*. Similarly, any field extension $\mathbb{E}/\mathbb{F}$ which satisfies point (3) above is called a *splitting field*. If $\mathbb{E}$ is a splitting field for A where $A_{\mathbb{E}} \cong \mathrm{M}_n(\mathbb{E})$, then since $\mathrm{M}_n(\mathbb{F}) \otimes \mathbb{E} \cong \mathrm{M}_n(\mathbb{E})$ we see that A is a twisted form of $\mathrm{M}_n(\mathbb{F})$. In this way, we see that central simple $\mathbb{F}$-algebras are exactly the twisted forms of the matrix algebra $\mathrm{M}_n(\mathbb{F})$.

Many splitting fields exist for any given central simple algebra, and every algebra has many splitting field which are Galois extensions of $\mathbb{F}$.

Proposition 1.1.4 ([GS17, Cor. 2.2.12])**.** *Let A be a finite dimensional $\mathbb{F}$-algebra. Then A is a central simple $\mathbb{F}$-algebra if and only if there exists a finite Galois extension $\mathbb{E}/\mathbb{F}$ and a natural number n such that $A_{\mathbb{E}} \cong \mathrm{M}_n(\mathbb{E})$.*

Since Galois extensions are separable, every finite Galois extension can be embedded into a separable closure of $\mathbb{F}$, denoted $\mathbb{F}_{\mathrm{sep}}$. Therefore, for any central simple $\mathbb{F}$-algebra A we have that $A_{\mathbb{F}_{\mathrm{sep}}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ for some natural number n. Furthermore, since $\dim_{\mathbb{F}}(A) = \dim_{\mathbb{F}_{\mathrm{sep}}}(A_{\mathbb{F}_{\mathrm{sep}}})$ we have that the dimension of A is n^2. We call n the *degree* of the central simple algebra A, denoted $\deg(A)$.

Two other useful tools which come from exploiting the similarities between arbitrary central simple algebras and matrix algebras are the reduced norm map and the reduced trace map. Let A be a central simple $\mathbb{F}$-algebra. The reduced norm is a multiplicative map $\mathrm{Nrd}\colon A \to \mathbb{F}$ which is an analogue of the determinant map $\det\colon \mathrm{M}_n(\mathbb{F}) \to \mathbb{F}$, and the reduced trace is a linear map $\mathrm{Trd}\colon A \to \mathbb{F}$ which is an analogue of the trace map $\mathrm{tr}\colon \mathrm{M}_n(\mathbb{F}) \to \mathbb{F}$. From [KMRT, pg. 5],

Definition 1.1.5. *Let A be a central simple $\mathbb{F}$-algebra. Since $A_{\mathbb{F}_{\mathrm{sep}}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ where $n = \deg(A)$, we can consider the composition*

$$A \to A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \overset{\det}{\to} \mathbb{F}_{\mathrm{sep}}$$

sending $a \mapsto \det(a \otimes 1)$. This will produce elements in $\mathbb{F}$, and we then define

$$\begin{aligned} \mathrm{Nrd}\colon A &\to \mathbb{F} \\ a &\mapsto \det(a \otimes 1). \end{aligned}$$

Definition 1.1.6. *Let A be a central simple $\mathbb{F}$-algebra of degree n. Consider the composition*

$$A \to A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \overset{\mathrm{tr}}{\to} \mathbb{F}_{\mathrm{sep}}$$

sending $a \mapsto \mathrm{tr}(a \otimes 1)$. This will produce elements in $\mathbb{F}$, and we then define

$$\begin{aligned} \mathrm{Trd}\colon A &\to \mathbb{F} \\ a &\mapsto \mathrm{tr}(a \otimes 1). \end{aligned}$$

We note that for matrix algebras $\mathrm{M}_n(\mathbb{F})$, the reduced norm coincides with the determinant and the reduced trace coincides with the usual trace.

1.1.1 The Brauer Group

Points (2) and (4) of Wedderburn's Theorem, and the fact that the associated division algebra is unique up to isomorphism, allow us to define the Brauer group of a field. First, we define Brauer equivalence of central simple algebras.

Definition 1.1.7. *Let A and B be two central simple $\mathbb{F}$-algebras. By Wedderburn's theorem there exist central division algebras D_1 and D_2 over $\mathbb{F}$, as well as natural numbers n_1 and n_2 such that*

$$A \cong M_{n_1}(D_1) \text{ and } B \cong M_{n_2}(D_2).$$

We call A and B Brauer equivalent *if $D_1 \cong D_2$ as $\mathbb{F}$-algebras. This defines an equivalence relation on the set of isomorphism classes of central simple $\mathbb{F}$-algebras.*

A convenient fact about Brauer equivalence is that it respects the tensor product of central simple algebras. If $A_1 \sim A_2$ and $B_1 \sim B_2$ are two pairs of Brauer equivalent central simple $\mathbb{F}$-algebras, then $A_1 \otimes B_1$ is Brauer equivalent to $A_2 \otimes B_2$ as well. Therefore we can define a binary operation on the set of central simple $\mathbb{F}$-algebras modulo Brauer equivalence by using the tensor product.

Definition 1.1.8. *Let $\mathbb{F}$ be a field. Let $\mathrm{Br}(\mathbb{F})$, called the* Brauer group of $\mathbb{F}$, *be the set of Brauer equivalence classes of central simple $\mathbb{F}$-algebras. Equipped with the binary operation coming from the tensor product it forms a commutative group (and so is often written additively, $[A]+[B]=[A\otimes B]$) where $0=[\mathbb{F}]$ and $-[A]=[A^{\mathrm{op}}]$.*

We note that $[\mathbb{F}] = [\mathrm{M}_n(\mathbb{F})]$ for any natural number n, and therefore the fact that the inverse of the class of an algebra is the class of its opposite algebra comes from Wedderburn's theorem point (2). If $\mathbb{E}/\mathbb{F}$ is a field extension and A is a central simple $\mathbb{F}$-algebra, then $A \otimes_{\mathbb{F}} \mathbb{E}$ will be a central simple $\mathbb{E}$-algebra. This extension of scalars respects the Brauer group structure in that if $[A] = [B] \in \mathrm{Br}(\mathbb{F})$ then $[A \otimes_{\mathbb{F}} \mathbb{E}] = [B \otimes_{\mathbb{F}} \mathbb{E}] \in \mathrm{Br}(\mathbb{E})$ and the map $\mathrm{Br}(\mathbb{F}) \to \mathrm{Br}(\mathbb{E})$ is a group homomorphism. Therefore, we can define the Brauer group functor.

Definition 1.1.9. *The* Brauer group functor *is defined as*

$$\begin{aligned} \mathrm{Br}\colon \mathbf{Fields}_{\mathbb{F}} &\to \mathbf{Ab} \\ \mathbb{E} &\mapsto \mathrm{Br}(\mathbb{E}). \end{aligned}$$

It is a covariant functor from the category of field extensions over $\mathbb{F}$ to the category of abelian groups.

1.1.2 Algebras with Involution

In this subsection we follow [KMRT, §2].

Definition 1.1.10. *If A is a ring, then an* involution *on A is a map $\sigma\colon A \to A$ such that for all $a, b \in A$,*

$$\begin{aligned} \sigma(a+b) &= \sigma(a)+\sigma(b) \\ \sigma^2(a) &= a \\ \sigma(ab) &= \sigma(b)\sigma(a). \end{aligned}$$

We will restrict our focus to involutions on central simple $\mathbb{F}$-algebras, noting that since involutions are defined for rings in general they are not required to be $\mathbb{F}$-linear. Simple examples include taking the transpose of a matrix, which provides an example of an involution on matrix algebras, or if $\mathbb{E}/\mathbb{F}$ is a degree two Galois extension then the non-trivial automorphism in $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is an involution of $\mathbb{E}$ as an $\mathbb{F}$-algebra. We denote an algebra A equipped with an involution $\sigma\colon A \to A$ by the pair (A, σ). If (A_1, σ_1) and (A_2, σ_2) are two algebras with involution then a homomorphism between them is a usual algebra homomorphism $\varphi\colon A_1 \to A_2$ such that we also have $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$.

Any involution on a central simple $\mathbb{F}$-algebra maps central elements to central elements, and so restricts to an automorphism of $\mathbb{F}$. We use the two possible behaviours of this automorphism to categorize involutions.

Definition 1.1.11. *Let (A, σ) be a central simple $\mathbb{F}$-algebra with involution. We call σ an* involution of the first kind *if $\sigma|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$. We call σ an* involution of the second kind *if $\sigma|_{\mathbb{F}} \neq \mathrm{id}_{\mathbb{F}}$ (and therefore is an order 2 automorphism of $\mathbb{F}$).*

For this thesis we will focus on involutions of the first kind. These involutions can be further categorized as being either orthogonal or symplectic. To describe this classification we recall some properties about the relationship between bilinear forms and involutions on matrix algebras from [KMRT, Ch. 1].

Let V be a finite dimensional $\mathbb{F}$-vector space and let $b\colon V \times V \to \mathbb{F}$ be a non-singular bilinear form. Further assume that b is either symmetric ($b(x, y) = b(y, x)$ for all $x, y \in V$) or skew-symmetric ($b(x, y) = -b(y, x)$ for all $x, y \in V$). Since b is non-singular the map

$$\begin{aligned} \hat{b}\colon V &\to V^* \\ x &\mapsto b(x, -) \end{aligned}$$

is an isomorphism. We then define an involution on $\mathrm{End}_{\mathbb{F}}(V)$ as

$$\begin{aligned} \sigma_b\colon \mathrm{End}_{\mathbb{F}}(V) &\to \mathrm{End}_{\mathbb{F}}(V) \\ f &\mapsto \hat{b}^{-1} \circ f^T \circ \hat{b} \end{aligned}$$

where $f^T \in \mathrm{End}_{\mathbb{F}}(V^*)$ is the transpose. The involution σ_b is an involution of the first kind and is called the *adjoint involution* of b due to the property that

$$b(x, f(y)) = b(\sigma_b(f)(x), y)$$

for all $x, y \in V$. In the case when $V = \mathbb{F}^n$ (or after choosing a basis of V) this appears as follows. For a non-singular bilinear form $b\colon \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ there exists a invertible matrix $B \in \mathrm{M}_n(\mathbb{F})$ such that $b(x, y) = x^T B y$ for all $x, y \in \mathbb{F}^n$ (considered as column vectors). Then the adjoint involution on $\mathrm{M}_n(\mathbb{F})$ takes the form

$$\sigma_b\colon \mathrm{M}_n(\mathbb{F}) \to \mathrm{M}_n(\mathbb{F})$$

$$A \mapsto B^{-1}A^TB.$$

Here we can observe that a second bilinear form which takes the form $b_2(x,y) = x^T(cB)y$ for some $c \in \mathbb{F}$ would produce the same adjoint involution. Accounting for this possibility of scaling produces the following theorem.

Theorem 1.1.12. *Let V be a finite dimensional $\mathbb{F}$-vector space and let $b\colon V \times V \to \mathbb{F}$ be a symmetric or skew-symmetric bilinear form. Let $[b] = \{cb \mid c \in \mathbb{F}^\times\}$. Then the map*

$$[b] \mapsto \sigma_b$$

which associates an equivalence class of bilinear forms on V to an involution of the first kind on $\operatorname{End}_{\mathbb{F}}(V)$, is a bijection.

Therefore, given an involution of the first kind on a matrix algebra, the involution is associated to a bilinear form, and whether that bilinear form is symmetric or skew-symmetric is how we classify the involution. The notion of skew-symmetry makes sense since our fields do not have characteristic 2, however analogous notions can be defined in the characteristic 2 case as in [KMRT, 2.5]. Since central simple algebras are twisted forms of matrix algebras, an algebra with involution becomes a matrix algebra with involution after a suitable extension of scalars, and therefore we can apply the same characterization in that setting.

Definition 1.1.13. *Let (A,σ) be a central simple $\mathbb{F}$-algebra with involution of the first kind. Let $\mathbb{E}/\mathbb{F}$ be a splitting field of A. Therefore $(A_{\mathbb{E}}, \sigma \otimes \mathrm{id}_{\mathbb{E}}) \cong (\mathrm{M}_n(\mathbb{E}), \sigma')$ for some natural number n and an involution of the first kind σ' on $\mathrm{M}_n(\mathbb{E})$. There is a bilinear form on $\mathbb{E}^n$ associated to σ'. If that bilinear form is symmetric we call σ an* orthogonal involution *on A. If the bilinear form is skew-symmetric we call σ a* symplectic involution *on A.*

We note here that in order to have a symplectic involution, $\mathrm{M}_n(\mathbb{E})$ must contain skew-symmetric matrices, and therefore $n = \deg(A)$ must be even. However, this is not the only restriction on which algebras can possess involutions of the first kind. A description of when an algebra has such an involution was given by Albert.

Theorem 1.1.14 ([KMRT, 3.1(1)])**.** *Let A be a central simple $\mathbb{F}$-algebra. There exists an involution of the first kind on A if and only if $A \otimes_{\mathbb{F}} A$ is split.*

The condition that $A \otimes_{\mathbb{F}} A$ is split can be rephrased in terms of the Brauer group of $\mathbb{F}$. Since split algebras are isomorphic to a matrix algebra, they belong to the 0 class in $\mathrm{Br}(\mathbb{F})$. Therefore A has an involution of the first kind if and only if $2[A] = 0 \in \mathrm{Br}(\mathbb{F})$.

As a brief word on notation, in the future we will prefer to reserve σ for elements of Galois groups and as such we will use (A,τ) for a central simple algebra with orthogonal involution and (A,ψ) for one with a symplectic involution when the nature of the involution is known.

1.1.3 Clifford Algebras

Whenever we have a central simple algebra with orthogonal involution, or in the split case a quadratic space, there are associated algebras called *Clifford algebras*. We will use these algebras in the future to define linear algebraic groups, and Clifford algebras will also appear when we discuss the cohomological invariants of linear algebraic groups. In this section, we start in the split case by defining the Clifford algebra related to a quadratic form, then define the Clifford algebra of a central simple algebra with orthogonal involution and detail the connection between the two constructions. We also introduce the Clifford bimodule which will be needed briefly in the future as well. This is all done following [KMRT].

Definition 1.1.15. *Let (V, q) be a nonsingular quadratic space over $\mathbb{F}$. The associated* Clifford Algebra *is*

$$C(V, q) = T(V)/I$$

where $T(V)$ is the tensor algebra of V over $\mathbb{F}$, and I is the ideal generated by all elements of the form $v \otimes v - q(v) \cdot 1$ for $v \in V$. The operations on $C(V, q)$ are those induced from $T(V)$, and we will write multiplication, which is given by the tensor product, simply as juxtaposition. The Clifford algebra comes with a canonical involution given by

$$\tau \colon C(V, q) \to C(V, q)$$
$$v_1 v_2 \dots v_m \mapsto v_m \dots v_2 v_1$$

and extended linearly. Furthermore, the tensor algebra $T(V)$ has a natural $\mathbb{Z}$-grading, and since the relations $vv = q(v) \cdot 1$ maintain even degrees, there is a $\mathbb{Z}_2$-grading of the Clifford algebra

$$C(V, q) = C_0(V, q) \oplus C_1(V, q)$$

where $C_0(V, q)$ is called the even Clifford algebra *and $C_1(V, q)$ is called the* odd part of the Clifford algebra.

Despite the use of τ for the involution on $C(V, q)$, this involution may or may not be orthogonal. The naming of $C_1(V, q)$ does reflect its properties, as it is not itself an algebra since it is not unital due to 1 being degree 0, and hence belonging to $C_0(V, q)$. The even portion is an algebra, and oftentimes is a central simple $\mathbb{F}$-algebra. To categorize when this occurs we need the discriminant of q.

Definition 1.1.16. *Let (V, q) be a nonsingular quadratic space over $\mathbb{F}$ with $\dim(V) = n$. Then there exists a unique symmetric matrix $B \in \mathrm{M}_n(\mathbb{F})$ such that $q(v) = v^T B v$, and the* discriminant of q *is defined to be*

$$\operatorname{disc}(q) = [(-1)^{\frac{n(n-1)}{2}} \det(B)] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2$$

If $\operatorname{disc}(q) = 1 \cdot (\mathbb{F}^\times)^2$ then we say q has trivial discriminant*, otherwise it has* non-trivial discriminant*.*

The possibilities for $C_0(V,q)$ can now be listed.

Theorem 1.1.17. *[KMRT, 8.2] Let (V,q) be a nonsingular quadratic space over $\mathbb{F}$ and consider $C_0(V,q)$. Then*

- *If $\dim(V) = 2n+1$ then $C_0(V,q)$ is a central simple $\mathbb{F}$-algebra of degree 2^n.*
- *If $\dim(V) = 2n$ and $\operatorname{disc}(q) = [\delta] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2$ is non-trivial, then $Z(C_0(V,q)) \cong \mathbb{F}[\sqrt{\delta}]$ and $C_0(V,q)$ is a central simple $\mathbb{F}[\sqrt{\delta}]$-algebra of degree 2^{n-1}.*
- *If $\dim(V) = 2n$ and $\operatorname{disc}(q)$ is trivial, then $Z(C_0(V,q)) \cong \mathbb{F}\times\mathbb{F}$ and there is a decomposition*
  $$C_0(V,q) = C^+(V,q) \times C^-(V,q)$$
  where each of $C^\pm(V,q)$ is a central simple $\mathbb{F}$-algebra of degree 2^{n-1}.

The odd portion of the Clifford algebra is not without use, it is part of the construction of the Clifford bimodule.

Definition 1.1.18. *Let (V,q) be a nonsingular quadratic space over $\mathbb{F}$ with $\dim(V) = n$. The* Clifford bimodule *is*

$$B(V,q) = V \otimes_{\mathbb{F}} C_1(V,q).$$

There are two module structures, given as follows.

- *It is a left $\mathrm{M}_n(\mathbb{F})$-module via $B(v\otimes c) = (Bv)\otimes c$ for $B \in \mathrm{M}_n(\mathbb{F})$, $v \in V$, and $c \in C_1(V,q)$.*
- *It is a $C_0(V,q)$-bimodule with actions given via $a * (v\otimes c) = v \otimes (ac)$ and $(v\otimes c)\cdot a = v\otimes(ca)$ for $a \in C_0(V,q)$, $v\in V$, and $c \in C_1(V,q)$.*

Important for later discussion of the Clifford algebras of central simple algebras is how the above Clifford bimodule contains a copy of $\mathrm{M}_n(\mathbb{F})$, where $\dim(V) = n$. Since q is nonsingular so is the associated symmetric bilinear form

$$\begin{aligned} b_q\colon V\times V &\to \mathbb{F} \\ (x,y) &\mapsto q(x+y)-q(x)-q(y) \end{aligned}$$

and therefore there is an isomorphism given by

$$\begin{aligned} V\otimes_{\mathbb{F}} V &\xrightarrow{\sim} \mathrm{End}(V) \\ v\otimes w &\mapsto (x \mapsto b_q(w,x)v). \end{aligned}$$

Once a basis of V is chosen, yielding an isomorphism $\mathrm{End}(V) \cong \mathrm{M}_n(\mathbb{F})$, we can extend the above isomorphism to construct

$$\varphi_q\colon V\otimes_{\mathbb{F}} V \xrightarrow{\sim} \mathrm{End}(V) \cong \mathrm{M}_n(\mathbb{F}).$$

For the purpose of this thesis, we will call φ_q the *standard isomorphism* between $V \otimes_{\mathbb{F}} V$ and $\mathrm{M}_n(\mathbb{F})$, keeping in mind that it does not only depend on q. Since $C_1(V,q)$ is a quotient of $\bigoplus_{i \text{ odd}} V^{\otimes i}$ by identities of even degree, it contains a copy of V. Therefore we have a vector space map

$$b\colon \mathrm{M}_n(\mathbb{F}) \hookrightarrow B(V,q)$$

given by $b\colon \mathrm{M}_n(\mathbb{F}) \overset{\varphi_q^{-1}}{\to} V \otimes_{\mathbb{F}} V \to V \otimes C_1(V,q)$. By [KMRT, 9.1] this is an injective left $\mathrm{M}_n(\mathbb{F})$-module homomorphism.

Now that we have the versions of our objects coming from quadratic forms, we move on to define them in the context of central simple algebras. The insight is that in $C_0(V,q)$, the factors come in pairs $V \otimes_{\mathbb{F}} V \cong \mathrm{M}_n(\mathbb{F})$ which is a central simple algebra. This suggests that the analogue of the even Clifford algebra should be a quotient of the tensor algebra of the central simple algebra. A family of vector space isomorphisms we will use comes from the sandwich maps. Since we are dealing with vector spaces, we use $\mathcal{V}(A)$ to denote the vector space underlying a central simple algebra A, i.e., $\mathcal{V}$ is the forgetful functor from central simple $\mathbb{F}$-algebras to $\mathbb{F}$-vector spaces.

Definition 1.1.19. *Let A be a central simple $\mathbb{F}$-algebra and let n be a natural number. The* sandwich map *is the map*

$$\mathrm{Sand}_n\colon \mathcal{V}(A)^{\otimes n} \to \mathrm{Hom}_{\mathbb{F}}(\mathcal{V}(A)^{\otimes n-1}, \mathcal{V}(A))$$

defined by $\mathrm{Sand}_n(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_{n-1}) := a_1 b_1 a_2 b_2 \ldots a_{n-1} b_{n-1} a_n$. *By [KMRT, 9.3] this is an isomorphism of $\mathbb{F}$-vector spaces.*

Let (A,τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. The map $\mathrm{Sand}_2\colon \mathcal{V}(A) \otimes_{\mathbb{F}} \mathcal{V}(A) \to \mathrm{End}_{\mathbb{F}}(\mathcal{V}(A))$ is used to define the following map. For any $u \in \mathcal{V}(A) \otimes \mathcal{V}(A)$, there is a linear map $\mathrm{Sand}_2(u) \circ \tau \in \mathrm{End}_{\mathbb{F}}(\mathcal{V}(A))$, and therefore a map

$$\begin{aligned} \phi\colon \mathcal{V}(A) \otimes_{\mathbb{F}} \mathcal{V}(A) &\to \mathcal{V}(A) \otimes_{\mathbb{F}} \mathcal{V}(A) \\ u &\mapsto \mathrm{Sand}_2^{-1}(\mathrm{Sand}_2(u) \circ \tau) \end{aligned}$$

Definition 1.1.20. *[KMRT, 8.7, pg. 94] Let (A,τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution and ϕ be as above. The* Clifford algebra of (A,τ) *is*

$$C(A,\tau) = T(\mathcal{V}(A))/I$$

where T denotes the tensor algebra and

$$I = \langle x - \tfrac{1}{2}\,\mathrm{Trd}(x) \cdot 1 \mid x \in \mathcal{V}(A), \tau(x) = x \rangle$$

$$+ \langle u - \tfrac{1}{2}\mathrm{Sand}_2(u)(1) \mid u \in \mathcal{V}(A) \otimes_{\mathbb{F}} \mathcal{V}(A), \phi(u) = u \rangle.$$

There is a canonical involution on $C(A, \tau)$ given by extending τ. We abuse notation and also denote it by τ, that is

$$\tau(a_1 \otimes \ldots \otimes a_n) = \tau(a_n) \otimes \ldots \otimes \tau(a_1).$$

This definition gives us an analogue to the even Clifford algebra of a quadratic form. If (V, q) is a nonsingular quadratic space with $\dim(V) = n$ and a chosen basis, the associated symmetric bilinear form b_q has a representing matrix $\Omega_q \in \mathrm{M}_n(\mathbb{F})$ which defines an orthogonal involution

$$\begin{aligned} \tau_q \colon \mathrm{M}_n(\mathbb{F}) &\to \mathrm{M}_n(\mathbb{F}) \\ B &\mapsto \Omega_q B^T \Omega_q^{-1}. \end{aligned}$$

We then have the following proposition.

Proposition 1.1.21. *[KMRT, 8.11] Let (V, q) be a nonsingular quadratic space with $\dim(V) = n$, and let $(\mathrm{M}_n(\mathbb{F}), \tau_q)$ be the associated central simple $\mathbb{F}$-algebra with orthogonal involution as above. The standard isomorphism $\varphi_q \colon V \otimes_F V \to \mathrm{M}_n(\mathbb{F})$ induces an isomorphism of algebras*

$$C_0(V, q) \cong C(\mathrm{M}_n(\mathbb{F}), \tau_q).$$

Furthermore, this is also an isomorphism of algebras with involution when each algebra is given its canonical involution.

There is an analogous structure theorem for our generalized Clifford algebras as theorem 1.1.17 was in the split case. It too relies on a notion of discriminant.

Definition 1.1.22. *Let (A, τ) be a central simple $\mathbb{F}$-algebra of degree $2n$ with orthogonal involution. The* discriminant *of τ is*

$$\mathrm{disc}(\tau) = [(-1)^n \,\mathrm{Nrd}(a - \tau(a))] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2$$

for any $a \in A$ such that $a - \tau(a) \in A^\times$. The discriminant does not depend on this choice.

Theorem 1.1.23. *[KMRT, 8.10] Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution and consider $C(A, \tau)$. Then*

- *If $\deg(A) = 2n+1$ then A is split by [KMRT, 2.8] and so $C(A, \tau) \cong C_0(V, q)$ for some quadratic space (V, q) over $\mathbb{F}$. Thus $C(A, \tau)$ is a central simple $\mathbb{F}$-algebra of degree 2^n.*

- *If* $\deg(A) = 2n$ *and* $\operatorname{disc}(\tau) = [\delta] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2$ *is non-trivial, then* $Z(C(A,\tau)) \cong \mathbb{F}[\sqrt{\delta}]$ *and* $C(A,\tau)$ *is a central simple* $\mathbb{F}[\sqrt{\delta}]$*-algebra of degree* 2^{n-1}*.*
- *If* $\deg(A) = 2n$ *and* $\operatorname{disc}(\tau)$ *is trivial, then* $Z(C(A,\tau)) \cong \mathbb{F} \times \mathbb{F}$ *and there is a decomposition*
$$C(A,\tau) = C^+(A,\tau) \times C^-(A,\tau)$$
where each of $C^\pm(A,\tau)$ *is a central simple* $\mathbb{F}$*-algebra of degree* 2^{n-1}*.*

The Clifford bimodule of a central simple algebra is a bit trickier to define since we do not have an odd portion to use, nor a quadratic space. This is navigated around by using the sandwich maps to define an action of the symmetric group S_{2n} on $\mathcal{V}(A)^{\otimes n}$.

Proposition 1.1.24. *[KMRT, 9.4] Let* (A,τ) *be a central simple* $\mathbb{F}$*-algebra with orthogonal involution, and let* n *be a natural number. There is a canonical representation* $\rho_n\colon S_{2n} \to \mathbf{GL}(\mathcal{V}(A)^{\otimes n})$*. It is defined on transpositions by*

$$\rho_n((i,i+1)) = \underset{1}{\mathrm{id}_{\mathcal{V}(A)}} \otimes \ldots \otimes \underset{k-1}{\mathrm{id}_{\mathcal{V}(A)}} \otimes \underset{k}{\tau} \otimes \underset{k+1}{\mathrm{id}_{\mathcal{V}(A)}} \otimes \ldots \otimes \underset{n}{\mathrm{id}_{\mathcal{V}(A)}}$$

if $i = 2k-1$ *is odd, and by*

$$\rho_n((i,i+1))(u) = \mathrm{Sand}_n^{-1}\Big(\mathrm{Sand}_n(u) \circ (\underset{1}{\mathrm{id}_{\mathcal{V}(A)}} \ldots \underset{k-1}{\mathrm{id}_{\mathcal{V}(A)}} \otimes \underset{k}{\tau} \otimes \underset{k+1}{\mathrm{id}_{\mathcal{V}(A)}} \otimes \ldots \otimes \underset{n-1}{\mathrm{id}_{\mathcal{V}(A)}})\Big)$$

if $i = 2k$ *is even. Since these transpositions generate* S_{2n}*, the representation* ρ_n *is then extended homomorphically.*

As a bit of motivation, the above representation is generalizing the following behaviour in the split case. When $A \cong \mathrm{M}_m(\mathbb{F})$ is split, we have the standard isomorphism $\mathcal{V}(A) \cong V \otimes_{\mathbb{F}} V$ for some V. Therefore, $A^{\otimes n} \cong V^{\otimes 2n}$ and S_{2n} acts by permuting the factors. In the case A is split, the involution τ corresponds to the transposition of the factors in $V \otimes_{\mathbb{F}} V$ and ρ_n recovers the action

$$\rho_n(\pi)(v_1 \otimes v_2 \otimes \ldots \otimes v_{2n}) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \ldots \otimes v_{\pi^{-1}(2n)}$$

for $\pi \in S_{2n}$. It is clear that $\rho_n((i,i+1)$ when i is odd simply performs τ to one of the factors $\mathcal{V}(A)$, and the complicated definition of $\rho_n((i,i+1)$ when i is even can be thought of as "performing τ between two factors $\mathcal{V}(A) \otimes_{\mathbb{F}} \mathcal{V}(A)$". This can be compared to the definition of ϕ above, where $\rho_2((2,3)) = \phi$.

The Clifford bimodule will be defined as a quotient of $\bigoplus_{i\geq 1} \mathcal{V}(A)^{\otimes i}$, but with a modified bimodule structure. To define this modified structure we use the following map

$$\begin{aligned}\gamma\colon T_+(\mathcal{V}(A)) &\to T_+(\mathcal{V}(A))\\ a_1 \otimes \ldots \otimes a_n &\mapsto \rho_n((1,2,\ldots,2n)^{-1})(a_1 \otimes \ldots \otimes a_n)\end{aligned}$$

which is assembled from the various ρ_n.

Definition 1.1.25. *Let $T_+(\mathcal{V}(A)) = \bigoplus_{i \geq 1} \mathcal{V}(A)^{\otimes i}$. It is a $T(\mathcal{V}(A))$-bimodule with the following module structures. For $c \in T(\mathcal{V}(A))$ and $v \in T_+(\mathcal{V}(A))$ we have*

- *The right action $v \cdot c = v \otimes c$ is the natural right module structure coming from multiplication within $T(\mathcal{V}(A))$.*
- *The left action is defined to be $c * v = \gamma^{-1}(c \otimes \gamma(v))$.*

Definition 1.1.26. *Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. Then the* Clifford bimodule of (A, τ) *is*

$$B(A, \tau) = \frac{T_+(\mathcal{V}(A))}{I * T_+(\mathcal{V}(A)) + T_+(\mathcal{V}(A)) \cdot I}$$

where $I = \langle x - \frac{1}{2} \operatorname{Trd}(x) \cdot 1 \mid x \in \mathcal{V}(A), \tau(x) = x \rangle$. By [KMRT, 9.7] this carries the structure of a $C(A, \tau)$-bimodule with actions being induced from the $T(\mathcal{V}(A))$-bimodule structure of $T_+(\mathcal{V}(A))$. They are similarly denoted by $*$ for left action and $\cdot$ for right action. There is also a natural left A-module structure coming from multiplication within $T(\mathcal{V}(A))$.

Here as well the Clifford bimodule contains a copy of the central simple algebra. Since there is an inclusion $\mathcal{V}(A) \hookrightarrow T_+(\mathcal{V}(A))$, there is an induced map

$$b \colon A \to B(A, \tau)$$

which we also denote with b. Theorem 9.7 of [KMRT] also contains a few more important facts.

Theorem 1.1.27. *[KMRT, 9.7] Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution.*

- *If $(A, \tau) \cong (\mathrm{M}_n(\mathbb{F}), \tau_q)$ is split, where (V, q) is some quadratic space, the standard isomorphism $A \cong V \otimes_{\mathbb{F}} V$ induces an isomorphism of Clifford bimodules*

  $$B(A, \tau) \cong V \otimes_{\mathbb{F}} C_1(V, q) = B(V, q).$$

- *The map $b \colon A \to B(A, \tau)$ is an injective left A-module homomorphism.*

The introduction of the Clifford bimodule will be needed for the definition of the non-split spin and half-spin groups, which, alongside many other linear algebraic groups, we define in the next chapter.

1.2 Linear Algebraic Groups

By a *linear algebraic group* over $\mathbb{F}$ (or simply algebraic group) we will mean an affine algebraic group scheme over $\mathbb{F}$. That is, an affine scheme whose ring of rational functions is a finitely generated $\mathbb{F}$-Hopf algebra whose Hopf structure defines the identity, inverse, and multiplication operations on the scheme. To describe this, we first recall the definition of a Hopf algebra.

Definition 1.2.1. *[KMRT, §20] A* Hopf algebra *over* $\mathbb{F}$ *is a unital, commutative, associative* $\mathbb{F}$*-algebra* H *together with algebra morphisms*

$$\begin{aligned} c&\colon H \to H \otimes_{\mathbb{F}} H \\ i&\colon H \to H \\ u&\colon H \to \mathbb{F} \end{aligned}$$

called the comultiplication, coinverse *(often also called the* antipode*), and* counit *respectively. We require that these maps make the following diagrams commute.*

$$\begin{tikzcd} H \arrow[r, "c"] \arrow[d, "c"] & H \otimes_{\mathbb{F}} H \arrow[d, "\mathrm{id} \otimes c"] \\ H \otimes_{\mathbb{F}} H \arrow[r, "c\otimes \mathrm{id}"] & H \otimes_{\mathbb{F}} H \otimes_{\mathbb{F}} H \end{tikzcd} \qquad \begin{tikzcd} H \otimes_{\mathbb{F}} H \arrow[r, "u\otimes \mathrm{id}"] & \mathbb{F} \otimes_{\mathbb{F}} H \arrow[d, equal] \\ H \arrow[u, "c"] \arrow[r, "\mathrm{id}"] & H \end{tikzcd}$$

$$\begin{tikzcd} H \otimes_{\mathbb{F}} H \arrow[rr, "i\otimes \mathrm{id}"] & & H \otimes_{\mathbb{F}} H \arrow[d, "m"] \\ H \arrow[u, "c"] \arrow[r, "u"] & \mathbb{F} \arrow[r, "\cdot 1"] & H \end{tikzcd}$$

where $m\colon H \otimes_{\mathbb{F}} H \to H$ *is the algebra multiplication of* H*. The first diagram asserts that* H *is* coassociative*, and therefore we use the notation* $c^n\colon H \to H^{\otimes n}$ *for any of the equal compositions of comultiplication. When performing calculations within Hopf algebras we will use* Sweedler notation *and write*

$$c^n(x) = \sum x_{(1)} \otimes \ldots \otimes x_{(n)}.$$

A homomorphism of Hopf algebras is an $\mathbb{F}$*-algebra morphism* $\varphi\colon H_1 \to H_2$ *such that*

$$\begin{aligned} c_2 \circ \varphi &= (\varphi \otimes \varphi) \circ c_1 \\ i_2 \circ \varphi &= \varphi \circ i_1 \\ u_2 \circ \varphi &= u_1. \end{aligned}$$

We will frequently choose to discuss algebraic groups using the language of functors of points. In this language, if $\mathbf{G}$ is an algebraic group with representing Hopf algebra H, then we consider $\mathbf{G}$ as a functor

$$\begin{aligned} \mathbf{G}\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbb{F}}(H, R) \end{aligned}$$

from unital, commutative, and associative $\mathbb{F}$-algebras to groups where the group structure on $\mathbf{G}(R) = \mathrm{Hom}_{\mathbb{F}}(H, R)$ is defined from the Hopf structure of H. For $f, g \in \mathrm{Hom}_{\mathbb{F}}(H, R)$, their product is given by

$$fg := m \circ (f \otimes g) \circ c$$

where we abuse notation and use m for the multiplication map $R \otimes_{\mathbb{F}} R \to R$ as well. The inverse of f is given by

$$f^{-1} = f \circ i$$

and the unit element in $\mathrm{Hom}_{\mathbb{F}}(H, R)$ is $u\colon H \to \mathbb{F} \hookrightarrow R$ where we once again abuse notation by reusing u.

From this point of view, a group scheme homomorphism from $\mathbf{G}$ to $\mathbf{H}$ is simply a natural transformation $\varphi\colon\ \mathbf{G} \to \mathbf{H}$ between the functors. Such a natural transformation corresponds to a map between representing Hopf algebras $\varphi^*\colon H_{\mathbf{H}} \to H_{\mathbf{G}}$, and we call φ injective if φ^* is surjective, and we call φ surjective if φ^* is injective. By [KMRT, 22.A], the injectivity condition is equivalent to requiring that $\varphi(R)\colon\ \mathbf{G}(R) \to \mathbf{H}(R)$ is injective for all $R \in \mathbf{Alg}_{\mathbb{F}}$. Just as for homomorphisms of abstract groups, group scheme homomorphisms have kernels and images.

Definition 1.2.2. *Let $\varphi\colon\ \mathbf{G} \to \mathbf{H}$ be a map of linear algebraic groups. The* kernel *of φ is the group scheme*

$$\begin{aligned} \ker(\varphi)\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \ker(\varphi(R)) = \{x \in \mathbf{G}(R) \mid \varphi(R)(x) = 1\}. \end{aligned}$$

Thus, the group of R-points of the kernel is the kernel of the morphism over R.

Definition 1.2.3. *Let $\varphi\colon\ \mathbf{G} \to \mathbf{H}$ be a map of linear algebraic groups with associated Hopf algebra map $\varphi^*\colon H_{\mathbf{H}} \to H_{\mathbf{G}}$. The* (scheme theoretic) image *of φ is the linear algebraic group represented by the Hopf algebra $H_{\mathbf{H}}/\ker(\varphi^*) \cong \mathrm{Img}(\varphi^*)$. That is,*

$$\begin{aligned} \mathrm{Img}(\varphi)\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbb{F}}(\mathrm{Img}(\varphi^*), R). \end{aligned}$$

Images of group scheme homomorphisms are not as well behaved as kernels in the following sense. It is not necessarily the case that the group of R-points of $\mathrm{Img}(\varphi)$

is the image of $\varphi(R)$, i.e., it may occur that $\operatorname{Img}(\varphi(R)) \subsetneq (\operatorname{Img}(\varphi))(R)$. In general, the image group scheme produces larger sets. To emphasize this difference, we will refer to such images as *scheme theoretic images.*

If $\mathbb{E}/\mathbb{F}$ is a field extension there is an inclusion of categories $\mathbf{Alg}_{\mathbb{E}} \to \mathbf{Alg}_{\mathbb{F}}$ which considers an $\mathbb{E}$-algebra as an $\mathbb{F}$-algebra through the $\mathbb{F}$-algebra structure of $\mathbb{E}$ itself. If $\mathbf{G}$ is an algebraic group over $\mathbb{F}$ we can then obtain an algebraic group over $\mathbb{E}$ by composing with this inclusion.

Definition 1.2.4. *Let* $\mathbf{G}$ *be an algebraic group over* $\mathbb{F}$ *and let* $\mathbb{E}/\mathbb{F}$ *be a field extension. The pullback of* $\mathbf{G}$ *to the category* $\mathbf{Alg}_{\mathbb{E}}$ *is denoted with* $\mathbf{G}_{\mathbb{E}}$ *and is called the* restriction of $\mathbf{G}$ to $\mathbb{E}$. *It is defined on* $R \in \mathbf{Alg}_{\mathbb{E}}$ *by* $\mathbf{G}_{\mathbb{E}}(R) = \mathbf{G}(R)$.

Similarly to central simple algebras, restricting to field extensions allows us to define the notion of twisted forms of algebraic groups.

Definition 1.2.5. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be algebraic groups over* $\mathbb{F}$. *We say that* $\mathbf{G}$ *is a* twisted form *of* $\mathbf{H}$ *(and symmetrically* $\mathbf{H}$ *is a twisted form of* $\mathbf{G}$*) if there exists a field extension* $\mathbb{E}/\mathbb{F}$ *such that* $\mathbf{G}_{\mathbb{E}} \cong \mathbf{H}_{\mathbb{E}}$ *as algebraic groups over* $\mathbb{E}$.

1.2.1 Initial Examples of Algebraic Groups

When defining algebraic groups we will often forego explicitly describing the underlying Hopf algebra, instead describing how they behave as a functor on input algebras. We now define some basic examples, all of which are group schemes over $\mathbb{F}$.

Definition 1.2.6. *The* additive group, $\mathbb{G}_a$, *is defined on* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by* $\mathbb{G}_a(R) = R$ *considered as an additive group.*

Definition 1.2.7. *The* multiplicative group, $\mathbb{G}_m$, *is defined on* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by* $\mathbb{G}_m(R) = R^\times$. *It sends an algebra to its group of units.*

Definition 1.2.8. *The* n^{th} roots of unity, $\boldsymbol{\mu}_n$, *is defined on* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by* $\boldsymbol{\mu}_n(R) = \{x \in R \mid x^n = 1\}$.

Since $\boldsymbol{\mu}_n(R)$ is a subgroup of $\mathbb{G}_m(R)$ for all $R \in \mathbf{Alg}_{\mathbb{F}}$ we have that $\boldsymbol{\mu}_n$ is a subgroup scheme of $\mathbb{G}_m$, denoted $\boldsymbol{\mu}_n \leq \mathbb{G}_m$.

Definition 1.2.9. *Let* A *be a central simple* $\mathbb{F}$*-algebra. The* general linear group, $\mathbf{GL}(A)$, *is defined on* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by*

$$\mathbf{GL}(A)(R) = (A \otimes_{\mathbb{F}} R)^\times$$

We note that $\mathbb{G}_m = \mathbf{GL}(\mathbb{F})$.

When our central simple algebra is split, so $\mathrm{M}_n(\mathbb{F})$ after choosing an isomorphism, the general linear group over $\mathrm{M}_n(\mathbb{F})$ is essentially the classical realization of the general linear group. Precisely, if $R \in \mathbf{Alg}_{\mathbb{F}}$ then

$$\mathbf{GL}(\mathrm{M}_n(\mathbb{F}))(R) = \{B \in \mathrm{M}_n(R) \mid \det(B) \neq 0 \in R\}.$$

For this case, when the central simple algebra is split, we will use the shorter notation $\mathbf{GL}_n := \mathbf{GL}(\mathrm{M}_n(\mathbb{F}))$.

To obtain the group scheme analogue of the special linear group, we first define an extension of the reduced norm map for a central simple algebra A to a group scheme homomorphism $\mathbf{GL}(A) \to \mathbb{G}_m$.

Definition 1.2.10. *Let A be a central simple $\mathbb{F}$-algebra. Let $R \in \mathbf{Alg}_{\mathbb{F}}$ and consider the composition*

$$\mathbf{GL}(A)(R) = (A \otimes_{\mathbb{F}} R)^\times \to (A_{\mathbb{F}_{\mathrm{sep}}} \otimes_{\mathbb{F}_{\mathrm{sep}}} R_{\mathbb{F}_{\mathrm{sep}}})^\times \cong \mathrm{GL}_{\deg(A)}(R_{\mathbb{F}_{\mathrm{sep}}}) \overset{\det}{\to} R^\times_{\mathbb{F}_{\mathrm{sep}}}.$$

When an element $x \in (A \otimes_{\mathbb{F}} R)^\times$ is considered as a matrix in $\mathrm{GL}_{\deg(A)}(R_{\mathbb{F}_{\mathrm{sep}}})$ it will still have entries in R, and therefore its determinant will also be in $R^\times = \mathbb{G}_m(R)$. In this way we have a group homomorphism $\mathbf{GL}(A)(R) \to \mathbb{G}_m(R)$ for each $R \in \mathbf{Alg}_{\mathbb{F}}$, and thus we define a group scheme homomorphism

$$\mathrm{Nrd}\colon \mathbf{GL}(A) \to \mathbb{G}_m$$

called the reduced norm character.

Definition 1.2.11. *Let A be a central simple $\mathbb{F}$-algebra. The* special linear group*, $\mathbf{SL}(A)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{SL}(A)(R) = \{x \in (A \otimes_{\mathbb{F}} R)^\times \mid \mathrm{Nrd}(x) = 1\}.$$

The special linear group scheme is the kernel of the reduced norm character and therefore $\mathbf{SL}(A) \leq \mathbf{GL}(A)$. Just as with the general linear group, we will use the shortened notation $\mathbf{SL}_n := \mathbf{SL}(\mathrm{M}_n(\mathbb{F}))$. As one would expect, for $R \in \mathbf{Alg}_{\mathbb{F}}$ we have that

$$\mathbf{SL}_n(R) = \{B \in \mathrm{M}_n(R) \mid \det(B) = 1\}$$

recovering the classical special linear group.

Next we present some examples of linear algebraic groups which arise from central simple algebras with involution.

Definition 1.2.12. *Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. The* orthogonal group*, $\mathbf{O}(A, \tau)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{O}(A,\tau)(R) = \{x \in (A \otimes_{\mathbb{F}} R)^\times \mid x \cdot (\tau \otimes \mathrm{id}_R)(x) = 1\}.$$

We have that $\mathbf{O}(A, \tau) \leq \mathbf{GL}(A)$.

Definition 1.2.13. *Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. Consider the restriction of the reduced norm character to $\mathbf{O}(A, \tau)$. The kernel of this restriction is the* special orthogonal group*, $\mathbf{SO}(A, \tau)$, which is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{SO}(A, \tau)(R) = \{x \in \mathbf{O}(A, \tau)(R) \mid \mathrm{Nrd}(x) = 1\}.$$

Clearly $\mathbf{SO}(A, \tau) \leq \mathbf{O}(A, \tau)$.

Definition 1.2.14. *Let (A, ψ) be a central simple $\mathbb{F}$-algebra with symplectic involution. The* symplectic group*, $\mathbf{Sp}(A, \psi)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{Sp}(A, \psi)(R) = \{x \in (A \otimes_{\mathbb{F}} R)^{\times} \mid x \cdot (\psi \otimes \mathrm{id}_R)(x) = 1\}.$$

We have that $\mathbf{Sp}(A, \psi) \leq \mathbf{GL}(A)$.

In the case when $A \cong \mathrm{M}_n(\mathbb{F})$ is split we fix preferred orthogonal and symplectic involutions. First we define matrices

$$\Omega_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathrm{M}_n(\mathbb{F}), \quad \Psi_{2n} = \begin{bmatrix} & \Omega_n \\ -\Omega_n & \end{bmatrix} \in \mathrm{M}_{2n}(\mathbb{F})$$

$$\Xi_{2n+1} = \begin{bmatrix} & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & \end{bmatrix} \in \mathrm{M}_{2n+1}(\mathbb{F})$$

and then define involutions

$$\begin{aligned} \tau_0 \colon \mathrm{M}_{2n}(\mathbb{F}) &\to \mathrm{M}_{2n}(\mathbb{F}) & \qquad \psi_0 \colon \mathrm{M}_{2n}(\mathbb{F}) &\to \mathrm{M}_{2n}(\mathbb{F}) \\ B &\mapsto \Omega_{2n} B^T \Omega_{2n} & B &\mapsto -\Psi_{2n} B^T \Psi_{2n} \end{aligned}$$

$$\begin{aligned} \xi_0 \colon \mathrm{M}_{2n+1}(\mathbb{F}) &\to \mathrm{M}_{2n+1}(\mathbb{F}) \\ B &\mapsto \Xi_{2n+1}^{-1} B^T \Xi_{2n+1}. \end{aligned}$$

The involutions τ_0 and ξ_0 are orthogonal and ψ_0 is symplectic. We then define the following notation

$$\begin{aligned} \mathbf{O}_{2n} &:= \mathbf{O}(\mathrm{M}_{2n}(\mathbb{F}), \tau_0) & \qquad \mathbf{O}_{2n+1} &:= \mathbf{O}(\mathrm{M}_{2n+1}(\mathbb{F}), \xi_0) \\ \mathbf{SO}_{2n} &:= \mathbf{SO}(\mathrm{M}_{2n}(\mathbb{F}), \tau_0) & \mathbf{SO}_{2n+1} &:= \mathbf{SO}_{2n+1}(\mathrm{M}_{2n+1}(\mathbb{F}), \xi_0) \end{aligned}$$

$$\mathbf{Sp}_{2n} := \mathbf{Sp}(\mathrm{M}_{2n}(\mathbb{F}), \psi_0).$$

The final collection of groups we will introduce here are automorphism groups of central simple algebras. As such, we recall the Skolem-Noether theorem.

Theorem 1.2.15 (Skolem-Noether, [KMRT, 1.4])**.** *Let A be a central simple $\mathbb{F}$-algebra and let $B \subseteq A$ be a simple subalgebra. If $\varphi\colon B \to A$ is an $\mathbb{F}$-algebra homomorphism then there exists $a \in A^\times$ such that $\varphi(b) = aba^{-1}$ for all $b \in B$. That is, there exists an inner automorphism of A which restricts to φ. In particular, taking $B = A$ implies that all automorphisms of A are inner.*

Thus if A is a central simple $\mathbb{F}$-algebra then any automorphism of A is of the form $x \mapsto axa^{-1}$ for some $a \in A^\times$, however not every invertible element of A produces a distinct automorphism. The kernel of the map

$$\begin{aligned} \mathrm{Inn}\colon A^\times &\to \mathrm{Aut}_{\mathbb{F}}(A) \\ a &\mapsto (x \mapsto axa^{-1}) \end{aligned}$$

is $\mathbb{F}^\times$ and therefore we have that $\mathrm{Aut}_{\mathbb{F}}(A) = A^\times/\mathbb{F}^\times$. We now extend the automorphism group of A to obtain a group scheme.

Definition 1.2.16. *Let A be a central simple $\mathbb{F}$-algebra. The* projective linear group*, $\mathbf{PGL}(A)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{PGL}(A)(R) = \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R).$$

If R has the property that all automorphisms of $A \otimes_{\mathbb{F}} R$ are inner, such as when R is a field, then $\mathbf{PGL}(A)(R) = (A \otimes_{\mathbb{F}} R)^\times/R^\times$.

As noted in [KMRT, §23], due to the fact that for field extensions $\mathbb{E}/\mathbb{F}$ we have that $\mathbf{PGL}(A)(\mathbb{E}) = \mathbf{GL}(A)(\mathbb{E})/\mathbb{G}_m(\mathbb{E})$, there is an exact sequence of group schemes

$$1 \to \mathbb{G}_m \to \mathbf{GL}(A) \to \mathbf{PGL}(A) \to 1.$$

Therefore the projective general linear group is the scheme theoretic factor group $\mathbf{PGL}(A) = \mathbf{GL}(A)/\mathbb{G}_m$. Keeping notation consistent, when $A \cong \mathrm{M}_n(\mathbb{F})$ is split we denote this group by $\mathbf{PGL}_n$.

Definition 1.2.17. *Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. The* projective general orthogonal group*, $\mathbf{PGO}(A, \tau)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{PGO}(A, \tau)(R) = \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R, \tau \otimes 1).$$

The projective general orthogonal group fits into an exact sequence of group schemes

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}(A, \tau) \to \mathbf{PGO}(A, \tau) \to 1$$

and therefore we have that $\mathbf{PGO}(A, \tau) \cong \mathbf{O}(A, \tau)/\boldsymbol{\mu}_2$.

There is a natural map $\mathbf{PGO}(A,\tau) \to \mathbf{PGL}(C(A,\tau))$ given by extending the automorphism of (A,τ) homomorphically to its Clifford algebra. Since the center $Z = Z(C(A,\tau))$ is either a degree 2 field extension of $\mathbb{F}$ or is $\mathbb{F}\times\mathbb{F}$, in either case $\mathbf{PGL}(Z) \cong \mathbb{Z}/2\mathbb{Z}$, where we view $\mathbb{Z}/2\mathbb{Z}$ as a constant group scheme. Hence there is a composite map

$$\gamma\colon\ \mathbf{PGO}(A,\tau) \to \mathbf{PGL}(C(A,\tau)) \to \mathbf{PGL}(Z) \cong \mathbb{Z}/2\mathbb{Z}$$

where the last map is the restriction of the automorphism to the center. We use this map to define our next group scheme.

Definition 1.2.18. *Let (A,τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. The* projective (special) orthogonal group*, $\mathbf{PSO}(A,\tau)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{PSO}(A,\tau)(R) = \{\varphi \in \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R, \tau \otimes \mathrm{id}_R) \mid \gamma(\varphi) = 0\}.$$

That is, $\mathbf{PSO}(A,\tau) = \ker(\gamma)$. Thus, the elements of $\mathbf{PSO}(A,\tau)$ are those automorphisms whose induced automorphism on the Clifford algebra fixes the center. The projective orthogonal group also fits into an exact sequence of group schemes

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{SO}(A,\tau) \to \mathbf{PSO}(A,\tau) \to 1$$

and therefore we have that $\mathbf{PSO}(A,\tau) = \mathbf{SO}(A,\tau)/\boldsymbol{\mu}_2$.

Definition 1.2.19. *Let (A,ψ) be a central simple $\mathbb{F}$-algebra with symplectic involution. The* projective symplectic group*, $\mathbf{PSp}(A,\psi)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{PSp}(A,\psi)(R) = \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R, \psi \otimes \mathrm{id}_R).$$

Similarly as above, there is an exact sequence of group schemes

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Sp}(A,\psi) \to \mathbf{PSp}(A,\psi) \to 1$$

and therefore $\mathbf{PSp}(A,\psi) = \mathbf{Sp}(A,\psi)/\boldsymbol{\mu}_2$.

As above we use brief notation to refer to these groups over a split central simple $\mathbb{F}$-algebra with our preferred involutions. Precisely, we set

$$\begin{aligned}\mathbf{PGO}_{2n} &:= \mathbf{PGO}(\mathrm{M}_{2n}(\mathbb{F}),\tau_0), & \mathbf{PGO}_{2n+1} &:= \mathbf{PGO}(\mathrm{M}_{2n+1}(\mathbb{F}),\xi_0),\\ \mathbf{PSO}_{2n} &:= \mathbf{PSO}(\mathrm{M}_{2n}(\mathbb{F}),\tau_0), & \mathbf{PSO}_{2n+1} &:= \mathbf{PSO}(\mathrm{M}_{2n+1}(\mathbb{F}),\xi_0),\end{aligned}$$

$$\mathbf{PSp}_{2n} := \mathbf{PSp}(\mathrm{M}_{2n}(\mathbb{F}),\psi_0).$$

1.2.2 The Spin and Half-spin Groups

There are two other linear algebraic groups which we will discuss throughout this thesis, called the Spin and Half-spin groups and denoted by **Spin** and **HSpin** respectively. These groups are related to a central simple algebra with orthogonal involution (A, τ), and lie over $\mathbf{PSO}(A, \tau)$. We define them following [KMRT, pg. 349-351]. First, we address the cases lying over $\mathbf{PSO}_d$.

Recall that $\mathbf{PSO}_d = \mathbf{PSO}(\mathrm{M}_d(\mathbb{F}), \tau_0)$ where the orthogonal involution is given by the matrix Ω_{2n} when $d = 2n$, and by Ξ_{2n+1} when $d = 2n + 1$. We consider Ω_{2n} and Ξ_{2n+1} as the matrices of quadratic forms on $V = \mathbb{F}^d$. In particular Ω_{2n} comes from the quadratic form

$$q_{2n}(x) = x_1x_{2n} + x_2x_{2n-1} + \ldots + x_nx_{n+1}.$$

and Ξ_{2n+1} comes from the quadratic form

$$q_{2n+1}(x) = x_1x_{2n} + x_2x_{2n-1} + \ldots + x_nx_{n+2} + x_{n+1}^2.$$

Let q be the appropriate quadratic form and consider the Clifford algebra $C(V, q)$.

Definition 1.2.20. *The* split spin group*,* $\mathbf{Spin}_d$*, is defined on* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by*

$$\mathbf{Spin}_d(R) = \{c \in C_0(V, q) \otimes_{\mathbb{F}} R \mid c(V \otimes_{\mathbb{F}} R)c^{-1} = V \otimes_{\mathbb{F}} R, c\tau(c) = 1\}$$

where $c(V \otimes_{\mathbb{F}} R)c^{-1}$ *is calculated inside* $C(V, q) \otimes_{\mathbb{F}} R$*.*

By definition, $\mathbf{Spin}_d$ is a subgroup of $\mathbf{GL}(C_0(V, q))$. The condition that $c(V \otimes_{\mathbb{F}} R)c^{-1} = V \otimes_{\mathbb{F}} R$, together with the definition of the Clifford algebra, means that we get an isometry

$$\begin{aligned} \chi(R)(c) \colon (V \otimes_{\mathbb{F}} R, q \otimes 1) &\to (V \otimes_{\mathbb{F}} R, q \otimes 1) \\ v &\mapsto cvc^{-1}. \end{aligned}$$

Since this is an isometry with respect to $q \otimes 1$, it will be given by a matrix in $\mathbf{SO}_d(R)$, and so we get a group scheme homomorphism

$$\chi \colon \mathbf{Spin}_d \to \mathbf{SO}_d$$

called the *vector representation* of **Spin**. This map is surjective and has kernel $\{t \cdot 1 \mid t \in \boldsymbol{\mu}_2(R)\} \cong \boldsymbol{\mu}_2(R)$ over R and so there is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}_d \xrightarrow{\chi} \mathbf{SO}_d \to 1.$$

The half-spin group occurs when $d = 2n$ and n is even. Consider the standard basis of V given by $e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$ and let $\bar{i} = 2n + 1 - i$. The center of

$\mathbf{Spin}_{2n}(R)$ will contain another subgroup isomorphic to $\boldsymbol{\mu}_2$, in particular a subgroup of the form

$$H(R) = \{\prod_{i=1}^{n}(1 + (t-1)e_i e_{\bar{i}}) \mid t \in \boldsymbol{\mu}_2(R)\} \cong \boldsymbol{\mu}_2(R).$$

Because the discriminant of q is trivial, the Clifford algebra decomposes as $C_0(V,q) = C^+(V,q) \times C^-(V,q)$, and so there is a composition

$$\mathbf{Spin}_{2n} \hookrightarrow \mathbf{GL}(C_0(V,q)) \twoheadrightarrow \mathbf{GL}(C^+(V,q))$$

where the second map is induced by the projection $C_0(V,q) \twoheadrightarrow C^+(V,q)$. The group $H(R)$ is the kernel of this composition, and the half-spin group is the quotient of $\mathbf{Spin}_{2n}$ by this subgroup.

Definition 1.2.21. *Let n be even. The* split half-spin group*, $\mathbf{HSpin}_{2n}$, is defined to be the scheme theoretic image of the composition*

$$\mathbf{Spin}_{2n} \hookrightarrow \mathbf{GL}(C_0(V,q)) \twoheadrightarrow \mathbf{GL}(C^+(V,q)).$$

That is, if H^+ is the Hopf algebra of $\mathbf{GL}(C^+(V,q))$ and $H_{\mathbf{Spin}_{2n}}$ is the Hopf algebra of $\mathbf{Spin}_{2n}$, then the composition above corresponds to a map

$$H^+ \to H_{\mathbf{Spin}_{2n}}.$$

Letting the kernel of that map be I, then $\mathbf{HSpin}_{2n}$ is the group represented by H^+/I. The half-spin group fits into an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}_{2n} \to \mathbf{HSpin}_{2n} \to 1.$$

Now we define the spin and half-spin groups lying over $\mathbf{PSO}(A,\tau)$. Here as well the spin group will be a subgroup of the Clifford algebra, but the defining condition references the Clifford bimodule, and in particular the injection $b\colon A \to B(A,\tau)$.

Definition 1.2.22. *Let (A,τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution. The* spin group*, $\mathbf{Spin}(A,\tau)$, is defined on $R \in \mathbf{Alg}_{\mathbb{F}}$ by*

$$\mathbf{Spin}(A,\tau)(R) = \{c \in (C(A,\tau)\otimes_{\mathbb{F}} R)^\times | c*(b(A)\otimes_{\mathbb{F}} R)\cdot c^{-1} = b(A)\otimes_{\mathbb{F}} R, c(\tau\otimes 1)(c) = 1\}$$

where $c * (b(A) \otimes_{\mathbb{F}} R) \cdot c^{-1}$ is calculated inside $B(A,\tau) \otimes_{\mathbb{F}} R$.

In this case as well, $\mathbf{Spin}(A,\tau)$ is a subgroup of $\mathbf{GL}(C(A,\tau))$. There is also a group scheme homomorphism

$$\chi\colon\ \mathbf{Spin}(A,\tau) \to \mathbf{SO}(A,\tau).$$

We abuse notation and reuse χ since, as we will explain once we have Galois cohomology, for $R \in Alg_{\mathbb{F}}$ the map $\chi(R)$ can be defined as the restriction of the vector representation $\chi(R\otimes_{\mathbb{F}}\mathbb{F}_{\mathrm{sep}})\colon\ \mathbf{Spin}_n(R\otimes_{\mathbb{F}}\mathbb{F}_{\mathrm{sep}}) \to \mathbf{SO}_n(R\otimes_{\mathbb{F}}\mathbb{F}_{\mathrm{sep}})$. The kernel of this new χ is also $\boldsymbol{\mu}_2$ and hence there is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}(A,\tau) \xrightarrow{\chi} \mathbf{SO}(A,\tau) \to 1.$$

Since $\mathbf{Spin}(A,\tau)$ lies over $\mathbf{SO}(A,\tau)$, it also lies over $\mathbf{PSO}(A,\tau)$ as promised, and the image of $\chi(R)(c)$ in $\mathbf{PSO}(A,\tau)(R)$ is the automorphism above. In the case that $(A,\tau)\cong(\mathrm{M}_d(\mathbb{F}),\tau_0)$ is split, using the same quadratic space (V,q) as was appropriate before, the standard isomorphisms

$$C(\mathrm{M}_d(\mathbb{F}),\tau_0)\cong C_0(V,q) \text{ and } B(\mathrm{M}_d(\mathbb{F}),\tau_0)\cong V\otimes C_1(V,q)$$

induce an isomorphism

$$\mathbf{Spin}(\mathrm{M}_d(\mathbb{F}),\tau_0)\cong \mathbf{Spin}_d\,.$$

The half-spin group of a central simple algebra with orthogonal involution (A,τ) can be defined when $\deg(A)=2n$ with n even, and the discriminant of τ is trivial. In this case, the Clifford algebra decomposes as $C(A,\tau)=C^+(A,\tau)\times C^-(A,\tau)$ and so we can consider a similar composition as in the previous case.

Definition 1.2.23. *Let* (A,τ) *be a central simple* $\mathbb{F}$*-algebra with orthogonal involution such that* $\deg(A)=2n$ *with* n *even, and* $\mathrm{disc}(\tau)$ *is trivial. The* half-spin group*,* $\mathbf{HSpin}(A,\tau)$*, is defined as the scheme theoretic image of the composition*

$$\mathbf{Spin}(A,\tau)\hookrightarrow \mathbf{GL}(C(A,\tau)) \twoheadrightarrow \mathbf{GL}(C^+(A,\tau)).$$

The kernel of this composition is isomorphic to $\boldsymbol{\mu}_2$ *and so* $\mathbf{HSpin}(A,\tau)$ *fits into an exact sequence*

$$1\to\boldsymbol{\mu}_2\to\mathbf{Spin}(A,\tau)\to\mathbf{HSpin}(A,\tau)\to 1.$$

In the case that (A,τ) is split, the isomorphisms $\mathbf{Spin}(\mathrm{M}_{2n}(\mathbb{F}),\tau_0)\cong\mathbf{Spin}_{2n}$ and $C(A,\tau)\cong C_0(V,q)$ for appropriate quadratic space (V,q), induce an isomorphism

$$\mathbf{HSpin}(\mathrm{M}_{2n}(\mathbb{F}),\tau_0)\cong\mathbf{HSpin}_{2n}\,.$$

While defining these groups, $\mathbf{Spin}_d$ and $\mathbf{HSpin}_{2n}$ were called *split*. We see what this means in the next section.

Remark 1.2.24. The notation used in [KMRT] for some groups is slightly different than the notation we choose to use here, as contrasted in the following table.

This thesis	[KMRT]
PSp	**PGSp**
SO	$\mathbf{O}^+$
PSO	$\mathbf{PGO}^+$
HSpin	$\mathbf{Spin}^+$

1.2.3 Semisimple Groups and Split Groups

In this subsection, following [KMRT, §25], we define the notions of semisimple algebraic group, of algebraic torus, and using properties of their maximal tori we also define split semisimple groups.

First, we recall that an algebraic group $\mathbf{G}$ is called *solvable* if the set of points $\mathbf{G}(\mathbb{F}_{\text{alg}})$ is solvable in the usual group-theoretic sense, where $\mathbb{F}_{\text{alg}}$ is an algebraic closure of $\mathbb{F}$. We then define semisimple algebraic groups.

Definition 1.2.25. *Let* $\mathbf{G}$ *be an algebraic group over* $\mathbb{F}$*. We say that* $\mathbf{G}$ *is* semisimple *if*

- $\mathbf{G} \neq 1$,
- $\mathbf{G}$ *is connected*
- $\mathbf{G}_{\mathbb{F}_{\text{alg}}}$ *does not contain a nontrivial, solvable, connected, normal subgroup.*

From the previous subsection, the groups $\mathbf{SL}(A)$, $\mathbf{PGL}(A)$, $\mathbf{SO}(A, \tau)$, $\mathbf{PSO}(A, \tau)$, $\mathbf{Sp}(A, \psi)$, $\mathbf{PSp}(A, \psi)$, $\mathbf{Spin}(A, \tau)$, and $\mathbf{HSpin}(A, \tau)$ are semisimple, as shown in the classification in [KMRT, §26].

At the core of the upcoming classification of semisimple linear algebraic groups is an understanding of their maximal tori.

Definition 1.2.26. *An algebraic group* $\mathbf{T}$ *over* $\mathbb{F}$ *is an* algebraic torus *(or simply a torus) if it is a twisted form of the group*

$$(\mathbb{G}_m)^n = \mathbb{G}_m \times \ldots \times \mathbb{G}_m \text{ (n times).}$$

Precisely, $\mathbf{T}$ *is a torus if* $\mathbf{T}_{\mathbb{F}_{\text{sep}}} \cong (\mathbb{G}_{m,\mathbb{F}_{\text{sep}}})^n$ *as algebraic groups over* $\mathbb{F}_{\text{sep}}$*. Furthermore, a torus is called* split *if* $\mathbf{T} \cong (\mathbb{G}_m)^n$ *as algebraic groups over* $\mathbb{F}$*.*

We note that this definition implies that all tori over $\mathbb{F}_{\text{sep}}$ are split, and therefore if $\mathbf{T}$ is a torus over $\mathbb{F}$, then $\mathbf{T}_{\mathbb{F}_{\text{sep}}}$ is split. For a semisimple algebraic group $\mathbf{G}$, we call a torus $\mathbf{T} \subseteq \mathbf{G}$ a *maximal torus* if it is maximal among subtori of $\mathbf{G}$ with respect to inclusion. We have two observations about maximal tori.

- If $\mathbb{E}/\mathbb{F}$ is a field extension and $\mathbf{T} \subseteq \mathbf{G}$ is a maximal torus, then $\mathbf{T}_{\mathbb{E}} \subseteq \mathbf{G}_{\mathbb{E}}$ is also a maximal torus.
- If $\mathbf{T}_1$ and $\mathbf{T}_2$ are two maximal tori in $\mathbf{G}$, then $\mathbf{T}_{1,\mathbb{F}_{\text{alg}}}$ and $\mathbf{T}_{2,\mathbb{F}_{\text{alg}}}$ are conjugate in $\mathbf{G}_{\mathbb{F}_{\text{alg}}}$ by an element of $\mathbf{G}(\mathbb{F}_{\text{alg}})$.

Mirroring the terminology for tori we define split semisimple groups.

Definition 1.2.27. *Let* $\mathbf{G}$ *be a semisimple algebraic group. We call* $\mathbf{G}$ split *if it contains a maximal torus which is a split torus.*

From the previous subsection, the groups $\mathbf{SL}_n$, $\mathbf{PGL}_n$, $\mathbf{SO}(\mathrm{M}_n(\mathbb{F}), \tau)$, $\mathbf{PSO}(\mathrm{M}_n(\mathbb{F}), \tau)$, $\mathbf{Sp}(\mathrm{M}_n(\mathbb{F}), \psi)$, $\mathbf{PSp}(\mathrm{M}_n(\mathbb{F}), \psi)$, $\mathbf{Spin}_d$, and $\mathbf{HSpin}_{2n}$ are split. Equivalently, if the central simple $\mathbb{F}$-algebra is split, the corresponding group is also split.

Since any torus splits over $\mathbb{F}_{\mathrm{sep}}$ and field extensions respect the maximality of tori, we have that any algebraic group over $\mathbb{F}_{\mathrm{sep}}$ is split and therefore if $\mathbf{G}$ is an algebraic group over $\mathbb{F}$, then $\mathbf{G}_{\mathbb{F}_{\mathrm{sep}}}$ is split. Because of this fact, we first classify split semisimple algebraic groups, and then include arbitrary semisimple groups into our classification based on the properties of $\mathbf{G}_{\mathbb{F}_{\mathrm{sep}}}$. This classification arises from the action of a maximal torus on the Lie algebra of $\mathbf{G}$, which we introduce in the next subsection.

1.2.4 The Lie Algebra of an Algebraic Group

Though there are various ways to define the Lie algebra associated to a group scheme, the easiest uses the representing Hopf algebra of the group. Here we follow the presentation in [KMRT, §21.A].

Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$ and let H be the representing Hopf algebra. Let $c\colon H \to H \otimes_{\mathbb{F}} H$ be the comultiplication of H and let $u\colon H \to \mathbb{F}$ be the counit. We call an $\mathbb{F}$-linear map $D\colon H \to H$ a *derivation* if it also satisfies $D(ab) = aD(b) + D(a)b$ for all $a, b \in H$. We call a derivation *left-invariant* if $c \circ D = (\mathrm{id}_H \otimes D) \circ c$. The set of such left-invariant derivations is an $\mathbb{F}$-vector space and is the Lie algebra of $\mathbf{G}$.

Definition 1.2.28. *Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$ with representing Hopf algebra H. The* Lie algebra of $\mathbf{G}$ *is defined by*

$$\mathrm{Lie}(\mathbf{G}) = \{D \in \mathrm{End}_{\mathbb{F}}(H) \mid D \textit{ is a left-invariant derivation}\}.$$

For $D_1, D_2 \in \mathrm{Lie}(\mathbf{G})$ the Lie bracket is given by $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Corollary 1.2.29 ([KMRT, 21.2])**.** *If $\mathbf{G}$ is an algebraic group over $\mathbb{F}$, then $\mathrm{Lie}(\mathbf{G})$ is a finite dimensional $\mathbb{F}$-vector space.*

While the above definition lends itself well to easily describing the Lie bracket, it does not lend itself to easily calculating the Lie algebra of algebraic groups. For that, we recall an equivalent definition of the Lie algebra. Denote by $\mathbb{F}[\varepsilon]$ the algebra of dual numbers over $\mathbb{F}$. The element ε satisfies $\varepsilon^2 = 0$, i.e., $\mathbb{F}[\varepsilon] \cong \mathbb{F}[x]/(x^2)$. In the category $\mathbf{Alg}_{\mathbb{F}}$ there is a unique map

$$\begin{aligned} \pi\colon \mathbb{F}[\varepsilon] &\to \mathbb{F} \\ a + b\varepsilon &\mapsto a. \end{aligned}$$

Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$. Since $\mathbf{G}$ is a functor, we obtain a group homomorphism $\mathbf{G}(\pi)\colon \mathbf{G}(\mathbb{F}[\varepsilon]) \to \mathbf{G}(\mathbb{F})$. The kernel of this homomorphism is given the structure of an $\mathbb{F}$-vector space in the following way.

- For $x, y \in \ker(\mathbf{G}(\pi))$, $x + y := xy$ using the product in $\mathbf{G}(\mathbb{F}[\varepsilon])$.
- For $a \in \mathbb{F}$ we have a map $m_a \colon \mathbb{F}[\varepsilon] \to \mathbb{F}[\varepsilon]$ which sends $z \mapsto az$. Then for $x \in \ker(\mathbf{G}(\pi))$ we define $a \cdot x := \mathbf{G}(m_a)(x)$.

This vector space is then also given a Lie bracket using a second $\mathbb{F}$-algebra, $\mathbb{F}[\varepsilon_1, \varepsilon_2] \cong \mathbb{F}[x, y]/(x^2, y^2)$ (so $\varepsilon_1^2 = 0$ and $\varepsilon_2^2 = 0$) and two algebra morphisms

$$\begin{aligned} i_1 \colon \mathbb{F}[\varepsilon] &\to \mathbb{F}[\varepsilon_1, \varepsilon_2] & \qquad i_2 \colon \mathbb{F}[\varepsilon] &\to \mathbb{F}[\varepsilon_1, \varepsilon_2] \\ a + b\varepsilon &\mapsto a + b\varepsilon_1 & a + b\varepsilon &\mapsto a + b\varepsilon_2. \end{aligned}$$

Since $\mathbf{G}(\mathbb{F}[\varepsilon]) = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F}[\varepsilon])$ and the identity element of $\mathbf{G}(\mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F})$ is the counit, elements in $\ker(\mathbf{G}(\pi))$ are of the form $x = u + f\varepsilon$ where $f \colon H \to \mathbb{F}$ is a derivation (considering $\mathbb{F}$ as an H-module via the counit). Now let $x, y \in \ker(\mathbf{G}(\pi))$ and let $a = \mathbf{G}(i_1)(x)$ and $b = \mathbf{G}(i_2)(y)$. The element $aba^{-1}b^{-1} \in \mathbf{G}(\mathbb{F}[\varepsilon_1, \varepsilon_2])$ will be of the form $u + z\varepsilon_1\varepsilon_2$ where $z \colon H \to \mathbb{F}$ is a derivation. We then set

$$[x, y] := u + z\varepsilon \in \ker(\mathbf{G}(\pi)).$$

By [KMRT, 21.1] and discussion on [KMRT, pg. 335] this recovers $\mathrm{Lie}(\mathbf{G})$.

Proposition 1.2.30. *Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$. The map*

$$\begin{aligned} \mathrm{Lie}(G) &\to \ker(\mathbf{G}(\pi)) \\ D &\mapsto u + (u \circ D)\varepsilon. \end{aligned}$$

is an isomorphism of Lie algebras.

For our purposes we will require concrete descriptions of the Lie algebras associated to the groups introduced in the previous subsection. We give those now from [KMRT, 21.5] and [KMRT, §23].

- $\mathrm{Lie}(\mathbb{G}_a) = \mathbb{F}$.
- $\mathrm{Lie}(\mathbb{G}_m) = \mathbb{F}$.
- $\mathrm{Lie}(\boldsymbol{\mu}_n) = \begin{cases} 0 & \mathrm{char}(\mathbb{F}) \not| n \\ \mathbb{F} & \mathrm{char}(\mathbb{F}) | n. \end{cases}$

The above three Lie algebras all have trivial Lie bracket. Now, let A be a central simple $\mathbb{F}$-algebra and let τ and ψ be orthogonal and symplectic involutions respectively on A as appropriate. For all following examples the Lie bracket is given by $[a, b] = ab - ba$.

$$\begin{aligned} \mathrm{Lie}(\mathbf{GL}(A)) &= A & \mathrm{Lie}(\mathbf{O}(A,\tau)) &= \{a \in A \mid \tau(a) = -a\} \\ \mathrm{Lie}(\mathbf{SL}(A)) &= \{a \in A \mid \mathrm{Trd}(a) = 0\} & \mathrm{Lie}(\mathbf{Spin}(A,\tau)) &= \mathrm{Lie}(\mathbf{O}(A,\tau)) \\ \mathrm{Lie}(\mathbf{PGL}(A)) &= A/\mathbb{F} & \mathrm{Lie}(\mathbf{HSpin}(A,\tau)) &= \mathrm{Lie}(\mathbf{O}(A,\tau)) \\ \mathrm{Lie}(\mathbf{Sp}(A,\psi)) &= \{a \in A \mid \psi(a) = -a\} & \mathrm{Lie}(\mathbf{SO}(A,\tau)) &= \mathrm{Lie}(\mathbf{O}(A,\tau)) \\ \mathrm{Lie}(\mathbf{PSp}(A,\psi)) &= \mathrm{Lie}(\mathbf{Sp}(A,\psi)) & \mathrm{Lie}(\mathbf{PSO}(A,\tau)) &= \mathrm{Lie}(\mathbf{O}(A,\tau)) \end{aligned}$$

When $\mathrm{char}(\mathbb{F})$ does not divide $\deg(A)$, we have $\mathrm{Lie}(\mathbf{PGL}(A)) \cong \mathrm{Lie}(\mathbf{SL}(A))$. This is due to the fact that for $1 \in A$ we have $\mathrm{Trd}(1) = \deg(A)$, and so when this is non-zero in $\mathbb{F}$ each equivalence class in $A/\mathbb{F}$ will have a unique representative with $\mathrm{Trd}(a) = 0$.

Since the Lie algebra of an algebraic group is a finite dimensional vector space we may use it to consider representations of the group. One such representation is the adjoint representation. This is defined using the alternate description of $\mathrm{Lie}(\mathbf{G})$, in particular the fact that for $R \in \mathbf{Alg}_{\mathbb{F}}$,

$$\mathrm{Lie}(\mathbf{G}) \otimes_{\mathbb{F}} R = \ker(\mathbf{G}(R[\varepsilon]) \to \mathbf{G}(R)).$$

Elements of this kernel are of the form $x = u' + f\varepsilon$ where u' is the composition $u' \colon H \xrightarrow{u} \mathbb{F} \hookrightarrow R$, and $f \colon H \to R$ is a derivation with respect to R as an H-module via u'. For any function $g \in \mathbf{G}(R)$ the map $gfg^{-1} \colon H \to R$ defined by "using the multiplication in $\mathbf{G}(R)$", i.e.,

$$gfg^{-1} := m \circ (\mathrm{id}_H \otimes m) \circ (g \otimes f \otimes g^{-1}) \circ (\mathrm{id}_H \otimes c) \circ c$$

is also a suitable derivation.

Definition 1.2.31. *Let* $\mathbf{G}$ *be an algebraic group over* $\mathbb{F}$. *The* adjoint representation*,* $\mathrm{ad} \colon \mathbf{G} \to \mathbf{GL}(\mathrm{End}(\mathrm{Lie}(\mathbf{G})))$*, is defined over* $R \in \mathbf{Alg}_{\mathbb{F}}$ *by*

$$\begin{aligned} \mathrm{ad}(R) \colon \mathbf{G}(R) &\to \mathrm{End}(\mathrm{Lie}(\mathbf{G}) \otimes_{\mathbb{F}} R)^{\times} \\ g &\mapsto (u' + f\varepsilon \mapsto u' + gfg^{-1}\varepsilon). \end{aligned}$$

In concrete examples this action often reduces to simple conjugation. Let A be a central simple $\mathbb{F}$-algebra and consider $\mathbf{GL}(A)$. Over $R \in \mathbf{Alg}_{\mathbb{F}}$ we have that $\mathbf{GL}(A)(R) = (A \otimes_{\mathbb{F}} R)^{\times}$ and $\mathrm{Lie}(G) \otimes_{\mathbb{F}} R = A \otimes_{\mathbb{F}} R$. Then for $g \in (A \otimes_{\mathbb{F}} R)^{\times}$ and $x \in (A \otimes_{\mathbb{F}} R)$ the adjoint action is given by $g \cdot x = gxg^{-1}$.

1.2.5 Root Systems

With both maximal tori and the adjoint representation in hand, we proceed to classifying algebraic groups via the adjoint action of their maximal torus as in [KMRT, §25], beginning with split groups. Let $\mathbf{G}$ be a split, semisimple algebraic group over $\mathbb{F}$

and let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. We consider the adjoint representation restricted to the chosen maximal torus of $\mathbf{G}$ to obtain a representation

$$\mathrm{ad}_{\mathbf{T}} \colon \mathbf{T} \to \mathbf{GL}(\mathrm{End}(\mathrm{Lie}(\mathbf{G}))).$$

Since $\mathbf{G}$ is split, $\mathbf{T}$ is also split and split tori are examples of diagonalizable groups schemes. To see the relevance of this, we briefly recall the definition of diagonalizable group schemes from [KMRT, pg. 332], and an important property of their representations from [KMRT, pg. 343].

Definition 1.2.32. *Let* $\mathbf{G}$ *be an algebraic group scheme over* $\mathbb{F}$ *and let* H *be its representing Hopf algebra.* $\mathbf{G}$ *is called* diagonalizable *if there exists a classical abelian group* S *such that* $H \cong \mathbb{F}\langle S \rangle$ *is the group algebra of* S *over* $\mathbb{F}$.

We also note that if $\mathbf{D}$ is a diagonalizable group with Hopf algebra $H \cong \mathbb{F}\langle S \rangle$, then when we consider the group of characters, $\mathbf{D}^* = \mathrm{Hom}(\mathbf{D}, \mathbb{G}_m)$, we have that $\mathbf{D}^* \cong S$. Furthermore, if we consider a representation $\rho \colon \mathbf{D} \to \mathbf{GL}(\mathrm{End}(V))$ where V is a finite dimensional $\mathbb{F}$-vector space, then ρ induces a decomposition

$$V = \bigoplus_{\chi \in S} V_\chi.$$

This decomposition is a simultaneous diagonalization of the image of $\rho(\mathbb{F})$, that is

$$V_\chi = \{x \in V \mid (\rho(\mathbb{F})(a)) \cdot v = (\chi(\mathbb{F})(a))v \quad \forall a \in \mathbf{D}(\mathbb{F})\}.$$

A character $\chi \in S$ is called a *weight* if $V_\chi \neq 0$. The *multiplicity* of a weight χ is $\dim_{\mathbb{F}}(V_\chi)$.

Returning to our split semisimple group $\mathbf{G}$, since $\mathbf{T}$ is a split torus, there exists a natural number n such that $\mathbf{T} \cong (\mathbb{G}_m)^n$, which has representing Hopf algebra $\mathbb{F}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \cong \mathbb{F}\langle \mathbb{Z}^n \rangle$. Hence $\mathbf{T}$ is diagonalizable. Therefore the adjoint representation restricted to $\mathbf{T}$ yields a decomposition

$$\mathrm{Lie}(\mathbf{G}) = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha.$$

If α is a non-zero weight of this representation, that is $\alpha \neq 0$ and $V_\alpha \neq 0$, then $\dim(V_\alpha) = 1$. Furthermore, the set of non-zero weights of this representation form a root system in $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$, which we define now following [KMRT, §24].

Definition 1.2.33. *Let* V *be a real Euclidean vector space. A set* $\Phi \subset V$ *is called a* root system *if*

- $0 \notin \Phi$ *and* $\mathrm{Span}_{\mathbb{R}}(\Phi) = V$,
- *For all* $\alpha \in \Phi$, $c\alpha \in \Phi$ *if and only if* $c = \pm 1$,

- *For all* $\alpha, \beta \in \Phi$, $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$*,*
- *For all* $\alpha, \beta \in \Phi$, $r_\alpha(\beta) \in \Phi$*,*

where $\langle -, - \rangle$ *is the inner product on* V *and* $r_\alpha(\beta) := \beta - 2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha$ *is the reflection of* β *across the hyperplane perpendicular to* α*. We call* Φ irreducible *if it can not be written as the direct sum of two roots systems* $\Phi_1 \oplus \Phi_2 \subset V_1 \oplus V_2$*.*

As suggested by the name, if Φ is a root system we call its elements *roots*. Due to the last point of the definition, each reflection r_α stabilizes the root system and so Φ is also stabilized by every element of the Weyl group.

Definition 1.2.34. *Let* $\Phi \subset V$ *be a root system. The* Weyl group *of* Φ *is*

$$W = \langle r_\alpha \mid \alpha \in \Phi \rangle \subset \mathrm{End}(V).$$

It is the group generated by the reflections r_α *and is a finite group.*

The classification of root systems is done by assigning a graph, called the Dynkin diagram, to each root system. The Dynkin diagram is defined from information contained in subsets of Φ called simple systems.

Definition 1.2.35. *Let* $\Phi \subset V$ *be a root system. A subset* $\Delta \subset \Phi$ *is called a* simple system *if for all* $\alpha \in \Phi$ *there exist integers* n_δ*, either all non-negative or all non-positive, such that*

$$\alpha = \sum_{\delta \in \Delta} n_\delta \delta.$$

Simple systems exist for any root system, and any two simple systems in Φ *are conjugate via the Weyl group. Furthermore, any simple system forms a basis of* V*.*

The size of any simple system is called the *rank* of the root system, denoted $\mathrm{rank}(\Phi)$.

Definition 1.2.36. *Let* Φ *be a root system with simple system* Δ*. The* Dynkin diagram*,* $\mathrm{Dyn}(\Phi)$*, is a graph defined as follows. The set of vertices is the simple system* Δ*. Then for* $\alpha \neq \beta \in \Delta$*, the vertices are joined by* $\frac{4\langle \alpha,\beta\rangle^2}{\langle \alpha,\alpha\rangle\langle \beta,\beta\rangle}$ *edges. These edges are*

- *Undirected if* $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$*,*
- *Directed towards* α *if* $\langle \alpha, \alpha \rangle < \langle \beta, \beta \rangle$*,*
- *Directed towards* β *if* $\langle \alpha, \alpha \rangle > \langle \beta, \beta \rangle$*,*

that is, all directed edges are directed towards short roots.

The Dynkin diagram of a root system is unique up to isomorphism. Conveniently, every Dynkin diagram can be categorized into one of finitely many families which we use to categorize root systems into types, either type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2. We refer to [KMRT, §24.A] for details and diagrams.

In this way, every split semisimple algebraic group $\mathbf{G}$ over $\mathbb{F}$ gives rise to a root system, and thus a Dynkin diagram of some type. If $\mathbf{G}$ is a semisimple algebraic group which is not split, then it becomes split over a separable closure of $\mathbb{F}$ and $\mathbf{G}_{\mathbb{F}_{\mathrm{sep}}}$ yields a Dynkin diagram.

Definition 1.2.37. *Let* $\mathbf{G}$ *be a semisimple algebraic group over* $\mathbb{F}$*. Since* $\mathbf{G}_{\mathbb{F}_{\mathrm{sep}}}$ *is a split group it contains a split maximal torus which yields a root system* Φ *of some type. We call* Φ *the* root system of $\mathbf{G}$ *and write* $\Phi(\mathbf{G}) = \Phi$*. Similarly the* type of $\mathbf{G}$ *is the type of* $\Phi(\mathbf{G})$*.*

Since non-split groups are classified by the type they take over $\mathbb{F}_{\mathrm{sep}}$, this classification fails to distinguish some algebraic groups. However, it does not only fail to distinguish between split and non-split groups, there are also non-isomorphic split groups which have the same type. To distinguish between these groups we take a closer look at the characters of their maximal tori.

1.2.6 Simply Connected and Adjoint Groups

In the construction of a root system from a split semisimple algebraic group $\mathbf{G}$ over $\mathbb{F}$, the roots that arise are characters of the maximal torus $\mathbf{T}$. We set $\Lambda_r = \mathrm{Span}_{\mathbb{Z}}(\Phi(\mathbf{G}))$ and call it the *root lattice of* $\mathbf{G}$. Clearly we have $\Lambda_r \subseteq \mathbf{T}^*$. The differences between split algebraic groups of the same type manifests in how far $\mathbf{T}^*$ is from Λ_r. To make this precise, and to describe the farthest $\mathbf{T}^*$ can differ from Λ_r, we introduce co-roots.

Definition 1.2.38. *Let* $\Phi \subset V$ *be a root system. For each* $\alpha \in \Phi$*, we define* $\check{\alpha} \in V^*$ *by* $\check{\alpha}(x) = 2\frac{\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}$ *for all* $x \in V$*. These* $\check{\alpha}$ *are called* co-roots*, and* $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\} \subset V^*$ *is a root system as well, called the* dual root system.

Root systems of most types are self dual, that is $\Phi \cong \check{\Phi}$, however a root system of type B_n is dual to one of type C_n and vice-versa. Since co-roots are linear functionals on V, they allow us to pick out a second distinguished lattice in V.

Definition 1.2.39. *Let* $\Phi \subset V$ *be a root system and* $\check{\Phi} \subset V^*$ *its dual root system. We define the* root lattice *of* Φ *as*

$$\Lambda_r = \mathrm{Span}_{\mathbb{Z}}(\Phi)$$

and we define the weight lattice *of* Φ *as*

$$\Lambda_w = \{x \in V \mid \check{\alpha}(x) \in \mathbb{Z} \quad \forall \check{\alpha} \in \check{\Phi}\}.$$

As a consequence of the definition of roots systems, we always have that $\Lambda_r \subseteq \Lambda_w$*.*

We note that since Λ_r and Λ_w are lattices of the same rank, that is the $\mathbb{R}$-span of both is V, we have that Λ_w/Λ_r is a finite group.

Let $\mathbf{G}$ be a split, semisimple algebraic group over $\mathbb{F}$. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus and $\Phi(\mathbf{G}) \subset \mathbf{T}^* \otimes_{\mathbb{Z}} \mathbb{R}$ the resulting root system. Let $\Lambda_r, \Lambda_w \subset \mathbf{T}^* \otimes_{\mathbb{Z}} \mathbb{R}$ be the root and weight lattices respectively. In the next series of statements we make precise the relation between $\mathbf{T}^*$, Λ_r, and Λ_w which we alluded to earlier.

Proposition 1.2.40 ([KMRT, 25.2])**.** *In the above setup, we have that* $\Lambda_r \subseteq \mathbf{T}^* \subseteq \Lambda_w$, *where we identify* $\mathbf{T}^*$ *with the elements* $\chi \otimes 1 \in \mathbf{T}^* \otimes_{\mathbb{Z}} \mathbb{R}$ *for* $\chi \in \mathbf{T}^*$.

Theorem 1.2.41 ([KMRT, 25.3])**.** *Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be split, semisimple algebraic groups over* $\mathbb{F}$ *with maximal tori* $\mathbf{T}_1, \mathbf{T}_2$ *and root systems* $\Phi(\mathbf{G}_1), \Phi(\mathbf{G}_2)$ *respectively. Then* $\mathbf{G}_1 \cong \mathbf{G}_2$ *if and only if there exists an isomorphism of* $\mathbb{R}$*-vector spaces*

$$f\colon\ \mathbf{T}_1^* \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbf{T}_2^* \otimes_{\mathbb{Z}} \mathbb{R}$$

such that $f(\Phi(\mathbf{G}_1)) = \Phi(\mathbf{G}_2)$ *and* $f(\mathbf{T}_1^*) = \mathbf{T}_2^*$.

Because Λ_w/Λ_r is finite, there are only finitely many options for subgroups of Λ_w which contain Λ_r, and therefore for any given type there are only finitely many isomorphism classes of split, semisimple algebraic groups of that type. Groups with the largest or smallest possible $\mathbf{T}^*$ are given the names in the title of this subsection.

Definition 1.2.42. *Let* $\mathbf{G}$ *be a split, semisimple algebraic group over* $\mathbb{F}$. *Let* $\mathbf{T} \subset \mathbf{G}$ *be a maximal torus and let* $\Lambda_r, \Lambda_w \subset \mathbf{T}^* \otimes_{\mathbb{Z}} \mathbb{R}$ *be the root and weight lattice respectively. The group* $\mathbf{G}$ *is called* simply connected *if* $\mathbf{T}^* = \Lambda_w$, *and it is called* adjoint *if* $\mathbf{T}^* = \Lambda_r$. *If* $\mathbf{G}$ *is any semisimple group, it is called simply connected if* $\mathbf{G}_{\mathbb{F}_{\mathrm{sep}}}$ *is, and likewise for adjoint.*

Furthermore, for each of the finitely many options for subgroups between Λ_r and Λ_w there is a corresponding group.

Theorem 1.2.43 ([KMRT, 25.5])**.** *Let* $\Phi \subset V$ *be a root system with root and weight lattice* Λ_r, Λ_w *respectively. For all additive subgroups* $L \leq V$ *such that* $\Lambda_r \subseteq L \subseteq \Lambda_w$ *there exists a split, semisimple algebraic group* $\mathbf{G}$ *over* $\mathbb{F}$ *such that* $(\Phi(\mathbf{G}), \mathbf{T}^*) \cong (\Phi, L)$. *That is, there exists an isomorphism of* $\mathbb{R}$*-vector spaces* $f : \mathbf{T}^* \otimes_{\mathbb{Z}} \mathbb{R} \to V$ *such that* $f(\Phi(\mathbf{G})) = \Phi$ *and* $f(\mathbf{T}^*) = L$.

Just as the character lattice of a group's maximal torus sits between the root and weight lattice, so too does the group itself sit between the simply connected and adjoint groups of its type. It does so in the following way.

Theorem 1.2.44 ([KMRT, 26.7])**.** *Let* $\mathbf{G}$ *be a semisimple algebraic group over* $\mathbb{F}$. *Then there exists a simply connected group* $\mathbf{G}_{\mathrm{sc}}$ *and an adjoint group* $\mathbf{G}_{\mathrm{ad}}$ *of the same type, each unique up to isomorphism, and there exist central isogenies*

$$\mathbf{G}_{\mathrm{sc}} \to \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}\,.$$

That is, the kernel of each map is a central subgroup and each map is surjective.

We end this chapter by summarizing the attributes, which type they are and where they lay on the simply connected vs adjoint scale, of each semisimple group introduced previously. The groups we introduced previously are of classical type, either A_n, B_n, C_n, or D_n.

	Simply Connected	Neither	Adjoint	
A_n	$\mathbf{SL}(A)$	$\mathbf{SL}(A)/\boldsymbol{\mu}_m$	$\mathbf{PGL}(A)$	$\deg(A) = n+1$
B_n	$\mathbf{Spin}(A,\tau)$	None	$\mathbf{PSO}(A,\tau)$	$\deg(A) = 2n+1$
C_n	$\mathbf{Sp}(A,\psi)$	None	$\mathbf{PSp}(A,\psi)$	$\deg(A) = 2n$
D_n	$\mathbf{Spin}(A,\tau)$	$\mathbf{SO}(A,\tau), \mathbf{HSpin}(A,\tau)$	$\mathbf{PSO}(A,\tau)$	$\deg(A) = 2n$

where m divides $n+1$, and n must be even to have **HSpin** occur. These groups fit together in the following diagrams, arranged by type.

$$\begin{tikzcd}
A_n & B_n & C_n & & D_n \\
& & & & n \text{ even} \\
\mathbf{SL}(A) \arrow[d, dashed, two heads] & \mathbf{Spin}(A,\tau) \arrow[dd, two heads] & \mathbf{Sp}(A,\psi) \arrow[dd, two heads] & & \mathbf{Spin}(A,\tau) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathbf{SL}(A)/\boldsymbol{\mu}_m \arrow[d, dashed, two heads] & & & \mathbf{SO}(A,\tau) \arrow[dr, two heads] & & \mathbf{HSpin}(A,\tau) \arrow[dl, two heads] \\
\mathbf{PGL}(A) & \mathbf{PSO}(A,\tau) & \mathbf{PSp}(A,\psi) & & \mathbf{PSO}(A,\tau) &
\end{tikzcd}$$

$$\begin{tikzcd}
D_n \\
n \text{ odd} \\
\mathbf{Spin}(A',\tau) \arrow[d, two heads] \\
\mathbf{SO}(A',\tau) \arrow[d, two heads] \\
\mathbf{PSO}(A',\tau)
\end{tikzcd}$$

where the degree of A in types A_n, B_n, C_n, and D_n is $n+1$, $2n+1$, $2n$, and $2n$ with n even respectively, and the degree of A' is $2n$ with n odd. The involutions τ are orthogonal on their respective algebras, and the involution ψ is symplectic. The topmost groups are simply connected and the bottommost are the adjoint groups. In type A_n we assume that m divides n, and the dashed arrows are to indicate that there may be multiple groups $\mathbf{SL}(A)/\boldsymbol{\mu}_{m_i}$ partially ordered by $\mathbf{SL}(A)/\boldsymbol{\mu}_{m_1} \twoheadrightarrow \mathbf{SL}(A)/\boldsymbol{\mu}_{m_2}$ if $m_1|m_2$.

1.3 Chevalley Groups

Chevalley groups, as in [Ste68], arise from representations $\mathrm{Lie}(\mathbf{G}) \to \mathrm{End}(V)$ of the Lie algebra of a split semisimple linear algebraic group $\mathbf{G}$ over $\mathbb{C}$. The Chevalley group, as a subgroup of $\mathbf{GL}(V)$, is generated by formal exponentials of images of a Chevalley basis in $\mathrm{Lie}(\mathbf{G})$. However, these representations can be somewhat sidestepped. Once one representation is used to calculate constants which describe the commutators in the Chevalley group, these constants, alongside the root system of $\mathbf{G}$, can be used to define the Chevalley groups, as well as other Chevalley groups of the same type, via generators and relations. In fact, since the constants which arise are all integers, this construction can be extended to produce a group for every $R \in \mathbf{Alg}_{\mathbb{F}}$, and so defines a *Chevalley group functor*. We utilize this latter approach, and will provide the constants that define the Chevalley group functors we are interested in. We introduce these Chevalley group functors because they form subfunctors of our linear algebraic groups which agree on stalks with respect to the étale topology on $\mathbf{Alg}_{\mathbb{F}}$. Therefore, we can consider a linear algebraic group as the sheafification of its Chevalley subfunctor. This allows us to use the generators and relations presentation of the Chevalley functors to compute properties of the linear algebraic groups, and of group scheme homomorphisms between them.

1.3.1 Generators and Relations

To describe Chevalley groups abstractly, we first need some constants arising from Lie algebras over $\mathbb{C}$. We describe them following [Ste68]. Let $\mathbf{G}$ be a split simple linear algebraic group over $\mathbb{C}$. Recall from section 1.2.5 that a choice of split maximal torus $\mathbf{T} \subset \mathbf{G}$ induces a decomposition of the Lie algebra $\mathrm{Lie}(\mathbf{G}) = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\alpha$. The root system Φ of $\mathbf{G}$ will be indecomposable since $\mathbf{G}$ is simple, and is comprised of those $\alpha \neq 0$ for which $V_\alpha \neq 0$ and so we may write

$$\mathrm{Lie}(\mathbf{G}) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} V_\alpha \right)$$

where $\mathfrak{h} = V_0$ is a Cartan subalgebra of $\mathrm{Lie}(\mathbf{G})$. Since the non-zero weight spaces are one dimensional we can find a convenient $\mathbb{C}$-basis of $\mathrm{Lie}(\mathbf{G})$ which is related to the root system. It is called a *Chevalley basis.*

Theorem 1.3.1 ([Ste68, Theorem 1])**.** *Let* $\mathbf{G}$ *and* $\mathrm{Lie}(\mathbf{G})$ *be as above. Consider a system of simple roots* $\{\alpha_1, \dots, \alpha_l\} \subset \Phi$*. There exists elements* $H_\alpha \in \mathfrak{h}$ *and* $X_\alpha \in V_\alpha$ *for* $\alpha \in \Phi$ *such that*

- $H_\alpha \in \mathrm{Span}_{\mathbb{Z}}\{H_{\alpha_1}, \dots, H_{\alpha_l}\}$,
- $[H_{\alpha_i}, H_{\alpha_j}] = 0$,

- $[H_{\alpha_i}, X_\alpha] = \langle \alpha, \alpha_i \rangle X_\alpha$,
- $[X_\alpha, X_{-\alpha}] = H_\alpha$,
- $[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}$ *if* $\alpha + \beta \in \Phi$,
- $[X_\alpha, X_\beta] = 0$ *if* $0 \neq \alpha + \beta \in \Phi$.

where $r \in \mathbb{Z}$ *is the maximal integer such that* $\beta, \beta - \alpha, \beta - 2\alpha, \ldots, \beta - r\alpha$ *are all in* Φ.

The constants we are looking for come from relations in the universal enveloping algebra of Lie($\mathbf{G}$), denoted by $\mathcal{U}$. Since Lie($\mathbf{G}$) injects into $\mathcal{U}$ we abuse notation and use X_α to also refer to the image of the basis element in $\mathcal{U}$. We are interested in the following result about commutators, which we denote by $(a, b) = aba^{-1}b^{-1}$.

Lemma 1.3.2 ([Ste68, Lemma 15])**.** *Fix an order on* Φ *and let* $\alpha, \beta \in \Phi$ *with* $\alpha + \beta \neq 0$. *Consider elements* $\exp(tX_\alpha) := \sum_{n=0}^{\infty} \frac{(tX_\alpha)^n}{n!} \in \mathcal{U}[[t,u]]$ *and* $\exp(uX_\beta) \in \mathcal{U}[[t,u]]$ *defined similarly. In this ring of formal power series we have that*

$$(\exp(tX_\alpha), \exp(uX_\beta)) = \prod_{i,j>0} \exp(c_{ij} t^i u^j X_{i\alpha + j\beta}).$$

where the product on the right is over $i\alpha + j\beta \in \Phi$ *with respect to the fixed order. The constants* c_{ij} *are integers which only depend on* α, β*, and the fixed order.*

Since this result holds over the universal enveloping algebra and with formal variables, it will also hold in any associative algebra in which Lie($\mathbf{G}$) is represented, and therefore the constants can be calculated using any convenient representation. In practice, we will be concerned with Lie algebras which sit naturally inside matrix algebras and so we can calculate the c_{ij} directly. With these constants in hand, we generalize [Ste68, Chapter 6], which defines an abstract group dependent on the choice of a field, to define group valued functors that we call the universal Chevalley groups.

Definition 1.3.3. *Let* $\mathbf{G}$ *and* Φ *be as above with some fixed order on* Φ*, and let* $\mathbb{F}$ *be a field. For any* $R \in \mathbf{Alg}_{\mathbb{F}}$*, consider formal elements* $x_\alpha(t)$ *for* $\alpha \in \Phi$ *and* $t \in R$*, as well as elements* $w_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ *and* $h_\alpha(t) := w_\alpha(t)w_\alpha(-1)$ *for* $t \in R^\times$*. The* R*-points of the* universal Chevalley group of type Φ over $\mathbb{F}$*, denoted* $\mathbf{E}_\Phi(R)$*, are defined to be the group generated by the elements* $x_\alpha(t)$ *subject to the following relations if* $\operatorname{rank}(\Phi) \geq 2$.

- $x_\alpha(t_1)x_\alpha(t_2) = x_\alpha(t_1 + t_2)$ *for all* $\alpha \in \Phi$ *and* $t_1, t_2 \in R$,
- $h_\alpha(t_1)h_\alpha(t_2) = h_\alpha(t_1 t_2)$ *for all* $\alpha \in \Phi$ *and* $t_1, t_2 \in R^\times$,

- *If $\alpha, \beta \in \Phi$ and $\alpha + \beta \neq 0$, then for all $t, u \in R$,*
$$(x_\alpha(t), x_\beta(u)) = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{ij}t^i u^j)$$
for c_{ij} as in lemma 1.3.2 with respect to the fixed order.

If instead we have that $\operatorname{rank}(\Phi) = 1$ *then the last relation does not apply and so we replace it with*

- *$w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^2u)$ for all $\alpha \in \Phi$, $t \in R^\times$, and $u \in R$.*

This defines a functor $\mathbf{E}_\Phi \colon \mathbf{Alg}_\mathbb{F} \to \mathbf{Grp}$, where an $\mathbb{F}$-algebra homomorphism $\varphi \colon R \to S$ is sent to

$$\begin{aligned} \mathbf{E}_\Phi(\varphi) \colon \mathbf{E}_\Phi(R) &\to \mathbf{E}_\Phi(S) \\ x_\alpha(t) &\mapsto x_\alpha(\varphi(t)) \end{aligned}$$

The set of points $\mathbf{E}_\Phi(\mathbb{F})$ recovers the group defined in [Ste68]. We choose to denote these groups with the letter $\mathbf{E}$ to match naming conventions in the literature. As we will discuss later, many of these new functors are subfunctors of the linear algebraic groups defined in section 1.2. Their images are called the *elementary subgroup* of the linear algebraic group. The utility of these Chevalley groups comes from the generators and relations based presentation lending itself to concrete calculations, referred to as *elementary calculations*. Toward that end, we now recall some useful identities which follow from the relations enforced by the definition.

Theorem 1.3.4. *Let $\mathbf{E}_\Phi$ be a universal Chevalley group over $\mathbb{F}$. For any $R \in \mathbf{Alg}_\mathbb{F}$ the following relations hold in $\mathbf{E}_\Phi(R)$:*

- $w_\alpha(1)h_\beta(t)w_\alpha(-1) = h_{r_\alpha(\beta)}(t)$
- $w_\alpha(1)x_\beta(u)w_\alpha(-1) = x_{r_\alpha(\beta)}(c(\alpha,\beta)u)$
- $h_\alpha(t)x_\beta(u)h_\alpha(t^{-1}) = x_\beta(t^{\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}}u).$

for $\alpha, \beta \in \Phi$, $u \in R$, $t \in R^\times$ and where $c(\alpha,\beta) = \pm 1$.

Proof: These relations follow from relations (R5), (R3), and (R6) in [Ste71, pg.975] which are stated for *Steinberg groups* over any commutative ring. Steinberg groups are subject to the same first and third relations as in definition 1.3.3, but are not subject to the relation enforcing multiplicativity of the $h_\alpha(t)$. The cited relations are

$$\text{(R5)}\quad w_\alpha(s)h_\beta(t)w_\alpha(s)^{-1} = h_{r_\alpha(\beta)}\left(c(\alpha,\beta)s^{-\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}}t\right)h_{r_\alpha(\beta)}\left(c(\alpha,\beta)s^{-\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}}\right)^{-1}$$

(R3) $w_\alpha(s)x_\beta(u)w_\alpha(s)^{-1} = x_{r_\alpha(\beta)}\left(c(\alpha,\beta)s^{-\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}}u\right)$

(R6) $h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta\left(t^{\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}}u\right)$.

After noting that $w_\alpha(s)^{-1} = w_\alpha(-s)$, setting $s = 1$, and enforcing the multiplicativity of the $h_\alpha(t)$ which holds in our Chevalley groups, these relations become the relations we desire. ∎

The universal Chevalley group is similar in spirit to the simply connected group of its type, it is the largest in the sense that any other Chevalley group of the same type is a central quotient of the universal Chevalley group. Because of this, we now recall a general description of the maximal torus and center of $\mathbf{E}_\Phi$.

Lemma 1.3.5 ([Ste68, Lemma 28]). *Let* $\mathbf{E}_\Phi$ *be a universal Chevalley group over* $\mathbb{F}$. *Choose a simple system* $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ *and set* $h_i(t) = h_{\alpha_i}(t)$ *for* $\alpha_i \in \Delta$ *and* $t \in R^\times$. *Let* $H(R) = \langle h_\alpha(t) \mid \alpha \in \Phi, t \in R^\times\rangle$. *Then,*

- $H(R)$ *is an abelian group generated by the* $h_i(t)$*'s,*
- *Each* $h \in H(R)$ *can be expressed uniquely as* $h = \prod_{i=1}^l h_i(t_i)$ *for some* $t_i \in R^\times$,
- $Z(\mathbf{E}_\Phi(R)) = \{\prod_{i=1}^l h_i(t_i) \mid \prod_{i=1}^l t_i^{\frac{2\langle\beta,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}} = 1$ *for all* $\beta \in \Delta\}$.

Proof: We note here that while our source, [Ste68, §3], is only written for the case when $R = k$ is a field, none of the proofs utilize the field structure. Therefore, the same statements and proofs hold over any commutative ring R. The only minor exception is that [Ste68, Lemma 28] states that $Z(\mathbf{E}_\Phi(k))$ is finite. This, in general, does not hold over commutative rings and $Z(\mathbf{E}_\Phi(R))$ may be infinite. ∎

In order to proceed in connecting these Chevalley groups to our previous linear algebraic groups, we now specify the structure constants c_{ij} for each type of classical irreducible root system by detailing the commutator relations in $\mathbf{E}_\Phi$. Each root system will be given in $\mathbb{R}^n$ with e_i being the i^{th} standard basis vector. The constants for types B_n, C_n, and D_n were summarized in [Rue20] and so we take their description from there, however these constants are well known. See [HO89], for instance.

Proposition 1.3.6. *Let* $\Phi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$ *which is a root system of type* A_{n-1}. *Denote the universal Chevalley group of this type by* $\mathbf{E}_{\mathbf{SL}_n}$. *It is defined by the following commutator relations. Let* $\alpha, \beta \in \Phi$,

- *If* $0 \neq \alpha + \beta \notin \Phi$ *then* $(x_\alpha(t), x_\beta(u)) = 1$.

- *Otherwise let $\alpha = \pm(e_i - e_j), \beta = \pm(e_k - e_l)$ with $i < j$ and $k < l$. Assume $i \leq k$, then we are in one of the following cases,*

$$\begin{aligned} j = k :& (x_{a(e_i-e_j)}(t), x_{a(-e_j+e_l)}(u)) = x_{a(e_i-e_l)}(atu) \\ i \neq k, j = l :& (x_{a(e_i-e_j)}(t), x_{a(-e_k+e_j)}(u)) = x_{a(e_i-e_k)}(atu) \\ i = k, j \neq l :& (x_{a(e_i-e_j)}(t), x_{a(-e_i+e_l)}(u)) = \begin{cases} x_{a(-e_j+e_l)}(-atu) & j < l \\ x_{a(e_l-e_j)}(-atu) & l < j \end{cases} \end{aligned}$$

where $a = \pm 1$. If $k < i$ then the appropriate commutator is the inverse of one above.

Proposition 1.3.7. *Let $\Phi = \{\pm e_i \pm e_j, \pm e_k \mid 1 \leq i, j, k \leq n, i < j\}$ which is a root system of type B_n. Denote the universal Chevalley group of this type by $\mathbf{E}_{\mathbf{Spin}_{2n+1}}$. It is defined by the following commutator relations. Let $\alpha, \beta \in \Phi$,*

- *If $0 \neq \alpha + \beta \notin \Phi$ then $(x_\alpha(t), x_\beta(u)) = 1$.*

Otherwise, for integers $i < j$, $k < l$, and $a_1, a_2, a_3, a_4 = \pm 1$, we are in one of the following cases if $i \leq k$.

- *$\alpha = a_1 e_i + a_2 e_j, \beta = a_3 e_k + a_4 e_l$ when*

$$\begin{aligned} j = k :& (x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j+a_4e_l}(u)) = x_{a_1e_i+a_4e_l}(-a_2tu) \\ i \neq k, j = l :& (x_{a_1e_i+a_2e_j}(t), x_{a_3e_k-a_2e_j}(u)) = x_{a_1e_i+a_3e_k}(-a_3tu) \\ i = k, j \neq l :& (x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i+a_4e_l}(u)) = \begin{cases} x_{a_2e_j+a_4e_l}(a_2tu) & j < l \\ x_{a_4e_l+a_2e_j}(-a_4tu) & l < j. \end{cases} \end{aligned}$$

- *$\alpha = a_1 e_i + a_2 e_j, \beta = a_3 e_k$ when*

$$\begin{aligned} i = k :& (x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i}(u)) = x_{-a_1e_i+a_2e_j}(-tu^2)x_{a_2e_j}(a_2tu) \\ j = k :& (x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j}(u)) = x_{a_1e_i-a_2e_j}(tu^2)x_{a_1e_i}(-a_2tu) \end{aligned}$$

- *$\alpha = a_1 e_i, \beta = a_2 e_k$, then*

$$(x_{a_1e_i}(t), x_{a_2e_k}(u)) = x_{a_1e_i+a_2e_k}(-2a_2tu).$$

If $k < i$ then the appropriate commutator is the inverse of one above.

Proposition 1.3.8. *Let $\Phi = \{\pm e_i \pm e_j, \pm 2e_k \mid 1 \leq i, j, k \leq n, i < j\}$ which is a root system of type C_n. Denote the universal Chevalley group of this type by $\mathbf{E}_{\mathbf{Sp}_{2n}}$. It is defined by the following commutator relations. Let $\alpha, \beta \in \Phi$,*

- *If* $0 \neq \alpha + \beta \notin \Phi$ *then* $(x_\alpha(t), x_\beta(u)) = 1$.

Otherwise, for integers $i < j$, $k < l$, *and* $a_1, a_2, a_3, a_4 = \pm 1$, *we are in one of the following cases if* $i \leq k$.

- $\alpha = a_1e_i + a_2e_j, \beta = a_3e_k + a_4e_l$ *when*

$$\begin{aligned} j = k, a_3 = -a_2 :& (x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j+a_4e_l}(u)) = x_{a_1e_i+a_3e_l}(a_2ctu) \\ i \neq k, j = l, a_4 = -a_2 :& (x_{a_1e_i+a_2e_j}(t), x_{a_3e_k-a_2e_j}(u)) = x_{a_1e_i+a_3e_k}(a_2ctu) \\ i = k, a_3 = -a_1, j \neq l :& (x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i+a_4e_l}(u)) = \begin{cases} x_{a_2e_j+a_4e_l}(a_1ctu) & j < l \\ x_{a_4e_l+a_2e_j}(a_1ctu) & l < j \end{cases} \end{aligned}$$

where $c = \min\{a_1a_2, a_3a_4\}$. *Furthermore, when* $i = k, j = l$,

$$\begin{aligned} (x_{a_1e_i+a_2e_j}(t), x_{a_1e_i-a_2e_j}(u)) &= x_{2a_1e_i}(-2a_2tu) \\ (x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i+a_2e_j}(u)) &= x_{2a_2e_j}(-2a_1tu) \end{aligned}$$

- $\alpha = a_1e_i + a_2e_j, \beta = 2a_3e_k$ *when*

$$\begin{aligned} i = k :& (x_{a_1e_i+a_2e_j}(t), x_{-2a_1e_i}(u)) = x_{-a_1e_i+a_2e_j}(a_2tu) \cdot x_{2a_2e_j}(-a_1a_2t^2u) \\ j = k :& (x_{a_1e_i+a_2e_j}(t), x_{-2a_2e_j}(u)) = x_{a_1e_i-a_2e_j}(a_1tu) \cdot x_{2a_1e_i}(-a_1a_2t^2u). \end{aligned}$$

If $k < i$ *then the appropriate commutator is the inverse of one above.*

Proposition 1.3.9. *Let* $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$ *which is a root system of type* D_n. *Denote the universal Chevalley group of this type by* $\mathbf{E}_{\mathbf{Spin}_{2n}}$. *It is defined by the following commutator relations. Let* $\alpha, \beta \in \Phi$,

- *If* $0 \neq \alpha + \beta \notin \Phi$ *then* $(x_\alpha(t), x_\beta(u)) = 1$.
- *Otherwise let* $\alpha = a_1e_i + a_2e_j, \beta = a_3e_k + a_4e_l$ *for integers* $i < j$, $k < l$, *and for* $a_1, a_2, a_3, a_4 = \pm 1$, *we are in one of the following cases if* $i \leq k$.

$$\begin{aligned} j = k :& (x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j+a_4e_l}(u)) = x_{a_1e_i+a_4e_l}(-a_2tu) \\ i \neq k, j = l :& (x_{a_1e_i+a_2e_j}(t), x_{a_3e_k-a_2e_j}(u)) = x_{a_1e_i+a_3e_k}(-a_3tu) \\ i = k, j \neq l :& (x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i+a_4e_l}(u)) = \begin{cases} x_{a_2e_j+a_4e_l}(a_2tu) & j < l \\ x_{a_4e_l+a_2e_j}(-a_4tu) & l < j. \end{cases} \end{aligned}$$

If $k < i$ *then the appropriate commutator is the inverse of one above.*

Proof: The proofs of all four of the above propositions are similar. As noted after lemma 1.3.2, the c_{ij} may be computed in any convenient representation of $\mathrm{Lie}(\mathbf{G})$, so we may choose the natural representations. An appropriate choice of Chevellay basis will cause the images of $\exp(tX_\alpha)$ under the natural representation to agree with the images of $x_\alpha(t)$ under the maps described below in section 1.3.2. The c_{ij} can then be extracted from matrix computations. ∎

As the notation suggests, we will call $\mathbf{E}_{\mathbf{SL}_n}, \mathbf{E}_{\mathbf{Spin}_d}$, and $\mathbf{E}_{\mathbf{Sp}_{2n}}$ the special linear, spin, and symplectic Chevalley groups respectively. With these concrete choices of root systems we can calculate the center of each of the above universal Chevalley groups over $R \in \mathbf{Alg}_{\mathbb{F}}$ using lemma 1.3.5.

Proposition 1.3.10. *For each root system Φ of types A_n, B_n, C_n, or D_n choose the standard simple system as in [KMRT, 24.A]. Then, for $R \in \mathbf{Alg}_{\mathbb{F}}$ we have*

$$\begin{aligned} Z(\mathbf{E}_{\mathbf{SL}_n}(R)) &= \left\{ \prod_{i=1}^{n-1} h_i(t^i) \mid t \in \boldsymbol{\mu}_n(R) \right\} \cong \boldsymbol{\mu}_n(R) \\ Z(\mathbf{E}_{\mathbf{Spin}_{2n+1}}(R)) &= \{h_n(t) \mid t \in \boldsymbol{\mu}_2(R)\} \cong \boldsymbol{\mu}_2(R) \\ Z(\mathbf{E}_{\mathbf{Sp}_{2n}}(R)) &= \left\{ \prod_{\substack{i=1 \\ i \text{ odd}}}^{n} h_i(t) \mid t \in \boldsymbol{\mu}_2(R) \right\} \cong \boldsymbol{\mu}_2(R) \end{aligned}$$

Type D_n depends on the parity of n. We have

$$Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)) = \left\{ \left(\prod_{\substack{i=1 \\ i \text{ odd}}}^{n-3} h_i(t_1 t_2) \right) h_{n-1}(t_1) h_n(t_2) \mid t_1, t_2 \in \boldsymbol{\mu}_2(R) \right\} \cong (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)(R)$$

when n is even, and

$$Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)) = \left\{ \left(\prod_{\substack{i=1 \\ i \text{ odd}}}^{n-2} h_i(t^2) \right) h_{n-1}(t) h_n(t^{-1}) \mid t \in \boldsymbol{\mu}_4(R) \right\} \cong \boldsymbol{\mu}_4(R)$$

when n is odd.

Because of the isomorphisms included in the previous proposition we can speak of the center of $\mathbf{E}_\Phi$ as a subfunctor, and then we have that

$$\begin{aligned} Z(\mathbf{E}_{\mathbf{SL}_n}) &\cong \boldsymbol{\mu}_n \\ Z(\mathbf{E}_{\mathbf{Spin}_{2n+1}}) &\cong \boldsymbol{\mu}_2 \end{aligned}$$

$$
\begin{aligned}
Z(\mathbf{E}_{\mathbf{Sp}_{2n}}) &\cong \boldsymbol{\mu}_2 \\
Z(\mathbf{E}_{\mathbf{Spin}_{2n}}) &\cong \begin{cases} \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 & n \text{ is even} \\ \boldsymbol{\mu}_4 & n \text{ is odd} \end{cases}
\end{aligned}
$$

As promised, the remaining Chevalley groups we will work with are central quotients of the universal Chevalley groups. We name these now.

Definition 1.3.11. *Let m be an integer diving n. We define a functor of type A_{n-1},*

$$
\begin{aligned}
\mathbf{E}_{\mathbf{SL}_n/\boldsymbol{\mu}_m} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\
R &\mapsto \mathbf{E}_{\mathbf{SL}_n}(R)/I
\end{aligned}
$$

where I is the image of the composition $\boldsymbol{\mu}_m(R) \hookrightarrow \boldsymbol{\mu}_n(R) \cong Z(\mathbf{E}_{\mathbf{SL}_n}(R))$. When $m = n$ we denote the functor by $\mathbf{E}_{\mathbf{PGL}_n}$.

Definition 1.3.12. *We define a functor of type B_n,*

$$
\begin{aligned}
\mathbf{E}_{\mathbf{SO}_{2n+1}} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\
R &\mapsto \mathbf{E}_{\mathbf{Spin}_{2n+1}}(R)/Z(\mathbf{E}_{\mathbf{Spin}_{2n+1}}(R)).
\end{aligned}
$$

Since we have removed the entire center, and to match behaviour among linear algebraic groups, we also use the notation $\mathbf{E}_{\mathbf{PSO}_{2n+1}} = \mathbf{E}_{\mathbf{SO}_{2n+1}}$.

Definition 1.3.13. *We define a functor of type C_n,*

$$
\begin{aligned}
\mathbf{E}_{\mathbf{PSp}_{2n}} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\
R &\mapsto \mathbf{E}_{\mathbf{Sp}_{2n}}(R)/Z(\mathbf{E}_{\mathbf{Sp}_{2n}}(R)).
\end{aligned}
$$

Definition 1.3.14. *We define a functor of type D_n,*

$$
\begin{aligned}
\mathbf{E}_{\mathbf{SO}_{2n}} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\
R &\mapsto \begin{cases} \mathbf{E}_{\mathbf{Spin}_{2n}}(R)/I & n \textit{ is even} \\ \mathbf{E}_{\mathbf{Spin}_{2n}}(R)/J & n \textit{ is odd} \end{cases}
\end{aligned}
$$

where when n is even, I is the image of the composition

$$
\begin{aligned}
\boldsymbol{\mu}_2(R) &\hookrightarrow (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)(R) \cong Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)) \\
a &\mapsto (a, a)
\end{aligned}
$$

and when n is odd, J is the image of the composition

$$
\boldsymbol{\mu}_2(R) \hookrightarrow \boldsymbol{\mu}_4(R) \cong Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)).
$$

Definition 1.3.15. *Let n be an even integer. We define a functor of type D_n,*

$$\begin{aligned} \mathbf{E}_{\mathbf{HSpin}_{2n}} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathbf{E}_{\mathbf{Spin}_{2n}}(R)/I \end{aligned}$$

where I is the image of the composition

$$\begin{aligned} \boldsymbol{\mu}_2(R) &\hookrightarrow (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)(R) \cong Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)) \\ a &\mapsto (a, 1). \end{aligned}$$

Definition 1.3.16. *We define a functor of type D_n,*

$$\begin{aligned} \mathbf{E}_{\mathbf{PSO}_{2n}} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathbf{E}_{\mathbf{Spin}_{2n}}(R)/Z(\mathbf{E}_{\mathbf{Spin}_{2n}}(R)). \end{aligned}$$

We call $\mathbf{E}_{\mathbf{PGL}_n}, \mathbf{E}_{\mathbf{SO}_d}, \mathbf{E}_{\mathbf{HSpin}_{2n}}, \mathbf{E}_{\mathbf{PSO}_d}$, and $\mathbf{E}_{\mathbf{PSp}_{2n}}$ the special orthogonal, half-spin, projective orthogonal, and projective symplectic Chevalley groups respectively. Stating that a Chevalley group is of some type A_n, B_n, C_n, or D_n communicates that the commutator relations will match those of the universal Chevalley group of the same type. By their construction, the above groups are also generated by symbols $x_\alpha(t)$ where the appropriate version of the relations in definition 1.3.3 together with the relations imposed by the central quotient form a complete set of relations for the group. The smallest group of each type, $\mathbf{E}_{\mathbf{PGL}_n}, \mathbf{E}_{\mathbf{PSO}_d}$, and $\mathbf{E}_{\mathbf{PSp}_{2n}}$ are called the *adjoint Chevalley groups.*

1.3.2 Connection to Linear Algebraic Groups

Our notation and naming conventions for Chevalley groups thus far have been obviously evocative of the linear algebraic groups of section 1.2. We now explain the connection between these two families by giving explicit morphisms which demonstrate that the Chevalley groups are subfunctors of the corresponding linear algebraic group. When considering the group $\mathbf{SO}_d$ we will use the notation $\bar{i} = d + 1 - i$ when discussing indices, and when considering the group $\mathbf{Sp}_{2n}$ we will use $\bar{i} = 2n + 1 - i$. This is the index of the i^{th} last row or column, as appropriate.

Proposition 1.3.17. *There is an injective natural transformation $\mathbf{E}_{\mathbf{SL}_n} \hookrightarrow \mathbf{SL}_n$. For $R \in \mathbf{Alg}_{\mathbb{F}}$ it is given on R-points as follows. Let $1 \leq i < j \leq n$ and let $t \in R$, then*

$$\begin{aligned} x_{e_i - e_j}(t) &\mapsto I_n + tE_{ij} \\ x_{-e_i + e_j}(t) &\mapsto I_n + tE_{ji}. \end{aligned}$$

This map restricts to the maximal torus as

$$h_{e_i - e_j}(u) \mapsto \operatorname{diag}(1, \ldots, \underset{i}{u}, \ldots, \underset{j}{u^{-1}}, \ldots, 1),$$

the diagonal matrix with $u, u^{-1} \in R^{\times}$ *in the* i^{th} *and* j^{th} *positions respectively and ones elsewhere.*

Proposition 1.3.18. *There is an injective natural transformation* $\mathbf{E}_{\mathbf{Spin}_{2n+1}} \hookrightarrow \mathbf{Spin}_{2n+1}$*. For* $R \in \mathbf{Alg}_{\mathbb{F}}$ *it is given on* R*-points as follows. Let* $1 \leq i, j, k \leq n$ *with* $i < j$ *and* $t \in R$*, then*

$$x_{e_i - e_j}(t) \mapsto 1 + t e_i e_{\bar{j}} \qquad x_{-e_i + e_j}(t) \mapsto 1 + e_j e_{\bar{i}}$$
$$x_{e_i + e_j}(t) \mapsto 1 + t e_i e_j \qquad x_{-e_i - e_j}(t) \mapsto 1 + e_{\bar{j}} e_{\bar{i}}$$

$$x_{e_k}(t) \mapsto 1 + t e_k e_{n+1}$$
$$x_{-e_k}(t) \mapsto 1 + t e_{n+1} e_{\bar{k}}.$$

where $\overline{y} = 2n + 2 - y$ *and* $\{e_i\}$ *is the standard basis of* $V = \mathbb{F}^{2n+1}$*.*

Corollary 1.3.19. *There is an injective natural transformation* $\mathbf{E}_{\mathbf{SO}_{2n+1}} \hookrightarrow \mathbf{SO}_{2n+1}$ *which makes the following diagram commute.*

$$\begin{tikzcd}
\mathbf{E}_{\mathbf{Spin}_{2n+1}} \arrow[r, hook] \arrow[d, two heads] & \mathbf{Spin}_{2n+1} \arrow[d, two heads, "\chi"] \\
\mathbf{E}_{\mathbf{SO}_{2n+1}} \arrow[r, hook] & \mathbf{SO}_{2n+1}
\end{tikzcd}$$

For $R \in \mathbf{Alg}_{\mathbb{F}}$ *it is given on* R*-points as follows. Let* $1 \leq i, j, k \leq n$ *with* $i < j$ *and let* $t \in R$*, then*

$$x_{e_i - e_j}(t) \mapsto I_{2n+1} + t(E_{ij} - E_{\bar{j}\,\bar{i}}) \qquad x_{-e_i + e_j}(t) \mapsto I_{2n+1} + t(E_{ji} - E_{\bar{i}\,\bar{j}})$$
$$x_{e_i + e_j}(t) \mapsto I_{2n+1} + t(E_{i\,\bar{j}} - E_{j\,\bar{i}}) \qquad x_{-e_i - e_j}(t) \mapsto I_{2n+1} + t(E_{\bar{j}\,i} - E_{\bar{i}\,j})$$

$$x_{e_k}(t) \mapsto I_{2n+1} + 2tE_{k,n+1} - t^2 E_{k\,\bar{k}} - tE_{n+1,\bar{k}}$$
$$x_{-e_k}(t) \mapsto I_{2n+1} + tE_{n+1,k} - t^2 E_{\bar{k}\,k} - 2tE_{\bar{k},n+1}.$$

This map restricts to the maximal torus as

$$h_{ae_i + be_j}(u) \mapsto \operatorname{diag}(1, \ldots, \underset{i}{u^a}, \underset{j}{u^b}, \ldots, \underset{\bar{j}}{u^{-b}}, \underset{\bar{i}}{u^{-a}}, \ldots, 1)$$
$$h_{ae_k}(u) \mapsto \operatorname{diag}(1, \ldots, \underset{k}{u^{2a}}, \ldots, \underset{\bar{k}}{u^{-2a}}, \ldots, 1)$$

for $u \in R^{\times}$ *and* $a, b = \pm 1$*.*

Proposition 1.3.20. *There is an injective natural transformation* $\mathbf{E}_{\mathbf{Sp}_{2n}} \hookrightarrow \mathbf{Sp}_{2n}$. *For* $R \in \mathbf{Alg}_{\mathbb{F}}$ *it is given on R-points as follows. Let* $1 \leq i, j, k \leq n$ *with* $i < j$ *and let* $t \in R$, *then*

$$\begin{aligned} x_{e_i-e_j}(t) &\mapsto I_{2n} + t(E_{ij} - E_{\bar{j}\,\bar{i}}) & x_{-e_i+e_j}(t) &\mapsto I_{2n} + t(E_{ji} - E_{\bar{i}\,\bar{j}}) \\ x_{e_i+e_j}(t) &\mapsto I_{2n} + t(E_{i\bar{j}} + E_{j\,\bar{i}}) & x_{-e_i-e_j}(t) &\mapsto I_{2n} + t(E_{\bar{j}\,i} + E_{\bar{i}\,j}) \\ x_{2e_k}(t) &\mapsto I_{2n} + tE_{k\,\bar{k}} & x_{-2e_k}(t) &\mapsto I_{2n} + tE_{\bar{k}\,k}. \end{aligned}$$

This map restricts to the maximal torus as

$$\begin{aligned} h_{ae_i+be_j}(u) &\mapsto \operatorname{diag}(1, \ldots, \underset{i}{u^a}, \underset{j}{u^b}, \ldots, \underset{\bar{j}}{u^{-b}}, \underset{\bar{i}}{u^{-a}}, \ldots, 1) \\ h_{a2e_k}(u) &\mapsto \operatorname{diag}(1, \ldots, \underset{k}{u^a}, \ldots, \underset{\bar{k}}{u^{-a}}, \ldots, 1) \end{aligned}$$

for $u \in R^\times$ *and* $a, b = \pm 1$.

Proposition 1.3.21. *There is an injective natural transformation* $\mathbf{E}_{\mathbf{Spin}_{2n}} \hookrightarrow \mathbf{Spin}_{2n}$. *For* $R \in \mathbf{Alg}_{\mathbb{F}}$ *it is given on R-points as follows. Let* $1 \leq i, j, k \leq n$ *with* $i < j$ *and* $t \in R$, *then*

$$\begin{aligned} x_{e_i-e_j}(t) &\mapsto 1 + te_ie_{\bar{j}} & x_{-e_i+e_j}(t) &\mapsto 1 + e_je_{\bar{i}} \\ x_{e_i+e_j}(t) &\mapsto 1_te_ie_j & x_{-e_i-e_j}(t) &\mapsto 1 + e_{\bar{j}}e_{\bar{i}} \end{aligned}$$

where $\overline{y} = 2n + 1 - y$ *and* $\{e_i\}$ *is the standard basis of* $V = \mathbb{F}^{2n}$.

Corollary 1.3.22. *There is an injective natural transformation* $\mathbf{E}_{\mathbf{SO}_{2n}} \hookrightarrow \mathbf{SO}_{2n}$ *which makes the following diagram commute.*

$$\begin{tikzcd} \mathbf{E}_{\mathbf{Spin}_{2n}} \arrow[r, hook] \arrow[d, two heads] & \mathbf{Spin}_{2n} \arrow[d, two heads, "\chi"] \\ \mathbf{E}_{\mathbf{SO}_{2n}} \arrow[r, hook] & \mathbf{SO}_{2n} \end{tikzcd}$$

For $R \in \mathbf{Alg}_{\mathbb{F}}$ *it is given on R-points as follows. Let* $1 \leq i < j \leq n$ *and let* $t \in R$, *then*

$$\begin{aligned} x_{e_i-e_j}(t) &\mapsto I_{2n} + t(E_{ij} - E_{\bar{j}\,\bar{i}}) & x_{-e_i+e_j}(t) &\mapsto I_{2n} + t(E_{ji} - E_{\bar{i}\,\bar{j}}) \\ x_{e_i+e_j}(t) &\mapsto I_{2n} + t(E_{i\bar{j}} - E_{j\,\bar{i}}) & x_{-e_i-e_j}(t) &\mapsto I_{2n} + t(E_{\bar{j}\,i} - E_{\bar{i}\,j}). \end{aligned}$$

This map restricts to the maximal torus as

$$h_{ae_i+be_j}(u) \mapsto \operatorname{diag}(1, \ldots, \underset{i}{u^a}, \underset{j}{u^b}, \ldots, \underset{\bar{j}}{u^{-b}}, \underset{\bar{i}}{u^{-a}}, \ldots, 1)$$

for $u \in R^\times$ *and* $a, b = \pm 1$.

Proof: All of the above propositions and corollaries can be verified by computing that the images satisfy the defining relations given for each group. For $\mathbf{SL}_n$, $\mathbf{Spin}_{2n}$, $\mathbf{Spin}_{2n+1}$, and $\mathbf{Sp}_{2n}$ these are just the relations of propositions 1.3.6, 1.3.9, 1.3.7, and 1.3.8. For $\mathbf{SO}_d$ the relations of $\mathbf{Spin}_d$ can be checked, as well as the fact that expressions in the kernel of $\mathbf{E}_{\mathbf{Spin}_d} \twoheadrightarrow \mathbf{E}_{\mathbf{SO}_d}$ map to 1 in $\mathbf{SO}_d$. It is also easy to calculate that the maps $\mathbf{E}_{\mathbf{Spin}_d} \twoheadrightarrow \mathbf{E}_{\mathbf{SO}_d}$ agree with the vector representation $\chi\colon \mathbf{Spin}_d \to \mathbf{SO}_d$ in the given diagrams. We mention the maps into $\mathbf{SO}_d$ explicitly because the structure constants given in the definitions of our Chevalley groups were initially computed within $\mathbf{SL}_n, \mathbf{SO}_d$, and $\mathbf{Sp}_{2n}$ as these are the Chevalley groups which arise from the natural representation of their Lie algebras. Therefore, their results follow immediately. ∎

The other quotient Chevalley groups also have injections into the corresponding linear algebraic group, induced by the maps above.

Proposition 1.3.23. *There are injective natural transformations of group functors*

$$\begin{aligned} \mathbf{E}_{\mathbf{SL}_n/\boldsymbol{\mu}_m} &\hookrightarrow \mathbf{SL}_n/\boldsymbol{\mu}_m \\ \mathbf{E}_{\mathbf{HSpin}_{2n}} &\hookrightarrow \mathbf{HSpin}_{2n} \quad \textit{when } n \textit{ is even} \\ \mathbf{E}_{\mathbf{PSO}_d} &\hookrightarrow \mathbf{PSO}_d \\ \mathbf{E}_{\mathbf{PSp}_{2n}} &\hookrightarrow \mathbf{PSp}_{2n} \end{aligned}$$

induced by the maps of propositions 1.3.17 to 1.3.22.

Proof: In type A_n the composition $\mathbf{E}_{\mathbf{SL}_n} \to \mathbf{SL}_n \twoheadrightarrow \mathbf{SL}_n/\boldsymbol{\mu}_m$ will have the same kernel as the map $\mathbf{E}_{\mathbf{SL}_n} \twoheadrightarrow \mathbf{E}_{\mathbf{SL}_n/\boldsymbol{\mu}_m}$ and therefore induces the desired map. The remaining cases follow similarly from the following compositions respectively

$$\begin{aligned} \mathbf{E}_{\mathbf{Spin}_{2n}} &\to \mathbf{Spin}_{2n} \to \mathbf{HSpin}_{2n} \\ \mathbf{E}_{\mathbf{SO}_d} &\to \mathbf{SO}_d \twoheadrightarrow \mathbf{PSO}_d \\ \mathbf{E}_{\mathbf{Sp}_{2n}} &\to \mathbf{Sp}_{2n} \twoheadrightarrow \mathbf{PSp}_{2n} \end{aligned}$$

∎

We now see how the Chevalley functors we have defined sit as subfunctors of our linear algebraic groups. A natural next question is whether they are actually the same functor. This would be desirable as then we could use the computational power of the generators and relations presentation of the Chevalley functor to speak about linear algebraic groups. Unfortunately, there are cases where the Chevalley functor is a proper subfunctor. As noted by Borel in [BCS$^+$69, A.§3.3(5)], if $\mathbf{G}$ is a linear algebraic group over $\mathbb{F}$ with corresponding Chevalley subfunctor $\mathbf{E}_\Phi$, then for field extensions $\mathbb{E}/\mathbb{F}$ we have $\mathbf{E}_\Phi(\mathbb{E}) = [\mathbf{G}(\mathbb{E}), \mathbf{G}(\mathbb{E})]$, the Chevalley group is the commutator subgroup of $\mathbf{G}(\mathbb{E})$. In some cases $\mathbf{G}(\mathbb{E})$ is its own commutator subgroup, but not always. For example, the special orthogonal group scheme over $\mathbb{Q}$ has that

$$\mathbf{E}_{\mathbf{SO}_d}(\mathbb{Q}) = [\mathbf{SO}_d(\mathbb{Q}), \mathbf{SO}_d(\mathbb{Q})] \subsetneq \mathbf{SO}_d(\mathbb{Q}),$$

and so the Chevalley group is distinct from the linear algebraic group. Luckily we are not without hope, as we will show in the next section, a linear algebraic group can be equivalently viewed as a representable functor $\widehat{\mathbf{G}} \colon \mathbf{Aff}_{\mathbb{F}} \to \mathbf{Sets}$ from the category of affine schemes over $\operatorname{Spec}(\mathbb{F})$ to the category of sets, represented by $\operatorname{Spec}(H)$ if it was previously represented by the Hopf algbera H. In this setting, $\widehat{\mathbf{G}}$ is a sheaf with respect to the étale site on $\mathbf{Aff}_{\mathbb{F}}$, and the stalks of this sheaf can be completely described with Chevalley generators. This will allow us to use Chevalley groups to investigate properties of homomorphisms between our linear algebraic groups.

1.4 Galois Cohomology

In order to further make use of the fact that any algebraic group is a twisted form of a split group, we seek both a way to classify a group's twisted forms and a way to recover our original group over $\mathbb{F}$ from its restriction to $\mathbb{F}_{\text{sep}}$. Galois cohomology provides both of these tools. In this section we introduce the general Galois cohomology of Galois modules before discussing how particular cohomology sets classify the central simple algebras appearing in section 1.1. With that classification, we then also discuss how non-split central simple algebras and their corresponding non-split semisimple algebraic groups can be recovered from their restriction to $\mathbb{F}_{\text{sep}}$ by Galois descent.

1.4.1 Cohomology Sets

Since Galois groups are examples of profinite groups, we begin by defining the Galois cohomology sets for a profinite group G and a group A with suitable G-action following [Ser02, §2], with some terminology borrowed from [KMRT].

Definition 1.4.1. *Let G be a profinite group and let A be a group considered with the discrete topology. We call A a G-*group *if there is a group action $G \curvearrowright A$ such that*

- *$g\cdot(ab) = (g\cdot a)(g\cdot b)$ for all $g \in G$ and $a, b \in A$ (G acts by group automorphisms),*
- *$\mathrm{Stab}_G(a) \subseteq G$ is open for all $a \in A$ (G acts continuously on A).*

*In the case that A is abelian we call it a G-*module*. Furthermore, If $H \leq G$ is a subgroup, we denote by A^H the subgroup of A which is fixed pointwise by H.*

Since G acts by automorphisms we employ notation which suggests this. We write $g(a)$ for $g \cdot a$.

Definition 1.4.2. *Let G be a profinite group and let A be a G-module. Denote the set of continuous maps from $G^n \to A$ by $C^n(G, A)$. These form a chain complex under the coboundary map $d\colon C^n(G, A) \to C^{n+1}(G, A)$ which for $f \in C^n(G, A)$ is given by*

$$\begin{aligned} df(g_1, \dots, g_{n+1}) =& g_1(f(g_2, \dots, g_{n+1})) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned}$$

The resulting cohomology sets are denoted $H^n(G, A)$ and are referred to as the cohomology sets of G with coefficients in A.

In fact, since G-modules are abelian, the cohomology sets $H^n(G, A)$ inherit their own abelian group structure by defining $[\alpha] + [\beta] = [\alpha + \beta]$ on cohomology classes. Due to this we may also refer to them as *cohomology groups*.

We mention an important extension here, in the above definition of df, the use of additive notation is granted by the assumption that A is abelian. If A is not abelian we do not necessarily have that $d^2 = 0$ and therefore the cohomology sets may not be defined. However, if A is a G-group we may still define $H^0(G, A)$, and with some modification we may also define $H^1(G, A)$. We note that if A is any G-group then

$$H^0(G, A) = A^G,$$

and the modification to the definition of $H^1(G, A)$ is as follows, where we write A multiplicatively since it need not be abelian. Furthermore, for a continuous map $\alpha\colon G \to A$ we denote the image of $g \in G$ under α by α_g.

Definition 1.4.3. *Let G be a profinite group and let A be a G-group. We call a continuous map $\alpha\colon G \to A$ a* 1-cocycle *if*

$$\alpha_{gh} = \alpha_g \, g(\alpha_h)$$

for all $g, h \in G$. Two such 1-cocycles α and β are cohomologous *if there exists $a \in A$ such that*

$$\alpha_g = a \, \beta_g \, g(a)^{-1}$$

for all $g \in G$. Being cohomologous is an equivalence relation and $H^1(G, A)$ is then the set of 1-cocycles modulo this relation.

If A is not a G-module there is no group structure on $H^1(G, A)$ and we refer to it as a cohomology set only. If A is a G-module, this definition is equivalent to the one above.

The cohomology sets we have defined enjoy functoriality in both arguments, they are contravariant in the first argument and covariant in the second. Explicitly, let $\varphi\colon G_1 \to G_2$ be a homomorphism of profinite groups and let A be both a G_1-group and G_2-group with actions compatible with φ. That is, $g(a) = \varphi(g)(a)$ for all $g \in G_1$ and $a \in A$. Then there is an induced map

$$\begin{aligned} \varphi^*\colon H^n(G_2, A) &\to H^n(G_1, A) \\ [\alpha] &\mapsto [\alpha \circ \varphi] \end{aligned}$$

when n is as appropriate for A. In the other argument, let G be a profinite group and let $\phi\colon A_1 \to A_2$ be a homomorphism of G-groups. That is, $g(\phi(a)) = \phi(g(a))$ for all $g \in G$ and $a \in A_1$. Then there is an induced map

$$\phi^*\colon H^n(G, A_1) \to H^n(G, A_2)$$

$$[\alpha] \mapsto [\phi \circ \alpha]$$

when n is appropriate for both A_1 and A_2.

Here we mention a tool which uses elements of lower cohomology sets to construct elements of higher cohomology sets. The cup product will define maps $H^i(G, A) \times H^j(G, B) \to H^{i+j}(G, A \otimes_{\mathbb{Z}} B)$. Since both higher cohomology and tensor products of $\mathbb{Z}$-modules appear in the construction, we will assume that A and B are G-modules for the profinite group G. The tensor product $A \otimes_{\mathbb{Z}} B$ is then given a G-module structure via $g(a \otimes b) := g(a) \otimes g(b)$ for all $g \in G$, $a \in A$, and $b \in B$.

Definition 1.4.4. *Let G be a profinite group with G-modules A and B. Consider elements $[\alpha] \in H^i(G, A)$ and $[\beta] \in H^j(G, B)$ represented by cocycles α and β respectively. Their* cup product *is defined to be the element represented by the cocycle*

$$\begin{aligned} G^{i+j} &\to A \otimes_{\mathbb{Z}} B \\ (g_1, \ldots, g_{i+j}) &\mapsto \alpha(g_1, \ldots, g_i) \otimes g_1 g_2 \ldots g_i(\beta(g_{i+1}, \ldots, g_{i+j})) \end{aligned}$$

and is denoted $[\alpha] \cup [\beta] \in H^{i+j}(G, A \otimes_{\mathbb{Z}} B)$.

The resulting map $\cup \colon H^i(G, A) \times H^j(G, B) \to H^{i+j}(G, A \otimes_{\mathbb{Z}} B)$ is a homomorphism of cohomology groups.

Lastly, we wish to take the machinery above and adapt it to support algebraic groups. The required adaptation is a minor one. While our cohomology functors only accept a profinite group acting on a classical group, an algebraic group $\mathbf{G}$ over $\mathbb{F}$ easily produces such pairs. If $\mathbb{E}/\mathbb{F}$ is a field extension then there is a natural action $\mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E}) \curvearrowright \mathbf{G}(\mathbb{E}_{\mathrm{sep}})$. We begin by defining this action.

Definition 1.4.5. *Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$ with representing Hopf algebra H. For a field extension $\mathbb{E}/\mathbb{F}$ we have that $\mathbf{G}(\mathbb{E}_{\mathrm{sep}}) = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{E}_{\mathrm{sep}})$. For $\sigma \in \mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E})$ and $p \in \mathbf{G}(\mathbb{E}_{\mathrm{sep}})$ we define $\sigma(p) \in \mathbf{G}(\mathbb{E}_{\mathrm{sep}})$ to be the composition*

$$\sigma(p) \colon H \xrightarrow{p} \mathbb{E}_{\mathrm{sep}} \xrightarrow{\sigma} \mathbb{E}_{\mathrm{sep}}.$$

This action is by automorphisms of $\mathbf{G}(\mathbb{E}_{\mathrm{sep}})$ and we have that the set of fixed points is isomorphic to $\mathbf{G}(\mathbb{E})$. If $\mathbf{G}$ is one of our previously defined split groups then an element of $\mathbf{G}(\mathbb{E}_{\mathrm{sep}})$ can be thought of as a matrix $[c_{ij}] \in \mathrm{M}_n(\mathbb{E}_{\mathrm{sep}})$ for some n, and then the action is given by $\sigma([c_{ij}]) = [\sigma(c_{ij})]$. With respect to this action we define the Galois cohomology sets.

Definition 1.4.6. *Let $\mathbf{G}$ be an algebraic group over $\mathbb{F}$. For a field extension $\mathbb{E}/\mathbb{F}$ and non-negative integer n we define the* Galois cohomology sets *to be*

$$H^n(\mathbb{E}, \mathbf{G}) = H^n(\mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E}), \mathbf{G}(\mathbb{E}_{\mathrm{sep}}))$$

with respect to the action $\mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E}) \curvearrowright \mathbf{G}(\mathbb{E}_{\mathrm{sep}})$ discussed above. If $\mathbf{G}$ is abelian then $H^n(\mathbb{E}, \mathbf{G})$ is a group. If $\mathbf{G}$ is not abelian we only consider $n = 0, 1$.

These cohomology sets also enjoy functoriality in both arguments, but now it is covariant in both arguments. If $\mathbb{K}/\mathbb{E}$ is a field extension, both of which are field extensions of $\mathbb{F}$, then there is a map $\mathrm{Gal}(\mathbb{K}_{\mathrm{sep}}/\mathbb{K}) \to \mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E})$ given by restriction to $\mathbb{E}_{\mathrm{sep}}$, as well as a map $\mathbf{G}(\mathbb{E}_{\mathrm{sep}}) \to \mathbf{G}(\mathbb{K}_{\mathrm{sep}})$. Together these induce the map

$$H^n(\mathbb{E}, \mathbf{G}) \to H^n(\mathbb{K}, \mathbf{G}).$$

when n is as appropriate for $\mathbf{G}$. In the second argument, if $\varphi\colon \mathbf{G}_1 \to \mathbf{G}_2$ is a homomorphism of algebraic groups over $\mathbb{F}$, then the map on $\mathbb{E}_{\mathrm{sep}}$-points, $\varphi(\mathbb{E}_{\mathrm{sep}})$, induces the map

$$H^n(\mathbb{E}, \mathbf{G}_1) \to H^n(\mathbb{E}, \mathbf{G}_2).$$

when n is as appropriate for $\mathbf{G}_1$ and $\mathbf{G}_2$.

1.4.2 Twisting by 1-Cocycles

In this subsection we will explain how central simple algebras can be constructed using, and classified by, elements of suitable first cohomology sets. We exploit the fact that central simple algebras become isomorphic to $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ over $\mathbb{F}_{\mathrm{sep}}$, and view the original algebra as a subset in $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$. The algebra is then the fixed points of some $\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$-action, and the difference between this action and the standard action on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ can be measured by a 1-cocycle. Conversely, we use a given 1-cocycle to modify the standard action of $\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$ on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$, which we call twisting the action, and then use Galois descent to produce a central simple algebra over $\mathbb{F}$. While we follow [GS17, §2.3], we note that the techniques discussed there apply beyond central simple algebras. Throughout this section we use $\Gamma = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$ as the absolute Galois group of $\mathbb{F}$.

First we cover the case of a central simple algebra without involution. Let A be a central simple $\mathbb{F}$-algebra of degree n. Since A is a twisted form of $\mathrm{M}_n(\mathbb{F})$ we have an isomorphism

$$\varphi\colon A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \to \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}).$$

Two Galois actions are at play here. The first is the action $\Gamma \curvearrowright A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$ given by $\sigma(a \otimes c) = a \otimes \sigma(c)$, and the second is the action $\Gamma \curvearrowright \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ given by $\sigma([c_{ij}]) = [\sigma(c_{ij})]$ for $a \in A$, $c, c_{ij} \in \mathbb{F}_{\mathrm{sep}}$, and $\sigma \in \Gamma$. To compare these actions we consider them both as actions on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ through the diagram

$$\begin{tikzcd}
\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \arrow[r, "\varphi^{-1}"] \arrow[d, "\sigma"] & A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \arrow[d, "\sigma"] \\
\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) & A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \arrow[l, "\varphi"]
\end{tikzcd}$$

i.e., for any $M \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ and $\sigma \in \Gamma$ we have

$$\sigma \cdot_1 M = \sigma(M) \text{ and } \sigma \cdot_2 M = \varphi \circ \sigma \circ \varphi^{-1}(M)$$

and each action is by automorphisms of $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$. Since we are now considering automorphisms of $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ we recall that $\mathrm{Aut}_{\mathbb{F}_{\mathrm{sep}}}(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})) = \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$. Here we make a clarification, if $B \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ then the class $[B] \in \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ corresponds to the inner automorphism

$$\begin{aligned} \mathrm{Inn}(B)\colon\ \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) &\to \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \\ M &\mapsto BMB^{-1}. \end{aligned}$$

If $\sigma \in \Gamma$ then $\sigma([B]) = [\sigma(B)]$ where σ acts on the entries of B, and this action appears as conjugation on the automorphism, that is

$$\sigma(\mathrm{Inn}(B)) = \mathrm{Inn}(\sigma(B)) = \sigma \circ \mathrm{Inn}(B) \circ \sigma^{-1}.$$

Now, to measure the difference between the two actions we consider the automorphism $(\varphi \circ \sigma \circ \varphi^{-1}) \circ \sigma^{-1}$. This automorphism depends on our choice of σ, and so we have produced a map $\alpha\colon \Gamma \to \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$. In fact, this map is a 1-cocycle and repeating the process with any other central simple $\mathbb{F}$-algebra which is isomorphic to A produces a cohomologous 1-cocycle. Conversely, given a 1-cocycle $\alpha\colon \Gamma \to \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ we define a new action $\Gamma \curvearrowright \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ given by

$$\sigma \cdot_\alpha B = \alpha_\sigma(\sigma(B)).$$

for all $\sigma \in \Gamma$ and $B \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$. This is called the *action twisted by* α and the set of fixed points, denoted $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$, will be a central simple $\mathbb{F}$-algebra. Cohomologous 1-cocycles will produce algebras which are isomorphic over $\mathbb{F}$, and in fact the isomorphism will be given by an inner automorphism of $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ which maps one set of fixed points onto the other. This provides an inverse to the process above and hence there is a bijection

$$\left\{ \begin{matrix} \text{Isomorphism classes of central} \\ \text{simple algebras of degree n over } \mathbb{F} \end{matrix} \right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PGL}_n).$$

The process of twisting the standard action $\Gamma \curvearrowright \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ by a 1-cocycle and then taking fixed points to obtain a central simple $\mathbb{F}$-algebra is called *Galois descent.*

A similar story classifies algebras with involution. Because of the fact that any two orthogonal involutions on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ are isomorphic we have that any central simple $\mathbb{F}$-algebra with orthogonal involution is a twisted form of $(\mathrm{M}_n(\mathbb{F}), \tau_0)$. Likewise, any central simple $\mathbb{F}$-algebra with a symplectic involution is a twisted form of $(\mathrm{M}_{2n}(\mathbb{F}), \psi_0)$. Here we recall our preferred involutions τ_0 and ψ_0 from section 1.2.1. Through the same processes as above, adapted so that the isomorphisms and actions respect the involutions, we obtain classifying bijections as in [KMRT, 29.B]

$$\left\{\begin{matrix}\text{Isomorphism classes of central}\\ \text{simple algebras of degree n over } \mathbb{F}\\ \text{with orthogonal involution}\end{matrix}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PGO}_n).$$

$$\left\{\begin{matrix}\text{Isomorphism classes of central}\\ \text{simple algebras of degree n over } \mathbb{F}\\ \text{with orthogonal involution}\\ \text{of trivial discriminant}\end{matrix}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PSO}_n).$$

$$\left\{\begin{matrix}\text{Isomorphism classes of central}\\ \text{simple algebras of degree 2n over } \mathbb{F}\\ \text{with symplectic involution}\end{matrix}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PSp}_{2n}).$$

Through these bijections we will identify isomorphism classes of algebras with cohomology classes and frequently write things such as $[A] \in H^1(\mathbb{F}, \mathbf{PGL}_n)$, $[(A, \tau)] \in H^1(\mathbb{F}, \mathbf{PSO}_n)$ or $[(A, \psi)] \in H^1(\mathbb{F}, \mathbf{PSp}_{2n})$.

The identifications we have just described are on the level of isomorphism classes and cohomology classes. However, in the future we will use this machinery to construct maps on the level of algebras. To allow for this, we now fix a standard identification with a subset of $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ for each central simple algebra A of degree n. This is equivalent to fixing a standard cocycle, denoted α_A, which represents $[A] \in H^1(\mathbb{F}, \mathbf{PGL}_n)$, and then considering $A = \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_{\alpha_A}}$. Similarly, for algebras with involution we fix a standard cocycle representing the algebra's isomorphism class in either $H^1(\mathbb{F}, \mathbf{PSp}_{2n})$ or $H^1(\mathbb{F}, \mathbf{PSO}_d)$ as appropriate, and then identify the algebra with the fixed points of the chosen cocycle. For split algebras, with our preferred involution when applicable, we choose the trivial cocycle.

1.4.3 Twisting for Linear Algebraic Groups

Now that we have identified non-split central simple $\mathbb{F}$-algebras as subsets of split $\mathbb{F}_{\mathrm{sep}}$-algebras, we would also like to have a similar method of Galois descent that can be used to construct non-split semisimple algebraic groups from their split counterparts. In order to use twisting, we need a standard Γ action to twist. For any linear algebraic group $\mathbf{G}$ over $\mathbb{F}$ and for any $R \in \mathbf{Alg}_{\mathbb{F}}$ we define an action $\Gamma \curvearrowright \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ as follows. Let H be the representing Hopf algebra of $\mathbf{G}$. Then an element $x \in \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is a map

$$x \colon H \to R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}.$$

For $\sigma \in \Gamma$, we define $\sigma(x)$ to be the composite map

$$\sigma(x) \colon H \to R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \overset{1 \otimes \sigma}{\to} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}.$$

This recovers the standard Γ action we are familiar with on groups where it was defined, for example this action corresponds to σ acting on matrices entry-wise for of elements $\mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$.

Since our non-split algebraic groups correspond to non-split central simple algebras, it is not surprising that the same 1-cocycle used to construct the algebra will be fruitful in constructing the group. We begin with the projective linear group. Let A be a central simple $\mathbb{F}$-algebra of degree n, and consider its standard cocycle $\alpha_A\colon \Gamma \to \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$. Just as the standard action $\Gamma \curvearrowright \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ can be twisted by α_A to produce an action whose fixed points are A, we twist the action $\Gamma \curvearrowright \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ to get an action whose fixed points are $\mathrm{Aut}_{\mathbb{F}}(A) = \mathbf{PGL}(A)(\mathbb{F})$. This new twisting is done via conjugation by α_A.

$$\sigma \cdot_{\alpha_A} \varphi := \alpha_{A,\sigma} \circ \sigma \circ \varphi \circ \sigma^{-1} \circ \alpha_{A,\sigma}^{-1}$$

for $\sigma \in \Gamma$ and $\varphi \in \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$. Now we notice the following,

$$\begin{aligned}\mathbf{PGL}(A)(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) &= \mathrm{Aut}_{R\otimes_{\mathbb{F}}\mathbb{F}_{\mathrm{sep}}}(A \otimes_{\mathbb{F}} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \\ &\cong \mathrm{Aut}_{R\otimes_{\mathbb{F}}\mathbb{F}_{\mathrm{sep}}}(\mathrm{M}_n(\mathbb{F}) \otimes_{\mathbb{F}} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) = \mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})\end{aligned}$$

since $A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \cong \mathrm{M}_n(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$. Here as well there is an action of Γ twisted by α_A,

$$\sigma \cdot_{\alpha_A} \varphi = (\alpha_{A,\sigma}\sigma \otimes \mathrm{id}_R) \circ \varphi \circ (\alpha_{A,\sigma}\sigma \otimes \mathrm{id}_R)^{-1}$$

for $\varphi \in \mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, and the fixed points of this action are $\mathrm{Aut}_R(A \otimes_{\mathbb{F}} R) = \mathbf{PGL}(A)(R)$.

To obtain $\mathbf{SL}(A)$ note that the canonical surjection $\mathbf{SL}_n(\mathbb{F}_{\mathrm{sep}}) \twoheadrightarrow \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ has a central quotient, and so any choices of $\mathfrak{a}_{A,\sigma} \in \mathbf{SL}_n(\mathbb{F}_{\mathrm{sep}})$ which sit above $\alpha_{A,\sigma} \in \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ will produce the same twisted action

$$\sigma \cdot_{\alpha_A} B = \mathfrak{a}_{A,\sigma}\sigma(B)\mathfrak{a}_{A,\sigma}^{-1}$$

and so we consider this the action of Γ on $\mathbf{SL}_n(\mathbb{F}_{\mathrm{sep}})$ twisted by α_A. Just as above, this extends to a twisted action on $\mathbf{SL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ for all $R \in \mathbf{Alg}_{\mathbb{F}}$ and the fixed points of this action are $\mathbf{SL}(A)(R)$.

Similarly, we can consider the case of a central simple $\mathbb{F}$-algebra (A, τ) of degree n with orthogonal involution of trivial discriminant. Let $\alpha = \alpha_{(A,\tau)}$ be the chosen 1-cocycle for (A, τ), therefore

$$A = \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \{B \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \mid \alpha_\sigma(\sigma(B)) = B,\ \forall \sigma \in \Gamma\}$$

with $\tau = \tau_0|_A$ and $\mathrm{Nrd} = \det|_A$. In an analogous way to the case above, the fixed points of the twisted action of Γ on $\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ will be $\mathbf{PSO}(A, \tau)(R)$. That is, letting α'_σ be the image of α_σ in $\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$,

$$\mathbf{PSO}(A, \tau)(R) = \{\varphi \in \mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \mid \alpha'_\sigma \circ \sigma \circ \varphi \circ \sigma^{-1} \circ \alpha'^{-1}_\sigma = \varphi\}.$$

Now choose $a_\sigma \in \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$ lying above each α_σ. First, we can also express A using these elements as

$$A = \{B \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \mid a_\sigma \sigma(B) a_\sigma^{-1} = B,\ \forall \sigma \in \Gamma\}$$

since the automorphism α_σ is the inner automorphism given by a_σ. In order to twist $\mathbf{SO}$, let $a'_\sigma \in \mathbf{SO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ be the images of a_σ. We will then have

$$A \otimes_{\mathbb{F}} R = \{B \in \mathrm{M}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \mid a'_\sigma \sigma(B) a'^{-1}_\sigma = n \ \forall \sigma \in \Gamma\}$$

and so when we twist the Γ-action on $\mathbf{SO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ the fixed points will be

$$\mathbf{SO}(A, \tau)(R) = \{B \in \mathbf{SO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \mid a'_\sigma \sigma(B) a'^{-1}_\sigma = B,\ \forall \sigma \in \Gamma\}.$$

An analogous story occurs for algebras (A, ψ) of degree $2n$ with symplectic involution and their corresponding cocycles to produce groups $\mathbf{PSp}(A, \psi)$ and $\mathbf{Sp}(A, \psi)$. Similar twisting techniques work for our two remaining groups, **Spin** and **HSpin**, which we show in the next chapter.

Chapter 2

Twisting Spin, Half-spin, and Hopf Algebras

The techniques of the previous section are also applicable to the spin and half-spin groups. In fact, the idea of using Galois cohomology to classify and construct twisted forms of linear algebraic groups is discussed in general in [Mil17, §3k]. In this chapter we describe in detail how $\mathbf{Spin}_n$ can be twisted to produce $\mathbf{Spin}(A,\tau)$ where (A,τ) is a central simple $\mathbb{F}$-algebra of degree n with orthogonal involution of trivial discriminant. The argument relies on similar twisting results relating the Clifford algebras of quadratic forms to those of central simple algebras. Then, on our way to twisting half-spin, we describe the analogous theory of twisting for the Hopf algebras which represent our linear algebraic groups. We find this necessary since **HSpin** is a scheme theoretic image, and therefore is most easily defined by its Hopf algebra. With our twisted Hopf algebras in hand, we end by detailing how twisting $\mathbf{HSpin}_{2n}$ behaves as expected and produces $\mathbf{HSpin}(A,\tau)$.

2.1 Twisting Spin

Let (A,τ) be a central simple $\mathbb{F}$-algebra of degree n with orthogonal involution of trivial discriminant. This algebra has a chosen cocycle $\alpha = \alpha_{(A,\tau)}\colon \Gamma \to \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$. For each $\sigma \in \Gamma$, we choose elements $\mathfrak{a}_\sigma \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ lying above elements $a_\sigma \in \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$ which in turn each lie above $\alpha_\sigma \in \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$. There is a standard Γ-action on the Clifford algebra $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}),\tau_0)$ given by

$$\sigma(B_1 \otimes \ldots \otimes B_k) = \sigma(B_1) \otimes \ldots \otimes \sigma(B_k)$$

extended linearly, where $B_i \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$. This is the action of Γ on the second factor of $C(\mathrm{M}_n(\mathbb{F}),\tau_0) \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}),\tau_0)$ and is therefore well-defined. Since $\mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}}) \subset C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}),\tau_0)$, we can use the $\mathfrak{a}_\sigma$ to twist this action.

Proposition 2.1.1. *Let* (A, τ)*,* $a_\sigma \in \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$ *and* $\mathfrak{a}_\sigma \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ *be as above. Define a twisted* Γ*-action on* $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)$ *by* $\sigma \cdot_\alpha c = \mathfrak{a}_\sigma \sigma(c) \mathfrak{a}_\sigma^{-1}$*. Then*

$$C(A, \tau) = C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha}.$$

Proof: Since we are working within the split case, we let $V = \mathbb{F}_{\mathrm{sep}}^n$ with the appropriate quadratic form q which gives rise to τ_0 on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ as in section 1.2.2. The standard isomorphism defined by q is

$$\begin{aligned} V \otimes_{\mathbb{F}_{\mathrm{sep}}} V &\to \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \\ e_i \otimes e_j &\mapsto E_{i\bar{j}} \end{aligned}$$

if n is even, and when $n = 2k+1$ is odd we have

$$e_i \otimes e_j \mapsto \begin{cases} E_{i\bar{j}} & j \neq k+1 \\ 2E_{i,k+1} & j = k+1 \end{cases}$$

where $\bar{j} = n + 1 - j$ and e_i are the standard basis vectors. The usual Γ-action on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ corresponds to Γ acting on the entries of vectors in V and $V \otimes_{\mathbb{F}_{\mathrm{sep}}} V$. In both cases, we have the following behaviour. Let $B, B' \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$, then $B' = \sum_{i=1}^k v_i \otimes w_i$ under the isomorphism, and we can express the matrix product as

$$\begin{aligned} BB' &= B\left(\sum_{i=1}^k v_i \otimes w_i\right) = \sum_{i=1}^k (Bv_i) \otimes w_i \\ B'B &= \left(\sum_{i=1}^k v_i \otimes w_i\right) B = \sum_{i=1}^k v_i \otimes (\tau_0(B) w_i). \end{aligned}$$

Note that for the $a_\sigma \in \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$ we have $a_\sigma \tau_0(a_\sigma) = I$, and so

$$a_\sigma \sigma\left(\sum_{i=1}^k v_i \otimes w_i\right) a_\sigma^{-1} = \sum_{i=1}^k a_\sigma \sigma(v_i) \otimes \tau_0(a_\sigma^{-1}) \sigma(w_i) = \sum_{i=1}^k a_\sigma v_i \otimes a_\sigma w_i.$$

Hence we can express $A \subseteq \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ in a third way,

$$A = \left\{ \sum_{i=1}^k v_i \otimes w_i \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \mid \sum_{i=1}^k a_\sigma \sigma(v_i) \otimes a_\sigma \sigma(w_i) = \sum_{i=1}^k v_i \otimes w_i, \ \forall \sigma \in \Gamma \right\}.$$

Now we recall some details of the vector representation $\chi \colon \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$. By definition, if $c \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ and $\sum_{i=1}^k v_i \otimes w_i \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ we have that

$$b^{-1}\left(c * b\left(\sum_{i=1}^k v_i \otimes w_i\right) \cdot c^{-1}\right) = b^{-1}\left(\sum_{i=1}^k v_i \otimes c \otimes w_i \otimes c^{-1}\right) = \sum_{i=1}^k v_i \otimes \chi(c) w_i.$$

Therefore, for $B = \sum_{i=1}^{k} v_i \otimes w_i \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$, we have in $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)$ where multiplication is given by $\otimes$, that

$$\begin{aligned} cBc^{-1} &= c \otimes B \otimes c^{-1} \\ &= \sum_{i=1}^{k} c \otimes v_i \otimes w_i \otimes c^{-1} \\ &= \sum_{i=1}^{k} c \otimes v_i \otimes c^{-1} \otimes c \otimes w_i \otimes c^{-1} \\ &= \sum_{i=1}^{k} \chi(c) v_i \otimes \chi(c) w_i. \end{aligned}$$

Hence if $B \in A \subset \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$, then for all $\sigma \in \Gamma$

$$\begin{aligned} \mathfrak{a}_\sigma \sigma(B) \mathfrak{a}_\sigma^{-1} &= \sum_{i=1}^{k} \chi(\mathfrak{a}_\sigma)\sigma(v_i) \otimes \chi(\mathfrak{a}_\sigma)\sigma(w_i) \\ &= \sum_{i=1}^{k} a_\sigma \sigma(v_i) \otimes a_\sigma \sigma(w_i) \\ &= \sum_{i=1}^{k} v_i \otimes w_i \\ &= B. \end{aligned}$$

This property extends to the Clifford algebra, so if $x \in C(A, \tau)$ then we can write

$$x = c_0 + \sum B_{i1} \otimes \ldots \otimes B_{ik}$$

where $c_0 \in \mathbb{F}$, each $B_{ij} \in A$, and k may vary. Then for each $\sigma \in \Gamma$,

$$\begin{aligned} \mathfrak{a}_\sigma \sigma(x) \mathfrak{a}_\sigma^{-1} &= \sigma(c_0) + \sum (\mathfrak{a}_\sigma \sigma(B_{i1}) \mathfrak{a}_\sigma^{-1}) \otimes \ldots \otimes (\mathfrak{a}_\sigma \sigma(B_{ik}) \mathfrak{a}_\sigma^{-1}) \\ &= c_0 + \sum B_{i1} \otimes \ldots \otimes B_{ik} \\ &= x. \end{aligned}$$

Therefore $C(A, \tau) \subseteq C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha}$. Now, $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)$ is an $\mathbb{F}_{\mathrm{sep}}$-vector space and the twisted Γ-action is semi-linear, therefore we may invoke a modified version of Speiser's Lemma, [GS17, 2.3.8], to get that

$$C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha} \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0).$$

Also, properties of the Clifford algebra give that $C(A, \tau) \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)$. In particular, this means that $\dim_{\mathbb{F}}(C(A, \tau)) = \dim_{\mathbb{F}}(C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha})$, and therefore

$$C(A, \tau) = C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha}$$

as desired. ∎

Corollary 2.1.2. *Let* (A, τ) *and* $\mathfrak{a}_\sigma \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ *be as above. Let* $R \in \mathbf{Alg}_{\mathbb{F}}$ *and let* $\mathfrak{a}'_\sigma \in \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ *be the images of the* $\mathfrak{a}_\sigma$*. Then, the* $\mathfrak{a}'_\sigma$ *define a twisted* Γ*-action on* $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R$*, and*

$$C(A, \tau) \otimes_{\mathbb{F}} R = (C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha}.$$

Proof: Letting $a_\sigma \in \mathbf{SO}_n(\mathbb{F}_{\mathrm{sep}})$ be as above and $a'_\sigma \in \mathbf{SO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ their images, we have that

$$((\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha} = (\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_\alpha} \otimes_{\mathbb{F}} R = (A, \tau) \otimes_{\mathbb{F}} R$$

where the twisted action is given by conjugation with a'_σ. Then since we also have

$$C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R \cong C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}) \otimes_{\mathbb{F}} R, \tau_0 \otimes 1),$$

and the elements $\mathfrak{a}'_\sigma$ will lie above each a'_σ, the same arguments as in the above proposition give the desired conclusion. ∎

Now we are ready to address the twisting of $\mathbf{Spin}_n$.

Theorem 2.1.3. *Let* (A, τ) *be a central simple* $\mathbb{F}$*-algebra of degree* n *with orthogonal involution of trivial discriminant, and let* α *be its chosen cocycle. Let* $\mathfrak{a}_\sigma \in \mathbf{Spin}_n(\mathbb{F}_{\mathrm{sep}})$ *be any elements above each* $\alpha_\sigma \in \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$ *and set* $\mathfrak{a}'_\sigma \in \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ *to be their images. These define a twisted* Γ*-action on* $\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ *by* $\sigma \cdot_\alpha c = \mathfrak{a}'_\sigma \sigma(c) \mathfrak{a}'^{-1}_\sigma$*. Then we have that*

$$\mathbf{Spin}(A, \tau)(R) = \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}.$$

Proof: Since $\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \subseteq C(\mathrm{M}_n(\mathbb{F}), \tau_0) \otimes_{\mathbb{F}} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes \mathbb{F}_{\mathrm{sep}}$ and the twisted Γ-action is the same on each, the fixed points of $\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ will simply be those points which are also fixed in the Clifford algebra. By corollary 2.1.2 this means that

$$\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \cap C(A, \tau) \otimes_{\mathbb{F}} R.$$

We now argue that any c in the above intersection also belongs to $\mathbf{Spin}(A, \tau)(R)$. Since the involution τ_0 on $C(\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \tau_0)$ restricts to the involution τ on $C(A, \tau)$, we have $c(\tau \otimes 1)(c) = 1$. Then we only need to check that $c * (b(A) \otimes_{\mathbb{F}} R) \cdot c^{-1} = b(A) \otimes_{\mathbb{F}} R$. Let $x = \sum_{i=1}^k v_i \otimes w_i \in b(A) \otimes_{\mathbb{F}} R$ with $v_i, w_i \in (\mathbb{F}_{\mathrm{sep}} \otimes_{\mathbb{F}} R)^n$. Then for each $\sigma \in \Gamma$,

$$x = \sum_{i=1}^k a'_\sigma \sigma(v_i) \otimes a'_\sigma \sigma(w_i)$$

$$= \sum_{i=1}^{k} a'_\sigma \sigma(v_i) \otimes \mathfrak{a}'_\sigma \otimes \sigma(w_i) \otimes \mathfrak{a}'^{-1}_\sigma$$

and so we have

$$\begin{aligned}
c * x \cdot c^{-1} &= \mathfrak{a}'_\sigma \sigma(c) \mathfrak{a}'^{-1}_\sigma * x \cdot \mathfrak{a}'_\sigma \sigma(c^{-1}) \mathfrak{a}'^{-1}_\sigma \\
&= \sum_{i=1}^{k} a'_\sigma \sigma(v_i) \otimes \mathfrak{a}'_\sigma \sigma(c) \mathfrak{a}'^{-1}_\sigma \mathfrak{a}'_\sigma \otimes \sigma(w_i) \otimes \mathfrak{a}'^{-1}_\sigma \mathfrak{a}'_\sigma \sigma(c^{-1}) \mathfrak{a}'^{-1}_\sigma \\
&= \sum_{i=1}^{k} a'_\sigma \sigma(v_i) \otimes \mathfrak{a}'_\sigma \sigma(c) \otimes \sigma(w_i) \otimes \sigma(c^{-1}) \mathfrak{a}'^{-1}_\sigma \\
&= \sum_{i=1}^{k} a'_\sigma \sigma(v_i) \otimes \mathfrak{a}'_\sigma \sigma(c) \otimes \sigma(w_i) \otimes \sigma(c^{-1}) \mathfrak{a}'^{-1}_\sigma \\
&= \sum_{i=1}^{k} a'_\sigma \sigma(v_i) \otimes a'_\sigma \sigma(c \otimes w_i \otimes c^{-1}) \\
&= \mathfrak{a}'_\sigma \sigma \left(\sum_{i=1}^{k} v_i \otimes c \otimes w_i \otimes c^{-1} \right) \mathfrak{a}'^{-1}_\sigma \\
&= \mathfrak{a}'_\sigma \sigma(c * x \cdot c^{-1}) \mathfrak{a}'^{-1}_\sigma .
\end{aligned}$$

Therefore $c * x \cdot c^{-1} \in b(A) \otimes_{\mathbb{F}} R$ as desired, and hence

$$\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} \subseteq \mathbf{Spin}(A, \tau)(R).$$

Conversely, any element $c \in \mathbf{Spin}(A, \tau)(R)$ is in $C(A, \tau) \otimes_{\mathbb{F}} R$ and so will be fixed, and because the involutions are compatible it will also have $c(\tau_0)(c) = 1$. We must check that it stabilizes $b(\mathrm{M}_n(\mathbb{F}_{\text{sep}}) \otimes_{\mathbb{F}} R)$. This follows from the fact that $A \otimes_{\mathbb{F}} R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\text{sep}}) \otimes_{\mathbb{F}} R$, which means that $A \otimes_{\mathbb{F}} R$ contains an $\mathbb{F}_{\text{sep}}$-basis of $\mathrm{M}_n(\mathbb{F}_{\text{sep}}) \otimes_{\mathbb{F}} R$. Therefore, if $B \in \mathrm{M}_n(\mathbb{F}_{\text{sep}}) \otimes_{\mathbb{F}} R$, we can write it as $B = \sum_{i=1}^{k} d_i x_i$ with $d_i \in \mathbb{F}_{\text{sep}}$ and $x_i \in A \otimes_{\mathbb{F}} R$, then

$$c * \left(\sum_{i=1}^{k} d_i x_i \right) \cdot c^{-1} = \sum_{i=1}^{k} d_i (c * x_i \cdot c^{-1})$$

which will be in $b(\mathrm{M}_n(\mathbb{F}_{\text{sep}}) \otimes_{\mathbb{F}} R)$ since $c \in \mathbf{Spin}(A, \tau)(R)$ means that each $c * x_i \cdot c^{-1}$ will be in $A \otimes_{\mathbb{F}} R$. Therefore

$$\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \mathbf{Spin}(A, \tau)(R).$$

as desired. ∎

This only leaves the non-split half-spin groups to be obtained via twisting. Since the half-spin groups are defined as scheme theoretic images, we do not have a concrete description of the functor of points. However, such scheme theoretic images are easily described by defining their Hopf algebra, and we therefore turn our attention to the representing Hopf algebras of twisted groups in the next section.

2.2 Twisting Hopf Algebras

In this section we develop an analogous theory of twisting and descent within Hopf algebras which mirrors the techniques of section 1.4.3. To begin, we recall the definition of Hopf algebras from section 1.2 and two lemmas we will need from [Wat79].

Lemma 2.2.1. *[Wat79, §1.5] Let H be an $\mathbb{F}$-Hopf algebra. Then the following diagrams commute.*

$$\begin{tikzcd}
H \otimes_{\mathbb{F}} H \arrow[r, "\mathrm{id} \otimes u"] & \mathbb{F} \otimes_{\mathbb{F}} H \arrow[d, equal] \\
H \arrow[u, "c"] \arrow[r, "\mathrm{id}"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes_{\mathbb{F}} H \arrow[rr, "\mathrm{id} \otimes i"] & & H \otimes_{\mathbb{F}} H \arrow[d, "m"] \\
H \arrow[u, "c"] \arrow[r, "u"] & \mathbb{F} \arrow[r, "\cdot 1"] & H
\end{tikzcd}$$

It also follows from the same discussion in [Wat79] that the coinverse satisfies the following identities.

Lemma 2.2.2. *Let H be an $\mathbb{F}$-Hopf algebra. Then,*

- $i^2 = \mathrm{id}$
- $c^n \circ i = (i \otimes \ldots \otimes i) \circ S \circ c^n$.

where $S\colon H^{\otimes n} \to H^{\otimes n}$ *is the swap* $S(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = x_n \otimes \ldots \otimes x_2 \otimes x_1$.

Now, since we are looking to mirror the process of twisting and descent among groups, we will consider the following situation. Let $\mathbf{G}$ be a split semisimple linear algebraic group over $\mathbb{F}$ which is represented by a Hopf algebra H. Let $\mathbf{G}(A)$ be another group over $\mathbb{F}$ which is a twisted form of $\mathbf{G}$. Here, the "A" is merely notational and meant to evoke the relationship between our examples in section 1.4.3 such as $\mathbf{SL}_n$ and $\mathbf{SL}(A)$ for a central simple algebra A, or $\mathbf{SO}_d$ and $\mathbf{SO}(A,\tau)$ for a central simple algebra (A,τ) with orthogonal involution. Recall that for $R \in \mathbf{Alg}_{\mathbb{F}}$ there is a standard Γ-action on $R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$ given by acting on the second factor and the fixed points of this action are $(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma} = R$. Composition by this action gives our standard action on $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, which for $p \in \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) = \mathrm{Hom}_{\mathbb{F}}(H, R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is given by

$$\sigma(p)\colon H \overset{p}{\to} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \overset{1 \otimes \sigma}{\to} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}.$$

The fixed points of this action are $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma} = \mathbf{G}(R)$. Elaborating the assumption that $\mathbf{G}(A)$ is a twisted form of $\mathbf{G}$, we assume there is a cocycle

$$\alpha\colon \Gamma \to \mathrm{Inn}(\mathbf{G}(\mathbb{F}_{\mathrm{sep}}))$$

of inner automorphisms of $\mathbf{G}(\mathbb{F}_{\text{sep}})$ with the following property. For $R \in \mathbf{Alg}_{\mathbb{F}}$, let $\alpha'_\sigma \in \operatorname{Inn}(\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}))$ be the image of α_σ induced by the canonical inclusion $\mathbf{G}(\mathbb{F}_{\text{sep}}) \to \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$, then define a twisted Γ-action on $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$ via

$$\sigma \cdot_\alpha x = \alpha'_\sigma(\sigma(x)).$$

Denoting the fixed points of this action by $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha}$ we have that

$$\mathbf{G}(A)(\mathbb{F}) = \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha}.$$

This is the story for all the group twisting done in section 1.4.3, and this is the story we will translate to Hopf algebras.

2.2.1 Galois Actions on H_{sep}

For brevity, we let $H_{\text{sep}} = H \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$. This is an $\mathbb{F}_{\text{sep}}$-Hopf algebra with structure induced from H, and so we use the same notation c, i, and u. As a starting point, we set the standard Γ-action on H_{sep} to be given by acting on the second factor so that $(H_{\text{sep}})^\Gamma = H$. Importantly, this Γ-action commutes with c, i, and u, so it is an action by $\mathbb{F}_{\text{sep}}$-Hopf automorphisms. We also define a Γ-action on $\operatorname{Aut}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}})$ via $\sigma(\varphi) = \sigma \circ \varphi \circ \sigma^{-1}$. Thus, for $\sigma \in \Gamma$, $\varphi \in \operatorname{Aut}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}})$, and $x \in H_{\text{sep}}$, we have $\sigma(\varphi(x)) = \sigma(\varphi)(\sigma(x))$.

We will make use of the following identification

$$\begin{aligned} \operatorname{Hom}_{\mathbb{F}}(H, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}}, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}) \\ p &\mapsto p_{\text{sep}} \\ q|_H &\leftarrow\!\shortmid q \end{aligned}$$

where the map p_{sep} is defined by $p_{\text{sep}}(x \otimes b) = bp(x)$, extended linearly. Since $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}) = \operatorname{Hom}_{\mathbb{F}}(H, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$ there is a Γ action on the above sets. A small thing to keep in mind about this action is the following lemma.

Lemma 2.2.3. *Let $p \in \operatorname{Hom}_{\mathbb{F}}(H, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$ and $\sigma \in \Gamma$. Under the identification above, we have that*

$$(\sigma(p))_{\text{sep}} = \sigma \circ p_{\text{sep}} \circ \sigma^{-1}$$

Proof: This can be easily verified by looking at pure tensors. We have

$$(\sigma(p))_{\text{sep}}(x \otimes b) = b(\sigma(p))(x) = b\sigma(p(x))$$

and we have

$$(\sigma \circ p_{\text{sep}} \circ \sigma^{-1})(x \otimes b) = (\sigma \circ p_{\text{sep}})(x \otimes \sigma^{-1}(b)) = \sigma(\sigma^{-1}(b)p(x)) = b\sigma(p(x)).$$

∎

Next, we build a cocycle of automorphisms of H_{sep} from the cocycle α of inner automorphisms of $\mathbf{G}(\mathbb{F}_{\text{sep}})$. However, we will need a lemma.

Lemma 2.2.4. *Let $h \in Z(\mathbf{G}(\mathbb{F}_{\text{sep}}))$. Then, if $x \in H_{\text{sep}}$ we have that*

$$\sum_i h_{\text{sep}}(x_{(1)})x_{(2)} = \sum_i x_{(1)}h_{\text{sep}}(x_{(2)}) \in H_{\text{sep}}.$$

Proof: Using the fact that $Z(\mathbf{G}(\mathbb{F}_{\text{sep}})) = Z(\mathbf{G})(\mathbb{F}_{\text{sep}})$, let h' be the image of h under the map

$$Z(\mathbf{G})(\mathbb{F}_{\text{sep}}) \to Z(\mathbf{G})(H_{\text{sep}}) \subseteq Z(\mathbf{G}(H_{\text{sep}})).$$

This means that for any $\mathbb{F}$-algebra morphism $\varphi\colon H \to H_{\text{sep}}$ we will have

$$m \circ (h' \otimes \varphi) \circ c = m \circ (\varphi \otimes h') \circ c.$$

In particular, applying this with φ being the injection $H \hookrightarrow H_{\text{sep}}$, we get for $y \in H$ that

$$\sum h'(y_{(1)})(y_{(2)} \otimes 1) = \sum (y_{(1)} \otimes 1)h'(y_{(2)}).$$

Since $h'(y) = h(y) \cdot 1$ this can be rewritten as

$$\sum h(y_{(1)})(y_{(2)} \otimes 1) = \sum (y_{(1)} \otimes 1)h(y_{(2)}).$$

Finally, let $x = y \otimes b \in H_{\text{sep}}$ with $y \in H$ and $b \in \mathbb{F}_{\text{sep}}$. Then

$$c(x) = \sum x_{(1)} \otimes x_{(2)} = \sum (y_{(1)} \otimes 1) \otimes (y_{(2)} \otimes 1)b$$

and so we can compute

$$\begin{aligned}
\sum h_{\text{sep}}(x_{(1)})x_{(2)} &= \sum_i h_{\text{sep}}(y_{(1)} \otimes 1)(y_{(2)} \otimes 1)b \\
&= \sum h(y_{(1)})(y_{(2)} \otimes 1)b \\
&= \sum (y_{(1)} \otimes 1)h(y_{(2)})b \\
&= \sum (y_{(1)} \otimes 1)h_{\text{sep}}(y_{(2)} \otimes 1)b \\
&= \sum x_{(1)}h_{\text{sep}}(x_{(2)}).
\end{aligned}$$

This extends linearly to produce the desired statement. ∎

Now we define the functions which will use in our cocycle.

Definition 2.2.5. *Consider the cocycle* $\alpha\colon \Gamma \to \mathbf{G}(\mathbb{F}_{\text{sep}})$ *and choose elements* $a_\sigma \in \mathbf{G}(\mathbb{F}_{\text{sep}})$ *representing* α_σ, *that is*

$$\alpha_\sigma(p) = a_\sigma p a_\sigma^{-1}.$$

We define functions

$$@_\sigma = (a_{\sigma,\text{sep}} \otimes \text{id} \otimes a_{\sigma,\text{sep}}^{-1}) \circ c^2\colon H_{\text{sep}} \to H_{\text{sep}}.$$

That is, for $x \in H_{\text{sep}}$ *we have*

$$@_\sigma(x) = \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)})$$

where we allow ourselves to write scalar multiplication by elements of $\mathbb{F}_{\text{sep}}$ *on the left and right as a convenience to help keep track of factors arising from comultiplication.*

Theorem 2.2.6. *Consider the functions* $@_\sigma$ *defined above. We have that*

- *Each* $@_\sigma$ *does not depend on the choice of* $a_\sigma \in \mathbf{G}(\mathbb{F}_{\text{sep}})$, *that is* $@_\sigma$ *is well-defined.*
- *Each* $@_\sigma$ *is an* $\mathbb{F}_{\text{sep}}$*-Hopf algebra automorphism.*
- *The map*

$$\begin{aligned}\mathfrak{A}\colon \Gamma &\to \text{Aut}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}})\\ \sigma &\mapsto \sigma(@_{\sigma^{-1}})\end{aligned}$$

  is a cocycle.

Proof: We begin with the first claim. Let $a_\sigma, b_\sigma \in \mathbf{G}(\mathbb{F}_{\text{sep}})$ be two elements whose inner automorphism is both α_σ. This means that there is a central element $h \in Z(\mathbf{G}(\mathbb{F}_{\text{sep}}))$ such that $a_\sigma = b_\sigma h$. Then we compute that for $x \in H_{\text{sep}}$,

$$\begin{aligned}&\sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)})\\ =&\sum (b_{\sigma,\text{sep}}h_{\text{sep}})(x_{(1)})x_{(2)}(b_{\sigma,\text{sep}}h_{\text{sep}})^{-1}(x_{(3)})\\ =&\sum b_{\sigma,\text{sep}}(x_{(1)})h_{\text{sep}}(x_{(2)})x_{(3)}h_{\text{sep}}^{-1}(x_{(4)})b_{\sigma,\text{sep}}^{-1}(x_{(5)})\\ =&\sum b_{\sigma,\text{sep}}(x_{(1)})h_{\text{sep}}(x_{(2)})h_{\text{sep}}^{-1}(x_{(3)})x_{(4)}b_{\sigma,\text{sep}}^{-1}(x_{(5)}) \text{ by lemma 2.2.4}\\ =&\sum b_{\sigma,\text{sep}}(x_{(1)})(h_{\text{sep}}h_{\text{sep}}^{-1})(x_{(2)})x_{(3)}b_{\sigma,\text{sep}}^{-1}(x_{(4)})\\ =&\sum b_{\sigma,\text{sep}}(x_{(1)})u(x_{(2)})x_{(3)}b_{\sigma,\text{sep}}^{-1}(x_{(4)})\\ =&\sum b_{\sigma,\text{sep}}(x_{(1)})x_{(2)}b_{\sigma,\text{sep}}^{-1}(x_{(3)})\end{aligned}$$

which verifies the first claim.

To see the second point, we begin by noting that $@_\sigma$ is an $\mathbb{F}_{\text{sep}}$-algebra morphism because it is a composition of algebra morphisms. It is an $\mathbb{F}_{\text{sep}}$-algebra automorphism because it has an inverse map given by $@_\sigma^{-1} = (a_{\sigma,\text{sep}}^{-1} \otimes \text{id} \otimes a_{\sigma,\text{sep}}) \circ c^2$, which we verify now.

$$\begin{aligned}
(@_\sigma^{-1} \circ @_\sigma)(x) &= @_\sigma^{-1}\left(\sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)})\right) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})@_\sigma^{-1}(x_{(2)})a_{\sigma,\text{sep}}^{-1}(x_{(3)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})a_{\sigma,\text{sep}}^{-1}(x_{(2)(1)})x_{(2)(2)}a_{\sigma,\text{sep}}(x_{(2)(3)})a_{\sigma,\text{sep}}^{-1}(x_{(3)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})a_{\sigma,\text{sep}}^{-1}(x_{(2)})x_{(3)}a_{\sigma,\text{sep}}(x_{(4)})a_{\sigma,\text{sep}}^{-1}(x_{(5)}) \\
&= \sum (a_{\sigma,\text{sep}}a_{\sigma,\text{sep}}^{-1})(x_{(1)})x_{(2)}(a_{\sigma,\text{sep}}a_{\sigma,\text{sep}}^{-1})(x_{(3)}) \\
&= \sum u(x_{(1)})x_{(2)}u(x_{(3)}) \\
&= \sum x_{(1)}u(x_{(2)}) \\
&= x.
\end{aligned}$$

The computation for $@_\sigma \circ @_\sigma^{-1} = \text{id}$ is symmetric. Therefore, to verify that $@_\sigma$ is an $\mathbb{F}_{\text{sep}}$-Hopf algebra automorphism, the only thing we have left to show is that it respects c, i, and u. Since we are discussing the Hopf operations on H_{sep}, they are $\mathbb{F}_{\text{sep}}$-linear. We begin with c,

$$\begin{aligned}
((@_\sigma \otimes @_\sigma) \circ c)(x) &= (@_\sigma \otimes @_\sigma)\left(\sum x_{(1)} \otimes x_{(2)}\right) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)})a_{\sigma,\text{sep}}(x_{(4)})x_{(5)}a_{\sigma,\text{sep}}^{-1}(x_{(6)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}(a_{\sigma,\text{sep}}^{-1}a_{\sigma,\text{sep}})(x_{(3)})x_{(4)}a_{\sigma,\text{sep}}^{-1}(x_{(5)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}u(x_{(3)})x_{(4)}a_{\sigma,\text{sep}}^{-1}(x_{(5)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}x_{(3)}a_{\sigma,\text{sep}}^{-1}(x_{(4)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)(1)}x_{(2)(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)})c(x_{(2)})a_{\sigma,\text{sep}}^{-1}(x_{(3)}) \\
&= c\left(\sum a_{\sigma,\text{sep}}(x_{(1)})x_{(2)}a_{\sigma,\text{sep}}^{-1}(x_{(3)})\right) \\
&= (c \circ @_\sigma)(x).
\end{aligned}$$

Therefore $@_\sigma$ respects c. Next, we consider i,

$$\begin{aligned}
(@_\sigma \circ i)(x) &= ((a_{\sigma,\text{sep}} \otimes \text{id} \otimes a_{\sigma,\text{sep}}^{-1}) \circ c^2 \circ i)(x) \\
&= ((a_{\sigma,\text{sep}} \otimes \text{id} \otimes a_{\sigma,\text{sep}}^{-1}) \circ (i \otimes i \otimes i) \circ S \circ c^2)(x)
\end{aligned}$$

using part two of lemma 2.2.2

$$= (a_{\sigma,\text{sep}} \otimes \text{id} \otimes a_{\sigma,\text{sep}}^{-1})\left(\sum i(x_{(3)}) \otimes i(x_{(2)}) \otimes i(x_{(1)})\right)$$

$$\begin{aligned}
&= \sum (a_{\sigma,\text{sep}} \circ i)(x_{(3)}) i(x_{(2)}) (a^{-1}_{\sigma,\text{sep}} \circ i)(x_{(1)}) \\
&= \sum a^{-1}_{\sigma,\text{sep}}(x_{(3)}) i(x_{(2)}) a_{\sigma,\text{sep}}(x_{(1)}) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)}) i(x_{(2)}) a^{-1}_{\sigma,\text{sep}}(x_{(3)}) \\
&= i\left(\sum a_{\sigma,\text{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\text{sep}}(x_{(3)})\right) \\
&= (i \circ @_\sigma)(x).
\end{aligned}$$

Therefore $@_\sigma$ respects i. Finally, we consider u.

$$\begin{aligned}
(u \circ @_\sigma)(x) &= u\left(\sum a_{\sigma,\text{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\text{sep}}(x_{(3)})\right) \\
&= \sum a_{\sigma,\text{sep}}(x_{(1)}) u(x_{(2)}) a^{-1}_{\sigma,\text{sep}}(x_{(3)}) \\
&= (a_{\sigma,\text{sep}} u a^{-1}_{\sigma,\text{sep}})(x) \\
&= (a_{\sigma,\text{sep}} a^{-1}_{\sigma,\text{sep}})(x) \\
&= u(x)
\end{aligned}$$

Therefore $@_\sigma$ respects u as well, and so it is an $\mathbb{F}_{\text{sep}}$-Hopf algebra automorphism as claimed.

Our last task is to verify that $\mathfrak{A}$ is a cocycle. To do this, we first show that the $@_\sigma$ satisfy the relationship

$$\sigma(@_\tau)@_\sigma = @_{\sigma\tau}$$

for all $\sigma, \tau \in \Gamma$. We compute

$$\begin{aligned}
&(\sigma(@_\tau)@_\sigma)(x) \\
=&\sigma(@_\tau)\left(\sum a_{\sigma,\text{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\text{sep}}(x_{(3)})\right) \\
=&(\sigma \circ @_\tau \sigma^{-1})\left(\sum a_{\sigma,\text{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\text{sep}}(x_{(3)})\right) \\
=&(\sigma \circ @_\tau)\left(\sum \sigma^{-1}(a_{\sigma,\text{sep}}(x_{(1)})) \sigma^{-1}(x_{(2)}) \sigma^{-1}(a^{-1}_{\sigma,\text{sep}}(x_{(3)}))\right) \\
=&\sigma\left(\sum \sigma^{-1}(a_{\sigma,\text{sep}}(x_{(1)})) a_{\tau,\text{sep}}(\sigma^{-1}(x_{(2)(1)}) \sigma^{-1}(x_{(2)(2)}) a^{-1}_{\tau,\text{sep}}(\sigma^{-1}(x_{(2)(3)}) \sigma^{-1}(a^{-1}_{\sigma,\text{sep}}(x_{(3)}))\right) \\
=&\sum a_{\sigma,\text{sep}}(x_{(1)}) (\sigma \circ a_{\tau,\text{sep}} \circ \sigma^{-1})(x_{(2)(1)}) x_{(2)(2)} (\sigma \circ a^{-1}_{\tau,\text{sep}} \circ \sigma^{-1})(x_{(2)(3)}) a^{-1}_{\sigma,\text{sep}}(x_{(3)}) \\
=&\sum a_{\sigma,\text{sep}}(x_{(1)}) (\sigma(a_\tau))_{\text{sep}}(x_{(2)(1)}) x_{(2)(2)} (\sigma(a^{-1}_\tau))_{\text{sep}}(x_{(2)(3)}) a^{-1}_{\sigma,\text{sep}}(x_{(3)})
\end{aligned}$$

using lemma 2.2.3, then

$$\begin{aligned}
=&\sum a_{\sigma,\text{sep}}(x_{(1)}) (\sigma(a_\tau))_{\text{sep}}(x_{(2)}) x_{(3)} (\sigma(a^{-1}_\tau))_{\text{sep}}(x_{(4)}) a^{-1}_{\sigma,\text{sep}}(x_{(5)}) \\
=&\sum (a_\sigma \sigma(a_\tau))_{\text{sep}}(x_{(1)}) x_{(2)} (\sigma(a^{-1}_\tau) a^{-1}_\sigma)_{\text{sep}}(x_{(3)})
\end{aligned}$$

Since the α_σ form a cocycle, the elements $a_\sigma \sigma(a_\tau)$ and $a_{\sigma\tau}$ will differ by a central element. Therefore, utilizing lemma 2.2.4 in the same way as we did above, we obtain

$$=\sum a_{\sigma\tau,\text{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma\tau,\text{sep}})(x_{(3)})$$

$=@_{\sigma\tau}(x).$

This verifies the identity. Expressing the identity in an equivalent way, we have that

$$@_{\sigma^{-1}}\tau(@_{\tau^{-1}}) = \tau(@_{(\sigma\tau)^{-1}})$$

for all $\sigma, \tau \in \Gamma$. Using this, we have that

$$\begin{aligned}\mathfrak{A}_\sigma \sigma(\mathfrak{A}_\tau) &= \sigma(@_{\sigma^{-1}})\sigma(\tau(@_{\tau^{-1}}))\\ &= \sigma(@_{\sigma^{-1}}\tau(@_{\tau^{-1}}))\\ &= \sigma(\tau(@_{(\sigma\tau)^{-1}}))\\ &= (\sigma\tau)(@_{(\sigma\tau)^{-1}})\\ &= \mathfrak{A}_{\sigma\tau}\end{aligned}$$

and hence $\mathfrak{A}$ is a cocycle, finalizing this long proof. ∎

Now that we have a cocycle in hand, we can twist the Γ-action on H_{sep}. We do this in the usual way, defining

$$\sigma \cdot_\alpha x = \mathfrak{A}_\sigma(\sigma(x)) = \sigma(@_{\sigma^{-1}}(x))$$

and denoting the fixed points by

$$(H_{\text{sep}})^{\Gamma_\alpha} = H(A)$$

Since this action is by $\mathbb{F}_{\text{sep}}$-Hopf algebra automorphisms, the fixed points are an $\mathbb{F}$-Hopf algebra. In addition, this action is semilinear since if $b \in \mathbb{F}_{\text{sep}}$,

$$\sigma \cdot_\alpha (bx) = \sigma(@_{\sigma^{-1}}(bx)) = \sigma(b)\sigma(@_{\sigma^{-1}}(x)) = \sigma(b)(\sigma \cdot_\alpha x).$$

Therefore, by Speiser's lemma the fixed points satisfy $H(A) \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong H_{\text{sep}}$. Just as before, this gives an identification

$$\begin{aligned}\operatorname{Hom}_{\mathbb{F}}(H(A), R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}}, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})\\ q &\mapsto q_{\text{sep}}\\ q'|_{H(A)} &\leftarrow\!\shortmid q'.\end{aligned}$$

Using this identification simultaneously with the one above, we view $\operatorname{Hom}_{\mathbb{F}}(H(A), R)$ as a subgroup of $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$ via

$$\begin{aligned}\operatorname{Hom}_{\mathbb{F}}(H(A), R) &\hookrightarrow \operatorname{Hom}_{\mathbb{F}}(H(A), R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})\\ &\cong \operatorname{Hom}_{\mathbb{F}_{\text{sep}}}(H_{\text{sep}}, R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})\end{aligned}$$

$$\begin{aligned}
&\cong \mathrm{Hom}_{\mathbb{F}}(H, R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \\
&= \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}),
\end{aligned}$$

and by construction $\mathbf{G}(A)(R) = \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$ is a subgroup of $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ as well. The fruit of our labour is that these two subgroups are equal, and so $H(A)$ is the Hopf algebra representing $\mathbf{G}(A)$.

Theorem 2.2.7. *Let $\mathbf{G}$ be a split semisimple linear algebraic group with representing Hopf algebra H. Let $\mathbf{G}(A)$ be a twisted form of $\mathbf{G}$ given by a cocycle α of inner automorphisms of $\mathbf{G}(\mathbb{F}_{\mathrm{sep}})$. Let $H(A)$ be the fixed points of the twisted Γ-action on H_{sep} defined from α as above. Then for all $R \in \mathbf{Alg}_{\mathbb{F}}$,*

$$\mathrm{Hom}_{\mathbb{F}}(H(A), R) = \mathbf{G}(A)(R)$$

as subgroups of $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$. In particular, $H(A)$ is the Hopf algebra representing $\mathbf{G}(A)$.

Proof: To begin, we note that under the above identifications $p_{\mathrm{sep}} \in \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ belongs to $\mathrm{Hom}_{\mathbb{F}}(H(A), R)$ if and only if $p_{\mathrm{sep}}|_{H(A)}$ takes values in R. Hence we need to show that such maps are fixed under the Γ action on $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, and that fixed maps have this property. Both inclusions are aided by the fact that

$$\begin{tikzcd}
H_{\mathrm{sep}} \arrow[r, "@_\sigma"] \arrow[rrrr, bend right, "(\sigma\cdot_\alpha p)_{\mathrm{sep}}"'] & H_{\mathrm{sep}} \arrow[r, "\sigma^{-1}"] & H_{\mathrm{sep}} \arrow[r, "p_{\mathrm{sep}}"] & R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \arrow[r, "\sigma"] & R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}
\end{tikzcd}$$

commutes, which we check now.

$$\begin{aligned}
(\sigma \circ p_{\mathrm{sep}} \circ \sigma^{-1} \circ @_\sigma)(x) &= (\sigma \circ p_{\mathrm{sep}} \circ \sigma^{-1}) \left(\sum a_{\sigma,\mathrm{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\mathrm{sep}}(x_{(3)}) \right) \\
&= \sigma(p)_{\mathrm{sep}} \left(\sum a_{\sigma,\mathrm{sep}}(x_{(1)}) x_{(2)} a^{-1}_{\sigma,\mathrm{sep}}(x_{(3)}) \right) \\
&= \sum a_{\sigma,\mathrm{sep}}(x_{(1)}) \sigma(p)_{\mathrm{sep}}(x_{(2)}) a^{-1}_{\sigma,\mathrm{sep}}(x_{(3)}) \\
&= (a'_{\sigma,\mathrm{sep}} \sigma(p)_{\mathrm{sep}} (a'_{\sigma,\mathrm{sep}})^{-1})(x) \\
&= (a'_\sigma \sigma(p) (a'_\sigma)^{-1})_{\mathrm{sep}}(x) \\
&= \alpha_\sigma(\sigma(p))_{\mathrm{sep}}(x) \\
&= (\sigma \cdot_\alpha p)_{\mathrm{sep}}(x).
\end{aligned}$$

where a'_σ is the image of a_σ in $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ and for $x \in H_{\mathrm{sep}}$ we have $a'_\sigma(x) = a_\sigma(x) \cdot 1 \in R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$. Hence the diagram commutes.

To address the first inclusion, assume $p_{\text{sep}} \in \operatorname{Hom}_{\mathbb{F}}(H(A), R)$. Since $H(A) \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong H_{\text{sep}}$, there is an $\mathbb{F}_{\text{sep}}$-basis of H_{sep} contained in $H(A)$. So, for $x \in H_{\text{sep}}$ we may write $x = \sum b_i x_i$ with $b_i \in \mathbb{F}_{\text{sep}}$ and $x_i \in H(A)$. Then

$$\begin{aligned}
(\sigma \cdot_\alpha p)_{\text{sep}}(x) &= (\sigma \circ p_{\text{sep}} \circ \sigma^{-1} \circ @_\sigma) \left(\sum b_i x_i\right) \\
&= (\sigma \circ p_{\text{sep}}) \left(\sum \sigma^{-1}(b_i)\sigma^{-1}(@_\sigma(x_i))\right) \\
&= (\sigma \circ p_{\text{sep}}) \left(\sum \sigma^{-1}(b_i)(\sigma^{-1} \cdot_\alpha x_i)\right) \\
&= (\sigma \circ p_{\text{sep}}) \left(\sum \sigma^{-1}(b_i) x_i\right) \\
&= \sigma \left(\sum \sigma^{-1}(b_i) p_{\text{sep}}(x_i)\right) \\
&= \sum b_i \sigma(p_{\text{sep}}(x_i)) \\
&= \sum b_i p_{\text{sep}}(x_i), \text{ since each } p_{\text{sep}}(x_i) \in R \\
&= p_{\text{sep}}(x)
\end{aligned}$$

therefore p_{sep} is a fixed point, meaning $p_{\text{sep}} \in \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \mathbf{G}(A)(R)$.

Conversely, let $p_{\text{sep}} \in \mathbf{G}(A)(R)$ be a fixed point. Then for any $x \in H(A)$ we have

$$\begin{aligned}
p_{\text{sep}}(x) &= (\sigma \circ p_{\text{sep}} \circ \sigma^{-1} \circ @_\sigma)(x) \\
&= (\sigma \circ p_{\text{sep}})(\sigma^{-1} \cdot_\alpha x) \\
&= (\sigma \circ p_{\text{sep}})(x) \\
&= \sigma(p_{\text{sep}}(x))
\end{aligned}$$

for each $\sigma \in \Gamma$. Hence $p_{\text{sep}}(x) \in (R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^\Gamma = R$, and therefore $p_{\text{sep}}|_{H(A)}$ only takes values in R. Thus $p_{\text{sep}} \in \operatorname{Hom}_{\mathbb{F}}(H(A), R)$.

Having shown both inclusions, we conclude that

$$\operatorname{Hom}_{\mathbb{F}}(H(A), R) = \mathbf{G}(A)(R)$$

as desired. ∎

2.3 Twisting Half-spin

We can now apply the results of the previous section to twist our last group which remains untwisted. We will be working with a central simple $\mathbb{F}$-algebra (A, τ) of degree $2n$ with orthogonal involution of trivial discriminant. Recall from theorem 1.1.17 that in this case the Clifford algebra decomposes as

$$C(A,\tau) = C^+(A,\tau) \times C^-(A,\tau)$$

where each of $C^\pm(A,\tau)$ is a central simple $\mathbb{F}$-algebra of degree 2^{n-1}. The central simple algebra (A,τ) has a corresponding cocycle α in $\mathbf{PSO}_{2n}(\mathbb{F}_{\mathrm{sep}})$, and we can choose elements $\mathfrak{a}_\sigma \in \mathbf{Spin}_{2n}(\mathbb{F}_{\mathrm{sep}})$ lying above each α_σ. For $R \in \mathbf{Alg}_{\mathbb{F}}$ we denote by $\mathfrak{a}'_\sigma$ the image of $\mathfrak{a}_\sigma$ in $\mathbf{Spin}_{2n}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$. We have seen in corollary 2.1.2 that these $\mathfrak{a}'_\sigma$ can twist the split Clifford algebra to yield

$$C(A,\tau) \otimes_{\mathbb{F}} R = (C(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha}.$$

By restricting to invertible elements, this construction immediately gives a twisting result for $\mathbf{GL}(C(A,\tau))$.

Lemma 2.3.1. *Let (A,τ) and $\mathfrak{a}'_\sigma$ be as above. For $R \in \mathbf{Alg}_{\mathbb{F}}$ there is a twisted Γ-action on $\mathbf{GL}(C(\mathrm{M}_{2n}(\mathbb{F}), \tau_0))(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ given by*

$$\sigma \cdot_\alpha x = \mathfrak{a}'_\sigma \sigma(x) (\mathfrak{a}'_\sigma)^{-1},$$

the fixed points of which are

$$\mathbf{GL}(C(A,\tau))(R) = (\mathbf{GL}(C(\mathrm{M}_{2n}(\mathbb{F}), \tau_0))(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}))^{\Gamma_\alpha}.$$

When dealing with half-spin we are concerned with $C^+(A,\tau)$. Since τ has trivial discriminant, so does τ_0 and so there is also a decomposition

$$C(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) = C^+(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \times C^-(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0).$$

Under this decomposition we may write the elements $\mathfrak{a}_\sigma$ as

$$\mathfrak{a}_\sigma = (\mathtt{a}_\sigma, \mathtt{a}^-_\sigma)$$

where $\mathtt{a}_\sigma \in \mathbf{HSpin}_{2n}(\mathbb{F}_{\mathrm{sep}})$ and each $\mathtt{a}_\sigma$ lies above α_σ. Similarly denoting the image of $\mathtt{a}_\sigma$ in $\mathbf{HSpin}_{2n}(R \otimes_{\mathbb{F}} R)$ by $\mathtt{a}'_\sigma$, we see that the twisted action on $C(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R$ restricts to give a twisting result for $C^+(A,\tau)$.

Lemma 2.3.2. *Let (A,τ) and $\mathtt{a}'_\sigma$ be as above. For $R \in \mathbf{Alg}_{\mathbb{F}}$ there is a twisted Γ-action on $C^+(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R$ given by*

$$\sigma \cdot_\alpha c = \mathtt{a}'_\sigma \sigma(c) (\mathtt{a}'_\sigma)^{-1},$$

the fixed points of which are

$$C^+(A,\tau) \otimes_{\mathbb{F}} R = (C^+(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha}.$$

Proof: For each $c \in C(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R$ we can write $c = (c_1, c_2)$ due to the decomposition of the Clifford algebra. The twisted action on the full Clifford algebra then appears as

$$\sigma \cdot_\alpha c = (\mathsf{a}'_\sigma \sigma(c_1)(\mathsf{a}'_\sigma)^{-1}, (\mathsf{a}^-_\sigma)'\sigma(c_2)(\mathsf{a}^-_\sigma)'^{-1})$$

and so when descending to fixed points we obtain

$$C(A, \tau) \otimes_{\mathbb{F}} R = (C^+(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha} \times (C^-(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha}.$$

Now, since each $C^\pm(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0)$ is a central simple $\mathbb{F}_{\mathrm{sep}}$-algebra of degree 2^{n-1}, the fixed points of their respective twisted actions will be central simple $\mathbb{F}$-algebras of degree 2^{n-1}. Hence this decomposition at the R level arises from the decomposition of theorem 1.1.17, and so

$$C^+(A, \tau) \otimes_{\mathbb{F}} R = (C^+(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \tau_0) \otimes_{\mathbb{F}} R)^{\Gamma_\alpha}$$

as desired. ∎

Corollary 2.3.3. *Let (A, τ) and a'_σ be as above. For $R \in \mathbf{Alg}_{\mathbb{F}}$ there is a twisted Γ-action on $\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}), \tau_0))(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ given by*

$$\sigma \cdot_\alpha x = \mathsf{a}'_\sigma \sigma(x)(\mathsf{a}'_\sigma)^{-1},$$

the fixed points of which are

$$\mathbf{GL}(C^+(A, \tau))(R) = (\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}), \tau_0))(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}))^{\Gamma_\alpha}.$$

These were the last pieces needed to twist half-spin.

Theorem 2.3.4. *Let (A, τ) be a central simple $\mathbb{F}$-algebra with orthogonal involution of trivial discriminant and of degree $2n$ where n is even. Let α be its chosen cocycle, and let $\mathsf{a}_\sigma \in \mathbf{HSpin}_n(\mathbb{F}_{\mathrm{sep}})$ be any elements above each $\alpha_\sigma \in \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$ and set $\mathsf{a}'_\sigma \in \mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ to be their images. These define a twisted Γ-action on $\mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ by $\sigma \cdot_\alpha c = \mathsf{a}'_\sigma \sigma(c) \mathsf{a}'^{-1}_\sigma$. Then we have that*

$$\mathbf{HSpin}(A, \tau)(R) = \mathbf{HSpin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}.$$

Proof: Recall that the projection $C(A, \tau) \twoheadrightarrow C^+(A, \tau)$ induces a group scheme morphism

$$\mathbf{GL}(C(A, \tau)) \twoheadrightarrow \mathbf{GL}(C^+(A, \tau))$$

and that $\mathbf{HSpin}(A, \tau)$ is the scheme theoretic image of the composition

$$\mathbf{Spin}(A, \tau) \hookrightarrow \mathbf{GL}(C(A, \tau)) \twoheadrightarrow \mathbf{GL}(C^+(A, \tau)).$$

Using traditional notation, we denote the representing Hopf algebra of a group $\mathbf{G}$ by $\mathbb{F}[\mathbf{G}]$ and also write $\mathbb{F}[\mathbf{G}]_{\mathrm{sep}} = \mathbb{F}_{\mathrm{sep}}[\mathbf{G}]$. The statement that $\mathbf{HSpin}(A,\tau)$ is the scheme theoretic image is equivalent to saying that $\mathbb{F}[\mathbf{HSpin}(A,\tau)]$ is the image of the induced map

$$\mathbb{F}[\mathbf{GL}(C^+(A,\tau))] \to \mathbb{F}[\mathbf{Spin}(A,\tau)].$$

Now we involve the results of theorem 2.2.7 and the lemmas above to situate these Hopf algebras inside their split counterparts. In the split case the map $\mathbf{Spin}_{2n} \to \mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau_0))$ corresponds to

$$\mathbb{F}[\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau_0))] \to \mathbb{F}[\mathbf{Spin}_{2n}].$$

Choose elements $\mathfrak{a}_\sigma \in \mathbf{Spin}_{2n}(\mathbb{F}_{\mathrm{sep}})$ lying above each $\mathsf{a}_\sigma \in \mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau_0))(\mathbb{F}_{\mathrm{sep}})$. Viewing these elements as maps, the a_σ are pullbacks of $\mathfrak{a}_\sigma$ along the above Hopf morphism. This means that the $\mathbb{F}_{\mathrm{sep}}$-map

$$\varphi\colon \mathbb{F}_{\mathrm{sep}}[\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau_0))] \to \mathbb{F}_{\mathrm{sep}}[\mathbf{Spin}_{2n}],$$

respects the twisted actions and therefore restricts to fixed points. Hence we have the following diagram.

$$\begin{tikzcd}
\mathbb{F}_{\mathrm{sep}}[\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau_0))] \arrow[r, "\varphi"] & \mathbb{F}_{\mathrm{sep}}[\mathbf{Spin}_{2n}] \\
\mathbb{F}[\mathbf{GL}(C^+(A,\tau))] \arrow[r] \arrow[u, hook] & \mathbb{F}[\mathbf{Spin}(A,\tau)] \arrow[u, hook]
\end{tikzcd}$$

Furthermore, $\mathbb{F}_{\mathrm{sep}}[\mathbf{HSpin}_{2n}]$ is the image of φ. Thus, our goal is to show that the fixed points of the image of φ are equal to the image along the bottom. This will show that $\mathbb{F}_{\mathrm{sep}}[\mathbf{HSpin}_{2n}]^{\Gamma_\alpha} = \mathbb{F}[\mathbf{HSpin}(A,\tau)]$ which via theorem 2.2.7 will yield the desired result.

Commutativity of the diagram gives that $\mathbb{F}[\mathbf{HSpin}(A,\tau)] \subseteq \mathbb{F}_{\mathrm{sep}}[\mathbf{HSpin}_{2n}]^{\Gamma_\alpha}$. For the converse, first note that since φ respects the twisted action, and

$$\sigma \cdot_\alpha 0 = 0 \in \mathbb{F}_{\mathrm{sep}}[\mathbf{Spin}_{2n}]$$

for all $\sigma \in \Gamma$, this means the twisted action stabilizes $\ker(\varphi)$. Therefore, Speiser's lemma applies to tell us that $\ker(\varphi)^{\Gamma_\alpha} \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \ker(\varphi)$. Furthermore, $\ker(\varphi)^{\Gamma_\alpha}$ is an $\mathbb{F}$-subspace of $\mathbb{F}[\mathbf{GL}(C^+(A,\tau))]$, so there is an $\mathbb{F}$-vector space decomposition

$$\mathbb{F}[\mathbf{GL}(C^+(A,\tau))] = W \oplus \ker(\varphi)^{\Gamma_\alpha}$$

for some other subspace $W \subset \mathbb{F}[\mathbf{GL}(C^+(A,\tau))]$. Tensoring with $\mathbb{F}_{\mathrm{sep}}$, we obtain an $\mathbb{F}_{\mathrm{sep}}$-vector space decomposition

$$\mathbb{F}_{\mathrm{sep}}[\mathbf{GL}(C^+(\mathrm{M}_{2n}(\mathbb{F}),\tau))] = (W \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \oplus \ker(\varphi).$$

where for elements $w \otimes c \in W \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$, the twisted action is given by $\sigma \cdot_\alpha (w \otimes c) = w \otimes \sigma(c)$, i.e., by acting on the second factor.

Now, consider $x \in \mathbb{F}_{\text{sep}}[\mathbf{HSpin}_{2n}]^{\Gamma_\alpha}$. Equivalently, $x \in \mathbb{F}[\mathbf{Spin}(A, \tau)]$ and $x \in \operatorname{Img}(\varphi)$. We need to find a fixed point in the preimage of x. The preimage of x is a coset

$$\varphi^{-1}(x) = w + \ker(\varphi)$$

for some unique $w \in W \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$. Then, $\sigma \cdot_\alpha (x) = x$ means that

$$w + \ker(\varphi) = (\sigma \cdot_\alpha w) + \ker(\varphi)$$

for all $\sigma \in \Gamma$. Since the coset representative in $W \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$ is unique and $\sigma \cdot_\alpha w \in W \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$, this means $w = \sigma \cdot_\alpha w$ for all $\sigma \in \Gamma$. Therefore w is a fixed point in $\varphi^{-1}(x)$ and hence

$$\mathbb{F}_{\text{sep}}[\mathbf{HSpin}_{2n}]^{\Gamma_\alpha} \subseteq \mathbb{F}[\mathbf{HSpin}(A, \tau)]$$

which finishes the proof. ∎

Chapter 3

Linear Algebraic Groups as Sheaves

The linear algebraic groups introduced in chapter 1 are so far just functors. In this chapter we will outline how these functors form sheaves on the category $\mathbf{Alg}_{\mathbb{F}}$ with respect to the étale topology. This viewpoint allows us to discuss a notion of stalks, which we will find to be the R-points of the group for local strictly Henselian rings $R \in \mathbf{Alg}_{\mathbb{F}}$. For a group scheme homomorphism, viewed as a morphism of sheaves, it is injective if and only if the induced map on each of these stalks is injective, and surjectivity on the stalks implies the homomorphism is surjective. Being able to focus only on the group's points over local strict Henselian rings has the advantage that we are able to completely describe the group's behaviour there in the language of Chevalley groups. In the second part of this chapter we introduce Chevalley group functors, which are subfunctors of our linear algebraic groups, but the two notions agree over local strictly Henselian rings. Thus, we can use the computational power of the Chevalley group's generators and relations presentation in order to define and analyze group scheme homomorphisms between linear algebraic groups.

3.1 Sites and Sheaves

Here we recount the general setup of sheaves on sites following The Stacks Project [Stacks]. What The Stacks Project calls sites are also known as categories with a Grothendieck topology. We use the former terminology in order to match our source. We must note, however, that some of the definitions we use here do not match with The Stacks Project exactly due to set theoretic concerns. The Stacks Project requires that the underlying category of any site forms a set, and that the coverings form a set as well, and they do this to ensure that many constructions they wish to consider exist. Since our use of sites and sheaves is limited in scope, we relax these definitions in order to allow a site to be a large category, with a class of coverings. All new constructions we consider will be local in some sense, i.e., only using a set's worth of information, so our relaxation is a safe one. If one wishes to formally address these issues, Grothendieck universes can be used. We begin with our relaxed definitions.

Definition 3.1.1. *[Stacks, Tag 03NH] A* site *is a category* $\mathcal{C}$ *together with a collection* $\mathrm{Cov}(\mathcal{C})$ *consisting of families of morphisms* $\{\varphi_i\colon U_i \to U\}_{i\in I}$ *where* I *is a set,* $U, U_i \in \mathrm{Ob}(\mathcal{C})$*, and* $\varphi_i \in \mathrm{Hom}(U_i, U)$*. Such a family is called a* covering of U*. We require that*

- *If* $\varphi\colon V \to U$ *is an isomorphism, then* $\{\varphi\colon V \to U\} \in \mathrm{Cov}(\mathcal{C})$*.*

- *If* $\{\varphi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ *and for all* $i \in I$ *we are given coverings* $\{\psi_{ij}\colon U_{ij} \to U_i\}_{j\in I_j}$*, then the family of compositions*

$$\{\varphi_i \circ \psi_{ij}\colon U_{ij} \to U\}_{(i,j)\in \prod_{i\in I}\{i\}\times I_i}$$

  is also a covering.

- *If* $\{\varphi\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ *and* $V \to U$ *is a morphism in* $\mathcal{C}$*, then*
  - *(i) for all* $i \in I$*, the fiber product* $U_i \times_U V$ *exists in* $\mathcal{C}$*, and*
  - *(ii)* $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$*.*

This definition is meant to extend the notion of open coverings from topology to categories. In fact, considering the open sets of a topological space and the subset inclusion maps between them as a category, we produce an example of a site by letting the coverings be exactly the usual open covers. The first point then says that a set should be open in itself, the second point says that open covers can be refined, and the third point says that an open cover of U can be restricted to a subset V via intersection. Just as a sheaf on a topological space is defined on open sets and respects open coverings, we define a sheaf on a site to be a functor which respects the coverings of the site.

Definition 3.1.2. *[Stacks, Tag 03NK] Let* $\mathcal{C}$ *be a site. A* presheaf of sets *on* $\mathcal{C}$ *is any contravariant functor* $\mathcal{F}\colon \mathcal{C} \to \mathbf{Sets}$*. A presheaf is considered a* sheaf *if for all coverings* $\{\varphi_i\colon U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$*, the diagram*

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\pi"] & \prod_{i\in I}\mathcal{F}(U_i) \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \prod_{i,j\in I}\mathcal{F}(U_i \times_U U_j) \end{tikzcd}$$

where $\pi(a) = (a|_{U_i})_{i\in I}$*,* $h_1((a_i)_{i\in I}) = (a_i|_{U_i\times_U U_j})_{i,j\in I}$*, and* $h_2((a_i)_{i\in I}) = (a_j|_{U_i\times_U U_j})_{i,j\in I}$ *is an equalizer diagram in* $\mathbf{Sets}$*.*

The restriction notation is also borrowed from sheaves on topological spaces, it means the following. For $a \in \mathcal{F}(U)$ and $\phi_i \colon U_i \to U$, we set $a|_{U_i} = \mathcal{F}(\varphi_i)(a)$. Next, there are canonical projections $\phi_i, \phi_j \colon U_i \times_U U_j \to U$, and using them we set $a|_{U_i \times_U U_j} = \mathcal{F}(\phi_i)(a)$ for $a \in \mathcal{F}(U_i)$, and $a|_{U_i \times_U U_j} = \mathcal{F}(\phi_j)(a)$ for $a \in \mathcal{F}(U_j)$. The property of being an equalizer diagram encodes all the usual properties we expect from a sheaf. In particular, π is injective, $h_1 \circ \pi = h_2 \circ \pi$, and any element of $\prod_{i \in I} \mathcal{F}(U_i)$ on which h_1 and h_2 agree is in the image of π. As with most things mathematical, we also consider maps between sheaves.

Definition 3.1.3. *[Stacks, Tag 00WM] Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a site $\mathcal{C}$. A* map of sheaves *is a natural transformation $\varphi \colon \mathcal{F} \to \mathcal{G}$. In addition, we say that*

- *φ is injective is $\varphi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each $U \in \mathrm{Ob}(\mathcal{C})$.*
- *φ is surjective if for all $U \in \mathrm{Ob}(\mathcal{C})$ and each $s \in \mathcal{G}(U)$, there is $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ such that for all i, $s|_{U_i}$ is in the image of $\varphi(U_i) \colon \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.*

There are two final small notions of the general setup which will be important for our applications. Since our linear algebraic groups are Hom-functors, that is each group $\mathbf{G}$ comes with a Hopf algebra H, and then for $R \in \mathbf{Alg}_{\mathbb{F}}$ we have $\mathbf{G}(R) = \mathrm{Hom}_{\mathbb{F}}(H, R)$, they are examples of *representable functors*.

Definition 3.1.4. *Let $\mathcal{F} \colon \mathcal{C} \to$ **Sets** be a covariant (resp. contravariant) functor. It is called a* representable functor *if there exists $F \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{F}(A) = \mathrm{Hom}_{\mathcal{C}}(F, A)$ (resp. $\mathcal{F}(A) = \mathrm{Hom}_{\mathcal{C}}(A, F)$). Then $\mathcal{F}$ is said to be* represented by *F.*

In a combination of terminology, we also refer to contravariant representable functors as *representable presheaves*. While representable presheaves are well behaved, they are not so well behaved as to be sheaves with respect to any site. Therefore, we introduce some terminology to distinguish those sites where this does happen. First, since the sheaf condition must hold for all coverings, we define a notion of coverings where representable presheaves satisfy the sheaf condition.

Definition 3.1.5. *[Stacks, Tag 00WP] A family of morphisms $\{U_i \to U\}_{i \in I}$ in a category $\mathcal{C}$ is called an* effective epimorphism *if*

- *For any other morphism $V \to U$ in $\mathcal{C}$, the fiber products $U_i \times_U V$ exist,*
- *For any $X \in \mathrm{Ob}(\mathcal{C})$ the sequence*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(U, X) \arrow[r] & \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(U_i, X) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j \in I} \mathrm{Hom}_{\mathcal{C}}(U_i \times_U U_j, X) \end{tikzcd}$$

  is an equalizer diagram. That is, the representable presheaf $\mathrm{Hom}_{\mathcal{C}}(-, X)$ satisfies the sheaf condition for this family.

Furthermore, family is called a universal effective epimorphism *if for any morphism* $V \to U$ *in* $\mathcal{C}$*, the family* $\{U_i \times_U V \to V\}_{i \in I}$ *is an effective epimorphism.*

As noted in [Stacks, Tag 00WO] below the above definition, a category $\mathcal{C}$ together with all universal effective epimorphisms as coverings forms a site.

Definition 3.1.6. *Let* $\mathcal{C}$ *be a category. The* canonical site *on* $\mathcal{C}$ *is the site where* $\mathrm{Cov}(\mathcal{C})$ *is the class of all universal effective epimorphisms in* $\mathcal{C}$*.*

By construction, all representable presheaves on this site will be sheaves. Conversely, if a site has the property that all representable presheaves are sheaves, then all its coverings must be universal effective epimorphisms. Hence we use the following terminology.

Definition 3.1.7. *We call a site* $\mathcal{C}$ subcanonical *if every covering in* $\mathrm{Cov}(\mathcal{C})$ *is a universal effective epimorphism. Equivalently,* $\mathcal{C}$ *is subcanonical if every representable presheaf on* $\mathcal{C}$ *is a sheaf.*

This definition is closely related to the definition of the *canonical topology* on a category, [Stacks, Tag 00ZA]. It is the finest topology for which all representable presheaves are sheaves. Any coarser topology also has this property and is called *subcanonical.* However, topologies on a category are a related but distinct notion to sites, so we include the above definitions.

3.2 The Étale Site

In this section we focus our attention on a particular site, namely the *étale site* on the category of affine schemes over $\mathrm{Spec}(\mathbb{F})$, which we denote by $\mathbf{Aff}_{\mathbb{F}}$. We focus on this category in particular since it is equivalent to $\mathbf{Alg}_{\mathbb{F}}$ via the Spec functor. Since the Spec functor is contravariant and our linear algebraic groups are covariant on $\mathbf{Alg}_{\mathbb{F}}$, we can view the groups as contravariant functors from $\mathbf{Aff}_{\mathbb{F}}$. Let $\mathbf{G}$ be a linear algebraic group scheme represented by a Hopf algebra H. It is a covariant functor

$$\begin{aligned} \mathbf{G}\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, R). \end{aligned}$$

We denote by $\widehat{\mathbf{G}}$ the equivalent functor on $\mathbf{Aff}_{\mathbb{F}}$, which is given by

$$\begin{aligned} \widehat{\mathbf{G}}\colon \mathbf{Aff}_{\mathbb{F}} &\to \mathbf{Sets} \\ \mathrm{Spec}(R) &\mapsto \mathrm{Hom}_{\mathbf{Aff}_{\mathbb{F}}}(\mathrm{Spec}(R), \mathrm{Spec}(H)) \\ &\qquad\qquad \cong \\ &\qquad \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, R). \end{aligned}$$

where we compose with the forgetful functor $\mathbf{Grp} \to \mathbf{Sets}$. In particular, our group schemes are representable presheaves on $\mathbf{Alg}_{\mathbb{F}}$. To begin, we recall the namesake notion of an étale morphism.

Definition 3.2.1. *A homomorphism of rings $R \to S$ is called* étale *if it is smooth and the module of differentials $\Omega_{S/R} = 0$. A homomorphism of affine schemes* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is called* étale *if the associated ring map $S \to R$ is étale.*

We will not need to deal with the internal workings of this definition. Instead, we will be able to work with the properties of étale morphisms. Now we can define the étale site.

Definition 3.2.2. *The* étale site *is the category $\mathbf{Aff}_{\mathbb{F}}$ equipped with coverings* $\mathrm{Cov}_{\text{étale}}(\mathbf{Aff}_{\mathbb{F}})$ *whose elements are all the finite families*

$$\{\varphi_i \colon U_i \to U \mid 1 \leq i \leq m\}$$

such that each φ_i is an étale morphism of affine schemes, and $\cup_{i=1}^{m} \varphi_i(U_i) = U$.

By the same argument as in [Stacks, Tag 021C], this does indeed form a site. As promised, this site is subcanonical.

Lemma 3.2.3. *The étale site $\mathbf{Aff}_{\mathbb{F}}$ is subcanonical.*

Proof: We combine two lemmas, [Stacks, Tag 03PH] which says that any étale covering is an fpqc covering (standing for faithfully flat and quasi-compact), and [Stacks, Tag 023Q] which says that fpqc coverings are universal effective epimorphisms. Therefore all étale covers are universal effective epimorphisms, and so the étale site is subcanonical. ▮

This result means that since our algebraic groups are representable presheaves on $\mathbf{Aff}_{\mathbb{F}}$, they are sheaves with respect to the étale site. This gives the following small lemma.

Lemma 3.2.4. *Let $\{\mathrm{Spec}(U_i) \to \mathrm{Spec}(R) \mid 1 \leq i \leq m\}$ be an étale covering in $\mathbf{Aff}_{\mathbb{F}}$. Then there is an equalizer diagram in $\mathbf{Alg}_{\mathbb{F}}$*

$$\begin{tikzcd} R \arrow[r] & \prod_{i=1}^{m} U_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j=1}^{m} U_i \otimes_R U_j \end{tikzcd}$$

Proof: Since the étale site is subcanonical, the presheaf $\mathrm{Hom}_{\mathbf{Aff}_F}(-, \mathrm{Spec}(\mathbb{F}[x]))$ is a sheaf. It therefore gives an equalizer diagram for the given cover, which when translated into algebras is the desired diagram. ▮

Ultimately, we want to determine properties of group scheme homomorphisms by looking at their sheaf properties. In particular, we have the following two lemmas which compare group scheme morphism injectivity and surjectivity to the respective property for sheaf morphisms.

Lemma 3.2.5. *Let $\varphi : \mathbf{G} \to \mathbf{H}$ be a homomorphism of linear algebraic group schemes. Then φ is injective as a group scheme homomorphism if and only if it is injective as a homomorphism of sheaves on the étale site.*

Proof: The morphism φ is injective as a group scheme homomorphism if and only if for all $R \in \mathbf{Alg}_{\mathbb{F}}$ we have $\varphi(R)\colon \mathbf{G}(R) \to \mathbf{H}(R)$ is injective. It is injective as an étale sheaf homomorphism if and only if for all $\mathrm{Spec}(R) \in \mathbf{Aff}_{\mathbb{F}}$ we have $\hat{\varphi}(R)\colon \widehat{\mathbf{G}}(\mathrm{Spec}(R)) \to \widehat{\mathbf{H}}(\mathrm{Spec}(R))$ is injective. These two notions are equivalent by definition. ∎

Lemma 3.2.6. *Let $\varphi\colon \mathbf{G}_1 \to \mathbf{G}_2$ be a homomorphism of linear algebraic group schemes. If φ is surjective as a map of étale sheaves, then it is surjective as a group scheme homomorphism.*

Proof: Let H_i be the representing Hopf algebra of $\mathbf{G}_i$. We aim to show that the map $\varphi^*\colon H_2 \to H_1$ is injective. Choose $\mathrm{Spec}(H_2) \in \mathbf{Aff}_{\mathbb{F}}$ and $\mathrm{id}_{H_2} \in \widehat{\mathbf{G}_2}(\mathrm{Spec}(H_2)) = \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H_2, H_2)$. Since φ is surjective as an étale sheaf morphism, there exists an étale cover $\{\phi_i\colon \mathrm{Spec}(U_i) \to \mathrm{Spec}(H_2) \mid 1 \le i \le m\}$ such that for each i, $\mathrm{id}_{H_2}|_{U_i} \in \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H_1, U_i)$ is in the image of $\varphi(U_i)$. Translated into algebras, this means that there is a commutative diagram

$$\begin{tikzcd}
H_1 \arrow[r, "\alpha"] & \prod_{i=1}^{m} U_i \\
H_2 \arrow[u, "\varphi^*"] \arrow[r, hook, "\mathrm{id}_{H_2}"] & H_2 \arrow[u, "\beta"]
\end{tikzcd}$$

where $\alpha = (\mathrm{id}_{H_2}|_{\mathrm{Spec}(U_i)})_{1 \le i \le m}$ and $\beta = (\phi_i^*)_{1 \le i \le m}$. In fact, β is also injective since it appears in the equalizer diagram

$$\begin{tikzcd}
H_2 \arrow[r, "\beta"] & \prod_{i=1}^{m} U_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j=1}^{m} U_i \otimes_{H_2} U_j
\end{tikzcd}$$

of the étale cover. Therefore since $\beta \circ \mathrm{id}_{H_2}$ is injective, φ^* must also be injective as desired. ∎

While injectivity of the two different notions is equivalent, we note that surjectivity as an étale sheaf morphism is strictly a stronger notion. See example 3.4.10.

3.3 Points and Stalks

Just like with sheaves on regular spaces, we can define a notion of stalks on which the behaviour of morphisms is reflected. Then, rather than having to check injectivity and surjectivity in full, it suffices to only check the induced maps on stalks. The stalks we define will be on *geometric points* of $\mathbf{Aff}_{\mathbb{F}}$. We model our definition off of [Stacks, Tag 03PO].

Definition 3.3.1. *Let $X \in \mathbf{Aff}_{\mathbb{F}}$.*

- *A* geometric point of $\mathbf{Aff}_{\mathbb{F}}$ over X *is a morphism*

$$p\colon \operatorname{Spec}(\mathbb{K}) \to X$$

  where $\mathbb{K} \in \mathbf{Alg}_{\mathbb{F}}$ is an algebraically closed field. We say p lies over $\operatorname{Img}(p) \in X$.

- *An* étale neighbourhood of p *is a diagram*

$$\begin{tikzcd}\operatorname{Spec}(\mathbb{K}) \arrow[r, "u"] \arrow[rr, bend left, "p"] & U \arrow[r, "\varphi"] & X\end{tikzcd}$$

  where φ is étale. We denote the neighbourhood by (U, u).

- *A morphism of étale neighbourhoods of p, $(U_1, u_1) \to (U_2, u_2)$, is a map $\varphi\colon U_1 \to U_2$ making the following diagram commute*

$$\begin{tikzcd} & U_1 \arrow[dd, "\varphi"] \arrow[dr] & \\ \operatorname{Spec}(\mathbb{K}) \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & & X \\ & U_2 \arrow[ur] & \end{tikzcd}$$

Since étale neighbourhoods of p have morphisms between them, they form a category. Our notion of the stalk of a functor at a geometric point p will come from taking a filtered colimit with respect to these étale neighbourhoods. We define stalks after verifying we can with the following lemma.

Lemma 3.3.2. *Let p be a geometric point of $\mathbf{Aff}_{\mathbb{F}}$ over X. The category of étale neighbourhoods of p is small, that is its objects form a set, and it is cofiltered.*

Proof: To see that the category is small, note that if $U \to X$ is an étale map of affine schemes, then it corresponds to an étale map of rings $R \to S$. Since such a ring map is smooth, it is finitely presented, and therefore there are, in a rough estimate, at most $(|R|^{\aleph_0})^{\aleph_0}$ such maps. The category is cofiltered by an analogous argument as in [Stacks, Tag 03PQ]. The referenced statement speaks about arbitrary étale neighbourhoods over an arbitrary scheme, i.e., not necessarily affine, however the proof only uses fiber products and the category $\mathbf{Aff}_{\mathbb{F}}$ is closed under fiber products. Therefore the proof applies here as well. ∎

Definition 3.3.3. *[Stacks, Tag 040R] Let p be a geometric point over $X \in \mathbf{Aff}_F$, and let $\mathcal{F}$ be a presheaf on $\mathbf{Aff}_{\mathbb{F}}$. The* stalk of $\mathcal{F}$ at p *is*

$$\mathcal{F}_p = \operatorname*{colim}_{(U,u)} \mathcal{F}(U)$$

where the colimit is indexed over all étale neighbourhoods of p.

As noted in [Stacks] following this definition, since the étale neighbourhoods are cofiltered, their images under a presheaf will be filtered. Therefore, elements of $\mathcal{F}_p$ are pairs $(s \in \mathcal{F}(U), (U, u))$ where two such pairs are equal if their restrictions agree on some common étale neighbourhood further down the filter.

Any morphism of presheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ induces a map on stalks. If p is a geometric point then we get a map

$$\begin{aligned} \varphi_p \colon \mathcal{F}_p &\to \mathcal{G}_p \\ (s, (U, u)) &\mapsto (\varphi(U)(s), (U, u)). \end{aligned}$$

The following proposition says that the geometric points of $\mathbf{Aff}_{\mathbb{F}}$ form a *conservative family*. That is, we can determine the injectivity or surjectivity of a sheaf morphism by looking at the behaviour at all such stalks.

Theorem 3.3.4. *Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $\mathbf{Aff}_{\mathbb{F}}$. Then φ is injective (resp. surjective) if and only if $\varphi_p \colon \mathcal{F}_p \to \mathcal{G}_p$ is injective (resp. surjective) for all geometric points p of $\mathbf{Aff}_{\mathbb{F}}$.*

Proof: We follow the argument of [Stacks, Tag 03PU]. First, assume that φ is injective. Let p be a geometric point and let $(s_1, (U_1, u_1)), (s_2, (U_2, u_2)) \in \mathcal{F}_p$. If we have that $\varphi_p(s_1, (U_1, u_1)) = \varphi_p(s_2, (U_2, u_2))$ then there exists a common étale neighbourhood $(V, v) \to (U_1, u_1), (U_2, u_2)$ such that

$$\varphi(V)(s_1|_V) = \varphi(V)(s_2|_V).$$

By assumption, $\varphi(V)$ is injective and so $s_1|_V = s_2|_V$, and therefore

$$(s_1, (U_1, u_1)) = (s_1|_V, (V, v)) = (s_2|_V, (V, v)) = (s_2, (U_2, u_2))$$

so φ_p is injective.

Now assume that φ_p is injective for all geometric points. Let $X \in \mathbf{Aff}_{\mathbb{F}}$ and consider $s_1, s_2 \in \mathcal{F}(X)$ such that $\varphi(X)(s_1) = \varphi(X)(s_2)$. For each point $x \in X$, choose a geometric point p which lies over x. For instance if $X = \mathrm{Spec}(R)$, we may choose $\mathbb{K} = (R_x/xR_x)_{\mathrm{alg}}$ and take $p\colon \mathrm{Spec}(\mathbb{K}) \to X$ to be the map coming from $R \to \mathbb{K}$. In the stalk at p we will have

$$\varphi_p(s_1, (X, p)) = (\varphi(X)(s_1), (X, p)) = (\varphi(X)(s_2), (X, p)) = \varphi_p(s_2, (X, p)).$$

Since φ_p is injective, $(s_1, (X, p)) = (s_2, (X, p))$ and so there exists some étale neighbourhood (U_p, u_p) of p on which $s_1|_{U_p} = s_2|_{U_p}$. Because we have done this at all points, and $x \in \mathrm{Img}(U_p \to X)$, we get an étale cover $\{U_p \to X\}_{x \in X}$ which we may refine to a finite cover

$$\{U_i \to X \mid 1 \leq i \leq m\}$$

since étale morphisms are open and X is quasi-compact. Since $\mathcal{F}$ is a sheaf we have an equalizer diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\alpha"] & \prod_{i=1}^{m} \mathcal{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j=1}^{m} \mathcal{F}(U_i \times_X U_j) \end{tikzcd}$$

where α is injective. From our construction of the étale cover we have

$$\alpha(s_1) = (s_1|_{U_i})_{i=1}^{m} = (s_2|_{U_i})_{i=1}^{m} = \alpha(s_2)$$

and so $s_1 = s_2$, meaning $\varphi(X)$ is injective. Hence φ is injective as desired.

To begin the second half, assume that φ is surjective. Let p be a geometric point and let $(s, (U, u)) \in \mathcal{G}_p$. Since φ is surjective, there exists an étale cover $\{U_i \to U \mid 1 \leq i \leq m\}$ such that each $s|_{U_i}$ is in the image of $\varphi(U_i)\colon \mathcal{F}(U_i) \to \mathcal{G}(U_i)$. Since the U_i cover U, at least one of them must be an étale neighbourhood of p, call it (U_j, u_j), and let y be a preimage of $s|_{U_j}$. Then we have

$$\varphi_p(y, (U_j, u_j)) = (\varphi(U_j)(y), (U_j, u_j)) = (s|_{U_j}, (U_j, u_j)) = (s, (U, u)).$$

Hence φ_p is surjective.

Finally, assume that φ_p is surjective for all geometric points. Let $X \in \mathbf{Aff}_{\mathbb{F}}$ and let $(s, (U, u)) \in \mathcal{G}(X)$. For each $x \in X$ choose a geometric point p lying over x. Since φ_p is surjective, there exists $(y_p, (U_p, u_p)) \in \mathcal{F}_p$ such that

$$(\varphi(U_p)(y_p), (U_p, u_p)) = (s, (U, u))$$

and therefore there exists a common étale neighbourhood V_p such that $\varphi(V_p)(y_p|_{V_p}) = s|_{V_p}$. Once again this produces an étale cover $\{V_p \to X\}_{x \in X}$ and we may take a finite subcover $\{V_i \to X \mid 1 \leq i \leq m\}$. By construction, this cover satisfies the property that each $s|_{V_i}$ is in the image of $\varphi(V_i)\colon \mathcal{F}(V_i) \to \mathcal{G}(V_i)$. Hence φ is surjective. ∎

Now that we know we can focus on stalks, we find the stalks of our linear algebraic groups in the next section.

3.4 Stalks of Linear Algebraic Groups

The aim of viewing linear algebraic groups as sheaves is to give us a convenient method for determining properties of group homomorphisms. As seen in the previous section we may focus on stalks, but to do so we need to be able to calculate the stalks of linear algebraic groups. Luckily, since the representing Hopf algebras of our groups are finitely presented, our group sheaves will commute with filtered colimits by [Stacks, Tag 00QO]. This causes their stalks to be their points on local strictly Henselian rings. We recall the definition of these rings and some useful facts about them.

Definition 3.4.1. *A ring R is called a local strictly Henselian ring if*

- *R is local,*
- *The residue field $R/\mathfrak{M}$ is a separably closed field,*
- *If $f(x) \in R[x]$ is a polynomial whose image in $R/\mathfrak{M}[x]$ factors, then that factorization lifts to one in $R[x]$.*

Starting with a local ring we can construct a local strictly Henselian ring.

Lemma 3.4.2. *[Stacks, Tag 04GP] Let R be a local ring with maximal ideal $\mathfrak{M}$ and residue field κ. Then there exists a local ring R^{sh} with residue field κ_{sep}, a separable closure of κ, which fits into the following commutative diagram*

$$\begin{tikzcd}
R \arrow[r] \arrow[d, two heads] & R^{sh} \arrow[d, two heads] \\
\kappa \arrow[r] & \kappa_{\text{sep}}
\end{tikzcd}$$

where

- *R^{sh} is a local strictly Henselian ring,*
- *The map $R \to R^{sh}$ is local,*
- *R^{sh} is a filtered colimit of étale R-algebras. In particular it is the filtered colimit over triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ is étale, $\mathfrak{q}$ lies over $\mathfrak{M}$, and $\alpha\colon \kappa(q) \to \kappa_{\text{sep}}$ is a field embedding. A morphism of such triples $(S, \mathfrak{q}, \alpha) \to (S', \mathfrak{q}', \alpha')$ is an R-algebra map $S \to S'$ such that $\mathfrak{q}'$ lies over $\mathfrak{q}$, and α factors through α'.*

Definition 3.4.3. *Let R be a local ring. The ring R^{sh} constructed above is called the* strict Henselization *of R.*

Starting with any ring we can also construct the strict Henslization at a prime ideal.

Lemma 3.4.4. *[Stacks, Tag 04GW] Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal with residue field $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ which has a separable closure $\kappa(\mathfrak{p})_{\mathrm{sep}}$. The category of triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ is étale, $\mathfrak{q}$ lies over $\mathfrak{p}$, and $\alpha\colon \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})_{\mathrm{sep}}$ is a field embedding, is a filtered category. Furthermore*

$$(R_{\mathfrak{p}})^{sh} = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} S = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} S_{\mathfrak{q}}$$

We will also use the following small result.

Lemma 3.4.5. *Let R be a local strictly Henselian ring. Then $R^{sh} = R$.*

Proof: Let R have residue field κ. Since κ is separably closed, the residue field of R^{sh} will also be κ. Therefore the diagram

$$\begin{tikzcd}
R^{sh} \arrow[r] & \kappa \\
R \arrow[u] \arrow[r, equal] & R \arrow[u] \arrow[ul, dotted, "\exists!"']
\end{tikzcd}$$

satisfies the conditions of [Stacks, Tag 08HT], which guarantees there is a unique isomorphism $R \overset{\sim}{\to} R^{sh}$. Therefore we make the identification $R = R^{sh}$. ∎

Our attention is drawn to the colimit construction of the strict Henselization of R at $\mathfrak{p}$. If $R \in \mathbf{Alg}_F$, it turns out that the filtered category defining $(R_{\mathfrak{p}})^{sh}$ is equivalent to the cofiltered category of étale neighbourhoods of a geometric point lying over $\mathfrak{p}$.

Proposition 3.4.6. *Let $R \in \mathbf{Alg}_{\mathbb{F}}$, let $\mathfrak{p} \subset R$ be a prime ideal, and let $p\colon \operatorname{Spec}(\mathbb{K}) \to \operatorname{Spec}(R)$ be a geometric point lying over $\mathfrak{p}$. Then the filtered category of triples $(S, \mathfrak{q}, \alpha)$ as above is equivalent via the* Spec *functor to the cofiltered category of étale neighbourhoods of p.*

Proof: Since p is a geometric point lying over $\mathfrak{p}$ its associated ring map factors through $\kappa(\mathfrak{p})$. Furthermore, since $\mathbb{K}$ is algebraically closed we may identify $\kappa(\mathfrak{p})_{\mathrm{sep}}$ with the separable closure of $\kappa(\mathfrak{p})$ in $\mathbb{K}$, and so we have

$$p^*\colon R \to \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})_{\mathrm{sep}} \to \mathbb{K}.$$

Now, given a triple $(S, \mathfrak{q}, \alpha)$, since $R \to S$ is an étale ring map, $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ will be étale. Then we obtain an étale neighbourhood of p by considering the map $S \to \kappa(\mathfrak{q}) \overset{\alpha}{\to} \kappa(\mathfrak{p})_{\mathrm{sep}} \to \mathbb{K}$ which gives a morphism $s\colon \operatorname{Spec}(\mathbb{K}) \to \operatorname{Spec}(S)$. This makes $(\operatorname{Spec}(S), s)$ an étale neighbourhood of p. A morphism of triples $(S, \mathfrak{q}, \alpha) \to (S', \mathfrak{q}', \alpha')$ will produce the morphism of neighbourhoods

$$(\operatorname{Spec}(S'), s') \to (\operatorname{Spec}(S), s)$$

where the morphisms s and s' are compatible because α and α' are.

Conversely, given an étale neighbourhood $(\mathrm{Spec}(U), u)$ of p, the associated ring map $R \to U$ is étale by definition. The point $\mathfrak{q} = \mathrm{Img}(u)$ will lie over $\mathfrak{p}$ since u is compatible with p. Since $R \to U$ is étale, we obtain from [Stacks, Tag 00U4] that the extension $\kappa(\mathfrak{p}) \subseteq \kappa(\mathfrak{q})$ is a finite separable extension, and therefore we have

$$\kappa(\mathfrak{p}) \subseteq \kappa(\mathfrak{q}) \subset \kappa(\mathfrak{p})_{\mathrm{sep}} \subseteq \mathbb{K}.$$

Choose $\alpha\colon \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})_{\mathrm{sep}}$ to be this inclusion, and then we have a triple $(U, \mathfrak{q}, \alpha)$. A morphism of neighbourhoods $(\mathrm{Spec}(U), u) \to (\mathrm{Spec}(U'), u')$ will give a map $U' \to U$ and will cause $\mathfrak{q}$ to lie over $\mathfrak{q}'$. This in turn will cause inclusions

$$\kappa(\mathfrak{q}) \subseteq \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{p})_{\mathrm{sep}} \subset \mathbb{K}$$

meaning that α factors through α'. Hence it induces a morphism of triples $(U', \mathfrak{q}', \alpha') \to (U, \mathfrak{q}, \alpha)$.

In summary, the categories are equivalent via Spec as desired. ∎

We can now prove our main result of this section.

Theorem 3.4.7. *Let* $\mathbf{G}$ *be a linear algebraic group. Let* $R \in \mathbf{Alg}_{\mathbb{F}}$ *and let* p *be a geometric point of* $\mathrm{Spec}(R)$ *lying over* $\mathfrak{p}$*. Then the stalk of* $\widehat{\mathbf{G}}$ *at* p *is*

$$\widehat{\mathbf{G}}_p = \mathbf{G}((R_{\mathfrak{p}})^{sh})$$

Furthermore, if $R' \in \mathbf{Alg}_{\mathbb{F}}$ *is any local strictly Henselian ring, then there exists a geometric point* p' *such that*

$$\widehat{\mathbf{G}}_{p'} = \mathbf{G}(R')$$

Proof: Let H be the Hopf algebra representing $\mathbf{G}$. As mentioned above, by [Stacks, Tag 00QO] the functor $\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, -)$ commutes with colimits since H is finitely presented. Therefore, using lemma 3.4.4 and proposition 3.4.6 we have that

$$\begin{aligned} \widehat{\mathbf{G}}_p &= \operatorname*{colim}_{(\mathrm{Spec}(U),u)} \widehat{\mathbf{G}}(\mathrm{Spec}(U)) = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} \mathbf{G}(S) = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, S) \\ &= \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} S) = \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{F}}}(H, (R_{\mathfrak{p}})^{sh}) = \mathbf{G}((R_{\mathfrak{p}})^{sh}) \end{aligned}$$

For the second claim, let R' have residue field κ, and then consider the geometric point $p'\colon \mathrm{Spec}(\kappa_{\mathrm{alg}}) \to \mathrm{Spec}(R')$ coming from the ring maps $R' \to \kappa \to \kappa_{\mathrm{alg}}$. Since R' is already local strictly Henselian, and p' lies above $\mathfrak{M} \subset R'$ we will have that

$$\widehat{\mathbf{G}}_{p'} = \mathbf{G}((R'_{\mathfrak{M}})^{sh}) = \mathbf{G}(R')$$

as desired. ∎

Conveniently for us, these stalks are completely described by Chevalley generators in almost all cases. The only exception is for the non-simply connected groups of type A_{n-1}, such as $\mathbf{SL}_n \,/\, \boldsymbol{\mu}_m$ and $\mathbf{PGL}_n$, when $\operatorname{char}(\mathbb{F})$ divides n.

Proposition 3.4.8. *Let* $\mathbf{G}$ *and* $\mathbf{E}_\Phi$ *be any of the linear algebraic group/Chevalley group pairs discussed in section 1.3 excluding the groups* $\mathbf{SL}_n \,/\, \boldsymbol{\mu}_m$ *and* $\mathbf{PGL}_n$ *when* $\operatorname{char}(\mathbb{F})$ *divides* n*. Let* $R \in \mathbf{Alg}_\mathbb{F}$ *be a local strictly Henselian ring. Then* $\mathbf{G}(R) = \mathbf{E}_\Phi(R)$*.*

Proof: To begin, we note the following property. If $\mathbf{G}' \twoheadrightarrow \mathbf{G}$ is a central isogeny between our linear algebraic groups, and $\mathbf{E}_\mathbf{G}(R) = \mathbf{G}(R)$, then $\mathbf{E}_{\mathbf{G}'}(R) = \mathbf{G}'(R)$ also. This occurs because the map $\mathbf{E}_{\mathbf{G}'}(R) \twoheadrightarrow \mathbf{E}_\mathbf{G}(R)$ is surjective and the kernel of $\mathbf{G}'(R) \to \mathbf{G}(R)$ is contained in $\mathbf{E}_{\mathbf{G}'}(R)$. Therefore, if $a \in \mathbf{G}(R) = \mathbf{E}_\mathbf{G}(R)$, there exists $b \in \mathbf{E}_{\mathbf{G}'}(R)$ such that $b \mapsto a$, and since these differ by an element of the kernel, $a \in \mathbf{E}_\mathbf{G}(R)$ as desired. Hence, it is sufficient to prove the claim for adjoint groups. As a starting point, the result already holds for simply connected groups by a result of Abe, [Abe69, Proposition 1.6].

We proceed by type, beginning with type A_{n-1} where we assume that $\operatorname{char}(\mathbb{F})$ does not divide n. The group $\mathbf{SL}_n$ is simply connected and so $\mathbf{E}_{\mathbf{SL}_n}(R) = \mathbf{SL}_n(R)$. Since the ring R is local, we have a version of Skolem-Noether over R. That is, all automorphisms of $\mathrm{M}_n(R)$ will be inner by [CD67, Theorem 1.2] and therefore the map $\mathbf{GL}_n(R) \to \mathbf{PGL}_n(R)$ is surjective. We show that the map $\mathbf{SL}_n(R) \to \mathbf{PGL}_n(R)$ is also surjective. Let $B \in \mathbf{GL}_n(R)$ with $\det(B) = b \in R^\times$. Because R is a local strictly Henselian ring, $R/\mathfrak{M}$ is separably closed and so the polynomial $x^n - b^{-1} \in R[x]$ will factor in $R/\mathfrak{M}[x]$. This factorization will lift back to $R[x]$, which means that R contains an n^{th} root of b^{-1}. Therefore $(\sqrt[n]{b^{-1}})B$ is in $\mathbf{SL}_n(R)$ and it has the same image in $\mathbf{PGL}_n(R)$ as B. Hence, the map $\mathbf{SL}_n(R) \twoheadrightarrow \mathbf{PGL}_n(R)$ is surjective. So, since $\mathbf{E}_{\mathbf{SL}_n}(R) = \mathbf{SL}_n(R)$, the surjectivity of the previous map means $\mathbf{E}_{\mathbf{PGL}_n}(R) = \mathbf{PGL}_n(R)$. Thus, $\mathbf{E}_{\mathbf{SL}_n/\boldsymbol{\mu}_m}(R) = \mathbf{SL}_n \,/\, \boldsymbol{\mu}_m(R)$ holds for the intermediate groups as well.

In type C, we know that $\mathbf{E}_{\mathbf{Sp}_{2n}}(R) = \mathbf{Sp}_{2n}(R)$ since $\mathbf{Sp}_{2n}$ is simply connected, so we show that $\mathbf{Sp}_{2n}(R) \to \mathbf{PSp}_{2n}(R)$ is surjective. Let $\varphi \in \mathbf{PSp}_{2n}(R)$ which, by Skolem-Noether, is the inner automorphism of some $B \in \mathbf{GL}_{2n}(R)$. Since the automorphism preserves the symplectic involution on $(\mathrm{M}_n(R), \psi_0)$, we have that for $T \in \mathrm{M}_n(R)$

$$B\psi_0(T)B^{-1} = \psi_0(BTB^{-1})$$

which is equivalent to requiring that $B\psi_0(B) = cI$ for some $c \in R^\times$. consider the polynomial $x^2 - c \in R[x]$. Since $R/\mathfrak{M}$ is separably closed and of characteristic not 2 (R is an $\mathbb{F}$-algebra and so $\mathbb{F} \hookrightarrow R/\mathfrak{M}$) this polynomial will factor in $R/\mathfrak{M}[x]$, and this factorization will lift to a factorization

$$(x-b)(x+b) = x^2 - c \in R[x].$$

Hence R contains square roots for all its units. In particular there is $\sqrt{c^{-1}} \in R$, and so $\sqrt{c^{-1}}B \in \mathbf{Sp}_{2n}(R)$ which also maps to φ. This means $\mathbf{Sp}_{2n}(R) \twoheadrightarrow \mathbf{PSp}_{2n}(R)$ is surjective and hence $\mathbf{E}_{\mathbf{PSp}_{2n}}(R) = \mathbf{PSp}_{2n}(R)$ as desired.

For types B and D, we show that the claim holds for $\mathbf{SO}_d$, first when d is even and then when d is odd. Identifying $\mathbf{E}_{\mathbf{SO}_{2n}}(R)$ with its image under the map of proposition 1.3.22, this means that the matrices

$$T_i(u) = \operatorname{diag}(1, \dots, \underset{i}{u}, \dots, \underset{\bar{i}}{u^{-1}}, \dots, 1) = \begin{cases} h_{e_i+e_{i+1}}(\sqrt{u})h_{e_i-e_{i+1}}(\sqrt{u}) & i < n \\ h_{e_{n-1}+e_n}(\sqrt{u})h_{-e_{n-1}+e_n}(\sqrt{u}) & i = n \end{cases}$$

are in $\mathbf{E}_{\mathbf{SO}_{2n+1}}(R)$ for all $u \in R^\times$. From here, we adapt the proof of [Dic98, Theorem 2], which deals with $\mathbf{SO}_{2n}$ over a finite field. Let $A \in \mathbf{SO}_{2n}(R)$, and consider the first row

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_{1n} & \dots & b_{11} \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix}.$$

Since $\det(A) \in R^\times$ and R is local, we cannot have that all a_{1i}, b_{1i} are in $\mathfrak{M}$, and so at least one of them is invertible. If $a_{1j} \in R^\times$ we may consider $Aw_{e_1-e_j}(1)$, which will have $-a_{1j}$ in the top-left position, and if $b_{1j} \in R^\times$ we may consider $Aw_{e_1+e_j}(1)$ which will have $-b_{1j}$ in the top left position. Because of this, we may assume without loss of generality that $a_{11} \in R^\times$. Now, let

$$B = \left(\prod_{k=2}^{n} x_{e_1-e_k}(a_{1k})x_{e_1+e_k}(b_{1k})\right) T_1(a_{11})$$

which has the same first row as A since the entries of A satisfy $\sum_{k=1}^{n} a_{1k}b_{1k} = 0$ by virtue of being in $\mathbf{SO}_{2n}(R)$. Then, keeping in mind that the last column of A^{-1} is $[b_{11}, \dots, b_{1n}, a_{1n}, \dots, a_{11}]^T$, computation shows that

$$AB^{-1} = \begin{bmatrix} 1 & 0 & \dots & & & \dots & 0 \\ & & & & & & \vdots \\ * & & & & & & \\ \vdots & & & & & & \vdots \\ * & & & & & & 0 \\ c_{11} & \dots & c_{1n} & d_{1n} & \dots & d_{12} & 1 \end{bmatrix}.$$

where $c_{11} + \sum_{k=2}^{n} c_{1k}d_{1k} = 0$. Then let

$$B' = \prod_{k=2}^{n} x_{-e_1-e_k}(-c_{1k})x_{-e_1+e_k}(-d_{1k})$$

which has the same last row as AB^{-1}, and then further computation will yield

$$AB^{-1}(B')^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \vdots \\ \vdots & & A' & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

where $A' \in \mathbf{SO}_{2n-2}(R)$. Repeating, we eventually reach a matrix of the form

$$\operatorname{diag}(1, \dots, 1, a, a^{-1}, 1, \dots, 1) = T_n(a) \in \mathbf{E}_{\mathbf{SO}_{2n}}(R).$$

Hence $A \in \mathbf{E}_{\mathbf{SO}_{2n}}(R)$ and so $\mathbf{E}_{\mathbf{SO}_{2n}}(R) = \mathbf{SO}_{2n}(R)$.

The $\mathbf{SO}_{2n+1}$ case proceeds similarly. We again identify $\mathbf{E}_{\mathbf{SO}_{2n+1}}(R)$ with its image under the map of proposition 1.3.19. In this case we have that for $u \in R^\times$

$$T_i(u) = \operatorname{diag}(1, \dots, \underset{i}{u}, \dots, \underset{n+1}{1}, \dots, \underset{\bar{i}}{u^{-1}}, \dots, 1) = h_{e_i}(\sqrt{u}) \in \mathbf{E}_{\mathbf{SO}_{2n+1}}(R).$$

Let $A \in \mathbf{SO}_{2n+1}(R)$ and consider its first row

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} & b & c_{1n} & \dots & c_{11} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}.$$

The entries of A will satisfy $2\left(\sum_{k=1}^n a_{1k}c_{1k}\right) + \frac{b^2}{2} = 0$. Again, since R is local at least one of these entries must be a unit. If b is a unit, then $\sum_{k=1}^n a_{1k}c_{1k}$ is a unit and so some a_{1j} or c_{1j} is also a unit. Considering $Aw_{e_1-e_j}(1)$ if a_{1j} is a unit or $Aw_{e_1+e_j}(1)$ if c_{1j} is a unit, we may assume without loss of generality that a_{11} is a unit. Let

$$B = \left(\prod_{k=2}^n x_{e_1-e_k}(a_{1k})x_{e_1+e_k}(c_{1k})\right) x_{e_1}\left(\frac{b}{2}\right) T_1(a_{11}),$$

which will have the same first row as A. Then we compute that

$$AB^{-1} = \begin{bmatrix} 1 & 0 & \dots & & & \dots & 0 \\ * & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ * & & & & & & 0 \\ d_{11} & \dots & d_{1n} & f & g_{1n} & \dots & g_{12} & 1 \end{bmatrix}.$$

where $2d_{11} + 2\sum_{k=2}^n d_{1k}g_{1k} + \frac{f^2}{2} = 0$. Using this relation, if we let

$$B' = \left(\prod_{k=2}^n x_{-e_1-e_k}(-d_{1k})x_{-e_1+e_k}(-g_{1k})\right) x_{-e_1}\left(-\frac{f}{2}\right),$$

then we can compute that

$$AB^{-1}(B')^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \vdots \\ \vdots & & A' & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

where $A' \in \mathbf{SO}_{2n-1}(R)$. Repeating, we eventually arrive at the identity matrix, which is in $\mathbf{E}_{\mathbf{SO}_{2n+1}}(R)$, and therefore $A \in \mathbf{E}_{\mathbf{SO}_{2n+1}}(R)$. Hence $\mathbf{E}_{\mathbf{SO}_{2n+1}}(R) = \mathbf{SO}_{2n+1}(R)$.

Now, in type B we have that $\mathbf{SO}_{2n+1} = \mathbf{PSO}_{2n+1}$ and so we are done. In type D we must show that $\mathbf{SO}_{2n}(R) \to \mathbf{PSO}_{2n}(R)$ is surjective. Just as in the symplectic case, if $\varphi \in \mathbf{PSO}_{2n+1}(R)$ there exists $B \in \mathbf{GL}_n(R)$ which maps to it, and because the automorphism respects the orthogonal involution on $(\mathrm{M}_{2n}(R), \tau_0)$ we have that $B\tau_0(B) = cI$ for some $c \in R^\times$. After rescaling, we may assume $B\tau_0(B) = I$ and so $B \in \mathbf{O}_{2n}(R)$. To show that B actually belongs to $\mathbf{SO}_{2n}(R)$, we must consider the Clifford algebra $C(\mathrm{M}_{2n}(R), \tau_0)$. Recall that $C(\mathrm{M}_{2n}(R), \tau_0) \cong C_0(\mathbb{F}^{2n}, q_{2n}) \otimes_{\mathbb{F}} R$ where q is the quadratic form

$$q_{2n}(x) = x_1x_{2n} + x_2x_{2n-1} + \ldots + x_nx_{n-1}.$$

Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis of $\mathbb{F}^{2n}$, and for $i \in 1, \ldots, 2n$ we denote $\bar{i} = 2n + 1 - i$. In terms of this basis, $C(\mathrm{M}_{2n}(R), \tau_0)$ consists of elements

$$\sum r_i e_{i_1} e_{i_2} \ldots e_{i_{2k_i}}$$

where $r_i \in R$, adjacency indicates the tensor product (i.e., $e_1e_2 = e_1 \otimes e_2$), and where multiplication is defined by the relations

$$\begin{aligned} e_ie_j &= -e_je_i \text{ if } i \neq j, \bar{j} \\ e_ie_i &= 0 \\ e_ie_{\bar{i}} &= 1 - e_{\bar{i}}e_i \end{aligned}$$

The automorphism of $C(\mathrm{M}_n(R), \tau_0)$ induced by φ acts as follows,

$$\begin{aligned} \sum r_i e_{i_1} e_{i_2} \ldots e_{i_{2k_i}} \mapsto & \sum r_i \varphi(e_{i_1} e_{i_2}) \ldots \varphi(e_{i_{2k_i-1}} e_{i_{2k_i}}) \\ &= \sum r_i (Be_{i_1})(Be_{i_2}) \ldots (Be_{i_{2k_i}}) \end{aligned}$$

because of the description of matrix multiplication in section 2.1.

Consider the central element $\omega = \prod_{k=1}^n (1 - 2e_k e_{\bar{k}})$. Computations using the above relations show that $\omega^2 = 1$ and so

$$\frac{1}{2}(1 + \omega) \text{ and } \frac{1}{2}(1 - \omega)$$

are idempotents. The discriminant of τ_0 is trivial and so $Z(C(\mathrm{M}_{2n}(R),\tau_0)) \cong R \times R$ with an isomorphism coming from the above idempotents

$$\begin{aligned} \frac{1}{2}(1+\omega) &\mapsto (1,0) \\ \frac{1}{2}(1-\omega) &\mapsto (0,1). \end{aligned}$$

Since $\varphi \in \mathbf{PSO}_{2n}(R)$, the induced automorphism fixes the center of the Clifford algebra. In fact, since $\mathbf{SO}_{2n}(R)$ maps into $\mathbf{PSO}_{2n}(R)$, the automorphism induced from the inner automorphism of any element in $\mathbf{SO}_{2n}(R)$ will also fix the center. Hence, to show that $B \in \mathbf{SO}_{2n}(R)$, we need only to show that the automorphisms coming from elements of $\mathbf{O}_{2n}(R) \backslash \mathbf{SO}_{2n}(R)$ negate ω. Consider the matrix

$$T = \begin{bmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & & & & 0 \end{bmatrix} \in \mathbf{O}_{2n}(R).$$

It has $\det(T) = -1$ and so any element of $\mathbf{O}_{2n}(R) \backslash \mathbf{SO}_{2n}(R)$ can be written as TT' for some $T' \in \mathbf{SO}_{2n}(R)$. Since the automorphism coming from T' will fix ω, all elements of $\mathbf{O}_{2n}(R) \backslash \mathbf{SO}_{2n}(R)$ will effect ω the same way T does. Therefore we compute

$$\begin{aligned} \prod_{k=1}^{n}(1-2e_k e_{\overline{k}}) &\mapsto (1-2(Te_1)(Te_{\overline{1}}))\prod_{k=2}^{n}(1-2(Te_k)(Te_{\overline{k}})) \\ &= \mapsto (1-2e_{\overline{1}}e_1)\prod_{k=2}^{n}(1-2e_k e_{\overline{k}}) \\ &= \mapsto (1-2(1-e_1e_{\overline{1}}))\prod_{k=2}^{n}(1-2e_k e_{\overline{k}}) \\ &= \mapsto (-1+2e_1e_{\overline{1}})\prod_{k=2}^{n}(1-2e_k e_{\overline{k}}) \\ &= -\omega. \end{aligned}$$

Hence $B \in \mathbf{SO}_{2n}(R)$ and so the map $\mathbf{SO}_{2n}(R) \twoheadrightarrow \mathbf{PSO}_{2n}(R)$ is surjective. That means $\mathbf{E}_{\mathbf{PSO}_{2n}}(R) = \mathbf{PSO}_{2n}(R)$, finishing the proof. ▮

The exclusion in type A_{n-1} is necessary, as illustrated by the following example.

Example 3.4.9. Consider $\mathbf{SL}_3$ over the field $\mathbb{F}_3$. Since $\mathbf{SL}_3$ is simply connected, $\mathbf{E}_{\mathbf{SL}_3}(\mathbb{F}_3(t)_{\mathrm{sep}}) = \mathbf{SL}_3(\mathbb{F}_3(t)_{\mathrm{sep}})$. However, $\mathbf{PGL}_3(\mathbb{F}_3(t)_{\mathrm{sep}})$ contains the inner automorphism coming from

$$\begin{bmatrix} t & & \\ & 1 & \\ & & 1 \end{bmatrix} \in \mathbf{GL}_3(\mathbb{F}_3(t)_{\mathrm{sep}}).$$

This matrix has determinant t, but there is no cubed root of t in $\mathbb{F}_3(t)_{\mathrm{sep}}$ since the polynomial $x^3 - t \in \mathbb{F}_3(t)[x]$ is not separable. Therefore, this matrix cannot be scaled to a matrix with determinant 1, and so the map $\mathbf{SL}_3(\mathbb{F}_3(t)_{\mathrm{sep}}) \to \mathbf{PGL}_3(\mathbb{F}_3(t)_{\mathrm{sep}})$ is not surjective. Thus, $\mathbf{E}_{\mathbf{PGL}_3}(\mathbb{F}_3(t)_{\mathrm{sep}}) \subsetneq \mathbf{PGL}_3(\mathbb{F}_3(t)_{\mathrm{sep}})$ despite $\mathbb{F}_3(t)_{\mathrm{sep}}$ being a local strictly Henselian ring by virtue of already being a separably closed field.

Proposition 3.4.8 will allow us to use the language of Chevalley groups and their generators and relations in order to check injectivity and often surjectivity, or compute kernels and images of group scheme homomorphisms, since these properties are determined by their behaviour on stalks where the linear algebraic group and the Chevalley group coincide. We say we may often check surjectivity since sheaf surjectivity is a strictly stronger notion than group scheme surjectivity, which we can now demonstrate in the following example.

Example 3.4.10. Consider the multiplicative group $\mathbb{G}_m$ over the field $\mathbb{F}_3$. The map

$$\begin{aligned} \varphi \colon \mathbb{G}_m &\to \mathbb{G}_m \\ a &\mapsto a^3 \end{aligned}$$

is surjective as a group scheme homomorphism since its associated Hopf algebra map

$$\begin{aligned} \varphi^* \colon \mathbb{F}_3[t, t^{-1}] &\to \mathbb{F}_3[t, t^{-1}] \\ t &\mapsto t^3 \\ t^{-1} &\mapsto t^{-3} \end{aligned}$$

is injective. However, $\hat{\varphi} \colon \widehat{\mathbb{G}_m} \to \widehat{\mathbb{G}_m}$ is not surjective as a sheaf homomorphism. If it were, then since the separable closure of the field of rational functions in one variable, $\mathbb{F}_3(t)_{\mathrm{sep}} \in \mathbf{Alg}_{\mathbb{F}_3}$, is a local strictly Henselian ring, the map

$$\begin{aligned} \hat{\varphi}(\mathbb{F}_3(t)_{\mathrm{sep}}) \colon (\mathbb{F}_3(t)_{\mathrm{sep}})^\times &\to (\mathbb{F}_3(t)_{\mathrm{sep}})^\times \\ a &\mapsto a^3 \end{aligned}$$

would be surjective. However, as shown in the previous example, t cannot be in the image of this map.

Chapter 4

Twisted Tensor Maps

Chapter 4 will be devoted to generalizing the Kronecker tensor product to the setting of our linear algebraic groups discussed in section 1.2. The techniques of this chapter, those of translating the classical Kronecker product into the language of Chevalley generators and using that description to define a map of group schemes, were inspired by S. Garibaldi who used these techniques in [Gar09, §7] to construct a map $\mathbf{PGL}_2 \times \mathbf{PSp}_8 \hookrightarrow \mathbf{HSpin}_{16}$. We proceed by first recalling the Kronecker product and mentioning some of its convenient properties which allows it to restrict to matrix subgroups.

4.1 The Kronecker Product for Matrix Subgroups

Let n and m be positive integers and let $R \in \mathbf{Alg}_{\mathbb{F}}$. The Kronecker tensor product is the map

$$\begin{aligned} \mathrm{M}_n(R) \times \mathrm{M}_m(R) &\to \mathrm{M}_{nm}(R) \\ (A, B) &\mapsto A \otimes B \end{aligned}$$

where $A \otimes B$ is the tensor product of A and B as endomorphisms of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Purely in terms of matrices, if $A = [a_{ij}]_{i,j=1}^n$ then the image of (A, B) is the block matrix $A \otimes B = [a_{ij}B]_{i,j=1}^n$. This product enjoys the following properties.

- $(AA', BB') \mapsto AA' \otimes BB' = (A \otimes B)(A' \otimes B)$,
- $\det(A \otimes B) = \det(A)^m \det(B)^n$.

Hence the Kronecker product restricts to a map $\mathbf{SL}_n(R) \times \mathbf{SL}_m(R) \to \mathbf{SL}_{nm}(R)$, which in turn extends to a homomorphism on the level of group schemes,

$$K_{\mathbf{SL}} \colon \mathbf{SL}_n \times \mathbf{SL}_m \to \mathbf{SL}_{nm} .$$

When there are involutions involved the Kronecker product may not directly restrict as we desire. Let σ_1 be an involution of the first kind on $\mathrm{M}_n(\mathbb{F})$ and let σ_2 be one on $\mathrm{M}_m(\mathbb{F})$. There is an induced involution $\sigma_1 \otimes 1$ on $\mathrm{M}_n(\mathbb{F}) \otimes_{\mathbb{F}} R \cong \mathrm{M}_n(R)$, and similarly an involution $\sigma_2 \otimes 1$ on $\mathrm{M}_m(R)$. Now consider $A \in \mathrm{M}_n(R)$ and $B \in \mathrm{M}_m(R)$ such that

$$A \cdot (\sigma_1 \otimes 1)(A) = I_n, \quad B \cdot (\sigma_2 \otimes 1)(B) = I_m.$$

Then since $\mathrm{M}_n(\mathbb{F}) \otimes_{\mathbb{F}} \mathrm{M}_m(\mathbb{F}) \cong \mathrm{M}_{nm}(\mathbb{F})$, we have that $\sigma_1 \otimes \sigma_2$ is an involution on $\mathrm{M}_{nm}(\mathbb{F})$ and

$$(A \otimes B) \cdot (\sigma_1 \otimes \sigma_2 \otimes 1)(A \otimes B) = (A \cdot (\sigma_1 \otimes 1)(A)) \otimes (B \cdot (\sigma_2 \otimes 1)(B)) = I_n \otimes I_m = I_{nm}.$$

Therefore, since the tensor of two orthogonal involutions remains orthogonal and the tensor of two symplectic involutions becomes orthogonal, there are restrictions of the Kronecker product,

$$\begin{aligned} \mathbf{SO}(\mathrm{M}_n(\mathbb{F}), \tau_1)(R) \times \mathbf{SO}(\mathrm{M}_m(\mathbb{F}), \tau_2)(R) &\to \mathbf{SO}(\mathrm{M}_{nm}(\mathbb{F}), \tau_1 \otimes \tau_2)(R) \\ \mathbf{Sp}(\mathrm{M}_n(\mathbb{F}), \psi_1)(R) \times \mathbf{Sp}(\mathrm{M}_m(\mathbb{F}), \psi_2)(R) &\to \mathbf{SO}(\mathrm{M}_{nm}(\mathbb{F}), \psi_1 \otimes \psi_2)(R) \end{aligned}$$

where τ_i are orthogonal involutions and ψ_i are symplectic involutions. However, some of these maps do not line up with our choices of $\mathbf{SO}_n$ and $\mathbf{Sp}_{2n}$ as in section 1.2.1. For instance, recall that the chosen involution governing $\mathbf{Sp}_{2n}$ is

$$\begin{aligned} \psi_{2n} \colon \mathrm{M}_{2n}(\mathbb{F}) &\to \mathrm{M}_{2n}(\mathbb{F}) \\ B &\mapsto -\Psi_{2n} B^T \Psi_{2n} \end{aligned}$$

where

$$\Omega_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathrm{M}_n(\mathbb{F}), \quad \Psi_{2n} = \begin{bmatrix} & \Omega_n \\ -\Omega_n & \end{bmatrix} \in \mathrm{M}_{2n}(\mathbb{F}).$$

Thus, we can compute that

$$\begin{aligned} \psi_{2n} \otimes \psi_{2m} \colon \mathrm{M}_{4nm}(\mathbb{F}) &\to \mathrm{M}_{4nm}(\mathbb{F}) \\ B &\mapsto (\Psi_{2n} \otimes \Psi_{2m}) B^T (\Psi_{2n} \otimes \Psi_{2m}) \end{aligned}$$

and

$$\Psi_{2n} \otimes \Psi_{2m} = \begin{bmatrix} & & & & & & & & & \Omega_m \\ & & & & & & & & -\Omega_m & \\ & & & & & & & \iddots & & \\ & & & & & & \Omega_m & & & \\ & & & & & -\Omega_m & & & & \\ & & & & -\Omega_m & & & & & \\ & & & \Omega_m & & & & & & \\ & & \iddots & & & & & & & \\ & -\Omega_m & & & & & & & & \\ \Omega_m & & & & & & & & & \end{bmatrix} \neq \Omega_{4nm}.$$

This can be corrected since the groups $\mathbf{SO}(\mathrm{M}_{4nm}, \Psi_{2n}\otimes\Psi_{2m})$ and $\mathbf{SO}_{4nm}$ are conjugate in $\mathbf{GL}_{4nm}$ by an element of $\mathbf{GL}_{4nm}(\mathbb{F})$. Using the matrix

$$P = \begin{bmatrix} I_m & & & & & & & & & \\ & -\Omega_m & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & I_m & & & & & & \\ & & & & -\Omega_m & & & & & \\ & & & & & \Omega_m & & & & \\ & & & & & & I_m & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \Omega_m & \\ & & & & & & & & & I_m \end{bmatrix}$$

we have $P\Omega_{4nm}P^T = \Psi_{2n} \otimes \Psi_{2m}$ and so there are maps

$$\begin{aligned} \mathbf{Sp}_{2n}(R) \times \mathbf{Sp}_{2m}(R) &\to \mathbf{SO}_{4nm}(R) \\ (A, B) &\mapsto (P')^{-1}(A \otimes B)P' \end{aligned}$$

where P' is the image of P under $\mathbf{GL}_{4nm}(\mathbb{F}) \to \mathbf{GL}_{4nm}(R)$. These maps collect into a homomorphism of group schemes

$$K_{\mathbf{Sp}} \colon \mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \to \mathbf{SO}_{4nm}\,.$$

Regarding special orthogonal groups, the restriction to $\mathbf{SO}_{2n} \times \mathbf{SO}_{2m}$ needs no further tweaking, and so there is a homomorphism of group schemes

$$K_{ee} \colon \mathbf{SO}_{2n} \times \mathbf{SO}_{2m} \to \mathbf{SO}_{4nm}\,.$$

We do require corrections for the restrictions involving $\mathbf{SO}_{2n+1}$. Recall again from section 1.2.1 that the involution governing this group is

$$\begin{aligned} \xi_0 \colon \mathrm{M}_{2n+1}(\mathbb{F}) &\to \mathrm{M}_{2n+1}(\mathbb{F}) \\ B &\mapsto \Xi_{2n+1}^{-1} B^T \Xi_{2n+1}. \end{aligned}$$

where

$$\Xi_{2n+1} = \begin{bmatrix} & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & \end{bmatrix}.$$

To correct for $\Omega_{2n} \otimes \Xi_{2m+1} \neq \Omega_{2n(2m+1)}$ we use the matrix

$$P = \begin{bmatrix} A & & & & & \\ & \ddots & & & & \\ & & A & & & \\ & & & I_{2m+1} & & \\ & & & & \ddots & \\ & & & & & I_{2m+1} \end{bmatrix}$$

where $A = \operatorname{diag}(1, \dots, 1, 1/2, 1, \dots, 1) \in \mathrm{M}_{2m+1}(\mathbb{F})$. This gives us that $P\Omega_{2n(2m+1)}P^T = (\Omega_{2n} \otimes \Xi_{2m+1})^{-1}$, and so we get a group scheme homomorphism

$$K_{eo}\colon\ \mathbf{SO}_{2n} \times \mathbf{SO}_{2m+1} \to \mathbf{SO}_{2n(2m+1)}\,.$$

Lastly, we wish to also restrict the Kronecker product to $\mathbf{SO}_{2n+1} \times \mathbf{SO}_{2m+1}$ by correcting for the fact that $\Xi_{2n+1} \otimes \Xi_{2m+1} \neq \Xi_{(2n+1)(2m+1)}$. However, since $\det(\Xi_{2n+1} \otimes \Xi_{2m+1}) = 4^{n+m+1}$ and $\det(\Xi_{(2n+1)(2m+1)}) = 2$ there does not exist a suitable matrix P with coefficients in $\mathbb{Z}$. What we need is a square root of two, and so we are unable to define our restriction over an arbitrary field $\mathbb{F}$. Nevertheless, in the case where $\mathbb{F}$ does contain a solution to $x^2 - 2$, which we will denote simply by $\sqrt{2}$, we can use the matrix

$$P = \begin{bmatrix} A & & & & & & \\ & \ddots & & & & & \\ & & A & & & & \\ & & & B & & & \\ & & & & I_{2m+1} & & \\ & & & & & \ddots & \\ & & & & & & I_{2m+1} \end{bmatrix}$$

where A is as above, and $B = \operatorname{diag}(1/2, \dots, 1/2, 1/\sqrt{2}, 1, \dots, 1) \in \mathrm{M}_{2m+1}(\mathbb{F})$. This choice of P gives that $P\Xi_{(2n+1)(2m+1)}^{-1}P^T = (\Xi_{2n+1} \otimes \Xi_{2m+1})^{-1}$ and so we obtain a group scheme homomorphism

$$K'_{oo}\colon\ \mathbf{SO}_{2n+1} \times \mathbf{SO}_{2m+1} \to \mathbf{SO}_{(2n+1)(2m+1)}\,.$$

In summary, after some tweaking we have obtained group scheme homomorphisms between our matrix subgroups. These are

$$\begin{aligned} K_{\mathbf{SL}}&\colon\ \mathbf{SL}_n \times \mathbf{SL}_m \to \mathbf{SL}_{nm} \\ K_{\mathbf{Sp}}&\colon\ \mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \to \mathbf{SO}_{4nm} \\ K_{ee}&\colon\ \mathbf{SO}_{2n} \times \mathbf{SO}_{2m} \to \mathbf{SO}_{4nm} \\ K_{eo}&\colon\ \mathbf{SO}_{2n} \times \mathbf{SO}_{2m+1} \to \mathbf{SO}_{2n(2m+1)} \\ K'_{oo}&\colon\ \mathbf{SO}_{2n+1} \times \mathbf{SO}_{2m+1} \to \mathbf{SO}_{(2n+1)(2m+1)}\,. \end{aligned}$$

The subscripts on the final three maps refer to the degree of the split special orthogonal groups in the domain, whether they are even (e) or odd (o) respectively. The prime adorning K'_{oo} is to remind us that this map is only defined when $\sqrt{2} \in \mathbb{F}$. The map K_{oe} can be obtained by swapping the inputs of K_{eo}, and so we omit it from our discussion.

4.2 Tensor Products for Split Simply Connected Groups

Our next task is to describe how these maps lift to the simply connected setting. For $K_{\mathbf{SL}}$ there is no lifting needed, as the special linear group is simply connected, however for all other maps the special orthogonal groups can be replaced by spin groups of the same degree. The existence of such liftings is guaranteed by [BT72, Proposition 2.24(i)]. We denote these new maps by $L_{\mathbf{Sp}}$, L_{ee}, L_{eo}, and L'_{oo}. They fit into the following commutative diagrams.

$$\begin{tikzcd}
& \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \arrow[ur, "L_{\mathbf{Sp}}"] \arrow[r, "K_{\mathbf{Sp}}"] & \mathbf{SO}_{4nm}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Spin}_{d_1} \times \mathbf{Spin}_{d_2} \arrow[r, "L"] \arrow[d, two heads] & \mathbf{Spin}_{d_1 d_2} \arrow[d, two heads] \\
\mathbf{SO}_{d_1} \times \mathbf{SO}_{d_2} \arrow[r, "K"] & \mathbf{SO}_{d_1 d_2}
\end{tikzcd}$$

where K and L in the right diagram are as appropriate based on the parity of d_1 and d_2. Since we wish to work computationally with these liftings, in this section we will describe how they behave on Chevalley generators. The first step is to describe the maps $K_{\mathbf{Sp}}$, K_{ee}, K_{eo}, and K_{oo} in terms of Chevalley generators, and then we will show that the same descriptions apply to the liftings. This was done for the maps $K_{\mathbf{Sp}}$, K_{ee}, and for slight variants of the maps K_{eo} and K_{oo} in [Rue20]. We take from there the results which apply here, and otherwise replicate the same procedure for our variants.

Proposition 4.2.1. *Consider the map $K_{\mathbf{Sp}}$, and let $R \in \mathbf{Alg}_{\mathbb{F}}$. The map $K_{\mathbf{Sp}}(R)$ restricts to Chevalley subgroups $\mathbf{E}_{\mathbf{Sp}_{2n}}(R) \times \mathbf{E}_{\mathbf{Sp}_{2m}}(R) \to \mathbf{E}_{\mathbf{SO}_{4nm}}(R)$ and the restriction is described on generators in the following table, where for an integer y we set $[y] = (2m)y$.*

(A, B)	$K_{\mathbf{Sp}}(R)(A, B)$
$(x_{e_i-e_j}(t), I)$	$\prod_{k=1}^{2m} x_{e_{[i-1]+k}-e_{[j-1]+k}}(t)$
$(x_{-e_i+e_j}(t), I)$	$\prod_{k=1}^{2m} x_{-e_{[i-1]+k}+e_{[j-1]+k}}(t)$
$(x_{e_i+e_j}(t), I)$	$\prod_{k=1}^{m} x_{e_{[i-1]+k}+e_{[j-1]+m+k}}(t) x_{e_{[i-1]+m+k}+e_{[j-1]+k}}(-t)$
$(x_{-e_i-e_j}(t), I)$	$\prod_{k=1}^{m} x_{-e_{[i-1]+k}-e_{[j-1]+m+k}}(t) x_{-e_{[i-1]+m+k}-e_{[j-1]+k}}(-t)$
$(x_{2e_i}(t), I)$	$\prod_{k=1}^{m} x_{e_{[i-1]+k}+e_{[i-1]+m+k}}(t)$
$(x_{-2e_i}(t), I)$	$\prod_{k=1}^{m} x_{-e_{[i-1]+k}-e_{[i-1]+m+k}}(t)$
$(I, x_{e_i-e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+j}}(u) x_{-e_{[k]+m+i}+e_{[k]+m+j}}(-u)$
$(I, x_{-e_i+e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+j}}(u) x_{e_{[k]+m+i}-e_{[k]+m+j}}(-u)$
$(I, x_{e_i+e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+m+j}}(-u) x_{e_{[k]+j}-e_{[k]+m+i}}(-u)$
$(I, x_{-e_i-e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+m+j}}(-u) x_{-e_{[k]+j}+e_{[k]+m+i}}(-u)$
$(I, x_{2e_i}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+m+i}}(-u)$
$(I, x_{-2e_i}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+m+i}}(-u)$

Since this also describes the behaviour on stalks, that is $K_{\mathbf{Sp}}(R)$ when R is a local strictly Henselian ring, it determines the group scheme homomorphism $K_{\mathbf{Sp}}$ in general.

Proposition 4.2.2. *Consider the map K_{ee}, and let $R \in \mathbf{Alg}_{\mathbb{F}}$. The map $K_{ee}(R)$ restricts to Chevalley subgroups $\mathbf{E}_{\mathbf{SO}_{2n}}(R) \times \mathbf{E}_{\mathbf{SO}_{2m}}(R) \to \mathbf{E}_{\mathbf{SO}_{4nm}}(R)$ and the restriction is described on generators in the following table, where for an integer y we set $[y] = (2m)y$ and $\overline{y} = 2m + 1 - y$.*

(A, B)	$K_{ee}(R)(A, B)$
$(x_{e_i - e_j}(t), I)$	$\prod_{k=1}^{2m} x_{e_{[i-1]+k} - e_{[j-1]+k}}(t)$
$(x_{-e_i + e_j}(t), I)$	$\prod_{k=1}^{2m} x_{-e_{[i-1]+k} + e_{[j-1]+k}}(t)$
$(x_{e_i + e_j}(t), I)$	$\prod_{k=1}^{2m} x_{e_{[i-1]+k} + e_{[j-1]+\overline{k}}}(t)$
$(x_{-e_i - e_j}(t), I)$	$\prod_{k=1}^{2m} x_{-e_{[i-1]+k} - e_{[j-1]+\overline{k}}}(t)$
$(I, x_{e_i - e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i} - e_{[k]+j}}(u) x_{e_{[k]+\overline{j}} - e_{[k]+\overline{i}}}(-u)$
$(I, x_{-e_i + e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i} + e_{[k]+j}}(u) x_{-e_{[k]+\overline{j}} + e_{[k]+\overline{i}}}(-u)$
$(I, x_{e_i + e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i} - e_{[k]+\overline{j}}}(u) x_{e_{[k]+j} - e_{[k]+\overline{i}}}(-u)$
$(I, x_{-e_i - e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i} + e_{[k]+\overline{j}}}(u) x_{-e_{[k]+j} + e_{[k]+\overline{i}}}(-u)$

This completely describes the behaviour on stalks, and therefore determines the group scheme homomorphism K_{ee}.

Both of the above tables are taken from [Rue20, Appendix 3] which describes the images of $K_{\mathbf{Sp}}(\mathbb{F}_{\text{sep}})$ and $K_{ee}(\mathbb{F}_{\text{sep}})$. These tables can be verified computationally by expressing the Chevalley generators as matrices as in section 1.3.2, and simplifying the results matrix products. This verification is tedious yet straightforward, so we omit the calculations.

Proposition 4.2.3. *Consider the map K_{eo}, and let $R \in \mathbf{Alg}_{\mathbb{F}}$. The map $K_{eo}(R)$ restricts to Chevalley subgroups $\mathbf{E}_{\mathbf{SO}_{2n}}(R) \times \mathbf{E}_{\mathbf{SO}_{2m+1}}(R) \to \mathbf{E}_{\mathbf{SO}_{2n(2m+1)}}(R)$ and the restriction is described on generators in the following table, where for an integer y we set $[y] = (2m + 1)y$ and $\overline{y} = 2m + 2 - y$.*

(A, B)	$K_{eo}(R)(A, B)$
$(x_{e_i - e_j}(t), I)$	$\prod_{k=1}^{2m+1} x_{e_{[i-1]+k} - e_{[j-1]+k}}(t)$
$(x_{-e_i + e_j}(t), I)$	$\prod_{k=1}^{2m+1} x_{-e_{[i-1]+k} + e_{[j-1]+k}}(t)$
$(x_{e_i + e_j}(t), I)$	$\left(\prod_{k=1}^{2m+1} x_{e_{[i-1]+k} + e_{[j-1]+\overline{k}}}(t)\right) x_{e_{[i-1]+m+1} + e_{[j-1]+m+1}}(t)$
$(x_{-e_i - e_j}(t), I)$	$\left(\prod_{k=1}^{2m+1} x_{-e_{[i-1]+k} - e_{[j-1]+\overline{k}}}(t)\right) x_{-e_{[i-1]+m+1} - e_{[j-1]+m+1}}(-{}^{t}\!/{2})$
$(I, x_{e_i - e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i} - e_{[k]+j}}(u) x_{e_{[k]+\overline{j}} - e_{[k]+\overline{i}}}(-u)$
$(I, x_{-e_i + e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i} + e_{[k]+j}}(u) x_{-e_{[k]+\overline{j}} + e_{[k]+\overline{i}}}(-u)$
$(I, x_{e_i + e_j}(u))$	$\prod_{k=0}^{n-1} x_{e_{[k]+i} - e_{[k]+\overline{j}}}(u) x_{e_{[k]+j} - e_{[k]+\overline{i}}}(-u)$
$(I, x_{-e_i - e_j}(u))$	$\prod_{k=0}^{n-1} x_{-e_{[k]+i} + e_{[k]+\overline{j}}}(u) x_{-e_{[k]+j} + e_{[k]+\overline{i}}}(-u)$
$(I, x_{e_i}(u))$	$\prod_{k=0}^{n-1} \big(x_{e_{[k]+i} - e_{[k]+(m+1)}}(u) x_{e_{[k]+(m+1)} - e_{[k]+\overline{i}}}(-2u) \cdot x_{e_{[k]+i} - e_{[k]+\overline{i}}}(u^2)\big)$
$(I, x_{-e_i}(u))$	$\prod_{k=0}^{n-1} \big(x_{-e_{[k]+i} + e_{[k]+(m+1)}}(2u) x_{-e_{[k]+(m+1)} + e_{[k]+\overline{i}}}(-u) \cdot x_{-e_{[k]+i} + e_{[k]+\overline{i}}}(-u^2)\big)$

This completely describes the behaviour on stalks, and therefore determines the group scheme homomorphism K_{eo}.

Proposition 4.2.4. *Assume that $\sqrt{2} \in \mathbb{F}$. Consider the map K'_{oo}, and let $R \in$ $\mathbf{Alg}_{\mathbb{F}}$. The map $K_{oo}(R)$ restricts to Chevalley subgroups $\mathbf{E}_{\mathbf{SO}_{2n+1}}(R) \times \mathbf{E}_{\mathbf{SO}_{2m+1}}(R) \to$ $\mathbf{E}_{\mathbf{SO}_{(2n+1)(2m+1)}}(R)$ and the restriction is described on generators in the following table, where for an integer y we set $[y] = (2m+1)y$ and $\overline{y} = 2m+2-y$.*

(A,B)	$K'_{oo}(A,B)$
$(x_{e_i-e_j}(t), I)$	$\prod_{k=1}^{2m+1} x_{e_{[i-1]+k}-e_{[j-1]+k}}(t)$
$(x_{-e_i+e_j}(t), I)$	$\prod_{k=1}^{2m+1} x_{-e_{[i-1]+k}+e_{[j-1]+k}}(t)$
$(x_{e_i+e_j}(t), I)$	$\left(\prod_{k=1}^{2m+1} x_{e_{[i-1]+k}+e_{[j-1]+\overline{k}}}(t)\right) x_{e_{[i-1]+m+1}+e_{[j-1]+m+1}}(t)$
$(x_{-e_i-e_j}(t), I)$	$\left(\prod_{k=1}^{2m+1} x_{-e_{[i-1]+k}-e_{[j-1]+\overline{k}}}(t)\right) x_{-e_{[i-1]+m+1}-e_{[j-1]+m+1}}(-{}^{t}/_{2})$
$(x_{e_i}(t), I)$	$\big(\prod_{k=1}^{m} x_{e_{[i-1]+k}-e_{[n]+k}}(t) x_{e_{[i-1]+\overline{k}}+e_{[n]+k}}(2t) \cdot x_{e_{[i-1]+k}+e_{[i-1]+\overline{k}}}(t^2)\big) \cdot x_{e_{[i-1]+(m+1)}}(\sqrt{2}t)$
$(x_{-e_i}(t), I)$	$\big(\prod_{k=1}^{m} x_{-e_{[i-1]+k}+e_{[n]+k}}(2t) x_{-e_{[i-1]+\overline{k}}-e_{[n]+k}}(t) \cdot x_{-e_{[i-1]+k}-e_{[i-1]+\overline{k}}}(-t^2)\big) \cdot x_{-e_{[i-1]+(m+1)}}({}^{t}/_{\sqrt{2}})$
$(I, x_{e_i-e_j}(u))$	$\big(\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+j}}(u) x_{e_{[k]+\overline{j}}-e_{[k]+\overline{i}}}(-u)\big) \cdot x_{e_{[n]+i}-e_{[n]+j}}(u)$
$(I, x_{-e_i+e_j}(u))$	$\big(\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+j}}(u) x_{-e_{[k]+\overline{j}}+e_{[k]+\overline{i}}}(-u)\big) \cdot x_{-e_{[n]+i}+e_{[n]+j}}(u)$
$(I, x_{e_i+e_j}(u))$	$\big(\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+\overline{j}}}(u) x_{e_{[k]+j}-e_{[k]+\overline{i}}}(-u)\big) \cdot x_{e_{[n]+i}+e_{[n]+j}}(2u)$
$(I, x_{-e_i-e_j}(u))$	$\big(\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+\overline{j}}}(u) x_{-e_{[k]+j}+e_{[k]+\overline{i}}}(-u)\big) \cdot x_{-e_{[n]+i}-e_{[n]+j}}(2u)$
$(I, x_{e_i}(u))$	$\big(\prod_{k=0}^{n-1} x_{e_{[k]+i}-e_{[k]+(m+1)}}(u) x_{e_{[k]+(m+1)}-e_{[k]+\overline{i}}}(-2u) \cdot x_{e_{[k]+i}-e_{[k]+\overline{i}}}(u^2)\big) \cdot x_{e_{[n]+i}}(\sqrt{2}u)$
$(I, x_{-e_i}(u))$	$\big(\prod_{k=0}^{n-1} x_{-e_{[k]+i}+e_{[k]+(m+1)}}(2u) x_{-e_{[k]+(m+1)}+e_{[k]+\overline{i}}}(-u) \cdot x_{-e_{[k]+i}+e_{[k]+\overline{i}}}(-u^2)\big) \cdot x_{-e_{[n]+i}}({}^{u}/_{\sqrt{2}})$

This completely describes the behaviour on stalks, and therefore determines the group scheme homomorphism K'_{oo}.

Both of the above propositions can be verified using the same method as the two propositions preceding them. Now we argue that the lifted maps behave the same way on the Chevalley generators of their respective groups.

Theorem 4.2.5. *Let $R \in \mathbf{Alg}_{\mathbb{F}}$. The images of $L_{\mathbf{Sp}}(R)$ restricted to $\mathbf{E}_{\mathbf{Sp}_{2n}}(R) \times$ $\mathbf{E}_{\mathbf{Sp}_{2m}}(R)$ on Chevalley generators are described by the same expressions as in proposition 4.2.1, where the $x_\alpha(t)$ in the images are now interpreted as generators in $\mathbf{Spin}_{4nm}$. The same is true for the restrictions of L_{ee}, L_{eo}, and L'_{oo} to Chevalley subgroups, they are described by the expressions in propositions 4.2.2, 4.2.3, and 4.2.4 respectively.*

Proof: Let L be any of the lifted maps we are considering. Let $x_\alpha(t)$ be a Chevalley generator in the domain of L, and let $z(t) \in \mathbf{E}_{\mathbf{Spin}_d}(R)$ be the element which is equal to the appropriate expression from the tables in propositions 4.2.1 to 4.2.4. It can be checked that in each case these expressions are additive in their arguments, and so

$z(t)z(u) = z(t+u)$ as well. Now we recall from definitions 1.3.12 and 1.3.14 that the kernel of the surjection

$$\mathbf{E}_{\mathbf{Spin}_d}(R) \twoheadrightarrow \mathbf{E}_{\mathbf{SO}_d}(R)$$

is a central subgroup isomorphic to $\boldsymbol{\mu}_2(R)$. Therefore, for each $t \in R$, let $h(t)$ be in the kernel such that $L(x_\alpha(t)) = z(t)h(t)$. Then, since R is an $\mathbb{F}$-algebra and $\operatorname{char}(\mathbb{F}) \neq 2$, we have $\frac{1}{2} \in R$. Therefore, for all $t \in R$,

$$\begin{aligned} L(x_\alpha(t)) &= L(x_\alpha({}^t\!/\!{}_2))L(x_\alpha({}^t\!/\!{}_2)) \\ &= z({}^t\!/\!{}_2)h({}^t\!/\!{}_2)z({}^t\!/\!{}_2)h({}^t\!/\!{}_2) \\ &= z({}^t\!/\!{}_2)z({}^t\!/\!{}_2)h({}^t\!/\!{}_2)^2 \\ &= z(t) \end{aligned}$$

as desired. Hence the tables in propositions 4.2.1 to 4.2.4 also describe the maps $L_{\mathbf{Sp}}$, L_{ee}, L_{eo}, and L'_{oo} on Chevalley subgroups. ∎

Now that we have a concrete description of the lifted maps, in particular a description of them on stalks, we can use this description to compute their restrictions to maximal tori.

Lemma 4.2.6. *Let $R \in \mathbf{Alg}_{\mathbb{F}}$ be a local strictly Henselian ring. The map $L_{\mathbf{Sp}}(R)$ behaves as follows on generators of the maximal torus of $\mathbf{Sp}_{2n}(R) \times \mathbf{Sp}_{2m}(R)$. For an integer y we set $[y] = (2m)y$.*

(A, B)	$L_{\mathbf{Sp}}(R)(A, B)$
$(h_{e_i - e_{i+1}}(t), I)$	$\prod_{k=1}^{2m} h_{e_{[i-1]+k} - e_{[i]+k}}(t)$
$(h_{2e_n}(t), I)$	$\prod_{k=1}^{m} h_{e_{[n-1]+k} + e_{[n-1]+m+k}}(t)$
$(I, h_{e_j - e_{j+1}}(u))$	$\prod_{k=0}^{n-1} h_{e_{([k]+j} - e_{[k]+j+1}}(u) h_{e_{[k]+m+j} - e_{[k]+m+j+1}}(u^{-1})$
$(I, h_{2e_m}(u))$	$\prod_{k=0}^{n-1} h_{e_{[k]+m} - e_{[k]+2m}}(u)$

Lemma 4.2.7. *Let $R \in \mathbf{Alg}_{\mathbb{F}}$ be a local strictly Henselian ring. The map $L_{ee}(R)$ behaves as follows on generators of the maximal torus of $\mathbf{Spin}_{2n}(R) \times \mathbf{Spin}_{2m}(R)$. For an integer y we set $[y] = (2m)y$ and $\overline{y} = 2m + 1 - y$.*

(A, B)	$L_{ee}(R)(A, B)$
$(h_{e_i - e_{i+1}}(t), 1)$	$\prod_{k=1}^{2m} h_{e_{[i-1]+k} - e_{[i]+k}}(t)$
$(h_{e_{n-1} + e_n}(t), 1)$	$\prod_{k=1}^{2m} h_{e_{[n-2]+k} + e_{[n-1]+\overline{k}}}(t)$
$(1, h_{e_j - e_{j+1}}(u))$	$\prod_{k=0}^{n-1} h_{e_{[k]+j} - e_{[k]+j+1}}(u) h_{e_{[k]+\overline{(j+1)}} - e_{[k]+\overline{j}}}(u)$
$(1, h_{e_{m-1} + e_m}(u))$	$\prod_{k=0}^{n-1} h_{e_{[k]+m-1} - e_{[k]+\overline{m}}}(u) h_{e_{[k]+m} - e_{[k]+\overline{(m-1)}}}(u)$

Lemma 4.2.8. *Let $R \in \mathbf{Alg}_{\mathbb{F}}$ be a local strictly Henselian ring. The map $L_{eo}(R)$ behaves as follows on generators of the maximal torus of $\mathbf{Spin}_{2n}(R) \times \mathbf{Spin}_{2m+1}(R)$. For an integer y we set $[y] = (2m+1)y$ and $\overline{y} = 2m+2-y$.*

(A, B)	$L_{eo}(R)(A, B)$
$(h_{e_i - e_{i+1}}(t), 1)$	$\prod_{k=1}^{2m+1} h_{e_{[i-1]+k} - e_{[i]+k}}(t)$
$(h_{e_{n-1}+e_n}(t), 1)$	$\prod_{k=1}^{2m+1} h_{e_{[n-2]+k} + e_{[n-1]+\overline{k}}}(t)$
$(1, h_{e_j - e_{j+1}}(u))$	$\prod_{k=0}^{n-1} h_{e_{[k]+j} - e_{[k]+j+1}}(u) h_{e_{[k]+\overline{(j+1)}} - e_{[k]+\overline{j}}}(u)$
$(1, h_{e_m}(u))$	$\prod_{k=0}^{n-1} h_{e_{[k]+m} - e_{[k]+\overline{m}}}(u^2)$

Lemma 4.2.9. *Assume $\sqrt{2} \in \mathbb{F}$ and let $R \in \mathbf{Alg}_{\mathbb{F}}$ be a local strictly Henselian ring. The map $L'_{oo}(R)$ behaves as follows on generators of the maximal torus of $\mathbf{Spin}_{2n+1}(R) \times \mathbf{Spin}_{2m+1}(R)$. For an integer y we set $[y] = (2m+1)y$ and $\overline{y} = 2m+2-y$.*

(A, B)	$L'_{oo}(R)(A, B)$
$(h_{e_i - e_{i+1}}(t), 1)$	$\prod_{k=1}^{2m+1} h_{e_{[i-1]+k} - h_{[i]+k}}(t)$
$(h_{e_n}(t), 1)$	$\prod_{k=1}^{2m+1} h_{e_{[n-1]+k}}(t)$
$(1, h_{e_j - e_{j+1}}(u))$	$\left(\prod_{k=0}^{n-1} h_{e_{[k]+j} - h_{[k]+j+1}}(u) h_{e_{[k]+\overline{(j+1)}} - e_{[k]+\overline{j}}}(u)\right)$ $\cdot h_{e_{[n]+j} - e_{[n]+j+1}}(u)$
$(1, h_{e_m}(u))$	$h_{e_{[n]+m}}(u) \prod_{k=0}^{n-1} h_{e_{[k]+m}}(u) h_{e_{[k]+\overline{m}}}(u^{-1})$

Proof: The images given in the above four lemmas can all be computed from the images given in propositions 4.2.1 to 4.2.4 using the relations in $\mathbf{Spin}_d$. As an example, and to highlight some of the important relations in **Spin**, we show the calculation of $L'_{oo}(h_{e_n}(t), 1)$. To compress notation, let $e_1 = e_{[n-1]+k}$, $e_2 = e_{[n-1]+\overline{k}}$, and $e_3 = e_{[n]+k}$. In this notation we have

$$L'_{oo}(x_{e_n}(t), 1) = \Big(\prod_{k=1}^{m} x_{e_1 - e_3}(t) x_{e_2 + e_3}(2t) x_{e_1 + e_2}(t^2)\Big) \cdot x_{e_{[n-1]+(m+1)}}(\sqrt{2}t)$$

$$L'_{oo}(x_{-e_n}(-t^{-1}), 1) = \Big(\prod_{k=1}^{m} x_{-e_1 + e_3}(-2t^{-1}) x_{-e_2 - e_3}(-t^{-1}) x_{-e_1 - e_2}(-t^{-2})\Big)$$

$$\cdot\, x_{-e_{[n-1]+(m+1)}}\left(\frac{-t^{-1}}{\sqrt{2}}\right)$$

We now compute the image of $(w_{e_n}(t), 1)$. Since the factors corresponding to different k, as well as the $e_{[n-1]+(m+1)}$ factors, will all commute past one another, we may deal with one value of k at a time. However, we first address the $e_{[n-1]+(m+1)}$ factors. In the image of $(w_{e_n}(t), 1)$ they will occur as

$$x_{e_{[n-1]+(m+1)}}(\sqrt{2}t) x_{-e_{[n-1]+(m+1)}}\left(\frac{-t^{-1}}{\sqrt{2}}\right) x_{e_{[n-1]+(m+1)}}(\sqrt{2}t)$$

$$=w_{e_{[n-1]+(m+1)}}(\sqrt{2}t).$$

Hence in the image of $(h_{e_n}(t),1)=(w_{e_n}(t),1)(w_{e_n}(-1),1)$ we will have

$$\begin{aligned}
&w_{e_{[n-1]+(m+1)}}(\sqrt{2}t)w_{e_{[n-1]+(m+1)}}(-\sqrt{2})\\
=&w_{e_{[n-1]+(m+1)}}(t)w_{e_{[n-1]+(m+1)}}(-1)w_{e_{[n-1]+(m+1)}}(\sqrt{2})w_{e_{[n-1]+(m+1)}}(-\sqrt{2})\\
=&h_{e_{[n-1]+(m+1)}}(t)
\end{aligned}$$

where the second line uses the observation that $w_\alpha(c)^{-1}=w_\alpha(-c)$ and [Rue20, Lemma 1(1)] which states that $w_\alpha(tu)=w_\alpha(t)w_\alpha(-1)w_\alpha(u)$. Now we consider the middle factors for some value of k. We use square brackets to indicate where we will apply a commutator relation from proposition 1.3.7. In the image of $(w_{e_n}(t),1)$ we will have

$$\begin{aligned}
&x_{e_1-e_3}(t)x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)x_{-e_1+e_3}(-2t^{-1})x_{-e_2-e_3}(-t^{-1})\left[x_{-e_1-e_2}(-t^{-2})x_{e_1-e_3}(t)\right]x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)x_{-e_1+e_3}(-2t^{-1})x_{-e_2-e_3}(-t^{-1})x_{-e_2-e_3}(t^{-1})x_{e_1-e_3}(t)x_{-e_1-e_2}(-t^{-2})\\
&\cdot x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{e_2+e_3}(2t)\left[x_{e_1+e_2}(t^2)x_{-e_1+e_3}(-2t^{-1})\right]x_{e_1-e_3}(t)x_{-e_1-e_2}(-t^{-2})x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{e_2+e_3}(2t)x_{e_2+e_3}(-2t)x_{-e_1+e_3}(-2t^{-1})x_{e_1+e_2}(t^2)x_{e_1-e_3}(t)x_{-e_1-e_2}(-t^{-2})x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})\left[x_{e_1+e_2}(t^2)x_{e_1-e_3}(t)\right]x_{-e_1-e_2}(-t^{-2})x_{e_2+e_3}(2t)x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{e_1+e_2}(t^2)\left[x_{-e_1-e_2}(-t^{-2})x_{e_2+e_3}(2t)\right]x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)\left[x_{e_1+e_2}(t^2)x_{-e_1+e_3}(-2t^{-1})\right]x_{e_2+e_3}(2t)x_{-e_1-e_2}(-t^{-2})x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{e_2+e_3}(-2t)x_{-e_1+e_3}(-2t^{-1})\left[x_{e_1+e_2}(t^2)x_{e_2+e_3}(2t)\right]x_{-e_1-e_2}(-t^{-2})x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{e_2+e_3}(-2t)\left[x_{-e_1+e_3}(-2t^{-1})x_{e_2+e_3}(2t)\right]x_{e_1+e_2}(t^2)x_{-e_1-e_2}(-t^{-2})x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{e_2+e_3}(-2t)x_{e_2+e_3}(2t)x_{-e_1+e_3}(-2t^{-1})x_{e_1+e_2}(t^2)x_{-e_1-e_2}(-t^{-2})x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1+e_2}(t^2)x_{-e_1-e_2}(-t^{-2})x_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})x_{e_1-e_3}(t)x_{-e_1+e_3}(-2t^{-1})w_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})x_{-e_1+e_3}(-t^{-1})x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})x_{-e_1+e_3}(-t^{-1})w_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})w_{-e_1+e_3}(-t^{-1})x_{-e_1+e_3}(-t^{-1})w_{e_1+e_2}(t^2)\\
\overset{i}{=}&x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})w_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})w_{e_1+e_2}(t^2)\\
=&x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})x_{e_1-e_3}(t)x_{-e_1+e_3}(-t^{-1})w_{e_1+e_2}(t^2)\\
=&w_{e_1-e_3}(t)w_{-e_1+e_3}(-t^{-1})w_{e_1+e_2}(t^2)\\
\overset{i}{=}&w_{e_1-e_3}(t)w_{e_1-e_3}(t)w_{e_1+e_2}(t^2)\\
=&w_{e_1-e_3}(t)^2w_{e_1+e_2}(t^2)\\
\overset{ii}{=}&h_{e_1-e_3}(-1)w_{e_1+e_2}(t^2)
\end{aligned}$$

where the marked equalities are using [Rue20, Lemma 1]. Those marked i are using part (2) which says $w_\alpha(t)=w_{-\alpha}(-t^{-1})$ in **Spin**, and the equality marked ii is using part (3) which says $w_\alpha(t)^2=h_\alpha(-1)$. Now in the image of $(h_{e_n}(t),1)$ we will have

$$h_{e_1-e_3}(-1)w_{e_1+e_2}(t^2)h_{e_1-e_3}(-1)w_{e_1+e_2}(1)$$

$$\begin{aligned}
&\overset{i}{=}h_{e_1-e_3}(-1)w_{e_1+e_2}(t^2)w_{e_1+e_2}(-1)h_{e_1-e_3}(-1)\\
&=h_{e_1-e_3}(-1)h_{e_1+e_2}(t^2)h_{e_1-e_3}(-1)\\
&=h_{e_1-e_3}(-1)h_{e_1-e_3}(-1)h_{e_1+e_2}(t^2)\\
&=h_{e_1+e_2}(t^2)\\
&\overset{ii}{=}h_{e_1}(t)h_{e_2}(t)\\
&=h_{e_{[n-1]+k}}(t)h_{e_{[n-1]+\bar{k}}}(t)
\end{aligned}$$

using theorem 1.3.4 at equality i, and [Rue20, Lemma 1(6)] at equality ii. Hence, when we assemble all pieces we obtain that

$$\begin{aligned}
L'_{oo}(h_{e_n}(t),1) &= \left(\prod_{k=1}^{m} h_{e_{[n-1]+k}}(t)h_{e_{[n-1]+\bar{k}}}(t)\right) h_{e_{[n-1]+(m+1)}}(t)\\
&= \prod_{k=1}^{2m+1} h_{e_{[n-1]+k}}(t)
\end{aligned}$$

as stated in the table. ∎

4.3 Tensor Products for Other Split Groups

With the explicit descriptions of the tensor product maps achieved in the previous section we are now ready to construct similar maps between non-simply connected groups. The technique is the same in all cases. If $L\colon \mathbf{G}_1 \to \mathbf{G}_2$ is one of our lifted maps, we will consider a composition

$$\varphi\colon\ \mathbf{G}_1 \overset{L}{\to} \mathbf{G}_2 \twoheadrightarrow \mathbf{H}$$

with a central quotient onto a non-simply connected group. Then using the description of L on maximal tori given in lemmas 4.2.6 to 4.2.9 we can compute the kernel of φ and therefore find a map ϕ which fits into the following commutative diagram.

$$\begin{tikzcd}
\mathbf{G}_1 \arrow[r, "L"] \arrow[d, two heads] & \mathbf{G}_2 \arrow[d, two heads] \\
\mathbf{G}_1 \,/\, \ker(\varphi) \arrow[r, hook, "\phi"] & \mathbf{H}
\end{tikzcd}$$

We consider such ϕ to be a tensor product map between non-simply connected groups. As the following lemma shows, the kernel of φ will be a central subgroup of $\mathbf{G}_1$ and we will choose $\mathbf{H}$ so that $\mathbf{G}_1 \,/\, \ker(\varphi)$ is a product of our non-simply connected linear algebraic groups, justifying the terminology.

Lemma 4.3.1. *Let $L\colon\ \mathbf{G}_1 \to \mathbf{G}_2$ be any of the maps $L_{\mathbf{Sp}}, L_{ee}, L_{eo}$, or L'_{oo}. For any central quotient $\mathbf{G}_2 \twoheadrightarrow \mathbf{H}$, the kernel of the composition $\mathbf{G}_1 \overset{L}{\to} \mathbf{G}_2 \twoheadrightarrow \mathbf{H}$ is a central subgroup of $\mathbf{G}_1$.*

Proof: Since any such L is a lifting of some tensor product between matrix subgroups K, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{G}_1 \arrow[r, "L"] \arrow[d, two heads] & \mathbf{G}_2 \arrow[d, two heads] \arrow[dr, two heads] & \\
\mathbf{G}'_1 \arrow[r, "K"] & \mathbf{G}'_2 \arrow[r, two heads] & \mathbf{G}_{2,\mathrm{ad}}
\end{tikzcd}$$

where $\mathbf{G}_{2,\mathrm{ad}}$ is the adjoint group in the family of $\mathbf{G}_2$. Over any $R \in \mathbf{Alg}_{\mathbb{F}}$, $K(R)$ is of the form $(A, B) \mapsto P^{-1}(A \otimes B)P$, and so its kernel consists only of pairs of the form $(\mathrm{diag}(c, \ldots, c), \mathrm{diag}(c^{-1}, \ldots, c^{-1})$ for certain $c \in R^{\times}$, all of which are central elements in $\mathbf{G}'_1(R)$. Thus, the $\ker(K)$ is a central subgroup of $\mathbf{G}'_1$ scheme theoretically. Then, the preimage of $Z(\mathbf{G}'_1)$ is $Z(\mathbf{G}_1)$, and $\mathbf{G}'_2 \twoheadrightarrow \mathbf{G}_{2,\mathrm{ad}}$ is a central quotient, so the kernel of composition

$$\mathbf{G}_1 \twoheadrightarrow \mathbf{G}'_1 \overset{K}{\to} \mathbf{G}'_2 \twoheadrightarrow \mathbf{G}_{2,\mathrm{ad}},$$

which is also the kernel of $\mathbf{G}_1 \xrightarrow{L} \mathbf{G}_2 \twoheadrightarrow \mathbf{G}_{2,\mathrm{ad}}$, is a central subgroup of $\mathbf{G}_1$. Since the surjection from the simply connected group to the adjoint group in a family factors through any intermediate group, we have that the composition

$$\mathbf{G}_1 \xrightarrow{L} \mathbf{G}_2 \twoheadrightarrow \mathbf{H} \twoheadrightarrow \mathbf{G}_{2,\mathrm{ad}}$$

is the same map as above, in particular with the same kernel. The kernel of the map into $\mathbf{H}$ is a subgroup of the full map's kernel, and so is a central subgroup of $\mathbf{G}_1$ as desired. ▮

Since we know where the kernels which arise will be, we take a moment now to describe the images of central elements under the lifted maps using the isomorphisms of proposition 1.3.10.

Lemma 4.3.2. *The map $L_{\mathbf{Sp}}$ restricts as follows to $Z(\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m})$.*

$$\begin{tikzcd}
L_{\mathbf{Sp}}\colon Z(\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{4nm}) \arrow[d, phantom, sloped, "\cong"] \\
\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \arrow[r] & \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \\
(t,u) \arrow[r, mapsto] & \begin{cases} (tu,1) & n \text{ even or } m \text{ even} \\ (u,t) & n \text{ odd and } m \text{ odd} \end{cases}
\end{tikzcd}$$

Lemma 4.3.3. *The restriction of L_{ee} to $Z(\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m})$ depends on the parity of n and m. First, when n and m are even we have*

$$\begin{tikzcd}
L_{ee}\colon Z(\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{4nm}) \arrow[d, phantom, sloped, "\cong"] \\
(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \times (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \arrow[r] & \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \\
((t_1,t_2),(u_1,u_2)) \arrow[r, mapsto] & (t_1 t_2 u_1 u_2, 1)
\end{tikzcd}$$

When n is odd and m is even,

$$\begin{tikzcd}
L_{ee}\colon Z(\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{4nm}) \arrow[d, phantom, sloped, "\cong"] \\
\boldsymbol{\mu}_4 \times (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \arrow[r] & \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \\
(t,(u_1,u_2)) \arrow[r, mapsto] & (t^2 u_1 u_2, 1)
\end{tikzcd}$$

When n is even and m is odd,

$$\begin{tikzcd}
L_{ee}\colon Z(\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{4nm}) \arrow[d, phantom, sloped, "\cong"] \\
(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \times \boldsymbol{\mu}_4 \arrow[r] & \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \\
((t_1,t_2),u) \arrow[r, mapsto] & (t_1 t_2 u^2, 1)
\end{tikzcd}$$

Finally, when n and m are both odd,

$$\begin{tikzcd}
L_{ee}\colon Z(\mathbf{Spin}_{2n}\times\mathbf{Spin}_{2m}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{4nm}) \arrow[d, phantom, sloped, "\cong"] \\
\boldsymbol{\mu}_4\times\boldsymbol{\mu}_4 \arrow[r] & \boldsymbol{\mu}_2\times\boldsymbol{\mu}_2 \\
(t,u) \arrow[r, mapsto] & (u^2,t^2)
\end{tikzcd}$$

Lemma 4.3.4. *The restriction of L_{eo} to $Z(\mathbf{Spin}_{2n}\times\mathbf{Spin}_{2m+1})$ depends on the parity of n. First, when n is even,*

$$\begin{tikzcd}
L_{eo}\colon Z(\mathbf{Spin}_{2n}\times\mathbf{Spin}_{2m+1}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{2n(2m+1)}) \arrow[d, phantom, sloped, "\cong"] \\
(\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2)\times\boldsymbol{\mu}_2 \arrow[r] & \boldsymbol{\mu}_2\times\boldsymbol{\mu}_2 \\
((t_1,t_2),u) \arrow[r, mapsto] & (t_1,t_2)
\end{tikzcd}$$

and when n is odd,

$$\begin{tikzcd}
L_{eo}\colon Z(\mathbf{Spin}_{2n}\times\mathbf{Spin}_{2m+1}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{2n(2m+1)}) \arrow[d, phantom, sloped, "\cong"] \\
\boldsymbol{\mu}_4\times\boldsymbol{\mu}_2 \arrow[r] & \boldsymbol{\mu}_4 \\
(t,u) \arrow[r, mapsto] & \begin{cases} t & m \text{ even} \\ t^{-1} & m \text{ odd} \end{cases}
\end{tikzcd}$$

Proof: These three lemmas are essentially corollaries to lemmas 4.2.6 to 4.2.9 respectively. Using those descriptions of the maps restricted to tori and the descriptions of the groups' centers from proposition 1.3.10, the stated images can be directly computed. ∎

Lemma 4.3.5. *Assume $\sqrt{2}\in\mathbb{F}$. The map L'_{oo} restricts to $Z(\mathbf{Spin}_{2n+1}\times\mathbf{Spin}_{2m+1})$ as follows.*

$$\begin{tikzcd}
L'_{oo}\colon Z(\mathbf{Spin}_{2n+1}\times\mathbf{Spin}_{2m+1}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & Z(\mathbf{Spin}_{(2n+1)(2m+1)}) \arrow[d, phantom, sloped, "\cong"] \\
\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2 \arrow[r] & \boldsymbol{\mu}_2 \\
(t,u) \arrow[r, mapsto] & tu
\end{tikzcd}$$

Proof: Let $R\in\mathbf{Alg}_{\mathbb{F}}$. The center of $\mathbf{Spin}_{2n+1}(R)\times\mathbf{Spin}_{2m+1}(R)$ is

$$\{(h_n(t),h_m(u)) \mid t,u\in\boldsymbol{\mu}_2(R)\}\cong(\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2)(R).$$

Using the information in 4.2.9, the image of a central element is

$$\begin{aligned}L'_{oo}(h_n(t), h_m(u)) = &\left(\prod_{k=0}^{n-2} h_{e_{[k]+m}}(u)h_{e_{[k]+\overline{m}}}(u)\right)\left(\prod_{j=1}^{m-1} h_{e_{[n-1]+j}}(t)\right)\\ &\cdot h_{e_{[n-1]+m}}(tu)h_{e_{[n-1]+m+1}}(t)h_{e_{[n-1]+m+2}}(tu)\\ &\cdot\left(\prod_{j=m+3}^{2m+1} h_{e_{[n-1]+j}}(t)\right) h_{e_{[n]+m}}(u).\end{aligned}$$

However, by [Rue20, Lemma 1 (6)], for all $1 \leq i \neq j \leq n(2m+1)+m$ we have $h_{\pm e_i}(t)h_{\pm e_j}(t) = h_{\pm e_i \pm e_j}(t^2)$ for all $t \in R^\times$. When $t \in \boldsymbol{\mu}_2(R)$ this gives that $h_{e_i}(t)h_{e_j}(t) = 1$, and so

$$h_{e_i}(t) = h_{e_j}(t)^{-1} = h_{e_j}(t^{-1}) = h_{e_j}(t).$$

Therefore all h_{e_i} in the product above can be replaced with $h_{e_{[n]+m}}$, and after collecting like terms we get

$$L'_{oo}(h_n(t), h_m(u)) = h_{e_{[n]+m}}(tu)$$

as claimed. ∎

With these computations complete we can now fulfil our promise and compose these maps with central quotients to define new tensor product maps. We will first focus on maps into **HSpin**.

Proposition 4.3.6. *There exists a map $N_{\mathbf{Sp}}$ when n or m is even, and a map $N'_{\mathbf{Sp}}$ when n and m are odd, which make the following diagrams commute.*

$$\begin{tikzcd}
\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \arrow[r, "L_{\mathbf{Sp}}"] \arrow[d, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] & \mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \arrow[r, "L_{\mathbf{Sp}}"] \arrow[d, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \arrow[r, hook, "N_{\mathbf{Sp}}"] & \mathbf{HSpin}_{4nm} & \mathbf{Sp}_{2n} \times \mathbf{PSp}_{2m} \arrow[r, hook, "N'_{\mathbf{Sp}}"] & \mathbf{HSpin}_{4nm}
\end{tikzcd}$$

Proof: By lemma 4.3.2 the kernel of the composition

$$\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \overset{L_{\mathbf{Sp}}}{\to} \mathbf{Spin}_{4nm} \twoheadrightarrow \mathbf{HSpin}_{4nm}\,.$$

is $Z(\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m})$ when n or m is even, and is $1 \times Z(\mathbf{Sp}_{2m})$ when n and m are odd. Taking the quotient of $\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m}$ by these kernels produces the desired maps. ∎

Proposition 4.3.7. *There exists a map N_{ee} when n or m is even, and a map N'_{ee} when n and m are odd, which make the following diagrams commute.*

$$\begin{tikzcd}
\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m} \arrow[r, "L_{ee}"] \arrow[d, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] & \mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m} \arrow[r, "L_{ee}"] \arrow[d, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSO}_{2n} \times \mathbf{PSO}_{2m} \arrow[r, hook, "N_{ee}"] & \mathbf{HSpin}_{4nm} & \mathbf{SO}_{2n} \times \mathbf{PSO}_{2m} \arrow[r, hook, "N'_{ee}"] & \mathbf{HSpin}_{4nm}
\end{tikzcd}$$

Proof: By lemma 4.3.3 the kernel of the composition

$$\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m} \overset{L_{ee}}{\to} \mathbf{Spin}_{4nm} \twoheadrightarrow \mathbf{HSpin}_{4nm} .$$

is $Z(\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m})$ when n or m is even, and is $\boldsymbol{\mu}_2 \times Z(\mathbf{Spin}_{2m})$ when n and m are odd. Taking the quotient of $\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m}$ by these kernels produces the desired maps. ∎

Proposition 4.3.8. *Assume n is even. Then there exists a map N_{eo} which makes the following diagram commute.*

$$\begin{tikzcd}
\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m+1} \arrow[r, "L_{eo}"] \arrow[d, two heads] & \mathbf{Spin}_{2n(2m+1)} \arrow[d, two heads] \\
\mathbf{HSpin}_{2n} \times \mathbf{PSO}_{2m+1} \arrow[r, hook, "N_{eo}"] & \mathbf{HSpin}_{2n(2m+1)}
\end{tikzcd}$$

Proof: By lemma 4.3.4 the kernel of the composition

$$\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m+1} \overset{L_{eo}}{\to} \mathbf{Spin}_{2n(2m+1)} \twoheadrightarrow \mathbf{HSpin}_{4nm} .$$

is $(\boldsymbol{\mu}_2 \times 1) \times \boldsymbol{\mu}_2$. Taking the quotient of $\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m+1}$ by this kernel produces the desired map. ∎

We now list similar results from maps into **PSO**. We note that these maps are not new. Because of this, and because the proofs are analogous to the ones above, we omit them for the following propositions.

Proposition 4.3.9. *There exists a map $K_{\mathbf{Sp},\mathrm{ad}}$ which makes the following diagram commute.*

$$\begin{tikzcd}
\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \arrow[r, "L_{\mathbf{Sp}}"] \arrow[d, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \arrow[r, hook, "K_{\mathbf{Sp},\mathrm{ad}}"] & \mathbf{PSO}_{4nm}
\end{tikzcd}$$

Proposition 4.3.10. *Let L be any of L_{ee}, L_{eo}, or L'_{oo}. Then there exists a map K_{ad} (denoted $K_{ee,\mathrm{ad}}$, $K_{eo,\mathrm{ad}}$, or $K'_{oo,\mathrm{ad}}$ respectively) which makes the following diagram commute.*

$$\begin{tikzcd}
\mathbf{Spin}_{d_1} \times \mathbf{Spin}_{d_2} \arrow[r, "L"] \arrow[d, two heads] & \mathbf{Spin}_{d_1 d_2} \arrow[d, two heads] \\
\mathbf{PSO}_{d_1} \times \mathbf{PSO}_{d_2} \arrow[r, hook, "K_{\mathrm{ad}}"] & \mathbf{PSO}_{d_1 d_2}
\end{tikzcd}$$

We end this section by noting that, by construction, the tensor maps into **HSpin** and the tensor maps between adjoint groups fit together into larger commutative diagrams where the maps N are under the lifted maps L, and the maps K_{ad} are under the maps N. For example, the following two diagrams.

$$\begin{tikzcd}
\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \arrow[r, "L_{\mathbf{Sp}}"] \arrow[dd, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
& \mathbf{HSpin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \arrow[r, hook, "K_{\mathbf{Sp},\mathrm{ad}}"] \arrow[ur, hook, "N_{\mathbf{Sp}}"] & \mathbf{PSO}_{4nm}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Spin}_{2n} \times \mathbf{Spin}_{2m} \arrow[r, "L_{ee}"] \arrow[dd, two heads] & \mathbf{Spin}_{4nm} \arrow[d, two heads] \\
& \mathbf{HSpin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSO}_{2n} \times \mathbf{PSO}_{2m} \arrow[r, hook, "K_{ee,\mathrm{ad}}"] \arrow[ur, hook, "N_{ee}"] & \mathbf{PSO}_{4nm}
\end{tikzcd}$$

Of course, there are similar diagrams containing the other maps into **HSpin** constructed above.

4.4 Tensor Products for Non-split Groups

In the previous section we used the Chevalley description of split groups to construct various tensor product maps between linear algebraic groups. We now show that these maps are compatible with the twisting construction of section 1.4.2. This occurs since the absolute Galois group acts on Chevalley generators by acting on their arguments. Therefore we can produce tensor product maps between non-split linear algebraic groups.

For this section we recall our notation $\Gamma = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$ for the absolute Galois group of $\mathbb{F}$. Recall that for $R \in \mathbf{Alg}_{\mathbb{F}}$ there is an action of Γ on $R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$ by acting on the second factor. This action extends to a Γ-action on our linear algebraic groups. Let $\mathbf{G}$ be a semisimple linear algebraic group with representing Hopf algebra H. Then a point of $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is a map $x\colon H \to R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$. For $\sigma \in \Gamma$, the action is defined by setting $\sigma(x)$ to be the map

$$\sigma(x)\colon H \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \xrightarrow{1\otimes\sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$$

For **SO** and **Sp** this action appears as σ acting entrywise on matrices. For **PSO** and **PSp**, the action is also entrywise on any matrix representing the equivalence class. For **Spin** and **HSpin**, which are defined within a Clifford algebra, σ acts on the coefficients of the elements. For example in **Spin**,

$$\sigma(1 + tv_iv_j) = \sigma(1) + \sigma(t)v_iv_j = 1 + \sigma(t)v_iv_j$$

for $t \in R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$. With this setup, any group scheme homomorphism will also be a Γ-morphisms over $R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$.

Lemma 4.4.1. *Let $\varphi\colon \mathbf{G}_1 \to \mathbf{G}_2$ be any homomorphism of group schemes, and let $R \in \mathbf{Alg}_{\mathbb{F}}$. Then the map $\varphi(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is a Γ-morphism. That is, for $\sigma \in \Gamma$ we have*

$$\varphi(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \circ (1 \otimes \sigma) = (1 \otimes \sigma) \circ \varphi(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}).$$

Proof: For brevity, we set $\sigma' = 1 \otimes \sigma$ and $R' = R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$. Let H_1 and H_2 be the representing Hopf algebras of $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively. Considering a point $x \in \mathbf{G}_1(R')$ as a map from H_1, we have the following diagram

$$\begin{tikzcd}
H_1 \arrow[r, "x"] & R' \arrow[r, "\sigma'"] & R' \\
H_2 \arrow[u, "\varphi^*"] \arrow[ur, "\varphi(R')(x)"'] \arrow[urr, bend right, "\sigma'(\varphi(R')(x))=\varphi(R')(\sigma'(x))"'] & &
\end{tikzcd}$$

From which it is clear that

$$\sigma' \circ \varphi(R') = \varphi(R') \circ \sigma'$$

as desired. ∎

The property of being a Γ-morphism will allow these maps to descent to maps on R points after twisting the Γ-action as in section 1.4.2. However, we first need to understand how the various maps K_{ad} effect Galois cohomology and cocycles. That is, we wish to understand the induced maps

$$\begin{aligned} K^*_{\mathbf{Sp},\mathrm{ad}} &\colon H^1(\mathbb{F}, \mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m}) \to H^1(\mathbb{F}, \mathbf{PSO}_{4nm}) \\ K^*_{--,\mathrm{ad}} &\colon H^1(\mathbb{F}, \mathbf{PSO}_{d_1} \times \mathbf{PSO}_{d_2}) \to H^1(\mathbb{F}, \mathbf{PSO}_{d_1 d_2}) \end{aligned}$$

We first address the $K^*_{\mathbf{Sp},\mathrm{ad}}$ case. First, we have that

$$H^1(\mathbb{F}, \mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m}) \cong H^1(\mathbb{F}, \mathbf{PSp}_{2n}) \times H^1(\mathbb{F}, \mathbf{PSp}_{2m})$$

and we recall from section 1.4.1 that $H^1(\mathbb{F}, \mathbf{PSp}_{2n})$ classifies isomorphism classes of central simple algebras with symplectic involution. It does so by associating the class of a cocycle $\alpha\colon \Gamma \to \mathbf{PSp}_{2n}(\mathbb{F}_{\mathrm{sep}})$ with the fixed points of the algebra $(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \psi_0)$ under the Γ-action twisted by α.

Proposition 4.4.2. *The group scheme homomorphism $K_{\mathbf{Sp},\mathrm{ad}}$ induces a map between first cohomology sets which behaves as*

$$\begin{aligned} K^*_{\mathbf{Sp},\mathrm{ad}} \colon H^1(\mathbb{F}, \mathbf{PSp}_{2n}) \times H^1(\mathbb{F}, \mathbf{PSp}_{2m}) &\to H^1(\mathbb{F}, \mathbf{PSO}_{4nm}) \\ ([A_1, \psi_1], [A_2, \psi_2]) &\to [A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2] \end{aligned}$$

where (A_1, ψ_1) and (A_2, ψ_2) are central simple $\mathbb{F}$-algebras with symplectic involutions of degrees $2n$ and $2m$ respectively.

Proof: Let (A_1, ψ_1) and (A_2, ψ_2) be as in the proposition, and let

$$\begin{aligned} \alpha_1 &\colon \Gamma \to \mathbf{PSp}_{2n}(\mathbb{F}_{\mathrm{sep}}) \\ \alpha_2 &\colon \Gamma \to \mathbf{PSp}_{2m}(\mathbb{F}_{\mathrm{sep}}) \end{aligned}$$

be the standard cocycles representing the algebras' classes in $H^1(\mathbb{F}, \mathbf{PSp}_{2n})$ and $H^1(\mathbb{F}, \mathbf{PSp}_{2m})$ respectively. The isomorphism classes $[A_1, \psi_1]$ and $[A_2, \psi_2]$ are then mapped by $K^*_{\mathbf{Sp},\mathrm{ad}}$ to the class represented by the cocycle

$$\begin{aligned} K_{\mathbf{Sp},\mathrm{ad}} \circ (\alpha_1, \alpha_2) \colon \Gamma &\to \mathbf{PSO}_{4nm}(\mathbb{F}_{\mathrm{sep}}) \\ \sigma &\to \overline{P}^{-1} (\alpha_{1,\sigma} \otimes \alpha_{2,\sigma}) \overline{P}. \end{aligned}$$

where $\overline{P}$ is the image in $\mathbf{PSO}_{4nm}(\mathbb{F}_{\mathrm{sep}})$ of the conjugating matrix P used in the definition of $K_{\mathbf{Sp}}$. Now we consider the fixed points of $(\mathrm{M}_{4nm}, \tau_0)$ after twisting the Γ-action by this new cocycle. First, we undo the conjugation by P which has the effect of exchanging the standard involution τ_0 for $\psi_1' \otimes \psi_2'$ where ψ_i' are our standard choice of symplectic involutions on $\mathrm{M}_{2n}(\mathbb{F})$ and $\mathrm{M}_{2m}(\mathbb{F})$ respectively.

$$(\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}), \tau_0)^{\Gamma_{K_{\mathbf{Sp},\mathrm{ad}} \circ (\alpha_1,\alpha_2)}} \cong (\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}), \psi_1' \otimes \psi_2')^{\Gamma_{\alpha_1 \otimes \alpha_2}}$$

Now we use the decomposition $\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}) \cong \mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}) \otimes_{\mathbb{F}_{\mathrm{sep}}} \mathrm{M}_{2m}(\mathbb{F}_{\mathrm{sep}})$ and see that in this description, the twisted action is

$$\sigma \cdot_{\alpha_1 \otimes \alpha_2} \sum_{i=1}^{k} C_i \otimes D_i = \sum_{i=1}^{k} (\alpha_{1,\sigma} \sigma(C_i) \alpha_{1,\sigma}^{-1}) \otimes (\alpha_{2,\sigma} \sigma(D_i) \alpha_{2,\sigma}^{-1}) = \sum_{i=1}^{k} (\sigma \cdot_{\alpha_1} C_i) \otimes (\sigma \cdot_{\alpha_2} D_i).$$

Then since $(\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}), \psi_1')^{\Gamma_{\alpha_1}} \cong (A_1, \psi_1)$ and $(\mathrm{M}_{2m}(\mathbb{F}_{\mathrm{sep}}), \psi_2')^{\Gamma_{\alpha_2}} \cong (A_2, \psi_2)$ it is clear that there is an injection

$$(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \hookrightarrow (\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}), \psi_1' \otimes \psi_2')^{\Gamma_{\alpha_1 \otimes \alpha_2}}.$$

Since the dimension over $\mathbb{F}$ is the same on each side, this map is also surjective and hence an isomorphism. Therefore the cocycle $K_{\mathbf{Sp},\mathrm{ad}} \circ (\alpha_1, \alpha_2)$ represents the isomorphism class $[A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2] \in H^1(\mathbb{F}, \mathbf{PSO}_{4nm})$, which means

$$K^*_{\mathbf{Sp},\mathrm{ad}}([A_1, \psi_1], [A_2, \psi_2]) = [A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2]$$

as desired. ▮

The proof of the following proposition, which deals with the cases $K^*_{--,\mathrm{ad}}$, is the same as the one above, just with all the symplectic content replaced with orthogonal content. Because of this, we omit the proof.

Proposition 4.4.3. *Let K be one of the group scheme homomorphisms $K_{ee,\mathrm{ad}}$, $K_{eo,\mathrm{ad}}$, or $K'_{oo,\mathrm{ad}}$. Then K induces a map between first cohomology sets which behaves as*

$$\begin{aligned} K^* \colon H^1(\mathbb{F}, \mathbf{PSO}_{d_1}) \times H^1(\mathbb{F}, \mathbf{PSO}_{d_2}) &\to H^1(\mathbb{F}, \mathbf{PSO}_{d_1 d_2}) \\ ([A_1, \tau_1], [A_2, \tau_2]) &\to [A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2] \end{aligned}$$

where (A_1, τ_1) and (A_2, τ_2) are central simple $\mathbb{F}$-algebras with orthogonal involutions of degrees d_1 and d_2 respectively.

Now that we know which algebras correspond to the images of cocycles, we are ready to construct our non-split tensor product maps. We note that the maps labelled by $\mathcal{K}$ or $\mathcal{K}_{\mathrm{ad}}$ in what follows are well known, and can be defined directly from the natural map

$$\begin{aligned} A_1 \times A_2 &\to A_1 \otimes_{\mathbb{F}} A_2 \\ (a, b) &\mapsto a \otimes b \end{aligned}$$

Theorem 4.4.4. *Let* (A_1, ψ_1) *and* (A_2, ψ_2) *be central simple algebras of degree* $2n$ *and* $2m$ *respectively with symplectic involutions. There are group scheme homomorphisms*

$$\begin{aligned}
\mathcal{L}_{\mathbf{Sp}} &\colon \mathbf{Sp}(A_1,\psi_1)\times\mathbf{Sp}(A_2,\psi_2)\to\mathbf{Spin}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)\\
\mathcal{K}_{\mathbf{Sp}} &\colon \mathbf{Sp}(A_1,\psi_1)\times\mathbf{Sp}(A_2,\psi_2)\to\mathbf{SO}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)\\
\mathcal{N}_{\mathbf{Sp}} &\colon \mathbf{PSp}(A_1,\psi_1)\times\mathbf{PSp}(A_2,\psi_2)\hookrightarrow\mathbf{HSpin}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)\ \textit{when } n \textit{ or } m \textit{ even}\\
\mathcal{N}'_{\mathbf{Sp}} &\colon \mathbf{Sp}(A_1,\psi_1)\times\mathbf{PSp}(A_2,\psi_2)\hookrightarrow\mathbf{HSpin}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)\ \textit{when } n \textit{ and } m \textit{ odd}
\end{aligned}$$

all lying above a map

$$\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}\colon \mathbf{PSp}(A_1,\psi_1)\times\mathbf{PSp}(A_2,\psi_2)\to\mathbf{PSO}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)$$

which for $R\in\mathbf{Alg}_{\mathbb{F}}$ *behaves on* R*-points as*

$$\begin{aligned}
\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}(R)\colon \mathbf{PSp}(A_1,\psi_1)(R)\times\mathbf{PSp}(A_2,\psi_2)(R) &\to \mathbf{PSO}(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)(R)\\
([a],[b]) &\mapsto [a\otimes b]
\end{aligned}$$

Proof: Let α_1 and α_2 be the standard cocycles representing (A_1,ψ_1) and (A_2,ψ_2) respectively, and let β be the standard cocycle representing $(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)$. First, we consider the Kronecker tensor product map on the level of split $\mathbb{F}_{\mathrm{sep}}$-algebras with the standard choices of involutions,

$$\begin{aligned}
K\colon (\mathrm{M}_{2n}(\mathbb{F}_{\mathrm{sep}}),\psi_1')\times(\mathrm{M}_{2m}(\mathbb{F}_{\mathrm{sep}}),\psi_2') &\to \mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}),\tau_0)\\
(B_1,B_2) &\mapsto P^{-1}(B_1\otimes B_2)P
\end{aligned}$$

where P is the matrix used to define $K_{\mathbf{Sp}}$. Since this map K is a Γ-morphism, it will restrict to fixed points

$$K'\colon (A_1,\psi_1)\times(A_2,\psi_2)\to(\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}),\tau_0)^{\Gamma_{K_{\mathbf{Sp},\mathrm{ad}}\circ(\alpha_1,\alpha_2)}}.$$

Now, it is unlikely that $K_{\mathbf{Sp},\mathrm{ad}}\circ(\alpha_1,\alpha_2)$ and β are equal, but in any case, since both cocycles represent the isomorphism class $[A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2]$, they will be cohomologous. This means that their respective fixed point sets differ by an automorphism of $(\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}),\tau_0)$, say by $[B]\in\mathbf{PSO}_{4nm}(\mathbb{F}_{\mathrm{sep}})$. Hence, we will have a map

$$[B]\circ K'\colon (A_1,\psi_1)\times(A_2,\psi_2)\to(A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2).$$

By the universal property of tensor products we will also be given a map ϕ fitting into the following diagram

$$\begin{tikzcd}
(A_1,\psi_1)\times(A_2,\psi_2) \arrow[r, "\otimes"] \arrow[dr, "{[B]\circ K'}"'] & (A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2) \arrow[d, "\phi"] \\
 & (A_1\otimes_{\mathbb{F}}A_2,\psi_1\otimes\psi_2)
\end{tikzcd}$$

and since $[B] \circ K' \neq 0$, then ϕ must be an automorphism of $(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$. Because this is an automorphism of a central simple algebra, it is an inner automorphism, and so it can be extended to an automorphism of $(\mathrm{M}_{4nm}(\mathbb{F}_{\mathrm{sep}}), \tau_0)$, say $[C] \in \mathbf{PSO}_{4nm}(\mathbb{F}_{\mathrm{sep}})$. Thus, we will have

$$\begin{aligned}[C^{-1}B] \circ K' \colon (A_1, \psi_1) \times (A_2, \psi_2) &\to (A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2) \\ (a, b) &\mapsto a \otimes b.\end{aligned}$$

We mention this since it will allow us to control the map between adjoint groups. Now let $R \in \mathbf{Alg}_{\mathbb{F}}$. Since $K_{\mathbf{Sp},\mathrm{ad}}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is a Γ-morphism, it will restrict to a map between fixed points

$$\mathbf{PSp}(A_1, \psi_1)(R) \times \mathbf{PSp}(A_2, \psi_2)(R) \to \mathbf{PSO}_{4nm}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_{K_{\mathbf{Sp},\mathrm{ad}} \circ (\alpha_1, \alpha_2)}}$$

This map as well can be corrected by using the element $1 \otimes [C^{-1}B]$ in the image of $\mathbf{PSO}_{4nm}(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{PSO}_{4nm}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$. Composing with conjugation by this element in $\mathbf{PSO}_{4nm}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ will give the desired map

$$\begin{aligned}\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}(R) \colon \mathbf{PSp}(A_1, \psi_1)(R) \times \mathbf{PSp}(A_2, \psi_2)(R) &\to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)(R) \\ ([a], [b]) &\mapsto [a \otimes b]\end{aligned}$$

Then if $\varphi \colon R \to S$ is an algebra homomorphism, we will have

$$1_R \otimes [C^{-1}B] \mapsto 1_S \otimes [C^{-1}B]$$

under the map $\mathbf{PSO}_{4nm}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \to \mathbf{PSO}_{4nm}(S \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, and so there will be a diagram

$$\begin{tikzcd}
\mathbf{PSp}(A_1, \psi_1)(R) \times \mathbf{PSp}(A_2, \psi_2)(R) \arrow[r, "\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}(R)"] \arrow[d] & \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)(R) \arrow[d] \\
\mathbf{PSp}(A_1, \psi_1)(S) \times \mathbf{PSp}(A_2, \psi_2)(S) \arrow[r, "\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}(S)"] & \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)(S)
\end{tikzcd}$$

Hence these maps combine into the desired group scheme homomorphism $\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}$.

Now we describe how to construct any of the maps lying over the map we just constructed. First, consider the split version N,

$$\begin{tikzcd}
\mathbf{G}_1 \times \mathbf{G}_2 \arrow[r, "N"] \arrow[d, two heads] & \mathbf{H} \arrow[d, two heads] \\
\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \arrow[r, "K_{\mathbf{Sp},\mathrm{ad}}"] & \mathbf{PSO}_{4nm}
\end{tikzcd}$$

Over $R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$ the map $N(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is a Γ-morphism, and any choice of elements lying over the cocycles α_1 and α_2 will be mapped by N to elements lying over the cocycle $K_{\mathbf{Sp},\mathrm{ad}} \circ (\alpha_1, \alpha_2)$, and so all maps in this diagram will restrict to fixed points. After that, to correct the image we may conjugate in $\mathbf{H}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ by any element lying over $1 \otimes [C^{-1}B]$, they will all produce the same inner automorphism since the kernel of $\mathbf{H}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \twoheadrightarrow \mathbf{PSO}_{4nm}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ is central. Say we choose $\mathfrak{C}$. This will then produce a map which lies over $\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}(R)$. Finally, for the ring homomorphism $\varphi \colon R \to S$, the element $\mathbf{H}(\varphi)(\mathfrak{C})$ will lie over $\mathbf{PSO}_{4nm}(\varphi)(1 \otimes [C^{-1}B])$ and so all these maps assemble together to produce the desired group scheme homomorphism which lies over $\mathcal{K}_{\mathbf{Sp},\mathrm{ad}}$. ∎

Theorem 4.4.5. *Let (A_1, τ_1) and (A_2, τ_2) be central simple algebras of degrees $2n$ and $2m$ respectively with orthogonal involutions. There are group scheme homomorphisms*

$$\begin{aligned}
\mathcal{L}_{ee} \colon \mathbf{Spin}(A_1,\tau_1) \times \mathbf{Spin}(A_2,\tau_2) &\to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathcal{K}_{ee} \colon \mathbf{SO}(A_1,\tau_1) \times \mathbf{SO}(A_2,\tau_2) &\to \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathcal{N}_{ee} \colon \mathbf{PSO}(A_1,\tau_1) \times \mathbf{PSO}(A_2,\tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ or } m \text{ even} \\
\mathcal{N}'_{ee} \colon \mathbf{SO}(A_1,\tau_1) \times \mathbf{PSO}(A_2,\tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ and } m \text{ odd}
\end{aligned}$$

all lying above a map

$$\mathcal{K}_{ee,\mathrm{ad}} \colon \mathbf{PSO}(A_1,\tau_1) \times \mathbf{PSO}(A_2,\tau_2) \to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

which for $R \in \mathbf{Alg}_{\mathbb{F}}$ behaves on R-points as

$$\begin{aligned}
\mathcal{K}_{ee,\mathrm{ad}} \colon \mathbf{PSO}(A_1,\tau_1)(R) \times \mathbf{PSO}(A_2,\tau_2)(R) &\to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)(R) \\
([a],[b]) &\mapsto [a \otimes b]
\end{aligned}$$

Theorem 4.4.6. *Let (A_1, τ_1) and (A_2, τ_2) be central simple algebras of degrees $2n$ and $2m+1$ respectively with orthogonal involutions. There are group scheme homomorphisms*

$$\begin{aligned}
\mathcal{L}_{eo} \colon \mathbf{Spin}(A_1,\tau_1) \times \mathbf{Spin}(A_2,\tau_2) &\to \mathbf{Spin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathcal{K}_{eo} \colon \mathbf{SO}(A_1,\tau_1) \times \mathbf{SO}(A_2,\tau_2) &\to \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\
\mathcal{N}_{eo} \colon \mathbf{HSpin}(A_1,\tau_1) \times \mathbf{PSO}(A_2,\tau_2) &\hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \text{ when } n \text{ even}
\end{aligned}$$

all lying above a map

$$\mathcal{K}_{eo,\mathrm{ad}} \colon \mathbf{PSO}(A_1,\tau_1) \times \mathbf{PSO}(A_2,\tau_2) \to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

which for $R \in \mathbf{Alg}_{\mathbb{F}}$ behaves on R-points as

$$\begin{aligned}
\mathcal{K}_{eo,\mathrm{ad}} \colon \mathbf{PSO}(A_1,\tau_1)(R) \times \mathbf{PSO}(A_2,\tau_2)(R) &\to \mathbf{PSO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)(R) \\
([a],[b]) &\mapsto [a \otimes b]
\end{aligned}$$

Proof: The proof of the above two theorems is analogous to the proof of theorem 4.4.4. By first considering the tensor product on the level of central simple algebras, appropriate elements can be found which will correct the images of the maps after they are restricted to fixed points. Then by conjugating the groups of $R\otimes_{\mathbb{F}}\mathbb{F}_{\text{sep}}$-points either by these elements, or elements lying over them, we will produce the desired group scheme homomorphisms as maps between fixed points in split groups. ∎

The following last theorem is isolated from the above two since we wish to emphasize that it requires no special assumptions about the field $\mathbb{F}$. Recall that the map L'_{oo} is only defined when $\sqrt{2} \in \mathbb{F}$. Despite using L'_{oo} in the following construction, we will demonstrate how we may drop the assumption on $\mathbb{F}$.

Theorem 4.4.7. *Let (A_1, τ_1) and (A_2, τ_2) be central simple algebras of degrees $2n+1$ and $2m+1$ respectively with orthogonal involutions. There are group scheme homomorphisms*

$$\begin{aligned} \mathcal{L}_{oo}\colon\ \mathbf{Spin}(A_1,\tau_1)\times\mathbf{Spin}(A_2,\tau_2) &\to \mathbf{Spin}(A_1\otimes_{\mathbb{F}} A_2,\tau_1\otimes\tau_2) \\ \mathcal{K}_{oo}\colon\ \mathbf{SO}(A_1,\tau_1)\times\mathbf{SO}(A_2,\tau_2) &\to \mathbf{SO}(A_1\otimes_{\mathbb{F}} A_2,\tau_1\otimes\tau_2) \end{aligned}$$

both lying above a map

$$\mathcal{K}_{oo,\text{ad}}\colon\ \mathbf{PSO}(A_1,\tau_1)\times\mathbf{PSO}(A_2,\tau_2) \to \mathbf{PSO}(A_1\otimes_{\mathbb{F}} A_2,\tau_1\otimes\tau_2)$$

which for $R \in \mathbf{Alg}_{\mathbb{F}}$ behaves on R-points as

$$\begin{aligned} \mathcal{K}_{oo,\text{ad}}\colon\ \mathbf{PSO}(A_1,\tau_1)(R)\times\mathbf{PSO}(A_2,\tau_2)(R) &\to \mathbf{PSO}(A_1\otimes_{\mathbb{F}} A_2,\tau_1\otimes\tau_2)(R) \\ ([a],[b]) &\mapsto [a\otimes b] \end{aligned}$$

Proof: We again begin by considering the tensor product map on the level of $\mathbb{F}_{\text{sep}}$-algebras. Since we work on this level and $\mathbb{F}_{\text{sep}}$ will contain $\sqrt{2}$ regardless of whether $\mathbb{F}$ does, we may still conjugate by the P used to define K'_{oo}. Since this is not a hindrance, the rest of the proof can proceed analogously to the situations above and produce the desired maps. ∎

We end this section by noting that for every commutative diagram composed of split tensor product maps and central isogenies, there is an analogous diagram comprised of these new maps. For example, there is a diagram

$$\begin{tikzcd}
\mathbf{Sp}(A_1,\psi_1)\times\mathbf{Sp}(A_2,\psi_2) \arrow[r, "\mathcal{L}_{\mathbf{Sp}}"] \arrow[dd, two heads] & \mathbf{Spin}(A_1\otimes_{\mathbb{F}} A_2,\psi_1\otimes\psi_2) \arrow[d, two heads] \\
 & \mathbf{HSpin}(A_1\otimes_{\mathbb{F}} A_2,\psi_1\otimes\psi_2) \arrow[d, two heads] \\
\mathbf{PSp}(A_1,\psi_1)\times\mathbf{PSp}(A_2,\psi_2) \arrow[ru, hook, "\mathcal{N}_{\mathbf{Sp}}"] \arrow[r, hook, "\mathcal{K}_{\mathbf{Sp},\text{ad}}"] & \mathbf{PSO}(A_1\otimes_{\mathbb{F}} A_2,\psi_1\otimes\psi_2).
\end{tikzcd}$$

Chapter 5

Application to Cohomological Invariants

In section 1.4.1 we defined Galois cohomology sets and saw how sets like $H^1(\mathbb{F}, \mathbf{PGL}_n)$ could classify algebraic objects, in this case classifying the isomorphism classes of central simple algebras over $\mathbb{F}$ of degree n. If $\mathbf{G}$ is any linear algebraic group, the set $H^1(\mathbb{F}, \mathbf{G})$ classifies the isomorphism classes of $\mathbf{G}$-*torsors* over $\mathbb{F}$. A $\mathbf{G}$-torsor, also known as a *principal homogeneous space for* $\mathbf{G}$, over $\mathbb{F}$ is an $\mathbb{F}$-scheme which is equipped with a simply transitive action of $\mathbf{G}$. Mirroring ideas in topology, one method of understanding such torsors is to assign invariants to these spaces, which we could view as a function from $H^1(\mathbb{F}, \mathbf{G})$ to wherever the invariant lives. Further, we would like the invariant to respect scalar extensions by being functorial with respect to the field, and so we consider natural transformations from the functor $H^1(-, \mathbf{G})$.

5.1 Cohomological Invariants

In this section, we recount this general setup following [GMS03].

Definition 5.1.1. *Let* $\mathbf{G}$ *be an algebraic group and recall that there is a Galois cohomology functor* $H^1(-, \mathbf{G})\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Sets}$. *Let* $\mathcal{F}\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Ab}$ *be another covariant functor to the category of abelian groups. A* cohomological invariant of $\mathbf{G}$ with coefficients in $\mathcal{F}$ *is a natural transformation*

$$\Delta\colon H^1(-, \mathbf{G}) \to \mathcal{F}.$$

The set of all such invariants is denoted $\operatorname{Inv}(\mathbf{G}, \mathcal{F})$. *It inherits the structure of an abelian group from the category* $\mathbf{Ab}$.

Since the Galois cohomology functor is also functorial in the second argument, we know that any algebraic group morphism $\mathbf{G}_1 \to \mathbf{G}_2$ induces a natural transformation

$H^1(-, \mathbf{G}_1) \to H^1(-, \mathbf{G}_2)$. This allows us to pullback invariants of $\mathbf{G}_2$ to an invariant of $\mathbf{G}_1$, as stated formally in the lemma below.

Lemma 5.1.2. *Let* $\mathcal{F}\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Ab}$ *be a functor. Then* $\mathrm{Inv}(-, \mathcal{F})$ *is a contravariant functor from the category of linear algebraic groups to* $\mathbf{Ab}$*.*

The abelian group structure on $\mathrm{Inv}(\mathbf{G}, \mathcal{F})$ as well as on $\mathcal{F}$ itself, allow us to also define *normalized invariants.* These are invariants which map the trivial element to zero.

Definition 5.1.3. *Let* $\Delta \in \mathrm{Inv}(\mathbf{G}, \mathcal{F})$*. Then* Δ *is called* normalized *if*

$$\Delta(\mathbb{F})([1]) = 0 \in \mathcal{F}(\mathbb{F}).$$

The subgroup of all normalized invariants is denoted $\mathrm{Inv}(\mathbf{G}, \mathcal{F})_{\mathrm{norm}}$*.*

If $\mathbb{E}/\mathbb{F}$ is a field extension, since the induced map $H^1(\mathbb{F}, \mathbf{G}) \to H^1(\mathbb{E}, \mathbf{G})$ sends $[1]_{\mathbb{F}} \mapsto [1]_{\mathbb{E}}$, and $\mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{E})$ is a group homomorphism, any normalized invariant Δ will also have $\Delta(\mathbb{E})([1]_{\mathbb{E}}) = 0 \in \mathcal{F}(\mathbb{E})$. Normalized invariants hold the majority of the information in $\mathrm{Inv}(G, \mathcal{F})$, as every invariant only differs by a constant invariant from a normalized one.

Definition 5.1.4. *Let* $\Delta \in \mathrm{Inv}(\mathbf{G}, \mathcal{F})$*. Then* Δ *is called* constant *if there exists* $c \in \mathcal{F}(\mathbb{F})$ *such that, for all field extensions* $\mathbb{E}/\mathbb{F}$

$$\Delta(\mathbb{E})(x) = c_{\mathbb{E}}$$

for all $x \in H^1(\mathbb{E}, \mathbf{G})$*, where* $c_{\mathbb{E}}$ *is the image of* c *under* $\mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{E})$*. The subgroup of constant invariants is denoted* $\mathrm{Inv}(\mathbf{G}, \mathcal{F})_{\mathrm{const}}$*, and its identity is called the* trivial invariant*.*

A constant invariant is determined uniquely by an element of $\mathcal{F}(\mathbb{F})$, and so $\mathrm{Inv}(\mathbf{G}, \mathcal{F})_{\mathrm{const}} \cong \mathcal{F}(\mathbb{F})$ as groups. If $\Delta \in \mathrm{Inv}(\mathbf{G}, \mathcal{F})$ is any invariant, let $c = \Delta(\mathbb{F})([1])$ and let Δ_c be the associated constant invariant. Then $\Delta - \Delta_c$ is a normalized invariant and trivially $\Delta = \Delta_c + (\Delta - \Delta_c)$. Therefore the group of invariants decomposes into a direct sum of these two subgroups,

$$\mathrm{Inv}(\mathbf{G}, \mathcal{F}) = \mathrm{Inv}(\mathbf{G}, \mathcal{F})_{\mathrm{const}} \oplus \mathrm{Inv}(\mathbf{G}, \mathcal{F})_{\mathrm{norm}}.$$

Since the structure of the constant invariants is already known, attention is usually restricted to considering the group of normalized invariants. In order to say more, we now focus on some particular choices of functors $\mathcal{F}$.

5.1.1 The Functor $H^{d+1}(-,\mathbb{Q}/\mathbb{Z}(d))$

A common choice for the functor $\mathcal{F}$ used to define cohomological invariants is

$$H^{d+1}(-,\mathbb{Q}/\mathbb{Z}(d))\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Ab}.$$

To define this functor we begin by defining the functor $\mathbb{Q}/\mathbb{Z}(d)\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Ab}$, taking our definition from both [KMRT, §31.B] and [GMS03].

Definition 5.1.5. *Let n and d be positive integers. First, we define a functor*

$$\begin{aligned} \boldsymbol{\mu}_n^{\otimes d}\colon \mathbf{Fields}_{\mathbb{F}} &\to \mathbf{Ab} \\ \mathbb{E} &\mapsto \boldsymbol{\mu}_n(\mathbb{E})^{\otimes d} \end{aligned}$$

where the groups $\boldsymbol{\mu}_n(\mathbb{E})$ are tensored as $\mathbb{Z}$-modules. Letting n vary over all integers, these functors form a directed set when considered with the canonical maps $\boldsymbol{\mu}_n^{\otimes d} \to \boldsymbol{\mu}_m^{\otimes d}$ coming from the inclusions $\boldsymbol{\mu}_n(\mathbb{E}) \to \boldsymbol{\mu}_m(\mathbb{E})$ when n divides m. We then define

$$\begin{aligned} \mathbb{Q}/\mathbb{Z}(d)\colon \mathbf{Fields}_{\mathbb{F}} &\to \mathbf{Ab} \\ \mathbb{E} &\mapsto \operatorname*{colim}_{n\in\mathbb{N}} \boldsymbol{\mu}_n^{\otimes d}(\mathbb{E}) \end{aligned}$$

When $d = 1$, the colimit in the definition of $\mathbb{Q}/\mathbb{Z}(1)$ has the effect of a union, and we have that

$$\mathbb{Q}/\mathbb{Z}(1)(\mathbb{E}) = \{x \in \mathbb{E} \mid x^m = 1 \text{ for some } m \in \mathbb{N}\}$$

that is, $\mathbb{Q}/\mathbb{Z}(1)$ returns all the roots of unity of the field. There is also a decomposition of $\mathbb{Q}/\mathbb{Z}(d)$ into p-components for primes p. We have that

$$\mathbb{Q}/\mathbb{Z}(d)(\mathbb{E})\{p\} = \operatorname*{colim}_{m\in\mathbb{N}} \boldsymbol{\mu}_{p^m}^{\otimes d}(\mathbb{E})$$

where appending $-\{p\}$ denotes the subgroup of elements of order p^m for some $m \in \mathbb{N}$. These subgroups form subfunctors.

Definition 5.1.6. *Let d be a positive integer and let p be prime. We define a functor*

$$\begin{aligned} \mathbb{Q}/\mathbb{Z}(d)\{p\}\colon \mathbf{Fields}_{\mathbb{F}} &\to \mathbf{Ab} \\ \mathbb{E} &\mapsto \operatorname*{colim}_{m\in\mathbb{N}} \boldsymbol{\mu}_{p^m}^{\otimes d}(\mathbb{E}) \end{aligned}$$

which is a subfunctor of $\mathbb{Q}/\mathbb{Z}(d)$.

Then, since all elements of $\mathbb{Q}/\mathbb{Z}(d)(\mathbb{E})$ have finite order, we obtain

$$\mathbb{Q}/\mathbb{Z}(d)(\mathbb{E}) = \bigoplus_{p \text{ prime}} \mathbb{Q}/\mathbb{Z}(d)(\mathbb{E})\{p\}.$$

and so we can view the functor as having a similar decomposition $\mathbb{Q}/\mathbb{Z}(d) = \bigoplus_{p \text{ prime}} \mathbb{Q}/\mathbb{Z}(d)\{p\}$. One issue, however, is that when $\mathbb{F}$ has positive characteristic, say $\operatorname{char}(\mathbb{F}) = p$, the group $\mathbb{Q}/\mathbb{Z}(d)\{p\}(\mathbb{E})$ will not have 'enough' elements since $\boldsymbol{\mu}_{p^m}(\mathbb{E}_{\text{sep}}) = \{1\}$. To fix this, we build our own p-component using Milnor's K-groups.

Definition 5.1.7. *Let d be a positive integer. Then the d^{th}* Milnor K-group *is a functor*

$$\begin{aligned} K_d \colon \mathbf{Fields}_{\mathbb{F}} &\to \mathbf{Ab} \\ \mathbb{E} &\mapsto (\mathbb{E}^{\times})^{\otimes d}/I \end{aligned}$$

where $I = \langle a_1 \otimes a_2 \otimes \ldots \otimes a_d \mid a_i + a_j = 1$ for some $i \neq j\rangle$.

In particular for our later use, $K_1(\mathbb{E}) = \mathbb{E}^{\times}$. Using these above two functors, we can finally define what we mean by $H^{d+1}(-, \mathbb{Q}/\mathbb{Z}(d))$.

Definition 5.1.8. *Let $\mathbb{F}$ be a field with $\operatorname{char}(\mathbb{F}) = p$. We define a functor*

$$H^{d+1}(-, \mathbb{Q}/\mathbb{Z}(d)) \colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Ab}$$

which for $\mathbb{E}/\mathbb{F}$ a field extension gives

$$\mathbb{E} \mapsto \begin{cases} H^{d+1}(\operatorname{Gal}(\mathbb{E}_{\text{sep}}/\mathbb{E}), \mathbb{Q}/\mathbb{Z}(d)(\mathbb{E}_{\text{sep}})) & p = 0 \\ H^{d+1}(\operatorname{Gal}(\mathbb{E}_{\text{sep}}/\mathbb{E}), \mathbb{Q}/\mathbb{Z}(d)(\mathbb{E}_{\text{sep}})) \oplus H^2(\operatorname{Gal}(\mathbb{E}_{\text{sep}}/\mathbb{E}), K_d(\mathbb{E}_{\text{sep}}))\{p\} & p > 0 \end{cases}$$

where $\operatorname{Gal}(\mathbb{E}_{\text{sep}}/\mathbb{E})$ acts naturally on $\mathbb{Q}/\mathbb{Z}(d)(\mathbb{E}_{\text{sep}})$ and $K_d(\mathbb{E}_{\text{sep}})$.

There is a convenient isomorphisms between $H^2(-, \mathbb{Q}/\mathbb{Z}(1))$ and the Brauer group.

Proposition 5.1.9. *[KMRT, §29] There is an isomorphism of functors*

$$H^2(-, \mathbb{Q}/\mathbb{Z}(1)) \cong \operatorname{Br}.$$

Proof: Let $\mathbb{E}/\mathbb{F}$ be a field extension. By [GS17, 4.4.5], for each positive integer m which is prime to $\operatorname{char}(\mathbb{E})$, there is an isomorphism $H^2(\mathbb{E}, \boldsymbol{\mu}_m) \cong \{x \in \operatorname{Br}(\mathbb{E}) \mid x^m = 1\}$. Therefore, for a prime $p \neq \operatorname{char}(\mathbb{E})$ we have

$$\begin{aligned} H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1))\{p\} &= H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1)\{p\}) \\ &= H^2(\mathbb{E}, \operatorname*{colim}_{m \in \mathbb{N}} \boldsymbol{\mu}_{p^m}) \\ &= \operatorname*{colim}_{m \in \mathbb{N}} H^2(\mathbb{E}, \boldsymbol{\mu}_{p^m}) \\ &\cong \operatorname*{colim}_{m \in \mathbb{N}} \{x \in \operatorname{Br}(\mathbb{E}) \mid x^{(p^m)} = 1\} \end{aligned}$$

$$= \mathrm{Br}(\mathbb{E})\{p\}$$

When $p = \mathrm{char}(\mathbb{F})$ we use [GS17, 4.4.3] which states that there is an isomorphism $H^2(\mathbb{E}, \mathbb{G}_m) \cong \mathrm{Br}(\mathbb{E})$. Then since $K_1(\mathbb{E}_{\mathrm{sep}}) = \mathbb{E}_{\mathrm{sep}}^\times$ it is immediate that

$$H^2(\mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E}), K_d(\mathbb{E}_{\mathrm{sep}}))\{p\} = H^2(\mathbb{E}, \mathbb{G}_m)\{p\} \cong \mathrm{Br}(\mathbb{E})\{p\}.$$

To put it all together we use [GS17, 2.8.5] which gives us that $\mathrm{Br}(\mathbb{E})$ is a torsion group. Hence, when $\mathrm{char}(\mathbb{F}) = 0$ we will have that

$$\begin{aligned} H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1)) &= H^2(\mathbb{E}, \bigoplus_{p \text{ prime}} \mathbb{Q}/\mathbb{Z}(1)\{p\}) \\ &= \bigoplus_{p \text{ prime}} H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1))\{p\} \\ &\cong \bigoplus_{p \text{ prime}} \mathrm{Br}(\mathbb{E})\{p\} \\ &= \mathrm{Br}(\mathbb{E}) \end{aligned}$$

and when $\mathrm{char}(\mathbb{F}) = p > 0$ we have

$$\begin{aligned} H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1)) &= H^2(\mathbb{E}, \bigoplus_{q \text{ prime}} \mathbb{Q}/\mathbb{Z}(1)\{q\}) \oplus H^2(\mathrm{Gal}(\mathbb{E}_{\mathrm{sep}}/\mathbb{E}), K_d(\mathbb{E}_{\mathrm{sep}}))\{p\} \\ &\cong \bigoplus_{q \text{ prime}} H^2(\mathbb{E}, \mathbb{Q}/\mathbb{Z}(1))\{q\} \oplus \mathrm{Br}(\mathbb{E})\{p\} \\ &\cong \bigoplus_{\substack{q \text{ prime} \\ q \neq p}} \mathrm{Br}(\mathbb{E})\{q\} \oplus \{1\} \oplus \mathrm{Br}(\mathbb{E})\{p\} \\ &= \mathrm{Br}(\mathbb{E}). \end{aligned}$$

Finally, since the proofs of 4.4.7 and 4.4.9 from [GS17] use the cohomological long exact sequence to construct the isomorphisms, these constructions will be functorial in $\mathbb{E}$ and so this all fits together into the desired isomorphism of functors. ▮

Now that we have the functor of coefficients we wish to use, we give the invariants coming from this functor their own name.

Definition 5.1.10. *Let* $\mathbf{G}$ *be an algebraic group. The group of* normalized degree d cohomological invariants of $\mathbf{G}$ *is*

$$\mathrm{Inv}^d(\mathbf{G}) = \mathrm{Inv}(\mathbf{G}, H^d(-, \mathbb{Q}/\mathbb{Z}(d-1)))_{\mathrm{norm}}.$$

These invariants act functorially, with Inv^d being a contravariant functor. That is, given a group scheme map $\varphi\colon \mathbf{G} \to \mathbf{H}$, there is a natural map

$$\varphi^*\colon \mathrm{Inv}^d(\mathbf{H}) \to \mathrm{Inv}^d(\mathbf{G})$$

which sends an invariant $\Delta \in \mathrm{Inv}^d(\mathbf{H})$ to the pullback

$$\varphi^*(\Delta)\colon H^1(-, \mathbf{G}) \xrightarrow{\varphi'} H^1(-, \mathbf{H}) \xrightarrow{\Delta} H^d(-, \mathbb{Q}/\mathbb{Z}(d-1)).$$

5.1.2 Degree 2 Cohomological Invariants

We now turn our attention to the degree d invariants of split groups. The main case of interest for us will be degree 3 cohomological invariants, but we begin by recalling the description of degree 2 invariants since they are extremely useful in understanding degree 3 invariants.

Proposition 5.1.11. *[KMRT, 31.21] Let* $\mathbf{G}$ *be a semisimple algebraic group and consider the surjection* $\pi\colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ *from the simply connected group of the same type. Letting* $Z = \ker(\pi)$ *there is an isomorphism*

$$Z^* \cong \operatorname{Inv}^2(\mathbf{G})$$

where $Z^* = \operatorname{Hom}(Z, \mathbb{G}_m)$ *is the character group of* Z*.*

This proposition makes it easy to describe the degree 2 invariants of our favourite groups since, knowing that these invariants take values in the Brauer group and that Z^* will be finite, we can write down all the invariants. We give some examples of this now.

Example 5.1.12. Let $\mathbf{G}$ be any simply connected group. Then Z and Z^* are trivial so $\operatorname{Inv}^2(\mathbf{G})$ has one element, namely the trivial invariant

$$\Delta\colon H^1(-, \mathbf{G}) \to \mathrm{Br}$$

where for all $\mathbb{E}/\mathbb{F}$ a field extension, $\Delta(\mathbb{E})(x) = 0 \in \mathrm{Br}(\mathbb{E})$ for all $x \in H^1(\mathbb{E}, \mathbf{G})$.

Example 5.1.13. Consider the group $\mathbf{PGL}_n$. The kernel of $\mathbf{SL}_n \twoheadrightarrow \mathbf{PGL}_n$ is $\boldsymbol{\mu}_n$, and so $\operatorname{Inv}^2(\mathbf{PGL}_n) \cong \boldsymbol{\mu}_n^* = \mathbb{Z}/n\mathbb{Z}$. This cyclic group of invariants is generated by the invariant Δ given over a field extension $\mathbb{E}/\mathbb{F}$ by

$$\begin{aligned} \Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{PGL}_n) &\to \mathrm{Br}(\mathbb{E}) \\ [A] &\mapsto [A] \end{aligned}$$

where A is a central simple $\mathbb{E}$-algebra of degree n. The remaining invariants are

$$\begin{aligned} (m\Delta)(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{PGL}_n) &\to \mathrm{Br}(\mathbb{E}) \\ [A] &\mapsto [A^{\otimes m}] \end{aligned}$$

for $0 \leq m \leq n-1$. These do in fact form a cyclic group since the order of $[A] \in \mathrm{Br}(\mathbb{E})$ divides the degree of A, and so $n\Delta$ is the trivial invariant.

Example 5.1.14. Consider the group $\mathbf{PSp}_{2n}$. The kernel of $\mathbf{Sp}_{2n} \twoheadrightarrow \mathbf{PSp}_{2n}$ is $\boldsymbol{\mu}_2$, and so $\operatorname{Inv}^2(\mathbf{PSp}_{2n}) \cong \boldsymbol{\mu}_2^* = \mathbb{Z}/2\mathbb{Z}$. Hence this group of invariants only contains one non-trivial invariant Δ which is given over a field extension $\mathbb{E}/\mathbb{F}$ by

$$\begin{aligned}\Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{PSp}_{2n}) &\to \operatorname{Br}(\mathbb{E})\\ [(A,\psi)] &\mapsto [A]\end{aligned}$$

where (A,ψ) is a central simple algebra of degree $2n$ with symplectic involution.

Example 5.1.15. Consider the group $\mathbf{SO}_{2n+1} = \mathbf{PSO}_{2n+1}$. The kernel of $\mathbf{Spin}_{2n+1} \twoheadrightarrow \mathbf{PSO}_{2n+1}$ is $\boldsymbol{\mu}_2$, and so $\operatorname{Inv}^2(\mathbf{PSO}_{2n}) \cong \boldsymbol{\mu}_2^* = \mathbb{Z}/2\mathbb{Z}$. Hence this group of invariants also only contains one non-trivial invariant Δ which is given over a field extension $\mathbb{E}/\mathbb{F}$ by

$$\begin{aligned}\Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{PSO}_{2n}) &\to \operatorname{Br}(\mathbb{E})\\ [(A,\tau)] &\mapsto [C_0(A,\tau)] - [C_0(\mathrm{M}_{2n+1}(\mathbb{E}),\tau_0)]\end{aligned}$$

where (A,τ) is a central simple algebra of degree $2n+1$ with orthogonal involution, and $C_0(A,\tau)$ is its even Clifford algebra. The class of $C_0(\mathrm{M}_{2n+1}(\mathbb{E}),\tau_0)$ is subtracted so that the invariant is normalized since $[(\mathrm{M}_{2n+1}(\mathbb{E}),\tau_0)]$ is the trivial class on the left. In this case we also have the invariant $[(A,\tau)] \mapsto [A]$, but this coincides with the trivial invariant since by [KMRT, 2.8] any central simple algebra of odd degree with involution of the first kind is split.

Example 5.1.16. Consider the group $\mathbf{SO}_{2n}$. The kernel of $\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{SO}_{2n}$ is $\boldsymbol{\mu}_2$, and so $\operatorname{Inv}^2(\mathbf{SO}_{2n}) \cong \boldsymbol{\mu}_2^* = \mathbb{Z}/2\mathbb{Z}$. The non-trivial invariant Δ is given over a field extension $\mathbb{E}/\mathbb{F}$ as follows. Under the induced map $H^1(\mathbb{E},\mathbf{SO}_{2n}) \to H^1(\mathbb{E},\mathbf{PSO}_{2n})$, any element $Y \in H^1(\mathbb{E},\mathbf{SO}_{2n})$ maps to the isomorphism class of a central simple algebra of degree $2n$ with orthogonal involution, say $Y \mapsto [(A(Y),\tau)]$. Then we have

$$\begin{aligned}\Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{SO}_{2n}) &\to \operatorname{Br}(\mathbb{E})\\ Y &\mapsto [A(Y)]\end{aligned}$$

Example 5.1.17. Let n be even and consider the group $\mathbf{HSpin}_{2n}$. The kernel of $\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{HSpin}_{2n}$ is $\boldsymbol{\mu}_2$, and so $\operatorname{Inv}^2(\mathbf{HSpin}_{2n}) \cong \boldsymbol{\mu}_2^* = \mathbb{Z}/2\mathbb{Z}$. The non-trivial invariant Δ is given over a field extension $\mathbb{E}/\mathbb{F}$ similarly as above. Any element $Y \in H^1(\mathbb{E},\mathbf{HSpin}_{2n})$ maps via $H^1(\mathbb{E},\mathbf{HSpin}_{2n}) \to H^1(\mathbb{E},\mathbf{PSO}_{2n})$ to an isomorphism class of a central simple algebra of degree $2n$ with orthogonal involution, say $Y \mapsto [(A(Y),\tau)]$. Then we have

$$\begin{aligned}\Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{HSpin}_{2n}) &\to \operatorname{Br}(\mathbb{E})\\ Y &\mapsto [A(Y)]\end{aligned}$$

For the next examples, we recall a fact about the class of Clifford algebras in the Brauer group.

Theorem 5.1.18. *Let* (A, τ) *be a central simple* $\mathbb{F}$*-algebra of degree* $2n$ *with orthogonal involution of trivial discriminant. Then, the Clifford algebra decomposes as* $C(A, \tau) = C^+(A, \tau) \times C^-(A, \tau)$ *and in the Brauer group* $\mathrm{Br}(\mathbb{F})$ *we have*

$$\begin{aligned} [C^+(A, \tau)] + [C^-(A, \tau)] &= [A] \\ 2[C^\pm(A, \tau)] &= 0 \end{aligned}$$

if n *is even. Then when* n *is odd we have*

$$\begin{aligned} [C^+(A, \tau)] + [C^-(A, \tau)] &= 0 \\ 2[C^\pm(A, \tau)] &= [A]. \end{aligned}$$

Proof: These statements all follow from [KMRT, 9.12], with the first and third equalities being directly stated there. As mentioned in the proof of 9.12, when n is even the canonical involution on $C(A, \tau)$ is type 1, therefore the involutions on $C^\pm(A, \tau)$ are type 1, and so they have order two in the Brauer group, yielding the second equality. The fourth equality is proved at the bottom of [KMRT, pg. 113]. ▮

Example 5.1.19. Consider the group $\mathbf{PSO}_{2n}$ where n is even. The kernel of $\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{PSO}_{2n}$ is $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, and so $\mathrm{Inv}^2(\mathbf{PSO}_{2n}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. An element $(a, b) \in \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ corresponds to the invariant $\Delta\colon H^1(-, \mathbf{PSO}_{2n}) \to \mathrm{Br}(-)$ which is given over a field extension $\mathbb{E}/\mathbb{F}$ by

$$[(A, \tau)] \mapsto a\Big([C^+(A, \tau)] - [C^+(\mathrm{M}_n(\mathbb{E}), \tau_0)]\Big) + b\Big([C^-(A, \tau)] - [C^-(\mathrm{M}_n(\mathbb{E}), \tau_0)]\Big).$$

Example 5.1.20. Consider the group $\mathbf{PSO}_{2n}$ where n is odd. The kernel of $\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{PSO}_{2n}$ is $\boldsymbol{\mu}_4$, and so $\mathrm{Inv}^2(\mathbf{PSO}_{2n}) \cong \mathbb{Z}/4\mathbb{Z}$. An element $a \in \mathbb{Z}/4\mathbb{Z}$ corresponds to the invariant which is given over a field extension $\mathbb{E}/\mathbb{F}$ by

$$\begin{aligned} \Delta(\mathbb{E})\colon H^1(\mathbb{E}, \mathbf{PSO}_{2n}) &\to \mathrm{Br}(\mathbb{E}) \\ [(A, \tau)] &\mapsto a([C^+(A, \tau)] - [C^+(\mathrm{M}_n(\mathbb{E}), \tau_0)]). \end{aligned}$$

5.1.3 Degree 3 Cohomological Invariants

We now move on to describe the group of degree 3 normalized invariants, $\mathrm{Inv}^3(\mathbf{G})$. Our thorough understanding of degree two invariants is used to define two groups related to $\mathrm{Inv}^3(\mathbf{G})$. The first uses the Galois cohomology cup product

$$\cup\colon H^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \times H^2(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \to H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2)).$$

Since there are inclusions $\boldsymbol{\mu}_n \to \mathbb{Q}/\mathbb{Z}$, there are also inclusions

$$\mathbb{F}^\times/(\mathbb{F}^\times)^n \cong H^1(\mathbb{F}, \boldsymbol{\mu}_n) \to H^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}).$$

We often use these inclusion to talk about cup products involving $(c) \in \mathbb{F}^\times/(\mathbb{F}^\times)^n$.

Definition 5.1.21. *An invariant* $\Delta \in \operatorname{Inv}^3(\mathbf{G})$ *is called* decomposable *if there exist* $c_i \in H^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})$ *and* $\Delta_i \in \operatorname{Inv}^2(\mathbf{G})$ *such that for all field extensions* $\mathbb{E}/\mathbb{F}$ *we have*

$$\Delta(\mathbb{E}) = \sum_{i=1}^{k} c_i \cup \Delta_i(\mathbb{E})$$

where we abuse notation and also write $c_i \in H^1(\mathbb{E}, \mathbb{Q}/\mathbb{Z})$ *for the images of the previous* c_i*. The group of all decomposable invariants is denoted* $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}}$.

The decomposable invariants are functorial in that any group scheme map $\mathbf{G} \to \mathbf{H}$ induced a map $\operatorname{Inv}^3(\mathbf{H})_{\mathrm{dec}} \to \operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}}$ which is the restriction of the usual map $\operatorname{Inv}^3(\mathbf{H}) \to \operatorname{Inv}^3(\mathbf{G})$ given by pullbacks. We now give some examples of the decomposable invariants for our semisimple groups.

Example 5.1.22. Let $\mathbf{G}$ be a simply connected group. Since $\operatorname{Inv}^2(\mathbf{G}) = 0$, we also have that $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}} = 0$.

The following examples involving adjoint groups are taken from [Mer16], and the case of **HSpin** is taken from [BR14]. All the following examples follow the pattern that $\operatorname{Inv}^3(\mathbf{G})_{\mathrm{dec}} \cong \operatorname{Inv}^2(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{F}^\times$.

Example 5.1.23. Consider the group $\mathbf{PGL}_n$. Then $\operatorname{Inv}^3(\mathbf{PGL}_n)_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^n$, and for an element $(c) \in \mathbb{F}^\times/(\mathbb{F}^\times)^n$ the invariant is given by

$$\begin{aligned} (c) \colon H^1(-, \mathbf{PGL}_n) &\to H^3(-, \mathbb{Q}/\mathbb{Z}(2)) \\ [A] &\mapsto (c) \cup [A] \end{aligned}$$

Note that by our definition the invariant $[A] \mapsto (c) \cup m[A]$ is also decomposable, but we have that $(c) \cup m[A] = (c^m) \cup [A]$, and so such an invariant corresponds to the element $(c^m) \in \mathbb{F}^\times/(\mathbb{F}^\times)^n$.

Example 5.1.24. Consider the group $\mathbf{PSp}_{2n}$. Then $\operatorname{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2$, and for an element (c) the invariant is given by

$$\begin{aligned} (c) \colon H^1(-, \mathbf{PSp}_{2n}) &\to H^3(-, \mathbb{Q}/\mathbb{Z}(2)) \\ [(A, \psi)] &\mapsto (c) \cup [A] \end{aligned}$$

Example 5.1.25. Consider the group $\mathbf{SO}_{2n+1} = \mathbf{PSO}_{2n+1}$. Then $\mathrm{Inv}^3(\mathbf{PSO}_{2n+1})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2$, and for an element (c) the invariant is given by

$$\begin{aligned}(c)\colon H^1(-,\mathbf{PSO}_{2n}) &\to H^3(-,\mathbb{Q}/\mathbb{Z}(2))\\ [(A,\tau)] &\mapsto (c)\cup([C(A,\tau)]-[C(\mathrm{M}_n(-),\tau_0)])\end{aligned}$$

Example 5.1.26. Consider the group $\mathbf{SO}_{2n}$. Then $\mathrm{Inv}^3(\mathbf{SO}_{2n})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2$, and for an element (c) the invariant is given by

$$\begin{aligned}(c)\colon H^1(-,\mathbf{SO}_{2n}) &\to H^3(-,\mathbb{Q}/\mathbb{Z}(2))\\ Y &\mapsto (c)\cup[A(Y)]\end{aligned}$$

where $[A(Y)]$ is the image of Y under $H^1(-,\mathbf{SO}_{2n}) \to H^1(-,\mathbf{PSO}_{2n})$.

Example 5.1.27. Consider $\mathbf{HSpin}_{2n}$. Then $\mathrm{Inv}^3(\mathbf{HSpin}_{2n})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2$, and for an element (c) the invariant is given by

$$\begin{aligned}(c)\colon H^1(-,\mathbf{HSpin}_{2n}) &\to H^3(-,\mathbb{Q}/\mathbb{Z}(2))\\ Y &\mapsto (c)\cup[A(Y)]\end{aligned}$$

where $[A(Y)]$ is the image of Y under $H^1(-,\mathbf{HSpin}_{2n}) \to H^1(-,\mathbf{PSO}_{2n})$.

Example 5.1.28. Consider the group $\mathbf{PSO}_{2n}$ when n is even. Then $\mathrm{Inv}^3(\mathbf{PSO}_{2n})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2$, and for an element $((c_1),(c_2))$ the invariant $((c_1),(c_2))\colon H^1(-,\mathbf{PSO}_{2n}) \to H^3(-,\mathbb{Q}/\mathbb{Z}(2))$ is given by

$$[(A,\tau)] \mapsto (c_1)\cup([C^+(A,\tau)]-[C^+(\mathrm{M}_{2n}(-),\tau_0)])+(c_2)\cup([C^-(A,\tau)]-[C^-(\mathrm{M}_{2n}(-),\tau_0)])$$

Example 5.1.29. Consider the group $\mathbf{PSO}_{2n}$ when n is odd. Then $\mathrm{Inv}^3(\mathbf{PSO}_{2n})_{\mathrm{dec}} \cong \mathbb{F}^\times/(\mathbb{F}^\times)^4$, and for an element (c) the invariant is given by

$$\begin{aligned}(c)\colon H^1(-,\mathbf{PSO}_{2n}) &\to H^3(-,\mathbb{Q}/\mathbb{Z}(2))\\ [(A,\tau)] &\mapsto (c)\cup([C^+(A,\tau)]-[C^+(\mathrm{M}_n(-),\tau_0)]).\end{aligned}$$

In order to understand the rest of the degree three invariants, we define the second related group.

Definition 5.1.30. *Let* $\mathbf{G}$ *be a semisimple groups. The group of* indecomposable invariants *is the quotient*

$$\mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}} = \frac{\mathrm{Inv}^3(\mathbf{G})}{\mathrm{Inv}^3(\mathbf{G})_{\mathrm{dec}}}.$$

Despite the name, elements of this group are not invariants in the sense of natural transformations, but are of course equivalence classes of invariants. This construction as well is functorial. Given a group scheme homomorphism $\mathbf{G} \to \mathbf{H}$, there will be a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Inv}^3(\mathbf{H})_{\mathrm{dec}} \arrow[r] \arrow[d] & \mathrm{Inv}^3(\mathbf{H}) \arrow[r] \arrow[d] & \mathrm{Inv}^3(\mathbf{H})_{\mathrm{ind}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Inv}^3(\mathbf{G})_{\mathrm{dec}} \arrow[r] & \mathrm{Inv}^3(\mathbf{G}) \arrow[r] & \mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}} \arrow[r] & 0
\end{tikzcd}$$

We now recall some known cases of the indecomposable invariants.

Example 5.1.31. Let $\mathbf{G}$ be a simply connected group. Then $\mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}} = \mathrm{Inv}^3(\mathbf{G})$. We mention that in this case, the group $\mathrm{Inv}^3(\mathbf{G})$ is a finite cyclic group generated by the *Rost Invariant* of $\mathbf{G}$. See [GMS03, §9].

Example 5.1.32. The following groups of indecomposable invariants were computed in [Mer16], with the **HSpin** case coming from [BR14].

$$\begin{aligned}
\mathrm{Inv}^3(\mathbf{PGL}_n)_{\mathrm{ind}} &= 0 \\
\mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{ind}} &\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \pmod 4 \\ 0 & n \not\equiv 0 \pmod 4 \end{cases} \\
\mathrm{Inv}^3(\mathbf{PSO}_{2n+1})_{\mathrm{ind}} &= 0 \\
\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}} &\cong \begin{cases} 0 & n = 2, \text{ or } n \text{ is odd and } n \neq 1 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4, n \neq 2 \\ \mathbb{Z}/4\mathbb{Z} & n \equiv 0 \pmod 4 \end{cases} \\
\mathrm{Inv}^3(\mathbf{PSO}_{2n})_{\mathrm{ind}} &\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \pmod 4 \\ 0 & n \not\equiv 0 \pmod 4 \end{cases}
\end{aligned}$$

In the cases above where $\mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}} = 0$, we obtain a complete description of the degree three normalized invariants since every such invariant is decomposable. For $\mathbf{PSp}_{2n}$ when $n \equiv 0 \pmod 4$, it was shown in [Mer16] that the exact sequence

$$1 \to \mathrm{Inv}^3(\mathbf{G})_{\mathrm{dec}} \to \mathrm{Inv}^3(\mathbf{G}) \to \mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}} \to 1$$

splits, and therefore we also have a description of the normalized invariants as $\mathrm{Inv}^3(\mathbf{PSp}_{2n}) \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{Z}/2\mathbb{Z}$. In [BR14] it was shown that the sequence involving $\mathbf{HSpin}_{16}$ splits, and so we have $\mathrm{Inv}^3(\mathbf{HSpin}_{16}) \cong \mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{Z}/4\mathbb{Z}$. Later, in section 5, we recall a result of [Rue20] which shows that there is a similar splitting and description of $\mathrm{Inv}^3(\mathbf{HSpin}_{4n})$ for all n.

5.2 Application to Half-spin

In this brief section, we retell the main result of [Rue20], which uses the existence of our new split tensor product maps to compute the normalized degree three invariants of $\mathbf{HSpin}_{4n}$ for all n. As just outlined in 5.1.3, the constructions $\mathrm{Inv}^3(-)$, $\mathrm{Inv}^3(-)_{\mathrm{dec}}$, and $\mathrm{Inv}^3(-)_{\mathrm{ind}}$ are covariant functors. Therefore, the split tensor product map

$$N_{\mathbf{Sp}} \colon \mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \to \mathbf{HSpin}_{4nm}$$

of proposition 4.3.6, which exists when n or m is even, induces maps between groups of degree three invariants. The known groups which play a roll here are the following

$$\begin{aligned}
\mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{dec}} &\cong \mathbb{F}^\times/(\mathbb{F}^\times)^2 \\
\mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{ind}} &\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \pmod 4 \\ 0 & n \not\equiv 0 \pmod 4 \end{cases} \\
\mathrm{Inv}^3(\mathbf{PSp}_{2n}) &= \mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{dec}} \oplus \mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{ind}} \\
\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{dec}} &\cong \mathbb{F}^\times/(\mathbb{F}^\times)^2 \\
\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}} &\cong \begin{cases} 0 & n = 2, \text{ or } n \text{ is odd and } n \neq 1 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4, n \neq 2 \\ \mathbb{Z}/4\mathbb{Z} & n \equiv 0 \pmod 4 \end{cases} .
\end{aligned}$$

In addition, we have that $\mathrm{Inv}^3(\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m}) = \mathrm{Inv}^3(\mathbf{PSp}_{2n}) \oplus \mathrm{Inv}^3(\mathbf{PSp}_{2m})$ and likewise for the decomposable and indecomposable invariants. We first describe the induced map on decomposable invariants.

Lemma 5.2.1. *[Rue20, Lemma 14] Let n or m be even and consider the map $N_{\mathbf{Sp}}$. The induced map on decomposable degree three invariants*

$$N^*_{\mathbf{Sp}} \colon \mathrm{Inv}^3(\mathbf{HSpin}_{4nm})_{\mathrm{dec}} \to \mathrm{Inv}^3(\mathbf{PSp}_{2n})_{\mathrm{dec}} \oplus \mathrm{Inv}^3(\mathbf{PSp}_{2m})_{\mathrm{dec}}$$

is the diagonal map

$$\begin{aligned}
\mathbb{F}^\times/(\mathbb{F}^\times)^2 &\to \mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2 \\
(c) &\mapsto ((c),(c))
\end{aligned}$$

In particular, $N^*_{\mathbf{Sp}}$ is injective.

Proof: The crux of the argument comes from the existence of the commutative diagram

$$\begin{tikzcd}
& \mathbf{HSpin}_{4nm} \arrow[d, two heads] \\
\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \arrow[ur, hook, "N_{\mathbf{Sp}}"] \arrow[r, "K_{\mathbf{Sp},\mathrm{ad}}"] & \mathbf{PSO}_{4nm}
\end{tikzcd}$$

which induced a commutative diagram on first cohomology sets

$$\begin{tikzcd}
 & H^1(-,\mathbf{HSpin}_{4nm}) \arrow[d] \\
H^1(-,\mathbf{PSp}_{2n}\times\mathbf{PSp}_{2m}) \arrow[r] \arrow[ur] & H^1(-,\mathbf{PSO}_{4nm}).
\end{tikzcd}$$

By proposition 4.4.2, the images of cohomology classes in the above diagram will be

$$\begin{tikzcd}
 & Y \arrow[d, mapsto] \\
([(A_1,\psi_1)],[(A_2,\psi_2)]) \arrow[r, mapsto] \arrow[ur, mapsto] & [(A_1\otimes_{\mathbb{F}} A_2, \psi_1\otimes\psi_2)].
\end{tikzcd}$$

Now let $(c)\in \mathrm{Inv}^2(\mathbf{HSpin}_{4nm})$ and consider its pullback $N^*_{\mathbf{Sp}}(c)$. Applying it to a pair in $H^1(-,\mathbf{PSp}_{2n}\times\mathbf{PSp}_{2m})$ we get

$$\begin{aligned}
N^*_{\mathbf{Sp}}(c)([(A_1,\psi_1)],[(A_2,\psi_2)]) &= (c)(Y)\\
&= (c)\cup[A_1\otimes_{\mathbb{F}} A_2]\\
&= (c)\cup([A_1]+[A_2])\\
&= (c)\cup[A_1]+(c)\cup[A_2]\\
&= ((c),(c))([(A_1,\psi_1)],[(A_2,\psi_2)]).
\end{aligned}$$

Therefore the map $N^*_{\mathbf{Sp}}$ is as claimed. ∎

This was the last ingredient needed for the following theorem. Beyond the existence of $N_{\mathbf{Sp}}$, the proof is entirely group theoretic.

Theorem 5.2.2. *[Rue20, Theorem 15] For an integer $n\geq 2$, the exact sequence*

$$0\to \mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{dec}}\to \mathrm{Inv}^3(\mathbf{HSpin}_{4n})\to \mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}}\to 0$$

is split. Therefore we can describe the degree three normalized invariants as

$$\mathrm{Inv}^3(\mathbf{HSpin}_{4n})\cong\begin{cases}\mathbb{F}^\times/(\mathbb{F}^\times)^2 & n \text{ is odd, or } n=2\\ \mathbb{F}^\times/(\mathbb{F}^\times)^2\oplus\mathbb{Z}/2\mathbb{Z} & n\equiv 2 \pmod 4, n\neq 2\\ \mathbb{F}^\times/(\mathbb{F}^\times)^2\oplus\mathbb{Z}/4\mathbb{Z} & n\equiv 0 \pmod 4\end{cases}.$$

Proof: In the case when n is odd or $n=2$ the result is immediate since $\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}}\cong 0$. For the remaining cases we use the commutative diagram coming from $N_{\mathbf{Sp}}$,

$$\begin{tikzcd}
\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{dec}} \arrow[r, hook] \arrow[d] & \mathrm{Inv}^3(\mathbf{HSpin}_{4n}) \arrow[r, two heads] \arrow[d] & \mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}} \arrow[d] \\
\mathrm{Inv}^3(\mathbf{G})_{\mathrm{dec}} \arrow[r, hook] & \mathrm{Inv}^3(\mathbf{G}) \arrow[r, two heads] & \mathrm{Inv}^3(\mathbf{G})_{\mathrm{ind}}
\end{tikzcd}$$

where $\mathbf{G} = \mathbf{PSp}_{2m_1} \times \mathbf{PSp}_{2m_2}$ and we choose m_1 and m_2 as are convenient.

For the $n \equiv 2 \pmod 4$ and $n \neq 2$ case, there is an odd integer m such that $n = 2m$, and we can use the map

$$N_{\mathbf{Sp}} \colon \mathbf{PSp}_4 \times \mathbf{PSp}_{2m} \to \mathbf{HSpin}_{4n}\,.$$

This gives the following version of the above commutative diagram

$$\begin{tikzcd}
\mathbb{F}^\times/(\mathbb{F}^\times)^2 \arrow[r, hook] \arrow[d, hook] & \mathrm{Inv}^4(\mathbf{HSpin}_{4n}) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[d] \\
\mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2 \arrow[r, hook] & \mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2 \arrow[r, two heads] & 0
\end{tikzcd}$$

Here we argue that $\Delta^2 = 0$ for all $\Delta \in \mathrm{Inv}^3(\mathbf{HSpin}_{4n})$, and thus any preimage of $1 \in \mathbb{Z}/2\mathbb{Z}$ produces a desired splitting. To see this, consider any such Δ and note that the image of $2\Delta \in \mathbb{Z}/2\mathbb{Z}$ will be 0, therefore by exactness of the top row we have that 2Δ is a decomposable invariant. Also, since every element in $\mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2$ has order dividing 2, we have

$$\begin{tikzcd}
2\Delta \arrow[r, equal] \arrow[d, mapsto] \arrow[dr, mapsto, dashed] & 2\Delta \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
Y \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

where the image down the dotted arrow is the image along an injection. Therefore $2\Delta = 0$ as desired.

In the last case, when $n \equiv 0 \pmod 4$ there is an even integer m such that $n = 2m$. We again use the map

$$N_{\mathbf{Sp}} \colon \mathbf{PSp}_4 \times \mathbf{PSp}_{2m} \to \mathbf{HSpin}_{4n}$$

but now we may have $\mathrm{Inv}^4(\mathbf{PSp}_{2m})_{\mathrm{ind}} \cong 0$ or $\mathrm{Inv}^4(\mathbf{PSp}_{2m})_{\mathrm{ind}} \cong \mathbb{Z}/2\mathbb{Z}$ depending on m. Our commutative diagram becomes

$$\begin{tikzcd}
\mathbb{F}^\times/(\mathbb{F}^\times)^2 \arrow[r, hook] \arrow[d, hook] & \mathrm{Inv}^4(\mathbf{HSpin}_{4n}) \arrow[r, two heads] \arrow[d] & \mathbb{Z}/4\mathbb{Z} \arrow[d] \\
\mathbb{F}^\times/(\mathbb{F}^\times)^2 \oplus \mathbb{F}^\times/(\mathbb{F}^\times)^2 \arrow[r, hook] & \mathrm{Inv}^3(\mathbf{PSp}_4 \times \mathbf{PSp}_{2m}) \arrow[r, two heads] & 0 \text{ or } \mathbb{Z}/2\mathbb{Z}.
\end{tikzcd}$$

The technique is the same, we show that $4\Delta = 0$ for any $\Delta \in \mathrm{Inv}^3(\mathbf{HSpin}_{4n})$, meaning that any preimage of $1 \in \mathbb{Z}/4\mathbb{Z}$ produces a splitting. Let Δ be such an invariant. Since 4Δ maps to 0 in $\mathbb{Z}/4\mathbb{Z}$, it is a decomposable invariant. Now, using the exactness of the bottom row, for any $\Delta' \in \mathrm{Inv}^3(\mathbf{PSp}_4 \times \mathbf{PSp}_{2m})$, in both cases $2\Delta'$ will be decomposable and so $4\Delta' = 0$. Therefore 4Δ also goes to 0 down the middle map. So we have a similar diagram of images

$$\begin{tikzcd}
4\Delta \arrow[r, equal] \arrow[d, mapsto] \arrow[dr, dashed, mapsto] & 4\Delta \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
Y \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

where the dotted arrow is the image along an injection, and so $4\Delta = 0$. All together, this completes the proof. ▮

Remark 5.2.3. The above theorem uses a split tensor product map to compute the normalized degree three invariants of the split group $\mathbf{HSpin}_{4n}$. The catalogue of results in section 5.1.3 also only deal with split groups, however analogues of many of these computation have been completed for non-split groups, for example in [Mer16]. However, a necessary ingredient for the above theorem was the computation of $\mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\mathrm{ind}}$ from [BR14], which was only done in the split case. The ambition is that, if the computation of $\mathrm{Inv}^3(\mathbf{HSpin}(A,\tau))_{\mathrm{ind}}$ is completed, then the existence of a twisted tensor product map

$$\mathcal{N}_{\mathbf{Sp}} \colon \mathbf{PSp}(A_1,\psi_1) \times \mathbf{PSp}(A_2,\psi_2) \to \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$$

or perhaps another twisted map, will be sufficient to complete the computation of $\mathrm{Inv}^3(\mathbf{HSpin}(A,\tau))$.

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List of Symbols

$\mathbf{Ab}$	Category of Abelian Groups	4
$\mathbf{Aff}_{\mathbb{F}}$	Category of Affine Schemes over $\mathbb{F}$	44
$\mathbf{Alg}_{\mathbb{F}}$	Category of Unital, Associative, Commutative $\mathbb{F}$-algebras	14
$\mathrm{Br}(-)$	Brauer Group	4
$\mathbf{E}$	Chevalley Group	33
$\mathrm{Cov}(\mathcal{C})$	Coverings of a Site	72
Δ	Cohomological Invariant	113
Δ	Simple System	28
disc	Discriminant	8
$\mathbb{E}$	Field Extension of $\mathbb{F}$	2
$\mathbb{F}$	Arbitrary Field of Characteristic Not 2.	1
$\mathbb{F}[\mathbf{G}]$	Hopf Algebra of $\mathbf{G}$	69
$\mathbb{F}_{\mathrm{alg}}$	Algebraic closure of $\mathbb{F}$	23
$\mathbf{Fields}_{\mathbb{F}}$	Category of Field Extensions of $\mathbb{F}$	4
$\mathbb{F}_{\mathrm{sep}}$	Separable Closure of $\mathbb{F}$	3
$\mathcal{F}$,$\mathcal{G}$	Functor	72
$\mathbb{G}_a$	Additive Group	15
Γ	Absolute Galois Group $\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$	48
$\mathbf{G}$,$\mathbf{H}$	Linear Algebraic Group	14
$\mathbf{GL}$	General Linear Group	16
$\mathbb{G}_m$	Multiplicative Group	15
$\mathbf{Grp}$	Category of Groups	14
H	Hopf Algebra	13
$\mathbf{HSpin}$	Half-spin Group	21
$\mathrm{Inv}(\mathbf{G},\mathcal{F})$	Group of Cohomological Invariants	113
$\mathrm{Inv}^d(\mathbf{G})$	Degree d Cohomological Invariants	117
K	Kronecker Product for Matrix Subgroups	90
K_{ad}	Adjoint Kronecker Product	104
$\mathrm{Lie}(-)$	Lie Algebra	24
L	Lifted Kronecker Product	93
$\mathfrak{M}$	Maximal Ideal	80
N	Induced Kronecker Product	103
Nrd	Reduced Norm	3

$\mathbf{O}$	Orthogonal Group	17
τ	Orthogonal Involution	6
$\mathbf{PGL}$	Projective General Linear Group	18
$\mathbf{PGO}$	Projective General Orthogonal Group	19
$\mathbf{PSO}$	Projective Special Orthogonal Group	19
$\mathbf{PSp}$	Projective Symplectic Group	19
$\mathbb{Q}/\mathbb{Z}$	Roots of Unity Functor	115
Λ_r	Root Lattice	30
Φ	Root System	28
$\boldsymbol{\mu}_n$	n^{th} Roots of Unity	15
Sand	Sandwich Map	9
$\mathbf{Sets}$	Category of Sets	44
$\mathbf{SL}$	Special Linear Group	16
$\mathbf{SO}$	Special Orthogonal Group	17
$\mathbf{Sp}$	Symplectic Group	17
$\mathbf{Spin}$	Spin Group	20
ψ	Symplectic Involution	6
$\mathbf{T}$	Algebraic Torus	23
$\mathcal{K}$	Twisted Kronecker Product	109
$\mathcal{K}_{\mathrm{ad}}$	Twisted Adjoint Kronecker Product	109
$\mathcal{L}$	Twisted Lifted Kronecker Product	109
$\mathcal{N}$	Twisted Induced Kronecker Product	109
Trd	Reduced Trace	3
$\mathcal{V}(A)$	Vector Space of a Central Simple Algebra	9
Λ_w	Weight Lattice	30
A	Central Simple Algebra	1
A^{op}	Opposite Algebra	2
$B(A,\tau)$	Clifford Bimodule of C.S.A. with Orthogonal Involution	12
$B(V,q)$	Clifford Bimodule of a Quadratic Space	8
$C(A,\tau)$	Clifford Algebra of C.S.A. with Orthogonal Involution	9
$C(V,q)$	Clifford Algebra of a Quadratic Space	7
$C_0(V,q)$	Even Clifford Algebra of a Quadratic Space	7
$C_1(V,q)$	Odd Clifford Algebra of a Quadratic Space	7
$H^n(-,-)$	Galois Cohomology Set	45
$h_\alpha(-)$	Chevalley Element	33
R^{sh}	Strict Henselization of a Local Ring	80
S_n	Symmetric Group on n Elements	11
$w_\alpha(-)$	Chevalley Element	33
X^Γ,X^{Γ_α}	Fixed Points of Γ-actions	49
$x_\alpha(-)$	Chevalley Generator	33
$x_{(1)} \otimes \ldots \otimes x_{(n)}$	Sweedler Notation for Comultiplication	13

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