THE EXACT SPANNING RATIO OF THE PARALLELOGRAM DELAUNAY GRAPH

SANDRINE NJOO

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of
Master of Computer Science
School of Electrical Engineering and Computer Science
Faculty of Engineering
University of Ottawa

Under the Supervision of Jean-Lou De Carufel (University of Ottawa) and Prosenjit Bose (Carleton University)
October 2023

To John

ABSTRACT

Finding the exact spanning ratio of a Delaunay graph has been one of the longstanding open problems in Computational Geometry. Currently there are only four convex shapes for which the exact spanning ratio of their Delaunay graph is known: the equilateral triangle, the square, the regular hexagon and the rectangle. In this thesis, we show the exact spanning ratio of the parallelogram Delaunay graph, making the parallelogram the fifth convex shape for which an exact bound is known. The worst-case spanning ratio is *exactly*

$$\frac{\sqrt{2}\sqrt{1+A^2+2A\cos(\theta_0)+(A+\cos(\theta_0))\sqrt{1+A^2+2A\cos(\theta_0)}}}{\sin(\theta_0)}.$$

where A is the aspect ratio and θ_0 is the non-obtuse angle of the parallelogram. Moreover, we show how to construct a parallelogram Delaunay graph whose spanning ratio matches the above mentioned spanning ratio.

ACKNOWLEDGMENTS

I would like to thank the following people who have helped me undertake this endeavor:

My supervisor, Jean-Lou De Carufel, for introducing me to research and to the field of Computational Geometry.

Both of my supervisors, Jean-Lou and Jit Bose, for their guidance, support, feedback, insight, and most importantly, for making research fun and rewarding.

The staff of the School of Electrical Engineering and Computer Science at the University of Ottawa, for their support during my studies.

My partner, John, for his unwavering support and encouragement.

Both of my brothers, Nicolas and Mathieu, and my parents, for their encouragement and support in studying Mathematics and Computer Science, ultimately setting me on this path.

CONTENTS

LIST OF FIGURES

1 INTRODUCTION

In computer science, many phenomena can be represented by graphs, such as computer networks. Often, these graphs have many edges, and computations can be simplified if fewer edges are retained, for example, in compact routing, broadcasting etc. Throwing away edges leaves a subgraph, and if it has certain distance preserving properties, then it is called a spanner. As described by Narasimhan and Smid [15], the goal is to have a *good spanner*, but what constitutes a good spanner depends on the application. In some cases, one may want a minimum weight spanning tree, whereas in other cases, it might be allowable to form cycles in order to keep all the points more connected, as long as the number of edges remains sufficiently small. Even more, some spanners are designed to be fault-tolerant, being able to lose edges without drastically affecting their ability to preserve distance.

A *geometric graph* G is an undirected and weighted graph whose vertices and edges are points and line segments in the plane, where the weight of an edge is the Euclidean distance (denoted by $d_2(\cdot, \cdot)$) between its two endpoints. When a geometric graph has certain distance preserving properties, it is called a *spanner*. More specifically, given a point set $\mathcal{P}$, a geometric graph with vertex set $\mathcal{P}$ is a *c-spanner* if, for any $a, b \in \mathcal{P}$, there is a path in G between a and b, whose length is less than or equal to $c \cdot d_2(a, b)$ [15]. The smallest constant c for which this is true is called the *spanning ratio* of G. A simple example of a 1-spanner is the complete graph. However, in the case of a complete graph, the number of edges is quadratic in the number of vertices. As such, using a subgraph with a linear number of edges is thus desirable.

We can also study the spanning ratio of a family $\mathcal{F}$ of geometric graphs. In this case, we say that c is an upper bound on the spanning ratio of $\mathcal{F}$ if all graphs in $\mathcal{F}$ are c-spanners. Moreover, we say that c' is a lower bound on the spanning ratio of $\mathcal{F}$ if there is at least one graph in $\mathcal{F}$ whose spanning ratio is c'. If we find matching upper and lower bounds on the spanning ratio of $\mathcal{F}$, we say that we have an exact (or tight) spanning ratio for this family.

For example, Delaunay triangulations, Θ_k-graphs and Yao graphs are spanners with a linear number of edges [15] (refer to Section 1.2). This thesis will focus specifically on the spanning ratio of Delaunay graphs. Delaunay graphs are a fundamental structure that have been intensely investigated in Computational Geometry [16]. There is an edge between two points a and b in the Delaunay graph provided there exists a circle with a and b on its boundary and no other points in its interior. Apart from being spanners, Delaunay graphs possess many interesting geometric properties. For example, among all triangulations of a set of points, the Delaunay graph is the one that maximizes the minimum angle [16].

Determining the exact spanning ratio of the Delaunay graph is a notoriously difficult problem. Dobkin et al. [13] were the first to show that the Delaunay graph has a spanning ratio of at most $\pi(1+\sqrt{5})/2 \approx 5.08$. This was later improved by Keil and Gutwin [14] who showed an upper bound of $4\pi/3\sqrt{3} \approx 2.42$. Currently, the best known upper bound of 1.998 was shown by Xia [19]. Chew [12] gave a lower bound of $\pi/2$ which was long believed to be optimal until Bose et al. [9] proved a lower bound of 1.5846. This was later improved to 1.5932 by Xia and Zhang [20]. A tight bound on the spanning ratio of the Delaunay graph remains elusive.

Several variations of the Delaunay triangulation come from generalizing the circle to other convex shapes, Bose et al. [5] showed that the Delaunay graph defined by the homothet of any convex shape C has a spanning ratio that is bounded by a constant times the ratio of the perimeter of C to its width. Intuitively, this suggests that when the empty region is a long and skinny shape, the spanning ratio is large. Moreover, Bose et al. [4] then showed that a Delaunay graph defined by any affine transformation C' of C is a constant spanner where the spanning ratio depends on the eigenvalues of the affine transformation. However, even if the spanning ratio for shape C is tight, the upper bound obtained on the spanning ratio for C' is not necessarily tight. In the search for tight bounds on the spanning ratio of Delaunay graphs, until recently, exact bounds were known for only 3 shapes, namely equilateral triangles [12], squares [3] and regular hexagons [17]. The spanning ratio of empty rectangle Delaunay graphs was first studied by Bose et al. [8] who showed a spanning ratio of $\sqrt{2}(2A+1)$, where A is the aspect ratio of the rectangle. This corroborates the intuition that long skinny rectangles have large spanning ratio. Recently, van Renssen et al. [18] found a way to generalize Bonichon et al.'s result to give a tight bound of $\sqrt{2}\sqrt{A^2+1+A\sqrt{1+A^2}}$ for empty rectangle Delaunay graphs. This is the fourth shape for which we now have an exact bound on the spanning ratio. Their result is a delicate case analysis where they study different cases depending on the aspect ratio of the empty rectangle and the type of edge that results from these empty rectangles. Our main result is that we push the envelope further by proving a tight bound on the spanning ratio of Delaunay graphs defined by empty parallelograms. We generalize ideas from Bonichon et al. [3], van Renssen et al. [18] and Bose et al. [4]. The key idea was to find a way to change basis vectors without applying an affine transformation as in [4]. The generalization can be seen as follows: in Bonichon et al.'s approach, there is no degree of freedom in the shape defining the Delaunay graph since the aspect ratio of the square is fixed. In van Renssen et al.'s case, the shape has one degree of freedom, namely the aspect ratio. In our setting, the difficulty that arises is that our shapes have two degrees of freedom: the aspect ratio and the angle between adjacent sides of the parallelogram. We obtain an exact worst-case bound, and in fact, if we set the angle to

$\pi/2$, we obtain van Renssen et al.'s result and in addition if we set the aspect ratio to 1, we obtain Bonichon et al.'s result.

1.1 PRELIMINARIES, NOTATIONS AND DEFINITIONS

The Delaunay triangulation of a point set $\mathcal{P}$ is a geometric graph $DT(\mathcal{P}) = (\mathcal{P}, E)$ defined as follows. There is an edge between two vertices $u, v \in \mathcal{P}$ if and only if there exists a circle, with u and v on the boundary, which does not contain any point of $\mathcal{P}$ in its interior. Equivalently, there is a triangle $\triangle uvw$ if there is a circle with u, v, w on its boundary which does not contain any point of $\mathcal{P}$ in its interior. To avoid degeneracies, we usually assume that the point set $\mathcal{P}$ is in *general position*. Specifically, we assume that no three points lie on a common line and no four points lie on the boundary of a common circle.

Variations of the Delaunay triangulation exist, for example when the convex shape used is no longer a circle. These variations have been shown to be spanners as well [6],[2],[17],[12],[18], etc. In this thesis, we consider the Delaunay graph where the convex shape is a parallelogram.

Consider an arbitrary parallelogram P. Let us denote the two side lengths by ℓ and s, respectively, where $\ell \geq s > 0$. We refer to the side with length ℓ as the *long side*, and to the side with length s as the *short side* (even though the case where $\ell = s$ is allowed). The *parallelogram Delaunay graph* is a geometric graph with $\mathcal{P}$ as the vertex set. Given two vertices $a, b \in \mathcal{P}$, there is an edge between a and b if and only if there exists a scaled translate of P with a and b on its boundary which contains no vertices of $\mathcal{P}$ in its interior. Observe that different parallelograms give different Delaunay graphs. Moreover, we emphasize the fact that rotations are *not* allowed, only scaled translate of P.

Without loss of generality, we assume that the long side is vertical and the short side has non negative slope. Let θ_0 be the non obtuse angle between the long and the short side of P. Note that any other orientation of the parallelogram can be seen as an isometry of the point set. The general position assumption we make on the point set $\mathcal{P}$ in this context is the following. We assume that no four vertices lie on the boundary of any scaled translate parallelogram of P and that no two vertices lie on a line parallel to the sides of P. This means that no two vertices lie on a vertical or a horizontal line. In general parallelogram Delaunay graphs are near-triangulations. A *near-triangulation* is a planar graph such that every bounded face is a triangle. As such, we denote them by $T_{\mathcal{P}}$ or simply T when the point set is clear from the context.

1.2 KNOWN RESULTS

Currently, a tight bound is not know for the classical Delaunay triangulation. The best known upper bound is 1.998 [19], while the best known lower bound is 1.5932 [20]. Variations of the Delaunay graph for which we know the tight spanning ratio are limited; there are only four convex shapes for which we know the exact spanning ratio of their Delaunay graph: the equilateral triangle, the square, the regular hexagon and the rectangle. When the equilateral triangle is used, this is often referred to as the TD Delaunay, for triangular distance, in literature. Chew [12] showed that in this case, the spanning ratio is exactly 2. When the convex shape is a square, Chew showed that the spanning ratio is at most $\sqrt{10}$, which was later improved by Bonichon et al. [3] who showed that the exact spanning ratio is $\sqrt{4+2\sqrt{2}} \approx 2.61$. When the hexagon is used, Perkovic et al. [17] showed the spanning ratio is 2. Finally, when the convex shaped is a rectangle, van Renssen et al. [18] showed that the spanning ratio is $\sqrt{2}\sqrt{A^2+1+A\sqrt{1+A^2}}$, where A is the aspect ratio of the rectangle.

Another example of geometric graphs defined on a point set $\mathcal{P}$ that are spanners are Θ_k-graphs. Θ_k-graphs are defined for any integer $k \geq 3$. First, for each i (where $0 \leq i < k$), let $\mathcal{R}_i$ be the ray emanating from the origin that forms an angle of $\frac{2i\pi}{k}$ with the negative y axis (by convention $\mathcal{R}_k = \mathcal{R}_0$). For a point $v \in \mathcal{P}$ and an index i (where $0 \leq i < k$), let R_i^v be the ray emanating from v that is parallel to R_i. Also define C_i^v to be the cone consisting of all the points in the plane that are strictly between R_i^v and R_{i+1}^v or on R_{i+1}^v. Then the Θ_k-graph of a point set $\mathcal{P}$ is the graph that has an edge (v, w_i) if vertex w_i lies in the cone C_i^v and the perpendicular projection of w_i onto the bisector of C_i^v is the closest to v compared to that of all other points in $(\mathcal{P}\backslash\{v\}) \cap C_i^v$. Note that any vertex has at most k outgoing edges. For example, the best known upper bound for the spanning ratio of the Θ_4-graph [6] was shown to be 17. In the case of the Θ_5-graph, the current best known upper bound has been shown to be $\frac{\sin(3\pi/10)}{\sin(3\pi/5)-\sin(3\pi/10)} < 5.70$ [10]. For the Θ_6-graph, Bonichon et al. [2] showed that the Θ_6-graph is the union of two spanning TD-Delaunay graphs and has a tight spanning ratio of 2. Moreover, Bose et al. [7] showed a tight bound of $1+2\sin(\pi/k)$ for Θ_k-graphs, where the number of cones is $k = 4m+2$, for $m \geq 1$. Next, they also show that when $k = 4m+4$ for $m \geq 1$, the Θ_k-graph has spanning ratio at most $\frac{1+2\sin(\pi/k)}{\cos(\pi/k)-sin(\pi/k)}$ and at least $1+2\tan(\pi/k)+2\tan^2(\pi/k)$ and for Θ_k-graphs with $k = 4m+3$ or with $k = 4m+5$ cones, the spanning ratio is at most $\frac{\cos(\pi/2k)}{\cos(\pi/k)-\sin(3\pi/2k)}$.

Yao graphs are defined in a similar way as the Θ_k-graphs, where we consider cones around each vertex. Formally, let Y_k denote the Yao graph with fixed integer $k > 0$. The point set is then partitioned such that around each vertex we have k equiangular cones of angle $2\pi/k$. Each vertex is then connected to its closest neighbour in each cone. For all integer values of k, it is known whether or not Y_k is a geometric spanner [1]. For example, Barba et al. [1], showed an upper bound on the spanning ratio for odd $k \geq 5$ of $1/(1-2\sin(\frac{3\cdot 2\pi}{8k}))$, for the case where $k = 5$, Barba et al. further improved the upper bound, thus showing that Y_5 has a spanning ratio of at most $2+\sqrt{3} \approx 3.74$. Next, the authors also show that in the case where $k = 6$, Y_k has a spanning ratio of at most 5.8. Barba et al., also give a lower bound of 2.87 for Y_5. While some bounds are known for the spanning ratio of Yao graphs, there are currently no known tight bound on their spanning ratios.

1.3 OUR CONTRIBUTIONS

This thesis further generalizes and extends the proof given by Bonichon et al. [3], and van Renssen et al. [18]. All together, we get an exact bound of

$$\frac{\sqrt{2}\sqrt{1 + A^2 + 2A\cos(\theta_0) + (A + \cos(\theta_0))\sqrt{1 + A^2 + 2A\cos(\theta_0)}}}{\sin(\theta_0)}$$

on the spanning ratio of the parallelogram Delaunay graph.

We summarize our proof technique here. To obtain an upper bound on the spanning ratio, we proceed by induction on the rank of the distances between pairs of points. To go from a start point a to a destination point b, we only consider the paths that use endpoints of segments that cross the line segment ab. If a point is above (respectively below) the line ab, we refer to that point as a high (resp. low) point. Moreover, we only consider the case where the start point a is on the bottom left corner and the destination point b is on the rightmost side of the parallelogram.

To bound the length of the path from a to b, we break the path into three subpaths. The first part is bounded using the so-called *Crossing Lemma* (see below). The second part considers the length of the subpath where the points all have y-coordinates that are significantly greater than that of b. On this path, the points are either all high points or all low points. The third part of the path is bounded using the induction hypothesis.

The aforementioned Crossing Lemma is a crucial part in the proof. The goal of the Crossing Lemma is to get a bound on the length of the path from a until we encounter a parallelogram that has an edge between a high and a low point that is not *too steep*. Using an inductive argument, we use techniques such as bounding monotone paths using the L_1-norm as well as bounding edge lengths in terms of their horizontal component.

2 MAIN RESULT

2.1 THE UPPER BOUND

Consider an arbitrary parallelogram P. Let us denote the two side lengths by ℓ and s, respectively, where $\ell \geq s > 0$. We refer to the side with length ℓ as the *long side*, and to the side with length s as the *short side* (even though the case $\ell = s$ is allowed). Without loss of generality, we assume that the long side is vertical and the short side has non-negative slope, let θ_0 be the non obtuse angle between the long and the short side of P. We define the aspect ratio of P, denoted by A, as $A := \frac{\ell}{s}$. For a point a in the plane, let x_a and y_a be the x- and y-coordinates of a, respectively. Since the parallelogram Delaunay graph of a point set is a near-triangulation, we denote it as T.

In this section, we prove that the worst-case spanning ratio of T is at most $h(A, \theta_0) :=$

$$\frac{\sqrt{2}\sqrt{1 + A^2 + 2A\cos(\theta_0) + (A + \cos(\theta_0))\sqrt{1 + A^2 + 2A\cos(\theta_0)}}}{\sin(\theta_0)}.$$

We obtain this result after a few preparatory lemmas and observations.

Let a, b be any two vertices in T. Without loss of generality, assume $a = (0, 0)$ and $x_b > 0$. Let $d_2^T(a, b)$ be the length of the shortest path in T between a and b. We prove that $d_2^T(a, b)$ is at most $h(A, \theta_0)\, d_2(a, b)$ with equality occurring in the worst case, where $d_2(a, b)$ is the Euclidean distance from a to b. We denote the slope of the segment ab as $S := \frac{y_b}{x_b}$. Our proof considers four different scenarios, based on the value of S. Each scenario is illustrated in Figure 2.1.

In each scenario, we write the coordinates of all points with respect to a new basis[1] $\{\hat{x}, \hat{y}\}$. We will denote the counterclockwise angle between $\hat{x}$ and $\hat{y}$ by θ. Using this new basis, P will be seen as a rectangle. We denote by $\hat{P}$ the parallelogram P written in the $\{\hat{x}, \hat{y}\}$ basis.

SCENARIO 1 In this scenario, there is a homothet of P with a on its top left corner and b on its lowest short side. Formally, the slope between a and b satisfies

$$S \in \left(-\infty, \frac{\cos(\theta_0) - A}{\sin(\theta_0)}\right].$$

In this scenario, the usual basis $\{(1, 0), (0, 1)\}$ will be sent to $\hat{x} := (0, -1)$, $\hat{y} := (\sin(\theta_0), \cos(\theta_0))$ (observe that $\frac{1}{A}\hat{x}_b \geq \hat{y}_b$). The interior counterclockwise angle θ between $\hat{x}$ and $\hat{y}$ is then $\theta = \pi - \theta_0$.

1 We do not perform a linear transformation. We simply write some objects with respect to the $\{\hat{x}, \hat{y}\}$ basis instead of the usual basis. All distances stay the same.

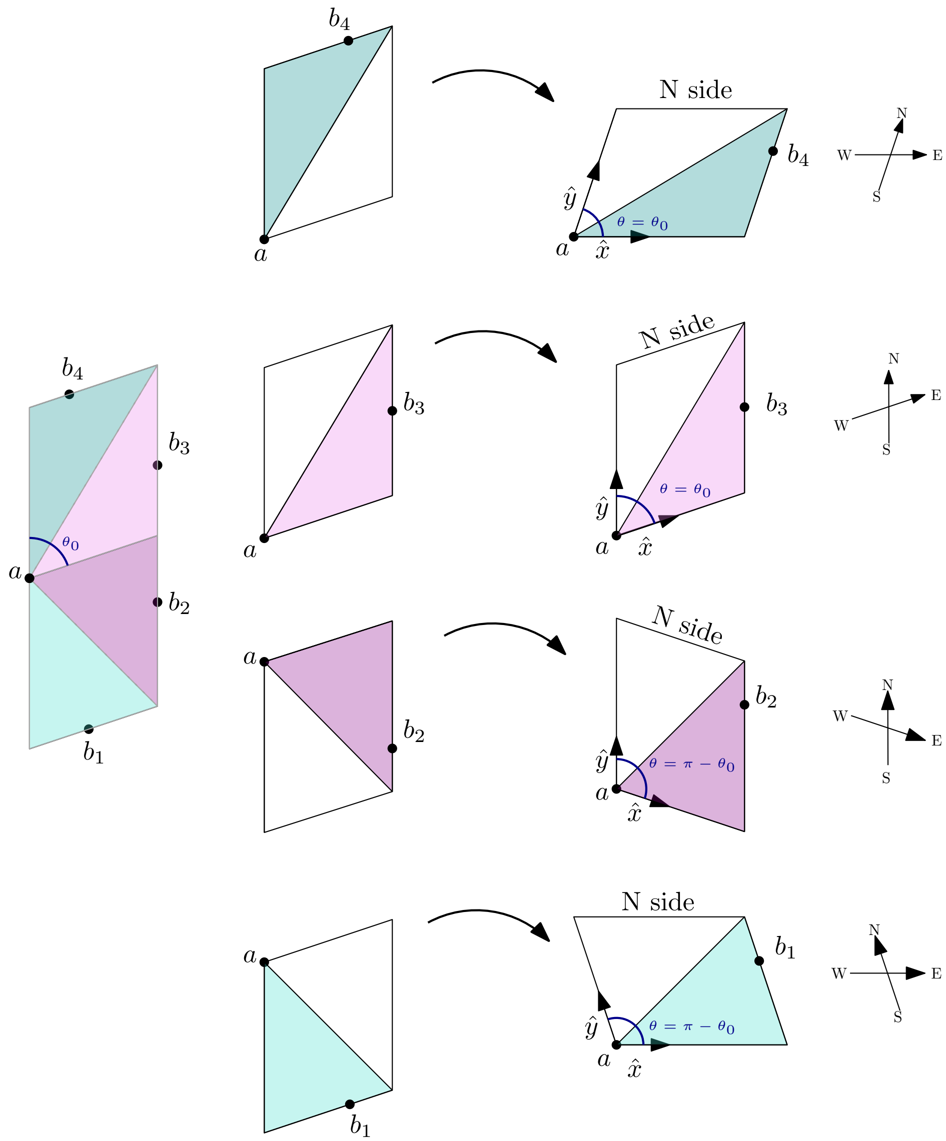

Figure 2.1: The four scenarios considered about the slope between a and b.

In Figure 2.1, the slope between a and b_1 falls into this case. As seen on the first column of Figure 2.1, there is a homothet of P with a on its top left corner and b_1 on its lowest short side.

As an example, suppose the vertices of P have coordinates $(0,0)$, $(\sin(\theta_0), \cos(\theta_0))$, $(0,-2)$ and $(\sin(\theta_0), \cos(\theta_0)-2)$, respectively. Then, in the new basis, the vertices of $\hat{P}$ have coordinates $(0,0)$, $(0,1)$, $(2,0)$ and $(2,1)$, respectively.

SCENARIO 2 In this scenario, there is a homothet of P with a on its top left corner and b on its furthest long side. Formally, the slope between a and b satisfies

$$S \in \left(\frac{\cos(\theta_0) - A}{\sin(\theta_0)}, \frac{\cos(\theta_0)}{\sin(\theta_0)} \right].$$

In this scenario, the usual basis $\{(1,0),(0,1)\}$ will be sent to $\hat{x} := (\sin(\theta_0), \cos(\theta_0))$, $\hat{y} := (0,-1)$ (observe that $A\hat{x}_b \geq \hat{y}_b$). The interior counterclockwise angle θ between $\hat{x}$ and $\hat{y}$ is then $\theta = \pi - \theta_0$.

In Figure 2.1, the slope between a and b_2 falls into this case. As seen on the first column of Figure 2.1, there is a homothet of P with a on its top left corner and b_2 on its furthest long side.

As an example, suppose the vertices of P have coordinates $(0,0)$, $(\sin(\theta_0), \cos(\theta_0))$, $(0,-2)$ and $(\sin(\theta_0), \cos(\theta_0)-2)$, respectively. Then, in the new basis, the vertices of $\hat{P}$ have coordinates $(0,0)$, $(1,0)$, $(0,2)$ and $(1,2)$, respectively.

SCENARIO 3 In this scenario, there is a homothet of P with a on its bottom left corner and b on its furthest long side. Formally, the slope between a and b satisfies

$$S \in \left(\frac{\cos(\theta_0)}{\sin(\theta_0)}, \frac{\cos(\theta_0) + A}{\sin(\theta_0)} \right].$$

In this scenario, the usual basis $\{(1,0),(0,1)\}$ will be sent to $\hat{x} := (\sin(\theta_0), \cos(\theta_0))$, $\hat{y} := (0,1)$ (observe that $A\hat{x}_b \geq \hat{y}_b$). The interior counterclockwise angle θ between $\hat{x}$ and $\hat{y}$ is then $\theta = \theta_0$.

In Figure 2.1, the slope between a and b_3 falls into this case. As seen on the first column of Figure 2.1, there is a homothet of P with a on its bottom left corner and b_3 on its furthest long side.

As an example, suppose the vertices of P have coordinates $(0,0)$, $(\sin(\theta_0), \cos(\theta_0))$, $(0,2)$ and $(\sin(\theta_0), \cos(\theta_0)+2)$, respectively. Then, in the new basis, the vertices of $\hat{P}$ have coordinates $(0,0)$, $(1,0)$, $(0,2)$ and $(1,2)$, respectively.

SCENARIO 4 In this scenario, there is a homothet of P with a on its bottom left corner and b on its highest short side. Formally, the slope between a and b satisfies

$$S \in \left(\frac{\cos(\theta_0) + A}{\sin(\theta_0)}, \infty \right).$$

In this scenario, the usual basis $\{(1,0),(0,1)\}$ will be sent to $\hat{x} := (0,1)$, $\hat{y} := \big(\sin(\theta_0), \cos(\theta_0)\big)$(observe that $\frac{1}{A}\hat{x}_b \geq \hat{y}_b$). The interior counterclockwise angle θ between $\hat{x}$ and $\hat{y}$ is then $\theta = \theta_0$.

In Figure 2.1, the slope between a and b_4 falls into this case. As seen on the first column of Figure 2.1, there is a homothet of P with a on its bottom left corner and b_4 on its highest short side.

As an example, suppose the vertices of P have coordinates $(0,0)$, $(\sin(\theta_0), \cos(\theta_0))$, $(0,2)$ and $(\sin(\theta_0), \cos(\theta_0)+2)$, respectively. Then, in the new basis, the vertices of $\hat{P}$ have coordinates $(0,0)$, $(0,1)$, $(2,0)$ and $(2,1)$, respectively.

In the second column of Figure 2.1, we highlight the homothets of $\hat{P}$ (with a and b on the boundary) on which our analysis will be based. In the third column we show the homothets of $\hat{P}$ in the new basis $\{\hat{x}, \hat{y}\}$ for each scenario. Notice that $\hat{P}$ is an axis aligned rectangle in the $\{\hat{x}, \hat{y}\}$ basis. We define the W and E sides of $\hat{P}$ to be the two sides parallel to $\hat{y}$ with the W side having smaller $\hat{x}$ coordinate. Similarly, we define the N and S sides of $\hat{P}$ to be the two sides parallel to $\hat{x}$ with the N side having larger $\hat{y}$ coordinate. A point on the east edge of $\hat{P}$ is said to be *eastern*. The *directions* N, S, E, W are defined accordingly, refer to the last column of Figure 2.1.

Observe that the distance between the origin and any point $\alpha\hat{x} + \beta\hat{y}$ in all scenarios is equal to

$$\|\alpha\hat{x} + \beta\hat{y}\|_2 = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\cos(\pi - \theta)} \leq \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta|\cos(\theta)|}.$$

Moreover, observe that the L_1-norm in the usual basis is equal to the L_1-norm in the $\{\hat{x}, \hat{y}\}$-basis.

To give an upper bound on the spanning ratio between any two vertices a and b, we consider the sequence of triangles $T_1, T_2, ..., T_k$ intersecting the line segment ab from a to b. Note that the triangulation of $\mathcal{P}$ may not contain its convex hull. As such, the sequence of triangles may not exist. For example, the sequence of triangles from l_1 to l_5 is undefined in Figure 2.2. In order to guarantee that this sequence of triangles is well defined, we augment the point set in the following way. Consider the triangulation of a larger point set $\mathcal{P}' = \mathcal{P} \cup \{p_1, p_2, p_3, p_4\}$, where $\{p_1, p_2, p_3, p_4\}$ is defined as follows. The points in $\{p_1, p_2, p_3, p_4\}$ are the vertices of a homothet of P containing $\mathcal{P}$ such that the distance from any point in $\mathcal{P}$ to any point in $\{p_1, p_2, p_3, p_4\}$ is arbitrarily large. Adding these points guarantees that the larger triangulation contains all the edges of the convex hull of $\mathcal{P}'$. Moreover, all the edges of the triangulation of $\mathcal{P}'$ that are not in the triangulation of $\mathcal{P}$, by construction, are arbitrarily long. Therefore, these edges will not be used in any bounded path. In what follows, we bound the length of the shortest path between points in $\mathcal{P}$.

We will denote the triangle with vertices u, v and w by $\triangle uvw$. Note that with the additional four points $\{p_1, p_2, p_3, p_4\}$, every face that intersects the line segment ab is a triangle. The order of this sequence is determined by the order in which these triangles are encountered when following the

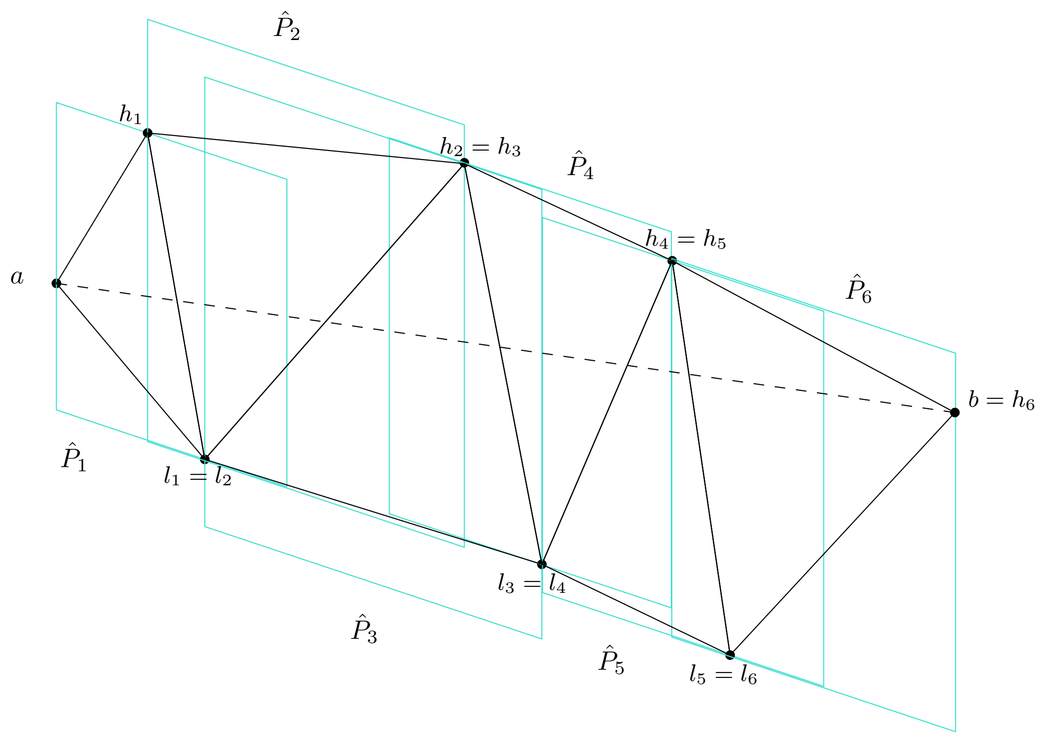

Figure 2.2: Illustration of the triangles intersecting ab.

line ab from a to b. The boundary of each triangle intersects ab twice. We refer to the intersection closer to a as the *first* intersection and the other as the *second*. Let h_i and l_i denote the endpoints of the edge containing the second intersection of T_i with ab. Note that h_i is above ab and l_i is below ab (with respect to the $\{\hat{x}, \hat{y}\}$ basis[2]). For every T_i and T_{i+1}, either $l_i = l_{i+1}$ or $h_i = h_{i+1}$. We define $h_0 = l_0 = a$, and we let $l_k = h_k = b$. Figure 2.2 shows the triangles as well as their respective parallelogram $\hat{P}_i$ with h_i and l_i in Scenario 2.

Each triangle T_i (or $\hat{T}_i$ in the $\{\hat{x}, \hat{y}\}$ basis) in the Delaunay graph has an associated parallelogram P_i (or $\hat{P}_i$ in the $\{\hat{x}, \hat{y}\}$ basis) which is a homothet of P (or $\hat{P}$ in the $\{\hat{x}, \hat{y}\}$ basis) that has the three vertices of T_i on its boundary. We define the parameter L as the positive slope of the diagonal of $\hat{P}$ expressed in the $\{\hat{x}, \hat{y}\}$ basis. The third column of Figure 2.1 illustrates these diagonals[3]. If the long side of $\hat{P}$ is vertical, then $L = A$, as in Scenarios 2 and 3. Otherwise, $L = 1/A$, as in Scenarios 1 and 4.

For any point u, we denote $\hat{x}_u$ and $\hat{y}_u$ to be the coordinates of u in its $\{\hat{x}, \hat{y}\}$ coordinate system. In other words, $u = \hat{x}_u\hat{x} + \hat{y}_u\hat{y}$.

Definition 2.1.1. An edge (u, v) is said to be *gentle* if it has slope within $[-L, L]$ with respect to the $\{\hat{x}, \hat{y}\}$ basis. In other words, $|\hat{y}_v - \hat{y}_u| \leq L|\hat{x}_v - \hat{x}_u|$. Otherwise, we say it is *steep*.

---

2 We do not use the notation $\hat{h}_i$ and $\hat{l}_i$ even though these two points are defined with respect to the $\{\hat{x}, \hat{y}\}$ basis. This would make expressions like $\hat{x}_{\hat{h}_i}$ too heavy.

3 Note that the slope in the $\{\hat{x}, \hat{y}\}$ basis corresponds to the ratio of the sides of $\hat{P}$.

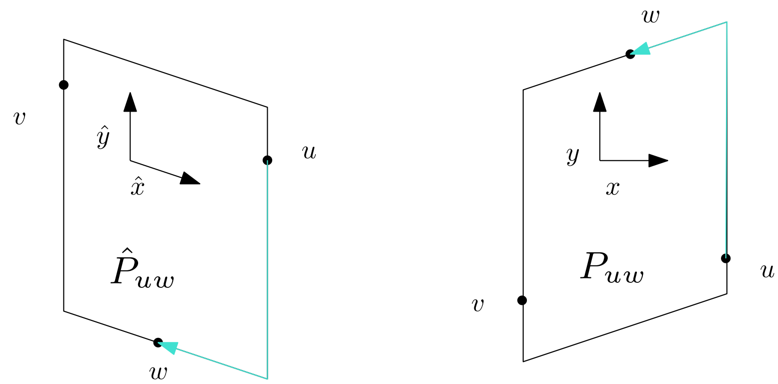

Figure 2.3: Illustration of the length of $d_{\hat{P}_{uw}}(u, w)$ in Scenario 2.

In order to bound the spanning ratio of the parallelogram Delaunay graph, we first define what it means for a parallelogram to be *inductive* and what it means to *have a potential*.

Definition 2.1.2. Parallelogram $\hat{P}_i$ is *inductive* if edge (h_i, l_i) is gentle. The *inductive point of* $\hat{P}_i$, which we denote by c, is the point with the larger $\hat{x}$ coordinate among h_i and l_i.

Let $d_{\hat{P}_i}(h_i, l_i)$ be the length of the path when moving clockwise from h_i to l_i along the sides of $\hat{P}_i$. Note that in the usual x, y-coordinate system, this path may be counter-clockwise. For example in Figure 2.3, we show in blue the length of $d_{\hat{P}_{uw}}(u, w)$ in Scenario 2. On the left, we have $\hat{P}_{uw}$ and on the right we have P_{uw} (in the usual $\{x, y\}$ basis). When in the $\{\hat{x}, \hat{y}\}$ basis, the length of $d_{\hat{P}_{uw}}(u, w)$ is the length of the path when going from u to w, whereas in the usual basis, this length is given by the counter clockwise path from u to w (or the clockwise path from w to u).

Definition 2.1.3. Parallelogram $\hat{P}_i$ *has a potential* if

$$d_2^T(a, h_i) + d_2^T(a, l_i) + d_{\hat{P}_i}(h_i, l_i) \leq (2 + 2L)\hat{x}_i,$$

where $\hat{x}_i$ is the $\hat{x}$-coordinate of the E side of $\hat{P}_i$.

Next, we define $P(a, b)$ as the parallelogram $\{\alpha\hat{x} + \beta\hat{y} : \hat{x}_a \leq \alpha \leq \hat{x}_b, \hat{y}_a \leq \beta \leq \hat{y}_b\}$ (refer to Figure 2.4). Observe that in general, $P(a, b)$ is not a homothet of P or of $\hat{P}$.

We now show that parallelograms that are not inductive pass on their potential.

Lemma 2.1.4. *If (a, b) is not an edge in T and parallelogram $P(a, b)$ contains no point of $\mathcal{P}$ other than a and b, then $\hat{P}_1$ has a potential. Furthermore, if, for any $1 \leq i < k$, $\hat{P}_i$ has a potential but is not inductive, then $\hat{P}_{i+1}$ has a potential.*

Proof. By our general position assumption, a, h_1, l_1 all lie on different sides of $\hat{P}_1$. Since $\triangle a h_1 l_1$ intersects the line segment ab, then h_1 and l_1 must have larger $\hat{x}$-coordinates than a. This implies that when $P(a, b)$ is empty,

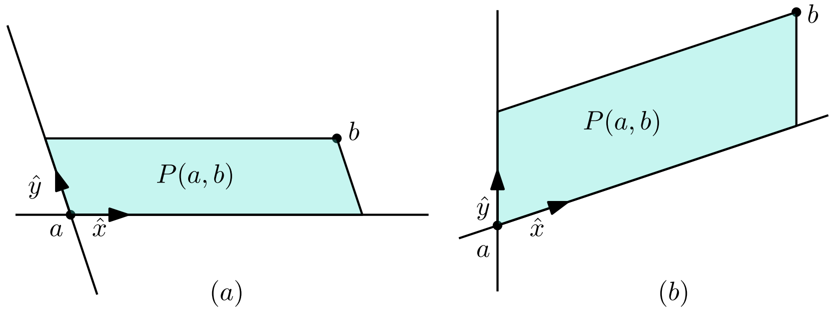

Figure 2.4: Illustration of $P(a,b)$ in Scenarios 1 and 3.

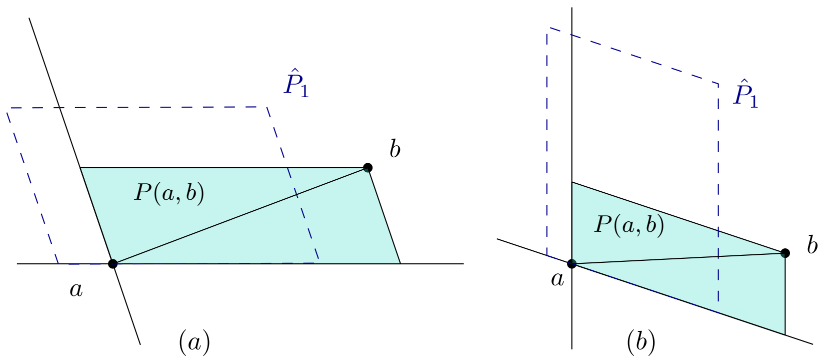

Figure 2.5: Example showing the positioning of $\hat{P}_1$ if a is on the S side in Scenario 1 on the left (a), and Scenario 2 on the right (b).

a cannot lie on the S side as it would mean that l_1 would lie on the E side of $\hat{P}_1$, which is inside $P(a,b)$, refer to Figure 2.5. Hence, a lies on the W side of $\hat{P}_1$ and $\hat{x}_1$ is the length of the south side of $\hat{P}_1$. Therefore, h_1 lies on the N or the E side, while l_1 lies on the S or E side. Notice that $d_2^T(a,h_1) + d_2^T(a,l_1) + d_{\hat{P}_1}(h_1,l_1)$ is bounded by the perimeter of $\hat{P}_1$, which is $(2+2L)\hat{x}_1$. Thus $\hat{P}_1$ has a potential.

Next, assume that for all $1 \leq i < k$, $\hat{P}_i$ has a potential but is not inductive. Since $\hat{P}_i$ is not inductive we know that (h_i, l_i) is steep. In the remainder of the proof we assume that $\hat{x}_{l_i} < \hat{x}_{h_i}$ as the case where $\hat{x}_{h_i} < \hat{x}_{l_i}$ can be shown using analogous arguments.

Since $\hat{x}_{l_i} < \hat{x}_{h_i}$, we know that l_i must be on the S side of $\hat{P}_i$ while h_i can be on the N or E side of $\hat{P}_i$. If h_i is on the N side of $\hat{P}_i$, then since $\hat{x}_{l_i} < \hat{x}_{h_i}$, then h_i must be on the N side of $\hat{P}_{i+1}$ and l_i is either on the W or S side of $\hat{P}_{i+1}$. If l_i is on the S side of $\hat{P}_{i+1}$ (refer to Figure 2.6 Case (a)), we have that $\hat{P}_{i+1}$ is just a translate of $\hat{P}_i$, as such we have that

$$d_{\hat{P}_{i+1}}(h_i,l_i) - d_{\hat{P}_i}(h_i,l_i) = 2(\hat{x}_{i+1} - \hat{x}_i). \tag{2.1.1}$$

Now, if l_i is on the W side of $\hat{P}_{i+1}$(refer to Figure 2.6 Case (b)), then we have that

$$d_{\hat{P}_{i+1}}(h_i,l_i) - d_{\hat{P}_i}(h_i,l_i) \leq (2+2L)(\hat{x}_{i+1} - \hat{x}_i). \tag{2.1.2}$$

Consider now the case where h_i is on the E side of $\hat{P}_i$. Because $\hat{x}_{l_i} < \hat{x}_{h_i}$, h_i must be on the N side of $\hat{P}_{i+1}$ and l_i is either on the S side or on the

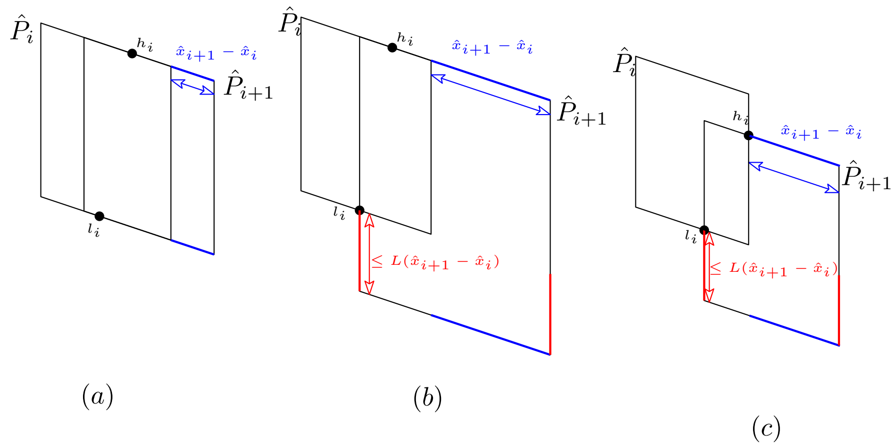

Figure 2.6: Illustration of the proof of Lemma 2.1.4 for Scenario 2.

W side of $\hat{P}_{i+1}$ (refer to Figure 2.6 Case (c)). If l_i is on the S side of $\hat{P}_{i+1}$ then (2.1.1) holds. If l_i is on the W side of $\hat{P}_{i+1}$ then (2.1.2) holds. In all cases we have that

$$d_{\hat{P}_{i+1}}(h_i, l_i) - d_{\hat{P}_i}(h_i, l_i) \le (2 + 2L)(\hat{x}_{i+1} - \hat{x}_i).$$

Since $\hat{P}_i$ has potential we know that

$$d_2^T(a, h_i) + d_2^T(a, l_i) + d_{\hat{P}_i}(h_i, l_i) \le (2 + 2L)\hat{x}_i.$$

Observe that

$$\begin{aligned} &d_2^T(a, h_i) + d_2^T(a, l_i) + d_{\hat{P}_i}(h_i, l_i) + d_{\hat{P}_{i+1}}(h_i, l_i) - d_{\hat{P}_i}(h_i, l_i) \\ &= d_2^T(a, h_i) + d_2^T(a, l_i) + d_{\hat{P}_{i+1}}(h_i, l_i). \end{aligned}$$

Putting the above equalities and inequalities together we get

$$\begin{aligned} d_2^T(a, h_i) + d_2^T(a, l_i) + d_{\hat{P}_{i+1}}(h_i, l_i) &\le (2 + 2L)\hat{x}_i + (2 + 2L)(\hat{x}_{i+1} - \hat{x}_i) \\ &= (2 + 2L)\hat{x}_{i+1} \end{aligned} \tag{2.1.3}$$

Assume $T_{i+1} = \triangle h_i h_{i+1} l_i$ (where $l_i = l_{i+1}$); in other words, (h_i, h_{i+1}) is an edge of T with h_{i+1} lying somewhere on the boundary of $\hat{P}_{i+1}$ between h_i and l_i, when moving clockwise from h_i to l_i. By the triangle inequality, we have $d_2(h_i, h_{i+1}) \le d_{\hat{P}_{i+1}}(h_i, h_{i+1})$ and

$$d_2^T(a, h_{i+1}) \le d_2^T(a, h_i) + d_2(h_i, h_{i+1}). \tag{2.1.4}$$

We also know that

$$d_{\hat{P}_{i+1}}(h_i, h_{i+1}) \le d_{\hat{P}_{i+1}}(h_i, l_i). \tag{2.1.5}$$

This gives us

$$\begin{aligned} d_2(h_i, h_{i+1}) + d_{\hat{P}_{i+1}}(h_{i+1}, l_i) &\le d_{\hat{P}_{i+1}}(h_i, h_{i+i}) + d_{\hat{P}_{i+1}}(h_{i+1}, l_i) \\ &= d_{\hat{P}_{i+1}}(h_i, l_i). \end{aligned} \tag{2.1.6}$$

Using inequalities (2.1.3), (2.1.4), (2.1.5) and (2.1.6), we get the following:

$$\begin{aligned} d_2^T(a,h_{i+1}) + d_2^T(a,l_{i+1}) + d_{\hat{P}_{i+1}}(h_{i+1},l_{i+1}) &\leq d_2^T(a,h_i) + d_2(h_i,h_{i+1}) \\ &\quad + d_2^T(a,l_i) + d_{\hat{P}_{i+1}}(h_{i+1},l_i) \\ &\leq d_2^T(a,h_i) + d_2^T(a,l_i) \\ &\quad + d_{\hat{P}_{i+1}}(h_i,l_i) \\ &\leq (2+2L)\hat{x}_{i+1}, \end{aligned}$$

as required to show $\hat{P}_{i+1}$ has a potential. Note that here $l_{i+1} = l_i$. The argument for the case when $T_{i+1} = \triangle h_{i+1} l_i l_{i+1}$, where $h_i = h_{i+1}$ is symmetric. □

Next, we can bound the distance from a to the inductive point of a parallelogram with potential when this inductive point lies on the E side of the parallelogram.

Lemma 2.1.5. *If parallelogram $\hat{P}_i$ has a potential and its inductive point c (either $c = h_i$ or $c = l_i$) lies on the E side of $\hat{P}_i$, then*

$$d_2^T(a,c) \leq (1+L)\hat{x}_c$$

Proof. Without loss of generality, assume $c = h_i$. Observe that, since h_i is eastern, we have

$$\hat{x}_c = \hat{x}_{h_i} = \hat{x}_i. \tag{2.1.7}$$

Moreover, since $\hat{P}_i$ has a potential, we have (1) $d_2^T(a,h_i) \leq (1+L)\hat{x}_i = (1+L)\hat{x}_{h_i}$ or (2) $d_2^T(a,l_i) + d_{\hat{P}_i}(h_i,l_i) \leq (1+L)\hat{x}_i$.

In the first case, we find

$$d_2^T(a,h_i) \leq (1+L)\hat{x}_i = (1+L)\hat{x}_c$$

by (2.1.7). In the second case, since (l_i, h_i) is an edge in T, by the triangle inequality we get

$$\begin{aligned} d_2^T(a,h_i) \leq d_2^T(a,l_i) + d_2(l_i,h_i) \leq d_2^T(a,l_i) + d_{\hat{P}_i}(h_i,l_i) &\leq (1+L)\hat{x}_i \\ &= (1+L)\hat{x}_c \end{aligned}$$

by (2.1.7). □

We now shift our focus to paths consisting of gentle edges.

Definition 2.1.6. Let $1 \leq j \leq k$. The *maximal high path ending at h_j* and the *maximal low path ending at l_j* are defined as follows:
If h_j is eastern in $\hat{P}_j$, the maximal high path ending at h_j is simply h_j; otherwise, it is the path $h_i, h_{i+1}, ..., h_j$ such that $h_{i+1}, ..., h_j$ are not eastern in $\hat{P}_{i+1}, .., \hat{P}_j$ and either $i = 0$ or h_i is eastern in $\hat{P}_i$.
If l_j is eastern in $\hat{P}_j$, the maximal low path ending at l_j is simply l_j; otherwise, it is the path $l_i, l_{i+1}, ..., l_j$ such that $l_{i+1}, ..., l_j$ are not eastern in $\hat{P}_{i+1}, .., \hat{P}_j$ and either $i = 0$ or l_i is eastern in $\hat{P}_i$.

Next, we bound the length of these maximal high and maximal low paths.

Lemma 2.1.7. *If the path $h_i, h_{i+1}, ..., h_j$ is a maximal high path then*

$$d_2^T(h_i, h_j) \leq (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}).$$

Similarly, if the path $l_i, l_{i+1}, ..., l_j$ is a maximal low path then

$$d_2^T(l_i, l_j) \leq (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_i} - \hat{y}_{l_j}).$$

Proof. By Definition 2.1.6, none of the $h_{i+1}, .., h_j$ are E and as such we have a succession of WN edges (by the general position assumption). As such, $\hat{y}_{h_i} < \hat{y}_{h_{i+1}} < ... < \hat{y}_{h_j}$. By the triangle inequality, we have the following for all $i \leq k \leq j$:

$$d_2^T(h_k, h_{k+1}) \leq (\hat{x}_{h_{k+1}} - \hat{x}_{h_k}) + (\hat{y}_{h_{k+1}} - \hat{y}_{h_k}).$$

By summing up the terms we get,

$$d_2^T(h_i, h_j) \leq (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}).$$

The bound on the length of the maximal low path can be shown using a symmetrical argument □

We now use the above lemmas to prove bounds on the path lengths from a to the inductive point on the first inductive parallelogram (if it exists) when the interior of $P(a, b)$ is empty.

Lemma 2.1.8. *(Crossing Lemma) Assume $P(a, b)$ does not contain any other vertices of $\mathcal{P}$ and (a, b) is not an edge in the parallelogram Delaunay graph. The following properties hold:*

(1) *If no parallelogram in $\hat{P}_1, \hat{P}_2, ..., \hat{P}_k$ is inductive then*

$$d_2^T(a, b) \leq \left(L + \sqrt{1 + L^2 + 2L|\cos(\theta)|}\right) \hat{x}_b + \hat{y}_b$$

(2) *Otherwise, let $\hat{P}_j$ be the first inductive parallelogram in the sequence $\hat{P}_1, \hat{P}_2, ..., \hat{P}_{k-1}$.*

   (a) *If $L = A$ (as in Scenarios 2 and 3) and h_j is the inductive point of $\hat{P}_j$ then,*

$$d_2^T(a, h_j) + (\hat{y}_{h_j} - \hat{y}_b) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{h_j}.$$

   (b) *If $L = A$ (as in Scenarios 2 and 3) and l_j is the inductive point of $\hat{P}_j$ then,*

$$d_2^T(a, l_j) - \hat{y}_{l_j} \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{l_j}.$$

(c) *If $L = 1/A$ (as in Scenarios 1 and 4) and h_j is the inductive point of $\hat{P}_j$ then,*

$$d_2^T(a,h_j) + A(\hat{y}_{h_j} - \hat{y}_b) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{h_j}.$$

(d) *If $L = 1/A$ (as in Scenarios 1 and 4) and l_j is the inductive point of $\hat{P}_j$ then,*

$$d_2^T(a,l_j) - A\hat{y}_{l_j} \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{l_j}.$$

Proof. (1) By Lemma 2.1.4, if no parallelogram in $\hat{P}_1, \hat{P}_2, ..., \hat{P}_k$ is inductive then the last parallelogram must have a potential since $\hat{P}_1$ has a potential. Since no two vertices have the same $\hat{y}$ coordinate, we have that b must lie on the E side of the last parallelogram $\hat{P}_k$. Thus by Lemma 2.1.5 we have:

$$\begin{aligned} d_2^T(a,b) &\leq (1+L)\hat{x}_k = (1+L)\hat{x}_b \\ &\leq \left(L + \sqrt{1 + L^2 + 2L|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b. \end{aligned}$$

(2) (a) In this case, we are either in Scenario 2 or 3.

Let $\hat{P}_j$ be the first inductive parallelogram in the sequence $\hat{P}_1, \hat{P}_2, ..., \hat{P}_{k-1}$ and we assume that the inductive point of $\hat{P}_j$ is $c = h_j$.

By Lemma 2.1.4, every parallelogram $\hat{P}_i$ with $i \leq j$ has a potential. Note that in this case, l_j is to the left of h_j. Since the edge (l_j, h_j) is gentle it follows that the vertical distance between l_j and h_j can be bounded:

$$(\hat{y}_{h_j} - \hat{y}_{l_j}) \leq A(\hat{x}_{h_j} - \hat{x}_{l_j}) \iff (\hat{y}_{h_j} - \hat{y}_{l_j}) + A\hat{x}_{l_j} \leq A\hat{x}_{h_j}. \tag{2.1.8}$$

Moreover, we can bound the length of the edge (l_j, h_j) by

$$\begin{aligned} d_2(l_j,h_j) &\leq \sqrt{1 + A^2 - 2A\cos(\pi - \theta)}(\hat{x}_{h_j} - \hat{x}_{l_j}) \\ &\leq \sqrt{1 + A^2 + 2A|\cos(\theta)|}(\hat{x}_{h_j} - \hat{x}_{l_j}). \end{aligned} \tag{2.1.9}$$

Let $l_i, l_{i+1}, ..., l_{j-1} = l_j$ be the maximal low path ending at l_j. Note that by Lemma 2.1.7 we have

$$d_2^T(l_i, l_j) \leq (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_i} - \hat{y}_{l_j}). \tag{2.1.10}$$

Next, note that either $l_i = l_0 = a$ or l_i is an eastern point in $\hat{P}_i$ that has a potential and Lemma 2.1.5 applies. In this case we have that since l_i is eastern in $\hat{P}_i$ we have:

$$\begin{aligned} d_2^T(a, l_i) &\leq (1+A)\hat{x}_i \\ &= (1+A)\hat{x}_{l_i} \end{aligned} \tag{2.1.11}$$

Using inequalities (2.1.8), (2.1.9), (2.1.10) and (2.1.11), together with the triangular inequality, we get

$$\begin{aligned} & d_2^T(a, h_j) + (\hat{y}_{h_j} - \hat{y}_b) \\ & \leq d_2^T(a, l_i) + d_2^T(l_i, l_j) + d_2(l_j, h_j) + (\hat{y}_{h_j} - \hat{y}_b) \\ & \leq (1+A)\hat{x}_{l_i} + (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_i} - \hat{y}_{l_j}) \\ & \quad + \sqrt{1 + A^2 + 2A|\cos(\theta)|}(\hat{x}_{h_j} - \hat{x}_{l_j}) + \hat{y}_{h_j} - \hat{y}_b \\ & = A\hat{x}_{l_i} + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{h_j} + (\hat{y}_{l_i} - \hat{y}_{l_j}) \\ & \quad - \left(1 + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{l_j} + \hat{y}_{h_j} - \hat{y}_b \\ & \leq A\hat{x}_{l_i} + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{h_j} + \hat{y}_{h_j} - \hat{y}_{l_j} \\ & \qquad \qquad \text{since } \hat{y}_{l_i} - \hat{y}_b < 0 \text{ and } -\hat{x}_{l_j} < 0 \\ & \leq \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{h_j} + A\hat{x}_{l_j} + \hat{y}_{h_j} - \hat{y}_{l_j} \\ & \leq \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{h_j} + A\hat{x}_{h_j} \\ & \qquad \qquad \text{since edge } (l_j, h_j) \text{ is gentle} \\ & = \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{h_j}. \end{aligned}$$

(b) In this case, we are either in Scenario 2 or 3.

Let $\hat{P}_j$ be the first inductive parallelogram in the sequence $\hat{P}_1, \hat{P}_2, ..., \hat{P}_{k-1}$ and we assume that the inductive point of $\hat{P}_j$ is $c = l_j$.

By Lemma 2.1.4, every parallelogram $\hat{P}_i$, for $i \leq j$ has a potential. Note that in this case, h_j is to the left of l_j. Since the edge (l_j, h_j) is gentle it follows that the vertical distance between l_j and h_j can be bounded:

$$(\hat{y}_{h_j} - \hat{y}_{l_j}) \leq A(\hat{x}_{l_j} - \hat{x}_{h_j}) \iff (\hat{y}_{h_j} - \hat{y}_{l_j}) + A\hat{x}_{h_j} \leq A\hat{x}_{l_j}. \tag{2.1.12}$$

Moreover, we can bound the length of the edge (l_j, h_j) by

$$\begin{aligned} d_2(l_j, h_j) &\leq \sqrt{1 + A^2 - 2A\cos(\theta)}(\hat{x}_{l_j} - \hat{x}_{h_j}) \\ &\leq \sqrt{1 + A^2 + 2A|\cos(\theta)|}(\hat{x}_{l_j} - \hat{x}_{h_j}) \end{aligned} \tag{2.1.13}$$

Let $h_i, h_{i+1}, ..., h_{j-1} = h_j$ be the maximal high path ending at h_j. Note that by Lemma 2.1.7 we have

$$d_2^T(h_i, h_j) \le (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}) \tag{2.1.14}$$

Next, note that either $h_i = h_0 = a$ or h_i is an eastern point in $\hat{P}_i$ that has a potential and Lemma 2.1.5 applies. In this case we have that since h_i is eastern in $\hat{P}_i$ we have:

$$\begin{aligned} d_2^T(a, h_i) &\le (1 + A)\hat{x}_i \\ &= (1 + A)\hat{x}_{h_i} \end{aligned} \tag{2.1.15}$$

Using inequalities (2.1.12), (2.1.13), (2.1.14) and (2.1.15), together with the triangular inequality, we get

$$\begin{aligned}
& d_2^T(a, l_j) - \hat{y}_{l_j} \\
\le\ & d_2^T(a, h_i) + d_2^T(h_i, h_j) + d_2(h_j, l_j) - \hat{y}_{l_j} \\
\le\ & (1 + A)\hat{x}_{h_i} + (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}) \\
& \quad + \sqrt{1 + A^2 + 2A|\cos(\theta)|}(\hat{x}_{l_j} - \hat{x}_{h_j}) - \hat{y}_{l_j} \\
=\ & A\hat{x}_{h_i} + \hat{x}_{h_j} + \sqrt{1 + A^2 + 2A|\cos(\theta)|}(\hat{x}_{l_j} - \hat{x}_{h_j}) + \hat{y}_{h_j} - \hat{y}_{l_j} \\
& \quad - \hat{y}_{h_i} \\
=\ & \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{l_j} + A\hat{x}_{h_i} + \hat{y}_{h_j} - \hat{y}_{l_j} \\
& \quad - \left(\sqrt{1 + A^2 + 2A|\cos(\theta)|} - 1\right)\hat{x}_{h_j} - \hat{y}_{h_i} \\
\le\ & \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{l_j} + A\hat{x}_{h_i} + \hat{y}_{h_j} - \hat{y}_{l_j} \\
& \qquad\qquad \text{since both } -\hat{y}_{h_i} < 0 \text{ and } -\hat{x}_{h_j} < 0 \\
\le\ & \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{l_j} + A\hat{x}_{h_j} + \hat{y}_{h_j} - \hat{y}_{l_j} \\
& \qquad\qquad \text{since } \hat{x}_{h_i} \le \hat{x}_{h_j} \\
\le\ & \sqrt{1 + A^2 + 2A|\cos(\theta)|}\hat{x}_{l_j} + A\hat{x}_{l_j} \\
& \qquad\qquad \text{since edge } (h_j, l_j) \text{ is gentle} \\
\le\ & \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{l_j}.
\end{aligned}$$

(c) In this case, we are either in Scenario 1 or 4.

Let $\hat{P}_j$ be the first inductive parallelogram in the sequence $\hat{P}_1, \hat{P}_2, ..., \hat{P}_{k-1}$ and we assume that the inductive point of $\hat{P}_j$ is $c = h_j$.

By Lemma 2.1.4, every parallelogram $\hat{P}_i$, for $i \le j$ has a potential. Note that in this case, l_j is to the left of h_j. Since the edge (l_j, h_j)

is gentle it follows that the vertical distance between l_j and h_j can be bounded:

$$(\hat{y}_{h_j} - \hat{y}_{l_j}) \leq \frac{1}{A}(\hat{x}_{h_j} - \hat{x}_{l_j}) \iff A(\hat{y}_{h_j} - \hat{y}_{l_j}) + \hat{x}_{l_j} \leq \hat{x}_{h_j}. \tag{2.1.16}$$

Moreover, we can bound the length of the edge (l_j, h_j) by

$$\begin{aligned} d_2(l_j, h_j) &\leq \sqrt{1 + \frac{1}{A^2} - \frac{2\cos(\pi - \theta)}{A}}(\hat{x}_{h_j} - \hat{x}_{l_j}) \\ &\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{h_j} - \hat{x}_{l_j}). \end{aligned} \tag{2.1.17}$$

Let $l_i, l_{i+1}, ..., l_{j-1} = l_j$ be the maximal low path ending at l_j. Note that by Lemma 2.1.7 we have

$$d_2^T(l_i, l_j) \leq (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_i} - \hat{y}_{l_j}). \tag{2.1.18}$$

Next, note that either $l_i = l_0 = a$ or l_i is an eastern point in $\hat{P}_i$ that has a potential and Lemma 2.1.5 applies. In this case we have that since l_i is eastern in $\hat{P}_i$ we have

$$\begin{aligned} d_2^T(a, l_i) &\leq (1 + 1/A)\hat{x}_i \\ &= (1 + 1/A)\hat{x}_{l_i}. \end{aligned} \tag{2.1.19}$$

Using inequalities (2.1.16), (2.1.17), (2.1.18) and (2.1.19), together with the triangular inequality, we get

$$\begin{aligned} & d_2^T(a, h_j) + A(\hat{y}_{h_j} - \hat{y}_b) \\ \leq\ & d_2^T(a, l_i) + d_2^T(l_i, l_j) + d_2(l_j, h_j) + A(\hat{y}_{h_j} - \hat{y}_b) \\ \leq\ & (1 + 1/A)\hat{x}_{l_i} + (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_i} - \hat{y}_{l_j}) \\ & + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{h_j} - \hat{x}_{l_j}) + A(\hat{y}_{h_j} - \hat{y}_b) \\ =\ & 1/A\hat{x}_{l_i} + \hat{x}_{l_j} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{h_j} - \hat{x}_{l_j}) \\ & + (\hat{y}_{l_i} - \hat{y}_{l_j}) + A(\hat{y}_{h_j} - \hat{y}_b) \\ \leq\ & 1/A\hat{x}_{l_i} + \hat{x}_{l_j} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{h_j} - \hat{x}_{l_j}) \\ & + A(\hat{y}_{l_i} - \hat{y}_{l_j}) + A(\hat{y}_{h_j} - \hat{y}_b) \\ & \qquad\qquad\qquad \text{since } A > 1 \\ =\ & 1/A\hat{x}_{l_i} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + A(\hat{y}_{l_i} - \hat{y}_{l_j}) \\ & + A(\hat{y}_{h_j} - \hat{y}_b) - (1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}})\hat{x}_{l_j} \\ \leq\ & 1/A\hat{x}_{l_i} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + A(\hat{y}_{l_i} - \hat{y}_{l_j}) \end{aligned}$$

$$
\begin{aligned}
&\quad + A(\hat{y}_{h_j} - \hat{y}_b) \\
&\qquad\qquad\qquad\qquad\qquad \text{since } -\hat{x}_{l_j} < 0 \\
&= 1/A\hat{x}_{l_i} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\quad + A(\hat{y}_{l_i} - \hat{y}_b) \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + 1/A\hat{x}_{l_i} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad\qquad\qquad\qquad \text{since } (\hat{y}_{l_i} - \hat{y}_b) < 0 \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + 1/A\hat{x}_{l_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad\qquad\qquad\qquad \text{since } \hat{x}_{l_i} \leq \hat{x}_{l_j} \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{h_j} + \hat{x}_{l_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad\qquad\qquad\qquad \text{since } A > 1 \\
&\leq \left(\sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{h_j} + \hat{x}_{h_j} \\
&\qquad\qquad\qquad\qquad\qquad \text{since edge } (l_j, h_j) \text{ is gentle} \\
&\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{h_j}.
\end{aligned}
$$

(d) In this case, we are either in Scenario 1 or 4.

Let $\hat{P}_j$ be the first inductive parallelogram in the sequence $\hat{P}_1, \hat{P}_2, ..., \hat{P}_{k-1}$ and we assume that the inductive point of $\hat{P}_j$ is $c = l_j$.

By Lemma 2.1.4, every parallelogram $\hat{P}_i$, for $i \leq j$ has a potential. Note that in this case, h_j is to the left of l_j. Since the edge (l_j, h_j) is gentle it follows that the vertical distance between l_j and h_j can be bounded:

$$(\hat{y}_{h_j} - \hat{y}_{l_j}) \leq \frac{1}{A}(\hat{x}_{l_j} - \hat{x}_{h_j}) \iff A(\hat{y}_{h_j} - \hat{y}_{l_j}) + \hat{x}_{h_j} \leq \hat{x}_{l_j}. \tag{2.1.20}$$

Moreover, we can bound the length of the edge (l_j, h_j) by

$$d_2(l_j, h_j) \leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{l_j} - \hat{x}_{h_j}). \tag{2.1.21}$$

Let $h_i, h_{i+1}, ..., h_{j-1} = h_j$ be the maximal low path ending at h_j. Note that by Lemma 2.1.7 we have

$$d_2^T(h_i, h_j) \leq (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}). \tag{2.1.22}$$

Next, note that either $h_i = h_0 = a$ or h_i is an eastern point in $\hat{P}_i$ that has a potential and Lemma 2.1.5 applies. In this case we have that since h_i is eastern in $\hat{P}_i$ we have

$$\begin{aligned} d_2^T(a, h_i) &\leq (1 + 1/A)\hat{x}_i \\ &= (1 + 1/A)\hat{x}_{h_i}. \end{aligned} \tag{2.1.23}$$

Using inequalities (2.1.20), (2.1.21), (2.1.22) and (2.1.23), together with the triangular inequality, we get

$$\begin{aligned}
& d_2^T(a, l_j) - A\hat{y}_{l_j} \\
&\leq d_2^T(a, h_i) + d_2^T(h_i, h_j) + d_2(h_j, l_j) - A\hat{y}_{l_j} \\
&\leq (1 + 1/A)\hat{x}_{h_i} + (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_j} - \hat{y}_{h_i}) \\
&\quad + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{l_j} - \hat{x}_{h_j}) - A\hat{y}_{l_j} \\
&= 1/A\hat{x}_{h_i} + \hat{x}_{h_j} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{l_j} - \hat{x}_{h_j}) \\
&\quad + (\hat{y}_{h_j} - \hat{y}_{h_i}) - A\hat{y}_{l_j} \\
&\leq 1/A\hat{x}_{h_i} + \hat{x}_{h_j} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{l_j} - \hat{x}_{h_j}) \\
&\quad + A(\hat{y}_{h_j} - \hat{y}_{h_i}) - A\hat{y}_{l_j} \\
&\qquad\qquad \text{since } A > 1 \\
&= 1/A\hat{x}_{h_i} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{l_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\quad - A\hat{y}_{h_i} - \left(\sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}} - 1\right)\hat{x}_{h_j} \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{l_j} + 1/A\hat{x}_{h_i} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad \text{since both } -\hat{y}_{h_i} < 0 \text{ and } -\hat{x}_{h_j} < 0 \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{l_j} + 1/A\hat{x}_{h_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad \text{since } \hat{x}_{h_i} \leq \hat{x}_{h_j} \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{l_j} + \hat{x}_{h_j} + A(\hat{y}_{h_j} - \hat{y}_{l_j}) \\
&\qquad\qquad \text{since } A > 1 \\
&\leq \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\hat{x}_{l_j} + \hat{x}_{l_j} \\
&\qquad\qquad \text{since edge } (h_j, l_j) \text{ is gentle} \\
&\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{l_j}.
\end{aligned}$$

□

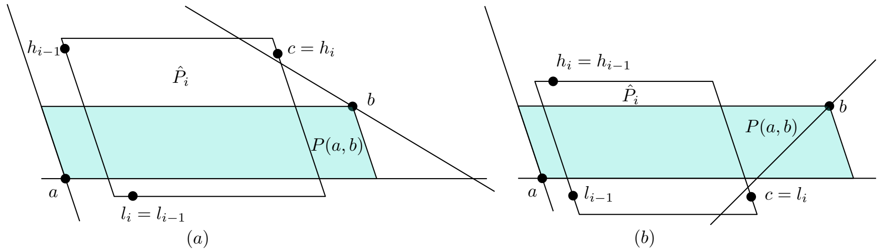

Figure 2.7: Illustration of Lemma 2.1.9 for Scenario 1. On the left (a), the inductive point is $c = h_i$ and on the right (b), the inductive point is $c = l_i$.

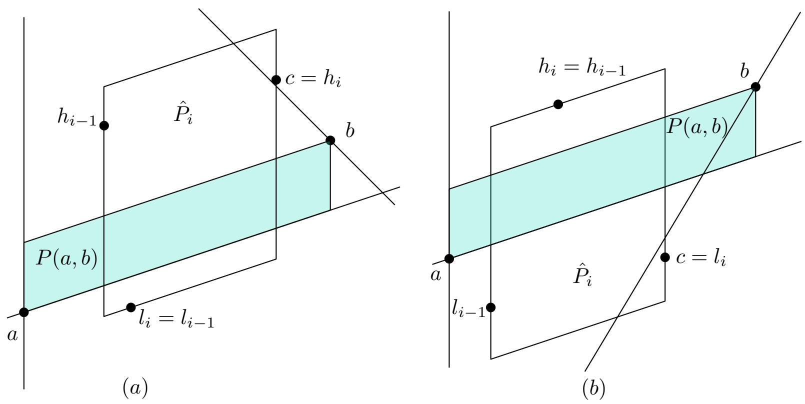

Figure 2.8: Illustration of Lemma 2.1.9 for Scenario 3. On the left (a), the inductive point is $c = h_i$ and on the right (b), the inductive point is $c = l_i$.

The final ingredient determines the type of edges we encounter when the $\hat{y}$-coordinate of a vertex differs significantly from that of b.

Lemma 2.1.9. *Assume $P(a,b)$ does not contain any vertices of $\mathcal{P}$. Let $1 < i < k$ and let the coordinates of the inductive point c of $\hat{P}_i$ be such that $0 < L(\hat{x}_b - \hat{x}_c) < |\hat{y}_b - \hat{y}_c|$.*

(i) If $c = h_i$, and thus $0 < L(\hat{x}_b - \hat{x}_c) < \hat{y}_c - \hat{y}_b$, then let j be the smallest index larger than i such that $L(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$. All edges on the path $h_i, ..., h_j$ are NE edges (refer to Figures 2.7(a) and 2.8(a)).

(ii) If $c = l_i$, and thus $0 < L(\hat{x}_b - \hat{x}_c) < \hat{y}_b - \hat{y}_c$, then let j be the smallest index larger than i such that $L(\hat{x}_b - \hat{x}_{l_j}) \geq \hat{y}_b - \hat{y}_{l_j} \geq 0$. All edges on the path $l_i, ..., l_j$ are SE edges (refer to Figures 2.7(b) and 2.8(b)).

Proof. We consider the case where $c = h_i$ as the case where $c = l_i$ can be shown using a similar argument.
Notice that such a j exists since choosing $h_j = b$ satisfies $L(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$. Next, let j be the smallest index larger than i such that $L(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$. Let h_m be any point on the path $h_i, ..h_{j-1}$, thus we have $0 < L(\hat{x}_b - \hat{x}_{h_m}) < \hat{y}_{h_m} - \hat{y}_b$. Observe that the edge (h_m, h_{m+1}), must be a WE, WN or NE edge in $\hat{P}_{m+1}$ since $\hat{x}_{h_{m+1}} > \hat{x}_{h_m}$ and neither

vertices can be below the line ab. If h_m lies on the W side of $\hat{P}_{m+1}$, then we will show that b must be inside $\hat{P}_{m+1}$, contradicting the fact that $\hat{P}_{m+1}$ must be empty. By construction we know that l_{m+1} must also be on the boundary of $\hat{P}_{m+1}$. Notice that the triangle formed by the WN, WS, SE corners of $\hat{P}_{m+1}$ can be expressed as the intersection of three half planes. We have that b is on the right side of the half plane given by the W side of $\hat{P}_{m+1}$, since $\hat{x}_{h_m} < \hat{x}_b$. Next, we also have that b is above the S side of $\hat{P}_{m+1}$, since $\hat{y}_{l_{m+1}} < \hat{y}_b$. Finally, by assumption we have that $\hat{y}_{h_m} > L(\hat{x}_b - \hat{x}_{h_m}) + \hat{y}_b$ which is below the diagonal of $\hat{P}_{m+1}$ as h_m is on the W side, hence b is below the diagonal given by the WN and SE corners. Since b is contained in the three half planes, then b is in the interior of $\hat{P}_{m+1}$. As such, we have that h_m cannot be on the W side and therefore, the edge (h_m, h_{m+1}) is NE. □

Next we prove the main theorem of this chapter. Recall that in all scenarios, the $\{\hat{x}, \hat{y}\}$ basis was defined so that $L(\hat{x}_b - \hat{x}_a) > \hat{y}_b - \hat{y}_a$, i.e. $L\hat{x}_b \geq \hat{y}_b$. Moreover, recall that at the beginning of the chapter, we assumed, without loss of generality, that the long side of P is vertical and the short side of P has non-negative slope.

Let $\Delta_{\hat{x}}(a,b)$ (respectively $\Delta_{\hat{y}}(a,b)$) denote the $\hat{x}$-coordinate difference (respectively the $\hat{y}$-coordinate difference) between a and b.

We now have all the ingredients needed to prove the main theorem.

Theorem 2.1.10. *Let a, b be two vertices in the parallelogram Delaunay graph. If $A\,\Delta_{\hat{x}}(a,b) \geq \Delta_{\hat{y}}(a,b)$ (i.e. we are in Scenario 2 or 3), then*

$$d_2^T(a,b) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b.$$

Otherwise, (i.e. we are in Scenario 1 or 4), we have

$$d_2^T(a,b) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b.$$

Proof. We consider all pairs of vertices (a,b) and order them by the size of the smallest scaled translate of $\hat{P}$ that has both a and b on its boundary. We perform induction based on the rank of this ordering.

The first pair (a,b) in this ordering has the smallest scaled translate of $\hat{P}$ and this can contain no vertices of $\mathcal{P}$. Indeed, the existence of a vertex would imply the existence of a smaller parallelogram with two vertices on its boundary. This would contradict that we are considering the smallest one. Hence, by construction there exists an edge between a and b, from which we get

$$d_2^T(a,b) = d_2(a,b) \leq \Delta_{\hat{x}}(a,b) + \Delta_{\hat{y}}(a,b) = \hat{x}_b + \hat{y}_b.$$

This satisfies the statement of the theorem for both cases.

Next, consider an arbitrary pair (a,b) and assume that the theorem holds for all pairs defining a smaller parallelogram. We consider two cases: (1) $P(a,b)$ does not contain any vertex of $\mathcal{P}$ or (2) $P(a,b)$ contains at least one vertex of $\mathcal{P}$.

(1) Assume there are no vertices of $\mathcal{P}$ inside $P(a,b)$. We distinguish (i) Scenarios 2 and 3 from (ii) Scenarios 1 and 4.

(i) Since we are in Scenario 2 or 3, we have $L = A$ and $A\hat{x}_b \geq \hat{y}_b$. If (a,b) is an edge in the parallelogram Delaunay graph, then

$$\begin{aligned} d_2^T(a,b) = d_2(a,b) &\leq \Delta_{\hat{x}}(a,b) + \Delta_{\hat{y}}(a,b) \\ &= \hat{x}_b + \hat{y}_b \\ &\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b. \end{aligned}$$

Otherwise, if no parallelogram in $\hat{P}_1, \hat{P}_2, .., \hat{P}_k$ is inductive then by property (1) of Lemma 2.1.8 we know that

$$d_2^T(a,b) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b.$$

Hence, we focus on the case where there is an inductive parallelogram. Let $\hat{P}_i$ be the first inductive parallelogram in the sequence $\hat{P}_1, ..., \hat{P}_k$. We distinguish between the case where the inductive point is h_i and where it is l_i. If the inductive point is h_i of $\hat{P}_i$, then by Property $(2)(a)$ of Lemma 2.1.8 we know that

$$d_2^T(a,h_i) + (\hat{y}_{h_i} - \hat{y}_b) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{h_i}$$

and thus that

$$d_2^T(a,h_i) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{h_i} - (\hat{y}_{h_i} - \hat{y}_b).$$

If $A(\hat{x}_b - \hat{x}_{h_i}) \geq \hat{y}_{h_i} - \hat{y}_b \geq 0$, we let $h_j = h_i$, else we let j be the smallest index greater than i such that $A(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$. By Property (i) of Lemma 2.1.9, h_j is eastern in $\hat{P}_j$ and all edges on the path $h_i, ..., h_j$ are NE edges. By the triangle inequality we get

$$d_2^T(h_m, h_{m+1}) \leq (\hat{x}_{h_{m+1}} - \hat{x}_{h_m}) + (\hat{y}_{h_m} - \hat{y}_{h_{m+1}})$$

for any edge (h_m, h_{m+1}) on this path. This implies that

$$d_2^T(h_i, h_j) \leq (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_i} - \hat{y}_{h_j}).$$

Since $A(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$ and the smallest scaled translate of $\hat{P}$ with h_j and b one its boundary is smaller than that of a and b we can use the induction hypothesis to get a bound on

$$\begin{aligned} d_2^T(h_j,b) \leq &\left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_{h_j}) \\ &+ (\hat{y}_{h_j} - \hat{y}_b). \end{aligned}$$

Putting everything together and using the triangular inequality we get that

$$\begin{aligned}
d_2^T(a,b) &\leq d_2^T(a,h_i) + d_2^T(h_i,h_j) + d_2^T(h_j,b) \\
&\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{h_i} - (\hat{y}_{h_i} - \hat{y}_b) \\
&\quad + (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_i} - \hat{y}_{h_j}) \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) (\hat{x}_b - \hat{x}_{h_j}) \\
&\quad + (\hat{y}_{h_j} - \hat{y}_b) \\
&= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{h_i} + (\hat{x}_{h_j} - \hat{x}_{h_i}) \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) (\hat{x}_b - \hat{x}_{h_j}) \\
&= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_b \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|} - 1\right) (\hat{x}_{h_i} - \hat{x}_{h_j}) \\
&\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_b \\
&\qquad\qquad\qquad\qquad \text{since } \hat{x}_{h_i} \leq \hat{x}_{h_j}. \\
&< \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_b + \hat{y}_b
\end{aligned}$$

Thus proving the theorem in the case where h_i is the inductive point of $\hat{P}_i$.

If l_i is the inductive point of $\hat{P}_i$, then by property $(2)(b)$ of Lemma 2.1.8 we know that

$$d_2^T(a,l_i) - \hat{y}_{l_i} \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{l_i}$$

and thus that

$$d_2^T(a,l_i) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right) \hat{x}_{l_i} + \hat{y}_{l_i}.$$

If $A(\hat{x}_b - \hat{x}_{l_i}) \geq \hat{y}_b - \hat{y}_{l_i} \geq 0$, we let $l_j = l_i$, else we let j be the smallest index greater than i such that $A(\hat{x}_b - \hat{x}_{l_j}) \geq \hat{y}_b - \hat{y}_{l_j} \geq 0$. By Property (ii) of Lemma 2.1.9, l_j is eastern in $\hat{P}_j$ and all edges on the path $l_i, ..., l_j$ are SE edges. By the triangle inequality we get

$$d_2^T(l_m, l_{m+1}) \leq (\hat{x}_{l_{m+1}} - \hat{x}_{l_m}) + (\hat{y}_{l_{m+1}} - \hat{y}_{l_m})$$

for any l_m and l_{m+1} on this path. This implies that

$$d_2^T(l_i, l_j) \leq (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_j} - \hat{y}_{l_i}).$$

Since $A(\hat{x}_b - \hat{x}_{l_j}) \geq \hat{y}_b - \hat{y}_{l_j} \geq 0$ and the smallest scaled translate of $\hat{P}$ with l_j and b on its boundary is smaller than that of a and b, we can use the induction hypothesis to get a bound of

$$d_2^T(l_j, b) \leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_{l_j}) + (\hat{y}_b - \hat{y}_{l_j}).$$

Putting everything together and using the triangular inequality we get

$$\begin{aligned}
d_2^T(a,b) &\leq d_2^T(a, l_i) + d_2^T(l_i, l_j) + d_2^T(l_j, b) \\
&\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{l_i} + \hat{y}_{l_i} \\
&\quad + (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_j} - \hat{y}_{l_i}) \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_{l_j}) \\
&\quad + (\hat{y}_b - \hat{y}_{l_j}) \\
&= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_{l_i} + (\hat{x}_{l_j} - \hat{x}_{l_i}) \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_{l_j}) + \hat{y}_b \\
&= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|} - 1\right)(\hat{x}_{l_i} - \hat{x}_{l_j}) \\
&\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b \\
&\qquad \text{since } \hat{x}_{l_i} \leq \hat{x}_{l_j}.
\end{aligned}$$

This completes the proof of Case (1)(i).

(ii) Since we are in Scenario 1 or 4, we have $L = \frac{1}{A}$ and $\frac{1}{A}\hat{x}_b \geq \hat{y}_b$. If (a, b) is an edge in the parallelogram Delaunay graph, then

$$\begin{aligned}
d_2^T(a,b) &= d_2(a,b) \\
&\leq d_{\hat{x}}(a,b) + d_{\hat{y}}(a,b) \\
&= \hat{x}_b + \hat{y}_b \\
&\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b.
\end{aligned}$$

Otherwise, if no parallelogram in $\hat{P}_1, \hat{P}_2, .., \hat{P}_k$ is inductive then by Property (1) of Lemma 2.1.8 we know that

$$d_2^T(a,b) \leq \left(\frac{1}{A} + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + \hat{y}_b$$

$$\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_b + A\hat{y}_b$$

since $A \geq 1$.

Hence, we focus on the case where there is an inductive parallelogram. Let $\hat{P}_i$ be the first inductive parallelogram in the sequence $\hat{P}_1, ..., \hat{P}_k$. We distinguish between the case where the inductive point is h_i and where is it l_i. If the inductive point is h_i of $\hat{P}_i$, then by property $(2)(c)$ of Lemma 2.1.8, we know that

$$d_2^T(a, h_i) + A(\hat{y}_{h_i} - \hat{y}_b) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{h_i},$$

and thus that

$$d_2^T(a, h_i) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{h_i} - A(\hat{y}_{h_i} - \hat{y}_b).$$

If $\frac{1}{A}(\hat{x}_b - \hat{x}_{h_i}) \geq \hat{y}_{h_i} - \hat{y}_b \geq 0$, we let $h_j = h_i$, else we let j be the smallest index greater than i such that $\frac{1}{A}(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$. By Property (i) of Lemma 2.1.9, h_j is eastern in $\hat{P}_j$ and all edges on the path $h_i, ..., h_j$ are NE edges. By the triangle inequality we get that

$$d_2^T(h_i, h_j) \leq (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_i} - \hat{y}_{h_j}).$$

Since $\frac{1}{A}(\hat{x}_b - \hat{x}_{h_j}) \geq \hat{y}_{h_j} - \hat{y}_b \geq 0$ and the smallest scaled translate of $\hat{P}$ with h_j and b on its boundary is smaller than that of a and b we can use the induction hypothesis to get a bound on

$$d_2^T(h_j, b) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) (\hat{x}_b - \hat{x}_{h_j}) + A(\hat{y}_{h_j} - \hat{y}_b).$$

Putting everything together and using the triangular inequality we get

$$\begin{aligned} d_2^T(a, b) &\leq d_2^T(a, h_i) + d_2^T(h_i, h_j) + d_2^T(h_j, b) \\ &\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{h_i} - A(\hat{y}_{h_i} - \hat{y}_b) \\ &\quad + (\hat{x}_{h_j} - \hat{x}_{h_i}) + (\hat{y}_{h_i} - \hat{y}_{h_j}) \\ &\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) (\hat{x}_b - \hat{x}_{h_j}) \\ &\quad + A(\hat{y}_{h_j} - \hat{y}_b) \\ &\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_{h_i} - A(\hat{y}_{h_i} - \hat{y}_b) \end{aligned}$$

$$+ (\hat{x}_{h_j} - \hat{x}_{h_i}) + A(\hat{y}_{h_i} - \hat{y}_{h_j})$$
$$+ \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_{h_j})$$
$$+ A(\hat{y}_{h_j} - \hat{y}_b)$$
$$\text{since } A \geq 1,$$
$$\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b$$
$$+ \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}(\hat{x}_{h_i} - \hat{x}_{h_j})$$
$$\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b$$
$$\text{since } \hat{x}_{h_i} \leq \hat{x}_{h_j}.$$
$$< \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b$$

Now, if l_i is the inductive point of $\hat{P}_i$ then by property $(2)(d)$ of Lemma 2.1.8 we know that

$$d_2^T(a, l_i) - A\hat{y}_{l_i} \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{l_i},$$

as such we have that

$$d_2^T(a, l_i) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{l_i} + A\hat{y}_{l_i}.$$

If $\frac{1}{A}(\hat{x}_b - \hat{x}_{l_i}) \geq \hat{y}_b - \hat{y}_{l_i} \geq 0$, we let $l_j = l_i$, else we let j be the smallest index greater than i such that $\frac{1}{A}(\hat{x}_b - \hat{x}_{l_j}) \geq \hat{y}_b - \hat{y}_{l_j} \geq 0$. By Property (ii) of Lemma 2.1.9, l_j is eastern in $\hat{P}_j$ and all edges on the path $l_i, ..., l_j$ are SE edges. By the triangle inequality we get that

$$d_2^T(l_i, l_j) \leq (\hat{x}_{l_j} - \hat{x}_{l_i}) + (\hat{y}_{l_j} - \hat{y}_{l_i}).$$

Since $\frac{1}{A}(\hat{x}_b - \hat{x}_{l_j}) \geq \hat{y}_b - \hat{y}_{l_j} \geq 0$ and the smallest scaled translate of $\hat{P}$ with l_j and b on its boundary is smaller than that of a and b we can use the induction hypothesis to get a bound on

$$d_2^T(l_j, b) \leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_{l_j})$$
$$+ A(\hat{y}_b - \hat{y}_{l_j}).$$

Putting everything together and using the triangular inequatity we get

$$d_2^T(a, b) \leq d_2^T(a, l_i) + d_2^T(l_i, l_j) + d_2^T(l_j, b)$$

$$
\begin{aligned}
&\leq \left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{l_i}+A\hat{y}_{l_i}\\
&\quad+(\hat{x}_{l_j}-\hat{x}_{l_i})+(\hat{y}_{l_j}-\hat{y}_{l_i})\\
&\quad+\left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b-\hat{x}_{l_j})\\
&\quad+A(\hat{y}_b-\hat{y}_{l_j})\\
&\leq \left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)\hat{x}_{l_i}+A\hat{y}_{l_i}\\
&\quad+(\hat{x}_{l_j}-\hat{x}_{l_i})+A(\hat{y}_{l_j}-\hat{y}_{l_i})\\
&\quad+\left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b-\hat{x}_{l_j})\\
&\quad+A(\hat{y}_b-\hat{y}_{l_j})\\
&\qquad\qquad\qquad\qquad\qquad\text{since } A\geq 1,\\
&\leq \left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b\\
&\quad+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}(\hat{x}_{l_i}-\hat{x}_{l_j})+A\hat{y}_b\\
&\leq \left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b+A\hat{y}_b\\
&\qquad\qquad\qquad\qquad\qquad\text{since } \hat{x}_{l_i}\leq\hat{x}_{l_j}.
\end{aligned}
$$

This completes the proof for Case (1)(ii), which completes the proof for Case (1).

(2) Assume $P(a,b)$ contains at least one vertex of $\mathcal{P}$. We distinguish (i) Scenarios 2 and 3 from (ii) Scenarios 1 and 4.

   (i) Assume we are in Scenario 2 or 3.

   We split $P(a,b)$ into three regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Informally, these regions can be constructed by considering the line through a and the line through b parallel to the longer diagonal of $\hat{P}$ and labelling the resulting regions as $\mathcal{A}, \mathcal{B}, \mathcal{C}$ from left to right (refer to Figure 2.9(a)). Formally, we have

$$
\begin{aligned}
\mathcal{A} =&\{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_p-\hat{x}_a)<\hat{y}_p-\hat{y}_a\},\\
\mathcal{B} =&\{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_p-\hat{x}_a)\geq\hat{y}_p-\hat{y}_a\\
&\quad \text{and } A(\hat{x}_b-\hat{x}_p)\geq\hat{y}_b-\hat{y}_p\},\\
\mathcal{C} =&\{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_b-\hat{x}_p)<\hat{y}_b-\hat{y}_p\}.
\end{aligned}
$$

   Observe that a point $p\in\mathcal{A}$ also satisfies $A(\hat{x}_b-\hat{x}_p)>\hat{y}_b-\hat{y}_p$. Moreover a point $p\in\mathcal{C}$ also satisfies $A(\hat{x}_p-\hat{x}_a)<\hat{y}_p-\hat{y}_a$.

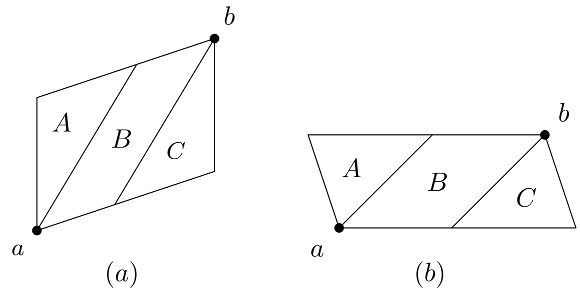

Figure 2.9: Illustration of the three regions in $P(a,b)$. Scenario 3 is shown on the left (a), and Scenario 1 is shown on the right (b).

If there exists a vertex p inside region $\mathcal{B}$, then we can apply the induction hypothesis on the pairs (a,p) and (p,b), which satisfies $A(\hat{x}_p - \hat{x}_a) \geq \hat{y}_p - \hat{y}_a$ and $A(\hat{x}_b - \hat{x}_p) \geq \hat{y}_b - \hat{y}_p$, respectively. We get

$$\begin{aligned} d_2^T(a,b) &\leq d_2^T(a,p) + d_2^T(p,b) \\ &\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_p - \hat{x}_a) \\ &\quad + (\hat{y}_p - \hat{y}_a) \\ &\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_p) \\ &\quad + (\hat{y}_b - \hat{y}_p) \\ &= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_p \\ &\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_p) + \hat{y}_b \\ &= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b. \end{aligned}$$

If there is no vertex inside region $\mathcal{B}$, we define $\hat{P}_a$ to be the smallest homothet of $\hat{P}$ that has a on its W side and some vertex $p \in \mathcal{A}$ in $P(a,b)$ on its boundary. Similarly, we define $\hat{P}_b$ to be the smallest homothet of $\hat{P}$ that has b on its E side and some vertex $q \in \mathcal{C}$ in $P(a,b)$ on its boundary. Since $P(a,b)$ is not empty, at least one of p or q must exist. Assume without loss of generality that p exists. In this case, we have that $A(\hat{x}_b - \hat{x}_p) > \hat{y}_b - \hat{y}_p$ and the smallest homothet of $\hat{P}$ with p and b on its boundary is smaller than that of a and b. Therefore, we can apply the induction hypothesis on the pair (p,b). If (a,p) is an edge in the parallelogram Delaunay graph we get:

$$\begin{aligned} d_2^T(a,b) &\leq d_2^T(a,p) + d_2^T(p,b) \\ &= d_2(a,p) + d_2^T(p,b) \end{aligned}$$

$$
\begin{aligned}
&\leq (\hat{x}_p - \hat{x}_a) + (\hat{y}_p - \hat{y}_a) \\
&\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_p) \\
&\quad + (\hat{y}_b - \hat{y}_p) \\
&= \hat{x}_p + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{x}_b - \hat{x}_p) \\
&\quad + \hat{y}_b \\
&\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b
\end{aligned}
$$

An analogous argument can be used if q exists and (q, b) in an edge in the parallelogram Delaunay graph.

It remains to consider the case where (a, p) is not an edge, in which case the parallelogram $\hat{P}_a$ is not empty. This implies that there exists $p' \in \mathcal{C}$ such that (a, p') is an edge. We have that $A(\hat{x}_b - \hat{x}_{p'}) < \hat{y}_b - \hat{y}_{p'}$ and the smallest homothet of $\hat{P}$ with p' and b on its boundary is smaller than that of a and b. By the induction hypothesis we have

$$
\begin{aligned}
d_2^T(p', b) \leq\ & A\left(\hat{x}_b - \hat{x}_{p'}\right) \\
&+ \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{y}_b - \hat{y}_{p'}).
\end{aligned}
$$

Since $p' \in \mathcal{C}$, we applied the induction hypothesis for instances of Scenarios 1 and 4 for the path from p' to b. This explains why we swapped the $\{\hat{x}, \hat{y}\}$ basis. We get

$$
\begin{aligned}
d_2^T(a, b) &\leq d_2^T(a, p') + d_2^T(p', b) \\
&= d_2(a, p') + d_2^T(p', b) \\
&\leq (\hat{x}_{p'} - \hat{x}_a) + (\hat{y}_{p'} - \hat{y}_a) + A(\hat{x}_b - \hat{x}_{p'}) \\
&\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{y}_b - \hat{y}_{p'}) \\
&= \hat{x}_{p'} + \hat{y}_{p'} + A(\hat{x}_b - \hat{x}_{p'}) \\
&\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{y}_b - \hat{y}_{p'}) \\
&\leq A\hat{x}_b + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{y}_b \\
&\qquad\qquad\qquad\qquad \text{since } A \geq 1.
\end{aligned}
$$

Since we are in Scenario 2 or 3, $A\hat{x}_b \geq \hat{y}_b$ and we know that $1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}} > 1$, we have[4]

$$\begin{aligned} d_2^T(a,b) &\leq A\hat{x}_b + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{y}_b \\ &\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)A\hat{x}_b + \hat{y}_b \\ &= \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b \end{aligned}$$

(ii) Assume we are in Scenario 1 or 4. We split $P(a,b)$ into three regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ (refer to Figure 2.9(b)).

$$\begin{aligned} \mathcal{A} =& \{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_b - \hat{x}_p) \geq \hat{y}_b - \hat{y}_p\}, \\ \mathcal{B} =& \{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_b - \hat{x}_p) < \hat{y}_b - \hat{y}_p \\ & \text{and } A(\hat{x}_p - \hat{x}_a) < \hat{y}_p - \hat{y}_a\}, \\ \mathcal{C} =& \{p : p \text{ is inside } P(a,b) \text{ such that } A(\hat{x}_p - \hat{x}_a) \geq \hat{y}_p - \hat{y}_a\}. \end{aligned}$$

Observe that a point $p \in \mathcal{A}$ also satisfies $A(\hat{x}_p - \hat{x}_a) > (\hat{y}_p - \hat{y}_a)$. Moreover a point $p \in \mathcal{C}$ also satisfies $A(\hat{x}_b - \hat{x}_p) < \hat{y}_b - \hat{y}_p$.

If there exists a vertex p inside region $\mathcal{B}$, then we can apply the induction hypothesis on the pairs (a,p) and (p,b), which satisfies $A(\hat{x}_p - \hat{x}_a) < \hat{y}_p - \hat{y}_a$ and $A(\hat{x}_b - \hat{x}_p) < \hat{y}_b - \hat{y}_p$, respectively. We get

$$\begin{aligned} d_2^T(a,b) &\leq d_2^T(a,p) + d_2^T(p,b) \\ &\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_p - \hat{x}_a) \\ &\quad + A(\hat{y}_p - \hat{y}_a) \\ &\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_p) \\ &\quad + A(\hat{y}_b - \hat{y}_p) \\ &= \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_p + A\hat{y}_p \\ &\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_p) \\ &\quad + A(\hat{y}_b - \hat{y}_p) \\ &= \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b. \end{aligned}$$

4 From the fact that if $n > m > 0$ and $k > 1$ then $kn + m > km + n$ (Proof: $kn + m > km + n \iff k(n-m) + (m-n) > 0 \iff k(n-m) > (n-m)$ which is true for $k \geq 1$). In our case, we have that $n = A\hat{x}_b, m = \hat{y}_b$ and $k = 1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}$.

If there is no vertex inside region $\mathcal{B}$, we define $\hat{P}_a$ to be the smallest homothet of $\hat{P}$ that has a on its W side and some vertex $p \in \mathcal{A}$ in $P(a,b)$ on its boundary. Similarly, we define $\hat{P}_b$ to be the smallest homothet of P that has b on its E side and some vertex $q \in \mathcal{C}$ in $P(a,b)$ on its boundary. Since $P(a,b)$ is not empty, at least one of p or q must exist. Assume without loss of generality that p exists. In this case, we have that $A(\hat{x}_b - \hat{x}_p) > \hat{y}_b - \hat{y}_p$ and the smallest homothet of $\hat{P}$ with p and b on its boundary is smaller than that of a and b. Therefore, we can apply the induction hypothesis on the pair (p,b). We get

$$\begin{aligned} d_2^T(p,b) \leq & \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{y}_b - \hat{y}_p) \\ & + (\hat{x}_b - \hat{x}_p). \end{aligned}$$

Since $p \in \mathcal{A}$, we applied the induction hypothesis from Scenarios 2 and 3 for the path from p to b. This explains why we swapped the $\{\hat{x}, \hat{y}\}$. If (a,p) is an edge in the parallelogram Delaunay graph we get:

$$\begin{aligned} d_2^T(a,b) &\leq d_2^T(a,p) + d_2^T(p,b) \\ &= d_2(a,p) + d_2^T(p,b) \\ &\leq (\hat{x}_p - \hat{x}_a) + (\hat{y}_p - \hat{y}_a) \\ &\quad + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{y}_b - \hat{y}_p) \\ &\quad + (\hat{x}_b - \hat{x}_p) \\ &= \hat{y}_p + \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)(\hat{y}_b - \hat{y}_p) \\ &\quad + \hat{x}_b \\ &\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{y}_b + \hat{x}_b. \end{aligned}$$

Since $\frac{1}{A}\hat{x}_b > \hat{y}_b$, we have $\hat{x}_b > A\hat{y}_b$, and since $1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}} > 1$, we find

$$\begin{aligned} d_2^T(a,b) &\leq \left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{y}_b + \hat{x}_b \\ &= \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) A\hat{y}_b + \hat{x}_b \\ &\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right) \hat{x}_b + A\hat{y}_b. \end{aligned}$$

An analogous argument can be used if q exists where (q,b) is an edge in the parallelogram Delaunay graph.

Hence, it remains to consider the case where (a,p) is not an edge, in which case $\hat{P}_a$ is not empty. This implies that there exists a $p' \in \mathcal{C}$ such that (a, p') is an edge. We have that $A(\hat{x}_b - \hat{x}_{p'}) < (\hat{y}_b - \hat{y}_{p'})$ and the smallest scaled translate of $\hat{P}$ with p' and b on its boundary is smaller than that of a and b. By the induction hypothesis we have

$$\begin{aligned}
d_2^T(a,b) &\leq d_2^T(a,p') + d_2^T(p',b) \\
&= d_2(a,p') + d_2^T(p',b) \\
&\leq (\hat{x}_{p'} - \hat{x}_a) + (\hat{y}_{p'} - \hat{y}_a) \\
&\quad + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_{p'}) \\
&\quad + A(\hat{y}_b - \hat{y}_{p'}) \\
&= \hat{x}_{p'} + \hat{y}_{p'} + \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)(\hat{x}_b - \hat{x}_{p'}) \\
&\quad + A(\hat{y}_b - \hat{y}_{p'}) \\
&\leq \left(1 + \sqrt{1 + \frac{1}{A^2} + \frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b
\end{aligned}$$

This completes the proof for case (2) and hence, for the theorem.

□

We can now use Theorem 2.1.10 to give an upper bound on the spanning ratio of the parallelogram Delaunay graph. For any pair of vertices a, b in the graph, if we are in Scenario 2 or 3, we have

$$\begin{aligned}
\frac{d_2^T(a,b)}{d_2(a,b)} &\leq \frac{\left(A + \sqrt{1 + A^2 + 2A|\cos(\theta)|}\right)\hat{x}_b + \hat{y}_b}{\sqrt{\hat{x}_b^2 + \hat{y}_b^2 - 2\hat{x}_b\hat{y}_b\cos(\pi - \theta)}} \\
&= \frac{A + \sqrt{1 + A^2 + 2A|\cos(\theta)|} + \frac{\hat{y}_b}{\hat{x}_b}}{\sqrt{1 + (\frac{\hat{y}_b}{\hat{x}_b})^2 + 2\frac{\hat{y}_b}{\hat{x}_b}\cos(\theta)}}.
\end{aligned} \tag{2.1.24}$$

Let $f_{2,3}(\hat{y}_b/\hat{x}_b)$ be this function.

The derivative of $f_{2,3}(\hat{y}_b/\hat{x}_b)$ is equal to 0 whenever

$$\frac{\hat{y}_b}{\hat{x}_b} = \frac{-\cos(\theta)\sqrt{1 + A^2 + 2A|\cos(\theta)|} - A\cos(\theta) + 1}{\sqrt{1 + A^2 + 2A|\cos(\theta)|} + A - \cos(\theta)},$$

in which case $f_{2,3}(\hat{y}_b/\hat{x}_b)$ is equal to

$$\frac{\sqrt{2}\sqrt{1+A^2+A(|\cos(\theta)|-\cos(\theta))+(A-\cos(\theta))\sqrt{1+A^2+2A|\cos(\theta)|}}}{\sin(\theta)}. \tag{2.1.25}$$

Therefore, the maximum of $f_{2,3}(\hat{y}_b/\hat{x}_b)$ is at least the expression in (2.1.25).

On the other hand, if we are in Scenario 1 or 4, we get that

$$\frac{d_2^T(a,b)}{d_2(a,b)} \leq \frac{\left(1+\sqrt{1+\frac{1}{A^2}+\frac{2|\cos(\theta)|}{A}}\right)\hat{x}_b + A\hat{y}_b}{\sqrt{\hat{x}_b^2+\hat{y}_b^2-2\hat{x}_b\hat{y}_b\cos(\pi-\theta)}} = \frac{A+\sqrt{1+A^2+2A|\cos(\theta)|}+A^2\frac{\hat{y}_b}{\hat{x}_b}}{A\sqrt{1+(\frac{\hat{y}_b}{\hat{x}_b})^2+2\frac{\hat{y}_b}{\hat{x}_b}\cos(\theta)}}. \tag{2.1.26}$$

Let $f_{1,4}(\hat{y}_b/\hat{x}_b)$ be this function.

The derivative of $f_{1,4}(\hat{y}_b/\hat{x}_b)$ is equal to 0 whenever

$$\frac{\hat{y}_b}{\hat{x}_b} = \frac{-\cos(\theta)\sqrt{1+A^2+2A|\cos(\theta)|}+A^2-A\cos(\theta)}{\sqrt{1+A^2+2A|\cos(\theta)|}-A^2\cos(\theta)+A},$$

in which case $f_{1,4}(\hat{y}_b/\hat{x}_b)$ is equal to

$$\frac{\sqrt{(1+A^2)^2+2A(|\cos(\theta)|-A^2\cos(\theta))+2A(1-A\cos(\theta))\sqrt{1+A^2+2A|\cos(\theta)|}}}{A\sin(\theta)}. \tag{2.1.27}$$

The three candidate values for the maximum of $f_{2,3}(\hat{y}_b/\hat{x}_b)$ (respectively $f_{1,4}(\hat{y}_b/\hat{x}_b)$) are

(1) $f_{2,3}(0)$ (respectively $f_{1,4}(0)$),

(2) $\lim_{\hat{y}_b/\hat{x}_b\to\infty} f_{2,3}(\hat{y}_b/\hat{x}_b) = 1$ (respectively $\lim_{\hat{y}_b/\hat{x}_b\to\infty} f_{1,4}(\hat{y}_b/\hat{x}_b) = A$) and,

(3) the value of $f_{2,3}$ (respectively $f_{1,4}$) when its derivative is 0, which we denote by $f_{2,3}^*$ (respectively $f_{1,4}^*$).

We can show that $f_{2,3}(0) \geq f_{1,4}(0)$, $f_{2,3}^* \geq A$ and $f_{2,3}^* \geq f_{1,4}^*$. Hence, the maximum occurs in Scenario 2 or 3. Moreover, we can show that the maximum value of $f_{2,3}(\hat{y}_b/\hat{x}_b)$ is obtained when its derivative is 0. As such, we find the following expression for the spanning ratio:

$$\frac{\sqrt{2}\sqrt{1+A^2+A(|\cos(\theta)|-\cos(\theta))+(A-\cos(\theta))\sqrt{1+A^2+2A|\cos(\theta)|}}}{\sin(\theta)}.$$

The worst case of the above expression occurs when $\theta = \pi - \theta_0$, i.e., when we are in Scenario 2. This allows us to conclude with the following.

Theorem 2.1.11. *The spanning ratio of a parallelogram Delaunay graph is at most*

$$\frac{\sqrt{2}\sqrt{1+A^2+2A\cos(\theta_0)+(A+\cos(\theta_0))\sqrt{1+A^2+2A\cos(\theta_0)}}}{\sin(\theta_0)}. \tag{2.1.28}$$

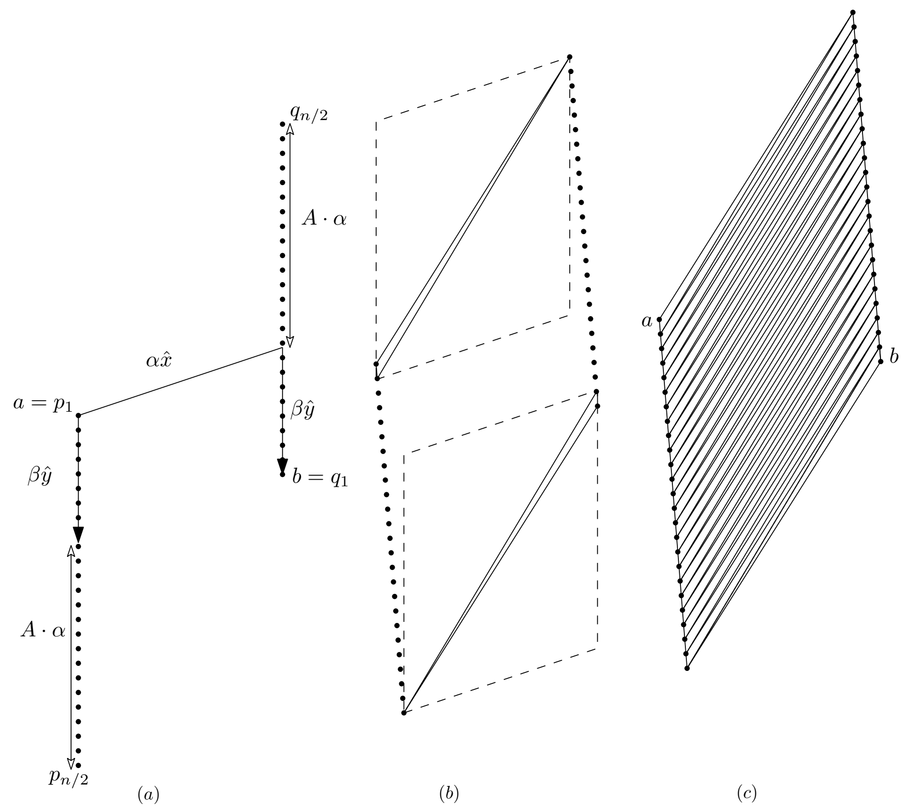

Figure 2.10: Lower bound construction.

2.2 THE LOWER BOUND

Theorem 2.2.1. *There are parallelogram Delaunay graphs that have spanning ratio arbitrarily close to*

$$\frac{\sqrt{2}\sqrt{1 + A^2 + 2A\cos(\theta_0) + (A + \cos(\theta_0))\sqrt{1 + A^2 + 2A\cos(\theta_0)}}}{\sin(\theta_0)}$$

We now construct a set of points whose parallelogram Delaunay graph achieves the lower bound. We are in Scenario 2 with $A \geq 1$, we have $L = A$ and $\pi - \theta = \theta_0 \leq \frac{\pi}{2}$. We construct this point set as follows, let a be the origin and let $b = \alpha\hat{x} + \beta\hat{y}$. We make two vertical columns of equidistant vertices $p_1, .., p_{n/2}$ and $q_1, .., q_{n/2}$ where $p_1 = a$ and $p_{n/2} = (\beta + \alpha A)\hat{y}$ and $q_1 = b$ and $q_{n/2} = \alpha\hat{x} - \alpha A\hat{y}$, refer to Figure 2.10($a$). Next, we move the vertices an arbitrary small distance in the horizontal direction, such that p_i lies to the left of p_{i+1} and q_i lies to the left of q_{i+1} for $1 \leq i \leq n/2$, refer to Figure 2.10(b). Moreover, we also show two triangles and their corresponding parallelograms.

We proceed to analyze the length of one of the shortest paths between a and b. Specifically the one via $p_{n/2}$, refer to Figure 2.10(c). Since all perturbations can be made arbitrarily small, this path has length

$$(\beta + \alpha A) + \sqrt{\alpha^2 + (-\alpha A)^2 - 2\alpha(-\alpha A)\cos(\pi - \theta)}$$

$$= \alpha(A + \sqrt{1 + A^2 + 2A\cos(\theta_0)}) + \beta.$$

The Euclidean distance between a and b is arbitrarily close to

$$\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\cos(\theta_0)}.$$

This implies that the spanning ratio is arbitrarily close to

$$\frac{\alpha(A + \sqrt{1 + A^2 + 2A|\cos(\theta_0)|}) + \beta}{\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\cos(\theta_0)}},$$

which matches $f_{2,3}(\beta/\alpha)$ defined at the end of the previous section.

3 CONCLUSION

This thesis generalized the approach by Bonichon et al. [3], and van Renssen et al. [18] to find the exact spanning ratio of the parallelogram Delaunay graph. This makes the parallelogram the fifth convex shape for which a tight spanning ratio is known. Note that the work of van Renssen et al. and of Bonichon et al. are special cases of this result. Indeed, by setting $\theta_0 = \pi/2$ in (2.1.28), we obtain the exact spanning ratio for the rectangle Delaunay graph [18]. If we also set $A = 1$, we obtain the exact spanning ratio for the square Delaunay graph [3]. Obviously, our bound also applies to diamonds.

Aside from the result itself, a significant technical contribution was the approach to generalizing the proof, which involves working with non-orthonormal bases. We believe this technique could be used to further broaden the class of convex shapes for which tight spanning ratios are known. For example, it may be feasible to apply this technique to the proof of Chew [11] for the TD Delaunay graph. This would lead to an exact spanning ratio for the triangle Delaunay graph, where the triangle can be arbitrary.

BIBLIOGRAPHY

[1] Luis Barba, Prosenjit Bose, Mirela Damian, Rolf Fagerberg, Wah Loon Keng, Joseph O'Rourke, André van Renssen, Perouz Taslakian, Sander Verdonschot, and Ge Xia. "New and improved spanning ratios for Yao graphs." In: *J. Comput. Geom.* 6.2 (2015), pp. 19–53.

[2] Nicolas Bonichon, Cyril Gavoille, Nicolas Hanusse, and David Ilcinkas. "Connections between Theta-Graphs, Delaunay Triangulations, and Orthogonal Surfaces." In: *WG*. Vol. 6410. Lecture Notes in Computer Science. 2010, pp. 266–278.

[3] Nicolas Bonichon, Cyril Gavoille, Nicolas Hanusse, and Ljubomir Perkovic. "Tight stretch factors for L_1- and L_∞-Delaunay triangulations." In: *Comput. Geom.* 48.3 (2015), pp. 237–250.

[4] Prosenjit Bose, Pilar Cano, and Rodrigo I. Silveira. "Affine invariant triangulations." In: *Comput. Aided Geom. Des.* 91 (2021), p. 102039.

[5] Prosenjit Bose, Paz Carmi, Sébastien Collette, and Michiel H. M. Smid. "On the Stretch Factor of Convex Delaunay Graphs." In: *J. Comput. Geom.* 1.1 (2010), pp. 41–56.

[6] Prosenjit Bose, Jean-Lou De Carufel, Darryl Hill, and Michiel H. M. Smid. "On the Spanning and Routing Ratio of Theta-Four." In: *SODA*. SIAM, 2019, pp. 2361–2370.

[7] Prosenjit Bose, Jean-Lou De Carufel, Pat Morin, André van Renssen, and Sander Verdonschot. "Towards tight bounds on theta-graphs: More is not always better." In: *Theor. Comput. Sci.* 616 (2016), pp. 70–93.

[8] Prosenjit Bose, Jean-Lou De Carufel, and André van Renssen. "Constrained generalized Delaunay graphs are plane spanners." In: *Comput. Geom.* 74 (2018), pp. 50–65.

[9] Prosenjit Bose, Luc Devroye, Maarten Löffler, Jack Snoeyink, and Vishal Verma. "Almost all Delaunay triangulations have stretch factor greater than pi/2." In: *Comput. Geom.* 44.2 (2011), pp. 121–127.

[10] Prosenjit Bose, Darryl Hill, and Aurélien Ooms. "Improved Bounds on the Spanning Ratio of the Theta-5-Graph." In: *WADS*. Vol. 12808. Lecture Notes in Computer Science. Springer, 2021, pp. 215–228.

[11] Paul Chew. "There are Planar Graphs Almost as Good as the Complete Graph." In: *Symposium on Computational Geometry*. 1986, pp. 169–177.

[12] Paul Chew. "There are Planar Graphs Almost as Good as the Complete Graph." In: *J. Comput. Syst. Sci.* 39.2 (1989), pp. 205–219.

[13] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. “Delaunay Graphs are almost as Good as Complete Graphs.” In: *Discret. Comput. Geom.* 5 (1990), pp. 399–407.

[14] J. Mark Keil and Carl A. Gutwin. “Classes of Graphs Which Approximate the Complete Euclidean Graph.” In: *Discret. Comput. Geom.* 7 (1992), pp. 13–28.

[15] Giri Narasimhan and Michiel H. M. Smid. *Geometric spanner networks*. Cambridge University Press, 2007.

[16] Atsuyuki Okabe, Barry Boots, Kokichi Sugihara, Sung Nok Chiu, and David G. Kendall. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, Second Edition*. Wiley Series in Probability and Mathematical Statistics. Wiley, 2000.

[17] Ljubomir Perkovic, Michael Dennis, and Duru Türkoglu. “The Stretch Factor of Hexagon-Delaunay Triangulations.” In: *J. Comput. Geom.* 12.2 (2021), pp. 86–125.

[18] André van Renssen, Yuan Sha, Yucheng Sun, and Sampson Wong. “The Tight Spanning Ratio of the Rectangle Delaunay Triangulation.” In: *ESA*. Vol. 274. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023, 99:1–99:15.

[19] Ge Xia. “The Stretch Factor of the Delaunay Triangulation Is Less than 1.998.” In: *SIAM J. Comput.* 42.4 (2013), pp. 1620–1659.

[20] Ge Xia and Liang Zhang. “Toward the Tight Bound of the Stretch Factor of Delaunay Triangulations.” In: *CCCG*. 2011.