Faculté des sciences sociales | Faculty of Social Sciences
Département de science économique
Department of Economics
uOttawa

Decision Making of Private Provision of Public Goods and Charitable Investment -A Game Theoretic Analysis

ECO 6999 Major Paper

Han Jiang 6200229

for the degree of

MA of Economics

Written under the direction of

Professor Vicky Barham

Table of Contents

Decision Making of Private Provision of Public Goods and Charitable Investment - A Game Theoretic Analysis

Abstract

This paper aims to provide theoretical insight into the relationship between private voluntary contribution and risky portfolio investments of charity organizations. To analyze this relationship, we will implement a three-stage non-cooperative game with two economic agents: the donor and the charity. The choices of the economic agents in the first two stages of the game are determined endogenously with the expectation that the opposite economic agents will react rationally in the following stages. We find that donors do in fact care about the investment strategies that charities employ, and that individuals' initial giving does not necessarily negatively correspond with the aggressiveness of a charity's investment strategy, which is measured by the proportion invested in the risky portfolio.

1. Introduction

The voluntary supply of public goods plays a pervasive role in many cases such as private donations to charitable organizations, political campaign funds, and lobbying capital. It is one of the most indispensible portions of social welfare. As Roberts (1984) points out, private donation to charities in the United States in 1981 totalled about 53 billion dollars. This is about 2 percent of GNP or about 225 dollars per capita.

Charities depend heavily on various sources of funding, such as private giving, government grants, and other sources. The availability of these funding sources is strongly positively correlated with the economic cycle. Consequently, charitable organizations have an incentive to invest the funds collected in order to reduce this dependence on unpredictable funding and smoothen the volatility of available resources. In 2006/2007, UK charities with £91 billion in assets made an income of £2.6 billion from investment (National Council for Voluntary Organizations(NCVO), 2009). Furthermore, due to the economic recession that occurred in 2007, many countries put forward "bail-out" plans to stimulate the real economy, which was financed by a high level of government debt. This will inevitably lead to the reduction of the current and future provision of government grants. Therefore, it is believed that the charities' investment in the financial markets will continue to increase.

There is extensive literature that focuses on the provision of public goods in simultaneous or sequential move games. As we will introduce in the literature review, most studies solely focus on either the donor or the charity. No existing literature examines the way in which charities invest funds, rather, once they receive donations the production of public goods is instantaneous. Two representative papers are Bergstrom, Blume and Varian (1986) and Varian (1992). An intriguing element in these papers is that the interaction between the donor and the charity is not incorporated in their models. This may be because they believe that only individuals' personal behaviours determine the total supply of public goods. The questions of whether and how the donor, one of these two economic agents, responds to the possible choices of investment strategies employed by the charity or vice versa in the formation of the provision of public goods has been mostly neglected. This oversight is most likely due to the fact that the exogenous determination of charitable investment strategies has the dominant position in the theory and practice of the private provision of public goods.

In sections 1.1.1 to 1.1.3 of chapter 1, we briefly examine the existing literature concerning the issues we are studying, and describe the model employed in this paper in chapter 2. In contrast with the existing literature, in order to fill the aforementioned gap in the determination of the provision of public goods, we employ a three-stage non-cooperative subgame perfect Nash equilibrium, and incorporate two economic agents; the representative individual and the charity, assuming that each of the economic agents has the incentive to maximize their own utility functions.

In chapter 3, we will solve this three-stage game by means of the backwards induction from stage 3 to stage 1. In chapter 4, we compare the 3 stage model with the 2 stage model where an individual's top-up contribution is not allowed in two different aspects; the stability of the provision of public goods and the consumption of private goods, and the possible existence of moral hazards in charities' behaviour. Our conclusions and suggestions for future work are presented in chapter 5.

Our results suggest that the charities' investment strategies do, in fact, affect individuals' contribution and vice versa. Moreover, the interaction between the charity and the individual jointly determines the supply of public goods and the level of private consumption.

.

1.1 Literature Review

1.1.1 Voluntary Provision of Public Goods

Since Olson's landmark paper in 1965 on the theory of voluntary provision of public goods, there have been many important studies concerning this topic of research. This includes papers by Bergstrom et al. (1986), Andreoni (1989, 1990), Bruce and Waldman (1990), and Varian (1994).

Bergstrom et al. (1986) examine the private supply of public goods in a simultaneous-move game. They assume that the individual i with endowment ω_i will choose private consumption x_i and personal giving g_i to contribute to the public goods G under the Nash assumption that the giving by other donors with notation G_{-i} is independent of his own g_i. Therefore, individual i's problem becomes the maximization of the utility function U (x_i, G) with a budget constraint of $x_i + G = \omega_i + G_{-i}$. Bergstrom et al. argue that the neutrality theory[1] is only true under the condition that the redistribution of income is not large enough to change the current contribution set. He also demonstrates that governmental grants of public goods will partially crowd out private contributions. Additionally, if the taxes that finance the governmental grant are collected from all the current contributors, this partial crowding out effect will become perfect. However, in their paper, Bergstrom et al. did not consider the impact of any agency issues with respect to the charity's behaviour in determining the individual's decision to make a contribution. Although he did research on the government's impact on donors, his entire conclusion is solely based on the

[1] When a single public good us provided at positive level by private individuals, its provision is unaffected by a redistribution of income. This holds regardless of differences in individual preferences and despite differences in marginal propensities to contribute to the public good.

utility maximization of individuals, rather than examining all of the involved economic agents.

James Andreoni (1989, 1990) extends the theory of the voluntary provision of public goods. He believes that donors are motivated by tangible or intangible private benefits, and the "warm glow[2]". Both warm glow and private benefits are incorporated in his model. Andreoni denoted the individual's donation function as:

$$\max_{x_i, g_i, G} \quad U_i = U_i(x_i, G_i, g_i)$$
$$\text{s.t.} \quad x_i + g_i = \omega_i$$
$$G_{-i} + g_i = G$$

where ω_i represents the individual's initial endowment, which can be allocated between the private consumption x_i and the gift to the public goods g_i. G_{-i} is the gifts of all individuals except individual i. Using comparative static analysis, he points out that people can gain utility from the action of giving, not only from increasing their total supply of public goods. Moreover, his impure altruism model leads to the conclusion that income redistribution is no longer neutral. Income redistribution from less altruistic people to more altruistic people will lead to the increase of the provision of public goods. What Andreoni missed in his paper is the same oversight in Bergstrom et al.'s research. He has again failed to consider the interaction between the charity and the donors.

Varian (1994) analyzes the voluntary provision of public goods in a sequential game. He finds several interesting and important conclusions. He argues that the individuals' heterogeneity of the ability or preference to provide public goods makes the free-ride problem worse in the case of the sequential game. This phenomenon leads to a smaller total supply of public goods than the total supply in the simultaneous-game. However, his study fails to show the mechanisms of a sequential game, in which different agents, the individual and the charity, move sequentially.

1.1.2 Charitable Investment Policy

On the charity's side, there are several theoretical and empirical studies

[2] Warm glow is the positive emotional feeling that people get from helping others.

concerning the investment strategies employed by charity organizations. The Mean-variance approach, one of the most celebrated models dealing with the portfolio optimization problem, was proposed by Henry Markowitz in his paper "Portfolio Selection" in 1952. In short, the risky portfolio problem is to allocate the initial capital among n assets with the uncertainty of assets return r, which is a $n \times 1$ vector, such that this allocation of asset vector z can maximize the following objective function:

$$\max_{z \in Z} f(r^T z) = \max_{z \in Z} \{ E(r^T z) - \gamma \operatorname{var}(r^T z)$$

where $Z = \{z \in R^n : \sum_{j=1}^{n} z_j \leq 1, z_j \geq 0, j = 1,2,n\}$.

Markowitz uses the variance of portfolio return as the risk measure, which is represented as the term $\operatorname{var}(r^T z)$. Non-negative parameter γ indicates the investor's tolerance for risk. However, the shortcoming of this model referred to as the "Markowitz Optimization Enigma" comes from the assumption that the mean and covariance of these n assets are known. In fact, we use historical data to estimate the real parameters of the mean and covariances of assets. If we plug these estimated values into the Markowitz efficient frontier, it will inevitably render the calculated optimal portfolio meaningless. In the real application of the mean-variance portfolio maximization problem, the future mean and covariance of the n assets are unknown. It is reasonable to use the conditional mean and covariance matrix of the future returns r_{n+1} given the historical data r_n, r_{n-1},…if employing the Bayesian model to predict the future (Tze Leung Lai, 2009). His research contributed much to the theory of portfolio selection, and made it more applicable. Even though the context of his research is beyond the scope of this paper, it provides us with evidence that theory of portfolio selection has been well developed to meet the requirement of real investment of portfolio and also become the general guidance for charitable investment.

Most of the empirical studies of charitable investment compare the average return of the funds with socially screened and unscreened universes, and also use historical data to calculate the standard deviation from the average return. Ros Havemann & Peter Webster (1999) point out that funds with ethical screens have lower total risk and lower return than funds with no ethical criteria based on the data collected from UK funds in 39 month period. They partially attribute this under-performance of funds to the expense of reduced risk and the higher charges by firms that provide such ethical funds. Using a data set containing 1300 stocks over the period of January 1987 to December 1994, John B. Guerard (1997) find that there is no statistically significant difference between the average return of socially screened and unscreened universes in the corresponding time period. However, higher return could be expected if the socially screened strategy is associated with better management. He argues that the performance of funds is highly positively correlated with the management level of the funds manager under the circumstance that the stock universe is socially screened.

We notice that the gap in these literature is that the investment behaviour of the charity is not seen as being influenced by donor behaviour.

1.1.3 The Hold-Up Problem

As discussed above, a gap in the existing literature is that it fails to recognize the interaction between donors and charities. In particular, it seems natural to imagine that charities may exploit the generosity of donors by choosing an overly aggressive investment policy. This is known as the "hold up" problem. An interesting study is that of Bruce and Waldman (1990). They extend the rotten-kid theorem[3] in the economic theory of the family. By using a two-time period model, they thoroughly examine the interaction between altruistic family members and a selfish "rotten" kid. Their research indicates that if the altruistic second-period transfer is applicable, the selfish child will attempt to maximize the whole family's income rather than to optimize his own income. However, the "rotten kid" may consume too much in the first period, and perform inefficiently, which can lead to the Samaritan's dilemma[4]. If a second-period transfer is not operative, the Samaritan's dilemma does not occur, but the "rotten-kid"

[3] If the transfer by altruistic family members is operative, the rotten kid will act unselfishly to maximize the whole family's income.

[4] A person will act in one of two ways: using the charity to improve their situation, or becoming to rely on charity as a means of survival.

will maximize his own income instead of the family's income. They conclude that the altruistic family members will encounter the trade off between these two kinds of inefficiencies by the "rotten-kid". The first inefficiency is the Samaritan dilemma. The second inefficiency occurs when the "rotten kid" maximizes his own income instead of the family's income. It is evident that Bruce and Waldman neglected to study the mutual interaction between the altruistic parents and the rotten kid. They acknowledge in their paper that their conclusion is based on the assumption that the altruistic family members do not commit the tit for tat strategy as the "rotten kid" performs inefficiently in at least one dimension.

It is evident that the important oversight in the existing literature is the interaction between the individual and the charity in the determination of the private provision of public goods and the level of consumption of private goods. We will employ the subgame perfect Nash equilibrium to show the donors' behaviours do affect the charity's investment strategy, and vice versa. Also, this interaction jointly determines the level of the supply of public goods and the consumption of private goods.

2. Description of the Model

In this paper, we focus on the canonical model in which private individuals only benefit from a private consumption good x_i and a total amount of public goods G that consists of proceeds R produced by a charitable investment and private contributions $g_i + \sum_{j \neq i} g_j$ made by all private individuals. We assume an economy with n consumers, each endowed with income ω_i and a strictly concave utility function ($u'(.)>0$, and $u''(.)<0$) defined over a private good x_i and public good G. Furthermore, we make an additional assumption that private good x_i and public good G are substitutes in consumption. Therefore, $\frac{\partial^2 U}{\partial x_i \, \partial G}$, the partial derivative of utility with respect to x_i and G is positive. In this paper, we do not take account of population heterogeneity among individuals. Therefore, we also assume that all individuals are identical in endowments and preferences.

On the charity organization side, we assume that the charity only derives its utility from the total amount of public goods G it possesses, and its utility function V (G) is quasi-concave (V' (G)>0, V'' (G) <0).

One of the main differences between previous studies and this paper is that we employ a three-stage non-cooperative game to observe the interaction between individuals and charities in the supply of public goods and the level of private consumption. The sequence of events in this game is in the following order:

Stage 1

Individual i chooses personal donation d_i under the anticipation that the charity will use the money $\sum_{i=1}^{n} d_i$ collected in this stage to invest, and also the individual will their top-up contribution based on the return of this investment.

Stage 2

Given the initial contribution $\sum d_i$ in Stage 1, the charity chooses β, the proportion of total assets, to invest in the risky portfolio with rate of return r. The remainder of the charity's assets are invested in the riskless asset, and maintain their value with certainty.

In order to simplify the problem, here we assume that there are only two states of nature, which are the success or failure of the charitable investment with probability π and $1 - \pi$ respectively. We define a success as the state where the investment pays a positive rate of return r, the corresponding top-up giving is $\overline{g_i}$, and the return of charitable investment is R_1. The failure occurs when the investment pays a return of zero, which means that the charity will lose the entire asset that is invested in the risky portfolio. Also, the corresponding top-up giving is $\widetilde{g_i}$, and the return of charitable investment is R_2.

Stage 3

Individual i chooses whether to make a top-up donation g_i to the charity, and this residually determines private good consumption x_i, which equals $\omega_i - d_i - g_i$. The charity uses the proceeds from the risky investment, the risk-free investment and the top-up donation received to produce the public goods. The game ends.

The critical element of this three-stage game is that we assume individual i prefers to provide funds to the charity organization rather than to invest by himself. There are several reasons that make this assumption feasible. First of all, charities usually have more financial instruments to employ. In addition, professional financial institutions will deliberately design particular products for a charity organization, which can not only meet the charity's financial needs, but also reduce the risk to an extent they can bear. Second, compared with individual investors, charities are more informed and specialized in some fields, which can make them react more rapidly to the potential opportunities or risks. Finally, the tax wedge is another important concern. Individuals must pay a tax if they have any capital gains from their investments. However, charitable investment can benefit from tax-free policy. It is immediately evident that there are certain costs associated with running a charity, including problems of inefficiency and scandal, which can stain the reputation of charity, and leads to a sharp decline in private voluntary contributions. These factors are not in the scope of this paper.

We will solve this three-stage game using the backwards induction from stage 3 to stage 1. The equilibrium path of the three stage model must be a subgame perfect Nash equilibrium.

3.0 Game Theoretic Analysis of the Three-Stage Giving Decision

3.1 Stage 3-Individual's Top-Up Decision Making Problem

In order to solve for the equilibrium of this three-stage game, we first analyze the last stage of the game to determine the amount of top-up donation g_i. This subgame we are analyzing looks very much like the problem treated in the literature on the voluntary supply of public goods. The difference is that charities have two sources of funds for producing the public goods, the proceeds R from risky and risk-free investments, and secondly, the top-up contributions by individuals.

Individual i's decision making problem in this stage can be written as

$$\max_{g_i} U_i = U(x_i, G) = U(x_i\,,\ g_i + \sum_{j \neq i} g_j + R(\beta, r, \pi, \sum d_i)) \qquad (1)$$

$$\text{s.t.} \quad x_i + g_i = \omega_i - d_i \tag{2}$$
$$g_i \geq 0$$

where $R(\beta, r, \pi, \sum d_i)$ is the realized return of the charity's investment portfolio, which depends on the proportion β of investment in risky assets, the state of nature π, and therefore the realized rate of return (either r to the risky asset or 0), and the sum of giving $\sum d_i$ in the first stage by individuals

Using (2), we can rewrite the optimization problem as

$$\max_{g_i} U_i = U(\omega_i - d_i - g_i,\ g_i + \textstyle\sum_{j \neq i} g_j + R(\beta, r, \pi, \sum d_i)) \tag{3}$$

$$g_i \geq 0$$
$$x_i = \omega_i - d_i - g_i \xrightarrow{\text{yields}} \frac{\partial x_i}{\partial g_i} = -1$$
$$G = g_i + \textstyle\sum_{j \neq i} g_j + R(\beta, r, \pi, \sum d_i) \xrightarrow{\text{yields}} \frac{\partial G}{\partial g_i} = 1$$

Utility maximization must therefore be the case for every individual i who belongs to the current contribution set C, the top-up giving g_i is greater than zero. The individual's first-order condition with respect to g_i can be written as:

$$U_{x_i}\left(\frac{\partial x_i}{\partial g_i}\right) + U_G\left(\frac{\partial G}{\partial g_i}\right) \geq 0 \qquad \text{for} \quad i \in C.$$
$$g_i\left[U_{x_i}\left(\frac{\partial x_i}{\partial g_i}\right) + U_G\left(\frac{\partial G}{\partial g_i}\right)\right] = 0 \qquad \text{(Kuhn-Tucker condition)}$$

Simplifying and rearranging, we have:

$$-U_{x_i} + U_G = 0 \qquad \text{for} \quad i \in C. \tag{4}$$

This first-order condition is interpreted as follows: an individual's marginal cost of giving an additional private good x_i is equal to their marginal benefit U_G from consuming public goods G. Also, equation (4) implicitly describes the single-valued reaction function for individual i, i.e., $f_i(.)$, which represents consumer i's demand function for public good G given the level of total voluntary contribution G_{-i} provided by the individuals except consumer i, with the properties of $G_{-i} = G - g_i$ and $0 < f_i'(.) < 1$ (See Bergstrom et al. 1986).

Therefore, we can expect the top-up donation function g_i^N :

$$g_i^N = \max\{f_i(\omega_i - d_i + \sum_{j\neq i} g_j + R(\beta, r, \pi, \sum d_i)) - G_{-i}, 0\} \qquad (5)$$

Bergstrom et al. (986) prove that if the reaction functions of individuals are continuous and the strategy space W={x in R^n: $0 \leq x_i \leq \omega_i$ for i=1…..n} is compact and convex, there exists a Nash equilibrium vector of gifts g_i^N. If

we also make a very weak assumption that an individual's utility function is quasi-concave and both private goods and public goods are normal goods, then this Nash equilibrium is unique and symmetric. Consequently, the quantity of top-up donation g_i^N at equilibrium by individual i is determined by d_i, $R(.)$, and can be simply denoted as:

$$g_i^N = \varphi(\omega_i, d_i, R(.)) \qquad \text{for } i \in C \qquad (6)$$

We have assumed in chapter 2 that individuals are assigned same endowment ω_i, and preference. Therefore, g_i^N is identical for $i \in C$, which implies that

$$\sum_{i=1}^n g_i = n\, g_i = n\, \varphi(\omega_i, d_i, R(.))$$

Theorem 1. *There exists a threshold $\overline{R}$ such that if the charitable investment returns R is less than $\overline{R}$, the individual's top-up giving g_i will be partially crowded out by R. Otherwise, g_i will be zero.*

Proof.

Take the derivative with respect to the returns from charitable investment R at both sides of equation (4), we get:

$$\frac{\partial(-U_{x_i}+U_G)}{\partial R} = 0$$

Extending the above equation, we have:

$$\left(U_{xx}\frac{\partial x}{\partial R} + U_{xG}\frac{\partial G}{\partial R}\right) + \left(U_{Gx}\frac{\partial x}{\partial R} + U_{GG}\frac{\partial G}{\partial R}\right) = 0 \qquad (7)$$

Using the fact that $x_i = w_i - d_i - g_i$, and $G = n\,g_i + R(.))$, we have the properties that $\frac{\partial x}{\partial R} = -\frac{\partial g_i}{\partial R}$, and $\frac{\partial G}{\partial R} = n\,\frac{\partial g_i}{\partial R} + 1$. Substituting them into (7), we can simplify and rewrite it as:

$$(U_{xx} - n\,U_{xG} - U_{Gx} + n\,U_{GG})\,\frac{\partial g_i}{\partial R} = -U_{GG} + U_{xG}$$

Isolating $\frac{\partial g_i}{\partial R}$, we have:

$$\frac{\partial g_i}{\partial R} = \frac{-U_{GG}+U_{xG}}{U_{xx}-n\,U_{xG}-U_{Gx}+n\,U_{GG}} = \frac{-U_{GG}+U_{xG}}{U_{xx}-(n+1)\,U_{xG}+n\,U_{GG}}$$

Therefore, we obtain:

$$n\,\frac{\partial g_i}{\partial R} = \frac{\partial\sum g_i}{\partial R} = n\,\frac{-U_{GG}+U_{xG}}{U_{xx}-n\,U_{xG}-U_{Gx}+n\,U_{GG}}$$

$$= n\frac{-U_{GG}+U_{xG}}{U_{xx}-(n+1)\,U_{xG}+n\,U_{GG}} < 0 \qquad (7\text{-}1)$$

We have already assumed that an individual's utility function is concave and both goods are normal and substitute in consumption. Therefore, $U_{GG} < 0, U_{xx} < 0,$ and $U_{xG} > 0$. It is immediately evident that the numerator $-U_{GG} + U_{xG}$ is positive, and the denominator $U_{xx} - n\,U_{xG} - U_{Gx} + n\,U_{GG}$ is negative. Therefore, (7-1) holds.

Moreover,

$$\begin{aligned}&|n(-U_{GG} + U_{xG})| - |U_{xx} - (n+1)\,U_{xG} + n\,U_{GG}|\\&= n(-U_{GG} + U_{xG}) - [-(U_{xx} - (n+1)U_{xG} + n\,U_{GG})]\\&= U_{xx} - U_{xG} < 0\end{aligned}$$

Which is equivalent to $\left| n \frac{-U_{GG}+U_{xG}}{U_{xx}-(n+1)\,U_{xG}+n\,U_{GG}} \right| < 1$. (7-2)

Combining (7-1), and (7-2), we directly obtain:

$$-1 < \frac{\partial \sum g_i}{\partial R} < 0 \tag{8}$$

Formula (8) clearly indicates that as the returns of charitable investment R increases, individuals' top-up giving $\sum g_i$ will be partially crowded out and diminish until R reaches the threshold $\overline{R}$ (if it is attainable) where g_i is zero. Especially, if the individual's utility function is quasi-linear in private consumption x, then $U_{xx} = U_{xG} = 0$. Consequently, $\frac{\partial \sum g_i}{\partial R} = -1$, which indicates this partial crowding out effect becomes perfect. **Q.E.D**

Theorem 1 is basically consistent with the result from BBV model, except that the individual's top-up giving is partially crowded out by the return to the investment portfolio, rather than by government's grant that is financed by the extra tax collection on individuals.

Theorem 2. *If the marginal return of the charity's investment portfolio with respect to initial giving* $\frac{\partial R}{\partial d_i}$ *is sufficiently high, then an increase of the donation in stage one will crowd out top-out giving, that is* $\frac{\partial g_i}{\partial d_i} < 0$.

Proof.

To establish this result, we consider how d_i affects the optimal choice of g_i. Differentiating the LHS and RHS of equation (4) with respect to d_i, we obtain:

$$\frac{\partial(-U_{x_i}+U_G)}{\partial d_i} = 0$$

Which can be extended and rewritten as:

$$\left(U_{xx}\frac{\partial x}{\partial d_i} + U_{xG}\frac{\partial G}{\partial d_i}\right) + \left(U_{Gx}\frac{\partial x}{\partial d_i} + U_{GG}\frac{\partial G}{\partial d_i}\right) = 0 \tag{9}$$

Using the fact that $x_i = \omega_i - d_i - g_i$, and $G = n\, g_i + R(.))$, we have:

$$\frac{\partial x}{\partial d_i} = -1 + \frac{\partial g_i}{\partial d_i}, \quad \text{and} \quad \frac{\partial G}{\partial d_i} = n\, \frac{\partial g_i}{\partial d_i} + \frac{\partial R}{\partial d_i} \tag{10}$$

Substituting (10) into (9), we can rewrite it as:

$$(-U_{xx} + (n+1)U_{xG} - n\, U_{GG})\, \frac{\partial g_i}{\partial d_i} + (U_{xG} - U_{GG})\frac{\partial R}{\partial d_i} = U_{xx} - U_{xG} \tag{11}$$

We notice that when d_i changes, the individual's marginal top-up giving and marginal returns of charitable investment have the same positive coefficient sign. In other words, these two marginal changes have the opposite directions.

It is immediately evident that if individual's utility function is quasi-linear, then $U_{xx} = U_{xG} = 0$. Equation (9) becomes:

$$\frac{\partial g_i}{\partial d_i} = -\frac{1}{n}\frac{\partial R}{\partial d_i}$$

which immediately implies that if the initial donation d_i changes, the total marginal top-up giving by individuals has the same magnitude as the marginal returns of charitable investment, but the direction is opposite.

Generally, from equation (11) we can obtain:

$$\frac{\partial g_i}{\partial d_i} = \frac{U_{xx} - U_{xG} - (U_{xG} - U_{GG})\frac{\partial R}{\partial d_i}}{(-U_{xx} + (n+1)U_{xG} - n\, U_{GG})} \tag{12}$$

Notice that since the individual's utility function is strictly concave in both private goods x_i and public goods G, both U_{xx} and U_{GG} are negative. And because these two goods are substitute in consumption, U_{xG} is positive. We immediately obtain:

$$-U_{xx} + (n+1)U_{xG} - n\, U_{GG} > 0\ ,\ \ U_{xG} - U_{GG} > 0, \text{ and } U_{xx} - U_{xG} < 0$$

Define $\hat{\Delta} = \frac{U_{xx} - U_{xG}}{U_{xG} - U_{GG}}$, which is obviously negative.

If $\frac{\partial \mathrm{R}}{\partial \mathrm{d_i}} < \hat{\Delta}$, then using this fact to sign (12), we can directly obtain that:

$$\frac{\partial \mathrm{g_i}}{\partial \mathrm{d_i}} > 0$$

In contrast, if $\frac{\partial \mathrm{R}}{\partial \mathrm{d_i}} > \hat{\Delta}$, then from (12) it must be true that:

$$\frac{\partial \mathrm{g_i}}{\partial \mathrm{d_i}} < 0$$

If $\frac{\partial \mathrm{R}}{\partial \mathrm{d_i}} = \hat{\Delta}$, then, from (12), we get:

$$\frac{\partial \mathrm{g_i}}{\partial \mathrm{d_i}} = 0$$

which implies that if $\mathrm{d_i}$ makes $\frac{\partial \mathrm{R}}{\partial \mathrm{d_i}} = \hat{\Delta}$ hold, then the same $\mathrm{d_i}$ will also make an individual's top-up giving achieve at least its local critical value.

Q.E.D

Theorem 2 is totally different from the conclusion of BBV model where simultaneous-game is employed. Because we adopt a sequential subgame perfect Nash equilibrium in this paper, it is inevitable that free-ride problem occurs. Although the sequence of contribution in our model is determined where individual is the first mover, as proved above, this free-ride effect between the individual and the charity is mutual, which is determined by the sign of $\frac{\partial R}{\partial d_i}$. If we do not take account of the moral hazards of the charity, it is evident that this mutual free-ride problem depends not only on the preference for public goods of the individual and the charity, but also on the parameters of the state, which is different from the results of Varian's sequential contribution model.

3.2 Stage 2- Charity's Investment Decision Making Problem

As explained in chapter 2, the charity derives utility from the total amount of public goods G it provides, and its utility function V (G) is quasi-concave (V' (G)>0, V'' (G) <0). Therefore, the charity's decision making problem can be written as:

$$\max_{0 \le \beta \le 1} E[V(G)] = \pi V\left[(1+r)\beta \sum d_i + (1-\beta)\sum d_i + n\, g_i|_{R_1}\right] + (1-\pi)V[(1+r)\beta \sum d_i + (1-\beta)\sum d_i + n\, g_i|_{R_2}] \quad (13)$$

Equation (13) can be simplified as:

$$\max_{0 \le \beta \le 1} E[V(G)] = \pi V[(1+r\beta)\sum d_i + n\, g_i|_{R_1}] + (1-\pi)V[(1-\beta)\sum d_i + n\, g_i|_{R_2}] \quad (14)$$

where $\begin{cases} R_1 = (1+r\beta)\sum d_i & G_1 = (1+r\beta)\sum d_i + n\, g_i|_{R_1} \\ R_2 = (1-\beta)\sum d_i & G_2 = (1-\beta)\sum d_i + n\, g_i|_{R_2} \end{cases}$

It is necessary to consider the concavity of the charity's expected utility with respect to β before furthering analysis. We make an additional assumption that the reaction function $g_i^N = \varphi(\omega_i, d_i, R(.))$, which we have discussed in section 3.1 equation (6), is strictly concave in $R(.)$. Implying that $\frac{\partial^2 g_i}{\partial R^2}$ is negative, which represents that the marginal top-up giving is decreasing when the return of investment R goes up.

From (14), the charity's marginal expected utility with respect to β and its corresponding second derivative become:

$$\frac{\partial E[V(G)]}{\partial \beta} = \pi V_1(r\sum d_i + n \frac{\partial g_i}{\partial \beta}|_{R_1}) + (1-\pi)V_2(-\sum d_i + n \frac{\partial g_i}{\partial \beta}|_{R_2}) \quad (15)$$

$$\frac{\partial^2 E[V(G)]}{\partial \beta^2} = \pi\left[V_{11}(r\sum d_i + n \frac{\partial g_i}{\partial \beta}|_{R_1})^2 + V_1 n \frac{\partial^2 g_i}{\partial \beta^2}|_{R_1}\right] + (1-\pi)\left[V_{22}(-\sum d_i + n \frac{\partial g_i}{\partial \beta}|_{R_2})^2 + V_2 n \frac{\partial^2 g_i}{\partial \beta^2}|_{R_2}\right] \quad (16)$$

If the charity's expected utility function is concave in β, then $\frac{\partial^2 E[V(G)]}{\partial \beta^2}$ must be negative. To check whether this is true, we compute:

Notice that $\frac{\partial g_i}{\partial \beta}|_{R_1} = \frac{\partial g_i}{\partial R}\frac{\partial R}{\partial \beta}|_{R_1} = \frac{\partial g_i}{\partial R}|_{R_1} \cdot \frac{\partial R}{\partial \beta}|_{R_1}$

$$= \frac{\partial g_i}{\partial R}|_{R_1} . r\sum d_i < 0 \quad (17)$$

$$\frac{\partial g_i}{\partial \beta}|_{R_2} = \frac{\partial g_i}{\partial R}\frac{\partial R}{\partial \beta}|_{R_2} = \frac{\partial g_i}{\partial R}|_{R_2}.\ \frac{\partial R}{\partial \beta}|_{R_2}$$
$$= \frac{\partial g_i}{\partial R}|_{R_2}.(-\sum d_i) > 0 \qquad (18)$$

and that given the assumption that optimal g_i is concave in β, therefore:

$$\frac{\partial^2 g_i}{\partial \beta^2}|_{R_1} = \frac{\partial^2 g_i}{\partial R^2}|_{R_1}.(r\sum d_i)^2 < 0 \qquad (19)$$

$$\frac{\partial^2 g_i}{\partial \beta^2}|_{R_2} = \frac{\partial g_i}{\partial R}|_{R_2}.(-\sum d_i)^2 < 0 \qquad (20)$$

Where $V_1 = \frac{\partial V}{\partial G_1}$ $V_{11} = \frac{\partial^2 V}{\partial {G_1}^2}$; $V_2 = \frac{\partial V}{\partial G_2}$ $V_{22} = \frac{\partial^2 V}{\partial {G_2}^2}$

We know that V_1 and V_2 are positive; V_{11}, and V_{22} are negative. Therefore, from (19), and (20), it is clear that both $\frac{\partial^2 g_i}{\partial \beta^2}|_{R_1}$ and $\frac{\partial^2 g_i}{\partial \beta^2}|_{R_2}$ are negative. Given the signs we have computed above, it now is evident that $\frac{\partial^2 E[V(G)]}{\partial \beta^2}$ is less than zero, which indicates that the charity's expected utility with respect to β is strictly concave.

Theorem 3. *If the risky asset has a higher expected rate of return than the risk-free asset (*$\pi > \frac{1}{1+r}$*), then the charity will invest a positive proportion of assets in the risky asset (*$\beta^* > 0$*). Otherwise, the charity will choose* $\beta^* = 0$.

Proof.

Using (17) and (18), and if there exists an internal solution to the charity's maximization problem, we can rewrite the first condition for the charity's optimization problem as:

$$\frac{\partial E[V(G)]}{\partial \beta} = r\sum d_i\, \pi\, V_1 \left(1 + n\ \frac{\partial g_i}{\partial R}|_{R_1}\right) -$$
$$\sum d_i (1-\pi) V_2 \left(1 + n\ \frac{\partial g_i}{\partial R}|_{R_2}\right) = 0 \qquad (21)$$

The expected utility function for the charity is continuous in β, which is compact. From theorem 3, we know that E[V(G)] is strictly concave in β. Then, by the Weierstrass theorem, since E[V(G)] is a continuous function of compact set β, it must attain a global maximization in the interval $\beta \in [0,1]$, which is either internal or boundary.

Before studying the behaviour of $\frac{\partial E[V(G)]}{\partial \beta}$ at $\beta = 0$, which means the charity chooses the completely conservative investment strategy, we need to verify the values of R and G at two states of nature.

Using the fact that $R_1 = (1 + r\beta)\sum d_i$, and $R_2 = (1 - \beta)\sum d_i$, if $\beta = 0$, we directly obtain that:

$$R_1 = R_2 = \sum d_i = R. \tag{21-1}$$

Also, we have already shown in section 3.1 that:

$$g_i^N = \varphi(\omega_i, d_i, R(.)) \qquad \text{for } i \in C$$

If ω_i, and d_i are given, then g_i^N only depends on $R(.)$. Therefore, if the return of investment is equal, the top-up contribution at two states of nature should be same. That is:

$$\overline{g_\imath} = \widetilde{g_\imath} = g_i. \tag{21-2}$$

We know the total supply of public goods is the sum of the return of investment and the top-up giving. It is evident that at these two states of nature the provision of public goods are same, and so are the marginal expected utilities. We simply denote:

$$G_1 = G_2 = G \tag{21-3}$$

$$\text{and} \quad V_1 = V_2 = V' \tag{21-4}$$

Using (21), (21-1), (21-2), (21-3), and (21-4), the expression $\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0}$ can be simplified as:

$$\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0} = [r\pi - (1-\pi)][V' \sum d_i(1+n\frac{\partial g_i}{\partial R}|_R)]$$

According to theorem 1, $(1+n\frac{\partial g_i}{\partial R}|_R)$ is greater than zero because of the partial crowding-out of $\sum g_i$ by R. Therefore, the sign of $\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0}$ is only determined by the sign of $r\pi - (1-\pi)$. We consider the following two cases:

Case 1: $\pi > \frac{1}{1+r}$, that is the expected rate of return on risky asset is greater than 1.

If $\pi > \frac{1}{1+r}$, that is $r\pi - (1-\pi) > 0$, then, we obtain:

$$\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0} > 0 \quad (22)$$

Equation (22) clearly indicates that expected utility $E[V(G)]$ cannot attain maximization at $\beta = 0$. This implies that the charity will invest a positive proportion of assets in the risky portfolio until reaching the maximization of the expected utility at the optimal choice of β^* , which is either internal or boundary at β^* =1.

Case 2: $\pi \leq \frac{1}{1+r}$, that is the expected rate of return on risky asset is not greater than 1.

Again evaluating $\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0}$, we directly note that:

$$\frac{\partial E[V(G)]}{\partial \beta}|_{\beta=0} < 0 \quad (23)$$

Which implies that the optimal choice of β is zero. This clearly shows that when the expected rate of return on the risky asset is not greater than on the risk-free asset, the charity will optimally invests all the funds in the risk-free asset. **Q.E.D**

Theorem 3 demonstrates that the charities are rational. They only choose to invest in risk asset if they should do so. Therefore, the charity does not need to be told or regulated to avoid bad investment opportunities. Admittedly, this conclusion is under the condition that the charity considers the maximization of the expected utility that is derived only from the total supply of public goods as their objective, not others.

We next investigate how the optimal choice of β responds to the variance of the initial contribution $\sum d_i$. The parameter β can be interpreted as an indicator of the aggressiveness by the charity investment. It is interesting, therefore, to see whether the charity becomes "more aggressive" when its initial endowment increases. Intuitively, it may be too complicated to assess this relationship without knowing the parameters of states, the economic agents' utility functions and their corresponding marginal effects.

Theorem 4. *There exists a threshold* $\frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$, *when the ratio* τ, *which is equal to* $[V_{11}\left(1+n\,\frac{\partial g_i}{\partial R_1}\right)^2+n\,V_1\frac{\partial^2 g_i}{\partial R_1^2}]\Big/[V_{22}\left(1+n\,\frac{\partial g_i}{\partial R_2}\right)^2+n\,V_1\frac{\partial^2 g_i}{\partial R_2^2}]$, *is greater than* $\frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$, *then the optimal choice of* β *will decrease as initial contribution* $\sum d_i$ *goes up.*

Proof.

Recall that, at an internal solution, the first order condition of the charity's expected utility can be written as:

$$\frac{\partial E[V(G)]}{\partial \beta}=r\sum d_i\,\pi\,V_1\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_1}\right)$$
$$-\sum d_i((1-\pi)V_2\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_2}\right)=0$$

which is equivalent to:

$$r\sum d_i\,\pi\,V_1\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_1}\right)=$$
$$\sum d_i((1-\pi)V_2\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_2}\right) \qquad (24)$$

Differentiating (24) with respect to $\sum d_i$, and rearranging, we obtain that:

$$r\pi[V_{11}\frac{\partial G_1}{\partial \sum d_i}\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_1}\right)+V_1 n\frac{\partial^2 g_i}{\partial R^2}\frac{\partial R}{\partial \sum d_i}|_{R_1}]=$$
$$(1-\pi)[V_{22}\frac{\partial G_2}{\partial \sum d_i}\left(1+n\,\frac{\partial g_i}{\partial R}|_{R_2}\right)+V_2 n\frac{\partial^2 g_i}{\partial R^2}\frac{\partial R}{\partial \sum d_i}|_{R_2}] \qquad (25)$$

We next need to evaluate this expression. Recall that:

$$G_1=(1+r\beta)\sum d_i+n\,g_i|_{R_1}$$
$$\text{and } R_1=(1+r\beta)\sum d_i$$

Therefore, $\frac{\partial R}{\partial \sum d_i}|_{R_1} = r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta)$

$$\begin{aligned} \text{and } \frac{\partial G_1}{\partial \sum d_i} &= r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta) + n \frac{\partial g_i}{\partial \sum d_i}|_{R_1} \\ &= r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta) + n \frac{\partial g_i}{\partial R} \frac{\partial R}{\partial \sum d_i}|_{R_1} \\ &= r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta) + n \frac{\partial g_i}{\partial R} [r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta)] \\ &= \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right) r \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 + r\beta) \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right) \\ &= \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right) (1 + r\beta\ r \sum d_i \frac{\partial \beta}{\partial \sum d_i}) \end{aligned} \quad (26)$$

Substituting (26) into equation (25), we can rewrite it as:

$$r\,\pi[V_{11} \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n \frac{\partial^2 g_i}{\partial R^2}|_{R_1}](1 + r\beta + r \sum d_i \frac{\partial \beta}{\partial \sum d_i}) =$$
$$(1 - \pi)[V_{22} \frac{\partial G_2}{\partial \sum d_i} \left(1 + n \frac{\partial g_i}{\partial R}|_{R_2}\right) + V_2 n \frac{\partial^2 g_i}{\partial R^2} \frac{\partial R}{\partial \sum d_i}|_{R_2}] \quad (27)$$

Similarly, recalling that:

$$G_2 = (1 - \beta) \sum d_i + n\, g_i|_{R_2}$$

$$\text{and } R_2 = (1 - \beta) \sum d_i$$

We can compute:

$$\begin{aligned} \frac{\partial R}{\partial \sum d_i}|_{R_2} &= - \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 - \beta) \\ \text{and } \frac{\partial G_2}{\partial \sum d_i} &= - \sum d_i \frac{\partial \beta}{\partial \sum d_i} + (1 - \beta) + n \frac{\partial g_i}{\partial \sum d_i}|_{R_2} \\ &= \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right) (1 - \beta - \sum d_i \frac{\partial \beta}{\partial \sum d_i}) \end{aligned} \quad (28)$$

Substituting (28) into (27), we can rewrite (27) as:

$$r\,\pi[V_{11} \left(1 + n \frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n \frac{\partial^2 g_i}{\partial R^2}|_{R_1}](1 + r\beta + r \sum d_i \frac{\partial \beta}{\partial \sum d_i}) =$$
$$(1 - \pi)[V_{22} \left(1 + n \frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n \frac{\partial^2 g_i}{\partial R^2}|_{R_2}](\,1 - \beta - \sum d_i \frac{\partial \beta}{\partial \sum d_i})$$

Solving $\sum d_i \frac{\partial\beta}{\partial\sum d_i}$, we obtain that:

$$\sum d_i \frac{\partial\beta}{\partial\sum d_i} = \frac{(1-\pi)(1-\beta)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2+V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right]-r\pi(1+r\beta)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2+V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}]}{(1-\pi)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2+V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right]+r^2\pi(1+r\beta)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2+V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}]} \quad (29)$$

Recall that the reaction function $g_i^N = \varphi(\omega_i, d_i, R(.))$ is strictly concave in $R(.)$. Therefore, $\frac{\partial^2 g_i}{\partial R^2}$ is negative, and $\frac{\partial g_i}{\partial R}$ is decreasing when the return of charitable investment R increase. Consequently, it is evident that the denominator of equation (29) is negative, and so that the sign of (29) depends on the sign of the numerator.

If $(1-\pi)(1-\beta)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2+V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right]- r\pi(1+r\beta)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2+V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1} > 0$

Consequently, if and only if:

$$\tau = \frac{V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2+V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}}{V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2+V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}} > \frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)} \quad (30)$$

then $\sum d_i \frac{\partial\beta}{\partial\sum d_i} < 0$

which explicitly indicates that under the condition of $\tau = \frac{V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2+V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}}{V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2+V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}} > \frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$, as the initial giving goes up, the charity will lower the proportion of its investment in risky portfolios in order to maximize its expected utility.

In contrast, if and only if:

$$\tau = \frac{V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}}{V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}} < \frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)} \tag{31}$$

then $\frac{\partial\beta}{\partial \sum d_i} > 0$ **Q.E.D**

We notice that the ratio $\tau = \frac{V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}}{V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}}$ is always positive even though both the numerator and denominator are negative. If the boundary solution $\beta^* = 1$ exists, condition (30) is always satisfied. Therefore, if individuals increase their initial giving, the charity will lower β, and become less aggressive.

π, the probability of having a successful investment, will also have an impact on the behaviour of charities' investment. If π goes up, the expected rate of return on risky asset increases. Intuitively, this will more likely lead the charity to become more conservative. From (30), we notice that if π increases, the threshold $\frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$ will decrease, which will make the condition (30) more easily be satisfied. Consequently, $\frac{\partial\beta}{\partial \sum d_i}$ will be more negative. This proves the intuition that charity will be more conservative.

We next consider the case where the charity's initial resources are so large that, even in the bad state when the investment is unsuccessful, it does not receive any top-up giving. Therefore, the first order condition of the charity's expected utility (24) can be rewritten as:

$$r\pi V_1 = (1-\pi)V_2 \tag{32}$$

Recalling that:

$$G_1 = (1+\gamma\beta)\sum d_i$$

and $$G_2 = (1-\beta)\sum d_i$$

Differentiating (32) with respect to $\sum d_i$, and rearranging, we obtain that:

$$\sum d_i \frac{\partial\beta}{\partial\sum d_i} = \frac{(1-\pi)(1-\beta)V_{22} - r\,\pi(1+r\beta)V_{11}}{(1-\pi)V_{22} + r^2\pi V_{11}} \tag{33}$$

The denominator $(1-\pi)V_{22} + r^2\pi V_{11}$ is negative. Therefore,

If $(1-\pi)(1-\beta)V_{22} - r\,\pi(1+r\beta)V_{11} > 0$, then $\frac{\partial\beta}{\partial\sum d_i} < 0$.

We can rewrite this condition as:

$$\tau = \frac{V_{11}}{V_{22}} > \frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}, \text{ if and only if } \frac{\partial\beta}{\partial\sum d_i} < 0.$$

Notice when the ratio $\tau = \frac{V_{11}}{V_{22}}$ is greater than the threshold $\frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$, the optimal choice of β will decrease as initial contribution $\sum d_i$ goes up. Obviously, the result in this special case is consistent with what Theorem 4 presents. However, compared to the earlier case where top-up giving exists, here the charity becomes less aggressive because the initial contribution is large enough that the charity does not need to increase β in order to produce adequate public goods.

Finally, we can check whether $\left|\sum d_i \frac{\partial\beta}{\partial\sum d_i}\right|$ is less than 1.

If $$\frac{(1-\pi)(1-\beta)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right] - r\pi(1+r\beta)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}]}{(1-\pi)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right] + r^2\pi(1+r\beta)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}]} < 1$$

Which implies that:

$$-\beta(1-\pi)\left[V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}\right] >$$
$$r\pi(1+r\beta+r)[V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}]$$

It is straightforward that the LS is greater than the RS because LS is positive and RS is negative. It is determined that $\sum d_i \frac{\partial\beta}{\partial\sum d_i}$, the total change of β caused by initial contribution $\sum d_i$, is less than 1. On the other hand, we can show if under the condition that the ratio

$$\tau = \frac{V_{11}\left(1+n\frac{\partial g_i}{\partial R}|_{R_1}\right)^2 + V_1\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_1}}{V_{22}\left(1+n\frac{\partial g_i}{\partial R}|_{R_2}\right)^2 + V_2\, n\frac{\partial^2 g_i}{\partial R^2}|_{R_2}} < \frac{(1-\pi)(2-\beta)}{r\pi(1+r\beta-r)} \quad (29)$$

then, if and only if:

$$\sum d_i \frac{\partial \beta}{\partial \sum d_i} > -1$$

As the number $\frac{(1-\pi)(2-\beta)}{r\pi(1+r\beta-r)}$ is greater than the threshold $\frac{(1-\pi)(1-\beta)}{r\pi(1+r\beta)}$, equation (29) is acceptable because it allows $\sum d_i \frac{\partial \beta}{\partial \sum d_i}$ to increase or decrease when $\sum d_i$ changes. However, $\sum d_i \frac{\partial \beta}{\partial \sum d_i}$ cannot be greater than -1, which means that the maximum of ratio τ is $\frac{(1-\pi)(2-\beta)}{r\pi(1+r\beta-r)}$. Consequently, the result that $\left|\sum d_i \frac{\partial \beta}{\partial \sum d_i}\right| < 1$ can be expected.

Intuitively, the increase of the donor's initial giving will make the charity perform less aggressively. However, the result of Theorem 4 explicitly indicates that the increase of individuals' initial contribution will lead the charities' investment strategy to become more aggressive or conservative. The outcome depends on not only the parameters of states, but also the value of τ, which is the ratio of utility and corresponding marginal utility of the individual and the charity at two states of nature.

3.3 Stage 1-Individual's Initial Contribution Decision Making

Theorems 1 through 4 have given us insight into the decision making processes by individuals and charities. Now, we move to stage 1. In this stage, the representative individual i chooses their initial donation d_i to maximize their expected utility, and believes the charity will use the money $\sum_{i=1}^{n} d_i$ collected in this stage to invest. Also, the individual knows that there will be an opportunity to make a top-up contribution decision g_i in stage 3 if the returns of the charitable investment R are not sufficiently higher than $\overline{R}$ (See theorem 1).

In stage 1, individuals' decision making problem can be written as:

$$\max_{d_i} E[U_i(x_i, G)] = \pi U(\omega_i - g_i|_{R_1} - d_i, (1 + r\beta)\sum d_i + \sum g_i|_{R_1}) + (1 - \pi)U(\omega_i - g_i|_{R_2} - d_i, (1 - \beta)\sum d_i + \sum g_i|_{R_2}) \quad (30)$$

$$\text{where} \quad \begin{cases} x = \omega_i - g_i - d_i \\ G_1 = (1 + r\beta)\sum d_i + n\, g_i|_{R_1} \\ G_2 = (1 - \beta)\sum d_i + n\, g_i|_{R_2} \end{cases}$$

Individual i's first order condition with respect to initial giving d_i becomes:

$$\pi\left[U_{x_1}\frac{\partial x_1}{\partial d_i} + U_{G_1}\frac{\partial G_1}{\partial d_i}\right] + (1 - \pi)\left[U_{x_2}\frac{\partial x_2}{\partial d_i} + U_{G_2}\frac{\partial G_2}{\partial d_i}\right] = 0 \quad (31)$$

Using the fact that $\frac{\partial x_1}{\partial d_i} = -\frac{\partial g_i}{\partial d_i}|_{R_1} - 1$, $\frac{\partial G_1}{\partial d_i} = r\sum d_i\frac{\partial \beta}{\partial d_i} + 1 + r\beta + n\frac{\partial g_i}{\partial d_i}|_{R_1}$, $\frac{\partial x_2}{\partial d_i} = -\frac{\partial g_i}{\partial d_i}|_{R_2} - 1$, and $\frac{\partial G_2}{\partial d_i} = -\sum d_i\frac{\partial \beta}{\partial d_i} + 1 - \beta + n\frac{\partial g_i}{\partial d_i}|_{R_2}$, we can rewrite equation (31) as :

$$\pi\left[U_{x_1}\left(-\frac{\partial g_i}{\partial d_i}|_{R_1} - 1\right) + U_{G_1}\left(r\sum d_i\frac{\partial \beta}{\partial d_i} + 1 + r\beta + n\frac{\partial g_i}{\partial d_i}|_{R_1}\right)\right] + (1 - \pi)\left[U_{x_2}\left(-\frac{\partial g_i}{\partial d_i}|_{R_2} - 1\right) + U_{G_2}\left(-\sum d_i\frac{\partial \beta}{\partial d_i} + 1 - \beta + n\frac{\partial g_i}{\partial d_i}|_{R_2}\right)\right] = 0 \quad (32)$$

Rearranging and simplifying the above equation, we can immediately get:

$$\pi U_{x_1}\left(1 + \frac{\partial g_i}{\partial d_i}|_{R_1}\right) + (1 - \pi)U_{x_2}\left(1 + \frac{\partial g_i}{\partial d_i}|_{R_2}\right) = \pi U_{G_1}\left(r\sum d_i\frac{\partial \beta}{\partial d_i} + n\frac{\partial g_i}{\partial d_i}|_{R_1} + 1 + r\beta\right) + (1 - \pi)U_{G_2}\left(-\sum d_i\frac{\partial \beta}{\partial d_i} + n\frac{\partial g_i}{\partial d_i}|_{R_2} + 1 - \beta\right) \quad (33)$$

By chain rule, $\frac{\partial g_i}{\partial d_i}|_R$ can be represented as $\frac{\partial g_i}{\partial R}\frac{\partial R}{\partial d_i}|_R$, which means that we can write:

$$\frac{\partial g_i}{\partial d_i}|_{R_1} = \frac{\partial g_i}{\partial R}\frac{\partial R}{\partial d_i}|_{R_1} = \frac{\partial g_i}{\partial R}|_{R_1}\left(r\sum d_i\frac{\partial \beta}{\partial d_i} + 1 + r\beta\right)$$

$$= \left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right)\frac{\partial g_i}{\partial R}|_{R_1} \quad (34)$$

Similarly,

$$\frac{\partial g_i}{\partial d_i}|_{R_2} = \frac{\partial g_i}{\partial R}\frac{\partial R}{\partial d_i}|_{R_2} = \frac{\partial g_i}{\partial R}|_{R_2}\left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right)$$
$$= \left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right)\frac{\partial g_i}{\partial R}|_{R_2} \quad (35)$$

Using (34), and (35), equation (33) can be written as:

$$\pi U_{x_1}\left(1 + \left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right)\frac{\partial g_i}{\partial R}|_{R_1}\right) + (1-\pi)U_{x_2}(1 + \left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right)\frac{\partial g_i}{\partial R}|_{R_2}\ ) =$$
$$\pi U_{G_1}\left(r\sum d_i \frac{\partial\beta}{\partial d_i} + \ n\ \left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right)\frac{\partial g_i}{\partial R}|_{R_1} + 1 + r\beta\right) + (1-\pi)U_{G_2}\left(-\sum d_i \frac{\partial\beta}{\partial d_i} + \right.$$
$$\left. n\ \left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right)\frac{\partial g_i}{\partial R}|_{R_2} + 1 - \beta\right) \quad (36)$$

Equation (36) explicitly indicates that with the expectation that the charity will use the money $\sum_{i=1}^{n} d_i$ collected in this stage to invest, and also the individual will top-up their contribution based on the return of this charitable investment, individual i will choose initial contribution d_i to maximize their expected utility. At this subgame perfect Nash equilibrium, such initial giving d_i will make an individual's expected marginal utility with respect to private goods and public goods equalized. This result is different from the interpretation of equation (4). The reason for this difference is that there is no uncertainty in stage 3. All the strategies that economic agents have chosen, except the individual's decision making of top-up giving, are realized before moving to stage 3.

4. The Determination of Optimal Policy

It is natural to wonder whether or not the fact that donors can make a top-up contribution leads to a higher or lower level of the public goods, and a higher or lower level of utility in equilibrium. Although we are not able to provide a definite comparison of either the equilibrium level of public goods, or of utility, we are able to provide some initial insights into how these two equilibrium paths differ. In this chapter, we will study the impact of possible top-up contribution policies on the determination of subgame perfect Nash equilibrium levels of optimal choices by the individual i and the charity.

4.1 No Top-Up Giving is Allowed

4.1.1 Stage 2- Charity's Investment Decision Making Problem

If the top-up giving in stage 3 is not allowed due to strict governmental regulation, stage 2 becomes the final stage. The previous 3 stage model simply becomes a 2 stage model.

Using (14), the charity's expected utility can be simplified as:

$$\max_{0 \le \beta \le 1} E\,[V\,(G)] = \pi\, V[(1+r\beta) \textstyle\sum d_i + n\, \bar{g_1}] + (1-\pi)V[(1-\beta) \textstyle\sum d_i + n\, \tilde{g_1}]$$

$$= \pi\, V[(1+r\beta) \textstyle\sum d_i] + (1-\pi)V[(1-\beta) \textstyle\sum d_i]$$

where
$$\begin{cases} R_1 = (1+r\beta)\sum d_i & G_1 = R_1 & V_1 = \frac{\partial V}{\partial G_1} \\ R_2 = (1-\beta)\sum d_i & G_2 = R_2 & V_2 = \frac{\partial V}{\partial G_2} \end{cases}$$

Using equation (21), we determine that when the charity maximizes its expected utility, there exists an optimal choice of β^* such that the charity's expected marginal utility is zero.

$$\frac{\partial E[V(G)]}{\partial \beta} = r \textstyle\sum d_i\, \pi\, V_1 - \textstyle\sum d_i((1-\pi)V_2$$
$$= \textstyle\sum d_i\, [r\ \pi\, V_1 - (1-\pi)V_2] \qquad (37)$$

If interior solution exists, the first order condition of E[V(G)] with respect to β becomes:

$$\frac{\partial E[V(G)]}{\partial \beta} = 0$$

Therefore, (37) can be rewritten as:

$$r\sum d_i\,\pi V_1 = \sum d_i\,(1-\pi)\,V_2$$

Rearranging, and simplifying, we obtain:

$$\frac{V_1}{V_2} = \frac{1-\pi}{r\pi}$$

It is obvious that $R_1 > R_2$. In addition, due to the assumed property of the strictly concave utility function, we directly get $0 < V_1 < V_2$.

Therefore, we directly have:

$$0 < \frac{V_1}{V_2} = \frac{1-\pi}{r\pi} < 1$$

Rearranging, we obtain the relation:

$$\pi > \frac{1}{1+r}$$

which indicates that if expected utility achieves its optimal value with an interior solution, the probability of the state of nature should be at least greater than $\frac{1}{1+r}$. From (19), we get the conclusion that the marginal expected utility at $\beta = 0$ is positive when $\pi > \frac{1}{1+r}$. That is:

$$\frac{\partial E[V(G)]}{\partial \beta}\Big|_{\beta=0} > 0 \qquad (38)$$

Also, we notice that the expected utility at $\beta = 0$ does not relate to the probability of the stage of nature. That is:

$$\begin{aligned} E[V(G)]|_{\beta=0} &= \{\pi\, V[(1+r\beta)\textstyle\sum d_i] + (1-\pi)V[(1-\beta)\textstyle\sum d_i]\}|_{\beta=0} \\ &= V[\textstyle\sum d_i] \qquad (39) \end{aligned}$$

Using (37), the second derivative of expected utility of the charity becomes:

$$\begin{aligned} \frac{\partial^2 E[V(G)]}{\partial \beta^2} &= \textstyle\sum d_i\,[r\ \pi\, V_{11} r \sum d_i + ((1-\pi)V_{22}\sum d_i] \\ &= (\textstyle\sum d_i)^2 (r^2\,\pi\, V_{11} + (1-\pi)\,V_{22}) \\ &< 0 \qquad (40) \end{aligned}$$

From (40), it is immediately evident that in the 2 stage model the charity's utility function is strictly concave with respect to β. Combining (38), (39), and (40), it is straightforward to show that, as in the three stage model, that $0 < \beta^* \leq 1$ when $\pi > \frac{1}{1+r}$; and $\beta^* = 0$ when $\pi < \frac{1}{1+r}$. That is, the charity invests in the risky asset only when the expected rate of return is great that the risk-free asset. Otherwise, the charities will not choose any risky portfolio to invest in, and prefer to employ completely conservative investment strategies in order to maintain their asset values with certainty.

4.1.2 Stage 1-Individual's Initial Contribution Decision Making

Using the fact that no top-up contribution exists, the individual's decision making problem becomes:

$$\max_{d_i} E[U_i(x_i, G)] = \pi U[\omega_i - d_i, (1 + r\beta)\sum d_i] + (1-\pi)U[\omega_i - d_i, (1-\beta)\sum d_i] \quad (41)$$

We can simplify (36) as:

$$\pi U_{x_1} + (1-\pi)U_{x_2} = \pi U_{G_1}\left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right) + (1-\pi)U_{G_2}\left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right) \quad (42)$$

Because in the 2 stage model, individuals consume the same private goods at two different states of nature, $x_1 = x_2 = \omega_i - d_i = x_i$, we express the first order condition (42) as:

$$\pi U_{x_i} + (1-\pi)U_{x_i} = \pi U_{G_1}\left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right) + (1-\pi)U_{G_2}\left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right)$$

Simplifying, we obtain:

$$U_{x_i} = \pi U_{G_1}\left(r\sum d_i \frac{\partial\beta}{\partial d_i} + 1 + r\beta\right) + (1-\pi)U_{G_2}\left(-\sum d_i \frac{\partial\beta}{\partial d_i} + 1 - \beta\right) \quad (43)$$

By interpreting equation (43), we can conclude that in a 2 stage model, an individual's expected marginal cost of giving an additional public goods is not affected by the state of the nature. Moreover, when an individual's expected utility is maximized, the optimal choice of giving d_i is implicitly determined by the first order condition (43), and the marginal cost of giving an additional private good x_i is equal to their expected marginal benefit from consuming public goods.

4.2 The Comparison Between 2 Stage Model and 3 Stage Model

Even though the theorems derived from the 3 stage model also fit the 2 stage model, it is difficult to directly identify which model can ensure that individuals and charities will achieve higher expected utility if without further assumptions. However, if we look further, we will find that there are several significant differences between these two models in the decision making processes.

4.2.1 The Moral Hazards of Charities

In the first stage of the two stage model, in order to maximize the expected utility, the individual chooses their contribution d_i with the anticipation that the charity will use the collected money $\sum d_i$ to invest. No second-period contribution is involved. In addition, in the second stage of the 2 stage model of our paper, we assume that the charity chooses β to maximize the expected utility that is derived from the total provision of the public goods, and no top-up contribution is allowed. Consequently, the hold up problem does not occur. However, this does not mean that there is no agency problem: the charity will not invest funds in the same way as the donors would.

The 3 stage model is relatively more reasonable. The mechanism of this model allows the individual to employ a second-period contribution based on the return of charitable investment, which makes the provision of public goods less fluctuated. Moreover, it can prevent the charity from behaving selfishly. However, the charity may exploit the generosity of individuals by choosing an overly aggressive investment policy by knowing that the individual will top up more if the return of investment decrease, which would otherwise likely lead to the "hold up problem", and other moral hazards.

In order to better characterize the possibility of moral hazard in the charities' behaviour, we look at the scenario that the individual were to choose the level that would maximize his own expected utility. we will get:

$$\max_{0 \le \beta \le 1} E[\, U_i(x_i, G)] = \pi U[\, \omega_i - d_i, (1 + r\beta) \sum d_i + \sum g_i |_{R_1}] + (1 - \pi) U[\, \omega_i - d_i, (1 - \beta) \sum d_i + \sum g_i |_{R_2}] \quad (44)$$

Compared with (13), it is clear that the optimal β^* for individuals will generally be different from the level that would maximize the expected utility of the charities. However, if the individual's utility function is additively separable, and the functions U(G) and V(G) are identical, and the initial contribution is given, then the first order condition for (44) is exactly same as that of (13). Consequently, there will be the same solution of β^* regardless of maximizing the individual's expected utility or the charity's expected utility.

4.2.2 The Stability of The Provision of Public Goods

In the 3 stage model, an individual chooses initial giving d_i to maximize their expected utility. The public goods are produced by the proceeds from the risky investment, the risk-free investment, and the top-up donation.

$$\max_{d_i} E[\, U_i(x_i, G)] = \pi U(\, \omega_i - g_i|_{R_1} - d_i, (1 + r\beta) \sum d_i + \sum g_i |_{R_1}) + (1 - \pi) U(\, \omega_i - g_i|_{R_2} - d_i, (1 - \beta) \sum d_i + \sum g_i |_{R_2}) \quad (45)$$

From theorem 1, the marginal total top-up contribution with respect to R will be within the interval $(-1, 0)$. Therefore, the increment of $\sum g_i$ will be less than the absolute value of the decrement of R, or the decrement of $\sum g_i$ will be less than the value of the increment of R. This compensation mechanism means that there is less fluctuation in the level of the provision of public goods.

In addition, from (44) we find that in the good and bad states, the consumption of private goods x_i is equal to $\omega_i - g_i|_{R_1} - d_i$, $\omega_i - g_i|_{R_2} - d_i$ respectively. However, in the 2 stage model, the consumption of the private goods x_i is equal to $\omega_i - d_i$, which is not correlated to the state of the nature. It is immediately evident that in the 3 stage model the stability of the provision of the public goods is at the expense of more fluctuation in the consumption of private goods.

To prevent the moral hazards of charities and to smoothen the provision of the public goods, the 3 stage model is better than the 2 stage model.

5. Conclusions and Implications for Future Study

We have studied the process of the formation and determination of the voluntary supply of public goods in the context of a three-stage subgame perfect Nash equilibrium, in which two different kinds of economic agents, the individual and the charity, are involved. Using backwards induction, we find several interesting and surprising results.

First, individuals' top –up contribution will be partially crowded out by the return to the charitable investment. In particular, if the individual's utility function is quasi-linear in private goods, this partial crowding-out effect becomes perfect. In addition, as we do not consider the borrowing of charities, the individual is always the first mover. However, the "free-ride" effect among the charitable investment and the individual's top-up giving is mutual.

Second, with the assumption that the charities maximize their own utility, which is derived from the total amount of public goods, charities do not need to be regulated to prevent them from investing in risky assets when the expected rate of return on the risky assets is low. They only choose to invest in the risky asset when it has a higher expected rate of return than that of the risk-free asset. However, there may be a role for regulation in determining the maximum proportion of the portfolio that can be invested in the risky asset if the government wishes to maximize the utility of charities. Moreover, the increase of an initial contribution will not necessarily lead the charity to invest more conservatively.

Finally, compared with the 2 stage model, the mechanism of the 3 stage model makes the provision of public goods more stable, which is at the expense of more volatile consumption of private goods. Also, the operative top-up contribution by the individual prevents the charity from performing unselfishly.

This paper's findings evoke several opportunities for future work. For instance, in this paper we do not evaluate which model, the 2 stage model or the 3 stage model, has a higher level of the provision of public goods and a higher expected utility. Also, we only assume that there are two states of nature. It may be interesting to assume that the rate of investment return is a stochastic variable in order to observe the variance of the expected utility of the economic agents in our future work.

References

[1] Hal R. Varian, 1994, Sequential contribution to public goods, *Journal of Public Economics,* 53 (1994) 165-186

[2] Harry Markowitz, 1952, Portfolio Selection, *The Journal of Finance*, 77- 91

[3] James Andreoni, 1989, Giving with Impure altruism: Application to charity and Ricardian equivalence, *Journal of Political Economy* 97, 1447-1458

[4] James Andreoni, 1990, Impure altruism and donations to public goods: A theory of warm-glow giving, *The Economic Journal*, 464-477

[5] Neil Bruce and Michael Waldman, 1990, The rotten-kid theorem meets the Samaritan's dilemma, *The Quarterly Journal of Economics* Vol.105, No.1, pp 155-165

[6] Russell D. Roberts, 1984, A positive model of private charity and public transfers, *Journal of Political Economy*, Vol 92, No.1

[7] Theodore Bergstrom, Lawrence Blume and Hal Varian, 1986, On the private provision of public goods, *Journal of Public Economics* 29, 25-49.

[8] Tze Leung Lai, Haipeng Xing, and Zehao Chen, 2009, Mean-variance portfolio optimization when means and covariances are unknown, *Technical Report* No. 2009-8