ON DUAL REPRESENTATIONS OF MULTIPLIERS AND BANACH MODULE HOMOMORPHISMS

A thesis submitted

by

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to

the Faculty of Science and Engineering of
the University of Ottawa

in partial fulfillment of the requirements

for the degree of

Master of Science

in the subject of

Mathematics

May, 1973

ACKNOWLEDGMENT

I wish to express my sincere thanks to Professor B.J. Tomiuk for suggesting the topic and for his invaluable guidance throughout the development of this thesis. I am also grateful to the University of Ottawa for providing me with financial support during my studies here for the past two years.

Special thanks should also go to Mrs. Heywood who typed this thesis with great patience.

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ABSTRACT

The aim of this thesis is to present L. Mâté's work on dual representations of multipliers and Banach-module homomorphisms. He showed that if a Banach algebra B has a weak left (right) identity then the algebra of left (right) continuous linear multipliers can be embedded isomorphically into the second conjugate space B^{**} of B. Now let W be a left Banach B-module and suppose B has a quasi-bounded approximate identity. Let $\mathrm{Hom}_B(B, W^*)$ be the space of all bounded linear operators $T : B \to W^*$ such that $T(ca) = T(c)a$ for all $a, c \in B$. Let $B \hat{\otimes} W$ be the projective (greatest) tensor product of B and W. Mâté has shown that $\mathrm{Hom}_B(B, W^*)$ is isometrically B-isomorphic to $(B \circ W)^*$, where $B \circ W$ is a quotient space of $B \hat{\otimes} W$.

As an application of the representation of $\mathrm{Hom}_B(B, W^*)$ as a conjugate space, we show that the algebra of left (right) continuous linear multipliers on the algebra $\tau c(H)$ of trace class operators is isometrically isomorphic to $L(H)$, the algebra of all bounded linear operators on a Hilbert space H.

INTRODUCTION

The main purpose of this thesis is to give a presentation of L. Máté's work on dual representations of multipliers and Banach-module homomorphisms.

Chapter I deals with definitions and basic properties of Banach modules, (topological) tensor products of locally convex spaces and Arens products on the second conjugate space of a Banach algebra. In Chapter II, we sketch L. Máté's work ([7], [8]) on the representation of (left, right) continuous linear multipliers on a Banach algebra B as elements of the second conjugate space B** and more generally as elements of the conjugate space of some Banach space.

In Chapter III we present a generalization of the multiplier problem as considered by L. Máté in [9]. More precisely, let B be a Banach algebra, W a left Banach B-module and W* its conjugate space. Let $\mathrm{Hom}_B(B, W^*)$ be the set of all bounded linear operators T from B into W* such that

$$T(ca) = T(c)a \qquad (c, a \in B).$$

$\mathrm{Hom}_B(B, W^*)$ is a Banach space under the operator bound norm. Let $B \hat{\otimes} W$ be the (greatest) projective tensor product of B and W and let $\phi : B \hat{\otimes} W \to W$ be given by $\phi(a \otimes w) = aw$; ϕ is a continuous homomorphism. Let $B \circ W = \phi(B \hat{\otimes} W)$ normed with the quotient norm. It follows that if B has a quasi-bounded approximate identity, then

$$\operatorname{Hom}_B(B, W^*) \cong (B \circ W)^*.$$

We also give necessary and sufficient conditions for $\operatorname{Hom}_B(B, W^*) \cong W^*$, when W is an essential Banach B-module.

Chapter IV is concerned mainly with the applications of results in Chapter III. For any Banach algebra B, let $M_\ell(B)$ be the algebra of all continuous linear left multipliers on B. We show that if B is a dual A*-algebra which is a two-sided ideal of its completion and such that B is isometrically B-isomorphic to the conjugate space W* of a left Banach B-module W, then $M_\ell(B)$ is isometrically isomorphic to $(B \circ W)^*$. From this it follows that if $B = \tau c(H)$, the algebra of trace class of operators on Hilbert space H, then $M_\ell(B)$ is isometrically isomorphic to L(H), the algebra of all continuous linear operators on H.

CHAPTER I

PRELIMINARIES

We shall assume that all algebras and vector spaces under consideration in this thesis are over the complex field C.

§1. Banach modules.

Let B be a Banach algebra and W a Banach space. W is called a left Banach B-module if it is a left B-module in the algebraic sense and there is a constant k, such that $||aw|| \leq k||a||\ ||w||$, for all $a \in B$, $w \in W$. Similarly, we define a right Banach B-module. If k = 1, we say that W is an isometric Banach B-module. From now on a left (right) Banach B-module will simply be called a left (right) B-module.

If W is a left B-module then its conjugate space W* is a right B-module; for $a \in B$ and $w^* \in W^*$, we define w^*a by $(w^*a)(x) = w^*(ax)$, for all $x \in W$. Similarly, W** is a left B-module.

Two left (resp. right) B-modules W_1 and W_2 are said to be B-isomorphic if there exists an isomorphism ϕ of W_1 onto W_2 such that $\phi(aw) = a\phi(w)$ (resp. $\phi(wa) = \phi(w)a$) for all $a \in B$ and $w \in W$.

Lemma (1.1.1). Let B be a Banach algebra and W a left

B-module. Then W can be renormed with an equivalent norm with respect to which it is an isometric left B-module.

<u>Proof.</u> [12; p. 452]. By the definition of a left B-module, there is a positive constant k such that $||aw|| \leq k||a||\ ||w||$, for all $a \in B$, $w \in W$. Define a new norm $||\cdot||'$ on W by letting $||w||'$ be the greater of $||w||$ and $\sup\{||aw|| : a \in B,\ ||a|| \leq 1\}$. Then

$$||w|| \leq ||w||' \leq k||w||,$$

so that the norm $||\cdot||'$ is equivalent to the given norm $||\cdot||$. Now, for all $0 \neq a \in B$, $w \in W$, we have

$$\frac{||aw||'}{||a||} = \left|\left|\frac{a}{||a||}w\right|\right|'$$

$$= \max\left(\left|\left|\frac{a}{||a||}w\right|\right|,\ \sup\left\{\left|\left|b\frac{a}{||a||}w\right|\right|,\ b \in B,\ ||b|| \leq 1\right\}\right).$$

But

$$\left|\left|\frac{a}{||a||}w\right|\right| \leq \sup\{||bw|| : b \in B,\ ||b|| \leq 1\} \leq ||w||'$$

$$\left(\text{since } \left|\left|\frac{a}{||a||}\right|\right| = 1\right)$$

and

$$\sup\left\{\left|\left|\frac{ba}{||a||}w\right|\right| : ||b|| \leq 1\right\} \leq \sup\{||bw||,\ ||b|| \leq 1\} \leq ||w||'$$

Hence $\frac{||aw||}{||a||} \leq ||w||'$ i.e. $||aw||' \leq ||a||\ ||w||'$, for all $a \in B$ and $w \in W$. This shows that W with the norm $||\cdot||'$ is an isometric B-module. This completes the proof.

In view of this result, we shall always assume from now on that all B-modules are isometric modules.

A net $\{e_\alpha\}$ of elements in B is called a left (resp. right) approximate identity if $e_\alpha a \to a$ (resp. $ae_\alpha \to a$), for all $a \in B$. A net $\{e_\alpha\}$ which is both a left and right approximate identity is called an approximate identity.

Definition (1.1.2). Let B be a Banach algebra and W a left B-module. The essential part W_1 of W is the closure of the linear hull of $\{aw : a \in B, w \in W\}$, if $W_1 = W$, i.e. BW is dense in W, then W is called an essential left B-module.

If B has a bounded approximate identity Rieffel [12] has obtained the following necessary and sufficient conditions for a B-module W to be essential.

Theorem (1.1.3). Let B be a Banach algebra with a bounded approximate identity $\{e_\alpha\}$ and W a left B-module. Then the following statements are equivalent:

(a) W is an essential B-module.

(b) $e_\alpha w \to w$ for every $w \in W$.

(c) Given $w \in W$, there exist $w' \in W$ and $a \in B$ such that $w = aw'$.

Proof. See [12; p. 453, Proposition 3.4].

§2. Tensor products.

In the presentation of tensor products in this section we follow [13] and [14].

Let B and W be vector spaces and let $\mathfrak{B}(B, W)$ be the vector space of all bilinear forms f on $B \times W$. For each $(a, w) \in B \times W$, the mapping $u_{a,w} : f \to f(a, w)$ is a linear form on $\mathfrak{B}(B, W)$ and hence an element $u_{a,w}$ of the algebraic dual $\mathfrak{B}(B, W)^*$. It is easily seen that the mapping $\chi : (a,w) \to u_{a,w}$ of $B \times W$ into $\mathfrak{B}(B, W)^*$ is bilinear.

<u>Definition (1.2.1)</u>. The linear hull of $\chi(B \times W)$ in $\mathfrak{B}(B, W)^*$ is denoted by $B \otimes W$ and is called the tensor product of B and W. χ is called the canonical bilinear mapping of $B \times W$ into $B \otimes W$.

The element $u_{a,w}$ of $B \otimes W$ will be denoted by $a \otimes w$, so that each element of $B \otimes W$ is a finite sum $\sum_i \lambda_i (a_i \otimes w_i)$, where $\lambda_i \in C$; $a_i \in B$, $w_i \in W$. One verifies without difficulty the following rules:

$$\lambda(a \otimes w) = (\lambda a) \otimes w = a \otimes (\lambda w) \qquad (\lambda \in C)$$

$$(a_1 + a_2) \otimes w = a_1 \otimes w + a_2 \otimes w$$

and

$$a \otimes (w_1 + w_2) = a \otimes w_1 + a \otimes w_2.$$

Hence each element $u \in B \otimes W$ is of the form

$$h = \sum_{i=1}^{n} a_i \otimes w_i, \quad a_i \in B, \; w_i \in W.$$

One of the principal advantages of tensor products lies in the fact that they permit us to consider vector spaces of bilinear maps as vector spaces of linear maps. More precisely, if G is any vector space and u is any element of the vector space $L(B \otimes W, G)$ of linear mappings from $B \otimes W$ into G, then the mapping $u \to u \circ \chi$ is an isomorphism of $L(B \otimes W, G)$ onto $\mathfrak{B}(B, W; G)$ [14; p. 92, Sec. 6.1].

When B, W and G are locally convex spaces, $B \otimes W$ can be given a topology in such a way that the correspondence between linear mappings u and bilinear mappings $u \circ \chi$ preserves the continuity.

Theorem (1.2.2). Let B and W be two locally convex spaces and χ the canonical bilinear mapping of $B \times W$ into $B \otimes W$. Then there is a finest locally convex topology τ_p on $B \otimes W$ under which χ is continuous. If $\mathcal{U}$ and $\mathcal{V}$ are neighborhood bases at 0 in B and W, then the family of convex, circled hulls $\{\Gamma(U \otimes V) : U \in \mathcal{U}, V \in \mathcal{V}\}$ (where $U \otimes V = \chi(U \times V)$) form a neighborhood base at 0 for this topology. The algebraic isomorphism $u \to u \circ \chi$ maps the space of continuous linear mappings $\mathcal{L}(B \otimes W, G)$ onto the space of continuous bilinear mappings $\mathcal{B}(B, W; G)$.

Proof. [13; p. 132]. Denote the family of convex, circled hulls $\{\Gamma(U \otimes V) : U \in \mathcal{U}, V \in \mathcal{V}\}$ by $\mathcal{W}$. Since $\mathcal{U}$ and $\mathcal{V}$ are neighborhood bases at 0 in B and W, it follows that there is a locally convex topology τ_p on $B \otimes W$ such that $\mathcal{W}$ is neighborhood base at 0 (cf. [13; p. 10, Theorem 2]). Let I be a

0-neighborhood in τ_p. Then there is a $\Gamma(U_1 \otimes V_1)$ such that $\Gamma(U_1 \otimes V_1) \subset I$, where $U_1 \in \mathcal{U}$ and $V_1 \in \mathcal{V}$. It follows that $\chi^{-1}(I)$ contains $U_1 \times V_1$, so that τ_p makes χ continuous at 0 and consequently continuous everywhere. Now, let τ be any locally convex topology on $B \otimes W$ under which χ is continuous. Let O be a τ-open set and let $x \in O$. Then there is a convex, circled neighborhood M of 0 in τ such that $x + M \subset O$. Since $\chi^{-1}(M)$ contains a set $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$, M contains the convex, circled hull $\Gamma(U \otimes V)$. It follows that $x + \Gamma(U \otimes V) \subset O$, and hence O is τ_p-open. Therefore τ_p is the finest locally convex topology on $B \otimes W$ under which χ is continuous.

It is now easily seen that if u is continuous in the topology τ_p on $B \otimes W$ then $u \circ \chi$ is continuous, since the topology τ_p makes χ continuous. Conversely, suppose $f = u \circ \chi$ is continuous. Let N be a convex, circled 0-neighborhood in G. Then $f^{-1}(N)$ contains a 0-neighborhood $U \times V$ ($U \in \mathcal{U}$, $V \in \mathcal{V}$) and hence $u \circ \chi\ (U \times V) = u(U \otimes V) \subset N$. Thus $u^{-1}(N)$ contains $U \otimes V$. Since N is convex and circled, it follows that $u^{-1}(N)$ is convex and circled, and hence contains $\Gamma(U \otimes V)$, which proves the continuity of u under the topology τ_p.

Definition (1.2.3). The topology τ_p of Theorem (1.2.2) defined on $B \otimes W$ is called the projective tensor product topology on $B \otimes W$.

Corollary (1.2.4). The dual of $B \otimes W$ for the projective topology can be identified with the space $\mathcal{B}(B, W)$ of all continuous bilinear forms on $B \times W$.

We shall list now several properties of the tensor product $B \otimes W$, where B and W are normed linear spaces, which will be useful to us in this thesis.

(1) Let U and V be closed unit balls in B and W, respectively. Then the closure of the convex, circled hull $\Gamma(U \otimes V)$ is the closed unit ball in $B \otimes W$ under the projective topology. Thus the norm in $B \otimes W$ is given by

$$||h|| = \inf\{\sum_k ||a_k|| \, ||w_k|| : h = \sum_k a_k \otimes w_k\}$$

(See [13; p. 137].).

(2) If B and W are complete, it is not in general true that $B \otimes W$ is complete. In applications it is often the completion of the topological tensor product arises (which is unique to within isomorphism). We denote this completion by $B \hat{\otimes} W$. The elements of $B \hat{\otimes} W$ can be written as sums of convergent series of the form

$$\sum_{k=1}^{\infty} a_k \otimes w_k \text{ such that } \sum_{k=1}^{\infty} ||a_k|| \, ||w_k|| < \infty,$$

where $a_k \in B$ and $w_k \in W$. (See [13; p. 133].).

(3) Let $(B \hat{\otimes} W)^*$ be the conjugate space of $B \hat{\otimes} W$. Then for each $F \in (B \hat{\otimes} W)^*$, we have

$$||F|| = \sup\{|F(h)| : h \in \Gamma(U \otimes V)\} = \sup\{|F(h)| : h \in \chi(U, V)\}$$

$$= \sup\{|(F \circ \chi)(a, w)| : a \in U \text{ and } w \in V\}.$$

where U, V are closed unit balls in B and W, respectively. (See [13, p. 137].)

§3. Arens products.

Let B be a Banach algebra. Let B^* and B^{**} be the first and second conjugate spaces of B. Arens [1] has defined two products on B^{**} under which B^{**} is a Banach algebra. These products (henceforth called Arens products) are defined in stages as follows: Let $x, y \in B$, $f, g \in B^*$ and $F, G \in B^{**}$.

(i) Define $f * x$ by $(f * x)(y) = f(xy)$; $f * x \in B^*$.

(ii) Define $G * f$ by $(G * f)(x) = G(f * x)$; $G * f \in B^*$.

(iii) Define $F * G$ by $(F * G)(f) = F(G * f)$, $F * G \in B^{**}$.

B^{**} with the product $*$ will be denoted by $(B^{**}, *)$.

(i)' Define $x *_1 f$ by $(x *_1 f)(y) = f(yx)$; $x *_1 f \in B^*$.

(ii)' Define $f *_1 F$ by $(f *_1 F)(x) = F(x *_1 f)$; $f *_1 F \in B^*$.

(iii)' Define $F *_1 G$ by $(F *_1 G)(f) = G(f *_1 F)$; $F *_1 G \in B^{**}$.

B^{**} with the product $*_1$ will be denoted by $(B^{**}, *_1)$.

It is easily seen that $||f * x|| \leq ||f||\ ||x||$, $||G * f|| \leq ||G||\ ||f||$ and $||F * G|| \leq ||F||\ ||G||$, similarly for the other product $*_1$.

When B is embedded canonically into B** each of the Arens products agrees with the given product on B [1].

Definition (1.3.1). A Banach algebra B is called Arens regular if the two Arens products coincide on B**.

Hennefeld [5] has given necessary and sufficient conditions for B to be Arens regular which are stated in the following theorem:

Theorem (1.3.2). Let B be a Banach algebra. Then the following conditions are equivalent:

(1) B is Arens regular.

(2) For each F, G ϵ B** and f ϵ B*, there exists F_α, $G_\beta \epsilon B$: $F_\alpha \to F$ and $G_\beta \to G$ in the weak*-topology on B** such that

$$\lim_\alpha \lim_\beta (F_\alpha * G_\beta)(f) = \lim_\beta \lim_\alpha (F_\alpha * G_\beta)(f).$$

(3) For each f ϵ B*, the mapping $T_f : B \to B^*$ given by $T_f(x) = f * x$, x ϵ B is weakly compact.

Proof. See [5; p. 117].

Definition (1.3.3). We say that a Banach algebra B has a weak right identity if there exists a net $\{e_\alpha\}$ in B and a constant M > 0 such that $||e_\alpha|| \leq M$, for all α and $\lim_\alpha f(xe_\alpha - x) = 0$, for all x ϵ B and f ϵ B*.

Lemma (1.3.4). A Banach algebra B has a weak right identity if and only if $(B^{**}, *)$ has a right identity.

Proof. [2] Suppose B has a weak right identity $\{e_\alpha\}$ with $||e_\alpha|| \leq M$, for all α. Since the natural embedding π is an isometry of B into B^{**}, we have $||\pi\, e_\alpha|| \leq M$, for all α. Since every closed bounded ball in B^{**} is compact in the weak*-topology on B^{**} [3; p. 423, Lemma 1], there exists a subnet $\{e_\beta\}$ of $\{e_\alpha\}$ such that $\lim_\beta \pi\, e_\beta = I \in B^{**}$ in the weak*-topology on B^{**} [20; p. 161, Theorem 1]. We claim that I is a right identity of $(B^{**}, *)$. In fact, let $f \in B^*$ and $x \in B$. Then

$$(I * f)(x) = I(f * x) = \lim_\beta (\pi\, e_\beta)(f * x) = \lim_\beta (f * x)(e_\beta)$$

$$= \lim_\beta f(xe_\beta) = f(x).$$

Hence $I * f = f$, for all $f \in B^*$. But for $F \in B^{**}$, we have

$$(F * I)(f) = F(I * f) = F(f) \qquad (f \in B^*).$$

Hence

$$F * I = F \qquad (F \in B^{**}).$$

Conversely, suppose that $(B^{**}, *)$ has a right identity I. By Goldstine's Theorem [3; p. 424, Theorem 5], there is a net $\{\pi\, e_\alpha\}$ with $||\pi\, e_\alpha|| \leq ||I||$, for all α, $\lim_\alpha \pi\, e_\alpha = I$ in the weak*-topology on B^{**}. Since I is a right identity in $(B^{**}, *)$, for all $F \in B^{**}$ and $f \in B^*$,

$$F(f) = (F * I)(f) = F(I * f).$$

Hence $I * f = f$, for all $f \in B^*$. Consequently, for all $x \in B$ and $f \in B^*$, we obtain that

$$f(x) = (I * f)(x) = I(f * x) = \lim_{\alpha} (\pi\ e_{\alpha})(f * x)$$

$$= \lim_{\alpha} (f * x)(e_{\alpha}) = \lim_{\alpha} f(xe_{\alpha}).$$

This shows that $\{e_{\alpha}\}$ is a weak right identity in B. This completes the proof.

Similarly, we can prove that $(B^{**}, *_1)$ has a left identity if and only if B has a weak left identity.

Corollary (1.3.5). $I \in (B^{**}, *)$ is a right identity if and only if $I * f = f$, for all $f \in B^*$.

It is easy to see that if B is Arens regular then B has a weak identity if and only if B^{**} has an identity. Thus, in particular, if B is a B^*-algebra then B is Arens regular and B^{**} is a B^*-algebra [2; p. 869, Theorem 7.1], so that B^{**} has an identity since B has an approximate identity $\{e_{\alpha}\}$ with $||e_{\alpha}||=1$, for all α [11; p. 245, Theorem (4.8.14)].

CHAPTER II

ARENS PRODUCTS AND MULTIPLIERS

§1. Multipliers.

Definition (2.1.1). Let B be a Banach algebra. A mapping T from B into itself is called a left (resp. right) multiplier if

$$T(xy) = T(x)y \quad (\text{resp. } T(xy) = xT(y)),$$

for all $x, y \in B$.

Example. For each $a \in B$ define the mapping $L_a : B \to B$ (resp. $R_a : B \to B$) by $L_a(x) = ax$ (resp. $R_a(x) = xa$). Then L_a (resp. R_a) is a left (resp. right) multiplier on B.

It follows that if B is a commutative Banach algebra without annihilators or B is a Banach algebra with a bounded (left or right) approximate identity then every multiplier (left or right) on B is a continuous linear operator. (See [19; p. 1132, Theorem 2.1] and [6].) Let $M_\ell(B)$ (resp. $M_r(B)$) denote the set of all continuous linear left (resp. right) multipliers on B. $M_\ell(B)$ and $M_r(B)$ are Banach algebras with identity under the usual algebraic operations for operators and the norm given by the operator bound. Clearly $L_a \in M_\ell(B)$ and $R_a \in M_r(B)$, for every $a \in B$. If B is commutative then $M_\ell(B) = M_r(B)$. If B has a minimal right (left) approximate identity $\{e_\alpha\}$ (i.e. $||e_\alpha|| \leq 1$, for all α) then $a \to L_a$ ($a \to R_a$) is an isometric isomorphism, so that B can be isometrically embedded into $M_\ell(B)$ ($M_r(B)$). If B has a left (right) identity then $B = M_\ell(B)$ ($B = M_r(B)$) under this identification.

Let B be a semi-simple commutative Banach algebra and let $\hat{B}$ be the algebra of all complex-valued functions on the carrier space Ω_B of B given by the Gelfand representation (cf. [11]). Then, for every multiplier T on B, there exists a continuous complex-valued function on Ω_B such that $\widehat{T(a)}(x) = g(x)\hat{a}(x)$, for all $a \in B$ and $x \in \Omega_B$. In particular, if B is a commutative B*-algebra, by [11; p. 190, Theorem (4.2.2)], the algebra of multipliers on B is mapped isometrically onto $C_b(\Omega_B)$, the algebra of all bounded continuous complex-valued functions on Ω_B (see [19]).

§2. Embedding of multipliers in the second conjugate space.

Let B be a Banach algebra. For each $F \in B^{**}$, the mapping $T : f \to F * f$, for all $f \in B$, is obviously a continuous linear operator from B^* into itself. We say that T is the operator represented by F.

If B has a weak right identity, L. Máté [7] has shown that for every $T \in M_r(B)$ there is a $F \in B^{**}$ such that the conjugate operator T^* of T is the operator represented by F; that is

$$(F * f)(x) = f(Tx) = (T^*f)(x) \quad (x \in B,\ f \in B^*).$$

He also showed that the operator represented by $F \in B^{**}$ is the conjugate operator of a $T \in M_r(B)$ if and only if it is continuous in the weak*-topology on B^*. Moreover, if T_i^* $(i = 1, 2)$ is represented by $F_i \in B^{**}$, where T_i^* is the conjugate operator of $T_i \in M_r(B)$, then $T_1^*T_2^*$ is represented by $F_1 * F_2$. From this it follows that when B has a weak right identity then the mapping $T \to F$ is an algebraic anti-isomorphism of $M_r(B)$ into B^{**}. However, if the weak right identity is not bounded by 1, then this embedding of $M_r(B)$ into B^{**} is not usually an isometry. In fact, from the inequality $||F * f|| \leq ||F||\ ||f||$, we have that $||F|| \geq ||T^*|| = ||T||$, where T^* is represented by F. To see that the equality does not necessarily hold, let us consider the subspace B' of elements $h \in B^*$ which are represented as follows:

$$h = \sum_{k=1}^{\infty} f_k * x_k; \quad \sum_{k=1}^{\infty} ||f_k||\ ||x_k|| < \infty,$$

where $f_k \in B^*$, $x_k \in B$, $k = 1, 2, \ldots$. B' is a Banach space with the norm

$$||h||' = \inf\{\sum_{k=1}^{\infty} ||f_k||\ ||x_k|| : h = \sum_{k=1}^{\infty} f_k * x_k\}.$$

Since $||F||' = \sup\{|F(h)| : ||h||' \leq 1\} \leq ||F||$, each $F \in B^{**}$ is a bounded linear functional on B', too. We shall show that if $T \in M_r(B)$ and T^* is represented by $F \in B^{**}$, then $||F||' = ||T||$, where $||\cdot||'$ is the norm on B'^*.

Let $T \in M_r(B)$ such that T^* is represented by F. Then for every $\varepsilon > 0$ and $h \in B'$ there are $f_k \in B^*$, $x_k \in B$, $k = 1, 2, \ldots$, such that

$$h = \sum_{k=1}^{\infty} f_k * x_k \quad \text{and} \quad \sum_{k=1}^{\infty} ||f_k||\ ||x_k|| \leq ||h||' + \varepsilon.$$

Since

$$|F(h)| = |F(\sum_{k=1}^{\infty} f_k * x_k)| = |\sum_{k=1}^{\infty} F(f_k * x_k)|$$

$$= |\sum_{k=1}^{\infty} (F * f_k)(x_k)| = |\sum_{k=1}^{\infty} (T^*f_k)(x_k)|$$

$$\leq ||T^*|| \sum_{k=1}^{\infty} ||f_k||\ ||x_k|| \leq ||T||\ (||h||' + \varepsilon),$$

it follows that $||F||' \leq ||T||$. On the other hand, if $f \in B^*$, $||f|| \leq 1$ and $x \in B$, $||x|| = 1$, then

$$|(T^*f)(x)| = |(F * f)(x)| = |F(f * x)| \leq ||F||',$$

so that

$$||T|| = ||T^*|| \leq ||F||'.$$

Hence

$$||T|| = ||F||'.$$

§3. Modified Arens product and multipliers.

In this section we shall consider the representation of continuous linear right multipliers on a Banach algebra with conditions other than the condition of having a weak right identity. Here we follow Máté [8].

Let B be a Banach algebra satisfying the following two conditions:

Condition 1. B is without a right annihilator.

Condition 2. From $f_k \in B^*$, $x_k \in B$, $\sum_{k=1}^{\infty} ||f_k||\ ||x_k|| < \infty$ and $\sum_{k=1}^{\infty} f_k * x_k = 0$ it follows that $\sum_{k=1}^{\infty} f_k(x_k) = 0$.

We note that if there is a weak right identity in B then Condition 2 is clearly satisfied. In fact, let $x_k \in B$, $f_k \in B^*$ $(k = 1, 2, \ldots)$ such that $\sum_{k=1}^{\infty} ||f_k||\ ||x_k|| < \infty$ and $\sum_{k=1}^{\infty} f_k * x_k = 0$. Since B has a weak right identity, $(B^{**}, *)$ has a right identity I and $(I * f)(x) = f(x)$, for all $x \in B$ and $f \in B^*$. Hence

$$\sum_{k=1}^{\infty} f_k(x_k) = \sum_{k=1}^{\infty} (I * f_k)(x_k) = \sum_{k=1}^{\infty} I(f_k * x_k)$$

$$= I(\sum_{k=1}^{\infty} f_k * x_k) = I(0) = 0,$$

which shows that B satisfies Condition 2. However, Conditions

1 and 2 together do not imply the condition that B has a weak right identity. As an example of such an algebra B we have the following: Let G be a compact and non-discrete group. Then $L^p(G)$ $(1 < p < \infty)$ is a Banach algebra which satisfies Conditions 1 and 2 but has no weak left (right) identity (see [8]). We shall discuss more fully at the end of this section another example of a Banach algebra having Conditions 1 and 2, without a weak left (right) identity.

Let $\pi[B]$ be the canonical image of B in B**. For each $F \in B^{**}$ such that $\pi[B] * F \subset \pi[B]$, let T_F be the operator on B given by

$$(1) \qquad T_F : \pi(T_F x) = \pi(x) * F \qquad (x \in B).$$

Then $T_F \in M_r(B)$. If $(\pi(x) * F)(f) = 0$, for all $f \in B^*$ and $x \in B$, then T_F is the zero operator and

$$(2) \quad F(f * x) = (F * f)(x) = \pi(x)(F * f) = (\pi(x) * F)(f) = 0.$$

<u>Definition (2.3.1)</u>. Let B be a Banach algebra. Define Y to be the linear hull of the set $\{f * x : f \in B^*, x \in B\}$ and $Y^\perp$ the orthogonal complement of Y in B**, i.e. $Y^\perp = \{F : F(h) = 0, h \in Y\}$.

It follows from (2) and Definition (2.3.1) that the operator T represented by F in the sense (1) is the zero operator if and only if $F \in Y^\perp$.

Lemma (2.3.2). The following two statements are equivalent:

(a) B is without a right annihilator.

(b) $\pi[B] \cap Y^{\perp} = (0)$.

Proof. (a) => (b). Assume (a) and let $x_o \in B$ such that $\pi(x_o) \in Y^{\perp}$. Then

$$\pi(x_o)(f * x) = (f * x)(x_o) = f(xx_o) = 0 \qquad (x \in B,\ f \in B^*).$$

Hence $xx_o = 0$, for all $x \in B$ and so $x_o = 0$. Thus $\pi(x_o) = 0$.

(b) => (a). Suppose (b) holds and let $x_o \in B$ such that $xx_o = 0$, for all $x \in B$. Then

$$f(xx_o) = (f * x)(x_o) = \pi(x_o)(f * x) = 0 \qquad (x \in B,\ f \in B^*).$$

Hence $\pi(x_o) \in Y^{\perp}$ and consequently $x_o = 0$, whence (a).

Definition (2.3.3). Let Y' be the linear hull of the set

$$(3) \quad \{h = \sum_{k=1}^{\infty} f_k * x_k : \sum_{k=1}^{\infty} ||f_k||\ ||x_k|| < \infty;\ f_k \in B^*,\ x \in B\}.$$

Lemma (2.3.4). Y' is a Banach space under the norm

$$||h||' = \inf\{\sum_{k=1}^{\infty} ||f_k||\ ||x_k|| : h = \sum_{k=1}^{\infty} f_k * x_k\}.$$

Moreover, Y' is a linear subset of B^* and $||h|| \leq ||h||'$, for all $h \in Y'$.

Proof. It is obvious that $||\cdot||'$ is a norm on Y' and that Y' is a linear subset of B^*. Now let $\{h_n\}$ be a Cauchy sequence in Y' with respect to the norm $||\cdot||'$. Then there is a subsequence $\{h_{n_k}\}$ such that

$$||h_{n_{k+1}} - h_{n_k}||' < \frac{1}{2^k} \qquad (k = 1, 2, \ldots).$$

Hence, if we let $h = h_{n_1} + \sum_{k=1}^{\infty} (h_{n_{k+1}} - h_{n_k})$, then

$$||h||' = ||h_{n_1} + \sum_{k=1}^{\infty}(h_{n_{k+1}} - h_{n_k})||' \leq$$

$$\leq ||h_{n_1}||' + \sum_{k=1}^{\infty} ||h_{n_{k+1}} - h_{n_k}||'$$

$$\leq ||h_{n_1}||' + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

h is in the form (3) and hence $h \in Y'$. Clearly $h = \lim h_n$ in the norm $||\cdot||'$. Thus Y' is a Banach space under the norm $||\cdot||'$. Since $||h|| = ||\sum_{k=1}^{\infty} f_k * x_k|| \leq \sum_{k=1}^{\infty} ||f_k||\,||x_k||$ for all representations (3) of h, we get $||h|| \leq ||h||'$.

Remark. It is easy to see that Y is a dense subset of the Banach space $(Y', ||\cdot||')$.

Our next objective is to define a product on the conjugate space Y'^* of Y' under which it is a Banach algebra. Such a product is called modified Arens product. Let $F \in Y'^*$, $h \in Y'$ and $x, y \in B$. Define $h * x$ by $(h * x)(y) = h(xy)$. Define $F * h$ by $(F * h)(x) = F(h * x)$.

Theorem (2.3.5). For every $x \in B$, $h \in Y'$ and $F \in Y'^*$. $h * x \in Y'$ and $F * h \in Y'$.

Proof. Since $||h|| \leq ||h||'$, we have $||h||\ ||x|| \leq ||h||'\,||x||$, so that $h * x \in Y'$.

Now for each $F \in Y'^*$ and $f \in B^*$, let Ψ be the linear form on B given by

$$\Psi(x) = F(f * x) \qquad (x \in B).$$

Since

$$(4) \quad |\Psi(x)| \leq ||F||\ ||f * x||' \leq ||F||\ ||f||\ ||x|| \qquad (x \in B),$$

it follows that $\Psi \in B^*$. Since, for all $x, x_o \in B$,

$$\begin{aligned}(\Psi * x)(x_o) &= \Psi(xx_o) = F(f * xx_o)\\ &= F[(f * x) * x_o] = [F * (f * x)](x_o),\end{aligned}$$

we have that

$$(5) \qquad \Psi * x = F * (f * x) \qquad (x \in B).$$

Now for each $F \in Y'^*$ and $h = \sum_{k=1}^{\infty} f_k * x_k \in Y'$, let Ψ_k be given by

$$\Psi_k(x) = F(f_k * x) \qquad (x \in B).$$

From (5), we have

$$\Psi_k * x_k = F * (f_k * x_k),$$

for each x_k, and from (4), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} ||\Psi_k||\ ||x_k|| &\leq \sum_{k=1}^{\infty} ||F||\ ||f_k||\ ||x_k||\\ &= ||F|| \sum_{k=1}^{\infty} ||f_k||\ ||x_k|| < \infty.\end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} \Psi_k * x_k = \sum_{k=1}^{\infty} F * (f_k * x_k) \in Y'.$$

But, for each positive integer n

$$\sum_{k=1}^{n} F * (f_k * x_k) = F * (\sum_{k=1}^{n} f_k * x_k).$$

Hence

$$\sum_{k=1}^{\infty} F * (f_k * x_k) = F * (\sum_{k=1}^{\infty} f_k * x_k) = F * h \in Y'.$$

Definition (2.3.6). For all F_1, $F_2 \in Y'^*$, let

$$F_1 * F_2 : (F_1 * F_2)(h) = F_1(F_2 * h) \qquad (h \in Y').$$

$F_1 * F_2$ is called a modified Arens product of F_1 and F_2. It is easy to see that $*$ is a multiplication on Y'^* and that $||F_1 * F_2|| \leq ||F_1|| \; ||F_2||$. Thus $(Y'^*, *)$ is a Banach algebra.

Theorem (2.3.7). Let B be a Banach algebra satisfying Condition 1. For each $x \in B$, let F_x be given by

$$(6) \qquad F_x(h) = h(x) \qquad (h \in Y').$$

Then $x \to F_x$ is an algebraic isomorphism from B into $(Y'^*, *)$.

Proof. From Lemma (2.3.2) we have that $x \to F_x$ is a one-to-one mapping. That $x \to F_x$ preserves all algebraic operations is obvious.

Lemma (2.3.8). If B is a Banach algebra satisfying Condition 1, then for every $T \in M_r(B)$ there is a unique multiplier extension T' onto Y'* with the following properties:

(a) $(T'F_x)(h) = h(Tx) \qquad (x \in B,\ h \in Y'),$

where F_x given in (6);

(b) $T'(F_1 * F_2) = F_1 * T'F_2 \qquad (F_1, F_2 \in Y'^*).$

Proof. See the proof of [8; p. 231, Theorem 2].

From the proof of Lemma (2.3.8) it follows that the conjugate operator T* of T is continuous on the Banach space Y'. Let T' be the conjugate operator of T* restricted to Y'. Then T' is the unique multiplier extension of T to Y'*.

Theorem (2.3.9). If B is a Banach algebra satisfying Conditions 1 and 2, then for every $T \in M_r(B)$ there is a unique $F_T \in Y'^*$ such that, by the isomorphism π' of Theorem (2.3.7),

(7) $\pi'(Tx) = F_x * F_T \qquad (x \in B).$

Proof. Let I be the linear functional on Y' given by

$$I(h) = \sum_{k=1}^{\infty} f_k(x_k),$$

for all $h = \sum_{k=1}^{\infty} f_k * x_k \in Y'$. It is easy to show that $I \in Y'^*$.

Since for all $x \in B$, we have

$$(I * h)(x) = I(h * x) = I[(\sum_{k=1}^{\infty} f_k * x_k) * x]$$

$$= I[\sum_{k=1}^{\infty}(f_k * x_k) * x] = I(\sum_{k=1}^{\infty} f_k * x_k x) = \sum_{k=1}^{\infty} f_k(x_k x)$$

$$= \sum_{k=1}^{\infty}(f_k * x_k)(x) = (\sum_{k=1}^{\infty} f_k * x_k)(x) = h(x),$$

it follows that $I * h = h$, for all $h \in Y'$ and consequently that

(8) $$F * I = F \qquad (F \in Y'^*).$$

Thus, if $T \in M_r(B)$ and T' is its multiplier extension onto Y'^*, then

$$T'F = T'(F * I) = F * T'I \qquad (F \in Y'^*).$$

(See Lemma (2.3.8).)

Therefore, if we let $F_T = T'I$, then (7) is satisfied.

To prove the uniqueness of F_T, let $T = 0 \in M_r(B)$ and let $F_T \in Y'^*$ be such that equation (7) is satisfied. Then $F_x * F_T = 0$, for all $x \in B$ and consequently, for any $h \in Y'$, we have

$$(F_T * h)(x) = F_x(F_T * h) = (F_x * F_T)(h) = 0 \qquad (x \in B).$$

Hence

(9) $$F_T * h = 0 \qquad (h \in Y').$$

Now, for each $h = \sum_{k=1}^{\infty} f_k * x_k \in Y'$ and $F \in Y'^*$, let Ψ_k be given by the relation

$$\Psi_k(x) = F(f_k * x), \quad k = 1, 2, \ldots .$$

Then, for all $h = \sum_{k=1}^{\infty} f_k * x_k \in Y'$, we have

$$(I * F)(h) = I(F * h) = I[F * (\sum_{k=1}^{\infty} f_k * x_k)] = I[\sum_{k=1}^{\infty} F * (f_k * x_k)]$$

$$= I(\sum_{k=1}^{\infty} \Psi_k * x_k) = \sum_{k=1}^{\infty} \Psi_k(x_k) = \sum_{k=1}^{\infty} F(f_k * x_k) = F(\sum_{k=1}^{\infty} f_k * x_k)$$

$$= F(h).$$

(See the proof of Theorem (2.3.5).)

Hence

$$I * F = F \qquad (F \in Y'^*),$$

and by (8), we obtain that

$$I * F = F * I = F \qquad (F \in Y'^*);$$

in particular

$$I * F_T = F_T * I = F_T \qquad (T \in M_r(B)).$$

Therefore,

$$(10) \quad I(F_T * h) = (I * F_T)(h) = (F_T * I)(h) = F_T(h),$$

for all $h \in Y'$. It follows now from (9) that $F_T = 0$ and consequently F_T is uniquely determined for each $T \in M_r(B)$.

Theorem (2.3.10). If B is a Banach algebra satisfying Conditions 1 and 2, then $M_r(B)$ is anti-isomorphic (algebraically) to the subalgebra K of Y'^* consisting of those F for which

$$(11) \qquad F_x * F \in \pi'[B] \qquad (x \in B),$$

where $\pi'[B]$ is the image of B in Y'^* by the isomorphism π' given in Theorem (2.3.7).

Proof. For each F in Y'^* such that (11) is satisfied, let T_F be the continuous linear operator on B given by

$$F_x * F = \pi'(T_F x) \qquad (x \in B).$$

Then $T_F \in M_r(B)$. In fact,

$$\pi'[T_F xy] = F_{xy} * F = (F_x * F_y) * F = F_x * (F_y * F)$$
$$= \pi'(x) * \pi'(T_F y) = \pi'[xT_F y] \qquad (x, y \in B).$$

Thus $T_F \in M_r(B)$. From application of Theorem (2.3.9) it now shows that the mapping $F \to T_F$ from K into $M_r(B)$ is one-to-one and onto. To complete the proof we observe that $F \to T_F$ is linear and that

$$\pi'[(T_1T_2)x] = \pi'[T_1(T_2x)] = F_{T_2x} * F_1 = (F_x * F_2) * F_1$$
$$= F_x * (F_2 * F_1),$$

for all $x \in B$.

We end this section with an example of a Banach algebra which satisfies Conditions 1 and 2 and does not have a weak right identity.

Let H be an infinite dimensional Hilbert space. Let B be the Banach algebra $\tau c(H)$ consisting of trace-class operators on H under the trace norm $\tau(\cdot)$ (see §1, IV). Let Y' be the linear hull of the set consisting of elements h of the form

$$\{h = \sum_{k=1}^{\infty} f_k * x_k : \sum_{k=1}^{\infty} ||f_k|| \ \tau(x_k) < \infty; \quad f_k \in B^*, x_k \in B\}.$$

By Lemma (2.3.4), Y' is a Banach space with the norm

$$||h||' = \inf\{\sum_{k=1}^{\infty} ||f_k||\ \tau(x_k) : h = \sum_{k=1}^{\infty} f_k * x_k\}.$$

Y' is a linear subset of B* and $||h|| \leq ||h||'$ for all $h \in Y'$. Moreover, under the modified Arens product * on Y'*, (Y'*, *) is a Banach algebra.

Since B is semisimple (B is an A*-algebra), B has no non-zero right (and left) annihilators. Hence B satisfies Condition 1.

Now let $\{f_k\}$ be a sequence in B* and $\{x_k\}$ a sequence in B such that

$$(*) \qquad \sum_{k=1}^{\infty} ||f_k||\ \tau(x_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} f_k * x_k = 0.$$

By [15; p. 47, Theorem 2], for each $f_k \in B^*$ there is a unique $a_k \in L(H)$, the Banach algebra of all continuous linear operators on H, such that $f_k(x) = tr(a_k x)$, for all $x \in B$, and $||a_k|| = ||f_k||$, where tr(z) is the trace of $z \in B$. Hence

$$(f_k * x_k)(y) = f_k(x_k y) = tr(a_k x_k y) \qquad (y \in B)$$

From (*) we have

$$\sum_{k=1}^{\infty} \tau(a_k x_k) \leq \sum_{k=1}^{\infty} ||a_k||\ \tau(x_k) = \sum_{k=1}^{\infty} ||f_k||\ \tau(x_k) < \infty$$

(cf. [15; p. 39, Lemma 8]), so that $\sum_{k=1}^{\infty} a_k x_k$ converges to an element $a \in B$. Hence

$$\left(\sum_{k=1}^{\infty} f_k * x_k\right)(y) = \mathrm{tr}(ay) \qquad (y \in B).$$

But from (*) we also have $\mathrm{tr}(ay) = 0$, for all $y \in B$ and, therefore, by [15; p. 45, Lemma 1], $\sum_{k=1}^{\infty} a_k x_k = a = 0$, which gives

$$\mathrm{tr}\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{\infty} \mathrm{tr}(a_k x_k) = \sum_{k=1}^{\infty} f_k(x_k) = 0.$$

Finally, we show that B has no weak right identity. On the contrary, suppose that B has a weak right identity. Then, by Lemma (1.3.4), there is a right identity I in $(B^{**}, *)$. It follows from [21; p. 831, Theorem 5.5 and its proof] that the radical R^{**} of $(B^{**}, *)$ is a non-zero two-sided ideal and $B^{**} * R^{**} = R^{**} * B^{**} = (0)$. Therefore, for every non-zero element F in R^{**}, we have $I * F = 0$. But this contradicts the fact that I is a right identity for $(B^{**}, *)$. Hence B has no weak right identity. Since the involution in B is continuous (isometric) [15, p. 39, Lemma 8], it follows that B has neither a weak left identity. For if $\{e_\alpha\}$ is a weak left identity then $\{e_\alpha^*\}$ is a weak right identity by the continuity of involution and the fact that f^*, defined by $f^*(x) = \overline{f(x^*)}$ for all $x \in A$, belongs to B^* for all $f \in B^*$.

CHAPTER III

DUAL REPRESENTATION OF BANACH MODULE-HOMOMORPHISMS

§1. Preliminaries.

Let B be a Banach algebra and W a left Banach B-module. A bounded linear operator T from B into W* such that

$$T(ca) = T(c)a \qquad (c, a \in B)$$

is called a (B, W*)-multiplier (or a Banach B-module homomorphism). It is easy to see that the set of all (B, W*)-multipliers is a Banach space under the operator bound norm. We denote this Banach space by $\underset{B}{\mathrm{Hom}}(B, W^*)$. Rieffel [12] has obtained the following relation between $\underset{B}{\mathrm{Hom}}(B, W^*)$ and W*.

Theorem (3.1.1). Let B be a Banach algebra with a bounded approximate identity, and let W be an essential left B-module. Then there is a natural isometric right B-module isomorphism mapping W* onto $\underset{B}{\mathrm{Hom}}(B, W_1^*)$, where W_1^* is the essential part of W*. This isomorphism is given by the mapping $w^* \to T_{w^*}$, where $w^* \in W^*$ and $T_{w^*}(a) = w^*a$ for all $a \in B$.

Proof. cf. [12; p. 473, Theorem 8.9].

Definition (3.1.2). Let B be a Banach algebra. An approdimate identity $\{e_\alpha\}$ in B for which $\{\sup_\alpha ||ae_\alpha|| : a \in B, ||a|| = 1\} < \infty$, is called a quasi-bounded approximate identity.

The purpose of this chapter is to extend Rieffel's result (Theorem (3.1.1)) to Banach algebras with a quasi-bounded approximate identity.

For the equivalence of $\operatorname{Hom}_B(B, W^*)$ and W^* to hold it is necessary that the following be true:

(*) $w^*a = 0$ for every $a \in B$ implies that $w^* = 0$.

For this reason we shall assume in the rest of this chapter that the right B-module W^* satisfies (*).

§2. Some Lemmas.

Let B be a Banach algebra and W a left Banach B-module. Then the projective (greatest) tensor product $B \hat{\otimes} W$ is a Banach space under the norm

$$||h|| = \inf\{\sum_{k=1}^{\infty} ||a_k||\ ||w_k|| : h = \sum_{k=1}^{\infty} a_k \otimes w_k\}.$$

(See §2, I.)

It is easy to see that there is a continuous homomorphism ϕ from $B \hat{\otimes} W$ into W defined by

$$\phi(a \otimes w) = aw \qquad (a \in B,\ w \in W).$$

Clearly the kernel, ker ϕ, of ϕ is given by

$$\{h = \sum_{k=1}^{\infty} a_k \otimes w_k : \sum_{k=1}^{\infty} (w^*, a_kw_k) = 0, \text{ for all } w^* \in W^*\}.$$

Definition (3.2.1). Let $B \circ W = \phi(B \hat{\otimes} W)$ with the quotient norm.

For each $w^* \in W^*$, define $F_{w^*} \in (B \hat{\otimes} W)^*$ by the relation

(1) $(F_{w^*}, a \otimes w) = (w^*, \phi(a \otimes w)) \quad (a \otimes w \in B \hat{\otimes} W).$

Since $F_{w^*} = 0$ implies that

$$(w^*, \phi(a \otimes w)) = (w^*, aw) = (w^*a, w) = 0 \qquad (w \in W),$$

it follows that $w^*a = 0$, for all $a \in B$, and therefore by (*) in §1, III, we have $w^* = 0$. Thus the linear mapping $w^* \to F_{w^*}$ from W^* into $(B \hat{\otimes} W)^*$ is a monomorphism. If we identify W^* with its image $\{F_{w^*} : w^* \in W^*\}$ then we have the following:

Lemma (3.2.2). Let $(B \circ W)^*$ be the conjugate space of $B \circ W$. Then

$$W^* \subseteq (B \circ W)^* \subseteq (B \hat{\otimes} W)^*.$$

Moreover, $(B \circ W)^*$ is the closure of W^* with respect to the weak*-topology on $(B \hat{\otimes} W)^*$.

Proof. We only need to verify the last part of the assertion. Let $h \in B \hat{\otimes} W$. Then $(F_{w^*}, h) = 0$ for all $w^* \in W^*$ if and only if $\phi(h) = 0$ (by (1)). This means that

$$\ker \phi = \bigcap_{w^* \in W^*} \ker F_{w^*} \text{ .}$$

Now the polar $(\ker \phi)^\circ$ of $\ker \phi$ ([14; p. 125]) is given by

$$(\ker \phi)^\circ = \{F \in (B \hat{\otimes} W)^* : \operatorname{Re}(F, h) \le 1, \text{ if } h \in \ker \phi\}$$
$$= \{F \in (B \hat{\otimes} W)^* : (F, h) = 0, \text{ if } h \in \ker \phi\},$$

and $(B \circ W)^*$ consists of exactly those elements F of $(B \hat{\otimes} W)^*$

for which $(F, h) = 0$, for all $h \in \ker \phi$. Hence $(B \circ W)^*$ can be identified with $(\ker \phi)^\circ$. Since

$$(\ker F_{w^*})^\circ = \{\alpha F_{w^*} : \alpha \in C\},$$

for each $w^* \in W^*$. It follows from the Bipolar Theorem [14; p. 126, Corollary 2] that $(\ker \phi)^\circ$ is the closed convex hull of $\bigcup_{w^* \in W^*} (\ker F_{w^*})^\circ$ in the weak*-topology on $(B \hat{\otimes} W)^*$. Hence $(B \circ W)^*$ is the closure of W^* in the weak*-topology on $(B \hat{\otimes} W)^*$. This completes the proof.

Now let $\mathcal{L}(B, W^*)$ be the usual Banach space of the bounded linear operators from B into W^*.

Lemma (3.2.3). The Banach spaces $(B \hat{\otimes} W)^*$ and $\mathcal{L}(B, W^*)$ are isometrically isomorphic.

Proof. For each $F \in (B \hat{\otimes} W)^*$, let $\mathfrak{I} F = T_F$ be defined by the following equation

(2) $(F, a \otimes w) = ((\mathfrak{I} F)a, w) = (T_F a, w) \qquad (a \in B, w \in W)$

Since $|(T_F a, w)| \leq ||F|| \; ||a|| \; ||w||$, we have that $T_F a \in W^*$, for each $a \in B$, and hence that T_F maps B into W^*. For any a_1, $a_2 \in B$, $w \in W$ and $\lambda_1, \lambda_2 \in C$, we have

$$(T_F(\lambda_1 a_1 + \lambda_2 a_2), w) = (F, (\lambda_1 a_1 + \lambda_2 a_2) \otimes w)$$

$$= (F, \lambda_1(a_1 \otimes w)) + (F, \lambda_2(a_2 \otimes w))$$

$$= \lambda_1(F, a_1 \otimes w) + \lambda_2(F, a_2 \otimes w) = \lambda_1(T_F a_1, w) + \lambda_2(T_F a_2, w)$$

$$= (\lambda_1 T_F a_1 + \lambda_2 T_F a_2, w).$$

Moreover, let χ be the canonical map from $B \times W$ into $B \hat{\otimes} W$ (cf. §2, I.). Then by (3) in §2, I, we have that

$$\begin{aligned}
||F|| &= \sup\{|F(h)| : h \in B \hat{\otimes} W, ||h|| \leq 1\} \\
&= \sup\{|(F \circ \chi)(a, w)| : a \in B, ||a|| \leq 1; w \in W, ||w|| \leq 1\} \\
&= \sup\{|(T_F a, w)| : a \in B, ||a|| \leq 1; w \in W, ||w|| \leq 1\} \\
&= \sup_{||a|| \leq 1} \{ \sup_{||w|| \leq 1} \{|(T_F a, w)|\}\} = \sup_{||a|| \leq 1} ||T_F a|| \\
&= ||T_F||.
\end{aligned}$$

Hence $T_F \in \mathfrak{X}(B, W^*)$.

Clearly T_F is uniquely determined for each $F \in (B \hat{\otimes} W)^*$ by (2), so that $\mathfrak{T}$ is a well-defined linear mapping from $(B \hat{\otimes} W)^*$ into $\mathfrak{X}(B, W^*)$.

In order to complete the proof, it remains to show that $\mathfrak{T}$ is one-to-one and onto.

Since $\mathfrak{T}$ is a linear isometry, it follows that it is one-to-one. To prove $\mathfrak{T}$ is onto, for each $T \in \mathfrak{X}(B, W^*)$, let F_T be the functional on $B \hat{\otimes} W$ given by

$$(F_T, h) = (F_T, \sum_{k=1}^{\infty} a_k \otimes w_k) = \sum_{k=1}^{\infty} (Ta_k, w_k),$$

where $h = \sum_{k=1}^{\infty} a_k \otimes w_k \in B \hat{\otimes} W$.

Clearly F_T is linear. Since

$$|(F_T,h)| = |\sum_{k=1}^{\infty} (Ta_k, w_k)| \leq \sum_{k=1}^{\infty} |(Ta_k, w_k)| \leq \sum_{k=1}^{\infty} ||Ta_k|| \; ||w_k||$$

$$\leq \sum_{k=1}^{\infty} ||T|| \; ||a_k|| \; ||w_k|| = ||T|| \sum_{k=1}^{\infty} ||a_k|| \; ||w_k||,$$

it follows that $||F_T|| \leq ||T||$ and consequently that $F_T \in (B \hat{\otimes} W)^*$. Moreover

$$(F_T, a \otimes w) = (Ta, w) \qquad (a \in B, w \in W).$$

Hence $\mathfrak{T} F_T = T$. which shows that $\mathfrak{T}$ is onto. This completes the proof.

Definition (3.2.4). The weak*-topology on $(B \hat{\otimes} W)^*$ is transferred by the isomorphism $\mathfrak{T}$ into a topology on $\mathcal{L}(B, W^*)$, called ultraweak topology.

Lemma (3.2.5). For each $w^* \in W^*$, let $T_{w^*} : B \rightarrow W^*$ be given by $T_{w^*}(a) = w^*a$, for all $a \in B$. Then $(B \circ W)^*$ is the ultraweak closure of $\{T_{w^*} : w^* \in W^*\}$. Furthermore, if $F \in (B \circ W)^*$ and T is the operator corresponding to F by (2), then $T \in \underset{B}{\mathrm{Hom}}(B, W^*)$.

Proof. Clearly $T_{w^*} \in \underset{B}{\mathrm{Hom}}(B, W^*)$, for all $w^* \in W^*$ and, by (2), T_{w^*} corresponds to F_{w^*} so that $F_{w^*} \rightarrow T_{w^*}$ is an isometry from $\{F_{w^*} : w^* \in W^*\}$ onto $\{T_{w^*} : w^* \in W^*\}$. Hence by Lemma (3.2.2), $(B \circ W)^*$ is the ultraweak closure of $\{T_{w^*} : w^* \in W^*\}$.

Let $F \in (B \circ W)^*$. By Lemma (3.2.2) $(B \circ W)^*$ is the weak*-closure of $\{F_{w^*} : w^* \in W^*\}$ in $(B \hat{\otimes} W)^*$. Hence we can find a net $\{F_{w^*}\}$ which weak*-converges to F in $(B \hat{\otimes} W)^*$. Therefore, if T is the operator corresponding to F by (2), then there corresponds a net $\{T_{w^*}\}$ which converges to T in the ultraweak topology on $\mathscr{L}(B, W^*)$ and since all $T_{w^*} \in \operatorname{Hom}_B(B, W^*)$, it follows that $T \in \operatorname{Hom}_B(B, W^*)$.

§3. A characterization of $\operatorname{Hom}_B(B, W^*)$.

In this section we will show that if B has a quasi-bounded approximate identity then $\operatorname{Hom}_B(B, W^*)$ is isometrically B-isomorphic to $(B \circ W)^*$. Thus Rieffel's result (Theorem (3.1.1)) extends to Banach algebras with a quasi-bounded approximate identity.

As above let B be a Banach algebra and W a left Banach B-module.

Theorem (3.3.1). If B has a quasi-bounded approximate identity, then $\operatorname{Hom}_B(B, W^*)$ is isometrically right B-module isomorphic to $(B \circ W)^*$. In symbols we write this as

$$\operatorname{Hom}_B(B, W^*) \cong (B \circ W)^*.$$

Proof. We first show that $\operatorname{Hom}_B(B, W^*)$ is isometrically isomorphic to $(B \circ W)^*$ as a Banach space. Since $\operatorname{Hom}_B(B,W^*) \subseteq \mathcal{L}(B,W^*)$ and $(B \circ W)^* \subseteq (B \hat{\otimes} W)^*$, by Lemma (3.2.3) and (3.2.5), it remains to show that if $T \in \operatorname{Hom}_B(B, W^*)$ and F corresponds to T by (2) in §2, III, then $F \in (B \circ W)^*$.

Let $\{e_\alpha\}$ be a quasi-bounded approximate identity in B and let T_{e_α} be given by $T_{e_\alpha}(a) = e_\alpha a$, for all $a \in B$. Then for $T \in \operatorname{Hom}_B(B, W^*)$, we have

$$\begin{aligned} \|TT_{e_\alpha}\| &\leq \|T\| \; \|T_{e_\alpha}\| \\ &\leq \|T\| \sup_\alpha \{\|e_\alpha a\| : a \in B, \|a\| = 1\} < \infty, \end{aligned}$$

for all α, so that the net $\{TT_{e_\alpha}\}$ is bounded. If F_α corresponds to TT_{e_α} by (2) in §2, III, then

$$(F_\alpha, a \otimes w) = (TT_{e_\alpha} a, w) = (T(e_\alpha a), w) = (T(e_\alpha)a, w),$$

for all $a \in B$ and $w \in W$. Since $T(e_\alpha) \in W^*$, it follows that each $F_\alpha \in \{F_{w^*} : w^* \in W^*\}$. Moreover $\{F_\alpha\}$ is bounded since $\{TT_{e_\alpha}\}$ is bounded. Thus, by Lemma (3.2.2), there is a weak*-convergent subnet of $\{F_\alpha\}$, still denoted by $\{F_\alpha\}$, which converges to some $F \in (B \circ W)^*$.

Let T_F be the operator corresponding to F by (2) in §2, III. To complete the proof it is enough to show that $T_F = T$.

Since $\{F_\alpha\}$ weak*-converges to F, it follows that $TT_{e_\alpha} \to T_F$ in the ultraweak topology on $\mathcal{L}(B, W^*)$. Thus, in particular,

$$(TT_{e_\alpha} a, w) \to (T_F a, w) \qquad (a \in B,\ w \in W).$$

On the other hand, we have

$$T(a) = T(\lim_\alpha (e_\alpha a)) = \lim_\alpha T(e_\alpha a) = \lim_\alpha T(T_{e_\alpha} a)$$
$$= \lim_\alpha (TT_{e_\alpha})(a) \qquad (a \in B)$$

That is, $TT_{e_\alpha} \to T$ in the strong operator topology and hence also in the weak operator topology [3; p. 476]; in particular

$$(TT_{e_\alpha} a, w) \to (Ta, w) \qquad (a \in B,\ w \in W).$$

Therefore $T_F = T$.

The proof that $\operatorname{Hom}_B(B, W^*)$ and $(B \circ W)^*$ are B-isomorphic is easy now if we observe that $\operatorname{Hom}_B(B, W^*)$ is a right B-module with the module operation defined by

$$(Ta)b = T(ab),$$

for all $T \in \operatorname{Hom}_B(B, W^*)$ and $a, b \in B$. This completes the proof.

Theorem (3.3.2). If B has a quasi-bounded approximate identity and W is essential, then the following statements are equivalent:

(a) The mapping $w^* \to T_{w^*}$ defines an isometric isomorphism between $\underset{B}{\mathrm{Hom}}(B, W^*)$ and W^*.

(b) The mapping $w^* \to T_{w^*}$ defines an isometric isomorphism between $\underset{B}{\mathrm{Hom}}(B, W_1^*)$ and W^*, where W_1^* is the essential part of W^*.

(c) $||T_{w^*}|| = ||w^*||$, for all $w^* \in W^*$.

Proof. (b) => (c). This is obvious.

(a) => (b). Clearly $\underset{B}{\mathrm{Hom}}(B, W^*) \supseteq \underset{B}{\mathrm{Hom}}(B, W_1^*)$. On the other hand, $T_{w^*} \in \underset{B}{\mathrm{Hom}}(B, W_1^*)$, for each $w^* \in W^*$. Hence if (a) is true so is (b).

(c) => (a). From Theorem (3.3.1) we see that (a) is the same as $(B \circ W)^* \cong W^*$.

Let the map $L : B \circ W \to W$ be given by

$$L(a \circ w) = \phi(a \otimes w) = aw,$$

where $a \circ w = a \otimes w + \ker \phi$ and $a \otimes w \in B \hat{\otimes} W$; L is a one-to-one linear map. Since, for all $a \otimes w \in B \hat{\otimes} W$ and $h \in \ker \phi$, we have

$$\begin{aligned} ||L(a \circ w)|| &= ||\phi(a \otimes w)|| \\ &= ||\phi(a \otimes w + h)|| \leq ||\phi|| \, ||a \otimes w + h||, \end{aligned}$$

it follows that $||L(a \circ w)|| \leq ||\phi|| \, ||a \circ w||$ and consequently that $||L|| \leq ||\phi||$. Thus L is continuous. Also, since W is essential, it follows that $L(B \circ W)$ is dense in W. Therefore, the conjugate operator L^* of L is a continuous one-to-one linear mapping of weak*-dense range from W^* into $(B \circ W)^*$ [4; p. 199, Proposition 1 and 2]. So if we show that L^*W^* is also weak*-closed, then it will follow that L^*W^* and $(B \circ W)^*$ coincide and that W^* is homeomorphic to $(B \circ W)^*$ by the open mapping theorem.

From the Krein-Šmulian Theorem [14; p. 152, Corollary] we see that it is enough to show that $L^*W^* \cap S^*$ is weak*-closed, where S^* is the closed unit ball in $(B \circ W)^*$. Let $\{L^*w^*_\alpha\}$ be a net in S^*. By [16; p. 228, Theorem 4.61 - A], S^* is weak*-compact and hence there exists a subnet of $\{L^*w^*_\alpha\}$, still denoted by $\{L^*w^*_\alpha\}$, such that $\{(a \circ w, L^*w^*_\alpha)\}$ converges for every $a \circ w \in B \circ W$. Thus from

$$(a \circ w, L^*w^*_\alpha) = (L(a \circ w), w^*_\alpha) = (aw, w^*_\alpha)$$

it follows that $\{w^*_\alpha\}$ is pointwise convergent on the dense subset $L(B \circ W)$ of W. Since

$$(a \circ w, L^*w^*_\alpha) = (aw, w^*_\alpha) = (w, w^*_\alpha a) = (T_{w^*_\alpha}\, a)(w),$$

we see that $\{(T_{w^*_\alpha}\, a)(w)\}$ converges for all $a \in B$ and $w \in W$. Therefore from the Banach-Steinhaus Theorem [16; p. 203, Theorem 4.4 - C] we obtain that $\{T_{w^*_\alpha}\, a\}$ is bounded. Using the Banach-Steinhaus Theorem again, we have that $\{T_{w^*_\alpha}\}$ is bounded. Therefore by (c) $\{w^*_\alpha\}$ is also bounded. Consequently, there is a subnet of $\{w^*_\alpha\}$, still denoted by $\{w^*_\alpha\}$, which weak*-converges to

$w^* \in W^*$. Thus, for all $a \circ w \in B \circ W$, we have that

$$\lim_{\alpha} (a \circ w, L^*w^*_{\alpha}) = \lim_{\alpha} (L(a \circ w), w^*_{\alpha}) = (L(a \circ w), w^*)$$
$$= (a \circ w, L^*w^*),$$

which shows that $L^*w^* \in L^*W^* \cap S^*$. Thus $L^*W^* \cap S^*$ is weak*-closed and consequently L^*W^* and $(B \circ W)^*$ coincide. Hence W^* is homeomorphic to $(B \circ W)^*$ and by (c), we have that

$$W^* \cong (B \circ W)^*.$$

This completes the proof.

Remark. (1) Theorem (3.3.2) gives necessary and sufficient conditions for $\operatorname{Hom}_B(B, W^*) \cong W^*$ in the case when B has a quasi-bounded approximate identity and W is essential.

(2) If we suppose merely the equivalence of the norms $||T_{w^*}||$ and $||w^*||$ in (c), Theorem (3.3.2) still remains true with the same proof if we replace $\cong$ by homeomorphism in (a) and (b).

CHAPTER IV

MULTIPLIERS ON DUAL A*-ALGEBRAS

The main purpose of this chapter is to apply the results of the previous chapter to multipliers on dual A*-algebras which are two-sided ideals of their completions. (See [18].)

§1. Preliminaries.

For any set S in a Banach algebra B, let $\ell(S)$ and $r(S)$ denote the left and right annihilators of S, respectively. If, for every closed right ideal I and for every closed left ideal J, $r(\ell(I)) = I$ and $\ell(r(J)) = J$, then B is called a dual algebra.

If B is a Banach *-algebra on which there is defined a second norm $|\cdot|$, which satisfies, in addition to the multiplicative condition $|xy| \leq |x||y|$, the B*-algebra condition $|x^*x| = |x|^2$, then B is called an A*-algebra. The norm $|\cdot|$ is called an auxiliary norm. It follows that the involution in B is continuous with respect to both norms and $|\cdot| \leq \beta||\cdot||$, for some constant β [11; p. 187, Theorem (4.1.15)]. Let $\mathcal{B}$ be the completion of an A*-algebra B with respect to the norm $|\cdot|$. Then $\mathcal{B}$ is a B*-algebra and if B is a two-sided ideal of $\mathcal{B}$ then $||xy|| \leq k||x|| \; |y|$, for all $x \in B$, $y \in \mathcal{B}$ and some constant k [10; p. 18, Lemma 4]. If B is a dual A*-algebra then B has a unique auxiliary norm ([17; p. 54, Theorem 5.4] and [10; p. 18, Theorem 3]) and the completion $\mathcal{B}$ of B in this norm is a dual

B*-algebra which is uniquely determined up to *-isomorphism [17; p. 54, Lemma 5.5].

§2. Multipliers on dual A*-algebras.

Theorem (4.2.1). Let B be a dual A*-algebra which is a two-sided ideal of its completion $\mathcal{B}$. Let $\{e_\alpha\}$ be a maximal orthogonal family of self-adjoint minimal idempotents in B. Then for all $x \in \mathcal{B}$, $x = \sum_\alpha xe_\alpha = \sum_\alpha e_\alpha x$ in the norm $|\cdot|$, and if $x \in B$, then $x = \sum_\alpha xe_\alpha = \sum_\alpha e_\alpha x$ in both norms $||\cdot||$ and $|\cdot|$. Moreover, if $e_{\alpha_1}, \ldots, e_{\alpha_n}$ are distinct elements of $\{e_\alpha\}$ then $||xe_{\alpha_1} + \ldots + xe_{\alpha_n}|| \leq k||x||$ and $||e_{\alpha_1}x + \ldots + e_{\alpha_n}x|| \leq k||x||$, for all $x \in B$.

Proof. This easily follows from [17; Theorems 6.1 and 7.1] and [10; p. 30, Theorem 16].

Let B be a dual A*-algebra which is a two-sided ideal of its completion $\mathcal{B}$. Let $\mathcal{K}$ be the set of all idempotents K in B that are of the form $K = e_{\alpha_1} + \ldots + e_{\alpha_n}$, where $e_{\alpha_1}, \ldots, e_{\alpha_n}$ are distinct elements of $\{e_\alpha\}$. Clearly $|K| = 1$, for all $K \in \mathcal{K}$. Let $E_K = \{e_{\alpha_1}, \ldots, e_{\alpha_n}\}$. For $K_1, K_2 \in \mathcal{K}$ write $K_1 \leq K_2$ if $E_{K_1} \subseteq E_{K_2}$. It is easy to see that $\leq$ is a partial ordering on $\mathcal{K}$. Let $E = E_{K_1} \cup E_{K_2}$ and let K be the sum of elements in E. Then $K \in \mathcal{K}$

and $K_1 \leq K$, $K_2 \leq K$. Thus $\mathcal{K}$ is a directed set under $\leq$ and consequently is a net when indexed over itself.

From the observations above and Theorem (4.1.1) it follows that $\mathcal{K}$ is an approximate identity for B with the property that, for every $x \in B$, we have

$$\sup\{||Kx|| : K \in \mathcal{K}\} \leq k||x||$$

and

$$\sup\{||xK|| : K \in \mathcal{K}\} \leq k||x||;$$

in particular

$$\sup_{K \in \mathcal{K}} \{||xK|| : ||x|| = 1, x \in B\} \leq k.$$

Thus $\mathcal{K}$ is a quasi-bounded approximate identity for B. As a consequence of these observations we have:

<u>Theorem (4.2.2)</u>. Let a_0 be any element of $\mathcal{B}$ and a_1, a_2, ..., a_n be any finite number of elements of B and let $\varepsilon > 0$. Then there exists $K \in \mathcal{K}$ such that $|a_0 - a_0K| < \varepsilon$ and $||a_i - Ka_i|| < \varepsilon$ for $i = 1, 2, \ldots, n$.

If B is isometrically B-isomorphic to the conjugate space W^* of a left B-module W, then $\mathrm{Hom}_B(B, W^*)$ is obviously isometrically isomorphic to $M_\ell(B)$. Thus, we have the following:

<u>Theorem (4.2.3)</u>. Let B be a dual A*-algebra which is a two-sided ideal of its completion $\mathcal{B}$ and such that B is

isometrically B-isomorphic to the conjugate space W^* of a left B-module W. Then $M_\ell(B)$ is isometrically isomorphic to $(B \circ W)^*$.

Proof. From the above considerations it follows that B has a quasi-bounded approximate identity and so by Theorem (3.3.1), $\underset{B}{\mathrm{Hom}}(B, W^*)$ is isometrically isomorphic to $(B \circ W)^*$. Hence $M_\ell(B)$ is isometrically isomorphic to $(B \circ W)^*$.

§3. Multipliers on the trace class operators.

For any Hilbert space H, L(H) will denote the Banach algebra of all continuous linear operators on H with the operator bound norm, LC(H) the subalgebra of L(H) consisting of all compact linear operators on H. Let $\{\phi_j\}$, $\{\psi_i\}$ be any two complete orthonormal families of vectors in H. Let $T \in L(H)$. Then, by [15; p. 29, Lemma 1], the three sums

$$\sum_j ||T\phi_j||^2, \quad \sum_{j,i} |(T\phi_j, \psi_i)|^2, \quad \sum_i ||T^*\psi||^2$$

are equal to one another (possibly to $+\infty$) and the common value $\sigma(T)^2$ of the sums is independent of the choice of $\{\phi_j\}$ and $\{\psi_i\}$. The operators T for which $\sigma(T) < \infty$ form the Schmidt-class $\sigma c(H)$. It follows that $\sigma c(H)$ is an H*-algebra under the norm $\sigma(T)$ [15; p. 29-34]. The trace class $\tau c(H)$ of operators on H consists of all operators of the form $T = T_1T_2$, where $T_1, T_2 \in \sigma c(H)$. $\tau c(H)$

is a Banach $*$-algebra under the trace norm $\tau(\cdot)$, which is defined as follows: Let $\{\phi_j\}$ be any complete orthonormal family in H and let $tr(T) = \sum_j (T\phi_j, \phi_j)$. Then $\tau(T) = tr((T^*T)^{1/2})$, for all $T \in \tau c(H)$. $\tau c(H)$ is a dual A*-algebra which is a dense two-sided ideal of LC(H) [17; p. 64]. Thus $\tau c(H)$ has a quasi-bounded approximate identity $\mathcal{H}$ constructed from a maximal orthogonal family of self-adjoint minimal idempotents. Since $||T|| \leq \sigma(T) \leq \tau(T)$, for all $T \in \tau c(H)$ [15; p. 42, Theorem 4], it follows that if the dimension of H is infinite then $\mathcal{H}$ is not bounded in the trace norm.

Let $B = \tau c(H)$ and $\mathcal{B} = LC(H)$. Since, for all $x \in B$ and $y \in \mathcal{B}$, we have $xy \in \mathcal{B}$ and $||xy|| \leq \tau(x)||y||$, it follows that $\mathcal{B}$ is a left Banach B-module with B isometrically B-isomorphic to $\mathcal{B}^*$ [15; p. 46, Theorem 1].

Theorem (4.3.1). If $B = \tau c(H)$ then $M_\ell(B)$ is isometrically isomorphic to L(H).

Proof. Let $\mathcal{B} = LC(H)$. By Theorem (4.2.3), $M_\ell(B)$ is isometrically isomorphic to $(B \circ \mathcal{B})^*$.

Let F(H) be the set of all operators of finite rank on H. F(H) is the socle of B and $\mathcal{B}$ [22; p. 289, Lemma 3.4]. Since every $h \in F(H)$ is of the form $h = x_1f_1 + \ldots + x_nf_n$, where $x_1, \ldots, x_n \in B$ and $f_1, \ldots, f_n$ are minimal idempotents in $\mathcal{B}$, it follows that $F(H) \subset B \circ \mathcal{B}$ and every $h \in F(H)$ belongs to a left ideal of finite order [22, p. 285]. Hence, by [22, p. 286, Lemma 2.3 and its proof], there exists an orthogonal family of self-adjoint minimal idempotents $\{e_1, \ldots, e_m\}$ such that

$$(1) \qquad h = h(e_1 + \ldots + e_m).$$

Now, define a norm on $B \circ \mathcal{B}$ by

$$||h||' = \inf\{\sum_{k=1}^{\infty} \tau(a_k)||w_k|| : h = \sum_{k=1}^{\infty} a_k w_k;\ a_k \in B,\ w_k \in \mathcal{B}\}.$$

By Lemma (2.3.4), $(B \circ \mathcal{B}, ||\cdot||')$ is a Banach space and from (1), we have that

$$||h||' \leq \tau(h)||e_1 + \ldots + e_m|| = \tau(h) \qquad (h \in F(H)).$$

On the other hand, if $h = \sum_{k=1}^{\infty} a_k w_k \in B \circ \mathcal{B}$, then

$$\tau(h) = \tau(\sum_{k=1}^{\infty} a_k w_k) \leq \sum_{k=1}^{\infty} \tau(a_k w_k) \leq \sum_{k=1}^{\infty} \tau(a_k)||w_k||,$$

so that $\tau(h) \leq ||h||'$, for all $h \in B \circ \mathcal{B}$. Hence

$$(2) \qquad ||h||' = \tau(h) \qquad (h \in F(H)).$$

By [15; p. 41, Theorem 3], $F(H)$ is dense in B with respect to the trace norm $\tau(\cdot)$. Therefore, for every $h \in B \circ \mathcal{B} \subseteq B$, there is a sequence $\{h_k\}$ in $F(H)$ such that $h_k \to h$ in the trace norm $\tau(\cdot)$. Since for all positive integers m, n, $m \leq n$, by (2), we have

$$||h_n - h_m||' = \tau(h_n - h_m),$$

it follows that $\{h_k\}$ is a Cauchy sequence in the norm $||\cdot||'$.

Let $h' \in B \circ \mathcal{B}$ such that $h_k \to h'$ in the norm $||\cdot||'$. Since

$$\tau(h' - h) \leq \tau(h' - h_n) + \tau(h_n - h)$$
$$\leq ||h' - h_n||' + \tau(h_n - h),$$

we have that $h' = h$ and consequently $F(H)$ is dense in $B \circ \mathcal{B}$ with respect to the norm $||\cdot||'$. Since $||h_k||' \to ||h||'$, $\tau(h_k) \to \tau(h)$ and $\tau(h_n) = ||h_n||'$, for each n, it follows that $||h||' = \tau(h)$. Hence $B \circ \mathcal{B} = B$.

Finally, let $||\cdot||''$ be the quotient norm defined on $B \circ \mathcal{B}$. Then $||\cdot||'' \leq ||\cdot||'$. By the Open-Mapping Theorem $||\cdot||'$ and $||\cdot||''$ are equivalent, so that $\ker \phi = (Q)$, where ϕ is the mapping from $B \hat{\otimes} \mathcal{B}$ onto $B \circ \mathcal{B}$ defined in §2, III and consequently $||\cdot||'' = ||\cdot||'$. Hence $||h||' = ||h||'' = \tau(h)$ for all $h \in B \circ \mathcal{B} = B$. The application of [15; p. 47, Theorem 2] completes the proof.

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UNIVERSITE D'OTTAWA / UNIVERSITY OF OTTAWA
École des études supérieures/School of Graduate Studies

Title of thesis ON DUAL REPRESENTATIONS OF MULTIPLIERS AND BANACH MODULE HOMOMORPHISMS

Name of candidate LAM, Kai-Tim

Degree M.Sc. Department MATHEMATICS

Date of defence June 13th, 1973.

We, the undersigned, certify that we have approved this thesis and that the candidate has defended it successfully.

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TITLE OF THESIS ON DUAL REPRESENTATIONS OF MULTIPLIERS AND BANACH MODULE HOMOMORPHISMS

DEGREE M.Sc. (Mathematics) YEAR GRANTED 1973

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