Overloading a Jackson Network of Shared Queues

By

Nadine Labrèche

Thesis submitted
to the School of Graduate Studies and Research
in partial fulfillment of the requirements for
the degree of Master of Science in Mathematics*

University of Ottawa
Ottawa, Ontario
Canada

September 1994

*The M.Sc. Program is a joint program with
Carleton University, administered by the Ottawa-Carleton
Institute of Mathematics and Statistics

ISBN 0-612-07875-2

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

Remerciements

Je voudrais tout d'abord exprimer ma sincère reconnaissance au professeur David McDonald qui a dirigé cette thèse. Ses encouragements et son précieux secours ont été essentiels à la réalisation de ce travail.

Je voudrais également remercier le Conseil de recherches en sciences naturelles et en génie du Canada ainsi que le Comité des bourses en sciences de l'Université d'Ottawa pour leur soutien financier.

Abstract

Jackson networks with finite buffers can be designed so that buffer overflows occur with very low probability. Still, queues can be filled beyond capacity so it is of interest to study the distribution of such occurrences and, as a measure of the reliability of the system, the mean time until an overload.

The approach we propose is based on a change of measure which twists the stationary Jackson network into a process for which overloads are frequent. For a Jackson network of shared queues with buffer of size $\ell - 1$, we show that the stationary overflow distribution of the twisted network can be used to determine both the limiting distribution (as $\ell \to \infty$) at the moment of overload and the mean time until overload of the *original* Jackson network.

Our estimation for the mean time until an overload is based on an article by Iscoe and McDonald (1994a) in which the mean time until the overflow of a Jackson network is estimated by the reciprocal of the smallest eigenvalue $\Lambda(B)$ of an associated Dirichlet problem. The distinct feature of this estimate is that it comes complete with error bounds so that, unlike other estimators, its accuracy can be evaluated. Of course, in practice such eigensystems of large dimensions cannot be solved. By applying our twist to the eigensystem, however, we obtain an estimate which we show to be asymptotically equivalent to the one proposed by Aldous (1989).

Contents

Notation

State space

- $S = \{\vec{x} \in Z^m : x_i \geq 0, i = 1, \ldots, m\}$
- $B = \{\vec{x} \in S : \sum_{i=1}^m x_i < \ell\}$
- $F = \{\vec{x} \in S : \sum_{i=1}^m x_i \geq \ell\}$
- $\partial F = \{\vec{x} \in S : \sum_{i=1}^m x_i = \ell\}$

- $S^\infty = \{\vec{x} \in Z^m : x_i \geq 0, i = 2, \ldots, m\}$
- $B^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i < \ell\}$
- $F^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i \geq \ell\}$
- $\partial F^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i = \ell\}$

Markov processes (Jackson networks)

- Q: original Jackson network on S with transition rates $q(\vec{x}, \vec{y})$
- Q^t: Q twisted (on S) with transition rates $q^t(\vec{x}, \vec{y})$
- Q^r: time reversal of Q (on S) with transition rates $q^r(\vec{x}, \vec{y})$

- Z: Q on S^∞ with transition rates $q(\vec{x}, \vec{y})$
- Z^t: Z twisted (on S^∞) with transition rates $q^t(\vec{x}, \vec{y})$
- Z^r: time reversal of Z *and* Z^t (on S^∞) with transition rates $q^r(\vec{x}, \vec{y})$

Generators

- $-L$: generator on S with rates $q(\vec{x}, \vec{y})$
- $-L^B$: generator $-L$ restricted on B with rates $q(\vec{x}, \vec{y})$
- $-L^\infty$: generator on S^∞ with rates $q(\vec{x}, \vec{y})$
- $-L^t$: generator on S^∞ with rates $q^t(\vec{x}, \vec{y})$

Chapter 1

Introduction

Queueing networks with finite buffers can be designed so that buffer overflows occur with very low probability. Still, queues can be filled beyond capacity so it is of interest to study the distribution of such occurrences and, as a measure of the reliability of the system, the mean time until an overload.

These quantities can theoretically be determined by linear-algebraic techniques. However, this approach involves inversion of nearly singular matrices whose dimensions grow with the size of the system, so it is not generally applicable. Calculations based on direct simulation turn out to be very costly. Because of their rarity, generation of overloads is a lengthy process which requires excessive computer resources. One has to rely on analytical approaches, simulation acceleration techniques or a combination of both methods.

Until now, most studies have focused on the evaluation of the mean overload time. For instance, Meyn and Frater (1993) studied the computation of the mean time between buffer overflows for a Jackson network. Parekh and Walrand (1989), Aldous (1989) and Frater, Lennon and Anderson (1991) examined the expected time until an overflow given the network is initially empty. This is the case we will consider. Furthermore, since Aldous' (1989) approach is closely related to our results, we will study his work attentively.

Both methods presented by Parekh and Walrand (1989) and Frater et al. (1991) rely on simulation acceleration techniques. Aldous (1989), on the other hand, proposed a heuristic for the mean overload time which involves the stationary distribution of the network. Though that measure is not readily available for most processes, it is certainly well-known and understood in the case of Jackson networks. In fact, Jackson networks are among the most studied queueing networks. It is to take advantage of the numerous available results that we will consider m-node Jackson networks of shared queues.

The solution we propose is based on a change of measure which twists the stationary Jackson network into a process for which overloads are frequent. This change of measure facilitates the estimation by simulation of the overflow distribution of the twisted Jackson network. We show that overflow distribution can be used to obtain estimates for the distribution of the *original* Jackson network at the moment of overload. Furthermore, that same distribution is used to estimate the mean time until overload.

Our estimation for the mean time until an overload is based on an article by Iscoe and McDonald (1994a) in which the mean time until the overflow of a Jackson network is estimated by the reciprocal of the smallest eigenvalue $\Lambda(B)$ of an associated Dirichlet problem. The distinct feature of this estimate is that it comes complete with error bounds so that, unlike other estimators, its accuracy can be evaluated. Of course, in practice such eigensystems of large dimensions cannot be solved. By applying our twist to the eigensystem, however, we obtain an estimate which we show to be asymptotically equivalent to the one proposed by Aldous (1989).

Our method can be applied as an algorithm which is given at the beginning of Chapter 4. Chapter 2 describes the Jackson network and its twist. In particular, Section 2.3 gives the distribution of the original Jackson network at the moment of overload. Chapter 3 presents the estimates put forward by Aldous (1989) and Iscoe and McDonald (1994a) for the mean time until a buffer overflow. Using results from Kesten (1974), the asymptotic equivalence of both estimates is shown. Finally, Chapter 4 illustrates the application of the twist with two examples of Jackson networks.

Chapter 2

A New Twist to Jackson Networks

2.1 Jackson networks

A Jackson network consists of m nodes or service stations that operate on a first come first served basis. Customers arrive at a typical node i from outside the system according to a Poisson process with rate $\bar{\lambda}_i$ and, if necessary, wait in a buffer to get served until the station gets free. Service time is exponentially distributed with mean $1/\mu_i(x_i)$, where x_i denotes the number of customers waiting or being served at node i. (Though the service rate may depend on the number of customers in the queue, we consider here the simple function $\mu_i(x_i) = \mu_i$ if $x_i \neq 0$ and 0 otherwise.) Once service is completed, the customer is routed to another node, say j, with probability r_{ij} ($r_{ii} = 0$) or leaves the system with probability $r_{i.} := 1 - \sum_{j=1}^{m} r_{ij}$.

We assume the network is both exogenously supplied and open. A network is exogenously supplied if each node i has an exogenous arrival rate $\bar{\lambda}_i \neq 0$ or can be fed by another node j for which $\bar{\lambda}_j \neq 0$. The network is also open if every node i has an exit probability $r_{i.} \neq 0$ or feeds a node j for which $r_{j.} \neq 0$. We say that node i feeds node j if there is a sequence $k_1, k_2, \ldots, k_q$ such that $r_{i\,k_1} r_{k_1\,k_2} \cdots r_{k_q\,j} > 0$.

A Jackson network can be described as a Markov jump process $(Q(s); s \geq 0)$ on $S = \{\vec{x} \in Z^m : x_i \geq 0, i = 1, \ldots, m\}$, where the state $\vec{x} = (x_1, x_2, \ldots, x_m) \in S$ depicts the system when there are x_i customers waiting or being served at node i. The process sojourns in state $\vec{x}$ for an exponential duration with mean $M(\vec{x}) := \left[\sum_{i=1}^{m}(\bar{\lambda}_i + \mu_i(x_i))\right]^{-1}$ and then jumps

$$\begin{array}{ll}
\text{to state} & \text{with probability} \\
T_{.i}\vec{x} := \vec{x} + \vec{e}_i & \bar{\lambda}_i M(\vec{x}) \\
T_{i.}\vec{x} := \vec{x} - \vec{e}_i & \mu_i(x_i) r_{i.} M(\vec{x}) \\
T_{ij}\vec{x} := \vec{x} - \vec{e}_i + \vec{e}_j & \mu_i(x_i) r_{ij} M(\vec{x}),
\end{array} \tag{2.1}$$

where $\vec{e}_i$ is the i-th canonical vector of R^m. The corresponding jump rates of this Markov process are given by

$$\begin{aligned}
q(\vec{x}, T_{.i}\vec{x}) &= \bar{\lambda}_i, \\
q(\vec{x}, T_{i.}\vec{x}) &= \mu_i(x_i) r_{i.}, \\
q(\vec{x}, T_{ij}\vec{x}) &= \mu_i(x_i) r_{ij}.
\end{aligned}$$

Considering the jump times $\{J_k; k \geq 0\}$ defined as

$$\begin{aligned}
J_0 &= 0, \\
J_k &= \min\{s : s \geq J_{k-1}, Q(s) \neq Q(J_{k-1})\}, \quad k \geq 1,
\end{aligned}$$

we obtain the embedded Markov chain $(C_k = Q(J_k); k \geq 0)$ on S with jumps as defined in (2.1). The chain $(C_k; k \geq 0)$ is irreducible by the assumption that the network is exogenously supplied and open.

Jackson (1957) gave an expression for the invariant measure of a Jackson network. His results are restated in the following theorem.

Theorem 2.1.1 *For an exogenously supplied and open Jackson network for which the solution* $(\lambda_1, \ldots, \lambda_m)$ *to the traffic equations*

$$\lambda_i = \bar{\lambda}_i + \sum_{j=1}^{m} \lambda_j r_{ji}, \quad i = 1, 2, \ldots, m \tag{2.2}$$

satisfies the light traffic conditions

$$\rho_i := \frac{\lambda_i}{\mu_i} < 1, \quad i = 1, 2, \ldots, m, \tag{2.3}$$

the stationary distribution $\pi(\vec{x})$ *of* $\vec{x} = (x_1, \ldots, x_m) \in S$ *is given by the product*

$$\pi(\vec{x}) = \prod_{i=1}^{m} \pi_i(x_i)$$

where

$$\pi_i(x_i) := (1 - \rho_i)\rho_i^{x_i}.$$

Call a Jackson network *stable* if the light traffic conditions (2.3) hold. Theorem 2.1.1 implies that, in the steady state, the queue sizes at the different nodes are independent. Furthermore, the queue size at node i has the stationary measure of a birth and death process with birth rate λ_i and death rate μ_i, $i = 1, 2, \ldots, m$.

If looked upon backwards in time, this stable process is still a Jackson network. The time reversed Jackson network $(Q^r(s); s \geq 0)$ has transition rates given by

$$\begin{aligned} q^r(\vec{x}, T_{.i}\vec{x}) &= \frac{\pi(T_{.i}\vec{x})}{\pi(\vec{x})} q(T_{.i}\vec{x}, \vec{x}) = \rho_i \mu_i(x_i) r_{i.}, \\ q^r(\vec{x}, T_{i.}\vec{x}) &= \frac{\pi(T_{i.}\vec{x})}{\pi(\vec{x})} q(T_{i.}\vec{x}, \vec{x}) = \frac{\bar{\lambda}_i}{\rho_i}, \\ q^r(\vec{x}, T_{ij}\vec{x}) &= \frac{\pi(T_{ij}\vec{x})}{\pi(\vec{x})} q(T_{ij}\vec{x}, \vec{x}) = \frac{\rho_j}{\rho_i} \mu_j(x_j) r_{ji}. \end{aligned}$$

If the light traffic conditions (2.3) do not hold for some node j, after some time the output rate at node j is not λ_j, but rather μ_j. Therefore, the traffic equations (2.2) no longer represent the average arrival rates at the other nodes (assuming of course that $r_{j.} \neq 1$). Goodman and Massey (1984) extended Theorem 2.1.1 to consider this case.

Theorem 2.1.2 *For an exogenously supplied and open Jackson network, there exists an unique solution $(\theta_1, \ldots, \theta_m)$ to the throughput equations*

$$\theta_i = \bar{\lambda}_i + \sum_{j=1}^{m} (\theta_j \wedge \mu_j) r_{ji}, \quad i = 1, 2, \ldots, m. \tag{2.4}$$

Let $\rho_i = \theta_i / \mu_i$ and $U = \{i : \theta_i < \mu_i\}$. The stationary distribution $\bar{\pi}(\vec{x})$ of those elements x_i of $\vec{x} = (x_1, \ldots, x_m)$ with $i \in U$ is given by the product

$$\bar{\pi}(\vec{x}) = \prod_{i \in U} \pi_i(x_i)$$

where

$$\pi_i(x_i) := (1 - \rho_i) \rho_i^{x_i}.$$

Note that when $U = \{1, 2, \ldots, m\}$, the statement in Theorem 2.1.2 is identical to Theorem 2.1.1. The ratio $\rho_i = \theta_i / \mu_i$ is called the *load* on node i.

In this study, we will consider a stable Jackson network of pooled queues in which all m nodes share a buffer of size $\ell - 1$. Without loss of generality, we can label the nodes from 1 to m so that, for $i = 2, \dots, m$,

$$\rho_i < \rho_1 < 1, \tag{2.5}$$

if we assume only one node has the maximum load value ρ_1 (which we do, for the remainder of this work).

2.2 A twisted Jackson network

Consider an extension of the Jackson network by allowing the queue at node 1 to hold a negative number of customers; let $(Z(s); s \geq 0)$ denote the extended Markov process. We extend the state space S to $S^\infty = \{\vec{x} \in Z^m : x_i \geq 0, i = 2, \dots, m\}$ and redefine the service rates as

$$\mu_i(x_i) = \begin{cases} \mu_i & \text{if } x_i > 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we declare the jump rates homogeneous in the first coordinate and denote the extended infinitesimal generator of Z by $-L^\infty$. Then, for a (bounded) function g on S^∞,

$$\begin{aligned} L^\infty g(\vec{x}) &= \sum_{i=1}^{m} \bar{\lambda}_i [g(\vec{x}) - g(T_{.i}\vec{x})] + \sum_{i=1}^{m} \mu_i(x_i) r_{i.} [g(\vec{x}) - g(T_{i.}\vec{x})] \\ &\quad + \sum_{i,j=1}^{m} \mu_i(x_i) r_{ij} [g(\vec{x}) - g(T_{ij}\vec{x})], \quad \vec{x} \in S^\infty \end{aligned}$$

where the operators $T_{.i}\vec{x} = \vec{x} + \vec{e}_i$, $T_{i.}\vec{x} = \vec{x} - \vec{e}_i$ and $T_{ij}\vec{x} = \vec{x} - \vec{e}_i + \vec{e}_j$ represent the jumps in S^∞.

Now, for a function k on S^∞ and positive constants $a_1, a_2, \dots, a_m$ define

$$\begin{aligned} L^t k(\vec{x}) &:= \frac{1}{\prod_{i=1}^m a_i^{x_i}} L^\infty \prod_{i=1}^{m} a_i^{x_i} k(\vec{x}) \\ &= \sum_{i=1}^{m} \bar{\lambda}_i [k(\vec{x}) - a_i k(T_{.i}\vec{x})] + \sum_{i=1}^{m} \mu_i(x_i) r_{i.} [k(\vec{x}) - a_i^{-1} k(T_{i.}\vec{x})] \\ &\quad + \sum_{i,j=1}^{m} \mu_i(x_i) r_{ij} [k(\vec{x}) - a_i^{-1} a_j k(T_{ij}\vec{x})]. \end{aligned} \tag{2.6}$$

Consider the partition $S^\infty = \text{int}(S^\infty) \bigcup \partial S^\infty$ where $\text{int}(S^\infty) = \{\vec{x} \in S^\infty : x_i \neq 0 \text{ for all } i = 2, \dots, m\}$ and $\partial S^\infty = \{\vec{x} \in S^\infty : x_i = 0 \text{ for some } i = 2, \dots, m\}$. If the constants

$a_1, a_2, \ldots, a_m$ are chosen so that, for $\vec{x} \in \mathrm{int}(S^\infty)$,

$$\sum_{i=1}^{m}(\bar{\lambda}_i + \mu_i) = \sum_{i=1}^{m} \bar{\lambda}_i a_i + \sum_{i=1}^{m} \frac{\mu_i r_{i.}}{a_i} + \sum_{i,j=1}^{m} \frac{\mu_i r_{ij} a_j}{a_i}$$

or, equivalently, $L^t 1 = 0$ on $\mathrm{int}(S^\infty)$, (2.6) can be rewritten as

$$\begin{aligned} L^t k(\vec{x}) &= \sum_{i=1}^{m} \bar{\lambda}_i a_i [k(\vec{x}) - k(T_{.i}\vec{x})] + \sum_{i=1}^{m} \mu_i r_{i.} a_i^{-1} [k(\vec{x}) - k(T_{i.}\vec{x})] \\ &\quad + \sum_{i,j=1}^{m} \mu_i r_{ij} a_i^{-1} a_j [k(\vec{x}) - k(T_{ij}\vec{x})] \end{aligned} \tag{2.7}$$

for $\vec{x} \in \mathrm{int}(S^\infty)$.

We recognize here the form of an infinitesimal generator of a Markov jump process on $\mathrm{int}(S^\infty)$. In order that $-L^t$ be an infinitesimal generator on all of S^∞, it must be correctly defined for $\vec{x} \in \partial S^\infty$ as well, i.e., we must have $-L^t 1 = 0$ on ∂S^∞. This is done by imposing additional constraints on the constants $a_1, a_2, \ldots, a_m$: for any subsample $\mathcal{I}$ of $\{2, \ldots, m\}$,

$$\sum_{i=1}^{m} \bar{\lambda}_i + \sum_{i \notin \mathcal{I}} \mu_i = \sum_{i=1}^{m} \bar{\lambda}_i a_i + \sum_{i=1}^{m} \frac{\mu_i r_{i.}}{a_i} \chi\{i \notin \mathcal{I}\} + \sum_{i,j=1}^{m} \frac{\mu_i r_{ij} a_j}{a_i} \chi\{i \notin \mathcal{I}\}. \tag{2.8}$$

Iscoe and McDonald (1994c) showed there exist unique positive constants $a_1, a_2, \ldots, a_m$ such that $-L^t$ is a generator. The constants are given as the solution of the $m \times m$ non-singular system of linear equations

$$\begin{pmatrix} 1 \; 0 \ldots 0 \\ (I - R)_{2\downarrow} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \rho_1^{-1} \\ r_{2.} \\ \vdots \\ r_{m.} \end{pmatrix} \tag{2.9}$$

where R denotes the matrix (r_{ij}) and $(I - R)_{2\downarrow}$ denotes the $(m-1) \times m$ matrix formed from $I - R$ by deleting the first row from the latter.

Since the elements of the new generator $-L^t$ consist of the parameters of the original Jackson network which have been *twisted*, we say that $-L^t$ is a twisted infinitesimal generator associated with the twisted extended Jackson network denoted by $(Z^t(s); s \geq 0)$. Returning to (2.7), we can express the parameters of the twisted process as

$$\begin{array}{ll} \bar{\lambda}_i^t = \bar{\lambda}_i a_i, & i = 1, \ldots, m \\ \mu_1^t = \frac{\mu_1}{a_1}\left(r_{1.} + \sum_{j=1}^{m} r_{1j} a_j\right), & \\ \mu_i^t = \frac{\mu_i}{a_i}\left(r_{i.} + \sum_{j=1}^{m} r_{ij} a_j\right) = \mu_i, & i = 2, \ldots, m \;\text{ by (2.9)} \\ r_{1j}^t = \frac{r_{1j} a_j}{r_{1.} + \sum_{i=1}^{m} r_{1i} a_i}, & j = 2, \ldots, m \\ r_{ij}^t = r_{ij} \frac{a_j}{a_i}, & i = 2, \ldots, m, \;\; j = 1, \ldots, m, \end{array} \tag{2.10}$$

and its transition rates as

$$\begin{aligned}
q^t(\vec{x}, T_{.i}\vec{x}) &= \bar{\lambda}_i^t = \bar{\lambda}_i a_i, \\
q^t(\vec{x}, T_{i.}\vec{x}) &= \mu_i^t(x_i) r_{i.}^t = \mu_i(x_i) r_{i.} a_i^{-1}, \\
q^t(\vec{x}, T_{ij}\vec{x}) &= \mu_i^t(x_i) r_{ij}^t = \mu_i(x_i) r_{ij} a_i^{-1} a_j,
\end{aligned}$$

for $i, j = 1, \ldots, m$ where the a_i's are given by (2.9).

If we suppose that, in the twisted process, the length of the queue at node 1 is transient while the queue sizes at the other nodes are recurrent to 0, the throughput equations (2.4) of Goodman and Massey (1984) expressed in matrix form become

$$\begin{pmatrix} 1 & \\ 0 & (I-R^t)'_{2\downarrow} \\ \vdots & \\ 0 & \end{pmatrix} \begin{pmatrix} \theta_1^t \\ \theta_2^t \\ \vdots \\ \theta_m^t \end{pmatrix} = \begin{pmatrix} a_1\bar{\lambda}_1 \\ a_2(\bar{\lambda}_2 + \mu_1 r_{12}/a_1) \\ \vdots \\ a_m(\bar{\lambda}_m + \mu_1 r_{1m}/a_1) \end{pmatrix}$$

where $R^t = (r_{ij}^t)$, $(I - R^t)_{2\downarrow}$ is defined as before, and $'$ denotes transposition. Recalling from (2.3) that $a_1 = \rho_1^{-1} = \mu_1/\lambda_1$, we can write

$$\begin{pmatrix} 1 & \\ 0 & (I-R^t)'_{2\downarrow} \\ \vdots & \\ 0 & \end{pmatrix} \begin{pmatrix} \theta_1^t \\ \theta_2^t \\ \vdots \\ \theta_m^t \end{pmatrix} = \begin{pmatrix} \bar{\lambda}_1/\rho_1 \\ a_2(\bar{\lambda}_2 + r_{12}\lambda_1) \\ \vdots \\ a_m(\bar{\lambda}_m + r_{1m}\lambda_1) \end{pmatrix}. \tag{2.11}$$

Lemma 2.2.1 *The solution to the $m \times m$ system of linear equations (2.11) is*

$$(\mu_1, a_2\lambda_2, \ldots, a_m\lambda_m)'.$$

Proof: Recalling from (2.10) that $r_{ij}^t = r_{ij}\frac{a_j}{a_i}$ for $i = 2, \ldots, m$, we get, by simple matrix multiplication,

$$\begin{aligned}
\begin{pmatrix} 1 & \\ 0 & (I-R^t)'_{2\downarrow} \\ \vdots & \\ 0 & \end{pmatrix} \begin{pmatrix} \mu_1 \\ a_2\lambda_2 \\ \vdots \\ a_m\lambda_m \end{pmatrix} &= \begin{pmatrix} \mu_1 - a_1\sum_j r_{j1}\lambda_j \\ a_2(\lambda_2 - \sum_{j\neq 1} r_{j2}\lambda_j) \\ \vdots \\ a_m(\lambda_m - \sum_{j\neq 1} r_{jm}\lambda_j) \end{pmatrix} \\
&= \begin{pmatrix} \mu_1 - a_1(\lambda_1 - \bar{\lambda}_1) \\ a_2(\bar{\lambda}_2 + r_{12}\lambda_1) \\ \vdots \\ a_m(\bar{\lambda}_m + r_{1m}\lambda_1) \end{pmatrix} \quad \text{by (2.2)}
\end{aligned}$$

$$= \begin{pmatrix} \bar{\lambda}_1/\rho_1 \\ a_2(\bar{\lambda}_2 + r_{12}\lambda_1) \\ \vdots \\ a_m(\bar{\lambda}_m + r_{1m}\lambda_1) \end{pmatrix}.$$

■

From Theorem 2.1.2, Lemma 2.2.1 and the equations given in (2.10), we can calculate the loads on the twisted system

$$\begin{aligned} \rho_1^t &= \frac{\theta_1^t}{\mu_1^t} = \frac{\mu_1}{\mu_1^t} = \frac{\rho_1^{-1}}{r_{1.} + \sum_{j=1}^m r_{1j}a_j} \\ \rho_i^t &= \frac{\theta_i^t}{\mu_i^t} = \frac{a_i\lambda_i}{\mu_i} = a_i\rho_i, \quad i = 2, \ldots, m. \end{aligned} \tag{2.12}$$

To verify our assertion that $\rho_1^t \geq 1$ and $\rho_i^t < 1$, $i = 2, \ldots, m$, we must hence show

$$\rho_1^{-1} \geq r_{1.} + \sum_{j=1}^m r_{1j}a_j \tag{2.13}$$

$$\rho_i^{-1} > a_i, \quad i = 2, \ldots, m, \tag{2.14}$$

given our initial assumption (2.5) that $1 > \rho_1 > \rho_i$, $i = 2, \ldots, m$. Note, however, that if $a_1 \geq a_i$, for all $i = 2, \ldots, m$, then

$$a_i \leq a_1 = \rho_1^{-1} < \rho_i^{-1}$$

giving (2.14). Moreover, if in addition $r_{1j} \neq 0$ for some j,

$$\begin{aligned} r_{1.} + \sum_{j=1}^m r_{1j}a_j &\leq r_{1.} + \left(\sum_{j=1}^m r_{1j}\right) a_1 \\ &\leq \left(r_{1.} + \sum_{j=1}^m r_{1j}\right) a_1 \quad \text{since } a_1 = \rho_1^{-1} > 1 \\ &= a_1 = \rho_1^{-1}, \end{aligned}$$

giving (2.13). Alternatively, if $r_{1j} = 0$, for all $j = 2, \ldots, m$,

$$r_{1.} + \sum_{j=1}^m r_{1j}a_j = r_{1.} = 1 < \rho_1^{-1},$$

giving (2.13). Thus we only need to show $a_1 \geq a_i$, $i = 2, \ldots, m$. We nevertheless prove an additional result which will be later used.

Lemma 2.2.2 *For the constants $a_1, \ldots, a_m$ defined in (2.9), the following inequalities hold for $i = 2, \ldots, m$:*

$$a_1 \geq a_i \geq 1.$$

Proof: We first show $a_1 \geq a_i$ for $i = 2, \ldots, m$, by contradiction. Let k be any index for which $a_k = \max_{i=1,\ldots,m}(a_i) > a_1$. Note that $r_{k.} \neq 1$ since otherwise, from (2.9), we obtain the contradiction $a_k = 1 < a_1$. Therefore $S_k := \{i \in \{1, \ldots, m\} : r_{ki} \neq 0\}$ is not empty. Suppose now there is a $j \in S_k$ such that $a_j < a_k$. Then, by (2.9),

$$\begin{aligned} a_k &= r_{k.} + \sum_{j \in S_k} r_{kj} a_j \\ &< r_{k.} + (\sum_{j \in S_k} r_{kj}) a_k \\ &\leq (r_{k.} + \sum_{j \in S_k} r_{kj}) a_k \quad \text{since } a_k > \rho_1^{-1} > 1 \\ &= a_k \end{aligned}$$

is a contradiction and it results that $a_j = a_k$ for all $j \in S_k$. (Note that since $a_k > a_1$ by assumption, $1 \notin S_k$.) Thus we can write

$$\begin{aligned} a_k &= r_{k.} + \sum_{j \in S_k} r_{kj} a_k \\ &= r_{k.} + (1 - r_{k.}) a_k \end{aligned}$$

or

$$r_{k.}(1 - a_k) = 0 \implies r_{k.} = 0 \tag{2.15}$$

since $a_k > a_1 > 1$. Moreover, since for all $j \in S_k$, $a_j = a_k > a_1$, (2.15) with j in place of k yields $r_{j.} = 0$. Continuing this pattern, we reach the conclusion that $r_{i.} = 0$ for all nodes i which node k feeds. This violates the open network assumption and thus $a_k \not> a_1$. This proves the first result.

We now proceed to show $a_i \geq 1$ for $i = 2, \ldots, m$, by contradiction. Let n be any index for which $a_n = \min_{i=1,\ldots,m}(a_i) < 1$. Again, by (2.9),

$$\begin{aligned} a_n &= r_{n.} + \sum_{i \in S_n} r_{ni} a_i \\ &\geq r_{n.} + (\sum_{i \in S_n} r_{ni}) a_n \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&= r_{n.} + (1 - r_{n.})a_n \\
&= a_n + r_{n.}(1 - a_n) \\
&\geq a_n, \text{ since } a_n < 1. \qquad (2.17)
\end{aligned}
$$

Thus we must have equality in (2.17), which implies

$$r_{n.}(1 - a_n) = 0 \implies r_{n.} = 0. \qquad (2.18)$$

Substituting (2.18) into (2.16),

$$a_n = \sum_{i \in S_n} r_{ni} a_i \geq \sum_{i \in S_n} r_{ni} a_n = a_n$$

or

$$\sum_{i \in S_n} r_{ni}(a_i - a_n) = 0.$$

Since for $i \in S_n$ $r_{ni} > 0$ and the differences $a_i - a_n$ are nonnegative, this equality can only be true if $a_i - a_n = 0, i \in S_n$. Thus, for all $i \in S_n$, $a_i = a_n < 1$ and we must have $r_{i.} = 0$ by (2.18), with i in place of n. In other words, $r_{j.} = 0$ for any node j which node n feeds, which violates the initial assumption that the Jackson network is open. This contradiction yields the desired result $a_n \geq 1$ and completes the proof. ∎

We summarize the above discussion in the following theorem.

Theorem 2.2.3 *For any m-node Jackson network of shared queues with parameters $\bar{\lambda}_i, \mu_i$ and r_{ij}, $i, j = 1, \ldots, m$, and loads $1 > \rho_1 > \rho_i$, $i = 2, \ldots, m$, the associated twisted Jackson network with parameters as defined in (2.10) has loads $\rho_1^t \geq 1, \rho_i^t < 1$, $i = 2, \ldots, m$. In other words, only the queue size at node 1 is transient.*

According to Theorem 2.2.3, in the twisted system, the queue sizes at nodes 2 to m are recurrent to 0; the queue size at node 1, however, is transient. The flow of customers from node 1 to node j, $j = 2, \ldots, m$, thus rapidly becomes $\mu_1^t r_{1j}^t$ and the net flow of customers at node j, θ_j^t, can be expressed from (2.4) with $U = \{2, \ldots, m\}$ as

$$\theta_j^t = \bar{\lambda}_j^t + \mu_1^t r_{1j}^t + \sum_{i=2}^{m} \theta_i^t r_{ij}^t, \quad j = 2, \ldots, m \qquad (2.19)$$

when the process is in equilibrium. Equations (2.19) are very similar to the traffic equations (2.2). In fact, if we consider the flow of customers from node 1 as exogenous arrivals and transfers of customers to node 1 as departures from the system composed of the nodes 2 through m, we obtain a stationary Jackson network with $(m-1)$ nodes. Consequently, calculations for the twisted Jackson network can be performed as for the original Jackson network, but by considering only nodes 2 to m and their associated *twisted* parameters. Now, for instance,

$$\begin{aligned}
& \bar{\lambda}_j^t + \mu_1^t r_{1j}^t \text{ denotes the exogenous arrival rate at node } j = 2, \ldots, m, \\
& r_{j.}^t + r_{j1}^t \text{ is the exit probability at node } j \\
\text{and } \pi^t(\vec{x}) \quad := \quad & \prod_{j=2}^{m} (1-\rho_j^t)(\rho_j^t)^{x_j} \text{ is the stationary distribution of the twisted network.}
\end{aligned}$$

The time reversal of the extended twisted process Z^t is precisely Z^r, the time reversal of the extended Jackson network. The stationary measure for the twisted process is $n \otimes \pi^t$ where n is a discrete uniform measure on the integers. Hence the transition rates of the time reversed twisted process are

$$\begin{array}{ll}
(q^t)^r(\vec{x}, T_{.1}\vec{x}) = \mu_1^t r_{1.}^t = \mu_1 r_{1.} \rho_1, & \\
(q^t)^r(\vec{x}, T_{1.}\vec{x}) = \bar{\lambda}_1^t = \bar{\lambda}_1 \rho_1^{-1}, & \\
(q^t)^r(\vec{x}, T_{.i}\vec{x}) = \mu_i^t(x_i) r_{i.}^t \rho_i^t = \mu_i(x_i) r_{i.} \rho_i, & i = 2, \ldots, m, \\
(q^t)^r(\vec{x}, T_{i.}\vec{x}) = \bar{\lambda}_i^t (\rho_i^t)^{-1} = \bar{\lambda}_i \rho_i^{-1}, & i = 2, \ldots, m, \\
(q^t)^r(\vec{x}, T_{1i}\vec{x}) = \mu_i^t(x_i) r_{i1}^t \rho_i^t = \mu_i(x_i) r_{i1} \rho_i \rho_1^{-1}, & i = 1, \ldots, m, \\
(q^t)^r(\vec{x}, T_{i1}\vec{x}) = \mu_1^t r_{1i}^t (\rho_i^t)^{-1} = \mu_1 r_{1i} \rho_1 \rho_i^{-1}, & i = 1, \ldots, m, \\
(q^t)^r(\vec{x}, T_{ij}\vec{x}) = \mu_j^t(x_j) r_{ji}^t (\rho_i^t)^{-1} \rho_j^t = \mu_j(x_j) r_{ji} \rho_j \rho_i^{-1}, & i, j = 2, \ldots, m.
\end{array}$$

2.3 A change of measure using multiplicative functionals

The twist developed in Section 2.2 may be obtained from a more conventional method which employs multiplicative functionals to change the measure. For a process $U(s)$ and some function g, the multiplicative functional

$$M_0^s = \frac{g(U(s))}{g(U(0))} \exp\left(-\int_0^s \frac{-Lg(U(r))}{g(U(r))} dr\right)$$

is an exponential martingale associated with $U(s)$. We quote Theorem B.5 from Shwartz and Weiss (1994).

Theorem 2.3.1 *If, under the measure $P_{\vec{x}}$, $U(s)$ is a multi-dimensional birth-death process with generator $-L$, step directions e_j and step rates $\phi_j(\vec{x})$, then under the measure $\tilde{P}_{\vec{x}}$ defined by*

$$\tilde{E}_{\vec{x}}W = E_{\vec{x}}M_0^T W, \quad W \in \mathcal{F}_T, \tag{2.20}$$

where $E_{\vec{x}}$ and $\tilde{E}_{\vec{x}}$ denote respectively the expectation with respect to the measures $P_{\vec{x}}$ and $\tilde{P}_{\vec{x}}$, the process $U(s)$ is again a multi-dimensional birth-death process with the same step directions and with the rates changed to

$$\phi_j^t(\vec{x}) = \phi_j(\vec{x})\frac{g(\vec{x}+e_j)}{g(\vec{x})}.$$

Applying the previous theorem to the Markov process Z with generator $-L^\infty$ on S^∞ and taking

$$g(\vec{x}) = \prod_{i=1}^m a_i^{x_i},$$

it results that, under the measure $\tilde{P}_{\vec{x}}$, Z is the twisted extended Markov process Z^t defined in the previous section. This approach, however, yields some additional information about the behaviour of the process. Noting that $Z(0) = \vec{0}$ and that, by (2.6), $L^\infty g(\vec{x}) = g(\vec{x})L^t 1 = 0$, we calculate

$$\begin{aligned} M_0^s &= \frac{g(Z(s))}{g(Z(0))} \\ &= \frac{\prod_{i=1}^m a_i^{Z_i(s)}}{\prod_{i=1}^m a_i^0} \\ &= \prod_{i=1}^m a_i^{Z_i(s)}. \end{aligned}$$

If we apply Theorem 2.3.1 for the random time τ which represents the time required to reach $\vec{y} \in \partial F^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i = \ell\}$ from $\vec{0}$ (recall the buffer is of size $\ell - 1$), and take $W = \frac{1}{M_0^\tau}\chi\{Z(\tau) = \vec{y}, \tau < \infty\}$, (2.20) becomes

$$\begin{aligned} E_{\vec{0}}\chi\{Z(\tau) = \vec{y}, \tau < \infty\} &= \tilde{E}_{\vec{0}}\frac{1}{M_0^\tau}\chi\{Z(\tau) = \vec{y}, \tau < \infty\} \\ &= \tilde{E}_{\vec{0}}\left[\chi\{Z(\tau) = \vec{y}\}\prod_{i=1}^m a_i^{-Z_i(\tau)}\right]. \end{aligned} \tag{2.21}$$

Similarly,

$$P_{\vec{0}}(\tau < \infty) = \tilde{E}_{\vec{0}}\prod_{i=1}^m a_i^{-Z_i(\tau)}. \tag{2.22}$$

Let $\vec{y} = (y_1, y_2, \cdots, y_m)$ be the first state entered in ∂F^∞ from $\vec{0}$. We can write $\vec{y} = (\ell - \sum_{i=2}^m y_i, y_2, \cdots, y_m)$ so that $\prod_{i=1}^m a_i^{-Z_i(\tau)}$ becomes

$$a_1^{-\ell} \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{y_i}.$$

By (2.21) and (2.22), the conditional hitting distribution is

$$\begin{aligned} P_{\vec{0}}(Z(\tau) = \vec{y} | \tau < \infty) &= \frac{\prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{y_i} \tilde{P}_{\vec{0}}(Z(\tau) = \vec{y})}{\tilde{E}_{\vec{0}} \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{Z_i(\tau)}} \\ &= \frac{\prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{y_i} \tilde{P}_{\vec{0}}(Z(\tau) = \vec{y})}{\sum_{\vec{z} \in \partial F^\infty} \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{z_i} \tilde{P}_{\vec{0}}(Z(\tau) = \vec{z})}. \end{aligned}$$

Hence we can obtain the hitting distribution of Z on the hyperplane ∂F^∞ if we know the hitting distribution of the twisted process on this hyperplane. We shall see later in Section 3.4 there is a limiting hitting distribution as $\ell \to \infty$. Let $\lim_{\ell \to \infty} \tilde{P}_{\vec{0}}(Z(\tau) = \vec{y}) = e(\vec{y}^-)$ where $\vec{y} = (\ell - \sum_{i=2}^m y_i, \vec{y}^-)$ and $\lim_{\ell \to \infty} P_{\vec{0}}(Z(\tau) = \vec{y} | \tau < \infty) = h(\vec{y})$. Then

$$h(\vec{y}) = \prod_{i=2}^m (a_1/a_i)^{y_i} e(\vec{y}^-) / [\sum_{\vec{z} \in \partial F^\infty} \prod_{i=2}^m (a_1/a_i)^{z_i} e(\vec{z}^-)].$$

The hitting distribution of interest, however, concerns the Jackson process Q on S (not S^∞). In this case, at overload time Q hits $\partial F = \{\vec{x} \in S : \sum_{i=1}^m x_i = \ell\}$. Redo the calculations by taking $U(s) = Z^t(s)$ in Theorem 2.3.1, where Z^t is the twisted process on S^∞ with associated infinitesimal generator $-L^t$. Denote by $P_{\vec{x}}$ the measure of Z^t on the space of trajectories and observe the process only on the interval $0 \le s \le T$, where T is fixed and finite. Again, let $\tilde{P}_{\vec{x}}$ be a new measure on $(S^\infty, \mathcal{F}_T)$ defined by

$$\tilde{E}_{\vec{x}} W = E_{\vec{x}} M_0^T W, \quad W \in \mathcal{F}_T, \tag{2.23}$$

where $E_{\vec{x}}$ and $\tilde{E}_{\vec{x}}$ denote respectively the expectation with respect to the measures $P_{\vec{x}}$ and $\tilde{P}_{\vec{x}}$, and

$$M_0^T = \frac{g(Z^t(T))}{g(Z^t(0))} \exp\left(-\int_0^T \frac{-L^t g(Z^t(r))}{g(Z^t(r))} dr\right)$$

with $g(\vec{x})$ defined as

$$g(\vec{x}) = \begin{cases} \prod_{i=1}^m a_i^{-x_i} & x_1 \ge 0 \\ 0 & x_1 < 0. \end{cases}$$

Note that, under the measure $\tilde{P}_{\vec{x}}$, Z^t is exactly the original Jackson network Q.

Now, by (2.6),

$$-L^t g(\vec{x}) = \frac{-1}{\prod_{i=1}^m a_i^{x_i}} L^\infty [\prod_{i=1}^m a_i^{x_i} \prod_{i=1}^m a_i^{-x_i} \chi\{x_1 \geq 0\}] = \frac{-1}{\prod_{i=1}^m a_i^{x_i}} L^\infty \chi\{x_1 \geq 0\}.$$

If $x_1 > 0$, $-L^t g(\vec{x}) = -1/(\prod_{i=1}^m a_i^{x_i}) L^\infty 1 = 0$. If $x_1 = 0$, note that $(T_{.i}\vec{x})_1 \geq 0$ is always true, but $(T_{i.}\vec{x})_1 < 0$ and $(T_{ij}\vec{x})_1 < 0$ if $i = 1$. Therefore, for $\vec{x} \in S^\infty$ with $x_1 = 0$ and $x_i = 0$, $i \in \mathcal{I}$ for some subsample $\mathcal{I}$ of $\{2, \ldots, m\}$,

$$\begin{aligned}
-L^t g(\vec{x}) &= \sum_{i=1}^m \bar{\lambda}_i^t [g(T_{.i}\vec{x}) - g(\vec{x})] + \sum_{i \notin \mathcal{I}} \mu_i^t r_{i.}^t [g(T_{i.}\vec{x}) - g(\vec{x})] + \sum_{i \notin \mathcal{I}} \sum_{j=1}^m \mu_i^t r_{ij}^t [g(T_{ij}\vec{x}) - g(\vec{x})] \\
&= \prod_{i=1}^m a_i^{-x_i} \left[\sum_{i=1}^m \bar{\lambda}_i a_i [a_i^{-1} - 1] + \sum_{i \notin \mathcal{I} \cup \{1\}} \frac{\mu_i r_{i.}}{a_i} [a_i - 1] + \sum_{i \notin \mathcal{I} \cup \{1\}} \sum_{j=1}^m \frac{\mu_i r_{ij} a_j}{a_i} [\frac{a_i}{a_j} - 1] \right] \\
&= \prod_{i=1}^m a_i^{-x_i} \left[\sum_{i=1}^m (\bar{\lambda}_i - \bar{\lambda}_i^t) + \sum_{i \notin \mathcal{I} \cup \{1\}} \mu_i - \sum_{i \notin \mathcal{I} \cup \{1\}} \mu_i \left(\frac{r_{i.} + \sum_{j=1}^m r_{ij} a_j}{a_i} \right) \right] \\
&= \prod_{i=1}^m a_i^{-x_i} \sum_{i=1}^m (\bar{\lambda}_i - \bar{\lambda}_i^t) \quad \text{by (2.9)} \\
&= \sum_{i=1}^m (\bar{\lambda}_i - \bar{\lambda}_i^t) g(\vec{x}).
\end{aligned}$$

Finally, if $x_1 < 0$, note that $(T_{i.}\vec{x})_1 < 0$ is always true, but that $(T_{.i}\vec{x})_1 = 0$ if $x_1 = -1$ and $i = 1$ and $(T_{ij}\vec{x})_1 = 0$ if $x_1 = -1$ and $j = 1$. Hence, for $\vec{x} \in S^\infty$ with $x_1 < 0$ and $x_i = 0$, $i \in \mathcal{I}$ for some subsample $\mathcal{I}$ of $\{2, \ldots, m\}$,

$$\begin{aligned}
-L^t g(\vec{x}) &= \sum_{i=1}^m \bar{\lambda}_i^t [g(T_{.i}\vec{x}) - g(\vec{x})] + \sum_{i \notin \mathcal{I}} \mu_i^t r_{i.}^t [g(T_{i.}\vec{x}) - g(\vec{x})] + \sum_{i \notin \mathcal{I}} \sum_{j=1}^m \mu_i^t r_{ij}^t [g(T_{ij}\vec{x}) - g(\vec{x})] \\
&= \chi\{x_1 = -1\} \prod_{i=1}^m a_i^{-x_i} [\bar{\lambda}_1^t a_1^{-1} + \sum_{i \notin \mathcal{I}} \mu_i^t r_{i1}^t a_i a_1^{-1}] \\
&= \chi\{x_1 = -1\} \prod_{i=1}^m a_i^{-x_i} [\bar{\lambda}_1 + \sum_{i \notin \mathcal{I}} \mu_i r_{i1}].
\end{aligned}$$

If, at some time $r < T$, Z^t wanders into the negative part of the x_1 hyperplane, i.e., $x_1 = -1$,

$$\frac{-L^t g(Z^t(r))}{g(Z^t(r))} = \frac{\chi\{Z_1^t(r) = -1\} \prod_{i=1}^m a_i^{-Z_i^t(r)} [\bar{\lambda}_1 + \sum_{i \notin \mathcal{I}} \mu_i r_{i1}]}{\prod_{i=1}^m a_i^{-Z_i^t(r)} \chi\{Z_1^t(r) \geq 0\}} = \infty$$

and thus, for that trajectory,

$$\exp\left(-\int_0^T \frac{-L^t g(Z^t(r))}{g(Z^t(r))}dr\right) = 0.$$

Consequently, noting that $Z^t(0) = \vec{0}$,

$$\begin{aligned} M_0^T &= \frac{g(Z^t(T))}{g(Z^t(0))}\exp\left(-\int_0^T \frac{-L^t g(Z^t(r))}{g(Z^t(r))}dr\right) \\ &= \frac{\prod_{i=1}^m a_i^{-Z_i^t(T)}}{\prod_{i=1}^m a_i^0}\exp\left(-\int_0^T \frac{\sum_{i=1}^m[\bar{\lambda}_i - \bar{\lambda}_i^t]\chi\{Z_1^t(r)=0\}g(Z^t(r))}{g(Z^t(r))}dr\right)\cdot\chi\{Z^t(r)\geq 0, r\leq T\} \\ &= \prod_{i=1}^m a_i^{-Z_i^t(T)}\exp\left(-\sum_{i=1}^m[\bar{\lambda}_i - \bar{\lambda}_i^t]\int_0^T \chi\{Z_1^t(r)=0\}dr\right)\cdot\chi\{Z^t(r)\geq 0, r\leq T\}. \end{aligned}$$

In other words, the contribution of trajectories which take negative x_1 values after the change of measure (2.23) is null.

We apply this result for the fixed time T by taking $W = \chi\{Z^t(T) = \vec{y}\}$. Then (2.23) becomes

$$\begin{aligned} \tilde{E}_{\vec{0}}\chi\{Z^t(T)=\vec{y}\} &= E_{\vec{0}}M_0^T\chi\{Z^t(T)=\vec{y}\} \\ &= E_{\vec{0}}\Bigg[\chi\{Z^t(T)=\vec{y}\}\prod_{i=1}^m a_i^{-Z_i^t(T)} \\ &\quad\cdot\exp\left(\sum_{i=1}^m[\bar{\lambda}_i^t - \bar{\lambda}_i]\int_0^T \chi\{Z_1^t(r)=0\}dr\right)\cdot\chi\{Z^t(r)\geq 0, r\leq T\}\Bigg]. \end{aligned} \tag{2.24}$$

Recalling that under the measure $\tilde{P}_{\vec{x}}$, Z^t is the original Jackson network Q, (2.24) shows that the trajectories of the Jackson network which hit the hyperplane ∂F can be simulated by calculating the above functional for the extended twisted process. Moreover, since the twisted process is transient, we expect that for large T

$$\exp\left(\sum_{i=1}^m[\bar{\lambda}_i^t - \bar{\lambda}_i]\int_0^T \chi\{Z_1^t(r)=0\}dr\right)\cdot\chi\{Z^t(r)\geq 0, r\leq T\}$$

is asymptotically independent of $Z^t(T)$. Hence, for large T,

$$\begin{aligned} \tilde{P}_{\vec{0}}(Z^t(T)=\vec{y}) &\approx E_{\vec{0}}\left[\chi\{Z^t(T)=\vec{y}\}\prod_{i=1}^m a_i^{-Z_i^t(T)}\right] \\ &\quad\cdot E_{\vec{0}}\left[\exp\left(\sum_{i=1}^m[\bar{\lambda}_i^t - \bar{\lambda}_i]\int_0^T \chi\{Z_1^t(r)=0\}dr\right)\cdot\chi\{Z^t(r)\geq 0, r\leq T\}\right]. \end{aligned}$$

We shall show later in Section 3.4 that $P_{\vec{0}}(Z^t(\tau) = \vec{y}) \to e(\vec{y}^-)$ so it follows that

$$\lim_{t\to\infty} \tilde{P}_{\vec{0}}(Z^t(\tau) = \vec{y}) = \prod_{i=2}^{m} (a_1/a_i)^{y_i} e(\vec{y}^-)/[\sum_{\vec{z}\in\partial F} \prod_{i=2}^{m} (a_1/a_i)^{z_i} e(\vec{z}^-)]. \qquad (2.25)$$

This heuristic expression will be shown to be true in Theorem 3.4.2.

Chapter 3

Overloading a Jackson Network of Shared Queues

A Jackson network of shared queues explodes when the sum of the queue sizes of all nodes exceeds the buffer size $\ell - 1$. As seen in Section 3.1, the exact distribution of the network at the moment of overload can be given. However, its evaluation is only possible when both m and ℓ are small. This explains the importance of the asymptotic expression for the hitting distribution given in Section 2.3.

The reliability of the network can be assessed by the first time $\tau \equiv \tau_\ell$ the Markov process Q describing the network hits the set of forbidden states $F = \{\vec{x} \in S : \sum_{i=1}^{m} x_i \geq \ell\}$. Section 3.2 shows this too has an exact expression which, as before, can only be calculated when both m and ℓ are small. For ℓ large and m arbitrary, Aldous (1989) presented a heuristic method which we show, in Section 3.3, to be asymptotically correct. In Section 3.5 we propose the application of a result by Iscoe and McDonald (1994a) on asymptotics of hitting times for Markov jump processes; this application yields an approximation for the mean hitting time τ along with error bounds which are shown to tend to 0 as $\ell \to \infty$. Sections 3.6 to 3.9 spell out the application of that result for a Jackson network of shared queues and, finally, Section 3.10 shows the asymptotic equivalence of the two estimation methods. These conclusions are made possible by the results shown in Section 3.4.

3.1 Exact hitting distribution of the forbidden set F

Consider the Markov process $(Q(s); s \geq 0)$ on the state space $S = \{\vec{x} \in Z^m : x_i \geq 0, i = 1, \ldots, m\}$ and its generator $-L$. Let $F := \{\vec{x} \in S : \sum_{i=1}^{m} x_i \geq \ell\}$ be a set of forbidden states and denote its complement as B. Since S is countable, we can express the operator $-L$ as a matrix of countable dimensions by using an appropriate indexing method $i = 1, 2, 3, \ldots$ to represent each $\vec{x} \in S$. Let $c : N \to S$ denote the indexing bijection; noting that $B = \{\vec{x} \in S : \sum_{i=1}^{m} x_i < \ell\}$ is finite ($\dim(B) < \infty$), we can restrict the indices $i = 1, 2, \ldots, \dim(B) = \sum_{x_1=0}^{\ell-1} \sum_{x_2=0}^{\ell-1-x_1} \cdots \sum_{x_m=0}^{\ell-1-\sum_{k=1}^{m-1} x_k} 1$ to correspond to states $\vec{x}$ in B and fix $c(1) = \vec{0}$.

We can now write $-L = (-L_{ij})$ where $q(i,j) := -L_{ij}$ denotes the transition rate from state $c(i)$ to state $c(j)$, $(i \neq j)$, and $q(i) := L_{ii}$ is the total transition rate from state $c(i)$. Defining the proportion of the transition rate directed from i to j, $(i \neq j)$, as $K_{ij} := q(i,j)/q(i)$ and setting $K_{ii} = 0$, we obtain the transition kernel K of the embedded Markov chain.

We can verify that

$$\text{diag}(q)\,(I - K) = L,$$

where $\text{diag}(q)$ denotes the matrix with entries $q(1), q(2), q(3), \ldots$ on the diagonal and 0 elsewhere. Similarly, for the Markov process restricted on B, we can write

$$\text{diag}^B(q)\,(I - K^B) = L^B$$

where $\text{diag}^B(q), I, K^B$ and L^B are now matrices with finite dimensions $\dim(B) \times \dim(B)$. By our assumption that the Jackson network is both open and exogenously supplied, $q(i) > 0$ for $i = 1, \ldots, \dim(B)$ and thus $[\text{diag}^B(q)]^{-1}$ exists. As shown in Appendix A, $(I - K^B)$ and L^B also are nonsingular. Consequently, we may write

$$(L^B)^{-1} = (I - K^B)^{-1}[\text{diag}^B(q)]^{-1}$$

or

$$(I - K^B)^{-1} = (L^B)^{-1}\text{diag}^B(q). \tag{3.1}$$

With these tools in hand, we can now calculate the exact hitting distribution of F. From $c(i) \in B$, the Jackson network Q wanders in B for some time and eventually hits F. The probability that it hits $c(k) \in F$ is given by

$$\begin{aligned}
P(Q(\tau) = c(k)) &= \sum_{n=0}^{\infty} \sum_{c(j) \in B} (K^B)^n_{ij} K_{jk} \\
&= \sum_{c(j) \in B} \sum_{n=0}^{\infty} (K^B)^n_{ij} K_{jk} \\
&= \sum_{c(j) \in B} (I - K^B)^{-1}_{ij} K_{jk} \\
&= \sum_{c(j) \in B} (L^B)^{-1}_{ij} q(j) \frac{q(j,k)}{q(j)} \quad \text{by (3.1)} \\
&= -\sum_{c(j) \in B} (L^B)^{-1}_{ij} L_{jk}.
\end{aligned}$$

3.2 Exact mean time to reach the forbidden set F

As in Section 3.1, consider the Markov process Q on the state space S and its generator $-L$. Consider also the transition kernel K of the embedded chain. Since S is countable, we can express the operators $-L$ and K as matrices of countable dimensions by using the indexing bijection $c : N \to S$ described in the previous section.

The Markov process Q sojourns at state $c(i)$ for an exponential duration, say D_i, of mean $1/q(i)$ and then jumps to state $c(j)$, $(i \neq j)$, with probability K_{ij}. Denote by $m(i)$ the mean time to reach F from state $c(i)$. Clearly, $m(i) = 0$ if $c(i) \in F$. Also, the time required to reach F from $c(i) \in B$ is $D_i + R(N)$ where N represents the next state entered after the sojourn in state $c(i)$ and $R(N)$ denotes the remaining time required to reach F after jumping to state $c(N)$. Consequently, for $c(i) \in B$,

$$\begin{aligned}
m(i) &= E(D_i + R(N)) \\
&= \frac{1}{q(i)} + \sum_{c(j) \in S} P(N = j) E(R(j) | N = j) \\
&= \frac{1}{q(i)} + \sum_{c(j) \in S} K_{ij} m(j) \\
&= \frac{1}{q(i)} + Km(i)
\end{aligned}$$

where we used the fact that the remaining time to reach F given we have just entered state $c(j)$ is independent of the past – the sojourn periods are memoryless. We can rewrite this as

$$(I - K)m(i) = \frac{1}{q(i)}$$

for $c(i) \in B$; if we multiply each of the $\dim(B)$ such equations by the appropriate $q(i)$ and simplify, we obtain

$$Lm = 1.$$

Recalling that $m(i) = 0$ for $c(i) \in F$, this is clearly equivalent to

$$L^B m^B = 1$$

where $-L^B$ is the generator of Q killed off B and m^B is the vector consisting of the first $\dim(B)$ elements of m. Since B is finite, $-L^B$ is a matrix of finite dimensions and, as shown in Appendix A, it is nonsingular. Therefore

$$\begin{array}{ll} m(i) = (L^B)^{-1}1(i) & \text{for } c(i) \in B, \\ m(i) = 0 & \text{for } c(i) \in F. \end{array} \tag{3.2}$$

This exact solution unfortunately cannot be used in general. Though (3.2) yields the exact mean time required to reach F from any state $\vec{x} \in B$, its calculation is only possible if the dimensions of the matrix $-L^B$ are of reasonable size, i.e., if m and ℓ are both small. In Section 4.2, for instance, (3.2) is used to assess the validity of our twist method in some examples with $m = 2$ and ℓ small.

3.3 Aldous' estimate for the mean overload time

Aldous (1989) proposed the following approximation for the time to hit F from $\vec{0}$ when $\pi(F)$ is small:

$$E_{\vec{0}}\tau \sim \left[\sum_{\vec{y} \in \partial F} \pi(\vec{y}) \sum_{\vec{x} \in B} q(\vec{y}, \vec{x}) f(\vec{x}) \right]^{-1} \tag{3.3}$$

where $f(\vec{x})$ denotes the probability that, starting at $\vec{x} \in B$, Q reaches $\vec{0}$ before $\partial F := \{\vec{x} \in F : \sum_{i=1}^{m} x_i = \ell\}$ and $q(\vec{y}, \vec{x})$ is the transition rate from state $\vec{y}$ to state $\vec{x}$. Though Aldous (1989) presented his method as heuristic, we can establish the validity of (3.3) by studying a process starting at $\vec{0}$ which regenerates when it returns to $\vec{0}$ after first hitting F. First consider the following result due to Keilson (1979).

Proposition 3.3.1 *Let R be the time to return to $\vec{0}$ after first hitting F and τ be the time to hit the forbidden set. Then*

$$\lim_{\pi(F)\to 0} \frac{E_{\vec{0}}R}{E_{\vec{0}}\tau} = 1.$$

An exact expression for $E_{\vec{0}}R$ is given by the next theorem the proof of which can be found in the chapter on renewal applications in McDonald (1994).

Theorem 3.3.2 *Let $f(\vec{x})$ denote the probability the Markov process Q starting from $\vec{x} \in B$ hits $\vec{0}$ before F. Then*

$$E_{\vec{0}}R = \left(\sum_{\vec{y}\in F} \pi(\vec{y}) \sum_{\vec{x}\in B} q(\vec{y},\vec{x}) f(\vec{x})\right)^{-1}.$$

Combining Proposition 3.3.1 and Theorem 3.3.2 yields the corollary from which (3.3) is established.

Corollary 3.3.3

$$\lim_{\pi(F)\to 0} E_{\vec{0}}\tau = \left(\sum_{\vec{y}\in F} \pi(\vec{y}) \sum_{\vec{x}\in B} q(\vec{y},\vec{x}) f(\vec{x})\right)^{-1}.$$

Returns to $\vec{0}$ can be achieved in one of two ways. From $\vec{0}$, the process may wander in B and return to $\vec{0}$ without ever hitting F; this we call a return of type I. On the other hand, the process may visit F before returning to $\vec{0}$; this is a return of type II. Let p denote the probability that a return to $\vec{0}$ is of type II. Keilson (1979) showed that

Proposition 3.3.4 *If the stationary probability $\pi(F)$ of a sequence of forbidden sets F tends to 0, the probability p of hitting F before a return to 0 tends to 0.*

A cycle of a process which regenerates after first hitting F is clearly composed of a sequence of returns to $\vec{0}$ of type I followed with a return of type II. Thus, if looked upon backward in time, this cycle consists of a return to $\vec{0}$ of type II followed with a sequence of returns to $\vec{0}$ of type I. The number of such returns follows a geometric distribution with mean $1/p$.

It is clear that, whether the cycle is observed forward or backward in time, it is of the same length R. If we apply Theorem 3.3.2 to the time reversal of the Jackson network, Q^r, it then follows that

$$E_{\vec{0}}R = \left(\sum_{\vec{y}\in F}\pi(\vec{y})\sum_{\vec{x}\in B}q^r(\vec{y},\vec{x})\rho_\ell(\vec{x})\right)^{-1} \tag{3.4}$$

where $\rho_\ell(\vec{x})$ denotes the probability that, starting at $\vec{x}\in B$, Q^r reaches $\vec{0}$ before ∂F. By the equality of (3.4) and the equation in Theorem 3.3.2, we see that Aldous' estimate can also be given by

$$E_{\vec{0}}\tau \sim \left(\sum_{\vec{y}\in F}\pi(\vec{y})\sum_{\vec{x}\in B}q^r(\vec{y},\vec{x})\rho_\ell(\vec{x})\right)^{-1}.$$

If we consider the time reversed Jackson network, the hitting distribution of ∂F is given in Lemma 3.3.5 the proof of which is given in McDonald (1994). This result will be used in the next section.

Lemma 3.3.5 *For the Jackson network Q and $\vec{y} = (\ell - \sum_{i=2}^m y_i, y_2, \ldots, y_m) \in \partial F$,*

$$\begin{aligned}
P_{\vec{0}}(Q(\tau)=\vec{y}) &= \frac{\pi(\vec{y})\sum_{\vec{x}\in B}q^r(\vec{y},\vec{x})\rho_\ell(\vec{x})}{\sum_{\vec{y}\in\partial F}\pi(\vec{y})\sum_{\vec{x}\in B}q^r(\vec{y},\vec{x})\rho_\ell(\vec{x})}\\
&= \frac{\prod_{i=1}^m(1-\rho_i)\rho_1^\ell\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar{\lambda}_i\rho_i^{-1}\rho_\ell(T_{i.}\vec{y})}{\prod_{i=1}^m(1-\rho_i)\rho_1^\ell\sum_{\vec{y}\in\partial F}\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar{\lambda}_i\rho_i^{-1}\rho_\ell(T_{i.}\vec{y})}\\
&= \frac{\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar{\lambda}_i\rho_i^{-1}\rho_\ell(T_{i.}\vec{y})}{\sum_{\vec{y}\in\partial F}\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar{\lambda}_i\rho_i^{-1}\rho_\ell(T_{i.}\vec{y})}.
\end{aligned}$$

For comparison purposes, we will consider Aldous' formula (3.3) and approximate its result by simulation. The method used is described below.

Define an Aldous trajectory as follows. Select randomly a coordinate $\vec{y}$ on ∂F. From $\vec{y}\in\partial F$, jump directly in B to $T_{i.}\vec{y}$, for some $i = 1, 2, \ldots, m$, and then let the Markov process Q wander on S until the origin $\vec{0}$ or ∂F is reached. Let

Y take value $\vec{y}\in\partial F$ with probability $\pi(\vec{y})/\pi(\partial F)$ where $\pi(\partial F) = \sum_{\vec{y}\in\partial F}\pi(\vec{y})$,

$D(Y) = \sum_{i=1}^m \mu_i(Y_i)r_{i.}$,

$J(Y)$ denote the state entered in B from Y where $J(\vec{y}) = T_{i.}\vec{y}$ with probability $\frac{\mu_i(y_i)r_{i.}}{D(\vec{y})} = \frac{q(\vec{y},T_{i.}\vec{y})}{D(\vec{y})}$, and $R(J(Y)) = \begin{cases} D(Y) & \text{if, from } J(Y),\ \vec{0} \text{ is reached before } \partial F, \\ 0 & \text{otherwise.} \end{cases}$

Generate N Aldous trajectories and calculate

$$A = \frac{\pi(\partial F)}{N} \sum_{n=1}^{N} R(J(Y(n))) \tag{3.5}$$

where $Y(n)$ denotes the value taken by Y at the n-th trajectory. By the law of large numbers,

$$\begin{aligned}
E[A] &= \pi(\partial F) E[R(J(Y))] \\
&= \pi(\partial F) E[E[R(J(Y))|Y]] \\
&= \pi(\partial F) \sum_{\vec{y} \in \partial F} \frac{\pi(\vec{y})}{\pi(\partial F)} E[E[R(J(\vec{y}))|J(\vec{y})]] \\
&= \sum_{\vec{y} \in \partial F} \pi(\vec{y}) \sum_{\vec{x} \in B} \frac{q(\vec{y}, \vec{x})}{D(\vec{y})} E[R(\vec{x})] \\
&= \sum_{\vec{y} \in \partial F} \pi(\vec{y}) \sum_{\vec{x} \in B} \frac{q(\vec{y}, \vec{x})}{D(\vec{y})} D(\vec{y}) f(\vec{x}) \\
&= \sum_{\vec{y} \in \partial F} \pi(\vec{y}) \sum_{\vec{x} \in B} q(\vec{y}, \vec{x}) f(\vec{x}),
\end{aligned} \tag{3.6}$$

where we note that, for $\vec{x} \in B$ and $\vec{y} \in \partial F$, $q(\vec{y}, \vec{x}) = 0$ if $\vec{x}$ is not of the form $\vec{x} = T_{i.}\vec{y}$ for some $i = 1, 2, \ldots, m$. We see (3.3) and the inverse of (3.6) are identical and thus, if N is large enough, the reciprocal of the random variable A given in (3.5) should be close to Aldous' estimate for the mean exit time. Note that this simulation is fast since Q drifts quickly from ∂F to $\vec{0}$.

3.4 Semi-Markov methods

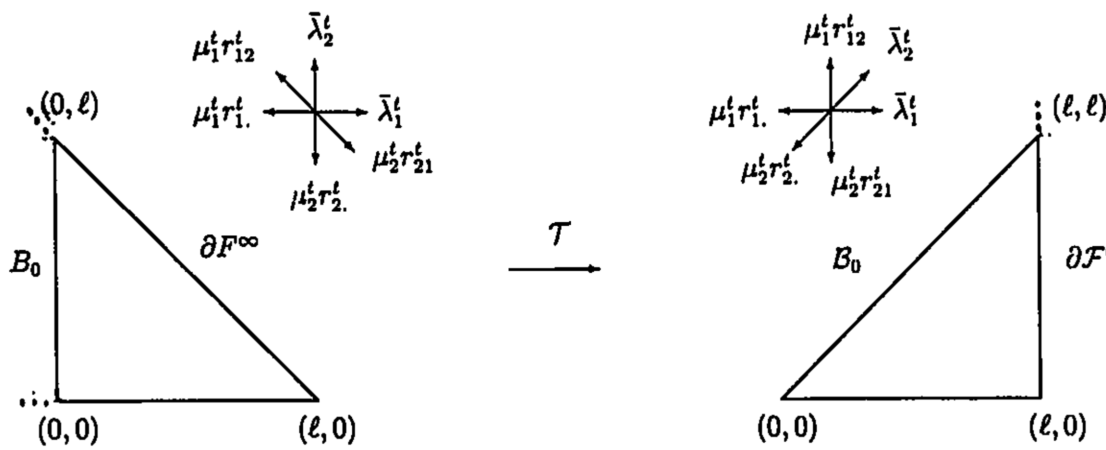
Consider the transformation $\mathcal{T} : S^\infty \to S^\infty$ defined by $\mathcal{T} : \vec{x} = (x_1, x_2, \ldots, x_m) \mapsto \vec{z} = (z_1, z_2, \ldots, z_m) := (\sum_{i=1}^m x_i, x_2, \ldots, x_m)$ and denote by $(\mathcal{Z}^t(s), s \geq 0)$, the twisted extended Markov process on $\mathcal{T}S^\infty$. Under this transformation, the sets $B^\infty, B_0 = \{\vec{x} \in B : x_1 = 0\}$ and ∂F^∞ are mapped to

$$\begin{aligned}
B^\infty &\longrightarrow \mathcal{B}^\infty := \{\vec{z} \in S^\infty : z_1 < \ell\} \\
B_0 \subset B^\infty &\longrightarrow \mathcal{B}_0 := \{\vec{z} \in \mathcal{B}^\infty : z_1 = \sum_{i=2}^{m} z_i\} \\
\partial F^\infty &\longrightarrow \partial\mathcal{F}^\infty := \{\vec{z} \in S^\infty : z_1 = \ell\}
\end{aligned}$$

and the jumps in this new set of coordinates can be easily verified to be the following: from $\vec{z} = (z_1, z_2, \ldots, z_m)$, $\mathcal{Z}$

$$\begin{array}{lll}
\text{jumps to} & \text{with rate} & \\
T_{.1}\vec{z} = \vec{z} + \vec{e}_1 & \bar{\lambda}_1^t & \\
T_{.\{1,i\}}\vec{z} = \vec{z} + \vec{e}_1 + \vec{e}_i & \bar{\lambda}_i^t, & i = 2, \ldots, m \\
T_{1.}\vec{z} = \vec{z} - \vec{e}_1 & \mu_1^t r_{1.}^t & \\
T_{\{1,i\}.}\vec{z} = \vec{z} - \vec{e}_1 - \vec{e}_i & \mu_i^t(z_i) r_{i.}^t, & i = 2, \ldots, m \\
T_{.i}\vec{z} = \vec{z} + \vec{e}_i & \mu_1^t r_{1i}^t, & i = 2, \ldots, m \\
T_{i.}\vec{z} = \vec{z} - \vec{e}_i & \mu_i^t(z_i) r_{i1}^t, & i = 2, \ldots, m \\
T_{ij}\vec{z} = \vec{z} - \vec{e}_i + \vec{e}_j & \mu_i^t(z_i) r_{ij}^t, & i, j = 2, \ldots, m
\end{array} \tag{3.7}$$

where we recall $\vec{e}_i$ is the i-th canonical vector of R^m and $\mu_i^t(z_i) = \mu_i^t$ if $z_i > 0$ and 0 otherwise. When $m = 2$, this transformation can be illustrated graphically as below:



Though the two representations are equivalent – they are isomorphic– the representation under $\mathcal{T}$ has the clear advantage that jumps towards (or away from) $\partial\mathcal{F}^\infty$ are necessarily function of increments (or decrements) in the z_1 coordinate. The position of the hit on $\partial\mathcal{F}^\infty$ (i.e., the hitting distribution), on the other hand, is a function of $z_2, \ldots, z_m$ alone.

Now recall, from the end of Section 2.2, the twisted Jackson network on S^∞ whose recurrent components 2 through m have stationary distribution $\pi^t(\vec{x}) = \prod_{i=2}^m (1 - \rho_i^t)(\rho_i^t)^{x_i}$. Let $\bar{\pi}$ denote the stationary distribution of the associated embedded recurrent chain and denote by X_n, $n = 0, 1, 2, \ldots$, the corresponding chain under the transformation $\mathcal{T}$. Clearly, since under the transformation $\mathcal{T}$ the last $(m-1)$ components of any state $\vec{x} \in S^\infty$ are unchanged, X_n has stationary measure $\bar{\pi}$.

(X_n, X_{n+1})	Distribution of u_n
$(\vec{z}^-, (T_{ij}\vec{z})^-)$	$P(u_n = 0) = 1$
$(\vec{z}^-, (T_{.i}\vec{z})^-)$	$P(u_n = 0) = \mu_1^t r_{1i}^t / (\bar{\lambda}_i^t + \mu_1^t r_{1i}^t)$ $P(u_n = 1) = \bar{\lambda}_i^t / (\bar{\lambda}_i^t + \mu_1^t r_{1i}^t)$
$(\vec{z}^-, (T_{i.}\vec{z})^-)$	$P(u_n = -1) = r_{i.}^t / (r_{i.}^t + r_{i1}^t)$ $P(u_n = 0) = r_{i1}^t / (r_{i.}^t + r_{i1}^t)$
$(\vec{z}^-, \vec{z}^-)$	$P(u_n = -1) = \mu_1^t r_{1.}^t / (\bar{\lambda}_1^t + \mu_1^t r_{1.}^t)$ $P(u_n = 1) = \bar{\lambda}_1^t / (\bar{\lambda}_1^t + \mu_1^t r_{1.}^t)$

(Notice the analogy between the values taken by u_n and the increments and decrements of z_1.)

Table 3.1: Distribution of u_n given (X_n, X_{n+1}): $P(u_n = \cdot | X_n, X_{n+1})$.

For $\vec{z} \in S^\infty$, define $\vec{z}^- = (z_2, \dots, z_m)$ and let $S^- = \{\vec{z}^- = (z_2, \dots, z_m) \in Z^{m-1} : z_i \geq 0, i = 2, \dots, m\}$. From the comments at the end of Section 2.2 and the jump rates given in (3.7), we see the Markov chain X_n jumps from $\vec{z}^-$ to

- $(T_{ij}\vec{z})^-$, $i, j = 2, \dots, m$, with rate $\mu_i^t(z_i) r_{ij}^t =: q^-(\vec{z}^-, (T_{ij}\vec{z})^-)$
- $(T_{.i}\vec{z})^-$, $i = 2, \dots, m$, with rate $\bar{\lambda}_i^t + \mu_1^t r_{1i}^t =: q^-(\vec{z}^-, (T_{.i}\vec{z})^-)$
- $(T_{i.}\vec{z})^-$, $i = 2, \dots, m$, with rate $\mu_i^t(z_i)(r_{i.}^t + r_{i1}^t) =: q^-(\vec{z}^-, (T_{i.}\vec{z})^-)$
- $\vec{z}^-$ with rate $\bar{\lambda}_1^t + \mu_1^t r_{1.}^t =: q^-(\vec{z}^-, \vec{z}^-)$.

Let $q^-(\vec{z}^-) := \sum_{i=2}^m [q^-(\vec{z}^-, (T_{.i}\vec{z})^-) + q^-(\vec{z}^-, (T_{i.}\vec{z})^-) + \sum_{j=2}^m q^-(\vec{z}^-, (T_{ij}\vec{z})^-)] + q^-(\vec{z}^-, \vec{z}^-)$ denote the total transition rate from $\vec{z}^-$.

Define, for $n = 0, 1, 2, \dots$ the random variable u_n with values in $\{-1, 0, 1\}$ whose distribution depends only on the state of the Markov chain X at times n and $n+1$ according to Table 3.1. Denote the conditional distribution as $P(u_n = \cdot | X_n, X_{n+1})$.

If we describe the random variable u_n as the "time" spent in state X_n, the pair $(X_n, Z_1^t(0) + V_n)$ forms a semi-Markov process where V_n, defined as

$$\begin{aligned} V_0 &= 0, \\ V_n &= \textstyle\sum_{i=0}^{n-1} u_i, \end{aligned} \tag{3.8}$$

describes the total time elapsed. The structure (X_n, u_n, V_n) corresponds to the setting established by Kesten (1974). We present here some of his results that will help us study the hitting distribution of $\partial\mathcal{F}^\infty$. We represent the variation in the z_1 coordinate after the n-th jump by the value taken by u_n. Hence the extended twisted Markov process $\mathcal{Z}^t$ takes, after n jumps, the value $(\mathcal{Z}_1^t(0) + V_n, X_n) = (z_1, z_2, \ldots, z_m)$. Since hitting $\partial\mathcal{F}^\infty$ corresponds to hitting $z_1 = \ell$, we conclude the buffer overflows when $V_n = \ell - \mathcal{Z}_1^t(0)$. Define $\ell^* := \ell - \mathcal{Z}_1^t(0)$ and let τ_ℓ denote the time at which the overflow occurs.

As in Kesten (1974), define ladder indices for the sequence $\{u_n\}_{n\geq 0}$ by

$$\begin{aligned} \nu_0 &= 0, \\ \nu_{i+1} &= \min\{n > \nu_i : V_n > V_{\nu_i}\}. \end{aligned} \tag{3.9}$$

It is clear, from Table 3.1, (3.8) and (3.9), that $V_{\nu_i} = i$, $i = 0, 1, 2, \ldots$ so that $\partial\mathcal{F}^\infty$ is first hit after ν_{ℓ^*} jumps from $(\mathcal{Z}_1^t(0), \vec{0}^-)$ at the coordinates $(\ell, X_{\nu_{\ell^*}})$.

Consider now a two sided process $\{X_n^\#, u_n^\#\}_{-\infty<n<\infty}$ with probability measure $P^\#$ determined by

$$\begin{aligned} &P^\#(X_{k+i}^\# \in \mathcal{A}_i, 0 \leq i \leq n, u_{k+i}^\# \in \mathcal{B}_i, 0 \leq i < n) \\ &= \sum_{\vec{z}_0^- \in \mathcal{A}_0} \bar{\pi}(\vec{z}_0^-) \sum_{\vec{z}_1^- \in \mathcal{A}_1} \frac{q^-(\vec{z}_0^-, \vec{z}_1^-)}{q^-(\vec{z}_0^-)} \cdots \sum_{\vec{z}_n^- \in \mathcal{A}_n} \frac{q^-(\vec{z}_{n-1}^-, \vec{z}_n^-)}{q^-(\vec{z}_{n-1}^-)} \\ &\quad \times \sum_{j \in \mathcal{B}_0} P(u_0 = j | \vec{z}_0^-, \vec{z}_1^-) \sum_{j \in \mathcal{B}_1} P(u_1 = j | \vec{z}_1^-, \vec{z}_2^-) \cdots \sum_{j \in \mathcal{B}_{n-1}} P(u_{n-1} = j | \vec{z}_{n-1}^-, \vec{z}_n^-) \end{aligned}$$

for $\mathcal{A}_i \subseteq S^-$, $\mathcal{B}_i \subseteq \{-1, 0, 1\}$ and any integer k. The pair $\{X_n^\#, u_n^\#\}_{-\infty<n<\infty}$ forms a stationary Markov chain, i.e.,

$$\begin{aligned} &P^\#(X_{k+i}^\# = \vec{z}_i^-, 0 \leq i \leq n, u_{k+i}^\# = c_i, 0 \leq i < n) \\ &= \sum_{\vec{z}^- \in S^-} \bar{\pi}(\vec{z}^-) P_{\vec{z}^-}(X_i = \vec{z}_i^-, 0 \leq i \leq n, u_i = c_i, 0 \leq i < n), \end{aligned}$$

and, in particular,

$$P^\#(X_k^\# = \vec{z}^-) = \bar{\pi}(\vec{z}^-).$$

In analogy with (3.8) and (3.9), define

$$V_n^\# = \begin{cases} \sum_{i=0}^{n-1} u_i^\# & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} u_i^\# & \text{if } n < 0, \end{cases}$$

and ladder indices for the sequence $\{V_n^{\#}\}_{-\infty<n<\infty}$:

$$\begin{aligned}\nu_0^{\#} &= \max\{n \le 0 : V_n^{\#} > \sup_{j<n} V_j^{\#}\},\\ \nu_{i+1}^{\#} &= \min\{n > \nu_i^{\#} : V_n^{\#} > V_{\nu_i}^{\#}\}.\end{aligned} \tag{3.10}$$

The index $\nu_0^{\#}$ represents the time at which occurred the last strict maximum of $V_n^{\#}$, $n \le 0$. Note that if $\nu_0^{\#} = 0$, (3.10) is exactly as (3.9).

We now construct the Markov chain

$$W_n = X_{\nu_n}, \quad n \ge 0, \tag{3.11}$$

which records the position of the chain X_n after each jump which brings $\mathcal{Z}^t$ closer than ever to $\partial\mathcal{F}^\infty$. Kesten (1974, Lemma 2) showed that, for $\vec{z}^- \in S^-$,

$$\psi(\vec{z}^-) = P^{\#}(\nu_0^{\#} = 0, X_0^{\#} = \vec{z}^-) = P^{\#}\left(\sup_{n<0} V_n^{\#} < 0, X_0^{\#} = \vec{z}^-\right) \tag{3.12}$$

is an invariant measure for the chain W_n. He also showed $q \equiv P^{\#}(\nu_0^{\#} = 0) > 0$ so $e(\vec{z}^-) := \psi(\vec{z}^-)/q$ is the stationary probability of W_n. By Theorem 1 in Kesten (1974) and in particular by (1.19) there,

$$\lim_{\ell\to\infty} P_{\vec{z}^-}(W_\ell = \vec{y}^-) = e(\vec{y}^-). \tag{3.13}$$

The identification of the limit in (1.19) with e is given at (3.10) in Kesten (1974).

Now return to the process Z^t on the untransformed state space S^∞ (i.e., in the original set of coordinates). Since all states $(x_1, \ldots, x_m)$ and $(z_1, \ldots, z_m)$ in the trajectories of Z^t and its equivalence $\mathcal{Z}^t$ in $\mathcal{T}S^\infty$ have common coordinates $x_2 = z_2, \ldots, x_m = z_m$, we can identify the last $(m-1)$ coordinates of $Z^t(s)$ with the state in which is the chain X at time s. In particular, $Z^t(\tau_\ell) = (\ell - \sum_{i=2}^m (X_{\nu_{\ell^*}})_i, X_{\nu_{\ell^*}})$. Hence, by (3.11) and (3.13),

$$\lim_{\ell\to\infty} P_{\hat{z}}(Z^t(\tau_\ell) = \vec{y}) = e(\vec{y}^-), \quad \vec{y} \in \partial F^\infty \tag{3.14}$$

where $\hat{z} = (0, \vec{z}^-) \in B_0$ so, even if we start from the origin, the hitting distribution of the extended twisted process tends to e as $\ell \to \infty$.

Returning to the transformed space, we can interpret $e(\vec{y}^-) = P^{\#}(\nu_0^{\#} = 0, X_0^{\#} = \vec{y}^-)/q$ by conditioning on $\{X_0^{\#} = \vec{y}^-\}$, i.e.,

$$e(\vec{y}^-) = \frac{1}{q} P^{\#}(\nu_0^{\#} = 0 | X_0^{\#} = \vec{y}^-) \bar{\pi}(\vec{y}^-). \tag{3.15}$$

Also,

$$P^{\#}(\nu_0^{\#} = 0 | X_0^{\#} = \vec{y}^-) = P^{\#}(-\sum_{i=n}^{-1} u_i^{\#} < 0, n = -1, -2, \ldots | X_0^{\#} = \vec{y}^-).$$

However, the distribution of $-u_i^{\#}$, $n = -1, -2, \ldots$ is obtained by the time reversal of the pair $\{X_n^{\#}, u_n^{\#}\}_{n \geq 0}$. Hence the above probability is equal to the probability the time reversal of $(V_n^{\#}, X_n^{\#})_{n \geq 0}$ drifts off to $-\infty$ in the first coordinate without hitting $\partial\mathcal{F}^\infty$. Equivalently, this is the probability the time reversal of the twisted process wanders off to $-\infty$ without hitting ∂F^∞. However, the time reversal of the twisted process Z^t was shown to be precisely the time reversal of the Jackson network extended to S^∞, Z^r. Consequently, $P^{\#}(-\sum_{i=n}^{-1} u_i^{\#} < 0, n = -1, -2, \ldots | X_0^{\#} = \vec{y}^-)$ is equal to the probability the time reversal of the extended Jackson network, Z^r, jumps from $\vec{y} \in \partial F^\infty$ into $\vec{x} \in B^\infty$ and wanders off to minus infinity without ever hitting ∂F^∞ again. Denote the probability that Z^r started at $\vec{x} \in B^\infty$ never hits the boundary by $\rho^\infty(\vec{x})$. Then,

$$P^{\#}(-\sum_{i=n}^{-1} u_i^{\#} < 0, n = -1, -2, \ldots | X_0^{\#} = \vec{y}^-) \quad = \quad \sum_{i=1}^{m} \frac{\bar{\lambda}_i \rho_i^{-1}}{\lambda(\vec{y}^-)} \rho^\infty(T_{i.}\vec{y})$$

where $\lambda(\vec{y}^-)$ is the total transition rate of the time reversal of the extended Jackson process, that is,

$$\lambda(\vec{y}^-) = \sum_{i=1}^{m} \bar{\lambda}_i^t + \mu_1^t + \sum_{i=2}^{m} \mu_i^t(y_i) \ \ (= q^-(\vec{z}^-)).$$

Finally we remark that

$$\frac{\bar{\lambda}_i \rho_i^{-1}}{\lambda(\vec{y}^-)} \bar{\pi}(\vec{y}^-) = \bar{\lambda}_i \rho_i^{-1} \lambda \pi^t(\vec{y}^-)$$

since

$$\pi^t(\vec{y}) \equiv \pi^t(\vec{y}^-) = \frac{\bar{\pi}(\vec{y}^-)/\lambda(\vec{y}^-)}{\lambda} \quad \text{where} \quad \lambda = \sum_{\vec{y}^- \in S^-} \bar{\pi}(\vec{y}^-)/\lambda(\vec{y}^-).$$

Putting all these remarks together we get, for any initial starting point $\hat{z} = (0, \vec{z}^-)$ and $\vec{y} \in \partial F^\infty$,

$$e(\vec{y}^-) = \lim_{\ell \to \infty} P_{\hat{z}}(Z^t(\tau_\ell) = \vec{y}) = \frac{1}{q} \sum_{i=1}^{n} \bar{\lambda}_i \rho_i^{-1} \lambda \rho^\infty(T_{i.}\vec{y}) \pi^t(\vec{y}^-). \tag{3.16}$$

The function ρ^∞ defined above is related to the function $\rho_\ell(\vec{x})$ defined in Section 3.3 to be the probability the time reversed Jackson network Q^r starting from $\vec{x} \in B$ hits $\vec{0}$ before hitting ∂F. Let $\vec{y} \in \partial F$ and consider any $\vec{x} = T_{i.}\vec{y} \in B$.

Lemma 3.4.1 *For $\vec{x} \in B \subset B^\infty$,*

$$\lim_{\ell \to \infty} |\rho^\infty(\vec{x}) - \rho_\ell(\vec{x})| = 0.$$

Proof: The time reversal of the Jackson network Q^r and the time reversal of the extended Jackson network Z^r may be coupled together until they hit the plane $x_1 = 0$ together. Hence, both processes have either crossed into F together or they hit the plane $x_1 = 0$ without reaching F first together. Consequently, the only difference in the values of $\rho^\infty(\vec{x})$ and $\rho_\ell(\vec{x})$ is the probability of two events. The first is that Z^r returns from the plane $x_1 = 0$ and hits F while Q^r is absorbed at $\vec{0}$. The second is that Q^r returns to F before hitting $\vec{0}$ while Z^r wanders off to $-\infty$.

The probability of the first event is bounded by the probability that Z^r hits the plane $x_1 = 0$ and then climbs back to F. We may apply the results in Kesten (1974) as above to conclude that, as $\ell \to \infty$ the process Z^r hits the plane $x_1 = 0$ according to a limiting distribution which we may call d. Hence, as $\ell \to \infty$, the probability of the above event is, asymptotically, $\sum_{\hat{z}} d(\hat{z}) P_{\hat{z}}(\tau < \infty)$ where, as above, $\hat{z}$ is a point of the form $(0, \vec{z}^-)$. Since the process Z^r is transient to minus infinity, it follows that $\lim_{\ell \to \infty} P_{\hat{z}}(\tau < \infty) = 0$ and we conclude the probability of the first event tends to 0.

The probability of the second event is bounded by the probability that Q^r hits the plane $x_1 = 0$ and then climbs back to F before hitting $\vec{0}$. As just remarked, the process Q^r hits the plane $x_1 = 0$ according to a limiting distribution d. Take any point $\hat{z}$ of the form $(0, \vec{z}^-)$ according to distribution d. We know from Proposition 3.3.4 that, starting from $\vec{0}$, the probability of hitting F before returning to $\vec{0}$ tends to 0 as $\ell \to \infty$. Since there is a nonzero probability of going from $\vec{0}$ to $\hat{z}$ independent of ℓ, it follows that the probability of starting from $\hat{z}$ and hitting F before hitting $\vec{0}$ tends to 0. Otherwise, from $\vec{0}$, Q^r could, with some fixed probability, first go to $\hat{z}$ without hitting $\vec{0}$ and then go to F, again without hitting $\vec{0}$. This is a contradiction.

We conclude the probabilities of both events tend to 0 as $\ell \to \infty$ and this gives the result. ■

We now return to the original Jackson network Q on S to confirm the heuristic expression (2.25) developed in Section 2.3.

Theorem 3.4.2 *For $\vec{y} \in \partial F$,*

$$\lim_{\ell\to\infty} P_{\vec{0}}(Q(\tau_\ell) = \vec{y}) = \frac{1}{D}\prod_{i=2}^{m}(\frac{a_1}{a_i})^{y_i} e(\vec{y}^-)$$

where $D = \sum_{\vec{w}\in\partial F}\prod_{i=2}^{m}(\frac{a_1}{a_i})^{w_i} e(\vec{w}^-)$.

Proof: We have already shown (see (3.16)) that, for $\vec{y} \in \partial F^\infty$,

$$\begin{aligned} e(\vec{y}^-) &= \lim_{\ell\to\infty} P_{\dot{z}}(Z^t(\tau_\ell) = \vec{y}) \\ &= \frac{1}{q}\sum_{i=1}^{n} \bar{\lambda}_i \rho_i^{-1} \lambda \rho^\infty (T_i.\vec{y}) \pi^t(\vec{y}^-). \end{aligned}$$

On the other hand, from Lemma 3.3.5 we know that the hitting distribution of the Jackson network on $\vec{y} \in \partial F$ is given exactly by:

$$P_{\vec{0}}(Q(\tau_\ell) = \vec{y}) = \frac{1}{E}\prod_{i=2}^{m}\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^{m}\bar{\lambda}_i\rho_i^{-1}\rho_\ell(T_i.\vec{y})$$

where E is a normalizing constant. From the above lemma it follows that for $\vec{y} \in \partial F \subset \partial F^\infty$,

$$\lim_{\ell\to\infty} P_{\vec{0}}(Q(\tau_\ell) = \vec{y}) = \frac{1}{E}\prod_{i=2}^{m}\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^{m}\bar{\lambda}_i\rho_i^{-1}\rho^\infty(T_i.\vec{y})$$

since ρ^∞ and ρ_ℓ are asymptotically the same. Now

$$\pi^t(\vec{y}^-) = \prod_{i=2}^{m}(1-\rho_i^t)(\rho_i^t)^{y_i}$$

so we can express

$$\begin{aligned} e(\vec{y}^-) &= \frac{1}{q}\sum_{i=1}^{m}\bar{\lambda}_i\rho_i^{-1}\lambda\rho^\infty(T_i.\vec{y})\pi^t(\vec{y}^-) \\ &= \left[\frac{1}{E}\prod_{i=2}^{m}\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^{m}\bar{\lambda}_i\rho_i^{-1}\rho^\infty(T_i.\vec{y})\right]\frac{\lambda E}{q}\left(\prod_{i=2}^{m}(1-\rho_i^t)(\rho_i^t)^{y_i}\right)\prod_{i=2}^{m}(\frac{\rho_1}{\rho_i})^{y_i} \\ &= \left[\frac{1}{E}\prod_{i=2}^{m}\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^{m}\bar{\lambda}_i\rho_i^{-1}\rho^\infty(T_i.\vec{y})\right]\frac{\lambda E}{q}\prod_{i=2}^{m}(1-\rho_i^t)\prod_{i=2}^{m}(\frac{a_i}{a_1})^{y_i}. \end{aligned}$$

Putting these results together, it follows that the hitting distribution $P_{\vec{0}}(Q(\tau_\ell) = \vec{y})$ is asymptotically proportional to $e(\vec{y}^-)\prod_{i=2}^{m}(\frac{a_1}{a_i})^{y_i}$ and this gives the result. ∎

We also need the following technical results for Section 3.9. Recalling from (3.12) that

$$\psi(\vec{z}^-) \leq P^\#(X_0^\# = \vec{z}^-) = \bar{\pi}(\vec{z}^-),$$

we have

$$e(\vec{z}^-) \leq c\bar{\pi}(\vec{z}^-) \tag{3.17}$$

for some constant c. We shall also need the inequality

$$\bar{\pi}(\vec{z}^-) \leq ce(\vec{z}^-) \tag{3.18}$$

for some constant c which we assume to be true without proof. From (3.15) which we may write as

$$e(\vec{z}^-) = \frac{1}{q}\sum_{i=1}^{m} \frac{\bar{\lambda}_i \rho_i^{-1}}{\lambda(\vec{y}^-)} \rho^\infty(T_{i.}\vec{y})\bar{\pi}(\vec{z}^-)$$

or

$$\left(\sum_{i=1}^{m} \frac{\bar{\lambda}_i \rho_i^{-1}}{\lambda(\vec{y}^-)} \rho^\infty(T_{i.}\vec{y})\right)^{-1} q = \frac{\bar{\pi}(\vec{z}^-)}{e(\vec{z}^-)},$$

this is equivalent to assuming that $\rho^\infty(\vec{z}) \geq c > 0$ for all $\vec{z} \in B$.

Finally, if we denote by $\mathcal{K}$ the kernel of W_n, we establish for any $\vec{z}^-$, $\vec{y}^- \in S^-$ and integer ℓ the inequality

$$\begin{aligned} \frac{\mathcal{K}^\ell_{\vec{z}^-\vec{y}^-}}{e(\vec{y}^-)} &= \frac{e(\vec{z}^-)}{e(\vec{z}^-)} \frac{\mathcal{K}^\ell_{\vec{z}^-\vec{y}^-}}{e(\vec{y}^-)} \\ &= \frac{1}{e(\vec{z}^-)} (\mathcal{K}^\ell_{\vec{y}^-\vec{z}^-})^* \\ &\leq \frac{1}{e(\vec{z}^-)}, \end{aligned} \tag{3.19}$$

where $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$.

3.5 Asymptotics of the hitting time

Iscoe and McDonald (1994a) proposed that the mean hitting time τ of the forbidden set $F = \{\vec{x} \in S : \sum_{i=1}^m x_i \geq \ell\}$ may be approximated by $[\Lambda(B)]^{-1}$, where $\Lambda(B)$ is the principal eigenvalue of the Dirichlet problem

$$\begin{aligned} L^B \varphi^B &= \Lambda(B)\varphi^B \qquad && \text{in } B := F^c, \\ \varphi^B &= 0 && \text{in } F, \end{aligned}$$

and $-L^B$ is the infinitesimal generator of the Markov jump process killed off B. The error bounds given for this approximation are quite complex and require some definitions, notation and calculations which are presented below. The precision of $[\Lambda(B)]^{-1}$ as an estimator for $E\tau$ is given at the end of the section.

The mean hitting time is approximated by performing calculations in the $L^2(S,\pi) \equiv L^2(\pi)$ space defined by

Definition 3.5.1 *A function u on S belongs to $L^2(S,\pi)$ if*

$$\sum_{\vec{x}\in S} u^2(\vec{x})\pi(\vec{x}) < \infty.$$

The following inner product on $L^2(\pi)$ is used.

Definition 3.5.2 *Let u and v be two functions on S. Define the inner product of the two functions to be*

$$< u, v >_\pi := \sum_{\vec{x}\in S} u(\vec{x})v(\vec{x})\pi(\vec{x}).$$

The vector norm in $L^2(\pi)$ associated with this inner product is

$$||u||_\pi = \sqrt{< u, u >_\pi} = \sqrt{\sum_{\vec{x}\in S} u^2(\vec{x})\pi(\vec{x})}.$$

The Markov process Q can be observed on the set $B = \{\vec{x} \in B : \sum_{i=1}^{m} x_i < \ell\}$. The restriction of Q to B requires the renormalization of the measures. Assuming $\pi(B) > 0$ (which we do for the remainder of this work), we denote the restriction of π to B, $\pi|B$, by the probability

$$\hat{\pi} \equiv \hat{\pi}^B := [\pi(B)]^{-1}(\pi|B).$$

The definition of the space, inner product and norm for the process killed off B then follows:

$$L^2(\hat{\pi}) \equiv L^2(B,\hat{\pi}), \quad < \cdot, \cdot >_{\hat{\pi}} \equiv < \cdot, \cdot >_{L^2(\hat{\pi})}, \quad || \cdot ||_{\hat{\pi}} \equiv || \cdot ||_{L^2(\hat{\pi})}.$$

Let $-L$ denote the infinitesimal generator of the Jackson network on $L^2(\pi)$ and $J(\vec{x}, \cdot)$ be the associated transition rate kernel. Since the transition rates of the network are constant (and supposed finite), $J(\vec{x}, \cdot)$ is uniformly bounded. Define

$$M := \sup_{\vec{x}} J(\vec{x}, \{\vec{x}\}^c) = \sum_{i=1}^{m}[\bar{\lambda}_i + \mu_i]. \tag{3.20}$$

The flows out and into of the set B can be described by the killing and resuscitation rates defined respectively by

$$K^B(\vec{x}) := J(\vec{x}, F) \quad \text{for } \vec{x} \in B \tag{3.21}$$

and

$$R^B(\vec{x}) := [\pi(\vec{x})]^{-1} \sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, \vec{x}) \quad \text{for } \vec{x} \in B.$$

Note that, in equilibrium, the rates of flow into and out of B coincide so that

$$\begin{aligned} \sum_{\vec{x} \in B} K^B(\vec{x})\pi(\vec{x}) &= \sum_{\vec{x} \in B} J(\vec{x}, F)\pi(\vec{x}) \quad \text{(flow out of } B) \\ &= \sum_{\vec{y} \in F} J(\vec{y}, B)\pi(\vec{y}) \quad \text{(flow into } B) \\ &= \sum_{\vec{x} \in B} [\pi(\vec{x})]^{-1} \sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, \vec{x})\pi(\vec{x}) \\ &= \sum_{\vec{x} \in B} R^B(\vec{x})\pi(\vec{x}), \end{aligned} \tag{3.22}$$

i.e., the mean killing and resuscitation rates are equal.

Consider now the three constants

$$\bar{\kappa} \equiv \bar{\kappa}^B := \sum_{\vec{x} \in B} K^B(\vec{x})\hat{\pi}(\vec{x}), \quad \kappa_1 := ||K^B - \bar{\kappa}||_{\hat{\pi}}, \quad \kappa_2 := ||R^B - \bar{\kappa}||_{\hat{\pi}} \tag{3.23}$$

which will be used in the error expression at the end of the section. Clearly, $\bar{\kappa}$ is the mean killing rate and thus, by (3.22), also the mean resuscitation rate with respect to the probability $\hat{\pi}$ on B. Therefore, κ_1 and κ_2 represent respectively the standard deviations of the killing and resuscitation rates with respect to the same probability.

We can give upper bounds for these constants. First,

$$\begin{aligned} \bar{\kappa} &= \sum_{\vec{x} \in B} K^B(\vec{x})\hat{\pi}(\vec{x}) \\ &= \sum_{\vec{x}: \sum_{i=1}^m x_i < \ell} J(\vec{x}, F)\hat{\pi}(\vec{x}) \\ &= \sum_{\vec{x}: \sum_{i=1}^m x_i = \ell - 1} \left(\sum_{i=1}^m \bar{\lambda}_i\right)\hat{\pi}(\vec{x}) \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i\right) \sum_{\vec{x}: \sum_{i=1}^m x_i = \ell - 1} \prod_{i=1}^m \frac{(1 - \rho_i)\rho_i^{x_i}}{\pi(B)} \end{aligned}$$

$$
\begin{aligned}
&< (\sum_{i=1}^{m} \bar{\lambda}_i) \prod_{i=1}^{m} \frac{(1-\rho_i)}{\pi(B)} \sum_{\vec{x}: \sum_{i=1}^{m} x_i = \ell - 1} \rho_1^{\sum_{i=1}^{m} x_i} \quad \text{by (2.5)} \\
&\leq \pi(B)^{-1} (\sum_{i=1}^{m} \bar{\lambda}_i) \prod_{i=1}^{m} (1-\rho_i) \ell^{m-1} \rho_1^{\ell-1}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\kappa_1^2 &= ||K^B - \bar{\kappa}||_{\hat{\pi}}^2 \\
&= \sum_{\vec{x} \in B} [K^B - \bar{\kappa}]^2(\vec{x}) \hat{\pi}(\vec{x}) \\
&= \sum_{\vec{x} \in B} [K^B(\vec{x})]^2 \hat{\pi}(\vec{x}) - 2\bar{\kappa} \sum_{\vec{x} \in B} K^B(\vec{x}) \hat{\pi}(\vec{x}) + \bar{\kappa}^2 \\
&= \sum_{\vec{x} \in B} [K^B(\vec{x})]^2 \hat{\pi}(\vec{x}) - \bar{\kappa}^2 \\
&\leq \sum_{\vec{x} \in B} [K^B(\vec{x})]^2 \hat{\pi}(\vec{x}) \\
&\leq M \sum_{\vec{x} \in B} K^B(\vec{x}) \hat{\pi}(\vec{x}) \quad \text{by (3.20) and (3.21)} \\
&= M\bar{\kappa}
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa_2^2 &= ||R^B - \bar{\kappa}||_{\hat{\pi}}^2 \\
&= \sum_{\vec{x} \in B} [R^B - \bar{\kappa}]^2(\vec{x}) \hat{\pi}(\vec{x}) \\
&= \sum_{\vec{x} \in B} [R^B(\vec{x})]^2 \hat{\pi}(\vec{x}) - 2\bar{\kappa} \sum_{\vec{x} \in B} R^B(\vec{x}) \hat{\pi}(\vec{x}) + \bar{\kappa}^2 \\
&= \sum_{\vec{x} \in B} [R^B(\vec{x})]^2 \hat{\pi}(\vec{x}) - \bar{\kappa}^2 \quad \text{by (3.22) and (3.23)} \\
&\leq \sum_{\vec{x} \in B} [R^B(\vec{x})]^2 \hat{\pi}(\vec{x}) \\
&= \sum_{\vec{x} \in B} [\pi(\vec{x})]^{-2} [\sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, \vec{x})]^2 \hat{\pi}(\vec{x}) \\
&\leq \sum_{\vec{x} \in B} [\pi(\vec{x})]^{-2} [\sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, \vec{x})]^2 [\hat{\pi}(\vec{x})]^2 \quad \text{since } \hat{\pi}(\vec{x}) \leq 1 \\
&\leq M \frac{\pi(F)}{[\pi(B)]^2} \sum_{\vec{x} \in B} \sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, \vec{x}) \\
&= M \frac{\pi(F)}{[\pi(B)]^2} \sum_{\vec{y} \in F} \pi(\vec{y}) J(\vec{y}, B)
\end{aligned}
$$

$$
\begin{aligned}
&= M\frac{\pi(F)}{[\pi(B)]^2}\sum_{\vec{x}\in B}\pi(\vec{x})J(\vec{x},F) \quad \text{(flows into and out of } B \text{ coincide)} \\
&= M\frac{\pi(F)}{\pi(B)}\sum_{\vec{x}\in B}\hat{\pi}(\vec{x})K^B(\vec{x}) \\
&= \frac{\pi(F)}{\pi(B)}M\bar{\kappa}
\end{aligned}
$$

where we have used in the last inequality the fact that $\sum_{\vec{y}\in F}\pi(\vec{y})J(\vec{y},\vec{x}) \le \sum_{\vec{y}\in F}\pi(\vec{y})M = M\pi(F)$.

To summarize,

$$
\begin{aligned}
&\bar{\kappa} < \pi(B)^{-1}(\textstyle\sum_{i=1}^m \bar{\lambda}_i)\prod_{i=1}^m(1-\rho_i)\ell^{m-1}\rho_1^{\ell-1}, \\
&\kappa_1 \le \sqrt{M\bar{\kappa}} \\
\text{and} \quad &\kappa_2 \le \sqrt{\tfrac{\pi(F)}{\pi(B)}M\bar{\kappa}}
\end{aligned}
\tag{3.24}
$$

where we note that $\pi(F) \to 0$ and $\pi(B) \to 1$ as $\ell \to \infty$.

Before we proceed to the evaluation of the accuracy of $\Lambda(B)$, we need one more definition.

Definition 3.5.3 *For $-L$ being the infinitesimal generator of the Jackson network on $L^2(\pi)$,*

$$Gap(L) := \inf\{(f, Lf)_\pi : ||f||_\pi = 1, (f,1)_\pi = 0\}.$$

In Iscoe and McDonald (1994b), it was shown that $Gap(L) > 0$.

Finally, from Corollary 2.10 of Iscoe and McDonald (1994a) we obtain

Theorem 3.5.4 *If $\pi(F)$ is sufficiently small, then*

$$|E_{\hat{\pi}}\tau - [\Lambda(B)]^{-1}| \le \beta(B)/\Lambda(B)$$

where

$$\beta(B) := \frac{4}{(Gap(L)-\bar{\kappa})^2 - 4\kappa_1\kappa_2}\left[1 + \frac{\sqrt{(Gap(L)-\bar{\kappa})^2 + 4\kappa_2^2}}{Gap(L)-\bar{\kappa}}\right]\kappa_1\kappa_2.$$

Conditions ensuring that $\pi(F)$ is sufficiently small are given in Theorem 2.8 in Iscoe and McDonald (1994a); $\bar{\kappa}$, κ_1, and κ_2 are described by (3.23) and (3.24). A similar result holds for an arbitrary starting measure π_0.

From (3.24) and since $Gap(L) > 0$, we see that the size of the relative error tends to 0 as $\ell \to \infty$.

3.6 An approximation b for $\Lambda(B)$

In Section 3.5, the expected time required to hit the forbidden state F was expressed in terms of $\Lambda(B)$, the Perron-Frobenius eigenvalue associated with the eigenvector φ^B where

$$\begin{array}{ll} L^B\varphi^B = \Lambda(B)\varphi^B & \text{in } B := F^c, \\ \varphi^B = 0 & \text{in } F, \end{array} \tag{3.25}$$

$B = \{\vec{x} \in S : \sum_{i=1}^m x_i < \ell\}$ and $F = \{\vec{x} \in S : \sum_{i=1}^m x_i \geq \ell\}$. The following theorem, which is a restriction to bounded operators of the Theorem 3.5 in Stewart (1971), can be used to construct an approximation of the Perron-Frobenius pair $(\Lambda(B), \varphi^B)$. It is stated as in Iscoe and McDonald (1994a).

Theorem 3.6.1 *Let A be a bounded linear operator on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{X}$ be a subspace and denote its orthogonal complement by $\mathcal{Y}$. Also denote by X and Y the injections of $\mathcal{X}$ and $\mathcal{Y}$, respectively, into $\mathcal{H}$ and set*

$$B = X^*AX, \qquad H = X^*AY,$$

$$G = Y^*AX, \qquad C = Y^*AY,$$

where * denotes the adjoint. Set $\gamma = ||G||$, $\eta = ||H||$ and $\delta = ||T^{-1}||^{-1}$, where T is the linear operator on the space $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ of bounded linear transformations from $\mathcal{X}$ to $\mathcal{Y}$ given by $TP = PB - CP$, and is assumed to be bijective. If $\gamma\eta/\delta^2 < 1/4$, then there is a $P \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that the range of $X + YP$ is an invariant subspace of A and $||P|| \leq 2\gamma/\delta$. Moreover, the spectrum of A is the disjoint union

$$\sigma(A) = \sigma(B + HP) \cup \sigma(C - PH) \equiv \sigma_1 \cup \sigma_2, \tag{3.26}$$

and $dist(\sigma_1, \sigma_2) > [\delta^2 - 4\gamma\eta]/\delta$.

Our application of Stewart's theorem occurs in the simple case where $\dim \mathcal{X} = 1$ and $A = L^B$ in $L^2(\hat{\pi})$. Let the vector $f \in \mathcal{X}$, $f \neq 0$, generate $\mathcal{X}$ so that every element in $\mathcal{X}$ can be expressed as rf, $r \in R$. Then, every element $h \in \mathcal{H}$ can be written as

$$h = \frac{< f, h >_{\hat{\pi}}}{< f, f >_{\hat{\pi}}} f \oplus \phi, \tag{3.27}$$

i.e., as the sum of its projection on f and a vector ϕ orthogonal to f. This last statement can be easily verified:

$$\begin{aligned} <\phi, f>_{\hat{\pi}} &= <h - \frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} f, f>_{\hat{\pi}} \\ &= <h,f>_{\hat{\pi}} - \frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} <f,f>_{\hat{\pi}} \\ &= 0. \end{aligned} \tag{3.28}$$

We can also write $h = X^*h + Y^*h$ since the two adjoints X^* and Y^* of X and Y, respectively, take the form

$$X^*h = \frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} f \quad \text{and} \quad Y^*h = \phi. \tag{3.29}$$

Indeed, by the properties of the inner product $<\cdot,\cdot>_{\hat{\pi}}$, (3.27) and (3.28),

$$\begin{aligned} <h, Xrf>_{\hat{\pi}} &= <rf, h>_{\hat{\pi}} \\ &= \frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} <rf,f>_{\hat{\pi}} + r <f,\phi>_{\hat{\pi}} \\ &= <\frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} f, rf>_{\hat{\pi}} \\ &= <X^*h, rf>_{\hat{\pi}} \end{aligned}$$

so that X^* is the adjoint of X and, for $\psi \in \mathcal{Y}$,

$$\begin{aligned} <h, Y\psi>_{\hat{\pi}} &= <h,\psi>_{\hat{\pi}} \\ &= \frac{<f,h>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} <f,\psi>_{\hat{\pi}} + <\phi,\psi>_{\hat{\pi}} \\ &= <\phi,\psi>_{\hat{\pi}} \\ &= <Y^*h,\psi>_{\hat{\pi}} \end{aligned}$$

which confirms that Y^* is the adjoint of Y.

We can now write

$$\begin{aligned} Brf &= X^*L^BXrf \\ &= rX^*L^Bf \\ &= \frac{<f,L^Bf>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}} rf \end{aligned}$$

and

$$
\begin{aligned}
(B+HP)rf &= X^* L^B X r f + X^* L^B Y P r f \\
&= \frac{< f, L^B f >_{\bar{*}}}{< f, f >_{\bar{*}}} r f + r X^* L^B P f \\
&= \frac{< f, L^B f >_{\bar{*}}}{< f, f >_{\bar{*}}} r f + \frac{< f, L^B P f >_{\bar{*}}}{< f, f >_{\bar{*}}} r f.
\end{aligned}
$$

Taking $b = \frac{<f,L^B f>_*}{<f,f>_*}$ and $\Lambda(B) = b + \frac{<f,L^B Pf>_*}{<f,f>_*}$, we conclude $B = bI$ and $B + HP = \Lambda(B)I$, where I denotes the identity matrix; $||I||_{\bar{*}} = 1$. Therefore $\sigma(B+HP) = \sigma(\Lambda(B)I) = \Lambda(B)$. We shall assume f is such that $\gamma\eta/\delta^2 < 1/4$ so, by (3.26), $\Lambda(B)$ is an isolated eigenvalue of L^B. Moreover,

$$
\begin{aligned}
|\Lambda(B) - b| &= |\Lambda(B) - b|\,||I||_{\bar{*}} = ||(\Lambda(B) - b)I||_{\bar{*}} \\
&= ||B + HP - B||_{\bar{*}} = ||HP||_{\bar{*}} \\
&\leq ||H||_{\bar{*}}||P||_{\bar{*}} \\
&\leq \frac{2\gamma\eta}{\delta}
\end{aligned}
$$

by Theorem 3.6.1. On the other hand, since the range of $X + YP$ is an invariant subspace of L^B, it follows that

$$L^B(X+YP)f = L^B(f + YPf) = \lambda(f + YPf) \tag{3.30}$$

for some λ, i.e., $\varphi^B := f \oplus YPf$ is an eigenvector of L^B with eigenvalue λ. However, note that

$$X^* \lambda \varphi^B = \lambda \frac{< f, \varphi^B >_{\bar{*}}}{< f, f >_{\bar{*}}} f = \frac{\lambda}{< f, f >_{\bar{*}}}[< f, f >_{\bar{*}} + < f, YPf >_{\bar{*}}] f = \lambda f \tag{3.31}$$

since $< f, YPf >_{\bar{*}} = 0$, and that

$$X^* L^B (X + YP) f = (X^* L^B X + X^* L^B Y P) f = (B + HP) f = \Lambda(B) f. \tag{3.32}$$

By (3.30), (3.31) and (3.32) must agree; this implies $\lambda = \Lambda(B)$. We can pick f such that b, thus $\Lambda(B)$, is very small. That, in addition to the fact that $\Lambda(B)$ is an isolated eigenvalue of L^B, indicates that $(\Lambda(B), \varphi^B)$ is the Perron-Frobenius pair of L^B. Finally, note that

$$||\varphi^B - f||_{\bar{*}} = ||YPf||_{\bar{*}} \leq ||Y||_{\bar{*}}\,||P||_{\bar{*}}\,||f||_{\bar{*}} \leq 2\frac{\gamma}{\delta}||f||_{\bar{*}}. \tag{3.33}$$

To complete our application of Theorem 3.6.1, we evaluate γ, η and δ.

$$\begin{aligned}
\gamma &= \|G\|_{\hat{\pi}} = \|Y^* L^B X\|_{\hat{\pi}} \\
&= \sup\{\|Y^* L^B X r f\|_{\hat{\pi}} : \|rf\|_{\hat{\pi}} = 1\} \\
&= \sup\{r\|Y^* L^B f\|_{\hat{\pi}} : r\|f\|_{\hat{\pi}} = 1\} \\
&= \sup\left\{r\left\|L^B f - \frac{< f, L^B f >_{\hat{\pi}}}{< f, f >_{\hat{\pi}}} f\right\|_{\hat{\pi}} : r\|f\|_{\hat{\pi}} = 1\right\} \quad \text{by (3.27) and (3.29)} \\
&= \sup\{r\|L^B f - bf\|_{\hat{\pi}} : r\|f\|_{\hat{\pi}} = 1\} \\
&= \frac{\|L^B f - bf\|_{\hat{\pi}}}{\|f\|_{\hat{\pi}}} \\
&\le \frac{\|L^B f\|_{\hat{\pi}}}{\|f\|_{\hat{\pi}}} + b, \qquad (3.34)
\end{aligned}$$

$$\begin{aligned}
\eta &= \|H\|_{\hat{\pi}} = \|X^* L^B Y\|_{\hat{\pi}} \\
&= \sup\{\|X^* L^B Y \phi\|_{\hat{\pi}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\} \\
&= \sup\{\|X^* L^B \phi\|_{\hat{\pi}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\} \\
&= \sup\left\{\left\|\frac{< f, L^B \phi >_{\hat{\pi}}}{< f, f >_{\hat{\pi}}} f\right\|_{\hat{\pi}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\right\} \\
&= \sup\left\{\frac{|< f, L^B \phi >_{\hat{\pi}}|}{< f, f >_{\hat{\pi}}}\|f\|_{\hat{\pi}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\right\} \\
&= \sup\left\{\frac{|< f, L^B \phi >_{\pi}|}{\|f\|_{\hat{\pi}}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\right\} \\
&= \sup\left\{\frac{|< L^{B*} f, \phi >_{\pi}|}{\|f\|_{\hat{\pi}}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\right\} \\
&= \sup\left\{\frac{|< L^{B*}(f-1), \phi >_{\pi}|}{\|f\|_{\hat{\pi}}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0\right\} \\
&\le \frac{\|L^B\|_{\pi}\|f-1\|_{\pi}}{\|f\|_{\hat{\pi}}}. \qquad (3.35)
\end{aligned}$$

From Iscoe and McDonald (1994a), it follows that $\delta \to \text{Gap}(L)$ as $\pi(B) \to 1$ (i.e., as $\ell \to \infty$). It was shown, in Iscoe and McDonald (1994b) that $\text{Gap}(L) > 0$ for generators $-L$ of Jackson networks.

Combining Theorem 3.6.1 and the results above, we obtain the following corollary.

Corollary 3.6.2 *The Perron-Frobenius pair* $(\Lambda(B), \varphi^B)$ *satisfies*

$$\begin{array}{ll} L^B\varphi^B = \Lambda(B)\varphi^B & \textit{in } B, \\ \varphi^B = 0 & \textit{in } F. \end{array}$$

Let $0 \neq f \in L^2(\hat{\pi})$ *and define* $b := \frac{<f, L^B f>_{\hat{\pi}}}{<f,f>_{\hat{\pi}}}$,

$$\begin{aligned} \gamma &:= \frac{\|L^B f - bf\|_{\hat{\pi}}}{\|f\|_{\hat{\pi}}} \\ \textit{and } \eta &:= \sup\left\{ \frac{|< f, L^B\phi >_{\hat{\pi}}|}{\|f\|_{\hat{\pi}}} : \|\phi\|_{\hat{\pi}} = 1, < \phi, f >_{\hat{\pi}} = 0 \right\}. \end{aligned}$$

If $b < Gap(L)$, $\gamma\eta/[Gap(L)]^2 < 1/4$ *and* $\pi(B)$ *is large enough, then*

(i) $\Lambda(B)$ *is an isolated eigenvalue of* L^B; $|\Lambda(B) - b| \leq 2\gamma\eta/Gap(L)$.

(ii) $\|\varphi^B - f\|_{\hat{\pi}} \leq 2\gamma\|f\|_{\hat{\pi}}/Gap(L)$.

Corollary 3.6.2 provides a measure of the accuracy of an approximation b of $\Lambda(B)$ which greatly depends on the precision of the initial guess f for φ^B. We construct our guess f by considering the system (3.25) with f in place of φ^B. In order to simplify (3.25), we can use $\Lambda(B) \approx 0$ to build f since $\Lambda(B)$ is small, but then $L^B f = 0$ admits the unique solution $f = 0$ (L^B has an inverse – see Appendix A). We can however replace $-L^B$ by the generator $-L^\infty$ of the extended process Z defined in Section 2.2 to avoid this problem.

Consider the function f^∞ as a guess for φ^B and say

$$\begin{aligned} L^\infty f^\infty(\vec{x}) = 0 &\quad \text{if } \vec{x} \in B^\infty, \\ f^\infty(\vec{x}) = 0 &\quad \text{if } \vec{x} \in F^\infty, \qquad (3.36) \\ \lim_{x_1 \to -\infty} f^\infty(\vec{x}) = 1 &\quad \text{if } \vec{x} \in B^\infty, \end{aligned}$$

where $B^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i < \ell\}$ and $F^\infty = \{\vec{x} \in S^\infty : \sum_{i=1}^m x_i \geq \ell\}$. The Dirichlet problem (3.36) has the probabilistic solution

$$f^\infty(\vec{x}) = P(\text{starting from } \vec{x}, Z \text{ never hits } F^\infty).$$

3.7 An application of the twist

The twist developed in Section 2.2 can be used to solve for f^∞. For some function k on S^∞ and the constants $a_1, a_2, \ldots, a_m$ we can express

$$f^\infty(\vec{x}) = 1 - (\prod_{i=1}^m a_i^{x_i})k(\vec{x}), \quad \vec{x} \in S^\infty. \tag{3.37}$$

Then

$$\begin{aligned} L^\infty f^\infty(\vec{x}) &= L^\infty(1 - \prod_{i=1}^m a_i^{x_i} k(\vec{x})) \\ &= L^\infty 1 - L^\infty \prod_{i=1}^m a_i^{x_i} k(\vec{x}) \\ &= -L^\infty \prod_{i=1}^m a_i^{x_i} k(\vec{x}) \end{aligned}$$

so that, by (2.6),

$$\begin{aligned} L^t k(\vec{x}) &:= \frac{1}{\prod_{i=1}^m a_i^{x_i}} L^\infty \prod_{i=1}^m a_i^{x_i} k(\vec{x}) \\ &= -\frac{1}{\prod_{i=1}^m a_i^{x_i}} L^\infty f^\infty(\vec{x}). \end{aligned} \tag{3.38}$$

Combining (3.36), (3.37) and (3.38), we can write

$$\begin{aligned} L^t k(\vec{x}) = 0 \quad &\text{if} \quad \vec{x} \in B^\infty, \\ k(\vec{x}) = \prod_{i=1}^m a_i^{-x_i} \quad &\text{if} \quad \vec{x} \in F^\infty. \end{aligned}$$

Now define

$$k^\ell := a_1^\ell k = \rho_1^{-\ell} k. \tag{3.39}$$

Then

$$\begin{aligned} L^t k^\ell(\vec{x}) = 0 \quad &\text{if} \quad \vec{x} \in B^\infty, \\ k^\ell(\vec{x}) = a_1^\ell \prod_{i=1}^m a_i^{-x_i} =: s(\vec{x}) \quad &\text{if} \quad \vec{x} \in F^\infty. \end{aligned} \tag{3.40}$$

Since we recognize in (3.40) a Dirichlet problem, we can express $k^\ell(\vec{x})$ for $\vec{x} \in B^\infty$ as:

$$k^\ell(\vec{x}) = E_{\vec{x}}^t s(Z^t(\tau)) = \sum_{\vec{y} \in F^\infty} s(\vec{y}) P_{\vec{x}}^t(Z^t(\tau) = \vec{y})$$

where $E^\ell_{\vec{x}}$ and $P^\ell_{\vec{x}}$ denote respectively the expectation and the probability with respect to the twisted process given we start at $\vec{x} \in B^\infty$.

Noting, however, that between any two events - arrivals, departures, transfers - the sum $\sum_{i=1}^m x_i$ varies by at most 1, it is clear that, starting in B^∞, F^∞ is hit on its boundary $\partial F^\infty = \{\vec{x} \in F^\infty : \sum_{i=1}^m x_i = \ell\}$. Every $\vec{x}$ on the boundary of F^∞ can be written as $\vec{x} = (\ell - \sum_{i=2}^m x_i, x_2, \ldots, x_m)$ and so

$$\begin{aligned} s(\vec{x}) &= s(\ell - \sum_{i=2}^m x_i, x_2, \ldots, x_m) \\ &= a_1^{\ell-\ell+\sum_{i=2}^m x_i} a_2^{-x_2} \cdots a_m^{-x_m} \\ &= a_1^{\sum_{i=2}^m x_i} a_2^{-x_2} \cdots a_m^{-x_m} \\ &= \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{x_i} \end{aligned} \tag{3.41}$$

is independent of ℓ and $k^\ell(\vec{x})$, rewritten as

$$k^\ell(\vec{x}) = \sum_{\vec{y} \in \partial F^\infty} s(\vec{y}) P^\ell_{\vec{x}}(Z^\ell(\tau) = \vec{y}), \tag{3.42}$$

depends on ℓ only through the hitting probabilities of ∂F^∞. This is a very desirable property that justifies the somewhat elaborate expression of f^∞ in (3.37). In fact, the definitions of k and k^ℓ were motivated by that result.

Now recall from (3.14) that

$$\lim_{\ell \to \infty} P_{\vec{z}}(Z^\ell(\tau) = \vec{y}) = e(\vec{y}^-), \quad \vec{y} \in \partial F^\infty$$

for all $\vec{z}^- \in S^-$ and define

$$\kappa := \sum_{\vec{y} \in \partial F^\infty} s(\vec{y}) e(\vec{y}^-). \tag{3.43}$$

Replacing $k(\vec{x})$ by $\rho_1^\ell \kappa$ in (3.37), we obtain a new function

$$f^\infty_\kappa(\vec{x}) := 1 - (\prod_{i=1}^m a_i^{x_i}) \rho_1^\ell \kappa.$$

For simplification, the function f^∞_κ will be used instead of the function f^∞ in later calculations.

3.8 Calculation of b for pooled queues

By definition,

$$b := \frac{< f^\infty, L^B f^\infty >_{\hat{\pi}}}{< f^\infty, f^\infty >_{\hat{\pi}}} = \frac{\sum_{\vec{x}\in B} f^\infty(\vec{x}) L^B f^\infty(\vec{x})\hat{\pi}(\vec{x})}{||f^\infty||^2_{\hat{\pi}}}$$

where $\hat{\pi}(\vec{x}) := \frac{\pi(\vec{x})\chi\{\vec{x}\in B\}}{\pi(B)}$, with $\pi(B) = \sum_{\vec{x}\in B}\pi(\vec{x})$, is the stationary measure of the process on B. Noting that the extended process is identical to the original Jackson network except for the movements past the $x_1 = 0$ hyperplane, we have that for $\vec{x} \in B \subset B^\infty$ with $x_1 \neq 0$,

$$L^B f^\infty(\vec{x}) = L^\infty f^\infty(\vec{x}) = 0$$

by (3.36). Hence we only need to consider the $\vec{x}$'s in the set $B_0 = \{\vec{x} \in B : x_1 = 0\}$.

Now, for $\vec{x}_0 \in B_0$,

$$\begin{aligned}
&L^B f^\infty(\vec{x}_0)\\
&= L^\infty f^\infty(\vec{x}_0) + \mu_1[r_{1.}(f^\infty(T_{1.}\vec{x}_0) - f^\infty(\vec{x}_0)) + \sum_{j=2}^m r_{1j}(f^\infty(T_{1j}\vec{x}_0) - f^\infty(\vec{x}_0))]\\
&= \mu_1(\prod_{i=2}^m a_i^{x_i})[r_{1.}(k(\vec{x}_0) - a_1^{-1}k(T_{1.}\vec{x}_0)) + \sum_{j=2}^m r_{1j}(k(\vec{x}_0) - a_j a_1^{-1}k(T_{1j}\vec{x}_0))] \text{ by (3.36) and (3.37)}\\
&= \mu_1(\prod_{i=2}^m a_i^{x_i})\rho_1^\ell[k^\ell(\vec{x}_0) - r_{1.}a_1^{-1}k^\ell(T_{1.}\vec{x}_0) - \sum_{j=2}^m r_{1j}a_j a_1^{-1}k^\ell(T_{1j}\vec{x}_0)] \quad \text{by (3.39).}
\end{aligned}$$

Similarly, we can write

$$f^\infty(\vec{x}_0) = 1 - \prod_{i=2}^m a_i^{x_i}\rho_1^\ell k^\ell(\vec{x}_0)$$

and thus calculate

$$\begin{aligned}
b &:= < f^\infty, L^B f^\infty >_{\hat{\pi}} / < f^\infty, f^\infty >_{\hat{\pi}}\\
&= \sum_{\vec{x}_0\in B_0} f^\infty(\vec{x}_0)L^B f^\infty(\vec{x}_0)\hat{\pi}(\vec{x}_0)/||f^\infty||^2_{\hat{\pi}}\\
&= \frac{1}{||f^\infty||^2_{\hat{\pi}}}\sum_{x_2=0}^{\ell-1}\sum_{x_3=0}^{\ell-1-x_2}\sum_{x_4=0}^{\ell-1-x_2-x_3}\cdots\sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1}x_i} f^\infty(\vec{x}_0)L^B f^\infty(\vec{x}_0)(1-\rho_1)\frac{\prod_{i=2}^m(1-\rho_i)\rho_i^{x_i}}{\pi(B)}\\
&= \frac{1}{||f^\infty||^2_{\hat{\pi}}}\frac{\prod_{i=1}^m(1-\rho_i)}{\pi(B)}\sum_{x_2=0}^{\ell-1}\sum_{x_3=0}^{\ell-1-x_2}\sum_{x_4=0}^{\ell-1-x_2-x_3}\cdots\sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1}x_i}\Bigg[(\prod_{i=2}^m\rho_i^{x_i})[1-\prod_{i=2}^m a_i^{x_i}\rho_1^\ell k^\ell(\vec{x}_0)]
\end{aligned}$$

$$\left. \cdot \mu_1 \rho_1^\ell (\prod_{i=2}^m a_i^{x_i}) [k^\ell(\vec{x}_0) - r_{1.}\rho_1 k^\ell(T_{1.}\vec{x}_0) - \sum_{j=2}^m r_{1j} a_j a_1^{-1} k^\ell(T_{1j}\vec{x}_0)] \right]$$

$$= \frac{1}{||f^\infty||_{\hat{\pi}}^2} \frac{\prod_{i=1}^m (1-\rho_i)}{\pi(B)} \mu_1 \rho_1^\ell \sum_{x_2=0}^{\ell-1} \sum_{x_3=0}^{\ell-1-x_2} \sum_{x_4=0}^{\ell-1-x_2-x_3} \cdots \sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1} x_i} \left[\prod_{i=2}^m (\rho_i^\ell)^{x_i} \right.$$

$$\left. \cdot [1 - \prod_{i=2}^m a_i^{x_i} \rho_1^\ell k^\ell(\vec{x}_0)] [k^\ell(\vec{x}_0) - r_{1.}\rho_1 k^\ell(T_{1.}\vec{x}_0) - \sum_{j=2}^m r_{1j} a_j a_1^{-1} k^\ell(T_{1j}\vec{x}_0)] \right].$$

For ℓ large, the evaluation of this expression is quite cumbersome. We thus proceed to gradual modifications that will yield a simple approximation for b. The error of that approximation will be bounded by the sum of the differences between the intermediate versions of b.

The first simplification results from the observation that $||f^\infty||_{\hat{\pi}} \approx 1$ (this is shown in (3.54)).

$$\begin{aligned} b_1 &= < f^\infty, L^B f^\infty >_{\hat{\pi}} \\ &= \frac{\prod_{i=1}^m (1-\rho_i)}{\pi(B)} \mu_1 \rho_1^\ell \sum_{x_2=0}^{\ell-1} \sum_{x_3=0}^{\ell-1-x_2} \sum_{x_4=0}^{\ell-1-x_2-x_3} \cdots \sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1} x_i} \left[\prod_{i=2}^m (\rho_i^\ell)^{x_i} \right. \\ &\quad \left. \cdot [1 - \prod_{i=2}^m a_i^{x_i} \rho_1^\ell k^\ell(\vec{x}_0)] [k^\ell(\vec{x}_0) - r_{1.}\rho_1 k^\ell(T_{1.}\vec{x}_0) - \sum_{j=2}^m r_{1j} a_j a_1^{-1} k^\ell(T_{1j}\vec{x}_0)] \right]. \end{aligned}$$

We can simplify this further by taking $f^\infty = 1$ altogether:

$$\begin{aligned} b_2 &= < 1, L^B f^\infty >_{\hat{\pi}} \\ &= \frac{\prod_{i=1}^m (1-\rho_i)}{\pi(B)} \mu_1 \rho_1^\ell \sum_{x_2=0}^{\ell-1} \sum_{x_3=0}^{\ell-1-x_2} \sum_{x_4=0}^{\ell-1-x_2-x_3} \cdots \sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1} x_i} \left[\prod_{i=2}^m (\rho_i^\ell)^{x_i} \right. \\ &\quad \left. \cdot [k^\ell(\vec{x}_0) - r_{1.}\rho_1 k^\ell(T_{1.}\vec{x}_0) - \sum_{j=2}^m r_{1j} a_j a_1^{-1} k^\ell(T_{1j}\vec{x}_0)] \right]. \end{aligned}$$

If we now substitute f_κ^∞ for f^∞, we obtain

$$\begin{aligned} b_3 &= < 1, L^B f_\kappa^\infty >_{\hat{\pi}} \\ &= \frac{\prod_{i=1}^m (1-\rho_i)}{\pi(B)} \mu_1 \rho_1^\ell \kappa [1 - r_{1.}\rho_1 - \sum_{j=2}^m r_{1j} a_j / a_1] \\ &\quad \cdot \sum_{x_2=0}^{\ell-1} \sum_{x_3=0}^{\ell-1-x_2} \sum_{x_4=0}^{\ell-1-x_2-x_3} \cdots \sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1} x_i} \left[\prod_{i=2}^m (\rho_i^\ell)^{x_i} \right]. \end{aligned}$$

Finally, an asymptotic approximation of b is obtained by letting $\ell \to \infty$:

$$b^{\infty} = \prod_{i=1}^{m}(1-\rho_i)\mu_1\kappa[1 - r_{1.}\rho_1 - \sum_{j=2}^{m} r_{1j}a_j/a_1]\prod_{i=2}^{m}\left(\frac{1}{1-\rho_i^t}\right)\rho_1^{\ell}. \tag{3.44}$$

Calculations for the differences $b - b_1$, $b_2 - b_1$ and $|b_2 - b_3|$, which are rather lengthy, are performed in Section 3.9. The results are

$$\begin{aligned} b - b_1 &\leq c\rho_1^{\ell} r^{\ell} && \text{from (3.56)} \\ b_2 - b_1 &\leq c\ell^m \delta^{m/2} \rho_1^{\ell} r^{\ell/2} && \text{from (3.57)} \\ |b_2 - b_3| &\leq c\rho_1^{\ell} r^{\ell} && \text{from (3.50)} \end{aligned}$$

for some canonical constant c, $\delta > 0$ small and some positive value $r < 1$.

The expression for b^{∞} was obtained by first noting that $\pi(B) \to 1$ as $\ell \to \infty$ and taking infinite sums. We note that

$$\begin{aligned} \sum_{x_2=0}^{\ell-1}\sum_{x_3=0}^{\ell-1-x_2}\sum_{x_4=0}^{\ell-1-x_2-x_3}\cdots\sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1}x_i}\prod_{i=2}^{m}(\rho_i^t)^{x_i} &< \sum_{x_2=0}^{\infty}\sum_{x_3=0}^{\infty}\sum_{x_4=0}^{\infty}\cdots\sum_{x_m=0}^{\infty}\prod_{i=2}^{m}(\rho_i^t)^{x_i} \\ &= \prod_{i=2}^{m}\left(\frac{1}{1-\rho_i^t}\right) \end{aligned} \tag{3.45}$$

since, by Theorem 2.2.3, $\rho_i^t < 1$, $i = 2, \ldots, m$, and the geometric series then converge. The difference between the expressions in the first and last lines of (3.45) is shown to be asymptotically small:

$$\begin{aligned} &\left(\sum_{x_2=0}^{\infty}\sum_{x_3=0}^{\infty}\cdots\sum_{x_m=0}^{\infty} - \sum_{x_2=0}^{\ell-1}\sum_{x_3=0}^{\ell-1-x_2}\cdots\sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1}x_i}\right)\prod_{i=2}^{m}(\rho_i^t)^{x_i} \\ \leq &\left(\sum_{x_2=0}^{\infty}\sum_{x_3=0}^{\infty}\cdots\sum_{x_m=0}^{\infty} - \sum_{x_2=0}^{\ell-1}\sum_{x_3=0}^{\ell-1-x_2}\cdots\sum_{x_m=0}^{\ell-1-\sum_{i=2}^{m-1}x_i}\right)(\rho_{max}^t)^{\sum_{i=2}^{m}x_i} \\ = &\sum_{s\geq\ell}\underbrace{\sum_{x_2}\cdots\sum_{x_m}}_{\bar{x}_0:\sum_{i=2}^{m}x_i=s}(\rho_{max}^t)^s \\ < &\sum_{s\geq\ell}(s+1)^{m-1}(\rho_{max}^t)^s. \end{aligned}$$

Since $\rho_{max}^{\ell} := \max_{i=2,\ldots,m}(\rho_i^{\ell}) < 1$ and m is fixed, the upper bound is small when $\ell \to \infty$.

We can finally write

$$b^{\infty} - b_3 < c\rho_1^{\ell} \sum_{s \geq \ell} (s+1)^{m-1} (\rho_{max}^{t})^{s}. \tag{3.46}$$

Combining (3.56), (3.57), (3.50) and (3.46),

$$\begin{aligned} |b - b^{\infty}| &\leq |b - b_1| + |b_1 - b_2| + |b_2 - b_3| + |b_3 - b^{\infty}| \\ &\leq c[\rho_1^{\ell} r^{\ell} + \ell^m \delta^{m/2} \rho_1^{\ell} r^{\ell/2} + \rho_1^{\ell} r^{\ell} + \rho_1^{\ell} \sum_{s \geq \ell} (s+1)^{m-1} (\rho_{max}^{t})^{s}]. \end{aligned} \tag{3.47}$$

This error tends to 0 quicker than b itself. Therefore, b^{∞} is asymptotically as good as b as an estimator for $\Lambda(B)$.

The accuracy of the estimator is given in Corollary 3.6.2 as

$$|\Lambda(B) - b| \leq \frac{2\gamma\eta}{\mathrm{Gap}(L)}.$$

Again, because of their complexity, calculations for γ and η are detailed in the next section. From (3.58) we get

$$\begin{aligned} |\Lambda(B) - b| &\leq \frac{2\gamma\eta}{\mathrm{Gap}(L)} \\ &\leq c[\rho_1^{\ell} r^{\ell/2} \ell^m \delta^{m/2} + \rho_1^{3\ell/2} \ell^{m/2}] \end{aligned}$$

for a canonical constant c and $0 < \delta, r < 1$. Thus, the relative error

$$\begin{aligned} \frac{|\Lambda(B) - b|}{b} &\leq \frac{c[\rho_1^{\ell} r^{\ell/2} \ell^m \delta^{m/2} + \rho_1^{3\ell/2} \ell^{m/2}]}{\rho_1^{\ell}} \\ &= c[r^{\ell/2} \ell^m \delta^{m/2} + \rho_1^{\ell/2} \ell^{m/2}] \end{aligned} \tag{3.48}$$

goes to 0 as $\ell \to \infty$.

3.9 Detailed calculations

We justify in this section the various operations that led to the simplified estimator b^{∞} for $\Lambda(B)$. We establish the asymptotics of the differences $|b - b_1|$, $|b_1 - b_2|$ and $|b_2 - b_3|$ which determine the validity of b^{∞}. Furthermore, we examine the factors η and γ defined in Corollary 3.6.2 whose product gives an upper bound for $|\Lambda(B) - b|$.

With the results from Section 3.4, we can show the convergence of b^∞ to b. To simplify the notation, we will consider the canonical constant c which will include all constant terms. For instance, the stationary probability of the chain X_n at state $\vec{z}^-$ is given by $\bar{\pi}(\vec{z}^-) = c\prod_{i=2}^m (\rho_i^t)^{z_i}$.

For our calculations, we consider again the state space S^∞ subject to the transformation $\mathcal{T}$ and the twisted extended Markov process $\mathcal{Z}^t$. The use of the transformation $\mathcal{T}$ requires a change of notation for the functions f^∞ and k^ℓ defined in (3.36) and (3.42) respectively. For $\vec{x} = (x_1, x_2, \ldots, x_m)$ and $\vec{z} = (z_1, z_2, \ldots, z_m)$ in S^∞ with $\vec{z} = \mathcal{T}\vec{x} = (\sum_{i=1}^m x_i, x_2, \ldots, x_m)$, define

$$
\begin{aligned}
f_{\mathcal{T}}^\infty(\vec{z}) &:= f^\infty(\vec{x}) \\
k_{\mathcal{T}}^\ell(\vec{z}) &:= k^\ell(\vec{x}).
\end{aligned}
$$

Using this notation,

$$
f_{\mathcal{T}}^\infty(\vec{z}) = 1 - a_1^{z_1 - \sum_{i=2}^m z_i} \prod_{i=2}^m a_i^{z_i} \rho_1^\ell k_{\mathcal{T}}^\ell(\vec{z})
$$

by (3.37) and (3.39). Note that the function s defined in (3.41) does not need any special notation since it only depends on the coordinates 2 through m.

We can now verify that the limit κ defined in (3.43) is finite:

$$
\begin{aligned}
\kappa &= \sum_{\vec{y}\in\partial F^\infty} s(\vec{y})e(\vec{y}^-) \\
&= \sum_{\vec{y}\in\partial\mathcal{F}^\infty} s(\vec{y})e(\vec{y}^-) \\
&\le c \sum_{\vec{y}\in\partial\mathcal{F}^\infty} \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{y_i} \bar{\pi}(\vec{y}^-) \quad \text{by (3.17)} \\
&= c \sum_{\vec{y}\in\partial\mathcal{F}^\infty} \prod_{i=2}^m \left(\frac{a_1}{a_i}\rho_i^t\right)^{y_i} \\
&= c \sum_{\vec{y}\in\partial\mathcal{F}^\infty} \prod_{i=2}^m \left(\frac{a_1}{a_i}a_i\rho_i\right)^{y_i} \\
&= c \sum_{\vec{y}\in\partial\mathcal{F}^\infty} \prod_{i=2}^m (\rho_1^{-1}\rho_i)^{y_i} \\
&< c \prod_{i=2}^m \sum_{y_i=0}^\infty (\rho_1^{-1}\rho_i)^{y_i} \\
&< \infty
\end{aligned}
$$

since $\rho_1^{-1}\rho_i < 1$ for $i = 2, \ldots, m$ by (2.5).

We now study the difference between $k_T^\ell(\vec{z})$ and κ for $\vec{z} = (z_1, \ldots, z_m) \in S^\infty$. We observe that the Markov process $\mathcal{Z}^\ell$ approaches the forbidden boundary $\partial\mathcal{F}^\infty = \{\vec{z} \in S^\infty : z_1 = \ell\}$ only through increases in the z_1 coordinate. Thus, from $\mathcal{Z}^\ell(0)$, $\ell - \mathcal{Z}_1^\ell(0) = \ell^*$ steps in the positive z_1 direction are required to reach $\partial\mathcal{F}^\infty$. From the existing analogy between increases in the z_1 coordinate and the variables u_n, we may say the process $X_{\nu_0}, X_{\nu_1}, X_{\nu_2}, \ldots$ (with kernel $\mathcal{K}$) reaches $\partial\mathcal{F}^\infty$ at "time" ν_{ℓ^*}.

Lemma 3.9.2 shows that, for $\vec{z} \in S^\infty$,

$$\begin{aligned} k_T^\ell(\vec{z}) &= \sum_{\vec{y} \in \partial\mathcal{F}^\infty} P_{\vec{z}}(X_{\nu_{\ell^*}} = \vec{y}^-) s(\vec{y}) \\ &\to \sum_{\vec{y} \in \mathcal{F}^\infty} e(\vec{y}^-) s(\vec{y}) = \kappa \end{aligned}$$

when the difference $\ell - z_1$ is large enough. This condition certainly holds for $\mathcal{Z}^\ell(0) = \vec{0}$ when ℓ is large. Thus, in practical applications, we shall approximate κ by $k^\ell(\vec{0})$. To prove the lemma, we will need the following result whose proof can be found in McDonald (1994).

Theorem 3.9.1 *[Mihail's Theorem] Let $||\alpha(\cdot)||_\Delta^2 = \sum_{\vec{z}^- \in S^-} |\alpha(\vec{z}^-)|$ denote the total variation of the unsigned measure α. Then the total variation distance between the distribution after n steps and the stationary distribution e is given by*

$$||\mathcal{K}_{\vec{z}^-\cdot}^n - e(\cdot)||_\Delta^2 \le (1 - Gap(\mathcal{K}))^n / e(\vec{z}^-).$$

We assume $Gap(\mathcal{K}) > 0$ without proof.

Pick $\epsilon > 0$ sufficiently small so that

$$\begin{aligned} \prod_{i=2}^m \left[\left(\frac{a_1}{a_i}\right)^{1+\epsilon} \rho_i^t \right] &= \prod_{i=2}^m \left[\left(\frac{a_1}{a_i}\right)^{(1+\epsilon)} a_i \rho_i \right] \\ &= \prod_{i=2}^m \left[\rho_1^{-1} \rho_i \left(\frac{a_1}{a_i}\right)^\epsilon \right] \\ &=: \beta < 1. \end{aligned} \tag{3.49}$$

Let $p = 1 + \epsilon$ and $q = (1+\epsilon)/\epsilon > 1$. Hence $1/p + 1/q = 1$.

Lemma 3.9.2 *For* $\vec{z} = (z_1, z_2, \ldots, z_m) \in S^\infty$,

$$|k_T^\ell(\vec{z}) - \kappa| = \left| \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left(\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1} - e(\vec{y}^-) \right) s(\vec{y}) \right| \le c \frac{e(\vec{z}^-)^{1/2q}}{e(\vec{z}^-)} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2q}.$$

Proof:

$$\begin{aligned}
\left| \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left(\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1} - e(\vec{y}^-) \right) s(\vec{y}) \right| &= \left| \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left(\frac{\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1}}{e(\vec{y}^-)} - 1 \right) s(\vec{y}) e(\vec{y}^-) \right| \\
&\le \left(\sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left| \frac{\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1}}{e(\vec{y}^-)} - 1 \right|^q e(\vec{y}^-) \right)^{1/q} \left(\sum_{\vec{y} \in \partial\mathcal{F}^\infty} |s(\vec{y})|^p e(\vec{y}^-) \right)^{1/p}
\end{aligned}$$

by Hölder's inequality. Now,

$$\begin{aligned}
\sum_{\vec{y} \in \partial\mathcal{F}^\infty} |s(\vec{y})|^p e(\vec{y}^-) &\le c \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left[\prod_{i=2}^m \left(\frac{a_1}{a_i} \right)^{y_i} \right]^{1+\epsilon} \bar{\pi}(\vec{y}^-) \quad \text{by (3.17)} \\
&= c \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \prod_{i=2}^m \left[\left(\frac{a_1}{a_i} \right)^{1+\epsilon} a_i \rho_i \right]^{y_i} \\
&= c \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \prod_{i=2}^m \beta^{y_i} \quad \text{by (3.49)} \\
&< c \prod_{i=2}^m \sum_{y_i=0}^\infty \beta^{y_i} < \infty.
\end{aligned}$$

Next,

$$\begin{aligned}
&\sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left| \frac{\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1}}{e(\vec{y}^-)} - 1 \right|^q e(\vec{y}^-) \\
&\le \sup_{\vec{y} \in \partial\mathcal{F}^\infty} \left| \frac{\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1}}{e(\vec{y}^-)} - 1 \right|^{q-1} \sum_{\vec{y} \in \partial\mathcal{F}^\infty} |\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1} - e(\vec{y}^-)| \\
&\le \left| \frac{1}{e(\vec{z}^-)} + 1 \right|^{q-1} \frac{1}{\sqrt{e(\vec{z}^-)}} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2} \quad \text{by (3.19) and Theorem 3.9.1} \\
&\le \left(\frac{2}{e(\vec{z}^-)} \right)^{q-1} \frac{1}{\sqrt{e(\vec{z}^-)}} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2} \\
&= c \frac{\sqrt{e(\vec{z}^-)}}{e(\vec{z}^-)^q} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2}
\end{aligned}$$

so that

$$\left(\sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left| \frac{\mathcal{K}_{\vec{z}^- \vec{y}^-}^{\ell - z_1}}{e(\vec{y}^-)} - 1 \right|^q e(\vec{y}^-) \right)^{1/q} \le c \frac{e(\vec{z}^-)^{1/2q}}{e(\vec{z}^-)} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2q}.$$

This proves the lemma. ■

We are now ready to justify the simplifications leading from b_2 to b_3.

3.9.1 Asymptotics of $|b_2 - b_3|$

From Section 3.8,

$$\begin{aligned} b_2 &= <1, L^B f^\infty>_{\hat{\pi}} \\ &= \sum_{\vec{x}_0 \in B_0} L^B f^\infty(\vec{x}_0)\hat{\pi}(\vec{x}_0) \\ &= c \sum_{\vec{x}_0 \in B_0} \prod_{i=2}^{m} \rho_i^{z_i} \mu_1 [r_{1.}(f^\infty(T_{1.}\vec{x}_0) - f^\infty(\vec{x}_0)) + \sum_{j=2}^{m} r_{1j}(f^\infty(T_{1j}\vec{x}_0) - f^\infty(\vec{x}_0))]. \end{aligned}$$

Since, by (3.7), $\mathcal{T}(T_{1.}\vec{x}) = T_{1.}\vec{z}$ and $\mathcal{T}(T_{1j}\vec{x}) = T_{1j}\vec{z}$ for any $\mathcal{T}\vec{x} = \vec{z}$,

$$b_2 = c \sum_{\vec{z}_0 \in \mathcal{B}_0} \prod_{i=2}^{m} \rho_i^{z_i} \mu_1 [r_{1.}(f_{\mathcal{T}}^\infty(T_{1.}\vec{z}_0) - f_{\mathcal{T}}^\infty(\vec{z}_0)) + \sum_{j=2}^{m} r_{1j}(f_{\mathcal{T}}^\infty(T_{1j}\vec{z}_0) - f_{\mathcal{T}}^\infty(\vec{z}_0))].$$

Separate this summation in two parts so that the first sum includes states in $\mathcal{B}_0$ with $z_1 = 0, \ldots, (\ell-1)(1-\delta)$ and the second sum combines states in $\mathcal{B}_0$ with $z_1 = (\ell-1)(1-\delta)+1, \ldots, \ell-1$ for some $\delta > 0$. Call the first set of states $\underline{\mathcal{B}}_0$ and the second set $\overline{\mathcal{B}}_0$. The second sum is bounded in absolute value by

$$\begin{aligned} & c \sum_{\vec{z}_0 \in \overline{\mathcal{B}}_0} \prod_{i=2}^{m} \rho_i^{z_i} \mu_1 [r_{1.}|f_{\mathcal{T}}^\infty(T_{1.}\vec{z}_0) - f_{\mathcal{T}}^\infty(\vec{z}_0)| + \sum_{j=2}^{m} r_{1j}|f_{\mathcal{T}}^\infty(T_{1j}\vec{z}_0) - f_{\mathcal{T}}^\infty(\vec{z}_0)|] \\ \leq\ & c \sum_{\vec{z}_0 \in \overline{\mathcal{B}}_0} \prod_{i=2}^{m} \rho_i^{z_i} \quad \text{since } f_{\mathcal{T}}^\infty(\cdot) \leq 1 \\ \leq\ & c(\ell^{m-1} - [(\ell-1)(1-\delta)]^{m-1})\rho_{max}^{(\ell-1)(1-\delta)+1} \end{aligned}$$

where $\rho_{max} = \max_{i=2,\ldots,m}(\rho_i)$.

The first sum is calculated by recalling $f_{\mathcal{T}}^\infty(\vec{z}_0) = 1 - \prod_{i=2}^{m} a_i^{z_i}\rho_1^\ell k_{\mathcal{T}}^\ell(\vec{z}_0)$. This gives

$$c \sum_{\vec{z}_0 \in \underline{\mathcal{B}}_0} \prod_{i=2}^{m} \rho_i^{z_i} \mu_1 \rho_1^\ell \prod_{i=2}^{m} a_i^{z_i} [r_{1.}(k_{\mathcal{T}}^\ell(\vec{z}_0) - a_1^{-1}k_{\mathcal{T}}^\ell(T_{1.}\vec{z}_0)) + \sum_{j=2}^{m} r_{1j}(k_{\mathcal{T}}^\ell(\vec{z}_0) - a_j a_1^{-1} k_{\mathcal{T}}^\ell(T_{1j}\vec{z}_0))].$$

Consider now any term, say

$$\begin{aligned} & c \sum_{\vec{z}_0 \in \underline{\mathcal{B}}_0} \prod_{i=2}^{m} (\rho_i a_i)^{z_i} \rho_1^\ell (k_{\mathcal{T}}^\ell(\vec{z}_0) - \kappa) \\ =\ & c\rho_1^\ell \sum_{\vec{z}_0 \in \underline{\mathcal{B}}_0} \prod_{i=2}^{m} (\rho_i^\dagger)^{z_i} \sum_{\vec{y} \in \partial\mathcal{F}^\infty} \left[\mathcal{K}_{\vec{z}_0^- \vec{y}^-}^{\ell - z_1} - e(\vec{y}^-)\right] s(\vec{y}) \\ \leq\ & c\rho_1^\ell \sum_{\vec{z}_0 \in \underline{\mathcal{B}}_0} \prod_{i=2}^{m} (\rho_i^\dagger)^{z_i} \frac{e(\vec{z}_0^-)^{1/2q}}{e(\vec{z}_0^-)} (1 - Gap(\mathcal{K}))^{(\ell - z_1)/2q} \quad \text{by Lemma 3.9.2} \\ \leq\ & c\rho_1^\ell [(\ell-1)(1-\delta)]^{m-1} (1 - Gap(\mathcal{K}))^{(\ell\delta + 1 - \delta)/2q} \quad \text{by (3.18).} \end{aligned}$$

Calculations for the other m terms clearly yield the same result. Therefore

$$\begin{aligned} & |b_2 - b_3| \\ \leq \quad & c[(\ell^{m-1} - [(\ell-1)(1-\delta)]^{m-1})\rho_{max}^{(\ell-1)(1-\delta)+1} + \rho_1^\ell[(\ell-1)(1-\delta)]^{m-1}(1 - Gap(\mathcal{K}))^{(\ell\delta+1-\delta)/2q}]. \end{aligned}$$

Noting that $\rho_{max} < \rho_1$ and that an exponential decreases faster than a polynomial, we can choose $\delta > 0$ so that

$$|b_2 - b_3| < c\rho_1^\ell r^\ell \tag{3.50}$$

for some $r < 1$. Consequently, the error induced by replacing $k^\ell(\cdot)$ by κ goes to 0 quicker that b.

3.9.2 Asymptotics of $b - b_1$ and $b_2 - b_1$

Before we can evaluate $b - b_1$ and $b_2 - b_1$, a few preliminary results are needed. Consider

$$\begin{aligned} \|L^B f^\infty\|_{\hat{\pi}}^2 &= \sum_{\vec{x}_0 \in B_0} [L^B f^\infty(\vec{x}_0)]^2 \hat{\pi}(\vec{x}_0) \\ &= c \sum_{\vec{x}_0 \in B_0} \prod_{i=2}^m \rho_i^{x_i} [\mu_1 r_{1.}(f^\infty(T_{1.}\vec{x}_0) - f^\infty(\vec{x}_0)) + \mu_1 \sum_{j=2}^m r_{1j}(f^\infty(T_{1j}\vec{x}_0) - f^\infty(\vec{x}_0))]^2 \\ &= c \sum_{\vec{z}_0 \in B_0} \prod_{i=2}^m \rho_i^{z_i} [\mu_1 r_{1.}(f_T^\infty(T_{1.}\vec{z}_0) - f_T^\infty(\vec{z}_0)) + \mu_1 \sum_{j=2}^m r_{1j}(f_T^\infty(T_{1j}\vec{z}_0) - f_T^\infty(\vec{z}_0))]^2. \end{aligned}$$

As before, split the summation over two sets. The first summation is performed on the set $\underline{B}_0 = \{\vec{z}_0 \in B_0 : z_1 = 0, \ldots, (\ell-1)(1-\delta')\}$ while the second is on the set $\overline{B}_0 = \{\vec{z}_0 \in B_0 : z_1 = (\ell-1)(1-\delta') + 1, \ldots, \ell - 1\}$. Since $f(\cdot) \leq 1$, the second sum is clearly bounded by

$$\begin{aligned} c \sum_{\vec{z}_0 \in \overline{B}_0} \prod_{i=2}^m \rho_i^{z_i} &\leq c \sum_{\vec{z}_0 \in \overline{B}_0} \rho_{max}^{z_1} \\ &< c[\ell^{m-1} - [(\ell-1)(1-\delta')]^{m-1}]\rho_{max}^{(\ell-1)(1-\delta')+1} \end{aligned}$$

where $\rho_{max} = \max_{i=2,\ldots,m}(\rho_i)$. Take $\delta' > 0$ such that

$$[\ell^{m-1} - [(\ell-1)(1-\delta')]^{m-1}]\rho_{max}^{(\ell-1)(1-\delta')+1} = (\rho_1 r_1)^\ell$$

for some $r_1 < 1$.

Now, the first sum

$$c\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}\rho_i^{z_i}[\mu_1 r_{1.}(f_T^{\infty}(T_{1.}\vec{z}_0)-f_T^{\infty}(\vec{z}_0))+\mu_1\sum_{j=2}^{m}r_{1j}(f_T^{\infty}(T_{1j}\vec{z}_0)-f_T^{\infty}(\vec{z}_0))]^2 \tag{3.51}$$

$$\leq c\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}\rho_i^{z_i}[2\mu_1^2 r_{1.}^2(f_T^{\infty}(T_{1.}\vec{z}_0)-f_T^{\infty}(\vec{z}_0))^2+2\mu_1^2\sum_{j=2}^{m}r_{1j}^2(f_T^{\infty}(T_{1j}\vec{z}_0)-f_T^{\infty}(\vec{z}_0))^2]$$

$$\leq c\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}\rho_i^{z_i}[r_{1.}^2|f_T^{\infty}(T_{1.}\vec{z}_0)-f_T^{\infty}(\vec{z}_0)|^{1+\epsilon'}+\sum_{j=2}^{m}r_{1j}^2|f_T^{\infty}(T_{1j}\vec{z}_0)-f_T^{\infty}(\vec{z}_0)|^{1+\epsilon'}]$$

for some $0<\epsilon'<1$ since $f(\cdot)\leq 1$. Again, $f_T^{\infty}(\vec{z}_0)=1-\prod_{i=2}^{m}a_i^{z_i}\rho_1^{\ell}k_T^{\ell}(\vec{z}_0)$ so, for instance,

$$\begin{aligned} f_T^{\infty}(T_{1.}\vec{z}_0)-f_T^{\infty}(\vec{z}_0) &= \rho_1^{\ell}\prod_{i=2}^{m}a_i^{z_i}[k_T^{\ell}(\vec{z}_0)-a_1^{-1}k_T^{\ell}(T_{1.}\vec{z}_0)] \\ &= \rho_1^{\ell}\prod_{i=2}^{m}a_i^{z_i}[(k_T^{\ell}(\vec{z}_0)-\kappa)-a_1^{-1}(k_T^{\ell}(T_{1.}\vec{z}_0)-\kappa)+(\kappa-a_1^{-1}\kappa)] \end{aligned}$$

and

$$\begin{aligned} &|f_T^{\infty}(T_{1.}\vec{z}_0)-f_T^{\infty}(\vec{z}_0)|^{1+\epsilon'} \\ &\leq c\rho_1^{\ell(1+\epsilon')}\prod_{i=2}^{m}a_i^{z_i(1+\epsilon')}[|k_T^{\ell}(\vec{z}_0)-\kappa|^{1+\epsilon'}+a_1^{-(1+\epsilon')}|k_T^{\ell}(T_{1.}\vec{z}_0)-\kappa|^{1+\epsilon'}+|\kappa-a_1^{-1}\kappa|^{1+\epsilon'}]. \end{aligned}$$

The sum

$$\begin{aligned} &\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}\rho_i^{z_i}\rho_1^{\ell(1+\epsilon')}\prod_{i=2}^{m}a_i^{z_i(1+\epsilon')}|\kappa-a_1^{-1}\kappa|^{1+\epsilon'} \\ &\leq (\ell^{m-1}-[(\ell-1)(1-\delta')]^{m-1})\rho_1^{\ell(1+\epsilon')}|\kappa-a_1^{-1}\kappa|^{1+\epsilon'}\prod_{i=2}^{m}\left(\frac{1}{1-a_i^{1+\epsilon'}\rho_i}\right) \end{aligned}$$

if

$$a_i^{1+\epsilon'}\rho_i=a_i^{\epsilon'}\rho_i^{\ell}<1 \quad \text{for } i=2,\ldots,m. \tag{3.52}$$

The sum for the term $|k_T^{\ell}(\vec{z}_0)-\kappa|^{1+\epsilon'}$ is

$$\begin{aligned} &\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}\rho_i^{z_i}\rho_1^{\ell(1+\epsilon')}\prod_{i=2}^{m}a_i^{z_i(1+\epsilon')}|k_T^{\ell}(\vec{z}_0)-\kappa|^{1+\epsilon'} \\ &\leq c\rho_1^{\ell(1+\epsilon')}\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}(\rho_i^{\ell})^{z_i}\prod_{i=2}^{m}a_i^{\epsilon' z_i}\frac{e(\vec{z}^-)^{(1+\epsilon')/2q}}{e(\vec{z}^-)^{1+\epsilon'}}(1-Gap(\mathcal{K}))^{(1+\epsilon')(\ell-z_1)/2q} \quad \text{by Lemma 3.9.2} \\ &\leq c\rho_1^{\ell(1+\epsilon')}\sum_{\vec{z}_0\in\underline{B}_0}\prod_{i=2}^{m}a_i^{\epsilon' z_i}(1-Gap(\mathcal{K}))^{(1+\epsilon')(\ell-z_1)/2q}\frac{e(\vec{z}^-)^{(1+\epsilon')/2q}}{e(\vec{z}^-)^{\epsilon'}} \quad \text{by (3.18)} \end{aligned}$$

$$
\begin{aligned}
&\leq c\rho_1^{\ell(1+\epsilon')} \sum_{\vec{z}_0 \in \underline{B}_0} g^\ell \prod_{i=2}^{m} \left(\frac{a_i^{\epsilon'}}{g(\rho_i^t)^{\epsilon'}}\right)^{z_i} \quad \text{where } 1 > g = (1 - Gap(\mathcal{K}))^{(1+\epsilon')/2q} \\
&= c\rho_1^{\ell(1+\epsilon')} g^\ell \sum_{\vec{z}_0 \in \underline{B}_0} \prod_{i=2}^{m} \left(\frac{1}{g\rho_i^{\epsilon'}}\right)^{z_i} \\
&\leq c\rho_1^{\ell(1+\epsilon')} g^\ell [(\ell-1)(\delta'-1)]^{m-1} \left(\frac{1}{g\rho_{min}^{\epsilon'}}\right)^{(\ell-1)(1-\delta')} \quad \text{where } \rho_{min} = \min_{i=2,\ldots,m}(\rho_i) \\
&= c\rho_1^{\ell(1+\epsilon')} g^{\ell\delta'+1-\delta'} [(\ell-1)(\delta'-1)]^{m-1} \rho_{min}^{-\epsilon'(\ell-1)(1-\delta')}.
\end{aligned}
$$

For ϵ' sufficiently small, $g^{\ell\delta'+1-\delta'}\rho_{min}^{-\epsilon'(\ell-1)(1-\delta')} < r_2^\ell$ for some $r_2 < 1$. Choose ϵ' so that r_2 can be found and (3.52) holds. All other sums in (3.51) can be done similarly and we obtain

$$
||L^B f^\infty||_{\dagger} \leq c[\rho_1^{\ell(1+\epsilon')/2} r_2^{\ell/2} [(\ell-1)(\delta'-1)]^{(m-1)/2} + (\rho_1 r_1)^{\ell/2}] \leq c\rho_1^{\ell/2} r^{\ell/2} (\ell\delta')^{m/2} \tag{3.53}
$$

for some $r < 1$.

From this, we can find an upper bound for $||f^\infty - 1||_{\dagger}$. By the triangle inequality,

$$
||f^\infty - 1||_{\dagger} \leq ||f^\infty - \varphi^B||_{\dagger} + ||1 - \varphi^B||_{\dagger}.
$$

Now,

$$
\begin{aligned}
||f^\infty - \varphi^B||_{\dagger} &\leq \frac{2\gamma||f^\infty||_{\dagger}}{Gap(L)} \quad \text{by Corollary 3.6.2} \\
&\leq 2\left(\frac{||L^B f^\infty||_{\dagger}}{||f^\infty||_{\dagger}} + b\right) \frac{||f^\infty||_{\dagger}}{Gap(L)} \quad \text{by (3.34)} \\
&= \frac{2||L^B f^\infty||_{\dagger}}{Gap(L)} + \frac{2b||f^\infty||_{\dagger}}{Gap(L)} \\
&\leq \frac{c\rho_1^{\ell/2} r^{\ell/2} (\ell\delta')^{m/2}}{Gap(L)} + \frac{2b}{Gap(L)} \quad \text{for } 0 < \delta', r < 1 \text{ by (3.53) and since } f(\cdot) \leq 1 \\
&\leq c[\rho_1^{\ell/2} r^{\ell/2} (\ell\delta')^{m/2} + \rho_1^\ell].
\end{aligned}
$$

Also,

$$
\begin{aligned}
||1 - \varphi^B||_{\dagger} &= \left\{\frac{2\sqrt{(Gap(L) - \bar{\kappa})^2 + 4\kappa_2^2}}{[Gap(L) - \bar{\kappa}]^2 - 4\kappa_1\kappa_2}\right\} \kappa_1 \\
&\qquad \text{by Theorem 2.7 in Iscoe and McDonald (1994a)} \\
&\leq c\ell^{m/2}\rho_1^{\ell/2} \quad \text{by (3.24).}
\end{aligned}
$$

Hence,

$$
||f^\infty - 1||_{\dagger} \leq c\ell^{m/2}\rho_1^{\ell/2}. \tag{3.54}
$$

On the other hand, $||1-f^\infty||_{\hat\pi} \ge |\, ||f^\infty||_{\hat\pi} - ||1||_{\hat\pi}|$ by the triangle inequality or

$$||1||_{\hat\pi} - ||1-f^\infty||_{\hat\pi} \le ||f^\infty||_{\hat\pi} \le ||1||_{\hat\pi} + ||1-f^\infty||_{\hat\pi}$$

or

$$\frac{1}{||1||_{\hat\pi} + ||1-f^\infty||_{\hat\pi}} \le \frac{1}{||f^\infty||_{\hat\pi}} \le \frac{1}{||1||_{\hat\pi} - ||1-f^\infty||_{\hat\pi}}.$$

From (3.54),

$$||f^\infty||_{\hat\pi} \ge ||1||_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}. \tag{3.55}$$

We can now calculate $b-b_1$ and b_2-b_1:

$$\begin{aligned} b-b_1 &= <f^\infty, L^B f^\infty>_{\hat\pi} \left[\frac{1}{||f^\infty||_{\hat\pi}^2} - 1\right] \qquad (3.56)\\ &\le c\rho_1^\ell \left[\frac{1-||f^\infty||_{\hat\pi}^2}{||f^\infty||_{\hat\pi}^2}\right] \\ &\le c\rho_1^\ell \left[\frac{1-(||1||_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2})^2}{(||1||_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2})^2}\right] \quad \text{by (3.55)} \\ &\le c\rho_1^\ell r^\ell \end{aligned}$$

for some $r<1$ and

$$\begin{aligned} b_2-b_1 &= <1, L^B f^\infty>_{\hat\pi} - <f^\infty, L^B f^\infty>_{\hat\pi} \qquad (3.57)\\ &= <1-f^\infty, L^B f^\infty>_{\hat\pi} \\ &\le ||1-f^\infty||_{\hat\pi} ||L^B f^\infty||_{\hat\pi} \quad \text{by the Cauchy-Schwarz inequality} \\ &\le c\ell^{m/2}\rho_1^{\ell/2}\rho_1^{\ell/2} r^{\ell/2}(\ell\delta)^{m/2}, \quad 0<r,\delta<1 \quad \text{by (3.54) and (3.53)} \\ &= c\ell^m \delta^{m/2}\rho_1^\ell r^{\ell/2}. \end{aligned}$$

3.9.3 Upper bounds for η and γ

From (3.34),

$$\begin{aligned} \gamma &\le \frac{||L^B f^\infty||_{\hat\pi}}{||f^\infty||_{\hat\pi}} + b \\ &\le \frac{c\rho_1^{\ell/2} r^{\ell/2}(\ell\delta)^{m/2}}{||1||_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}} + c\rho_1^\ell \quad \text{by (3.53) and (3.55).} \end{aligned}$$

Also, from (3.35),

$$\begin{aligned}\eta &\le \frac{\|L^B\|_{\hat\pi}\|f^\infty - 1\|_{\hat\pi}}{\|f^\infty\|_{\hat\pi}} \\ &\le \frac{c\ell^{m/2}\rho_1^{\ell/2}}{\|1\|_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}} \quad \text{by (3.54) and (3.55)}\end{aligned}$$

where we used the result from Lemma 2.2 in Iscoe and McDonald (1994a) that $\|L^B\|_{\hat\pi} \le 2\sum_{i=1}^m(\bar\lambda_i + \mu_i)$, a constant. Therefore,

$$\begin{aligned}\eta\gamma &\le \frac{c\rho_1^{\ell/2} r^{\ell/2}(\ell\delta)^{m/2}\ell^{m/2}\rho_1^{\ell/2}}{[\|1\|_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}]^2} + \frac{c\rho_1^{\ell}\ell^{m/2}\rho_1^{\ell/2}}{\|1\|_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}} \\ &= \frac{c\rho_1^{\ell} r^{\ell/2}\ell^{m}\delta^{m/2}}{[\|1\|_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}]^2} + \frac{c\rho_1^{3\ell/2}\ell^{m/2}}{\|1\|_{\hat\pi} - c\ell^{m/2}\rho_1^{\ell/2}} \\ &\le c[\rho_1^{\ell} r^{\ell/2}\ell^{m}\delta^{m/2} + \rho_1^{3\ell/2}\ell^{m/2}]. \end{aligned} \tag{3.58}$$

3.10 Asymptotic equivalence of Aldous' estimate and b

By equation (3.4), the estimate for $1/E\tau$ proposed by Aldous (1989) is

$$\begin{aligned}[E_{\vec 0}R]^{-1} &= \sum_{\vec y\in\partial F}\pi(\vec y)\sum_{\vec x\in B} q^r(\vec y,\vec x)\rho_\ell(\vec x) \\ &= \sum_{\vec y\in\partial F}\prod_{i=1}^m(1-\rho_i)\rho_1^\ell\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar\lambda_i\rho_i^{-1}\rho_\ell(T_i\vec y).\end{aligned}$$

However, it was shown in Lemma 3.4.1 that $\rho_\ell(T_i\vec y) \to \rho^\infty(T_i\vec y)$ as $\ell\to\infty$ (for $i = 1,\dots,m$) so

$$\begin{aligned}\rho_1^{-\ell}[E_{\vec 0}R]^{-1} &\sim \prod_{i=1}^m(1-\rho_i)\sum_{\vec y\in\partial F}\prod_{i=2}^m\left(\frac{\rho_i}{\rho_1}\right)^{y_i}\sum_{i=1}^m\bar\lambda_i\rho_i^{-1}\rho^\infty(T_i\vec y) \\ &= \frac{\prod_{i=1}^m(1-\rho_i)}{\prod_{i=2}^m(1-\rho_i^t)}\frac{1}{\lambda}\sum_{\vec y\in\partial F}\prod_{i=2}^m\left(\frac{1}{\rho_1 a_i}\right)^{y_i}\left[\prod_{i=2}^m(1-\rho_i^t)(\rho_i^t)^{y_i}[\sum_{i=1}^m\bar\lambda_i\rho_i^{-1}\lambda\rho^\infty(T_i\vec y)]\right] \\ &= \frac{\prod_{i=1}^m(1-\rho_i)}{\prod_{i=2}^m(1-\rho_i^t)}\frac{1}{\lambda}\sum_{\vec y\in\partial F}\prod_{i=2}^m\left(\frac{a_1}{a_i}\right)^{y_i}\left[\pi^t(\vec y^-)\sum_{i=1}^m\bar\lambda_i\rho_i^{-1}\lambda\rho^\infty(T_i\vec y)\right] \\ &= \frac{\prod_{i=1}^m(1-\rho_i)}{\prod_{i=2}^m(1-\rho_i^t)}\frac{q}{\lambda}\sum_{\vec y\in\partial F}\prod_{i=2}^m\left(\frac{a_1}{a_i}\right)^{y_i} e(\vec y^-) \quad \text{by (3.16)} \\ &\sim \frac{\prod_{i=1}^m(1-\rho_i)}{\prod_{i=2}^m(1-\rho_i^t)}\frac{q}{\lambda}\sum_{\vec y\in\partial F^\infty}\prod_{i=2}^m\left(\frac{a_1}{a_i}\right)^{y_i} e(\vec y^-) \\ &= \frac{\prod_{i=1}^m(1-\rho_i)}{\prod_{i=2}^m(1-\rho_i^t)}\frac{q}{\lambda}\kappa \quad \text{by (3.43).}\end{aligned}$$

An exact expression for q is derived in Kesten (1974). In (1.3) and (3.22) respectively, Kesten (1974) defines

$$\alpha \equiv \lim_{n\to\infty} \frac{1}{n} V_n,$$

$$\lim_{n\to\infty} \frac{\nu_n}{n} = \beta \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} V_{\nu_n} = \gamma$$

and, in (3.23), shows $\beta = 1/q$. In the discussion following these equations, he also concludes $\gamma = \alpha\beta$. Clearly, from (3.8) and (3.9), $\gamma = 1$ in our case so $q = \alpha$. Also, given the analogy established in Section 3.4 between V_n and the total variation in the first coordinate of $\mathcal{Z}^\ell$, α may be taken as the average drift of $\mathcal{Z}^\ell$ in the positive z_1 direction with respect to the measure $\bar{\pi}$. Thus, using the notation developed in Section 3.4, we calculate

$$\begin{aligned} q &= \sum_{\vec{y}^- \in S^-} \frac{\bar{\pi}(\vec{y}^-)}{\lambda(\vec{y}^-)} \left(\sum_{i=1}^m \bar{\lambda}_i^\ell - \mu_1^\ell r_{1.}^\ell - \sum_{i=2}^m \mu_i^\ell(y_i) r_{i.}^\ell \right) \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i^\ell - \mu_1^\ell r_{1.}^\ell - \sum_{i=2}^m \sum_{\vec{y}^- \in S^-} \pi^\ell(\vec{y}^-) \mu_i^\ell(y_i) r_{i.}^\ell \right) \lambda \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i^\ell - \mu_1^\ell r_{1.}^\ell - \sum_{i=2}^m [\mu_i^\ell r_{i.}^\ell - (1-\rho_i^\ell) \mu_i^\ell r_{i.}^\ell] \right) \lambda \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i^\ell - \mu_1^\ell r_{1.}^\ell - \sum_{i=2}^m \rho_i^\ell \mu_i^\ell r_{i.}^\ell \right) \lambda. \end{aligned}$$

Now, as $\ell \to \infty$, $\rho_1^{-\ell} |[E_{\vec{0}}\tau]^{-1} - b| \to 0$ by Theorem 3.5.4 taking π_0 to be a point probability at $\vec{0}$ and (3.48). Also, $\rho_1^{-\ell}|b - b^\infty| \to 0$ by (3.47) so

$$\rho_1^{-\ell}[E_{\vec{0}}\tau]^{-1} \sim \rho_1^{-\ell} b \sim \rho_1^{-\ell} b^\infty = \frac{\prod_{i=1}^m (1-\rho_i)}{\prod_{i=2}^m (1-\rho_i^\ell)} \mu_1 [1 - r_{1.}\rho_1 - \sum_{i=2}^m r_{1i} a_i a_1^{-1}] \kappa$$

by (3.44). However, $E_{\vec{0}}R \sim E_{\vec{0}}\tau$ from Proposition 3.3.1. Also, by the calculations above

$$\rho_1^{-\ell}[E_{\vec{0}}R]^{-1} \to \frac{\prod_{i=1}^m (1-\rho_i)}{\prod_{i=2}^m (1-\rho_i^\ell)} \frac{q}{\lambda} \kappa.$$

This implies $\rho_1^{-\ell} b^\infty = \lim_{\ell\to\infty} \rho_1^{-\ell} [E_{\vec{0}}R]^{-1}$ is established if we show

$$\frac{q}{\lambda} = \mu_1 [1 - r_{1.}\rho_1 - \sum_{i=2}^m r_{1i} a_i a_1^{-1}].$$

We can verify this directly in the following lemma.

Lemma 3.10.1

$$\mu_1[1 - r_{1.}\rho_1 - \sum_{i=2}^{m} r_{1i}a_i a_1^{-1}] = \sum_{i=1}^{m} \bar{\lambda}_i^t - \mu_1^t r_{1.}^t - \sum_{i=2}^{m} \rho_i^t \mu_i^t r_{i.}^t$$

Proof: First, notice that

$$\mu_1 r_{1.}\rho_1 = \frac{\mu_1 r_{1.}}{a_1} = \mu_1^t r_{1.}^t$$

by (2.10) so that all we have to do is show the equality between

$$\mu_1 - \mu_1 \sum_{i=2}^{m} r_{1i}a_i a_1^{-1} \tag{3.59}$$

and

$$\sum_{i=1}^{m} \bar{\lambda}_i^t - \sum_{i=2}^{m} \rho_i^t \mu_i^t r_{i.}^t. \tag{3.60}$$

Now,

$$\begin{aligned} \sum_{i=1}^{m} \bar{\lambda}_i^t - \sum_{i=2}^{m} \rho_i^t \mu_i^t r_{i.}^t &= \sum_{i=1}^{m} \bar{\lambda}_i a_i - \sum_{i=2}^{m} \frac{a_i \lambda_i}{\mu_i} \frac{\mu_i r_{i.}}{a_i} \quad \text{by (2.12) and (2.10)} \\ &= \sum_{i=1}^{m} \bar{\lambda}_i a_i - \sum_{i=2}^{m} \lambda_i r_{i.} \end{aligned} \tag{3.61}$$

where

$$\lambda_i r_{i.} = \lambda_i (1 - \sum_{j=1}^{m} r_{ij})$$

so that

$$\begin{aligned} \sum_{i=2}^{m} \lambda_i r_{i.} &= \sum_{i=2}^{m} \lambda_i - \sum_{i=2}^{m} \sum_{j=1}^{m} \lambda_i r_{ij} \\ &= \sum_{i=2}^{m} \lambda_i - \sum_{j=1}^{m} \sum_{i=1}^{m} \lambda_i r_{ij} + \sum_{j=1}^{m} \lambda_1 r_{1j} \\ &= \sum_{i=2}^{m} \lambda_i - \sum_{j=1}^{m} (\lambda_j - \bar{\lambda}_j) + \sum_{j=1}^{m} \lambda_1 r_{1j} \quad \text{by (2.2)} \\ &= \sum_{i=2}^{m} \lambda_i - \sum_{j=1}^{m} \lambda_j + \sum_{j=1}^{m} \bar{\lambda}_j + \sum_{j=1}^{m} \lambda_1 r_{1j} \\ &= -\lambda_1 + \sum_{j=1}^{m} \bar{\lambda}_j + \sum_{j=1}^{m} \lambda_1 r_{1j} \\ &= \lambda_1 (\sum_{j=1}^{m} r_{1j} - 1) + \sum_{j=1}^{m} \bar{\lambda}_j \\ &= \sum_{i=1}^{m} \bar{\lambda}_i - \lambda_1 r_{1.}. \end{aligned}$$

Hence (3.61) becomes

$$\sum_{i=1}^{m} \bar{\lambda}_i a_i + \lambda_1 r_{1.} - \sum_{i=1}^{m} \bar{\lambda}_i.$$

On the other hand, (3.59) is

$$\mu_1 - \frac{\mu_1}{a_1} \sum_{i=2}^{m} r_{1i} a_i = \mu_1 - \lambda_1 \sum_{i=1}^{m} r_{1i} a_i$$

since $a_1^{-1} = \rho_1 = \lambda_1/\mu_1$ and $r_{11} = 0$. Therefore, (3.59) and (3.60) are equal if

$$\mu_1 - \lambda_1 \sum_{i=1}^{m} r_{1i} a_i = \sum_{i=1}^{m} \bar{\lambda}_i a_i + \lambda_1 r_{1.} - \sum_{i=1}^{m} \bar{\lambda}_i$$

or

$$\begin{aligned} \sum_{i=1}^{m} \bar{\lambda}_i + \mu_1 &= \sum_{i=1}^{m} \bar{\lambda}_i a_i + \lambda_1 (r_{1.} + \sum_{i=1}^{m} r_{1i} a_i) \\ &= \sum_{i=1}^{m} \bar{\lambda}_i a_i + \frac{\mu_1}{a_1} (r_{1.} + \sum_{i=1}^{m} r_{1i} a_i). \end{aligned}$$

This is exactly (2.8) with $\mathcal{I} = \{2, \ldots, m\}$, so the equality is verified. ■

Chapter 4

Application

4.1 Algorithm

If we make abstraction of the error calculations, our method can be presented in a simple algorithmic form.

For an m-node stable Jackson network with

- exogenous arrival rates $\bar{\lambda}_i$, $i = 1, \ldots, m$
- service rates μ_i, $i = 1, \ldots, m$
  and
- transition probabilities r_{ij}, $i, j = 1, \ldots, m$, $i \neq j$,

where the labelling from 1 to m of the nodes respects the load ordering condition $\rho_i < \rho_1$ for $i = 2, \ldots, m$, find the solution $a_1, \ldots, a_m$ to the matrix system

$$\begin{pmatrix} 1 \; 0 \ldots 0 \\ (I - R)_{2\downarrow} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \rho_1^{-1} \\ r_{2.} \\ \vdots \\ r_{m.} \end{pmatrix}$$

where R denotes the matrix (r_{ij}) and $(I - R)_{2\downarrow}$ denotes the $(m-1) \times m$ matrix formed from $I - R$ by deleting the first row from the latter.

Next, construct a twisted Jackson network with

- exogenous arrival rates $\bar{\lambda}_i^t := \bar{\lambda}_i a_i$, $i = 1, \ldots, m$

- service rates $\mu_1^t := \frac{\mu_1}{a_1}\left(r_{1.} + \sum_{j=1}^m r_{1j}a_j\right)$, $\mu_i^t := \mu_i$, $i = 2, \ldots, m$

  and

- transition probabilities $r_{1j}^t = r_{1j}a_j/(r_{1.} + \sum_{i=1}^m r_{1i}a_i)$, $j = 2, \ldots, m$,

  $r_{ij}^t := r_{ij}a_j/a_i$, $i = 2, \ldots, m$, $j = 1, \ldots, m$.

Extend the twisted process by allowing the first node to serve imaginary customers so that the queue size at node 1 may be negative; let the m-tuple $\vec{x} = (x_1, \ldots, x_m)$ record the queue size at each node.

By simulation, estimate the distribution of the extended twisted Jackson network at the moment of overload. Starting with empty queues $\vec{0} = (0, 0, \ldots, 0)$, generate traffic according to the arrival, transfer and departure rates of the twisted network and record the state $\vec{x} = (x_1, x_2, \ldots, x_m)$ at which the sum of the queue sizes $\sum_{i=1}^m x_i$ reaches the critical overload value ℓ for the first time. Repeating this many times, we obtain an estimate for

$$P_{\vec{0}}^t(\text{overload state is } \vec{x}) \tag{4.1}$$

for $\vec{x} \in \{\vec{x} : \sum_{i=1}^m x_i = \ell\}$, from which we can calculate

$$\kappa_e = \sum_{\{\vec{x}: \sum_{i=1}^m x_i = \ell\}} \prod_{i=2}^m \left(\frac{a_1}{a_i}\right)^{x_i} P_{\vec{0}}^t(\text{overload state is } \vec{x}).$$

An estimate for the mean time until overload given the queues are initially empty in the *original* Jackson network is then given by $1/b^\infty$ where

$$b^\infty = \frac{\prod_{i=1}^m (1-\rho_i)}{\prod_{i=2}^m (1-\rho_i^t)} \mu_1 \kappa_e [1 - r_{1.}\rho_1 - \sum_{j=2}^m r_{1j}a_j a_1^{-1}]\rho_1^\ell.$$

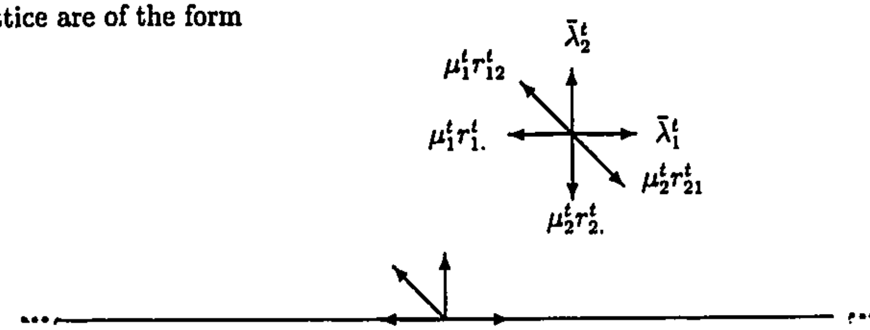
The estimates obtained for (4.1) can also be used to approximate the distribution of the system at the moment of overload. If we denote these estimates by $\hat{e}(\vec{x}^-)$,

$$P_{\vec{0}}(\text{overload state is } \vec{x} \text{ in the } \textit{original} \text{ network}) \approx \prod_{i=2}^m (a_1/a_i)^{x_i} \hat{e}(\vec{x}^-) / [\sum_{\vec{z} \in \partial F} \prod_{i=2}^m (a_1/a_i)^{z_i} \hat{e}(\vec{z}^-)].$$

4.2 Two examples

We illustrate our method with two examples of a Jackson network with two nodes. Calculations for problems of higher dimension are similar.

Using a simulation program, 50 000 trajectories starting at the origin $(0,0)$ are generated on the lattice composed of the coordinates $\{(x_1, x_2) : x_1 + x_2 \leq \ell, x_2 \geq 0\}$. Movements on the lattice are of the form



with rates as shown in the diagram. These movements represent the jumps of the Markov chain embedded in the twisted extended Jackson network Z^ℓ.

Each trajectory ends when a coordinate on $\partial F^\infty = \{(x_1, x_2) : x_1 + x_2 = \ell, x_2 \geq 0\}$ is reached. The coordinates $(\ell - x_2, x_2)$ hit are recorded and used to calculate the empirical hitting distribution P_e^ℓ of ∂F^∞ starting from $(0,0)$. The empirical value of κ can then be evaluated by recalling that when ℓ is large, κ may be approximated by $k^\ell(\vec{0})$:

$$\kappa_e = \sum_{i=0}^{\infty} \left(\frac{a_1}{a_2}\right)^i P_{e_{\vec{0}}}^\ell (Z_2^\ell(\tau) = i)$$

and from (3.44) we can calculate the approximate asymptotic value of b:

$$b_e^\infty = \frac{\prod_{i=1}^2 (1 - \rho_i)}{\pi(B)} \mu_1 \kappa_e [1 - r_{1.}\rho_1 - r_{12} a_2 / a_1] \left(\frac{1}{1 - \rho_2^\ell}\right) \rho_1^\ell.$$

The factor $\pi(B)$ is kept for simulations with ℓ small to improve the estimate.

We can obtain another measure of the mean exit time by performing the simulation described in Section 3.3. Aldous' simulation estimate (3.5) can then be compared with the values of b_e^∞ for different values of ℓ. Moreover, for ℓ small, it is possible to compare the values b_e^∞ and A with the smallest real eigenvalue $\Lambda(B)$ of L^B. These can, in turn, be compared to the exact mean time to reach F from $(0,0)$ given in (3.2). The C and *Mathematica* programs used to compute these values are given in Appendix B.

4.2.1 First example

The parameters of the first Jackson network examined are

$$\begin{array}{lll} \bar{\lambda}_1 = 2 & \mu_1 = 4 & r_{12} = 0.15 \\ \bar{\lambda}_2 = 3 & \mu_2 = 6 & r_{21} = 0.3 \end{array}$$

with corresponding loads $\rho_1 = 0.759162$ and $\rho_2 = 0.575916$ which agree with the initial assumption (2.5). The twist parameters

$$a_1 = 1.317241 \qquad a_2 = 1.095172$$

obtained from (2.9), yield, by (2.10), the following parameters for the twisted process:

$$\begin{array}{lll} \bar{\lambda}_1^t = 2.634483 & \mu_1^t = 3.08 & r_{12}^t = 0.161964 \\ \bar{\lambda}_2^t = 3.285517 & \mu_2^t = 6 & r_{21}^t = 0.360831 \end{array}$$

to which correspond the loads $\rho_1^t = 1.298701$ and $\rho_2^t = 0.630728$. These values agree with the statements of Lemma 2.2.2 and Theorem 2.2.3.

Table 4.1 presents the values of $\pi(B)$, κ_ℓ, b_ℓ^∞ and A computed from the simulation for different values of ℓ. These numbers seem to confirm that $\kappa_\ell(= k^\ell(\vec{0}))$ converges to some value κ as $\ell \to \infty$. Also, the estimates b_ℓ^∞ and A are close, and thus agree with our statement that they are asymptotically equivalent. Note, however, that $A < b_\ell^\infty$. This follows from the fact that $A^{-1} \approx E_{\vec{0}}R > E_{\vec{0}}\tau$, that is, A^{-1} gives an upper bound for the mean overload time. On the other hand, b_ℓ^∞ approximates $\Lambda(B)$ whose difference from $E_{\vec{0}}\tau$ is given in absolute values. Part of the differences between the two estimates may be also imputed to the Monte Carlo error resulting from the simulations.

ℓ	$\pi(B)$	κ_ℓ	b_ℓ^∞	A
15	0.9720235	2.017463	8.467698×10^{-3}	7.843819×10^{-3}
20	0.9929092	2.075365	2.150279×10^{-3}	1.994045×10^{-3}
35	0.9998861	2.102398	3.468116×10^{-5}	3.199883×10^{-5}
100	1	2.108264	5.794361×10^{-13}	5.305142×10^{-13}
500	1	2.120460	7.931670×10^{-61}	7.333558×10^{-61}
1000	1	2.120341	$1.165890 \times 10^{-120}$	$1.059399 \times 10^{-120}$

Table 4.1: Outcome of the simulations for example 1.

ℓ	$m(1)$	$[\Lambda(B)]^{-1}$	$[b_\ell^\infty]^{-1}$	A^{-1}
15	104.28	99.46	118.10	127.49
20	442.72	434.14	465.06	501.49
35	28608.60	28585.10	28834.10	31251.14

Table 4.2: Comparison table for example 1.

For ℓ small, we can compare $m(1) = (L^B)^{-1}1(1)$, the exact mean time to reach F^∞ from $(0,0)$ (see (3.2)), $[\Lambda(B)]^{-1}$, the inverse of the smallest real eigenvalue of L^B, $[b_\ell^\infty]^{-1}$ and A^{-1}. This is shown in Table 4.2. This table reveals that the values of $m(1)$, $[\Lambda(B)]^{-1}$ and $[b_\ell^\infty]^{-1}$ get relatively closer as ℓ gets larger, even when ℓ is as small as 35. In this example, Aldous' estimate A does not seem to perform as well as b_ℓ^∞ when ℓ is small.

4.2.2 Second example

The parameters of the second Jackson network and the associated twisted network are

$$\begin{aligned} \bar{\lambda}_1 &= 3 & \bar{\lambda}_1^t &= 3.88 \\ \bar{\lambda}_2 &= 3 & \bar{\lambda}_2^t &= 3.22 \\ \mu_1 &= 5 & \mu_1^t &= 3.9 \\ \mu_2 &= 8 & \mu_2^t &= 8 \\ r_{12} &= 0.12 & r_{12}^t &= 0.127676 \\ r_{21} &= 0.25 & r_{21}^t &= 0.301242 \end{aligned}$$

with corresponding loads

$$\begin{aligned} \rho_1 &= 0.773196 & \rho_1^t &= 1.282051 \\ \rho_2 &= 0.432990 & \rho_2^t &= 0.464742 \end{aligned}$$

where the overload of the system is accelerated with the twist parameters

$$a_1 = 1.293333 \qquad a_2 = 1.073333.$$

Table 4.3 presents the values of $\pi(B)$, κ_ℓ, b_ℓ^∞ and A computed from the simulations for different values of ℓ. Table 4.4 shows, for ℓ small, $m(1)$, the exact mean time to reach F^∞ from $(0,0)$ (see (3.2)), $[\Lambda(B)]^{-1}$, the inverse of the smallest real eigenvalue of L^B, $[b_\ell^\infty]^{-1}$ and A^{-1}. These results are similar to those in the previous example and hence the same comments apply.

ℓ	$\pi(B)$	κ_ϵ	b_ϵ^∞	A
15	0.9728065	1.488334	8.532740×10^{-3}	8.263375×10^{-3}
20	0.9924850	1.493692	2.319526×10^{-3}	2.214115×10^{-3}
35	0.9998414	1.493991	4.859841×10^{-5}	4.679650×10^{-5}
100	1	1.496150	2.666857×10^{-12}	2.557029×10^{-12}
500	1	1.489209	5.492776×10^{-57}	5.352138×10^{-57}
1000	1	1.491620	$7.678192 \times 10^{-113}$	$7.441498 \times 10^{-113}$

Table 4.3: Outcome of simulations for example 2.

ℓ	$m(1)$	$[\Lambda(B)]^{-1}$	$[b_\epsilon^\infty]^{-1}$	A^{-1}
15	104.47	100.15	117.20	121.01
20	414.07	406.41	431.12	451.65
35	20542.50	20522.30	20576.81	21365.12

Table 4.4: Comparison table for example 2.

4.3 Hitting distributions

Denote by $P_{\vec{0}}(Q(\tau) = \vec{y})$ the probability that the original Jackson network Q hits the set of forbidden states at $\vec{y} \in \partial F$ given it is at state $\vec{0}$ at time 0. Similarly, denote by $e(\vec{y}^-)$ the limiting probability that the extended twisted Jackson network Z^t hits $\vec{y} \in \partial F^\infty$. From (2.25),

$$\lim_{\ell \to \infty} P_{\vec{0}}(Q(\tau) = \vec{y}) = \prod_{i=2}^{m} (a_1/a_i)^{y_i} e(\vec{y}^-) / [\sum_{\vec{z} \in \partial F} \prod_{i=2}^{m} (a_1/a_i)^{z_i} e(\vec{z}^-)].$$

We consider here the twisted process Q^t on S (not S^∞) with hitting distribution at $\vec{y} \in \partial F$ denoted by $P_{\vec{0}}(Q^t(\tau) = \vec{y})$ and define

$$\hat{P}_{\vec{0}}(Q(\tau) = \vec{y}) = \prod_{i=2}^{m} \left(\frac{a_1}{a_i}\right)^{y_i} P_{\vec{0}}(Q^t(\tau) = \vec{y}) / \sum_{\vec{z} \in \partial F} \prod_{i=2}^{m} \left(\frac{a_1}{a_i}\right)^{z_i} P_{\vec{0}}(Q^t(\tau) = \vec{z})$$

since, with *Mathematica*, it is possible to compute exact values for $P_{\vec{0}}(Q(\tau) = \vec{y})$ and $P_{\vec{0}}(Q^t(\tau) = \vec{y})$ for all $\vec{y} \in \partial F$ when ℓ and m are small. Using the parameters of the Jackson networks in examples 1 and 2 of Section 4.2, we show the values of $P_{\vec{0}}(Q(\tau) = \vec{y})$ and $\hat{P}_{\vec{0}}(Q(\tau) = \vec{y})$ when $\ell = 35$. Tables 4.5 and 4.6 show the results for examples 1 and 2 respectively. Note that the results in the second table agree up to the 5-th decimal.

$\vec{y}$	$P_{\vec{\delta}}(Q(\tau) = \vec{y})$	$\hat{P}_{\vec{\delta}}(Q(\tau) = \vec{y})$
(0,35)	0.0398735	0.0400235
(1,34)	0.113081	0.113507
(2,33)	0.133928	0.134432
(3,32)	0.130103	0.130592
(4,31)	0.115652	0.116083
(5,30)	0.0978427	0.0982025
(6,29)	0.0802591	0.0805459
(7,28)	0.0644917	0.0647095
(8,27)	0.0510807	0.0512355
(9,26)	0.0400407	0.0401386
(10,25)	0.0311476	0.0311942
(11,24)	0.0240914	0.0240915
(12,23)	0.018553	0.0185109
(13,22)	0.0142404	0.0141601
(14,21)	0.0109023	0.0107878
(15,20)	0.00833019	0.00818567
(16,19)	0.00635505	0.00618507
(17,18)	0.00484236	0.00465189
(18,17)	0.00368624	0.0034806
(19,16)	0.00280403	0.00258877
(20,15)	0.00213167	0.00191237
(21,14)	0.00161974	0.00140174
(22,13)	0.00123024	0.00101843
(23,12)	0.000934051	0.000732646
(24,11)	0.000708908	0.000521275
(25,10)	0.000537803	0.000366397
(26,9)	0.000407766	0.000254111
(27,8)	0.000308912	0.000173667
(28,7)	0.000233708	0.000116783
(29,6)	0.000176409	0.000077126
(30,5)	0.000132623	0.0000498962
(31,4)	0.000098971	0.0000314957
(32,3)	0.0000728099	0.0000192606
(33,2)	0.0000519763	0.0000112395
(34,1)	0.0000344236	0.00000600074
(35,0)	0.0000147642	0.00000216169

Table 4.5: Showing the convergence of $\hat{P}_{\vec{\delta}}(Q(\tau) = \vec{y}$ to $P_{\vec{\delta}}(Q(\tau) = \vec{y})$ in example 1.

$\vec{y}$	$P_{\vec{0}}(Q(\tau) = \vec{y})$	$\hat{P}_{\vec{0}}(Q(\tau) = \vec{y})$
(0, 35)	0.129235	0.129236
(1, 34)	0.252874	0.252877
(2, 33)	0.214965	0.214968
(3, 32)	0.152157	0.152158
(4, 31)	0.0991786	0.0991796
(5, 30)	0.0617274	0.061728
(6, 29)	0.0373193	0.0373195
(7, 28)	0.0221262	0.0221261
(8, 27)	0.0129391	0.0129388
(9, 26)	0.00749133	0.00749082
(10, 25)	0.00430507	0.00430443
(11, 24)	0.00246012	0.00245938
(12, 23)	0.00139977	0.00139896
(13, 22)	0.000793793	0.000792939
(14, 21)	0.000448978	0.000448116
(15, 20)	0.000253428	0.000252587
(16, 19)	0.000142818	0.000142024
(17, 18)	0.0000803818	0.0000796537
(18, 17)	0.0000451952	0.0000445472
(19, 16)	0.0000253908	0.0000248307
(20, 15)	0.0000142554	0.000013785
(21, 14)	7.99946×10^{-6}	7.61516×10^{-6}
(22, 13)	4.48704×10^{-6}	4.1816×10^{-6}
(23, 12)	2.51602×10^{-6}	2.27964×10^{-6}
(24, 11)	1.41041×10^{-6}	1.23218×10^{-6}
(25, 10)	7.90444×10^{-7}	6.59385×10^{-7}
(26, 9)	4.42886×10^{-7}	3.48827×10^{-7}
(27, 8)	2.48077×10^{-7}	1.82131×10^{-7}
(28, 7)	1.38893×10^{-7}	9.36895×10^{-8}
(29, 6)	7.76956×10^{-8}	4.73836×10^{-8}
(30, 5)	4.33801×10^{-8}	2.34975×10^{-8}
(31, 4)	2.41156×10^{-8}	1.13793×10^{-8}
(32, 3)	1.32665×10^{-8}	5.34247×10^{-9}
(33, 2)	7.10364×10^{-9}	2.39193×10^{-9}
(34, 1)	3.51055×10^{-9}	9.7191×10^{-10}
(35, 0)	1.08162×10^{-9}	2.56142×10^{-10}

Table 4.6: Showing the convergence of $\hat{P}_{\vec{0}}(Q(\tau) = \vec{y})$ to $P_{\vec{0}}(Q(\tau) = \vec{y})$ in example 2.

Chapter 5

Conclusion

For the well known Jackson network, our method, which consists of estimating the reciprocal of the mean overflow time through a twisted change of measure, gives asymptotically good results which are in fact equivalent to those obtained with the (simpler) method proposed by Aldous (1989). Moreover, while Aldous' estimate is an upper bound on the mean overflow time, our results give both upper and lower bounds. The power of this method, however, resides in its potential application in a more general context.

For an arbitrary queueing network with generator $-L$ on S, we can construct the extended process on S^{∞} with generator $-L^{\infty}$ and define the operator

$$L^t k(\vec{x}) = \frac{1}{\prod_{i=1}^m a_i^{x_i}} L^{\infty} \prod_{i=1}^m a_i^{x_i} k(\vec{x}), \quad \vec{x} \in S^{\infty}.$$

By choosing constants $a_1, \ldots, a_m$ so that $-L^t$ is a generator on S^{∞}, we obtain the extended twisted process with generator $-L^t$. Using Stewart's (1971) result as stated in Corollary 3.6.2, we get an estimator for $1/E_{\vec{0}}\tau$

$$b = \frac{< f^{\infty}, L^B f^{\infty} >_{\hat{\pi}}}{< f^{\infty}, f^{\infty} >_{\hat{\pi}}} \to < 1, L^B f_{\kappa}^{\infty} >_{\hat{\pi}} = \sum_{\vec{x}_0 \in B_0} L^B f_{\kappa}^{\infty}(\vec{x}_0) \hat{\pi}(\vec{x}_0) \tag{5.1}$$

as $\ell \to \infty$, where B_0, B, f^{∞} and f_{κ}^{∞} are defined as before, and $\hat{\pi}$ is the stationary distribution of that (more general) process on B.

Many aspects of the calculations still have to be established. For instance, the relationship between the a_i's given in Lemma 2.2.2 may not hold, we ignore the behaviour of the twisted process; more than one queue size may be transient, κ may not tend to a limit! Furthermore, the stationary measure $\hat{\pi}$ is generally unknown. Note, however, that the summation in (5.1) is over the set B_0 only so if everything works as in the Jackson case, (5.1) tends to

$$-a_1^{-\ell}\left(\sum_{\vec{x}_0\in B_0} L(\prod_{i=2}^{m} a_i^{x_i})\pi(\vec{x}_0)\right)\kappa$$

where

$$\kappa = \lim_{\ell\to\infty} E_{\vec{0}}\prod_{i=2}^{m}\left(\frac{a_1}{a_i}\right)^{Z^i(\tau)}.$$

Now, as before, κ is quickly estimated by κ_ϵ and $\sum_{\vec{x}_0\in B_0} L(\prod_{i=2}^{m} a_i^{x_i})\pi(\vec{x}_0)$ can be estimated quickly by simulating Q near $\vec{0}$. This is a clear advantage over Aldous' (1989) method where $1/E_{\vec{0}}\tau$ is estimated by

$$\sum_{\vec{y}\in\partial F}\pi(\vec{y})\sum_{\vec{x}\in B} q(\vec{y},\vec{x})f(\vec{x})$$

and the stationary measure must be known for all $\vec{y}\in\partial F$. Thus, although proving the asymptotical validity of $1/b^\infty$ as an estimator for $E_{\vec{0}}\tau$ was decidedly harder than establishing the asymptotic quality of Aldous' (1989) estimate, the twist method proposed here is assuredly destined to wider applications.

Appendix A

The nonsingularity of L^B

Since $B = \{\vec{x} \in S : \sum_{i=1}^{m} x_i < \ell\}$ is finite, we can express the generator of the Jackson network Q killed off B, $-L^B$, as a matrix of finite dimensions by using the appropriate indexing method $i = 1, 2, 3, \ldots, \dim(B) = \sum_{x_1=0}^{\ell-1} \sum_{x_2=0}^{\ell-1-x_1} \cdots \sum_{x_m=0}^{\ell-1-\sum_{k=1}^{m-1} x_k} 1$. For instance, consider the bijection $c^B : \{1, 2, \ldots, \dim(B)\} \to B$ defined from the index $c : N \to S$ described in Section 3.1.

Similarly, the kernel of the embedded Markov chain killed off B, K^B, can be viewed as a $\dim(B) \times \dim(B)$ matrix with entries $K_{ij} = q(i,j)/q(i)$, if $i \neq j$, and $K_{ii} = 0$, where we recall $q(i,j) := -L_{ij}$ and $q(i) := L_{ii}$. Defining $\mathrm{diag}^B(q)$ as the $\dim(B) \times \dim(B)$ diagonal matrix with (i,i)-th entry $q(i)$, we obtain

$$\mathrm{diag}^B(q)\,(I - K^B) = L^B.$$

By our assumption that the Jackson network is both open and exogenously supplied, $q(i) > 0$ for $i = 1, \ldots, \dim(B)$ so that $\mathrm{diag}^B(q)$ is invertible. Consequently, L^B is nonsingular if and only if $(I - K^B)$ is nonsingular. Now, since

$$(I - K^B)^{-1} = \sum_{n=0}^{\infty} (K^B)^n,$$

$(I - K^B)^{-1}$ exists only if the sum on the right-hand side converges. Consider the (i,j)-th entry of the infinite sum,

$$\sum_{n=0}^{\infty} (K^B)^n_{ij} = \sum_{n=0}^{\infty} P_{c^B(i) \in B}(Q \text{ reaches } c^B(j) \in B \text{ after } n \text{ jumps while always remaining in } B).$$

This sum certainly converges, and thus $(I - K^B)$ is nonsingular, if

$$\sum_{c^B(j)\in B} \sum_{n=0}^{\infty} (K^B)^n_{ij} < \infty. \qquad (A.1)$$

Now, $\sum_{c^B(j)\in B}(K^B)^n_{ij} = P_{c^B(i)\in B}(Q$ is still in B after n jumps). Consider the state in B which is the farthest away from F, i.e., the origin $\vec{0} = c^B(1)$, and assume for now that states in F are absorbing. Since, by the assumption that the network is exogenously supplied $\bar{\lambda}_i > 0$ for some $i = 1, \ldots, m$, Q can reach ∂F from $\vec{0}$ after ℓ exogenous arrivals at node i (i.e. ℓ jumps – we suppose ℓ fixed). Hence, if we denote by $J_k(s)$ the time at which will occur the k-th jump after the present time s, we have

$$P_{\vec{0}}(Q(J_\ell(s)) \in F) > \epsilon$$

for some $\epsilon > 0$ since

$$P_{\vec{0}}(Q(J_\ell(s)) = \ell\vec{e}_i \in F) = \frac{\bar{\lambda}_i}{\sum_{j=1}^m \bar{\lambda}_j} \left(\frac{\bar{\lambda}_i}{\sum_{j=1}^m \bar{\lambda}_j + \mu_i} \right)^{\ell-1} > 0.$$

Hence, in general, for any $c^B(i) \in B$,

$$P_{c^B(i)}(Q(J_\ell(s)) \in F) = 1 - \sum_{c^B(j)\in B} (K^B)^\ell_{ij} > \epsilon,$$

or

$$\sum_{c^B(j)\in B} (K^B)^\ell_{ij} < (1-\epsilon),$$

and, for any integer k,

$$\sum_{c^B(j)\in B} (K^B)^{k\ell}_{ij} = \sum_{c^B(j_1)\in B} \sum_{c^B(j_2)\in B} \cdots \sum_{c^B(j)\in B} (K^B)^\ell_{ij_1} (K^B)^\ell_{j_1 j_2} \cdots (K^B)^\ell_{j_{k-1} j} < (1-\epsilon)^k.$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{c^B(j)\in B} (K^B)^n_{ij} &= \sum_{k=1}^{\infty} \sum_{n=(k-1)\ell}^{k\ell-1} \sum_{c^B(j)\in B} (K^B)^n_{ij} \\ &< \sum_{k=1}^{\infty} \ell \sum_{c^B(j)\in B} (K^B)^{(k-1)\ell}_{ij} \\ &\leq \ell \sum_{k=0}^{\infty} (1-\epsilon)^k < \infty \end{aligned}$$

since $(1-\epsilon) < 1$, i.e., (A.1) holds and $(I - K^B)$, thus L^B, are nonsingular. Note however, that as $\ell \to \infty$, $B \to S$ so that $L^B \to L$ where $-L$, a generator, is singular. Hence, as ℓ gets larger, L^B gets nearly singular.

Appendix B

Computer programs

B.1 Mathematica programs: exact results

This *Mathematica* program computes the exact mean time until an overflow given the process started at $\vec{0}$ ($E_{\vec{0}}\tau$) and the Perron-Frobenius eigenvalue $\Lambda(B)$ of the matrix $-L^B$.

```
(* T I M E T O F . M A *)
(* Critical number of customers for overflow. *)
L=35;

(* Parameters of the original Jackson network. *)
ilam1 = 3;
ilam2 = 3;
moo1  = 5;
moo2  = 8;
r12   = .12;
r21   = .25;

coord[i1_,i2_,j1_,j2_]:=
Which[j1==i1+1  && j2==i2,ilam1,
      j2==i2+1 && j1==i1,ilam2,
      j1==i1-1 && j2==i2 &&i1 >0, (1-r12) moo1,
      j2==i2-1 && j1==i1 && i2 >0,(1-r21) moo2,
      j1==i1-1 && j2==i2+1 && i1>0,r12 moo1,
      j2==i2-1 && j1==i1+1 && i2>0,r21 moo2,
      i1==0    && j1==0  && i2==0 && j2==0,-(ilam1+ilam2),
      i1==0    && j1==0  && i2==j2,-(ilam1+ilam2+moo2),
      i2==0    && j2==0  && i1==j1,-(ilam1+ilam2+moo1),
      j1==i1   && j2==i2, -(ilam1+ilam2+moo1+moo2),
      True,0]

bigmatrix[x_,y_]:=coord[Quotient[x-1,L+1], Mod[x-1,L+1],
                        Quotient[y-1,L+1], Mod[y-1,L+1]]

bigjump=Array[bigmatrix, {(L+1)^2, (L+1)^2}];

sumqueue[n_] := Quotient[n-1,L+1] + Mod[n-1,L+1]
```

```
good[i_] := If[sumqueue[i] < L, i, 0]
mat =Array[good,{ (L+1)^2}];
mat0 = Position[mat,0];
indexkeep = Delete[mat, mat0];

(* The generator of the process on B. *)
gb=bigjump[[ indexkeep, indexkeep]];

gbinv=Inverse[gb];
Dimensions[%]
L (L+1)/2     (*These two should be the same.*)

{630, 630}
630

un[x_]=1;
vecteurun = Array[un,L (L+1)/2];

m=-gbinv . vecteurun;

vals=Eigenvalues[gb];

First[m]
-Last[vals]^(-1)

20542.5
20522.3
```

This *Mathematica* program computes the hitting distribution of ∂F from $\vec{0}$ for the original and the twisted Jackson networks. From these, we can evaluate the calculations described in Section 4.3.

```
(* H I T D I S T B O T H . M A *)
(* Critical number of customers for overflow. *)
L=35;

(* Parameters of the original Jackson network. *)
ilam1 = 3;
ilam2 = 3;
mu1   = 5;
mu2   = 8;
r12   = .12;
r21   = .25;

(* Load on the system *)
lam1 = (ilam1 + r21 ilam2) /(1 - r21 r12);
lam2 = (ilam2 + r12 ilam1)/(1-r12 r21);
rho1 = lam1/mu1
rho2 = lam2/mu2

0.773196
0.43299

(* Parameters of the twisted Jackson network. *)
a1 = rho1 ^-1
a2 = ((1-r21) rho1 + r21)/rho1

1.29333
1.07333

ilam1t = ilam1 a1
ilam2t = ilam2 a2
mu1t   = mu1 ((1-r12)/a1 + r12 a2 / a1)
mu2t   = mu2
r12t   = r12 a2 / (1-r12 + r12 a2)
r21t   = r21 a1 / a2

3.88
3.22
3.9
8
0.127676
0.301242

(* Creation of the generator for the original process. *)
coord[i1_,i2_,j1_,j2_]:=
Which[j1==i1+1  && j2==i2,ilam1,
      j2==i2+1 && j1==i1,ilam2,
      j1==i1-1 && j2==i2 &&i1 >0, (1-r12) mu1,
      j2==i2-1 && j1==i1 && i2 >0,(1-r21) mu2,
```

```
      j1==i1-1 && j2==i2+1 && i1>0,r12 mu1,
      j2==i2-1 && j1==i1+1 && i2>0,r21 mu2,
      i1==0    && j1==0  && i2==0 && j2==0,-(ilam1+ilam2),
      i1==0    && j1==0  && i2==j2,-(ilam1+ilam2+mu2),
      i2==0    && j2==0  && i1==j1,-(ilam1+ilam2+mu1),
      j1==i1   && j2==i2, -(ilam1+ilam2+mu1+mu2),
      True,0]

bigmatrix[x_,y_]:=coord[Quotient[x-1,L+1], Mod[x-1,L+1],
                        Quotient[y-1,L+1], Mod[y-1,L+1]]

bigjump=Array[bigmatrix, {(L+1)^2, (L+1)^2}];

(* Creation of the generator for the twisted process. *)
coordtwist[i1_,i2_,j1_,j2_]:=
Which[j1==i1+1  && j2==i2,ilam1t,
      j2==i2+1 && j1==i1,ilam2t,
      j1==i1-1 && j2==i2 &&i1 >0, (1-r12t) mu1t,
      j2==i2-1 && j1==i1 && i2 >0,(1-r21t) mu2t,
      j1==i1-1 && j2==i2+1 && i1>0,r12t mu1t,
      j2==i2-1 && j1==i1+1 && i2>0,r21t mu2t,
      i1==0    && j1==0  && i2==0 && j2==0,-(ilam1t+ilam2t),
      i1==0    && j1==0  && i2==j2,-(ilam1t+ilam2t+mu2t),
      i2==0    && j2==0  && i1==j1,-(ilam1t+ilam2t+mu1t),
      j1==i1   && j2==i2, -(ilam1t+ilam2t+mu1t+mu2t),
      True,0]

bigmatrixtwist[x_,y_]:=coordtwist[Quotient[x-1,L+1], Mod[x-1,L+1],
                        Quotient[y-1,L+1], Mod[y-1,L+1]]

bigjumptwist=Array[bigmatrixtwist, {(L+1)^2, (L+1)^2}];

sumqueue[n_] := Quotient[n-1,L+1] + Mod[n-1,L+1]
good[i_] := If[sumqueue[i] < L, i, 0]
mat =Array[good,{ (L+1)^2}];
mat0 = Position[mat,0];
indexkeep = Delete[mat, mat0];

(* The generator of the original process on B. *)
gb=bigjump[[ indexkeep, indexkeep]];

gbinv=Inverse[gb];
Dimensions[%]
L (L+1)/2     (*These two should be the same.*)

{630, 630}
630

(* The generator of the twisted process on B. *)
gbtwist=bigjumptwist[[ indexkeep, indexkeep]];

gbtwistinv=Inverse[gbtwist];
```

```
Dimensions[%]
L (L+1)/2     (*These two should be the same.*)

{630, 630}
630

(* Probability the original process hits F at (L-y,y) given *)
(* it starts at (0,0). *)
probhit[y_] := - Sum[gbinv[[1,k]]
                 bigjump[[indexkeep[[k]],(L+1) (L-y) + y + 1]],
                        {k, 1, L (L+1)/2} ]
disthit = Table[probhit[k-1], {k,L+1}]

{0.129235, 0.252874, 0.214965, 0.152157, 0.0991786, 0.0617274, 0.0373193,

 0.0221262, 0.0129391,

  0.00749133, 0.00430507, 0.00246012, 0.00139977, 0.000793793, 0.000448978,

  0.000253428, 0.000142818,
                                                                       -6
  0.0000803818, 0.0000451952, 0.0000253908, 0.0000142554, 7.99946 10   ,
             -6            -6
  4.48704 10   , 2.51602 10   ,
             -6            -7             -7             -7             -7
  1.41041 10   , 7.90444 10   , 4.42886 10   , 2.48077 10   , 1.38893 10   ,
             -8            -8
  7.76956 10   , 4.33801 10   ,
             -8            -8             -9             -9             -9
  2.41156 10   , 1.32665 10   , 7.10364 10   , 3.51055 10   , 1.08162 10   }


(* Probability the twisted process hits F at (L-y,y) given *)
(* it starts at (0,0). *)
probhittwist[y_] := -Sum[gbtwistinv[[1,k]] bigjumptwist[[indexkeep[[k]],
                        (L+1) (L-y) + y + 1]], {k, 1, L (L+1)/2} ]
disthittwist = Table[probhittwist[k-1], {k,L+1}]

{0.193111, 0.313585, 0.22123, 0.129954, 0.0702977, 0.0363099, 0.0182181,

  0.00896388, 0.0043502,

  0.00209011, 0.000996734, 0.000472621, 0.000223109, 0.000104948,

  0.0000492211, 0.0000230248,
                          -6             -6             -6             -7
  0.0000107441, 5.00079 10   , 2.32101 10   , 1.07367 10   , 4.94665 10   ,
             -7            -7
  2.26782 10   , 1.03347 10   ,
             -8            -8             -9            -9             -9
  4.67568 10   , 2.09737 10   , 9.31462 10   , 4.0894 10   , 1.77198 10   ,
            -10           -10
```

```
 7.56466 10    , 3.17505 10    ,
           -10           -11            -11           -12            -12
 1.30668 10    , 5.25156 10    , 2.04615 10    , 7.6027 10    , 2.56371 10    ,
           -13
  5.60722 10   }

numerator = Table[disthittwist[[k]] (a1/a2)^(k-1), {k, L+1}];
kappa = Sum[numerator[[i+1]], {i,0,L}];
disthitorig = 1/kappa numerator

{0.129236, 0.252877, 0.214968, 0.152158, 0.0991796, 0.061728, 0.0373195,

  0.0221261, 0.0129388,

  0.00749082, 0.00430443, 0.00245938, 0.00139896, 0.000792939, 0.000448116,

  0.000252587, 0.000142024,
                                                                      -6
  0.0000796537, 0.0000445472, 0.0000248307, 0.000013785, 7.61516 10    ,
          -6           -6
  4.1816 10   , 2.27964 10   ,
           -6           -7            -7           -7            -8
  1.23218 10   , 6.59385 10   , 3.48827 10   , 1.82131 10   , 9.36895 10   ,
           -8           -8
  4.73836 10   , 2.34975 10   ,
           -8           -9            -9          -10            -10
  1.13793 10   , 5.34247 10   , 2.39193 10   , 9.7191 10    , 2.56142 10    }
```

B.2 C programs: simulations

This *C* program is used to simulate the extended twisted Jackson network and to calculate the estimate for κ.

```
/* T W I S T E D . H */

#ifndef _TWISTED_

   #define _TWISTED_

   /* Definition of the structure to hold the current position
      of the trajectory.  */

      typedef struct tag_point
      {
       int x,y;
      } Point;

   /* Critical sum of the length of the queues. */
      #define L      1000

   /* Value assumed to be at least bigger than the highest y value
      hit in F.*/
      #define highy 50

   /* Prototypes */

      extern void    InitializeGlobal();
      extern void    WriteResults();
      extern Point   GenerateJump(Point Pos);
      extern int     AtBorder(Point Pos);

#endif


/* T W I S T E D . C */

/*********************************************************************
 *
 *  Hitting distribution of a twisted extended Jackson network.
 *
 *  For a twisted Jackson network with 2 nodes, we study the
 *  hitting distribution of the process Z(t)= (X(t), Y(t)) on the
 *  set of forbidden states F={(x,y): x+y=L}.
 *
 *********************************************************************/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
```

```
#include "twisted.h"

/*
 * Parameters of the simulation:
 *
 * NTraject : Number of trajectories that must reach the boundary.
 * ilam1, ilam2, mu1, mu2, r12, r21: Transition parameters
 *     of the original Jackson network.
 * Warning !!!
 * Do not define a constant like  r12=1/2
 * Use decimal points i.e.  r12=1./2.
 */

const int    NTraject = 50000;
const double ilam1 = 2.0;
const double ilam2 = 3.0;
const double mu1 = 4.0;
const double mu2 = 6.0;
const double r12 = 0.15;
const double r21 = 0.3;

/*  Name of the output file. */

const char *outputfile="programmes_c/twisted.res";

/*
 * Seed used by the random function.
 * It consists of an array of three 16-bit integers {0-65535}
 */
const short unsigned seed[]={ 23201, 878, 15955 };

/*
 * Declaration of global variables.
 *
 * a1,a2: Twist parameters.
 * ilam1t,ilam2t,mu1t,mu2t,r12t,r21t: Transition parameters of the
 *   twisted process.
 * LastStep: Array to record where trajectories hit the boundary.
 * We assume that the trajectory will not hit F at a y-value > highy.
 */
double a1, a2, ilam1t, ilam2t, mu1t, mu2t, r12t, r21t;
int LastStep[1 + highy];

/*****************************************************************************
 * This initialization routine is called before the start of the simulation.
 *****************************************************************************/
void InitializeGlobal()
{
  int i;
  double rho1;

  rho1 = (ilam1 + r21 * ilam2)/((1-r21*r12) * mu1);
```

```
  a1 = 1/rho1;
  a2 = ( (1-r21) * rho1 + r21) /rho1;
  ilam1t = ilam1 * a1;
  ilam2t = ilam2 * a2;
  mu1t = mu1 * ((1-r12)/a1 + r12 * a2/a1);
  mu2t = mu2;
  r12t = r12 * a2/(1-r12+r12 * a2);
  r21t = r21 * a1/a2;

  for (i=0;i<=highy;i++)
      LastStep[i]=0;
}

/***************************************************************************
 * Use this routine to generate tables of results. The output is sent
 * to the file specified by 'outputfile'.
 *
 * This routine writes, for each (L-y,y) in F, the number (and proportion)
 * of trajectories that hit F at that coordinate.
 *
 * The format is the following:
 *       "y #hits proportion"
 ***************************************************************************/
void WriteResults()
{
  int i;
  double prop;
  FILE *outFile;

  outFile=fopen(outputfile,"w");

  for (i=0;i<=highy;i++)
  {
      if (NTraject > 0)
         prop = LastStep[i]/(double)NTraject;
      else prop=0.0;
      fprintf(outFile,"%3d %3d %9.6f\n",i,LastStep[i],prop);
  }
  fclose(outFile);
}

/****************************************************************************
 *  This function is used to generate a transition from the current position.
 *  The value returned is the new position.
 *  Since movements past the x=0 line are allowed, there are only
 *  2 cases to consider:
 *
 *  - The current position is not on the x-axis.
 *  - The current position is along the x-axis.
 *
 ****************************************************************************/
Point GenerateJump(Point Pos)
```

```
{
 double R,Sum;
 double ratio1,ratio2,ratio3,ratio4,ratio5;

/*
 *  R is a pseudo-uniform random number (double precision) on (0,1).
 */

  R=drand48();

/*
 *  First case:  the current position is not
 *  on the x-axis. (6 possible transitions)
 */
 if (Pos.y>0)
  {
      Sum=ilam1t+ilam2t+mu1t+mu2t;
      ratio1=ilam1t/Sum;
      ratio2=ratio1+(ilam2t/Sum);
      ratio3=ratio2+(mu1t*r12t/Sum);
      ratio4=ratio3+(mu1t*(1-r12t)/Sum);
      ratio5=ratio4+(mu2t*(1-r21t)/Sum);

      if (R<=ratio1)
Pos.x++;
      else if (R<=ratio2)
Pos.y++;
      else if (R<=ratio3)
      {
        Pos.x--; Pos.y++;
      }
      else if (R<=ratio4)
Pos.x--;
      else if (R<=ratio5)
Pos.y--;
      else
      {
Pos.x++; Pos.y--;
      }
   }
  /*
   *  Second case: the current position is on the x-axis.
   */
    else if (Pos.y==0)
    {
      Sum=ilam1t+ilam2t+mu1t;
      ratio1=ilam1t/Sum;
      ratio2=ratio1+(ilam2t/Sum);
      ratio3=ratio2+(mu1t*r12t/Sum);

      if (R<=ratio1)
        Pos.x++;
```

```
    else if (R<=ratio2)
      Pos.y++;
    else if (R<=ratio3)
    {
      Pos.x--; Pos.y++;
    }
    else
      Pos.x--;
  }
/*
 * If the boundary is reached, record the event in LastStep.
 */
  if (Pos.x+Pos.y==L)
     {
     if (Pos.y > highy)
printf("W A R N I N G!!  Y(hit) = %5d > highy\n",Pos.y);
     else LastStep[Pos.y]++;
     }
  return Pos;
}

/************************************************************************
 * This function checks if the current position is on the frontier.
 * If it is true the function returns 1, else 0.
 ***********************************************************************/
int AtBorder(Point Pos)
{
  if (Pos.x+Pos.y==L) return 1;
  return 0;
}

/**************************************************************************
 *                    M A I N   P R O G R A M
 *************************************************************************/

void main()
{
  Point Pos;
  int i,count=0;
  double K=0.0;

  InitializeGlobal();
  seed48(seed);
  printf("Simulation running...\n\n");
  printf("The parameters for the twisted process are...\n");
  printf("a1     : %9.6f  a2     :  %9.6f\n",a1,a2);
  printf("ilam1t: %9.6f  ilam2t:  %9.6f\n",ilam1t,ilam2t);
  printf("mu1t:   %9.6f  mu2t   :  %9.6f\n",mu1t,mu2t);
  printf("r12t:   %9.6f  r21t:     %9.6f\n\n",r12t,r21t);

 /* Generate trajectories until 'NTraject' of them reach the boundary. */
  while (count<NTraject)
```

```
  {
      Pos.x=0;
      Pos.y=0;

      do
        Pos=GenerateJump(Pos);
      while (!AtBorder(Pos));

      count++;

/* Print a message for every 10000 trajectories that reached the boundary */
      if ((count%10000)==0) printf("Boundary reached %d times\n",count);
  }
  WriteResults();

  printf("\n  y     Proportion that hits y\n");
  for (i=0;i<=highy;i++)
  {
      K += pow(a1/a2,(double) i)*LastStep[i]/(double)NTraject;
      if (i%2)
      printf("   %3d      %9.6f",i,LastStep[i]/(double)NTraject);
      else
      printf("   %3d      %9.6f\n",i,LastStep[i]/(double)NTraject);
  }
  printf("\n kl(0,0)= %9.6f\n",K);
}
```

Given the value of κ_e obtained from *twisted.c*, this *C* program computes the value of b_e^∞ which corresponds to a certain value of ℓ.

```
/* W H A T S B . H */

#ifndef _WHATSB_
   #define _WHATSB_

   /* Prototypes */

   extern double pie   (double rho1, double rho2, int x, int y);
   extern double pie_B (int L, double rho1, double rho2);

#endif

/* W H A T S B . C */

/****************************************************************
 *
 * We compute the guess b for Lambda(B) in the extremely
 * stable case by using the limit K obtained from twisted.c.
 *
 ***************************************************************/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "whatsb.h"

/* Transition parameters of the original Jackson network.
 * W A R N I N G ! ! !
 * Do not define a constant like r12=1/2
 * Use decimal points, i.e. r12=1./2.
 */

const double ilam1 = 2.0;
const double ilam2 = 3.0;
const double m001  = 4.0;
const double m002  = 6.0;
const double r12   = 0.15;
const double r21   = 0.3;

/* This function computes the invariant distribution of the system
 * with x customers in node 1 and y customers in node 2.
 */
double pie(double rho1, double rho2, int x, int y)
{
return (1-rho1)*pow(rho1,(double) x)*(1-rho2)*pow(rho2,(double) y);
}

/* This function sums the invariant distribution of all states in
 * the "good" set B.
```

```
 */
double pie_B(int L, double rho1, double rho2)
{
int i,j;
double temp=0.0;

for (i=0; i<L; i++)
    for (j=0; i+j < L; j++)
        temp += pie(rho1, rho2, i, j);
return temp;
}

/*                        M A I N   P R O G R A M
 *
 * Prompts for the critical length L, limit K and displays
 * the value b.
 */
void main()
{
double lam1,lam2,r001,r002,a1,a2,K,bass;
double  r011, pi_B;
int    L, i;

/* Parameters */
lam1  = (ilam1 + r21*ilam2)/(1-r21*r12);
lam2  = (ilam2 + r12*ilam1)/(1-r21*r12);
r001  = lam1/m001;
r002  = lam2/m002;
a1    = 1/r001;
a2    = (r001 * (1-r21) + r21)/r001;

printf("\n\nLoad on the system:  r001=%9.6f   r002=%9.6f\n",r001,r002);
printf("Twist parameters:     a1=%9.6f      a2=%9.6f\n",a1,a2);

/* Input more parameters */
printf("\nEnter the critical length L:  ");
scanf("%d", &L);
printf("\nEnter the value K:  ");
scanf("%lf", &K);
printf("\n... computing ... \n");

/* Compute asymptotic value of b */
r011 = pow(r001, (double) L);

pi_B  = pie_B(L,r001,r002);

bass = (1/pi_B)*(1-r001)*(1-r002) * m001 * K * r011
       * (1 - (1-r12)*r001 - r12*a2/a1) *(1/(1-r002*a2));

printf("pi_B = %e\n",pi_B);
printf("Result:  bass=%e\n",bass);
}
```

This C program simulates the original Jackson network in order to compute the value of A (described in Section 3.3), where A^{-1} approximates Aldous' heuristic estimator of $E_{\vec{0}}\tau$.

```
/* A L D O U S . H */

#ifndef _ALDOUS_

   #define _ALDOUS_

   /* Definition of the structure to hold the current position
      of the trajectory.  */

      #define L  1000

      typedef struct tag_point
      {
       int x,y;
      } Point;

   /* Prototypes */

      extern int    startwhereinF(double *);
      extern Point  jumpwhereinB(int);
      extern Point  GenerateJump(int,
                                 Point,
                                 int *);
      extern int    AtBorder(Point);

#endif


/* A L D O U S . C */

/*********************************************************************
 *
 *  What does Aldous' method say?
 *
 *********************************************************************/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

#include "aldous.h"

/*
 * Parameters of the simulation:
 *
 * NTraject : Number of trajectories that must reach the boundary.
 * ilam1, ilam2, mu1, mu2, r12, r21: Transition parameters
 *     of the original Jackson network.
 * Warning !!!
```

```
 * Do not define a constant like  r12=1/2
 * Use decimal points i.e.  r12=1./2.
 */

const int    NTraject =50000;
const double ilam1 = 2.0;
const double ilam2 = 3.0;
const double mu1 = 4.0;
const double mu2 = 6.0;
const double r12 = 0.15;
const double r21 = 0.3;

/*
 * Seed used by the random function.
 * It consists of an array of three 16-bit integers {0-65535}
 */
const short unsigned seed[]={ 23201, 878, 15955 };

/*****************************************************************************
 * This function determines the starting point onthe boundary of F.
 *****************************************************************************/
int startwhereinF(double *pis)
{
double R;
int    y=-1;

R=drand48();

do
  ++y;
while (R > pis[y]/pis[L]);
return y;
}

/*****************************************************************************
 * This function determines where we jump in B from F.
 *****************************************************************************/
Point jumpwhereinB(int y)
{
double R;
Point  Pos;

R = drand48();

if (y==0)
  {
  Pos.x=L-1;
  Pos.y=0;
  return Pos;
  }
 if (y==L)
  {
```

```
  Pos.x=0;
  Pos.y=L-1;
  return Pos;
  }
if (R <= mu1*(1-r12)/(mu1*(1-r12)+mu2*(1-r21)))
  {
  Pos.x=L-y-1;
  Pos.y=y;
  return Pos;
  }
else
  {
  Pos.x=L-y;
  Pos.y=y-1;
  return Pos;
  }
}

/*****************************************************************************
 *  This function is used to generate a transition from the current position.
 *  The value returned is the new position.
 *  There are four cases to consider:
 *
 *  - The current position is not on an axis.
 *  - The current position is along the x-axis.
 *  - The current position is along the y-axis.
 *  - The current position is the origin.
 *
 *****************************************************************************/
Point GenerateJump(int ystart, Point Pos,  int *Reach0)
{
 double R,Sum;
 double ratio1,ratio2,ratio3,ratio4,ratio5;

/*
 *  R is a pseudo-uniform random number (double precision) on (0,1).
 */

  R=drand48();

/*
 *  First case:  the current position is not
 *  on  an axis. (6 possible transitions)
 */
 if ((Pos.x>0) && (Pos.y>0))
  {
     Sum=ilam1+ilam2+mu1+mu2;
     ratio1=ilam1/Sum;
     ratio2=ratio1+(ilam2/Sum);
     ratio3=ratio2+(mu1*r12/Sum);
     ratio4=ratio3+(mu1*(1-r12)/Sum);
     ratio5=ratio4+(mu2*(1-r21)/Sum);
```

```
      if (R<=ratio1)
Pos.x++;
      else if (R<=ratio2)
Pos.y++;
      else if (R<=ratio3)
      {
        Pos.x--; Pos.y++;
      }
      else if (R<=ratio4)
Pos.x--;
      else if (R<=ratio5)
Pos.y--;
      else
      {
Pos.x++; Pos.y--;
      }
  }
/*
 *  Second case: the current position is on the x-axis,
 *  but not at the origin.
 */
  else if ((Pos.x>0) && (Pos.y==0))
  {
    Sum=ilam1+ilam2+mu1;
    ratio1=ilam1/Sum;
    ratio2=ratio1+(ilam2/Sum);
    ratio3=ratio2+(mu1*r12/Sum);

    if (R<=ratio1)
      Pos.x++;
    else if (R<=ratio2)
      Pos.y++;
    else if (R<=ratio3)
    {
      Pos.x--; Pos.y++;
    }
    else
      Pos.x--;
  }
 /*
  * Third case:  the current position is on the y-axis,
  * but not at the origin.
  */
  else if ((Pos.x==0) && (Pos.y>0))
  {
   Sum=ilam1+ilam2+mu2;
   ratio1=ilam1/Sum;
   ratio2=ratio1+(ilam2/Sum);
   ratio3=ratio2+(mu2*(1-r21)/Sum);

   if (R<=ratio1)
```

```
     Pos.x++;
  else if (R<=ratio2)
     Pos.y++;
  else if (R<=ratio3)
     Pos.y--;
  else
     {
     Pos.x++; Pos.y--;
     }
  }
/*
 * Fourth case: the current position is the origin.
 */
 else if ((Pos.x==0) && (Pos.y==0))
 {
  Sum=ilam1+ilam2;
  ratio1=ilam1/Sum;
  if (R<=ratio1)
     Pos.x++;
  else
     Pos.y++;
 }
/*
 * If (0,0) is reached, record the event.
 */
  if ( (Pos.x==0) && (Pos.y==0) )
    {
     if (ystart==0)
     *Reach0 += mu1*(1-r12);
     else if (ystart==L)
     *Reach0 += mu2*(1-r21);
     else
     *Reach0 += mu1*(1-r12)+mu2*(1-r21);
    }

  return Pos;
}

/*****************************************************************************
 * This function checks if the current position is on the frontier or (0,0).
 * If it is true the function returns 1, else 0.
 ****************************************************************************
*/
int AtBorder(Point Pos)
{
   if ((Pos.x+Pos.y==L) ||
      ( (Pos.x==0) && (Pos.y==0) ))
      return 1;
   return 0;
}

/**************************************************************************
```

```
 *                    M A I N   P R O G R A M
 ***********************************************************************/
void main()
{
  Point Pos;
  int   count=0,Reach0=0,y,i;
  double pis[L+1],rho1,rho2,pi=0.0,Aldous;

  rho1=(ilam1+r21*ilam2)/((1-r21*r12)*mu1);
  rho2=(ilam2+r12*ilam1)/((1-r21*r12)*mu2);

  for (i=0; i<=L; i++)
  {
  pi += (1-rho1)*(1-rho2)
        * pow(rho1, (double) L-i)
        * pow(rho2, (double) i);
  pis[i] = pi;
  }

  seed48(seed);
  printf("Simulation running...\n\n");

/* Generate trajectories until 'NTraject' of them reach the boundary
 * or (0,0).
 */
  while (count<NTraject)
  {
      y=startwhereinF(pis);
      Pos=jumpwhereinB(y);

      do
         Pos=GenerateJump(y,Pos,&Reach0);
      while (!AtBorder(Pos));

      count++;

/* Print a message for every 10000 trajectories that terminated. */
      if ((count%10000)==0) printf("Boundary or (0,0) reached %d times\n",count);
  }

  Aldous = pis[L] * (double) Reach0/NTraject;
  printf("Aldous says: %e\n",Aldous);
}
```

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