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Categorical Models of Computation: Partially Traced Categories and Presheaf Models of Quantum Computation

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CATEGORICAL MODELS OF COMPUTATION: PARTIALLY TRACED CATEGORIES AND PRESHEAF MODELS OF QUANTUM COMPUTATION

By

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Abstract

This dissertation has two main parts. The first part deals with questions relating to Haghverdi and Scott's notion of partially traced categories. The main result is a representation theorem for such categories: we prove that every partially traced category can be faithfully embedded in a totally traced category. Also conversely, every monoidal subcategory of a totally traced category is partially traced, so this characterizes the partially traced categories completely. The main technique we use is based on Freyd's paracategories, along with a partial version of Joyal, Street, and Verity's Int construction. Along the way, we discuss some new examples of partially traced categories, mostly arising in the context of quantum computation.

The second part deals with the construction of categorical models of higher-order quantum computation. We construct a concrete semantic model of Selinger and Valiron's quantum lambda calculus, which has been an open problem until now. We do this by considering presheaf categories over appropriate base categories arising from first-order quantum computation. The main technical ingredients are Day's convolution theory and Kelly and Freyd's notion of continuity of functors. We first give an abstract description of the properties required of the base categories for the model construction to work; then exhibit a specific example of base categories satisfying these properties.

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To Magdalena

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Chapter 1

Introduction

Quantum computers are computing devices which are based on the laws of quantum physics. While no actual general-purpose quantum computer has yet been built, research in the last two decades indicates that quantum computers would be vastly more powerful than classical computers. For instance, Shor proved in 1994 that the integer factoring problem can be solved in polynomial time on a quantum computer, while no efficient classical algorithm is known.

The goal of this research is to extend existing connections between logic and computation, and to apply them to the field of quantum computation. Logic has been applied to the study of classical computation in many ways. For instance, the lambda calculus, a prototypical programming language invented by Church and Curry in the 1930's, can be simultaneously regarded as a programming language and as a formalism for writing mathematical proofs. This observation has become the basis for the development of several modern programming languages, including ML, Haskell, and Lisp.

Recent research by Selinger, Valiron, and others has shown that the logical system which corresponds most closely to quantum computation is the so-called "linear logic" of Girard. Linear logic, a resource sensitive logic, formalizes one of the central principles of quantum physics, the so-called "no-cloning property", which asserts that a given quantum state cannot be replicated. This property is reflected on the logical side by the requirement that a given logical assumption (or "resource") can

only be used once. However, the correspondence between linear logic and quantum computation has only been established at the syntactic level; it is an important open question how to construct semantic models of higher-order quantum computation.

In a series of fundamental works, Girard has examined dynamical models of proofs in linear logic and their evaluation under normalization, using C*-algebras and functional analysis. This program, which he calls "The Geometry of Interaction", has recently received increased attention as having deep connections with quantum computation and quantum protocols. See especially the work of Abramsky and Coecke [3] and of Haghverdi and Scott [34], [36], who have given categorical descriptions of it. Using the work of Joyal, Street and Verity they organize these ideas systematically into a theoretical framework based on the abstract notion of a traced monoidal category. Scott and Haghverdi showed how these techniques could be re-introduced and extended to handle a typed categorical version of Girard's "Geometry of Interaction" through the notion of a partially traced category.

One of the objectives of this thesis is to systematically explore this new notion of partially traced category by providing a representation theorem which establishes a precise correspondence between partially traced categories and totally traced categories. Also, we want to use this framework to elucidate how to build new partially traced categories in connection with some standard models of quantum computation.

A second objective of this thesis is to construct mathematical semantical models of higher-order quantum computation. While the algorithmic aspects of quantum computation have been analyzed extensively, the consideration of quantum computation as a programming paradigm in need of a programming language has been explored far less.

One of the most fruitful methods used to explore the general idea of computational effect in computer science has been the use of computational monads in the sense of Moggi. We study models that exhibit this feature based on linear logic, taking insights and inspiration from Day's characterization of convolution in presheaf categories. In addition we use Freyd-Kelly's notion of continuous functors, as well as Selinger's models for first-order quantum computation.

The basic idea is to start from existing low level models of quantum computation,

such as the category of superoperators, and to use a Yoneda type construction to adapt and extend these models to a higher order quantum situation. The tool used to lift this category is Day's theory for obtaining monoidal structure in presheaf categories. Also, this work partly builds on previous research by Benton et al. on categorical models of linear logic. More precisely, we give a method for constructing models that depends on a family of possible choices.

Specifically, the model construction depends on a sequence of categories and functors $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$, and on a family Γ of cones in $\mathcal{D}$. We use this data to obtain a pair of adjunctions

$$\begin{tikzcd} {[\mathcal{B}^{op}, \mathbf{Set}]} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & {[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[l, shift left, "\Phi^*"] \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & {[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma} \arrow[l, shift left, "G"] \end{tikzcd}$$

and give sufficient conditions on $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ and Γ so that the resulting structure is a model of the quantum lambda calculus.

This provides a general framework in which one can describe various classes of models that depend on the concrete choice of the parameters $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and Γ.

Chapter 2

Some mathematical background

The aim of this chapter is to review some basic categorical background material that is needed to understand this thesis. For a more detailed discussion, see [54], [15], and [52]. The reader who is already familiar with category theory can skip this chapter initially, and refer back to it when needed.

2.1 Monads and adjunctions

In what follows, $Id_{\mathcal{C}}$ is the identity functor on a category $\mathcal{C}$ and 1_G is the identity natural transformation on a functor G. Given a category $\mathcal{C}$, the symbol $\mathcal{C}(A, B)$ denotes the set of morphisms from A to B.

Definition 2.1.1 (Adjunction)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An *adjunction* from $\mathcal{A}$ to $\mathcal{B}$ is a quadruple $(F, G, \eta, \varepsilon)$ where $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors and $\eta : Id_{\mathcal{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow Id_{\mathcal{B}}$ are natural transformations such that: $(G\varepsilon) \circ (\eta G) = 1_G$ and $(\varepsilon F) \circ (F\eta) = 1_F$. The functor F is said to be a *left adjoint* for G or G a *right adjoint* for F and we use the following notation: $F \dashv G$ or $(F, G, \eta, \varepsilon) : \mathcal{A} \rightharpoonup \mathcal{B}$ or even more graphically

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B}. \arrow[l, shift left, "G"] \end{tikzcd}$$

Definition 2.1.2 (Monads). A *monad* or a *triple* on a category $\mathcal{C}$ is a 3-tuple (T, η, μ) where $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor and $\eta : Id_{\mathcal{C}} \Rightarrow T$ (unit law), $\mu : T^2 \Rightarrow T$ (multiplication law) are two natural transformations, satisfying the following conditions:

$$\begin{tikzcd} T \circ Id_{\mathcal{C}} \arrow[r, Rightarrow, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & Id_{\mathcal{C}} \circ T \arrow[l, Rightarrow, "\eta T"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

Theorem 2.1.3 (Huber). *If $F \dashv G$ with unit $\eta : Id_{\mathcal{A}} \Rightarrow GF$ and co-unit $\varepsilon : FG \Rightarrow Id_{\mathcal{B}}$, then $(GF, \eta, G\varepsilon F)$is a monad on $\mathcal{A}$.*

Proof. See Lambek and Scott [52]. □

Suppose we have two adjunctions: $(F, G, \eta, \varepsilon) : \mathcal{A} \rightharpoonup \mathcal{B}$ and $(F', G', \eta', \varepsilon') : \mathcal{B} \rightharpoonup \mathcal{C}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "G'"] \end{tikzcd}$$

We can consider the composite: $(F'F, GG', G\eta'F \circ \eta, \varepsilon' \circ F'\varepsilon G') : \mathcal{A} \rightharpoonup \mathcal{C}$ yielding an adjunction from $\mathcal{A}$ to $\mathcal{C}$. Therefore, by Theorem 2.1.3 $(T, \tilde{\eta}, \tilde{\mu})$ with $T = GG'F'F$, $\tilde{\eta} = G\eta'F \circ \eta$ and $\tilde{\mu} = GG'(\varepsilon' \circ F'\varepsilon G')F'F$ is a monad defined by this new adjunction.

Next we recall the comparison theorem for the Kleisli category.

Definition 2.1.4. Given a monad (T, η, μ) on a category $\mathcal{C}$, the *Kleisli category* $\mathcal{C}_T$ is determined by the following conditions:

- $Obj(\mathcal{C}_T) = Obj(\mathcal{C})$
- $\mathcal{C}_T(A, B) = \mathcal{C}(A, TB)$
- $f^K \circ_K g^K = \mu_C \circ T(g) \circ f$ when $A \xrightarrow{f^K} B$ and $B \xrightarrow{g^K} C$ are arrows in $\mathcal{C}_T$. The identity is given by $1_C^K = \eta_C : C \to TC$.

There is an adjunction between the category $\mathcal{C}$ and the Kleisli category $\mathcal{C}_T$, given by the following:

- $F_T(A) = A$ and $F_T(f) = \eta_B \circ f$ if $A \xrightarrow{f} B$ is an arrow in $\mathcal{C}$.

- $G_T(B) = T(B)$ and $G_T(f^K) = \mu_B \circ T(f)$ if $A \xrightarrow{f^K} B$ is an arrow in $\mathcal{C}_T$.

The adjunction $F_T \dashv G_T$ has the following universal property: given any other adjunction $F \dashv G$ such that $G \circ F = T$, there exists a unique functor $C : \mathcal{C}_T \to \mathcal{D}$, called *the comparison functor*, with the following properties $C \circ F_T = F$ and $G \circ C = G_T$.

- $C(A) = F(A)$ on objects and $F_T(f) = \eta_B \circ f$ if $A \xrightarrow{f} B$ is an arrow in $\mathcal{C}$.
- $C(f) = \varepsilon_{FB} \circ F(f)$ when $A \xrightarrow{f^K} B$ is an arrow in $\mathcal{C}_T$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "F_T"'] & \mathcal{D} \arrow[l, shift left, "G"] \\
\mathcal{C}_T \arrow[u, shift right, "G_T"'] \arrow[ur, "C"'] \arrow[u, phantom, "\vdash"] &
\end{tikzcd}$$

First we evaluate the identity: $C(1_A^K) = C(\eta_A) = \varepsilon_{FA} \circ F(\eta_A) = 1_{FA}$ by definition of the adjoint pair.

Now, suppose we have $A \xrightarrow{f^K} B$ and $B \xrightarrow{g^K} C$ a pair of arrows in $\mathcal{C}_T$ i.e. a pair $A \xrightarrow{f} GFB$ and $B \xrightarrow{g} GFC$ in $\mathcal{C}$. We want to prove that $C(g \circ_K f) = C(g) \circ C(f)$. We have that:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[ddr, "F(g\circ_K f)=F(\mu_C T(g) f)"'] & FGFB \arrow[r, "\varepsilon_{FB}"] \arrow[d, "FTg=FGFg"'] & FB \arrow[d, "Fg"] \\
& FGFGFC \arrow[r, "\varepsilon_{FGFC}"] \arrow[d, "F(\mu_C)=FG\varepsilon_{FC}"] & FGFC \arrow[d, "\varepsilon_{FC}"] \\
& FGFC \arrow[r, "\varepsilon_{FC}"'] & FC
\end{tikzcd}$$

Where the top square commutes by naturality of ε with $F(g)$ and the bottom square by naturality of ε with ε_{FC}. The top leg of the diagram is $C(g) \circ C(f)$ since $C(f) = \varepsilon_{FB} \circ Ff$ and $C(g) = \varepsilon_{FC} \circ Fg$. The bottom leg is $C(g \circ_K f) = \varepsilon_{FC} \circ F(g \circ_K f)$.

Notice that the comparison functor is fully faithful. The definition of the functor between hom-sets is given by

$$C : \mathcal{C}_T(A, B) \to \mathcal{D}(C(A), C(B))$$
$$(A \xrightarrow{f} TB) \longmapsto (FA \xrightarrow{F(f)} FGFB \xrightarrow{\varepsilon_{FB}} FB)$$

Therefore define a function C^{-1} by

$$C^{-1} : \mathcal{D}(FA, FB) \to \mathcal{C}_T(A, B)$$
$$(FA \xrightarrow{g} FB) \longmapsto (A \xrightarrow{\eta_A} GFA \xrightarrow{G(g)} GFB)$$

i.e., $C^{-1}(g) = G(g) \circ \eta_A$.

2.2 Monoidal categories

Definition 2.2.1. A *monoidal* category, also often called *tensor* category, is a category $\mathcal{V}$ with a unit object $I \in \mathcal{V}$ together with a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and natural isomorphisms $\rho : A \otimes I \xrightarrow{\cong} A$, $\lambda : I \otimes A \xrightarrow{\cong} A$, $\alpha : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$, satisfying the following coherence axioms:

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\lambda"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho\otimes 1"] \\
& A \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "\alpha"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "\alpha"] \\
(A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D.
\end{tikzcd}$$

Definition 2.2.2. A *symmetric* monoidal category consists of a monoidal category $(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda)$ with a chosen natural isomorphism $\sigma : A \otimes B \xrightarrow{\cong} B \otimes A$, called *symmetry*, which satisfies the following coherence axioms:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\sigma"] \arrow[dr, "id"'] & B \otimes A \arrow[d, "\sigma"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "\sigma"] \arrow[dr, "\rho"'] & I \otimes A \arrow[d, "\lambda"] \\
& A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\sigma"] & (A \otimes B) \otimes C \arrow[r, "\sigma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"] & (A \otimes C) \otimes B \arrow[r, "\sigma\otimes 1"] & (C \otimes A) \otimes B.
\end{tikzcd}$$

Definition 2.2.3. A *symmetric monoidal closed category* category is a symmetric monoidal category $\mathcal{V}$ for which each functor $- \otimes B : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[B, -] : \mathcal{V} \to \mathcal{V}$, i.e. :

$$\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, [B, C]).$$

Definition 2.2.4. A *monoidal functor* $(F, m_{A,B}, m_I)$ between monoidal categories $(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda)$ and $(\mathcal{W}, \otimes', I', \alpha', \rho', \lambda')$ is a functor $F : \mathcal{V} \to \mathcal{W}$ equipped with:

- morphisms $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$ natural in A and B,
- a morphism $m_I : I' \to F(I)$,

which satisfy the following coherence axioms:

$$\begin{tikzcd}
FA \otimes' (FB \otimes' FC) \arrow[r, "1\otimes' m"] \arrow[d, "\alpha'"] & FA \otimes' F(B \otimes C) \arrow[r, "m"] & F(A \otimes (B \otimes C)) \arrow[d, "F\alpha"] \\
(FA \otimes' FB) \otimes' FC \arrow[r, "m\otimes' 1"] & F(A \otimes B) \otimes' FC \arrow[r, "m"] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes' I' \arrow[r, "\rho'"] \arrow[d, "1\otimes' m"'] & FA \\
FA \otimes' FI \arrow[r, "m"'] & F(A \otimes I) \arrow[u, "F\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes' FA \arrow[r, "\lambda'"] \arrow[d, "m\otimes' 1"] & FA \\
FI \otimes' FA \arrow[r, "m"'] & F(I \otimes A). \arrow[u, "F(\lambda)"']
\end{tikzcd}$$

A monoidal functor is *strong* when m_I and for every A and B $m_{A,B}$ are isomorphisms. It is said to be *strict* when all the $m_{A,B}$ and m_I are identities.

Remark 2.2.5. Throughout the remainder of this exposition whenever we write (F, m) we symbolize a monoidal functor where m not only represents the natural transformation $m_{A,B} : FA \otimes FB \to F(A \otimes B)$ but also $m_I : I \to FI$ relating the units of the two monoidal categories.

Definition 2.2.6. If $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories with natural symmetry maps σ and σ', a *symmetric monoidal functor* is a monoidal functor $(F, m_{A,B}, m_I)$ satisfying the following axiom:

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "\sigma'"] \arrow[d, "m"'] & FB \otimes' FA \arrow[d, "m"] \\ F(A \otimes B) \arrow[r, "F(\sigma)"'] & F(B \otimes A) \end{tikzcd}$$

Definition 2.2.7. A *monoidal natural transformation* $\theta : (F, m) \to (G, n)$ between monoidal functors is a natural transformation $\theta_A : FA \to GA$ such that the following axioms hold:

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "m"] \arrow[d, "\theta_A \otimes' \theta_B"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\ GA \otimes' GB \arrow[r, "n"'] & G(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} I' \arrow[r, "m_I"] \arrow[dr, "n_I"'] & FI \arrow[d, "\theta_I"] \\ & GI. \end{tikzcd}$$

2.3 Monoidal adjunctions and monoidal monads

Definition 2.3.1. A *monoidal adjunction*

$$(\mathcal{V}, \otimes, I) \underset{(G,n)}{\overset{(F,m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{W}, \otimes', I')$$

between two monoidal categories $\mathcal{V}$ and $\mathcal{W}$ consists of an adjunction $(F, G, \eta, \varepsilon)$ in which (F, m) and (G, n) are monoidal functors and the unit $\eta : Id \Rightarrow G \circ F$ and the counit $\varepsilon : F \circ G \Rightarrow Id$ are monoidal natural transformations, as defined in Definition 2.2.7.

Definition 2.3.2. Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A *monoidal monad* (T, η, μ, m) on $\mathcal{V}$ is a monad (T, η, μ) such that the endofunctor $T : \mathcal{V} \to \mathcal{V}$ is a monoidal functor (T, m) with $m_{A,B} : TA \otimes TB \to T(A \otimes B)$ and $m : I \to TI$ as coherence maps, and the natural transformations $\eta : Id \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are monoidal natural transformations.

Lemma 2.3.3. *Let T be a monoidal monad. Consider the Kleisli adjunction* $\mathcal{C} \underset{G_T}{\overset{F_T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}_T$ *as in Definition 2.1.4. Then $\mathcal{C}_T$ is a monoidal category and $F_T \dashv G_T$ is a monoidal adjunction, where*

- $m^T_{A,B} : F_T(A) \otimes F_T(B) \to F_T(A \otimes B)$ *is given by* $\eta : A \otimes B \to T(A \otimes B)$,
- $m^T_I = \eta_I : I \to T(I)$,
- $n^T_{A,B} : G_T(A) \otimes G_T(B) \to G_T(A \otimes B)$ *is given by* $m_{A,B} : T(A) \otimes T(B) \to T(A \otimes B)$, *and*
- $n^T_I = \eta_I : I \to T(I)$.

Definition 2.3.4. A *strong monad* (T, η, μ, t) is a monad (T, η, μ) and a natural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$ called a *strength* satisfying the following axioms:

$$\begin{tikzcd} I \otimes TA \arrow[r, "t_{I,A}"] \arrow[dr, "\lambda"'] & T(I \otimes A) \arrow[d, "T(\lambda)"] \\ & T(A) \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, "1 \otimes \eta_B"] \arrow[dr, "\eta_{A \otimes B}"'] & A \otimes TB \arrow[d, "t_{A,B}"] \\ & T(A \otimes B) \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes TC \arrow[rr, "t_{A \otimes B, C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \otimes B) \otimes C) \arrow[d, "T(\alpha_{A,B,C})"] \\ A \otimes (B \otimes TC) \arrow[r, "1 \otimes t_{B,C}"'] & A \otimes T(B \otimes C) \arrow[r, "t_{A, B \otimes C}"'] & T(A \otimes (B \otimes C)) \end{tikzcd}$$

$$\begin{tikzcd} A \otimes T^2B \arrow[r, "t_{A,TB}"] \arrow[d, "1 \otimes \mu_B"] & T(A \otimes TB) \arrow[r, "T(t_{A,B})"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\ A \otimes TB \arrow[rr, "t_{A,B}"] & & T(A \otimes B). \end{tikzcd}$$

Remark 2.3.5. Let (T, η, μ, m) be a symmetric monoidal monad. A strong monad can be defined in which the strength $t_{A,B}$ is given by the following formula:

$$A \otimes TB \xrightarrow{\eta \otimes 1} TA \otimes TB \xrightarrow{m_{A,B}} T(A \otimes B)$$

see Theorem 2.1 in [49].

We conclude this section with a theorem by Kelly.

Proposition 2.3.6 (Kelly). *Let $(F, m) : \mathcal{C} \to \mathcal{C}'$ be a monoidal functor. Then F has a right adjoint G for which the adjunction $(F, m) \dashv (G, n)$ is monoidal if and only if F has a right adjoint $F \dashv G$ and F is strong monoidal.*

Proof. Here we give a sketch; see [42], [44] or [55] for a detailed proof. Since we have that $\mathcal{C}'(FA, B) \cong \mathcal{C}(A, GB)$ then there is a unique $n_{A,B}$ and n_I such that:

$$\begin{tikzcd} F(GA \otimes GB) \arrow[r, "F(n_{A,B})"] \arrow[d, "m^{-1}_{GA,GB}"'] & FG(A \otimes' B) \arrow[d, "\epsilon_{A\otimes B}"] \\ FGA \otimes' FGB \arrow[r, "\epsilon_A \otimes \epsilon_B"'] & A \otimes' B \end{tikzcd} \qquad \begin{tikzcd} FI \arrow[r, "F(n_I)"] \arrow[dr, "m_I^{-1}"'] & FGI' \arrow[d, "\epsilon_{I'}"] \\ & I' \end{tikzcd}$$

Then using the adjunction we check that this candidates satisfy the definition. □

2.4 The finite coproduct completion of a category

We recall some properties of the finite coproduct completion of a category. A reference can be found in [17].

Definition 2.4.1. Let us consider the category **FinSet** whose objects are finite sets $A = \{a_1, \ldots, a_n\}$ and whose arrows are functions. To avoid any problem about the size of this category, we assume without loss of generality that all objects of **FinSet** are subsets of a given fixed infinite set; thus **FinSet** can be regarded as a small category.

Note that **FinSet** has finite coproducts and products.

Definition 2.4.2. Let $\mathcal{C}$ be a category. The category $\mathcal{C}^+$ has as its objects finite families of objects of $\mathcal{C}$: $V = \{V_a\}_{a\in A}$, with A a finite set. A morphism from $V = \{V_a\}_{a\in A}$ to $W = \{W_b\}_{b\in B}$ consists of the following two items:

- a function $\phi : A \to B$
- a family $f = \{f_a\}_{a\in A}$ of morphisms of $\mathcal{C}$
  $f_a : V_a \to W_{\phi(a)}$.

`Notation`: We shall denote a morphism of $\mathcal{C}^+$ as a pair $F = (\phi, f)$. Moreover, sometimes we write V_b^a instead of $(V_a)_b$ to emphasize some particular set index subscript, and in the same way for arrows.

Before we study any possible structure in $\mathcal{C}^+$ we observe that this is really a category. The identity map is given by taking $\phi = id_A$ the identity function on A and $f_a = 1_{V_a}$, the identity map in $\mathcal{C}$, for every $a \in A$.
Composition is defined by the following rule: if $F = (\phi, f)$ and $G = (\psi, g)$ then $G \circ_{\mathcal{C}^+} F = (\psi \circ \phi, \{g_{\phi(a)} \circ f_a\}_{a \in A})$.

To verify the associative law for the composition we have that if $F = (\phi, f)$, $G = (\psi, g)$ and $H = (\lambda, h)$ then:

$$H \circ (G \circ F) = H \circ (\psi \circ \phi, \{g_{\phi(a)} \circ f_a\}_{a \in A}) = (\lambda \circ (\psi \circ \phi), \{h_{\psi \circ \phi(a)} \circ (g_{\phi(a)} \circ f_a)\}_{a \in A}) =$$
$$((\lambda \circ \psi) \circ \phi, \{(h_{\psi(\phi(a))} \circ g_{\phi(a)}) \circ f_a\}_{a \in A}) = (\lambda \circ \psi, \{h_{\psi(b)} \circ g_b\}_{b \in B}) \circ F = (H \circ G) \circ F.$$

Lemma 2.4.3. *$\mathcal{C}^+$ has finite coproducts.*

Proof. On objects we have that if $V = \{V_a\}_{a \in A}$, $W = \{W_b\}_{b \in B}$ then $V \oplus W = \{Z_c\}_{c \in C}$ where $C = A + B$ is the coproduct in **FinSet**. We take $Z_{in_1(a)} = V_a$ and $Z_{in_2(b)} = W_b$ for every $a \in A$, $b \in B$. Thus, $V \oplus W$ is just a concatenation of families of objects of $\mathcal{C}$.

Injections maps are defined in the following way:

$$\{V_a\}_{a \in A} \xrightarrow{i^1} \{Z_c\}_{c \in C} \quad \text{and} \quad \{W_b\}_{b \in B} \xrightarrow{i^2} \{Z_c\}_{c \in C}$$

where $i^1 = (in_1, Id_A^V)$, $i^2 = (in_2, Id_B^W)$ are given by:

$$A \xrightarrow{in_1} A + B \quad B \xrightarrow{in_2} A + B \text{ injections in } \mathbf{FinSet}$$

and $Id_A^V = \{1_a^V\}_{a \in A}$, $Id_B^W = \{1_b^W\}_{b \in B}$ where $V_a \xrightarrow{1_a^V} V_a$ and $W_b \xrightarrow{1_b^W} W_b$ are identities in $\mathcal{C}$.

`Notation`: Sometimes we shall use $V \oplus W$ for Z, so we have the following notation $V \oplus W = \{(V \oplus W)_c\}_{c \in A+B}$.

There is also an initial object that we shall denote by ϵ. It is the empty family of objects. The unique morphism $\epsilon \xrightarrow{\epsilon_W} \{W_b\}_{b \in B}$ is given by $\epsilon_W = (\emptyset, \emptyset)$.

□

With any category $\mathcal{C}$, we associate a functor $I : \mathcal{C} \to \mathcal{C}^+$ as follows, $I(V) = \{V_*\}_{*\in 1}$, $V_* = V$ and when there is a $f : V \to W$ in $\mathcal{C}$ then $I(f) = (id_1, \{f_*\}_{*\in 1})$ with $f_* = f$.

Proposition 2.4.4. *Given any category $\mathcal{A}$ with finite coproducts $\coprod$ and any functor $F : \mathcal{C} \to \mathcal{A}$, there is a unique finite coproduct preserving functor $G : \mathcal{C}^+ \to \mathcal{A}$, up to natural isomorphism, such that $G \circ I = F$.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{A} \\ \mathcal{C}^+ \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. We shall begin by considering the definition of the functor $G : \mathcal{C}^+ \to \mathcal{A}$ that assigns to each object $V = \{V_a\}_{a\in A}$ the coproduct $G(\{V_a\}_{a\in A}) = \coprod_{a\in A} F(V_a)$ in the category $\mathcal{A}$. For any arrow $\{V_a\}_{a\in A} \xrightarrow{(\phi,f)} \{W_b\}_{b\in B}$ we define $G(\phi, f) = [i_{F(W_{\phi(a)})} \circ F(f_a)]_{a\in A}$ as the unique arrow in $\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd} F(V_a) \arrow[r, "F(f_a)"] \arrow[d, "i_{F(V_a)}"'] & F(W_{\phi(a)}) \arrow[d, "i_{F(W_{\phi(a)})}"] \\ \coprod_{a\in A} F(V_a) \arrow[r, dashed, "G(\phi,f)"'] & \coprod_{b\in B} F(W_b) \end{tikzcd}$$

We must show that G is a functor. To see this, suppose we have

$$\{V_a\}_{a\in A} \xrightarrow{(\phi,f)} \{W_b\}_{b\in B} \xrightarrow{(\psi,g)} \{Z_c\}_{c\in C}$$

then by hypothesis

$$\begin{tikzcd} F(W_b) \arrow[r, "F(g_b)"] \arrow[d, "i_{F(W_b)}"'] & F(Z_{\psi(b)}) \arrow[d, "i_{F(Z_{\psi(b)})}"] \\ \coprod_{b\in B} F(W_b) \arrow[r, "G(\psi,g)"'] & \coprod_{c\in C} F(Z_c) \end{tikzcd}$$

therefore using the case $b = \phi(a)$ we obtain

$$\begin{tikzcd} F(V_a) \arrow[r, "F(f_a)"] \arrow[d, "i_{F(V_a)}"'] & F(W_{\phi(a)}) \arrow[r, "F(g_{\phi(a)})"] \arrow[d, "i_{F(W_{\phi(a)})}"] & F(Z_{\psi(\phi(a))}) \arrow[d, "i_{F(Z_{\psi(\phi(a))})}"] \\ \coprod_{a\in A} F(V_a) \arrow[r, "G(\phi,f)"'] & \coprod_{b\in B} F(W_b) \arrow[r, "G(\psi,g)"'] & \coprod_{c\in C} F(Z_c) \end{tikzcd}$$

then by unique existence property of coproducts we have that $G((\psi, g) \circ (\phi, f)) = G(\psi, g) \circ G(\phi, f)$. Also by uniqueness it is easily to check that $G(id_A, id_A^V) = id_{\coprod_{a\in A} F(V_a)}$.

The functor G preserves coproducts. To see this let us consider $V^i = \{V_a^i\}_{a\in A_i}$, $i \in I$ then

$$G(\oplus_{i\in I} V^i) = G(\oplus_{i\in I}\{V_a^i\}_{a\in A_i}) = G(\{Z_c\}_{c\in\oplus_{i\in I}A_i}) = \coprod_{c\in\oplus_{i\in I}A_i} F(Z_c) \cong \coprod_{i\in I}(\coprod_{a\in A_i} F(V_a^i)) =$$

$$= \coprod_{i\in I} G(\{V_a^i\}_{a\in A_i}) = \coprod_{i\in I} G(V^i)$$

with $Z_c = V_a^i$ if $in_{A_i}(a) = c$. It remains to verify that G is unique up to natural isomorphism. Suppose there is another H preserving coproducts such that $H \circ I = F$. Therefore, using the definitions given above of coproduct in $\mathcal{C}^+$, the functor G and the fact that by hypothesis H preserves coproducts, we calculate on objects

$$H(\{V_a\}_{a\in A}) \cong H(\oplus_{a\in A}\{V_*^a\}_{*\in 1}) \cong \coprod_{a\in A} H(\{V_*^a\}_{*\in 1}) =$$

$$= \coprod_{a\in A} H(I(V_a)) = \coprod_{a\in A} F(V_a) = G(\{V_a\}_{a\in A})$$

Suppose we have a morphism $\{V_a\}_{a\in A} \xrightarrow{(\phi,f)} \{W_b\}_{b\in B}$ with $\phi : A \to B$ and $f = \{f_a\}_{a\in A}$ then using the coproduct in $\mathcal{C}^+$ we consider a decomposition of it, up to isomorphism, in the following way

$$\oplus_{a\in A}\{V_*^a\}_{*\in 1} \xrightarrow{[i_{W_{\phi(a)}} \circ I(f_a)]_{a\in A}} \oplus_{b\in B}\{W_*^b\}_{*\in 1}$$

these morphisms are explicitly given by

$$\{V_*^a\}_{*\in 1} \xrightarrow{I(f_a)} \{W_*^{\phi(a)}\}_{*\in 1} \xrightarrow{i_{W_{\phi(a)}}} \oplus_{b\in B}\{W_*^b\}_{*\in 1}$$

where $I(f_a) = (id_1, \{f_*^a\}_{*\in 1})$, $i_{W_{\phi(a)}} = (in_{\phi(a)}, \{1_*^{W^{\phi(a)}}\}_{*\in 1})$ with $1 \xrightarrow{in_{\phi(a)}} \oplus_B 1$, $W_{\phi(a)} \xrightarrow{1_*^{W^{\phi(a)}}=1} W_{\phi(a)}$ and $\oplus_{b\in B}\{V_*^b\}_{*\in 1} = \{Z_c\}_{c\in\oplus_B 1}$ with $Z_{in_b(*)} = W_*^b = W_b$.

Since H preserves coproducts

$$H([i_{W_{\phi(a)}} \circ I(f_a)]_{a\in A}) \cong [H(i_{W_{\phi(a)}}) \circ H(I(f_a))]_{a\in A} = [i_{F(W_{\phi(a)})} \circ F(f_a)]_{a\in A} = G(\phi, f)$$

where the second equality is justified by the following

$$H(\{W_*^{\phi(a)}\}_{*\in 1}) \xrightarrow{H(\iota_{W_{\phi(a)}})} H(\oplus_{b\in B}\{W_*^b\}_{*\in 1})$$

hence using again that H preserves coproducts, up to isomorphism, we have

$$H(\{W_*^{\phi(a)}\}_{*\in 1}) \xrightarrow{H(\iota_{W_{\phi(a)}})} \coprod_{b\in B} H(\{W_*^b\}_{*\in 1})$$

this means by definition of the functor I,

$$H(I(W_{\phi(a)})) \xrightarrow{H(\iota_{W_{\phi(a)}})} \coprod_{b\in B} H(I(W_b))$$

but, by hypothesis we know that $H \circ I = F$,

$$F(W_{\phi(a)}) \xrightarrow{\iota_{F(W_{\phi(a)})}} \coprod_{b\in B} F(W_b)$$

□

Corollary 2.4.5. *$\mathcal{C}^+$ is the free finite coproduct completion generated by $\mathcal{C}$.*

Proposition 2.4.6. *If $\mathcal{C}$ is a symmetric monoidal category then $\mathcal{C}^+$ is also a symmetric monoidal category.*

Proof. Assume that $V = \{V_a\}_{a\in A}$ and $W = \{W_b\}_{b\in B}$ are objects in $\mathcal{C}^+$ then we take $V \otimes_{\mathcal{C}^+} W = \{V_a \otimes W_b\}_{(a,b)\in A\times B}$ where $A \times B$ is the finite product of sets.

The tensor extends to morphisms, if $V \xrightarrow{F} X$, $W \xrightarrow{G} Y$, with $X = \{X_c\}_{c\in C}$, $Y = \{Y_d\}_{d\in D}$, $F = (\phi, f)$, $G = (\psi, g)$ then $F \otimes G = (\phi \times \psi, f\bar{\otimes}g)$ is given by the following data:

- $\phi \times \psi : A \times B \to C \times D$, $(\phi \times \psi)(a, b) = (\phi(a), \psi(b))$

- $f\bar{\otimes}g = \{(f\bar{\otimes}g)_{(a,b)}\}_{(a,b)\in A\times B}$ where we have that $(f\bar{\otimes}g)_{(a,b)} : (V \otimes W)_{(a,b)} \longrightarrow (X \otimes Y)_{(\phi\times\psi)(a,b)}$ is defined by:

$$f_a \otimes g_b : V_a \otimes W_b \longrightarrow X_{\phi(a)} \otimes Y_{\psi(b)}$$

To prove that $-\otimes_{\mathcal{C}^+}- : \mathcal{C}^+ \times \mathcal{C}^+ \to \mathcal{C}^+$ is a bifunctor one first calculates the definition by using that $1_{A\times B} = 1_A \times 1_B$ and $1_{V_a} \otimes 1_{W_a} = 1_{V_a \otimes W_b}$.

Next, we shall prove that $(F \circ F') \otimes (G \circ G') = (F \otimes G) \circ (F' \otimes G')$. Suppose: $F' = (\phi, f)$, $F = (\eta, h)$, $G' = (\psi, g)$, $G = (\xi, k)$ where

$$\{V_a\}_{a\in A} \xrightarrow{(\phi,f)} \{X_c\}_{c\in C} \xrightarrow{(\eta,h)} \{Z_e\}_{e\in E}$$

and

$$\{W_b\}_{b\in B} \xrightarrow{(\psi,g)} \{Y_d\}_{d\in D} \xrightarrow{(\xi,k)} \{H_f\}_{f\in F}$$

Therefore, $(F\circ F')\otimes(G\circ G') = ((\eta\circ\phi)\times(\xi\circ\psi), \{(h_{\phi(a)}\circ f_a)\otimes(k_{\psi(b)}\circ g_b)\}_{(a,b)\in A\times B}) = ((\eta\times\xi)\circ(\phi\times\psi), \{(h_{\phi(a)}\otimes k_{\psi(b)})\circ(f_a\otimes g_b)\}_{(a,b)\in A\times B}) = (F\otimes G)\circ(F'\otimes G')$ where we simplify the notation of the tensor symbol. The unit of the tensor is given by $I = \{I_*\}_{*\in\{*\}}$. The tensor functor is equipped with the following set of isomorphisms:

- $V \otimes I \xrightarrow{\bar{\rho}} V$, and $I \otimes V \xrightarrow{\bar{\lambda}} V$ where $V = \{V_a\}_{a\in A}$, $I = \{I_*\}_{*\in\{*\}}$ then $V \otimes I = \{V_a \otimes I_*\}_{(a,*)\in A\times 1}$.
  These maps are given by: $\bar{\rho} = (\rho, r)$ with $\rho : A \times \{*\} \to A$, $\rho(a,*) = a$ and with $r = \{r_{(a,*)}\}_{(a,*)\in A\times 1}$ where $r_{(a,*)} = r_{V_a}$, $V_a \otimes I \xrightarrow{r_{V_a}} V_a$. In an analogous way is defined $\bar{\lambda} = (\lambda, l)$.

- If $V = \{V_a\}_{a\in A}$ and $W = \{W_b\}_{b\in B}$ then $\bar{\sigma} = (\sigma, s)$ with $\sigma : A \times B \to B \times A$, $\sigma(x,y) = (y,x)$ and $s = \{s_{(x,y)}\}_{(x,y)\in A\times B}$, where $s_{(x,y)} = s$ i.e,

$$V_x \otimes W_y \xrightarrow{s} W_y \otimes V_x$$

- If $V = \{V_a\}_{a\in A}$, $W = \{W_b\}_{b\in B}$, $Z = \{Z_c\}_{c\in C}$, then $\bar{\alpha} = (\alpha, a)$ with $\alpha : A\times(B\times C) \to (A\times B)\times C$, $\alpha(x,(y,z)) = ((x,y),z)$ and $a = \{a_{(x,(y,z))}\}_{(x,(y,z))\in A\times(B\times C)}$, where $a_{(x,(y,z))} = a$ i.e.,

$$V_x \otimes (W_y \otimes Z_z) \xrightarrow{a} (V_x \otimes W_y) \otimes Z_z$$

Coherence follows by definition, coherence in **FinSet** and coherence in the symmetric monoidal category $\mathcal{C}$. □

Remark 2.4.7. Notice that the distributivity condition $V \otimes (W \oplus Z) \cong (V \otimes W) \oplus (V \otimes Z)$ is satisfied with the map:

$$D : V \otimes (W \oplus Z) \to (V \otimes W) \oplus (V \otimes Z)$$

where $V = \{V_a\}_{a \in A}$, $W = \{W_b\}_{b \in B}$, $Z = \{Z_c\}_{c \in C}$, $D = (\delta, Id)$ in which δ is the bijective function $\delta : (A + B) \times C \to (A \times C) + (B \times C)$ and $Id = \{1_d\}_{d \in (A+B) \times C}$.

Example 2.4.8. If $\mathbf{1}$ is the one object, one arrow strict symmetric monoidal category with the evident monoidal structure then $\mathbf{1}^+ \cong \mathbf{FinSet}$ and $\otimes_{\mathbf{1}^+} = \times$ and $I = 1$.

Proposition 2.4.9. *Under the hypotheses of Proposition 2.4.4, assume that the categories $\mathcal{C}$ and $\mathcal{A}$ are symmetric monoidal. Then I is a symmetric monoidal functor. If moreover F is a symmetric monoidal functor and tensor distributes over coproducts in A, then G is a symmetric monoidal functor. Moreover, if F is strong monoidal then so is G.*

Proof. We first show that I is a monoidal functor by considering:

$$I(V) \otimes I(W) \xrightarrow{u} I(V \otimes W)$$

where $V = \{V_*\}_{*\in 1}$, $W = \{W_*\}_{*\in 1}$ and $u = (\mu, \{1_*^V \otimes 1_*^W\}_{(*,*)\in 1\times 1})$ with $\mu : 1 \times 1 \to 1$ and $1_*^V \otimes 1_*^W = 1_V \otimes 1_W$. It is easy to check that all the axioms of the definition are satisfied. As an example we have that by routine calculations the following axiom is satisfied:

$$\begin{tikzcd}
\{V_* \otimes I_*\}_{(*,*)\in 1\times 1} \arrow[r, "{(\rho, \{r_{*,*}\}_{(*,*)\in 1\times 1})}"] \arrow[d, "1\otimes 1"'] & \{V_*\}_{*\in 1} \\
\{V_* \otimes I_*\}_{(*,*)\in 1\times 1} \arrow[r, "u"] & \{(V \otimes I)_*\}_{*\in 1} \arrow[u, "I(r)"']
\end{tikzcd}$$

since $\rho_I = \mu$ and $(r_V)_{\mu(*,*)} = (r_V)_* = r_V = r_{*,*}$.

Next assuming that (F, m) is monoidal we wish to show that G is also a monoidal functor.

Since we also assumed that the category $(\mathcal{A}, \rho^\otimes, \lambda^\otimes, \alpha^\otimes, \rho^\oplus, \lambda^\oplus, \alpha^\oplus, \delta, \sigma^\otimes, \sigma^\oplus, \lambda^0, \rho^0, I, 0)$ is symmetric distributive then there exists isomorphisms of type:

$$\phi : (\coprod_{a\in A} F(V_a)) \otimes (\coprod_{b\in B} F(W_b)) \xrightarrow{\cong} \coprod_{a\in A, b\in B} F(V_a) \otimes F(W_b)$$

generated by these isomorphisms.

We consider the unique arrow ξ given by the universal property of the coproduct:

$$\begin{tikzcd}
F(V_a) \otimes F(W_b) \arrow[r, "i_{a,b}"] \arrow[d, "m_{V_a,W_b}"'] & \coprod_{a\in A, b\in B} F(V_a) \otimes F(W_b) \arrow[d, "\xi = [m_{V_a,W_b} \circ j_{a,b}]_{a\in A, b\in B}"] \\
F(V_a \otimes W_b) \arrow[r, "j_{a,b}"] & \coprod_{a\in A, b\in B} F(V_a \otimes W_b)
\end{tikzcd} \tag{1}$$

Using these maps we define the mediating arrow $\vartheta : G(V) \otimes G(W) \to G(V \otimes W)$ as the composition $\vartheta_{V,W} = \xi \circ \phi$. We also have that $\vartheta_I : I \to G(I)$ is given by m_I.

To show that ϑ satisfies the axioms of a symmetric monoidal functor we shall only provide the proof of one of the diagrams. This is justified by obvious coproduct properties: the exterior diagram commutes for every $a \in A$ and this implies that the interior diagram commutes by pre-composing with injections i and using the universal property of coproducts:

$$\begin{tikzcd}
F(V_a) \otimes I \arrow[rrr, "\rho"] \arrow[dr, "i"] \arrow[ddd, "1\otimes m_I"'] & & & F(V_a) \arrow[dl, "i"'] \\
& \coprod_{a\in A}(F(V_a) \otimes I) \arrow[r, "{[i_{F(V_a)} \circ \rho]_{a\in A}}"] \arrow[d, "{[i \circ (1\otimes m_I)]_{a\in A}}"'] & \coprod_{a\in A} F(V_a) & \\
& \coprod_{a\in A}(F(V_a) \otimes FI) \arrow[r, "\xi"] & \coprod_{a\in A} F(V_a \otimes I) \arrow[u, "{[i_{F(V_a)} \circ F(\rho)]_{a\in A}}"'] & \\
F(V_a) \otimes FI \arrow[ur, "i"] \arrow[rrr, "m_{V_a,I}"'] & & & F(V_a \otimes I) \arrow[ul, "i"] \arrow[uuu, "F(\rho)"']
\end{tikzcd}$$

Then by coherence [53], distributivity of the tensor through coproduct:

$$\begin{tikzcd}
(A \coprod B) \otimes I \arrow[r, "\delta"] \arrow[dr, "\rho"'] & (A \otimes I) \coprod (B \otimes I) \arrow[d, "\rho \coprod \rho"] \\
& A \coprod B
\end{tikzcd}$$

naturality and by definition of ϑ we may infer that:

$$\begin{tikzcd}
(\coprod F(V_a)) \otimes I \arrow[dr, "\bar{\delta}"'] \arrow[drr, bend left, "\rho"] \arrow[ddd, "(\coprod_{a\in A} 1)\otimes m_I"'] & & \\
& \coprod_{a\in A}(F(V_a)\otimes I) \arrow[r, "\coprod_{a\in A}\rho_{F(V_a)}"'] \arrow[d, "\coprod_{a\in A}(1\otimes m_I)"'] & \coprod_{a\in A} F(V_a) \\
& \coprod_{a\in A}(F(V_a)\otimes FI) \arrow[r, "\xi"] & \coprod_{a\in A} F(V_a\otimes I) \arrow[u, "\coprod_{a\in A} F(\rho_{V_a})"'] \\
(\coprod F(V_a)) \otimes FI \arrow[ur, "\phi"] \arrow[urr, bend right, "\vartheta_{V,I}"'] & &
\end{tikzcd}$$

commutes, which turns to be:

$$\begin{tikzcd}
G(V)\otimes I \arrow[r, "\rho"] \arrow[d, "1\otimes\vartheta_I"] & G(V) \\
G(V)\otimes G(I) \arrow[r, "\vartheta_{V,I}"] & G(V\otimes I) \arrow[u, "G(\rho)"']
\end{tikzcd}$$

Similarly one could prove the rest of the axioms.

Notice that if the mediating arrows m_{V_a,W_b} are isomorphisms in diagram (1) above then ξ is an isomorphism. Therefore this implies that $\vartheta_{V,W}$ is an isomorphism for every V and W i.e., G is a strong functor. □

2.5 The functor Φ : FinSet $\to \mathcal{C}^+$.

Now we turn to prove that when $\mathcal{C}$ is affine, there exists a functor Φ : **FinSet** $\to \mathcal{C}^+$ which is fully faithful and preserves tensor and coproduct.

Definition 2.5.1. A monoidal category $\mathcal{C}$ is called *affine* if the tensor unit I is a terminal object.

Lemma 2.5.2. *Let $\mathcal{C}$ be an affine category. Then there exists a fully-faithful strong monoidal functor* $\Phi : (\mathbf{FinSet}, \times, 1) \to (\mathcal{C}^+, \otimes_{\mathcal{C}^+}, I)$ *that preserves coproducts.*

Proof. We shall begin by considering the functor Φ which assigns to each finite set A a family $\Phi(A) = \{C_a\}_{a\in A}$, such that for every $a \in A$, $C_a = I$ is the unit of the

category $\mathcal{C}$.

Now let $A \xrightarrow{\phi} B$ be a function in **FinSet**, then

$$\Phi(A) \xrightarrow{\Phi(\phi)} \Phi(B) \text{ with } \Phi(\phi) = (\phi, Id_A) \text{ and } Id_A = \{1_a\}_{a\in A}, \quad I \xrightarrow{1_a = id_I} I.$$

The kind of functor obtained in this way has been motivated in order to satisfy the following properties which are essential for the model.

Φ *is faithful:* The way we define morphisms in $\mathcal{C}^+$ allows us to infer that if $\Phi(\phi) = \Phi(\psi)$ then $\phi = \psi$.

Φ *is full:* Suppose we have a pair $(\phi, f) \in \mathcal{C}^+(\Phi(A), \Phi(B))$ then $f = \{f_a\}_{a\in A}$ with $I \xrightarrow{f_a} I$; since I is a terminal object this implies that $f_a = 1_a = !$ for every $a \in A$. Therefore $\Phi(\phi) = (\phi, f)$.

Φ *preserves coproducts:*

Take objects A and B; then by definition we have that

$$\Phi(A \oplus B) = \{C_c\}_{c\in A\oplus B} = \{C_a\}_{a\in A} \oplus \{C_b\}_{b\in B} = \Phi(A) \oplus \Phi(B).$$

Suppose we have two arrows $A \xrightarrow{\phi} C$, $B \xrightarrow{\psi} D$ then:

$$\Phi(\phi\oplus\psi) = (\phi\oplus\psi, Id_{A\oplus B}) = (\phi\oplus\psi, Id_A \oplus Id_B) \overset{def.}{=} (\phi, Id_A)\oplus(\psi, Id_B) = \Phi(\phi)\oplus\Phi(\psi)$$

Φ *preserves tensor product:*

Assuming A and B are finite sets then

$$\Phi(A\times B) = \{C_{(a,b)}\}_{(a,b)\in A\times B} = \{C_a \otimes C_b\}_{(a,b)\in A\times B} = \{C_a\}_{a\in A} \otimes \{C_b\}_{b\in B} = \Phi(A)\otimes\Phi(B)$$

at the level of objects. If $A \xrightarrow{\phi} C$, $B \xrightarrow{\psi} D$ then we have that naturality is satisfied:

$$\Phi(\phi \times \psi) = (\phi \times \psi, Id_{A\times B}) = (\phi \times \psi, Id_A \bar{\otimes} Id_B) = (\phi, Id_A) \otimes (\psi, Id_B) = \Phi(\phi) \otimes \Phi(\psi)$$

since $Id_A \bar{\otimes} Id_B = \{(1 \otimes 1)_{(a,b)}\}_{(a,b)\in A\times B} = \{1_a \otimes 1_b\}_{(a,b)\in A\times B} = \{1_{(a,b)}\}_{(a,b)\in A\times B} = Id_{A\times B}$.

Also, $\Phi(1) = \Phi(\{*\}) = \{C_*\}_{*\in 1} = I_{\mathcal{C}^+}$.

This implies that Φ is a monoidal functor with identity $id : \Phi(A) \otimes \Phi(B) \to \Phi(A \times B)$, $id : I \to \Phi(1)$ as mediating natural transformations. It is a routine

exercise to show that the remaining equations of a monoidal functor, involving the structural maps α, ρ and λ, are satisfied.

For example, the diagram

$$\begin{tikzcd} \Phi(B) \otimes I \arrow[r, "\bar{\rho}"] \arrow[d, "1\otimes 1"'] & \Phi(B) \\ \Phi(B) \otimes \Phi(1) \arrow[r, "1"] & \Phi(B \times 1) \arrow[u, "\Phi(\rho)"'] \end{tikzcd}$$

is satisfied. To see this, we calculate $\Phi(\rho) = (\rho, \{1_{(a,*)}\}_{(a,*)\in A\times 1})$. On the other hand by definition we have that $\bar{\rho} = (\rho, r)$ with $\rho : A \times \{*\} \to A$, $\rho(a,*) = a$ and with $r = \{r_{(a,*)}\}_{(a,*)\in A\times 1}$ where $r_{(a,*)} = r_{V_a}$, $V_a \otimes I \xrightarrow{r_{V_a}} V_a$ but since $V_a = I$ this implies $I \xrightarrow{r_{V_a}=1_I} I$. Hence, these two arrows are equal. □

2.6 Affine monoidal categories

Recall from Definition 2.5.1 that a monoidal category is affine when the tensor unit I is a terminal object. The following construction is well-known.

Definition 2.6.1 (Free affine symmetric monoidal category). Let $\mathcal{K}$ be a category. The *free affine symmetric monoidal category* $\mathcal{F}wm(\mathcal{K})$ is the category defined as follows:

(a) objects are finite sequences of objects of $\mathcal{K}$:

$$\{V_i\}_{i\in[n]} = \{V_1, \ldots, V_n\}$$

(b) maps $(\phi, \{f_i\}_{i\in[m]}) : \{V_i\}_{i\in[n]} \longrightarrow \{W_i\}_{i\in[m]}$ are determined by:

- an injective function $\phi : [m] \to [n]$
- a family of morphism $f_i : V_{\phi(i)} \to W_i$ in the category $\mathcal{K}$

(c) composition $(\phi, \{f_i\}_{i\in[m]}) \circ (\psi, \{g_i\}_{i\in[s]}) = (\psi \circ \phi, \{f_i \circ g_{\phi(i)}\}_{i\in[s]})$

(d) the unit is given by the empty sequence.

(e) the tensor $\otimes$ is given by concatenation of sequences of objects and arrows:

$$\{V_i\}_{i\in[n]} \otimes \{W_i\}_{i\in[m]} = \{Z_i\}_{i\in[n+m]}$$

where $Z_i = V_i$ if $1 \leq i \leq n$ and $Z_i = W_{i-n}$ if $n+1 \leq i \leq n+m$

$$\{V_i\}_{i\in[n]} \otimes \{W_i\}_{i\in[m]} = \{P_i\}_{i\in[n+m]} \xrightarrow{(\phi,f)\otimes(\psi,g)} \{Q_i\}_{i\in[s+t]} = \{X_i\}_{i\in[s]} \otimes \{Y_i\}_{i\in[t]}$$

given by $(\phi, f) \otimes (\psi, g) = (\phi + \psi, f + g)$ where $\phi + \psi : [s+t] \to [n+m]$ is defined by $(\phi+\psi)(i) = \phi(i)$ if $1 \leq i \leq s$ and $(\phi+\psi)(i) = \psi(i-s) + n$ if $s+1 \leq i \leq s+t$ and $f + g = \{(f+g)_j\}_{j\in[s+t]}$ where $(f+g)_j : P_{(\phi+\psi)(j)} \to Q_j$ is defined by $(f+g)_j = f_j$ if $1 \leq j \leq s$ and $(f+g)_j = g_{j-s}$ if $s+1 \leq j \leq s+t$

(f) the canonical isomorphisms are strict given by $l = r = 1$, $a = 1$ and symmetries by $s = (\sigma, 1)$ with $\sigma : [n+m] \to [n+m]$ such that $\sigma(i) = i + n$ if $1 \leq i \leq m$ and $\sigma(i) = i - m$ if $m+1 \leq i \leq n+m$.

Remark 2.6.2. The tensor unit of $\mathcal{F}wm(\mathcal{K})$ is a terminal object:

$$\{V_i\}_{i\in[n]} \xrightarrow{(\emptyset,\emptyset)} \{\}$$

for every $\{V_i\}_{i\in[n]}$ object in $\mathcal{K}$. In addition, notice that $\mathcal{F}wm(\mathcal{K})(\{\}, \{V_i\}_{i\in[n]}) = \emptyset$ if $\{V_i\}_{i\in[n]} \neq \{\}$.

Proposition 2.6.3. *Given any symmetric monoidal category $\mathcal{A}$ whose tensor unit is terminal and any functor $F : \mathcal{K} \to \mathcal{A}$, there is a unique strong monoidal functor $G : \mathcal{F}wm(\mathcal{K}) \to \mathcal{A}$, up to isomorphism, such that $G \circ I = F$.*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{A} \\ \mathcal{F}wm(\mathcal{K}) \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. (sketch) The functor G is defined on objects by: $G(\{\}) = I$ and $G(\{V_i\}_{i\in[n]}) = (\ldots(F(V_1) \otimes F(V_2) \otimes F(V_3))\ldots \otimes F(V_n))$.

Let $(\phi, \{f_i\}_{i\in[m]}) : \{V_i\}_{i\in[n]} \to \{W_i\}_{i\in[m]}$ be a map in $\mathcal{F}wm(\mathcal{K})$ then

$$G(\phi, \{f_i\}_{i\in[m]}) : G(\{V_i\}_{i\in[n]}) \to G(\{W_i\}_{i\in[m]})$$

is given by

$$\begin{tikzcd}
(F(V_1)\otimes F(V_2))\otimes\ldots\otimes F(V_n) = G(\{V_i\}_{i\in[n]}) \arrow[r, dotted] \arrow[d, "(x_1\otimes x_2)\ldots\otimes x_n"] & (F(W_1)\otimes F(W_2))\ldots F(W_m) = G(\{W_i\}_{i\in[m]}) \\
(X_1\otimes X_2)\otimes\ldots\otimes X_n \arrow[r, "\cong"'] & (F(V_{\phi(1)})\otimes F(V_{\phi(2)}))\ldots F(V_{\phi(m)}) \arrow[u, "(F(f_1)\otimes F(f_2))\ldots\otimes F(f_m)"']
\end{tikzcd}$$

where $x_i = 1_{F(V_i)} : F(V_i) \to F(V_i)$ if $i \in \phi([m])$ and $x_i = ! : F(V_i) \to I$ if $i \in [n] - \phi([m])$.

Using coherence of the category $\mathcal{A}$ we prove that G is a strong functor: the mediating isomorphism is given by the unique morphism that shifts all the parenthesis to the left:

$$G(\{V_i\}_{i\in[n]}) \otimes G(\{W_i\}_{i\in[m]}) \xrightarrow{m} G(\{V_i\}_{i\in[n]} \otimes \{W_i\}_{i\in[m]})$$

and

$$I \xrightarrow{m^0=1} G(\{\}).$$

To prove uniqueness we use the fact that $x_i = ! : F(V_i) \to I$ transforms into $x_i = ! : G \circ I(V_i) \to G\{\}$ if $i \in [n] - \phi([m])$ and also that the coherence structure is preserved, up to isomorphism, for any functor satisfying these conditions. □

Corollary 2.6.4. *$Fwm(\mathcal{K})$ is the free affine symmetric monoidal category generated by $\mathcal{K}$.*

Example 2.6.5. To illustrate the definition of the functor G in the proof of Proposition 2.6.3, let us consider $(\phi, \{f_i\}_{i\in[2]}) : \{V_1, V_2, V_3\} \to \{W_1, W_2\}$ with $\phi : [2] \to [3]$, $\phi(1) = 3, \phi(2) = 1$ then

$$G(\phi, \{f_i\}_{i\in[2]}) : G(\{V_1, V_2, V_3\}) \to G(\{W_1, W_2\})$$

is given by

$$\begin{tikzcd}
(F(V_1)\otimes F(V_2))\otimes F(V_3) = G(\{V_1, V_2, V_3\}) \arrow[r, dotted, "G(\phi,\{f_i\}_{i\in[2]})"] \arrow[d, "F(f_2)\otimes!\otimes F(f_1)"'] & F(W_1)\otimes F(W_2) = G(\{W_1, W_2\}) \\
(F(W_2)\otimes I)\otimes F(W_1) \arrow[r, "\rho\otimes 1"'] & F(W_2)\otimes F(W_1). \arrow[u, "\sigma"']
\end{tikzcd}$$

2.7 Traced monoidal categories

We recall the definition of a trace from [41].

Definition 2.7.1. A *trace* for a symmetric monoidal category $(\mathcal{C}, \otimes, I, \rho, \lambda, s)$ consists of a family of functions

$$\mathrm{Tr}^U_{A,B} : \mathcal{C}(A \otimes U, B \otimes U) \to \mathcal{C}(A, B)$$

natural in A, B, and dinatural in U, satisfying the following axioms:

Vanishing I:
$\mathrm{Tr}^I_{X,Y}(f) = f$,

Vanishing II:
$\mathrm{Tr}^{U\otimes V}_{X,Y}(g) = \mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g))$,

Superposing:
$\mathrm{Tr}^U_{A\otimes C, B\otimes D}((1_B \otimes \sigma^{-1}_{D,U}) \circ (f \otimes g) \circ (1_A \otimes \sigma_{C,U})) = \mathrm{Tr}^U_{A,B}(f) \otimes g =$
$\mathrm{Tr}^U_{A\otimes C, B\otimes D}((1_B \otimes \sigma_{U,D}) \circ (f \otimes g) \circ (1_A \otimes \sigma^{-1}_{U,C}))$,

Yanking:
For every U, we have $\mathrm{Tr}^U_{U,U}(\sigma_{U,U}) = 1_U$.

Explicitly, naturality and dinaturality mean the following

Naturality in A and B:
For any $g : X' \to X$ and $h : Y \to Y'$ we have that

$$\mathrm{Tr}^U_{X',Y'}((h \otimes 1_U) \circ f \circ (g \otimes 1_U) = h \circ \mathrm{Tr}^U_{X,Y}(f) \circ g.$$

Dinaturality in U:
For any $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$ we have that

$$\mathrm{Tr}^U_{X,Y}((1_Y \otimes g) \circ f) = \mathrm{Tr}^{U'}_{X,Y}(f \circ (1_X \otimes g)).$$

Definition 2.7.2. Suppose we have two traced monoidal categories $(\mathcal{V}, \mathrm{Tr})$ and $(\mathcal{W}, \widehat{\mathrm{Tr}})$. We say that a strong monoidal functor $(F, m) : \mathcal{V} \to \mathcal{W}$ is *traced monoidal* when it preserves the trace operator in the following way: for $f : A \otimes U \to B \otimes U$

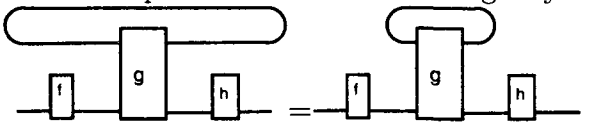
$$\widehat{\mathrm{Tr}}^{FU}_{FA,FB}(m^{-1}_{A,U} \circ F(f) \circ m_{A,U}) = F(\mathrm{Tr}^{U}_{A,B}(f)) : FA \to FB.$$

2.8 Graphical language

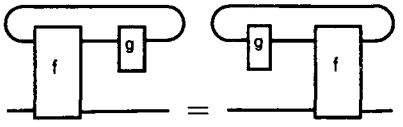
Graphical calculi are an important tool for reasoning about monoidal categories, dating back at least to the work of Penrose [59]. There are various graphical languages which are provably complete for reasoning about diagrams in different kinds of monoidal categories. They allow efficient geometrical and topological insights to be used in a kind of calculus of "wirings", which simplifies diagrammatic reasoning. See [66] for a detailed survey of such graphical languages.

In particular, there is a graphical language for traced monoidal categories, which was already used in the original paper of Joyal, Street, and Verity [41]. The axioms of traced monoidal categories are represented in the following way.

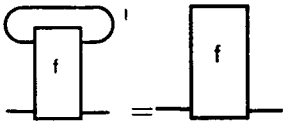
Naturality:



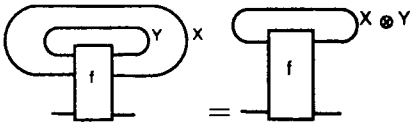
Dinaturality:

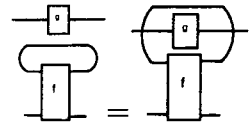


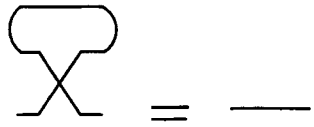
Vanishing I:

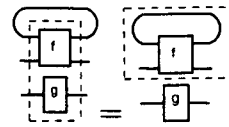


Vanishing II:



Superposing (equivalent formulation): 

Yanking: 

Strength (equivalent formulation of superposing): 

The following theorem shows the validity of such diagrammatic reasoning in compact closed categories:

Theorem 2.8.1 (Coherence, see [66]). *A well-formed equation between morphisms in the language of symmetric traced categories follows from the axioms of symmetric traced categories if and only if it holds, up to isomorphism of diagrams, in the graphical language.*

Here by isomorphism of diagrams we mean a bijective correspondence between wires and boxes in which the structure of the graph is preserved.

2.9 Compact closed categories

Definition 2.9.1. A *compact closed* category is a symmetric monoidal category $\mathcal{V}$ for which every object A has assigned another object A^*, called the dual, and a pair of arrows $\eta : I \to A^* \otimes A$ (unit), $\varepsilon : A \otimes A^* \to I$ (counit) such that the following diagrams commute:

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "1"'] & A \otimes I \arrow[r, "1\otimes\eta"] & A \otimes (A^* \otimes A) \arrow[d, "\alpha"] \\
A & I \otimes A \arrow[l, "\lambda^{-1}"] & (A \otimes A^*) \otimes A \arrow[l, "\varepsilon\otimes 1"]
\end{tikzcd}$$

and also,

$$\begin{tikzcd}
A^* \arrow[r, "\lambda"] \arrow[d, "1"'] & I \otimes A^* \arrow[r, "\eta\otimes 1"] & (A^* \otimes A) \otimes A^* \arrow[d, "\alpha^{-1}"] \\
A^* & A^* \otimes I \arrow[l, "\rho^{-1}"] & A^* \otimes (A \otimes A^*). \arrow[l, "1\otimes\varepsilon"]
\end{tikzcd}$$

In a compact closed category we can define a functor $(-)^* : \mathcal{V}^{op} \to \mathcal{V}$ where if $f : A \to B$ then $f^* : B^* \to A^*$ is given by:

$$B^* \xrightarrow{\lambda} I \otimes B^* \xrightarrow{\eta\otimes 1} A^* \otimes A \otimes B \xrightarrow{1\otimes f\otimes 1} A^* \otimes B \otimes B^* \xrightarrow{1\otimes\varepsilon} A^* \otimes I \xrightarrow{\rho^{-1}} A^*.$$

Proposition 2.9.2. *Let $(\mathcal{V}, \otimes, \eta, \varepsilon)$ be a compact closed category. There exists a trace, which we call the canonical trace, defined by:*

$$\mathrm{Tr}^U_{A,B}(f) = (1 \otimes \varepsilon\sigma)(f \otimes 1)(1 \otimes \eta).$$

Moreover every symmetric strong monoidal functor between compact categories is traced monoidal with respect to the canonical trace.

Proof. See [41]. □

Proposition 2.9.3. *Let $\mathcal{C}$ be a compact closed category. Then $\mathcal{C}$ has a unique trace, i.e., the canonical trace*

$$\mathrm{Tr}^U_{A,B}(f) = (1 \otimes \varepsilon\sigma)(f \otimes 1)(1 \otimes \eta).$$

Proof. Appendix B of [38]. □

Chapter 3

Categories of completely positive maps

3.1 Completely positive maps

Definition 3.1.1. Let H be a finite dimensional Hilbert space, i.e., a finite dimensional complex inner product space. Let us write $\mathcal{L}(H)$ for the space of linear functions $\rho : H \to H$. Equivalently, we can write $\mathcal{L}(H) = H^* \otimes H$.

Recall that the *adjoint* of a linear function $F : H \to K$ is defined to be the unique function $F^\dagger : K \to H$ such that $\langle F^\dagger v, w\rangle = \langle v, Fw\rangle$, for all $v \in K$ and $w \in H$.

Definition 3.1.2. Let H, K be finite dimensional Hilbert spaces. A linear function $F : \mathcal{L}(H) \to \mathcal{L}(K)$ is said to be *completely positive* if it can be written in the form

$$F(\rho) = \sum_{i=1}^{m} F_i \rho F_i^\dagger,$$

where $F_i : H \to K$ is a linear function for $i = 1, \ldots, m$.

Definition 3.1.3. The category $\mathbf{CPM}_s$ of *simple completely positive maps* has finite dimensional Hilbert spaces as objects, and the morphisms $F : H \to K$ are completely positive maps $F : \mathcal{L}(H) \to \mathcal{L}(K)$.

Definition 3.1.4. The category **CPM** of *completely positive maps* is defined as $\mathbf{CPM} = \mathbf{CPM}_s^{\oplus}$, the biproduct completion of $\mathbf{CPM}_s$. Specifically, the objects of **CPM** are finite sequences $(H_1, \ldots, H_n)$ of finite-dimensional Hilbert spaces, and a morphism $F : (H_1, \ldots, H_n) \to (K_1, \ldots, K_m)$ is a matrix (F_{ij}), where each $F_{ij} : H_j \to K_i$ is a completely positive map. Composition is defined by matrix multiplication.

Remark 3.1.5. In quantum mechanics, completely positive maps correspond to general transformations between quantum systems. Two special cases are of note: first, $F(\rho) = U\rho U^\dagger$, where U is a unitary transformation. This represents the unitary evolution of an isolated quantum system. Second,

$$F(\rho) = (P_1 \rho P_1^\dagger, \ldots, P_m \rho P_m^\dagger),$$

where $P_1, \ldots, P_m$ is a system of commuting self-adjoint projections. This corresponds to measurement with possible outcomes $1, \ldots, m$. For more details on the physical interpretation, see e.g. [58] or [63].

Remark 3.1.6. Note that the category **CPM** is the same (up to equivalence) as the category **W** of [63] and the category $\mathbf{CPM}(\mathbf{FdHilb})^{\oplus}$ of [65].

Note that for any two finite dimensional Hilbert spaces V and W, there is a canonical isomorphism $\phi_{V,W} : \mathcal{L}(V \otimes W) \to \mathcal{L}(V) \otimes \mathcal{L}(W)$.

Remark 3.1.7. The categories $\mathbf{CPM}_s$ and **CPM** are symmetric monoidal. For $\mathbf{CPM}_s$, the tensor product is given on objects by the tensor product defined on Hilbert spaces $V \bar{\otimes} W = V \otimes W$, and on morphisms by the following map $f \bar{\otimes} g$:

$$\begin{tikzcd}
\mathcal{L}(V \otimes W) \arrow[r, "f\bar{\otimes}g"] \arrow[d, "\phi_{V,W}"'] & \mathcal{L}(X \otimes Y) \arrow[d, "\phi_{X,Y}"] \\
\mathcal{L}(V) \otimes \mathcal{L}(W) \arrow[r, "f\otimes g"] & \mathcal{L}(X) \otimes \mathcal{L}(Y).
\end{tikzcd}$$

The left and right unit, associativity, and symmetry maps are inherited from the symmetric monoidal structure of Hilbert spaces. For the symmetric monoidal structure on **CPM**, define

$$(V_i)_{i \in I} \otimes (W_j)_{j \in J} = (V_i \otimes W_j)_{i \in I, j \in J}.$$

This extends to morphisms in an obvious way. For details, see [63].

3.2 Superoperators

Definition 3.2.1. We say that a linear map $F : \mathcal{L}(V) \to \mathcal{L}(W)$ is a *trace preserving* linear function when it satisfies

$$\mathrm{tr}_W(F(\rho)) = \mathrm{tr}_V(\rho) \tag{2}$$

for all positive $\rho \in \mathcal{L}(V)$. F is called *trace non-increasing* when it satisfies

$$\mathrm{tr}_W(F(\rho)) \leq \mathrm{tr}_V(\rho) \tag{3}$$

for all positive $\rho \in \mathcal{L}(V)$.

Definition 3.2.2. A linear function $F : \mathcal{L}(V) \to \mathcal{L}(W)$ is called a *trace preserving superoperator* if it is completely positive and trace preserving, and it is called a *trace non-increasing superoperator* if it is completely positive and trace non-increasing.

Definition 3.2.3. A completely positive map $F : (H_1, \ldots, H_n) \to (K_1, \ldots, K_m)$ in the category **CPM** is called a *trace preserving superoperator* if for all j and all positive $\rho \in \mathcal{L}(H_j)$,

$$\sum_i \mathrm{tr}(F_{ij}(\rho)) = \mathrm{tr}(\rho),$$

and a *trace non-increasing superoperator* if for all j and all positive $\rho \in \mathcal{L}(H_j)$,

$$\sum_i \mathrm{tr}(F_{ij}(\rho)) \leq \mathrm{tr}(\rho).$$

Definition 3.2.4. We define four symmetric monoidal categories of superoperators. All of them are symmetric monoidal subcategories of **CPM**.

- $\mathbf{Q}$ and $\mathcal{Q}'$ have the same objects as **CPM**, and $\mathbf{Q}_s$ and $\mathcal{Q}'_s$ have the same objects as $\mathbf{CPM}_s$.
- The morphisms of $\mathbf{Q}$ and $\mathbf{Q}_s$ are trace non-increasing superoperators, and the morphisms of $\mathcal{Q}'$ and $\mathcal{Q}'_s$ are trace preserving superoperators.

The six categories defined in this chapter are summarized in the following table:

	simple	non-simple
no trace condition	$\mathbf{CPM}_s$	$\mathbf{CPM}$
trace non-increasing	$\mathbf{Q}_s$	$\mathbf{Q}$
trace preserving	$\mathcal{Q}'_s$	$\mathcal{Q}'$

Remark 3.2.5. The categories $\mathbf{Q}$, $\mathbf{Q}_s$, $\mathcal{Q}'$, and $\mathcal{Q}'_s$ are all symmetric monoidal. The symmetric monoidal structure is as in $\mathbf{CPM}$ and $\mathbf{CPM}_s$, and it is easy to check that all the structural maps are trace preserving.

Lemma 3.2.6. $\mathbf{Q}$ *and* $\mathcal{Q}'$ *have finite coproducts.*

Proof. The injection and copairing maps are as in $\mathbf{CPM}$; we only need to show that they are trace preserving. But this is trivially true. □

Chapter 4

Partially traced categories

Traced monoidal categories were introduced by Joyal, Street and Verity [41] as an attempt to organize properties from different fields of mathematics, such as algebraic topology and computer science. This abstraction has been useful in formulating new insights in concrete topics of theoretical computer science such as feedback, fixed-point operators, the execution formula in Girard's Geometry of Interaction (GoI) [27], etc. In this spirit, an axiomatization for partially traced symmetric monoidal categories was introduced by Haghverdi and Scott [34] providing an appropriate framework for a typed version of the Geometry of Interaction.

An important part of the treatment of the dynamics of proofs in the Geometry of Interaction relies on the expressiveness of its model: proofs are interpreted as linear operators in Hilbert spaces and an invariant for the cut-elimination process is modelled by a convergent sum in some linear space. Haghverdi and Scott [34] have demonstrated that the categorical notion of partially traced category is a useful tool for capturing the dynamic behavior of all of these conceptual ideas as described by Girard. The word "partial" here refers to the fact that the trace operator is defined on a subset of the set of morphisms $Hom(A \otimes U, B \otimes U)$ called the *trace class*. A large portion of Haghverdi and Scott's work is concerned with constructing the appropriate abstract notion of a typed GoI aided by the idea of orthogonality in the sense of Hyland and Schalk. Partial traces play a central role in Haghverdi and Scott's work. For example, their analysis of the idea of an abstract algorithm concerns the

interplay with the execution formula defined in terms of a partially traced category. The categorical formula agrees with the original formula of Girard in some concrete Hilbert spaces and the execution formula in this new setting is an invariant of the cut-elimination process.

In this chapter, we give some examples of partially traced categories, including an example in the context of quantum computation. We also provide a method for constructing more examples by proving that each monoidal subcategory of a (totally or partially) traced category is partially traced.

4.1 Partially traced categories

We recall the definition of a monoidal partially traced category from [34].

Definition 4.1.1. Let f and g be partially defined operations. We write $f(x)\downarrow$ if $f(x)$ is defined, and $f(x)\uparrow$ if it is undefined. Following Freyd and Scedrov [25], we also write $f(x) \eqsim g(x)$ if $f(x)$ and $g(x)$ are either both undefined, or else they are both defined and equal. The relation "$\eqsim$" is known as *Kleene equality*. We also write $f(x) \succeq g(x)$ if either $f(x)$ is undefined, or else $f(x)$ and $g(x)$ are both defined and equal. The relation "$\succeq$" is known as *directed Kleene equality*.

Definition 4.1.2. Suppose $(\mathcal{C}, \otimes, I, \rho, \lambda, s)$ is a symmetric monoidal category. A *partial trace* is given by a family of partial functions $\mathrm{Tr}^U_{X,Y} : \mathcal{C}(X \otimes U, Y \otimes U) \rightharpoonup \mathcal{C}(X, Y)$, satisfying the following axioms:

Naturality:

For any $f : X \otimes U \to Y \otimes U$, $g : X' \to X$ and $h : Y \to Y'$ we have that

$$h\,\mathrm{Tr}^U_{X,Y}(f)g \succeq \mathrm{Tr}^U_{X',Y'}((h \otimes 1_U)f(g \otimes 1_U)).$$

Dinaturality:

For any $f : X \otimes U \to Y \otimes U'$, $g : U \to U'$ we have

$$\mathrm{Tr}^U_{X,Y}((1_Y \otimes g)f) \eqsim \mathrm{Tr}^{U'}_{X,Y}(f(1_X \otimes g)).$$

Vanishing I:
For every $f : X \otimes I \to Y \otimes I$ we have

$$\mathrm{Tr}^I_{X,Y}(f) \cong \rho_Y f \rho_X^{-1}.$$

Vanishing II:
For every $g : X \otimes U \otimes V \to Y \otimes U \otimes V$, if

$$\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g) \downarrow,$$

then

$$\mathrm{Tr}^{U\otimes V}_{X,Y}(g) \cong \mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g)).$$

Superposing:
For any $f : X \otimes U \to Y \otimes U$ and $g : W \to Z$,

$$g \otimes \mathrm{Tr}^U_{X,Y}(f) \cong \mathrm{Tr}^U_{W\otimes X, Z\otimes Y}(g \otimes f).$$

Yanking:
For any U,

$$\mathrm{Tr}^U_{U,U}(\sigma_{U,U}) \cong 1_U.$$

Definition 4.1.3. A *partially traced* category is a symmetric monoidal category with a partial trace.

Remark 4.1.4. Comparing this to the definition of a traced monoidal category in Section 2.7, we see that a traced monoidal category is exactly the same as a partially traced category where the trace operation happens to be total. We sometimes refer to traced monoidal categories as *totally* traced monoidal categories, when we want to emphasize that they are not partial.

Definition 4.1.5. The subset of $\mathcal{C}(X \otimes U, Y \otimes U)$ where $\mathrm{Tr}^U_{X,Y}$ is defined is sometimes called the *trace class*, and is written

$$\mathbb{T}^U_{X,Y} = \{f : X \otimes U \to Y \otimes U \mid \mathrm{Tr}^U_{X,Y}(f) \downarrow\}.$$

Lemma 4.1.6. *Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a partially traced category. The superposition axioms is equivalent to the following axiom (called* strength*):*
For $f : A \otimes U \to B \otimes U$ and $g : C \to D$,

$$\mathrm{Tr}^U_{A,B}(f) \otimes g \rightleftharpoons \mathrm{Tr}^U_{A\otimes C, B\otimes D}((1_B \otimes s_{U,D}) \circ (f \otimes g) \circ (1_A \otimes s_{C,U})).$$

Proof. ($\Rightarrow$) First, from the original version we shall prove this second version. By hypothesis and by naturality of the symmetries we have:

$$g \otimes f \in \mathbb{T}^U_{C\otimes A, D\otimes B} \text{ and}$$
$$s_{DB} \circ \mathrm{Tr}^U_{C\otimes A, D\otimes B}(g \otimes f) \circ s_{AC} = s_{DB} \circ (g \otimes \mathrm{Tr}^U_{A,B}(f)) \circ s_{AC} = \mathrm{Tr}^U_{AB}(f) \otimes g.$$

Thus by the naturality axiom we have that since $g \otimes f \in \mathbb{T}^U_{C\otimes A, D\otimes B}$:

$$(s_{DB} \otimes 1_U) \circ (g \otimes f) \circ (s_{AC} \otimes 1_U) \in \mathbb{T}^U_{A\otimes C, B\otimes D} \text{ and}$$
$$\mathrm{Tr}^U_{A\otimes C, B\otimes D}(s_{DB} \otimes 1_U) \circ (g \otimes f) \circ (s_{AC} \otimes 1_U) = s_{DB} \circ \mathrm{Tr}^U_{C\otimes A, D\otimes B}(g \otimes f) \circ s_{AC}.$$

Finally by coherence we obtain:

$$(s_{DB} \otimes 1_U) \circ (g \otimes f) \circ (s_{AC} \otimes 1_U) = (1_B \otimes s_{UD}) \circ (f \otimes g) \circ (1_A \otimes s_{CU})$$

($\Leftarrow$) Conversely by hypothesis and composing with symmetries we get:

$$(1_B \otimes s_{UD}) \circ (f \otimes g) \circ (1_A \otimes s_{CU}) \in \mathbb{T}^U_{A\otimes C, B\otimes D} \text{ and}$$
$$s_{BD} \circ \mathrm{Tr}^U_{A\otimes C, B\otimes D}((1_B \otimes s_{U,D}) \circ (f \otimes g) \circ (1_A \otimes s_{C,U})) \circ s_{CA} = s_{BD} \circ (\mathrm{Tr}^U_{A,B}(f) \otimes g) \circ s_{BD}.$$

Which implies by the naturality axiom that:

$$\alpha = (s_{BD} \otimes 1_U) \circ (1_B \otimes s_{U,D}) \circ (f \otimes g) \circ (1_A \otimes s_{C,U}) \circ (s_{CA} \otimes 1_U) \in \mathbb{T}^U_{C\otimes A, D\otimes B} \text{ and}$$
$$\mathrm{Tr}^U_{C\otimes A, D\otimes B}(\alpha) = g \otimes \mathrm{Tr}^U_{A,B}(f).$$

But by coherence $\alpha = g \otimes f$. □

4.2 Examples of partially traced categories

4.2.1 Finite dimensional vector spaces

Among the examples that motivated this notion of partially traced category in Definition 4.1.3 a particularly important one [34], [36] is the category $(\mathbf{Vect}_{fn}, \oplus, \mathbf{0})$ of

finite dimensional vector spaces and linear transformations, with biproduct $\oplus$ as the tensor product.

We recall that in an additive category a morphism $f : X \oplus U \to Y \oplus V$ is characterized by compositions with injections and projections: $f_{ij} = \pi_i \circ f \circ in_j$, $1 \leq i, j \leq 2$. We denote f by a matrix of morphisms of type $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ where composition corresponds to multiplication of matrices.

Definition 4.2.1. The *trace class in* $(\mathbf{Vect}_{fn}, \oplus, \mathbf{0})$ is defined as follows: we say that $f : X \oplus U \to Y \oplus U \in \mathbb{T}^U_{X,Y}$ iff $I - f_{22}$ is invertible, where $I = id$ on U.
When this is the case we define $\mathrm{Tr}^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21}$.

Proposition 4.2.2. *With the operation defined in Definition 4.2.1, the category of finite dimensional vector spaces is partially traced.*

Proof. [34], [36]. □

4.2.2 Stochastic relations

In order to capture classical probabilistic computation (as a stepping stone towards quantum computation), we now describe a trace class in the category **Srel** of stochastic relations. In fact, this partial trace arises from the canonical total trace on $(\mathbf{Vect}_{fn}, \otimes)$ by a general construction that we will examine in detail in Section 4.3. Note that it differs from the trace on **Srel** given by Abramsky [2], [31]. Abramsky's trace is with respect to the coproduct structure $\oplus$ and is total; here we discuss a partial trace with respect to the tensor structure $\otimes$.

The category of stochastic relations attempts to model the probability of a bit being in states 0 or 1, or more generally, of a variable taking a specific value in a finite set of possible values. Morphisms in this category correspond to the behaviours of finitary probabilistic systems. The general category of stochastic relations, **Srel**, is described in [2] and [4]. It arises as the Kleisli category of the Giry Monad [30]. We look at the special case where the objects are finite sets.

Definition 4.2.3. The category $\mathbf{Srel}_{fn}$ of *finite stochastic relations* consists of the following data:

- objects are finite sets: A, $B\ldots$
- morphisms: $A \xrightarrow{f} B$ are finite matrices $f : B \times A \to [0,1]$ such that $\forall a \in A$

$$\sum_{b \in B} f(b,a) \leq 1.$$

The composite of two morphisms is defined by matrix multiplication:
If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ then $g \circ f : C \times A \to [0,1]$ is:

$$(g \circ f)(c,a) = \sum_{b \in B} g(c,b).f(b,a).$$

It is immediate that composition as defined above is associative, with identities $1_A : A \times A \to [0,1]$, defined $1_A(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

Remark 4.2.4. Note that we allow $\sum_{b \in B} f(b,a) \leq 1$, rather than requiring equality. This is also called a "partial" stochastic relation. A probability that is less than 1 corresponds to a computational process that may not terminate.

One obtains a symmetric monoidal category $(\mathbf{Srel}_{fn}, \otimes, I)$ where the tensor product on objects is given by the set product $A \otimes B = A \times B$. For arrows $f : A \to B$ and $g : C \to D$, i.e., $f : B \times A \to [0,1]$ and $g : D \times C \to [0,1]$ then we have $f \otimes g : A \otimes C \to B \otimes D$ is given by a map of type $f \otimes g : B \times D \times A \times C \to [0,1]$, where

$$(f \otimes g)(b,d,a,c) = f(b,a) \cdot g(d,c)\,.$$

Let A, B be finite sets. There is a canonical way to encode a function $f : A \to B$ as a stochastic map: we write $\hat{f} : B \times A \to [0,1]$ where $\hat{f}(b,a) = 1$ if $f(a) = b$ and $\hat{f}(b,a) = 0$ otherwise. We define the symmetric monoidal coherence isomorphisms by applying this codification to the coherence structure of the cartesian category **FinSet** of finite sets.

Definition 4.2.5. Let $f : X \otimes U \to Y \otimes U$ be a stochastic map. We define the following trace class $\mathbb{T}^U_{X,Y} \subseteq \mathbf{Srel}_{fn}(X \otimes U, Y \otimes U)$ for all X and Y:

$$f \in \mathbb{T}^U_{X,Y} \text{ iff } \textstyle\sum_{y \in Y} \sum_{u \in U} f(y,u,x,u) \leq 1, \forall x \in X$$

and a partial trace:

$$\mathrm{Tr}^U_{X,Y} : \mathbb{T}^U_{X,Y} \to \mathbf{Srel}_{fn}(X,Y) \text{ with } \mathrm{Tr}^U_{X,Y}(f)(y,x) = \textstyle\sum_{u\in U} f(y,u,x,u).$$

Proposition 4.2.6. *The formula given in Definition 4.2.5 defines a partial trace on* $\mathbf{Srel}_{fn}$.

Proof. We check the axioms of partial trace.

Naturality:

Let $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \to X$ and $h : Y \to Y'$ be stochastic maps, first we want to prove that

$$(h \otimes 1_U)f(g \otimes 1_U) \in \mathbb{T}^U_{X',Y'} \text{ with } (h \otimes 1_U)f(g \otimes 1_U) : X' \otimes U \to Y' \otimes U.$$

Since we have a map of type $(h \otimes 1_U)f : X \otimes U \to Y' \otimes U$ we evaluate:

$$(h \otimes 1_U)f(y',u,x,v) = \textstyle\sum_{y\in Y,u'\in U}(h \otimes 1_U)(y',u,y,u')f(y,u',x,v) =$$
$$\textstyle\sum_{y\in Y,u'\in U} h(y',y)1(u,u')f(y,u',x,v) = \sum_{y\in Y} h(y',y)f(y,u,x,v).$$

Now we compose again:

$$(h \otimes 1_U)f(g \otimes 1_U)(y',u,x',v) = \textstyle\sum_{x\in X,u'\in U}(h \otimes 1_U)f(y',u,x,u')(g \otimes 1_U)(x,u',x',v) =$$
$$\textstyle\sum_{x\in X,u'\in U}(\sum_{y\in Y} h(y',y)f(y,u,x,u')).g(x,x').1_U(u',v) =$$
$$\textstyle\sum_{x\in X,y\in Y} h(y',y)f(y,u,x,v)g(x,x').$$

Thus $(h \otimes 1_U)f(g \otimes 1_U) \in \mathbb{T}^U_{X',Y'}$ iff $\sum_{y'\in Y',u\in U}(h \otimes 1_U)f(g \otimes 1_U)(y',u,x',u) \leq 1, \forall x' \in X'$.

We know by hypothesis that $f \in \mathbb{T}^U_{X,Y}$ which implies that $\sum_{y\in Y,u\in U} f(y,u,x,u) \leq 1, \forall x \in X$. On the other hand we also know that $\sum_{x\in X} g(x,x') \leq 1\ \forall x' \in X$ and $\sum_{y'\in Y'} h(y',y) \leq 1\ \forall y \in Y$ since $g : X' \to X$ and $h : Y \to Y'$ are stochastic maps. Thus,

$$\textstyle\sum_{x\in X}(\sum_{y\in Y,u\in U} f(y,u,x,u))g(x,x') \leq \sum_{x\in X} 1.g(x,x') \leq 1\ \forall x' \in X'.$$

Therefore,

$$\textstyle\sum_{x\in X,y\in Y,u\in U} f(y,u,x,u))g(x,x') \leq 1\ \forall x' \in X'.$$

Now using this and the fact that $\sum_{y'\in Y'} h(y',y) \leq 1$:

$$\sum_{x\in X,y\in Y,u\in U}(\sum_{y'\in Y'} h(y',y)).f(y,u,x,u))g(x,x') \leq$$
$$\sum_{x\in X,y\in Y,u\in U} 1.f(y,u,x,u))g(x,x') \leq 1 \ \forall x' \in X'.$$

This implies the following:

$$\sum_{x\in X,y\in Y,u\in U,y'\in Y'} h(y',y))f(y,u,x,u))g(x,x') \leq 1 \ \forall x' \in X'.$$

Therefore,

$$\sum_{y'\in Y',u\in U}(\sum_{x\in X,y\in Y} h(y',y))f(y,u,x,u))g(x,x') =$$
$$\sum_{y'\in Y',u\in U}(h\otimes 1_U)f(g\otimes 1_U)(y',u,x',u) \leq 1 \ \forall x' \in X'$$

which implies that the following assertion holds:

$$(h\otimes 1_U)f(g\otimes 1_U) \in \mathbb{T}^U_{X',Y'}.$$

Next, we preliminary compute the partial trace. For that purpose, we first need some previous calculations:

$$\mathrm{Tr}^U_{X',Y'}((h\otimes 1_U)f(g\otimes 1_U))(y',x') = \sum_{u\in U}(\sum_{x\in X,y\in Y} h(y',y)f(y,u,x,u))g(x,x')).$$

If we apply the definition of partial trace to f and compose with h then this comes down to

$$h\circ \mathrm{Tr}^U_{X,Y}(f)(y',x) = \sum_{y\in Y} h(y',y).(\sum_{u\in U} f(y,u,x,u)) =$$
$$\sum_{y\in Y,u\in U} h(y',y)f(y,u,x,u).$$

Similarly, we compose with g

$$((h\,\mathrm{Tr}^U_{X,Y}(f))g)(y',x') = \sum_{x\in X}(h\,\mathrm{Tr}^U_{X,Y}(f))(y',x).g(x,x') =$$
$$\sum_{x\in X}(\sum_{y\in Y,u\in U} h(y',y)f(y,u,x,u)).g(x,x') =$$
$$\sum_{x\in X,y\in Y,u\in U} h(y',y)f(y,u,x,u)g(x,x')$$

which proves that both previous calculations are equal.

Yanking:

Let $\sigma : A\otimes B \to B\otimes A$ be defined as the matrix $\sigma : B\times A\times A\times B \to [0,1]$ with

$$\sigma(b,a,a',b') = 1 \text{ iff } b=b' \text{ and } a=a' \text{ otherwise is } 0.$$

It may be seen immediately that if $\sigma : U\otimes U \to U\otimes U$

$$\mathrm{Tr}^U_{U,U}(\sigma)(u,v) = \sum_{x\in U}\sigma(u,x,v,x) = 1 \text{ if and only if } u=x=v \text{ otherwise is } 0.$$

Then, since $1_U(u,v) = 1$ if and only if $u = v$, otherwise it is 0 we obtain that

$$\mathrm{Tr}^U_{U,U}(\sigma)(u,v) = 1_U(u,v) \text{ for every } u \text{ and } v.$$

Dinaturality:

Consider the stochastic maps $f : X \otimes U \to Y \otimes U'$ and $g : U' \to U$. First we want to prove that

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \text{ if and only if } f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y}.$$

By definition of trace class we know that

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \text{ if and only if } \textstyle\sum_{y\in Y, u\in U}(1_Y \otimes g)f(y,u,x,u) \leq 1 \ \forall x \in X.$$

Also, by definition of composition in the category $\mathbf{Srel}_{fn}$:

$$(1_Y \otimes g)f(y,u,x,v) = \textstyle\sum_{y'\in Y, u'\in U'}(1_Y \otimes g)(y,u,y',u')f(y',u',x.v) =$$
$$\textstyle\sum_{y'\in Y, u'\in U'} 1_Y(y,y')g(u,u')f(y',u',x,v) = \sum_{u'\in U'} g(u,u')f(y,u',x,v).$$

Thus, we have seen that

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \text{ if and only if } \textstyle\sum_{y\in Y, u\in U}(\sum_{u'\in U'} g(u,u')f(y,u',x,u)) \leq 1 \ \forall x \in X.$$

Following a similar argument we have that

$$f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y} \text{ if and only if } \textstyle\sum_{y\in Y, u'\in U'} f(1_X \otimes g)(y,u',x,u') \leq 1 \ \forall x \in X.$$

But, again by definition of composition

$$f(1_X \otimes g)(y,u',x,v') = \textstyle\sum_{x'\in X, u\in U} f(y,u',x',u)(1_X \otimes g)(x',u,x,v') =$$
$$\textstyle\sum_{x'\in X, u\in U} f(y,u',x',u)1_X(x',x)g(u,v') = \sum_{u\in U} f(y,u',x,u)g(u,v').$$

This means that

$$f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y} \text{ if and only if } \textstyle\sum_{y\in Y, u'\in U'}(\sum_{u\in U} f(y,u',x,u)g(u,u')) \leq 1 \ \forall x \in X.$$

This implies that the condition on the trace class is satisfied. Next, it remains to calculate the corresponding partial traces.

$$\mathrm{Tr}^U_{X,Y}((1_Y \otimes g)f)(y,x) = \textstyle\sum_{u\in U}(1_Y \otimes g)f(y,u,x,u) =$$
$$\textstyle\sum_{u\in U}(\sum_{u'\in U'} g(u,u')f(y,u',x,u)) = \sum_{u\in U, u'\in U'} g(u,u')f(y,u',x,u) =$$
$$\textstyle\sum_{u'\in U', u\in U} f(y,u',x,u))g(u,u') = \sum_{u'\in U'} f(1_X \otimes g)(y,u',x,u') =$$
$$\mathrm{Tr}^{U'}_{X,Y}(f(1_X \otimes g))(y,x).$$

Vanishing I:

Let $f : X \otimes I \to Y \otimes I$ be a stochastic map. Therefore, this implies by definition

$$\sum_{y\in Y, u\in\{*\}} f(y,u,x,*) = \sum_{y\in Y} f(y,*,x,*) \leq 1 \text{ for every } x \in X.$$

Thus, this is equivalent to

$$\sum_{y\in Y, u\in\{*\}} f(y,u,x,u) \leq 1 \text{ for every } x \in X$$

which is the condition $f \in \mathbb{T}^I_{X,Y}$.

Now, we compute the partial traces. Let us consider the following composition

$$X \overset{\rho_X^{-1}}{\to} X \otimes I \overset{f}{\to} Y \otimes I \overset{\rho_Y}{\to} Y.$$

We have

$$\begin{aligned} f\rho_X^{-1}(y,*,x) &= \sum_{x'\in X, u\in I} f(y,*,x',u)\rho_X^{-1}(x',u,x) \\ &= f(y,*,x,*) \end{aligned}$$

$$f\rho_X^{-1}(y,*,x) = \sum_{x'\in X, u\in I} f(y,*,x',u)\rho_X^{-1}(x',u,x) = f(y,*,x,*).$$

Now, we compose with ρ_Y to get:

$$\begin{aligned} \rho_Y(f\rho_X^{-1})(y,x) &= \sum_{y'\in Y, u\in I} \rho_Y(y,y',u)(f\rho_X^{-1})(y',u,,x) \\ &= f\rho_X^{-1}(y,*,x) = f(y,*,x,*) \end{aligned}$$

$$\rho_Y(f\rho_X^{-1})(y,x) = \sum_{y'\in Y, u\in I} \rho_Y(y,y',u)(f\rho_X^{-1})(y',u,,x) = f\rho_X^{-1}(y,*,x) = f(y,*,x,*)$$

which clearly means that

$$\mathrm{Tr}^I_{X,Y}(f)(y,x) = f(y,*,x,*) = \rho_Y f\rho_X^{-1}(y,x) \text{ for every } x \in X \text{ and } y \in Y.$$

Thus, we proved that $\mathrm{Tr}^I_{X,Y}(f) = \rho_Y f\rho_X^{-1}$.

Vanishing II:

Suppose we have a stochastic map $g : X\otimes U\otimes V \to Y\otimes U\otimes V$ such that $g \in \mathbb{T}^V_{X\otimes U, Y\otimes U}$. We need to check that

$$g \in \mathbb{T}^{U\otimes V}_{X,Y} \text{ if and only if } \mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g) \in \mathbb{T}^U_{X,Y}.$$

By definition, it follows that

$$g \in \mathbb{T}^{U\otimes V}_{X,Y} \text{ if and only if } \textstyle\sum_{y\in Y,(u,v)\in U\times V} g(y,u,v,x,u,v) \leq 1.$$

On the other hand we have

$$\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g)(y,u,x,u') = \textstyle\sum_{v\in V} g(y,u,v,x,u',v).$$

We obtain

$$\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g) \in \mathbb{T}^U_{X,Y} \text{ if and only if}$$
$$\textstyle\sum_{y\in Y}(\sum_{u\in U} \mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g)(y,u,x,u) = \sum_{y\in Y, u\in U, v\in V} g(y,u,v,x,u,v) \leq 1.$$

Thus, we have shown that both conditions are equivalent. Now we move to the calculation of the partial traces.

$$\begin{aligned}
\mathrm{Tr}^{U\otimes V}_{X,Y}(g)(y,x) &= \sum_{(u,v)\in U\times V} g(y,u,v,x,u,v) \\
&= \sum_{u\in U}\sum_{v\in V} g(y,u,v,x,u,v) \\
&= \sum_{u\in U} \mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g)(y,u,x,u) \\
&= \mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g))(y,x).
\end{aligned}$$

In conclusion we obtain that

$$\mathrm{Tr}^{U\otimes V}_{X,Y}(g) = \mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\otimes U, Y\otimes U}(g)).$$

Superposing:

Consider the stochastic maps $f : X \otimes U \to Y \otimes U$ with $f \in \mathbb{T}^U_{X,Y}$ and $g : W \to Z$. First, we want to prove that

$$g \otimes f \in \mathbb{T}^U_{W\otimes X, Z\otimes Y}.$$

In order to prove this we have that

$g \otimes f \in \mathbb{T}^U_{W\otimes X, Z\otimes Y}$	
if and only if	$\sum_{(z,y)\in Z\times Y, u\in U} g\otimes f(z,y,u,w,x,u) \leq 1 \forall w \in W, \forall x \in X$
if and only if	$\sum_{z\in Z, y\in Y, u\in U} g(z,w) f(y,u,x,u) \leq 1 \forall w \in W, \forall x \in X$
if and only if	$\sum_{z\in Z} g(z,w) \sum_{y\in Y, u\in U} f(y,u,x,u) \leq 1, \forall w \in W, \forall x \in X.$

Here the last equivalence is true since g is stochastic i.e., $\sum_{z\in Z} g(z,w) \leq 1$, $\forall w \in W$. Since we have that $f \in \mathbb{T}^U_{X,Y}$ this implies $\sum_{y\in Y, u\in U} f(y,u,x,u) \leq 1$, $\forall x \in X$. We show now that the partial traces are equal.

$$\mathrm{Tr}^U_{W\otimes X, Z\otimes Y}(g\otimes f)(z,y,w,x) = \sum_{u\in U}(g\otimes f)(z,y,u,w,x,u) = \sum_{u\in U} g(z,w).f(y,u,x,u) = g(z,w).\sum_{u\in U} f(y,u,x,u) = g(z,w).\,\mathrm{Tr}^U_{X,Y}(f)(y,x) = g\otimes \mathrm{Tr}^U_{X,Y}(f)(z,y,w,x).$$

This means that

$$\mathrm{Tr}^U_{W\otimes X, Z\otimes Y}(g\otimes f) = g\otimes \mathrm{Tr}^U_{X,Y}(f).$$

□

4.2.3 Total trace on completely positive maps with $\otimes$

In this section, we define a total trace on the category $\mathbf{CPM}_s$ of simple completely positive maps (see Section 3.1). As a matter of fact, this category is compact closed, and therefore is has a unique total trace. Here, we describe it explicitly via a Kraus operator-sum representation.

Recall that the category **fHilb** of finite dimensional Hilbert spaces and linear maps is compact closed, and therefore (totally) traced. Let, $\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B}$ and $\mathcal{H}_\mathcal{C}$ be finite dimensional Hilbert spaces with orthonormal bases $\{e_i\}, \{f_i\}$ and $\{w_i\}$, respectively, and let $F : \mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B} \to \mathcal{H}_\mathcal{C}\otimes\mathcal{H}_\mathcal{B}$ be a linear function, i.e.,

$$F = \sum_{j,l,k,m} F_{j,l,k,m}|w_j, f_k\rangle\langle e_l, f_m|.$$

Then $\mathrm{tr}_B(F) = \sum_{j,l,k} F_{j,l,k,k}|w_j\rangle\langle e_l|$ defines a total trace on **fHilb**.

Proposition 4.2.7. *Let $F : \mathcal{L}(\mathcal{H}_\mathcal{A})\otimes\mathcal{L}(\mathcal{H}_\mathcal{B}) \to \mathcal{L}(\mathcal{H}_\mathcal{C})\otimes\mathcal{L}(\mathcal{H}_\mathcal{B})$ be a complete positive map with representation $F = \sum_{j=1}^n F_j\rho F_j^\dagger$. Then $\mathrm{Tr}^{A,C}_B(F)(\rho) = \sum_{j=1}^n \mathrm{tr}_B F_j\rho\ \mathrm{tr}_B F_j^\dagger$ defines a (total) trace on the category $\mathbf{CPM}_s$.*

Proof. Suppose we take two representations of

$$F(\rho) = \sum_{i=1}^n E_i\rho E_i^\dagger = \sum_{j=1}^n F_j\rho F_j^\dagger.$$

Then

$$\mathrm{Tr}_B^{A,C}(F)(\rho)=\sum_{i=1}^n \mathrm{tr}_B\, F_i\, \rho\, \mathrm{tr}_B\, F_i^\dagger=\sum_{i=1}^n \mathrm{tr}_B(\sum_j U_{i,j}F_j)\rho\ \mathrm{tr}_B(\sum_j U_{i,j}F_j)^\dagger = \sum_{i=1}^n(\sum_j U_{i,j}\,\mathrm{tr}_B\, F_j)\rho\ (\sum_j U_{i,j}^*\,\mathrm{tr}_B\, F_j^\dagger) = \sum_{i,j,k} U_{i,j}U_{i,k}^*\,\mathrm{tr}_B\, F_j\ \rho\ \mathrm{tr}_B\, F_k^\dagger = \sum_{j,k}(\sum_i U_{i,j}U_{i,k}^*)\,\mathrm{tr}_B\, F_j\,\rho\,\mathrm{tr}_B\, F_k^\dagger = \sum_{j,k}(\sum_i U_{k,i}^\dagger U_{i,j})\,\mathrm{tr}_B\, F_j\rho\ \mathrm{tr}_B\, F_k^\dagger = \sum_{j,k}\delta_{k,j}\,\mathrm{tr}_B\, F_j\rho\ \mathrm{tr}_B\, F_k^\dagger = \sum_j \mathrm{tr}_B\, F_j\rho\ \mathrm{tr}_B\, F_j^\dagger$$

since U is unitary.

Now we check all the axioms.

Naturality:

Let us consider $f = \sum_i U_i - U_i^\dagger$ and $g = \sum_j V_j - V_j^\dagger$ where $f\colon \mathcal{L}(\mathcal{H}_A)\otimes\mathcal{L}(\mathcal{H}_B)\to\mathcal{L}(\mathcal{H}_C)\otimes\mathcal{L}(\mathcal{H}_B)$ and $g\colon \mathcal{L}(\mathcal{H}_{A'})\to\mathcal{L}(\mathcal{H}_A)$.

Since $f(g\otimes id) = (\lambda\rho\sum_i U_i\rho U_i^\dagger)(\lambda\rho\sum_j V_j\rho V_j^\dagger\otimes id) = \lambda\rho\sum_{i,j} U_i(V_j\otimes I)\rho(V_j^\dagger\otimes I)U_i^\dagger$ therefore, we have:

$$\mathrm{Tr}_{A'C}^B(f(g\otimes id)) = \lambda\rho\sum_{i,j}\mathrm{tr}_B(U_i(V_j\otimes I))\rho\ \mathrm{tr}_B((V_j^\dagger\otimes I)U_i^\dagger) = \lambda\rho\sum_{i,j}(\mathrm{tr}_B\, U_i)V_j\rho V_j^\dagger(\mathrm{tr}_B\, U_i^\dagger) = \lambda\rho\sum_{i,j}(\mathrm{tr}_B\, U_i)\rho(\mathrm{tr}_B\, U_i^\dagger)\circ\lambda\rho\sum_j V_j\rho V_j^\dagger = \mathrm{Tr}_{AC}^B(f)\circ g.$$

Dinaturality:

Suppose we have $f = \sum_i U_i - U_i^\dagger$ and $g = \sum_j V_j - V_j^\dagger$ where $f\colon \mathcal{L}(\mathcal{H}_A)\otimes\mathcal{L}(\mathcal{H}_B)\to\mathcal{L}(\mathcal{H}_C)\otimes\mathcal{L}(\mathcal{H}_{B'})$ and $g\colon \mathcal{L}(\mathcal{H}_{B'})\to\mathcal{L}(\mathcal{H}_B)$.

Then

$$\mathrm{Tr}_{AC}^B((1\otimes g)f) = \mathrm{Tr}_{AC}^B((\lambda\rho\sum_j(I\otimes V_j)\rho(I\otimes V_j^\dagger))\circ(\lambda\rho\sum_i U_i\rho U_i^\dagger)) = \mathrm{Tr}_{AC}^B(\lambda\rho\sum_{i,j}(I\otimes V_j)(U_i\rho U_i^\dagger)(I\otimes V_j^\dagger)) = \sum_{i,j}\mathrm{tr}_B((I\otimes V_j)U_i)\rho\ \mathrm{tr}_B(U_i^\dagger(I\otimes V_j^\dagger)) = \sum_{i,j}\mathrm{tr}_{B'}(U_i(I\otimes V_j))\rho\ \mathrm{tr}_{B'}((I\otimes V_j^\dagger)U_i^\dagger) = \mathrm{Tr}_{AC}^{B'}(\lambda\rho\sum_{i,j}U_i(I\otimes V_j)\rho(I\otimes V_j^\dagger)U_i^\dagger) = \mathrm{Tr}_{AC}^{B'}((\lambda\rho\sum_i U_i\rho U_i^\dagger)\circ(\lambda\rho\sum_j(I\otimes V_j)\rho(I\otimes V_j^\dagger))) = \mathrm{Tr}_{AC}^{B'}(f(1\otimes g)).$$

Vanishing I:

Consider the map $f\colon \mathcal{L}(\mathcal{H}_A)\otimes\mathcal{L}(\mathcal{H}_I)\to\mathcal{L}(\mathcal{H}_B)\otimes\mathcal{L}(\mathcal{H}_I)$ with the following representation $f = \sum_i U_i - U_i^\dagger$, so

$\mathrm{Tr}_{A,B}^I(f) = \lambda\rho\sum_i \mathrm{tr}_I\, U_i\rho\ \mathrm{tr}_I\, U_i^\dagger = \sum_i U_i\rho\, U_i^\dagger.$

Vanishing II:

Let us consider $g : \mathcal{L}(\mathcal{H}_\mathcal{X}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{U}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{V}) \to \mathcal{L}(\mathcal{H}_\mathcal{Y}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{U}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{V})$ with representation $g = \sum_i E_i - E_i^\dagger$ then:

$$T^U_{X,Y}(T^V_{X\otimes U, Y\otimes U}(g)) = T^U_{X,Y}(\lambda\rho \sum_i \operatorname{tr}_V E_i \rho \operatorname{tr}_V E_i^\dagger) =$$
$$\lambda\rho \sum_i \operatorname{tr}_U(\operatorname{tr}_V(E_i))\rho \operatorname{tr}_U(\operatorname{tr}_V(E_i^\dagger)) = \lambda\rho \sum_i \operatorname{tr}_{U\otimes V}(E_i))\rho \operatorname{tr}_{U\otimes V}(E_i^\dagger) = T^{U\otimes V}_{X,Y}(g).$$

Yanking:

Before we study the proof of this axiom we consider a representation of the symmetric isomorphism:

$$\sigma_{N,M} : \mathcal{L}(\mathcal{H}_\mathcal{N}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{M}) \to \mathcal{L}(\mathcal{H}_\mathcal{M}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{N}).$$

Let $\{e_i^n\}$, $\{e_j^m\}$ be an orthonormal basis for $\mathcal{H}_\mathcal{N}$ and $\mathcal{H}_\mathcal{M}$ respectively. Then $\{E^n_{i,j}\}$ and $\{E^m_{k,l}\}$ are orthonormal basis for $\mathcal{L}(\mathcal{H}_\mathcal{N})$ and $\mathcal{L}(\mathcal{H}_\mathcal{N})$ respectively with $E^n_{i,j} = e_i^n e_j^{n\dagger}$, $E^m_{k,l} = e_k^m e_l^{n\dagger}$ and $\langle A, B\rangle = \operatorname{tr}(A^\dagger B)$ as a inner product.
Thus we have:

$$\sigma(E^n_{i,j} \otimes E^m_{k,l}) = \sigma(|e_i^n\rangle\langle e_j^n| \otimes |e_k^m\rangle\langle e_l^m|) = \sigma(|e_i^n\rangle|e_k^m\rangle \otimes \langle e_j^n|\langle e_l^m|) =$$
$$U(|e_i^n\rangle|e_k^m\rangle \otimes \langle e_j^n|\langle e_l^m|)U^\dagger = U|e_i^n, e_k^m\rangle \otimes (U|e_j^n, e_l^m\rangle)^\dagger = |e_k^m, e_i^n\rangle \otimes (|e_l^m, e_j^n\rangle)^\dagger =$$
$$|e_k^m, e_i^n\rangle \otimes \langle e_l^m, e_j^n| = |e_k^m\rangle\langle e_l^m| \otimes |e_i^n\rangle\langle e_j^n| = E^m_{k,l} \otimes E^n_{i,j}$$

for every vector basis where the action U is defined by $U|e_i^n \otimes e_j^m\rangle = |e_j^m \otimes e_i^n\rangle$ on the basis of the tensor space. This implies that $\sigma(A) = UAU^\dagger$ for every $A \in \mathcal{L}(\mathcal{H}_\mathcal{N}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{M})$.

Now, let $\sigma_{N,N} : \mathcal{L}(\mathcal{H}_\mathcal{N}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{N}) \to \mathcal{L}(\mathcal{H}_\mathcal{N}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{N})$ be the symmetric natural isomorphism with the representation $\sigma_{N,N} = U - U^\dagger$, $\sigma_{N,N} : \mathcal{H}_\mathcal{N} \otimes \mathcal{H}_\mathcal{N} \to \mathcal{H}_\mathcal{N} \otimes \mathcal{H}_\mathcal{N}$ where $U = \sum_{k,l} |e_l^m\rangle\langle e_k^n| \otimes |e_k^n\rangle\langle e_l^m|$ and $U^\dagger = \sum_{k,l} |e_k^n\rangle\langle e_l^n| \otimes |e_l^n\rangle\langle e_k^n|$. Thus we have that $\operatorname{tr}_N U = \sum_{k,l} |e_l^n\rangle\langle e_k^n| \otimes \operatorname{tr}(|e_k^n\rangle\langle e_l^n|) = \sum_{k,l} |e_l^n\rangle\langle e_k^n| \otimes \langle e_k^n|e_l^n\rangle = \sum_{k=l} |e_l^n\rangle\langle e_k^n| = \sum_l |e_l^n\rangle\langle e_l^n| = Id_N$. In an analogous way we trace $U^\dagger$ obtaining the identity. Hence $\operatorname{Tr}^N_{N,N}(\sigma_{N,N})(\rho) = \operatorname{tr}_N U\rho \operatorname{tr}_N U^\dagger = Id_N \rho Id_N = \rho$. □

Remark 4.2.8. The category $\mathbf{CPM}_s$ is compact closed, due to the existence of a monoidal functor $F : \mathbf{fHilb} \to \mathbf{CPM}_s$ which is onto objects. (This functor takes each object to itself, and each linear map f to $F(\rho) = f\rho f^\dagger$). This already implies that

this category is traced, and moreover that the trace is unique by Proposition 2.9.3. It is easy to check that the trace is indeed computed as above.

4.2.4 Partial trace in the category Vect

In Definition 4.2.1, we considered a partial trace on the category of finite dimensional vector spaces with $\oplus$ as a tensor product. Now, we relax conditions on the definition of the trace class and we define another partial trace on vector spaces for not necessarily finite dimensions.

Definition 4.2.9. Let $(\mathbf{Vect}, \oplus, 0)$ be the symmetric monoidal category of vector spaces and linear transformations with the monoidal tensor taken to be the direct sum. We define a trace class in the following way. Given a map $f : V \oplus U \to W \oplus U$ we say $f \in \mathbb{T}^U_{V,W}$ iff

- $imf_{21} \subseteq im(I - f_{22})$ and
- $ker(I - f_{22}) \subseteq kerf_{12}$,

where I is the identity map. Whenever these conditions are satisfied we define $\mathrm{Tr}^U_{V,W}(f) : V \to W$:

$$\mathrm{Tr}^U_{V,W}(f)(v) = f_{11}(v) + f_{12}(u) \text{ for some } u \in U \text{ such that } (I - f_{22})(u) = f_{21}(v).$$

To show that this is well-defined, suppose u' is another candidate satisfying

$$(I - f_{22})(u') = f_{21}(v).$$

Then $(I - f_{22})(u - u') = 0$ which implies by the second condition of Definition 4.2.9 that $f_{12}(u) = f_{12}(u')$. This shows that the value of the trace does not depend on the choice of the pre-image, but on its existence.

Remark 4.2.10. Notice that the partial trace of Definition 4.2.9 generalizes that of Definition 4.2.1. Indeed, if $I - f_{22}$ is invertible, then $im(I - f_{22}) = U$ and $ker(I - f_{22}) = 0$, which implies that Definition 4.2.9 is trivially satisfied and in this case, $\mathrm{Tr}^U_{V,W}(f) =$

$f_{11} + f_{12}(I - f_{22})^{-1}f_{21}$ (where $u = (I - f_{22})^{-1}f_{21}(v)$). Moreover, Definition 4.2.9 is strictly more general than Definition 4.2.1, because the identity maps are traceable in Definition 4.2.9, but not in Definition 4.2.1.

Theorem 4.2.11. *The formula given in Definition 4.2.9 is a partial trace.*

Proof. **Naturality:**
Let $f \in \mathbb{T}^U_{X,Y}$, $g : X' \to X$ and $h : Y \to Y'$ be linear maps. First, we want to prove that

$$(h \oplus 1_U)f(g \oplus 1_U) \in \mathbb{T}^U_{X',Y'} \text{ with } (h \oplus 1_U)f(g \oplus 1_U) : X' \oplus U \to Y' \oplus U.$$

The following equations are satisfied by naturality on injections and projections:

- $((h \oplus 1_U)f(g \oplus 1_U))_{11} = hf_{11}g$
- $((h \oplus 1_U)f(g \oplus 1_U))_{12} = hf_{12}$
- $((h \oplus 1_U)f(g \oplus 1_U))_{21} = f_{21}g$
- $((h \oplus 1_U)f(g \oplus 1_U))_{22} = f_{22}$.

Thus, we have

$$im((h\oplus 1_U)f(g\oplus 1_U))_{21} = imf_{21}g \subseteq imf_{21} \subseteq im(I-f_{22}) = im(I-((h\oplus 1_U)f(g\oplus 1_U))_{22})$$

by the hypotheses, properties of the image, and the equations above.
Also,

$$ker(I-((h\oplus 1_U)f(g\oplus 1_U))_{22}) = ker(I-f_{22}) \subseteq kerf_{12} \subseteq ker\, hf_{12} = ker((h\oplus 1_U)f(g\oplus 1_U))_{12}$$

by the equations above, by the hypothesis, the properties of the kernel and the equations above again.
Now, we want to check the value of the trace. In view of the definition, we may write:

$$\mathrm{Tr}^U_{X',Y'}((h \oplus 1_U)f(g \oplus 1_U))(x) = ((h \oplus 1_U)f(g \oplus 1_U))_{11}(x) + ((h \oplus 1_U)f(g \oplus 1_U))_{12}(u)$$

for some $u \in U$ such that $(I - ((h \oplus 1_U)f(g \oplus 1_U))_{22})(u) = (h \oplus 1_U)f(g \oplus 1_U))_{21}(x)$. But, this implies using the equations above that:

$$\mathrm{Tr}^U_{X',Y'}((h \oplus 1_U)f(g \oplus 1_U))(x) = hf_{11}g(x) + hf_{12}(u) = h(f_{11}(g(x)) + f_{12}(u)) = h\,\mathrm{Tr}^U_{X,Y}(f)(g(x)) = h\,\mathrm{Tr}^U_{X,Y}(f)g(x)$$

for some $u \in U$ such that $(I - f_{22})(u) = f_{21}(g(x))$.

Dinaturality :

For any $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$ we must prove that

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \text{ iff } f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y}$$

and also we need to check: $\mathrm{Tr}^U_{X,Y}((1_Y \otimes g)f) = \mathrm{Tr}^{U'}_{X,Y}(f(1_X \otimes g))$.

On the one hand, we know by naturality on injections and projections that we have the following equations:

- $((1_Y \oplus g)f)_{11} = f_{11}$
- $((1_Y \oplus g)f)_{12} = f_{12}$
- $((1_Y \oplus g)f)_{21} = gf_{21}$
- $((1_Y \oplus g)f)_{22} = gf_{22}$.

On the other hand we know:

- $(f(1_X \oplus g))_{11} = f_{11}$
- $(f(1_X \oplus g))_{12} = f_{12}g$
- $(f(1_X \oplus g))_{21} = f_{21}$
- $(f(1_X \oplus g))_{22} = f_{22}g$.

First, let us now prove the following equivalence:

$im((1_Y \oplus g)f)_{21} \subseteq im(I - ((1_Y \oplus g)f)_{22})$ iff $im(f(1_X \oplus g))_{21} \subseteq im(I - (f(1_X \oplus g))_{22})$.

By the equations above, it corresponds to the following equivalence:

$$im\, gf_{21} \subseteq im(I - gf_{22}) \text{ iff } imf_{21} \subseteq im(I - f_{22}g).$$

($\Rightarrow$) Given $x = f_{21}(z)$ for some z we want to prove that $x \in im(I - f_{22}g)$.
Since, by hypothesis $g(x) = g(f_{21}(z)) \in im(I - gf_{22})$ then $g(f_{21}(z)) = z' - g(f_{22}(z'))$ for some z', which implies that $g(f_{21}(z) + f_{22}(z')) = z'$. Thus, now choose $v = f_{21}(z) + f_{22}(z')$ allowing us to obtain:

$$v - f_{22}(g(v)) = f_{21}(z) + f_{22}(z') - f_{22}(g(v)) = f_{21}(z) + f_{22}(z') - f_{22}(z') = f_{21}(z) = x$$

($\Leftarrow$) Given $y = g(f_{21}(u))$ for some u we want to prove $y \in im(I - gf_{22})$.
Since by hypothesis there is a z such that $f_{21}(u) = z - f_{22}(g(z))$ consider $v = g(z)$; then we get the following:

$$(I - gf_{22})(v) = g(z) - g(f_{22}(g(z))) = g(z - f_{22}(g(z))) = g(f_{21}(u)) = y.$$

Next, we want to check the following:

$$ker(I - ((1_Y \oplus g)f)_{22}) \subseteq ker((1_Y \oplus g)f)_{12} \text{ iff}$$
$$ker(I - (f(1_X \oplus g))_{22}) \subseteq ker(f(1_X \oplus g))_{12}$$

which by the equations above is equivalent to:

$$ker(I - gf_{22}) \subseteq kerf_{12} \text{ iff } ker(I - f_{22}g) \subseteq kerf_{12}g.$$

($\Rightarrow$) If $z = f_{22}g(z)$ then $g(z) = g(f_{22}(g(z)))$ which implies that $g(z) \in ker(I - gf_{22})$ and by hypothesis that $f_{12}(g(z)) = 0$ i.e., $z \in kerf_{12}g$.
($\Leftarrow$) If $v - gf_{22}(v) = 0$ then choosing $z = f_{22}(v)$ there is a z such that $g(z) = v$. But, clearly $z \in ker(I - f_{22}g)$ since $v = gf_{22}(v)$ implies:

$$(I - f_{22}g)(z) = f_{22}(v) - f_{22}g(f_{22}(v)) = f_{22}(v) - f_{22}(g(f_{22}(v))) = f_{22}(v) - f_{22}(v) = 0.$$

Then by hypothesis $z \in kerf_{12}g$, which means that $f_{12}g(z) = 0$ i.e., $f_{12}(v) = 0$. Hence, we proved that if $v - gf_{22}(v) = 0$ then $f_{12}(v) = 0$.
Now we are ready to check the values of the traces.

$$\mathrm{Tr}^U_{X,Y}((1_Y \oplus g)f)(u) = ((1_Y \oplus g)f)_{11}(u) + ((1_Y \oplus g)f)_{12}(v) \text{ for some } v \text{ with}$$
$$(I - ((1_Y \oplus g)f)_{22})(v) = ((1_Y \oplus g)f)_{21}(u)$$

which by the equations above we get:

$\mathrm{Tr}^U_{X,Y}((1_Y \oplus g)f)(u) = f_{11}(u) + f_{12}(v)$ for some v such that $(I - gf_{22})(v) = gf_{21}(u)$.

On the other hand we have that:

$$\mathrm{Tr}^{U'}_{X,Y}(f(1_X \oplus g))(u) = (f(1_X \oplus g))_{11}(u) + (f(1_X \oplus g))_{12}(v') \text{ for some } v' \text{ such that } (I - (f(1_X \oplus g))_{22})(v') = (f(1_X \oplus g))_{21}(u)$$

and again by the equations above:

$$\mathrm{Tr}^{U'}_{X,Y}(f(1_X \oplus g))(u) = f_{11} + f_{12}g(v') \text{ for some } v' \text{ such that } (I - f_{22}g)(v') = f_{21}(u).$$

($\Rightarrow$) Given v as above there is a v' such that $g(v') = v$ since we have $(I - gf_{22})(v) = gf_{21}(u)$ then $v = g(f_{22}(v) + f_{21}(u))$ so choose $v' = f_{22}(v) + f_{21}(u)$ and this vector satisfies the condition required since $(I - f_{22}g)(v') = v' - f_{22}g(v') = f_{22}(v) + f_{21}(u) - f_{22}g(v') = f_{21}(u)$.
($\Leftarrow$) Choose $v = g(v')$ and then we get $(I - gf_{22})(v) = (I - gf_{22})(g(v')) = g(v') - gf_{22}g(v') = g(v' - f_{22}g(v')) = g(I - f_{22}g)(v')) = g(f_{21}(u)) = gf_{21}(u)$.

Vanishing I :

Now, we want to check that:

$$\widehat{\mathbb{T}}^I_{X,Y} = C(X \otimes I, Y \otimes I) \text{ and } \mathrm{Tr}^I_{X,Y}(f) = \rho_Y f \rho_X^{-1}.$$

Let us consider $f : X \oplus I \to Y \oplus I$, we notice first that $im f_{21} = im\, 0 = 0 \subseteq im(I - f_{22})$ and $ker(I - f_{22}) = ker I = 0 \subseteq ker f_{12}$ since f_{12}, f_{21}, f_{22} are constant 0 functions. Next, we move to the value of the trace:

$$\mathrm{Tr}^0_{X,Y}(f) = f_{11}(u) + f_{12}(v) \text{ for some } v \text{ such that } (I - f_{22})(v) = f_{21}(u).$$

Therefore, since $f_{21} = 0$ we choose $v = 0$ as a representative and we obtain:

$$\mathrm{Tr}^0_{X,Y}(f) = f_{11}(u) = \pi_{11} f\, in_{11}(u) = \rho_Y\, f\, \rho_X^{-1}(u).$$

since injection, projection and ρ isomorphism coincide in this case.

Vanishing II :

For any $g : X \otimes U \otimes V \to Y \otimes U \otimes V$, with $g \in \mathbb{T}^V_{X \otimes U, Y \otimes U}$ we want to prove the following equivalence:

$$g \in \mathbb{T}^{U \otimes V}_{X,Y} \text{ iff } \mathrm{Tr}^V_{X \otimes U, Y \otimes U}(g) \in \mathbb{T}^U_{X,Y}.$$

We are going to represent g using matrix notation:

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33}. \end{pmatrix}$$

First, we translate the general hypothesis $g \in \mathbb{T}^V_{X\otimes U, Y\otimes U}$ in terms of this matrix representation.

- $\widehat{g_{11}} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : X \oplus U \to Y \oplus U$
- $\widehat{g_{21}} = \begin{pmatrix} g_{31} & g_{32} \end{pmatrix} : X \oplus U \to V$
- $\widehat{g_{12}} = \begin{pmatrix} g_{13} \\ g_{23} \end{pmatrix} : V \to Y \oplus U$
- $\widehat{g_{22}} = \begin{pmatrix} g_{33} \end{pmatrix} : V \to V.$

Thus the condition $im\,\widehat{g_{21}} \subseteq im\,(I - \widehat{g_{22}})$ is actually $im \begin{pmatrix} g_{31} & g_{32} \end{pmatrix} \subseteq im\,(I - g_{33})$ which implies that: $\forall x \in X, \forall u \in U, \exists v \in V : g_{31}(x) + g_{32}(u) + g_{33}(v) = v$. On the other hand, the condition $ker(I - \widehat{g_{22}}) \subseteq ker\,\widehat{g_{12}}$ is $ker(I - g_{33}) \subseteq ker \begin{pmatrix} g_{13} \\ g_{23} \end{pmatrix}$ which implies that: $\forall v \in V$ such that $g_{33}(v) = v$ then $g_{13}(v) + g_{23}(v) = 0$.
We are now ready to translate the condition $g \in \mathbb{T}^{U\otimes V}_{X,Y}$ in terms of the matrix representation of g.

- $\tilde{g}_{11} = \begin{pmatrix} g_{11} \end{pmatrix} : X \to Y$
- $\tilde{g}_{21} = \begin{pmatrix} g_{21} \\ g_{31} \end{pmatrix} : X \to U \oplus V$
- $\tilde{g}_{12} = \begin{pmatrix} g_{12} & g_{13} \end{pmatrix} : U \oplus V \to Y$
- $\tilde{g}_{22} = \begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix} : U \oplus V \to U \oplus V.$

Thus the condition $im\,\tilde{g}_{21} \subseteq im\,(I - \tilde{g}_{22})$ is actually $im\begin{pmatrix} g_{21} \\ g_{31} \end{pmatrix} \subseteq im\,(I - \begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix})$ which implies that: $\forall x \in X, \exists u \in U, \exists v \in V$: $\left\{ \begin{array}{l} g_{21}(x) + g_{22}(u) + g_{23}(v) = u \\ g_{31}(x) + g_{32}(u) + g_{33}(v) = v \end{array} \right)$

On the other hand, the condition $ker(I - \tilde{g}_{22}) \subseteq ker\,\tilde{g}_{12}$ is $ker(I - \begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix}) \subseteq ker\begin{pmatrix} g_{12} & g_{13} \end{pmatrix}$ which implies that: $\forall u \in U, \forall v \in V$ such that $u = g_{22}(u) + g_{23}(v)$ and $v = g_{32}(u) + g_{33}(v)$ then $g_{12}(u) + g_{13}(v) = 0$.

Now we express $\mathrm{Tr}^V_{X\oplus U, Y\oplus U}(g) \in \mathbb{T}^U_{X,Y}$ in terms of the components of g $\mathrm{Tr}^V_{X\oplus U, Y\oplus U}(g)(x,u) = \widehat{g_{11}}(x,u) + \widehat{g_{12}}(v)$ for some $v \in V$ such that $(I - \widehat{g_{22}})(v) = \widehat{g_{21}}(x,u)$ which implies:

$$\mathrm{Tr}^V_{X\oplus U, Y\oplus U}(g)(x,u) = (g_{11}(x) + g_{12}(u), g_{21}(x) + g_{22}(u)) + (g_{13}(v), g_{23}(v)) \text{ for some } v \in V \text{ such that } v - g_{33}(v) = g_{31}(x) + g_{32}(u).$$

Now we renamed $\bar{g} = \mathrm{Tr}^V_{X\oplus U, Y\oplus U}(g)$ and compose with injections and projections.

- $\bar{g}_{11} = \pi_1 \bar{g}\, in_1 : X \to Y$, $\bar{g}_{11}(x) = g_{11}(x) + g_{13}(v_1)$ with v_1 such that $v_1 - g_{33}(v_1) = g_{31}(x)$
- $\bar{g}_{21} = \pi_2 \bar{g}\, in_1 : X \to U$, $\bar{g}_{21}(x) = g_{21}(x) + g_{23}(v_1)$ with v_1 such that $v_1 - g_{33}(v_1) = g_{31}(x)$
- $\bar{g}_{12} = \pi_1 \bar{g}\, in_2 : U \to Y$, $\bar{g}_{12}(u) = g_{12}(u) + g_{13}(v_2)$ with v_2 such that $v_2 - g_{33}(v_2) = g_{32}(u)$
- $\bar{g}_{22} = \pi_2 \bar{g}\, in_2 : U \to U$, $\bar{g}_{22}(u) = g_{22}(u) + g_{23}(v_2)$ with v_2 such that $v_2 - g_{33}(v_2) = g_{32}(u)$.

Thus we have that:

$$\bar{g} \in \mathbb{T}^U_{X,Y} \text{ iff } im\,\bar{g}_{21} \subseteq im\,(I - \bar{g}_{22}) \text{ and } ker(I - \bar{g}_{22}) \subseteq ker\bar{g}_{12}.$$

By the equations above the condition $im\,\bar{g}_{21} \subseteq im\,(I - \bar{g}_{22})$ implies that

$$\forall x \in X, \forall v_1 \in V \text{ such that } v_1 - g_{33}(v_1) = g_{31}(x), \exists u \in U, \exists v_2 \in V \text{ such that } v_2 - g_{33}(v_2) = g_{32}(u) \text{ and } g_{21}(x) + g_{23}(v_1) + g_{22}(u) + g_{23}(v_2) = u.$$

On the other hand, the condition $ker(I - \bar{g}_{22}) \subseteq ker\bar{g}_{12}$ implies by the equations above that

$$\forall u \in U, \forall v_2 \in V \text{ such that } v_2 - g_{33}(v_2) = g_{32}(u), \text{ if } g_{22}(u) + g_{23}(v_2) = u \text{ then } g_{12}(u) + g_{13}(v_2) = 0.$$

Now since we have all the conditions in term of g we can prove the equivalence.

($\Rightarrow$) We have by general hypothesis that the condition $im\,\widehat{g_{21}} \subseteq im\,(I - \widehat{g_{22}})$ is actually $\forall x \in X, \forall u \in U, \exists v \in V : g_{31}(x) + g_{32}(u) + g_{33}(v) = v$. We also have now as hypothesis that the condition $im\,\tilde{g}_{21} \subseteq im\,(I - \tilde{g}_{22})$ is $\forall x \in X, \exists u \in U, \exists v \in V : g_{21}(x) + g_{22}(u) + g_{23}(v) = u$ and $g_{31}(x) + g_{32}(u) + g_{33}(v) = v$.
By the equations above we want to prove that:

$$\forall x \in X, \forall v_1 \in V \text{ such that } v_1 - g_{33}(v_1) = g_{31}(x) \qquad (\alpha)$$

then $\exists u \in U, \exists v_2 \in V$ such that the following two equations hold:

$$v_2 - g_{33}(v_2) = g_{32}(u) \qquad (\beta)$$

$$g_{21}(x) + g_{23}(v_1) + g_{22}(u) + g_{23}(v_2) = u \qquad (\gamma).$$

By hypothesis given $x \in X$, let us consider u, v such that $g_{21}(x) + g_{22}(u) + g_{23}(v) = u$ and $g_{31}(x) + g_{32}(u) + g_{33}(v) = v$. Now choose $v_2 = v - v_1$; then we have that $g_{21}(x) + g_{23}(v_1) + g_{22}(u) + g_{23}(v - v_1) = g_{21}(x) + g_{22}(u) + g_{23}(v) = u$ which proves equation (γ) using the first of the equations above. It can be seen that:
$v_2 = v - v_1 = g_{31}(x) + g_{32}(u) + g_{33}(v) - v_1 = g_{31}(x) + g_{32}(u) + g_{33}(v) - (g_{33}(v_1) + g_{31}(x)) = g_{32}(u) + g_{33}(v) - g_{33}(v_1) = g_{32}(u) + g_{33}(v - v_1) = g_{32}(u) + g_{33}(v_2)$ which proves equation (β) using the equations above.
($\Leftarrow$) Now assume the same general hypothesis as before: $\forall x \in X, \forall u \in U, \exists v \in V : g_{31}(x) + g_{32}(u) + g_{33}(v) = v$. We know by hypothesis that:
$\forall x \in X, \forall v_1 \in V$ such that $v_1 - g_{33}(v_1) = g_{31}(x), \exists u \in U, \exists v_2 \in V$ such that

$v_2 - g_{33}(v_2) = g_{32}(u)$ and $g_{21}(x) + g_{23}(v_1) + g_{22}(u) + g_{23}(v_2) = u$.
We want to prove that: $\forall x \in X, \exists u \in U, \exists v \in V$such that

$$g_{21}(x) + g_{22}(u) + g_{23}(v) = u \qquad (\star)$$

$$g_{31}(x) + g_{32}(u) + g_{33}(v) = v \qquad (\star\star)$$

Using the general hypothesis with $u = 0$ we obtain: $\forall x \in X, \exists v_1 \in V$:

$$g_{31}(x) + g_{32}(0) + g_{33}(v_1) = v_1. \qquad (\star\star\star)$$

Now by hypothesis we have: given $x \in X, v_1 \in V$, since $v_1 = g_{31}(x) + g_{33}(v_1)$ we have that $\exists u \in U, \exists v_2 \in V$ such that

$$v_2 - g_{33}(v_2) = g_{32}(u) \qquad (\mathbf{1})$$

$$g_{21}(x) + g_{23}(v_1) + g_{22}(u) + g_{23}(v_2) = u \qquad (\mathbf{2}).$$

Now consider $v = v_1 + v_2$ we have by the equation above $(\mathbf{2})$ that: $g_{21}(x) + g_{23}(v_1 + v_2) + g_{22}(u) = u$ which proves $(\star)$. We also have that $v_1 + v_2 = g_{31}(x) + g_{33}(v_1) + g_{33}(v_2) + g_{32}(u)$ by adding equations $(\mathbf{1})$ and $(\star\star\star)$. Thus $v_1 + v_2 = g_{31}(x) + g_{33}(v_1 + v_2) + g_{32}(u)$ which proves $(\star\star)$.

Now we move to checking that the condition on kernels is also satisfied. It follows from the general hypothesis that: $\forall v \in V$ such that $g_{33}(v) = v$ then $g_{13}(v) + g_{23}(v) = 0$.

$(\Rightarrow)$ By hypothesis we know that the two equations

$$\forall u \in U, \forall v \in V u = g_{22}(u) + g_{23}(v) \qquad (\star)_1$$

$$v = g_{32}(u) + g_{33}(v) \qquad (\star)_2$$

imply $g_{12}(u) + g_{13}(v) = 0$. We want to prove that: $\forall u \in U$ if $\exists v_2 \in V : v_2 = g_{33}(v_2) + g_{32}(u)$ with $g_{22}(u) + g_{23}(v_2) = u$ then $g_{12}(u) + g_{13}(v_2) = 0 \qquad (\star\star)$.
So, given $u \in U, v_2 \in V : v_2 = g_{33}(v_2) + g_{32}(u)$ with $g_{22}(u) + g_{23}(v_2) = u$ then by hypothesis since $u = g_{22}(u) + g_{23}(v_2)$ and $v_2 = g_{32}(u) + g_{33}(v_2)$ so $(\star)_1$ and $(\star)_2$ are satisfied with $v = v_2$ which implies $g_{12}(u) + g_{13}(v_2) = 0$ $(\star\star)$.
$(\Leftarrow)$By a similar argument with $v = v_2$.

The values of the trace are conditioned by the implications above. According to these equations we have that: $\mathrm{Tr}^V_{X\oplus U,Y\oplus V}(g)(x,u) = \widehat{g_{11}}(x,u)+\widehat{g_{12}}(v)$ for some v such $(I-\widehat{g_{22}})(v) = \widehat{g_{21}}(x,u)$. If we apply $\mathrm{Tr}^U_{X,Y}$ to this function, it is equivalent in terms of the g to $\mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\oplus U,Y\oplus V}(g))(x) = g_{11}(x) + g_{13}(v_1+v_2) + g_{12}(u)$ with $u \in U$, $v_1 \in V$, $v_2 \in V$ such that: $u = g_{21}(x) + g_{23}(v_1+v_2) + g_{22}(u)$, $v_1 = g_{31}(x) + g_{33}(v_1)$ and $v_2 = g_{32}(u) + g_{33}(v_2)$.

On the other hand, we may also calculate $\mathrm{Tr}^{U\otimes V}_{X,Y}(g)(x) = \tilde{g}_{11}(x) + \tilde{g}_{12}(u,v)$ for some $u \in U$, $v \in V$ such that $(I - \tilde{g}_{22}(u,v)) = \tilde{g}_{21}(x)$ and we get by the equations above: $\mathrm{Tr}^{U\otimes V}_{X,Y}(g)(x) = g_{11}(x) + g_{13}(v) + g_{12}(u)$ with $u = g_{21}(x) + g_{23}(v) + g_{22}(u)$, $v = g_{31}(x) + g_{33}(v) + g_{32}(u)$. In both implications we obtain the same value of the trace. Notice that the value is independent of the choice of the vectors that satisfy the auxiliary conditions. When we chose $v = v_1 + v_2$ we have:
$\mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\oplus U,Y\oplus V}(g))(x) = g_{11}(x) + g_{13}(v_1+v_2) + g_{12}(u) = \mathrm{Tr}^{U\otimes V}_{X,Y}(g)(x)$
and when we chose $v_2 = v - v_1$ we have
$\mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^V_{X\oplus U,Y\oplus V}(g))(x) = g_{11}(x) + g_{13}(v_1+v_2) + g_{12}(u) = g_{11}(x) + g_{13}(v_1+v-v_1) + g_{12}(u) = g_{11}(x) + g_{13}(v) + g_{12}(u) = \mathrm{Tr}^{U\otimes V}_{X,Y}(g)(x)$.

Superposing:
Suppose now that $f \in \widehat{\mathbb{T}}^U_{X,Y}$ and $g : W \to Z$; we want to prove that $g\oplus f \in \widehat{\mathbb{T}}^U_{W\oplus X,Z\oplus Y}$. First, we start writing the matrix representation of $g \oplus f$ in terms of g.

- $(g\oplus f)_{11} = g \oplus f_{11} : W \oplus X \to Z \oplus Y$
- $(g\oplus f)_{21} = \begin{pmatrix} 0 & f_{21} \end{pmatrix} : W \oplus X \to U$
- $(g\oplus f)_{12} = \begin{pmatrix} 0 \\ f_{12} \end{pmatrix} : U \to Z \oplus Y$
- $(g\oplus f)_{22} = f_{22} : U \to U$.

If $z \in im\ (g\oplus f)_{21}$ then $z = 0w + f_{21}(x)$ for some $w \in W$, $x \in X$ which by hypothesis and the equation above implies that $f_{21}(x) \in im\ (I - f_{22}) = im\ (I - (g\oplus f)_{22})$.
On the other hand, we have $ker(I - (g\oplus f)_{22}) = ker(I - f_{22}) \subseteq ker f_{12} \subseteq$

$ker \begin{pmatrix} 0 \\ f_{12} \end{pmatrix} = ker(g \oplus f)_{12}$ by hypothesis and properties of kernels.
Now we evaluate the traces:

$$\mathrm{Tr}^U_{W\oplus X, Z\oplus Y}(g \oplus f)(w,x) = (g \oplus f)_{11}(w,x) + (g \oplus f)_{12}(u) =$$
$$g \oplus f_{11}(w,x) + \begin{pmatrix} 0 \\ f_{12} \end{pmatrix}(u) = (g(w), f_{11}(x)) + (0, f_{12}(u)) = (g(w), f_{11}(x) + f_{12}(u)) =$$
$$(g(w), \mathrm{Tr}^U_{X,Y}(f)(x) = (g \oplus \mathrm{Tr}^U_{X,Y}(f))(w,x)$$

with $u - f_{22}(u) = \begin{pmatrix} 0 & f_{21} \end{pmatrix}(w,x)$ which by the equations above is equivalent to $u - f_{22}(u) = f_{21}(x)$. Thus

$$\mathrm{Tr}^U_{W\oplus X, Z\oplus Y}(g \oplus f) = g \oplus \mathrm{Tr}^U_{X,Y}(f).$$

Yanking:
We want to prove that $\sigma_{U,U} \in \mathbb{T}^U_{U,U}$, and also $\mathrm{Tr}^U_{U,U}(\sigma_{U,U}) = 1_U$ where $\sigma_{U,U} : U \oplus U \to U \oplus U$ is the coherent isomorphism.

- $\sigma_{11} = \pi_1 \sigma_{UU}\, in_1 : U \to U$, with $\sigma_{11} = 0$
- $\sigma_{21} = \pi_2 \sigma_{UU}\, in_1 : U \to U$, with $\sigma_{21} = id_U$
- $\sigma_{12} = \pi_1 \sigma_{UU}\, in_2 : U \to U$, with $\sigma_{12} = id_U$
- $\sigma_{22} = \pi_2 \sigma_{UU}\, in_2 : U \to U$, with $\sigma_{22} = 0$.

Thus, we have $\sigma_{21}(u) = u = (I - \sigma_{22})(u)$ which means that $im\ \sigma_{21} \subseteq im\ (I - \sigma_{22})$. On the other hand we have that if $u = \sigma_{22}(u)$ then $u = 0$. This means that $ker(I - \sigma_{22}) \subseteq ker\ \sigma_{12}$.
The value of the trace is the following:

$$\mathrm{Tr}^U_{U,U}(\sigma_{UU})(u) = \sigma_{11}(u) + \sigma_{12}(v) = 0 + v = v$$

with the condition: $(I - \sigma_{22})(v) = \sigma_{21}(u)$ for some $v \in U$. But this implies by the equations above that $v = u$. Thus $\mathrm{Tr}^U_{U,U}(\sigma_{UU})(u) = u$, i.e., $\mathrm{Tr}^U_{U,U}(\sigma_{UU}) = id_U$.

□

4.2.5 Completely positive maps with $\oplus$

Definition 4.2.12. On the category **CPM** with monoidal structure $\oplus$, we define a partial trace as follows. We say that $f \in \widetilde{\mathbb{T}}^U_{X,Y}$ for some objects X, Y, U iff

(a) $(I - f_{22})$ is invertible as linear function and

(b) the inverse map $(I - f_{22})^{-1}$ is a completely positive map.

We define $\widetilde{\mathrm{Tr}}^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21}$ where I is the identity map.

Thus, we are demanding that $(I - f_{22})^{-1}$ should be regarded as an inverse in the category **CPM**.

Lemma 4.2.13. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a partitioned matrix with sub-block* $A \in Mat_{m\times m}$, $B \in Mat_{m\times n}$, $C \in Mat_{n\times m}$ *and* $D \in Mat_{n\times n}$. *Assume* D *is invertible. Then* M *is invertible if and only if* $A - BD^{-1}C$ *is invertible.*

Lemma 4.2.14. *Let us consider* $A \in Mat_{m\times n}$ *and* $B \in Mat_{n\times m}$. *Then* $(I_m - AB)$ *is invertible if and only if* $(I_n - BA)$ *is invertible and* $(I_m - AB)^{-1}A = A(I_n - BA)^{-1}$.

Proposition 4.2.15. $(\mathbf{CPM}, \oplus, 0)$ *is a partially traced category with respect to Definition 4.2.12.*

Proof. The partial trace axioms, restricted to condition (a) of Definition 4.2.12, are basically proved in [34]. This picture is completed by adding the proof of the trace axioms for the positiveness condition (b) of Definition 4.2.12.

Vanishing I:

This follows from the definition of the unit I as the empty list and the fact that the identity map is an invertible map where its inverse is a completely positive map. Thus $\widetilde{\mathbb{T}}^I_{X,Y} = \mathbf{CPM}(X, Y)$ and $\widetilde{\mathrm{Tr}}^I_{X,Y}(f) = f$ for every $f \in \widetilde{\mathbb{T}}^I_{X,Y}$.

Superposing:

Let us consider $f \in \widetilde{\mathbb{T}}^U_{X,Y}$ and $g : W \to Z$ then $g \oplus f \in \widetilde{\mathbb{T}}^U_{W\oplus X, Z\oplus Y}$ since $(g \oplus f)_{22} = f_{22}$.

We also have: $\widetilde{\mathrm{Tr}}^U_{W\oplus X, Z\oplus Y}(g \oplus f) = \begin{bmatrix} g & 0 \\ 0 & f_{11} + f_{12}(I - f_{22})^{-1} f_{21} \end{bmatrix} = g \oplus \widetilde{\mathrm{Tr}}^U_{X,Y}(f)$.

Naturality:

If $f \in \widetilde{\mathbb{T}}^U_{X,Y}$ and we have two arrows $g : X' \to X$, $h : Y \to Y'$ then since

$$((h \oplus id_U) f (g \oplus id_U))_{22} = f_{22}$$

always is satisfied for linear maps since composition computes as matrix product i.e.,

$$\begin{bmatrix} h & 0 \\ 0 & 1_u \end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \cdot \begin{bmatrix} g & 0 \\ 0 & 1_u \end{bmatrix} = \begin{bmatrix} hf_{11}g & hf_{12} \\ f_{21}g & f_{22} \end{bmatrix}$$

Thus then the conditions remain exactly the same, meaning that

$$((h \oplus id_u) f (g \oplus id_u))_{22} = f_{22} \in \widetilde{\mathbb{T}}^U_{X',Y'}.$$

Moreover

$$\widetilde{\mathrm{Tr}}^U_{X',Y'}((h \oplus 1_u) f (g \oplus 1_u) = hf_{11}g + (hf_{12})(f_{22})(f_{21}g) = h(f_{11} + f_{12}f_{22}f_{21})g = h\widetilde{\mathrm{Tr}}^U_{X,Y}(f)g.$$

Yanking :

Note that $s_{U,U} \in \widetilde{\mathbb{T}}^U_{U,U}$ since $(s_{U,U})_{2,2} = 0$ which implies that $I - (s_{U,U})_{2,2}$ is invertible and $(I - 0)^{-1}$ is a completely positive map. Moreover $\widetilde{\mathrm{Tr}}^U_{U,U}(\sigma_{U,U}) = 1_u$ since $(s_{U,U})_{1,1} = (s_{U,U})_{2,2} = 0$ and $(s_{U,U})_{1,2} = (s_{U,U})_{2,1} = 1_u$.

Vanishing II :

Let us consider $g : X \oplus U \oplus V \to Y \oplus U \oplus V$, we write using matrix notation $g = \begin{bmatrix} a & b & c \\ d & e & f \\ m & n & p \end{bmatrix}$.

Now, assuming by hypothesis that $g \in \widetilde{\mathbb{T}}^V_{X\oplus U, Y\oplus U}$, i.e., $I - p$ is invertible and $(I-p)^{-1}$ is a completely positive map we must show that $g \in \widetilde{\mathbb{T}}^{U\oplus V}_{X,Y}$ iff $\widetilde{\mathrm{Tr}}^V_{X\oplus U, Y\oplus U}(g) \in \widetilde{\mathbb{T}}^U_{X,Y}$.

First, we analyze the conditions of definition 4.2.12 in terms of its matrix term components. If we represent functions using matrix notation we have:

$$\widetilde{\mathrm{Tr}}^V_{X\oplus U,Y\oplus U}(g)\begin{pmatrix} x \\ u \end{pmatrix} = \left[\begin{pmatrix} a & b \\ d & e \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix} \circ (1-p)^{-1} \circ \begin{pmatrix} m & n \end{pmatrix}\right]\begin{pmatrix} x \\ u \end{pmatrix} \text{ and}$$

we obtain

$$(\widetilde{\mathrm{Tr}}^V_{X\oplus U,Y\oplus U}(g))_{22}(u) = (e + f(1-p)^{-1}n)(u)$$

by composing with the second injection and the second projection.
Thus we know by definition:
$\widetilde{\mathrm{Tr}}^V_{X\oplus U,Y\oplus U}(g) \in \widetilde{\mathbb{T}}^U_{X,Y}$ i.e., $I - e - f(1-p)^{-1}n$ is invertible and $(I - e - f(1-p)^{-1}n)^{-1}$ is a complete positive map.
On the other hand, $g \in \widetilde{\mathbb{T}}^{U\oplus V}_{X,Y}$ i.e., $I - \begin{pmatrix} e & f \\ n & p \end{pmatrix}$ is invertible and $(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1}$ is a complete positive map. Also we obtain the explicit inverse by

$$(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1} = \begin{bmatrix} (I-e-fqn)^{-1} & (I-e-fqn)^{-1}(f)q \\ qn(I-e-fqn)^{-1} & q + qn(I-e-fqn)^{-1}fq \end{bmatrix}$$

where $q = (I-p)^{-1}$. Now we prove the equivalence on the trace class:

First of all, by Lemma 4.2.13 above we get: if we know that $(I-p)$ is invertible then $(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})$ is invertible iff $I - e - f(1-p)^{-1}n$ is invertible, which means that the first part of the definition is satisfied. Also from Lemma 4.2.13 we have that the equation on traces $\widetilde{\mathrm{Tr}}^{U\oplus V}_{X,Y}(g) = \widetilde{\mathrm{Tr}}^U_{X,Y}(\widetilde{\mathrm{Tr}}^V_{X\oplus U,Y\oplus U}(g))$ is satisfied by using matrix multiplication and the explicit inverse $(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1}$ written above.

Positiveness condition (b):
($\Leftarrow$) Injections and projections are completely positive maps and by the fact that g is a completely positive map this implies by definition that n and f are complete positive maps. Also, $(I-e-f(1-p)^{-1}n)^{-1} = (I-(\widetilde{\mathrm{Tr}}^V_{X\oplus U,Y\oplus U}(g))_{22})^{-1}$ is a completely positive map by conditional hypothesis and $(I-p)^{-1}$ is also a completely positive map by the general hypothesis. This implies that $(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1}$ is a completely positive since each component of the matrix is obtained by sum and composition of completely positive maps.

($\Rightarrow$) If $(I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1}$ is a completely positive map then $\pi_1 \circ (I - \begin{pmatrix} e & f \\ n & p \end{pmatrix})^{-1} \circ i_1 = (I - e - f(I-p)^{-1}n)^{-1}$ is a completely positive map where π_1 and i_1 are the first projection and first injection. Therefore, we showed that $(I - e - f(I-p)^{-1}n)^{-1} = (I - (e + f(1-p)^{-1}n))^{-1} = (I - (\widetilde{\mathrm{Tr}}^v_{X\oplus U, Y\oplus U}(g))_{22})^{-1}$ is a completely positive map.

Dinaturality:

Now, suppose $f : X \oplus U \to Y \oplus U'$ and $g : U' \to U$ are completely positive maps then we want to prove that $(id_y \oplus g) \circ f \in \widetilde{\mathbb{T}}^U_{X,Y}$ iff $f \circ (id_x \oplus g) \in \widetilde{\mathbb{T}}^{U'}_{X,Y}$ which means that $((id_y \oplus g) \circ f)_{22} = g \circ f_{22}$ satisfies conditions (a) and (b) of Definition 4.2.12 if and only if $(f \circ (id_x \oplus g))_{22} = f_{22} \circ g$ does.

Given $f : X \oplus U \to Y \oplus U'$ and $g : U' \to U$ by Lemma 4.2.14 above, $I - g \circ f_{22}$ is invertible if and only if $I - f_{22} \circ g$ is invertible and we have that

$$\widetilde{\mathrm{Tr}}^U_{X,Y}((1_y \oplus g)f) = f_{11} + f_{12}(I - gf_{22})^{-1}gf_{21} = f_{11} + f_{12}g(I - f_{22}g)^{-1}f_{21} = \widetilde{\mathrm{Tr}}^{U'}_{X,Y}(f(1_x \oplus g)).$$

Therefore, it suffices to prove the following: if $I_u - g \circ f$ is invertible and $(I_u - g \circ f)^{-1}$ is a completely positive map then $(I_{u'} - f \circ g)^{-1}$ is a completely positive map, where $f : U \to U'$ and $g : U' \to U$.

We know by hypothesis that

$$\forall \tau, \forall A' \in V_\tau \otimes V_u, \text{ if } A' \geq 0 \text{ then } (Id_\tau \otimes (I_u - g \circ f)^{-1})(A') \geq 0$$

and we want to prove that

$$\forall \tau, \forall A \in V_\tau \otimes V_{u'}, \text{ if } A \geq 0 \text{ then } (Id_\tau \otimes (I_{u'} - f \circ g))^{-1})(A) \geq 0.$$

Suppose we name $(Id_\tau \otimes (I_{u'} - f \circ g)^{-1})(A) = B$ then by hypothesis

$$A = (Id_\tau \otimes (I_{u'} - f \circ g))(B) \geq 0. \tag{4}$$

Since g is a completely positive map this implies that: if $A \geq 0$ then $(Id_\tau \otimes g)(A) \geq 0$; next we apply this property to equation (4).

So, we get:
$0 \leq (Id_\tau \otimes g)\circ(Id_\tau \otimes (I_{u'} - f\circ g))(B) = (Id_\tau \otimes g(I_{u'} - f\circ g))(B) =$
$(Id_\tau \otimes (g - g\circ f\circ g))(B) = (Id_\tau \otimes (I_u - g\circ f)g)(B) = (Id_\tau \otimes (I_u - g\circ f))\circ(Id_\tau \otimes g)(B)$
which implies (rename it C)

$$(Id_\tau \otimes (I_u - g\circ f))((Id_\tau \otimes g)(B)) = C \geq 0.$$

Thus we have

$$(Id_\tau \otimes g)(B) = (Id_\tau \otimes (I_u - g\circ f))^{-1}(C) = (Id_\tau \otimes (I_u - g\circ f)^{-1})(C) \geq 0$$

since $(I_u - g\circ f)^{-1}$ is a completely positive map by hypothesis. Therefore, $(Id_\tau \otimes g)(B) \geq 0$ and on the other hand f is a completely positive map which implies $(Id_\tau \otimes f)((Id_\tau \otimes g)(B)) \geq 0$, which means $(Id_\tau \otimes f\circ g)(B) \geq 0$.

Finally, since if $A \geq 0$ implies $(Id_\tau \otimes I_{u'})(B) - (Id_\tau \otimes f\circ g)(B) \geq 0$ by equation (4) this implies that $B \geq (Id_\tau \otimes f\circ g)(B)$ hence $B = (Id_\tau \otimes (I_{u'} - f\circ g)^{-1})(A) \geq 0$ by transitivity for every τ. For the converse implication we repeat this argument interchanging f and g. □

4.3 Partial trace in a monoidal subcategory of a partially traced category

The aim of this section is to provide a general construction of partially traced categories as subcategories of other partially (or totally) traced categories.

Suppose $(\mathcal{D}, \otimes, I, s, \mathrm{Tr})$ is a partially traced category with trace

$$\mathrm{Tr}^U_{X,Y} : \mathcal{D}(X \otimes U, Y \otimes U) \rightharpoonup \mathcal{D}(X, Y).$$

Given a monoidal subcategory $\mathcal{C} \subseteq \mathcal{D}$, we get a partial trace on $\mathcal{C}$, defined by $\widehat{\mathrm{Tr}}^U_{X,Y}(f) = \mathrm{Tr}^U_{X,Y}(f)$ if $\mathrm{Tr}^U_{X,Y}(f) \downarrow$ and $\mathrm{Tr}^U_{X,Y}(f) \in \mathcal{C}(X, Y)$, and undefined otherwise.

More generally, we shall show a method of constructing one partially traced category from another in such a way that the first one is faithfully embedded in the second.

Proposition 4.3.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful strong symmetric monoidal functor with $(\mathcal{D}, \otimes, I, s, \mathrm{Tr})$ a partially traced category and $(\mathcal{C}, \otimes, I, s)$ a symmetric monoidal category. Then we obtain a partial trace $\widehat{\mathrm{Tr}}$ on $\mathcal{C}$ as follows. For $f : X \otimes U \to Y \otimes U$, we define $\widehat{\mathrm{Tr}}^U_{X,Y}(f) = g$ if there exists some (necessarily unique) $g : X \to Y$ such that $F(g) = \mathrm{Tr}^{FU}_{FX,XY}(m^{-1}_{Y,U} \circ F(f) \circ m_{X,U})$ is defined, and $\widehat{\mathrm{Tr}}^U_{X,Y}(f)$ undefined otherwise.*

Proof. To clarify the notation used here we recall that there are two partial functions:

$$\widehat{\mathrm{Tr}}^U_{X,Y} : \mathcal{C}(X \otimes U, Y \otimes U) \rightharpoonup \mathcal{C}(X, Y)$$

and

$$\mathrm{Tr}^U_{X,Y} : \mathcal{D}(X \otimes U, Y \otimes U) \rightharpoonup \mathcal{D}(X, Y).$$

Then we have two maps

$$\widehat{\mathrm{Tr}}^U_{X,Y} : \widehat{\mathbb{T}}^U_{X,Y} \to \mathcal{C}(X, Y)$$

where $\widehat{\mathbb{T}}^U_{X,Y} \subseteq \mathcal{C}(X \otimes U, Y \otimes U)$ and we also have

$$\mathrm{Tr}^U_{X,Y} : \mathbb{T}^U_{X,Y} \to \mathcal{D}(X, Y)$$

where $\mathbb{T}^U_{X,Y} \subseteq \mathcal{D}(X \otimes U, Y \otimes U)$.

Naturality :

For any X, Y, U objects in $\mathcal{C}$, $f \in \widehat{\mathbb{T}}^U_{X,Y}$ and $g : X' \to X$, $h : Y \to Y'$ arrows in $\mathcal{C}$. We want to prove that the two conditions given above hold:
(1) we must prove that

$$m^{-1}_{Y'U} \circ F(h \otimes 1_U)F(f)F(g \otimes 1_U) \circ m_{X',U} \in \mathbb{T}^{FU}_{FX',FY'}$$

By naturality of the map m^{-1} with h, g and identities we have:

$$m^{-1}_{Y'U} \circ F(h \otimes 1_U) = (F(h) \otimes 1_{FU}) \circ m^{-1}_{YU} \text{ and also } F(g \otimes 1_U) \circ m_{X',U} = m_{X,U} \circ (F(g) \otimes 1_{FU}). \tag{5}$$

Consequently, we need to prove that

$$(F(h) \otimes 1_{FU}) \circ m_{YU}^{-1} \circ F(f) \circ m_{X,U} \circ (F(g) \otimes 1_{FU}) \in \mathbb{T}_{FX',FY'}^{FU}.$$

Notice that by hypothesis

$$m_{YU}^{-1} \circ F(f) \circ m_{X,U} \in \mathbb{T}_{FX,FY}^{FU}.$$

Then by the naturality axiom in the category $\mathcal{D}$ we have that

$$(F(h) \otimes 1_{FU}) \circ m_{YU}^{-1} \circ F(f) \circ m_{X,U} \circ (F(g) \otimes 1_{FU}) \in \mathbb{T}_{FX',FY'}^{FU}$$

and also

$\mathrm{Tr}_{FX',FY'}^{FU}((F(h) \otimes 1_{FU}) \circ m_{YU}^{-1} \circ F(f) \circ m_{X,U} \circ (F(g) \otimes 1_{FU})) =$
$= F(h) \circ \mathrm{Tr}_{FX,FY}^{FU}(m_{YU}^{-1} \circ F(f) \circ m_{X,U}) \circ F(g).$

(2) Since by hypothesis there exists an arrow $p_1 : X \to Y$ such that

$$F(p_1) = \mathrm{Tr}_{FX,FY}^{FU}(m_{YU}^{-1} \circ F(f) \circ m_{X,U})$$

then

$\mathrm{Tr}_{FX',FY'}^{FU}(m_{Y'U}^{-1} \circ F((h \otimes 1_U) \circ f \circ (g \otimes 1_U)) \circ m_{X',U}) =$ (equation (5) above)
$= \mathrm{Tr}_{FX',FY'}^{FU}((F(h) \otimes 1_{FU}) \circ m_{YU}^{-1} \circ F(f) \circ m_{X,U} \circ (F(g) \otimes 1_{FU})) =$ (naturality axiom in $\mathcal{D}$ and hyp.)
$= F(h) \circ F(p_1) \circ F(g) = F(h \circ p_1 \circ g).$

This means that we can choose $p_2 = h \circ p_1 \circ g$.

Now we are able to compute the trace:

$$\widehat{\mathrm{Tr}}_{X',Y'}^{U}((h \otimes 1_U) f (g \otimes 1_U)) = p_2 = h \circ p_1 \circ g = h \circ \widehat{\mathrm{Tr}}_{X,Y}^{U}(f) \circ g.$$

Dinaturality:

For any $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$ where f and g are in $\mathcal{C}$ we must prove that

$$(1_Y \otimes g) f \in \widehat{\mathbb{T}}_{X,Y}^{U} \text{ iff } f(1_X \otimes g) \in \widehat{\mathbb{T}}_{X,Y}^{U'}.$$

We must check condition (1) and (2).

(1) By definition we have

$$(1_Y \otimes g) f \in \widehat{\mathbb{T}}_{X,Y}^{U} \text{ implies } m_{Y,U}^{-1} \circ F((1_Y \otimes g) f) \circ m_{X,U} \in \mathbb{T}_{FX,FY}^{FU}. \tag{6}$$

But in view of the naturality of m it follows that $m^{-1}_{Y,U} \circ F(1_Y \otimes g) = (F(1) \otimes F(g)) \circ m^{-1}_{Y,U'}$. Then we can replace it in (6) obtaining:

$$m^{-1}_{Y,U} \circ F((1_Y \otimes g)f) \circ m_{X,U} = m^{-1}_{Y,U} \circ F(1_Y \otimes g) \circ Ff \circ m_{X,U} =$$

$$(1_{FY} \otimes F(g)) \circ m^{-1}_{Y,U'} \circ Ff \circ m_{X,U} \in \mathbb{T}^{FU}_{FX,FY}.$$

It now follows by the dinaturality axiom of the category $\mathcal{D}$ that this condition is equivalent to proving:

$$m^{-1}_{Y,U'} \circ F(f) \circ m_{X,U} \circ (1_{FX} \otimes F(g)) \in \mathbb{T}^{FU'}_{FX,FY}$$

and again by naturality of m we have that $m_{X,U} \circ (1_{FX} \otimes F(g)) = (F(1 \otimes g)) \circ m_{X,U'}$ and we replace it:

$$m^{-1}_{Y,U'} \circ F(f) \circ F((1 \otimes g) \circ m_{X,U'} = m^{-1}_{Y,U'} \circ F(f(1 \otimes g)) \circ m_{X,U'} \in \mathbb{T}^{FU'}_{FX,FY}$$

which is condition (1) in the definition $f \circ (1_X \otimes g) \in \widehat{\mathbb{T}}^{U'}_{X,Y}$. In the same way we prove the converse.

(2) Also there is an arrow p_1 such that $F(p_1) = \mathrm{Tr}^{FU'}_{FX,FY}(m^{-1}_{Y,U'} \circ F((1_Y \otimes g)f) \circ m_{X,U})$ if and only if there is an arrow $F(p_2) = \mathrm{Tr}^{FU'}_{FX,FY}(m^{-1}_{Y,U'} \circ F(f(1_X \otimes g)) \circ m_{X,U'})$. Since the value of the trace remains invariant under the dinaturality axiom and all the transformations made in part (1) then it is enough to take $p_1 = p_2$.

Vanishing I:

Now we want to check that: $\widehat{\mathbb{T}}^I_{X,Y} = \mathcal{C}(X \otimes I, Y \otimes I)$. Given any $f : X \otimes I \to Y \otimes I$ we want to prove that $f \in \widehat{\mathbb{T}}^I_{X,Y}$ by verifying conditions (1) and (2).

(1) Let us consider $g = (1_{FY} \otimes m_I^{-1}) \circ m^{-1}_{Y,I} \circ F(f) \circ m_{X,I} \circ (1_{FX} \otimes m_I)$. By the vanishing I axiom in the category $\mathcal{D}$ we know that $g \in \widehat{\mathbb{T}}^I_{FX,FY}$. Then, since $(1_{FY} \otimes m_I^{-1}) \circ (1_{FY} \otimes m_I) \circ g = g \in \widehat{\mathbb{T}}^I_{FX,FY}$ we can apply the dinaturality axiom in $\mathcal{D}$ to conclude that $(1_{FY} \otimes m_I) \circ g \circ (1_{FX} \otimes m_I^{-1}) \in \widehat{\mathbb{T}}^{FI}_{FX,FY}$ but we have that $(1_{FY} \otimes m_I) \circ g \circ (1_{FX} \otimes m_I^{-1}) = m^{-1}_{Y,I} \circ F(f) \circ m_{X,I}$. So we proved that $m^{-1}_{Y,I} \circ F(f) \circ m_{X,I} \in \widehat{\mathbb{T}}^{FI}_{FX,FY}$.

(2) Since $g \in \mathcal{D}(FX \otimes I, FY \otimes I)$ we can say also, by the dinaturality axiom, that

$$\mathrm{Tr}^I_{FX,FY}(g) = \mathrm{Tr}^{FI}_{FX,FY}(m^{-1}_{Y,I} \circ F(f) \circ m_{X,I})$$

but on the other hand we know that

$$\mathrm{Tr}^{I}_{FX,FY}(g) = \rho_{FY} \circ g \circ \rho^{-1}_{FX}$$

by vanishing I in $\mathcal{D}$ which implies that

$\mathrm{Tr}^{I}_{FX,FY}(g) = \rho_{FY} \circ (1_{FY} \otimes m_I^{-1}) \circ m^{-1}_{Y,I} \circ F(f) \circ m_{X,I} \circ (1_{FX} \otimes m_I) \circ \rho^{-1}_{FX} =$ (since F is monoidal) $F(\rho_Y) \circ F(f) \circ F(\rho_X^{-1}) = F(\rho_Y \circ f \circ \rho_X^{-1})$.

Thus there exists a $p = \rho_Y \circ f \circ \rho_X^{-1}$ such that $F(p) = \mathrm{Tr}^{FI}_{FX,FY}(m^{-1}_{Y,I} \circ F(f) \circ m_{X,I})$. Also notice that we prove that $\widehat{\mathrm{Tr}}^{I}_{X,Y}(f) = p = \rho_Y \circ f \circ \rho_X^{-1}$, which is the equation of the trace value in the category $\mathcal{C}$.

Vanishing II:

Let $g : X \otimes U \otimes V \to Y \otimes U \otimes V$ be an arrow in the category $\mathcal{C}$. By hypothesis, we are given $g \in \widehat{\mathbb{T}}^{V}_{X\otimes U, Y\otimes U}$ (general hypothesis) and we want to prove the following equivalence:

$$g \in \widehat{\mathbb{T}}^{U\otimes V}_{X,Y} \text{ iff } \widehat{\mathrm{Tr}}^{V}_{X\otimes U, Y\otimes U}(g) \in \widehat{\mathbb{T}}^{U}_{X,Y}.$$

According to the general hypothesis there is a map:

$$F(X\otimes U)\otimes FV \xrightarrow{m_{X\otimes U,V}} F(X\otimes U\otimes V) \xrightarrow{Fg} F(Y\otimes U\otimes V) \xrightarrow{m^{-1}_{Y\otimes U,V}} F(Y\otimes U)\otimes FV \in \widehat{\mathbb{T}}^{FV}_{F(X\otimes U),F(Y\otimes U)}$$

and also there exists $p_1 : X \otimes U \to Y \otimes U$ such that

$F(p_1) = \mathrm{Tr}^{FV}_{F(X\otimes U),F(Y\otimes U)}(m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V})$ i.e., by definition $p_1 = \widehat{\mathrm{Tr}}^{V}_{X\otimes U, Y\otimes U}(g)$.

($\Rightarrow$) We have a conditional hypothesis $g \in \widehat{\mathbb{T}}^{U\otimes V}_{X,Y}$ which asserts that the map:

$$FX\otimes F(U\otimes V) \xrightarrow{m_{X,U\otimes V}} F(X\otimes U\otimes V) \xrightarrow{Fg} F(Y\otimes U\otimes V) \xrightarrow{m^{-1}_{Y,U\otimes V}} FY\otimes F(U\otimes V) \in \widehat{\mathbb{T}}^{F(U\otimes V)}_{FX,FY}$$

and also that there exists an $p_2 : X \to Y$ such that

$F(p_2) = \mathrm{Tr}^{F(U\otimes V)}_{FX,FY}(m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V})$ i.e., by definition $p_2 = \widehat{\mathrm{Tr}}^{U\otimes V}_{X,Y}(g)$.

Recalling that $p_1 = \widehat{\mathrm{Tr}}^{V}_{X\otimes U, Y\otimes U}(g)$, we want to prove that $p_1 \in \widehat{\mathbb{T}}^{U}_{X,Y}$. For that purpose, we shall prove the two conditions that characterize the trace class definition which are the following:

(1) the map

$$FX \otimes FU \xrightarrow{m_{X,U}} F(X \otimes U) \xrightarrow{F(p_1)} F(Y \otimes U) \xrightarrow{m^{-1}_{Y,U}} FY \otimes FU) \in \mathbb{T}^{FU}_{FX,FY}$$

(2) there exists an $p_3 : X \to Y$ such that

$$F(p_3) = \mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ F(p_1) \circ m_{X,U}) \text{ i.e., by definition}$$

$$p_3 = \widehat{\mathrm{Tr}}^{U}_{X,Y}(p_1) = \widehat{\mathrm{Tr}}^{U}_{X,Y}(\widehat{\mathrm{Tr}}^{V}_{X\otimes U,Y\otimes U}(g)).$$

(1) To prove condition (1) we notice that since by definition

$$F(p_1) = \mathrm{Tr}^{FV}_{F(X\otimes U),F(Y\otimes U)}(m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V})$$

then we must prove that

$$FX\otimes FU \xrightarrow{m_{X,U}} F(X\otimes U) \xrightarrow{\mathrm{Tr}^{FV}_{F(X\otimes U),F(Y\otimes U)}(m^{-1}\circ F(g)\circ m)} F(Y\otimes U) \xrightarrow{m^{-1}_{Y,U}} FY\otimes FU) \in \widehat{\mathbb{T}}^{FU}_{FX,FY}.$$

But since

$$m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V} \in \mathbb{T}^{FV}_{F(X\otimes U),F(Y\otimes U)}$$

this condition allows us to apply the naturality axiom in the category $\mathcal{D}$:

$$FX\otimes FU\otimes FV \xrightarrow{m_{X,U}\otimes 1_{FV}} F(X\otimes U)\otimes FV \xrightarrow{m^{-1}_{Y\otimes U,V}\circ F(g)\circ m_{X\otimes U,V}} F(Y\otimes U)FV \xrightarrow{m^{-1}_{Y,U}\otimes FV} FY\otimes FU\otimes FV$$
$$\in \widehat{\mathbb{T}}^{FV}_{FX\otimes FU,FY\otimes FU}.$$

And also by the same axiom we have that:

$$\mathrm{Tr}^{FV}_{FX\otimes FU,FY\otimes FU}((m^{-1}_{Y,U} \otimes 1_{FV}) \circ m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V} \circ (m_{X,U} \otimes 1_{FV})) =$$

$$= m^{-1} \circ \mathrm{Tr}^{FV}_{F(X\otimes U),F(Y\otimes U)}(m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V}) \circ m.$$

Hence, this is equivalent to proving that:

$$\mathrm{Tr}^{FV}_{FX\otimes FU,FY\otimes FU}((m^{-1}_{Y,U} \otimes 1_{FV}) \circ m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V} \circ (m_{X,U} \otimes 1_{FV})) \in \mathrm{Tr}^{FU}_{FX,FY}.$$

Consequently, by vanishing II in the category $\mathcal{D}$, it would be enough that the map

$$\lambda = (m^{-1}_{Y,U} \otimes 1_{FV}) \circ m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V} \circ (m_{X,U} \otimes 1_{FV}) \in \mathbb{T}^{FU\otimes FV}_{FX,FY}$$

since we know that $\lambda \in \mathbb{T}^{FV}_{FX\otimes FU, FY\otimes FU}$. But by coherence of monoidal functors we have:

$$(m^{-1}_{Y,U} \otimes 1_{FV}) \circ m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V} \circ (m_{X,U} \otimes 1_{FV}) =$$

$$= (1_{FX} \otimes m^{-1}_{U,V}) \circ m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V} \circ (1_{FX} \otimes m_{U,V}).$$

Therefore, by the dinaturality axiom in the category $\mathcal{D}$:

$$(1_{FX} \otimes m^{-1}_{U,V}) \circ m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V} \circ (1_{FX} \otimes m_{U,V}) \in \mathbb{T}^{FU\otimes FV}_{FX,FY}$$

if and only if

$$(m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V}) \circ (1_{FX} \otimes m_{U,V}) \circ (1_{FX} \otimes m^{-1}_{U,V}) \in \mathbb{T}^{F(U\otimes V)}_{FX,FY}$$

which is valid since this is the conditional hypothesis.

(2) We shall prove that there exists an arrow $p_3 : X \to Y$ in $\mathcal{C}$ such that

$$F(p_3) = \mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ F(p_1) \circ m_{X,U}).$$

For that purpose, take $p_3 = p_2$. Hence by the conditional hypothesis if $g \in \widehat{\mathbb{T}}^{U\otimes V}_{X,Y}$ holds then there is p_2 with

$$F(p_2) = \mathrm{Tr}^{F(U\otimes V)}_{FX,FY}(m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V}).$$

Therefore, this is equal to,

$$\mathrm{Tr}^{F(U\otimes V)}_{FX,FY}((m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V}) \circ (1_{FX} \otimes m_{U,V}) \circ (1_{FX} \otimes m^{-1}_{U,V})) =$$

$$\mathrm{Tr}^{FU\otimes FV}_{FX,FY}((1_{FX} \otimes m^{-1}_{U,V}) \circ m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V} \circ (1_{FX} \otimes m_{U,V})) = \text{(dinaturality)}$$

$$\mathrm{Tr}^{FU}_{FX,FY}(\mathrm{Tr}^{FV}_{FX\otimes FU,FY\otimes FU}((1_{FX} \otimes m^{-1}_{U,V}) \circ m^{-1}_{Y,U\otimes V} \circ F(g) \circ m_{X,U\otimes V} \circ (1_{FX} \otimes m_{U,V}))) =$$

(vanishing II)

$\mathrm{Tr}^{FU}_{FX,FY}(\mathrm{Tr}^{FV}_{FX\otimes FU,FY\otimes FU}((m^{-1}_{Y,U}\otimes 1_{FV})\circ m^{-1}_{Y\otimes U,V}\circ F(g)\circ m_{X\otimes U,V}\circ(m_{X,U}\otimes 1_{FV}))) =$
(coherence)

$\mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ \mathrm{Tr}^{FV}_{F(X\otimes U),F(Y\otimes U)}(m^{-1}_{Y\otimes U,V} \circ F(g) \circ m_{X\otimes U,V}) \circ m_{X,U}) =$
(naturality axiom)

$\mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ F(p_1) \circ m_{X,U}) =$ (definition of general hypothesis).

So we have proved that:

$$F(p_2) = \mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ F(p_1) \circ m_{X,U})$$

which means that

$$\widehat{\mathrm{Tr}}^{U}_{X,Y}(\widehat{\mathrm{Tr}}^{V}_{X\otimes U,Y\otimes U}(g)) = \widehat{\mathrm{Tr}}^{U}_{X,Y}(p_1) = p_3 = p_2 = \widehat{\mathrm{Tr}}^{U\otimes V}_{X,Y}(g).$$

($\Leftarrow$) Similarly, we prove the converse. The proof is just a matter of using the converse hypothesis of vanishing II in the category $\mathcal{D}$.

Superposing:

Suppose $f \in \widehat{\mathbb{T}}^{U}_{X,Y}$ and $g : W \to Z$ with $g \in \mathcal{C}$ we want to prove that $g\otimes f \in \widehat{\mathbb{T}}^{U}_{W\otimes X,Z\otimes Y}$ by checking conditions (1) and (2). Also we want to show that

$$\widehat{\mathrm{Tr}}^{U}_{W\otimes X,Z\otimes Y}(g \otimes f) = g \otimes \widehat{\mathrm{Tr}}^{U}_{X,Y}(f).$$

(1) By hypothesis we know that

$$FX \otimes FU \xrightarrow{m_{X,U}} F(X \otimes U) \xrightarrow{F(f)} F(Y \otimes U) \xrightarrow{m^{-1}_{Y,U}} FY \otimes FU) \in \widehat{\mathbb{T}}^{FU}_{FX,FY}$$

and also there exists an arrow $p_1 : X \to Y$ such that

$$F(p_1) = \mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U} \circ F(f) \circ m_{X,U}).$$

Then by the superposing axiom in the category $\mathcal{D}$ it follows that

$$F(g) \otimes (m^{-1}_{Y,U} \circ F(f) \circ m_{X,U}) \in \mathbb{T}^{FU}_{FW\otimes FX,FW\otimes FY}$$

and the trace value turns out to be

$$\mathrm{Tr}^{FU}_{FW\otimes FX,FZ\otimes FY}(F(g)\otimes(m^{-1}_{Y,U}\circ F(f)\circ m_{X,U})) = F(g)\otimes\mathrm{Tr}^{FU}_{FX,FY}(m^{-1}_{Y,U}\circ F(f)\circ m_{X,U}).$$

But by functoriality of the tensor we obtain

$$F(g)\otimes(m^{-1}_{Y,U}\circ F(f)\circ m_{X,U}) = (1_{FZ}\otimes m^{-1}_{Y,U})\circ(F(g)\otimes F(f))\circ(1_{FW}\otimes m_{X,U}) = \beta$$

(To simplify notation, we name this equation β).
We can apply the naturality axiom in the category $\mathcal{D}$ and we obtain:

$$(m_{Z,Y}\otimes 1_{FU})\circ\beta\circ(m^{-1}_{W,X}\otimes 1_{FU}) \in \mathbb{T}^{FU}_{F(W\otimes X),F(Z\otimes Y)}$$

and

$$\mathrm{Tr}^{FU}_{F(W\otimes X),F(Z\otimes Y)}((m_{Z,Y}\otimes 1_{FU})\circ\beta\circ(m^{-1}_{W,X}\otimes 1_{FU})) = m_{Z,Y}\circ\mathrm{Tr}^{FU}_{FW\otimes FX,FZ\otimes FY}(\beta)\circ m^{-1}_{W,X}$$

but by naturality and monoidal functor axioms we have that

$$(m_{Z,Y}\otimes 1_{FU})\circ\beta\circ(m^{-1}_{W,X}\otimes 1_{FU}) = m_{Z\otimes Y,U}\circ F(g\otimes f)\circ m_{W\otimes X,U}.$$

Therefore, we proved that $m^{-1}\circ F(g\otimes f)\circ m \in \mathbb{T}^{FU}_{F(W\otimes X),F(Z\otimes Y}$.

(2) Let us consider $\beta = (1_{FZ}\otimes m^{-1}_{Y,U})\circ(Fg\otimes Ff)\circ(1_{FW}\otimes m_{X,U})$. It follows that $\mathrm{Tr}^{FU}_{F(W\otimes X),F(Z\otimes Y)}((m_{Z,Y}\otimes 1_{FU})\circ\beta\circ(m^{-1}_{W,X}\otimes 1_{FU})) =$ (naturality axiom)

$= m_{Z,Y}\circ\mathrm{Tr}^{FU}_{FW\otimes FX,FZ\otimes FY}(\beta)\circ m^{-1}_{W,X} =$ (functoriality of the tensor)

$= m_{Z,Y}\circ\mathrm{Tr}^{FU}_{FW\otimes FX,FZ\otimes FY}(F(g)\otimes(m^{-1}_{Y,U}\circ Fg\circ m_{X,U}))\circ m^{-1}_{W,X} =$ (superposing)

$= m_{Z,Y}\circ(F(g)\otimes\mathrm{Tr}^{FU}_{FW\otimes FX,FZ\otimes FY}(m^{-1}_{Y,U}\circ Fg\circ m_{X,U}))\circ m^{-1}_{W,X} =$ (by hypothesis)

$= m_{Z,Y}\circ(F(g)\otimes F(p_1))\circ m^{-1}_{W,X} =$ (by naturality of m)

$= F(g\otimes p_1)$.

Thus, we proved that there exists an arrow $p_2 = g\otimes p_1$ such that

$$F(p_2) = \mathrm{Tr}^{FU}_{F(W\otimes X),F(Z\otimes Y)}((m_{Z,Y}\otimes 1_{FU})\circ\beta\circ(m^{-1}_{W,X}\otimes 1_{FU})).$$

On the other hand, we have by naturality and the fact that F is a monoidal functor:

$$(m_{Z,Y} \otimes 1_{FU}) \circ \beta \circ (m^{-1}_{W,X} \otimes 1_{FU}) = m^{-1}_{Z\otimes Y,U} \circ F(g \otimes f) \circ m_{W\otimes X,U}$$

which means, according to our definition, that

$$\widehat{\mathrm{Tr}}^{U}_{W\otimes X,Z\otimes Y}(g \otimes f) = p_2 = g \otimes p_1 = g \otimes \widehat{\mathrm{Tr}}^{U}_{X,Y}(f).$$

Yanking:

Let us consider $\sigma : U \otimes U \to U \otimes U$; we want to prove that $\sigma_{U,U} \in \widehat{\mathbb{T}}^{U}_{U,U}$ and $\widehat{\mathrm{Tr}}^{U}_{U,U}(\sigma_{U,U}) = 1_U$. To show that $\sigma_{U,U} \in \widehat{\mathbb{T}}^{U}_{U,U}$ we recall from the trace class definition that we must check two conditions:

(1) First, we notice that since F is a symmetric monoidal functor and by the yanking axiom in the category $\mathcal{D}$:

$$m^{-1}_{U,U} \circ F(\sigma_{U,U}) \circ m_{U,U} = \sigma_{FU,FU} \in \mathbb{T}^{U}_{U,U}.$$

From which it follows that $\mathrm{Tr}^{FU}_{FU,FU}(\sigma_{FU,FU}) = 1_{FU} = F(1_U)$.

(2) Therefore there exists an arrow $p = 1_U$ such that

$$F(1_U) = \mathrm{Tr}^{FU}_{FU,FU}(m^{-1}_{U,U} \circ F(\sigma_{U,U}) \circ m_{U,U}).$$

Hence, we are saying that $\widehat{\mathrm{Tr}}^{U}_{U,U}(\sigma_{U,U}) = p = 1_U$. □

4.4 Another partial trace on completely positive maps with $\oplus$

Definition 4.4.1. Consider the forgetful functor $F : (\mathbf{CPM}, \oplus) \to (\mathbf{Vect}_{fn}, \oplus)$, where $(\mathbf{Vect}_{fn}, \oplus, \mathbf{0}, \mathbb{T})$ is partially traced by Definition 4.2.1, i.e., **CPM** is a monoidal subcategory of $\mathbf{Vect}_{fn}$. We define a partial trace $\widehat{\mathrm{Tr}}$ with trace class given by $\widehat{\mathbb{T}}$ on **CPM** by the method of Section 4.3.

Remark 4.4.2. Comparing this with the partial trace (on the same category) defined in Section 4.2.5, we note that if f and $(I - f_{22})^{-1}$ are completely positive then

$$f_{11} + f_{12}(I - f_{22})^{-1} f_{21}$$

is a completely positive map. This implies that $\widetilde{\mathbb{T}}^U_{X,Y}$ as in Definition 4.2.12 satisfies: $\widetilde{\mathbb{T}}^U_{X,Y} \subseteq \widehat{\mathbb{T}}^U_{X,Y}$. However, consider the **CPM**-map $f : U \oplus U \to U \oplus U$ given by the following matrix:

$$\begin{pmatrix} I & 0 \\ 0 & 2I \end{pmatrix}$$

We have $f_{11} = I$, $f_{21} = f_{12} = 0$ and $f_{22} = 2I$. Then $I - f_{22} = I - 2I = (-1)I$ is an invertible map with inverse $(-1)I$ but is not a positive map. On the other hand, $f_{11} + f_{12}(I - f_{22})^{-1} f_{21} = I + 0((-1)I)0 = I$ is a **CPM**-map, i.e., $f \in \widehat{\mathbb{T}}^U_{U,U}$ but $f \notin \widetilde{\mathbb{T}}^U_{U,U}$.

4.5 Partial trace on superoperators with $\oplus$ and $\otimes$

As an application of the construction of Section 4.3, we now focus on the category $\mathbf{Q}$ which is not a compact closed category. We discuss examples of partial traces in connection with its two monoidal structures.

Example 4.5.1. $(\mathbf{Q}, \oplus)$ has a total trace operator $\mathrm{Tr}^u_{x,y} : \mathbf{Q}(x \oplus u, y \oplus u) \to \mathbf{Q}(x, y)$ defined by $\mathrm{Tr}^u_{x,y}(f) = f_{11} + \sum_{i=0}^{\infty} f_{12} f_{22}^i f_{21}$, see [63] for details.

Example 4.5.2. By Proposition 4.3.1, $(\mathbf{Q}, \oplus)$ has a partial trace $\mathrm{Tr}^u_{x,y} : \mathbf{Q}(x \oplus u, y \oplus u) \rightharpoonup \mathbf{Q}(x, y)$, given by $\mathrm{Tr}^u_{x,y}(f) \cong f_{11} + f_{12}(I - f_{22})^{-1} f_{21}$.

Example 4.5.3. Another partial trace on $(\mathbf{Q}, \oplus)$ is given by considering the forgetful functor from $\mathbf{Q}$ to the category of vector spaces $(\mathbf{Vect}, \oplus)$ with the kernel-image partial trace of Definition 4.2.9 given in Section 4.2.4. Notice that the identity is a superoperator satisfying Definition 4.2.9 which implies that these two partial traces still remain different on $\mathbf{Q}$.

Example 4.5.4. We can consider the category $(\mathbf{Q}_s, \otimes)$ of simple superoperators as a subcategory of the compact closed category $(\mathbf{CPM}_s, \otimes)$, see Definition 3.2.4. It has

a partial trace Tr given by Proposition 4.3.1 where $\mathrm{Tr}^U_{X,Y} : \mathbf{Q}_s(X \otimes U, Y \otimes U) \rightharpoonup \mathbf{Q}_s(X, Y)$ is the canonical trace on $\mathbf{CPM}_s$. Since linear maps f in the category of finite dimensional vector spaces are continuous functions we can prove that for every completely positive map there exists a $0 < \lambda \leq 1$ such that λf is a superoperator. Then, for every unit map $\eta_U : I \to U^* \otimes U$ in $\mathbf{CPM}$ there exists a λ_U such that $\lambda_U . \eta_U$ is a superoperator. Therefore, if $\lambda_U^{-1} . f$ is a superoperator then $f \in \mathbb{T}^U_{X,Y}$.

Chapter 5

A representation theorem for partially traced categories

The goal of this chapter is to prove a strong converse to Proposition 4.3, i.e.: every partially traced category arises as a monoidal subcategory of a totally traced category. More precisely, we show that every partially traced category can be faithfully embedded in a compact closed category in such a way that the trace is preserved.

Our construction uses a partial version of the "*Int*" construction of Joyal, Street, and Verity [41]. When we try to apply the *Int* construction to a partially traced category $\mathcal{C}$, we find that the composition operation in $\mathrm{Int}(C)$ is a well-defined operation only if the trace is total. We therefore consider a notion of "categories" with partially defined composition, namely, Freyd's paracategories [39]. Specifically, we introduce the notion of a strict symmetric compact closed paracategory.

We first show that every partially traced category can be fully and faithfully embedded in a compact closed paracategory, by an analogue of the *Int* construction. We then show that every compact closed paracategory can be embedded (faithfully, but not necessarily fully) in a compact closed (total) category, using a construction similar to Freyd's. Finally, every compact closed category is (totally) traced, yielding the desired result.

5.1 Paracategories

The aim of this section is to recall Freyd's notion of paracategory. A reference on this subject is [39]. Informally, a paracategory is a category with partially defined composition.

Definition 5.1.1. A *(directed) graph* $\mathcal{C}$ consists of:
- a class of elements called objects $obj(\mathcal{C})$
- for every pair of objects A, B a set $\mathcal{C}(A, B)$ called arrows from A to B. Let $Arrow(\mathcal{C})$ be the class of all the arrows in $\mathcal{C}$.

Definition 5.1.2. Let $\mathcal{C}$ be a graph. We define $\mathcal{P}(\mathcal{C})$, the *path category* of $\mathcal{C}$, by $obj(\mathcal{P}(\mathcal{C})) = obj(\mathcal{C})$ and arrows from A_0 to A_n are finite sequences $(A_0, f_0, A_1, f_1, \ldots, A_n)$ of alternating objects and arrows of the graph $\mathcal{C}$, where $n \geq 0$. We say that n is the *length* of the path. Two arrows are equal when the sequences coincide. Composition is defined by concatenation and the identity arrow at A is the path of zero length (A) with an object A. We write $\epsilon_A = (A)$ for the identity arrow.

`Notation:` For the sake of simplification, we often write $\vec{f} = f_1, f_2, \ldots, f_n$ for a path and the symbol "$;$" or "$,$" for concatenation.

Recall the definition of Kleene equality "$\asymp$" and directed Kleene equality "$\succeq$" from Definition 4.1.1.

We write $\phi(f) \downarrow$ to say a partial function ϕ is defined on input f.

Definition 5.1.3. A *paracategory* $(\mathcal{C}, [-])$ consists of a directed graph $\mathcal{C}$ and a partial operation $[-] : Arrow(\mathcal{P}(\mathcal{C})) \rightharpoonup Arrow(\mathcal{C})$ called *composition*, which satisfies the following axioms:

(a) for all A, $[\epsilon_A] \downarrow$, i.e., $[-]$ is a total operation on empty paths

(b) for paths of length one, $[f] \downarrow$ and $[f] = f$

(c) for all paths $\vec{r} : A \to B$, $\vec{f} : B \to C$, and $\vec{s} : C \to D$, if $[\vec{f}] \downarrow$ then

$$[\vec{r}, [\vec{f}], \vec{s}] \asymp [\vec{r}, \vec{f}, \vec{s}].$$

We introduce the following notation:

- $1_A = [(A)] = [\epsilon_A]$ for every object A in $\mathcal{C}$.
- for a path $\vec{f} = f_1, f_2, \ldots, f_n$ and an operation $\otimes$, defined on $\mathcal{C}$ (see Definition 5.2.1), we extend it to the category of paths using the following notation:

$$1 \otimes_p \vec{f} = 1 \otimes f_1, 1 \otimes f_2, \ldots, 1 \otimes f_n$$

  and in the same way: $\vec{f} \otimes_p 1$. We drop the symbol p when it is clear from the context.

Definition 5.1.4. Let $(\mathcal{C}, [-])$ and $(\mathcal{D}, [-]')$ be two paracategories. A *functor* between paracategories is a graph morphism $F : Obj(\mathcal{C}) \to Obj(\mathcal{D})$, $F : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ such that when $[\vec{p}] \downarrow$ then $F[\vec{p}] = [\vec{Fp}]'$. Let **PCat** be the category of (small) paracategories and functors.
We say that such a functor is *faithful* if it is faithful as a morphism of graphs.

Remark 5.1.5. Every category $\mathcal{C}$ can be regarded as a paracategory with $[f_1, \ldots, f_n] = f_n \circ \ldots \circ f_1$. In this case, composition is a totally defined operation. This yields a forgetful functor **Cat** $\to$ **PCat**.

5.2 Symmetric monoidal paracategories

Definition 5.2.1. A *strict symmetric monoidal paracategory* $(\mathcal{C}, [-], \otimes, I, \sigma)$, also called an *ssmpc*, consists of:

- a paracategory $(\mathcal{C}, [-])$
- a total operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies:
  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ on objects, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ on arrows (associative); there is an object I such that $A \otimes I = I \otimes A = A$ and $f \otimes 1_I = 1_I \otimes f = f$ for every object A and arrow f (unit). Subject to the following conditions:

(a) $1_A \otimes 1_B = 1_{A\otimes B}$.

(b) $[f, f'] \otimes [g, g'] \coloneqq [f \otimes g, f' \otimes g']$ where f, g, f', g' are arrows of $\mathcal{C}$ and $\coloneqq$ denotes Kleene directed equality.

(c) $1 \otimes [\vec{p}] \coloneqq [1 \otimes \vec{p}]$ and $[\vec{p}] \otimes 1 \coloneqq [\vec{p} \otimes 1]$

- for all objects A and B there is an arrow $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that:
  - for every $f : B \otimes A \to X$, $g : Y \to A \otimes B$, $[\sigma_{A,B}, f] \downarrow$ and $[g, \sigma_{A,B}] \downarrow$
  - for every $f : A \to A'$ and $g : B \to B'$: $[f \otimes 1_B, \sigma] = [\sigma, 1_B \otimes f]$ and $[1_A \otimes g, \sigma] = [\sigma, g \otimes 1_A]$
  - for every A and B: $[\sigma_{A,B}, \sigma_{B,A}] = 1_{A\otimes B}$
  - for every A, B, and C: $[\sigma_{A,B} \otimes 1_C, \sigma_{B,A\otimes C}] = 1_A \odot \sigma_{B,C}$.

Remark 5.2.2. Conditions (b) and (c) are equivalent to the condition $[f_1, \ldots, f_n] \odot [g_1 \ldots, g_n] \coloneqq [f_1 \otimes g_1, \ldots, f_n \otimes g_n]$ for all natural numbers n.

Proposition 5.2.3. *Let $(\mathcal{C}, [-], \otimes, I, \sigma)$ be a ssmpc. Then for paths p, q of length one we have that $[p \otimes 1_C, 1_B \otimes q] \downarrow$, $[1_A \otimes q, p \otimes 1_D] \downarrow$ and are equal to $p \odot q$. Moreover, for paths $\vec{p}$ and $\vec{q}$ $[1_A \otimes \vec{q}, \vec{p} \otimes 1_D] \coloneqq [\vec{p} \otimes 1_C, 1_B \otimes \vec{q}]$.*

Proof. Let us first prove the result for paths of length 1, say $p : A \to B$, $q : C \to D$. Observe that $[p, 1_B] = [p, [(B)]] = [p, (B)] = [p] = p$ since the last equation is defined and by the axioms. In the same way $p = [1_A, p]$, $[1_C, q] = q = [q, 1_D]$. Then $p \otimes q = [p, 1_B] \otimes [1_C, q] \coloneqq [p \otimes 1_C, 1_B \otimes q]$ and $p \otimes q = [1_A, p] \odot [q, 1_D] \coloneqq [1_A \odot q, p \odot 1_D]$, by condition (b) of Definition 5.2.1, which implies that $[1_A \odot q, p \odot 1_D] \downarrow$, $[p \odot 1_C, 1_B \odot q] \downarrow$ and $[1_A \otimes q, p \otimes 1_D] = [p \otimes 1_C, 1_B \otimes q] = p \otimes q$.
Now since we have already proved that $[1_A \otimes q, p \otimes 1_D] \downarrow$, $[p \otimes 1_C, 1_B \otimes q] \downarrow$ and that they are equal we can use the axioms of paracategories and extend this to $[1_A \otimes \vec{q}, \vec{p} \otimes 1_D] \coloneqq [\vec{p} \otimes 1_C, 1_B \otimes \vec{q}]$ by iterating this procedure in the following way:

$$
\begin{aligned}
[1_A \otimes \vec{q}, \vec{p} \otimes 1_D] &= [1_A \otimes q_1, \ldots, 1_A \otimes q_n, p_1 \otimes 1_D, \ldots, p_m \otimes 1_D] \\
&\asymp [1_A \otimes q_1, \ldots, [1_A \otimes q_n, p_1 \otimes 1_D], \ldots, p_m \otimes 1_D] \\
&\asymp [1_A \otimes q_1, \ldots, [p_1 \otimes 1_C, 1_B \otimes q_n], \ldots, p_m \otimes 1_D] \\
&\asymp [1_A \otimes q_1, \ldots, p_1 \otimes 1_C, 1_B \otimes q_n, \ldots, p_m \otimes 1_D] \\
&\asymp \ldots \text{we move } p_1 \otimes 1_C \text{ to the first position} \\
&\asymp [p_1 \otimes 1_C, 1_B \otimes q_1, \ldots, 1_A \otimes q_n, p_2 \otimes 1_C, \ldots, p_m \otimes 1_D] \\
&\asymp \ldots \text{we iterate this procedure } m-1 \text{ times} \\
&\asymp [\vec{p} \otimes 1_C, 1_B \otimes \vec{q}].
\end{aligned}
$$

□

Definition 5.2.4. Let $(\mathcal{C}, [-], \otimes, I, \sigma)$ and $(\mathcal{D}, [-]', \otimes', I', \sigma')$ be two ssmpcs. A functor between them is *strict monoidal* when $F(A) \otimes' F(B) = F(A \otimes B)$, $F(I) = I'$ on objects, $F(f) \otimes' F(g) = F(f \otimes g)$ and $F(\sigma) = \sigma'$ on arrows.

5.3 The completion of symmetric monoidal paracategories

From now on $\mathcal{C}$ denotes a ssmpc. We wish to prove the following theorem:

Theorem 5.3.1. *Every strict symmetric monoidal paracategory can be faithfully embedded in a strict symmetric monoidal category.*

Definition 5.3.2. A *congruence relation* $\mathcal{S}$ on $\mathcal{P}(\mathcal{C})$ is given as follows: for every pair of objects A, B, an equivalence relation $\sim_{\mathcal{S}}^{A,B}$ on the hom-set $\mathcal{P}(\mathcal{C})(A, B)$, satisfying the following axioms. We usually omit the superscripts when they are clear from the context.

(1) If $\vec{p} \sim_{\mathcal{S}} \vec{p}'$ and $\vec{q} \sim_{\mathcal{S}} \vec{q}'$, then $\vec{p}; \vec{q} \sim_{\mathcal{S}} \vec{p}'; \vec{q}'$.

(2) Whenever $[\vec{p}] \downarrow$, then $\vec{p} \sim_{\mathcal{S}} [\vec{p}]$.

(3) If $\vec{p} \sim_{\mathcal{S}} \vec{q}$, then $\vec{p} \otimes 1 \sim_{\mathcal{S}} \vec{q} \otimes 1$ and $1 \otimes \vec{p} \sim_{\mathcal{S}} 1 \otimes \vec{q}$.

Remark 5.3.3. Technically Definition 5.3.2 can be regarded as a "congruence subcategory" on $\mathcal{P}(\mathcal{C})$, i.e., $\mathcal{S}$ is a subcategory of $\mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C})$ satisfying axioms (2) and (3).

Definition 5.3.4. We define a particular congruence relation $\hat{\mathcal{S}}$ as follows: $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{q}$ if and only if $\forall \vec{r}, \vec{s}, \forall A, B \in Obj(\mathcal{C})$ $[\vec{r}, 1_A \otimes \vec{p} \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{q} \otimes 1_B, \vec{s}]$.

Remark 5.3.5. It should be observed that $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{q}$ implies $[\vec{p}] \eqcirc [\vec{q}]$ by letting $\vec{r}, \vec{s}$ be empty lists and $A = B = I$.

Let us check that $\hat{\mathcal{S}}$ is a congruence relation.

Lemma 5.3.6. *$\hat{\mathcal{S}}$ is a congruence relation.*

Proof. We need to show axioms (1), (2), and (3). To show (1), assume $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{q}$ and $\vec{u} \sim_{\hat{\mathcal{S}}} \vec{t}$ we have to check that $\vec{p}; \vec{u} \sim_{\hat{\mathcal{S}}} \vec{q}; \vec{t}$. Consider arbitrary $\vec{r}$, $\vec{s}$, A, B. We have:

$$[\vec{r}, 1_A \otimes (\vec{p}; \vec{u}) \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{p} \otimes 1_B, 1_A \otimes \vec{u} \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{q} \otimes 1_B, 1_A \otimes \vec{u} \otimes 1_B, \vec{s}].$$

The first equation is by definition of the tensor $\otimes_p$ on paths, the second equation is because by hypothesis we have that: $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{q}$ implies $[\vec{r'}, 1_A \otimes \vec{p} \otimes 1_B, \vec{s'}] \eqcirc [\vec{r'}, 1_A \otimes \vec{q} \otimes 1_B, \vec{s'}]$ with $\vec{r} = \vec{r'}$ and $\vec{s'} = 1_A \otimes \vec{u} \otimes 1_B, \vec{s}$. In a similar way we have that:

$$[\vec{r}, 1_A \otimes (\vec{q}; \vec{t}) \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{q} \otimes 1_B, 1_A \otimes \vec{t} \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{q} \otimes 1_B, 1_A \otimes \vec{u} \otimes 1_B, \vec{s}].$$

It follows that $\vec{p}; \vec{u} \sim_{\hat{\mathcal{S}}} \vec{q}; \vec{t}$. To prove (2), assume $[\vec{p}] \downarrow$, and let $\vec{r}, \vec{s}, A, B$ be given. We observe first that $1_A \otimes [\vec{p}] \otimes 1_B \eqcirc [1_A \otimes \vec{p} \otimes 1_B]$ by (c) in the definition of a ssmpc. Then $[\vec{p}] \downarrow$ implies that $1_A \otimes [\vec{p}] \otimes 1_B \downarrow$ and then $[1_A \otimes \vec{p} \otimes 1_B] \downarrow$ and and they are equal. Thus we have by one of the axioms of paracategory that:

$$[\vec{r}, 1_A \otimes [\vec{p}] \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, [1_A \otimes \vec{p} \otimes 1_B], \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{p} \otimes 1_B, \vec{s}].$$

To prove (3), assume $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{p'}$. We observe that this implies for every $C \in Obj(\mathcal{C})$, $[\vec{r}, 1_A \otimes \vec{p} \otimes 1_C \otimes 1_B, \vec{s}] \eqcirc [\vec{r}, 1_A \otimes \vec{p'} \otimes 1_C \otimes 1_B, \vec{s}]$, $\forall \vec{r}, \vec{s}, \forall A, B \in Obj(\mathcal{C})$, therefore $\vec{p} \otimes 1 \sim_{\hat{\mathcal{S}}} \vec{p'} \otimes 1$. In a similar way we get the other equation. $\square$

Definition 5.3.7. Let $\sim$ be the smallest congruence relation on $\mathcal{P}(\mathcal{C})$, i.e., the intersection of all congruence relations.

Proposition 5.3.8. *$\vec{p} \sim \vec{q}$ implies $[\vec{p}] \asymp [\vec{q}]$.*

Proof. Since $(\vec{p}, \vec{q})$ is in the intersection of all congruence relations then in particular $\vec{p} \sim_{\hat{\mathcal{S}}} \vec{q}$ which implies that $[\vec{p}] \asymp [\vec{q}]$ by Remark 5.3.5.

□

Corollary 5.3.9. *For paths $p, q : A \to B$ of length 1, $p \sim q$ iff $p = q$.*

Proof. Obvious from Proposition 5.3.8 and axiom (b) of paracategories. □

We now introduce the following notation:

$$\vec{f} \otimes_p \vec{g} = (\vec{f} \otimes_p 1), (1 \otimes_p \vec{g}).$$

Note that, as a path, this is not equal to $(1 \otimes_p \vec{g}), (\vec{f} \otimes_p 1)$. However, we will show that they are congruent. When it is clear from the context we drop the letter p.

Lemma 5.3.10. *Let $\mathcal{S}$ be a congruence relation of $\mathcal{P}(\mathcal{C})$. Then if $\vec{f} \sim_{\mathcal{S}} \vec{f'}$ and $\vec{g} \sim_{\mathcal{S}} \vec{g'}$ then $\vec{f} \otimes \vec{g} \sim_{\mathcal{S}} \vec{f'} \otimes \vec{g'}$.*

Proof. By assumption $\vec{f} \sim_{\mathcal{S}} \vec{f'}$ therefore by (3) we have $\vec{f} \otimes 1 \sim_{\mathcal{S}} \vec{f'} \otimes 1$. Similarly $1 \otimes \vec{q} \sim_{\mathcal{S}} 1 \otimes \vec{q'}$. Therefore by (1), we have: $\vec{f} \otimes 1, 1 \otimes \vec{q} \sim_{\mathcal{S}} \vec{f'} \otimes 1, 1 \otimes \vec{q'}$.

□

Lemma 5.3.11. *Let $\mathcal{S}$ be a congruence relation of $\mathcal{P}(\mathcal{C})$. Then*

$$\vec{f} \otimes 1, 1 \otimes \vec{g} \sim_{\mathcal{S}} 1 \otimes \vec{g}, \vec{f} \otimes 1.$$

Proof. Given $\vec{f} = f_1, \ldots, f_n$ and $\vec{g} = g_1, \ldots, g_m$ we have that by Proposition 5.2.3 above $[f_n \otimes 1, 1 \otimes g_1] \downarrow$, $[1 \otimes g_1, f_n \otimes 1] \downarrow$ and are equal to $f_n \otimes g_1$. This yields, by Definition 5.3.2 of congruence relation, the following sequence of equivalences:

$$f_n \otimes 1, 1 \otimes g_1 \quad \sim_{\mathcal{S}} \quad [f_n \otimes 1, 1 \otimes g_1] = [1 \otimes g_1, f_n \otimes 1] \quad \sim_{\mathcal{S}} \quad 1 \otimes g_1, f_n \otimes 1.$$

Which implies by composition:

$$f_1 \otimes 1, \ldots, f_{n-1} \otimes 1, \overbrace{(f_n \otimes 1, 1 \otimes g_1)}, 1 \otimes g_2, \ldots 1 \otimes g_m \sim_{\mathcal{S}}$$
$$f_1 \otimes 1, \ldots, f_{n-1} \otimes 1, \underbrace{(1 \otimes g_1, f_n \otimes 1)}, 1 \otimes g_2, \ldots 1 \otimes g_m.$$

By iterating this procedure we end up moving $1 \otimes g_1$ into the first place. We finish the proof by repeating this $m-1$ times. □

From now, ";" denotes composition in the quotient category written in diagrammatic order (here this means concatenation of paths).

Lemma 5.3.12. *Let $\mathcal{S}$ be a congruence relation defined on a strict symmetric monoidal paracategory $(\mathcal{C}, [-], \otimes, I, \sigma)$. Then the quotient $(\mathcal{P}(\mathcal{C})/\mathcal{S}, \hat{\otimes}, I, s)$ is a strict symmetric monoidal category, where $\hat{\otimes}$ is the obvious tensor and $s = \overline{\sigma}$.*

Proof. Let $(\mathcal{C}, [-], \otimes, I, \sigma)$ be a strict symmetric monoidal paracategory. It induces a strict symmetric monoidal category $(\mathcal{P}(\mathcal{C})/\mathcal{S}, \hat{\otimes}, I, s)$ in the following way:
The objects of $\mathcal{P}(\mathcal{C})/\mathcal{S}$ are the same as the objects of the graph $\mathcal{C}$ and the arrows $\overline{\vec{f}} = \overline{f_1 \ldots, f_n}$ are $\mathcal{S}$-equivalence classes of paths. Composition on classes is induced by composition on paths by axiom (1) of congruences. The identity is the class of the identity of the path category.
A bifunctor $\hat{\otimes} : P(C)/\mathcal{S} \times P(C)/\mathcal{S} \to P(C)/\mathcal{S}$ is defined by $\overline{\vec{f}} \hat{\otimes} \overline{\vec{g}} = \overline{\vec{f} \otimes_p 1, 1 \otimes_p \vec{g}}$.
The tensor is well-defined by the Lemma 5.3.10 above.
We must check the interchange law:

$$(\overline{\vec{f}} \hat{\otimes} \overline{\vec{g}}); (\overline{\vec{f}'} \hat{\otimes} \overline{\vec{g}'}) = (\overline{\vec{f}}; \overline{\vec{f}'}) \hat{\otimes} (\overline{\vec{g}}; \overline{\vec{g}'})$$

We have:

$$(\overline{\vec{f}} \hat{\otimes} \overline{\vec{g}}); (\overline{\vec{f}'} \hat{\otimes} \overline{\vec{g}'}) = (\overline{\vec{f} \otimes 1, 1 \otimes \vec{g}}); (\overline{\vec{f}' \otimes 1, 1 \otimes \vec{g}'}) = \overline{\vec{f} \otimes 1, 1 \otimes \vec{g}, \vec{f}' \otimes 1, 1 \otimes \vec{g}'} =$$
$$\overline{\vec{f} \otimes 1}; \overline{1 \otimes \vec{g}, \vec{f}' \otimes 1}; \overline{1 \otimes \vec{g}'} \overset{(*)}{=} \overline{\vec{f} \otimes 1}; \overline{\vec{f}' \otimes 1, 1 \otimes \vec{g}}; \overline{1 \otimes \vec{g}'} =$$
$$\overline{\vec{f} \otimes 1, \vec{f}' \otimes 1, 1 \otimes \vec{g}, 1 \otimes \vec{g}'} = \overline{(\vec{f}, \vec{f}') \otimes 1, 1 \otimes (\vec{g}, \vec{g}')} = \overline{(\vec{f}, \vec{f}') \otimes_p (\vec{g}, \vec{g}')} =$$
$$\overline{(\vec{f}, \vec{f}')} \hat{\otimes} \overline{(\vec{g}, \vec{g}')} = (\overline{\vec{f}}; \overline{\vec{f}'}) \hat{\otimes} (\overline{\vec{g}}; \overline{\vec{g}'}).$$

Where in $(*)$ we used the property of the Lemma 5.3.11 above: $\overline{1 \otimes \vec{g}, \vec{f}' \otimes 1} = \overline{\vec{f}' \otimes 1, 1 \otimes \vec{g}}$.
Also we want to check that $\overline{\epsilon_{A \otimes B}} = \overline{\epsilon_A} \hat{\otimes} \overline{\epsilon_B}$.

$$\overline{\epsilon_A}\hat{\otimes}\overline{\epsilon_B} = \overline{\epsilon_A \otimes_p 1_B, 1_A \otimes_p \epsilon_B} = \overline{\epsilon_{A\otimes B}, \epsilon_{A\otimes B}} = \overline{\epsilon_{A\otimes B}}$$

since $[\,1_{A\otimes B}, 1_{A\otimes B}\,]\downarrow$.

Given paths $\vec{f} : A \to B$, $\vec{g} : C \to D$ and $\vec{h} : E \to F$ we check the associative property:

$$(\overline{\vec{f}}\hat{\otimes}\overline{\vec{g}})\hat{\otimes}\overline{\vec{h}} = (\overline{\vec{f} \otimes_p \vec{g}})\hat{\otimes}\overline{\vec{h}} = \overline{(\vec{f} \otimes 1_C, 1_B \otimes \vec{g})}\hat{\otimes}\overline{\vec{h}} = \overline{(\vec{f} \otimes 1_C, 1_B \otimes \vec{g}) \otimes_p \vec{h}} =$$

$$\overline{(\vec{f} \otimes 1_C, 1_B \otimes \vec{g}) \otimes 1_E, 1_{B\otimes D} \otimes \vec{h}} = \overline{\vec{f} \otimes 1_C \otimes 1_E, 1_B \otimes \vec{g} \otimes 1_E, 1_{B\otimes D} \otimes \vec{h}} =$$

$$\overline{\vec{f} \otimes 1_{C\otimes E}, 1_B \otimes \vec{g} \otimes 1_E, 1_B \otimes 1_D \otimes \vec{h}} = \overline{\vec{f} \otimes 1_{C\otimes E}, 1_B \otimes (\vec{g} \otimes 1_E, 1_D \otimes \vec{h})} =$$

$$\overline{\vec{f} \otimes_p (\vec{g} \otimes 1_E, 1_D \otimes \vec{h})} = \overline{\vec{f} \otimes_p (\vec{g} \otimes_p \vec{h})} = \overline{\vec{f}}\hat{\otimes}(\overline{\vec{g} \otimes_p \vec{h}}) = \overline{\vec{f}}\hat{\otimes}(\overline{\vec{g}}\hat{\otimes}\overline{\vec{h}}).$$

Also if $\vec{f} : A \to B$ and $1_I : I \to I$ then:

$$\overline{\vec{f}}\hat{\otimes}\overline{1_I} = \overline{\vec{f} \otimes_p 1_I} = \overline{\vec{f} \otimes 1_I, 1_B \otimes 1_I} = \overline{\vec{f}, 1_B} = \overline{\vec{f}}.$$

Since $\vec{f} \otimes 1_I = \overrightarrow{f \otimes 1_I} = \vec{f}$ and $1_B \otimes 1_I = 1_B$. In the same way we get $\overline{1_I}\hat{\otimes}\overline{\vec{f}} = \overline{\vec{f}}$.

The symmetry is defined as $s_{A,B} : A\hat{\otimes}B \to B\hat{\otimes}A$, $s_{A,B} = \overline{\sigma_{A,B}}$. This arrow is an isomorphism since $[\sigma_{A,B}, \sigma_{B,A}]\downarrow$ implies $\sigma_{A,B}, \sigma_{B,A} \sim_S [\sigma_{A,B}, \sigma_{B,A}]$ and then:

$$s_{A,B}; s_{B,A} = \overline{\sigma_{A,B}}; \overline{\sigma_{B,A}} = \overline{\sigma_{A,B}, \sigma_{B,A}} = \overline{[\sigma_{A,B}, \sigma_{B,A}]} = \overline{1_{A\otimes B}}.$$

Similarly $s_{B,A}; s_{A,B} = \overline{1_{B\otimes A}}$.

Next, we check the following coherence diagram: $(s_{A,B} \otimes 1_C); s_{B,A\otimes C} = 1_A \otimes s_{B,C}$.

$$(s_{A,B} \otimes 1_C); s_{B,A\otimes C} = (\overline{\sigma_{A,B}}\hat{\otimes}\overline{1_C}); \overline{\sigma_{B,A\otimes C}} = \overline{(\sigma_{A,B} \otimes 1_C)}; \overline{\sigma_{B,A\otimes C}} =$$

$$\overline{(\sigma_{A,B} \otimes 1_C), \sigma_{B,A\otimes C}} = \overline{[(\sigma_{A,B} \otimes 1_C), \sigma_{B,A\otimes C}]} = \overline{1_A \otimes \sigma_{B,C}} = \overline{1_A}\hat{\otimes}\overline{\sigma_{B,C}} = 1_A\hat{\otimes}s_{B,C}.$$

Next we prove naturality of the map $s_{A,B} : A\hat{\otimes}B \to B\hat{\otimes}A$. To see this, it is enough to prove it on simple path of length one and then extend it by composition. Let us consider $\overline{f} : A \to A'$ and since $\overline{[\sigma_{A,B}, 1 \otimes f]}\downarrow$

$s_{A,B}; (\overline{1}\hat{\otimes}\overline{f}) = \overline{\sigma_{A,B}}; (\overline{1}\hat{\otimes}\overline{f}) = \overline{\sigma_{A,B}}; \overline{1 \otimes f} = \overline{\sigma_{A,B}, 1 \otimes f} = \overline{[\sigma_{A,B}, 1 \otimes f]} =$

$\overline{[f \otimes 1, \sigma_{A',B}]} = \overline{f \otimes 1, \sigma_{A',B}} = \overline{f \otimes 1}; \overline{\sigma_{A',B}} = (\overline{f}\hat{\otimes}\overline{1}); s_{A',B}$.

For the general case we iterate this, applying the above equation several times.

□

Proof of Theorem 5.3.1

Proof. A functor between paracategories $F : (\mathcal{C}, [-], \otimes, I, \sigma) \to (\mathcal{P}(\mathcal{C})/\mathcal{S}, \hat{\otimes}, I, s)$, where the category $\mathcal{P}(\mathcal{C})/\mathcal{S}$ is taken as a (total) paracategory, is defined in the following way:

- on objects as the identity and
- on arrows $F(f) = \overline{f}$ as the projection on classes.

Observe that F preserves identities and composition when $[\vec{f}]$ is defined:

$$F[\vec{f}] = \overline{[\vec{f}]} = \overline{\vec{f}} = \overline{f_1, \ldots, f_n} = \overline{f_1}; \ldots; \overline{f_n} = Ff_1; \ldots; Ff_n.$$

Following the definition, we have that F preserves symmetries: $F(\sigma) = \overline{\sigma} = s$.
In addition, if $f : A \to C$ and $g : B \to D$ then

$$F(f \otimes g) = \overline{f \otimes g} = \overline{[f \otimes 1_B, 1_C \otimes g]} = \overline{f \otimes 1_B, 1_C \otimes g} = \overline{f \otimes_p g} = Ff \hat{\otimes} Fg$$

where the last sequence of equations is justified by Proposition 5.2.3, the property above, axioms and by definition of congruence relation.
Moreover, if $\mathcal{S}$ is the smallest congruence relation, or indeed any congruence relation satisfying $\mathcal{S} \subseteq \hat{\mathcal{S}}$, then F is faithful by Corollary 5.3.9. □

5.4 Compact closed paracategories

Definition 5.4.1. A *(strict symmetric) compact closed paracategory* $(\mathcal{C}, [-], \otimes, I, \sigma, \eta, \epsilon)$ is a strict symmetric monoidal paracategory such that for every object A there is an object A^* and arrows $\eta_A : I \to A \otimes A^*$, $\epsilon_A : A^* \otimes A \to I$ such that $[\eta_A \otimes 1_A, 1_A \otimes \epsilon_A] \downarrow$, $[1_{A^*} \otimes \eta_A, \epsilon_A \otimes 1_{A^*}] \downarrow$ and $[\eta_A \otimes 1_A, 1_A \otimes \epsilon_A] = 1_A$, $[1_{A^*} \otimes \eta_A, \epsilon_A \otimes 1_{A^*}] = 1_{A^*}$.

Theorem 5.4.2. *Every compact closed paracategory can be faithfully embedded in a compact closed category.*

Proof. Let us consider the paracategory $(\mathcal{C}, [-], \otimes, I, \sigma, \eta, \epsilon)$.
As a result of the proof of Theorem 5.3.1 above, it suffices to show that $(\mathcal{P}(\mathcal{C})/\mathcal{S}, \hat{\otimes}, I, s, \eta', \epsilon')$ is compact closed, where $\eta' = \overline{\eta}$ and $\epsilon' = \overline{\epsilon}$. Notice that by definition the functor F preserves η and ϵ. Consequently, the compactness diagrams are satisfied, since the condition $[\eta \otimes 1, 1 \otimes \epsilon] \downarrow$ implies:

$$\begin{aligned}
\overline{\eta}\hat{\otimes}\overline{1_A}; \overline{1_A}\hat{\otimes}\overline{\epsilon} &= \overline{\eta \otimes 1_A}; \overline{1_A \otimes \epsilon} \\
&= \overline{\eta \otimes 1_A, 1_A \otimes \epsilon} \\
&= \overline{[\eta \otimes 1_A, 1_A \otimes \epsilon]} \\
&= \overline{1_A}.
\end{aligned}$$

In the same way, $\overline{1_{A^*}}\hat{\otimes}\overline{\eta}; \overline{\epsilon}\hat{\otimes}\overline{1_{A^*}} = \overline{1_{A^*}}$

□

5.5 Freeness

We can strengthen Theorem 5.3.1 by noting that the faithful embedding satisfies a universal property.

Theorem 5.5.1. *The category $(\mathcal{P}(\mathcal{C})/\sim, \hat{\otimes}, I, s)$ satisfies the following property: for any strict symmetric monoidal category $\mathcal{D}$ and any strict symmetric monoidal functor $G : \mathcal{C} \to \mathcal{D}$ between paracategories, there exists a unique strict symmetric monoidal functor $L : \mathcal{P}(\mathcal{C})/\sim \to \mathcal{D}$ such that $L \circ F = G$, where F is the inclusion map defined in Theorem 5.3.1 above.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{P}(\mathcal{C})/\sim \arrow[d, "L"] \\
& \mathcal{D}
\end{tikzcd}$$

Proof. Consider the set $\mathcal{S}$:

$$\{(\vec{f},\vec{g}) \in \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C}) : \ G(f_1) \circ \cdots \circ G(f_n) = G(g_1) \circ \cdots \circ G(g_m)\}$$

where $\vec{f} = f_1, \ldots, f_n$ and $\vec{g} = g_1, \ldots, g_m$.

We claim that $\mathcal{S}$ is a congruence relation in the sense of Definition 5.3.2 stipulated above. Clearly, it is an equivalence relation. To show that it satisfies axiom (1), assume

$p_1, \ldots p_n \sim_{\mathcal{S}} q_1, \ldots, q_m$ and $f_1, \ldots f_s \sim_{\mathcal{S}} q_1, \ldots, g_t$, then by hypothesis $G(p_1) \circ \cdots \circ G(p_n) = G(q_1) \circ \cdots \circ G(q_m)$ and $G(f_1) \circ \cdots \circ G(f_s) = G(g_1) \circ \cdots \circ G(g_t)$.

Then by composing the left hand side and the right hand side we get the condition

$$p_1, \ldots, p_n, f_1, \ldots, f_s \sim_{\mathcal{S}} q_1, \ldots, q_m, g_1, \ldots, g_t.$$

To show (3), assume $p_1, \ldots, p_n \sim_{\hat{\mathcal{S}}} q_1, \ldots, q_m$, then $G(p_1) \circ \cdots \circ G(p_n) = G(q_1) \circ \cdots \circ G(q_m)$ which in $\mathcal{C}$ implies that $(G(p_1) \circ \cdots \circ G(p_n)) \otimes G1 = (G(q_1) \circ \cdots \circ G(q_m)) \otimes G1$ and by the tensor property of G and functoriality we obtain $G(p_1 \otimes 1) \circ \cdots \circ G(p_n \otimes 1) = G(q_1 \otimes 1) \circ \cdots \circ G(q_m \otimes 1)$, which means $p_1 \otimes 1, \ldots, p_n \otimes 1 \sim_{\mathcal{S}} q_1 \otimes 1, \ldots, q_m \otimes 1$. In the same way $\vec{p} \sim_{\mathcal{S}} \vec{q}$ implies $1 \otimes \vec{p} \sim_{\mathcal{S}} 1 \otimes \vec{q}$.

To show (2), since G is a functor between paracategories, we have $G(p_1) \circ \ldots G(p_n) = G([\vec{p}])$ when $[\vec{p}] \downarrow$ hence $\vec{p} \sim_{\mathcal{S}} [\vec{p}]$.

Now we define the functor L in the following way:

$L(A) = G(A)$ on objects and $L(\overline{\vec{p}}) = G(p_1) \circ \cdots \circ G(p_n)$, where $\vec{p} = p_1, \ldots, p_n$.

It should be apparent that F is well-defined since when $\vec{p} \sim \vec{q}$ then in particular it is true that $\vec{p} \sim_{\mathcal{S}} \vec{q}$ and this implies $L(\overline{\vec{p}}) = L(\overline{\vec{q}})$.

We check functoriality:

$$\begin{aligned} L(\overline{p_1, \ldots, p_n}; \overline{q_1, \ldots, q_m}) &= L(\overline{p_1, \ldots, p_n, q_1, \ldots, q_m}) \\ &= G(p_1) \circ \cdots \circ G(p_n) \circ G(q_1) \circ \cdots \circ G(q_m) \\ &= L(\overline{p_1, \ldots, p_n}) \circ L(\overline{q_1, \ldots, q_m}) \end{aligned}$$

and

$$L(\overline{(A)}) = L(\overline{1_A}) = G(1_A) = 1_{GA}.$$

Furthermore L is strict symmetric monoidal:

$$\begin{aligned}
L(\overline{p}\hat{\otimes}\overline{q}) &= L(\overline{p \otimes 1, 1 \otimes q}) \\
&= G(p_1 \otimes 1) \circ \cdots \circ G(p_n \otimes 1) \circ G(1 \otimes q_1) \circ \cdots \circ G(1 \otimes q_m) \\
&= (G(p_1) \otimes G1) \circ \cdots \circ (G(p_n) \otimes G1) \circ (G1 \otimes G(q_1)) \circ \cdots \circ (G1 \otimes G(q_m)) \\
&= (G(p_1) \otimes 1) \circ \cdots \circ (G(p_n) \otimes 1) \circ (1 \otimes G(q_1)) \circ \cdots \circ (1 \otimes G(q_m)) \\
&= (G(p_1) \circ \cdots \circ G(p_n)) \otimes 1) \circ (1 \otimes (G(q_1) \circ \cdots \circ G(q_m)) \\
&= (G(p_1) \circ \cdots \circ G(p_n)) \otimes (G(q_1) \circ \cdots \circ G(q_m)) \\
&= L(\overline{p}) \otimes L(\overline{q})
\end{aligned}$$

Finally, since G is strict symmetric $L(s) = L(\overline{\sigma}) = G(\sigma) = \sigma'$ where σ' is the symmetry of the category $\mathcal{D}$.

□

5.6 Partially traced categories and the partial Int construction

Joyal, Street, and Verity proved in [41] that every (totally) traced monoidal category $\mathcal{C}$ can be faithfully embedded in a compact closed category $Int(\mathcal{C})$. Here, we give a similar construction for *partially* traced categories. We call the corresponding construction the *partial Int construction*, or the Int^p construction for short. When $\mathcal{C}$ is a partially traced category, $Int^p(\mathcal{C})$ will be a compact closed *para*category.

Definition 5.6.1. Let $(\mathcal{C}, \otimes, I, \sigma)$ be a symmetric monoidal category. There is a graph $Int^p(\mathcal{C})$ associated to this category defined in the following way:

- objects: are a pair of object (A^+, A^-) of the category $\mathcal{C}$.

- arrows: $f^{Int^p} : (A^+, A^-) \to (B^+, B^-)$ are arrows of type $f : A^+ \otimes B^- \to B^+ \otimes A^-$ in the category $\mathcal{C}$.

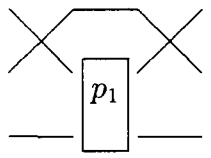
When it is clear from the context we drop the symbol Int^p on the arrows of $Int^p(\mathcal{C})$.

We want to define a partial composition on this graph. For that purpose, consider the following natural transformation, uniquely induced by the symmetric monoidal structure, for $n \geq 0$:

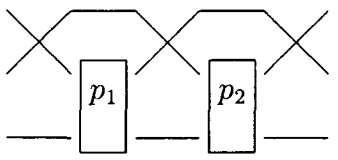
$$\gamma_n : A_1 \otimes A_2 \otimes \ldots A_{n-1} \otimes A_n \to A_n \otimes A_{n-1} \ldots A_2 \otimes A_1.$$

Also, given a path $\vec{p} = p_1, \ldots, p_m \in \mathcal{P}(Int^p(\mathcal{C}))$, using graphical language of symmetric monoidal categories, we shall define an arrow $\epsilon(\vec{p}) \in \mathcal{C}$ in the following way: if $\vec{p} = p_1, \ldots, p_m$ then $\epsilon(\vec{p})$ pictorially is equal to:

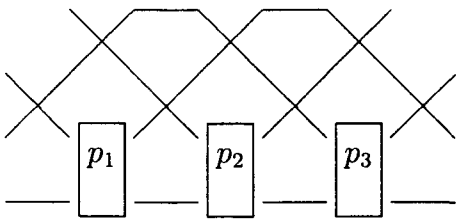
For $m = 1$ arrow:



For $m = 2$ arrows:



For $m = 3$ arrows:



and so on.

In order to get $\epsilon(p_1, \ldots, p_m)$ we form a pyramid of $m-1$ layers of symmetries.

Definition 5.6.2. Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a symmetric monoidal partially traced category. We turn the graph $Int^p(C)$ into a paracategory by defining a partial composition operation $[\vec{p}]$. First of all, when it is applied to an empty path it will be defined as the identity arrow i.e., $[(A^+, A^-)] = 1_{A^+ \otimes A^-}$. On path of length one it will be by definition the same arrow, i.e., $[f] = \mathrm{Tr}^U(\epsilon(f)(1_{X_1^+} \otimes \sigma_{X_2^-, X_1^-})) = f$ with $U = X_1^-$. Suppose that we have a family of arrows $f_i^{Int^p} : (X_i^+, X_i^-) \to (X_{i+1}^+, X_{i+1}^-)$ with $1 \leq i \leq n$ $(n \geq 2)$ in the graph $Int^p(\mathcal{C})$ such that $dom(f_{i+1}) = cod(f_i)$ and $1 \leq i \leq n-1$. Let $U = X_n^- \otimes X_{n-1}^- \otimes \cdots \otimes X_3^- \otimes X_2^-$ and the permutation γ

$$X_n^- \otimes X_{n-1}^- \otimes \cdots \otimes X_3^- \otimes X_2^- \xrightarrow{\gamma} X_2^- \otimes X_3^- \otimes \cdots \otimes X_{n-1}^- \otimes X_n^-.$$

We define the following operation for $n \geq 2$:

$$[f_1, \ldots, f_n] \cong \mathrm{Tr}^U(\epsilon(f_1, \ldots, f_n)(1_{X_1^+} \otimes 1_{X_{n+1}^-} \otimes \gamma_{n-1})).$$

Note that therefore, $[f_1, \ldots, f_n]$ is defined if and only if

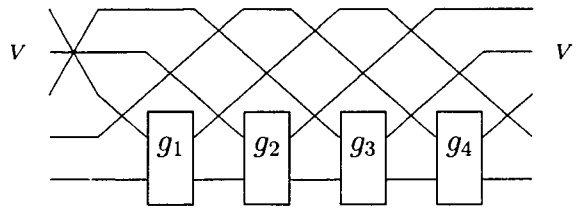
$$\epsilon(f_1, \ldots, f_n)(1_{X_1^+} \otimes 1_{X_{n+1}^-} \otimes \gamma) \in \mathbb{T}^U.$$

We show now that the operation $[-]$ satisfies the axioms required in order to be a paracategory.

Lemma 5.6.3. *Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a strict symmetric monoidal partially traced category. The operation defined in Definition 5.6.2 determines a paracategory $(Int^p(\mathcal{C}), [-])$.*

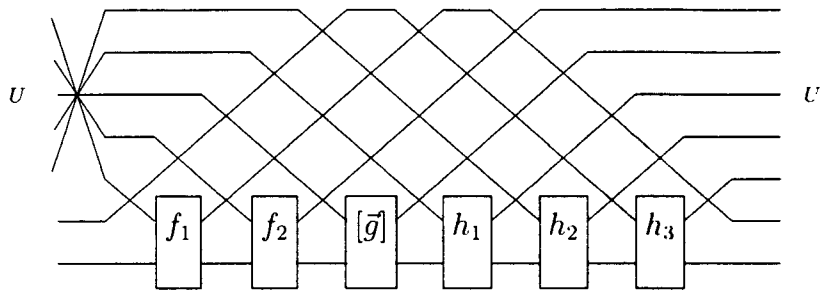
Proof. Properties (a) and (b) of Definition 5.1.3 hold by definition. The goal is to prove (c), i.e., if $[\vec{g}] \downarrow$ then $[\vec{f}, [\vec{g}], \vec{h}] \cong [\vec{f}, \vec{g}, \vec{h}]$ for every $\vec{f}$ and $\vec{h}$. The value of the trace remains always invariant or follows the variations that the axioms trace dictate. Without loss of generality we are going to represent these paths using graphical language in a concrete situation. Therefore, suppose we have $\vec{f} = f_1, f_2$, $\vec{g} = g_1, g_2, g_3, g_4$ and $\vec{h} = h_1, h_2, h_3$. The most general case follows the same pattern.

The fact that $[\vec{g}]\downarrow$ means that the map:



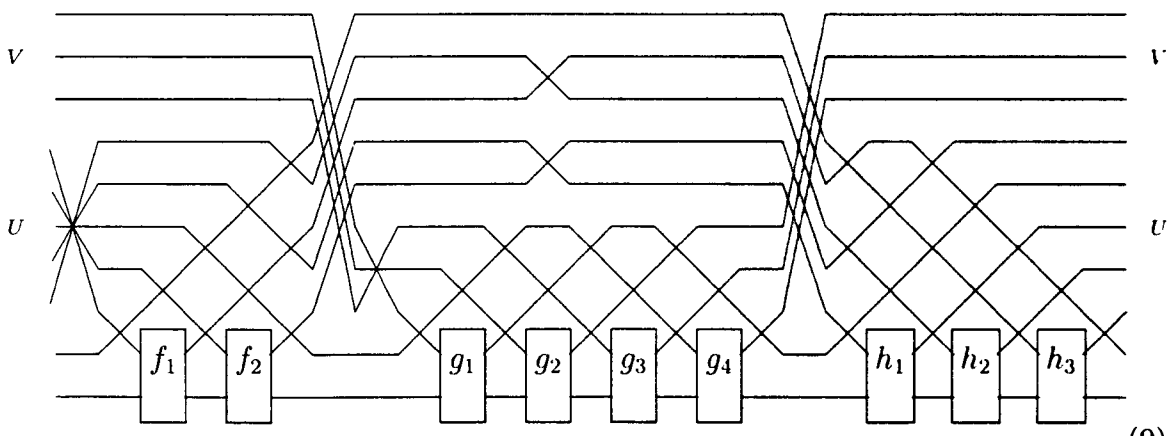
(7)

(without the dotted lines) is in the trace class $\mathbb{T}^V$. We symbolize that it is in the trace class of this type with these dotted lines. Moreover, $[\vec{f},[\vec{g}],\vec{h}]\downarrow$ means that:



(8)

without the dotted lines is in trace class $\mathbb{T}^U$. We want to obtain $[\vec{f},\vec{g},\vec{h}]$. So, for that purpose, we start by replacing the first diagram (7) traced on V into the second diagram (8). Then we apply superposition, and the naturality axiom and we get the following diagram:

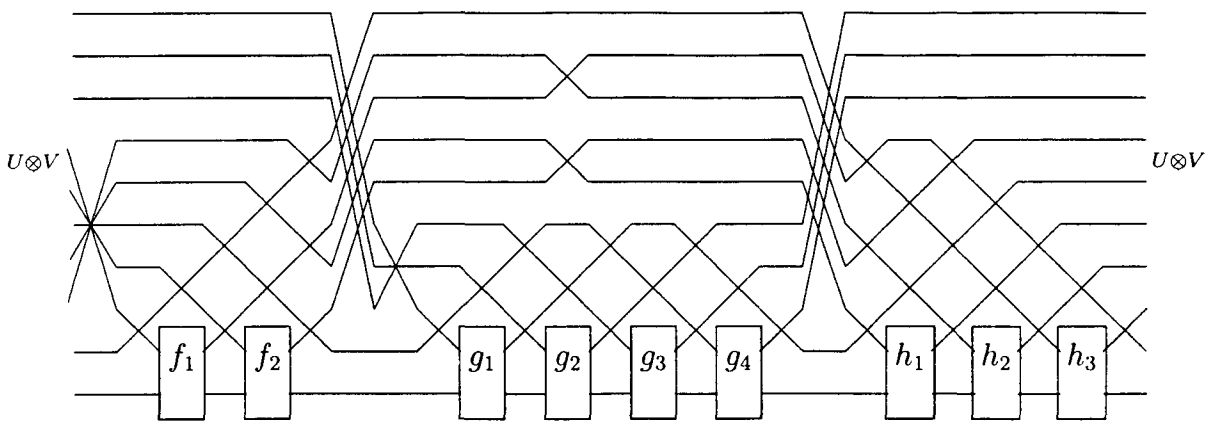


(9)

Let us call this map α (without the dotted lines). Notice that since $[\vec{g}] \downarrow$ and after applying superposing and the naturality axioms we have that $\alpha \in \mathbb{T}^V$. This turns out to be the general condition that we need in order to use the Vanishing II axiom, i.e., if we consider $\alpha \in \mathbb{T}^V$ as a general hypothesis then the equivalence

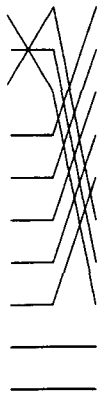
$$\alpha \in \mathbb{T}^{U \otimes V} \;\Leftrightarrow\; \mathrm{Tr}^V(\alpha) \in \mathbb{T}^U$$

is precisely the condition required to apply the Vanishing II axiom in which the condition $[\vec{f}, [\vec{g}], \vec{h}]$ translates into $\mathrm{Tr}^V(\alpha) \in \mathbb{T}^U$ and $[\vec{f}, \vec{g}, \vec{h}]$ into $\alpha \in \mathbb{T}^{U \otimes V}$. Thus we can replace the previous diagram by the next one:

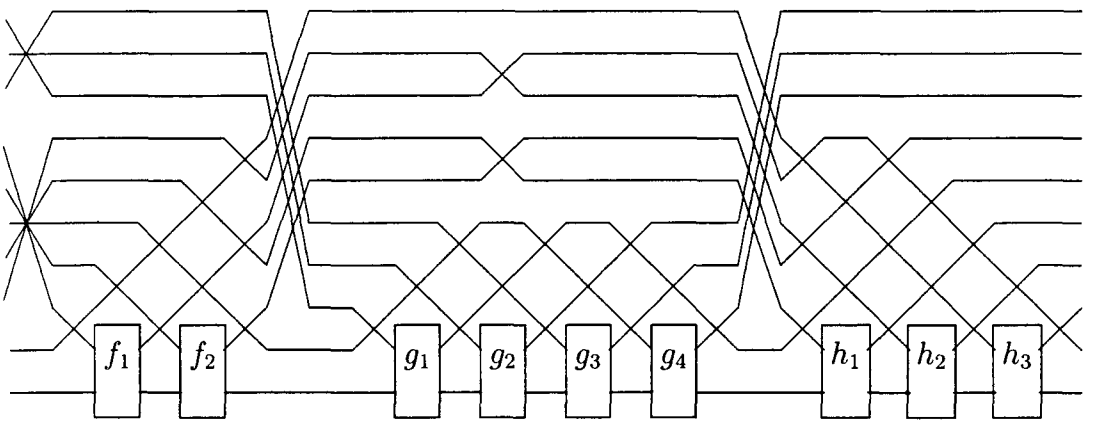


By coherence we can replace this part of the diagram:

by this one



So, by this substitution and functoriality we get:



From now, we are going to permute the objects that are traced in order to get the formula $[\vec{f}, \vec{g}, \vec{h}]$. The dinaturality axiom allows us to commute the objects that are traced by composing with a permutation and pre-composing with its inverse. For that purpose we define a permutation which will impose an order at the level of objects in such a way that creates a sequence where the objects that are connected to $\vec{g}$ follow the objects connected to $\vec{f}$ and the objects of $\vec{h}$ follow the objects of $\vec{g}$: .

$$\tau : A_1 \otimes A_2 \cdots \otimes A_n \otimes B_m \otimes C_1 \otimes \cdots \otimes C_{s-1} \otimes B_1 \otimes \cdots \otimes B_{m-1} \rightarrow$$
$$A_1 \otimes A_2 \cdots \otimes A_n \otimes B_1 \otimes \cdots \otimes B_{m-1} B_m \otimes C_1 \otimes \cdots \otimes C_{s-1}.$$

Also, by definition of our product we have the permutations associated with $[\vec{f},[\vec{g}],\vec{h}]$:

$$\gamma' : C_{s-1}\otimes C_{s-2}\otimes\ldots C_2\otimes C_1\otimes B_m\otimes A_n\ldots A_2\otimes A_1 \to A_1\otimes A_2\ldots A_n\otimes B_m\otimes C_1\otimes C_2\ldots C_{s-1}$$

and with $[\vec{g}]$:

$$\gamma'' : B_{m-1}\otimes B_{m-2}\otimes\cdots\otimes B_2\otimes B_1 \to B_1\otimes B_2\otimes\cdots\otimes B_{m-2}\otimes B_{m-1}$$

and with $[\vec{f},\vec{g},\vec{h}]$:

$$C_{s-1}\otimes C_{s-2}\otimes\ldots C_1\otimes B_m\otimes B_{m-1}\otimes\ldots B_1\otimes A_n\otimes A_{n-1}\cdots\otimes A_1 \xrightarrow{\gamma} A_1\otimes\ldots A_n\otimes B_1\cdots\otimes B_m\otimes C_1\ldots C_{s-}$$

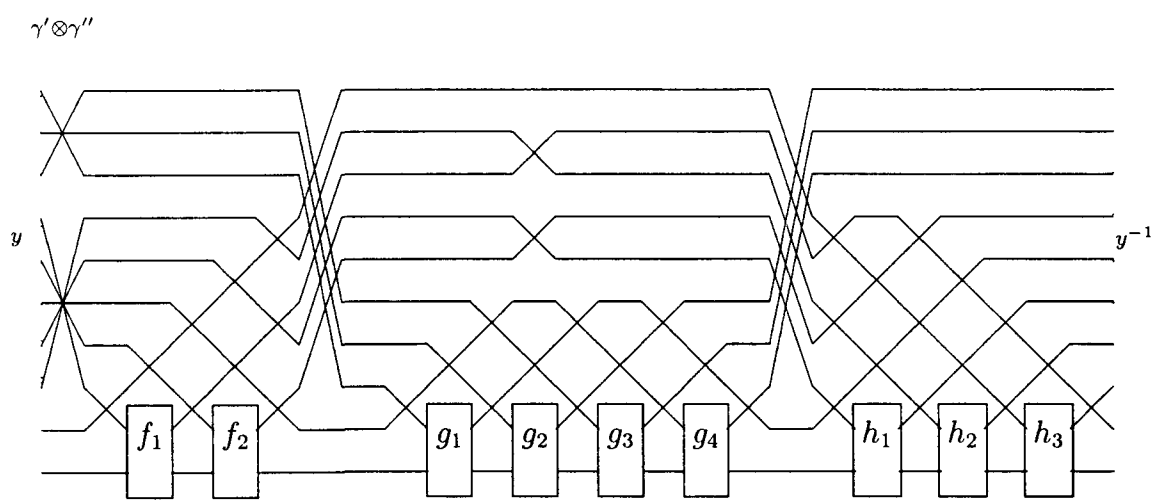
As we said, we want to compose and pre-compose with a permutation, let us call it y, for our purpose this permutation should satisfy:

$$y;(\gamma'\otimes\gamma'') = \gamma;\tau^{-1}.$$

Thus, since all this map are invertible we define:

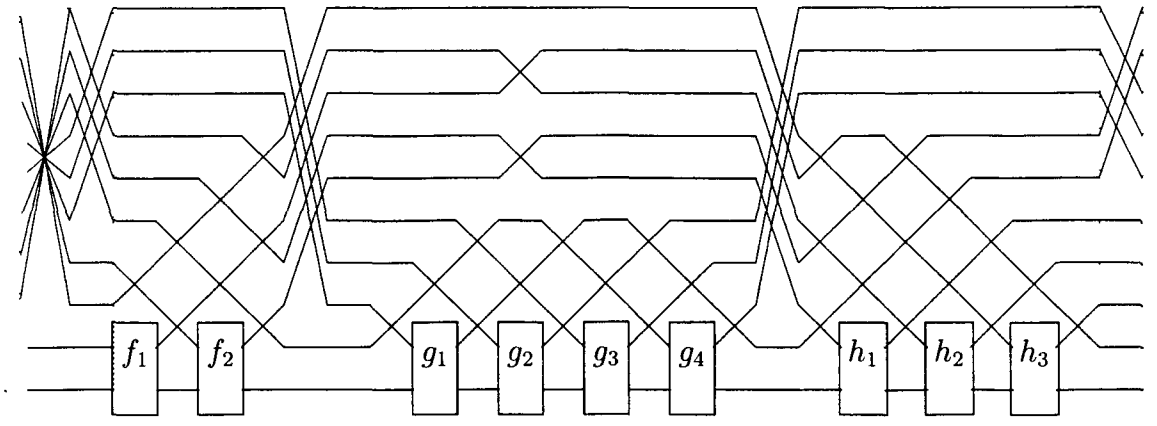
$$y \doteq \gamma;\tau^{-1};(\gamma'^{-1}\otimes\gamma''^{-1}).$$

In our concrete graphical description after applying dinaturality we get:

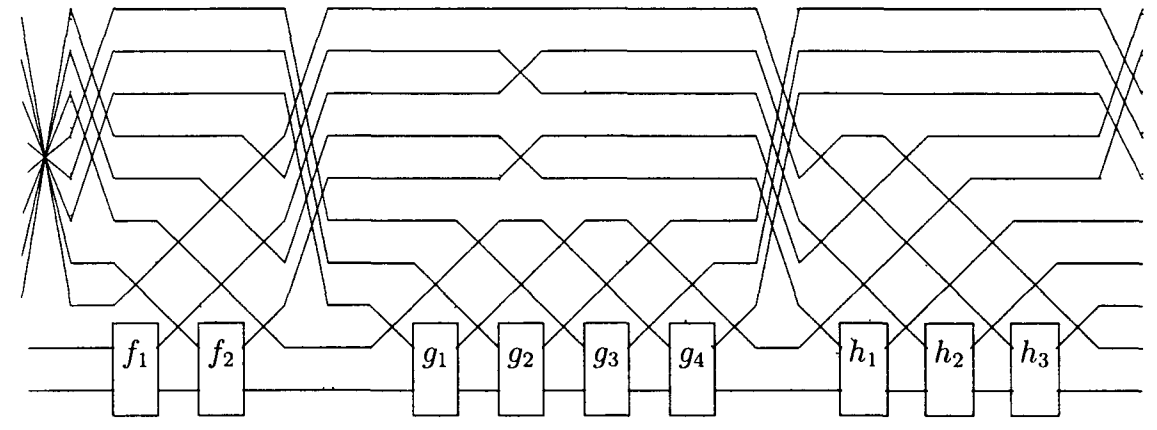


and using the equation that we defined above $y;(\gamma'\otimes\gamma'') = \gamma;\tau^{-1}$ we replace it and we obtain:

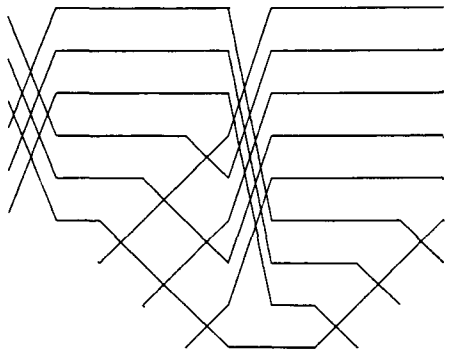
$\gamma \quad \tau^{-1}$



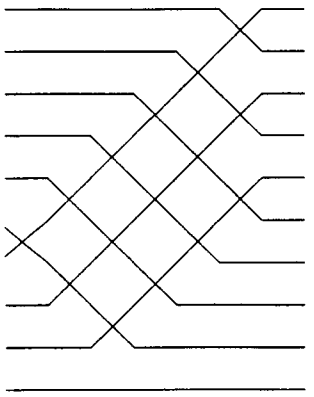
Now we split the diagram in two sets of different types of symmetries, those which are functorially free from the set of arrows $\{f_i, g_j, h_k : i, j, k\}$ and those that are not. Here, in the next diagram, the dotted boxes contain part of the free ones:



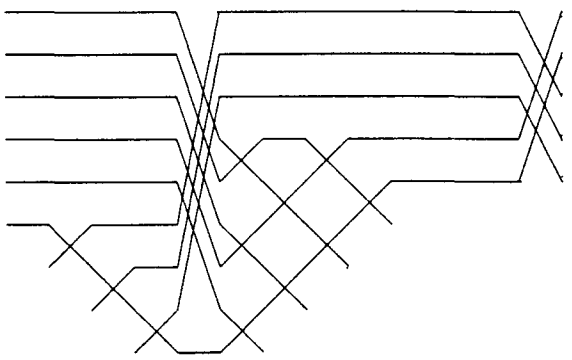
So, we replace this box:



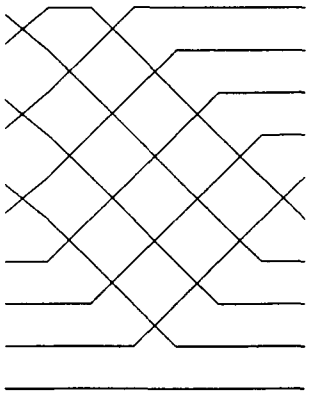
By this one:



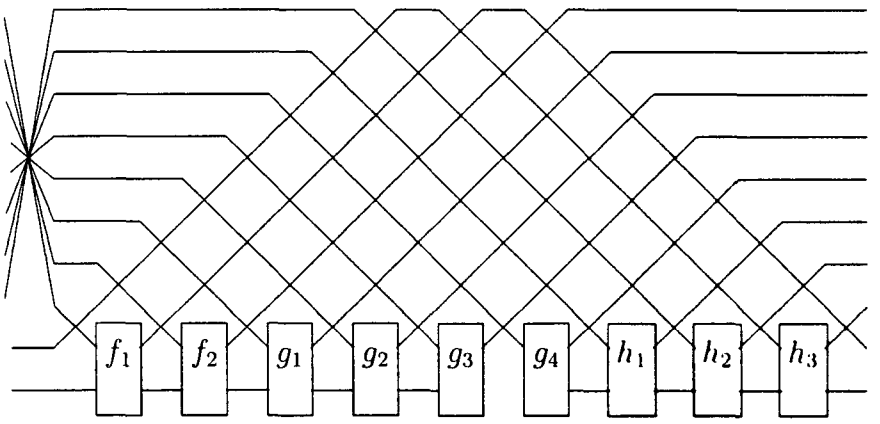
and this one:



By this other one:



Finally, we get the desired diagram:



To go from $[\vec{f}, \vec{g}, \vec{h}]$ to $[\vec{f}, [\vec{g}], \vec{h}]$ we use the same arguments in the reverse order since $[\vec{g}] \downarrow$. □

Next, we wish to show that the paracategory $Int^p(C)$ is strict symmetric monoidal.

Definition 5.6.4. Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a symmetric monoidal partially traced category, the tensor in the graph $Int^p(\mathcal{C})$ is defined as follows:

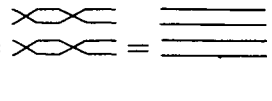
- The unit is (I, I)
- on objects: $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, B^- \otimes A^-)$

- on arrows: given $f^{Int^p} : (A^+, A^-) \to (C^+, C^-)$ and $g^{Int^p} : (B^+, B^-) \to (D^+, D^-)$ then $(f \otimes g)^{Int^p} : (A^+, A^-) \otimes (B^+, B^-) \to (C^+, C^-) \otimes (D^+, D^-)$ is defined by

$$A^+ \otimes B^+ \otimes D^- \otimes C^- \overset{s\otimes s}{\to} B^+ \otimes A^+ \otimes C^- \otimes D^- \overset{1\otimes f\otimes 1}{\to} B^+ \otimes C^+ \otimes A^- \otimes D^- \overset{s\otimes s}{\to} C^+ \otimes B^+ \otimes D^- \otimes A^- \overset{1\otimes g\otimes 1}{\to} C^+ \otimes D^+ \otimes B^- \otimes A^-.$$

Let us derive some immediate consequences of this definition:

(i)

$1_{(A^+,A^-)} \otimes 1_{(B^+,B^-)} =$  $= 1_{(A^+\otimes B^+, B^-\otimes A^-)}$

(ii) $(A^+, A^-) \otimes (I, I) = (A^+ \otimes I, I \otimes A^-) = (A^+, A^-)$ and $(I, I) \otimes (A^+, A^-) = (A^+, A^-)$.

(iii) $(A^+, A^-) \otimes ((B^+, B^-) \otimes (C^+, C^-)) = (A^+ \otimes B^+ \otimes C^+, C^- \otimes B^- \otimes A^-) = ((A^+, A^-) \otimes (B^+, B^-)) \otimes (C^+, C^-)$.

Definition 5.6.5. The symmetry $(A^+, A^-) \otimes (B^+, B^-) \overset{\sigma}{\to} (B^+, B^-) \otimes (A^+, A^-)$ is defined in $Int^p(\mathcal{C})$ by the following formula: $\sigma = s_{A^+B^+} \otimes s_{A^-B^-}$.

Lemma 5.6.6. *Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a symmetric monoidal partial traced category. Given $f^{Int^p} : (Y^+, Y^-) \to (C^+, C^-) \otimes (D^+, D^-)$, and $g^{Int^p} : (A^+, A^-) \otimes (B^+, B^-) \to (X^+, X^-)$ then $[f, \sigma] \downarrow$ and $[\sigma, g] \downarrow$.*

Proof. To simplify the notation we use the symbol ";" for the composition in the category $\mathcal{C}$ with the order given by graphical concatenation.

We first consider the composition of $f : Y^+ \otimes D^- \otimes C^- \to C^+ \otimes D^+ \otimes Y^-$ with the following symmetries and identities in the category $\mathcal{C}$: $(1_{Y^+} \otimes s_{C^-D^-}); f; (s_{C^+D^+} \otimes 1_{Y^-})$. Next, since by the yanking axiom $s_{D^-\otimes C^-, D^-\otimes C^-} \in \mathbb{T}^{D^-\otimes C^-}_{D^-\otimes C^-, D^-\otimes C^-}$ and $\mathrm{Tr}^{D^-\otimes C^-}_{D^-\otimes C^-, D^-\otimes C^-}(s_{D^-\otimes C^-, D^-\otimes C^-}) = 1_{D^-\otimes C^-}$ then by superposing axiom we have that $1_{Y^+} \otimes s_{D^-\otimes C^-, D^-\otimes C^-} \in \mathbb{T}^{D^-\otimes C^-}_{Y^+\otimes D^-\otimes C^-, Y^+\otimes D^-\otimes C^-}$ and

$$1_{Y^+} \otimes \mathrm{Tr}^{D^-\otimes C^-}_{D^-\otimes C^-, D^-\otimes C^-}(s_{D^-\otimes C^-, D^-\otimes C^-}) = \mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes D^-\otimes C^-, Y^+\otimes D^-\otimes C^-}(1_{Y^+} \otimes (s_{D^-\otimes C^-, D^-\otimes C^-})).$$

Therefore, by naturality we have that:
$(1_{Y^+} \otimes s_{C^-D^-}); \mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes D^-\otimes C^-, Y^+\otimes D^-\otimes C^-}(1_{Y^+} \otimes (s_{D^-\otimes C^-, D^-\otimes C^-})); f; (s_{C^+D^+} \otimes 1_{Y^-}) =$
$\mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes C^-\otimes D^-, Y^+\otimes D^-\otimes C^-}((1_{Y^+} \otimes s_{C^-D^-} \otimes 1_{D^-\otimes C^-}); (1_{Y^+} \otimes s_{D^-\otimes C^-D^-\otimes C^-})); f; (s_{C^+D^+} \otimes 1_{Y^-}) =$
by coherence:
$\mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes C^-\otimes D^-, Y^+\otimes D^-\otimes C^-}((1_{Y^+} \otimes s_{C^-\otimes D^-, D^-\otimes C^-}); (1_{Y^+} \otimes 1_{D^-\otimes C^-} \otimes s_{C^-D^-})); f; (s_{C^+D^+} \otimes 1_{Y^-}) =$

by the naturality axiom:
$\mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes C^-\otimes D^-, D^+\otimes C^+\otimes Y^-}((1_{Y^+} \otimes s_{C^-\otimes D^-, D^-\otimes C^-}); (1_{Y^+} \otimes 1_{D^-\otimes C^-} \otimes s_{C^-D^-}); f \otimes 1_{D^-\otimes C^-}; (s_{C^+D^+} \otimes 1_{Y^-} \otimes 1_{D^-\otimes C^-})) =$

and by functoriality:
$\mathrm{Tr}^{D^-\otimes C^-}_{Y^+\otimes C^-\otimes D^-, D^+\otimes C^+\otimes Y^-}((1_{Y^+} \otimes s_{C^-\otimes D^-, D^-\otimes C^-}); (f \otimes 1_{C^-\otimes D^-}); (s_{C^+D^+} \otimes 1_{Y^-} \otimes s_{C^-D^-})) =$

Now by coherence, we can replace:

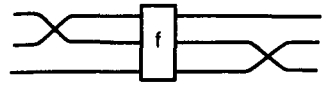
$$s_{C^+D^+} \otimes 1_{Y^-} \otimes s_{C^-D^-}$$

by the following

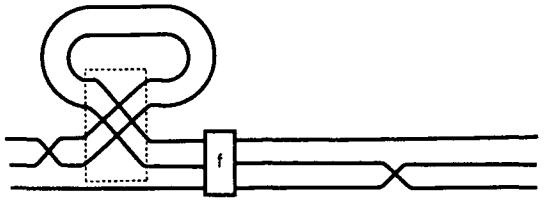
$$(1_{C^+\otimes D^+} \otimes s_{Y^-, C^-\otimes D^-}); (s_{C^+D^+} \otimes s_{C^-D^-} \otimes 1_{Y^-}); (1_{D^+\otimes C^+} \otimes s_{D^-\otimes C^-, Y^-}).$$

Which, by definition, is $[f, s_{C^+D^+} \otimes s_{C^-D^-}]$, i.e., we proved that $[f, \sigma_{(C^+,C^-),(D^+,D^-)}] \downarrow$. After repeating a similar argument as above, we have: $[s_{A^+B^+} \otimes s_{A^-B^-}, f] \downarrow$.

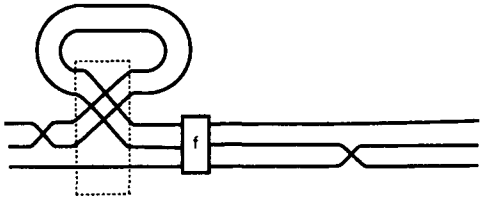
Now we repeat the proof using graphical language. The purpose of this is to persuade the reader of the advantages of using this methodology. We start with the following diagram



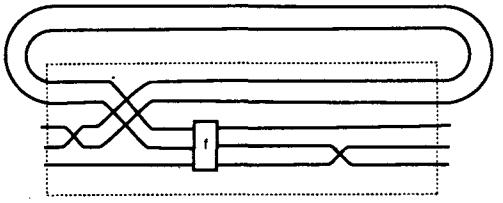
by the yanking axiom the graphic inside the box is in the trace class



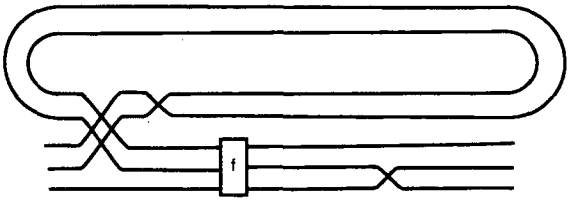
by the superposition axiom



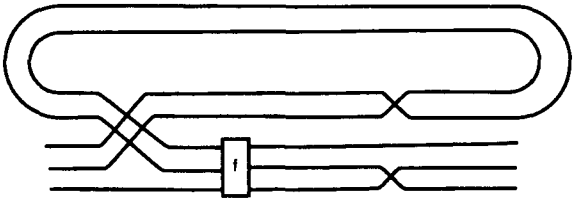
naturality axiom



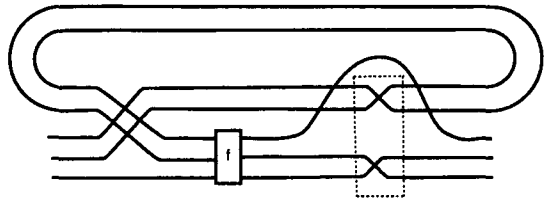
naturality of the symmetry σ



functoriality



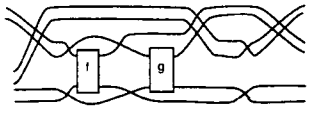
by coherence given by naturality of σ and coherence axiom in $\mathcal{C}$



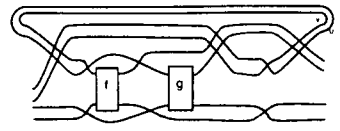
From now we use the graphical language systematically. □

Lemma 5.6.7. *σ is a natural transformation.*

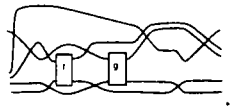
Proof. We want to prove that $\sigma \circ (f \otimes g) = (f \otimes g) \circ \sigma$. Notice that we have already proved that $(f \otimes g) \circ \sigma \downarrow$ and $\sigma \circ (f \otimes g) \downarrow$ by Lemma 5.6.6. We have by assumption that $\sigma \circ (f \otimes g)$ is defined. In the graphical language this means that $h \in \mathbb{T}^{U \otimes V}$, where h is the following diagram:



the trace given by $\mathrm{Tr}^{U \otimes V}(h) =$

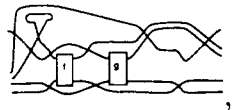


Here the issue is to justify the use of the Vanishing II axiom. Putting the matter schematically without to much emphasis on the name of the objects, we want to split the trace over $U \otimes V$ by using a general hypothesis of type $h \in \mathbb{T}^V$ and a conditional hypothesis of type $h \in \mathbb{T}^{U \otimes V}$ and we must prove that $\mathrm{Tr}^V(h) \in \mathbb{T}^U$. This is the kind of back and forward process of proof that we have repeatedly used before where the justification of the use of the axiom is also the proof that we need. Let us start by considering the following diagram:



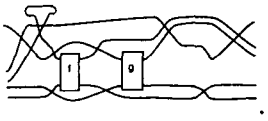
(10)

Then by the yanking axiom, which is totally defined: $\sigma \in \mathbb{T}^V$ and we can replace the former graph by this one

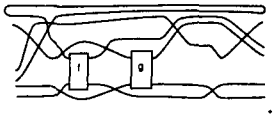


,

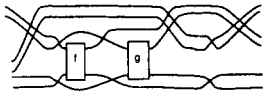
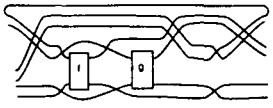
and by the superposition axiom (both versions) we have that locally the diagram satisfies that it is part of the trace class $\mathbb{T}^V$ and the graph, after tracing it out is given by



Then the naturality axiom allows us to include the full diagram in the trace class $\mathbb{T}^V$ and we are allowed also to trace it:

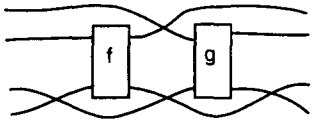


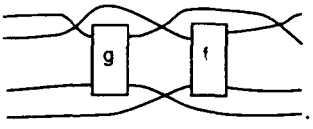
Finally, by coherence

 $\in \mathbb{T}^V$ and the trace is represented by  .

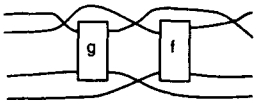
Now we are in a position where we can use the vanishing II axiom and to conclude that $\mathrm{Tr}^V(h) \in \mathbb{T}^U$ and of course the value of the trace is given by $\mathrm{Tr}^U(\mathrm{Tr}^V(h)) = \mathrm{Tr}^{U\otimes V}(h)$.

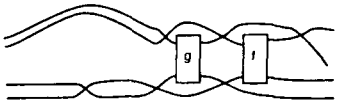
After justifying the use of the vanishing II axiom we move to ensure that both diagram are equal. First notice that the following diagrams are equivalent :

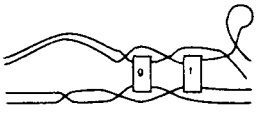


by coherence 
.

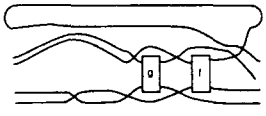
Starting with the last diagram and applying the axioms, where the existence of the trace is justified by the axiom that we are mentioning, we obtain:

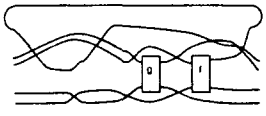


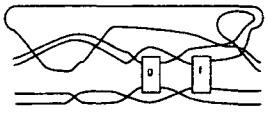
coh. 

yank. 

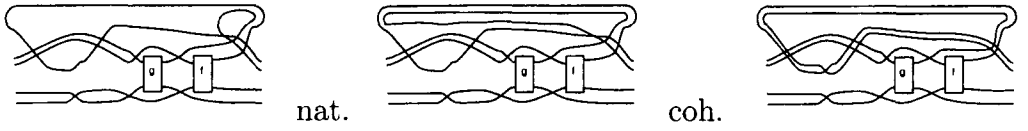
nat.



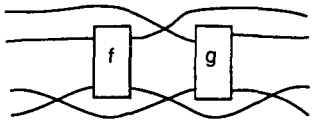
coh. 

yank., sup. 

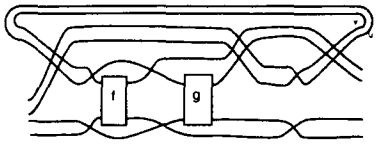
coh.



where the last diagram is of type $\mathrm{Tr}^U(\mathrm{Tr}^V(h'))$. Therefore, by the same reasoning as given at the beginning of the proof we find that $\mathrm{Tr}^{U\otimes V}(h') = \mathrm{Tr}^U(\mathrm{Tr}^V(h'))$ also for this diagram. As before we repeat our arguments to justify the existence and value of the trace for the case when we start with the graph



obtaining the following diagram:

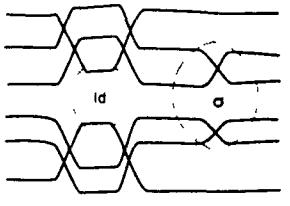


□

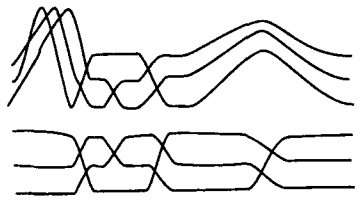
Lemma 5.6.8. $[\sigma \otimes 1, \sigma] = 1 \otimes \sigma$.

Proof. Here again, as in Lemma 5.6.7, the key point is to justify the use of the Vanishing II axiom. We will apply this strategy twice. Since by Lemma 5.6.6 $[\sigma \otimes 1, \sigma] \downarrow$ we want to be able to use an scheme proof of type: $g \in \mathbb{T}^{U\otimes V\otimes W}$ iff $\mathrm{Tr}^W(g) \in \mathbb{T}^{U\otimes V}$, but for this we need an hypothesis of type $g \in \mathbb{T}^W$. To justify this, we want to split the trace over $U \otimes V$ by using a general hypothesis of type $h \in \mathbb{T}^V$ and a conditional hypothesis of type $h \in \mathbb{T}^{U\otimes V}$ and we must prove that $\mathrm{Tr}^V(h) \in \mathbb{T}^U$.

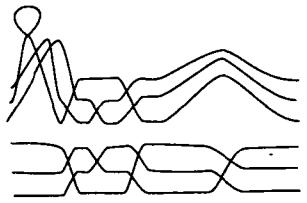
We start with the following diagram that represents $1 \otimes \sigma$:



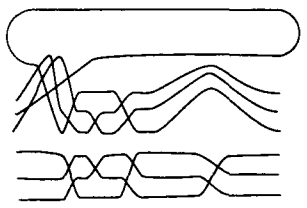
which by coherence is equivalent to the following



Therefore, by yanking, (let us name the variable U) we have

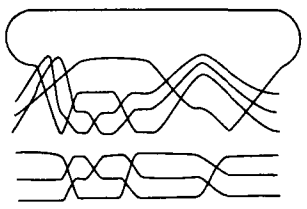


Then by the naturality and the superposition axioms we obtain that it is equal to the trace represented by:



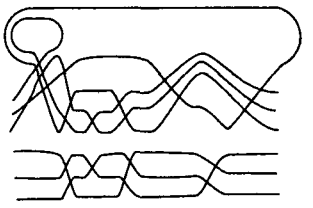
in which the diagram below the trace, let us call it h, satisfies $h \in \mathbb{T}^U$. Notice that this is true because our axioms of partially traced category allow us to entail this last statement.

Now, by coherence we have that is equal to

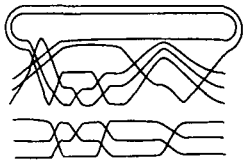


Let us still call h the new graph below the trace. By yanking with respect to a variable

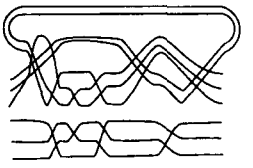
V



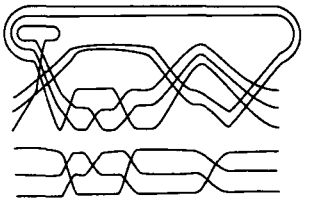
Now again by naturality, superposition and coherence we conclude that the graph below the trace, name it h', is in the trace class $\mathbb{T}^V$. Moreover, the value of the trace along V is equal to h, i.e., $\mathrm{Tr}^V(h') = h$, which implies that is in the trace class $\mathbb{T}^U$, this means that we are allowed to use vanishing II and to conclude that $h' \in \mathbb{T}^{U \otimes V}$:



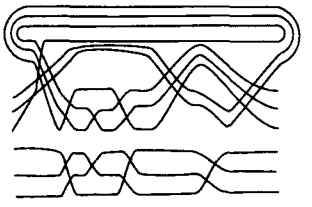
and coherence



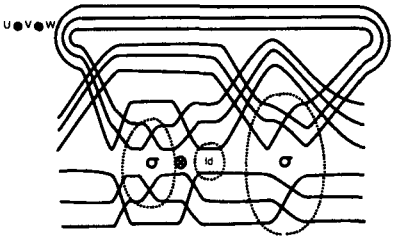
Now we repeat the idea with a new parameter W.



and



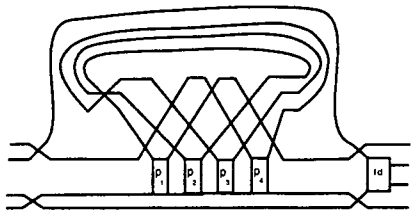
Hence, this yields after applying vanishing II again



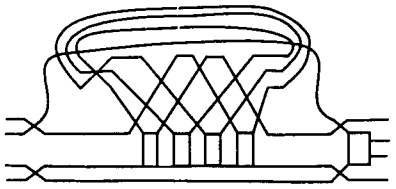
which represents $[\sigma \otimes 1, \sigma]$. □

Lemma 5.6.9. $1 \otimes [\vec{p}] \coloneqq [1 \otimes \vec{p}]$ *and* $[\vec{p}] \otimes 1 \coloneqq [\vec{p} \otimes 1]$.

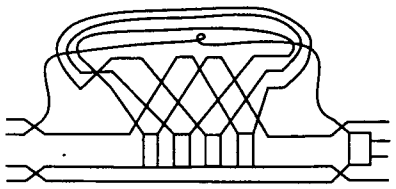
Proof. Without loss of generality we consider the case when $\vec{p} = p_1, p_2, p_3, p_4$. By definition $[\vec{p}] \otimes 1$ is equal to:



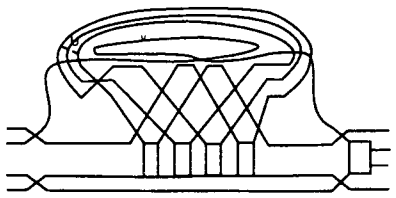
Then using superposing axiom we obtain:



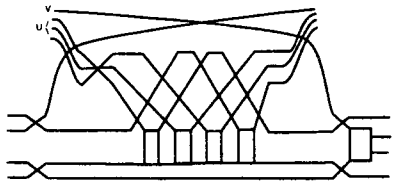
and since by the yanking axiom $\mathrm{Tr}(\sigma) = 1$, we have that:



Now by the fact that the trace is defined on symmetries this is the hypothesis that I need in order to apply superposing (equivalent version) axiom, thus by the same reason we can apply also the naturality axiom:

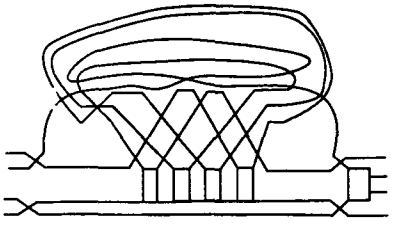


We name g the diagram without being traced, i.e., g is

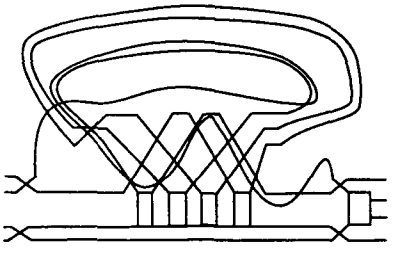


Then $g \in \mathbb{T}^V$ by the reasons given above and if we reverse this procedure in fact we are showing that $\mathrm{Tr}^V(g) \in \mathbb{T}^U$ (after applying superposition, yanking and

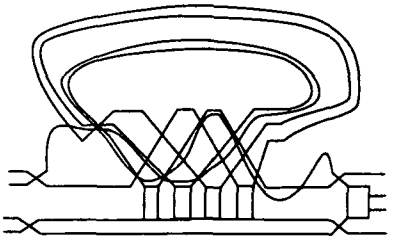
naturality and returning to the very beginning of the proof) thus we are satisfying the hypothesis of Vanishing II which means that $g \in \mathbb{T}^{U \otimes V}$.
Now we are allowed to apply the dinaturality axiom in order to permute the order of the objects that are going to be traced out:



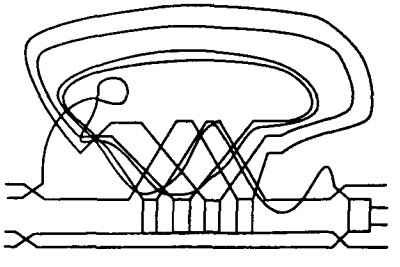
Thus by coherence we have that:



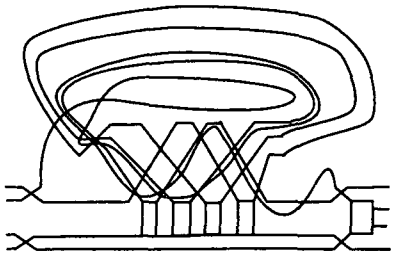
Again by coherence:



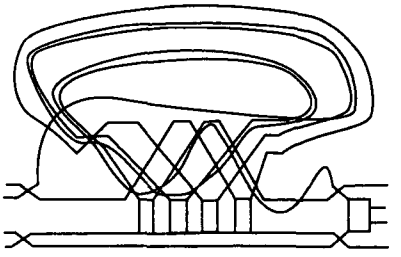
Now by coherence and the yanking axiom:



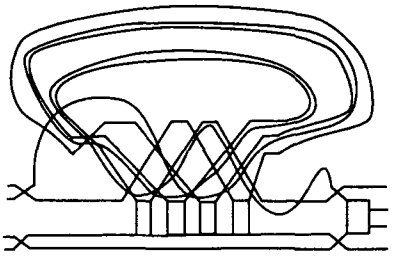
Again since the trace is total on symmetries, and after applying superposing (equivalent version), the naturality axiom shows that:



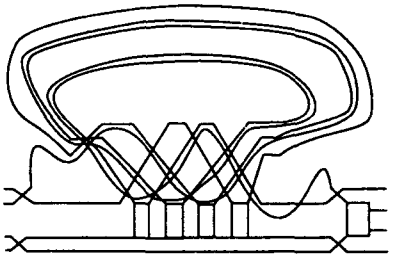
In the same way as before we repeat what we did but now applied to the second line:



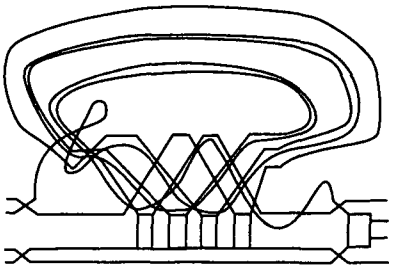
Because the map involve are coherence maps:



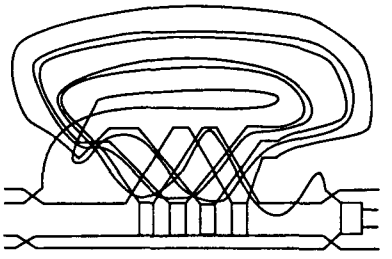
Coherence:



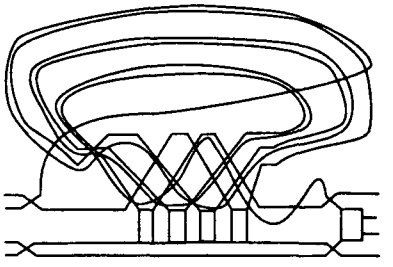
Coherence and the yanking axiom:



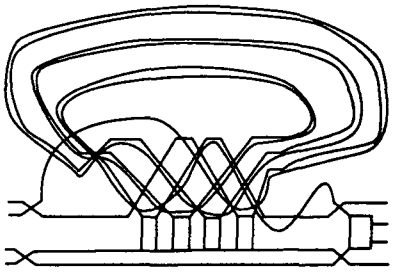
Superposition and naturality:



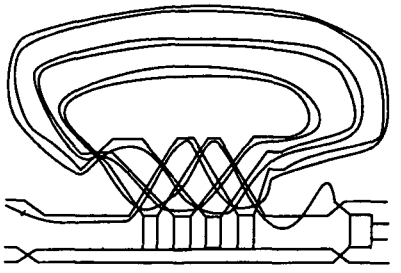
Same argument as before applied to the third line:



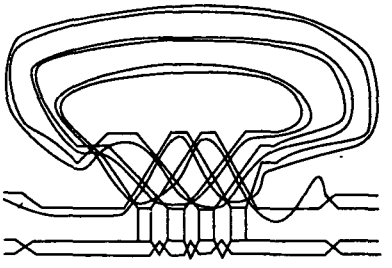
Coherence allows us to express:



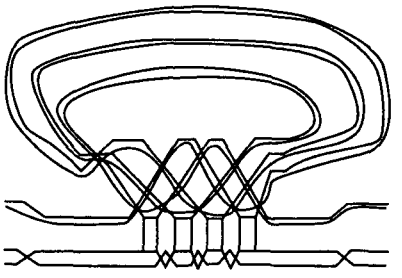
We therefore have again by coherence:



Since $\sigma \circ \sigma^{-1} = 1$:



Finally by coherence we get:



Which is by definition $[\vec{p} \otimes 1]$. □

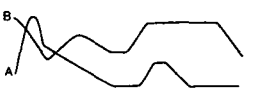
Theorem 5.6.10. *Let $(\mathcal{C}, \otimes, I, \mathrm{Tr}, s)$ be a symmetric monoidal partially traced category. The operation defined above $[-]$ determines a ssmpc $(Int^p(\mathcal{C}), [-], \otimes, I, \sigma)$.*

Proof. It follows from the previous lemmas. □

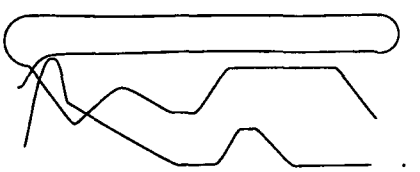
Next, we wish to show that $Int^p(\mathcal{C})$ is a compact closed paracategory. Let $(I, I) \xrightarrow{\eta} (A, B) \otimes (A, B)^*$ and $(A, B)^* \otimes (A, B) \xrightarrow{\varepsilon} (I, I)$ be the unit and counit associated to the paracategory $Int^p(\mathcal{C})$. Actually, since $\mathcal{C}$ is a strict category, we can regard these morphisms as $id : I \otimes A \otimes B \to A \otimes B \otimes I$ and $id : B \otimes A \otimes B \to I \otimes B \otimes A$ respectively.

Lemma 5.6.11. $[\eta\otimes 1, 1\otimes\varepsilon]\downarrow$, $[1\otimes\eta, \varepsilon\otimes 1]\downarrow$ *and* $[\eta\otimes 1, 1\otimes\varepsilon] = 1$, $[1\otimes\eta, \varepsilon\otimes 1] = 1$.

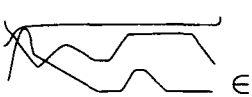
Proof. Notice that $\sigma_{A,I} = id_A$ for every object $A \in \mathcal{C}$. We start with the identity map $(A, B) \xrightarrow{1} (A, B)$ which is the map $1 : A \otimes B \to A \otimes B$ in $\mathcal{C}$. Since,

$1_{A\otimes B} =$ 

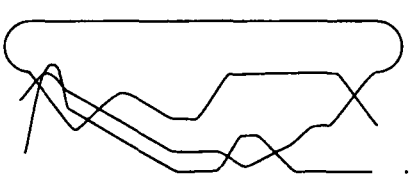
holds by coherence, using the yanking axiom

and naturality 

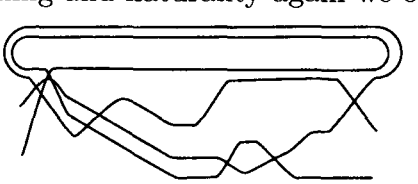
Notice that, all along this proof, we implicitly claim that the graph below the trace is in the corresponding trace class. For instance, in the last diagram from the naturality axiom it follows that

 $\in \mathbb{T}^B$.

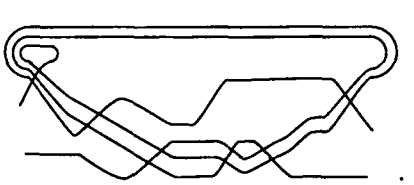
Then by superposing axiom and coherence



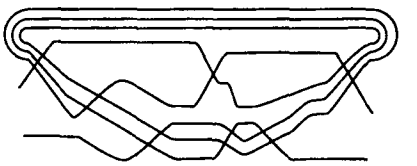
Therefore, by applying yanking and naturality again we obtain



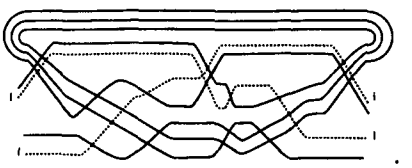
where the graph below this new trace is in the trace class $\mathbb{T}^A$ and this, of course, will be preserved by any coherent modification of the graph. We have from superposition and coherence axioms that



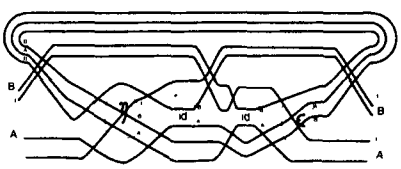
Again, naturality, superposition and coherence gives us



where the graph below the trace is in the trace class $\mathbb{T}^B$. Since $\mathcal{C}$ is a strict category then is equal to



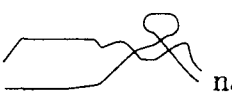
Finally, since the trace class conditions for applying vanishing II are satisfied, we apply the vanishing II axiom twice and we obtain that is equal to

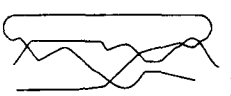
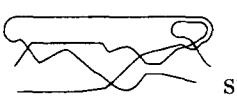
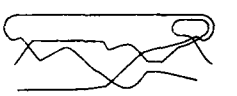


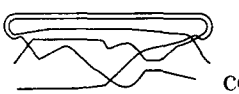
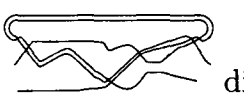
$= [\eta \otimes 1, 1 \otimes \varepsilon]$.

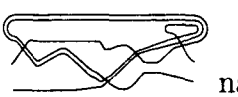
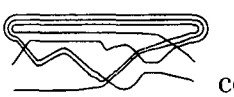
In the same way as before we prove that $[1 \otimes \eta, \varepsilon \otimes 1] = 1$.

We sketch schematically the rest of the proof leaving details to the reader. We start with the identity $1_{B \otimes A}$.

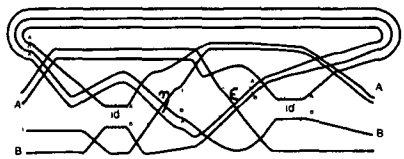
coherence:  yanking:  naturality: 

coh.:  yanking:  superposing: 

nat.:  coh.:  dinaturality: 

yanking:  naturality:  coherence: 

vanishing II:



□

Corollary 5.6.12. *Let $\mathcal{C}$ be partially traced. Then $Int^p(\mathcal{C})$ is a compact closed paracategory.*

Proof. This is a consequence of Lemma 5.6.11. □

Our final result for this section is that there exists a full and faithful, trace preserving functor from $\mathcal{C}$ to $Int^p(\mathcal{C})$.

Definition 5.6.13. In a similar way as done in [41], we define a fully faithful functor between paracategories $N : \mathcal{C} \to Int^p(\mathcal{C})$ defined by $N(A) = (A, I)$ and $N(f) = f$ by strictness of the category $\mathcal{C}$.

Lemma 5.6.14. *N is a well-defined, full and faithful functor of paracategories.*

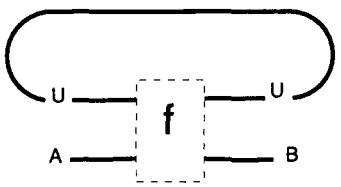
Proof. To prove well-definedness, note that we are considering the category $\mathcal{C}$ as a paracategory with composition $[f_1, \ldots, f_n] = f_n \circ \ldots \circ f_1$ as its partial operation, and $[-]'$ the partial composition defined in $Int^p(\mathcal{C})$. Thus, $N([\vec{f}]) = [N(\vec{f})]'$, since by the Vanishing I axiom, the trace operator is totally defined when we restrict it to this type of arrows i.e., $\mathbb{T}^I_{A,B} = \mathcal{C}(A \otimes I, B \otimes I)$ and $\mathrm{Tr}^I_{A,B}(f) = f$.

By definition, $N(f) = N(g)$ implies $f = g$, which proves faithfulness. If we take and arrow in $Int^p((A, I), (B, I))$, let us say for example $f : (A, I) \to (B, I)$, which really means in $\mathcal{C}$ an arrow of type $f : A \otimes I \to B \otimes I$, then we just choose the same f obtaining $Nf = f$. This proves fullness. □

Lemma 5.6.15. *The functor $N : \mathcal{C} \to Int^p(\mathcal{C})$ preserves the trace, i.e., if $f : A \otimes U \to B \otimes U$ is in $\mathbb{T}^U_{A,B}$ then $N(\mathrm{Tr}^U_{A,B}(f)) = \mathrm{Tr}^{NU}_{NA,NB}(Nf) : (A, I) \to (B, I)$ which means*

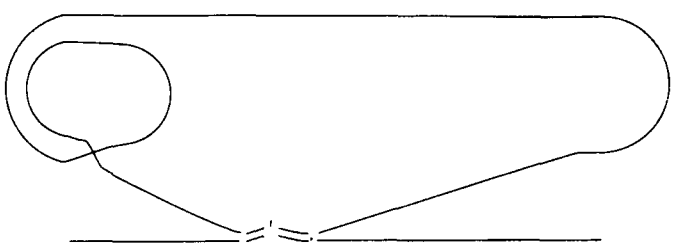
$$N(\mathrm{Tr}^U_{A,B}(f)) = [1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon].$$

Proof. Let us start with $N(\mathrm{Tr}^U_{A,B}(f)) : A \otimes I \to B \otimes I$ in $\mathcal{C}$ which is represented by



Notice that by hypothesis we have $f \in \mathbb{T}^U$. Let us call this hypothesis: condition (A).

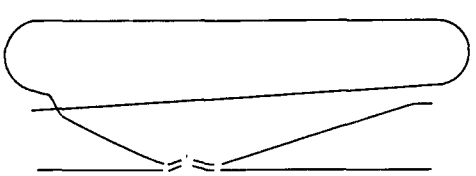
By the yanking axiom $\sigma_{U,U} \in \mathbb{T}^U$ where the trace is locally represented by



and by applying superposing axiom $\sigma \otimes 1_A \in \mathbb{T}^U$ and then by applying the naturality axiom we obtain that the full diagram below this trace is in $\mathbb{T}^U$ (let us call it condition (B)), i.e.,

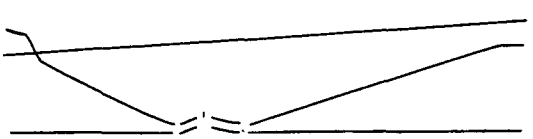
$\in \mathbb{T}^U$.

The trace of this graph is equal to f which implies by condition (A) that is in $\mathbb{T}^U$ i.e.,



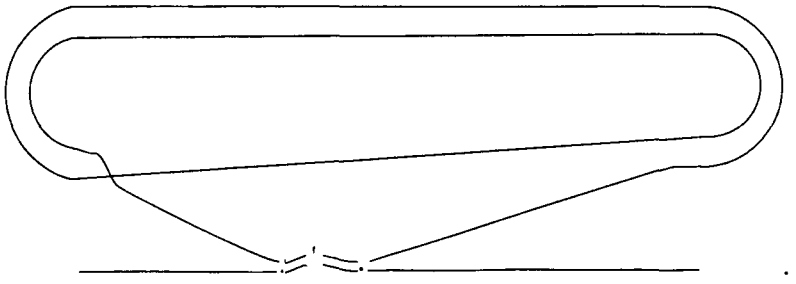
$\in \mathbb{T}^U$.

From condition (A) and (B) and the vanishing II axiom we conclude that

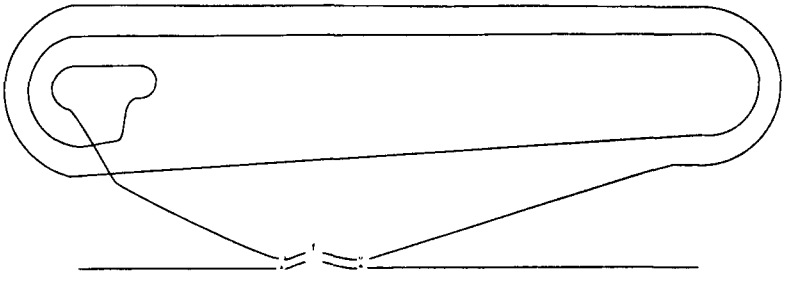


$\in \mathbb{T}^{U \otimes U}$

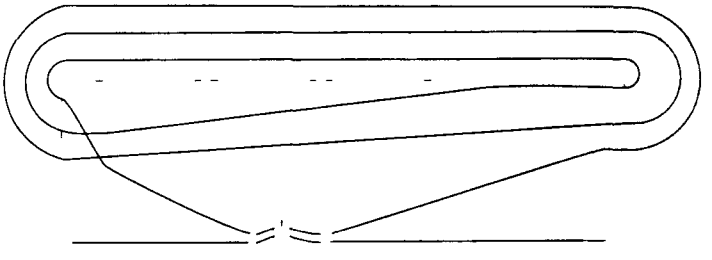
(let us call it condition $A + B$) and the trace is represented by:



We repeat this operation, by yanking:

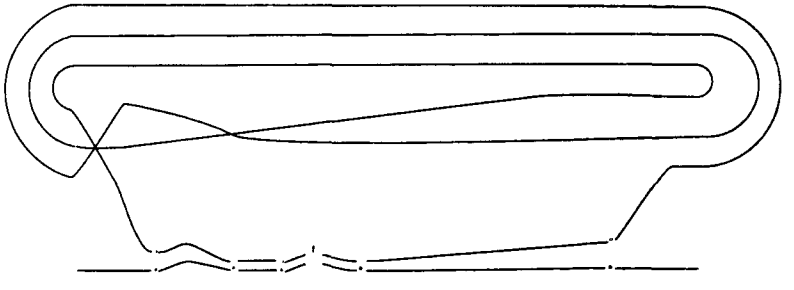


and naturality we obtain that the diagram in the dotted box:

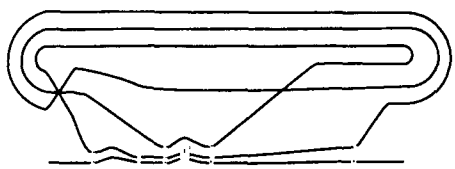
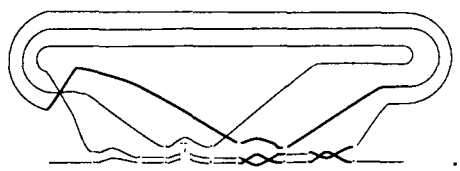


is in $\mathbb{T}^U$.

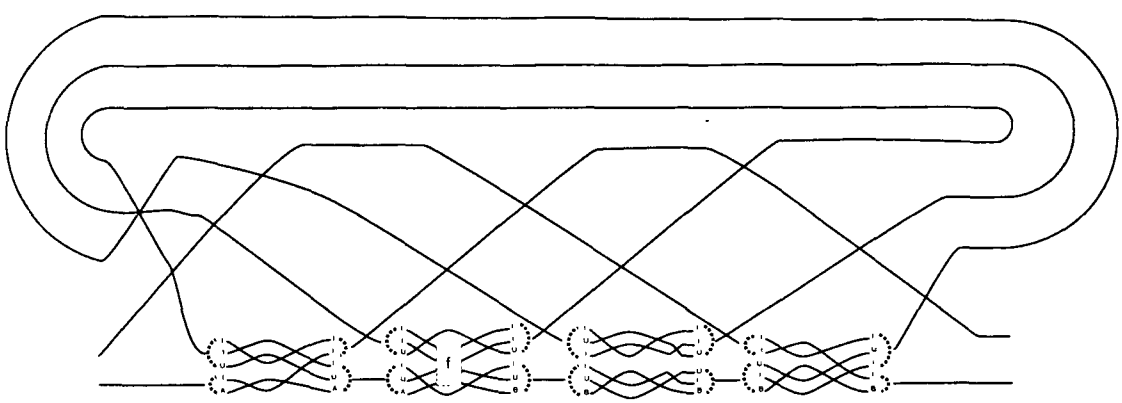
Hence, after any further coherent change we made in the graph, it will remain in the trace class $\mathbb{T}^U$. Let us call it condition (C); where the trace will be represented by



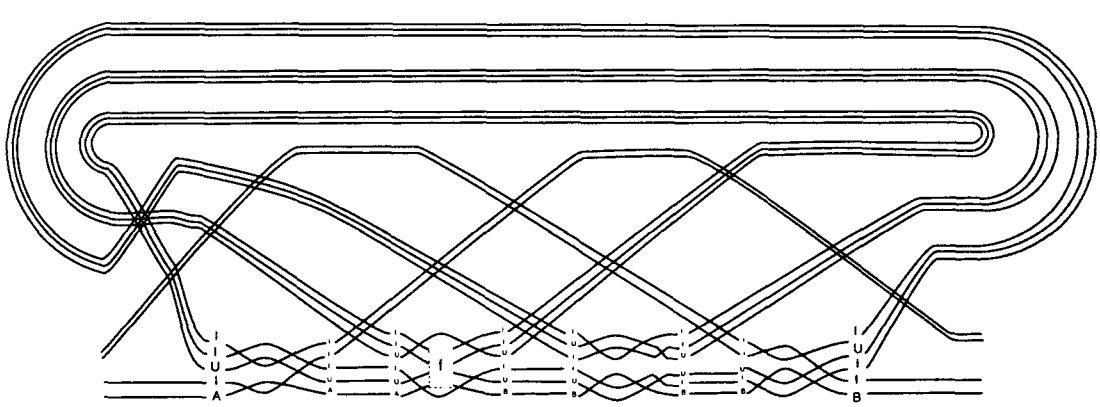
coherence:

 and  .

In the same way as above: by condition $A + B$, C and the vanishing II axiom we obtain that the graph is in the trace class $\mathbb{T}^{U \otimes U \otimes U}$ and the trace given by



Now, since $\mathcal{C}$ is a strict category we can represent the last diagram in the following way:



which is equal to $[1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon]$.

□

5.7 Representation theorem for partially traced categories

Theorem 5.7.1. *Every (strict) symmetric partially traced category can be faithfully embedded in a totally traced category.*

Proof. This follows from the various lemmas. Let $\mathcal{C}$ be a strict symmetric partially traced category. By Lemmas 5.6.14 and 5.6.15, $\mathcal{C}$ can be faithfully embedded in a compact closed paracategory $Int^p(\mathcal{C})$, and the embedding is trace preserving. By Lemma 5.4.2, $Int^p(\mathcal{C})$ can be faithfully embedded in a compact closed category $\mathcal{P}(Int^p(\mathcal{C}))/\sim$ (and the embedding preserves the compact closed structure, hence the trace). Since $\mathcal{P}(Int^p(\mathcal{C}))/\sim$ is compact closed, it is totally traced, which proves the theorem. □

Remark 5.7.2. Notice that by the Lemma 5.6.15 above if $f : A \otimes U \to B \otimes U$ is in $\mathbb{T}^U_{A,B}$ then $[1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon] \downarrow$; therefore the projection functor

$$F : Int^p(\mathcal{C}) \to \mathcal{P}(Int^p(\mathcal{C}))/\sim$$

also preserves the trace $F(\mathrm{Tr}^U_{A,B}(f)) = \mathrm{Tr}^{FU}_{FA,FB}(Ff)$ since we have that
$F(\mathrm{Tr}^U_{A,B}(f)) = F[1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon] = \overline{[1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon]} = \overline{1 \otimes \eta; f \otimes 1; 1 \otimes \sigma; 1 \otimes \varepsilon} = \overline{1 \otimes \eta} \circ \overline{f \otimes 1} \circ \overline{1 \otimes \sigma} \circ \overline{1 \otimes \varepsilon} = \overline{1}\hat{\otimes}\overline{\eta} \circ \overline{f}\hat{\otimes}\overline{1} \circ \overline{1}\hat{\otimes}\overline{\sigma} \circ \overline{1}\hat{\otimes}\overline{\varepsilon} = \mathrm{Tr}^{FU}_{FA,FB}(Ff)$.

5.8 Universal property

The category $(\mathcal{P}(Int^p(\mathcal{C}))/\sim, \hat{\otimes}, I, s)$ satisfies the following universal property.

Proposition 5.8.1. *Let $\mathcal{C}$ be a partially traced category and $\mathcal{D}$ a compact closed category. If $F : \mathcal{C} \to \mathcal{D}$ is a strict monoidal traced functor then there exists a unique monoidal functor $L : \mathcal{P}(Int^p(\mathcal{C}))/\sim \to \mathcal{D}$ such that*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\hat{N}"] \arrow[dr, "F"'] & \mathcal{P}(Int^p(\mathcal{C}))/\sim \arrow[d, "L"] \\ & \mathcal{D} \end{tikzcd}$$

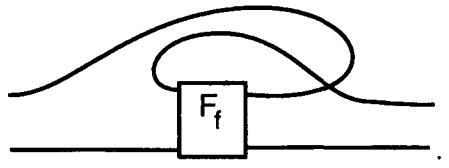
where $\hat{N}$ *is* $\mathcal{C} \xrightarrow{N} Int^p(\mathcal{C}) \xrightarrow{\pi} \mathcal{P}(Int^p(\mathcal{C}))/\sim$

Proof. We first construct a monoidal functor $K : Int^p(\mathcal{C}) \to \mathcal{D}$ such that $K \circ N = F$. This functor is defined in the same way as in [41], and is in fact unique.

On objects $K(A, U) = FA \otimes (FU)^*$ and given $(A, U) \xrightarrow{f} (B, V)$ we define $K(f)$ as

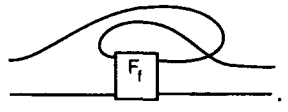
$$FA \otimes FU^* \xrightarrow{1\otimes\eta\otimes 1} FA \otimes FV \otimes FV^* \otimes FU^* \xrightarrow{Ff\otimes 1} FB \otimes FU \otimes FV^* \otimes FU^* \xrightarrow{(1\otimes\sigma\otimes 1)\circ(1\otimes\varepsilon\sigma)} FB \otimes FV^*$$

Graphically this is represented by the following diagram



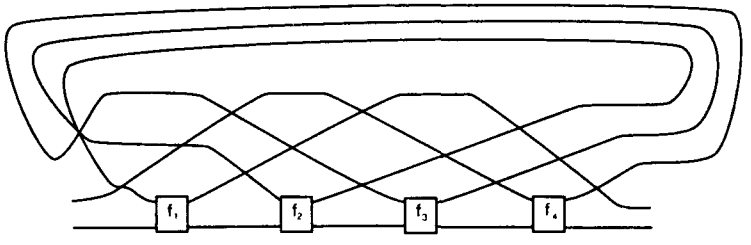
We need to prove that K is a functor between paracategories, i.e., if $[f_1, \ldots f_n] \downarrow$ then $K[f_1, \ldots f_n] = [Kf_1, \ldots Kf_n]$. The remaining properties of K are proved as in [41].

Without loss of generality we take $n = 4$. Therefore we have

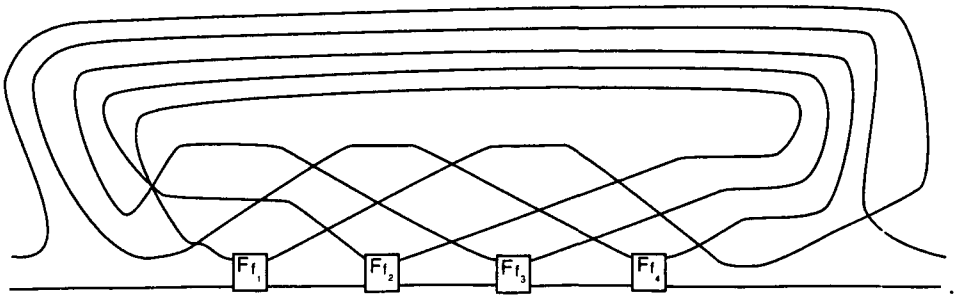
$K[f_1, \ldots f_4] =$ 

. (11)

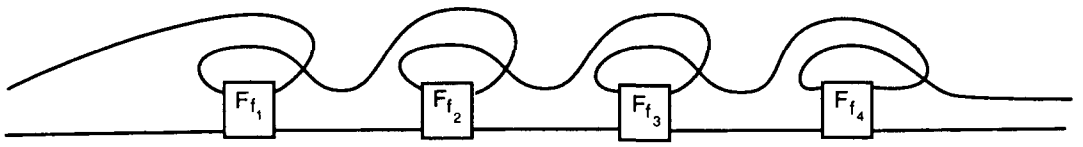
where

$f =$ 

Since F preserves the trace, composition and symmetries we have that equation (11) is equal to the following diagram



Notice that the category $\mathcal{D}$ is compact closed and its trace is totally defined and given by composition of unit η, counit ε, symmetries σ and arrows Ff_i in $\mathcal{D}$, $i = 1 \ldots 4$. Therefore, by coherence in $\mathcal{D}$, we transform the previous diagram into



i.e., $[Kf_1, \ldots Kf_4]$.

Given K, we use Theorem 5.5.1 to obtain a unique L such that:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "N"] \arrow[drr, "F"'] & Int^p(\mathcal{C}) \arrow[r, "\pi"] \arrow[dr, "K"] & \mathcal{P}(Int^p(\mathcal{C}))/\sim \arrow[d, "L"] \\ & & \mathcal{D} \end{tikzcd}$$

Uniqueness: Suppose $L' : \mathcal{P}(Int^p(\mathcal{C}))/\sim \to \mathcal{D}$ is another monoidal functor such that $L' \circ \pi \circ N = F$. Then $K' = L' \circ \pi$ satisfies $K' \circ N = F$ so by uniqueness of K, it follows that $K = K'$. But then $L' \circ \pi = K$, and by uniqueness of L, we have $L = L'$.

$\square$

Chapter 6

Background material on presheaf categories

Here we review some of the basic and advanced concepts of functor categories that will be used in Chapters 7 and 8. For additional details, see [54], [15], [51], [46].

6.1 Universal arrows, representable functors, and the Yoneda Lemma

Definition 6.1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and $B \in \mathcal{B}$. A pair (A, f) where $A \in \mathcal{A}$ and $f : B \to F(A)$ is said to be a *universal arrow* from B to F when for every arrow $f' : B \to F(A')$ there is a unique arrow $g : A \to A'$ in the category $\mathcal{A}$ such that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "f'"'] & F(A) \arrow[d, "F(g)"] \\ & F(A') \end{tikzcd}$$

is a commutative diagram.

Definition 6.1.2. A *universal element* of the functor $F : \mathcal{A} \to \mathbf{Set}$ is an object $A \in \mathcal{A}$ and an element $x \in F(A)$ such that for any other pair $A' \in \mathcal{A}$ and $x' \in F(A')$ there exists a unique $f : A \to A'$ that satisfies $F(f)(x) = x'$.

Definition 6.1.3. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. The category $El(F)$ of *elements* is given by the following data:

(a) objects of $El(F)$ are pairs (C, x) where $x \in FC$ and $C \in |\mathcal{C}|$.

(b) morphisms $f : (C, x) \to (D, y)$ are arrows $f : C \to D$ in the category $\mathcal{C}$ such that $Ff(x) = y$.

Definition 6.1.4. An object $A \in \mathcal{A}$ is said to be the *representing object* of a functor $F : \mathcal{A} \to \mathbf{Set}$ when there is a natural isomorphism ϕ:

$$\mathcal{A}(A, -) \xrightarrow{\phi} F.$$

When this occurs we said that F is a *representable functor.* There is a distinguished element of this isomorphism $\phi_A(1_A) \in F(A)$, which is called the *unit* of the representation.

Theorem 6.1.5 (The Yoneda Lemma). *Let $F : \mathcal{A} \to \mathbf{Set}$ be a functor, $A \in \mathcal{A}$. There exists a bijection*

$$\vartheta_{F,A} : [\mathcal{A}, \mathbf{Set}](\mathcal{A}(A, -), F) \to F(A)$$

which is natural in A and if $\mathcal{A}$ is a small category ϑ is natural in F.

Proof. [15] □

Theorem 6.1.6. *Let $F : \mathcal{A} \to \mathbf{Set}$ be a functor, F is representable iff it has a universal element.*

Proof. [54] □

6.2 Limits and colimits

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. For every object $A \in \mathcal{A}$ the constant functor is defined to be $\Delta_A : \mathcal{B} \to \mathcal{A}$ with $\Delta_A(B) = A$ and $\Delta_A(f) = 1_A$ when $B \xrightarrow{f} B'$. If $A \xrightarrow{g} A'$ is an arrow in $\mathcal{A}$ there is a natural transformation $\Delta(g) : \Delta_A \Rightarrow \Delta_{A'}$

defined $(\Delta(g))(B) = g$. These functors and natural transformations define a functor $\Delta : \mathcal{A} \to [\mathcal{B}, \mathcal{A}]$.

Let $F : \mathcal{J} \to \mathcal{A}$ be a functor. The definition of limits and colimits can be characterized by objects that represent the following functors:

$$\mathcal{A}(-, \lim F) \cong [\mathcal{J}, \mathcal{A}](\Delta -, F) : \mathcal{A}^{op} \to \mathbf{Set} \tag{12}$$

and

$$\mathcal{A}(\operatorname{colim} F, -) \cong [\mathcal{J}, \mathcal{A}](F, \Delta -) : \mathcal{A} \to \mathbf{Set}. \tag{13}$$

To see this, suppose we have $\mathcal{A}(-, \lim F) \xrightarrow{\phi} [\mathcal{J}, \mathcal{A}](\Delta -, F)$. Then $\phi_{limF}(1_{limF}) : \Delta_{limF} \Rightarrow F$ is a cone determined by the universal element. If $\Delta \overset{\alpha}{\Rightarrow} F$ is another cone then $\phi_A^{-1}(\alpha) : A \to lim\, F$ is an arrow on the category $\mathcal{A}$ such that by naturality we have:

$$\begin{tikzcd}
\mathcal{A}(\lim F, \lim F) \arrow[r, "\phi_{limF}"] \arrow[d, "\mathcal{A}(\phi_A^{-1}(\alpha), \lim F)"'] & {[\mathcal{J}, \mathcal{A}](\Delta \lim F, F)} \arrow[d, "{[\mathcal{J}, \mathcal{A}](\Delta(\phi_A^{-1}(\alpha)), F)}"] \\
\mathcal{A}(A, \lim F) \arrow[r, "\phi_A"] & {[\mathcal{J}, \mathcal{A}](\Delta A, F)}
\end{tikzcd}$$

which implies by evaluating at $1_{\lim F}$ that:

$$\phi_{limF}(1_{limF}) \circ \Delta(\phi_A^{-1}(\alpha)) = \phi_A(\phi_A^{-1}(\alpha)) : \Delta A \Rightarrow F.$$

Graphically:

$$\begin{tikzcd}
\Delta A \arrow[r, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\Delta(\phi_A^{-1}(\alpha))"'] & F \\
& \Delta \lim F \arrow[u, Rightarrow, "\phi_{limF}(1_{limF})"']
\end{tikzcd}$$

Therefore, evaluating at $i \in \mathcal{J}$:

$$\begin{tikzcd}
A \arrow[r, "\alpha_i"] \arrow[dr, "\phi_A^{-1}(\alpha)"'] & F(i) \\
& \lim F \arrow[u, "(\phi_{limF}(1_{limF}))(i)"']
\end{tikzcd}$$

6.3 Dinatural transformations, ends, and co-ends

Next, we recall the notion of dinatural transformation. The case which interests us the most is when one of the functors involved is a constant functor.

Definition 6.3.1 (Dinatural transformation). Suppose we have two functors $F, G : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$, a family of maps $\alpha : F \overset{\cdot\cdot}{\longrightarrow} G = \{\alpha_A : F(A,A) \to G(A,A)\}_{A \in |\mathcal{A}|}$ is called a *dinatural transformation* when for every arrow $f : A \to B$ the following holds:

$$\begin{tikzcd}
 & F(A,A) \arrow[r, "\alpha_A"] & G(A,A) \arrow[dr, "G(1,f)"] & \\
F(B,A) \arrow[ur, "F(f,1)"] \arrow[dr, "F(1,f)"'] & & & G(A,B) \\
 & F(B,B) \arrow[r, "\alpha_B"'] & G(B,B) \arrow[ur, "G(f,1)"'] &
\end{tikzcd}$$

Example 6.3.2. Let $S : \mathcal{A}^{op} \to \mathbf{Set}$ be a functor, and let $B \in |\mathcal{A}|$. There are two functors $F, G : \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Set}$ defined by $F(A', A) = S(A') \times \mathcal{A}(B, A)$, and $G = \Delta(S(B))$, the constant functor. Let us consider maps of type $\lambda_A : S(A) \times \mathcal{A}(B, A) \to S(B)$ with $\lambda_A(x, f) = S(f)(x)$. Then $\lambda : F \to G$ is a dinatural transformation: for all $f : A' \to A$,

$$\begin{tikzcd}
 & S(A) \times \mathcal{A}(B,A) \arrow[dr, "\lambda_A"] & \\
S(A) \times \mathcal{A}(B,A') \arrow[ur, "1 \times \mathcal{A}(B,f)"] \arrow[dr, "S(f) \times 1"'] & & S(B) \\
 & S(A') \times \mathcal{A}(B,A') \arrow[ur, "\lambda_{A'}"'] &
\end{tikzcd}$$

Definition 6.3.3 (Wedge). Given a functor $F : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$, a *wedge* is a dinatural transformation from a constant functor to F,

$$\lambda : \Delta(E) \overset{\cdot\cdot}{\longrightarrow} F.$$

Definition 6.3.4 (End). Given a functor $F : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$, an *end* is a wedge

$$\lambda : \Delta(E) \overset{\cdot\cdot}{\longrightarrow} F$$

satisfying a universal property: if there is another wedge $\alpha : \Delta(A) \overset{\cdot\cdot}{\longrightarrow} F$ then there is a unique $g : A \to E$ with $\lambda_A \circ g = \alpha_A$ for every $A \in \mathcal{A}$.

In an analogous way we define the notion of co-end.

Example 6.3.5. In the example above we have that $S(B)$ with component λ is a co-end for the functor F. Given a dinatural transformation $\alpha_A : S(A) \times \mathcal{A}(B, A) \longrightarrow X$ there is a unique $g : S(B) \to X$ given by $g(y) = \alpha_B(y, 1_B)$ that satisfies the definition.

From the uniqueness of the universal property we conclude that, up to isomorphism, all the ends are equal. This justifies the following notation to indicate an end E with components λ_A:

$$\int_A F(A, A) \xrightarrow{\lambda_A} F(A, A)$$

and in the same way the co-end:

$$F(A, A) \xrightarrow{\lambda_A} \int^A F(A, A).$$

Theorem 6.3.6. *Let $\alpha : F \Rightarrow G : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ be a natural transformation. Suppose also that there exists the ends induced by F and G:*

$$\int_A F(A, A) \xrightarrow{\lambda_A} F(A, A) \quad \textit{and} \quad \int_A G(A, A) \xrightarrow{\mu_A} G(A, A) \tag{14}$$

then there is a unique map $\int_A \alpha_{A,A}$ in the category $\mathcal{B}$ such that:

$$\begin{tikzcd}
\int_A F(A, A) \arrow[r, "\lambda_A"] \arrow[d, "\int_A \alpha_{A,A}"] & F(A, A) \arrow[d, "\alpha_{A,A}"] \\
\int_A G(A, A) \arrow[r, "\mu_A"'] & G(A, A)
\end{tikzcd}$$

Proof. [54] □

Theorem 6.3.7. *Let $F : \mathcal{A} \times \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{C}$ be a functor such that for each $A \in |\mathcal{A}|$ there exists an end*

$$\int_{B,B} F(A, B, B) \xrightarrow{\lambda_B^A} F(A, B, B).$$

Then there is a unique functor $U : \mathcal{A} \to \mathcal{C}$ with $U(A) = \int_B F(A, B, B)$ making λ_B^A natural in $A \in |\mathcal{A}|$.

Proof. [54] □

6.4 Indexed limits and colimits

Definition 6.4.1. Let $\mathcal{A}$ be a small category and $G : \mathcal{A} \to \mathcal{B}$ be functors. We define a functor $\hat{G} : \mathcal{B}^{op} \to [\mathcal{A}, \mathbf{Set}]$ whose values on objects are functors

$$\hat{G}(B) = \mathcal{B}(B, G-) : \mathcal{A} \to \mathbf{Set}$$

and whose value on a morphism $B \xrightarrow{f} B'$ is a natural transformation

$$\mathcal{B}(f, G-) : \mathcal{B}(B', G-) \to \mathcal{B}(B, G-).$$

Let $F : \mathcal{A} \to \mathbf{Set}$ be a functor. Thus we have a composition of functors:

$$\mathcal{B}^{op} \xrightarrow{\hat{G}} [\mathcal{A}, \mathbf{Set}] \xrightarrow{[\mathcal{A},\mathbf{Set}](F,-)} \mathbf{Set}.$$

Suppose now that this composition admits a representation:

$$\phi : \mathcal{B}(-, C) \cong [\mathcal{A}, \mathbf{Set}](F, \hat{G}(-)).$$

Definition 6.4.2 (Indexed limit)**.** Let us denote $C = \{F, G\}$, so we have that

$$\mathcal{B}(B, \{F, G\}) \cong [\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(B, G-))$$

natural in B with counit $\mu = \phi_{\{F,G\}}(1_{\{F,G\}}) : F \to \mathcal{B}(\{F, G\}, G-)$ which has the property of being a universal element. Following Kelly's definition [46], we name this pair $(\{F, G\}, \mu)$ the limit of G indexed by F.

Thus $\mu \in [\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(\{F, G\}, G-))$ and if there is another $\lambda \in [\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(B', G-))$ then there exists a unique $\{F, G\} \xrightarrow{g^{op}} B'$ in the category $\mathcal{B}^{op}$ such that $([\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(g, G-)))(\mu) = \lambda$ which means that $\mathcal{B}(g, G-) \circ \mu = \lambda$.

Therefore,

$$\begin{tikzcd} F(A) \arrow[r, "\mu_A"] \arrow[dr, "\lambda_A"'] & \mathcal{B}(\{F, G\}, G(A)) \arrow[d, "\mathcal{B}(g, G(A))"] \\ & \mathcal{B}(B', G(A)) \end{tikzcd}.$$

Thus, after evaluating at $x \in F(A)$ we obtain:

$$\begin{tikzcd} B' \arrow[r, "g"] \arrow[dr, "\lambda_A(x)"'] & \{F,G\} \arrow[d, "\mu_A(x))"] \\ & G(A) \end{tikzcd}$$

There is a bijection:

$$[\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(B, G-)) \cong [El(F), \mathcal{B}](\Delta B, G \circ \pi_F)$$

natural in B, with the projection $\pi_F : El(F) \to \mathcal{A}$.
By equation (12):

$$\mathcal{B}(B, \lim G \circ \pi_F) \cong [El(F), \mathcal{B}](\Delta B, G \circ \pi_F)$$

we conclude that:

Proposition 6.4.3.

$$lim\, G \circ \pi_F = \{F, G\}$$

Proof. To see this bijection we have that every natural transformation $\alpha \in [\mathcal{A}, \mathbf{Set}](F, \mathcal{B}(B, G-))$ and for every $f : A \to A'$ there is a diagram:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & \mathcal{B}(B, G(A)) \arrow[d, "\mathcal{B}(B,G(f))"] \\ F(A') \arrow[r, "\alpha_{A'}"] & \mathcal{B}(B, G(A')) \end{tikzcd}$$

which translates into a diagram:

$$\begin{tikzcd} B \arrow[d, "\alpha_A(a)"'] \arrow[dr, "\alpha_{A'}(F(f)(a))"] & \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

for every $a \in F(A)$.

$\square$

Remark 6.4.4. When we choose $F = \Delta 1$

$$\mathcal{B}(B, \lim G) \cong [\mathcal{A}, \mathcal{B}](\Delta B, G) \cong [\mathcal{A}, \mathbf{Set}](\Delta 1, \mathcal{B}(B, G-))$$

we obtain by definition that

$$\lim G = \{\Delta 1, G\}$$

Definition 6.4.5 (Indexed colimit)**.** In the same way as above by duality we define the colimit of $G : \mathcal{A} \to \mathcal{B}$ indexed by $F : \mathcal{A}^{op} \to \mathbf{Set}$ as the representing pair $(F \star G, \lambda)$ of the functor:

$$[\mathcal{A}^{op}, \mathbf{Set}](F, \tilde{G}(-)) : \mathcal{B} \to \mathbf{Set}$$

where $\tilde{G} : \mathcal{B} \to [\mathcal{A}^{op}, \mathbf{Set}]$ whose values on objects are functors

$$\tilde{G}(B) = \mathcal{B}(G-, B) : \mathcal{A}^{op} \to \mathbf{Set}$$

and whose value on a morphism $B \xrightarrow{f} B'$ is a natural transformation

$$\mathcal{B}(G-, f) : \mathcal{B}(G-, B) \to \mathcal{B}(G-, B').$$

Therefore, we have that

$$\mathcal{B}(F \star G, B) \cong [\mathcal{A}^{op}, \mathbf{Set}](F, \mathcal{B}(G-, B)) \tag{15}$$

and after evaluating the representation isomorphism on the identity with $B = F \star G$ we obtain a unit $\lambda : F \to \mathcal{B}(G-, F \star G)$.

Remark 6.4.6. With enough conditions, for example when $\mathcal{B}$ in cocomplete, there is a functor $\bullet \star G : [\mathcal{A}^{op}, \mathbf{Set}] \to \mathcal{B}$. Also, from equation (15) we conclude that $\bullet \star G$ is left adjoint of the functor $\mathcal{B}(G-, \bullet) : \mathcal{B} \to [\mathcal{A}^{op}, \mathbf{Set}]$ where $\mathcal{B}(G-, \bullet)(B) = \mathcal{B}(G-, B) : \mathcal{A}^{op} \to \mathbf{Set}$. We write $\bullet \star G \dashv \mathcal{B}(G-, \bullet)$.

The functor $\bullet \star G$ is the unique, up to isomorphism, colimit preserving functor such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Y"] \arrow[dr, "G"'] & {[\mathcal{A}^{op}, \mathbf{Set}]} \arrow[d, "\bullet\star G"] \\ & \mathcal{B} \end{tikzcd}$$

In the next section we shall discuss this construction in more detail in the context of a coproduct preserving Yoneda embedding.

Proposition 6.4.7. *If $F : \mathcal{A}^{op} \to \mathbf{Set}$ and $G : \mathcal{A} \to \mathcal{B}$ then*

$$colim\, G \circ \pi_F^{op} \cong F \star G$$

Proof. Analogously, there is a bijection:

$$[\mathcal{A}^{op}, \mathbf{Set}](F, \mathcal{B}(G-, B)) \cong [El(F)^{op}, \mathcal{B}](G \circ \pi_F^{op}, \Delta B)$$

natural in B, with projection $\pi_F^{op} : El(F)^{op} \to \mathcal{A}$.
From this since by equation (13):

$$\mathcal{B}(\mathrm{colim}\, G \circ \pi_F^{op}, B) \cong [El(F)^{op}, \mathcal{B}](G \circ \pi_F^{op}, \Delta B)$$

we conclude that:

$$\mathrm{colim}\, G \circ \pi_F^{op} \cong F \star G.$$

□

Remark 6.4.8. Since all colimits may be expressed in terms of coproducts and coequalizers we have the following explicit formula:

$$\coprod_{x \in F(A), f : A' \to A} G(A') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{A, x \in F(A)} G(A) \xrightarrow{\lambda} F \star G$$

where λ is a coequalizer of the unique maps τ and θ:

$$\begin{tikzcd} G(A') \arrow[r, "id"] \arrow[d, "\iota_{(x,f)}"'] & G(A') \arrow[d, "\iota_{(A', F(f)(x))}"] \\ \coprod_{x \in F(A), A' \xrightarrow{f} A} G(A') \arrow[r, "\theta"'] & \coprod_{A, x \in F(A)} G(A) \end{tikzcd} \qquad \begin{tikzcd} G(A') \arrow[r, "G(f)"] \arrow[d, "\iota_{(x,f)}"'] & G(A) \arrow[d, "\iota_{(A,x)}"] \\ \coprod_{x \in F(A), A' \xrightarrow{f} A} G(A') \arrow[r, "\tau"'] & \coprod_{A, x \in F(A)} G(A) \end{tikzcd}$$

obtained by the coproduct definition.

Now, suppose we take $F = \mathcal{A}(-, A) : \mathcal{A}^{op} \to \mathbf{Set}$, then for every B we have that:

$$\mathcal{B}(\mathcal{A}(-, A) \star G, B) \cong [\mathcal{A}^{op}, \mathbf{Set}](\mathcal{A}(-, A), \mathcal{B}(G-, B)) = \mathcal{B}(G(A), B)$$

by the Yoneda Lemma. Therefore $\mathcal{A}(-, A) \star G \cong G(A)$. In the same way we obtain that $\{\mathcal{A}(A, -), G\} \cong G(A)$.

Proposition 6.4.9.

$$\int^A F(A) \otimes G(A) \cong F \star G$$

Proof. Let $F : \mathcal{A} \to \mathbf{Set}$ and $G : \mathcal{A} \to \mathcal{B}$ be functors and suppose now that the category $\mathcal{B}$ has copowers[1]. We denote by $X \otimes A = \coprod_X A \in |\mathcal{B}|$ where X is a set and $A \in |\mathcal{B}|$. Then we have

$$\mathcal{B}(\textstyle\int^A F(A) \otimes G(A), B) \cong \int_A \mathcal{B}(F(A) \otimes G(A), B) \cong \int_A [F(A), \mathcal{B}(G(A), B)] \cong \\ [\mathcal{A}^{op}, \mathbf{Set}](F, \mathcal{B}(G(-), B))$$

by properties of ends, copowers, hom as end in the functor category.

Thus, by definition this implies that

$$\int^A F(A) \otimes G(A) \cong F \star G.$$

□

In particular when $G = Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Set}]$ we have that:

$$\int^A F(A) \otimes \mathcal{A}(-, A) \cong F \star Y \cong F$$

as we already have proved (Example 6.3.2).

6.5 Idempotent adjunctions

Proposition 6.5.1. *Let* $\mathcal{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{B}$ *be an adjunction with unit* $\eta : 1_{\mathcal{A}} \Rightarrow GF$ *and counit* $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$. *Then (i)* F *is full and faithful if and only if (ii)* η *is an isomorphism. When these conditions are satisfied,* $\varepsilon * G$ *and* $F * \varepsilon$ *are isomorphisms. Dually,* G *is full and faithful iff and only if* ε *is an isomorphism. When this happens* $\eta * G$ *and* $F * \eta$ *are isomorphism as well.*

[1] If X is a set and B an object, the *copower* $X \times B$ is defined to be a coproduct of X copies of B, i.e., $\coprod_{x \in X} B$.

Proof. (i)⇒(ii): We have that $\phi : \mathcal{B}(FA, B) \to \mathcal{A}(A, GB)$, where $\phi^{-1}(g) = \varepsilon_B \circ F(g)$. Since F is full there is an f such that $F(f) = \varepsilon_{FA}$. Hence since F is faithful, $F(f \circ \eta_A) = F(f)F(\eta_A) = \varepsilon_{FA}F(\eta_A) = 1_{FA} = F(1_A)$ implies $f \circ \eta_A = 1_A$ has a left inverse.
Therefore we have: $\phi^{-1}(\eta_A \circ f) = \varepsilon_{FB} \circ F(\eta_A \circ f) = \varepsilon_{FB} \circ F(\eta_A) \circ F(f) = \varepsilon_{FB} \circ F(\eta_A) \circ \varepsilon_{FB} = 1_{FA} \circ \varepsilon_{FB} = \varepsilon_{FB} \circ F(1_{GFA}) = \phi^{-1}(1_{GFA})$. This implies that $\eta_A \circ f = 1_{GFA}$ is also a right inverse.
(ii)⇒ (i): Consider the following isomorphism

$$\mathcal{A}(A, A') \xrightarrow{\mathcal{A}(A,\eta_{A'})} \mathcal{A}(A, GFA') \xrightarrow{\phi^{-1}} \mathcal{B}(FA, FA').$$

When we evaluate at $g : A \to A'$ we obtain that:

$$\phi^{-1}(\mathcal{A}(A, \eta_{A'})(g)) = \phi^{-1}(\eta_{A'} \circ g)) = \varepsilon_{FA'} \circ F(\eta_A \circ g) = \varepsilon_{FA'} \circ F(\eta_A) \circ F(g) = F(g)$$

by definition of adjunction. Thus $\phi^{-1} \circ \mathcal{A}(A, \eta_{A'}) = F$, is an isomorphism.

□

6.6 Lambek's completion for small categories

In this section, we review some material from [51] relevant to the following question: how to embed a small category as a full subcategory of a complete and cocomplete category in which the embedding preserves existing limits and colimits.

Definition 6.6.1. Let $G : \mathcal{A} \to \mathcal{B}$ be a functor, $\mathcal{A}$ a small category. Recall the functor $\tilde{G}$ defined in Definition 6.4.1 by $\tilde{G}(B) = \mathcal{B}(G(-), B)$ on objects and $\tilde{G}(f) = \mathcal{B}(G(-), f)$ on arrows. We say that G is *left adequate* for the category $\mathcal{B}$ if the functor $\tilde{G} : \mathcal{B} \to [\mathcal{A}^{op}, \mathbf{Set}]$ is fully faithful.

Proposition 6.6.2. *Suppose we have a functor $G : \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$ a small category, $\mathcal{B}$ a co-complete category. If G is a left adequate functor then for every $B \in \mathcal{B}$ there exists a small category $\mathcal{I}$ and a functor $H : \mathcal{I} \to \mathcal{A}$ such that $\operatorname{colim} GH = B$.*

Proof. For every $B \in \mathcal{B}$ let us consider $F = \mathcal{B}(G(-), B) : \mathcal{A}^{op} \to \mathbf{Set}$. Also consider the category $El(F)^{op}$ of elements of F, defined in Definition 6.1.3. We claim that $H = \pi^{op} : El(F)^{op} \to \mathcal{A}$, i.e.,

$$\operatorname{colim}(El(F)^{op} \xrightarrow{\pi^{op}} \mathcal{A} \xrightarrow{G} \mathcal{B}) \cong B.$$

If $(A', x') \xrightarrow{f^{op}} (A, x)$ then $(A, x) \xrightarrow{f} (A', x')$ with

$$\begin{tikzcd} G(A') \arrow[r, "x'"] \arrow[d, "G(f)"'] & B \\ G(A) \arrow[ur, "x"] & \end{tikzcd}$$

since $x' = F(f^{op})(x)$.
We define the following set of arrows $G\pi^{op}(A, x) \xrightarrow{\beta_{(A,x)}} B$ with $\beta_{(A,x)} = x$. Naturality follows from the previous diagram:

$$\begin{tikzcd} G\pi^{op}(A, x) \arrow[r, "\beta_{(A,x)}"] \arrow[d, "G\pi^{op}(f^{op})"'] & \Delta B(A', x') \arrow[d, "\Delta B(f^{op})"] \\ G\pi^{op}(A, x) \arrow[r, "\beta_{(A,x)}"] & \Delta B(A, x) \end{tikzcd}$$

for every $(A', x') \xrightarrow{f^{op}} (A, x)$. Now since $\mathcal{B}$ is co-complete we have that there exists a co-cone $(C, u_{(A,x)} : G\pi^{op}(A, x) \to C)$ such that $\operatorname{colim} G\pi^{op} = C$. This implies, by definition of colimit, that there exists a unique $p : C \to B$ such that the following diagram commutes:

$$\begin{tikzcd} G\pi^{op}(A, x) \arrow[r, "u_{(A,x)}"] \arrow[dr, "\beta_{(A,x)}"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

Actually p is an epimorphism. If $fp = gp$ with $f : B \to B'$ and $g : B \to B'$ then we have that $fpu_{(A,x)} = gpu_{(A,x)}$ for every $g(A) \xrightarrow{x} B$. This implies $fx = f\beta_{(A,x)} = g\beta_{(A,x)} = gx$ for every $g(A) \xrightarrow{x} B$. Now we use the fact that by hypothesis $\tilde{G}$ is faithful. By definition we have $\tilde{G}(f) = \tilde{G}(g) : \mathcal{B}(G-, B) \to \mathcal{B}(G-, B')$ since $\tilde{G}(f)(A)(x) = fx = gx = \tilde{G}(g)(A)(x)$, which implies $f = g$.

Now we define $\alpha_A : \mathcal{B}(G(A), B) \to \mathcal{B}(G(A), C)$ with $\alpha_A(x) = u_{(A,x)}$ for every $A \in \mathcal{A}$ and $g(A) \xrightarrow{x} B$. We check that α is a natural transformation:

$$\begin{tikzcd}
\mathcal{B}(G(A'), B) \arrow[r, "\alpha_{A'}"] \arrow[d, "\mathcal{B}(G(f),B)"'] & \mathcal{B}(G(A'), C) \arrow[d, "\mathcal{B}(G(f),C)"] \\
\mathcal{B}(G(A), B) \arrow[r, "\alpha_A"] & \mathcal{B}(G(A), C)
\end{tikzcd}$$

for every $A \xrightarrow{f} A'$.

$$\mathcal{B}(G(f), C)(\alpha_{A'}(x')) = \mathcal{B}(G(f), C)(u_{(A',x')}) = u_{(A',x')}G(f) = \qquad (*)$$

$$u_{(A,x)} = \alpha_A(x'G(f)) = \alpha_A(\mathcal{B}(G(f), C)(x')).$$

This equality $(*)$ is justified because u is a co-cone, i.e., for every $(A, x) \xrightarrow{f^{op}} (A', x')$

$$\begin{tikzcd}
G\pi^{op}(A, x) \arrow[rr, "G\pi^{op}(f^{op})"] \arrow[dr, "u_{(A,x)}"'] & & G\pi^{op}(A', x') \arrow[dl, "u_{(A',x')}"] \\
& C &
\end{tikzcd}$$

since we have that

$$\begin{tikzcd}
G(A) \arrow[rr, "G(f)"] \arrow[dr, "u_{(A,x)}"'] & & G(A') \arrow[dl, "u_{(A',x')}"] \\
& C &
\end{tikzcd}$$

The rest of the proof follows now from the fact that $\tilde{G}$ is a full functor. Hence there exists a morphism $b : B \to C$ such that $\alpha = \mathcal{B}(G-, b) : \mathcal{B}(G-, B) \to \mathcal{B}(G-, C)$. Therefore using this representation we get that $u_{(A,x)} = \alpha_A(x) = \mathcal{B}(G(A), b)(x) = bx$ for every $(A, x) \in \mathrm{El}(F)^{op}$. Thus by definition of colimits we get that $bpu_{(A,x)} = bx = u_{(A,x)}$ for every $(A, x) \in \mathrm{El}(F)^{op}$ implies that $bp = 1_C$. But p is an epimorphism, so we cancel to obtain $pbp = p1 = p$ and thus $pb = 1_B$, which means it is an isomorphism. Therefore $\operatorname{colim} G\pi^{op} = (C, u_{(A,x)})_{(A,x)\in El(F)} \cong (B, \beta_{(A,x)})_{(A,x)\in El(F)}$. □

Corollary 6.6.3. *For every $F \in [\mathcal{A}^{op}, \mathbf{Set}]$*

$$F = colim\,(El(F)^{op} \xrightarrow{\pi^{op}} \mathcal{A} \xrightarrow{Y} [\mathcal{A}^{op}, \mathbf{Set}]).$$

Proof. The Yoneda functor $A \xrightarrow{Y} [\mathcal{A}^{op}, \mathbf{Set}]$ is left adequate since we have that: $\tilde{Y} : [\mathcal{A}^{op}, \mathbf{Set}] \to [\mathcal{A}^{op}, \mathbf{Set}]$ is defined $\tilde{Y}(F) = [\mathcal{A}^{op}, \mathbf{Set}](Y-, F) = F$ on objects and $\tilde{Y}(\alpha) = [\mathcal{A}^{op}, \mathbf{Set}](Y-, F) \to [\mathcal{A}^{op}, \mathbf{Set}](Y-, F') = Id(\alpha)$ on arrows by the Yoneda Lemma. □

Definition 6.6.4. A functor $F : \mathcal{A} \to \mathcal{B}$ *reflects limits* when for each functor $G : \mathcal{I} \to \mathcal{A}$ with $\mathcal{I}$ small and given a cone $(A, u_i)_{i \in \mathcal{I}}$, $u_i : A \to G(i)$, if $(F(A), F(u_i))_{i \in \mathcal{I}}$ is a limit of FG then $(A, u_i)_{i \in \mathcal{I}}$ is a limit of G.

Proposition 6.6.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. F preserves colimits if and only if $\mathcal{B}(F-, B) : \mathcal{A}^{op} \to \mathbf{Set}$ preserves limits for every $B \in \mathcal{B}$.*

Proof. ($\Rightarrow$) Let us first observe that we have a composition of functors $\mathcal{B}(F-, B) = \mathcal{B}(-, B) \circ F^{op}$ where $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ preserves limits since F preserves colimits and $\mathcal{B}(-, B) : \mathcal{B}^{op} \to \mathbf{Set}$ preserves limits [15].
($\Leftarrow$) Now consider the functor $G : \mathcal{I} \to \mathcal{A}$ with $\operatorname{colim} G = (A, u_i)_{i \in \mathcal{I}}$, $u_i : G(i) \to A$. Thus $\lim G^{op} = (A, u_i^{op})_{i \in \mathcal{I}^{op}}$ where $G^{op} : \mathcal{I}^{op} \to \mathcal{A}^{op}$. By hypothesis we know that $\mathcal{B}(F-, B) : \mathcal{A}^{op} \to \mathbf{Set}$ preserves limits, hence for every $B \in \mathcal{B}$ the limit takes the form $\lim \mathcal{B}(F-, B) \circ G^{op} = (\mathcal{B}(F(A), B), \mathcal{B}(F(u_i^{op}), B))_{i \in \mathcal{I}^{op}}$, so we have:

$$\mathcal{I}^{op} \xrightarrow{G^{op}} \mathcal{A}^{op} \xrightarrow{F^{op}} \mathcal{B}^{op} \xrightarrow{Y} [\mathcal{B}, \mathbf{Set}]$$
$$i \mapsto G(i) \mapsto F(G(i)) \mapsto \mathcal{B}(F(G(i)), -),$$

where $Y(B') = \mathcal{B}(B', -) : \mathcal{B} \to \mathbf{Set}$.
Therefore for any $B \in \mathcal{B}$ it may be verified that $Y \circ F^{op} \circ G^{op}(-)(B) : \mathcal{I}^{op} \to \mathbf{Set}$ has a limit by hypothesis, since $\forall B \in \mathcal{B}$:

$$\lim Y \circ F^{op} \circ G^{op}(-)(B) = (\mathcal{B}(F(A), B), \mathcal{B}(F(u_i^{op}), B))_{i \in \mathcal{I}^{op}}.$$

Then, by proposition 2.15.1 of [15] we have $Y \circ F^{op} \circ G^{op} : \mathcal{I}^{op} \to [\mathcal{B}, \mathbf{Set}]$ has a limit being compute pointwise. Which means we have:

$$\lim Y \circ F^{op} \circ G^{op} = (\mathcal{B}(F(A), -), \mathcal{B}(F(u_i^{op}), -))_{i \in \mathcal{I}^{op}}.$$

But Y is a full and faithful functor, it reflects limits (see proposition 2.9.9 [15]) which implies that (see definition 2.9.6 [15]) since $(Y(F(A)), Y((F(u_i^{op}))_{i \in \mathcal{I}^{op}}$ is the limit of $Y \circ F^{op} \circ G^{op}$ then $(F(A), F(u_i^{op}))_{i \in \mathcal{I}^{op}}$ is the limit of $F^{op} \circ G^{op}$ in $\mathcal{B}^{op}$. Equivalently,

in view of this we are saying that $(F(A), F(u_i))_{i\in\mathcal{I}}$ is the colimit of $F \circ G$ in $\mathcal{B}$. Summarizing, we started with $\operatorname{colim} G = (A, u_i)_{i\in\mathcal{I}}$ and we end with $\operatorname{colim} FG = (FA, Fu_i)_{i\in\mathcal{I}}$, i.e., F preserves colimits. □

Proposition 6.6.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. F preserves coproducts if and only if $\mathcal{B}(F-, B) : \mathcal{A}^{op} \to$ **Set** preserves products for every $B \in \mathcal{B}$.*

Proposition 6.6.7. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. F preserves limits if and only if $\mathcal{B}(B, F-) : \mathcal{A} \to$ **Set** preserves limits for every $B \in \mathcal{B}$.*

Let $F : \mathcal{A} \to \mathcal{C}$ be a fully faithful functor. Consider the full subcategories $\mathcal{B}$ of $\mathcal{C}$ such that $|F(\mathcal{A})| \subseteq |\mathcal{B}| \subseteq |\mathcal{C}|$ and define:

$$\mathcal{A} \overset{F_{\mathcal{B}}}{\to} \mathcal{B} \overset{j_{\mathcal{B}}}{\to} \mathcal{C}$$

with $F = j_{\mathcal{B}} F_{\mathcal{B}}$, $F_{\mathcal{B}}(A) = F(A)$, $F_{\mathcal{B}}(f) = F(f)$ and j the inclusion functor. Define $\mathcal{B}_0$ a full subcategory of $\mathcal{C}$ in the following way:

$$|\mathcal{B}_0| = \{B \in |\mathcal{C}| : \mathcal{C}(F(-), B) : \mathcal{A}^{op} \to \textbf{Set} \text{ preserves limits}\}.$$

Remark 6.6.8. If $F : \mathcal{A} \to \mathcal{C}$ is a fully faithful functor then $|F(\mathcal{A})| \subseteq |\mathcal{B}_0|$. To see this we have that $\mathcal{C}(F-, F(A)) \cong \mathcal{A}(-, A)$ are naturally isomorphic which implies that $\mathcal{C}(F-, F(A))$ preserves limits.

Proposition 6.6.9. *Let $F : \mathcal{A} \to \mathcal{C}$ be a fully faithful functor. Then:*
(a) if $j_{\mathcal{B}} F_{\mathcal{B}}$ preserves colimits then $\mathcal{B} \subseteq \mathcal{B}_0$
(b) let $\mathcal{J}$ be a small category, and consider the following composition of functors:

$$\mathcal{J} \overset{\Delta}{\to} \mathcal{B}_0 \overset{j_{\mathcal{B}_0}}{\to} \mathcal{C}$$

if $\lim j_{\mathcal{B}_0} \Delta = (C, v_j)$ then $C \in |\mathcal{B}_0|$.

Proof. (a) Take $B \in |\mathcal{B}|$, since $j_{\mathcal{B}} F_{\mathcal{B}}$ preserves colimits then by Proposition 6.6.5 $\mathcal{C}(j_{\mathcal{B}}(F_{\mathcal{B}}(-)), B) = \mathcal{C}(F(-), B)$ preserves limits, which by definition means that $B \in |\mathcal{B}_0|$.

(b) We are going to prove that $|\mathcal{B}'| = |\mathcal{B}_0| \cup \{C\}$ also satisfies property of part (a) above. This implies that $\mathcal{B}' \subseteq \mathcal{B}_0$ i.e., $C \in |\mathcal{B}_0|$. We have that

$$\mathcal{A} \xrightarrow{F_{\mathcal{B}'}} \mathcal{B}' \xrightarrow{J_{\mathcal{B}'}} \mathcal{C}$$

and we want to show that if $\mathcal{I} \xrightarrow{\Gamma} \mathcal{A}$ with $\operatorname{colim} \Gamma = (A, u)$, with $\Gamma(i) \xrightarrow{u_i} A$, $i \in \mathcal{I}$ then $\operatorname{colim} F_{\mathcal{B}'}\Gamma = (F_{\mathcal{B}'}(A), F_{\mathcal{B}'}(u_i))_{i\in\mathcal{I}} = (F(A), F(u_i))_{i\in\mathcal{I}}$.
Let $t : F_{\mathcal{B}'}\Gamma \Rightarrow C$ be a co-cone. Without loss of generality, we assume that $F_{\mathcal{B}'}(\Gamma(i)) \neq C$ for every $i \in \mathcal{I}$. If there exists a i_0 with $F_{\mathcal{B}'}(\Gamma(i_0)) = C$ then since $|F(\mathcal{A})| \subseteq |\mathcal{B}|$ this implies that $C \in |\mathcal{B}|$.
We fix an object $j \in |\mathcal{J}|$. Therefore since t is a co-cone we consider the following co-cone:

$$F(\Gamma(i)) \xrightarrow{t_i} C \xrightarrow{v_j} \Delta(j).$$

These arrows are contained in the category $\mathcal{B}$ because $F(\Gamma(i))$ and $\Delta(j)$ are object of $\mathcal{B}$. We know by part (a) that $F_{\mathcal{B}_0}$ has the property of preserving colimits:

$$\operatorname{colim} F_{\mathcal{B}_0}\Gamma = (F_{\mathcal{B}_0}(A), F_{\mathcal{B}_0}(u_i))_{i\in\mathcal{I}} = (F(A), F(u_i))_{i\in\mathcal{I}}.$$

For that reason there exists a unique $x_j : F(A) \to \Delta(j)$ such that

$$\begin{tikzcd} F(\Gamma(i)) \arrow[r, "t_i"] \arrow[d, "F(u_i)"'] & C \arrow[d, "v_j"] \\ F(A) \arrow[r, "x_j"] & \Delta(j) \end{tikzcd}$$

for every $i \in |\mathcal{I}|$. We will show that x_j is a cone in order to use the universal property of the limit. Let $f : j \to j'$ be an arrow in $\mathcal{J}$. We want to prove that $\Delta(f)x_j = x'_j$. This follows from the fact that x_j is defined using $\operatorname{colim} F_{\mathcal{B}_0}\Gamma$. We must check that $v'_j t_i = \Delta(f)x_j F(u_i)$ for every $i \in |\mathcal{I}|$. Then by uniqueness of the colimit definition we get that $\Delta(f)x_j = x'_j$.
But we know by definition of x_j that: $x_j F(u_i) = v_j t_i$ for every $i \in |\mathcal{I}|$, then composing with $\Delta(f)$ we obtain $\Delta(f)x_j F(u_i) = \Delta(f)v_j t_i$ for every $i \in |\mathcal{I}|$. Therefore, it will be enough to prove that $\Delta(f)v_j = v'_j$, but this follows from the naturality of the cone $C \Rightarrow \Delta$.

We have proved that $F(A) \overset{x}{\Rightarrow} \Delta$ is a cone in $\mathcal{B}_0$. Then by definition of $\lim\Delta = (C, v)$ there exists a unique $y : F(A) \to C$ such that:

$$\begin{tikzcd} F(A) \arrow[r, "y"] \arrow[dr, "x_j"'] & C \arrow[d, "v_j"] \\ & \Delta(j) \end{tikzcd}$$

We therefore put all the equations together: $v_j t_i = x_j F(u_i) = v_j y F(u_i)$ for every $j \in |\mathcal{J}|$. Thus since this is true for every $j \in |\mathcal{J}|$, by definition of limit we have that $t_i = yF(u_i)$.

So now suppose there exists another y' satisfying the same property as above: $t_i = y'F(u_i)$. We want to prove that $y = y'$. It will be enough to prove that: $v(j)y' = x_j$ for every $j \in |\mathcal{J}|$. For that purpose, we know by hypothesis that $t_i = y'F(u_i)$ for every $i \in \mathcal{I}$. Then by composing we get $v(j)y'F(u_i) = v(j)t_i$ for every $i \in \mathcal{I}$, and since $v_j t_i = x_j F(u_i)$ we replace it: $v_j y' F(u_i) = x_j F(u_i)$ for every $i \in \mathcal{I}$. This implies by uniqueness of the colimit that $v_j y' = x_j$.

We proved that $\operatorname{colim} F_{\mathcal{B}'}\Gamma = (F_{\mathcal{B}'}(A), F_{\mathcal{B}'}(u_i))_{i\in\mathcal{I}}$ where $|\mathcal{B}'| = |\mathcal{B}_0| \cup \{C\}$, i.e., for an arbitrary co-cone in $\mathcal{B}'$, $(F(A), F(u_i))_{i\in\mathcal{I}}$ is still a limit co-cone and this implication is the the property that characterizes the set $|\mathcal{B}_0|$. □

Corollary 6.6.10. *Let $F : \mathcal{A} \to \mathcal{C}$ be a fully faithful functor such that for every $C \in \mathcal{C}$ there exists a functor $G : \mathcal{I} \to \mathcal{A}$ with $\lim FG = C$. Then F preserves colimits.*

Proof. We consider $\mathcal{B}_0$ as above. Since $\lim FG = C$ for some G, then by part (b) of the Proposition 6.6.9 above we have that $C \in \mathcal{B}_0$, therefore $F = F_{\mathcal{B}_0}$ and it preserves colimits by Proposition 6.6.5. □

Remark 6.6.11. To prove that $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Set}]$ preserves limits is equivalent to proving that $Y^{op} : \mathcal{A}^{op} \to [\mathcal{A}^{op}, \mathbf{Set}]^{op}$ preserves colimits and since by Corollary 6.6.3:

$$F = \operatorname{colim}\left(El(F)^{op} \overset{\pi^{op}}{\to} \mathcal{A} \overset{Y}{\to} [\mathcal{A}^{op}, \mathbf{Set}]\right)$$

for every $F \in [\mathcal{A}^{op}, \mathbf{Set}]$ this implies that:

$$F = \lim\left(El(F) \overset{\pi}{\to} \mathcal{A}^{op} \overset{Y^{op}}{\to} [\mathcal{A}^{op}, \mathbf{Set}]^{op}\right)$$

for every $F \in [\mathcal{A}^{op}, \mathbf{Set}]^{op}$. But we know by the Corollary 6.6.10 above that this implies that $\mathcal{A}^{op} \xrightarrow{Y^{op}} [\mathcal{A}^{op}, \mathbf{Set}]^{op}$ preserves colimits.

Definition 6.6.12. Let $[\mathcal{A}, \mathbf{Set}]_{inf} \subseteq [\mathcal{A}, \mathbf{Set}]$ be the full subcategory of limit preserving functors. Since the representable functors $\mathcal{A}(-, A) : \mathcal{A}^{op} \to \mathbf{Set}$ preserve limits, we can define a functor $\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ by co-restriction induced by the Yoneda embedding.

Remark 6.6.13. Let $\mathcal{A}$ be a small category. The functor $\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is left adequate since the induced functor $[\mathcal{A}, \mathbf{Set}]_{inf} \xrightarrow{\tilde{Y}_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]$ is fully faithful. To see this, we check that we have on objects:

$$\tilde{Y}_{inf}(F) = [\mathcal{A}^{op}, \mathbf{Set}]_{inf}(Y_{inf}-, F) = [\mathcal{A}^{op}, \mathbf{Set}](Y_-, F) \cong F$$

since is a full subcategory and $Y_{inf}- = Y- \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$. Thus we have that

$$[\mathcal{A}^{op}, \mathbf{Set}]_{inf}(F, G) = [\mathcal{A}^{op}, \mathbf{Set}](F, G) \cong [\mathcal{A}^{op}, \mathbf{Set}](\tilde{Y}_{inf}(F), \tilde{Y}_{inf}(G))$$

which means that $\tilde{Y}_{inf}$ is fully faithful, i.e., Y_{inf} left adequate. Therefore, using the same argument we get that $\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ preserves limits.

Proposition 6.6.14. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$ such that for every $C \in \mathcal{C}$ there exists functor $G : \mathcal{I} \to \mathcal{B}$ with $\operatorname{colim} j_{\mathcal{B}} G = C$. If $\mathcal{B}$ is a co-complete category then $\mathcal{B}$ is a left reflective subcategory of $\mathcal{C}$. Conversely, suppose $\mathcal{B}$ is a left reflective subcategory of $\mathcal{C}$. If $\mathcal{C}$ is co-complete then $\mathcal{B}$ is co-complete.*

Proof. We want to prove that the inclusion functor $\mathcal{B} \xhookrightarrow{j} \mathcal{C}$ has a left adjoint $\mathcal{C} \xrightarrow{R} \mathcal{B}$. It is enough to prove that for every $C \in \mathcal{C}$ there is an object $R(C) \in \mathcal{B}$, a map $C \xrightarrow{\eta_C} R(C)$ such that for every $f : C \to B'$ with $B' \in \mathcal{B}$ there is a unique $g : R(C) \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "f"'] & R(C) \arrow[d, "g"] \\ & B'. \end{tikzcd}$$

Let us consider $C \in \mathcal{C}$. By hypothesis we have that there exists a functor $G : \mathcal{I} \to \mathcal{B}$ with $\operatorname{colim} j_{\mathcal{B}} G = C$. But since $\mathcal{B}$ is a co-complete category then there is an object $B \in \mathcal{B}$ and a co-cone $\{G(i) \xrightarrow{u_i} B\}_{i \in I}$ with $\operatorname{colim} G = (B, u)$.
We define $R(C) = B$, and since $\{jG(i) = G(i) \xrightarrow{u_i} B\}_{i \in I}$ is a co-cone of jG in the category $\mathcal{C}$ therefore there exists a unique $C \xrightarrow{\eta_C} R(C)$, such that:

$$\begin{tikzcd} G(i) \arrow[r, "v_i"] \arrow[dr, "u_i"'] & C \arrow[d, "\eta_C"] \\ & B = R(C) \end{tikzcd}$$

commutes for every $i \in \mathcal{I}$. Now suppose we have a map $f : C \to B'$ with $B' \in \mathcal{B}$. Then since $G(i)$ is an object of $\mathcal{B}$ for every $i \in \mathcal{I}$ and $\{v_i\}_{i \in \mathcal{I}}$ is a co-cone in $\mathcal{C}$ this implies that $\{jG(i) = G(i) \xrightarrow{v_i} C \xrightarrow{f} B'\}_{i \in I}$ is a co-cone in the category $\mathcal{B}$. Therefore by definition of $\operatorname{colim} G = (B, u)$ there is a unique $g : R(C) \to B'$, $g \in \mathcal{B}$ with $fv_i = gu_i$ for every $i \in \mathcal{I}$. Hence $fv_i = gu_i = g\eta_C v_i$ for every $i \in \mathcal{I}$, and this implies by definition (uniqueness) of colimit that $f = g\eta_C$.
If there is a morphism $\tilde{g} : R(C) \to B'$ such that $f = \tilde{g}\eta_C$ then by composing with v_i we get $fv_i = g\eta_C v_i$ for every $i \in \mathcal{I}$ which means that $fv_i = \tilde{g}u_i$ for every $i \in \mathcal{I}$ therefore $g = \tilde{g}$. If $C \xrightarrow{f} C'$ a morphism in $\mathcal{C}$ then $R(f)$ is defined as the unique arrow such that:

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"] & R(C) \arrow[d, "R(f)"] \\ C' \arrow[r, "\eta_{C'}"] & R(C') \end{tikzcd}$$

commutes. By uniqueness we obtain that R is a functor and naturality of $Id \xrightarrow{\eta} j \circ R$ follows from the diagram.
Conversely, let $G : \mathcal{I} \to \mathcal{B}$ be a functor. Since $\mathcal{C}$ is co-complete there exists $\operatorname{colim} jG = (C, v)$ with $j : \mathcal{B} \to \mathcal{C}$ the inclusion functor and $G(i) \xrightarrow{v_i} C$. By hypothesis we know that R is a reflection of j, which means $\mathcal{B}(R(A), B) \cong \mathcal{C}(A, j(B))$ for every $A \in \mathcal{C}$, and $B \in \mathcal{B}$. When $A \in \mathcal{B}$ then since $\mathcal{B}$ is a full subcategory we have that $\mathcal{B}(R(A), B) \cong \mathcal{B}(A, B)$ for every $B \in \mathcal{B}$. By the Yoneda Lemma this implies that $R(A) \cong A$. On the other hand R preserves colimits because is a left adjoint. Thus $G(i) \cong R(G(i)) = RjG(i) \xrightarrow{R(v_i)} R(C)$ is a colimit of G with $R(C) \in \mathcal{B}$. $\square$

Remark 6.6.15. Notice that, from the proof above, colimits in $\mathcal{B}$ are induced by the reflection, i.e., if $G : \mathcal{I} \to \mathcal{B}$ is a functor with $\mathcal{I}$ small then:

$$\operatorname{colim}_{\mathcal{B}} G \cong R(\operatorname{colim}_{\mathcal{C}} j \circ G).$$

Remark 6.6.16. For every $F \in [\mathcal{A}^{op}, \mathbf{Set}]$ there exists a functor $\mathcal{I} \xrightarrow{G} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ such that $\operatorname{colim} jG = F$:

$$F = \operatorname{colim}\,(El(F)^{op} \xrightarrow{\pi^{op}} \mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf} \xhookrightarrow{j} [\mathcal{A}^{op}, \mathbf{Set}]).$$

Proposition 6.6.17. *Let $\mathcal{A}$ be a small category. Then $[\mathcal{A}, \mathbf{Set}]_{inf}$ is a reflective subcategory of $[\mathcal{A}, \mathbf{Set}]$.*

Proof. [47]. □

Remark 6.6.18. This implies that $[\mathcal{A}, \mathbf{Set}]_{inf}$ is a co-complete category.

Proposition 6.6.19. *Let $\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ be the restricted Yoneda embedding from Definition 6.6.12 above. Then Y_{inf} is a full and faithful, limit and colimit preserving functor such that for every $F \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ there exists a functor $G : \mathcal{I} \to \mathcal{A}$ with $\lim Y_{inf} G = F$. Moreover, $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is a complete and co-complete category.*

Proof. First, $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is a co-complete category by Remark 6.6.18 above. In view of the Remark 6.6.13 above $\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ preserves limits.
Using Proposition 6.6.5:

$$\mathcal{A} \xrightarrow{Y_{inf}} [\mathcal{A}^{op}, \mathbf{Set}]_{inf} \text{ preserves co-limits if and only if}$$
$$[\mathcal{A}^{op}, \mathbf{Set}]_{inf}(Y_{inf}-, F) : \mathcal{A}^{op} \to \mathbf{Set} \text{ preserves limits for all } F \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}.$$

But by the Yoneda Lemma we have that

$$[\mathcal{A}^{op}, \mathbf{Set}]_{inf}(Y_{inf}-, F) = [\mathcal{A}^{op}, \mathbf{Set}](Y_{-}, F) \cong F$$

which is the condition that defines the subcategory. Notice that we used the fact that $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is a full subcategory.
Now, in view of Proposition 6.6.9, consider the fully faithful functor $F : \mathcal{A} \to \mathcal{C}$, with $F = Y$, $\mathcal{B}_0 = [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ and $\mathcal{C} = [\mathcal{A}^{op}, \mathbf{Set}]$. By part (b) when there is a functor

$$\mathcal{J} \xrightarrow{\Delta} \mathcal{B}_0 \xrightarrow{j_{\mathcal{B}_0}} \mathcal{C}$$

since $[\mathcal{A}^{op}, \mathbf{Set}]$ is a complete category then $\lim j_{\mathcal{B}_0}\Delta = (C, v_j)$ exists. But this implies that $C \in |\mathcal{B}_0|$ which means that $\mathcal{B}_0 = [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is complete.
To see why $\mathcal{B}_0 = [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$, consider $\mathcal{B} = [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ and $j_{\mathcal{B}}F_{\mathcal{B}} = Y$ with $F_{\mathcal{B}} = Y_{inf}$. Since it preserves colimits then $\mathcal{B} \subseteq \mathcal{B}_0$. On the other hand if $B \in \mathcal{B}_0$ such that $Y_{B_0} : \mathcal{A} \to \mathcal{B}_0$ preserves colimits then by Proposition 6.6.5 this implies that: $B_0(Y_{B_0}-, B) : \mathcal{A}^{op} \to \mathbf{Set}$ preserves limits. But

$$B_0(Y_{B_0}-, B) = [\mathcal{A}^{op}, \mathbf{Set}](Y-, B) \cong B.$$

Thus it means that B preserves limits, i.e., $B \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$.
It remains to show that if $F \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ then there exists a functor $G : \mathcal{I} \to \mathcal{A}$ with $\lim Y_{inf}G = F$. For this, it is enough to prove that Y_{inf} is left adequate, which was done on Remark 6.6.13. □

Remark 6.6.20. This amounts to proving that for every $F \in [\mathcal{A}^{op}, \mathbf{Set}]$ there is an object $R(F) \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$, a co-cone $Y_{inf}\pi_F \overset{u}{\Rightarrow} \Delta R(F)$, and a co-cone $jY_{inf}\pi_F \overset{v}{\Rightarrow} \Delta F$ such that $\operatorname{colim} Y_{inf}\pi_F = (R(F), u)$ and $\operatorname{colim} jY_{inf}\pi_F = (F, v)$. Therefore there is a unique $F \overset{\eta_F}{\to} R(F)$ such that

$$\begin{tikzcd}
F \arrow[rr, "\eta_F"] & & R(F) \\
& jY_{inf}\pi_F(A,a) = Y_{inf}\pi_F(A,a) = \mathcal{A}(-,A) \arrow[ul, "v_{(A,a)}"] \arrow[ur, "u_{(A,a)}"'] &
\end{tikzcd}$$

commutes for every $i \in \mathcal{I}$.

To conclude this section, we briefly comment on the reflective adjoint pair $i \vdash R$ of Proposition 6.6.17. Since $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ is a co-complete category, all small colimits exists and we are in a position to consider co-powers $A \otimes_{inf} B$ where $A \in \mathbf{Set}$ and $B \in [\mathcal{A}^{op}, \mathbf{Set}]_{inf}$. On the other hand, co-powers in the category $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$ are induced by copowers in $[\mathcal{A}^{op}, \mathbf{Set}]$ using the reflection above:

$$A \otimes_{inf} B = R(A \otimes i(B)).$$

Therefore, since R preserves coends we have that we can express $R(F) = F \star Y_{inf}$ as an indexed colimit where the definition of the operation $\star$, taken from [46] (see

Definition 6.4.5) is given by the next first equation:

$$F \star Y_{inf} = \int^x F(x) \otimes_{int} Y_{inf}(x) = \int^x R(F(x) \otimes A(-,x)) \cong R(\int^x F(x) \otimes A(-,x)) \cong R(F)$$

Notice that we are using the fact that every representable functor is included in the category $[\mathcal{A}^{op}, \mathbf{Set}]_{inf}$. Thus, in terms of left Kan extension (see Section 6.7) or indexed colimits we have the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "Y"] \arrow[dr, "Y_{inf}"'] & {[\mathcal{A}^{op}, \mathbf{Set}]} \arrow[d, shift left, "\tilde{Y}_{inf}"] \arrow[d, phantom, "\vdash"] \\
& {[\mathcal{A}^{op}, \mathbf{Set}]_{inf}} \arrow[u, shift left, "\hat{Y}_{inf}"]
\end{tikzcd}$$

where $\tilde{Y}_{inf} = R = - \star Y_{inf} = Lan_Y(Y_{inf})$ since

$$Lan_Y(Y_{inf})(F) = \int^A [\mathcal{A}^{op}, \mathbf{Set}](Y(A), F) \otimes_{inf} Y_{inf}(A) \cong \int^A F(A) \odot_{inf} Y_{inf}(A)$$

and $\hat{Y}_{inf} \cong i$ the inclusion functor since

$$\hat{Y}_{inf}(F) = [\mathcal{A}^{op}, \mathbf{Set}](Y(-), F) \cong F.$$

6.7 Kan extensions

This section provides a brief overview of the left Kan extension. A large portion of Chapter 7 depends on this central notion. To mention two examples: the definition of a left adjoint of a certain functor and the monoidal enrichment of the functor category.

Definition 6.7.1. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{C}$ be two functors. The *left Kan extension* of the functor G along F, if it exists, is a functor $K : \mathcal{B} \to \mathcal{C}$ together with a natural transformation $\alpha : G \Rightarrow KF$ satisfying the following universal property: if $H : \mathcal{B} \to \mathcal{C}$ and $\beta : G \Rightarrow HF$ then there is a unique natural transformation $\gamma : K \Rightarrow H$ satisfying $(\gamma * F) \circ \alpha = \beta$.

`Notation:` We denote the functor K by $Lan_F(G)$.

Let $F : \mathcal{A} \to \mathcal{B}$, and consider the functor between functor categories

$$[\mathcal{B}, \mathcal{C}] \xrightarrow{F^*} [\mathcal{A}, \mathcal{C}] \tag{16}$$

defined by precomposition with F, i.e., $F^*(G) = G \circ F$ for any functor $G : \mathcal{B} \to \mathcal{C}$.

Corollary 6.7.2. *If $Lan_F(G)$ exists for all G, then $Lan_F \dashv F^*$.*

Proof. The definition above turns out to be the following: for every $\beta : G \Rightarrow F^*(K)$ there exists a unique $\gamma : Lan_F(G) \to H$ such that:

$$\begin{tikzcd} G \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\ F^*(Lan_F(G)) \arrow[r, "F^*(\gamma)"'] & F^*(H) \end{tikzcd}$$

which means that:

$$[\mathcal{B}, \mathcal{C}](Lan_F(G), H] \cong [\mathcal{A}, \mathcal{C}](G, F^*(H))$$

with unit $\alpha = \eta_G : G \Rightarrow F^*(Lan_F(G))$.

□

Proposition 6.7.3. *If $\mathcal{A}$ is a small category and $\mathcal{C}$ is co-complete then the left Kan extension of G along F exists.*

Remark 6.7.4. We can also formulate the left Kan extension as a coend. If $\forall a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$ the copowers $\mathcal{B}(F(a'), b) \times G(a)$ exist in $\mathcal{C}$; and the following coend exists $\forall b \in \mathcal{B}$ then:

$$Lan_F(G)(b) = \int^a \mathcal{B}(F(a), b) \times G(a).$$

`Notation:` For the sake of brevity we sometimes write Lan_F instead of $Lan_{F^{op}}$ when the extension is along the opposite functor $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$.

Remark 6.7.5. Notice that for a functor $\Phi : \mathcal{A} \to \mathcal{B}$ we can express the adjunction $Lan_\Phi \dashv \Phi^*$ as a left Kan extension of $Y \circ \Phi : \mathcal{A} \to [\mathcal{B}^{op}, \mathbf{Set}]$ along $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathbf{Set}]$ in the following way: for some $F : \mathcal{A}^{op} \to \mathbf{Set}$ we have

$$Lan_Y(Y \circ \Phi)(F) = (\int^a [\mathcal{A}^{op}\mathbf{Set}](Y(a), -) \times Y \circ \Phi(a))(F) =$$
$$\int^a [\mathcal{A}^{op}\mathbf{Set}](Y(a), F) \times Y \circ \Phi(a) \cong \int^a F(a) \times \mathcal{B}(-, \Phi(a)) = Lan_{\Phi^{op}}(F)$$

and also for some $G : \mathcal{B}^{op} \to \mathbf{Set}$:

$$[\mathcal{B}^{op}, \mathbf{Set}](Y(\Phi(-)), G) \cong G(\Phi(-)) = \Phi^*(G).$$

6.8 Day's closed monoidal convolution

A symmetric monoidal category can be fully and faithfully embedded in a symmetric monoidal closed category in such a way that the tensor is preserved. This construction is a particular instance of a more general notion called promonoidal categories defined by Day [18]. In fact there is a correspondence between promonoidal categories and biclosed monoidal structures defined on the functor categories.

Proposition 6.8.1. *Let $\mathcal{A}$ be a symmetric monoidal category. Then $[\mathcal{A}^{op}, \mathbf{Set}]$ can be equipped with a symmetric monoidal structure (called the* Day tensor *[18]), such that the Yoneda embedding $Y : \mathcal{A} \to [\mathcal{A}^{op}, \mathbf{Set}]$ is a strong monoidal functor. Moreover, $[\mathcal{A}^{op}, \mathbf{Set}]$ is monoidal closed.*

Proof. (sketch)

We consider the monoidal closed case on functor categories

$$([\mathcal{A}^{op}, \mathbf{Set}], \otimes_D, I_D, \multimap).$$

This structure is obtained by using the Kan extension to closed functor categories:

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{A} \arrow[r, "Y \times Y"] \arrow[d, "\otimes"'] & {[\mathcal{A}^{op}, \mathbf{Set}] \times [\mathcal{A}^{op}, \mathbf{Set}]} \arrow[d, "Lan_{Y \times Y}(-\otimes-)"] \\
\mathcal{A} \arrow[r, "Y"] & {[\mathcal{A}^{op}, \mathbf{Set}]}
\end{tikzcd}$$

In more detail the following data is obtained:

- $- \otimes_D - : [\mathcal{A}^{op}, \mathbf{Set}] \times [\mathcal{A}^{op}, \mathbf{Set}] \to [\mathcal{A}^{op}, \mathbf{Set}]$ is defined by

$$S \otimes_D T = \int^a S(a) \times \int^b T(b) \times \mathcal{A}(-, a \otimes b)$$

This operation is also called the *convolution* of S and T.

- $I_D = \mathcal{A}(-, I)$

- $l : I_D \otimes_D T \to T$ is given by:
  $\int^x I_D(x) \times (\int^a T(a) \times \mathcal{A}(-, x \otimes a)) \cong \int^a (\int^x I_D(x) \times \mathcal{A}(-, x \otimes a)) \times T(a) \overset{\int \lambda^* \times 1}{\to}$
  $\int^a \mathcal{A}(-, a) \times T(a) \cong T$
  where $\lambda^* : \int^x \mathcal{A}(x, I) \times \mathcal{A}(-, x \otimes a)) \cong \mathcal{A}(-, I \otimes a) \overset{\mathcal{A}(-,\lambda)}{\to} \mathcal{A}(-, a)$

- $r : T \otimes_D I_D \to T$: analogous.

- $a : (R \otimes_D S) \otimes_D T \to R \otimes_D (S \otimes_D T)$
  $(R \otimes_D S) \otimes_D T =$
  $\int^x (\int^a R(a) \times (\int^b S(b) \times \mathcal{A}(x, a \otimes b)) \times (\int^c T(c) \times \mathcal{A}(-, x \otimes c)))$
  $\underset{\to}{\cong \circ (\int 1 \times \int 1 \times \int (1 \times \alpha)) \circ \cong}$
  $\int^a R(a) \times \int^x (\int^b S(b) \times (\int^c T(c) \times \mathcal{A}(x, b \otimes c))) \times \mathcal{A}(-, a \otimes x)) = R \otimes_D (S \otimes_D T)$

- $c : S \otimes_D T \to T \otimes_D S$ is

$$S \otimes_D T = \int^a S(a) \times (\int^b T(b) \times \mathcal{A}(-, a \otimes b)) \cong \int^b T(b) \times (\int^a S(a) \times \mathcal{A}(-, a \otimes b))$$
$$\overset{\int 1 \times (\int 1 \times \sigma)}{\to} \int^b T(b) \times (\int^a S(a) \times \mathcal{A}(-, b \otimes a)) = T \otimes_D S$$

- the internal hom is:
$$[S, T]_D \cong \int_b [S(b), T(- \otimes b)]$$

□

For more details on this construction we refer the reader to [18].

6.9 The reflective subcategory $[\mathcal{C}, \mathcal{A}]_\Gamma$

In this section we give a brief overview the methodology of Freyd and Kelly [24] in order to build reflections in a more general way using the notion of orthogonality. In particular, we are interested in some full subcategories of presheaves. This construction generalizes Lambek's presentation in Section 6.6 by regarding the condition of

preserving limits as a special case of the continuity of functors over a certain class of cylinders.

Given an object $A \in \mathcal{A}$, we define a preorder among the class of monomorphisms with codomain A: if $f : B \to A$, $g : C \to A$ are two monomorphisms f is said to be smaller than g ($f \leq g$) when f factors through g i.e., $f = gk$ for some $k : B \to C$. Note that k is unique and also a monomorphism.

We have an equivalence relation $f \equiv g$ iff $f \leq g$ and $g \leq f$.

Definition 6.9.1. A *subobject* of A is an equivalence class of these monomorphisms.

The class of subobjects is partially ordered by the order induced by the representatives.

Definition 6.9.2. We say that a category $\mathcal{A}$ is *well-powered* when for every $A \in \mathcal{A}$ the class of subobjects of A is a set.

The dual notions applied to epimorphisms are called *quotient* for an equivalence class of epimorphisms, and *co-well-powered.*

Definition 6.9.3. Let A be an object. The *intersection* of a family of subobjects of A, if it exists, is the greatest lower bound defined in the partially ordered class of subobjects of A. Analogously, by the *union* we mean the least upper bound, if it exists.

Concretely, we mean the following: if $\{A_i \xrightarrow{f_i} A\}_{i \in I}$ are subobjects of A then there exists an arrow $\cap_{i \in I} A_i \xrightarrow{f} A$ satisfying the following properties:

- $f \leq f_i$ $\forall i \in I$, i.e., for every $i \in I$ there exists an arrow $\cap_{i \in I} A_i \xrightarrow{t_i} A_i$ such that $f_i \circ t_i = f$.

- if there exists a p such that $p \leq f_i$ $\forall i \in I$ then $p \leq f$, i.e., if there are maps $B \xrightarrow{p} A$ and $B \xrightarrow{p_i} A_i$ with the property $f_i \circ p_i = p$ $\forall i \in I$ then there exists a unique $h : B \to \cap_{i \in I} A_i$ such that $p = f \circ h$.

Definition 6.9.4. An infinite limit cardinal α is *regular* when it is equal to its cofinality: $\mathrm{cf}(\alpha) = \alpha$. Here $\mathrm{cf}(\alpha)$ is the least limit ordinal β such that there exists an increasing sequence $\{\alpha_\eta\}_{\eta < \beta}$ with $\lim_{\eta \to \beta} \alpha_\eta = \alpha$.

The fact that α is regular means cannot be written as a sum of a lesser number of cardinals less than α.

Definition 6.9.5. Let α be a regular cardinal. An ordered set J is *α-directed* when for every subset $I \subseteq J$ with $|I| \leq \alpha$ there exists an upper bound in J.

Definition 6.9.6. Let $S = \{f_\xi : C_\xi \to B\}$ be a family of subobjects of B with the monotonic property: $f_\xi \leq f_\zeta$ whenever $\xi \leq \zeta$. The family S is called *α-directed* provided that the set J is *α-directed.*

Definition 6.9.7. We say that an object $A \in \mathcal{A}$ is *bounded* by a regular cardinal α when for every morphism from A to a α-directed union $\cup_{\xi \in J} C_\xi$ factors through a union $\cup_{\xi \in K} C_\xi$ for some $K \subseteq J$ with $|K| < \alpha$. We call $\mathcal{A}$ *bounded* if each $A \in \mathcal{A}$ is bounded.

Definition 6.9.8. Let $E, M \subseteq \mathrm{Mor}(\mathcal{A})$ be two classes of morphisms. A *factorization system* (E, M) on a category $\mathcal{A}$ consists of the following data:

- $\mathrm{Isos}(\mathcal{A}) \subseteq E \cap M$, isomorphisms belong to the intersection of the two classes
- E and M are closed under composition
- for every morphism f there is a factorization $f = m \circ e$ with $e \in E$ and $m \in M$
- for every f and g if $m' \circ e' \circ f = g \circ m \circ e$ with $e, e' \in E$ and $m, m' \in M$ then there exists a unique w making the whole diagram

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[r, "m"] \arrow[d, dashed, "w"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "e'"'] & \bullet \arrow[r, "m'"'] & \bullet \end{tikzcd}$$

commutative. A factorization system (E, M) is called a *proper factorization* when $E \subseteq \mathrm{Epis}(\mathcal{A})$, $M \subseteq \mathrm{Monos}(\mathcal{A})$ where $\mathrm{Epis}(\mathcal{A})$ is the class of all epimorphisms of $\mathcal{A}$ and $\mathrm{Monos}(\mathcal{A})$ is the class of all monomorphisms of $\mathcal{A}$.

Definition 6.9.9. An epimorphism p is called *extremal* provided that whenever we have $p = m \circ g$, where m is a monomorphism then m is also an isomorphism. Dually we define the notion of *extremal monomorphism.* Epi* denotes the class of extremal epimorphism and Mon* the class of extremal monomorphism.

Proposition 6.9.10. *If one of these two conditions below are satisfied*

- *the category $\mathcal{A}$ is finitely complete and has arbitrary intersections of monomorphisms*
- *the category $\mathcal{A}$ is finitely co-complete and has arbitrary co-intersections of extremal epimorphisms*

then (Epi*, Mon) *is a proper factorization system.*

Proof. [24] □

In the case of the category of sets a direct calculation shows that Epi* = Epi and Mon* = Mon since we have: if $p \in$ Epi with $p = m \circ g$ then $a \circ m = b \circ m$ implies $a \circ m \circ g = b \circ m \circ g$ which is $a \circ p = b \circ p$ and this $a = b$.

Definition 6.9.11. Given factorization system (E, M) a *generator* of the category $\mathcal{A}$ is a small full subcategory $\mathcal{G}$ such that for each $A \in \mathcal{A}$, $\cup_{G \in \mathcal{G}} \mathcal{A}(G, A) \subseteq E$.

When a factorization system (E, M) is proper and $\mathcal{G}$ a generator then given any pair of morphisms $f, g : A \to B$ then for every $p : G \to A$ with $G \in \mathcal{G}$ we have that $f \circ p = g \circ p \;\Rightarrow\; f = g$.

If $\mathcal{A}$ has coproducts then $\mathcal{G}$ is a generator if and only if for every $A \in \mathcal{A}$ the map:

$$k_A : \coprod_{G \in \mathcal{G}} (\coprod_{\mathcal{A}(G,A)}) \to A$$

is in E; where k_A is defined by the universal property of the coproduct, i.e., $k_A \circ i_{G,f} = f : G \to A$ and $i_{G,f} : G \to \coprod_{G \in \mathcal{G}} (\coprod_{\mathcal{A}(G,A)})$ is the coproduct injection.

Definition 6.9.12. Let $P, Q : \mathcal{K} \to \mathcal{C}$ be functors with $\mathcal{K}$ a small category. A *cylinder* in $\mathcal{C}$ is just a natural transformation $\alpha : P \to Q$.

Definition 6.9.13. A functor $T : \mathcal{C} \to \mathcal{A}$ is *continuous* with respect to the cylinder α when:

- there exists $\lim TP$ and $\lim TQ$ as cones in $\mathcal{A}$.
- the unique morphism $\lim T\alpha : \lim TP \to \lim TQ$ determined by the definition of limit $(\lim TQ, \pi_K^Q)$:

$$\begin{tikzcd} TPK \arrow[r, "T\alpha_K"] & TQK \\ \lim TP \arrow[u, "\pi_K^P"] \arrow[r, dashed, "\lim T\alpha"] & \lim TQ \arrow[u, "\pi_K^Q"'] \end{tikzcd}$$

  is an isomorphism.

Remark 6.9.14. In the case when $P = \Delta C$ is a constant functor, $C \in \mathcal{C}$, then α is just a cone in the usual sense and continuity is the standard definition of continuity of functors.

Definition 6.9.15. Let Γ be a class of cylinders in the category $\mathcal{C}$. Then $[\mathcal{C}, \mathcal{A}]_\Gamma$ is the full subcategory of $[\mathcal{C}, \mathcal{A}]$ of functors T that are continuous w.r.t. each $(P, Q, \alpha) \in \Gamma$.

Definition 6.9.16. Consider an arrow $f : A \to B$ and an object $C \in \mathcal{A}$. We say that f is *orthogonal* to C, and we write $C \perp f$, if for every morphism $y : A \to C$ there exists a unique $x : B \to C$ such that $x \circ f = y$.

This definition is basically the definition of a bijective function since is equivalent to the fact that the representables $\mathcal{A}(B, C) \xrightarrow{\mathcal{A}(f,C)} \mathcal{A}(A, C)$ are isos in the category of sets.

Dually we consider $f \perp C$.

Definition 6.9.17. Given a class Δ of morphisms in a category $\mathcal{A}$, let us consider the full subcategory of $\mathcal{A}$ defined by the following object: $\Delta^\perp = \{B \in \mathcal{A} : B \perp f, \forall f \in \Delta\}$.

Definition 6.9.18. Let us consider $X \in \mathbf{Set}$, where $A \in \mathcal{A}$. The tensor product $X \otimes A \in \mathcal{A}$ is the co-power, i.e., the coproduct of $|X|$ copies of the object A in the category $\mathcal{A}$ characterized by the following natural isomorphism:

$$\mathcal{A}(X \otimes A, B) \cong \mathbf{Set}(X, \mathcal{A}(A, B)).$$

Now, to each cylinder $\alpha : P \to Q : \mathcal{K} \to \mathcal{C}$ we associate an arrow $\tilde{\alpha} : \tilde{Q} \to \tilde{P}$ in the presheaf category $[\mathcal{C}, \mathcal{A}]$ in the following way.

First we consider the functor $\hat{P} : \mathcal{K}^{op} \to [\mathcal{C}, \mathbf{Set}]$ defined by:

$$\mathcal{K}^{op} \xrightarrow{P^{op}} \mathcal{C}^{op} \xrightarrow{Y} [\mathcal{C}, \mathbf{Set}]$$

thus $\hat{P}(K) = \mathcal{C}(P(K), -)$, and $\mathcal{C}(f, -)$ if $f^{op} \in \mathcal{K}^{op}$.

Then, we take $\tilde{P} = colim\, \hat{P}$ the pointwise colimit in the category $[\mathcal{C}, \mathbf{Set}]$, i.e., $\tilde{P} \in [\mathcal{C}, \mathbf{Set}]$.

In the same way, at the level of arrows we get:

$$\hat{\alpha_K} = \mathcal{C}(\alpha_K, -) : \mathcal{C}(QK, -) \longrightarrow \mathcal{C}(PK, -)$$

and then we obtain:

$$\begin{tikzcd}
\hat{Q}(K) \arrow[r, "\hat{\alpha}_K"] \arrow[d, "\pi_K^{\mathcal{Q}}"] & \hat{P}(K) \arrow[d, "\pi_K^{\mathcal{P}}"] \\
\tilde{Q} \arrow[r, dashed, "\tilde{\alpha}"] & \tilde{P}
\end{tikzcd}$$

by definition of colimit $(\tilde{Q}, \pi_K^{\mathcal{Q}})$, since $\pi_K^{\mathcal{P}} \circ \hat{\alpha}_K$ is natural in K. So, $\tilde{\alpha}$ is given as the unique arrow in $[\mathcal{C}, \mathbf{Set}]$ making the previous diagram commute. Now we consider the class of morphisms $\Delta \subseteq [\mathcal{C}, \mathcal{A}]$ depending on a choice of a class of cylinders Γ:

$$\Delta = \{\tilde{\alpha} \otimes A : \tilde{Q} \otimes A \to \tilde{P} \otimes A, \text{with } A \in \mathcal{A},\ \alpha \in \Gamma\}$$

where $\tilde{Q} \otimes A : \mathcal{C} \to \mathcal{A}$ and $\tilde{\alpha} \otimes A$ are defined using the pointwise co-power as $(\tilde{Q} \otimes A)(C) = \tilde{Q}(C) \otimes A$.

Proposition 6.9.19. *Let $\mathcal{A}$ be a complete and co-complete category and let Γ be a class of cylinders in the small category $\mathcal{C}$. Then $[\mathcal{C}, \mathcal{A}]_\Gamma = \Delta^\perp$.*

Proof. Since both categories are full it is enough to check that they contain the same objects. We want to prove that $T \in \Delta^\perp$ if and only if $T \in [\mathcal{C}, \mathcal{A}]_\Gamma$.

By definition of the orthogonal class, $T \in \Delta^\perp$ if and only if for every $\tilde{\alpha} \otimes A \in \Delta$ we have that $[\mathcal{C}, \mathcal{A}](\tilde{\alpha} \otimes A, T)$ is a bijective map, i.e., for every $\mu : \tilde{Q} \otimes A \to T$ there

exists a unique ν such that,

$$\begin{tikzcd} \tilde{Q} \otimes A \arrow[r, "\mu"] \arrow[d, "\tilde{\alpha}\otimes A"'] & T \\ \tilde{P} \otimes A \arrow[ur, "\nu"'] & \end{tikzcd}$$

But since when $\mathcal{A}(F(X), B) \cong \mathbf{Set}(X, G(B))$ is natural with $F : \mathbf{Set} \to \mathcal{A}$, $F(X) = X \otimes A$ and $G : \mathcal{A} \to \mathbf{Set}$, $G(B) = \mathcal{A}(A, B)$. Then we have that:

$$\begin{tikzcd} \mathcal{A}(F(X), B) \arrow[r, "\phi_{X,B}"] \arrow[d, "\mathcal{A}(F(f),g)"'] & \mathbf{Set}(X, G(B)) \arrow[d, "\mathbf{Set}(f,G(g))"] \\ \mathcal{A}(F(X'), B') \arrow[r, "\phi_{X',B'}"'] & \mathbf{Set}(X', G(B')) \end{tikzcd}$$

with $X' \xrightarrow{f} X$ and $B \xrightarrow{g} B'$.

This implies that $G(g) \circ \phi_{X,B}(x) \circ f = \phi_{X',B'}(g \circ x \circ F(f))$ for every $x : F(X) \to B$. Therefore choosing $g = 1$, $x = \nu$, $X = \tilde{P}$, $X' = \tilde{Q}$, $f = \tilde{\alpha}$, $B = T$ we have that since $\nu \circ F(\tilde{\alpha}) = \mu$ then $\phi_{X',B'}(\nu \circ F(\tilde{\alpha})) = \phi_{X',B'}(\mu)$ and then $G(1) \circ \phi_{X,B}(\nu) \circ \tilde{\alpha} = \phi_{X',B'}(\mu)$.

Using the natural isomorphism let us call $\nu' = \phi_{X,B}(\nu) : \tilde{P} \to \mathcal{A}(A, T-)$ where $\nu : F(\tilde{P}) \to T$ and $\mu' = \phi_{X',B'}(\mu) : \tilde{Q} \to \mathcal{A}(A, T-)$ where $\mu : F(\tilde{Q}) \to T$. So this turns out to be $\nu' \circ \tilde{\alpha} = \mu$,

$$\begin{tikzcd} \tilde{Q} \arrow[r, "\mu'"] \arrow[d, "\tilde{\alpha}"'] & \mathcal{A}(A, T-) \\ \tilde{P} \arrow[ur, "\nu'"'] & \end{tikzcd}$$

Then by definition of $\tilde{Q} = \operatorname{colim} \hat{Q}$ with injection $\hat{Q}K \xrightarrow{i_K^Q} \tilde{Q}$ and $\tilde{P} = \operatorname{colim} \hat{P}$ with injection $\hat{P}K \xrightarrow{i_K^P} \tilde{P}$ we define μ'' and ν'' by the following compositions: $\mu'' = \mu' \circ i_K^Q$ where $\mathcal{C}(QK, -) = \hat{Q}K \xrightarrow{i_K^Q} \tilde{Q} \xrightarrow{\mu'} \mathcal{A}(A, T-)$ and $\nu'' = \nu' \circ i_K^P$ where $\mathcal{C}(PK, -) = \hat{P}K \xrightarrow{i_K^P} \tilde{P} \xrightarrow{\mu'} \mathcal{A}(A, T-)$. Therefore we have

$$\begin{tikzcd} \mathcal{C}(QK, -) \arrow[r, "i_K^Q"] \arrow[d, "\mathcal{C}(\alpha_K,-)"'] & \tilde{Q} \arrow[r, "\mu'"] \arrow[d, "\tilde{\alpha}"'] & \mathcal{A}(A, T-) \\ \mathcal{C}(PK, -) \arrow[r, "i_K^P"'] & \tilde{P} \arrow[ur, "\nu'"'] & \end{tikzcd}$$

Let us call $F = \mathcal{A}(A, T-)$, then by naturality of the Yoneda Lemma with respect to α_K we have that:

$$\begin{tikzcd}
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(PK, -), F)} \arrow[r, "\theta_P"] \arrow[d, "{[\mathcal{C},\mathbf{Set}](\mathcal{C}(\alpha_K,-),F)}"'] & F(PK) \arrow[d, "F(\alpha_K)"'] \\
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(QK, -), F)} \arrow[r, "\theta_Q"] & F(QK)
\end{tikzcd}$$

Thus if we evaluate $\nu : \mathcal{C}(PK, -) \to F$ we obtain:

$$\theta_Q(\nu \circ \mathcal{C}(\alpha_K, -)) = F(\alpha_K)(\theta_P(\nu))$$

and since $F = \mathcal{A}(A, T-)$ then we get

$$\theta_Q(\nu \circ \mathcal{C}(\alpha_K, -)) = T(\alpha_K) \circ \theta_P(\nu)$$

Therefore since $\mu'' = \nu'' \circ \mathcal{C}(\alpha_K, -)$ we have by choosing $\nu = \nu''$:

$$\theta_Q(\mu'') = T(\alpha_K) \circ \theta_P(\nu'')$$

where $\theta_Q(\mu'') \in F(QK) = \mathcal{A}(A, TQK)$, $\theta_Q(\mu'') : A \to TQK$ and $\theta_Q(\nu'') \in F(PK) = \mathcal{A}(A, TPK)$, $\theta_P(\nu'') : A \to TPK$.

So by naturality of K and the definition of limit we obtain the following diagram:

$$\begin{tikzcd}
 & & \lim TQ \arrow[dl, "\pi^Q"] \arrow[dd, "\lim T\alpha"] \\
A \arrow[urr, "\tilde{\mu}"] \arrow[r, "\theta_Q(\mu'')"'] \arrow[d, equal] & TQK & \\
A \arrow[r, "\theta_P(\nu'')"] \arrow[drr, "\tilde{\nu}"'] & TPK \arrow[u, "T(\alpha_K)"'] & \\
 & & \lim TP \arrow[ul, "\pi^P"']
\end{tikzcd}$$

Thus the condition of $T \in [\mathcal{C}, \mathcal{A}]_\Gamma$ (continuity) is by definition that $\lim T\alpha$ is an isomorphism and $T \in \Delta^\perp$ (orthogonality) iff $[\mathcal{C}, \mathcal{A}](\tilde{\alpha} \otimes A, T)$ is an isomorphism. $\square$

Theorem 6.9.20. *Let $\mathcal{A}$ be a complete and co-complete category with a given proper factorization system (E, M). Let $\mathcal{A}$ be bounded and co-well-powered. Let us consider the class $\Delta = \Phi \cup \Psi$ where Φ is small and where $\Psi \subseteq E$. Then $\Delta^\perp$ is a reflective subcategory of $\mathcal{A}$.*

Proof. [24] □

Theorem 6.9.21. *Let $\mathcal{A}$ be a complete and co-complete category with a given proper factorization system (E, M). Let $\mathcal{A}$ be bounded with a generator, and co-well-powered. Let Γ be a class of cylinders in the small category $\mathcal{C}$, and let all but a set of these cylinders be cones. Then $[\mathcal{C}, \mathcal{A}]_\Gamma$ is a reflective subcategory of $[\mathcal{C}, \mathcal{A}]$.*

Proof. [24] □

6.10 Day's reflection theorem

Let $\mathcal{B}$ be a symmetric monoidal closed category. Day's so-called reflection theorem [19] can be used to derive a monoidal closed structure in a reflective subcategory of $[\mathcal{B}^{op}, \mathbf{Set}]$. In Chapter 7, we shall utilize this to determine a strong monoidal functor which, in turns, determines a monoidal adjunction. Here, we review Day's reflection theorem.

Definition 6.10.1. A class of objects $\mathcal{A} \subseteq |\mathcal{B}|$ is *strongly generating* when $\mathcal{B}(1, f) : \mathcal{B}(A, B) \to \mathcal{B}(A, B')$ is an isomorphism for every $A \in \mathcal{A}$ implies that $f : B \to B'$ is an isomorphism in $\mathcal{B}$.

Dually we define the notion of *strongly cogenerating* class of object by considering the maps $\mathcal{B}(f, 1)$.

Example 6.10.2. The class $\mathcal{A} \subseteq [\mathcal{B}^{op}, \mathbf{Set}]$, where $\mathcal{A} = \{\mathcal{B}(-, B) : B \in |\mathcal{B}|\}$ are representables, is strongly generating. To see this we must prove that if $(1, \alpha) : [\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-, B), F) \to [\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-, B), G)$ is an isomorphism for every $B \in \mathcal{B}$, where $(1, \alpha) = [\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-, B), \alpha)$ acts on natural transformations as $(1, \alpha)(\beta) = \alpha \circ \beta$, then $\alpha : F \Rightarrow G$ is an isomorphism. To prove this, consider the following diagram:

$$\begin{tikzcd}
{[\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-, B), F)} \arrow[r, "(1,\alpha)"] & {[\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-, B), G)} \arrow[d, "\phi_G"'] \\
F(B) \arrow[u, "\phi_F^{-1}"] \arrow[r, "\alpha_B"'] & G(B)
\end{tikzcd}$$

where $\phi_F^{-1} : F(B) \to [\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-,B), F)$ is defined $\phi_F^{-1}(x) : \mathcal{B}(-,B) \Rightarrow F$ as $(\phi_F^{-1}(x))_C(g) = F(g)(x)$ and $\phi_G : [\mathcal{B}^{op}, \mathbf{Set}](\mathcal{B}(-,B), G) \to G(B)$ is defined as $\phi_G(\beta) = \beta_B(1_B)$.

Therefore, we have

$$(\phi_G \circ (1,\alpha) \circ \phi_F^{-1})(x) = \phi_G((1,\alpha)((\phi_F^{-1}(x)))) =$$

$$= \phi_G(\alpha \circ \phi_F^{-1}(x)) = (\alpha \circ \phi_F^{-1}(x))_B(1_B) = \alpha_B \circ (\phi_F^{-1}(x))_B(1_B) =$$

$$= \alpha_B((\phi_F^{-1}(x))_B(1_B)) = \alpha_B(F(1_B)(x)) = \alpha_B(1_{FB})(x)) = \alpha_B(x),$$

which means $\phi_G \circ (1,\alpha) \circ \phi_F^{-1} = \alpha_B$.

Theorem 6.10.3. *(Day's reflection theorem) Let $(\mathcal{B}, \otimes, I, [-])$ be a symmetric monoidal closed category, and let $\mathcal{B} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}$ be an adjunction from $\mathcal{B}$ to $\mathcal{C}$, where G is full and faithful. Let $\mathcal{A} \subseteq |\mathcal{B}|$ be a strongly generating class in $\mathcal{B}$ and $\mathcal{D} \subseteq |\mathcal{C}|$ be a strongly cogenerating class in $\mathcal{C}$. Then the following are equivalent:*

(0) there exists a monoidal closed structure on $\mathcal{C}$ for which F is a monoidal strong functor.

(a) $\eta : [B, GC] \to GF[B, GC]$, is an isomorphism for all $C \in \mathcal{C}$, $B \in \mathcal{B}$.

(b) $\eta : [A, GD] \to GF[A, GD]$, is an isomorphism for all $A \in \mathcal{A}$, $D \in \mathcal{D}$

(c) $[\eta, 1] : [GFB, GC] \to [B, GC]$, is an isomorphism for all $C \in \mathcal{C}$, $B \in \mathcal{B}$.

(d) $F(\eta \otimes 1) : F(B \otimes B') \to F(GFB \otimes B')$, is an isomorphism for all $B, B' \in \mathcal{B}$.

(e) $F(\eta \otimes 1) : F(B \otimes A) \to F(GFB \otimes A)$, is an isomorphism for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

(f) $F(\eta \otimes \eta) : F(B \otimes B') \to F(GFB \otimes GFB')$, is an isomorphism for all $B, B' \in \mathcal{B}$.

Proof. $(a) \Rightarrow (b)$ Since $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{C}$.
$(b) \Rightarrow (e)$

$$\begin{tikzcd}
\mathcal{C}(F(GFB \otimes A), D) \arrow[r, "\mathcal{C}(F(\eta\otimes 1),1)"] \arrow[d, "\text{adjunction}"'] \arrow[dr, phantom, "(4)"] & \mathcal{C}(F(B \otimes A), D) \arrow[d, "\text{adjunction}"'] \\
\mathcal{B}(GFB \otimes A, GD) \arrow[r, "\mathcal{B}(\eta\otimes 1,1)"] \arrow[d, "\text{closed}"'] \arrow[dr, phantom, "(3)"] & \mathcal{B}(B \otimes A, GD) \arrow[d, "\text{closed}"'] \\
\mathcal{B}(GFB, [A, GD]) \arrow[r, "\mathcal{B}(\eta,1)"] \arrow[d, "\eta \text{ iso by hypothesis}"', "\mathcal{B}(1\otimes\eta)"] \arrow[dr, phantom, "(2)"] & \mathcal{B}(B, [A, GD]) \arrow[d, "\eta \text{ iso by hypothesis}"', "\mathcal{B}(1\otimes\eta)"] \\
\mathcal{B}(GFB, GF[A, GD]) \arrow[r, "\mathcal{B}(\eta,1)"] \arrow[d, "G \text{ fully faithful}"'] \arrow[dr, phantom, "(1)"] & \mathcal{B}(B, GF[A, GD]) \\
\mathcal{C}(FB, F[A, GD]) \arrow[ur, "\text{adjunction}"'] &
\end{tikzcd}$$

(1) commutes since we have $\theta(f) = G(f) \circ \eta_B = (\mathcal{B}(\eta, 1) \circ G_{FB,F[A,GD]})(f)$

$$\begin{tikzcd}
\mathcal{B}(GFB, GF[A, GD]) \arrow[r, "\mathcal{B}(\eta,1)"] & \mathcal{B}(B, GF[A, GD]) \\
\mathcal{C}(FB, F[A, GD]) \arrow[u, "G"'] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

(2) by functoriality; (3) and (4) by naturality. The vertical and bottom arrows are isos then the top is an isomorphism. Hence since $D \subseteq \mathcal{C}$ is strongly cogenerating we have that $F(\eta \otimes 1) : F(B \otimes A) \to F(GFB \otimes A)$ is an isomorphism for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

$(e) \Rightarrow (c)$

$$\begin{tikzcd}
\mathcal{C}(F(GFB \otimes A), C) \arrow[r, "\mathcal{C}(F(\eta\otimes 1),1)"] \arrow[d, "\text{adjunction}"'] \arrow[dr, phantom, "(2)"] & \mathcal{C}(F(B \otimes A), C) \arrow[d, "\text{adjunction}"'] \\
\mathcal{B}(GFB \otimes A, GC) \arrow[r, "\mathcal{B}(\eta\otimes 1,1)"] \arrow[d, "\text{closed}"'] \arrow[dr, phantom, "(1)"] & \mathcal{B}(B \otimes A, GC) \arrow[d, "\text{closed}"'] \\
\mathcal{B}(A, [GFB, GC]) \arrow[r, "\mathcal{B}(1,[\eta,1])"] & \mathcal{B}(A, [B, GC])
\end{tikzcd}$$

(1) and (2) commute by naturality. The top arrow is an isomorphism by hypothesis, also the vertical arrows are isomorphism, this implies that the bottom arrow is an iso and since $\mathcal{A}$ is strongly generating then $[\eta, 1] : [GFB, GC] \to [B, GC]$ is an isomorphism as well.

$(c) \Rightarrow (d)$ We use the same diagram with $A \in \mathcal{B}$.

$(d) \Rightarrow (f)$
By functoriality

$$\begin{tikzcd}
F(B \otimes B') \arrow[rr, "F(\eta\otimes\eta)"] \arrow[dr, "F(\eta\otimes 1)"'] & & F(GFB \otimes GFB') \\
& F(GFB \otimes B') \arrow[ur, "F(1\otimes\eta)"'] &
\end{tikzcd}$$

$(f) \Rightarrow (a)$
We want to find an arrow $\nu : GF[B, GC] \to [B, GC]$ such that $\eta \circ \nu = \nu \circ \eta = 1$. From naturality of the following diagram

$$\begin{tikzcd}
\mathcal{B}(GF[B, GC] \otimes B, GC) \arrow[r, "\phi"] \arrow[d, "\mathcal{B}(\eta\otimes 1, 1)"] & \mathcal{B}(GF[B, GC], [B, GC]) \arrow[d, "\mathcal{B}(\eta, 1)"] \\
\mathcal{B}([B, GC] \otimes B, GC) \arrow[r, "\phi"] & \mathcal{B}([B, GC], [B, GC])
\end{tikzcd}$$

we obtain $\mathcal{B}(\eta \otimes 1, 1)(\phi^{-1}(\nu)) = \phi^{-1}(\mathcal{B}(\eta, 1)(\nu))$ which implies that $\phi^{-1}(\nu) \circ (\eta \otimes 1) = \phi^{-1}(\nu \circ \eta)$. On the other hand we have that

$$1 = \nu \circ \eta \text{ if and only if } \phi^{-1}(1) = \phi^{-1}(\nu \circ \eta) \text{ if and only if } ev = \phi^{-1}(\nu) \circ (\eta \otimes 1).$$

Therefore by uniqueness it is enough to find an arrow x of the correct type which is a solution of the following equation

$$ev = x \circ (\eta \otimes 1)$$

for then $x = \phi^{-1}(\nu)$, i.e., $\phi(x) = \nu$. We choose $x = G(\theta^{-1}(ev))GF(\eta \otimes \eta)\eta(1 \otimes \eta)$ satisfying the following diagram

$$\begin{tikzcd}
{[B,GC]\otimes B} \arrow[rr, "ev"] \arrow[dr, "\eta\otimes\eta"] \arrow[drr, "\eta"] \arrow[dd, "\eta\otimes 1"'] & & GC \\
& & GF([B,GC]\otimes B) \arrow[u, "G(\theta^{-1}(ev))"'] \arrow[u, phantom, shift left=4ex, "(1)"] \\
GF[B,GC]\otimes B \arrow[r, "1\otimes\eta"'] & GF[B,GC]\otimes GFB \arrow[r, "\eta"'] & GF(GF[B,GC]\otimes GFB) \arrow[u, "GF(\eta\otimes\eta)"']
\end{tikzcd}$$

To justify (1), let $\phi : \mathcal{B}([B,GC]\otimes B, GC) \to \mathcal{B}([B,GC],[B,GC])$ be the tensor adjunction. By definition we have $ev = \phi^{-1}(1_{[B,GC]})$. Now consider the adjunction between functors F and G,

$$\theta : \mathcal{C}(F([B,GC]\otimes B), C) \to \mathcal{B}([B,GC]\otimes B, GC)$$

and take $e' = \theta^{-1}(ev)$. Then we have that $G(e')\circ\eta_{[B,GC]\otimes B} = \theta(e') = \theta(\theta^{-1}(ev)) = ev$. It remains to prove that $\eta\circ\nu = 1$. Since $G : \mathcal{C} \to \mathcal{B}$ is a fully faithful functor, there is a unique f such that $G(f) = \eta\circ\nu$. Also we know that $\nu\circ\eta = 1$ Hence, we have

$$G(f)\circ\eta = (\eta\circ\nu)\circ\eta = \eta\circ(\nu\circ\eta) = \eta\circ(1) = \eta = G(1)\circ\eta.$$

Finally, from the adjunction $\theta : \mathcal{C}(F[B,GC], F[B,GC] \to \mathcal{B}([B,GC], GF[B,GC])$ we obtain $\theta(f) = G(f)\circ\eta_{[B,GC]}$, which implies that $\theta(f) = \theta(1)$, i.e., $f = 1$. Therefore $\eta\circ\nu = G(1) = 1$.

$(0) \Rightarrow (f)$ See [43].

$(f) \Rightarrow (0)$

The monoidal closed structure induced on $\mathcal{C}$:

Now using Theorem 6.10.3 we are able to induce a monoidal structure on the category $\mathcal{C}$. Define $C\tilde{\otimes}C' = F(GC\otimes GC')$ and $f\tilde{\otimes}g = F(Gf\otimes Gg)$. Also define $\tilde{I} = FI$ and (F,m) is monoidal functor, where $m_{A,B} : F(A)\tilde{\otimes}F(B) \to F(A\otimes B)$ is given by: $m_{A,B} = (F(\eta\otimes\eta))^{-1}$ with $(F(\eta\otimes\eta))^{-1} : F(GFA\otimes GFB) \to F(A\otimes B)$.

The tensor has right adjoint given by the following formula $[C,E]_{\mathcal{C}} = F[GC,GE]$, $C,E \in |\mathcal{C}|$

$$\mathcal{C}(D\tilde{\otimes}C, E) = \mathcal{C}(F(GD\otimes GC), E) \cong \mathcal{B}(GD\otimes GC, GE)$$

$$\cong \mathcal{B}(GD, [GC, GE]) \cong \mathcal{B}(GD, GF[GC, GE]) \cong \mathcal{C}(D, F[GC, GE]) \cong \mathcal{C}(D, [C, E]_{\mathcal{C}}).$$

In order to obtain a monoidal structure on the category $\mathcal{C}$ we define natural isomorphisms $\tilde{\lambda}$, $\tilde{\rho}$ and $\tilde{\alpha}$ determined by the following diagrams:

$$\begin{tikzcd}
\tilde{I}\tilde{\otimes}C = F(GFI \otimes GC) \arrow[r, "\tilde{\lambda}"] \arrow[d, "F(\eta_I \otimes 1)^{-1}"'] & C \\
F(I \otimes GC) \arrow[r, "F(\lambda)"'] & FGC \arrow[u, "\epsilon_C"']
\end{tikzcd}$$

$$\begin{tikzcd}
C\tilde{\otimes}\tilde{I} = F(GC \otimes GFI) \arrow[r, "\tilde{\rho}"] \arrow[d, "F(1 \otimes \eta_I)^{-1}"'] & C \\
F(GC \otimes I) \arrow[r, "F(\rho)"'] & FGC \arrow[u, "\epsilon_C"']
\end{tikzcd}$$

$$\begin{tikzcd}
C\tilde{\otimes}(C'\tilde{\otimes}C'') = F((GC \otimes GF(GC' \otimes GC')) & F((GC \otimes (GC' \otimes GC'')) \arrow[l, "F(1 \otimes \eta)"'] \\
(C\tilde{\otimes}C')\tilde{\otimes}C'' = F(GF(GC \otimes GC') \otimes GC'') \arrow[u, "\tilde{\alpha}"] \arrow[r, "F(\eta \otimes 1)^{-1}"'] & F((GC \otimes GC') \otimes GC'') \arrow[u, "F(\alpha)"']
\end{tikzcd}$$

For example we want to check that:

$$\begin{tikzcd}
(C\tilde{\otimes}\tilde{I})\tilde{\otimes}C' \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\tilde{\rho}\tilde{\otimes}1"'] & & C\tilde{\otimes}(\tilde{I}\tilde{\otimes}C') \arrow[dl, "1\tilde{\otimes}\tilde{\lambda}"] \\
& C\tilde{\otimes}C' &
\end{tikzcd}$$

this diagram is the center (F) of the following diagram:

$$\begin{tikzcd}
 & & & F(GC \otimes GC') \arrow[dd, bend left, "1"] \arrow[dd, "F(1\otimes\eta_{GC'})"] \\
F(GC \otimes (I \otimes GC')) \arrow[rr, "F(1\otimes\eta_{I\otimes GC'})"'] \arrow[rrrr, bend left, "F(1\otimes\lambda)"] \arrow[d, "F(1\otimes(\eta\otimes 1))"] \arrow[dddd, bend right=60, "F(\alpha)"'] \arrow[ddddrr, bend right=60, "F(\rho\otimes 1)"'] & E & F(GC \otimes GF(I \otimes GC')) \arrow[rr, "F(1\otimes GF(\lambda))"] \arrow[dr, "G"] & & F(GC \otimes GF(GC')) \arrow[d, "F(1\otimes G(\epsilon_{C'}))"] \\
 & & & I & \\
F(GC \otimes (GFI \otimes GC')) \arrow[rr, "F(1\otimes\eta)"'] \arrow[d, "F(\alpha)"] \arrow[dd, "B"] & C & F(GC \otimes GF(GFI \otimes GC')) \arrow[u, "F(1\otimes G(F(\eta\otimes 1)^{-1}))"] \arrow[rr, "1\tilde{\otimes}\tilde{\lambda}"] & F & F(GC \otimes GC') \\
F((GC \otimes GFI) \otimes GC') \arrow[d, "F((1\otimes\eta)\otimes 1)^{-1}"] \arrow[dd, bend right=60, "F(\alpha)"'] & A & F(G(F(GC \otimes GFI)) \otimes GC') \arrow[l, "F(\eta\otimes 1)^{-1}"'] \arrow[u, "\hat{\alpha}"] \arrow[d, "F(G(F(1\otimes\eta_I)^{-1})\otimes 1)"] \arrow[urr, "\tilde{\rho}\tilde{\otimes}1"] & H & F(G(F(GC)) \otimes GC') \arrow[u, "F(G(\epsilon_C)\otimes 1)"] \arrow[uu, bend right=60, "1"'] \\
F((GC \otimes I) \otimes GC') \arrow[r, "F(\eta_{GC\otimes I}\otimes 1)"] \arrow[uuuu, bend left=60] \arrow[rrr, "F(\rho\otimes 1)"'] & & F(G(F(GC \otimes I)) \otimes GC') \arrow[urr, "F(G(F(\rho))\otimes 1)"] & D & F(GC \otimes GC') \arrow[u, "F(\eta_{GC}\otimes 1)"] \arrow[uu, bend right=60]
\end{tikzcd}$$

Diagram A: By naturality of η with $1 \otimes \eta_I : GC \otimes I \to GC \otimes GFI$, then by functoriality of $- \otimes GC'$ and F we obtain:

$$\begin{tikzcd}
F((GC \otimes I) \otimes GC') \arrow[r, "F(\eta_{GC\otimes I}\otimes 1)"] \arrow[d, "F((1\otimes\eta_I)\otimes 1)"'] & F(GF(GC \otimes I) \otimes GC') \arrow[d, "F(GF(1\otimes\eta_I)\otimes 1)"] \\
F((GC \otimes GFI) \otimes GC') \arrow[r, "F(\eta_{GC\otimes GFI}\otimes 1)"'] & F(GF(GC \otimes GFI) \otimes GC')
\end{tikzcd}$$

Since $F(1_{GC} \otimes \eta_I)$, $F(\eta_{GC\otimes I} \otimes 1_{GC'})$ and $F(\eta_{GC\otimes GFI} \otimes 1_{GC'})$ are invertible map this implies that $F((1_{GC} \otimes \eta_I) \otimes 1_{GC'})$ is invertible as well.

Diagram D: by naturality of η with $\rho : GC\otimes I \to GC$ we have that $GF(\rho)\circ\eta_{GC\otimes I} = \eta_{GC} \circ \rho$ then by functoriality of $- \otimes GC'$ and F.

Diagram H: by definition we have $\tilde{\rho} = F(1 \otimes \eta)^{-1}; F(\rho); \varepsilon$, then we apply functor $-\tilde{\otimes}- = F(G(-) \otimes G(-))$ to the pair of arrows $(\tilde{\rho}, 1_{C'})$.

Diagram C: by definition of $\tilde{\alpha}$.

Diagram B: by considering the diagram A, the map $F((1_{GC} \otimes \eta_I) \otimes 1_{GC'})^{-1}$ makes sense, also by naturality of α with 1_{GC}, η_I and $1_{GC'}$, and then compose with F.

Diagram E: this is analogous to diagram A. We consider naturality of η with the map $\eta_I \otimes 1 : I \otimes GC' \to GFI \otimes GC'$, then compose with the functor $GC \otimes -$ and F. Since $F(\eta_I \otimes 1)$ is invertible then $F(1 \otimes GF(\eta_I \otimes 1))$ is invertible and we have that:

$$F(1 \otimes \eta_{I \otimes GC'}) = F(1 \otimes G((F(\eta_I \otimes 1)^{-1})) \circ F(1 \otimes \eta_{GFI \otimes GC'}) \circ F(1 \otimes (\eta_I \otimes 1))$$

Diagram G: this is analogous to diagram D. Naturality of η with $\lambda : I \otimes GC' \to GC'$ then compose with $GC \otimes -$ and F.

At the bottom of the diagram we have an adjoint equation: $\eta_G \circ G(\varepsilon) = 1$.

We can also define ρ on the image of F in the following way:

$$\begin{tikzcd}
FB \tilde{\otimes} \tilde{I} = F(GFB \otimes GFI) \arrow[r, "\tilde{\rho}"] \arrow[d, "F(\eta_B \otimes \eta_I)^{-1}"'] & FB \\
F(B \otimes I) \arrow[ur, "F(\rho)"'] &
\end{tikzcd}$$

This coincides with the above definition:

$$\begin{tikzcd}
F(GFB \otimes GFI) \arrow[r, "F(1_{GFB} \otimes \eta_I)^{-1}"] \arrow[dd, "F(\eta_B \otimes \eta_I)^{-1}"'] & F(GFB \otimes I) \arrow[d, "F(\rho)"'] \\
 & F(G(F(B))) \arrow[d, "\varepsilon_{FB}"'] \\
F(B \otimes I) \arrow[r, "F(\rho)"'] & FB
\end{tikzcd}$$

To see this we have that:

$F(\rho) \circ F(\eta_B \otimes \eta_I)^{-1} = \varepsilon_{FB} \circ F(\rho) \circ F(1_{GFB} \otimes \eta_I)^{-1}$ iff

$F(\rho) = \varepsilon_{FB} \circ F(\rho) \circ F(1_{GFB} \otimes \eta_I)^{-1} \circ F(\eta_B \otimes \eta_I)$ iff

$F(\rho) = \varepsilon_{FB} \circ F(\rho) \circ F(1_{GFB} \otimes \eta_I)^{-1} \circ F(1_{GFB} \otimes \eta_I) \circ F(\eta_B \otimes 1_I)$ iff

$F(\rho) = \varepsilon_{FB} \circ F(\rho) \circ F(\eta_B \otimes 1_I)$ iff

$\varepsilon_{FB}^{-1} \circ F(\rho) = F(\rho) \circ F(\eta_B \otimes 1_I)$ iff

$F(\eta_B) \circ F(\rho) = F(\rho) \circ F(\eta_B \otimes 1_I)$ iff

$F(\eta_B \circ \rho) = F(\rho \circ (\eta_B \otimes 1_I))$

where the last two equations are justified by naturality of ρ with $\eta_B : B \to GFB$ and, since G is full and faithful, we have that ε is an isomorphism and $\varepsilon_{FB}^{-1} = F(\eta_B)$.

We can also define an associativity isomorphism on the image of G

$$\tilde{\alpha} : (GC\tilde{\otimes}GC')\tilde{\otimes}GC'' \to GC\tilde{\otimes}(GC'\tilde{\otimes}GC'')$$

in the following way:

$$\begin{tikzcd}
F(G(F(G(FB) \otimes G(FB'))) \otimes G(FB'')) \arrow[r, "\tilde{\alpha}"] \arrow[d, "F(G(F(\eta\otimes\eta)^{-1})\otimes 1)"'] & F(GFB \otimes GF(GFB' \otimes GFB'')) \\
F(GF(B \otimes B') \otimes GFB'') \arrow[d, "F(\eta\otimes\eta)^{-1}"'] & F(GFB \otimes GF(B' \otimes B'')) \arrow[u, "F(1\otimes GF(\eta\otimes\eta))"'] \\
F((B \otimes B') \otimes B'') \arrow[r, "F(\alpha)"'] & F(B \otimes (B' \otimes B'')) \arrow[u, "F(\eta\otimes\eta)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(G(F(G(FB) \otimes G(FB'))) \otimes G(FB'')) \arrow[r, "F(\eta\otimes 1)^{-1}"] \arrow[d, "F(G(F(\eta\otimes\eta)^{-1})\otimes 1)"'] & F((GFB \otimes GFB') \otimes GFB'') \arrow[d, "F(\alpha)"'] \\
F(GF(B \otimes B') \otimes GFB'') \arrow[d, "F(\eta\otimes\eta)^{-1}"'] \arrow[ur, "F((\eta\otimes\eta)\otimes\eta)"'] \arrow[dr, phantom, "A"] & F(GFB \otimes (GFB' \otimes GFB'')) \arrow[d, "F(1\otimes\eta)"'] \\
F((B \otimes B') \otimes B'') \arrow[d, "F(\alpha)"'] \arrow[ur, "F(\eta\otimes(\eta\otimes\eta))"'] \arrow[dr, phantom, "B"] & F(GFB \otimes GF(GFB' \otimes GFB'')) \arrow[d, phantom, "C"] \\
F(B \otimes (B' \otimes B'')) \arrow[r, "F(\eta\otimes\eta)"'] & F(GFB \otimes GF(B' \otimes B'')) \arrow[u, "F(1\otimes GF(\eta\otimes\eta))"']
\end{tikzcd}$$

Diagram A commutes by naturality of η with $\eta \otimes \eta : B \otimes B' \to GFB \otimes GFB'$: we apply $- \otimes \eta_{B''}$

$$\begin{tikzcd}
(B \otimes B') \otimes B'' \arrow[r, "\eta\otimes\eta"'] \arrow[d, "(\eta\otimes\eta)\otimes\eta"'] & GF(B \otimes B') \otimes GFB'' \arrow[d, "GF(\eta\otimes\eta)\otimes 1"'] \\
(GFB \otimes GFB') \otimes GFB'' \arrow[r, "\eta\otimes 1"'] & GF(GFB \otimes GFB') \otimes GFB''
\end{tikzcd}$$

and then we apply functor F.

Diagram B commutes by naturality of the isomorphism α

Diagram C is analogous to diagram A: it commutes by naturality of η with $\eta \otimes \eta : B' \otimes B'' \to GFB' \otimes GFB''$, then we apply $\eta \otimes -$ and finally we evaluate the functor F on this diagram. □

6.11 Application of Day's reflection theorem to presheaves

Now we consider a particular case of Theorem 6.10.3 studied in [20]. Let us consider $[\mathcal{B}^{op}, \mathbf{Set}] \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}$ with G fully faithful and where $([\mathcal{B}^{op}, \mathbf{Set}], \otimes, I)$ has the monoidal structure induced by the convolution product (defined in Proposition 6.8.1). When $A = \mathcal{B}(-, B)$ is a representable functor, by the Yoneda Lemma we have that:

$$[A, G(C)] = \int_{B'} [\mathcal{B}(B', B), G(C)(- \otimes B')] \cong G(C)(- \otimes B) \tag{17}$$

Now suppose there exists $C' \in \mathcal{C}$ such that

$$G(C)(- \otimes B) \cong G(C') \tag{18}$$

is a natural isomorphism between functors. Let us explicitly call ϕ the composition of these two isomorphisms (17) and (18) above: $\phi : [A, G(C)] \to G(C')$. Then we have:

$$\begin{tikzcd}
{[A, G(C)]} \arrow[r, "\phi"] \arrow[d, "\eta_{[A,G(C)]}"] & G(C') \arrow[d, "\eta_{G(C')}"] \\
GF[A, G(C)] \arrow[r, "GF(\phi)"] & GF(G(C'))
\end{tikzcd}$$

From this diagram we conclude that the condition of $\eta_{[A,G(C)]}$ being an isomorphism is equivalent to the condition of $\eta_{G(C')}$ of being an isomorphism. Thus, since G is fully faithful we have by Proposition 6.5.1 that $\eta * G$ is always an isomorphism which implies that $\eta_{[A,G(C)]}$ is an isomorphism as well. Therefore, the adjunction is monoidal if and only if condition (18) is satisfied.

In the particular case when G is an inclusion this translates to the condition that there exists an isomorphism $C(- \otimes B) \cong C'$ where $C \in \mathcal{C} \subseteq [\mathcal{B}^{op}, \mathbf{Set}]$, $B \in \mathcal{B}$ for some $C' \in \mathcal{C}$.

Remark 6.11.1. Consider $\mathcal{C} = [\mathcal{B}^{op}, \mathbf{Set}]_{inf}$. Suppose we have two functors F and H isomorphic in $[\mathcal{B}^{op}, \mathbf{Set}]$. Then F preserves limits if and only if H preserves limits. Therefore the condition $C(- \otimes B) \cong C' \in \mathcal{C}$ implies that $C(- \otimes B)$ preserves limits, i.e., $C(- \otimes B) \in [\mathcal{B}^{op}, \mathbf{Set}]_{inf}$. We have by hypothesis that $C \in \mathcal{C}$ and hence it depends on whether the functor $- \otimes B : \mathcal{B}^{op} \to \mathcal{B}^{op}$ preserves limits. The same is valid if we consider not all but some specific limits: a certain class Γ.

Chapter 7

Presheaf models of a quantum lambda calculus

In this chapter we study a categorical model for the quantum lambda calculus of Selinger and Valiron [67]. We focus on exploring the existence of such a model using presheaf categories.

In [63], Selinger defined an elementary quantum flow chart language and gave a denotational model in terms of superoperators. This axiomatic framework captures the behavior and interconnection between the basic quantum computation concepts such as the manipulation of quantum bits by considering two basic operations: measurement and unitary transformation in a lower-level language. In particular, the semantics of this framework is very well understood: each program corresponds to a concrete superoperator.

Higher-order functions are functions that can input or output other functions. In order to deal with higher-order functions, Selinger and Valiron introduced, in several papers [68], [69], [70] a typed lambda calculus for quantum computation and investigated several aspects of its semantics. In this context, they combined two very well-established languages in the literature of computer science: the intuitionistic fragment of Girard's linear logic [26] and the computational monads introduced by Moggi in [56].

The type system of Selinger and Valiron's quantum lambda calculus is based on

intuitionistic linear logic, where the rules of weakening and contraction are controlled in a sensitive way by an operator "!" called "of course" or "exponential". This operator creates a bridge between two different kinds of computation. More precisely, a value of a general type A can only be used once, whereas a value of type $!A$ can be copied and used multiple times. The impossibility of copying quantum information is one of the fundamental differences between quantum information and classical information, and is known as the *no-cloning property.* From the logical perspective, it therefore seems natural to relate quantum computation and linear logic. Note that the operator "!" satisfies the properties of a *comonad.*

Since we have higher-order functions, as well as probabilistic operations (namely quantum measurement), the language needs to address the question of evaluation strategies. Otherwise, in some concrete situation, it would be impossible to give a coherent outcome every time for identical circumstances. In order to deal with this issue, Selinger and Valiron chose to incorporate a methodology *à la* Moggi by making the distinction between values and computations. Moggi [56] proposed the notion of a monad as an appropriate tool for interpreting computational behavior. At the level of the denotational model, this will be reflected by a strong monad.

To summarize, let us say that the exponential operator ! will be modelled by a monoidal comonad arising from an adjunction between a cartesian category (accounting for classical duplicability) and a symmetric monoidal category (accounting for quantum non-duplicability) while the manipulation of the probabilistic aspect of the quantum computation is handled by a monoidal monad. The result of combining these two methodologies is what Selinger and Valiron call a *linear category for duplication.*

This is not the first time that this interaction between a monad and a comonad has been invoked in order to express denotational aspects of a system in computer science (see [10] for example). But what is new in Selinger and Valiron's work, is putting this interaction in the context of quantum computation.

In this thesis, we will focus exclusively on the categorical aspects of the model construction. Thus, we will not review the syntax of the quantum lambda calculus itself. Instead, we will take as our starting point Selinger and Valiron's definition of

a *categorical model of the quantum lambda calculus* [70]. It was already proven in [70] that the quantum lambda calculus forms an internal language for the class of such models. This is similar to the well-known interplay between typed lambda calculus and cartesian closed categories [52]. What was left open in [70] was the construction of a concrete such model (other than that given by the syntax itself). This is the question we answer here.

The use of category theory to model and to explain formal languages has an established tradition in logic, but in quantum computation it constitutes a relatively recent trend. We finish this introduction by stressing that the field of quantum computation in connection with category theory is fast-growing. The ability to create bridges among these different branches of mathematics that are apparently far from one another is one of the motivating goals of this thesis and we hope to contribute in this direction.

7.1 Definition of a categorical model for quantum lambda calculus

In the introduction we informally described the main ideas and motivation of what should be a categorical model for quantum lambda calculus. Here we shall take the formal definition in [70] as our starting point. However, before presenting it, we will give some preliminary definitions and we shall make some remarks about how to simplify its presentation. Several of the definitions sketched here will be made more precise in Section 7.3 and beyond.

Let $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ be a symmetric monoidal category.

Definition 7.1.1. A symmetric *monoidal comonad* $(!, \delta, \varepsilon, m_{A,B}, m_I)$ is a comonad $(!, \delta, \varepsilon)$ where the functor ! is a monoidal functor $(!, m_{A,B}, m_I)$, i.e., with natural transformations $m_{A,B} : !A \otimes !B \to !(A \otimes B)$ and $m_I : I \to !I$ satisfying the coherence axioms of Definition 2.2.4, such that δ and ε are symmetric monoidal natural transformations.

Definition 7.1.2. A *linear exponential comonad* is a symmetric monoidal comonad $(!, \delta, \varepsilon, m_{A,B}, m_I)$ in which the following conditions hold:

- for every $A \in \mathcal{C}$ there exists a commutative comonoid, with $d_A : !A \to !A \otimes !A$ and $e_A : !A \to I$ as associated maps,
- d_A and e_A are monoidal natural transformation with respect to the natural transformations m,
- d_A and e_A are coalgebra morphisms when we consider $(!A, \delta_A)$, $(!A \otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$, and (I, m_I) as coalgebras,
- the maps $\delta_A : (!A, e_A, d_A) \to (!!A, e_{!A}, d_{!A})$ are comonoid morphisms.

Definition 7.1.3. Let (T, η, μ) be a strong monad. We say that $\mathcal{C}$ has *Kleisli exponentials* if there exists a functor $[-,-]_k : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and a natural isomorphism:

$$\mathcal{C}(A \otimes B, TC) \cong \mathcal{C}(A, [B, C]_k))$$

Remark 7.1.4. When the category $(\mathcal{C}, \otimes, [-,-])$ is a monoidal closed category then it certainly has Kleisli exponentials just by putting $[B, C]_k = [B, TC]$.

Definition 7.1.5 (Linear category for duplication [70]). A *linear category for duplication* consists of a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ satisfying the following data:

- an idempotent, strongly monoidal, linear exponential comonad $(!, \delta, \varepsilon, d, e)$,
- a strong monad (T, μ, η, t),
- $\mathcal{C}$ has Kleisli exponentials.

Further, if the unit I is a terminal object we shall speak of an *affine linear category for duplication*, cf. Definition 2.5.1.

Remark 7.1.6. The definition of a linear category for duplication (Definition 7.1.5) is equivalent to the existence of a pair of monoidal adjunctions ([9], [55] and [49]):

$$(\mathcal{B}, \times, 1) \underset{(I,i)}{\overset{(L,l)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{C}, \otimes, I) \underset{(G,n)}{\overset{(F,m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{D}, \otimes, I)$$

where the category $\mathcal{B}$ has finite products and $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal closed categories. The monoidal adjoint pair of functors on the left represents a linear-non-linear model in the sense of Benton [9] in which we obtain a monoidal comonad by $! = L \circ I$. The monoidal adjoint on the right gives rise to $T = G \circ F$ a strong monad in the sense of Kock [48], [49] which is also a computational monad in the sense of Moggi [56].

We now state the main definition of a model of the quantum lambda calculus.

Definition 7.1.7 (Model of the quantum lambda calculus [70]). An *abstract model of the quantum lambda calculus* is an affine linear category for duplication $\mathcal{C}$ with finite coproducts, preserved by the comonad !. Moreover, a *concrete model of the quantum lambda calculus* is an abstract model of the quantum lambda calculus such that there exists a full and faithful embedding $\mathbf{Q} \hookrightarrow \mathcal{C}_T$, preserving tensor $\otimes$ and coproduct $\oplus$, from the category $\mathbf{Q}$ of norm non-increasing superoperators (see Definition 3.2.4) into the Kleisli category generated by the monad T.

Remark 7.1.8. To make the connection to quantum lambda calculus: the category $\mathcal{C}$, the Kleisli category $\mathcal{C}_T$, and the co-Kleisli category $\mathcal{C}_!$ all have the same objects, which correspond to *types* of the quantum lambda calculus. The morphism $f : A \to B$ of $\mathcal{C}$ correspond to *values* of type B (parameterized by variables of type A). A morphism $f : A \to B$ in $\mathcal{C}_T$, which is really a morphism $f : A \to TB$ in $\mathcal{C}$, corresponds to a *computation* of type B (roughly, a probability distribution of values). Finally, a morphism $f : A \to B$ in $\mathcal{C}_!$, which is really a morphism $f : !A \to B$ in $\mathcal{C}$, corresponds to a *classical value* of type B, i.e., one which only depends on classical variables. The idempotence of "!" implies that morphisms $!A \to B$ are in one-to-one correspondence with morphisms $!A \to !B$, i.e., classical values are duplicable. For details, see [70].

7.2 Outline of the procedure for obtaining the model

Our complete process for obtaining a categorical model of the quantum lambda calculus consists of two stages. In the first stage, we will construct *abstract* models of

the quantum lambda calculus by applying a certain presheaf construction to suitable sequences of functors $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$. This construction is very general, and the base categories $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ can be viewed as parameters. We will identify the precise conditions required of the base categories (and the functors connecting them) in order to obtain a valid abstract model. This is the content of Chapter 7.

In the second stage, we will construct a *concrete* model of the quantum lambda calculus by identifying particular base categories so that the remaining conditions of Definition 7.1.7 are satisfied. This is the content of Chapter 8.

We briefly outline the main steps of the construction; full details will be given in later sections.

- The basic idea of the construction is to lift a sequence of functors

$$\mathcal{B} \xrightarrow{\Phi} \mathcal{C} \xrightarrow{\Psi} \mathcal{D}$$

  into a pair of adjunctions between presheaf categories

$$\begin{tikzcd} {[\mathcal{B}^{op}, \mathbf{Set}]} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & {[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[l, shift left, "\Phi^*"] \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] & {[\mathcal{D}^{op}, \mathbf{Set}]} \arrow[l, shift left, "\Psi^*"] \end{tikzcd}$$

  Here, Φ^* and Ψ^* are the precomposition functors, and L and F_1 are their left Kan extensions. By Remark 7.1.6, such a pair of adjunctions potentially yields a linear category for duplication, and therefore, with additional conditions, an abstract model of quantum computation. Our goal is to identify the particular conditions on $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, Φ, and Ψ, that make this construction work correctly.

- By Day's construction, the requirement that $[\mathcal{C}^{op}, \mathbf{Set}]$ and $[\mathcal{D}^{op}, \mathbf{Set}]$ are monoidal closed can be achieved by requiring $\mathcal{C}$ and $\mathcal{D}$ to be monoidal. The requirement that the adjunctions $L \dashv \Phi^*$ and $F_1 \dashv \Psi^*$ are monoidal is directly related to the fact that the functors Ψ and Φ are strong monoidal. More precisely, this implies that the left Kan extension is a strong monoidal functor which in turn determines the enrichment of the adjunction. We also note that the category $\mathcal{B}$ must be cartesian.

- One important complication with the model, as discussed so far, is the following. The Yoneda embedding $Y : \mathcal{D} \to [\mathcal{D}^{op}, \mathbf{Set}]$ is full and faithful, and by Day's result, also preserves the monoidal structure $\otimes$. Therefore, if one takes $\mathcal{D} = \mathbf{Q}$, all but one of the conditions of a concrete model (from Definition 7.1.7) are automatically satisfied. Unfortunately, the Yoneda embedding does not preserve coproducts, and therefore the remaining condition of Definition 7.1.7 fails. For this reason, we modify the construction and use the modified presheaf category and coproduct-preserving Yoneda embedding from Section 6.9. Our adjunctions, and the associated Yoneda embeddings, now look like this:

$$\begin{tikzcd}
{[\mathcal{B}^{op}, \mathbf{Set}]} \arrow[r, "L \dashv \Phi^*"] & {[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[r, "F \dashv G"] & {[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma} \\
\mathcal{B} \arrow[u, "Y"] \arrow[r, "\Phi"] & \mathcal{C} \arrow[u, "Y"] \arrow[r, "\Psi"] & \mathcal{D} \arrow[u, "Y_\Gamma"]
\end{tikzcd}$$

The second pair of adjoint functors $F \dashv G$ is generated by the composition of two adjunctions:

$$\begin{tikzcd}
{[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] & {[\mathcal{D}^{op}, \mathbf{Set}]} \arrow[l, shift left, "\Psi^*"] \arrow[r, shift left, "F_2"] \arrow[r, phantom, "\perp"] & {[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma} \arrow[l, shift left, "G_2"]
\end{tikzcd}$$

Here, the pair of functors $F_2 \dashv G_2$ arises as a reflection of $[\mathbf{Q}^{op}, \mathbf{Set}]_\Gamma$ in $[\mathbf{Q}^{op}, \mathbf{Set}]$, and depends on a choice of a certain class Γ of cones. The structural aspects of the modified Yoneda embedding $\mathbf{Q} \to [\mathbf{Q}^{op}, \mathbf{Set}]_\Gamma$ depend crucially on general properties of the functor categories, which go back to the study of continuous functors by Lambek (see Section 6.6) and Freyd and Kelly (see Section 6.9).

But, as we mentioned before, at the same time we still require that the reflection functor remain strongly monoidal. Here will will use Day's results (see Section 6.10) on the conditions that are needed for the reflection to be strong monoidal, by inducing a monoidal structure from the category $[\mathbf{Q}^{op}, \mathbf{Set}]$ into its subcategory $[\mathbf{Q}^{op}, \mathbf{Set}]_\Gamma$ (see Section 6.10). In particular, this induces a constraint on the choice of Γ considered above: all the cones considered in Γ

must be preserved by the opposite functor of the tensor function in $\mathcal{D}$ (see Remark 6.11.1).

- Notice that the above adjunctions are examples of what in topos theory is named an *essential geometric* morphism, in which both functors are left adjoint to some other two functors: $L \dashv \Phi^* \dashv \Phi_*$. Therefore, this shows that the comonad "!" obtained will preserve finite coproducts.

- The condition for the comonad "!" to be idempotent turns out to depend on the fact that the functor Φ is full and faithful.

- In addition to the requirement that "!" preserves coproducts, we also need "!" to preserve the tensor, i.e., to be strongly monoidal, as required in Definition 7.1.7. This property is unusual for models of intuitionistic linear logic and puts some restriction on the range of possible choices we have for the category $\mathcal{C}$. In brief, since the left Kan extension along Φ is a strong monoidal functor we find that a concrete condition in the category $\mathcal{C}$ is necessary to ensure that this property holds when we lift the functor Φ to the category of presheaves; see Section 7.6.

- Once we have constructed this categorical environment our next task is to translate these properties to the Kleisli category. To achieve this we use the comparison Kleisli functor for passing from the framework we have already established to the Kleisli monoidal adjoint pair of functors. Also, at the same time in this context, we shall find it convenient to characterize the functor $H : \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]_T$ as a strong monoidal functor.

  All of the above steps yield an abstract model of quantum computation, parametric on the sequence of functors $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$.

- Finally, as we shall see in Section 8.2, we will identify specific categories $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ that yield a concrete model of quantum computation. We let $\mathcal{D} = \mathbf{Q}$, the category of superoperators. The categories $\mathcal{B}$ and $\mathcal{C}$ must be chosen in such a way as to satisfy all of the properties outlined above. For $\mathcal{B}$, we take the category of finite sets.

Identifying a suitable candidate for the category $\mathcal{C}$ is more tricky. For example, here are two of the requirements directly concerning the semantics: $\mathcal{C}$ must be affine monoidal and must satisfy the condition of equation (19) in Section 7.6.

In a series of intermediate steps, with the help of some universal constructions, we introduce a category $\mathcal{C} = \mathbf{Q}''$ related to the category $\mathbf{Q}$ of superoperators.

As we have noted, the category $\mathbf{Q}''$ plays a central role in our construction. It is in some sense the "barycenter" of our model. While the basic structural properties occur at the level of the functor categories, providing a general mathematical setting, the development of the concrete quantum meaning of the model occurs mostly at this base level.

7.3 Categorical models of linear logic

The first definition of a categorical model of linear logic was given by Seely [62]. Other pioneering studies in this area were Lafont's thesis [50] and Abramsky's paper [1]. Also, Melliès' survey [55] is an excellent introduction to the topic.
Now we formulate Bierman's definition of linear category [14] which is based upon the above-mentioned previous work on the Topic. We also state an equivalent alternative simplified version that we take from Benton [9] (this is the notion we outlined in Remark 7.1.6). For the purpose of this thesis, since it is clear that the linear fragment of Definition 7.1.7 does not impose any constraints on the rest of the definition, it follows that it will be more helpful to work with Benton's version representing the underlying linear fragment. In any case, to appreciate the details behind these categorical models, Bierman's definition will occupy the rest of the present section.

Definition 7.3.1 (Bierman). A *linear category* $\mathcal{C}$ consists of a symmetric monoidal closed category $(C, I, \otimes, \multimap, \alpha, \lambda, \rho, \gamma)$ with a symmetric monoidal comonad $(!, \varepsilon, \delta, m_I, m_{A,B})$ defined on $\mathcal{C}$ and monoidal natural transformations $e : !(-) \to I$, $d : !(-) \to !(-) \otimes !(-)$ such that:

- $e_A : !(A) \to I$, $d_A : !(A) \to !(A) \otimes !(A)$ are coalgebra morphisms for each A;

- $((!A, \delta_A), e_A, d_A)$ is a commutative comonoid for every free coalgebra $(!A, \delta_A)$ and

- morphisms between free coalgebras $f : (!A, \delta_A) \to (!B, \delta_B)$ are also comonoid commutative morphisms.

We will now consider the meaning of each of these conditions:

- for every $A \in \mathcal{C}$ there exists a commutative comonoid, with $d_A : !A \to !A \otimes !A$ and $e_A : !A \to I$ as associated maps. This means the following:
  The assumption that $((!A, \delta_A), e_A, d_A)$ is a commutative comonoid for every free coalgebra $(!A, \delta_A)$ means that:

$$\begin{tikzcd}
!A \arrow[rr, "d_A"] \arrow[d, "d_A"'] & & !A \otimes !A \arrow[d, "d_A \otimes 1_{!A}"] \\
!A \otimes !A \arrow[r, "1_{!A} \otimes d_A"'] & !A \otimes (!A \otimes !A) \arrow[r, "\alpha"] & (!A \otimes !A) \otimes !A
\end{tikzcd}$$

$$\begin{tikzcd}
!A & !A \arrow[l, "1_{!A}"'] \arrow[r, "1_{!A}"] \arrow[d, "d_A"'] & !A \\
I \otimes !A \arrow[u, "\lambda_{!A}"] & !A \otimes !A \arrow[l, "e_A \otimes 1_{!A}"'] \arrow[r, "1_{!A} \otimes e_A"] & !A \otimes I \arrow[u, "\rho_{!A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "d_A"] \arrow[dr, "d_A"'] & !A \otimes !A \arrow[d, "\gamma_{!A,!A}"] \\
& !A \otimes !A
\end{tikzcd}$$

- d_A and e_A are monoidal natural transformation with respect to the natural transformation m.

  The transformations $e : !(-) \to I$ and $d : !(-) \to !(-) \otimes (-)$ are monoidal natural transformations between monoidal functors; if $f : A \to B$ then $e : (!, m_{A,B}, m_I) \to (I, \lambda_I, 1_I)$ is the statement that the following diagrams commute:

$$\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[dr, "e_A"'] & !B \arrow[d, "e_B"] \\
& I
\end{tikzcd}$$

$$\begin{tikzcd} !A\otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "e_A\otimes e_B"'] & !(A\otimes B) \arrow[d, "e_{A\otimes B}"] \\ I\otimes I \arrow[r, "\lambda_I"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "m_I"] \arrow[dr, "id_I"'] & !I \arrow[d, "e_I"] \\ & I \end{tikzcd}$$

and $d : (!, m_{A,B}, m_I) \to (!\otimes !, t_{A,B}, t_I)$:

$$\begin{tikzcd} !A \arrow[r, "d_A"] \arrow[d, "!f"'] & !A\otimes !A \arrow[d, "!f\otimes !f"] \\ !B \arrow[r, "d_B"'] & !B\otimes !B \end{tikzcd}$$

$$\begin{tikzcd} !A\otimes !B \arrow[rr, "m_{A,B}"] \arrow[d, "d_A\otimes d_B"'] & & !(A\otimes B) \arrow[d, "d_{A\otimes B}"] \\ !A\otimes !A\otimes !B\otimes !B \arrow[r, "Id_{!A}\otimes\gamma_{!A,!B}\otimes Id_{!B}"'] & !A\otimes !B\otimes !A\otimes !B \arrow[r, "m_{A,B}\otimes m_{A,B}"] & !(A\otimes B)\otimes !(A\otimes B) \end{tikzcd}$$

with $t_{A,B} = (m_{A,B} \otimes m_{A,B}) \circ Id_{!A} \otimes \gamma_{!A,!B} \otimes Id_{!B}$ and

$$\begin{tikzcd} I \arrow[r, "m_I"] \arrow[d, "\lambda_I^{-1}"'] & !I \arrow[d, "d_I"] \\ I\otimes I \arrow[r, "m_I\otimes m_I"'] & !I\otimes !I \end{tikzcd}$$

with $t_I = (m_I \otimes m_I) \circ \lambda_I^{-1}$.

- d_A and e_A are coalgebra morphisms when we consider $(!A, \delta_A)$, $(!A\otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$, and (I, m_I) as coalgebras:

  The definition of linear category characterizes $e_A : (!A, \delta_A) \to (I, m_I)$ and $d_A : (!A, \delta_A) \to (!A\otimes !A, m_{!A,!A} \circ (\delta_A \otimes \delta_A))$ as coalgebra morphisms which means that the following diagrams commute:

$$\begin{tikzcd} !A \arrow[r, "\delta_A"] \arrow[d, "e_A"'] & !!A \arrow[d, "!e_A"] \\ I \arrow[r, "m_I"'] & !I \end{tikzcd} \qquad \begin{tikzcd} !A \arrow[r, "\delta_A"] \arrow[d, "d_A"'] & !!A \arrow[d, "!d_A"] \\ !A\otimes !A \arrow[r, "m_{!A,!A}\circ(\delta_A\otimes\delta_A)"'] & !(!A\otimes !A) \end{tikzcd}$$

- Morphisms between free coalgebras $f : (!A, \delta_A) \to (!B, \delta_B)$ are also comonoid commutative morphisms. This means that if $f : !A \to !B$ is an arrow with $!f \circ \delta_A = \delta_B \circ f$ then is also true that $f : (!A, d_A, e_A) \to (!B, d_B, e_B)$ is a map between commutative comonoids that is $f : !A \to !B$ is an arrow that satisfies:

$$\begin{tikzcd} !A \arrow[r, "d_A"] \arrow[d, "f"'] & !A\otimes !A \arrow[d, "f\otimes f"] \\ !B \arrow[r, "d_B"'] & !B\otimes !B \end{tikzcd} \qquad \begin{tikzcd} !A \arrow[r, "f"] \arrow[dr, "e_A"'] & !B \arrow[d, "e_B"] \\ & I \end{tikzcd}$$

To complete the list of conditions let us show the structural conditions. The natural transformations $\varepsilon : !(-) \to I$ and $\delta : !(-) \to !!(-)$ are monoidal. If $(!, m_{A,B}, m_I)$ and $(Id, 1_{A\otimes B}, 1_I)$ are monoidal functors then $\varepsilon : (!, m_{A,B}, m_I) \to (Id, 1_{A\otimes B}, 1_I)$ is a monoidal natural transformation which is compatible in the sense that the following diagrams commute:

$$\begin{tikzcd} !A\otimes !B \arrow[r, "m_{A,B}"] \arrow[d, "\varepsilon_A\otimes\varepsilon_B"'] & !(A\otimes B) \arrow[d, "\varepsilon_{A\otimes B}"] \\ A\otimes B \arrow[r, "1"'] & A\otimes B \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "m_I"] \arrow[dr, "1"'] & !I \arrow[d, "\varepsilon_I"] \\ & I \end{tikzcd}$$

Also $\delta : (!, m_{A,B}, m_I) \to (!!, t_{A,B}, t_I)$ is a monoidal natural transformation between monoidal functors; with $t_{A,B} = !(m_{A,B}) \circ m_{!A,!B}$ and $t_I = !(m_I) \circ m_I$:

$$\begin{tikzcd} !A\otimes !B \arrow[rr, "m_{A,B}"] \arrow[d, "\delta_A\otimes\delta_B"'] & & !(A\otimes B) \arrow[d, "\delta_{A\otimes B}"] \\ !!A\otimes !!B \arrow[r, "m_{!A,!B}"'] & !(!A\otimes !B) \arrow[r, "!(m_{A,B})"] & !!(A\otimes B) \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "m_I"] \arrow[dr, "t_I"'] & !I \arrow[d, "\delta_I"] \\ & !!I \end{tikzcd}$$

Recalling that a symmetric monoidal comonad $(!, \varepsilon, \delta, m_{A,B}, m_I)$ is a comonad $(!, \varepsilon, \delta)$ equipped with a symmetrical monoidal functor $(!, m_{A,B}, m_I)$, where: $! : C \to C$ is a functor, for every object A and B there is a morphism $m_{A,B} : !A \otimes !B \to !(A \otimes B)$ natural in A and B, for the unit I there is a morphism $m_I : I \to !I$.

These morphisms with the structural maps $\alpha, \lambda, \rho, \gamma$ must make the following diagrams commute:

$$\begin{tikzcd} !A \otimes (!B \otimes !C) \arrow[r, "id_{!A} \otimes m_{B,C}"] \arrow[d, "\alpha"'] & !A \otimes !(B \otimes C) \arrow[r, "m_{A,B \otimes C}"] & !(A \otimes (B \otimes C) \arrow[d, "!\alpha"] \\ (!A \otimes !B) \otimes !C \arrow[r, "m_{A,B} \otimes id_{!C}"'] & !(A \otimes B) \otimes !C \arrow[r, "m_{A \otimes B, C}"] & !((A \otimes B) \otimes C) \end{tikzcd}$$

$$\begin{tikzcd} !B \otimes I \arrow[r, "\rho_{!B}"] \arrow[d, "id_{!B} \otimes m_I"'] & !B \\ !B \otimes !I \arrow[r, "m_{B,I}"'] & !(B \otimes I) \arrow[u, "!(\rho_B)"'] \end{tikzcd} \qquad \begin{tikzcd} I \otimes !B \arrow[r, "\lambda_{!B}"] \arrow[d, "m_I \otimes id_{!B}"'] & !B \\ !I \otimes !B \arrow[r, "m_{I,B}"'] & !(I \otimes B) \arrow[u, "!(\lambda_B)"'] \end{tikzcd}$$

$$\begin{tikzcd} !A \otimes !B \arrow[r, "\gamma_{!A,!B}"] \arrow[d, "m_{A,B}"'] & !B \otimes !A \arrow[d, "m_{B,A}"] \\ !(A \otimes B) \arrow[r, "!(\gamma_{A,B})"'] & !(B \otimes A) \end{tikzcd}$$

Definition 7.3.2 (Benton [9]). A *linear-non-linear category* consists of:

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, I, \multimap)$

(2) a category $(\mathcal{B}, \times, 1)$ with finite product

(3) a symmetric monoidal adjunction:

$$(\mathcal{B}, \times, 1) \underset{(G,n)}{\overset{(F,m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{C}, \otimes, I) .$$

Note that Definition 7.3.2 is far simpler than Definition 7.3.1. Its significance is in the following:

Proposition 7.3.3. *Every linear-non-linear category gives rise to a linear category. Every linear category defines a linear-non-linear category, where* $(\mathcal{B}, \times, 1)$ *is the category of coalgebras of the comonad* $(!, \varepsilon, \delta)$.

Proof. See [9] or [55]. □

Remark 7.3.4. Kelly's characterization of monoidal adjunctions (see Proposition 2.3.6) allows us to replace condition (3) in the definition of linear-non-linear categories by the following new statement in Definition 7.3.2:

(3') an adjunction:

$$(\mathcal{B}, \times, 1) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{C}, \otimes, I)$$

and there exist isomorphisms

$$m_{A,B} : FA \otimes FB \to F(A \times B) \ , \ m_I : I \to F(1)$$

making $(F, m_{A,B}, m_I) : (\mathcal{B}, \times, 1) \to (\mathcal{C}, \otimes, I)$ a strong symmetric monoidal functor.

7.4 Linear-non-linear models on presheaf categories

Our purpose here is to characterize Benton's linear-non-linear models of intuitionistic linear logic, in the sense of Definition 7.3.2, on presheaf categories using Day's monoidal structure from Section 6.8. This is an application of monoidal enrichment of the Kan extension see [22]. We use Kelly's equivalent formulation of monoidal adjunctions from Proposition 2.3.6.

Proposition 7.4.1. *Suppose we have a strong monoidal functor* $\Phi : (\mathcal{A}, \times, 1) \to (\mathcal{B}, \otimes, I)$ *from a cartesian category to a monoidal category, i.e., we have a natural isomorphism* $\Phi(a) \otimes \Phi(b) \cong \Phi(a \times b)$ *and* $I \cong \Phi(1)$.

Let us consider the left Kan extension along Φ in the functor category $[\mathcal{B}^{op}, \mathbf{Set}]$ where the copower is product on sets:

$$Lan_\Phi(F) = \int^a \mathcal{B}(-, \Phi(a)) \times F(a)$$

Then Lan_Φ is strong monoidal.

Proof. By the Yoneda Lemma, the strong functor Φ, Fubini and coend properties: $Lan_\Phi(F \times G) = Lan_\Phi(\int^a \mathcal{A}(-, a) \times F(a) \times \int^b \mathcal{A}(-, b) \times G(b))$ by the Yoneda Lemma and pointwise product

$= Lan_\Phi(\int^{a\,b} \mathcal{A}(-, a) \times \mathcal{A}(-, b) \times F(a) \times G(b)) = Lan_\Phi(\int^{a\,b} \mathcal{A}(-, a \times b) \times F(a) \times G(b))$ cartesian product

$= \int^c \mathcal{B}(-, \Phi(c)) \times \int^{a\,b} \mathcal{A}(c, a \times b) \times F(a) \times G(b) = \int^{a\,b}(\int^c \mathcal{B}(-, \Phi(c)) \times \mathcal{A}(c, a \times b)) \times F(a) \times G(b)$ definition of Kan extension

$= \int^{a\,b} \mathcal{B}(-, \Phi(a \times b)) \times F(a) \times G(b) = \int^{a\,b} \mathcal{B}(-, \Phi(a) \otimes \Phi(b)) \times F(a) \times G(b)$ Φ strong functor

$= \int^{a\,b}(\int^y \mathcal{B}(y, \Phi(a)) \times \mathcal{B}(-, y \otimes \Phi(b))) \times F(a) \times G(b) = \int^{a\,b} \int^y \mathcal{B}(y, \Phi(a)) \times (\int^z \mathcal{B}(-, y \otimes z) \times \mathcal{B}(z, \Phi(b))) \times F(a) \times G(b)$ by the Yoneda Lemma

$= \int^{y\,z}(\int^a \mathcal{B}(y, \Phi(a)) \times F(a)) \times (\int^b \mathcal{B}(z, \Phi(b)) \times G(b)) \times \mathcal{B}(-, y \otimes z) = \int^{y\,z}((Lan_\Phi(F))(y)) \times ((Lan_\Phi(G))(z)) \times \mathcal{B}(-, y \otimes z)$ by Fubini and copower preserves colimits

$= Lan_\Phi(F) \otimes_D Lan_\Phi(G)$ by definition of Kan extension and convolution

and also the units:

$Lan_\Phi(I_D^{\mathcal{A}}) = Lan_\Phi(\mathcal{A}(-, 1)) = \int^a \mathcal{B}(-, \Phi(a)) \times \mathcal{A}(a, 1) = \mathcal{B}(-, \Phi(1)) = \mathcal{B}(-, I) = I_D^{\mathcal{B}}$. □

Remark 7.4.2. Note that, in view of the line of arguments used above, the case where $\mathcal{A}$ is monoidal has the same proof, i.e., if we have $\Phi(a) \otimes \Phi(b) \cong \Phi(a \otimes b)$ and $I \cong \Phi(I')$ we start directly from the convolution product:

$$Lan_\Phi(F \otimes_D G) = Lan_\Phi(\int^{a\,b} \mathcal{A}(-, a \otimes b) \times F(a) \times G(b))$$

and we repeat the same proof. Also notice that when we have a product in $\mathcal{A}$ the convolution is a pointwise product of functors:

$$F \times G = \int^{a\,b} \mathcal{A}(-, a \times b) \times F(a) \times G(b)).$$

Remark 7.4.3. If the unit of a monoidal category $\mathcal{C}$ is a terminal object then the unit of the convolution is also terminal. Let us consider a morphism $\alpha : F \to \mathcal{C}(-, I)$ in the functor category $[\mathcal{C}^{op}, \mathbf{Set}]$. Then for every V there is only one way to define the map $\alpha_V : F(V) \to \mathcal{C}(V, I)$ which is $\alpha_V(x) = !$ for every $x \in F(V)$ in the category of sets. Hence there is a unique α. Therefore it is a terminal object in the functor category.

7.5 Idempotent comonad in the functor category

A comonad $(!, \epsilon, \delta)$ is said to be *idempotent* if $\delta : ! \Rightarrow !!$ is an isomorphism. Let $(!, \epsilon, \delta)$ be the comonad generated by the adjunction:

$$(\mathcal{D}, \times, 1) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{V}, \otimes, I, \multimap)$$

then $\delta = F\eta_G$ with $\eta : A \to GFA$. Thus if η is an isomorphism then δ is also an isomorphism. Now consider the unit of the Kan extension:

$$G \Rightarrow F^*(Lan_F(G)).$$

It is given by:

$$\begin{tikzcd} G(a) \arrow[r, "i_{1_{F(a)}}"] \arrow[dr, "(\eta_G)(a)"'] & \mathcal{B}(F(a), F(a)) \times G(a) \arrow[d, "(w_a)_{F(a)}"] \\ & \int^{a'} \mathcal{B}(F(a'), F(a)) \times G(a') \end{tikzcd} ,$$

where i is the injection of the copower and w is the wedge of the coend.

Proposition 7.5.1. *If F is a full and faithful functor then $\eta_G : G \Rightarrow F^*(Lan_F(G))$ is an isomorphism.*

Proof. [15] □

7.6 A strong comonad

In this section we study conditions that allow us to force the idempotent comonad to be a strong monoidal functor. This property, part of the model we are building, is a main difference with other previously intuitionistic linear models. In order to achieve this, consider a full and faithful functor $\Phi : \mathcal{A} \to \mathcal{B}$ as in Proposition 7.5.1. Let Φ^* be the functor we had seen earlier in Section 6.7:

$$[\mathcal{B}^{op}, \mathbf{Set}] \xrightarrow{\Phi^*} [\mathcal{A}^{op}, \mathbf{Set}],$$

i.e., the right adjoint of the left Kan extension.

Lemma 7.6.1. *If there exists a natural isomorphism:*

$$\mathcal{B}(\Phi(a), b) \times \mathcal{B}(\Phi(a), b') \cong \mathcal{B}(\Phi(a), b \otimes b') \tag{19}$$

where $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ and Φ is a fully faithful, strong monoidal functor then Φ^* is a strong monoidal functor.

Proof. To see this: $\Phi^*(F) \times \Phi^*(G) = F(\Phi(-)) \times G(\Phi(-)) \cong \int^b F(b) \times \mathcal{B}(\Phi(-), b) \times \int^{b'} G(b') \times \mathcal{B}(\Phi(-), b') \cong$ by the Yoneda Lemma, definition of Φ^* and the fact that convolution in $[\mathcal{A}^{op}, \mathbf{Set}]$ is pointwise cartesian product

$\cong \int^{b\,b'} F(b) \times G(b') \times \mathcal{B}(\Phi(-), b) \times \mathcal{B}(\Phi(-), b') \cong$ by properties of coends (preservation)

$\cong \int^{b\,b'} F(b) \times G(b') \times \mathcal{B}(\Phi(-), b \otimes b') =$ by hypothesis (19) and Lemma 6.3.6

$= (F \otimes G)(\Phi(-)) = \Phi^*(F \otimes G)$ by definition of convolution in $[\mathcal{B}^{op}, \mathbf{Set}]$ and definition of Φ^*.

Moreover the units are isomorphic,

$\Phi^*(\mathcal{B}(-, I)) = \mathcal{B}(\Phi(-), I) \cong$ by definition of Φ^*

$\cong \mathcal{B}(\Phi(-), \Phi(1))$ since Φ is strong

$\mathcal{A}(-, 1)$ since Φ is fully faithful. □

Remark 7.6.2. At this point it is useful to mention that the conditions of Lemma 7.6.1 are an example of a *multiplicative kernel* $K : \mathcal{B} \times \mathcal{A}^{op} \to \mathbf{Set}$ from

the monoidal category $\mathcal{B}$ to $\mathcal{A}$ in the sense of [21]. In fact, K is explicitly defined as $K(b,a) = \mathcal{B}(\Phi(a), b)$ satisfying the two following equations as part of the definition:

$$\int^{yz} K(a,y) \times K(b,z) \times \mathcal{A}(x, y \times z) \cong \int^{c} K(c,x) \times \mathcal{B}(c, a \otimes b)$$

$$\int^{b} \mathcal{B}(\Phi(x), b) \times \mathcal{B}(b, I) \cong \mathcal{A}(x, 1)$$

In Section 8.2 we shall built a category satisfying this specific requirement among others. More precisely, from our viewpoint this will depend on the construction of a certain category that we will name $\mathbf{Q}''$ which is a modification of the category $\mathbf{Q}$ of superoperators. Also we consider the functor Φ of Section 2.5 where $\mathcal{C}^+ = \mathbf{Q}''$.

7.7 If $\mathcal{C}$ has finite coproducts then $\mathcal{C}_T$ has finite coproducts

An important property of the Kleisli construction is that if we assume that the original category has finite coproducts then we can define finite coproducts in the Kleisli category.

Proposition 7.7.1. *Kleisli categories inherit coproducts, i.e., if $\mathcal{C}$ has finite coproducts then $\mathcal{C}_T$ also has finite coproducts.*

Proof. Suppose we have that $A \xrightarrow{f^K} C$ and $B \xrightarrow{g^K} C$ two arrows in the category $\mathcal{C}_T$. We take $A \oplus_K B = A \oplus B$ on objects, and

$$\begin{tikzcd} A \arrow[rr, "i_A^T"] \arrow[dr, "i_A"'] & & T(A \oplus B) \\ & A \oplus B \arrow[ur, "\eta_{A \oplus B}"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "i_B^T"] \arrow[dr, "i_B"'] & & T(A \oplus B) \\ & A \oplus B \arrow[ur, "\eta_{A \oplus B}"'] & \end{tikzcd}$$

as injections in the category $\mathcal{C}_T$.

We want to find a unique $A \oplus_K B \xrightarrow{[f^K, g^K]_K} C$ such that $f^K = [f^K, g^K]_K \circ_K i_A^T$ and $g^K = [f^K, g^K]_K \circ_K i_B^T$ commute. This is verified by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, "f"'] & A \oplus B \arrow[r, "\eta_{A\oplus B}"] \arrow[d, "{[f,g]}"] & T(A \oplus B) \arrow[d, "T[f,g]"] & A \oplus B \arrow[l, "\eta_{A\oplus B}"'] \arrow[d, "{[f,g]}"] & B \arrow[l, "i_B"'] \arrow[dl, "g"] \\
& TC \arrow[r, "\eta_{TC}"] \arrow[dr, "1_{TC}"'] & T^2C \arrow[d, "\mu_C"] & TC \arrow[l, "\eta_{TC}"'] \arrow[dl, "1_{TC}"] & \\
& & TC & &
\end{tikzcd}$$

where $[f, g]$ is the unique morphism that defines coproduct in $\mathcal{C}$. This last diagram commutes by naturality of η with respect to $[f, g]$ and by definition of monad.
Uniqueness follows from the following reasoning: suppose there is an arrow $A \oplus_K B \xrightarrow{h_K} C$, i.e., $A \oplus B \xrightarrow{h} TC$, such that $\mu_C \circ T(h) \circ \eta_{A\oplus B} \circ i_A = f$ and $\mu_C \circ T(h) \circ \eta_{A\oplus B} \circ i_B = g$ then by naturality and monad definition we have that $h \circ i_A = f$ and $h \circ i_A = g$, thus by uniqueness in $\mathcal{C}$ we have that $h = [f, g]$. □

We notice that $C : \mathcal{C}_T \to \mathcal{D}$ preserves finite coproducts. To see this, by definition we have that $i_A^T = \eta_{A\oplus B} \circ i_A$ and $i_B^T = \eta_{A\oplus B} \circ i_B$. Then

$$C(i_A^T) = c(\eta_{A\oplus B} \circ i_A) = \varepsilon_{F(A\oplus B)} \circ F(\eta_{A\oplus B} \circ i_A) =$$

$$= \varepsilon_{F(A\oplus B)} \circ F(\eta_{A\oplus B}) \circ F(i_A) = 1_{A\oplus B} \circ F(i_A) = F(i_A).$$

In the same way $C(i_B^T) = F(i_B)$.
Given that right adjoint preserves coproducts then $C(A \oplus_K B) = C(A \oplus_{\mathcal{C}} B) = F(A \oplus_K B) = FA \oplus_{\mathcal{D}} FB$ and

$$C(A \xrightarrow{i_A^T} A \oplus_K B) = C(A) \xrightarrow{C(i_A^T)} C(A \oplus_K B) = F(A) \xrightarrow{F(i_A)} F(A) \oplus_{\mathcal{D}} F(B)$$

which is a coproduct in $\mathcal{D}$.
In the same way we can apply a similar reasoning with B.

7.8 The functor $H : \mathcal{D} \to \hat{\mathcal{C}}_T$

7.8.1 Preliminaries

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\hat{\mathcal{C}}$ and $T = G \circ F$ be defined as in Section 7.2.

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$$

In this section we consider the construction of a coproduct and tensor preserving functor $H : \mathcal{D} \to \hat{\mathcal{C}}_T$ with properties similar to the Yoneda embedding. We investigate the role of a general category $\mathcal{D}$ fully embedded into a Kleisli category $\hat{\mathcal{C}}_T$. Certain properties of this functor are introduced in order to apply this to the category of superoperators $\mathbf{Q}$ as well as to develop a methodology for obtaining higher-order models in the sense of Section 7.1.

Let $F_1 \dashv G_1$ and $F_2 \dashv G_2$ be two monoidal adjoint pairs with associated natural transformations (F_1, m_1), (G_1, n_1) and (F_2, m_2), (G_2, n_2). We shall use the following notation $F = F_2 \circ F_1$, $G = G_1 \circ G_2$, $T = G \circ F$. We now describe a typical situation of this kind generated by a functor $\Phi : \mathcal{C} \to \mathcal{D}$.

Let us consider $F_1 = Lan_\Phi$ and $G_1 = \Phi^*$. With some co-completeness condition assumed, we can express $F_1(A) = \int^c \mathcal{D}(-, \Phi(c)) \otimes A(c)$ and $G_1 = \Phi^*$.

On the other hand we consider

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "Y"] \arrow[dr, "Y_\Gamma"'] & {[\mathcal{D}^{op}, \mathbf{Set}]} \arrow[d, shift left, "F_2"] \arrow[d, phantom, "\vdash"] \\
& {[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma} \arrow[u, shift left, "G_2"]
\end{tikzcd}$$

where we take $F_2 = Lan_Y(Y_\Gamma) : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$, and $Y_\Gamma : \mathcal{D} \to [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ is given by $Y_\Gamma(d) = \mathcal{D}(-, d)$. Thus we have that $F_2(D) = D \star Y_\Gamma = \int^d D(d) \otimes Y_\Gamma(d)$.

Assuming that $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ is co-complete and contains the representable presheaves then the right adjoint is given by

$$G_2(F) = [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma(Y_\Gamma -, F) = [\mathcal{D}^{op}, \mathbf{Set}](Y -, F) \cong F \qquad .$$

since it is a full subcategory and by the Yoneda Lemma. Therefore we consider G_2 as the inclusion functor up to isomorphism.

7.8.2 Definition of H.

We want to study the following situation:

$$\begin{tikzcd}
 & \hat{\mathcal{C}} \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] \arrow[dd, shift left, "F_T"] \arrow[dr, shift left, "C"] & \hat{\mathcal{D}} \arrow[l, shift left, "G_1"] \arrow[r, shift left, "F_2"] \arrow[r, phantom, "\perp"] & \hat{\mathcal{D}}_\Gamma \arrow[l, shift left, "G_2"] \\
\mathcal{C} \arrow[ur, bend left, "Y"] \arrow[dr, bend right, "\Phi"'] & \hat{\mathcal{C}}_T \arrow[u, shift left, "G_T"] \arrow[u, phantom, "\vdash"] & & \\
 & \mathcal{D} \arrow[u, dashed, "H"] \arrow[uur, "Y_\Gamma"'] & &
\end{tikzcd}$$

The goal is to determine a fully faithful functor, H in this diagram, that preserves tensor and coproduct.

First, notice that the perimeter of this diagram commutes on objects:

$$F_1(\mathcal{C}(-,c)) = \int^{c'} \mathcal{D}(-,\Phi(c')) \otimes \mathcal{C}(c',c) = \mathcal{D}(-,\Phi(c))$$

When we evaluate again we obtain:

$$F_2(\mathcal{D}(-,\Phi(c))) = \int^{d'} \mathcal{D}(d',\Phi(c)) \otimes Y_\Gamma(d') = Y_\Gamma(\Phi(c)) = \mathcal{D}(-,\Phi(c))$$

Summing up we have that $F(\mathcal{C}(-,c)) = \mathcal{D}(-,\Phi(c))$ up to isomorphism.

Suppose now that Φ is onto on objects. We have that:

$$\mathcal{D}(-,d) = \mathcal{D}(-,\Phi(c))$$

for some $c \in \mathcal{C}$, i.e., we can make a choice, for every $d \in |\mathcal{D}|$, of some $c \in |\mathcal{C}|$ such that $\Phi(c) \cong d$. Let us call this choice a "choice of preimages". We can therefore define a map $H : |\mathcal{D}| \to |\hat{\mathcal{C}}_T|$ by $H(d) = \mathcal{C}(-,c)$ on objects.

Hence, we can define a functor $H : \mathcal{D} \to \hat{\mathcal{C}}_T$ in the following way:
let $d \xrightarrow{f} d'$ be an arrow in the category $\mathcal{D}$, then we apply Y_Γ obtaining $\mathcal{D}(-,d) \xrightarrow{Y_\Gamma(f)} \mathcal{D}(-,d')$. This arrow is equal to $\mathcal{D}(-,\Phi(c)) \xrightarrow{Y_\Gamma(f)} \mathcal{D}(-,\Phi(c'))$ for some $c, c' \in \mathcal{C}$ and

for the reason stipulated above is equal to $F(\mathcal{C}(-,c)) \overset{Y_\Gamma(f)}{\to} F(\mathcal{C}(-,c'))$. Now we use the fact that the comparison functor $C : \hat{\mathcal{C}}_T \to \hat{\mathcal{D}}_\Gamma$,

$$C : \hat{\mathcal{C}}_T(\mathcal{C}(-,c), \mathcal{C}(-,c')) \to \hat{\mathcal{D}}_\Gamma(F(\mathcal{C}(-,c)), F(\mathcal{C}(-,c')))$$

is fully faithful, i.e, there is a unique $\gamma : \mathcal{C}(-,c) \to \mathcal{C}(-,c')$ such that $C(\gamma) = Y_\Gamma(f)$. Then we define: $H(f) = \gamma$ on morphisms and $H(d) = \mathcal{C}(-,c)$ on objects, where c is given by our choice of preimages.

Explicitly on arrows we have that $H : \mathcal{D} \to \hat{\mathcal{C}}_T$ is given by $H(f) = G(Y_\Gamma(f)) \circ \eta_{\mathcal{C}(-,c)}$ i.e.,

$$\mathcal{C}(-,c) \overset{\eta_{\mathcal{C}(-,c)}}{\longrightarrow} GF(\mathcal{C}(-,c)) \overset{G(Y_\Gamma(f))}{\longrightarrow} GF(\mathcal{C}(-,c'))$$

Remark 7.8.1. We notice that:

$$C \circ H(d) = C(\mathcal{C}(-,c)) = F(\mathcal{C}(-,c)) = \mathcal{D}(-,\Phi(c)) = \mathcal{D}(-,d) = Y_\Gamma(d)$$

Also since $\Phi(c) = d \xrightarrow{f} d' = \Phi(c')$ then

$$F(\mathcal{C}(-,c)) = \mathcal{D}(-,\Phi(c)) \overset{Y_\Gamma(f))}{\longrightarrow} \mathcal{D}(-,\Phi(c')) = F(\mathcal{C}(-,c'))$$

Moreover,

$$C \circ H(f) = C(\ \mathcal{C}(-,c) \overset{\eta_{\mathcal{C}(-,c)}}{\longrightarrow} GF(\mathcal{C}(-,c)) \overset{G(Y_\Gamma(f))}{\longrightarrow} GF(\mathcal{C}(-,c'))\ ) = Y_\Gamma(f)$$

since

$$\begin{tikzcd}
F(\mathcal{C}(-,c)) \arrow[r, "F(\eta_{\mathcal{C}(-,c)})"] \arrow[dr, "1"'] & FGF(\mathcal{C}(-,c)) \arrow[r, "FG(Y_\Gamma(f))"] \arrow[d, "\varepsilon_{F(\mathcal{C}(-,c))}"] & FGF(\mathcal{C}(-,c')) \arrow[d, "\varepsilon_{F(\mathcal{C}(-,c'))}"] \\
& F(\mathcal{C}(-,c)) \arrow[r, "Y_\Gamma(f)"] & F(\mathcal{C}(-,c'))
\end{tikzcd}$$

Thus $C \circ H = Y_\Gamma$.

Remark 7.8.2. Suppose that we are in the above situation where $(F_2, m_2) \dashv (G_2, n_2)$ is a monoidal adjunction. The Yoneda embedding is a strong monoidal functor respecting the Day's convolution monoidal structure. Then we have:

$$\begin{tikzcd}
\hat{\mathcal{D}} \arrow[r, shift left, "(F_2,m_2)"] \arrow[r, phantom, "\perp"] & \hat{\mathcal{D}}_\Gamma \arrow[l, shift left, "G_2"] \\
\mathcal{D} \arrow[u, "(Y,y)"] \arrow[ur, "Y_\Gamma"'] &
\end{tikzcd}$$

Since the adjunction is monoidal F_2 is a strong monoidal functor. This implies that Y_Γ is a strong monoidal functor by composition.

7.8.3 $C : \hat{\mathcal{C}}_T \to \hat{\mathcal{D}}_\Gamma$ is a strong monoidal functor

We define $C(A) \otimes_{\hat{\mathcal{D}}_\Gamma} C(B) \xrightarrow{u_{AB}} C(A \otimes_{\mathcal{C}_T} B)$ by the following arrow: $F(A) \otimes_{\hat{\mathcal{D}}_\Gamma} F(B) \xrightarrow{m_{AB}} F(A \otimes B)$. We want to check naturality: for every $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$, where $f, g \in \hat{\mathcal{C}}_T$

$$\begin{tikzcd}
C(A) \otimes_{\hat{\mathcal{D}}_\Gamma} C(B) \arrow[r, "u_{AB}"] \arrow[d, "C(f) \otimes_{\hat{\mathcal{D}}_\Gamma} C(g)"] & C(A \otimes_{\mathcal{C}_T} B) \arrow[d, "C(f \otimes_{\mathcal{C}_T} g)"] \\
C(A') \otimes_{\hat{\mathcal{D}}_\Gamma} C(B') \arrow[r, "u_{A'B'}"'] & C(A' \otimes_{\mathcal{C}_T} B')
\end{tikzcd} .$$

This turns out to be

$$\begin{tikzcd}
F(A) \otimes_{\hat{\mathcal{D}}_\Gamma} F(B) \arrow[r, "m_{AB}"] \arrow[d, "\varepsilon_{FA'} F(f) \otimes \varepsilon_{FB'} F(g)"] & F(A \otimes_{\mathcal{C}} B) \arrow[d, "\varepsilon_{F(A' \otimes B')} F(G(m_{A'B'}) n(f \otimes g))"] \\
F(A') \otimes_{\hat{\mathcal{D}}_\Gamma} F(B') \arrow[r, "m_{A'B'}"'] & F(A' \otimes_{\mathcal{C}} B')
\end{tikzcd}$$

where $f^K \otimes_{\mathcal{C}_T} g^K$ is equal to

$$A \otimes B \xrightarrow{f \otimes g} GFA' \otimes_{\mathcal{C}} GFB' \xrightarrow{n_{FA'FB'}} G(FA' \otimes_{\hat{\mathcal{D}}_\Gamma} FB') \xrightarrow{G(m_{A'B'})} GF(A' \otimes_{\hat{\mathcal{C}}} B').$$

We define $I \xrightarrow{u_I = m_I} C(I) = F(I)$.

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "m_{AB}"] \arrow[d, "F(f)\otimes F(g)"] & F(A\otimes B) \arrow[r, "F(f\otimes g)"] & F(TA'\otimes TB') \arrow[r, "F(n)"] & F(G(FA'\otimes FB')) \arrow[d, "F(G(m_{A'B'}))"] \\
FGF(A')\otimes FGF(B') \arrow[urr, "m_{GFA'GFB'}"] \arrow[d, "\varepsilon_{FA'}\otimes\varepsilon_{FB'}"] \arrow[drr, phantom, "(a)"] & & & FG(F(A'\otimes B')) \arrow[dll, "\varepsilon_{FA'\otimes FB'}"] \arrow[d, "\varepsilon_{F(A'\otimes B')}"] \arrow[dll, phantom, "(b)"] \\
FA'\otimes FB' \arrow[rrr, "m_{A'B'}"'] & & & F(A'\otimes B')
\end{tikzcd}$$

(a) commutes since ε is a monoidal natural transformation of the monoidal adjunction $(F,m)\dashv(G,n)$.
(b) ε is natural with m.

Since m_{AB} and m_I are invertible in $\hat{\mathcal{D}}_\Gamma$ then u_{AB} and u_I are invertible. This implies that (C,m) is a strong functor.

Now we want to check that

$$\begin{tikzcd}
C(A)\otimes_{\hat{\mathcal{D}}_\Gamma} I \arrow[r, "\rho"] \arrow[d, "1\otimes u_I"] & C(A) \\
C(A)\otimes_{\hat{\mathcal{D}}_\Gamma} C(I) \arrow[r, "u_{AI}"] & C(A\otimes_{\mathcal{C}_T} I) \arrow[u, "C(\rho^T)"']
\end{tikzcd}$$

since $A\otimes_{\mathcal{C}_T} I \xrightarrow{\rho^T} A$ is by definition $A\otimes_{\mathcal{C}} I \xrightarrow{\rho} A \xrightarrow{\eta} GFA$ this implies that $C(\rho^T)$ is

$$F(A\otimes I) \xrightarrow{F(\rho)} FA \xrightarrow{F(\eta)} FGFA \xrightarrow{\varepsilon_{FA}} FA$$

i.e., $C(\rho^T)=F(\rho)$. Thus we obtain that

$$\begin{tikzcd}
F(A)\otimes I \arrow[r, "\rho"] \arrow[d, "1\otimes m_I"] & F(A) \\
F(A)\otimes F(I) \arrow[r, "m_{AI}"] & F(A\odot I) \arrow[u, "F(\rho)"']
\end{tikzcd}$$

and this is satisfied since (F,m) is a monoidal functor. The same is true for the λ axiom.

$$\begin{tikzcd}
C(A) \otimes (C(A') \otimes C(A'')) \arrow[r, "\alpha"] \arrow[d, "1 \otimes u_{A'A''}"] & (C(A) \otimes C(A')) \otimes C(A'') \arrow[d, "u_{AA'} \otimes 1"] \\
C(A) \otimes C(A' \otimes A'') \arrow[d, "u_{A,A' \otimes A''}"] & C(A \otimes A') \otimes C(A'') \arrow[d, "u_{A \otimes A',A''}"] \\
C(A \otimes (A' \otimes A'')) \arrow[r, "C(\alpha)"'] & C((A \otimes A') \otimes A'')
\end{tikzcd}$$

For the same reasons as above we have that $C(\alpha^T) = F(\alpha)$, since $\alpha^T = \eta_{A \otimes A',A''} \circ \alpha$ by definition.

7.8.4 H is a strong monoidal functor

We want to define a natural transformation $H(A) \otimes_{\mathcal{C}^T} H(B) \xrightarrow{\psi_{A,B}} H(A \otimes_{\mathcal{D}} B)$ that makes H into a strong monoidal functor.

Definition of ψ.

We begin by recalling that (C, u) and (Y_Γ, y) are strong monoidal functors, i.e., u and y are isomorphisms, and since C is a fully faithful functor this allows us to define $\psi_{A,B}$ as the unique map making the following diagram commute:

$$\begin{tikzcd}
Y_\Gamma(A) \otimes Y_\Gamma(B) \arrow[rr, "y_{A,B}"] \arrow[dr, "u_{HA,HB}"'] & & Y_\Gamma(A \otimes B) = C \circ H(A \otimes B) \\
& C(H(A) \otimes H(B)) \arrow[ur, "C(\psi_{A,B})"'] &
\end{tikzcd}$$

In the same way we define ψ_I as the unique map $\psi_I : I \to H(I)$ making the following diagram commute:

$$\begin{tikzcd}
I \arrow[rr, "y_I"] \arrow[dr, "u_I"'] & & Y_\Gamma(I) = C \circ H(I) \\
& C(I) \arrow[ur, "C(\psi_I)"'] &
\end{tikzcd}$$

i.e., since C is fully faithful the unique ψ_I such that $C(\psi_I) = y_I \circ u_I^{-1}$.

Notice that since $\mathcal{C}_T \xrightarrow{C} \mathcal{D}_\Gamma$ is fully faithful and u and y are invertible maps this implies that ϕ is an invertible map.

We shall prove naturality of ϕ.

$$\begin{tikzcd}
CH(A)\otimes CH(B) = Y_\Gamma(A)\otimes Y_\Gamma(B) \arrow[dr, "u_{HA,HB}"'] \arrow[drr, "y_{A,B}"] \arrow[ddd, "CH(f)\otimes CH(g)= Y_\Gamma(f)\otimes Y_\Gamma(g)"'] & & \\
& C(H(A)\otimes H(B)) \arrow[r, "C(\psi_{A,B})"'] \arrow[d, "C(H(f)\otimes H(g))"] \arrow[dr, phantom, "(a)"] \arrow[dr, phantom, "(c)"'] & CH(A\otimes B) = Y_\Gamma(A\otimes B) \arrow[d, "CH(f\otimes g)=Y_\Gamma(f\otimes g)"] \arrow[dl, phantom, "(d)"] \\
& C(H(A')\otimes H(B')) \arrow[r, "C(\psi_{A',B'})"] \arrow[dr, phantom, "(b)"] & CH(A'\otimes B') = Y_\Gamma(A'\otimes B') \\
CH(A')\otimes CH(B') = Y_\Gamma(A')\otimes Y_\Gamma(B') \arrow[ur, "u_{HA',HB'}"] \arrow[urr, "y_{A',B'}"'] & &
\end{tikzcd}$$

(a) and (b) by definition of ψ.

(c) naturality of u where (C, u) is a monoidal functor.

The perimeter of the diagram commutes by naturality of y where (Y_Γ, y) is a monoidal functor. Using the fact that $u_{HA,HB}$ is an iso, all this implies that the interior square (d) commutes. Thus we obtain that $C(\psi_{A',B'})\circ C(H(f)\otimes H(g)) = CH(f\otimes g)\circ C(\psi_{A,B})$ therefore since C is faithful $\psi_{A',B'} \circ (H(f)\otimes H(g)) = H(f\otimes g)\circ \psi_{A,B}$.

Now we want to prove that this natural transformation satisfies all the axioms of a monoidal structure. We start with the following axiom:

$$\begin{tikzcd}
H(A)\otimes H(I) \arrow[r, "\rho"] \arrow[d, "1\otimes\Psi_I"] & H(A) \\
H(A)\otimes H(I) \arrow[r, "\psi_{AI}"] & H(A\otimes I) \arrow[u, "H(\rho)"']
\end{tikzcd} \tag{20}$$

This turns to be the following diagram:

$$
\begin{tikzcd}
CH(A)\otimes CH(I) \arrow[ddr, "1"'] & & & \\
& CH(A)\otimes C(I) \arrow[ul, "C1\otimes C\psi_I"'] \arrow[dd, "C1\otimes C\psi_I"] \arrow[dr, "u_{HA,I}"] \arrow[r, phantom, "(d)"] & & CH(A)\otimes I \arrow[ull, "1\otimes y_I"'] \arrow[l, "1\otimes u_I"] \arrow[d, "\rho"] \arrow[dl, phantom, "(a)"] \\
& & C(H(A)\otimes I)) \arrow[r, "C(\rho)"] \arrow[d, "C(1\otimes\psi_I)"] \arrow[dr, phantom, "(c)"] & CH(A) \\
& & C(H(A)\otimes H(I)) \arrow[r, "C\psi_{A,I}"] \arrow[dr, phantom, "(e)"] & CH(A\otimes I) \arrow[u, "Y_{\Gamma(\rho)}", "CH(\rho)="'] \\
& CH(A)\otimes CH(I) \arrow[ur, "u_{HA,HI}"] \arrow[urr, "y_{A,I}"'] \arrow[ur, phantom, "(b)"] & &
\end{tikzcd}
$$

We use the same argument again and we show that it satisfies the required equation.

(a) C is a monoidal functor.

(b) by definition of $\psi_{A,I}$.

(c) naturality of u where (C, u) is a monoidal functor.

(d) definition of ψ_I.

The exterior diagram commutes because (Y_Γ, y) is a monoidal functor. Using the fact that $u_{HA,I}$ is an iso, all this implies that the interior square (e) commutes. Again, since C is faithful we get $H(\rho) \circ \psi_{A,I} \circ (1 \otimes \psi_I) = \rho$ which is diagram (20).

In the same way we can verify that $H(\lambda) \circ \psi_{I,A} \circ (\psi_I \otimes 1) = \lambda$.

Now we move to proving the associativity axiom.

$$
\begin{tikzcd}
CHA \otimes CH(A' \otimes A'') \arrow[dddd, "1"'] \arrow[ddddrr, bend right, "y_{A,A'\otimes A''}"'] & & & CHA \otimes (CHA' \otimes CHA'') \arrow[lll, "1\otimes y_{A',A''}"'] \arrow[dl, "1\otimes u"'] \arrow[d, "1\otimes u"] \arrow[dr, "\alpha"] & & \\
& CHA \otimes C(HA'' \otimes HA'') \arrow[dl, "1\otimes C(\psi_{A',A''})"'] \arrow[dr, "u"] \arrow[rr, phantom, "(f)"] & & CHA \otimes C(HA' \otimes HA'') \arrow[dl, "u"'] \arrow[rr, phantom, "(a)"] & & (CHA \otimes CHA') \otimes CHA'' \arrow[dl, "u\otimes 1"'] \arrow[ddd, "y_{A,A'}\otimes 1"] \arrow[ddll, bend left, "u\otimes 1"] \\
CHA \otimes CH(A' \otimes A'') \arrow[dr, "u"] \arrow[rr, phantom, "(b)"] & & C(HA \otimes (HA' \otimes HA'')) \arrow[dl, "C(1\otimes\psi_{A',A''})"'] \arrow[dr, "C(\alpha)"] & & C(HA \otimes HA') \otimes CHA'') \arrow[dl, "u"'] & \\
& C(HA \otimes H(A' \otimes A'')) \arrow[dl, "C(\psi_{A,A'\otimes A''})"'] \arrow[rr, phantom, "(k)"] & & C((HA \otimes HA') \otimes HA'') \arrow[dl, "C(\psi_{A,A'}\otimes 1)"'] \arrow[rr, phantom, "(j)"] & & \\
CH(A \otimes (A' \otimes A'')) \arrow[dr, "CH(\alpha)"'] & & C(H(A \otimes A') \otimes HA'') \arrow[dl, "C(\psi_{A\otimes A',A''})"'] \arrow[rr, phantom, "(d)"] & & C(HA \otimes HA') \otimes CHA'' \arrow[dl, "C(\psi_{A,A'})\otimes 1"'] \arrow[ull, "u"'] \arrow[rr, phantom, "(h)"] & & \\
& CH((A \otimes A') \otimes A'') & & CH(A \otimes A') \otimes CHA'' \arrow[ll, "y_{A\otimes A',A''}"'] & & CH(A \otimes A') \otimes CHA'' \arrow[ll, "1"'] \arrow[uuuurr, phantom, "(g)"] \arrow[uuuu, phantom, "(i)"] \arrow[uuuu, phantom, "(c)"] \arrow[uuuu, phantom, "(e)"]
\end{tikzcd}
$$

The goal is to prove that the diagram (k) commutes. We have that:

- (a): (C, u) is a monoidal functor.
- (b) and (h): u is natural with Ψ and 1.
- (c) and (d): definition of $\psi_{A,A'\otimes A''}$ and $\psi_{A\otimes A',A''}$.
- (i) and (g): definition of $\psi_{A',A''}$ and $\psi_{A,A'}$ and functoriality of the tensor.
- (f) and (j): are equal.
- The exterior diagram commutes because (Y_Γ, y) is a monoidal functor.

Since $1 \otimes u_{HA',HA''}$ and $u_{HA,HA'\otimes HA''}$ are isos it is enough to check that (top leg of (h)) $\circ\, u \circ (1 \otimes u)$ =(bottom leg of (h)) $\circ\, u \circ (1 \otimes u)$. Then we use the fact that C is a faithful functor.

Remark 7.8.3. Notice that since C and Y_Γ are fully faithful functor, H is fully-faithful as well.

7.8.5 H preserves coproducts

In this section we focus on the specific problem of the preservation of finite coproducts of the functor H defined in Section 7.8.2. First, we notice that the category $[\mathcal{C}^{op}, \mathbf{Set}]$ has finite coproducts. These coproducts are computed pointwise: if F and G are in $[\mathcal{C}^{op}, \mathbf{Set}]$ then $(F \oplus G)(C) = F(C) \oplus G(C)$ for every $C \in \mathcal{C}$ and with injections as in the category $\mathbf{Set}$.

Also these coproducts are preserved going to the category $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ via the left adjoint $F = F_2 \circ F_1$, where $F_2 = R$ is the left adjoint of the reflection determined by the class Γ. The coproducts in $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ are induced by this reflection $i \vdash R$. More precisely: $A \oplus_\Gamma B = R(i(A) \oplus i(B))$ and $in_\Gamma = R(in)$ where A and B are in $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$. Then, it makes sense to think about finite coproducts in $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$.

Finally, the Kleisli category $\hat{\mathcal{C}}_T$ inherits the coproduct structure from $\hat{\mathcal{C}}$ as we proved in Section 7.7. Therefore, $[\mathcal{C}^{op}, \mathbf{Set}]_T$ has finite coproducts. Recall that the comparison functor

$$C : [\mathcal{C}^{op}, \mathbf{Set}]_T \to [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$$

is fully faithful. Also, by Corollary 6.6.6, $H : \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]_T$ preserves coproducts iff $[\mathcal{C}^{op}, \mathbf{Set}]_T(H-, A) : \mathcal{D}^{op} \to \mathbf{Set}$ preserves products for every $A \in [\mathcal{C}^{op}, \mathbf{Set}]_T$.

But, we have

$$[\mathcal{C}^{op}, \mathbf{Set}]_T(H-, A) \cong [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma(CH-, CA).$$

More precisely, since C is fully faithful then the following functors:

$$\mathcal{D}^{op} \overset{H^{op}}{\to} \hat{\mathcal{C}}_T^{op} \overset{\hat{\mathcal{C}}_T(-,A)}{\to} \mathbf{Set}$$

and

$$\mathcal{D}^{op} \overset{(CH)^{op}}{\to} \hat{\mathcal{D}}_T^{op} \overset{\hat{\mathcal{D}}_\Gamma(-,CA)}{\to} \mathbf{Set}$$

are naturally isomorphic.

Therefore, we have:

$$[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma(CH-, CA) = [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma(Y_\Gamma -, CA)$$

because $CH = Y_\Gamma$. Also,

$$[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma(Y_\Gamma -, CA) \cong [\mathcal{D}^{op}, \mathbf{Set}](Y-, CA)$$

because $[\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ is a full subcategory of $[\mathcal{D}^{op}, \mathbf{Set}]$ and $Y_\Gamma - = Y-$ evaluated on $-$. Finally,

$$[\mathcal{D}^{op}, \mathbf{Set}](Y-, CA) \cong CA$$

holds because by the Yoneda Lemma 6.1.5 there is a bijection which is natural in $-$, i.e., these functors are naturally isomorphic. But $CA \in [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$ for every $A \in [\mathcal{C}^{op}, \mathbf{Set}]_T$, which by definition means that CA satisfies the property of continuity i.e., preserves all the cylinders and limit cones that are in Γ. In particular, since natural isomorphisms preserve limits, it will be enough to impose that condition on the class Γ. From this, we conclude that Γ contains all the finite products. This is another requirement to obtain a model.

7.9 $F_T \dashv G_T$ is a monoidal adjunction

In this section we show how a monoidal adjoint pair $(F, m) \dashv (G, n)$ induces a monoidal structure for the adjunction $F_T \dashv G_T$ associated with the Kleisli construction, where $T = GF$.

Lemma 7.9.1. *Let $F \dashv G$ be a monoidal adjunction, let $T = GF$, and consider the Kleisli adjunction $\mathcal{C} \underset{G_T}{\overset{F_T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}_T$ as in Definition 2.1.4. Then $\mathcal{C}_T$ is a monoidal category and $F_T \dashv G_T$ is a monoidal adjunction.*

Proof. Since $F \dashv G$ is a monoidal adjunction, it follows that $T = GF$ is a monoidal monad. The result then follows from Lemma 2.3.3. □

7.10 Abstract model of the quantum lambda calculus

To sum up the sections of this chapter we have the following theorem.

Theorem 7.10.1. *Given categories $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, and functors $\Phi : \mathcal{B} \to \mathcal{C}$, and $\Psi : \mathcal{C} \to \mathcal{D}$, satisfying*

- *$\mathcal{B}$ has finite products, $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal,*
- *$\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ have coproducts, and they are distributive w.r.t. tensor,*
- *$\mathcal{C}$ is affine,*
- *Φ and Ψ are strong monoidal,*
- *Φ and Ψ preserve coproducts,*
- *Φ is full and faithful,*
- *Ψ is essentially surjective on objects,*
- *for every $b \in \mathcal{B}$, $c, c' \in \mathcal{C}$ we have*

$$\mathcal{C}(\Phi(b), c) \times \mathcal{C}(\Phi(b), c') \cong \mathcal{C}(\Phi(b), c \otimes c').$$

Let Γ be any class of cones preserved by the opposite tensor functor, including all the finite product cones and Lan_Φ, Φ^*, F and G be defined as in Section 7.2. Then

$$[\mathcal{B}^{op}, \mathbf{Set}] \underset{\Phi^*}{\overset{Lan_\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{C}^{op}, \mathbf{Set}] \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{D}^{op}, \mathbf{Set}]_\Gamma$$

forms an abstract model of the quantum lambda calculus.

Proof. Relevant propositions from sections 6.7, 6.8, 6.9, 6.10, 6.11, 7.3, 7.4, 7.5, 7.6. □

Chapter 8

A concrete model of the quantum lambda calculus

8.1 An example: $\mathbf{Srel}_{fn}$

Before we give the main model for higher-order quantum computation, it is instructive to consider a simpler model for higher-order *probabilistic* computation. In the sense of Section 7.2, we let $\mathcal{D}$ be the category $\mathbf{Srel}_{fn}$ of sets and stochastic relations, see Definition 4.2.3, and we let $\mathcal{C}$ be the category of finite sets and functions. In this setting, we let "!" be the identity comonad, i.e., $\mathcal{B} = \mathcal{C}$. The latter is justified because in the context of classical probabilistic computation, there are no quantum types and no no-cloning property; all types are classical and hence $!A = A$.

Lemma 8.1.1. $\mathbf{Srel}_{fn}$ *has finite coproducts, satisfying distributivity* $(A \oplus B) \otimes C \cong A \otimes C \oplus B \otimes C$.

Proof. The coproduct of two objects is given by their disjoint union, $A \oplus B = (A \times 1) \cup (B \times 2)$. Injections are given by the following stochastic maps: $i_1 : A \to A \oplus B$ and $i_2 : B \to A \oplus B$, where

$$i_1((x,j),y) = \begin{cases} 1 & \text{if } j = 1 \text{ and } x = y \\ 0 & \text{otherwise.} \end{cases}$$

and

$$i_2((x,j),y) = \begin{cases} 1 & \text{if } j = 2 \text{ and } x = y \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that these satisfy the required universal property. The natural map

$$d_{A,B,C} : A \otimes C \oplus B \otimes C \to (A \oplus B) \otimes C$$

is defined as $[i_1 \otimes C, i_2 \otimes C]$. The map d is easily seen to be a natural isomorphism by precomposing with injections i_1, i_2 and using the universal property for coproducts. □

Definition 8.1.2. Let $\Psi : \mathbf{FinSet} \to \mathbf{Srel}_{fn}$ be the functor that is the identity on objects, and defined on morphisms by

$$\Psi(f)(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.1.3. The functor Ψ is strong monoidal and preserves coproducts.

Theorem 8.1.4. *The choice $\mathcal{B} = \mathbf{FinSet}$, $\mathcal{C} = \mathbf{FinSet}$, $\mathcal{D} = \mathbf{Srel}_{fn}$ with $\Phi = id$ and Ψ as in Definition 8.1.2. Let Γ be the class of all finite product cones in $\mathcal{D}^{op}$. This choice satisfies all the properties required by the Theorem 7.10.1. Therefore, this gives an abstract model of the quantum lambda calculus.*

Proof. By Lemma 8.1.1and Remark 8.1.3. □

Remark 8.1.5. Such a model could be considered to be a concrete model of "probabilistic lambda calculus", i.e., of higher-order probabilistic computation.

Remark 8.1.6. By Lemma 8.1.1, the functor $- \otimes X$ preserves finite coproducts for all $X \in \mathbf{Srel}_{fn}$. It is possible to show that this functor in fact preserves all existing colimits (due to the natural isomorphism $A \otimes X \cong A \oplus A \oplus \ldots \oplus A$, $|X|$ times for any fixed X). Therefore, in Theorem 8.1.4, we could have alternatively defined Γ to be the class of all limit cones. In fact, any class of limit cones that contains at least all finite product ones would do. Each such choice yields an a priori different model.

8.2 The category $\mathbf{Q}''$ and the functors Φ and Ψ

Recall the definition of the category $\mathbf{Q}$ of superoperators from Section 3.2. In this section, we discuss a category $\mathbf{Q}''$ related to superoperators $\mathbf{Q}$, together with functors $\mathbf{FinSet} \xrightarrow{\Phi} \mathbf{Q}'' \xrightarrow{\Psi} \mathbf{Q}$. Here, the goal is to choose $\mathbf{Q}''$ and the functors Φ and Ψ carefully so as to satisfy the requirement of Theorem 7.10.1.

Recall the definition of the free affine monoidal category $\mathcal{F}wm(\mathcal{K})$ from Section 2.6. We apply this universal construction to situation where $\mathcal{K}$ is a discrete category. For later convenience, we let $\mathcal{K}$ be the discrete category with finite dimensional Hilbert spaces as objects. Then $\mathcal{F}wm(\mathcal{K})$ has sequences of Hilbert spaces as objects and dualized, compatible, injective functions as arrows:

- objects: finite sequences of finite dimensional Hilbert spaces
- a morphism from $\{V_1, \ldots, V_n\}$ to $\{W_1, \ldots, W_m\}$ is given by an injective function $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, such that for all i, $V_{f(i)} = W_i$.

Remark 8.2.1. Since the objects of $\mathbf{Q}$ and $\mathcal{F}wm(\mathcal{K})$ are finite sequences of finite-dimensional Hilbert spaces, and there are only countably many finite-dimensional Hilbert spaces up to isomorphism, we may w.l.o.g. assume that $\mathbf{Q}$ and $\mathcal{F}wm(\mathcal{K})$ are small categories.

Now consider the identity-on-objects inclusion functor $F : \mathcal{K} \to \mathcal{Q}'_s$ where $\mathcal{Q}'_s$ is the category of simple trace-preserving superoperator defined in Section 3.2. Since $\mathcal{Q}'_s$ is affine, by Proposition 2.6.3 there exists a unique (up to natural isomorphism) strong monoidal functor $\hat{F}$ such that:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{Q}'_s \\ \mathcal{F}wm(\mathcal{K}) \arrow[ur, "\hat{F}"'] & \end{tikzcd}$$

Remark 8.2.2. This reveals the purpose of using the equality instead of $\leq$ in the definition of a trace-preserving superoperator (Definition 3.2.4). When the codomain is the unit, there is only one map $f(\rho) = \mathrm{tr}(\rho)$, and therefore $\mathcal{Q}'_s$ is affine.

Remark 8.2.3. By definition, $\mathbf{Q}_s$ is a full subcategory of $\mathbf{Q}$, and the inclusion functor $In : \mathbf{Q}_s \to \mathbf{Q}$ is strong monoidal. Also, since every trace preserving superoperator is trace non-increasing, $\mathcal{Q}'_s$ is a subcategory of $\mathbf{Q}_s$, and the inclusion functor $E : \mathcal{Q}'_s \to \mathbf{Q}_s$ is strong monoidal as well.

Then we apply the machinery of Proposition 2.4.9 to the functor:

$$\mathcal{F}wm(\mathcal{K}) \xrightarrow{\hat{F}} \mathcal{Q}'_s \xrightarrow{E} \mathbf{Q}_s \xrightarrow{In} \mathbf{Q}.$$

where In and E are as defined in Remark 8.2.3.

Definition 8.2.4. Let $\mathbf{Q}'' = (\mathcal{F}wm(\mathcal{K}))^+$ and let Ψ be the unique finite coproduct preserving functor making the following diagram commute:

$$\begin{tikzcd} \mathcal{F}wm(\mathcal{K}) \arrow[r, "\hat{F}"] \arrow[d, "I"'] & \mathcal{Q}'_s \arrow[r, "E"] & \mathbf{Q}_s \arrow[r, "In"] & \mathbf{Q} \\ (\mathcal{F}wm(\mathcal{K}))^+ \arrow[urrr, "\Psi"'] & & & \end{tikzcd} \tag{21}$$

Note that such a functor exists by Proposition 2.4.4, and it is strong monoidal by Proposition 2.4.9.

Remark 8.2.5. Since

$$\Psi\{\{V_i^a\}_{i\in[n_a]}\}_{a\in A} = \coprod_{a\in A}\{(V_1^a \otimes \ldots \otimes V_{n_a}^a)_*\}_{*\in 1}$$

the functor Ψ is essentially onto objects. Specifically, given any object $\{V_a\}_{a\in A} \in |\mathbf{Q}|$, we can choose a preimage (up to isomorphism) as follows:

$$\Psi\{\{V_i^a\}_{i\in[1]}\}_{a\in A} = \coprod_{a\in A}\{(V_1^a)_*\}_{*\in 1} \cong \{V_a\}_{a\in A}. \tag{22}$$

Here is the full picture of categories and functors:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{Q}'_s \arrow[r, "E"] & \mathbf{Q}_s \arrow[d, "In"] \\ \mathcal{F}wm(\mathcal{K}) \arrow[ur, "\hat{F}"'] \arrow[d, "I"'] & & \mathbf{Q} \\ (\mathcal{F}wm(\mathcal{K}))^+ \arrow[urr, "\Psi"'] & & \end{tikzcd}$$

Remark 8.2.6. Since $Fwm(\mathcal{K})$ is an affine category and $\mathbf{Q}'' = Fwm(\mathcal{K})^+$, let us consider the functor

$$\Phi : \mathbf{Finset} \to \mathbf{Q}''$$

defined by Lemma 2.5.2.

Theorem 8.2.7. *The choice $\mathcal{B} = \mathbf{FinSet}$, $\mathcal{C} = \mathbf{Q}''$, $\mathcal{D} = \mathbf{Q}$ with the functors Φ as in Remark 8.2.6 and Ψ as in Definition 8.2.4 satisfies all the properties required by Theorem 7.10.1.*

Proof. By relevant propositions from Section 8.2. □

8.3 A concrete model

Theorem 8.3.1. *Let $\mathbf{Q}$ and $\mathbf{Q}''$ be defined as in Sections 3.2 and 8.2. Let Γ be the class of all finite product cones in $\mathcal{D}^{op}$ where $\mathcal{D} = \mathbf{Q}$. Then*

$$[\mathbf{FinSet}^{op}, \mathbf{Set}] \underset{\Phi^*}{\overset{Lan_\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathbf{Q}''^{op}, \mathbf{Set}] \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathbf{Q}^{op}, \mathbf{Set}]_\Gamma$$

forms a concrete model of the quantum lambda calculus.

Proof. The proof is by Theorem 8.2.7, by and by Theorem 7.10.1. □

Chapter 9

Conclusions and future work

In the first part of this thesis, we established that the partially traced categories, in the sense of Haghverdi and Scott, are precisely the monoidal subcategories of totally traced categories. This was proved by a partial version of Joyal, Street, and Verity's "Int"-construction, and by considering a strict symmetric compact closed version of Freyd's paracategories.

We also introduced some new examples of partially traced categories, in connection with some standard models of quantum computation such as completely positive maps and superoperators.

One question that we did not answer is whether specific partially traced categories can always be embedded in totally traced categories in a "natural" way. For example, the category of finite dimensional vector spaces, with the biproduct $\oplus$ as the tensor, carries a partial trace. By our proof, it follows that it can be faithfully embedded in a totally traced category. However, we do not know any concrete "natural" example of such a totally traced category (i.e., other than the free one constructed in our proof) in which it can be faithfully embedded.

In the second part, we constructed mathematical (semantical) models of higher-order quantum computation, and more specifically, for the quantum lambda calculus of Selinger and Valiron. The central idea of our model construction was to apply the presheaf construction to a sequence of three categories and two functors, and to find a set of sufficient conditions for the resulting structure to be a valid model.

The construction depends crucially on properties of presheaf categories, using Day's convolution theory, Lambek's modified Yoneda embedding, and Kelly and Freyd's notion of continuity of functors.

We then identified specific base categories and functors which satisfy these abstract conditions, based on the category of superoperators. Thus, our choice of base categories ensures that the resulting model has the "correct" morphisms at base types, whereas the presheaf construction ensures that it has the "correct" structure at higher-order types.

Our work has concentrated solely on the existence of such a model. One question that we have not yet addressed is specific properties of the interpretation of quantum lambda calculus in this model. It would be interesting, in future work, to analyze whether this particular interpretation yields new insights into the nature of higher-order quantum computation, or to use this model to compute properties of programs.

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