CENTRAL LIMIT THEOREMS FOR SET-INDEXED STRONG MARTINGALES

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To my family, friends and instructors

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Abstract

In [24], Ivanoff and Merzbach introduced the notion of set-indexed strong martingales, a generalization of the planar strong martingales introduced by Cairoli and Walsh in [10]. A set-indexed strong martingale is a special case of a set-indexed process by which we mean a collection $X = \{X_A : A \in \mathcal{A}\}$ of random variables where $\mathcal{A}$ is some collection of subsets of a fixed set T. In this thesis, T is always a compact metric space and $\mathcal{A}$ is a semilattice of closed subsets of T.

In this thesis, we obtain limit theorems for sequences of set-indexed strong martingales and develop two general tools that are useful in obtaining such limit theorems. These limit theorems establish convergence to set-indexed Gaussian processes in one of two modes, functional or semi-functional. Whereas the former mode of convergence is classic, the latter is entirely new and is particularly well-suited to set-indexed strong martingale central limit theorems. The first tool developed is the establishment of sufficient conditions for compactness in the function space $D(\mathcal{A})$ defined in [22] under a Skorokhod J_2-like metric. $D(\mathcal{A})$ serves as a set-indexed generalization of the classic function space $D[0,1]$. The resulting compact sets lead to tightness criteria for set-indexed processes with sample paths in $D(\mathcal{A})$. These results extend those found in Section 3 of [5]. For the second tool, quadratic variation for set-indexed strong martingales is defined and conditions ensuring its existence are given. The general role of quadratic variation in set-indexed strong martingale central limit theorems is similar to that played by quadratic variation processes in the classical theory. Namely, under certain conditions, convergence of quadratic variations to a continuous deterministic limit implies convergence of the underlying sequence of set-indexed strong martingales to a suitably scaled set-indexed Gaussian process. As an application, we derive a central limit theorem for set-indexed weighted empirical processes.

Contents

Chapter 1

Introduction

A substantial portion of the probabilistic literature concerns *continuous parameter stochastic processes*, families of random variables indexed by the points of $\mathbf{R}^+$. Of particular fame are the *Brownian processes* and the *Poisson processes.* In most cases, the indexing parameter t represents time.

A natural generalization of the continuous parameter process is the *multiparameter process* by which we mean a family X of random variables indexed by the points $\mathrm{s} = (s_1, \cdots, s_n) \in (\mathrm{R}^+)^n$ for some fixed $n \in \mathrm{N}$. These processes, on which there is considerably less literature, model many natural phenomena. For example, X_{s} can denote some physical measurement taken at the spatial location s.

A motivation for set-indexed processes

A well studied class of continuous parameter processes are the *empirical processes.* Given a sequence $(Y_n)_n$ of non-negative independent identically distributed (i.i.d.) random variables with common distribution function F, we define for each $i \in \mathrm{N}$ the i-th empirical process,

$$X_t^{(i)} = i^{-1/2} \cdot \left[\sum_{j=1}^{i} 1_{[Y_j \leq t]} - i\,F(t)\right], \quad (t \in \mathbf{R}^+) \tag{1.1}$$

where 1_B denotes the indicator of an event B. Given a sequence $(\mathrm{Y}_n)_n$ of non-negative i.i.d. random vectors with common k-dimensional distribution $\mathbf{F}$, one constructs multiparameter empirical processes in an analogous fashion. However, in the multiparameter case it is often more convenient to replace the summands in (1.1) by random variables of the form

$$\mathbf{1}_{[\mathrm{Y}_j \in A]} - \imath\,\mathbf{F}(A)\,, \quad (A \in \mathcal{A})$$

where $\mathcal{A}$ is an appropriate collection of closed and bounded subsets of $(\mathbf{R}^+)^n$, usually a *Vapnik-Červonenkis class.* In this sense, the multiparameter empirical process can be viewed as a "set-indexed" process.

Now, take any set T and any collection $\mathcal{A}$ of subsets of T. Generalizing the above scenario, one defines a *set-indexed process* to be any collection $X = \{X_A : A \in \mathcal{A}\}$ of random variables defined on a common probability space. The most common situation is where T is a metric space and $\mathcal{A}$ is some collection of closed and bounded subsets of T. As is true for its classical counterparts, this class of processes can model a vast array of natural phenomena. For example, take T to be the unit sphere in $\mathbf{R}^3$ with the metric

$$d(\mathrm{s}, \mathrm{s}') = \text{length of shortest arc between s and s}'\,, \quad (\mathrm{s}, \mathrm{s}' \in T)$$

and let $\mathcal{A} = \{A \subseteq T : A \text{ closed and simply connected}\}$. A set-indexed process of considerable interest would be

$$X_A = \text{average thickness of ozone layer over the region } A\,, \quad (A \in \mathcal{A}).$$

Here, T represents the surface of the Earth modulo a scaling factor.

Two perspectives in the set-indexed theory

Set-indexed processes have been studied by various authors. The following works are of particular interest here:

- Chapter 1 of [1] which discusses general Gaussian processes, a special case of which are the set-indexed Gaussian processes,
- [5] and [36] in which limit theorems for set-indexed partial sum-processes are derived,
- [16] in which a Doob-Meyer decomposition for set-indexed submartingales is obtained and
- [26] which contains a martingale characterization for set-indexed Brownian motion.

This list can be divided into two slightly overlapping categories.

(1) In [1], [5] and [36], the size of $\mathcal{A}$ measured via the notion of *metric entropy* plays a central role. Apart from this, few (in the case of [5] and [36]) or no (in the case of [1]) extra conditions are required on $\mathcal{A}$.

(2) Except for Theorem 1 in [26], nowhere in [16] and [26] are explicit size limits placed on $\mathcal{A}$. However, [16] and [26] require $\mathcal{A}$ to satisfy certain "geometric" conditions. Other papers which exhibit this perspective include [23], [24] and [28].

Category (1) reflects what Adler referred to as the "modern attitude" in the study of general Gaussian processes in which "the precise geometric structure of the parameter space is of relatively little importance" (see p.1 of [1]). On the other hand, the geometric conditions mentioned in category (2) allow for the definition of set-indexed filtrations and set-indexed submartingales.

Besides closure under intersections, many of the papers listed in category (2) require $\mathcal{A}$ to be a lattice and to satisfy the so called *shape property*. The latter states that, given any sets $A, A_1, \cdots, A_n \in \mathcal{A}$,

$$A \subseteq \bigcup_{i=1}^{n} A_1 \implies \exists 1 \leq i \leq n \text{ s.t. } A \subseteq A_i.$$

Neither shape nor a lattice structure on $\mathcal{A}$ is required for the definition of *set-indexed strong submartingales*, set-indexed processes which generalize the 2-parameter strong submartingales introduced in [10].

Thesis objective

The goal of this thesis is to obtain limit theorems for sequences of set-indexed strong martingales and to develop tools (tightness criteria in Chapter 2, quadratic variation in Chapter 3) that are useful in obtaining such limit theorems. In this thesis, $\mathcal{A}$ will always be a collection of closed subsets of a compact metric space T. The indexing classes used will be hybrids of the two previously mentioned categories. In particular,

(a) Other than Theorems 4.4.4 and 4.5.8, nowhere in this thesis is $\mathcal{A}$ required to be small with respect to the notion of metric entropy.

(b) Except for a few specialized results, only the minimal geometric conditions are placed on $\mathcal{A}$. Of particular mention, only those results which depend on Appendix B require $\mathcal{A}$ to be a lattice and nowhere in this thesis is $\mathcal{A}$ required to satisfy the shape assumption.

By working with such $\mathcal{A}$, most of the results in this thesis will hold for the case in which $\mathcal{A}$ is the collection $\mathcal{LL}_k$ of *lower layers* on $T = [0,1]^k$ (see Example 2.8.1). Note that $\mathcal{LL}_k$ does not satisfy the shape assumption. Furthermore, $\mathcal{LL}_k$ has an arbitrarily large *exponent of metric entropy* for arbitrarily large k (see Proposition 4.6.25).

Highlights and major results

The thesis is divided into three main chapters.

Chapter 2 studies the function space $D(\mathcal{A})$ which serves as a set-indexed generalization of $D[0,1]$. A typical element of $D(\mathcal{A})$ has the form $x : \mathcal{A} \to \mathbf{R}$ and is *inner-continuous with outer-limits* on $\mathcal{A}$, a set-indexed generalization of "right-continuous with left-limits". After equipping $D(\mathcal{A})$ with a Skorokhod J_2-like metric denoted d_D, a partial characterization of d_D-compactness in $D(\mathcal{A})$ is developed. The resulting compacts lead to tightness criteria for set-indexed processes with sample paths in $D(\mathcal{A})$. The main original achievements in Chapter 2 include:

- A sufficient condition under which a *purely atomic* set-function lies in $D(\mathcal{A})$ (Theorem 2.3.11). The function space $D(\mathcal{A})$ has been studied by various authors (see [5] and [22]), however, this sufficient condition appears to be the first of its kind.

- Sufficient conditions for compactness in $(D(\mathcal{A}), d_D)$ (Theorems 2.5.12, 2.6.5 and 2.6.7). Even for the case of $T = [0,1]^k$ ($k \geq 1$), the compactness criterion given in Theorem 2.5.12 is more general than that given in Theorem 3.4 of [5].

Chapter 3 defines set-indexed filtrations and set-indexed strong submartingales. Using admissible functions, decomposition theorems for set-indexed strong submartingales are obtained. A suitable form of quadratic variation for set-indexed square integrable strong martingales is defined and studied. The main original achievements in Chapter 3 include:

- Existence and uniqueness of Doob-Meyer-type decompositions for set-indexed strong submartingales (Theorems 3.4.11 and 3.4.14). Unlike the proof of the corresponding decomposition for set-indexed weak submartingales presented in [16], the argument in Theorem 3.4.11 does not assume that the indexing class $\mathcal{A}$ satisfies the shape property or is a distributive lattice.

- The definition of *$*$-predictable quadratic variation* ($*$-PQV) for square integrable set-indexed strong martingales (Definition 3.5.6), a generalization of the predictable quadratic variation for square integrable continuous parameter martingales. This definition overcomes the inherent dilemma associated with strong martingales, namely that the square of an L_2 strong martingale is not necessarily a strong submartingale (Observation 3.5.1).

- Sufficient conditions for the existence and uniqueness of $*$-PQV (Theorem 3.5.2, Corollary 3.6.11) and conditions under which a $*$-PQV can be approximated in L_2-norm by discrete sums (Theorem 3.7.7).

The general role played by $*$-PQV in obtaining limit theorems for sequences of set-indexed strong martingales is best illustrated by Propositions 4.4.2 and 4.5.6. In both results, convergence of $*$-PQV to a determistic limit implies convergence of the corresponding sequence of set-indexed strong martingales.

Chapter 4 contains several limit theorems for sequences of set-indexed strong martingales. The main original achievements in Chapter 4 include:

- The formulation of a new mode of convergence for sequences of set-indexed processes termed *semi-functional convergence* (Definition 4.3.15). This new mode of convergence is ideally suited to strong martingale central limit theorems (CLTs).

- Semi-functional CLTs for sequences of continuous strong martingales (Theorem 4.4.1, Proposition 4.4.2) and general strong martingales (Theorem 4.5.5, Proposition 4.5.6).

- Functional CLTs for sequences of continuous strong martingales (Theorem 4.4.4) and strong martingales in $D(\mathcal{A})$ (Theorem 4.5.8).

- A semi-functional CLT for k-dimensional set-indexed weighted empirical processes (Theorem 4.6.22). Of particular interest is the case in which the indexing class is taken to be the lower layers, $\mathcal{LL}_k$ (Corollary 4.6.24). Note that $\mathcal{LL}_k$ is not a Vapnik-Červonenkis class. In fact, when indexed by $\mathcal{LL}_k$ ($k \geq 2$), the above mentioned empirical processes converge semi-functionally to a set-indexed Gaussian process for which almost all sample paths are necessarily Hausdorff discontinuous everywhere on $\mathcal{LL}_k$ (Example 4.6.26).

For more detailed summaries of the above chapters, see the corresponding introductory sections. The thesis also contains four appendices.

Appendix A contains assorted technical results which, although used in the above mentioned chapters, are not themselves of primary interest.

Appendix B develops several technical results under a conditional independence assumption. These technical results are used in Chapter 3 to establish, among other things, adaptedness in certain decompositions (Theorem 3.4.14) and a Rosenthal-like inequality for square integrable set-indexed strong martingales (Lemma 3.6.7).

Appendix C introduces an entirely new function space denoted $D_p(\mathcal{A})$. Athough not used in this thesis, the space $D_p(\mathcal{A})$ may be of independent interest. Indeed, as demonstrated by Theorem C.2.13 and Example C.2.12, $D_p(\mathcal{A})$ is a more accurate generalization of the classical multiparameter function space $D([0,1]^k)$ than is $D(\mathcal{A})$.

Appendix D contains a complete list of the assumptions used occasionally in the thesis. Included is a list of the conventions adopted in the thesis.

Additional notes

The end of a proof for any Theorem, Proposition or Lemma is marked by □. The end of a proof for any Claim is marked by Ω. The end of a proof for any Subclaim is marked by ω.

This thesis is rich in new terminology and notation. See the end of the thesis for complete lists of acronyms and symbols as well as an index.

Chapter 2

Function Spaces

2.1 Introduction

Recall the metric spaces, $(D[0,1], \rho)$ where $D[0,1]$ consists of all functions, $f : [0,1] \to \mathbf{R}$ which are right-continuous with left-limits on $[0,1]$ and ρ is one of the four Skorokhod metrics, J_1, J_2, M_1 or M_2. In this chapter, a set-indexed analogue of $(D[0,1], J_2)$, denoted $(D(\mathcal{A}), d_D)$, will be constructed and analyzed. Attempts at defining such analogues have already been made in [5], [22] and [25]. This chapter is strongly motivated by these earlier attempts.

This chapter, which is completely non-probabilistic, is divided into seven sections. In Section 2.2 we introduce the concept of an *indexing collection* which serves as a natural generalization of the classical parameter space, $[0,1]^k$ ($k \in \mathbf{N}$). Specifically, given a compact metric space (T, d), an indexing collection $\mathcal{A}$ on (T, d) is any class of d-closed subsets of T which is closed under countable intersections and satisfies a certain separability condition resembling that found in Assumption 1 of [26].

Two criteria governed our choice of the above mentioned separability condition. On one hand, this condition had to be weak enough to allow for a variety of diverse and useful examples of indexing collections. On the other hand, this condition had to be sufficiently strong so as to ensure that each resulting indexing collection, $\mathcal{A}$ on (T, d) possessed certain fundamental properties, the most important being the compactness of $\mathcal{A}$ w.r.t. the Hausdorff metric generated by (T, d) (see Theorem 2.2.13).

In Section 2.3, given any indexing collection $\mathcal{A}$, we define the function space $D(\mathcal{A})$ to be the class of all set-functions, $x : \mathcal{A} \to \mathbf{R}$ which are *outer-continuous* (a generalization of right-continuity) with *inner-limits* (a generalization of left-limits) at every $A \in \mathcal{A}$. The space $D(\mathcal{A})$ has appeared in [22] and [25] and also in [5] for the case of $T = [0,1]^k$. When $T = [0,1]$ and $\mathcal{A} = \{[0,t] : 0 \leq t \leq 1\}$, there is a natural 1-1 correspondence between $D[0,1]$ and $D(\mathcal{A})$. [1]

In this same section, we construct two important subspaces, PA and $D_0(\mathcal{A})$ of $D(\mathcal{A})$. PA consists of all *purely atomic* set-functions, i.e., those set-functions with a finite number of "jump points" in T. Although more restrictive than that given in [5], our definition of PA implies $PA \subseteq D(\mathcal{A})$ (compare with Example 2.3.6). The subspace $D_0(\mathcal{A})$ consists of all uniform limits of sequences of the form $(j_n + c_n)_n$ where each $c_n : \mathcal{A} \to \mathbf{R}$ is continuous on $\mathcal{A}$ with respect to the Hausdorff metric and $j_n \in PA$ $\forall n$. In Theorem 2.3.17, it is shown that $D_0(\mathcal{A}) \subseteq D(\mathcal{A})$, an inclusion which is not valid under the definition of $D_0(\mathcal{A})$ given in [5].

In Section 2.4 we define a metric, denoted d_D, on the function space $D(\mathcal{A})$. This same metric has appeared in equivalent form in [22] and [25] and also in [5] for the case of $T = [0,1]^k$. When $T = [0,1]$ and $\mathcal{A} = \{[0,t] : 0 \leq t \leq 1\}$, there is a natural homeomorphism between $(D(\mathcal{A}), d_D)$ and $(D[0,1], J_2)$ (see Remark 1 on p.4 of [22]). Although the Skorokhod J_1 metric is the one most used when working with $D[0,1]$, there are inherent difficulties in defining a J_1-type metric on $D(\mathcal{A})$ (see Remark 4 on p.882 of [5]).

In Section 2.5, a sufficient condition for d_D-compactness in $D_0(\mathcal{A})$ is given. Bass and Pyke have obtained a similar result for the case of $T = [0,1]^k$ (see Theorem 3.4 in [5]). However, even for this case, our result is more general.

In Section 2.6, a sufficient condition for d_D-compactness in PA is given (see Theorem 2.6.5). This sufficient condition resembles that found in Proposition 3.2 of [5], deviating enough to ensure that the resulting compacta are contained in $D(\mathcal{A})$. Since the compacta in (PA, d_D) generated by Theorem 2.6.5 will have an explicit form, they can be used in conjunction with Theorem 2.5.12 to generate *explicit compacts* in $D_0(\mathcal{A})$ (see Theorem 2.6.7).

In Section 2.7 we present the proof of the main compactness result of Section 2.6. By isolating its rather lengthy proof in a separate section, we preserve the flow of Section 2.6.

Section 2.8 closes the chapter with several examples of indexing collections. Also included are the counterexamples referred to in the various sec-

[1] This assumes the convention, y continuous at $t = 1$ $\forall y \in D[0,1]$.

tions of this chapter.

Many interesting topological questions concerning spaces of the form $(D(\mathcal{A}), d_D)$ remain. Problems of particular interest include: sufficient conditions on $\mathcal{A}$ and (T, d) under which $D(\mathcal{A})$ is d_D-separable and the possibility of a complete characterization of d_D-compactness in $D(\mathcal{A})$ and $D_0(\mathcal{A})$. Independent of the answers to these questions, Chapter 2 will provide the tools necessary for studying set-indexed processes in upcoming chapters.

2.2 Indexing Collections and their Essential Properties

We begin by recalling the concept of *Hausdorff distance*. Specifically, given a metric space (T, d), let

$$\mathcal{K}_T = \{\text{non-empty, } d\text{-closed and bounded subsets of } T\}.$$

The Hausdorff distance between any $A, B \in \mathcal{K}_T$ is defined by

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq B^\epsilon \text{ and } B \subseteq A^\epsilon\} \tag{2.1}$$

where, given any $C \subseteq T$, $C^\epsilon = \{t \in T : \exists c \in C \text{ s.t. } d(c, t) < \epsilon\}$. On p.279 of [32] it is stated that d_H defines a metric on $\mathcal{K}_T$.

If we replace each set, C^ϵ in (2.1) by the set

$$C^{\overline{\epsilon}} = \{t \in T : \exists c \in C \text{ s.t. } d(c, t) \leq \epsilon\},$$

we once again obtain a metric on $\mathcal{K}_T$. [2] By the basic properties of infimum, it is clear that the metric on $\mathcal{K}_T$ obtained via the sets $C^{\overline{\epsilon}}$ coincides with our original metric defined in (2.1). In this thesis, we adopt the C^ϵ-formulation of the Hausdorff metric but will occasionally revert to the $C^{\overline{\epsilon}}$-formulation when more convenient.

Several basic properties of the set operations $(\cdot)^\epsilon$ and $(\cdot)^{\overline{\epsilon}}$ will be used frequently in the upcoming sections. For convenience, we list them here.

Proposition 2.2.1 *Given $C, D \subseteq T$ and $\epsilon, \epsilon_1, \epsilon_2 > 0$:*

(i) *C^ϵ is open and $C^{\overline{\epsilon}}$ is closed,*

[2] This formulation of the Hausdorff metric is presented in [22].

(ii) $\overline{C} \subseteq C^{\epsilon} \subseteq C^{\overline{\epsilon}}$,

(iii) $C \subseteq D \Rightarrow C^{\epsilon} \subseteq D^{\epsilon}$ *and* $C^{\overline{\epsilon}} \subseteq D^{\overline{\epsilon}}$,

(iv) $\epsilon_1 \leq \epsilon_2 \Rightarrow C^{\epsilon_1} \subseteq C^{\epsilon_2}$ *and* $C^{\overline{\epsilon_1}} \subseteq C^{\overline{\epsilon_2}}$,

(v) $(\forall \delta > 0)\ C^{\overline{\epsilon}} \subseteq C^{\epsilon+\delta}$,

(vi) $(\overline{C})^{\epsilon} = C^{\epsilon}$,

(vii) $\overline{C^{\epsilon}} = C^{\overline{\epsilon}}$, and

(viii) $(C^{\epsilon_1})^{\epsilon_2} \subseteq C^{\epsilon_1+\epsilon_2}$.

Proof (i) Consider the function

$$f : T \to \mathbf{R} \text{ where } f(t) = \inf\{d(c,t) : c \in C\}.$$

As stated in Appendix 1 of [6], f is continuous on T. Moreover,

$$\begin{aligned} t \in C^{\epsilon} &\iff \exists\, c \in C \text{ s.t. } d(c,t) < \epsilon \\ &\iff f(t) < \epsilon \\ &\iff t \in f^{-1}[(-\infty, \epsilon)]. \end{aligned}$$

Thus, since f is continuous, $C^{\epsilon} = f^{-1}[(-\infty, \epsilon)]$ is open in T. Similarly, $C^{\overline{\epsilon}} = f^{-1}[(-\infty, \epsilon]]$ is closed in T.

(ii) Clearly, $C^{\epsilon} \subseteq C^{\overline{\epsilon}}$. To show the first inclusion, take $t \in \overline{C}$. By the definition of closure, $\exists\, c \in C$ s.t. $d(c,t) < \epsilon$ hence, $t \in C^{\epsilon}$.

(iii), (iv) and (v) are all obvious consequences of the definitions.

(vi) As $C \subseteq \overline{C}$, it follows by property (iii) above that $C^{\epsilon} \subseteq (\overline{C})^{\epsilon}$. To show the opposite inclusion, take $t \in (\overline{C})^{\epsilon}$ thus, $\exists\, c' \in \overline{C}$ s.t. $d(c',t) < \epsilon$. Let $\gamma = \epsilon - d(c',t) > 0$, then

$$c' \in \overline{C} \Rightarrow \exists\, c \in C \text{ s.t. } d(c',c) < \gamma,$$

which gives us

$$d(c,t) \leq d(c,c') + d(c',t) < \gamma + d(c',t) = \epsilon$$

implying that $t \in C^\epsilon$.

(vii) As $C^\epsilon \subseteq C^{\overline{\epsilon}}$ where $C^{\overline{\epsilon}}$ is closed, it follows that $\overline{C^\epsilon} \subseteq \overline{C^{\overline{\epsilon}}} = C^{\overline{\epsilon}}$. For the opposite inclusion note that, by definition,

$$C^\epsilon = \cup_{c \in C} B(c, \epsilon) \quad \text{and} \quad C^{\overline{\epsilon}} = \cup_{c \in C} \overline{B(c, \epsilon)}$$

hence, $C^{\overline{\epsilon}} = \cup_{c \in C} \overline{B(c, \epsilon)} \subseteq \overline{\cup_{c \in C} B(c, \epsilon)} = \overline{C^\epsilon}$.

(viii) If $t \in (C^{\epsilon_1})^{\epsilon_2}$, then $\exists\, s \in C^{\epsilon_1}$ s.t. $d(s,t) < \epsilon_2$. As $s \in C^{\epsilon_1}$, $\exists\, c \in C$ s.t. $d(c,s) < \epsilon_1$ hence,

$$d(c,t) \leq d(c,s) + d(s,t) < \epsilon_1 + \epsilon_2,$$

which implies $t \in C^{\epsilon_1 + \epsilon_2}$. □

In the upcoming chapters, our set-indexed processes will not be indexed by the entire collection $\mathcal{K}_T$ but rather by a special subcollection of $\mathcal{K}_T$ appropriately termed an *indexing collection.* Among other things, an indexing collection will satisfy the following strong separablility condition.

Definition 2.2.2 *Let (T,d) be a compact metric space and let $\mathcal{A}$ be a non-empty subcollection of $\mathcal{K}_T$ which contains T and is closed under countable intersections. $\mathcal{A}$ is said to be uniformly separable from above provided*

(1) *$\exists$ a sequence, $(\mathcal{A}_n)_n$ of subcollections of $\mathcal{A}$ s.t., given any $n, m \in \mathbf{N}$,*

(i) *$\mathcal{A}_n$ is finite and closed under intersections,*

(ii) *$T \in \mathcal{A}_n$,*

(iii) *$n \leq m \Rightarrow \mathcal{A}_n \subseteq \mathcal{A}_m$, and*

(2) *$\exists$ a sequence, $(g_n)_n$ of set-functions of the form, $g_n : \mathcal{A} \to \mathcal{A}_n$ s.t., given any $A, A' \in \mathcal{A}$,*

(i') *$(g_n(A))_n$ is decreasing in n w.r.t. $\subseteq$ with $\bigcap_n g_n(A) = A$,*

(ii') *$A \subseteq [g_n(A)]^\circ \ \forall n$,*

(iii') *if $A \subset A'$ then, $A \subset g_n(A) \cap A' \ \forall n$,* (iv′) *if* $A \cup A' \in \mathcal{A}$, *then* $g_n(A \cup A') = g_n(A) \cup g_n(A')$ $\forall n$,

(v′) *given* $(A_k)_k$ *in* $\mathcal{A}$, $g_n(\bigcap_k A_k) = \bigcap_k g_n(A_k)$ $\forall n$

and $\exists$ *a sequence* $(\epsilon_n)_n$ *of non-negative constants s.t.* $\epsilon_n \downarrow 0$ *and*

$$d_H(A, g_n(A)) \leq \epsilon_n, \quad (\forall A \in \mathcal{A}) \tag{2.2}$$

for any given $n \in \mathrm{N}$.

Remark 2.2.3 If $\mathcal{A}$ is uniformly separable from above, then by (2.2), $\mathcal{A}^* = \bigcup_n \mathcal{A}_n$ is a countable d_H-dense subset of $\mathcal{A}$. That is, $\mathcal{A}$ is d_H-separable.

The following class of objects will be the center of study in this section.

Definition 2.2.4 *Given a compact metric space* (T, d), *an indexing collection on* T *is any non-empty subcollection,* $\mathcal{A}$ *of* $\mathcal{K}_T$ *for which*

(i) $T \in \mathcal{A}$,

(ii) $\mathcal{A}$ *is closed under countable intersections and*

(iii) $\mathcal{A}$ *is uniformly separable from above.*

Remark 2.2.5 (a) The conditions on $\mathcal{A}$ given in Definition 2.2.4 represent the core assumptions which we will require in all chapters. In future sections, additional assumptions may be added to indexing collections in order to obtain specific results.

(b) With two exceptions, all conditions in Definition 2.2.2 have appeared earlier in at least one of [16], [22] or [26]. The first exception is that in most of the earlier works, each g_n had {all finite unions in $\mathcal{A}_n$} as a range instead of $\mathcal{A}_n$. The second is the addition of the *uniform approximation* condition, (2.2). Both of these changes are needed to ensure that every indexing collection $\mathcal{A}$ is d_H-closed in $\mathcal{K}_T$ (Theorem 2.2.10).

(c) By definition, $\phi \notin \mathcal{K}_T$ and hence, $\phi \notin \mathcal{A}$ for any indexing collection $\mathcal{A}$ on T. We purposely exclude ϕ because of its bad behaviour with respect to the Hausdorff metric. In particular, $(\phi)^\epsilon = \phi$ $\forall \epsilon > 0$ which implies $\inf\{\epsilon > 0 : A \subseteq (\phi)^\epsilon\} = \infty$ $\forall A \neq \phi$. On the other hand, using the extention of d_H presented in the Note on p.5 of [13], we can always adjoin ϕ to $(\mathcal{A}, d_H)$ as an isolated point by setting $d_H(\phi, \phi) = 0$ and $d_H(\phi, A) = M + 1$ $\forall A \in \mathcal{A}$

where M is the diameter of (T, d) (recall that (T, d) is taken to be compact). In this case, $\mathcal{A}$ should be replaced by $\mathcal{A} \setminus \phi$ in Definition 2.2.4 (iii). [3]

(d) In [16] and several other set-indexed papers, the so-called *shape* assumption is included in the definition of $\mathcal{A}$. This assumption states that, for any $A, A_1, \cdots, A_n$ in $\mathcal{A}$, if $A \subseteq \bigcup_{i=1}^{n} A_i$, then $\exists\, 1 \leq i \leq n$ s.t. $A \subseteq A_i$. We avoid shape since it is rather restrictive. In particular, the indexing collection $\mathcal{LL}_2$ of *lower layers* on $[0,1]^2$ do not satisfy shape (see Example 2.8.1).

(e) In general, $\mathcal{K}_T$ is itself not an indexing collection on T since there may be disjoint non-empty elements in $\mathcal{K}_T$, implying that $\mathcal{K}_T$ is not closed under finite intersection.

As mentioned in the introduction to this chapter, an indexing collection is intended to serve as a generalization of the classical parameter space, $[0,1]^k$ ($k \in \mathbb{N}$). Indeed, $[0,1]^k$ is captured by the following indexing collection.

Example 2.2.6 Fix $k \in \mathbb{N}$ and let $(T, d) = ([0,1]^k, \|\cdot\|_\infty)$ where

$$\|\mathbf{t}\|_\infty = \max_{1 \leq i \leq k} |t_i|, \quad (\forall\, \mathbf{t} = (t_1, \cdots, t_k) \in [0,1]^k).$$

Given $\mathbf{t} = (t_1, \cdots, t_k) \in [0,1]^k$, define

$$[\mathbf{0}, \mathbf{t}] = \prod_{i=1}^{k} [0, t_i].$$

and define

$$\mathcal{I}_k = \{[\mathbf{0}, \mathbf{t}] : \mathbf{t} \in [0,1]^k\}.$$

If $\mathbf{s}, \mathbf{t} \in [0,1]^k$, then $[\mathbf{0}, \mathbf{t}] \cap [\mathbf{0}, \mathbf{s}] = [\mathbf{0}, \mathbf{s} \wedge \mathbf{t}] \in \mathcal{I}_k$. Hence, $\mathcal{I}_k$ is a sub-collection of $\mathcal{K}_{[0,1]^k}$ which contains I^k and is closed under finite intersection. Furthermore, by the nested-intervals theorem, $\mathcal{I}_k$ is closed under countable intersection.

Given any $n \in \mathbb{N}$, let

$$D_k^{(n)} = \{\, \mathbf{t} \in [0,1]^k : t_i = \tfrac{m_i}{2^n} \text{ for some } m_i \in \{0, 1, \cdots, 2^n\} \ \forall\, 1 \leq i \leq k\}$$

and define

$$\mathcal{I}_k^{(n)} = \{[\mathbf{0}, \mathbf{t}] \in \mathcal{I}_k : t \in D_k^{(n)}\}.$$

[3]Most of the earlier papers, [16] and [26] for example, include ϕ in $\mathcal{A}$ from the onset and thus require the added condition, $A \cap B \neq \phi \ \forall A, B \in \mathcal{A} \setminus \{\phi\}$.

Since $(D_k^{(n)})_n$ is an increasing sequence (w.r.t. $\subseteq$) of finite sub-semilattices of $([0,1]^k, \wedge)$, $(\mathcal{I}_k^{(n)})_n$ is an increasing sequence of finite subcollections of $\mathcal{I}_k$. Moreover, given any n, it follows from the definition of $D_k^{(n)}$ that $\mathcal{I}_k^{(n)}$ contains $[0,1]^k$ and is closed under intersection. If we define $g_n : \mathcal{I}_k \to \mathcal{I}_k^{(n)}$ by letting

$$g_n(A) = \bigcap \{B \in \mathcal{I}_k^{(n)} : A \subseteq B^\circ\}, \quad (\forall A \in \mathcal{I}_k),$$

then it is straightforward to show that $(g_n)_n$ satisfies the properties listed in (2) of Definition 2.2.2. In particular, we can take $\epsilon_n = 1/2^n \ \forall n$ in (2.2). Therefore, $\mathcal{I}_k$ is an indexing collection on $[0,1]^k$.

Via the natural bijection, $\mathrm{t} \mapsto [0,\mathrm{t}]$ ($\mathrm{t} \in [0,1]^k$), every $[0,1]^k$-indexed process can be viewed as an $\mathcal{I}_k$-indexed process and vice-versa.

Remark 2.2.7 See Section 2.8 for additional examples of indexing collections on $[0,1]^k$ and see [16] for indexing collections on T where T is either a function space or a graph.

Throughout this thesis, unless otherwise stated, $\mathcal{A}$ will always denote a generic indexing collection on a generic compact metric space, (T, d).

In the remainder of this section, we derive four important consequences of Definition 2.2.4. Specifically, given any indexing collection $\mathcal{A}$,

- $\mathcal{A}$ is closed under arbitrary intersections,
- the domain of each g_n can be "appropriately" extended,
- $\mathcal{A}$ is a d_H-compact subspace of $\mathcal{K}_T$ and
- $\exists$ a smallest non-empty element in $(\mathcal{A}, \subseteq)$.

These four intrinsic properties of indexing collections will be applied frequently in the thesis. We begin at the top of the list.

Proposition 2.2.8 *If $\mathcal{A}$ is an indexing collection on T, then $\mathcal{A}$ is closed under arbitrary intersections.*

Proof Let $(A_j)_{j\in J}$ be any collection of sets in $\mathcal{A}$. By Definition 2.2.2 (2)(i′),

$$\bigcap_{j\in J} A_j \;=\; \bigcap_{j\in J}\bigcap_n g_n(A_j) \;=\; \bigcap_n\bigcap_{j\in J} g_n(A_j). \qquad (2.3)$$

But, given any n, $\{g_n(A_j) : j \in J\} \subseteq \mathcal{A}_n$ and hence by Definition 2.2.2 (1)(i), $\bigcap_{j\in J} g_n(A_j) \in \mathcal{A}_n \subseteq \mathcal{A}$. Therefore, since $\mathcal{A}$ is closed under countable intersections, (2.3) implies $\bigcap_{j\in J} A_j \in \mathcal{A}$. □

Next, we extend the domain of definition for each $g_n : \mathcal{A} \to \mathcal{A}_n$ to the set of all finite unions in $\mathcal{A}$. These extensions, which have already appeared in [22], will not be used until Chapter 3 when we define the so-called "strong past". In particular, given any n and any finite union, $B = \bigcup_{i=1}^{n} A_i$ ($A_i \in \mathcal{A}$), we define

$$g_n(B) \;=\; \bigcup_{\substack{A\in\mathcal{A}\\ A\subseteq B}} g_n(A). \qquad (2.4)$$

Since each g_n is increasing on $\mathcal{A}$ w.r.t. $\subseteq$, $g_n(A') = \bigcup_{A\in\mathcal{A}, A\subseteq A'} g_n(A)\ \ \forall A' \in \mathcal{A}$. Thus, our use of g_n in (2.4) is unambiguous. Since each $\mathcal{A}_n$ is finite, each $g_n(B)$ is a finite union of sets in $\mathcal{A}_n$.

Our next result contains two important properties of the extended g_n.

Proposition 2.2.9 *Let $\mathcal{A}$ be an indexing-collection on T. If $B = \bigcup_{i=1}^{k} A_i$ and $B' = \bigcup_{j=1}^{k'} A'_j$ with $A_i, A'_j \in \mathcal{A}\ \forall i, j$, then for any n*

(a) $g_n(B \cup B') = g_n(B) \cup g_n(B')$ *and*

(b) $g_n(B) = \bigcup_{i=1}^{k} g_n(A_i)$.

Proof Since $B, B' \subseteq B \cup B'$, it is clear from (2.4) that

$$g_n(B),\ g_n(B') \;\subseteq\; g_n(B \cup B')$$

and hence, $g_n(B) \cup g_n(B') \subseteq g_n(B \cup B')$.

To establish the opposite inclusion in (a), take $A \in \mathcal{A}$ s.t. $A \subseteq B \cup B'$. Then, $A = (\bigcup_{i=1}^{k} A \cap A_i) \cup (\bigcup_{j=1}^{k'} A \cap A'_j)$ where $A, A \cap A_i, A \cap A'_j \in \mathcal{A}$ $\forall i, j$. By an inductive extension of condition (2)(iv′) in Definition 2.2.2, this implies

$$g_n(A) \;=\; \bigcup_{i=1}^{k} g_n(A \cap A_i) \;\cup\; \bigcup_{j=1}^{k'} g_n(A \cap A'_j)$$

where, by (2.4), $g_n(A \cap A_i) \subseteq g_n(B)$ $\forall i$ and $g_n(A \cap A'_j) \subseteq g_n(B')$ $\forall j$. Therefore, $g_n(A) \subseteq g_n(B) \cup g_n(B')$ for our arbitrarily chosen $A \subseteq B \cup B'$ in $\mathcal{A}$ and hence by (2.4),

$$g_n(B \cup B') \subseteq g_n(B) \cup g_n(B')$$

which establishes (a). (b) follows by a trivial inductive extension of (a). □

In the upcoming sections of this chapter, we will often require a d_H-continuous set-function, $x : \mathcal{A} \to \mathbf{R}$ to be d_H-uniformly continuous. A classic sufficient condition ensuring this property is the d_H-compactness of $\mathcal{A}$ which, as we will see, is a consequence of Definition 2.2.4. We begin with the following result.

Theorem 2.2.10 *If $\mathcal{A}$ is any indexing collection on T, then $\mathcal{A}$ is d_H-closed in $\mathcal{K}_T$.*

Proof Take $(A_n)_n$ in $\mathcal{A}$ s.t. $A_n \to_{d_H} C$ for some $C \in \mathcal{K}_T$. The theorem will follow if we can show $C \in \mathcal{A}$.

Using the sequence $(\delta_n)_n$ from Lemma A.2.1, we construct a subsequence, $(A_{k_n})_n$ as follows: given $n \in \mathbf{N}$, choose $k_n \in \mathbf{N}$ such that $d_H(A_{k_n}, C) < \delta_n$. (Clearly, we can choose $(k_n)_n$ to be strictly increasing in n.) Therefore, since $d_H(A_{k_n}, C) < \delta_n$ $\forall n$, Lemma A.2.1 implies

$$C \subseteq A_{k_n}^{\delta_n} \subseteq [g_n(A_{k_n})]^{\delta_n} \subseteq g_n^2(A_{k_n}), \quad (\forall n) \tag{2.5}$$

where g_n^2 denotes $g_n \circ g_n$. Moreover, given any n, condition (2.2) in Definition 2.2.2 implies

$$\begin{aligned} d_H(g_n^2(A_{k_n}), C) &\leq d_H(g_n^2(A_{k_n}), g_n(A_{k_n})) + d_H(g_n(A_{k_n}), A_{k_n}) + d_H(A_{k_n}, C) \\ &< \epsilon_n + \epsilon_n + \delta_n \end{aligned}$$

where $2\epsilon_n + \delta_n \to 0$ as $n \to \infty$.

Since $g_n^2(A_{k_n}) \to_{d_H} C$ and, by (2.5), $C \subseteq g_n^2(A_{k_n})$ $\forall n$, Lemma A.1.3 implies $C = \bigcap_n g_n^2(A_{k_n})$. Therefore, since $g_n^2(A_{k_n}) \in \mathcal{A}_{k_n} \subseteq \mathcal{A}$ $\forall n$ and $\mathcal{A}$ is closed under countable intersections, we have shown $C \in \mathcal{A}$. □

Remark 2.2.11 As a consequence of Theorem 2.2.10, any indexing collection $\mathcal{A}$ on the Euclidean space $[0,1]^k$ ($k \in \mathbf{N}$) satisfies Assumption (A1) in [5]. In particular, given any $\delta > 0$, we can define the finite subcollection $\mathcal{A}(\delta)$ from [5] to be $\mathcal{A}_{\lceil \delta^{-1} \rceil}$ where $\lceil \delta \rceil$ denotes the the least integer greater than δ.

The following result, a consequence of the compactness of (T, d), can be found on p.279 of [32].

Proposition 2.2.12 *$(\mathcal{K}_T, d_H)$ is compact.*

Combining Theorem 2.2.10 and Proposition 2.2.12, we obtain the following result, third on our list of consequences of Definition 2.2.4.

Theorem 2.2.13 *If $\mathcal{A}$ is any indexing collection on T, then $(\mathcal{A}, d_H)$ is a compact metric space.*

The fourth consequence listed on p.17 is in fact the most basic. Given an indexing collection $\mathcal{A}$ on T, define the set

$$\phi' = \bigcap_{A \in \mathcal{A}} A. \tag{2.6}$$

By Proposition 2.2.8, $\phi' \in \mathcal{A}$. Since $\phi \notin \mathcal{A}$, $\phi' \neq \phi$. If we adjoin ϕ to $\mathcal{A}$ according to Remark 2.2.5 (c), then the above intersection must be taken over all $A \in \mathcal{A}$ s.t. $A \neq \phi$ to ensure that $\phi' \neq \phi$. In either case, the set ϕ' plays the role of 0 in the classical theory in that ϕ' is the smallest non-empty element in $(\mathcal{A}, \dot{\subseteq})$.

We close this section with a result illustrating certain restrictions that uniform separability from above places on the cardinality of indexing collections. An additional discussion on the size of indexing collections, measured by means other than cardinality, is given in Subsection 4.2.4.

Proposition 2.2.14 *If $\mathcal{A}$ is an indexing collection on T, then*

(a) *the cardinality of $\mathcal{A}$ cannot exceed that of $\mathcal{P}(\mathbf{R})$ and*

(b) *if (T, d) is connected, then $\mathcal{A}$ is either infinite or $\mathcal{A} = \{T\}$.*

Proof It is well-known that the cardinality of any separable Hausdorff space X cannot exceed that of $\mathcal{P}(\mathbf{R})$. Since $(\mathcal{A}, d_H)$ is a metric space, it is Hausdorff. As mentioned in Remark 2.2.3, $\mathcal{A}$ is d_H-separable.

To show (b), assume (T, d) is connected and $\mathcal{A} \neq \{T\}$. Selecting a fixed set, $A_0 \neq T$ in $\mathcal{A}$, $\exists\, K \in \mathbf{N}$ s.t. $g_n(A_0) \neq T$ $\forall n \geq K$. (Otherwise, condition

(2)(i′) of Definition 2.2.2 would imply $A_0 = T$, resulting in a contradiction.) Therefore, since T is connected,

$$[g_n(A_0)]^\circ \subset g_n(A_0), \;(\forall n \geq K). \tag{2.7}$$

(Recall that each $g_n(A)$ is d-closed and that $\phi \notin \mathcal{K}_T$.)

Now, assume $\mathcal{A}$ has finite cardinality. Since $(g_n(A_0))_n$ is a decreasing sequence in $\mathcal{A}$ with $\bigcap_n g_n(A_0) = A_0$, the finiteness of $\mathcal{A}$ implies $g_n(A_0) = A_0$ for all large n. Hence, by (2.7) and condition (2)(ii′) of Definition 2.2.2,

$$\exists\, k_0 \geq K \text{ s.t. } g_{k_0}(A_0) = A_0 \subseteq [g_{k_0}(A_0)]^\circ \subset g_{k_0}(A_0)$$

which gives a contradiction. Therefore, $\mathcal{A}$ must be an infinite set. □

2.3 The Function Space $D(\mathcal{A})$

In this section we define the function space $D(\mathcal{A})$ and two important subspaces thereof denoted PA and $D_0(\mathcal{A})$. Among other things, $D(\mathcal{A})$ is a collection of real-valued *set-functions*, that is, functions of the form $x : \mathcal{A} \to \mathbf{R}$. As commented in [5], $D(\mathcal{A})$ is large enough to contain the sample paths of all set-indexed processes of interest. In fact, as we will see in upcoming chapters, PA and $D(\mathcal{A})$ are sufficiently rich for most applications. We begin with some terminology.

Definition 2.3.1 *For any $A, (A_n)_n$ in $\mathcal{A}$, we write*

(a) $A_n \searrow A$ *if* $A \subseteq A_n$ $\forall n$ *and* $A_n \xrightarrow{d_H} A$ *and*

(b) $A_n \nearrow A$ *if* $A_n \subseteq A^\circ$ $\forall n$ *and* $A_n \xrightarrow{d_H} A$.

Given $x : \mathcal{A} \to \mathbf{R}$, we say

(c) *x is <u>outer-continuous</u> at A if $A_n \searrow A$ implies $x(A_n) \to x(A)$ and*

(d) *x has <u>inner-limits</u> at A if $A_n \nearrow A$ implies $(x(A_n))_n$ converges.*

If x is outer-continuous (has inner-limits) at every $A \in \mathcal{A}$, then we simply say that x is outer-continuous (respectively, has inner-limits).

Remark 2.3.2 It is important to note that neither $A_n \searrow A$ nor $A_n \nearrow A$ require $(A_n)_n$ to be monotone w.r.t. $\subseteq$. An alternate form of continuity, defined via monotone sequences, can be found in Definition 3.2.44.

Given any set-function, $x:\mathcal{A} \to \mathbf{R}$ define the *supremum norm* of x by

$$\|x\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}} |x(A)| \in [0,\infty]$$

and let

$$B(\mathcal{A}) = \{x : \mathcal{A} \to \mathbf{R} \text{ s.t. } \|x\|_{\mathcal{A}} < \infty\}.$$

Any element of $B(\mathcal{A})$ is said to be a *bounded set-function.*

Definition 2.3.3 *$D(\mathcal{A})$ consists of all set-functions, $x \in B(\mathcal{A})$ that are outer-continuous with inner-limits on $\mathcal{A}$.*

The space $D(\mathcal{A})$ has already been defined in [5] for the case of $T = [0,1]^k$ and in [26] for the case of a general compact metric space. $D(\mathcal{A})$ is intended to mimic the classical function space $D[0,1]$ consisting of all real-valued functions on $[0,1]$ which are right continuous with left limits at each $t \in [0,1]$. In fact, when $T = [0,1]$ and $\mathcal{A} = \{[0,x] : 0 \leq x \leq 1\}$, $D(\mathcal{A})$ and $D[0,1]$ are naturally homeomorphic under appropriate metrics (see Remark 2.4.2 (c)).

In words, the following result implies that $D(\mathcal{A})$ is a $\|\cdot\|_{\mathcal{A}}$-closed linear subspace of $B(\mathcal{A})$ containing all d_H-continuous set-functions.

Proposition 2.3.4 (a) *Given $x, y \in D(\mathcal{A})$ and $\alpha \in \mathbf{R}$, $x + y, \alpha{\cdot}x \in D(\mathcal{A})$.*
(b) *If $x : \mathcal{A} \to \mathbf{R}$ is continuous on $(\mathcal{A}, d_H)$, then $x \in D(\mathcal{A})$.*
(c) *Given $(y_n)_n$ in $D(\mathcal{A})$ and $y \in B(\mathcal{A})$, if $\|y_n - y\|_{\mathcal{A}} \to 0$, then $y \in D(\mathcal{A})$.*

Proof Both (a) and (b) are obvious consequences of Definition 2.3.1.

To show that the set-function, y in (c) is outer-continuous, take $A_n \searrow A$ in $\mathcal{A}$. Then, for each k, n and $A \in \mathcal{A}$,

$$|y(A_n) - y(A)| \leq |y(A_n) - y_k(A_n)| + |y_k(A_n) - y_k(A)| + |y_k(A) - y(A)|.$$

Given any $\epsilon > 0$, $\|y_n - y\|_{\mathcal{A}} \to 0$ implies $\exists\, K \in \mathbf{N}$ s.t. $\sup_{B\in\mathcal{A}} |y_K(B) - y(B)| \leq \epsilon/3$. But y_K is outer-continuous and hence, $\exists\, N \in \mathbf{N}$ s.t. $|y_K(A_n) - y_K(A)| \leq \epsilon/3\ \forall n \geq N$. Therefore, by the above inequality, $|y(A_n) - y(A)| \leq 3(\epsilon/3)$ $\forall n \geq N$. The argument for inner-limits is similar. □

Before we can define the subspaces PA and $D_0(\mathcal{A})$ of $D(\mathcal{A})$, we must first examine the following class of set-functions.

Definition 2.3.5 *A set-function $x : \mathcal{A} \to \mathbf{R}$ is purely atomic if, for some $n \in \mathbf{N}$, $\exists\ t_1, \cdots, t_n \in T$ and $a_1, \cdots, a_n \in \mathbf{R}$ s.t.*

$$x(A) = \sum_{j: t_j \in A} a_j, \quad (\forall A \in \mathcal{A}).$$

The t_j are called the locations of atoms and the a_j are called the masses of atoms. The variation of x is the number, $\nu(x) = \sum_{i=1}^{n} |a_i|$.

When the term "atoms" is used alone, it will always refer to the locations of atoms.

If one is not careful in selecting the locations of atoms, the resulting purely atomic set-functions may not have inner-limits on $\mathcal{A}$. This is illustrated by the following example.

Example 2.3.6 Let $\mathcal{A} = \mathcal{I}_2$, the indexing collection defined in Example 2.2.6. By the geometry of $([0,1]^2, \|\cdot\|_\infty)$, it is clear that $d_H([0, \mathbf{u}], [0, \mathbf{v}]) = \|\mathbf{u} - \mathbf{v}\|_\infty$ $\forall\, \mathbf{u}, \mathbf{v} \in [0,1]^2$.

Given $0 \le x_0 < x'_0 < 1$, let $x : \mathcal{I}_2 \to \mathbf{R}$ be purely atomic with one atom of mass 1 located at $(x_0, 1)$. To show that x does not have inner-limits at $A = [0, (x_0{}', 1)]$, take a sequence $(x_n)_{n \ge 1}$ s.t. $x_0 < x_n < x'_0$ $\forall n$ and $x_n \uparrow x'_0$. If we define

$$A_n = \begin{cases} [0, (x_n, 1)] & , \text{if } n \text{ is even} \\ [0, (x_n, 1 - 1/n)] & , \text{if } n \text{ is odd}, \end{cases}$$

then $A_n \subseteq A^\circ$ $\forall n$ and $d_H(A_n, A) = \max\{|x_n - x'_0|, 1/n\} \xrightarrow{n} 0$. But $(x_0, 1) \in A_n$ if and only if n is even and therefore,

$$x(A_n) = \begin{cases} 1 & , \text{if } n \text{ is even} \\ 0 & , \text{if } n \text{ is odd} \end{cases}$$

which is to say x does not have inner-limits at A. [4]

Repeating the above argument, if $x : \mathcal{I}_2 \to \mathbf{R}$ is purely atomic with one atom of mass 1 located at $(1, y_0)$ for some $0 < y_0 < 1$, then x does not have inner-limits on $\mathcal{A}$. The same is true for x purely atomic with one atom of mass 1 at $(1, 1)$.

[4] In [5], Bass and Pyke mistakenly claimed that all purely atomic set-functions had inner-limits on $\mathcal{A}$.

In view of the above example, we require a restriction on the location of atoms (for any (T, d)) which will ensure that the corresponding purely atomic functions lie in $D(\mathcal{A})$. For this, we turn to the concept of *proper sets* which was originally defined in [25].

Definition 2.3.7 (a) *Given $C \subseteq T$ and $\epsilon > 0$, define $C^{-\epsilon} := \bigcap_{B \in \mathcal{A}, C \subseteq B^\epsilon} B$.*
(b) *$A \in \mathcal{A}$ is proper provided $\exists\, \epsilon_0 > 0$ s.t. $(A^\epsilon)^{-\epsilon} = A$ $\forall\, 0 < \epsilon < \epsilon_0$.*

Remark 2.3.8 (a) Given $\epsilon > 0$ and $C \subseteq D$, if $D \subseteq B^\epsilon$, then $C \subseteq B^\epsilon$. Therefore, $C \subseteq D$ implies $C^{-\epsilon} \subseteq D^{-\epsilon}$.

(b) By Proposition 2.2.1, we can replace $(\cdot)^\epsilon$ by $(\cdot)^{\overline{\epsilon}}$ in Definition 2.3.7 to obtain an equivalent definition of a proper set.

(c) It is clear from Definition 2.3.7 (a) that $(A^\epsilon)^{-\epsilon} \subseteq A$ for every $A \in \mathcal{A}$ and $\epsilon > 0$. Therefore, since ϕ' is the smallest element in $(\mathcal{A}, \subseteq)$, $\phi' \subseteq (\phi'^{\epsilon})^{-\epsilon}$ $\forall\, \epsilon > 0$ which implies ϕ' is a proper set in $\mathcal{A}$.

The proof of the following Lemma is trivial.

Lemma 2.3.9 *Let $A \in \mathcal{A}$ be given. If, for any $\epsilon > 0$, $\exists\, B \in \mathcal{A}$ s.t.*

$$A^\epsilon \subseteq B^\epsilon \text{ and } A \not\subseteq B,$$

then A is not a proper set in $\mathcal{A}$.

For each $t \in T$, define the set $A_t \subseteq T$ by letting

$$A_t = \bigcap_{\substack{B \in \mathcal{A} \\ t \in B}} B. \tag{2.8}$$

By Proposition 2.2.8, $A_t \in \mathcal{A}$ $\forall\, t \in T$. (When $T = [0,1]^k$ and $\mathcal{A} = \mathcal{I}_k$, it is clear that $A_{\mathrm{t}} = [0, \mathrm{t}]$ $\forall\, \mathrm{t} \in [0,1]^k$.) Now, define

$$\omega(\mathcal{A}) = \{t \in T : A_t \text{ is not proper in } \mathcal{A}\}. \tag{2.9}$$

To better understand proper sets and the set $\omega(\mathcal{A})$, we now determine all $\mathrm{t} \in [0,1]^2$ for which A_{t} is not proper in $\mathcal{I}_2$.

Proposition 2.3.10 *If $(T, d) = ([0,1]^2, \|\cdot\|_\infty)$ and $\mathcal{A} = \mathcal{I}_2$, then*

$$\{\mathrm{t} \in [0,1]^2 : A_{\mathrm{t}} \textit{ not proper}\} = \{(x,1), (1,y) : 0 \leq x, y \leq 1\}. \tag{2.10}$$

Therefore, $\omega(\mathcal{I}_2) = \{(x,y) \in [0,1]^2 : x = 1 \text{ or } y = 1\}$

Proof The following basic observation will be needed: given $A = [0, (x, y)] \in \mathcal{I}_2$ and $\delta > 0$, $A^{\overline{\delta}} = [0, (\min\{1, x + \delta\}, \min\{1, y + \delta\})]$.

Take $\mathbf{t} = (x, 1) \in [0, 1]^2$. As mentioned above, $A_{\mathbf{t}} = [0, (x, 1)]$. Given any $\epsilon > 0$, if $B = [0, (x, 1 - \epsilon/2)] \in \mathcal{I}_2$, then it is clear that $A_{\mathbf{t}}^{\epsilon} = B^{\epsilon}$ while $A \not\subseteq B$. Therefore, by Lemma 2.3.9, $A_{\mathbf{t}}$ is not proper in $\mathcal{I}_2$. The argument for $\mathbf{t} = (1, y) \in [0, 1]^2$ is identical.

We establish the opposite inclusion in (2.10) by contrapositive. Take $\mathbf{t} = (x, y) \in [0, 1]^2$ with $0 \leq x, y < 1$. Once again, $A_{\mathbf{t}} = [0, (x, y)]$. If we define $\epsilon_0 = \min\{1 - x, 1 - y\} > 0$, then for any $0 < \epsilon < \epsilon_0$, the above observation yields the implication,

$$A_{\mathbf{t}}^{\overline{\epsilon}} \subseteq B^{\overline{\epsilon}} \implies A_{\mathbf{t}} \subseteq B$$

Therefore, $A_{\mathbf{t}} \subseteq (A_{\mathbf{t}}^{\overline{\epsilon}})^{-\overline{\epsilon}} \subseteq A_{\mathbf{t}}$ $\forall\, 0 < \epsilon < \epsilon_0$ which, by Remark 2.3.8 (b), implies $A_{\mathbf{t}}$ is proper in $\mathcal{I}_2$. □

Given any $k \in \mathbf{N}$, the result in Proposition 2.3.10 generalizes to

$$\begin{aligned} \omega(\mathcal{I}_k) &= \{\mathbf{t} \in [0, 1]^k : t_i = 1 \text{ for some } 1 \leq i \leq k\} \\ &= \bigcup \{\text{back faces of } [0, 1]^k\}. \end{aligned} \tag{2.11}$$

(Note that each of the k back faces of the k-dimensional cube is a set of the form, $\{\mathbf{t} \in [0, 1]^k : (t_1, \cdots, t_{i-1}, 1, t_{i+1}, \cdots, t_k)\}$ where $1 \leq i \leq k$.)

In general, $\omega(\mathcal{A})$ has the following nice property.

Theorem 2.3.11 *Let $x : \mathcal{A} \to \mathbf{R}$ be a purely atomic set-function with atoms located at $t_1, \cdots, t_n \in T$. If $t_i \notin \omega(\mathcal{A})$ $\forall\, 1 \leq i \leq n$, then $x \in D(\mathcal{A})$.*

Proof For any $S \subseteq T$, define $P_S = \{j : t_j \in S\}$. To establish outer-continuity, fix $A \in \mathcal{A}$ and take $A_n \searrow A$ in $\mathcal{A}$. Since A is d-closed in T, $\exists\, \epsilon_0 > 0$ s.t.

$$d(A, t_j) \geq \epsilon_0, \quad (\forall j \notin P_A).$$

By the definition of P_S, this implies $P_A = P_{A^{\epsilon_0}}$. Since $A_n \to_{d_H} A$, $\exists\, K \in \mathbf{N}$ s.t. $A_n \subseteq A^{\epsilon_0}$ $\forall n \geq K$ and thus,

$$P_{A_n} \subseteq P_{A^{\epsilon_0}} = P_A, \quad (\forall n \geq K).$$

But $A \subseteq A_n$ $\forall n$ implies $P_A \subseteq P_{A_n}$ $\forall n$. Therefore, $P_{A_n} = P_A$ $\forall n \geq K$ which yields the limit, $x(A_n) \to x(A)$, establishing outer-continuity of x at

$\mathcal{A}$. (The above argument works for any purely atomic x, regardless of the locations of its atoms.)

To establish that x has inner-limits on $\mathcal{A}$, first consider the case in which x has a single atom of mass 1 at some $t \in T \setminus \omega(\mathcal{A})$. Defining the *set-interval*,

$$[A_t, T) = \{A \in \mathcal{A} : A_t \subseteq A \subseteq T^\circ\},$$

it is clear from the definition of A_t that $x = \mathbf{1}_{[A_t,T)}$ as set-functions on $\mathcal{A}$. (Note that $T^\circ = T$.) Furthermore, since A_t is a proper set, $[A_t, T)$ is a *proper interval* and thus, by Lemma 2.5 in [25], $\mathbf{1}_{[A_t,T)}$ has inner-limits on $\mathcal{A}$. (See Section C.2 for additional material on proper intervals.) Therefore, x has inner-limits on $\mathcal{A}$.

Given any purely atomic x with atoms $t_1 \cdots, t_n \notin \omega(\mathcal{A})$ and respective masses $a_1, \cdots, a_n \in \mathbf{R}$, $x \in D(\mathcal{A})$ follows from Proposition 2.3.4 (a) and the identity, $x = \sum_{i=1}^{n} a_i x_i$ where x_i is a purely atomic set-function with a single atom of mass 1 at $t_i \in T$. □

Remark 2.3.12 (a) As shown in the first paragraph of the above proof, any purely atomic set-function is outer-continuous on $\mathcal{A}$.

(b) By Proposition 2.3.10 and Example 2.3.6, the converse of Theorem 2.3.11 holds for the case of $T = [0, 1]^2$ and $\mathcal{A} = \mathcal{I}_2$. Namely, if x is purely atomic and $x \in D(\mathcal{I}_2)$, then all atoms of x must lie outside of $\omega(\mathcal{I}_2)$. (In fact, this is true for any $\mathcal{I}_k$ ($k \in \mathbf{N}$).) It is presently unclear as to whether the converse of Theorem 2.3.11 holds for a general indexing collection $\mathcal{A}$ on a general compact metric space (T, d).

In many applications, one requires set-functions $x : \mathcal{A} \to \mathbf{R}$ which are smooth in the sense that $x(A) = 0$ for every $A \in \mathcal{A}$ with empty d-interior. (See Remark 2 on p.6 of [22] for a situation in which this restriction is required.) If we define

$$\alpha(\mathcal{A}) = \{t \in T : \exists\, A \in \mathcal{A} \text{ s.t. } A^\circ = \phi \text{ and } t \in A\}, \tag{2.12}$$

then for purely atomic set-functions, the above restriction is equivalent to requiring that all atoms be located outside the set $\alpha(\mathcal{A})$. Given $k \in \mathbf{N}$ and $\mathcal{A} = \mathcal{I}_k$, it is clear that

$$\begin{aligned} \alpha(\mathcal{I}_k) &= \{\mathrm{t} \in [0,1]^k : t_i = 0 \text{ for some } 1 \leq i \leq n\} \\ &= \bigcup \{\text{front faces of } [0,1]^k\}. \end{aligned} \tag{2.13}$$

(Note that each of the k front faces of the k-dimensional cube is a set of the form, $\{t \in [0,1]^k : (t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_k)\}$ where $1 \leq i \leq k$.)

Combining this with the preceding restriction on the location of atoms, we define the *edge* of $\mathcal{A}$, to be the set

$$E(\mathcal{A}) = \alpha(\mathcal{A}) \cup \omega(\mathcal{A}).$$

The term "edge" is geometrically motivated by (2.11) and (2.13) for the case of $T = [0,1]^2$. Via $E(\mathcal{A})$, we can define a class of well-behaved purely atomic set-functions as follows.

Definition 2.3.13 *PA consists of all purely atomic set-functions, $x : \mathcal{A} \to \mathbf{R}$ whose atoms are located outside the set $E(\mathcal{A})$.*

Remark 2.3.14 Theorem 2.3.11 implies $PA \subseteq D(\mathcal{A})$.

The following result establishes an important smoothness property for the set-functions in PA.

Lemma 2.3.15 *Each $x \in PA$ is d_H-continuous at ϕ' and T.*

Proof Let $x \in PA$ be given. To establish d_H-continuity of x at ϕ', take an arbitrary sequence, $A_n \to_{d_H} \phi'$. By minimality, $\phi' \subseteq A_n$ $\forall n$. Therefore, since x is outer-continuous at ϕ', it follows that $x(A_n) \to x(\phi')$, implying d_H-continuity of x at ϕ'.

To establish d_H-continuity at T, first consider the case in which x has a single atom of mass 1 at $t \in T \setminus E(\mathcal{A})$. Since $\omega(\mathcal{A}) \subseteq E(\mathcal{A})$, $A_t \in \mathcal{A}$ is a proper set which, by Definition 2.3.7(b), implies $\exists\, \epsilon > 0$ s.t. $(A_t^\epsilon)^{-\epsilon} = A_t$.

Now, given $B_n \to_{d_H} T$, $\exists\, K \in \mathbf{N}$ s.t.

$$n \geq K \Rightarrow T \subseteq (B_n)^\epsilon.$$

Thus, for each $n \geq K$, Remark 2.3.8(a) implies

$$A_t = (A_t^\epsilon)^{-\epsilon} \subseteq T^{-\epsilon} \subseteq (B_n^\epsilon)^{-\epsilon} \subseteq B_n.$$

But $t \in A_t$ therefore, $x(B_n) = 1 = x(T)$ $\forall n \geq K$, i.e., $x(B_n) \xrightarrow{n} x(T)$. Since the sequence, $B_n \to_{d_H} T$ was arbitrarily chosen, this implies x is d_H-continuous at T.

For the general case, assume x has atoms $t_1, \cdots, t_k \in T \setminus E(\mathcal{A})$ with respective masses, $a_1, \cdots, a_n \in \mathbf{R}$. Let $x_j \in PA$ have a single atom of mass 1 at t_j $\forall 1 \leq j \leq k$. Then, given an arbitrary sequence, $B_n \to_{d_H} T$, the above case implies

$$x(B_n) = \sum_{j=1}^{k} a_j \cdot x_j(B_n) \xrightarrow{n} \sum_{j=1}^{k} a_j \cdot x_j(T) = x(T),$$

establishing d_H-continuity of x at T. □

Finally, we define the subspace $D_0(\mathcal{A})$ of $B(\mathcal{A})$. Let

$$C(\mathcal{A}) = \{x \in B(\mathcal{A}) : x \text{ is } d_H\text{-continuous on } \mathcal{A}\}.$$

The following definition is analogous to Definition 3.2 in [5].

Definition 2.3.16 *$D_0(\mathcal{A})$ consists of all $x \in B(\mathcal{A})$ for which $\exists\, C_n \in C(\mathcal{A})$ and $J_n \in PA, (n = 1, 2, \cdots)$ s.t. $\|x - (C_n + J_n)\|_{\mathcal{A}} \to 0$ as $n \to \infty$.*

In other words, $D_0(\mathcal{A})$ consists of all set-functions x which can be uniformly approximated by set-functions that are continuous with a "pure-jump part". Trivially, $C(\mathcal{A}), PA \subseteq D_0(\mathcal{A})$. See Remark 3 on p.881 of [5] for an example of a set-function, $x : \mathcal{A} \to \mathbf{R}$ which is in $D(\mathcal{A})$ but not in $D_0(\mathcal{A})$ for some indexing collection $\mathcal{A}$ on $[0, 1]^3$.

Note that in Definition 2.3.16, the J_n are selected from PA and hence are not allowed to possess atoms in $E(\mathcal{A})$. [5] If we did not restrict our choice of J_n, the resulting $D_0(\mathcal{A})$ may not have been contained in $D(\mathcal{A})$ (see Example 2.3.6). As it is,

Theorem 2.3.17 $D_0(\mathcal{A}) \subseteq D(\mathcal{A})$.

Proof As mentioned in Remark 2.3.14, $PA \subseteq D(\mathcal{A})$. Theorem 2.3.17 now follows by applying the various parts of Proposition 2.3.4. □

[5] In contrast, the definition of $D_0(\mathcal{A})$ given in Definition 3.2 of [5] places no restriction on the locations of the atoms of J_n.

2.4 A J_2-type Metric on $D(\mathcal{A})$

In this section, we define a probabilistically meaningful metric on $B(\mathcal{A})$. As the title suggests, this metric, when restricted to the subspace $D(\mathcal{A})$, will yield a topology which is analogous to the Skorokhod J_2 topology on $D[0,1]$ (see Remark 2.4.2 (c)). Our approach is the same as that found in Section 2 of [22] and in [5] for the case of $T = [0,1]^k$ ($k \in \mathbf{N}$).

Given an indexing collection, $\mathcal{A}$ on (T, d), define a metric $\rho_{\mathcal{A}}$ on $\mathcal{A} \times \mathbf{R}$ by letting

$$\rho_{\mathcal{A}}((A_1, r_1), (A_2, r_2)) \;=\; d_H(A_1, A_2) \,+\, |r_1 - r_2|$$

for any $(A_1, r_1), (A_2, r_2) \in \mathcal{A} \times \mathbf{R}$. Note that $\rho_{\mathcal{A}}$ generates the product topology on $\mathcal{A} \times \mathbf{R}$. Now, let

$$\mathcal{K}_{\mathcal{A}\times\mathbf{R}} \;=\; \{\text{non-empty, } \rho_{\mathcal{A}}\text{-closed and bounded subsets of } \mathcal{A} \times \mathbf{R}\}$$

and let d_G be the Hausdorff metric on $\mathcal{K}_{\mathcal{A}\times\mathbf{R}}$. By Theorem 2.2.13, $(\mathcal{A}, d_H)$ is compact and hence complete. Furthermore, by Remark 2.2.3, $(\mathcal{A}, d_H)$ is separable. Therefore, $(\mathcal{A} \times \mathbf{R}, \rho_{\mathcal{A}})$ is complete and separable which, as mentioned on p.864 of [5], implies

Lemma 2.4.1 *$(\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$ is complete and separable.*

As done in [5], we define the map $G : B(\mathcal{A}) \to \mathcal{K}_{\mathcal{A}\times\mathbf{R}}$ by letting

$$G(x) \;=\; cl(\{(A, x(A)) : A \in \mathcal{A}\}), \quad (\forall\, x \in B(\mathcal{A})),$$

the closure taken w.r.t. $\rho_{\mathcal{A}}$. Since d_G is a metric,

$$d_D(x, y) \;:=\; d_G(G(x), G(y)), \quad (x, y \in B(\mathcal{A}))$$

defines a pseudometric on $B(\mathcal{A})$. Thus, under the equivalence relation

$$x \sim y \text{ in } B(\mathcal{A}) \iff G(x) = G(y),$$

the set of equivalence classes of $B(\mathcal{A})$ (modulo $\sim$) forms a metric space under d_D. For the sake of simplicity, we will denote the equivalence class of each $x \in B(\mathcal{A})$ by x and denote the set of equivalence classes by $B(\mathcal{A})$. An important subspace of $(B(\mathcal{A}), d_D)$ is $D(\mathcal{A})$.

Remark 2.4.2 (a) By definition, $G : (B(\mathcal{A}), d_D) \to (\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$ is an isometric embedding.

(b) By the definition of $\rho_{\mathcal{A}}$-closure, if $x \in B(\mathcal{A})$, then $(B, r) \in G(x)$ if and only if $\exists\,(B_n)_n$ in $\mathcal{A}$ such that $B_n \to_{d_H} B$ and $x(B_n) \to r$ as $n \to \infty$.

(c) As mentioned in Remark 1 on p.4 of [22], $(D(\mathcal{I}_1), d_D)$ is homeomorphic to $(D[0,1], J_2)$ (see the footnote on p.11).

The next two results are motivated by Proposition 3.1 in [5].

Lemma 2.4.3 *Given any $R > 0$, $\mathcal{E}_R = \{E \in \mathcal{K}_{\mathcal{A}\times\mathbf{R}} : E \subseteq \mathcal{A} \times [-R, R]\}$ is compact in $(\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$.*

Proof If we take

$$(X, \sigma) = (\mathcal{A} \times \mathbf{R}, \rho_{\mathcal{A}}) \quad \text{and} \quad Y = \mathcal{A} \times [-R, R]$$

in Lemma A.1.1, then

$$(\mathcal{E}_R, d_G|_{\mathcal{E}_R}) \;=\; (\mathcal{K}_{\mathcal{A}\times[-R,R]}, d_{G'})$$

where $d_{G'}$ is the Hausdorff metric generated by $\rho_{\mathcal{A}}|_{\mathcal{A}\times[-R,R]}$ on $\mathcal{A} \times [-R, R]$.

Since $[-R, R]$ is compact and, by Theorem 2.2.13, $(\mathcal{A}, d_H)$ is compact, it follows that $(\mathcal{A} \times [-R, R], \rho_{\mathcal{A}}|_{\mathcal{A}\times[-R,R]})$ is compact. Therefore, by Proposition 2.2.12, $(\mathcal{K}_{\mathcal{A}\times[-R,R]}, d_{G'})$ is compact, which establishes the result. □

Lemma 2.4.4 *If Θ is bounded in $(B(\mathcal{A}), d_D)$, then $\sup_{x\in\Theta} \|x\|_{\mathcal{A}} < \infty$.*

Proof Let $R_1 = \sup_{x,y\in\Theta} d_D(x, y)$. Since Θ is d_D-bounded and G is an isometric embedding,

$$\sup_{x,y\in\Theta} d_G(G(x), G(y)) \;=\; R_1 \;<\; \infty. \tag{2.14}$$

Next, select a fixed $x_0 \in \Theta$. Since $x_0 \in B(\mathcal{A})$, $\exists\, 0 < R_2 < \infty$ s.t.

$$|x_0(B)| \;\le\; R_2\,, \quad (\forall\, B \in \mathcal{A}). \tag{2.15}$$

Now, take $x \in \Theta$ and $A \in \mathcal{A}$. By (2.14), $d_G(G(x_0), G(x)) \le R_1$, which implies that

$$\exists\, B \in \mathcal{A} \text{ s.t. } \rho_{\mathcal{A}}((A, x(A)), (B, x_0(B))) \;<\; R_1.$$

Hence, $|x(A) - x_0(B)| < R_1$ which, along with (2.15), implies that

$$|x(A)| \le |x(A) - x_0(B)| + |x_0(B)| \le R_1 + R_2.$$

Therefore, since both $A \in \mathcal{A}$ and $x \in \Theta$ were arbitrarily chosen,

$$\sup_{x \in \Theta} \|x\|_{\mathcal{A}} = \sup_{x \in \Theta}(\sup_{A \in \mathcal{A}} |x(A)|) \le R_1 + R_2 < \infty,$$

which establishes the Lemma. □

Recall that any compact subspace of a metric space is bounded. This fact, when combined with the above lemma, implies the following result.

Corollary 2.4.5 *If Θ is compact in $(B(\mathcal{A}), d_D)$, then* $\sup_{x \in \Theta} \|x\|_{\mathcal{A}} < \infty$.

The following result will be useful in establishing that a subset of $B(\mathcal{A})$ is d_D-compact.

Lemma 2.4.6 *Let $\Theta \subseteq B(\mathcal{A})$ be such that $\sup_{x \in \Theta} \|x\|_{\mathcal{A}} < \infty$. If $G[\Theta]$ is closed in $(\mathcal{K}_{\mathcal{A} \times \mathbf{R}}, d_G)$, then Θ is compact in $(B(\mathcal{A}), d_D)$.*

Proof Let $R = \sup_{x \in \Theta} \|x\|_{\mathcal{A}} < \infty$. If we are given $x \in \Theta$, then $|x(A)| \le R$ $\forall A \in \mathcal{A}$ which implies that $(A, x(A)) \in \mathcal{A} \times [-R, R]$ $\forall A \in \mathcal{A}$. Furthermore, $\mathcal{A} \times [-R, R]$ is a $\rho_{\mathcal{A}}$-closed subset of $\mathcal{A} \times \mathbf{R}$ and hence,

$$G(x) = cl(\{(A, x(A)) : A \in \mathcal{A}\}) \subseteq \mathcal{A} \times [-R, R].$$

Since $G(x) \in \mathcal{K}_{\mathcal{A} \times \mathbf{R}}$, this implies $G(x) \in \mathcal{E}_R$ where $\mathcal{E}_R$ is defined in Lemma 2.4.3. Therefore, since $x \in \Theta$ was arbitrarily chosen, $G[\Theta] \subseteq \mathcal{E}_R$.

Now, by Lemma 2.4.3, $\mathcal{E}_R$ is compact in $(\mathcal{K}_{\mathcal{A} \times \mathbf{R}}, d_G)$. Thus, $G[\Theta]$ is a closed, hence compact, subspace of $\mathcal{E}_R$. Therefore, applying Lemma A.1.6 twice,

$$\begin{aligned} G[\Theta] \text{ compact in } \mathcal{E}_R &\implies G[\Theta] \text{ compact in } \mathcal{K}_{\mathcal{A} \times \mathbf{R}} \\ &\implies G[\Theta] \text{ compact in } G[B(\mathcal{A})]. \end{aligned}$$

But, by Remark 2.4.2(a), $G : B(\mathcal{A}) \to G[B(\mathcal{A})]$ is a homeomorphism. Therefore, $\Theta = G^{-1}[G[\Theta]]$ is compact in $(B(\mathcal{A}), d_D)$. □

We now examine the relation between the topologies on $B(\mathcal{A})$ generated by d_D and the norm, $\|\cdot\|_{\mathcal{A}}$. The following result implies that the topology generated by $\|\cdot\|_{\mathcal{A}}$ is stronger than that generated by d_D.

Proposition 2.4.7 *Let $x, y \in B(\mathcal{A})$ and $\epsilon > 0$ be given. If $\|x-y\|_{\mathcal{A}} \leq \epsilon$, then $d_D(x,y) \leq \epsilon$.*

Proof Let $x, y \in B(\mathcal{A})$ be such that $\|x-y\|_{\mathcal{A}} < \epsilon$. Given $A \in \mathcal{A}$,

$$\rho_{\mathcal{A}}((A, x(A)), (A, y(A))) \;=\; 0 + |x(A) - y(A)| \;\leq\; \|x-y\|_{\mathcal{A}} \;\leq\; \epsilon$$

which implies that $(A, x(A)) \in [G(y)]^{\overline{\epsilon}}$. Thus, as $A \in \mathcal{A}$ was arbitrarily chosen and $[G(y)]^{\overline{\epsilon}}$ is closed in $\mathcal{A} \times \mathbf{R}$,

$$G(x) \;=\; cl(\{(A, x(A)) : A \in \mathcal{A}\}) \;\subseteq\; [G(y)]^{\overline{\epsilon}}.$$

Furthermore, given any $\delta > 0$, Proposition 2.2.1 (v) implies that

$$[G(y)]^{\overline{\epsilon}} \;\subseteq\; [G(y)]^{\epsilon+\delta}.$$

Reversing the roles of x and y in the above argument, we obtain that $G(y) \subseteq [G(x)]^{\epsilon+\delta}$ $\forall\, \delta > 0$. Therefore,

$$d_D(x,y) \;=\; d_G(G(x), G(y)) \;\leq\; \epsilon + \delta\,, \quad (\forall\, \delta > 0)$$

which implies that $d_D(x,y) \leq \epsilon$. □

Corollary 2.4.8 *Given $(x_n)_n$ in $B(\mathcal{A})$, if $\|x_n - x\|_{\mathcal{A}} \to 0$ for some $x \in B(\mathcal{A})$, then $d_D(x_n, x) \to 0$.*

The next result is a partial converse of Corollary 2.4.8. A simple proof for the case of $T = [0,1]^k$ is given in Theorem 3.5 of [5]. The argument therein clearly carries over to any compact metric space (T, d).

Proposition 2.4.9 *If $(x_n)_n$ in $D(\mathcal{A})$ is such that $d_D(x_n, x) \to 0$ for some $x \in C(\mathcal{A})$, then $\|x_n - x\|_{\mathcal{A}} \to 0$.*

Combining Corollary 2.4.8 and Proposition 2.4.9, we have

Corollary 2.4.10 *$(C(\mathcal{A}), d_D)$ is homeomorphic to $(C(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$.*

Since d_D is not generated by a norm on $B(\mathcal{A})$, it need not follow that

$$x_n \xrightarrow{d_D} x \text{ and } y_n \xrightarrow{d_D} y \;\Longrightarrow\; x_n + y_n \xrightarrow{d_D} x + y.$$

However, we have the following result whose proof, for the case of $T = [0,1]^k$, is given in Lemma 3.3 of [5]. Once again, the proof in [5] clearly carries over to any compact metric space.

Proposition 2.4.11 *Take $(x_n)_n, (y_n)_n, x$ and y in $D(\mathcal{A})$ such that y is continuous on $(\mathcal{A}, d_H)$. If $x_n \rightarrow_{d_D} x$ and $\|y_n - y\|_{\mathcal{A}} \rightarrow 0$, then $x_n + y_n \rightarrow_{d_D} x + y$.*

Finally, we have

Proposition 2.4.12 *Both $D(\mathcal{A})$ and $D_0(\mathcal{A})$ are closed, and hence complete, subspaces of $(B(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$.*

Proof In Proposition 2.3.4 it has been shown $D(\mathcal{A})$ is $\|\cdot\|_{\mathcal{A}}$-closed in $B(\mathcal{A})$.
Take $(x_n)_n$ in $D_0(\mathcal{A})$ such that $\|x_n - x\|_{\mathcal{A}} \rightarrow 0$ for some $x \in B(\mathcal{A})$. By Definition 2.3.16, for each $n \in \mathrm{N}$, $\exists$ sequences $(J_k(x_n))_k$ in PA and $(C_k(x_n))_k$ in $C(\mathcal{A})$ such that

$$\|x_n - (J_k(x_n) + C_k(x_n))\|_{\mathcal{A}} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By a diagonalization argument, we can select a sequence, $(k_n)_n$ in N, such that $k_n \rightarrow \infty$ fast enough that

$$\|x - (J_{k_n}(x_n) + C_{k_n}(x_n))\|_{\mathcal{A}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, $x \in D_0(\mathcal{A})$ which implies that $D_0(\mathcal{A})$ is closed in $(B(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$.
$(B(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$ is complete. Therefore, since completeness is closed hereditary, both $D(\mathcal{A})$ and $D_0(\mathcal{A})$ are $\|\cdot\|_{\mathcal{A}}$ -complete. □

In contrast, even for the simplest of examples, namely $\mathcal{A} = \mathcal{I}_1$, neither $G[D(\mathcal{I}_1)]$ nor $G[D_0(\mathcal{I}_1)]$ is closed in $(\mathcal{K}_{\mathcal{I}_1 \times \mathbf{R}}, d_G)$ (see p.864 in [5]), a consequence being that neither $(D(\mathcal{I}_1), d_D)$ nor $(D_0(\mathcal{I}_1), d_D)$ is complete. This is unfortunate since many useful results in the theory of weak convergence in metric spaces require the underlying metric space to be both complete and separable. (Fortunately, the classical weak convergence results that are needed in Section 4.3 hold for any metric space.)

This leads to an interesting question. Are there general conditions on $\mathcal{A}$ and (T, d) which guarantee the separability of $(D(\mathcal{A}), d_D)$ or, at least, the separability of $(D_0(\mathcal{A}), d_D)$? The answer for $D(\mathcal{A})$ is not clear. For $D_0(\mathcal{A})$, it is simpler and perhaps necessary to first ask for conditions under which PA is d_D-separable.

2.5 d_D-Compactness in the Space $D_0(\mathcal{A})$

In this section, we give a sufficient condition under which a subset of $D_0(\mathcal{A})$ is compact with respect to the metric d_D. For this purpose, we define the following collection of subspaces of $D_0(\mathcal{A})$.

Definition 2.5.1 *Given a triple,* $(\Gamma, \Sigma, \Delta) = ((\Gamma_n)_n, (\Sigma_n)_n, (\Delta_n)_n)$, *where*

$$\Gamma_n \subseteq PA \quad \text{and} \quad \Sigma_n \subseteq C(\mathcal{A}), \; (\forall n)$$

and $(\Delta_n)_n$ *is a sequence of constants s.t.* $\Delta_n \to 0$ *and* $\Delta_n \geq 0 \; \forall n$, *define* $\Xi(\Gamma, \Sigma, \Delta)$ *to be the collection of all* $x \in B(\mathcal{A})$ *such that, for each* $n \in \mathrm{N}$, *there exists* $J_n(x) \in \Gamma_n$ *and* $C_n(x) \in \Sigma_n$ *satisfying*

$$\|x - (J_n(x) + C_n(x))\|_{\mathcal{A}} \leq \Delta_n. \tag{2.16}$$

Remark 2.5.2 Clearly, $\Xi(\Gamma, \Sigma, \Delta) \subseteq D_0(\mathcal{A})$ for any (Γ, Σ, Δ).

The main result of this section is that, when the d-boundaries of the sets in $\mathcal{A}$ are well-behaved, $\Xi(\Gamma, \Sigma, \Delta)$ is compact in $(D(\mathcal{A}), d_D)$ whenever each Γ_n is compact in (PA, d_D) and each Σ_n is compact in $(C(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$. A similar result has already been shown in Theorem 3.4 of [5] for the case of $T = [0,1]^k$. However, even in this case our result is more general in that we allow Γ to be any sequence of compacta in (PA, d_D) whereas Theorem 3.4 in [5] only allows each Γ_n to be selected from a specific, pre-constructed class of compact sets in (PA, d_D).

We begin by defining a special class of indexing collections. (Recall that, given any $S \subseteq T$ and any $t \in T$, $d(t, S) = \inf\{d(t, s) : s \in S\}$.)

Definition 2.5.3 *An indexing collection,* $\mathcal{A}$ *on* (T, d) *is said to be* <u>*consistent with boundaries*</u> *(c.w.b.) if, given any* $A, B \in \mathcal{A} \setminus \{T\}$ *and any* $\epsilon > 0$,

(a) $\partial A \neq \phi$,

(b) $d_H(A, B) < \epsilon$ *implies* $d_H(\partial A, \partial B) < \epsilon$, *and*

(c) *given* $t \in T \setminus A$, $d(t, \partial A) \geq \epsilon$ *implies* $B_d(t, \epsilon) \cap A = \phi$.

Remark 2.5.4 (a) and (b) give "geometric" conditions on the sets in $\mathcal{A}$. From (b), when two sets, $A, B \in \mathcal{A}$, are close with respect to d_H, so are their boundaries. From (c), if t lies outside of A, then the points in A closest to t must lie in ∂A. All three conditions are satisfied when $T = [0,1]^k$ ($k \in \mathrm{N}$) and $\mathcal{A}$ is either $\mathcal{I}_k$ or $\mathcal{LL}_k$ (see Examples 2.2.6 and 2.8.1 respectively).

Not every indexing collection is c.w.b. In particular, Example 2.8.2 in Section 2.8 contains an indexing collection on $[0, 1]$ for which property (b) in the above definition does not hold. Also note that (a) and (c) fail when d is the discrete metric on $T = \{t_1, \cdots, t_k\}$ $(k \geq 2)$ and $\mathcal{A} = \mathcal{P}(T)$ since, for this T, $\partial A = \phi \;\forall A \subseteq T$ where, by convention, $d(t, \phi) = \infty$.

We now introduce some basic notation and terminology. Given a purely atomic set-function, $x : \mathcal{A} \to \mathbf{R}$, denote the set of all locations of atoms of x by $T[x]$. Also, given any $S \subseteq T$, define

$$S[x] = T[x] \cap S,$$

i.e., $S[x]$ is the set of all atoms of x lying in S.

Definition 2.5.5 *Given a purely atomic set-function, $x : \mathcal{A} \to \mathbf{R}$, a set $A \in \mathcal{A}$ is a continuity set of x provided $(\partial A)[x] = \phi$, i.e., no atoms of x lie on the boundary of A.*

Remark 2.5.6 Since $\partial T = \phi$, T is a continuity set for every purely-atomic set-function.

Before stating and proving the main theorem of this section, we present several technical results, the first of which does not require $\mathcal{A}$ to be c.w.b.

Lemma 2.5.7 *Given a purely atomic set-function, $x : \mathcal{A} \to \mathbf{R}$ and a set $A \in \mathcal{A}$, if x is d_H-continuous at A, then $\exists\, \delta > 0$ s.t.*

$$B \in \mathcal{A} \text{ and } d_H(A, B) < \delta \implies x(A) = x(B).$$

Proof Clearly, $\{x(A) : A \in \mathcal{A}\}$ is a finite subset of $\mathbf{R}$ hence,

$$\epsilon_0 = \inf\{|x(A) - x(B)| : A, B \in \mathcal{A} \text{ and } x(A) \neq x(B)\} > 0. \qquad (2.17)$$

As x is d_H-continuous at $A \in \mathcal{A}$, $\exists\, \delta > 0$ such that

$$B \in \mathcal{A} \text{ and } d_H(A, B) < \delta \implies |x(A) - x(B)| < \epsilon_0.$$

Moreover, by (2.17), $|x(A) - x(B)| < \epsilon_0$ implies $|x(A) - x(B)| = 0$, which establishes the lemma. □

Lemma 2.5.8 *If $\mathcal{A}$ is c.w.b., then given any $A \in \mathcal{A} \setminus \{T\}$ and $t \in T \setminus A$*

$$B \in \mathcal{A} \text{ and } d_H(A, B) < d(t, \partial A) \implies t \notin B.$$

Proof Let $\epsilon = d(t, \partial A)$. Since $\partial A \neq \phi$ is d-closed in T, $t \notin \partial A$ implies that $\epsilon > 0$. Take $B \in \mathcal{A}$ s.t. $d_H(A, B) < \epsilon$. If $t \in B$, then

$$d_H(A,B) < \epsilon \implies t \in B \subseteq A^{\epsilon} \implies \exists\, \tilde{a} \in A \text{ s.t. } d(\tilde{a}, t) < \epsilon.$$

Hence, $B_d(t, \epsilon) \cap A \neq \phi$ which contradicts (c) of Definition 2.5.3. □

Lemma 2.5.9 *Assume that $\mathcal{A}$ is c.w.b. Given $x \in PA$ and a continuity set $A \in \mathcal{A}$ of x, $\exists\, \delta > 0$ s.t.*

$$B \in \mathcal{A} \text{ and } d_H(A,B) < \delta \implies x(A) = x(B), \tag{2.18}$$

i.e., x is d_H-continuous at A.

Proof Given $x \in PA$, Lemma 2.3.15 implies that x is d_H-continuous at T. Thus, by Lemma 2.5.7, $\exists\, \delta_1 > 0$ s.t.

$$C \in \mathcal{A} \text{ and } d_H(C,T) < \delta_1 \implies x(C) = x(T). \tag{2.19}$$

Now, take a continuity set, A of x. If $A = T$, then, by (2.19), $\delta = \delta_1 > 0$ satisfies (2.18). If $A \in \mathcal{A} \setminus \{T\}$, then, since $\partial A \cap T(x) = \phi$ and $\partial A \neq \phi$ is d-closed in T,

$$\delta_2 = 1/2 \cdot \min_{t \in T(x)} d(t, \partial A) > 0. \tag{2.20}$$

Let $\delta = \delta_1 \wedge \delta_2 > 0$ and take $B \in \mathcal{A}$ such that $d_H(A, B) < \delta$. If $B = T$, then by (2.19), $x(A) = x(B)$. This takes care of the "easy" cases.

Now, assume $A, B \in \mathcal{A} \setminus \{T\}$. To show $x(A) = x(B)$, it is clearly sufficient to show $A[x] = B[x]$. First, for the inclusion $B[x] \subseteq A[x]$, take $t \in T[x]$ such that $t \notin A$. Since $d_H(A, B) < \delta \leq \delta_2 < d(t, \partial A)$, Lemma 2.5.8 implies $t \notin B$, i.e., $t \notin B[x]$.

To establish the opposite inclusion, take $t \in T[x]$ such that $t \notin B$. There are two cases. First, if $d(t, \partial B) \geq \delta_2$, then, by Lemma 2.5.8 (with the roles of A and B interchanged), $d_H(A, B) < \delta \leq \delta_2 < d(t, \partial B)$ implies $t \notin A$, i.e., $t \notin A[x]$.

On the other hand, if $d(t, \partial B) < \delta_2$, then $\exists\, \tilde{b} \in \partial B$ s.t. $d(t, \tilde{b}) < \delta_2$. Since $\mathcal{A}$ is c.w.b.,

$$d_H(A,B) < \delta \leq \delta_2 \implies d_H(\partial A, \partial B) < \delta_2$$

and thus, $\partial B \subseteq (\partial A)^{\delta_2}$. This implies $\exists\, \tilde{a} \in \partial A$ s.t. $d(\tilde{b}, \tilde{a}) < \delta_2$ which, by (2.20), implies

$$d(t, \partial A) = \inf_{a \in \partial A} d(t, a) \leq d(t, \tilde{a}) \leq d(t, \tilde{b}) + d(\tilde{b}, \tilde{a}) < 2 \cdot \delta_2 \leq d(t, \partial A),$$

resulting in a contradiction. Therefore, if $d_H(A,B) < \delta$, it is impossible for $t \in T[x]$ to satisfy $d(t, \partial B) < \delta_2$. This establishes the inclusion, $A[x] \subseteq B[x]$, completing the proof of the lemma. □

Remark 2.5.10 As illustrated by Example 2.8.3, Lemma 2.5.9 need not follow if $\mathcal{A}$ is not c.w.b.

Our final technical result relates d_D-convergence in $B(\mathcal{A})$ (to a limit in $P\mathcal{A}$) to a weak form of pointwise convergence on $\mathcal{A}$ — assuming $\mathcal{A}$ is c.w.b. A special case of this result will be used in the proof of Theorem 2.5.12.

Proposition 2.5.11 *Assume that $\mathcal{A}$ is c.w.b. and let $(x_n)_n$ be any sequence of set-functions on $\mathcal{A}$. If $x_n \to_{d_D} x$ for some $x \in P\mathcal{A}$, then*

$$x_n(A) \to x(A), \quad (\forall \textit{ continuity sets, } A \in \mathcal{A} \textit{ of } x).$$

Proof (The proof is by contrapositive.) Assume that that $\exists$ a continuity set, A of x, such that $x_n(A) \not\to x(A)$. Then, $\exists\ \alpha > 0$ and a subsequence, $(x_{k_n})_n$ such that

$$|x_{k_n}(A) - x(A)| \geq \alpha, \quad (\forall n). \tag{2.21}$$

Via Lemma 2.5.9 we can assume, without a loss of generality, that α is small enough so that $d_H(A,B) < \alpha$ implies $x(A) = x(B)$.

The following claim will establish that $x_n \not\to x$ with respect to d_D.

<u>Claim</u>: *For any $n \in \mathbf{N}$, $G(x_{k_n}) \not\subseteq [G(x)]^\alpha$.*

<u>Proof:</u> (The proof is by contradiction.) Assume $\exists\ n \in \mathbf{N}$ such that $G(x_{k_n}) \subseteq [G(x)]^\alpha$. Then, by Proposition 2.2.1(vi) and the definition of $G(x)$,

$$[G(x)]^\alpha = [\{(B, x(B)) : B \in \mathcal{A}\}]^\alpha$$

and hence, $\exists\ A \in \mathcal{A}$ such that $\rho_{\mathcal{A}}((A, x_{k_n}(A)), (B, x(B))) < \alpha$. Therefore,

$$d_H(A,B) < \alpha \text{ and } |x_{k_n}(A) - x(B)| < \alpha. \tag{2.22}$$

But, as mentioned above, α has been chosen small enough to insure that $d_H(A,B) < \alpha$ implies $x(A) = x(B)$. Therefore, by (2.21) and (2.22),

$$\alpha \leq |x_{k_n}(A) - x(A)| = |x_{k_n}(A) - x(B)| < \alpha,$$

which gives us a contradiction and thus, establishes the claim. □

We now present the main result of this section. Even though our hypotheses are more general, our proof remains close to that of Theorem 3.4 in [5]. We assume $\mathcal{A}$ is c.w.b. but, as mentioned in Remark 2.5.13, this assumption can be lifted in special cases.

Theorem 2.5.12 *Assume that $\mathcal{A}$ is c.w.b. Given a triple, (Γ, Σ, Δ) as in Definition 2.5.1, if*

(a) *Γ_n is compact in (PA, d_D) $\forall n$ and*

(b) *Σ_n is compact in $(C(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$ $\forall n$,*

then $\Xi\,(\Gamma, \Sigma, \Delta)$ is compact in $(D(\mathcal{A}), d_D)$.

Proof In the following proof, we will frequently replace subsequences by their original sequence. *This operation is solely intended to simplify notation and is performed only when generality is not compromised.*

Take a triple, (Γ, Σ, Δ) satisfying conditions (a) and (b) of Theorem 2.5.12 and let $\Xi = \Xi\,(\Gamma, \Sigma, \Delta)$ be as described in Definition 2.5.1. Given $x \in \Xi$, Definition 2.5.1 implies $\exists\ J_1(x) \in \Gamma_1$ and $\exists\ C_1(x) \in \Sigma_1$ s.t.

$$\begin{aligned} \|x\|_{\mathcal{A}} &\leq \|J_1(x)\|_{\mathcal{A}} + \|C_1(x)\|_{\mathcal{A}} + \|x - (J_1(x) + C_1(x))\|_{\mathcal{A}} \\ &\leq \sup_{j\in\Gamma_1}\|j\|_{\mathcal{A}} + \sup_{c\in\Sigma_1}\|c\|_{\mathcal{A}} + \Delta_1. \end{aligned} \tag{2.23}$$

Taking the supremum over all $x \in \Xi$ in (2.23), we obtain

$$\sup_{x\in\Xi}\|x\|_{\mathcal{A}} \;\leq\; \sup_{j\in\Gamma_1}\|j\|_{\mathcal{A}} + \sup_{c\in\Sigma_1}\|c\|_{\mathcal{A}} + \Delta_1. \tag{2.24}$$

Since Σ_1 is compact in $(C(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$, $\sup_{c\in\Sigma_1}\|c\|_{\mathcal{A}} < \infty$ and since Γ_1 is compact in (PA, d_D), Corollary 2.4.5 implies $\sup_{j\in\Gamma_1}\|j\|_{\mathcal{A}} < \infty$. Therefore, by Lemma 2.4.6 and (2.24), we have the following

Reduction: *It is sufficient to show $G[\Xi]$ is closed in $(\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$.*

With this reduction in mind, take $(x_n)_n$ in Ξ such that $G(x_n) \to_{d_G} G_0$ for some $G_0 \in \mathcal{K}_{\mathcal{A}\times\mathbf{R}}$. We need to find a function, $y \in \Xi$ such that $G(y) = G_0$ (see Steps 1, 2 and 3 below).

Given a fixed $n \in \mathbf{N}$, Definition 2.5.1 implies the existence of sequences, $(J_m(x_n))_m$ in PA and $\exists\ (C_m(x_n))_m$ in $C(\mathcal{A})$ s.t.

$$J_m(x_n) \in \Gamma_m \text{ and } C_m(x_n) \in \Sigma_m, \ (\forall m).$$

(Also, $\|x_n - (J_m(x_n) + C_m(x_n))\|_{\mathcal{A}} \leq \Delta_m \ \forall m$, but this is not needed here.) Now, fix $m \in \mathrm{N}$. Since $(J_m(x_n))_n$ lies in the d_D-compact subspace, Γ_m of PA, there exists a subsequence, $(J_m(x_{k_n}))_n$, such that

$$J_m(x_{k_n}) \xrightarrow{d_D} j_m \text{ as } n \to \infty \text{ for some } j_m \in \Gamma_m.$$

Furthermore, since $(C_m(x_{k_n}))_n$ lies in the $\|\cdot\|_{\mathcal{A}}$-compact subspace, Σ_m of $C(\mathcal{A})$, there exists a subsequence, $(C_m(x_{r_{k_n}}))_n$ such that

$$\|C_m(x_{r_{k_n}}) - c_m\|_{\mathcal{A}} \to 0 \text{ as } n \to \infty \text{ for some } c_m \in \Sigma_m.$$

Replacing r_{k_n} by n, we can simply write

$$J_m(x_n) \xrightarrow{d_D} j_m \text{ and } \|C_m(x_n) - c_m\|_{\mathcal{A}} \to 0 \text{ as } n \to \infty. \tag{2.25}$$

Since the above $m \in \mathrm{N}$ was arbitrarily chosen we have thus constructed sequences, $(j_m)_m$ and $(c_m)_m$ such that

$$j_m \in \Gamma_m \text{ and } c_m \in \Sigma_m \quad (\forall m) \tag{2.26}$$

where j_m and c_m satisfy (2.25) for each $m \in \mathrm{N}$. We divide the remainder of the proof into three key steps.

Step 1: $\exists\, y \in B(\mathcal{A})$ s.t. $\|y - (j_m + c_m)\|_{\mathcal{A}} \to 0$ as $m \to \infty$.

Since $(B(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$ is complete, Step 1 is equivalent to showing $(j_m + c_m)_m$ is $\|\cdot\|_{\mathcal{A}}$-Cauchy.

Assume, to the contrary, that $(j_m + c_m)_m$ is not $\|\cdot\|_{\mathcal{A}}$-Cauchy. Then, $\exists\, \epsilon_0 > 0$ such that, for every $M \in \mathrm{N}$, $\exists\, m, k \geq M$ and $\exists\, A \in \mathcal{A}$ s.t.

$$|(j_m(A) + c_m(A)) - (j_k(A) + c_k(A))| \geq 5 \cdot \epsilon_0. \tag{2.27}$$

Also, by (2.16) of Definition 2.5.1, $\exists\, M_0 \in \mathrm{N}$ s.t.

$$p \geq M_0 \implies \|x - (J_p(x) + C_p(x))\|_{\mathcal{A}} < \epsilon_0, \ (\forall x \in \Xi). \tag{2.28}$$

Selecting $k, m \geq M_0$ and $A \in \mathcal{A}$ so as to satisfy (2.27) for $M = M_0$, the d_H-continuity of c_m and c_k at $A \in \mathcal{A}$ implies $\exists\, \delta > 0$ s.t.

$$d_H(A, B) \leq \delta \implies |c_m(A) - c_m(B)|,\ |c_k(A) - c_k(B)| \leq \epsilon_0. \tag{2.29}$$

Furthermore, applying Corollary 2.7.3 to j_m, j_k and $\delta > 0$, $\exists\, B \in \mathcal{A}$ possessing the following three properties:

(P1) $d_H(A, B) \leq \delta$,
(P2) B is a continuity set of j_m and j_k and
(P3) $j_m(A) = j_m(B)$ and $j_k(A) = j_k(B)$.

Applying (P1) to (2.29), (2.27) becomes

$$\begin{aligned} 5 \cdot \epsilon_0 &\leq |(j_m(A) + c_m(A)) - (j_k(A) + c_k(A))| \\ &\leq |(j_m(A) + c_m(B)) - (j_k(A) + c_k(B))| + \\ &\quad |c_m(A) - c_m(B)| + |c_k(B) - c_k(A)| \\ &\leq |(j_m(A) + c_m(B)) - (j_k(A) + c_k(B))| + 2 \cdot \epsilon_0 \end{aligned}$$

and hence,

$$3 \cdot \epsilon_0 \leq |(j_m(A) + c_m(B)) - (j_k(A) + c_k(B))|. \tag{2.30}$$

Moreover, since $J_r(x_n) \rightarrow_{d_D} j_r$ $(r = m, k)$, Proposition 2.5.11 implies

$$J_r(x_n)(C) \xrightarrow{n} j_r(C), \quad (r = m, k) \tag{2.31}$$

where $C \in \mathcal{A}$ is any continuity set of both j_m and j_k. Thus, by (P2), $\exists\, K_1 \in \mathbf{N}$ s.t.

$$n \geq K_1 \implies |J_r(x_n)(B) - j_r(B)| \leq \epsilon_0/4, \quad (r = m, k). \tag{2.32}$$

Also, since $\|C_r(x_n) - c_r\|_{\mathcal{A}} \to 0$ as $n \to \infty$ $(r = m, k)$, $\exists\, K_2 \in \mathbf{N}$ s.t.

$$n \geq K_2 \implies |C_r(x_n)(B) - c_r(B)| \leq \epsilon_0/4, \quad (r = m, k). \tag{2.33}$$

Now, take $n = K_1 \vee K_2$. Combining (2.32), (2.33) and (P3), (2.30) becomes

$$\begin{aligned} 3 \cdot \epsilon_0 &\leq |(j_m(B) + c_m(B)) - (j_k(B) + c_k(B))| \\ &\leq |(J_m(x_n)(B) + C_m(x_n)(B)) - (J_k(x_n)(B) + C_k(x_n)(B))| + \\ &\quad |j_m(B) - J_m(x_n)(B)| + |J_k(x_n)(B) - j_k(B)| + \\ &\quad |c_m(B) - C_m(x_n)(B)| + |C_k(x_n)(B) - c_k(B)| \\ &\leq |(J_m(x_n)(B) + C_m(x_n)(B)) - (J_k(x_n)(B) + C_k(x_n)(B))| + 4 \cdot \epsilon_0/4. \end{aligned}$$

But, since $m, k \geq M_0$, (2.28) implies

$$\begin{aligned} 2\cdot\epsilon_0 &\leq |(J_m(x_n)(B)+C_m(x_n)(B)) - (J_k(x_n)(B)+C_k(x_n)(B))| \\ &\leq |x_n(B) - (J_m(x_n)(B)+C_m(x_n)(B))| + \\ &\quad |x_n(B) - (J_k(x_n)(B)+C_k(x_n)(B))| \\ &< \epsilon_0 + \epsilon_0 \end{aligned}$$

which results in a contradiction. Therefore, it must be that $(j_m + c_m)_m$ is Cauchy in $(B(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$. This completes Step 1.

Step 2: The function y from Step 1 lies in Ξ.

If we can show $\|y - (j_m + c_m)\|_{\mathcal{A}} \leq \Delta_m \;\; \forall m$, then Step 2 will follow by (2.26). For this purpose, fix $m \in \mathrm{N}$ and note that for any $k \in \mathrm{N}$,

$$\|y - (j_m + c_m)\|_{\mathcal{A}} \;\leq\; \|y - (j_k + c_k)\|_{\mathcal{A}} + \|(j_k + c_k) - (j_m + c_m)\|_{\mathcal{A}}.$$

Since $\overline{\lim}_k \|y - (j_k + c_k)\|_{\mathcal{A}} = 0$, the above inequality implies

$$\|y - (j_m + c_m)\|_{\mathcal{A}} \;\leq\; \overline{\lim}_k \|(j_k + c_k) - (j_m + c_m)\|_{\mathcal{A}}. \tag{2.34}$$

To show $\|y - (j_m + c_m)\|_{\mathcal{A}} \leq \Delta_m \;\; \forall m$, we need the following inequality.

Claim: *Given any* $m, k \in \mathrm{N}$,
$$\|(j_m+c_m) - (j_k+c_k)\|_{\mathcal{A}} \;\leq\; \overline{\lim}_n \|(J_m(x_n)+C_m(x_n)) - (J_k(x_n)+C_k(x_n))\|_{\mathcal{A}}.$$

Proof: Take $\epsilon > 0$ and $A \in \mathcal{A}$. Since c_k and c_m are d_H-continuous at $A \in \mathcal{A}$, $\exists\, \delta > 0$ s.t.

$$A' \in \mathcal{A} \text{ and } d_H(A, A') \leq \delta \implies |c_r(A) - c_r(A')| \leq \epsilon/2, \;\; (r = m, k). \tag{2.35}$$

By Corollary 2.7.3, $\exists\, B \in \mathcal{A}$ satisfying properties (P1), (P2) and (P3) on p.40. In particular, $d_H(A, B) \leq \delta$ and $j_r(A) = j_r(B)$ $(r = m, k)$ which, by (2.35), implies

$$\begin{aligned} &|(j_k(A)+c_k(A)) - (j_m(A)+c_m(A))| \\ &\quad= |(j_k(B)+c_k(A)) - (j_m(B)+c_m(A))| \\ &\quad\leq |(j_k(B)+c_k(B)) - (j_m(B)+c_m(B))| + \\ &\quad\quad |c_k(A)-c_k(B)| + |c_m(B)-c_m(A)| \\ &\quad\leq |(j_k(B)+c_k(B)) - (j_m(B)+c_m(B))| + 2\cdot\epsilon/2. \end{aligned} \tag{2.36}$$

Now, since $J_r(x_n) \to_{d_D} j_r$ $(r = m, k)$, (P2) and Propostion 2.5.11 imply

$$J_r(x_n)(B) \to j_r(B) \text{ as } n \to \infty, \quad (r = m, k).$$

Furthermore, since $\|C_r(x_n) - c_r\|_{\mathcal{A}} \to 0$ as $n \to \infty$ $(r = m, k)$, we have

$$C_r(x_n)(B) \to c_r(B) \text{ as } n \to \infty, \quad (r = m, k)$$

and therefore,

$$\begin{aligned} &|(j_k(B)+c_k(B)) - (j_m(B)+c_m(B))| \\ &\quad = \overline{\lim}_n |(J_k(x_n)(B)+C_k(x_n)(B)) - (J_m(x_n)(B)+C_m(x_n)(B))| \\ &\quad \le \overline{\lim}_n \|(J_k(x_n) + C_k(x_n)) - (J_m(x_n) + C_m(x_n))\|_{\mathcal{A}}. \end{aligned}$$

Substituting the above inequality into (2.36), noting that $\epsilon > 0$ and $A \in \mathcal{A}$ were both arbitrarily chosen, we obtain the claim. Ω

We now return to the verification of Step 2. Given $k, n \in \mathrm{N}$, (2.16) and the triangle inequality imply

$$\|(J_k(x_n) + C_k(x_n)) - (J_m(x_n) + C_m(x_n))\|_{\mathcal{A}} \le \Delta_k + \Delta_m.$$

Therefore, applying (2.34) and the above Claim (in that order), we obtain

$$\|y - (j_m + c_m)\|_{\mathcal{A}} \le \overline{\lim}_k \left[\overline{\lim}_n(\Delta_k + \Delta_m)\right] = (\lim_k \Delta_k) + \Delta_m = 0 + \Delta_m$$

which establishes Step 2.

Step 3: $G(y) = G_0$.

Step 3 will follow if we can show $d_G(G(y), G_0) < \epsilon \ \forall \epsilon > 0$. For this purpose, take $\epsilon > 0$. First, by Proposition 2.4.7 and (2.16),

$$d_G(G(x), G(J_m(x) + C_m(x))) \le \Delta_m \quad (\forall x \in \Xi \text{ and } \forall m \in \mathrm{N}). \tag{2.37}$$

Since $\Delta_m \to 0$ with $\Delta_n \ge 0 \ \forall n$, $\exists m_0 \in \mathrm{N}$ s.t. $\Delta_{m_0} < \epsilon/4$. Moreover, since $G(x_n) \to_{d_G} G_0$, $\exists K_1 \in \mathrm{N}$ s.t.

$$d_G(G(x_n), G_0) < \epsilon/4, \quad (\forall n \ge K_1) \tag{2.38}$$

and, since $J_{m_0}(x_n) \to_{d_D} j_{m_0}$ and $\|C_{m_0}(x_n) - c_{m_0}\|_{\mathcal{A}} \to 0$, Proposition 2.4.11 implies $\exists\, K_2 \in \mathbf{N}$ s.t.

$$d_G(\, G(j_{m_0}+c_{m_0})\,,\, G(J_{m_0}(x_n)+C_{m_0}(x_n))\,) < \epsilon/4, \;\; (\forall n \geq K_2). \tag{2.39}$$

If we take $n_0 = K_1 \vee K_2$, then (2.37), (2.38) and (2.39) imply

$$\begin{aligned} d_G(G(y), G_0) \;\; &\leq \;\; d_G(\, G(y)\,,\, G(j_{m_0} + c_{m_0})\,) \; + \\ & \quad\; d_G(\, G(j_{m_0}+c_{m_0})\,,\, G(J_{m_0}(x_{n_0})+C_{m_0}(x_{n_0}))\,) \; + \\ & \quad\; d_G(\, G(J_{m_0}(x_{n_0})+C_{m_0}(x_{n_0}))\,,\, G(x_{n_0})\,) \; + \; d_G(G(x_{n_0}), G_0) \\ &\leq \;\; \Delta_{m_0} + \epsilon/4 + \Delta_{m_0} + \epsilon/4. \end{aligned} \tag{2.40}$$

Since m_0 was selected so that $\Delta_{m_0} < \epsilon/4$, (2.40) implies $d_G(G(y), G_0) < \epsilon$.
Therefore, since $\epsilon > 0$ was arbitrarily chosen, $d_G(G(y), G_0) = 0$ which completes the proof of Step 3 and thus also of Theorem 2.5.12. □

The above proof used the c.w.b. property exactly twice: once in the paragraph containing equation (2.31) and once again in the paragraph following equation (2.36). In both cases, it was only the consequence, Proposition 2.5.11 of the c.w.b. property which was needed. We record this observation for future reference.

Remark 2.5.13 Given an indexing collection $\mathcal{A}$, not necessarily c.w.b., the proof presented for Theorem 2.5.12 still holds provided each Γ_m possesses the following property: *if $x_n \to_{d_D} x$ in Γ_m, then $\exists$ a subsequence, $(k_n)_n$ s.t.*

$$x_{k_n}(A) \to x(A)\,, \;\; (\forall \text{ continuity sets, } A \in \mathcal{A} \text{ of } x). \tag{2.41}$$

A class of compact sets in (PA, d_D) possessing property (2.41) will be given in Theorem 2.6.5.

2.6 A Class of d_D-Compact Subsets of PA

In this section we give a sufficient condition, similar to that found in Proposition 3.2 of [5], under which a subset of PA is d_D-compact. The resulting d_D-compact subsets of PA are not merely of independent interest for they will also be used in conjunction with Theorem 2.5.12 to generate *explicit* (in the sense of Remark 2.6.8) d_D-compact subsets of $D_0(\mathcal{A})$.

First, we need to introduce some new terminology. Given a purely atomic function $x : \mathcal{A} \to \mathbf{R}$ — not necessarily in PA — define

$$\text{at}(x) = \text{number of atoms of } x$$

and, given any $S \subseteq T$, define

$$\begin{aligned}\text{at}(x, S) &= \text{number of atoms of } x \text{ that lie in } S \\ &= \text{the cardinality of } S[x].\end{aligned}$$

For the latter quantity, we will sometimes write $\text{at}(S)$ when there is no danger of ambiguity. We also define the non-negative number,

$$\text{gap}(x) = \min_{i \neq j} d(t_i, t_j)$$

where the t_i denote the distinct atoms of x, When $\text{at}(x) = 1$, we adopt the convention, $\text{gap}(x) = \text{diam}(T)$.

Remark 2.6.1 Given x purely atomic and $S_1, S_2 \subseteq T$, it is clear that $\text{at}(S_1) \leq \text{at}(S_2)$ whenever $S_1 \subseteq S_2$, and that $\text{at}(S_1 \cup S_2) = \text{at}(S_1) + \text{at}(S_2)$ whenever S_1 and S_2 are disjoint.

The above mentioned sufficient condition is embedded in the following definition (see Theorem 2.6.5).

Definition 2.6.2 *Given*

- *positive constants,* η *and* R*,*
- $N : (0, \infty) \to \mathbf{N}$*,*
- $h : (0, \infty) \to (0, \infty)$ *s.t.* $h(\delta) \leq \delta \ \forall \delta > 0$ *and*
- *a compact subspace,* T' *of* (T, d) *s.t.* $T' \cap E(\mathcal{A}) = \phi$*,*

define $\Gamma(T', h, N, \eta, R)$ *to be the set of all* $x \in PA$ *for which*

(i) $\text{gap}(x) \geq \eta$*,*

(ii) $\nu(x) \leq R$ *(see Definition 2.3.5 for* ν*),*

(iii) *for any* $\delta > 0$*,* $\exists A_1, \cdots, A_{N(\delta)} \in \mathcal{A}$ *s.t.:*
  (a) $\forall (B, r) \in G(x)$*,* $\exists 1 \leq i \leq N(\delta)$ *s.t.* $\rho_{\mathcal{A}}((B, r), (A_i, x(A_i))) \leq \delta$*,*
  (b) $A_i^{h(\delta)} \setminus A_i$ *contains no atoms of* x $\forall 1 \leq i \leq N(\delta)$

and

(iv) *all atoms of x lie in T'.*

Remark 2.6.3 Given any $\delta > 0$, the sets $A_1, \cdots, A_{N(\delta)}$ in (iii) may depend on $x \in \Gamma(T', h, N, \eta, R)$ but the numbers $h(\delta)$ and $N(\delta)$ do not.

As our next result illustrates, the collection of subspaces generated by Definition 2.6.2 is rich enough to account for all well-behaved purely atomic set-functions, namely those in PA.

Lemma 2.6.4 $\bigcup \Gamma(T', h, N, \eta, R) = PA$ *where the union is over all lists* (T', h, N, η, R) *as given in Definition 2.6.2.*

Proof Take $x \in PA$ with atoms $t_1, \cdots, t_n \in T \setminus E(\mathcal{A})$ and let

$$\eta = \text{gap}(x)\,,\; R = \nu(x) + 1 \quad \text{and} \quad T' = \{t_1, \cdots, t_n\}.$$

It is clear that x satisfies (i), (ii) and (iv) of Definition 2.6.2 for these parameters. We still need appropriate functions, h and N so that x will also satisfy (iii) of Definition 2.6.2.

We construct $N : (0, \infty) \to \mathbf{N}$ and $h : (0, \infty) \to (0, \infty)$ as follows. Take $\delta > 0$. If $r = \|x\|_{\mathcal{A}}$, then by Theorem 2.2.13, $\mathcal{A} \times [-r, r]$ is a $\rho_{\mathcal{A}}$-compact subspace of $\mathcal{A} \times \mathbf{R}$. Thus, since $G(x)$ is a $\rho_{\mathcal{A}}$-closed subspace of $\mathcal{A} \times [-r, r]$, it is compact, implying $\exists$ a finite subset, $\{(C_i, r_i) : 1 \le i \le k\}$ of $G(x)$ s.t.

$$G(x) \subseteq \cup_{i=1}^{k} B_{\rho_{\mathcal{A}}}((C_i, r_i), \delta/2). \tag{2.42}$$

Furthermore, since $G(x) = cl(\{(A, x(A)) : A \in \mathcal{A}\})$,

$$\exists\, A_1, \cdots, A_k \in \mathcal{A} \text{ s.t. } \rho_{\mathcal{A}}((C_i, r_i), (A_i, x(A_i))) < \delta/2 \;\; \forall\, 1 \le i \le k.$$

Therefore, given any $(B, r) \in G(x)$, (2.42) implies $\exists\, 1 \le i \le k$ s.t.

$$\begin{aligned} \rho_{\mathcal{A}}((B, r), (A_i, x(A_i))) &\le \rho_{\mathcal{A}}((B, r), (C_i, r_i)) + \rho_{\mathcal{A}}((C_i, r_i), (A_i, x(A_i))) \\ &< 2(\delta/2), \end{aligned}$$

i.e., each $(B, r) \in G(x)$ lies within $\rho_{\mathcal{A}}$-distance δ of some $(A_i, x(A_i))$.

So far, we have shown that $A_1, \cdots, A_k \in \mathcal{A}$ satisfy condition (iii)(a) of Definition 2.6.2 for our δ and for this reason, we define $N(\delta) = k$. Now, for each $1 \le i \le N(\delta)$, define

$$\gamma_i = \min\{d(A_i, t_j) : t_j \notin A_i\} \quad \text{and} \quad \gamma = \min_{1 \le i \le N(\delta)} \gamma_i.$$

Since each A_i is d-closed, it is clear that $\gamma > 0$. It is also clear that each $A_i^\gamma \setminus A_i$ contains no atoms of x. Thus, if we define $\tau = \gamma \wedge \delta > 0$, then

$$0 < \tau \leq \delta \text{ and } A_i^\tau \setminus A_i \text{ contains no atoms of } x, \quad (\forall 1 \leq i \leq N(\delta)).$$

Therefore, the sets $A_1, \cdots, A_{N(\delta)}$ and the number $\tau > 0$ satisfy condition (iii)(b) of Definition 2.6.2, suggesting the assignment $h(\delta) = \tau$.

Repeating the above procedure for every $\delta > 0$, we obtain functions N and h satisfying condition (iii) of Definition 2.6.2 for x, ensuring that $x \in \Gamma(T', h, N, \eta, R)$. □

And now for the main result of this section. The proof, which is long and requires additional technical results, has been placed in Section 2.7 so as not to interrupt the flow of the present discussion.

Theorem 2.6.5 *Each* $\Gamma(T', h, N, \eta, R) \subseteq PA$ *is compact in* $(D(\mathcal{A}), d_D)$.

Remark 2.6.6 Unfortunately, the conditions in Definition 2.6.2 do not fully characterize the d_D-compact subsets of PA. See Example 2.8.6 for a d_D-compact set, $\Gamma \subseteq PA$ for which $\inf_{x \in \Gamma} \text{gap}(x) = 0$, implying that there is no $\eta > 0$ satisfying condition (i) of Definition 2.6.2 for this Γ.

Our next task is to use the compacta generated by Theorem 2.6.5 to obtain explicit d_D-compact subsets in $D_0(\mathcal{A})$. But first, we recall a classic characterization of compactness in $(C(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$.

Given $x \in B(\mathcal{A})$, the *modulus of continuity* of x is the function $w(x, \cdot) : (0, \infty) \to [0, \infty)$ defined by

$$w(x, \delta) \;=\; \sup\{|x(A) - x(B)| : A, B \in \mathcal{A} \text{ and } d_H(A, B) \leq \delta\}. \tag{2.43}$$

Given a constant $M > 0$ and an increasing function $W : (0, \infty) \to [0, \infty)$ s.t. $W(\delta) \to 0$ as $\delta \to 0$, define $\Sigma(W, M)$ to be the set of all $x \in C(\mathcal{A})$ s.t.

(a) $\|x\|_{\mathcal{A}} \leq M$ and

(b) $w_x(\delta) \leq W(\delta) \;\; \forall \delta > 0$.

It is easy to show that $\Sigma(W, M)$ is $\|\cdot\|_{\mathcal{A}}$-closed in $C(\mathcal{A})$. Moreover, by the Arzela-Ascoli characterization of compactness

$$\Sigma(W, M) \text{ is compact in } (C(\mathcal{A}), \|\cdot\|_{\mathcal{A}}). \tag{2.44}$$

The following result is an analogue of Theorem 3.4 in [5].

Theorem 2.6.7 *For each $n \in \mathbf{N}$, take:*

- *a subspace $\Gamma(T'_n, h_n, N_n, \eta_n, R_n)$ (see Definition 2.6.2),*
- *a constant $M_n > 0$ and*
- *a function $W_n : (0, \infty) \to \mathbf{R}$ increasing, with $\lim_{\delta \downarrow 0} W_n(\delta) = 0$.*

Given a sequence $(\Delta_n)_n$ of constants s.t. $\Delta_n \to 0$ and $\Delta_n \geq 0$ $\forall n$, define Ξ_0 to consist of all $x \in D_0(\mathcal{A})$ such that, for each $n \in \mathbf{N}$, $\exists J_n(x) \in PA$ and $\exists C_n(x) \in C(\mathcal{A})$ s.t.:

(i) $J_n(x) \in \Gamma(T'_n, h_n, N_n, \eta_n, R_n)$,

(ii) (a) $\|C_n(x)\|_{\mathcal{A}} \leq M_n$,
(b) $w(C_n(x), \delta) \leq W_n(\delta)$ $\forall \delta > 0$ *and*

(iii) $\|x - (J_n(x) + C_n(x))\|_{\mathcal{A}} \leq \Delta_n$.

Then, Ξ_0 is compact in $(D(\mathcal{A}), d_D)$.

Proof Take $(\mathbf{\Gamma}, \mathbf{\Sigma}, \mathbf{\Delta}) = ((\Gamma_n)_n, (\Sigma_n)_n, (\Delta_n)_n)$ where

$$\Gamma_n = \Gamma(T'_n, h_n, N_n, \eta_n, R_n) \quad \text{and} \quad \Sigma_n = \Sigma(W_n, M_n), \quad (\forall n).$$

By Theorem 2.6.5 and (2.44), the triple $(\mathbf{\Gamma}, \mathbf{\Sigma}, \mathbf{\Delta})$ satisfies the conditions of Theorem 2.5.12, although $\mathcal{A}$ is not assumed to be c.w.b. However, by (2.49) in the upcoming proof of Theorem 2.6.5, each $\Gamma(T'_n, h_n, N_n, \eta_n, R_n)$ satisfies property (2.41) and therefore, by Remark 2.5.13, Theorem 2.5.12 implies $\Xi(\mathbf{\Gamma}, \mathbf{\Sigma}, \mathbf{\Delta})$ is compact in $(D(\mathcal{A}), d_D)$. The present theorem thus follows from the identity, $\Xi_0 = \Xi(\mathbf{\Gamma}, \mathbf{\Sigma}, \mathbf{\Delta})$. □

Remark 2.6.8 Theorem 2.6.7 enables us to construct d_D-compact subsets in $D_0(\mathcal{A})$ from simple objects such as real-functions and sequences of positive numbers. In this sense, Theorem 2.6.7 generates *explicit compacts.*

If the compacta generated via Theorem 2.6.7 are to be of any use, it is essential that they contain a variety of "natural" set-functions. Clearly, each $x \in C(\mathcal{A})$ lies in some Ξ_0 and, by Lemma 2.6.4, each $x \in PA$ lies in some Ξ_0. In our next result, we show that any $x : \mathcal{A} \to \mathbf{R}$ with countably many atoms in $T \setminus E(\mathcal{A})$ also lies in some Ξ_0 provided the corresponding masses of x satisfy a certain condition.

Lemma 2.6.9 *Given sequences, $(t_n)_n$ in $T \setminus E(\mathcal{A})$ and $(a_n)_n$ in $\mathbf{R}$, define*

$$\alpha_n = |\{k \in \mathbf{N} : 1/(n+1) < |a_k| \le 1/n\}|, \quad (\forall n).$$

If $\{k : |a_k| > 1\}$ is a finite set and $\sum_{n=1}^{\infty} \frac{\alpha_n}{n} < \infty$, then the set-function $x : \mathcal{A} \to \mathbf{R}$ defined by

$$x(A) = \sum_{n:t_n \in A} a_n, \quad (\forall A \in \mathcal{A})$$

lies in some subspace, Ξ_0 as defined in Theorem 2.6.7.

Proof Given any $n \in \mathbf{N}$, define $J_n(x) : \mathcal{A} \to \mathbf{R}$ to be purely atomic with atoms, $\{t_k : |a_k| > 1/n\} \subseteq T \setminus E(\mathcal{A})$ and masses, $\{a_k : |a_k| > 1/n\} \subseteq \mathbf{R}$. That is, if $k \in \mathbf{N}$ satisfies $|a_k| > 1/n$, then t_k is an atom of $J_n(x)$ with mass a_k. Of course, by the above conditions, $\{k : |a_k| > 1/n\}$ is a finite set $\forall n$.

Now, working with a fixed n, define $P_k = \{j : 1/(1+k) < |a_j| \le 1/k\}$ $\forall k \ge n$. Note that $|P_k| = \alpha_k$ $\forall k$. Since

$$\{j : |a_j| \le 1/n\} = \cup_{k \ge n} P_k$$

is a disjoint union, we have the inequality,

$$\begin{aligned} \|x - J_n(x)\|_{\mathcal{A}} &\le \textstyle\sum_{j:|a_j| \le 1/n} |a_j| \\ &= \textstyle\sum_{k \ge n} \left(\sum_{j \in P_k} |a_j| \right) \\ &\le \textstyle\sum_{k \ge n} \alpha_k / k. \end{aligned} \tag{2.45}$$

Furthermore, by Lemma 2.6.4, we can find appropriate parameters so that

$$J_n(x) \in \Gamma(T_n', h_n, N_n, \eta_n, R_n), \quad (\forall n).$$

Thus, if we take

$$W_n(\delta) = \delta, \;\; M_n = 1 \;\text{ and }\; \Delta_n = \textstyle\sum_{k \ge n} \alpha_k / k, \quad (\forall n),$$

then by (2.45), choosing $C_n(x) = 0$ $\forall n$ implies

$$\|x - (J_n(x) + C_n(x))\|_{\mathcal{A}} \le \Delta_n, \quad (\forall n)$$

where, by assumption, $\Delta_n \to 0$ as $n \to \infty$. Therefore, $x \in \Xi_0$ where Ξ_0 is the subspace of $D_0(\mathcal{A})$ corresponding to the above parameters. □

Remark 2.6.10 In order for the condition $\sum_{n=1}^{\infty} \frac{\alpha_n}{n} < \infty$ to hold, it is necessary that $\alpha_n = 0$ for infinitely many n.

We close this section with a comparison between our results, in particular Theorems 2.6.5 and 2.6.7, and their counterparts in [5] for the case of $T = [0,1]^k$ ($k \in \mathrm{N}$) and $\mathcal{A}$ any indexing collection on $[0,1]^k$. By Remark 2.2.11, any indexing collection on $[0,1]^k$ satisfies assumption (A1) in [5], making such a comparison is possible.

Given numbers $\eta, R > 0$ and functions h and N as described in Definition 2.6.2, Bass and Pyke define $\mathcal{F}_{PA}(h, N, \eta, R)$ to be the collection of all purely atomic set-functions — not necessarily in PA — which satisfy properties (i), (ii) and (iii) of Definition 2.6.2. Since this permits the elements of $\mathcal{F}_{PA}(h, N, \eta, R)$ to have atoms with locations in $\omega(\mathcal{A})$, the inclusion, $\mathcal{F}_{PA}(h, N, \eta, R) \subseteq D(\mathcal{A})$ does not necessarily hold (see Example 2.3.6).

By Proposition 3.2 in [5], each $\mathcal{F}_{PA}(h, N, \eta, R)$ is compact in $(B(\mathcal{A}), d_D)$. By contrast, even when relativized to PA, $\mathcal{F}_{PA}(h, N, \eta, R) \cap PA$ is not necessarily compact in $(D(\mathcal{A}), d_D)$ as illustrated by Example 2.8.4. However, since

$$\mathcal{F}_{PA}(h, N, \eta, R) \cap PA \;=\; \bigcup_n \Gamma\left(\,[1/n\,,\, 1-1/n]^k, h, N, \eta, R\right),$$

each $\mathcal{F}_{PA}(h, N, \eta, R)$ is σ-compact in $(D(\mathcal{A}), d_D)$.

Given a sequence of subspaces, $(\mathcal{F}_{PA}(h_n, N_n, \eta_n, R_n))_n$ as defined above, Bass and Pyke (in Definition 3.6 of [5]) define $\mathcal{F} \subseteq B(\mathcal{A})$ to be the collection of all $x \in B(\mathcal{A})$ which satisfy the conditions of Theorem 2.6.7, only now with $\mathcal{F}_{PA}(h_n, N_n, \eta_n, R_n)$ replacing the subspaces $\Gamma\,(T_n', h_n, N_n, \eta_n, R_n)$. Using an argument identical to that of Lemma 2.6.4, it can be shown that each purely atomic set-function lies in some $\mathcal{F}_{PA}(h, N, \eta, R)$ and therefore, by Example 2.3.6, $\exists\, \mathcal{F}_0 = \mathcal{F}_{PA}(h, N, \eta, R)$ s.t. $\mathcal{F}_0 \not\subseteq D(\mathcal{A})$. Thus, if we take

$$\mathcal{F}_{PA}(h_n, N_n, \eta_n, R_n) = \mathcal{F}_0, \;\; W_n(\delta) = \delta, \;\; M_n = 1 \;\text{ and }\; \Delta_n = 1/n\,, \;\; (\forall\, n)$$

the subspace $\mathcal{F}$ generated by these parameters contains $\mathcal{F}_0$ and as a consequence, $\mathcal{F} \not\subseteq D(\mathcal{A})$.

In further contrast, while Theorem 3.4 in [5] implies that each such $\mathcal{F}$ is compact in $(B(\mathcal{A}), d_D)$, Example 2.8.5 illustrates that the subspaces $\mathcal{F}$ are not necessarily compact in $(D(\mathcal{A}), d_D)$, even when relativized to $D(\mathcal{A})$.

2.7 The Proof of Theorem 2.6.5

Before proceeding with the proof of Theorem 2.6.5, we need three technical results, the first of which illustrates the restriction that $\text{gap}(x)$ places on $\text{at}(x)$. See Section 2.6 for the definitions of $\text{gap}(x)$ and $\text{at}(x)$.

Lemma 2.7.1 *Given $\eta > 0$, $\exists\, M \in \mathbf{N}$ s.t.*

$$x \text{ purely atomic with } \text{gap}(x) \geq \eta \implies \text{at}(x) \leq M.$$

Proof Let $\eta > 0$ be given and take x to be purely atomic with atoms $t_1, \cdots, t_n \in T$ s.t. $\text{gap}(x) \geq \eta > 0$. Since (T, d) is compact, $\exists\, s_1, \cdots, s_M \in T$ s.t. $T = \bigcup_{i=1}^{M} B_d(s_i, \eta/2)$. If $n > M$, then, by the Pigeonhole Principle, $\exists\, 1 \leq k_0 \leq M$ and $1 \leq i_0 < j_0 \leq n$ s.t. $t_{i_0}, t_{j_0} \in B_d(x_{k_0}, \eta/2)$. But this implies

$$\eta \;\leq\; \text{gap}(x) \;\leq\; d(t_{i_0}, t_{j_0}) \;\leq\; d(t_{i_0}, s_{k_0}) + d(s_{k_0}, t_{j_0}) \;<\; 2 \cdot \eta/2$$

which gives a contradiction. Therefore, $\text{at}(x) = n \leq M$. □

Given x purely atomic and $A \in \mathcal{A}$, recall the terminology, $T[x]$, $A[x]$ and *continuity set* of x introduced in Section 2.6.

Lemma 2.7.2 *Let $x : \mathcal{A} \to \mathbf{R}$ be a purely atomic set-function. Given $A \in \mathcal{A}$, for each $\epsilon > 0$, $\exists\, B \in \mathcal{A}$ such that:*

(i) $A \subseteq B^{\circ} \subseteq B \subseteq A^{\epsilon}$ *(thus $d_H(A, B) \leq \epsilon$),*

(ii) $x(A) = x(B)$ *and*

(iii) *B is a continuity set of x.*

Proof Let x be purely atomic with atoms $t_1, \cdots, t_n \in T$ and let $A \in \mathcal{A}$ and $\epsilon > 0$ be given. Define $P = \{j : t_j \notin A\}$. Without a loss of generality, we can assume that $P = \{1, 2, \cdots, m\}$ for some $1 \leq m \leq n$. (If $P = \phi$, we can take $\epsilon_0 = \epsilon$ in (2.47) and proceed from there.)

Since A is closed in T, for each $1 \leq j \leq m$, $\exists\, \epsilon_j > 0$ such that $d(A, t_j) > \epsilon_j$. If we let $\epsilon_0 = \epsilon \wedge [\min_{1 \leq j \leq m} \epsilon_j] > 0$, then

$$d(A, t_j) > \epsilon_0\,, \quad (\forall\, 1 \leq j \leq m). \tag{2.46}$$

But Definition 2.2.2 implies $g_n(A) \rightarrow_{d_H} A$ where $A \subseteq [g_n(A)]^\circ$ $\forall n$. Thus, $\exists$ $K \in \mathbf{N}$ s.t.

$$A \subseteq [g_K(A)]^\circ \subseteq g_K(A) \subseteq A^{\epsilon_0} \subseteq A^\epsilon. \tag{2.47}$$

Selecting $B = g_K(A)$, we have established (i).

By Remark 2.6.1 and (2.46), $\text{at}(A^{\epsilon_0}) = \text{at}(A)$. Hence, applying Remark 2.6.1 to (2.47) yields

$$\text{at}(A) \leq \text{at}(B^\circ) \leq \text{at}(B) \leq \text{at}(A^{\epsilon_0}) = \text{at}(A).$$

But $A \subseteq B$ implies $A[x] \subseteq B[x]$. Therefore, since $\text{at}(A) = \text{at}(B)$, it must be that $A[x] = B[x]$ which establishes (ii).

By Remark 2.6.1, $\text{at}(B) = \text{at}(B^\circ) + \text{at}(\partial B)$. Therefore, since $\text{at}(B) = \text{at}(B^\circ)$, (iii) follows from the definition of continuity set. □

The construction of the set $B \in \mathcal{A}$ in the above proof relied solely on the fact that the set of all atoms of x is a finite subset of T (see the definition of ϵ_0). Hence, given purely atomic functions $x_1, \cdots, x_n$ on $\mathcal{A}$, the following corollary results from applying the above argument to the finite subset, $\bigcup_{i=1}^n T[x_i]$ of T where $T[x_i] = \{\text{all atoms of } x_i\}$.

Corollary 2.7.3 *Let $x_1, x_2, \cdots, x_n : \mathcal{A} \rightarrow \mathbf{R}$ be purely atomic set-functions. Given $A \in \mathcal{A}$, for each $\epsilon > 0$, $\exists$ $B \in \mathcal{A}$ such that:*

(i) *$A \subseteq B^\circ \subseteq B \subseteq A^\epsilon$ (thus $d_H(A, B) \leq \epsilon$),*

(ii) *$x_k(A) = x_k(B)$ $\forall 1 \leq k \leq n$ and*

(iii) *B is a continuity set of x_k $\forall 1 \leq k \leq n$.*

For our third and final technical result we need the following set-up. For each $n \in \mathbf{N}$, let x_n be purely atomic with atoms $t_{1,n}, \cdots, t_{M,n} \in T$ and respective masses $a_{1,n}, \cdots, a_{M,n} \in \mathbf{R}$ where $M \in \mathbf{N}$ is independent of n. Also, assume that for each $1 \leq i \leq M$, $\exists\, t_i \in T$ and $\exists\, a_i \in \mathbf{R}$ s.t.

$$t_{i,n} \rightarrow t_i \quad \text{and} \quad a_{i,n} \rightarrow a_i \quad \text{as } n \rightarrow \infty.$$

If we define x to be purely atomic with atoms $t_1, \cdots, t_M$ and respective masses $a_1, \cdots, a_M$, then we have the following

Lemma 2.7.4 *Let $(x_n)_n$ and x be as defined above. If $B \in \mathcal{A}$ is a continuity set of x, then $x_n(B) \rightarrow x(B)$ as $n \rightarrow \infty$.*

Proof Since $T = B^\circ \cup \partial B \cup [B^c]^\circ$ and ∂B contains no atoms of x,

$$t_i \in B^\circ \text{ or } t_i \in [B^c]^\circ, \quad (\forall\, 1 \le i \le n).$$

Given $1 \le i \le M$, if $t_i \in B^\circ$ (or $t_i \in [B^c]^\circ$), then

$$t_{i,n} \to t_i \implies \exists\, K_i \in \mathbf{N} \text{ s.t. } t_{i,n} \in B^\circ \text{ (or } t_{i,n} \in [B^c]^\circ \text{) } \forall n \ge K_i.$$

If we select $K = \max_{1 \le i \le M} K_i$, then, for each $n \ge K$,

$$t_i \in (\notin) B \iff t_{i,n} \in (\notin) B.$$

Thus, given $n \ge K$,

$$\begin{aligned} |x_n(B) - x(B)| &= \left| \sum\nolimits_{i:t_{i,n} \in B} a_{i,n} - \sum\nolimits_{i:t_i \in B} a_i \right| \\ &= \left| \sum\nolimits_{i:t_i \in B} a_{i,n} - \sum\nolimits_{i:t_i \in B} a_i \right| \\ &\le \sum\nolimits_{i=1}^{M} |a_{i,n} - a_i|, \end{aligned}$$

where $\sum_{i=1}^{M} |a_{i,n} - a_i| \to 0$ as $n \to \infty$. □

And now for the proof of Theorem 2.6.5. Our argument is similar to that found in Proposition 3.2 of [5].

Theorem *Each $\Gamma(T', h, N, \eta, R) \subseteq PA$ is compact in $(D(\mathcal{A}), d_D)$.*

Proof To make nested brackets more readable, we will occasionally write $\langle a, b\rangle$ in place of (a, b).

Let $\Gamma = \Gamma(T', h, N, \eta, R)$ for some list, (T', h, N, η, R) (see Definition 2.6.2). By condition (ii) in Definition 2.6.2

$$\|x\|_{\mathcal{A}} \le \nu(x) \le R, \quad (\forall\, x \in \Gamma).$$

Therefore, by Lemma 2.4.6, we have the following

Reduction: *It is sufficient to show $G[\Gamma]$ is closed in $(\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$.*

With this reduction in mind, take $(x_n)_n$ in Γ s.t. $G(x_n) \to_{d_G} G_0$ for some $G_0 \in \mathcal{K}_{\mathcal{A}\times\mathbf{R}}$. Since $\text{gap}(x_n) \ge \eta\ \forall n$, Lemma 2.7.1 implies $\exists\ M_0 \in \mathbf{N}$ s.t.

$\mathrm{at}(x_n) \leq M_0 \ \forall n$. Hence, we can extract a subsequence, which we also denote $(x_n)_n$, s.t., for some $1 \leq M \leq M_0$, $\mathrm{at}(x_n) = M \ \forall n$.

For each $n \in \mathrm{N}$, denote the locations and respective masses of the atoms of x_n by

$$t_{1,n}, \cdots, t_{M,n} \text{ and } a_{1,n}, \cdots, a_{M,n}.$$

Given any $1 \leq i \leq M$, Definition 2.6.2 implies that $(t_{i,n})_n$ lies in T' and

$$|a_{i,n}| \leq \nu(x_n) \leq R, \quad (\forall n).$$

Therefore, since T' and $[-R, R]$ are both compact, we can extract a further subsequence, $(x_{k_n})_n$, s.t., for each $1 \leq i \leq M$, $\exists\, t_i \in T'$ and $\exists\, a_i \in \mathrm{R}$ s.t.

$$t_{i,k_n} \to t_i \text{ and } a_{i,k_n} \to a_i \text{ as } n \to \infty \tag{2.48}$$

For notational convenience, we replace k_n by $n \ \forall n$.

Now, consider the purely atomic function, x with atoms $t_1, \cdots, t_M$ and respective masses $a_1, \cdots, a_M$. By (2.48) and Lemma 2.7.4,

$$x_n(C) \to x(C), \quad (\forall \text{ continuity sets, } C \text{ of } x). \tag{2.49}$$

Via the above reduction, *the d_D-compactness of Γ will follow if we can show that $x \in \Gamma$ and $G(x) = G_0$.* We divide this into three steps.

Step 1: $G(x) \subseteq G_0$.

Since G_0 is $\rho_{\mathcal{A}}$-closed and $G(x) = cl(\{(A, x(A)) : A \in \mathcal{A}\})$, it is sufficient to show that, for each $A \in \mathcal{A}$ and $\delta > 0$,

$$\exists\, (B, r) \in G_0 \text{ s.t. } \rho_{\mathcal{A}}((A, x(A)), (B, r)) < \delta. \tag{2.50}$$

Let $A \in \mathcal{A}$ and $\delta > 0$ be given. By Lemma 2.7.2, $\exists$ a continuity set, B of x such that,

$$d_H(A, B) < \delta \text{ and } x(A) = x(B). \tag{2.51}$$

Thus, by (2.49), $(B, x_n(B)) \to_{\rho_{\mathcal{A}}} (B, x(B))$. Furthermore, since $G(x_n) \to_{d_G} G_0$, Lemma A.1.4 implies that $(B, x(B)) \in G_0$.

If we let $r = x(B)$, then by (2.51),

$$\rho_{\mathcal{A}}((A, x(A)), (B, r)) = d_H(A, B) + 0 < \delta.$$

Since $A \in \mathcal{A}$ and $\delta > 0$ were arbitrarily chosen, this establishes (2.50), completing Step 1.

Step 2: $G_0 \subseteq G(x)$.

Let $(B, r) \in G_0$ be given. Since $G(x)$ is $\rho_{\mathcal{A}}$-closed, it suffices to such that, for any $\delta > 0$,

$$\exists\, C \in \mathcal{A} \text{ s.t. } \rho_{\mathcal{A}}((C, x(C)), (B, r)) < 3\delta. \tag{2.52}$$

For this purpose, take $\delta > 0$. By Lemma A.1.4, $\exists\, (B_n)_n$ in $\mathcal{A}$ s.t.

$$(B_n, x_n(B_n)) \xrightarrow{\rho_{\mathcal{A}}} (B, x(B)) \text{ as } n \to \infty. \tag{2.53}$$

Hence, by (iii) in Definition 2.6.2, for each $n \in \mathrm{N}$, $\exists\, A_n \in \mathcal{A}$ s.t.:

(a)$_1$ $\rho_{\mathcal{A}}\langle(A_n, x_n(A_n)), (B_n, x_n(B_n))\rangle \leq \delta$ and
(b)$_1$ $A_n^{h(\delta)} \setminus A_n$ contains no atoms of x_n.

Since $(\mathcal{A}, d_H)$ is compact (see Theorem 2.2.13), $\exists$ a subseqence of $(A_n)_n$ converging to some $A \in \mathcal{A}$ w.r.t. d_H. For notational convenience, *this subsequence is also denoted by* $(A_n)_n$. Furthermore, applying Lemma 2.7.2 to x at A, $\exists\, C \in \mathcal{A}$ s.t.:

(a)$_2$ $A \subseteq C^\circ \subseteq C \subseteq A^{h(\delta)/2}$ (hence $d_H(A, C) \leq h(\delta)/2$),
(b)$_2$ $x(A) = x(C)$ and
(c)$_2$ C is a continuity set of x.

The following claim will establish (2.52).

Claim I: *The set C, as defined above, satisfies $\rho_{\mathcal{A}}((C, x(C)), (B, r)) < 3\delta$.*

Proof: First, note that

$$d_H(C, B) \leq d_H(C, A) + d_H(A, A_n) + d_H(A_n, B_n) + d_H(B_n, B), \quad (\forall n).$$

Therefore, by (a)$_1$ and (a)$_2$,

$$d_H(C, B) \leq h(\delta)/2 + d_H(A, A_n) + \delta + d_H(B_n, B), \quad (\forall n).$$

Since $A_n \to_{d_H} A$, $B_n \to_{d_H} B$ and $h(\delta) \leq \delta$, this implies $d_H(C, B) < 2\delta$. If we can show

$$|x(C) - r| \leq \delta, \tag{2.54}$$

then, by the definition of ρ_A, Claim I will follow.

En route to establishing (2.54), we need the following result.

Subclaim I: $\exists K \in \mathrm{N}$ *s.t.* $x_n(C) = x_n(A_n)$ $\forall n \geq K$.

Proof: Since $A \subseteq C^\circ$, Lemma A.1.2 implies $\exists\, 0 < \beta < h(\delta)/2$ s.t. $A^\beta \subseteq C$. Moreover, since $A_n \to_{d_H} A$, $\exists K \in \mathrm{N}$ s.t.

$$A_n \subseteq A^\beta \text{ and } A \subseteq (A_n)^\beta, \quad (\forall n \geq K).$$

Thus, given any $n \geq K$, Proposition 2.2.1 and $(a)_2$ imply

$$A_n \subseteq A^\beta \subseteq C \subseteq A^{h(\delta)/2} \subseteq (A_n^\beta)^{h(\delta)/2} \subseteq A_n^{\beta + h(\delta)/2} \subseteq A_n^{h(\delta)}.$$

In particular, $A_n \subseteq C \subseteq A_n^{h(\delta)}$ $\forall n \geq K$ which, by $(b)_1$, implies $x_n(C) = x_n(A_n)$ $\forall n \geq K$. This completes the proof of Subclaim I. ω

Now, to establish (2.54), let $n \geq K$ be given. By Subclaim I and $(a)_1$,

$$\begin{aligned} |x(C) - r| &\leq |x(C) - x_n(C)| + |x_n(C) - x_n(A_n)| + \\ &\quad |x_n(A_n) - x_n(B_n)| + |x_n(B_n) - r| \\ &\leq |x(C) - x_n(C)| + 0 + \delta + |x_n(B_n) - r|. \end{aligned} \tag{2.55}$$

Since C is a continuity set of x, (2.49) implies $x_n(C) \to x(C)$. We also have $x(B_n) \to r$ as $n \to \infty$. Therefore, letting $n \to \infty$ in (2.55), we obtain (2.54). This completes the proof of Claim I, thus completing Step 2. Ω

Step 3: $x \in \Gamma$.

Here, we need to show that x satisfies conditions (i) – (iv) of Definition 2.6.2. Condition (iii) is by far the most difficult.

(i): Take any $1 \leq i, j \leq n$ such that $i \neq j$ and let $\epsilon > 0$ be given. By (2.48), $\exists\, n_0 \in \mathrm{N}$ s.t. $d(t_i, t_{i,n_0}), d(t_j, t_{j,n_0}) < \epsilon/2$. Since t_{i,n_0}, t_{j,n_0} are distinct atoms of x_{n_0} and $\mathrm{gap}(x_{n_0}) \geq \eta$,

$$\eta \leq d(t_{i,n_0}, t_{j,n_0}) \leq d(t_{i,n_0}, t_i) + d(t_i, t_j) + d(t_j, t_{i,n_0}) < d(t_i, t_j) + \epsilon.$$

Letting $\epsilon \to \infty$, this implies $\eta \leq d(t_i, t_j)\ \forall i \neq j$ and therefore, $\eta \leq \min_{i \neq j} d(t_i, t_j) = \text{gap}(x)$.

(ii): By (2.48), for each $1 \leq i \leq M$, $a_{i,n} \to a_i$ as $n \to \infty$. Thus,

$$\nu(x) \;=\; \sum_{i=1}^{M} |a_i| \;=\; \lim_n \sum_{i=1}^{M} |a_{i,n}| \;=\; \lim_n \nu(x_n).$$

But $x_n \in \Gamma\ \forall n$ implies $\nu(x_n) \leq R\ \forall n$. Therefore, $\nu(x) \leq R$.

(iv): By the argument prior to (2.48), we have $T(x) = \{t_1, \cdots t_M\} \subseteq T'$.

(iii): Take $\delta > 0$. Given any n, $x_n \in \Gamma$ implies $\exists\, A_{1,n}, \cdots A_{N(\delta),n} \in \mathcal{A}$ s.t.

$$A_{1,n}, \cdots, A_{N(\delta),n} \text{ satisfy (iii) of Definition 2.6.2 for } x_n \text{ and } \delta. \tag{2.56}$$

In this way, we construct sequences, $(A_{i,n})_n$ $(1 \leq i \leq N(\delta))$ in $\mathcal{A}$. But by Theorem 2.2.13, $(\mathcal{A}, d_H)$ is compact and hence, $\exists$ a common subsequence $(k_n)_n$ and sets $A_1, \cdots, A_{N(\delta)} \in \mathcal{A}$ s.t.

$$A_{i,k_n} \xrightarrow{d_H} A_i \text{ as } n \to \infty, \quad (\forall\, 1 \leq i \leq N(\delta)). \tag{2.57}$$

For notational convenience, we replace k_n by n in (2.57).

To complete Step 3, *the sets* $A_1, \cdots, A_{N(\delta)}$ *defined in* (2.57) *will be shown to satisfy conditions* (iii)(a) *and* (iii)(b) *of Definition 2.6.2 for our* x *and* δ.

(iii)(b): We need to show $A_i^{h(\delta)} \setminus A_i$ contains no atoms of x $\forall\, 1 \leq i \leq N(\delta)$. Assume to the contrary that

$$\exists\, 1 \leq i_0 \leq N(\delta) \text{ and } \exists\, 1 \leq j_0 \leq M \text{ s.t. } t_{j_0} \in A_{i_0}^{h(\delta)} \setminus A_{i_0}, \tag{2.58}$$

i.e., $t_{j_0} \in A_{i_0}^{h(\delta)}$ and $t_{j_0} \notin A_{i_0}$. Then, since A_{i_0} is d-closed in T, $\exists\, \epsilon_1 > 0$ s.t. $d(t_{j_0}, a) \geq \epsilon_1\ \forall a \in A_{i_0}$ which implies

$$t_{j_0} \notin A_{i_0}^{\epsilon_1}.$$

Moreover, $t_{j_0} \in A_{i_0}^{h(\delta)}$ implies $d(A_{i_0}, t_{j_0}) < h(\delta)$. Therefore, if we let $\epsilon_2 = 1/2 \cdot [h(\delta) - d(A_{i_0}, t_{j_0})] > 0$, then $d(A_{i_0}, t_{j_0}) < 1/2 \cdot [h(\delta) + d(A_{i_0}, t_{j_0})] = h(\delta) - \epsilon_2$ which implies

$$t_{j_0} \in A_{i_0}^{h(\delta) - \epsilon_2}.$$

In total, if we define $\epsilon = 1/2 \cdot \min\{\epsilon_1, \epsilon_2\} > 0$, then, by Proposition 2.2.1 (iv)

$$t_{j_0} \in A_{i_0}^{h(\delta)-2\epsilon} \setminus A_{i_0}^{2\epsilon} \tag{2.59}$$

(note that $2\epsilon < h(\delta)$).

En route to obtaining a contradiction, we present a basic result.

Claim II: $\exists\, K_1 \in \mathrm{N}$ *s.t.* $A_{i_0}^{h(\delta)-\epsilon} \setminus A_{i_0}^{\epsilon} \subseteq A_{i_0,n}^{h(\delta)} \setminus A_{i_0,n}\ \forall n \geq K_1$.

Proof: By (2.57), $A_{i_0,n} \to_{d_H} A_{i_0}$ as $n \to \infty$. Therefore, $\exists\, K_1 \in \mathrm{N}$ s.t.

$$A_{i_0,n} \subseteq A_{i_0}^{\epsilon} \text{ and } A_{i_0} \subseteq A_{i_0,n}^{\epsilon}, \quad (\forall n \geq K_1). \tag{2.60}$$

By Proposition 2.2.1, this implies

$$A_{i_0}^{h(\delta)-\epsilon} \subseteq (A_{i_0,n}^{\epsilon})^{h(\delta)-\epsilon} \subseteq A_{i_0,n}^{h(\delta)}, \quad (\forall n \geq K_1). \tag{2.61}$$

Claim II follows by combining (2.60) and (2.61). Ω

Now, by Proposition 2.2.1, $A_{i_0}^{\epsilon} \subseteq A_{i_0}^{\bar{\epsilon}} \subseteq A_{i_0}^{2\epsilon}$. Hence, by Proposition 2.2.1(i) and (2.59), $A_{i_0}^{h(\delta)-2\epsilon} \setminus A_{i_0}^{\bar{\epsilon}}$ is a d-open neighborhood of t_{j_0}. Since $t_{j_0,n} \to t_{j_0}$ as $n \to \infty$, this implies $\exists\, K_2 \in \mathrm{N}$ s.t.

$$t_{j_0,n} \in A_{i_0}^{h(\delta)-2\epsilon} \setminus A_{i_0}^{\bar{\epsilon}} \subseteq A_{i_0}^{h(\delta)-\epsilon} \setminus A_{i_0}^{\epsilon}, \quad (\forall n \geq K_2). \tag{2.62}$$

If we let $K = K_1 \vee K_2$, then (2.62) and Claim II yield

$$t_{j_0,K} \in A_{i_0}^{h(\delta)-\epsilon} \setminus A_{i_0}^{\epsilon} \subseteq A_{i_0,K}^{h(\delta)} \setminus A_{i_0,K}$$

which implies $A_{i_0,K}^{h(\delta)} \setminus A_{i_0,K}$ contains an atom of x_K, namely $t_{j_0,K}$. This contradicts (2.56). Since this contradiction resulted from assuming (2.58), $A_1, \cdots, A_{N(\delta)} \in \mathcal{A}$ satisfy (iii)(b) of Definition 2.6.2 for our x and δ.

(iii)(a): We need to show that for any $(C, r) \in G(x)$,

$$\exists\, 1 \leq i_0 \leq n \text{ s.t. } \rho_{\mathcal{A}}\langle (C, r), (A_{i_0}, x(A_{i_0}))\rangle \leq \delta. \tag{2.63}$$

The following claim establishes this condition for many $(C, r) \in G(x)$.

Claim III: $\forall A \in \mathcal{A}, \exists 1 \le i_0 \le N(\delta)$ *s.t.* $\rho_{\mathcal{A}}\langle (A, x(A)), (A_{i_0}, x(A_{i_0}))\rangle \le \delta$.

Proof: Take $A \in \mathcal{A}$. Claim III will follow if we can show $\exists 1 \le i_0 \le N(\delta)$ s.t.

$$\rho_{\mathcal{A}}\langle (A, x(A)), (A_{i_0}, x(A_{i_0}))\rangle \le \delta + 3\gamma, \quad (\forall \gamma > 0). \tag{2.64}$$

For this purpose, take $\gamma > 0$. By Lemma 2.7.2, $\exists$ a continuity set, $B \in \mathcal{A}$ of x such that $d_H(A, B) < \gamma$ and $x(A) = x(B)$. Since B is a continuity set of x, (2.49) implies $\exists K_1 \in \mathrm{N}$ s.t. $|x_n(B) - x(B)| < \gamma \ \forall n \ge K_1$ and therefore,

$$\rho_{\mathcal{A}}\langle (A, x(A)), (B, x_n(B))\rangle = d_H(A, B) + |x(B) - x_n(B)| < 2\gamma, \quad (\forall n \ge K_1). \tag{2.65}$$

In view of (2.56), for each $n \in \mathrm{N}$

$$\exists 1 \le i_n \le N(\delta) \text{ s.t. } \rho_{\mathcal{A}}\langle (B, x_n(B)), (A_{i_n,n}, x_n(A_{i_n,n}))\rangle \le \delta. \tag{2.66}$$

Since $1 \le i_n \le N(\delta) \ \forall n$, $\exists 1 \le i_0 \le N(\delta)$ and a subsequence $(k_n)_n$ s.t. $i_{k_n} = i_0 \ \forall n$. Therefore, by (2.57),

$$A_{i_{k_n},k_n} = A_{i_0,k_n} \xrightarrow{d_H} A_{i_0} \text{ as } n \to \infty. \tag{2.67}$$

Since (2.67) is the only place where $i_{k_n} = i_0$ is used, *we will not replace* i_{k_n} *by* i_0 *in what is to follow so as to emphasize that* (2.66) *holds for each* i_{k_n}.

Our goal is to show that the above i_0 satisfies (2.64) for our given $A \in \mathcal{A}$. For this, we need the following result.

Subclaim II: *If* $1 \le j \le M$, *then* $t_j \in A_{i_0}$ *if and only if* $\exists H \in \mathrm{N}$ *such that* $t_{j,k_n} \in A_{i_{k_n},k_n} \ \forall n \ge H$.

Proof: First, assume that $\exists H \in \mathrm{N}$ s.t. $t_{j,k_n} \in A_{i_{k_n},k_n} \ \forall n \ge H$. Since $A_{i_{k_n},k_n} \to_{d_H} A_{i_0}$ and $t_{j,k_n} \to t_j$ as $n \to \infty$ (by (2.67) and (2.48) respectively), Lemma A.1.4 implies that $t_j \in A_{i_0}$.

Conversely, assume $t_j \in A_{i_0}$. Since $A_{i_{k_n},k_n} \to_{d_H} A_{i_0}$, $\exists H_1 \in \mathrm{N}$ s.t.

$$t_j \in A_{i_0} \subseteq (A_{i_{k_n},k_n})^{h(\delta)/2}, \quad (\forall n \ge H_1). \tag{2.68}$$

Furthermore, since $t_{j,k_n} \to t_j$, $\exists H_2 \in \mathrm{N}$ s.t.

$$d(t_{j,k_n}, t_j) < h(\delta)/2, \quad (\forall n \ge H_2). \tag{2.69}$$

If we let $H = H_1 \vee H_2$ and take $n \geq H$, then by (2.68), $\exists\, \bar{a} \in A_{i_{k_n},k_n}$ s.t. $d(t_j, \bar{a}) < h(\delta)/2$. Thus, by (2.69),

$$d(t_{j,k_n}, \bar{a}) \;\leq\; d(t_{j,k_n}, t_j) + d(t_j, \bar{a}) \;<\; 2 \cdot [h(\delta)/2]$$

which implies $t_{j,k_n} \in A^{h(\delta)}_{i_{k_n},k_n}$. But by (2.56), $A^{h(\delta)}_{i_{k_n},k_n} \setminus A_{i_{k_n},k_n}$ contains no atoms of x_{k_n}. Therefore, since t_{j,k_n} is an atom of x_{k_n}, $t_{j,k_n} \in A_{i_{k_n},k_n}$. This establishes Subclaim II. ω

Applying Subclaim II to each $1 \leq j \leq M$ for which $t_j \in A_{i_0}$, we have

$$\{j : t_j \in A_{i_0}\} \;=\; \{j : t_{j,k_n} \in A_{i_{k_n},k_n}\}$$

for all large n. Furthermore, since $a_{j,k_n} \to a_j \;\forall\, 1 \leq j \leq M$,

$$\lim_n x_{k_n}(A_{i_{k_n},k_n}) \;=\; \lim_n \sum_{j\,:\,t_j \in A_{i_0}} a_{j,k_n} \;=\; \sum_{j: t_j \in A_{i_0}} \left(\lim_n a_{j,k_n}\right) \;=\; x(A_{i_0}).$$

Combining this limit with (2.67), $\exists\, K_2 \in \mathbb{N}$ s.t.

$$\rho_{\mathcal{A}}\langle (A_{i_{k_n},k_n}, x_n(A_{i_{k_n},k_n})), (A_{i_0}, x(A_{i_0}))\rangle \;<\; \gamma, \quad (\forall n \geq K_2). \tag{2.70}$$

Thus, if we take $K = K_1 \vee K_2$, then by (2.65), (2.66) and (2.70),

$$\begin{aligned} \rho_{\mathcal{A}}\langle (A, x(A)), (A_{i_0}, x(A_{i_0}))\rangle \;&\leq\; \rho_{\mathcal{A}}\langle (A, x(A)), (B, x_K(B))\rangle \;+ \\ &\quad\; \rho_{\mathcal{A}}\langle (B, x_K(B)), (A_{i_K,K}, x_K(A_{i_K,K}))\rangle \;+ \\ &\quad\; \rho_{\mathcal{A}}\langle (A_{i_K,K}, x_K(A_{i_K,K})), (A_{i_0}, x(A_{i_0}))\rangle \\ &<\; 2\gamma + \delta + \gamma. \end{aligned}$$

This establishes (2.64), completing the proof of Claim III. Ω

Finally, take any $(C, r) \in G(x)$ — we need to establish (2.63). By Remark 2.4.2(b), $\exists\, (B_n)_n$ in $\mathcal{A}$ s.t.

$$\rho_{\mathcal{A}}\langle (C, r), (B_n, x(B_n))\rangle \to 0 \;\text{ as }\; n \to \infty \tag{2.71}$$

Applying Claim III to each B_n, noting that $N(\delta) < \infty$, $\exists\, 1 \leq i_0 \leq N(\delta)$ s.t.

$$\rho_{\mathcal{A}}\langle (B_n, x(B_n)), (A_{i_0}, x(A_{i_0}))\rangle \;\leq\; \delta \quad \text{for infinitely many } n.$$

Therefore, we can select a subsequence $(k_n)_n$ s.t.

$$\rho_{\mathcal{A}}\langle (B_{k_n}, x(B_{k_n})), (A_{i_0}, x(A_{i_0}))\rangle \leq \delta, \quad (\forall n). \tag{2.72}$$

Since $\rho_{\mathcal{A}}\langle \cdot , (A_{i_0}, x(A_{i_0}))\rangle : (\mathcal{A} \times \mathbf{R}, \rho_{\mathcal{A}}) \to [0, \infty)$ is continuous, (2.71) and (2.72) imply

$$\rho_{\mathcal{A}}\langle (C, r), (A_{i_0}, x(A_{i_0}))\rangle = \lim_n \rho_{\mathcal{A}}\langle (B_{k_n}, x(B_{k_n})), (A_{i_0}, x(A_{i_0}))\rangle \leq \delta.$$

Hence, (2.63) holds for our arbitrary $(C, r) \in G(x)$,

In total, the sets $A_1, \cdots, A_{N(\delta)}$ satisfy condition (iii) of Definition 2.6.2 for our $\delta > 0$ and x. Since $\delta > 0$ was arbitrary, this implies $x \in \Gamma$, establishing Step 3 and therefore completing the proof of Theorem 2.6.5. □

2.8 Examples for Chapter 2

This section contains some of the examples and counterexamples referred to in earlier sections of Chapter 2. A brief description is given prior to each example. See Section 2.2 for terminology.

- *The so called "lower layers" in* $[0,1]^k$ $(k \in \mathbf{N})$.

Example 2.8.1 Fix $k \in \mathbf{N}$. The reader is assumed to be familiar with the indexing collection $\mathcal{I}_k$ on $[0,1]^k$ defined in Example 2.2.6.

A set $L \subseteq [0,1]^k$ is said to be a *lower layer* in $[0,1]^k$ if for any $\mathbf{t} \in [0,1]^k$,

$$\mathbf{t} \in L \implies [\mathbf{0}, \mathbf{t}] \subseteq L.$$

See Figure 2.1 for some examples of lower layers. As suggested by the left-most illustration, any finite union in $\mathcal{I}_k$ is a lower layer. Furthermore, it is clear from the right-most illustration that a lower layer need not be convex.

Let $\mathcal{LL}_k$ denote the collection of all lower layers in $[0,1]^k$. Clearly, $\mathcal{LL}_k$ is a non-empty subcollection of $\mathcal{K}_{[0,1]^k}$ which contains $[0,1]^k$ and is closed under countable intersection. Moreover, if we define

$$\mathcal{LL}_k^{(n)} = \{\text{all finite unions in } \mathcal{I}_k^{(n)}\}, \quad (\forall n),$$

then each $\mathcal{I}_k^{(n)}$ is a finite subcollection of $\mathcal{LL}_k$ satisfying the conditions in (1) of Definition 2.2.2. What can also be shown is that $g_n : \mathcal{LL}_k \to \mathcal{LL}_k^{(n)}$, where

$$g_n(L) = \bigcap \{M \in \mathcal{LL}_k^{(n)} : L \subseteq M^\circ\}, \quad (\forall L \in \mathcal{LL}_k)$$

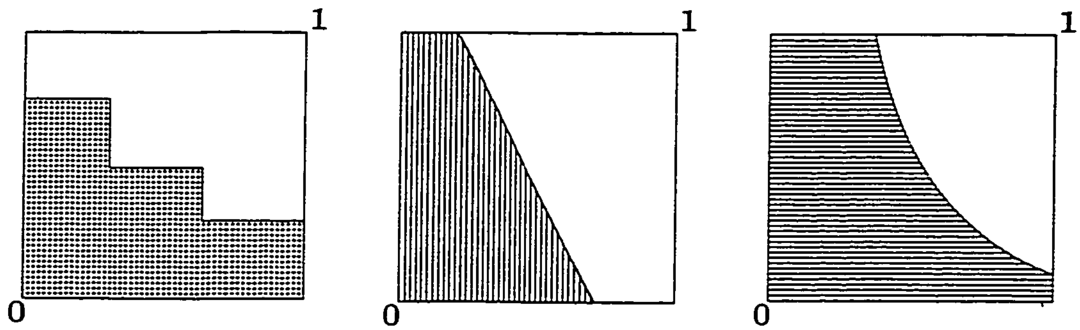

Figure 2.1: Some lower layers in $[0,1]^2$.

satisfies all properties listed in (2) of Definition 2.2.2. Therefore, $\mathcal{LL}_k$ is an indexing collection on $[0,1]^k$. Note that $\phi' = \{0\}$.

Each $\mathcal{LL}_k$ is closed under finite unions and because of this, $\mathcal{LL}_k$ does not satisfy the shape assumption described in Remark 2.2.5 (d) whenever $k \geq 2$. In particular, given $L_1, L_2 \in \mathcal{LL}_2$ s.t. $L_1 \not\subseteq L_2$ and $L_2 \not\subseteq L_1$, the set $L = L_1 \cup L_2 \in \mathcal{LL}_2$ satisfies the inclusion $L \subseteq L_1 \cup L_2$ while $L \not\subseteq L_i$ $(i = 1, 2)$.

- *An indexing collection on* $[0,1]$ *which is not c.w.b. (see Definition 2.5.3).*

Example 2.8.2 Take a collection of "good" sets,

$$\mathcal{A}_g = \{[0,t] : 1/2 \leq t \leq 1\},$$

and a collection of "bad" sets,

$$\mathcal{A}_b = \{S_{(a,b,c,d)} : 0 \leq a \leq b \leq 1/4 \leq c \leq d \leq 1/2\}$$

where $S_{(a,b,c,d)} = [0,a] \cup [b,c] \cup [d,1/2]$. Clearly, $\mathcal{A} = \mathcal{A}_g \cup \mathcal{A}_b$ is a subcollection of $\mathcal{K}_{[0,1]}$ which contains $[0,1]$ and is closed under countable intersections. Note that $\phi' = \bigcap_{A \in \mathcal{A}} A = \{0, 1/4, 1/2\}$.

Given any $n \in \mathbb{N}$, define $\mathcal{A}_n = \mathcal{A}_g^{(n)} \cup \mathcal{A}_b^{(n)}$ where

$$\mathcal{A}_g^{(n)} = \{[0,t] : t \in D^{(n)}\} \cap \mathcal{A} \quad \text{and} \quad \mathcal{A}_b^{(n)} = \{S_{(a,b,c,d)} : a,b,c,d \in D^{(n)}\} \cap \mathcal{A}.$$

Here, $D^{(n)} = \{\frac{m}{2^n} : 0 \leq m \leq 2^n\}$. Also, define $g_n : \mathcal{A} \to \mathcal{A}_n$ by

$$g_n(A) = \bigcap \{B \in \mathcal{A}_n : A \subseteq B^\circ\}, \quad (\forall A \in \mathcal{A}).$$

It is straightforward (but tedious) to show that $(\mathcal{A}_n)_n$ and $(g_n)_n$ satisfy the conditions of Definition 2.2.2. From this, it follows that $\mathcal{A}$ is an indexing collection on $[0,1]$.

But $\mathcal{A}$ is not c.w.b. In particular, if we select $\frac{1}{8} < a < \frac{1}{4}$, then

$$d_H([0,1/2], S_{(a,\frac{1}{4},\frac{1}{4},\frac{1}{4})}) = d_H([0,1/2], [0,a] \cup [1/4,1/2]) < 1/8$$

while

$$d_H(\partial([0,1/2]), \partial(S_{(a,\frac{1}{4},\frac{1}{4},\frac{1}{4})})) = d_H(\{1/2\}, \{a, 1/4, 1/2\}) > 1/8,$$

implying that condition (b) of Definition 2.5.3 fails.

• *An example of a purely atomic set-function which is not continuous at all of its continuity sets (a result of omitting c.w.b.).*

Example 2.8.3 Let $\mathcal{A}$ be the indexing collection defined in Example 2.8.2 and let $x : \mathcal{A} \to \mathbf{R}$ be purely atomic with one atom of mass 1 at $t = 1/8$. Since $1/8 \notin \partial A = \{1/2\}$, $A = [0,1/2] \in \mathcal{A}$ is a continuity set of x.

But the sequence $(A_n)_{n\geq 8}$ in $\mathcal{A}$ where $A_n = S_{(\frac{1}{8}-\frac{1}{n},\frac{1}{8}+\frac{1}{n},\frac{1}{4},\frac{1}{4})}$ $\forall n \geq 8$ is such that

$$A_n \xrightarrow{d_H} A \quad \text{and} \quad 0 = x(A_n) \not\to x(A) = 1.$$

Therefore, x is not d_H-continuous at A. This does not contradict Lemma 2.5.9 since $\mathcal{A}$ is not c.w.b.

• *A list (h, N, η, R) for which $\mathcal{F}_{PA}(h, N, \eta, R) \cap PA$ is not compact in $(D(\mathcal{I}_1), d_D)$ (see the end of Section 2.6 for terminology).*

Example 2.8.4 Take a sequence $(r_n)_n$ in $\mathbf{R}$ s.t.

$$1/2 < r_n < r_{n+1} < 1 \; \forall n \quad \text{and} \quad r_n \to 1 \text{ as } n \to \infty.$$

Consider $(x_n)_n$ where $x_n : \mathcal{I}_1 \to \mathbf{R}$ is purely atomic with one atom of mass 1 at r_n. Since $r_n \notin \{0,1\} = E(\mathcal{I}_1)$, each x_n lies in PA.

Now, let $x : \mathcal{I}_1 \to \mathbf{R}$ be purely atomic with one atom of mass 1 at $t = 1$. Since the locations and corresponding masses of atoms of x_n converge to those of x, Lemma 2.7.4 combined with Theorem 2.9 in [22] implies

$$x_n \xrightarrow{d_D} x \text{ as } n \to \infty.$$

Note that $\mathcal{I}_1$ satisfies the conditions presented in [22].

But $t \in E(\mathcal{I}_1)$ implies $x \notin PA$. In fact, $x \notin D(\mathcal{I}_1)$. Therefore, no subsequence of $(x_n)_n$ converges in $(D(\mathcal{A}), d_D)$, a consequence being that *any subspace of $D(\mathcal{I}_1)$ which contains $(x_n)_n$ cannot be compact in $(D(\mathcal{I}_1), d_D)$.*

Consider $\Gamma_0 = \mathcal{F}_{PA}(h, N, \eta, R)$ with $\eta = R = 1$ and

$$h(\delta) = \delta/2, \quad (\forall \delta > 0).$$

(We will define N later.) If the sequence $(x_n)_n$ lies in Γ_0, then by the above italicized observation, $\Gamma_0 \cap PA$ is not compact in $(D(\mathcal{I}_1), d_D)$.

By convention, if x is purely atomic with a single atom, then $\text{gap}(x) = \text{diam}(T)$. This implies

$$\text{gap}(x_n) = \text{diam}([0,1]) = 1, \quad (\forall n).$$

Furthermore, $\nu(x_n) = 1 \ \forall n$.

Take $0 < \delta < 1$. Given any $n \in \mathrm{N}$, define $t_n{}' = r_n - \delta/2 > 0$ and select a partition, $\{0 = t_1 < \cdots < t_{i_0} = t_n{}' < r_n = t_{i_0+1} < \cdots < t_k = 1\}$ of $[0,1]$ satisfying

$$t_{i+1} - t_i \leq \delta, \quad (\forall 1 \leq i < k).$$

Clearly, we can choose this k large enough so as to work for every r_n. For example, we can take $k = 2 + [1/\delta]$ where $[1/\delta]$ is the integer part of $1/\delta$.

Now, define $N(\delta) = k$ and take $A_i = [0, t_i] \ \forall 1 \leq i \leq N(\delta)$. Clearly,

$$A_i^{h(\delta)} \setminus A_i \text{ contains no atoms of } x_n, \quad (\forall 1 \leq i \leq N(\delta))$$

since the only "interesting" set difference is $A_{i_0}^{h(\delta)} \setminus A_{i_0} = (t_{i_0}, t_{i_0} + \delta/2) = (t_{i_0}, r_n)$ which does not contain r_n, the only atom of x_n. Moreover, given any $A = [0, t] \in \mathcal{I}_1$, if we choose $1 \leq i < N(\delta)$ s.t. $t_i \leq t < t_{i+1}$, then

$$\rho_A((A_i, x_n(A_i)), (A, x_n(A))) = d_H(A_i, A) + 0 \leq \delta.$$

Since both n and $0 < \delta < 1$ were arbitrarily chosen, Definition 3.5 in [5] implies $x_n \in \mathcal{F}_{PA}(h, N, \eta, R) \cap PA \ \forall n$. Therefore, by our earlier comment, $\mathcal{F}_{PA}(h, N, \eta, R) \cap PA$ cannot be compact in $(D(\mathcal{I}_1), d_D)$.

• *A subspace $\mathcal{F} \subseteq B(\mathcal{I}_1)$ of the type generated by Definition 3.6 of* [5] *for which $\mathcal{F} \cap D(\mathcal{I}_1)$ is not compact in $(D(\mathcal{I}_1), d_D)$.*

Example 2.8.5 Let $\mathcal{A} = \mathcal{I}_1$. As demonstrated in Example 2.8.4, $\exists$ a list (h, N, η, R) and a sequence $(x_n)_n$ in $\mathcal{F}_{PA}(h, N, \eta, R) \cap D(\mathcal{I}_1)$ such that no subsequence of $(x_n)_n$ converges in $(D(\mathcal{I}_1), d_D)$.

If we select suitable parameters, $(M_m)_m$, $(W_m)_m$ and $(\Delta_m)_m$ as described in Definition 3.6 of [5] and take

$$\mathcal{F}_{PA}(h_m, N_m, \eta_m, R_m) = \mathcal{F}_{PA}(h, N, \eta, R), \quad (\forall m),$$

the subspace $\mathcal{F}$ of $B(\mathcal{I}_1)$ generated by these parameters will contain the above mentioned $(x_n)_n$. Since no subsequence of $(x_n)_n$ converges in $(D(\mathcal{I}_1), d_D)$, the subspace, $\mathcal{F} \cap D(\mathcal{I}_1)$ cannot be d_D-compact.

- *A d_D-compact subspace, $\Gamma_0 \subseteq PA$ which is not of the form $\Gamma(T', h, N, \eta, R)$ for any list, (T', h, N, η, R) (see Definition 2.6.2).*

Example 2.8.6 For any $n \geq 3$, let $x_n : \mathcal{I}_1 \to \mathbf{R}$ be purely atomic with three atoms,

$$t_1^{(n)} = 1/2 - 1/n, \quad t_2^{(n)} = 1/2 \quad \text{and} \quad t_3^{(n)} = 1/2 + 1/n$$

with corresponding masses,

$$a_1^{(n)} = 1, \quad a_2^{(n)} = -1 \quad \text{and} \quad a_3^{(n)} = 1.$$

Also, let $x_\infty : \mathcal{I}_1 \to \mathbf{R}$ to be purely atomic with one atom of mass 1 at $t = 1/2$. Note that $\{x_n : 3 \leq n \leq \infty\} \subseteq PA$.

For each $3 \leq n \leq \infty$, define $f_n : [0,1] \to \mathbf{R}$ by $f_n(t) = x_n([0,t]) \ \forall t$. On p.5 of [35], it is shown that

$$f_n \to f_\infty \quad \text{in } (D[0,1], J_2)$$

which, by Remark 2.4.2(c), implies $x_n \to x_\infty$ in $(D(\mathcal{I}_1), d_D)$. Therefore, $\Gamma_0 = \{x_n : 3 \leq n \leq \infty\}$ is a compact subspace of $(D(\mathcal{I}_1), d_D)$.

But, given any $\eta > 0$, if we select $n_0 \geq 3$ s.t. $\frac{1}{n_0} < \eta$, then

$$\text{gap}(x_{n_0}) \leq |t_1^{(n_0)} - t_2^{(n_0)}| = 1/n_0 < \eta.$$

Therefore, no $\eta > 0$ satisfies the condition

$$\text{gap}(y) \geq \eta, \quad (\forall y \in \Gamma_0),$$

implying that Γ_0 cannot be generated via Definition 2.6.2.

Chapter 3

Decomposition Theorems

3.1 Introduction

An important and well studied problem in classical probability theory concerns the decomposition of continuous parameter submartingales. In particular, given a continuous parameter submartingale $X = (X_t)_{t\geq 0}$ defined w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$, one seeks decompositions of the form $X = M + V$ where M is a martingale and V is a non-negative increasing process. The most celebrated such result is the Doob-Meyer decomposition theorem. In this theorem, it is shown that any cadlag submartingale X of class D can be uniquely decomposed into the above mentioned sum, $X = M + V$ so that V is non-negative, right-continuous, increasing and predictable. Recall that a cadlag submartingale X is of *class D* if $\{X_\tau : \tau \text{ a finite stopping time}\}$ is uniformly integrable, a condition equivalent to the uniform integrability of

$$\left\{\sum_{i=1}^{n} \left|E[X(t_{i-1}, t_i] \,|\, \mathcal{F}_{t_{i-1}}]\right| : n \in \mathrm{N},\ 0 \leq t_0 < \cdots < t_n\right\}.$$

Here, given any $0 \leq s < t$, $X(s,t] := X_t - X_s$. Analogous decompositions for planar strong submartingales have been obtained in [15].

The simplest and perhaps most insightful proof of the Doob-Meyer decomposition theorem involves discrete approximations. In particular, under the above mentioned conditions, Rao in [37] has shown that the predictable process V can be expressed as

$$\lim_n \sum_{i=1}^{n2^n} E\left[X(t_{i-1,n}, t_{i,n} \wedge t] \,\Big|\, \mathcal{F}_{t_{i-1,n}}\right] \;=\; V_t\,, \quad (\forall t \geq 0) \qquad (3.1)$$

where $t_{i,n} = \frac{i}{2^n}$ $(0 \leq i \leq n2^n)$ and the limit is taken w.r.t. the weak L_1 topology. If X has continuous sample paths, the mode of convergence in (3.1) can be upgraded to L_1-norm convergence.

When $X = (X_t)_{t \geq 0}$ is a cadlag martingale in L_2, i.e., $X_t \in L_2$ $\forall\, t \geq 0$, it is well known that X^2 is a submartingale of class D. Hence, the Doob-Meyer decomposition theorem applied to X^2 yields a non-negative, right-continuous, increasing, predictable process $\langle X \rangle$ s.t. $X^2 - \langle X \rangle$ is a martingale. $\langle X \rangle$ is commonly termed the *predictable quadratic variation* of X.

In this chapter, we duplicate the above development in the set-indexed setting. We begin in Section 3.2 by defining *set-indexed processes*, $X = (X_A)_{A \in \mathcal{A}}$ and *set-indexed filtrations*, $(\mathcal{F}_A)_{A \in \mathcal{A}}$. Here, $\mathcal{A}$ denotes a generic indexing collection. Adapted processes are then defined to be those for which X_A is $\mathcal{F}_A$-measurable $\forall\, A \in \mathcal{A}$. Of particular interest are the adapted *set-indexed strong submartingales* (see Definition 3.2.38) and the not necessarily adapted (L_1-) *right-continuous* processes (see Definition 3.2.44).

In Section 3.3, we define and study three important concepts, the first being *class* $(D')^*$ set-indexed processes which serve as analogues of class D processes. Secondly, we define the *admissible function* associated to a set-indexed strong submartingale. Finally, we define the *$*$-predictable σ-algebra*, $\mathcal{P}^*$. In Theorem 3.3.14, it is shown that the admissible function of any L_1-right-continuous class $(D')^*$ strong submartingale possesses a unique extension to a measure on $\mathcal{P}^*$.

In Section 3.4, after defining an appropriate set-indexed analogue of predictability termed *$*$-predictability*, a Doob-Meyer decomposition for L_1-right-continuous class $(D')^*$ strong submartingales is established (see Theorem 3.4.11). The key to this decomposition is the extention theorem for admissible functions mentioned in the preceding paragraph. Our proof is similar to that of Theorem 5.1 in [16] wherein a Doob-Meyer decomposition was obtained for set-indexed *weak submartingales*. As done in [16] and [37], we employ a discrete approximation approach. On the other hand, the $*$-predictable process $V = (V_A)_{A \in \mathcal{A}}$ in Theorem 3.4.11 is not necessarily adapted. [1] However, as demonstrated by Theorem 3.4.14, under a certain conditional independence assumption (see Assumption B.3.1), any L_1-right-continuous class $(D')^*$ strong martingale $X = (X_A)_{A \in \mathcal{A}}$ can be uniquely decomposed into a sum $X = M + V$ where $M = (M_A)_{A \in \mathcal{A}}$ is a strong martingale and $V = (V_A)_{A \in \mathcal{A}}$ is a non-negative, adapted, $*$-predictable, right-continuous,

[1] Unlike its classical counterpart, $*$-predictable processes are not necessarily adapted.

increasing process.

In contrast to the classical situation, the square of a set-indexed strong martingale in L_2 is not necessarily a strong submartingale (see Observation 3.5.1). Just the same, Section 3.5 presents a suitable set-indexed analogue of predictable quadratic variation termed *$*$-predictable quadratic variation* ($*$-PQV). In Theorem 3.5.2, conditions are given under which a set-indexed strong martingale in L_2 possesses a $*$-PQV. However, it is important to note that the $*$-PQV, say Q, of a set-indexed strong martingale X in L_2 does not necessarily "compensate" X^2. That is, $X^2 - Q$ is not necessarily a strong martingale. This will not be a problem for us and in fact this discrepancy is the same as that found in [18].

Section 3.6 contains two square function inequalities for set-indexed strong martingales in L_2. They resemble and are in fact based on the classic square function inequalities of Burkholder and Rosenthal. Several applications of these inequalities are given. Of particular interest is Corollary 3.6.11 which gives a simple moment condition under which a strong martingale possesses a $*$-PQV. The topic of Section 3.7 is yet another application of the square function inequalities of Section 3.6. Precisely, Theorem 3.7.7 gives sufficient conditions under which a $*$-PQV can be approximated in L_2-norm by discrete sums analogous to those found in (3.1).

3.2 Preliminaries

In this section, we present the technical machinery required for the definition of set-indexed strong submartingales and related processes. Many of the notions presented in this section have already appeared in either [23] or [26]. We will presuppose a knowledge of Section 2.2. As in Chapter 2, $\mathcal{A}$ denotes a generic indexing collection on a generic compact metric space (T, d).

3.2.1 Extensions of indexing collections

The following extensions of the family $\mathcal{A}$ will be encountered frequently in the sequel.

Definition 3.2.1 *Given a subcollection $\mathcal{A}'$ of $\mathcal{A}$, define*

(a) $\mathcal{A}'(u) = \{$ *all finite unions in* $\mathcal{A}'\}$,

(b) $\mathcal{C}' = \{A \setminus B : A \in \mathcal{A}' \text{ and } B \in \mathcal{A}'(u)\}$ *and*
(c) $\mathcal{C}'(u) = \{$ *all finite unions in* $\mathcal{C}'\}$.

Remark 3.2.2 (a) In particular, we can take $\mathcal{A}' = \mathcal{A}$ in which case we write $\mathcal{C}$ for $\mathcal{C}'$. The sets in $\mathcal{C}$ are intended to mimic left-open, right-closed intervals in $\mathbf{R}$. In fact, when $\mathcal{A} = \mathcal{I}_1$ (see Example 2.2.6), $\mathcal{C}$ coincides with the collection of all left-open, right-closed subintervals of $[0,1]$.

(b) Since ϕ can be viewed as a finite union in $\mathcal{A}$, i.e., the union of zero elements of $\mathcal{A}$, we have the inclusions $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A}(u) \subseteq \mathcal{C}(u)$.

Because of disagreements within the literature, it is important that we agree on the following terminology. A collection $\mathcal{S}$ of subsets of some fixed set S is said to be a *semi-algebra* (on S) provided

(i) $\phi, S \in \mathcal{S}$,

(ii) $\mathcal{S}$ is closed under finite intersections and

(iii) if $A \subseteq B$ in $\mathcal{S}$, then $B - A$ is a finite disjoint union of sets in $\mathcal{S}$.

Given a semi-algebra $\mathcal{S}$ on S,

$$\mathcal{S}(u) = \{\text{ all finite unions in } \mathcal{S}\}$$

is an algebra of sets on S and hence $\mathcal{S}(u)$ must coincide with the algebra generated by $\mathcal{S}$. Also note that if $\mathcal{S}$ is a semi-algebra, then every $U \in \mathcal{S}(u)$ can be expressed as a finite union of disjoint sets in $\mathcal{S}$. These basic facts on semi-algebras will be used frequently in this chapter.

The proof of the following result is trivial.

Lemma 3.2.3 *Let $\mathcal{A}'$ be a subcollection of $\mathcal{A}$. If $\mathcal{A}'$ is closed under finite intersection and contains T, then $\mathcal{C}'$ (see Definition 3.2.1*(b)*) is a semi-algebra on T.*

Remark 3.2.4 In particular, the family $\mathcal{C} = \{A \setminus B : A \in \mathcal{A} \text{ and } B \in \mathcal{A}(u)\}$ is a semi-algebra on T.

Since it is possible to have $\bigcup_{i=1}^{k_1} A_i = \bigcup_{j=1}^{k_2} A'_j$ for distinct subcollections $\{A_1, \cdots, A_{k_1}\}$ and $\{A'_1, \cdots, A'_{k_2}\}$ of $\mathcal{A}$, a set $B \in \mathcal{A}(u)$ (or $C \in \mathcal{C}$) may have several distinct representations, not all of which are useful. We present three important types of representations below.

Definition 3.2.5 *Given $B \in \mathcal{A}(u)$, a subcollection $\{A_1, A_2, \cdots, A_k\}$ of $\mathcal{A}$ is an* extremal representation *of B if $B = \bigcup_{i=1}^{k} A_i$ and, given any $1 \leq i, j \leq k$,*

$$i \neq j \implies A_i \not\subseteq A_j.$$

In such a case, $\bigcup_{i=1}^{k} A_i$ is also termed an extremal representation.

In other words, an extremal representation is one which does not contain any "redundant" sets. By a trivial induction, any $B \in \mathcal{A}(u)$ possesses an extremal representation.

Definition 3.2.6 *Take $A, A_1, \cdots, A_n \in \mathcal{A}$, $B \in \mathcal{A}(u)$ and let $C = A \setminus B$.*

(a) *$A \setminus \bigcup_{i=1}^{n} A_i$ is a* minimal representation *of C if $\bigcup_{i=1}^{n} A_i$ is an extremal representation of B and $A_i \subseteq A$ for each $1 \leq i \leq m$.*

(b) *$A \setminus \bigcup_{i=1}^{n} A_i$ is a* maximal representation *of C if $\bigcup_{i=1}^{n} A_i$ is an extremal representation of B and, given any $A' \in \mathcal{A}$,*

$$A' \cap C = \phi \implies A' \subseteq \bigcup_{i=1}^{n} A_i.$$

Since $\mathcal{A}$ is closed under finite intersection, it is clear that each $C \in \mathcal{C}$ possesses a minimal representation. For the examples we have in mind, every $C \in \mathcal{C}$ also possesses a maximal representation. Just the same, we only assume the existence of maximal representation when they are needed.

The remainder of this subsection deals with subcollections $\mathcal{A}'$ of $\mathcal{A}$ which are both finite and closed under intersection. Of course, $\mathcal{A}$ itself is closed under intersection and hence forms a semilattice under $\wedge = \cap$. For this reason, any such $\mathcal{A}'$ can be viewed as a *finite sub-semilattice* of $(\mathcal{A}, \cap)$ which we shall abbreviate throughout by f.s.s.l. **No generality is lost if we consider only those f.s.s.l. which contain both T and ϕ'.**

Definition 3.2.7 *Let $\mathcal{A}'$ be a f.s.s.l. of $\mathcal{A}$. Given $A \in \mathcal{A}'$, the* left-neighborhood *of A in $\mathcal{A}'$ is the set $C_A \in \mathcal{C}'$ defined by*

$$C_A := A \setminus \bigcup_{A' \in \mathcal{A}', A \not\subseteq A'} A'. \tag{3.2}$$

We also define $\mathcal{N}' = \{C_A : A \in \mathcal{A}'\} \setminus \{\phi\}$, the collection of all non-empty left-neighborhoods generated by $\mathcal{A}'$.

Remark 3.2.8 (a) Since $\mathcal{A}'$ is closed under finite intersections, we can clearly write $C_A = A \setminus \bigcup_{A' \in \mathcal{A}', A' \subset A} A'$ for any $A \in \mathcal{A}'$.

(b) Since $\phi' \subseteq A \ \forall A \in \mathcal{A}$, the union $\bigcup_{A' \in \mathcal{A}', \phi' \not\subseteq A'} A'$ is empty. Therefore, $C_{\phi'} = \phi'$ in any f.s.s.l. $\mathcal{A}'$.

(c) Recall the sequence $(\mathcal{A}_n)_n$ of f.s.s.l. of $\mathcal{A}$ given in Definition 2.2.2 (1). When $\mathcal{A}' = \mathcal{A}_n$ ($n \in \mathrm{N}$), we denote $\mathcal{N}'$ by $\mathcal{N}_n$. (In the earlier set-indexed literature, this collection of left-neighborhoods was denoted by $\mathcal{C}_n$.)

Given a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$, the left-neighborhoods are, by definition, the smallest sets in $\mathcal{C}'$ w.r.t. $\subseteq$. This is illustrated through the following example.

Example 3.2.9 Let $\mathcal{A} = \mathcal{I}_2$ on $T = [0,1]^2$ and take $\mathcal{A}' = \mathcal{I}_2^{(n)}$ ($n \in \mathrm{N}$) (see Example 2.2.6). Then, given $A = [0, \frac{i}{2^n}] \times [0, \frac{j}{2^n}] \in \mathcal{I}_2^{(n)}$ $(1 \le i, j \le 2^n)$,

$$C_A = \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right] \times \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right].$$

If $i = 0$ and $j > 0$, then $C_A = \{0\} \times (\frac{j-1}{2^n}, \frac{j}{2^n}]$ whereas $C_A = (\frac{i-1}{2^n}, \frac{i}{2^n}] \times \{0\}$ when $j = 0$ and $i > 0$. When $i = j = 0$, $A = \{(0,0)\} = \phi'$ and thus, $C_A = \{(0,0)\}$.

As demonstrated in the above example, the collection of all left-neighborhoods in $\mathcal{I}_2^{(n)}$ (any fixed n) form a partition of $[0,1]^2$. This useful fact is true in general.

Lemma 3.2.10 *If $\mathcal{A}'$ is a f.s.s.l. of $\mathcal{A}$, then, given $A_1, A_2 \in \mathcal{A}'$,*

$$A_1 \neq A_2 \implies C_{A_1} \cap C_{A_2} = \phi. \tag{3.3}$$

Furthermore, given any $A \in \mathcal{A}'$, $\bigcup_{C \in \mathcal{N}', C \subseteq A} C = A$.

Proof If $A_1 \neq A_2$ in $\mathcal{A}'$, then w.l.o.g., $A_2 \not\subseteq A_1$. By (3.2), this implies $C_{A_1} \cap C_{A_2} \subseteq A_1 \cap C_{A_2} = \phi$ and (3.3) is established.

Now, take $A \in \mathcal{A}'$. Clearly, $\bigcup_{C \in \mathcal{N}', C \subseteq A} C \subseteq A$. To show the opposite inclusion, take any $t \in A$ and define the set

$$A(t) = \bigcap \{A \in \mathcal{A}' : t \in A\}.$$

Since $\mathcal{A}'$ is a f.s.s.l., $A(t) \in \mathcal{A}'$ and $C_{A(t)} \subseteq A(t) \subseteq A$. If $t \notin C_{A(t)}$, then by (3.2), $\exists B \in \mathcal{A}'$ s.t. $A(t) \not\subseteq B$ and $t \in B$. But this contradicts the definition of $A(t)$. Therefore, $t \in C_{A(t)} \subseteq \bigcup_{C \in \mathcal{N}', C \subseteq A} C$, completing the proof. □

If $\mathcal{A}'$ is a f.s.s.l. of $\mathcal{A}$, we refer to any listing $\{A_1, A_2, \cdots, A_k\}$ of the elements of $\mathcal{A}'$ as a *numbering* of $\mathcal{A}'$. In the following definition, we present a useful numbering of a f.s.s.l. Both the name and purpose of this type of numbering is explained in Lemma 3.2.26.

Definition 3.2.11 *Let $\mathcal{A}'$ be a f.s.s.l. of $\mathcal{A}$. A numbering $\{A_1, A_2, \cdots, A_k\}$ of $\mathcal{A}'$ is said to be consistent with the strong past (c.w.s.p.) if $A_1 = \phi'$ and, given any $2 \leq i \leq k$ and $A' \in \mathcal{A}'$,*

$$A' \subset A_i \implies \exists 1 \leq j \leq i-1 \text{ s.t. } A' = A_j. \tag{3.4}$$

In fact, any f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ can be numbered in a manner c.w.s.p. by applying the following algorithm. To avoid trivialities, we assume $\mathcal{A} \neq \{T\}$.

- First, let $A_1 = \phi'$.
- Next, select $A_2 \in \mathcal{A}' \setminus \{A_1\}$ s.t. $A' \not\subset A_2 \ \forall A' \in \mathcal{A}' \setminus \{A_1\}$. (Although such an A_2 exists, it may not be unique.)
- If $\mathcal{A}' \setminus \{A_1, A_2\} = \phi$, we are done. Otherwise, select A_3 to be any element in $\mathcal{A}' \setminus \{A_1, A_2\}$ s.t. $A' \not\subset A_3 \ \forall A' \in \mathcal{A}' \setminus \{A_1, A_2\}$.
- Continue the above selection process until $\mathcal{A}' \setminus \{A_1, A_2, \cdots, A_k\} = \phi$ for some $k \in \mathbb{N}$. It is clear that $A_k = T$.

By a simple induction, one can show

$$A' \not\subset A_i, \quad (\forall A' \in \mathcal{A}' \setminus \{A_1, \cdots, A_{i-1}\})$$

for any $2 \leq i \leq k$, i.e., the numbering $\{A_1, \cdots, A_k\}$ of $\mathcal{A}'$ is c.w.s.p.

The following application illustrates the usefulness of numberings that are c.w.s.p. The simple representations for left-neighborhoods obtained via this result will be of great importance in Section 3.6.

Lemma 3.2.12 *If $\mathcal{A}' = \{A_1, A_2, \cdots, A_k\}$ is a f.s.s.l. of $\mathcal{A}$ numbered in a manner c.w.s.p. then, given any $2 \leq i \leq k$, $C_{A_i} = A_i \setminus \bigcup_{j=1}^{i-1} A_j$.*

Proof In view of (3.2) and Remark 3.2.8 (a), it is sufficient to show

$$\{A' \in \mathcal{A}' : A' \subset A_i\} \subseteq \{A_1, \cdots, A_{i-1}\} \subseteq \{A' \in \mathcal{A}' : A_i \not\subseteq A'\}. \tag{3.5}$$

The left-most inclusion in (3.5) follows automatically from condition (3.4).

To establish the right-most inclusion, take $1 \leq l \leq i-1$. (We need to show that $A' = \mathcal{A}_l$ satisfies $\mathcal{A}_i \not\subseteq A_l$.) If $A_i \subseteq A_l$, then since $i \neq l$, it follows that $A_i \subset A_l$. But by condition (3.4), this strict inclusion implies $i \leq l-1$ which is clearly impossible. Therefore, it must be that $A_i \not\subseteq A_l$, completing the proof. □

We close this subsection with a basic construction and application.

Definition 3.2.13 *Given a finite subcollection $\mathcal{A}_0$ of $\mathcal{A}$, the f.s.s.l. generated by $\mathcal{A}_0$ is the subcollection $\mathcal{A}' = \{$ all finite intersections in $\mathcal{A}_0 \} \cup \{\phi', T\}$.*

The promised application concerns special subcollections of $\mathcal{C}$.

Definition 3.2.14 *A collection $\mathcal{N}_0 = \{D_1, \cdots, D_n\}$ of disjoint sets in $\mathcal{C}$ is said to be a finite left-neighborhood subcollection (f.n.s.) of $\mathcal{A}$ if $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ s.t. $\mathcal{N}_0 \subseteq \mathcal{N}'$.*

In other words, the elements of an f.n.s. are left-neighborhoods, all of which are generated by a common f.s.s.l. The following result will be used frequently in the sequel. In essence, it states that f.n.s. are universal among finite disjoint unions in $\mathcal{C}$.

Lemma 3.2.15 *Given any $C \in \mathcal{C}(u)$, $\exists$ a f.n.s. $\mathcal{N}_0$ of $\mathcal{A}$ s.t. $C = \bigcup_{D \in \mathcal{N}_0} D$.*

Proof Assume $C = \bigcup_{j=1}^{m} C_j$ where $C_j = A^{(j)} \setminus \bigcup_{l=1}^{k(j)} A_l^{(j)} \in \mathcal{C}\ \forall 1 \leq j \leq m$. If we define $\mathcal{A}'$ to be the f.s.s.l. of $\mathcal{A}$ generated by the collection

$$\mathcal{A}_0 = \{A^{(j)}, A_l^{(j)} : 1 \leq l \leq k(j),\ 1 \leq j \leq m\},$$

then $C_j \in \mathcal{C}'\ \forall 1 \leq j \leq m$. The present lemma now follows by applying Lemma A.3.3 to each $C_j \in \mathcal{C}'$. □

The reader is referred to Section A.3 for additional results on f.s.s.l.

3.2.2 Set-indexed filtrations and their extensions

Fix a complete probability space $(\Omega, \mathcal{F}, P)$ for use throughout this subsection. The following definition has already appeared in [16].

Definition 3.2.16 *A family $(\mathcal{F}_A)_{A\in\mathcal{A}}$ of sub-σ-algebras of $\mathcal{F}$ is said to be an* <u>*$\mathcal{A}$-indexed filtration*</u> *on $(\Omega, \mathcal{F}, P)$ provided*

(i) *for each $A \in \mathcal{A}$, $\mathcal{F}_A$ contains all P-null subsets of $\mathcal{F}$,*

(ii) *if $A, B \in \mathcal{A}$ are s.t. $A \subseteq B$, then $\mathcal{F}_A \subseteq \mathcal{F}_B$ and*

(iii) *if $(A_n)_n$ decreasing in $\mathcal{A}$, then $\bigcap_n \mathcal{F}_{A_n} = \mathcal{F}_{\bigcap_n A_n}$.*

We refer to any such tuple, $(\Omega, \mathcal{F}, P, (\mathcal{F}_A)_A, \mathcal{A})$ as a <u>*stochastic base*</u>.

Remark 3.2.17 (a) In other words, an $\mathcal{A}$-indexed filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ is an increasing family of P-complete sub-σ-algebras of $\mathcal{F}$ which is right-continuous w.r.t. decreasing intersections in $\mathcal{A}$.

(b) Note that Definition 3.2.16 is entirely classic. See p.83 in [30] for the definition of filtrations indexed by the points of $\mathbf{R}^+$.

In practice, one frequently encounters families of sub-σ-algebras which are complete and increasing but not necessarily right-continuous in the sense of Definition 3.2.16 (iii). Through the following result, we can always extend such a family in a minimal way so as to produce a filtration.

Proposition 3.2.18 *If $(\mathcal{H}_A)_{A\in\mathcal{A}}$ is an increasing family of P-complete sub-σ-algebras of $\mathcal{F}$, then the family $(\mathcal{H}_A^+)_{A\in\mathcal{A}}$ where*

$$\mathcal{H}_A^+ := \bigcap_k \mathcal{H}_{g_k(A)}\,, \quad (\forall\, A \in \mathcal{A}) \tag{3.6}$$

is a filtration on $(\Omega, \mathcal{F}, P)$.

Proof From (3.6), it is clear that $(\mathcal{H}_A^+)_{A\in\mathcal{A}}$ is an increasing family of P-complete sub-σ-algebras of $\mathcal{F}$. Moreover, given any $A \in \mathcal{A}$ and any sequence $(A_n)_n$ in $\mathcal{A}$ s.t. $A_n \downarrow A$,

$$\begin{aligned} \bigcap_n \mathcal{H}_{A_n}^+ &= \bigcap_n \bigcap_k \mathcal{H}_{g_k(A_n)} \quad \text{(by (3.6))} \\ &= \bigcap_k \bigcap_n \mathcal{H}_{g_k(A_n)} \\ &= \bigcap_k \mathcal{H}_{g_k(A)} \quad \text{(by Lemma A.2.5)} \\ &= \mathcal{H}_A^+ \quad \text{(by (3.6))} \end{aligned} \tag{3.7}$$

which establishes the right-continuity of $(\mathcal{H}_A^+)_{A\in\mathcal{A}}$. □

Remark 3.2.19 From array (3.7), it is clear that $\mathcal{H}_A^+ = \bigcap_n \mathcal{H}_{A_n}^+$ whenever $A_n \searrow A$ in $\mathcal{A}$. In this sense, the filtration $(\mathcal{H}_A^+)_{A\in\mathcal{A}}$ is also "outer-continuous" (see Definition 2.3.1).

Take a filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ on $(\Omega, \mathcal{F}, P)$ and a set $B = \bigcup_{i=1}^n A_i \in \mathcal{A}(u)$ s.t. $A_i \in \mathcal{A}\ \forall i$. As was done in [22] and [23], given any $B \in \mathcal{A}(u)$, we associate to B the sub-σ-algebra

$$\mathcal{F}_B^\circ = \bigvee_{\substack{A\in\mathcal{A}\\ A\subseteq B}} \mathcal{F}_A \tag{3.8}$$

and the possibly larger sub-σ-algebra,

$$\mathcal{F}_B = \bigcap_k \mathcal{F}_{g_k(B)}^\circ . \tag{3.9}$$

Since $B \in \mathcal{A}(u)$, $g_k(B)$ is defined via (2.4).

Remark 3.2.20 Given any $A \in \mathcal{A}$, note that $\mathcal{F}_{g_k(A)} = \mathcal{F}_{g_k(A)}^\circ\ \forall k$ which by Definition 2.2.2 (2)(i') and the right-continuity of our filtration implies $\mathcal{F}_A = \cap_n \mathcal{F}_{g_k(A)}^\circ$. For this reason, there is no danger of ambiguity in writing $\mathcal{F}_B$ for any $B \in \mathcal{A}(u)$.

Let $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ be a classical filtration on $(\Omega, \mathcal{F}, P)$ as defined in [30]. Since $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$ is increasing and $(\mathbf{R}^+, \leq)$ is linearly ordered, it is natural to define "the past" or "history" at any point $t \in \mathbf{R}^+$ to be the sub-σ-algebra $\mathcal{F}_t$. On the other hand, given a filtration $(\mathcal{F}_z)_{z\in(\mathbf{R}^+)^2}$, the situation becomes more complicated. Since $(\mathbf{R}^+)^2$ is not linearly ordered w.r.t. the co-ordinate-wise partial order relation, there is no unique, natural way to define the past at a given point $z \in (\mathbf{R}^+)^2$. For this reason, earlier authors defined two distinct pasts at z, the *weak past* and the *strong past* which lead to the definitions of weak and strong planar submartingales respectively. For example, see [10].

The same is true for the set-indexed situation. Taking their cue from the existing multiparameter theory, Dozzi et. al. in [16] presented set-indexed analogues of the weak and strong past. In this thesis we will only work with the strong past. It is defined below.

Definition 3.2.21 *Let $(\mathcal{F}_A)_{A\in\mathcal{A}}$ be a filtration on $(\Omega, \mathcal{F}, P)$. Given $C \in \mathcal{C}(u)$, the strong past, $\mathcal{G}_C^*$ at C is given by*

$$\mathcal{G}_C^* = \bigvee_{\substack{B\in\mathcal{A}(u)\\ B\cap C=\phi}} \mathcal{F}_B \tag{3.10}$$

whenever $C \notin \mathcal{A}(u)$ and $\mathcal{G}_C^* = \mathcal{F}_{\phi'}$ otherwise. (Note that $\mathcal{G}_\phi^* = \bigvee_{B\in\mathcal{A}(u)} \mathcal{F}_B$.)

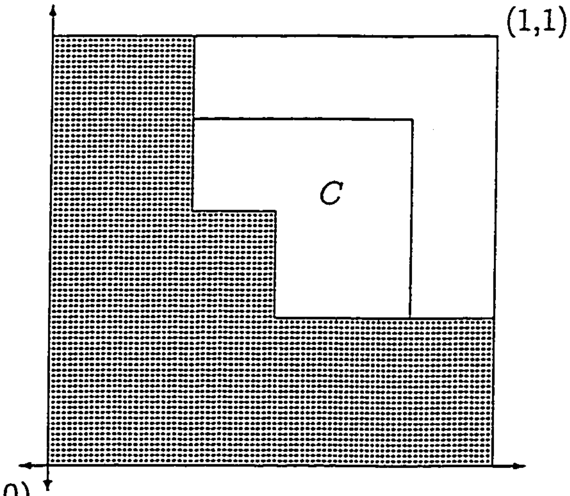

Figure 3.1: The strong past at $C \in \mathcal{C}$ when $\mathcal{A} = \mathcal{I}_2$.

Example 3.2.22 Let $\mathcal{A} = \mathcal{I}_1$ and take an $\mathcal{I}_1$-indexed filtration $(\mathcal{F}_a)_{a \in [0,1]}$. (Here, we are identifying the point $a \in [0,1]$ with the set $[0,a] \in \mathcal{I}_1$.) Given any non-empty set $C \in \mathcal{C}$, $\exists\, 0 \leq a < b \leq 1$ s.t.

$$C = [0,b] \setminus [0,a] = (a,b]$$

in which case it is clear that $\mathcal{G}_C^* = \mathcal{F}_a$.

More instructive is the case of $\mathcal{A} = \mathcal{I}_2$ on $T = [0,1]^2$. Given an $\mathcal{I}_2$ indexed filtration $(\mathcal{F}_A)_{A \in \mathcal{I}_2}$ and a set $C \in \mathcal{C}$, the strong past $\mathcal{G}_C^*$ at C is generated by the events in all σ-algebras $\mathcal{F}_B$ ($B \in \mathcal{A}(u)$) which satisfy $B \cap C = \phi$. In this sense, the shaded region in Figure 3.1 represents the strong past at C.

Remark 3.2.23 (a) See Proposition 3.2.25 for conditions under which $\mathcal{G}_C^*$ takes on a simple form.

(b) As mentioned above, there is another type of past in the set-indexed theory, namely the *weak past.* Given any $C \in \mathcal{C}$, it is defined to be the sub-σ-algebra $\mathcal{G}_C = \bigcap_{A \in \mathcal{A},\, A \cap C \neq \phi} \mathcal{F}_A$. It has been studied in [16] and [28].

We now record some important although obvious properties for the above families of sub-σ-algebras. In particular, note that $(\mathcal{F}_B^\circ)_{B \in \mathcal{A}(u)}$ and $(\mathcal{F}_B)_{B \in \mathcal{A}(u)}$ are increasing on $\mathcal{A}(u)$ while $(\mathcal{G}_C^*)_{C \in \mathcal{C}(u)}$ is decreasing on $\mathcal{C}(u)$.

Lemma 3.2.24 *Let $(\mathcal{F}_A)_{A \in \mathcal{A}}$ be a filtration on $(\Omega, \mathcal{F}, P)$.*

(a) *If* $B_1, B_2 \in \mathcal{A}(u)$ *are s.t.* $B_1 \subseteq B_2$, *then* $\mathcal{F}^\circ_{B_1} \subseteq \mathcal{F}^\circ_{B_2}$ *and* $\mathcal{F}_{B_1} \subseteq \mathcal{F}_{B_2}$.

(b) $\mathcal{F}^\circ_B \subseteq \mathcal{F}_B \ \forall B \in \mathcal{A}(u)$ *and* $\mathcal{F}^\circ_A = \mathcal{F}_A \ \forall A \in \mathcal{A}$.

(c) *If* $A_1, \cdots, A_n \in \mathcal{A}$, *then* $\bigvee_{i=1}^n \mathcal{F}_{A_i} \subseteq \mathcal{F}^\circ_{\bigcup_{i=1}^n A_i}$.

(d) *If* $C_1, C_2 \in \mathcal{C}(u)$ *are s.t.* $C_1 \subseteq C_2$, *then* $\mathcal{G}^*_{C_2} \subseteq \mathcal{G}^*_{C_1}$.

(e) *If* $B \in \mathcal{A}(u)$ *is s.t.* $B \cap C = \phi$, *then* $\mathcal{F}_B \subseteq \mathcal{G}^*_C$.

(f) $\mathcal{F}^\circ_B$, $\mathcal{F}_B$ *and* $\mathcal{G}^*_C$ *are P-complete* $\forall B \in \mathcal{A}(u)$ *and* $C \in \mathcal{C}(u)$.

When maximal representations of sets in $\mathcal{C}$ exist, our next result simplifies the expression in (3.10).

Proposition 3.2.25 *If* $A \setminus B$ $(A \in \mathcal{A}, B \in \mathcal{A}(u))$ *is a maximal representation of a set* $C \in \mathcal{C}$ *(see Definition 3.2.6* (b)*), then* $\mathcal{G}^*_C = \mathcal{F}_B = \bigcap_k \mathcal{F}^\circ_{g_k(B)}$.

Proof By Lemma 3.2.24 (e), $\mathcal{F}_B \subseteq \mathcal{G}^*_C$. For the opposite inclusion, take any $B' \in \mathcal{A}(u)$ s.t. $B' \cap C = \phi$. Since $A \setminus B$ is a maximal representation of C, $B' \subseteq B$ and thus by Lemma 3.2.24 (a), $\mathcal{F}_{B'} \subseteq \mathcal{F}_B$. Since this B' was arbitrarily chosen, (3.10) implies $\mathcal{G}^*_C \subseteq \mathcal{F}_B$. □

We end this subsection with two results concerning the strong past. The first result explains our use of the term "consistent with the strong past" in Definition 3.2.11. Its proof follows automatically from Lemma 3.2.12 and Lemma 3.2.24 (e).

Lemma 3.2.26 *If* $\mathcal{A}' = \{A_1, \cdots, A_k\}$ *is a f.s.s.l. of* $\mathcal{A}$ *numbered in a manner c.w.s.p., then given any* $2 \leq i \leq k$,

$$\mathcal{F}_{A_j} \subseteq \mathcal{G}^*_{C_{A_i}}, \quad (\forall 1 \leq j \leq i-1)$$

where $C_{A_i} \in \mathcal{N}'$ *denotes the left-neighborhood of* A_i *in* $\mathcal{A}'$. *(In other words, the strong past at* C_{A_i} *contains all events in* $\mathcal{F}_{A_1}, \cdots, \mathcal{F}_{A_{i-2}}$ *and* $\mathcal{F}_{A_{i-1}}$.*)*

The second result is a consequence of the first.

Lemma 3.2.27 *Given a f.s.s.l.* $\mathcal{A}'$ *of* $\mathcal{A}$, *if* $A, A' \in \mathcal{A}'$ *are s.t.* $A \neq A'$, *then*

$$\mathcal{F}_A \subseteq \mathcal{G}^*_{C'} \quad \text{or} \quad \mathcal{F}_{A'} \subseteq \mathcal{G}^*_C \tag{3.11}$$

where $C, C' \in \mathcal{N}'$ *denote the left-neighborhoods of* A *and* A' *respectively.*

Proof Take $A, A' \in \mathcal{A}'$. By the comment following Definition 3.2.11, $\mathcal{A}'$ admits a numbering $\{A_1, \cdots, A_k\}$ which is c.w.s.p. Let $1 \leq i, j \leq k$ be s.t. $A = A_i$ and $A' = A_j$. If $A \neq A'$, then either $i < j$ or $j < i$. (3.11) now follows by applying Lemma 3.2.26 to either case. □

Remark 3.2.28 Given disjoint sets $C = A \setminus B$ and $C' = A' \setminus B'$ in $\mathcal{C}$ with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{A}(u)$, if $\mathcal{A}$ satisfies the shape property, then (3.11) remains valid (see Lemma 1.2 in [23]).

3.2.3 Set-indexed stochastic processes

Fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$. In this subsection, we formally define set-indexed stochastic processes. Afterwords, we will develop and discuss several concepts related to set-indexed processes.

Definition 3.2.29 *Any collection $X = (X_A)_{A\in\mathcal{A}}$ of random variables on $(\Omega, \mathcal{F}, P)$ is referred to as a* <u>*set-indexed process*</u>*. If in addition,*

$$X_A \text{ is } \mathcal{F}_A\text{-measurable}, \quad (\forall\, A \in \mathcal{A}),$$

the process X is said to be <u>*adapted*</u> *(to $(\mathcal{F}_A)_{A\in\mathcal{A}}$).*

We will sometimes write $X(A)$ instead of X_A when the set $A \in \mathcal{A}$ is notationally complicated or X has additional subscripts.

Given any process $X = (X_A)_{A\in\mathcal{A}}$ on $(\Omega, \mathcal{F}, P)$, there is always a non-trivial filtration $(\mathcal{F}_A^X)_{A\in\mathcal{A}}$ on $(\Omega, \mathcal{F}, P)$ to which X is adapted. First, we associate the sub-σ-algebra

$$\mathcal{H}_A = \sigma(\{X_{A'} : A' \in \mathcal{A} \text{ and } A' \subseteq A\}) \vee \mathcal{F}_0$$

to each $A \in \mathcal{A}$ where $\mathcal{F}_0$ denotes the sub-σ-algebra of $\mathcal{F}$ generated by all P-null subsets of Ω. Clearly, $(\mathcal{H}_A)_{A\in\mathcal{A}}$ is an increasing family of P-complete sub-σ-algebras of $\mathcal{F}$. Therefore, if we define

$$\mathcal{F}_A^X = \bigcap_k \mathcal{H}_{g_k(A)}, \quad (\forall\, A \in \mathcal{A}),$$

then by Lemma 3.2.18, $(\mathcal{F}_A^X)_{A\in\mathcal{A}}$ is a filtration on $(\Omega, \mathcal{F}, P)$. Clearly, X is adapted to $(\mathcal{F}_A^X)_{A\in\mathcal{A}}$. Moreover, if $(\mathcal{F}_A)_{A\in\mathcal{A}}$ is any filtration to which X is adapted, then

$$\mathcal{F}_A^X \subseteq \mathcal{F}_A, \quad (\forall\, A \in \mathcal{A}).$$

For this reason, we call $(\mathcal{F}_A^X)_{A\in\mathcal{A}}$ the *minimal filtration* generated by X.

Any $\mathcal{A}$-indexed process X can be viewed as a "random function" denoted $\hat{X} : \Omega \to \mathbf{R}^{\mathcal{A}}$ where $\mathbf{R}^{\mathcal{A}} = \{x : \mathcal{A} \to \mathbf{R}\}$ and

$$[\hat{X}(\omega)](A) := X_A(\omega), \quad (\forall A \in \mathcal{A}).$$

We refer to $\hat{X}(\omega)$ as the *sample path* of X at ω. When $\hat{X}(\omega) \in D(\mathcal{A})$ $\forall \omega \in \Omega$, $\hat{X}$ is in fact measurable w.r.t. the Borel σ-algebra on $D(\mathcal{A})$ generated by the metric d_D defined on p.29 (see Proposition 4.3.6). Such measurablility properties for set-indexed processes will not be required in this chapter.

The following definition is entirely classic.

Definition 3.2.30 *Let $X = (X_A)_{A\in\mathcal{A}}$ and $Y = (Y_A)_{A\in\mathcal{A}}$ be processes.*

(a) *X and Y are indistinguishable if $\hat{X}(\omega) = \hat{Y}(\omega)$ for P-a.e. $\omega \in \Omega$.*

(b) *X is a modification of Y if for every $A \in \mathcal{A}$, $X_A = Y_A$ a.s.*

Remark 3.2.31 If X and Y are indistinguishable set-indexed processes, then it is clear that X is a modification of Y. See Lemma 3.2.45 for conditions on X and Y under which the converse is true.

Given a planar process $X = \{X(z) : z \in (\mathbf{R}^+)^2\}$ and two points $z = (s,t)$ and $z' = (s',t')$ in $(\mathbf{R}^+)^2$ with $s \le s'$ and $t \le t'$, one defines the *increment* of X by the formula

$$X(z,z'] = X(s',t') - X(s',t) - X(s,t') + X(s,t).$$

By way of set-indexed analogue,

Definition 3.2.32 *Let $X = (X_A)_{A\in\mathcal{A}}$ be a process. Given a set $C \in \mathcal{C}$, a random variable X_C is said to be the increment of X at C if*

$$X_C = \sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot X(A \cap \textstyle\bigcap_{i\in I} A_i) \tag{3.12}$$

for any representation $C = A \setminus \bigcup_{i=1}^k A_i$ with $A, A_1, \cdots A_k \in \mathcal{A}$ and $k \in \mathbf{N}$.

Since a set $C \in \mathcal{C}$ possesses many different representations, the quantity on the right-hand side of (3.12) may not be unique. That is, a general set-indexed process does not necessarily have increments defined at each $C \in \mathcal{C}$. However, (3.12) is none other than the inclusion-exclusion formula. For this reason, the following result is trivial. Recall that $\mathcal{C}$ is a semi-algebra on T and hence, $\mathcal{C}(u)$ is an algebra on T.

Proposition 3.2.33 *Let $X = (X_A)_{A \in \mathcal{A}}$ be a process. If X possesses a unique finitely additive extension to a $\mathcal{C}(u)$-indexed process, then X has an increment defined at every $C \in \mathcal{C}$.*

Comment 3.2.34 If $X = (X_A)_{A \in \mathcal{A}}$ possesses a unique finitely additive extension to a $\mathcal{C}(u)$-indexed process, then since $\phi = T \setminus T$, $X_\phi = X_T - X_T = 0$. Therefore, if we adjoin ϕ to $\mathcal{A}$ via Remark 2.2.5 (c), $X_\phi = 0$ is a necessary condition for the existence of such an extension.

Next, we explore conditions under which an $\mathcal{A}$-indexed process possesses a unique finitely additive extension to a $\mathcal{C}(u)$-indexed processes. Our first result, which has already appeared as Proposition 2.1 in [16], gives a condition on the indexing collection itself.

Proposition 3.2.35 *If $\mathcal{A}$ satisfies the shape property as defined in Remark 2.2.5* (d)*, then any $\mathcal{A}$-indexed process X possesses a unique finitely additive extension to a $\mathcal{C}(u)$-indexed process. (See Comment 3.2.34.)*

By Proposition 3.2.35, any process indexed by $\mathcal{I}_k$ ($k \in \mathrm{N}$) will possess a finitely additive extension to $\mathcal{C}(u)$. Even if $\mathcal{A}$ does not possess the shape property (e.g.: $\mathcal{A} =$ the lower layers, $\mathcal{LL}_k$ on $[0,1]^k$ for any $k \geq 2$), an $\mathcal{A}$-indexed process may still possess a finitely additive exension to $\mathcal{C}(u)$. For example,

Proposition 3.2.36 *Let $\mathcal{A}$ be any indexing collection. If $X = (X_A)_{A \in \mathcal{A}}$ is a process with purely atomic sample paths (see Definition 2.3.5), then X possesses a unique finitely additive extension to $\mathcal{C}(u)$.*

Proof It is sufficient to establish this result for the deterministic case. First, assume $x : \mathcal{A} \to [0, \infty)$ is purely atomic with one atom of mass 1 located at some $t \in T$. Clearly, such an x can be uniquely defined at any $S \subseteq T$. Since x generates a point-mass measure on $\sigma(\mathcal{A})$, $x : \mathcal{C}(u) \to [0, \infty)$ is finitely additive.

For the general case, if x is purely atomic with atoms $t_1, \cdots, t_k \in T$ having respective masses $a_1, \cdots, a_k \in \mathrm{R}$, define x_i to be purely atomic with one atom of mass a_i at t_i $\forall\, 1 \leq i \leq k$. Since $x = \sum_{i=1}^{k} a_i \cdot x_i$, the present case follows from the previous case. □

Given an adapted process $X = (X_A)_{A\in\mathcal{A}}$ with increments defined at every $C \in \mathcal{C}$, it is important to note that the resulting family $(X_C)_{C\in\mathcal{C}}$ is not necessarily "adapted" to the family $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$, i.e., X_C is not necessarily $\mathcal{G}^*_C$-measurable $\forall\, C \in \mathcal{C}$. However, if $C = A\setminus B \in \mathcal{C}$ ($A \in \mathcal{A}$, $B \in \mathcal{A}(u)$), then by (3.12), X_C is $\mathcal{F}_A$-measurable. This observation leads to the following result.

Lemma 3.2.37 *Let $X = (X_A)_{A\in\mathcal{A}}$ be an adapted process with increments defined at every $C \in \mathcal{C}$. If $\mathcal{A}'$ is a f.s.s.l. of $\mathcal{A}$ and $C_1, C_2 \in \mathcal{N}'$ are distinct (hence disjoint) left-neighborhoods, then X_{C_1} is $\mathcal{G}^*_{C_2}$-measurable or X_{C_2} is $\mathcal{G}^*_{C_1}$-measurable.*

Proof If C_i is the left-neighborhood of $A_i \in \mathcal{A}'$ ($i = 1, 2$), then by the above observation, X_{C_i} is $\mathcal{F}_{A_i}$-measurable ($i = 1, 2$). But $C_1 \neq C_2$ implies $A_1 \neq A_2$. Therefore, by Lemma 3.2.27, $\mathcal{F}_{A_1} \subseteq \mathcal{G}^*_C$ or $\mathcal{F}_{A_2} \subseteq \mathcal{G}^*_C$, completing the proof of the present lemma. □

We close this subsection by mentioning a basic operation on set-indexed processes. Given $1 \leq p < \infty$ and a process $X = (X_A)_{A\in\mathcal{A}}$, we define $X^p = (X^p_A)_{A\in\mathcal{A}}$ to be the process with

$$X^p_A = (X_A)^p, \quad (\forall\, A \in \mathcal{A}). \tag{3.13}$$

Clearly, if X is adapted to $(\mathcal{F}_A)_{A\in\mathcal{A}}$, then so is X^p.

3.2.4 Set-indexed strong martingales

In this subsection, we define set-indexed strong (sub)martingales, a set-indexed analogue of the planar strong (sub)martingales introduced in [10]. A partial discussion on sample path regularity will follow.

Fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$. For any $1 \leq p \leq \infty$, let $L_p = L_p(\Omega, \mathcal{F}, P)$ denote the classical Lebesgue space. Also, given any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, let

$$E[\,\cdot\,|\,\mathcal{G}] : L_1 \to L_1(\Omega, \mathcal{G}, P)$$

denote the *conditional expectation operator*. By way of terminology, we say a process $X = (X_A)_{A\in\mathcal{A}}$ is in L_p ($1 \leq p \leq \infty$) provided

$$X_A \in L_p\,, \quad (\forall\, A \in \mathcal{A}).$$

When $p = 1$, X is said to be *integrable*. The following definition has already appeared in [26].

Definition 3.2.38 *An adapted, integrable process $M = (M_A)_{A\in\mathcal{A}}$ is said to be a (set-indexed) strong submartingale if M possesses a unique finitely additive extension to $\mathcal{C}(u)$ and*

$$E[M_C \,|\, \mathcal{G}_C^*] \geq 0, \quad (\forall C \in \mathcal{C}). \tag{3.14}$$

If equality holds in (3.14) for every $C \in \mathcal{C}$, then M is said to be a (set-indexed) strong martingale.

Remark 3.2.39 (a) Recall the weak past $\mathcal{G}_C$ ($C \in \mathcal{C}$) as defined in Remark 3.2.23 (b). If we replace (3.14) by the condition $E[M_C \,|\, \mathcal{G}_C] \geq 0$ $\forall C \in \mathcal{C}$, then M is said to be a *weak submartingale*. Since $\mathcal{G}_C \subseteq \mathcal{G}_C^*$ $\forall C \in \mathcal{C}$, the tower property implies that every strong submartingale is a weak submartingale.

(b) The set-indexed Poisson process is an example of a set-indexed strong submartingale (see [23]). For an example of a set-indexed strong martingale, see Proposition 4.6.8.

The following example gives the connection between set-indexed strong (sub)martingales and planar strong (sub)martingales. The definition of the latter concept can be found in [10].

Example 3.2.40 Let $X = (X_z)_{z\in[0,1]^2}$ be a planar process adapted to a filtration $(\mathcal{F}_z)_{z\in[0,1]^2}$. If X is a planar strong (sub)martingale and

$$X_{(s,t)} = 0, \quad (\forall (s,t) \in [0,1]^2 \text{ s.t. } s = 0 \text{ or } t = 0),$$

i.e., X vanishes on the axes, then as commented on p.522 of [16], X is an $\mathcal{I}_2$-indexed strong (sub)martingale when we identify the points in $[0,1]^2$ with sets in $\mathcal{I}^2$. Under the same correspondence, it is clear that any $\mathcal{I}_2$-indexed strong (sub)martingale constitutes a planar strong (sub)martingale on $[0,1]^2$.

As our next result illustrates, we can always extend the (sub)martingale property in (3.14) to the sets in $\mathcal{C}(u)$.

Lemma 3.2.41 *If $M = (M_A)_{A\in\mathcal{A}}$ is a strong (sub)martingale, then*

$$E[M_D \,|\, \mathcal{G}_D^*] \;(\geq) = 0, \quad (\forall D \in \mathcal{C}(u)). \tag{3.15}$$

Proof Let $D \in \mathcal{C}(u)$ be given. Since $\mathcal{C}$ is a semi-algebra (see Remark 3.2.4), $\exists$ disjoint sets $C_1, \cdots, C_k \in \mathcal{C}$ s.t. $D = \bigcup_{i=1}^k C_i$. Therefore,

$$E[M_D \,|\, \mathcal{G}_D^*] = \sum_{i=1}^k E[M_{C_i} \,|\, \mathcal{G}_D^*].$$

But $(\mathcal{G}_C^*)_{C \in \mathcal{C}(u)}$ is decreasing on $\mathcal{C}(u)$. Therefore, $\mathcal{G}_D^* \subseteq \mathcal{G}_{C_i}^*$ $\forall 1 \leq i \leq k$ which by (3.14) and the tower property implies

$$E[M_D \,|\, \mathcal{G}_D^*] = \sum_{i=1}^k E(\, E[M_{C_i} \,|\, \mathcal{G}_{C_i}] \,|\, \mathcal{G}_D^* \,) \geq 0.$$

The argument for set-indexed strong martingales is identical. □

In addition to strong (sub)martingales, we will also encounter the following types of set-indexed processes in this thesis.

Definition 3.2.42 *Let $X = (X_A)_{A \in \mathcal{A}}$ be an integrable adapted process possessing a unique finitely additive extension to $\mathcal{C}(u)$. Given $1 \leq p \leq \infty$,*

(a) *X is Lp-bounded if $\sup_{C \in \mathcal{C}} E|X_C|^p < \infty$ and*

(b) *X is increasing if $X_C \geq 0$ a.s. $\forall C \in \mathcal{C}$.*

Remark 3.2.43 Clearly, any increasing process is a strong submartingale.

We will sometimes require the sample paths of our set-indexed processes to possess certain continuity properties. We list three such properties below. Additional path properties will be discussed in Chapter 4.

Definition 3.2.44 *Let $X = (X_A)_{A \in \mathcal{A}}$ be an integrable process.*

(a) *X is L_p-right-continuous $(1 \leq p \leq \infty)$ provided*

$$A_n \downarrow A \text{ in } \mathcal{A} \implies X_{A_n} \to X_A \text{ in } L_p \text{ norm.}$$

(b) *X is right-continuous provided $\exists\, \Omega_0 \in \mathcal{F}$ s.t. $P(\Omega_0) = 1$ and*

$$A_n \downarrow A \text{ in } \mathcal{A} \implies X_{A_n}(\omega) \to X_A(\omega) \;\; \forall \omega \in \Omega_0.$$

(c) *X is left-continuous provided* $\exists\, \Omega_0 \in \mathcal{F}$ *s.t.* $P(\Omega_0) = 1$ *and*

$$(A_n)_n \text{ increasing in } \mathcal{A} \implies X_{A_n}(\omega) \to X_B(\omega)\ \forall\, \omega \in \Omega_0$$

where $B = \overline{\cup_n A_n}$. *(By Lemma A.2.2, any such B lies in $\mathcal{A}$.)*

The following result complements the discussion in Remark 3.2.31. Its proof, which relies on the d_H-separability of $\mathcal{A}$ (see Remark 2.2.3), has already appeared on p.507 of [16].

Lemma 3.2.45 *Let* $X = (X_A)_{A\in\mathcal{A}}$ *and* $Y = (Y_A)_{A\in\mathcal{A}}$ *be two right-continuous processes. If X is a modification of Y, then X and Y are indistinguishable.*

As our next result illustrates, L_p-right-continuity is not a restriction for strong martingales in L_p.

Lemma 3.2.46 *If* $M = (M_A)_{A\in\mathcal{A}}$ *is a strong martingale in* L_p $(1 \le p < \infty)$, *then M is L_p-right-continuous.*

Proof Given any $A, B \in \mathcal{A}$, if we can show

$$A \subseteq B \implies E[M_B \,|\, \mathcal{F}_A] = M_A, \tag{3.16}$$

the lemma will follow by Proposition 2.5 of [16] (see Remark 3.2.47). For this purpose, take $A, B \in \mathcal{A}$ s.t. $A \subseteq B$. By Lemma 3.2.24 (e),

$$A \cap (B\backslash A) = \phi \implies \mathcal{F}_A \subseteq \mathcal{G}^*_{B\backslash A}.$$

Therefore, by (3.12) and the tower property,

$$E[M_B - M_A \,|\, \mathcal{F}_A] = E(\, E[M_{B\backslash A} \,|\, \mathcal{G}^*_{B\backslash A}] \,|\, \mathcal{F}_A) = 0$$

which establishes (3.16). □

Remark 3.2.47 In [16], any set-indexed process satisfying condition (3.16) was referred to as a *set-indexed martingale.* Proposition 2.5 of [16] states that any L_p-martingale is L_p-right-continuous $(1 \le p < \infty)$.

We close this section with a useful property for L_1-right-continuous processes, a proof for which can be found on p.p. 507-508 of [16]. (Neither shape nor the weak submartingale property was used in this proof.)

Lemma 3.2.48 *If $X = (X_A)_{A\in\mathcal{A}}$ is an L_1-right-continuous process possessing a unique finitely additive extension to $\mathcal{C}(u)$, then given any $C \in \mathcal{C}$, $\exists$ an increasing sequence $(C_n)_n$ in $\mathcal{C}$ s.t.*

(a) $C_n \uparrow C$,

(b) $\overline{C}_n \subseteq C \ \forall n$ *and*

(c) $E|X_{C\setminus C_n}| \to 0$ *as* $n \to \infty$.

In particular, if $C = A \setminus \bigcup_{i=1}^{k} A_i$, *we can take* $C_n = A \setminus \bigcup_{i=1}^{k} g_n(A_i) \ \forall n$.

Remark 3.2.49 Under the assumptions in Lemma 3.2.48, if C_n is replaced by $g_n(A) \setminus \bigcup_{i=1}^{k} g_n(A_i) \ \forall n$, then (c) continues to hold. This involves only a minor adjustment to proof of Proposition 2.6 (i) in [16].

3.3 Admissible Functions and Measures

In this section, we define the $*$-predictable σ-algebra $\mathcal{P}^*$ on $\Omega \times T$ w.r.t. a fixed stochastic base. Our definition will be similar in spirit to that of the predictable σ-algebra $\mathcal{P}$ on $\Omega \times T$ defined in [16] and [28]. As well, we will introduce the concept of *admissible function*, a function μ_X associated to an integrable set-indexed process X.

The main goal of this section is to obtain sufficient conditions on a set-indexed strong submartingale X under which the corresponding admissible function μ_X extends to a measure on $\mathcal{P}^*$. In Section 3.4, these measures will be used to obtain a Doob-Meyer-type decomposition for set-indexed strong submartingales. Our development is close to that of [16].

Now, fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$.

Definition 3.3.1 *The collection of* <u>*$*$-predictable rectangles*</u>, *denoted* $\mathcal{P}_0^*$ *is defined by*

$$\mathcal{P}_0^* = \{F \times C : C \in \mathcal{C},\ F \in \mathcal{G}_C^*\}.$$

The <u>*$*$-predictable σ-algebra*</u> *on* $\Omega \times T$, *denoted* $\mathcal{P}^*$ *is then defined to be*

$$\mathcal{P}^* = \sigma(\mathcal{P}_0^*).$$

Any set in $\mathcal{P}^*$ *is referred to as a* <u>*$*$-predictable set.*</u>

Remark 3.3.2 (a) The $*$-predictable σ-algebra is a set-indexed analogue of the σ-algebra $\underline{\underline{P}}^*$ on $\Omega \times (\mathbf{R}^+)^2$ defined in [15]. In [15], $\underline{\underline{P}}^*$ played a central role in the decomposition of strong submartingales on $(\mathbf{R}^+)^2$ and likewise, $\mathcal{P}^*$ will play an important role in the decomposition of set-indexed strong submartingales.

(b) By replacing the strong past $\mathcal{G}^*_C$ in Definition 3.3.1 with the weak past $\mathcal{G}_C$ for each $C \in \mathcal{C}$, we obtain a σ-algebra $\mathcal{P}$ on $\Omega \times T$ termed the *predictable σ-algebra*. This σ-algebra has been studied extensively in [28].

Lemma 3.3.3 *$\mathcal{P}^*_0$ is a semi-algebra on $\Omega \times T$.*

Proof Since $T \in \mathcal{C}$ and $\phi, \Omega \in \mathcal{G}^*_T$, both ϕ and $\Omega \times T$ are in $\mathcal{P}^*_0$.

Take $R_i = F_i \times C_i \in \mathcal{P}^*_0$ $(i = 1, 2)$. By basic set-theory,

$$R_1 \cap R_2 \;=\; (F_1 \cap F_2) \times (C_1 \cap C_2). \tag{3.17}$$

Since $\mathcal{C}$ is a semi-algebra on T, $C_1 \cap C_2 \in \mathcal{C}$. Moreover, by Lemma 3.2.24 (d),

$$C_1 \cap C_2 \subseteq C_i \implies \mathcal{G}^*_{C_i} \subseteq \mathcal{G}^*_{C_1 \cap C_2}, \quad (i = 1, 2).$$

Therefore, $F_1 \cap F_2 \in \mathcal{G}^*_{C_1 \cap C_2}$ which, by (3.17), implies $R_1 \cap R_2 \in \mathcal{P}^*_0$.

Next, assume $R_1 \subseteq R_2$. Then $F_1 \subseteq F_2$ and $C_1 \subseteq C_2$ which implies

$$R_2 - R_1 \;=\; [(F_2 - F_1) \times C_1] \cup [F_2 \times (C_2 - C_1)], \tag{3.18}$$

the above union being disjoint. Lemma 3.3.3 will follow if both summands in (3.18) are themselves disjoint unions in $\mathcal{P}^*_0$.

First, since $C_1 \subseteq C_2$, Lemma 3.2.24 (d) implies $\mathcal{G}^*_{C_2} \subseteq \mathcal{G}^*_{C_1}$. Therefore, $F_2 - F_1 \in \mathcal{G}^*_{C_1}$ which yields $(F_2 - F_1) \times C_1 \in \mathcal{P}^*_0$. For the other summand in (3.18), since $\mathcal{C}$ is a semi-algebra on T, $\exists$ disjoint sets $D_1, \cdots, D_k \in \mathcal{C}$ s.t. $C_2 - C_1 = \bigcup_{i=1}^k D_i$. But given any $1 \leq i \leq k$,

$$D_i \subseteq C_2 \implies \mathcal{G}^*_{C_2} \subseteq \mathcal{G}^*_{D_i}.$$

Therefore, $F_2 \times D_i \in \mathcal{P}^*_0$ $\forall\, 1 \leq i \leq k$. Since $D_1, \cdots, D_k$ are disjoint, it follows that $F_2 \times (C_2 - C_1)$ is the disjoint union of the sets $F_2 \times D_1, \cdots, F_2 \times D_k \in \mathcal{P}^*_0$ which completes the proof. □

Since $\mathcal{P}^*_0$ is a semi-algebra, $\mathcal{P}^*_0(u)$ is an algebra on $\Omega \times T$ where

$$\begin{aligned} \mathcal{P}^*_0(u) &:= \{\text{ finite unions in } \mathcal{P}^*_0\} \\ &= \{\text{ finite disjoint unions in } \mathcal{P}^*_0\}. \end{aligned}$$

Furthermore, we can express each $R \in \mathcal{P}_0^*(u)$ as a disjoint union over a f.n.s. (see Definition 3.2.14).

Lemma 3.3.4 *Given $R \in \mathcal{P}_0^*(u)$, $\exists$ a f.n.s., $\mathcal{N}_0$ s.t. $R = \bigcup_{D\in\mathcal{N}_0} G_D \times D$ where $G_D \times D \in \mathcal{P}_0^*$ $\forall D \in \mathcal{N}_0$.*

Proof Assume $R = \bigcup_{i=1}^k F_i \times C_i$ is a disjoint union in $\mathcal{P}_0^*$ and define $C = \bigcup_{i=1}^k C_i \in \mathcal{C}(u)$. By Lemma 3.2.15, $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ and a subcollection $\mathcal{N}_0$ of $\mathcal{N}'$ s.t. $C = \bigcup_{D\in\mathcal{N}_0} D$. Moreover, it is clear from the proof of Lemma 3.2.15 that $C_i \in \mathcal{C}'$ $\forall 1 \leq i \leq k$. Therefore,

$$\begin{aligned} R &= \bigcup_{i=1}^k \bigcup_{D\in\mathcal{N}_0} F_i \times (D \cap C_i) \\ &= \bigcup_{D\in\mathcal{N}_0} \bigcup_{i=1}^k F_i \times (D \cap C_i) \\ &= \bigcup_{D\in\mathcal{N}_0} \bigcup_{i:D\subseteq C_i} F_i \times D \quad \text{(by Lemma A.3.3)} \\ &= \bigcup_{D\in\mathcal{N}_0} \left(\bigcup_{i:D\subseteq C_i} F_i \right) \times D. \end{aligned}$$

If we define $G_D = \bigcup_{i:D\subseteq C_i} F_i$ $\forall D \in \mathcal{N}_0$, then by Lemma 3.2.24 (d), $G_D \in \mathcal{G}_D$ $\forall D \in \mathcal{N}_0$, completing the proof. □

The following definition has already appeared in [16].

Definition 3.3.5 *Given an integrable but not necessarily adapted process $X = (X_A)_{A\in\mathcal{A}}$ with increments defined at every $C \in \mathcal{C}$, the admissible function associated to X is the function*

$$\mu_X : \{F \times C : F \in \mathcal{F}, C \in \mathcal{C}\} \to \mathbf{R}$$

where, given any $F \in \mathcal{F}$ and any $C \in \mathcal{C}$,

$$\mu_X(F \times C) = E[\mathbf{1}_F X_C]. \tag{3.19}$$

(Occasionally, we will take $\mathcal{P}_0^*$ as the domain of μ_X.)

The admissible function provides a useful characterization for set-indexed strong (sub)martingales and increasing processes.

Lemma 3.3.6 *Given a process* $X = (X_A)_{A\in\mathcal{A}}$,

(a) X *is a strong martingale if and only if* $\mu_X = 0$ *on* $\mathcal{P}_0^*$,

(b) X *is a strong submartingale if and only if* $\mu_X \geq 0$ *on* $\mathcal{P}_0^*$ *and*

(c) X *is increasing if and only if* $\mu_X \geq 0$ *on* $\{F \times C : F \in \mathcal{F}, C \in \mathcal{C}\}$.

Proof We will show (a) and (b) simultaneously. Let $C \in \mathcal{C}$ be given. By the definition of conditional expectation,

$$\begin{aligned} E(X_C \,|\, \mathcal{G}_C^*)\ (\geq) = 0 &\iff E(1_F X_C)\ (\geq) = 0\ \ \forall F \in \mathcal{G}_C^* \\ &\iff \mu_X(F\times C)\ (\geq) = 0\ \ \forall F \in \mathcal{G}_C^*. \end{aligned} \tag{3.20}$$

Applying (3.20) to every $C \in \mathcal{C}$, we obtain (a) (or (b)).

Next, take any $C \in \mathcal{C}$. It is clear that

$$\begin{aligned} X_C \geq 0 &\iff E(1_F X_C) \geq 0\ \ \forall F \in \mathcal{F} \\ &\iff \mu_X(F\times C) \geq 0\ \ \forall F \in \mathcal{F}. \end{aligned} \tag{3.21}$$

Part (c) now follows by applying (3.21) to every $C \in \mathcal{C}$. □

From this point forward, unless otherwise mentioned, we take $\mathcal{P}_0^*$ **as the domain for all admissible functions.**

The first step toward extending μ_X to a measure on $\mathcal{P}^*$ is to show that μ_X is finitely additive on $\mathcal{P}_0^*$.

Lemma 3.3.7 *let* $X = (X_A)_{A\in\mathcal{A}}$ *be an integrable process with increments defined at every* $C \in \mathcal{C}$. *If* $F_1 \times C_1, \cdots F_n \times C_n$ *are disjoint sets in* $\mathcal{P}_0^*$ *s.t.* $\bigcup_{i=1}^n F_i \times C_i \in \mathcal{P}_0^*$, *then*

$$\mu_X(\cup_{i=1}^n F_i \times C_i) \;=\; \sum_{i=1}^k \mu_X(F_i \times C_i). \tag{3.22}$$

Proof If $F_1 \times C_1, \cdots, F_n \times C_n \in \mathcal{P}_0^*$ are as described above, then $\exists\, F \times C \in \mathcal{P}_0^*$ s.t. $\bigcup_{i=1}^n F_i \times C_i = F \times C$. Motivated by an argument of Gushchin (see Theorem 9 in [18]), we argue by cases.

First, consider the case in which $F_1 = F_2 = \cdots = F_n$. Clearly, this implies $F_i = F\ \forall 1 \leq i \leq n$ and hence $F \times C = \bigcup_{i=1}^n F_i \times C_i = F \times (\bigcup_{i=1}^n C_i)$ where

$\bigcup_{i=1}^n C_i$ is a disjoint union in $\mathcal{C}$. Therefore, since X is finitely additive on $\mathcal{C}(u)$,

$$\mu_X(F \times C) = \sum_{i=1}^n E[\mathbf{1}_F X(C_i)]$$

which is precisely (3.22).

Next, consider the case in which C_i and C_j are either disjoint or equal $\forall\, 1 \le i, j \le n$. If we let $\Delta_1, \cdots, \Delta_k$ denote all distinct (hence disjoint) sets among the C_i, then since $F \times C = \bigcup_{i=1}^n F_i \times C_i$, it is clear that

$$F = \bigcup_{i: C_i = \Delta_j} F_i, \quad (\forall\, 1 \le j \le k)$$

where the above union is disjoint $\forall\, 1 \le j \le k$. This implies

$$\sum_{i=1}^n \mu_X(F_i \times C_i) = \sum_{j=1}^k \left(\sum_{i: C_i = \Delta_j} E[\mathbf{1}_{F_i} X_{\Delta_j}] \right) = \sum_{j=1}^k E[\mathbf{1}_F X_{\Delta_j}].$$

Therefore, by applying the first case to the disjoint union $F \times C = \bigcup_{j=1}^k F \times \Delta_j$, we obtain

$$\sum_{i=1}^n \mu_X(F_i \times C_i) = \sum_{j=1}^k E[\mathbf{1}_F X_{\Delta_j}] = \mu_X(F \times C)$$

which is precisely (3.22).

Finally, consider the case in which $F \times C = \bigcup_{i=1}^n F_i \times C_i$ is any disjoint union in $\mathcal{P}_0^*$. Applying Lemma 3.2.15 to the set $\bigcup_{i=1}^n C_i \in \mathcal{C}(u)$, $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ s.t. for each $1 \le i \le n$,

$$\exists\, D_1^i, \cdots D_{k(i)}^i \in \mathcal{N}' \text{ s.t. } C_i = \bigcup_{k=1}^{k(i)} D_k^i.$$

Since $(\mathcal{G}_D^*)_{D \in \mathcal{C}}$ is decreasing on $\mathcal{C}$, it is clear that

$$F_i \times D_k^i \in \mathcal{P}_0^*, \quad (\forall\, 1 \le k \le k(i),\ 1 \le i \le n). \tag{3.23}$$

But $F \times C = \bigcup_{i=1}^n \bigcup_{k=1}^{k(i)} F_i \times D_k^i$ where the sets $D_k^i \in \mathcal{N}'$ are either disjoint or equal, a basic property of left-neighborhoods. Therefore,

$$\begin{aligned} \mu_X(F \times C) &= \sum_{i=1}^n \left(\sum_{k=1}^{k(i)} \mu_X(F_i \times D_k^i) \right) \quad \text{(by previous case)} \\ &= \sum_{i=1}^n \mu_X(F_i \times C_i) \quad \text{(by first case)}. \end{aligned}$$

This completes the proof of Lemma 3.3.7. □

Next, we extend μ_X to a finitely additive set function on the algebra $\mathcal{P}_0^*(u)$. Instead of showing this directly, we establish a slight generalization which will be applicable in Sections 3.5 and A.4.

Lemma 3.3.8 *Let $\mathcal{S}$ be a semi-algebra on a set S. If $\mu : \mathcal{S} \to \mathrm{R}$ is finitely additive on $\mathcal{S}$, then μ can be uniquely extended to a finitely additive set-function on the algebra $\mathcal{S}(u)$.*

Proof Given two disjoint unions $\bigcup_{i=1}^n A_i = \bigcup_{j=1}^m B_j$ with $A_i, B_j \in \mathcal{S}\ \forall\, i, j$, we need to show

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{m} \mu(B_j). \tag{3.24}$$

First, take $1 \leq i \leq n$. Then,

$$A_i = \bigcup_{j=1}^m A_i \cap B_j$$

is a disjoint union. Since $\mathcal{S}$ is a semi-algebra, $A_i \cap B_j \in \mathcal{S}\ \forall 1 \leq j \leq m$. Therefore, by the finite additivity of μ on $\mathcal{S}$,

$$\mu(A_i) = \sum_{j=1}^{m} \mu(A_i \cap B_j).$$

Summing over all i, we obtain the identity,

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(A_i \cap B_j).$$

But, by symmetry,

$$\sum_{j=1}^{m} \mu(B_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(B_j \cap A_i)$$

which establishes (3.24). □

Since $\mathcal{P}_0^*$ is a semi-algebra on $\Omega \times T$, we have the following application.

Corollary 3.3.9 *If $X = (X_A)_{A \in \mathcal{A}}$ is an integrable process with increments defined at every $C \in \mathcal{C}$, then μ_X can be uniquely extended to a finitely additive set-function, also denoted μ_X on $\mathcal{P}_0^*(u)$.*

We now define an important class of set-indexed processes.

Definition 3.3.10 *A process $X = (X_A)_{A\in\mathcal{A}}$ with increments defined at every $C \in \mathcal{C}$ is said to be of* class $(D')^*$ *if the family*

$$\left\{ \sum_{D\in\mathcal{N}_0} |E(X_D \,|\, \mathcal{G}^*_D)| \; : \; \mathcal{N}_0 \text{ a f.n.s. of } \mathcal{A} \right\}$$

is uniformly integrable.

Our definition of class $(D')^*$ is intended to mimic that of class D' set-indexed processes (see [16]). Similar classes have been defined for continuous parameter and planar processes (see [30] and [8] respectively).

Now, let $\mathcal{U}$ be an algebra on a set U. Proofs for the following three measure theoretic results can be found in most measure theory texts.

Lemma 3.3.11 *If $\mu : \mathcal{U} \to [0,\infty)$ is a finitely additive set-function with $\mu(\phi) = 0$, then given $R_1, R_2, \cdots, R_n \in \mathcal{U}$,*

(a) $R_1 \subseteq R_2$ *implies* $\mu(R_1) \leq \mu(R_2)$ *and*

(b) $\mu(\cup_{i=1}^n R_i) \leq \sum_{i=1}^n \mu(R_i)$,

i.e., μ is monotone and finitely sub-additive on $\mathcal{U}$.

Lemma 3.3.12 *Let $\mu : \mathcal{U} \to [0,\infty)$ be a finitely additive set-function on $\mathcal{U}$ with $\mu(\phi) = 0$. If for any sequence $(R_n)_n$ in $\mathcal{U}$*

$$R_n \downarrow \phi \implies \mu(R_n) \to 0,$$

then μ is σ-additive on $\mathcal{U}$, i.e., μ is a measure on $\mathcal{U}$.

Lemma 3.3.13 *If $\mu : \mathcal{U} \to [0,\infty)$ is a measure on $\mathcal{U}$, then μ can be uniquely extended to a measure on $\sigma(\mathcal{U})$.*

We now state and prove the main result of this section. Our proof is close to that of Theorem 3.1 in [16].

Theorem 3.3.14 *If $X = (X_A)_{A\in\mathcal{A}}$ is an L_1-right-continuous strong submartingale of class $(D')^*$, the admissible function μ_X extends to a measure, also denoted μ_X on $\mathcal{P}^*$.*

Proof We will employ a proof by contradiction. First of all, by Corollary 3.3.9, the set-function $\mu_X : \mathcal{P}_0^* \to [0, \infty)$ possesses a unique, finitely additive extension, also denoted μ_X to the algebra $\mathcal{P}_0^*(u)$. Therefore, if we assume μ_X does not possess an extension to a measure on $\mathcal{P}^*$, then by Lemmas 3.3.12 and 3.3.13, $\exists$ a sequence $(R_n)_n$ in $\mathcal{P}_0^*(u)$ s.t. $R_n \downarrow \phi$ and $(\mu_X(R_n))_n$ does not converge to 0. Since μ_X is non-negative and monotone on $\mathcal{P}_0^*(u)$ (see Lemma 3.3.11 (a)), this implies $\exists\, a > 0$ s.t.

$$\mu_X(R_n) \;\geq\; 3a, \quad (\forall n). \tag{3.25}$$

Now, since $\mathcal{P}_0^*$ is a semi-algebra on $\Omega \times T$, given any $n \in \mathbf{N}$, $\exists\, b_n \in \mathbf{N}$ and $\exists$ disjoint sets, $F_1^n \times C_1^n, \cdots F_{b_n}^n \times C_{b_n}^n \in \mathcal{P}_0^*$ s.t.

$$R_n \;=\; \textstyle\bigcup_{i=1}^{b_n} F_i^n \times C_i^n.$$

By Lemma 3.2.48, given any n and any $1 \leq i \leq b_n$, $\exists\, E_i^n \in \mathcal{C}$ s.t.

$$\overline{E_i^n} \subseteq C_i^n \quad \text{and} \quad E[X(C_i^n \setminus E_i^n)] \leq \frac{a}{2^n\, b_n}. \tag{3.26}$$

Moreover, since $C_k^n \setminus E_k^n$ is a disjoint union of sets in the semi-algebra $\mathcal{C}$ and X is finitely additive on $\mathcal{C}(u)$,

$$\mu_X[\Omega \times (C_i^n \setminus E_i^n)] \;=\; E[1_\Omega \cdot X(C_i^n \setminus E_i^n)] \;\leq\; \frac{a}{2^n\, b_n} \tag{3.27}$$

for each n and $1 \leq i \leq b_n$.

The following series of sets was previously defined on p.513 of [16]:

$$\begin{aligned}
H_n &:= \textstyle\bigcup_{i=1}^{b_n} F_i^n \times E_i^n \\
\widetilde{H}_n &:= \textstyle\bigcup_{i=1}^{b_n} F_i^n \times \overline{E_i^n} \\
D_n &:= \textstyle\bigcap_{k=1}^{n} H_k \\
\widetilde{D}_n &:= \textstyle\bigcap_{k=1}^{n} \widetilde{H}_k \\
S_n &:= \textstyle\bigcup_{i=1}^{b_n} \Omega \times [C_i^n \setminus E_i^n].
\end{aligned}$$

Since $(\mathcal{G}_C^*)_{C \in \mathcal{C}}$ is decreasing on $\mathcal{C}$, the inclusion in (3.26) implies

$$F_i^n \in \mathcal{G}_{E_i^n}^*, \quad (\forall\, 1 \leq i \leq b_n,\; n \in \mathbf{N})$$

and therefore, $H_n, D_n \in \mathcal{P}_0^*(u)$ $\forall n$. It is also clear that $S_n \in \mathcal{P}_0^*(u)$ $\forall n$.

We now isolate a useful set-inclusion.

Claim I: $R_n \subseteq D_n \cup (\bigcup_{k=1}^{n} S_k)\ \forall n$.

Proof: Take $(\omega, t) \in R_n$ and assume $(\omega, t) \notin D_n$. (We must show that $(\omega, t) \in \bigcup_{k=1}^{n} S_k$.) By definition, $(\omega, t) \notin D_n$ implies $\exists\, 1 \leq k' \leq n$ s.t. $(\omega, t) \notin H_{k'}$. That is,

$$(\omega, t) \notin F_i^{k'} \times E_i^{k'}, \quad (\forall 1 \leq i \leq b_{k'}). \tag{3.28}$$

Since $(R_n)_n$ is decreasing, $(\omega, t) \in R_n$ implies $(\omega, t) \in R_{k'}$ and therefore,

$$\exists\, 1 \leq i' \leq b_{k'} \text{ s.t. } (\omega, t) \in F_{i'}^{k'} \times C_{i'}^{k'}. \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$(\omega, t) \in F_{i'}^{k'} \times [C_{i'}^{k'} \setminus E_{i'}^{k'}]$$

where $F_{i'}^{k'} \times [C_{i'}^{k'} \setminus E_{i'}^{k'}] \subseteq S_{k'} \subseteq \bigcup_{k=1}^{n} S_k$. As mentioned above, this completes the proof of Claim I. Ω

Since μ_X is finitely subadditive on $\mathcal{P}_0^*(u)$ (see Lemma 3.3.11 (b)), Claim I implies

$$\mu_X(R_n) \;\leq\; \mu_X(D_n) + \sum_{k=1}^{n} \left(\sum_{i=1}^{b_k} \mu_X \left(\Omega \times [C_i^k \setminus E_i^k] \right) \right), \quad (\forall n)$$

Therefore, by (3.27),

$$\mu_X(R_n) \;\leq\; \mu_X(D_n) + a, \quad (\forall n). \tag{3.30}$$

In view of (3.30) and (3.25), a contradiction will follow if we can find $N \in \mathbf{N}$ s.t. $\mu_X(D_n) \leq a\ \forall n \geq N$. To obtain such an N, we need the following limit.

Claim II: *If we define* $D_n(\omega) = \{t \in T : (\omega, t) \in D_n\}$ $(\omega \in \Omega)$, *then* $P(\{\omega \in \Omega : D_n(\omega) \neq \phi\}) \to 0$ *as* $n \to \infty$.

Proof: First, note that $\widetilde{D}_n \subseteq \widetilde{H}_n\ \forall n$ and, by (3.26), $\widetilde{H}_n \subseteq R_n\ \forall n$. Therefore, since $(\widetilde{D}_n)_n$ is decreasing and $R_n \downarrow \phi$, we have $\widetilde{D}_n \downarrow \phi$.

Next, if we define the set $\widetilde{D}_n(\omega) = \{t \in T : (\omega, t) \in \widetilde{D}_n\}$ $(\omega \in \Omega)$, then

$$\widetilde{D}_n(\omega) \downarrow \phi \text{ as } n \to \infty.$$

Moreover, given any n, it is also clear that

$$\widetilde{D}_n(\omega) = \bigcap_{k=1}^{n} \bigcup_{\substack{1 \leq i \leq b_k \\ \omega \in F_i^k}} \overline{E_i^k},$$

a closed set in (T, d). Therefore, by Lemma A.1.5, $\exists n(\omega) \in \mathrm{N}$ s.t. $\widetilde{D}_n(\omega) = \phi$ $\forall n \geq n(\omega)$ and hence

$$D_n(\omega) = \phi, \quad (\forall n \geq n(\omega)). \tag{3.31}$$

By Lemma 3.3.4, each $D_n \in \mathcal{P}_0^*(u)$ can be expressed as a union of finitely many non-empty disjoint sets in $\mathcal{P}_0^*$, say

$$D_n = \cup_{j=1}^{a_n} G_j^n \times B_j^n \tag{3.32}$$

so that $\{B_1^n, \cdots, B_{a_n}^n\}$ is a f.n.s. of $\mathcal{A}$. Clearly,

$$\{\omega : D_n(\omega) \neq \phi\} = \cup_{j=1}^{a_n} G_j^n, \quad (\forall n) \tag{3.33}$$

which, among other things, implies $\{\omega : D_n(\omega) \neq \phi\} \in \mathcal{F}$ $\forall n$.

If, for each $n \in \mathrm{N}$, we let $A_n = \{\omega : D_n(\omega) \neq \phi\}$, then by (3.31), $\limsup_n A_n = \phi$. Therefore, by Fatou's lemma,

$$0 \leq \limsup_n P(\{\omega : D_n(\omega) \neq \phi\}) \leq P(\limsup_n A_n)$$

which completes the proof of Claim II. Ω

Now, for each n,

$$\begin{aligned}
\mu_X(D_n) &= \sum_{j=1}^{a_n} \mu_X(G_i^n \times B_i^n) \quad (\text{by } (3.32)) \\
&= \sum_{j=1}^{a_n} \left(\int_{G_j^n} X_{B_j^n} \, dP \right) \\
&= \sum_{j=1}^{a_n} \left(\int_{G_j^n} E[X_{B_j^n} \mid \mathcal{G}_{B_j^n}^*] \, dP \right) \\
&\leq \int_{\bigcup_j G_j^n} \left(\sum_{j=1}^{a_n} \left| E[X_{B_j^n} \mid \mathcal{G}_{B_j^n}^*] \right| \, dP \right).
\end{aligned} \tag{3.34}$$

But, by Claim II and (3.33), $P(\bigcup_{j=1}^{a_n} G_i^n) \to 0$ as $n \to \infty$. Therefore, since X is of class $(D')^*$, the above array implies $\exists\, N \in \mathbf{N}$ s.t.

$$\mu_X(D_n) \leq a, \ (\forall n \geq N). \tag{3.35}$$

(Recall that $\{B_1^n, \cdots, B_{a_n}^n\}$ is a f.n.s. of $\mathcal{A}$ for each $n \in \mathbf{N}$.)

Substituting (3.35) into (3.30), we obtain

$$\mu_X(R_n) \leq 2a, \ (\forall n \geq N)$$

which contradicts (3.25). This contradiction resulted from assuming μ_X could not be extended to a measure on $\mathcal{P}^*$. □

The next result follows immediately from the proof of Theorem 3.3.14. In particular, see (3.34).

Corollary 3.3.15 *If X is a L_1-right-continuous strong submartingale with*

$$\left\{ \sum_{D \in \mathcal{N}_0} X_D \ : \ \mathcal{N}_0 \text{ a f.n.s. of } \mathcal{A} \right\}$$

uniformly integrable, then the admissible function μ_X extends to a measure, also denoted μ_X on $\mathcal{P}^*$.

We close this section with some terminology. Given a strong submartingale X, if the admissible function of X extends to a measure on $\mathcal{P}^*$, we refer to this extension as an *admissible measure* of X. Clearly, all such measures will be finite on $\mathcal{P}^*$. Moreover, since $\mathcal{P}_0^*$ is a π-system on $\Omega \times T$, Theorem 10.3 in [7] implies that the admissible measure of a strong submartingale, if it exists, is unique.

3.4 A Doob-Meyer Decomposition for Set-Indexed Strong Submartingales

In this section, after defining the class of $*$-predictable set-indexed processes, we will give sufficient conditions on a set-indexed strong submartingale X so as to guarantee a Doob-Meyer-type decomposition. In particular, if X is L_1-right-continuous and of class $(D')^*$, it will be shown in Theorem 3.4.11

that M can be decomposed into a sum $X = M + V$ where $E[M_C \mid \mathcal{G}^*_C] = 0$ $\forall C \in \mathcal{C}$ and V is a $*$-predictable process satisfying $V_C \geq 0$ $\forall C \in \mathcal{C}$. (Neither V nor M will be necessarily adapted.) Moreover, in Proposition 3.4.12, such a decomposition will be shown to be unique up to indistinguishablity. The section closes with sufficient conditions on $\mathcal{A}$ and $(\mathcal{F}_A)_{A\in\mathcal{A}}$ under which both M and V are adapted (see Theorem 3.4.14). This will result in a true Doob-Meyer decomposition in that X will be the sum of a strong martingale and a $*$-predictable increasing process.

Our development will follow the "discrete approximation" approach used earlier in connection with set-indexed weak submartingales (see Section 5 of [16]). This approach has two main advantages. First, via such an approach, we can avoid some messy technical devices such as p-stochastic measures (see p.515 of [16]). Secondly, the discrete approximants used to define V can be exploited in various applications. For example, see Proposition 3.6.12.

Fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ for use throughout the section.

3.4.1 Technical preliminaries

Take any $t \in T$. Recall that for each $n \in \mathrm{N}$, the collection $\mathcal{N}_n$ of non-empty left-neighborhoods in $\mathcal{A}_n$ forms a partition of T and hence, for each $n \in \mathrm{N}$, $\exists$ a unique left-neighborhood, say $C^n_t \in \mathcal{N}_n$ s.t. $t \in C^t_n$. In fact, if we define $A^n_t \in \mathcal{A}_n$ to be the set $A^n_t = \bigcap_{A\in\mathcal{A}_n,\, t\in A} A$, then it is easy to show that C^n_t is the left-neighborhood of A^n_t.

Clearly, $C^n_t \cap C^m_t \neq \phi$ $\forall n, m$. Thus, if $n \leq m$, Proposition A.3.1 (a) implies $C^t_m \subseteq C^t_n$. Since $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is a decreasing family of σ-algebras, we have established the following result.

Lemma 3.4.1 *Given any $t \in T$, $(\mathcal{G}^*_{C^n_t})_n$ is a increasing family of σ-algebras.*

The proof of the classic Doob-Meyer decomposition theorem found in [30] and others relies on the existence of predictable projections for processes indexed by the points of $\mathbf{R}^+$. At the present time, it is unclear as to whether such a projection theorem holds in the set-indexed setting. Therefore, in the absence of such a result, we will simply assume such projections exist. This is the content of the following

Assumption 3.4.2 *To each $t \in T$, associate the σ-algebra $\mathcal{H}_t = \bigvee_n \mathcal{G}^*_{C^n_t}$. Then, given any set $F \in \mathcal{F}$, $\exists$ a collection $Y(F) = \{Y(F,t) : t \in T\}$ of random variables (i.e., a T-indexed process) s.t.*

(i) *the map* $(\omega, t) \mapsto Y(F,t)(\omega)$ *is* $\mathcal{P}^*$*-measurable and*

(ii) *for each* $t \in T$, $Y(F,t)$ *is a version of* $E[\mathbf{1}_F \mid \mathcal{H}_t]$.

Moreover, the process $Y(F)$ *is unique up to indistinguishability on* T.

An analogous assumption was required in Section 5 of [16] for the Doob-Meyer decomposition of weak submartingales.

In this subsection, we will show that, given any $F \in \mathcal{F}$, the T-indexed process $Y(F)$ from Assumption 3.4.2 can be expressed as a pointwise limit of explicit $\mathcal{P}^*$-measurable maps on $\Omega \times T$. This approximation of $Y(F)$ will play a central role in our proof of Theorem 3.4.11.

First, let $\text{Sub}(\mathcal{F})$ be the collection of all sub-σ-algebras of $\mathcal{F}$. By the axiom of choice, there exists a collection of random variables,

$$\mathcal{R} = \{R(F,\mathcal{G}) : F \in \mathcal{F},\ \mathcal{G} \in \text{Sub}(\mathcal{F})\}$$

s.t. for each $F \in \mathcal{F}$ and $\mathcal{G} \in \text{Sub}(\mathcal{F})$, $R(F,\mathcal{G})$ is a version of $E[\mathbf{1}_F \mid \mathcal{G}]$ satisfying $R(F,\mathcal{G})(\omega) \geq 0\ \forall \omega \in \Omega$. Given any $n \in \mathrm{N}$, $F \in \mathcal{F}$ and $t \in T$, define the random variable

$$Y^n(F,t) := R(F, \mathcal{G}^*_{C^n_t}). \tag{3.36}$$

Our present goal is to show that, for any $F \in \mathcal{F}$,

- the map $(\omega, t) \mapsto Y^n(F,t)(\omega)$ is $\mathcal{P}^*$-measurable $\forall n$ and
- $\exists$ a set, $\Omega_0 \in \mathcal{F}$ s.t. $P(\Omega_0) = 1$ and $\lim_n Y^n(F,t)(\omega) = Y(F,t)(\omega)$ $\forall (\omega,t) \in \Omega_0 \times T$

where $\{Y(F,t) : t \in T\}$ is the collection defined in Assumption 3.4.2.

Recall that $C^n_t \in \mathcal{N}_n$ is, by definition, the unique left-neighborhood containing t. Therefore, given any $D \in \mathcal{N}_n$, $t \in D$ if and only $D = C^n_t$. By (3.36), this implies

$$Y^n(F,t) = \sum_{D \in \mathcal{N}_n} R(F, \mathcal{G}^*_D) \cdot \mathbf{1}_D(t). \tag{3.37}$$

Lemma 3.4.3 *For any* $n \in \mathrm{N}$ *and* $F \in \mathcal{F}$, *the map* $(\omega, t) \mapsto Y^n(F,t)(\omega)$ *is* $\mathcal{P}^*$*-measurable.*

Proof In light of (3.37), it is sufficient to show

$$(\omega, t) \mapsto R(F, \mathcal{G}^*_D)(\omega) \cdot \mathbf{1}_D(t), \quad ((\omega, t) \in \Omega \times T)$$

is $\mathcal{P}^*$-measurable for any $D \in \mathcal{N}_n$. For this purpose, take $D \in \mathcal{N}_n$. Since $R(F, \mathcal{G}^*_D) \geq 0$ on Ω,

$$\{(\omega, t) : R(F, \mathcal{G}^*_D)(\omega)\, \mathbf{1}_D(t) \geq a\} = \begin{cases} \{\omega : R(F, \mathcal{G}^*_D)(\omega) \geq a\} \times D & \text{, if } a > 0 \\ \Omega \times T & \text{, if } a \leq 0 . \end{cases}$$

Furthermore, since $R(F, \mathcal{G}^*_D)$ is a version of $E[\mathbf{1}_F \,|\, \mathcal{G}^*_D]$,

$$\{\omega : R(F, \mathcal{G}^*_D)(\omega) \geq a\} \in \mathcal{G}^*_D, \quad (\forall\, a \in \mathbf{R}).$$

Therefore,

$$\{(\omega, t) : R(F, \mathcal{G}^*_D)(\omega)\, \mathbf{1}_D(t) \geq a\} \in \mathcal{P}^*_0, \quad (\forall\, a \in \mathbf{R}), \tag{3.38}$$

i.e., $(\omega, t) \mapsto R(F, \mathcal{G}^*_D)(\omega) \cdot \mathbf{1}_D(t)$ is $\mathcal{P}^*$-measurable. □

Lemma 3.4.4 *Given $F \in \mathcal{F}$, $\exists$ a set $\Omega^+ \in \mathcal{F}$ of full P-measure s.t. for any $(\omega, t) \in \Omega^+ \times T$, $\lim_n Y^n(F, t)(\omega) = Y(F, t)(\omega)$.*

Proof Let $F \in \mathcal{F}$ be given. If we define the function $Z^* : \Omega \times T \to \mathbf{R}^+$ by

$$Z^*(\omega, t) := \limsup_n Y^n(F, t)(\omega), \quad (\forall\, (\omega, t) \in \Omega \times T),$$

then by Lemma 3.4.3, Z^* is $\mathcal{P}^*$-measurable.

Given any $t \in T$, $Y^n(F, t)$ is, by definition, a version of $E[\mathbf{1}_F \,|\, \mathcal{G}^*_{C^n_t}]$ $\forall n$. Furthermore, by Lemma 3.4.1, $\mathcal{G}^*_{C^n_t} \uparrow \mathcal{H}_t$. Therefore, by the martingale convergence theorem, $Z^*(\cdot, t)$ is a version of $E(\mathbf{1}_F \,|\, \mathcal{H}_t)$. Applying this argument to each $t \in T$, $Z^*(\cdot, t)$ is a version of $E(\mathbf{1}_F \,|\, \mathcal{H}_t)$ $\forall t \in T$. Therefore, by the uniqueness of $Y(F)$ (see Assumption 3.4.2), $Y(F)$ and Z^* are indistinguishable as T-indexed processes, i.e., $\exists$ a set $\Omega^* \in \mathcal{F}$ of full P-measure s.t. for every $\omega \in \Omega^*$,

$$Z^*(w, t) = Y(F, t)(\omega), \quad (\forall\, t \in T). \tag{3.39}$$

Likewise, if we define the function $Z_* : \Omega \times T \to \mathbf{R}^+$ by

$$Z_*(\omega, t) := \liminf_n Y^n(F, t)(\omega), \quad (\forall\, (\omega, t) \in \Omega \times T),$$

then by repeating the above argument, $\exists$ a set $\Omega_* \in \mathcal{F}$ with $P(\Omega_*) = 1$ s.t. for every $\omega \in \Omega_*$,

$$Z_*(\omega, t) = Y(F, t)(\omega), \quad (\forall\, t \in T). \tag{3.40}$$

To complete the proof of Lemma 3.4.4, take $\Omega^+ = \Omega^* \cap \Omega_*$. □

Corollary 3.4.5 *If μ_X is an admissible measure on $\mathcal{P}^*$ (see p. 94) corresponding to some strong submartingale X, then given any $F \in \mathcal{F}$,*

$$\lim_n Y^n(F,t)(\omega) = Y(F,t)(\omega) \text{ for } \mu_X\text{-a.e. } (\omega,t) \in \Omega \times T.$$

Furthermore, $Y(F,t)(\omega) \geq 0$ for μ_X-a.e. $(\omega,t) \in \Omega \times T$.

Proof Let $\Omega^+ \in \mathcal{F}$ be the set of full P-measure defined in Lemma 3.4.4. Since $\mathcal{G}_T^* = \mathcal{F}_{\phi'}$ is a P-complete σ-algebra, $\Omega^+ \times T \in \mathcal{P}_0^*$. Furthermore,

$$\mu_X(\Omega^+ \times T) = E[1_{\Omega_+} X_T] = E[1_\Omega X_T] = \mu_X(\Omega \times T).$$

Since μ_X is a finite measure on $\mathcal{P}^*$, the result follows by Lemma 3.4.4. □

3.4.2 The definition of $*$-predictable processes

In Theorem 3.3.14, sufficient conditions were given under which the admissible function of a set-indexed strong submartingale extended to an admissible measure on $\mathcal{P}^*$. In order to define $*$-predictable set-indexed processes, we will need an additional extension theorem, the proof of which can be found in Proposition 4.1 of [16].

Proposition 3.4.6 *If $V = (V_A)_{A \in \mathcal{A}}$ is s.t.*

(i) *V is right-continuous but not necessarily adapted,*

(ii) *for each $C \in \mathcal{C}$, V has an increment defined at C and*

(iii) *$V_C \geq 0 \ \forall C \in \mathcal{C}$,*

then the admissible function μ_V of V has a unique extension to a measure on the product space $(\Omega \times T, \mathcal{F} \times \sigma(\mathcal{A}))$.

We now turn to the task of defining a class of set-indexed processes which are in some sense $*$-predictable. To begin with, note that the $*$-predictable σ-algebra $\mathcal{P}^*$ is a σ-algebra on the set $\Omega \times T$ whereas $\mathcal{A}$-indexed processes have domain $\Omega \times \mathcal{A}$. Therefore, we cannot simply define a $*$-predictable processes to be a $\mathcal{P}^*$-measurable map as is done in the classical theory. Instead, we define $*$-predictablility in an indirect fashion through Assumption 3.4.2 and Proposition 3.4.6.

Definition 3.4.7 *Given a process $V = (V_A)_{A \in \mathcal{A}}$ s.t.*

(i) *V is right-continuous but not necessarily adapted,*

(ii) *for each $C \in \mathcal{C}$, V has an increment defined at C and*

(iii) *$V_C \geq 0 \;\forall C \in \mathcal{C}$,*

V is said to be <u>*$*$-predictable*</u> *if for any $F \in \mathcal{F}$ and any $C \in \mathcal{C}$,*

$$E[1_F V_C] = \int_{\Omega \times C} Y(F,t)(\omega)\, d\mu_V(\omega,t) \tag{3.41}$$

where the integral on the right-hand side of (3.41) *is taken w.r.t. the measure space $(\Omega \times T,\, \mathcal{P}^*,\, \mu_V)$. (Note that $\Omega \times C \in \mathcal{P}_0^* \;\forall C \in \mathcal{C}$.)*

Remark 3.4.8 Our definition of $*$-predictability is motivated by Definition 4.1 of [16] in which Dozzi et. al. defined the class of predictable set-indexed processes (see Remark 3.3.2 (b)). In contrast, our definition does not require $*$-predictable processes to be adapted.

3.4.3 The Doob-Meyer decomposition

Before presenting the said decomposition, we need to make two assumptions on our indexing collection $\mathcal{A}$. The first assumption has already appeared in several set-indexed papers (see [16], [23] and [28]).

Assumption 3.4.9 *Given any $C = A \setminus \bigcup_{i=1}^{n} A_i \in \mathcal{C}$, $\exists$ a maximal representation $A \setminus \bigcup_{i=1}^{k} B_i$ of C (see Definition 3.2.6* (b)*).*

On the other hand, our second assumption is new.

Assumption 3.4.10 *If $A \setminus \bigcup_{i=1}^{k} A_i = A' \setminus \bigcup_{j=1}^{k'} A'_j \in \mathcal{C}$, then $\exists\, N \in \mathbf{N}$ s.t.*

$$g_n(A) \setminus \bigcup_{i=1}^{k} g_n(A_i) = g_n(A') \setminus \bigcup_{j=1}^{k'} g_n(A'_j)$$

for each $n \geq N$.

In the absence of the shape property, Assumption 3.4.10 will ensure that the process V in Theorem 3.4.11 has increments defined at every $C \in \mathcal{C}$. Both Assumption 3.4.9 and Assumption 3.4.10 are satisfied by Examples 2.2.6 and 2.8.1.

We now state and prove the main result of this section, a Doob-Meyer-type decomposition for set-indexed strong submartingales. Our proof is similar to that of Theorem 5.1 in [16]. For (3.42) and (3.43), it is important to recall that the sequence $(\mathcal{A}_n)_n$ from Definition 2.2.2 is increasing in n w.r.t. $\subseteq$ and that $\{g_m(A) : A \in \mathcal{A}\} \subseteq \mathcal{A}_m \; \forall m$.

Theorem 3.4.11 *Under Assumptions 3.4.2, 3.4.9 and 3.4.10, given an L_1-right-continuous strong submartingale $X = (X_A)_{A\in\mathcal{A}}$ of class $(D')^*$, $\exists$ processes $M = (M_A)_{A\in\mathcal{A}}$ and $V = (V_A)_{A\in\mathcal{A}}$ s.t.*

(i) $X_A = M_A + V_A \;\; \forall A \in \mathcal{A}$,

(ii) *for each $C \in \mathcal{C}$, both M and V have increments defined at C,*

(iii) $E[M_C \,|\, \mathcal{G}_C^*] = 0 \;\; \forall C \in \mathcal{C}$ *and*

(iv) *V is right-continuous and $*$-predictable with $V_C \geq 0 \;\; \forall C \in \mathcal{C}$.*

(M and V are not necessarily adapted and do not necessarily possess finitely additive extensions to $C(u)$.) Moreover, given any $A \in \mathcal{A}$,

$$V_A \;=\; \lim_{m\to\infty} \lim_{n\to\infty,\, n\geq m} V^{(n)}_{g_m(A)} \quad \text{weakly in } L_1 \tag{3.42}$$

where, given any $k \in \mathbb{N}$ and $B \in \mathcal{A}_n$, $V_B^{(k)}$ is the random variable

$$V_B^{(k)} \;=\; \sum_{\substack{D\in\mathcal{N}_k \\ D\subseteq B}} E[X_D \,|\, \mathcal{G}_D^*]. \tag{3.43}$$

Proof We start by establishing the limit in (3.42). We will then show that the processes V and $M = X - V$ satisfy properties (ii), (iii) and (iv). The following technical results will be used several times in the proof.

Result I: *Given random variables $Z_1 \in L_\infty$, $Z_2 \in L_1$ and a sub-σ-algebra, $\mathcal{G}$ of $\mathcal{F}$, $E[\,E(Z_1 \,|\, \mathcal{G}) \cdot Z_2] = E[Z_1 \cdot E(Z_2 \,|\, \mathcal{G})\,]$.*

Proof: Conditioning w.r.t. $\mathcal{G}$, we can write

$$\begin{aligned} E[E(Z_1 \,|\, \mathcal{G}) \cdot Z_2] &= E\{E[E(Z_1 \,|\, \mathcal{G}) \cdot Z_2 \,|\, \mathcal{G}]\} \\ &= E[E(Z_1 \,|\, \mathcal{G}) \cdot E(Z_2 \,|\, \mathcal{G})]. \end{aligned}$$

Result I follows by symmetry. Ω

For the second technical result, note that by Theorem 3.3.14, the admissible function μ_X extends to a measure, also denoted μ_X on $\mathcal{P}^*$.

Result II: *Let $D \in \mathcal{C}$ be given. If $g \in L_\infty$ is s.t.*

$$\{(\omega, t) : g(\omega)\, \mathbf{1}_D(t) \geq a\} \in \mathcal{P}_0^*, \quad (\forall\, a \in \mathbf{R}), \tag{3.44}$$

then $\int_{\Omega\times D} g\, d\mu_X = E(g X_D)$. (Note that under condition (3.44), the map $(\omega, t) \mapsto g(\omega)\, \mathbf{1}_D(t)$ is $\mathcal{P}^*$-measurable.)

Proof: First, assume $g = \mathbf{1}_G$ for some $G \in \mathcal{F}$. If (3.44) holds, then

$$G \times D = \{(\omega, t) : g(\omega)\, \mathbf{1}_D(t) \geq 1\} \in \mathcal{P}_0^*$$

and therefore,

$$\int_{\Omega\times D} \mathbf{1}_G\, d\mu_X = \mu_X(G \times D) = E(\mathbf{1}_G X_D). \tag{3.45}$$

The result for a general $g \in L_\infty$ follows by the linearity of integration and the monotone convergence theorem applied to (3.45). Ω

To establish the limit in (3.42), fix $m \in \mathbf{N}$ and $A \in \mathcal{A}_m$. Since $\Omega \times A \in \mathcal{P}_0^*$, we can define the number

$$\sigma_A(F) := \int_{\Omega\times A} Y(F)\, d\mu_X$$

for any $F \in \mathcal{F}$ where $Y(F)$ is the $\mathcal{P}^*$-measurable process in Assumption 3.4.2.

Given any $F \in \mathcal{F}$, recall the sequence $(Y^n(F))_n$ of $\mathcal{P}^*$-measurable maps defined in (3.36). Since $|Y^n(F)| \leq 1 \ \forall n$ and $Y^n(F) \to Y(F)$ μ_X-a.e. (see Corollary 3.4.5), the dominated convergence theorem implies

$$\sigma_A(F) = \lim_{n\to\infty} \int_{\Omega\times A} Y^n(F)\, d\mu_X. \tag{3.46}$$

We now isolate an important identity.

Claim I: *Given $F \in \mathcal{F}$ and $n \geq m$, $\int_{\Omega\times A} Y^n(F)\, d\mu_X = E[\mathbf{1}_F\, V_A^{(n)}]$.*

Proof: Since $n \geq m$, $A \in \mathcal{A}_n$. Therefore, given any $D \in \mathcal{N}_n$, $D \subseteq A$ if and only if $D \cap A \neq \phi$. By (3.37), this implies

$$Y^n(F,t)\,\mathbf{1}_A(t) = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} R(F,\mathcal{G}_D^*)\,\mathbf{1}_D(t) \tag{3.47}$$

which further implies

$$\int_{\Omega\times A} Y^n(F)\, d\mu_X = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} \int_{\Omega\times D} R(F,\mathcal{G}_D^*)\, d\mu_X. \tag{3.48}$$

But by (3.38), $g = R(F,\mathcal{G}_D^*)$ satisfies (3.44) $\forall\, D \in \mathcal{N}_n$. Therefore, by Result II, (3.48) becomes

$$\begin{aligned}\int_{\Omega\times A} Y^n(F)\, d\mu_X &= \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} E[\,E(\mathbf{1}_F \,|\, \mathcal{G}_D^*)\cdot X_D] \\ &= \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} E[\mathbf{1}_F \cdot E(X_D \,|\, \mathcal{G}_D^*)\,] \quad \text{(by Result I).}\end{aligned}$$

Claim I now follows by (3.43). Ω

Working with our fixed $m \in \mathrm{N}$ and $A \in \mathcal{A}_m$, Claim I and (3.46) imply

$$\sigma_A(F) = \lim_{\substack{n\to\infty \\ n \geq m}} E[\mathbf{1}_F\, V_A^{(n)}]\,, \quad (\forall\, F \in \mathcal{F}).$$

Therefore, by the Hahn-Vitali-Saks Theorem, $\exists\, \overline{V}_A \in L_1$ s.t.

$$\sigma_A(F) = E[\mathbf{1}_F\, \overline{V}_A], \quad (\forall\, F \in \mathcal{F}). \tag{3.49}$$

That is, $(V_A^{(n)})_{n\geq m}$ converges to $\overline{V}_A$ in the weak L_1 topology.

Applying the above argument to each $A \in \mathcal{A}_m$ (with m still fixed), we obtain a collection $\{\overline{V}_A : A \in \mathcal{A}_m\}$ of random variables with each $\overline{V}_A$ satisfying (3.49). By Corollary A.4.4, this collection can be extended to a collection

$\{\overline{V}_C : C \in \mathcal{C}_m\}$ s.t. given any $C \in \mathcal{C}_m$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C with $A, A_i \in \mathcal{A}_m$,

$$\overline{V}_C(\omega) = \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{V}(A \cap \textstyle\bigcap_{i\in I} A_i)(\omega)\,, \quad (\forall \omega \in \Omega') \tag{3.50}$$

where Ω' is an event of full P-measure which is independent of both C and m. $\{\overline{V}_C : C \in \mathcal{C}_m\}$ satisfies the following property.

Claim II: *Given any $C \in \mathcal{C}_m$, $E[\mathbf{1}_F \overline{V}_C] = \int_{\Omega\times C} Y(F)\, d\mu_X \;\; \forall F \in \mathcal{F}$.*

Proof: All limits in the following proof are taken as $n \to \infty$ with $n \geq m$.

Let $A \setminus \bigcup_{i=1}^k A_i$ be a representation of C with $A, A_i \in \mathcal{A}_m$. Since $C \in \mathcal{C}_n$ $\forall n \geq m$,

$$\begin{aligned}
E[\mathbf{1}_F \overline{V}_C] &= \lim \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot E\left[\mathbf{1}_F \cdot V^{(n)}(A \cap \textstyle\bigcap_{i\in I} A_i)\right] \quad \text{(by (3.50))} \\
&= \lim E\left\{ \mathbf{1}_F \cdot \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C}} E[X_D \,|\, \mathcal{G}_D^*] \right\} \quad \text{(by Lemma A.4.1)} \\
&= \lim \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C}} E[\, E(\mathbf{1}_F \,|\, \mathcal{G}_D^*) \cdot X_D \,] \quad \text{(by Result I).}
\end{aligned}$$

By (3.38), we can apply Result II to obtain

$$E[\, E(\mathbf{1}_F \,|\, \mathcal{G}_D^*) \cdot X_D] = \int_{\Omega\times D} R(F, \mathcal{G}_D^*)\, d\mu_X\,, \quad (\forall D \in \mathcal{N}_n \text{ s.t. } D \subseteq C)$$

for any $n \geq m$. Substituting these identities into the above array,

$$\begin{aligned}
E[\mathbf{1}_F \overline{V}_C] &= \lim \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C}} \int_{\Omega\times D} R(F, \mathcal{G}_D^*)\, d\mu_X \\
&= \lim \int_{\Omega\times C} Y^n(F)\, d\mu_X \quad \text{(by Lemma A.3.1 (c))} \\
&= \int_{\Omega\times C} Y(F)\, d\mu_X \quad \text{(by dominated convergence)}
\end{aligned}$$

which establishes Claim II. Ω

Given any $C \in \mathcal{C}_m$, Claim II and Corollary 3.4.5 imply

$$E[\mathbf{1}_F \overline{V}_C] \geq 0\,, \quad (\forall F \in \mathcal{F})$$

and therefore, $\overline{V}_C \geq 0$ a.s. Applying this argument to every $m \in \mathrm{N}$ and every $C \in \mathcal{C}_m$, we have thus shown

$$\overline{V}_C \geq 0 \text{ a.s.}\,, \quad (\forall C \in \mathcal{C}_m,\, m \in \mathrm{N}). \tag{3.51}$$

Define the collections

$$\mathcal{A}^* := \textstyle\bigcup_m \mathcal{A}_m \quad \text{and} \quad \mathcal{C}^* := \bigcup_m \mathcal{C}_m. \tag{3.52}$$

Since $\mathcal{C}^*$ is countable, (3.51) implies $\exists\, \Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ s.t.

$$\overline{V}_C(\omega) \geq 0, \quad (\forall C \in \mathcal{C}^*,\, \omega \in \Omega_0). \tag{3.53}$$

We can assume w.l.o.g. that $\Omega_0 \subseteq \Omega'$ where Ω' is the event of full P-measure introduced in (3.50). (In this way, (3.50) holds for every $C \in \mathcal{C}^*$ and $\omega \in \Omega_0$.) Now, take any $A, A' \in \mathcal{A}^*$ s.t. $A \subseteq A'$. Since $(\mathcal{A}_m)_m$ is increasing w.r.t. $\subseteq$, $\exists\, m \in \mathrm{N}$ s.t. $A, A' \in \mathcal{A}_m$. By (3.50) and (3.53), this implies

$$\overline{V}_{A'} - \overline{V}_A(\omega) = \overline{V}_{A' \setminus A}(\omega) \geq 0\,, \quad (\forall \omega \in \Omega_0).$$

Therefore, for each $\omega \in \Omega_0$, $(\overline{V}_A(\omega))_{A \in \mathcal{A}^*}$ is increasing on $\mathcal{A}^*$. But given any $A \in \mathcal{A}$, $(g_m(A))_m$ is decreasing in $\mathcal{A}^*$. Therefore, we can define a random variable V_A by letting

$$V_A = \lim\nolimits_m \left(\mathbf{1}_{\Omega_0} \cdot \overline{V}_{g_m(A)} \right). \tag{3.54}$$

Furthermore, by the monotone convergence theorem,

$$\lim\nolimits_m E\left(\mathbf{1}_F \cdot \overline{V}_{g_m(A)} \right) = E(\mathbf{1}_F V_A)\,, \quad (\forall F \in \mathcal{F}),$$

i.e., $(\overline{V}_{g_m(A)})_m$ converges to V_A in the weak L_1 topology. This establishes the limit in (3.42).

Now, by Lemma A.4.6, V can be extended to a collection $(V_C)_{C \in \mathcal{C}}$ of random variables s.t. for any $C \in \mathcal{C}$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C,

$$V_C = \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V(A \cap \textstyle\bigcap_{i \in I} A_i), \tag{3.55}$$

i.e., V has increments defined at every $C \in \mathcal{C}$. This is precisely (ii). (The process $M = X - V$ inherits increments from X and V.) Furthermore,

Claim III: *Given any $C \in \mathcal{C}$, $V_C \geq 0$ a.s.*

Proof: Take a representation $A \setminus \bigcup_{i=1}^k A_i$ of C and consider $(C^m)_m$ where $C^m \in \mathcal{C}_m$ is defined by $C^m = g_m(A) \setminus \bigcup_{i=1}^k g_m(A_i)\ \ \forall m$. Since each g_m preserves finite intersections, we can write

$$\begin{aligned} V_C &= \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V(A \cap \textstyle\bigcap_{i \in I} A_i) \\ &= \lim_m \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V(g_m(A) \cap \textstyle\bigcap_{i \in I} g_m(A_i)) \\ &= \lim_m \overline{V}_{C^m}. \end{aligned}$$

But by (3.53), $\overline{V}_{C^m} \geq 0$ a.s. $\forall m$. Therefore, $V_C \geq 0$ a.s. Ω

To complete the verification of (iv), it remains to be shown that V is right-continuous and $*$-predictable. We begin with right-continuity.

Claim IV: *Given any $\omega \in \Omega_0$, if $A_n \downarrow A$ in $\mathcal{A}$, then $V_{A_n}(\omega) \to V_A(\omega)$.*

Proof: Take a point $\omega \in \Omega_0$ and a sequence $A_n \downarrow A$ in $\mathcal{A}$. The proof will require two subclaims. (We will suppress the point ω with the understanding that all random variables in Subclaims I and II are evaluated at ω.)

Subclaim I: $V_A = \inf_{B \in \mathcal{A}^*, A \subseteq B^\circ} \overline{V}_B$.

Proof: Since $A \subseteq [g_m(A)]^\circ\ \ \forall m$ (see Definition 2.2.2 (2)(ii′)), (3.54) implies

$$\inf_{\substack{B \in \mathcal{A}^* \\ A \subseteq B^\circ}} \overline{V}_B \leq V_A.$$

For the opposite inequality, take any $B \in \mathcal{A}^*$ s.t. $A \subseteq B^\circ$. By Corollary A.2.4, $\exists\, m_0 \in \mathrm{N}$ s.t. $g_{m_0}(A) \subseteq B$. Therefore, since $\overline{V}_{g_m(A)} \downarrow V_A$,

$$V_A = \lim_{m \geq m_0} \overline{V}_{g_m(A)} \leq \overline{V}_{g_{m_0}(A)} \leq \overline{V}_B.$$

Subclaim I follows by taking the infimum over all $B \in \mathcal{A}^*$ s.t. $A \subseteq B^\circ$. ω

Subclaim II: *If $(B_m)_m$ in $\mathcal{A}^*$ is s.t. $A \subseteq (B_m)^\circ$ $\forall m$ and $B_m \downarrow A$, then* $V_A = \lim_m \overline{V}_{B_m}$.

Proof: Since $A \subseteq (B_m)^\circ$ $\forall m$, Subclaim I implies $V_A \leq \lim_m \overline{V}_{B_m}$. To obtain the opposite inequality, we argue as in Subclaim I, only now using Lemmas A.2.2 and A.2.3 in place of Corollary A.2.4. ω

Returning to the proof of Claim IV, note that (3.54) implies

$$\lim_n V_{A_n} = \lim_n \left[\lim_m \overline{V}_{g_m(A_n)}\right].$$

Thus, by a diagonalization argument, $\exists$ a subsequence $(m_n)_n$ s.t.

$$\lim_n \left[\lim_m \overline{V}_{g_m(A_n)}\right] = \lim_n \overline{V}_{g_{m_n}(A_n)}. \tag{3.56}$$

Clearly, $A \subseteq [g_{m_n}(A_n)]^\circ$ $\forall n$ and $g_{m_n}(A_n) \downarrow A$. Therefore,

$$\begin{aligned} V_A &= \lim_n \overline{V}_{g_{m_n}(A_n)} \quad \text{(by Subclaim II)} \\ &= \lim_n V_{A_n} \quad \text{(by (3.56))} \end{aligned}$$

which completes the proof of Claim IV. Ω

So far, by Claims III and IV, the process V is right-continuous with non-negative increments defined at every $C \in \mathcal{C}$. In view of Definition 3.4.7, this makes V a candidate for a $*$-predictable process. Indeed, given any $F \in \mathcal{F}$ and $C \in \mathcal{C}$,

$$\begin{aligned} E[\mathbf{1}_F V_C] &= \lim_m E[\mathbf{1}_F \overline{V}_{C^m}] \\ &= \lim_m \int_{\Omega\times C^m} Y(F)\, d\mu_X \quad \text{(by Claim II)} \\ &= \int_{\Omega\times C} Y(F)\, d\mu_X \quad \text{(by dominated convergence).} \end{aligned}$$

Although similar in appearance to (3.41), the above identity does not imply that V is $*$-predictable. In particular, the last integral is taken w.r.t. the measure μ_X, not μ_V as required.

On the other hand, if we can show

$$\mu_X(F\times C) = \mu_V(F\times C), \quad (\forall F\times C \in \mathcal{P}_0^*), \tag{3.57}$$

then it will follow that $\mu_X = \mu_V$ on $\mathcal{P}^*$ and property (iv) will be established. Moreover, if we define the process $M = (M_A)_{A\in\mathcal{A}}$ by

$$M_A := X_A - V_A, \quad (\forall A \in \mathcal{A}),$$

then properties (i) and (ii) are trivially satisfied and (3.57) will imply

$$\mu_M(F\times C) = 0, \quad (\forall F\times C \in \mathcal{P}_0^*),$$

a condition which is equivalent to property (iii) (see the proof of Lemma 3.3.6 (a)). In other words,

Reduction: *If we can establish* (3.57), *the proof of Theorem 3.4.11 will be complete.*

With this reduction in mind, take $F\times C \in \mathcal{P}_0^*$. By Assumption 3.4.9, $\exists$ a representation $A \setminus \bigcup_{i=1}^k B_i$ of C s.t.

$$B \in \mathcal{A}(u) \text{ and } B\cap C = \phi \implies B \subseteq \bigcup_{i=1}^k B_i. \tag{3.58}$$

Once again, take $(C^m)_m$ in $\mathcal{C}^*$ to be s.t. $C^m = g_m(A) \setminus \bigcup_{i=1}^k g_m(A_i)$ $\forall m$.

Claim V: $\mathcal{G}_C^* \subseteq \mathcal{G}_{C^m}^*$ $\forall m$.

Proof: Given any $B \in \mathcal{A}(u)$,

$$\begin{aligned} B\cap C = \phi &\implies B \subseteq \textstyle\bigcup_{i=1}^k B_i \quad \text{(by (3.58))} \\ &\implies B \subseteq \textstyle\bigcup_{i=1}^k g_m(B_i) \;\; \forall m \\ &\implies B \subseteq C^m = \phi \;\; \forall m. \end{aligned}$$

Thus, Claim V follows by applying Definition 3.2.21 (b) to each $m \in \mathbf{N}$. Ω

Working with the above $F \times C \in \mathcal{P}_0^*$, since X is L_1-right-continuous, Remark 3.2.49 implies

$$E[\mathbf{1}_F X_C] = \lim_m E[\mathbf{1}_F X_{C^m}]. \tag{3.59}$$

Furthermore, given m, Corollary A.3.2 implies $X_{C^m} = \sum_{D\in\mathcal{N}_n, D\subseteq C^m} X_D$ $\forall n \geq m$, allowing us to write

$$E[\mathbf{1}_F X_{C^m}] = \lim_{\substack{n\to\infty \\ n\geq m}} \sum_{\substack{D\in\mathcal{N}_n \\ D\subseteq C^m}} E[\mathbf{1}_F X_D], \quad (\forall m). \tag{3.60}$$

Since $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is a decreasing and $F \in \mathcal{G}^*_C$, Claim V implies $F \in \mathcal{G}^*_D \ \forall D \subseteq C^m$ s.t. $D \in \mathcal{N}_n$ $(m, n \in \mathrm{N},\, n \geq m)$. Therefore,

$$
\begin{aligned}
E[1_F X_C] &= \lim_m \lim_{\substack{n\to\infty \\ n\geq m}} \sum_{\substack{D\in\mathcal{N}_n \\ D\subseteq C^m}} E[1_F X_D] \quad \text{(by (3.59) and (3.60))} \\
&= \lim_m \lim_{\substack{n\to\infty \\ n\geq m}} \sum_{\substack{D\in\mathcal{N}_n \\ D\subseteq C^m}} E[1_F E(X_D \mid \mathcal{G}^*_D)] \quad \text{(by conditioning)} \\
&= \lim_m \lim_{\substack{n\to\infty \\ n\geq m}} E\left[1_F \cdot \sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot V^{(n)}_{g_m(A)\cap(\cap_{i\in I} g_m(A_i))}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma A.4.1)} \\
&= E[1_F V_C] \quad \text{(by (3.42) and (3.55))}.
\end{aligned}
$$

This establishes (3.57), thus completing the proof of Theorem 3.4.11. □

Next, we show that the decomposition in Theorem 3.4.11 is unique up to indistinguishability. The proof is essentially that of Theorem 4.1 on p.518 of [16], nonetheless, we repeat it here for the sake of completeness.

Proposition 3.4.12 *Let $X = (X_A)_{A\in\mathcal{A}}$ be a strong submartingale. Under Assumptions 3.4.2, 3.4.9 and 3.4.10, if $\exists$ processes $M = (M_A)_{A\in\mathcal{A}}$ and $V = (V_A)_{A\in\mathcal{A}}$ s.t.*

(i) $X_A = M_A + V_A \quad \forall A \in \mathcal{A}$,

(ii) *for each $C \in \mathcal{C}$, both M and V have increments defined at C,*

(iii) $E[M_C \mid \mathcal{G}^*_C] = 0 \ \forall C \in \mathcal{C}$ *and*

(iv) *V is right-continuous and $*$-predictable with $V_C \geq 0 \ \forall C \in \mathcal{C}$,*

then both V and M are unique up to indistinguishablity.

Proof Assume such a decomposition exists and take two additional processes $M' = (M'_A)_{A\in\mathcal{A}}$ and $V' = (V'_A)_{A\in\mathcal{A}}$ satisfying conditions (i) through (iv). By (i), it is sufficient to show V and V' are indistinguishable. Furthermore, since V and V' are right-continuous, if we can show

$$V_A = V'_A \text{ a.e.}, \quad (\forall A \in \mathcal{A}), \tag{3.61}$$

then Lemma 3.2.45 will imply that V and V' are indistinguishable.

By the linearity of integration, it is clear that $\mu_X = \mu_M + \mu_V$ on $\mathcal{P}_0^*$. Furthermore, (iii) implies

$$\mu_M(F \times C) = E[\mathbf{1}_F M_C] = 0\,, \quad (\forall F \times C \in \mathcal{P}_0^*).$$

Therefore, $\mu_X = \mu_V$ on $\mathcal{P}_0^*$ and, by an identical argument, $\mu_X = \mu_{V'}$ on $\mathcal{P}_0^*$, implying $\mu_V = \mu_{V'}$ on $\mathcal{P}_0^*$. But by Proposition 3.4.6, both μ_V and $\mu_{V'}$ extend to finite measures on $\mathcal{P}^*$. Therefore, since μ_V and $\mu_{V'}$ coincide on the π-system $\mathcal{P}_0^*$, μ_V and $\mu_{V'}$ coincide on the σ-algebra $\mathcal{P}^* = \sigma(\mathcal{P}_0^*)$.

Now, take any $C \in \mathcal{C}$. Since V and V' are $*$-predictable,

$$\begin{aligned} E[\mathbf{1}_F V_C] &= \int_{\Omega\times C} Y(F,t)(\omega)\, d\mu_V(\omega,t) \\ &= \int_{\Omega\times C} Y(F,t)(\omega)\, d\mu_{V'}(\omega,t) \;=\; E[\mathbf{1}_F V'_C] \end{aligned}$$

for every $F \in \mathcal{F}$. Therefore, $V_C = V'_C$ a.s. Since $\mathcal{A} \subseteq \mathcal{C}$ (see Remark 3.2.2 (b), this establishes (3.61) and completes the proof. □

3.4.4 Additional comments and results

Following the development in Section 4 of [16], it would appear as though a "direct" proof to Theorem 3.4.11, i.e., one which avoids discrete approximations, is possible. Although such an approach may require additional technical machinery such as p-stochastic measures (see the Remark on p.515 of [16]), it would have one advantage, a weakening of Assumption 3.4.2. Specifically, in such an approach we could replace the family $(\mathcal{H}_t)_{t\in T}$ in Assumption 3.4.2 by any family $(\mathcal{F}_t^-)_{t\in T}$ satisfying the condition

$$t \in C \implies \mathcal{G}_C^* \subseteq \mathcal{F}_t^-\,, \quad (\forall C \in \mathcal{C}). \tag{3.62}$$

This is analogous to Assumption (A3) in [16]. Indeed,

Proposition 3.4.13 *Given any $C \in \mathcal{C}$, if $t \in C$, then $\mathcal{G}_C^* \subseteq \mathcal{H}_t$.*

Proof By Assumption 3.4.9, $\exists$ a maximal representation $A \setminus \cup_{i=1}^m B_i$ of C. For each $n \in \mathbf{N}$, define

$$C^n := g_n(A) \setminus \cup_{i=1}^m g_n(B_i) \;\in\; \mathcal{C}_n.$$

If $t \in C$, then by Lemma A.2.8, $\exists n_0 \in \mathbb{N}$ s.t. $t \in C^{n_0}$.

As done earlier, let $C_t^{n_0}$ denote the unique left-neighborhood in $\mathcal{N}_{n_0}$ containing t. Since both $C_t^{n_0} \in \mathcal{N}_{n_0}$ and $C^{n_0} \in \mathcal{C}_{n_0}$ contain t, Lemma A.3.3 implies $C_t^{n_0} \subseteq C^{n_0}$. Thus, since $(\mathcal{G}_C^*)_{C \in \mathcal{C}}$ is decreasing,

$$\mathcal{G}_{C^{n_0}}^* \subseteq \mathcal{G}_{C_t^{n_0}}^* \subseteq \mathcal{H}_t.$$

Furthermore, by Claim V in the proof of Theorem 3.4.11,

$$\mathcal{G}_C^* \subseteq \mathcal{G}_{C^{n_0}}^*.$$

Combining these inclusions yields $\mathcal{G}_C^* \subseteq \mathcal{H}_t$. □

We close this section with a result which, more than Theorem 3.4.11, resembles the classic Doob-Meyer decomposition theorem. This result depends heavily on the development in Appendix B.

Theorem 3.4.14 *Let $X = (X_A)_{A \in \mathcal{A}}$ be an L_1-right-continuous strong submartingale of class $(D')^*$. Under Assumption Groups D.1 and D.3, $\exists$ processes $M = (M_A)_{A \in \mathcal{A}}$ and $V = (V_A)_{A \in \mathcal{A}}$ s.t.*

(i) $X_A = M_A + V_A \quad \forall A \in \mathcal{A}$,

(ii) *M is a strong martingale and*

(iii) *V is right-continuous, increasing and $*$-predictable.*

Moreover, V satisfies (3.42) *and is unique up to indistinguishablility.*

Proof Under Assumption Group D.3, Theorem 3.4.11 holds and both M and V possess unique finitely additive extensions to $\mathcal{C}(u)$. Therefore, all that remains to be shown is that both M and V are adapted to $(\mathcal{F}_A)_{A \in \mathcal{A}}$. By (i), it is sufficient to show V is adapted.

For this purpose, take any $A \in \mathcal{A}$ and let $L_1(\mathcal{F}_A)$ denote the linear subspace of L_1 consisting of all $X \in L_1$ s.t. X is $\mathcal{F}_A$-measurable. Since $L_1(\mathcal{F}_A)$ is closed w.r.t. the L_1-norm, Corollary 1.5 on p.126 of [12] implies $L_1(\mathcal{F}_A)$ is closed w.r.t. the weak L_1 topology. Therefore, if we can show

$$\{V_A^{(n)} : A \in \mathcal{A}_n,\, n \in \mathbb{N}\} \subseteq L_1(\mathcal{F}_A), \tag{3.63}$$

then (3.42) will imply $V_A \in L_1(\mathcal{F}_A)$ and the proof will be complete.

With this reduction in mind, take any $n \in \mathbb{N}$ and $A \in \mathcal{A}_n$. By (3.43),

$$V_A^{(n)} = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} E[X_D \,|\, \mathcal{G}_D^*].$$

But given any $D \in \mathcal{N}_n$, $\exists A' \in \mathcal{A}_n$ s.t. $D = A' \setminus \bigcup_{A'' \in \mathcal{A}_n, A' \not\subseteq A''} A''$. Thus, if $D \subseteq A$, Lemma A.3.6 implies $A' \subseteq A$ which by (3.12) implies $X_D \in L_1(\mathcal{F}_A)$. Therefore, by Proposition B.3.10,

$$E[X_D \,|\, \mathcal{G}_D^*] \in L_1(\mathcal{F}_A), \quad (\forall D \in \mathcal{N}_n \text{ s.t. } D \subseteq A).$$

This implies $V_A^{(n)} \in L_1(\mathcal{F}_A)$, completing the proof of Theorem 3.4.14. □

3.5 The Existence of ∗-Predictable Quadratic Variation for Set-Indexed Strong Martingales

Let $M = (M_t)_{t \geq 0}$ be a cadlag martingale in L_2 w.r.t. some filtration $(\mathcal{F}_t)_{t \geq 0}$. As mentioned in Section 3.1, the process M^2 is a submartingale of class D and as such possesses a unique predictable quadratic variation $\langle M \rangle$. Since $M^2 - \langle M \rangle$ is a martingale,

$$E[M_t^2 - M_s^2 | \mathcal{F}_s] = E[\langle M \rangle_t - \langle M \rangle_s | \mathcal{F}_s], \quad (\forall s < t \text{ in } [0, \infty)). \qquad (3.64)$$

Recall that $\langle M \rangle$ is non-negative, right-continuous, increasing and predictable. We wish to duplicate this development for set-indexed strong martingales.

3.5.1 The problem

Given a strong martingale $M = (M_A)_{A \in \mathcal{A}}$ in L_2, the process M^2 defined by $M_A^2 = (M_A)^2$ $(A \in \mathcal{A})$ is not necessarily a set-indexed strong submartingale. This shortcoming for planar strong martingales was observed by Gushchin in Section 6 of [18].

For example, if $C = A \setminus A' \in \mathcal{C}$ where $A, A' \in \mathcal{A}$ are s.t. $A' \subseteq A$, then (3.12) implies $M_C^2 = M_A^2 - M_{A'}^2$ so that

$$E[M_C^2 \,|\, \mathcal{G}_C^*] = E[M_A^2 \,|\, \mathcal{G}_C^*] - M_{A'}^2.$$

But the relation $M^2_{A'} \leq E[M^2_A \,|\, \mathcal{G}^*_C]$ $(A' \subseteq A)$ is not a consequence of Definition 3.2.38 or any of the earlier assumptions. We highlight this shortcoming for future reference.

Observation 3.5.1 *The square of a set-indexed strong martingale in L_2 need not be a set-indexed strong submartingale.*

Furthermore, it is clear from (3.12) that M^2_C is not necessarily non-negative so that we could conceivably have

$$E[M^2_C \,|\, \mathcal{G}^*_C] \not\geq 0\,, \quad (\text{some } C \in \mathcal{C}).$$

This is unacceptable in view of (3.64). In particular, a set-indexed quadratic variation process should have non-negative increments over the sets in $\mathcal{C}$, a condition which is equivalent to increasing when $\mathcal{A} = \{[0,x] : x \in [0,1]\}$.

3.5.2 The solution

However, note that (3.64) is equivalent to

$$E[(M_t - M_s)^2 | \mathcal{F}_s] \;=\; E[\langle M\rangle_t - \langle M\rangle_s | \mathcal{F}_s]\,, \quad (\forall\, s < t \text{ in } [0,\infty)).$$

Therefore, following the lead of Gushchin in [18], we work with the collection of squared increments,

$$\{(M_C)^2 : C \in \mathcal{C}\} \tag{3.65}$$

rather that the collection of increments of the square, $\{M^2_C : C \in \mathcal{C}\}$. Thus, given a strong martingale $M = (M_A)_{A\in\mathcal{A}}$ in L_2, we are no longer interested in obtaining a quadratic variation process in the sense of (3.64). Instead, we have the following result whose proof is the goal of the present section.

Theorem 3.5.2 *Under Assumptions 3.4.2, 3.4.9 and 3.4.10, if*

$$\left\{ \sum_{D\in\mathcal{N}_0} E[\,(M_D)^2 \,|\, \mathcal{G}^*_D] \;:\; \mathcal{N}_0 \text{ a f.n.s. of } \mathcal{A} \right\} \tag{3.66}$$

is uniformly integrable, $\exists$ a process $Q = (Q_A)_{A\in\mathcal{A}}$ s.t.

(i) *Q has increments defined at every $C \in \mathcal{C}$,*

(ii) $Q_C \geq 0 \;\; \forall C \in \mathcal{C}$,

(iii) $E[(M_C)^2 \mid \mathcal{G}_C^*] = E[Q_C \mid \mathcal{G}_C^*] \;\; \forall C \in \mathcal{C}$ *and*

(iv) Q *is right-continuous and* $*$*-predictable.*

(Note that Q is not necessarily adapted nor does Q necessarily possess a finitely additive extension to $\mathcal{C}(u)$.) Moreover,

$$Q_A \;=\; \lim_{m\to\infty} \lim_{n\to\infty,\, n\geq m} Q^{(n)}_{g_m(A)} \;\; \text{weakly in } L_1 \tag{3.67}$$

where, given any $k \in \mathrm{N}$ and $B \in \mathcal{A}_k$, $Q_B^{(k)}$ is the random variable

$$Q_B^{(k)} \;=\; \sum_{\substack{D \in \mathcal{N}_k \\ D \subseteq B}} E[(M_D)^2 \mid \mathcal{G}_D^*]. \tag{3.68}$$

Comment 3.5.3 Unlike the classical situation, the $\mathcal{A}$-indexed process $M^2 - Q$ is not necessarily a strong martingale. There are three possible shortcomings. First, as mentioned above, Q is not necessarily adapted. Secondly, without additional assumptions on $\mathcal{A}$ (see Proposition 3.2.35), Q does not necessarily possess a finitely additive extension to $\mathcal{C}(u)$. Finally, even if Q is adapted and possesses a finitely additive extension to $\mathcal{C}(u)$, property (iii) does not necessarily imply $E[(M^2 - Q)_C \mid \mathcal{G}_C^*] = 0 \;\; \forall C \in \mathcal{C}$ since we do not necessarily have $E[M_C^2 - (M_C)^2 \mid \mathcal{G}_C^*] = 0 \;\; \forall C \in \mathcal{C}$.

The proof of Theorem 3.5.2 will be close to that of Theorem 3.4.11. In the latter result, it was shown that any L_1-right-continuous strong submartingale X of class $(D')^*$ possessed a Doob-Meyer-type decomposition. The key to this decomposition was the extension of the admissible function μ_X to a measure on the σ-algebra $\mathcal{P}^*$ (see Theorem 3.3.14).

Take a strong martingale $M = (M_A)_{A\in\mathcal{A}}$ in L_2 for which (3.66) is uniformly integrable. Define the function $\mu_{(M)^2} : \mathcal{P}_0^* \to \mathrm{R}^+$ by letting

$$\mu_{(M)^2}(F \times C) \;=\; E[\,1_F\,(M_C)^2]\,, \;\; (\forall\, F \times C \in \mathcal{P}_0^*). \tag{3.69}$$

Note that $\mu_{(M)^2}$ is not the admissible function of the process M^2. Our first task is to show that $\mu_{(M)^2}$ extends to a measure on $\mathcal{P}^*$. However, $C \mapsto (M_C)^2$ $(C \in \mathcal{C}(u))$ is not necessarily finitely additive on $\mathcal{C}(u)$. Because of this shortcoming, we will frequently require the following technical result.

Lemma 3.5.4 *Let $\mathcal{A}'$ be a f.s.s.l. of $\mathcal{A}$ and let $D_1, D_2 \in \mathcal{N}'$ be distinct left-neighborhoods in $\mathcal{A}'$. If $g \in L_\infty$ is s.t. g is $\mathcal{G}_{D_i}^*$-measurable $(i = 1,2)$, then $E[\,g\,X_{D_1}\,X_{D_2}] = 0$ for any strong martingale $X = (X_A)_{A\in\mathcal{A}}$ in L_2.*

Proof By Lemma 3.2.37, since D_1 and D_2 are disjoint sets in $\mathcal{N}'$, we can assume w.l.o.g. that X_{D_1} is $\mathcal{G}^*_{D_2}$-measurable. Since g is also $\mathcal{G}^*_{D_2}$-measurable, conditioning w.r.t. $\mathcal{G}^*_{D_2}$ implies

$$E[\,g X_{D_1} X_{D_2}] \;=\; E[\,g X_{D_1} \cdot E(X_{D_2} \,|\, \mathcal{G}^*_{D_2})\,]$$

where, by the strong martingale property, $E(X_{D_2} \,|\, \mathcal{G}^*_{D_2}) = 0$. □

Toward extending $\mu_{(M)^2}$ to a measure on $\mathcal{P}^*$, we have the following analogue of Lemma 3.3.7.

Result *If* $F_1 \times C_1, \cdots F_n \times C_n$ *are disjoint sets in* $\mathcal{P}_0^*$ *s.t.* $\bigcup_{i=1}^n F_i \times C_i \in \mathcal{P}_0^*$, *then*

$$\mu_{(M)^2}(\textstyle\bigcup_{i=1}^n F_i \times C_i) \;=\; \displaystyle\sum_{i=1}^n \mu_{(M)^2}(F_i \times C_i). \tag{3.70}$$

Proof Let $F \times C \in \mathcal{P}_0^*$ be s.t. $\bigcup_{i=1}^n F_i \times C_i = F \times C$. First, we consider the case in which $F = F_1 = \cdots = F_n$.

As shown in the proof of Lemma 3.3.7, under this case, $F \times C \;=\; \bigcup_{i=1}^n F_i \times C_i \;=\; F \times (\bigcup_{i=1}^n C_i)$ where $C = \bigcup_{i=1}^n C_i$ is a disjoint union. Therefore, by the finite additivity of M on $\mathcal{C}(u)$,

$$\mu_{(M)^2}(F \times C) \;=\; E\left[\mathbf{1}_F \left(\textstyle\sum_{i=1}^n M_{C_i}\right)^2\right]. \tag{3.71}$$

Unlike the proof of Lemma 3.3.7, we require the following

Claim: $E[\mathbf{1}_F\, M_{C_i}\, M_{C_j}] = 0 \;\; \forall\, i \neq j$.

Proof: For the sake of notation, take $i = 1$ and $j = 2$. Since $C_1 \cup C_2 \in \mathcal{C}(u)$, Lemma 3.2.15 implies $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ and disjoint left-neighborhoods $D_1, \cdots, D_n \in \mathcal{N}'$ s.t. $C_1 = \bigcup_{i=1}^m D_i$ and $C_2 = \bigcup_{j=m+1}^n D_j$. Since $F \in \mathcal{G}^*_C$ and $(\mathcal{G}^*_D)_{D \in \mathcal{C}}$ is decreasing, $\mathbf{1}_F$ is $\mathcal{G}^*_{D_k}$-measurable $\forall\, 1 \leq k \leq n$. By Lemma 3.5.4, this implies

$$E[\,\mathbf{1}_F\, M_{D_i}\, M_{D_j}] = 0\,, \;\; (\forall\, 1 \leq i \leq m \text{ and } m{+}1 \leq j \leq n).$$

Therefore, since M is finitely additive on $\mathcal{C}(u)$,

$$E[\,\mathbf{1}_F\, M_{C_1}\, M_{C_2}] \;=\; \sum_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq n}} E[\,\mathbf{1}_F\, M_{D_i}\, M_{D_j}] \;=\; 0,$$

completing the proof of the claim. Ω

Returning to (3.71),

$$\begin{aligned}\mu_{(M)^2}(F \times C) &= \sum_{1\leq i,j\leq n} E[\,1_F\, M_{C_i}\, M_{C_j}] \\ &= \sum_{i=1}^{n} E[\,1_F\,(M_{C_i})^2] \quad \text{(by previous Claim)}\end{aligned}$$

which is precisely (3.70) for the case of $F_1 = F_2 = \cdots = F_n$.

The case in which $F \times C = \bigcup_{i=1}^{n} F_i \times C_i$ is any disjoint union in $\mathcal{P}_0^*$ follows word for word from the argument for the last two cases in the proof of Lemma 3.3.7; we simply replace μ_X by $\mu_{(M)^2}$ throughout. □

Applying Lemma 3.3.8 to $\mu_{(M)^2} : \mathcal{P}_0^* \to \mathbf{R}^+$, we obtain an analogue of Corollary 3.3.9.

Result *$\mu_{(M)^2}$ has a unique extension to a finitely additive set-function, also denoted $\mu_{(M)^2}$ on the algebra $\mathcal{P}_0^*(u)$.*

Finally, we have an analogue of Theorem 3.3.14.

Result *$\mu_{(M)^2}$ extends to a measure, also denoted $\mu_{(M)^2}$ on $\mathcal{P}^*$.*

The proof of the above result follows by replacing μ_X by $\mu_{(M)^2}$ throughout the proof of Theorem 3.3.14. The only non-trivial difference occurs at (3.26). Here, we can no longer use Lemma 3.2.48. Instead, we need the following

Lemma 3.5.5 *Let $X = (X_A)_{A\in\mathcal{A}}$ be a strong martingale in L_2. Given any $C \in \mathcal{C}$, $\exists$ an increasing sequence $(C_n)_n$ in $\mathcal{C}$ s.t.*

(i) $C_n \uparrow C$,

(ii) $\overline{C}_n \subseteq C$ $\forall n$ *and*

(iii) $E(X_{C\setminus C_n})^2 \to 0$ *as* $n \to \infty$.

In particular, if $C = A \setminus \bigcup_{i=1}^{k} A_i$, we can take $C_n = A \setminus \bigcup_{i=1}^{k} g_n(A_i)$ $\forall n$.

Proof Since X is in L_2, Lemma 3.2.46 implies X is L_2-right-continuous. Using this property, we can repeat the argument from the proof of Proposition 2.6 (i) in [16]. □

With $\mu_{(M)^2}$ extended to $\mathcal{P}^*$, we can now complete our main objective.

Proof of Theorem 3.5.2 Assuming our stochastic base satisfies Assumptions 3.4.2, 3.4.9 and 3.4.10, the proof, for the most part, follows by replacing X_C and μ_X in the proof of Theorem 3.4.11 by $(M_C)^2$ and $\mu_{(M)^2}$ respectively. The only arguments in Section 3.4 which do not carry over are those which rely on the finite additivity of $C \mapsto X_C$ on $\mathcal{C}(u)$. (As mentioned earlier, $C \mapsto (M_C)^2$ is not necessarily finitely additive on $\mathcal{C}(u)$.)

First, recall the T-indexed processes $Y(F)$ and $Y^n(F)$ $(F \in \mathcal{F})$ defined in Section 3.4. To establish the limit in (3.67), fix $m \in \mathrm{N}$ and $A \in \mathcal{A}_m$ and define

$$\tilde{\sigma}_A(F) := \int_{\Omega\times A} Y(F)\, d\mu_{(M)^2}, \quad (\forall\, F \in \mathcal{F}).$$

Once again, we have

$$\tilde{\sigma}_A(F) = \lim_{n\to\infty} \int_{\Omega\times A} Y^n(F)\, d\mu_{(M)^2}, \quad (\forall\, F \in \mathcal{F}). \tag{3.72}$$

Applying a suitable analogue of Result II, we obtain the following analogue of Claim I. (See the proof of Theorem 3.4.11 for the statements of Result II and Claim I.)

Claim $\tilde{\mathrm{I}}$: *Given* $n \in \mathrm{N}$ *s.t.* $n \geq m$, $E[1_F\, Q_A^{(n)}] = \int_{\Omega\times A} Y^n(F)\, d\mu_{(M)^2}\ \forall\, F \in \mathcal{F}$

Combining Claim $\tilde{\mathrm{I}}$ and (3.72),

$$\tilde{\sigma}_A(F) = \lim_{\substack{n\to\infty\\ n\geq m}} E[1_F\, Q_A^{(n)}], \quad (\forall\, F \in \mathcal{F}).$$

Therefore, by the Hahn-Vitali-Saks Theorem, $\exists\, \overline{Q}_A \in L_1$ s.t.

$$\tilde{\sigma}_A(F) = E[1_F\, \overline{Q}_A], \quad (\forall\, F \in \mathcal{F}), \tag{3.73}$$

i.e., $(Q_A^{(n)})_{n\geq m}$ converges to $\overline{Q}_A$ in the weak L_1 topology. Applying this argument to each $A \in \mathcal{A}_m$, we obtain a collection $\{\overline{Q}_A : A \in \mathcal{A}_m\}$ of random variables, each satisfying (3.73). (We will keep m fixed for the time being.)

Before proceeding with the present proof, we make a key observation. Given any $n \in \mathrm{N}$, if we define the function $\tilde{f} : \mathcal{N}_n \to \mathrm{R}$ by

$$\tilde{f}(D) = E[(M_D)^2 \,|\, \mathcal{G}_D^*], \quad (\forall D \in \mathcal{N}_n)$$

and replace $\overline{f}$ by $\tilde{f}$ in the proof of Lemma A.4.1, then

Claim $\widetilde{\text{II}}$: *Every Lemma and Corollary in Section A.4 remains valid when we replace $V^{(n)}$, $\overline{V}$ and V by $Q^{(n)}$, $\overline{Q}$ and Q respectively. (See (3.77) for the definition of Q.)*

In particular, applying the obvious analogue of Corollary A.4.4, the collection $\{\overline{Q}_A : A \in \mathcal{A}_m\}$ can be extended to $\{\overline{Q}_C : C \in \mathcal{C}_m\}$ so that

$$\overline{Q}_C(\omega) = \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{Q}(A \cap \textstyle\bigcap_{i \in I} A_i)(\omega), \quad (\forall \omega \in \Omega') \tag{3.74}$$

for any $C \in \mathcal{C}_m$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C with $A, A_i \in \mathcal{A}_m$. Here, Ω' is an event of full P-measure which is independent of C and m.

Next, recall the countable subcollections

$$\mathcal{A}^* = \textstyle\bigcup_m \mathcal{A}_m \quad \text{and} \quad \mathcal{C}^* = \bigcup_m \mathcal{C}_m$$

of $\mathcal{A}$ and $\mathcal{C}$ respectively. Applying the above argument to each $m \in \mathrm{N}$, we obtain a collection of random variables $(\overline{Q}_C)_{C \in \mathcal{C}^*}$ satisfying (3.74). Furthermore, by repeating the argument from Claim II in the proof of Theorem 3.4.11, we have the following result.

Claim $\widetilde{\text{III}}$: *Given any $C \in \mathcal{C}^*$, $E[1_F\, \overline{Q}_C] = \int_{\Omega \times C} Y(F)\, d\mu_{(M)^2}$ $\forall F \in \mathcal{F}$.*

Using Claim $\widetilde{\text{III}}$ in place of Claim II, we obtain

$$\overline{Q}_C(\omega) \geq 0, \quad (\forall C \in \mathcal{C}^* \text{ and } \omega \in \Omega_0) \tag{3.75}$$

where $\Omega_0 \in \mathcal{F}$ is an event of full P-measure. As done in the proof of Theorem 3.4.11, we insist that $\Omega_0 \subseteq \Omega'$ so that (3.74) holds for every $C \in \mathcal{C}^*$ and $\omega \in \Omega_0$. Given any $A, A' \in \mathcal{A}^*$ s.t. $A \subseteq A'$, (3.75) implies

$$\overline{Q}_A \leq \overline{Q}_{A'} \text{ on } \Omega_0. \tag{3.76}$$

Therefore, we can define a process $Q = (Q_A)_{A\in\mathcal{A}}$ by letting

$$Q_A \;=\; \lim_m \left(\mathbf{1}_{\Omega_0} \cdot \overline{Q}_{g_m(A)}\right), \quad (\forall\, A \in \mathcal{A}). \tag{3.77}$$

Since the limit in (3.77) is mononote decreasing on Ω, $(\overline{Q}_{g_m(A)})_m$ converges to Q_A in the weak L_1 topology $\forall\, A \in \mathcal{A}$. This establishes the limit in (3.67).

Applying the obvious analogue of Lemma A.4.6 (see Claim $\widetilde{\text{II}}$), we can extend Q to a collection $(Q_C)_{C\in\mathcal{C}}$ of random variables s.t.

$$Q_C \;=\; \sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot Q(A \cap \textstyle\bigcap_{i\in I} A_i)$$

for any $C \in \mathcal{C}$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C. This is precisely property (i) in the statement of Theorem 3.5.2.

We now establish properties (ii), (iii) and (iv) in that order. First, by repeating the argument from Claim III in the proof of Theorem 3.4.11, we obtain

Claim $\widetilde{\text{IV}}$: $Q_C \geq 0$ *a.s.* $\forall\, C \in \mathcal{C}$.

This is precisely property (ii).

To establish property (iii), it is sufficient to show

$$E[\mathbf{1}_F\,(M_C)^2] \;=\; E[\mathbf{1}_F\,Q_C], \quad (\forall C \in \mathcal{C} \text{ and } F \in \mathcal{G}^*_C). \tag{3.78}$$

With this reduction in mind, select $F \times C \in \mathcal{P}^*_0$ and let $A \setminus \bigcup_{i=1}^k B_i$ be a maximal representation of C. Such a representation exists by Assumption 3.4.9. Since M is a strong martingale in L_2, Lemma 3.2.46 implies M is L_2-right-continuous. Therefore, if we define

$$C^m = g_m(A) \setminus \textstyle\bigcup_{i=1}^k g_m(A_i)\,, \quad (\forall\, m),$$

it is clear from (3.12) that $M_{C^m} \to M_C$ in L_2-norm and thus

$$E[\mathbf{1}_F\,(M_C)^2] \;=\; \lim_m E[\mathbf{1}_F\,(M_{C^m})^2]. \tag{3.79}$$

Since $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is decreasing on $\mathcal{C}$ and $F \in \mathcal{G}^*_C$, Claim V in Theorem 3.4.11 implies

$$F \in \mathcal{G}^*_D\,, \quad (\forall\, D \in \mathcal{N}_n \text{ s.t. } D \subseteq C^m) \tag{3.80}$$

where $m, n \in \mathbb{N}$ is any pair satisfying $n \geq m$. Furthermore, for any $m \in \mathbb{N}$, Corollary A.3.2 implies $C^m = \bigcup_{D \in \mathcal{N}_n,\, D \subseteq C^m} D \;\; \forall n \geq m$. Therefore, by Lemma 3.5.4 and (3.80),

$$E[\mathbf{1}_F (M_{C^m})^2] = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C^m}} E[\mathbf{1}_F (M_D)^2], \quad (\forall n \geq m). \tag{3.81}$$

Combining (3.79) and (3.81), we obtain

$$\begin{aligned} E[\mathbf{1}_F (M_C)^2] &= \lim_{m} \lim_{\substack{n \to \infty \\ n \geq m}} \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C^m}} E[\mathbf{1}_F (M_D)^2] \\ &= \lim_{m} \lim_{\substack{n \to \infty \\ n \geq m}} \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C^m}} E\{\mathbf{1}_F E[(M_D)^2 \,|\, \mathcal{G}_D^*]\} \quad \text{(by (3.80))} \\ &= \lim_{m} \lim_{\substack{n \to \infty \\ n \geq m}} E[\mathbf{1}_F Q^{(n)}_{C^m}] \quad \text{(by Lemma A.4.1 via Claim } \widetilde{\text{II}}) \\ &= E[\mathbf{1}_F Q_C] \quad \text{(by (3.67))} \end{aligned}$$

which establishes (3.78), completing the proof of (iii).

For property (iv), note that the right-continuity of V as established in Claim IV of Theorem 3.4.11 only required $(\overline{V}_A(\omega))_{A \in \mathcal{A}^*}$ to be increasing on $\mathcal{A}^*$ for every $\omega \in \Omega_0$. Since this property is shared by $(\overline{Q}_A)_{A \in \mathcal{A}^*}$ (see (3.76)), Q is also right-continuous. Furthermore, repeating the argument from the proof of Theorem 3.5.2 (see (3.57)), a sufficient condition for $*$-predictability of Q is

$$\mu_{(M)^2}(F \times C) = \mu_Q(F \times C), \quad (\forall F \times C \in \mathcal{P}_0^*).$$

But this is clearly equivalent to (3.78). Therefore, (iv) is established and the proof of Theorem 3.5.2 is complete. □

3.5.3 General $*$-predictable quadratic variation

Theorem 3.5.2 motivates the following terminology.

Definition 3.5.6 *Let $M = (M_A)_{A \in \mathcal{A}}$ be a strong martingale in L_2. Under Assumption 3.4.2, a process $Q = (Q_A)_{A \in \mathcal{A}}$ is said to be a <u>$*$-predictable quadratic variation</u> ($*$-PQV) of M if*

(i) *Q has increments defined at every $C \in \mathcal{C}$,*

(ii) $Q_C \geq 0 \ \forall C \in \mathcal{C}$,

(iii) $E[(M_C)^2 \,|\, \mathcal{G}^*_C] = E[Q_C \,|\, \mathcal{G}^*_C] \ \forall C \in \mathcal{C}$ *and*

(iv) Q *is right-continuous and* $*$*-predictable.*

(Note that Q *is not necessarily adapted nor does* Q *necessarily possess a finitely additive extension to* $\mathcal{C}(u)$*.)*

Remark 3.5.7 (a) Since Assumption 3.4.2 was required in the definition of $*$-predictability, it is also required in Definition 3.5.6.

(b) Under Assumptions 3.4.2, 3.4.9 and 3.4.10, any strong martingale in L_2 for which (3.66) is uniformly integrable possesses a $*$-PQV. This is the content of Theorem 3.5.2.

We close this section with two general results concerning $*$-predictable quadratic variation. Whereas Theorem 3.5.2 gives sufficient conditions for the existence of $*$-PQV, our first result implies uniqueness.

Proposition 3.5.8 *Let* $M = (M_A)_{A \in \mathcal{A}}$ *be a strong martingale in* L_2*. If* $Q = (Q_A)_{A \in \mathcal{A}}$ *is a* $*$*-PQV of* M*, then* Q *is unique up to indistinguishability.*

Proof Assume both Q and Q' are $*$-PQV of M. In light of properties (ii) and (iv), Proposition 3.4.6 implies both μ_Q and $\mu_{Q'}$ extend to measures on $\mathcal{P}^*$. But by property (iii), μ_Q and $\mu_{Q'}$ coincide on $\mathcal{P}^*_0$, a π-system. Therefore, μ_Q and $\mu_{Q'}$ coincide on $\mathcal{P}^* = \sigma(\mathcal{P}^*_0)$. Repeating the argument in Proposition 3.4.12, replacing V and V' by Q and Q' respectively, this coincidence of measures implies $Q_A = Q'_A \ \forall A \in \mathcal{A}$. Since Q is right-continuous, the proof now follows by Lemma 3.2.45. □

Let Q be the $*$-PQV of some L_2 strong martingale M. If Q possesses a finitely additive extension to $\mathcal{C}(u)$, then we trivially have $Q_C \geq 0 \ \forall C \in \mathcal{C}(u)$. Furthermore,

Proposition 3.5.9 *Let* $M = (M_A)_{A \in \mathcal{A}}$ *be a strong martingale in* L_2*. If* Q *is a* $*$*-PQV of* M *possessing a finitely additive extension to* $\mathcal{C}(u)$*, then* $E[(M_C)^2 \,|\, \mathcal{G}^*_C] = E[Q_C \,|\, \mathcal{G}^*_C] \ \forall C \in \mathcal{C}(u)$.

Proof Take any $C \in \mathcal{C}(u)$. Clearly, it is sufficient to show

$$E[1_F (M_C)^2] = E[1_F Q_C], \quad (\forall F \in \mathcal{G}^*_C). \tag{3.82}$$

With this reduction in mind, take any $F \in \mathcal{G}^*_C$. By Lemma 3.2.15, $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ and distinct left-neighborhoods $D_1, \cdots, D_n \in \mathcal{N}'$ s.t. $C = \bigcup_{i=1}^n D_i$. Since $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is decreasing on $\mathcal{C}$, $F \in \mathcal{G}^*_{D_i}$ $\forall 1 \leq i \leq n$. Therefore,

$$\begin{aligned} E[1_F (M_C)^2] &= \sum_{i=1}^n E[1_F (M_{D_i})^2] \quad \text{(by Lemma 3.5.4)} \\ &= \sum_{i=1}^n E[1_F Q_{D_i}] \quad \text{(by property (iii))} \\ &= E[1_F Q_C] \end{aligned}$$

which establishes (3.82). □

Remark 3.5.10 Note that the argument in Proposition 3.5.9 only required the relation $E[(M_C)^2 \mid \mathcal{G}^*_C] = E[Q_C \mid \mathcal{G}^*_C]$ $\forall C \in \mathcal{C}$.

A more general type of set-indexed quadratic variation termed *$*$-quadratic variation* will be defined in Subsection 4.2.3. While these $*$-quadratic variations will not necessarily be $*$-predictable, they will always possess a unique finitely additive extension to $\mathcal{C}(u)$.

3.6 Square Function Inequalities for Set-Indexed Strong Martingales

In this section, we develop set-indexed analogues of Burkholder's and Rosenthal's square function inequalities (see Section 2.4 of [19]). These inequalities will lead to refinements and useful consequences of Theorem 3.5.2, some of which will be required in Section 3.7.

3.6.1 Classical inequalities

The following result appeared as Theorem A.8 on p.282 of [19].

Lemma 3.6.1 *Given non-negative random variables $Z_1, \cdots, Z_r \in L_1$ and sub-σ-algebras $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \cdots \subseteq \mathcal{H}_r$ of $\mathcal{F}$,*

$$E\left\{\left[\sum_{i=1}^r E(Z_i \mid \mathcal{H}_{i-1})\right]^p\right\} \ \leq \ K_2 \cdot E\left[\left(\sum_{i=1}^r Z_i\right)^p\right] \qquad (3.83)$$

for any $p \in [1, \infty)$ where K_2 is a constant which depends only on p.

(3.83) was used in Hall and Heyde's proof of Rosenthal's inequality (see p.29 in [19]) and will play a similar role in the next subsection.

Before presenting another classical result, we must first agree on some terminology.

Definition 3.6.2 *Given sub-σ-algebras $\mathcal{H}_1, \cdots, \mathcal{H}_r$ of $\mathcal{F}$ and random variables $X_1, \cdots, X_r \in L_1$ ($r \in \mathrm{N}$), the collection $\{X_i, \mathcal{H}_i : 1 \leq i \leq r\}$ is said to be a finite martingale difference array (f.m.d.a.) if*

(i) $\mathcal{H}_i \subseteq \mathcal{H}_{i+1} \ \forall 1 \leq i \leq r-1$,

(ii) X_i *is* $\mathcal{H}_i$*-measurable* $\forall 1 \leq i \leq r$ *and*

(iii) $E(X_{i+1} \mid \mathcal{H}_i) = 0 \ \forall 1 \leq i \leq r-1$.

The following inequality is due to Burkholder (see Theorem 2.10 in [19]).

Lemma 3.6.3 *If $\{X_i, \mathcal{H}_i : 1 \leq i \leq r\}$ is a f.m.d.a., then*

$$E\left[\left(\sum_{i=1}^r X_i^2\right)^{p/2}\right] \ \leq \ K_1 \cdot E\left[\left|\sum_{i=1}^r X_i\right|^p\right] \qquad (3.84)$$

for any $p \in (1, \infty)$ where K_1 is a constant which depends only on p.

3.6.2 Set-indexed inequalities

We begin with a key observation.

Lemma 3.6.4 *Let $M = (M_A)_{A \in \mathcal{A}}$ be a strong martingale and let $\mathcal{A}' = \{A_1, \cdots A_n\}$ be a f.s.s.l. of $\mathcal{A}$ numbered in a manner c.w.s.p. If we are given indices $1 \leq k_1 < \cdots < k_r \leq n$ in N s.t.*

$$C_{k_i} \ := \ A_{k_i} \setminus \bigcup_{A \in \mathcal{A}',\, A_{k_i} \not\subseteq A} A \ \neq \ \phi, \ \ (\forall 1 \leq i \leq r)$$

(i.e., $C_{k_i} \in \mathcal{N}' \ \forall i$), then $\{X_i, \mathcal{H}_i : 1 \le i \le r\}$ is a f.m.d.a. when we take

$$\mathcal{H}_i = \begin{cases} \bigcap_m \bigvee_{j=1}^{(k_{i+1})-1} \mathcal{F}_{g_m(A_j)} & , \text{if } 1 \le i \le r-1 \\ \mathcal{F} & , \text{if } i = r \end{cases} \tag{3.85}$$

and $X_i = M(C_{k_i}) \ \forall 1 \le i \le r$.

Proof Since k_i increases in i, $\mathcal{H}_i \subseteq \mathcal{H}_{i+1} \ \forall 1 \le i \le r-1$. By (3.12), $M(C_{k_i})$ is $\mathcal{F}_{A_{k_i}}$-measurable $\forall 1 \le i \le r$. Therefore, since $k_i \le k_{i+1} - 1$ $\forall 1 \le i \le r-1$, it is clear that X_i is $\mathcal{H}_i$-measurable $\forall 1 \le i \le r-1$. (Trivially, X_r is $\mathcal{H}_r$-measurable.)

To establish (iii) of Definition 3.6.2, take any $1 \le i \le r-1$. Since $\mathcal{A}'$ is numbered in a manner c.w.s.p., Lemma 3.2.12 implies

$$C_{k_{j+1}} = A_{k_{j+1}} \setminus \bigcup_{j=1}^{(k_{i+1})-1} A_j. \tag{3.86}$$

If we let $B = \bigcup_{j=1}^{(k_{i+1})-1} A_j$, then by Lemma 3.2.24 (e), $\mathcal{F}_B \subseteq \mathcal{G}_C^*$. Furthermore, by Lemma 3.2.24 (c) and (3.9), $\mathcal{H}_i \subseteq \mathcal{F}_B$. Therefore,

$$\begin{aligned} E\left(X_{i+1} | \mathcal{H}_i\right) &= E\left[E\left[M(C_{k_{j+1}}) \,\middle|\, \mathcal{G}^*_{C_{k_{j+1}}}\right] \,\middle|\, \mathcal{H}_i\right] \quad \text{(by the tower property)} \\ &= 0 \quad \text{(by the strong martingale property)} \end{aligned}$$

which completes the proof of the present lemma. □

Remark 3.6.5 Note that $\{M_{C_{k_i}}, \mathcal{G}^*_{C_{k_i}} : 1 \le i \le r\}$ is not necessarily a f.m.d.a. since $\{\mathcal{G}^*_{C_{k_i}} : 1 \le i \le r\}$ is not necessarily increasing in i nor is $M_{C_{k_i}}$ necessarily $\mathcal{G}^*_{C_{k_i}}$-measurable.

We now present two lemmas containing inequalities resembling those of Burkholder and Rosenthal respectively.

Lemma 3.6.6 *Let $M = (M_A)_{A \in \mathcal{A}}$ be a strong martingale in L_2. If $\mathcal{N}_0$ is a f.n.s. of $\mathcal{A}$ (see Definition 3.2.14), then*

$$E\left[\left(\sum_{C \in \mathcal{N}_0} (M_C)^2\right)^{p/2}\right] \le \kappa_1 \cdot E\left[|M_{C_0}|^p\right] \tag{3.87}$$

where $C_0 = \bigcup_{C \in \mathcal{N}_0} C$ and κ_1 is a positive constant depending only on p.

Proof By the definition of f.n.s., $\exists$ a f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ s.t. $\mathcal{N}_0 \subseteq \mathcal{N}'$. Let $\{A_1, \cdots, A_n\}$ be a numbering of $\mathcal{A}'$ which is c.w.s.p. — note that $A_1 = \phi'$. Then, $\exists$ indices $1 \leq k_i < \cdots < k_r \leq n$ in N s.t.

$$\mathcal{N}_0 = \{C_{k_1}, \cdots, C_{k_r}\}$$

where C_{k_i} denotes the left-neighborhood generated by A_{k_i} in $\mathcal{A}'$ $(1 \leq i \leq r)$. By definition, each $C \in \mathcal{N}'$ is non-empty.

Now, let $p \in (1, \infty)$ be given. If we take $\{X_i, \mathcal{H}_i : 1 \leq i \leq r\}$ to be the f.m.d.a. defined in Lemma 3.6.4, then

$$\begin{aligned} E\left[\left(\sum_{C\in\mathcal{N}_0}(M_C)^2\right)^{p/2}\right] &= E\left[\left(\sum_{i=1}^r X_i^2\right)^{p/2}\right] \\ &\leq K_1 \cdot E\left[\left|\sum_{i=1}^r X_i\right|^p\right] \quad \text{(by (3.84))} \\ &= K_1 \cdot E\left[|M_{C_0}|^p\right]. \end{aligned}$$

(The last equality follows by the additivity of M on $\mathcal{C}(u)$, noting that distinct left-neighborhoods are disjoint.) To complete (3.87), we select $\kappa_1 = K_1$. □

Unlike Lemma 3.6.6, our set-indexed Rosenthal-like inequality relies heavily on the development in Appendix B.

Lemma 3.6.7 *Let $M = (M_A)_{A\in\mathcal{A}}$ be a strong martingale in L_2. Under Assumption Group D.1, if $\mathcal{N}_0$ is a f.n.s. of $\mathcal{A}$, then given any $2 \leq p < \infty$,*

$$E\left[\left(\sum_{C\in\mathcal{N}_0} E\left[(M_C)^2 \middle| \mathcal{G}_C^*\right]\right)^{p/2}\right] \leq \kappa_2 \cdot E\left[|M_{C_0}|^p\right] \qquad (3.88)$$

where $C_0 = \bigcup_{C\in\mathcal{N}_0} C$ and κ_2 is a positive constant depending only on p.

Proof Take $p \in [2, \infty)$. Let $\mathcal{A}' = \{A_1, \cdots, A_n\}$ and $\mathcal{N}_0 = \{C_{k_1}, \cdots, C_{k_r}\}$ be as described in the proof of Lemma 3.6.6 and let $\mathcal{H}_1 \subseteq \cdots \subseteq \mathcal{H}_r$ be as defined in (3.85). If in addition, we define

$$\mathcal{H}_0 = \begin{cases} \bigcap_m \bigvee_{j=1}^{k_1-1} \mathcal{F}_{g_m(A_j)} & \text{, if } k_1 \geq 2 \\ \mathcal{F}_{\phi'} & \text{, if } k_1 = 1\,, \end{cases}$$

then $\mathcal{H}_0 \subseteq \mathcal{H}_1$.

Since $\mathcal{A}'$ is numbered in a manner c.w.s.p., (3.86) holds for $j = 0, \cdots, r-1$. By Proposition B.3.9, this implies

$$E\left[(M_{C_{k_i}})^2 \,\middle|\, \mathcal{G}^*_{C_{k_i}} \right] = E\left[(M_{C_{k_i}})^2 \,\middle|\, \mathcal{H}_{i-1} \right], \quad (\forall\, 1 \le i \le r).$$

(In the case where $k_1 = 1$, $C_{k_1} = C_{\phi'} = \phi'$ and hence $\mathcal{G}^*_{C_{k_1}} = \mathcal{F}_{\phi'} = \mathcal{H}_0$.) Therefore, since $p/2 \in [1, \infty)$,

$$\begin{aligned} E\left[\left(\sum_{C \in \mathcal{N}_0} E\left[(M_C)^2 \,\middle|\, \mathcal{G}^*_C \right] \right)^{p/2} \right] &= E\left[\left(\sum_{i=1}^r E\left[(M_{C_{k_i}})^2 \,\middle|\, \mathcal{H}_{i-1} \right] \right)^{p/2} \right] \\ &\le K_2 \cdot E\left[\left(\sum_{i=1}^r (M_{C_{k_i}})^2 \right)^{p/2} \right] \\ &\qquad\qquad \text{(by (3.83))} \\ &= K_2 \cdot E\left[\left(\sum_{C \in \mathcal{N}_0} (M_C)^2 \right)^{p/2} \right] \\ &\le (K_2 \cdot \kappa_1) \cdot E\left[|M_{C_0}|^p \right] \quad \text{(by (3.87))}. \end{aligned}$$

By selecting $\kappa_2 = K_2 \cdot \kappa_1$, we obtain (3.88). □

Remark 3.6.8 Since the summands on the left-hand side of either (3.87) or (3.88) are non-negative and since $\bigcup_{C \in \mathcal{N}'} C = T$ for any f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$ (see Lemma 3.2.10), both (3.87) and (3.88) remain valid when we replace $E[|M_{C_0}|^p]$ by $E[|M_T|^p]$.

3.6.3 Applications

In this subsection, we present four applications of our set-indexed square function inequalities. We begin with a simple sufficient condition under which a set-indexed strong martingale is in L_p.

Proposition 3.6.9 *Given $p \in (1, \infty)$ and an $\mathcal{A}$-indexed strong martingale M, $M_A \in L_p$ $\forall A \in \mathcal{A}$ if and only if $M_T \in L_p$.*

Proof Take $A \in \mathcal{A} \setminus \{T\}$ and define $C = T \setminus A \in \mathcal{C}$. Since C is a left neighborhood in the f.s.s.l. $\mathcal{A}' = \{\phi', A, T\}$, Remark 3.6.8 implies

$$E[|M_C|^p] = E\left\{ \left[(M_C)^2 \right]^{p/2} \right\} \le \kappa_1 \cdot E[|M_T|^p].$$

Therefore, since $M_C = M_T - M_A$, $M_T \in L_p$ implies $M_A \in L_p$. Since $T \in \mathcal{A}$, the opposite implication is trivial. □

In our second application, we give a moment condition on M under which the family in (3.66) is uniformly integrable.

Proposition 3.6.10 *Under Assumption Group D.1, if M is an $\mathcal{A}$-indexed strong martingale with $E[|M_T|^{2+\delta}] < \infty$ for some $\delta > 0$, then*

$$\left\{ \sum_{D \in \mathcal{N}_0} E[\,(M_D)^2 \,|\, \mathcal{G}^*_D] \,:\, \mathcal{N}_0 \text{ a f.n.s. of } \mathcal{A} \right\}$$

is uniformly integrable.

Proof Given any f.n.s. $\mathcal{N}_0$ of $\mathcal{A}$, Remark 3.6.8 implies

$$\left\| \sum_{C \in \mathcal{N}_0} E\left[(M_C)^2 \,\middle|\, \mathcal{G}^*_C \right] \right\|_{1+\delta/2} \leq \left(\kappa_2 \cdot E\left[|M_T|^{2+\delta} \right] \right)^{\frac{2}{2+\delta}} < \infty . \quad (3.89)$$

where κ_2 is a constant depending only on δ. Therefore, the above family is $L_{1+\delta/2}$-bounded, a sufficient condition for uniform integrability (see Exercise 1.2.7 (1) in [30]). □

By Theorem 3.5.2, Proposition 3.6.10 and Proposition 3.5.8 yield

Corollary 3.6.11 *Let M be an $\mathcal{A}$-indexed strong martingale. Under Assumption Groups D.1 and D.2, if $\exists\, \delta > 0$ s.t. $E[|M_T|^{2+\delta}] < \infty$, then M possesses a unique $*$-PQV satisfying* (3.68).

Another interesting consequence of Proposition 3.6.10 is given below.

Proposition 3.6.12 *Let $(M_n)_n$ be a sequence of L_2 strong martingales defined on a common stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ satisfying Assumption Groups D.1 and D.2. If*

$$\sup_n E\left[|M_n(T)|^{2+\delta} \right] < \infty \quad (3.90)$$

for some $\delta > 0$, then $\{Q_n(T) : n \geq 1\}$ is uniformly integrable. Here, Q_n denotes the unique $*$-PQV of M_n given by Theorem 3.5.2.

Proof First note that by (3.90), Corollary 3.6.11 implies the existence of the above mentioned $*$-PQV processes, $Q_1, Q_2, \cdots$. The uniform integrability of $\{Q_n(T) : n \geq 1\}$ will follow from the weak L_1 approximations in (3.67) and the following result.

Claim: *The collection $\mathcal{S}$ consisting of all random variables of the form*

$$\sum_{C \in \mathcal{N}_k} E\left[[M_n(C)]^2 \,\middle|\, \mathcal{G}_C^*\right], \quad (n, k \in \mathrm{N})$$

is uniformly integrable.

Proof: Combining (3.90) and the inequality in (3.89), noting that the constant κ_2 therein depends only on δ, $\mathcal{S}$ is $L_{1+\delta/2}$-bounded and as such, is uniformly integrable. Ω

Property (iii) of Theorem 3.5.2 implies $E[M_n^2(T) \,|\, \mathcal{G}_T^*] = E[Q_n(T) \,|\, \mathcal{G}_T^*]$ $\forall n$ while property (ii) of Theorem 3.5.2 implies $Q_n(T) \geq 0$ $\forall n$. Therefore, since (3.90) implies $\sup_n E[M_n^2(T)] < \infty$, $\{Q_n(T) : n \geq 1\}$ is L_1-bounded. We complete the proof of Proposition 3.6.12 by employing the ϵ-δ-characterization of uniform integrability given in Proposition 1.2.4 of [30].

Let $\epsilon > 0$ be given. By the above Claim, $\exists\, \rho > 0$ s.t.

$$F \in \mathcal{F} \text{ and } P(F) < \rho \implies \sup_{f \in \mathcal{S}} \int_F f\, dP < \epsilon/2. \tag{3.91}$$

(Note that $f \in \mathcal{S}$ implies $f \geq 0$.) Take any set $F \in \mathcal{F}$ s.t. $P(F) < \rho$. Given any n, (3.67) implies $\exists\, f_n \in \mathcal{S}$ s.t.

$$\int_F Q_n(T)\, dP \leq \int_F f_n\, dP + \epsilon/2.$$

Therefore,

$$\begin{aligned} \sup_n \int_F Q_n(T)\, dP &\leq \sup_n \int_F f_n\, dP + \epsilon/2 \\ &\leq \sup_{f \in \mathcal{S}} \int_F f_n\, dP + \epsilon/2 \\ &< \epsilon \quad \text{(by (3.91))}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\{Q_n(T) : n \geq 1\}$ is uniformly integrable. □

As mentioned above, given a strong martingale $M = (M_A)_{A\in\mathcal{A}}$ in L_2, the random variables

$$Q_A^{(k)} = \sum_{\substack{C\in\mathcal{N}_k\\ C\subseteq A}} E\left[(M_C)^2 \,\middle|\, \mathcal{G}_C^*\right], \quad (k\in\mathrm{N},\, A\in\mathcal{A}_k)$$

have appeared in Theorem 3.5.2 as weak L_1 approximants for the $*$-PQV, $Q = (Q_A)_{A\in\mathcal{A}}$ of M. The following proposition will be used in Section 3.7 to show that these $Q_A^{(k)}$ are in fact L_2-norm approximants of Q under certain conditions.

Proposition 3.6.13 *Under Assumption Groups D.1 and D.2, if M is an $\mathcal{A}$-indexed strong martingale with $E[M_T^4] < \infty$, then given any $k \in \mathrm{N}$ and any $A \in \mathcal{A}_k$,*

$$\left\|Q_A^{(n)} - Q_A^{(m)}\right\|_2^2 \le C_1\cdot\alpha_n + C_2\cdot\alpha_m \tag{3.92}$$

for every $n, m \ge k$ where

$$\alpha_r := E\left[\max_{C\in\mathcal{N}_r}(M_C)^4\right], \quad (\forall r\in\mathrm{N}) \tag{3.93}$$

and C_1, C_2 are positive constants depending only on $E[M_T^4]$.

Proof Since $E[M_T^4] < \infty$, Proposition 3.6.9 implies $M_A \in L_4\ \forall A \in \mathcal{A}$ and thus by (3.12), $M_C \in L_4\ \forall C \in \mathcal{C}$. Therefore, $Q_A^{(k)} \in L_2\ \forall k$ and $A \in \mathcal{A}_k$.

Take $k, m, n \in \mathrm{N}$ s.t. $m, n \ge k$. Without loss of generality, we may assume $n < m$ so that, given any $C \in \mathcal{N}_n$, Corollary A.3.2 implies

$$E\left[(M_C)^2 \,\middle|\, \mathcal{G}_C^*\right] = \sum_{\substack{D_1,D_2\in\mathcal{N}_m\\ D_1,D_2\subseteq C}} E\left[M_{D_1} M_{D_2} \,|\, \mathcal{G}_C^*\right]. \tag{3.94}$$

Furthermore, by Lemma 3.2.37, given any $D_1 \ne D_2$ in $\mathcal{N}_m$, we may assume w.l.o.g. that M_{D_1} is $\mathcal{G}_{D_2}^*$-measurable. Therefore, since $\mathcal{G}_C^* \subseteq \mathcal{G}_D^*\ \forall D \in \mathcal{N}_m$ s.t. $D \subseteq C$, a simple conditioning argument applied to (3.94) yields

$$E\left[(M_C)^2 \,\middle|\, \mathcal{G}_C^*\right] = \sum_{\substack{D\in\mathcal{N}_m\\ D\subseteq C}} E\left[(M_D)^2 \,|\, \mathcal{G}_C^*\right] \tag{3.95}$$

by the strong martingale property of M.

Now, select a set $A \in \mathcal{A}_k$. Applying (3.95) to each $C \in \mathcal{N}_n$ s.t. $C \subseteq A$, we obtain

$$Q_A^{(n)} - Q_A^{(m)} = \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} E[(M_D)^2 \,|\, \mathcal{G}_C^*] - \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq A}} E[(M_D)^2 \,|\, \mathcal{G}_D^*]. \quad (3.96)$$

If for each $D \in \mathcal{N}_m$ and $C \in \mathcal{N}_n$ we define

$$d_D^{(C)} = E[(M_D)^2 \,|\, \mathcal{G}_C^*] - E[(M_D)^2 \,|\, \mathcal{G}_D^*], \quad (3.97)$$

then by Lemma A.3.5, (3.96) becomes

$$Q_A^{(n)} - Q_A^{(m)} = \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} d_D^{(C)}.$$

Therefore, by a basic calculation,

$$\begin{aligned}
\left(Q_A^{(n)} - Q_A^{(m)}\right)^2 &= \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} \left(d_D^{(C)}\right)^2 \\
&\quad + \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D_1, D_2 \in \mathcal{N}_m \\ D_1, D_2 \subseteq C \\ D_1 \neq D_2}} d_{D_1}^{(C)} \cdot d_{D_2}^{(C)} \\
&\quad + \sum_{\substack{C_1, C_2 \in \mathcal{N}_n \\ C_1, C_2 \subseteq A \\ C_1 \neq C_2}} \sum_{\substack{D_1, D_2 \in \mathcal{N}_m \\ D_i \subseteq C_i}} d_{D_1}^{(C_1)} \cdot d_{D_2}^{(C_2)}. \quad (3.98)
\end{aligned}$$

Employing this expansion, we will obtain the upper bound in (3.92) in three steps. In the first step, we dispose of the last sum in (3.98).

<u>Step 1:</u> *Given* $C_1, C_2 \in \mathcal{N}_n$ *s.t.* $C_1 \neq C_2$ *and* $D_1, D_2 \in \mathcal{N}_m$ *s.t.* $D_i \subseteq C_i$ $(i = 1, 2)$, $E[\, d_{D_1}^{(C_1)} \cdot d_{D_2}^{(C_2)}\,] = 0$.

Select such sets C_1, C_2, D_1 and D_2. By (3.97),

$$\begin{aligned}
d_{D_1}^{(C_1)} \cdot d_{D_2}^{(C_2)} &= E[(M_{D_1})^2 \,|\, \mathcal{G}_{C_1}^*] \cdot E[(M_{D_2})^2 \,|\, \mathcal{G}_{C_2}^*] \\
&\quad - E[(M_{D_1})^2 \,|\, \mathcal{G}_{C_1}^*] \cdot E[(M_{D_2})^2 \,|\, \mathcal{G}_{D_2}^*] \\
&\quad - E[(M_{D_1})^2 \,|\, \mathcal{G}_{D_1}^*] \cdot E[(M_{D_2})^2 \,|\, \mathcal{G}_{C_2}^*] \\
&\quad + E[(M_{D_1})^2 \,|\, \mathcal{G}_{D_1}^*] \cdot E[(M_{D_2})^2 \,|\, \mathcal{G}_{D_2}^*]. \quad (3.99)
\end{aligned}$$

Assume C_i is the left-neighborhood generated by $A_i \in \mathcal{A}_n$ $(i = 1, 2)$ and likewise, assume D_i is the left-neighborhood generated by $B_i \in \mathcal{A}_m$ $(i = 1, 2)$. By Lemma 3.2.27, we can assume w.l.o.g. that $\mathcal{F}_{A_1} \subseteq \mathcal{G}^*_{C_2}$. Moreover, $D_1 \subseteq C_1$ implies $B_1 \subseteq A_1$. (Otherwise $D_1 = D_1 \cap C_1 = \phi$, a contradiction.) Therefore, since $(\mathcal{F}_A)_{A\in\mathcal{A}}$ is increasing and $(\mathcal{G}^*_C)_{C\in\mathcal{C}}$ is decreasing,

$$\mathcal{F}_{B_1} \subseteq \mathcal{F}_{A_1} \subseteq \mathcal{G}^*_{C_2} \subseteq \mathcal{G}^*_{D_2}. \tag{3.100}$$

Furthermore, since D_1 is the left-neighborhood of B_1, (3.12) implies $(M_{D_1})^2$ is $\mathcal{F}_{B_1}$-measurable. Therefore, since $\mathcal{F}_{B_1} \subseteq \mathcal{F}_{A_1}$, Proposition B.3.10 implies

$$E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{C_1}] \text{ is } \mathcal{F}_{A_1} \text{ measurable.} \tag{3.101}$$

En route to establishing Step 1, we present the following general result.

Claim I: *Given sub-σ-algebras $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{F}$ and random variables $X, Y \in L_2$ s.t. X is $\mathcal{H}_i$-measurable $(i = 1, 2)$, $E[X \cdot E(Y \,|\, \mathcal{H}_1)\,] = E[X \cdot E(Y \,|\, \mathcal{H}_2)\,]$.*

Proof: Since $X \cdot E(Y \,|\, \mathcal{H}_i) = E(XY \,|\, \mathcal{H}_i)$ $(i = 1, 2)$, both expectations equal $E[XY]$ by conditioning. Ω

If we take

$$\mathcal{H}_1 = \mathcal{G}^*_{C_2},\ \mathcal{H}_2 = \mathcal{G}^*_{D_2},\ X = E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{C_1}] \quad \text{and} \quad Y = (M_{D_2})^2,$$

then by (3.100) and (3.101), we can apply Claim I to obtain

$$E\left\{E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{C_1}] \cdot E[\,(M_{D_2})^2 \,|\, \mathcal{G}^*_{C_2}]\right\} = E\left\{E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{C_1}] \cdot E[\,(M_{D_2})^2 \,|\, \mathcal{G}^*_{D_2}\,]\right\}.$$

By an identical argument,

$$E\left\{E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{D_1}] \cdot E[\,(M_{D_2})^2 \,|\, \mathcal{G}^*_{C_2}]\right\} = E\left\{E[\,(M_{D_1})^2 \,|\, \mathcal{G}^*_{D_1}] \cdot E[\,(M_{D_2})^2 \,|\, \mathcal{G}^*_{D_2}\,]\right\}.$$

In view of (3.99), these two identities imply $E[\,d^{(C_1)}_{D_1} \cdot d^{(C_2)}_{D_2}\,] = 0$, establishing Step 1.

Step 2: *$\exists$ a positive constant C_2 depending only on $E[M_T^4]$ s.t.*

$$\sum_{\substack{C\in\mathcal{N}_n\\ C\subseteq A}}\ \sum_{\substack{D_1,D_2\in\mathcal{N}_m\\ D_1,D_2\subseteq C\\ D_1\neq D_2}} E\left[d_{D_1}^{(C)}\cdot d_{D_2}^{(C)}\right] \ \le\ C_2\cdot\alpha_n\,.$$

Fix $C\in\mathcal{N}_n$ s.t. $C\subseteq A$ and let $D_1,D_2\in\mathcal{N}_m$ be s.t. $D_1,D_2\subseteq C$ and $D_1\neq D_2$. Since $\mathcal{G}_C^*\subseteq\mathcal{G}_{D_i}^*$ $(i=1,2)$, we obtain

$$E\left\{E[(M_{D_1})^2\,|\,\mathcal{G}_C^*]\cdot E[(M_{D_2})^2\,|\,\mathcal{G}_{D_2}^*]\right\} = E\left\{E[(M_{D_1})^2\,|\,\mathcal{G}_C^*]\cdot E[(M_{D_2})^2\,|\,\mathcal{G}_C^*]\right\}$$

when we condition w.r.t. $\mathcal{G}_C^*$ and then apply the tower property.

Now, consider the identity in (3.99) with $C_1=C_2=C$. Since all four products in (3.99) are non-negative, the previous identity implies

$$E\left[d_{D_1}^{(C)}\cdot d_{D_2}^{(C)}\right] \ \le\ E\left\{E[(M_{D_1})^2\,|\,\mathcal{G}_{D_1}^*]\cdot E[(M_{D_2})^2\,|\,\mathcal{G}_{D_2}^*]\right\}.$$

Therefore,

$$\begin{aligned}\sum_{\substack{D_1,D_2\in\mathcal{N}_m\\ D_1,D_2\subseteq C\\ D_1\neq D_2}} E\left[d_{D_1}^{(C)}\cdot d_{D_2}^{(C)}\right] &\le E\left[\left(\sum_{\substack{D\in\mathcal{N}_m\\ D\subseteq C}} E\left[(M_D)^2\,\middle|\,\mathcal{G}_D^*\right]\right)^2\right]\\ &\le \kappa_2\cdot E\left\{\left[M(\textstyle\bigcup_{D\in\mathcal{N}_m,\,D\subseteq C}D)\right]^4\right\}\quad\text{(by (3.88))}\\ &= \kappa_2\cdot E\left[(M_C)^4\right]\quad\text{(by Lemma A.3.3)}\end{aligned}$$

where κ_2 is the universal constant in Lemma 3.6.7. Thus, Step 2 will follow when we take $r=n$ in Claim II below.

Claim II: *$\exists$ a positive constant C_2' depending only on $E[M_T^4]$ s.t.*

$$\sum_{C\in\mathcal{N}_r} E\left[(M_C)^4\right] \ \le\ C_2'\cdot\alpha_r\,,\quad (\forall r\in\mathrm{N}).$$

Proof: Take $r\in\mathrm{N}$. In what is to follow, all sums and supremums will range over the sets $C\in\mathcal{N}_r$. Since $(M_C)^2\in L_2$ $\forall C\in\mathcal{N}_r$,

$$\begin{aligned}E\left\{\sum(M_C)^4\right\} &= E\left\{\left[\sup(M_C)^2\right]\cdot\left[\sum(M_C)^2\right]\right\}\\ &\le \sqrt{E\left[\sup(M_C)^4\right]}\cdot\sqrt{E\left\{\left[\sum(M_C)^2\right]^2\right\}}.\quad(3.102)\end{aligned}$$

(The last line is due to Hölder's inequality.) But by (3.87), $\exists$ a universal constant $\kappa_1 > 0$ s.t.

$$E\left\{\left[\sum (M_C)^2\right]^2\right\} \;\leq\; \kappa_1 \cdot \sqrt{E[M_T^4]}. \tag{3.103}$$

Therefore, substituting (3.103) into (3.102), we obtain Claim II. Ω

Finally,

Step 3: $\exists$ *a positive constant* C_1 *depending only on* $E[M_T^4]$ *s.t.*

$$\sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} E\left[\left(d_D^{(C)}\right)^2\right] \;\leq\; C_1 \cdot \alpha_m.$$

Given any $C \in \mathcal{N}_n$ s.t. $C \subseteq A$ and any $D \in \mathcal{N}_n$ s.t. $D \subseteq C$,

$$\begin{aligned} E\left[\left(d_D^{(C)}\right)^2\right] \;&=\; E\left\{\left(E[(M_D)^2 \,|\, \mathcal{G}_C^*] - E[(M_D)^2 \,|\, \mathcal{G}_D^*]\right)^2\right\} \\ &\leq\; E\left(E[(M_D)^2 \,|\, \mathcal{G}_C^*]^2\right) + E\left(E[(M_D)^2 \,|\, \mathcal{G}_D^*]^2\right) \\ &\leq\; 2 \cdot E\left[(M_D)^4\right] \quad \text{(by Jensen's inequality).} \end{aligned}$$

(The second to last line follows from the deterministic inequality $(a-b)^2 \leq a^2 + b^2$ where $a, b \geq 0$.) Therefore, since $E\left[(M_D)^4\right] \geq 0 \;\forall D \in \mathcal{N}_m$,

$$\sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} E\left[\left(d_D^{(C)}\right)^2\right] \;\leq\; 2 \cdot \sum_{D \in \mathcal{N}_m} E\left[(M_D)^4\right] \tag{3.104}$$

which, by Claim II, completes Step 3

Applying Steps 1, 2 and 3 to (3.98), we obtain (3.92), completing the proof of Proposition 3.6.13. □

3.7 L_2-norm Approximations of $*$-PQV

In this section, we give conditions on $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ and M under which the mode of convergence in (3.67) can be upgraded to convergence in L_2-norm. To start, we require the following

Assumption 3.7.1 *$\exists$ a constant $K \in \mathrm{N}$ s.t., given any $n \in \mathrm{N}$ and any $A \in \mathcal{A}_n$, $\exists\ \{A_1, \cdots, A_k\} \subseteq \mathcal{A}_n$ s.t.*

(i) *$\bigcup_{i=1}^k A_k$ is an extremal representation of $\bigcup\{A' \in \mathcal{A}_n : A' \subset A\}$ in the sense of Definition 3.2.5,*

(ii) *$k \leq K$ and*

(iii) *for any $A' \in \mathcal{A}_n$, $A' \subset A$ implies $\exists\, 1 \leq j \leq k$ s.t. $A' \subseteq A_j$.*

Remark 3.7.2 By Remark 3.2.8 (a), the set $A \setminus \bigcup_{i=1}^k A_i \in \mathcal{C}_n$ described in Assumption 3.7.1 is a minimal representation of the left-neighborhood generated by A in $\mathcal{A}_n$.

Examples 2.2.6 and 2.8.1 satisfy Assumption 3.7.1. For instance, when $\mathcal{A} = \mathcal{I}_n$ ($n \in \mathrm{N}$), we can take $K = n$. Moreover, if $\mathcal{A}$ satisfies the shape property (see Remark 2.2.5 (d)), condition (iii) follows from condition (i).

We begin with a few technical results.

Lemma 3.7.3 *Let M be an $\mathcal{A}$-indexed strong martingale for which*

$$E\left[\sup_{A\in\mathcal{A}} M_A^4\right] < \infty. \tag{3.105}$$

Under Assumption 3.7.1, if

$$\max_{C\in\mathcal{N}_n} |M_C| \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{3.106}$$

then $\alpha_n \to 0$ where $(\alpha_n)_n$ is the sequence defined in (3.93).

Proof For the sake of notation, let $T_n = \max_{C\in\mathcal{N}_n}(M_C)^4\ \forall n$.

<u>Claim:</u> *$\exists$ a universal constant $\beta > 0$ s.t. $T_n \leq \beta \cdot (\sup_{A\in\mathcal{A}} M_A^4)\ \forall n$.*

<u>Proof:</u> Fix $n \in \mathrm{N}$. Given $C \in \mathcal{N}_n$, $\exists\, A \in \mathcal{A}_n$ s.t. C is the left-neighborhood generated by A in $\mathcal{A}_n$. If $A \setminus \bigcup_{i=1}^k A_i$ is the minimal representation of C described in Assumption 3.7.1 (see Remark 3.7.2), then by (3.12),

$$\begin{aligned}|M_C| &\leq \sum_{I\subseteq\{1,\cdots,k\}} |M(A \cap \textstyle\bigcap_{i\in I} A_i)| \\ &\leq 2^K \cdot (\sup_{B\in\mathcal{A}} M_B)\end{aligned}$$

Since the right-hand side of this inequality is independent of n and $C \in \mathcal{N}_n$, the Claim follows by taking $\beta = 2^{4K}$. Ω

The above Claim implies $(T_n)_n$ is uniformly integrable. Therefore, if (3.106) holds, then by dominated convergence, $\alpha_n = E(T_n) \to 0$. □

Combining Lemma 3.7.3 and Proposition 3.6.13, we obtain

Corollary 3.7.4 *If M is an $\mathcal{A}$-indexed strong martingale satisfying* (3.105) *and* (3.106), *then under Assumption 3.7.1 and Assumption Groups D.1 and D.2,*

$$\left\{Q^{(n)}_{g_m(A)} : n \geq m\right\} \text{ is Cauchy in } L_2$$

for any fixed $m \in \mathbf{N}$ and $A \in \mathcal{A}$.

In the upcoming lemma, we will show that condition (3.106) holds for strong martingales whose sample paths are smooth in the following sense.

Definition 3.7.5 *A process $X = (X_A)_{A\in\mathcal{A}}$ defined on $(\Omega, \mathcal{F}, P)$ is said to be* <u>*d_H-continuous*</u> *if $\exists$ an event Ω_0 of full P-measure s.t.*

$$A_n \xrightarrow{d_H} A \text{ in } \mathcal{A} \implies X_{A_n}(\omega) \to X_A(\omega), \quad (\forall\, \omega \in \Omega_0).$$

Lemma 3.7.6 *Let M be an $\mathcal{A}$-indexed strong martingale satisfying* (3.105). *Under Assumption 3.7.1, if M is d_H-continuous, then* (3.106) *holds.*

Proof Before proceeding, we recall some terminology from the theory of functions. In particular, given a set-function $x : \mathcal{A} \to \mathbf{R}$, the *modulus of continuity* of x, denoted $w(x,\cdot) : (0,\infty) \to (0,\infty]$ is defined by

$$w(x,\epsilon) = \sup\{|x(A) - x(B)| : d_H(A,B) \leq \epsilon,\ A, B \in \mathcal{A}\}, \quad (\forall\, \epsilon > 0).$$

Since $(\mathcal{A}, d_H)$ is compact (see Theorem 2.2.13),

$$x \in C(\mathcal{A}) \iff \lim_{\epsilon\to 0} w(x,\epsilon) = 0. \tag{3.107}$$

Given any $\omega \in \Omega$, recall that the *sample path* of M at ω is the set-function $\hat{M}(\omega) : \mathcal{A} \to \mathbf{R}$ defined by letting

$$\hat{M}(\omega)(A) = M_A(\omega), \quad (\forall\, A \in \mathcal{A}).$$

If $(\epsilon_n)_n$ is the sequence of constants introduced in (2.2), then

Claim: $\max_{C \in \mathcal{N}_n} |M_C(\omega)| \;\leq\; 2^K \cdot w\left(\hat{M}(\omega)\,,\, 2\,\epsilon_n\right) \quad \forall n \in \mathbf{N}$ *and* $\omega \in \Omega$.

Proof: Fix $n \in \mathbf{N}$. Given $C \in \mathcal{N}_n$, $\exists\, A \in \mathcal{A}_n$ s.t. C is the left-neighborhood generated by A. Let $A \backslash \bigcup_{i=1}^{k} A_i$ be the minimal representation of C described in Assumption 3.7.1 (see Remark 3.7.2). If we define

$$\mathcal{E} \;=\; \{I \subseteq \{1, \cdots k\} : |I| \text{ is even}\} \quad \text{and} \quad \mathcal{O} \;=\; \{I \subseteq \{1, \cdots k\} : |I| \text{ is odd}\}\,,$$

then by the binomial theorem, $\exists$ a bijection $f : \mathcal{E} \to \mathcal{O}$. Clearly, $|\mathcal{E}| \leq 2^K$ where K is the constant defined in Assumption 3.7.1. Hence, by (3.12),

$$\begin{aligned} |M_C(\omega)| \;&\leq\; \sum_{I \in \mathcal{E}} \Big| M(A \cap \textstyle\bigcap_{i \in I} A_i)(\omega) \;-\; M(A \cap \textstyle\bigcap_{j \in f(I)} A_j)(\omega) \Big| \\ &\leq\; 2^K \cdot w\left(\hat{M}(\omega)\,,\, 2\,\epsilon_n\right) \quad \text{(by Lemma A.2.7)} \end{aligned} \tag{3.108}$$

for each $\omega \in \Omega$. Taking the maximum over all $C \in \mathcal{N}_n$ in the left-hand side of (3.108), the Claim is established. Ω

Assuming M is d_H-continuous, $\exists$ an event Ω_0 of full P-measure s.t. $\hat{M}(\omega)$ is d_H-continuous on $\mathcal{A}$ $\forall \omega \in \Omega_0$. Therefore, since $\epsilon_n \to 0$, (3.107) and the above claim imply

$$\lim_{n \to \infty} \max_{C \in \mathcal{N}_n} |M_C(\omega)| \;\leq\; 2^K \cdot \lim_{n \to \infty} w(\hat{M}(\omega)\,,\, 2\,\epsilon_n) \;=\; 0\,, \quad (\forall \omega \in \Omega_0),$$

i.e., $\max_{C \in \mathcal{N}_n} |M_C(\omega)| \to 0$ a.s. □

And now for the main result of this section,

Theorem 3.7.7 *Let M be an $\mathcal{A}$-indexed strong martingale for which*

$$E\left[\sup_{A \in \mathcal{A}} M_A^4\right] \;<\; \infty.$$

Under Assumption 3.7.1 and Assumption Groups D.1 and D.2, if either

(a) *M is d_H-continuous or*

(b) *M is right-continuous and* $\max_{C \in \mathcal{N}_n} |M_C| \xrightarrow{P} 0$ *as $n \to \infty$,* *then M possesses a unique $*$-PQV (see Definition 3.5.6) denoted Q s.t.*

$$Q_A \;=\; \lim_{m\to\infty} \lim_{n\to\infty,\, n\geq m} Q^{(n)}_{g_m(A)} \;\; in\ L_2\,, \quad (\forall\, A \in \mathcal{A}) \tag{3.109}$$

where $Q^{(k)}_B$ ($k \in \mathbb{N}$, $B \in \mathcal{A}_k$) is the random variable defined in (3.68).

Proof By Lemma 3.7.6 and Definition 3.7.5, it is clear that (a) implies (b). Therefore, we need only establish Theorem 3.7.7 under condition (b).

By Corollary 3.6.12, M possesses a unique $*$-PQV denoted $Q = (Q_A)_{A\in\mathcal{A}}$ s.t., given any $A \in \mathcal{A}$,

$$\overline{Q}_{g_m(A)} \xrightarrow{m} Q_A \text{ weakly in } L_1 \tag{3.110}$$

where

$$Q^{(n)}_{g_m(A)} \xrightarrow{n} \overline{Q}_{g_m(A)} \text{ weakly in } L_1\,, \quad (\forall m \in \mathbb{N}). \tag{3.111}$$

(See the proof of Theorem 3.5.2 for details.) To establish L_2 convergence in (3.110) and (3.111), we need the following result.

<u>Claim I:</u> *Let $(X_n)_n$ be a Cauchy sequence in L_2. If $X_n \to X$ weakly in L_1 to some $X \in L_1$, then $X \in L_2$ and $X_n \to X$ in L_2.*

<u>Proof:</u> The proof follows from three well known facts: L_2 is complete, convergence in L_2 implies convergence in the weak L_1 topology and weak L_1 limits are unique up to a.s.-equivalence. Ω

Now, take $A \in \mathcal{A}$. Given $m \in \mathbb{N}$, Corollary 3.7.4 implies $(Q^{(n)}_{g_m(A)})_{n\geq m}$ is Cauchy in L_2. Therefore, by Claim I and (3.111),

$$Q^{(n)}_{g_m(A)} \xrightarrow{n} \overline{Q}_{g_m(A)} \text{ in } L_2. \tag{3.112}$$

In light of Claim I and (3.110), the limit in (3.109) will follow if we can show $(\overline{Q}_{g_m(A)})_m$ is Cauchy in L_2. For this purpose, we need two technical results.

<u>Claim II:</u> *Given $k, m, n \in \mathbb{N}$ s.t. $k, m \leq n$,*

$$\left\| Q^{(n)}_{g_k(A)} - Q^{(n)}_{g_m(A)} \right\|_2 \;\leq\; \gamma \cdot \left\| M_{g_k(A)} - M_{g_m(A)} \right\|_4^2 \tag{3.113}$$

where γ is a positive constant depending only on $E\,[M_T^4]$.

Proof: If we assume $k \leq m$, then by Definition 2.2.2, $g_k(A),\ g_m(A) \in \mathcal{A}_n$ and $g_m(A) \subseteq g_k(A)$. Therefore,

$$\begin{aligned} Q^{(n)}_{g_k(A)} - Q^{(n)}_{g_m(A)} &= \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq g_k(A)}} E\left[(M_C)^2 \,|\, \mathcal{G}^*_C\right] - \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq g_m(A)}} E\left[(M_C)^2 \,|\, \mathcal{G}^*_C\right] \\ &= \sum_{\substack{C \in \mathcal{N}_n \\ C \subseteq g_k(A) \setminus g_m(A)}} E\left[(M_C)^2 \,|\, \mathcal{G}^*_C\right] \quad \text{(by Lemma A.3.4).} \end{aligned}$$

Applying (3.88) to the above array, $\exists$ a positive constant κ_2 depending only on $E\left[M_T^4\right]$ s.t.

$$\begin{aligned} \left\| Q^{(n)}_{g_k(A)} - Q^{(n)}_{g_m(A)} \right\|_2^2 &\leq \kappa_2 \cdot \left\| M\left(\bigcup\{C \in \mathcal{C}_n : C \subseteq g_k(A) \setminus g_m(A)\}\right)\right\|_4^4 \\ &\leq \kappa_2 \cdot \left\| M_{g_k(A)\setminus g_m(A)} \right\|_4^4 \quad \text{(by Lemma A.3.4).} \end{aligned}$$

Since M is finitely additive on $\mathcal{C}(u)$, $M_{g_k(A)\setminus g_m(A)} = M_{g_k(A)} - M_{g_m(A)}$ $\forall m$ so that Claim II follows by selecting $\gamma = \sqrt{\kappa_2}$. Ω

Claim III: $M_{g_m(A)} \xrightarrow{m} M_A$ *in* L_4.

Proof: Since $g_m(A) \downarrow A$ and M is right-continuous, $M_{g_m(A)} \to M_A$ a.s. Therefore, since

$$\left(M_{g_k(A)} - M_A\right)^4 \leq 2^4 \cdot \left(\sup_{A \in \mathcal{A}} M_A^4\right) \in L_1,$$

Claim III follows by the dominated convergence theorem. Ω

To complete the proof of Theorem 3.7.7 (in particular (3.109)), take any $\epsilon > 0$. For any $k, m, n \in \mathrm{N}$,

$$\begin{aligned} \left\| \overline{Q}_{g_k(A)} - \overline{Q}_{g_m(A)} \right\|_2 \leq\ & \left\| \overline{Q}_{g_k(A)} - Q^{(n)}_{g_k(A)} \right\|_2 + \left\| Q^{(n)}_{g_k(A)} - Q^{(n)}_{g_m(A)} \right\|_2 + \\ & \left\| Q^{(n)}_{g_m(A)} - \overline{Q}_{g_m(A)} \right\|_2 . \end{aligned} \tag{3.114}$$

By Claims II and III, $\exists\, M \in \mathrm{N}$ s.t. for any $k, m \geq M$,

$$\left\| Q^{(n)}_{g_k(A)} - Q^{(n)}_{g_m(A)} \right\|_2 < \epsilon/3, \quad (\forall n \in \mathrm{N} \text{ s.t. } n \geq k, m). \tag{3.115}$$

But given $k, m \geq M$, (3.112) implies $\exists\, n_0 \geq k, m$ s.t.

$$\left\| \overline{Q}_{g_k(A)} - Q^{(n_0)}_{g_k(A)} \right\|_2, \left\| \overline{Q}_{g_m(A)} - Q^{(n_0)}_{g_m(A)} \right\|_2 < \epsilon/3 . \tag{3.116}$$

Substituting (3.115) (with $n = n_0$) and (3.116) into (3.114), we obtain $\|\overline{Q}_{g_k(A)} - \overline{Q}_{g_m(A)}\|_2 < \epsilon$ for our arbitrary $k, m \geq M$. Therefore, $(\overline{Q}_{g_m(A)})_m$ is Cauchy in L_2 which, by Claim I and (3.110), implies

$$\overline{Q}_{g_m(A)} \xrightarrow{m} Q_A \text{ in } L_2,$$

establishing (3.109). □

Chapter 4

Convergence Theorems

4.1 Introduction

In this chapter, we obtain limit theorems for sequences of set-indexed strong martingales. In all limit theorems, the limiting process will be an appropriately scaled set-indexed Gaussian process. The convergence will primarily be *semi-functional*, an entirely new mode of convergence ideally suited to strong martingale CLTs. As well, this chapter contains two functional CLTs and an application to set-indexed weighted empirical processes. Throughout this chapter, unless otherwise mentioned, $\mathcal{A}$ denotes a generic indexing collection on a generic compact metric space (T, d).

After reviewing the relevant aspects of the classical theory, Section 4.2 presents three key concepts. The first is that of *flows*, maps of the form $f : [0,1] \to \mathcal{A}(u)$ which are capable of transforming set-indexed problems into classical problems. Next is the concept of $*$-*quadratic variation* for L_2 strong martingales. A $*$-quadratic variation differs from a $*$-PQV only in that the former is not necessarily $*$-predictable. For this very reason, $*$-quadratic variations are generally easier to work with in applications. Finally, Section 4.2 formally defines set-indexed Gaussian processes and presents a sufficient condition on $\mathcal{A}$ under which an $\mathcal{A}$-indexed Gaussian process possesses a Hausdorff continuous version. A special class of set-indexed Gaussian processes termed *Gaussian white noise* are also defined.

In Section 4.3, we define two modes of convergence for sequences of set-indexed processes. The first, termed *functional convergence* is entirely classic being defined via the notion of weak convergence of probability measures.

The other mode of convergence termed *semi-functional convergence* depends heavily on the flows introduced in Subsection 4.3.2. Unlike functional convergence, semi-functional convergence does not require the $\mathcal{A}$-indexed processes we work with to possess versions in $D(\mathcal{A})$. This omission is crucial when $\mathcal{A}$ is large w.r.t. the notion of *metric entropy*. Under a mild condition which is met in each of the subsequent limit theorems, semi-functional convergence implies convergence in finite dimensional distribution (see Proposition 4.3.17).

Section 4.4 contains our first limit theorem. Specifically, given a sequence $(X_n)_n$ of $\mathcal{A}$-indexed strong martingales, each possessing a weak form of sample path continuity, Theorem 4.4.1 gives sufficient conditions under which $(X_n)_n$ converges semi-functionally to an appropriately scaled Gaussian white noise. These sufficient conditions include the existence of $*$-quadratic variation Λ_n for each X_n such that $(\Lambda_n)_n$ converges pointwise in probability to a continuous deterministic limit. When each X_n possesses $*$-PQV, these conditions can be weakened (see Proposition 4.4.2). A related functional CLT for continuous strong martingales is given in Theorem 4.4.4.

Section 4.5 contains the second and most general limit theorem (see Theorem 4.5.5). This limit theorem is similar in form to the one mentioned in the previous paragraph. Theorem 4.5.5 does not require the X_n to have any type of sample path continuity, however, the discontinuities of the X_n must become negligible as $n \to \infty$ in the sense of the J-L_2-AS condition for asymptotic rarefaction of jumps (see Definition 4.5.1). This condition, which depends heavily on flows, is satisfied by many natural set-indexed processes (see Examples 4.5.2, 4.5.3 and 4.5.4 and Proposition 4.6.21). A related functional CLT for set-indexed strong martingales in $D(\mathcal{A})$ is given in Theorem 4.5.8.

In Section 4.6, we apply Theorem 4.5.5 to obtain a semi-functional CLT for the d-dimensional weighted empirical processes $(U_n)_n$ defined in (4.56). In particular, when the random weights have finite fourth moments and each U_n is indexed by a suitable indexing collection $\mathcal{A}$ on $T = [0,1]^d$ $(d \geq 1)$, $(U_n)_n$ converges semi-functionally to an appropriately scaled $\mathcal{A}$-indexed Gaussian white noise (see Theorem 4.6.22). For the case of $\mathcal{A} = \mathcal{LL}_d$ $(d \geq 2)$, it is shown that a.e. sample path of the limiting Gaussian white noise is necessarily Hausdorff discontinuous everywhere on $\mathcal{LL}_d$ (see Example 4.6.26).

In this chapter, with one exception, **all $\mathcal{A}$-indexed processes and set-functions we work with are assumed to possess unique finitely additive extensions to $\mathcal{C}(u)$.** This one exception is the general set-indexed Gaussian process (see Comment 4.2.29).

4.2 Fundamental Concepts

In this section, we present the concepts that are essential in the formulation and proof of central limit theorems for sequences of set-indexed strong martingales. In what is to follow, the adjective *classical* will refer either to filtrations or processes which are indexed by the points of $[0,\infty)$ or some compact subinterval thereof.

4.2.1 Classical martingales and $D[0,a]$

The upcoming limit theorems for set-indexed strong martingales will require results from the theory of classical processes. This subsection contains the required portion of that theory. The reader is assumed to be familiar with the following concepts:

- weak convergence in $D[0,a]$ $(a > 0)$ w.r.t. the Skorokhod metric, [1]
- classical Brownian motion,
- classical filtrations and classical martingales and
- the Doob-Meyer decomposition for classical submaringales.

For good references, see [6] for the first topic, [38] for the second topic and Chapter 3 of [30] for the third and fourth. As is usual practice, given a property **P** concerning sample paths, we say that a process (classical or set-indexed) possesses **P** provided a.e. sample path of that process possesses **P**. For example, a continuous process is one for which a.e. sample path is continuous.

Let (S,τ) be any metric space (not necessarily complete or separable) and let $\mathcal{S}$ denote the Borel σ-algebra on S generated by τ. The following definition appears on p.7 of [6].

Definition 4.2.1 *Let $\mu, \mu_1, \mu_2 \cdots$ be be probability measures on $\mathcal{S}$. $(\mu_n)_n$ is said to converge weakly to μ, denoted $\mu_n \Rightarrow \mu$, provided*

$$\int f\,d\mu_n \;\to\; \int f\,d\mu \tag{4.1}$$

for every bounded, τ-continuous function, $f : S \to \mathbf{R}$.

[1]In this chapter, *Skorokhod metric* refers exclusively to the Skorokhod J_1 metric

Fix a number, $a > 0$ and let $D[0,a]$ consist of real-functions on $[0,a]$ which are right-continuous with left-limits on $[0,a]$. So as to invite an application of the theory of weak convergence in metric spaces, endow $D[0,a]$ with the Skorokhod metric. As mentioned in [6], $D[0,a]$ is both complete and separable under this metric. We let $\mathcal{D}[0,a]$ denote the Borel σ-algebra on $D[0,a]$ generated by the Skorokhod metric.

A classical process $Y = \{Y_t : t \in [0,a]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ is said to be in $D[0,a]$ (or in $C[0,a]$) if a.e. sample path of Y lies in $D[0,a]$ (respectively, in $C[0,a]$). As mentioned on p.128 of [6], the $\mathcal{F}$-measurability of each Y_t is enough to ensure the $\mathcal{D}[0,a]$-measurability of $Y : \Omega \to D[0,a]$ viewed as a random function. This fact is recorded below for future reference.

Lemma 4.2.2 *If Y is a process in $D[0,a]$, then Y is $\mathcal{D}[0,a]$-measurable.*

The analogous statement is true for processes in $C[0,a]$.

In view of Lemma 4.2.2, any processes Y in $D[0,a]$ induces a probability measure on $(D[0,a], \mathcal{D}[0,a])$. A sequence $(Y_n)_n$ of processes in $D[0,a]$ is said to *converge in distribution* to a process Y in $D[0,a]$ if the sequence of probability measures induced by the Y_n converges weakly in the sense of Definition 4.2.1 to the probability measure induced by Y. In such a case, we will write

$$Y_n \xrightarrow{\mathcal{L}} Y \text{ in } D[0,a].$$

Although each Y_n (viewed as a random function) must be $D[0,a]$-valued, it is not necessary that each be defined on a common probability space.

Define the *jump functional*, $J_a : D[0,a] \to [0,\infty)$ by letting

$$J_a(y) = \sup_{0 \le t \le a} |\Delta y(t)|, \quad (\forall\, y \in D[0,a]) \tag{4.2}$$

where $\Delta y(t) = y(t) - y(t-)$ $\forall\, t \in (0,a]$ and $\Delta y(0) = 0$. Given any $y \in D[0,a]$, $J_a(y)$ represents the largest vertical jump in the graph of y. In particular, note that

$$J_a(y) = 0, \quad (\forall\, y \in C[0,a]). \tag{4.3}$$

As mentioned on p.179 of [34], J_a is $\mathcal{D}[0,a]$-measurable. Also, as mentioned in Proposition 2.4 on p.303 of [29], given any $y \in D[0,a]$,

$$\Delta y(a) = 0 \implies J_a \text{ is continuous at } y \tag{4.4}$$

where continuity is w.r.t. the Skorokhod metric on $D[0,a]$. The following result appears as Proposition 3.26 on p.315 of [29].

Proposition 4.2.3 *Assume $(Y_n)_n$ is a tight sequence of processes in $D[0,a]$ w.r.t. the Skorokhod topology (i.e., the induced measures are tight on $\mathcal{D}[0,a]$) and that*

$$J_a(Y_n) \to 0 \text{ in probability as } n \to \infty. \tag{4.5}$$

If $Y_{n'} \to_{\mathcal{L}} Y$ in $D[0,a]$ for some subsequence $(n')_n$, then Y is in $C[0,a]$.

By (4.3), Proposition 4.2.3 yields the following result.

Corollary 4.2.4 *Assume $(Y_n)_n$ is a sequence of processes in $C[0,a]$ which is tight in $D[0,a]$ w.r.t. the Skorokhod topology. If $Y_{n'} \to_{\mathcal{L}} Y$ in $D[0,a]$ for some subsequence $(n')_n$, then Y is in $C[0,a]$.*

At this point, we define a famous family of classical processes.

Definition 4.2.5 *Let $\mathbf{T}$ be a subinterval of $\mathbf{R}$ either of the form $[0,a]$ $(a>0)$ or $[0,\infty)$. A process $B = \{B_t : t \in \mathbf{T}\}$ is said to be a standard Brownian motion if*

(i) $B_0 = 0$ *a.s.,*

(ii) *for any $s < t$ in $\mathbf{T}$, $B_t - B_s \sim N(0, t-s)$,*

(iii) *for any n and any $t_1 < \cdots < t_n$ in $\mathbf{T}$, $B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$ are mutually independent and*

(iv) *the sample paths of B are continuous on $[0,\infty)$.*

Given any standard Brownian motion B and any continuous increasing function $\lambda : \mathbf{T} \to [0,\infty)$ with $\lambda(0) = 0$, the process $B_\lambda = \{B(\lambda(t)) : t \in \mathbf{T}\}$ is said to be a stretched-out Brownian motion.

Comment 4.2.6 For most authors, [34] for example, a classical process Y is a stretched-out Brownian motion if and only if $Y_t = B(\lambda(t))\ \forall t \in \mathbf{T}$ where B is a standard Brownian motion and $\lambda : \mathbf{T} \to [0,\infty)$ is a continuous strictly increasing function with $\lambda(0) = 0$. The omission of "strictly increasing" in our definition will be important in the sequel (see Proposition 4.2.36).

We now turn our attention to classical martingales. In this thesis, any classical stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{H}_t), \mathbf{T})$ is assumed to be such that

- $(\Omega, \mathcal{F}, P)$ is complete,
- $\mathbf{T}$ is either of the form $[0, a]$ $(a > 0)$ or $[0, \infty)$ and
- $(\mathcal{H}_t)_{t \in \mathbf{T}}$ is increasing, right-continuous and complete.

Let $D[0, \infty)$ denote the set of all functions $y : [0, \infty) \to \mathbf{R}$ which are right-continuous with left-limits on $[0, \infty)$. For our purposes, it is not necessary to topologize $D[0, \infty)$. The following is a well-known consequence of the Doob-Meyer decomposition theorem.

Theorem 4.2.7 *If $Y = \{Y_t : t \in [0, \infty)\}$ is an L_2 martingale in $D[0, \infty)$ w.r.t. some stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{H}_t), [0, \infty))$, then $\exists$ a unique (up to indistinguishability) process, $\langle Y \rangle = \{\langle Y \rangle(t) : t \in [0, \infty)\}$ on $(\Omega, \mathcal{F}, P)$ s.t.*

(P1) *$\langle Y \rangle$ is predictable,*

(P2) *$\langle Y \rangle$ is a non-negative increasing process in $D[0, \infty)$ and*

(P3) *$Y^2 - \langle Y \rangle$ is a martingale w.r.t. $(\mathcal{H}_t)_{t \in [0,\infty)}$ with $Y^2(0) - \langle Y \rangle(0) = 0$.*

Furthermore, if Y is continuous, then so is $\langle Y \rangle$. In any case, the process $\langle Y \rangle$ is called the predictable quadratic variation of Y.

Given a process $Y = \{Y_t : t \in [0, \infty)\}$ and any number $a > 0$, let $Y|_a$ denote the process Y *truncated at a*. That is,

$$Y|_a(t) = Y(t), \quad (\forall t \in [0, a]). \tag{4.6}$$

By Lemma 4.2.2, the composition $J_a(Y|_a)$ is a random variable for every $a > 0$. The following CLT appears as Theorem 13 on p.179 of [34]. We have made some minor changes in its statement to suit our upcoming applications.

Theorem 4.2.8 *For each n, let $Y_n = \{Y_n(t) : t \in [0, \infty)\}$ be an L_2 martingale in $D[0, \infty)$. If $\lambda : [0, \infty) \to [0, \infty)$ is a continuous increasing function with $\lambda(0) = 0$ s.t.*

(i) *$Y_n(0) \to 0$ in probability,*

(ii) *for each $t \geq 0$, $\langle Y_n \rangle(t) \to \lambda(t)$ in probability and*

(iii) *for each $a > 0$, $E[J_a(Y_n|_a)^2] \to 0$ as $n \to \infty$,* *then* $\exists$ *a standard Brownian motion,* $B = \{B_t : t \in [0, \infty)\}$ *s.t.* $Y_n|_a \xrightarrow{\mathcal{L}} B_\lambda|_a$ *in* $D[0, a]$ *for every* $a > 0$ *where* $B_\lambda(t) = B(\lambda(t))$ $\forall t$.

Comment 4.2.9 Although stated for the homogeneous case, neither Lemma 11 nor its consequence, Theorem 13 in Chapter VIII of [34] require each martingale Y_n to be defined w.r.t. a common stochastic base. Also, nowhere in the proof of Theorem 13 in [34] is λ required to be strictly increasing on $[0, \infty)$.

We close this subsection with a result concerning sample path properties. In general, a right-continuous function $y : [0, a] \to \mathbf{R}$ need not have left-limits on $[0, a]$ and thus, a classical process with right-continuous paths is not necessarily in $D[0, a]$. However, as stated in Theorem 2.9 on p.61 of [38],

Proposition 4.2.10 *If* $Y = \{Y_t : t \in [0, a]\}$ *is a martingale, then* Y *has a modification* Y' *in* $D[0, a]$, *i.e.,* $Y'_t = Y_t$ *a.e.* $\forall t$.

Since any two right-continuous modifications of a classical process are indistinguishable (and hence equal as processes), Proposition 4.2.10 yields

Corollary 4.2.11 *If* $Y = \{Y_t : t \in [0, a]\}$ *is a right-continuous martingale, then* Y *is in* $D[0, a]$.

Remark 4.2.12 By Theorem 2.8 on p.61 of [38], the conclusion in Corollary 4.2.11 also holds for right-continuous classical submartingales.

4.2.2 Flows

Let $\mathcal{A}$ be an indexing collection on a compact metric space (T, d). The following definition has appeared in [26].

Definition 4.2.13 *Given* $a < b$ *in* $\mathbf{R}$, *a function* $f : [a, b] \to \mathcal{A}(u)$ *is said to be a* <u>*flow*</u> *provided*

(i) $a \le t \le s \le b$ *implies* $f(t) \subseteq f(s)$

(ii) $f(s) = \bigcap_{v>s} f(v) = \overline{\bigcup_{u<s} f(u)}$ $\forall a < s < b$ *and*

(iii) $f(a) = \bigcap_{v>a} f(v)$, $f(b) = \overline{\bigcup_{u<b} f(u)}$

where the closures in (ii) *and* (iii) *are taken in* (T,d).

Condition (i) states that f is increasing on $[a,b]$ w.r.t. $\subseteq$ whereas conditions (ii) and (iii) determine a form of continuity for f.

Our interest in flows lies in their potential to transform set-indexed problems into classical problems. However, set-indexed filtrations and processes are parameterized by $\mathcal{A}$ whereas the range of an arbitrary flow is contained in $\mathcal{A}(u)$. One could naively overcome this problem by working only with flows whose range lies in $\mathcal{A}$ but this would be too restrictive. Between these two extremes is the following class of flows.

Definition 4.2.14 *A flow,* $f:[0,1]\to\mathcal{A}(u)$ *is said to be* <u>*simple*</u> *provided* $\exists\, 0=t_0<t_1<\cdots<t_k=1$ *(some* $k\in\mathbb{N}$*) and corresponding flows*

$$f_i:[t_{i-1},t_i]\to\mathcal{A},\quad (1\le i\le k)$$

s.t. $f(0)=\phi'$ *(as defined in* (2.6)*),* $f(1)=T$ *and for each fixed* $1\le i\le k$*,*

$$f(t) \;=\; [\textstyle\bigcup_{j=1}^{i-1} f_j(t_j)] \cup f_i(t)\,,\quad (\forall\, t\in[t_{i-1},t_i]). \tag{4.7}$$

The collection of all simple flows is denoted by $S(\mathcal{A})$.

Remark 4.2.15 (a) In other words, a flow is simple if there is a partition of $[0,1]$ s.t. f is $\mathcal{A}$-valued, modulo a fixed set in $\mathcal{A}(u)$, on each piece of the said partition. Note that all simple flows begin at ϕ' and end at T.

(b) (4.7) forces the inclusion, $f_i(t_{i-1})\subseteq\bigcup_{j=0}^{i-1}f_j(t_j)\ \forall\, 1\le i\le k$.

(c) Since any flow can be linearly rescaled without losing properties (i), (ii) and (iii) of Definition 4.2.13, we can always take $t_i=\frac{i}{k}\ \forall\, 0\le i\le k$ in Definition 4.2.14.

The richness of $S(\mathcal{A})$ is illustrated by the following result which appears as Lemma 3 in [26].

Proposition 4.2.16 *Let* $\mathcal{A}'=\{A_0,\cdots,A_k\}$ *be a f.s.s.l. of* $\mathcal{A}$ *numbered in a manner c.w.s.p.* [2] *(Recall that* $A_0=\phi'$ *and* $A_k=T$.*) Then,* $\exists\, f\in S(\mathcal{A})$ *with* $t_i=\frac{i}{k}\ \forall\, i$ *such that*

(a) $f(\frac{i}{k})=\bigcup_{j=0}^{i}A_j\ \ \forall\, 0\le i\le k$ *and*

[2] See Subsection 3.2.1 for information on f.s.s.l. of $\mathcal{A}$ and their numberings

(b) $C_i = f(\frac{i}{k}) \setminus f(\frac{i-1}{k})\ \forall 1 \leq i \leq k$

where C_i denotes the left-neighborhood generated by A_i in $\mathcal{A}'$.

In other words, every f.s.s.l. of $\mathcal{A}$ is accounted for by some simple flow.

Comment 4.2.17 As previously mentioned, Proposition 4.2.16 appears as Lemma 3 in [26]. We comment briefly on the assumptions on (T, d) and $\mathcal{A}$ in [26]. First of all, Ivanoff and Merzbach assumed the underlying metric space (T, d) to be complete and separable. Neither of these properties were used in the proof of Lemma 3. (Just the same, our (T, d) is compact, hence complete.) In addition, the indexing collection $\mathcal{A}$ in [26] was assumed to satisfy

(i) $(A_n)_n$ an increasing sequence in $\mathcal{A}$ implies $\overline{\bigcup_n A_n} \in \mathcal{A}$ and

(ii) $\mathcal{B}(T) \subseteq \sigma(\mathcal{A})$

where $\mathcal{B}(T)$ denotes the Borel σ-algebra on T generated by d. Whereas (i) is a consequence of Lemma A.2.2, condition (ii) was not used in the proof of Lemma 4.2.16. Other than these points, our (T, d) and $\mathcal{A}$ satisfy all conditions found in [26].

We now explore the effect that simple flows have on certain set-functions. Recall by Lemma A.2.2 that $(A_n)_n$ increasing in $\mathcal{A}$ implies $\overline{\bigcup_n A_n} \in \mathcal{A}$. Of course, by the definition of indexing collection, (A_n) decreasing in $\mathcal{A}$ implies $\bigcap_n A_n \in \mathcal{A}$.

Lemma 4.2.18 (a) *If $x : \mathcal{A} \to \mathbf{R}$ is s.t.*

$$(A_n)_n \textit{ decreasing in } \mathcal{A} \implies x(A_n) \to x(\textstyle\bigcap_n A_n), \tag{4.8}$$

then $x \circ f : [0,1] \to \mathbf{R}$ is right-continuous on $[0,1]$ $\forall f \in S(\mathcal{A})$.
(b) *If $x : \mathcal{A} \to \mathbf{R}$ is s.t.*

$$(A_n)_n \textit{ increasing in } \mathcal{A} \implies x(A_n) \to x(\overline{\textstyle\bigcup_n A_n}), \tag{4.9}$$

then $x \circ f : [0,1] \to \mathbf{R}$ is left-continuous on $[0,1]$ $\forall f \in S(\mathcal{A})$.

Proof Select $f \in S(\mathcal{A})$. By Remark 4.2.15 (c), $\exists\, k \in \mathbf{N}$ and flows, $f_i : [\frac{i-1}{k}, \frac{i}{k}] \to \mathcal{A}$ $(1 \leq i \leq k)$ satisfying (4.7).

We begin by showing part (a). To establish the right-continuity of $x \circ f$ on $[0,1]$, take any $s \in [0,1)$ and let $1 \leq i \leq k$ be s.t. $s \in [\frac{i-1}{k}, \frac{i}{k})$. Given any sequence $(s_n)_n$ in $[s, \frac{i}{k})$ s.t. $s_n \downarrow s$, the finite additivity of x on $C(u)$ yields

$$\begin{aligned} x(f(s_n)) - x(f(s)) &= x(f(s_n) \setminus f(s)) \\ &= x(f_i(s_n) \setminus f_i(s)) \quad \text{(by (4.7))} \\ &= x(f_i(s_n)) - x(f_i(s)) \end{aligned}$$

for any n. Since f_i is an $\mathcal{A}$-valued flow, $(f_i(s_n))_n$ decreases in $\mathcal{A}$ to $\cap_n f_i(s_n) = f_i(s)$. Therefore, (4.8) implies

$$\lim_n x(f(s_n)) - x(f(s)) \;=\; \lim_n x(f_i(s_n)) - x(f_i(s)) \;=\; 0$$

which is sufficient for right-continuity of $x \circ f$ at s. The argument for part (b) is identical. □

Remark 4.2.19 By a similar argument, $x : \mathcal{A} \to \mathbf{R}$ outer-continuous implies $x \circ f : [0,1] \to \mathbf{R}$ right-continuous $\forall\, f \in S(\mathcal{A})$. However, under our present definition of $D(\mathcal{A})$, $x \in D(\mathcal{A})$ need not imply $x \circ f \in D[0,1]$ for an arbitrary simple flow f. In particular, given $t \in (0,1]$ and $(s_n)_n$ in $[0,1]$ s.t. $s_n < t\ \forall\, n$ and $s_n \to t$, it need not follow that $f(s_n) \subseteq [f(t)]^\circ\ \forall\, n$ as required in the definition of inner-limits.

Fix a stochastic base, $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$. To any flow, $f : [0,1] \to \mathcal{A}(u)$, we can associate a family $(\mathcal{H}_t)_{t \in [0,1]}$ of sub-σ-algebras by defining

$$\mathcal{H}_t \;=\; \mathcal{F}_{f(t)}\,, \quad (\forall\, t \in [0,1]). \tag{4.10}$$

where $\mathcal{F}_{f(t)}$ is defined via (3.9). Since f is increasing, Lemma 3.2.24 (e) implies

$$\mathcal{H}_s \;\subseteq\; \mathcal{G}^*_{f(t) \setminus f(s)}\,, \quad (\forall\, s < t). \tag{4.11}$$

Although the family in (4.10) is complete and increasing, it may not be right-continuous. On the other hand,

Lemma 4.2.20 *Given $f \in S(\mathcal{A})$, the family $(\mathcal{H}_t)_{t \in [0,1]}$ defined in (4.10) is a classical filtration (see the third bullet on p.144). Moreover,*

(a) *X adapted to $(\mathcal{F}_A)_{A\in\mathcal{A}}$ implies $X\circ f$ is adapted to $(\mathcal{H}_t)_{t\in[0,1]}$ and*

(b) *X a strong (sub)martingale implies $X\circ f$ a (sub)martingale.*

Here, $X\circ f$ denotes the classical process $\{X_{f(t)} : t\in[0,1]\}$.

Proof As mentioned above, $(\mathcal{H}_t)_{t\in[0,1]}$ is an increasing family of complete sub-σ-algebras of $\mathcal{F}$. To show right-continuity, take $t\in[0,1)$ and let $1\le i\le k$ be s.t. $t\in[\frac{i-1}{k},\frac{i}{k})$. Then,

$$\bigcap_{t<s}\mathcal{H}_s = \bigcap_{t<s<i/k}\mathcal{F}_{f(s)} = \bigcap_{t<s<i/k}\bigcap_{n}\mathcal{F}^{\circ}_{g_n(f(s))} \tag{4.12}$$

where $g_n:\mathcal{A}(u)\to\mathcal{A}_n(u)$ is the extension of g_n defined in (2.4).

Fix n. By (4.7) and Proposition 2.2.9 (b),

$$g_n(f(s)) = [\cup_{j=1}^{i-1}g_n(f_j(j/k))]\cup g_n(f_i(s)), \quad (\forall\, s\in(t,i/k)). \tag{4.13}$$

However, since

- $f_i(s)\downarrow f_i(t)$ as $s\downarrow t$ and
- g_n is decreasing, preserves countable intersections and has a finite range,

$\exists\, s_0\in(t,i/k)$ s.t. $g_n(f_i(s))=g_n(f_i(t))\ \ \forall\, s\in(t,s_0)$ which, by (4.13), implies

$$g_n(f(s)) = g_n(f(t)), \quad (\forall\, s\in(t,s_0)). \tag{4.14}$$

Since n was arbitrary, (4.12) and (4.14) yield

$$\bigcap_{t<s}\mathcal{H}_s = \bigcap_{n}\bigcap_{t<s<i/k}\mathcal{F}^{\circ}_{g_n(f(s))} = \bigcap_{n}\mathcal{F}^{\circ}_{g_n(f(t))} = \mathcal{F}_{f(t)} = \mathcal{H}_t.$$

For part (a), note that $\mathcal{F}^{\circ}_{f(t)}\subseteq\mathcal{F}_{f(t)}\ \forall\, t$ so that the finite additivity of X on $\mathcal{C}(u)$ implies $X_{f(t)}$ is $\mathcal{F}^{\circ}_{f(t)}$-measurable $\forall\, t$. For part (b), take any $s<t$ in $[0,1]$. By (4.11) and the tower property,

$$E\left(X_{f(t)}-X_{f(s)}\middle|\mathcal{H}_t\right) = E\left[E\left(X_{f(t)\setminus f(s)}\middle|\mathcal{G}^*_{f(t)\setminus f(s)}\right)\middle|\mathcal{H}_t\right]$$

which is (greater than) equal to zero if X is a strong (sub)martingale. □

Concerning the behaviour of path properties under flows, we have

Lemma 4.2.21 *Let X be an $\mathcal{A}$ indexed process and $f \in S(\mathcal{A})$.*

(a) *X (left-) right-continuous in the sense of Definition 3.2.44 implies $X \circ f$ is (left-) right-continuous on $[0,1]$.*

(b) *X a right-continuous strong submartingale implies $X \circ f$ is in $D[0,1]$.*

(c) *If X is right-continuous with $X_C \geq 0 \ \forall C \in \mathcal{C}$, then $X \circ f$ is an increasing process in $D[0,1]$.*

Proof (a) follows from Lemma 4.2.18 and Definition 3.2.44. For part (b), Lemma 4.2.20 (b) implies $X \circ f$ a submartingale whereas part (a) implies $X \circ f$ is right-continuous on $[0,1]$. Therefore, by Remark 4.2.12, $X \circ f$ is in $D[0,1]$.

Let Ω_0 be the event of full P-measure described in Definition 3.2.44 (b) and, for each $s < t$ in $[0,1]$, define the set

$$\Omega_{s,t} = \{\omega \in \Omega : X_{f(s)}(\omega) \leq X_{f(t)}(\omega)\}.$$

Since $X_C \geq 0 \ \forall C \in \mathcal{C}$, $\Omega' = \Omega_0 \cap (\bigcap_{s,t} \Omega_{s,t})$ is an event of full P-measure provided the intersection is taken over all pairs $s < t$ in $[0,1] \cap \mathbf{Q}$. By part (a) and the density of $\mathbf{Q}$, $t \mapsto X \circ f(\omega, t)$ is right-continuous and increasing on $[0,1]$ $\forall \omega \in \Omega'$, establishing part (c). □

The above lemmas will be exploited in Sections 4.4 and 4.5 where convergence results for sequences of set-indexed strong martingales will be obtained by studying associated sequences of classical martingales.

4.2.3 $*$-quadratic variation

Fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ to be used throughout this subsection. We begin with a definition.

Definition 4.2.22 *Given a strong martingale, $X = (X_A)_{A \in \mathcal{A}}$ in L_2, a process $\tilde{X} = (\tilde{X}_A)_{A \in \mathcal{A}}$ is said to be a <u>$*$-quadratic variation</u> of X provided*

(i) *$\tilde{X}$ is right-continuous in the sense of Definition 3.2.44* (b),

(ii) *$\tilde{X}_C \geq 0$ a.s. $\forall C \in \mathcal{C}$ and*

(iii) $E[(X_C)^2 | \mathcal{G}_C^*] = E[\tilde{X}_C | \mathcal{G}_C^*] \ \forall C \in \mathcal{C}$.

(Since $\mathcal{A} \subseteq \mathcal{C}$, (ii) *implies $\tilde{X}$ is non-negative.)*

Remark 4.2.23 For any $*$-quadratic variation, condition (ii) extends to all $C \in \mathcal{C}(u)$ by finite additivity whereas condition (iii) extends to all $C \in \mathcal{C}(u)$ by the argument in Proposition 3.5.9 (see Remark 3.5.10).

Similar processes have already appeared in Chapter 3. Specifically, when $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ satisfied Assumption 3.4.2, a process Q was said to be the *$*$-predictable quadratic variation* ($*$-PQV) of a strong L_2 martingale X if Q satisfied conditions (i), (ii) and (iii) of Definition 4.2.22 and Q was $*$-predictable in the sense of Definition 3.4.7. [3] Corollary 3.6.11 gave sufficient conditions on $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ and X under which such Q existed. We record these facts below.

Proposition 4.2.24 *Let X be a strong martingale in L_2. Under Assumption Group D.3, if Q is a $*$-PQV of X, then it is a $*$-quadratic variation of X. Therefore, under Assumption Groups D.1 and D.3, if $E[\,|X_T|^{2+\delta}] < \infty$ for some $\delta > 0$, then X possesses $*$-quadratic variation.*

If X is an L_2 strong martingale w.r.t. a filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$, note that a $*$-quadratic variation $\tilde{X}$ of X need not be adapted to $(\mathcal{F}_A)_{A\in\mathcal{A}}$. Furthermore, even if $\tilde{X}$ were adapted to $(\mathcal{F}_A)_{A\in\mathcal{A}}$, $X^2 - \tilde{X}$ need not be a strong martingale since $(X_C)^2$ is not necessarily equal to X^2_C and, as a result, we may have

$$E[(X^2 - \tilde{X})_C |\, \mathcal{G}^*_C] \neq 0, \quad (\text{some } C \in \mathcal{C})$$

Here, $(X_C)^2$ is the square of the increment of X at C while X^2_C denotes the increment of X^2 at C.

$*$-quadratic variation will play a key role in the strong martingale CLTs presented in Sections 4.4 and 4.5. In particular, given a sequence $(X_n)_n$ of well-behaved L_2 strong martingales with corresponding $*$-quadratic variations $(\tilde{X}_n)_n$, pointwise convergence in probability of $(\tilde{X}_n)_n$ to a continuous deterministic process Λ will imply convergence of $(X_n)_n$ (in some sense) to a Gaussian process. By not demanding adaptedness or $*$-predictability for $*$-quadratic variation, the range of applicability for the said CLTs is increased. This is demonstrated in Section 4.6 where $*$-quadratic variations are explicitly calculated (see Proposition 4.6.14) and later applied to obtain a CLT for set-indexed weighted empirical processes (see Theorem 4.6.22).

[3] However, unlike $*$-quadratic variation, a $*$-PQV does not necessarily possess a finitely additive extension to $\mathcal{C}(u)$. See Definition 3.5.6.

4.2.4 Set-indexed Gaussian processes

In this subsection, we define and study the class of set-indexed Gaussian processes. Sufficient conditions ensuring d_H-continuous versions for set-indexed Gaussian processes will be given. These processes, continuous or otherwise, will serve as the limits in the upcoming strong martingale CLTs. The reader is referred to Chapter 1 of [1] for additional material on the subject.

Due to the generality inherent in the set-indexed setting, there is no clear definition of a "standard" set-indexed Gaussian process. For this reason, all of our set-indexed Gaussian processes will be "stretched-out". The following class of set-functions plays the role of the continuous increasing functions in Definition 4.2.5.

Definition 4.2.25 *$\Lambda : \mathcal{A} \to [0, \infty)$ is said to be a variance function on $\mathcal{A}$ if $\Lambda(\phi') = 0$, $\Lambda(C) \geq 0$ $\forall C \in \mathcal{C}$ and Λ is left- and right-continuous on $\mathcal{A}$ in the sense of Definition 3.2.44.*

Variance functions behave nicely under simple flows. This is a consequence of Lemma 4.2.18 and the monotonicity of flows.

Lemma 4.2.26 *If Λ is a variance function on $\mathcal{A}$ and $f \in S(\mathcal{A})$, then $\Lambda \circ f : [0, 1] \to [0, \infty)$ is continuous and increasing on $[0, 1]$ with $\Lambda \circ f(0) = 0$.*

The following result is a special case of Corollary 2.2 in [16].

Proposition 4.2.27 *Any variance function Λ on $\mathcal{A}$ can be uniquely extended to a finite measure, also denoted Λ, on $\sigma(\mathcal{A})$.*

We now define set-indexed Gaussian processes.

Definition 4.2.28 *Given a variance function Λ on $\mathcal{A}$, a process $G = \{G_A : A \in \mathcal{A}\}$ is said to be an $\mathcal{A}$-indexed Gaussian process based on Λ if for any sets $A, B, B_1, \cdots, B_n \in \mathcal{A}$,*

(i) $G_A \sim N(0, \Lambda(A))$,

(ii) $\text{cov}(G_A, G_B) = \Lambda(A \cap B)$ *and*

(iii) $(G_{B_1}, \cdots, G_{B_n})$ *is distributed multivariate normal.*

Comment 4.2.29 If X is an $\mathcal{A}$-indexed process and X' is a version of X, then, even if X possesses a unique finitely additive extension to $\mathcal{C}(u)$, there is no guarantee that X' will possesses such an extension. For this reason, contrary to the convention on set-indexed processes adopted in this chapter, **general set-indexed Gaussian processes are not assumed to possess unique finitely additive extensions to $\mathcal{C}(u)$.** By not restricting ourselves in this way, we can be assured that any version of an $\mathcal{A}$-indexed Gaussian process is itself an $\mathcal{A}$-indexed Gaussian process.

By its very definition, the sample paths of a standard Brownian motion are continuous. To guarantee the existence of d_H-continuous versions of $\mathcal{A}$-indexed Gaussian processes, one must place additional conditions on $\mathcal{A}$. In this section, we present one such condition which we term the *canonical entropy condition* (see (4.17)). We will discuss a related condition, the *metric entropy condition* in Subsection 4.6.4.

To any variance function Λ on $\mathcal{A}$, we can associate a pseudo-metric d_Λ on $\mathcal{A}$ by letting

$$d_\Lambda(A,B) \;=\; [\Lambda(A\triangle B)]^{1/2}\,, \quad (\forall\, A,B \in \mathcal{A}) \tag{4.15}$$

where $\triangle$ denotes the symmetric difference of sets. [4] The next few results appear in some form in Chapter 1 of [1]. Our "proofs" will merely indicate the location of the required elements in [1]. Given any $\mathcal{A}$-indexed process X on $(\Omega, \mathcal{F}, P)$ and any metric γ on $\mathcal{A}$, the statement "X is γ-continuous" is to be read, "P-a.e. sample path of X is γ-continuous on $\mathcal{A}$".

Lemma 4.2.30 *Given an $\mathcal{A}$-indexed process X, define the covariance function,* $\mathrm{cov}_X : \mathcal{A}\times\mathcal{A} \to \mathbf{R}$ *by* $\mathrm{cov}_X(A,B) = E(X_A X_B)$ $(A, B \in \mathcal{A})$*. If* cov_X *is d_H-continuous on the diagonal,* $\{(A,A) : A \in \mathcal{A}\}$ *then* cov_X *is $d_H \times d_H$-continuous on all of $\mathcal{A}\times\mathcal{A}$.*

Proof This is precisely Exercise 1.1 on p.37 of [1]. □

Lemma 4.2.31 *Let G be a Gaussian process based on Λ. If Λ is d_H-continuous as a set-function on $\mathcal{A}$, then G is d_H-continuous if and only if G is d_Λ-continuous.*

[4] This is well-defined since Λ is extends to $\mathcal{C}(u)$ and $A\triangle B \in \mathcal{C}(u)$ for any $A, B \in \mathcal{A}$.

Proof By Definition 4.2.28 (ii), $\mathrm{cov}_G(A, A) = \Lambda(A)\ \forall A \in \mathcal{A}$. Therefore, by assuming Λ is d_H-continuous, Lemma 4.2.30 implies cov_G is $d_H \times d_H$-continuous on $\mathcal{A} \times \mathcal{A}$.

According to the second paragraph on p.3 of [1], the continuity of cov_G and the compactness of $(\mathcal{A}, d_H)$ (see Theorem 2.2.13) imply

$$G \text{ is } d_H\text{-continuous} \iff G \text{ is } d_0\text{-continuous}$$

where $d_0(A,B) = E[(G_A - G_B)^2]^{1/2}$ for all $A, B \in \mathcal{A}$; note that G is a process in L_2. But, as calculated on p.7 of [1],

$$d_0(A,B) = [\Lambda(A \triangle B)]^{1/2} = d_\Lambda(A,B)\,, \quad (\forall A, B \in \mathcal{A}). \tag{4.16}$$

Therefore, G is d_H-continuous if and only if G is d_Λ-continuous. □

Recall that a pseudo-metric space (E, τ) is *totally bounded* if for every $\epsilon > 0$, $\exists$ a finite subset $\{e_1, \cdots e_n\}$ of E s.t. any $e \in E$ is within τ-distance ϵ of some e_i. Any such $\{e_1, \cdots e_n\}$ is called a *(finite) ϵ-net* of (E, τ). For example, any compact metric space is totally bounded.

Theorem 4.2.32 *Let Λ be a d_H-continuous variance function on $\mathcal{A}$ for which $(\mathcal{A}, d_\Lambda)$ is totally bounded and let G be a Gaussian process based on Λ. For each $\epsilon > 0$, define the number*

$$N_{d_\Lambda}(\epsilon) = \min\{\,\#\mathcal{A}(\epsilon) : \mathcal{A}(\epsilon) \text{ an } \epsilon\text{-net for } (\mathcal{A}, d_\Lambda)\}$$

where # denotes cardinality. If

$$\lim_{\epsilon \to 0} \int_\epsilon^1 [\ln N_{d_\Lambda}(\epsilon)]^{1/2}\, d\epsilon < \infty, \tag{4.17}$$

then $\exists$ a d_H-continuous version, G' of G.

Proof Clearly, without assuming $(\mathcal{A}, d_\Lambda)$ is totally bounded, the function N_{d_Λ} would not be defined for all sufficiently small $\epsilon \in (0, 1]$. Under condition (4.17), Theorem 1.1 on p.4 of [1] and the identity in (4.16) imply the existence of a d_Λ-continuous version, G' of G. (Note that $\ln N_{d_\Lambda}(\epsilon) = 0\ \forall \epsilon > \mathrm{diam}_\Lambda(\mathcal{A})$.) Since Λ is assumed to be d_H-continuous on $\mathcal{A}$, Lemma 4.2.31 implies G' is d_H-continuous as well. □

Remark 4.2.33 In particular, condition (4.17) is satisfied whenever $\mathcal{A}$ is a *Vapnik-Červonenkis class*. See [1] for the definition and basic properties of these special subcollections of $\mathcal{P}(T)$.

We now define a special class of set-indexed Gaussian processes.

Definition 4.2.34 *Given a variance function Λ on $\mathcal{A}$, an $\mathcal{A}$-indexed Gaussian process W is said to be a* <u>*Gaussian white noise*</u> *based on Λ provided*

(i) *W has a unique finitely additive extension to a $\mathcal{C}(u)$-indexed process,*

(ii) $W_C \sim N(0, \Lambda(C))$ $\forall C \in \mathcal{C}(u)$,

(iii) $\text{cov}(W_A, W_B) = \Lambda(A \cap B)$ $\forall A, B \in \mathcal{A}$ *and*

(iv) *$C, D \in \mathcal{C}(u)$ disjoint implies W_C and W_D are independent.*

Remark 4.2.35 Given any variance function Λ on $\mathcal{A}$, the existence of a Gaussian white noise W based on Λ is guaranteed by Proposition 4.2.27 and the Kolmogorov extension theorem. See p.6 of [1] for details.

Since a Gaussian white noise possesses a unique finitely additive extension to $\mathcal{C}(u)$, it can be composed with any flow. When the flow is simple, such a composition is rather well-behaved.

Proposition 4.2.36 *Let W be an $\mathcal{A}$-indexed Gaussian white noise based on a variance function Λ on $\mathcal{A}$ and let $f \in S(\mathcal{A})$. If we define $W_f = W \circ f$ and $\lambda = \Lambda \circ f$, then*

(a) $W_f(0) = 0$,

(b) *for any $s < t$ in $[0,1]$, $W_f(t) - W_f(s) \sim N(0, \lambda(t) - \lambda(s))$,*

(c) *for any n and any $t_1 < \cdots < t_n$ in $[0,1]$, the random variables, $W_f(t_2) - W_f(t_1), \cdots, W_f(t_n) - W_f(t_{n-1})$ are mutually independent and*

(d) *W_f has a modification in $C[0,1]$.*

In other words, $W \circ f$ has a modification which is a standard Brownian motion stretched out by the continuous increasing function $\Lambda \circ f : [0,1] \to [0, \infty)$.

Proof Select $f \in S(\mathcal{A})$. In view of Definition 4.2.34, W_f satisfies properties (a), (b) and (c). (Note that $\Lambda \circ f(0) = \Lambda(\phi') = 0$.)

By Lemma 4.2.26, λ is continuous and increasing on $[0,1]$. Therefore, λ maps $[0,1]$ onto $[0,a]$ where $a = \Lambda(f(1)) \geq 0$. If $a = 0$, we are done. Otherwise, define $\mu : [0,a] \to [0,1]$ by letting

$$\mu(u) \; = \; \min\{s \in [0,1] : \lambda(s) = u\}.$$

(The continuity of λ allows us to write "min" in place of "inf".) By its very definition,

$$\lambda(\mu(u))) \; = \; u\,, \quad (\forall u \in [0,a]). \tag{4.18}$$

Now, define the process $W_{f\circ\mu} = \{W_f(\mu(u)) : u \in [0,a]\}$. We have the

Claim I: *$W_{f\circ\mu}$ has a continuous modification $Y = \{Y_t : t \in [0,a]\}$.*

Proof: By (b) and (4.18),

$$W_{f\circ\mu}(t) - W_{f\circ\mu}(s) \;\sim\; N(0, t-s)\,, \quad (\forall s < t \text{ in } [0,a]).$$

Therefore, the existence of the said modification follows by Kolmogorov's continuity criterion as presented in Theorem 1.8 on p.18 of [38]. Ω

Since λ is continuous on $[0,1]$, $Y_\lambda = \{Y(\lambda(t)) : t \in [0,1]\}$ is continuous. What is more,

Claim II: *Y_λ is a continuous modification of W_f.*

Proof: Take $t \in [0,1]$. By (a), $Y_\lambda(0) = 0 = W_f(0)$ a.e. If $t \in (0,1]$, there are two cases. First, if $\lambda(s) < \lambda(t)\ \ \forall s \in [0,t)$, then

$$\mu(\lambda(t)) \; = \; \min\{s : \lambda(s) = \lambda(t)\} \; = \; t$$

which, by Claim I, implies $Y_\lambda(t) \overset{a.e}{=} W_{f\circ\mu}(\lambda(t)) = W_f(t)$.

On the other hand, if $t \in (0,1]$ is s.t. $\lambda(s) = \lambda(t)$ for some $s < t$, define

$$t_0 \; = \; \min\{s \in [0,1] : \lambda(s) = \lambda(t)\}.$$

Then, $t_0 < t$, $\lambda(t_0) = \lambda(t)$ and by the definition of μ, $\mu(\lambda(t)) = t_0$. Therefore, Claim I implies $Y_\lambda(t) \overset{a.e}{=} W_{f\circ\mu}(\lambda(t)) = W_f(t_0)$. But by part (b),

$$\lambda(t_0) = \lambda(t) \implies W_f(t) - W_f(t_0) \sim N(0,0)$$

which is to say $W_f(t) \overset{a.e}{=} W_f(t_0)$. This completes the proof of Claim II and of Proposition 4.2.36 (d). □

4.3 Modes of Convergence for Set-Indexed Processes

4.3.1 Functional convergence

In this subsection, we define and give sufficient conditions for convergence in distribution (also termed *functional convergence*) for sequences of set-indexed processes. Our development is close to that found in Section 4 of [5]. Indeed, the arguments therein apply to any compact metric space (T, d) and any collection $\mathcal{E}$ of closed subsets of T which is closed and separable w.r.t. the Hausdorff metric. [5]

Recall the metric space $(D(\mathcal{A}), d_D)$ defined in Section 2.4 and let $\mathcal{D}$ denote the Borel σ-algebra on $D(\mathcal{A})$ generated by d_D. Applying Definition 4.2.1 to $(D(\mathcal{A}), d_D)$, we say that probability measures $(\mu_n)_n$ on $\mathcal{D}$ *converge weakly* to a probability measure μ on $\mathcal{D}$ whenever

$$\int f\, d\mu_n \;\to\; \int f\, d\mu$$

for every bounded d_D-continuous function $f : D(\mathcal{A}) \to \mathbf{R}$. In such a case, we write $\mu_n \Rightarrow \mu$.

Definition 4.3.1 *Let $\mathcal{M}$ be a collection of probability measures on $\mathcal{D}$.*

(a) *$\mathcal{M}$ is tight if for any $\epsilon > 0$, $\exists$ a d_D-compact subset K_ϵ of $D(\mathcal{A})$ s.t. $\mu(K_\epsilon) \geq 1 - \epsilon$ $\forall \mu \in \mathcal{M}$.*

(b) *$\mathcal{M}$ is relatively compact if for any sequence $(\mu_n)_n$ in $\mathcal{M}$, $\exists$ a subsequence $(\mu_{n'})_n$ and a probability measure μ on $\mathcal{D}$ s.t. $\mu_{n'} \Rightarrow \mu$.*

As mentioned at the end of Section 2.4, there are at present no known conditions on (T, d) and $\mathcal{A}$ which imply completeness and separability of $(D(\mathcal{A}), d_D)$. Since many results in the theory of weak convergence in metric spaces assume these two properties, care is needed in its application to $(D(\mathcal{A}), d_D)$. Fortunately, the direct half of Prohorov's theorem requires neither completeness nor separability. See Theorem 6.1 in [6].

Theorem 4.3.2 *Let $\mathcal{M}$ be a collection of probability measures on $\mathcal{D}$. If $\mathcal{M}$ is tight, then $\mathcal{M}$ is relatively compact.*

[5] By Theorem 2.2.10 and Remark 2.2.3, any indexing collection has these two properties.

Similarly, the following result, which combines Theorem 4.3.2 and Theorem 2.3 in [6], applies to any metric space, Polish or otherwise.

Theorem 4.3.3 *Let $\mu, \mu_1, \mu_2, \cdots$ be probability measures on $\mathcal{D}$ s.t. $(\mu_n)_n$ is tight. If every weakly converging subsequence of $(\mu_n)_n$ has μ as a weak limit, then $\mu_n \Rightarrow \mu$.*

Next, we examine the role of the *finite dimensional projections.* Given any $n \in \mathbf{N}$ and any $A_1, \cdots, A_n \in \mathcal{A}$, define $\pi_{A_1,\cdots,A_n} : D(\mathcal{A}) \to \mathbf{R}^n$ by

$$\pi_{A_1,\cdots,A_n}(x) = (x(A_1), \cdots, x(A_n)), \quad (\forall x \in D(\mathcal{A})). \tag{4.19}$$

Proposition 4.3.4 *Let $\mathcal{A}^*$ be any d_H-dense subset of $\mathcal{A}$ and define $\mathcal{D}^* = \sigma(\{\pi_A : A \in \mathcal{A}^*\})$, i.e., $\mathcal{D}^*$ is the smallest σ-algebra on $D(\mathcal{A})$ to which each 1-dimensional projection π_A $(A \in \mathcal{A}^*)$ is measurable. Then, $\mathcal{D} = \mathcal{D}^*$.*

Proof Recall the metric space $(\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$ defined on p.29 and the isometry $G : (D(\mathcal{A}), d_D) \to (\mathcal{K}_{\mathcal{A}\times\mathbf{R}}, d_G)$. If $\mathcal{B}$ denotes the Borel σ-algebra on $\mathcal{K}_{\mathcal{A}\times\mathbf{R}}$ generated by d_G, then by Proposition 4.1 in [5],

$$G^{-1}[\mathcal{B}] := \{G^{-1}[R] : R \in \mathcal{B}\} = \mathcal{D}^*.$$

Therefore, it is sufficient to show $\mathcal{D} = G^{-1}[\mathcal{B}]$.

Since G is continuous, G is $\mathcal{D}$-measurable and hence $G^{-1}[\mathcal{B}] \subseteq \mathcal{D}$. For the opposite inclusion, it is sufficient that $G^{-1}[\mathcal{B}]$ contain all d_D-open subsets U of $D(\mathcal{A})$. Take such a U. Since G is an isometry, $G[U]$ is open in the subspace $G[D(\mathcal{A})]$. Therefore, $\exists$ a d_G-open subset V of $\mathcal{K}_{\mathcal{A}\times\mathbf{R}}$ s.t. $G[U] = V \cap G[D(\mathcal{A})]$. Since G is injective, this implies $U = G^{-1}[V]$ where $V \in \mathcal{B}$, hence completing the proof. □

Proposition 4.3.5 *Given a d_H-dense subset $\mathcal{A}^*$ of $\mathcal{A}$, define the class*

$$\mathcal{D}_0^* = \{\pi^{-1}_{A_1,\cdots,A_n}[\Pi_{i=1}^n R_i] : n \geq 1,\ A_i \in \mathcal{A}^*,\ R_i \in \mathcal{B}(\mathbf{R})\}$$

where $\mathcal{B}(\mathbf{R})$ denotes the Borel σ-algebra on $\mathbf{R}$. Given any two probability measures μ_1 and μ_2 on $\mathcal{D}$, if

$$\mu_1(D) = \mu_2(D), \quad (\forall D \in \mathcal{D}_0^*), \tag{4.20}$$

then $\mu_1 = \mu_2$ on $\mathcal{D}$. That is, $\mathcal{D}_0^*$ is a probability determining subclass of $\mathcal{D}$.

Proof Given any such $\mathcal{A}^*$, $\mathcal{D}_0^*$ is clearly a π-class on $D(\mathcal{A})$. Therefore, by Proposition 4.3.4 and Theorem 3.3 in [7], (4.20) implies $\mu_1 = \mu_2$ on $\mathcal{D}$. □

Let X be an $\mathcal{A}$-indexed process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Given any $\omega \in \Omega$, let $\hat{X}(\omega)$ denote the sample path of X at ω. That is,

$$\hat{X}(\omega)(A) \;=\; X_A(\omega)\,, \quad (\forall\, A \in \mathcal{A}).$$

We say that X is in $D(\mathcal{A})$ whenever $\hat{X}(\omega) \in D(\mathcal{A})$ for P-a.e. ω, i.e., a.e. sample path of X lies in $D(\mathcal{A})$.

Proposition 4.3.6 *Any process $X = (X_A)_{A\in\mathcal{A}}$ in $D(\mathcal{A})$ is $\mathcal{D}$-measurable when viewed as a random function $\hat{X} : \Omega \to D(\mathcal{A})$.*

Proof Since X_A is measurable $\forall\, A \in \mathcal{A}$,

$$\hat{X}^{-1}\left[\pi_A^{-1}[B]\right] \;=\; (\pi_A \circ \hat{X})^{-1}[B] \;=\; [X_A \in B] \;\in\; \mathcal{F}$$

for any $A \in \mathcal{A}$ and any Borel set $B \in \mathcal{B}(\mathbf{R})$. Therefore, by Proposition 4.3.4, $\hat{X}^{-1}[\mathcal{D}] \subseteq \mathcal{F}$. □

By Proposition 4.3.6, any $\mathcal{A}$-indexed process X in $D(\mathcal{A})$ generates a probability measure $L(X)$ on $\mathcal{D}$ termed the *law* of X. It is defined by

$$L(X)(D) \;:=\; P(\hat{X} \in D)\,, \quad (\forall\, D \in \mathcal{D}). \tag{4.21}$$

This leads to the following terminology.

Definition 4.3.7 *Let $X, X_1, X_2, \cdots$ be processes in $D(\mathcal{A})$.*

(a) *$(X_n)_n$ is said to converge in finite dimensional distribution to X if $(X_n(A_1), \cdots, X_n(A_m)) \to (X(A_1), \cdots, X(A_m))$ in distribution (as random vectors) for any $m \in \mathbf{N}$ and $A_1, \cdots, A_m \in \mathcal{A}$.*

(b) *$(X_n)_n$ is said to converge in distribution (or functionally) to X, denoted $X_n \xrightarrow{\mathcal{L}} X$, provided $L(X_n) \Rightarrow L(X)$.*

Remark 4.3.8 The above definitions extend to any metric topology $\mathcal{T}$ on $D(\mathcal{A})$ although measurability of $\mathcal{A}$-indexed processes w.r.t. the Borel σ-algebra generated by $\mathcal{T}$ may no longer be trivial. As it is, unless otherwise mentioned, functional convergence of processes in $D(\mathcal{A})$ is understood to be w.r.t. the topology generated by d_D.

Our next result follows classic lines.

Theorem 4.3.9 *Let $X, X_1, X_2, \cdots$ be processes in $D(\mathcal{A})$. If*

(i) *$X_n \to X$ in finite dimensional distribution and*

(ii) *$(L(X_n))_n$ is tight,*

then $X_n \to_{\mathcal{L}} X$.

Proof Take any subsequence $(n')_n$. If $(L(X_{n'}))_n$ converges weakly to a probability measure μ' on $\mathcal{D}$, then condition (i) and Proposition 4.3.5 imply $\mu' = L(X)$ on $\mathcal{D}$. Therefore, by (ii) and Theorem 4.3.3, $L(X_n) \Rightarrow L(X)$. □

Combining Theorem 4.3.9 with Theorem 2.6.5, we obtain a generic functional limit theorem for processes in $D(\mathcal{A})$.

Theorem 4.3.10 *Given processes $X, X_1, X_2, \cdots$ in $D(\mathcal{A})$, if $X_n \to X$ in finite dimensional distribution and, for every $\epsilon > 0$, $\exists$ constants η_ϵ and R_ϵ, functions N_ϵ and h_ϵ and a d-compact subset T_ϵ of T — all as described in Definition 2.6.2 — such that*

$$P\left[X_n \in \Gamma(T_\epsilon, h_\epsilon, N_\epsilon, \eta_\epsilon, R_\epsilon)\right] \geq 1 - \epsilon, \quad (\forall n), \tag{4.22}$$

then $X_n \to_{\mathcal{L}} X$.

Remark 4.3.11 (a) For the case of $T = [0,1]^k$ ($k \geq 1$), specific examples of set-indexed processes satisfying (4.22) can be found in Theorem 4.4 of [5]. Note that any indexing collection on $[0,1]^k$ satisfies Assumption (A1) therein (see Remark 2.2.11).

(b) If we replace each $\Gamma(T_\epsilon, h_\epsilon, N_\epsilon, \eta_\epsilon, R_\epsilon)$ in (4.22) by a subset of the form Ξ_0 (the latter defined on p.47), then by Theorem 2.6.7 and Theorem 4.3.9, Theorem 4.3.10 remains valid.

We close this subsection with a classic sufficient condition for functional convergence for processes with sample paths in $C(\mathcal{A})$. For example, see Theorem 8.2 in [6].

Theorem 4.3.12 *Let $X, X_1, X_2, \cdots$ be $\mathcal{A}$-indexed processes in $C(\mathcal{A})$. If*

(i) *$X_n \to X$ in finite dimensional distribution,*

(ii) $\forall \eta > 0 \ \exists a > 0$ *s.t.* $P[\,\|X_n\|_{\mathcal{A}} > a] < \eta \ \forall n$ *and*

(iii) $\forall \eta > 0$ *and* $\epsilon > 0 \ \exists \delta > 0$ *s.t.* $P[w(X_n, \delta) \geq \epsilon] \ < \eta \ \forall n$

where $w(X_n, \delta) = \sup\{|X_n(A) - X_n(B)| : d_H(A, B) \leq \delta\,,\ A, B \in \mathcal{A}\}$, *then* $X_n \to_{\mathcal{L}} X$.

Proof Since $C(\mathcal{A}) \subseteq D(\mathcal{A})$, each X_n is in $D(\mathcal{A})$. For the sake of notation, write μ_n for $L(X_n)\ \forall n$. By Theorem 4.3.9 and assumption (i), it is sufficient to show $(\mu_n)_n$ is a tight family of measures on $\mathcal{D}$.

Select $\eta > 0$. By (ii), $\exists\, a > 0$ s.t.

$$\mu_n(\{x \in C(\mathcal{A}) : \|x\|_{\mathcal{A}} \leq a\}) \ \geq\ 1 - \eta/2\,,\ \ (\forall n) \tag{4.23}$$

whereas (iii) implies that for every $k \in \mathbf{N}$, $\exists\, \delta_k > 0$ s.t.

$$\mu_n(\{x \in C(\mathcal{A}) : w(x, \delta_k) < 1/k\}) \ \geq\ 1 - \eta/2^{k+1}\,,\ \ (\forall n). \tag{4.24}$$

If we define $\Sigma \subseteq C(\mathcal{A})$ by

$$\Sigma \ =\ \{x : \|x\|_{\mathcal{A}} \leq a\} \cap \textstyle\bigcap_k \{x : w(x, \delta_k) < 1/k\},$$

then Σ is uniformly bounded and uniformly equicontinuous. Therefore, by the Arzelà-Ascoli characterization of compactness in $C(\mathcal{A})$ (see p.279 in [32]) and Corollary 2.4.10, the $\|\cdot\|_{\mathcal{A}}$-closure, $\overline{\Sigma}$ of Σ is d_D-compact in $D(\mathcal{A})$. Furthermore, by (4.23) and (4.24),

$$\mu_n(\overline{\Sigma}) \ \geq\ 1 - \eta\,,\ \ (\forall n).$$

Since $\eta > 0$ was arbitrarily chosen, this implies $(\mu_n)_n$ is a tight family of probability measures on $\mathcal{D}$. □

4.3.2 Semi-functional convergence

The main goal of this chapter is to obtain CLTs — convergence of some type to Gaussian processes — for sequences of set-indexed strong martingales. Ideally, this convergence is functional convergence in $D(\mathcal{A})$ as defined in the preceding subsection. However, there are inherent difficulties and limitations associated to functional CLTs.

For example, Pyke in [36] obtained a functional CLT for set-indexed partial-sum processes by first smoothing the said processes and then establishing weak convergence w.r.t. the uniform topology on $C(\mathcal{A})$. Among other things, this required the limiting Gaussian process to be continuous. Switching to weak convergence w.r.t. d_D on $D(\mathcal{A})$ would remove the need for sample path continuity of the limiting process. However, one would still require the existence of $\mathcal{A}$-indexed Gaussian processes with sample paths in $D(\mathcal{A})$. When our indexing collection is as large as $\mathcal{LL}_2$, the lower layers on $[0,1]^2$, such versions may not exist.

For this reason, we define a new mode of convergence, termed *semi-functional convergence*, defined on a wide class of set-indexed process which includes all strong martingales and all Gaussian white noise (see Proposition 4.3.14). As will be illustrated in Example 4.6.26, it is possible for a sequence of partial sum-processes to converge semi-functionally to a discontinuous Gaussian process.

Definition 4.3.13 *The set $D[S(\mathcal{A})]$ consists of all $\mathcal{A}$-indexed processes X s.t. $X \circ f$ has a modification in $D[0,1]$ $\forall f \in S(\mathcal{A})$.*

$D[S(\mathcal{A})]$ contains many useful processes, some of which are listed below.

Proposition 4.3.14 *Let X be an $\mathcal{A}$-indexed process. If*

(i) *X has purely atomic sample paths,*

(ii) *X left- and right-continuous in the sense of Definition 3.2.44,*

(iii) *X a right-continuous strong submartingale,*

(iv) *X a strong martingale or*

(v) *X a Gaussian white noise,*

then $X \in D[S(\mathcal{A})]$.

Proof Fix a simple flow $f : [0,1] \to \mathcal{A}(u)$ for use in all parts of Proposition 4.3.14. Parts (ii) and (v) are due to Lemma 4.2.21 (a) and Proposition 4.2.36 (d) respectively.

For (i), note that if $x : \mathcal{A} \to \mathbf{R}$ is any purely-atomic set-function, then $x \circ f : [0,1] \to \mathbf{R}$ is pure-jump and as such, lies in $D[0,1]$.

Parts (iii) and (iv) follow by applying Lemma 4.2.20 (b) and Remark 4.2.12 (for (iii)) or Proposition 4.2.10 (for (iv)). □

Given any $f \in S(\mathcal{A})$, since cadlag modifications are unique up to indistinguishability, we can define a map M_f on $D[S(\mathcal{A})]$ by letting

$$M_f(X) := \text{the unique cadlag modification of } X \circ f. \tag{4.25}$$

And now for our new mode of convergence,

Definition 4.3.15 *Given processes $X, X_1, X_2, \cdots$ in $D[S(\mathcal{A})]$, $(X_n)_n$ is said to converge semi-functionally to X if*

$$M_f(X_n) \xrightarrow{\mathcal{L}} M_f(X) \text{ in } D[0,1], \ (\forall f \in S(\mathcal{A})). \tag{4.26}$$

Remark 4.3.16 Since the convergence in (4.26) is distributional, it is not necessary that the $\mathcal{A}$-indexed processes $X, X_1, X_2, \cdots$ be defined on a common probability space.

Under a basic continuity assumption, semi-functional convergence implies convergence in finite dimensional distribution.

Proposition 4.3.17 *Take $X, X_1, X_2, \cdots \in D[S(\mathcal{A})]$. If $X_n \to X$ semi-functionally and $M_f(X)$ is in $C[0,1]$ $\forall f \in S(\mathcal{A})$, then*

$$(X_n(C_0), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (X(C_0), \cdots, X(C_k)) \text{ as } n \to \infty \tag{4.27}$$

for any finite subcollection $\{C_0, \cdots, C_k\}$ of $C(u)$. Since $\mathcal{A} \subseteq C(u)$, this implies $X_n \to X$ in finite dimensional distribution.

Proof First, assume $C_0, \cdots, C_k$ are the left-neighborhoods generated by a f.s.s.l. $\mathcal{A}' = \{A_0, \cdots, A_k\}$ of $\mathcal{A}$ which is numbered in a manner c.w.s.p. Then, by Proposition 4.2.16, $\exists f \in S(\mathcal{A})$ s.t.

$$C_i = f(i/k) \setminus f((i-1)/k), \ (\forall 1 \leq i \leq k) \tag{4.28}$$

and $C_0 = \phi' = f(0)$.

By assumption, $M_f(X_n) \to_{\mathcal{L}} M_f(X)$ in $D[0,1]$ where $M_f(X)$ is a continuous modification of $X \circ f$. Among other things, this implies

$$(X^n_{f(0)}, X^n_{f(1/k)}, \cdots, X^n_{f(0)}) \xrightarrow{\mathcal{L}} (X_{f(0)}, X_{f(1/k)}, \cdots, X_{f(1)}). \tag{4.29}$$

(Here, $X_A^n = X_n(A)$, *not* exponentiation.) Define $h : \mathbf{R}^{k+1} \to \mathbf{R}^{k+1}$ by

$$h(x_0, \cdots, x_k) = (x_0, x_1 - x_0, \cdots, x_k - x_{k-1}), \quad (\forall (x_0, \cdots, x_k) \in \mathbf{R}^{k+1})$$

Since h is continuous on $\mathbf{R}^{k+1}$, (4.28), (4.29) and the continuous mapping theorem imply

$$(X_n(C_0), X_n(C_1), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (X(C_0), X(C_1), \cdots, X(C_k)).$$

So far, we have established (4.27) for the case in which $\{C_0, \cdots, C_k\}$ is the collection of left-neighborhoods generated by an arbitrary f.s.s.l. of $\mathcal{A}$. The proof for a general finite subcollection $\{C_0, \cdots, C_k\}$ of $\mathcal{C}(u)$ now follows from Proposition A.6.2. □

4.4 A Semi-Functional CLT for Continuous Strong Martingales

This section contains our first semi-functional limit theorem and some of its consequences. Specifically, conditions on a sequence of left- and right-continuous strong martingales [6] are given so as to ensure semi-functional convergence to a Gaussian white noise based on an appropriate variance function. Semi-functional convergence under weaker sample path conditions will be established in Theorem 4.5.5.

In what is to follow, we will frequently write $X(A)$ for X_A so as to simplify notation. Recall that $\mathcal{A}$ contains the sets ϕ' and T which serve as the left and right "endpoints" of $(\mathcal{A}, \subseteq)$ respectively.

Theorem 4.4.1 *Let $X_1, X_2, \cdots$ be a sequence of $\mathcal{A}$-indexed, left- and right-continuous L_2 strong martingales s.t. $X_n(\phi') \to 0$ in probability and*

$$\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty \tag{4.30}$$

for some $\delta > 0$. If $\exists$ a sequence of $\mathcal{A}$-indexed processes $(\Lambda_n)_n$ s.t.

(C1) *$\{\Lambda_n(T) : n \geq 1\}$ is uniformly integrable,*

(C2) *Λ_n is a $*$-quadratic variation of X_n $\forall n$ and*

[6] This is less restrictive than requiring the processes to be d_H-continuous.

(C3) *for every* $A \in \mathcal{A}$, $\Lambda_n(A) \to \Lambda(A)$ *in probability*

for some variance function Λ *on* $\mathcal{A}$, *then* $\exists$ *an* $\mathcal{A}$*-indexed Gaussian white noise* W *based on* Λ *s.t.* $X_n \to W$ *semi-functionally.*

Proof So as to simplify notation, **we assume each X_n is defined on a common stochastic base,** $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$. The argument which follows is directly applicable to the "non-homogeneous" case.

By Remark 4.2.35, $\exists$ an $\mathcal{A}$-indexed Gaussian white noise W based on Λ. To show that $(X_n)_n$ converges semi-functionally to W, select a simple flow $f \in S(\mathcal{A})$ — we need to show $M_f(X_n) \to_{\mathcal{L}} M_f(W)$ in $D[0,1]$. Note by Proposition 4.3.14 that $W, X_1, X_2, \cdots$ are all elements of $D[S(\mathcal{A})]$ as required in the definition of semi-functional convergence. In fact, by left- and right-continuity, Lemma 4.2.21 (a) implies

$$M_f(X_n) = X_n \circ f\,, \quad (\forall n). \tag{4.31}$$

For the sake of notation, write

$$Y_n = M_f(X_n)\,, \quad (\forall n).$$

Also, define classical processes $(\lambda_n)_n$ on $[0,1]$ by letting

$$\lambda_n = \Lambda_n \circ f\,, \quad (\forall n)$$

and define the function, $\lambda : [0,1] \to [0,\infty)$ by letting $\lambda = \Lambda \circ f$. Since $f(0) = \phi'$ by the definition of a simple flow, Lemmas 4.2.20 and 4.2.21 imply:

(a1) $Y_n(0) \xrightarrow{P} 0$ as $n \to \infty$,

(a2) for each n, Y_n is a continuous classical L_2 martingale w.r.t. the filtration $\{\mathcal{H}_t : t \in [0,1]\}$ defined by $\mathcal{H}_t = \mathcal{F}_{f(t)}$ $\forall\, t \in [0,1]$,

(a3) for each n, λ_n is a non-negative, increasing, integrable (but not necessarily adapted) process in $D[0,1]$ and

(a4) λ is increasing and continuous on $[0,1]$ with $\lambda(0) = 0$.

Furthermore, since Λ and each Λ_n possess finitely additive extensions to the algebra $\mathcal{C}(u)$, condition (C3) extends to all sets $B \in \mathcal{A}(u)$. Therefore,

$$\lambda_n(t) \xrightarrow{P} \lambda(t)\,, \quad (\forall\, t \in [0,1]). \tag{4.32}$$

To simplify our application of Theorem 4.2.8, we trivially extend the filtration $(\mathcal{H}_t)_{t\in[0,1]}$ and each process Y_n ($n \in \mathrm{N}$) to $[0,\infty)$ by defining

$$\mathcal{H}'_t = \mathcal{H}_1 \quad \text{and} \quad Y'_n(t) = Y_n(1)\,, \quad (\forall\, t > 1) \tag{4.33}$$

and letting $\mathcal{H}'_t = \mathcal{H}_t$, $Y'_n(t) = Y_n(t)$ $\forall\, t \in [0,1]$. Combining (a2), (4.33) and Theorem 4.2.7, we obtain

(a5) for each n, Y'_n is a continuous L_2 martingale w.r.t. $\{\mathcal{H}'_t : t \in [0,\infty)\}$ with continuous predictable quadratic variation $\langle Y'_n \rangle$.

Just like Y'_n, $\langle Y'_n \rangle$ is trivial beyond $t = 1$. To be precise,

<u>Claim 1:</u> For each n, $\langle Y'_n \rangle(t) = \langle Y'_n \rangle(1)$ a.s. $\forall\, t > 1$.

<u>Proof:</u> Fix n and $t > 1$. By (4.33) and (P3) in Theorem 4.2.7,

$$E[\,\langle Y'_n \rangle(t) - \langle Y'_n \rangle(1) \,|\, \mathcal{H}'_1] \;=\; E[\,Y'_n(t)^2 - Y'_n(1)^2 \,|\, \mathcal{H}'_1] \;=\; 0.$$

Since $\mathcal{H}'_t = \mathcal{H}'_1$ and $\langle Y'_n \rangle$ is adapted to $(\mathcal{H}'_t)_{t\in[0,\infty)}$, $\langle Y'_n \rangle(t) - \langle Y'_n \rangle(1)$ is $\mathcal{H}'_1$-measurable. Therefore, $\langle Y'_n \rangle(t) - \langle Y'_n \rangle(1) = 0$ a.e. Ω

Now, assume we have shown

$$\langle Y'_n \rangle(t) \xrightarrow{P} \lambda(t)\,, \quad (\forall\, t \in [0,1]). \tag{4.34}$$

If λ' denotes the trivial extension of λ to $[0,\infty)$, then by Claim 1, $\langle Y'_n \rangle(t) \to_P \lambda'(t)$ $\forall\, t$. Furthermore, property (a1) implies $Y'_n(0) \to_P 0$ while (a5) and (4.3) imply $E[J_a(Y'_n|_a)^2] = 0$ $\forall\, n$ and $\forall\, a > 0$. Therefore, by Theorem 4.2.8, $\exists$ a standard Brownian motion B s.t.

$$M_f(X_n) \;=\; Y'_n|_1 \xrightarrow{\mathcal{L}} B_\lambda \;\text{ in }\; D[0,1] \tag{4.35}$$

where $B_\lambda(t) = B(\lambda(t))$ $\forall\, t \in [0,1]$ (see (4.6) for the definition of $|_1$.) But by Proposition 4.2.36, $M_f(W)$ is also a Brownian motion stretched-out by λ. Since weak convergence of processes in $D[0,1]$ is equivalent to tightness plus convergence of the finite dimensional distributions, we can thus replace B_λ by $M_f(W)$ in (4.35). Therefore, since our goal is semi-functional convergence of $(X_n)_n$ to W, Reduction: *The proof of Theorem 4.4.1 will be complete if we can show* (4.34).

(4.34) will be obtained in three steps. First, it will be shown that $(\langle Y_n' \rangle|_1)_n$ is tight in $D[0,1]$ w.r.t. the Skorokhod topology. It will then be argued that any weakly convergent subsequence of $(\langle Y_n' \rangle|_1)_n$ must have λ, viewed as a deterministic process, as its weak limit. At this point, we will have $\langle Y_n' \rangle|_1 \to_{\mathcal{L}} \lambda$ in $D[0,1]$ which, by Lemma A.5.8, will yield (4.34). We begin with a technical result.

Claim 2: *Given any* $n \in \mathrm{N}$ *and any* $s < t$ *in* $[0,1]$,
(a) $E[Y_n^2(t) - \lambda_n(t) \,|\, \mathcal{H}_s] = E[Y_n^2(s) - \lambda_n(s) \,|\, \mathcal{H}_s]$ *and*
(b) $E[\langle Y_n' \rangle(t) - \lambda_n(t) \,|\, \mathcal{H}_s'] = E[\langle Y_n' \rangle(s) - \lambda_n(s) \,|\, \mathcal{H}_s']$.

Proof: Fix $n \in \mathrm{N}$ and $s < t$ in $[0,1]$. If we let $U = Y_n$ and $V = \lambda_n$, then

$$\begin{aligned} E[(U_t^2 - U_s^2) - (V_t - V_s) \,|\, \mathcal{H}_s] &= E[(U_t - U_s)^2 - (V_t - V_s) \,|\, \mathcal{H}_s] + \\ &\quad + 2U_s \cdot E[U_t - U_s \,|\, \mathcal{H}_s] \\ &= E[(U_t - U_s)^2 - (V_t - V_s) \,|\, \mathcal{H}_s] \end{aligned}$$

where the last equality is due to (a2). Furthermore, since Λ_n is a $*$-quadratic variation of X_n,

$$\begin{aligned} &E[(U_t - U_s)^2 - (V_t - V_s) \,|\, \mathcal{H}_s] \\ &\qquad = E[X_n(f(t)\backslash f(s))^2 - \Lambda_n(f(t)\backslash f(s)) \,|\, \mathcal{H}_s] \\ &\qquad = 0 \end{aligned}$$

when we apply the tower property to the sub-σ-algebras $\mathcal{H}_s \subseteq \mathcal{G}^*_{f(t)\backslash f(s)}$ (see (4.11)). This completes part (a). Part (b) follows from part (a) and (P3) in Theorem 4.2.7. Ω

Claim 3: $\{\langle Y_n' \rangle|_1 : n \geq 1\}$ *is tight in* $D[0,1]$ *w.r.t. the Skorokhod topology. (See* (4.6) *for the definition of* $|_1$.*)*

Proof: By Theorem 1 in [2], this will follow if

(i) $\{\langle Y_n' \rangle(t) : n \geq 1\}$ is tight for every fixed $t \in [0,1]$ and

(ii) $\{\langle Y_n' \rangle|_1 : n \geq 1\}$ satisfies condition (A) of Aldous (see (4.36) below).

To establish (i), fix $t \in [0,1]$. Given any $K > 0$, since each $\langle Y_n' \rangle$ is a non-negative increasing process, Markov's inequality implies

$$\begin{aligned} P(\,|\langle Y_n'\rangle(t)| > K) &\leq K^{-1} \cdot E[\,\langle Y_n'\rangle(1)\,] \\ &= K^{-1} \cdot E[\,Y_n'(1)^2\,] \quad \text{(by Theorem 4.2.7)} \\ &= K^{-1} \cdot E[\,X_n(T)^2\,] \quad \text{(since } f(1) = T\text{)}. \end{aligned}$$

But by (4.30), $\sup_n E[\,X_n(T)^2\,] < \infty$. Therefore, given any $\epsilon > 0$, we can choose $K > 0$ large enough so that $P(\,|\langle Y_n'\rangle(t)| > K) < \epsilon \;\forall n$. In other words, $\{\langle Y_n'\rangle(t) : n \geq 1\}$ is a tight collection of random variables.

To establish condition (A) of Aldous, select

- for each n, a stopping time $\tau_n : \Omega \to [0,1]$ w.r.t. the natural filtration generated by $\langle Y_n'\rangle|_1$ and
- a sequence of constants, $(\delta_n)_n$ s.t. $0 \leq \delta_n \leq 1 \;\forall n$ with $\delta_n \to 0$.

We need to show

$$\langle Y_n'\rangle(\sigma_n) - \langle Y_n'\rangle(\tau_n) \xrightarrow{P} 0 \quad \text{as } n \to \infty \tag{4.36}$$

where $\sigma_n = 1 \wedge (\tau_n + \delta_n) \;\forall n$. (Note that $\langle Y_n'\rangle = \langle Y_n'\rangle|_1$ on $\Omega \times [0,1]$.)

For this purpose, select $\epsilon > 0$. Given any n, $\langle Y_n'\rangle$ is increasing and $\tau_n \leq \sigma_n$ on Ω. Therefore, Markov's inequality implies

$$P(\,|\langle Y_n'\rangle(\sigma_n) - \langle Y_n'\rangle(\tau_n)| > \epsilon) \;\leq\; \epsilon^{-1} \cdot E[\langle Y_n'\rangle(\sigma_n) - \langle Y_n'\rangle(\tau_n)]\,, \quad (\forall n).$$

Since each $\langle Y_n'\rangle$ is adapted to $(\mathcal{H}_t')_{t\in[0,\infty)}$, τ_n and σ_n are stopping times w.r.t. $(\mathcal{H}_t')_{t\in[0,\infty)}$ $\forall n$. If we define $M = \{M(t) : t \in [0,\infty]\}$ by letting

$$M(t) \;=\; \begin{cases} \langle Y_n'\rangle(t) - \lambda_n(t) & \text{, if t} \in [0,1] \\ \langle Y_n'\rangle(1) - \lambda_n(1) & \text{, if t} \in (1,\infty]\,, \end{cases}$$

then Claim 2 (b) and the "generalized stopping theorem" in Lemma A.5.1 imply

$$E[\langle Y_n'\rangle(\sigma_n) - \langle Y_n'\rangle(\tau_n)\,|\,\mathcal{H}_{\tau_n}'] \;=\; E[\lambda_n(\sigma_n) - \lambda_n(\tau_n)\,|\,\mathcal{H}_{\tau_n}']\,, \quad (\forall n).$$

Therefore, (4.36) will follow if we can show

$$E[\lambda_n(\sigma_n) - \lambda_n(\tau_n)] \to 0 \quad \text{as } n \to \infty. \tag{4.37}$$

Given any n, the triangle inequality implies

$$\begin{aligned}
&|\lambda_n(\sigma_n) - \lambda_n(\tau_n)| \\
&\quad \leq\ |\lambda_n(\sigma_n) - \lambda(\sigma_n)| + |\lambda_n(\tau_n) - \lambda(\tau_n)| + |\lambda(\sigma_n) - \lambda(\tau_n)| \\
&\quad \leq\ 2 \cdot \sup_{0 \leq t \leq 1} |\lambda_n(t) - \lambda(t)| + |\lambda(\sigma_n) - \lambda(\tau_n)|.
\end{aligned}$$

In view of (4.32), Lemma A.5.8 implies $\sup_{0\leq t\leq 1} |\lambda_n(t) - \lambda(t)| \rightarrow_P 0$. Also, since $P(|\sigma_n - \tau_n| > \epsilon) = 0\ \forall n$ s.t. $\delta_n < \epsilon$, the continuity of λ on $[0,1]$ implies $|\lambda(\sigma_n) - \lambda(\tau_n)| \rightarrow_P 0$. In total,

$$|\lambda_n(\sigma_n) - \lambda_n(\tau_n)| \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore, since each λ_n is non-negative and increasing,

$$|\lambda_n(\sigma_n) - \lambda_n(\tau_n)| \ \leq\ 2 \cdot \lambda_n(1) \ =\ 2 \cdot \Lambda_n(T)\,, \quad (\forall n)$$

which, by assumption (C1), implies $\{\lambda_n(\sigma_n) - \lambda_n(\tau_n) : n \geq 1\}$ is uniformly integrable. By dominated convergence this yields (4.37), establishing condition (A) of Aldous for $\{\langle Y'_n\rangle|_1 : n \geq 1\}$. Therefore, $\{\langle Y'_n\rangle|_1 : n \geq 1\}$ is tight w.r.t. the Skorokhod topology on $D[0,1]$. Ω

Next, assume $\langle Y'_{r_n}\rangle|_1 \rightarrow_{\mathcal{L}} Z$ in $D[0,1]$ for some subsequence $(r_n)_n$ and some random element $Z : (\Omega', \mathcal{F}', P') \rightarrow D[0,1]$ where $(\Omega', \mathcal{F}', P')$ is taken to be complete. Since each $\langle Y'_n\rangle$ is continuous, Claim 3 and Corollary 4.2.4 imply Z is in $C[0,1]$. As mentioned prior to Claim 2, we need to show that Z and λ are equal when viewed as classical processes.

Toward this goal, let $M = Z - \lambda$ and define a family $(\mathcal{G}^\circ_t)_{t\in[0,1]}$ of sub-σ-algebras of $\mathcal{F}'$ by letting

$$\mathcal{G}^\circ_t \ =\ \sigma(\{M(u) : 0 \leq u \leq t\})\,, \quad (\forall t \in [0,1]).$$

$(\mathcal{G}^\circ_t)_{t\in[0,1]}$ is increasing but not necessarily complete or right-continuous. To remedy this shortcoming, define

$$\mathcal{G}_t \ =\ \mathcal{G}^\circ_t \vee \mathcal{F}'_0\,, \quad (\forall t \in [0,1])$$

where $\mathcal{F}'_0$ denotes the sub-σ-algebra of $\mathcal{F}'$ generated by all P'-null subsets of Ω' and define

$$\mathcal{G}^+_t \ =\ \bigcap_{s>t} \mathcal{G}_s\,, \quad (\forall t \in [0,1))$$

with $\mathcal{G}_1^+ = \mathcal{G}_1$. $(\mathcal{G}_t^+)_{t\in[0,1]}$ is commonly referred to as the *minimal filtration* generated by M. M is clearly adapted to $(\mathcal{G}_t^+)_{t\in[0,1]}$. Furthermore,

Claim 4: *M is a martingale w.r.t. its minimal filtration.*

Proof: In what is to follow, we will replace r_n by n so as to simplify notation. We begin by showing

$$E[M_t|\,\mathcal{G}_s] = M_s\,, \quad (\forall\, s < t \text{ in } [0,1]) \tag{4.38}$$

which, by the definition of conditional expectation, is equivalent to showing $E[M_t|\,\mathcal{G}_s^\circ] = M_s \;\; \forall\, s < t$ in $[0,1]$.

Since $\lambda(u) \in [0,\infty)$ and $M(u) = Z(u) - \lambda(u) \;\; \forall\, u \in [0,1]$,

$$\mathcal{G}_t^\circ \;=\; \sigma(\{Z(u) : 0 \le u \le t\})\,, \quad (\forall\, t \in [0,1]).$$

Furthermore, since the set Γ_u of P'-continuity points of $Z(u)$ is dense in $\mathbf{R}$ $\forall\, u$, it is clear that for any $s \in [0,1]$,

$$\mathcal{E}_s \;=\; \{\cap_{i=1}^m [Z(s_i) \le x_i] : m \in \mathbf{N},\; 0 \le s_i \le s \;\forall\, i \text{ and } x_i \in \Gamma_{s_i} \forall\, i\}$$

is a π-class generating $\mathcal{G}_s^\circ$. Therefore, given any $s < t$ in $[0,1]$, if we can show

$$\int_A (M_t - M_s)\, dP' \;=\; 0\,, \quad (\forall\, A \in \mathcal{E}_s), \tag{4.39}$$

then by Theorem 34.1 in [7] we will obtain $E[M_t|\,\mathcal{G}_s] = E[M_t|\,\mathcal{G}_s^\circ] = M_s$.

To this end, fix $s < t$ in $[0,1]$ and select a set $A = \bigcap_{i=1}^m [Z(s_i) \le x_i]$ in $\mathcal{E}_s$. We will establish (4.39) via an argument which is similar to that found in the proof of Theorem 3.1 in [21]. For each n, define the set

$$A_n \;=\; \cap_{i=1}^m \,[\,\langle Y_n' \rangle (s_i) \le x_i]$$

and the process

$$M_n \;=\; \langle Y_n' \rangle - \lambda_n.$$

To save space, given any process or function U, we will write $U(s,t]$ for $U(t) - U(s)$, the *increment* of U over $(s,t]$. To obtain (4.39), we need the following distributional limits.

Subclaim: *As $n \to \infty$,*

(a) $\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} \to_{\mathcal{L}} Z(s,t] \cdot \mathbf{1}_A$ *and*
(b) $\lambda_n(s,t] \cdot \mathbf{1}_{A_n} \to_{\mathcal{L}} \lambda(s,t] \cdot \mathbf{1}_A$.

Proof: Since we assume $\langle Y_n' \rangle|_1 \to_{\mathcal{L}} Z$ where Z is continuous,

$$(\langle Y_n' \rangle(u_1), \cdots, \langle Y_n' \rangle(u_r)) \xrightarrow{\mathcal{L}} (Z(u_1), \cdots, Z(u_r)) \tag{4.40}$$

for all choices of $r \in \mathrm{N}$ and $u_1, \cdots, u_r \in [0,1]$. Therefore, by a simple application of the continuous mapping theorem,

$$(\langle Y_n' \rangle(s,t],\ \langle Y_n' \rangle(s_1), \cdots, \langle Y_n' \rangle(s_m)) \xrightarrow{\mathcal{L}} (Z(s,t],\ Z(s_1), \cdots, Z(s_m)). \tag{4.41}$$

To show part (a), select a P'-continuity point, x of $Z(s,t] \cdot \mathbf{1}_A$. We consider two cases. First, assume $x < 0$. Since each $\langle Y_n' \rangle$ is increasing, Z is also increasing (see Lemma A.5.7). Therefore,

$$P(\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} \le x) \ = \ P'(Z(s,t] \cdot \mathbf{1}_A \le x) \ = \ 0\,, \ \ (\forall n).$$

On the other hand, if we assume $x \ge 0$, then

$$\begin{aligned} P'(Z(s,t] \cdot \mathbf{1}_A > x) \ &= \ P'([Z(s,t] > x] \cap A) \quad \text{and} \\ P(\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} > x) \ &= \ P\left([\langle Y_n' \rangle(s,t] > x] \cap A_n\right)\,, \ \ (\forall n). \end{aligned}$$

Therefore, by (4.41) and Lemma A.6.3,

$$\begin{aligned} &P(\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} > x) \\ &\qquad = \ P([\langle Y_n' \rangle(s,t] > x] \cap A_n) \\ &\qquad = \ P(\langle Y_n' \rangle(s,t] > x,\ \langle Y_n' \rangle(s_1) \le x_1,\ \cdots,\ \langle Y_n' \rangle(s_m) \le x_m) \\ &\qquad \xrightarrow{n} \ P'(Z(s,t] > x,\ Z(s_1) \le x_1,\ \cdots,\ Z(s_m) \le x_m) \\ &\qquad = \ P'([Z(s,t] > x] \cap A) \\ &\qquad = \ P'(Z(s,t] \cdot \mathbf{1}_A > x). \end{aligned}$$

In total, we have shown

$$P(\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} \le x) \ \to \ P'(Z(s,t] \cdot \mathbf{1}_A \le x) \ \text{ as } n \to \infty$$

for our arbitrary continuity point, x of $Z(s,t] \cdot \mathbf{1}_A$.

For part (b), note that (4.40) implies $\mathbf{1}_{A_n} \to_{\mathcal{L}} \mathbf{1}_A$ while (4.32) implies $\lambda_n(s,t] \to_P \lambda(s,t]$ where $\lambda(s,t] \in \mathrm{R}$. ω

We now return to the proof of Claim 4, in particular, the verification of (4.39) for our chosen set $A \in \mathcal{E}_s$. By Proposition A.5.5, (4.30) implies $\{\langle Y_n' \rangle(1) : n \geq 1\}$ is uniformly integrable. Furthermore, since each $\langle Y_n' \rangle$ is increasing,

$$|\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n}| \leq 2 \cdot \langle Y_n' \rangle(1), \quad (\forall n)$$

and hence,

$$\{\langle Y_n' \rangle(s,t] \cdot \mathbf{1}_{A_n} : n \geq 1\} \text{ uniformly integrable.}$$

Likewise, since each λ_n is increasing, assumption (C1) implies

$$\{\lambda_n(s,t] \cdot \mathbf{1}_{A_n} : n \geq 1\} \text{ uniformly integrable.}$$

Therefore, by the above Subclaim and dominated convergence (the form given in Theorem 25.12 of [7]),

$$\begin{aligned} \int_A M(s,t]\, dP' &= \int_A Z(s,t]\, dP' - \int_A \lambda(s,t]\, dP' \\ &= \lim_n \int_{A_n} \langle Y_n' \rangle(s,t]\, dP - \lim_n \int_{A_n} \lambda_n(s,t]\, dP \\ &= \lim_n \int_{A_n} M_n(s,t]\, dP. \end{aligned}$$

But $\langle Y_n' \rangle$ is adapted to $(\mathcal{H}_t')_{t\in[0,\infty)}$ and hence, $A_n \in \mathcal{H}_s'\ \forall n$. By Claim 2 (b), this implies $\int_{A_n} M_n(s,t]\, dP = 0\ \forall n$, establishing (4.39) and thus (4.38) also.

To complete the proof of Claim 4, take any $s < t$ and choose $N \in \mathbb{N}$ large enough so that $s < s + \frac{1}{n} < t\ \forall n \geq N$. Then, for every $n \geq N$,

$$\begin{aligned} E[M_t | \mathcal{G}_s^+] &= E[E(M_t | \mathcal{G}_{s+1/n}) | \mathcal{G}_s^+] \quad \text{(tower propery)} \\ &= E[M_{s+1/n} | \mathcal{G}_s^+] \quad \text{(by (4.38))} \\ &\xrightarrow{a.s.} M_s. \end{aligned}$$

The last line is due to dominated convergence since $\lim_n M_{s+1/n} = M_s$ a.s. and $|M_{s+1/n}| \leq Z(1) + \lambda(1)\ \forall n \geq N$. (Recall that $M = Z - \lambda$ with Z and λ both increasing and continuous on $[0,1]$.) Ω

Claim 5: *Z is indistinguishable from the deterministic process λ.*

Proof: Being the weak limit of increasing processes in $D[0,1]$, Z is itself an increasing process (see Lemma A.5.7). Since λ is also increasing and

$M = Z - \lambda$, Claim 4 and the continuity of Z and λ imply M is a continuous martingale of finite variation. Therefore, by Proposition 1.2 on p.114 of [38], M is a constant process, i.e., M is indistinguishable from a constant, say $\alpha \in \mathbf{R}$.

To find the value of α, note that (P3) in Theorem 4.2.7 implies $\langle Y_n' \rangle(0) = Y_n'^2(0) = Y_n^2(0)$ a.s. $\forall n$ so that (a1) yields

$$\langle Y_n' \rangle(0) \xrightarrow{P} 0.$$

Furthermore, since $\langle Y_{r_n}' \rangle|_1 \to Z$ in $D[0,1]$ and Z is continuous,

$$\langle Y_n' \rangle(0) \xrightarrow{\mathcal{L}} Z(0).$$

Combining these two limits, we obtain $Z(0) = 0$ a.s. Since $\lambda(0) = 0$, this implies $M(0) = Z(0) - \lambda(0) = 0$ a.s. Therefore, $\alpha = 0$ which is to say that Z and λ are indistinguishable. Ω

Combining Claims 3 and 5, $(\langle Y_n \rangle|_1)_n$ is a tight sequence of processes in $D[0,1]$, every weakly convergent subsequence of which converges weakly to λ. Therefore, by Theorem 2.3 in [6], $\langle Y_n' \rangle|_1 \to_{\mathcal{L}} \lambda$ in $D[0,1]$. By applying Lemma A.5.8 to this distributional limit, we obtain (4.34) and thus, in view of the Reduction on p.167, the proof of Theorem 4.4.1 is complete. □

4.4.1 Some consequences, including a functional CLT

In this complementary subsection, we list several quick consequences of the above semi-functional CLT. Included in this list is a related functional CLT for d_H-continuous strong martingales indexed by a small class $\mathcal{A}$.

To begin with, note that Theorem 4.4.1 required no additional assumptions on the stochastic bases. In particular, the processes $\Lambda_1, \Lambda_2, \cdots$ were not required to be $*$-predictable in the sense of Definition 3.4.7, eliminating the need for Assumption 3.4.2 among others. On the other hand, by imposing certain conditions on the said stochastic bases, Theorem 4.4.1 admits a simplification.

Proposition 4.4.2 *For each n, let X_n be an $\mathcal{A}$-indexed left- and right-continuous L_2 strong martingale defined on a stochastic base satisfying Assumption Groups D.1 and D.3. If*

(i) $\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty$ *for some* $\delta > 0$,

(ii) $X_n(\phi') \to 0$ *in probability*

and $\exists$ *a variance function* Λ *on* $\mathcal{A}$ *s.t.*

$$Q_n(A) \to \Lambda(A) \quad \textit{in probability}, \quad (\forall A \in \mathcal{A})$$

where Q_n *denotes the unique* $*$*-PQV of* X_n *(see Theorem 3.5.2), then* $\exists$ *an* $\mathcal{A}$*-indexed Gaussian white noise* W *based on* Λ *s.t.* $X_n \to W$ *semi-functionally.*

Proof For the sake of notation, assume each X_n is defined w.r.t. a common stochastic base, $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$ satisfying Assumption Groups D.1 and D.3.

By Corollary 3.6.11, condition (i) guarantees the existence of $*$-predictable quadratic variation ($*$-PQV) Q_n for each strong martingale X_n. As stated in Proposition 4.2.24, any $*$-PQV is a $*$-quadratic variation.

Furthermore, by Proposition 3.6.12 and (i), the sequence $(Q_n)_n$ satisfies condition (C1) in Theorem 4.4.1. Proposition 4.4.2 now follows from Theorem 4.4.1. □

If W is any Gaussian white noise, then by Proposition 4.2.36, $M_f(W)$ is in $C[0,1]$ $\forall f \in S(\mathcal{A})$. Thus, by Proposition 4.3.17, Theorem 4.4.1 implies

Corollary 4.4.3 *If* $X_1, X_2, \cdots$ *and* $\Lambda, \Lambda_1, \Lambda_2, \cdots$ *are as described in Theorem 4.4.1, then* $\exists$ *an* $\mathcal{A}$*-indexed Gaussian white noise* W *with variance function* Λ *s.t.*

$$(X_n(C_0), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (W(C_0), \cdots, W(C_k)) \quad \textit{as } n \to \infty \tag{4.42}$$

for any finite subcollection $\{C_0, \cdots, C_k\}$ *of* $\mathcal{C}(u)$*. In particular,* $X_n \to W$ *in finite dimensional distribution.*

We close with a functional CLT for d_H-continuous strong martingales. To simplify notation, we assume all processes are defined on a common probability space $(\Omega, \mathcal{F}, P)$ — the result clearly carries over to the heterogeneous case.

Theorem 4.4.4 *Assume* $\mathcal{A}$ *satisfies condition* (4.17)*. Let* $X_1, X_2, \cdots$ *be a sequence of* $\mathcal{A}$*-indexed,* d_H*-continuous strong martingales in* L_2 *s.t.*

(i) $X_n(\phi') \xrightarrow{P} 0$,

(ii) $\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty$ *for some* $\delta > 0$,

(iii) $\forall \eta > 0 \ \exists a > 0$ *s.t.* $P[\,\|X_n\|_{\mathcal{A}} > a] \leq \eta \ \forall n$ *and*

(iv) $\forall \eta, \epsilon > 0 \ \exists \delta > 0$ *s.t.* $P[w(X_n, \delta) \geq \epsilon] \leq \eta \ \forall n$

where $w(X_n, \delta) = \sup\{|X_n(A) - X_n(B)| : d_H(A, B) \leq \delta\,,\ A, B \in \mathcal{A}\}$. *If* $\exists$ *a sequence of* $\mathcal{A}$*-indexed processes* $(\Lambda_n)_n$ *s.t.*

(i') $\{\Lambda_n(T) : n \geq 1\}$ *is uniformly integrable,*

(ii') Λ_n *is a* $*$*-quadratic variation of* X_n $\forall n$ *and*

(iii') *for every* $A \in \mathcal{A}$, $\Lambda_n(A) \xrightarrow{P} \Lambda(A)$

for some d_H*-continuous variance function* Λ *on* $\mathcal{A}$*, then* $\exists$ *a* d_H*-continuous Gaussian process* G *based on* Λ *such that* $X_n \rightarrow_{\mathcal{L}} G$ *in the sense of Definition 4.3.7*(b).

Proof Since $(X_n)_n$ satisfies all the conditions of Theorem 4.4.1, Corollary 4.4.3 implies the existence of an $\mathcal{A}$-indexed Gaussian white noise W based on Λ s.t. $X_n \to W$ in finite dimensional distribution. In particular, note that d_H-continuity implies left- and right-continuity for each X_n.

Since $\mathcal{A}$ satisfies condition (4.17) and Λ is d_H-continuous on $\mathcal{A}$, Theorem 4.2.32 implies the existence of a d_H-continuous version G of W which, as mentioned in Comment 4.2.29, is also a Gaussian process based on Λ. Since G is a version of W, we have

$$X_n \to G \ \text{ in finite dimensional distribution.}$$

Therefore, by conditions (iii) and (iv), the present theorem follows from Theorem 4.3.12. □

4.5 Semi-Functional CLTs for General Strong Martingales

In Theorem 4.4.1, conditions were given under which a sequence of $\mathcal{A}$-indexed strong martingales with smooth sample paths (in the sense of left-and right-continuity) converged semi-functionally to an $\mathcal{A}$-indexed Gaussian white noise. This assumption on the sample paths is too restrictive for most applications. Indeed, most CLTs for sequences $(Y_n)_n$ of classical martingales allow for discontinuities of the first kind, i.e., sample paths in $D[0,\infty)$, so long as the jumps in Y_n become asymptotically neglegible as n becomes large. Unfortunately, due to the complexity of the set-indexed framework, there is at present no explicit and useful asymptotic rarefaction of jumps condition for set-indexed processes. In its place, we define the following class of sequences of set-indexed processes.

Definition 4.5.1 *A sequence $(X_n)_n$ of L_2 processes in $D[S(\mathcal{A})]$ is said to be J-L_2-asymptotically smooth, abbreviated J-L_2-AS, if for every $f \in S(\mathcal{A})$,*

$$E[J(M_f(X_n))^2] \to 0 \text{ as } n \to \infty \tag{4.43}$$

where $J : D[0,1] \to [0,\infty)$ denotes the jump functional defined in (4.2) *and M_f is defined in* (4.25).

Several generic examples of J-L_2-AS sequences are presented below. A specific example can be found in Proposition 4.6.21.

Example 4.5.2 Assume $(X_n)_n$ is a sequence of $\mathcal{A}$-indexed left- and right-continuous process in L_2. Given any $f \in S(\mathcal{A})$, Lemma 4.2.21 (a) implies $M_f(X_n) = X_n \circ f$ is in $C[0,1]$ and hence, $X_n \in D[S(\mathcal{A})]$ $\forall n$. Furthermore, (4.3) implies $E[J(X_n \circ f)^2] = 0$ $\forall n$ so that $(X_n)_n$ is trivially J-L_2-AS.

Example 4.5.3 Let $(X_n)_n$ be a sequence of $\mathcal{A}$-indexed purely atomic processes. By Proposition 4.3.14 (i), $X_n \in D[S(\mathcal{A})]$ $\forall n$. In fact, as mentioned in the proof thereof, $M_f(X_n) = X_n \circ f$ $\forall n$ and $f \in S(\mathcal{A})$.

For the sake of notation, assume each X_n is defined on a common probability space $(\Omega, \mathcal{F}, P)$. Given n and ω, denote the atoms of the sample path $A \mapsto X_n(A,\omega)$ by

$$t_1(\omega,n), \cdots, t_k(\omega,n) \in T$$

and the respective masses by

$$a_1(\omega,n), \cdots, a_k(\omega,n) \in \mathbf{R}.$$

The dependence of k on n and ω is suppressed from the notation. A condition under which $(X_n)_n$ is J-L_2-AS is the existence of random variables, $(W_n)_n$ on $(\Omega, \mathcal{F}, P)$ s.t. $E(W_n^2) \to 0$ and

$$\sup_{0\le t\le 1} \left| \sum \{ a_i(\omega,n) \ : \ t_i(\omega,n) \in \Delta f(t) \} \right| \le W_n \text{ a.e.}, \quad (\forall n) \tag{4.44}$$

for any fixed $f \in S(\mathcal{A})$ where $\Delta f(t) = f(t) \setminus \bigcup_{s<t} f(s)$ $\forall t \in (0,1]$ and $\Delta f(0) = \phi$. Indeed, Lemma A.8.1 implies

$$\sup_{0\leq t\leq 1} |\Delta X_n \circ f(t)| = \sup_{0\leq t\leq k} |X_n(\Delta f(t))| \text{ on } \Omega, \ (\forall n)$$

with the right-hand side equaling the left-hand side of (4.44).

Example 4.5.4 Assume $(X_n)_n$ is a sequence of $\mathcal{A}$-indexed processes in L_2 s.t. $X_n = U_n + V_n$ $\forall n$ where each U_n is left- and right-continuous and each V_n is purely atomic. One can view such X_n as being smooth, in the sense of left- and right-continuity, with a purely atomic part. Since both $(U_n)_n$ and $(V_n)_n$ are sequences in $D[S(\mathcal{A})]$, so is (X_n).

If the atoms and masses of the sample paths of V_n satisfy (4.44), then by the arguments in Examples 4.5.2 and 4.5.3, $(X_n)_n$ is J-L_2-AS. In particular, given any $f \in S(\mathcal{A})$, $\Delta X_n \circ f = \Delta V_n \circ f$ $\forall n$ and hence,

$$E[J(M_f(X_n))^2] = E[J(V_n \circ f)^2] \to 0 \text{ as } n \to \infty.$$

For our second semi-functional CLT, we no longer require the strong martingales $X_1, X_2, \cdots$ to have sample paths which are left- and right-continuous. In fact, we only request that the discontinuities of the X_n become negligible as $n \to \infty$ according to the J-L_2-AS condition.

Theorem 4.5.5 *Let $(X_n)_n$ be a J-L_2-AS sequence of $\mathcal{A}$-indexed strong martingales for which $X_n(\phi') \to 0$ in probability and*

$$\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty \tag{4.45}$$

for some $\delta > 0$. If $\exists$ a sequence of $\mathcal{A}$-indexed processes $(\Lambda_n)_n$ s.t.

(C1) *$\{\Lambda_n(T) : n \geq 1\}$ is uniformly integrable,*

(C2) *Λ_n is a $*$-quadratic variation of X_n $\forall n$ and*

(C3) *for every $A \in \mathcal{A}$, $\Lambda_n(A) \to \Lambda(A)$ in probability*

for some variance function Λ on $\mathcal{A}$, then $\exists$ an $\mathcal{A}$-indexed Gaussian white noise W based on Λ s.t. $X_n \to W$ semi-functionally.

Proof Unless otherwise labeled, **all Claims, Reductions or properties referred to in this proof are to be found in the proof of Theorem 4.4.1.** As mentioned in the proof of Theorem 4.4.1, it is sufficient to consider the case in which each X_n is defined on a common stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$.

Select $f \in S(\mathcal{A})$ and define $(\mathcal{H}_t)_{t\in[0,1]}$, (Y_n), $(\lambda_n)_n$ and λ as done in the proof of Theorem 4.4.1. Note that properties (a1) through (a4) on p.165 continue to hold with the sole exception that Y_n need not have continuous sample paths. [7] Despite this discrepency, large portions of the proof of Theorem 4.4.1 will apply to the present case.

To begin with, if we extend $(\mathcal{H}_t)_{t\in[0,1]}$ to $(\mathcal{H}'_t)_{t\in[0,\infty)}$ and each Y_n to $\{Y'_n(t) : t \in [0,\infty)\}$ according to (4.33), then by Theorem 4.2.7, each Y'_n possesses predictable quadratic variation $\langle Y'_n\rangle$. Repeating the argument in Claim 3, we have

Claim A: $\{\langle Y'_n\rangle|_1 : n \geq 1\}$ *is tight in* $D[0,1]$ *w.r.t. the Skorokhod topology.*

Since the Y'_n are not necessarily continuous, neither are the $\langle Y'_n\rangle$. However,

Claim B: *If* $\langle Y'_{r_n}\rangle|_1 \to_{\mathcal{L}} Z$ *in* $D[0,1]$ *for some subsequence* $(r_n)_n$*, then* Z *is a continuous and increasing process.*

Proof: For the sake of notation, take $r_n = n\ \forall n$. Since each $\langle Y'_n\rangle$ is increasing, Lemma A.5.7 implies Z is increasing. To establish sample path continuity of Z, it is sufficient by Proposition 4.2.3 and Claim A to show

$$J(\langle Y'_n\rangle|_1) \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{4.46}$$

To this end, define for any n and $\epsilon > 0$ the random variable

$$S_n = \inf\{s > 0 : \Delta\langle Y'_n\rangle_s > \epsilon\} \tag{4.47}$$

with the convention $\inf \phi = \infty$. By Claim 1 and the right-continuity of $\langle Y'_n\rangle$,

$$S_n = \inf\{0 < s \leq 1 : \Delta\langle Y'_n\rangle_s > \epsilon\}. \tag{4.48}$$

Also note that

$$S_n(\omega) \leq 1 \iff J(\langle Y'_n\rangle)(\omega) = \sup_{0\leq s\leq 1} \Delta\langle Y'_n\rangle(s,\omega) > \epsilon.$$

[7] However, since $X_n \in D[S(\mathcal{A})]\ \forall n$, each Y_n has sample paths in $D[0,1]$

Therefore, (4.46) will follow if we can show $P(S_n \le 1) \to 0$ as $n \to \infty$ for every $\epsilon > 0$. This will be accomplished via an argument resembling that found on p.268 of [21].

Fix $\epsilon > 0$. Given any n, since $\langle Y_n' \rangle$ is non-negative and increasing, the sample path $t \mapsto \langle Y_n' \rangle(t, \omega)$ will have at most a finite number of jumps of size greater than ϵ on $[0,1]$ for a.e. ω. Therefore, we can replace inf by min in (4.48) and so doing, obtain $(\Delta \langle Y_n' \rangle)_{S_n} > \epsilon$ on $[S_n \le 1]$. This implies

$$P(S_n \le 1) \;\le\; \epsilon^{-1} \cdot E\left[\mathbf{1}_{[S_n \le 1]} \cdot (\Delta \langle Y_n' \rangle)_{S_n} \right]. \tag{4.49}$$

By Proposition A.5.4, S_n is a predictable stopping time w.r.t. $(\mathcal{H}_t')_{t \in [0,\infty)}$. Therefore by Claim 2 (b), we can apply Lemma A.5.3 to the process

$$M(t) \;=\; \begin{cases} \langle Y_n' \rangle(t) - \lambda_n(t) & \text{, if t} \in [0,1] \\ \langle Y_n' \rangle(1) - \lambda_n(1) & \text{, if t} \in (1,\infty] \end{cases}$$

to obtain

$$E[\, (\Delta \langle Y_n' \rangle)_{S_n} \,|\, \mathcal{H}_{S_n -}' \,] \;=\; E[\, (\Delta \lambda_n)_{S_n} \,|\, \mathcal{H}_{S_n -}' \,]. \tag{4.50}$$

(Note that $\Delta M = \Delta(\langle Y_n' \rangle - \lambda_n) = \Delta \langle Y_n' \rangle - \Delta \lambda_n$.) Furthermore, since S_n is $\mathcal{H}_{S_n -}'$-measurable (a consequence of predictablility), conditioning w.r.t. $\mathcal{H}_{S_n -}'$ yields

$$\begin{aligned} P(S_n \le 1) \;&\le\; \epsilon^{-1} \cdot E\left[\mathbf{1}_{[S_n \le 1]} \cdot E[\, (\Delta \langle Y_n' \rangle)_{S_n} \,|\, \mathcal{H}_{S_n -}' \,] \right] \quad \text{(by (4.49))} \\ &=\; \epsilon^{-1} \cdot E\left[\mathbf{1}_{[S_n \le 1]} \cdot E[\, (\Delta \lambda_n)_{S_n} \,|\, \mathcal{H}_{S_n -}' \,] \right] \quad \text{(by (4.50))} \\ &=\; \epsilon^{-1} \cdot E\left[\mathbf{1}_{[S_n \le 1]} \cdot (\Delta \lambda_n)_{S_n} \right] \quad \text{(by conditioning)} \\ &\le\; \epsilon^{-1} \cdot E\left[\sup_{0 \le t \le 1} \Delta \lambda_n(t) \right] \\ &=\; \epsilon^{-1} \cdot E[J(\lambda_n)] \quad \text{(since } \lambda_n \text{ is increasing).} \end{aligned} \tag{4.51}$$

Since condition (C3) implies (4.32), Lemma A.5.8 implies $\lambda_n \to_{\mathcal{L}} \lambda$ in $D[0,1]$. Thus, since J is continuous at $\lambda \in C[0,1]$ (see (4.4)), the continuous mapping theorem implies $J(\lambda_n) \to_{\mathcal{L}} J(\lambda) = 0$. Furthermore, since each λ_n is increasing on $[0,1]$,

$$J(\lambda_n) \;\le\; 2 \cdot \lambda_n(1)\,, \quad (\forall n).$$

In view of condition (C1), this implies $\{J(\lambda_n) : n \ge 1\}$ is uniformly integrable. Therefore, by dominated convergence $E[J(\lambda_n)] \to 0$ as $n \to \infty$

which, by (4.51), implies $P(S_n \leq 1) \to 0$. This establishes (4.46), completing the proof of Claim B. Ω

With Claim B established, we can repeat the entire argument following Claim 3 to obtain

$$\langle Y_n' \rangle(t) \xrightarrow{P} \lambda(t)\,, \quad (\forall\, t \in [0,1]). \tag{4.52}$$

In fact, by Claim 1, $\langle Y_n' \rangle(t) \to_P \lambda(t)$ $\forall\, t \geq 0$. Since $(X_n)_n$ is J-L_2-AS and $\Delta Y_n'(t) = 0$ $\forall\, t > 1$,

$$0 \leq E[J_a(Y_n'|_a)^2] \leq E[J(M_f(X_n))^2] \;\to\; 0 \text{ as } n \to \infty, \quad (\forall\, a > 0).$$

Therefore by (a1), Theorem 4.2.8 implies the existence of a standard Brownian motion B s.t.

$$Y_n \xrightarrow{\mathcal{L}} B_\lambda \text{ in } D[0,1] \tag{4.53}$$

where $B_\lambda = \{B(\lambda(t)) : t \in [0,1]\}$.

Finally, by Remark 4.2.35, $\exists$ a Gaussian white noise W based on Λ. [8] Since $M_f(W)$ is in $C[0,1]$ and is a version of B_λ (see Proposition 4.2.36), (4.53) implies $M_f(X_n) \to_{\mathcal{L}} M_f(W)$ in $D[0,1]$, completing the proof of Theorem 4.5.5. $\square$

4.5.1 Some consequences, including a functional CLT

This subsection, which is similar in nature to Subsection 4.4.1, contains several consequences of Theorem 4.5.5 including a generic functional CLT for set-indexed strong martingales. We begin with an analogue of Proposition 4.4.2.

Proposition 4.5.6 *For each n, let X_n be an $\mathcal{A}$-indexed strong martingale defined on a stochastic base satisfying Assumption Groups D.1 and D.3. If $(X_n)_n$ is such that*

(i) $\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty$ *for some* $\delta > 0$,

(ii) $X_n(\phi') \to 0$ *in probability,*

(iii) $(X_n)_n$ *is J-L_2-AS*

[8]Note that W is independent of the simple flow f.

and $\exists$ a variance function Λ on $\mathcal{A}$ s.t.

$$Q_n(A) \to \Lambda(A) \text{ in probability}, \quad (\forall A \in \mathcal{A})$$

where Q_n denotes the unique $*$-PQV of X_n (see Theorem 3.5.2), then $\exists$ an $\mathcal{A}$-indexed Gaussian white noise W based on Λ s.t. $X_n \to W$ semi-functionally.

Proof Identical to that of Proposition 4.4.2 with Theorem 4.5.5 playing the role of Theorem 4.4.1. □

As was the case for Corollary 4.4.3, the following limit theorem is an automatic consequence of Theorem 4.5.5 and Proposition 4.3.17.

Corollary 4.5.7 *If $X_1, X_2, \cdots$ and $\Lambda, \Lambda_1, \Lambda_2, \cdots$ are as described in Theorem 4.5.5, then $\exists$ an $\mathcal{A}$-indexed Gaussian white noise W with variance function Λ s.t.*

$$(X_n(C_0), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (W(C_0), \cdots, W(C_k)) \text{ as } n \to \infty \tag{4.54}$$

for any finite subcollection $\{C_0, \cdots, C_k\}$ of $\mathcal{C}(u)$. In particular, $X_n \to W$ in finite dimensional distribution.

We close this section with a functional CLT for J-L_2-AS sequences of strong martingales in $D(\mathcal{A})$. As was the case for Theorem 4.4.4, this will involve an application of Theorem 4.3.2. However, since there is at present no modulus of continuity for $D(\mathcal{A})$, we must rely directly on the rich class of d_D-compact subsets of $D(\mathcal{A})$ obtained in Chapter 2 for tightness. To simplify notation, we assume all processes are defined on a common probability space $(\Omega, \mathcal{F}, P)$ — the result clearly carries over to the heterogeneous case.

Theorem 4.5.8 *Assume $\mathcal{A}$ satisfies condition* (4.17). *Let $(X_n)_n$ be a J-L_2-AS sequence of strong martingales in $D(\mathcal{A})$ s.t.*

(i) $X_n(\phi') \xrightarrow{P} 0$,

(ii) $\sup_n E\left[|X_n(T)|^{2+\delta}\right] < \infty$ *for some $\delta > 0$*

and, for any $\epsilon > 0$, $\exists$ constants η_ϵ and R_ϵ, functions N_ϵ and h_ϵ and a d-compact subset T_ϵ of T — all as described in Definition 2.6.2 — such that

$$P\left[X_n \in \Gamma(T_\epsilon, h_\epsilon, N_\epsilon, \eta_\epsilon, R_\epsilon)\right] \geq 1 - \epsilon, \quad (\forall n). \tag{4.55}$$

If $\exists$ a sequence of $\mathcal{A}$-indexed processes $(\Lambda_n)_n$ s.t.

(i') *$\{\Lambda_n(T) : n \geq 1\}$ is uniformly integrable,*

(ii') *Λ_n is a $*$-quadratic variation of X_n $\forall n$ and*

(iii') *for every $A \in \mathcal{A}$, $\Lambda_n(A) \xrightarrow{P} \Lambda(A)$*

for some d_H-continuous variance function Λ on $\mathcal{A}$, then $\exists$ a d_H-continuous Gaussian process G based on Λ such that $X_n \to_{\mathcal{L}} G$ in the sense of Definition 4.3.7(b).

Proof Identical to that of Theorem 4.4.4, only now we use Theorem 4.3.10 in place of Theorem 4.3.12. Note that the continuous limiting Gaussian process G, whose existence is guaranteed by Theorem 4.2.32, is in $D(\mathcal{A})$ by Proposition 2.3.4 (b). □

Remark 4.5.9 If we replace each $\Gamma(T_\epsilon, h_\epsilon, N_\epsilon, \eta_\epsilon, R_\epsilon)$ in (4.55) by a subset of the form Ξ_0 (the latter defined in Theorem 2.6.7), then by Remark 4.3.11 (b), Theorem 4.5.8 remains valid.

4.6 An Application to Set-Indexed Weighted Empirical Processes

Take $d \in \mathbf{N}$ and let $\mathbf{F} : \mathbf{R}^d \to [0, 1]$ be a continuous distribution function with $\mathbf{F}([0, 1]^d) = 1$. Take two sequences,

(I) i.i.d. random vectors, $(\mathbf{Y}_n)_n$ distributed $\mathbf{F}$ and

(II) i.i.d. random variables $(Z_n)_n$ which are independent of $(\mathbf{Y}_n)_n$ with $P(Z_1 = 0) = 0$, $E(Z_1) = 0$ and $\mathrm{var}(Z_1) = 1$.

Assume the probability space $(\Omega, \mathcal{F}, P)$ carrying these sequences is complete.

Let $\mathcal{A}$ be an indexing collection on $T = [0, 1]^d$. Since the elements of $\mathcal{A}$ are closed subsets of $[0, 1]^d$, we have $\mathcal{A} \subseteq \mathcal{B}_d$ where $\mathcal{B}_d$ denotes the Borel σ-algebra on $[0, 1]^d$. Given any $n \in \mathbf{N}$, the *n-th ($\mathcal{A}$-indexed) weighted empirical process based on* $\mathbf{F}$, $U_n = \{U_n(A) : A \in \mathcal{A}\}$ is defined by

$$U_n(A) \; = \; n^{-1/2} \cdot \sum_{i=1}^{n} 1_{[\mathbf{Y}_i \in A]} \, Z_i \, , \quad (\forall A \in \mathcal{A}). \qquad (4.56)$$

For the case of $\mathcal{A} = \mathcal{I}_d$ and $Z_i \sim N(0,1)$, Burke has shown weak convergence of $(U_n)_n$, viewed as multiparameter processes, to a Brownian sheet with variance function $\mathbf{F}$ (see Theorem 1.1 in [9]). The statistical advantages of weighted empirical processes over the classical empirical process are discussed in [9].

The main goal of this section is a semi-functional CLT for $(U_n)_n$. Specifically, in Theorem 4.6.22 we will show that the $\mathcal{A}$-indexed weighted empirical processes based on $\mathbf{F}$ converge semi-functionally to an $\mathcal{A}$-indexed Gaussian white noise based on $\mathbf{F}$ provided the random weights have finite fourth moments, i.e., $E(Z_i^4) < \infty$, and $\mathcal{A}$ satisfies some additional conditions beyond those given in Definition 2.2.4. [9]

Before proceeding, it is important to note that the compact support condition, $\mathbf{F}([0,1]^d) = 1$ in no way impairs the generality of our results. In particular, if $\mathbf{F} : \mathbf{R}^d \to [0,1]$ is any continuous distribution, define the *marginal transform* $M : \mathbf{R}^d \to [0,1]^d$ by

$$M(x_1, \cdots, x_d) = (\mathbf{F}^{(1)}(x_1), \cdots, \mathbf{F}^{(d)}(x_d)), \quad (\forall (x_1, \cdots, x_d) \in \mathbf{R}^d) \qquad (4.57)$$

where $\mathbf{F}^{(i)} : \mathbf{R} \to [0,1]$ denotes the i-th marginal of $\mathbf{F}$. Then, $(M(\mathbf{Y}_n))_n$ is a i.i.d. sequence whose common distribution function $\mathbf{G}$ is s.t. $\mathbf{G}([0,1]^d) = 1$. (Although the marginals of $M(\mathbf{Y}_n)$ are distributed Uniform $([0,1])$, $M(\mathbf{Y}_n)$ is not necessarily distributed Uniform $([0,1]^d)$ since the components of $M(\mathbf{Y}_n)$ are not necessarily independent. This point has been made in [9].)

4.6.1 Technical preliminaries

Define a strict ordering, $\prec$ of $[0,1]^d$ by

$$\mathbf{x} \prec \mathbf{y} \iff x_i < y_i \ \forall 1 \le i \le d$$

and a partial ordering, $\le$ of $[0,1]^d$ by

$$\mathbf{x} \le \mathbf{y} \iff x_i \le y_i \ \forall 1 \le i \le d$$

where $\mathbf{x}$ and $\mathbf{y}$ denote arbitrary elements of $[0,1]^d$. Given $\mathbf{x} \le \mathbf{y}$, define

$$[\mathbf{x}, \mathbf{y}] = \prod_{i=1}^{d} [x_i, y_i]$$

[9] Under these additional conditions, any such $\mathbf{F}$ will be a variance function on $\mathcal{A}$.

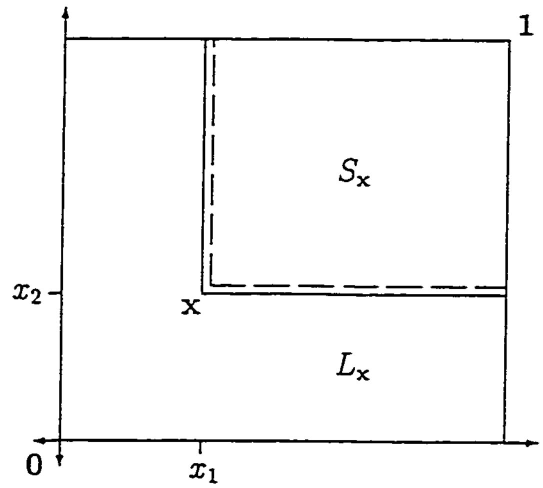

Figure 4.1: The sets $L_{\mathbf{x}}$ and $S_{\mathbf{x}}$ for $d = 2$.

and, given $\mathbf{x} \prec \mathbf{y}$ in $[0,1]^d$, define

$$(\mathbf{x}, \mathbf{y}] = \prod_{i=1}^{d} \langle x_i, y_i]$$

where $\langle a, b] = [0, b]$ if $a = 0$ and $(a, b]$ otherwise. Also, for any $\mathbf{0} \prec \mathbf{x} \prec \mathbf{1}$, define the sets

$$\begin{aligned} S_{\mathbf{x}} &= \textstyle\prod_{i=1}^{d}(x_i, 1]\,, \\ S_{\mathbf{x}-} &= \textstyle\prod_{i=1}^{d}[x_i, 1] \quad \text{and} \\ L_{\mathbf{x}} &= [0,1]^d \setminus S_{\mathbf{x}}. \end{aligned}$$

For the case of $d = 2$, see Figure 4.1.

We will often need the following

Assumption 4.6.1 *Every set $C \in \mathcal{C}$ possesses a maximal representation (see Definition 3.2.6). Furthermore,*

(i) $\mathcal{I}_d \subseteq \mathcal{A}$,

(ii) *for every n,* $\mathcal{A}_n = \begin{cases} \mathcal{I}_d^{(n)}(u) & , \text{if } \mathcal{I}_d(u) \subseteq \mathcal{A} \\ \mathcal{I}_d^{(n)} & , \text{otherwise} \end{cases}$ *and*

(iii) *for every* n, $g_n(A) = \bigcap\{B \in \mathcal{A}_n : A \subseteq B^\circ\}$ $\forall A \in \mathcal{A}$.

See Example 2.2.6 for the definition of $\mathcal{I}_d$, $\mathcal{I}_d^{(n)}$ *and* $\mathcal{I}_d^{(n)}(u)$.

Remark 4.6.2 (a) Similar assumptions have appeared in [27]. Clearly, both $\mathcal{I}_d$ and the lower layers, $\mathcal{LL}_d$ (see Example 2.8.1) satisfy Assumption 4.6.1. Recall that the lower layers require $\mathcal{LL}_d^{(n)} = \mathcal{I}_d^{(n)}(u)$ $\forall n$.

(b) In view of (i), it is clear that $\phi' = \bigcap_{A\in\mathcal{A},\, A\neq\phi} A = \{0\}$; a set of zero Lebesgue measure. Also note that condition (i) implies $\sigma(\mathcal{A}) = \mathcal{B}_d$.

By default, any indexing collection $\mathcal{A}$ on $[0,1]^d$ satisfying Assumption 4.6.1 has the indexing collection $\mathcal{I}_d$ as a subset. Furthermore,

Lemma 4.6.3 *Let* $\mathcal{A}$ *be an indexing collection on* $[0,1]^d$ *(some* $d \in \mathbf{N}$*). If* $\mathcal{A}$ *satisfies Assumption 4.6.1, then* $\mathcal{A}(u) \subseteq \mathcal{LL}_d$.

Proof Take $A \in \mathcal{A}$ and a point $\mathbf{x} \in A$. Since $\mathcal{I}_d(u) \subseteq \mathcal{LL}_d$, Assumption 4.6.1 (ii) implies $g_n(A) \in \mathcal{LL}_d$ $\forall n$. Since $A \subseteq g_n(A)$ $\forall n$, this implies $[0,\mathbf{x}] \subseteq g_n(A)$ $\forall n$. But by the defining properties of $(g_n)_n$,

$$[0,\mathbf{x}] \subseteq \bigcap_n g_n(A) = A.$$

Therefore, since $\mathbf{x} \in A$ was arbitrarily chosen, A is a lower layer. The inclusion $\mathcal{A}(u) \subseteq \mathcal{LL}_d$ now follows from the fact that $\mathcal{LL}_d$ is closed under finite unions. □

In the sequel, it will be essential that $\mathbf{F}$, when restricted to $\mathcal{A} \subseteq \mathcal{B}_d$, constitutes a variance function on $\mathcal{A}$. Toward this property, we have a basic result concerning lower layers.

Lemma 4.6.4 *If* $(A_n)_n$ *is an increasing sequence in* $\mathcal{LL}_d$ *(some* $d \in \mathbf{N}$*) and* $B = \overline{\bigcup_n A_n}$, *then* $B \setminus \bigcup_n A_n \subseteq \partial B \cup$ *faces of* $[0,1]^d$. [10] *Here, both the boundary and closure are w.r.t. the subspace topology on* $[0,1]^d$.

Proof We begin with two geometric facts concerning lower layers in $[0,1]^d$. Given $A \in \mathcal{LL}_d$ and $\mathbf{0} \prec \mathbf{x} \prec \mathbf{1}$,

(GF-1) $\mathbf{x} \in A$ and $\mathbf{x} \notin \partial A$ implies $A \cap L_{\mathbf{x}} \subset A$ and

[10] faces = back faces ∪ front faces; see p.25 and p.26.

(GF-2) $\mathbf{x} \notin A$ implies $A \subseteq L_{\mathbf{x}}$.

For (GF-1), it is helpful to note that $\partial A = A \setminus A^\circ$. (GF-2) reflects the maximal nature of the lower layer $L_{\mathbf{x}}$.

Take $\mathbf{x} \in B \setminus \bigcup_n A_n$. If $\mathbf{x} \notin$ faces of $[0,1]^d$, then $\mathbf{0} \prec \mathbf{x} \prec \mathbf{1}$. Applying (GF-2) to each A_n, we obtain $(\bigcup_n A_n) \bigcap L_{\mathbf{x}} = \bigcup_n A_n$. If in addition $\mathbf{x} \notin \partial B$, then (GF-1) implies $L_{\mathbf{x}} \bigcap B \subset B$. Therefore, since B and $L_{\mathbf{x}}$ are both closed,

$$B \;=\; \overline{(\bigcup_n A_n) \cap L_{\mathbf{x}}} \;\subseteq\; B \cap L_{\mathbf{x}} \;\subset\; B$$

which gives a contradiction, implying $\mathbf{x} \in \partial B \bigcup$ faces of $[0,1]^d$. □

Proposition 4.6.5 *Assume $\mathcal{A}$ satisfies Assumption 4.6.1. If $\mathbf{F} : \mathbf{R}^d \to [0,1]$ is the continuous distribution introduced on p.182, then*

$$(A_n)_n \text{ increasing in } \mathcal{A}(u) \;\Longrightarrow\; \lim_n \mathbf{F}(A_n) \;=\; \mathbf{F}\left(\overline{\bigcup_n A_n}\right). \qquad (4.58)$$

In particular, $\mathbf{F}|_{\mathcal{A}}$ is a variance function on $\mathcal{A}$ in the sense of Definition 4.2.25.

Proof Take an increasing sequence $(A_n)_n$ in $\mathcal{A}(u)$ and let $B = \overline{\bigcup_n A_n}$. Since $\mathbf{F}$ is a probability measure on $\mathcal{B}_d$, $\lim_n \mathbf{F}(A_n) = \mathbf{F}(\bigcup_n A_n)$. Furthermore, Lemma 4.6.3 implies $(A_n)_n$ is in $\mathcal{LC}_d$ while Lemma A.2.2 implies $B \in \mathcal{LC}_d$. Therefore, since $\mathbf{F}$ is absolutely continuous w.r.t. Lebesgue measure λ and $\lambda(B \setminus \bigcup_n A_n) \le \lambda(\partial B)$ (see Lemma 4.6.4),

$$\lambda(\partial B) \;=\; 0 \qquad (4.59)$$

will imply $\lim_n \mathbf{F}(A_n) = \mathbf{F}(B)$. (Note that $\lambda(\text{faces of } [0,1]^d) = 0$.)

To establish (4.59), define an *inner approximator*, $g_m^- : \mathcal{LC}_d \to \mathcal{I}_d^{(m)}(u)$ for each $m \in \mathbf{N}$ by letting

$$g_m^-(A) \;=\; \bigcup \{A' \in \mathcal{I}_d^{(m)}(u) : A' \subseteq A^\circ\}\,, \quad (\forall\, A \in \mathcal{LC}_d).$$

Working with our set $B \in \mathcal{LC}_d$,

$$g_m^-(B) \subseteq B^\circ \;\text{ and }\; B \subseteq [g_m(B)]\,, \quad (\forall\, m)$$

and thus, since $\partial B = B \setminus B^\circ$,

$$\partial B \;\subseteq\; g_m(B) \setminus g_m^-(B)\,, \quad (\forall\, m). \qquad (4.60)$$

However, by the uniform separability condition presented on p.82 of [27],

$$\lambda\left(g_m(B)\setminus g_m^-(B)\right)\to 0 \quad \text{as } m\to\infty.$$

In view of (4.60), this establishes (4.59) which implies (4.58) for our arbitrary increasing sequence $(A_n)_n$ in $\mathcal{A}$.

That $\mathrm{F}|_{\mathcal{A}}$ is a variance function on $\mathcal{A}$ follows from (4.58), Remark 4.6.2 (b) and the fact that F is a finite measure on $\mathcal{B}_d$. □

By Proposition 2.2.9 (b), each g_n admits an extension to $\mathcal{A}(u)$ satisfying

$$g_n(\cup_{i=1}^k A_i) = \cup_{i=1}^k g_n(A_i)\,, \quad (A_i\in\mathcal{A}).$$

The proofs of the following properties, although elementary, are rather tedious and are thus omitted.

Lemma 4.6.6 *Under Assumption 4.6.1, given $B, B'\in\mathcal{A}(u)$ and $m\in\mathrm{N}$,*

(a) $\exists\, N\in\mathrm{N}$ *s.t.* $g_n(B)\subseteq[g_m(B)]^\circ\ \forall n\geq N$ *and*

(b) *if* $B\subseteq[g_m(B')]^\circ$*, then* $\exists\, K\in\mathrm{N}$ *s.t.* $g_k(B)\subseteq[g_m(B')]^\circ\ \forall k\geq K$.

We now construct a filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ to which each U_n is adapted. Given any $n\in\mathrm{N}$, define

$$M_n(A) = \mathbf{1}_{[\mathrm{Y}_n\in A]}\cdot Z_n\,, \quad (\forall A\in\mathcal{A}). \tag{4.61}$$

Note that $U_n = n^{-1/2}\cdot\sum_{i=1}^n M_i$. To any $B\in\mathcal{A}(u)$, associate the σ-algebra

$$\mathcal{H}_B = \sigma(\,M_n(A) : n\in\mathrm{N},\ A\subseteq B,\ A\in\mathcal{A})\vee\mathcal{F}_0 \tag{4.62}$$

where $\mathcal{F}_0$ denotes the σ-algebra generated by all P-null subsets of Ω. Clearly, $(\mathcal{H}_B)_{B\in\mathcal{A}(u)}$ is an increasing family of complete sub-σ-algebras of $\mathcal{F}$. Therefore, by Proposition 3.2.18, the family $(\mathcal{F}_A)_{A\in\mathcal{A}}$ given by

$$\mathcal{F}_A = \cap_n\mathcal{H}_{g_n(A)}\,, \quad (\forall A\in\mathcal{A}), \tag{4.63}$$

is a filtration on $(\Omega,\mathcal{F},P)$.

By Proposition 3.2.25, if $C = A\setminus B\in\mathcal{C}$ ($A\in\mathcal{A}$, $B\in\mathcal{A}(u)$) is a maximal representation, then the strong past at C can be expressed as

$$\mathcal{G}_C^* = \cap_n\,\mathcal{F}_{g_n(B)}^\circ \tag{4.64}$$

where $\mathcal{F}_{g_n(B)}^\circ$ is as defined in (3.8). The following result further simplifies this expression.

Lemma 4.6.7 *Take a set $C \in \mathcal{C}$. Under Assumption 4.6.1, given any maximal representation $A \setminus B$ of C ($A \in \mathcal{A}$, $B \in \mathcal{A}(u)$),*

$$\mathcal{G}_C^* \;=\; \textstyle\bigcap_m \mathcal{H}_{g_m(B)}. \tag{4.65}$$

Proof Since $\mathcal{H}_B \subseteq \mathcal{F}^{\circ}_{g_n(B)}$ $\forall B \in \mathcal{A}(u)$ and $n \in \mathrm{N}$, (4.64) implies

$$\textstyle\bigcap_n \mathcal{H}_{g_n(B)} \;\subseteq\; \mathcal{G}_C^*.$$

We will complete the lemma by showing $\mathcal{G}_C^* \subseteq \mathcal{H}_{g_m(B)}$ $\forall m$.

For this purpose, fix $m \in \mathrm{N}$. By Lemma 4.6.6 (a), $\exists n \in \mathrm{N}$ s.t.

$$g_n(B) \;\subseteq\; [g_m(B)]^{\circ}\,, \quad (\forall n \geq N). \tag{4.66}$$

For a given $n \geq N$ and $A \in \mathcal{A}$ s.t. $A \subseteq [g_n(B)]^{\circ}$, (4.66) implies $A \subseteq g_m(B)$. By Lemma 4.6.6 (b), this implies $\exists K \in \mathrm{N}$ s.t.

$$g_K(A) \;\subseteq\; g_m(B). \tag{4.67}$$

Therefore,

$$\begin{aligned} F \in \mathcal{F}_A \;&\Longrightarrow\; F \in \mathcal{H}_{g_K(A)} \quad \text{(by (4.63))} \\ &\Longrightarrow\; F \in \mathcal{H}_{g_m(B)} \quad \text{(by (4.67))}. \end{aligned}$$

Since $A \in \mathcal{A}$ s.t. $A \subseteq g_n(B)$ was arbitrarily chosen, (3.8) implies $\mathcal{F}^{\circ}_{g_n(B)} \subseteq \mathcal{H}_{g_m(B)}$. Furthermore, since $n \geq N$ was arbitrarily chosen, (4.64) implies $\mathcal{G}_C^* = \bigcap_{n \geq N} \mathcal{F}^{\circ}_{g_n(B)} \subseteq \mathcal{H}_{g_m(B)}$, completing the proof of the lemma. □

Theorem 4.5.5 states sufficient conditions under which well-behaved sequence of strong martingales satisfy a semi-functional CLT. If we hope to make use of this result, it is essential that each U_n be a strong martingale w.r.t. some filtration. Indeed,

Proposition 4.6.8 *If $\mathcal{A}$ satisfies Assumption 4.6.1, then each $\mathcal{A}$-indexed weighted empirical process U_n ($n \in \mathrm{N}$) is a strong martingale in L_2 w.r.t. the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ defined in* (4.63).

Proof Since $U_n = n^{-1/2} \cdot \sum_{i=1}^{n} M_i$ $\forall n$, it is sufficient to show that each M_i is a strong martingale in L_2 w.r.t. $(\mathcal{F}_A)_{A \in \mathcal{A}}$. With this in mind, fix $i \in \mathrm{N}$. Clearly, M_i is in L_2 and is adapted to $(\mathcal{F}_A)_{A \in \mathcal{A}}$.

According to Definition 3.2.38, we need to show $E[M_i(C)\,|\,\mathcal{G}^*_C] = 0\ \forall C \in \mathcal{C}$. First, take $C = (\mathbf{x}, \mathbf{y}]$ where $\mathbf{x}, \mathbf{y} \in [0,1]^d$ and $\mathbf{0} \prec \mathbf{x} \prec \mathbf{y}$. By Proposition A.7.1,

$$\begin{aligned} E[M_i(C)\,|\,\mathcal{G}^*_C] &= E[1_{[\mathbf{Y}_i \in C]}\, Z_i |\, \mathcal{G}^*_C] \\ &= 1_{[\mathbf{Y}_i \in S_{\mathbf{x}}]} \cdot E(Z_i)\, F(C)\, h_F(\mathbf{x}) \\ &= 0 \quad (\text{since } E(Z_i) = 0). \end{aligned} \tag{4.68}$$

Next, take any $C \in \mathcal{C}$ s.t. $C = \bigcup_{j=1}^k C_i$, a disjoint union, where each $C_j \in \mathcal{C}$ is of the form $(\mathbf{x}_j, \mathbf{y}_j]$ with $\mathbf{0} \prec \mathbf{x}_j \prec \mathbf{y}_j$. By the finite additivity of M_i over $\mathcal{C}(u)$, $M_i(C) = \sum_{j=1}^k M_i(C_j)$. Furthermore, Lemma 3.2.24 (d) implies $\mathcal{G}^*_C \subseteq \mathcal{G}^*_{C_j}\ \forall 1 \le j \le k$ which yields

$$E[M_i(C)\,|\,\mathcal{G}^*_C] \;=\; E[\,E(M_i(C_j)\,|\,\mathcal{G}^*_{C_j})\,|\,\mathcal{G}^*_C] \;=\; 0 \tag{4.69}$$

when we apply the tower property and (4.68).

Finally, take any $C \in \mathcal{C}$ and any maximal representation $A \setminus \bigcup_{i=1}^k B_i$ of C. For each $m \in \mathbf{N}$, define

$$D_m = g_m(A) \setminus \textstyle\bigcup_{i=1}^k g_m(B_i). \tag{4.70}$$

As shown in Claim V in the proof of Theorem 3.4.11,

$$\mathcal{G}^*_C \;\subseteq\; \mathcal{G}^*_{D_m}\,, \quad (\forall m). \tag{4.71}$$

Since $g_m(A), g_m(B_i) \in \mathcal{I}_d^{(m)}\ \forall m$ and i, each D_m can be expressed as a finite disjoint union of sets of the form $(\mathbf{x}, \mathbf{y}]$ with $\mathbf{0} \prec \mathbf{x} \prec \mathbf{y}$. Therefore, since $\lim_m M_i(D_m) = \lim_m 1_{[\mathbf{Y}_i \in D_m]} \cdot Z_i = 1_{[\mathbf{Y}_i \in C]} \cdot Z_i = M_i(C)$ (see Lemma A.2.8) and $|M_i(D_m)| \le |Z_i| \in L_1\ \forall m$, dominated convergence implies

$$\begin{aligned} E[M_i(C)\,|\,\mathcal{G}^*_C] &= \lim_m E[M_i(D_m)\,|\,\mathcal{G}^*_C] \;\text{ a.e.} \\ &= \lim_m E[\,E[M_i(D_m)\,|\,\mathcal{G}^*_{D_m}]\,|\,\mathcal{G}^*_C] \quad (\text{by } (4.71)) \\ &= 0 \quad (\text{by } (4.69)) \end{aligned}$$

which completes the proof. □

In the course of the above proof, we have established the following result.

Corollary 4.6.9 *If $\mathcal{A}$ satisfies Assumption 4.6.1, then for each n, the process M_n defined in* (4.61) *is a strong martingale in L_2 w.r.t. the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ defined in* (4.63).

Unless otherwise mentioned, **for the remainder of this chapter, $\mathcal{A}$ is assumed to satisfy Assumption 4.6.1.**

4.6.2 A $*$-quadratic variation for U_n

For each $k \in \mathbb{N}$, define the $\mathcal{A}$-indexed process $\tilde{M}_k$ by

$$\tilde{M}_k(A) = \int_A \mathbf{1}_{[0,\mathbf{Y}_k]} h_{\mathbf{F}} \, d\mathbf{F}, \quad (\forall A \in \mathcal{A}) \tag{4.72}$$

where $h_{\mathbf{F}} : [0,1]^d \to [0,\infty)$ is as defined in (A.40) and $[0, \mathbf{Y}_i]$ denotes the "random rectangle",

$$[0,\mathbf{Y}_k](\omega) = [0, \mathbf{Y}_k(\omega)] \subseteq [0,1]^d, \quad (\omega \in \Omega).$$

The unique finitely additive extension of $\tilde{M}_k$ to $\mathcal{C}(u)$ is the obvious one, namely, $C \in \mathcal{C}(u)$ replaces $A \in \mathcal{A}$ on the right hand side of (4.72). In addition, given any $n \in \mathbb{N}$, define the process $\tilde{U}_n$ by

$$\tilde{U}_n(A) = n^{-1} \cdot \sum_{k=1}^{n} \tilde{M}_k(A), \quad (\forall A \in \mathcal{A}). \tag{4.73}$$

The goal of this subsection is to show that $\tilde{U}_n$ is a $*$-quadradic variation of U_n. Adaptedness and $*$-predictablility of $\tilde{U}_n$ w.r.t. $(\mathcal{F}_A)_{A \in \mathcal{A}}$ is not of concern to us (see Remark 4.6.15).

$\tilde{M}_k$ has already appeared in [27] under the label $\tilde{X}_k$. In Theorem 2.2 of this particular paper, it was shown that

$$E[\tilde{X}_k(C) \,|\, \mathcal{G}_C^{**}] = E[\mathbf{1}_{[\mathbf{Y}_k \in C]} | \mathcal{G}_C^{**}], \quad (\forall C \in \mathcal{C}). \tag{4.74}$$

The extra $*$ on $\mathcal{G}_C^*$ signifies that the definition of strong past in [27] differs slightly from ours. Nonetheless, conditioning yields

Lemma 4.6.10 *For any k, $E[\tilde{M}_k(C)] = \mathbf{F}(C)$ $\forall C \in \mathcal{C}$.*

Remark 4.6.11 Since $\mathcal{A} \subseteq \mathcal{C}$ and $\tilde{M}_n(A) \geq 0$ $\forall A \in \mathcal{A}$, Lemma 4.6.10 implies $\tilde{M}_k$ is a process in L_1.

To show $\tilde{U}_n$ is a $*$-quadratic variation of U_n, we need several technical results, the first of which originates from the proof of Theorem 2.2 in [27].

Lemma 4.6.12 *If $\mathbf{u}, \mathbf{v}, \mathbf{t} \in [0,1]^d$ are s.t. $\mathbf{0} \prec \mathbf{u} \prec \mathbf{v} \prec \mathbf{t}$, then*

$$E\left[\mathbf{1}_{[\mathbf{Y}_k \in S_\mathbf{u}]} \tilde{M}_k(C)\right] = \mathbf{F}(C), \quad (\forall k) \tag{4.75}$$

where $S_\mathbf{u}$ is as defined on page 184 and $C = (\mathbf{v}, \mathbf{t}] \in \mathcal{C}$.

Proof In the proof of Theorem 2.2 in [27], it is shown that $[\mathbf{Y}_k \in S_\mathbf{u}] \in \mathcal{G}^{**}_{(\mathbf{u},\mathbf{t}]}$. However, the family $(\mathcal{G}^{**}_C)_{C\in\mathcal{C}(u)}$ in [27] is s.t.

$$C \subseteq D \text{ in } \mathcal{C}(u) \implies \mathcal{G}^{**}_D \subseteq \mathcal{G}^{**}_C. \tag{4.76}$$

Therefore, $[\mathbf{Y}_k \in S_\mathbf{u}] \in \mathcal{G}^{**}_C$ which implies

$$\begin{aligned} E[\mathbf{1}_{[\mathbf{Y}_k\in S_\mathbf{u}]}\, \tilde{M}_k(C)] &= E[\mathbf{1}_{[\mathbf{Y}_k\in S_\mathbf{u}]}\cdot \mathbf{1}_{[\mathbf{Y}_k\in C]}] \quad (\text{by } (4.74)) \\ &= E[\mathbf{1}_{[\mathbf{Y}_k\in C]}] \quad (\text{since } C \subseteq (\mathbf{u},\mathbf{t}] \subseteq S_\mathbf{u}) \\ &= \mathbf{F}(C). \end{aligned}$$

This is precisely (4.75). □

In view of Definition 4.2.22 (i), $\tilde{U}_n$ must be right-continuous on $\mathcal{A}$. The following result goes one step further.

Lemma 4.6.13 *Given k, a.e. sample path of $\tilde{M}_k$ is outer-continuous on $\mathcal{A}$.*

Proof By (4.74), $E[\tilde{M}_k([0,1]^d)] < \infty$. Therefore, $\exists$ an event $\Omega' \in \mathcal{F}$ of full P-measure s.t.

$$\int_{[0,1]^d} \mathbf{1}_{[0,\mathbf{Y}_k(\omega')]}\, h_\mathbf{F}\, d\mathbf{F} < \infty\,, \quad (\forall\, \omega' \in \Omega'). \tag{4.77}$$

We will show that for every $\omega' \in \Omega'$, the sample path of $\tilde{M}_k$ at ω' is outer-continuous on $\mathcal{A}$.

Fix $\omega' \in \Omega'$ and let $\mathbf{x} = \mathbf{Y}_k(\omega') \in [0,1]^d$. Take $A \in \mathcal{A}$ and a sequence $(A_n)_n$ in $\mathcal{A}$ for which $A \subseteq A_n$ $\forall n$ and $A_n \to_{d_H} A$. If we define $(f_n)_n$ by

$$f_n(\mathbf{u}) = \mathbf{1}_{A_n\cap[0,\mathbf{x}]}\, h_\mathbf{F}(\mathbf{u})\,, \quad (\forall\, \mathbf{u} \in [0,1]^d),$$

then $|f_n| \le \mathbf{1}_{[0,\mathbf{x}]}\, h_\mathbf{F}$ $\forall n$ where, by (4.77), $\mathbf{1}_{[0,\mathbf{x}]}\, h_\mathbf{F} \in L_1([0,1]^d, d\mathbf{F})$.

<u>Claim:</u> $\mathbf{1}_{A_n} \to \mathbf{1}_A$ *on* $[0,1]^d$ *as* $n \to \infty$.

<u>Proof:</u> Fix $\mathbf{u} \in [0,1]^d$. If $\mathbf{u} \in A$, then $\mathbf{u} \in A_n$ $\forall n$. On the other hand, assume $\mathbf{u} \notin A$. If $\exists$ a subsequence $(k_n)_n$ s.t. $\mathbf{u} \in A_{k_n}$ $\forall n$, then since $A_{k_n} \to_{d_H} A$, Lemma A.1.4 implies $\mathbf{u} \in A$; a contradiction. Therefore, $\mathbf{u} \notin A_n$ for all large n which establishes the Claim. Ω

By the above Claim, $\lim_n f_n = 1_{A\cap[0,\mathbf{x}]}\, h_{\mathbf{F}}$ on $[0,1]^d$. Therefore, by dominated convergence,

$$\tilde{M}_k(A_n)(\omega') \;=\; \int_{[0,1]^d} f_n \, d\mathbf{F} \;\xrightarrow{\,n\,}\; \int_A 1_{[0,\mathbf{x}]}\, h_{\mathbf{F}} \, d\mathbf{F} \;=\; \tilde{M}_k(A)(\omega'),$$

i.e., the sample path of $\tilde{M}_k$ at ω' is outer-continuous at A. □

And now for the main result.

Proposition 4.6.14 *For every n, $\tilde{U}_n$ is a $*$-quadratic variation of U_n w.r.t. the filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$ defined in* (4.63).

Proof We begin with something simpler.

<u>Claim 1:</u> *For each k, $\tilde{M}_k$ is a $*$-quadratic variation w.r.t. $(\mathcal{F}_A)_{A\in\mathcal{A}}$ of the process M_k defined in* (4.61).

<u>Proof:</u> Clearly, $\tilde{M}_k$ is non-negative with $\tilde{M}_k(C) \geq 0\ \forall\, C \in \mathcal{C}$. Furthermore, by Lemma 4.6.13 and Lemma A.2.2 (a), a.e. sample path of $\tilde{M}_k$ is right-continuous on $\mathcal{A}$. Therefore, by Definition 4.2.22, all that remains to be shown is

$$E[M_k(C)^2 \,|\, \mathcal{G}^*_C] \;=\; E[\tilde{M}_k(C) \,|\, \mathcal{G}^*_C]\,, \quad (\forall\, C \in \mathcal{C}). \tag{4.78}$$

We begin with the simplest C.

Take $C = (\mathbf{x}, \mathbf{y}]$ where $\mathbf{x}, \mathbf{y} \in [0,1]^d$ are s.t. $\mathbf{0} \prec \mathbf{x} \prec \mathbf{y}$. Following the development found in the proof of Proposition A.7.1, define the sequence $(\mathbf{x}_n)_n$ in $[0,1]^d$ to be s.t. $g_n([0,\mathbf{x}]) = [0,\mathbf{x}_n]\ \forall\, n$. (As mentioned on p.233, we can assume w.l.o.g. that $\mathbf{0} \prec \mathbf{x}_n \prec \mathbf{y}\ \forall\, n$.) Also, define the subsets

$$C_n \;=\; (\mathbf{x}_n, \mathbf{y}]\,, \quad (\forall\, n).$$

As stated in (A.50), $C_n \uparrow C$ and $S_{\mathbf{x}_n} \uparrow S_{\mathbf{x}}$ as $n \to \infty$.

First, assume $\mathbf{F}(S_{\mathbf{x}}) = 0$. Since $C = (\mathbf{x}, \mathbf{y}] \subseteq S_{\mathbf{x}}$, this implies $M_k = 1_{[\mathbf{Y}_k \in C]} \cdot Z_k = 0$ a.e. Furthermore,

$$\begin{aligned} \mathbf{u} \in (\mathbf{x}, \mathbf{y}] \;&\Longrightarrow\; S_{\mathbf{u}-} \subseteq S_{\mathbf{x}} \\ &\Longrightarrow\; h_{\mathbf{F}}(\mathbf{u}) = 0 \quad (\text{by } (A.40)) \end{aligned}$$

so that $\tilde{M}_k(C) = \int_{(\mathbf{x},\mathbf{y}]} 1_{[0,\mathbf{Y}_k]}\, h_{\mathbf{F}} \, d\mathbf{F} = 0$. This trivially implies (4.78).

On the other hand, if we assume $\mathbf{F}(S_{\mathbf{x}}) > 0$, then w.l.o.g. $\mathbf{F}(S_{\mathbf{x}_n}) > 0$ $\forall n$. Take $n \in \mathrm{N}$. Since $C_m \subseteq C_n \subseteq S_{\mathbf{x}_n}$ $\forall m \leq n$,

$$\begin{aligned} \omega \notin [\mathbf{Y}_k \in S_{\mathbf{x}_n}] &\implies S_{\mathbf{x}_n} \cap [0, \mathbf{Y}_k(\omega)] = \phi \\ &\implies C_m \cap [0, \mathbf{Y}_k(\omega)] = \phi \\ &\implies \tilde{M}_k(C_m, \omega) = 0 \end{aligned}$$

for every $m \leq n$. Therefore, since $[\mathbf{Y}_k \in S_{\mathbf{x}_n}]$ is an *atomic event* in $\mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}$ (see Lemma A.7.5), Corollary A.7.7 implies

$$\begin{aligned} E\left[\tilde{M}_k(C_m) \,\middle|\, \mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}\right] &= \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot E\left[\mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \tilde{M}_k(C_m)\right] \cdot [\mathbf{F}(S_{\mathbf{x}_n})]^{-1} \\ &= \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot \mathbf{F}(C_m) \cdot [\mathbf{F}(S_{\mathbf{x}_n})]^{-1} \quad \text{(by Lemma 4.6.12)} \end{aligned}$$

for every $m \leq n$. The sub-σ-algebra $\mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}$ of $\mathcal{F}$ is defined in (A.46). Furthermore, since $\tilde{M}_k(C_m)$ is $\sigma(\mathbf{Y}_k, Z_k)$-measurable $\forall m$, the argument for (A.52) (with $\tilde{M}_k(C_m)$ replacing V_m) implies

$$\sigma(\tilde{M}_k(C_m)) \vee \mathcal{H}^{(k)}_{L_{\mathbf{x}_n}} \text{ is independent of } \textstyle\bigvee_{r \neq k} \mathcal{H}^{(r)}_{L_{\mathbf{x}_n}}, \quad (\forall m, n).$$

Thus, by (A.47), Lemma A.7.8 and the previous array, we can write

$$E\left[\tilde{M}_k(C_m) \,\middle|\, \mathcal{H}_{L_{\mathbf{x}_n}}\right] = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot \mathbf{F}(C_m) \cdot [\mathbf{F}(S_{\mathbf{x}_n})]^{-1}, \quad (\forall m \leq n).$$

Repeating the limiting procedures which follow (A.53), we thus obtain

$$E[\tilde{M}_k(C) \,|\, \mathcal{G}^*_C] = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} \cdot \mathbf{F}(C) \cdot h_{\mathbf{F}}(\mathbf{x}). \tag{4.79}$$

On the other hand, since $E(Z_k^2) = 1$, Proposition A.7.1 implies

$$E\left[M_k(C)^2 \middle| \mathcal{G}^*_C\right] = E\left[\mathbf{1}_{[\mathbf{Y}_k \in C]} Z_k^2 \middle| \mathcal{G}^*_C\right] = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} \cdot \mathbf{F}(C) \cdot h_{\mathbf{F}}(\mathbf{x}) \tag{4.80}$$

and therefore, $E[M_k(C)^2 | \mathcal{G}^*_C] = E[\tilde{M}_k(C) \,|\, \mathcal{G}^*_C]$ for our $C = (\mathbf{x}, \mathbf{y}]$.

Next, assume $C \in \mathcal{C}$ is a disjoint union, $\bigcup_{i=1}^r C_i$ where $C_i = (\mathbf{x}_i, \mathbf{y}_i]$ with $\mathbf{0} \prec \mathbf{x}_i \prec \mathbf{y}_i$ $\forall 1 \leq i \leq r$. By Lemma 3.2.24 (d), $\mathcal{G}^*_C \subseteq \mathcal{G}^*_{C_i}$ $\forall 1 \leq i \leq r$. Thus, given any $1 \leq i \leq r$,

$$\begin{aligned} E[M_k(C_i)^2 | \mathcal{G}^*_C] &= E(E[M_k(C_i)^2 | \mathcal{G}^*_{C_i}] \,|\, \mathcal{G}^*_C) \quad \text{(by tower property)} \\ &= E(E[\tilde{M}_k(C_i) \,|\, \mathcal{G}^*_{C_i}] \,|\, \mathcal{G}^*_C) \quad \text{(by previous case)} \\ &= E[\tilde{M}_k(C_i) \,|\, \mathcal{G}^*_C] \quad \text{(by tower property).} \end{aligned} \tag{4.81}$$

Furthermore,

<u>Subclaim:</u> *Given any* $1 \le i, j \le r$ *s.t.* $i \ne j$, *either*

$$\mathcal{F}_{[0,\mathbf{y}_i]} \subseteq \mathcal{G}^*_{C_j} \quad or \quad \mathcal{F}_{[0,\mathbf{y}_j]} \subseteq \mathcal{G}^*_{C_i}.$$

<u>Proof:</u> Since the indexing collection $\mathcal{I}_d$ satisfies the shape property (see Remark 2.2.5 (d)) and C_i, C_j are generated by the sets in $\mathcal{I}_d$, the Subclaim follows by Remark 3.2.28. ω

Thus, given any $1 \le i, j \le r$ s.t. $i \ne j$, since $M_k(C_i)$ is $\mathcal{F}_{[0,\mathbf{y}_i]}$-measurable and $M_k(C_j)$ is $\mathcal{F}_{[0,\mathbf{y}_j]}$-measurable (see (3.12)), we can assume w.l.o.g. that

$$\begin{aligned} E[M_k(C_i)\, M_k(C_j) \,|\, \mathcal{G}^*_C] &= E(M_k(C_i) \cdot E[M_k(C_j) \,|\, \mathcal{G}^*_{C_j}] \,|\, \mathcal{G}^*_C) \quad \text{(by Subclaim)} \\ &= 0 \quad \text{(by Corollary 4.6.9).} \end{aligned} \tag{4.82}$$

Combining (4.81) and (4.82), the finite additivity of M_k and $\tilde{M}_k$ yields (4.78) for this particular $C \in \mathcal{C}$.

Finally, consider the case in which C is an arbitrary element of $\mathcal{C}$. By Assumption 4.6.1, C possesses a maximal representation $A \setminus \bigcup_{i=1}^{j} B_i$. Let $(D_m)_m$ be the sequence in $\mathcal{C}$ defined in (4.70) and, for each m, define the random variable

$$V_m = M_k(D_m)^2 - \tilde{M}_k(D_m).$$

By Lemma A.2.8,

$$\lim_m M_k(D_m) = M_k(C) \text{ on } \Omega \quad \text{and} \quad \lim_m \tilde{M}_k(D_m) = \tilde{M}_k(C) \text{ a.e.}$$

(The latter limit is due to dominated covergence in $L([0,1]^d, d\mathrm{F})$.) Also,

$$|M_k(D_m)^2| \le Z_k^2 \text{ and } |\tilde{M}_k(D_m)| \le \tilde{M}_k([0,1]^d)\,, \quad (\forall\, m)$$

where $Z_k \in L_2$ and $\tilde{M}_k([0,1]^d) \in L_1$. Thus, by dominated convergence,

$$\begin{aligned} E[M_k(C)^2 - \tilde{M}_k(C) \,|\, \mathcal{G}^*_C] &= \lim_m E(V_m \,|\, \mathcal{G}^*_C) \text{ a.e.} \\ &= \lim_m E[\,E(V_m |\, \mathcal{G}^*_{D_m}) \,|\, \mathcal{G}^*_C] \quad \text{(by (4.71)).} \end{aligned}$$

But, given any m, $g_m(A), g_m(B_i) \in \mathcal{I}_d^{(m)}$ $\forall 1 \le i \le j$. Therefore, each D_n can be written as a disjoint union of sets of the form $(\mathbf{x}, \mathbf{y}]$ where $0 \prec \mathbf{x} \prec \mathbf{y}$. By the previous case, this implies

$$E(V_m |\, \mathcal{G}^*_{D_m}) = 0\,, \quad (\forall\, m)$$

and hence by the above limit, $E[M_k(C)^2 - \tilde{M}_k(C) \mid \mathcal{G}^*_C] = 0$, completing (4.78) and the proof of the Claim 1. Ω

To complete the proof of Proposition 4.6.14, fix $n \in \mathbb{N}$ and select $1 \leq i, j \leq n$ s.t. $i \neq j$. Enroute to showing $E[U_n(C) \mid \mathcal{G}^*_C] = E[\tilde{U}_n(C) \mid \mathcal{G}^*_C]$ $\forall C \in \mathcal{C}$, we need to show

$$E\left[M_i(C) \cdot M_j(C) \mid \mathcal{G}^*_C\right] = 0, \quad (\forall C \in \mathcal{C}). \tag{4.83}$$

This will be accomplished via the three cases mentioned above.

First, take $C = (\mathbf{x}, \mathbf{y}]$ with $0 \prec \mathbf{x} \prec \mathbf{y}$, Proposition A.7.9 (with $r = 2$ and $W = Z_i Z_j$) implies

$$E\left[M_i(C) \cdot M_j(C) \mid \mathcal{G}^*_C\right] = E\left[1_{[\mathbf{Y}_i \in C]} 1_{[\mathbf{Y}_j \in C]} Z_i Z_j \,\middle|\, \mathcal{G}^*_C\right] = 0, \quad (\forall i \neq j)$$

since $E(Z_i Z_j) = E(Z_i)\, E(Z_j) = 0$ by independence.

Next, take $C = \bigcup_{l=1}^r C_l \in \mathcal{C}$, a disjoint union where $C_l = (\mathbf{x}_l, \mathbf{y}_l]$ with $0 \prec \mathbf{x}_l \prec \mathbf{y}_l$ $\forall 1 \leq l \leq r$.

Claim 2: *Given any* $1 \leq l_1 < l_2 \leq r$, $E[M_i(C_{l_1}) \cdot M_j(C_{l_2}) \mid \mathcal{G}^*_C] = 0$.

Proof: For the sake of notation, take $l_1 = 1$ and $l_2 = 2$. Since $M_i(C_1)$ is $\mathcal{F}_{[0,\mathbf{y}_1]}$-measurable and $M_j(C_2)$ is $\mathcal{F}_{[0,\mathbf{y}_2]}$-measurable, Claim 2 follows from the previous Subclaim and the argument in (4.82). Ω

Therefore, since $i \neq j$ and both M_i and M_j are finitely additive on $\mathcal{C}(u)$, the previous case and Claim 2 imply (4.83) for this particular $C \in \mathcal{C}$.

Finally, take any $C \in \mathcal{C}$. If we approximate C by the sets $(D_m)_m$ defined in (4.70) and then repeat the argument found in the last paragraph of the proof of Claim 1, we obtain (4.83). Therefore,

$$\begin{aligned} E[U_n(C)^2 \mid \mathcal{G}^*_C] &= n^{-1} \cdot \sum_{k=1}^n E[M_k(C)^2 \mid \mathcal{G}^*_C] \quad \text{(by (4.83))} \\ &= n^{-1} \cdot \sum_{k=1}^n E[\tilde{M}_k(C)^2 \mid \mathcal{G}^*_C] \quad \text{(by Claim 1)} \\ &= E[\tilde{U}_n(C) \mid \mathcal{G}^*_C] \end{aligned}$$

$\forall C \in \mathcal{C}$. Since $\tilde{U}_n$ is non-negative, increasing and right-continuous (being a sum of such processes), it is indeed a $*$-quadratic variation of U_n. □

Remark 4.6.15 It is presently unclear if $\tilde{U}_n$ is $*$-predictable in the sense of Definition 3.4.7. As it is, we do not require any form of predictability for the $\tilde{U}_n$ to establish a semi-functional CLT for $(U_n)_n$.

4.6.3 Asymptotic rarefaction of jumps

In this subsection, $(U_n)_n$ will be shown to satisfy the J-L_2-AS condition for asymptotic rarefaction of jumps. To begin with, since each U_n is a strong martingale (see Theorem 4.6.8), Proposition 4.3.14 (iv) implies

$$U_n \in D[S(\mathcal{A})], \quad (\forall n).$$

In fact, since each U_n has purely atomic sample paths,

$$M_f(U_n) = U_n \circ f, \quad (\forall f \in S(\mathcal{A}))$$

(see the proof of Proposition 4.3.14 (i)). Thus, to show $(U_n)_n$ is J-L_2-AS, we require

$$E[J(U_n \circ f)^2] \to 0, \quad (\forall f \in S(\mathcal{A})) \tag{4.84}$$

where $J : D[0,1] \to [0,\infty)$ is the jump functional defined in (4.2). As it turns out, (4.84) is a consequence of Assumption 4.6.1 and the continuity of **F**. Before we can prove this fact, we need several technical results, the first of which appears as Problem 21.3 in [7].

Lemma 4.6.16 *If $(W_n)_n$ is identically distributed with $E|W_1| < \infty$, then $E\left[\max_{1\leq i\leq n}|W_i|\right] = o(n)$.*

Lemma 4.6.17 *If $(W_n)_n$ is i.i.d. with continuous distribution G, then*

$$\sup_t \sum_{i=1}^{n} \mathbf{1}_{[W_i=t]} \leq 1 \ \ a.e., \quad (\forall n). \tag{4.85}$$

Proof Since G is continuous, $P(W_i = W_j) = 0 \ \forall i \neq j$. Therefore, $\bigcup_{i\neq j}[W_i = W_j]$ is a P-null event containing all $\omega \in \Omega$ for which the inequality in (4.85) fails. □

The first step toward establishing J-L_2-AS for $(U_n)_n$ is given below.

Lemma 4.6.18 *Take $f \in S(\mathcal{A})$ and define the sets $\Delta f(t)$ ($t \in [0,1]$) where*

$$\Delta f(t) = f(t) \setminus \cup_{s<t} f(s), \quad (\forall t \in (0,1])$$

and $\Delta f(0) = \phi$. If

$$\sup_{0 \le t \le 1} \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{Y}_i \in \Delta f(t)]} \le 1 \ \ a.e., \quad (\forall n), \tag{4.86}$$

then $E[J(U_n \circ f)^2] \to 0$ as $n \to \infty$.

Proof Given any n, since $U_n(B) = n^{-1/2} \cdot \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{Y}_i \in B]} Z_i \ \forall B \in \mathcal{B}_d$,

$$\begin{aligned} J(U_n \circ f) &= \sup_{0 \le t \le 1} |\Delta\, U_n \circ f(t)| \\ &= \sup_{0 \le t \le 1} |U_n(\Delta f(t))| \quad \text{(by Lemma A.8.1)} \\ &\le n^{-1/2} \cdot \max_{1 \le i \le n} |Z_i| \quad \text{(by (4.86))} \end{aligned}$$

Therefore, since $(Z_n^2)_n$ is identically distributed with $E(Z_1^2) = \mathrm{var}(Z_1) = 1$, Lemma 4.6.16 implies

$$E[J(U_n \circ f)^2] \le n^{-1} \cdot E\left[\max_{1 \le i \le n} Z_i^2\right] \to 0 \quad \text{as } n \to \infty$$

which completes the proof. □

Take $f \in S(\mathcal{A})$. If $\exists$ i.i.d. random variables $(W_n)_n$ with continuous underlying distribution s.t.

$$\sup_{0 \le t \le 1} \sum_{i=1}^{n} \mathbf{1}_{[W_i = t]} = \sup_{0 \le t \le 1} \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{Y}_i \in \Delta f(t)]} \ \text{a.e.}, \quad (\forall n), \tag{4.87}$$

then by Lemmas 4.6.17 and 4.6.18, $E[J(U_n \circ f)^2] \to 0$. The existence of such W_n is the content of Lemma 4.6.20.

Given $f \in S(\mathcal{A})$, define the *flow induced distribution*, $\mathbf{F}_f : \mathbf{R} \to [0,1]$ by

$$\mathbf{F}_f(t) = \begin{cases} 0 & , \text{if } t < 0 \\ \mathbf{F}(f(t)) & , \text{if } 0 \le t \le 1 \\ 1 & , \text{if } t > 1 \ . \end{cases} \tag{4.88}$$

By Definition 4.2.13, $\mathbf{F}_f$ is indeed a distribution function. Furthermore,

Lemma 4.6.19 *For any $f \in S(\mathcal{A})$, $\mathbf{F}_f$ is a continuous distribution function.*

Proof Take a simple flow, $f : [0,1] \to \mathcal{A}(u)$. By the definition of $\mathbf{F}_f$, it is sufficient to show left-continuity on $(0,1]$.

With this in mind, take $t \in (0,1]$ and a sequence $(t_n)_n$ in $[0,t)$ s.t. $t_n \uparrow t$. By the definition of flow, $(f(t_n))_n$ is increasing in $\mathcal{A}(u)$ and $\overline{\bigcup_n f(t_n)} = f(t)$. Therefore, by (4.58),

$$\lim_n \mathbf{F}_f(t_n) \;=\; \lim_n \mathbf{F}(f(t_n)) \;=\; \mathbf{F}(f(t)) \;=\; \mathbf{F}_f(t)$$

which implies left-continuity of $\mathbf{F}_f$ at t. □

Lemma 4.6.20 *Given $f \in S(\mathcal{A})$, if we define random variables,*

$$W_n \;=\; \inf\{t \in [0,1] : \mathbf{Y}_n \in f(t)\}\,, \quad (\forall n),$$

then $(W_n)_n$ is i.i.d. with distribution $\mathbf{F}_f$. Furthermore, $(W_n)_n$ satisfies (4.87).

Proof Given any n, since $\mathbf{F}([0,1]^d) = 1$, we can assume w.l.o.g. that $\mathbf{Y}_n(\omega) \in [0,1]^d = f(1)$ for every $\omega \in \Omega$ and hence, $W_n(\omega) \in [0,1]\ \forall\, \omega \in \Omega$.

<u>Claim 1:</u> *For any n, $W_n = \min\{t \in [0,1] : \mathbf{Y}_n \in f(t)\}$ on Ω.*

<u>Proof:</u> Fix $\omega_0 \in \Omega$ and let $t_0 = \inf\{t \in [0,1] : \mathbf{Y}_n(\omega_0) \in f(t)\}$. If $t_0 = 1$, we are done. On the other hand, assume $0 \le t_0 < 1$. Since f is increasing w.r.t. $\subseteq$ on $[0,1]$, this clearly implies $\mathbf{Y}_n(\omega_0) \in f(s)\ \forall\, t_0 < s \le 1$. But by Definition 4.2.13 (ii), $f(t_0) = \bigcap_{s>t_0} f(s)$. Therefore, $\mathbf{Y}_n(\omega_0) \in f(t_0)$ which completes the proof of Claim 1. Ω

Given any n, Claim 1 and the monotonicity of f yield

$$[W_n \le t] \;=\; [\mathbf{Y}_n \in f(t)]\,, \quad (\forall\, t \in [0,1]). \tag{4.89}$$

Since $f(t) \in \mathcal{B}_d\ \forall\, t$ and each $\mathbf{Y}_n$ is a random vector, (4.89) implies each W_n is a random variable with distribution $\mathbf{F}_f$. Furthermore, given $n_1 < \cdots < n_k$ and any $t_1, \cdots, t_k \in [0,1]$, (4.89) and the independence of $\mathbf{Y}_{n_1}, \cdots, \mathbf{Y}_{n_k}$ imply

$$P\left(\bigcap_{i=1}^k [W_{n_i} \le t_i]\right) \;=\; \prod_{i=1}^k P[W_{n_i} \le t_i].$$

In total, $(W_n)_n$ is i.i.d. with distribution $\mathbf{F}_f$.

Finally, we must show (4.87). Recall the sets $\Delta f(t)$ $(t \in [0,1])$ defined in Lemma 4.6.18. Given any n, Claim 1 yields the identity

$$[W_n = t] \;=\; [\mathbf{Y}_n \in \Delta f(t)]\,, \quad (\forall\, t \in (0,1]). \tag{4.90}$$

(Specifically, $W_n(\omega) = t$ if and only if $\mathbf{Y}_n(\omega) \in f(t)$ and $\mathbf{Y}_n(\omega) \notin f(s)$ $\forall\, s < t$.) Therefore,

$$\sup_{0<t\leq 1} \sum_{i=1}^{n} \mathbf{1}_{[W_i=t]} \;=\; \sup_{0<t\leq 1} \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{Y}_i \in \Delta f(t)]} \;\text{ on } \Omega\;, \quad (\forall\, n) \tag{4.91}$$

Furthermore, Remark 4.6.2 (b) and the continuity of $\mathbf{F}$ yield $P(W_n = 0) = P[\mathbf{Y}_n \in \phi'] = 0$ $\forall\, n$. When combined with (4.91), this implies (4.87), completing the proof of Lemma 4.6.20. □

And now for the main result of this subsection,

Proposition 4.6.21 *$(U_n)_n$ is J-L_2-AS.*

Proof Take $f \in S(\mathcal{A})$. By Lemma 4.6.20,

$$\sup_{0\leq t\leq 1} \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{Y}_i \in \Delta f(t)]} \;=\; \sup_{0\leq t\leq 1} \sum_{i=1}^{n} \mathbf{1}_{[W_i=t]} \;\text{ a.e. }, \quad (\forall\, n)$$

where $(W_n)_n$ is i.i.d. with common distribution $\mathbf{F}_f$. Since $\mathbf{F}_f$ is continuous (see Lemma 4.6.19), Lemma 4.6.17 implies

$$\sum_{i=1}^{n} \mathbf{1}_{[W_i=t]} \leq 1 \;\text{ a.e. }, \quad (\forall\, n)$$

which, by Lemma 4.6.18, implies $E[J(U_n \circ f)^2] \to 0$. Since $f \in S(\mathcal{A})$ was arbitrarily chosen, this establishes J-L_2-AS for $(U_n)_n$. □

4.6.4 The proof of semi-functional convergence

In this subsection, we combine the elements found in the earlier subsections to obtain a semi-functional CLT for the weighted empirical processes $(U_n)_n$. In particular, when $\mathcal{A}$ satisfies Assumption 4.6.1 and the random weights $(Z_n)_n$ have finite fourth moments, we will show that $U_n \to W$ semi-functionally to some Gaussian white noise W. Our tool in this task is the strong martingale semi-functional CLT in Theorem 4.5.5.

Theorem 4.6.22 *Let $\mathcal{A}$ be an indexing collection on $[0,1]^d$ satisfying Assumption 4.6.1 and let $(U_n)_n$ be the sequence of $\mathcal{A}$-indexed weighted empirical processes defined in* (4.56). *If $E(Z_1^4) < \infty$, then $\exists$ an $\mathcal{A}$-indexed Gaussian white noise W based on* $\mathbf{F}$ *s.t. $U_n \to W$ semi-functionally.*

Proof Given any n, note that $(\sum_{i=1}^n Z_i)^4$ contains n terms of the form Z_i^4 and $3n(n-1)$ terms of the form $Z_i^2 Z_j^2$. Therefore, since $(Z_n)_n$ is i.i.d. with $E(Z_1) = 0$ and $E(Z_1^2) = 1$,

$$\begin{aligned} E\left[U_n([0,1]^d)^4\right] &= n^{-2} \cdot \left[\sum_{i=1}^n Z_i\right]^4 \\ &= n^{-2} \cdot \left[nE(Z_1^4) + 3n(n-1)\left[E(Z_1^2)\right]^2\right] \\ &\le E(Z_1^4) + 3. \end{aligned}$$

Since $E(Z_1^4) < \infty$, this establishes (4.45) for $\delta = 2$.

Since $\mathbf{F}$ is continuous, Remark 4.6.2 (b) implies $U_n(\phi') = 0 \ \forall n$ and hence

$$U_n(\phi') \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Moreover, by Propositions 4.6.8 and 4.6.21, $(U_n)_n$ is a J-L_2-AS sequence of strong martingales. Therefore, in view of Theorem 4.5.5, all that remains to be shown is

$$\{\tilde{U}_n([0,1]^d) : n \ge 1\} \text{ uniformly integrable} \tag{4.92}$$

and, for the variance function $\mathbf{F}|_{\mathcal{A}}$ (see Proposition 4.6.5),

$$\tilde{U}_n(A) \xrightarrow{P} \mathbf{F}(A), \quad (\forall A \in \mathcal{A}) \tag{4.93}$$

where $\tilde{U}_n$ is the $*$-quadratic variation of U_n defined in (4.73) (see Proposition 4.6.14). Both parts will follow from L_2 convergence in (4.93). In fact,

Reduction: *If for every $A \in \mathcal{A}$, $E[\tilde{U}_n(A)] \to \mathbf{F}(A)$ and $E[(\tilde{U}_n(A))^2] \to (\mathbf{F}(A))^2$ as $n \to \infty$, then $\tilde{U}_n(A) \to \mathbf{F}(A)$ in L_2,* implying (4.92) *and* (4.93) *and hence completing the proof of Theorem 4.6.22.*

To this end, fix $A \in \mathcal{A}$. Recall that $\tilde{U}_n = n^{-1}\sum_{k=1}^n \tilde{M}_k \ \forall n$, where $\tilde{M}_k$ is defined in (4.72). Thus, by Lemma 4.6.10 and the inclusion $\mathcal{A} \subseteq \mathcal{C}$,

$$E[\tilde{U}_n(A)] = n^{-1}\sum_{i=1}^n E\left[\tilde{M}_k(A)\right] = \mathbf{F}(A), \quad (\forall n),$$

trivially implying $E[\tilde{U}_n(A)] \to \mathbf{F}(A)$.

For convergence of second moments, define

$$\alpha_k = E[\tilde{M}_k(A)^2], \quad (\forall k \in \mathrm{N}).$$

On p.86 of [27], it is shown that $\exists\, 0 < \alpha < \infty$ s.t. $E[\tilde{M}_k([0,1]^d)^2] < \alpha \ \forall k$. But clearly, $\sum_{k=1}^n \alpha_k \leq \sum_{k=1}^n E[\tilde{M}_k([0,1]^d)^2] \leq n\alpha \ \forall n$. Thus,

$$n^{-2} \cdot \sum_{k=1}^{n} \alpha_k \to 0 \text{ as } n \to \infty. \tag{4.94}$$

Furthermore, $(\mathbf{Y}_n)_n$ independent implies $(\tilde{M}_k(A))_k$ is independent. Therefore,

$$\begin{aligned} E[(\tilde{U}_n(A))^2] &= n^{-2} \sum_{k=1}^{n} E\left[(\tilde{M}_k(A))^2\right] + n^{-2} \sum_{i \neq j} E\left[\tilde{M}_i(A)\, \tilde{M}_j(A)\right] \\ &= n^{-2} \cdot \sum_{k=1}^{n} \alpha_k + n^{-2}[n(n-1)(\mathbf{F}(A))^2] \quad \text{(by Lemma 4.6.10)} \\ &\xrightarrow{n} (\mathbf{F}(A))^2 \quad \text{(by (4.94))}. \end{aligned}$$

In view of the above Reduction, the proof of Theorem 4.6.22 is complete. □

By Remark 4.6.2 (a), Theorem 4.6.22 yields

Corollary 4.6.23 *If $E(Z_1^4) < \infty$ and each U_n is $\mathcal{I}_d$-indexed, then $\exists$ a Gaussian white noise W based on $\mathbf{F}$ s.t. $U_n \to W$ semi-functionally.*

In fact, Corollary 4.6.23 follows from Theorem 1.1 of [9] On the other hand, the following consequence of Theorem 4.6.22 is entirely new.

Corollary 4.6.24 *If $E(Z_1^4) < \infty$ and each U_n is $\mathcal{LL}_d$-indexed, then $\exists$ a Gaussian white noise W based on $\mathbf{F}$ s.t. $U_n \to W$ semi-functionally.*

Whereas Burke in [9] has shown functional convergence of $(U_n)_n$, when indexed by $\mathcal{I}_d$, to a suitable Gaussian process, it is doubtful that the mode of convergence in Corollary 4.6.24 can be upgraded to functional. The reason is that in the existing literature, functional CLTs for set-indexed processes generally require restrictions to the size of the indexing class, restrictions which exclude the use of the lower layers, $\mathcal{LL}_d$ for any $d \geq 2$. For example, Pyke in [36] obtained a uniform CLT for set-indexed partial-sum processes indexed by a class $\mathcal{E}$ of closed subsets of $[0,1]^d$ which satisfied a certain metric entropy condition which, among other things, ensured the existence of an $\mathcal{E}$-indexed Gaussian limiting process with d_H-continuous sample paths. This size condition, which differs from that found in Theorem 4.2.32, is discussed below.

Let $\mathcal{E}$ be any non-empty collection of closed subsets of $[0,1]^d$ which is totally bounded w.r.t. the Hausdorff metric d_H. As done in Subsection 4.2.4, define $N_{d_H} : (0,\infty) \to \mathrm{N}$ by letting

$$N_{d_H}(\epsilon) \;=\; \min\{\#\mathcal{E}(\epsilon) : \mathcal{E}(\epsilon) \text{ an } \epsilon\text{-net for } (\mathcal{E}, d_H)\}$$

where # denotes cardinality. Define the *exponent of metric entropy* of $\mathcal{E}$ by

$$\text{e.m.e.}(\mathcal{E}) \;=\; \inf\{r > 0 : \ln N_{d_H}(\epsilon) = O(\epsilon^{-r})\}. \tag{4.95}$$

(Given functions $f, g : (0,\infty) \to \mathbf{R}$, we write $f(\epsilon) = O(g(\epsilon))$ if $\exists K > 0$ s.t. $|f(\epsilon)/g(\epsilon)| \le K$ for all sufficiently small ϵ.) In a certain sense, the exponent of metric entropy measures the size of $\mathcal{E}$; the larger the exponent, the larger the indexing collection. For the above mentioned limit theorem, Pyke required

$$\text{e.m.e.}(\mathcal{E}) \;<\; 1. \tag{4.96}$$

While e.m.e.$(\mathcal{I}_d) < 1 \ \forall d$, the same cannot be said for $\mathcal{LC}_d$.

Proposition 4.6.25 *For each $d \ge 2$,* e.m.e.$(\mathcal{LC}_d) = d - 1$.

Proof Fix $d \ge 2$ and define $\psi : (0,\infty) \to [0,\infty)$ by letting

$$\psi(\epsilon) \;=\; \sup\{n : \exists A_1, \cdots A_n \in \mathcal{LC}_d \text{ s.t. } d_H(A_i, A_j) > \epsilon \ \forall i \ne j\}.$$

By Theorem 6.0.1 on p.39 of [17],

$$\psi(2\epsilon) \;\le\; \nu(\epsilon) \;\le\; \psi(\epsilon), \quad (\forall \epsilon > 0).$$

Furthermore, by Theorem 7.2.1 on p.57 of [17],

$$\ln \psi(\epsilon) \;\asymp\; \epsilon^{-(d-1)}.$$

(Given functions $f, g : (0,\infty) \to \mathbf{R}$, we write $f(\epsilon) \asymp g(\epsilon)$ if $f(\epsilon) = O(g(\epsilon))$ and $g(\epsilon) = O(f(\epsilon))$.) Therefore, $\nu(\epsilon) = O(\epsilon^{-(d-1)})$ and $\nu(\epsilon) \ne O(\epsilon^{-\alpha})$ for any $0 < \alpha < d-1$ which is to say e.m.e.$(\mathcal{LL}_d) = d-1$. □

Proposition 4.6.25 rules out the use of Pyke's argument in [36] as a means of establishing functional convergence of $(U_n)_n$ to an $\mathcal{LL}_2$-indexed Gaussian process — if such a limit theorem is even possible. On the other hand,

Example 4.6.26 If we take

- $\mathbf{F} = \text{Uniform}([0,1]^2)$,
- $Z_n \sim N(0,1)$ and
- $\mathcal{A} = \mathcal{LL}_2$,

then by Corollary 4.6.24, ∃ an $\mathcal{LL}_2$-indexed Gaussian white noise W based on $\mathbf{F}$ s.t. $U_n \to W$ semi-functionally. However, by Proposition 1.3 on p.9 of [1] (also see the second paragraph in Section VI.6 of [1]), if W is any Gaussian white noise based on Lebesgue measure, then for a.e. ω

$$A \mapsto W_A(\omega) \quad (A \in \mathcal{LL}_2) \quad \text{is } d_H\text{-discontinuous at every } A \in \mathcal{LL}_2.$$

That is, a.e. sample path of W is d_H-discontinuous everywhere on $\mathcal{LL}_2$!

Appendix A

Miscellaneous Technical Results

A.1 Some Basic Results for Metric Spaces

Throughout this section, (T, d) denotes a compact metric space and $(\mathcal{K}_T, d_H)$ denotes the corresponding metric space of all non-empty d-closed and bounded subsets of T equipped with the Hausdorff metric. The first result follows automatically from the definition of the subspace metric.

Lemma A.1.1 *Take any set $Y \in \mathcal{K}_T$. If $\mathcal{Y} = \{E \in \mathcal{K}_T : E \subseteq Y\}$ and $\mathcal{K}_Y = \{$ non-empty $d|_Y$-closed and bounded subsets of $Y\}$, then*

$$(\mathcal{Y}, d_H|_{\mathcal{Y}}) = (\mathcal{K}_Y, d_{H'})$$

where $d_{H'}$ is the Hausdorff metric generated by $d|_Y$ on Y.

Lemma A.1.2 *Given $A, B \in \mathcal{K}_T$, if $A \subseteq B^\circ$, then $\exists\, \beta > 0$ s.t. $A^\beta \subseteq B$.*

Proof If no such β exists, then, in particular, $A^{\frac{1}{n}} \cap [B^\circ]^c \neq \phi \; \forall\, n$. Choosing a sequence, $(t_n)_n$ in T s.t. $t_n \in A^{\frac{1}{n}} \cap [B^\circ]^c \; \forall\, n$, we have that for every $n \in \mathbb{N}$, $\exists\, a_n \in A$ s.t.

$$d(a_n, t_n) < 1/n. \tag{A.1}$$

Furthermore, since (T, d) is sequentially compact and A, $[B^\circ]^c$ are both d-closed in T, there is a subsequence $(k_n)_n$ s.t.

$$a_{k_n} \xrightarrow{d} a \text{ for some } a \in A \quad \text{and} \quad t_{k_n} \xrightarrow{d} t \text{ for some } t \in [B^\circ]^c.$$

Applying (A.1),

$$d(a,t) \le d(a, a_{k_n}) + 1/k_n + d(t_{k_n}, t), \quad (\forall n).$$

Letting $n \to \infty$, yields $d(a,t) = 0$, i.e., $a = t$. But then,

$$a \in A \subseteq B^\circ \text{ and } a = t \in [B^\circ]^c,$$

a contradiction. Therefore, there must be a $\beta > 0$ s.t. $A^\beta \subseteq B$. □

The next three results deal with sequences in $\mathcal{K}_T$. In the first, note that $\mathcal{K}_T$ is closed under arbitrary intersections.

Lemma A.1.3 *Given A and $(A_n)_n$ in $\mathcal{K}_T$, if $A \subseteq A_n \; \forall n$ and $A_n \to_{d_H} A$, then $d_H(\bigcap_n A_n, A) = 0$, i.e., $\bigcap_n A_n = A$.*

Proof We will show that $d_H(\bigcap_n A_n, A) \le \delta \; \forall \delta > 0$. For this purpose, take $\delta > 0$. Since $A \subseteq A_n \; \forall n$, the inclusion

$$A \subseteq (\textstyle\bigcap_n A_n)^\delta \tag{A.2}$$

follows by the definition of $(\cdot)^\delta$. Furthermore, since $A_n \to_{d_H} A$, $\exists\, n_0 \in \mathrm{N}$ s.t. $d_H(A_{n_0}, A) < \delta$ and thus,

$$\textstyle\bigcap_n A_n \subseteq A_{n_0} \subseteq A^\delta. \tag{A.3}$$

Combining (A.2) and (A.3), $d_H(\bigcap_n A_n, A) \le \delta$. Since $\delta > 0$ was arbitrary, the proof is complete. □

Lemma A.1.4 *Let A and $(A_n)_n$ in $\mathcal{K}_T$ be s.t. $A_n \to_{d_H} A$. Then, given any $t \in T$, t.f.a.e.:*

(i) *$t \in A$ and*

(ii) *$\exists\, (a_n)_n$ in T s.t. $a_n \in A_n \; \forall n$ and $a_n \xrightarrow{d} t$.*

Proof If we define $\epsilon_n = d_H(A_n, A) \; \forall n$, then $A \subseteq (A_n)^{\epsilon_n} \; \forall n$. Hence, if $t \in A$, then for every $n \in \mathrm{N}$, $\exists\, a_n \in A_n$ s.t. $d(t, a_n) < \epsilon_n$. Since $\epsilon_n \to 0$, (ii) follows.

Conversely, assume (ii) holds. Since A is d-closed in T, it is sufficient to show that for any $\epsilon > 0$,

$$\exists\, a \in A \text{ s.t. } d(a,t) < \epsilon.$$

For this purpose, take $\epsilon > 0$. Since $a_n \to_d t$, $\exists\, n_1 \in \mathrm{N}$ s.t.

$$d(a_n, t) < \epsilon/2, \quad (\forall n \geq n_1).$$

Since $A_n \to_{d_H} A$, $\exists\, n_2 \in \mathrm{N}$ s.t.

$$A_n \subseteq A^{\epsilon/2}, \quad (\forall n \geq n_2).$$

Therefore, if we take $n_0 = n_1 \vee n_2$,

$$a_{n_0} \in A_{n_0} \subseteq A^{\epsilon/2} \implies \exists\, a \in A \text{ s.t. } d(a, a_{n_0}) < \epsilon/2$$

which implies $d(a,t) \leq d(a, a_{n_0}) + d(a_{n_0}, t) < 2\,(\epsilon/2)$.

Since $\epsilon > 0$ was arbitrarily chosen, this establishes $t \in \overline{A} = A$. □

Lemma A.1.5 *Let $(A_n)_n$ be a sequence in $\mathcal{K}_T$ which is decreasing w.r.t. $\subseteq$. If $\bigcap_n A_n = \phi$, then $\exists\, N \in \mathrm{N}$ s.t. $A_n = \phi\ \forall n \geq N$.*

Proof (T,d), being compact, possesses the finite intersection property. Thus, $\bigcap_n A_n = \phi$ implies $\exists\, n_1 < \cdots < n_k \in \mathrm{N}$ s.t. $\bigcap_{i=1}^k A_{n_i} = \phi$.

Since $(A_n)_n$ is decreasing, $A_{n_k} = \bigcap_{i=1}^k A_{n_i}$. Therefore, the lemma follows by selecting $N = n_k$. □

Applying the sequential characterization of compactness in metric spaces,

Lemma A.1.6 *Given subspaces, Z and Y of T s.t. $Z \subseteq Y$, Z is compact (w.r.t. the subspace topology) in Y if and only if Z is compact in T.*

A.2 Some Additional Properties for Indexing Collections

In Section 2.2, four frequently used properties of indexing collections were developed. In this section, we will develop additional properties for indexing collections, most of which concern sequences of the form $(g_n(A))_n$

$(A \in \mathcal{A})$ (see Definition 2.2.2). Since these additional properties are only used occasionally in the thesis, they are better suited to an appendix than to Section 2.2.

Throughout this section, $\mathcal{A}$ **denotes an indexing collection on a compact metric space** (T, d). Recall that the elements of $\mathcal{A}$ are non-empty d-closed subsets of T. Our first result is an integral part of the proof of Theorem 2.2.10.

Lemma A.2.1 *For every $n \in \mathbb{N}$, $\exists \delta_n > 0$ s.t. $A^{\delta_n} \subseteq g_n(A)$ $\forall A \in \mathcal{A}_n$. Moreover, $\delta_n \downarrow 0$ as $n \to \infty$.*

Proof Fix $n \in \mathbb{N}$ and take $A \in \mathcal{A}_n$. By Lemma A.1.2,

$$A \subseteq [g_n(A)]^\circ \implies \exists \delta'_A > 0 \text{ s.t. } A^{\delta'_A} \subseteq g_n(A).$$

Define $\delta'_n = \inf\{\delta'_A : A \in \mathcal{A}_n\}$. Since $\mathcal{A}_n$ is a finite subcollection of $\mathcal{A}$, $\delta'_n > 0$. Moreover, by Proposition 2.2.1 (iv),

$$A^{\delta'_n} \subseteq A^{\delta'_A} \subseteq g_n(A), \quad (\forall A \in \mathcal{A}_n).$$

Now, define $(\delta_n)_n$ recursively as follows:

$$\delta_1 = \min\{\delta'_1, 1\}, \quad \delta_{n+1} = \min\{\delta_n, \delta'_n, \tfrac{1}{n}\} \ (n = 1, 2, \cdots).$$

Clearly, $\delta_n \downarrow 0$ and $\delta_n \leq \delta'_n$ $\forall n$. Thus, given any n, Proposition 2.2.1 (iv) implies

$$A^{\delta_n} \subseteq A^{\delta'_n} \subseteq g_n(A), \quad (\forall A \in \mathcal{A}_n)$$

which establishes the lemma. □

The following result appeared in a preprint of [24]. As commented therein, part (b) is not valid when (T, d) is not compact.

Lemma A.2.2 *Given $(A_n)_n$ in $\mathcal{A}$,*

(a) *if $(A_n)_n$ is decreasing w.r.t. $\subseteq$, then $A_n \xrightarrow{d_H} \bigcap_n A_n$ and*

(b) *if $(A_n)_n$ is increasing w.r.t. $\subseteq$, then $A_n \xrightarrow{d_H} cl(\bigcup_n A_n)$*

where the closure in (b) is taken w.r.t. d. Thus, since $\mathcal{A}$ is d_H-closed in $\mathcal{K}_T$ (see Theorem 2.2.10), (b) implies $cl(\bigcup_n A_n) \in \mathcal{A}$ for every increasing sequence, $(A_n)_n$ in $\mathcal{A}$.

Given a sequence $(x_n)_n$ in $\mathbf{R}$ s.t. $x_n \to x$ for some $x \in \mathbf{R}$, recall that $x < y$ implies $x_n \leq y$ for all large n. Even though $(\mathcal{A}, \subseteq)$ is not necessarily totally ordered, there is an analogous result for convergence in $(\mathcal{A}, d_H)$.

Lemma A.2.3 *Let A and $(A_n)_n$ in $\mathcal{A}$ be s.t. $A_n \to_{d_H} A$. If $B \in \mathcal{A}$ is s.t. $A \subseteq B^\circ$, then $\exists\, N \in \mathbf{N}$ s.t. $A_n \subseteq B$ $\forall n \geq N$.*

Proof By Lemma A.1.2, $\exists\, \beta > 0$ s.t. $A^\beta \subseteq B$. Furthermore, since $A_n \to_{d_H} A$, $\exists\, N \in \mathbf{N}$ s.t. $d_H(A_n, A) < \beta$ $\forall n \geq N$. This implies $A_n \subseteq A^\beta$ $\forall n \geq N$, completing the proof. □

Since, for any $A \in \mathcal{A}$, $g_n(A) \to_{d_H} A$, we have the following consequence of Lemma A.2.3.

Corollary A.2.4 *If $A, B \in \mathcal{A}$ are s.t. $A \subseteq B^\circ$, then $\exists\, N \in \mathbf{N}$ s.t. $g_n(A) \subseteq B$ $\forall n \geq N$.*

Note that the approximators, $g_m : \mathcal{A} \to \mathcal{A}_m$ $(m \in \mathbf{N})$ need not be d_H-continuous. For example, the sequence $(A_n)_n$ in $\mathcal{I}_1$ (see Example 2.2.6) with $A_n = [0, \frac{1}{2} + (-1)^n \frac{1}{n}]$ $\forall n$ is s.t. $A_n \to_{d_H} [0, \frac{1}{2}]$ whereas

$$g_1(A_n) = \begin{cases} [0,1] & \text{, if } n \text{ is even} \\ [0, \frac{1}{2}] & \text{, if } n \text{ is odd .} \end{cases}$$

On the other hand, each g_m is outer-continuous in the following sense.

Lemma A.2.5 *Let $m \in \mathbf{N}$ be given. If $A_n \searrow A$ in $\mathcal{A}$, then $\exists\, N \in \mathbf{N}$ s.t. $g_m(A_n) = g_m(A)$ $\forall n \geq N$.*

Proof Take $A_n \searrow A$ in $\mathcal{A}$ and assume to the contrary that there exists a subsequence $(k_n)_n$ s.t.

$$g_m(A_{k_n}) \neq g_m(A)\,, \quad (\forall n). \tag{A.4}$$

Since $\{g_m(A_{k_n}) : n \in \mathbf{N}\} \subseteq \mathcal{A}_m$ and $\mathcal{A}_m$ is finite, there exists a further subsequence, $(r_{k_n})_n$ s.t. $g_m(A_{r_{k_n}}) = A_0$ $\forall n$ for some $A_0 \in \mathcal{A}_m$.

Since $A_n \searrow A$, Lemma A.1.3 implies $\bigcap_n A_{r_{k_n}} = A$. Thus, since g_m preserves countable intersections,

$$A_0 = \textstyle\bigcap_n g_m(A_{r_{k_n}}) = g_m(\bigcap_n A_{r_{k_n}}) = g_m(A).$$

But this contradicts (A.4). Therefore, no subsequence $(k_n)_n$ can satisfy condition (A.4), i.e., $g_m(A_n) = g_m(A)$ for all large n. □

As a consequence of Lemma A.2.5,

Corollary A.2.6 *Given $A \in \mathcal{A}$ and $m \in \mathrm{N}$, $g_m(g_n(A)) = g_m(A)$ for all large n.*

The following technical result was needed in Section 3.7.

Lemma A.2.7 *Take $n \in \mathrm{N}$ and $A \in \mathcal{A}_n$. If $\{A_1, \cdots, A_k\}$ is any subcollection of $\mathcal{A}_n$ satisfying conditions* (i), (ii) *and* (iii) *of Assumption 3.7.1, then*

$$d_H\left(\bigcap_{i\in I} A_i\,,\,\bigcap_{j\in J} A_j\right) \;\leq\; 2\cdot\epsilon_n \quad (\forall\, I, J \subseteq \{1,\cdots,k\})$$

where ϵ_n is the number defined in (2.2) *and, by convention, $\bigcap_{i\in\phi} A_i = A$.*

Proof The bulk of the proof is contained in the following two results.

<u>Claim I:</u> $A \subseteq g_n(A_i)\ \forall\, 1 \leq i \leq k$.

<u>Proof:</u> If $\exists\, 1 \leq i_0 \leq k$ s.t. $A \not\subseteq g_n(A_{i_0})$, then $g_n(A_{i_0}) \cap A \subset A$. Furthermore, by condition (2) (iii′) of Definition 2.2.2,

$$A_{i_0} \subset A \implies A_{i_0} \subset g_n(A_{i_0}) \cap A. \tag{A.5}$$

Now, let $A' = g_n(A_{i_0}) \cap A \in \mathcal{A}_n$. Since $A' \subset A$, condition (iii) of Assumption 3.7.1 implies $\exists\, 1 \leq j_0 \leq k$ s.t. $A' \subseteq A_{j_0}$. By (A.5), this implies $A_{i_0} \subset A_{j_0}$, resulting in a contradiction to condition (i) of Assumption 3.7.1. Therefore, no such i_0 exists. Ω

<u>Claim II:</u> $d_H(A\,, \cap_{i\in I} A_i) \leq \epsilon_n\ \forall I \subseteq \{1,\cdots,k\}$.

<u>Proof:</u> Take $I \subseteq \{1,\cdots,k\}$. Since g_n preserves finite intersections,

$$\begin{aligned} A \;&\subseteq\; g_n\left(\bigcap_{i\in I} A_i\right) \quad \text{(by Claim I)} \\ &\subseteq\; \left(\bigcap_{i\in I} A_i\right)^{\epsilon_n} \quad \text{(by (2.2))}. \end{aligned}$$

Therefore, Claim II follows by the inclusion $\bigcap_{i\in I} A_i \subseteq A$.

To establish the present lemma, take any pair $I, J \subseteq \{1, \cdots, k\}$, apply the triangle inequality for d_H and then apply Claim II twice. □

We close this section with a consequence of Definition 2.2.2 (2)(i').

Lemma A.2.8 *Given sets $A, A_1, \cdots, A_k \in \mathcal{A}$, $t \in A \setminus \bigcup_{i=1}^{k} A_i$ if and only if $\exists N \in \mathbf{N}$ s.t. $t \in g_n(A) \setminus \bigcup_{i=1}^{k} g_n(A_i)$ $\forall n \geq N$.*

Proof If $t \in A \setminus \bigcup_{i=1}^{k} A_i$, then $t \in g_n(A)$ $\forall n$. Assume $\exists$ a subsequence $(n')_n$ s.t. $t \in \bigcup_{i=1}^{k} g_{n'}(A_i)$ $\forall n$. Then, $\exists\, 1 \leq i_0 \leq k$ and a further subsequence $(n'')_n$ s.t. $t \in g_{n''}(A_{i_0})$ $\forall n$. But $A_{i_0} = \bigcap_n g_n(A_{i_0}) = \bigcap_n g_{n''}(A_{i_0})$ implies $t \in A_{i_0}$, a contradiction. Therefore, no such subsequence $(n')_n$ exists.

Conversely, suppose $\exists N \in \mathbf{N}$ s.t. $t \in g_n(A) \setminus \bigcup_{i=1}^{k} g_n(B_i)$ $\forall n \geq N$. Then, $t \in \bigcap_n g_n(A) = \bigcap_{n \geq N} g_n(A) = A$ and $t \notin \bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{k} g_N(A_i)$ □

A.3 On Finite Sub-Semilattices of Indexing Collections

In this section, we present technical results on f.s.s.l. which are used frequently throughout Chapters 3 and 4. See Section 3.2 for the necessary terminology.

Let $\mathcal{A}$ be an indexing collection on a compact metric space (T, d) and let $(\mathcal{A}_n)_n$ be the corresponding sequence of f.s.s.l. of $\mathcal{A}$ given via Definition 2.2.4. Recall that $(\mathcal{A}_n)_n$ is increasing in n w.r.t. $\subseteq$. Given any f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$, recall the associated collections,

- $\mathcal{A}'(u) = \{$ all finite unions in $\mathcal{A}'\}$,
- $\mathcal{C}' = \{A \setminus B : A \in \mathcal{A}',\ B \in \mathcal{A}'(u)\}$ and
- $\mathcal{N}' = \{C_{A'} : A' \in \mathcal{A}'\} \setminus \{\phi\}$ where $C_{A'} = A' \setminus \bigcup_{A \in \mathcal{A}',\, A' \not\subseteq A} A$.

We write $\mathcal{A}_n(u)$, $\mathcal{C}_n$ and $\mathcal{N}_n$ respectively when $\mathcal{A}' = \mathcal{A}_n$ $(n \in \mathbf{N})$.

As mentioned in Section 3.2, we will only consider those f.s.s.l. of $\mathcal{A}$ which contain both ϕ' and T. Under this condition, $\mathcal{N}'$ is a partition of T (see Lemma 3.2.10). Now, for our first result,

Proposition A.3.1 *Let $\mathcal{A}'$ and $\mathcal{A}''$ be f.s.s.l. of $\mathcal{A}$ satisfying $\mathcal{A}' \subseteq \mathcal{A}''$. If $\mathcal{N}'$ and $\mathcal{N}''$ denote the collections of non-empty left-neighborhoods generated by $\mathcal{A}'$ and $\mathcal{A}''$ respectively, then*

(a) *given any* $D \in \mathcal{N}''$, $\exists$ *a unique* $C \in \mathcal{N}'$ *s.t.* $D \subseteq C$,

(b) $(\{D \in \mathcal{N}'' : D \subseteq C\})_{C \in \mathcal{N}'}$ *is a partition of* $\mathcal{N}''$ *and*

(c) *given any* $C \in \mathcal{N}'$, $\bigcup_{D \in \mathcal{N}'',\, D \subseteq C} D \;=\; C$.

Proof To show (a), take $D = C_{B''} \in \mathcal{N}''$ $(B'' \in \mathcal{A}'')$ and define

$$B' \;=\; \bigcap_{\substack{A \in \mathcal{A}' \\ B'' \subseteq A}} A. \tag{A.6}$$

Since $\mathcal{A}'$ is a f.s.s.l., $B' \in \mathcal{A}'$. (Note that $B'' \subseteq B'$.) Given any $A \in \mathcal{A}'$, if $B' \not\subseteq A$, then, by (A.6), $B'' \not\subseteq A$. This implies

$$\bigcup_{\substack{A \in \mathcal{A}' \\ B' \not\subseteq A}} A \;\subseteq\; \bigcup_{\substack{A \in \mathcal{A}' \\ B'' \not\subseteq A}} A.$$

Furthermore, since $\mathcal{A}' \subseteq \mathcal{A}''$, we also have the inclusion

$$\bigcup_{\substack{A \in \mathcal{A}' \\ B'' \not\subseteq A}} A \;\subseteq\; \bigcup_{\substack{A \in \mathcal{A}'' \\ B'' \not\subseteq A}} A.$$

Now, define $C = C_{B'} \in \mathcal{N}'$. Since $B'' \subseteq B'$, combining the above inclusions yields

$$D \;=\; B'' \setminus \bigcup_{\substack{A \in \mathcal{A}'' \\ B'' \not\subseteq A}} A \;\subseteq\; B' \setminus \bigcup_{\substack{A \in \mathcal{A}' \\ B' \not\subseteq A}} A \;=\; C,$$

establishing (a). The uniqueness of C follows from the disjointness of distinct left-neighborhoods (see Lemma 3.2.10).

Part (b) is a direct consequence of part (a). To show (c), take $C \in \mathcal{N}'$ and assume $\bigcup_{D \in \mathcal{N}'',\, D \subseteq C} D \;\subset\; C$. Since $\mathcal{N}''$ is a partition of T, this assumption implies

$$\exists\, D'' \in \mathcal{N}'' \text{ s.t. } D'' \not\subseteq C \text{ and } D'' \cap C \neq \phi. \tag{A.7}$$

But by part (a), $\exists C' \in \mathcal{N}'$ s.t. $D'' \subseteq C'$. Since $\mathcal{N}'$ is a partition of T, this implies $C \cap C' = \phi$, contradicting (A.7). Therefore, it must be that $\bigcup_{D \in \mathcal{N}'',\, D \subseteq C} D = C$. □

All remaining results in this section are stated for the special case of $\mathcal{A}' = \mathcal{A}_n$ $(n \in \mathrm{N})$ since this is the most frequently encounter situation in the thesis. By examining their proofs, it is clear that each of these results remains valid when $\mathcal{A}_n$ is replaced by a general f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$. The first such result follows automatically from Proposition A.3.1 (c) when applied to the two f.s.s.l. $\mathcal{A}_n \subseteq \mathcal{A}_m$ $(n \leq m)$.

Corollary A.3.2 *Let $x : \mathcal{C}(u) \to \mathbf{R}$ be a finitely-additive set-function on the algebra $\mathcal{C}(u)$. If $n \leq m$ and $C \in \mathcal{N}_n$, then*

$$x(C) = \sum_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} x(D).$$

Next, take $n \in \mathbf{N}$ and $C = A \setminus \bigcup_{i=1}^{k} A_i \neq \phi$ in $\mathcal{C}_n$. If we define $\mathcal{A}'$ to be the f.s.s.l. generated by $\{A, A_1, \cdots, A_k\}$, then it is clear that C is the left-neighborhood of A in $\mathcal{A}'$ and hence $C \in \mathcal{N}'$. But $\mathcal{A}' \subseteq \mathcal{A}_n$. Therefore, by Proposition A.3.1 (c), we have established the following result.

Lemma A.3.3 *Given $C \in \mathcal{C}_n$ $(n \in \mathbf{N})$, $C = \bigcup_{D \in \mathcal{N}_n, D \subseteq C} D$.*

The next result deals with a special case from Lemma A.3.3.

Lemma A.3.4 *Given $n \in \mathbf{N}$ and $A', A'' \in \mathcal{A}_n$ s.t. $A' \subseteq A''$,*

$$A'' \setminus A' = \bigcup \{D \in \mathcal{N}_n : D \subseteq A'' \text{ and } D \not\subseteq A'\}.$$

Proof If $A' = A''$, the result is trivial. If $A' \subset A''$, then by Lemma A.3.3,

$$A'' \setminus A' = \bigcup \{D \in \mathcal{N}_n : D \subseteq A'' \setminus A'\}. \tag{A.8}$$

Now, take $D \in \mathcal{N}_n$. If $D \subseteq A'' \setminus A'$, then $D \subseteq A''$ and $D \not\subseteq A'$. Thus, (A.8) implies

$$A'' \setminus A' \subseteq \bigcup \{D \in \mathcal{N}_n : D \subseteq A'' \text{ and } D \not\subseteq A'\}.$$

For the opposite inclusion, take $D \in \mathcal{N}_n$ s.t. $D \subseteq A''$ and $D \not\subseteq A'$. By the definition of left-neighborhoods, it must be that $D \cap A' = \phi$, i.e., $D \subseteq (A')^c$. This implies $D \subseteq A'' \setminus A'$ which completes the proof. □

Lemma A.3.5 *Given $n \leq m$ in $\mathbf{N}$ and $A \in \mathcal{A}_n$,*

$$\bigcup_{\substack{D \in \mathcal{N}_m \\ D \subseteq A}} D = \bigcup_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \bigcup_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} D.$$

Proof Recall that $\mathcal{N}_n \subseteq \mathcal{N}_m$. It is sufficient to show

$$\bigcup_{\substack{D \in \mathcal{N}_m \\ D \subseteq A}} D \subseteq \bigcup_{\substack{C \in \mathcal{N}_n \\ C \subseteq A}} \bigcup_{\substack{D \in \mathcal{N}_m \\ D \subseteq C}} D$$

since the opposite inclusion is trivial. For this purpose, take $D \in \mathcal{N}_m$ s.t. $D \subseteq A$. Since $\mathcal{A}_n \subseteq \mathcal{A}_m$, Proposition A.3.1 (a) implies $\exists\, C \in \mathcal{N}_n$ s.t. $D \subseteq C$. The above inclusion will follow if we can show $C \subseteq A$.

Since $C \in \mathcal{N}_n$, $\exists\, A' \in \mathcal{A}_n$ s.t. $C = A' \setminus \bigcup_{A'' \in \mathcal{A}_n, A' \not\subseteq A''} A''$. If $C \not\subseteq A$, then $A' \not\subseteq A$ which implies $C \cap A = \phi$. Since $D \subseteq C \cap A$, this implies $D = \phi$ which contradicts the definition of $\mathcal{N}_m$. Therefore, it must be that $C \subseteq A$. □

The proof of the next result follows immediately from the definition of left-neighborhoods.

Lemma A.3.6 *Let $n \in \mathbf{N}$ be given. If $C \in \mathcal{N}_n$ is the left-neighborhood of $A \in \mathcal{A}_n$, then given any $B \in \mathcal{A}_n$,*

$$C \subseteq B \iff A \subseteq B.$$

Our final result is an identity which will be needed in Chapter 3 to obtain the Doob-Meyer decomposition theorem for set-indexed strong submartingales.

Lemma A.3.7 *Let $\overline{f} : \mathcal{N}_n \to \mathbf{R}$ be any function and define $f : \mathcal{A}_n \to \mathbf{R}$ by*

$$f(A) = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A}} \overline{f}(D), \quad (\forall\, A \in \mathcal{A}_n). \tag{A.9}$$

Then, given $C = A \setminus \bigcup_{i=1}^{k} A_i$ with $A, A_1, \cdots, A_k \in \mathcal{A}_n$,

$$\sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot f(A \cap \textstyle\bigcap_{i\in I} A_i) = \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C}} \overline{f}(D) \tag{A.10}$$

(Recall the adopted convention, $\bigcap_{i\in\phi} A_i = T$).

Proof To each $D \in \mathcal{N}_n$, we associate the sets

$$p(D) := \{\, I \subseteq \{1, \cdots k\} : D \subseteq A \cap \textstyle\bigcap_{i\in I} A_i \text{ and } |I| \text{ is even} \,\}$$

and

$$n(D) := \{ I \subseteq \{1, \cdots k\} : D \subseteq A \cap \bigcap_{i \in I} A_i \text{ and } |I| \text{ is odd} \}.$$

By (A.9),

$$\sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot f(A \cap \bigcap_{i\in I} A_i) = \sum_{I \subseteq \{1,\cdots,k\}} \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq A \cap \bigcap_{i \in I} A_i}} (-1)^{|I|} \cdot \overline{f}(D).$$

Therefore, (A.10) will clearly follow if we can show

$$|p(D)| - |n(D)| = \begin{cases} 1 & \text{, if } D \subseteq C \\ 0 & \text{, otherwise} \end{cases} \tag{A.11}$$

for every $D \in \mathcal{N}_n$.

For this purpose, take $D \in \mathcal{N}_n$ and assume $D \subseteq C$. Since $D \neq \phi$ and $C \cap A_i = \phi \ \forall 1 \leq 1 \leq k$, it is clear that

$$D \subseteq A \cap \bigcap_{i \in I} A_i \iff I = \phi$$

which is to say that $p(D) = \{\phi\}$ and $n(D) = \phi$, i.e, $|p(C)| - |n(C)| = 1$.

Next, assume $D \not\subseteq C$. There are two cases. For the first case, assume $D \not\subseteq A_i \ \forall 1 \leq i \leq k$. This clearly implies

$$D \not\subseteq A \cap \bigcap_{i \in I} A_i$$

for all non-empty subsets, I of $\{1, \cdots, k\}$. Furthermore, $D \not\subseteq A$. (If $D \subseteq A$, then $D \subseteq C$, a contradiction.) In total, $p(D) = n(D) = \phi$, i.e., $|p(D)| = |n(D)| = 0$.

For the remaining case, assume $\exists 1 \leq i_0 \leq k$ s.t. $D \subseteq A_{i_0}$. Given any $I \subseteq \{1, \cdots, k\}$, define $J \subseteq \{1, \cdots, k\}$ as follows:

$$J := \begin{cases} I \setminus \{i_0\} & \text{, if } i_0 \in I \\ I \cup \{i_0\} & \text{, if } i_0 \notin I. \end{cases}$$

By basic set theory, given any $I \subseteq \{1, \cdots, k\}$,

$$D \subseteq A \cap \bigcap_{i \in I} A_i \implies D \subseteq A \cap \bigcap_{j \in J} A_j.$$

From this implication, we can define a map $\eta : p(D) \to n(D)$ s.t. $\eta(I) = J$ $\forall I \in p(D)$. Clearly, η is injective and therefore $|p(D)| \leq |n(D)|$. By an identical argument, $|n(D)| \leq |p(D)|$. This establishes (A.11), completing the proof of Lemma A.3.7. □

A.4 Extending V in Theorem 3.4.11

Let $X = (X_A)_{A\in\mathcal{A}}$ be an L_1-right-continuous strong submartingale of class $(D')^*$ and let $V = (V_A)_{A\in\mathcal{A}}$ be the associated process defined in (3.42) of Theorem 3.4.11. Even though X, being a strong submartingale, possesses a unique finitely additive extension to $\mathcal{C}(u)$, it must be shown that the derived process V possesses increments (in the sense of Definition 3.2.32) at every $C \in \mathcal{C}$. Recall that $\mathcal{A}$ does not necessarily possess the shape property so that Proposition 3.2.35 cannot be applied. The present section, which complements Section 3.4, is devoted to showing that this V does indeed possess increments at every $C \in \mathcal{C}$. A condition under which V possesses a unique finitely additive extention to $\mathcal{C}(u)$ is given in Comment A.4.7.

First, take $n \in \mathrm{N}$ and recall the finite collection $\{V_A^{(n)} : A \in \mathcal{A}_n\}$ defined in (3.43). In particular, note that

$$V_A^{(n)} = \sum_{\substack{D\in\mathcal{N}_n \\ D\subseteq A}} E[X_D \,|\, \mathcal{G}_D^*], \quad (\forall A \in \mathcal{A}_n).$$

We begin with the following technical result. Recall that $\mathcal{A}_n$ is closed under finite intersection.

Lemma A.4.1 *Given any $C \in \mathcal{C}_n$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C with $A, A_i \in \mathcal{A}_n$,*

$$\sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot V^{(n)}(A \cap \textstyle\bigcap_{i\in I} A_i) = \sum_{\substack{D\in\mathcal{N}_n \\ D\subseteq C}} E[X_D \,|\, \mathcal{G}_D^*]. \tag{A.12}$$

Proof Let $\omega \in \Omega$ be given. If we define the set-function $\overline{f} : \mathcal{N}_n \to \mathbf{R}$ by

$$\overline{f}(D) = E[X_D \,|\, \mathcal{G}_D^*](\omega), \quad (\forall D \in \mathcal{N}_n)$$

and $f : \mathcal{A}_n \to \mathbf{R}$ by $f(A) = V_A^{(n)}(\omega)$ $\forall A \in \mathcal{A}_n$, then (A.12) follows from (A.10) in Lemma A.3.7. □

The following result is the first step toward extending V to $\mathcal{C}$.

Lemma A.4.2 *Let $C \in \mathcal{C}_n$ be given. If $A \setminus \bigcup_{i=1}^k A_i$ and $A' \setminus \bigcup_{j=1}^{k'} A'_j$ are any two representations of C with $A, A', A_i, A'_j \in \mathcal{A}_n$, then*

$$\sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot V^{(n)}(A \cap \textstyle\bigcap_{i\in I} A_i) = \sum_{J\subseteq\{1,\cdots,k'\}} (-1)^{|J|} \cdot V^{(n)}(A' \cap \textstyle\bigcap_{j\in J} A'_j). \tag{A.13}$$

Moreover, given disjoint unions $\bigcup_{i=1}^{k} C_i = \bigcup_{j=1}^{k'} C'_j$ *in* $\mathcal{C}_n(u)$*,*

$$\sum_{i=1}^{k} V_{C_i}^{(n)} = \sum_{j=1}^{k'} V_{C'_j}^{(n)} \tag{A.14}$$

where, given any $C \in \mathcal{C}_n$*,* $V_C^{(n)}$ *denotes the random variable uniquely determined by* (A.13). *(Recall by Lemma 3.2.3 that* $\mathcal{C}_n$ *is a semi-algebra.)*

Proof Since the right-hand side of (A.12) is independent of the representation of C, (A.13) follows. Therefore, given any $C = A \setminus \bigcup_{i=1}^{k} A_i \in \mathcal{C}_n$ $(A, A_i \in \mathcal{A}_n)$, we can define the random variable,

$$V_C^{(n)} := \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V^{(n)}(A \cap \bigcap_{i \in I} A_i). \tag{A.15}$$

By Lemma 3.3.8, (A.14) will follow if we can show $C \mapsto V_C^{(n)}(\omega)$ $(C \in \mathcal{C}_n)$ is finitely additive on the semi-algebra $\mathcal{C}_n$. For this purpose, take $C = \bigcup_{i=1}^{k} C_i \in \mathcal{C}_n$ with $C_i \in \mathcal{C}_n \, \forall 1 \leq i \leq k$. Since any set in $\mathcal{C}_n$ is a disjoint union of sets in $\mathcal{N}_n$ (see Lemma A.3.3),

$$\sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C}} E[X_D \,|\, \mathcal{G}_D^*] = \sum_{i=1}^{k} \sum_{\substack{D \in \mathcal{N}_n \\ D \subseteq C_i}} E[X_D \,|\, \mathcal{G}_D^*].$$

Therefore, by (A.12) and (A.15),

$$V_C^{(n)} = \sum_{i=1}^{k} V_{C_i}^{(n)},$$

establishing the finite additivity of $C \mapsto V_C^{(n)}(\omega)$ $(C \in \mathcal{C}_n)$. □

Next, take any $m \in \mathbf{N}$ and let $\{\overline{V}_A : A \in \mathcal{A}_m\}$ be the collection of random variables defined in Section 3.4. Recall that

$$\overline{V}_A = \lim_{n \to \infty, n \geq m} V_A^{(n)} \text{ weakly in } L_1, \quad (\forall A \in \mathcal{A}_m). \tag{A.16}$$

For each $C \in \mathcal{C}_m$, select a representation $A(C) \setminus \bigcup_{i=1}^{k(C)} A(C,i)$ of C in $\mathcal{A}_m$. Using these fixed representations, $\{\overline{V}_A : A \in \mathcal{A}_m\}$ can be extended to a collection $\{\overline{V}_C : C \in \mathcal{C}_m\}$ of random variables by defining

$$\overline{V}_C := \sum_{I \subseteq \{1,\cdots,k(C)\}} (-1)^{|I|} \cdot \overline{V}[A(C) \cap \bigcap_{i \in I} A(C,i)] \tag{A.17}$$

for each $C \in \mathcal{C}_m$. This leads to the following result.

Lemma A.4.3 *Given any $C \in \mathcal{C}_m$ and any representation $A \setminus \bigcup_{i=1}^{k} A_i$ of C with $A, A_i \in \mathcal{A}_m$,*

$$\overline{V}_C = \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{V}(A \cap \textstyle\bigcap_{i \in I} A_i). \tag{A.18}$$

Moreover, given finite disjoint unions $\bigcup_{i=1}^{k} C_i = \bigcup_{j=1}^{k'} C'_j$ in $\mathcal{C}_m(u)$,

$$\textstyle\sum_{i=1}^{k} \overline{V}_{C_i} = \sum_{j=1}^{k'} \overline{V}_{C'_j}. \tag{A.19}$$

Proof Let C and $A \setminus \bigcup_{i=1}^{k} A_i$ be as defined in Lemma A.4.3. Then, given any $F \in \mathcal{F}$, (A.16) implies

$$\begin{aligned} E[\mathbf{1}_F \overline{V}_C] &= \lim E\left(\mathbf{1}_F \sum_{I \subseteq \{1,\cdots,k(C)\}} (-1)^{|I|} \cdot V^{(n)}[A(C) \cap \textstyle\bigcap_{i \in I} A(C,i)]\right) \\ &= \lim E\left(\mathbf{1}_F \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V^{(n)}(A \cap \textstyle\bigcap_{i \in I} A_i)\right) \quad \text{(by (A.13))} \\ &= E\left(\mathbf{1}_F \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{V}(A \cap \textstyle\bigcap_{i \in I} A_i)\right). \end{aligned}$$

where the above limits are taken as $n \to \infty$ with $n \geq m$. Since $F \in \mathcal{F}$ was arbitrarily chosen, (A.18) follows.

Once again, take any $F \in \mathcal{F}$. Applying the linearity of E, it is clear the relation in (A.16) can be extended to each $\overline{V}_C$ ($C \in \mathcal{C}_m$). Therefore,

$$\begin{aligned} E\left[\mathbf{1}_F \cdot \textstyle\sum_{i=1}^{k} \overline{V}_{C_i}\right] &= \lim E\left[\mathbf{1}_F \cdot \textstyle\sum_{i=1}^{k} V_{C_i}^{(n)}\right] \\ &= \lim E\left[\mathbf{1}_F \cdot \textstyle\sum_{j=1}^{k'} V_{C'_j}^{(n)}\right] \quad \text{(by (A.14))} \\ &= E\left[\mathbf{1}_F \cdot \textstyle\sum_{j=1}^{k'} \overline{V}_{C'_j}\right] \end{aligned}$$

where the above limits are taken as $n \to \infty$ with $n \geq m$. Since $F \in \mathcal{F}$ was arbitrary, this establishes (A.19). □

Since $\mathcal{A}_m$ is a finite subcollection of $\mathcal{A}$, each $C \in \mathcal{C}_m$ has at most a countable number of distinct representations in $\mathcal{A}_m$. Therefore, Lemma A.4.3 can be stengthened as follows.

Corollary A.4.4 *$\exists\,\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ s.t. for any $m \in \mathrm{N}$, any $C \in \mathcal{C}_m$ and any representation $A \setminus \bigcup_{i=1}^k A_i$ of C with $A, A_i \in \mathcal{A}_m$,*

$$\overline{V}_C(\omega) \;=\; \sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{V}(A \cap \textstyle\bigcap_{i\in I} A_i)(\omega)\,, \quad (\forall\,\omega \in \Omega').$$

Moreover, given any $m \in \mathrm{N}$ and any two disjoint unions $\bigcup_{i=1}^k C_i = \bigcup_{j=1}^{k'} C'_j$ with $C_i, C'_j \in \mathcal{C}_m\ \ \forall\, i, j$,

$$\textstyle\sum_{i=1}^{k} \overline{V}_{C_i}(\omega) \;=\; \sum_{j=1}^{k'} \overline{V}_{C'_j}(\omega)\,, \quad (\forall\,\omega \in \Omega').$$

Remark A.4.5 Recall the collections $\mathcal{A}^* = \bigcup_m \mathcal{A}_m$ and $\mathcal{C}^* = \bigcup_m \mathcal{C}_m$. By Corollary A.4.4, given any $C \in \mathcal{C}^*$, $\overline{V}_C$ can be defined independently of a fixed representation, say $C = A \setminus \bigcup_{i=1}^k A_i$ with $A, A_i \in \mathcal{A}^*$. This improves the original definition of $\overline{V}_C$ given in (A.17).

Finally, we extend the process $V = (V_A)_{A\in\mathcal{A}}$ defined in (3.54) of Section 3.4 to a $\mathcal{C}$-indexed process satisfing (3.12). Recall that

$$V_A \;=\; \lim_m \left(\mathbf{1}_{\Omega_0} \cdot \overline{V}_{g_m(A)}\right)\,, \quad (\forall\, A \in \mathcal{A}) \tag{A.20}$$

where $\Omega_0 \subseteq \Omega'$ is the event of full P-measure defined above (3.53). It is here that we require our indexing collection $\mathcal{A}$ to satisfy Assumption 3.4.10. For the sake of reference, we restate it: *if $A \setminus \bigcup_{i=1}^k A_i = A' \setminus \bigcup_{j=1}^{k'} A'_j$ in $\mathcal{C}$, then $\exists\, N \in \mathrm{N}$ s.t. $g_m(A) \setminus \bigcup_{i=1}^k g_m(A_i) = g_m(A) \setminus \bigcup_{i=1}^{k'} g_m(A'_j)\ \ \forall\, m \geq N$.*

Lemma A.4.6 *Given any $C \in \mathcal{C}$ and any two representations $A \setminus \bigcup_{i=1}^k A_i$ and $A' \setminus \bigcup_{j=1}^{k'} A'_j$ of C, the process $V = (V_A)_{A\in\mathcal{A}}$ defined in* (A.20) *satsifies*

$$\sum_{I \subseteq \{1,\cdots,k\}} (-1)^{|I|} \cdot V(A \cap \textstyle\bigcap_{i\in I} A_i)(\omega) \;=\; \displaystyle\sum_{J \subseteq \{1,\cdots,k'\}} (-1)^{|J|} \cdot V(A' \cap \textstyle\bigcap_{j\in J} A'_j)(\omega) \tag{A.21}$$

for all $\omega \in \Omega_0$. (Recall that the event Ω_0 does not depend on C.) Therefore, the process V has increments defined at every $C \in \mathcal{C}$.

Proof Let N be the number described in Assumption 3.4.10 and take any $m \geq N$. Applying Corollary A.4.4 to the set

$$g_m(A) \setminus \textstyle\bigcup_{i=1}^k g_m(A_i) \;=\; g_m(A') \setminus \bigcup_{j=1}^{k'} g_m(A'_j) \;\in\; \mathcal{C}_m,$$

we have that

$$\sum_{I\subseteq\{1,\cdots,k\}} (-1)^{|I|} \cdot \overline{V}\left[g_m(A) \cap \bigcap_{i\in I} g_m(A_i)\right](\omega)$$

equals

$$\sum_{J\subseteq\{1,\cdots,k'\}} (-1)^{|J|} \cdot \overline{V}\left[g_m(A') \cap \bigcap_{j\in J} g_m(A'_j)\right](\omega)$$

for each $\omega \in \Omega_0$. (Recall that $\Omega_0 \subseteq \Omega'$.) The proof now follows by (A.20), noting that each g_m preserves finite intersections. □

Comment A.4.7 Unlike the earlier results in this section, Lemma A.4.6 offers no finitely additive extension of V to $\mathcal{C}(u)$. Examining the above proofs, such an extension would clearly require that each g_m preserve disjoint unions in $\mathcal{C}$ for all large m. Specifically, given disjoint unions $\bigcup_{i=1}^{k} C_i = \bigcup_{i=1}^{r} D_i$ with $C_i, D_i \in \mathcal{C}$ and representations

$$A^i \setminus \bigcup_{j=1}^{k(i)} A_j^i \text{ of } C_i \;\; (1 \le i \le k) \quad \text{and} \quad B^i \setminus \bigcup_{j=1}^{r(i)} B_j^i \text{ of } D_i \;\; (1 \le i \le r),$$

we would require

$$\bigcup_{i=1}^{k} \left[g_m(A^i) \setminus \bigcup_{j=1}^{k(i)} g_m(A_j^i)\right] \;=\; \bigcup_{i=1}^{r} \left[g_m(B^i) \setminus \bigcup_{j=1}^{r(i)} g_m(B_j^i)\right]$$

for all sufficiently large $m \in \mathbf{N}$. This property is satisfied by $\mathcal{I}_k$ for any $k \in \mathbf{N}$ (see Example 2.2.6). As it is, finitely additive extensions of the process V to the algebra $\mathcal{C}(u)$ are not required anywhere in Chapter 3.

A.5 Concerning Classical Processes

This section contains assorted technical results concerning continuous parameter processes, the majority of which could not be located in any earlier publication. These results are used extensively in Section 4.4. The reader is referred to [30] for terminology.

Let $\mathbf{T} = [0, \infty)$ and let $(\Omega, \mathcal{F}, P, (\mathcal{H}_t), \mathbf{T})$ be a classical stochastic base. We begin by studying processes on $\mathbf{T}$ which, although not necessarily adapted, behave in some sense like closed martingales. These processes will possess a sufficiently nice form so as to allow for an extension of the classical martingale stopping theorems. The first result extends Doob's stopping theorem (see Theorem 3.3.9 in [30]) to such processes.

Lemma A.5.1 *Let $M = \{M_t : t \in [0,\infty]\}$ be a collection of random variables in L_1 satisfying*

(i) $E(M_t - M_s \,|\, \mathcal{H}_s) = 0 \;\forall s < t$ *in* $[0,\infty]$ *and*

(ii) $M_t = N_t - \lambda_t \;\forall t \in [0,\infty]$

where N and λ are non-negative, right-continuous, integrable processes with

- *N a closed submartingale w.r.t. $(\mathcal{H}_t)_{t\in\mathbf{T}}$ and*
- *λ increasing (but not necessarily adapted).*

If τ and σ are any two stopping times with $\tau \le \sigma$, then $M_\tau, M_\sigma \in L_1$ and $E[M_\sigma - M_\tau \,|\, \mathcal{H}_\tau] = 0$.

Proof We consider two cases. First, if $\exists r \in \mathbf{N}$ and $0 = s_0 < s_1 < \cdots < s_r = \infty$ s.t.

$$\tau[\Omega] \cup \sigma[\Omega] \;\subseteq\; \{s_0, s_1, \cdots, s_r\},$$

then it is clear that $M_\mathcal{T}, M_\mathcal{O} \in L_1$ and

$$M_\mathcal{O} - M_\mathcal{T} \;=\; \sum_{i=1}^{r} \mathbf{1}_{[\mathcal{T} < s_i \le \mathcal{O}]} \cdot (M_{s_i} - M_{s_{i-1}}) \;\text{ on } \Omega.$$

Since $\tau \le \sigma$, $\mathcal{H}_\mathcal{T} \subseteq \mathcal{H}_\mathcal{O}$ and thus, given any $F \in \mathcal{H}_\mathcal{T}$, $F \cap [\tau < s_i \le \sigma] \in \mathcal{H}_{s_{i-1}}$ $\forall 1 \le i \le r$. Therefore, by assumption (i) and the above identity,

$$\int_F (M_\mathcal{O} - M_\mathcal{T})\, dP \;=\; \sum_{i=1}^{r} E\left[\mathbf{1}_{[\mathcal{T} < s_i \le \mathcal{O}] \cap F} \cdot E(M_{s_i} - M_{s_{i-1}} | \mathcal{H}_{s_{i-1}})\right] \;=\; 0.$$

Since $F \in \mathcal{H}_\mathcal{T}$ was arbitrary, this implies $E[M_\mathcal{O} - M_\mathcal{T} \,|\, \mathcal{H}_\mathcal{T}] = 0$.

For the second case, assume that $\tau \le \sigma$ are any two stopping times. For each $n \in \mathbf{N}$, define

$$\tau_n \;:=\; \sum_{j=1}^{n\cdot 2^n} (j/2^n) \cdot \mathbf{1}_{B_j} \;+\; \infty \cdot \mathbf{1}_{B_\infty}$$

where $B_j = [\frac{j-1}{2^n} \le \tau < \frac{j}{2^n}]$ $\forall 1 \le j \le n\,2^n$ and $B_\infty = [\tau \ge n]$. It is straightforward to show that each τ_n is a stopping time w.r.t. $(\mathcal{H}_t)_{t\in\mathbf{T}}$ and $\tau_n \downarrow \tau$ on Ω. Repeating this procedure, we can also construct such a sequence $(\sigma_n)_n$ for σ. Note that $\tau_n \le \sigma_n$ $\forall n$.

Now, take $F \in \mathcal{H}_{\mathcal{T}}$. Since $\tau \leq \tau_n \ \forall n$, $F \in \mathcal{H}_{\mathcal{T}_n} \ \forall n$. Therefore, since τ_n and σ_n assume only finitely many values in $[0, \infty]$, we can apply the earlier case to obtain

$$\int_F M_{\sigma_n} \, dP \;=\; \int_F M_{\mathcal{T}_n} \, dP \,, \quad (\forall n). \tag{A.22}$$

Lemma A.5.1 will follow from (A.22) if we can show that $M_{\mathcal{T}_n} \to M_{\mathcal{T}}$ in L_1 and $M_{\sigma_n} \to M_{\sigma}$ in L_1.

First, since a.e. sample path of λ is right-continuous and increasing, $\tau_n \downarrow \tau$ implies $\lambda_{\mathcal{T}_n} \downarrow \lambda_{\mathcal{T}}$ a.s. Therefore, since $\lambda_{\mathcal{T}} \geq 0$ a.s., the monotone convergence theorem implies

$$E|\lambda_{\mathcal{T}_n} - \lambda_{\mathcal{T}}| \;=\; E(\lambda_{\mathcal{T}_n} - \lambda_{\mathcal{T}}) \;\longrightarrow\; 0.$$

Secondly, since N is a closed submartingale, Doob's stopping theorem implies that $(N_{\mathcal{T}_n})_n$ is a reverse submartingale w.r.t. $(\mathcal{H}_{\mathcal{T}_n})_n$. Since

$$E(N_{\mathcal{T}_n}) \;=\; E[E(N_0 \,|\, \mathcal{H}_{\mathcal{T}_n})] \;\geq\; E(N_0)\,, \quad (\forall n),$$

Corollary 2.10.2 in [30] implies

$$\exists\, N' \in L_1 \text{ s.t. } N_{\mathcal{T}_n} \to N' \text{ a.s. and in } L_1. \tag{A.23}$$

But a.e. path of N is right-continuous. Therefore, $\tau_n \downarrow \tau$ implies $N_{\mathcal{T}_n} \to N_{\mathcal{T}}$ a.s. which, by (A.23), implies $N_{\mathcal{T}_n} \to N_{\mathcal{T}}$ in L_1.

By assumption (ii), we have thus shown that $M_{\mathcal{T}_n} \to M_{\mathcal{T}}$ in L_1. The argument for $M_{\sigma_n} \to M_{\sigma}$ in L_1 is identical. As mentioned below (A.22), this completes the proof of Lemma A.5.1. □

Remark A.5.2 (a) Take $a > 0$. If we replace τ by $\tau \wedge a$ and σ by $\sigma \wedge a$ we obtain, as a simple corollary, Lemma A.5.1 for the case of $\mathbf{T} = [0, a]$.

(b) If M is L_1-right continuous, i.e., $t_n \downarrow t$ implies $M_{t_n} \to M_t$ in L_1, or if τ and σ both have finite ranges, we can eliminate assumption (ii) and the bulleted assumptions which follow in Lemma A.5.1.

Let σ be a predictable stopping time and let $(\sigma_n)_n$ be any announcing sequence for σ. Recall that $\sigma_n \uparrow \sigma$ on Ω and $\sigma_n < \sigma$ on $[0 < \sigma] \ \forall n$ which, by Exercise 3.4.6. (2) in [30], implies

$$\mathcal{H}_{\sigma-} \;=\; \vee_n \, \mathcal{H}_{\sigma_n}. \tag{A.24}$$

If $X = (X_t)_{t \in \mathbf{T}}$ is any cadlag process, we can thus define a random variable $X_{\sigma-}$ by letting

$$X_{\sigma-} = \lim_n X_{\sigma_n}. \tag{A.25}$$

(Since a.e. sample path of X has left-limits, $X_{\sigma-}$ is clearly independent of the announcing sequence.) This leads to a useful application of Lemma A.5.1.

Lemma A.5.3 *If M is as described in the statement of Lemma A.5.1, then for any predictable stopping time σ, $M_{\sigma-} \in L_1$ and $E[(\Delta M)_\sigma | \mathcal{H}_{\sigma-}] = 0$. Here, $\Delta M_t = M_t - M_{t-}$ $\forall t > 0$ and $\Delta M_0 = 0$.*

Proof Since λ is right-continuous and increasing, it is cadlag. Since N is a right-continuous submartingale, it too is cadlag (see Remark 4.2.12). Therefore, M itself is cadlag.

Let $(\sigma_n)_n$ be an announcing sequence for σ. Since M is cadlag,

$$(\Delta M)_\sigma = M_\sigma - \lim_n M_{\sigma_n} = M_\sigma - M_{\sigma-}$$

on Ω. Therefore, it is sufficient to show $E[M_\sigma - M_{\sigma-} | \mathcal{H}_{\sigma-}] = 0$.

Since $\sigma_n \leq \sigma$ $\forall n$, Lemma A.5.1 implies $M_\sigma, M_{\sigma_n} \in L_1$ $\forall n$ and

$$E[M_\sigma - M_{\sigma_n} | \mathcal{H}_{\sigma_n}] = 0, \quad (\forall n). \tag{A.26}$$

<u>Claim:</u> *$M_{\sigma_n} \to M_{\sigma-}$ in L_1.*

<u>Proof:</u> Since λ is increasing and cadlag, $\lambda_{\sigma_n} \uparrow \lambda_{\sigma-}$ a.s. where $\lambda_{\sigma-} \geq 0$ a.s. Therefore, by the monotone convergence theorem, $\lambda_{\sigma_n} \to \lambda_{\sigma-}$ in L_1.

Next, since N is a submartingale, Doob's stopping theorem implies $(N_{\sigma_n})_n$ is a submartingale w.r.t. $(\mathcal{H}_{\sigma_n})_n$. Moreover,

$$0 \leq N_{\sigma_n} \leq E[N_\sigma | \mathcal{H}_{\sigma_n}], \quad (\forall n)$$

implies $(N_{\sigma_n})_n$ is uniformly integrable which, by Theorem 2.6.4 in [30], implies

$$\exists N' \in L_1 \text{ s.t. } N_{\sigma_n} \to N' \text{ a.s. and in } L_1. \tag{A.27}$$

But since N is cadlag, $N_{\sigma_n} \to N_{\sigma-}$ a.s. Therefore, (A.27) implies $N_{\sigma_n} \to N_{\sigma-}$ in L_1 which by assumption (ii) of Lemma A.5.1 completes the proof of the Claim. Ω

Take any $F \in \bigcup_n \mathcal{H}_{\sigma_n}$. Since $(\sigma_n)_n$ is increasing, $\exists\, n_0 \in \mathrm{N}$ s.t. $F \in \mathcal{H}_{\sigma_n}$ $\forall n \geq n_0$. Therefore,

$$\begin{aligned} \int_F M_\sigma \, dP &= \lim_n \int_F M_{\sigma_n} \, dP \quad \text{(by (A.26))} \\ &= \int_F M_{\sigma-} \, dP \end{aligned} \tag{A.28}$$

But by (A.24), $\bigcup_n \mathcal{H}_{\sigma_n}$ is a π-class generating $\mathcal{H}_{\sigma-}$. Therefore, the relation in (A.28) extends to all $F \in \mathcal{H}_{\sigma-}$ which implies $E[M_\sigma - M_{\sigma-} \,|\, \mathcal{H}_{\sigma-}] = 0$, completing the proof. □

Lemma A.5.4 *If $X = (X_t)_{t \in \mathrm{T}}$ is a predictable cadlag process and $\epsilon > 0$, then*

$$S = \inf\{s > 0 : \Delta X_s > \epsilon\}, \tag{A.29}$$

where $\inf \phi = \infty$, *defines a predictable stopping time.*

Proof By Proposition 2.6 on p.17 of [29], ΔX is predictable. Therefore, by Proposition 2.13 on p.18 of [29], it is sufficient to show

$$\{(\omega, S(\omega)) : \omega \in \Omega\} \cap (\Omega \times [0, \infty)) \subseteq [\Delta X > \epsilon]. \tag{A.30}$$

With this in mind, take $(\omega, t) \in \Omega \times [0, \infty)$ s.t. $S(\omega) = t$. Since $S(\omega) < \infty$ and X is cadlag, $\{s > 0 : \Delta X_s > \epsilon\}$ has a smallest element. By (A.29), this smallest element is $S(\omega)$. In other words,

$$\Delta X(\omega, t) = \Delta X_{S(\omega)}(\omega) > \epsilon,$$

i.e., $(\omega, t) \in [\Delta X > \epsilon]$ which establishes (A.30). □

The following result could not be located in exact form in the literature. See Theorem 4.2.7 for the defining properties of $\langle Y \rangle$.

Proposition A.5.5 *For each n, let $Y_n = \{Y_n(t) : t \in [0, \infty)\}$ be an L_2 martingale w.r.t. a filtration $\mathcal{H} = (\mathcal{H}_t)_{t \in [0,\infty)}$. If $\exists\, \delta > 0$ s.t.*

$$\sup_n E\left[|Y_n(1)|^{2+\delta}\right] < \infty, \tag{A.31}$$

then $\{\langle Y_n \rangle(1) : n \geq 1\}$ is uniformly integrable.

Proof Identical to that of Proposition 3.6.12. Indeed, as mentioned on p.87 of [20], given any n, $\exists$ a sequence of finite partitions $\{\Delta^{(k)} : k \geq 1\}$ of $[0,1]$, say $\Delta^{(k)} = \{0 = t_0^{(k)} < \cdots < t_{m_k}^{(k)} = 1\}$ s.t.

$$\sum_{i=1}^{m_k} E\left[\left(Y_n(t_i^{(k)}) - Y_n(t_{i-1}^{(k)})\right)^2 \middle| \mathcal{H}(t_{i-1}^{(k)})\right] \longrightarrow \langle Y_n \rangle(1)$$

weakly in L_1 as $k \to \infty$. (The dependence of $\Delta^{(k)}$ on n has been suppressed from the notation.) Applying Rosenthal's inequality (see Section 2.4 of [19]), the $L_{1+\delta/2}$-norm of any such approximant (ranging over all k and n) is bounded by $C \cdot \sup_n[|Y_n(1)|^{2+\delta}]$ where $C = C(\delta) \in (0,\infty)$ is a constant depending only on $\delta > 0$. As in Proposition 3.6.12, this implies $\{\langle Y_n \rangle(1) : n \geq 1\}$ is uniformly integrable. □

Remark A.5.6 Proposition A.5.5 continues to holds even when each Y_n is a martingale w.r.t. some filtration $\mathcal{H}^{(n)}$ possibly depending on n. In particular, the above mentioned collection of approximants continues to be $L_{1+\delta/2}$-norm bounded since $C(\delta)$ depends only on $\delta > 0$.

The next two results concern the weak convergence of processes in $D[0,a]$ (i.e., with sample paths in $D[0,a]$) w.r.t. the Skorokhod J_1 topology. See Subsection 4.2.1 and [6] for additional material on this topic.

Lemma A.5.7 *If $A, A_1, A_2, \cdots$ are processes in $D[0,a]$ s.t.*

(i) *for each n, A_n is increasing and*

(ii) *$A_n \xrightarrow{\mathcal{L}} A$ in $D[0,a]$ w.r.t. the Skorokhod topology,*

then A is also increasing.

Proof For the sake of notation, assume A and each A_n are defined on a common probability space $(\Omega, \mathcal{F}, P)$ and denote A_n by A^n $\forall n$. Since $A_n \to_{\mathcal{L}} A$ in $D[0,a]$, $\exists$ a countable dense subset $\mathbf{D}$ of $[0,a]$ s.t.

$$(A^n_{d_1}, \cdots, A^n_{d_m}) \xrightarrow{\mathcal{L}} (A_{d_1}, \cdots A_{d_m}) \text{ in distribution} \qquad \text{(A.32)}$$

for any $d_1, \cdots, d_m \in \mathbf{D}$ $(m \in \mathrm{N})$.

Take $d_1 < d_2$ in $\mathbf{D}$ and let Γ^- denote the set of all negative continuity points of $A_{d_2} - A_{d_1}$. By the continuous mapping theorem, (A.32) implies

$$P(A^n_{d_2} - A^n_{d_1} > \alpha) \longrightarrow P(A_{d_2} - A_{d_1} > \alpha)\,, \quad (\forall\, \alpha \in \Gamma^-). \tag{A.33}$$

Since each A_n is increasing, (A.33) implies

$$P(A_{d_2} - A_{d_1} > \alpha) \;=\; 1\,, \quad (\forall\, \alpha \in \Gamma^-)$$

and therefore, since Γ^- is dense in $[0, -\infty)$, $P(A_{d_2} - A_{d_1} \geq 0) = 1$. Applying this argument to every pair $d_1 < d_2$ in $\mathbf{D}$, the right-continuity of A on $[0, a]$ implies A is increasing on $[0, a]$. (This is similar to the argument used in Lemma 4.2.21 (c).) □

Lemma A.5.8 *Given processes $A, A_1, A_2, \cdots$ in $D[0, a]$ s.t. each A_n is increasing and A is continuous, deterministic and increasing, t.f.a.e.:*

(a) $A_n(t) \to A(t)$ *in probability* $\forall\, t \in [0, a]$,

(b) $\sup_{t\in[0,a]} |A_n(t) - A(t)| \to 0$ *in probability and*

(c) $A_n \xrightarrow{\mathcal{L}} A$ *in* $D[0, a]$ *w.r.t. the Skorokhod topology.*

Proof For the sake of notation, assume A and each A_n are defined on a common probability space $(\Omega, \mathcal{F}, P)$. By Lemma 1 in [31], (a) implies (b). Since A is continuous, (c) implies

$$A_n(t) \xrightarrow{\mathcal{L}} A(t)\,, \quad (\forall\, t \in [0, a])$$

and therefore, since $A(t)$ is constant $\forall\, t$, (a) follows. Finally, if we let ρ denote the uniform metric on $D[0, a]$, then (b) can be read:

$$P[\,\rho(A_n, A) > \epsilon\,] \longrightarrow 0\,, \quad (\forall \epsilon > 0),$$

a condition which yields $A_n \to_{\mathcal{L}} A$ in $D[0, a]$ w.r.t. the uniform topology on $D[0, a]$ (see the top of p.25 in [6]). Since the uniform topology on $D[0, a]$ is stronger than the Skorokhod topology on $D[0, a]$, this implies (c). □

A.6 Convergence of Random Vectors

In this small section, we establish a few technical results concerning convergence in distribution of random vectors. Our main goal is Proposition A.6.2, a result which reduces our work in establishing convergence in finite dimensional distribution for sequences of set-indexed processes.

In our first result, any upper case X, regardless of subscripts and superscripts, will denote a random variable on a fixed probability space $(\Omega, \mathcal{F}, P)$. To simplify the notation in part (b), we have employed the following device: given $m \in \mathbf{N}$ and $x \in \mathbf{R}$, $m \odot x$ represents m copies of x separated by commas. For example, $3 \odot x$ represents x, x, x.

Lemma A.6.1 *Given $1 < i_1 < i_2 < \cdots < i_m \leq k$ in $\mathbf{N}$, $m_1, \cdots, m_k \in \mathbf{N} \cup \{0\}$ and any permutation π on $\{1, \cdots, k\}$, if*

$$(X_{n,1}, \cdots, X_{n,k}) \xrightarrow{\mathcal{L}} (X_1, \cdots, X_k) \quad as\ n \to \infty, \tag{A.34}$$

then the following distributional limits hold as $n \to \infty$:

(a) $\left(\Sigma_{j=1}^{i_1-1} X_{n,j}, \cdots, \Sigma_{j=i_m}^{k} X_{n,j}\right) \xrightarrow{\mathcal{L}} \left(\Sigma_{j=1}^{i_1-1} X_j, \cdots, \Sigma_{j=i_m}^{k} X_j\right)$,

(b) $(m_1 \odot X_{n,1}, \cdots, m_k \odot X_{n,k}) \xrightarrow{\mathcal{L}} (m_1 \odot X_1, \cdots, m_k \odot X_k)$ *and*

(c) $\left(X_{n,\pi(1)}, \cdots, X_{n,\pi(k)}\right) \xrightarrow{\mathcal{L}} \left(X_{\pi(1)}, \cdots, X_{\pi(k)}\right)$.

Proof Consider the function $f : \mathbf{R}^k \to \mathbf{R}^{m+1}$ where

$$f(x_1, \cdots x_k) = \left(\Sigma_{j=1}^{i_1-1} x_j, \cdots, \Sigma_{j=i_m}^{k} x_j\right),$$

the function $g : \mathbf{R}^k \to \mathbf{R}^r$ $(r = \sum_{j=1}^{k} m_j)$ where

$$g(x_1, \cdots x_k) = (m_1 \odot x_1, \cdots, m_k \odot x_k)$$

and the function $h : \mathbf{R}^k \to \mathbf{R}^k$ where

$$h(x_1, \cdots, x_k) = \left(x_{\pi(1)}, \cdots, x_{\pi(k)}\right).$$

f, g and h are continuous on $\mathbf{R}^k$. Therefore, (a), (b) and (c) follow from (A.34) and the continuous mapping theorem. □

The next result will require several applications of Lemma A.6.1.

Proposition A.6.2 *Given $\mathcal{A}$-indexed processes $X, X_1, X_2, \cdots$, if*

$$(X_n(C_1), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (X(C_1), \cdots, X(C_k)) \quad \text{as } n \to \infty \tag{A.35}$$

$\forall$ f.n.s. $\{C_1, \cdots, C_k\}$ of $\mathcal{A}$, then (A.35) *holds $\forall \{C_1, \cdots, C_k\} \subseteq \mathcal{C}(u)$. (See Definition 3.2.14 for the definition of f.n.s.)*

Proof Take any finite subcollection $\{C_1, \cdots, C_k\}$ of $\mathcal{C}(u)$. Applying Lemma 3.2.15 to the set $C = \bigcup_{i=1}^k C_i \in \mathcal{C}(u)$, $\exists$ a f.n.s. $\mathcal{N}_0 = \{D_1, \cdots, D_r\}$ of $\mathcal{A}$ s.t.

$$C_1 = \bigcup_{j=1}^{r_1} D_j^1 \,,\ C_2 = \bigcup_{j=1}^{r_2} D_j^2 \,,\ \cdots \,,\ C_k = \bigcup_{j=1}^{r_k} D_j^k \tag{A.36}$$

where, given any $1 \le i \le k$, $D_1^i, \cdots, D_{r_i}^i \in \mathcal{N}_0$ are disjoint. (This is not precisely the statement of Lemma 3.2.15 but the proof thereof clearly implies (A.36).)

By (A.35),

$$(X_n(D_1), \cdots, X_n(D_r)) \xrightarrow{\mathcal{L}} (X(D_1), \cdots, X(D_r)) \quad \text{as } n \to \infty.$$

For each fixed $1 \le q \le r$, define $m_q \in \mathbb{N} \cup \{0\}$ to be the number of occurrences of D_q among the D_j^i. Then, by Lemma A.6.1 (b),

$$(m_1 \odot X_n(D_1) \cdots, m_r \odot X_n(D_r)) \xrightarrow{\mathcal{L}} (m_1 \odot X(D_1) \cdots, m_r \odot X(D_r)) \,. \tag{A.37}$$

Considering the way in which $m_1, \cdots, m_r$ were defined, we can rearrange the components on the left-hand side of (A.37) in accordance with (A.36) so as to read

$$\Big(X_n(D_1^1), \ \cdots \ , X_n(D_{r_1}^1), \ X_n(D_1^2), \ \cdots \ , X_n(D_{r_k}^k)\Big) \,.$$

Therefore, if we apply Lemma A.6.1 (c) to (A.37) and then apply Lemma A.6.1 (a) (the latter using the additivity of X, X_n on $\mathcal{C}(u)$), we can obtain

$$(X_n(C_1), \cdots, X_n(C_k)) \xrightarrow{\mathcal{L}} (X(C_1), \cdots, X(C_k)) \quad \text{as } n \to \infty$$

which completes the proof. □

We close this section with a technical result which is used in the proof of Theorem 4.4.1. Given a random vector $\mathbf{X} = (X_1, \cdots, X_n)$, recall that a set $R \in \mathcal{B}(\mathbb{R}^n)$ is a *continuity set* of $\mathbf{X}$ if $P(\mathbf{X} \in \partial R) = 0$.

Lemma A.6.3 *Given $m \in \mathbf{N}$, random variables $U, U_1, \cdots, U_m$ on $(\Omega, \mathcal{F}, P)$ and continuity points x_i of U_i $(1 \le i \le m)$, define the event*

$$A = \cap_{i=1}^{m} [U_i \le x_i]. \tag{A.38}$$

If $x \in \mathbf{R}$ is a continuity point of $U \cdot \mathbf{1}_A$, then $C = (x, \infty) \times \prod_{i=1}^{m}(-\infty, x_i]$ is a continuity set of the random vector $\mathbf{U} = (U, U_1, \cdots, U_m)$.

Proof Since

$$\partial C = (\{x\} \times \Pi_{i=1}^{m}(-\infty, x_i]) \cup \cup_{i=1}^{m} ([x, \infty) \times \{x_i\} \times \Pi_{j \neq i}(-\infty, x_j]),$$

i.e., the union of the $m+1$ faces of $cl(C)$, we have

$$[\mathbf{U} \in \partial C] = [U = x, A] \cup \cup_{i=1}^{m} [U \ge x, U_i = x_i, U_j \le x_j \, (\forall j \neq i)]. \tag{A.39}$$

We will show that each summand in (A.39) is P-null.

If $x \neq 0$, then

$$\begin{aligned} P([U = x] \cap A) &= P(U = x, \mathbf{1}_A = 1) \\ &= P(U \cdot \mathbf{1}_A = x) \\ &= 0 \end{aligned}$$

since x is a continuity point of $U \cdot \mathbf{1}_A$. On the other hand, if $x = 0$, then

$$P([U = 0] \cap A) \le P(U = 0) \le P(U \cdot \mathbf{1}_A = 0) = 0.$$

Either way, $P([U = x] \cap A) = 0$. Moreover, given any $1 \le i \le m$,

$$P[U \ge x, U_i = x_i, U_j \le x_j \, (\forall j \neq i)] \le P(U_i = x_i) = 0$$

since x_i is a continuity point of U_i. Therefore, in view of (A.39), $P(\mathbf{U} \in \partial C) = 0$, i.e., C is a continuity set of $\mathbf{U}$. □

A.7 A Technical Result for Section 4.6

Let $\mathbf{F} : \mathbf{R}^d \to [0, 1]$ (some $d \in \mathbf{N}$) be a continuous distribution function with $\mathbf{F}([0, 1]^d) = 1$ and let $(\mathbf{Y}_n)_n$ and $(Z_n)_n$ be as described in (I) and (II) respectively on p.182. **Throughout this subsection,** $(\Omega, \mathcal{F}, P)$ **will**

denote the complete probability space on which $(\mathbf{Y}_n)_n$ and $(Z_n)_n$ are defined.

Recall the relations $\leq$ and $\prec$ on $[0,1]^d$ and the subsets $L_{\mathbf{x}}$, $S_{\mathbf{x}}$ and $S_{\mathbf{x}-}$ of $[0,1]^d$ associated to any $\mathbf{x} \in (0,1)^d$, all of which are defined at the start of Subsection 4.6.1. Define the function $h_{\mathbf{F}} : [0,1]^d \to [0,\infty)$ by letting

$$h_{\mathbf{F}}(\mathbf{x}) = \begin{cases} [\mathbf{F}(S_{\mathbf{x}-})]^{-1} & , \text{if } \mathbf{F}(S_{\mathbf{x}-}) > 0 \\ 0 & , \text{otherwise}. \end{cases} \tag{A.40}$$

Given random variables $X_1, \cdots, X_n$, let $\sigma(X_1, \cdots, X_n)$ denote the smallest sub-σ-algebra of $\mathcal{F}$ for which $X_1, \cdots, X_n$ are measurable. The following technical result is used several times in Section 4.6.

Proposition A.7.1 *Under Assumption 4.6.1, given $k \in \mathrm{N}$ and $\mathbf{x}, \mathbf{y} \in [0,1]^d$ s.t. $\mathbf{0} \prec \mathbf{x} \prec \mathbf{y}$, if $W \in L_1$ is $\sigma(Z_k)$-measurable, then*

$$E\left[\mathbf{1}_{[\mathbf{Y}_k \in C]} W \middle| \mathcal{G}_C^*\right] = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} \cdot E(W) \cdot \mathbf{F}(C) \cdot h_{\mathbf{F}}(\mathbf{x}) \tag{A.41}$$

where $C = (\mathbf{x}, \mathbf{y}] \in \mathcal{C}$ and $\mathcal{G}_C^*$ denotes the strong past at C generated by the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ defined in (4.63).

To establish the above lemma, we will employ the so-called "smoothing property" of conditional expectation in relation to *atomic events* in σ-algebras. These atomic events correspond to the "atoms" found in Lemma 6.5.8 of [3]. Recall that the term *atom* already has an assigned meaning in this thesis (see Section 2.3).

Definition A.7.2 *Given a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$, an event $H \in \mathcal{H}$ is an atomic event in $\mathcal{H}$ w.r.t. P if*

$$F \in \mathcal{H} \text{ and } F \subseteq H \implies P(F) = 0 \text{ or } P(H - F) = 0. \tag{A.42}$$

Note that all P-null events in $\mathcal{H}$ are atomic according to our definition. In contrast, [3] requires $P(H) > 0$.

Given any two sets, $H, F \subseteq \Omega$, we have the relations $H \cap F \subseteq H$ and $H \setminus F = H - (H \cap F)$. Therefore,

Lemma A.7.3 *Given a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$, H is an atomic event in $\mathcal{H}$ if and only if*

$$P(H \cap F) = 0 \text{ or } P(H \setminus F) = 0, \quad (\forall F \in \mathcal{H}).$$

The next result will reduce our work in showing an event is atomic.

Lemma A.7.4 *Given a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$ and an event $H \in \mathcal{H}$, if $\exists\, \mathcal{E} \subseteq \mathcal{H}$ s.t. $\sigma(\mathcal{E}) = \mathcal{H}$ and*

$$P(H \cap E) = 0 \text{ or } P(H \setminus E) = 0\,, \quad (\forall\, E \in \mathcal{E}),$$

then H is an atomic event in $\mathcal{H}$.

Proof Define

$$\mathcal{G} = \{F \in \mathcal{H} : P(H \cap F) = 0 \text{ or } P(H \setminus F) = 0\}.$$

Clearly, $\phi, \Omega \in \mathcal{G}$ and $\mathcal{G}$ is closed under complementation.

Take $(F_n)_n$ in $\mathcal{G}$ and let $F = \bigcup_n F_n$. If $P(H \cap F_n) = 0\ \forall n$, then

$$P(H \cap F) \leq \sum_n P(H \cap F_n) = 0.$$

On the other hand, if $\exists\, n_0 \in \mathrm{N}$ s.t. $P(H \setminus F_{n_0}) = 0$, then

$$P(H \setminus F) \leq P(H \setminus F_{n_0}) = 0.$$

Therefore, $\mathcal{G}$ is a σ-algebra containing $\mathcal{E}$. By assumption, this implies $\mathcal{H} = \mathcal{G}$. In light of Lemma A.7.3 and the definition of $\mathcal{G}$, H is indeed an atomic event in $\mathcal{H}$. □

Given any r.v. X, it is clear that

$$\mathcal{E} = \{[X > x] : x \geq 0\} \cup \{[X \leq x] : x < 0\} \implies \sigma(\mathcal{E}) = \sigma(X). \tag{A.43}$$

With future applications of Lemma A.7.4 in mind, define

$$\begin{aligned} \mathcal{E}_B^{(k)} = \; & \{[\mathbf{Y}_k \in A] \cap [Z_k > x] : x \geq 0,\ A \subseteq B \text{ and } A \in \mathcal{A}\} \cup \\ & \{[\mathbf{Y}_k \in A] \cap [Z_k \leq x] : x < 0,\ A \subseteq B \text{ and } A \in \mathcal{A}\} \cup \\ & \{P\text{-null subsets of } \Omega\} \end{aligned} \tag{A.44}$$

for any $k \in \mathrm{N}$ and any $B \in \mathcal{A}(u)$. Also, define sub-σ-algebras

$$\mathcal{H}_B^{(k)} = \sigma(\, M_k(A) : A \subseteq B \text{ and } A \in \mathcal{A}\,) \vee \mathcal{F}_0 \tag{A.45}$$

for any $k \in \mathrm{N}$ and $B \in \mathcal{A}(u)$ where M_k is the $\mathcal{A}$-indexed process

$$M_k(A) = 1_{[\mathbf{Y}_k \in A]} Z_k, \quad (\forall A \in \mathcal{A})$$

and $\mathcal{F}_0$ is the σ-algebra generated by all P-null subsets of Ω. Since

$$\begin{aligned} [M_k(A) > x] &= [\mathbf{Y}_k \in A] \cap [Z_k > x], \quad (\forall x \geq 0) \text{ and} \\ [M_k(A) \leq x] &= [\mathbf{Y}_k \in A] \cap [Z_k \leq x], \quad (\forall x < 0) \end{aligned}$$

for every $A \in \mathcal{A}$, (A.44) and (A.45) imply

$$\mathcal{H}_B^{(k)} = \sigma(\mathcal{E}_B^{(k)}), \quad (\forall B \in \mathcal{A}(u),\ k \in \mathrm{N}) \tag{A.46}$$

when we apply the principle in (A.43). Note that

$$\mathcal{H}_B = \vee_k \mathcal{H}_B^{(k)}, \quad (\forall B \in \mathcal{A}(u)) \tag{A.47}$$

where $\mathcal{H}_B$ is the sub-σ-algebra of $\mathcal{F}$ defined in (4.62).

The following consequence of Lemma A.7.4 will be needed in the proof of Proposition A.7.1. See Subsection 4.6.1 for the definition of the subsets $L_{\mathbf{t}}$ and $S_{\mathbf{t}}$ of $[0,1]^d$.

Lemma A.7.5 *Under Assumption 4.6.1, given any* $\mathbf{t} \in (0,1)^d$, $[\mathbf{Y}_k \in S_{\mathbf{t}}]$ *is an atomic event in* $\mathcal{H}_{L_{\mathbf{t}}}^{(k)}$ $\forall k$.

Proof Fix $\mathbf{t} \in (0,1)^d$ and $k \in \mathrm{N}$. Our first task is to show $[\mathbf{Y}_k \in S_{\mathbf{t}}] \in \mathcal{H}_{L_{\mathbf{t}}}^{(k)}$. In this direction, note that for any $A \in \mathcal{A}$,

$$\begin{aligned} [\mathbf{Y}_k \in A] = {} & ([\mathbf{Y}_k \in A] \cap [Z_k > 0]) \cup ([\mathbf{Y}_k \in A] \cap [Z_k = 0]) \cup \\ & \cup_{r \in \mathrm{N}} ([\mathbf{Y}_k \in A] \cap [Z_k \leq -1/r]) \end{aligned}$$

which, by (A.46), implies

$$[\mathbf{Y}_k \in A] \in \mathcal{H}_{L_{\mathbf{t}}}^{(k)}, \quad (\forall A \in \mathcal{A} \text{ s.t. } A \subseteq L_{\mathbf{t}}). \tag{A.48}$$

(Since $P(Z_1 = 0) = 0$ by assumption, $[\mathbf{Y}_k \in A] \cap [Z_k = 0]$ is P-null.) Moreover, as remarked on p.83 of [27],

$$S_{\mathbf{t}} = \bigcap \{ [0, \mathbf{s}]^C : \mathbf{s} \in ([0,1] \cap \mathbf{Q})^d \text{ and } \mathbf{s} \in L_{\mathbf{t}} \}$$

which implies

$$[\mathbf{Y}_k \in S_t] \;=\; \bigcap \{\, [\mathbf{Y}_k \notin [0,\mathbf{s}]\,] : \mathbf{s} \in ([0,1] \cap \mathbf{Q})^d \text{ and } \mathbf{s} \in L_t\}.$$

Therefore, since $[0,\mathbf{s}] \subseteq L_t \;\; \forall\, \mathbf{s} \in L_t$, (A.48) yields $[\mathbf{Y}_k \in S_t] \in \mathcal{H}^{(k)}_{L_t}$.

Since $L_t \cap S_t = \phi$, it is clear from the definition of $\mathcal{E}^{(k)}_{L_t}$ that

$$P([\mathbf{Y}_k \in S_t] \cap E) = 0\,, \;\; (\forall\, E \in \mathcal{E}^{(k)}_{L_t}). \tag{A.49}$$

Therefore, by (A.46) and Lemma A.7.4, $[\mathbf{Y}_k \in S_t]$ is atomic in $\mathcal{H}^{(k)}_{L_t}$. □

The following result which appears as Theorem 6.5.9 (a) in [3] is commonly referred to as the *smoothing property* of the conditional expectation operator.

Proposition A.7.6 *Given a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$, an atomic event H in $\mathcal{H}$ and any random variable $Y \in L_1$, if $P(H) > 0$, then*

$$E(Y|\,\mathcal{H}) \;=\; E(\mathbf{1}_H Y)\cdot[P(H)]^{-1} \;\; \textit{a.e. on } H.$$

In the above Proposition, if $Y = 0$ a.e. on H^C, then $E(Y|\mathcal{H})\cdot \mathbf{1}_{H^C} = E(Y \cdot \mathbf{1}_{H^C}|\mathcal{H}) = 0$ a.e. Therefore,

Corollary A.7.7 *In Proposition A.7.6, if Y is s.t. $Y = 0$ a.e. on H^C, then*

$$E(Y|\,\mathcal{H}) \;=\; \mathbf{1}_H \cdot E(\mathbf{1}_H Y)\cdot[P(H)]^{-1}.$$

The following property of conditional expectation can be found in most probability texts. For example, see 9.7 (k) on p.88 of [40].

Lemma A.7.8 *Given sub-σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ and an r.v. $V \in L_1$, if $\sigma(V) \vee \mathcal{F}_1$ is independent of $\mathcal{F}_2$, then $E(V|\,\mathcal{F}_1 \vee \mathcal{F}_2) = E(V|\,\mathcal{F}_1)$.*

We can now present the following:

Proof (of Proposition A.7.1)

Let $C = (\mathbf{x},\mathbf{y}]$ be as described in the statement of Proposition A.7.1. Define the sequence $(\mathbf{x}_n)_n$ in $[0,1]^d$ to be s.t. $g_n([0,\mathbf{x}]) = [0,\mathbf{x}_n] \;\; \forall\, n$ and let

$$C_n \;=\; (\mathbf{x}_n,\mathbf{y}]\,, \;\; (\forall\, n).$$

Clearly, $\mathbf{0} \prec \mathbf{x}_n\ \forall n$. Furthermore, since $\mathbf{x} \prec \mathbf{y}$, we must have $\mathbf{x}_n \prec \mathbf{y}$ for all large n. Therefore, it can be assumed w.l.o.g. that $\mathbf{0} \prec \mathbf{x}_n \prec \mathbf{y}\ \forall n$. It is also clear that

$$C_n \uparrow C \text{ and } S_{\mathbf{x}_n} \uparrow S_{\mathbf{x}} \text{ as } n \to \infty. \tag{A.50}$$

Select a fixed $k \in \mathbb{N}$. There are two cases. First, if $P(\mathbf{Y}_k \in S_{\mathbf{x}}) = 0$, then $\mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} = 0$ a.e. Furthermore, since $(\mathbf{x}, \mathbf{y}] \subseteq S_{\mathbf{x}}$, $\mathbf{1}_{[\mathbf{Y}_k \in C]} \cdot W = 0$ a.e. In total, both sides of (A.41) are equal to zero a.e.

On the other hand, if $P(\mathbf{Y}_k \in S_{\mathbf{x}}) > 0$, then by (A.50), we can assume w.l.o.g. that $P(\mathbf{Y}_k \in S_{\mathbf{x}_n}) > 0\ \forall n$. For each $m \in \mathbb{N}$, define the random variable,

$$V_m = \mathbf{1}_{[\mathbf{Y}_k \in C_m]} W.$$

Given any $m \leq n$, $C_m \subseteq C_n \subseteq S_{\mathbf{x}_n}$ implies $V_m = 0$ a.e. on $[\mathbf{Y}_k \notin S_{\mathbf{x}_n}]$. Therefore, since $[\mathbf{Y}_k \in S_{\mathbf{x}_n}]$ is an atomic event in $\mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}$ (see Lemma A.7.5), Corollary A.7.7 yields

$$E(V_m | \mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}) = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot E(V_m) \cdot [P(\mathbf{Y}_k \in S_{\mathbf{x}_n})]^{-1}, \quad (\forall m \leq n). \tag{A.51}$$

(Note that $C_m \subseteq S_{\mathbf{x}_n}$ implies $V_m \cdot \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} = V_m$.)

Furthermore, since $\sigma(V_m) \subseteq \sigma(\mathbf{Y}_k, Z_k)\ \forall m$ and $\mathcal{H}^{(i)}_B \subseteq \sigma(\mathbf{Y}_i, Z_i)\ \forall i \in \mathbb{N}$ and $B \in \mathcal{A}(u)$, the independence assumptions concerning $(\mathbf{Y}_n)_n$ and $(Z_n)_n$ (see (I) and (II) on p.182) imply

$$\sigma(V_m) \vee \mathcal{H}^{(k)}_{L_{\mathbf{x}_n}} \text{ is independent of } \bigvee_{r \neq k} \mathcal{H}^{(r)}_{L_{\mathbf{x}_n}}, \quad (\forall m, n). \tag{A.52}$$

Therefore, by (A.47) and Lemma A.7.8,

$$E(V_m | \mathcal{H}_{L_{\mathbf{x}_n}}) = E(V_m | \mathcal{H}^{(k)}_{L_{\mathbf{x}_n}}), \quad (\forall m, n)$$

which, by (A.51), yields

$$E(V_m | \mathcal{H}_{L_{\mathbf{x}_n}}) = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot E(V_m) \cdot [P(\mathbf{Y}_k \in S_{\mathbf{x}_n})]^{-1}, \quad (\forall m \leq n).$$

Since W is assumed to be $\sigma(Z_k)$ measurable — and hence independent of $\mathbf{Y}_k$ — this further implies

$$E(V_m | \mathcal{H}_{L_{\mathbf{x}_n}}) = \mathbf{1}_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot E(W) \cdot \mathrm{F}(C_m) \cdot [\mathrm{F}(S_{\mathbf{x}_n})]^{-1}, \quad (\forall m \leq n). \tag{A.53}$$

Now, fix $m \in \mathbb{N}$. Clearly, $g_n(L_{\mathbf{x}}) = L_{\mathbf{x}_n}\ \forall n$ and hence by Lemma 4.6.7,

$$\mathcal{G}^*_C = \bigcap_n \mathcal{H}_{L_{\mathbf{x}_n}}.$$

(Note that $[0, \mathbf{y}] \setminus L_{\mathbf{x}}$ is a maximal representation of $C = (\mathbf{x}, \mathbf{y}]$ as required.) Therefore, by the reverse martingale convergence theorem (as presented on p.136 of [40]),

$$\begin{aligned} E(V_m | \mathcal{G}_C^*) &= \lim_n E(V_m | \mathcal{H}_{L_{\mathbf{x}_n}}) \text{ a.s.} \\ &= \lim_n 1_{[\mathbf{Y}_k \in S_{\mathbf{x}_n}]} \cdot E(W) \cdot \mathbf{F}(C_m) \cdot [F(U_{\mathbf{x}_n})]^{-1} \quad \text{(by (A.53))} \\ &= 1_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} \cdot E(W) \cdot \mathbf{F}(C_m) \cdot [\mathbf{F}(S_{\mathbf{x}})]^{-1} \quad \text{(by (A.50))}. \end{aligned}$$

Finally, since $|V_m| \leq |W| \in L_1 \ \forall m$ and, by (A.50), $\lim_m V_m = 1_{[\mathbf{Y}_k \in C]} W$, dominated convergence and the above array imply

$$\begin{aligned} E\left[1_{[\mathbf{Y}_k \in C]} W \middle| \mathcal{G}_C^*\right] &= \lim_m E(V_m | \mathcal{G}_C^*) \text{ a.s.} \\ &= 1_{[\mathbf{Y}_k \in S_{\mathbf{x}}]} \cdot E(W) \cdot \mathbf{F}(C) \cdot [\mathbf{F}(S_{\mathbf{x}})]^{-1} \quad \text{(by (A.50))} \end{aligned}$$

where, by the continuity of $\mathbf{F}$, $\mathbf{F}(S_{\mathbf{x}}) = \mathbf{F}(S_{\mathbf{x}-})$. This gives us (A.41) for the case of $P(\mathbf{Y}_k \in S_{\mathbf{x}}) > 0$, completing the proof of Proposition A.7.1. □

We end this section with an application of Corollary A.7.7 which is needed in Subsection 4.6.4 to calculate a $*$-quadratic variation for the weighted empirical process. (In fact, we only require this result for the case of $n = 2$ and $E(W) = 0$.) Its proof is close to that of Proposition A.7.1, give or take a few minor modifications.

Proposition A.7.9 *Under Assumption 4.6.1, given $r \in \mathbf{N}$, distinct indices $k_1, \cdots, k_r \in \mathbf{N}$ and $\mathbf{x}, \mathbf{y} \in [0,1]^d$ s.t. $\mathbf{0} \prec \mathbf{x} \prec \mathbf{y}$, if $W \in L_1$ is $\sigma(Z_{k_1}, \cdots, Z_{k_r})$-measurable, then*

$$E\left[\prod_{i=1}^r 1_{[\mathbf{Y}_{k_i} \in C]} W \middle| \mathcal{G}_C^*\right] = \prod_{i=1}^r 1_{[\mathbf{Y}_{k_i} \in S_{\mathbf{x}}]} \cdot E(W) \cdot \mathbf{F}(C)^r \cdot [h_{\mathbf{F}}(\mathbf{x})]^r \quad \text{(A.54)}$$

where $C = (\mathbf{x}, \mathbf{y}] \in \mathcal{C}$ and $\mathcal{G}_C^*$ denotes the strong past at C generated by the filtration $(\mathcal{F}_A)_{A \in \mathcal{A}}$ defined in (4.63).

Proof To simplify notation, choose $k_i = i$ for each $i = 1, \cdots, r$. Let $C = (\mathbf{x}, \mathbf{y}] \in \mathcal{C}$ be as described above. The present proof will employ the elements found in the first paragraph of the proof of Proposition A.7.1. For the reasons mentioned in the second paragraph of the same proof, we need only consider the case in which $P(\bigcap_{i=1}^r [\mathbf{Y}_i \in S_{\mathbf{x}}]) > 0$. As before, this condition enables us to assume, w.l.o.g., that $P(\bigcap_{i=1}^r [\mathbf{Y}_i \in S_{\mathbf{x}_n}]) > 0 \ \forall n$.

The following result extends Lemma A.7.5.

Claim: $\bigcap_{i=1}^r[\mathbf{Y}_i \in S_{\mathbf{x}_n}]$ is an atomic event in $\bigvee_{i=1}^r \mathcal{H}^{(i)}_{L_{\mathbf{x}_n}}$ $\forall n$.

Proof: Fix $n \in \mathbf{N}$ and define $H = \bigcap_{i=1}^r[\mathbf{Y}_i \in S_{\mathbf{x}_n}]$. By Lemma A.7.5, $[\mathbf{Y}_i \in S_{\mathbf{x}_n}]$ is an atomic event in $\mathcal{H}^{(i)}_{L_{\mathbf{x}_n}}$ $\forall 1 \le i \le r$ which implies $H \in \bigvee_{i=1}^r \mathcal{H}^{(i)}_{L_{\mathbf{x}_n}}$.

Moreover, given any $1 \le i \le r$ and any $E_i \in \mathcal{E}^{(i)}_{L_{\mathbf{x}_n}}$, (A.49) implies

$$P(H \cap E_i) \;\le\; P([\mathbf{Y}_i \in S_{\mathbf{x}_n}] \cap E_i) \;=\; 0,$$

i.e., $P(H \cap E) = 0$ $\forall E \in \bigcup_{i=1}^r \mathcal{E}^{(i)}_{L_{\mathbf{x}_n}}$. Therefore, since $\sigma(\bigcup_{i=1}^r \mathcal{E}^{(i)}_{L_{\mathbf{x}_n}}) = \bigvee_{i=1}^r \mathcal{H}^{(i)}_{L_{\mathbf{x}_n}}$, Lemma A.7.4 implies H is an atomic event in $\bigvee_{i=1}^r \mathcal{H}^{(i)}_{L_{\mathbf{x}_n}}$. Ω

For each $m \in \mathbf{N}$, define the random variable

$$V_m \;=\; \prod_{i=1}^r \mathbf{1}_{[\mathbf{Y}_i \in C_m]}\, W.$$

Given any $m \le n$, $C_m \subseteq C_n \subseteq S_{\mathbf{x}_n}$ implies $V_m = 0$ a.e. outside the set $\bigcap_{i=1}^r[\mathbf{Y}_i \in S_{\mathbf{x}_n}]$. Therefore, since

$$\sigma(V_m) \vee \bigvee_{i=1}^r \mathcal{H}^{(i)}_{L_{\mathbf{x}_n}} \text{ is independent of } \bigvee_{j>r} \mathcal{H}^{(j)}_{L_{\mathbf{x}_n}}, \quad (\forall m, n)$$

(a consequence of (I) and (II) on p.182), the argument for (A.53) — with the role of Lemma A.7.5 played by the above Claim — implies

$$E(V_m|\,\mathcal{H}_{L_{\mathbf{x}_n}}) \;=\; \prod_{i=1}^r \mathbf{1}_{[\mathbf{Y}_i \in S_{\mathbf{x}_n}]} \cdot E(W) \cdot \mathbf{F}(C_m)^r \cdot [\mathbf{F}(S_{\mathbf{x}_n})]^{-r}, \quad (\forall m \le n).$$

(Note that $P(\bigcap_{i=1}^r[\mathbf{Y}_i \in S_{\mathbf{x}_n}]) = [\mathbf{F}(S_{\mathbf{x}_n})]^r$ by independence.) By (A.50), $\lim_n \prod_{i=1}^r \mathbf{1}_{[\mathbf{Y}_i \in S_{\mathbf{x}_n}]} = \prod_{i=1}^r \mathbf{1}_{[\mathbf{Y}_i \in S_{\mathbf{x}}]}$ and $\lim_m V_m = \prod_{i=1}^r \mathbf{1}_{[\mathbf{Y}_i \in C]}\, W$ on Ω, allowing us to repeat the limiting procedures which followed (A.53) and thus completing the proof of Proposition A.7.9. □

A.8 Additional Results on Flows

This section contains two miscellaneous results concerning the flows defined in Subsection 4.2.2. Lemma A.8.1 is used in Subsection 4.6.3 to show asymptotic rarefaction of jumps for sequences of set-indexed weighted empirical processes (see Lemma 4.6.18). Lemma A.8.2, although never used in the thesis, has not appeared in the earlier literature, warranting its inclusion here.

We begin with some terminology. Let $\mathcal{A}$ be an indexing collection on a compact metric space (T, d) and let $f : [a, b] \to \mathcal{A}(u)$ be any flow. Define

$$f(t-) = \bigcup_{a<s<t} f(s), \quad (\forall t \in (a, b]) \tag{A.55}$$

and adopt the convention, $f(a-) = f(a)$. Given any $t \in [a, b]$, the *jump of f at t* is defined to be the subset

$$\Delta f(t) = f(t) \setminus f(t-). \tag{A.56}$$

Let $x : \mathcal{A} \to \mathbf{R}$ be a purely atomic set-function (see Definition 2.3.5). Note that the domain of x can be extended to all subsets of T. Also note that the function $x \circ f : [a, b] \to \mathbf{R}$ is pure-jump and hence lies in $D[0, a]$.

Lemma A.8.1 *If f and x are as described above, then*

$$\Delta\, x \circ f(t) = x(\Delta f(t)), \quad (\forall t \in [a, b])$$

where $\Delta\, x \circ f(t)$ denotes the jump of the function $x \circ f : [a, b] \to \mathbf{R}$ at t.

Proof Take $t \in [a, b]$. If $t = a$, then

$$\Delta\, x \circ f(t) = 0 = x(\phi) = x(\Delta f(a)).$$

On the other hand, if $t \in (a, b]$, we have the following:

Claim: $\lim_{s<t} x(f(s)) = x(f(t-))$.

Proof: Let $\xi_1, \cdots, \xi_n \in T$ denote the atoms of x that lie in $f(t-)$. By the definition of flow, f is increasing w.r.t. $\subseteq$ on $[a, b]$. Therefore, for each $1 \le i \le n$, $\exists\, s_i \in (a, t)$ s.t.

$$\xi_i \in f(s), \quad (\forall\, s_i < s < t).$$

Define $s_0 = \max_{1 \le i \le n} s_i < t$. Clearly,

$$\{\xi_i : 1 \le i \le n\} \subseteq f(s), \quad (\forall\, s_0 < s < t).$$

Furthermore, since $f(s) \subseteq f(t-)\ \forall s < t$,

$$\{\text{atoms of } x \text{ in } f(s)\} \subseteq \{\xi_i : 1 \le i \le n\}, \quad (\forall s_0 < s < t).$$

Combining these two inclusions, we obtain $x(f(s)) = x(f(t-))\ \forall s_0 < s < t$ which completes the proof of the Claim. Ω

Finally, we have

$$\begin{aligned} \Delta x \circ f(t) &= x(f(t)) - \lim_{s<t} x(f(s)) \\ &= x(f(t)) - x(f(t-)) \quad \text{(by above Claim)} \\ &= x(f(t) \setminus f(t-)) \\ &= x(\Delta f(t)) \end{aligned}$$

which completes the proof of Lemma A.8.1. □

By definition, every flow is increasing w.r.t. $\subseteq$. As illustrated by the following result, the flows given via Lemma 4.2.16 are, in fact, strictly increasing.

Lemma A.8.2 *Given any f.s.s.l. $\mathcal{A}'$ of $\mathcal{A}$, the simple flow $f : [0,1] \to \mathcal{A}(u)$ satisfying conditions* (a) *and* (b) *of Lemma 4.2.16 can be chosen so that $0 \le s < t \le k$ implies $f(s) \subset f(t)$.*

Proof Take a f.s.s.l. $\mathcal{A}' = \{A_0, \cdots, A_k\}$ of $\mathcal{A}$ and let f be the associated simple flow discussed in Lemma 4.2.16. By (4.7), f has the form

$$f(t) = \left[\cup_{r=1}^{j-1} f_r(r/k)\right] \cup f_j(t), \quad (\forall t \in \left(\tfrac{j-1}{k}, \tfrac{j}{k}\right], 1 \le j \le k)$$

where each $f_j : [\frac{j-1}{k}, \frac{j}{k}] \to \mathcal{A}$ is a flow. Therefore, it is clearly sufficient that $f_j : [\frac{j-1}{k}, \frac{j}{k}] \to \mathcal{A}$ be stricly increasing $\forall j$. The proof of this fact draws heavily from the development of Lemma 2 in [26].

Given $1 \le j \le k$, a linear transformation allows us to write $f_j : [0,1] \to \mathcal{A}$. Following the proof of Lemma 2 in [26] — from which Lemma 4.2.16 originates — we can select f_j so that $f_j(1) = A_j$ and $f_j(0) = A'$. We comment no further on A' other than to say that $A' \in \mathcal{A}'$ and $A' \subset A_j$. Define $\mathcal{R}$ to be the subset of $[0,1]$ consisting of all i/r $(i, r \in \mathrm{N})$ for which $(i+1)/r \in [0,1]$ and $\exists\, S, T \in \mathcal{A}$ s.t.

$$f_j(i/r) = S, \ \ f_j((i+1)/r) = T \ \text{ and } \ S \subset T. \qquad \text{(A.57)}$$

From the proof of Lemma 2 in [26], $\mathcal{R}$ is dense in $[0,1]$. (In particular, $\mathcal{J} \subseteq \mathcal{R}$ where $\mathcal{J}$ is the dense subset of $[0,1]$ defined on p.907 of [26].)

Now, take $s < t$ in $[0,1]$ and select $\epsilon_0 > 0$ small enough so that $s < t - \epsilon_0$. Since $\mathcal{R}$ is dense in $[0,1]$, $\exists\, i/r \in \mathcal{R}$ s.t. $s < i/r < t - \epsilon_0$. Moreover, i/r can be chosen so that $1/r < \epsilon_0$. (If this were not so, $\mathcal{R} \cap (s, t + \epsilon_0)$ would be a finite set, contradicting the density of $\mathcal{R}$.) This implies $s < i/r < (i+1)/r < t$ which by (A.57) implies $f_j(s) \subset f_j(t)$, hence completing the proof. □

Appendix B

On Conditional Independence

B.1 Introduction

In this appendix, after defining and studying conditional independence for sub-σ-algebras of a fixed σ-algebra, we will present a class of stochastic bases for which the following is true: *given any set* $C \in \mathcal{C}$, *any representation,* $A \setminus \bigcup_{i=1}^{n} A_i$ *of* C *and any* $\mathcal{F}_A$*-measurable random variable* $X \in L_1$,

- $E(X \mid \mathcal{G}_C^*)$ *is* $\mathcal{F}_A$*-measurable and*
- $E(X \mid \mathcal{G}_C^*) = E(X \mid \bigcap_k \bigvee_{i=1}^{n} \mathcal{F}_{g_k(A_i)})$.

As the title suggests, conditional independence will play a central role in the proof of both results.

B.2 On General Conditional Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, define the linear subspace $L_1(\mathcal{G}) := \{X \in L_1 : X \text{ is } \mathcal{G}\text{-measurable}\}$. Our definition of conditional independence is taken from p.36-II of [14].

Definition B.2.1 *Let* $\mathcal{F}_1$, $\mathcal{F}_2$ *and* $\mathcal{F}_3$ *be sub-*σ*-algebras of* $\mathcal{F}$. *We say that* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are* <u>*conditionally independent*</u> *given* $\mathcal{F}_3$ *if*

$$E(X_1 \cdot X_2 \mid \mathcal{F}_3) = E(X_1 \mid \mathcal{F}_3) \cdot E(X_2 \mid \mathcal{F}_3) \tag{B.1}$$

for every $X_1 \in L_1(\mathcal{F}_1)$ *and* $X_2 \in L_1(\mathcal{F}_2)$. *In such a case, we will write* $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \mid \mathcal{F}_3 \rangle$ *(referred to as a "bracket").*

Remark B.2.2 As commented on p.36-II of [14], it is sufficient to test (B.1) for the case in which $X_i = \mathbf{1}_{F_i}$ $(i = 1, 2)$ where $F_i \in \mathcal{F}_i$ $(i = 1, 2)$.

We will make frequent use of the following characterization of conditional independence. Its proof can be found on p.36-II of [14].

Lemma B.2.3 *If $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are sub-σ-algebras of $\mathcal{F}$, then t.f.a.e.:*

(a) $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{F}_3 \rangle$,

(b) $E(X \,|\, \mathcal{F}_3) = E(X \,|\, \mathcal{F}_1 \vee \mathcal{F}_3) \quad \forall X \in L_1(\mathcal{F}_2)$.

Next, we present some important properties for brackets. The first Lemma is an obvious consequence of Definition B.2.1, the second is a special case of Lemma 2.2 in [11].

Lemma B.2.4 *Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be sub-σ-algebras of $\mathcal{F}$ for which $\mathcal{G}_i \subseteq \mathcal{F}_i$ $(i = 1, 2)$. If $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{F}_3 \rangle$, then*

(a) $\langle \mathcal{F}_2 \perp \mathcal{F}_1 \,|\, \mathcal{F}_3 \rangle$ *and*

(b) $\langle \mathcal{G}_1 \perp \mathcal{G}_2 \,|\, \mathcal{F}_3 \rangle$.

Lemma B.2.5 *Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{G}$ be sub-σ-algebras of $\mathcal{F}$ for which $\mathcal{G} \subseteq \mathcal{F}_1 \vee \mathcal{F}_2$. If $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{F}_3 \rangle$, then*

(a) $\langle (\mathcal{F}_1 \vee \mathcal{F}_3) \perp (\mathcal{F}_2 \vee \mathcal{F}_3) \,|\, \mathcal{F}_3 \rangle$ *and*

(b) $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, (\mathcal{F}_3 \vee \mathcal{G}) \rangle$.

We close this section with a simple application of Lemmas B.2.4 and B.2.5.

Lemma B.2.6 *Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$. If*

(i) $\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{H} \rangle$ *and*

(ii) $\mathcal{G} \subseteq \mathcal{F}_2$,

then $\langle (\mathcal{F}_1 \vee \mathcal{G}) \perp \mathcal{F}_2 \,|\, (\mathcal{H} \vee \mathcal{G}) \rangle$.

Proof Since $\mathcal{G} \subseteq \mathcal{F}_1 \vee \mathcal{F}_2$, we have the following chain of implications:

$$\begin{aligned}
\langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, \mathcal{H} \rangle \;&\Rightarrow\; \langle \mathcal{F}_1 \perp \mathcal{F}_2 \,|\, (\mathcal{H} \vee \mathcal{G}) \rangle && \text{(by Lemma B.2.5 (b))} \\
&\Rightarrow\; \langle (\mathcal{F}_1 \vee \mathcal{H} \vee \mathcal{G}) \perp (\mathcal{F}_2 \vee \mathcal{H} \vee \mathcal{G}) \,|\, (\mathcal{H} \vee \mathcal{G}) \rangle \\
& && \text{(by Lemma B.2.5 (a))} \\
&\Rightarrow\; \langle (\mathcal{F}_1 \vee \mathcal{G}) \perp \mathcal{F}_2 \,|\, (\mathcal{H} \vee \mathcal{G}) \rangle && \text{(by Lemma B.2.4 (b))}\,.
\end{aligned}$$

This is precisely Lemma B.2.6. □

B.3 Applications

In this section, after presenting a few technical lemmas, we will establish the two itemized results listed in the introduction to this appendix. Both results require some additional assumptions.

Fix a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$. Our first additional assumption concerns the filtration $(\mathcal{F}_A)_{A\in\mathcal{A}}$.

Assumption B.3.1 *Given any $A_1, A_2 \in \mathcal{A}$ and any $X \in L_1$,*

$$E[\,E(X\,|\,\mathcal{F}_{A_1})\,|\,\mathcal{F}_{A_2}] \;=\; E(X\,|\,\mathcal{F}_{A_1\cap A_2}). \tag{B.2}$$

Assumption B.3.1 has already appeared in [16]. As commented therein, (B.2) reduces to the classic F4 property when $\mathcal{A} = \mathcal{I}_2$.

Lemma B.3.2 *Let $A_1, \cdots, A_n \in \mathcal{A}$ be given. Under Assumption B.3.1,*

(a) $\bigcap_{i=1}^{n} \mathcal{F}_{A_i} = \mathcal{F}_{\bigcap_{i=1}^{n} A_i}$ *and*

(b) $\langle \mathcal{F}_{A_1} \perp \mathcal{F}_{A_2} \,|\, \mathcal{F}_{A_1\cap A_2}\rangle$.

Proof Part (a) is a trivial inductive extension of Lemma 2.3 (i) in [16].

For part (b), take events $F_i \in \mathcal{F}_{A_i}$ $(i = 1, 2)$ with corresponding indicator functions f_i $(i = 1, 2)$. Then,

$$\begin{aligned} E(f_1 f_2\,|\,\mathcal{F}_{A_1\cap A_2}) &= E[\,E(f_1 f_2\,|\,\mathcal{F}_{A_1})\,|\,\mathcal{F}_{A_2}] \quad \text{(by (B.2))} \\ &= E[\,f_1 \cdot E(f_2\,|\,\mathcal{F}_{A_1})\,|\,\mathcal{F}_{A_2}]. \end{aligned} \tag{B.3}$$

Since f_1 is $\mathcal{F}_{A_1}$-measurable, (B.2) implies $E(f_1\,|\,\mathcal{F}_{A_2}) = E(f_1\,|\,\mathcal{F}_{A_1\cap A_2})$. Similarly, $E(f_2\,|\,\mathcal{F}_{A_1}) = E(f_2\,|\,\mathcal{F}_{A_1\cap A_2})$. Applying these identities to (B.3),

$$\begin{aligned} E(f_1 f_2\,|\,\mathcal{F}_{A_1\cap A_2}) &= E[\,f_1 \cdot E(f_2\,|\,\mathcal{F}_{A_1\cap A_2})\,|\,\mathcal{F}_{A_2}] \\ &= E(f_2\,|\,\mathcal{F}_{A_1\cap A_2}) \cdot E(f_1\,|\,\mathcal{F}_{A_2}) \\ &= E(f_2\,|\,\mathcal{F}_{A_1\cap A_2}) \cdot E(f_1\,|\,\mathcal{F}_{A_1\cap A_2}) \end{aligned}$$

which, by Remark B.2.2, establishes (b). □

Recall that any indexing collection is a sub-semilattice under $\wedge = \cap$. In addition, we will need the following assumption on $\mathcal{A}$.

Assumption B.3.3 *$\exists$ a binary operation, $\vee$ on $\mathcal{A}$ s.t. $\mathcal{A}$ is a distributive lattice under $\vee$ and $\wedge = \cap$.*

This assumption was common in earlier set-indexed papers such as [16] and [28]. The indexing collections $\mathcal{I}_k$ and $\mathcal{LL}_k$ on $[0,1]^k$ satisfy Assumption B.3.3 (see Examples 2.2.6 and 2.8.1 respectively).

We now present the key technical result of this section. Its proof requires several judicious applications of Lemma B.2.6.

Lemma B.3.4 *Under Assumptions B.3.1 and B.3.3, if $A, A_1, \cdots, A_n \in \mathcal{A}$ are s.t. $A_i \cap A_j \subseteq A \;\forall i \neq j$, then*

$$E(X \mid \vee_{i=1}^n \mathcal{F}_{A_i}) \;=\; E(X \mid \vee_{i=1}^n \mathcal{F}_{A\cap A_i})$$

for every $X \in L_1(\mathcal{F}_A)$.

Proof Since $\vee$ is distributive over $\cap$ and $A_1 \cap A_i \subseteq A_1 \cap A \;\forall 2 \leq i \leq n$,

$$A_1 \cap [A \vee (\vee_{i=2}^n A_i)] \;=\; (A_1 \cap A) \vee (\vee_{i=2}^n A_1 \cap A_i) \;=\; A_1 \cap A \qquad \text{(B.4)}$$

which, by Lemma B.3.2 (b), implies

$$\langle \mathcal{F}_{A_1} \perp \mathcal{F}_{A\vee(\vee_{i=2}^n A_i)} \mid \mathcal{F}_{A\cap A_1} \rangle.$$

Therefore, if we define $\mathcal{G} = \vee_{i=2}^n \mathcal{F}_{A_i}$, then by Lemma B.2.6,

$$\langle (\vee_{i=1}^n \mathcal{F}_{A_i}) \perp \mathcal{F}_{A\vee(\vee_{i=2}^n A_i)} \mid \mathcal{F}_{A\cap A_1} \vee (\vee_{i=2}^n \mathcal{F}_{A_i}) \rangle$$

which by Lemma B.2.4 (b) implies

$$\langle (\vee_{i=1}^n \mathcal{F}_{A_i}) \perp \mathcal{F}_A \mid \mathcal{F}_{A\cap A_1} \vee (\vee_{i=2}^n \mathcal{F}_{A_i}) \rangle. \qquad \text{(B.5)}$$

Next, take any $2 \leq j \leq n-1$. By an argument similar to (B.4),

$$A_j \cap [A \vee (\vee_{i=j+1}^n A_i)] \;=\; A_j \cap A$$

which by Lemma B.3.2 (b) implies

$$\langle \mathcal{F}_{A_j} \perp \mathcal{F}_{A\vee(\vee_{i=j+1}^n A_i)} \mid \mathcal{F}_{A\cap A_j} \rangle.$$

If we let $\mathcal{G} = (\vee_{i=1}^{j-1} \mathcal{F}_{A\cap A_i}) \vee (\vee_{i=j+1}^n \mathcal{F}_{A_i})$, then by Lemma B.2.6,

$$\langle\, (\vee_{i=1}^{j-1} \mathcal{F}_{A\cap A_i}) \vee (\vee_{i=j}^n \mathcal{F}_{A_i}) \;\perp\; \mathcal{F}_A \mid (\vee_{i=1}^{j} \mathcal{F}_{A\cap A_i}) \vee (\vee_{i=j+1}^n \mathcal{F}_{A_i}) \,\rangle \qquad \text{(B.6)}$$

when we replace $\mathcal{F}_{A\vee(\vee_{i=j+1}^{n}A_i)}$ by $\mathcal{F}_A$ via Lemma B.2.4 (b).

Finally, since $\langle \mathcal{F}_{A_n} \perp \mathcal{F}_A \mid \mathcal{F}_{A\cap A_n}\rangle$, Lemma B.2.6 implies

$$\langle\, ((\vee_{i=1}^{n-1}\mathcal{F}_{A\cap A_i}) \vee \mathcal{F}_{A_n}) \perp \mathcal{F}_A \mid \vee_{i=1}^{n}\mathcal{F}_{A\cap A_i}\rangle \tag{B.7}$$

when we add $\mathcal{G} = \vee_{i=1}^{n-1}\mathcal{F}_{A\cap A_i}$ throughout the previous bracket.

Now, take any $X \in L_1(\mathcal{F}_A)$. By Lemma B.2.3, (B.5) implies

$$E(X \mid \vee_{i=1}^{n}\mathcal{F}_{A_i}) \;=\; E[\,X \mid \mathcal{F}_{A\cap A_1} \vee (\vee_{i=2}^{n}\mathcal{F}_{A_i})\,] \tag{B.8}$$

while for each $2 \le j \le n-1$, (B.6) implies

$$E[\,X \mid (\vee_{i=1}^{j-1}\mathcal{F}_{A\cap A_i})\vee(\vee_{i=j}^{n}\mathcal{F}_{A_i})\,] \;=\; E[\,X \mid (\vee_{i=1}^{j}\mathcal{F}_{A\cap A_i})\vee(\vee_{i=j+1}^{n}\mathcal{F}_{A_i})\,] \tag{B.9}$$

and (B.7) implies

$$E[\,X \mid (\vee_{i=1}^{n-1}\mathcal{F}_{A\cap A_i}) \vee \mathcal{F}_{A_n}] \;=\; E(X \mid \vee_{i=1}^{n}\mathcal{F}_{A\cap A_i}). \tag{B.10}$$

Linking the n indentities from (B.8), (B.9) and (B.10), we obtain

$$E(X \mid \vee_{i=1}^{n}\mathcal{F}_{A_i}) \;=\; E(X \mid \vee_{i=1}^{n}\mathcal{F}_{A\cap A_i})$$

which completes the proof. □

Like the first, our third additional assumption concerns the filtration.

Assumption B.3.5 *Given sets $A, A_1, \cdots, A_n \in \mathcal{A}$, if $A \subseteq \cup_{i=1}^{n} A_i$, then $\mathcal{F}_A \subseteq \vee_{i=1}^{n}\mathcal{F}_{A_i}$.*

Remark B.3.6 By (3.9), Assumption B.3.5 implies $\mathcal{F}^{\circ}_{\cup_{i=1}^{n}A_i} \subseteq \vee_{i=1}^{n}\mathcal{F}_{A_i}$. In conjunction with Lemma 3.2.24 (c), this implies $\mathcal{F}^{\circ}_{\cup_{i=1}^{n}A_i} = \vee_{i=1}^{n}\mathcal{F}_{A_i}$.

If $\mathcal{A}$ satisfies the shape property (see Remark 2.2.5 (d)), then any $\mathcal{A}$-indexed filtration automatically satisfies Assumption B.3.5. Moreover, given any indexing collection $\mathcal{A}$ and any process $X = (X_A)_{A\in\mathcal{A}}$ possessing a finitely additive extension to $\mathcal{C}(u)$, the family $(\mathcal{H}_A)_{A\in\mathcal{A}}$ where

$$\mathcal{H}_A := \sigma(\{X_B : B \in \mathcal{A} \text{ s.t. } B \subseteq A\}), \quad (\forall\, A \in \mathcal{A})$$

can be easily shown to satisfy Assumption B.3.5.

Lemma B.3.7 *Under Assumption B.3.5, if $A \setminus \bigcup_{i=1}^m B_i$ is a maximal representation of a set $C \in \mathcal{C}$, then $\mathcal{G}_C^* = \bigcap_k \bigvee_{i=1}^m \mathcal{F}_{g_k(B_i)}$.*

Proof Letting $B = \bigcup_{i=1}^m B_i$,

$$\begin{aligned} \mathcal{G}_C^* &= \bigcap_n \mathcal{F}^\circ_{g_k(B)} \quad \text{(by Proposition 3.2.25)} \\ &= \bigcap_n \bigvee_{i=1}^m \mathcal{F}_{g_k(B_i)} \quad \text{(by Remark B.3.6)} \end{aligned}$$

which completes the proof. □

Our final additional assumption concerns the indexing collection. Examples for which it is satisfied include $\mathcal{I}_2$ and $\mathcal{LL}_2$.

Assumption B.3.8 *Given any $n \in \mathbf{N}$ and any set $C = A \setminus \bigcup_{i=1}^n A_i \in \mathcal{C}$, $\exists$ a maximal representation $A \setminus \bigcup_{i=1}^m B_i$ of C s.t. $B_i \cap B_j \subseteq A \;\; \forall i \neq j$.*

And now for the main result of Appendix B,

Proposition B.3.9 *Take any $n \in \mathbf{N}$ and any $C = A \setminus \bigcup_{i=1}^n A_i \in \mathcal{C}$. Under Assumptions B.3.1, B.3.3, B.3.5 and B.3.8, if $X \in L_1(\mathcal{F}_A)$, then*

$$E(X \mid \mathcal{G}_C^*) = E(X \mid \bigcap_k \bigvee_{i=1}^n \mathcal{F}_{g_k(A_i)}).$$

Proof Let $A \setminus \bigcup_{i=1}^m B_i$ be the maximal representation of C described in Assumption B.3.8. Since each g_k is monotone increasing and preserves finite intersections,

$$g_k(B_i) \cap g_k(B_j) \subseteq g_k(A), \quad (\forall k, \; i \neq j).$$

Hence by Theorem B.3.4,

$$E[X \mid \bigvee_{i=1}^n \mathcal{F}_{g_k(B_i)}] = E[X \mid \bigvee_{i=1}^n \mathcal{F}_{g_k(A \cap B_i)}], \quad (\forall k).$$

Since $(g_k(B))_k$ is decreasing in $\mathcal{A}$ $\forall B \in \mathcal{A}$, the reverse martingale convergence theorem and Lemma B.3.7 thus imply

$$E[X \mid \bigcap_k \bigvee_{i=1}^n \mathcal{F}_{g_k(A \cap B_i)}] = E[X \mid \bigcap_k \bigvee_{i=1}^n \mathcal{F}_{g_k(B_i)}] = E[X \mid \mathcal{G}_C^*]. \quad \text{(B.11)}$$

Therefore, if we can show

$$\bigcap_k \bigvee_{i=1}^m \mathcal{F}_{g_k(A \cap B_i)} \subseteq \bigcap_k \bigvee_{i=1}^n \mathcal{F}_{g_k(A_i)} \subseteq \mathcal{G}_C^*, \quad \text{(B.12)}$$

Proposition B.3.9 will follow from (B.11) and the tower property.

We quickly dispose of the right-most inclusion in (B.12) by noting that $\bigcap_k \bigvee_{i=1}^n \mathcal{F}_{g_k(A_i)} \subseteq \mathcal{F}_{\bigcup_{i=1}^n A_i}$ and $C \cap [\bigcup_{i=1}^n A_i] = \phi$ (see Lemma 3.2.24, parts (c) and (e) respectively). For the left-most inclusion in (B.12), note that

$$\begin{aligned} A \setminus \bigcup_{j=1}^m A \cap B_j \;=\; A \setminus \bigcup_{i=1}^n A \cap A_i \;&\Longrightarrow\; \bigcup_{j=1}^m A \cap B_j \;=\; \bigcup_{i=1}^n A \cap A_i \\ &\Longrightarrow\; A \cap B_j \;\subseteq\; \bigcup_{i=1}^n A_i \;\; \forall j \end{aligned}$$

which, by Proposition 2.2.9 (b) and the monotonicity of g_k, implies

$$g_k(A \cap B_j) \;\subseteq\; \bigcup_{i=1}^n g_k(A_i)\,, \quad (\forall\, k,\ 1 \le j \le m).$$

Therefore, by Assumption B.3.5,

$$\mathcal{F}_{g_k(A \cap B_j)} \;\subseteq\; \bigvee_{i=1}^n \mathcal{F}_{g_k(A_i)}\,, \quad (\forall\, k,\ 1 \le j \le m)$$

which is to say

$$\bigvee_{i=1}^m \mathcal{F}_{g_k(A \cap B_i)} \;\subseteq\; \bigvee_{i=1}^n \mathcal{F}_{g_k(A_i)}\,, \quad (\forall\, k).$$

This completes (B.12) and the proof of Proposition B.3.9. □

Given any sets $A, B_1, \cdots, B_k \in \mathcal{A}$, the monotonicity of g_k implies

$$\bigvee_{i=1}^k \mathcal{F}_{g_k(A \cap B_j)} \;\subseteq\; \mathcal{F}_{g_k(A)}\,, \quad (\forall\, k \in \mathrm{N})$$

whereas the right-continuity of $(\mathcal{F}_A)_{A \in \mathcal{A}}$ implies

$$\bigcap_n \bigvee_{i=1}^k \mathcal{F}_{g_k(A \cap B_j)} \;\subseteq\; \mathcal{F}_A.$$

Therefore, (B.11) yields the following consequence.

Corollary B.3.10 *Under Assumptions B.3.1, B.3.3, B.3.5 and B.3.8, given any $C = A \setminus B \in \mathcal{C}$ ($A \in \mathcal{A}$, $B \in \mathcal{A}(u)$),*

$$X \in L_1(\mathcal{F}_A) \;\Longrightarrow\; E(X \mid \mathcal{G}_C^*) \;\in\; L_1(\mathcal{F}_A). \tag{B.13}$$

Appendix C

An Alternative to $D(\mathcal{A})$

C.1 Introduction

The function space $D(\mathcal{A})$ defined in Section 2.3 has already been successful in representing sample paths of set-indexed processes. For example, see [5] and [26]. Unfortunately, it is not without shortcomings. In particular, when $\mathcal{A}$ is the "natural" indexing collection, $\mathcal{I}_k$ on $[0,1]^k$ ($k \geq 2$), $D(\mathcal{I}_k)$ is too large to properly represent the classical multiparameter function space D_k introduced by Neuhaus in [33] (see Definition C.2.3 and Example C.2.12).

In this appendix, we present an alternative to $D(\mathcal{A})$, denoted $D_p(\mathcal{A})$, which is defined for any indexing collection $\mathcal{A}$ on any compact metric space (T, d). This new function space will be small enough so that $D_p(\mathcal{I}_k) = D_k \ \forall\, k$ (up to identification) while being large enough to ensure $PA, D_0(\mathcal{A}) \subseteq D_p(\mathcal{A})$ for any $\mathcal{A}$ on any (T, d). Although not used directly in this thesis, the function space $D_p(\mathcal{A})$ may be of independent interest.

Throughout this appendix, unless otherwise stated, $\mathcal{A}$ denotes a generic indexing collection on a generic compact metric space (T, d).

C.2 The Function Space $D_p(\mathcal{A})$

The classical multiparameter function spaces, D_k ($k = 1, 2, \cdots$) defined by Neuhaus in [33] serve as multidimensional analogues of $D[0,1]$. Whereas $D[0,1]$ is the collection of all functions, $f : [0,1] \to \mathbf{R}$ which are right-continuous with left-limits at each $t \in [0,1]$, D_k consists of all functions

$f : [0,1]^k \to \mathbf{R}$ which, for any $\mathbf{t} \in [0,1]^k$, have "limits in all quadrants" at $\mathbf{t}$ with "continuity from above" at $\mathbf{t}$ (see Definition C.2.3).

As will be illustrated in Example C.2.12, $D(\mathcal{I}_k)$ and D_k do not coincide when $k \geq 2$. The goal of this section is to construct a set-indexed analogue of D_k, denoted $D_p(\mathcal{A})$ which coincides with D_k when $\mathcal{A} = \mathcal{I}_k$ ($k \in \mathrm{N}$). For this purpose, we must characterize "limits in all quadrants" strictly in terms of the sets in $\mathcal{I}_k$ so as to free ourselves from the co-ordinate notation in [33] which only applies to $T = [0,1]^k$. At the heart of this characterization lies in the concepts of *proper intervals* and *finite interval partitions*. Their definitions, which originally appeared in [25], are given below.

Definition C.2.1 (a) *Given sets, $A, A' \in \mathcal{A}$, the set-interval $[A, A')$ in $\mathcal{A}$ is the subcollection,*

$$[A, A') = \{B \in \mathcal{A} : A \subseteq B \subseteq (A')^\circ\}.$$

If $A \subseteq (A')^\circ$ and A is proper (see Definition 2.3.7), $[A, A')$ is said to be a proper interval in $\mathcal{A}$.

(b) *A finite interval partition, Δ of $\mathcal{A}$ is any finite collection of disjoint proper intervals in $\mathcal{A}$ s.t.*

$$\mathcal{A} = \bigcup \{ [A, A') : [A, A') \in \Delta\}.$$

(c) *A set-function $x : \mathcal{A} \to \mathbf{R}$ is simple provided $\exists$ a finite interval partition, Δ of $\mathcal{A}$ s.t. x is constant on each $[A, A') \in \Delta$.*

Recall the indexing collection $\mathcal{I}_k$ on $[0,1]^k$ ($k \in \mathrm{N}$) defined in Example 2.2.6. Given any $\mathbf{t} \in [0,1]^k$, to save space, we will often write $A_\mathbf{t}$ for the set $[0, \mathbf{t}]$. As mentioned in Section 2.3, this does not conflict with the definition of $A_\mathbf{t}$ given in (2.8). The following properties of proper intervals in $\mathcal{I}_k$ will be needed later in this section.

Proposition C.2.2 *Let $\mathbf{u} = (u_1, \cdots, u_k)$, $\mathbf{v} = (v_1, \cdots, v_k) \in [0,1]^k$ be given.*

(i) $[A_\mathbf{u}, A_\mathbf{v}) = \{A_\mathbf{w} \in \mathcal{I}_k : \mathbf{w} \in \prod_{i=1}^k [u_i, v_i\rangle\}$ *where* $[a, b\rangle = [a, b)$ *if* $b < 1$ *and* $[a, b\rangle = [a, b]$ *if* $b = 1$.

(ii) $[A_\mathbf{u}, A_\mathbf{v})$ *is proper if and only if* $u_i < v_i \ \forall 1 \leq i \leq n$.

(iii) *If one of $[C, C')$ or $[D, D')$ is proper in $\mathcal{I}_k$, then $[C, C') \cap [D, D')$ is either proper or empty.*

Proof (i) follows from Definition C.2.1 (a), (ii) follows from (2.11) and (iii) follows from (i) and (ii). □

We now present the framework (as found in [33]) required for the definition of the classical function space, D_k ($k \in \mathbf{N}$). Let

$$\wp = \{\rho \in [0,1]^k : \rho_i = 1 \text{ or } \rho_i = 0 \ \forall 1 \le i \le k\}.$$

Given $\mathbf{t} = (t_1, \cdots, t_k) \in [0,1]^k$, the finite collection of *quadrants at* $\mathbf{t}$, denoted

$$\tilde{\mathcal{Q}}(\mathbf{t}) = \{\tilde{Q}(\rho, \mathbf{t}) : \rho \in \wp\}$$

is defined by $\tilde{Q}(\rho, \mathbf{t}) = \prod_{i=1}^k \tilde{I}(\rho_i, t_i)$ ($\rho \in \wp$) where

$$\tilde{I}(\rho_i, t_i) = \begin{cases} \phi & \text{, if } \rho_i = t_i = 1 \\ [0, t_i\rangle & \text{, if } \rho_i = 0 \\ [t_i, 1] & \text{, if } \rho_i = 1 \text{ and } t_i < 1. \end{cases}$$

Note that $\bigcup_{\rho \in \wp} \tilde{Q}(\rho, \mathbf{t}) = [0,1]^k$.

Given any $\mathbf{t} = (t_1, \cdots, t_k) \in [0,1]^k$, the sets in $\tilde{\mathcal{Q}}(\mathbf{t})$ determine a finite interval partition, $\Delta_{\mathbf{t}}$ of $\mathcal{I}_k$ when we identify points in $[0,1]^k$ with sets in $\mathcal{I}_k$. To be precise, take any $\rho = (\rho_1, \cdots, \rho_k) \in \wp$ and define

$$\rho \wedge \mathbf{t} = (\rho_1 \wedge t_1, \cdots, \rho_k \wedge t_k) \quad \text{and} \quad \rho \vee \mathbf{t} = (\rho_1 \vee t_1, \cdots, \rho_k \vee t_k),$$

By Proposition C.2.2(ii), the set $[\tilde{Q}(\rho, \mathbf{t})\rangle \subseteq \mathcal{A}$ defined by

$$[\tilde{Q}(\rho, \mathbf{t})\rangle = \begin{cases} \phi & \text{, if } \rho_i = t_i \text{ for some } 1 \le i \le k \\ [A_{\rho \wedge \mathbf{t}}, A_{\rho \vee \mathbf{t}}\rangle & \text{, otherwise} \end{cases}$$

is either a proper interval in $\mathcal{I}_k$ or else it is empty. Furthermore,

$$\mathbf{u} \in \tilde{Q}(\rho, \mathbf{t}) \iff A_{\mathbf{u}} \in [\tilde{Q}(\rho, \mathbf{t})\rangle \tag{C.1}$$

and hence, $\bigcup_{\rho \in \wp} \tilde{Q}(\rho, \mathbf{t}) = [0,1]^k$ implies $\bigcup_{\rho \in \wp} [\tilde{Q}(\rho, \mathbf{t})\rangle = \mathcal{I}_k$. In total,

$$\Delta_{\mathbf{t}} = \{[\tilde{Q}(\rho, \mathbf{t})\rangle : \rho \in \wp\} \tag{C.2}$$

is a finite interval partition of $\mathcal{I}_k$.

Now, take $f : [0,1]^k \to \mathbf{R}$, $\mathbf{t} \in [0,1]^k$ and $\rho \in \wp$. If f satisfies the property,

$$(\mathbf{t}_n)_n \text{ in } \tilde{Q}(\rho, \mathbf{t}) \text{ and } \|\mathbf{t}_n - \mathbf{t}\|_\infty \to 0 \implies (f(\mathbf{t}_n))_n \text{ converges,} \tag{C.3}$$

we say that f has *ρ-quadrant limits* at $\mathbf{t}$. In such a case, it is clear that the limit of each such $(f(\mathbf{t}_n))_n$ is common. As in [33], we denote this common limit by $f(\mathbf{t}+O_\rho)$.

Given any $\mathbf{t} \in [0,1]^k$, since $[0,1]^k$ is the disjoint union of the sets in $\bar{\mathcal{Q}}(\mathbf{t})$, there is a unique $\sigma \in \wp$ such that $\mathbf{t} \in \bar{Q}(\sigma, \mathbf{t})$. As in [33], we refer to this $\bar{Q}(\sigma, \mathbf{t})$ as the *continuity quadrant* of $\mathbf{t}$.

The following definition has already appeared in [33].

Definition C.2.3 *Given $k \in \mathbf{N}$, D_k consists of all functions $f : [0,1]^k \to \mathbf{R}$ s.t., for any $\mathbf{t} \in [0,1]^k$, $f(\mathbf{t}+O_\rho)$ exists $\forall\, \rho \in \wp$ and $f(\mathbf{t}+O_\sigma) = f(\mathbf{t})$.*

Remark C.2.4 (a) The definition of D_k given in [33] uses "open" quadrants. As mentioned in [33], this definition is equivalent to that given in C.2.3. We use the sets $\bar{Q}(\rho, \mathbf{t})$ because of their close relation to proper intervals in $\mathcal{I}_k$.

(b) Given $\mathbf{t} \in [0,1]^k$, if f has ρ-quadrant limits at $\mathbf{t}$ $\forall\, \rho \in \wp$, then, taking $\mathbf{t}_n = \mathbf{t}$ $\forall n$, $f(\mathbf{t}+O_\sigma) = \lim_n f(\mathbf{t}_n) = f(\mathbf{t})$. In other words, if for every $\mathbf{t} \in [0,1]^k$, f has ρ-quadrant limits at $\mathbf{t}$ $\forall\, \rho \in \wp$, then $f \in D_k$.

(c) In general, we have the inclusion $[A_\mathbf{t}, T) \subseteq [\bar{Q}(\sigma, \mathbf{t})\rangle$. Furthermore, by (2.11), $\mathbf{t} \notin \omega(\mathcal{I}^k)$ implies $[\bar{Q}(\sigma, \mathbf{t})\rangle = [A_\mathbf{t}, T)$.

(d) When $k = 1$, D_k coincides with the space of all cadlag functions, $f : [0,1] \to \mathbf{R}$ which are continuous at $t = 1$.

Returning to the set-indexed setting, the following class of sequences will be central in the definition of $D_p(\mathcal{A})$.

Definition C.2.5 *A sequence $(A_n)_n$ in $\mathcal{A}$ is <u>proper</u> if, given any proper interval $[D, D')$ in $\mathcal{A}$, $\exists K \in \mathbf{N}$ s.t. either $A_n \in [D, D')$ $\forall n \geq K$ or $A_n \notin [D, D')$ $\forall n \geq K$.*

Remark C.2.6 By Lemma 2.5 in [25], any sequence $(A_n)_n$ in $\mathcal{A}$ for which $A_n \searrow A$ or $A_n \nearrow A$ (some $A \in \mathcal{A}$) is proper in $\mathcal{A}$.

The key property possessed by proper sequences is given below.

Proposition C.2.7 *Given a finite interval partition, Δ of $\mathcal{A}$ and a proper sequence, $(A_n)_n$ in $\mathcal{A}$, $\exists [D, D') \in \Delta$ s.t. $A_n \in [D, D')$ for all sufficiently large n.*

Proof If no such $[D, D') \in \Delta$ existed, then, since $(A_n)_n$ is a proper sequence and $|\Delta| < \infty$, $\exists K \in \mathrm{N}$ s.t. $A_n \notin \bigcup\{[A, A') : [A, A') \in \Delta\}$ $\forall n \geq K$ which contradicts Definition C.2.1 (b). □

Let d_∞ denote the metric on $[0,1]^k$ ($k \in \mathrm{N}$) generated by the norm $\|\cdot\|_\infty$. Given $S \subseteq [0,1]^k$ and $\mathbf{t} \in [0,1]^k$, define $d_\infty(S, \mathbf{t}) = \inf\{d_\infty(\mathbf{s}, \mathbf{t}) : \mathbf{s} \in S\}$. The following result generates a useful class of proper sequences in $\mathcal{I}_k$.

Lemma C.2.8 *If $(\mathbf{t}_n)_n$ in $[0,1]^k$ is s.t.*

(i) $\mathbf{t}_n \xrightarrow{d_\infty} \mathbf{t}$ *for some* $\mathbf{t} \in [0,1]^k$ *and*

(ii) $\mathbf{t}_n \in \tilde{Q}(\rho, \mathbf{t})$ $\forall n$ *for some* $\rho \in \wp$,

then $(A_{\mathbf{t}_n})_n$ is a proper sequence in $\mathcal{I}_k$.

Proof Let $[A_{\mathbf{u}}, A_{\mathbf{v}})$ be a proper interval in $\mathcal{I}_k$ with $\mathbf{u} = (u_1, \cdots, u_k)$ and $\mathbf{v} = (v_1, \cdots, v_k)$. We will show $A_{\mathbf{t}_n} \in (\notin) [A_{\mathbf{u}}, A_{\mathbf{v}})$ $\forall$ large n by considering three cases, the first of which is trivial in light of (C.1) and assumption (ii).

Case I: $[\tilde{Q}(\rho, \mathbf{t})\rangle \cap [A_{\mathbf{u}}, A_{\mathbf{v}}) = \phi$.

The two remaining cases involve the number ϵ_0 defined by

$$\epsilon_0 = d_\infty\Big(\Pi_{i=1}^k [u_i, v_i\rangle, \mathbf{t}\Big).$$

Case II: $[\tilde{Q}(\rho, \mathbf{t})\rangle \cap [A_{\mathbf{u}}, A_{\mathbf{v}}) \neq \phi$ and $\epsilon_0 > 0$.

Since $\mathbf{t}_n \rightarrow_{d_\infty} \mathbf{t}$, $\exists K \in \mathrm{N}$ s.t. $d_\infty(\mathbf{t}_n, \mathbf{t}) < \epsilon_0$ $\forall n \geq K$ which, by the definition of ϵ_0, implies $\mathbf{t}_n \notin \Pi_{i=1}^k [u_i, v_i\rangle$ $\forall n \geq K$ Therefore, by Proposition C.2.2(i), $A_{\mathbf{t}_n} \notin [A_{\mathbf{u}}, A_{\mathbf{v}})$ $\forall n \geq K$.

Case III: $[\tilde{Q}(\rho, \mathbf{t})\rangle \cap [A_{\mathbf{u}}, A_{\mathbf{v}}) \neq \phi$ and $\epsilon_0 = 0$.

As will be shown in Proposition C.3.1, this case implies the existence of a d_∞-open neighborhood, U of $\mathbf{t}$ s.t.

$$\tilde{Q}(\rho, \mathbf{t}) \cap U \subseteq \prod_{i=1}^k [u_i, v_i\rangle. \qquad \text{(C.4)}$$

Since assumptions (i) and (ii) imply $\mathbf{t}_n \in \tilde{Q}(\rho, \mathbf{t}) \cap U$ for all large n, Proposition C.2.2 (i) and (C.4) imply $A_{\mathbf{t}_n} \in [A_{\mathbf{u}}, A_{\mathbf{v}})$ for all large n. This completes Case III and the proof of Lemma C.2.8. □

Definition C.2.9 *Given $x : \mathcal{A} \to \mathbf{R}$ and $A \in \mathcal{A}$,*

(a) *x has proper-limits at A if for any proper sequence $(A_n)_n$ in $\mathcal{A}$, $A_n \xrightarrow{d_H} A$ implies $(x(A_n))_n$ converges and*

(b) *x is properly-continuous at A if for any proper sequence $(A_n)_n$ in $\mathcal{A}$, $A_n \xrightarrow{d_H} A$ implies $x(A_n) \to x(A)$.*

Recalling the concept of outer-continuity as given in Definition 2.3.1 (b),

Definition C.2.10 *$D_p(\mathcal{A})$ consists of all $x \in B(\mathcal{A})$ s.t. x has proper-limits and outer-continuity at each $A \in \mathcal{A}$ whereas $C_p(\mathcal{A})$ consists of all $x \in B(\mathcal{A})$ s.t. x is properly-continuous at each A in $\mathcal{A}$.*

Remark C.2.11 Clearly, $C_p(\mathcal{A}) \subseteq D_p(\mathcal{A})$ for any $\mathcal{A}$.

There is a natural correspondence between set-functions on $\mathcal{I}_k$ and real-valued functions on $[0,1]^k$ ($k \in \mathbf{N}$). In particular, given any $f : [0,1]^k \to \mathbf{R}$, the associated set-function, $x_f : \mathcal{I}_k \to \mathbf{R}$ is defined by

$$x_f(A_{\mathbf{t}}) = f(\mathbf{t}), \quad (\forall \mathbf{t} \in [0,1]^k). \tag{C.5}$$

Similarly, given any $x : \mathcal{I}_k \to \mathbf{R}$, the associated multiparameter function, $f_x : [0,1]^k \to \mathbf{R}$ is defined by

$$f_x(\mathbf{t}) = x(A_{\mathbf{t}}), \quad (\forall \mathbf{t} \in [0,1]^k). \tag{C.6}$$

Recall that $\mathbf{t} \mapsto A_{\mathbf{t}}$ ($\mathbf{t} \in [0,1]^k$) is a bijection.

As mentioned earlier, $D(\mathcal{I}_k)$ is too large to coincide with D_k when $k \geq 2$. This is illustrated by the following example which appears as Example 4.1 in [25].

Example C.2.12 Consider the closed subregion, S of $[0,1]^2$ defined by

$$S = \{\mathbf{t} \in [0,1]^2 : t_1 + t_2 \geq 1\} \cap [0,1]^2.$$

Geometrically, S is the "upper-half" triangle in $[0,1]^2$. Define the set-function, $x : \mathcal{I}_2 \to \mathbf{R}$ by letting

$$x(A_{\mathbf{t}}) = \begin{cases} 1 & , \text{if } \mathbf{t} \in S \\ 0 & , \text{if } \mathbf{t} \notin S \end{cases}, \quad (\forall \mathbf{t} \in [0,1]^k).$$

It is straightforward to show that x has inner-limits and outer-continuity at every $A \in \mathcal{I}_2$, i.e., $x \in D(\mathcal{I}_2)$. However, the associated multivariate function f_x defined in (C.6) is not in D_2. In particular, f_x does not have $(0,1)$- or $(1,0)$-quadrant limits at any $\mathbf{t} = (t_1, t_2) \in [0,1]^2$ for which $t_1 + t_2 = 1$.

In general, given any $k \in \mathbf{N}$, if

$$S = \{\mathbf{t} \in [0,1]^k : t_1 + \cdots + t_k \geq 1\} \cap [0,1]^k,$$

a similar example implies $D_k \subset D(\mathcal{I}_k)$ in the sense of (C.5) and (C.6).

On the other hand,

Theorem C.2.13 *For any $k \in \mathbf{N}$, $D_p(\mathcal{I}_k) = D_k$ up to the identification $x \mapsto f_x$ ($x \in D_p(\mathcal{I}_k)$).*

Proof In this proof, we will use the following basic geometric property of the metric d_∞: given $(\mathbf{t}_n)_n$ and $\mathbf{t}$ in $[0,1]^k$,

$$A_{\mathbf{t}_n} \xrightarrow{d_H} A_{\mathbf{t}} \iff \mathbf{t}_n \xrightarrow{d_\infty} \mathbf{t}. \tag{C.7}$$

This fact has already been mentioned in the proof of Proposition 2.3.10.

To establish $D_k \subseteq D_p(\mathcal{I}_k)$, take $f \in D_k$ and let x_f be the associated set-function defined in (C.5). Given an arbitrary $\mathbf{t} \in [0,1]^k$, we need to show x_f is outer-continuous with proper-limits at $A_{\mathbf{t}}$.

For proper-limits, take any proper sequence $(A_{\mathbf{t}_n})_n$ in $\mathcal{I}_k$ s.t. $A_{\mathbf{t}_n} \to_{d_H} A_{\mathbf{t}}$. By Proposition C.2.7, since $\Delta_{\mathbf{t}}$ in (C.2) is a finite interval partition of $\mathcal{I}_k$, $\exists$ $[D, D') \in \Delta_{\mathbf{t}}$ and $K \in \mathbf{N}$ s.t.

$$A_{\mathbf{t}_n} \in [D, D'), \quad (\forall n \geq K).$$

But, by the definition of $\Delta_{\mathbf{t}}$, $\exists\, \rho \in \wp$ s.t. $[D, D') = [\tilde{Q}(\rho, \mathbf{t})\rangle$. Therefore, by (C.1), $(\mathbf{t}_{n+K})_n$ lies in $\tilde{Q}(\rho, \mathbf{t})$ while (C.7) implies $\mathbf{t}_n \to_{d_\infty} \mathbf{t}$. Since f has ρ-quadrant limits at $\mathbf{t}$ (see (C.3)), this implies $(x_f(A_{\mathbf{t}_n}))_n$ converges, confirming that x_f has proper-limits at $A_{\mathbf{t}}$.

To show that x is outer-continuous at $A_{\mathbf{t}}$, take any sequence $(A_{\mathbf{t}_n})_n$ in $\mathcal{A}$ s.t. $A_{\mathbf{t}_n} \searrow A_{\mathbf{t}}$. By the inclusion in Remark C.2.4 (c) and (C.1), $(\mathbf{t}_n)_n$ lies in $\tilde{Q}(\sigma, \mathbf{t})$. Furthermore, (C.7) implies $\mathbf{t}_n \to_{d_\infty} \mathbf{t}$. Therefore, since $f \in D_k$,

$$x_f(A_{\mathbf{t}_n}) = f(\mathbf{t}_n) \longrightarrow f(\mathbf{t}) = x_f(A_{\mathbf{t}}) \text{ as } n \to \infty,$$

i.e., x is outer-continuous at $A_{\mathbf{t}}$ which establishes $x_f \in D_p(\mathcal{I}_k)$.

For the opposite inclusion, take $x \in D_p(\mathcal{I}_k)$ and let f_x be as defined in (C.6). To show $f_x \in D_k$, take an arbitrary point, $\mathbf{t} \in [0,1]^k$. By Remark C.2.4 (b), it is sufficient to show f_x has ρ-quadrant limits at $\mathbf{t}$ $\forall \rho \in \wp$.

For this purpose, take $\rho \in \wp$ and a sequence $(\mathbf{t}_n)_n$ in $[0,1]^k$ s.t. $\mathbf{t}_n \to_{d_\infty} \mathbf{t}$ and $\mathbf{t}_n \in \tilde{Q}(\rho, \mathbf{t})$ $\forall n$. Then, by (C.7), $A_{\mathbf{t}_n} \to_{d_H} A_{\mathbf{t}}$ whereas by Lemma C.2.8, $(A_{\mathbf{t}_n})_n$ is a proper sequence in $\mathcal{I}_k$. Therefore, since x has proper-limits at $A_{\mathbf{t}}$, $(f_x(\mathbf{t}_n))_n$ converges, i.e., f_x has ρ-quadrant limits at $\mathbf{t}$. This establishes that $f_x \in D_k$, completing the proof of Theorem C.2.13. □

It is clear from Definition C.2.10 that $C(\mathcal{A}) \subseteq C_p(\mathcal{A})$ for any $\mathcal{A}$. When $\mathcal{A} = \mathcal{I}_k$, the opposite inclusion is also valid.

Theorem C.2.14 *For any $k \in \mathrm{N}$, $C_p(\mathcal{I}_k) = C(\mathcal{I}_k)$.*

Proof As mentioned above, we need only show $C_p(\mathcal{I}_k) \subseteq C(\mathcal{I}_k)$. With this in mind, take any $x \in C_p(\mathcal{I}_k)$. If $x \notin C(\mathcal{I}_k)$, then $\exists A_{\mathbf{t}}$ and $(A_{\mathbf{t}_n})_n$ in $\mathcal{I}_k$ s.t. $A_{\mathbf{t}_n} \to_{d_H} A_{\mathbf{t}}$ and

$$|x(A_{\mathbf{t}_n}) - x(A_{\mathbf{t}})| \geq \eta, \quad (\forall n) \tag{C.8}$$

for some $\eta > 0$.

Since $|\wp| < \infty$, $\exists \rho \in \wp$ and a subsequence $(\mathbf{t}_{k_n})_n$ s.t.

$$\mathbf{t}_{k_n} \in \tilde{Q}(\rho, \mathbf{t}), \quad (\forall n).$$

Thus, by (C.7) and Lemma C.2.8, $(A_{\mathbf{t}_{k_n}})_n$ is a proper sequence in $\mathcal{I}_k$. Since x is properly-continuous at $A_{\mathbf{t}}$, this implies $x(A_{\mathbf{t}_n}) \to x(A_{\mathbf{t}})$ which contradicts (C.8). Therefore, it must be that $x \in C(\mathcal{I}_k)$. □

By Theorem C.2.13, $D_p(\mathcal{A})$ is small enough to ensure that D_k coincides with $D_p(\mathcal{I}_k)$ for every k. Next, we show that $D_p(\mathcal{A})$ (any $\mathcal{A}$) is sufficiently large in the sense that it contains the subspaces PA and $D_0(\mathcal{A})$ of $B(\mathcal{A})$: But first, we present a result whose statement and proof are identical to that of Proposition 2.3.4.

Proposition C.2.15 (a) *$D_p(\mathcal{A})$ is a linear subspace of $B(\mathcal{A})$.*
(b) *$C(\mathcal{A}) \subseteq D_p(\mathcal{A})$.*
(c) *$D_p(\mathcal{A})$ is $\|\cdot\|_{\mathcal{A}}$-closed in $B(\mathcal{A})$.*

Theorem C.2.16 *$D_0(\mathcal{A}) \subseteq D_p(\mathcal{A}) \subseteq D(\mathcal{A})$.*

Proof To establish the left-most inclusion, first take $x \in D_0(\mathcal{A})$ to be purely atomic with one atom of mass 1 at $t \in T \backslash E(\mathcal{A})$. Here, $E(\mathcal{A})$ denotes the edge of $\mathcal{A}$ as defined on p.27. Clearly, $x = \mathbf{1}_{[A_t,T)}$ on $\mathcal{A}$ where $A_t = \bigcap_{A \in \mathcal{A},\, t \in A} A$.

By the definition of edge, $t \notin E(\mathcal{A})$ implies $[A_t, T)$ is a proper interval in $\mathcal{A}$. Thus, given any proper sequence $(A_n)_n$ in $\mathcal{A}$, $\exists\, K \in \mathrm{N}$ s.t.

$$A_n \in (\notin)\, [A_t, T)\,, \quad (\forall n \geq K),$$

implying $x(A_n) = \mathbf{1}_{[A_t,T)}(A_n) = 1$ (respectively, 0) $\forall\, n \geq K$. This establishes proper-limits for x. By Proposition 2.3.11, x is also outer-continuous and thus, $x \in D_p(\mathcal{A})$. The left-most inclusion now follows by applying the various parts of Proposition C.2.15 as was done in the proof of Theorem 2.3.17.

The right-most inclusion follows automatically from Definition C.2.10 and Remark C.2.6. □

Remark C.2.17 Since we essentially work with the subspace $D_0(\mathcal{A})$ in this thesis, Theorem C.2.16 illustrates that there is no advantage in replacing $D(\mathcal{A})$ by $D_p(\mathcal{A})$ in the earlier chapters. Just the same, the space $D_p(\mathcal{A})$ may be of independent interest.

Combining Example C.2.12 and Theorem C.2.13, $D_p(\mathcal{I}_k) \subset D(\mathcal{I}_k)$ whenever $k \geq 2$. Recalling the set $\phi' = \bigcap_{A \in \mathcal{A}} A$ in $\mathcal{A}$, the following definition contains a sufficient condition on $\mathcal{A}$ under which $D_p(\mathcal{A}) = D(\mathcal{A})$.

Definition C.2.18 *$\mathcal{A}$ is said to be <u>one-dimensional</u> if $\mathcal{A} = [\phi', A) \cup [A, T)$ $\forall\, A \in \mathcal{A}$.*

Remark C.2.19 (a) Clearly, $\mathcal{I}_k$ is one-dimensional if and only if $k = 1$.

(b) If $\mathcal{A}$ is one-dimensional, $\Delta = \{[\phi', A), [A, T)\}$ is a finite interval partition of $\mathcal{A}$ whenever A is a proper set in $\mathcal{A}$. By Remark 2.3.8 (c), ϕ' is always a proper set.

Theorem C.2.20 *If $\mathcal{A}$ is one-dimensional, then $D_p(\mathcal{A}) = D(\mathcal{A})$.*

Proof By Theorem C.2.16, we only need to show that $D(\mathcal{A}) \subseteq D_p(\mathcal{A})$. With this in mind, take $x \in D(\mathcal{A})$. Since x is outer-continuous, all that remains to be shown is that x has proper-limits on $\mathcal{A}$.

To this end, take a proper sequence, $(A_n)_n$ in $\mathcal{A}$ s.t. $A_n \to_{d_H} A$ for some $A \in \mathcal{A}$. As mentioned in Remark C.2.19 (b), $[\phi', A)$ is a proper interval in $\mathcal{A}$ and hence, $\exists K \in \mathbb{N}$ s.t. $A_n \in (\notin) [\phi', A)$ $\forall n \geq K$. We need to show that $(x(A_n))_n$ converges. There are two cases.

First, if $A_n \in [\phi', A)$ $\forall n \geq K$, then $A_{n+K} \nearrow A$. Since x has inner-limits, $(x(A_n))_n$ converges.

On the other hand, if $A_n \notin [\phi', A)$ $\forall n \geq K$, then, since $\mathcal{A}$ is one-dimensional, $A_n \in [A, T)$ $\forall n \geq K$, implying $A_{n+K} \searrow A$. Since x is outer-continuous, this implies $x(A_n) \to x(A)$.

In total, x has proper-limits on $\mathcal{A}$ which completes the proof of Theorem C.2.20. □

C.3 A Technical Result for Lemma C.2.8

The following technical result was required in the proof of Lemma C.2.8.

Proposition C.3.1 *Let* $\mathbf{t}, \mathbf{u}, \mathbf{v} \in [0,1]^k$ *be s.t.* $[A_{\mathbf{u}}, A_{\mathbf{v}})$ *is a proper interval in* $\mathcal{I}_k$. *If* $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathbf{t}) = 0$ *and* $[\tilde{Q}(\rho, \mathbf{t})\rangle \cap [A_{\mathbf{u}}, A_{\mathbf{v}}) \neq \phi$ *for some* $\rho \in \wp$, *then* $\exists$ *a* d_∞*-open neighborhood,* U *of* $\mathbf{t}$ *s.t.* $\tilde{Q}(\rho, \mathbf{t}) \cap U \subseteq \prod_{i=1}^k [u_i, v_i\rangle$.

Take $\mathbf{t} = (t_1, \cdots, t_k), \mathbf{u} = (u_1, \cdots, u_k), \mathbf{v} = (v_1, \cdots v_k) \in [0,1]^k$ and $\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in \wp$ such that $[A_{\mathbf{u}}, A_{\mathbf{v}})$ is proper in $\mathcal{I}_k$. To establish Proposition C.3.1, we need three technical results.

Lemma C.3.2 *If* $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathbf{t}) = 0$, *then* $u_i \leq t_i \leq v_i$ $\forall 1 \leq i \leq k$.

Proof If $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathbf{t}) = 0$, then $\mathbf{t} \in cl\,(\prod_{i=1}^k [u_i, v_i\rangle)$, the closure taken w.r.t. d_∞. But $cl\,(\prod_{i=1}^k [u_i, v_i\rangle) = \prod_{i=1}^k [u_i, v_i]$, establishing the Lemma. □

Lemma C.3.3 *Assume* $[A_{\mathbf{u}}, A_{\mathbf{v}})$ *is a proper interval in* $\mathcal{I}_k$. *If* $[A_{\mathbf{u}}, A_{\mathbf{v}}) \subseteq [\tilde{Q}(\rho, \mathbf{t})\rangle$ *and* $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathbf{t}) = 0$, *then for every* $1 \leq i \leq k$, *either* $t_i = u_i$ *(if* $\rho_i = 1$*) or* $t_i = v_i$ *(if* $\rho_i = 0$*).*

Proof Since $[A_{\mathrm{u}}, A_{\mathrm{v}})$ is a proper interval in $\mathcal{I}_k$, it is non-empty, implying $[\tilde{Q}(\rho, \mathrm{t})\rangle \neq \phi$. Thus, by definition, $[\tilde{Q}(\rho, \mathrm{t})\rangle = [A_{\rho\wedge\mathrm{t}}, A_{\rho\vee\mathrm{t}})$. Since $[A_{\mathrm{u}}, A_{\mathrm{v}}) \subseteq [\tilde{Q}(\rho, \mathrm{t})\rangle$ Proposition C.2.2 (i) implies

$$\rho_i \wedge t_i \leq u_i \text{ and } v_i \leq \rho_i \vee t_i\,, \quad (\forall\, 1 \leq i \leq k). \tag{C.9}$$

Now, take $1 \leq i \leq k$. By Lemma C.3.2, $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathrm{t}) = 0$ implies

$$u_i \leq t_i \leq v_i. \tag{C.10}$$

If $\rho_i = 1$, then $\rho_i \wedge t_i = t_i$ and hence, by (C.9) and (C.10), we have $u_i \leq t_i = \rho_i \wedge t_i \leq u_i$. On the other hand, if $\rho_i = 0$, then $\rho_i \vee t_i = t_i$ which similarly implies $v_i \leq \rho_i \wedge t_i = t_i \leq v_i$. □

The third technical result follows by the definition of $\tilde{Q}(\rho, \mathrm{t})$.

Lemma C.3.4 *If $\mathbf{w} = (w_1, \cdots, w_k) \in \tilde{Q}(\rho, \mathrm{t})$, then for every $1 \leq i \leq k$, either $t_i \leq w_i$ (if $\rho_i = 1$) or $w_i \leq t_i$ (if $\rho_i = 0$). Furthermore, $t_i < 1$ implies $w_i < t_i$.*

Now, to prove Proposition C.3.1, assume $[A_{\mathrm{u}}, A_{\mathrm{v}})$ is a proper interval s.t. $d_\infty(\prod_{i=1}^k [u_i, v_i\rangle, \mathrm{t}) = 0$ and $[\tilde{Q}(\rho, \mathrm{t})\rangle \cap [A_{\mathrm{u}}, A_{\mathrm{v}}) \neq \phi$. Without a loss of generality, we can assume $[A_{\mathrm{u}}, A_{\mathrm{v}}) \subseteq [\tilde{Q}(\rho, \mathrm{t})\rangle$. (Otherwise, replace $[A_{\mathrm{u}}, A_{\mathrm{v}})$ by $[\tilde{Q}(\rho, \mathrm{t})\rangle \cap [A_{\mathrm{u}}, A_{\mathrm{v}})$ which is a proper interval by Proposition C.2.2(iii).)

Select $\epsilon_0 > 0$ s.t. $\epsilon_0 < \min_{1 \leq i \leq k} (v_i - u_i)$. Since $[A_{\mathrm{u}}, A_{\mathrm{v}})$ is a proper interval in $\mathcal{I}_k$, Proposition C.2.2 (ii) implies such an $\epsilon_0 > 0$ exists. To show that the d_∞-open ball, $U = B_{d_\infty}(\mathrm{t}, \epsilon_0)$ satisfies

$$\tilde{Q}(\rho, \mathrm{t}) \cap U \;\subseteq\; \prod_{i=1}^k [u_i, v_i\rangle,$$

take $\mathbf{w} = (w_1, \cdots, w_k) \in B_{d_\infty}(\mathrm{t}, \epsilon) \cap \tilde{Q}(\rho, \mathrm{t})$. We need to show

$$w_i \in [u_i, v_i\rangle\,, \quad (\forall\, 1 \leq i \leq k). \tag{C.11}$$

For this purpose, take $1 \leq i \leq k$. If $\rho_i = 0$, then by Lemma C.3.3, $t_i = v_i$. Thus, by Lemma C.3.4, $w_i \leq t_i = v_i$. Moreover, if $v_i < 1$, then $t_i < 1$ and hence, by Lemma C.3.4, $w_i < t_i = v_i$. This takes care of the "$\rangle$" in (C.11).

Since $d_\infty(\mathbf{w}, \mathrm{t}) < \epsilon_0$, it follows from the definition of ϵ_0 that

$$|w_i - t_i| \;<\; v_i - u_i,. \tag{C.12}$$

Substituting v_i for t_i in (C.12), we obtain $u_i < w_i$. In total, we have shown that $w_i \in [u_i, v_i\rangle$. (The case where $\rho_i = 1$ is similar to that of $\rho_i = 0$.)

This completes the proof of Proposition C.3.1. □

Appendix D

Assumptions and Conventions

At a fundamental level, this thesis can be viewed as a study of set-indexed stochastic bases. The two main constituents of a stochastic base are the indexing collection, generically denoted $\mathcal{A}$, and the filtration, generically denoted $(\mathcal{F}_A)_{A\in\mathcal{A}}$ (see Definitions 2.2.4 and 3.2.16 respectively). In certain applications, additional assumptions on $\mathcal{A}$ and/or $(\mathcal{F}_A)_{A\in\mathcal{A}}$ were required. Since these assumptions are scattered throughout the thesis, they are listed here for the sake of easy reference.

This appendix closes with a list of the conventions adopted in certain segments of the thesis.

Assumptions on Indexing Collections

Assumption 3.4.9 *Given any* $C = A \setminus \bigcup_{i=1}^{n} A_i \in \mathcal{C}$, $\exists$ *a maximal representation* $A \setminus \bigcup_{i=1}^{m} B_i$ *of* C.

Assumption 3.4.10 *If* $A \setminus \bigcup_{i=1}^{k} A_i = A' \setminus \bigcup_{j=1}^{k'} A'_j$ *in* $\mathcal{C}$, *then* $\exists\, N \in \mathbf{N}$ *s.t.*

$$g_n(A) \setminus \bigcup_{i=1}^{k} g_n(A_i) \;=\; g_n(A') \setminus \bigcup_{j=1}^{k'} g_n(A'_j)$$

for each $n \geq N$.

Assumption 3.7.1 $\exists$ *a constant* $K \in \mathbf{N}$ *s.t., given any* $n \in \mathbf{N}$ *and any* $A \in \mathcal{A}_n$, $\exists\ \{A_1, \cdots, A_k\} \subseteq \mathcal{A}_n$ *s.t.*

(i) $\bigcup_{i=1}^{k} A_k$ *is an extremal representation of* $\bigcup\{A' \in \mathcal{A}_n : A' \subset A\}$ *in the sense of Definition 3.2.5,*

(ii) $k \leq K$ *and*

(iii) *for any* $A' \in \mathcal{A}_n$, $A' \subset A$ *implies* $\exists 1 \leq j \leq k$ *s.t.* $A' \subseteq A_j$.

Assumption 4.6.1 Every set $C \in \mathcal{C}$ possesses a maximal representation (see Definition 3.2.6). Furthermore,

(i) $\mathcal{I}_d \subseteq \mathcal{A}$,

(ii) $\mathcal{A}_n = \begin{cases} \mathcal{I}_d^{(n)}(u) & \text{, if } \mathcal{I}_d(u) \subseteq \mathcal{A} \\ \mathcal{I}_d^{(n)} & \text{, otherwise} \end{cases}$ and

(iii) $g_n(A) = \bigcap\{B \in \mathcal{A}_n : A \subseteq B^\circ\}$ $\forall A \in \mathcal{A}$ and $n \in \mathrm{N}$.

Assumption B.3.3 $\exists$ *a binary operation* $\vee$ *on* $\mathcal{A}$ *s.t.* $\mathcal{A}$ *is a distributive lattice under* $\vee$ *and* $\wedge = \cap$.

Assumption B.3.8 *Given any set* $C = A \setminus \bigcup_{i=1}^{n} A_i \in \mathcal{C}$, $\exists$ *a maximal representation* $A \setminus \bigcup_{i=1}^{m} B_i$ *of* C *s.t.* $B_i \cap B_j \subseteq A$ $\forall i \neq j$.

Assumptions on Set-Indexed Filtrations

Assumption B.3.1 *Given any* $A_1, A_2 \in \mathcal{A}$ *and any* $X \in L_1$,

$$E[E(X \mid \mathcal{F}_{A_1}) \mid \mathcal{F}_{A_2}] = E(X \mid \mathcal{F}_{A_1 \cap A_2}).$$

Assumptions B.3.5 *Given sets* $A, A_1, \cdots, A_n \in \mathcal{A}$, *if* $A \subseteq \bigcup_{i=1}^{n} A_i$, *then* $\mathcal{F}_A \subseteq \bigvee_{i=1}^{n} \mathcal{F}_{A_i}$.

Other Assumptions

Assumption 3.4.2 *To each* $t \in T$, *associate the* σ*-algebra* $\mathcal{H}_t = \bigvee_n \mathcal{G}^*_{C_t^n}$. *Then, given any set* $F \in \mathcal{F}$, $\exists$ *a collection* $Y(F) = \{Y(F,t) : t \in T\}$ *of random variables (i.e., a* T*-indexed process) s.t.*

(i) *the map* $(\omega, t) \mapsto Y(F,t)(\omega)$ *is* $\mathcal{P}^*$*-measurable and*

(ii) *for each* $t \in T$, $Y(F,t)$ *is a version of* $E(\mathbf{1}_F \mid \mathcal{H}_t)$.

Moreover, the process $Y(F)$ *is unique up to indistinguishability on* T.

Assumption Groups

Some results rely on several distinct assumptions which are related in some way. We list a few such groups below.

Assumption Group D.1 *Comprised of Assumptions B.3.1, B.3.3, B.3.5 and B.3.8.*

Assumption Group D.2 *Comprised of Assumptions 3.4.2, 3.4.9 and 3.4.10.*

The assumptions in Assumption Group D.2 ensure the existence of $*$-PQV for well-behaved $\mathcal{A}$-indexed strong martingales (see Theorem 3.5.2).

Assumption Group D.3 *Comprised of Assumption Group D.2 and the following assumption on the indexing collection* $\mathcal{A}$*:*

- *any process* $X = (X_A)_{A\in\mathcal{A}}$ *which has increments defined at every* $C \in \mathcal{C}$ *possesses a unique finitely additive extension to* $\mathcal{C}(u)$.

The bulleted assumption in Assumption Group D.3 ensures that any $*$-PQV of a strong martingale (in particular, the one generated by Theorem 3.5.2) possesses a unique finitely additive extention to $\mathcal{C}(u)$.

Conventions

- For any set T and any indexed family $\{A_i : i \in I\}$ of subsets of T, $\bigcup_{i\in\phi} A_i = \phi$ and $\bigcap_{i\in\phi} A_i = T$. The former is considered a finite union.
- $\inf \phi = \infty$, the infimum taken in $(\mathbf{R}, \leq)$.
- We assume the axiom of choice.

- In Chapter 4 and Sections A.6 and A.8 of Appendix A, all $\mathcal{A}$-indexed processes and set-functions we work with are assumed to possess finitely additive extensions to $\mathcal{C}(u)$. (To the contrary, we do not assume every $\mathcal{A}$-indexed process possesses such an extension since this requires a rather strong hypothesis on $\mathcal{A}$ such as the shape property.) The only exception to this convention is in the definition of general set-indexed Gaussian processes (see Comment 4.2.29).

- From Chapter 3 onward, all finite sub-semilattices (f.s.s.l.) $\mathcal{A}'$ of $\mathcal{A}$ are assumed to contain ϕ' and T. See (2.6) for the definition of ϕ'.

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List of Acronyms

$*$-PQV	$*$-predictable quadratic variation, 119
J-L_2-AS	J-L_2 asymptotically smooth, 176
a.s. (a.e.)	almost surely (almost everywhere)
cadlag	(French) right-continuous with left-limits
CLT	central limit theorem
c.w.b.	consistent with boundaries, 34
c.w.s.p.	consistent with strong past, 71
f.m.d.a.	finite martingale difference array, 122
f.n.s.	finite left-neighborhood subcollection, 72
f.s.s.l.	finite sub-semilattice (of an indexing collection $\mathcal{A}$), 69
i.i.d.	independent identically distributed
s.t.	such that
t.f.a.e.	the following are equivalent
w.l.o.g.	without loss of generality
w.p. 1	with probability 1
w.r.t.	with respect to

List of Symbols

$\mathbf{N}$	the natural numbers (starting at 1)
$\mathbf{Q}$	the rational numbers
$\mathbf{R}$	the real numbers
$\mathbf{R}^+$	the non-negative real numbers
$[a, b\rangle$	, 247
$\mathbf{R}^k$	k-dimensional Euclidean space ($k \in \mathbf{N}$)
$\mathcal{B}(\mathbf{R}^k)$	Borel σ-algebra on $\mathbf{R}^k$
$\mathcal{B}_k$	Borel σ-algebra on $[0, 1]^k$
$\|\cdot\|_\infty$	supremum norm on $\mathbf{R}^k$, 16
d_∞	metric generated by $\|\cdot\|_\infty$
$\mathbf{x}$	generic element of $\mathbf{R}^k$
$\mathbf{0}$	the zero vector in $\mathbf{R}^k$
$\mathbf{1}$	the vector $(1, 1, \cdots, 1)$ in $\mathbf{R}^k$
$\mathbf{x} \leq \mathbf{y}$	, 183
$\mathbf{x} \prec \mathbf{y}$	, 183
$[\mathbf{x}, \mathbf{y}]$	closed interval in $(\mathbf{R}^+)^k$, 183
$(\mathbf{x}, \mathbf{y}]$	half-open interval in $(\mathbf{R}^+)^k$, 184
$L_\mathbf{x}$	, 184
$S_\mathbf{x}$	, 184
$S_{\mathbf{x}-}$	, 184
$\odot$	, 226
$\mathbf{s} \vee \mathbf{t}$	coordinate-wise maximimum of $\mathbf{s}, \mathbf{t} \in \mathbf{R}^k$
$\mathbf{s} \wedge \mathbf{t}$	coordinate-wise minimum of $\mathbf{s}, \mathbf{t} \in \mathbf{R}^k$
$\overline{\lim}$	limit superior
$\underline{\lim}$	limit inferior
$\exists$	there exists
$\forall$	for every (or, for all)
$\Longrightarrow$	implies
$\Longleftrightarrow$	if and only if
$(a_n)_n$	sequence with n-th term a_n
$\xrightarrow{n}$	converges as n tends to infinity
$\xrightarrow{\tau}$	converges w.r.t. τ, τ any metric
$\uparrow$	monotone increasing limit
$\downarrow$	monotone decreasing limit
ϵ	element of

$\subseteq$	set inclusion
$\subset$	strict set inclusion
S^c	complement of the set S
$A \triangle B$	symmetric difference of sets
$\lvert S \rvert$	cardinality of a set S
$\mathcal{P}(S)$	power set of the set S
$\mathbf{1}_S$	indicator function of a set S
$f\vert_A$	the function f restricted to the set A
$\prod$	(cartesian) product of numbers (sets)
(T, d)	generic metric space
$B_d(t, \epsilon)$	d-open ball with radius ϵ and center t
$d(t, S)$	d-distance of $t \in T$ from $S \subseteq T$
$\text{diam}(S)$	d-diameter of $S \subseteq T$
$\mathcal{K}_T$	the non-empty d-closed and bounded subsets of T, 12
d_H	the Hausdorff metric on $\mathcal{K}_T$, 12
$\overline{A}$, $cl(A)$	topological closure of the set A
A°	topological interior of the set A
∂A	topological boundary of the set A
A^ϵ	, 12
$A^{\overline{\epsilon}}$	, 12
$A^{-\epsilon}$	, 24
$\mathcal{A}$	generic indexing collection, 15
$\mathcal{A}_n$	n-th approximating f.s.s.l. of $\mathcal{A}$, 14
$\mathcal{A}^*$	the union of all $\mathcal{A}_n$
g_n	n-th set-approximator on $\mathcal{A}$, 14
$\emptyset'$	minimal element in $\mathcal{A}$, 20
$[A, A')$	set-interval in $\mathcal{A}$, 247
$\mathcal{A}'(u)$	all finite unions in $\mathcal{A}'$ ($\mathcal{A}' \subseteq \mathcal{A}$)
$\mathcal{C}$, $\mathcal{C}(u)$	, 68
$\mathcal{C}'$, $\mathcal{C}'(u)$	, 68
$\mathcal{C}_n$	, 210
$\mathcal{C}^*$	the union of all $\mathcal{C}_n$
C_A	left-neighborhood of A, 69
$\mathcal{N}'$	left-neighborhoods generated by $\mathcal{A}'$, 69
$\mathcal{N}_n$	left-neighborhoods generated by $\mathcal{A}_n$, 210
C_t^n	unique left neighborhood in $\mathcal{N}_n$ containing t, 95
$\mathcal{N}_0$	generic f.n.s., 72

$\mathcal{I}_k$	, 16
$\mathcal{I}_k^{(n)}$	, 16
$\mathcal{LL}_k$	the lower layers in $[0,1]^k$ ($k \in \mathrm{N}$), 60
$\mathcal{LL}_k^{(n)}$	, 60
$A_n \uparrow A$	increasing limit of sets
$A_n \downarrow A$	decreasing limit of sets
$A_n \nearrow A$	, 21
$A_n \searrow A$	, 21
$\|\cdot\|_{\mathcal{A}}$	supremum norm on $B(\mathcal{A})$, 22
$\nu(x)$	the variation of a purely atomic x, 23
A_t	smallest set in $\mathcal{A}$ containing t, 24
$\omega(\mathcal{A})$	, 24
$\alpha(\mathcal{A})$	, 26
$E(\mathcal{A})$	the edge of $\mathcal{A}$, 27
$\rho_{\mathcal{A}}$	the product metric on $\mathcal{A} \times \mathbf{R}$, 29
$\mathcal{K}_{\mathcal{A}\times\mathbf{R}}$	, 29
d_G	Hausdorff metric on $\mathcal{K}_{\mathcal{A}\times\mathbf{R}}$, 29
d_D	Skorokhod J_2-type metric on $D(\mathcal{A})$, 29
$G(x)$	the closed graph of the set function x, 29
$C_n(x)$	n-th continuous approximation of x, 47
$J_n(x)$	n-th purely atomic approximation of x, 47
$S[x]$	set of all atoms of x lying in the set S, 35
$\mathrm{at}(x)$	number of atoms of x, 44
$\mathrm{at}(x,S)$, $\mathrm{at}(S)$	number of atoms of x in the set S, 44
$\mathrm{gap}(x)$	gap of a purely atomic x, 44
$w(x,\delta)$	the modulus of continuity of x at $\delta > 0$, 46
$\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$	generic σ-algebras
$\sigma(\mathcal{E})$	σ-algebra generated by $\mathcal{E}$
$\mathcal{G} \vee \mathcal{H}$	σ-algebra generated by $\mathcal{G} \cup \mathcal{H}$
$(\Omega, \mathcal{F}, P)$	generic complete probability space
$L_p = L_p(\Omega, \mathcal{F}, P)$	classical Lebesgue space ($1 \le p \le \infty$)
$L_p(\mathcal{G})$	subspace of $\mathcal{G}$-measurable elements of L_p
$\sigma(X_i : i \in I)$	smallest σ-algebra to which each X_i is measurable
$X \sim N(\mu, \sigma^2)$	X distributed normal, mean μ, variance σ^2
$E(X)$	expectation of X
$\mathrm{var}(X)$	variance of X
$\mathrm{cov}(X,Y)$	covariance of X and Y

$E[X\|\mathcal{G}]$	conditional expectation of X given $\mathcal{G}$
$(\Omega, \mathcal{F}, P, (\mathcal{H}_t), \mathbf{T})$	classical stochastic base ($\mathbf{T} \subseteq \mathbf{R}^+$), 143
$Y(s,t]$	$Y_t - Y_s$, Y a classical processes, $0 \leq s < t$
Y_σ	the classical process Y stopped at σ
$(\Delta Y)_\sigma$	, 222
$\mathcal{H}_\sigma$	events that occur by time σ
$Y_{\sigma-}$	, 222
$\mathcal{H}_{\sigma-}$	, 221
$(\mathcal{F}_A)_{A \in \mathcal{A}}$	set-indexed filtration, 73
$(\Omega, \mathcal{F}, P, (\mathcal{F}_A), \mathcal{A})$	set-indexed stochastic base, 73
$\mathcal{F}_B^\circ$ $(B \in \mathcal{A}(u))$	, 74
$\mathcal{F}_B$ $(B \in \mathcal{A}(u))$	, 74
$\mathcal{G}_C^*$ $(C \in \mathcal{C}(u))$	the strong past at C, 74
$\mathcal{G}_C$ $(C \in \mathcal{C})$	the weak past at C, 75
$X = (X_A)_{A \in \mathcal{A}}$	set-indexed process, 77
$\hat{X}(\omega)$	sample path of X at ω, 78
X_C $(C \in \mathcal{C})$	increment of X at C, 78
X^p	p-th power of the process X, 80
$(\mathcal{F}_A^X)_{A \in \mathcal{A}}$	minimal filtration generated by X, 77
$\mathcal{P}_0^*$	the semi-algebra of $*$-predictable rectangles, 84
$\mathcal{P}_0^*(u)$	the algebra of finite unions in $\mathcal{P}_0^*(u)$
$\mathcal{P}^*$	the $*$-predictable σ-algebra, 84
μ_X	the admissible function of the process X, 86
$\mu_{(M^2)}$	, 113
$Y(F)$	, 95
$Y^n(F,t)$	, 96
$\underline{V}_B^{(k)}$	, 100
$\overline{V}_A$	, 102
$\underline{Q}_B^{(k)}$	, 113
$\overline{Q}_A$	, 116
$\langle \cdot \perp \cdot \| \cdot \rangle$	conditional independence bracket, 239
$\Rightarrow$	weak convergence, 141
$\xrightarrow{\mathcal{L}}$	convergence in distribution
$\mathcal{D}[0,a]$	, 142
$\mathcal{D}$	, 157
$\pi_{A_1,\cdots,A_n}$	finite dimensional projection on $D(\mathcal{A})$, 158

List of Function Spaces

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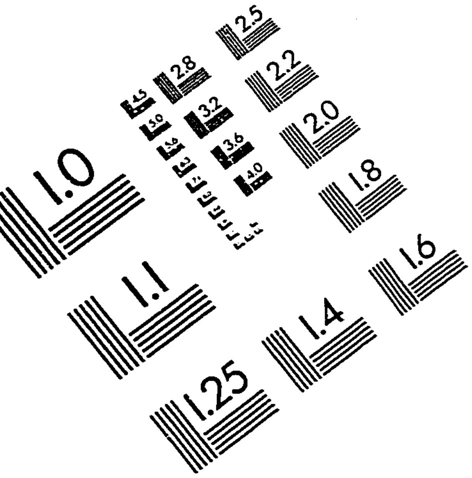
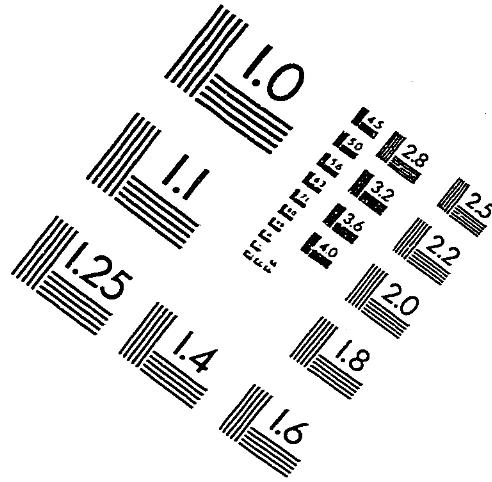
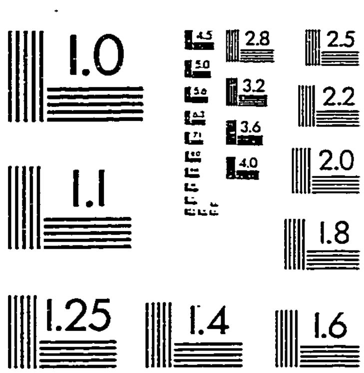
IMAGE EVALUATION TEST TARGET (QA-3)







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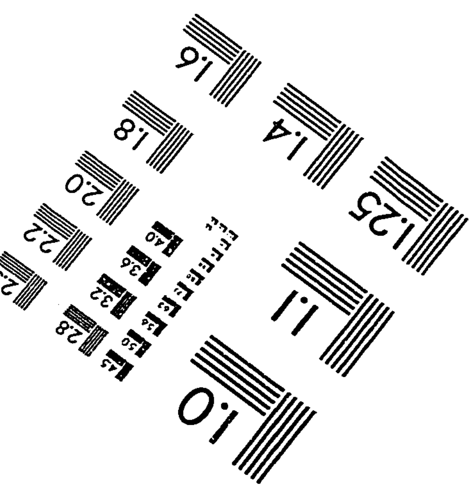

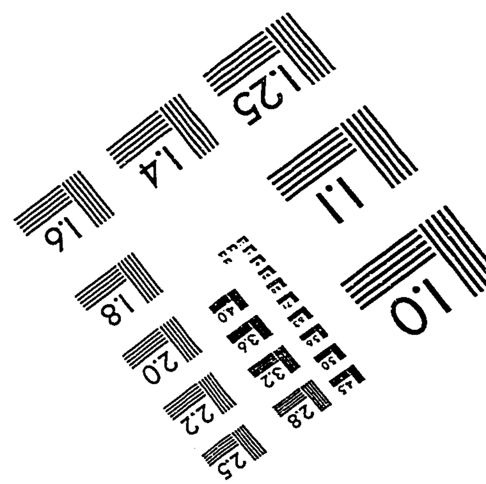

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