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Gaël Giordano

AUTEUR DE LA THÈSE / AUTHOR OF THESIS

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Realization of Critical Eigenvalues for Systems of Linear Delay Differential Equations

TITRE DE LA THÈSE / TITLE OF THESIS

V. Leblanc

DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

R. Vaillancourt

D. Amundsen

Gary W. Slater

Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

Realization of Critical Eigenvalues for Systems of Linear Delay Differential Equations

Gaël Giordano

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in partial fulfilment of the requirements for the degree of Master of Science in Mathematics [1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

This thesis is a step forward in the generalization of the *Realization Theorem* in the paper [15] by Buono and LeBlanc. In that theorem, the two authors study the link between the number of critical eigenvalues and the number of delays in a scalar delay differential equation of the form:

$$\dot{y}(t) = \sum_{j=1}^{l} a_j y(t - \tau_j), \quad a_j \in \mathbb{R}.$$

In this thesis, we shall consider a system of p ($p \in \mathbb{N}$) scalar delay-differential equations. That system can be written as:

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j), \quad M_j \in \mathbb{M}_p(\mathbb{R}).$$

The goal is therefore to study the links between the number of critical eigenvalues, the number of delays and the number p of equations. We study these links in three particular cases.

First of all, we are interested in the case $l = 1$. That is $\dot{y}(t) = My(t - \tau)$. Secondly, we consider the equation $\dot{y}(t) = M_1 y(t - \tau_1) + M_2 y(t - \tau_2)$, $M_j \in \mathbb{M}_2(\mathbb{R})$. Finally, we study the case:

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j)$$

where the matrices $M_j \in \mathbb{M}_p(\mathbb{R})$ are diagonal, for $1 \leq j \leq l$.

Résumé

Cette thèse représente un pas en avant vers la généralisation du *théorème de réalisation* de Buono et LeBlanc [15]. Dans ce papier, les deux auteurs étudient le lien entre le nombre de valeurs propres critiques et le nombre de délais dans une équation différentielle à délais scalaire de type:

$$\dot{y}(t) = \sum_{j=1}^{l} a_j y(t - \tau_j), \quad a_j \in \mathbb{R}.$$

Dans cette thèse, on considère un système de p ($p \in \mathbb{N}$) équations différentielles à délais scalaires qui s'exprime mathématiquement par

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j), \quad M_j \in \mathbb{M}_p(\mathbb{R}).$$

Le but est donc d'étudier les liens entre le nombre de valeurs propres critiques, le nombre de délais et le nombre p d'équations. Ces liens sont étudiés dans trois cas particuliers.

Premièrement, on s'intéresse au cas $l = 1$, c'est-à-dire à $\dot{y}(t) = My(t-\tau)$. Deuxièmement, on considère l'équation $\dot{y}(t) = M_1 y(t - \tau_1) + M_2 y(t - \tau_2)$, $M_j \in \mathbb{M}_2(\mathbb{R})$, et finalement, on étudie le cas

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j)$$

où les matrices M_j sont diagonales, pour $1 \leq j \leq l$.

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Cette thèse n'aurait jamais été écrite sans l'aide de plusieurs personnes: tout d'abord Victor LeBlanc, mon superviseur qui m'a inculqué les bases de la théorie des équations différentielles à délais et m'a proposé le sujet très intéressant de cette thèse; mes parents et ma sœur, dont l'aide plus ou moins mathématique (plutôt moins que plus) a été fort appréciée; et pour finir, toute personne qui, de près ou de loin a pu m'aider au cours de ces dernières années. A tous, un grand merci.

Dedication

Je dédie cette thèse à la phrase "Cette fois, ça devrait fonctionner." tant utilisée et si souvent prématurée; à toutes ces idées qui finissent en cul-de-sac et celles qui ne vont simplement nul part; à ce théorème qu'on voit déjà dans *Invenciones* et qui termine en remarque d'une sous-proposition; ou à ce pas en avant qui en précède toujours trois en arrière. Je dédie aussi cette thèse à la satisfaction de se sentir suffisamment à l'aise dans un domaine pour pouvoir dire "mon domaine"; à la fierté d'avoir créé des raisonnements clairs et structurés; et surtout, au plaisir d'avoir pu donner les réponses à des questions qui n'en avaient pas. Bref, je dédie cette thèse à mes premiers pas en recherche mathématique.

Contents

Chapter 1

Introduction

It is not exaggerated to say that any scientist or engineer is familiar with first order explicit ordinary differential equations (ODE). A first order explicit ODE is of the type

$$\dot{y}(t) = f(t, y(t)), \quad t \geq t_0, \tag{1.0.1}$$

where the function $y(t)$ is called the state variable and t the independent variable. Under some regularity conditions, such an ODE is governed by the principle of causality; that is the future state of the system is independent of the past states and it is determined solely by the present [1, p. 1]. The modification of the right-hand side of (1.0.1) to include the dependence of the derivative $\dot{y}$ on past values of the state variable y, yields a retarded functional differential equation.

Let $X = C([-r, 0], \mathbb{R}^p)$ denote the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^p$ and let $\Omega \subset \mathbb{R} \times X$. A retarded functional differential equation (RFDE) is of the form

$$\dot{y}(t) = f(t, y_t), \quad t \geq t_0, \tag{1.0.2}$$

where $y_t = y(t + \theta) \in X$, $\theta \in [-r, 0]$ and $f : \Omega \longrightarrow \mathbb{R}^p$ is a given function.

Equation (1.0.2) includes both distributed delay differential equations and discrete delay differential equations. In the former, f depends on y computed on a continuum, possibly unbounded ($r \to +\infty$), set of past values. As for the latter, f depends on only a finite number of past values of the state variable y. One can note that a discrete delay differential equation is a special case of a distributed delay differential equation.

Distributed delay differential equations are also called Volterra-type RFDE, after the Italian mathematician Vito Volterra (1860–1940). In his research on predator-prey models and viscoelasticity, Volterra formulated some rather general differential equations incorporating the past state of the system. Also, because of the close connections between the equations and specific physical systems, Volterra attempted to introduce a concept of energy function for these models. He then exploited the behaviour of the energy function to study the asymptotic behaviour of the system in the distant future [1, p. 1].

Indeed, distributed delay differential equations are not only mathematically interesting, they are also very useful to build more realistic models in population dynamics. Here are two examples of phenomena that were first modeled using ODE's and then made more realistic using distributed delay differential equations:

The first model helps describing species population struggling for a common resource; it is the logistic model. Rewritten with a delay, we obtain the *Hutchinson–Wright* equation [2] & [3]:

$$\dot{y}(t) = ry(t)\left[\frac{1 - y(t-\tau)}{K}\right].$$

In this case, we consider y as a population size. The constant K is the carrying capacity while r is the growth rate.

The second model was used by Gurney *et al.* [4] in 1980 to describe the periodic oscillations in Nicholson's classic laboratory experiments with the Australian sheep blowfly [5]. It is a model with delayed recruitment and instantaneous death:

$$\dot{y}(t) = \alpha y(t-\tau)e^{-\beta y(t-\tau)} - \delta y(t).$$

In this model, y is also a population size. The constant α and δ are the birth and death rate,respectively. The expression $\alpha y(t-\tau)e^{-\beta y(t-\tau)}$ is a density-dependent per capita reproductive rate.

The French mathematician Charles Emile Picard had already understood that need for more realistic models in his address at the Fourth ICM in 1908:

Les équations différentielles de la mécanique classique sont telles qu'il en résulte que le mouvement est déterminé par la simple connaissance des positions et des vitesses, c'est-à-dire par l'état à un instant donné et à l'instant infiniment voisin. Les états antérieurs n'y intervenant pas, l'hérédité y est un vain mot. L'application de ces équations où le passé ne se distingue pas de l'avenir, où les mouvements sont de nature réversible, sont donc inapplicables aux êtres vivants. Nous pouvons rêver d'équations fonctionnelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre un temps passé très éloigné et le temps actuel, qui apporteront la part de l'hérédité.

(La mathématique dans ses rapports avec la physique, Actes du IVe congrès international des Mathématiques, 1908. [6, p. 1]).

In fact, the effectiveness of modelization with RFDE is not limited to living beings. It is also used in economics (see the model by J.B.S. Haldane [7]), in biology (see the *sunflower model* [8, 9, 10]), in medical research (see the Mackay–Glass model [11, p. 72] and its numerical treatment [12]), and in many other fields (see [11]).

In this thesis the focus is confined to specific types of discrete delay differential equations. For convenience, discrete delay differential equations will be referred to as DDEs. A general DDE can be written:

$$\dot{y}(t) = F\left(t, y\left(t-\tau_1(t,y(t))\right), \ldots, y\left(t-\tau_l(t,y(t))\right)\right), \quad t \geq 0, \tag{1.0.3}$$

where

$$y : [-\tau; \infty) \to \mathbb{R}^p, \qquad \tau > 0;$$
$$F : [0; \infty) \times \mathbb{R}^p \times \cdots \times \mathbb{R}^p \to \mathbb{R}^p;$$
$$\tau_i : [0; \infty) \times \mathbb{R}^p \to [-\tau; \infty), \qquad 1 \leq i \leq l.$$

For $1 \leq i \leq l$, the functions $\tau_i(t, y(t))$ are called the delays (or lags). By definition, if a delay $\tau(t, y(t))$ depends on $y(t)$, it is said to be *state dependent.* If not, it is *state independent.* Moreover, a state independent delay $\tau(t)$ is *variable* if it depends on t. If not, it is said to be *constant.*

In this thesis, we are interested in particular autonomous DDE's with constant, state independent delays τ_i,where $0 < \tau_i < \tau$:

$$\dot{y}(t) = A\left(y(t-\tau_1), \ldots, y(t-\tau_l))\right) + N\left(y(t-\tau_1), \ldots, y(t-\tau_l)\right), \quad t \geq 0, \quad (1.0.4)$$

where $A : \mathbb{R}^p \times \cdots \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ is a linear map,
and $N : \mathbb{R}^p \times \cdots \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ is a smooth nonlinear map with $F(0) = 0$ and $DF(0) = 0$.
The linearization of equation (1.0.4) about the trivial equilibrium is given by:

$$\dot{y}(t) = A\left(y(t-\tau_1), \ldots, y(t-\tau_l)\right), \quad t \geq 0. \qquad (1.0.5)$$

Under a standard basis for $\mathbb{R}^p \times \cdots \times \mathbb{R}^p$, $A \in \mathrm{End}(\mathbb{R}^p \times \cdots \times \mathbb{R}^p, \mathbb{R}^p)$ will always be of the form :

$$A = \sum_{j=1}^{l} M_j \qquad \text{where } M_j \in \mathrm{End}(\mathbb{R}^p).$$

Therefore, the DDE (1.0.5) becomes:

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j), \qquad t \geq 0, \tag{1.0.6}$$

where $\tau_j \in [0, \tau]$ for all $1 \leq j \leq l$.

Associated with the DDE (1.0.6) or with the linear operator $A = \sum_{j=1}^{l} M_j$ is its *characteristic matrix*

$$\Delta(\alpha) = \alpha I_p - \sum_{j=1}^{l} e^{-\alpha\tau_j} M_j, \qquad \alpha \in \mathbb{C}.$$

The roots of the *characteristic equation* $\det \Delta(\alpha) = 0$ are the eigenvalues of the DDE (1.0.6). If λ is an eigenvalue of (1.0.6) and $v \in \mathbb{C}^p$ is in $\ker \Delta(\alpha)$, then, as we shall show in Theorem 2.0.3, $y(t) = e^{\lambda t} v$ is a solution of (1.0.6).

Now, let us make the correspondence with the definition of linear autonomous DDE given in Hale and Verduyn Lunel [1, p. 193]:
Let $\tau > 0$ and $C = C([-\tau, 0], \mathbb{R}^p)$ be the Banach space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^p$. For $t \geq 0$ and $y : [-\tau, \infty) \to \mathbb{R}^p$, the function y_t is given by $y_t(\theta) = y(t + \theta)$, for $\theta \in [-\tau, 0]$. For $0 < \tau_1 < \cdots < \tau_l < \tau$, and $M_1 \in \mathbb{M}_p(\mathbb{R}), \ldots,$ $M_l \in \mathbb{M}_p(\mathbb{R})$, let $\eta : \mathbb{R} \to \mathbb{M}_p(\mathbb{R})$ be the function of bounded variation defined by

$$\eta(\theta) = \sum_{j=1}^{l-1} (-j)\chi_{(-\tau_{j+1}, -\tau_j]}(\theta) M_j - l\chi_{(-\infty, -\tau_l]}(\theta) M_l$$

where χ_A is the *indicator function* of the interval A.
Then the linear operator $L : C \to \mathbb{R}^p$ defined by

$$L\varphi = \int_{-r}^{0} d(\eta(\theta))\varphi(\theta)$$

is equal to

$$L\varphi = \int_{-r}^{0} \sum_{j=1}^{l} \delta_{-\tau_j}(\theta) M_j \varphi(\theta) = \sum_{j=1}^{l} M_j \varphi(-\tau_j).$$

Hence, for $\varphi = y_t$, we get

$$Ly_t = \sum_{j=1}^{l} M_j y(t - \tau_j).$$

Therefore, the DDE defined by Hale and Lunel by $\dot{y}(t) := Ly_t$ corresponds to the DDE (1.0.6). In his thesis, B.F. Redmond [13] explains perfectly why, in the study of bifurcations from the trivial equilibrium of (1.0.4), understanding the structure of the eigenvalues of the linear part (1.0.6) is crucial.

The main question addressed in this thesis is related to the number of purely imaginary eigenvalues of the DDE (1.0.6). The question falls within the category of "realization problems". Note that the two Portuguese mathematicians T. Faria and L.T. Magalhães were the first to study these so-called realization problems [14]. The realization question addressed here is the following: Consider n non-zero pure imaginary numbers $i\omega_1, \ldots, i\omega_n$ with positive frequencies. Then what is the minimal number l of delays needed such that the DDE (1.0.6) has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k$ for $1 \leq k \leq n$. The scalar case ($p = 1$) with n nonzero delays has been studied by Buono and LeBlanc [15]. Their theorem states the following:

Theorem 1.0.1 Suppose $\omega_1 > 0$, $\omega_2 > 0$, ... , $\omega_n > 0$ are linearly independent over the rationals. Then there exist $\tau_1 > 0$, $\tau_2 > 0$, ... , $\tau_n > 0$ and $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, ... , $a_n \in \mathbb{R}$ such that the linear delay differential equation

$$\dot{x}(t) = a_1\, x(t - \tau_1) + a_2\, x(t - \tau_2) + \cdots + a_n\, x(t - \tau_n) \tag{1.0.7}$$

has solutions $x_k^{\pm}(t) = e^{\pm i\omega_k t}$, for $1 \leq k \leq n$.

Answering the"realization question" for a DDE of the type (1.0.6) for any p proved to be too difficult. Therefore, this thesis is mostly a collection of "particular cases": for specific systems of p equations and n frequencies (with particular conditions), we are able to find a small number of delays such that the DDE (1.0.6) has solutions $y_k^{pm}(t) = e^{\pm \imath \omega_k t} v_k$. Hence the structure of the thesis is straightforward: Each chapter studies system (1.0.6) for a particular p and a particular l with particular conditions on the frequencies.

Chapter 2

Terminology and Basic Results

In this chapter we introduce terminology and basic results. Their definitions and the results of this sections are used in each chapter of this thesis.

Definition 2.0.1 Let $\tau > 0$, and let l, p be two integers ≥ 1.
Consider the linear delay differential equation

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{2.0.1}$$

where $M_j \in \mathbb{M}_p(\mathbb{R})$ and $\tau_j \in [0, \tau]$, for $1 \leq j \leq l$.

(a) For $\alpha \in \mathbb{C}$, we will denote by $\Delta(\alpha) \in \mathbb{M}_p(\mathbb{C})$ the *characteristic matrix* of (2.0.1) given by

$$\Delta(\alpha) = \alpha I_p - \sum_{j=1}^{l} e^{-\alpha\tau_j} M_j.$$

(b) The *characteristic equation* of (2.0.1) is given by $\det \Delta(\alpha) = 0$. Following [15], the roots of the characteristic equation are the *eigenvalues* of the equation (2.0.1).

In Definition 2.0.2 below, $\delta_{0,i}$ denotes the Kronecker symbol, i.e. $\delta_{0,i} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$

Definition 2.0.2 Let $\mathcal{M} = \{M_0, M_1, \ldots, M_l\}$ be a set of $l+1$ matrices in $\mathbb{M}_p(\mathbb{R})$ where M_0 is the identity matrix. Let us associate with $\mathcal{M}$ the homogeneous polynomial $\rho \in \mathbb{R}[x_0, \cdots, x_l]$ of degree p given by

$$\rho(x_0, \ldots, x_l) = \sum_{i_1=0}^{l} \cdots \sum_{i_p=0}^{l} \det(c_1^{(i_1)}, \ldots, c_p^{(i_p)})(-1)^{\gamma(i_1, \ldots, i_p)} x_{i_1} \cdots x_{i_p}$$

where $c_r^{(j)}$ is the r^{th} column of the matrix M_j and $\gamma(i_1, \ldots, i_p) = \sum_{j=1}^{p} 1 - \delta_{0,i_j}$.

Proposition 2.0.2 Let $\Delta(\alpha)$ be the characteristic matrix of (2.0.1) and $\rho \in \mathbb{R}[x_0, \ldots, x_l]$ be the homogeneous polynomial associated with the matrices of (2.0.1).
Then $\det \Delta(\alpha) = \rho(\alpha, e^{-\alpha\tau_1}, \ldots, e^{-\alpha\tau_l})$.

Proof: In order to use Proposition A.1.1 of Appendix A, let us introduce the following notation: First, let $A_0 \in \mathbb{M}_p(\mathbb{C})$ denote the matrix αI_p and, for $1 \le j \le l$, let $A_j \in \mathbb{M}_p(\mathbb{C})$ denote the matrices $M_j e^{-\alpha\tau_j}$. Now if $c_k^{(j)}$, for $0 \le j \le l$ and $1 \le k \le p$, denote the k^{th} column of M_j, let $\beta_0 c_k^{(0)}$ denote the k^{th} column of A_0 and let, for $1 \le j \le l$, $\beta_j c_k^{(j)}$ denote the k^{th} column of A_j.
By Definiton 2.0.2 and Proposition A.1.1 of Appendix A, we have:

$$\begin{aligned}
\rho(\alpha, e^{-\alpha\tau_1}, \ldots, e^{-\alpha\tau_l}) &= \rho(\beta_0, \ldots, \beta_l) \\
&= \sum_{i_1=0}^{l} \cdots \sum_{i_p=0}^{l} \det(c_1^{(i_1)}, \ldots, c_p^{(i_p)})(-1)^{\gamma(i_1, \ldots, i_p)} \beta_{i_1} \cdots \beta_{i_p} \\
&= \sum_{i_1=0}^{l} \cdots \sum_{i_p=0}^{l} \det((-1)^{1-\delta_{0,i_1}} \beta_{i_1} c_1^{(i_1)}, \ldots, (-1)^{1-\delta_{0,i_p}} \beta_{i_p} c_p^{(i_p)}) \\
&= \det \sum_{j=0}^{l} A_j \\
&= \det \left(\alpha I_p - \sum_{j=1}^{l} e^{-\alpha\tau_j} M_j \right) \\
&= \det \Delta(\alpha) \qquad \square
\end{aligned}$$

Theorem 2.0.3 For $\tau > 0$, consider the linear system of delay differential equations

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{2.0.2}$$

where $M_j \in \mathbb{M}_p(\mathbb{R})$ and $\tau_j \in [0, \tau]$, for $1 \le j \le l$.

For $\alpha \in \mathbb{C}$, the following two conditions are equivalent:

(a) There exists $v \in \mathbb{C}^p$, $v \neq 0$, such that $y_\alpha(t) = e^{\alpha t} v$ is a solution of (2.0.2),

(b) $\rho(\alpha, e^{-\alpha\tau_1}, \dots, e^{-\alpha\tau_l}) = 0$, where ρ is defined in Definition 2.0.2.

Moreover, $y_\alpha(t) = e^{\alpha t} v$ is a solution of (2.0.2) if and only if $v \in \ker \Delta(\alpha)$.

Proof:

Let $y_\alpha(t) = e^{\alpha t} v$ with $v \in \mathbb{C}^p$, $v \neq 0$, and $\alpha \in \mathbb{C}$.

As the characteristic equation for the DDE

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{2.0.3}$$

is obtained by substituting $y_\alpha(t) = e^{\alpha t} v$, $v \in \mathbb{C}^p$, into the equation, we get:

$$\begin{aligned} \alpha e^{\alpha t} v &= \sum_{j=1}^{l} M_j e^{\alpha(t-\tau_j)} v \\ &= \sum_{j=1}^{l} M_j e^{-\alpha\tau_j} e^{\alpha t} v. \end{aligned}$$

By rearranging the terms we obtain

$$\left(\alpha I_p - \sum_{j=1}^{l} M_j e^{-\alpha\tau_j}\right) y_\alpha(t) = 0.$$

which is equivalent to

$$\Delta(\alpha) v = 0, \qquad \text{where} \quad \Delta(\alpha) = \alpha I_p - \sum_{j=1}^{l} M_j e^{-\alpha\tau_j}.$$

Hence, $y_\alpha(t)$ is a nonzero solution of (2.0.3) if and only if $v \in \ker \Delta(\alpha)$, $v \neq 0$. But $\ker \Delta(\alpha) \neq \{0\}$ if and only if $\det \Delta(\alpha) = 0$. By Proposition 2.0.2, the theorem is verified. $\square$

Proposition 2.0.4 For $\tau > 0$, consider the scalar delay differential equation

$$\dot{y}(t) = \sum_{j=1}^{l} a_j y(t - \tau_j) \tag{2.0.4}$$

where $a_j \in \mathbb{R}^*$ and $\tau_j \in [0, \tau]$, for $1 \leq j \leq l$.
Let $\Delta(\alpha) = \alpha - \sum_{j=1}^{l} a_j e^{-\alpha\tau_j}$, $\alpha \in \mathbb{C}$.
Then, $y_\alpha(t) = e^{\alpha t}$ is a solution of (2.0.4) if and only if $\Delta(\alpha) = 0$.

Proof: Let $y_\alpha(t) = e^{\alpha t}$, with $\alpha \in \mathbb{C}$.
Substituting $y_\alpha(t)$ in equation (2.0.4) and rearranging the terms yields:

$$\left(\alpha - \sum_{j=1}^{l} a_j e^{-\alpha\tau_j}\right) y_\alpha(t) = 0.$$

This equation is equivalent to $\Delta(\alpha) = 0$ where $\Delta(\alpha) = \alpha - \sum_{j=1}^{l} a_j e^{-\alpha\tau_j}$. $\square$

Chapter 3

Single-Delay Linear DDEs

It is always convenient to start from the beginning. For this thesis, the beginning would be to study the simplest linear DDE: delay differential equations with one delay. Hence, we are interested in the single-delay linear system of DDEs:

$$\dot{y}(t) = My(t-\tau) \tag{3.0.1}$$

where $M \in \mathbb{M}_p(\mathbb{R})$ and $\tau > 0$.

This chapter is divided into 2 sections. In Section 1, we consider n non-zero frequencies. We are able to find the smallest natural number $p \geq 1$ for which there exists $M \in \mathbb{M}_p(\mathbb{R})$ such that the single-delay linear system of DDEs (3.0.1) has solutions $y_k^{\pm}(t) = e^{\pm \imath \omega_k t} v_k$, for $1 \leq k \leq n$.

In Section 2, the perspective is different: we suppose that the matrix $M \in \mathbb{M}_p(\mathbb{R})$ is given. Then, we find the largest natural number $n \geq 1$ such that the single-delay linear system of DDEs (3.0.1) has solutions $y_k^{\pm}(t) = e^{\pm \imath \omega_k t} v_k^{\pm}$, for $1 \leq k \leq n$.

3.1 Realization Theorem for a System of Linear Single-Delay DDEs

The results in this section allow us to determine, for n given non-zero frequencies, the smallest natural number $p \geq 1$ for which there exists $M \in \mathbb{M}_p(\mathbb{R})$ such that the single-delay linear system of DDEs (3.0.1) has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$, for $1 \leq k \leq n$.

The proofs of these theorems require a result deduced from Theorem 2.0.3. This result is presented as Proposition 3.1.1.

Proposition 3.1.1 For $\tau > 0$ and $M \in \mathbb{M}_p(\mathbb{R})$, consider the single-delay linear system of DDEs:

$$\dot{y}(t) = My(t - \tau). \tag{3.1.1}$$

For $\omega \in \mathbb{R}$, the following two conditions are equivalent:

(a) There exist non-zero vectors $v^+, v^- \in \mathbb{C}^p$ such that $y^+(t) = e^{i\omega t} v^+$ and $y^-(t) = e^{-i\omega t} v^-$ are solutions of (3.1.1),

(b) $z^+ = i\omega e^{i\omega\tau}$ and $z^- = -i\omega e^{-i\omega\tau}$ are roots of the characteristic polynomial of M.

Proof: For $\alpha \in \mathbb{C}$, recall that the characteristic matrix $\Delta(\alpha)$ of (3.1.1) is given by

$$\Delta(\alpha) = \alpha I_p - e^{-\alpha\tau} M = e^{-\alpha\tau}(\alpha e^{\alpha\tau} I_p - M).$$

By Theorem 2.0.3, $y(t) = e^{i\omega t} v$, with $v \in \mathbb{C}^2$, is a non-trivial solution of (3.1.1) if and only if $v \in \ker \Delta(i\omega)$, $v \neq 0$.

Hence $y^+(t) = e^{i\omega t} v^+$ and $y^-(t) = e^{-i\omega t} v^-$ are non-trivial solutions of (3.1.1) if and only if $\det \Delta(i\omega) = \det \Delta(-i\omega) = 0$ and $v^{\pm} \in \ker \Delta(i\omega)$.

Let $z^+ = i\omega e^{i\omega\tau}$ and $z^- = \overline{z}^+ = -i\omega e^{-i\omega\tau}$. As $\Delta(i\omega) = e^{-i\omega\tau}(z^+ I_p - M)$ and

$\Delta(-i\omega) = e^{i\omega\tau}(z^- I_p - M)$, the necessary and sufficient condition for condition (a) to be satisfied is that z^+ and z^- are roots of the characteristic polynomial of M. $\square$

Remark 3.1.2 Keeping the same notation as in Proposition 3.1.1, recall that $\overline{z}^+ = -i\omega e^{-i\omega\tau} = z^-$. Then, as $M \in \mathbb{M}_p(\mathbb{R})$,

$$\det(z^+ I_p - M) = 0 \text{ if and only if } \det(z^- I_p - M) = 0.$$

Hence, condition (b) of (3.1.1) is equivalent to:

$$z^+ = i\omega e^{i\omega\tau} \text{ or } z^- = -i\omega e^{-i\omega\tau} \text{ is a root of the characteristic polynomial of M.}$$

Theorem 3.1.3 Let $\omega_1 > 0, \ldots, \omega_n > 0$ be n distinct real numbers and let $A = \bigcup_{j=1}^{n} \frac{\pi}{2\omega_j}\mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Then, for any $\tau \notin A$, there exist $M \in \mathbb{M}_p(\mathbb{R})$, with $p = 2n$, and $v_j^+, v_j^- \in \mathbb{C}^p$ such that, for $1 \leq j \leq n$, $y_j^+(t) = e^{i\omega_j t} v_j^+$ and $y_j^-(t) = e^{-i\omega_j t} v_j^-$ are $2n$ distinct solutions of the single-delay linear system of DDEs

$$\dot{y}(t) = My(t - \tau). \tag{3.1.2}$$

Proof: To simplify the notation, let $z^{(j)} = i\omega_j e^{i\omega_j\tau}$, for $1 \leq j \leq n$. Let us first check that if $\tau \notin A$, then $\{z^{(1)}, \ldots, z^{(n)}\} \bigcap \mathbb{R} = \emptyset$. If not, there exist j_0 such that

$$e^{i\omega_{j_0}\tau} = \pm i, \text{ ie. } \omega_{j_0}\tau + 2k\pi = \frac{\pi}{2} + l\pi, \text{ for some } k, l \in \mathbb{Z}.$$

Hence $\omega_{j_0}\tau = \frac{\pi}{2} + m\pi$, for some $m \in \mathbb{Z}$. Thus $\tau \in \frac{\pi}{2\omega_{j_0}}(1 + 2\mathbb{Z}) \subset A$. Since $\{z^{(1)}, \ldots, z^{(n)}\} \bigcap \mathbb{R} = \emptyset$, the polynomial ρ, defined as

$$\rho(x) \;=\; \prod_{j=1}^{n}(x - z^{(j)})(x - \overline{z}^{(j)})$$

$$= \prod_{j=1}^{n} \left[(\mathrm{Re}(z^{(j)}) - x)^2 + \mathrm{Im}(z^{(j)})^2 \right],$$

is an element of $\mathbb{R}[x]$ of degree $2n = p$.

By Proposition A.2.1 in Appendix A, there exist $M \in \mathbb{M}_p(\mathbb{R})$ such that ρ is the characteristic polynomial of M. Then, the proof of Theorem 3.1.3 follows from Proposition 3.1.1. □

Theorem 3.1.4 Let $\omega_0 > 0, \ldots, \omega_n > 0$ be $n+1$ distinct real numbers such that $\omega_j \notin \omega_0\mathbb{Q}$, for $1 \leq j \leq n$.

Then, there exist $M \in \mathbb{M}_p(\mathbb{R})$, with $p = 2n+1$, and $v_j^+, v_j^- \in \mathbb{C}^p$ such that $y_j^+(t) = e^{i\omega_j t}v_j^+$ and $y_j^-(t) = e^{-i\omega_j t}v_j^-$, $1 \leq j \leq n$, are distinct solutions of the single-delay linear system of DDEs

$$\dot{y}(t) = My(t-\tau) \tag{3.1.3}$$

with $\tau \in \dfrac{\pi}{2\omega_0}(-1+4\mathbb{Z})$.

Proof: To simplify the notation, let $z^{(j)} = i\omega_j e^{i\omega_j \tau}$, for $1 \leq j \leq n$. Now, let $\tau \in \dfrac{\pi}{2\omega_0}(-1+4\mathbb{Z})$. Thus, there exist $k \in \mathbb{Z}$ such that $\tau = \frac{\pi}{2\omega_0}(-1+4k)$. With such a value for τ we have:

$$\omega_0\tau = -\frac{\pi}{2} + 2k\pi \quad \text{thus} \quad z^{(0)} = i\omega_0 e^{i\omega_0\tau} = i\omega_0(-i) = \omega_0.$$

Moreover, since $\frac{\omega_j}{\omega_0} \notin \mathbb{Q}$, we have

$$\omega_j\tau = \frac{\omega_j}{\omega_0}\frac{\pi}{2}(-1+4k) \notin \frac{\pi}{2}\mathbb{Z}.$$

Thus, $e^{i\omega_j\tau} \notin \{\pm i\}$ and therefore $z^{(j)} = i\omega_j e^{i\omega_j\tau} \notin \mathbb{R}$.

Let $\rho \in \mathbb{R}[x]$ be the polynomial of degree p given by

$$\rho(x) = (x - z^{(0)}) \prod_{j=1}^{n} \left[(\mathrm{Re}(z^{(j)}) - x)^2 + \mathrm{Im}(z^{(j)})^2 \right].$$

By Proposition A.2.1 in Appendix A, there exist $M \in \mathbb{M}_p(\mathbb{R})$ whose ρ is the characteristic polynomial of M.

Thus, the proof of Theorem 3.1.4 follows from Proposition 3.1.1. $\square$

Remark 3.1.5 Keeping the same notations as in the proof of Theorem 3.1.4, we have that $z^{(0)} = \omega_0$ is a real eigenvalue of M. If $v_0 \in \mathbb{R}^p$ is an eigenvector corresponding to the eigenvalue $z^{(0)}$, then $y_0(t) = e^{i\omega_0 t}v_0$ is the unique solution of (3.1.3) associated with the real number ω_0.

Theorem 3.1.6 Let p be an integer ≥ 1 and $M \in \mathbb{M}_p(\mathbb{R})$. For any $\tau \in \mathbb{R}$, the single-delay linear system of DDEs

$$\dot{y}(t) = My(t-\tau) \tag{3.1.4}$$

has at most $n = [\frac{p+1}{2}]$ pairs of solutions $y_j^{\pm}(t) = e^{\pm i\omega_j t}v_j^{\pm}$, where $v_j^{\pm} \in \mathbb{C}^p$, $v_j^{\pm} \neq 0$ and $\omega_j > 0$, for $1 \leq j \leq n$.

Proof: Let $\rho \in \mathbb{R}[x]$ denote the characteristic polynomial of M. By Proposition 3.1.1, for a given $\omega > 0$ and a given $\tau > 0$, $y_\omega^{\pm}(t) = e^{\pm i\omega t}v^{\pm}$, with $v^{\pm} \in \mathbb{C}^p$, are non-trivial solutions of (3.1.4) if and only if $z^{\pm} = \pm i\omega e^{\pm i\omega\tau}$ are roots of ρ.

If ρ is of degree p, and since $2n = 2[\frac{p+1}{2}]$ is the largest integer $\leq p$, Theorem 3.1.6 follows. $\square$

3.2 Propositions

This section's perspective is different: we suppose that the matrix $M \in \mathbb{M}_p(\mathbb{R})$ is given. Then, we find the largest natural number $n \geq 1$ for which the single-delay linear system of DDEs (3.0.1) has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t}v_k^{\pm}$, for $1 \leq k \leq n$. In particular, we define necessary and sufficient conditions on the positive frequencies

$\omega_1, \dots, \omega_n$ and on the delay τ such that, for $1 \le k \le n$, $y_k^{\pm}(t) = e^{\pm \imath\omega_k t} v_k^{\pm}$, are solutions of the linear system of DDEs (3.0.1).

There are three different propositions for three different types of matrices: the first two propositions have, respectively, 2×2 and $2n \times 2n$ matrices with non-real eigenvalues and the last proposition has a 3×3 matrix with one real eigenvalue.

Proposition 3.2.1 Let $M \in \mathbb{M}_2(\mathbb{R})$ be a matrix with two non-real eigenvalues z and $\bar{z}$. Then there exist $\omega > 0$, $\tau > 0$ and two non-zero vectors v^+, v^- in $\mathbb{C}^2$ such that $y^{\pm}(t) = e^{\pm \imath\omega\tau} v^{\pm}$ are solutions of $\dot{y}(t) = My(t-\tau)$.

Proof: Let us write the eigenvalue z as $|z|e^{\imath(\frac{\pi}{2}+\theta)}$ with $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$ and $\theta \neq -\frac{\pi}{2}$. By Proposition 3.1.1, it is sufficient to show that there exist $\omega > 0$ and $\tau > 0$ such that $z = \imath\omega e^{\imath\omega\tau} \notin \mathbb{R}$. Set $\omega = |z|$ and choose $k \in \mathbb{N}$ such that $\tau = \dfrac{\theta + 2k\pi}{\omega} > 0$. Hence we have

$$z = |z|e^{\imath(\frac{\pi}{2}+\theta)} = \omega \imath e^{\imath(\theta+2k\pi)} = \imath\omega e^{\imath\omega\tau}. \qquad \square$$

Proposition 3.2.2 Let $M \in \mathbb{M}_{2n}(\mathbb{R})$ be a matrix with non-real eigenvalues $\{z_\jmath, \bar{z}_\jmath\,; 1 \le \jmath \le n\}$. For $1 \le j \le n$, we can write $z_\jmath = |z_\jmath|e^{\imath(\frac{\pi}{2}+\theta_\jmath)}$ with $-\frac{3\pi}{2} < \theta_\jmath < \frac{\pi}{2}$, $\theta_\jmath \neq -\frac{\pi}{2}$. If there are $k_\jmath \in \mathbb{N}$, for $1 \le j \le n$, such that

$$\theta_\jmath + 2k_\jmath\pi > 0 \text{ and } \frac{|z_\jmath|}{|z_1|} = \frac{\theta_\jmath + 2k_\jmath\pi}{\theta_1 + 2k_1\pi},$$

then there exist $\omega_1 > 0, \dots, \omega_n > 0$, $\tau > 0$ and non-zero vectors $v_\jmath^+$, $v_\jmath^- \in \mathbb{C}^{2n}$ such that $y_\jmath^{\pm}(t) = e^{\pm \imath\omega_\jmath t} v_\jmath^{\pm}$ are solutions of $\dot{y}(t) = My(t-\tau)$.

Proof: By Proposition 3.1.1, it is sufficient to show that there exist $\omega_1 > 0, \dots \omega_n > 0$ and $\tau > 0$ such that $z_\jmath = \imath\omega_\jmath e^{\imath\omega_\jmath\tau} \notin \mathbb{R}$.

One can set $\omega_\jmath = |z_\jmath|$ and $\tau = \dfrac{\theta_1 + 2k_1\pi}{|z_1|}$.

Now, it remains to show that, for $1 \le j \le n$, $z_\jmath = \imath\omega_\jmath e^{\imath\omega_\jmath\tau}$.

Note that

$$\omega_j \tau = |z_j| \frac{\theta_1 + 2k_1\pi}{|z_1|} = \frac{\theta_j + 2k_j\pi}{\theta_1 + 2k_1\pi}(\theta_1 + 2k_1\pi) = \theta_j + 2k_j\pi.$$

Hence we obtain:

$$z_j = |z_j| e^{\imath(\frac{\pi}{2}+\theta_j)} = i\omega_j e^{\imath\theta_j} = i\omega_j e^{\imath(\theta_j + 2k_j\pi)} = i\omega_j e^{\imath\omega_j\tau}. \qquad \square$$

Proposition 3.2.3 Let $M \in \mathbb{M}_3(\mathbb{R})$ be a matrix with two non-real eigenvalues $z_1, \bar{z}_1$. Let $z_0 \in \mathbb{R}$ denote the third eigenvalue. For z_0, we can write $z_0 = |z_0| e^{\imath(\frac{\pi}{2}+\theta_0)}$ with $\theta_0 = \pm\frac{\pi}{2}$; and for z_1, we can write $z_1 = |z_1| e^{\imath(\frac{\pi}{2}+\theta_1)}$ with $-\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$, $\theta_1 \neq -\frac{\pi}{2}$.
If there are $k_0, k_1 \in \mathbb{N}$ such that $\theta_0 + 2k_0\pi > 0$, $\theta_1 + 2k_1\pi > 0$, and

$$\frac{|z_1|}{|z_0|} = \frac{\theta_1 + 2k_1\pi}{\theta_0 + 2k_0\pi},$$

then there exist $\omega_0 \geq 0, \omega_1 > 0$, $\tau > 0$ and $v_0 \in \mathbb{R}$, $v_1^+, v_1^- \in \mathbb{C}^3$ such that $y_1^\pm(t) = e^{\pm\imath\omega_1 t} v_1^\pm$ and $y_0(t) = e^{\pm\imath\omega_0 t} v_0$ are solutions of $\dot{y}(t) = My(t-\tau)$.

Proof: By Proposition 3.1.1, it is sufficient to show that there exist $\omega_0 \geq 0$, $\omega_1 > 0$ and $\tau > 0$ such that $z_0 = i\omega_0 e^{\imath\omega_0\tau}$ and $z_1 = i\omega_1 e^{\imath\omega_1\tau} \notin \mathbb{R}$.
One can set $\omega_1 = |z_1|$, $\omega_0 = |z_0|$ and $\tau = \frac{\theta_0 + 2k_0\pi}{|z_0|}$.
Now, it remains to show that $z_0 = i\omega_0 e^{\imath\omega_0\tau}$ and that $z_1 = i\omega_1 e^{\imath\omega_1\tau}$. Let us compute z_0:

$$z_0 = |z_0| e^{\imath(\frac{\pi}{2}+\theta_0)} = i\omega_0 e^{\imath\theta_0} = i\omega_0 e^{\imath(\theta_0 + 2k_0\pi)} = i\omega_0 e^{\imath\omega_0\tau}.$$

Now, note that, for $j = 0, 1$,

$$\omega\tau = |z_j| \frac{\theta_0 + 2k_0\pi}{|z_0|} = \frac{\theta_1 + 2k_1\pi}{\theta_0 + 2k_0\pi}(\theta_0 + 2k_0\pi).$$

Hence we obtain:

$$z_1 = |z_1| e^{\imath(\frac{\pi}{2}+\theta_1)} = i\omega_1 e^{\imath\theta_1} = i\omega_1 e^{\imath(\theta_1 + 2k_1\pi)} = i\omega_1 e^{\imath\omega_1\tau}. \qquad \square$$

Chapter 4

Realization Theorem for a DDE with Two Delays in $\mathbb{R}^2$

In this chapter, we intend to prove a realization theorem for a system of two linear DDE's. Thus, for two frequencies ω_1, ω_2, we would like to find l delays and l matrices in $\mathbb{M}_2(\mathbb{R})$ such that the DDE

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j)$$

has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k$, for $k = 1, 2$.

The proofs of the realization theorem uses a long technical theorem and a proposition. We present both the technical theorem and the proposition in the first section of this chapter. Section 2 contains the realization and openness theorems for such a system.

4.1 Existence Theorem of Specific 2×2 Matrices and a Lemma of Linear Algebra

This section contains a technical theorem and a proposition. They are necessary for the main proof of this chapter. The theorem proves the existence of two 2×2

matrices with very specific conditions and the proposition shows a short result in linear algebra.

Theorem 4.1.1 Let A_2, B_2, D_2, E, F be real numbers such that either $D_2^2 - 4FA_2 \geq 0$, $B_2^2 - 4A_2 > 0$ or $E^2 - 4F > 0$.

Then there exist two matrices in $\mathbb{M}_2(\mathbb{R})$, $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ such that :

$$A_2 = \det M_2, \qquad B_2 = \text{-Tr } M_2$$

$$E = \text{-Tr } M_1, \qquad F = \det M_1$$

$$\text{and} \qquad D_2 = \det K_{1,2} + \det K_{2,1}\,, \qquad \text{where } K_{i,j} = \begin{pmatrix} a_i & b_j \\ c_i & d_j \end{pmatrix}.$$

Proof: The proof will be divided in the following three cases: $D_2^2 - 4FA_2 > 0$, $D_2^2 - 4FA_2 = 0$ and $D_2^2 - 4FA_2 < 0$. In the first two, we will construct explicitly matrices M_1 and M_2 which satisfy the five equalities of the theorem. In the third case, we prove the existence of M_1 and M_2.

<u>Case 1</u>: $D_2^2 - 4FA_2 > 0$.

This case is divided in four subcases:

(a) $F = A_2 = 0$,

(b) $F = 0$ and $A_2 \neq 0$,

(c) $F \neq 0$ and $A_2 = 0$,

(d) $F \neq 0$ and $A_2 \neq 0$.

Note that since $D_2^2 - 4FA_2 > 0$, subcases (a), (b) and (c) imply that $D_2 \neq 0$.

Subcase (a): Let M_1 and M_2 denote the two matrices:

$$M_1 = \begin{pmatrix} 0 & b_1 \\ 0 & -E \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 \\ \frac{-D_2}{b_1} & -B_2 \end{pmatrix}$$

where $b_1 \neq 0$.

It is easy to verify that:

$$\det M_2 = 0 = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = 0 = F,$$

$$\text{and} \qquad \det K_{1,2} + \det K_{2,1} = -c_2 b_1 = -\frac{-D_2}{b_1} b_1 = D_2.$$

Subcase (b): Let M_1 and M_2 denote the two matrices:

$$M_1 = \begin{pmatrix} 0 & \frac{D_2}{A_2} b_2 \\ 0 & -E \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & b_2 \\ \frac{-A_2}{b_2} & -B_2 \end{pmatrix}$$

where $b_2 \neq 0$.

Then we have:

$$\det M_2 = A_2, \qquad \text{-Tr } M_2 = B_2$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = 0 = F$$

$$\text{and} \qquad \det K_{1,2} + \det K_{2,1} = -c_2 b_1 = \frac{A_2}{b_2} \frac{D_2}{A_2} b_2 = D_2.$$

Subcase (c): Let M_1 and M_2 denote the two matrices:

$$M_1 = \begin{pmatrix} 0 & b_1 \\ \frac{-F}{b_1} & -E \end{pmatrix}, M_2 = \begin{pmatrix} 0 & \frac{D_2 b_1}{F} \\ 0 & -B_2 \end{pmatrix}$$

where $b_1 \neq 0$.

$$\det M_2 = 0 = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = F,$$

and $$\det K_{1,2} + \det K_{2,1} = -c_1 b_2 = \frac{F}{b_1}\frac{D_2 b_1}{F} = D_2.$$

Subcase (d): Let M_1 and M_2 denote the two matrices:

$$M_1 = \begin{pmatrix} 0 & \alpha \\ -\frac{F}{\alpha} & -E \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & \alpha^2 \\ -\frac{A_2}{\alpha^2} & -B_2 \end{pmatrix}$$

where

$$\alpha = \frac{D_2 + \sqrt{D_2^2 - 4A_2F}}{2F}.$$

Since A_2 and F are non-zero, $D_2^2 - 4A_2F \neq D_2^2$ and therefore $\alpha \neq 0$. By direct computations, one verifies that :

$$\det M_2 = \frac{A_2}{\alpha^2}\alpha^2 = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \qquad \det M_1 = \frac{F}{\alpha}\alpha = F,$$

$$\begin{aligned}
\det K_{1,2} + \det K_{2,1} &= -c_1 b_2 - c_2 b_1 \\
&= \frac{F}{\alpha}\alpha^2 + \frac{A_2}{\alpha^2}\alpha \\
&= \frac{F\alpha^2 + A_2}{\alpha} \\
&= \frac{F\left(D_2^2 + 2D_2\sqrt{D_2^2 - 4A_2F} + (D_2^2 - 4A_2F)\right) + 4A_2F^2}{4F^2}\frac{1}{\alpha} \\
&= 2D_2\alpha\frac{1}{2\alpha} \\
&= D_2.
\end{aligned}$$

Case 2: $D_2^2 - 4FA_2 = 0$.

In this case, there are two subcases: (a) $D_2 = 0$ or (b) $D_2 \neq 0$.

Subcase (a): If $D_2 = 0$, then $-4FA_2 = 0$. That is either $F = 0$ or $A_2 = 0$.

If $F = 0$, then consider the two matrices M_1 and M_2:

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & -E \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & b_2 \\ \frac{-A_2}{b_2} & -B_2 \end{pmatrix}$$

where $b_2 \neq 0$.

It is easy to verify that:

$$\det M_2 = \frac{A_2}{b_2} b_2 = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = 0 = F,$$

$$\text{and} \qquad \det K_{1,2} + \det K_{2,1} = a_1 d_2 - c_1 b_2 + a_2 d_1 - c_2 b_1 = 0 = D_2.$$

Likewise, if $A_2 = 0$, then consider the two matrices M_1 and M_2:

$$M_1 = \begin{pmatrix} 0 & b_1 \\ \frac{-F}{b_1} & -E \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & -B_2 \end{pmatrix}$$

where $b_1 \neq 0$. Indeed we have:

$$\det M_2 = 0 = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = \frac{F}{b_1} b_1 = F,$$

$$\text{and} \qquad \det K_{1,2} + \det K_{2,1} = a_1 d_2 - c_1 b_2 + a_2 d_1 - c_2 b_1 = 0 = D_2.$$

Subcase (b): Since $D_2^2 - 4FA_2 = 0$ and $D_2 \neq 0$, we have that both $F \neq 0$ and $A_2 \neq 0$.

Consider the two matrices:

$$M_1 = \begin{pmatrix} 0 & \frac{2F}{D_2} b_2 \\ -\frac{D_2}{2b_2} & -E \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & b_2 \\ \frac{-A_2}{b_2} & -B_2 \end{pmatrix}$$

where $b_2 \neq 0$. Now we can verify that:

$$\det M_2 = \frac{A_2}{b_2} = A_2, \qquad \text{-Tr } M_2 = B_2,$$

$$\text{-Tr } M_1 = E, \quad \det M_1 = -\frac{D_2}{2b_2}\frac{2Fb_2}{D_2} = F,$$

$$\begin{aligned}
\det K_{1,2} + \det K_{2,1} &= -c_1b_2 - c_2b_1 \\
&= \frac{D_2}{2b_2}b_2 + \frac{A_2}{b_2}\frac{2F}{D_2}b_2 \\
&= \frac{D_2}{2} + \frac{2A_2F}{D_2} \\
&= \frac{D_2}{2} + \frac{1}{2}\frac{D_2^2}{D_2} \\
&= D_2.
\end{aligned}$$

<u>**Case 3:**</u> $D_2^2 - 4FA_2 < 0$.

In this case, we will not exhibit M_1 and M_2 but simply show that there exist, for $i = 1, 2$, a choice of $a_i, b_i, c_i, d_i \in \mathbb{R}$, satisfying the conditions:

$$A_2 = \det M_2, \qquad B_2 = \text{-Tr } M_2, \tag{4.1.1}$$

$$E = \text{-Tr } M_1, \qquad F = \det M_1, \tag{4.1.2}$$

$$\text{and} \quad D_2 = a_1d_2 - c_1b_2 + a_2d_1 - c_2b_1. \tag{4.1.3}$$

Adding the conditions $b_1, b_2 \in \mathbb{R}^*$, (4.1.2) and (4.1.3) becomes:

$$c_1 = \frac{-F + a_1d_1}{b_1}, \qquad c_2 = \frac{-A_2 + a_2d_2}{b_2}$$

and

$$\begin{aligned}
D_2 &= \det K_{1,2} + \det K_{2,1} \\
&= a_1d_2 - c_1b_2 + a_2d_1 - c_2b_1 \\
&= a_1d_2 - a_1d_1\frac{b_2}{b_1} + a_2d_1 - a_2d_2\frac{b_1}{b_2} + F\frac{b_2}{b_1} + A_2\frac{b_1}{b_2}.
\end{aligned}$$

Setting $r = \frac{b_2}{b_1}$, rearranging the terms and simplifying yields

$$(-F + a_1d_1)r^2 + (D_2 - a_1d_2 - a_2d_1)r + (-A_2 + a_2d_2) = 0. \tag{4.1.4}$$

Then using (4.1.1), we obtain

$$(F+a_1(E+a_1))r^2-(D_2+a_1(B_2+a_2)+a_2(E+a_1))r+(A_2+a_2(B+a_2)) = 0. \quad (4.1.5)$$

Therefore, to prove the existence of $M_1, M_2 \in \mathbb{M}_2(\mathbb{R})$ satisfying (4.1.1) to (4.1.3), it is sufficient to show that for a good choice of a_1, a_2, the quadratic equation (4.1.5) has a non-zero real solution r. This follows if the discriminant of (4.1.5) is greater than zero i.e.,

$$(D_2 + a_1(B_2 + a_2) + a_2(E + a_1))^2 - 4\left(F + a_1(E + a_1)\right)\left(A_2 + a_2(B_2 + a_2)\right) > 0. \quad (4.1.6)$$

Or equivalently, if for a good choice of a_1, a_2,

$$\begin{aligned}(B_2^2 - 4A_2)a_1^2 + (E^2 - 4F)a_2^2 + 2(2D_2 - EA_2)a_1a_2 + 2(B_2D_2 - 2EA_2)a_1 \\ +2(ED_2 - FB_2)a_2 + (D_2^2 - 4FA_2) > 0\end{aligned} \quad (4.1.7)$$

As by assumption either $B_2^2 - 4A_2 > 0$ or $E^2 - 4F > 0$, such a choice of a_1, a_2 exits. □

As said above, the proposition below is a result of linear algebra that we will be use in section 4.2. Note that this lemma has no link with the preceding theorem.

Proposition 4.1.2 Let V be a real vector space. For $n \geq 3$, let $S = \{v_1, \ldots, v_n\}$ be a linearly dependent set of n non-zero vectors such that $S_{n,n-1} = S \setminus \{v_n, v_{n-1}\}$ forms a basis of $W = \text{span}(S)$.
Then there exist $\lambda_1 \in \mathbb{R}, \ldots, \lambda_{n-1} \in \mathbb{R}$, not all zero, such that

$$\sum_{j=1}^{n-1} \lambda_j v_j + v_n = 0 \quad \text{and} \quad \lambda_{n-1}^2 - 4\lambda_{n-2} > 0.$$

Proof: As $S_{n,n-1}$ is a basis of W, there exist $\alpha_1 \in \mathbb{R}, \ldots, \alpha_{n-2} \in \mathbb{R}$, not all zero, such that

$$\sum_{j=1}^{n-2} \alpha_j v_j + v_n = 0.$$

Likewise, there exist $\beta_1 \in \mathbb{R}, \ldots, \beta_{n-2} \in \mathbb{R}$, not all zero, such that

$$\sum_{j=1}^{n-2} t\beta_j v_j + tv_{n-1} = 0.$$

Thus, for $\alpha_1 \in R, \ldots, \alpha_{n-2} \in \mathbb{R}$, not all zero, and $\beta_1 \in \mathbb{R}, \ldots, \beta_{n-2} \in \mathbb{R}$, not all zero,we have

$$\sum_{j=1}^{n-2} (\alpha_j + t\beta_j)v_j + tv_{n-1} + v_n = 0.$$

Now, for given α_{n-2} and β_{n-2}, we can choose $t_\circ$ large enough such that $t_\circ^2 - 4(\alpha_{n-2} + t_\circ\beta_{n-2}) > 0$. Then, setting $\lambda_j = \alpha_j + t_\circ\beta_j$, for $1 \leq j \leq n-2$, and $\lambda_{n-1} = t_\circ^2$ finishes the proof. $\square$

4.2 Realization Theorem for a System of Two DDE's with Two Delays

This section contains the Realization Theorem 4.2.3 and the Openness Theorem 4.2.6. The realization theorem (the most important one) states that for two rationally independent frequencies, there exist a linear system of DDEs

$$\dot{y}(t) = M_1 y(t - \tau_j) + M_2 y(t - \tau_2) \tag{4.2.1}$$

such that the DDE has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t}v$, for $k = 1, 2$. The proof of the realization theorem uses the result of Theorem 4.2.1 below that establishes an equivalence between a solution to the system (4.2.1) and a point on a conic in $\mathbb{C}^2$. The openness theorem shows that the realization of two imaginary numbers (not necessarily rationally independent) as eigenvalues of (4.2.1) is valid in a neighbourhood of any set of two rationally independent imaginary numbers.

Let us recall that, if $M_i = \left(a_{n,m}^i\right) \in \mathbb{M}_2(\mathbb{R})$, for $1 \leq i \leq l$, then we denote by $c_1^{(i)}$ (respectively $c_2^{(i)}$) its first (respectively second) column.

Theorem 4.2.1 For $\tau > 0$, consider the linear system of DDEs

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{4.2.2}$$

where $M_j \in \mathbb{M}_2(\mathbb{R})$ and $\tau_j \in [0, \tau]$, for $1 \leq j \leq l$.

For $\omega \in \mathbb{R}$, let $\lambda(\omega) = (i\omega e^{i\omega\tau_1}, e^{i\omega(\tau_2-\tau_1)}, \dots, e^{i\omega(\tau_l-\tau_1)}) \in \mathbb{C}^l$. The following two conditions are equivalent:

(a) There exist $v \in \mathbb{C}^2$, $v \neq 0$, such that $y_\omega(t) = e^{i\omega t}v$ is a solution of (4.2.2),

(b) λ is on the conic C of equation

$$\phi(x_1, \dots, x_l) = x_1^2 + \sum_{j=2}^{l} A_j x_j^2 + \sum_{j=2}^{l} B_j x_1 x_j + \sum_{\substack{i,j=2 \\ i<j}}^{l} C_{i,j} x_i x_j + \sum_{j=2}^{l} D_j x_j + E x_1 + F$$

with

$$A_j = \det M_j, \qquad B_j = -\mathrm{Tr}\ M_j,$$

$$E = -\mathrm{Tr}\ M_1, \qquad F = \det M_1,$$

$$\text{and } D_j = \det K_{1,j} + \det K_{j,1},$$

$$C_{i,j} = \det K_{i,j} + \det K_{j,i}$$

$$\text{where } K_{i,j} = \left(c_1^{(i)}, c_2^{(j)}\right).$$

Moreover, if $y_\omega(t) = e^{i\omega t}v$ is a solution of (4.2.2), then $v \in \ker \Delta(i\omega)$, where Δ denotes the characteristic matrix of (4.2.2).

Proof: Recall (see the proof of Theorem 2.0.3) that for condition (a) to hold, it is necessary and sufficient that $\det \Delta(i\omega) = 0$, where Δ denotes the characteristic matrix of (4.2.1). As

$$\det \Delta(i\omega) = \det \left(i\omega I_2 - \sum_{j=1}^{l} M_j e^{-i\omega\tau_j} \right) = e^{-2i\omega\tau_1} \det \left(i\omega e^{i\omega\tau_1} I_2 - \sum_{j=1}^{l} M_j e^{-i\omega(\tau_j-\tau_1)} \right),$$

condition (a) is equivalent to

$$\det\left(i\omega e^{i\omega\tau_1} I_2 - \sum_{j=1}^{l} M_j e^{-i\omega(\tau_j-\tau_1)}\right) = 0. \tag{4.2.3}$$

Setting $z = i\omega e^{i\omega\tau_1}$ and $y_j = e^{i\omega(\tau_1-\tau_j)}$, we obtain:

$$\det\left(zI_2 - \sum_{j=1}^{l} M_j y_j\right) = 0. \tag{4.2.4}$$

Therefore, condition (a) is equivalent to

$$\left(z - a_{11}^{(1)} - \sum_{j=2}^{l} a_{11}^{(j)} y_j\right)\left(z - a_{22}^{(1)} - \sum_{j=2}^{l} a_{22}^{(j)} y_j\right) - \left(a_{21}^{(1)} + \sum_{j=2}^{l} a_{21}^{(j)} y_j\right)\left(a_{12}^{(1)} + \sum_{j=2}^{l} a_{12}^{(j)} y_j\right) = 0.$$

Expanding and reorganizing the terms yields:

$$z^2 + \sum_{j=2}^{l}(\det M_j)y_j^2 + \sum_{j=2}^{l}(-\mathrm{Tr}\ M_j)zy_j + \sum_{\substack{i,j=2\\ i<j}}^{l}(\det K_{i,j} + \det K_{j,i})y_i y_j$$
$$+ \sum_{j=2}^{l}(\det K_{1,j} + \det K_{j,1})y_j + (-\mathrm{Tr}\ M_1)z + \det M_1 = 0$$

where $K_{i,j} = \left(c_1^{(i)}, c_2^{(j)}\right)$.

Using the notations defined in the statement of the theorem, one obtains

$$\det\left(zI_2 - \sum_{j=1}^{l} M_j y_j\right) = \phi(z, y_2, \ldots, y_l),$$

Hence, condition (a) is equivalent to $\phi(z, y_2, \ldots, y_l) = 0$. Let $\lambda = (i\omega e^{-i\omega\tau_1}, e^{i\omega(\tau_2-\tau_1)}, \ldots, e^{i\omega(\tau_l-\tau_1)})$. By statement (4.2.6), λ is on the conic C of equation $\phi(x_1, \ldots, x_l)$. Moreover, by Theorem 2.0.3, if $y_\omega = e^{i\omega t}v$, $v \in \mathbb{C}^2$, is a solution of (4.2.2), then $v \in \ker \Delta(i\omega)$. $\square$

Remark 4.2.2 The proof of Theorem 4.2.2 can be also deduced from Theorem 2.0.3 as follows: For $p = 2$, let ρ denote the homogeneous polynomial of degree 2 defined in Definition 2.0.2. Then, by Theorem 2.0.3, condition (a) of Theorem 4.2.2 is equivalent to $\rho(i\omega, e^{i\omega\tau_1}, \ldots, e^{i\omega\tau_l}) = 0$. As ρ is a homogeneous polynomial and $e^{i\omega\tau_1} \neq 0$, we have

$$\rho(i\omega, e^{i\omega\tau_1}, \ldots, e^{i\omega\tau_l}) = 0 \text{ if and only if } \rho(i\omega e^{-i\omega\tau_1}, 1, e^{i\omega(\tau_2-\tau_1)}, \ldots, e^{i\omega(\tau_l-\tau_1)}) = 0. \tag{4.2.5}$$

By Theorem B.0.2 in Appendix B, the statement (4.2.5) can be equivalently written as:

$$\rho(i\omega, e^{i\omega\tau_1}, \ldots, e^{i\omega\tau_l}) = 0 \text{ if and only if } \phi(i\omega e^{-i\omega\tau_1}, e^{i\omega(\tau_2-\tau_1)}, \ldots, e^{i\omega(\tau_l-\tau_1)}) = 0. \tag{4.2.6}$$

Let $\lambda = (i\omega e^{-i\omega\tau_1}, e^{i\omega(\tau_2-\tau_1)}, \ldots, e^{i\omega(\tau_l-\tau_1)})$. By statement (4.2.6), λ is on the conic C of equation $\phi(x_1, \ldots, x_l)$. $\square$

We can now state and prove the central theorem of the chapter:

Theorem 4.2.3 Let ω_1, ω_2 be two positive real numbers such that $\{\omega_1, \omega_2\}$ are linearly independent over the rationals.
Then, there exist $\tau_1 > 0$, $\tau_2 > 0$, $M_1 \in \mathbb{M}_2(\mathbb{R})$, $M_2 \in \mathbb{M}_2(\mathbb{R})$ such that the linear system of DDEs

$$\dot{y}(t) = M_1\, y(t - \tau_1) + M_2\, y(t - \tau_2) \tag{4.2.7}$$

has non-trivial solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$ with $v_k^{\pm} \in \mathbb{C}^2, v_k^{\pm} \neq 0$, for $k = 1, 2$.

Proof: By Theorems 4.1.1 and 4.2.1, it is sufficient to prove that there exist $\tau_1, \tau_2 > 0$ such that the following statement holds:
There exists a conic C of equation

$$\phi\,(x_1,\, x_2) = x_1^2 + A_2 x_2^2 + B_2 x_1 x_2 + E x_1 + D_2 x_2 + F$$

where A_2, B_2, E, D_2, F are real numbers and $B_2^2 - 4A_2 > 0$, that contains the two pairs of complex conjugate points $(i\omega_k e^{i\omega_k \tau_1}, e^{i\omega_k(\tau_1-\tau_2)})$ and $(-i\omega_k e^{-i\omega_k \tau_1}, e^{-i\omega_k(\tau_1-\tau_2)})$, for $k = 1, 2$.

To simplify notation, let $z^{(k)} = i\omega_k e^{i\omega_k \tau_1}$ and $y_2^{(k)} = e^{i\omega_k(\tau_1-\tau_2)}$, for $k = 1, 2$. It is therefore sufficient to show the following claim:

Claim 4.2.4 Let us consider the matrix $M \in \mathbb{M}_{2,6}(\mathbb{C})$ defined by

$$\begin{pmatrix} (z^{(1)})^2 & (y_2^{(1)})^2 & z^{(1)}y_2^{(1)} & z^{(1)} & y_2^{(1)} & 1 \\ (z^{(2)})^2 & (y_2^{(2)})^2 & z^{(2)}y_2^{(2)} & z^{(2)} & y_2^{(2)} & 1 \end{pmatrix}. \tag{4.2.8}$$

Now, let $\tilde{P} = \begin{pmatrix} M \\ \overline{M} \end{pmatrix}$ and $R = \begin{pmatrix} R_1 \\ \vdots \\ R_6 \end{pmatrix} \in \mathbb{R}^6$. Then there exist $\tau_1 >$ and $\tau_2 > 0$ such that, the homogeneous linear system $\tilde{P}R = 0$ has a nontrivial real solution with $R_1 = 1$ and $R_3^2 - 4R_2 > 0$.

Recall that $N = \left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in \mathbb{C}^4;\ z \in \mathbb{C}^2 \right\}$ is a vector subspace of $\mathbb{C}^4$ of real dimension 4, and that the map

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in N \longmapsto \begin{pmatrix} \mathrm{Re}\ z \\ \mathrm{Im}\ z \end{pmatrix} \in \mathbb{R}^4$$

is an isomorphism. Therefore $\{R \in \mathbb{R}^6;\ \tilde{P}R = 0\} = \{R \in \mathbb{R}^6;\ PR = 0\}$ where $P \in \mathbb{M}_{4,6}(\mathbb{R})$ is the matrix given by

$$P = \begin{pmatrix} \mathrm{Re}\ M \\ \mathrm{Im}\ M \end{pmatrix}.$$

That is

$$P = \begin{pmatrix} \mathrm{Re}(z^{(1)})^2 & \mathrm{Re}(y_2^{(1)})^2 & \mathrm{Re}\, z^{(1)}y_2^{(1)} & \mathrm{Re}\, z^{(1)} & \mathrm{Re}\, y_2^{(1)} & 1 \\ \mathrm{Re}(z^{(2)})^2 & \mathrm{Re}(y_2^{(2)})^2 & \mathrm{Re}\, z^{(2)}y_2^{(2)} & \mathrm{Re}\, z^{(2)} & \mathrm{Re}\, y_2^{(2)} & 1 \\ \mathrm{Im}(z^{(1)})^2 & \mathrm{Im}(y_2^{(1)})^2 & \mathrm{Im}\, z^{(1)}y_2^{(1)} & \mathrm{Im}\, z^{(1)} & \mathrm{Im}\, y_2^{(1)} & 0 \\ \mathrm{Im}(z^{(2)})^2 & \mathrm{Im}(y_2^{(2)})^2 & \mathrm{Im}\, z^{(2)}y_2^{(2)} & \mathrm{Im}\, z^{(2)} & \mathrm{Im}\, y_2^{(2)} & 0 \end{pmatrix} \tag{4.2.9}$$

Setting $x_1 = \tau_1$ and $x_2 = \tau_1 - \tau_2$, we have

$$P = \begin{pmatrix} -\omega_1^2 \cos(2\omega_1 x_1) & \cos(2\omega_1 x_2) & -\omega_1 \sin(\omega_1 x_1 + \omega_1 x_2) & -\omega_1 \sin(\omega_1 x_1) & \cos(\omega_1 x_2) & 1 \\ -\omega_2^2 \cos(2\omega_2 x_1) & \cos(2\omega_2 x_2) & -\omega_2 \sin(\omega_2 x_1 + \omega_2 x_2) & -\omega_2 \sin(\omega_2 x_1) & \cos(\omega_2 x_2) & 1 \\ -\omega_1^2 \sin(2\omega_1 x_1) & \sin(2\omega_1 x_2) & -\omega_1 \cos(\omega_1 x_1 + \omega_1 x_2) & -\omega_1 \cos(\omega_1 x_1) & \sin(\omega_1 x_2) & 0 \\ -\omega_2^2 \sin(2\omega_2 x_1) & \sin(2\omega_2 x_2) & -\omega_2 \cos(\omega_2 x_1 + \omega_2 x_2) & -\omega_2 \cos(\omega_2 x_1) & \sin(\omega_2 x_2) & 0 \end{pmatrix} \tag{4.2.10}$$

Therefore, Claim 4.2.4 follows from:

Claim 4.2.5

$$P \begin{pmatrix} R_1 \\ \vdots \\ R_6 \end{pmatrix} = 0$$

has a non-trivial solution with $R_1 = 1$ and $R_3^2 - 4R_2 > 0$.

For $1 \le j \le 6$, let c_j denote the j-th column of P and let $S = \{c_1, \ldots, c_6\}$. Note that, for $1 \le j \le 6$, each c_j is in $\mathbb{R}^4$ hence $\mathrm{span}(S) \subset \mathbb{R}^4$. Thus, if there exist $x_1 > 0$ and $x_2 < x_1$ such that $S_1 = \{c_2, c_4, c_5, c_6\}$ is linearly independent, then S_1 is a basis for $\mathrm{span}(S) = \mathbb{R}^4$. Then, by Proposition 4.1.2, there would exist $R_1 \in \mathbb{R}, \ldots, R_6 \in \mathbb{R}$, not all zero, such that $c_1 + R_2c_2 + R_3c_3 + R_4c_4 + R_5c_5 + R_6c_6 = 0$ and $R_3^2 - 4R_2 > 0$. Therefore, to prove Claim 4.2.5, it is sufficient to show that there exist $x_1 > 0$ and

$x_2 < x_1$ such that the set $S_1 = \{c_2, c_4, c_5, c_6\}$ is linearly independent.

Let us define $\omega \in \mathbb{R}^2$ by $\omega = (\omega_1, \omega_2)^T$; then let

$$F_\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

be a map given by

$$F_\omega(x_1, x_2) = \det X \tag{4.2.11}$$

where $X = (c_2\, c_4\, c_5\, c_6) \in \mathbb{M}_4(\mathbb{R})$.

To prove Claim 4.2.5, we therefore must show that there exist $x_1 > 0$ and $x_2 < x_1$ such that $F_\omega(x_1, x_2) \neq 0$.

Let $\tilde{A} = (\tilde{c_1}\, \tilde{c_2}\, \tilde{c_3}\, \tilde{c_4}\, \tilde{c_5}\, \tilde{c_6}) \in \mathbb{M}_{4,6}(\mathbb{R})$ denote the matrix

$$\begin{pmatrix} -\omega_1^2 \cos(2\varphi_1^1) & \cos(2\varphi_1^2) & -\omega_1 \sin(\varphi_1^1 + \varphi_1^2) & -\omega_1 \sin(\varphi_1^1) & \cos(\varphi_1^2) & 1 \\ -\omega_2^2 \cos(2\varphi_2^1) & \cos(2\varphi_2^2) & -\omega_2 \sin(\varphi_1^2 + \varphi_2^2) & -\omega_2 \sin(\varphi_2^1) & \cos(\varphi_2^2) & 1 \\ -\omega_1^2 \sin(2\varphi_1^1) & \sin(2\varphi_1^2) & -\omega_1 \cos(\varphi_1^1 + \varphi_1^2) & -\omega_1 \cos(\varphi_1^1) & \sin(\varphi_1^2) & 0 \\ -\omega_2^2 \sin(2\varphi_2^1) & \sin(2\varphi_2^2) & -\omega_2 \cos(\varphi_1^2 + \varphi_2^2) & -\omega_2 \cos(\varphi_2^1) & \sin(\varphi_2^2) & 0 \end{pmatrix} \tag{4.2.12}$$

For $\Phi_j = (\varphi_j^1, \varphi_j^2)^T$, $j = 1, 2$, let $\tilde{F_\omega}$ be a map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\tilde{F_\omega}(\Phi_1, \Phi_2) = \det \tilde{X}(\Phi_1, \Phi_2) = \det(\tilde{c_2}\, \tilde{c_4}\, \tilde{c_5}\, \tilde{c_6}). \tag{4.2.13}$$

Note that $\tilde{F_\omega}(\Phi_1, \Phi_2) = \tilde{F_\omega}\left(\Phi_1 + 2\pi \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \Phi_2 + 2\pi \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\right)$. Thus, we shall consider $\tilde{F_\omega}$ as a function from $\mathbb{T} \times \mathbb{T}$ to $\mathbb{R}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

For $\hat{\Phi}_1 = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{4} \end{pmatrix} \in \mathbb{T}$ and $\hat{\Phi}_2 = \begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix} \in \mathbb{T}$, we have

$$\tilde{F}_\omega(\hat{\Phi}_1, \hat{\Phi}_2) = \det \begin{pmatrix} 0 & -\omega_1 & \frac{\sqrt{2}}{2} & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\omega_2 & 1 & 0 \end{pmatrix} = \omega_1 \neq 0. \tag{4.2.14}$$

As $\tilde{F}_\omega$ is continuous, there exist an open neighbourhood N of $(\hat{\Phi}_1, \hat{\Phi}_2)$ such that, for all $(\hat{\Phi}_1, \hat{\Phi}_2) \in N$, $\tilde{F}_\omega(\hat{\Phi}) \neq 0$.

As $\{\theta_1 \times \theta_2;\ \theta_i$ open set of $\mathbb{T}\}$ forms a basis for the topology of $\mathbb{T} \times \mathbb{T}$ (see [16], Theorem 15.1), there exists an open neighbourhood N_i of $\hat{\Phi}_i$ in $\mathbb{T}$, i=1,2, such that $N_1 \times N_2 \subset N$. Since $\{\omega_1, \omega_2\}$ is linearly independent over $\mathbb{Q}$, the integral curve $\left\{ x_1 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \bmod 2\pi \mid x_1 \in \mathbb{R} \right\}$ and the integral curve $\left\{ x_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \bmod 2\pi \mid x_2 \in \mathbb{R} \right\}$ are both dense on the 2-torus $\mathbb{T} \times \mathbb{T}$.

Hence there exist $x_1 > 0$ such that $\begin{pmatrix} \omega_1 x_1 \\ \omega_2 x_1 \end{pmatrix} \bmod 2\pi \in N_1$ and $x_2 < 0 < x_1$ such that $\begin{pmatrix} \omega_1 x_2 \\ \omega_2 x_2 \end{pmatrix} \bmod 2\pi \in N_2$.

Therefore, $F_\omega(x_1, x_2) = \tilde{F}_\omega \left(\begin{pmatrix} \omega_1 x_1 \\ \omega_2 x_1 \end{pmatrix}, \begin{pmatrix} \omega_1 x_2 \\ \omega_2 x_2 \end{pmatrix} \right) \neq 0$.

Setting $\tau_1 = x_1$ and $\tau_2 = \tau_1 - x_2$ completes the proof. $\square$

To close this section and this chapter, we prove the openness theorem.

Theorem 4.2.6 Let ω_1, ω_2 be two positive numbers such that $\{\omega_1, \omega_2\}$ are linearly independent over the rationals. Then there exist $\tau_1 > 0$, $\tau_2 > 0$ and an open neighbourhood $\mathcal{N}$ of $\tilde{\omega} = (\omega_1, \omega_2)^T \in \mathbb{R}^2_+$ with the following property:

For every $\omega = (\omega_a, \omega_b)^T \in \mathcal{N}$, there exist $M_1 \in \mathbb{M}_2(\mathbb{R})$, $M_2 \in \mathbb{M}_2(\mathbb{R})$, and $v_k^\pm \in \mathbb{C}^2$, $k = a, b$, such that the linear system of DDEs

$$\dot{y}(t) = M_1\, y(t - \tau_1) + M_2\, y(t - \tau_2) \tag{4.2.15}$$

has solutions $x_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$ with $v_k^{\pm} \in \mathbb{C}^2$, $v_k^{\pm} \neq 0$ for $k = a, b$.

Proof: Keeping the notation of the proof of Theorem 4.2.3, it is enough to show that there exist $\tau_1 > 0, \tau_2 > 0$ and an open neighbourhood $\mathcal{N}$ of $\tilde{\omega} \in \mathbb{R}_+^2$ such that $F(\tau_1, \tau_2; \omega) \neq 0$ for all $\omega \in \mathcal{N}$. By Theorem 4.2.3, there exist $\tau_1 > 0$, $\tau_2 > 0$ such that $F(\tau_1, \tau_2; \tilde{\omega}) \neq 0$. Now, consider the map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

defined by

$$f(\tilde{\omega}) = F(\tau_1, \tau_2; \tilde{\omega}), \tag{4.2.16}$$

By construction, $f(\tilde{\omega}) \neq 0$. Since f is continuous, there exist $\mathcal{N}$, an open neighbourhood of $\tilde{\omega} \in \mathbb{R}_+^2$ such that for all $\omega \in \mathcal{N}$, $f(\tilde{\omega}) \neq 0$. By the construction of f, it is equivalent to $F(\tau_1, \tau_2; \omega) \neq 0$ for all $\omega \in \mathcal{N}$. $\square$

Chapter 5

Realization theorem

This last chapter contains the most serious attempt to obtain a realization theorem as general as possible for a system of linear DDE's with fixed delays. Thus, we consider a system of linear DDE's written as :

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{5.0.1}$$

where $M_j \in \mathbb{M}_p(\mathbb{R})$.

One restriction remains: the matrices M_j will be invertible and diagonal. We write: $M_j = \operatorname{Diag}(a_{1j}, \ldots, a_{pj})$ where $a_{kj} \neq 0$ for $1 \leq k \leq p$.

This chapter also contains an openness theorem for the system (5.0.1).

The proof of the realization theorem requires a result presented in Proposition 5.0.1.

5.1 Propositions

To any l positive distinct numbers $\tau_1, \ldots, \tau_l$, we associate the subset $\Omega \in \mathbb{R}_+$ given by

$$\Omega = \left(\bigcap_{j=2}^{l} \left\{ \frac{k\pi}{\tau_j - \tau_1}; \ k \in \mathbb{Z} \right\}^{\complement} \right) \bigcap \left(\bigcap_{j=3}^{l} \left\{ \frac{k\pi}{\tau_j - \tau_2}; \ k \in \mathbb{Z} \right\}^{\complement} \right). \tag{5.1.1}$$

where $\complement$ denotes the complement. Note that for $\omega \in \Omega$, $\sin(\omega(\tau_1 - \tau_2)) \neq 0$, as well as $\sin(\omega(\tau_1 - \tau_{2+r})) \neq 0$ and $\sin(\omega(\tau_2 - \tau_{2+r})) \neq 0$, for $1 \leq r \leq l-2$. We can now write the following proposition:

Proposition 5.1.1 Let $\tau_1 > 0, \dots, \tau_l > 0$ be l distinct delays, $l \geq 3$.
Then, for $\omega \in \Omega \cap \mathbb{R}_+$, there exist $a_1 \in \mathbb{R}^*, \dots, a_l \in \mathbb{R}^*$ such that $y_\omega^\pm(t) = e^{\pm\imath\omega t}$ is a solution of the scalar differential equation

$$\dot{y}(t) = \sum_{\jmath=1}^{l} a_\jmath y(t - \tau_\jmath). \tag{5.1.2}$$

Proof: Let $\omega \in \Omega \cap \mathbb{R}_+$. By Proposition 2.0.4, $y_\omega^+(t) = e^{\imath\omega t}$ and $y_\omega^-(t) = e^{-\imath\omega t}$ are solutions of (5.1.2) if and only if

$$\begin{aligned} -i\omega - \sum_{\jmath=1}^{l} a_\jmath\, e^{-\imath\omega\tau_\jmath} &= 0 \\ i\omega - \sum_{\jmath=1}^{l} a_\jmath\, e^{\imath\omega\tau_\jmath} &= 0 \end{aligned}$$

In matrix notation we obtain:

$$\begin{pmatrix} P(\tau;\omega) \\ \\ P(-\tau;\omega) \end{pmatrix} A = \begin{pmatrix} i\omega \\ \\ -i\omega \end{pmatrix} \tag{5.1.3}$$

where $A = (a_1, \dots, a_l)^T \in \mathbb{R}^l$ and $P(\tau, \omega)$ is the $1 \times l$ matrix whose entry at column $\jmath$ is $e^{-\imath\omega\tau_\jmath}$.

As the real part of $P(t;\omega)$ is $\mathrm{Re}(P(t;\omega)) = (\cos\omega\tau_1, \dots, \cos\omega\tau_l)$ and the imaginary part of $P(t;\omega)$ is $\mathrm{Im}(P(t;\omega)) = (-\sin\omega\tau_1, \dots, -\sin\omega\tau_l)$, the system (5.1.3) is equivalent to

$$\begin{pmatrix} \cos\omega\tau_1 & \dots & \cos\omega\tau_l \\ -\sin\omega\tau_1 & \dots & -\sin\omega\tau_l \end{pmatrix} A = \begin{pmatrix} 0 \\ \omega \end{pmatrix}. \tag{5.1.4}$$

Let $c_\imath$ denote the i^{th} column of the coefficient matrix. We can rewrite (5.1.4) as :

$$\begin{pmatrix} c_1 & \dots & c_l \end{pmatrix} A = \begin{pmatrix} 0 \\ \omega \end{pmatrix}. \tag{5.1.5}$$

Since $\omega \in \Omega$, we know that, for $1 \leq r \leq l-2$, any subset of two vectors of the set $\{c_1, c_2, c_{2+r}\}$ is linearly independent. Thus, by Lemma C.0.3, the system (5.1.4) has a solution $A = (a_1, \ldots, a_l)^T \in \mathbb{R}^l$ where all $a_i \neq 0$, for $1 \leq i \leq l$. □

Proposition 5.1.2 Let $\tau > 0$ be a positive delay and let $\Omega_\tau = \left\{\frac{\pi(1+2k)}{2\tau}; k \in \mathbb{Z}\right\}$. Then, for $\omega > 0$ such that $\omega \in \Omega_\tau$, there exist $a \in \mathbb{R}^*$ such that the scalar differential equation

$$\dot{y}(t) = ay(t-\tau) \tag{5.1.6}$$

has solutions $y_\omega^\pm(t) = e^{\pm i\omega t}$.

Proof: By Proposition 2.0.4, $y_\omega^+(t) = e^{i\omega t}$ and $y_\omega^-(t) = e^{-i\omega t}$ are solutions of (5.1.6) if and only if, for $a \neq 0$,

$$\begin{aligned} -i\omega - ae^{i\omega\tau} &= 0, \\ i\omega - ae^{-i\omega\tau} &= 0. \end{aligned}$$

That is, if and only if $a\cos\omega\tau = 0$ and $a\sin\omega\tau = \omega$. If $\omega \in \Omega_\tau$, then $\omega\tau = \frac{\pi}{2}(1+2k)$, for some $k \in \mathbb{Z}$. Hence $\cos\omega\tau = 0$ and $a\sin\omega\tau = a(-1)^k$. Hence, when $a = (-1)^k\omega$, the functions $y_\omega^\pm(t) = e^{\pm i\omega t}$ are solutions of (5.1.6). □

5.2 Realization Theorem for a System of p DDE's with ℓ Delays

Given l $\mathbb{Q}$-linearly independent frequencies $\omega_1 > 0, \ldots, \omega_l > 0$, recall that by the *Realization Theorem 2.1* (see [15]), there exist l delays $\tau_1 > 0, \ldots, \tau_l > 0$ and real coefficients a_{1j}, $1 \leq j \leq l$, such that $e^{\pm i\omega_k t}$ are solutions of the delay differential equation

$$\dot{y}(t) = \sum_{j=1}^{l} a_{1j} y(t-\tau_j), \qquad \text{for } 1 \leq k \leq l. \tag{5.2.1}$$

We shall call $\tau_1, \ldots, \tau_l$ (respectively $a_{11}, \ldots, a_{1l}$) the delays (respectively, the coefficients) associated with $\omega_1, \ldots, \omega_l$.

Theorem 5.2.1 Let $\omega_1 > 0$, $\omega_2 > 0$, $\ldots$, $\omega_l > 0$ be $\mathbb{Q}$-linearly independent frequencies and $\tau_1, \tau_2, \ldots, \tau_l$ be their associated delays. For any $p \geq 1$ and any $\omega_{l+1}, \ldots, \omega_{l+p-1}$ belonging to the subset Ω (defined in (5.1.1)), there exist l diagonal matrices $M_1 \in \mathbb{M}_p(\mathbb{R})$, $\ldots$, $M_l \in \mathbb{M}_p(\mathbb{R})$ such that the linear system of DDEs

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{5.2.2}$$

has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$ with $v_k^{\pm} \in \mathbb{C}^p$, $v_k^{\pm} \neq 0$, for $1 \leq k \leq l + p - 1$.

Proof: Recall that the characteristic matrix of the linear DDE in (5.2.1) is given by

$$\Delta(\alpha) = \alpha I - \sum_{j=1}^{l} a_j e^{-\alpha\tau_j}, \qquad \alpha \in \mathbb{C}. \tag{5.2.3}$$

By Theorem 2.0.3, we have that $y_k^+(t)$ (respectively, $y_k^-(t)$) is a solution of (5.2.3) if and only if $0 \neq v_k^+ \in \ker \Delta(i\omega_k)$ (respectively, $0 \neq v_k^- \in \ker \Delta(-i\omega_k)$), for $1 \leq k \leq l$. But $\ker \Delta(\pm i\omega) \neq \{0\}$ if and only if $\det \Delta(\pm i\omega) = 0$. Denoting the diagonal matrices M_j by $\mathrm{Diag}(a_{1j}, \ldots, a_{pj})$, we have:

$$\det \Delta(\alpha) = \prod_{n=1}^{p} \left(\alpha - \sum_{j=1}^{l} a_{nj}\, e^{\alpha\tau_j} \right), \qquad \alpha \in \mathbb{C}. \tag{5.2.4}$$

Hence, the l pairs of equations $\det \Delta(\pm i\omega_k) = 0$, for $1 \leq k \leq l$, are given by:

$$\left(i\omega_k - \sum_{j=1}^{l} a_{1j}\, e^{-i\omega_k\tau_j} \right) \cdots \left(i\omega_k - \sum_{j=1}^{l} a_{pj}\, e^{-i\omega_k\tau_j} \right) = 0 \tag{5.2.5}$$

$$\left(-i\omega_k - \sum_{j=1}^{l} a_{1j}\, e^{i\omega_k\tau_j} \right) \cdots \left(-i\omega_k - \sum_{j=1}^{l} a_{pj}\, e^{i\omega_k\tau_j} \right) = 0 \tag{5.2.6}$$

where a_{nj} is the n^{th} element on the diagonal matrix M_j, i.e., $a_{nj} = [M_j]_{nn}$, for $1 \leq n \leq p$ and $1 \leq j \leq l$. for $1 \leq n \leq p$ and $1 \leq j \leq l$.

We can rewrite (5.2.5) and (5.2.6) as:

$$\prod_{n=1}^{p}\left(i\omega_k - \sum_{j=1}^{l} a_{nj}\, e^{-i\omega_k \tau_j}\right) = 0 \tag{5.2.7}$$

$$\prod_{n=1}^{p}\left(-i\omega_k - \sum_{j=1}^{l} a_{nj}\, e^{i\omega_k \tau_j}\right) = 0 \tag{5.2.8}$$

where $a_{nj} = [M_j]_{nn}$, for $1 \leq n \leq p$ and $1 \leq j \leq l$.

A sufficient condition to satisfy the equations (5.2.7) and (5.2.8) (and hence for the theorem to hold) is that the algebraic system of $2l$ equations below has solutions in the $(p+1)l$ unknowns $(\tau_1, \ldots, \tau_l, a_{11}, \ldots, a_{1l}, \ldots, a_{p1}, \ldots, a_{pl})$. The system is the following:

$$\begin{aligned} \sum_{j=1}^{l} a_{1j}\, e^{-i\omega_k \tau_j} &= i\omega_k, \quad 1 \leq k \leq l \\ \sum_{j=1}^{l} a_{1j}\, e^{i\omega_k \tau_j} &= -i\omega_k, \quad 1 \leq k \leq l \end{aligned} \tag{5.2.9}$$

The proof of Theorem 2.1 in the Buono and LeBlanc paper [15] (given in Appendix D) allows to claim that there exist $\tau_1 > 0, \ldots, \tau_l > 0$ such that system (5.2.9) has a nontrivial solution $A_1 = (\tilde{a}_{11}, \ldots, \tilde{a}_{1l})^T$ where $\tilde{a}_{11} \in \mathbb{R}^*, \ldots, \tilde{a}_{1l} \in \mathbb{R}^*$. Thus $\tau_1 > 0, \tau_2 > 0, \ldots, \tau_l > 0$ are associated with $\omega_1 > 0, \omega_2 > 0, \ldots, \omega_l > 0$.

Now, for $1 \leq r \leq p-1$, consider ω_{l+r} such that $\omega_{l+r} \in \Omega$. By Proposition 5.1.1, there exist $\tilde{a}_{r+1,1} \in \mathbb{R}^*, \ldots, \tilde{a}_{r+1,l} \in \mathbb{R}^*$ such that the scalar DDE

$$\dot{y}(t) = \sum_{j=1}^{l} \tilde{a}_{r+1,j}\, y(t - \tau_j)$$

has the two conjugates solutions $y_{l+r}^{+}(t) = e^{\imath\omega_{l+r}t}$ and $y_{l+r}^{-}(t) = e^{-\imath\omega_{l+r}t}$. Equivalently, Proposition 5.1.1 states that the systems

$$\begin{aligned} \sum_{\jmath=1}^{l} a_{r+1,\jmath}\, e^{-\imath\omega_{l+r}\tau_\jmath} &= i\omega_{l+r}, \quad 1 \le r \le p-1 \\ \sum_{\jmath=1}^{l} a_{r=1,\jmath}\, e^{\imath\omega_{l+r}\tau_\jmath} &= -i\omega_{l+r}, \quad 1 \le r \le p-1 \end{aligned} \tag{5.2.10}$$

have solutions $A_{r+1} = (\tilde{a}_{r+1,1}, \ldots, \tilde{a}_{r+1,l})^T$ where $\tilde{a}_{r+1,1} \in \mathbb{R}^*, \ldots, \tilde{a}_{r+1,l} \in \mathbb{R}^*$. Thus equations (5.2.7) and (5.2.8) hold.

Now, we can consider the system of linear DDEs

$$\dot{y}(t) = \sum_{\jmath=1}^{l} \tilde{M}_\jmath y(t - \tau_\jmath) \tag{5.2.11}$$

where $\tilde{M}_\jmath = \mathrm{Diag}(\tilde{a}_{1\jmath}, \ldots, \tilde{a}_{p\jmath})$, for $1 \le j \le l$. Let $(b_k)_{1\le k\le p}$ be the canonical basis of $\mathbb{R}^p$. To finish the proof, we need to show that $y_k^{\pm}(t) = e^{\pm\imath\omega_k t}b_1$, for $1 \le k \le l$, and $y_{l+r}^{\pm}(t) = e^{\pm\imath\omega_{l+r}t}b_{r+1}$, for $1 \le r \le p-1$, are solutions of (5.2.11). We have, for $1 \le k \le l$,

$$\begin{aligned} \sum_{\jmath=1}^{l} \tilde{M}_\jmath y_k^{\pm}(t - \tau_\jmath) &= \sum_{\jmath=1}^{l} e^{\pm\imath\omega_k(t-\tau_\jmath)}\tilde{M}_\jmath b_1 \\ &= \sum_{\jmath=1}^{l} \tilde{a}_{1\jmath} e^{\pm\imath\omega_k(t-\tau_\jmath)} \\ &= \sum_{\jmath=1}^{l} \tilde{a}_{1\jmath} e^{\mp\imath\omega_k\tau_\jmath} e^{\pm\imath\omega_k t} \\ &= \pm i\omega_k e^{\pm\imath\omega t} \\ &= \dot{y}_k^{\pm}(t). \end{aligned}$$

Moreover, for $k = l + r$ with $1 \leq r \leq p-1$, we have:

$$\begin{aligned}
\sum_{j=1}^{l} \tilde{M}_j y_k^{\pm}(t-\tau_j) &= \sum_{j=1}^{l} e^{\pm\imath\omega_k(t-\tau_j)} \tilde{M}_j b_{r+1} \\
&= \sum_{j=1}^{l} \tilde{a}_{r+1,j} e^{\pm\imath\omega_k(t-\tau_j)} \\
&= \sum_{j=1}^{l} \tilde{a}_{r+1,j} e^{\mp\imath\omega_k\tau_j} e^{\pm\imath\omega_k t} \\
&= \pm\imath\omega_k e^{\pm\imath\omega t} \\
&= \dot{y}_k^{\pm}(t). \qquad \square
\end{aligned}$$

We will now use Proposition 5.1.1, Theorem 5.2.1 and the *Realization Theorem* [15] to show that that the previous realization result holds for open sets near solutions found in Theorem 5.2.1.

Corollary 5.2.2 Let $p \geq 1$ and let $\tilde{\omega}_1, \ldots, \tilde{\omega}_l$ be l $\mathbb{Q}$-linearly independent positive frequencies. For $\tilde{\omega} = (\tilde{\omega}_1, \ldots \tilde{\omega}_l)^T \in \mathbb{R}_+^l$, there exists an open neighbourhood $\mathcal{N}$ of $\tilde{\omega}$ in $\mathbb{R}_+^l$ and a smooth mapping

$$\begin{aligned}
H: \ V &\longrightarrow \mathbb{R}^l \\
\omega &\longmapsto \tau(\omega) = (\tau_1(\omega), \ldots, \tau_n(\omega))
\end{aligned}$$

with the following property:
For all $\omega = (\omega_1, \ldots, \omega_l)^T \in \mathbb{N}^l$ and all $\omega' = (\omega_{l+1}, \ldots, \omega_{l+p-1})^T$ with $\omega_{l+r} \in \Omega(\tau(\omega))$, for $1 \leq r \leq p-1$, there exist l matrices $M_j \in \mathbb{M}_p(\mathbb{R})$ such that, for $1 \leq j \leq l$, (respectively, for $1 \leq r \leq p-1$) $y_j^{\pm}(t) = e^{\pm\imath\omega_j t} b_1$ (respectively $y_j^{\pm}(t) = e^{\pm\imath\omega_{l+r} t} b_{r+1}$) are solutions of the system of linear DDEs

$$\dot{y}(t) = \sum_{k=1}^{l} M_k y(t-\tau_k).$$

Proof: By the *Realization Theorem* D.1.1, there exists a neighbourhood $\mathcal{N}$ of $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_l)^T$ in $\mathbb{R}^l_+$ with the following property:
For all $\omega = (\omega_1, \ldots, \omega_l)^T \in \mathcal{N}$, there exist $\tau = (\tau_1, \ldots, \tau_l)^T \in \mathbb{R}^l$ and $A_1 = (a_{11}, \ldots, a_{1l})^T \in \mathbb{R}^l$ such that, for $1 \leq j \leq l$, $y^{\pm}_{\omega_j}(t) = e^{\pm i\omega_j t}$ are solutions of the scalar DDE

$$\dot{y}(t) = \sum_{j=1}^{l} a_{1j} y(t - \tau_j).$$

Let $\Omega(\tau) = \{x \in \mathbb{R}_+;\ \sin(x(\tau_i - \tau_j)) \neq 0, \text{ for some } 1 \leq i < j \leq l\}$. Then, applying Proposition 5.1.1 $p-1$ times to ω, τ and $\Omega(\tau)$ ends the proof. $\square$

Chapter 6

Conclusion

6.1 Summary

The starting point of this thesis was the *Realization theorem* of Buono and LeBlanc [15]:

Theorem 6.1.1 Suppose $\omega_1 > 0$, $\omega_2 > 0$, $\dots$, $\omega_n > 0$ are linearly independent over the rationals. Then there exist $\tau_1 > 0$, $\tau_2 > 0$, $\dots$, $\tau_n > 0$ and $a_1 \in \mathbb{R}^*$, $a_2 \in \mathbb{R}^*$, $\dots$, $a_n \in \mathbb{R}^*$ such that the linear delay differential equation

$$\dot{x}(t) = a_1\, x(t-\tau_1) + a_2\, x(t-\tau_2) + \cdots + a_n\, x(t-\tau_n) \tag{6.1.1}$$

has solutions $x_k^{\pm}(t) = e^{\pm\, i\omega_k t}$, for all $k = 1, \dots, n$.

That theorem is, to the best knowledge of the authors, the first general result linking the number of purely imaginary eigenvalues of scalar linear DDE's with the number of delays.

In this thesis, we tried to study as thoroughly as possible the link between the number of critical eigenvalues and the number of discrete delays for a system of p scalar linear

DDEs. Recall that a system of p linear DDEs with l delays is written:

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j), \quad M_j \in \mathbb{M}_p(\mathbb{R}).$$

We started (in Chapter 3) by considering systems of p linear DDE's with a unique discrete delay. Such systems are written

$$\dot{y}(t) = My(t - \tau), \quad M \in \mathbb{M}_p(\mathbb{R}). \tag{6.1.2}$$

We showed that there exist $\tau > 0$ and $M \in \mathbb{M}_p(\mathbb{R})$ (where $[\frac{p}{2}] = l$) such that $y_j^{\pm}(t) = e^{\pm i\omega_j t} v_j^{\pm}$ (with $0 \neq v_j^{\pm} \in \mathbb{C}^p$) are solutions of (6.1.2).
But we were able to show an important nuance. Indeed, if p is even (i.e. $p = 2l$) equation 6.1.2 has solutions for all $\tau > 0$ but countably many. On the other hand, if p is odd (i.e. $p = 2l + 1$) there are only countably many possible τ.

Secondly, we showed that, for $\tau > 0$ and $M \in \mathbb{M}_p(\mathbb{R})$, equation (6.1.2) has at most $l = [\frac{p+1}{2}]$ pairs of solutions $y_j^{\pm}(t) = e^{\pm i\omega_j t} v_j^{\pm}$ (with $0 \neq v_j^{\pm} \in \mathbb{C}^p$).

In the following chapter (Chapter 4) we proved that for two $\mathbb{Q}$-linearly independent positive frequencies, there exist $\tau_1 > 0$, $\tau_2 > 0$ and $M_1 \in \mathbb{M}_2(\mathbb{R}), M_2 \in \mathbb{M}_2(\mathbb{R})$ such that the system of linear DDEs

$$\dot{y}(t) = M_1\, y(t - \tau_1) + M_2\, y(t - \tau_2) \tag{6.1.3}$$

has solutions $y_k^{\pm}(t) = e^{\pm\, i\omega_k t} v_k^{\pm}$ with $v_k^{\pm} \in \mathbb{C}^2$, for $k = 1, 2$.
We thus have a realization theorem in the case of a 2×2 system. We also proved an openness theorem stating that the realization of two imaginary numbers $i\omega_1$ and $i\omega_2$ (not necessarily $\mathbb{Q}$-linearly independent) as eigenvalues of the DDEs (6.1.3) is possible in a neighbourhood of any set of two $\mathbb{Q}$-linearly independent purely imaginary numbers $\{i\tilde{\omega}_1, i\tilde{\omega}_2\}$.

In the last chapter (Chapter 5), we attempted to obtain a realization theorem and an openness theorem for a system of linear DDE's written as

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j)$$

where $M_j \in \mathbb{M}_p(\mathbb{R})$ are diagonal matrices. We considered $\omega_1 > 0, \dots, \omega_l > 0$, $\mathbb{Q}$-linearly independent frequencies and $\tau_1, \dots, \tau_l$ their associated delays. We showed that, for any $p \geq 1$ and any $\omega_{l+1}, \dots, \omega_{l+p-1}$ in the subset Ω (see equation (5.1.1)), there exist l diagonal matrices $M_1 \in \mathbb{M}_p(\mathbb{R})$, $\dots$, $M_l \in \mathbb{M}_p(\mathbb{R})$ such that the system of linear delay differential equations

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{6.1 4}$$

has solutions $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$ with $v_k^{\pm} \in \mathbb{C}^p$, $v_k^{\pm} \neq 0$, for $1 \leq k \leq l + p - 1$.

6.2 Discussion

The aim of this section is to foresee possible future research that could be undertaken to extend the results of this thesis. Let l be an integer ≥ 2, p be an integer ≥ 1 and let $S_{l,p}$ be the set of positive natural numbers $f(l, p)$ such that the following statement holds:

Suppose $\omega_1 > 0, \omega_2 > 0, \dots, \omega_l$ are $\mathbb{Q}$-linearly independent frequencies. Then there exist $\tau_1 > 0, \tau_2 > 0, \dots, \tau_{f(l,p)} > 0$ and $M_1 \in \mathbb{M}_p(\mathbb{R}), M_2 \in \mathbb{M}_p(\mathbb{R}), \dots, M_{f(l,p)} \in \mathbb{M}_p(\mathbb{R})$ such that the system of linear DDEs

$$\dot{y}(t) = \sum_{j=1}^{f(l,p)} M_j y(t - \tau_j) \tag{6.2.1}$$

has l solutions of the type $y_k^{+}(t) = e^{i\omega_k t} v_k^{+}$ (respectively, $y_k^{-}(t) = e^{-i\omega_k t} v_k^{-}$) with $v_k^{+} \in \mathbb{C}^p$, $v_k^{+} \neq 0$ (respectively, $v_k^{-} \in \mathbb{C}^p$, $v_k^{-} \neq 0$), for $1 \leq k \leq l$.

In our opinion, the long-term goal is to determine the smallest positive number $f_m(l,p)$ of $S_{l,p}$, i.e. to find $f_m(l,p)$ such that $f_m(l,p) = \min S_{l,p}$.

Note that, in Chapter 3, the approach was different since we considered a system of DDEs with a single delay. Thus we fixed the number of delays (to 1) instead of the number of frequencies. We showed that for $p \geq 1$, $\left[\frac{p}{2}\right]$ is the largest number of purely imaginary numbers for which there exists $\tau > 0$ and $M \in \mathbb{M}_p(\mathbb{R})$ such that $\dot{y}(t) = My(t-\tau)$ has a spectrum containing those imaginary numbers.

For any $l \geq 1$ and $p \geq 1$, it seems very difficult to find a "candidate" $f_c(l,p)$ and even more difficult to prove that this "candidate" is indeed the smallest positive number of $S_{l,p}$, i.e., to prove that $f_c(l,p) = f_m(l,p)$. For example, consider the case $p = 1, l \geq 2$; it is the Buono and LeBlanc *Realization Theorem*. It states that for l purely imaginary numbers which are linearly independent over the rationals, there exists a scalar DDE depending on l fixed delays whose spectrum contains at least those l purely imaginary numbers. It would be interesting to be able to show that l is indeed the smallest number of delays whose spectrum contains those l purely imaginary numbers. Thus we have $f_c(l,1) = l$, but we would like to show that $f_m(l,1) = l$.

Likewise, in Chapter 4, we found a function $f \in S_{2,2}$ such that $f(2,2) = 2$. It would also be interesting to verify whether $f_m(2,2) = 2$ or $f_m(2,2) = 1$.

In Chapter 5, some questions need to be addressed before trying to find $f_m(l,p)$. Recall that for $p \geq 1$, there exist l real diagonal matrices and l fixed delays associated with $\omega_1, \omega_2, \ldots, \omega_l$ such that the system of linear DDEs

$$\dot{y}(t) = \sum_{j=1}^{l} M_j y(t - \tau_j) \tag{6.2.2}$$

has $l+p-1$ pairs of solutions of the type $y_k^{\pm}(t) = e^{\pm i\omega_k t} v_k^{\pm}$, for $1 \leq k \leq l+p-1$. But the choice of the frequencies $\omega_{l+1}, \ldots, \omega_{l+p-1}$ depend on $\tau_1, \ldots, \tau_l$.

First, it would be interesting to work on the proof of the same theorem with non-diagonal matrices $M_1, \ldots, M_l$. In order to achieve a similar result, one would have to find a way to solve $\det \Delta(\pm i\omega_k) = 0$ with $\Delta(\pm i\omega_k)$ not diagonal.

As well, it would be interesting to be able to show that, for any $\omega_{l+1} > 0, \ldots, \omega_{l+p-1} > 0$, there is enough freedom to choose the delays $\tau_1, \ldots, \tau_l$ such that $\omega_k \in \Omega$, for $l+1 \leq k \leq l+p-1$. It would allow for the following more powerful statement:

Let $\omega_1 > 0$, $\omega_2 > 0$, $\ldots$, $\omega_l > 0$, be $\mathbb{Q}$-linearly independent frequencies and let $p < l$.
Then there exist $\tau_1 > 0$, $\ldots$, $\tau_{l-p+1} > 0$, and $M_1 \in \mathbb{M}_p(\mathbb{R})$, $\ldots$, $M_{l-p+1} \in \mathbb{M}_p(\mathbb{R})$ where $M_j = Diag(a_{1j}, \ldots, a_{pj})$, for $1 \leq j \leq l-p+1$, such that the system of linear DDEs

$$\dot{y}(t) = \sum_{j=1}^{l-p+1} M_j y(t - \tau_j) \tag{6.2.3}$$

has solutions $y_k^+(t) = e^{i\omega_k t} v_k^+$ (respectively, $y_k^-(t) = e^{-i\omega_k t} v_k^-$) with $v_k^+ \in \mathbb{C}^p$, $v_k^+ \neq 0$ (respectively, $v_k^- \in \mathbb{C}^p$, $v_k^- \neq 0$), for $1 \leq k \leq l$.

As the reader can notice, there are many ways in which the results presented in this thesis could be extended. Moreover, some new tools will be required as the ones used in this thesis seem to work for specific values of l and p. We hope that this thesis will give ideas to researchers to continue what has been modestly started.

Appendix A

Properties and Definitions of Matrix Theory

In this Appendix, we recall standard definitions and properties of matrix theory. Let K be a field and $\mathbb{M}_n(K)$ be the K-algebra of $(n \times n)$ matrices.

A.1 Useful Property of the Determinant

For $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{M}_n(K)$, we denote by $c_r = (a_{1r}, \ldots, a_{nr})^T$ the r^{th} column of the matrix A and by $\det A = \det(c_1, \ldots, c_n)$ the determinant of A. We then have:

Proposition A.1.1 Let $A_0 \in \mathbb{M}_n(K), \cdots, A_k \in \mathbb{M}_n(K)$ be $k+1$ $(n \times n)$-matrices over K. For $0 \leq i \leq k$, let us write $A_i = (c_1^{(i)}, \cdots, c_n^{(i)})$ where $c_r^{(i)}$ is the r^{th} column of the matrix A_i. Then

$$\det \sum_{i=0}^{k} A_i = \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \det(c_1^{(i_1)}, \ldots, c_n^{(i_n)}).$$

Proof: Using the multilinearity of the determinant we have:

$$\det \sum_{i=0}^{k} A_i = \det \left(\sum_{i_1=0}^{k} c_1^{(i_1)}, \ldots, \sum_{i_n=0}^{k} c_n^{(i_n)} \right)$$

$$
\begin{aligned}
&= \sum_{i_1=0}^{k} \det\left(c_1^{(i_1)}, \sum_{i_2=0}^{k} c_2^{(i_2)}, \dots, \sum_{i_n=0}^{k} c_n^{(i_n)}\right) \\
&= \sum_{i_1=0}^{k}\sum_{i_2=0}^{k} \det\left(c_1^{(i_1)}, c_2^{(i_2)}, \sum_{i_3=0}^{k} c_3^{(i_3)}, \dots, \sum_{i_n=0}^{k} c_n^{(i_n)}\right) \\
&\ \vdots \\
&= \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \det(c_1^{(i_1)}, \dots, c_n^{(i_k)}) \qquad \square
\end{aligned}
$$

A.2 Companion Matrix

Proposition A.2.1 Let $\rho \in K[x]$ be a monic polynomial of degree n.

Then there exist $M \in \mathbb{M}_n(K)$ such that ρ is the characteristic polynomial of M.

Proof: Recall that for any monic polynomial $\rho \in K[x]$ of degree n, $\rho(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, there exists an $n \times n$ *companion matrix* $C(\rho)$ defined as

$$
C(\rho) = \begin{pmatrix}
0 & 0 & \dots & 0 & -a_0 \\
1 & 0 & \dots & 0 & -a_1 \\
0 & 1 & \dots & 0 & -a_2 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & -a_{n-1}
\end{pmatrix}.
$$

It is easy to see that the characteristic polynomial, as well as the minimal polynomial, of $C(\rho)$ are equal to ρ. $\square$

Appendix B

Equation of a Conic Associated with a Finite Set of 2×2 Real Matrices

Let $\mathcal{M} = \{M_0, M_1, \ldots, M_l\}$, be a set of $l+1$ matrices in $\mathbb{M}_2(\mathbb{R})$ where M_0 is the identity matrix. Let us recall the definition of the homogeneous polynomial $\rho \in \mathbb{R}[x_0, \cdots, x_l]$ associated with $\mathcal{M}$, given in Definition 2.0.2:

$$\rho(x_0, \ldots, x_l) = \sum_{i_1=0}^{l} \cdots \sum_{i_p=0}^{l} \det\left(c_1^{(i_1)}, \ldots, c_p^{(i_p)}\right) (-1)^{\gamma(i_1, \ldots, i_p)} x_{i_1} \ldots x_{i_p}$$

where $c_r^{(j)}$ is the r^{th} column of the matrix M_j and $\gamma(i_1, \ldots, i_p) = \sum_{j=1}^{p}(1 - \delta_{0,i_j})$, with $\delta_{0,i}$ denoting the Kronecker delta.

Theorem B.0.2 Let $\{M_0, M_1, \ldots, M_l\}$ be a set of $l+1$ matrices in $\mathbb{M}_2(\mathbb{R})$ where M_0 is the identity matrix and let $\phi(x_1, \ldots, x_l)$ be the equation of the conic C defined in Theorem 4.2.1. Then we have

$$\rho(x_0, 1, x_2, \ldots, x_l) = \phi(x_0, x_2, \ldots, x_l).$$

Proof: The equation of ρ associated to the set $\mathcal{M}$ of 2×2 matrices is given by

$$\rho(x_0, \dots, x_l) = \sum_{\imath_1=0}^{l}\sum_{\imath_2=0}^{l} \det\left(c_1^{(\imath_1)}, c_2^{(\imath_2)}\right)(-1)^{2-(\delta_{\imath_1}+\delta_{\imath_2})} x_{\imath_1} x_{\imath_2}$$

where $c_r^{\jmath}$ is the r^{th} column of the matrix $M_\jmath$.

Expressed in a more convenient form, ρ is

$$\rho(x_0, \dots, x_l) = \sum_{\imath_1=0}^{l} \det\left(c_1^{(\imath_1)}, c_2^{(\imath_1)}\right) x_{\imath_1}^2 + \sum_{\substack{\imath_1,\imath_2=0\\ \imath_1<\imath_2}}^{l} \left[\det\left(c_1^{(\imath_1)}, c_2^{(\imath_2)}\right) + \det\left(c_1^{(\imath_2)}, c_2^{(\imath_1)}\right)\right](-1)^{2-(\delta_{\imath_1}+\delta_{\imath_2})} x_{\imath_1} x_\imath$$

Now, let $\Gamma_{\imath_1,\imath_2} = \det\left(c_1^{(\imath_1)}, c_2^{(\imath_2)}\right) + \det\left(c_1^{(\imath_2)}, c_2^{(\imath_1)}\right)$. We obtain:

$$\rho(x_0, \dots, x_l) = \sum_{\imath_2=0}^{l} \det M_{\imath_2} x_{\imath_2}^2 - \sum_{\imath_2=1}^{l} \Gamma_{0,\imath_2} x_0 x_{\imath_2} - \sum_{\imath_2=2}^{l} \Gamma_{1,\imath_2} x_1 x_{\imath_2} + \sum_{\substack{\imath_1,\imath_2=2\\ \imath_1<\imath_2}}^{l} \Gamma_{\imath_1,\imath_2} x_{\imath_1} x_{\imath_2}.$$

Reorganizing the terms yields:

$$\begin{aligned}\rho(x_0, \dots, x_l) = x_0^2 &+ \sum_{\imath_2=1}^{l} \det M_{\imath_2} x_{\imath_2}^2 + \sum_{\imath_2=0}^{l} \Gamma_{0,\imath_2} x_0 x_{\imath_2} \\ &+ \sum_{\imath_2=2}^{l} \Gamma_{1,\imath_2} x_1 x_{\imath_2} + \sum_{\substack{\imath_1,\imath_2=2\\ \imath_1<\imath_2}}^{l} \Gamma_{\imath_1,\imath_2} x_{\imath_1} x_{\imath_2}\end{aligned}$$

Now, let $x_1 = 1$ and note that since $M_0 = (c_1^{(0)}, c_2^{(0)}) = I_2$, $\Gamma_{0,\imath_2} = \det\left(c_1^{(0)}, c_2^{(\imath_2)}\right) + \det\left(c_1^{(\imath_2)}, c_2^{(0)}\right) = \mathrm{Tr}\ M_{\imath_2}$. Hence we have:

$$\begin{aligned}\rho(x_0, 1, x_2 \dots, x_l) = x_0^2 &+ \sum_{\imath_2=2}^{l} \det M_{\imath_2} x_{\imath_2}^2 - \sum_{\imath_2=1}^{l} \mathrm{Tr}\ M_{\imath_2} x_0 x_{\imath_2} + \sum_{\imath_2=2}^{l} \Gamma_{1,\imath_2} x_1 x_{\imath_2} \\ &+ \sum_{\substack{\imath_1,\imath_2=2\\ \imath_1<\imath_2}}^{l} \Gamma_{\imath_1,\imath_2} x_{\imath_1} x_{\imath_2} - \mathrm{Tr}\ M_1 x_0 + \det M_1.\end{aligned}$$

Let us use the notation:

$$A_{\imath_2} = \det M_{\imath_2}, \qquad B_{\imath_2} = -\mathrm{Tr}\ M_{\imath_2},$$

$$E = -\mathrm{Tr}\ M_1, \qquad F = \det M_1,$$

$$\text{and} \quad D_j = \Gamma_{1,i_2} \quad C_{i_1,i_2} = \Gamma_{i_1,i_2}$$

$$\text{where } K_{i_1,i_2} = \left(c_1^{(i_1)}, c_2^{(i_2)}\right).$$

Thus, we get:

$$\rho(x_0, 1, x_2 \dots, x_l) = x_0^2 + \sum_{i_1=2}^{l} A_{i_2} x_{i_2}^2 + \sum_{i_2=1}^{l} B_{i_2} x_0 x_{i_2} + \sum_{\substack{i_1,i_2=2 \\ i_1<i_2}}^{l} C_{i_1,i_2} x_{i_1} x_{i_2} + \sum_{i_2=2}^{l} D_{i_2} x_1 x_{i_2} + E x_0 + F.$$

Hence the intended result: $\rho(x_0, 1, x_2, \dots, x_l) = \phi(x_0, x_2, \dots, x_l)$. $\square$

Appendix C

Useful Lemma of Linear Algebra

Lemma C.0.3 Let V be a vector space of dimension n over the field K of characteristic zero. Let $u \in V$.
Let $v_1, v_2, \ldots, v_m$ be $m > n$ non-zero vectors such that, for $1 \leq i \leq m-n$, any subset of n vectors of the set $\{v_1, \ldots, v_n, v_{n+i}\}$ is linearly independent.
Then there exist non-zero numbers $\gamma_1, \ldots, \gamma_m$ such that $\sum_{j=1}^{m} \gamma_j v_j = u$.

Proof: Since $\{v_1, \ldots v_n\}$ is a basis of V, there exist $a_1 \in K, \ldots, a_n \in K$ such that

$$u = \sum_{j=1}^{n} a_j v_j. \tag{C.0.1}$$

As well, we can write v_{n+i}, for $1 \leq i \leq m-n$, as a linear combination of the vectors $v_1, \ldots, v_n$. Thus,

$$v_{n+i} = k_1^{(i)} v_1 + \cdots + k_n^{(i)} v_n \tag{C.0.2}$$

with $k_1^{(i)} \neq 0, \ldots, k_n^{(i)} \neq 0$, for all $1 \leq i \leq m-n$.
Indeed, suppose there exists $1 \leq l \leq n$ such that $k_l^{(i)} = 0$. Then

$$v_{n+i} = k_1^{(i)} v_1 + \cdots + k_{l-1}^{(i)} v_{l-1} + k_{l+1}^{(i)} v_{l+1} + \cdots + k_n^{(i)} v_n,$$

which is a contradiction since $\{v_1, \ldots, v_{l-1}, v_{l+1}, \ldots, v_n, v_{n+i}\}$ is linearly independent. Hence, for $1 \leq i \leq m-n$, equation (C.0.2) yields:

$$0 = t^{(i)} k_1^{(i)} v_1 + \cdots + t^{(i)} k_n^{(i)} v_n - t^{(i)} v_{n+i}. \tag{C.0.3}$$

Thus, by (C.0.1) and (C.0.3), we have

$$u = \sum_{j=1}^{n} \left(a_j + \sum_{i=1}^{m-n} t^{(i)} k_j^{(i)} \right) v_j - \sum_{j=1}^{m-n} t^{(j)} v_{n+j}. \tag{C.0.4}$$

Now, choose $t_o^{(1)} \in \mathbb{R}^*, t_o^{(2)} \in \mathbb{R}^*, \ldots, t_o^{(m-n)} \in \mathbb{R}^*$ such that $a_j + \sum_{i=1}^{m-n} t^{(i)} k_j^{(i)} \neq 0$, for $1 \leq i \leq m-n$. Then, to finish the proof, let $\gamma_j = a_j + \sum_{i=1}^{m-n} t^{(i)} k_j^{(i)}$, for $1 \leq i \leq n$, and $\gamma_j = t_o^{(j-n)}$, for $n+1 \leq j \leq m$. $\square$

Appendix D

Realization and Openness Theorems for a Scalar DDE

The two theorems in this Appendix are the *Realization Theorem* and the *Openness Theorem* in the paper by Buono and LeBlanc [15]. The proofs for both theorems are similar to the proof in [15], but contain more details.

D.1 Realization Theorem

The first theorem is the *Realization Theorem.* We keep the same notation used in [15]. In particular, the torus $\mathbb{T}$ is identified with $\mathbb{R}/2\pi\mathbb{Z}$.

Before the theorem, we start with the definition of transversality. This definition can be found in Wiggins' book [19]. In geometry, transversality is a notion that deals with the intersection of surfaces or manifolds. Let M and N be differentiable (at least $\mathbf{C^1}$) manifolds in $\mathbb{R}^n$ and let T_aM and T_aN be the tangent spaces of M and N, respectively, at the point a.

Definition D.1.1 Let p be a point in $\mathbb{R}^n$. Then, M and N are said to be transversal at p if :

(a) $p \notin M \cap N$,

(b) $p \in M \cap N$ and $T_pM + T_pN = \mathbb{R}^n$.

The manifolds M and N are transversal if they are transversal at every point $p \in \mathbb{R}^n$.

We can now state the theorem:

Theorem D.1.1 (Realization Theorem) Suppose $\omega_1 > 0, \omega_2 > 0, \dots, \omega_n > 0$ are linearly independent over the rationals. Then there exists $\tau_1 > 0, \tau_2 > 0, \dots, \tau_n > 0$ and $a_1 \in \mathbb{R}^*$, $a_2 \in \mathbb{R}^*$, $\dots$, $a_n \in \mathbb{R}^*$ such that the linear delay differential equation

$$\dot{x}(t) = a_1\, x(t - \tau_1) + a_2\, x(t - \tau_2) + \cdots + a_n\, x(t - \tau_n) \qquad \text{(D.1.1)}$$

has solutions $x_k^{\pm}(t) = e^{\pm i\omega_k t}$, for all $k = 1, \dots, n$.

Proof: As shown in the proof of Proposition 2.0.4, for $1 \leq k \leq n$, respectively $x_k^{\pm}(t)$ are solutions of (D.1.1) if and only if $\Delta(\pm i\omega_k) = 0$ where $\Delta(\alpha) = \alpha - \sum_{j=1}^{l} a_k e^{-i\alpha\tau_k}$.

The $2n$ pairs of equations $\Delta(i\omega_j)$ and $\Delta(-i\omega_j)$ are given by:

$$\begin{aligned} \sum_{k=1}^{n} a_k\, e^{-i\omega_j\tau_k} &= i\omega_j, \qquad j = 1, \dots, n \\ \sum_{k=1}^{n} a_k\, e^{i\omega_j\tau_k} &= -i\omega_j, \qquad j = 1, \dots, n \end{aligned} \qquad \text{(D.1.2)}$$

Therefore, a necessary and sufficient condition for Theorem D.1.1 to hold is for system (D.1.2) to have solutions in the $2n$ unknowns $(\tau_1, \tau_2, \dots, \tau_n, a_1, a_2, \dots, a_n)$. It is useful to use the matrix notation for system (D.1.2). We obtain:

$$\begin{pmatrix} P(\tau;\omega) \\ \\ P(-\tau;\omega) \end{pmatrix} A = \begin{pmatrix} i\omega \\ \\ -i\omega \end{pmatrix} \tag{D.1.3}$$

where $\omega = (\omega_1, \ldots, \omega_n)^T$, $\tau = (\tau_1, \ldots, \tau_n)$, $A_m = (a_{1m}, \ldots, a_{lm})^T$ and $P(\tau;\omega)$ is the $n \times n$ matrix whose (j,k) entry is

$$[P(\tau;\omega)]_{jk} = e^{-i\omega_j \tau_k}.$$

Note that $\overline{P(\tau;\omega)} = P(-\tau;\omega)$.

Now, recall that $N = \left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in \mathbb{C}^{2n};\ z \in \mathbb{C}^n \right\}$ is a vector subspace of $\mathbb{C}^n$ of real dimension n, and that the map

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in N \longmapsto \begin{pmatrix} \mathrm{Re}(z) \\ \mathrm{Im}(z) \end{pmatrix} \in \mathbb{R}^n \tag{D.1.4}$$

is an isomorphism. Thus system (D.1.3) has the same solutions as the system:

$$\begin{pmatrix} \mathrm{Re}\,(P(\tau;\omega)) \\ \\ \mathrm{Im}\,(P(\tau;\omega)) \end{pmatrix} A = \begin{pmatrix} 0 \\ \\ \omega \end{pmatrix}. \tag{D.1.5}$$

Instead of attempting to solve (D.1.5) directly, the idea is to adopt an approach based on the fallowing fact. For each j, consider the exponents $\omega_j \tau_k$ in $P(\tau;\omega)$ taken modulo 2π, $k = 1, \ldots, n$. Since the ω_j are independent over the rationals, for $\tau_k \geq 0$, the vector $\tau_k \omega \bmod 2\pi$ generates a dense orbit, denoted by $\mathcal{O}_k$ on an n-torus $\mathbb{T}^n$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. If (τ, A) is a solution of (D.1.5), then τ produces a point on $V := (\mathbb{T}^n)^n$, via the dense orbits.

Thus, the idea is to embed the problem of finding solutions of (D.1.5) into a problem of finding solutions of a mapping F_ω defined on V and which is an extension of (D.1.5). If an explicit solution of $F_\omega = 0$ can be obtained, we can then use the Implicit

Function Theorem to find a submanifold of solutions of $F_\omega = 0$. Finally, using the density of the orbits $\mathcal{O}_k$ on the submanifold yields an infinite number of solutions to $F_\omega = 0$ and therefore to (D.1.5).

Let us define $V := (\mathbb{T}^n)^n$ with coordinates on V as follows:

$$V := \{\, \tilde{\Phi} = (\Phi^1, \ldots, \Phi^n) \mid \Phi^j = (\varphi_1^j, \ldots, \varphi_n^j)^T \in \mathbb{T}^n,\ j = 1, \ldots, n \,\}$$

and consider the following mapping associated to (D.1.3):

$$\mathcal{F}_\omega : V \times \mathbb{R}^n \to \mathbb{R}^{2n}$$

defined by

$$\mathcal{F}_\omega(\tilde{\Phi}, A) = \begin{pmatrix} \widetilde{P}(\tilde{\Phi}) \\ \\ \widetilde{P}(-\tilde{\Phi}) \end{pmatrix} A - i \begin{pmatrix} \omega \\ \\ -\omega \end{pmatrix}, \tag{D.1.6}$$

where A and ω are as previously defined, and $\widetilde{P}(\tilde{\Phi})$ is the $n \times n$ matrix whose (j,k)-entry is

$$\left[\widetilde{P}(\tilde{\Phi})\right]_{jk} = \left[\widetilde{P}(\Phi^1, \ldots, \Phi^n)\right]_{jk} = e^{-\imath\varphi_j^k}.$$

Therefore, by the isomorphism (D.1.4), we obtain:

$$\tilde{F}_\omega : V \times \mathbb{R}^n \to \mathbb{R}^{2n}$$

defined by

$$\tilde{F}_\omega(\tilde{\Phi}, A) = \begin{pmatrix} \operatorname{Re}\left(\widetilde{P}(\tilde{\Phi})\right) \\ \\ \operatorname{Im}\left(\widetilde{P}(-\tilde{\Phi})\right) \end{pmatrix} A - \begin{pmatrix} 0 \\ \\ \omega \end{pmatrix}. \tag{D.1.7}$$

Now, for future use, we single out the last torus in V. Writing $V = V_{n-1} \times \mathbb{T}^n$, we obtain:

$$\begin{aligned} F_\omega : V_{n-1} \times \mathbb{T}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^{2n} \\ (\Phi, \Psi, A) &\longmapsto F(\Phi, \Psi, A) \end{aligned} \tag{D.1.8}$$

where

$$V_\Phi = \{\, \Phi = (\Phi^1, \dots, \Phi^{n-1}) \,|\, \Phi^j \text{ as defined above}\} \text{ and } V_\Psi = \Phi^n.$$

For notational simplicity, we re-label φ_j^n by ψ_j, for $1 \le j \le n$.

We now need to find an explicit solution $(\widehat{\Phi}, \widehat{\Psi}) \in V_{n-1} \times \mathbb{T}^n$ for the equation $F_\omega(\Phi, \Psi, A; \omega) = 0$ in order to apply the Implicit Function Theorem at $(\widehat{\Phi}, \widehat{\Psi})$.

Consider the particular set of vectors in $\mathbb{R}^n$:

If $\mathcal{E} = \{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, we define the vectors $v_1 \in \mathbb{R}^n, \dots, v_q \in \mathbb{R}^n$ by

$$v_1 = \sum_{k=1}^{n} e_k \quad \text{and for } j = 2, \dots, n, \quad v_j = v_1 - \sum_{\ell=0}^{j-2} 2\, e_{n-\ell}. \tag{D.1.9}$$

By construction, the set $\{v_1, \dots, v_n\}$ is linearly independent and every coordinates of each vector v_j, for $1 \le j \le n$, are non-zero. This set of vectors allows to define, for $1 \le j \le n$, the matrices $\mathcal{U}_j$ whose k^{th} diagonal element is the k^{th} component of the vector v_j. In particular, $\mathcal{U}_1$ is the $n \times n$ identity matrix. Let us also define the matrix L which is the $n \times n$ matrix whose j^{th} column is composed of the coefficients of the vector v_j in the basis $\mathcal{E}$. That is

$$L = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & -1 \\ 1 & 1 & -1 & & -1 \\ 1 & -1 & -1 & \cdots & -1 \end{pmatrix}. \tag{D.1.10}$$

Note that L is invertible in $\mathbb{M}_n(\mathbb{Q})$. Indeed, by construction, $L \in \mathbb{M}_n(\mathbb{Z})$ and $\det L \neq 0$ (the set $\{v_1, \dots, v_n\}$ is linearly independent). As $L^{-1} = (\det L)^{-1} \operatorname{adj} L$, where $\operatorname{adj} L$ denotes the adjugate matrix of L (see, for example, [17]), $L^{-1} \in \mathbb{M}_n(\mathbb{Q})$.

Consider the following point in $V_{n-1} \times \mathbb{T}^n$:

$$(\widehat{\Phi}, \widehat{\Psi}) = -\frac{\pi}{2} L.$$

Then, we have that $\widetilde{P}(\widehat{\Phi}, \widehat{\Psi}) = iL$. We define $\widehat{A} = L^{-1}\omega$. Note that each coordinate a_j, for $1 \leq j \leq n$, of the column vector $\hat{A}$ is non-zero. Indeed, suppose there exist $a_{j_\circ} = 0$. Then we have

$$\sum_{k=1}^{n} l_{j_\circ k}\omega_k = 0 \qquad \text{where } l_{j_\circ k} = [L^{-1}]_{j_\circ k}.$$

But by hypothesis, $\omega_1, \ldots, \omega_n$ are linearly independent over the rationals. Hence there exist $k_\circ$ such that $l_{j_\circ k_\circ} \notin \mathbb{Q}$, which is a contradiction with the fact that $L^{-1} \in \mathbb{M}_n(\mathbb{Q})$.

Hence we have

$$\begin{pmatrix} \operatorname{Re}\left(\widetilde{P}(\widehat{\Phi}, \widehat{\Psi})\right) \\ \operatorname{Im}\left(\widetilde{P}(\widehat{\Phi}, \widehat{\Psi})\right) \end{pmatrix} A - \begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ L \end{pmatrix} A - \begin{pmatrix} 0 \\ \omega \end{pmatrix}. \tag{D.1.11}$$

Therefore, there exist a unique $\widehat{A} \in \mathbb{R}^n$ such that $F_\omega\left(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It is $\widehat{A} = L^{-1}\omega$.

We now need to verify that $J \equiv D_{(\Psi,A)}F_\omega(\widehat{\Phi}, \widehat{\Psi}, \widehat{A})$, is invertible. The computation yields:

$$J \equiv D_{(\Psi,A)}F_\omega(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}) = \begin{pmatrix} \hat{a}_n \mathcal{U}_n & 0 \\ 0 & L \end{pmatrix}$$

and therefore

$$J^{-1} = \begin{pmatrix} \frac{1}{\hat{a}_n}\mathcal{U}_n & 0 \\ 0 & L^{-1} \end{pmatrix}.$$

Since $\det J = \hat{a}_n \mathcal{U}_n L \neq 0$, by the Implicit Function Theorem, there exist a neighbourhood N of $\widehat{\Phi}$ in V_{n-1} and a unique smooth mapping

$$\begin{aligned} G_\omega : N &\longrightarrow \mathbb{T}^n \times \mathbb{R}^n \\ \Phi &\longmapsto G_\omega(\Phi) \end{aligned} \tag{D.1.12}$$

such that

$$G(\widehat{\Phi}) = (\widehat{\Psi}, \widehat{A}) \quad \text{and} \quad F_\omega(\Phi, G_\omega(\Phi)) \equiv 0, \ \forall\, \Phi \in N. \tag{D.1.13}$$

Now, let G_Ψ and G_A be, respectively, the projection of G_ω onto $\mathbb{T}^n$ and the projection of G_ω onto $\mathbb{R}^n$. Hence:

$$G_\Psi = \mathrm{proj}_1 \circ G_\omega : N \longrightarrow \mathbb{T}^n \quad \text{and} \quad G_A = \mathrm{proj}_2 \circ G_\omega : N \longrightarrow \mathbb{R}^n.$$

Recall that $\mathcal{O}_k$ is the dense orbit generated by $\tau_k\omega$ mod 2π on the n-torus $\mathbb{T}^n$. Let $\mathcal{O}_\Phi \subset V_{n-1}$ be the direct product of the dense orbits $\mathcal{O}_k$, for $1 \le k \le n-1$, and $\mathcal{O}_\Psi$ be the dense orbit in V_Ψ. By the construction of the function G_Ψ, if $\Phi \in \mathcal{O}_\Phi$ and $\Psi = G_\Psi(\Phi) \in \mathcal{O}_\Psi$, then $A = G_A(\Phi)$ yields a solution to the original system of equations (D.1.5).

That is, we are looking for $(\tau_1\omega, \ldots, \tau_{n-1}\omega) = \Phi \in V_{n-1} \cap N$such that $G_\Psi(\Phi) \in V_\Psi$

We now need to show that G_Ψ is regular at $\widehat{\Phi}$.

Implicit differentiation of (D.1.13) yields that

$$D_{(\Psi,A)}F_\omega(\Phi, \Psi, A) \begin{pmatrix} DG_\Psi(\Phi) \\ \\ DG_A(\Phi) \end{pmatrix} = D_\Phi F_\omega(\Phi, \Psi, A), \tag{D.1.14}$$

for all $\Phi \in N$.

Therefore, for $\Phi = \hat{\Phi}$, we have

$$DG(\widehat{\Phi}) \;=\; \begin{pmatrix} DG_\Psi(\widehat{\Phi}) \\ \\ DG_A(\widehat{\Phi}) \end{pmatrix} = -J^{-1}K \tag{D.1.15}$$

where J^{-1} is the inverse of the $2n \times 2n$ matrix J and the $2n \times (n-1)n$ matrix K is defined as $K \equiv D_\Phi F_\omega(\widehat{\Phi}, \widehat{\Psi}, \widehat{A})$.

Note that

$$D_\Phi F_\omega(\Phi, \Psi, A) = \begin{pmatrix} D_{\Phi^1}\left(\mathrm{Re}(\tilde{P})A\right) & D_{\Phi^2}\left(\mathrm{Re}(\tilde{P})A\right) & \cdots & D_{\Phi^{n-1}}\left(\mathrm{Re}(\tilde{P})A\right) \\ D_{\Phi^1}\left(\mathrm{Im}(\tilde{P})A\right) & D_{\Phi^2}\left(\mathrm{Im}(\tilde{P})A\right) & \cdots & D_{\Phi^{n-1}}\left(\mathrm{Im}(\tilde{P})A\right). \end{pmatrix}$$

Computations yields the following for K:

$$K \equiv D_{\Phi}F_{\omega}(\widehat{\Phi},\widehat{\Psi},\widehat{A}) = \begin{pmatrix} \hat{a}_1\mathcal{U}_1 & \hat{a}_2\mathcal{U}_2 & \cdots & \hat{a}_{n-1}\mathcal{U}_{n-1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}.$$

Hence after computing the product $-J^{-1}K$ we obtain:

$$DG(\widehat{\Phi}) = \begin{pmatrix} -\frac{\hat{a}_1}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_1 & -\frac{\hat{a}_2}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_2 & \cdots & -\frac{\hat{a}_{n-1}}{\hat{a}_q}\mathcal{U}_n\mathcal{U}_{n-1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \tag{D.1.16}$$

where $\mathbf{0}$ denotes the $n \times n$ zero matrix. Consequently,

$$DG_{\Psi}(\widehat{\Phi}) = \begin{pmatrix} -\frac{\hat{a}_1}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_1 & -\frac{\hat{a}_2}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_2 & \cdots & -\frac{\hat{a}_{n-1}}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_{n-1} \end{pmatrix}. \tag{D.1.17}$$

Since $\mathcal{U}_1$ is the identity matrix and $\mathcal{U}_n \in \mathbb{GL}_n(\mathbb{R})$, the rank of $DG(\widehat{\Phi})$ is maximum. It follows that the mapping

$$G_{\Psi} : N \longrightarrow V_{\Psi}$$

is regular at $\widehat{\Phi}$.

Since $\mathcal{O}_{\Phi}$ is dense in V_{n-1}, for $1 \leq k \leq n-1$, there exist, for any $\epsilon > 0$, $(\tau_{1,\epsilon}, \dots, \tau_{n-1,\epsilon})$ such that

$$(\tau_{1,\epsilon}\omega, \dots, \tau_{n-1,\epsilon}\omega) \bmod 2\pi = \Phi^*_{\epsilon} \text{ is in a } \epsilon\text{-neighbourhood of } \hat{\Phi}.$$

Let us define a small $(n-1)$-dimensional manifold in V_{n-1} based at Φ^*_{ϵ} by

$$S^h_{\Phi^*_{\epsilon}} = \{(\tau_1\omega, \dots, \tau_{n-1}\omega) \bmod 2\pi \mid \tau_j \in (\tau_{j,\epsilon} - h, \tau_{j,\epsilon} + h)\}$$

with ϵ, h small enough so that $S^h_{\Phi^*_{\epsilon}} \subset N$.

Now it is left to show that $G_{\Phi}(S^h_{\Phi^*_{\epsilon}})$, image of $S^h_{\Phi^*_{\epsilon}}$ by G_{Ψ} has a nontrivial transversal intersection with $\mathcal{O}_{\Psi}$. To do this, we consider the following set of $n-1$ vectors in $(\mathbb{R}^n)^{n-1}$:

$$W_1 \;=\; (\omega, 0, \dots, 0, 0)^T$$

$$\begin{aligned} W_2 &= (0,\omega, \dots ,0,0)^T \\ &\vdots \\ W_{n-2} &= (0,0, \dots ,\omega,0)^T \\ W_{n-1} &= (0,0, \dots ,0,\omega)^T \end{aligned}$$

where 0 represents the zero vector in $\mathbb{R}^n$, and $\omega = (\omega_1, \dots, \omega_n)^T$. The set $\{W_1, \dots , W_{n-1}\}$ is linearly independent. For $\Phi^*_\epsilon \in V_{n-1}$, consider the set

$$P_{\widehat{\Phi}} = \left\{ \widehat{\Phi} + \sum_{r=1}^{n-1} \sigma_r W_r \,(\mathrm{mod} V_\Phi) \,|\, 0 \leq |\sigma_r| << 1,\; 1 \leq r \leq n-1 \right\} \tag{D.1.18}$$

where we choose σ_r, for $1 \leq r \leq n-1$, such that $P_{\widehat{\Phi}} \subset N$. The set $P_{\widehat{\Phi}}$ is a small $n-1$ dimensional manifold in V_{n-1}. We are interested in showing that the image of $P_{\widehat{\Phi}}$ by G_Ψ in V_Ψ is transverse to $\mathcal{O}_\Psi$ (see Definition D.1.1).

To show the transversality of $G_\Psi(P_{\widehat{\Phi}})$ to $\mathcal{O}_\Psi$, we need the tangent space to $G_\Psi(P_{\widehat{\Phi}})$ at $G_\Psi(\widehat{\Phi})$ and the image of the tangent space to $\mathbb{R}$ at zero under $DH_{\widehat{\Phi}}(0)$ to be transversal, where H is defined as

$$\begin{aligned} H_\Phi : \mathbb{R} &\longrightarrow V_\Psi \\ t &\longmapsto G_\Psi(\Phi) + t\omega. \end{aligned}$$

This is equivalent to show that the following equality holds:

$$DH_{\widehat{\Phi}}(0)\, T_0\mathbb{R} + T_{H_{\widehat{\Phi}}(0)}\, G_\Psi(P_{\widehat{\Phi}}) = TH_{\widehat{\Phi}}(0)\, \mathbb{R}^n. \tag{D.1.19}$$

First, note that $T_{H_{\widehat{\Phi}}(0)}\, G_\Psi(P_{\widehat{\Phi}}) = T_{G_\Psi(\widehat{\Phi})}(P_{\widehat{\Phi}})$ is well defined. Since G_Ψ is regular at $\widehat{\Phi}$, $G_\Psi(P_{\widehat{\Phi}})$ is a $n-1$ dimensional submanifold in V_Ψ and $T_{G_\Psi(\widehat{\Phi})}(P_{\widehat{\Phi}})$ is a $n-1$ dimensional hyperplane in V_Ψ. Locally around $G_\Psi(\widehat{\Phi})$, we have that $G_\Psi(P_{\widehat{\Phi}})$ is generated by $\{G_\Psi(W_1), \dots , G_\Psi(W_{n-1})\}$. Hence $T_{G_\Psi(\widehat{\Phi})}(P_{\widehat{\Phi}})$ is generated by $\{DG_\Psi.W_1, \dots , DG_\Psi.W_{n-1}\}$.
Secondly, note that $DH_{\widehat{\Phi}}(0)\, T_0\mathbb{R} \equiv \mathrm{span}\{\omega\}$. Thus for (D.1.19) to be satisfied, it is

sufficient to show that the n vectors of $\mathbb{R}^n$, $DG_\Psi.W_1, \ldots, DG_\Psi.W_{n-1}, \omega$, are linearly independent. To show this, we consider the function

$$\mathcal{T} : N \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{T}(\Phi) = \det\left(DG_\Psi(\Phi)\cdot W_1 \quad DG_\Psi(\Phi)\cdot W_2 \quad \cdots \quad DG_\Psi(\Phi)\cdot W_{n-1} \quad \omega \right). \quad \text{(D.1.20)}$$

Clearly, $\mathcal{T}$ is continuous, and $\mathcal{T}$ evaluated at $\widehat{\Phi}$ yields:

$$\begin{aligned}
\mathcal{T}(\widehat{\Phi}) &= \det\left(-\tfrac{\hat{a}_1}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_1\omega \quad -\tfrac{\hat{a}_2}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_2\omega \quad \cdots \quad -\tfrac{\hat{a}_{n-1}}{\hat{a}_n}\mathcal{U}_n\mathcal{U}_{n-1}\omega \quad \mathcal{U}_n\mathcal{U}_n\omega \right) \\
&= \frac{(-1)^{n-1}}{\hat{a}_n^{n-1}} \det\mathcal{U}_n \det\left(\hat{a}_1\mathcal{U}_1\omega \quad \hat{a}_2\mathcal{U}_2\omega \quad \cdots \quad \hat{a}_{n-1}\mathcal{U}_{n-1}\omega \quad \mathcal{U}_n\omega \right) \\
&= \frac{(\omega_1\,\omega_2\,\cdots\,\omega_n)\,(\hat{a}_1\,\hat{a}_2\,\cdots\,\hat{a}_{n-1})}{\hat{a}_n^{n-1}} \det L \\
&\neq 0.
\end{aligned}$$

It follows that there is a neighbourhood $N' \subseteq N$ of $\widehat{\Phi}$ in which $\mathcal{T} \neq 0$. Thus, by choosing ϵ, h small enough such that $S^h_{\Phi^*_\epsilon} \subset N'$, the image of $S^h_{\Phi^*_\epsilon}$ by G_Ψ is transverse to $\mathcal{O}_\Psi$. Note that since the orbit of $\mathcal{O}_\Psi \subset V_\Psi$ is dense, there are infinitely many intersections with $G_\Psi(S^h_{\Phi^*_\epsilon})$ near the point $\widehat{\Psi} = G_{\widehat{\Psi}}(\widehat{\Phi})$. Therefore, there exist $\mathring{\tau}_1 > 0, \mathring{\tau}_2 > 0, \ldots, \mathring{\tau}_n > 0$ such that $\mathring{\Phi} = (\mathring{\tau}_1\omega, \mathring{\tau}_2\omega, \ldots, \mathring{\tau}_{n-1}\omega) \bmod V_{n-1}$ and $G_\Psi(\mathring{\Phi}) = \mathring{\tau}_n\omega$. Thus, the linear system (D.1.3) has solutions. $\square$

D.2 Openness Theorem

The second theorem is the *Openness Theorem* in the paper by Buono and LeBlanc [15]. Again, we keep the same notations used in [15]. This theorem shows that the previous realization result holds for open sets near solutions found in Theorem (D.1.1). The theorem states the following:

Theorem D.2.1 *Openness Theorem.* Suppose $\omega_1 > 0, \omega_2 > 0, \ldots, \omega_n > 0$ are linearly independent over the rationals. There exists a neighbourhood $\mathcal{N}$ of $\tilde{\omega} = (\omega_1, \ldots, \omega_n)$ in $\mathbb{R}^n$ and a smooth mapping

$$\begin{aligned} H:\ V &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ \omega &\longmapsto H(\omega) = (\tau(\omega), A(\omega)) = ((\tau_1(\omega), \ldots, \tau_n(\omega)), (a_1(\omega), \ldots, a_n(\omega))) \end{aligned}$$

such that

$$\begin{aligned} \sum_{k=1}^{n} a_k(\omega)\, e^{-i\omega_j \tau_k(\omega)} &= i\omega_j, \quad j = 1, \ldots, n \\ \sum_{k=1}^{n} a_k(\omega)\, e^{i\omega_j \tau_k(\omega)} &= -i\omega_j, \quad j = 1, \ldots, n \end{aligned} \tag{D.2.1}$$

for all $\omega \in \mathcal{N}$.

Proof: We consider the system $F_\omega = 0$ given by (D.1.8). We have already shown in the previous proof that for the n frequencies defined in Theorem 5.2.1, there exist infinitely many solutions to $F_\omega = 0$. We use the solution $\left(\mathring{\Phi}, \mathring{\Psi}, G_A(\mathring{\Phi}); \tilde{\omega}\right)$ that was constructed in the previous proof. Recall that if the point $\left(\mathring{\Phi}, \mathring{\Psi}, G_A(\mathring{\Phi}); \tilde{\omega}\right)$ is a zero of F_ω then, equivalently, the $2n$ equations of system (D.1.2) are satisfied.
Using the mapping F_ω, we can build the function Q defined as

$$\begin{aligned} Q\ &: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (\omega, \tau, A) &\longmapsto Q(\omega, \tau, A) = F_\omega((\tau_1\omega, \ldots, \tau_{n-1}\omega), \tau_n\omega, A), \end{aligned} \tag{D.2.2}$$

The goal is now to apply the Implicit Function Theorem at the point $(\tilde{\omega}, \mathring{\tau}, \mathring{A})$ where $Q(\tilde{\omega}, \mathring{\tau}, \mathring{A}) = 0$ and the $2n \times 2n$ matrix

$$D_{(\tau,A)}Q(\tilde{\omega}, \mathring{\tau}, \mathring{A}) = \begin{pmatrix} D_\tau Q(\mathring{\tau}, \mathring{A}, \tilde{\omega}) & D_A Q(\mathring{\tau}, \mathring{A}, \tilde{\omega}) \end{pmatrix}. \tag{D.2.3}$$

is invertible.

Classic computation of derivatives yields that the $2n \times n$ matrix $D_\tau Q$ is equal to

$$D_\tau Q(\tau, \omega, A) = D_{((\Phi^1,\ldots,\Phi^{n-1}),\Psi)} F_\omega((\tau_1\omega, \ldots, \tau_{n-1}\omega), \tau_n\omega, A) \cdot \begin{pmatrix} \omega & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega \end{pmatrix} \tag{D.2.4}$$

where 0 is an n-dimensional zero column vector; and the $2n \times n$ matrix $D_A Q$ is equal to

$$D_A Q(\omega, \tau, A) = D_A F_\omega((\tau_1\omega, \ldots, \tau_{q-1}\omega), \tau_n\omega, A). \tag{D.2.5}$$

Now we need to show that $D_{(\tau,A)}Q(\tilde{\omega}, \mathring{\tau}, \mathring{A}) \neq 0$. To do so, consider the following mappings associated to (D.2.3), (D.2.5) and (D.2.4):

$$\mathcal{R}_\tau : V_{n-1} \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{M}_{2n,n}(\mathbb{C})$$

defined by

$$\mathcal{R}_1(\Phi, \Psi, A, \omega) = D_{(\Phi,\Psi)} F_\omega(\Phi, \Psi, A) \cdot \begin{pmatrix} \omega & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega \end{pmatrix};$$

$$\mathcal{R}_A : V_{n-1} \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{M}_{2n,n}$$

defined by

$$\mathcal{R}_A(\Phi, \Psi, A, \omega) = D_A F_\omega(\Phi, \Psi, A);$$

and

$$\mathcal{R} : V_{n-1} \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{M}_{2n,n}$$

defined by

$$\mathcal{R}(\Phi, \Psi, A, \omega) = \left(\mathcal{R}_\tau(\Phi, \Psi, A, \omega) \quad \mathcal{R}_A(\Phi, \Psi, A, \omega) \right).$$

Using the same particular set $\{v_1, \dots, v_n\}$ constructed in the previous proof, we can compute $\mathcal{R}(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}, \omega)$ where $(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}, \omega) = \left(-\frac{\pi}{2}(v_1, \dots, v_n),\ L^{-1}\omega\right)$.
Recall that L is the $n \times n$ matrix whose j^{th} column is the vector v_j and that $\mathcal{U}_j$ is the $n \times n$ matrix whose k^{th} diagonal element is the k^{th} component of the vector v_j.
After computations we obtain:

$$\mathcal{R}\left(-\frac{\pi}{2}(v_1, \dots, v_q), \widehat{A}, \tilde{\omega}\right) = \begin{pmatrix} \mathcal{Z} & i\,L \\ & \\ \mathcal{Z} & -i\,L \end{pmatrix}$$

where

$$\mathcal{Z} = \begin{pmatrix} a_1 \mathcal{U}_1 \tilde{\omega} & a_2 \mathcal{U}_2 \tilde{\omega} & \cdots & a_n \mathcal{U}_n \tilde{\omega} \end{pmatrix}.$$

Recall that each a_j is non-zero thus the $n \times n$ matrix $\mathcal{Z}$ is invertible, since its determinant is

$$\det \mathcal{Z} = \prod_{j=1}^{n} a_j \omega_j \ \det L \neq 0.$$

Quick matrix computations yields:

$$\mathcal{R}\left(-\frac{\pi}{2}(v_1, \dots, v_n), \widehat{A}, \tilde{\omega}\right)^{-1} = \begin{pmatrix} \frac{1}{2}\mathcal{Z}^{-1} & \frac{1}{2}\mathcal{Z}^{-1} \\ & \\ -\frac{i}{2}\,L^{-1} & \frac{i}{2}\,L^{-1} \end{pmatrix}.$$

Hence $\mathcal{R}$ is clearly invertible.
By continuity of the determinant, there is a neighbourhood E of the point $(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}, \tilde{\omega}) \in V_{n-1} \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n$ in which $\mathcal{R}$ is invertible. As stated in Theorem 5.2.1,

because of the density of $\mathcal{O}_\Phi$ and $\mathcal{O}_\Psi$, there are infinitely many solutions to $F_\omega = 0$ near $(\widehat{\Phi}, \widehat{\Psi}, \widehat{A}, \tilde{\omega})$. Hence there exist a point $(\tilde{\omega}, \mathring{\tau}, \mathring{A}) \in E$ such that $Q(\tilde{\omega}, \mathring{\tau}, \mathring{A}) = 0$ and Q is invertible at that point. Therefore, by the Implicit Function Theorem, there exist a neighbourhood $\mathcal{N} \subset \mathbb{R}^n$ of $\tilde{\omega}$ and a unique smooth mapping

$$\begin{aligned} H: \ \mathcal{N} &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ \omega &\longmapsto H(\omega) = (\tau(\omega), A(\omega)) = ((\tau_1(\omega), \ldots, \tau_n(\omega)), (a_1(\omega), \ldots, a_n(\omega))) \end{aligned}$$

such that

$$H(\tilde{\omega}) = Q(\tilde{\omega}, \mathring{\tau}, \mathring{A}) \quad \text{and} \quad Q(\tau(\omega), A(\omega); \omega) \equiv 0, \quad \forall \omega \in \mathcal{N}. \qquad \square$$

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