Random Walks on Products of Hyperbolic Groups

Oleksii Volkov

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

The subject area of this thesis is the theory of random walks on groups. First, we study random walks on products of hyperbolic groups and show that the Poisson boundary can be identified with an appropriate geometric boundary (the skeleton). Second, we show that in the particular case of free and free-product factors, the Hausdorff dimension of the conditional measures on product fibers of the Poisson boundary is related to the asymptotic entropy and the rate of escape of the corresponding conditional random walks via a generalized entropy-dimension formula.

Dedications

To my family.

Acknowledgement

I am especially grateful to my advisor, Vadim Kaimanovich, whose guidance and generosity have been invaluable. From the very beginning, his door was always open to me, and I consider myself fortunate to have studied under someone with as much patience, breadth, and understanding. Without Vadim's constant support and meaningful encouragement this thesis would not have happened.

I would also like to thank the University of Ottawa for their financial support during my Ph.D. Finally, I would like to thank the many people who have helped and befriended me along the way. The list of names is so long that, were I to attempt to write it, I would inevitably leave someone out and suffer a pang of regret upon realizing my mistake. I hope, therefore, that those whom I would like to thank know who they are and know that they have my gratitude.

Contents

Chapter 1

Introduction

The subject of this thesis lies at the crossroads of several mathematical disciplines: graph theory, group theory, ergodic theory and probability. More specifically, we study the boundary behavior of random walks on the products of the hyperbolic groups and, in particular, the Hausdorff dimension of the arising harmonic measures.

1. Random walks: historical background

Random walks on groups have been the topic of numerous research papers and many books since the beginning of the last century, their history can be traced back to the very origins of probability. The term ”Random walk” was originally proposed by Karl Pearson (1905). In a letter to the *Nature*, he gave a simple model to describe the mosquito infestation in a forest. The approximation of diffusion processes by random walks goes back even further to Bachelier (1900, 1901). It is remarkable that in the same year as Pearson’s letter, Albert Einstein also published his seminal paper on the Brownian motion—the complicated path of a large dust particle in air—which he modeled as a random walk, driven by collisions with gas molecules. A further application was to potential theory, where in the 1920s a method of discrete approximation was devised, admitting a probabilistic interpretation in terms of a simple symmetric random walk.

The modern theory of random walks began with the works of Poincaré (1912) on card deck shuffling (i.e., random compositions in the symmetric group), and Pólya's (1921) discovery that a simple symmetric random walk on $\mathbb{Z}^d$ is recurrent for $d \leq 2$ and transient otherwise.

Although disparate examples of random walks existed well before Markov, it was he who provided a general theoretical background by introducing in 1906 what is now known as Markov chains. Originally his work remained relatively unknown, and it is only in the 1920s and 1930s that its significance in this rapidly developing area was universally recognized. The boundary theory of Markov chains in general and of random walks in particular originates from the works of Blackwell (1955), Feller (1956), Doob (1957), Dynkin (1961), Furstenberg (1963).

2. Random walks on groups with hyperbolic properties

Free and similar classes of groups (Fuchsian and, more recently, hyperbolic) have always served as examples of boundary behaviour. These groups can be considered as "rank 1 groups[1]" (and, indeed, this class includes discrete subgroups of rank 1 semi-simple Lie groups). The discrete subgroups of higher rank groups were in this context considered by Ledrappier and Kaimanovich who, in particular, identified their Poisson boundaries. Products of hyperbolic groups can be naturally considered

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[1] Informally, the rank of a group is the maximal rank of its free abelian subgroups.

as being of ”higher rank ” as well.

3. Random walks on higher rank and product groups

During the centenary history of random walks only a handful of authors have considered products of non-abelian groups despite the fact that these groups quite naturally arise in many situations. For example, if a Lie group is a product group itself (e.g., the product of two copies of $\mathrm{SL}(2,\mathbb{R})$ corresponding to the bi-disk), then the product of discrete subgroups in each copy is a discrete subgroup in the product Lie group. However, there are very few results concerning the step distributions that are not product measures. A recent exception is the paper [4] devoted to noise sensitivity, where the convex combinations of the diagonal and product measures are considered.

4. Boundary convergence and identification of the Poisson boundary for product groups

For random walks on hyperbolic groups it is known (see for example [22]) that under quite general conditions the sample paths almost surely converge to the hyperbolic boundary, and, moreover, the hyperbolic boundary endowed with the resulting hitting distribution is isomorphic to the Poisson boundary of the random walk.

More precisely, let $\mathbf{P}$ be the measure on the path space of the random walk on a hyperbolic group G with a step distribution μ issued from the group identity. Then for a.e. sample path (g_n) the limit

$$\lim g_n = g_\infty \in \partial G$$

exists. The image ν of the measure $\mathbf{P}$ under the boundary map

$$\mathbf{bnd} : (g_n) \to (g_\infty)$$

is called the harmonic (hitting) measure of the random walk (G, μ). The fact that the Poisson boundary of the random walk (G, μ) coincides with the hyperbolic boundary ∂G endowed with the harmonic measure ν means that the stochastically significant behavior of the random walk at infinity is completely described by the limits, i.e., the Poisson partition ρ of the path space coincides with the preimage partition of the boundary map.

In this work, we consider a random walk on a product $\hat{G} = G_1 \times G_2$ of two hyperbolic groups. The resulting boundary behavior may be quite different depending on the chosen measure. For an illustration, let us look at a number of particular cases.

Product step distribution. In this case the random walk on the product group $\hat{G}$ is the product of two independent random walks on the factors G_1, G_2, and the harmonic measure is also a product measure. The Poisson boundary then coincides with the product of the Poisson boundaries of the quotient random walks (see Proposition 4.4.2).

Diagonal step distribution. Another possible (if degenerate) situation is when the factors G_1 and G_2 coincide with the same group G, and the step distribution is

$$\hat{\mu} = \mathrm{diag}(\mu) = \sum_{\hat{G}} \mu(g)\delta_{(g,g)}$$

for a certain measure μ on G. In this case the random walk $(\hat{G}, \hat{\mu})$ is concentrated on the diagonal of the product group $\hat{G}$.

Mixed step distribution. However, already the convex combinations

$$\hat{\mu} = \lambda(\mu \times \mu) + (1 - \lambda)\mathrm{diag}(\mu),\ 0 < \lambda < 1,$$

of product and diagonal measures (in the case when $G_1 \cong G_2$) turn out to be quite interesting and non-trivial. Even if μ is finitely supported, the resulting harmonic measure is then not Markov and, apparently, not even a Gibbs one. The step distributions of this kind were recently considered in relation to the noise sensitivity[2] of random walks on groups [4].

We will be interested in the convergence of the trajectories of random walks on products of hyperbolic groups to an appropriate geometric boundary, and the Hausdorff dimension of the harmonic measures of the arising conditional Markov operators.

5. The Poisson boundary of products of hyperbolic groups

We define the boundary skeleton of the product $\hat{G} = G_1 \times G_2$ of hyperbolic groups as the product $\partial\hat{G} = \partial G_1 \times \partial G_2$ of their hyperbolic boundaries (cf. [45]). Figure 1.1 gives a naive visualization of the boundary skeleton for the direct product of two line segments.

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[2] In probability theory, the noise of an event $E(x_1, ..., x_n)$ depending on a large number of variables can be modeled as the effect of replacing a small proportion $\lambda \in (0, 1)$ of the variables by random entries. An event is noise sensitive if the realization of E gives very little information on its realization for corresponding noised entries. The authors of the aforementioned paper adapt the notion of noise sensitivity to random walks on groups and touch upon several interesting topics.

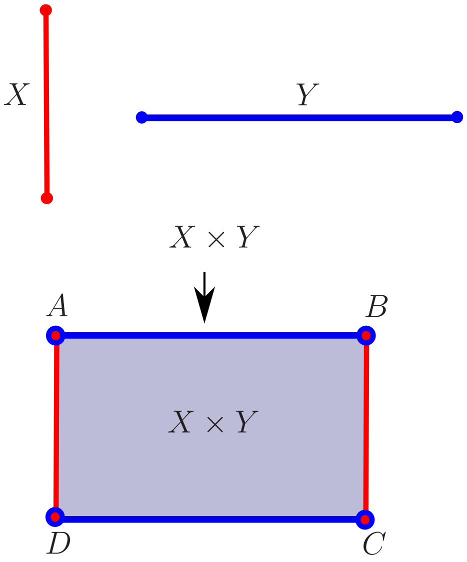

Figure 1.1: The boundary of the product $X \times Y$ of line segments X and Y is formed by the set of line segments $\{AB, BC, CD, DA\}$. The set of points $\{A, B, C, D\}$ forms the skeleton of the boundary of the product $X \times Y$.

Theorem (4.4.2). Let G_1 and G_2 be hyperbolic groups. Let $\hat{G} = G_1 \times G_2$ be the product group endowed with a probability measure $\hat{\mu}$ with its first moment finite, and non-degenerate projections μ_1 and μ_2. Then almost every sample path of the random walk $(\hat{G}, \hat{\mu})$ converges to the boundary skeleton $\partial\hat{G} = \partial G_1 \times \partial G_2$, and this boundary skeleton with the resulting limit measure $\hat{\nu}$ is isomorphic to the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$.

Our proof of the above Theorem is based on the same entropy theory approach that was earlier used in the study of random walks on hyperbolic groups ([19], [20], [21], [22]).

Once we have identified the Poisson boundary with the skeleton $\partial\hat{G}$, i.e., with the product $\partial G_1 \times \partial G_2$, it is natural to ask about the properties of the conditional measures on the fibers of the coordinate projections.

6. Conditional Markov chains on a product group

The projection

$$\mathbf{bnd}_2 : \hat{G}^{\mathbb{Z}_+} \to \partial G_2$$

of the sample path space onto ∂G_2 is given by the composition

$$\big(X_n, Y_n\big) \mapsto \big(X_\infty, Y_\infty\big) \mapsto Y_\infty = \gamma \in \partial G_2.$$

The measures are projected in the same way $\mathbf{P} \mapsto \hat{\nu} \mapsto \nu_2$.

The conditional measure $\mathbf{P}^\gamma$ is Markov and corresponds to the Markov chain (a Doob transform of the original random walk) with the one step transition probabilities

$$p^\gamma[(x, x'), (xh, x'h')] = \hat{\mu}(h, h')\frac{dh'\nu_2}{d\nu_2}(\gamma),$$

for the initial distribution concentrated on $\mathbf{1}_{\hat{G}}$. The time n one dimensional distributions of the measure $\mathbf{P}^\gamma$ are

$$\pi_n^\gamma(g, h) = p_n^\gamma\big[1_{\hat{G}}, (g, h)\big]$$

We denote the transition operator of this chain by P^γ. The conditional measures ν^γ of $\hat{\nu}$ on the fibers of the projection $\partial\hat{G} \to \partial G_2$ can be then interpreted as the harmonic measures of the conditional Markov operator P^γ on $\hat{G}$. Since the projection $\mathbf{bnd}_2$ is equivariant with respect to the action of the product group, the quotient space

(Γ_2, ν_2) can be considered as a $\hat{\mu}$-boundary.

7. Entropy

Let us recall that the *entropy* of a discrete probability distribution $p = \{p_1, p_2, ...\}$ is defined as

$$H(p) = -\sum_i p_i \log p_i.$$

If the step distribution μ of a random walk on a group G has a finite entropy, then the entropies $H(\mu^{*n})$ of its n-fold convolutions μ^{*n} (i.e., of the time n distributions of the random walk issued from the group identity) are also finite, the limit

$$h\,(G,\mu) = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}.$$

exists, and it is called the *asymptotic entropy of the random walk* (G, μ) ([2], [8], [19]). It plays a crucial role in understanding the asymptotic properties of the random walk (G, μ).

We shall also use the notion of the entropy of a measurable partition. If (X, m) is a Lebesgue probability space, and $\xi = \{X_i\}$ is a countable measurable partition of X, then we define $H(\xi) = H_m(\xi)$ as the entropy of the probability distribution $p_i = m(X_i)$. Suppose we have another measurable partition η of X. The conditional entropy of ξ with respect to η is defined as

$$H(\xi \mid \eta) = -\int H_{\eta(x)}(\xi) dm(x), = -\int \log m^{\eta(x)}(\xi(x)) dm(x).$$

Kaimanovich and Vershik showed in [19] that the *conditional entropy* of the time $\leq k$ partition ζ_1^k $(k \geq 1)$ with respect to the Poisson partition ρ is

$$H(\zeta_1^k|\rho) = kH(\zeta|\rho) = k[H(\mu) - h(G,\mu)].$$

This formula implies that the asymptotic entropy $h(G,\mu)$ coincides with the differential entropy $\mathbf{E}(\partial G,\nu)$ of the boundary. Moreover, the above formula is also valid for an arbitrary equivariant quotient of the Poisson boundary

$$H(\zeta_1^k|\rho_\xi) = kH(\zeta|\rho_\xi).$$

Here ξ is a G-invariant partition of the Poisson boundary, and ρ_ξ is the associated partition of the path space. For $\zeta = \zeta_1^1$ we have

$$H(\zeta|\rho_\xi) = H(\mu) - \mathbf{E}(\partial G_\xi, \nu_\xi).$$

Returning to the random walks on product groups, ∂G_2 is an equivariant quotient of the Poisson boundary. For $\gamma \in \partial G_2$, let $(\partial\hat{G}^\gamma, \nu^\gamma)$ be the fiber of $(\hat{\Gamma}, \hat{\nu})$ above (Γ_2, ν_2). Then for ν_2-a.e. $\gamma \in \partial G_2$, the asymptotic entropy of the conditional measure $\mathbf{P}^\gamma$ on the path space $\hat{G}^{\mathbb{Z}_+}$ exists and is equal

$$\mathbf{h}(\mathbf{P}^\gamma) = h(\hat{G}, \hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2).$$

8. The Hausdorff dimension

The Hausdorff dimension $\mathbf{HD}\,\mu$ of a measure μ on a metric space (X, d) plays a fundamental role in fractal analysis and is defined as the minimal Hausdorff dimension of the sets of full measure for μ (showing the "degree of singularity" of this measure). Even if the support of the measure μ is the whole space, $\mathbf{HD}\,\mu$ does not have to coincide with $\mathbf{HD}\,X$. The Hausdorff dimension $\mathbf{HD}\,\mu$ characterizes the polynomial rate of decrease of the measures μ of the balls of the metric d around typical (with respect to μ) points of X. In particular, if the ball measures decrease regularly, i.e. the limit

$$\lim_{r\to 0} \frac{\log \mu\mathbf{B}(x,r)}{\log r} = \alpha$$

exists and is the same for μ-a.e. $x \in X$, we say that a measure μ is *dimensionally homogeneous*. In this case $\mathbf{HD}\,\mu = \alpha$, and all other reasonable definitions of the dimension of a measure give the same result [41], [52].

In our work we take inspiration from the well known results of Kaimanovich ([19], [21], [22])*, which were in turn partially based on the series of very important (but unpublished at that time) results of Ledrappier* ([30], [31])*. The work of these authors was recently generalized by Tanaka* [49]*.*

Finding or estimating the Hausdorff dimension of various measures is one of the central problems of fractal analysis.

In the context of random walks, Ledrappier established (in a somewhat weaker sense) that for random walks on discrete subgroups of $SL(2, \mathbb{C})$, the dimension of the harmonic measure on the boundary sphere of the hyperbolic 3-space coincides with the ratio $h/(2\lambda)$, where λ is the Lyapunov exponent [30], and h is the asymptotic entropy. For finitely generated free groups, he further showed that the harmonic measure ν is dimensionally homogeneous and $\mathbf{HD}\,\nu = h/\ell$ when μ has finite first moment [31, Theorem 4.15].

Kaimanovich strengthened this result to arbitrary free groups (including an infinitely generated case) and established the dimension formula for a general class of processes on trees [21].

Even further generalization of the above results was achieved in the recent work of Tanaka. He showed (under appropriate conditions) the dimensional homogeneity of the harmonic measure ν for a countable group G of isometries of a hyperbolic space and, in particular, that $\mathbf{HD}\,\nu = h/\ell$ [49, Theorem 1.1].

9. The Hausdorff dimension of conditional harmonic measures on product groups.

We establish our main result on the Hausdorff dimension of the conditional harmonic measures under the additional assumption that the hyperbolic factors G_1, G_2 of the product $\hat{G} = G_1 \times G_2$ are actually free-product groups.

Theorem (5.4.1). Let $(\hat{\Gamma}, \hat{\nu})$ be the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$. For a fixed $\gamma \in \partial G_2$, we denote by $(\partial \hat{G}^{\gamma}, \nu^{\gamma})$ the fiber $(\partial G_1 \times \{\gamma\}, \nu_1)$ of $(\hat{\Gamma}, \hat{\nu})$ above γ. Then for a.e. $\gamma \in \partial G_2$, the conditional measure ν^{γ} is dimensionally homogeneous and

$$\mathbf{HD}\ \nu^{\gamma} = \frac{\hat{h} - h_2}{\ell_1}, \tag{1.0.1}$$

where $\hat{h} - h_2$ is the entropy (5.4.1) of the corresponding conditional random walk, $h_2 = h(G_2, \mu_2)$, and ℓ_1 is the rate of escape along the first (non-fixed) component, respectively. In other words, for ν^γ-a.e. $(\eta, \gamma) \in \partial\hat{G}^\gamma$ and $\ell_1 > 0$

$$\frac{-\log \nu^\gamma\big(\boldsymbol{B}_\eta^k\big)}{k} \to \frac{\hat{h} - h_2}{\ell_1}, \text{ as } k \to \infty, \tag{1.0.2}$$

where B_η^k are the balls (5.2.5) of the restriction of the metric (5.2.4) to the boundary. The proof of the above theorem consists of two separate results for the upper and lower bounds (Theorems 5.4.2 and 5.4.3).

Chapter 2

Boundary behavior of Markov chains

2.1 Lebesgue spaces

Throughout the present work we are using the machinery of Rokhlin's theory (see [43], [44]) of Lebesgue probability spaces (canonical systems of conditional measures, the correspondence between complete sub-σ-algebras and quotient spaces, etc.). This is a standard tool in ergodic theory, but is less popular in the probabilistic context. For the sake of completeness we begin with reminding the basic facts of the Rokhlin theory. A morphism of measure spaces is a map

$$\phi : (X, \mathcal{A}, m) \to (\bar{X}, \bar{\mathcal{A}}, \bar{m}),$$

which is measurable, i.e.

$$\phi^{-1}(\bar{A}) \in \mathcal{A}, \ \forall \bar{A} \in \bar{\mathcal{A}},$$

and measure preserving, i.e.

$$\bar{m}(\bar{A}) = m(\phi^{-1}(\bar{A})),\ \forall \bar{A} \in \bar{\mathcal{A}}.$$

A morphism mod 0 is a partially defined map

$$\phi : (X, \mathcal{A}, m) \to (\bar{X}, \bar{\mathcal{A}}, \bar{m}),$$

such that there exist null sets $O \in \mathcal{A}$ and $\bar{O} \in \bar{\mathcal{A}}$ with the property that the induced map

$$(X \backslash O) \to (\bar{X} \backslash \bar{O})$$

is a morphism. According to the above definition, when dealing with a morphism mod 0, we disregard subsets of measure 0, hence we disregard individual points of X (their measure is 0 because we are considering non-atomic measures) and focus on the typical behavior of the morphism.

Definition 2.1.1. *A basis of a probability measure space* $(X, \mathcal{A}, m)$ *is a countable collection of measurable sets* $\{B_n\} \subset \mathcal{A}$ $(n \in \mathbb{N})$ *such that* $\sigma(\{B_n\}) = \mathcal{A} \mod 0$*, and for all* $x \neq x' \in X$ *there exists* B_n *such that* $\mathbf{1}_{B_n}(x) \neq \mathbf{1}_{B_n}(x')$.

We call a basis $\{B_n\}_{n\in\mathbb{N}}$ complete if

$$\bigcap_{n\in\mathbb{N}} B_n^{\varepsilon_n} \neq 0 \tag{2.1.1}$$

for all binary sequences $(\varepsilon_n)_{n\in\mathbb{N}}$. We denote $B_n^0 = B_n, B_n^1 = X \setminus B_n, \ldots.$ Since (by Definition 2.1.1) a basis separates points, every intersection (2.1.1) contains precisely one point. We can now define a Lebesgue space as follows

Definition 2.1.2. *A Lebesgue space is a measure space, which admits a complete basis.*

A very important property of such spaces is that any Lebesgue space has the form $[0, a] \cup (m_1, m_2, ...)$ with $a + m_1 + m_2 + \cdots = 1$. In plain words this means that any Lebesgue space is isomorphic $\mod 0$ to a union of a certain interval $[0, a]$ (equipped with a Lebesgue measure) and a countable set of atoms of weights m_i. Whence, every purely non-atomic Lebesgue space is isomorphic to the interval $[0, 1]$ with Lebesgue measure on it.

A separable topological space whose topology can be generated by a complete metric is called a Polish topological space. Any such space endowed with a Borel probability measure on it is a Lebesgue space. In particular, all Polish spaces endowed with a purely non-atomic Borel probability measure are isomorphic (as measure spaces) to the unit interval with the Lebesgue measure on it (e.d. see Figure 2.1).

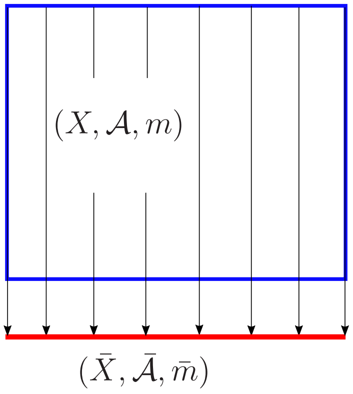

Figure 2.1: Line segment as a quotient space of a square

Consider a morphism mod 0

$$\phi : (X, \mathcal{A}, m) \to (\bar{X}, \bar{\mathcal{A}}, \bar{m}).$$

A partition ξ of X into the preimages of points of $\bar{X}$ under ϕ is called a measurable partition of a measure space X. We also have the σ-algebra $\mathcal{A}_\phi$ associated to ϕ, which is the completion of the σ-algebra $\phi^{-1}(\bar{\mathcal{A}})$ with respect to m.

A cornerstone result by V.A. Rokhlin asserts the existence of the one-to-one correspondence between measurable partitions, complete sub-σ-algebras and quotient spaces for any Lebesgue measure space.

Theorem 2.1.1. (Rokhlin) For a Lebesgue space $(X, \mathcal{A}, m)$ there is a one-to-one correspondence between

(i) quotient spaces,

(ii) complete sub-σ-algebras of $\mathcal{A}$,

(iii) measurable partitions ξ of X.

2.2 Canonical system of measures

An important property of Lebesgue spaces is the existence of canonical systems of measures (or system of conditional measures) which can be viewed as an extension of the Fubini theorem.

Theorem 2.2.1. (Fubini) Consider an integrable function $f : [0,1] \times [0,1] \to \mathbb{R}$ and the projection $\pi : (x, y) \mapsto x$. Denote the Lebesgue measure on the fiber $\pi^{-1}(\{x\}) \cong$

$[0,1]$ by m_x. Then for almost every $x \in [0,1]$ the function $y \mapsto f(x,y)$ is integrable with respect to m_x and

$$\int f(x,y)dm(x,y) = \int \left(\int f(x,y)dm_x(y) \right) dm(x)$$

The canonical system of measures on a Lebesgue space $(X, \mathcal{A}, m)$ is a system of measures $m(\cdot|x_\xi)$ on the elements x_ξ of the partition ξ which enables us to consider integration with respect to measure m as repeated integration. We first integrate over the x_ξ with respect to appropriate $m(\cdot|x_\xi)$, and then we must integrate the result (which can be viewed as a function on the quotient space X/ξ), with respect to the natural measure m_ξ on the space X/ξ. By definition, X/ξ has the elements of ξ as points, and its measurable subsets are those with measurable pre-images under the natural projection

$$\pi : X \to X/\xi,$$

where the measure is defined to be $m_\xi(S) = m(\pi^{-1}(S))$.

Theorem 2.2.2. [43] Consider a morphism mod 0 of Lebesgue spaces

$$\phi : (X, \mathcal{A}, m) \to (\bar{X}, \bar{\mathcal{A}}, \bar{m}).$$

For $\bar{m}$-a.e. $\bar{x} \in \bar{X}$ there exists a measure $m_{\bar{x}}$ defined on a σ-algebra $\mathcal{A}_{\bar{x}}$ on the fiber $\phi^{-1}(\bar{x})$ such that for all $f \in \mathcal{A}$ we have

$$f|_{\phi^{-1}(\bar{x})} \in \mathcal{A}_{\bar{x}} \text{ for } \bar{m}\text{-a.e. } \bar{x} \in \bar{X}$$

and

$$\int f(x)dm(x) = \int \left(\int f|_{\phi^{-1}(\bar{x})} dm_{\bar{x}} \right) d\bar{m}(\bar{x}).$$

The system $\{m_{\bar{x}}\}_{\bar{x}\in\bar{X}}$ is unique $\mod 0$, and is called the canonical system of measures associated to ϕ.

The canonical systems of measures are transitive

Corollary 2.2.1. *Consider two morphisms* $\mod 0$ *of Lebesgue spaces*

$$\phi : (X, \mathcal{A}, m) \xrightarrow{\phi} (\bar{X}_1, \bar{\mathcal{A}}_2, \bar{m}_1) \xrightarrow{\psi} (\bar{X}_2, \bar{\mathcal{A}}_2, \bar{m}_2).$$

The canonical system $\bar{\mu}$ *associated to* $\psi \circ \phi$ *can be constructed using canonical systems associated to* ψ *and* ϕ *as follows*

$$\bar{\mu}_{\bar{x}_2} = \int\limits_{\psi^{-1}(\bar{x}_2)} m_{\bar{x}_1} dm_{\bar{x}_2}(\bar{x}_1).$$

2.3 Conditional expectations and Martingales

Consider a Lebesgue space $(X, \mathcal{A}, m)$. As we have mentioned in Section 2.1, a sub-σ-algebra $\bar{\mathcal{A}} \subset \mathcal{A}$ corresponds to a quotient space $\bar{X}$ of X. Choose a canonical system of measures $\{m_{\bar{x}}\}_{\bar{x}\in\bar{X}}$ associated to the quotient map $\pi : X \to \bar{X}$.

Definition 2.3.1. *The* conditional expectation $E[f|\bar{\mathcal{A}}]$ *is the function on the quotient space* $(\bar{X}, \bar{\mathcal{A}}, \bar{m})$ *of a Lebesgue space* $(X, \mathcal{A}, m)$ *given by*

$$E[f|\bar{\mathcal{A}}](\bar{x}) = \int f|_{\pi^{-1}(\bar{x})} dm_{\bar{x}}, \quad \bar{x} \in \bar{X}.$$

The conditional expectation $E[f|\bar{\mathcal{A}}]$ is well-defined due to the uniqueness of the canonical system of measures. The composition $E[f|\bar{\mathcal{A}}] \circ \pi$ is $\bar{\mathcal{A}}$-measurable and for every set $A \in \bar{\mathcal{A}}$ we have

$$\int\limits_A E[f|\bar{\mathcal{A}}] \circ \pi dm = \int\limits_A f dm.$$

Consider an increasing sequence of sub-σ-algebras

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A},$$

such that $\mathcal{A}_n \nearrow \mathcal{A}$ (i.e., this sequence exhausts $\mathcal{A}$). A sequence with the above properties is called an exhausting filtration of sub-σ-algebras.

Definition 2.3.2. *A martingale (with respect to an exhausting filtration) is a sequence of functions $f_n \in \mathcal{A}_n$ that satisfies*

$$E[f_{n+1}|\mathcal{A}_n] = f_n.$$

An important for us result from the martingale theory is the following

Theorem 2.3.1 (Doob martingale convergence theorem, [47])**.** Let $(X, \mathcal{A}, m)$ be a Lebesgue space and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \nearrow \mathcal{A}$ be an exhausting filtration of sub-σ-algebras of $\mathcal{A}$. Suppose that $f_n \in \mathcal{A}_n$ is a uniformly bounded martingale, i.e., $\|f_n\|_\infty \leq C$ for all n. Then there exists a function $f \in L^\infty(X)$ such that $f_n \to f$ almost everywhere and in L^1. Moreover

$$f_n = E[f|\mathcal{A}_n].$$

Conversely, if $f \in L^\infty(X)$ then $f_n = E[f|\mathcal{A}_n]$ is a martingale and $f_n \to f$.

The above theorem can be seen as a measure theoretic analogue of the Lebesgue differentiation theorem. Let $\mathcal{A}_n$ be the σ-algebra generated by the sets $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \subset [0,1]$. Let $f \in L^\infty([0,1])$ be a bounded function. For $x \in [0,1] \setminus \mathbb{Q}$, let $I_n(x)$ be the dyadic interval of length 2^{-n} containing x. Set

$$f_n(x) = 2^n \int\limits_{I_n(x)} f dm = E[f|\mathcal{A}_n](x).$$

The Lebesgue differentiation theorem then says that $f_n \to f$ almost everywhere.

2.4 Markov Chains

A Markov chain on a countable state space X is defined by its transition probabilities

$$p(y|x) = p(x,y) \geq 0, \quad x, y \in X, \quad \sum_y p(x,y) = 1.$$

Alternatively, on a Lebesgue space, we can denote the transition probability measure from a point $x \in X$ as $\pi_x(\cdot)$. Then

$$p(x,y) = \pi_x(y).$$

One can also define Markov chain in terms of the associated Markov operator (on the space $\ell^\infty(X)$)

$$Pf(x) = \sum_y p(x,y)f(y) = \int f(z_1) d\mathbf{P}_x(\boldsymbol{z}) = \mathbf{E}_x f(z_1), \tag{2.4.1}$$

where $\mathbf{P}_x$ is the measure on the trajectory space $X^{\mathbb{Z}_+}$ corresponding to the starting point $x \in X$ (see (2.4.3)).

Definition 2.4.1. *A linear operator $P : \ell^\infty(X) \hookleftarrow$ is called* Markov *if*

(i) $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}(x) \equiv \mathbf{1}$, $\|P\|_{\ell^\infty} = 1$;

(ii) $Pf(x) \geq 0$ whenever $f(x) \geq 0$;

(iii) P is continuous in the sense that $Pf_n \downarrow 0$ pointwise whenever $f_n \downarrow 0$.

The pre-dual operator in this case acts on the space $\ell^1(X)$ of measures θ on X by

$$\theta P(y) = \sum_x \theta(x) p(x, y). \tag{2.4.2}$$

Definition 2.4.2. *A triple (X, P, θ), where $P : \ell^\infty(X) \hookleftarrow$ is a Markov operator and θ is a probability measure on a space X is called a (time homogeneous) Markov chain on the state space X with transition operator P and initial distribution θ.*

Let $X^{\mathbb{Z}_+}$ be the infinite Cartesian product of X with itself ($\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$). The elements of this product

$$\boldsymbol{x} = (x_0, x_1, x_2, ...) \in X^{\mathbb{Z}_+}$$

will be called trajectories or sample paths. The set $X^{\mathbb{Z}_+}$ will be called the trajectory space (or sample path space). The space $X^{\mathbb{Z}_+}$ carries a natural topology—the product of discrete topologies on every factor.

One can define paths over finite sets of indices as well: $X^{[0,n]}$. Hence, for $m < n$ there exists a natural projection

$$\phi_m^n : X^{[0,n]} \to X^{[0,m]}$$

defined by the truncation map

$$(x_0, ..., x_m, x_{m+1}, ..., x_n) \mapsto (x_0, ..., x_m).$$

The finite-dimensional spaces $X^{[0,n]}$ are endowed with the probability measures

$$\mathbf{P}_{x_0}^{[0,n]}(x_0, x_1, ..., x_n) = \pi_{x_0}(x_1) \cdot \pi_{x_1}(x_2) \cdots \pi_{x_{n-1}}(x_n). \tag{2.4.3}$$

corresponding to starting the Markov chain at x_0. Then we have the following consistency relation

$$\mathbf{P}_{x_0}^{[0,m]} = \phi_m^n \Bigg(\mathbf{P}_{x_0}^{[0,n]} \Bigg) \tag{2.4.4}$$

The trajectory space

$$X^{\mathbb{Z}_+} = \big\{ (x_0, x_1, x_2, ...) \big\};\ x_i \in X \tag{2.4.5}$$

is the projective limit of finite-dimensional spaces $X^{[0,n]} = \big\{ (x_0, ..., x_n) \big\},\ n \in \mathbb{Z}_+$. By the Kolmogorov's extension theorem, the projective system of measures (2.4.3) produces a measure $\mathbf{P}_{x_0}$ on the trajectory space $X^{\mathbb{Z}_+}$. The $\mathbf{P}_x$-distribution of x_n is the n step transition probability $\pi_x^n = p^{(n)}(x, \cdot)$.

Given an initial distribution θ and a Markov operator $P = \{p(x,y) : x, y \in X\}$ on X, we denote by

$$\mathbf{P}_\theta = \sum_{x \in X} \theta(x)\, \mathbf{P}_x \tag{2.4.6}$$

the corresponding probability measure on the trajectory space $X^{\mathbb{Z}_+}$. The coordinates x_n can be viewed as X-valued random variables with

$$\mathbf{P}_\theta(x_0 = x) = \theta(x)$$

and

$$\mathbf{P}_\theta(x_{n+1} = y | x_n = x) = p(x,y) = \pi_x(y), \;\; x, y \in X.$$

We denote by

$$\mathbf{c}_n : \boldsymbol{x} \mapsto x_n, \;\; n \geq 0$$

the n-th coordinate map from $X^{\mathbb{Z}_+}$ into X. The subsets of $X^{\mathbb{Z}_+}$ of the form

$$C_x^n = \{\boldsymbol{y} \in X^{\mathbb{Z}_+} : \mathbf{c}_n(\boldsymbol{y}) = x\}$$

are called (one-dimensional) cylinder sets. Their finite intersections (cylinder sets)

$$\bigcap_{i=0}^{k} C_{x_i}^{n_i} = C_{x_0,\ldots,x_k}^{n_0,\ldots,n_k} \tag{2.4.7}$$

consist of the trajectories that pass through fixed finite set of points of X at fixed set of times. Then, for any fixed trajectory $\boldsymbol{y} \in X^{\mathbb{Z}_+}$, we denote by

$$C_{\boldsymbol{y}}^{n_0,\dots,n_k} = \bigcap_{i=0}^{k} C_{x_i}^{n_i}, \quad x_i = \mathbf{c}_{n_i}(\boldsymbol{y}),$$

the set of trajectories that pass through the same points as $\boldsymbol{y}$ at fixed times n_i.

2.5 σ-Algebras on the path space

Denote by α_k the time k coordinate partition in the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ of a Markov chain (X, P, θ), so that two paths $\boldsymbol{x}$ and $\boldsymbol{x'}$ belong to the same element of the partition α_k if and only if $x_k = x'_k$. One can see that the corresponding k-th coordinate σ-algebra is $\mathcal{A}_k$ and the corresponding quotient space is

$$(X^{\mathbb{Z}_+}, \mathbf{P}_\theta)/\alpha_k = (X, \theta_k), \quad \text{where } \ \theta_k = \theta P^k \tag{2.5.1}$$

By

$$\mathcal{A}_k^n = \bigvee_{i=k}^{n} \mathcal{A}_i \tag{2.5.2}$$

we denote the σ-algebra of the path space generated by the positions of the Markov chain at times $k \le i \le n$, where n is allowed to take the value $+\infty$.

The intersection

$$\mathcal{A}^\infty = \bigcap_{n} \mathcal{A}_n^\infty \tag{2.5.3}$$

of the decreasing sequence of σ-algebras $\mathcal{A}_n^\infty$ of the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ is called the **tail σ-algebra** corresponding to the partition α^∞ of a sample path space of the Markov chain (X, P, θ). Informally, a measurable subset $A \subset X^{\mathbb{Z}_+}$ belongs to $\mathcal{A}^\infty$ if and only if the fact that a trajectory $\boldsymbol{x} \in X^{\mathbb{Z}_+}$ belongs to A does not depend on any finite set of coordinates $x_0, ..., x_n$ and is determined by its "behavior at infinity" only. Denote by **tail** the quotient map

$$\mathbf{tail} : (X^{\mathbb{Z}_+}, \mathbf{P}_\theta) \to (X^{\mathbb{Z}_+}, \mathbf{P}_\theta)/\alpha^\infty. \tag{2.5.4}$$

The quotient space $(E, \varepsilon_\theta) = (X^{\mathbb{Z}_+}, \mathbf{P}_\theta)/\alpha^\infty$ is called the tail boundary of the chain (X, P, θ), and the measure $\varepsilon_\theta = \mathbf{tail}(\mathbf{P}_\theta)$ is the (tail) harmonic measure corresponding to the initial distribution θ.

Let T be the time shift on the path space $X^{\mathbb{Z}_+}$:

$$T : \{x_n\} \to \{x_{n+1}\}. \tag{2.5.5}$$

Denote by $\sim$ the orbit equivalence relation of the shift T on the path space $X^{\mathbb{Z}_+}$:

$$\boldsymbol{x} \sim \boldsymbol{x}' \Leftrightarrow \exists\, n, n' \geq 0 : T^n \boldsymbol{x} = T^{n'} \boldsymbol{x}' \tag{2.5.6}$$

A measurable subset A of the trajectory space $X^{\mathbb{Z}_+}$ is called shift-invariant ($\mathbf{P}_m$ mod 0) if it contains simultaneously with almost every trajectory $\boldsymbol{x}$ all trajectories $\boldsymbol{y}$ such that

$$\boldsymbol{x} \sim \boldsymbol{y},$$

i.e., is a union mod 0 of the equivalence classes of the relation $\sim$

Remark: According to another, equivalent definition, a set A is T-invariant if it coincides ($\mathbf{P}_m \mod 0$) with its T-preimage i.e.,

$$A = T^{-1}A.$$

Definition 2.5.1. *The σ-algebra $\mathcal{E}$ of ($\mathbf{P}_m \mod 0$) T-invariant sets is called the exit σ-algebra.*

Since for any trajectory $\boldsymbol{x} \in X^{\mathbb{Z}_+}$, the fact that $\boldsymbol{x}$ belongs to T-shift-invariant set A does not depend on first $n < \infty$ elements of $\boldsymbol{x}$, we have

$$\mathcal{E} \subseteq \mathcal{A}^\infty$$

Generally speaking, the σ-algebras $\mathcal{E}$ and $\mathcal{A}^\infty$ are different. The simplest example is provided by any periodic random walk, for instance, the simple random walk on $\mathbb{Z}_+$, where the exit σ-algebra $\mathcal{E}$ is always trivial, but the tail σ-algebra $\mathcal{A}^\infty$ is non-trivial (for parity considerations). Nonetheless, these σ-algebras always coincide $\mod 0$ with respect to a single-point initial distribution [19, Kaimanovich, Vershik].

2.6 The Poisson boundary

We assume that the state space X is embedded in a topological space $\mathcal{B} \supset X$, and assume that almost all trajectories of the Markov chain converge in the space $\mathcal{B}$, i.e.

$$\mathbf{P}_\theta(\{\boldsymbol{x_n} \in X^{\mathbb{Z}_+} \ : \ \boldsymbol{x_n} \to \boldsymbol{x_\infty}\}) = 1, \tag{2.6.1}$$

then the mapping

$$\mathbf{bnd} : X^{\mathbb{Z}_+} \to \mathcal{B} \tag{2.6.2}$$

has the following properties:

(i) **bnd** is measurable because pointwise limits of measurable maps are also measurable.

(ii) Equivalent trajectories in the sense of (2.5.6) are merged by **bnd**.

Let λ denote the image of the measure $\mathbf{P}_\theta$ under the map **bnd**, i.e.,

$$\lambda(A) = \mathbf{P}_\theta(\mathbf{bnd}^{-1}(A)), \quad A \subset \mathcal{B}$$

There exists a maximal covering space (Γ, ν) for all quotient spaces $(\mathcal{B}, \lambda)$, which satisfies properties (i) and (ii).

$$\begin{tikzcd}
\left(X^{\mathbb{Z}_+}, \mathbf{P}\right) \arrow[dr] \arrow[dd] & \\
& (\Gamma, \nu) \arrow[dl] \\
(\mathcal{B}, \lambda) &
\end{tikzcd}$$

Figure 2.2: The Poisson Boundary as maximal covering space for all quotient spaces.

This space (Γ, ν) completely describes the boundary behavior of random walk (X, P) and is called the Poisson boundary. In terms of the exit σ-algebra $\mathcal{E}$, the Poisson boundary is the quotient space associated to it by Rokhlin's theorem (2.1.1).

For the time shift $T : \{x_n\} \mapsto \{x_{n+1}\}$ in the path space $X^{\mathbb{Z}_+}$, we have

$$T\mathbf{P}_\theta = \mathbf{P}_{\theta P} \tag{2.6.3}$$

for an arbitrary initial distribution θ on X.

By definition of Γ as the quotient space with respect to the exit σ-algebra $\mathcal{E}$, for an arbitrary initial distribution θ we have $\mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(T\mathbf{P}_\theta)$, so that by (2.6.3)

$$\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(T\mathbf{P}_\theta) = \mathbf{bnd}(\mathbf{P}_{\theta P}) = \nu_{\theta P}. \tag{2.6.4}$$

We shall be interested in describing the Poisson boundary of the initial distribution δ_e, i.e., in describing the measure space (Γ, ν).

2.7 Boundaries of product Markov chains

Let (X, P) be a Markov chain. As defined in (2.5), let $\mathcal{A}_n^m$ ($n \geq 0$, m is allowed to take value $+\infty$) be coordinate σ-algebras and $\mathcal{A}^\infty$ be the tail σ-algebra in the path space $X^{\mathbb{Z}_+}$ (see (2.4.5)). For the conditional probability, with respect to coordinate σ-algebra $\mathcal{A}_n^\infty$, of the cylinder-set $C_0^k = C_{x_0,\ldots,x_k}^{t_0,\ldots,t_k}$ (see (2.4.7)) we have (due to the Martingale convergence theorem)

$$\mathbf{P}[C_0^k|\mathcal{A}_n^\infty] \longrightarrow \mathbf{P}[C_0^k|\mathcal{A}^\infty], \text{ as } n \to \infty. \tag{2.7.1}$$

Theorem 2.7.1. ([19], [40]) The tail σ-algebra $\mathcal{A}^\infty$ of a Markov chain (X, P) (with a single point initial distribution concentrated at x_0) is trivial if and only if

$$\frac{p_{n-k}(x_k, x_n)}{p_n(x_0, x_n)} \to 1, \ \forall\, k, \quad \text{as } n \to \infty, \tag{2.7.2}$$

for a.e. trajectory $\boldsymbol{x} \in X^{\mathbb{Z}_+}$.

Proof: Consider the conditional probability $\mathbf{P}(C_0^k|\mathcal{A}_n^\infty)$, which can be rewritten as follows (using the Markov property)

$$\begin{aligned}
\mathbf{P}[C_0^k|\mathcal{A}_n^\infty] &= \mathbf{P}[x_0, x_1, ..., x_k|x_n, x_{n+1}, x_{n+1}, ...] \\
&= \mathbf{P}[x_0, x_1, ..., x_k|x_n] \\
&= \frac{\mathbf{P}[x_0, x_1, ..., x_k, x_n]}{\mathbf{P}[x_n]} \\
&= \frac{p(x_0, x_1)p(x_1, x_2)\cdots p(x_{k-1}, x_k)p_{n-k}(x_k, x_n)}{p_n(x_0, x_n)} \\
&= \frac{\mathbf{P}[C_0^k]\cdot p_{n-k}(x_k, x_n)}{p_n(x_0, x_n)}.
\end{aligned}$$

Now suppose that $\mathcal{A}^\infty = \{\cdot\}$. Then $\mathbf{P}[C_0^k|\mathcal{A}^\infty] = \mathbf{P}[C_0^k]$ and we have (by (2.7.1))

$$\frac{\mathbf{P}[C_0^k]\cdot p_{n-k}(x_k, x_n)}{p_n(x_0, x_n)} \longrightarrow \mathbf{P}[C_0^k],$$

which implies (2.7.2).

In the other direction, presuming (2.7.2) we get

$$\mathbf{P}[C_0^k|\mathcal{A}_n^\infty] \longrightarrow \mathbf{P}[C_0^k],$$

which, using the limit (2.7.1), gives $\mathbf{P}[C_0^k|\mathcal{A}^\infty] = \mathbf{P}[C_0^k]$. Which in turn implies that all cylinder sets (which generate the entire space of σ-algebras on $X^{\mathbb{Z}_+}$) are independent of the tail σ-algebra $\mathcal{A}^\infty$, whence that $\mathcal{A}^\infty$ is trivial.

∎

Let (X, P) and (Y, Q) be two Markov chains (defined by Markov operators P and Q respectively) with single point initial distributions. Let π_x and π'_y be the families of transition probabilities on X and Y respectively. Define the product Markov chain $(X \times Y, P \times Q)$ by its one-step transition probabilities for $k \geq 0$

$$\hat{\pi}_{(x_k,y_k)}(x_{k+1}, y_{k+1}) = \pi_{x_k}(x_{k+1})\pi'_{y_k}(y_{k+1}) \tag{2.7.3}$$

or, equivalently

$$\hat{p}[(x_k, y_k), (x_{k+1}, y_{k+1})] = p(x_k, x_{k+1})p'(y_k, y_{k+1}) \tag{2.7.4}$$

Lemma 2.7.1. *If the tail σ-algebras of Markov chains (X, P) and (Y, Q) (with single point initial distributions) are trivial then the tail σ-algebra of the product Markov chain $(X \times Y, P \times Q)$ is also trivial.*

Proof: Consider the conditional probability, with respect to σ-algebra $\hat{\mathcal{A}}_n^\infty$ (which describes the behavior of Markov chain $(X \times Y, P \times Q)$ from time n, onwards), of a cylinder set $\hat{C}_0^k$. Using (2.7.1), we have

$$\mathbf{P}[\hat{C}_0^k|\hat{\mathcal{A}}_n^\infty] \longrightarrow \mathbf{P}[\hat{C}_0^k|\hat{\mathcal{A}}^\infty], \text{ as } n \to \infty. \tag{2.7.5}$$

At the same time

$$\mathbf{P}[\hat{C}_0^k|\hat{\mathcal{A}}_n^\infty] = \mathbf{P}[(x_0,y_0),(x_1,y_1),...,(x_k,y_k) \mid (x_n,y_n),(x_{n+1},y_{n+1}),...],$$

which by the Markov property is

$$\begin{aligned} &= \mathbf{P}[(x_0,y_0),(x_1,y_1),...,(x_k,y_k) \mid (x_n,y_n)] \\ &= \frac{\mathbf{P}[(x_0,y_0),(x_1,y_1),...,(x_k,y_k),(x_n,y_n)]}{\mathbf{P}[(x_n,y_n)]} \\ &= \frac{\hat{p}[(x_0,y_0),(x_1,y_1)]\cdots\hat{p}[(x_{k-1},y_{k-1}),(x_k,y_k)]\cdot\hat{p}_{n-k}[(x_k,y_k),(x_n,y_n)]}{\hat{p}_n[(x_0,y_0),(x_n,y_n)]}, \end{aligned}$$

which by (2.7.4) is

$$= \mathbf{P}[C_0^k|\mathcal{A}_n^\infty]\cdot\mathbf{P}[C_0'^k|\mathcal{A}_n'^\infty],$$

where $\mathcal{A}_n^\infty$ (resp. $\mathcal{A}_n'^\infty$) are the coordinate σ-algebras of Markov chain (X,P) (resp. (Y,Q)), and C_0^k (resp. $C_0'^k$) are projections of a cylinder set $\hat{C}_0^k$ to X (resp. Y). Comparing with the limit (2.7.5), we get

$$\mathbf{P}[\hat{C}_0^k|\hat{\mathcal{A}}_n^\infty] \longrightarrow \mathbf{P}[C_0^k|\mathcal{A}^\infty]\cdot\mathbf{P}[C_0'^k|\mathcal{A}'^\infty]$$

Assuming that $\mathcal{A}^\infty = \{\cdot\}$ and $\mathcal{A}'^\infty = \{\cdot\}$, we have (using (2.7.4))

$$\mathbf{P}[\hat{C}_0^k|\hat{\mathcal{A}}_n^\infty] \longrightarrow \mathbf{P}[C_0^k]\cdot\mathbf{P}[C_0'^k] = \mathbf{P}[\hat{C}_0^k],$$

which implies

$$\frac{\hat{p}_{n-k}[(x_k,y_k),(x_n,y_n)]}{\hat{p}_n[(x_0,y_0),(x_n,y_n)]} \longrightarrow 1, \text{ as } n\to\infty$$

Thus, the result follows from the Theorem 2.7.1.

∎

Let $((X \times Y), (P \times Q))$ be a product Markov chain, as defined above (2.7.3), with the tail boundary $\mathcal{T} = \mathcal{T}((X \times Y), (P \times Q))$. Let $\mathcal{T}_X$ (resp. $\mathcal{T}_Y$) be the tail boundary of the Markov chain (X, P) (resp. (Y, Q)). Since $\mathcal{T}$ has natural projections onto $\mathcal{T}_X$ and $\mathcal{T}_Y$ (given by considering only the first or the second coordinate), any tail event of the quotient Markov chain (X, P) (resp. (Y, Q)) is also a tail event of the product Markov chain $((X \times Y), (P \times Q))$, so that the product $\mathcal{T}_X \times \mathcal{T}_Y$ is (by default) only a quotient space of $\mathcal{T}$. We now prove that $\mathcal{T} \cong \mathcal{T}_X \times \mathcal{T}_Y$.

Theorem 2.7.2. [40] The product $\mathcal{T}_X \times \mathcal{T}_Y$ of tail-boundaries of Markov chains (X, P) and (Y, Q) with initial distributions concentrated at $x_0 \in X$ (resp. $y_0 \in Y$) coincides with the tail-boundary $\mathcal{T}$ of Markov chain on the product space $(X \times Y)$ defined by the Markov operator $(P \times Q)$ with initial distribution concentrated at $(x_0, y_0) \in (X \times Y)$.

$$\mathcal{T} \cong \mathcal{T}_X \times \mathcal{T}_Y \tag{2.7.6}$$

Proof: Consider conditional Markov chains (X, P^α) and (Y, Q^β) with respect to boundary points $\alpha \in \partial(X, P)$ and $\beta \in \partial(Y, Q)$ (excluding sets of measure zero).

A well known result (see for example [22]) states that the quotient $\overline{\partial(\Omega, \mathcal{L})}$ of the tail-boundary $\partial(\Omega, \mathcal{L})$ of Markov chain $(\Omega, \mathcal{L})$ coincides with the whole tail-boundary if and only if conditional Markov chains $(\Omega, \mathcal{L}^\alpha)$, for a.e. $\alpha \in \overline{\partial(\Omega, \mathcal{L})}$, have trivial tail σ-algebras.

Applying this result for our set up, since $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the tail boundaries, we have that both conditional chains (X, P^α) and (Y, Q^β) have trivial tail σ-algebras for almost all boundary points $\alpha \in \mathcal{T}_X,\ \beta \in \mathcal{T}_Y$. Then, by Lemma 2.7.1, the tail

σ-algebra of the conditional product Markov chain $((X \times Y), (P \times Q)^{(\alpha,\beta)})$ is trivial for a.e. boundary point $(\alpha, \beta) \in \mathcal{T}_X \times \mathcal{T}_Y$. Whence, applying the above mentioned result in the other direction, we obtain

$$\mathcal{T}_X \times \mathcal{T}_Y \cong \mathcal{T}((X \times Y), (P \times Q)) = \mathcal{T}.$$

∎

2.8 Random walks on discrete groups

Random walks on countable groups are a special case of Markov chains for which the state space X is a group G and the transition probabilities are adapted to the group structure.

Let G be a countable group with the identity element e. The pair (G, μ), where μ is a probability measure on G, will be called a group with measure. The support of the measure μ is

$$\mathbf{supp}(\mu) = \{g \in G : \mu(g) > 0\}. \tag{2.8.1}$$

The reflection of the measure μ will be defined

$$\check{\mu}(g) = \mu(g^{-1}), g \in G. \tag{2.8.2}$$

The measure μ is called:

nondegenerate - if the *semigroup* generated by its support is all of G, irreducible - if the *group* generated by its support is all of G,

finitely supported - if $\mathbf{supp}(\mu)$ is finite,

symmetric - if $\mu = \check{\mu}$.

Definition 2.8.1. *The right random walk* (G, μ) *on a group* G *determined by a probability measure* μ *is the time homogeneous Markov chain with the state space* G *and transition probabilities from* x *to* y

$$p(y|x) = p(x, y) = \mu(x^{-1}y), \;\; \pi_g = g\mu, \;\; x, y \in G,$$

which are invariant under the left action of G *on itself by translations.*

In other words, the transitions of the random walk consist of the right multiplication by a random element of G (increment of the random walk), sampled from the distribution μ

$$g \rightsquigarrow gh, \;\; g, h \in G, \quad h \sim \mu. \tag{2.8.3}$$

If the initial distribution of the random walk is concentrated on an element $g \in G$, then the associated measure $\mathbf{P}_g^\mu$ on the trajectory space is the image of the product measure $\mu^\infty = \mu \times \mu \times ...$ on the space of increments $\prod\limits_{i=1}^{\infty} G$ given by the map

$$(h_1, h_2, h_3, ...) \mapsto (g, gh_1, gh_1h_2, gh_1h_2h_3, ...) \tag{2.8.4}$$

If the initial distribution is simply δ_e, the corresponding probability measure $\mathbf{P}_e^\mu$ will be denoted $\mathbf{P}^\mu$. Then, (specific for the group case) considering the left group action on itself

$$g(g_0, g_1, g_2, ...) = (gg_0, gg_1, gg_2, ...),$$

we have the following formula

$$\mathbf{P}^{\mu}_{g} = g\mathbf{P}^{\mu} \tag{2.8.5}$$

The measure determined on $G^{\mathbb{Z}_+}$ by an arbitrary distribution θ (using (2.4.6) and (2.8.5)) is

$$\mathbf{P}^{\mu}_{\theta} = \sum_{g\in G} \theta(g)\, \mathbf{P}^{\mu}_{g} = \sum_{g\in G} \theta(g) g\mathbf{P}^{\mu} = \theta\mathbf{P}^{\mu}. \tag{2.8.6}$$

Define the convolution $\mu * \mu'$ of two measures on G to be the image of the product measure $\mu \times \mu'$ on $G \times G$ by the map

$$(x_1, x_2) \mapsto x_1x_2$$

from $G \times G$ to G. Denoting the n-fold convolution as μ^{*n} and considering δ_e as μ^{*0}, we can write the one dimensional distribution of the measure $\mathbf{P}^{\mu}_{\theta}$ at a time $n \geq 0$ as

$$\mathbf{c}^{n}(\mathbf{P}^{\mu}_{\theta}) = \theta * \mu^{*n}. \tag{2.8.7}$$

The Markov operator P^{μ} acts on functions on G (well-defined)

$$P^{\mu} f(g) = \sum_{x} f(gx)\mu(x) = \int f(y_1) d\mathbf{P}^{\mu}_{g}(\boldsymbol{y}), \tag{2.8.8}$$

which is the special case of (2.4.1) Which establishes the operator P^{μ}, acting on the space $\ell^{\infty}(G)$, as the *Markov operator* of the random walk (G, μ). By the properties of Markov operators, P^{μ} is positive and the constant function $\mathbf{1}$ is invariant under P^{μ}. In the sequel, we shall simply use P and $\mathbf{P}$ to denote P^{μ} and $\mathbf{P}^{\mu}$ correspondingly, when the group G and the measure μ are fixed.

2.9 Poisson boundary of random walks on groups

We shall now specialize the definitions and facts from Section 2.6 to the case of random walks on groups. By using (2.6.3) and the formulas

$$\theta P(y) = \sum_x \theta(x)p(x,y) = \sum_x \theta(x)\mu(x^{-1}y) = \theta * \mu(y)$$

$$\mathbf{P}_\theta(C^{i_0,i_1,\dots,i_n}_{g_0,g_1,\dots,g_n}) = \theta(g_0)\mu(g_0^{-1}g_1)\dots\mu(g_{n-1}^{-1}g_n)$$

we have

$$T\mathbf{P}_\theta = \mathbf{P}_{\theta P} = \mathbf{P}_{\theta*\mu}, \tag{2.9.1}$$

therefore if θ has full support, $\theta * \mu$ also has for an arbitrary initial distribution θ on G. Hence all measures $\mathbf{P}_\theta$ with $\mathbf{supp}(\theta) = G$ are quasi-invariant with respect to T. The counting measure m on G is stationary with respect to the Markov operator P, i.e., $mP = m$, consequently the σ-finite measure $\mathbf{P}_m$ is invariant with respect to shift T for a random walk on G.

Recall that $\sim$ is the orbit equivalence relation of the shift T on the path space $G^{\mathbb{Z}_+}$:

$$\boldsymbol{x} \sim \boldsymbol{x}' \Leftrightarrow \exists\, n, n' \geq 0 : T^n\boldsymbol{x} = T^{n'}\boldsymbol{x}'$$

We denote by $\mathcal{E}$ the exit σ-algebra of all measurable unions of equivalence classes in the space $(G^{\mathbb{Z}_+}, \mathbf{P}_m)$. Since $(G^{\mathbb{Z}_+}, \mathbf{P}_m)$ is a Lebesgue space, there is a (unique up to isomorphism) measurable space Γ (called the space of ergodic components) and a map

$$\mathbf{bnd} : G^{\mathbb{Z}_+} \to \Gamma$$

such that the exit σ-algebra $\mathcal{E}$ coincides ($\mathbf{P} - \mod 0$) with the σ-algebra of **bnd**-preimages of measurable subsets of Γ.

Definition 2.9.1. *The space of ergodic components Γ of the time shift T on the path space $(G^{\mathbb{Z}_+}, \mathbf{P}_m)$ is called the* ***Poisson boundary*** *of the random walk (G, μ).*

Definition 2.9.2. *The corresponding measurable partition ρ of the path space $(G^{\mathbb{Z}_+}, \mathbf{P}_m)$ into the* **bnd***-preimages of points from Γ is called the* ***Poisson partition*** *of $G^{\mathbb{Z}_+}$.*

Definition 2.9.3. *For an initial probability distribution θ on G the measure*

$$\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta)$$

is called the ***harmonic measure*** *determined by θ.*

Remark: The measure **bnd** $\mathbf{P}_m$ is a trivially infinite measure. However, one can still define the corresponding measure class as the class of the measure $\mathbf{P}_m$, called the harmonic measure class. In other words, $[\nu_m]$ is the class of all measures $\mathbf{bnd}\mathbf{P}_\theta$, where θ is the finite measure on G equivalent to m. Any harmonic measure is absolutely continuous with respect to the harmonic measure class but not necessarily belongs to it (cf. [23]).

By definition of Γ as the space of ergodic components of the shift T, for an arbitrary initial distribution θ we have $\mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(T\mathbf{P}_\theta)$, so that by (2.9.1)

$$\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(T\mathbf{P}_\theta) = \mathbf{bnd}(\mathbf{P}_{\theta P}) = \nu_{\theta P} = \nu_{\theta * \mu}. \tag{2.9.2}$$

The above observation is actually valid for an arbitrary Markov chain (see Section 2.6).

The coordinate-wise action of G on the path space commutes with the shift T, hence it projects to a canonical G-action on Γ (this is because the orbit equivalence relation $\sim$ is G-invariant), and

$$\nu_{g\theta} = g\nu_\theta \tag{2.9.3}$$

By G-invariance of the measure m, the harmonic measure type is quasi-invariant with respect to the action of G.

Denote by $\nu = \nu_e = \mathbf{bnd}(\mathbf{P})$ the harmonic measure of the group identity. Then, using the formula

$$\mathbf{P}_\theta = \sum_{g\in G} \theta(g)\mathbf{P}_g = \sum_{g\in G} \theta(g) g\mathbf{P} = \theta * \mathbf{P}, \tag{2.9.4}$$

for an arbitrary initial distribution θ we have

$$\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta) = \mathbf{bnd}(\theta\mathbf{P}) = \theta * \mathbf{bnd}(\mathbf{P}) = \theta * \nu. \tag{2.9.5}$$

Since $\nu_\theta = \nu_{\theta P}$ (2.9.2), the harmonic measure $\nu = \nu_e$ is μ-stationary, i.e.,

$$\nu = \mu * \nu = \sum_{g\in G} \mu(g) g\nu. \tag{2.9.6}$$

The formula (2.9.6) implies that $g\nu \prec \nu$ for all $g \in \mathrm{sgr}(\mu)$, the semigroup generated by $\mathrm{supp}(\mu)$. Therefore, if μ is non-degenerate, then the measure ν is quasi-invariant and belongs to the harmonic measure class $[\nu_m]$. However, this is not necessarily true under the weaker assumption that μ is irreducible (cf. [23]).

The Bernoulli shift on the space of increments (2.8.4) of the random walk determines the measure preserving ergodic transformation

$$(U\boldsymbol{x})_n = x_1^{-1}x_{n+1} \tag{2.9.7}$$

of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$. Since the paths $\boldsymbol{x}$ and $x_1(U\boldsymbol{x}) = T\boldsymbol{x}$ are $\sim$ - equivalent (in the sense of (2.5.6)) we have that for $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = \{x_n\} \in G^{\mathbb{Z}_+}$

$$\mathbf{bnd}(\boldsymbol{x}) = x_1\mathbf{bnd}(U\boldsymbol{x}). \tag{2.9.8}$$

2.10 Bounded harmonic functions and the Poisson formula.

Let $P = P^\mu$ be the Markov operator (see Definition 2.4.1 and (2.8.8)) of the random walk (G, μ). We shall call the harmonic functions of the operator P^μ μ-harmonic. Denote by $H^\infty(G, \mu)$ the Banach space of bounded μ-harmonic functions on G with the sup-norm. The formulas

$$F(\mathbf{bnd\ x}) = \lim_{n\to\infty} f(x_n), \quad f(g) = \langle F, g\nu\rangle\,;\ g \in G,\ f \in H^\infty(G, \mu) \tag{2.10.1}$$

state an isometric isomorphism between the spaces $H^\infty(G, \mu)$ with nondegenerate μ, and $\ell^\infty(\Gamma, \nu)$ (*V. Kaimanovich*, [22]).

Since $g\nu \prec \nu$ for all $g \in \operatorname{sgr}(\mu)$ (2.9.6), the Poisson formula can be rewritten using the **Poisson kernel** $\Pi(g,\gamma) = \frac{dg\nu}{d\nu}(\gamma)$ as

$$f(g) = \langle F, g\nu\rangle = \int F(\gamma)\Pi(g,\gamma)d\nu(\gamma), \tag{2.10.2}$$

or, denoting $\Pi(\cdot,\gamma) = \varphi_\gamma$

$$f = \int F(\gamma)\varphi_\gamma d\nu(\gamma) = \int \varphi_\gamma d\nu_f(\gamma), \tag{2.10.3}$$

where φ_γ are μ-harmonic functions on $\operatorname{sgr}(\mu)$ given by the Radon-Nikodym derivatives of the translates of the measure ν, and $\nu_f = F\nu$ is the **representing measure** of f.

Denote by $H_1^+(G,\mu)$ the convex set of all non-negative harmonic functions on $\operatorname{sgr}(\mu)$ normalized by the condition $f(\mathbf{1}_G) = 1$. Any function $f \in H_1^+(G,\mu)$ determines a new Markov chain (the **Doob transform**) on $\operatorname{sgr}(\mu)$ with the transition probabilities

$$p^f(x,y) = \mu(x^{-1}y)\frac{f(y)}{f(x)}.$$

For any cylinder subset (2.4.7) of the path space the Markov measure $\mathbf{P}^f$ on the space of sample paths of the Doob transform (with the initial distribution $\delta_{\mathbf{1}_G}$) is related with the measure $\mathbf{P}$ by the formula

$$\mathbf{P}^f(C_{\mathbf{1}_G,g_1,\dots,g_n}) = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})f(g_n), \tag{2.10.4}$$

i.e., the map (2.10.4) is a convex embedding of $H_1^+(G,\mu)$ into the space of Markov measures on $G^{\mathbb{Z}_+}$.

If A is a measurable subset of the Poisson boundary with $\nu(A) > 0$, then by the Markov property for any cylinder set $C_{\mathbf{1}_G,g_1,\dots,g_n}$:

$$\mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n} \cap \mathbf{bnd}^{-1}A) = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})\mathbf{P}_{g_n}(\mathbf{bnd}^{-1}A) = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})g_n\nu(A),$$

and

$$\mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n} \mid \mathbf{bnd}^{-1}A) = \frac{\mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})g_n\nu(A)}{\mathbf{P}(\mathbf{bnd}^{-1}A)} = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})\frac{g_n\nu(A)}{\nu(A)},$$

Hence, the conditional measure $\mathbf{P}^A(\cdot) = \mathbf{P}(\cdot \mid \mathbf{bnd}^{-1}A)$ is the Doob transform of the measure $\mathbf{P}$ determined by the normalized harmonic function $\varphi_A(x) = \frac{x\nu(A)}{\nu(A)}$. We also have

$$\varphi_A = \frac{1}{\nu(A)}\int_A \varphi_\gamma d\nu(\gamma), \quad \varphi_\gamma(x) = \frac{dx\nu}{d\nu}(\gamma),$$

cf. (2.10.3). By the convexity of the Doob transform:

$$\mathbf{P}^A = \frac{1}{\nu(A)}\int_A \mathbf{P}^\gamma d\nu(\gamma),$$

where $\mathbf{P}^\gamma$ are Doob transforms determined by the functions φ_γ. The above consideration yields the following theorem (cf. [22, formula (6.4)]).

Theorem 2.10.1. The measures

$$\mathbf{P}^\gamma(C_{\mathbf{1}_G,g_1,\dots,g_n}) = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n} \mid \gamma) = \mathbf{P}(C_{\mathbf{1}_G,g_1,\dots,g_n})\frac{dg_n\nu}{d\nu}(\gamma)$$

corresponding to the Markov operators P_γ on $\mathrm{sgr}(\mu)$ with the transition probabilities

$$p_\gamma(x,y) = \mu(x^{-1}y)\frac{dy\nu}{dx\nu}(\gamma)$$

are the canonical system[1] of conditional measures of the measure $\mathbf{P}$ with respect to the Poisson boundary.

Corollary 2.10.1. *The Radon-Nikodym derivatives* $\varphi_\gamma(x) = \frac{dx\nu}{d\nu}(\gamma)$, $x \in sgr(\mu)$, $\gamma \in \Gamma$ *separate points of the space* Γ, ν.

Proof: Since the conditional measures of the path space corresponding to different points $\gamma \in \Gamma$ are pairwise singular different points $\gamma \in \Gamma$ determine different functions φ_γ. ∎

Corollary 2.10.2. *The harmonic functions* $\varphi_\gamma(x) = \frac{dx\nu}{d\nu}(\gamma)$ *are a.e. minimal, i.e., cannot be decomposed into a non-trivial linear combinations of positive harmonic functions.*

Proof: The measures $\mathbf{P}^\gamma = \mathbf{P}(\cdot \mid \gamma)$ are conditional measures on ergodic components of the time shift, so that they are ergodic themselves. By convexity of the Doob transform (2.10.4) it implies minimality of φ_γ. ∎

---

[1] Refer to the section 2.2 for description and properties of the canonical systems of measures.

2.11 Examples of random walks with trivial Poisson boundary

We begin with an almost trivial example. Let $G = \mathbb{Z}/2\mathbb{Z}, \ \ \mu = \delta_1$. Then we have two sample paths, $(0, 1, 0, 1, ...)$ and $(1, 0, 1, 0, ...)$. These correspond to the two points on the Tail boundary. However, under the asymptotic equivalence relation these sample paths are identified, so the Poisson boundary is reduced to a single point.

Another classical example is a random walk (G, μ) on abelian group G. Observe the following property of the center $Z(G)$ of a group G regarding its action on the boundary ∂G of random walk (G, μ).

Proposition 2.11.1. [10] *If $\mu(z) > 0$ for all $z \in Z(G)$, then the center $Z(G)$ of the group G acts trivially on the boundary of random walk (G, μ) i.e.,*

$$\forall z \in Z(G), z\gamma = \gamma, \ \ \textit{for almost all} \ \ \gamma \in \Gamma(G, \mu). \tag{2.11.1}$$

Proof: Assume that for $z \in Z(G)$, $\mu(z) > 0$. Now consider a harmonic function f. We have

$$f(x_n) \to \hat{f}(x_\infty).$$

Since

$$x_{n+1} = x_n h_{n+1}, \ \ h_n \sim \mu, \ \ \text{where} \ \ h_n \ \ \text{are i.i.d.},$$

we must have that $h_n = z$ infinitely often. Then, there exists a subsequence $f(x_{n_k})$, such that

$$f(x_{n_k} z) \to \hat{f}(x_\infty),$$

or, equivalently

$$f(zx_{n_k}) \to \hat{f}(x_\infty).$$

Consider y_n such that $y_1 = z, y_2 = zx_1, ...$ Then

$$f(zx_{n_k}) \to \hat{f}(y_\infty).$$

Hence,

$$\hat{f}(y_\infty) = \hat{f}(x_\infty).$$

Denoting $\gamma = x_\infty$ and $z\gamma = y_\infty$, we obtain

$$\hat{f}\gamma = \hat{f}(z\gamma) \text{ or } \hat{f} = z\hat{f}$$

i.e., the element $z \in Z(G)$ acts trivially on the boundary of random walk (G, μ).

∎

2.12 Examples of random walk with non-trivial Poisson boundary

A classical example of random walk with non-trivial Poisson boundary is given by the random walk (F, μ) on the free group F with two free generators a and b and

$$\mu = \frac{1}{4}(a + b + a^{-1} + b^{-1}). \tag{2.12.1}$$

Definition 2.12.1. (Cayley graph) *Let G be a group and $\mathcal{A} = \{a_i : i \in I\}$ a symmetric finite generating set for G. The Cayley graph of G constructed with respect to $\mathcal{A}$ is the graph Δ whose vertex set is identified with G and such that two elements g and g' are connected with an edge directed from g to g' and labeled with the generator $a_i \in \mathcal{A}$ if and only if $ga_i = g'$.*

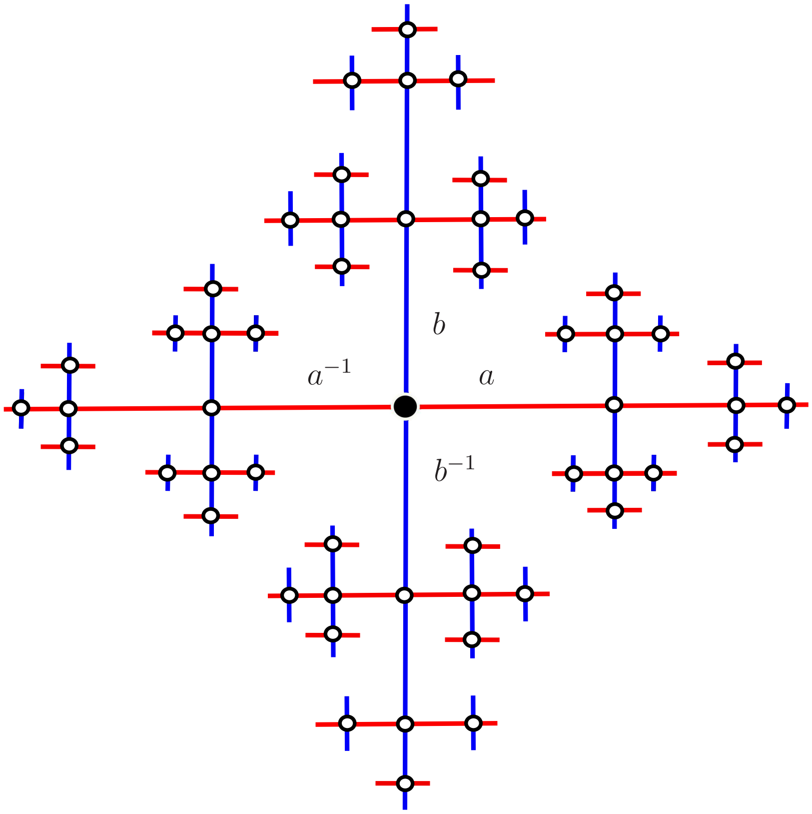

Figure 2.3: The Cayley Graph of the free group with two free generators a, b

Figure 2.3 illustrates the Cayley graph of the free group with two free generators. Since the random walk (F, μ) (with μ defined as in (2.12.1)) is transient, almost any of its trajectories $x_1, x_2, ..., x_n, ...$ visits the identity of the group only a finite number of times with probability 1. Thus, there exists $N_1 \in \mathbb{N}$ such that the irreducible form of all x_n with $n > N_1$ starts with the generator a_{i_1}. After the time N_1, any trajectory visits a_{i_1} only on a finite number of occurrences. Hence, after the time $N_2 \in \mathbb{N}$, the second coordinate a_{i_2} is fixed. Continuing this process, one obtains an infinite word $\gamma = a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ which one naturally calls a **boundary point** and concludes that the sample path (x_n) converges to γ. This shows the non-triviality of the Poisson boundary.

Moreover, for the random walk on the monoid (F_+, μ), (i.e., $\mu = \frac{1}{2}(a+b)$, see Figure 2.4) two paths from the path space $(F_+^{\mathbb{Z}_+}, \mathbf{P})$ are $\approx$ - equivalent if and only if they coincide. Thus, the Poisson partition ρ coincides with the point partition of the path space, and the Poisson boundary (Γ, ν) is the set of infinite words in the alphabet $\{a, b\}$.

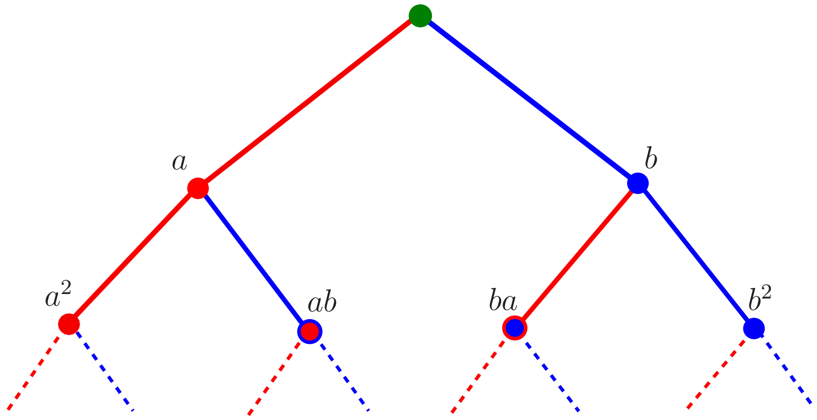

Figure 2.4: The Cayley Graph of the free monoid F_+ with the generating set $\{a, b\}$

Chapter 3

Entropy theory of random walks

3.1 Entropy

In many cases, the triviality or non-triviality of the Poisson boundary can be established using the methods developed in ergodic theory. We shall briefly review some of these methods in this chapter. We begin by reminding the Fekete subadditive lemma and the Kingman subadditive ergodic theorem. Fekete's lemma is formulated for a non-random setup.

Lemma 3.1.1. (Fekete) *Let* f_n *be a subadditive sequence on the set of positive integers. Then the* $\lim\limits_{n\to\infty} \frac{f_n}{n}$ *exists and is equal to* $\inf\limits_n \frac{f_n}{n}$ *for* $n \geq 1$*.*

The Kingman theorem is a simultaneous generalization of the Birkhoff ergodic theorem and Fekete's lemma.

Theorem 3.1.1. (Kingman) Let U be a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $f_n \geq 0$ be a sequence of L^1 functions such that a.e.

$$f_{n+m}(x) \leq f_n(x) + f_m(U^n x).$$

Then the limit

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = f(x) \geq 0 \tag{3.1.1}$$

exists for **P**- a.e. x and in L^1. The limit function f is U-invariant, therefore f is constant if U is ergodic.

Let $p = \{p_1, p_2, ...\}$ be a discrete probability distribution. Its entropy is defined as [44]

$$H(p) = -\sum_i p_i \log p_i. \tag{3.1.2}$$

Three important properties of H are

$$H(p \otimes q) = H(p) + H(q), \text{ for } p \text{ and } q; \tag{3.1.3}$$

$$H(q) \leq H(p), \text{ for any mapping } p \to q; \tag{3.1.4}$$

$$\text{If } |\mathrm{supp}(p)| = N, \text{ then } H(p) \leq \log N. \tag{3.1.5}$$

Let us now return to random walks on groups. As before, we denote by $H(\mu)$ the entropy of the step distribution μ on a group G. Consider the random walk (G, μ) with $H(\mu) < \infty$. Denote the entropy of n-fold convolution of μ as $H(\mu^{*n})$. Then for μ^{*n} and μ^{*m}, by property (3.1.3), we have

$$H(\mu^{*n} \otimes \mu^{*m}) = H(\mu^{*n}) + H(\mu^{*m}). \tag{3.1.6}$$

On the other hand we can obtain the $(n+m)$- fold convolution $\mu^{*(n+m)}$ as a result of the group operation from the product $\mu^{*n} \otimes \mu^{*m}$ by the map $(g, g') \mapsto gg'$. Then, by the property (3.1.4), we have

$$H(\mu^{*(n+m)}) \leq H(\mu^{*n}) + H(\mu^{*m}), \tag{3.1.7}$$

which means that the sequence $h_n = H(\mu^{*n})$ satisfies the subadditivity condition. Hence, by Lemma 3.1.1 the limit

$$h\,(G, \mu) = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}.$$

exists. It is called the asymptotic entropy of the random walk (G, μ) ([2], [8], [19]), and plays a crucial role in understanding the asymptotic properties of the random walk (G, μ).

Theorem 3.1.2. ([50], [8], [19]) The asymptotic entropy $h(G, \mu)$ vanishes if and only if the tail σ-algebra, with respect to a single point initial distribution, of the random walk (G, μ) is trivial.

This result means that the "remote future" of the random walk (G, μ) is trivial if and only if the amount of information about the first increment h_1 in the random product $g_n = h_1 h_2 \cdots h_n$ asymptotically vanishes.

Definition 3.1.2. [50] *A probability measure* Λ *on* $G^{\mathbb{Z}_+}$ *has asymptotic entropy* $\mathbf{h}(\Lambda)$ *if it has the following Shannon-Breiman-McMillan type equidistribution property:*

$$-\frac{1}{n} \log \Lambda(C^n_{x_n}) \to \mathbf{h}(\Lambda)$$

for Λ*-a.e.* $\mathbf{x} = \{x_n\} \in G^{\mathbb{Z}_+}$ *and in the space* $L^1(\Lambda)$*.*

If λ_n is the one-dimensional distribution of the measure Λ at time n, then

$$-\int \log \Lambda(C^n_{x_n}) d\Lambda(\mathbf{x}) = -\sum_{x_n} \log[\lambda_n(x_n)]\lambda_n(x_n) = H(\lambda_n), \tag{3.1.8}$$

so that L^1-convergence in the above definition implies that

$$\frac{H(\lambda_n)}{n} \to \mathbf{h}(\Lambda).$$

Consider the functions $f_n(\boldsymbol{g}) = -\log \mu^{*n}(\boldsymbol{g}_n)$ on the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$. By the formulas (3.1.7) and (3.1.8) these functions are integrable. Also

$$\mu^{*(n+m)}(g_{n+m}) = p_{n+m}(e, g_{n+m}) \geq p_n(e, g_n)p_m(g_n, g_{n+m}) = \mu^{*n}(g_n)\mu^{*m}(g_n^{-1}g_{n+m}).$$

Hence we have the subadditivity property

$$f_{n+m}(\boldsymbol{g}) \leq f_n(\boldsymbol{g}) + f_m(U^n \boldsymbol{g}),$$

where U is the measure-preserving transformation of $(G^{\mathbb{Z}_+}, \mathbf{P})$ introduced in (2.9.7). Then, by the Kingman subadditive theorem 3.1.1 we have

Theorem 3.1.3. ([8], [19]) Suppose the entropy $H(\mu)$ of a probability measure μ on a countable group G is finite. Then the asymptotic entropy $\mathbf{h}(\mathbf{P})$ of the measure $\mathbf{P}$ exists, and

$$\mathbf{h}(\mathbf{P}) = h(G, \mu).$$

Corollary 3.1.3. [19] *The Poisson boundary is trivial if and only if there exist* $\varepsilon > 0$

and a sequence of sets A_n *such that* $\mu_n(A_n) > \varepsilon$ *and* $\log |A_n| = o(n)$.

3.2 The rate of escape and the entropy bounds.

For a finitely generated group G with a measure μ on it, the first moment of μ is

$$|\mu| = \sum_g d(e, g)\mu(g), \tag{3.2.1}$$

where the length $d(e, g)$ is defined as the distance to the identity element.

Remark 3.2.1. ([22, Lemma 12.2], [9]) If G is a finitely generated group with probability measure μ such that $|\mu| < \infty$, then the entropy $H(\mu)$ is also finite.

For any trajectory $\boldsymbol{g}_{n+m}$, by the triangle inequality for $d(e, g_n) = |\boldsymbol{g}_n|$, we have

$$|\boldsymbol{g}_{n+m}| \leq |\boldsymbol{g}_n| + |(T^n\boldsymbol{g})_m|, \tag{3.2.2}$$

where T is the shift defined in (2.5.5) and therefore $(T^n\boldsymbol{g})_m = \boldsymbol{g}_n^{-1}\boldsymbol{g}_{n+m}$. On the space $(G^{\mathbb{Z}_+}, \mathbf{P})$, we define the sequence of integrable functions $\{f_n\}$ as $f_n(\boldsymbol{g}) = |\boldsymbol{g}_n|$. Then we have

$$f_{n+m}(\boldsymbol{g}) \leq f_n(\boldsymbol{g}) + f_m(T^n\boldsymbol{g}), \tag{3.2.3}$$

and by the Kingman subadditive theorem 3.1.1, we obtain for almost every trajectory $\boldsymbol{g} \in (G^{\mathbb{Z}_+}, \mathbf{P})$

$$\frac{|\boldsymbol{g}_n|}{n} \to \ell, \tag{3.2.4}$$

where the constant $\ell \geq 0$ is called the rate of escape to infinity.

The n-fold convolution μ^{*n} of measure μ can be obtained by integrating $|\boldsymbol{g}_n|$ with respect to all possible distances to the identity element. If μ has a finite first moment, there exists a non-negative constant ℓ, such that

$$\frac{|\mu^{*n}|}{n} \to \ell$$

The constant ℓ is the same as in (3.2.4).

For any $g \in G$, denote its distance to the identity element $e \in G$ by $d(e, g)$. Recall that for balls $B_n = \{g \in G \ : \ d(e, g) \leq n\}$, we have

$$|B_{n+m}| \leq |B_n| \times |B_m|$$

and, by the Lemma (3.1.1), there exists a value v such that

$$v = \lim_{n\to\infty} \frac{\log |B_n|}{n}.$$

This value is called the exponential growth rate of G.

Theorem 3.1.3 implies the following important result (sometimes called the "fundamental inequality"). We include its proof as we will use a similar argument in the proof of our main result.

Theorem 3.2.1. [15] Let G be a finitely generated group with the probability measure μ such that $|\mu| < \infty$. Then the following inequality holds

$$h(G, \mu) \leq \ell v$$

Proof: At time n almost all trajectories $\boldsymbol{g}_n$ will belong to the ball $B_{n(\ell+\varepsilon)}$ with the radius $n(\ell+\varepsilon)$ (where $\varepsilon > 0$ is arbitrarily small constant). Then

$$\mu^{*n}\{\boldsymbol{g}_n : |\boldsymbol{g}_n| \leq n(\ell+\varepsilon)\} \geq 1-\varepsilon. \tag{3.2.5}$$

Denote $h = h(G,\mu)$. By Shannon's theorem, for trajectories $\boldsymbol{g}_n$, there exists an integer N such that

$$e^{-n(h-\varepsilon)} \leq \mu^{*n}(\boldsymbol{g}_n) \leq e^{-n(h+\varepsilon)}, \;\; \forall n > N \tag{3.2.6}$$

Combining, (3.2.5) and (3.2.6), we can write that there exists an integer N such that

$$\mu^{*n}(\{\boldsymbol{g}_n : -\log(\mu^{*n}(\boldsymbol{g}_n)) \geq h-\varepsilon, \;\; |\boldsymbol{g}_n| \leq n(\ell+\varepsilon)\}) \geq 1-\varepsilon, \;\; \forall n > N. \tag{3.2.7}$$

Therefore

$$|B_{n(\ell+\varepsilon)}| \times e^{-n(h-\varepsilon)} \geq 1-\varepsilon. \tag{3.2.8}$$

Taking logarithms and dividing by n:

$$\frac{\log|B_{n(\ell+\varepsilon)}|}{n} - (h-\varepsilon) \geq \frac{\log(1-\varepsilon)}{n}. \tag{3.2.9}$$

Now let $\varepsilon \to 0$ as $n \to \infty$ to obtain

$$\lim_{n\to\infty} \frac{\log|B_{n\ell}|}{n} \geq h. \tag{3.2.10}$$

Using the property

$$|B_{n\ell}| \leq |B_n|^{\ell},$$

We write:

$$\begin{aligned}\lim_{n\to\infty}\frac{\log|B(e,n\ell)|}{n} \le \lim_{n\to\infty}\frac{\log|B_n|^\ell}{n} &= \lim_{n\to\infty}\ell\cdot\frac{\log|B_n|}{n}\\ &= \ell\cdot\lim_{n\to\infty}\frac{\log|B_n|}{n} = \ell v.\end{aligned}$$

Hence the claim. ∎

Corollaries: The above theorem implies triviality of Poisson Boundary for groups with sub-exponential growth ($v = 0$) and for groups with rate of escape to infinity equal to zero ($\ell = 0$).

3.3 Entropy of measurable partitions of Lebesgue spaces

Along with the entropy of random walk, we shall use the entropy properties of measurable partitions of Lebesgue spaces (see [43], [44]) and apply this theory to the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$.

If (X, m) is a Lebesgue probability space, and $\xi = \{X_i\}$ is a countable measurable partition of X, then we define $H(\xi) = H_m(\xi)$ as the entropy of the probability distribution $p_i = m(X_i)$. Or, equivalently

$$H(\xi) = H(m_\xi) = -\int \log m(\xi(x))dm(x), \tag{3.3.1}$$

where $\xi(x)$ denotes the element (of the partition ξ) that contains x.

Suppose we have another measurable partition η of X. Denote the projection from the space (X, m) onto its quotient by η as

$$x \mapsto \eta(x) \in X. \tag{3.3.2}$$

Observe that (3.3.2) identifies points of the quotient space with the corresponding elements of the partition η. Denote the conditional measures of this projection by $m^{\eta(x)}$, and the entropy of the trace of ξ on $\eta(x)$ with respect to the measure $m^{\eta(x)}$–by $H_{\eta(x)}(\xi)$. Then, using (3.3.1), we can define the conditional entropy of ξ with respect to η as follows

$$H(\xi \mid \eta) = -\int H_{\eta(x)}(\xi) dm(x), = -\int \log m^{\eta(x)}(\xi(x)) dm(x). \tag{3.3.3}$$

Proposition 3.3.1. [44] *Let ξ and η be measurable partitions of a Lebesgue space (X, m). If ξ is countable with $H(\xi) < \infty$, then*

(i) $0 \leq H(\xi|\eta) \leq H(\xi)$, and $H(\xi|\eta) = 0$ (resp., $H(\xi|\eta) = H(\xi)$) if and only if η is a refinement of ξ (resp., η and ξ are independent);

(ii) if η' is a refinement of η, then $H(\xi|\eta') \leq H(\xi|\eta)$, and the equality holds if and only if $m^{\eta'(x)}(\xi(x)) = m^{\eta(x)}(\xi(x))$ for m-a.e. $x \in X$;

(iii) if η is the limit of a monotonously decreasing sequence of measurable partitions[1] $\{\eta_n\}$, then $H(\xi|\eta_n) \nearrow H(\xi|\eta)$.

---

[1] A sequence of measurable partitions $\{\eta_n\}$ is monotonously decreasing if $\eta_1 \succcurlyeq \eta_2 \succcurlyeq \eta_3 \succcurlyeq ...$, (i.e., if η_i is a refinement of η_{i+1} for $i = 1, 2, 3, ...$).

Let $(G^{\mathbb{Z}_+}, \mathbf{P})$ be a path space with a probability measure on it. Denote by α_1^k the partition of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ determined by the positions of the random walk at times $1, 2, ..., k$, i.e., we say that two sample paths $\boldsymbol{x} = (x_i)$ and $\boldsymbol{x'} = (x'_i)$ belong to the same class of α_1^k if and only if $x_i = x'_i$ for all $i = 1, 2, ..., k$. The quotient of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ determined by the partition α_1^k is the space of initial segments (up to time k) of sample paths, and it is isomorphic to the space of the first k increments of the random walk (see Section 2.5). Denote $\alpha = \alpha_1^1$. Since the increments are independent and μ-distributed, using (3.1.3), we obtain

$$H(\alpha_1^k) = kH(\mu) = kH(\alpha).$$

As mentioned before, the asymptotic entropy of the random walk (G, μ) can be used to determine if the Poisson boundary of this random walk is trivial. For this purpose one has to apply the above properties of the conditional entropies of partitions to the coordinate partitions of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ of the random walk (G, μ).

Lemma 3.3.2. [22] *The conditional entropy of a partition* α_1^k, $k \geq 1$ *with respect to the Poisson partition* ρ *is*

$$H(\alpha_1^k|\rho) = kH(\alpha|\rho) = k[H(\mu) - h(G, \mu)].$$

The above lemma can be used to prove Theorem 3.1.2.

Proof of Theorem 3.1.2: Suppose that the asymptotic entropy $h(G, \mu) = 0$. Then by Lemma 3.3.2 and Proposition 3.3.1, the Poisson partition ρ is independent of all coordinate partitions α_1^k, which by the Martingale convergence theorem (2.3.1) is only possible if ρ is trivial. In the other direction, if the Poisson partition ρ is trivial, we have

$$H(\alpha_1^k|\rho) = H(\alpha_1^k) = kH(\alpha) = kH(\mu),$$

so applying Lemma 3.3.2 we conclude that $h(G,\mu) = 0$. Hence, if the entropy $H(\mu)$ of the measure μ is finite, then the Poisson boundary (Γ,ν) of the random walk (G,μ) is trivial $\mathbf{P} \mod 0$ if and only if $h\,(G,\mu) = 0$.

3.4 Quotients of the Poisson boundary

Definition 3.4.1. [22] *The quotient (Γ_ξ, ν_ξ) of the Poisson boundary (Γ,ν) with respect to a certain G-invariant measurable partition ξ is called a μ-boundary*

In other words, a μ-boundary is an equivariant quotient of the Poisson boundary [13]. The Poisson boundary itself is the maximal μ-boundary. We shall denote by $\mathbf{bnd}_\xi$ the projection

$$\mathbf{bnd}_\xi : (G^{\mathbb{Z}_+}, \mathbf{P}) \to (\Gamma,\nu) \to (\Gamma_\xi, \nu_\xi),$$

and by ρ_ξ the corresponding partition of the path space. The measure ν_ξ and almost all conditional measures on the fibers of the projection $\Gamma \to \Gamma_\xi$ are purely non-atomic (unless $\Gamma_\xi = \{\cdot\}$ or $\Gamma_\xi = \Gamma$, respectively) [20].

Any G-space which is a $\sim$-measurable image of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ is a μ-boundary (recall that $\sim$ is the orbit equivalence relation (2.5.6) of the time shift T (2.5.5)). If π is a T-invariant equivariant measurable map from the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ to G-space $\mathcal{B}$, then $(\mathcal{B}, \pi(\mathbf{P}))$ is a μ-boundary. For example, such a map arises in the situation when G is embedded into a topological G-space X, and $\mathbf{P}$-a.e. sample path $\mathbf{x} = (x_n)$ converges to a limit $x_\infty = \pi(\mathbf{x}) \in X$ (cf. section 2.6). In this case the space X endowed with the resulting harmonic measure is a μ-boundary.

Another example of a μ-boundary arises from taking the quotient of the group G by a normal subgroup $H < G$. Denote by μ' the image of the measure μ on the quotient group $G' = G/H$. Then the Poisson boundary (Γ', ν') of the random walk (G', μ') is the space of ergodic components of the Poisson boundary (Γ, ν) of the random walk (G, μ) with respect to the action of H [20], which is an equivariant image of the Poisson boundary. Therefore, (Γ', ν') is a μ-boundary.

In terms of μ-boundaries, the problem of describing the Poisson boundary of (G, μ) consists of the two following parts.

(i) To find a μ-boundary $(\mathcal{B}, \lambda)$ which by definition is just a quotient (Γ_ξ, ν_ξ) of the Poisson boundary with respect to certain G-invariant partition ξ.

(ii) To show that this μ-boundary is maximal, i.e., that ξ is in fact the point partition of the Poisson boundary.

A particular case of the problem of describing the Poisson boundary is proving its triviality in which case the role of the "candidate boundary" is played by the singleton.

A compactification of the group G is called μ-**maximal** if the sample paths of the random walk (G, μ) converge a.e. in this compactification (so that it is a μ-boundary), and this μ-boundary is isomorphic to the Poisson boundary of (G, μ).

Conditional measures

We have already identified the conditional random walks determined by the points of the Poisson boundary. This identification can be extended to μ-boundaries as well Let (Γ_ξ, ν_ξ) be a μ-boundary. Then for ν_ξ-a.e. $\gamma_\xi \in \Gamma_\xi$

$$\frac{dg\nu_\xi}{d\nu_\xi}(\gamma_\xi) = \int \frac{dg\nu}{d\nu}(\gamma) d\nu(\gamma|\gamma_\xi), \tag{3.4.1}$$

where $\nu(\cdot|\gamma_\xi)$ are the conditional measures of the measure ν on the fibers of the projection $\Gamma \to \Gamma_\xi,\ \gamma \to \gamma_\xi$. Then Theorem 2.10.1 and the convexity of the Doob transform (2.10.4) imply.

Theorem 3.4.1. [22] The conditional measures of the measure $\mathbf{P}$ with respect to the μ-boundary (Γ_ξ, ν_ξ) are

$$\mathbf{P}^{\gamma_\xi}(C_{1_G, g_1, \ldots, g_n}) = \mathbf{P}(C_{1_G, g_1, \ldots, g_n}|\gamma_\xi) \frac{dg_n\nu_\xi}{d\nu_\xi}(\gamma_\xi),\ \gamma_\xi \in \Gamma_\xi$$

and they correspond to the Markov operators P^{γ_ξ} on $\mathrm{sgr}(\mu)$ with transition probabilities

$$p^{\gamma_\xi}(x, y) = \mu(x^{-1}y) \frac{dy\nu_\xi}{dx\nu_\xi}(\gamma_\xi).$$

3.5 Entropy of conditional walks and maximality of μ-boundary

Let ξ be a G-invariant partition of the Poisson boundary, and (Γ_ξ, ν_ξ) - the corresponding μ-boundary.

Lemma 3.5.1. [22] *For any* $k \geq 1$

$$H(\alpha_1^k|\rho_\xi) = kH(\alpha|\rho_\xi) = k\left[H(\mu) - \int \log\left[\frac{dx_1\nu_\xi}{d\nu_\xi}(\boldsymbol{bnd}_\xi\mathbf{x})\right]d\mathbf{P}(\mathbf{x})\right].$$

It follows from the Lemma 3.5.1 that the integral

$$\mathbf{E}(\Gamma_\xi, \nu_\xi) = \int \log\left[\frac{dx_1\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\mathbf{x})\right]d\mathbf{P}(\mathbf{x}). \tag{3.5.1}$$

is finite. For two probability measures $\lambda \prec \lambda'$ on a same space Ω, this integral can be expressed in terms of Kullback-Leibler divergence [27] of λ from λ'

$$\mathbf{I}(\lambda|\lambda') = \int \log\left[\frac{d\lambda'}{d\lambda}(x)\right]d\lambda'(x). \tag{3.5.2}$$

Using the change of variables $\mathbf{x} \mapsto (g, \mathbf{x}'),\ g = x_1, \mathbf{x}' = U\mathbf{x}$ we get from (3.5.1)

$$\begin{aligned}\mathbf{E}(\Gamma_\xi, \nu_\xi) &= \sum_g \mu(g)\int \log\left[\frac{dg\nu_\xi}{d\nu_\xi}(g\mathbf{bnd}_\xi\mathbf{x}')\right]d\mathbf{P}(\mathbf{x}')\\ &= \sum_g \mu(g)\int \log\left[\frac{dg\nu_\xi}{d\nu_\xi}(g\gamma_\xi)\right]d\nu_\xi(\gamma_\xi)\\ &= \sum_g \mu(g)\mathbf{I}(g^{-1}\nu_\xi|\nu_\xi)\\ &= \sum_g \mu(g)\mathbf{I}(\nu_\xi|g\nu_\xi).\end{aligned} \tag{3.5.3}$$

Hence, by Lemma 3.5.1

$$H(\alpha|\rho_\xi) = H(\mu) - \mathbf{E}(\Gamma_\xi, \nu_\xi) = H(\mu) - \sum_g \mu(g)\mathbf{I}(\nu_\xi|g\nu_\xi). \tag{3.5.4}$$

Comparing (3.5.4) with Lemma 3.3.2 we get

$$h(G,\mu) = \mathbf{E}(\Gamma,\nu) = \sum_g \mu(g)\mathbf{I}(\nu|g\nu). \tag{3.5.5}$$

Thus, the entropy $h(G,\mu)$ coincides with the average Kullback-Leibler divergence from the harmonic measure ν on the Poisson boundary to its translates [19].

Theorem 3.5.1. [22] Let $\xi \preccurlyeq \xi'$ be two G-invariant measurable partitions of the Poisson boundary (Γ,ν). Then $H(\alpha|\rho_\xi) \geq H(\alpha|\rho_{\xi'})$, and equality holds if and only if $\xi = \xi'$.

Consider the case when ξ' is the point partition of the Poisson boundary. Then, using formulas (3.5.4), (3.5.5) and Theorem 3.5.1 we get

Theorem 3.5.2. [22] A μ-boundary (Γ_ξ,ν_ξ) coincides with the Poisson boundary if and only if $\mathbf{E}(\Gamma_\xi,\nu_\xi) = h(G,\mu)$.

In view of Definition 3.1.2 and Theorem 3.1.3 we have

Theorem 3.5.3. [22] Let ξ be a measurable G-invariant partition of the Poisson boundary (Γ,ν). Then for ν_ξ-a.e. point $\gamma_\xi \in \Gamma_\xi$ the asymptotic entropy of the conditional measure $\mathbf{P}^{\gamma_\xi}$ exists and is equal

$$\mathbf{h}(\mathbf{P}^{\gamma_\xi}) = h(G,\mu) - \mathbf{E}(\Gamma_\xi,\nu_\xi) = H(\alpha|\rho_\xi) - H(\alpha|\rho).$$

Now combining Theorems 3.5.2 and 3.5.3 we get the following results.

Theorem 3.5.4. [22] A μ-boundary $(\mathcal{B},\lambda) \cong (\Gamma_\xi,\nu_\xi)$ is the Poisson boundary if and only if the asymptotic entropy $\mathbf{h}(\mathbf{P}^{\gamma_\xi})$ of almost all conditional measures of the measure $\mathbf{P}$ with respect to Γ_ξ vanishes.

Corollary 3.5.2. *A μ-boundary $(\mathcal{B}, \lambda) \cong (\Gamma_\xi, \nu_\xi)$ is the Poisson boundary if and only if for ν_ξ-a.e. point $\gamma_\xi \in \Gamma_\xi$ there exist $\varepsilon > 0$ and a sequence of sets $A_n = A_n(\gamma_\xi) \subset G$ such that*

(i) $\log |A_n| = o(n)$;

(ii) $p_n^{\gamma_\xi}(A_n) > \varepsilon$ *for all sufficiently large* n, *where* $p_n^{\gamma_\xi}(g) = \mathbf{P}^{\gamma_\xi}(C_g^n)$ *are the one-dimensional distributions of the measures* $\mathbf{P}^{\gamma_\xi}$.

The definition of asymptotic entropy in this case is a generalization of the classic definition of $h(G, \mu)$. For any boundary point $\beta \in \mathcal{B}$, the conditional transition probability with respect to β is defined as

$$p_\beta(x, y) = \frac{dy\lambda}{dx\lambda}(\beta) \cdot p(x, y),$$

where $\dfrac{dy\lambda}{dx\lambda}$ is the Radon-Nikodym derivative. Then, for the induced conditional measure $\mathbf{P}_\beta$ (defined by the transition probabilities $p_\beta(x, y)$), the observed boundary space $(\mathcal{B}, \lambda)$ coincides with the Poisson boundary (Γ, ν) if and only if

$$h(\mathbf{P}_\beta) = 0, \quad \text{for } \lambda\text{-a.e. } \beta.$$

3.6 Geometric criteria of boundary maximality

Gauges are a generalization of the word distance.

Definition 3.6.1. *An increasing sequence $\mathcal{G} = (\mathcal{G}_k)_{k \geq 1}$ of sets exhausting a countable group G is called a gauge on G. By*

$$\mathfrak{g}(g) = \mathfrak{g}_{\mathcal{G}}(g) = \min\{k : g \in \mathcal{G}_k\}$$

we denote the corresponding gauge function.

We shall say that a gauge $\mathcal{G}$ is

(i) symmetric if $\mathfrak{g}(g) = \mathfrak{g}(g^{-1}), \forall g \in G$ (i.e., all gauge sets $\mathcal{G}_k$ are symmetric);

(ii) sub-additive if $\mathfrak{g}(g_1 g_2) \le \mathfrak{g}(g_1) + \mathfrak{g}(g_2), \forall g_1, g_2 \in G$;

(iii) finite if all gauge sets are finite;

(iv) temperate if it is finite and the gauge sets grow at most exponentially:

$$\sup\nolimits_k \left(\frac{1}{k} \log(\sharp \mathcal{G}_k) \right) < \infty,$$

where $\sharp \mathcal{G}_k$ denotes cardinality of $\mathcal{G}_k$.

A family of gauges $\mathcal{G}^\alpha$ is uniformly temperate if

$$\sup\nolimits_{\alpha,k} \left(\frac{1}{k} \log(\sharp \mathcal{G}_k^\alpha) \right) < \infty.$$

For example, a family of translations $g\mathcal{G} = (g\mathcal{G}_k),\ g \in G$ of any temperate gauge $\mathcal{G}$ is uniformly temperate. In what follows, unless otherwise specified, the gauges considered are not assumed to be finite or sub-additive.

An important class of gauges consists of word gauges (the word distance in the terminology of (3.2.2)), i.e., such gauges $(\mathcal{G}_k)$ that $\mathcal{G}_1$ is a set generating G as a semigroup, and $\mathcal{G}_k = (\mathcal{G}_1)^k$ is the set of words of length $\le k$ in the alphabet $\mathcal{G}_1$. Any word gauge is sub-additive. It is symmetric if and only if the set $\mathcal{G}_1$ is symmetric, and finite if and only if $\mathcal{G}_1$ is finite. In the latter case, the gauge is temperate. Any two finite word gauges $\mathcal{G}$, $\mathcal{G}'$ on a finitely generated group G are equivalent (quasi-isometric) if there is a constant $C > 0$ such that

$$\frac{1}{C}\mathfrak{g}_{\mathcal{G}'}(g) \leq \mathfrak{g}_{\mathcal{G}}(g) \leq C\mathfrak{g}_{\mathcal{G}'}(g), \ \forall \ g \in G.$$

Thus, for a probability measure μ on a finitely generated group G finiteness of its first moment $\sum_g(\mathfrak{g}(g)\mu(g))$ or of its first logarithmic moment $\sum_g(\log \mathfrak{g}(g)\mu(g))$ are invariant properties of the measure μ, being independent of the choice of a finite word gauge $\mathfrak{g}(\cdot)$ on G. An important relation between the finiteness of the first moment and finiteness of the entropy of the probability measure μ is proven in the following proposition.

By the Kingman Subadditive Ergodic Theorem (3.1.1), as a generalization of the case (3.2.4), we have the following proposition for sub-additive gauges

Proposition 3.6.2. [22] *If $\mathcal{G}$ is a sub-additive gauge on a countable group G, then for any probability measure μ on G with finite first moment with respect to $\mathcal{G}$ the limit (rate of escape)*

$$\ell(G, \mu, \mathcal{G}) = \lim_{n\to\infty} \frac{\mathfrak{g}_{\mathcal{G}}(x_n)}{n}$$

exists for $\mathbf{P}$-a.e. sample path $\{x_n\}$ and in the space $L^1(\mathbf{P})$.

Ray approximation

Let $\pi_n : \mathcal{B} \to G$ be a sequence of measurable maps from a μ-boundary $\mathcal{B}$ to the group G. Geometrically, we can think of the sequences $\pi_n(\beta)$ ($\beta \in \mathcal{B}$), as "rays" in G corresponding to points from $\mathcal{B}$.

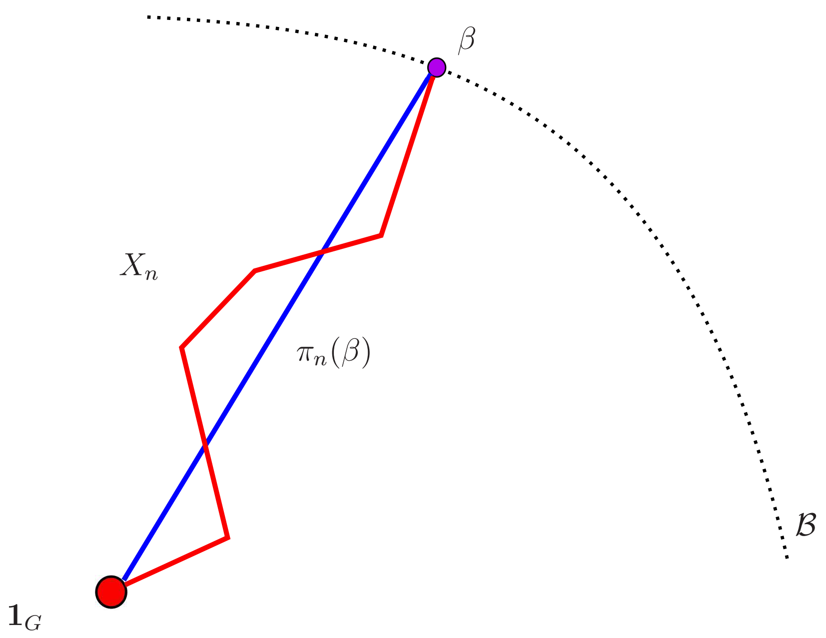

Figure 3.1: Ray $\pi_n(\beta)$ in G corresponding to $\beta \in \mathcal{B}$ and trajectory X_n such that $X_n \to \beta$

Theorem 3.6.1. [22] Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and let $(\mathcal{B}, \lambda) = \mathbf{bnd}_\xi(G^{\mathbb{Z}_+}, \mathbf{P})$ be a μ-boundary. If there exists a temperate gauge $\mathcal{G}$ and a sequence of measurable maps $\pi_n : \mathcal{B} \to G$ such that

$$\frac{\mathfrak{g}_{\mathcal{G}}((\pi_n(\mathbf{bnd}_\xi \mathbf{x}))^{-1} x_n)}{n} \to 0, \text{ as } n \to \infty \tag{3.6.1}$$

for $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = (x_n)$, then $(\mathcal{B}, \lambda)$ is the Poisson boundary of the random walk (G, μ).

Proof: Consider a family of translates $\pi_n(\beta)\mathcal{G}$, where $\mathcal{G}$ is a fixed temperate gauge on G. The condition (3.6.1) is equivalent to

$$\frac{\mathfrak{g}_{\mathcal{G}}((\pi_n(\beta))^{-1} x_n)}{n} \to 0, \text{ as } n \to \infty,$$

for λ-a.e. $\beta \in \mathcal{B}$ and $\mathbf{P}^{\beta}$-a.e. sample path of the random walk conditioned by β (see Theorem 3.4.1). Thus, for λ-a.e. $\beta \in \mathcal{B}$ and any $\varepsilon > 0$ there exists a sequence of sets $A_n = A_n(\beta, \varepsilon) \subset G$ such that

$$\log(\sharp A_n) = o(n), \quad \mathbf{P}^{\beta}[x_n \in A_n] \geq \varepsilon.$$

Hence, $(\mathcal{B}, \lambda)$ is the Poisson boundary of the random walk (G, μ) by the Corollary 3.5.2.

∎

Taking $\pi_n(\beta) \equiv \mathbf{1}$ for the one point μ-boundary we get the following result (e.g., see [19])

Corollary 3.6.3. *If μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and $\ell(G, \mu, \mathcal{G}) = 0$ for a certain temperate gauge $\mathcal{G}$, then the Poisson boundary of the pair (G, μ) is trivial.*

Chapter 4

Random walks on product groups

In this chapter we are looking at hyperbolic groups and group products as the fundamental objects for our work. We begin with a description of a hyperbolic space, using the theory developed in [18] to define a hyperbolic group and describe some of its properties. We then remind several fundamental facts about random walks on hyperbolic groups, and quote in Theorem 4.2.1 a very well known result of Kaimanovich [22]. Our proof for this result is along the lines of the original proof, however it provides more detailed explanations for several arguments.

We then employ the geometric criteria of boundary maximality from the previous chapter. In particular, for a probability measure μ (with a finite first moment) on a hyperbolic group G such that the group $\mathrm{gr}(\mu)$ is non-elementary, we show (Theorem 4.3.3) that a.e. sample path of the random walk (G, μ) converges in the hyperbolic compactification, and the hyperbolic boundary ∂G with the resulting limit measure is isomorphic to the Poisson boundary of (G, μ). For the direct product $\hat{G} = G \times G'$ of hyperbolic groups we define the boundary skeleton and show in Theorem 4.4.2

that almost every sample path of any non-degenerate random walk converges to the boundary skeleton $\partial\hat{G}$ and this boundary skeleton with the resulting limit measure is isomorphic to the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$.

At the end of this chapter, we adopt the general theory of conditional Markov chains to the special case of products of hyperbolic groups and set the grounds for our main result, which will be presented in the following chapter.

4.1 Hyperbolic groups

Hyperbolic groups are of great interest for our work because they exemplify the relationship between the Poisson boundaries and the geometrical properties of groups. We begin by reminding the definition and several basic properties of hyperbolic groups. Our exposition is based on a finitely generated group G with a finite generating set S. Let S be symmetric, meaning that if $a \in S$, then $a^{-1} \in S$. We assume that $\mathbf{1}_G \notin S$. As before (e.g. see 3.2.2), for any element $g \in G$ we define its length $|g|$ to be the minimal number of elements of S needed to write g. For any two elements $g_1 \neq g_2 \in G$, we shall define the associated left-invariant, integer-valued distance d between g_1 and g_2 by

$$d(g_1, g_2) = |g_1^{-1} g_2|. \tag{4.1.1}$$

For $g_1 = g_2$ we set $d(g_1, g_2) = 0$.

The distance function defined above depends on the choice of the generating set. To avoid this inconvenience, we shall use the notion of quasi-isometry as the mathematical reflection of the fact that two metric spaces are similar when viewed from (large) distance.

Definition 4.1.1. (e.g., [18]) *Two metric spaces* (X, d_X) *and* (Y, d_Y) *are called quasi-isometric if there exist two maps*

$$f : X \to Y \ \textit{and} \ g : Y \to X,$$

and two constants $K > 0$ *and* $C \geq 0$*, such that*

$$\begin{gathered} d_Y(f(x_2), f(x_1)) \leq K d_X(x_2, x_1) + C; \ \forall x_1, x_2 \in X \\ d_X(g(y_2), g(y_1)) \leq K d_Y(y_2, y_1) + C; \ \forall y_1, y_2 \in Y \\ d_X(x, g(f(x))) \leq C; \ \forall x \in X \\ d_Y(y, f(g(y))) \leq C; \ \forall y \in Y \end{gathered}$$

This definition means that (for large distances) f and g satisfy the Lipschitz condition, and that f and g are almost inverses of each other.

One can define **geodesic segments** to be isometric embeddings of integer or real intervals. Since we are dealing with integer-valued metrics only, it is more convenient here to define geodesic segments as follows.

Definition 4.1.2. *Let* $(X, d(\cdot))$ *be a metric space. A* **parametrized geodesic segment** *that connects two points* $x, y \in X$ *(such that* $d(x, y) = t \in \mathbb{Z}_+$*) is an isometric embedding of the integer interval* $f : [0, t] \to X$*, such that* $f(0) = x$*,* $f(t) = y$*.*

Similarly, a **geodesic ray** γ is an isometric (integer-valued) embedding

$$\gamma : [0, +\infty) \to X.$$

We call two geodesic rays equivalent $\gamma \sim \gamma'$ if there exist constants $t, t', C \in \mathbb{Z}_+$ such that

$$d(\gamma(t+n), \gamma'(t'+n)) \leq C, \ \forall n = 0, 1, 2, 3...$$

A metric space X is called geodesic if for any $x, y \in X$, there exists a geodesic segment that connects x and y. A geodesic triangle in X with vertices $x, y, z \in X$ is the union of three geodesic segments that connect x, y, z pairwise. The uniqueness of such geodesic segments is not required. We shall denote by $[x, y]$ geodesic segments that connect x and y.

A geodesic triangle $T = [x, y] \cup [y, z] \cup [z, x]$ in a metric geodesic space X is called δ-*thin* if for any point $a \in [y, z]$ we have

$$d([x, y] \cup [z, x], a) \leq \delta$$

for a constant $\delta \geq 0$ (see Figure 4.1).

Definition 4.1.3. *A geodesic metric space X is called Gromov-hyperbolic if there exists a number $0 \leq \delta < \infty$ such that any geodesic triangle $T = [x, y] \cup [y, z] \cup [z, x]$ in X is $\delta-$thin.*

We now define a hyperbolic group.

Definition 4.1.4. *A finitely generated group G is (word) hyperbolic if its Cayley graph is a Gromov-hyperbolic space.*

Note that if the Cayley graph $\Delta(G, S)$ is hyperbolic for some fixed finite generating set S, it is also hyperbolic for any other finite generating set. Moreover, if two groups G_1 and G_2 are quasi-isometric and one of them is hyperbolic, then the other one is also hyperbolic [18, Theorem 29].

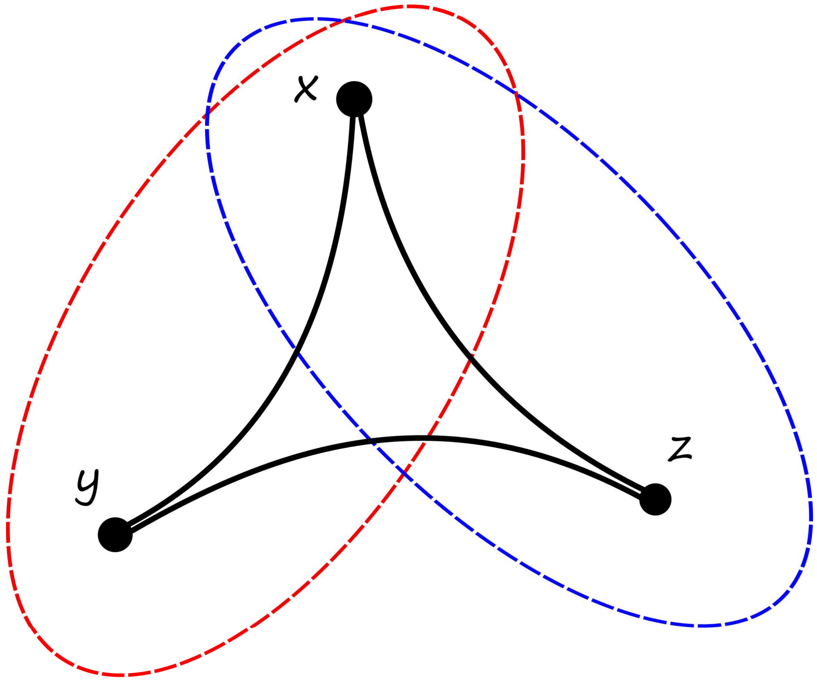

Figure 4.1: δ-thin geodesic triangle

Another (more abstract and widely used) way to define hyperbolic space involves the Gromov product. Let (X, d) be a proper geodesic metric space with a chosen reference point $o \in X$. For a point $x \in X$ put $|x|_o = d(o, x)$, and denote by

$$(x \mid y)_o = \frac{1}{2}\Big[|x|_o + |y|_o - d(x, y)\Big] \tag{4.1.2}$$

the Gromov product on X. Then the space (X, d) is Gromov hyperbolic if there exists $\delta > 0$ such that the δ-ultrametric inequality

$$(x \mid y)_o \geq \min\{(x \mid z)_o, (x \mid y)_o\} - \delta \tag{4.1.3}$$

is satisfied for all o, x, y, $z \in X$ [17].

The **hyperbolic boundary** ∂X of a hyperbolic space X is defined as the space of equivalence classes of asymptotic geodesic rays in X (i.e., those which lie at a finite distance from one another). The boundary ∂X can be endowed with a natural topology in such a way that it is the boundary of the hyperbolic compactification of X [18]. The group G acts on ∂G by homeomorphisms. The definition of the Gromov product can be extended to the case when one or both arguments belong to ∂X. In these terms a sequence $x_n \in X$ converges to a boundary point γ if and only if $(x_n \mid \gamma) \to \infty$. The hyperbolic boundary of an infinite hyperbolic group either consists of two points or is infinite uncountable. In the latter case the group is called non-elementary and the boundary action is minimal (i.e., has no non-trivial closed invariant subsets). In particular, the boundary action of a non-elementary hyperbolic group has no finite fixed subsets. Any non-elementary hyperbolic group is non-amenable.

4.2 Random walks on hyperbolic groups

Let us endow a hyperbolic group G with a probability measure μ. Since G acts on ∂G, it also acts on the probability measures on ∂G by

$$\int f(\gamma) d(x\nu)(\gamma) = \int f(x\gamma) d\nu(\gamma), \tag{4.2.1}$$

where ν is a measure on ∂G and $x \in G$. We say that a probability measure ν on ∂G is *μ-stationary* if

$$\nu = \sum_{x \in S} \mu(x) x\nu. \tag{4.2.2}$$

Definition 4.2.1. [19] *A subgroup $G' \subset G$ is called elementary with respect to a compactification $\overline{G} = G \cup \partial G$ if G' fixes a finite subset of ∂G.*

Lemma 4.2.2. [19] *Let G be a non-elementary hyperbolic group with a non-degenerate probability measure μ. Let λ be a μ-stationary measure on the boundary ∂G of random walk (G, μ). Then λ is purely non-atomic.*

Proof: Suppose that λ has atoms and let A be the set of atoms of maximal measure. Then

$$\forall \gamma \in A : \lambda(\gamma) = C > 0$$

By the μ-stationarity we have

$$\lambda(\gamma) = \sum_{g \in S} \mu(g) g\lambda(\gamma) = \sum_{g \in S} \mu(g) \lambda(g^{-1}\gamma) = C,$$

since $\lambda(g^{-1}\gamma) \leq C$ and μ is a probability measure. It follows that, for all $g \in S$, $\lambda(g^{-1}\gamma) = C$. Hence, we have obtained a finite G-invariant set, which is impossible for a non-elementary group G. ∎

Lemma 4.2.3. [22] *Let λ be a purely non-atomic measure on ∂G, and g_n be a sequence of elements of a hyperbolic group G such that $g_n \to \gamma \in \partial G$. Then $g_n \lambda \to \delta_\gamma$, where δ_γ is the delta-measure at the point γ.*

Corollary 4.2.4. *Let G be a hyperbolic group, λ - a non-atomic probability measure on ∂G, and $g_n \lambda \to \theta$ weakly [1]for a sequence g_n of elements of G such that $|g_n| \to \infty$. Then the limit θ is a point measure δ_γ, for $\gamma \in \partial G$, and $g_n \to \gamma$.*

Let us now consider a sequence of independent identically μ- distributed increments h_i. Recall that the sequence $(X_0, X_1, X_2, ...)$, defined by

$$X_0 = 1_S;\ X_1 = h_1;\ X_2 = h_1 h_2;\ ...\ ; X_n = h_1 h_2 \cdots h_n;\ ... \tag{4.2.3}$$

---

[1] This condition is discussed in the proof of Theorem 4.2.1 below.

(with $X_n \sim \mu^{*n}$, where μ^{*n} is the n-fold convolution of μ) is a realization of the random walk (G, μ) (cf. Definition 2.8.1).

For the sake of completeness, we give here a full argument for the following important result. It is based on the outline in [22] which in turn is based on an idea from [12].

Theorem 4.2.1. [22] Let (X_n) be a realization of random walk (G, μ) as defined by (4.2.3). Then (X_n) converges to a limit $X_\infty : G^{\mathbb{Z}_+} \to \partial G$ almost surely with respect to the associated probability measure $\mathbf{P}$ on the path space $G^{\mathbb{Z}_+}$. The arising limit distribution is μ-stationary and is the only μ-stationary probability on ∂G, called the harmonic measure of the random walk (G, μ).

Proof:

(I) We shall prove that there exists a μ-stationary probability measure on ∂G (using the standard Krylov-Bogolyubov argument). Indeed, for a sequence of measures (Cesaro averages)

$$\lambda_n = \frac{\mu * \theta + \mu^{*2} * \theta + \ldots + \mu^{*n} * \theta}{n},$$

where θ is a probability measure on ∂G, we have

$$\begin{aligned} ||\lambda_n - \mu * \lambda_n|| &= \left\| \frac{\mu * \theta + \mu^{*2} * \theta + \ldots + \mu^{*n} * \theta - \mu^{*2} * \theta - \mu^{*3} * \theta - \mu^{*(n+1)} * \theta}{n} \right\| \\ &= \left\| \frac{\mu * \theta - \mu^{*(n+1)} * \theta}{n} \right\| \leq \frac{2}{n}. \end{aligned}$$

Consequently $||\lambda_n - \mu * \lambda_n|| \to 0$ as $n \to \infty$. We remind that $\lambda \in \mathcal{P}(\partial G)$ is a weak limit of the sequence of measures (λ_n) if and only if the corresponding sequence of integrals $\langle \lambda_n, f \rangle$ converges to $\langle \lambda, f \rangle$. Since the space of probability measures on a compact space is compact in the weak topology, and the hyperbolic boundary of G is compact, one can pass to a convergent subsequence $(\lambda_{n_k}) \to \lambda$. Passing to limits in $||\lambda_{n_k} - \mu * \lambda_{n_k}||$ we obtain $||\lambda - \mu * \lambda|| = 0$. Hence, λ is a μ-stationary probability measure on ∂G.

(II) Since the boundary action is minimal and ∂G is the only closed G-invariant non-empty subset, the support of any stationary probability measure must be all of ∂G. Note that by Lemma 4.2.2, μ-stationary measure λ is purely non-atomic.

(III) Let $\hat{f}$ be a function from $C(\partial G)$. Let us consider the Poisson integrals

$$f(g) = \left\langle \hat{f}, g\lambda \right\rangle .$$

We observe that $f(g)$ is a bounded μ-harmonic function. Indeed, $f(1_G) = \left\langle \hat{f}, \lambda \right\rangle$, whereas for the average

$$\sum_g \mu(g) f(g) = \sum_g \mu(g) \left\langle \hat{f}, g\lambda \right\rangle = \left\langle \hat{f}, \lambda \right\rangle = f(1_G),$$

due to the fact that λ is μ-stationary. In the same way, for any $g \in G$, we have

$$\sum_h \mu(h) f(gh) = \sum_h \mu(h) \left\langle \hat{f}, gh\lambda \right\rangle = \left\langle \hat{f}, g\lambda \right\rangle = f(g).$$

(IV) Consider the sequence of measurable bounded (since $\hat{f} \in C(\partial G, \lambda)$) functions

(F_n) defined on the path space by

$$F_n(X) = f(X_n) = \left\langle \hat{f}, X_n\nu \right\rangle .$$

We recall that $\mathcal{A}_0^n$ is the σ-algebra on the path space generated by the positions of the random walk at times $0, 1, 2, ..., n$. By Definition 2.3.2, the sequence of functions (F_n) is a bounded martingale with respect to the increasing sequence (filtration) of coordinate σ-algebras $(\mathcal{A}_0^n)$ if and only if

$$\mathbf{E}(F_{n+1}|\mathcal{A}_0^n) = F_n,$$

which is precisely the μ-harmonicity condition for the function f. Hence, we can apply the Martingale Convergence Theorem 2.3.1 to see that the sequence of functions $F_n = \left\langle \hat{f}, X_n\lambda \right\rangle$ is a.e. convergent. The boundary ∂G is separable, hence taking $\hat{f}$ from a dense countable subset of $C(\partial G)$ one obtains that almost every sequence of measures $X_n\lambda$ converges weakly to a probability measure λ_X.

(V) The group G is non-elementary, hence non-amenable, which means the transience of the random walk (G, μ) [22], and we have $|X_n| \to +\infty$. By the Corollary 4.2.4, $|X_n| \to +\infty$ implies that almost every $X_n \to X_\infty$ valued in ∂G, and $\lambda_{X_\infty} = \delta_{X_\infty}$. We denote by **bnd** the arising map from the path space to ∂G. Let $\nu = \mathbf{bnd}(\mathbf{P})$ be the distribution of limit points X_∞. Then $(\partial G, \nu)$ is a μ-boundary (see Definition 3.4.1). Let us now show that ν is the unique stationary measure. Indeed, let λ be an arbitrary μ-stationary measure, then

$$\lambda = \mu^{*n}\lambda = \sum_g \mu^{*n}(g)g\lambda = \int x_n \lambda d\mathbf{P}(X_n), \quad \forall n \geq 0,$$

and passing to the limit as $n \to \infty$, we obtain

$$\lambda = \int \lambda_{X_\infty} d\mathbf{P}(X_n) = \int \delta_{X_\infty} d\mathbf{P}(X_n) = \mathbf{bnd}(\mathbf{P}) = \nu.$$

∎

4.3 The Poisson boundary of a hyperbolic group

By Theorem 4.2.1, the hyperbolic boundary endowed with the unique μ-stationary measure is a μ-boundary. Under additional conditions on the measure μ one can show that it actually coincides with the Poisson boundary (for general measures the question is completely open even for free groups).

Definition 4.3.1. [22] *A sequence of points (x_n) in a Gromov hyperbolic space X is called regular if there exists a geodesic ray α and a number $\ell \geq 0$ such that $d(x_n, \alpha(n\ell)) = o(n)$. If $\ell > 0$, we call (x_n) a non-trivial regular sequence.*

The idea of the proof for the following result of [22, V. Kaimanovich] belongs to T. Delzant.

Theorem 4.3.1. [22] A sequence (x_n) in a Gromov hyperbolic space X is regular if and only if

(i) $d(x_n, x_{n+1}) = o(n)$;

(ii) $\frac{|x_n|}{n} \to \ell \geq 0$.

We can now describe the Poisson boundary of a hyperbolic group G and its relation to the hyperbolic boundary by applying the general ray criterion from Theorem 3.6.1. The proofs of two following theorems are taken from [22]. We present these arguments in full because they form a base of the analogous result for product groups (see Theorem 4.4.2).

Theorem 4.3.2. [22] Let μ be a non-degenerate probability measure with a finite first moment on a hyperbolic group G such that the group $\mathrm{gr}(\mu)$ is non-elementary. Then almost every sample path of the random walk (G, μ) is a non-trivial regular sequence in G.

Proof: Fix a word distance determined by a generating set S (with the gauge function $\mathfrak{g}(\cdot)$ defined in 3.6.1) on G and denote by ℓ the corresponding rate of escape (Lemma 3.6.2). We assumed that the group $\mathrm{gr}(\mu)$ is non-elementary (hence, not amenable) so that the Poisson boundary is not trivial. Then, it follows from inequality (3.2.1) that $\ell > 0$. Since the measure μ has finite first moment, and this measure can be thought of as the distribution of the lengths of increments $|h_n| = |x_{n-1}^{-1}x_n| = d(x_{n-1}, x_n)$, we obtain that $d(x_{n-1}, x_n) = o(n)$. Then, by Theorem 4.3.1, the claim follows. ∎

Theorem 4.3.3. [22] Let μ be a non-degenerate probability measure with a finite first moment on a hyperbolic group G such that the group $\mathrm{gr}(\mu)$ is non-elementary. Then almost every sample path of the random walk (G, μ) converges in the hyperbolic compactification, and the hyperbolic boundary ∂G with the resulting limit measure is isomorphic to the Poisson boundary of (G, μ).

Proof: For all points $\xi \in \partial G$ choose a geodesic ray α_ξ from $\mathbf{1}_G$ to ξ (by taking α_ξ to be the lexicographically minimal geodesic ray among all rays that connect $\mathbf{1}_G$ and

ξ). Then the map $\xi \mapsto \alpha_\xi$ is measurable. Let $\pi_n(\xi) = \alpha_\xi([n\ell])$, where ℓ is the rate of escape of the random walk (G, μ) (see Lemma 3.6.2) and $[t]$ is the integer part of the number t. Then by Theorem 4.3.2 for $\mathbf{P}$-a.e. sample path $\{x_n\}$ we have

$$d(x_n, \pi_n(x_\infty)) = o(n), \tag{4.3.1}$$

and by Theorem 3.6.1 we obtain the claim. ∎

Recall from Section 3.2 that for random walk (G, μ) there exists a constant $\ell > 0$, called the rate of escape to infinity, such that

$$\frac{|x_n|}{n} \to \ell, \quad \text{as} \quad n \to \infty, \tag{4.3.2}$$

for almost every sample path $\{x_n\} \in G^{\mathbb{Z}_+}$.

A useful property that connects the notions of the rate of escape (3.2.4) and the Gromov product (4.1.2) can be formulated as follows. Let x_n be a sequence in a Gromov hyperbolic space. Suppose x_n has the property $\frac{|x_n|}{n} \to \ell$, where ℓ is a constant. Then, the statements

(i) $d(x_{n-1}, x_n) = o(n)$,

(ii) $\dfrac{(x_{n-1} \mid x_n)}{n} \to \ell$

are equivalent. To see this, use the Gromov product formula (4.1.2)

$$\big(x_{n-1} \mid x_n\big) = \frac{1}{2}\big[|x_{n-1}| + |x_n| - d(x_{n-1}, x_n)\big].$$

Dividing by n and evaluating the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{(x_{n-1} \mid x_n)}{n} = \lim_{n\to\infty} \frac{1}{2}\left[\frac{|x_{n-1}|}{n} + \frac{|x_n|}{n} - \frac{d(x_{n-1}, x_n)}{n}\right] = \ell.$$

The above property also holds in the setup of random walks on hyperbolic groups (see Theorem 4.3.2) and (as a special case) free groups.

4.4 Product of groups

For simplicity we consider the product of two groups only. Let G and G' be two groups with generating sets S and S', and group operations " $*$ " and " $\star$ " respectively.

Definition 4.4.1. [35] *The set of ordered pairs $(g; g')$ where $g \in G$ and $g' \in G'$ form a group, called the* **product** *of G and G' with the binary operation defined coordinate-wise*

$$(g_1; g_1')(g_2; g_2') = (g_1 * g_1'; g_2 \star g_2').$$

Setting a generating set for the direct product $\hat{G} = G \times G'$ to be

$$\hat{S} = \left\{(s; s') \mid s \in S, s' \in S'\right\},$$

we can define the set of "words" $W \subset \hat{S}^*$ as

$$W = \left\{(s_1 \cdots s_m; s_1' \cdots s_n') \in \hat{S}^*\right\}. \tag{4.4.1}$$

Then the product

$$\hat{G} = G \times G' = \left\{(g; g') \mid g \in G, g' \in G'\right\}. \tag{4.4.2}$$

is a group with the set of elements W, the unit element $\mathbf{1}_{\hat{G}} = (1_S; 1_{S'})$ and the group operation defined coordinate-wise as in Definition 4.4.1.

We define the sequence $((X_0; Y_0), (X_1; Y_1), (X_2; Y_2), ...)$ by (2.8.1) coordinate-wise. Then, the sequence $(X_n; Y_n)$ is a random walk on $\hat{G}$ with $(X_n; Y_n) \sim \hat{\mu}^{*n}$. Although the product of hyperbolic groups is not hyperbolic, one can still consider the skeleton (cf. [45]) $\partial\hat{G} = \partial G_1 \times \partial G_2$ of the boundary of random walk $(\hat{G}, \hat{\mu})$. In the sequel, unless otherwise specified, when talking about the boundary $\partial\hat{G}$ of the random walk on direct product group $\hat{G} = G_1 \times G_2$, we mean the skeleton of this boundary.

Let $\hat{\mu}$ be a probability measure on $\hat{G} = G_1 \times G_2$. Assume that the projections μ_1 and μ_2 of $\hat{\mu}$ are nondegenerate. In particular, this is satisfied if $\hat{\mu}$ is nondegenerate. For the boundary skeleton $\partial\hat{G} = \partial G_1 \times \partial G_2$ of the random walk $(\hat{G}, \hat{\mu})$, the following theorem is a direct corollary of Theorem 4.2.1.

Theorem 4.4.1. Let $\hat{G} = G_1 \times G_2$ be a product of hyperbolic groups with nondegenerate probability measure $\hat{\mu}$. For almost every sample path $(X_n; Y_n)$, there exist $X_\infty \in \partial G_1$ and $Y_\infty \in \partial G_2$, such that

$$\lim_{n\to\infty} X_n = X_\infty, \quad \lim_{n\to\infty} Y_n = Y_\infty.$$

If ν_1 (resp. ν_2) is the distribution of X_∞(resp. Y_∞), it is μ_1 (resp. μ_2)-stationary and is the only μ_1 (resp. μ_2)-stationary probability on ∂G_1 (resp. ∂G_2), called the harmonic measure of (G_1, μ_1) (resp. (G_2, μ_2)).

Denote by **bnd** the map from the space of sample paths $\hat{G}^{\mathbb{Z}_+}$ to the skeleton $\partial\hat{G} = \partial G_1 \times \partial G_2$, and by

$$\hat{\nu} = \mathbf{bnd}(\hat{\mathbf{P}}),$$

the resulting image of the probability measure $\hat{\mathbf{P}}$ on the space of sample paths (trajectories) $\hat{G}^{\mathbb{Z}_+}$ (cf. (2.6.2)).

If a non-degenerate probability measure μ on a group G has a finite first moment, then fixing a word gauge $\mathcal{G}$ (see 3.6.1) on G and recalling the notation ℓ for the corresponding rate of escape (see (3.6.2)), we have that $\ell > 0$ by the Corollary 3.6.3. Moreover, for the lengths of the increments

$$|h_n| = |x_{n-1}^{-1} x_n| = d(x_{n-1}, x_n) = o(n),$$

since the measure μ has a finite first moment. Applying Theorem 4.3.2, we obtain that almost every sample path of the random walk (G, μ) is a non-trivial regular sequence in G.

Theorem 4.4.2. Let G_1 and G_2 be hyperbolic groups. Let $\hat{G} = G_1 \times G_2$ be the product group endowed with a probability measure $\hat{\mu}$ with the finite first moment and nondegenerate projections μ_1 and μ_2. Then almost every sample path of the random walk $(\hat{G}, \hat{\mu})$ converges to the boundary skeleton $\partial\hat{G} = \partial G_1 \times \partial G_2$ and this boundary skeleton with the resulting limit measure is isomorphic to the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$.

Proof: For every point $\hat{\gamma} = (\gamma_1; \gamma_2) \in \partial\hat{G}$ we choose a pair of lexicographically minimal geodesic rays $\hat{\alpha}_{\hat{\gamma}_1} = (\alpha_{\gamma}; \alpha_{\gamma_2})$ from $\mathbf{1}_{\hat{G}}$ to $\hat{\gamma}$. Then the map $\hat{\gamma} \mapsto \hat{\alpha}_{\hat{\gamma}}$ is measurable. We define a sequence of measurable maps $\hat{\pi}_n = (\pi_n; \pi'_n)$ from $\hat{\mu}$-boundary $\partial\hat{G}$ to $\hat{G}$ by

$$\hat{\pi}_n(\hat{\gamma}) = \hat{\alpha}_{\hat{\gamma}}(\lfloor n\hat{\ell} \rfloor),$$

where $\hat{\ell} = (\ell_1; \ell_2)$ is the pair of corresponding rates of escape of two independent random walks (G_1, μ_1), (G_2, μ_2), and $\lfloor \cdot \rfloor$ is a coordinate-wise "floor" function. Since all elements of $\hat{G}$ are pairs of elements of G_1 and G_2 and the statement is true for each individual random walk (G_1, μ_1), (G_2, μ_2) (by Theorem 4.3.2), then almost every sample path of the random walk $(\hat{G}, \hat{\mu})$ is a non-trivial regular sequence in $\hat{G}$. Applying the Definition 4.3.1 we deduce that for $\mathbf{P}$-a.e. sample path $(\hat{\chi}_n) = (x_n; y_n)$ the coordinate-wise distance

$$d(\hat{\chi}_n, \hat{\pi}_n(\chi_\infty)) = o(n).$$

By Theorem 3.6.1 (applied coordinate-wise) claim follows. ∎

Special cases

Consider some special cases of our general setup.

1. Let $\hat{\mu} = \mu_1 \times \mu_2$, where μ_1 and μ_2 are probability measures on hyperbolic groups G_1 and G_2. For $\hat{G} = G_1 \times G_2$ consider the random walk $(\hat{G}, \hat{\mu})$ and its $\hat{\mu}$-boundary $\partial\hat{G}$ with harmonic measure $\hat{\nu}$ on it. Since in this case the random walk $(\hat{G}, \hat{\mu})$ is the product of two independent random walks (G_1, μ_1) and (G_2, μ_2), we have

$$\hat{\nu} = \nu_1 \times \nu_2,$$

where $\nu_i = \mathbf{bnd}(\mathbf{P}_i)$, $(i = 1, 2)$ is a harmonic measure on the Poisson boundary (Γ_i, ν_i) of random walk (G_i, μ_i). As before, we have the stationarity

$$\hat{\mu} * \hat{\nu} = \sum_{\hat{G}} \hat{\mu}(g) g\hat{\nu} = \hat{\nu}.$$

Since the tail-boundary of a random walk with a single point initial distribution coincides with the Poisson Boundary ([19]), the following result is a corollary of Theorems 2.7.2 and 4.4.2.

Proposition 4.4.2. *Let (Γ, ν) be the Poisson boundary of the random walk $(G_1 \times G_2, \mu_1 \times \mu_2)$ with a single point initial distribution (concentrated at $(g_0, h_0) \in G_1 \times G_2$) on the product group $G_1 \times G_2$ defined by the product measure $\mu_1 \times \mu_2$. If (G_1, μ_1) and (G_2, μ_2) are random walks (with initial distributions concentrated at $g_0 \in G_1$ and $h_0 \in G_2$) with Poisson boundaries (Γ_1, ν_1) and (Γ_2, ν_2) respectively, then*

$$(\Gamma, \nu) \cong (\Gamma_1, \nu_1) \times (\Gamma_2, \nu_2).$$

2. Another possible situation is when the factors G_1 and G_2 coincide with the same group G, and the step distribution is

$$\hat{\mu} = \operatorname{diag}(\mu) = \sum_{\hat{G}} \mu(g) \delta_{(g,g)}$$

for a certain measure μ on G. In this case the random walk $(\hat{G}, \hat{\mu})$ is concentrated on the diagonal of the product group $\hat{G}$.

3. However, already the convex combinations

$$\hat{\mu} = \lambda(\mu \times \mu) + (1 - \lambda)\mathrm{diag}(\mu), \ 0 < \lambda < 1,$$

of product and diagonal measures (in the case when $G_1 \cong G_2$) turn out to be quite interesting and non-trivial. Even if μ is finitely supported, the resulting harmonic measure is then not Markov and, apparently, not even a Gibbs one. The step distributions of this kind were recently considered in relation to the noise sensitivity of random walks on groups [4].

4.5 Conditional Markov chains on product groups

Let $\hat{\mu}$ be a probability measure on the product $\hat{G} = G_1 \times G_2$ of hyperbolic groups G_1, G_2. Assume as before that the quotient measures μ_1 and μ_2 on corresponding groups G_1 and G_2 are nondegenerate (see Section 2.8). Consider a sample path $\big(X_n, Y_n\big)$ of the random walk $(\hat{G}, \hat{\mu})$ issued from the identity, i.e.,

$$\big(X_n, Y_n\big) = \big((x_1, y_1), \cdots, (x_n, y_n)\big), \tag{4.5.1}$$

where the increments (x_i, y_i) are i.i.d. $\hat{\mu}$-distributed. We have component-wise convergence $X_n \to X_\infty \in \partial G_1$ and $Y_n \to Y_\infty \in \partial G_2$ (see Theorem 4.2.1). Moreover, the corresponding rates of escape to infinity ℓ_1, ℓ_2 (3.2.4) exist for each of the two components. We call the pair

$$(\ell_1, \ell_2) = \Big(\lim_{n\to\infty} (|X_n|/n), \lim_{n\to\infty} (|Y_n|/n) \Big) \tag{4.5.2}$$

the escape vector. The projection

$$\mathbf{bnd}_2 : \hat{G}^{\mathbb{Z}_+} \to \partial G_2 \tag{4.5.3}$$

of the sample path space onto ∂G_2 is given by the composition

$$\big(X_n, Y_n\big) \mapsto \big(X_\infty, Y_\infty\big) \mapsto Y_\infty = \gamma \in \partial G_2, \tag{4.5.4}$$

where its first arrow is the projection from the sample path space onto the boundary skeleton of the product group

$$\mathbf{bnd} : \hat{G}^{\mathbb{Z}_+} \to \partial \hat{G} \cong \partial G_1 \times \partial G_2, \tag{4.5.5}$$

and the second arrow is the coordinate projection of the boundary skeleton of the product group onto the second component

$$\partial \hat{G} \to \partial G_2. \tag{4.5.6}$$

The measures are projected in the same way

$$\mathbf{P} \mapsto \hat{\nu} \mapsto \nu_2. \tag{4.5.7}$$

Since the projection (4.5.3) is equivariant with respect to the action of the product group, the quotient space (Γ_2, ν_2) can be considered as a $\hat{\mu}$-boundary (see Definition 3.4.1) for the projection $\mathbf{bnd}_2$ (given by the composition (4.5.4), see also the considerations in Section 3.4). For any fixed $\gamma \in \partial G_2$ the corresponding fiber of $(\hat{\Gamma}, \hat{\nu})$ above (Γ_2, ν_2) is $(\partial G_1 \times \{\gamma\}, \nu^\gamma)$, where ν^γ is the corresponding conditional measure. The fiber

$$\partial \hat{G}^\gamma = \partial G_1 \times \{\gamma\} \tag{4.5.8}$$

of the projection (4.5.6) consists of the limit points (X_∞, Y_∞) of all sample paths (X_n, Y_n) with $Y_n \to Y_\infty = \gamma$. Consider the conditional measures $\mathbf{P}^\gamma$ and ν^γ on the fibers of the projections (4.5.4) and (4.5.6), respectively. The measure $\mathbf{P}^\gamma$ is Markov and corresponds to the Markov chain with the one step transition probabilities (see Theorem 3.4.1)

$$p^\gamma[(x, x'), (xh, x'h')] = \hat{\mu}(h, h') \frac{d(h, h')\nu_2}{d\nu_2}(\gamma),$$

where ν_2 is the harmonic measure on ∂G_2. Since $h \in G_1$, it acts trivially on ν_2 and we can write

$$p^\gamma[(x, x'), (xh, x'h')] = \hat{\mu}(h, h') \frac{dh'\nu_2}{d\nu_2}(\gamma), \tag{4.5.9}$$

for the initial distribution concentrated on $\mathbf{1}_{\hat{G}}$. Then the n-step transition probabilities, for the starting point (x, x'), can be written as

$$p_n^\gamma[(x, x'), (xh, x'h')] = \hat{\mu}_n(h, h') \frac{dh'\nu_2}{d\nu_2}(\gamma). \tag{4.5.10}$$

We denote by

$$\pi_n^\gamma(g, h) = p_n^\gamma\big[1_{\hat{G}}, (g, h)\big] \tag{4.5.11}$$

the one-dimensional distribution of the measure $\mathbf{P}^\gamma$ at time n. Further, we denote the transition operator of this chain by P^γ. Recall that the conditional measures of $\hat{\nu}$ on the fibers (4.5.8) of the projection (4.5.6) were denoted by ν^γ. In terms of the projections on fibers, we have

$$\mathbf{P}^\gamma \to \nu^\gamma \to \delta_\gamma.$$

The measure ν^γ can also be considered as the harmonic measure of the conditional Markov operator P^γ on $\hat{G}$ (cf. Theorem 3.4.1), i.e., for ν_2-a.e. $\gamma \in \partial G_2 \cong \Gamma_2$, we have $\mathbf{bnd}(\mathbf{P}^\gamma) = \nu^\gamma$.

4.6 Entropy of conditional walks on product groups

Let ρ be the Poisson partition of the path space $(\hat{G}^{\mathbb{Z}_+}, \mathbf{P})$ (see Definition 2.9.2). Similarly to the considerations in Section 3.4, we denote by ρ_2 the partition of $(\hat{G}^{\mathbb{Z}_+}, \mathbf{P})$, that corresponds to the canonical projection

$$\mathbf{bnd}_2 : (\hat{G}^{\mathbb{Z}_+}, \mathbf{P}) \to (\hat{\Gamma}, \hat{\nu}) \to (\Gamma_2, \nu_2).$$

By Lemma 3.5.1, for any $k \geq 1$ we have

$$H(\zeta_1^k|\rho_2) = kH(\zeta|\rho_2) = k\left[H(\hat{\mu}) - \int \log\left[\frac{d(x_1,y_1)\nu_2}{d\nu_2}(\mathbf{bnd}_2(\boldsymbol{x},\boldsymbol{y}))\right] d\mathbf{P}(\boldsymbol{x},\boldsymbol{y})\right],$$

where ζ_1^k is the time $\leq k$ partition of the path space $(\hat{G}^{\mathbb{Z}_+}, \mathbf{P})$, and ζ denotes ζ_1^1. Since $H(\hat{\mu}) < \infty$ (due to the assumption $|\hat{\mu}| < \infty$), the integral

$$\mathbf{E}(\partial G_2, \nu_2) = \int \log\left[\frac{d(x_1,y_1)\nu_2}{d\nu_2}(\mathbf{bnd}_2(\boldsymbol{x},\boldsymbol{y}))\right] d\mathbf{P}(\boldsymbol{x},\boldsymbol{y}) \qquad (4.6.1)$$

is finite. Moreover, since $x_1 \in G_1$, it acts trivially on ν_2 and we can write

$$\mathbf{E}(\partial G_2, \nu_2) = \int \log \left[\frac{dy_1\nu_2}{d\nu_2}(\gamma) \right] d\mathbf{P}(\boldsymbol{x}, \boldsymbol{y}) \tag{4.6.2}$$

Using the coordinate-wise change of variables as described in (3.5.3), we get

$$\begin{aligned} \mathbf{E}(\partial G_2, \nu_2) &= \sum_{g,h} \hat{\mu}(g,h)\mathbf{I}(h^{-1}\nu_2|\nu_2) \\ &= \sum_{g,h} \hat{\mu}(g,h)\mathbf{I}(\nu_2|h\nu_2), \end{aligned} \tag{4.6.3}$$

where

$$\mathbf{I}(\nu_2|h\nu_2) = \int \log \left[\frac{dh\nu_2}{d\nu_2}(g\boldsymbol{x'}_\infty, h\gamma) \right] d\nu_2(\boldsymbol{x'}_\infty, \gamma) = \int \log \left[\frac{dh\nu_2}{d\nu_2}(h\gamma) \right] d\nu_2(\gamma)$$

is the Kullback-Leibler divergence of ν_2 from its translates $h\nu_2$ (3.5.2) Hence, by Lemma 3.5.1

$$H(\zeta|\rho_2) = H(\hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2) = H(\hat{\mu}) - \sum_{g,h} \hat{\mu}(g,h)\mathbf{I}(\nu_2|h\nu_2). \tag{4.6.4}$$

The following theorem is a special case of Theorem 3.5.3 (Kaimanovich [22]) and the proof is an adaptation of the proof of the above mentioned theorem. We give it here for the sake of completeness.

Theorem 4.6.1. For $\gamma \in \partial G_2$, let $(\partial\hat{G}^\gamma, \nu^\gamma)$ be the fiber of $(\hat{\Gamma}, \hat{\nu})$ above (Γ_2, ν_2). Then for ν_2-a.e. $\gamma \in \partial G_2$, the asymptotic entropy of the conditional measure $\mathbf{P}^\gamma$ on the path space $\hat{G}^{\mathbb{Z}_+}$ exists and is equal

$$\mathbf{h}(\mathbf{P}^{\gamma}) = h(\hat{G}, \hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2).$$

Proof: We want for ν_2-a.e. $\gamma \in \partial G_2$

$$-\frac{1}{n} \log \mathbf{P}^{\gamma}(C^n_{\alpha_n}) \to h(\hat{G}, \hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2)$$

for $\mathbf{P}^{\gamma}$-a.e. sample path $\boldsymbol{\alpha} = (\boldsymbol{x}, \boldsymbol{y})$ and in the space $L^1(\mathbf{P}^{\gamma})$ (see Definition 3.1.2). Due to the fact that the measures $\mathbf{P}^{\gamma}$ are the conditional measures of the measure $\mathbf{P}$, we need to show that

$$-\frac{1}{n} \log \mathbf{P}^{\mathbf{bnd}_2\, \boldsymbol{\alpha}}(C^n_{\alpha_n}) \to h(\hat{G}, \hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2)$$

for $\mathbf{P}$-a.e. sample path $\boldsymbol{\alpha}$ and in the space $L^1(\mathbf{P})$. By Theorem 3.4.1, we have

$$\mathbf{P}^{\mathbf{bnd}_2\, \boldsymbol{\alpha}}(C^n_{\alpha_n}) = \mathbf{P}(C^n_{\alpha_n}) \frac{d\boldsymbol{y}_n \nu_2}{d\nu_2}(\mathbf{bnd}_2\, \boldsymbol{\alpha})$$

Now, since the Bernoulli shift on the space of increments of the random walk determines the measure preserving ergodic transformation $(U\boldsymbol{y})_n = y_1^{-1} y_{n+1}$ of the sample path space (see 2.9.7), we have $\mathbf{bnd}_2(\boldsymbol{\alpha}) = y_1 \mathbf{bnd}_2(U\boldsymbol{\alpha})$, where $U\boldsymbol{\alpha} = U(\boldsymbol{x}, \boldsymbol{y}) =$

$(U\boldsymbol{x}, U\boldsymbol{y})$ (see 2.9.8). By telescoping,

$$\frac{d\boldsymbol{y}_n\nu_2}{d\nu_2}(\mathbf{bnd}_2\,\boldsymbol{\alpha}) = \frac{dy_1\cdots y_n\nu_2}{d\nu_2}(\mathbf{bnd}_2\,\boldsymbol{\alpha})$$

$$= \prod_{i=1}^{n} \frac{dy_i\nu_2}{d\nu_2}(y_{i-1}^{-1}\mathbf{bnd}_2\,\boldsymbol{\alpha})$$

$$= \prod_{i=1}^{n} \frac{d(U^{i-1}\boldsymbol{y})_1\nu_2}{d\nu_2}(\mathbf{bnd}_2\,U^{i-1}\boldsymbol{\alpha})$$

Taking logarithms on both sides, we get

$$\log\left[\frac{d\boldsymbol{y}_n\nu_2}{d\nu_2}(\mathbf{bnd}_2\,\boldsymbol{\alpha})\right] = \log\left[\prod_{i=1}^{n} \frac{d(U^{i-1}\boldsymbol{y})_1\nu_2}{d\nu_2}(\mathbf{bnd}_2\,U^{i-1}\boldsymbol{\alpha})\right]$$

$$= \sum_{i=1}^{n} \log\left[\frac{d(U^{i-1}\boldsymbol{y})_1\nu_2}{d\nu_2}(\mathbf{bnd}_2\,U^{i-1}\boldsymbol{\alpha})\right]$$

We apply the Birkhoff Ergodic Theorem to the transformation U of the sample path space and the function

$$f(\boldsymbol{\alpha}) = \log\left[\frac{d(U\boldsymbol{y})_1\nu_2}{d\nu_2}(\mathbf{bnd}_2\,U\boldsymbol{\alpha})\right]$$

to obtain for ν_2-a.e. $\gamma \in \partial G_2$

$$\frac{1}{n}\log\left[\frac{d\boldsymbol{y}_n\nu_2}{d\nu_2}(\mathbf{bnd}_2\,\boldsymbol{\alpha})\right] \underset{n\to\infty}{\longrightarrow} \int \log\left[\frac{d(U\boldsymbol{y})_1\nu_2}{d\nu_2}(\mathbf{bnd}U\boldsymbol{\alpha})\right] d\mathbf{P}^{\gamma}(\boldsymbol{\alpha}) \tag{4.6.5}$$

for $\mathbf{P}^{\gamma}$-a.e. sample path $\boldsymbol{\alpha} = (\boldsymbol{x}, \boldsymbol{y})$ and in the space $L^1(\mathbf{P}^{\gamma})$. Writing $\boldsymbol{\alpha} = (\boldsymbol{x}, \boldsymbol{y})$ and applying the coordinate-wise change of variables $\boldsymbol{x} \mapsto (g, \boldsymbol{x}'),\ g = x_1,\ \boldsymbol{x}' = U\boldsymbol{x}$ and $\boldsymbol{y} \mapsto (h, \boldsymbol{y}'),\ h = y_1,\ \boldsymbol{y}' = U\boldsymbol{y}$, similarly to (4.6.3), we obtain

$$\begin{aligned}
&\int \log \left[\frac{d(U\boldsymbol{y})_1 \nu_2}{d\nu_2} (\mathbf{bnd}_2\, U\boldsymbol{\alpha}) \right] d\mathbf{P}^\gamma(\boldsymbol{\alpha}) \\
&= \int \log \left[\frac{dh\nu_2}{d\nu_2} ((g,h)\mathbf{bnd}_2\, (\boldsymbol{x'}, \boldsymbol{y'})) \right] d\mathbf{P}^\gamma(\boldsymbol{x'}, \boldsymbol{y'}) \\
&= \sum_{g,h} \hat{\mu}(g,h) \int \log \left[\frac{dh\nu_2}{d\nu_2} (g\boldsymbol{x'}_\infty, h\gamma) \right] d\nu_2(\boldsymbol{x'}_\infty, \gamma) \\
&= \sum_{g,h} \hat{\mu}(g,h) \int \log \left[\frac{dh\nu_2}{d\nu_2} (h\gamma) \right] d\nu_2(\gamma) = \mathbf{E}(\partial G_2, \nu_2).
\end{aligned} \tag{4.6.6}$$

Consequently, for ν_2-a.e. $\gamma \in \partial G_2$

$$\begin{aligned}
&-\frac{1}{n} \log \mathbf{P}(C^n_{\alpha_n}) \frac{d\boldsymbol{y}_n \nu_2}{d\nu_2} (\mathbf{bnd}_2\, \boldsymbol{\alpha}) \\
&= -\frac{1}{n} \log \mathbf{P}^{\mathbf{bnd}_2\, \boldsymbol{\alpha}} (C^n_{\alpha_n}) \longrightarrow h(\hat{G}, \hat{\mu}) - \mathbf{E}(\partial G_2, \nu_2),
\end{aligned}$$

for $\mathbf{P}^\gamma$-a.e. sample path $\boldsymbol{\alpha} = (\boldsymbol{x}, \boldsymbol{y})$ and in the space $L^1(\mathbf{P}^\gamma)$. The claim follows. ∎

Chapter 5

The Hausdorff dimension of harmonic measures

In this chapter we shall state and prove our main result about the Hausdorff dimension of the conditional harmonic measure on products of free-product groups (Theorem 5.4.1). We begin with reminding the basic definitions and some technical properties of the free-product groups. We then discuss several definitions of the Hausdorff dimension and single out the case when all reasonable definitions of the Hausdorff dimension coincide. Our main result is proven in the setup of the direct product of free-product groups. The starting point is the work of V. Kaimanovich [21] who established that the Hausdorff dimension of the harmonic measure on free groups is precisely the asymptotic entropy normalized by the rate of escape. Our work is a generalization and extension of the above result. One should mention here another direction stemming from [21], in which an important recent contribution is due to Tanaka [49] (published in 2019). He showed the exact dimensionality of the harmonic measure of random walks on hyperbolic groups (under appropriate assumptions). We plan to amalgamate our approach with that of Tanaka in a future work.

5.1 Free products

We begin with a brief survey of principal facts concerning free-product groups their properties, in which we follow [36]. Let $(G_i)_{i\in\mathcal{I}}$ ($|\mathcal{I}| \geq 2$) be a finite family of finite groups with unit elements $\mathbf{1}_i \in G_i$ (we impose the finiteness assumption for the sake of simplicity as it is satisfied for the examples we are working with). Let $S_i = G_i \setminus \{\mathbf{1}_i\}$ and $S = \bigsqcup_i S_i$. We define the index map

$$\psi : S \to \mathcal{I}$$

by $\psi(s) = k$ if $s \in S_k$.

Let S^* be the set of all finite words in the alphabet S (including the empty word denoted by $\mathbf{1}_S$). In other terms, S^* is the free monoid generated by the set S. The set of words $W \subset S^*$, defined by

$$W = \{x_1 \cdots x_n \in S^* \mid \forall i \in \{1, ..., n-1\}, \psi(x_i) \neq \psi(x_{i+1})\} \tag{5.1.1}$$

is called the set of normal forms. Observe that $\mathbf{1}_S \in W$ and the set W consists of the words over the alphabet S whose consecutive letters belong to different subalphabets S_i.

The free product $G = *_{i\in\mathbf{I}}G_i$ is the group with the set of elements W, the unit element 1_S and the group operation "$\cdot$" defined recursively by:

$$x_1...x_m \cdot y_1...y_n = \begin{cases} x_1...x_{m-1}x_m y_1 y_2...y_n, & \text{if } \psi(x_m) \neq \psi(y_1) \\ x_1...x_{m-1}(x_m \cdot y_1)y_2...y_n, & \text{if } \psi(x_m) = \psi(y_1), x_m \neq y_1^{-1} \\ x_1...x_{m-1} \cdot y_2...y_n, & \text{if } x_m = y_1^{-1} \end{cases} \quad (5.1.2)$$

As before, the length $|g|$ of an element $g \in G$ is the number of letters of the word g in W.

Remark 5.1.1. Free groups are free products of infinite cyclic groups. Nevertheless, all our arguments work in the case with the modified setup for infinite cyclic multipliers (we use a single generator and its inverse).

Remark 5.1.2. A free product is hyperbolic if all multipliers are either finite or $\mathbb{Z}$; otherwise it need not be hyperbolic (example - if one of the multipliers is $\mathbb{Z}^2$). In the sequel, we assume that all multipliers are either finite or $\mathbb{Z}$.

Let the set of infinite words $S^{\mathbb{Z}_+}$ be equipped with the product topology. Cylinders of order n will be denoted

$$s_1 \cdots s_n[S^{\mathbb{Z}_+}]. \quad (5.1.3)$$

Following [36], we define the set of infinite normal forms $W^\infty \subset S^{\mathbb{Z}_+}$ as

$$W^\infty = \{x_1x_2x_3 \cdots \in S^{\mathbb{Z}_+} \mid \psi(x_i) \neq \psi(x_{i+1}), \ \ \forall i \in \mathbb{Z}_+\} \quad (5.1.4)$$

Informally, we shall say that a word $x_1x_2x_3...$ belongs to W^∞ if all of its finite prefixes belong to W. The boundary ∂G of a free product group G can be represented by

W^{∞}. We remind that since G is a product of finite groups, it is a hyperbolic group (see Remark 5.1.2). The geometric boundary ∂G can be identified with the boundary of the hyperbolic compactification of G.

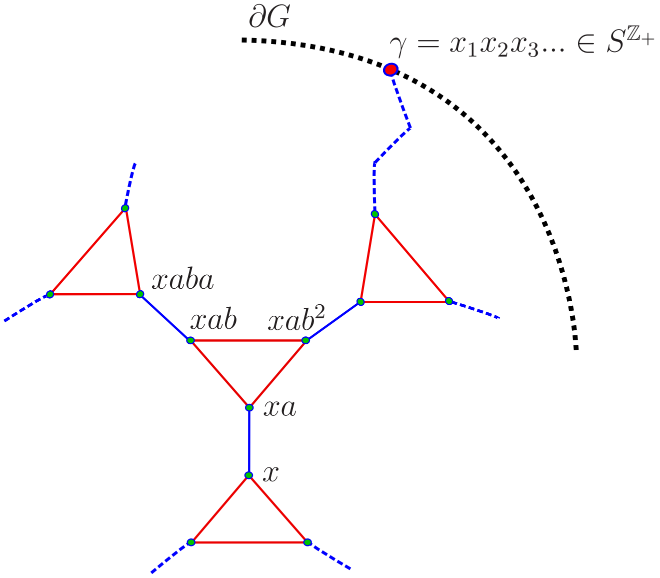

Figure 5.1: Free product $G = \mathbb{Z}_2 * \mathbb{Z}_3$ and its boundary ∂G

Remark 5.1.3. A free-product group is non-elementary (see def. 4.2.1) unless it is the free product of two copies of $\mathbb{Z}/2\mathbb{Z} \equiv \mathbb{Z}_2$.

Any hyperbolic group acts on its boundary. This extends the left continuous group action of G on itself to the natural action of G on ∂G. In the particular case of a free product, in view of the identification of its boundary with W^{∞}, the boundary action is given by the map

$$\phi : S \times \partial G \to \partial G, \quad (x, \gamma) \mapsto x\gamma, \tag{5.1.5}$$

with

$$x\gamma = \begin{cases} x\gamma_0\gamma_1\cdots, & \text{if } \psi(x) \neq \psi(\gamma_0) \\ (x\cdot\gamma_0)\gamma_1\cdots, & \text{if } \psi(x) = \psi(\gamma_0), x \neq \gamma_0^{-1} \\ \gamma_1\gamma_2\cdots, & \text{if } x = \gamma_0^{-1} \end{cases} \tag{5.1.6}$$

Remark 5.1.4. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ is amenable, and any nearest neighbor random walk on it is recurrent. Other than $\mathbb{Z}_2 * \mathbb{Z}_2$, all free-product groups are non-amenable, therefore any random walk defined by a nondegenerate probability measure is transient. Moreover, if this probability measure has a finite first moment, the random walk has a strictly positive rate of escape to infinity $\ell > 0$ (see [15] and [51]).

5.2 Approximation on hyperbolic free products

The Cayley graphs of all groups we consider here have a "tree-like" structure. One of their properties is that removal of any vertex disconnects the graph. Another useful property is that all group elements of such free-product group $G = *_{i\in\mathbf{I}}G_i$ have a unique geodesic representative with respect to $S = \bigsqcup_i S_i$, where $S_i = G_i \setminus \{\mathbf{1}_i\}$, $\forall i$ [36].

Before defining the Hausdorff dimension and stating our main result, we give several approximation properties for free-product groups. These properties also hold in a more general setup (e.g. see [52]).

Recall from Section 4.1 that for a finite group G with a generating set S we define the length of $g \in G$ to be the minimal number of elements of S needed to write g (or by assigning length 1 to each edge of the corresponding Cayley graph). For any two elements $g_1, g_2 \in G$, the integer-valued distance d between g_1 and g_2 is defined in (4.1.1). The confluent

$$(g_1 \wedge g_2)_{1_G},\ g_1, g_2 \in G. \tag{5.2.1}$$

is the length of the common part of the geodesic segments $[1_G, g_1]$ and $[1_G, g_2]$. Due to the aforementioned uniqueness of geodesics, the confluent is also uniquely defined. For example, in Figure 5.1 we observe that

$$(xaba \wedge xab^2)_x = 1 \text{ and } (xaba \wedge xab)_x = 2.$$

Similarly, for any two boundary elements $\xi_1, \xi_2 \in G \cup \partial G$, we denote by $(\xi_1 \wedge \xi_2)$ the length of the common part of the geodesics $[1_G, \xi_1]$ and $[1_G, \xi_2]$.

Remark 5.2.1. On trees and in particular free groups (endowed with a free generating set) the notions of the confluent and the Gromov product (4.1.2) coincide. However, in our more general setup they are somewhat different. This can be explained by observing the presence of loops in the Cayley graphs of free-product groups. For instance, it can be easily seen in Figure 5.1 that

$$(xab \mid xab^2)_x = \frac{1}{2}\Big[2 + 2 - 1\Big] = \frac{3}{2} \neq 1 = (xab \wedge xab^2)_x.$$

However, for the free-product groups we consider in this work, it is always true that

$$(g_1 \mid g_2)_o \geq (g_1 \wedge g_2)_o \tag{5.2.2}$$

The next property is not generally true for hyperbolic groups, but holds for free groups and free-product groups because of the "tree-like" structure of their associated Cayley graphs and the uniqueness of the geodesic representative for any group element. For the Gromov product the inequality (4.1.3) holds for $\delta > 0$, whereas for the above defined confluent one does have the ultrametric inequality without the constant delta.

$$(\xi_1 \wedge \xi_2) \geq \min\{(\xi_1 \wedge \xi_3), (\xi_2 \wedge \xi_3)\} \quad \forall\, \xi_1, \xi_2, \xi_3 \in G \cup \partial G, \tag{5.2.3}$$

Analogous to the boundary metric (see [18]), using inequality (5.2.3) we can define the metric κ on $G \cup \partial G$

$$\kappa(\xi_1, \xi_2) = e^{-(\xi_1 \wedge \xi_2)}, \quad \xi_1 \neq \xi_2. \tag{5.2.4}$$

We denote the e^{-m}-ball (of the restriction of the metric κ to the boundary) centered at a point $\xi \in \partial G$ by

$$\boldsymbol{B}_\xi^m = \{\xi' \in \partial G : (\xi \wedge \xi') \geq m\} \subset \partial G. \tag{5.2.5}$$

For $x_\infty \in \partial G$ we denote by $[x_\infty]_m$ the point on the geodesic $[1_G, x_\infty]$ uniquely determined by the condition

$$\big|[x_\infty]_m\big| = m.$$

Inequality (5.2.3) yields

Proposition 5.2.2. *Let (x_n) be a sequence of elements of a free-product group G such that there exist a number $\ell > 0$ and an integer $N > 0$ with the property that for all $n \geq N$*

$$(x_{n-1} \wedge x_n) \geq \ell n.$$

Then the sequence x_n converges to a point $x_\infty \in \partial G$ in the metric κ (5.2.4) and

$$[x_n]_{\ell n} = [x_\infty]_{\ell n}, \;\; \forall n \geq N.$$

Another approximation property we shall use in the sequel is

Proposition 5.2.3. *Let (x_n) be a sequence of elements of a free-product group G such that there exist numbers $\epsilon, \ell > 0$ and an integer $N > 0$ with the property that for all $n \geq N$*

$$d(x_{n-1}, x_n) \leq \epsilon n, \; ||x_n| - \ell n| \leq \epsilon n. \tag{5.2.6}$$

Then the sequence x_n converges to a point $x_\infty \in \partial G$ in the metric κ (5.2.4) and

$$d(x_n, [x_\infty]_{\ell n}) \leq 5\epsilon n, \;\; \forall n \geq N.$$

Proof: For $n \geq N$, we have

$$(x_{n-1} \wedge x_n) \geq |x_n| - d(x_{n-1}, x_n).$$

It follows from conditions (5.2.6) that

$$|x_n| \geq \ell n - \epsilon n \;\text{ and }\; d(x_{n-1}, x_n) \leq \epsilon n.$$

Hence, we have

$$(x_{n-1} \wedge x_n) \geq |x_n| - d(x_{n-1}, x_n) \geq \ell n - 2\epsilon n,$$

and conditions of Proposition 5.2.2 are met for $\boldsymbol{x}$. Thus $x_n \to x_\infty$, and we can write

$$\big[x_n\big]_{(\ell-2\epsilon)n} = \big[x_\infty\big]_{(\ell-2\epsilon)n}, \quad \forall n \geq N.$$

Now, using the triangle inequality, for any $n \geq N$

$$\begin{aligned} d\big(x_n, \big[x_\infty\big]_{\ell n}\big) &\leq d\big(x_n, \big[x_\infty\big]_{(\ell-2\epsilon)n}\big) + d\big(\big[x_\infty\big]_{(\ell-2\epsilon)n}, \big[x_\infty\big]_{ln}\big) \\ &= d\big(x_n, \big[x_n\big]_{(\ell-2\epsilon)n}\big) + d\big(\big[x_\infty\big]_{(\ell-2\epsilon)n}, \big[x_\infty\big]_{\ell n}\big) \\ &= \big|x_n\big| - (\ell-2\epsilon)n + \ell n - (\ell-2\epsilon)n \\ &= \big(\big|x_n\big| - \ell n\big) + 4\epsilon n \leq 5\epsilon n. \end{aligned} \tag{5.2.7}$$

■

5.3 Definitions of the Hausdorff dimension

The content of this section is mostly adopted from [41] and [52]. We are using it to set up the background for the notion of a dimensionally homogeneous measure (Definition 5.3.1), which is central for our main result (Theorem 5.4.1). For a metric space $(\mathcal{X}, d)$ the Hausdorff measure of dimension $\tau > 0$ of a subset $\mathcal{K} \subset \mathcal{X}$ is defined as

$$\mathbf{HM}_\tau(\mathcal{K}) = \lim_{\epsilon \to 0} \left(\inf_{\{B_i\}} \sum_i (\mathrm{diam}\ B_i)^\tau \right), \tag{5.3.1}$$

where the infimum is taken over all countable covers of the space $\mathcal{X}$ by balls B_i with diam $B_i \leq \epsilon$. The Hausdorff dimension of a set $\mathcal{K} \subset \mathcal{X}$ is defined as

$$\mathbf{HD}\ \mathcal{K} = \inf\{\tau \geq 0\ :\ \mathbf{HM}_\tau(\mathcal{K}) = 0\}. \tag{5.3.2}$$

The Hausdorff dimension of a σ-finite measure μ on $\mathcal{X}$ is the infimum of Hausdorff dimensions of the sets of full measure μ

$$\mathbf{HD}\ \mu = \inf\{\mathbf{HD}(\mathcal{K})\ :\ \mu(\mathcal{X} \setminus \mathcal{K}) = 0\}. \tag{5.3.3}$$

The Hausdorff dimension of a measure μ can be also described by the following formula

$$\mathbf{HD}\ \mu = \operatorname*{ess\,sup}_x \liminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r}, \tag{5.3.4}$$

where $B(x,r) \subset \mathcal{X}$ is the ball of radius r centered at $x \in \mathcal{X}$. The $\operatorname{ess\,sup}_x$ is taken with respect to the measure μ [52].

Definition 5.3.1. *We say that a measure μ is* ***dimensionally homogeneous*** *if*

$$\lim_{r\to 0} \frac{\log \mu B(x,r)}{\log r} = \tau \text{ for } \mu\text{-a.e. } x \in \mathcal{X}, \tag{5.3.5}$$

*In this case **HD** $\mu = \tau$.*

Remark 5.3.2. All reasonable definitions of the Hausdorff dimension in this case coincide [41].

For the boundary ∂G of a free group G equipped with the metric (5.2.4) we can rewrite formula (5.3.4) as

$$\mathbf{HD}\ \nu = \operatorname{ess\,sup}_{\xi} \liminf_{k\to\infty} \frac{-\log \nu(B_\xi^k)}{k}, \tag{5.3.6}$$

where B_ξ^k are the balls (5.2.5) of the metric (5.2.4).

5.4 The Hausdorff dimension of conditional harmonic measures on product groups

Before formulating and proving our main result (Theorem 5.4.1), let us remind several facts from previous chapters. Let $\hat{\mu}$ be a probability measure (with a finite first moment) on a direct product $\hat{G} = G_1 \times G_2$ of free-product groups G_1, G_2. We assume that the quotient measures μ_1 and μ_2 on corresponding groups G_1 and G_2 are nondegenerate (see Section 2.8).

Similarly to Section 4.4, we use Definition 4.4.1 for the direct product of free-product groups, with the free-product conditions (5.1.1) and (5.1.2) imposed on multipliers. Free products that we cinsider in our work are a particular case of hyperbolic groups, and therefore all considerations from the Section 4.4 carry over. In particular, Theorems 4.4.1 and 4.4.2 hold in this setup.

For a sample path $\big(X_n, Y_n\big)$ of the random walk $(\hat{G}, \hat{\mu})$ issued from the identity we have the component-wise convergence

$$X_n \to X_\infty \in \partial G_1 \text{ and } Y_n \to Y_\infty \in \partial G_2.$$

The projection $\mathbf{bnd}_2$ (4.5.3) of the sample path space onto ∂G_2 is given by the composition

$$\big(X_n, Y_n\big) \mapsto \big(X_\infty, Y_\infty\big) \mapsto Y_\infty = \gamma \in \partial G_2,$$

The measures are projected in the same way (4.5.7).

The quotient space (Γ_2, ν_2) can be considered as a $\hat{\mu}$-boundary (see Definition 3.4.1) for the projection $\mathbf{bnd}_2$. For any fixed $\gamma \in \partial G_2$ the corresponding fiber of $(\hat{\Gamma}, \hat{\nu})$ above (Γ_2, ν_2) is $(\partial G_1 \times \{\gamma\}, \nu^\gamma)$, where ν^γ is the corresponding conditional measure. The conditional measure $\mathbf{P}^\gamma$ on the fibers of the projection (4.5.4) is Markov and corresponds to the Markov chain with the one step transition probabilities (see Theorem 3.4.1)

$$p^\gamma[(x, x'), (xh, x'h')] = \hat{\mu}(h, h') \frac{d(h, h')\nu_2}{d\nu_2}(\gamma),$$

where ν_2 is the harmonic measure on ∂G_2. Due to the triviality of the action of $h \in G_1$ on ν_2 we can use (4.5.9) for the initial distribution concentrated on $\mathbf{1}_{\hat{G}}$. Then n-step transition probabilities, for the starting point (x, x'), are given by (4.5.10). Referring to (4.5.10), we denote by (4.5.11) the one-dimensional distribution of the measure $\mathbf{P}^\gamma$ at time n. The conditional measures of $\hat{\nu}$ on the fibers (4.5.8) of the projection (4.5.6) were denoted by ν^γ. In terms of the projections on fibers, we have

$$\mathbf{P}^\gamma \to \nu^\gamma \to \delta_\gamma.$$

The measure ν^γ can also be considered as the harmonic measure of the conditional Markov operator P^γ on $\hat{G}$, i.e. for ν_2-a.e. $\gamma \in \partial G_2 \cong \Gamma_2$, we have $\mathbf{bnd}(\mathbf{P}^\gamma) = \nu^\gamma$.

We recall that the differential entropy (see also (4.6.3)) of the space ∂G_2 with respect to the measure $\hat{\mu}$ is given by

$$E(\partial G_2, \nu_2, \hat{\mu}) = \sum_{(g,h)} \hat{\mu}(g,h) I(\nu_2 | h\nu_2),$$

where in the right-hand side $I(\nu_2|h\nu_2)$ is the Kullback-Leibler divergence (3.5.2) between ν_2 and its translate $h\nu_2$. In our setup, the differential entropy $E(\partial G_2, \nu_2, \hat{\mu})$ is precisely the asymptotic entropy $h(G_2, \mu_2)$ (see [19]) because $(\partial G_2, \nu_2)$ is the Poisson boundary of the random walk (G_2, μ_2).

By Theorem 4.6.1, the asymptotic entropy

$$\mathbf{h}(\mathbf{P}^\gamma) = \lim_{n\to\infty} \frac{-\log \pi_n^\gamma(\alpha_n)}{n} < \infty \tag{5.4.1}$$

of the conditional measure $\mathbf{P}^\gamma$ on the path space $\hat{G}^{\mathbb{Z}_+}$ exists for ν_2-a.e. $\gamma \in \partial G_2$, and we have shown that

$$\mathbf{h}(\mathbf{P}^\gamma) = h(\hat{G}, \hat{\mu}) - h(G_2, \mu_2)$$

Remark 5.4.1. For notational simplicity we shall use $\hat{h}$ and h_2 to denote the respective asymptotic entropies $h(\hat{G}, \hat{\mu})$ and $h(G_2, \mu_2)$.

The decomposition

$$\mathbf{P} = \int \mathbf{P}^\gamma d\nu_2(\gamma)$$

implies that the escape vector is the same for almost all sample paths of a.e. operator P^γ and coincides with the deterministic vector (ℓ_1, ℓ_2) of the rates of escape (4.5.2) for the corresponding quotient random walks (G_1, μ_1) and (G_2, μ_2).

By Theorem 4.4.2, almost every sample path of the random walk $(\hat{G}, \hat{\mu})$ converges to the boundary skeleton $\partial\hat{G}$, and this boundary skeleton with the resulting limit measure is isomorphic to the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$.

Theorem 5.4.1. Let $(\hat{\Gamma}, \hat{\nu})$ be the Poisson boundary of the random walk $(\hat{G}, \hat{\mu})$. For a fixed $\gamma \in \partial G_2$, we denote by $(\partial\hat{G}^\gamma, \nu^\gamma)$ the fiber $(\partial G_1 \times \{\gamma\}, \nu_1)$ of $(\hat{\Gamma}, \hat{\nu})$ above γ. Then for a.e. $\gamma \in \partial G_2$, the measure ν^γ is dimensionally homogeneous and

$$\mathbf{HD}\ \nu^\gamma = \frac{\hat{h} - h_2}{\ell_1}, \tag{5.4.2}$$

where $\hat{h} - h_2$ is the entropy (5.4.1) of the corresponding conditional random walk, and ℓ_1 is the rate of escape along the first (non-fixed) component, respectively. In other words, for ν^γ-a.e. $(\eta, \gamma) \in \partial\hat{G}^\gamma$ and $\ell_1 > 0$

$$\frac{-\log \nu^\gamma\big(\boldsymbol{B}_\eta^k\big)}{k} \to \frac{\hat{h} - h_2}{\ell_1}, \text{ as } k \to \infty, \tag{5.4.3}$$

where B_η^k are the balls (5.2.5) of the restriction of the metric (5.2.4) to the boundary.

The proof of the above theorem consists of the two following separate results, establishing the upper and the lower bounds, respectively.

Theorem 5.4.2. Under the conditions of Theorem 5.4.1

$$\mathbf{HD}\ \nu^\gamma \le \frac{\hat{h} - h_2}{\ell_1}.$$

Theorem 5.4.3. Under the conditions of Theorem 5.4.1

$$\mathbf{HD}\ \nu^\gamma \ge \frac{\hat{h} - h_2}{\ell_1}.$$

5.5 Proofs of Theorems 5.4.2. and 5.4.3.

Before giving the proofs, let us remind several notations. For a sample path $\boldsymbol{\alpha} = (\alpha_n) = (X_n, Y_n) \in \hat{G}^{\mathbb{Z}_+}$ and an index set $J \subset \mathbb{Z}_+$, we denote by $\mathcal{C}^J_{\boldsymbol{\alpha}}$-the set of paths which pass through the same points as $\boldsymbol{\alpha}$ at times $j \in J$. The σ-algebra of the path space determined by the coordinates $\alpha_j (j \in J)$ will be denoted by $\mathcal{A}^J$. As usual, we denote the tail σ-algebra of the chain (α_n) by $\mathcal{A}^\infty$. Recall that $\mathcal{A}^\infty$ is the limit of the decreasing sequences of σ-algebras $\mathcal{A}^{[n,\infty)}$. We will use the notations $\mathbf{P}^\gamma\big(\cdot|\mathcal{C}^{[n,\infty)}_{\boldsymbol{\alpha}}\big)$, $\mathbf{P}^\gamma\big(\cdot|\mathbf{tail}\boldsymbol{\alpha}\big) = \mathbf{P}\left(\cdot|\mathbf{tail}\boldsymbol{\alpha}\right)$ (where $\mathbf{tail}\boldsymbol{\alpha} = (\eta, \gamma)$ with $\eta \sim \nu^\gamma$) for the conditional probabilities at the path $\boldsymbol{\alpha}$ with respect to the $\sigma-$algebras $\mathcal{A}^{[n,\infty)}$ and $\mathcal{A}^\infty$. Recall that in our case the tail boundary coincides with the Poisson boundary (Theorem 4.4.2) and we can use the maps **tail** and **bnd** interchangeably.

Proof of Theorem 5.4.2.

We need to prove that

$$\limsup_{k\to\infty} -\frac{\log \nu^\gamma\big(\boldsymbol{B}^k_\eta\big)}{k} \le \frac{\hat{h} - h_2}{\ell_1}$$

for ν^γ-a.e. point (η, γ) such that $\partial G_1 \ni \eta \sim \nu^\gamma$. For $\epsilon > 0$ and $J \subset \mathbb{Z}_+$ denote by U^J_ϵ the subset of the trajectory space $\hat{G}^{\mathbb{Z}_+}$, consisting of all paths $\boldsymbol{\alpha} = (\alpha_n) = (X_n, Y_n)$, which for any $n \in J$ satisfy the conditions:

$$(X_{n-1} \wedge X_n) > (\ell_1 - \epsilon)n, \tag{5.5.1}$$

$$(Y_{n-1} \wedge Y_n) > (\ell_2 - \epsilon)n, \tag{5.5.2}$$

$$-\log \pi^\gamma_n(\alpha_n) < (\bar{h} + \epsilon)n, \text{ with } \bar{h} = \hat{h} - h_2. \tag{5.5.3}$$

Since we are considering the conditional measure $\mathbf{P}^\gamma$ defined in (4.5.7), we always have that $Y_n \to \gamma$ for a fixed $\gamma \in \partial G_2$. Using the notation (5.4.1) for $\bar{h}$ and applying Theorem 4.6.1 in view of conditions (5.5.1), (5.5.2), (5.5.3), we obtain

$$\mathbf{P}^\gamma\big(U_\epsilon^{[N,\infty)}\big) \to 1 \text{ for any given } \epsilon > 0 \ \ (\text{as } N \to \infty),$$

and there exists a minimal number N_ϵ such that $\mathbf{P}^\gamma\big(U_\epsilon^{[N_\epsilon,\infty)}\big) > 1-\epsilon$. Assume $n > N_\epsilon$. Then for $\mathbf{P}^\gamma$−a.e. $\boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)}$ we have

$$\begin{aligned}
\mathbf{P}^\gamma\left(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) &= \mathbf{P}^\gamma\left(U_\epsilon^{[N_\epsilon,n-1]} \cap U_\epsilon^{[n,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) \\
&= \frac{\mathbf{P}^\gamma\left(\left(U_\epsilon^{[N_\epsilon,n-1]} \cap U_\epsilon^{[n,\infty)}\right) \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)}{\mathbf{P}^\gamma\left(\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)} \\
&= \frac{\mathbf{P}^\gamma\left(\left(U_\epsilon^{[N_\epsilon,n-1]} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) \bigcap \left(U_\epsilon^{[n,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)\right)}{\mathbf{P}^\gamma\left(\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)}
\end{aligned}$$

By the Markov property, we can rewrite the numerator (for $\mathbf{P}^\gamma$−a.e. $\boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)}$)

$$\begin{aligned}
&\mathbf{P}^\gamma\left(\left(U_\epsilon^{[N_\epsilon,n-1]} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) \bigcap \left(U_\epsilon^{[n,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)\right) \\
&= \mathbf{P}^\gamma\left(U_\epsilon^{[N_\epsilon,n-1]} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) \cdot \mathbf{P}^\gamma\left(U_\epsilon^{[n,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right),
\end{aligned}$$

which gives

$$\mathbf{P}^\gamma\left(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right) = \mathbf{P}^\gamma\left(U_\epsilon^{[N_\epsilon,n-1]}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right)\mathbf{P}^\gamma\left(U_\epsilon^{[n,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\right).$$

Again by the Markov property, for $\mathbf{P}^\gamma$-a.e. $\boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)}$

$$\mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,n-1]}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\Big) = \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{[n,\infty)}\Big).$$

By the Martingale Convergence Theorem

$$\lim_{n\to\infty} \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{[n,\infty)}\Big) = \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathbf{tail}\boldsymbol{\alpha}\Big).$$

Also, for $\mathbf{P}^\gamma$-a.e. $\boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)}$

$$\mathbf{P}^\gamma\Big(U_\epsilon^{[n,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\Big) = \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{[0,n)}\Big),$$

and again by the Martingale Convergence Theorem

$$\lim_{n\to\infty} \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{[0,n)}\Big) = \mathbf{1}_{U_\epsilon^{[N_\epsilon,\infty)}}(\boldsymbol{\alpha}) = 1.$$

Thus we have obtained

$$\mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}}\Big) \to \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathbf{tail}\boldsymbol{\alpha}\Big). \tag{5.5.4}$$

Put

$$Q_\epsilon = \Big\{\boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)} \ : \ \mathbf{P}^\gamma\Big(U_\epsilon^{[N_\epsilon,\infty)}|\mathbf{tail}\boldsymbol{\alpha}\Big) > 0\Big\},$$

and observe that

$$Q_\epsilon = Q^+ \bigcap U_\epsilon^{[N_\epsilon,\infty)},$$

where

$$Q^+ = \left\{ \boldsymbol{\alpha} \in \hat{G}^{\mathbb{Z}_+} \ : \ \mathbf{P}^\gamma \Big(U_\epsilon^{[N_\epsilon,\infty)} | \mathbf{tail}\boldsymbol{\alpha} \Big) > 0 \right\}.$$

Therefore, $\mathbf{P}^\gamma(Q_\epsilon) > 1 - 2\epsilon$ and by (5.5.4) for $\mathbf{P}^\gamma$-a.e. $\boldsymbol{\alpha} \in Q_\epsilon$ there exists a limit

$$\lim_{n\to\infty} \frac{\mathbf{P}^\gamma \left(U_\epsilon^{[N_\epsilon,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}} \right)}{\mathbf{P}^\gamma \left(\mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}} \right)} = \lim_{n\to\infty} \frac{\mathbf{P}^\gamma \left(U_\epsilon^{[N_\epsilon,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}} \right)}{\pi_n^\gamma(\alpha_n)} > 0. \tag{5.5.5}$$

We denote the set of all sample paths $\boldsymbol{\alpha} \in \hat{G}^{\mathbb{Z}_+}$ whose limit points belong to the e^{-k} neighborhood of a boundary point $\eta \in \partial G_1$ in the metric (5.2.4) by

$$S_{\alpha_\infty}^k = \left\{ \boldsymbol{\alpha}' \in \hat{G}^{\mathbb{Z}_+} \ : \ (\alpha'_\infty \wedge \eta) \geq k \right\}, \tag{5.5.6}$$

where $\alpha_\infty = (\eta, \gamma) \in \partial \hat{G}^\gamma$. Let $k = (\ell_1 - \epsilon)n$ to obtain (by Proposition 5.2.2)

$$\left(U_\epsilon^{[N_\epsilon,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}} \right) \subset S_{\alpha_\infty}^k \quad \forall n \geq N_\epsilon. \tag{5.5.7}$$

Then, by (5.5.5) for $\mathbf{P}^\gamma$-a.e. $\boldsymbol{\alpha} \in Q_\epsilon$ (with $\alpha_\infty = (\eta, \gamma) \in \partial \hat{G}^\gamma$) we have

$$\begin{aligned}
\limsup_{k\to\infty} -\frac{\log \nu^\gamma \left(\boldsymbol{B}_\eta^k \right)}{k} &= \limsup_{k\to\infty} -\frac{\log \mathbf{P}^\gamma \left(S_{\alpha_\infty}^k \right)}{k} \\
&\leq \limsup_{n\to\infty} -\frac{\log \mathbf{P}^\gamma \left(U_\epsilon^{[N_\epsilon,\infty)} \cap \mathcal{C}_{\boldsymbol{\alpha}}^{\{n\}} \right)}{(\ell_1 - \epsilon)n} \\
&= \limsup_{n\to\infty} -\frac{\log \pi_n^\gamma(\alpha_n)}{(\ell_1 - \epsilon)n} \\
&\leq \frac{\hat{h} - h_2 + \epsilon}{\ell_1 - \epsilon},
\end{aligned}$$

where we use (5.5.3) to arrive at the last inequality. Letting $\epsilon \to 0$ and taking into account that $\mathbf{P}^\gamma(Q_\epsilon) > 1 - 2\epsilon$, we obtain

$$\limsup_{k\to\infty} -\frac{\log \nu^\gamma\left(\boldsymbol{B}^k_\eta\right)}{k} \leq \frac{\hat{h} - h_2}{\ell_1} \tag{5.5.8}$$

for ν^γ-a.e. point (η, γ) with $\partial G_1 \ni \eta \sim \nu^\gamma$.

■

Proof of Theorem 5.4.3

For $\gamma \in \partial G_2$, $\epsilon > 0$, and $J \in \mathbb{Z}_+$ denote by U^J_ϵ the subset of the trajectory space $\hat{G}^{\mathbb{Z}_+}$, consisting of all paths $\boldsymbol{\alpha} = (\alpha_n) = (X_n, Y_n)$, which for any $n \in J$ and fixed point $\gamma \in \partial G_2$ satisfy the conditions:

$$d(X_{n-1}, X_n) \leq \epsilon n, \;\; d(Y_{n-1}, Y_n) \leq \epsilon n; \tag{5.5.9}$$

$$\big||X_n| - \ell_1 n\big| \leq \epsilon n, \;\; \big||Y_n| - \ell_2 n\big| \leq \epsilon n; \tag{5.5.10}$$

$$\left|\log \pi^\gamma_n(\alpha_n) + \bar{h} n\right| \leq \epsilon n, \text{ with } \bar{h} = \hat{h} - h_2. \tag{5.5.11}$$

Since we are considering the conditional measure $\mathbf{P}^\gamma$ defined in (4.5.7), we always have $Y_n \to \gamma \in \partial G_2$. Then by Theorem 4.6.1 we have

$$\mathbf{P}^\gamma\left(U^{[N,\infty)}_\epsilon\right) \to 1 \text{ for any given } \epsilon > 0,$$

and there exists a minimal number N_ϵ such that $\mathbf{P}^\gamma\left(U^{[N_\epsilon,\infty)}_\epsilon\right) > 1 - \epsilon$. Note that the boundary $\partial\hat{G}^\gamma$ is totally disconnected, and for any k the balls $\boldsymbol{B}^k_\xi$ form a partition of $\partial\hat{G}^\gamma$. In the same way as in (5.5.6), we denote by $S^k_{\alpha_\infty}$ the set of all sample paths whose limit points belong to the e^{-k} neighborhood of a point $\alpha_\infty = (\eta, \gamma) \in \partial \hat{G}^\gamma$ in the metric (5.2.4). By the Martingale Convergence Theorem (Theorem 2.3.1, also see [47], [48]) for $\mathbf{P}^\gamma$-a.e. sample path $\boldsymbol{\alpha}$ there exists the limit

$$\lim_{k\to\infty} \mathbf{P}^\gamma \Big(U_\epsilon^{[N_\epsilon,\infty)} | S^k_{\alpha_\infty} \Big) = \mathbf{P}^\gamma \Big(U_\epsilon^{[N_\epsilon,\infty)} | \mathbf{tail}\boldsymbol{\alpha} \Big). \tag{5.5.12}$$

In the same way as in the proof of Theorem 5.4.2, let

$$Q_\epsilon = \Big\{ \boldsymbol{\alpha} \in U_\epsilon^{[N_\epsilon,\infty)} \ : \ \mathbf{P}^\gamma \Big(U_\epsilon^{[N_\epsilon,\infty)} | \mathbf{tail}\boldsymbol{\alpha} \Big) > 0 \Big\}.$$

Then $\mathbf{P}^\gamma(Q_\epsilon) > 1 - 2\epsilon$ and by (5.5.12) for $\mathbf{P}^\gamma$-a.e. $\boldsymbol{\alpha} \in Q_\epsilon$

$$\lim_{k\to\infty} -\frac{\log \mathbf{P}^\gamma \left(S^k_{\alpha_\infty} \right)}{k} = \lim_{k\to\infty} -\frac{\log \mathbf{P}^\gamma \left(U_\epsilon^{[N_\epsilon,\infty)} \bigcap S^k_{\alpha_\infty} \right)}{k}. \tag{5.5.13}$$

Now, setting $k = \ell_1 n$ and applying Proposition 5.2.3, we obtain the inclusion

$$\Big(U_\epsilon^{[N_\epsilon,\infty)} \cap S^{\ell_1 n}_{\alpha_\infty} \Big) \subset \bigcup_{\beta\in\Lambda} U_\epsilon^{[N_\epsilon,\infty)} \cap \mathcal{C}^{\{n\}}_\beta, \tag{5.5.14}$$

where

$$\Lambda = \big\{ \beta = (u, v) \in \hat{G} \ : \ d(u, [X_\infty]_{\ell_1 n}) \le 5\epsilon n; \ d(v, [\gamma]_{\ell_2 n}) \le 5\epsilon n \big\},$$

The relation (5.5.14) follows from the definitions of the sets $U_\epsilon^{[N_\epsilon,\infty)}$, $S^k_{\alpha_\infty}$, and $\mathcal{C}^{\{n\}}_\beta$.

The probability of the right hand side of (5.5.14) does not exceed $(MK)^{5\epsilon n} e^{-n(\bar{h}-\epsilon)}$ (from (5.5.11)), where M, K are upper bounds on cardinalities of generating sets of

G_1, G_2 respectively. Therefore, we can write

$$\mathbf{P}^{\gamma}\left(U_{\epsilon}^{[N_{\epsilon},\infty)} \cap S_{\alpha_\infty}^{\bar{\ell}n}\right) \leq (MK)^{5\epsilon n} e^{-n(\bar{h}-\epsilon)}. \tag{5.5.15}$$

Taking the logarithm of the right-hand side and dividing by $k = \ell_1 n$, we obtain

$$\frac{\log(MK)^{5\epsilon n} e^{-n(\bar{h}-\epsilon)}}{\ell_1 n} = -\frac{\bar{h}}{\ell_1} + \frac{\epsilon}{\ell_1}(5\log MK + 1).$$

Thus, for a.e. $\boldsymbol{\alpha} \in Q_{\epsilon}$

$$\lim_{k\to\infty} -\frac{\log \mathbf{P}^{\gamma}\left(U_{\epsilon}^{[N_{\epsilon},\infty)} \cap S_{\alpha_\infty}^{\ell_1 n}\right)}{k} \geq \frac{\bar{h}}{\ell_1} - \frac{\epsilon}{\ell_1}\left(5\log MK + 1\right).$$

Using (5.5.13) and recalling the notation in (5.5.11), we can write

$$\lim_{k\to\infty} -\frac{\log \mathbf{P}^{\gamma}(S_{\alpha_\infty}^{k})}{k} \geq \frac{\hat{h} - h_2}{\ell_1} - \frac{\epsilon}{\ell_1}(5\log MK + 1).$$

Finally, we let $\epsilon \to 0$, whence for ν^{γ}-a.e. $\eta \in \partial G_1$

$$\lim_{k\to\infty} -\frac{\log \nu^{\gamma}\left(\boldsymbol{B}_{\eta}^{k}\right)}{k} \geq \frac{\hat{h} - h_2}{\ell_1}. \tag{5.5.16}$$

■

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