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MATHEMATICAL THEORY OF ELECTRICAL NETWORKS

by

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ABSTRACT

In this thesis an attempt has been made to present a mathematical model of an electrical network based on Kron's work and further to use this model, for analyzing electrical network problems.

To begin with, we shall represent an electrical network as a linear graph. The electrical network problem is presented in an abstract topological form, and the condition on the property of branch impedance matrix for the unique existence of the solution is developed. The various contradictions of the linear graph model are then pointed out.

To remove these contradictions, a topological model, consisting of branches only, which is called the branch network or 1-network by Kron is introduced. The various concepts of 1-networks are generalized to any dimension p. The properties of p-networks are described with the help of elementary concepts of algebraic topology.

Finally, diakoptic and co-diakoptic property of p-network is described as a direct outcome of p-network theory.

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LINEAR GRAPH MODEL

To begin, an electrical network is represented by a linear graph. The required properties of linear graph are discussed in part 1.1. The electrical network problem is presented in an abstract topological form, thereby establishing the condition on the property of branch impedance matrix for the unique existence of the solution of network problem. The various properties of linear graph which contradict the properties of electrical network are then discussed.

Finally, to remove the contradictions, a new topological model is introduced which consists of branches only and is called the branch network or 1-network by Kron.

1.1 The Elements of Linear Graph Theory.

For developing a linear graph model of an electrical network, a brief study of the properties of linear graphs is required. We have tried to make this chapter self-contained. Any property of linear graph which is used in our work, is described .

Definition 1.1.1(12)

1 - simplex: In any n-dimensional Euclidean (real) space, the points collinear with and between two distinct points constitute a 1-simplex. The ends of 1-simplex are the given points but are not regarded as points of 1-simplex.

1 - cell or Branch: Any set of objects in one to one correspondence with the points of the 1-simplex and its two ends is called 1-cell or branch.

Definition 1.1.2 (12)

0-cell or Junction: Any single object is referred as 0-cell or junction.

Objects corresponding to ends of 1-simplex are called the boundaries of a branch.

Definition 1.1.3 (12)

Oriented Junction: It is a junction associated with the number +1 or -1. If number +1 is associated, the junction is said to be positively oriented. If number -1 is associated, the junction is said to be negatively oriented.

Oriented 0-complex: It is a set of the finite number of oriented junctions.

Definition 1.1.4 (12)

Oriented Branch: A branch is oriented by ordering its bounding junctions. Fig. 1.1.1 shows the oriented branch a_1^1 i.e. $(a_1^o; a_2^o)$.

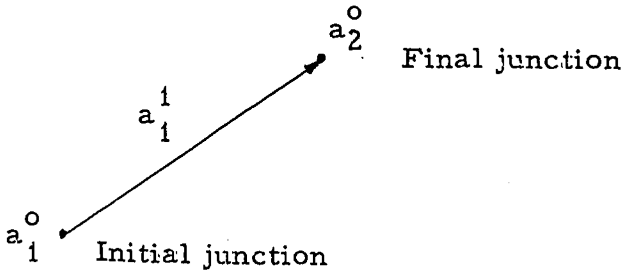


Fig. 1.1.1 Oriented Branch.

In the graph, instead of mentioning the order of bounding junctions, orientation can be shown by an arrow, pointing from the initial junction to the final junction.

Definition 1.1.5$^{(12)}$

Linear Graph: A 1-complex or linear graph is a 0-complex together with a finite number of branches each bounded by a pair of its junctions, such that no two of the branches have a point in common, and each junction is an end of at least one branch.

When the branches of linear graph are assigned an orientation, then linear graph is said to be oriented. In our study, a linear graph is always assumed to be oriented, unless otherwise stated. Fig. 1.1.2 shows some examples of linear graph.

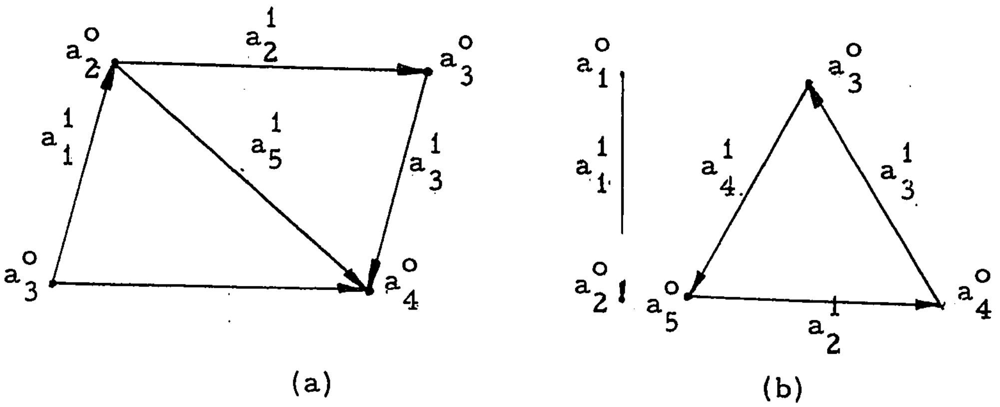


Fig. 1.12 Examples of Linear Graph.

Definition 1.1.3$^{(12)}$

Incidence: A junction is said to be incident with a branch if and only if it is the end of a branch, and under the same conditions, a branch is said to be incident with the junction.

Definition 1.1.7$^{(1,12)}$

Connectedness: Two branches are said to be connected if they are incident on at least one common junction. A set of branches is said to be connected if all the branches are at least pairwise connected to one another. Similarly two or more junctions are said to be connected if they are incident on a connected set of branches.

A linear graph is said to be connected if all its branches and junctions are connected.

Definition 1.1.8$^{(1)}$

Simple Path: It is an alternating sequence of connected junctions and branches such as sequence of symbols a^o_1, a^1_1, a^o_2, a^1_2, $a^o_3 \ldots$, where a^o_i is the symbol for the i-th junction and a^1_j is the symbol for j-th branch. The tranversal of any branch first in one direction, then immediately in opposite direction is excluded.

Open Path: It is a sequence of symbols such that no junction is included twice.

Closed-path or 1-circuit$^{(10)}$

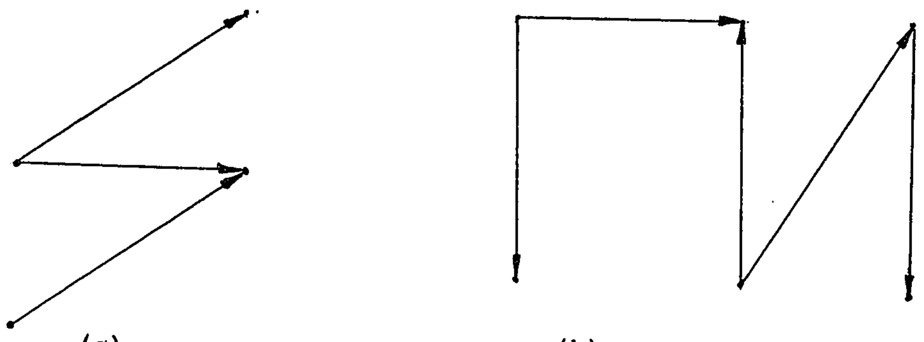
It is a connected linear graph, each junction of which is an end of two and only two branches.

In general, any path may be open or closed or may consist of open and closed paths. We shall call 1-circuit simply a circuit or a mesh.

Another important concept of a linear graph theory is a tree.

Definition 1.1.9$^{(10)}$

Tree: Any connected linear graph containing no circuit is called a tree, Fig. 1.1.3.



(a) (b)

Fig. 1.1.3 Examples of Tree.

Theorem 1.1.1(10,12)

A finite connected linear graph is a tree if and only if there exists exactly one path between any two junctions of the graph.

Proof,

If a graph is a tree, there always exists at least one path betweeen any two junctions of the graph, since a tree is a connected graph. Let us assume there exist two paths o_1 and o_2 between any two junctions of the tree. The existence of paths o_1 and o_2 between two junctions creates a closed path in the tree which is impossible by definition 1.9. Thus a tree contains exactly one path between any two of its junctions.

If a graph contains exactly one path between any two junctions, it is connected and contains no closed paths, hence it is a tree.

Theorem 1.1.2(1)

In a tree containing j-junctions, there are exactly (j-1) branches.

Proof:

The proof of this theorem is by induction. Take a tree containing j-junctions. To start with, remove all the branches except one. This branch will be incident with two junctions forming a unique path between the two junctions. Introduce another branch from the removed branches, such that it is incident with one of the junctions already joined. Other end of the introduced branch has to be incident with a junction not considered so far. If it is incident with the junction considered earlier, this will form a closed path which is not possible. Thus we have three junctions, and two branches; We can introduce another branch which is incident with one of the junctions joined earlier, the other end of this branch is incident with a junction not considered so far. If it were incident with another junction considered earlier then it will form a closed path which is contradictory. So there are three branches, and four junctions connected together without any closed paths, thus forming a tree. We can generalise to any number of junctions, say j-junctions, proving that a tree with j-junctions will have exactly (j-1) branches.

Theorem 1.1.3(10)

Every finite connected graph contains a tree.

Proof:

If a graph itself is not a tree, then it contains a circuit. Removal of an element (branch) from the circuit leaves the graph connected and does not remove any junction. The graph is still connected, though the circuit is destroyed. Repeating this, we can destroy all the circuits keeping the graph still connected, thus forming the required tree of the graph.

Definition 1.1.10(1.10.12)

Co-tree: A finite connected graph can be reduced to a tree, by removing some of the branches which destroy the closed paths. The set of branches removed from the connected graph is called a co-tree. Co-tree is also called a complement of tree and branches of co-tree are called links. The sum of the branches of a tree and the branches of corresponding co-tree of a given graph is equal to the total number of branches of the graph.

For example, if a graph has b-branches with j-junctions, then its tree will have (j-1) branches. The corresponding co-tree has b-(j-1) branches.

Addition of a link to a tree introduces a circuit. This is a unique circuit and the only circuit of the graph. The addition of all the links to the corresponding tree, introduces as many unique closed paths as there are links in the corresponding co-tree. The closed paths thus formed are called basic closed-paths or basic circuits. They are also referred to as fundamental or independent circuits.

Basic closed-paths of a connected graph G for a chosen tree T are the (b-j+1) closed-paths formed by each branch of the corresponding co-tree, and its unique tree-path. The number of basic closed paths is also called nullity, cyclomatic number, or 1st Betty number of the graph. If a graph contains p-connected sulegraphs, then the basic closed-paths are (b-j+p) in number.

Incidence Relations of a Linear Graph.

The fundamental characteristics of a linear graph are the interconnections among the branches and junctions. A graph is fully described, once the incidence relations of junctions and branches are specified. The incidence relations of the junctions and branches can be completely represented by a matrix instead of showing this by a graph. The matrix showing the incidence relations of the graph is called a junction-branch-incidence matrix.

Definition 1.1.12(1, 10)

Junction-Branch incidence Matrix B_a.

$B_a = (b_{ij})$ is a matrix of j-columns and b-rows for a graph of j-junctions and b-branches where

$b_{ij} = (+1, -1, 0)$ if i-th branch is (positively, negatively, or not) incident with j-th junction.

According to the above rule, the elements of a row corresponding to a branch which form self loops will be zero. Thus all those branches which form self loops are not included, since this would give a row of zeros. In our work, we shall be using the reduced incidence matrix B instead of B_a, which contains (j-1) columns, obtained by deleting any one column of B_a. The junction corresponding to the deleted column is termed a datum junction.

The matrix B_a of Fig. 1.1.2a is given as:

$$B_a = \begin{array}{c|c|c|c|c|} & a_1^o & a_2^o & a_3^o & a_4^o \\ \hline a_1' & 1 & -1 & & \\ \hline a_2' & & 1 & -1 & \\ \hline a_3' & & & 1 & -1 \\ \hline a_4' & 1 & & & -1 \\ \hline a_5' & & 1 & & -1 \\ \hline \end{array} \qquad 1.1.1$$

Matrix B_a has exactly one +1 and one -1 in each row. This is the direct result of our definition for b_{ij}. Since the sum of all the columns of B_a is equal to zero, B_a has a linear relation among its columns.

<u>Theorem 1.1.4(a)</u>$^{(10)}$

The rank of junction-branch-incidence matrix of a connected graph is at most (j-1) where j is the number of junctions of the graph.

<u>Proof:</u>

Add all the first (j-1) columns to the last column. This matrix operation does not change the rank of matrix. Since each row has exactly one +1 and one -1, the last column will become a column of zeros. Since the matrix has only $(j-1)$ non-zero columns, the rank cannot exceed $(j-1)$.

Corollary 1.1.1 For a connected graph G, the sum of any r-columns of B_a, $r < j$ contains at least one non-zero element.

Proof:

For $r < j$, let any r columns of B_a add to a column of zeros. Arrange the columns so that the r columns taken above occupy first r-columns. The r-columns add up to zero, showing that each row contains exactly one +1, and one -1 in these columns, or contains zeros only. Permute the rows of B_a so that the rows containing only zeros occupy the last rows. There will be rows like this otherwise the last $(j-r)$ columns will have all the elements as zeros showing the existence of $(j-r)$ isolated junctions which cannot exist. The rows which have one +1 and one -1 in first r-columns will have zero elements in the last $(j-r)$ columns. So the matrix partitioned in this way takes the form

$$B_a = \begin{array}{|c|c|} \hline B_{11} & 0 \\ \hline 0 & B_{22} \\ \hline \end{array} \qquad 1.1.2$$

The partitioned matrix B_a shows that first r-junctions are not connected to the last $(j-r)$ junctions, proving that the given graph is not a connected graph, contradicting our hypothesis. Hence the results for a connected graph G, the sum of any r-columns of B_a, $r < j$, contains at least one non-zero element.

Theorem 1.1.4b(10)

The rank of matrix B_a of a connected graph is exactly (j-1) where j is the number of junctions of the graph.

Proof:

Let $B_1, B_2, \ldots, B_j$ be the j-columns of B_a.

Any relation among the columns $B_1, B_2, \ldots, B_j$ of the form given in equation 1.1.3 will show a linear dependence among the columns. The equation is

$$\sum_{i=1}^{j} C_i B_i = 0 \qquad 1.1.3$$

where C_i's are either 1's or 0's.

There exists only one solution to this equation which is derived using the results of theorem 1.1.4a and corollary 1.1.1. By theorem 1.1.4a, the sum of all the columns B_i's is equal to zero, i.e.

$$B_1 + B_2 \ldots, + B_j = 0 \qquad 1.1.4$$

By the corollary 1.1.1, the sum of any r-columns of B_a is never equal to zero, as long as $r < j$ i.e.

$$B_1 + B_2 \ldots, + B_r \neq 0 \qquad r < j \qquad 1.1.5$$

This means that the only solution to the equation 1.1.3 is when all the C_i's are equal to one, and corollary 1.1.1 does not allow the existence of any other solution.

Thus there exists only one relation among the columns of B_a, thereby proving that the rank of B_a is (j-1).

If anyone column of B_a is deleted, the matrix thus obtained having (j-1) - columns is called the reduced-incidence-matrix B. The junction corresponding to the deleted column is called a datum junction.

The matrix B_T for the tree of a connected graph will have (j-1) rows and (j-1) columns where j is the number of junctions of the given graph. The rank of B for a connected graph is (j-1). The tree of a connected graph with j junctions is itself a connected graph having j-junctions and (j-1) branches. The matrix B_T will have (j-1) rows and (j-1) columns, and its rank is (j-1) since the rank of B for a connected graph is (j-1). Thus the matrix B_T is square and nonsingular.

Any connected graph will have many trees. Let us choose a tree for Fig. 1.1.2a

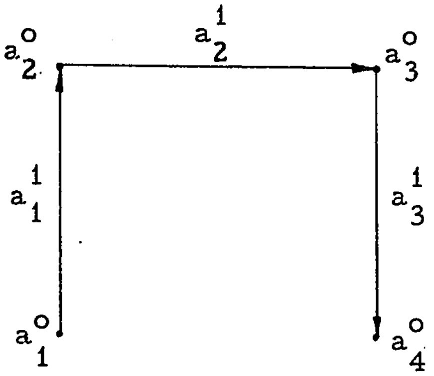


Fig. 1.1.4 Tree for Fig. 1.1.2a

Choosing a_4^o as the datum junction, the matrix is

$$B_T = \begin{array}{c|c|c|c|} & a_1^o & a_2^o & a_3^o \\ \hline a_1^1 & 1 & -1 & \\ \hline a_2^1 & & 1 & -1 \\ \hline a_3^1 & & & 1 \\ \hline \end{array} \qquad 1.1.6$$

There are different B_T's for different trees of the graph. But all the B_T's for the graph will be square and nonsingular.

Space of Junction-Pairs.

In matrix B_a, the columns represent the j-junctions of the given graph. In a graph, any branch is incident with two junctions called the bounding junctions. If it is assumed that columns of B_a represent the junction pairs instead of junctions, there shall be (j-1) independent columns since there are (j-1) independent junction pairs. Thus the columns of matrix B can be assumed to represent the (j-1) independent junction pairs of the graph. In the language of graph theory junction pairs are called the 0-circuits, a term similar to 1-circuit.

Definition 1.1.13(12)

0-circuit: A pair of junctions is called 0-circuit. If any two junctions bound a branch or a set of branches, it is called a bounding 0-circuit. A 0-circuit is oriented if to one junction a number +1 is attached, and to the other a number -1 is attached. An oriented bounded 0-circuit will be called junction-pair. Generally one junction is taken as datum junction and junction pairs are formed by pairing non-datum (j-1) junctions with the datum junction. It should be mentioned that in forming junction pairs, we need not pick up a datum junction and form junction pairs as was first suggested by Maxwell[7]. Any two arbitrary junctions of a connected graph will form a junction pair, and the concept of datum junction is not a necessity but is only convenient in some cases.

As matrix B_T is square and nonsingular (from the property of tree) the $(B_T)^{-1}$ exists. Now $(B_T)^{-1}$ shall be defined and we shall see what does it signify in a linear graph.

Definition 1.1.14(1)

Matrix P_T: Junction to Datum-Path-Matrix.

P_T = (p_{ij}) is a matrix of (j-1) rows and (j-1) columns for a tree of a graph with (j-1) branches and (j-1) non-datum junctions where

p_{ij} = (+1, -1, 0) if i-th branch is (positively, negatively, not) included in the unique path formed between j-th and datum junction.

The path is oriented from non-datum to datum junction. If any branch is oriented along the orientation of the path, it is said to be positively included, if the branch is oriented in the opposite direction, it is said to be negatively included. The branch which does not exist in the path from j-th junction to datum junction is not included in that path.

Fig. 1.1.3 shows the tree of a given Fig. 1.1.2a. It is redrawn in Fig. 1.1.5 with the orientation assigned to the chosen paths.

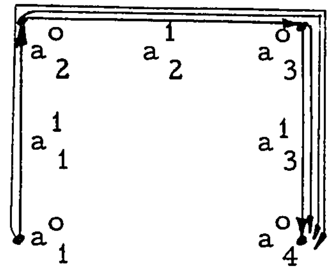


Fig. 1.1.5 Orientation of Paths.

The junction to datum path matrix is given as

$$P_T = \begin{array}{c|c|c|c|} & a^o_1 & a^o_2 & a^o_3 \\ \hline a^1_1 & 1 & & \\ \hline a^1_2 & 1 & 1 & \\ \hline a^1_3 & 1 & 1 & 1 \\ \hline \end{array} \qquad 1.1.7$$

Also we have

$$P_T . (B_T)_t = U_T = B_T . (P_T)_t \qquad 1.1.8$$

$$P_T = (B_T)_t^{-1} \qquad 1.1.9$$

The matrix P_T is inverse of B_T(12).

There is another matrix which describes the circuits or closed paths of a linear graph.

Definition 1.1.15(1, 10, 12)

Branch-Circuit Matrix C.

The structure of each basic closed path or circuit of a graph can be described algebraically with the help of matrix C. An orientation to the basic circuit is assigned by the orientation of the identifying branches of the co-tree. When the branch of the co-tree is tranversed from initial to final junction, the path is completed by traversing along the associated tree path from the final junction back to the initial junction as shown in Fig. 1.1.6

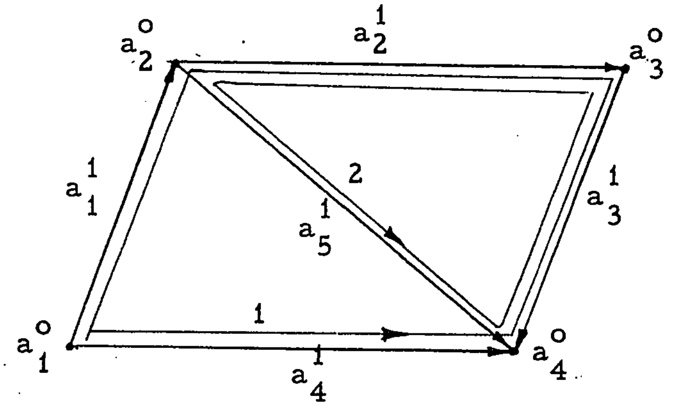


Fig. 1.1.6 Orientation of Closed-paths.

Any branch of the associated tree is positively included in a closed path if it is directed along the closed path, negatively included if directed against the orientation of the closed path, and not included at all if it does not form the part of tree path chosen.

Matrix C is defined as

C = A matrix of b-rows and b-j+1 columns for a connected graph of b-branches and b-j+1 basic circuits such that

c_{ij} = (+1, -1, 0) if i-th branch is (positively, negatively, not) included in the j-th basic circuit.

The rows of matrix C can be arranged in such a way that first (b-j+1) rows correspond to the branches of the co-tree and last (j-1) rows correspond to the branches of the associated tree. The matrix C then splits into

$$C = \begin{array}{|c|}\hline C_L \\ \hline C_T \\ \hline \end{array} \tag{1.1.10}$$

C_L is a unity matrix with rows and columns equal to (b-j+1). Matrix C for the Fig. 1.1.6 is written as

$$C = \begin{array}{c|c|c|} & 1 & 2 \\ \hline a^1_4 & 1 & \\ \hline a^1_5 & & 1 \\ \hline a^1_1 & -1 & \\ \hline a^1_2 & -1 & -1 \\ \hline a^1_3 & -1 & -1 \\ \hline \end{array} = \begin{array}{|c|} \hline C_L \\ \hline C_T \\ \hline \end{array} \qquad 1.1.11$$

The rank of C is obviously equal to b-j+1, i.e. equal to the number of independent basic closed paths.

There exists an important relation between C and B

Theorem 1.1.5(1)

If rows of B and C matrices for a given graph are arranged in the same element order then

$$B_t . C = 0$$

or $$C_t . B = 0 \qquad 1.1.12$$

Proof:

To prove this, we have to show that the scalar product of i-th column B_i of B, and j-th column C_j of C is zero for all i and j, or $(B_i . C_j) = 0$.

Column B_i has non-zero elements only in those rows which correspond to the branches incident with i-th junction. Similarly, C_j has no-zero elements only in those rows which correspond to the branches included in the j-th closed path. If i-th junction is not incident with any of the branches included in the j-th closed path, B_i will have zero elements corresponding to the nonzero elements of C_j giving $B_i . C_j = 0$.

In the contrary case, the i-th junction will be incident with exactly two adjacent branches of j-th closed path, so there will be nonzero elements in just two matching rows of B_i and C_j. If both of these elements are of the same sign in B_i, then they must be of opposite sign in C_j and vice versa by the rules already given for determining b_{ij} and c_{ij}. In either case,

$B_i . C_j = +1 - 1 = 0$, thus

proving the theorem that

$$B_t . C = 0$$

or $$C_t . B = 0$$

If matrices B and C are partitioned along the branches of co-tree and the tree, then

$$B_t . C = \left[(B_T)_t \quad (B_L)_t \right] \begin{bmatrix} C_T \\ C_L \end{bmatrix} = 0 \qquad 1.1.13$$

$$(B_T)_t \, C_T + (B_L)_t \, C_L = 0 \qquad 1.1.14$$

$$C_T = -(B_T)_t^{-1} (B_L)_t \qquad 1.1.15$$

$$C_T = - P_T (B_L)_t \qquad 1.1.16$$

Relation 1.1.16 is very important in computation of C_T directly from P_T and B_L.

Cut-Set(10)

Another important term in graph theory is the cut-set. This concept is complementary to the basic closed-path.

Definition 1.1.16(10)

Basic Cut-set: A cut-set is a set of branches of a connected graph G such that the removal of these branches from G splits the graph G into two separate graphs provided that no proper subset of this set changes the graph G into two separate graphs.

A basic cut-set is defined by choosing a tree of the graph. Each basic cut-set is identified by the branches of the tree. There are as many basic cut-sets in a graph as there are number of the branches of the tree, i.e. (j-1). Each basic cut-set of the graph will contain a branch of the tree and exactly those links of the corresponding co-tree for which the branch is in each of the circuits formed by the selected links. To assign an orientation to the branches of each basic cut-set, we assume that the identifying branch of the tree is positively oriented for the cut-set.

Definition 1.1.17(1,10)

Cut-set orientation: An orientation is assigned to the cut-set by assuming that the identifying tree branch is positively oriented. If any link selected has its initial junction in the same sub-tree (a part of the tree) in which the initial junction of the identifying tree branch is, it is said to be positively included, if otherwise it is negatively included.

Definition 1.1.18(1,10)

Basic Cut-Set Matrix D: It is a matrix with (j-1) columns and b rows where columns correspond to the basic cut-sets and rows to the number of branches of the graph such that $(d_{ij}) = (+1, -1, 0)$ if i-th branch is(positively, negatively, not) included in the j-th basic cut-set.

Each column of D is numbered as the corresponding identifying tree branch is numbered. D for the graph of Fig. 1. 1. 2a with the tree shown in Fig. 1. 1. 3 is

$$D = \begin{array}{c|c|c|c|} & a^1_1 & a^1_2 & a^1_3 \\ \hline a^1_1 & 1 & & \\ \hline a^1_2 & & 1 & \\ \hline a^1_3 & & & 1 \\ \hline a^1_4 & 1 & 1 & 1 \\ \hline a^1_5 & & 1 & 1 \\ \hline \end{array} = \begin{array}{|c|} \hline D_T \\ \hline D_L \\ \hline \end{array} \qquad 1.1.17$$

where $D_T = U_T$ 1. 1. 18

Theorem 1. 1. 7(1)

In a connected graph with a tree,

$D_L = -(C_T)_t$

Proof:

From the definition, if the i-th link is positively included in the j-th basic cut-set, then its initial junction lies in the same subtree in which the initial junction of j-th tree branch is, and the final junctions of the i-th link and j-th tree branch lie in same subtree. Therefore, the path from the final junction back to the initial junction of the i-th link, which completes the i-th basic closed path must include the j-th tree branch in negative sense, since this is the only tree branch which connects the two subtrees in question .

If the link is negatively included, then its final junction and the initial junction of the j-th tree branch will be in the same subtree, and the initial junction of i-th link and final junction of j-th tree branch will be in the same subtree. Therefore, the path from the final junction back to the initial junction of the i-th link, which completes the i-th closed path must include the j-th tree branch in a positive sense, since this is the only tree branch which connects the two subtrees in question. If i-th link is not included in the same j-th basic cut-set, then both the initial and the final junctions of the link will be in the same sub-tree and i-th basic closed path will be completed without including j-th tree branch.

∴ This proves the theorem that if, in a connected graph with a specified tree, the i-th link is (positively, negatively, not) included in the j-th basic cut-set, then the j-th tree branch is (negatively, positively, not) included in the i-th basic closed path i.e.

$d_{ij} = - c_{ji}$ or

$D_L = - (C_T)_t.$

Theorem 1.1.8(1)

The basic cut-set matrix D corresponding to a tree of a given connected graph is equal to the branch junction incidence matrix post-multiplied by the transpose of the junction to datum path matrix P_T of the same tree.

i.e. $D = B.\,(P_T)_t$ 1.1.19

Proof $$D = \begin{array}{|c|}\hline D_T \\ \hline D_L \\ \hline\end{array} = \begin{array}{|c|}\hline U_T \\ \hline -(C_T)_t \\ \hline\end{array} \qquad 1.1.20$$

$$D = \begin{array}{|c|}\hline U_T \\ \hline B_L\,(P_T)_t \\ \hline\end{array} \qquad \text{(From equation 1.1.16)} \qquad 1.1.21$$

$$D = \begin{array}{|c|} \hline B_T(P_T)_t \\ \hline B_L(P_T)_t \\ \hline \end{array} = \begin{array}{|c|} \hline B_T \\ \hline B_L \\ \hline \end{array} [P_T]_t \qquad 1.1.22$$

$$D = B.\ (P_T)_t \qquad 1.1.23$$

Theorem 1.1.9$^{(1)}$

$$C_t\, D = 0 \quad \text{or} \quad D.C_t = 0$$

Proof:

$$D = B[P_T]_t$$

$$C_t . D = C_t . B\ [P_T]_t$$

$$C_t .\ D = 0\ [P_T]_t = 0$$

This completes the necessary review of the linear graph theory. Various definitions and relations described above will be used in the following section.

1.2 The Linear Graph as a Model of Electrical Network.

In this chapter, we select a linear graph as a model for the electrical network. The branches of a linear graph are each assigned an impedance operator which is non-zero and finite. Various properties of a linear graph are used to describe the quantities and relations that exist in an electrical network. The electrical network problem is formulated in an abstract form to find the general condition on the impedance operator of the network for the unique existence of the solution. This approach also helps us in finding which non electrical problems can be solved by the electrical network approach.

a. The Assignment of Electrical Quantities to the Various Spaces of a Linear Graph.(8,9,22)

We choose a linear graph as the topological structure of the underlying material network. The properties and relations of a linear graph are fully known to us. Various quantities are assigned to the spaces of linear graph in a way that when properly interpreted, they should satisfy all the relations and quantities that exist in an electrical network.

In a linear graph, three spaces are recognized,

i Junction space

ii Branch space

iii Closed-path space

The elements of an arbitrary Abelian group are assigned to the spaces of a linear graph. The practice of assigning group structure to the various spaces of a complex is very familar to the algebraic topologist. In our approach, we shall not be rigorous and formal, but shall be more intuitive and descriptive.

To start with, a set of arbitrary elements of an Abelian group is assigned to the basic closed-path space. The matrix notation is used to represent the assigned set of elements to facilitate the computation work. Assigned set of elements to the closed path space is represented by a column matrix $[i']$ having as many rows as there are number of basic closed paths.

Assignment of a set of quantities to the circuit space will induce an assignment of corresponding set of quantities to the branch space by the relation

$$[i] = C[i'] \qquad 1.2.1$$

Another set of arbitrary elements of an Abelian group can be assigned over and above the induced set of elements [i] . Let the arbitrarily assigned elements be represented by column matrix [I] . The complete set of branch quantities are shown by [J] .

$$[J] = [i] + [I] \qquad 1.2.2$$

Branch quantities [J] will induce a set of quantities on the non-datum junctions by the rule

$$[I'] = B_t [J] \qquad 1.2.3$$

The matrices B and C are defined in the section 1.1 for a linear graph.

$$[I'] = B_t [I] + B_t [i] \qquad 1.1.4$$

$$[I'] = B_t [I] + B_t C [i'] \qquad 1.1.5$$

And $B_t C = 0$ (From equation 1.1, 12)

$$[I'] = B_t [I] \qquad 1.1.6$$

This completes the assignment of the required quantities to the space of a linear graph. This set of quantities is called the primary set of quantities or elements. The algebraic diagram for primary quantities is shown in Fig. 1.2.1a, showing various transformations.

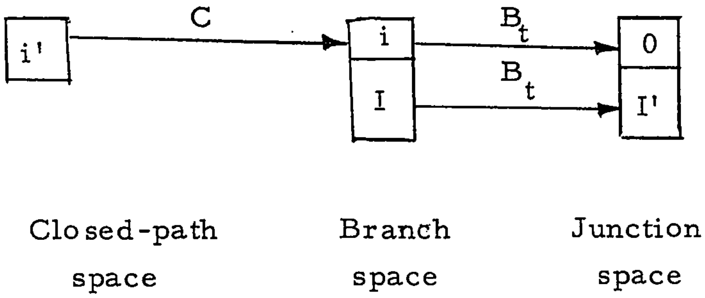


Fig. 1.2.1(a) Algebraic Diagram for Primary Quantities.

Fig. 1.2.1(a) can be redrawn where dots are used to identify the different spaces.

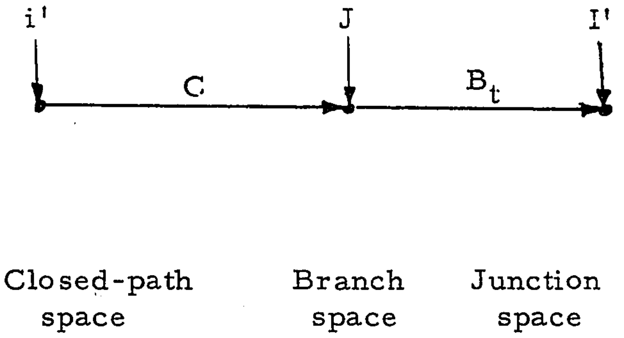


Fig. 1.2.1(b)

Similarly a dual set of quantities can be attached to the spaces of linear graph. The dual set of quantities is taken again as elements of Abelian group. The name "dual set of quantities" signifies the dual laws of transformation as compared to those of the primary quantities.

Let an arbitrary set of quantities be attached to the junction space and represented by column matrix $[\bar{E}]$. If one of the junctions is taken as datum junction, the assigned set of quantities to the various non-datum junctions can be referred with respect to the quantity assigned to the datum junction. Denoting the datum junction by n, the assigned quantity to the datum junction is written as $\bar{E}_n$. The non-datum quantities, with respect to the datum quantity are

$$[E'] = [\bar{E} - E_n] \qquad 1.2.6$$

The set of induced relative branch quantities $[E]$ is given by

$$[E] = B[E'] \qquad 1.2.7$$

Another set of arbitrary elements $[e]$ of an Abelian group can be assigned to the branches over and above the induced set. The total branch quantities are thus

$$[V] = [E] + [e] \tag{1.2.8}$$

The set of branch quantities $[V]$ induces a set of quantities $[e']$ to the closed-path space by the rule

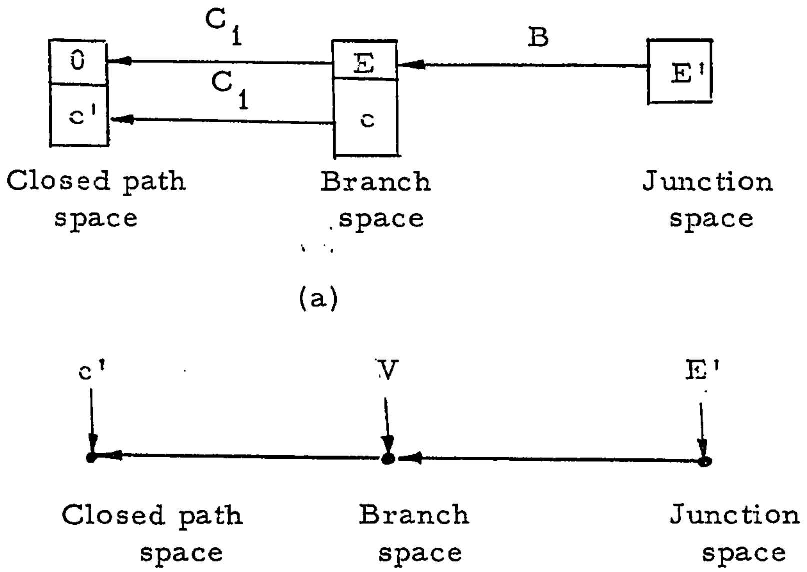
$$[e'] = C_t[V] \tag{1.2.9}$$

$$[e'] = C_t[E] + C_t[e] \tag{1.2.10}$$

$$[e'] = C_tB\,[E'] + C_t[e] \quad (C_tB = 0) \tag{1.2.11}$$

$$[e'] = C_t[e]. \tag{1.2.12}$$

Various relations for dual set of quantities can be summarized in Fig. 1.2.2.



(a)

(b)

Fig. 1.2.2. Transformation Diagram for Dual Quantities.

b. Kirchhoff's Laws - Dependent on the Topology of the Network.

The set of various primary and dual quantities attached to the spaces of linear graph is a set of elements of an arbitrary Abelian group. The various assigned quantities shall be interpreted so that they correspond to the quantities that exist in an electrical network. All the relations introduced above become the corresponding relations of an electrical network.

$[i']$ = Closed-path current.

$[I']$ = Current across junction-pairs or open-path current.

$[I]$ = Current generators across the branches.

$[J]$ = Total branch-Current.

$[e']$ = Total impressed voltage in closed paths.

$[e]$ = Voltage generators in the branches

$[E']$ = Voltages across junction - pairs or open paths voltage.

$[V]$ = Voltages across branches.

The interpretation of the various quantities of a linear graph in terms of electrical quantities, describes the two important laws of electricity, called:

i Kirchhoff's current law.

ii Kirchhoff's voltage law.

The current law gives a relationship between the currents of various branches, and voltage law gives the relationship between the voltages of various branches. All the relations among the primary quantities are the codification of current law in various forms. The relations among dual quantities codify the voltage law in various forms.

The current law in various forms is as below:

$$
\begin{aligned}
[i] &= C[i'] \\
[I'] &= B_t[J] \qquad\qquad 1.2.13 \\
[0] &= B_t[i]
\end{aligned}
$$

The voltage law can be written as below,

$$
\begin{aligned}
[\bar{E}] &= B[E'] \\
[e'] &= C_t[V] \qquad\qquad 1.2.14 \\
[0] &= C_t[E]
\end{aligned}
$$

The current and voltage laws are dependent on the topology of the given network, i.e. how the various branches of the network are connected. So far, to describe the current and voltage laws, the concept of impedance has not been introduced.

The currents and voltages are existing independently in the network so far, and no dependence is introduced among currents and voltages. Such types of relations are very common in algebraic topology.

c. Ohm's Law - A distinct property of the Electrical Network(2, 3, 9)

The electrical network introduces a new concept, so far unknown to the algebraic topologists . Now, there exists a relationship between the primary and dual quantities of a network, and two sets of quantities can no longer assume values independent of each other.

In an electrical network, by Ohm's Law, there exists a one to one linear transformation between the set of primary branch quantities and set of dual branch quantities. Representing the primary and dual branch quantities by [J] and [V], the isomorphism between [J] and [V] is shown in Fig. 1.2.3. The relation between [J] and [V] is

$$\begin{aligned} [V] &= Z\,[J] \\ [J] &= Y\,[V] \end{aligned} \qquad 1.2.15$$

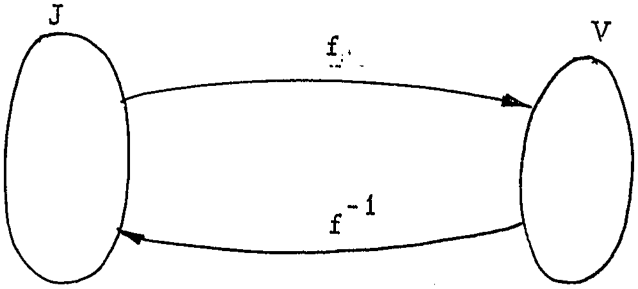


Fig. 1.2.3 Isomorphism between Two Abelian Groups.

Function f represents a one to one mapping from group [J] onto group [V]. f^{-1} represent an inverse one to one mapping from the group [V] onto the group [J]. Since the elements of the group [V] and [J] are represented in matrix form, function f is represented by the square and nonsingular matrix Z. Z is bxb matrix where b is the number of branches of the network. f^{-1} is then obtained from Z^{-1} and written as $Y = Z^{-1}$.

In electrical terms, f is called the impedance function and f^{-1} the admittance function. With the introduction of Ohm's law, the independent diagrams of Fig. 1.2.1b and Fig. 1.2.2b are combined into one diagram Fig. 1.2.4.

The impedance matrix is called the primitive impedance matrix of the network

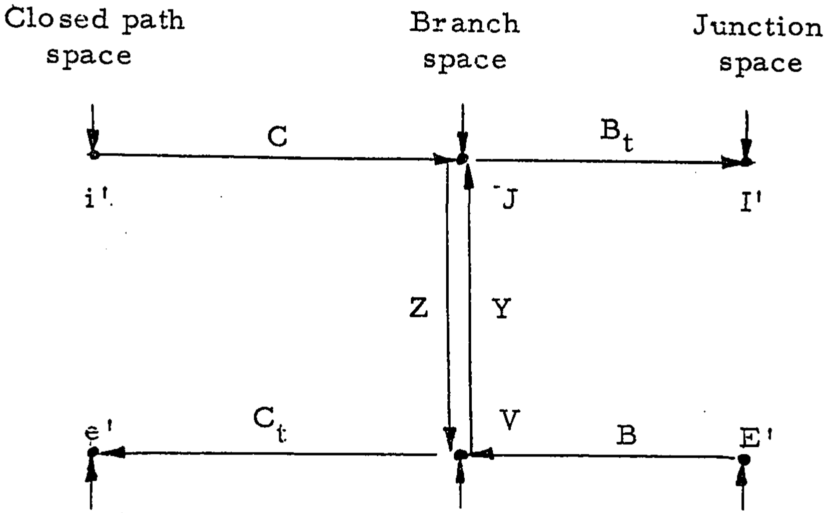


Fig. 1.2.4 Relations of an Electrical Network.

When a linear graph is given, the matrices B and C are fully described. Existence of the matrices Z and Y induces a transformation from [i'] to [e'] and from [E'] to [I'].

The transformation from [i'] to [e'] can be traced as follows:

i. Take an element [i'] of the closed-path space. With the help of matrix C, transform it to the branch-space, [I].

ii. Transform the above element into the V-space by transformation Z.

iii. The element of V-space can be then transformed to the e'-space by C.

Thus

$$[i] = C\,[i'] \tag{1.2.16}$$

From equation 1.2.15

$$[V] = Z[J]$$

$$[V] = ZC\,[i'] \tag{1.2.17}$$

$$[e'] = C_t\,[V] \tag{1.2.18}$$

Putting equation 1.2.17 into equation 1.2.18, we have

$$[e'] = C_t Z C [i'] \qquad 1.2.19$$

Thus the transformation $(C_t ZC)$ is being induced from $[i']$ to $[e']$.

Similarly a transformation from $[E']$ to $[I']$ is induced as

$$[I'] = B_t Y B [E'] \qquad 1.2.20$$

A complete transformation diagram can be drawn as in Fig. 1.2.5 with all the induced transformations indicated.

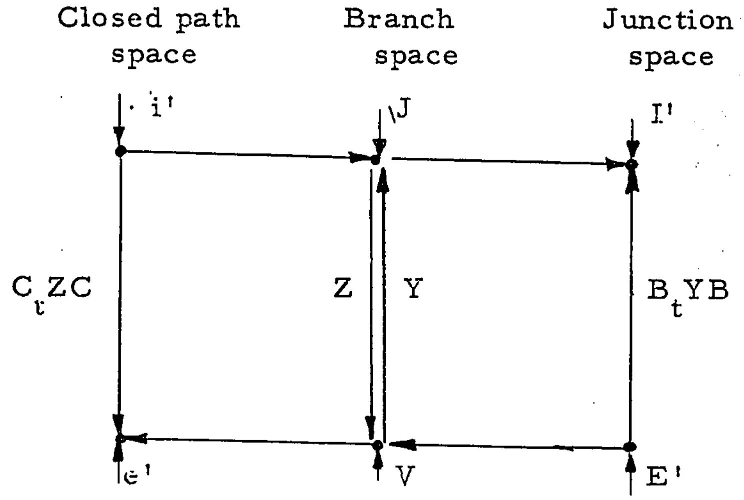


Fig. 1.2.5 Complete Transformation Diagram.

With each branch of the network, three distinct concepts are connected i.e. current, voltage, impedance. Currents among various branches are related by the current law, and the voltages are related by the voltage law, the branch-voltages and branch-currents are related by the Ohm's law. By now, all the relations and quantities that exist in an electrical network are fully described. Various relations developed so far will be now used in finding unknown quantities of an electrical network from a given set of known quantities.

d. Formulation of an Electrical Network Problem (1, 8)

Out of all the quantities introduced in an electrical network, some quantities are known and others are to be found out. We shall now formulate the electrical network problem in a most general way.

Given

i A Linear graph of the network, and the matrices C and B.

ii A set of voltages $[e']$ impressed in series with the basic closed-paths and a set $[I']$ of the currents impressed from the outside on the junction pairs.

iii The impedance function Z and its inverse function Y

Find:

The branch currents and the branch voltages $[J]$ and $[V]$ respectively such that

i $C_t [V] = [e']$

ii $B_t [J] = [I']$

iii $[V] = Z [J]$ and $[J] = Y [V]$ 1.2.21

The solution of this problem can be easily established with the help of Fig. 1.2.5. In Fig. 1.2.5, $[e']$ and $[I']$ are given, the $[V]$ and $[J]$ can be found out so that the relations of equation 1.2.21 are satisfied. The relations of equation 1.2.21 are already codified by the transformations of Fig. 1.2.5. The solution is found as follows:

From Fig. 1.2.5

$[e'] = (C_t Z C) [i']$

$[I'] = (B_t Y B) [E']$

$[J] = C [i'] + Y.B [E']$ 1.2.22

And

$$[i'] = (C_t Z C)^{-1} [e'] \qquad 1.2.23$$

$$[E'] = (B_t Y B)^{-1} [I'] \qquad 1.2.24$$

Putting the values of $[i']$ and $[E']$ in equation 1.2.22,

$$[J] = C(C_t Z C)^{-1} [e'] + Y B(B_t Y B)^{-1} [I'] \qquad 1.2.25$$

$$[V] = Z [J]$$

$$[V] = Z C (C_t Z C)^{-1} [e'] + B(B_t Y B)^{-1} [I'] \qquad 1.2.26$$

The equations 1.2.25 and 1.2.26 give the solution of the electrical network in an explicit form. For solving the network, the inverses of $(C_t ZC)$ and $(B_t YB)$ matrices are required .

In actual network problems, it is not required to find both inverses of $(C_t ZC)$ and $(B_t YB)$. The network problem can be solved(8,9) by simply finding either $(C_t ZC)^{-1}$ or $(B_t YB)^{-1}$. In actual problems, $[e]$ and $[I]$ are always given instead of $[e']$ and $[i']$ where

$$[e'] = C_t [e]$$

$$[I'] = B_t [I] \qquad 1.2.27$$

Even if $[e']$ and $[I']$ are given, it is always possible to find out $[e]$ and $[I]$ by equation 1.2.27. Thus if $[e]$ and $[I]$ are given, the network problem can be stated as follows

Given

i A linear graph of an electrical network and matrices C and B.

ii The voltage $[e]$ and the current $[I]$.

iii The branch (primitive)(3) matrices $[Z]$ and $[Y]$

Find

$[V]$ and $[J]$ such that equation 1.2.21 is satisfied.

To solve this problem, either $(C_tZC)^{-1}$ or $(B_tYB)^{-1}$ is required. If the network is solved by finding $(C_tZC)^{-1}$, the method is called the closed-path or mesh method of solving the network. If $(B_tYB)^{-1}$ is required to solve the given network, the method is called the junction pair method of solving the network.

i <u>Mesh or Closed-Path Method.</u>

In this method, the network is solved by finding the value of closed-path variables. The other unknown quantities are found by the routine transformations without any inverse, once the closed-path variables are solved.

We know

$$C_t[V] = [e'] \qquad 1.2.28$$

$$[V] = [e] + [E] \qquad 1.2.29$$

So

$$C_t[V] = C_t[e] = [e'] \qquad 1.2.30$$

since $C_t[E] = 0$ (from equation 1.2.14)

As $[e]$ is given, $[e']$ can be found. The transformation diagram for the closed path method is given in Fig. 1.2.7.

Closed path space Branch space

$$\begin{tikzcd}
i' \arrow[r, "C"] & J \arrow[d, "Z"] \\
e' \arrow[u, "(C_tZC)^{-1}"] & V \arrow[l, "C_t"]
\end{tikzcd}$$

Fig. 1.2.7 Transformation Diagram for the Closed-path Method.

From Fig. 1.2.7, the following equations can be set up.

$$[V] = Z[J]$$

$$[E] + [e] = Z([i] + [I]) \qquad 1.2.31$$

Transforming the equation 2.31 to the closed path space, we get

$$C_t[E] + C_t[e] = C_t Z([i] + [I]) \qquad 1.2.32$$

But $C_t[E] = 0$

so

$$C_t[e] - C_t Z[I] = C_t Z[i] \qquad 1.2.33$$

And $[i] = C[i']$

$$C_t Z C[i'] = C_t([e] - Z[I]) \qquad 1.2.34$$

$$[i'] = (C_t ZC)^{-1} C_t([e] - Z[I]) \qquad 1.2.35$$

In equation 1.2.35, [e] and [I] are known. Once $(C_t ZC)^{-1}$ is found, [i'] can be solved.

$$[J] = [I] + [i]$$

$$[J] = [I] + C[i'] \qquad 1.2.36$$

From equation 2.36,

$$[J] = [I] + (C_t ZC)^{-1} C_t([e] - Z[I]) \qquad 1.2.37$$

$$[J] = C(C_t ZC)^{-1} C_t[e] + (U - C(C_t ZC)^{-1} C_t Z)[I] \qquad 1.2.38$$

And $[V] = Z[J]$

From equation 1.2.38,

$$[V] = ZC(C_t ZC)^{-1} C_t[e] + Z(U - C(C_t ZC)^{-1} C_t Z)[I] \qquad 1.2.39$$

Equations 1.2.38 and 1.2.39 give the complete solution to the network problem by the closed path method. Once the vvalue of [i'] is found, the rest of the unknowns can be found by routine transformations without finding any inverse.

ii **Junction - pair Method.**

Any given electrical network can also be solved by junction pair method. To find [V] and [J] , it is required to find the value of [E'] , and other unknown quantities can be then easily derived. The required transformation diagram for the junction-pair method can be shown in Fig. 2.8.

Branch space Junction space

$$\begin{tikzcd} J \arrow[r, "B_t"] & I' \arrow[d, "(B_tYB)^{-1}"] \\ V \arrow[u, "Y"] & E' \arrow[l, "B"] \end{tikzcd}$$

Fig. 1.2.8 Transformation Diagram for the Junction-Method.

From the Fig. 1.2.8,

$$[J] = Y[V]$$

$$[I] + [i] = Y[E] + Y[e] \qquad 1.2.40$$

Transforming the quantities to the junction space,

$$B_t[I] + B_t[i] = B_t Y [E] + B_t Y [C] \qquad 1.2.41$$

And $B_t[i] = 0$

So

$$B_t[I] - B_tY[e] = B_t Y [E] \qquad 1.2.42$$

$$[E] = B[E']$$

Putting this value in 1.2.42 we get

$$B_t[I] - B_t Y[e] = B_t YB[E'] \qquad 1.2.43$$

$$[E'] = (B_t YB)^{-1}(B_t[I] - B_t Y[e]) \qquad 1.2.44$$

$$[V] = B[E'] + [e]$$

$$[V] = B(B_t YB)^{-1} B_t([I] - Y[e]) + [e] \qquad 1.2.45$$

$$[V] = (U - B(B_t YB)^{-1} B_t)[e] + B(B_t YB)^{-1} B_t[I] \qquad 1.2.46$$

$$[J] = Y[V]$$

$$[J] = Y(U - B(B_t YB)^{-1} B_t)[e] + YB(B_t YB)^{-1} B_t[I] \qquad 1.2.47$$

The equations 1.2.46 and 1.2.47 give the complete solution to the network problem and the inverse of $(B_t YB)$ need only be calculated.

If the network is to be solved by the closed path method, $(C_t ZC)^{-1}$ is to be calculated and is called a closed path solution matrix. If the network is solved by the junction method, $(B_t YB)^{-1}$ is to be calculated, and is called a junction solution matrix.

There is another method of solving the network which is almost equivalent to the junction method, called the Tree Method.

iii <u>Tree Method:</u> In this method matrix B_T is used to find the tree voltages from $[E']$ by the rule:

$$[E_T] = B_T[E'] \qquad 1.2.48$$

$$[E'] = B_T^{-1}[E_T]$$

$$[E'] = (P_T)_t[E_T] \qquad 1.2.49$$

From equation 1.2.44, put the value of $[E']$ in equation 1.2.49

$$(B_tYB)^{-1}(B_t[I] - B_tY[e]) = (P_T)_t[E_T] \tag{1.2.50}$$

$$B_t([I] - Y[e]) = (B_tYB)(P_T)_t[E_T] \tag{1.2.51}$$

Premultiplying both sides by P_T, we have

$$P_TB_t([I] - Y[e]) = P_T(B_tYB)(P_T)_t\ [E_T] \tag{1.2.52}$$

But $P_TB_t = D_t$ (From equation 1.1.19)

$$D_t([I] - Y[e]) = (P_TB_t)\,Y\,(B(P_T)_t)\,[E_T]$$

$$D_t([I] - Y[e]) = (D_tYD)[E_T] \tag{1.2.53}$$

$$[E_T] = (D_tYD)^{-1}D_t([I] - Y[e]) \tag{1.2.54}$$

And $[E] = D[E_T]$

$$[V] = [E] + [e]$$

$$[V] = (U - D(D_tYD)^{-1}D_tY)[e] + D(D_tYD)^{-1}D_t[I] \tag{1.2.55}$$

$$[J] = Y(U - D(D_tYD)^{-1}D_tY)[e] + YD(D_tYD)^{-1}D_t[I] \tag{1.2.56}$$

Thus to solve the network, $(D_tYD)^{-1}$ is to be calculated, and $(D_tYD)^{-1}$ is called the tree solution matrix. Any network can be solved by either of the three methods.

To solve the network, inverse of either (C_tZC) or (B_tYB) matrix is required. The next question is: does the inverse of these matrices always exist? In the next section, we shall find out the condition for the unique existence of the solution to the network problem: under what conditions does the solution to the network problem exist and is unique?

c. The Existence of the Unique Solution to the Network Problem.(8, 9)

Given a graph of the network, the existence of the unique solution depends on the nature of impedance operator Z. To start with, a term ohmicness will be defined.

Definition 1.2.1(1, 8)

Ohmicness: Let J - be any complex space of dimension b, and V- a dual space of linear function of J, , then a linear transformation Z of J onto V is termed ohmic, if for any [J] $\neq$ 0 in J

$$(Z\,[J])_t \cdot [J] \neq 0 \qquad \rightarrow \quad J_t \quad {}_t \quad J_t^{*} Z J$$

or

$[J]_t^{*} \cdot Z \cdot [J] \neq 0$ where $[J]_t^{*}$ is the complex conjugate of [J] .

Theorem 1.2.1(8)

If Z is ohmic, Y is also ohmic.

Proof:

$$\text{Let } [V] = Z[J]$$

$$[J] = Y[V]$$

Y is ohmic, if for [V] $\neq$ 0,

$$(Y[V])_t \cdot [V] \neq 0$$

$$\text{or} \quad [J]_t^{*} \cdot [V] \neq 0$$

$$\text{or} \quad [J]_t^{*} \cdot Z \cdot [J] \neq 0 \qquad 1.2.57$$

By definition, for [V] $\neq$ 0, it implies [J] $\neq$ 0

Since Z is ohmic,

$$[J]_t^{*} \cdot Z \cdot [J] \neq 0 \quad \text{for } [J] \neq 0 \qquad 1.2.58$$

This implies that equation 1.2.57 is true, so Y is ohmic.

Theorem 1.2.2(1),(8)

If the primitive impedance matrix is ohmic, then $(C_t ZC)^{-1}$ and $(B_t YB)^{-1}$ exist, and the network problem has unique solution.

Proof:

Let $[i']$ be an element of i'-space, and

$$(C_t ZC)\ [i'] = 0 \qquad 1.2.59$$

$$[i']_t^* \ (C_t ZC)\ [i'] = 0 \qquad 1.2.60$$

$$([i']_t^* C_t)\ Z\ (C\,[i']) = 0 \qquad [i^*]_t = [i']_t^* C_t$$

$$[i_t]^* \ Z\ [i] = 0 \qquad 1.2.61$$

Since Z is ohmic,

then equation 1.2.61 is true if $[i] = 0$

Transformation C transforms each element of i'-space to an element of J-space and it is an isomorphism into. So when $[i] = 0$, it implies $[i'] = 0$ 1.2.62

To prove that $(C_t ZC)$ is one to one mapping, we have to prove that Kernel of this mapping is zero, i.e. to a null element in a domain, there corresponds one and only one element in the range which is also the null element. It is required to prove that for all

$$f\,[i'] = 0$$

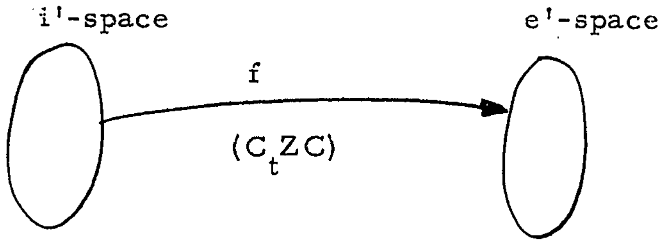


Fig. 1.2.10 Mapping from i'-space to e'-space.

$$[i'] = 0 \qquad 1.2.62$$

or for $(C_tZC)[i'] = 0, \quad [i'] = 0$ 1.2.63

We have already proved in equation 1.2.59 that if $(C_tZC)[i'] = 0$

then $[i'] = 0$ which is equation 1.2.63.

Thus we have proved that (C_tZC) is one-to-one and its inverse exists i.e. $(C_tZC)^{-1}$ exists. Thus the solution to the network problem exists as long as Z is ohmic.

As Z is ohmic, Y is also ohmic and it can be proved that B_tYB is one-to-one and its inverse exists and the solution of network problem exists.

It has to be shown that if Z is ohmic, the solution to the network problem is unique.

Let us assume that for a given network problem, two different solutions exist, i.e. [V], [J] and $[V]^*$, $[J]^*$

Let

$$[V_C] = [V] - [V]^*$$
$$[J_e] = [J] - [J]^* \qquad 1.2.64$$

For a given [I'] and [e'], two different solutions are assumed.

$$B_t[J] = [I'] \qquad C_t[V] = [e']$$

$$B_t[J^*] = [I'] \qquad C_t[V^*] = [e'] \qquad 1.2.65$$

Since [I'] and [e'] are given, therefore

$$B_t[J_e] = 0 \qquad C_t[V_e] = 0. \qquad 1.2.66$$

$B_t[J_e] = 0$ means that

either $[J_e] = 0$ or $[J_e]$ is an element of i'-space.

As we have assumed $[J_e]$ not to be equal to zero, therefore

$$[J_e] = C\,[i'] \qquad 1.2.67$$

Similarly

$$[V_e] = B\,[E'] \qquad 1.2.68$$

$$[V_e] = Z\,[J_e]$$

$$C_t\,[V_e] = C_t\,Z\,C\,[i'] \qquad 1.2.69$$

From equation 1.2.66 $C_t\,[V_c] = 0$

So $(C_t Z C)\,[i'] = 0$ 1.2.70

Therefore

$$[i'] = 0$$

and $[J_e] = 0$ (from equation 1.2.67)

It means that $[J] = [J]$

and $[V] = [V]$

Thus we have shown that whenever Z is ohmic, the solution to the network problem exists and is unique. Studying the network problem with the help of linear graph has helped in finding the most general condition for the unique existence of the solution to the network problem.

The matrices B, and C are very easy to find and they are very easy to handle on a computer.

This completes the study of the electrical networks by the linear graph approach. In the following chapter, the various drawbacks of the linear graph model of the electrical network will be pointed out. An attempt will be made to rectify the drawbacks and present another model of the electrical network which will overcome these drawbacks of linear graph model. The development of the new model was initiated by Kron.(4,5,6, 8,9)

1.3 The Linear Graph - A Wrong Model of the Electrical Network.

In the previous chapter, an electrical network was modelled by a linear graph. Various relations and properties of the linear graph were used to describe an electrical network. The question now arises why is the linear graph chosen as a model? Is linear graph the right choice? If not, then what is the right model which can represent an electrical network. The answer to all these questions will be attempted in this chapter.

a. The Choice of the Linear Graph.

The reasons for choosing a linear graph as a model for the electrical network may be the following:

1. In the statements of Kirchhoff's two laws of electricity, there is a clear mention of junctions and circuits. From this, one may immediately conclude that any model of electrical network needs the presence of junctions and circuits. The linear graph is the simplest structure which contains both

junctions and circuits. Such an interpretation of the Kirchhoffs' laws leads us to the choice of a linear graph as a model.

2. An electrical network drawn on a paper looks more like a linear graph with the presence of all those branches and junctions.

The above factors lead us to the choice of a linear graph as a model. After choosing the linear graph, we try to fit the properties and relations of an electrical network into those of a linear graph. Whenever the properties of the two contradict, the contradictions are removed by introducing some constraints into the properties of a linear graph. In the following section, we shall compare the properties of a linear graph with those of an electrical network.

b <u>Properties of a Linear Graph.</u>(5,6,7)

In a linear graph, an arbitrary set of primary and dual quantities (elements of an Abelian group) can be attached to the junction space over and above the induced set of quantities.

This does not require that the sum of all the quantities attached to the junction space be equal to zero. Also there does not exist any relation whatsoever between the total branch and junction quantities. Take for example the set of primary quantities. Fig. 1.3.1 shows the branch and junction spaces with the attached primary quantities.

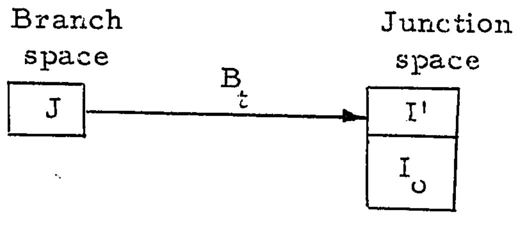


Fig. 1.3.1 Branch - Junction Space.

The branch quantities are shown by [J]. [I'] is a set of the induced quantities and $[I_o]$ is the set of arbitrary quantities assigned to the junction-space. The total set of junction quantities is represented by $[J_o]$. By the property of a linear graph, there is no relation such as

$$\sum_{1}^{j} [J_o] = 0 \tag{1.3.1}$$

Since an arbitrary set of quantities can be attached to the junction space. Also the quantities [J] and $[J_o]$ are not related to each other in a linear graph.

Similarly in a linear graph, an arbitrary set of quantities can be attached to the closed-path space. The quantities of closed-path space and branch-space have no relation between each other. Let us take the dual set of quantities for example. Fig. 1.3.2 shows the branch and closed-path dual set of quantities.

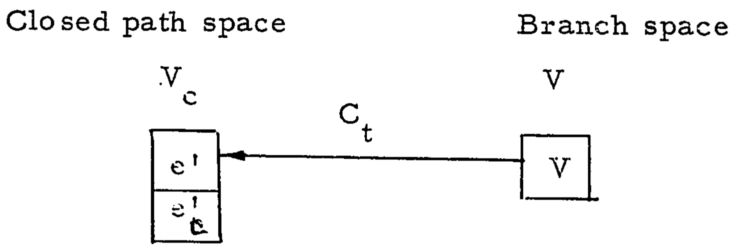


Fig. 1.3.2 Branch - Closed-path Space.

The branch quantities are shown by [V]. [e'] is the set of induced closed-path quantities, and $[e'_c]$ is the set of arbitrary assigned quantities to the closed-path space. The total closed-path quantities are shown by $[V_c]$. By the property of the linear graph, there is no relation such as

$$\sum_{1}^{C} [V_c] = 0 \tag{1.3.2}$$

since one can attach an arbitrary set of quantities to the closed-path space, the sum of which need not be zero. Also, the quantities [V] and $[V_c]$ are not related to each other.

c. Kirchhoff's Laws of Electricity. (16,17,18)

With an electrical network, the three basic concepts are connected, i.e. current, voltage and impedance. Kirchhoff's laws are concerned with the relation of currents among the various branches of the network, and the relation of the voltages among the various branches. The two laws of Kirchhoff are

i Current Law.
ii Voltage Law.

i Current Law.

If there are no current generators, the current law can be stated as follows:

"The sum of the currents of the branches incident at the same junction is equal to zero"

The current law defines a relationship among the currents in the branches of the network. The path of the flow of currents is supplied by the branches, and the current law gives the relation among the currents of those branches which meet at the same point. If there exist current generators also then the current law states!

"The sum of the currents of the branches incident at a junction is equal to the sum of the currents impressed across these branches by the current generators (taking into account the proper orientation). The current law can be written algebraically once a proper orientation is assigned to the branch currents and current generators. Take for example, Fig. 1.3.3. An

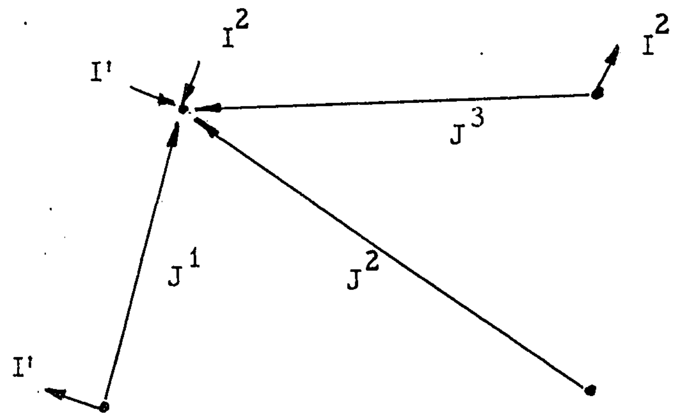


Fig. 1.3.3 Branch Currents and the Current Generators of the Branches incident at a Junction.

orientation is assigned to the currents by showing arrows as in Fig. 1.3.3. We take the current entering the junction as positive and leaving the junction as negative. By the current law,

$$-J^1 + J^2 + J^3 = I^1 + I^2 \qquad 1.3.3$$

I^1 and I^2 are taken as positive, since when these impressed currents flow in the branches, they flow away from the junction considered. Note that the sum of all the currents entering the entire network from outside is equal to zero, since whichever current enters the network from outside, same current leaves it.

ii Voltage Law

The voltage law relates the various branch voltages of the network by the voltage law, "The sum of the total branch voltages existing in a closed path is equal to the sum of the impressed voltages existing in the branches of the closed-path".

To write the voltage law algebraically, an orientation has to be assigned. Let us refer to the Fig. 1.3.3.

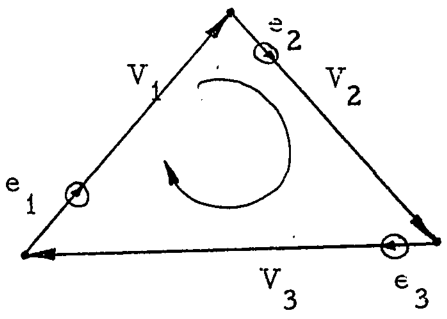


Fig. 1.3.3 Voltages in the Closed-path.

$$V_1 + V_2 + V_3 = e_1 + e_2 + e_3 \qquad 1.3.4$$

The equation 1.3.4 is written by assigning an orientation to the closed-path and the branches. If the branch voltage is along the closed-path orientation, it is positive, otherwise negative. The same rule is applied to the impressed voltages. The voltage law is a statement of the relation among the branch voltages of the network.

d. The Linear Graph - A Violation of the Kirchhoff's Current Laws.(4,5,6)

If the properties of the electric network and the linear graph are compared, we find that the linear graph does not recognise the current law.

The current law demands that the sum of all the currents at a junction must be equal to zero and the sum of the total currents entering the entire network from outside must be also equal to zero.

But in a linear graph, the sum of the quantities attached to the junctions can be arbitrary and need not be equal to zero. If the junction-space is to be recognized, then it will always violate Kirchhoff's current law unless an additional constraint is introduced to the theory of linear graph . Actually the current law statesthat an electrical network model does not require the recognition of the junction space. In an unexcited electrical network, thejunctions always exist, and one cannot ignore them. As soon as the network is excited, the presence of the junction space cannot be recognized by the currents. Any introduction of junction-space in a model of electrical network will always contradict Kirchhoff's current law, and to remove the contradiction one has to introduce the extra constraint equation.

The linear graph contains the junction space and branch-space; It will always violate the current law unless a constraint equation is introduced. The graph theorists avoid the contradiction by introducing the condition that the junctions do not have arbitrary assigned primary quantities but only the induced primary quantities. In fact, a linear graph has more general structure than the electrical network needs. A linear graph is an"oversized" model of an electrical network.

e. 2-Complex - A Violation of Current and Voltage Laws(4, 5, 6, 13)

One might think of representing an electrical network by 2-complex(13), i.e. by a topological structure containing junctions, branches and planes. As 2-complex contains the junctions, so it will violate the current law. Now with the introduction of planes, one can assume that the closed-paths define the boundary of planes.

In an electrical network, whichever voltage exists in the closed-path is being due to the impressed branch voltages. The total branch-voltages in a closed-path must be equal to the impressed branch voltages in that closed-path. In a 2-complex, one can attach an arbitrary set of dual quantities to the closed-paths or planes over and above the induced set of quantities. This property of 2-complex will violate the voltage law which does not allow the existence of arbitrary voltages in the closed-paths, over and above the induced voltages.

The voltage law states the relation between the branch-voltages of the network and does not demand the presence of planes. Any recognition of planes in a model of electrical network will violate the voltage law and we have to introduce another constraint which prohibits the attachment of arbitrary set of dual quantities to the planes. Thus the introduction of planes and junctions in a model of the electrical network is not allowed by the current and voltage laws of electricity.

f. The Branch-Network - A Correct Kirchhoffian Model.(4, 5, 6, 15)

In the previous two sections, it is shown that the linear graph violates the current law, and the 2-complex violates both the current and voltage laws.

Thus any model of an electrical network must not contain junctions and planes. The implication is that an electrical network can be modelled by a topological structure composed only of branches. This conclusion is arrived at simply with the help of Kirchhoff's two laws of electricity which must be satisfied at all times in an electrical network. We may call the branch-network, also the Kirchhoffian model, since it is being logically developed with the help of Kirchhoff's two laws of electricity.

After discovering the fact, that an electrical network can be modelled correctly only by the branches, we have to find out how the branches should be organized so as to satisfy the properties of an electrical network at all times and, further, how to use the branch-model for solving the various network problems.

f. Conclusions

The electrical network is modelled by a linear graph. The various properties of linear graph are used to study an electrical network. A comparison of the properties of the linear graph and electrical network reveals that the electrical network cannot be modelled correctly by a linear graph. The properties of linear graph violate the laws of electricity and the linear graph is not the correct model for an electrical network.

An electrical network can be represented correctly by a model composed of branches only. In the following work, the topological structure composed of branches will be organized and its various properties will be discussed.

CHAPTER 2

P-NETWORK THEORY.

The concepts of 1-network or branch network are generalized to any dimension p. The various relations and quantities that exist in a p-network are described by using the terminology of algebraic topology. The p-network problem is formulated and its solution is given in general form. The electrical network problem which is a special case of p-network problem is described. The concepts of algebraic topology used in our work are elementary and defined wherever used.

2.1 1-Network - The Correct Model.

In chapter 1, it was concluded that an electrical network can be correctly modelled by branch or 1-network. The branch network is orthogonally organized and is tearable. The significance of these terms will be clear as the complete organization of 1-network and its generalisation will be given.

a. Unexcited Network.(4,5,6)

We already know that the branch is an element of a network to which a finite value of impedance is attached. It is assumed that an unexcited network is lying in an abstract n-dimensional space. Take the topological structure of Fig. 2.1.1. It defines five branches, four junctions, two planes. It may be imagined to define three dimensional elements also.

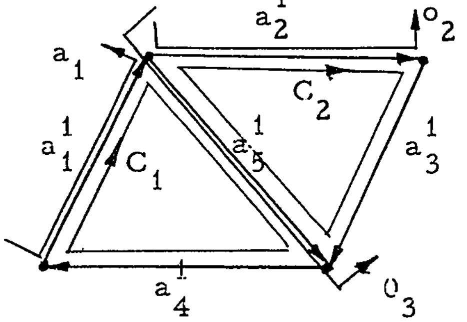


Fig. 2.1.1 Unexcited Network - A Polytope.

An unexcited material network can be assumed to define various different dimensional spaces. From the knowledge of the number of junctions and branches that exist in a network, the number of independent closed 1-paths and open 1-paths can be found. The independent oriented open and closed 1-paths are shown in Fig. 4.1.1. The 1-paths are defined by the branches which are considered its building blocks. The various independent 1-paths can be written as a matrix Q having as many rows as the number of branches and as many columns as the number of independent 1-paths, which is also equal to the number of branches. The matrix Q is called the connection matrix by Kron. In Fig. 2.1.1., there are three independent open 1-paths, and two independent closed 1-paths. (By paths we shall mean independent 1-path in this section).

Definition 2.1.1.

Connection Matrix Q: It is a matrix having as many rows and columns as the number of branches. The columns represent the closed and open 1-paths, the rows represent the branches of the network such that

$(Q_{ij}) = (+1, -1, 0)$ if the ith branch is (positively, negatively, not) included in the j-th 1-path.

Q for Fig. 2.1.1 is:

$$Q = \begin{array}{c|c|c|c|c|c|} & C_1 & C_2 & 0_1 & 0_2 & 0_3 \\ \hline a^1_1 & 1 & & 1 & & \\ \hline a^1_2 & & 1 & & 1 & \\ \hline a^1_3 & & 1 & & & \\ \hline a^1_4 & 1 & & & & \\ \hline a^1_5 & 1 & -1 & & & 1 \\ \hline \end{array} \qquad 2.1.1$$

The choice of independent closed and open 1-path is quite arbitrary.

Properties of Q.

Matrix Q is square and non-singular. Its inverse $A = (Q^{-1})_t$ exists as will be shown later in this chapter. Matrix A can be easily written from the topology of the network instead of obtaining it from the inverse of Q . To write down A from the given network; Once the open and closed 1-paths are given, we note the 1-paths or the combination of 1-paths by which just one branch is traversed.

Defintion 2.1.2

Matrix A has as many rows and columns as there are number of branches. The columns represent the number of independent open and closed 1-paths, the rows the number of branches such that

$(a_{ij}) = (+1, -1, 0)$ if the i-th branch is(positively, negatively, not) traversed by a 1-path .

The matrix A for the Fig. 2.1.1 is:

$$A = \begin{array}{c|c|c|c|c|c|} & C_1 & C_2 & 0_1 & 0_2 & 0_3 \\ \hline a_1^1 & & & 1 & & \\ \hline a_2^1 & & & & 1 & \\ \hline a_3^1 & & 1 & & -1 & 1 \\ \hline a_4^1 & 1 & & -1 & & -1 \\ \hline a_5^1 & & & & & 1 \\ \hline \end{array} \tag{2.1.2}$$

Matrices Q and A can be partitioned as

$$Q = \begin{array}{|c|c|}\hline Q^c & Q^o \\ \hline \end{array} \qquad A = \begin{array}{|c|c|}\hline A^c & A^o \\ \hline \end{array} \qquad 2.1.3$$

And

$$(Q^{-1})_t = A \qquad (A^{-1})_t = Q \qquad 2.1.4$$

Also

$$(Q_t).\ (A) = (A_t)\,(Q) = U \qquad 2.1.5$$

$$\begin{array}{|c|}\hline Q_t^c \\ \hline Q_t^o \\ \hline \end{array} \quad \begin{array}{|cc|}\hline A^c & A^o \\ \hline \end{array} = \begin{array}{|c|c|}\hline Q_t^c A^c & Q_t^c A^o \\ \hline Q_t^o A^c & Q_t^o A^o \\ \hline \end{array} = \begin{array}{|c|c|}\hline U & 0 \\ \hline 0 & U \\ \hline \end{array} \qquad 2.1.5$$

i.e.

$$Q_t^c A^c = Q_t^o A^o = U \qquad 2.1.7$$

$$Q_t^o A^c = Q_t^c A^o = 0 \qquad 2.1.8$$

From these relations, it can be shown that the matrices Q^c and A^o, Q^o and A^c are orthogonal to each other. That is the reason for calling the 1-network an orthogonal network.

b. The Existence of the Currents and Voltages in an Excited Network.

With the help of Kirchhoff's laws, the paths of currents and voltages shall be found in an excited network. When currents and voltages exist in a network, they must satisfy Kirchhoff's laws at all times. We have to discover the paths along which currents and voltage exist so that Kirchhoff's laws are always satisfied.

To achieve this, let us assume that the only paths along which currents and voltages can exist in a network are the open and closed paths. For currents and voltages, an electrical network is a set of closed and open paths as shown in Fig. 2.1.2.

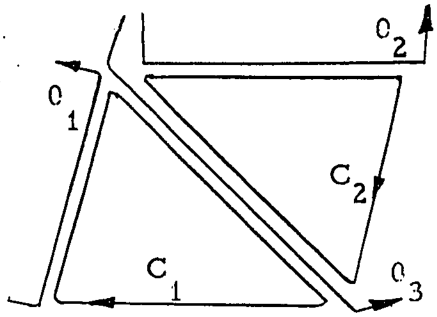


Fig. 2.1.2 Paths for the Currents and Voltages.

As long as the currents and voltages exist along closed and open paths, the Kirchhoff's laws are always satisfied. The open and closed paths are algebraically shown by the matrix Q in terms of its building blocks i.e. branches. The choice of independent open and closed paths depends on the problem at hand.

Current Law

As long as the currents exist along the closed and open paths, the current law is always satisfied. Along the closed paths, whichever current enters the junctions of dead network, the same current leaves the junction. It means that the currents entering the junctions from the closed-paths are zero. This is actually the statement of the current law, if there are no current generators.

Again along the open paths, whichever current enters the network, the same current leaves the network. Thus the sum of all the currents entering the network from the outside is equal to zero. This can be taken as another statement of current law when the current generators exist. So as long as the currents exist along open and closed-paths, the current law is always satisfied.

Voltage Law.

The existence of the current in open and closed paths can be easily visualized as a fluid flow. For the voltages, there is no such analogy. Along the closed-paths, if there is no impressed branch voltage, the total voltage is zero. If the impressed voltages exist, then the sum of the voltages existing along the closed-paths is equal to the impressed branch voltage in that path. This is the statement of the voltage law which is automatically satisfied.

The properties of the branch-network were so far discussed without any mathematical rigor. In the next section, an algebraic topological approach will be given to the branch-network theory. The various concepts of 1-network will be directly generalized to p-dimensional network with the help of algebraic topological concepts.

c. The Algebraic-Topological Representation of p-Networks.

For our purpose, only the elementary concepts of algebraic topology are required.

c.i The Elementary Concepts of Algebraic Topology.

Defintion 2.1.2⁽¹¹⁾

A finite abstract complex $k^{(n)}$ is the set of objects called vertices $b^1, b^2, \ldots, b^n$ and a set k of the subsets a_p^i of the vertices where $(p+1)$ is the number of the vertices of the simplex a_p^i, i is the indexing superscript.

Definition 2.1.3(11)

p-Chain. A p-chain of complex k is defined as

$$C^p = \Sigma\; m^i a_i^p \qquad 2.1.9$$

where the summation is over the oriented p-simplexes a_i^p, and m^i are the arbitrary elements of the Abelian group G. The chain is a set of oriented p-simplexes taken with certain multiplicities.

Incidence Matrix $B_i^{j,p}$ is the same matrix as defined in Chapter 1 by B. In general, $B_i^{j,p}$ gives the incidence relations of p, and (p-1) simplexes.

$$C^p = \Sigma\; B_i^{j,p+1}\; a_j^p \qquad 2.1.10$$

C^p represents the collection of p-simplexes which are faces of a_j^{p+1}. The chain, so defined, is called the boundary chain.

Definition 2.1.4(11, 14)

p-Chain Group: The set of p-chains of k form an additive Abelian group by the rule:

$$\Sigma\; m^i a_i^p + \Sigma n^i\, a_i^p = \Sigma\; (m^i + n^i)\, a_i^p \qquad 2.1.11$$

By the equation 2.1.11, a group structure is assigned to the set of p-chains, and is called p-chain group.

Definition 2.1.5

Generators(28) of a group are those elements of the group from which all the other elements can be generated. It is assumed that by generators, it is meant independent generators, i.e. no element of the set of generators can be generated from the other elements of the set of generators.

Free Group(28) A group is called a free group if there exists no relation among the generators of the group. The p-chain group is a free group.

Definition 2.1.6

When there exists a mapping(28) between two groups G_a and G_b, the typical feature of the mapping is the conservation of the group structure. This can be expressed as follows:

Let f be a mapping from Abelian additive group G_a to G_b i.e.

$$f \quad G_a \longrightarrow G_b \tag{2.1.12}$$

If C_a, D_a are two elements of G_a, then the conservation of structure means

$$f\,(C_a + D_a) = f(C_a) + f(D_a) \tag{2.1.13}$$

Any mapping $f \quad G_a \rightarrow G_b$ which satisfies the equation 2.1.13 is called homomorphism. B^p defined by the equation 2.1.10 may be extended uniquely to the right hand homomorphism,

$$B^p \qquad C^p \longrightarrow C^{p-1} \tag{2.1.14}$$

by the rule

$$\Sigma\ (m^i a_i^p)\ B^p \qquad \Sigma(m^i)(a_i^p B^p) \tag{2.1.15}$$

$$\Sigma\ (m^i a_i^p)\ B^p \qquad \Sigma(\Sigma m^i, B_i^j\ p)\ a_j^{p-1} \tag{2.1.16}$$

Definition 2.1.7(23)

p-Cycles: We shall represent independent set of p-closed paths of complex k by a_i^{pc}. The number of independent closed-paths can be easily found from the topology of K.

A p-cycle or p-closed chain is defined as

$$C^{pc'} = \Sigma\ m^i a_i^{pc} \tag{2.1.17}$$

where the summation is over the oriented p-closed paths, and m^i's are the arbitrary elements of an Abelian group.

p-Closed Chain Group

The set of p-closed chains can be assigned a group structure by the rule

$$\Sigma\, m^i a_i^{pc} + \Sigma n^i a_i^{pc} = \Sigma(m^i + n^i)\; a_i^{pc} \qquad 2.1.18$$

The p-closed chain group is a free group.

In the books on algebraic topology (11,14), p-closed chains are defined as sets of the elements of p-chain group which are mapped to zero element by the homomorphism B^p. The set of p-closed chains represents the Kernel of the homomorphism B^p, and forms a subgroup of closed-chains.

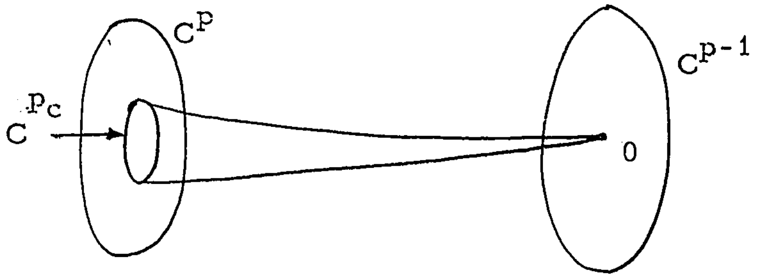


Fig. 2.1.3 Homomorphism B^p between C^p and C^{p-1} Groups.

Definition 2.4.8

p-Open Chain: With a given oriented complex k, the independent set of open-paths a_i^{po} can be found as (11)

(Number of independent p-open-paths)
+ (Number of independent p-closed-paths)
= (Number of p-simplexes). 2.1.19

The p-open chain is defined as

$$C^{po'} = \Sigma\, m^i\; a_i^{po} \qquad 2.1.20$$

where the a_i^{po} are the oriented p-open paths and m^i's are the arbitrary elements of an Abelian group.

p-Open Chain Group: The set of p-open chains form an additive Abelian group by the rule

$$\Sigma\, m^i a_i^{po} + \Sigma\, n^i a_i^{po} = \Sigma(m^i + n^i)\, a_i^{po} \qquad 2.1.21$$

To clarify the concepts of a_i^p, a_i^{pc}, a_i^{po}, take the Fig. 2.1.4.

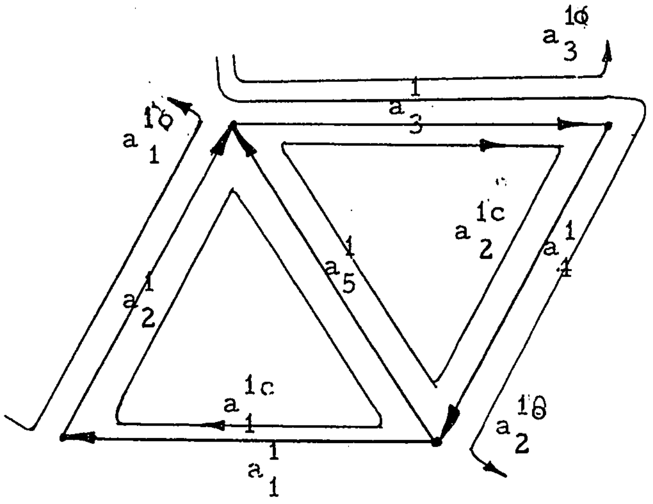
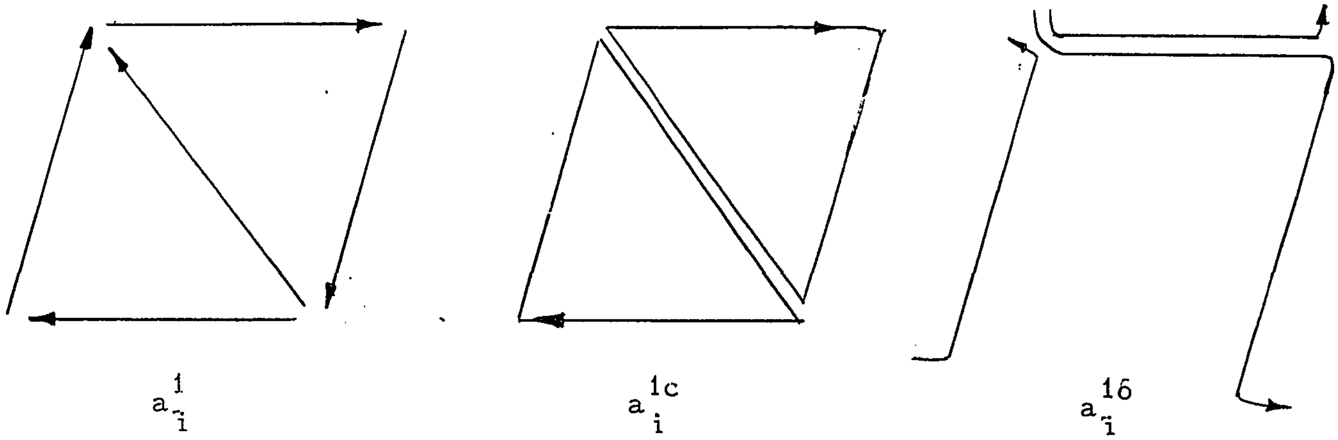


Fig. 2.1.4 The complex K with a_i^1, a_i^{1c}, a_i^{1o}.

The complex K has four o-simplexes, five 1-simplexes and two 2-simplexes. There are three independent open 1-paths and two independent closed 1-paths. In the Fig. 2.1.4, the various 1-paths are drawn.

The groups C^p, C^{pc}, C^{po} can be generated by the rules already given. Fig. 2.1.4 can be split more clearly into three distinct components a_i^p, a_i^{pc} and a_i^{po} as in Fig. 2.1.5.



a_i^1 a_i^{1c} a_i^{16}

Fig. 2.1.5 The Basis for the Groups C^p, C^{pc}, C^{po}

The Matrix Representation of Chains.

The matrix notation will be used to represent the elements of C^p, C^{pc}, C^{po}. The matrix representation helps us in using the matrix algebra for the computation work. Any chain

$$C^p = \Sigma m^i a_i^p = m^1 a_1^p + m^2 a_2^p \ldots, + m^n a_n^p \qquad 2.1.22$$

can be written as(26) product of two matrices

$$(C^p) = (m^1 \; m^2 \ldots\ldots m^n). \begin{bmatrix} a_1^p \\ a_2^p \\ . \\ . \\ . \\ a_n^p \end{bmatrix} \qquad 2.1.23a$$

The row matrix shows the coefficients of each basis, and the column matrix the corresponding basis. To facilitate our work, we shall be simplifying the matrix representation of equation 2.1.23a as below!

$$(C^p) = \begin{bmatrix} m^1 \\ m^2 \\ \cdot \\ \cdot \\ m^n \end{bmatrix} \begin{matrix} a^p_1 \\ a^p_2 \\ \cdot \\ \cdot \\ a^p_n \end{matrix} \qquad 2.1.23b$$

It can be imagined that m_i multiplies a^p_i and adds giving the chain (C^p) of equation 2.1.22. The equation 2.1.23b shall be used throughout to represent various chains, cycles, and open chains. In our study, we specify the bases, and we are required to find the coefficients m's in some set of bases, given the coefficient in some other set of basis. All our transformations are concerned with the transformation of coefficients to different set of bases.

In equation 2.1.23b, the column matrix is the same as the row matrix of equation 2.1.23b. There is no reason why the equation 2.1.23b be not called the matrix representation of chain C^p, since we are using the $\begin{bmatrix} m^1 \\ m^2 \\ \vdots \\ m^n \end{bmatrix}$ throughout as a column matrix.

Similarly the elements of group $(C^{pc'}) \cdot (C^{pc'}) = \Sigma a^i a^{pc}_i = m^1 a^{pc}_1 + m^2 a^{pc}_2 \ldots\ldots, + m^{n_c} a^{pc}_{n_c}$ 2.1.24

can be written

$$(C^{pc'}) \triangleq \begin{bmatrix} m^1 \\ m^2 \\ \cdot \\ \cdot \\ m^{n_c} \end{bmatrix} \begin{matrix} a^{pc}_1 \\ a^{pc}_2 \\ \cdot \\ \cdot \\ a^{pc}_{n_c} \end{matrix} \qquad 2.1.25$$

The elements of group$(C^{po'})$ are

$$(C^{po'}) = \Sigma\, m^i a^{po}_i = m^1 a^{po}_1 + m^2 a^{po}_2 \ldots., + n^{n_c} a^{po}_{n_o} \qquad 2.1.26$$

$$(C^{po'}) \quad \triangleq \quad \begin{bmatrix} m^1 \\ m^2 \\ \vdots \\ m^{n_o} \end{bmatrix} \begin{matrix} a^{po}_1 \\ a^{po}_2 \\ \\ a^{p}_{n_o} \end{matrix} \qquad 2.1.27$$

Any p-chain is either a closed p-chain, or open p-chain, or a combination of the open and closed chains. Any p-chain can be thus written as the direct sum of a p-open chain and p-closed chain. If all the p-chains are represented in terms of the basis a^p_i, then

$$(C^p) = (C^{pc}) + (C^{po}) \qquad 2.1.28$$

Each matrix here has as many rows as the number of p-simplexes. (C^{pc}) forms a subgroup of (C^p) whose elements are mapped to zero by the boundary homomorphism and (C^{po}) covers those elements of (C^p) which are not in (C^{pc}).

Any element (C^{pc}) can be written in terms of $(C^{pc'})$ which has as many rows as there are a^{pc}_i. Any element (C^{po}) can be written in terms of $(C^{po'})$ which has rows equal to the number of a^{po}_i. Thus any chain (C^p) can be written also as a direct sum of $(C^{pc'})$ and $(C^{po'})$.

We have

$$(C^p) = \begin{bmatrix} m^1 \\ m^2 \\ \vdots \\ m^n \end{bmatrix} \begin{matrix} a^p_1 \\ a^p_2 \\ \vdots \\ a^p_n \end{matrix} \qquad 2.1.29$$

$$(C^{po'}) + (C^{pc'}) = (C^{p'}) \qquad 2.1.30$$

$$(C^{p'}) = \begin{bmatrix} m^{1'} \\ m^{2'} \\ \vdots \\ m^{n'_c} \end{bmatrix} \begin{matrix} a^{pc}_1 \\ a^{pc}_2 \\ \vdots \\ a^{pc}_{n_c} \end{matrix} + \begin{bmatrix} m^{1''} \\ m^{2''} \\ \vdots \\ m^{n''_o} \end{bmatrix} \begin{matrix} a^{po}_1 \\ a^{po}_2 \\ \vdots \\ a^{po}_{n_o} \end{matrix} \qquad 2.1.31$$

$$(C^{p'}) = \left[\begin{array}{c} m^{1'} \\ m^{2'} \\ \vdots \\ m^{n'_c} \\ \hline m^{1''} \\ m^{2''} \\ \vdots \\ m^{n''_o} \end{array}\right] \begin{matrix} a^{pc}_1 \\ a^{pc}_2 \\ \vdots \\ a^{pc}_{n_c} \\ a^{po}_1 \\ a^{po}_2 \\ \vdots \\ a^{po}_{n_o} \end{matrix} \qquad 2.1.32$$

The same chain is first represented in terms of bases a^p_i in equation 2.1.29 and then it is represented in terms of bases a^{pc}_i and a^{po}_i in equation 2.1.32. The number of bases for (C^p) and $(C^{p'})$ is the same, the only change is that the same chain is represented in terms of two different bases. A chain is represented by two different set of bases, the number of bases being the same. We shall call the bases for (C^p) as the natural or primitive bases, and for $(C^{p'})$ as actual bases.

As the same chain is represented in terms of two independent set of bases, a transformation matrix can be found which transforms the coefficients of the chain from one set of bases to the other set.

Let the transformation be

$$(C^{p}) = Q^{p}(C^{p'}) \tag{2.1.33}$$

where Q^{p} is non-singular matrix which transforms the coefficients of $(C^{p'})$ to those of (C^{p}). The method of finding Q^{p} is the same as that of Q, except that instead of 1-paths, we have p-paths.

The inverse transformation A^{p} is

$$(C^{p'}) = A_t^{p}(C^{p}) \tag{2.1.34}$$

where $\quad A_t^{p} = (Q^{p})^{-1}$

Rewriting the equations 2.1.33 and 2.1.34,

$$(C^{p}) = \begin{bmatrix} Q^{pc} & Q^{po} \end{bmatrix} \begin{bmatrix} C^{pc'} \\ C^{po'} \end{bmatrix} \tag{2.1.35}$$

$$\begin{bmatrix} C^{pc'} \\ C^{po'} \end{bmatrix} = \begin{bmatrix} A_t^{pc} \\ A_t^{po} \end{bmatrix} \begin{bmatrix} C^{p} \end{bmatrix} \tag{2.1.36}$$

From equations 2.1.35 and 2.1.36,

$$\begin{bmatrix} A_t^{pc} \\ \hline A_t^{po} \end{bmatrix} \begin{bmatrix} Q^{pc} & Q^{po} \end{bmatrix} = \left[\begin{array}{c|c} A_t^{pc}\,Q^{pc} & A_t^{pc}Q^{po} \\ \hline A_t^{po}\,Q^{pc} & A_t^{po}\,Q^{po} \end{array}\right] = \left[\begin{array}{c|c} U & O \\ \hline O & U \end{array}\right] \tag{2.1.37}$$

Thus

$$\begin{aligned} A_t^{pc} \cdot Q^{pc} &= A_t^{po} \cdot Q^{po} = U \\ A_t^{pc} \cdot Q^{po} &= A_t^{po} \cdot Q^{pc} = O \end{aligned} \tag{2.1.38}$$

The equation 2.1.38 shows that the closed-path and open-path bases are orthogonal to each other. Any element of the closed-path bases is transformed to zero element in the open-path bases and vice versa.

All the transformations of equation 2.1.35 and 2.1.36 can be summerized as in Fig. 2.1.6.

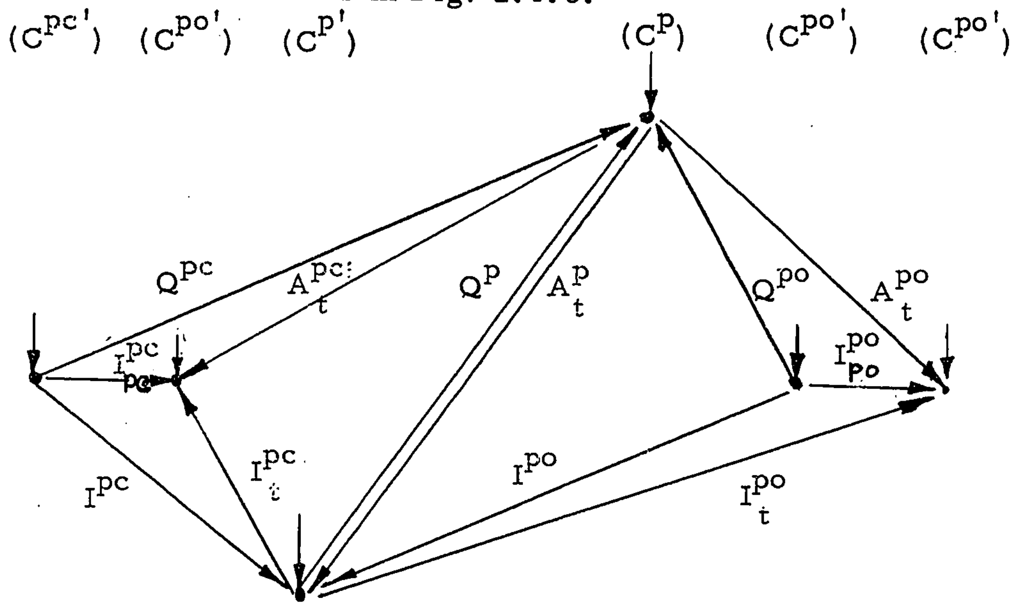


Fig. 2.1.6 Transformation Diagram for Chains.

The various dots show the different spaces representing the different set of bases. The space corresponding to $(C^{p'})$ may be called an orthogonal or actual space.

Definition 2.4.9$^{(11, 14)}$

p-Cochains It is the function defined on the oriented simplexes of k with values in an Abelian group G.

$$(a_i^p)\ C_p \longrightarrow g_i \qquad 2.1.39$$

g_i is the value of p-cochain on the oriented simplex a_i^p. The coefficients g_i can be as well assigned to determine a p-chain. Thus for each set of g_i, there exist a unique co-chain and chain, and a one-to-one correspondence exists between p-cochains and p-chains. Every cochain C_p determines a unique homomorphism

$$C^p(k) \xrightarrow{C_p} G$$

given by

$$\Sigma m^i \; a_i^p \xrightarrow{C_p} \Sigma m^i (a_i^p C_p) \qquad 2.1.40$$

The cochain C_p is identified with the above homomorphism. If A is taken as any group, the set of homomorphisms of A into G is written

$$A_1(\phi + \psi) = A_1 \; \phi + A_1 \psi \qquad A_1 \in A \qquad 2.1.41$$

ϕ, $\psi \in$ Homomorphism (A, G). If the above group structure is assigned to the set of homomorphism $(C^p(k), G_1)$, then it is called the group of cochains $C_p(k)$ with values in G, and is written as $C_p(k)$. If C_p, D_p are two different cochains, then due to the group structure attached to the set of cochains we have

$$a^p (C_p + D_p) = (a^p, C_p) + (a^p, D_p) \qquad 2.1.42$$

Product (a^p, C_p) is called the Kromecker Index of a^p and C_p.

Definition 2.1.10$^{(11,23)}$

Same basis: Let us take a cochain so that

$$(a_i^p, C_p) = \delta_i^j \qquad 2.1.43$$

The cochain satisfying the equation 2.1.43 is written a_p^j, and called the "same basis" for the p-cochains C_p as compared to the bases a_i^p of the p-chains.

Definition 2.1.11(11)

Coboundary b_p is defined as

$$(a^p, B_{p-1} d_{p-1}) = (a^p B^p, d_{p-1}) \qquad 2.1.44$$

The equation 2.1.44 is the most important relation of algebraic topology which, when properly interpreted, states the generalized Stoke's Theorem.

$B_{p-1}\, d_{p-1}$ is the coboundary of d_{p-1}, and B_{p-1} is adjoint to A^p. The coboundary homomorphism is defined as the left homomorphism.

$$B_{p-1} \quad C_{p-1} \longrightarrow C_p \qquad 2.1.45$$

Choice of Same Bases for Cochains

For every set of bases chosen for chains, we can choose the "same" bases for cochains. Once the bases for C^p, C^{pc} C^{po} are chosen, we can choose the same bases for cochains. Corresponding to the same bases a_i^p, a_i^{pc}, a_i^{po}, we have the same cochain bases a_p^i, a_{pc}^i, a_{po}^i as defined by equation 2.1.43. We shall be using the same notation to represent the various cochains as is used for chains. The cochain (C_p) is represented in terms of a_p^i bases.

$$(C_p) \triangleq \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} \begin{array}{c} a_p^1 \\ a_p^2 \\ \vdots \\ a_p^n \end{array} \qquad 2.1.46$$

The same cochain can be represented in terms of the bases a^i_{pc}, a^i_{po} as below:

$$(C'_p) = \begin{bmatrix} m'_1 \\ m'_2 \\ \vdots \\ m'_{n_c} \end{bmatrix} \begin{matrix} a^1_{po} \\ a^2_{pc} \\ \\ a^{n_c}_{pc} \end{matrix} + \begin{bmatrix} m''_1 \\ m''_2 \\ \vdots \\ m''_{n_o} \end{bmatrix} \begin{matrix} a^1_{po} \\ a^2_{po} \\ \\ a^{n_o}_{po} \end{matrix} \qquad 2.1.47$$

$$(C'_p) = \left[\begin{array}{c} m'_1 \\ m'_2 \\ \vdots \\ m'_{n_c} \\ \hline m''_1 \\ m''_2 \\ \vdots \\ m''_{n_o} \end{array}\right] \begin{matrix} a^1_{pc} \\ a^2_{pc} \\ \vdots \\ a^{n_c}_{pc} \\ a^1_{po} \\ a^2_{po} \\ \vdots \\ a^{n_o}_{po} \end{matrix} \qquad 2.1.48$$

The bases for representing cochains are the "same" bases with respect to the bases chosen for the chains . Once the cochains are represented by the "same" bases, the laws of transformation for the elements of cochains in different set of bases are dual to those of chains (11, 23). As the transformations are already found for the chains, the same are used for the cochains in a dual way.

$$(C_p) = A^p (C'_p) \qquad 2.1.49$$

$$(C'_p) = Q^p_t (C_p) \qquad 2.1.50$$

$$(C_p) = \begin{bmatrix} A^{pC} & A^{po} \end{bmatrix} \begin{bmatrix} C'_{pc} \\ \hline C'_{po} \end{bmatrix} \qquad 2.1.51$$

$$\begin{bmatrix} C'_{pc} \\ \hline C'_{po} \end{bmatrix} = \begin{bmatrix} Q^{pc}_t \\ \hline Q^{po}_t \end{bmatrix} \cdot (C_p) \qquad 2.1.52$$

The transformations for cochains can be summarized in Fig. 2.1.7.

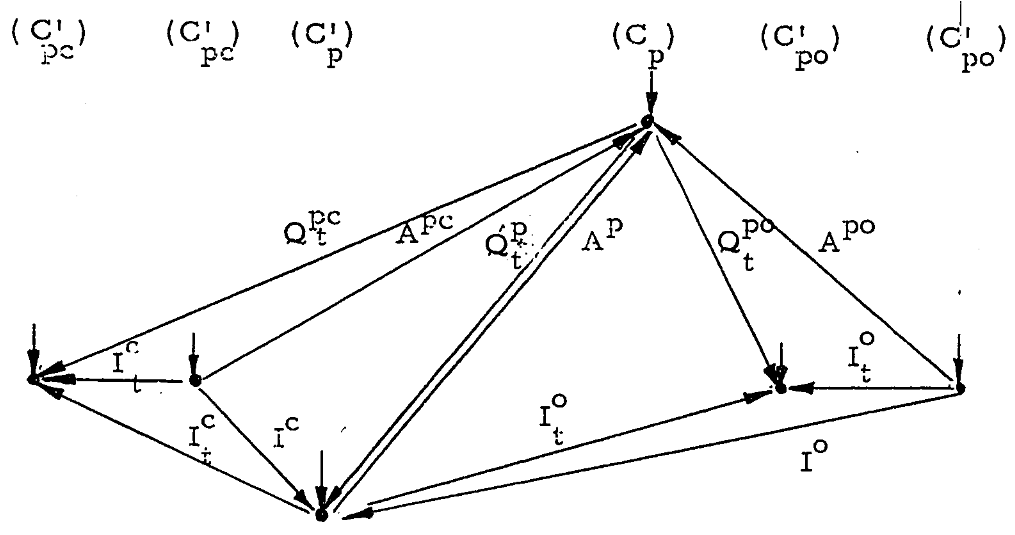


Fig. 2.1.7 Transformation Diagram for Cochains.

Also we have a set of relation between Q^p and A^p.

$$Q^p_t \cdot A^p = U \qquad 2.1.53$$

$$\begin{bmatrix} Q^{pc}_t \\ \hline Q^{po}_t \end{bmatrix} \begin{bmatrix} A^{pc} & A^{po} \end{bmatrix} = U \qquad 2.1.54$$

$$\begin{array}{|cc|cc|} \hline Q_t^{pc} & A^{pc} & Q_t^{pc} & A^{po} \\ \hline Q_t^{po} & A^{pc} & Q_t^{po} & A^{po} \\ \hline \end{array} = \begin{array}{|c|c|} \hline U & O \\ \hline O & U \\ \hline \end{array} \qquad 2.1.55$$

The equation 2.1.55 again proves the orthogonality of the closed and open paths.

$$\begin{aligned} Q_t^{po} \; A^{po} &= Q_t^{pc} \; A^{pc} = U \\ Q_t^{po} \; A^{pc} &= Q_t^{pc} \; A^{po} = O \end{aligned} \qquad 2.1.56$$

c. ii Generalized Kirchhoff's Laws.

Fig. 2.1.6 and 2.1.7 codify Kirchhoff's Laws for chains and cochains of p-dimensional structure which we are calling p-network. The study of chains and cochains from this point of view has discovered new properties of p-dimensional structure . So far, the algebraic topologists are not aware(17,18) of the Kirchhoff's laws in their study, though recently they have discovered the Stoke's theorem as the property of complex K, which deals with the simplexes of two different dimensions.

The Kirchhoff's laws deal with simplexes of the same dimensions, and do not deal with simplexes of different dimensions. Instead of calling Kirchhoff's two laws, the current law and voltage law, we shall call them in general,

i Chain law.

ii Co-chain law.

i Chain Law. This law states that any chain represented in terms of closed-path bases has zero value in terms of open-paths bases and vice-versa. From fig. 2.1.6, the value of $(C^{pc'})$ in $(C^{po'})$ is given by

$$(C^{po'}) = A_t^{po} \; Q^{pc} \; (C^{pc'}) \qquad 2.1.57$$

From the equation 2.1.38

$A_t^{po} \; Q^{pc} = 0,$ giving

$$(C^{po'}) = 0. \; (C^{pc'}) = 0 \qquad 2.1.58$$

Similarly

$$(C^{pc'}) = A_t^{pc} \, Q^{po} \; (C^{po'}). \qquad 2.1.59$$

From the equation 2.1.38,

$A_t^{pc} \, Q^{po} = 0$

So

$$(C^{pc'}) = 0. \; (C^{po'}) = 0 \qquad 2.1.60$$

When the equations 2.1.57 and 2.1.59 are interpreted for electrical network with p = 1, we have the familiar form of the current law. The chain law can be written in various other forms from Fig. 2.1.6.

ii <u>Co-chain Law.</u> This law states that any cochain represented in (C'_{po}) has zero value in (C'_{pc}) and vice-versa.

Written algebraically from Fig. 2.1.7

$$\begin{aligned} (C'_{pc}) &= Q_t^{pc} A^{po} \; (C'_{po}) \\ (C'_{po}) &= Q_t^{po} A^{pc} \; (C'_{pc}) \end{aligned} \qquad 2.1.61$$

where $Q_t^{pc} \; A^{po} = Q_t^{po} \; A^{pc} = 0$ (from equation 2.1.56.)

Thus

$$\begin{aligned} (C'_{pc}) &= 0. \; (C'_{po}) = 0 \\ (C'_{po}) &= 0. \; (C'_{pc}) = 0 \end{aligned} \qquad 2.1.62$$

From the Fig. 2.1.7, one can write the cochain law in various forms. Any p-network will always satisfy the chain and cochain laws.

c iii. Ohm's Law

So far, the elements of chain and cochain group are assumed to have independent existence. Now we shall introduce a correspondence between the chain group and the cochain group. The elements of these two groups no longer have independent existence. The correspondence between the two groups is stated as follows:

"There exists an isomorphism between the elements of p-chain group and p-cochain group."

The above statement is also called Ohm's law.

As the elements of C_p, C^p are written in matrix form, the isomorphism can be written in a matrix form, which is square and non-singular. The function f shows an isomorphism from (C^p) to (C_p) in Fig. 2.1.8.

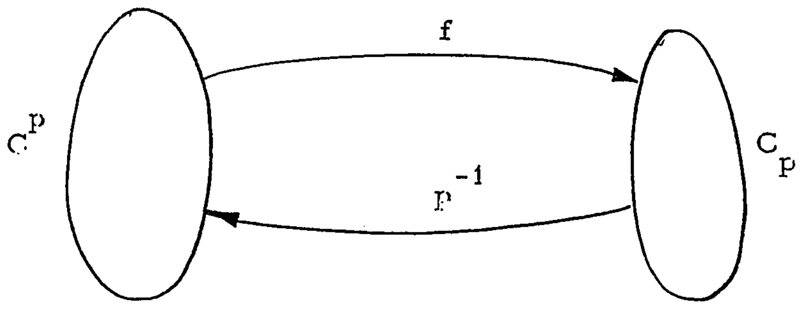


Fig. 2.1.8 Isomorphism between p-Chain and p-Cochain Groups.

If the function f is written in matrix form, we write

$$f = Z_p \tag{2.1.63a,}$$

$$f^{-1} = Y^p = (Z_p)^{-1} \tag{2.1.63b.}$$

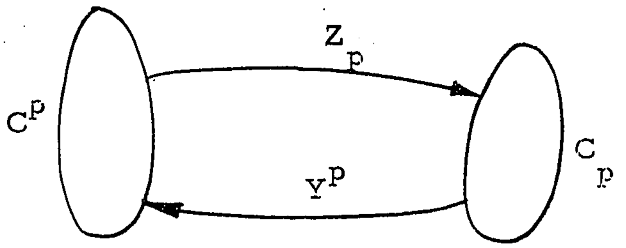


Fig. 2.1.9 Transformation between C^p and C^p

Algebraically we have

$$(C_p) = Z_p (C^p) \tag{2.1.64a,}$$

$$(C^p) = Y^p (C_p) \tag{2.1.64b.}$$

Z_p and Y^p can be written as left isomorphism. With the introduction of Z_p and Y^p, the Fig. 2.1.6 and 2.1.7 can be combined into Fig. 2.1.10. Various other transformations are shown among the various spaces of chains and cochains. The induced transformations can be easily traced from Fig. 2.1.10. Fig. 2.1.11 gives the complete algebraic diagram of a p-network with all the induced transformations. From Fig. 2.1.11,

$$Z'_p = Q^p_t \cdot Z_p \cdot Q^p \tag{2.1.65}$$

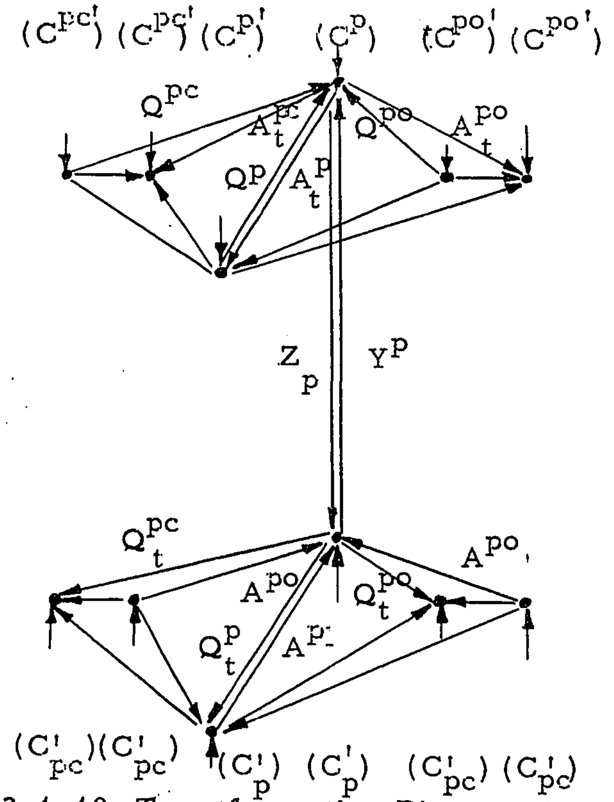


Fig. 2.1.10 Transformation Diagram in a p-Network.

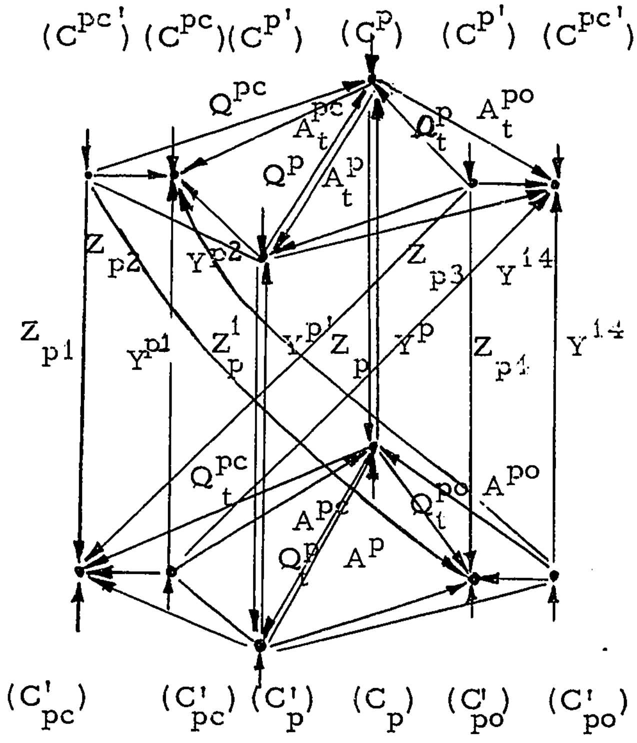


Fig. 2.1.11 Complete Algebraic Diagram of p-Network.

$$Z'_p = \left[\begin{array}{c} Q_t^{pc} \\ \hline Q_t^{po} \end{array}\right] . \quad Z_p \; . \; \left[\begin{array}{cc} Q^{pc} & Q^{po} \end{array}\right]$$

$$Z'_p = \begin{array}{c|c|c|} & c & 0 \\ c & Q_t^{pc} Z_p Q^{pc} & Q_t^{pc} Z_p Q^{po} \\ \hline 0 & Q_t^{po} Z_p Q^{pc} & Q_t^{po} Z_p Q^{po} \end{array} = \begin{array}{|c|c|} Z_{p1} & Z_{p2} \\ \hline Z_{p3} & Z_{p4} \end{array} \qquad 2.1.66$$

Similarly

$$Y^{p'} = A_t^p \, . \, Y^p . \, A^p \qquad 2.1.67$$

$$Y^{p'} = \left[\begin{array}{c} A_t^{pc} \\ \hline A_t^{po} \end{array}\right] . \quad Y^p \; . \; \left[\begin{array}{cc} A^{pc} & A^{po} \end{array}\right] \qquad 2.1.68$$

$$Y^{p'} = \begin{array}{|c|c|} A_t^{pc} \; Y^p \, A^{pc} & A_t^{pc} \; Y^p \; A^{po} \\ \hline A_t^{po} \; Y^p \, A^{pc} & A_t^{po} \; Y^p \; A^{po} \end{array} = \begin{array}{|c|c|} Y^{p1} & Y^{p2} \\ \hline Y^{p3} & Y^{p4} \end{array} \qquad 2.1.69$$

We have defined all the quantities and relations that can exist in a p-network. In the next section, p-network problem will be described .

d. <u>Abstract Formulation of p-Network Problem.</u>

A p-network problem can be formulated as follows

Given

1. The topology of a p-network so that Q^p and A^p are defined.
2. Isomorphism Z_p and Y^p.
3. Values of $(C^{po'}_{\;p})$ and (C'_{pc})

Find

(C_p) and (C^p) so that the Ohm's law and generalized Kirchhoff's laws are satisfied. In formulating the problem, the maximum number of unknowns are assumed which can be solved from the given set of equations. The problem described above is in the most general form.

Solution

In terms of primitive bases.

$$(C_p) = Z_p\,(C^p)\ ;\ (C^p) = Y^p(C_p) \qquad 2.1.70$$

Transforming these relations to the actual bases,

$$(C'_p) = Z'_p\,(C^{p'})$$

$$\left[\begin{array}{c} C'_{pc} \\ \hline C'_{po} \end{array}\right] = \left[\begin{array}{c|c} Z_{p1} & Z_{p2} \\ \hline Z_{p3} & Z_{p4} \end{array}\right] \left[\begin{array}{c} C'^{pc} \\ \hline C'^{po} \end{array}\right] \qquad 2.1.71a$$

$$\begin{aligned} (C'_{pc}) &= Z_{p1}(C^{pc'}) + Z_{p2}\,(C^{po'}) \\ (C'_{po}) &= Z_{p3}\,(C^{pc'}) + Z_{p4}\,(C^{po'}) \end{aligned} \qquad 2.1.72a$$

$$(C^{p'}) = Y^{p'}\,(C'_p)$$

$$\left[\begin{array}{c} C^{pc'} \\ \hline C^{po'} \end{array}\right] = \left[\begin{array}{c|c} Y^{p1} & Y^{p2} \\ \hline Y^{p3} & Y^{p4} \end{array}\right] \left[\begin{array}{c} C'_{pc} \\ \hline C'_{po} \end{array}\right] \qquad 2.1.61b$$

$$\begin{aligned} (C^{pc'}) &= Y^{p1}\,(C'_{pc}) + Y^{p2}\,(C'_{po}) \\ (C^{po'}) &= Y^{p3}\,(C'_{pc}) + Y^{p4}\,(C'_{po}) \end{aligned} \qquad 2.1.72b$$

When the equations are written in the form of equations 2.1.72a and 2.1.72b, the known and unknown quantities are separated. In the equations of the form 2.1.70 and 2.1.71, the known and unknown quantities are not separate and it is not possible to find the unknown quantities from these equations. In equation 2.1.72, the quantities (C'_{pc}) and $(C^{po'})$ are known, and the unknown quantities can be found.

Solving equation 2.1.72 for the unknowns

$$(C^{pc'}) = (Z_{p1})^{-1}\Big((C'_{pc}) - Z_{p2}\,(C^{po'})\Big) \qquad 2.1.73a$$

$$(C'_{po}) = (Y^{p4})^{-1}\Big((C^{po'}) - Y^{p3}\,(C'_{pc})\Big) \qquad 2.1.73b$$

Putting these values of equations 2.1.73 into the equation 2.1.72,

$$(C'_{pc}) = Z_{p3}\,(Z_{p1})^{-1}\Big((C'_{pc}) - Z_{p2}(C^{po'})\Big) + Z_{p4}(C^{po'}) \qquad 2.1.74a$$

$$(C^{pc'}) = Y^{p1}\,(C'_{pc}) + Y^{p2}\,(Y^{p4})^{-1}\Big((C^{po'}) - Y^{p3}(C'_{pc})\Big) \qquad 2.1.74b$$

From equation 2.1.74 and the relations

$$(C^{p}) = Q^{p}\,(C^{p'}) \qquad 2.1.75a$$

$$(C_{p}) = A^{p}\,(C'_{p}) \qquad 2.1.75b$$

$$(C^{p}) = Q^{pc}\,(C^{pc'}) + Q^{po}\,(C^{po'}) \qquad 2.1.76a$$

$$(C_{p}) = A^{pc}\,(C'_{po}) + A^{po}\,(C'_{po}) \qquad 2.1.76b$$

We have

$$(C^{p}) = Q^{pc}\,(Z_{p1})^{-1}\Big((C'_{pc}) - Z_{p2}\,(C^{po'})\Big) + Q^{po}(C^{po'}) \qquad 2.1.77a$$

$$(C_{p}) = (A^{pc} - A^{po}\,(Y^{p4})^{-1}Y^{p3})\,(C'_{pc}) + A^{po}(Y^{p4})^{-1}(C^{po'}) \qquad 2.1.77b$$

Rearranging equations 2.1.77a and 2.1.77b

$$(C^{p}) = Q^{pc}\,(Z_{p1})^{-1}(C'_{pc}) + (Q^{po} - Q^{pc}(Z_{p1})^{-1}Z_{p2})\,(C^{po'}) \qquad 2.1.78a$$

$$(C_{p}) = (A^{pc} - A^{po}(Y^{p4})^{-1}Y^{p3})\,(C'_{pc}) + A^{po}(Y^{p4})^{-1}(C^{po'}) \qquad 2.1.78b$$

And $(C_{p}) = Z_{p}\,(C^{p})$

$$(C^{p}) = Y^{p}(C_{p})$$

Thus

$$(C_p) = Z_p Q^{pc}(Z_{p1})^{-1}(C'_{pc}) + Z_p(Q^{po} - Q^{pc}(Z_{p1})^{-1} Z_{p2})(C^{po'}) \qquad 2.1.79a$$

$$C^p) = Y^p(A^{pc} - A^{po}(Y^{p4})^{-1} Y^{p3})(C'_{po}) + Y^p A^{po}(Y^{p4})^{-1}(C^{po'}) \qquad 2.1.79b$$

Equations with the symbol "a" give the solution by chain or closed-path method and equations with the symbol "b" give the solution by cochain or open path method, which is dual to the chain method. Equations 2.1.78 and 2.1.79 give the solution to the network problem in an explicit form. The solution of network by closed-path method requires the inverse $(Z_{p1})^{-1}$, by open-path method the inverse $(Y^{p4})^{-1}$ is required. Matrices $(Z_{p1})^{-1}$ and $(Y^{p4})^{-1}$ are called the solution matrices. The solution of a network problem actually means that one has to find the solution matrices, the rest of the quantities are then found easily by the routine transformations. Also $(Y^{p4})^{-1}$ can be found from $(Z_{p1})^{-1}$ and vice versa. If it is easier to find one inverse, then the other can be found from it.

Comparing the coefficients of 2.1.78a and 2.1.79b,

$$Q^{pc}(Z_{p1})^{-1} = Y^p (A^{pc} - A^{po}(Y^{p4})^{-1} Y^{p3}) \qquad 2.1.80$$

Multiplying both sides by A_t^{pc} and simplifying,

$$(Z_{p1})^{-1} = (Y^{p1} - Y^{p2} (Y^{p4})^{-1} Y^{p3}) \qquad 2.1.81$$

Similarly, comparing the coefficient of equations 2.1.78b and 2.1.79a, and multiplying by Q_t^{po}, we get after simplification,

$$(Y^{p4})^{-1} = (Z_{p4} - Z_{p3} (Z_{p1})^{-1} Z_{p2}) \qquad 2.1.82$$

Equations 2.1.81 and 2.1.82 give us the expressions for finding $(Z_{p1})^{-1}$ from $(Y^{p4})^{-1}$ and $(Y^{p4})^{-1}$ from $(Z_{p1})^{-1}$ respectively. This completes the solution of p-network problem. We are interested in using this method to the solution of electrical network which is a 1-network. The various relations derived above shall be interpreted in terms of electrical quantities.

c. Formulation of Electrical Network Problem

The various quantities of p-networks will be interpreted for an electrical network. An electrical network is modelled by a p-network with $p = 1$.

The various relations of 1-network are written in electrical terms,

Branch Current = $(J) = (C^{1})$

Branch Voltage = $(V) = (C_{1})$

Closed-path Current = $(i') = (C^{1c'})$

Closed-path Voltage = $(e') = (C'_{1c})$

Open-path Current = $(I') = (C^{1o'})$

Open-path Voltage = $(E') = (C_{1o})$

Branch Impedance = $(Z) = (Z_{1})$

Branch Admittance = $(Y) = (Y^{1})$

Actual Current = $(J') = (C^{1'})$

Actual Voltage = $(V') = (C'_{1})$

From the relations of 1-network, the following relations are obtained after identifying 1-network quantities with electrical quantities.

$$(J) = Q^{1} (J')$$

(We shall be omitting p=1 subscript or superscript with symbols Q and A)

$$(J) = \begin{bmatrix} Q^{c} & Q^{o} \end{bmatrix} \left[\begin{array}{c} i' \\ \hline I' \end{array}\right] \qquad 2.1.83$$

$$(J) = Q^{c}(i') + Q^{o}(I') \qquad 2.1.84$$

$$(J) = (i) + (I) \qquad 2.1.85$$

as $(i) = Q^{c}(i')$

$(I) = Q^{o}(I')$

Similarly

$$(V) = A\ (V') \qquad 2.1.86$$

$$(V) = \begin{bmatrix} A^c & A^o \end{bmatrix} \begin{bmatrix} e' \\ \hline E' \end{bmatrix} \qquad 2.1.87$$

$$(V) = A^c\,(e') + A^o\,(E') \qquad 2.1.88$$

$$(V) = (e) + (E) \qquad 2.1.89$$

as

$$(e) = A^c(e') \text{ and } (E) = A^o\,(E')$$

All the above relations can be directly taken from Fig. 2.1.11, if properly interpreted. Next the electrical network problem will be formulated directly from p-network problem.

Given:

i. The topology of 1-network with matrices Q and A defined.

ii. The quantities (e') and (I')

iii. Transformation matrix or branch impedance matrix (Z) and admittance matrix (Y).

Find

(V) and (J) so that the Kirchhoff's laws and Ohm's law are satisfied.

The solution is directly taken from the equations 2.1.78 and 2.1.79.

By the closed-path method,

$$(J) = Q^c\,(Z_1)^{-1}\,(e') + (Q^o - Q^c(Z_1)^{-1}\,Z_2)\,(I') \qquad 2.1.90a$$

$$(V) = ZQ^c(Z_1)^{-1}\,(e') + Z(Q^o - Q^c(Z_1)^{-1}Z_2)\,(I') \qquad 2.1.91a$$

By the open path method,

$$(V) = (A^c - A^o (Y^4)^{-1} Y^3)(e') + A^o (Y^4)^{-1} (I') \qquad 2.1.90b$$

$$(J) = Y(A^c - A^o (Y^4)^{-1} Y^3)(e') + YA^o (Y^4)^{-1} (I') \qquad 2.1.91b$$

The equations 2.1.90 and 2.1.91 give us the complete solution of electrical network problem under the most general excitation.

In the above equations,

$$\begin{array}{|c|c|} \hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline Q^c Z Q^c & Q^c Z Q^o \\ \hline Q^o Z Q^c & Q^o Z Q^o \\ \hline \end{array} \qquad 2.1.92$$

And

$$\begin{array}{|c|c|} \hline Y^1 & Y^2 \\ \hline Y^3 & Y^4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline A^c Y A^c & A^c Y A^o \\ \hline A^o Y A^o & A^o Y A^o \\ \hline \end{array} \qquad 2.1.93$$

The solution matrices $(Z_1)^{-1}$ and $(Y^4)^{-1}$ are related to each other as given in equations 2.1.81 and 2.1.82.

f. Existence of Unique Solution of p-Network Problem.

It was already shown in part I that there exists a unique solution to the electrical network if Z or Y is Ohmic. The same condition can be directly generalized to any p-network. So, for any p-network, the existence of unique solution is guaranteed if Z_p or Y^p is ohmic.

Whenever Z_p or Y^p is ohmic, the $(Z_{p1})^{-1}$ and $(Y^{p4})^{-1}$ exist and are unique, thus giving a unique solution to the p-network problem.

g. Conclusions

In this part, an algebraic topological approach is given for the electrical network. The branch-theory of electrical network has been generalized to any dimension p, and all the relations and quantities are described in elementary algebraic topological terms. By this approach, we are able to use the elementary concepts of algebraic topology for solving electrical network problems in particular, and p-network problems in general. So far no attempt has been made for using p-network theory to engineering and physical problems. In this work, only the theory has been described, the application of this theory to various problems is still to be described. This is a problem of great potential value. It is expected that a large class of multivariable problems in various field of engineering can be analysed by this approach. Thus, for the student interested in this field, the application and extension of p-network theory to the engineering problems offers wide opportunity for research.

In the part 3, one application of p-network theory will be given. An algebraic topological approach to p-network enables to discover a new property, i.e., Diakoptic property, it means that given a p-network, then it can be torn into smaller parts, each part solved separately, and the solution of untorn network obtained from the torn parts. This property will be described in details and its application to electrical networks will be illustrated by examples.

CHAPTER 3

DIAKOPTICS AND CO-DIAKOPTICS

Diakoptic and co-diakoptic property of p-network has been described in details which is direct outcome of p-network theory. The application of diakoptic and codiakoptic property has been shown to the electrical networks. Two examples of electrical networks are solved to illustrate the diakoptic and co-diakoptic method for solving electrical network. The method is given in the most general form and does not restrict itself to any particular class of problems. It can be applied to any type of electrical networks, though it may be debatable whether it will be advantageous or not in some cases.

3.1 Diakoptics and Co-diakoptics of p-Network.

The most important property of p-network is its diakoptic property. Diakoptic is a greek word, where "dia" means back and forth, and 'koptic' means to tear or to dissect. By the diakoptic property of p-network, we mean that a given p-network can be torn into small subdivisions, each subdivision solved separately, and the solution of given network can be obtained from the solved small subdivisions. In other words, given a set of p-networks, having the same number of p-simplexes, connected differently, the solution of one network can be obtained from the solution of some other network of the set by routine transformations(21, 24, 25). The various p-networks obtained by connecting the same number of p-simplexes in different ways can be thought of as offering different set of bases for the same (C^p) and (C_p) for a given Z_p or Y^p.

The diakoptic property of p-networks can be used to solve very large complicated p-networks. To start with, the mathematical representation of the diakoptic theory will be given.

a. Diakoptic Property of p-Network.

The diakoptic property is the natural outcome of p-network theory of chapter 2. The p-chains, p-cochains and the relation between them are defined as

$$(C^p) = Y^p (C_p) \qquad (C_p) = Z_p (C^p) \qquad 3.1.1$$

where the bases chosen are primitive bases. The primitive bases can be assumed to correspond to a p-network with all the p-simplexes isolated, called a primitive network.

The same p-simplexes of the primitive network are connected into a network $k^{(1)}$. The network $k^{(1)}$ offers another set of bases to represent the quantities of equation 3.1.1.

In network $k^{(1)}$,

$$(C^{(1)}_p) = Z^{(1)}_p (C^p_{(1)}) \qquad (C^p_{(1)}) = Y^p_{(1)} (C^{(1)}_p) \qquad 3.1.2$$

where

$$(C^{(1)}_p) = Q^{p(1)}_t (C_p) \text{ or } (C_p) = A^{p(1)} (C^{(1)}_p) \qquad 3.1.3$$

$$(C^p_{(1)}) = A^{p(1)}_t (C^p) \text{ or } (C^p) = Q^{p(1)} (C^p_{(1)}) \qquad 3.1.4$$

$$Z^{(1)}_p = Q^{p(1)}_t . Z_p . Q^{p(1)} \qquad 3.1.5$$

$$Y^p_{(1)} = Q^{p(1)}_t . Y^p . A^{p(1)} \qquad 3.1.6$$

If the same p-simplexes are connected differently into a network $k^{(2)}$, it offers a different set of bases to represent the quantities of equation 3.1.1 as below:

$$(C^{(2)}_{p}) = Q_{t}^{p(2)} \; (C_{p}) \quad \text{or} \quad (C_{p}) = A^{p(2)} \, (C^{(2)}_{p} \qquad 3.1.7$$

$$(C^{p}_{(2)}) = A_{t}^{p(2)} \; (C^{p}) \quad \text{or} \quad (C^{p}) = A^{p(2)} \, (C^{p}_{(2)}) \qquad 3.1.8$$

$$Z^{(2)}_{p} = Q_{t}^{p(2)} \, . \, Z_{p} \, . \, Q^{p} \qquad 3.1.9$$

$$Y^{p}_{(2)} = A_{t}^{\;p(2)} \, . \, Y^{p}_{(2)} \, . \, A^{p(2)} \qquad 3.1.10$$

From the equations for $k^{(1)}$ and $k^{(2)}$, the relations between the quantities of $k^{(1)}$ and $k^{(2)}$ can be established.

$$(C^{p}_{(2)}) = A_{t}^{\;p(2)} \; Q^{p(1)} \, (C^{p}_{(1)}) \quad \text{or} \quad (C^{p}_{(1)}) = A_{t}^{p(1)} Q^{p(2)} (C^{p}_{(2)} \qquad 3.1.11$$

$$(C^{(2)}_{p}) = Q_{t}^{\;p(2)} \; A^{p(1)} (C^{(1)}_{p}) \quad \text{or} \quad (C^{(1)}_{p}) = A_{t}^{p(2)} \; A^{p(2)} (C^{(2)}_{p} \qquad 3.1.12$$

The above relations are shown in Fig. 3.1.1. Once the network $k^{(2)}$ is solved, the solution of network $k^{(1)}$ can be obtained from the solution of network $k^{(2)}$ by the routine transformations shown in Fig. 3.1.1. The relations between the differently connected networks having the same number of simplexes lead us to a new method of solving the networks.

To solve any network, the inverse of Y^{p4} or Z_{pl} is required. In very large networks, the inverse of Y^{p4} or Z_{pl} is very difficult to find. The inverse is very laborious to find and too much time is required on a digital computer.

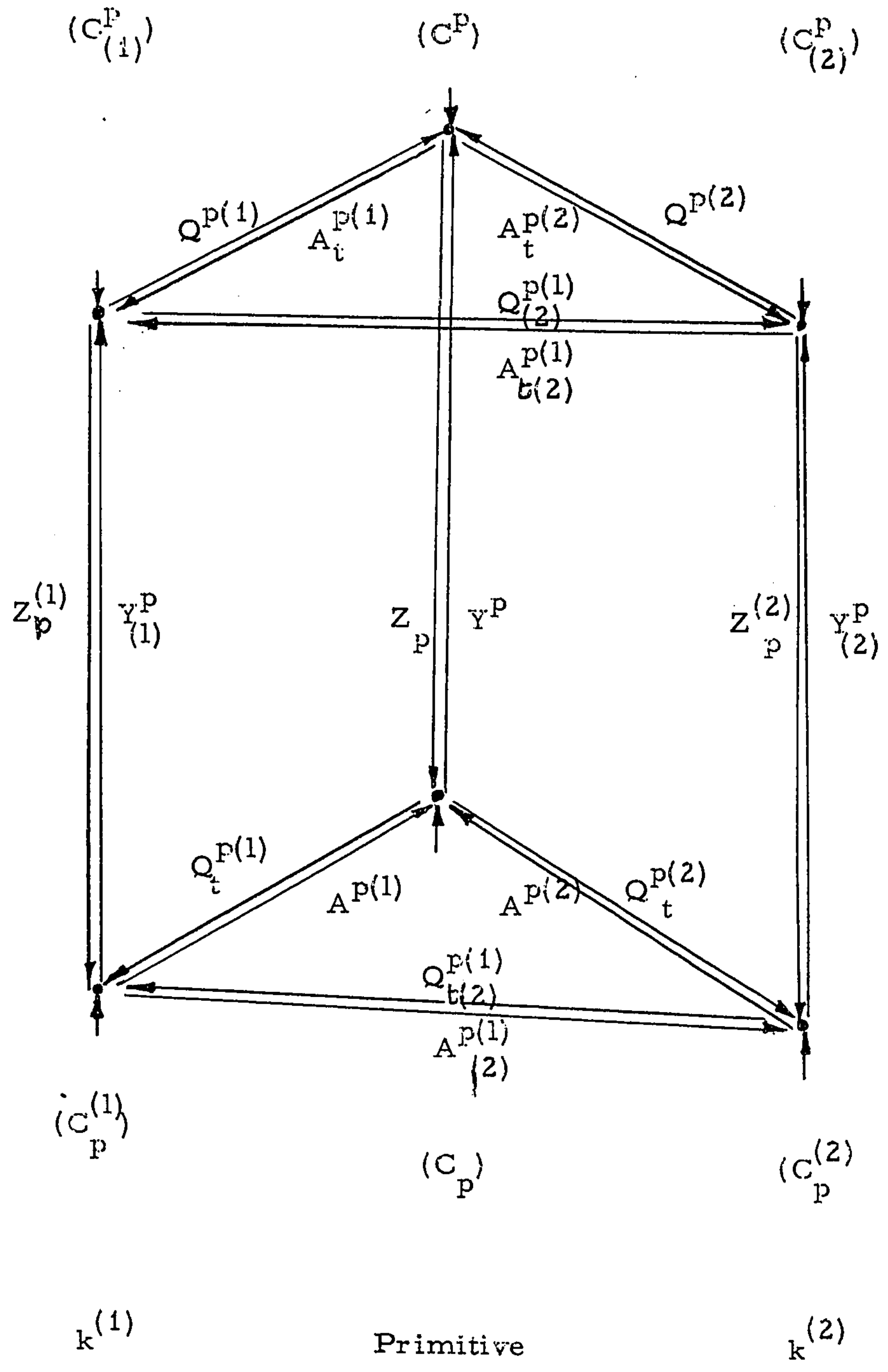


Fig. 3.1.2 Transformation Diagram for $k^{(1)}$ and $k^{(2)}$.

The diakoptic property of p-network offers a new method for solving very large networks. By this property, the solution of a given network $k^{(1)}$ can be obtained from the solution of another but much simpler network $k^{(2)}$ composed of the same number of p-simplexes.

We are given a network $k^{(1)}$. The given network $k^{(1)}$ is torn into small subdivisions in such a way that the torn network $k^{(2)}$ has no inductive coupling among its subdivisions. The solution of $k^{(1)}$ is then obtained from the solution of torn network $k^{(2)}$ much simpler to solve.

There are two ways of tearing the network $k^{(1)}$. In one case, when the network $k^{(1)}$ is torn, some of the closed paths of $k^{(1)}$ are opened. In the second case, by tearing the closed-paths which are opened in each subdivision are closed. The two different methods of tearing named:

i. Diakoptic Method.(2)

ii. Co-diakoptic Method(13).

b.i The Solution of Given Network $k^{(1)}$ by Diakoptic Method.

In this method, by tearing, new open paths are created in torn network $k^{(2)}$ which do not exist in network $k^{(1)}$. The network $k^{(1)}$ is assumed to have the most general excitation. If the network $k^{(1)}$ is solved by taking the inverse $(Y^{p4}_{(1)})^{-1}$, by the equation 2.1.73b,

$$(C^{(1)'}_{po}) = (Y^{p4}_{(1)})^{-1}\Big((C^{po'}_{(1)}) - Y^{p3}_{(1)}\,(C^{(1)'}_{pc})\Big) \qquad 3.1.13$$

Once $(C_{po}^{(1)})$ is found, the various other quantities can be easily obtained by simple transformations. Let us now try to solve the given network $k^{(1)}$ by tearing. The network $k^{(1)}$ is torn in such a way that the torn subdivisions are inductively isolated from each other. The solution of network $k^{(2)}$ requires the inverse $(Y_{(2)}^{p4})^{-1}$. It should be noted that if it is much easier to find $(Z_{p1}^{(2)})^{-1}$, then $(Y_{(2)}^{p4})^{-1}$ can be found from $(Z_{p1}^{(2)})^{-1}$ by equation 2.1.82.

If the torn network $k^{(2)}$ has n isolated subdivisions with no inductive coupling between them, then,

$$Y_{(2)}^{p4} = \begin{bmatrix} Y_{(2)}^{1} & & & \\ & Y_{(2)}^{2} & & \\ & & \ddots & \\ & & & Y_{(2)}^{n} \end{bmatrix} \qquad 3.1.14$$

The solution of $k^{(2)}$ requires $(Y_{(2)}^{p4})^{-1}$. Since $Y_{(2)}^{p4}$ is torn into n-subdivisions, for finding $(Y_{(2)}^{p4})^{-1}$, it is simply required to find the inverse of $Y_{(2)}^{1}, Y_{(2)}^{2}, \ldots, Y_{(2)}^{n}$ separately. Thus the inverse $(Y_{(2)}^{p4})^{-1}$ is very easy to find. We shall establish the solution of $k^{(1)}$ from the solution of $k^{(2)}$.

Tearing of $k^{(1)}$ involves the opening of some closed paths of $k^{(1)}$. The torn network $k^{(2)}$ contains all the open paths of $k^{(1)}$ and also some new open paths which are created by tearing. In going from network $k^{(2)}$ to $k^{(1)}$, all the closed paths of $k^{(2)}$ are carried to the closed paths of $k^{(1)}$, open-paths of $k^{(2)}$ are changed into open-paths of $k^{(1)}$ and some closed paths of $k^{(1)}$. If the values of the chains and cochains of $k^{(1)}$ and $k^{(2)}$ are related, the matrix

$$Q^{p(1)'}_{(2)} = \begin{array}{c|cc|c} & 0^{(1)} & C_1^{(1)} & C_2^{(1)} \\ \hline 0^{(2)} & \multicolumn{2}{c|}{Q^{p(1)}_{(2)}} & 0 \\ \hline C^{(2)} & \multicolumn{2}{c|}{0} & U \\ \hline \end{array} \qquad 3.1.15$$

is obtained. Matrix $Q^{p(1)'}_{(2)}$ transforms the value of chain in $k^{(1)}$ to the corresponding value in $k^{(2)}$. The matrix $Q^{p(1)'}_{(2)}$ is split into four subdivisions where the two subdivisions have zero elements. $C_2^{(1)}$ corresponds to all those closed paths of $k^{(1)}$ which are also in $k^{(2)}$, since in going from $k^{(2)}$ to $k^{(1)}$, all the closed paths of $k^{(2)}$ are carried to the closed paths of $k^{(1)}$ which are shown by subdivision $C_2^{(1)}$ and $C^{(2)}$. The closed paths $C_1^{(1)}$ of $k^{(1)}$ are the newly created closed paths of $k^{(1)}$ which do not exist in $k^{(2)}$. From the above argument, the subdivisions $C^{(2)}$ and $(0^{(1)}, C_1^{(1)})$, and $0^{(2)}, C_2^{(1)}$ have zero elements since they have nothing in common.

The solution of $k^{(2)}$ is obtained in terms of open path bases, i.e. by $(Y^{p4}_{(2)})^{-1}$. The solution of $k^{(1)}$ will be found from the solution of network $k^{(2)}$. For our purpose only the submatrix $Q^{p(1)}_{(2)}$ will be used. The matrix $Q^{p(1)}_{(2)}$ transforms the values of chains of $k^{(1)}$ and $k^{(2)}$ as

$$(C^{po'}_{(2)}) = Q^{p(1)}_{(2)} \left[\begin{array}{c} C^{po'}_{(1)} \\ \hline C^{pc'}_{(1)} \end{array}\right] \qquad 3.1.15a$$

or

$$(C^{po'}_{(2)}) = \begin{bmatrix} Q^{po(1)}_{(2)} & Q^{pc(1)}_{(2)} \end{bmatrix} \begin{bmatrix} C^{po'}_{(1)} \\ C^{pc'}_{(1)} \end{bmatrix} \qquad 3.1.16b$$

Once the transformation for chains is found, the transformation for various other quantities can be easily established, as shown in Chapter 2.1. We have

$$Z^{(1)}_{p} = Q^{p(1)}_{(2)t} \cdot (Y^{p4}_{(2)})^{-1} \cdot Q^{p(1)}_{(2)} \qquad 3.1.17$$

$$\begin{bmatrix} Z^{(1)}_{p1} & Z^{(1)}_{p2} \\ Z^{(1)}_{p3} & Z^{(1)}_{p4} \end{bmatrix} = \begin{bmatrix} Q^{pc(1)}_{t(2)} (Y^{p4}_{(2)})^{-1} Q^{pc(1)}_{(2)} & Q^{pc(1)}_{t(2)} (Y^{p4}_{(2)})^{-1} Q^{po(1)}_{(2)} \\ Q^{po(1)}_{t(2)} (Y^{p4}_{(2)})^{-1} Q^{pc(1)}_{(2)} & Q^{po(1)}_{t(2)} (Y^{p4}_{(2)})^{-1} Q^{po(1)}_{(2)} \end{bmatrix} \qquad 3.1.18$$

The orthogonal set of equations for $k^{(1)}$ can be written as

$$(C^{(1)'}_{pc}) = Z^{(1)}_{p1} (C^{pc'}_{(1)}) + Z^{(1)}_{(2)} (C^{po'}_{(1)})$$

$$3.1.19$$

$$(C^{(1)'}_{po}) = Z^{(1)}_{p3} (C^{pc'}_{(1)}) + Z^{(1)}_{p4} (C^{po'}_{(1)})$$

The closed-path quantities correspond to the newly created closed paths and open path quantities correspond to the open paths of $k^{(1)}$. From the equation 3.1.19

$$(C^{(1)'}_{po}) = Z^{(1)}_{p3} (Z^{(1)}_{p1})^{-1} (C^{(1)}_{pc}) + (Z^{(1)}_{p4} - Z^{(1)}_{p3} (Z^{(1)}_{p4})^{-1})(C^{po'}_{(1)}) \qquad 3.1.20$$

Since in finding $(C^{(1)'}_{po})$, only the newly created closed paths are considered, the effect of the excitation for the closed paths of $k^{(1)}$ not considered in equation 3.1.20 has to be taken into account by modifying the values of $(C^{po'}_{(1)})$ and $(C^{(1)'}_{po})$.

Once $(C^{(1)'}_{po})$ is solved, the other quantities of $k^{(1)}$ can be found by simple transformations. To solve $k^{(1)}$, we need $(Y^{p4}_{(2)})^{-1}$ and $(Z^{(1)}_{p1})^{-1}$. The matrix $Z^{(1)}_{p1}$ has the rank equal to the number of newly created closed paths of $k^{(1)}$ which is very small. Instead of finding the inverse $(Y^{p4}_{(1)})^{-1}$ which is too laborious, we find $(Y^{p4}_{(2)})^{-1}$ and $(Z^{(1)}_{p1})^{-1}$ which are much easier to find. The various transformations used are summarized in the Fig. 3.1.2. With this, the basic mathematical representation of diakoptic method of solving p-network is complete. The actual details will be shown more clearly by a numerical example given later.

b.ii Solution of p-Network by Co-diakoptics

This method is dual to the diakoptic method. When the network $k^{(1)}$ is torn into the network $k^{(2)}$, all the p-simplexes which are cut are shorted together in each subdivision.

In this method, from the network $k^{(2)}$ to $k^{(1)}$, all the open paths of $k^{(2)}$ are carried to the open-paths of $k^{(1)}$, and closed-paths of $k^{(2)}$ are carried to the closed-paths and some open paths of $k^{(1)}$. In fact, from $k^{(2)}$ to $k^{(1)}$ we are creating some new open paths not present in $k^{(2)}$. If the network $k^{(1)}$ is solved untorn the inverse $(Z^{(1)}_{p1})^{-1}$ or $(Y^{p4}_{(1)})^{-1}$ is required. For our purpose it is assumed that $(Z^{(1)}_{p1})^{-1}$ is required for solving the network. The torn network $k^{(2)}$ is solved firstly by the closed-path method i.e. the inverse $(Z^{(2)}_{p1})^{-1}$ is found out. The tearing is done

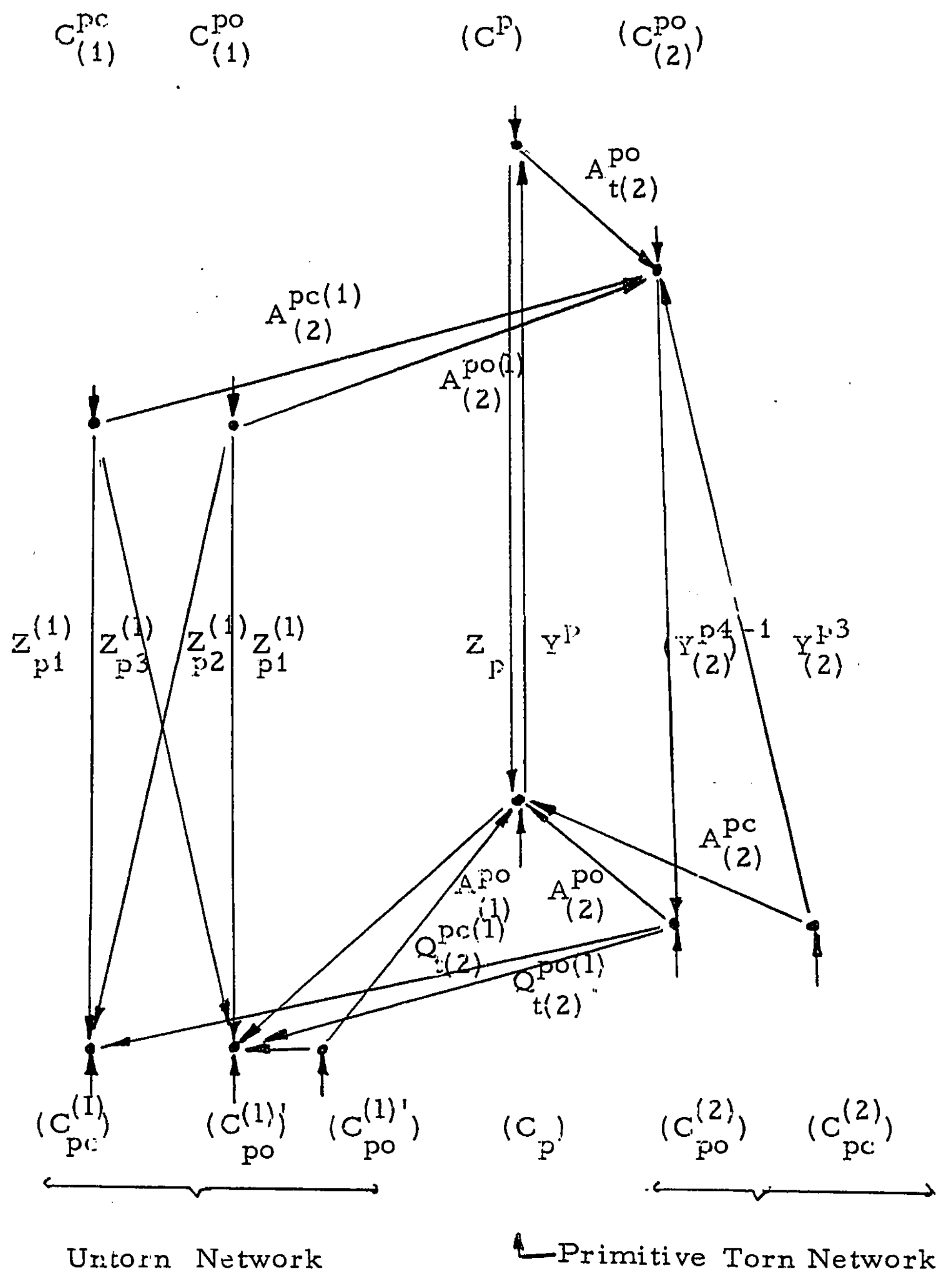


Fig. 3.1.1 Transformation Diagram for the Diakoptic Method.

in such a way that the torn subdivisions of $k^{(2)}$ are inductively isolated from each other. The matrix $Z^{(2)}_{p1}$ take the following form:

$$Z^{(2)}_{p1} = \begin{bmatrix} Z^1 & & & \\ & Z^2 & & \\ & & \ddots & \\ & & & Z^n \end{bmatrix} \qquad 3.1.21$$

The inverse $(Z^{(2)}_{p1})^{-1}$ requires the inverse of submatrices Z^1, $Z^2, \ldots,$ Z^n, separately. After solving $k^{(2)}$, i.e. finding $(Z^{(2)}_{p1})^{-1}$, the matrix $A^{p(1)}_{(2)}$ is set up which transforms the coefficients of chains of $k^{(2)}$ to those of $k^{(1)}$.

$$A^{p(1)'}_{(2)} = \begin{array}{c|c|c|} & C^{(1)} \quad 0^{(1)}_1 & 0^{(2)}_2 \\ \hline C^{(1)} & A^{p(1)}_{(2)} & 0 \\ \hline 0^{(2)} & 0 & U \\ \hline \end{array} \qquad 3.1.22$$

The open paths of $k^{(1)}$ are split into $0^{(1)}_1$ and $0^{(1)}_2$ where $0^{(1)}_2$ are those open paths of $k^{(1)}$ which exist in $k^{(2)}$ also. None of the open-paths of $k^{(2)}$ is changed into $C^{(1)}$ and $0^{(1)}_1$ and no closed path of $k^{(2)}$ is changed into $0^{(1)}_2$. This implies that the nondiagonal terms are zero. For our purpose the submatrix $A^{p(1)}_{(2)}$ shall be required, since $k^{(2)}$ is already in closed-path bases and the nondiagonal terms are zero in the matrix $A^{p(1)'}_{(2)}$.

We have

$$(C_{p}^{(2)'}) = A_{(2)}^{p(1)} \quad (C_{p}^{(1)'}) \qquad 3.1.23$$

Splitting $A_{(2)}^{p(1)}$ into closed and open path subdivisions,

$$(C_{p}^{(2)'}) = \begin{array}{|c|c|}\hline A_{(2)}^{pc(1)} & A_{(2)}^{po(1)} \\ \hline\end{array} \quad \begin{bmatrix} C_{pc}^{(1)'} \\ \hline C_{po}^{(1)'} \end{bmatrix} \qquad 3.1.24$$

Once the cochain transformation is set up, the transformation for the various other quantities are easily set up.

$$(Y_{(1)}^{p}) = A_{t(2)}^{p(1)} \cdot (Z_{pl}^{(2)})^{-1} \cdot A_{(2)}^{p(1)} \qquad 3.1.25$$

$$\begin{array}{|c|c|}\hline Y_{(1)}^{p1} & Y_{(1)}^{p2} \\ \hline Y_{(1)}^{p3} & Y_{(1)}^{p4} \\ \hline\end{array} = \begin{array}{|c|c|}\hline A_{t\,(2)}^{pc(1)} \quad Z_{p1}^{(2)})^{-1} \quad A_{(2)}^{pc(1)} & A_{t\,(2)}^{pc(1)} \quad (Z_{p1}^{(2)})^{-1} A_{(2)}^{po(1)} \\ \hline A_{t\,(2)}^{po(1)} \quad (Z_{p1}^{(2)})^{-1} \quad A_{(2)}^{pc(1)} & A_{t\,(2)}^{po(1)} \quad (Z_{p1}^{(2)})^{-1} A_{(2)}^{po(1)} \\ \hline\end{array}$$

3.1.26

If the cochain form of the orthogonal equations of $k^{(1)}$ is written where the open paths refer only to the newly created open paths of $k^{(1)}$, we have

$$\begin{aligned}(C_{(1)}^{pc'}) &= Y_{(1)}^{p1} \; (C_{pc}^{(1)'}) + Y_{(1)}^{p2} \; (C_{po}^{(1)'}) \\ (C_{(1)}^{po'}) &= Y_{(1)}^{p3} \; (C_{pc}^{(1)'}) + Y_{(1)}^{p4} \; (C_{po}^{(1)'})\end{aligned} \qquad 3.1.27$$

From the equations 3.1.27

$$(C_{(1)}^{po'}) = (Y_{(1)}^{p1} - Y_{(1)}^{p2} (Y_{(1)}^{p4})^{-1} Y_{(1)}^{p3}) (C_{pc}^{(1)'}) + Y_{(1)}^{p2} (Y_{(1)}^{p4})^{-1} (C_{(1)}^{po'})$$

3.1.28

Once $(C_{(1)}^{pc'})$ is found, the rest of the quantities of $k^{(1)}$ can be found as shown in Chapter 2.1. The solution of network $k^{(1)}$ via tearing requires the inverses $(Z_{pl}^{(2)})^{-1}$ and $(Y_{(1)}^{p4})^{-1}$. The inverse of $Z_{p1}^{(2)}$ is very easy to find, and $Y_{(1)}^{p4}$ is matrix of very small rank equal to the number of newly created open paths.

The excitation in those open paths of $k^{(1)}$ which are not taken into account is to be considered by its equivalent effect on the considered closed-paths and newly created open paths. The various relation described above can be summarized in Fig. 3.1.3.

The co-diakoptic method of solving a network will be illustrated with the help of a numerical example. Then it will become clear how powerful this method is in solving large networks.

We have made an attempt to mathematically describe the diakoptic and co-diakoptic property of p-networks. In this study no attempt is made to use this property to analyse p-networks. Only the application of this property to 1-network i.e. electrical networks is described in details.

c. Application of Tearing to 1-Networks or Electrical Networks.

The diakoptic and co-diakoptic methods will be now illustrated with the help of two examples. The two examples are taken from electrical networks and the various relations and terms are in electrical terms.

Example I (Diakoptic Method)

The electrical network chosen is given in fig. 3.1.4. The various branches of the network are oriented as shown. The network is assumed to have general excitation with both current and voltage

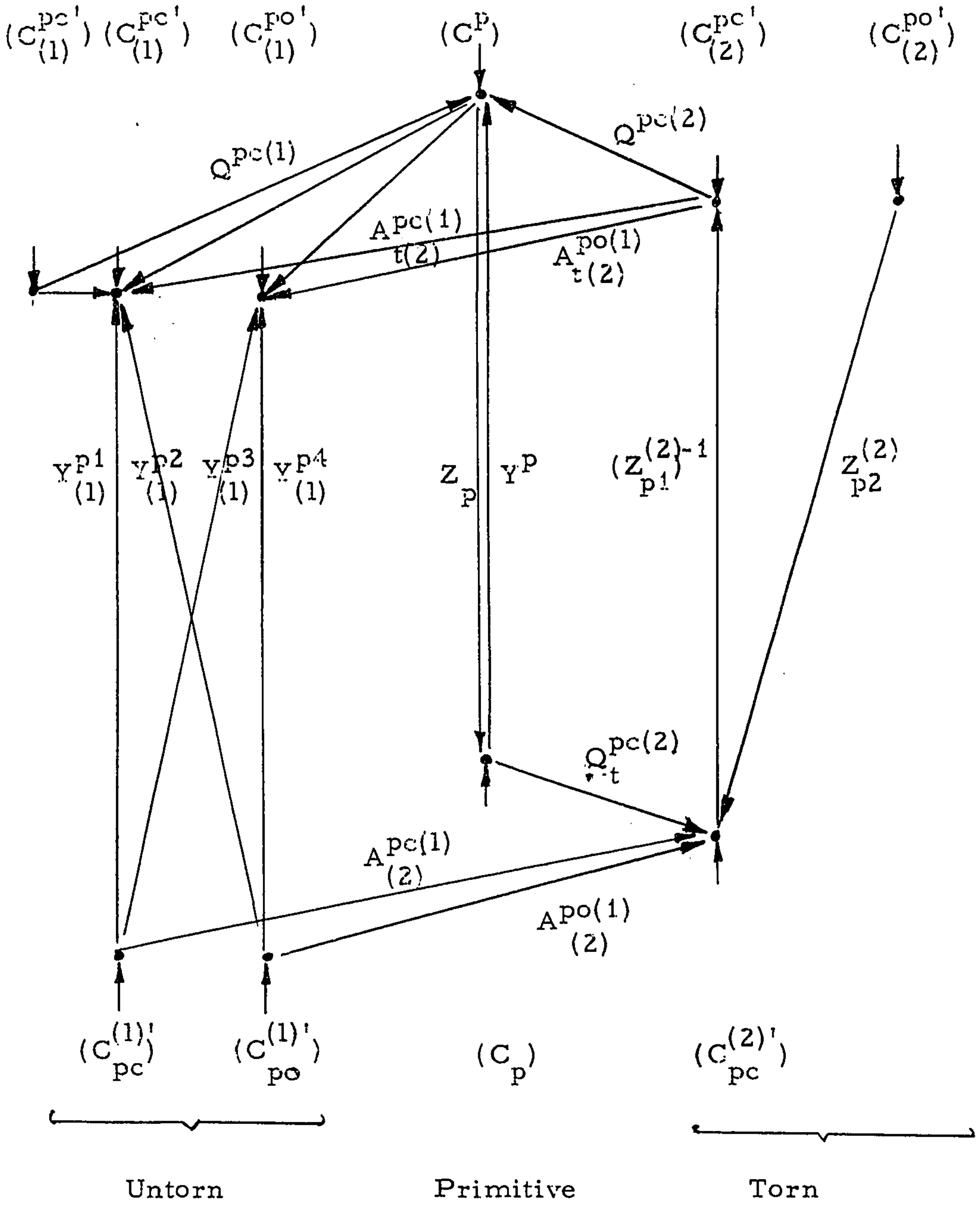


Fig. 3.1.3 Transformation Diagram for Co-diakoptic Method.

generators in the branches. All the branches are assumed to have an impedance of one ohm each with no mutual coupling among the branches. The primitive or branch impedance matrix is

$$
Z = \begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & \cdot & \cdot & \cdot & 22 \\
\hline
1 & 1 & & & & & & \\
2 & & 1 & & & & & \\
3 & & & 1 & & & & \\
\cdot & & & & \cdot & & & \\
\cdot & & & & & \cdot & & \\
\cdot & & & & & & \cdot & \\
22 & & & & & & & 1 \\
\hline
\end{array}
\qquad 3.1.29
$$

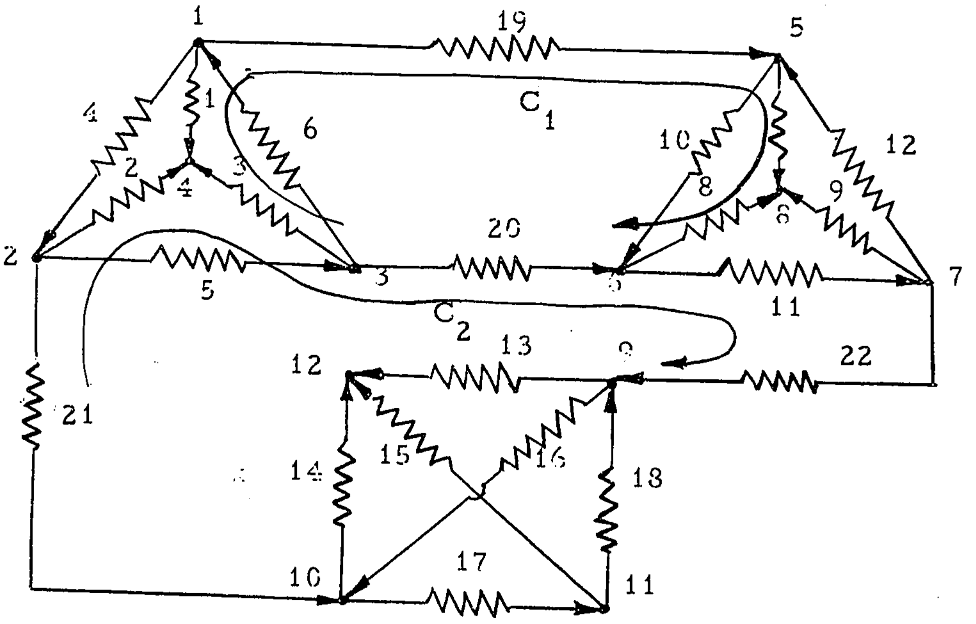


Fig. 3.1.4 Given Network. (Untorn) $k^{(1)}$

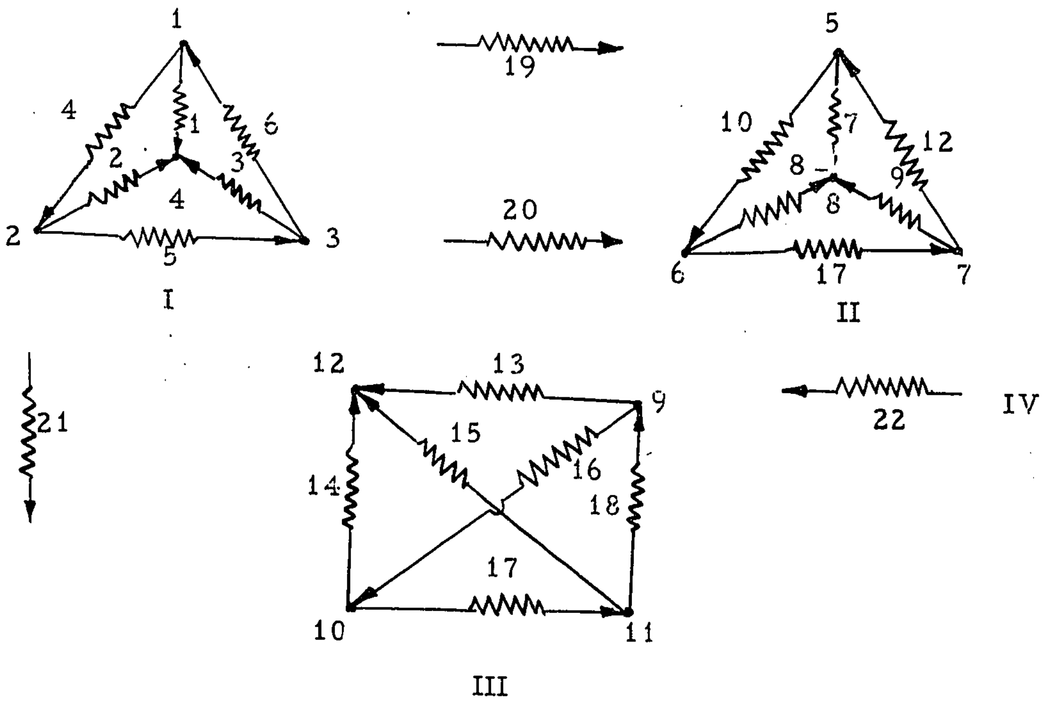


Fig. 3.1.5 Torn Network $k^{(2)}$

The untorn network $k^{(1)}$ has 22 branches and 11 independent open-paths. Thus it has (22-11) = 11 independent closed paths. To solve $k^{(1)}$ without tearing, either the closed-path or the open-path method can be used. In either case, the inverse of 11×11 matrix is required to solve the network. In this example, the network $k^{(1)}$ shall be solved via tearing and it will be shown how much labor is saved in this way. In fact, when the networks are very large, the saving in labor for solving the networks is tremendous and the power of tearing method is quite evident.

Let the given network $k^{(1)}$ be torn into four subdivisions as shown in Fig. 3.1.5 i.e. the network $k^{(2)}$. For solving $k^{(1)}$ via tearing, the network $k^{(2)}$ is solved first by taking the inverse $(Y^{p4}_{(2)})^{-1}$. To facilitate the application of our method, Fig. 3.1.2 is redrawn as Fig. 3.1.6 interpreting the various quantities of Fig. 3.1.5 in electrical terms. From Fig. 3.1.6, to find $(Y^{p4}_{(2)})^{-1}$, matrix $A^{o}_{(2)}$ is to be set up. The matrix $A^{o}_{(2)}$ for the network $k^{(2)}$ is given in equation 3.1.30.

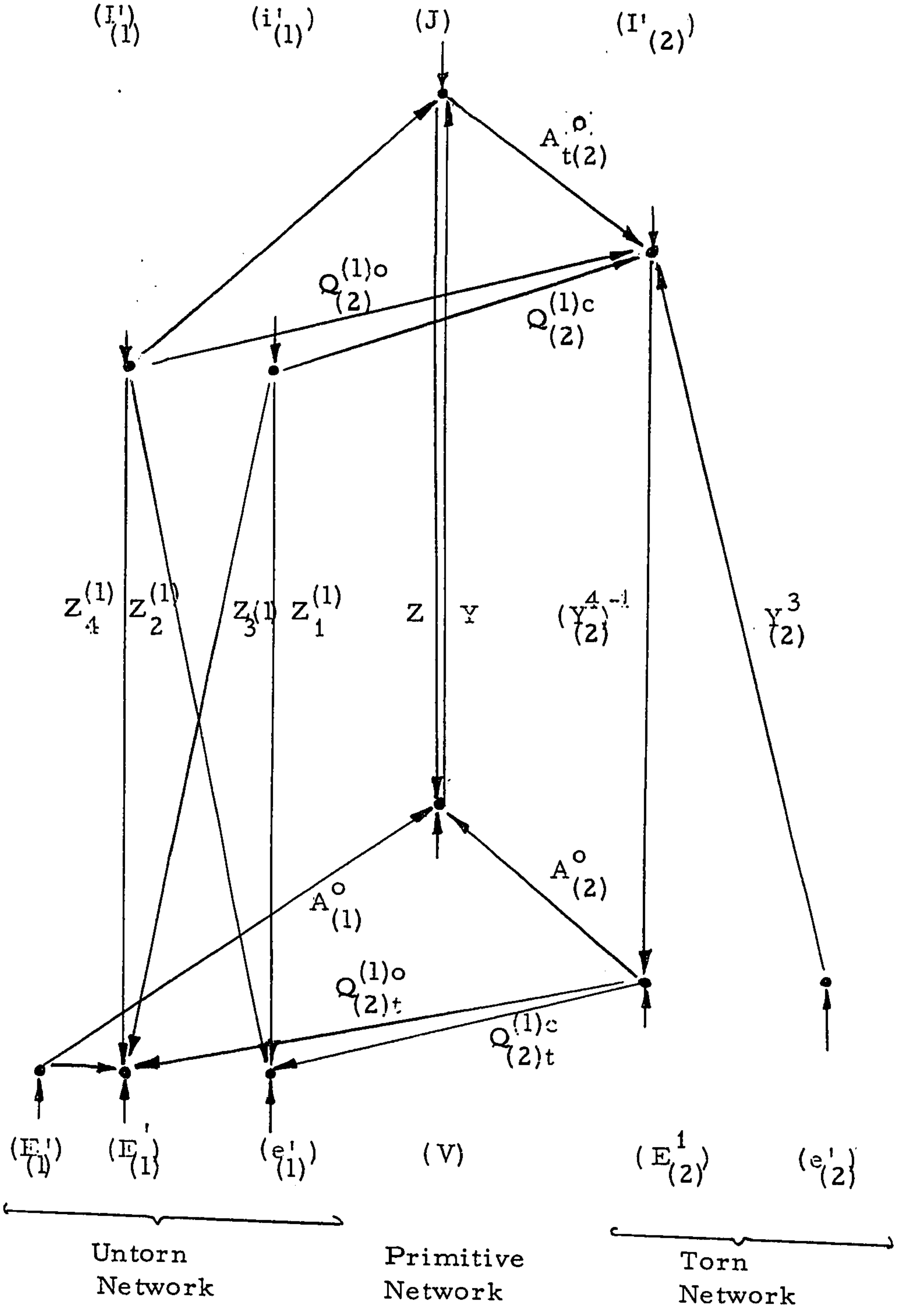


Fig. 3.1.6 Transformation Diagram for Solution of an Electrical Network by Diakoptic Method.

The typical method used in these examples for finding A's and Q's is described in full details in Reference 3.

$$A^{o}_{(2)} = \begin{array}{r|ccccccccccccc|}
 & {}^{0}1\text{-}4 & {}^{0}2\text{-}4 & {}^{0}5\text{-}4 & {}^{0}5\text{-}8 & {}^{0}6\text{-}8 & {}^{0}7\text{-}8 & {}^{0}9\text{-}12 & {}^{0}10\text{-}12 & {}^{0}11\text{-}12 & {}^{0}19 & {}^{0}20 & {}^{0}21 & {}^{0}22 \\
\hline
\text{Branches } 1 & 1 & & & & & & & & & & & & \\
2 & & 1 & & & & & & & & & & & \\
3 & & & 1 & & & & & & & & & & \\
4 & 1 & -1 & & & & & & & & & & & \\
5 & & 1 & -1 & & & & & & & & & & \\
6 & -1 & & 1 & & & & & & & & & & \\
7 & & & & 1 & & & & & & & & & \\
8 & & & & & 1 & & & & & & & & \\
9 & & & & & & 1 & & & & & & & \\
10 & & & & +1 & -1 & & & & & & & & \\
11 & & & & & 1 & -1 & & & & & & & \\
12 & & & & -1 & & 1 & & & & & & & \\
13 & & & & & & & 1 & & & & & & \\
14 & & & & & & & & 1 & & & & & \\
15 & & & & & & & & & 1 & & & & \\
16 & & & & & & & 1 & -1 & & & & & \\
17 & & & & & & & & 1 & -1 & & & & \\
18 & & & & & & & -1 & & 1 & & & & \\
19 & & & & & & & & & & 1 & & & \\
20 & & & & & & & & & & & 1 & & \\
21 & & & & & & & & & & & & 1 & \\
22 & & & & & & & & & & & & & 1 \\
\hline
\end{array} \qquad 3.1.30$$

For the network $k^{(2)}$,

$$Y^{4}_{(2)} = A^{o}_{(2)t} \; Y \; A^{o}_{(2)}$$

$$Y^{4}_{(2)} = \begin{array}{ccccccccccccc}
 & \text{I} & & & \text{II} & & & \text{III} & & & \text{IV} & & \\
3 & -1 & -1 & & & & & & & & & & \\
-1 & 3 & -1 & & & & & & & & & & \\
-1 & -1 & 3 & & & & & & & & & & \\
 & & & 3 & -1 & -1 & & & & & & & \\
 & & & -1 & 3 & -1 & & & & & & & \\
 & & & -1 & -1 & 3 & & & & & & & \\
 & & & & & & 3 & -1 & -1 & & & & \\
 & & & & & & -1 & 3 & -1 & & & & \\
 & & & & & & -1 & -1 & 3 & & & & \\
 & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & 1
\end{array} \qquad 3.1.31$$

In finding $(Y^4_{(2)})^{-1}$, it is simply required to invert each subdivision separately, since all the subdivisions are inductively isolated from each other

$$(Y^4_{(2)})^{-1} = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|cccc}
2 & 1 & 1 & & & & & & & & & & \\
1 & 2 & 1 & & & & & & & & & & \\
1 & 1 & 2 & & & & & & & & & & \\
\hline
 & & & 2 & 1 & 1 & & & & & & & \\
 & & & 1 & 2 & 1 & & & & & & & \\
 & & & 1 & 1 & 2 & & & & & & & \\
\hline
 & & & & & & 2 & 1 & 1 & & & & \\
 & & & & & & 1 & 2 & 1 & & & & \\
 & & & & & & 1 & 1 & 2 & & & & \\
\hline
 & & & & & & & & & 4 & & & \\
 & & & & & & & & & & 4 & & \\
 & & & & & & & & & & & 4 & \\
 & & & & & & & & & & & & 4
\end{array}\right] \qquad 3.1.33$$

$(Y^4_{(2)})^{-1}$ is the open-path solution matrix of network $k^{(2)}$. To find the solution of network $k^{(1)}$ from the solution of the network $k^{(2)}$, the matrix $Q^{(1)}_{(2)}$ is required.

$$Q^{(1)}_{(2)} = \begin{array}{c|cc|ccccccccccc|}
 & C_1 & C_2 & 0_{1-4} & 0_{2-4} & 0_{3-4} & 0_{5-8} & 0_{6-8} & 0_{7-8} & 0_{9-12} & 0_{10-12} & 0_{11-12} & 0_{19} & 0_{21} \\
\hline
0_{1-4} & -1 & & 1 & & & & & & & & & & \\
0_{2-4} & & 1 & & 1 & & & & & & & & & \\
0_{3-4} & 1 & -1 & & & 1 & & & & & & & & \\
0_{5-8} & 1 & & & & & 1 & & & & & & & \\
0_{6-8} & -1 & 1 & & & & & 1 & & & & & & \\
0_{7-8} & & -1 & & & & & & 1 & & & & & \\
0_{9-12} & & 1 & & & & & & & 1 & & & & \\
0_{10-12} & & -1 & & & & & & & & 1 & & & \\
0_{11-12} & & & & & & & & & & & 1 & & \\
0_{19} & 1 & & & & & & & & & & & 1 & \\
0_{20} & & -1 & & & & & & & & & & & 1 \\
0_{21} & -1 & 1 & & & & & & & & & & & \\
0_{22} & & 1 & & & & & & & & & & & \\
\hline
\end{array} \qquad 3.1.34$$

$Q^{(1)}_{(2)}$ relates the open-paths quantities of $k^{(2)}$ to the open-paths and newly created closed-paths quantities of $k^{(1)}$.

$$Z^{(1)} = Q^{(1)}_{t(2)} \; (Y^{4}_{(2)})^{-1} \; Q^{(1)}_{(2)} \qquad 3.1.35$$

where $Q^{(1)}_{(2)}$ has two parts as

$$Q^{(1)}_{(2)} = \begin{bmatrix} Q^{(1)c}_{(2)} & Q^{(1)o}_{(2)} \end{bmatrix} \qquad 3.1.36$$

$$Z^{(1)} = \begin{array}{c|cc|ccccccccccc|}
 & C_1 & C_2 & 0_{1-4} & 0_{2-4} & 0_{3-4} & 0_{5-8} & 0_{6-8} & 0_{7-8} & 0_{9-12} & 0_{10-12} & 0_{11-12} & 0_{19} & 0_{21} \\
\hline
C_1 & 12 & -6 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 4 & 0 \\
C_2 & -6 & 18 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & -4 \\
\hline
0_{1-4} & -1 & 0 & 2 & 1 & 1 & & & & & & & & \\
0_{2-4} & 0 & 1 & 1 & 2 & 1 & & & & & & & & \\
0_{3-4} & 1 & -1 & 1 & 1 & 2 & & & & & & & & \\
0_{5-8} & 1 & 0 & & & & 2 & 1 & 1 & & & & & \\
0_{6-8} & -1 & 1 & & & & 1 & 2 & 1 & & & & & \\
0_{7-8} & 0 & -1 & & & & 1 & 1 & 2 & & & & & \\
0_{9-12} & 0 & 1 & & & & & & & 2 & 1 & 1 & & \\
0_{10-12} & 0 & 0 & & & & & & & 1 & 2 & 1 & & \\
0_{11-12} & 0 & 0 & & & & & & & 1 & 1 & 2 & & \\
0_{19} & 4 & 0 & & & & & & & & & & 4 & \\
0_{21} & 0 & -4 & & & & & & & & & & & 4 \\
\hline
\end{array} = \begin{array}{|c|c|} \hline Z^{(1)}_1 & Z^{(1)}_2 \\ \hline Z^{(1)}_3 & Z^{(1)}_4 \\ \hline \end{array} \qquad 3.1.37$$

$$(Z^{(1)}_1)^{-1} = \frac{2}{15} \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \qquad 3.1.38$$

$$Z_3^{(1)}(Z_1^{(1)})^{-1} = \frac{1}{30}\left[\begin{array}{r|r} 3 & -1 \\ 1 & 2 \\ 2 & -1 \\ 3 & 1 \\ -2 & 1 \\ -1 & -2 \\ 1 & 2 \\ -1 & -2 \\ 0 & 0 \\ 12 & 4 \\ 4 & -8 \end{array}\right] \tag{3.1.39}$$

$$Z_3^{(1)}(Z_1^{(1)})^{-1}Z_2^{(1)} = \frac{1}{120}\left[\begin{array}{rrr|rrr|rrr|rr} 3 & -1 & -2 & -3 & 2 & 1 & -1 & 1 & 0 & -12 & 4 \\ -1 & 2 & -1 & 1 & 1 & -2 & 2 & -2 & 0 & 4 & -8 \\ -2 & -1 & 3 & 2 & -3 & 1 & -1 & 1 & 0 & 8 & 4 \\ \hline -3 & 1 & 2 & 3 & -2 & -1 & 1 & -1 & 0 & 12 & -4 \\ 2 & 1 & -3 & -2 & 3 & -1 & +1 & -1 & 0 & -8 & -4 \\ 1 & -2 & 1 & -1 & -1 & 2 & -2 & 2 & 0 & -4 & 8 \\ \hline -1 & 2 & -1 & 1 & 1 & -2 & 2 & -2 & 0 & 4 & -8 \\ 1 & -2 & 1 & -1 & -1 & 2 & -2 & 2 & 0 & -4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -12 & 4 & 8 & 12 & -8 & -4 & 4 & -4 & 0 & 48 & -16 \\ 4 & -8 & 4 & -4 & -4 & 8 & -8 & 8 & 0 & -16 & 32 \end{array}\right] \tag{3.1.40}$$

$$(Z_4^{(1)} - Z_3^{(1)}(Z_1^{(1)})^{-1}Z_2^{(1)}) = \frac{1}{120}\left[\begin{array}{rrr|rrr|rrr|rr} 57 & 31 & 32 & 3 & -2 & -1 & 1 & -1 & 0 & 12 & -4 \\ 31 & 58 & 31 & -1 & -1 & -2 & 2 & 2 & 0 & -4 & 8 \\ 31 & 31 & 57 & 2 & 3 & -1 & 1 & -1 & 0 & -8 & -4 \\ \hline 3 & -1 & 2 & 57 & 32 & 31 & -1 & 1 & 0 & -12 & 4 \\ -2 & -1 & 3 & 31 & 57 & 31 & -1 & 1 & 0 & -8 & 4 \\ -1 & 2 & -1 & 31 & 31 & 58 & 2 & -2 & 0 & 4 & -8 \\ \hline 1 & -2 & 1 & -1 & -1 & 2 & 58 & 32 & 0 & -4 & 8 \\ -1 & 2 & -1 & 1 & 1 & -2 & 32 & 58 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 12 & -4 & -8 & -14 & 4 & 4 & -4 & 4 & 0 & 72 & 16 \\ -4 & 8 & -4 & 4 & 4 & 8 & 8 & 8 & 0 & 16 & 88 \end{array}\right] \text{ ohms} \tag{3.1.41}$$

All the matrix multiplications are complete. Let us assume the network is excited as follows:

$$I = \begin{array}{c@{}c} \begin{matrix} 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12\\ 13\\ 14\\ 15\\ 16\\ 17\\ 18\\ 19\\ 20\\ 21\\ 22 \end{matrix} & \begin{bmatrix} 1\\ 1\\ 0\\ -1\\ 0\\ -1\\ 0\\ 1\\ 0\\ 0\\ -1\\ 0\\ 0\\ 0\\ 0\\ 1\\ 0\\ 0\\ 0\\ -1\\ 0\\ 0 \end{bmatrix} \end{array} \text{ amps} \qquad 3.1.42\text{a}$$

$$e = \begin{bmatrix} -1\\ 1\\ 0\\ 0\\ 0\\ 1\\ 0\\ 0\\ 1\\ 0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 1\\ 0 \end{bmatrix} \text{ volts} \qquad 3.1.42\text{b}$$

Also the matrix $A^{o}_{(1)}$ is required which is given in equation 3.1.43.

$$A^{o}_{(1)} = \begin{array}{c|ccccccccccc|}
 & O_{1-4} & O_{2-4} & O_{3-4} & O_{5-8} & O_{6-8} & O_{7-8} & O_{9-12} & O_{10-12} & O_{11-12} & O_{19} & O_{21} \\
\hline
1 & 1 & & & & & & & & & & \\
2 & & 1 & & & & & & & & & \\
3 & & & 1 & & & & & & & & \\
4 & 1 & -1 & & & & & & & & & \\
5 & & 1 & -1 & & & & & & & & \\
6 & -1 & & 1 & & & & & & & & \\
7 & & & & 1 & & & & & & & \\
8 & & & & & 1 & & & & & & \\
9 & & & & & & 1 & & & & & \\
10 & & & & 1 & -1 & & & & & & \\
11 & & & & & 1 & -1 & & & & & \\
12 & & & & -1 & & 1 & & & & & \\
13 & & & & & & & 1 & & & & \\
14 & & & & & & & & 1 & & & \\
15 & & & & & & & & & 1 & & \\
16 & & & & & & & 1 & -1 & & & \\
17 & & & & & & & & 1 & -1 & & \\
18 & & & & & & & -1 & & 1 & & \\
19 & & & & & & & & & & 1 & 0 \\
20 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
22 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 1 \\
\hline
\end{array} \qquad 3.1.43$$

$$(I'_{(1)}) = A^{o}_{(1)}(I) = \left[\begin{array}{c} 2 \\ 2 \\ -2 \\ \hline -1 \\ 1 \\ 0 \\ \hline 1 \\ -1 \\ 0 \\ \hline -1 \\ 0 \end{array}\right] \qquad 3.1.44$$

To solve $k^{(1)}$, the equivalent effect of the voltage sources in the closed-path not taken into account by $(e'_{(1)})$ is represented by the open-path and considered closed-path voltages. From the Fig. 3.1.6, the equation for the equivalent effect is

$$\begin{bmatrix} e'_{(1)eq.} \\ E'_{(1)eq.} \end{bmatrix} = - \begin{bmatrix} Q^{(1)c}_{(2)t} \\ Q^{(1)o}_{(2)t} \end{bmatrix} (Y^{4}_{(2)})^{-1} \; Y^{3}_{(2)} \; (e'_{(2)}) \qquad 3.1.45$$

$$\begin{bmatrix} e'_{(1)eq.} \\ E'_{(1)eq.} \end{bmatrix} = - \begin{bmatrix} Q^{(1)c}_{(2)t} \\ Q^{(1)o}_{(2)t} \end{bmatrix} (Y^{4}_{(2)})^{-1} \; A^{o}_{(2)} \; Y\,(e) \qquad 3.1.46$$

$$\begin{bmatrix} e'_{(1)eq.} \\ E'_{(1)eq.} \end{bmatrix} = - \frac{1}{4} \left[\begin{array}{c} 2 \\ -3 \\ \hline -2 \\ 1 \\ 1 \\ 0 \\ 1 \\ 3 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array}\right] \qquad 3.1.47$$

$$(e'_{(1)}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

From the equation 3.1.20

$$(E'_{(1)}) = Z_3^{(1)}(Z_1^{(1)})^{-1}\Big((e'_{(1)}) - (e'_{(1)eq.})\Big) + (Z_4^{(1)} - Z_3^{(1)}(Z_1^{(1)})^{-1}Z_2^{(1)})(I'_{(1)}) + (E'_{(1)eq.}) \qquad 3.1.48$$

The equation 3.1.48 is a modification of Equation 3.1.20 since the effect of the voltage sources along the closed-paths not considered by $(e'_{(1)})$ is taken into account explicitly. The equation 3.1.48 is obvious from Figure 3.1.6.

$$\underset{\text{Total}}{(E'_{(1)})} = \frac{1}{120}\begin{bmatrix} -8 \\ -24 \\ 32 \\ 8 \\ -32 \\ 24 \\ -24 \\ 24 \\ 0 \\ 32 \\ 90 \end{bmatrix} + \frac{1}{120}\begin{bmatrix} 60 \\ -30 \\ -30 \\ 0 \\ -30 \\ -90 \\ 0 \\ -30 \\ 30 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{120}\begin{bmatrix} 60 \\ -30 \\ -30 \\ 0 \\ -30 \\ -90 \\ 0 \\ -30 \\ 30 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{120}\begin{bmatrix} 149 \\ 66 \\ 29 \\ 1 \\ -59 \\ -58 \\ 2 \\ -32 \\ 30 \\ 4 \\ 112 \end{bmatrix} \text{volts} \qquad 3.1.49$$

As $(E'_{(1)})$ is solved, all the other quantities are found directly by routine transformations given in Figure 3.1.6.

$$(V) = A^{o}_{(1)}\ (E'_{(1)}) + (e) \qquad 3.1.50a$$

And $$(I) = Y\ (V) \qquad 3.1.50b$$

$$(V) = \frac{1}{120} \begin{bmatrix} 29 \\ 186 \\ 29 \\ 83 \\ 37 \\ 0 \\ 1 \\ -59 \\ 62 \\ 60 \\ -1 \\ 61 \\ 2 \\ -32 \\ 30 \\ 34 \\ 58 \\ 28 \\ 4 \\ -56 \\ 232 \\ 148 \end{bmatrix} \quad \text{Volts} \qquad 3.1.51$$

This gives a complete solution to network problem $k^{(1)}$. All the branch current and voltage are found. The above results are also checked by solving $k^{(1)}$ by the classical closed path method. If $k^{(1)}$ is solved without tearing, inversion of 11 x 11 matrix is required. If we assume that the total multiplications involved in inversion of a matrix are of the order of the $(\text{rank})^3$ of the given matrix, then $11^3=1331$ multiplications are required for solving the network without tearing. Network $k^{(1)}$, when solved with tearing requires the inversion of three 3 x 3 and one 2 x 2 matrices. The total multiplications involved are 89. Thus the saving involved by tearing is quite evident. If the network is very large and has some subdivisions repetive then the saving in computation is really tremendous and the method of tearing speaks for itself.

Example 2.

Solution of an Electrical Network by Co-diakoptic Method.

We take an electrical network of Fig. 3.1.7 as an example. In Fig. 3.1.7, all the branches are oriented and values of voltage and current sources are given. The branch or primitive impedance matrix is taken as

$$Z = \begin{array}{c|cccccc} & 1 & 2 & \cdot & \cdot & 20 & 21 \\ \hline 1 & 1 & & & & & \\ 2 & & 1 & & & & \\ \cdot & & & \ddots & & & \\ \cdot & & & & \ddots & & \\ 20 & & & & & 1 & \\ 21 & & & & & & 1 \end{array} \quad \Omega \qquad 3.1.52$$

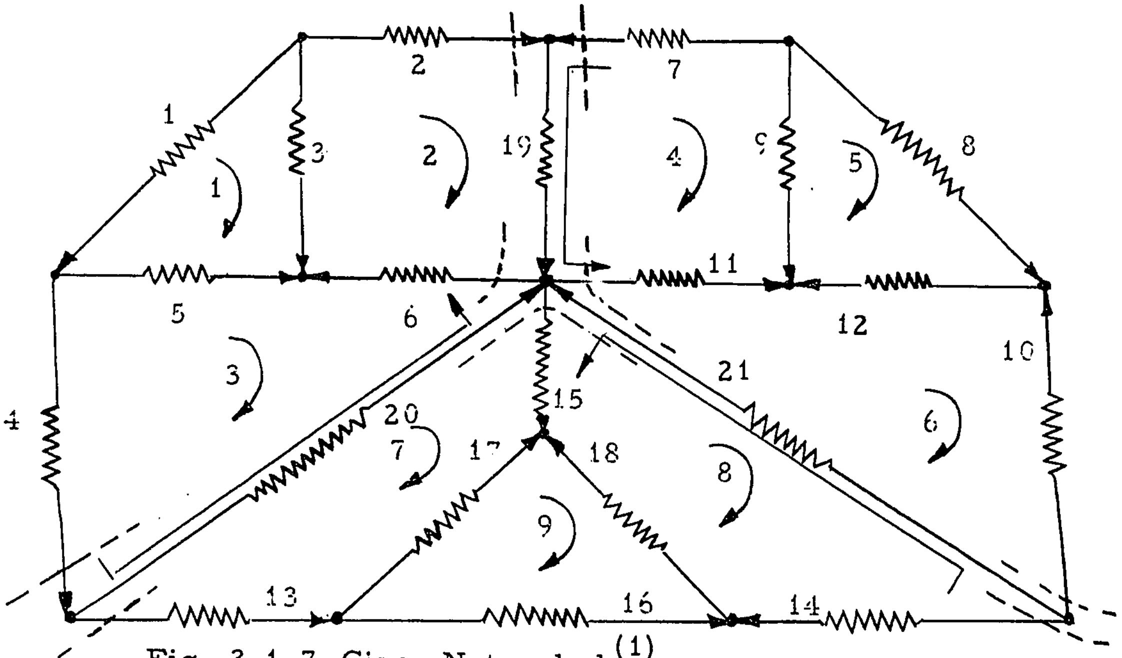


Fig. 3.1.7 Given Network $k^{(1)}$

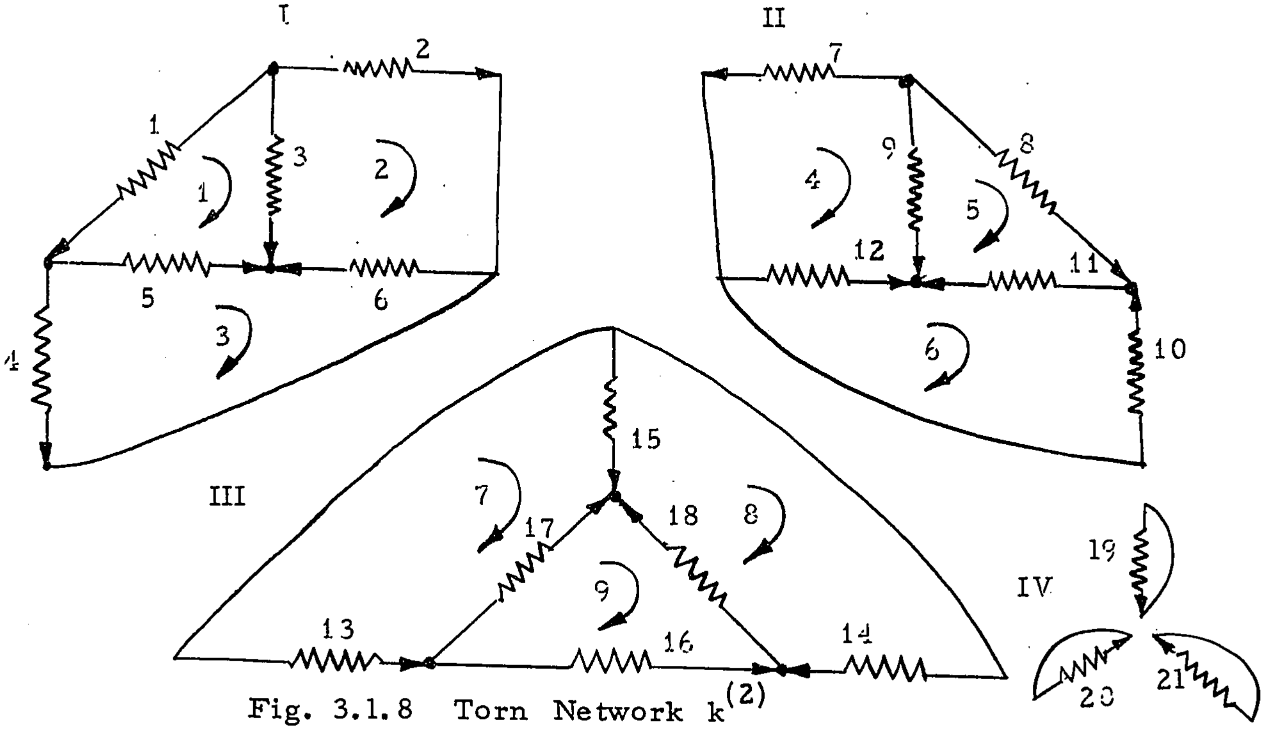


Fig. 3.1.8 Torn Network $k^{(2)}$

The given network, has 21 branches and 12 independent open-paths, so it has 9 independent closed-paths. To solve $k^{(1)}$ without tearing, an inverse of 9×9 matrix is required. In this example network $k^{(1)}$ will be solved by tearing method. The given network $k^{(1)}$ is torn into network $k^{(2)}$, Fig. 3.1.8, and has four linductively isolated subdivisions. For solving $k^{(1)}$, $k^{(2)}$ is solved first by finding $(Z_1^{(2)})^{-1}$. Fig. 3.1.3 is redrawn as Fig. 3.1.9 with all the quantities interpreted by electrical terms. To find $(Z_1^{(2)})^{-1}$, $Q^c_{(2)}$ is to be set up which is

$$Q^{c}_{(2)} = \begin{array}{c|cccccccccccc|}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{20} & C_{21} \\
\hline
1 & -1 & & & & & & & & & & & \\
2 & & 1 & & & & & & & & & & \\
3 & 1 & & -1 & & & & & & & & & \\
4 & & -1 & & & & & & & & & & \\
5 & -1 & & 1 & & & & & & & & & \\
6 & & 1 & -1 & & & & & & & & & \\
7 & & & & -1 & & & & & & & & \\
8 & & & & & 1 & & & & & & & \\
9 & & & & 1 & & -1 & & & & & & \\
10 & & & & & -1 & & & & & & & \\
11 & & & & -1 & & 1 & & & & & & \\
12 & & & & & 1 & -1 & & & & & & \\
13 & & & & & & & -1 & & & & & \\
14 & & & & & & & & 1 & & & & \\
15 & & & & & & & 1 & & -1 & & & \\
16 & & & & & & & & -1 & & & & \\
17 & & & & & & & -1 & & 1 & & & \\
18 & & & & & & & & 1 & -1 & & & \\
19 & & & & & & & & & & +1 & & \\
20 & & & & & & & & & & & 1 & \\
21 & & & & & & & & & & & & 1 \\
\hline
\end{array} \qquad 3.1.53$$

$$(Z_1^{(2)}) = Q^{c}_{(1)t} \quad Z \, Q^{c}_{(2)} \qquad 3.1.54$$

$$Z_1^{(2)} = \begin{array}{|ccccccccccccc|}
 & \mathrm{I} & & & \mathrm{II} & & & \mathrm{III} & & \mathrm{IV} & & & \\
\hline
3 & -1 & -1 & & & & & & & & & & \\
-1 & 3 & -1 & & & & & & & & & & \\
-1 & -1 & 3 & & & & & & & & & & \\
 & & & 3 & -1 & -1 & & & & & & & \\
 & & & -1 & 3 & -1 & & & & & & & \\
 & & & -1 & -1 & 3 & & & & & & & \\
 & & & & & & 3 & -1 & -1 & & & & \\
 & & & & & & -1 & 3 & -1 & & & & \\
 & & & & & & -1 & -1 & 3 & & & & \\
 & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & 1 \\
\hline
\end{array} \qquad 3.1.55$$

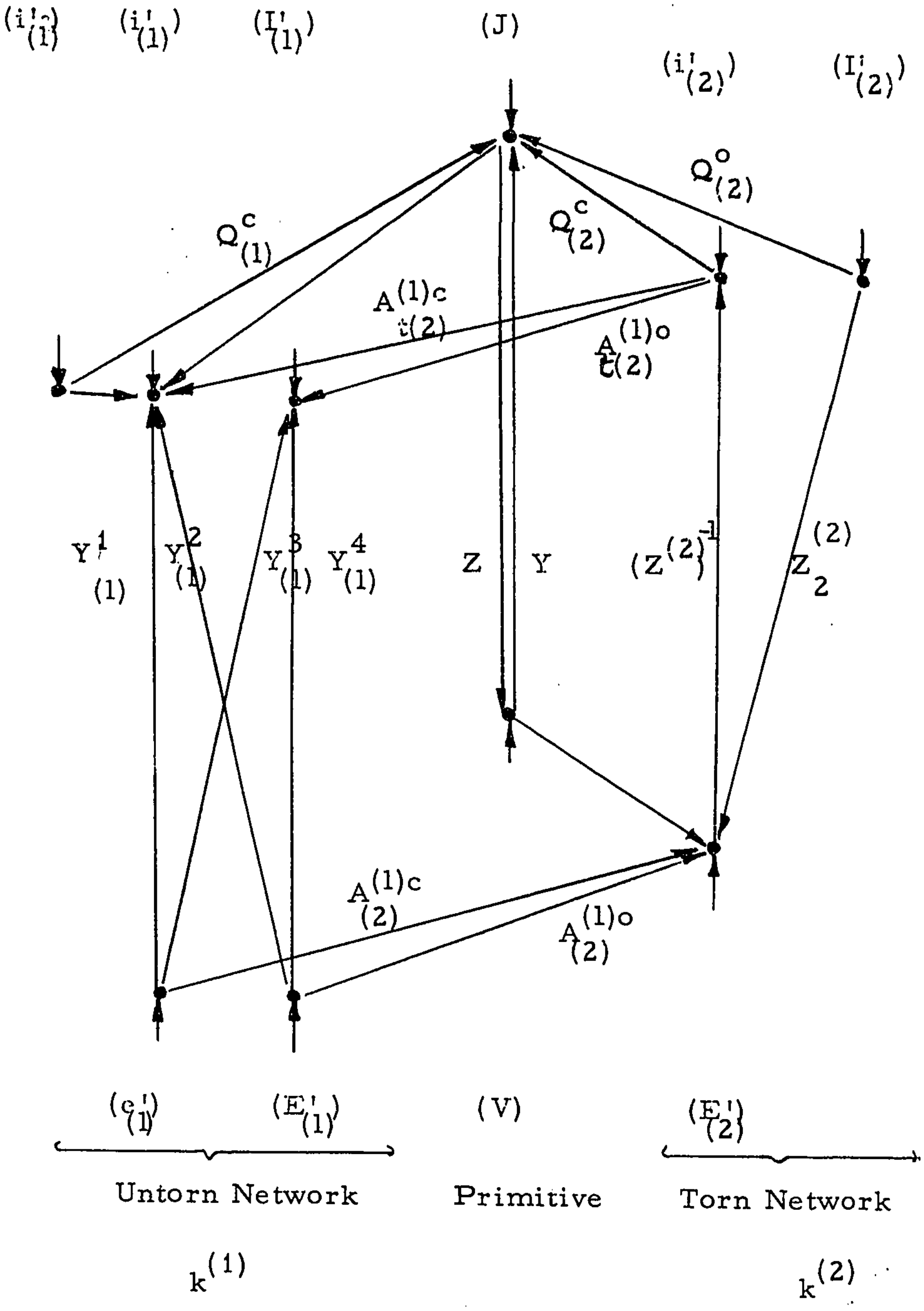


Fig. 3.1.8. Transformation Diagram for the Co-diakoptic Method.

To find $(Z_1^{(2)})^{-1}$, it is simply required to find the inverse of each subdivision separately, since each subdivision is inductively isolated from each other.

$$(Z_1^{(2)})^{-1} = \frac{1}{4} \quad \begin{array}{c|ccc|ccc|ccc|ccc|}
C_1 & 2 & 1 & 1 & & & & & & & & & \\
C_2 & 1 & 2 & 1 & & & & & & & & & \\
C_3 & 1 & 1 & 2 & & & & & & & & & \\
\hline
C_4 & & & & 2 & 1 & 1 & & & & & & \\
C_5 & & & & 1 & 2 & 1 & & & & & & \\
C_6 & & & & 1 & 1 & 2 & & & & & & \\
\hline
C_7 & & & & & & & 2 & 1 & 1 & & & \\
C_8 & & & & & & & 1 & 2 & 1 & & & \\
C_9 & & & & & & & 1 & 1 & 2 & & & \\
\hline
C_{19} & & & & & & & & & & 4 & & \\
C_{20} & & & & & & & & & & & 4 & \\
C_{21} & & & & & & & & & & & & 4 \\
\end{array} \qquad 3.1.56$$

With the inverse $(Z_1^{(2)})^{-1}$, the network $k^{(2)}$ is completely solved. To solve network $k^{(1)}$ from $k^{(2)}$, the matrix $A_{(2)}^{(1)}$ is set up

$$A_{(2)}^{(1)} = \begin{array}{c|ccccccccc|ccc|}
 & C_1^{(1)} & C_2^{(1)} & C_3^{(1)} & C_4^{(1)} & C_5^{(1)} & C_6^{(1)} & C_7^{(1)} & C_8^{(1)} & C_9^{(1)} & 0_1^{(1)} & 0_2^{(1)} & 0_3^{(1)} \\
\hline
C_1 & 1 & & & & & & & & & & & \\
C_2 & & 1 & & & & & & & & -1 & & \\
C_3 & & & 1 & & & & & & & & 1 & \\
C_4 & & & & 1 & & & & & & 1 & & \\
C_5 & & & & & 1 & & & & & & & \\
C_6 & & & & & & 1 & & & & & & -1 \\
C_7 & & & & & & & 1 & & & & -1 & \\
C_8 & & & & & & & & 1 & & & & 1 \\
C_9 & & & & & & & & & 1 & & & \\
C_{19} & & & & & & & & & & 1 & & \\
C_{20} & & & & & & & & & & & 1 & \\
C_{21} & & & & & & & & & & & & 1 \\
\hline
\end{array} \qquad 3.1.57$$

$$Y_{(1)} = A^{(1)}_{(2)t} \; (Z^{(1)}_{1})^{-1} \; A^{(1)}_{(2)} \qquad 3.1.58$$

And

$$A^{(1)}_{(2)} = \begin{bmatrix} A^{(1)c}_{(2)} & A^{(1)o}_{(2)} \end{bmatrix}$$

$$Y_{(1)} = \frac{1}{4} \left[\begin{array}{ccccccccc|ccc}
2 & 1 & 1 & & & & & & & 1 & -1 & 0 \\
1 & 2 & 1 & & & & & & & 1 & -2 & 0 \\
1 & 1 & 2 & & & & & & & 2 & -1 & 0 \\
 & & & 2 & 1 & 1 & & & & 0 & 2 & -1 \\
 & & & 1 & 2 & 1 & & & & 0 & 1 & -1 \\
 & & & 1 & 1 & 2 & & & & 0 & 1 & -2 \\
 & & & & & & 2 & 1 & 1 & -2 & 0 & 1 \\
 & & & & & & 1 & 2 & 1 & -1 & 0 & 2 \\
 & & & & & & 1 & 1 & 2 & -1 & 0 & 1 \\
\hline
1 & 1 & 2 & 0 & 0 & 0 & -2 & -1 & -1 & 8 & -1 & -1 \\
-1 & -2 & -1 & 2 & 1 & 1 & 0 & 0 & 0 & -1 & 8 & -1 \\
0 & 0 & 0 & -1 & -1 & -2 & 1 & 2 & 1 & -1 & -1 & 8
\end{array}\right] = \left[\begin{array}{cc} Y^{1}_{(1)} & Y^{2}_{(1)} \\ \hline Y^{3}_{(1)} & Y^{4}_{(1)} \end{array}\right] \qquad 3.1.59$$

$$(Y^{4}_{(1)})^{-1} = \frac{2}{27} \begin{bmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{bmatrix} \text{ ohms} \qquad 3.1.60$$

$$Y^{2}_{(1)} \; (Y^{4}_{(1)})^{-1} = \frac{1}{54} \begin{array}{ccc}
0^{(1)}_{1} & 0^{(1)}_{2} & 0^{(1)}_{3} \\
\hline
-6 & 6 & 0 \\
-13 & 5 & -1 \\
-5 & 13 & 1 \\
13 & 1 & -5 \\
6 & 0 & -6 \\
5 & -1 & -13 \\
-1 & -13 & 5 \\
1 & -5 & 13 \\
0 & -6 & 6
\end{array} \qquad 3.1.61$$

$$(Y^{1}_{(1)} - Y^{2}_{(1)}(Y^{4}_{(1)})^{-1} Y^{3}_{(1)} = \frac{1}{216} \begin{array}{c|ccccccccc|} & C^{(1)}_1 & C^{(1)}_2 & C^{(1)}_3 & C^{(1)}_4 & C^{(1)}_5 & C^{(1)}_6 & C^{(1)}_7 & C^{(1)}_8 & C^{(1)}_9 \\ \hline C^{(1)}_1 & 96 & 36 & 36 & 12 & 6 & 6 & 12 & 6 & 6 \\ C^{(1)}_2 & 36 & 77 & 31 & 25 & 12 & 11 & 11 & 7 & 6 \\ C^{(1)}_3 & 36 & 31 & 77 & 11 & 6 & 7 & 25 & 11 & 12 \\ C^{(1)}_4 & 12 & 25 & 11 & 77 & 36 & 31 & 7 & 11 & 6 \\ C^{(1)}_5 & 6 & 12 & 6 & 36 & 96 & 36 & 6 & 12 & 6 \\ C^{(1)}_6 & 0 & 11 & 7 & 31 & 36 & 77 & 11 & 25 & 12 \\ C^{(1)}_7 & 12 & 11 & 25 & 7 & 6 & 11 & 77 & 31 & 36 \\ C^{(1)}_8 & 6 & 7 & 11 & 12 & 12 & 25 & 31 & 77 & 36 \\ C^{(1)}_9 & 6 & 0 & 12 & 6 & 6 & 12 & 36 & 36 & 96 \\ \hline \end{array} \text{ mhos. } 3.1.62$$

All the matrix inversions are complete and once the current and voltage generators are defined numerically, the network $k^{(1)}$ is solved with the use of routine transformations.

Let us assume the excitation as follows

$$(I) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ Amp.} \quad 3.1.63a \qquad (e') = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \\ 1 \\ -4 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ Volts} \quad 3.1.63b$$

$$(I'_{(1)}) = \begin{array}{c} 0_1^{(1)} \\ 0_2^{(1)} \\ 0_3^{(1)} \end{array} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ . Amps.} \qquad 3.1.64$$

To find $e_1^{(1)}$, the matrix $Q^c_{(1)}$ is to be set up which is given as:

$$Q^c_{(1)} = \begin{array}{c|ccc|ccc|ccc|} & C_1^{(1)} & C_2^{(1)} & C_3^{(1)} & C_4^{(1)} & C_5^{(1)} & C_6^{(1)} & C_7^{(1)} & C_8^{(1)} & C_9^{(1)} \\ \hline 1 & -1 & & & & & & & & \\ 2 & & 1 & & & & & & & \\ 3 & 1 & -1 & & & & & & & \\ 4 & & & -1 & & & & & & \\ 5 & -1 & & 1 & & & & & & \\ 6 & & 1 & -1 & & & & & & \\ \hline 7 & & & & -1 & & & & & \\ 8 & & & & & 1 & & & & \\ 9 & & & & 1 & -1 & & & & \\ 10 & & & & & & -1 & & & \\ 11 & & & & -1 & & 1 & & & \\ 12 & & & & & 1 & -1 & & & \\ \hline 13 & & & & & & & -1 & & \\ 14 & & & & & & & & 1 & \\ 15 & & & & & & & 1 & -1 & \\ 16 & & & & & & & & & -1 \\ 17 & & & & & & & 1 & & 1 \\ 18 & & & & & & & & 1 & -1 \\ \hline 19 & & 1 & & -1 & & & & & \\ 20 & & & -1 & & & & 1 & & \\ 21 & & & & & & 1 & & -1 & \\ \hline \end{array} \qquad 3.1.65$$

$$(e^{(1)'}) = Q^c_{(1)t}\,(e) = \begin{bmatrix} -2 \\ +3 \\ -1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Volts} \qquad 3.1.65$$

The equivalent effect of current sources in those open-paths which are not considered is also to be taken into account.

$$\begin{bmatrix} i'^{(1)}_{eq.} \\ I'^{(1)}_{eq.} \end{bmatrix} = - \begin{bmatrix} A^{(1)c}_{(2)t} \\ A^{(1)o}_{(2)t} \end{bmatrix} (Z_1^{(2)})^{-1} \quad Z_2^{(2)} (I'_{(2)}) \qquad 3.1.66$$

$$\begin{bmatrix} i'^{(1)'}_{eq.} \\ I^{(1)'}_{eq.} \end{bmatrix} = - \frac{1}{4} \begin{bmatrix} -2 \\ 0 \\ -2 \\ 2 \\ 4 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ 2 \end{bmatrix} \quad \text{Amps.} \qquad 3.1.67$$

From the equation 3.1

$$(i'_{(1)}) = Y^2_{(1)} (Y^4_{(1)})^{-1} \left((I'_{(1)}) - (I'_{eq(1)})\right) + (Y^1_{(1)} - Y^2_{(1)} (Y^4_{(1)})^{-1} Y^3_{(1)}) (e'_{(1)})$$

$$+ (i'_{(1)eq.}) \qquad 3.1.68$$

Actually equation 3.1.68 is a modification of the equation 3.1.28. The effect of the current sources along the open-paths not considered is taken into account in the above equation.

$$(i'_{(1)}) = \frac{1}{216}\begin{bmatrix}0\\-18\\18\\18\\0\\-18\\-18\\18\\0\end{bmatrix} + \frac{1}{216}\begin{bmatrix}89\\203\\-23\\271\\126\\107\\3\\31\\12\end{bmatrix} - \frac{1}{216}\begin{bmatrix}-108\\0\\-108\\+108\\246\\108\\0\\0\\0\end{bmatrix} = \frac{1}{216}\begin{bmatrix}197\\185\\103\\181\\-90\\-19\\-15\\49\\12\end{bmatrix} \qquad 3.1.69$$

$$J = Q^{c}_{(1)}(i'_{(1)})+(I) = \frac{1}{216}\begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\end{array}\begin{bmatrix}-197\\185\\12\\-103\\-94\\82\\-181\\-90\\271\\-19\\-172\\-109\\15\\49\\-64\\-12\\27\\37\\4\\-118\\30\end{bmatrix} + \frac{1}{216}\begin{bmatrix}216\\216\\0\\216\\0\\216\\0\\216\\0\\0\\0\\216\\0\\0\\0\\0\\0\\0\\0\\216\\0\end{bmatrix} = \frac{1}{216}\begin{bmatrix}19\\401\\12\\113\\-94\\298\\-181\\126\\271\\-19\\-172\\107\\15\\49\\-64\\-12\\27\\37\\4\\108\\-30\end{bmatrix} \quad \text{Amps } 3.1.70$$

And

$$(V) = Z\,(J)$$

Thus all the branch currents and branch voltages are solved via tearing for the network $k^{(1)}$. For solving $k^{(1)}$ without tearing a matrix of 9×9 is to be inverted requiring about $9^3 = 729$ multiplications. By tearing we need the inverse of four 3×3 matrices requiring about 108 multiplications. The saving in computation is quite clear by this example. When the networks are very large and complicated, the method of tearing is a very useful method in (21, 24) solving the networks and the saving in computation is tremendous.

In many problems,(13) we can make use of both the diakoptic and co-diakoptic methods in solving the networks. For example we are given a very large network $k^{(1)}$. It is torn into subdivisions for solving $k^{(1)}$ by diakoptic method. The torn network $k^{(2)}$ may have the subdivision which are still very large to solve directly. In that case, we can solve each subdivision of $k^{(1)}$ by tearing method. It may happen that the subdivisions of $k^{(2)}$ may be simplified by co-diakoptic rather than by diakoptic method. Thus in solving $k^{(1)}$, both the diakoptic and co-diakoptic methods can be used to solve the network. Whether one should apply the diakoptic method or co-diakoptic method or both for solving a given network depends on the type of problem.

It should be noted that the tearing method is advantageous in general if the torn subdivisions are inductively isolated, from each other. If the subdivisions are not inductively solved, then the tearing method may not bring much saving.

d. Conclusions.

In this section, the application of p-network theory to the solution of large p-network by tearing was given. The method of tearing was illustrated by applying it to 1-networks i.e. electrical networks. The advantages of tearing method were clearly indicated.

The application of p-network theory to multi-variable systems is expected. But it is still a long way before the application of this theory can be materialized in solving the large p-networks. In this thesis no attempt whatsoever was made in applying p-network theory to any practical problems except to the 1-network . Though Kron[15, 16, 17, 18, 19; 25] has written a few papers about his polytope approach to the multi-dimensional problems such as calculus of finite differences, multi-dimensional curve fitting, etc. his works simply give the philosophy of the technique and lack all the mathematical rigor.

It is the firm belief of the authors that any further work along these lines will develope new techniques in solving multi-variable systems. Any further study in this field demands a good knowledge of algebraic and differential topology. It can be confidently stressed that any study in Kron's polytope theory will undoubtedly yield fruitful results.

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