THE DEVELOPMENT OF DISPLACEMENT BASED TOROIDAL ELASTICITY

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Abstract

Toroidal Elasticity (TE) is a three-dimensional theory used for the elastostatic analysis of thick-walled curved pipes, elbows, and toroidal shells. A displacement-based approach to TE is used in the present study, and analytical solutions are set up using the method of successive approximations. Three problems are solved analytically for the first time to illustrate the approach. The first problem concerns a 90° cantilevered elbow subjected to an out-of-plane couple at the free end. A real form general solution is developed up to the third order approximation, and results obtained are compared with boundary element results. The second problem concerns an axisymmetric sectorial toroidal ring subjected to loads on its radial surfaces. A complex form solution is developed with the aid of homogeneous eigenfunctions. Numerical results obtained are compared with finite element results. The third problem concerns a toroidal shell subjected to band loads of normal pressure. The applied loading is expanded in a Fourier series, and analytical solutions are developed for each term of the series. Numerical results are computed and again are compared with finite element results. The present work demonstrates the use of displacement based TE for the analytical prediction of displacement and stress in toroidal geometries, and the agreement between the displacement based TE solution and numerical results.

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Nomenclature

a, b	inside and outside cross-sectional radius of the toroidal shell
c	a reference length
$a_n, b_n, c_n, d_n, e_n, f_n$	constants of the nth order complementary solution
A_{ni}, B_{ni}, C_{ni}	constants of the nth order particular solution
$\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{k}$	unit vectors of polar coordinates
$\mathbf{e}_\xi, \mathbf{e}_\phi, \mathbf{e}_\theta$	unit vectors of toroidal coordinates
E	Young's modulus
f_x, f_y, f_z	components of the body force
F_x, F_y, F_z	force resultants
G	shear modulus
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors of cartesian coordinates
M_x, M_y, M_z	bending moments
n_x, n_y, n_z	normal unit vector of boundary surface
R	mean toroidal radius
T_o	twisting moment
u_ξ, u_ϕ, u_θ	displacement components in toroidal coordinates
u, v, w	non-dimensional displacement components
u_k, v_k, w_k	the kth order non-dimensional displacement components
U, V, W	the first part of the Navier functions
$\bar{U}, \bar{V}, \bar{W}$	the second part of the Navier functions
$\hat{U}, \hat{V}, \hat{W}$	the third part of the Navier functions
U_k, V_k, W_k	the first part of the Navier functions of the kth order
$\bar{U}_k, \bar{V}_k, \bar{W}_k$	the second part of the Navier functions of the kth order
$\hat{U}_k, \hat{V}_k, \hat{W}_k$	the third part of the Navier functions of the kth order
β_n	the nth eigenvalue of toroidal ring
ϵ	b/R
ϑ	dilatation
ϑ_k	$\psi_k + \frac{c}{\rho}\chi_k$, the kth order dilatation
ν	Poisson ratio
ξ, ϕ, θ	toroidal coordinates
$\sigma_x, \sigma_y, \sigma_z$	normal stress components
$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$	shear stress components
$\tau_{\xi\xi}, \tau_{\phi\phi}, \tau_{\theta\theta}$	non-dimensional normal stress components
$\tau_{\xi\phi}, \tau_{\xi\theta}, \tau_{\phi\theta}$	non-dimensional shear stress components
$\tau_{\xi\xi k}, \tau_{\phi\phi k}, \tau_{\theta\theta k}$	the kth order non-dimensional normal stress components
$\tau_{\xi\phi k}, \tau_{\xi\theta k}, \tau_{\phi\theta k}$	the kth order non-dimensional shear stress components
χ_k	$u_k \cos\phi - v_k \sin\phi + \frac{\partial w_k}{\partial\theta}$
ψ_k	$\frac{\partial u_k}{\partial\xi} + \frac{u_k}{\xi} + \frac{\partial v_k}{\xi\partial\phi}$

Chapter 1

INTRODUCTION

High pressure and large size are two of the characteristics of the modern pressure vessel industry. The present condition is the results of a century of development. It was shown in 1888 that the chemical reaction rate is much faster under a high pressure environment than under atmospheric pressure. The German chemical scientist, Haber, conducted extensive research on the pressurized reaction process and set up the first chemical plant, which operated at a pressure of about 20 MPa(2820 psi). This work served to stimulate the analysis of pressure vessels, and the development of shell theories. For the thin shell (eg. cylindrical, spherical, conical, elliptical and toroidal shell), the shell membrane and bending theories were developed, and for thick shells, the axisymmetric and three-dimensional theories of elasticity were developed. Such theories are essential for thr modern pressure vessel industry where large vessels and pressures up to 200 MPa (28.2 $Kpsi$) are common.

Toroidal shells are frequently used in the pressure vessel industry and other branches of engineering. They form parts of high pressure containers which are

widely used in the modern chemical and nuclear industries. Specific applications arise in chemical reactors (Figs. 1.1, 1.2), nuclear heat exchangers (Fig. 1.3), steam generators (Fig. 1.4), rotational gas compressors, and high-pressure enclosed-rotor pumps. These vessels all have a part of a toroidal ring, which is used as a part of the containers. Elbows and U-bends (part of a toroidal shell) are common components in these plants. Estimates of the total number of elbows in a nuclear plant are 1300 for one reactor unit and 2500 for two reactor units. Many of these elbows are small, less than 50.8 mm (2 in) in diameter. Some larger elbows may have a 762 mm (30 in) cross-sectional diameter and a thickness of about 25.4 mm (1 in). The complete toroidal shell is used as a basic structural element in Tokamak-type fusion reactor (Fig. 1.5), in the rotating space station, and in the liquid storage containers of a space vehicle.

The stress-based toroidal elasticity (TE) is a method for calculating the stress field in a thick-walled elbow. Extensive work on the stress-based TE was begun by Lang [1] in 1984. The load cases covered were in-plane loads, out-of-plane loads and seismic forces. The displacement field, if given, was found using a cumbersome integration process. The toroidal ring was not included in his study. The displacement-based TE was developed by Redekop [2] in 1992. Only the solution for the in-plane load case was presented. The method of successive approximations (MSA) was used to decompose the governing equations into different orders in these two TE solutions. In both of the TE solutions, generally the precision of the solution is proportional to the number of orders taken in the MSA.

In the absence of convenient analytical solutions for the thick-walled pressure

vessel, the finite difference method (FDM), the finite element method (FEM), or the boundary element method (BEM) may be used. The FDM is less accurate than the other two methods for the same mesh design. There are many FEM codes available. For a three-dimensional pressure vessel problem, the preparation of the large input file is a major task. The cost of solution is very high. For the BEM, it is necessary to make a mesh only on the surface of the domain. The input file is much smaller, but there are fewer codes of this kind available.

The objective of this thesis is to demonstrate a displacement-based successive approximation method which can be used to analyse toroidal structures. The thick-walled toroidal elbow subjected to out-of-plane twist bending is considered. Also considered are an axisymmetric sectorial toroidal ring subjected to loading on its radial surfaces, and a complete toroidal shell subjected to band pressure loads. The solutions are compared with the numerical results obtained by the FEM and BEM. The study is divided into four parts.

In chapter three, the governing equations are set up in the toroidal coordinate system. The method of successive approximations is presented. Then the governing equations are decomposed into different orders, based on a small parameter ϵ which is equal to the ratio of outside shell radius to the mean toroidal radius. The formulas for calculating different order stress components are derived. All the equations needed for further study are set up.

In chapter four, the analysis of a thick-walled 90° elbow subjected to an out-of-plane loads is conducted. One end of the thick-walled elbow is fixed. The

out-of-plane loads are applied on the free end. A real form displacement-based complementary solution for the general out-of-plane load case is formulated. A solution for a 90° thick-walled elbow subjected to out-of-plane twist couple is developed up to the third order. Results obtained are compared with a BEM solution.

In chapter five, an axisymmetric sectorial toroidal ring subjected to self-balanced loads on the radial (edge) surfaces is treated. The complementary solution for general axisymmetric load cases is formulated. This solution is in complex number form, and complex eigenvalues are calculated. The governing equations required for axisymmetric analysis are derived. Boundary conditions on radial surfaces are satisfied at collocation points by the linear least squares (LLSQ) method. The example of a 45° sectorial toroidal ring subjected to pure bending moments on two edge surfaces is considered. Results obtained are compared with a FEM solution.

The chapter six of the study is concerned with a thick-walled complete toroidal shell subjected to band loads of normal pressure. The applied band loads are symmetric about XOZ and YOZ planes. The loads are expressed in a Fourier series in the longitudinal direction. The load is uniform in the circumferential direction for each term of the series. This problem is treated as an in-plane deformation problem. An analytical solution is developed for each term of the load series. Numerical results are compared with an FEM solution.

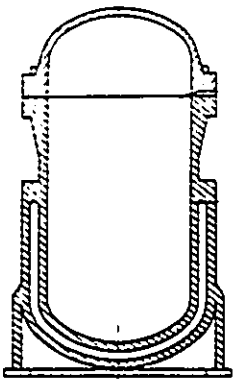

Figure 1.1: Typical jacketed chemical reactor

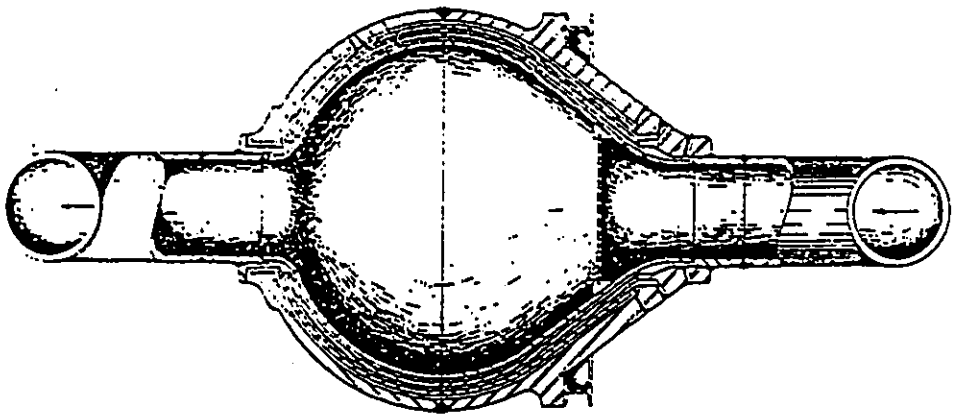

Figure 1.2: Typical single-phase chemical reactor
(Adapted from reference [75]).

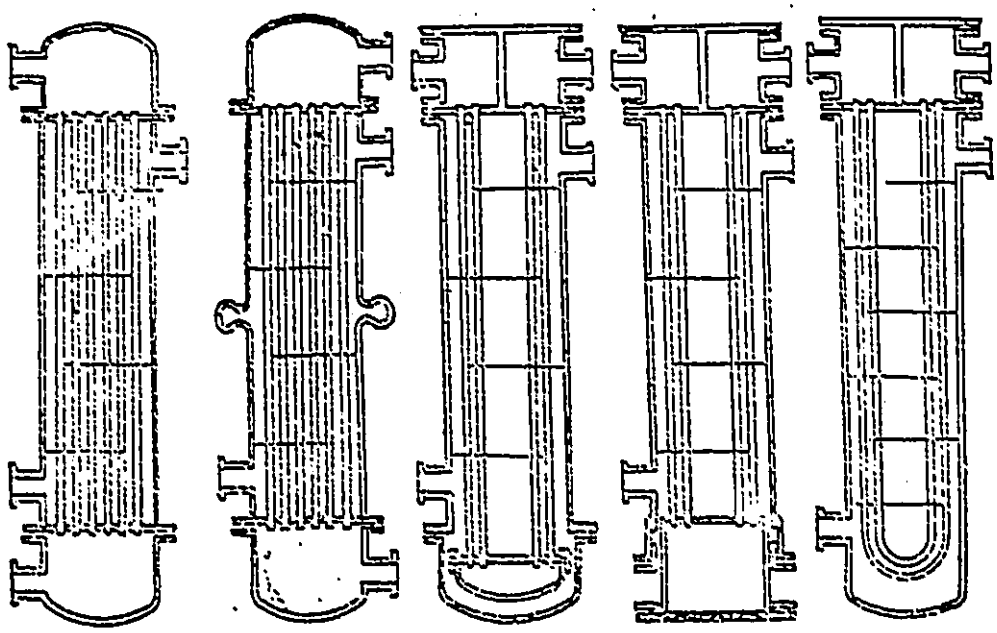

Figure 1.3: Typical heat exchanger

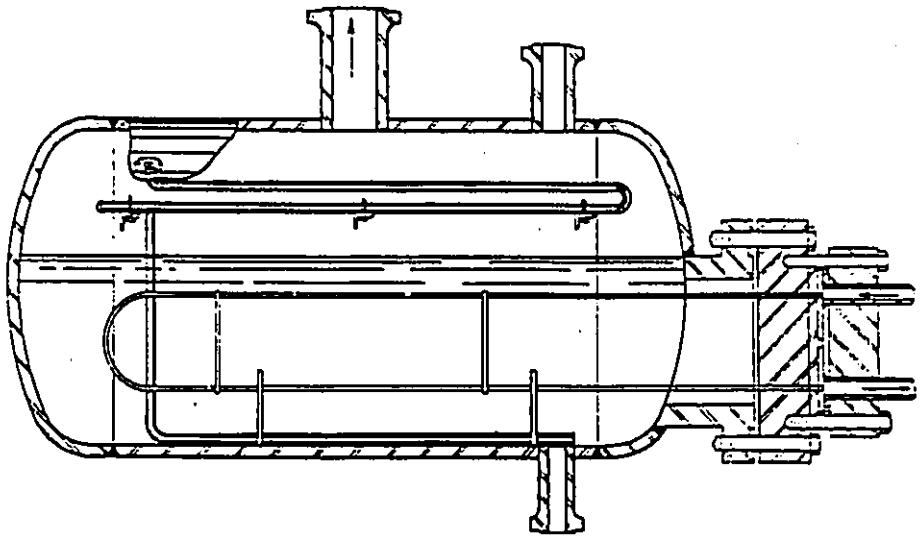

Figure 1.4: Typical steam generator
(Adapted from reference [75]).

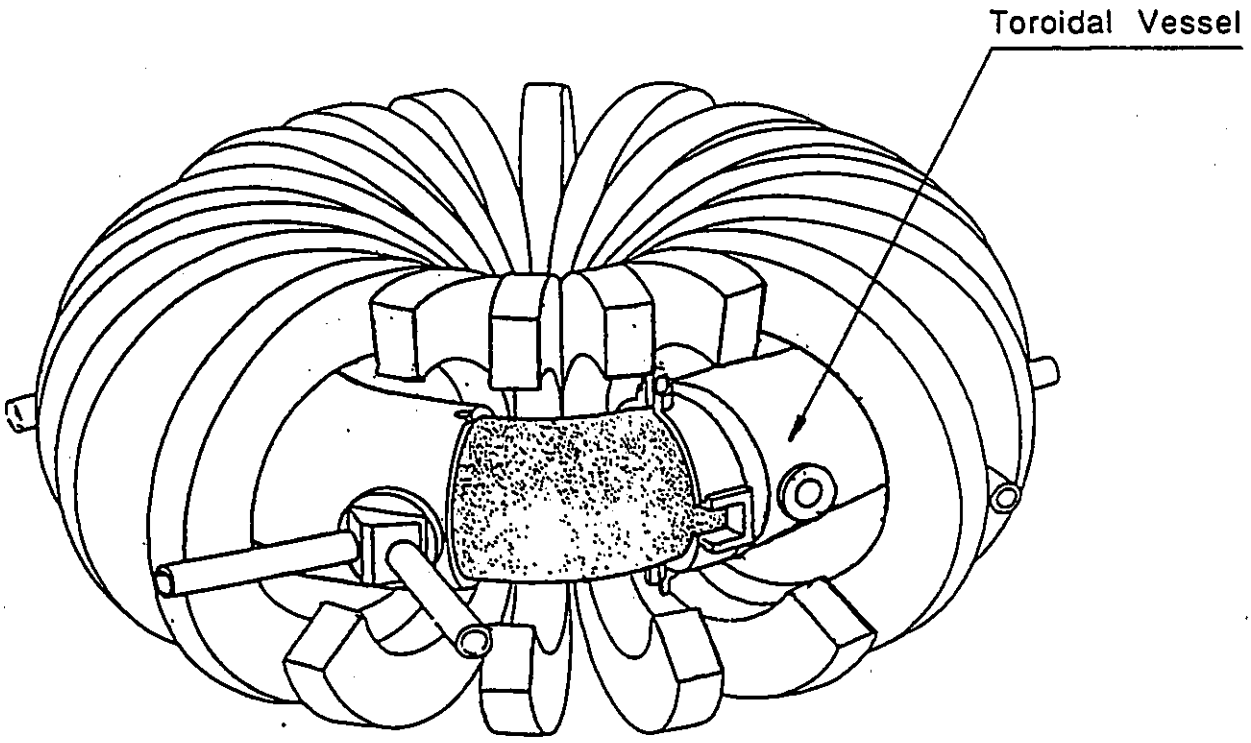

Figure 1.5: Tokamak-type fusion reactor
(Adapted from reference [76]).

Chapter 2

LITERATURE SURVEY

In 1911, Von Karman [3] gave a theoretical explanation for the phenomenon of a curved tube having more flexibility in bending than a straight one. His work and its extension provided the foundations for much of the subsequent tube analysis. Extensive literature related to the elbows, bends, and curved pipes has been published in recent decades. The studies have provided different methods to determine the stress and/or displacement fields in thin-walled and thick-walled elbows under different load cases. For a compound-shell vessel subjected to internal pressure, the deformation at the junctions is different. Thus, the problem of a shell subjected to edge loads is encountered. Shell bending theory may be used to solve this edge load problems. The membrane theory may be used to solve the problem of thin-walled complete toroidal shell under internal and external pressure load. Some papers published are concerned with the problems of thick-walled cylindrical and spherical shell under edge loads. The stress-based TE theory has been investigated extensively by Lang since 1984. The thick-walled problem in toroidal geometry was investigated

by Redekop in 1992. Because of the difficulty of solving the governing equations, the literature concerning the toroidal ring subjected to edge loads and thick-walled complete toroidal shell subjected to local pressure load is less abundant than that of the elbow, cylindrical and spherical shell under similar loads. The literature on these topics are reviewed separately below.

2.1 Theory of Toroidal Elasticity

The developing theory of toroidal elasticity is employed to determine the complete stress and displacement fields in toroidal rings, elbows, and vessels which are used in chemical, nuclear and power supply plant. Göhner [4] apparently was first to investigate the technical problem of a curved solid circular ring sector subjected to pure twist and bending moments. Thus, he was the first to develop the TE theory. He chose a special co-ordinate system (Fig. 2.1) for his study and gave two excellent solutions. The first, for the pure twist of a solid circular ring sector, has only two shear stress components, one equilibrium equation, and two compatibility equations. The second problem, for the pure bending by end couples of a solid circular ring sector, has four stress components, two equilibrium equations and four compatibility equations. He didn't encompass a complete theory, which would include six stress components, six compatibility equations and three equilibrium equations. His theory applied only to axisymmetric cases. His co-ordinate system is good for the boundary conditions of a solid circular cross-section, but is not adaptable to the boundary conditions of an annular circular cross-sections.

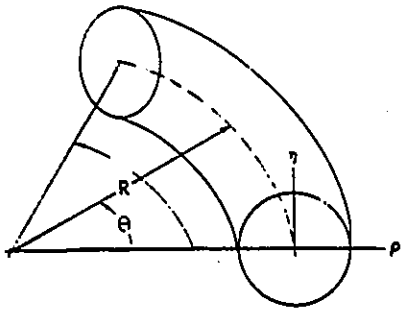

Figure 2.1: Göhner's co-ordinate system.

Ancker and Goodier [5] studied the problem of helical springs subjected to tension and torsion loading by the "thin-slice" method. They assumed that the springs had the same cross-section and same resultant force and moments on each cross section. Thus a typical slice could be taken from the spring as a free element. The displacements were expressed as a doubly infinite power series in the small parameters a/R and p, where a is the radius of the cross section, R is the mean coil radius, and p is the tangent of the pitch angle γ. These series were substituted into the governing equations. The stresses were obtained from the displacements. The constants of the series were determined from the boundary conditions.

A constant-thickness elbow can be represented easily by the non-classical toroidal coordinate system r, ϕ, θ, where r and ϕ are polar coordinates in the plane of the elbow's cross-section and θ defines the position of the elbow's cross-section (Fig. 2.2). Kornecki [6] and McGill [7] developed the theory of toroidal elasticity

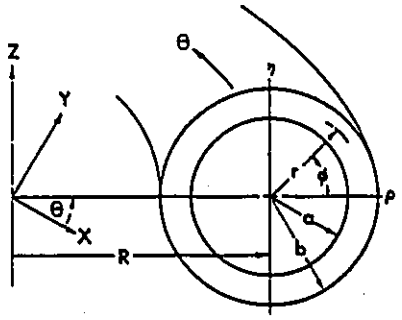

Figure 2.2: Non-classical toroidal coordinate system.

by extending Göhner's work. Kornecki used the method of successive approximation to solve the governing equations. The finite difference method was used by McGill. Both of the studies were based on the assumption of axisymmetry. Only the axisymmetric problem could be solved.

A major contribution to the theory of toroidal elasticity was made by Lang [1]. He summarized prior research work and developed the theory of toroidal elasticity in the non-classical toroidal coordinate system in which the boundary conditions of a hollow elbow could be represented easily. He recast the three equilibrium equations and the six compatibility equations into this suitable curvilinear coordinate system. Solutions were found using the method of successive approximations. This theory can be used to solve either axisymmetric problems (for internal pressure and pure bending) or non-axisymmetric problems (for in-plane and out-of-plane bending). In Lang's work, the stress approach method is used to perform the successive

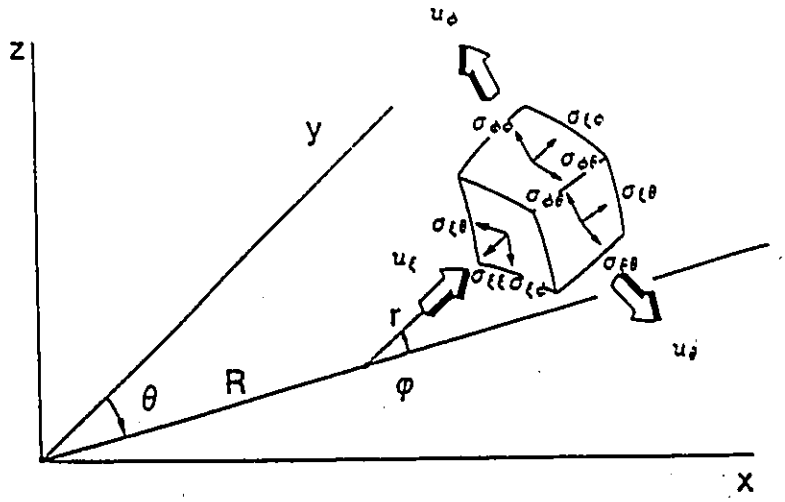

Figure 2.3: Displacement and stress components.

approximation analysis, thus, he didn't give deformation fields or the ovalization of the shell in most of his studies.

The work of Lang has been extended by Redekop [2]. He chose the displacement components u_ξ, u_ϕ, u_θ as the basic variables (Fig. 2.3) and developed the governing Navier equations in this coordinate system. Solutions were found by the method of successive approximations. All these equations were developed up to and including the third order. This displacement based toroidal elasticity has the advantage of yielding immediately the displacements as well as the stresses. Also, the zeroth order displacement functions required for starting the approach of successive approximations are easily set up based on the general mechanical knowledge. The displacement components are important information for considering special restrictions, and for fracture analysis.

2.2 Elbows and Pipe Bends Solutions

Curved tubes in the form of elbows, U-bends, etc., play an important role in piping networks, mainly in providing necessary structural flexibility. The models for curved tubes over the years have been developed by taking into account the information required and the computing resources available. Separate models have been developed for thin-wall and thick-wall tubes in the linear and non-linear ranges. Mathematically the models may be divided into one, two and three variables. Axelrad [8] and Redekop [9] have presented reviews about curved tubes.

2.2.1 Thin-Wall Models

The shell membrane theory may be used to solve the problem of a complete toroidal shell subjected to internal and external pressure load (Fig. 2.4). This solution is generally considered to be invalid for a thick-walled toroidal shell and part of a toroidal shell, such as a toroidal ring (Fig. 2.5), elbows, U-bends and curved pipes (Fig. 2.6), etc.. For solving these partial toroidal shell problems, different models are assumed.

The first model was set up by Von Karman [3]. He considered a circular cross-section curved tube under an in-plane bending load. His model was based on the bending theory. Later researchers made the same basic assumptions to simplify the original shell theory, and thus reduce the computational requirements.

The out-of-plane bending of a curved tube was considered by Vigness [10] in

1943. Pardue and Vigness [11] investigated the flexibility and stress intensification factor for U-bends and 90° elbows under in-plane and out-of-plane bending using a theoretical analysis, and investigated experimentally the effect of flanges on the stress intensification factors as well as the flexibility factors for 45°, 90° and 180° bends under in-plane and out-of-plane loads.

Recent work has dealt with non-classical geometries. Sutcliffe [12] considered shells having an initially elliptical cross-section. Boyle and Spence [13] studied an oval pressurized pipe bends under external bending. Spence and Findlay [14] treated the case of variation of thickness of the tube wall. Arav [15] investigated the effect of circumferential corrugations.

Eric [16] considered the linear problem of pure bending of thin-walled curved pipes. The solution of a circular cylindrical shell under a uniformly pressure was employed as the first approximation to the solution of this problem. He found a correcting part as the second approximation. No numerical results were presented.

Bushnell [17] investigated an initially uniformly curved pipe. The pipe was treated as part of a toroidal shell. Bending in the plane of the curvature of the pipe centerline was applied by means of an appropriate temperature distribution over the pipe cross section. The ovalization of the cross-section introduced an axial strain expression. This expression was based on the assumption that the centerline of the elbow is inextensible. Comparison was made with experimental data. The agreement was satisfactory.

Two-variable models are required to account for longitudinal variation of stresses in addition to the circumferential. Such models can be used to study the effect of end restraints in sectorial toroidal shells, such as elbow and U-bends with terminal flanges or tangent pipes. The series solution is an effective method for a tube with circular cross-section and constant wall thickness. A complete set of such series solutions was presented by Whatham [18-21]. The governing equations were presented in the first of these papers, while the end effect was covered in the second paper. The results and discussion of a general computer program based on this theory were given in the last two papers.

With the increase of computer capacity, numerical solutions are often more convenient than series solutions. For this reason, the FEM and FDM have obtained wide application. The linear, two-variable FEM models for curved tubes are of two main types. One is the "elbow" element, which has a shell-type variation in the circumferential direction and a beam-type variation in the longitudinal direction. The other is the "shell" element. A number of "elbow" and "shell" elements have been developed in the recent years.

Experimentation is the best way to check the theoretical and numerical solutions. Smith and Ford [22] investigated experimentally the stress distribution in a thin-walled pipe bend subjected to in-plane and out-of-plane bending loads. The experimental results were compared with previous published analytical data. The comparison showed that there was good agreement between the experimental results and previous published results for in-plane bending case, but not for out-of-plane bending case.

2.2.2 Thick-Wall Models

These models are required when the assumption that the stress component in the radial (r) direction is negligible is no longer valid. Thus the number of variables in the problem is increased to two for an axisymmetric tube, and to three for a non-symmetric tube. Because of the difficulty of the elasticity equations in toroidal coordinates, no models had been developed even by the 1960's [23], and designers relied on extensions of the thin shell models [24].

A few of the early investigators attempted to find solutions in the classical toroidal coordinate system ξ, η, θ. Solov'ev [25] considered axisymmetric problems while Podil'chuck and Kirilyuk [26] investigated improper agreement fully three-dimensional problems. The problem with this coordinate system is that the constant wall thickness of the tube cannot be represented easily.

A constant-thickness tube can be represented easily in the non-classical toroidal system r, ϕ, θ. Early work using this system was conducted by Kornecki [6] and Zak [27], who used the method of successive approximations to solve the governing equations. In McGill's works [7, 28, 29], the finite difference method was used. All these studies were based on the assumption of axisymmetry. Thus their models contained two variables. An example of a thick-walled torus under an internal pressure load was given by McGill [28]. There was a good agreement between McGill's solution and Kornecki's solution.

Lang [30-43] conducted extensive theoretical research on thick-walled elbows

under different loads. For axisymmtric problems, the load cases included internal and external pressure. For in-plane bending problems, the load cases included end couples, end shear forces, and end normal forces. For out-of-plane bending problems, the load cases included out-of-plane shear forces, twist moments, and seismic forces.

The problem of a torus subjected to internal and external pressure was solved by Lang [30,31]. The zeroth, first and second order fields of stress, displacement and ovalization of series were developed. He gave one set of numerical results for a thick-walled elbow under internal pressure load. In reference [32] he discussed a toroidal tube or pipe bend under an end in-plane bending moment load. The analysis was limited to the first order of stress and displacement fields. The solution was extended to the second order in reference [33]. No numerical results were presented. The thick-walled 90° elbow subjected to twist (out-of-plane couple) bending [39], out-of-plane bending by an end shear force [38], in-plane end shear force [34] and an end normal force [35] were subsequently considered. For these four load cases, the zeroth and first order stress fields were developed, and no numerical results were produced. The solution of the twist bending problem was extended to the third order.

In Lang's work, the stress approach method was used. Deformation fields and the ovalization of the cross-section of the elbow were only given for a few cases.

Redekop [2] investigated elbows under in-plane loading by the displacement based toroidal elasticity. Numerical solutions for a thick-walled torus subjected to internal pressure load were given in this reference. The stress and displacement fields were given up to and including the thrid order. The numerical solution obtained for

this case was in excellent agreement with finite element results.

2.2.3 The Numerical Methods

For the full three-dimensional problem of thick-walled curved tubes, numerical solutions are easier to formulate than analytic ones. Because of the thickness, the BEM method is more suitable than the FEM method. Discretization is required only on the surface of the tube in BEM. The dimension of the problem is reduced to two. Results for thick-walled curved tubes were obtained, using the BEM method by Colle, Redekop and Tan [44, 45, 46]. Load cases solved were internal pressure loading, in-plane, out-of-plane and twist bending. The results were compared with results obtained using Lang's stress based TE. For the bending problems, the check was for values corresponding to a cross-section far from the end. In general, good agreement was obtained by the author's results with numerical results.

2.3 Edge Loads Solutions

A pressure vessel is usually composed of several different shell segments. A typical example is a long cylinder with two heads. The solution for such a vessel subjected to internal pressure load is composed of two parts. The first part corresponds directly to the applied internal pressure. The second part corresponds to a bending moment and shear force arising at the junction of segments, which are required to satisfy the displacement continuity conditions at the junction. These additional effects originate at the edge surfaces of the individual shell segments. Their contri-

bution to the total stress field vanishes at positions far from the junction surfaces. When solving an edge load problem, a set of homogeneous equations are obtained by forcing the stress components to satisfy the stress free boundary conditions on the shell surfaces. An eigenvalue problem arises from these equations. The roots of the characteristic equation are eigenvalues for these shell segments. There is one free constant for each eigenvalue, which is determined by satisfying the edge boundary conditions.

2.3.1 The Cylindrical Problem

The thick-walled cylinder is the most common component in the field of pressure vessel engineering. It is usually used as the body of chemical reactors, the shell of the heat exchanger, or the chamber or the cylinder of the compressor. Lamé developed a method to calculate the stress fields of a thick-walled cylinder under a uniformly distributed pressure load. The behavior of a thick-walled cylinder under different kinds of edge loads has been a subject of study in recent decades.

A finite length, hollow cylinder subjected to an axisymmetric edge load was investigated by Bazarenko and Vorovich [47] in 1965. The case of a relatively thin cylinder was studied. The approximate eigenvalues of this cylinder were obtained by the perturbation method. The accuracies of this solution was examined using the three-dimensional solution.

Grinchenko [48] in 1967 investigated a short thick-walled cylinder subjected to a uniformly distributed axisymmetric load on the surface of the cylinder. He

used Lamé's solution of a thick cylinder under internal and external pressure loads as the first part of displacement field, then added two additional parts which were employed to adjust the stress components.

One of the major papers on thick-walled cylinders is that by Balabanov [49] (1968). He used the Papkovich-Neuber solution to study the problem of a cylinder subjected to edge loads. He calculated a set of complex roots of the well-known characteristic equation which was set up from the homogeneous boundary conditions on the surfaces of cylinder. He then gave the stress and the displacement fields which satisfied the governing equations and the boundary conditions on the cylindrical surface in terms of free constants. The free constants were determined by forcing the stress components to satisfy the given applied loads on the edge surface of the cylinder.

Widera and Wu [50] discussed the nature of an approximate solution for the hollow cylinder whose fixed ends are given a uniform axial displacement and whose cylindrical surfaces are stress free from traction. The solution of this problem is given by a superposition of the following two problems for a finite length cylinder; the first includes an axial load and allows a radial displacement; the second removes the radial displacement of problem one.

An elastostatic problem of a semi-infinite cylinder with the long sides free from stress subjected to self-equilibrated loads on the end was investigated by Steven [51], and Klemm and Little [52]. The Papkovich-Neuber solution was employed. The solution of the biharmonic equations for a hollow circular cylinder was presented in

terms of Bessel functions. Because the surfaces of the cylinder were free from stress, an eigenvalue problem arise. A method of establishing these eigenvalues and tables of the eigenvalues for various types of hollow cylinders were presented. The following two special cases were studied: (i) the ratio of the inner and outer radii tends to unity, that is a "thin shell", (ii) the ratio tends to infinity, which can either be regarded as the inner radius tending to zero for a fixed outer radius or as the outer radius tending to infinity for a fixed inner radius. The effect of the edge loading on a semi-infinite length cylinder was calculated. These numerical results were compared with thin shell solutions.

2.3.2 The Spherical Problem

The asymptotic behavior of the solution for a thin spherical shell was discussed by Vilenskaia and Vorovich [53]. Sternberg and Rosenthal [54] studied the solid sphere under diametrically opposite concentrated loads, Eubanks [55] investigated the hemispherical pit in a semi-infinite solid under uniform loading at infinity, Goldberg, Salerno and Sadowsky [56] considered the rotating spherical shell. An especially interesting solution of the spherical shell was obtained using a stress function approach method by Levine and Klosner [57]. They combined Lure's [66] homogeneous functions, which yielded stress-free conditions on the spherical surfaces, and the full-sphere functions used by Sternberg, Eubanks and Goldberg [54, 55, 56]. This combination was used to form the stress field of a spherical segment shell under an ordinary axisymmetric surface traction. The edge boundary conditions were satisfied using the boundary-point-least-squares technique. Numerical solutions to

the following problems were presented in their paper:

1. A spherical segment with a stress-free edge, subjected to a centrifugal force field.
2. A spherical segment (having openings of 30° and 60° degree) under an external pressure varying as $\cos^{2n}\theta$ supported on a rigid surface with no shear resistance.
3. A hemisphere having zero traction on its spherical surface, subjected to constant and parabolically varying shear stresses on the edge surface.

2.3.3 The Numerical Methods

A number of numerical methods have been applied to predict the stress and displacement fields in axisymmetric elastostatics. These can be classified as domain methods (FDM, FEM) or boundary methods (BEM, etc.). The boundary methods have received more attention than the domain method in recent years because of their greater computational efficiency. Redekop and Thompson [58] employed another kind of boundary method (boundary-point-least-square method) to study the axisymmetric problem. The stress and displacement fields were expressed in terms of Legendre functions. All boundary conditions were satisfied at collocation points. Two sets of numerical results of an annular disk under radial tension and a circular cylinder with a spherical cavity under axial tension were presented. The results were compared with previously published data. The agreement was satisfactory.

2.4 Local Loads Solutions

To solve the problem of a shell subjected to local loads on the shell surfaces (pad loads or band loads), one usually expresses the applied load in a Fourier series, then uses a suitable theory to formulate the solution for each term of the series.

A theoretical study of the problem of a long, thick-walled cylindrical tube subjected to two equal and diametrically opposite concentrated normal loads was carried out by Yao [59] in 1969. The three-dimensional theory of elasticity was used to set up the solution. The local loads were expressed in a Fourier series in the circumferential direction and a Fourier integral in the axial direction. The stress components were constructed using the Papkovich and Neuber stress functions. Numerical results were compared with experimental data obtained by the photoelasticity technique. This solution could be used solely for the problem with local distributed pressure load.

In Sundara and Sebastian's work [60], a finite hollow cylinder subjected to a band of pressure load on its outer rim was analyzed. Three-dimensional elasticity and classical shell theories were used in their study. For the elasticity solution, Love's stress function was formulated in the form of Fourier-Bessel series. Numerical results were obtained for shells with the ratio of inner to outer radii equal to 0.80, 0.85, 0.90, and 0.93 and for the ratio of outer shell diameter to the length of the shell equal to 0.5, 1.0, and 2.0. The ratio of the band width to the length of the shell was 0.2. The shell theory solution was compared with the elasticity solution. The maximum

deviation of the shell theory solution from elasticity solution was about 20% for a shell with $R/l = 0.5$, and 30% for the one with $R/l = 1.0$.

Kasano and Matsumoto [61] extended Yao's solution [59]. They solved two problems. The first concerned an infinite solid cylinder subjected to two diametrically opposite loads partially distributed on the outside surface. The other concerned an infinite hollow cylinder subjected to distributed load on the inside surface. Superposition of these two solutions gave the solution for an infinite hollow cylinder under the same load as the first problem, but the stress-free on the inside surface. With this treatment, a fast convergence was obtained. Similar work had been done previously by Okumura and Miyako [62]. They studied both long and short hollow cylinders subjected to a partial pressure on the side surface. This solution was based on the extended Neuber solution.

For the cases of a toroidal shell that is stiffened by several solid rings, and a complete toroidal shell that is fixed on a static base by several saddle supports ($120° - 180°$) or on a movable base by circular supports, the toroidal shell is subjected to local or band pressure loads. To date no papers have been published about the practically important problem of a thick-walled toroidal shell under local pressure loads.

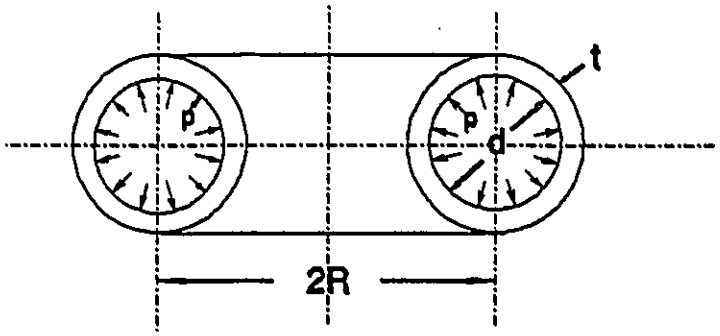

Figure 2.4: Complete toroidal shell subjected to internal pressure

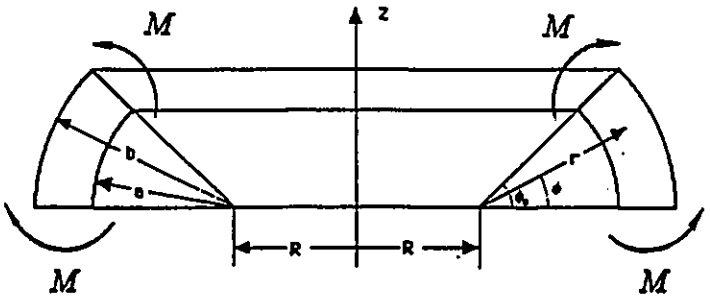

Figure 2.5: Sectional toroidal ring subjected to edge bending moments.

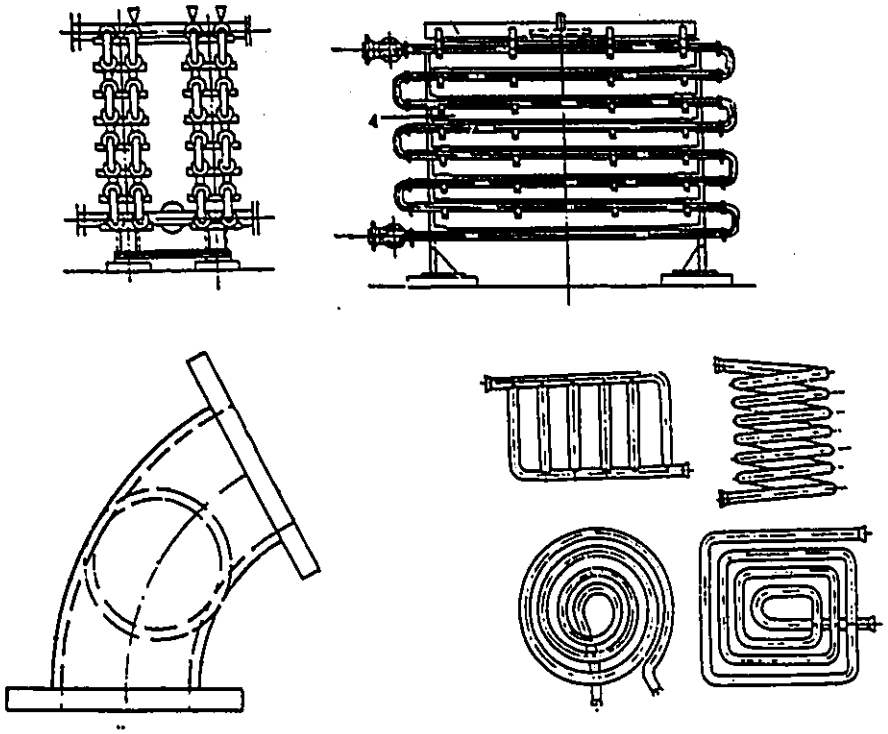

Figure 2.6: Different type of elbows and pipe bends.

Chapter 3

DISPLACEMENT BASED TOROIDAL ELASTICITY

3.1 Introduction

Toroidal Elasticity is a three-dimensional (3D) theory used for the elastostatic analysis of thick-walled curved pipes, elbows, and complete toroidal shells. Göhner [4] was the first person to investigate the technical problem of a curved toroidal structure considering solid elements. Kornecki [6] extended previous work by considering thick-walled shell elements. Extensive work on TE was carried out by Lang [1]. In his work, a stress approach was used, satisfying the equilibrium and Beltrami-Michell equations. Solutions were set up for a number of specific loading cases, using the method of successive approximations. Redekop [2] used a displacement approach to perform the TE analysis. The governing Navier equations were employed. A displacement approach has the advantage of the problem where both displacements and stresses are considered. The approach again relied on the method

of successive approximations. The full development of the displacement based TE is presented in the following sections.

3.2 Governing Equations in Cartesian Coordinates

For a three-dimensional linearly elastic, isotropic problem, there are three normal stress components σ_x, σ_y, σ_z and six shear stress components $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, $\sigma_{zx} = \sigma_{xz}$ at a point P(x,y,z). The displacement based governing equations for a 3D problem are presented in eqn. (3.1).

$$\begin{aligned}(E_o + G)\frac{\partial \vartheta}{\partial x} + \nabla^2 u + f_x = 0 \\ (E_o + G)\frac{\partial \vartheta}{\partial y} + \nabla^2 v + f_y = 0 \\ (E_o + G)\frac{\partial \vartheta}{\partial z} + \nabla^2 w + f_z = 0\end{aligned} \tag{3.1}$$

where

$$E_o = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

f_x, f_y and f_z are body force components in x, y and z direction, G is the shear modulus, ν is the Poisson ratio, E is the elastic modulus and $\vartheta = div\mathrm{u}$ is the dilatation. The stress displacement relations are

$$\begin{aligned}\sigma_x = E_o\vartheta + 2G\frac{\partial u}{\partial x};\ \sigma_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \sigma_y = E_o\vartheta + 2G\frac{\partial v}{\partial y};\ \sigma_{yz} = G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \sigma_z = E_o\vartheta + 2G\frac{\partial w}{\partial z};\ \sigma_{zx} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\end{aligned} \tag{3.2}$$

where

$$\vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The stress boundary conditions in Cartesian coordinates are expressed as

$$\begin{aligned} \sigma_x n_x + \sigma_{xy} n_y + \sigma_{xz} n_z &= F_{nx} \\ \sigma_{yx} n_x + \sigma_y n_y + \sigma_{yz} n_z &= F_{ny} \\ \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_z n_z &= F_{nz} \end{aligned} \tag{3.3}$$

where n_x, n_y and n_z are components of the unit normal vector of the boundary surface, and F_{nx}, F_{ny} and F_{nz} are components of the applied boundary force. The displacement components u, v and w must satisfy the Navier equations (3.1) in the domain and the displacement boundary conditions on the surface of the domain, and the six corresponding stress components have to satisfy the boundary conditions (3.3) on the boundary surface.

The expressions of eqn. (3.3) are suitable for a domain having boundary surfaces parallel to the coordinate planes. It is not convenient for a cylindrical, spherical or toroidal domain. For such domains, the equations should be recast in the coordinate systems of that domain.

3.3 Governing Equations in Toroidal Coordinates

The non-classical toroidal coordinate system is a suitable coordinate system for the analysis of toroidal structures. The non-classical toroidal coordinate system,

and the related cylindrical and cartesian coordinate system are shown in Fig. 3.1. In the figure, R is the mean toroidal radius, r is the distance from the toroidal center line, ϕ is the vertical angle in the toroidal cross-section, θ is the horizontal angle indicating the position of the toroidal cross-section.

A set of unit vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is based on the cartesian frame, $\{\mathbf{e}_\rho,\mathbf{k},\mathbf{e}_\theta\}$ is a set of unit vectors based on the cylindrical frame, and $\{\mathbf{e}_\xi,\mathbf{e}_\phi,\mathbf{e}_\theta\}$ is a set of unit vectors based on the toroidal frame. The relations among these three sets of unit vectors are

$$\begin{aligned}
\mathbf{e}_\rho &= \sin\theta\mathbf{i} + \cos\theta\mathbf{j} \\
\mathbf{e}_\xi &= \cos\phi\mathbf{e}_\rho + \sin\phi\mathbf{k} \\
\mathbf{e}_\phi &= -\sin\phi\mathbf{e}_\rho + \cos\phi\mathbf{k} \\
\mathbf{e}_\theta &= \cos\theta\mathbf{i} - \sin\theta\mathbf{j}
\end{aligned} \tag{3.4}$$

where $\xi = r/c$ is a non-dimensional radial coordinate in the toroidal coordinate system, and c is a reference length to be specified later. A vector from the origin of the Cartesian space to a point P in the domain defined in toroidal coordinates is presented by, $\mathbf{P}$, where

$$\begin{aligned}
\mathbf{P} &= R\mathbf{e}_\rho + r\mathbf{e}_\xi \\
&= (R\sin\theta + c\xi\sin\theta\cos\phi)\mathbf{i} + (R\cos\theta + c\xi\cos\theta\cos\phi)\mathbf{j} + c\xi\sin\phi\mathbf{k}
\end{aligned} \tag{3.5}$$

The expression, $\rho = R + r\cos\phi$ is the radial cylindrical coordinate of the point P. The components of the displacement vector $\mathbf{u}$ and stress tensor $\mathbf{T}$ in the toroidal coordinate system are presented in Fig. 3.2. The displacement vector, $\mathbf{u}$, stress

tensor, $\mathbf{T}$, and strain tensor, $\mathbf{E}$, are given by Lure [55] as

$$\mathbf{u} = \mathbf{e}_\alpha u_\alpha;\ \mathbf{T} = \mathbf{e}_\alpha \mathbf{e}_\beta \sigma_{\alpha\beta};\ \mathbf{E} = \mathbf{e}_\alpha \mathbf{e}_\beta \varepsilon_{\alpha\beta}$$

where α, β are subscripts with values 1, 2, 3 which correspond in toroidal coordinates to ξ, ϕ, θ. The dilatation, ϑ, strain tensor, $\mathbf{E}$, and stress-strain relations in a general curvilinear coordinate system [2] are given as

$$\vartheta = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial(H_2 H_3 u_\xi)}{\partial \xi} + \frac{\partial(H_3 H_1 u_\phi)}{\partial \phi} + \frac{\partial(H_1 H_2 u_\theta)}{\partial \theta}\right] \tag{3.6}$$

$$\mathbf{E} = \frac{1}{2}\mathbf{e}_\alpha \mathbf{e}_\beta \left[\frac{1}{H_\beta}\frac{\partial u_\alpha}{\partial q_\beta} + \frac{1}{H_\alpha}\frac{\partial u_\beta}{\partial q_\alpha} + \frac{u_\gamma}{H_\beta}\begin{pmatrix} \alpha \\ \gamma\beta \end{pmatrix} + \frac{u_\gamma}{H_\alpha}\begin{pmatrix} \beta \\ \gamma\alpha \end{pmatrix}\right] \tag{3.7}$$

$$\mathbf{T} = 2G[\mathbf{E} + \nu(1-2\nu)^{-1}\vartheta \mathbf{U}] \tag{3.8}$$

where $\mathbf{U}$ is an identity tensor, subscript γ is same as α and β, coordinate q_α is dependent on the type of curvilinear coordinate system used, and the bracketed quantities of the right hand side of the equations are the Cristoffel symbols of the second kind. The Cristoffel symbols have the values:

$$\begin{pmatrix} \gamma \\ \alpha\beta \end{pmatrix} = 0; for \alpha \neq \beta \neq \gamma;\ \begin{pmatrix} \alpha \\ \alpha\beta \end{pmatrix} = 0;\ \begin{pmatrix} \alpha \\ \alpha\alpha \end{pmatrix} = 0$$
$$\begin{pmatrix} \gamma \\ \alpha\alpha \end{pmatrix} = -\frac{1}{H_\gamma}\frac{\partial H_\alpha}{\partial q_\gamma};\ \begin{pmatrix} \gamma \\ \alpha\gamma \end{pmatrix} = \frac{1}{H_\alpha}\frac{\partial H_\gamma}{\partial q_\alpha} \tag{3.9}$$

where

$$H_\alpha = \left(\frac{\partial \mathbf{P}}{\partial q_\alpha}\cdot\frac{\partial \mathbf{P}}{\partial q_\alpha}\right)^{1/2} \tag{3.10}$$

In the case of toroidal coordinates, one has

$$q_1 = \xi;\ q_2 = \phi;\ q_3 = \theta$$

Substituting eqn. (3.5) into eqn. (3.10) and expanding the result, one obtains

$$H_1 = \left(\frac{\partial \mathbf{P}}{\partial \xi} \cdot \frac{\partial \mathbf{P}}{\partial \xi}\right)^{1/2} = (\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi)^{1/2} = 1 \tag{3.11}$$

$$H_2 = \left(\frac{\partial \mathbf{P}}{\partial \phi} \cdot \frac{\partial \mathbf{P}}{\partial \phi}\right)^{1/2} = c\xi(\sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\phi)^{1/2} = c\xi = r$$

$$H_3 = \left(\frac{\partial \mathbf{P}}{\partial \theta} \cdot \frac{\partial \mathbf{P}}{\partial \theta}\right)^{1/2} = \{(\sin^2\theta + \cos^2\theta)(R + c\xi\cos\phi)\}^{1/2} = R + r\cos\phi = \rho$$

For convenience, non-dimensional displacements u, v, and w are defined as

$$u = \frac{u_\xi}{c};\ v = \frac{u_\phi}{c};\ w = \frac{u_\theta}{c} \tag{3.12}$$

Non-dimensional stress components are defined as

$$\tau_{\xi\xi} = \frac{1+\nu}{E}\sigma_{\xi\xi};\ \tau_{\phi\phi} = \frac{1+\nu}{E}\sigma_{\phi\phi};\ \tau_{\theta\theta} = \frac{1+\nu}{E}\sigma_{\theta\theta} \tag{3.13}$$
$$\tau_{\xi\phi} = \frac{1+\nu}{E}\sigma_{\xi\phi};\ \tau_{\xi\theta} = \frac{1+\nu}{E}\sigma_{\xi\theta};\ \tau_{\phi\theta} = \frac{1+\nu}{E}\sigma_{\phi\theta}$$

By substituting the quantities H_α ($\alpha = 1, 2, 3$) into eqn. (3.6), one obtains the dilatation ϑ as

$$\begin{aligned}\vartheta &= \varepsilon_\xi + \varepsilon_\phi + \varepsilon_\theta \\ &= \left(\frac{\partial u}{\partial \xi} + \frac{u}{\xi} + \frac{1}{\xi}\frac{\partial v}{\partial \phi}\right) + \frac{c}{\rho}\left(u\cos\phi - v\sin\phi + \frac{\partial w}{\partial \theta}\right) = \psi + \frac{c}{\rho}\chi\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\psi &= \frac{\partial u}{\partial \xi} + \frac{u}{\xi} + \frac{1}{\xi}\frac{\partial v}{\partial \phi} \\ \chi &= u\cos\phi - v\sin\phi + \frac{\partial w}{\partial \theta}\end{aligned} \tag{3.15}$$

By substituting eqn. (3.7) into eqn. (3.8) and using eqn. (3.9-3.15), the non-dimensional stress components may be expressed in terms of the three non-dimensional

displacement components as

$$\begin{aligned}
\tau_{\xi\xi} &= \frac{\partial u}{\partial \xi} + \frac{\nu}{1-2\nu}\vartheta \\
\tau_{\phi\phi} &= \left(\frac{u}{\xi} + \frac{\partial v}{\partial \phi}\right) + \frac{\nu}{1-2\nu}\vartheta \\
\tau_{\theta\theta} &= \frac{c}{\rho}\chi + \frac{\nu}{1-2\nu}\vartheta \\
\tau_{\xi\phi} &= \frac{1}{2}\left(\frac{1}{\xi}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \xi} - \frac{v}{\xi}\right) \\
\tau_{\xi\theta} &= \frac{1}{2}\left(\frac{\partial w}{\partial \xi} + \frac{c}{\rho}\frac{\partial u}{\partial \theta} - \frac{c}{\rho}w\cos\phi\right) \\
\tau_{\phi\theta} &= \frac{1}{2}\left(\frac{1}{\xi}\frac{\partial w}{\partial \phi} + \frac{c}{\rho}\frac{\partial v}{\partial \theta} + \frac{c}{\rho}w\sin\phi\right)
\end{aligned} \tag{3.16}$$

The equilibrium equation in a general curvilinear coordinate system is given as

$$div\mathbf{T} + \mathbf{f} = \mathbf{e}_\beta\left[\frac{1}{H_\alpha}\frac{\partial \sigma_{\alpha\beta}}{\partial q_\alpha} + \frac{\sigma_{\alpha\beta}}{H_\delta}\begin{pmatrix}\delta\\ \alpha\delta\end{pmatrix} + \frac{\sigma_{\alpha\beta}}{H_\alpha}\begin{pmatrix}\beta\\ \delta\alpha\end{pmatrix} + f_\beta\right] = 0 \tag{3.17}$$

where f_β are the components of the body force in toroidal coordinates. Neglecting the body force, f, expanding eqn. (3.17) using eqn. (3.9-3.15), and setting each component of this equation to zero separately yields three equilibrium equations in stress form as

$$\begin{aligned}
&\frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{1}{\xi}\frac{\partial \tau_{\xi\phi}}{\partial \phi} + \frac{1}{\xi}(\tau_{\xi\xi} - \tau_{\phi\phi}) + \frac{c}{\rho}\left[\frac{\partial \tau_{\xi\theta}}{\partial \theta} + \cos\phi(\tau_{\xi\xi} - \tau_{\phi\phi}) - \sin\phi(\tau_{\xi\phi})\right] = 0 \\
&\frac{\partial \tau_{\xi\phi}}{\partial \xi} + \frac{1}{\xi}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + 2\frac{\tau_{\xi\phi}}{\xi} + \frac{c}{\rho}\left[\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \cos\phi(\tau_{\xi\phi}) - \sin\phi(\tau_{\phi\phi} - \tau_{\theta\theta})\right] = 0 \\
&\frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{1}{\xi}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{\tau_{\xi\theta}}{\xi} + \frac{c}{\rho}\left[\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\cos\phi(\tau_{\xi\theta}) - 2\sin\phi(\tau_{\phi\theta})\right] = 0
\end{aligned} \tag{3.18}$$

Substituting the stress-displacement relations (3.16) into the equilibrium equations (3.18), one obtains the governing Navier equations in toroidal coordinates as

$$U = -\frac{c}{\rho}\bar{U} - \left(\frac{c}{\rho}\right)^2\hat{U}$$

$$V = -\frac{c}{\rho}\bar{V} - \left(\frac{c}{\rho}\right)^2 \hat{V} \tag{3.19}$$
$$W = -\frac{c}{\rho}\bar{W} - \left(\frac{c}{\rho}\right)^2 \hat{W}$$

where

$$\begin{aligned}
U &= \nabla^2 u - \frac{u}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi}{\partial \xi} \\
\bar{U} &= \cos\phi\frac{\partial u}{\partial \xi} - \sin\phi\left(\frac{1}{\xi}\frac{\partial u}{\partial \phi} - \frac{v}{\xi}\right) + \frac{1}{1-2\nu}\frac{\partial \chi}{\partial \xi} \\
\hat{U} &= \frac{\partial^2 u}{\partial \theta^2} - \cos\phi\frac{\partial w}{\partial \theta} - \frac{2(1-\nu)}{1-2\nu}\cos\phi\chi \\
V &= \nabla^2 v - \frac{v}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi}{\partial \phi} \\
\bar{V} &= \cos\phi\frac{\partial v}{\partial \xi} - \sin\phi\left(\frac{u}{\xi} + \frac{1}{\xi}\frac{\partial v}{\partial \phi}\right) + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \chi}{\partial \phi} \\
\hat{V} &= \frac{\partial^2 v}{\partial \theta^2} + \sin\phi\frac{\partial w}{\partial \theta} + \frac{2(1-\nu)}{1-2\nu}\sin\phi\chi \\
W &= \nabla^2 w \\
\bar{W} &= \cos\phi\frac{\partial w}{\partial \xi} - \sin\phi\frac{1}{\xi}\frac{\partial w}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi}{\partial \theta} \\
\hat{W} &= -\frac{\partial^2 w}{\partial \theta^2} - w + \frac{3-4\nu}{1-2\nu}\frac{\partial \chi}{\partial \theta} \\
\nabla^2(\ldots) &= \frac{\partial^2(\ldots)}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial(\ldots)}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial^2(\ldots)}{\partial \phi^2}
\end{aligned} \tag{3.20}$$

The three Navier equations serve as the fundamental equations for the displacement-based TE. These equations are composed of three parts. The first part is independent of c/ρ. The second and the third parts are the linear and non-linear parts of c/ρ. The Navier equations in toroidal coordinate system are much more complicated than those in the cartesian coordinate system. As it is impossible to find an exact solution for the Navier equations, recourse is made to the method of successive approximations to obtain an approximate solution.

3.4 Method of Successive Approximations

The method of successive approximations is employed to solve the Navier equations in the toroidal coordinate system. The solution for each component of the displacement is assumed to be a series in terms of a small parameter ϵ. The parameter ϵ is chosen as

$$\epsilon = \frac{c}{R} \tag{3.21}$$

Thus the non-dimensional displacement components take the form

$$\begin{aligned} u &= u_o + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \dots \\ v &= v_o + \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \dots \\ w &= w_o + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \dots \end{aligned} \tag{3.22}$$

where the subscripts of the right hand side terms indicate the order of the displacement components in the series. Substituting the series form displacement functions (3.22) into eqn. (3.16), one obtains the corresponding non-dimensional stress components as

$$\begin{aligned} \tau_{\xi\xi} &= \tau_{\xi\xi o} + \epsilon\tau_{\xi\xi 1} + \epsilon^2\tau_{\xi\xi 2} + \epsilon^3\tau_{\xi\xi 3} + \dots \\ \tau_{\phi\phi} &= \tau_{\phi\phi o} + \epsilon\tau_{\phi\phi 1} + \epsilon^2\tau_{\phi\phi 2} + \epsilon^3\tau_{\phi\phi 3} + \dots \\ \tau_{\theta\theta} &= \tau_{\theta\theta o} + \epsilon\tau_{\theta\theta 1} + \epsilon^2\tau_{\theta\theta 2} + \epsilon^3\tau_{\theta\theta 3} + \dots \\ \tau_{\xi\phi} &= \tau_{\xi\phi o} + \epsilon\tau_{\xi\phi 1} + \epsilon^2\tau_{\xi\phi 2} + \epsilon^3\tau_{\xi\phi 3} + \dots \\ \tau_{\xi\theta} &= \tau_{\xi\theta o} + \epsilon\tau_{\xi\theta 1} + \epsilon^2\tau_{\xi\theta 2} + \epsilon^3\tau_{\xi\theta 3} + \dots \\ \tau_{\xi\phi} &= \tau_{\phi\theta o} + \epsilon\tau_{\phi\theta 1} + \epsilon^2\tau_{\phi\theta 2} + \epsilon^3\tau_{\phi\theta 3} + \dots \end{aligned} \tag{3.23}$$

The functions of Navier equations are expanded as

$$
\begin{aligned}
U &= U_o + \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \dots \\
V &= V_o + \epsilon V_1 + \epsilon^2 V_2 + \epsilon^3 V_3 + \dots \\
W &= W_o + \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 + \dots \\
\bar{U} &= \bar{U}_o + \epsilon \bar{U}_1 + \epsilon^2 \bar{U}_2 + \epsilon^3 \bar{U}_3 + \dots \\
\bar{V} &= \bar{V}_o + \epsilon \bar{V}_1 + \epsilon^2 \bar{V}_2 + \epsilon^3 \bar{V}_3 + \dots \\
\bar{W} &= \bar{W}_o + \epsilon \bar{W}_1 + \epsilon^2 \bar{W}_2 + \epsilon^3 \bar{W}_3 + \dots \\
\hat{U} &= \hat{U}_o + \epsilon \hat{U}_1 + \epsilon^2 \hat{U}_2 + \epsilon^3 \hat{U}_3 + \dots \\
\hat{V} &= \hat{V}_o + \epsilon \hat{V}_1 + \epsilon^2 \hat{V}_2 + \epsilon^3 \hat{V}_3 + \dots \\
\hat{W} &= \hat{W}_o + \epsilon \hat{W}_1 + \epsilon^2 \hat{W}_2 + \epsilon^3 \hat{W}_3 + \dots
\end{aligned}
\tag{3.24}
$$

The quantities $\frac{c}{\rho}$ and $(\frac{c}{\rho})^2$ which appear in the Navier equations are functions of ξ since $\rho/c = R/c + r/c\cos\phi$ and $r/c = \xi$. These quantities may be expanded in a series of the parameter ϵ using the Taylor Theorem as

$$
\begin{aligned}
\frac{c}{\rho} &= \epsilon - \epsilon^2 \xi \cos\phi + \epsilon^3 \xi^2 \cos^2\phi - \dots \\
\left(\frac{c}{\rho}\right)^2 &= \epsilon^2 - 2\epsilon^3 \xi \cos\phi + 3\epsilon^4 \xi^2 \cos^2\phi - \dots
\end{aligned}
\tag{3.25}
$$

Substituting the expanded Navier functions and eqn. (3.25) into the Navier equations (3.19), matching terms in ϵ, and setting each order of ϵ to zero, one obtains the governing equations for the different orders. The equations may be expressed for the zeroth order as

$$
U_o = 0;\ V_o = 0;\ W_o = 0 \tag{3.26}
$$

for the first order as

$$U_1 = -\bar{U}_o;\ V_1 = -\bar{V}_o;\ W_1 = -\bar{W}_o \tag{3.27}$$

for the second order as

$$\begin{aligned} U_2 &= -\bar{U}_1 + \xi\cos\phi\bar{U}_o - \hat{U}_o \\ V_2 &= -\bar{V}_1 + \xi\cos\phi\bar{V}_o - \hat{V}_o \\ W_2 &= -\bar{W}_1 + \xi\cos\phi\bar{W}_o - \hat{W}_o \end{aligned} \tag{3.28}$$

and for the third order as

$$\begin{aligned} U_3 &= -\bar{U}_2 + \xi\cos\phi\bar{U}_1 - \xi^2\cos^2\phi\bar{U}_o - \hat{U}_1 + 2\xi\cos\phi\hat{U}_o \\ V_3 &= -\bar{V}_2 + \xi\cos\phi\bar{V}_1 - \xi^2\cos^2\phi\bar{V}_o - \hat{V}_1 + 2\xi\cos\phi\hat{V}_o \\ W_3 &= -\bar{W}_2 + \xi\cos\phi\bar{W}_1 - \xi^2\cos^2\phi\bar{W}_o - \hat{W}_1 + 2\xi\cos\phi\hat{W}_o \end{aligned} \tag{3.29}$$

The Navier functions of order k ($k = 0, 1, 2, \ldots$) in eqns. (3.26-3.29) are defined as

$$\begin{aligned} U_k &= \nabla^2 u_k - \frac{u_k}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_k}{\partial\phi} + \frac{1}{1-2\nu}\frac{\partial\psi_k}{\partial\xi} \\ \bar{U}_k &= \cos\phi\frac{\partial u_k}{\partial\xi} - \sin\phi\left(\frac{1}{\xi}\frac{\partial u_k}{\partial\phi} - \frac{v_k}{\xi}\right) + \frac{1}{1-2\nu}\frac{\partial\chi_k}{\partial\xi} \\ \hat{U}_k &= \frac{\partial^2 u_k}{\partial\theta^2} - \cos\phi\frac{\partial w_k}{\partial\theta} - \frac{2(1-\nu)}{1-2\nu}\cos\phi\chi_k \\ V_k &= \nabla^2 v_k - \frac{v_k}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_k}{\partial\phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial\psi_k}{\partial\phi} \\ \bar{V}_k &= \cos\phi\frac{\partial v_k}{\partial\xi} - \sin\phi\left(\frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial\phi}\right) + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial\chi_k}{\partial\phi} \\ \hat{V}_k &= \frac{\partial^2 v_k}{\partial\theta^2} + \sin\phi\frac{\partial w_k}{\partial\theta} + \frac{2(1-\nu)}{1-2\nu}\sin\phi\chi_k \\ W_k &= \nabla^2 w_k \\ \bar{W}_k &= \cos\phi\frac{\partial w_k}{\partial\xi} - \sin\phi\frac{1}{\xi}\frac{\partial w_k}{\partial\phi} + \frac{1}{1-2\nu}\frac{\partial\psi_k}{\partial\theta} \\ \hat{W}_k &= -\frac{\partial^2 w_k}{\partial\theta^2} - w_k + \frac{3-4\nu}{1-2\nu}\frac{\partial\chi_k}{\partial\theta} \end{aligned} \tag{3.30}$$

where

$$\begin{aligned}\nabla^2(\ldots) &= \frac{\partial^2(\ldots)}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial(\ldots)}{\partial\xi} + \frac{1}{\xi^2}\frac{\partial^2(\ldots)}{\partial\phi^2} \\ \psi_k &= \frac{\partial u_k}{\partial\xi} + \frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial\phi} \\ \chi_k &= u_k\cos\phi - v_k\sin\phi + \frac{\partial w_k}{\partial\theta}\end{aligned} \tag{3.31}$$

Combining eqns. (3.16, 3.22 and 3.25), one obtains the formulas for calculating the stress components of order k, valid for $k = 0, 1, 2$ and 3

$$\begin{aligned}
\tau_{\xi\xi k} &= \frac{\partial u_k}{\partial\xi} + \frac{\nu}{1-2\nu}(\psi_k + \chi_{k-1} - \xi\cos\phi\chi_{k-2} + \xi^2\cos^2\phi\chi_{k-3}) \\
\tau_{\phi\phi k} &= \left(\frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial\phi}\right) + \frac{\nu}{1-2\nu}(\psi_k + \chi_{k-1} - \xi\cos\phi\chi_{k-2} + \xi^2\cos^2\phi\chi_{k-3}) \\
\tau_{\theta\theta k} &= \frac{\nu}{1-2\nu}\psi_k + \frac{1-\nu}{1-2\nu}(\chi_{k-1} - \xi\cos\phi\chi_{k-2} + \xi^2\cos^2\phi\chi_{k-3}) \\
\tau_{\xi\phi k} &= \frac{1}{2}\left(\frac{1}{\xi}\frac{\partial u_k}{\partial\phi} + \frac{\partial v_k}{\partial\xi} - \frac{v_k}{\xi}\right) \\
\tau_{\xi\theta k} &= \frac{1}{2}\{\frac{\partial w_k}{\partial\xi} + \left(\frac{\partial u_{k-1}}{\partial\theta} - w_{k-1}\cos\phi\right) - \xi\cos\phi\left(\frac{\partial u_{k-2}}{\partial\theta} - w_{k-2}\cos\phi\right) \\
&\quad +\xi^2\cos^2\phi\left(\frac{\partial u_{k-3}}{\partial\theta} - w_{k-3}\cos\phi\right)\} \\
\tau_{\phi\theta k} &= \frac{1}{2}\{\frac{1}{\xi}\frac{\partial w_k}{\partial\phi} + \left(\frac{\partial v_{k-1}}{\partial\theta} - w_{k-1}\sin\phi\right) - \xi\cos\phi\left(\frac{\partial v_{k-2}}{\partial\theta} - w_{k-2}\sin\phi\right) \\
&\quad +\xi^2\cos^2\phi\left(\frac{\partial v_{k-3}}{\partial\theta} - w_{k-3}\sin\phi\right)\}
\end{aligned} \tag{3.32}$$

The displacement functions on the right hand members with negative subscripts are to be taken as zero.

The method of successive approximations is a useful and powerful one for finding approximate solutions to equations, but it is difficult to justify rigorously. This method is a collection of methods for the systematic analysis of the global

behavior of the governing equations. The solution series is local in ϵ and global in x. When ϵ is very small, one obtains a good approximation to the solution by summing the first few terms of the series. When ϵ is not small, it may still be possible to obtain a good approximation to the solution from a slowly converging or divergent series [61]. Sometimes it is actually advisable to introduce a parameter ϵ temporarily into a difficult problem having no small parameter, and then finally to set $\epsilon = 1$ to recover the original problem. This apparently artificial conversion to the difficult problem may be the only way to make progress.

Selecting the parameter ϵ requires judgment, and sometimes there may be several tractable choices for ϵ. In general, it is desirable to introduce an ϵ so that the zeroth-order solution is obtained as a closed-form analytic expression. In practice, only the first few terms of the solution series can be conveniently calculated because the iteration procedure becomes increasingly cumbersome as the order increase. It is necessary to find a suitable parameter ϵ which gives a solution with fast convergence. The solution obtained by the method of successive approximations converges only for those values of $|\epsilon|$ less than the radius of convergence of the displacement series.

It is evident from eqns. (3.26-3.29) that the zeroth order governing equations are homogenous equations. The right hand side terms for orders greater than zero contain contributions from the lower orders, and the complexity of the right hand side terms increases with the order. There is only a complementary solutions for the zeroth order governing equations. For orders greater than the zeroth, the solutions are composed of a complementary and a particular part. The complementary part is determined from the homogeneous part of that order's governing equations, which

has the same form as that of the zeroth order. The particular part is determined to satisfy the right hand side terms of the governing equations. The particular part may be found by using trial displacement functions with free constants, to match the right hand side terms of the governing equations of that order.

The procedure for solving a specific problem follows that of the method of successive approximations. Solutions are developed consecutively for the various orders, starting from the zeroth order. The zeroth order solution is chosen from the complementary solution. Only the terms corresponding to the principal deformation of the structure are taken. For orders greater than zero, the right hand side members of the Navier equations are evaluated. The particular part is set up using trial functions. The displacement functions for that order are constructed by summing the complementary and particular parts. The constants in the complementary solution of each order are found separately from the boundary conditions. The total solution is the sum of the particular and complementary solutions of all the orders factored by the appropriate power of ϵ. Any constants remaining in the total solution are found from end or edge boundary conditions. This solution procedure is demonstrated for some specific problems in the following chapters.

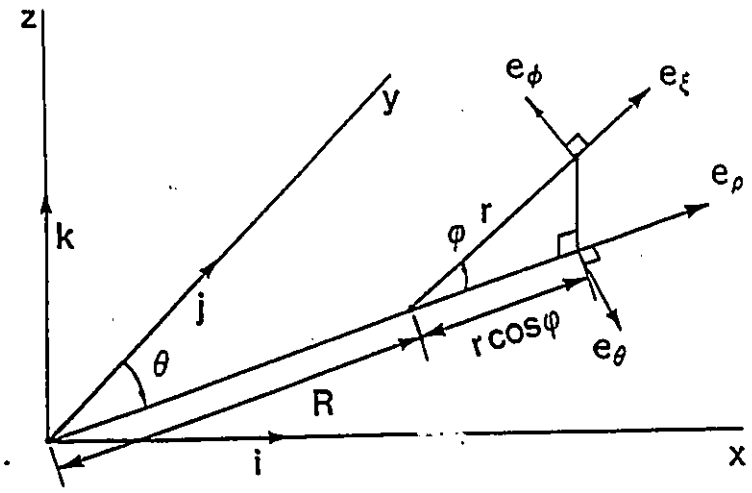

Figure 3.1: The relations among three coordinate systems

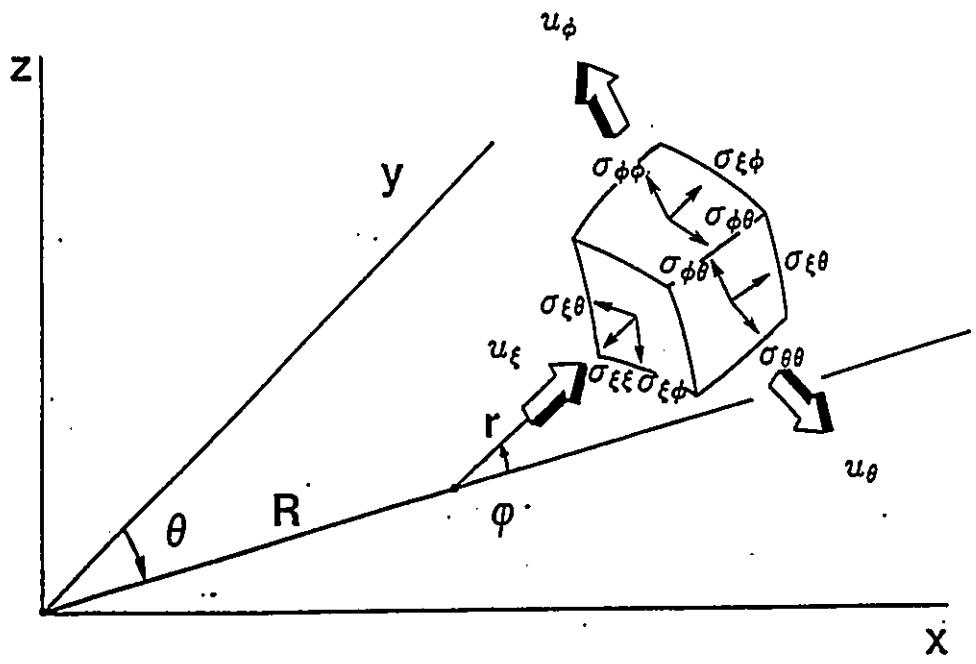

Figure 3.2: Components of displacement vector u and stress tensor T

Chapter 4

OUT-OF-PLANE LOADING OF AN ELBOW

4.1 Introduction

Elbows are very common components in chemical, nuclear and power plants. They are employed to change the pipe line direction and to increase the flexibility. A detailed study of stress in elbows subjected to different kinds of load cases was conducted by Lang, using the stress based TE. In the present chapter, a displacement approach is used and the out-of-plane loading case is considered. The Navier equations presented in chapter 3 are employed to solve this problem. A general solution, valid for out-of-plane loading is then presented. Finally an example is considered covering a case of out-of-plane loading of a 90° elbow. Contributions from the zeroth to the third order in the method of successive approximations are considered. Numerical results are determined, and these are compared with results obtained from the stress-based TE, and the boundary element method (BEM).

4.2 General Solution for Out-of-Plane Loading

The solution is sought for a curved tube with loading that produces bending that is out-of-plane with respect to the xy plane (Fig. 4.1). A direct solution is formulated to avoid the complications which arise from the use of stress functions. The complementary solution for the non-dimensional displacements (Fig. 4.2) is thus sought in the form

$$\begin{aligned} u_o &= C_1\xi^{\gamma}\sin n\phi F(\theta) \\ v_o &= C_2\xi^{\gamma}\cos n\phi F(\theta) \\ w_o &= C_3\xi^{\bar{\gamma}}\sin n\phi G(\theta) \end{aligned} \tag{4.1}$$

where the functions $F(\theta)$ and $G(\theta)$ are arbitrary at this point. They can be used to satisfy the displacement boundary conditions for a specific problem. The quantities $C_1, C_2, C_3, \gamma, \bar{\gamma}$, and n are constants and parameters which are determined from the zeroth order Navier equations and stress free boundary conditions on the shell surfaces. There are four roots for γ and two roots for $\bar{\gamma}$. Using superposition and adopting a convenient notation for the constants, one obtains the non-dimensional full solution as

$$\begin{aligned} u_o &= [a_o\xi^{-1} + b_o\xi + \{a_1 + b_1(1-4\nu)\xi^2 + c_1\xi^{-2} + d_1(3-4\nu)\ln\xi\}\sin\phi \\ &+ \sum_{n=2}^{\infty}\{a_n\xi^{n-1} + b_n(2-n-4\nu)\xi^{n+1} + c_n\xi^{-(n+1)} \\ &+ d_n(-2-n+4\nu)\xi^{-(n-1)}\}\sin n\phi]F(\theta) \\ v_o &= [c_o\xi^{-1} + d_o\xi + \{a_1 - b_1(5-4\nu)\xi^2 - c_1\xi^{-2} + d_1(3-4\nu)\ln\xi + d_1\}\cos\phi \\ &+ \sum_{n=2}^{\infty}\{a_n\xi^{n-1} - b_n(4+n-4\nu)\xi^{n+1} - c_n\xi^{-(n+1)} \end{aligned} \tag{4.2}$$

$$- \; d_n(4-n-4\nu)\xi^{-(n-1)}\}\cos n\phi]F(\theta)$$

$$w_o = [e_o + f_o \ln\xi + (e_1\xi^{-1} + f_1\xi)\sin\phi + \sum_{n=2}^{\infty}\{e_n\xi^{-n} + f_n\xi^n\}\sin n\phi]G(\theta)$$

where a_n, b_n, c_n, d_n, e_n and f_n $(n = 0, 1, 2, \ldots)$ are constants to be found from the boundary conditions.

4.3 Solution of Elbow Under Out-of-plane Bending

A detailed solution is presented for a 90° elbow with one end fixed, and the other end subjected to an out-of-plane twist moment (Fig. 4.1). The boundary conditions satisfied for the problem are as follows:

on the curved surfaces $\xi = 1.0$ and $\xi = a/b$

$$\tau_{\xi\xi k} = \tau_{\xi\phi k} = \tau_{\xi\theta k} = 0 \tag{4.3}$$

on the end surface $\theta = 0°$

$$\tau_{\xi\phi k} = \tau_{\phi\theta k} = 0;\ (F_x)_k = (M_z)_k = 0$$

$$M_y = (M_y)_o + \epsilon(M_y)_1 + \epsilon^2(M_y)_2 + \epsilon^3(M_y)_3 + \ldots = T_o \tag{4.4}$$

and on the end surface $\theta = 90°$

$$\tau_{\theta\theta k} = 0;\ (F_x)_k = (F_z)_k = 0$$

$$M_y = (M_y)_o + \epsilon(M_y)_1 + \epsilon^2(M_y)_2 + \epsilon^3(M_y)_3 + \ldots = T_o \tag{4.5}$$

where $(F_x)_k, (F_z)_k$ are the resultant force components of the kth order in the x and z directions, $(M_y)_k$, $(M_z)_k$ are the resultant moment components of the kth

order in the y and z directions, and T_o is the applied twist moment. The boundary conditions (4.3) on the curved surfaces are satisfied by each order separately. The total bending moment, M_y, on the fixed end, and the total twist moment, M_y, on the free end must be equal to T_o. In order to satisfy these boundary conditions on the end surfaces, the functions $F(\theta)$ and $G(\theta)$ must be taken as

$$F(\theta) = \cos\theta;\ G(\theta) = \sin\theta \tag{4.6}$$

The contributions from the various orders to the total solution will now be developed, starting from the zeroth order.

4.3.1 The Zeroth Order Solution

The zeroth order governing equations are homogeneous equations, and thus only the complementary solution is required. Considering the expected deformation, the displacement components are selected from the functions of eqns.(4.2) as

$$\begin{aligned} u_o &= 0 \\ v_o &= d_o \xi \cos\theta \\ w_o &= 0 \end{aligned} \tag{4.7}$$

These displacement functions represent a 'rigid-body' rotation of the cross-section. Constant d_o of this order is to be determined later from the boundary conditions on the ends of the elbow. The zeroth order stress components are found from eqns.(3.32) as

$$\tau_{\xi\xi o} = \tau_{\phi\phi o} = \tau_{\theta\theta o} = \tau_{\xi\phi o} = \tau_{\xi\theta o} = \tau_{\phi\theta o} = 0 \tag{4.8}$$

4.3.2 The First Order Solution

For the first order analysis, the right hand side terms of the governing equations are first evaluated. These can be found from eqns. (3.27, 3.30) and the zeroth order displacement components. The governing equations of the first order then are

$$\begin{aligned}
&\nabla^2 u_1 - \frac{u_1}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_1}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_1}{\partial \xi} = \frac{2\nu}{1-2\nu} d_o \sin\phi\cos\theta \\
&\nabla^2 v_1 - \frac{v_1}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_1}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_1}{\partial \phi} = \frac{2\nu}{1-2\nu} d_o \cos\phi\cos\theta \qquad (4.9) \\
&\nabla^2 w_1 = 0
\end{aligned}$$

The first order solutions comprise of a complementary and a particular part. The complementary part is selected from eqns. (4.2). The particular part is found using trial functions selected to match the right hand side of the governing equations. Adding together the complementary and particular parts one obtains the first order displacement components

$$\begin{aligned}
u_1 &= \{a_1 + b_1(1-4\nu)\xi^2 + c_1\xi^{-2} + d_1(3-4\nu)\ln\xi + A_{11}^*\xi^*\}\sin\phi\cos\theta \\
v_1 &= \{a_1 - b_1(5-4\nu)\xi^2 - c_1\xi^{-2} + d_1(3-4\nu)\ln\xi + d_1 + B_{11}^*\xi^2\}\cos\phi\cos\theta \\
w_1 &= (e_1\xi^{-1} + f_1\xi)\sin\phi\sin\theta \qquad (4.10)
\end{aligned}$$

where

$$A_{11}^* = B_{11}^* = \frac{\nu}{3-4\nu} d_o$$

This displacement field corresponds to an expansion and a 'rigid body' motion of the cross-section. Using eqns. (3.32), one obtains the first order stress components

as

$$\begin{aligned}
\tau_{\xi\xi 1} &= [2b_1\xi - 2c_1\xi^{-3} + (3-2\nu)d_1\xi^{-1} - \frac{\nu}{3-4\nu}d_o\xi]\sin\phi\cos\theta \\
\tau_{\phi\phi 1} &= [6b_1\xi + 2c_1\xi^{-3} - (1-2\nu)d_1\xi^{-1} - \frac{3\nu}{3-4\nu}d_o\xi]\sin\phi\cos\theta \\
\tau_{\theta\theta 1} &= [8\nu b_1\xi + 2\nu d_1\xi^{-1} - \frac{3-\nu}{3-4\nu}d_o\xi]\sin\phi\cos\theta \\
\tau_{\xi\phi 1} &= [-2b_1\xi + 2c_1\xi^{-3} + (1-2\nu)d_1\xi^{-1} + \frac{\nu}{3-4\nu}d_o\xi]\cos\phi\cos\theta \\
\tau_{\xi\theta 1} &= \frac{1}{2}[-e_1\xi^{-2} + f_1]\sin\phi\sin\theta \\
\tau_{\phi\theta 1} &= \frac{1}{2}[e_1\xi^{-2} + f_1]\cos\phi\sin\theta - \frac{1}{2}d_o\xi\sin\theta
\end{aligned} \tag{4.11}$$

The free constants are determined by forcing the stress components $\tau_{\xi\xi 1}, \tau_{\xi\phi 1}$ and $\tau_{\xi\theta 1}$ to satisfy the conditions (4.3). The constants c_1, d_1, e_1 and f_1 are zero, while $b_1 = \nu d_o/(6-8\nu)$. Substituting these constants into eqns. (4.16, 4.17), one obtains the first order displacement and stress fields as

$$\begin{aligned}
u_1 &= (a_1 - \frac{\nu}{2}d_o\xi^2)\sin\phi\cos\theta \\
v_1 &= (a_1 + \frac{\nu}{2}d_o\xi^2)\cos\phi\cos\theta \\
w_1 &= 0 \\
\tau_{\xi\xi 1} &= \tau_{\phi\phi 1} = \tau_{\xi\phi 1} = \tau_{\xi\theta 1} = 0 \\
\tau_{\theta\theta 1} &= -(1+\nu)d_o\xi\sin\phi\cos\theta \\
\tau_{\phi\theta 1} &= -\frac{1}{2}d_o\xi\sin\theta
\end{aligned} \tag{4.12}$$

Constant a_1 is to be determined later from the conditions on the ends of the elbow.

4.3.3 The Second Order Solution

The second order governing equations are determined as

$$\begin{aligned}
\nabla^2 u_2 - \frac{u_2}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_2}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_2}{\partial \xi} &= -\frac{1+\nu}{1-2\nu} d_o \xi \sin 2\phi \cos\theta \\
\nabla^2 v_2 - \frac{v_2}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_2}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_2}{\partial \phi} &= -\frac{1+\nu}{1-2\nu} d_o \xi \cos 2\phi \cos\theta \qquad (4.13) \\
\nabla^2 w_2 &= (1+2\nu) d_o \xi \sin\phi \sin\theta
\end{aligned}$$

The right hand sides of the first two second-order governing equations are functions of 2ϕ, while the right hand side of the third equation is a function of ϕ. The second order displacement components are thus taken as

$$\begin{aligned}
u_2 &= \{a_2\xi + b_2(-4\nu)\xi^3 + c_2\xi^{-3} + d_2(-4+4\nu)\xi^{-1} + A_{21}^*\xi^3\}\sin 2\phi\cos\theta \\
v_2 &= \{a_2\xi - b_2(6-4\nu)\xi^3 - c_2\xi^{-3} - d_2(2-4\nu)\xi^{-1} + B_{21}^*\xi^3\}\cos 2\phi\cos\theta \\
w_2 &= \{e_2\xi^{-2} + f_2\xi^2 + C_{21}^*\xi^3\}\sin\phi\sin\theta \qquad (4.14)
\end{aligned}$$

with the stress components taking the form

$$\begin{aligned}
\tau_{\xi\xi 2} &= \{a_2 - 3c_2\xi^{-4} + 4d_2\xi^{-2} - \frac{1+\nu}{4} d_o\xi^2\}\sin 2\phi\cos\theta \\
\tau_{\phi\phi 2} &= \{-a_2 + 12b_2\xi^2 + 3c_2\xi^{-4} + \frac{(1+\nu)(1+4\nu)}{4(3-4\nu)} d_o\xi^2\}\sin 2\phi\cos\theta \\
\tau_{\theta\theta 2} &= \{12\nu b_2\xi^2 + 4\nu d_2\xi^{-2} + \frac{(1+\nu)(3-2\nu)}{2(3-4\nu)} d_o\xi^2\}\sin 2\phi\cos\theta \qquad (4.15) \\
\tau_{\xi\phi 2} &= \{a_2 - 6b_2\xi^2 + 3c_2\xi^{-4} - 2d_2\xi^{-2} - \frac{1+\nu}{2(3-4\nu)} d_o\xi^2\}\cos 2\phi\cos\theta \\
\tau_{\xi\theta 2} &= \frac{1}{2}\{(f_2 - a_1) - e_2\xi^{-2} + \frac{3+2\nu}{8} d_o\xi^2\}\sin\phi\sin\theta \\
\tau_{\phi\theta 2} &= \frac{1}{2}\{(f_2 - a_1) + e_2\xi^{-2} + \frac{9+6\nu}{8} d_o\xi^2\}\cos\phi\sin\theta
\end{aligned}$$

where a_2, b_2, c_2, d_2, e_2, and f_2 are free constants, A_{21}^*, B_{21}^*, and C_{21}^* are constants of the second order particular solution. Enforcement of the governing equations (4.13)

and the boundary conditions (4.3) leads to the following values of the constants

$$\begin{aligned}
a_2 &= \frac{1+\nu}{4}\frac{1-\lambda^6}{(1-\lambda^2)^3}(1+\lambda^2)d_o \\
b_2 &= \frac{1+\nu}{24}\{2\frac{1-\lambda^6}{(1-\lambda^2)^3} - \frac{5-4\nu}{3-4\nu}\}d_o \\
c_2 &= -\frac{1+\nu}{4}\frac{1-\lambda^4}{(1-\lambda^2)^3}\lambda^4 d_o \\
d_2 &= -\frac{1+\nu}{4}\frac{1+\lambda^6}{(1-\lambda^2)^3}\lambda^2 d_o \\
e_2 &= -\frac{3+2\nu}{8}\lambda^2 d_o \\
f_2 &= -\frac{3+2\nu}{8}(1-\lambda^2)d_o + a_1 \\
A^*_{21} &= B^*_{21} = -\frac{1+\nu}{4(3-4\nu)}d_o \\
C^*_{21} &= \frac{1+2\nu}{8}d_o
\end{aligned} \tag{4.16}$$

4.3.4 The Third Order Solution

The third-order governing equations are developed from eqns. (3.29, 3.30). On the right hand sides of the first two of these equations, there are two sets of terms. The first set contains functions which are first harmonics in ϕ, while the second set contain third harmonics in ϕ. Most contribution of the first harmonics in ϕ has been considered in the first order solution. In order to avoid cumbersome procedure, only the second part of these two sets is considered here. Thus the governing equations for the third order are

$$\begin{aligned}
\nabla^2 u_3 \; - \; & \frac{u_3}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_3}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_3}{\partial \xi} = \\
- \; & \frac{(1+\nu)d_o}{4(1-2\nu)(1-\lambda^2)^3}\{\left[2(1-\nu)(1-\lambda^6) - 3(2-\nu)(1-\lambda^2)^3\right]\xi^2
\end{aligned}$$

$$
\begin{aligned}
&+ 6(1-\nu)\lambda^4(1-\lambda^4)\xi^{-4} - 2(1-\nu)(3-4\nu)\lambda^2(1-\lambda^6)\xi^{-2}\}\sin 3\phi\cos\theta \\
\nabla^2 v_3 &- \frac{v_3}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_3}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_3}{\partial \phi} = \qquad (4.17)\\
&- \frac{(1+\nu)d_o}{4(1-2\nu)(1-\lambda^2)^3}\{\left[2(1-\nu)(1-\lambda^6) - 3(2-\nu)(1-\lambda^2)^3\right]\xi^2 \\
&- 6(1-\nu)\lambda^4(1-\lambda^4)\xi^{-4} + 2(1-\nu)(1+4\nu)\lambda^2(1-\lambda^6)\xi^{-2}\}\cos 3\phi\cos\theta \\
\nabla^2 w_3 &= \frac{(1+\nu)d_o)}{4(1-\lambda^2)^3}\{\left[(1-\lambda^6) + \frac{21-28\nu-10\nu^2}{2(1+\nu)(1-2\nu)}(1-\lambda^2)^3\right]\xi^2 \\
&- \left[(1-\lambda^6) + \frac{3+2\nu}{2(1+\nu}(1-\lambda^2)^3\right]\lambda^2\xi^{-2}\}\sin 2\phi\sin\theta
\end{aligned}
$$

For the complementary parts of u_3 and v_3, the third harmonic solution is taken, while for w_3 the second harmonic. The third order solution then takes the form

$$
\begin{aligned}
u_3 &= \{a_3\xi^2 - b_3(1+4\nu)\xi^4 + c_3\xi^{-4} - d_3(5-4\nu)\xi^{-2} \\
&+ A^*_{31}\xi^4 + A^*_{32}\xi^{-2} + A^*_{33}\}\sin 3\phi\cos\theta \\
v_3 &= \{a_3\xi^2 - b_3(7-4\nu)\xi^4 - c_3\xi^{-4} - d_3(1-4\nu)\xi^{-2} \\
&+ B^*_{31}\xi^4 + B^*_{32}\xi^{-2} + B^*_{33}\}\cos 3\phi\cos\theta \\
w_3 &= \{e_3\xi^{-3} + f_3\xi^3 + C^*_{31}\xi^4 + C^*_{32}\}\sin 2\phi\sin\theta \\
\tau_{\xi\xi 3} &= \{2a_3\xi - 4b_3\xi^3 - 4c_3\xi^{-5} + 10d_3\xi^{-3} - h[\left(\frac{1-2\nu}{3}(1-\lambda^6) - \frac{3-\nu}{3}(1-\lambda^2)^3\right)\xi^3 \\
&+ \lambda^4(1-\lambda^4)\xi^{-3} + \nu\lambda^2(1-\lambda^6)\xi^{-1}]\}\sin 3\phi\cos\theta \\
\tau_{\phi\phi 3} &= \{-2a_3\xi + 20b_3\xi^3 + 4c_3\xi^{-5} - 2d_3\xi^{-3} - h[\left(\frac{1-2\nu}{3}(1-\lambda^6) - \frac{3-\nu}{3}(1-\lambda^2)^3\right)\xi^3 \\
&+ \lambda^4(1-\lambda^4)\xi^{-3} - \nu\lambda^2(1-\lambda^6)\xi^{-1}]\}\sin 3\phi\cos\theta \qquad (4.18)\\
\tau_{\theta\theta 3} &= \{16\nu b_3\xi^3 + 8\nu d_3\xi^{-3} + h[\left((1-\nu)(1-\lambda^6) - \frac{11-5\nu}{6}(1-\lambda^2)^3\right)\xi^3 \\
&- (1-\nu)\lambda^4(1-\lambda^4)\xi^{-3} + (1+\nu)\lambda^2(1-\lambda^6)\xi^{-1}]\}\sin 3\phi\cos\theta \\
\tau_{\xi\phi 3} &= \{2a_3\xi - 12b_3\xi^3 + 4c_3\xi^{-5} - 6d_3\xi^{-3} - h(1-\nu)\lambda^2(1-\lambda^6)d_o\xi^{-1}\}\cos 3\phi\cos\theta \\
\tau_{\xi\theta 3} &= \frac{1}{2}[-3e_3\xi^{-4} + 3f_3\xi^2 + h_1\xi^3 + h_2\xi^{-3} + h_3\xi + h_4\xi^{-1}]\sin 2\phi\sin\theta
\end{aligned}
$$

$$\tau_{\phi\theta 3} = \frac{1}{2}[e_3\xi^{-4} + f_3\xi^2 + g_1\xi^3 + g_2\xi^{-3} + g_3\xi + g_4\xi^{-1}]\cos 2\phi \sin\theta$$

where a_3, b_3, c_3, d_3, e_3 and f_3 are free constants. A^*_{3i}, B^*_{3i}, and C^*_{3i} (i=1, 2, 3) are constants of the third order particular solution. These are determined from the boundary conditions (4.3) and governing equations (4.17). Quantities h, h_1, h_2, h_3, h_4, g, g_1, g_2, g_3 and g_4 are known constants. These constants are functions of the material property, ν, and the ratio, λ, of the inside and outside radii. Evaluation of the constants leads to

$$\begin{aligned}
b_3 &= \frac{1}{8g}\{I\left[4\lambda^4(1-\lambda^2)+(1+\lambda^2)(1-\lambda^8)\right]-3J(1-\lambda^4) \\
&+ K\left[4(1-2\nu)\lambda^2(1-\lambda^2)+(1+\lambda^2)(1-\lambda^4)\right]\} \\
d_3 &= -\frac{1}{4g}\{I\left[3\lambda^4(1-\lambda^8)\right]+J\left[2\lambda^4(1-\lambda^2)-(1+\lambda^2)(1-\lambda^8)\right] \\
&+ J\left[2\lambda^4(1-\lambda^4)+(1-2\nu)\lambda^2(1-\lambda^8)\right]\} \\
c_3 &= b_3+2d_3+\frac{1}{8}(I+J+K) \\
a_3 &= 4b_3-d_3-\frac{1}{4}[I+J-(1-2\nu)K] \\
e_3 &= \frac{\lambda}{3(1-\lambda^6)}[-\lambda^5(1-\lambda)h_1+(1-\lambda^5)h_2+\lambda^4(1-\lambda)h_3+\lambda^2(1-\lambda^3)h_4] \\
f_3 &= -\frac{\lambda}{3(1-\lambda^6)}[(1-\lambda^7)h_1+(1-\lambda)h_2+(1-\lambda^5)h_3+(1-\lambda^3)h_4] \\
h &= \frac{(1+\nu)d_o}{4(1-\lambda^2)^3} \qquad (4.19) \\
h_1 &= h\left[\frac{1+4\nu}{3}(1-\lambda^6)+\frac{43-56\nu-32\nu^2+16\nu^3}{12(1+\nu)(1-2\nu)}(1-\lambda^2)^3\right] \\
h_2 &= h\lambda^4(1-\lambda^4) \\
h_3 &= h\left[-(1+\lambda^2)(1-\lambda^6)+\frac{3+2\nu}{4(1+\nu)}(1+\lambda^2)(1-\lambda^2)^3+\frac{2a_1}{(1+\nu)d_o}(1-\lambda^2)^3\right] \\
h_4 &= h\left[-4(1-\nu)\lambda^2(1-\lambda^6)+\frac{3+2\nu}{4(1+\nu)}\lambda^2(1-\lambda^2)^3\right] \\
g &= 8\lambda^4(1-\lambda^2)-(1+\lambda^2)(1-\lambda^8)
\end{aligned}$$

$$
\begin{aligned}
g_1 &= h\left[\frac{13-8\nu}{12}(1-\lambda^6)-\frac{11-16\nu-25\nu^2+8\nu^3}{12(1+\nu)(1-2\nu)}(1-\lambda^2)^3\right]\\
g_2 &= -\frac{1}{2}h\lambda^4(1-\lambda^4)\\
g_3 &= h\left[-\frac{1}{2}(1+\lambda^2)(1-\lambda^6)+\frac{3+2\nu}{8(1+\nu)}(1+\lambda^2)(1-\lambda^2)^3+\frac{a_1}{(1+\nu)d_o}(1-\lambda_2)^3\right]\\
g_4 &= h\left[\frac{1}{4}(1-\lambda^6)-(1-2\nu)\lambda^2(1-\lambda^6)+\frac{3+2\nu}{8(1+\nu)}(1+\lambda^2)(1-\lambda^2)^3\right]\\
I &= h\left[-\frac{1-2\nu}{3}(1-\lambda^6)+\frac{3-\nu}{3}(1-\lambda^2)^3\right]\\
J &= h\lambda^4(1-\lambda^4)\\
K &= h\lambda^2(1-\lambda^6)
\end{aligned}
$$

Integrating the stress components up to the third order over the two end surfaces, one finds that the force and moment resultants are all zero, except M_y on the surface $\theta = 0°$, and M_y on the surface $\theta = 90°$. These two non-zero resultants can be expressed as

$$
\begin{aligned}
\theta = 0°;\ M_y &= -\frac{1+\nu}{4}\epsilon\pi d_o(1-\lambda^4)\\
\theta = 90°;\ M_y &= -\frac{1}{4}\epsilon\pi d_o(1-\lambda^4)-\epsilon^3\frac{\pi d_o}{96}[3(3+2\nu)(1+\lambda^2)(1-\lambda^4) \qquad (4.20)\\
&+ 6(3+2\nu)\lambda^2(1-\lambda^2)+2(7+2\nu)(1-\lambda^6)-24\frac{a_1}{d_o}(1-\lambda^4)]
\end{aligned}
$$

By equating these two resultants to the applied twist moment T_o and solving these two equations, the two constants d_o and a_1 are determined.

Substituting the solutions for the various orders into eqns. (3.22. 3.23) yields total solution for the problem is obtained.

4.4 Numerical Results

Numerical results are presented for the following values of the geometric and material parameters; $b/a = 2$, $R/a = 5$, $\phi = 90°$, $\nu = 0.3$ $c = b$, and $E = 0.206 \times 10^6 MPa$. The FORTRAN code is given in the appendix program 1. The numerical results obtained using the present method (labelled DBTE), are compared with results from the stress-based TE [39] (labelled SBTE), and results from the BEM [71].

The radius of convergence of a series is defined as the limit value of u_{k+1}/u_k when k tends to infinity. It can be proved that the radius of convergence of these three series of displacement components is larger than one. The small parameter ϵ is the ratio of out-side radius of the cross-section to the mean toroidal radius. It is smaller than the radius of convergence, so the solution of the successive approximation converges. The stress factors for each stress component are defined as

$$\sigma_f = \sigma / \frac{T_o b}{I_p}$$

where σ is the actual stress, and I_p is the polar second moment of area determined as $\pi(b^4 - a^4)/2$. The maximum stress factors of different orders on the shell surfaces around $\theta = 45°$ are presented in table 4.1. It is obvious that the first order term gives a much larger contribution than the higher order terms.

A total solution up to and including the third order is represented in the following figures. The deformation of the elbow cross-section at $\theta = 45°$ is shown

Table 4.1: The contributions from different orders

	$0th-order$	$1st-order$	$2nd-order$	$3rd-order$
$\sigma_\phi \mid \xi = 0.5$	0.0	0.00000	1.13147	0.17771
$\sigma_\theta \mid \xi = 0.5$	0.0	0.70717	0.24751	0.05965
$\sigma_{\phi\theta} \mid \xi = 0.5$	0.0	0.27199	0.14687	0.05275
$\sigma_\phi \mid \xi = 1.0$	0.0	0.00000	0.42430	0.06501
$\sigma_\theta \mid \xi = 1.0$	0.0	1.41433	0.31183	0.10703
$\sigma_{\phi\theta} \mid \xi = 1.0$	0.0	0.54397	0.14687	0.12160

in Fig. 4.3. The cross-section moves upward and outward under a negative twist moment. This deformation can be decomposed into rotation, 'rigid body' motion, and expansion of the cross-section. Applying the out-of-plane twist moment causes a point p on the outside surface of the elbow to deform to a point p'. It is evident that the rotation of the cross-section is the largest part of the deformation.

The circumferential, longitudinal and shear stress factors on the surface of $\theta = 45°$ up to and including the third order are given in Figs. 4.4, 4.5, and 4.6. These results are compared with the results obtained using the stress based TE and the BEM. The circumferential stress factor σ_ϕ is largest on the inside surface (Fig. 4.4), near $\phi = 45°$, 135°, 225° and 315°. It indicates that the circular cross-section of the elbow deformes to an elliptical cross-section. The largest longitudinal stress factor σ_θ on the outside surface is located at $\phi = 112.5°$ and 247.5° (Fig. 4.5). This is different from a straight pipe under a bending moment for which the maximum stresses are located at the top and bottom points of the pipe cross-section. The shear stress in the circumferential direction is not uniform for a curved tube. The maximum and minimum values of the shear stress factor $\sigma_{\phi\theta}$ are located on the line $\phi = 180°$. The maximum value is obtained on the outside surface, and the minimum value on the inside surface (Fig. 4.6). The values obtained using the displacement

based TE differ slightly from the value obtained using the stress based TE. There is generally good agreement between the TE and BEM results.

The stress components on the inner and outer surfaces at different cross-section of θ are shown in the Figs. 4.7 - 4.13 (The boundary conditions on cross-sections $\theta = 0°, 90°$ are satisfied globally, and boundary layer solution is not considered in these figures. The stress components, $\sigma_{\phi\phi}, \sigma_{\theta\theta}$, and $\sigma_{\phi\theta}$, are presented by letters, SQ, ST, and SQT). On the $\theta = 0°$ cross-section, the shear stress component is zero, and the stress components $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ reach their maximum values. On the $\theta = 90°$ cross-section, there is only one non-zero stress component ($\sigma_{\phi\theta}$) . The variation of the stress components at $\phi = 45°, 90°, 135°$ along the elbow length (θ direction) on the elbow surface are presented in Figs. 4.14 - 4.16. As θ increases, the normal stress components $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ reduce in value, and become zero at the $\theta = 90°$ cross-section. The shear stress component, $\sigma_{\phi\theta}$, reaches a maximum value at the $\theta = 90°$ cross-section and becomes zero at the $\theta = 0°$ cross-section.

Figs. 4.17 - 4.22 present $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ and $\sigma_{\phi\theta}$ at $\theta = 45°$ with different values of parameter ϵ and the thickness ratio λ. The parameter ϵ varies from 0.2 to 0.4, and the thickness ratio, λ, takes the value 0.5 to 0.8. The maximum stress factor of the normal stress components $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ with different values of ϵ and λ are presented in Figs. 4.23 - 4.24. For elbows with the same cross-section, the larger the mean toroidal radius is, the smaller the maximum stress factors are, and the solution of the elbow is closer to the solution of a cylindrical shell. For same dimension elbows, the thin shell elbow gives larger stress factors. There is a significant difference between the solution of toroidal shell and cylindrical shell for thin shell problem.

The large stresses appear on the fixed end and on the inside surface of the elbow. The maximum longitudinal stress factor is located around $\phi = 135°$, and $225°$ on the outside surface, and the maximum circumferential stress factors are located around $\phi = 45°, 153°, 225°, 315°$.

4.5 Conclusions

A general series solution has been presented for the out-of-plane loading case in the displacement-based TE. An example concerning a 90° elbow subjected to an out-of-plane end couple was considered in detail. The development of the solution was up to the third order. Numerical results for displacements obtained using this method serve to indicated clearly the deformation pattern that occurs. Numerical values for stresses obtained using this procedure were compared with values obtained using the stress-based TE and the BEM. Excellent agreement was obtained with the stress-based TE values, and good agreement with the BEM values.

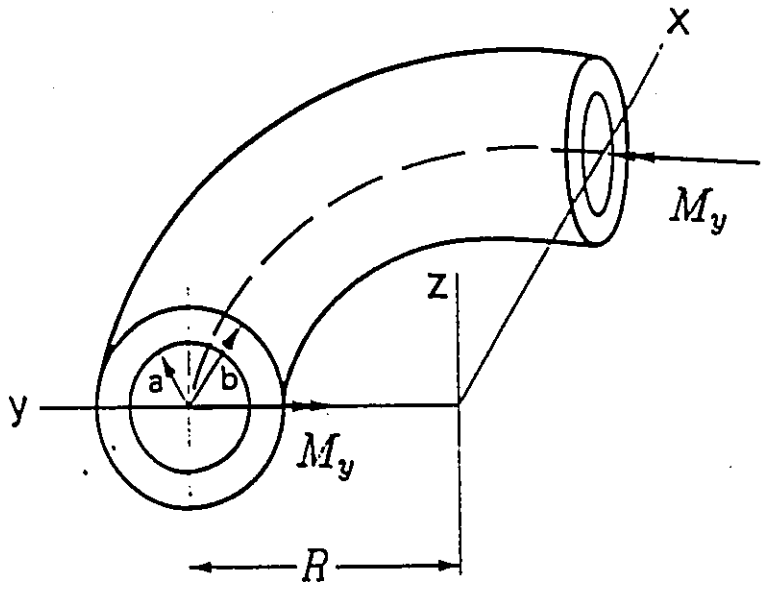

Figure 4.1: 90° elbow and applied loads

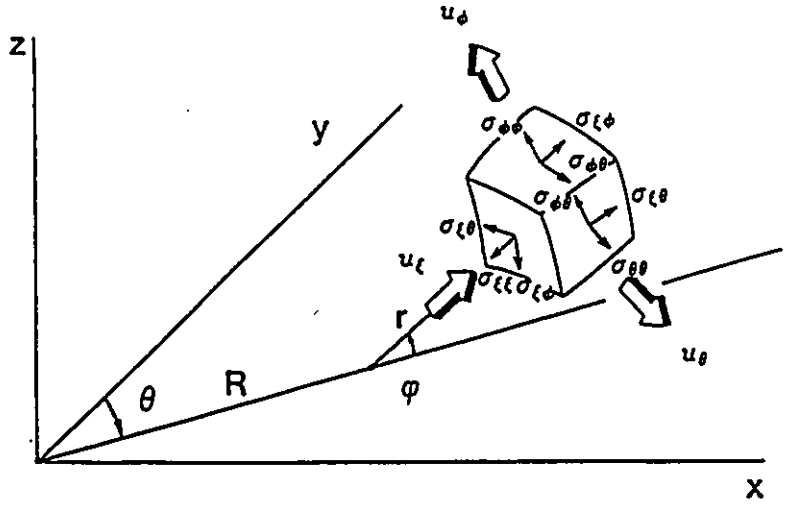

Figure 4.2: Displacement and stress components

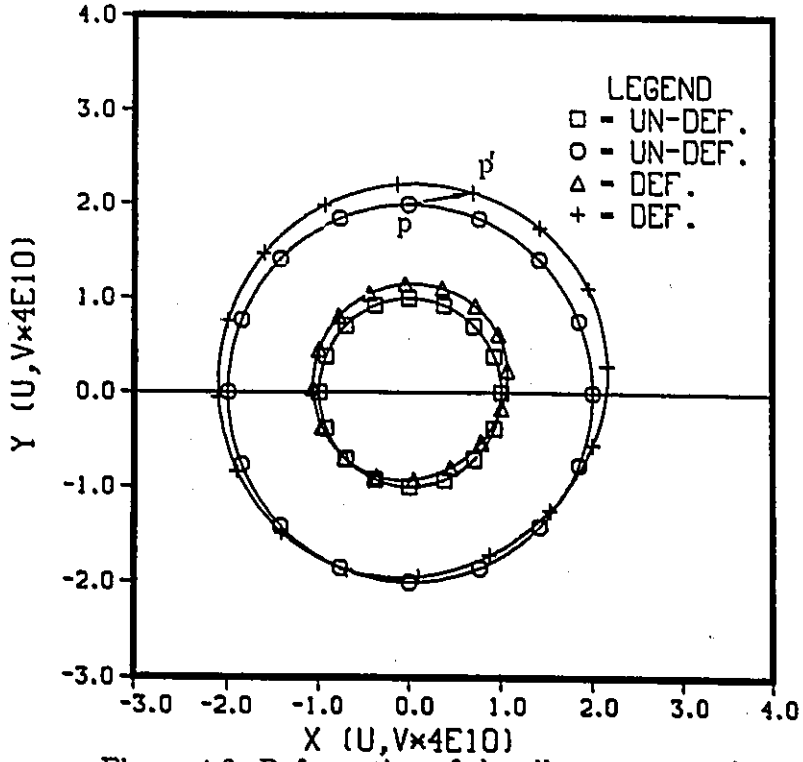

Figure 4.3: Deformation of the elbow cross-section

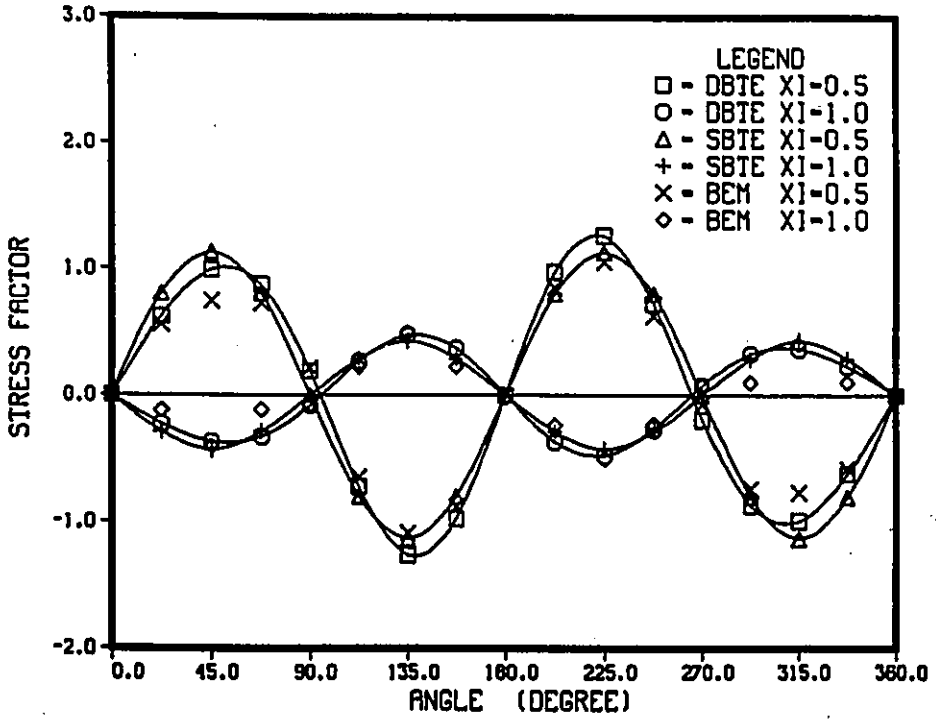

Figure 4.4: Comparison of circumferential stress factor on $\theta = 45°$ surface

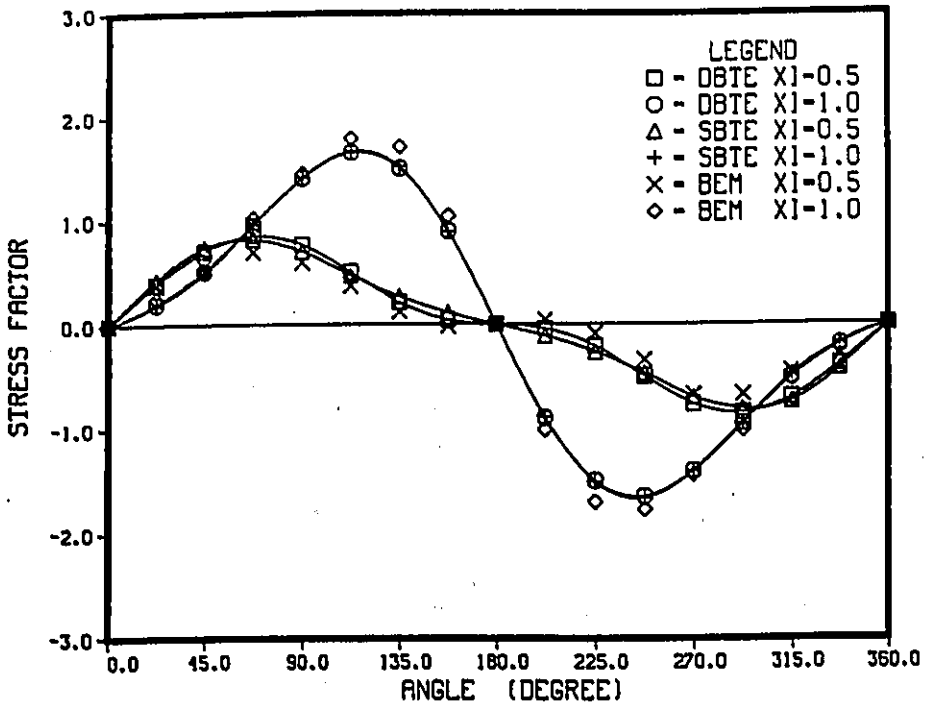

Figure 4.5: Comparison of longitudinal stress factor on $\theta = 45°$ surface

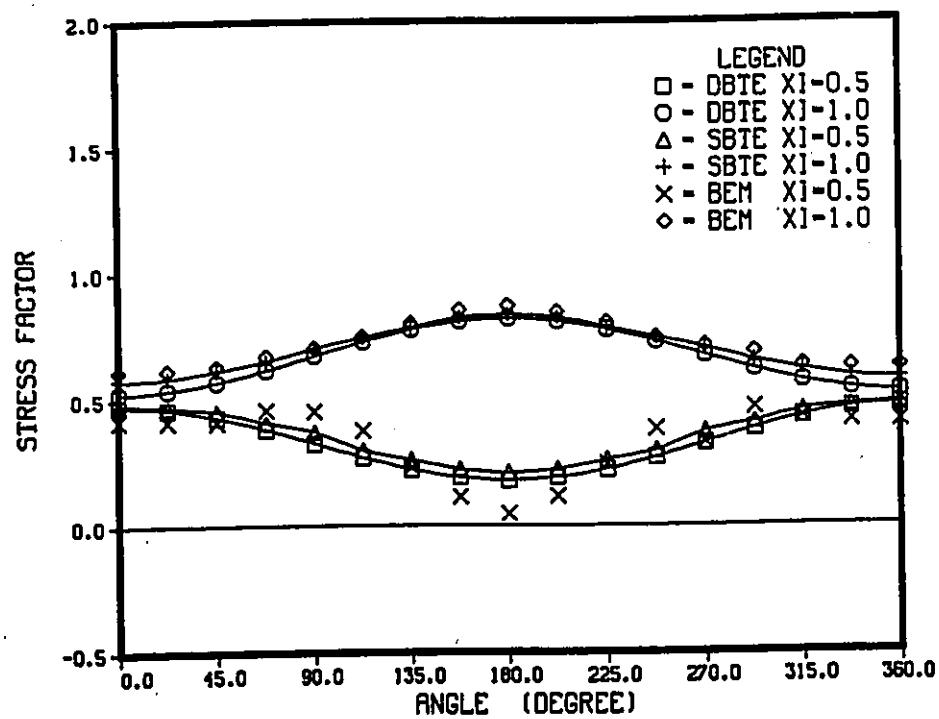

Figure 4.6: Comparison of shear stress factor on $\theta = 45°$ surface

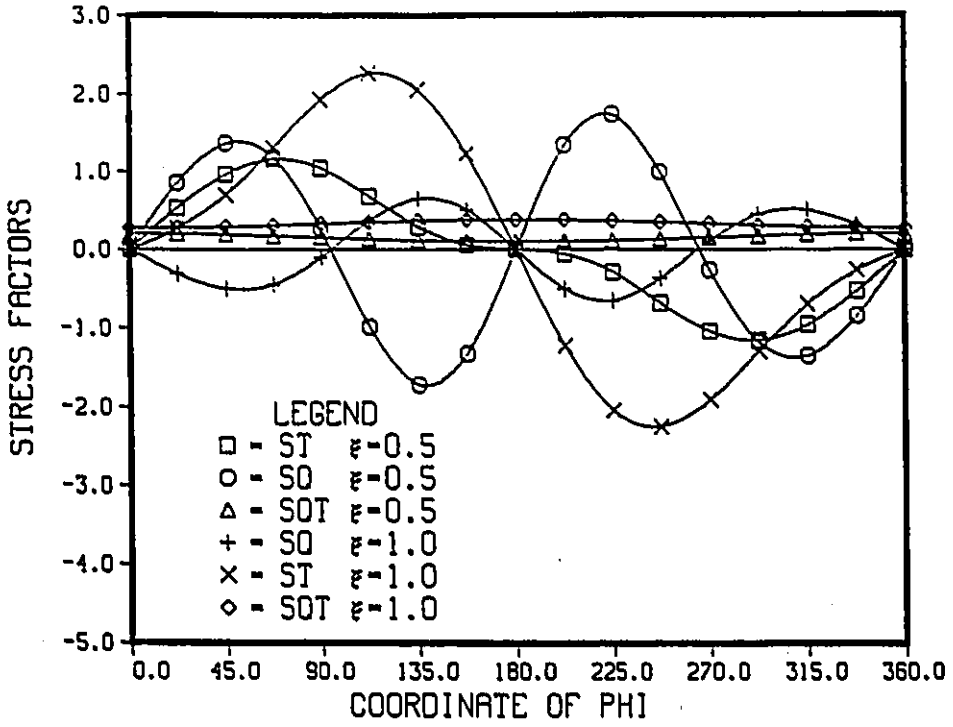

Figure 4.7: Stress factors for θ of 15 degree

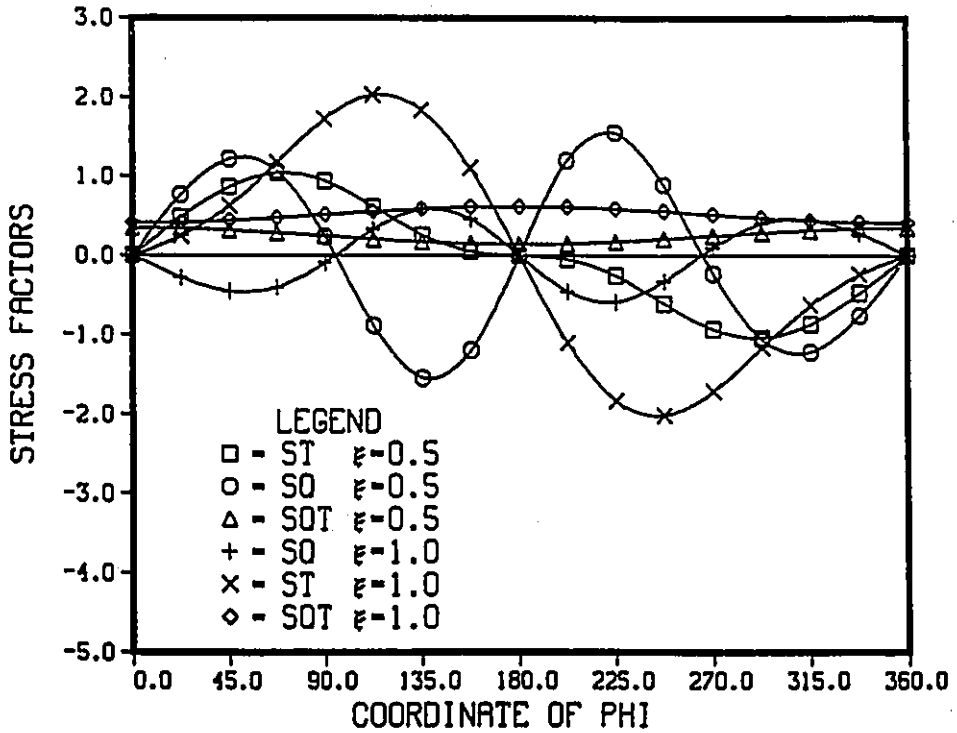

Figure 4.8: Stress factors for θ of 30 degree

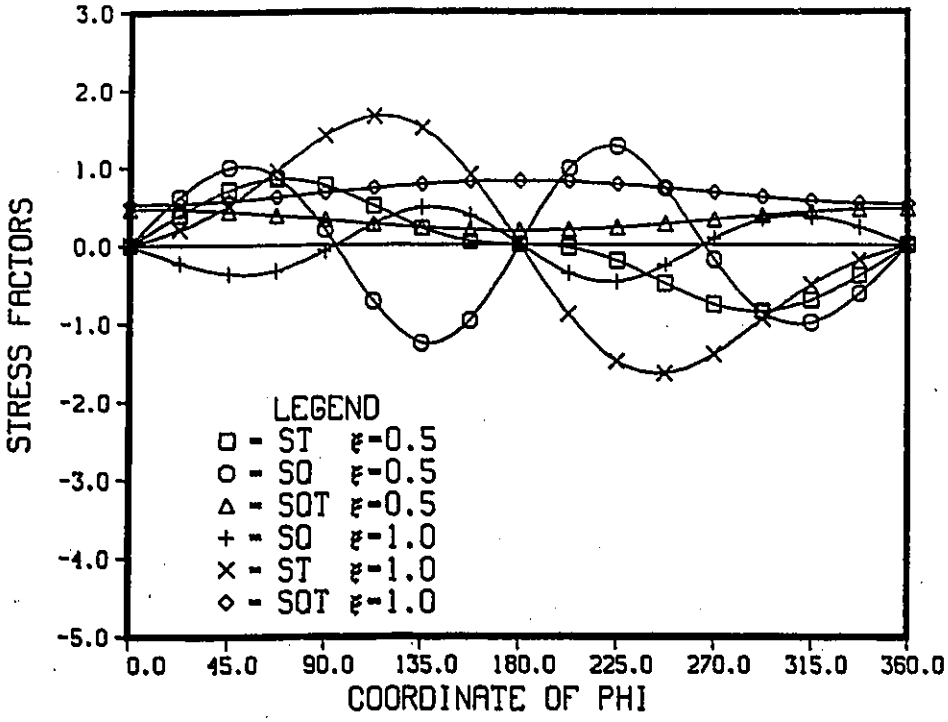

Figure 4.9: Stress factors for θ of 45 degree

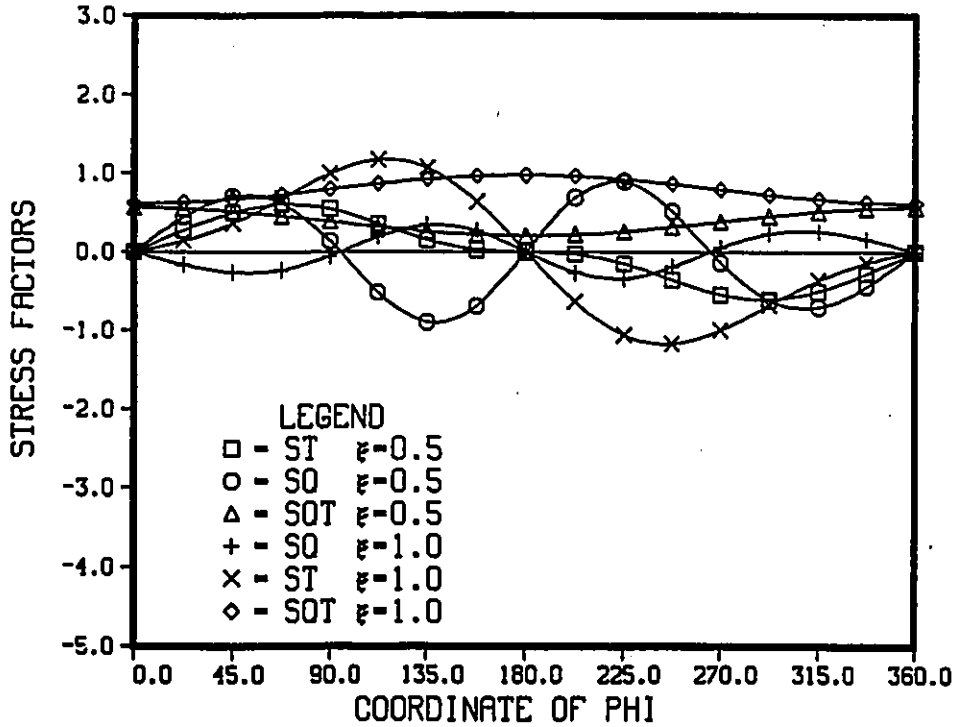

Figure 4.10: Stress factors for θ of 60 degree

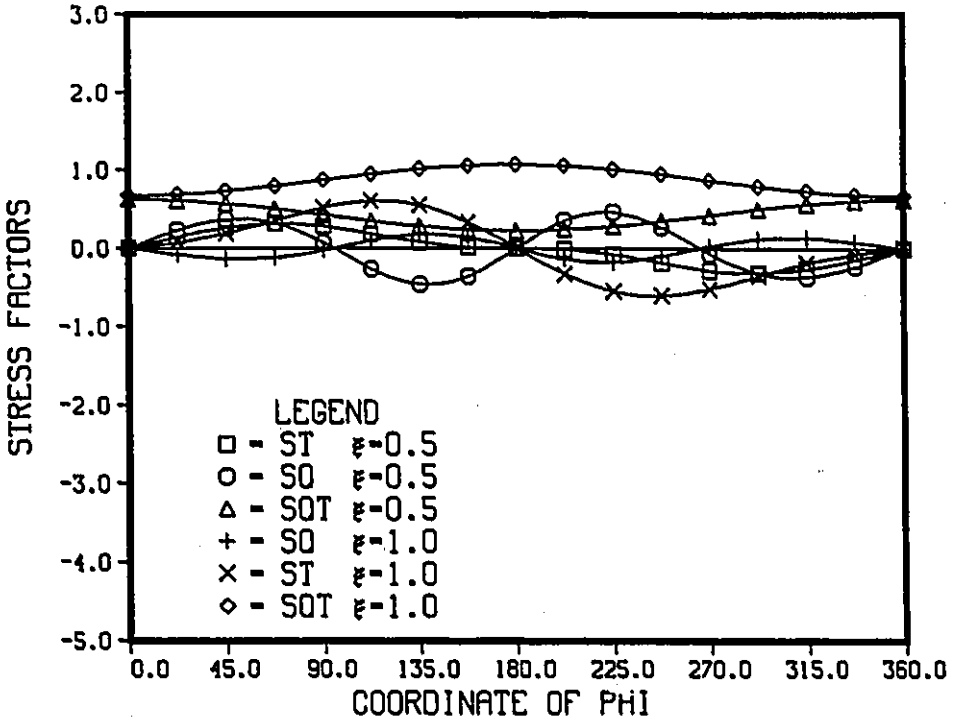

Figure 4.11: Stress factors for θ of 75 degree

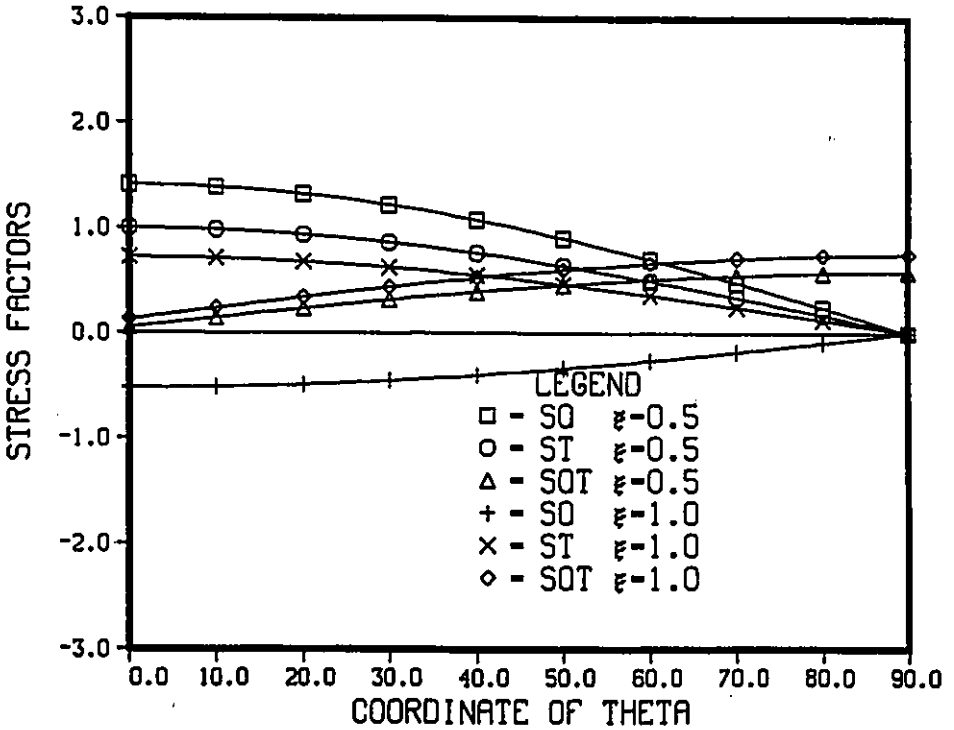

Figure 4.12: Stress factors for ϕ of 45 degree

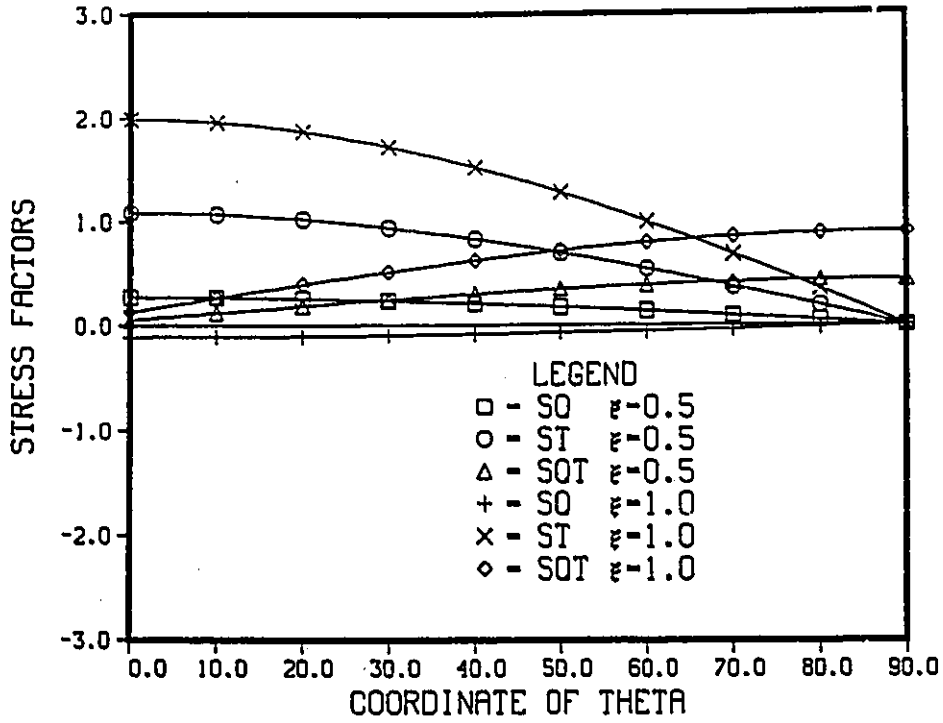

Figure 4.13: Stress factors for ϕ of 90 degree

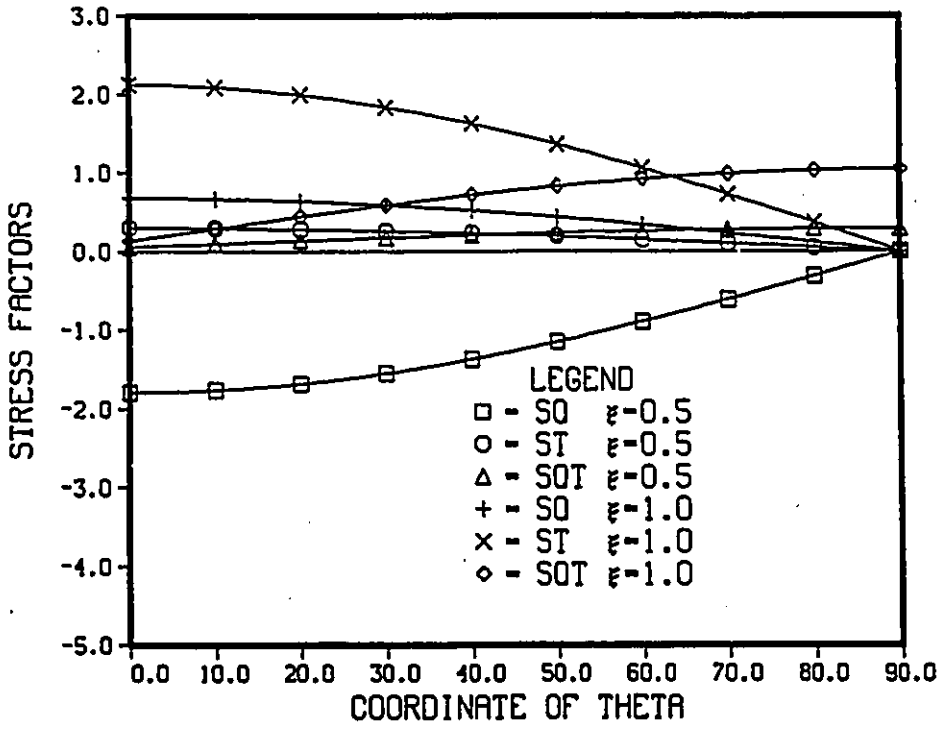

Figure 4.14: Stress factors for ϕ of 135 degree

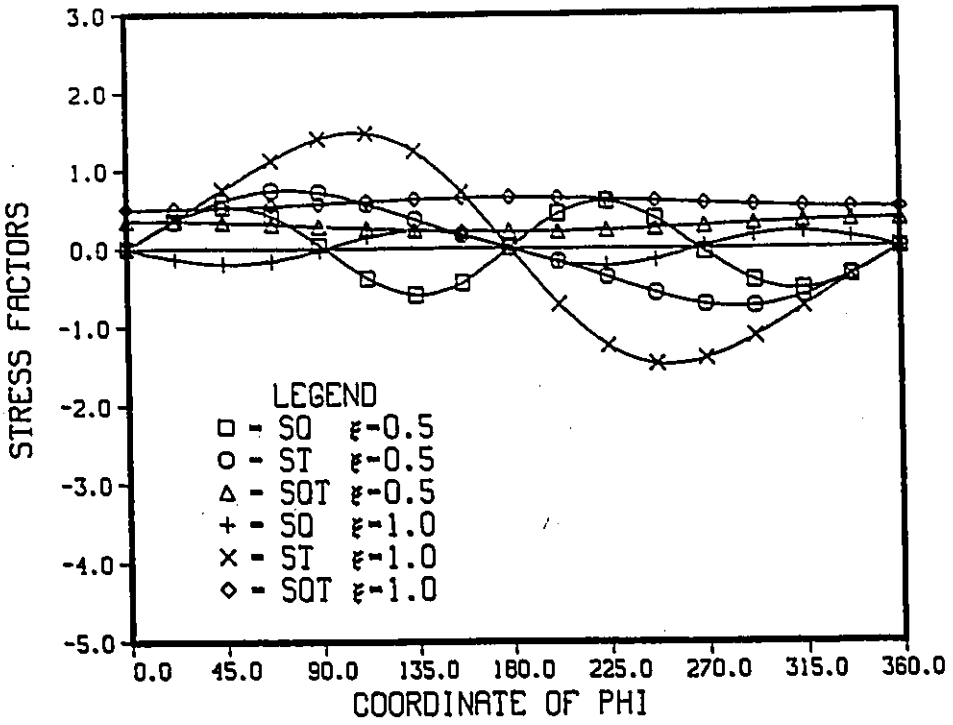

Figure 4.15: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.2$

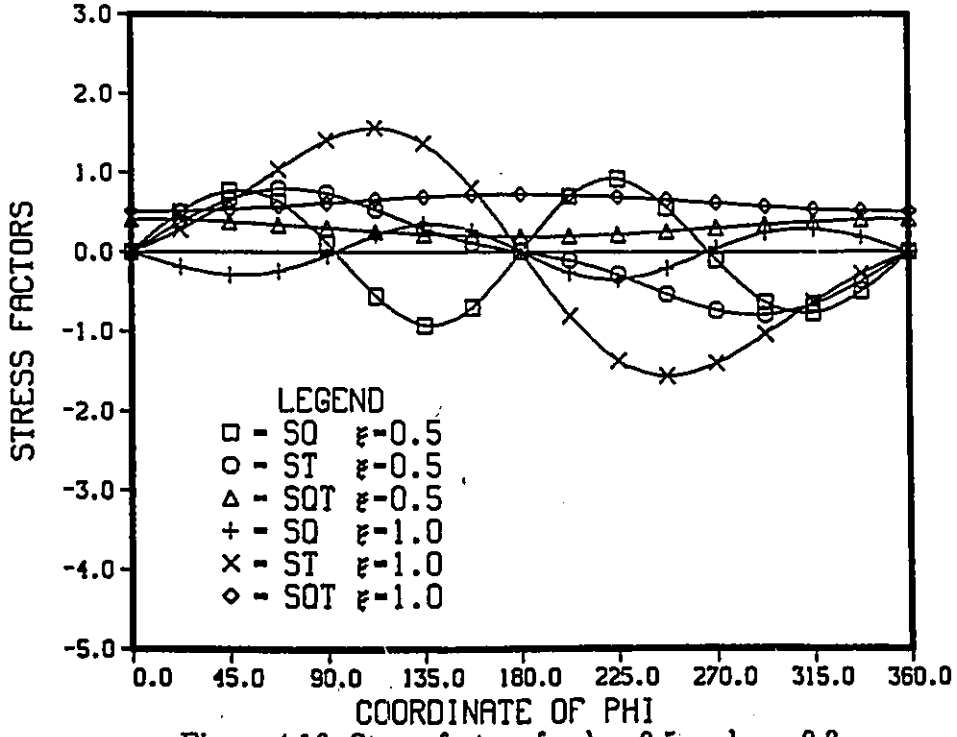

Figure 4.16: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.3$

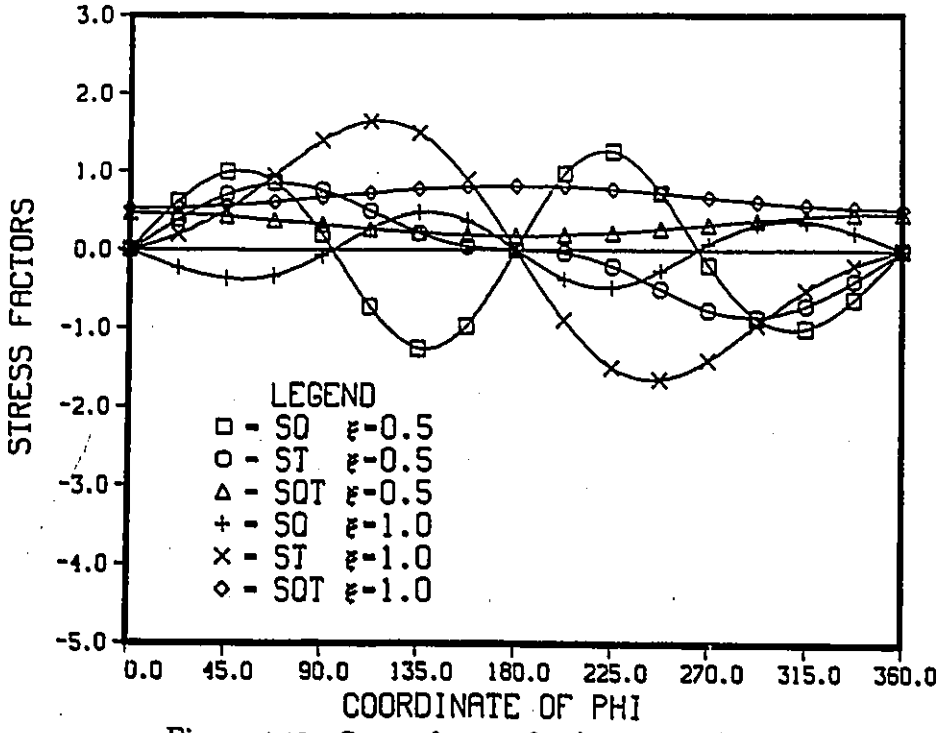

Figure 4.17: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.4$

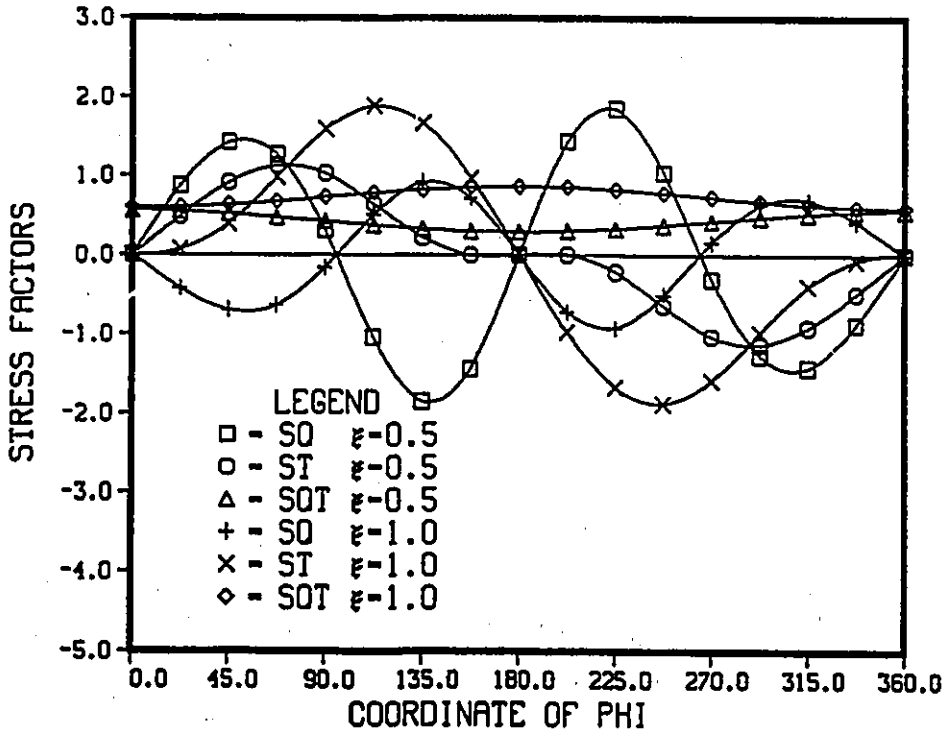

Figure 4.18: Stress factors for $\lambda = 0.6$ and $\epsilon = 0.4$

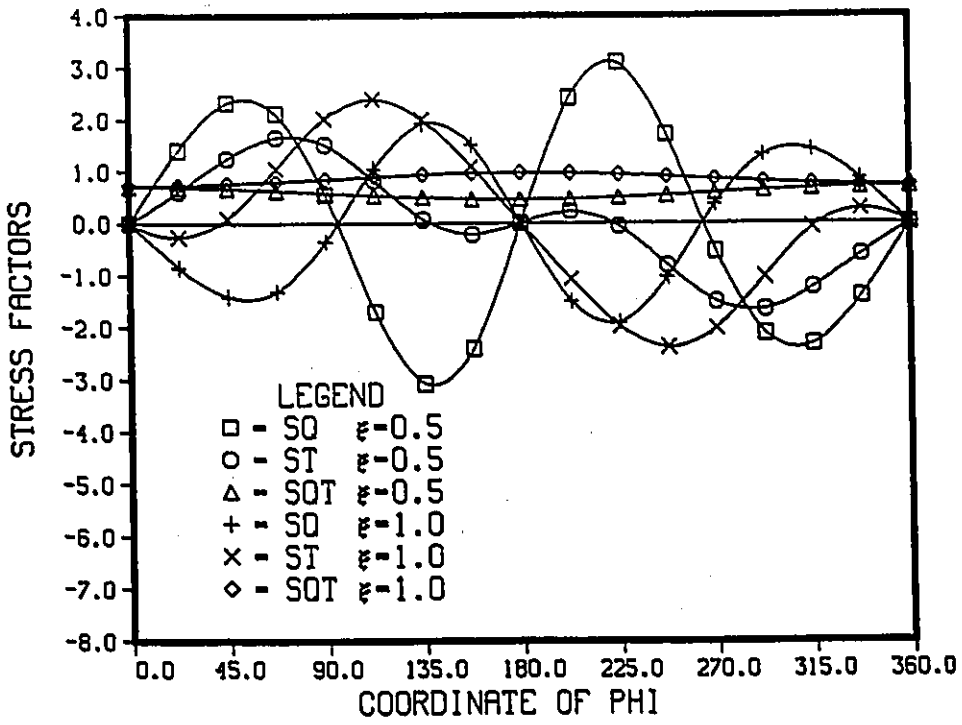

Figure 4.19: Stress factors for $\lambda = 0.7$ and $\epsilon = 0.4$

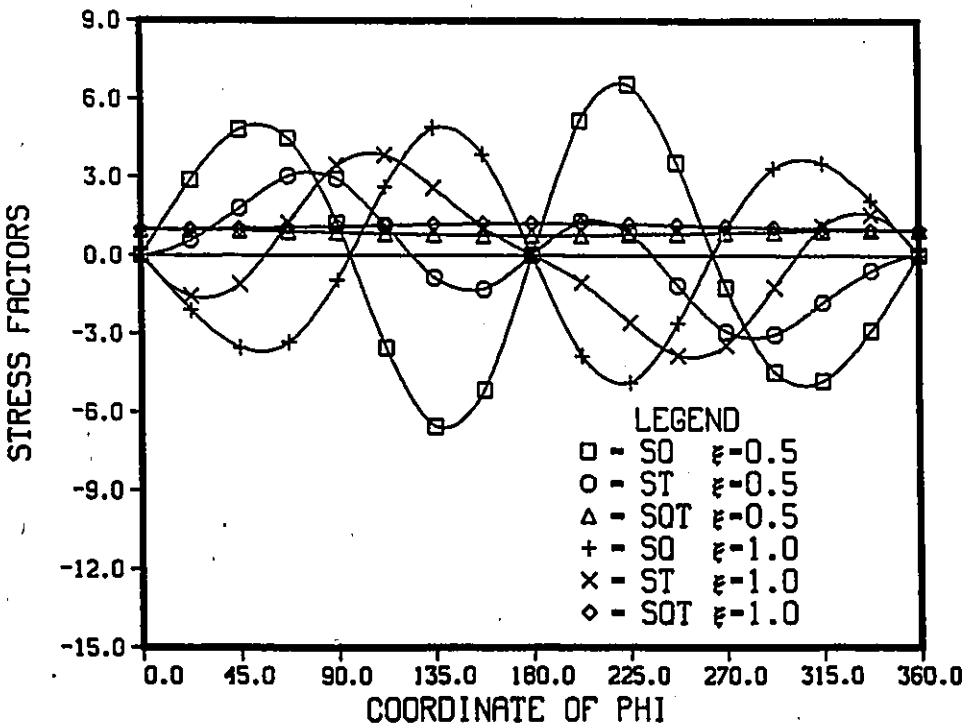

Figure 4.20: Stress factors for $\lambda = 0.8$ and $\epsilon = 0.4$

Chapter 5

AXISYMMETRIC SOLUTION FOR A TOROIDAL RING

5.1 Introduction

In pressure vessel engineering, torispherical heads often provide the best means of closing cylindrical vessels [63]. Such heads contain a toroidal segment (toroidal ring or 'knuckle') inserted between the cylindrical and spherical components. Theoretical solutions for such vessels can provide insight not available from purely numerical solutions. In carrying out theoretical solutions, homogeneous eigenfunctions are required for each of the components. The theory of eigenfunctions for cylindrical [47, 49] and spherical [53, 56] components has been available in the literature for some time. Recent developments in toroidal elasticity now permit setting up the eigenfunctions for toroidal components.

Toroidal elasticity is a three-dimensional theory used for the elastostatic analysis of thick-walled curved pipes, complete toroidal shells, and toroidal rings. Exten-

sive work on the stress based TE has been carried out by Lang [1]. The displacement based toroidal elasticity (DBTE) theory has been developed by Redekop [2]. The latter approach has been used to solve several three-dimensional problems [67, 68]. Preliminary work on toroidal eigenfunctions was reported in ref. [69], but a full solution has not been presented to date. Some related work has also been done by Zak [70], dealing with solid toroidal geometries.

In this chapter the DBTE theory is used to set up homogeneous eigenfunctions for an axisymmetric toroidal ring. The theory presented complements similar theory given previously for cylindrical and spherical geometries [47, 49, 53, 56]. The governing elastostatic equations are solved approximately by the method of successive approximations (MSA). A new method is introduced which includes adjusting the zeroth order solution to produce a faster convergence in the MSA approach. Contributions up to and including the second order term in the MSA are included. Eigenfunctions are developed which satisfy homogenous boundary conditions for the stresses on the inner and outer curved surfaces of the ring. Comparisons are made between the eigenvalues in the toroidal, cylindrical and spherical geometries. Finally, numerical results are computed for the case of pure bending of the edge surfaces of the ring, and comparisons are made with results obtained from the finite element method.

5.2 Geometry and Loading

An axisymmetric toroidal ring is shown in Fig. 5.1. The toroidal coordinates of a typical point in the cross-section are r, ϕ, where r is the radial distance from the toroidal center line, and ϕ is the vertical angle in the toroidal cross-section. The mean toroidal radius is represented by R, while the radii to the inner and outer curved surfaces by a and b. A non-dimensional radial coordinate, ξ, is defined through $\xi = r/c$ where c is a characteristic cross-sectional dimension, which will be specified as b. The ring extends between the edge surfaces $\phi = 0$ and $\phi = \phi_o$.

For the axisymmetric problem, there is no deformation in the circumferential direction thus there are only two displacement components, u_ξ, u_ϕ, and four stress components $\sigma_{\xi\xi}, \sigma_{\phi\phi}$, $\sigma_{\theta\theta}$ and $\sigma_{\xi\phi}$ Fig. 4.2. In the development of homogeneous eigenfunctions, the curved surfaces of the ring are assumed stress free, i. e.

$$\sigma_{\xi\xi} = \sigma_{\xi\phi} = 0 \; on \; r = a, b \tag{5.1}$$

On the edge surfaces of the ring, the stresses are specified as follows:
on the surface $\phi = 0^o$

$$\sigma_{\phi\phi} = F_1^s(\xi); \; \sigma_{\xi\phi} = G_1^s(\xi) \tag{5.2}$$

on the surface $\phi = \phi_o$

$$\sigma_{\phi\phi} = F_2^s(\xi); \; \sigma_{\xi\phi} = G_2^s(\xi) \tag{5.3}$$

The functions F_1^s, G_1^s, F_2^s and G_2^s correspond to the applied loads on the edge surfaces.

5.3 Development of Governing Equations

Non-dimensional displacements and stresses are defined as

$$\begin{aligned} u &= \frac{u_\xi}{c};\ v = \frac{u_\phi}{c} \\ \tau_{\xi\xi} &= \frac{1+\nu}{E}\sigma_{\xi\xi};\ \tau_{\phi\phi} = \frac{1+\nu}{E}\sigma_{\phi\phi} \\ \tau_{\theta\theta} &= \frac{1+\nu}{E}\sigma_{\theta\theta};\ \tau_{\xi\phi} = \frac{1+\nu}{E}\sigma_{\xi\phi} \end{aligned} \tag{5.4}$$

The governing Navier equations [2] for the axisymmetric problem in toroidal coordinates take the form

$$\begin{aligned} U &= -\frac{c}{\rho}\bar{U} - \left(\frac{c}{\rho}\right)^2 \hat{U} \\ V &= -\frac{c}{\rho}\bar{V} - \left(\frac{c}{\rho}\right)^2 \hat{V} \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} U &= \nabla^2 u - \frac{u}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi}{\partial \xi} \\ \bar{U} &= \cos\phi\frac{\partial u}{\partial \xi} - \sin\phi\left(\frac{1}{\xi}\frac{\partial u}{\partial \phi} - \frac{v}{\xi}\right) + \frac{1}{1-2\nu}\frac{\partial \chi}{\partial \xi} \\ \hat{U} &= -\frac{2(1-\nu)}{1-2\nu}\cos\phi\chi \\ V &= \nabla^2 v - \frac{v}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi}{\partial \phi} \\ \bar{V} &= \cos\phi\frac{\partial v}{\partial \xi} - \sin\phi\left(\frac{u}{\xi} + \frac{1}{\xi}\frac{\partial v}{\partial \phi}\right) + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \chi}{\partial \phi} \\ \hat{V} &= \frac{2(1-\nu)}{1-2\nu}\sin\phi\chi \end{aligned} \tag{5.6}$$

and

$$\nabla^2(\ldots) = \frac{\partial^2(\ldots)}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial(\ldots)}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial^2(\ldots)}{\partial \phi^2}$$

$$\psi = \frac{\partial u}{\partial \xi} + \frac{u}{\xi} + \frac{1}{\xi}\frac{\partial v}{\partial \phi} \tag{5.7}$$
$$\chi = u\cos\phi - v\sin\phi$$

For the MSA [2], expansions are written in terms of a small parameter ϵ, where $\epsilon = c/R$. Thus the displacement components are expressed as

$$u = u_o + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \ldots$$
$$v = v_o + \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \ldots \tag{5.8}$$

the stress components as

$$\tau_{\xi\xi} = \tau_{\xi\xi o} + \epsilon\tau_{\xi\xi 1} + \epsilon^2\tau_{\xi\xi 2} + \epsilon^3\tau_{\xi\xi 3} + \ldots$$
$$\tau_{\phi\phi} = \tau_{\phi\phi o} + \epsilon\tau_{\phi\phi 1} + \epsilon^2\tau_{\phi\phi 2} + \epsilon^3\tau_{\phi\phi 3} + \ldots$$
$$\tau_{\theta\theta} = \tau_{\theta\theta o} + \epsilon\tau_{\theta\theta 1} + \epsilon^2\tau_{\theta\theta 2} + \epsilon^3\tau_{\theta\theta 3} + \ldots \tag{5.9}$$
$$\tau_{\xi\phi} = \tau_{\xi\phi o} + \epsilon\tau_{\xi\phi 1} + \epsilon^2\tau_{\xi\phi 2} + \epsilon^3\tau_{\xi\phi 3} + \ldots$$

and the ratios c/ρ, $(c/\rho)^2$ as

$$\frac{c}{\rho} = \epsilon - \epsilon^2\xi\cos\phi + \epsilon^3\xi^2\cos^2\phi - \ldots$$
$$(\frac{c}{\rho})^2 = \epsilon^2 - 2\epsilon^3\xi\cos\phi + 3\epsilon^4\xi^2\cos^2\phi - \ldots \tag{5.10}$$

The governing Navier equations for the axisymmetric toroidal ring may then be expressed for the zeroth order as

$$U_o = 0;\ V_o = 0 \tag{5.11}$$

for the first order as

$$U_1 = -\bar{U}_o;\ V_1 = -\bar{V}_o \tag{5.12}$$

for the second order as

$$U_2 = -\bar{U}_1 + \xi \cos\phi \bar{U}_o - \hat{U}_o$$
$$V_2 = -\bar{V}_1 + \xi \cos\phi \bar{V}_o - \hat{V}_o \tag{5.13}$$

and finally for the third order as

$$U_3 = -\bar{U}_2 + \xi \cos\phi \bar{U}_1 - \xi^2 \cos^2\phi \bar{U}_o - \hat{U}_1 + 2\xi \cos\phi \hat{U}_o$$
$$V_3 = -\bar{V}_2 + \xi \cos\phi \bar{V}_1 - \xi^2 \cos^2\phi \bar{V}_o - \hat{V}_1 + 2\xi \cos\phi \hat{V}_o \tag{5.14}$$

The kth order functions of the Navier equations are given by

$$\begin{aligned}
U_k &= \nabla^2 u_k - \frac{u_k}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_k}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_k}{\partial \xi} \\
\bar{U}_k &= \cos\phi \frac{\partial u_k}{\partial \xi} - \sin\phi \left(\frac{1}{\xi}\frac{\partial u_k}{\partial \phi} - \frac{v_k}{\xi}\right) + \frac{1}{1-2\nu}\frac{\partial \chi_k}{\partial \xi} \\
\hat{U}_k &= -\frac{2(1-\nu)}{1-2\nu}\cos\phi \chi_k \\
V_k &= \nabla^2 v_k - \frac{v_k}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_k}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_k}{\partial \phi} \\
\bar{V}_k &= \cos\phi \frac{\partial v_k}{\partial \xi} - \sin\phi \left(\frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial \phi}\right) + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \chi_k}{\partial \phi} \\
\hat{V}_k &= \frac{2(1-\nu)}{1-2\nu}\sin\phi \chi_k
\end{aligned} \tag{5.15}$$

where

$$\psi_k = \frac{\partial u_k}{\partial \xi} + \frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial \phi};\ \chi_k = u_k \cos\phi - v_k \sin\phi \tag{5.16}$$

By applying the stress-displacement relations, the kth order stress components are given as

$$\tau_{\xi\xi k} = \frac{\partial u_k}{\partial \xi} + \frac{\nu}{1-2\nu}(\psi_k + \chi_{k-1} - \xi \cos\phi \chi_{k-2} + \xi^2 \cos^2\phi \chi_{k-3})$$

$$\tau_{\phi\phi k} = \left(\frac{u_k}{\xi} + \frac{1}{\xi}\frac{\partial v_k}{\partial \phi}\right) + \frac{\nu}{1-2\nu}(\psi_k + \chi_{k-1} - \xi\cos\phi\chi_{k-2} + \xi^2\cos^2\phi\chi_{k-3})$$
$$\tau_{\theta\theta k} = \frac{\nu}{1-2\nu}\psi_k + \frac{1-\nu}{1-2\nu}(\chi_{k-1} - \xi\cos\phi\chi_{k-2} + \xi^2\cos^2\phi\chi_{k-3}) \qquad (5.17)$$
$$\tau_{\xi\phi k} = \frac{1}{2}\left(\frac{1}{\xi}\frac{\partial u_k}{\partial \phi} + \frac{\partial v_k}{\partial \xi} - \frac{v_k}{\xi}\right)$$

In eqns. (5.15-5.17), k indicates the order of the solution ($k = 0, 1, 2, 3$). Displacement functions on the right hand side with negative subscripts are to be taken as zero.

5.4 Eigenvalue Solution

The solution for the zeroth-order displacement components is first sought in the form

$$u_o = A\xi^\gamma e^{\beta\phi} \qquad (5.18)$$
$$v_o = B\xi^\gamma e^{\beta\phi}$$

where $e = 2.718...$, and the quantities A, B, γ and β are constants and parameters which are to be determined from the governing equations and boundary conditions. Substituting eqn. (5.18) into the homogeneous part of the governing equations (5.11) yields the two homogeneous linear equations

$$[2(1-\nu)(\gamma^2-1) + (1-2\nu)\beta^2]A + [(\gamma - 3 + 4\nu)\beta]B = 0 \qquad (5.19)$$
$$[(\gamma + 3 - 4\nu)\beta]A + [(1-2\nu)(\gamma^2-1) + 2(1-\nu)\beta^2]B = 0$$

Table 5.1: The four sets of constants A, B

γ	A	B
$1+\beta i$	$-[2(1-2\nu)-\beta i]C_1$	$-[4(1-\nu)i-\beta]C_1$
$1-\beta i$	$-[2(1-2\nu)+\beta i]C_2$	$[4(1-\nu)i+\beta]C_2$
$-(1+\beta i)$	$-[4(1-\nu)+\beta i]C_3$	$-[4(1-\nu)i-\beta]C_3$
$-(1-\beta i)$	$-[4(1-\nu)-\beta i]C_4$	$[4(1-\nu)i+\beta]C_4$

For a non-trivial solution the determinant of these equations must be zero. The resulting characteristic equation may be written as

$$\gamma^4 + 2(\beta^2-1)\gamma^2 + (\beta^2+1)^2 = 0 \tag{5.20}$$

which relates γ to β. Solving this equation analytically, one obtains four complex roots $\gamma = \pm(1 \pm \beta i)$, where $i = \sqrt{-1}$. Substituting these roots into eqns. (5.19) separately yields four sets of non-trivial solutions for the constants A and B. It is convenient to express these in terms of a set of free complex constants C_i, $i = 1,2,3,4$. The full set of solutions for A and B is given in Table 5.1. The free constants C_i and the parameter β are to be found from boundary conditions.

Superposing the four sets of solutions, one obtains expressions for the displacements from (5.18) in complex form as

$$\begin{aligned} u_o &= \{[-2(1-2\nu)+\beta i]C_1\xi^{1+\beta i} - [2(1-2\nu)+\beta i]C_2\xi^{1-\beta i} \\ &- [4(1-\nu)+\beta i]C_3\xi^{-(1+\beta i)} - [4(1-\nu)-\beta i]C_4\xi^{-(1-\beta i)}\}e^{\beta\phi} \\ v_o &= \{[-4(1-\nu)i+\beta]C_1\xi^{1+\beta i} + [4(1-\nu)i+\beta]C_2\xi^{1-\beta i} \\ &+ [-4(1-\nu)i+\beta]C_3\xi^{-(1+\beta i)} + [4(1-\nu)i+\beta]C_4\xi^{-(1-\beta i)}\}e^{\beta\phi} \end{aligned} \tag{5.21}$$

where β, C_1, C_2, C_3 and C_4 are parameter and constants. The real and imaginary parts separately satisfy the governing equations. The stress components may then

be derived from eqns. (5.17). Enforcing the stress boundary conditions (5.1), one obtains the following four homogeneous linear equations for the determination of the parameter β and the set of constants C_i:

$$\begin{aligned}
&-(\beta^2+2+\beta i)\lambda^{\beta i}C_1-(\beta^2+2-\beta i)\lambda^{-\beta i}C_2+[-\beta^2+4(1-\nu)\\
+\;&(5-4\nu)\beta i]\lambda^{-(2+\beta i)}C_3+[-\beta^2+4(1-\nu)-(5-4\nu)\beta i]\lambda^{-(2-\beta i)}C_4=0\\
&-(\beta^2+2+\beta i)C_1-(\beta^2+2-\beta i)C_2+[-\beta^2+4(1-\nu)+(5-4\nu)\beta i]C_3\\
+\;&[-\beta^2+4(1-\nu)-(5-4\nu)\beta i]C_4=0\\
&(\beta+\beta^2 i)\lambda^{\beta i}C_1+(\beta-\beta^2 i)\lambda^{-\beta i}C_2-[(5-4\nu)\beta+(\beta^2-4+4\nu)i]\lambda^{-(2+\beta i)}C_3\\
-\;&[(5-4\nu)\beta-(\beta^2-4+4\nu)i]\lambda^{-(2-\beta i)}C_4=0\\
&(\beta+\beta^2 i)C_1+(\beta-\beta^2 i)C_2-[(5-4\nu)\beta+(\beta^2-4+4\nu)i]C_3\\
-\;&[(5-4\nu)\beta-(\beta^2-4+4\nu)i]C_4=0
\end{aligned} \tag{5.22}$$

where $\lambda = a/b$. For a nontrivial solution, the determinant of these equations must be zero. The characteristic equation may be written as

$$(1+\beta^2)^2[16(1-\nu)^2+\beta^2][4\sin^2(\beta\ln\lambda)-(\lambda-\lambda^{-1})^2\beta^2]=0 \tag{5.23}$$

This nonlinear equation may be solved using the complex form of the Muller method [65]. The roots to the characteristic equation (5.23) appear in four groups:

1) the single root $\beta = 0$ from last part,

2) the imaginary roots $\beta = \pm 4(1-\nu)i$ from the second part,

3) the triple imaginary roots $\beta = \pm i$ from $(1+\beta^2)^2$ and the last part,

4) the complex roots appearing in sets of four from the last part .

Calculating the stress components for roots from the first two groups, and requiring them to satisfy the stress boundary conditions (5.1), one finds that there are no

contributions to the displacement and stress components from these groups. The eigenfunctions of the roots of the third group are trigonometric functions. The eigenfunctions corresponding to each set of roots of the fourth group exhibit decay characteristic. For the fourth group, two of each set of four eigenfunctions exhibit decay characteristics from the end $\phi = 0°$, while the other two decay from the end $\phi = \phi_o$. The total solution for a perturbation problem thus contains the contributions from group three and group four. An initial solution using the eigenvalues indicated led to a very slowly converging series. Adjustments were subsequently made which eventually led to acceptable results.

The reason for the slow convergence of the initial approach is that the eigenfunctions corresponding to the third group eigenvectors are trigonometric functions. This kind of function has a very slow convergence rate. To simulate a linear function with satisfactory precision, a large number of terms of the trigonometric series is required. For example, sixty terms of the trigonometric series are taken for solving a thick-walled toroidal shell subjected to band loads in chapter 6. The MSA is an asymptotic method for the systematic analysis of the global behavior of the solutions to a set of differential equations. In practical applications, only the first few orders of the MSA solutions can be conveniently calculated, because the procedure becomes increasingly cumbersome as the order increases. It is impossible to obtain a satisfactory approximation in a few orders of the MSA approach using solely the trigonometric function.

The convergence rate of the MSA approach depends on the parameter ϵ and the functions selected for the zeroth order solution. A small parameter ϵ corresponds

to a fast convergence rate. The smaller the difference between the selected zeroth order solution and the exact solution, the faster the MSA approach converges. When the governing Navier equation is discretized into different order governing equations by the MSA, some changes are introduced into the system eigenvalues. Usually, there is a significant change for the first few eigenvalues. An accurate approximation to the exact solution may be obtained even from a few orders of the MSA approach, if the first few eigenvalues can be adjusted using some analytic information about the exact solution [72]. Considering this information, a new method is introduced which includes adjusting the zeroth order solution to produce a fast convergence rate in the MSA approach. In this new method, the third group of eigenvalues are adjusted based on the thin shell theory. The final solution is the sum of the adjusted third group of eigenfunctions and the fourth group of eigenfunctions.

The assumption that 'the lines which were normal to the middle surface before deformation will remain normal to the deformed middle surface during the process of deformation' [71] is made in most shell theories. This assumption implies that the displacement components are proportional to the distance from the mid-surface. Based on this assumption, the terms with ξ^{-1} in eqn. (5.21) may be expressed approximately by the first few terms of a Taylor expansion in $(1-\xi)$ as

$$\xi^{-1} = 1 + (1-\xi) + (1-\xi)^2 + (1-\xi)^3 + \ldots\ldots \tag{5.24}$$

The adjusted displacements are expressed as

$$\begin{aligned} u_o &= \{[-2(1-2\nu)+\alpha i]C_1\xi^{1+\alpha i} - [2(1-2\nu)+\alpha i]C_2\xi^{1-\alpha i} \\ &- [4(1-\nu)+\alpha i][1+(1-\xi)+(1-\xi)^2+\ldots]C_3\xi^{-\alpha i} \end{aligned}$$

$$
\begin{aligned}
&- [4(1-\nu)-\alpha i][1+(1-\xi)+(1-\xi)^2+\ldots]C_4\xi^{\alpha i}\}e^{\alpha\phi} \qquad (5.25)\\
v_o &= \{[-4(1-\nu)i+\alpha]C_1\xi^{1+\alpha i}+[4(1-\nu)i+\alpha]C_2\xi^{1-\alpha i}\\
&+ [-4(1-\nu)i+\alpha][1+(1-\xi)+(1-\xi)^2+\ldots]C_3\xi^{-\alpha i}\\
&+ [4(1-\nu)i+\alpha][1+(1-\xi)+(1-\xi)^2+\ldots]C_4\xi^{\alpha i}\}e^{\alpha\phi}
\end{aligned}
$$

where α is the adjusted eigenvalue. The more terms of eqn. (5.24) one takes, the smaller is the deviation introduced into the governing equation. The complexity of the approach increases with the number of terms one takes. In order to obtain an accurate approximation to the exact solution and retain reasonable simplicity of procedure, the first three terms of eqn. (5.24) are taken in this study. There is about a three percent error caused by this adjustment for the case of $\lambda = 0.5$. The error becomes smaller as the ratio of a/b increases. Following the same procedure used for developing the eqn. (5.23), one obtains the adjusted characteristic equation as

$$(1-\lambda)^2(3-\lambda)^2\alpha^2 - 4\sin^2(\alpha\ln\lambda) = 0 \qquad (5.26)$$

The adjusted third group eigenvalues are obtained by solving equation (5.26). These eigenvalues appear in a pair of repeat pure imaginary roots (α_1, α_2) and a pair of real roots (α_3, α_4). The first twenty four eigenvalues for the case $\nu = 0.3$ and $\lambda = a/b = 0.5$ are shown in Table 5.2. The first four eigenvalues are adjusted third group eigenvalues. Others are the fourth group eigenvalues obtained by solving equation (5.23).

For each eigenvalue, the corresponding eigenvector can be computed from boundary condition (5.1). As there are only three independent relations left, the

Table 5.2: The first twenty-four nontrivial eigenvalues

Number	*Real*	*Imag*	*Number*	*Real*	*Imag*
1	0.0000	1.0000	13	15.4449	4.5931
2	0.0000	−1.0000	14	−15.4449	−4.5931
3	2.7010	0.0000	15	15.4449	−4.5931
4	−2.7010	0.0000	16	−15.4449	4.5931
5	6.0541	3.3707	17	20.0455	4.9516
6	−6.0541	−3.3707	18	−20.0455	−4.9516
7	6.0541	−3.3707	19	20.0455	−4.9516
8	−6.0541	3.3707	20	−20.0455	4.9516
9	10.8029	4.1123	21	24.6252	5.2380
10	−10.8029	−4.1123	22	−24.6252	−5.2380
11	10.8029	−4.1123	23	24.6252	−5.2380
12	−10.8029	4.1123	24	−24.6252	5.2380

first three constants are solved in terms of the fourth, leaving one free constant. Upon substitution of these constants into the stress expressions (5.17), one obtains the homogeneous eigenfunctions, i. e. stresses automatically satisfying the boundary conditions (5.1). Free constants are present in the eigenfunctions for the satisfying of the boundary conditions on the end surfaces $\phi = 0, \phi_o$ of the ring.

5.5 Eigenfunctions for Toroidal Ring

Using the 'adjusted' procedure for the axisymmetric toroidal ring, three types of eigenvalues appear in the solution. For simplicity in notation they will all be represented by the symbol β_n. The first type of the eigenvalues is a repeat purely iamginary roots ($\beta_n = \pm i; n = 1,2$). The second type of eigenvalues is a pair of purely real roots ($\beta_n = \alpha_n; n = 3,4$). These two types of eigenvalues are obtained

from the adjusted third group eigenvalues. The third type of eigenvalues are obtained from the original fourth group ($\beta_n; n > 4$). They are complex and appear in sets of four. The eigenfunctions corresponding to these three types of roots are discussed in detail in the following.

5.5.1 Eigenfunctions for roots $\beta_n = \pm i (n = 1, 2)$

These roots are the first two of the four adjusted eigenvalues of the original third group. The eigenfunctions are purely real for these imaginary roots. There are only two independent eigenfunctions corresponding to these repeated eigenvalues. One is a function of $sin\phi$ for u and $cos\phi$ for v, while the other a function of $cos\phi$ for u and $sin\phi$ for v. The zeroth order solution is shown in eqn. (5.27). The terms a_{nk}, b_{nk}, c_{nk}, and d_{nk} are kth order free constants corresponding to the nth eigenvalue. The eigenfunctions consist of complementary and particular solutions of the governing equations, and are set up order by order in the MSA. The theory presented in chapter 4 can be applied to these roots. There is one free constant associated with each part.

$$\begin{aligned} u_{1o} + u_{2o} &= [a_{1o} + b_{1o}(1-4\nu)\xi^2 + c_{1o}\xi^{-2} + d_{1o}(3-4\nu)\ln\xi]\sin\phi \\ &+ [a_{2o} + b_{2o}(1-4\nu)\xi^2 + c_{2o}\xi^{-2} + d_{2o}(3-4\nu)\ln\xi]\cos\phi \qquad (5.27) \\ v_{1o} + v_{2o} &= [a_{1o} - b_{1o}(5-4\nu)\xi^2 - c_{1o}\xi^{-2} + d_{1o}(3-4\nu)\ln\xi + d_{1o}]\cos\phi \\ &+ [-a_{2o} + b_{2o}(5-4\nu)\xi^2 + c_{2o}\xi^{-2} - d_{2o}(3-4\nu)\ln\xi - d_{2o}]\sin\phi \end{aligned}$$

The solution to the first part is derived in detail. The zeroth order displacements functions is taken from eqn. (5.27) as

$$\begin{aligned} u_{1o} &= [a_{1o} + b_{1o}(1-4\nu)\xi^2 + c_{1o}\xi^{-2} + d_{1o}(3-4\nu)\ln\xi]\sin\phi \\ v_{1o} &= [a_{1o} - b_{1o}(5-4\nu)\xi^2 - c_{1o}\xi^{-2} + d_{1o}(3-4\nu)\ln\xi + d_{1o}]\cos\phi \end{aligned} \tag{5.28}$$

The corresponding zeroth order stress components are obtained from eqn. (5.17). By satisfying the stress boundary conditions (5.1), one finds that the constants b_{1o}, c_{1o}, d_{1o} and four stress components an zero. The free constant a_{1o} corresponds to the rigid body motion in axial direction, and it will be determined from displacement boundary conditions. The right hand members of the first order governing equations (5.12) are found by substituting eqns. (5.28) into eqns. (5.15) to obtain

$$\bar{U}_o = 0;\ \bar{V}_o = 0 \tag{5.29}$$

There is only the complementary solution for the first order governing equation. To satisfy the stress boundary condition (5.1), one obtains the first order displacement functions as

$$\begin{aligned} u_{11} &= 0 \\ v_{11} &= c_{11}\xi + d_{11}\xi^{-1} \end{aligned} \tag{5.30}$$

Calculating the first order stress components using eqn. (5.17), and enforcing the stress boundary conditions (5.1), one finds the constants c_{11} to be zero. The first order stress components are still zero. The remaining constant, $d_{11} = D_1$, is to be determined by satisfy the stress boundary conditions (5.2, 5.3). The right hand side members of the second order governing equations (5.13) take the form

$$-\bar{U}_1 + \xi\cos\phi\bar{U}_o - \hat{U}_o = \frac{2\nu}{1-2\nu}D_1\sin\phi \tag{5.31}$$

$$-\bar{V}_1 + \xi\cos\phi\bar{V}_o - \hat{V}_o = \frac{2\nu}{1-2\nu}D_1\cos\phi$$

The particular solution can be obtained from theory presented in chapter 4. Adding the particular solution with the special term of the complementary solution which has the same harmonic in ϕ, one get the second order displacement functions as

$$\begin{aligned} u_{12} &= [a_{12} + b_{12}(1-4\nu)\xi^2 + c_{12}\xi^{-2} + d_{12}(3-4\nu)\ln\xi + A_{12}^{1*}\xi^2]\sin\phi \\ v_{12} &= [a_{12} - b_{12}(5-4\nu)\xi^2 - c_{12}\xi^{-2} - d_{12}(3-4\nu)\ln\xi + d_{12} + B_{12}^{1*}\xi^2]\cos\phi \end{aligned} \tag{5.32}$$

where

$$A_{12}^{1*} = B_{12}^{1*} = \frac{\nu}{3-4\nu}D_1 \tag{5.33}$$

The upper case letters A_{nk}^{i*} and B_{nk}^{i*} are constants of the kth order particular solution corresponding to the nth eigenvalue. The i indicates the number of the constants. a_{12}, b_{12}, c_{12} and d_{12} are free constants. The corresponding second order stress components are obtained from eqns. (5.17) as

$$\begin{aligned} (\tau_{\xi\xi})_{12} &= [2b_{12}\xi - 2c_{12}\xi^{-3} + (3-4\nu)d_{12}\xi^{-1} + \frac{1}{1-2\nu}(2(1-\nu) * A_{12}^{1*} - \nu d_{11})\xi]\sin\phi \\ (\tau_{\phi\phi})_{12} &= [-a_{12} + 12b_{12}\xi^2 + 3c_{12}\xi^{-4} - \frac{1}{4\nu}(12A_{12}^{1*} + (1+\nu)^2 D_1)\xi^2]\sin\phi \\ (\tau_{\theta\theta})_{12} &= [12\nu b_{12}\xi^2 + 4\nu d_{12}\xi^{-2} - \frac{1}{4}(12A_{12}^{1*} - (1+\nu)D_1)\xi^2]\sin\phi \\ (\tau_{\xi\phi})_{12} &= [-2b_{12}\xi + 2c_{12}\xi^{-3} + (1-2\nu)d_{12}\xi^{-1} + A_{12}^{1*}\xi]\cos\phi \end{aligned} \tag{5.34}$$

By satisfying the stress free boundary conditions on the curved surfaces, one finds that the constants c_{12}, d_{12} are zero, while b_{12} as $A_{12}^{1*}/2$. The term a_{12} corresponds to the out-of-plane rigid body motion, and is covered in the zeroth order solution. The constant a_{12} is set to zero here. Substituting these constants into eqns. (5.32,

5.34), one obtains the non-zero terms of the second order solution as

$$
\begin{aligned}
u_{12} &= \frac{\nu}{2} D_1 \xi^2 \sin\phi \\
v_{12} &= -\frac{\nu}{2} D_1 \xi^2 \cos\phi \\
(\tau_{\xi\xi})_{12} &= (\tau_{\phi\phi})_{12} = (\tau_{\xi\phi})_{12} = 0 \\
(\tau_{\theta\theta})_{12} &= -(1+\nu) D_1 \xi \sin\phi
\end{aligned}
\tag{5.35}
$$

Following a procedure similar to that used for the zeroth, first and second order analysis, one obtains the third order solutions as

$$
\begin{aligned}
u_{13} &= [a_{13}\xi + b_{13}(-4\nu)\xi^3 + c_{13}\xi^{-3} + d_{13}(-4+4\nu)\xi^{-1} + A_{13}^{1*}\xi^3]\sin 2\phi \\
v_{13} &= [a_{13}\xi - b_{13}(6-4\nu)\xi^3 - c_{13}\xi^{-3} - d_{13}(2-4\nu)\xi^{-1} + B_{13}^{1*}\xi^3]\cos 2\phi
\end{aligned}
\tag{5.36}
$$

The corresponding stress components are

$$
\begin{aligned}
(\tau_{\xi\xi})_{13} &= [a_{13} - 3c_{13}\xi^{-4} + 4d_{13}\xi^{-2} - \frac{1-\nu}{4} D_1 \xi^2]\sin 2\phi \\
(\tau_{\phi\phi})_{13} &= [-a_{13} + 12 b_{13}\xi^2 + 3c_{13}\xi^{-4} - \frac{1}{4\nu}(12 A_{13}^{1*} + (1+\nu)^2 D_1)\xi^2]\sin 2\phi \\
(\tau_{\theta\theta})_{13} &= [12\nu b_{13}\xi^2 + 4\nu d_{13}\xi^{-2} - \frac{1}{4}(12 A_{13}^{1*} - (1+\nu)D_1)\xi^2]\sin 2\phi \\
(\tau_{\xi\phi})_{13} &= [a_{13} - 6 b_{13}\xi^2 + 3c_{13}\xi^{-4} - 2d_{13}\xi^{-2} + \frac{1}{8\nu}(12 A_{13}^{1*} + (1+\nu)D_1)\xi^2]\cos 2\phi
\end{aligned}
\tag{5.37}
$$

where

$$
\begin{aligned}
A_{13}^{1*} &= -B_{13}^{1*} = -\frac{1+\nu}{12} D_1 \\
a_{13} &= \frac{(1+\lambda^2)(1+\lambda^2+\lambda^4)}{(1-\lambda^2)^2}\frac{1+\nu}{4} D_1 \\
b_{13} &= [\frac{1+\lambda^2+\lambda^4}{(1-\lambda^2)^2} + \frac{1-2\nu}{4\nu}]\frac{1+\nu}{12} D_1 + \frac{1}{4\nu} A_{13}^{1*} \\
c_{13} &= -\frac{\lambda^4+\lambda^6}{(1-\lambda^2)^2}\frac{1+\nu}{4} D_1 \\
d_{13} &= -\frac{\lambda^2(1+\lambda^2+\lambda^4)}{(1-\lambda^2)^2}\frac{1+\nu}{4} D_1
\end{aligned}
\tag{5.38}
$$

The second part of this eigenfunction for the roots $\beta = \pm i$ is generated by the same procedure. The second part of eqn. (5.25) is employed to start the MSA approach. A second free constant D_2 is introduced. The second part of the eigenfunction is given directly for the zeroth order as

$$\begin{aligned} u_{2o} &= a_{2o}\cos\phi \\ v_{2o} &= -a_{2o}\sin\phi \\ (\tau_{\xi\xi})_{2o} &= (\tau_{\phi\phi})_{2o} = (\tau_{\theta\theta})_{2o} = (\tau_{\xi\phi})_{2o} = 0 \\ a_{2o} &= D_2 \end{aligned} \tag{5.39}$$

for the first order as

$$\begin{aligned} u_{21} &= a_{21}\xi \\ v_{21} &= 0 \\ (\tau_{\xi\xi})_{21} &= (\tau_{\phi\phi})_{21} = (\tau_{\xi\phi})_{21} = 0 \\ (\tau_{\theta\theta})_{21} &= (1+\nu)D_2 \\ a_{21} &= -\nu D_2 \end{aligned} \tag{5.40}$$

for the second order as

$$\begin{aligned} u_{22} &= \frac{\nu(1+\nu)}{2} D_2\xi^2\cos\phi \\ v_{22} &= \frac{\nu(1+\nu)}{2} D_2\xi^2\sin\phi \\ (\tau_{\xi\xi})_{22} &= (\tau_{\phi\phi})_{22} = (\tau_{\xi\phi})_{22} = 0 \\ (\tau_{\theta\theta})_{22} &= -(1+\nu)^2 D_2\xi\cos\phi \end{aligned}$$

and for the third order as

$$u_{23} = [a_{23}\xi + b_{23}(-4\nu)\xi^3 + c_{23}\xi^{-3} + d_{23}(-4+4\nu)\xi^{-1} + A^{1*}_{23}\xi^3]\cos 2\phi$$

$$
\begin{aligned}
v_{23} &= [-a_{23}\xi + b_{23}(6-4\nu)\xi^3 + c_{23}\xi^{-3} + d_{23}(2-4\nu)\xi^{-1} + B_{23}^{1*}\xi^3]\sin 2\phi \\
(\tau_{\xi\xi})_{23} &= [a_{23} - 3c_{23}\xi^{-4} + 4d_{23}\xi^{-2} + \frac{\nu(1+\nu)^2-1}{2(1-2\nu)}D_2\xi^2]\cos 2\phi \\
(\tau_{\phi\phi})_{23} &= [-a_{23} + 12b_{23}\xi^2 + 3c_{23}\xi^{-4} + \frac{1}{1-2\nu}(3A_{23}^{1*} + \frac{\nu(1+\nu)^2}{2}D_2)\xi^2]\cos 2\phi \\
(\tau_{\theta\theta})_{23} &= [12\nu b_{23}\xi^2 + 4\nu d_{23}\xi^{-2} + \frac{1}{1-2\nu}(6\nu A_{23}^{1*} + \frac{1}{2}(1-\nu)(1+\nu)^2 D_2)\xi^2]\cos\phi \\
(\tau_{\xi\phi})_{23} &= [-a_{23} + 6b_{23}\xi^2 - 3c_{23}\xi^{-4} + 2d_{23}\xi^{-2}]\cos 2\phi
\end{aligned}
\tag{5.41}
$$

where

$$
\begin{aligned}
A_{23}^{1*} &= B_{23}^{1*} = -\frac{1}{6}D_2 \\
a_{23} &= \frac{1-\nu(1+\nu)^2}{2(1-2\nu)}\frac{(1+\lambda^2)(1-\lambda^6)}{(1-\lambda^2)^3}D_2 \\
b_{23} &= \frac{1-\nu(1+\nu)^2}{2(1-2\nu)}[\frac{2(1-\lambda^6)}{(1-\lambda^2)^3} - 1]D_2 \\
c_{23} &= -\frac{1-\nu(1+\nu)^2}{2(1-2\nu)}\frac{\lambda^4(1-\lambda^4)}{(1-\lambda^2)^3}D_2 \\
d_{23} &= -\frac{1-\nu(1+\nu)^2}{2(1-2\nu)}\frac{\lambda^2(1-\lambda^6)}{(1-\lambda^2)^3}D_2
\end{aligned}
\tag{5.42}
$$

A complex form for the eigenfunction is constructed from the two parts, with the second part being stored in the imaginary area. By superposing the contributions up to the third order, one obtains the total solution for the displacement components corresponding to the eigenvalues $\beta = \pm i$ as

$$
\begin{aligned}
u_1 + iu_2 &= (u_{1o} + \epsilon u_{11} + \epsilon^2 u_{12} + \epsilon^3 u_{13}) + i(u_{2o} + \epsilon u_{21} + \epsilon^2 u_{22} + \epsilon^3 u_{23}) \\
v_1 + iv_2 &= (v_{1o} + \epsilon v_{11} + \epsilon^2 v_{12} + \epsilon^3 v_{13}) + i(v_{2o} + \epsilon v_{21} + \epsilon^2 v_{22} + \epsilon^3 v_{23})
\end{aligned}
\tag{5.43}
$$

The corresponding stress components are

$$
\begin{aligned}
(\tau_{\xi\xi})_1 + i(\tau_{\xi\xi})_2 &= [(\tau_{\xi\xi})_{1o} + \epsilon(\tau_{\xi\xi})_{11} + \epsilon^2(\tau_{\xi\xi})_{12} + \epsilon^3(\tau_{\xi\xi})_{13}] \\
&+ i[(\tau_{\xi\xi})_{2o} + \epsilon(\tau_{\xi\xi})_{21} + \epsilon^2(\tau_{\xi\xi})_{22} + \epsilon^3(\tau_{\xi\xi})_{23}]
\end{aligned}
$$

$$
\begin{aligned}
(\tau_{\phi\phi})_1 + i(\tau_{\phi\phi})_2 &= [(\tau_{\phi\phi})_{1o} + \epsilon(\tau_{\phi\phi})_{11} + \epsilon^2(\tau_{\phi\phi})_{12} + \epsilon^3(\tau_{\phi\phi})_{13}] \\
&+ i[(\tau_{\phi\phi})_{2o} + \epsilon(\tau_{\phi\phi})_{21} + \epsilon^2(\tau_{\phi\phi})_{22} + \epsilon^3(\tau_{\phi\phi})_{23}] \qquad (5.44) \\
(\tau_{\theta\theta})_1 + i(\tau_{\theta\theta})_2 &= [(\tau_{\theta\theta})_{1o} + \epsilon(\tau_{\theta\theta})_{11} + \epsilon^2(\tau_{\theta\theta})_{12} + \epsilon^3(\tau_{\theta\theta})_{13}] \\
&+ i[(\tau_{\theta\theta})_{2o} + \epsilon(\tau_{\theta\theta})_{21} + \epsilon^2(\tau_{\theta\theta})_{22} + \epsilon^3(\tau_{\theta\theta})_{23}] \\
(\tau_{\xi\phi})_1 + i(\tau_{\xi\phi})_2 &= [(\tau_{\xi\phi})_{1o} + \epsilon(\tau_{\xi\phi})_{11} + \epsilon^2(\tau_{\xi\phi})_{12} + \epsilon^3(\tau_{\xi\phi})_{13}] \\
&+ i[(\tau_{\xi\phi})_{2o} + \epsilon(\tau_{\xi\phi})_{21} + \epsilon^2(\tau_{\xi\phi})_{22} + \epsilon^3(\tau_{\xi\phi})_{23}]
\end{aligned}
$$

The constants D_1 and D_2 remain available for the satisfying of the boundary conditions on the end surfaces of the ring.

5.5.2 Eigenfunctions for roots $\beta_n (n = 3, 4)$

These roots are the last two of the four adjusted eigenvalues of the original third group. The eigenfunctions corresponding to these two real eigenvalues are still in complex form. It is understood that the real and imaginary parts represent separate solutions. The positive eigenvalue ($n = 3$) exhibits decay characteristic from the end $\phi = \phi_o$, while the other ($n = 4$) exhibits decay characteristic from the end $\phi = 0°$. For convenience in analysis, $e^{\beta_n(\phi-\phi_o)}$ is employed to replace $e^{\beta_n\phi}$ in the eigenfunctions with $n = 3$. The zeroth order displacement functions of the real eigenvalues are taken as

$$
\begin{aligned}
(u)_{no} &= (U_\xi)_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \qquad (5.45) \\
(v)_{no} &= (U_\phi)_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

where D_n is a free complex constant and

$$\begin{aligned}
(U_\xi)_{no} &= [-2(1-2\nu)+\beta_n i]a_{no}\xi^{1+\beta_n i} - [2(1-2\nu)+\beta_n i]b_{no}\xi^{1-\beta_n i} \\
&- [4(1-\nu)+\beta_n i]c_{no}(3-3\xi+\xi^2)\xi^{-\beta_n i} \\
&- [4(1-\nu)-\beta_n i]d_{no}(3-3\xi+\xi^2)\xi^{\beta_n i} \qquad (5.46)\\
(U_\phi)_{no} &= [-4(1-\nu)i+\beta_n]a_{no}\xi^{1+\beta_n i} + [4(1-\nu)i+\beta_n]b_{no}\xi^{1-\beta_n i} \\
&+ [-4(1-\nu)i+\beta_n]c_{no}(3-3\xi+\xi^2)\xi^{-\beta_n i} \\
&+ [4(1-\nu)i+\beta_n]d_{no}(3-3\xi+\xi^2)\xi^{\beta_n i}
\end{aligned}$$

where a_{no}, b_{no}, c_{no} and d_{no} are two set of constants. Using eqn. (5.17) to derive the stress components, and enforcing the boundary conditions (5.1), one obtains these constants. The zeroth order stress components are written as

$$\begin{aligned}
(\tau_{\xi\xi})_{no} &= \{-(\beta_n^2+2+\beta_n i)a_{no}\xi^{\beta_n i} - (\beta_n^2+2-\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [4(1-\nu)-\beta_n^2+(5-4\nu)\beta_n i]c_{no}(3-2\xi)\xi^{-\beta_n i} \\
&+ [4(1-\nu)-\beta_n^2-(5-4\nu)\beta_n i]d_{no}(3-2\xi)\xi^{\beta_n i}\} \\
&\quad D_n e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
(\tau_{\phi\phi})_{no} &= \{(\beta_n^2-2-3\beta_n i)a_{no}\xi^{\beta_n i} + (\beta_n^2-2+3\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [-4(1-\nu)+\beta_n^2-(5-4\nu)\beta_n i]c_{no}(3-2\xi)\xi^{-\beta_n i} \qquad (5.47)\\
&+ [-4(1-\nu)+\beta_n^2+(5-4\nu)\beta_n i]d_{no}(3-2\xi)\xi^{\beta_n i}\} \\
&\quad D_n e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
(\tau_{\theta\theta})_{no} &= \{(-4\nu)[(1+\beta_n i)a_{no}\xi^{\beta_n i} + (1-\beta_n i)b_{no}\xi^{-\beta_n i}]\} \\
&\quad D_n e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
(\tau_{\xi\phi})_{no} &= \{(\beta_n+\beta_n^2 i)a_{no}\xi^{\beta_n i} + (\beta_n-\beta_n^2 i)b_{no}\xi^{-\beta_n i}
\end{aligned}$$

$$
\begin{aligned}
&- [(5-4\nu)\beta_n + (\beta_n^2 - 4 + 4\nu)i]c_{no}(3-2\xi)\xi^{-\beta_n i} \\
&- [(5-4\nu)\beta_n - (\beta_n^2 - 4 + 4\nu)i]d_{no}(3-2\xi)\xi^{\beta_n i}\} \\
&D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{no} &= \frac{[\beta_n^2 - 4(1-\nu) + (5-4\nu)\beta_n i][(3-2\lambda)(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{-\beta_n i}]}{-(\beta_n^2 - \beta_n i)[(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{-\beta_n i}]} \\
b_{no} &= \frac{[\beta_n^2 - 4(1-\nu) + (5-4\nu)\beta_n i][(3-2\lambda)(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{\beta_n i}]}{-(\beta_n^2 + \beta_n i)[(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{-\beta_n i}]} \\
c_{no} &= \frac{[\beta_n^2 - 4(1-\nu) + (5-4\nu)\beta_n i][(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{\beta_n i}]}{[\beta_n^2 - 4(1-\nu) - (5-4\nu)\beta_n i][(\lambda^{-\beta_n i} - \lambda^{\beta_n i}) + 2(1-\lambda)\beta_n i\lambda^{-\beta_n i}]} \qquad (5.48) \\
d_{no} &= 1
\end{aligned}
$$

The first order governing equations are non-homogenous equations. The solution is composed of a complementary part and a particular part. Substituting the zeroth order solution (5.45) into the first order governing equations (5.12) yields the right hand side members

$$
\begin{aligned}
-\bar{U}_{no} &= \frac{2(1-\nu)}{1-2\nu}\{(\beta_n^2 + 1 - 4\nu - 4\nu\beta_n i)a_{no}\xi^{\beta_n i} + (1-\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [\beta_n^2 - 4(1-\nu) + (5-4\nu)\beta_n i]d_{no}(3-2\xi)\xi^{\beta_n i}\}D_n e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \frac{2(1-\nu)}{1-2\nu}\{(1+\beta_n i)a_{no}\xi^{\beta_n i} + (\beta_n^2 + 1 - 4\nu + 4\nu\beta_n i)b_{no}\xi^{-\beta_n i} \qquad (5.49) \\
&+ [\beta_n^2 - 4(1-\nu) - (5-4\nu)\beta_n i]c_{no}(3-2\xi)\xi^{-\beta_n i}\}D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
-\bar{V}_{no} &= \frac{2(1-\nu)}{1-2\nu}\{[-2(1-2\nu)\beta_n + (1-4\nu-\beta_n^2)i]a_{no}\xi^{\beta_n i} + (\beta_n + i)b_{no}\xi^{-\beta_n i} \\
&+ [(5-4\nu)\beta_n - (\beta_n^2 - 4 + 4\nu)i]d_{no}(3-2\xi)\xi^{\beta_n i}\}D_n e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \frac{2(1-\nu)}{1-2\nu}\{(\beta_n - i)a_{no}\xi^{\beta_n i} + [-2(1-2\nu)\beta_n + (\beta_n^2 - 1 + 4\nu)i]b_{no}\xi^{-\beta_n i} \\
&+ [(5-4\nu)\beta_n + (\beta_n^2 - 4 + 4\nu)i]c_{no}(3-2\xi)\xi^{-\beta_n i}\}D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

It is evident from eqns. (5.49) that the particular solution for these two roots can be written in two separate parts (5.50). Substituting the particular solution into eqn. (5.12), and matching coefficients of similar terms, the free constants in the particular solutions are determined. The particular solution takes the form

$$\begin{aligned} u^*_{n1} &= (A^{1*}_{n1}\xi^{2+\beta_n i} + A^{2*}_{n1}\xi^{2-\beta_n i} + A^{3*}_{n1}\xi^{3+\beta_n i})D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ &+ (A^{4*}_{n1}\xi^{2+\beta_n i} + A^{5*}_{n1}\xi^{2-\beta_n i} + A^{6*}_{n1}\xi^{3-\beta_n i})D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ v^*_{n1} &= (B^{1*}_{n1}\xi^{2+\beta_n i} + B^{2*}_{n1}\xi^{2-\beta_n i} + B^{3*}_{n1}\xi^{3+\beta_n i})D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ &+ (B^{4*}_{n1}\xi^{2+\beta_n i} + B^{5*}_{n1}\xi^{2-\beta_n i} + B^{6*}_{n1}\xi^{3-\beta_n i})D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \end{aligned} \tag{5.50}$$

where

$$A^{i*}_{n1} = \frac{[(1-2\nu)(\gamma_i^2-1)+2(1-\nu)\alpha_i^2]R_1^{i*} - [(\gamma_i-3+4\nu)\alpha_i]R_2^{i*}}{2(1-\nu)[\gamma_i^4+2(\alpha_i^2-1)\gamma_i^2+(\alpha_i^2+1)^2]}$$

$$B^{i*}_{n1} = \frac{[2(1-\nu)(\gamma_i^2-1)+(1-2\nu)\alpha_i^2]R_2^{i*} - [(\gamma_i+3-4\nu)\alpha_i]R_1^{i*}}{2(1-\nu)[\gamma_i^4+2(\alpha_i^2-1)\gamma_i^2+(\alpha_i^2+1)^2]}$$

$i = 1, 2, ..., 6$

$$\begin{aligned} R_1^{1*} &= \frac{2(1-\nu)}{1-2\nu}\{[\beta_n^2+(1-4\nu)-4\nu\beta_n i]a_{no} \\ &\quad +3[\beta_n^2-4(1-\nu)+(5-4\nu)\beta_n i]d_{no}\} \\ R_2^{1*} &= \frac{2(1-\nu)}{1-2\nu}\{[-2(1-2\nu)\beta_n+(1+4\nu-\beta_n^2)i]a_{no} \\ &\quad +3[(5-4\nu)\beta_n-(\beta_n^2-4+4\nu)i]d_{no}\} \\ R_1^{2*} &= \frac{2(1-\nu)}{1-2\nu}(1-\beta_n i)b_{no} \\ R_2^{2*} &= \frac{2(1-\nu)}{1-2\nu}(\beta_n+i)b_{no} \\ R_1^{3*} &= -\frac{4(1-\nu)}{1-2\nu}[\beta_n^2-4(1-\nu)+(5-4\nu)\beta_n i]d_{no} \\ R_2^{3*} &= -\frac{4(1-\nu)}{1-2\nu}[(5-4\nu)\beta_n-(\beta_n^2-4+4\nu)i]d_{no} \\ R_1^{4*} &= \frac{2(1-\nu)}{1-2\nu}(1+\beta_n i)a_{no} \end{aligned} \tag{5.51}$$

Table 5.3: Coefficients for α_i and γ_i

i	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
α_i	β_n+i	β_n+i	β_n+i	β_n-i	β_n-i	β_n-i
γ_i	$2+\beta_n i$	$2-\beta_n i$	$3+\beta_n i$	$2+\beta_n i$	$2-\beta_n i$	$3-\beta_n i$

$$
\begin{aligned}
R_2^{4*} &= \frac{2(1-\nu)}{1-2\nu}(\beta_n - i)a_{no} \\
R_1^{5*} &= \frac{2(1-\nu)}{1-2\nu}\{[\beta_n^2 + (1-4\nu) + 4\nu\beta_n i]b_{no} \\
&\quad +3[\beta_n^2 - 4(1-\nu) - (5-4\nu)\beta_n i]c_{no}\} \\
R_2^{5*} &= \frac{2(1-\nu)}{1-2\nu}\{[-2(1-2\nu)\beta_n + (\beta_n^2 - 1 + 4\nu)i]b_{no} \\
&\quad +3[(5-4\nu)\beta_n + (\beta_n^2 - 4 + 4\nu)i]c_{no}\} \\
R_1^{6*} &= -\frac{4(1-\nu)}{1-2\nu}[\beta_n^2 - 4(1-\nu) - (5-4\nu)\beta_n i]c_{no} \\
R_2^{6*} &= -\frac{4(1-\nu)}{1-2\nu}[(5-4\nu)\beta_n + (\beta_n^2 - 4 + 4\nu)i]c_{no}
\end{aligned}
\tag{5.52}
$$

The parameters α_i and γ_i corresponding to different set of constants are given in Table 5.3.

The first order complementary solution takes the same form as the zeroth order complementary solution, except for the value of β_n. The complementary solution and particular solution for the root β_n are written in two separate parts. Adding the complementary and particular solutions together, one obtains the first order displacement functions as

$$
\begin{aligned}
u_{n1} &= \{[-(3-4\nu) + \beta_n i]a_{n1}^1 \xi^{\beta_n i} - [(1-4\nu) + \beta_n i]b_{n1}^1 \xi^{2-\beta_n i} \\
&- [(3-4\nu) + \beta_n i]c_{n1}^1 \xi^{-\beta_n i} - [(5-4\nu) - \beta_n i]d_{n1}^1 (3-2\xi)\xi^{\beta_n i} \\
&+ A_{n1}^{1*}\xi^{2+\beta_n i} + A_{n1}^{2*}\xi^{2-\beta_n i} + A_{n1}^{3*}\xi^{3+\beta_n i}\} D_n e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \{[-(1-4\nu)+\beta_n i]a^2_{n1}\xi^{2+\beta_n i} - [(3-4\nu)+\beta_n i]b^2_{n1}\xi^{-\beta_n i} \\
&- [(5-4\nu)+\beta_n i]c^2_{n1}(3-2\xi)\xi^{-\beta_n i} - [(3-4\nu)-\beta_n i]d^2_{n1}\xi^{\beta_n i} \\
&+ A^{4*}_{n1}\xi^{2+\beta_n i} + A^{5*}_{n1}\xi^{2-\beta_n i} + A^{6*}_{n1}\xi^{3-\beta_n i}\}D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \qquad (5.53) \\
v_{n1} &= \{[-(3-4\nu)i+\beta_n]a^1_{n1}\xi^{\beta_n i} - [(1-4\nu)i+\beta_n]b^1_{n1}\xi^{2-\beta_n i} \\
&- [(3-4\nu)i+\beta_n]c^1_{n1}\xi^{-\beta_n i} + [(5-4\nu)i-\beta_n]d^1_{n1}(3-2\xi)\xi^{\beta_n i} \\
&+ B^{1*}_{n1}\xi^{2+\beta_n i} + B^{2*}_{n1}\xi^{2-\beta_n i} + B^{3*}_{n1}\xi^{3+\beta_n i}\}D_n e^{(\beta_n + i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \{[-(5-4\nu)i+\beta_n]a^2_{n1}\xi^{2+\beta_n i} + [(3-4\nu)i+\beta_n]b^2_{n1}\xi^{-\beta_n i} \\
&- [(5-4\nu)i+\beta_n]c^2_{n1}(3-2\xi)\xi^{-\beta_n i} + [(3-4\nu)i-\beta_n]d^2_{n1}\xi^{\beta_n i} \\
&+ B^{4*}_{n1}\xi^{2+\beta_n i} + B^{5*}_{n1}\xi^{2-\beta_n i} + B^{6*}_{n1}\xi^{3-\beta_n i}\}D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

where a^i_{n1}, b^i_{n1}, c^i_{n1}, and d^i_{n1} (i=1, 2) are two sets of the first order free constants. Substituting the first order displacement functions into eqns. (5.17), one obtains the first order stress field as

$$
\begin{aligned}
(\tau_{\xi\xi})_{n1} &= [(C_{\xi\xi})^1_{n1}e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\xi})^2_{n1}e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
(\tau_{\phi\phi})_{n1} &= [(C_{\phi\phi})^1_{n1}e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\phi\phi})^2_{n1}e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
(\tau_{\theta\theta})_{n1} &= [(C_{\theta\theta})^1_{n1}e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\theta\theta})^2_{n1}e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \qquad (5.54) \\
(\tau_{\xi\phi})_{n1} &= [(C_{\xi\phi})^1_{n1}e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\phi})^2_{n1}e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n
\end{aligned}
$$

where $(C_{ij})^1_{n1}$ and $(C_{ij})^2_{n1}$ ($ij = \xi\xi,\ \phi\phi,\ \theta\theta,\ \xi\phi$) are functions of the nth eigenvalues β_n, the constants $a^i_{n1}, b^i_{n1}, c^i_{n1}, d^i_{n1} (i = 1, 2)$, and non-dimensional radial coordinate ξ. Full details are given in Appendix 5.2. The two set of constants $a^i_{n1}, b^i_{n1}, c^i_{n1}, d^i_{n1}$ are determined from the boundary conditions on the inner and outer surface of the ring. The boundary conditions lead to two sets of four non-homogeneous equations, from which two sets of constants are obtained.

Superposing the contributions up to the second order one obtains the total solutions for the two real eigenvalue β_n $(n = 3, 4)$ as

$$
\begin{aligned}
u_n &= u_{no} + \epsilon u_{n1} \\
v_n &= v_{no} + \epsilon v_{n1} \\
(\tau_{\xi\xi})_n &= [(C_{\xi\xi})_{no} e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon (C_{\xi\xi})^1_{n1} e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} + \epsilon (C_{\xi\xi})^2_{n1} e^{(\beta_n-i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}] D_n \\
(\tau_{\phi\phi})_n &= [(C_{\phi\phi})_{no} e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon (C_{\phi\phi})^1_{n1} e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} + \epsilon (C_{\phi\phi})^2_{n1} e^{(\beta_n-i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}] D_n \quad (5.55) \\
(\tau_{\theta\theta})_n &= [(C_{\theta\theta})_{no} e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon (C_{\theta\theta})^1_{n1} e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} + \epsilon (C_{\theta\theta})^2_{n1} e^{(\beta_n-i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}] D_n \\
(\tau_{\xi\phi})_n &= [(C_{\xi\phi})_{no} e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon (C_{\xi\phi})^1_{n1} e^{(\beta_n+i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} + \epsilon (C_{\xi\phi})^2_{n1} e^{(\beta_n-i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}] D_n
\end{aligned}
$$

where D_n $(n = 3, 4)$ are two remained free complex constants.

5.5.3 Eigenfunctions for general roots $\beta_n (n > 4)$

For brevity of representation, the displacement and stress components of root β_n $(n > 4)$ are written in complex form. The eigenvalues with positive real part (n takes an odd number) exhibit decay characteristic from the end of $\phi = \phi_o$. A constant $e^{\beta_n \phi_o}$ is employed to multiply and divide the displacement function at the same time, and combining the free constant D_n with $e^{\beta_n \phi_o}$ to form a new free constant D_n. The zeroth-order displacement functions corresponding to the general eigenvalue β_n are

taken as

$$(u)_{no} = (U_\xi)_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)}$$
$$(v)_{no} = (U_\phi)_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \tag{5.56}$$

where D_n is a free complex constant and

$$\begin{aligned}
(U_\xi)_{no} &= \{[-2(1-2\nu)+\beta_n i]a_{no}\xi^{1+\beta_n i} - [2(1-2\nu)+\beta_n i]b_{no}\xi^{1-\beta_n i} \\
&- [4(1-\nu)+\beta_n i]c_{no}\xi^{-(1+\beta_n i)} - [4(1-\nu)-\beta_n i]d_{no}\xi^{-(1-\beta_n i)}\} \qquad (5.57)\\
(U_\phi)_{no} &= \{[-4(1-\nu)i+\beta_n]a_{no}\xi^{1+\beta_n i} + [4(1-\nu)i+\beta_n]b_{no}\xi^{1-\beta_n i} \\
&+ [-4(1-\nu)i+\beta_n]c_{no}\xi^{-(1+\beta_n i)} + [4(1-\nu)i+\beta_n]d_{no}\xi^{-(1-\beta_n i)}\}
\end{aligned}$$

Constants a_{no}, b_{on}, c_{no} and d_{no} are components of the eigenvector from the zeroth order, representing a solution for the constants C_i in the relations (5.22). The zeroth order stress components are written as

$$\begin{aligned}
(\tau_{\xi\xi})_{no} &= (C_{\xi\xi})_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
(\tau_{\phi\phi})_{no} &= (C_{\phi\phi})_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \\
(\tau_{\theta\theta})_{no} &= (C_{\theta\theta})_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)} \qquad (5.58)\\
(\tau_{\xi\phi})_{no} &= (C_{\xi\phi})_{no} D_n e^{\beta_n(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}$$

where

$$\begin{aligned}
(C_{\xi\xi})_{no} &= \{-(\beta_n^2+2+\beta_n i)a_{no}\xi^{\beta_n i} - (\beta_n^2+2-\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [4(1-\nu)-\beta_n^2+(5-4\nu)\beta_n i]c_{no}\xi^{-(2+\beta_n i)} \\
&+ [4(1-\nu)-\beta_n^2-(5-4\nu)\beta_n i]d_{no}\xi^{-(2-\beta_n i)}\} \\
(C_{\phi\phi})_{no} &= \{(\beta_n^2-2-3\beta_n i)a_{no}\xi^{\beta_n i} + (\beta_n^2-2+3\beta_n i)b_{no}\xi^{-\beta_n i}
\end{aligned}$$

$$
\begin{aligned}
&+ [-4(1-\nu)+\beta_n^2-(5-4\nu)\beta_n i]c_{no}\xi^{-(2+\beta_n i)} \\
&+ [-4(1-\nu)+\beta_n^2+(5-4\nu)\beta_n i]d_{no}\xi^{-(2-\beta_n i)}\} \\
(C_{\theta\theta})_{no} &= \{(-4\nu)[(1+\beta_n i)a_{no}\xi^{\beta_n i}+(1-\beta_n i)b_{no}\xi^{-\beta_n i}]\} \qquad (5.59)\\
(C_{\xi\phi})_{no} &= \{(\beta_n+\beta_n^2 i)a_{no}\xi^{\beta_n i}+(\beta_n-\beta_n^2 i)b_{no}\xi^{-\beta_n i} \\
&- [(5-4\nu)\beta_n+(\beta_n^2-4+4\nu)i]c_{no}\xi^{-(2+\beta_n i)} \\
&- [(5-4\nu)\beta_n-(\beta_n^2-4+4\nu)i]d_{no}\xi^{-(2-\beta_n i)}\}
\end{aligned}
$$

The first order governing equations are non-homogenous equations. Substituting the zeroth order solution (5.56) into the first order governing equations (5.12) yields the right hand side members

$$
\begin{aligned}
-\bar{U}_{no} &= \frac{2(1-\nu)}{1-2\nu}\{(\beta_n^2+1-4\nu-4\nu\beta_n i)a_{no}\xi^{\beta_n i}+(1-\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [\beta_n^2-4(1-\nu)+(5-4\nu)\beta_n i]d_{no}\xi^{-(2-\beta_n i)}\}D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
&+ \frac{2(1-\nu)}{1-2\nu}\{(1+\beta_n i)a_{no}\xi^{\beta_n i}+(\beta_n^2+1-4\nu+4\nu\beta_n i)b_{no}\xi^{-\beta_n i} \\
&+ [\beta_n^2-4(1-\nu)-(5-4\nu)\beta_n i]c_{no}\xi^{-(2+\beta_n i)}\}D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \qquad (5.60)\\
-\bar{V}_{no} &= \frac{2(1-\nu)}{1-2\nu}\{[-2(1-2\nu)\beta_n+(1-4\nu-\beta_n^2)i]a_{no}\xi^{\beta_n i}+(\beta_n+i)b_{no}\xi^{-\beta_n i} \\
&+ [(5-4\nu)\beta_n-(\beta_n^2-4+4\nu)i]d_{no}\xi^{-(2-\beta_n i)}\}D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
&+ \frac{2(1-\nu)}{1-2\nu}\{(\beta_n-i)a_{no}\xi^{\beta_n i}+[-2(1-2\nu)\beta_n+(\beta_n^2-1+4\nu)i]b_{no}\xi^{-\beta_n i} \\
&+ [(5-4\nu)\beta_n+(\beta_n^2-4+4\nu)i]c_{no}\xi^{-(2+\beta_n i)}\}D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

It is evident from eqns. (5.60) that the particular solution for the general root β_n can be written in two separate parts. One obtains

$$
u_{n1}^{*} = (A_{n1}^{1*}\xi^{2+\beta_n i}+A_{n1}^{2*}\xi^{2-\beta_n i}+A_{n1}^{3*}\xi^{\beta_n i})D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}
$$

$$
\begin{aligned}
&+ (A^{4*}_{n1}\xi^{2+\beta_n i} + A^{5*}_{n1}\xi^{2-\beta_n i} + A^{6*}_{n1}\xi^{-\beta_n i}) D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \qquad (5.61)\\
v^*_{n1} &= (B^{1*}_{n1}\xi^{2+\beta_n i} + B^{2*}_{n1}\xi^{2-\beta_n i} + B^{3*}_{n1}\xi^{\beta_n i}) D_n e^{(\beta_n + i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}\\
&+ (B^{4*}_{n1}\xi^{2+\beta_n i} + B^{5*}_{n1}\xi^{2-\beta_n i} + B^{6*}_{n1}\xi^{-\beta_n i}) D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

where A^{i*}_{n1}, B^{i*}_{n1}, ($i = 1, 2, \ldots, 6$, the number of the constants) are constants. Substituting the particular solution into eqn. (5.12), and matching coefficients of same terms, the constants are determined. The details are given in Appendix 5.1. Adding the complementary and particular solutions together, one obtains the first order displacement functions as

$$
\begin{aligned}
u_{n1} &= \{[-(3-4\nu) + \beta_n i]a^1_{n1}\xi^{\beta_n i} - [(1-4\nu) + \beta_n i]b^1_{n1}\xi^{2-\beta_n i}\\
&- [(3-4\nu) + \beta_n i]c^1_{n1}\xi^{-\beta_n i} - [(5-4\nu) - \beta_n i]d^1_{n1}\xi^{-(2-\beta_n i)}\\
&+ A^{1*}_{n1}\xi^{2+\beta_n i} + A^{2*}_{n1}\xi^{2-\beta_n i} + A^{3*}_{n1}\xi^{\beta_n i}\} D_n e^{(\beta_n + i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}\\
&+ \{[-(1-4\nu) + \beta_n i]a^2_{n1}\xi^{2+\beta_n i} - [(3-4\nu) + \beta_n i]b^2_{n1}\xi^{-\beta_n i}\\
&- [(5-4\nu) + \beta_n i]c^2_{n1}\xi^{-(2+\beta_n i)} - [(3-4\nu) - \beta_n i]d^2_{n1}\xi^{\beta_n i}\\
&+ A^{4*}_{n1}\xi^{2+\beta_n i} + A^{5*}_{n1}\xi^{2-\beta_n i} + A^{6*}_{n1}\xi^{-\beta_n i}\} D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)} \qquad (5.62)\\
v_{n1} &= \{[-(3-4\nu)i + \beta_n]a^1_{n1}\xi^{\beta_n i} - [(1-4\nu)i + \beta_n]b^1_{n1}\xi^{2-\beta_n i}\\
&- [(3-4\nu)i + \beta_n]c^1_{n1}\xi^{-\beta_n i} + [(5-4\nu)i - \beta_n]d^1_{n1}\xi^{-(2-\beta_n i)}\\
&+ B^{1*}_{n1}\xi^{2+\beta_n i} + B^{2*}_{n1}\xi^{2-\beta_n i} + B^{3*}_{n1}\xi^{\beta_n i}\} D_n e^{(\beta_n + i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}\\
&+ \{[-(5-4\nu)i + \beta_n]a^2_{n1}\xi^{2+\beta_n i} + [(3-4\nu)i + \beta_n]b^2_{n1}\xi^{-\beta_n i}\\
&- [(5-4\nu)i + \beta_n]c^2_{n1}\xi^{-(2+\beta_n i)} + [(3-4\nu)i - \beta_n]d^2_{n1}\xi^{\beta_n i}\\
&+ B^{4*}_{n1}\xi^{2+\beta_n i} + B^{5*}_{n1}\xi^{2-\beta_n i} + B^{6*}_{n1}\xi^{-\beta_n i}\} D_n e^{(\beta_n - i)(\phi - \frac{1-(-1)^n}{2}\phi_o)}
\end{aligned}
$$

where a^i_{n1}, b^i_{n1}, c^i_{n1}, and d^i_{n1} (i=1, 2) are two sets of the first order free constants.

Substituting the first order displacement functions into eqns. (5.17), one obtains the first order stress field as

$$\begin{aligned}
\tau_{\xi\xi_{n1}} &= [(C_{\xi\xi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\xi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
\tau_{\phi\phi_{n1}} &= [(C_{\phi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\phi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
\tau_{\theta\theta_{n1}} &= [(C_{\theta\theta})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\theta\theta})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
\tau_{\xi\phi_{n1}} &= [(C_{\xi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n
\end{aligned} \tag{5.63}$$

where $(C_{ij})^1_{n1}$ and $(C_{ij})^2_{n1}$ $(ij = \xi\xi,\ \phi\phi,\ \theta\theta,\ \xi\phi\ )$ are functions of the nth eigenvalues β_n, the constants $a^i_{n1}, b^i_{n1}, c^i_{n1}, d^i_{n1} (i = 1,2)$, and non-dimensional radial coordinate ξ. Full details are given in Appendix 5.2. The two set of constants $a^i_{n1}, b^i_{n1}, c^i_{n1}, d^i_{n1}$ are determined from the boundary conditions on the inner and outer surface of the ring. The boundary conditions lead to two sets of four non-homogeneous equations, from which two sets of constants are obtained.

Superposing the contributions up to the second order, one obtains the total solutions for the general eigenvalue β_n as

$$\begin{aligned}
u_n &= u_{no} + \epsilon u_{n1} \\
v_n &= v_{no} + \epsilon v_{n1} \\
(\tau_{\xi\xi})_n &= [(C_{\xi\xi})_{no} e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon(C_{\xi\xi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + \epsilon(C_{\xi\xi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
(\tau_{\phi\phi})_n &= [(C_{\phi\phi})_{no} e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\
&+ \epsilon(C_{\phi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + \epsilon(C_{\phi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n \\
(\tau_{\theta\theta})_n &= [(C_{\theta\theta})_{no} e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)}
\end{aligned} \tag{5.64}$$

$$+ \quad \epsilon(C_{\theta\theta})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + \epsilon(C_{\theta\theta})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n$$

$$(\tau_{\xi\phi})_n = [(C_{\xi\phi})_{no} e^{\beta_n(\phi-\frac{1-(-1)^n}{2}\phi_o)}$$

$$+ \quad \epsilon(C_{\xi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + \epsilon(C_{\xi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}]D_n$$

where $D_n (n > 4)$ remains a free complex constant.

The total solution for the toroidal ring comprises of the sum of the eigenfunctions corresponding to the eigenvalues $\beta_n (n = 1, 2, \ldots, n)$. This total solution satisfies the homogeneous boundary conditions (5.1) on the curved surfaces of the ring. As well, it satisfies the Navier equations in the domain. D_n $(n = 1, 2, 3, ..., n)$ are n free constants which are available to satisfy the end boundary conditions (5.2, 5.3). The approximate boundary point matching technique [64] is employed to enforce these latter conditions. In this technique the boundary conditions are satisfied only at certain selected collocation points. The number of selected points is taken as so as to produce a linear equation set which contains about twice as many equations as unknowns. The linear least squares method is employed to solve this equation set. In this way, numerical values for the free constants D_n are obtained to complete the solution. An example for a specific load case is given in the following section.

5.6 Numerical Results

Numerical results of a toroidal ring subjected to pure bending moments are found in this section for the following values of the material and geometric parameters; $\nu = 0.3$, $R = 0.7m$, $a = 0.2m$, $b = 0.4m$, $c = 0.4m$, $\phi_o = 45°$. The detail of the

Table 5.4: Decay coefficients for thick-walled shells ($\lambda = 0.5$)

	Toroidal		*Cylindrical*		*Spherical*	
Set	*Real*	*Imag*	*Real*	*Imag*	*Real*	*Imag*
1*	2.7010	1.0000	1.7925	1.4256	1.6023	1.5418
2	6.0541	3.3707	6.4478	3.3760	3.4105	6.0986
3	10.8029	4.1123	11.3112	4.1529	4.1291	10.8147
4	15.4449	4.5931	16.1120	4.6543	4.6022	15.4498
5	20.0455	4.9516	20.8824	5.0278	4.9573	20.0481
6	24.6252	5.2380	25.6361	5.3261	5.2419	24.6267
7	29.1925	5.4768	30.3796	5.5747	5.4796	29.1935
8	33.7519	5.6815	35.1163	5.7878	5.6837	33.7526
9	38.3059	5.8607	39.8483	5.9743	5.8624	38.3064
10	42.8560	6.0202	44.5769	6.1402	6.0215	42.8563

* Real and Imag parts are separate roots.

FORTRAN program is given in appendix program 2. The initial solution using the unadjusted eigenvalues of eqn. 5.23 is presented in Fig. 5.2 and is compared with the FEM solution. It shows that there is a significant difference between these two solutions.

The adjusted third group and the initial fourth group eigenvalues in the first quadrant are presented in increasing order of magnitude in Table 5.4, along with corresponding values for cylindrical and spherical shells. The first set of roots in the first columns is the adjusted third group eigenvalues of the toroidal ring. They are two real eigenvalues ± 2.7010 and two imaginary eigenvalues $\pm i$ located on the real and imaginary axes. The latter two columns of values were found using theory presented in [64]. The eigenvalues are indicative of the rate of decay into the shell of the St. Venant end effects. Significant differences occur only for the lower eigenvalues.

Sample plots for the stress components corresponding to specific eigenvalues on the $\phi = \phi_o$ surface are given in Figs. 5.3-5.6. The stress components, $\sigma_{\xi\xi}, \sigma_{\phi\phi}, \sigma_{\theta\theta}$, and $\sigma_{\phi\theta}$, are presented by letters, SR, SQ, ST, and SQT. The real and imaginary parts of the stress components corresponding to the first four eigenvalues are plotted in these figures.

The specified values of the stresses on the end surfaces of the ring are on the surface $\phi = 0$

$$F_1^s(\xi) = \frac{1640.0}{1-\lambda}(1-\xi) - 800.0;\ G_1^s(\xi) = 0 \tag{5.65}$$

and on the surface $\phi = \phi_o$

$$F_2^s(\xi) = \frac{1900.0}{1-\lambda}(1-\xi) - 880.0;\ G_2^s(\xi) = 0 \tag{5.66}$$

These functions correspond to a pure bending moment acting on the two end surfaces. These boundary conditions of the edge surfaces are enforced at twice the number of eigenvalues collocation points.

The stress component σ_ϕ is the largest in this problem. The maximum values of σ_ϕ on the inner and outer surfaces of the toroidal ring corresponding to different numbers of the eigenvalues taken are shown in Table 5.5. The first two columns are maximum value of σ_ϕ on $\phi = 0°$ cross-section. The next two columns show the maximum value of σ_ϕ on $\phi = 22.5°$ cross-section. The last two columns show the maximum value of σ_ϕ on $\phi = 45°$ cross-section. It is clear that taking about twenty four eigenvalues provides sufficient accuracy, which agrees with the experience of Balabanov[49]. The data and figures presented in this section correspond to a solution containing twenty-four eigenvalues.

Table 5.5: σ_ϕ vs number of eigenvalues

	$\phi = 0°$		$\phi = 22.5°$		$\phi = 45°$	
Eigenvalues	$\sigma_{\phi\phi} - in$	$\sigma_{\phi\phi} - out$	$\sigma_{\phi\phi} - in$	$\sigma_{\phi\phi} - out$	$\sigma_{\phi\phi} - in$	$\sigma_{\phi\phi} - out$
8	840.059	-798.681	796.681	-670.552	1019.948	-879.352
12	839.899	-799.483	817.436	-735.933	1019.908	-879.789
16	840.004	-799.955	821.921	-764.849	1020.002	-879.986
20	840.000	-799.999	798.194	-793.958	1020.000	-879.997
24	840.000	-800.000	845.794	-802.611	1020.000	-879.999
28	840.000	-800.000	824.039	-811.890	1020.000	-880.000
32	840.000	-800.000	849.479	-800.020	1020.000	-880.000
36	840.000	-800.000	840.470	-774.966	1020.000	-880.000

The numerical results obtained using the present method are compared with the results obtained from the finite element method. The software MSC-NASTRAN was used. The cross-section of the toroidal ring is discreted by 20 × 20 same-size triangle ring elements. The variation of stress components on the surface of $\phi = 9°, 18°, 22.5°$ are shown in Fig. 5.7-5.9. There is a general good agreement between the DBTE and the FEM. The variation of the stress component σ_ϕ on the inner and outer surfaces of the ring is shown in Fig. 5.10. Even if the DBTE theory is developed only to the first order, one still obtains a reasonable agreement with FEM results.

The distributions of the four stress components on the surface $\phi = 22.5°$, and the variation of the maximum stress component $\sigma_{\phi\phi}$ vs ϕ with different values for the parameter ϵ (0.2-0.5) are shown in Fig. 5.11-5.18. The stress components $\sigma_{\xi\xi}, \sigma_{\theta\theta}$ and $\sigma_{\xi\phi}$ with small parameter ϵ are much larger than those with large ϵ. The solution of a toroidal ring with small ϵ is close to the solution of cylindrical ring. Because the rotational stiffness of the cross section is relatively small, a large hoop stress component, $\sigma_{\theta\theta}$, is obtained for the case of a small ϵ.

5.7 Conclusions

An outline of the theory for determining the homogeneous eigenfunctions for an axisymmetric toroidal ring has been presented. Results for eigenvalues have been calculated and these have been compared with corresponding results for cylindrical and spherical shells. A sample problem of a 45° toroidal ring subjected to a pure bending moment on two edge surfaces was presented. The results obtained compare well with results from the FEM.

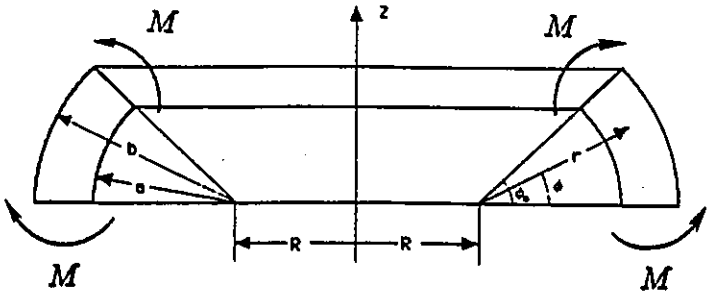

Figure 5.1: The cross-sectional view of toroidal ring

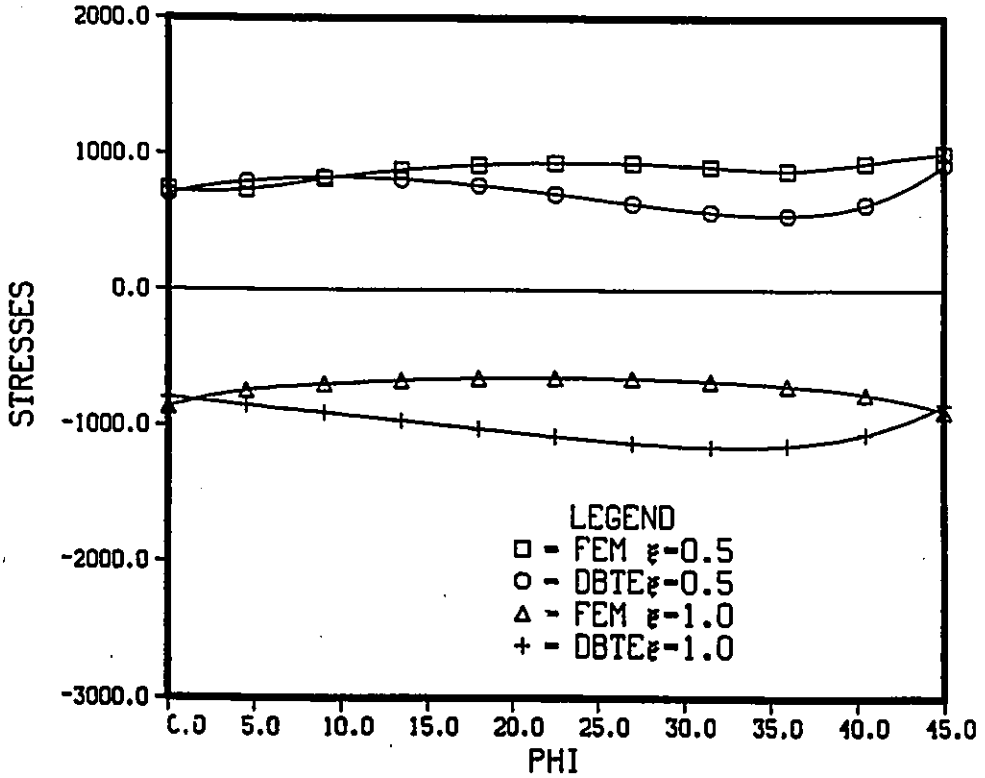

Figure 5.2: Comparison of $\sigma_{\phi\phi}$ on inner and outer surfaces of the ring of initial (unadjusted) solution

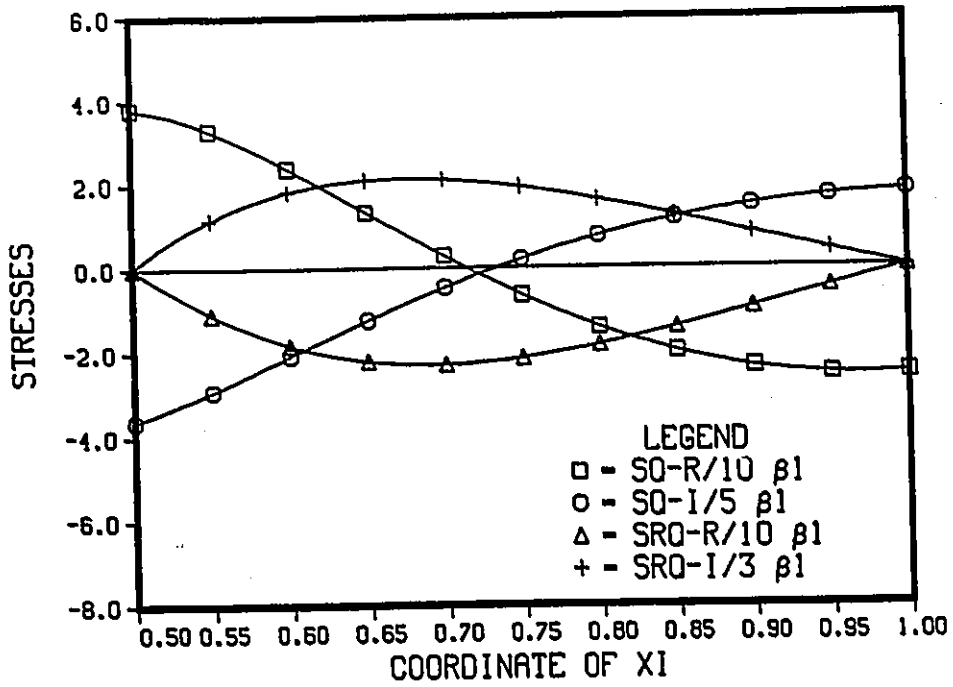

Figure 5.3: Plot of stress eigenfunctions for β_1

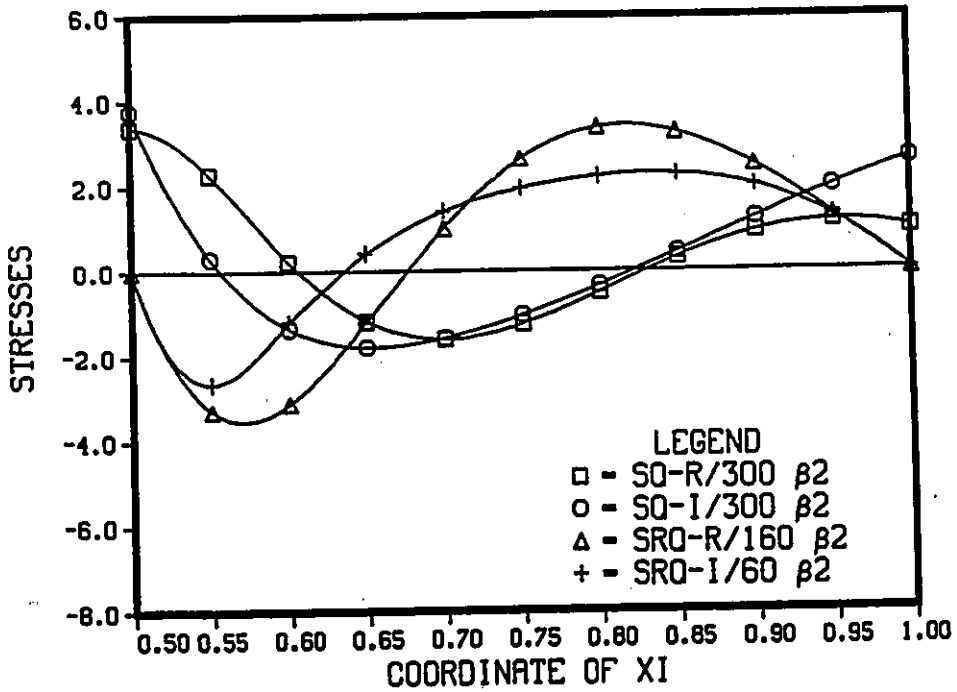

Figure 5.4: Plot of stress eigenfunctions for β_2

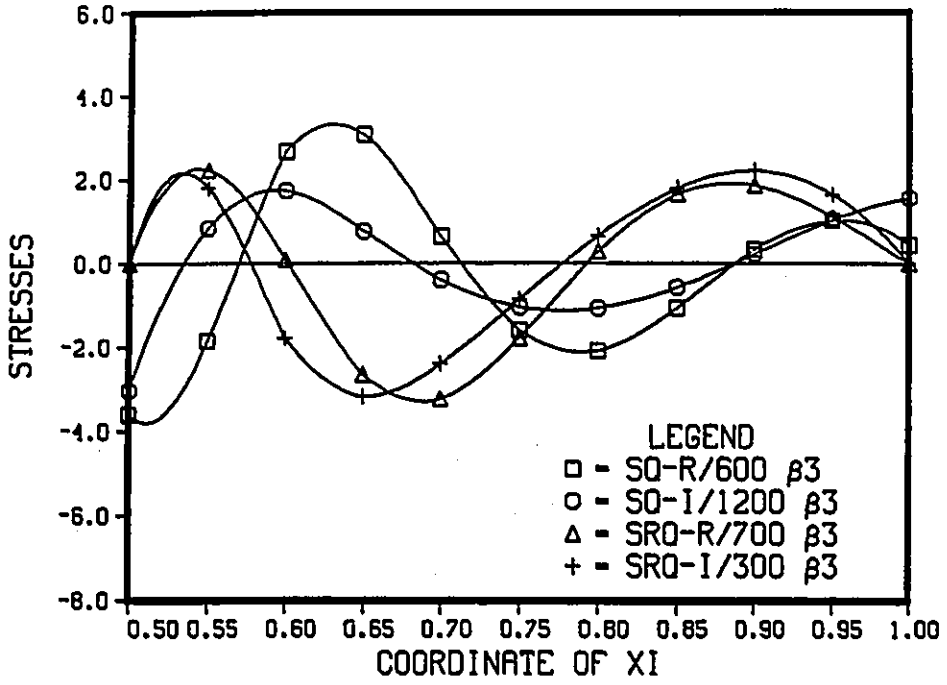

Figure 5.5: Plot of stress eigenfunctions for β_3

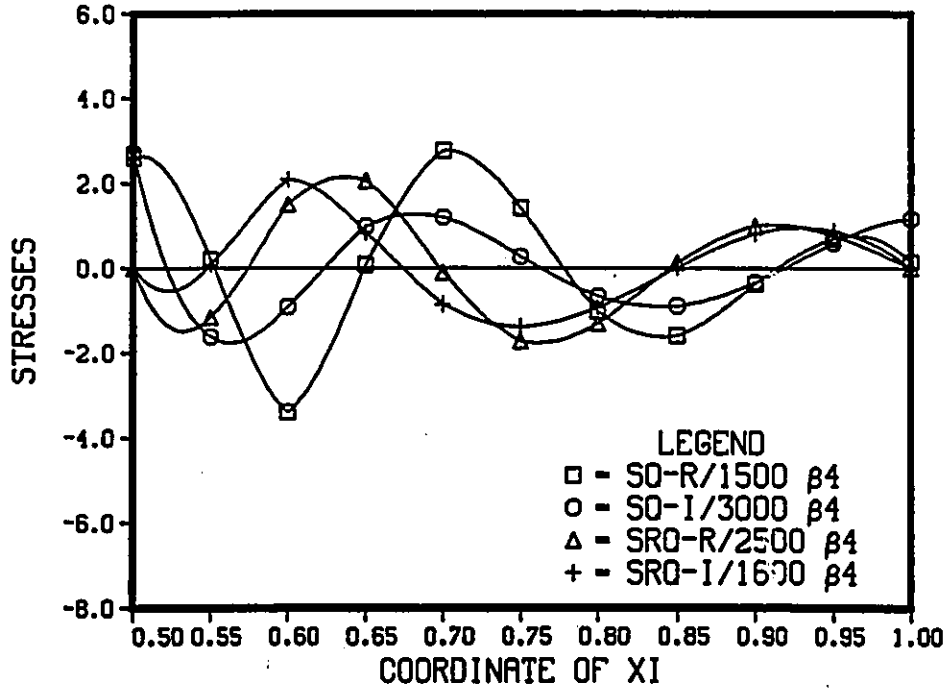

Figure 5.6: Plot of stress eigenfunctions for β_4

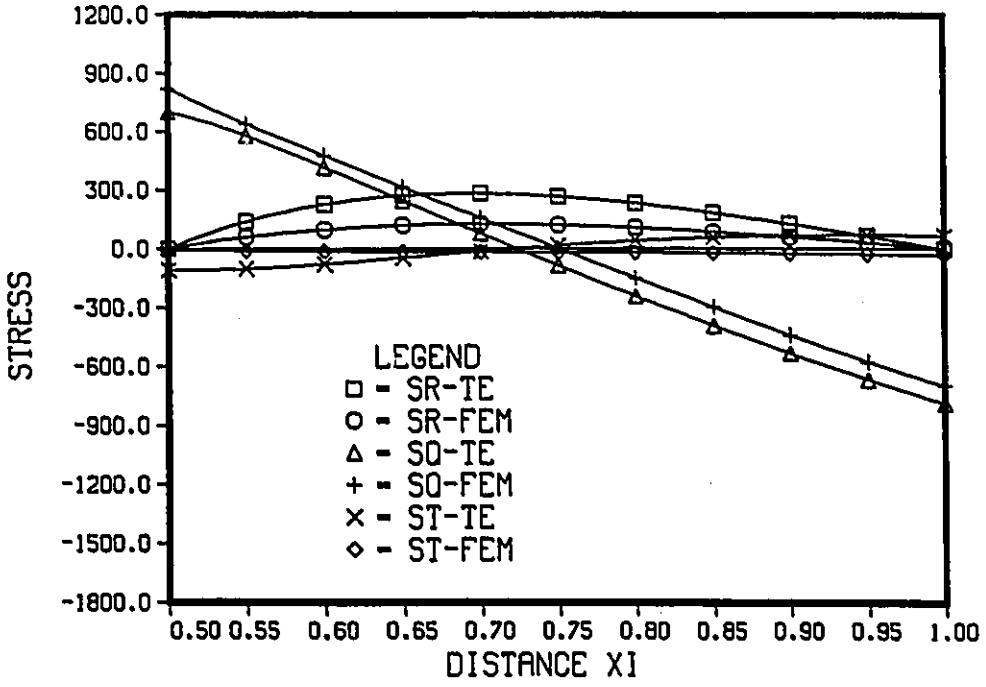

Figure 5.7: Comparison of stress components on $\phi = 9°$

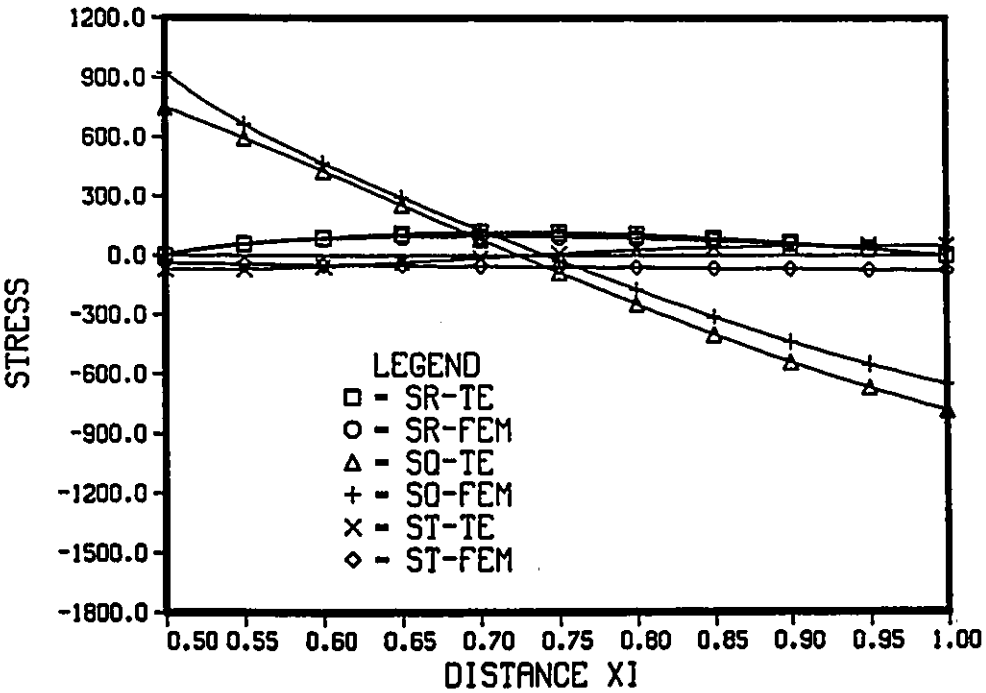

Figure 5.8: Comparison of stress components on $\phi = 18°$

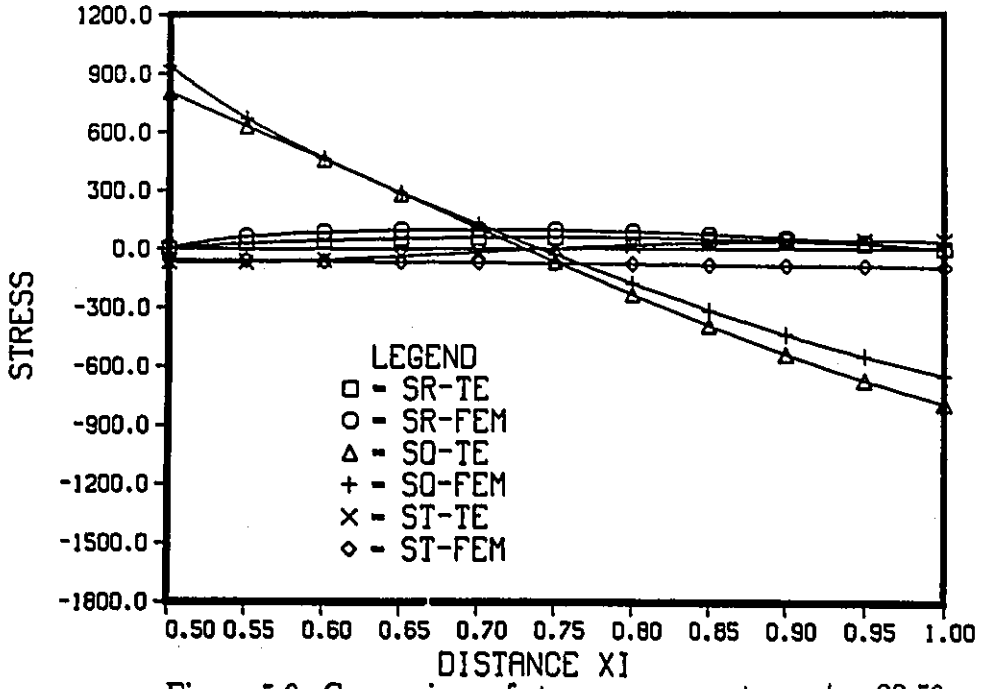

Figure 5.9: Comparison of stress components on $\phi = 22.5°$

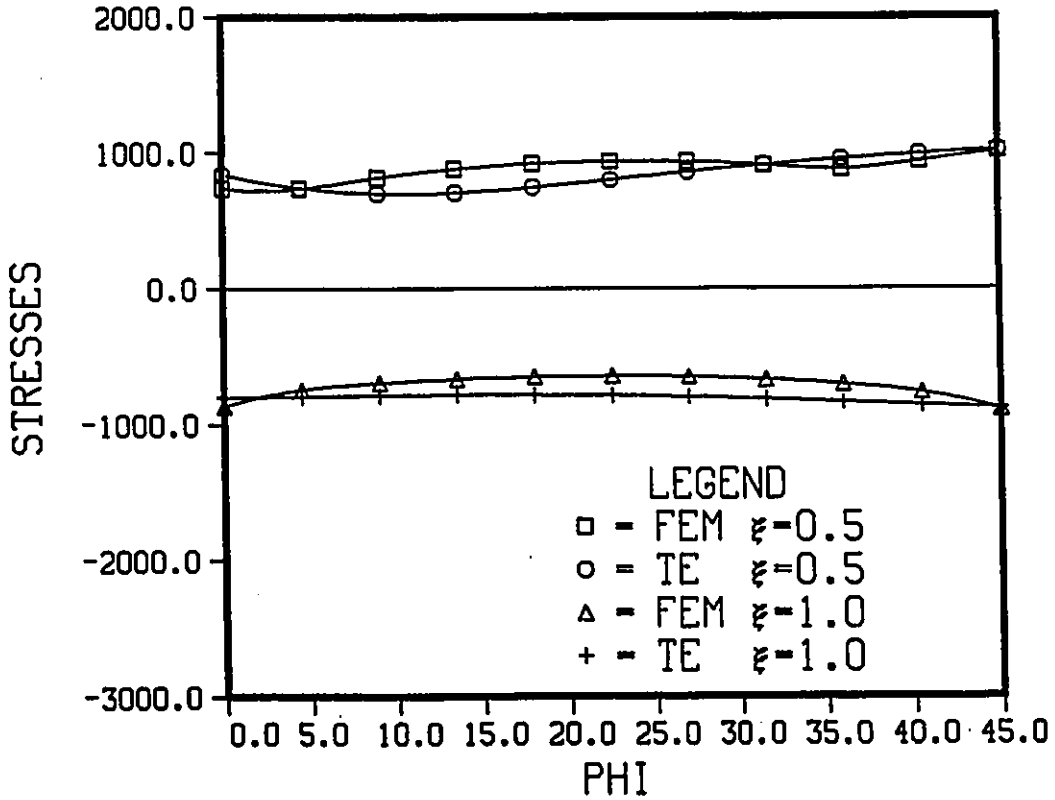

Figure 5.10: Comparison of $\sigma_{\phi\phi}$ on inner and outer surfaces of the ring

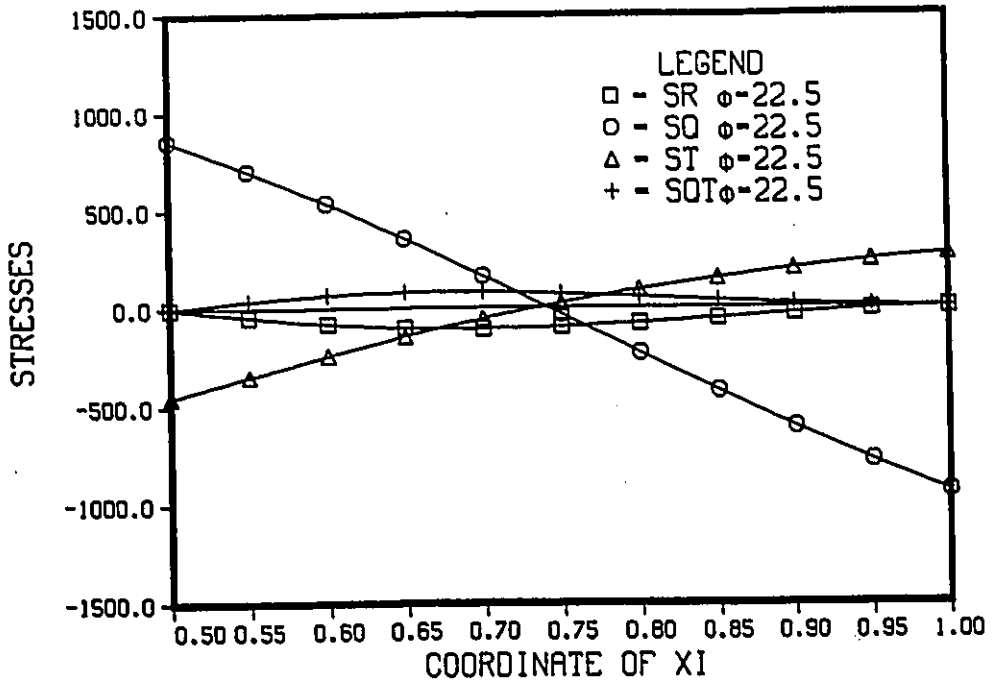

Figure 5.11: Stress factors for ϵ of 0.2 at $\theta = 22.5°$

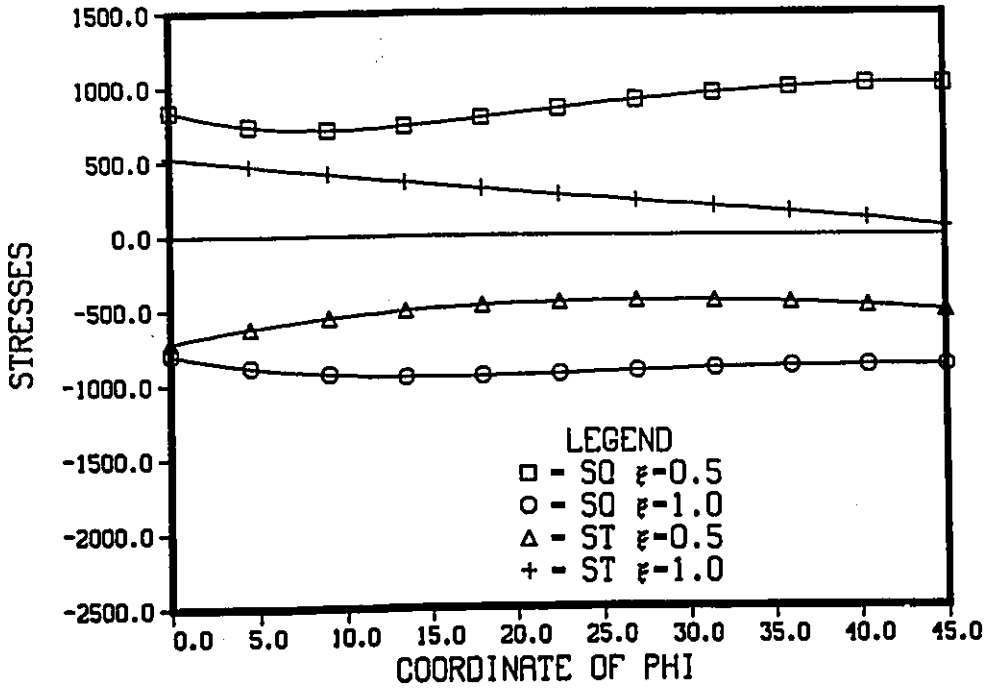

Figure 5.12: Stress factors for ϵ of 0.2 on inner and outer surface

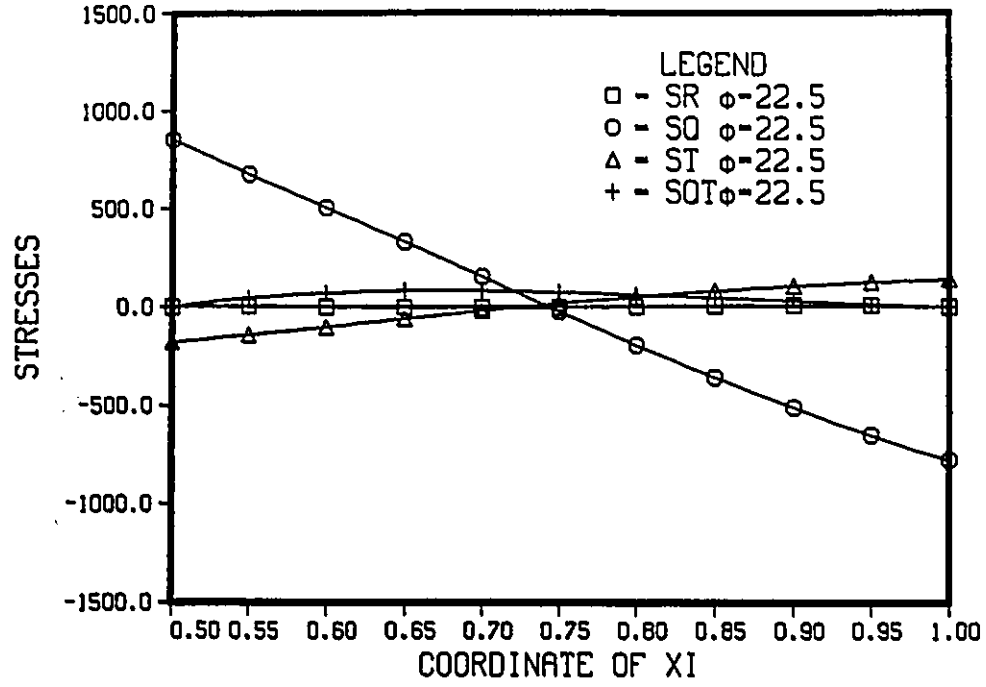

Figure 5.13: Stress factors for ϵ of 0.3 at $\theta = 22.5°$

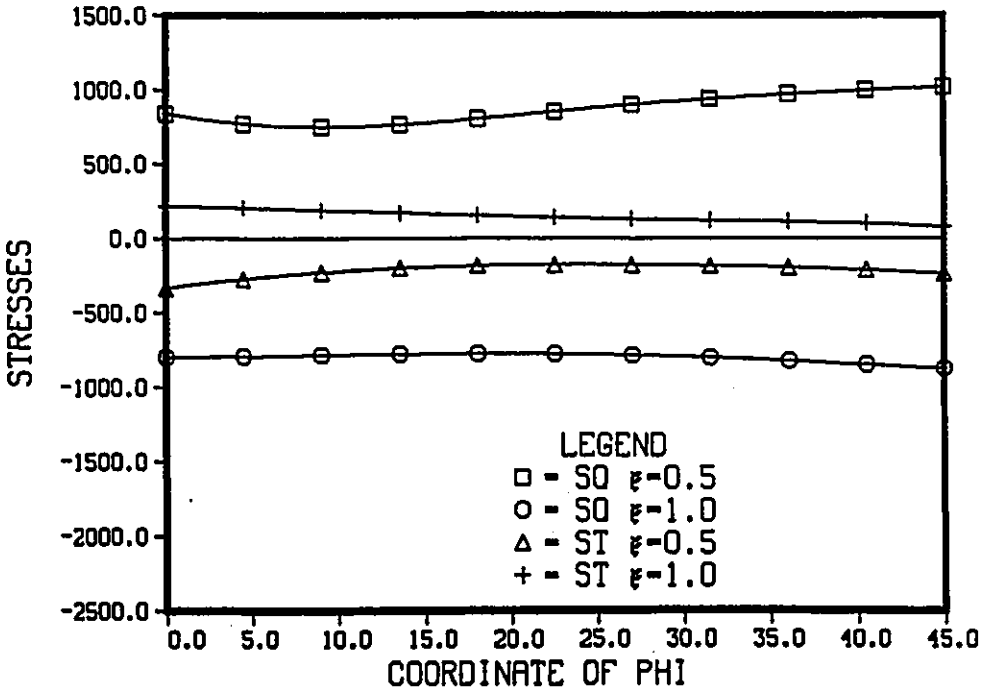

Figure 5.14: Stress factors for ϵ of 0.3 on inner and outer surface

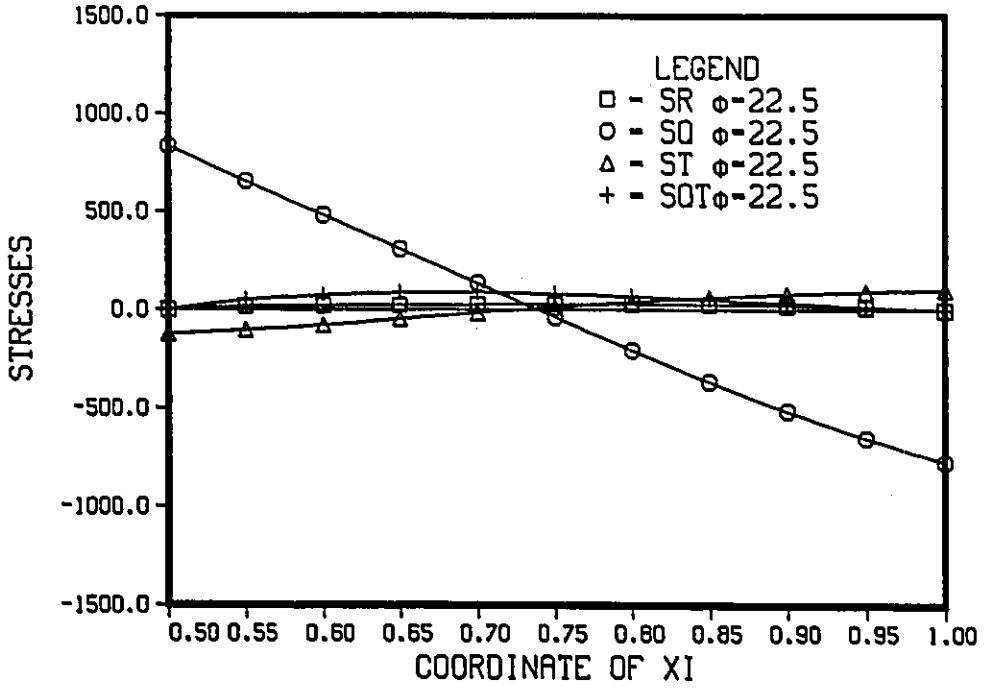

Figure 5.15: Stress factors for ϵ of 0.4 at $\theta = 22.5^{o}$

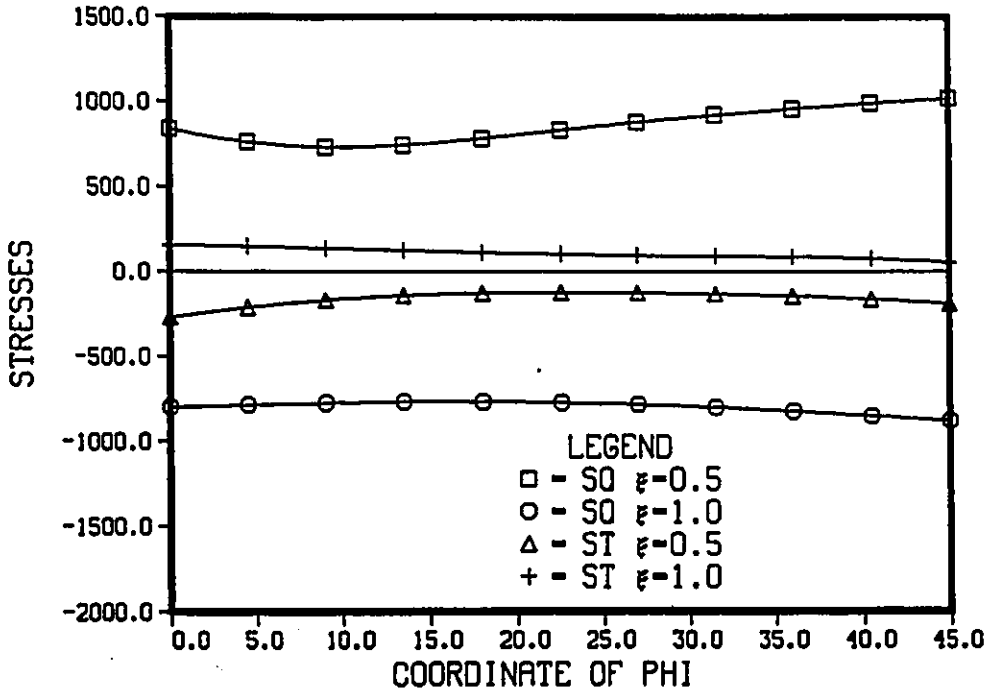

Figure 5.16: Stress factors for ϵ of 0.4 on inner and outer surface

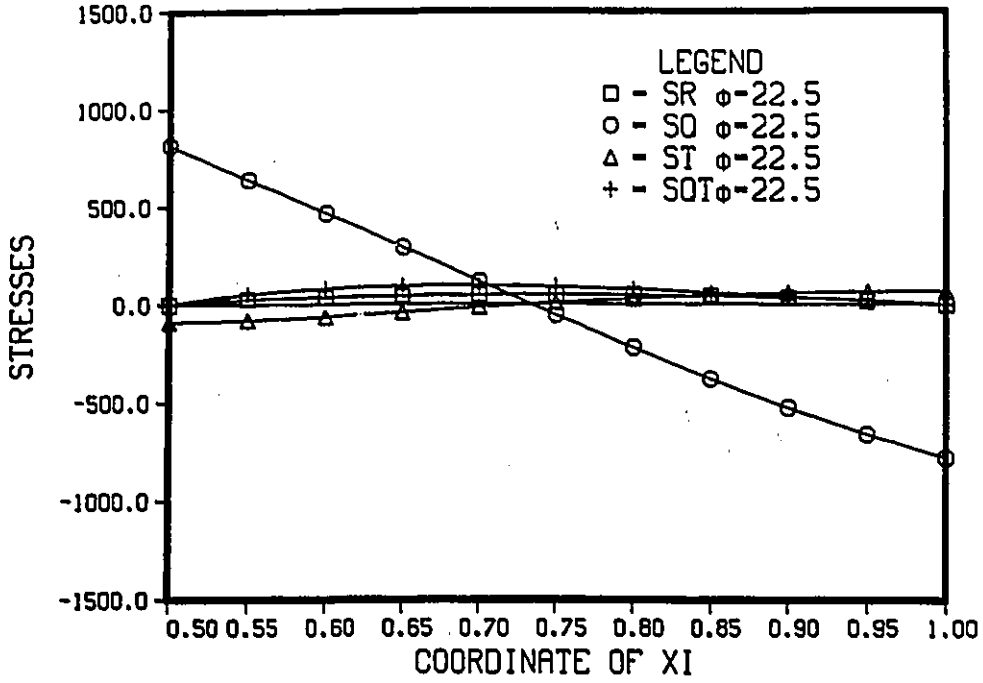

Figure 5.17: Stress factors for ϵ of 0.5 at $\theta = 22.5°$

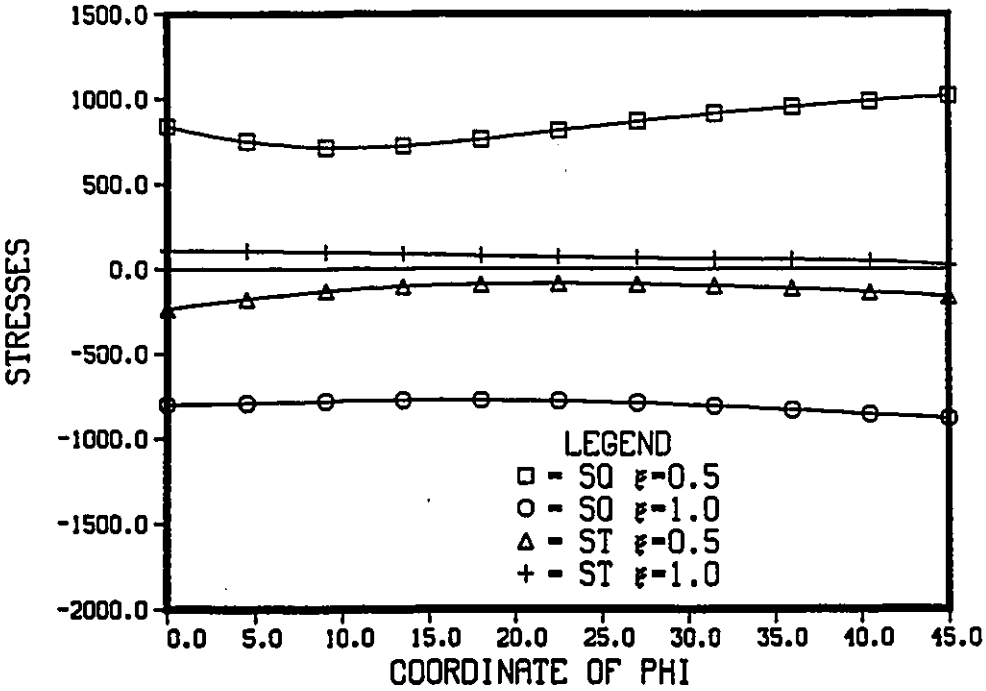

Figure 5.18: Stress factors for ϵ of 0.5 on inner and outer surface

5.8 Appendix 5.1

The particular solution for the first order governing equations can be expressed in two parts as

$$\begin{aligned} u^*_{n1} &= (A^{1*}_{n1}\xi^{2+\beta_n i} + A^{2*}_{n1}\xi^{2-\beta_n i} + A^{3*}_{n1}\xi^{\beta_n i})D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ &+ (A^{4*}_{n1}\xi^{2+\beta_n i} + A^{5*}_{n1}\xi^{2-\beta_n i} + A^{6*}_{n1}\xi^{-\beta_n i})D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ v^*_{n1} &= (B^{1*}_{n1}\xi^{2+\beta_n i} + B^{2*}_{n1}\xi^{2-\beta_n i} + B^{3*}_{n1}\xi^{\beta_n i})D_n e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \\ &+ (B^{4*}_{n1}\xi^{2+\beta_n i} + B^{5*}_{n1}\xi^{2-\beta_n i} + B^{6*}_{n1}\xi^{-\beta_n i})D_n e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} \end{aligned} \tag{5.67}$$

Substituting the above particular solution into the first order governing equations, and matching coefficients of corresponding terms, one obtains the following equations

$$\begin{aligned} &[(5-4\nu-\beta_n^2)+2(5-6\nu)\beta_n i]A^{1*}_{n1} + [-2(1-2\nu)\beta_n + (\beta_n^2-1+4\nu)i]B^{1*}_{n1} \\ =\ & 2(1-\nu)(\beta_n^2+1-4\nu-4\nu\beta_n i)a_{no} \\ &[4(1-\nu)\beta_n + (\beta_n^2+5-4\nu)i]A^{1*}_{n1} + [\beta_n^2+1-4\nu+4(2-3\nu)\beta_n i]B^{1*}_{n1} \\ =\ & 2(1-\nu)[-2(1-2\nu)\beta_n + (1-4\nu-\beta_n^2)i]a_{no} \\ &[(5-4\nu-\beta_n^2)-2(3-2\nu)\beta_n i]A^{2*}_{n1} + [-4\nu)\beta_n - (\beta_n^2+1-4\nu)i]B^{2*}_{n1} \\ =\ & 2(1-\nu)(1-\beta_n i)b_{no} \\ &[2(3-2\nu)\beta_n + (-\beta_n^2+1-4\nu)i]A^{2*}_{n1} + [\beta_n^2+1-4\nu+4\nu\beta_n i]B^{2*}_{n1} \\ =\ & 2(1-\nu)(\beta_n+i)b_{no} \\ &[-(3-4\nu+\beta_n^2)+2(1-2\nu)\beta_n i]A^{3*}_{n1} + [-4(1-\nu)\beta_n + (\beta_n^2-3+4\nu)i]B^{3*}_{n1} \\ =\ & 2(1-\nu)[\beta_n^2-4+4\nu+(5-4\nu)\beta_n i]d_{no} \end{aligned} \tag{5.68}$$

$$
\begin{aligned}
& [2(1-2\nu)\beta_n + (\beta_n^2 + 3 - 4\nu)i]A_{n1}^{3*} + [\beta_n^2 - 3 + 4\nu + 4(1-\nu)\beta_n i]B_{n1}^{3*} \\
= \; & 2(1-\nu)[(5-4\nu)\beta_n - (\beta_n - 4 + 4\nu)i]d_{no}
\end{aligned}
$$

and

$$
\begin{aligned}
& [(5-4\nu-\beta_n^2) + 2(3-2\nu)\beta_n i]A_{n1}^{4*} + [4\nu\beta_n + (\beta_n^2 + 1 - 4\nu)i]B_{n1}^{4*} \\
= \; & (1+\beta_n i)a_{no} \\
& [2(3-2\nu)\beta_n + (\beta_n^2 - 5 + 4\nu)i]A_{n1}^{4*} + [\beta_n^2 + 1 - 4\nu - 4\nu\beta_n i]B_{n1}^{4*} \\
= \; & (\beta_n - i)a_{no} \\
& [(5-4\nu-\beta_n^2) - 2(5-6\nu)\beta_n i]A_{n1}^{5*} + [4\nu\beta_n - (\beta_n^2 - 1 + 4\nu)i]B_{n1}^{5*} \\
= \; & (\beta_n^2 + 1 - 4\nu + 4\nu\beta_n i)b_{no} \qquad (5.69) \\
& [4(1-\nu)\beta_n - (\beta_n^2 + 5 - 4\nu)i]A_{n1}^{5*} + [\beta_n^2 + 1 - 4\nu - 4(2-3\nu)\beta_n i]B_{n1}^{5*} \\
= \; & [-2(1-2\nu)\beta_n - (4 - 4\nu - \beta_n^2)i]b_{no} \\
& [-(3-4\nu+\beta_n^2) - 2(1-2\nu)\beta_n i]A_{n1}^{6*} + [-4(1-\nu)\beta_n - (\beta_n^2 - 3 + 4\nu)i]B_{n1}^{6*} \\
= \; & [\beta_n^2 - 4(1-\nu) - (5-4\nu)\beta_n i]c_{no} \\
& [2(1-2\nu)\beta_n - (\beta_n^2 + 3 - 4\nu)i]A_{n1}^{6*} + [\beta_n^2 - 3 + 4\nu - 4(1-\nu)\beta_n i]B_{n1}^{6*} \\
= \; & [(5-4\nu)\beta_n - (4 - 4\nu - \beta_n^2)i]c_{no}
\end{aligned}
$$

where a_{no}, b_{no}, c_{no}, and d_{no} are the zeroth order eigenvactors. Solving these two-equation sets, one obtains the constants A_{n1}^{i*}, and B_{n1}^{i*} $(i = 1, ..., 6)$ in terms of $a_{no}, b_{no}, c_{no}, d_{no}$, and β_n.

5.9 Appendix 5.2

The general terms of the first order stress fields are

$$
\begin{aligned}
\tau_{\xi\xi n1} &= [(C_{\xi\xi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\xi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}] D_n \\
\tau_{\phi\phi n1} &= [(C_{\phi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\phi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}] D_n \\
\tau_{\theta\theta n1} &= [(C_{\theta\theta})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\theta\theta})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}] D_n \\
\tau_{\xi\phi n1} &= [(C_{\xi\phi})^1_{n1} e^{(\beta_n+i)(\phi-\frac{1-(-1)^n}{2}\phi_o)} + (C_{\xi\phi})^2_{n1} e^{(\beta_n-i)(\phi-\frac{1-(-1)^n}{2}\phi_o)}] D_n
\end{aligned}
\tag{5.70}
$$

where

$$
\begin{aligned}
(C_{\xi\xi})^1_{n1} &= -(\beta_n^2+3\beta_n i)a^1_{n1}\xi^{\beta_n i-1} - (\beta_n^2+2+\beta_n i)b^1_{n1}\xi^{1-\beta_n i} \\
&- [\beta_n^2-(3-4\nu)\beta_n i]c^1_{n1}\xi^{-(1+\beta_n i)} + [2(5-4\nu)-\beta_n^2-(7-4\nu)\beta_n i]d^1_{n1}\xi^{-(3-\beta_n i)} \\
+\frac{\nu}{1-2\nu} &\{ [\frac{1-2\nu}{\nu}(2+\beta_n i)A^{1*}_{n1} + (3+\beta_n i)A^{1*}_{n1} + (\beta_n+i)B^{1*}_{n1} + (1+\beta_n i)a_{no}]\xi^{(1+\beta_n i)} \\
&+ [\frac{1-2\nu}{\nu}(2-\beta_n i)A^{2*}_{n1} + (3-\beta_n i)A^{2*}_{n1} + (\beta_n+i)B^{2*}_{n1} - (3-4\nu)b_{no}]\xi^{(1-\beta_n i)} \\
&+ [\frac{1-2\nu}{\nu}\beta_n i A^{3*}_{n1} + (1+\beta_n i)A^{3*}_{n1} + (\beta_n+i)B^{3*}_{n1} - (4-4\nu-\beta_n i)d_{no}]\xi^{-(1-\beta_n i)}\} \\
(C_{\phi\phi})^1_{n1} &= (\beta_n^2-\beta_n i)a^1_{n1}\xi^{-(1-\beta_n i)} + (\beta_n^2-6+5\beta_n i)b^1_{n1}\xi^{1-\beta_n i} \\
&+ [\beta_n^2-(3-4\nu)\beta_n i]c^1_{n1}\xi^{-(1+\beta_n i)} \\
&+ [\beta_n^2-2(5-4\nu)+(7-4\nu)\beta_n i]d^1_{n1}\xi^{-(3-\beta_n i)} \\
+\frac{\nu}{1-2\nu} &\{ [\frac{1-2\nu}{\nu}A^{1*}_{n1} + \frac{1-2\nu}{\nu}(\beta_n+i)B^{1*}_{n1} + (3+\beta_n i)A^{1*}_{n1} + (\beta_n+i)B^{1*}_{n1} \\
&+ (1+\beta_n i)a_{no}]\xi^{(1+\beta_n i)} + [\frac{1-2\nu}{\nu}A^{2*}_{n1} + \frac{1-2\nu}{\nu}(\beta_n+i)B^{2*}_{n1} + (3-\beta_n i)A^{2*}_{n1} \\
&+ (\beta_n+i)B^{2*}_{n1} - (3-4\nu)b_{no}]\xi^{(1-\beta_n i)} + [\frac{1-2\nu}{\nu}A^{3*}_{n1} + \frac{1-2\nu}{\nu}(\beta_n+i)B^{3*}_{n1} \\
&+ (1+\beta_n i)A^{3*}_{n1} + (\beta_n+i)B^{3*}_{n1} - (4-4\nu-\beta_n i)d_{no}]\xi^{-(1-\beta_n i)}\} \\
(C_{\theta\theta})^1_{n1} &= (-4\nu)\{(\beta_n i)a^1_{n1}\xi^{\beta_n i-1} + (2-\beta_n i)b^1_{n1}\xi^{1-\beta_n i}\}
\end{aligned}
$$

$$
\begin{aligned}
+\frac{\nu}{1-2\nu} \quad \{ \quad & [(1-\nu)(1+\beta_n i)a_{no} + (3+\beta_n i)\nu A^{1*}_{n1} + (\beta_n + i)\nu B^{1*}_{n1}]\xi^{(1+\beta_n i)} \\
+ \quad & [-(1-\nu)(3-4\nu)b_{no} + (3-\beta_n i)\nu A^{2*}_{n1} + (\beta_n + i)\nu B^{2*}_{n1}]\xi^{(1-\beta_n i)} \\
+ \quad & [(1-\nu)(-4+4\nu+\beta_n i)d_{no} + (1+\beta_n i)\nu A^{3*}_{n1} + (\beta_n + i)\nu B^{3*}_{n1}]\xi^{-(1-\beta_n i)}\} \\
(C_{\xi\phi})^1_{n1} \quad = \quad & (-\beta_n + \beta_n^2 i)a^1_{n1}\xi^{-(1-\beta_n i)} + (3\beta_n - \beta_n^2 i + 2i)b^1_{n1}\xi^{(1-\beta_n i)} \\
- \quad & [(3-4\nu)\beta_n + \beta_n^2 i]c^1_{n1}\xi^{-(1+\beta_n i)} \\
- \quad & [(7-4\nu)\beta_n + (\beta_n^2 - 10 + 8\nu)i]d^1_{n1}\xi^{-(3-\beta_n i)} \qquad (5.71) \\
+ \quad & [(\beta_n + i)A^{1*}_{n1} + (1+\beta_n i)B^{1*}_{n1}]\xi^{(1+\beta_n i)} \\
+ \quad & [(\beta_n + i)A^{2*}_{n1} + (1-\beta_n i)B^{2*}_{n1}]\xi^{(1-\beta_n i)} + [(\beta_n + i)A^{3*}_{n1} + (1-\beta_n i)B^{3*}_{n1}]\xi^{-(1-\beta_n i)} \\
(C_{\xi\xi})^2_{n1} \quad = \quad & -(\beta_n^2 + 2 - \beta_n i)a^2_{n2}\xi^{1+\beta_n i} - (\beta_n^2 - 3\beta_n i)b^2_{n2}\xi^{-(1+\beta_n)} \\
+ \quad & [2(5-4\nu) - \beta_n^2 + (7-4\nu)\beta_n i]c^2_{n2}\xi^{-(3+\beta_n i)} - [\beta_n^2 + (3-4\nu)\beta_n i]d^2_{n2}\xi^{-(1-\beta_n i)} \\
+\frac{\nu}{1-2\nu} \quad \{ \quad & [\frac{1-2\nu}{\nu}(2+\beta_n i)A^{1*}_{n2} + (3+\beta_n i)A^{1*}_{n2} + (\beta_n - i)B^{1*}_{n2} - (3-4\nu)a_{no}]\xi^{(1+\beta_n i)} \\
+ \quad & [\frac{1-2\nu}{\nu}(2-\beta_n i)A^{2*}_{n2} + (3-\beta_n i)A^{2*}_{n2} + (\beta_n - i)B^{2*}_{n2} + (1-\beta_n i)b_{no}]\xi^{(1-\beta_n i)} \\
+ \quad & [-\frac{1-2\nu}{\nu}A^{3*}_{n2} + (1-\beta_n i)A^{3*}_{n2} + (\beta_n - i)B^{3*}_{n2} - (4-4\nu+\beta_n i)c_{no}]\xi^{-(1+\beta_n i)}\} \\
(C_{\phi\phi})^2_{n1} \quad = \quad & (\beta_n^2 - 6 - 5\beta_n i)a^2_{n2}\xi^{(1+\beta_n i)} + (\beta_n^2 + 3\beta_n i)b^2_{n2}\xi^{-(1+\beta_n i)} \\
+ \quad & [\beta_n^2 - 2(5-4\nu) - (5-4\nu)\beta_n i]c^2_{n2}\xi^{-(3+\beta_n i)} + [\beta_n^2 + (3-4\nu)\beta_n i]d^2_{n2}\xi^{-(1-\beta_n i)} \\
+ \quad & \frac{\nu}{1-2\nu}[(3+\beta_n i)A^{1*}_{n2} + (\beta_n - i)B^{1*}_{n2} - (3-4\nu)a_{no}]\xi^{(1+\beta_n i)} \\
+ \quad & \frac{\nu}{1-2\nu}[(3-\beta_n i)A^{2*}_{n2} + (\beta_n - i)B^{2*}_{n2} + (1-\beta_n i)b_{no}]\xi^{(1-\beta_n i)} \\
+ \quad & \frac{\nu}{1-2\nu}[(1-\beta_n i)A^{3*}_{n2} + (\beta_n - i)B^{3*}_{n2} - (4-4\nu+\beta_n i)c_{no}]\xi^{-(1+\beta_n i)} \\
+ \quad & [A^{1*}_{n2} + (\beta_n - i)B^{1*}_{n2}]\xi^{(1+\beta_n i)} + [A^{2*}_{n2} + (\beta_n - i)B^{2*}_{n2}]\xi^{(1-\beta_n i)} \\
+ \quad & [A^{3*}_{n2} + (\beta_n - i)B^{3*}_{n2}]\xi^{-(1+\beta_n i)} \\
(C_{\theta\theta})^2_{n1} \quad = \quad & (-4\nu)\{(2+\beta_n i)\nu a^2_{n2}\xi^{(1+\beta_n i)} + (\beta_n i)b^2_{n2}\xi^{-(1+\beta_n i)}\}
\end{aligned}
$$

$$
\begin{aligned}
+\frac{\nu}{1-2\nu} \quad \{ \quad & [\nu)(3+\beta_n i)A^{1*}_{n2} + (\beta_n - i)\nu B^{1*}_{n2} - (1-\nu)(3-4\nu)a_{no}]\xi^{(1+\beta_n i)} \\
+ \quad & [\nu(3-\beta_n i)A^{2*}_{n2} + (\beta_n - i)\nu B^{2*}_{n2} + (1-\nu)(1-\beta_n i)b_{no}]\xi^{(1-\beta_n i)} \\
+ \quad & [(1-\beta_n i)\nu A^{3*}_{n2} + (\beta_n - i)\nu B^{3*}_{n2} - (1-\nu)(4-4\nu+\beta_n i)c_{no}]\xi^{-(1+\beta_n i)}\} \\
(C_{\xi\phi})^2_{n1} \quad = \quad & (3\beta_n + \beta_n^2 i - 2i)a^2_{n2}\xi^{(1+\beta_n i)} - (\beta_n + \beta_n^2 i)b^2_{n2}\xi^{-(1+\beta_n i)} \\
- \quad & [(7-4\nu)\beta_n + (\beta_n^2 - 10 + 8\nu)i]c^2_{n2}\xi^{-(3+\beta_n i)} \\
- \quad & [(3-4\nu)\beta_n - \beta_n^2 i]d^2_{n2}\xi^{-(1-\beta_n i)} \\
+ \quad & [(\beta_n - i)A^{1*}_{n2} + (1+\beta_n i)B^{1*}_{n2}]\xi^{(1+\beta_n i)} + [(\beta_n - i)A^{2*}_{n2} + (1-\beta_n i)B^{2*}_{n2}]\xi^{(1-\beta_n i)} \\
+ \quad & [(\beta_n - i)A^{3*}_{n2} - (1+\beta_n i)B^{3*}_{n3}]\xi^{-(1+\beta_n i)}
\end{aligned}
$$

Chapter 6

TOROIDAL SHELL UNDER BAND LOADS

6.1 Introduction

Complete toroidal shells may be used as containers supported by several saddle supports, with a 'wrap' angle from 120° to 180°. If the containers are mounted on a movable base such as a space vehicle, 360° supports are required. In these two cases, the toroidal shells are subjected to pad or band loads. In the present chapter, the displacement-based TE is employed to solve the elastostatic problem of a complete toroidal shell subjected to band loads. The governing Navier equations were given up to and including the third order in the third chapter. The band loads are expanded in a Fourier series. A general solution for a typical term of the loading is then presented. Finally, an example including a thick-walled toroidal shell under four band loads is considered. Contributions up to and including the third order term in the successive approximation series are included. Numerical results

are computed, and these are compared with results obtained from the finite element method (FEM).

6.2 Band Loading

Fig. 6.1 indicates that the complete toroidal shell is axisymmetric about the Z axis, while the applied band loads are symmetric about the XOZ and YOZ planes. The displacement and stress components are periodic functions of $\pi/2$, and thus the solution need be set up only for a typical quadrant of the shell. The band loads make contributions at two positions in each quadrant. In the first quadrant, the band loads may be expressed

$$p = \begin{cases} 0 & \alpha < \theta < \pi/2 - \alpha \\ q & \text{otherwise} \end{cases}$$

where α is the semi-angular length of an entire band, and q is the intensity of the loading in the band. The loads can be expanded as a Fourier series in the θ (longitudinal) direction. For each term of the Fourier series, the amplitude is uniform in the ϕ (circumferential) direction. The condition of symmetry with respect to the X and Y axis is satisfied by taking the load as

$$p = \frac{p_o}{2} + \sum_{m=1}^{M} p_m \cos \frac{m\pi\theta}{\theta_o} \tag{6.1}$$

where

$$\begin{aligned} p_o &= \frac{4\alpha q}{\theta_o} \\ p_m &= \frac{2q}{m\pi}(1 + (-1)^m) \sin \frac{m\pi\alpha}{\theta_o};\ m = 1, 2, \ldots \end{aligned} \tag{6.2}$$

and M is the truncation integer. The angle θ_o equals $\pi/2$ for a shell quadrant. The first term of the load represents a uniform distribution and is an axisymmetric term.

6.3 Complementary Solution for In-Plane Case

It is evident from eqns. (3.26-3.29) that the solutions for orders greater than zero contain contributions from the lower orders, and that the complexity of solution increases with the order. For orders greater than the zeroth one, the solution will comprise of a complementary and a particular part. The complementary part is determined from equations which have the same form as those of the zeroth order. It is thus worthwhile to derive a general solution for this order. As the solution will represent the complete solution for the zeroth order, the notation of this order is used.

For a typical term of the applied load (except the first term), the magnitude varies in the longitudinal direction. This means that the full set of three displacement and six stress components is present. Because the applied loads are symmetric about the XOY plane, the shell deformation must be symmetric about the XOY plane. The tangential displacement v is zero at $\phi = 0°$ and 180°. Thus the non-dimensional displacements (Fig. 6.2) are taken as

$$\begin{aligned} u_o &= C_1 \xi^\gamma \cos n\phi F(\theta) \\ v_o &= C_2 \xi^\gamma \sin n\phi F(\theta) \\ w_o &= C_3 \xi^\gamma \cos n\phi G(\theta) \end{aligned} \tag{6.3}$$

where the functions $F(\theta)$ and $G(\theta)$ are arbitrary at this point. The quantities C_1, C_2, C_3, γ, and n are constants and parameters which are determined from the zeroth-order Navier equations and stress boundary conditions on the shell surfaces. There are four roots for γ which are obtained through the Navier equations. Using superposition and summing all possible solutions with respect to n, one obtains the complementary solution as

$$\begin{aligned}
u &= [a_o\xi^{-1} + b_o\xi + \{a_1 + b_1(1-4\nu)\xi^2 + c_1\xi^{-2} + d_1(3-4\nu)\ln\xi\}\cos\phi \\
&+ \sum_{n=2}^{\infty}\{a_n\xi^{n-1} + b_n(2-n-4\nu)\xi^{n+1} + c_n\xi^{-(n+1)} \\
&+ d_n(-2-n+4\nu)\xi^{-(n-1)}\}\cos n\phi]F(\theta) \\
v &= [+\{-a_1 + b_1(5-4\nu)\xi^2 + c_1\xi^{-2} - d_1(3-4\nu)\ln\xi - d_1\}\sin\phi \\
&+ \sum_{n=2}^{\infty}\{-a_n\xi^{n-1} + b_n(4+n-4\nu)\xi^{n+1} + c_n\xi^{-(n+1)} \\
&+ d_n(4-n-4\nu)\xi^{-(n-1)}\}\sin n\phi]F(\theta) \\
w &= [e_o + f_o\ln\xi + (e_1\xi^{-1} + f_1\xi)\cos\phi + \sum_{n=2}^{\infty}\{e_n\xi^{-n} + f_n\xi^n\}\cos n\phi]G(\theta)
\end{aligned} \tag{6.4}$$

where a_n, b_n, c_n, d_n, e_n and f_n $(n = 0, 1, 2, \ldots)$ are constants to be found from the boundary conditions.

The procedure for solving the problem is that of the method of successive approximation. Solutions are developed consecutively for the various orders starting from the zeroth order. For orders greater than zero, the solutions consist of a complementary and a particular part. The particular part is most easily set up by using trial functions for the displacements, selected so as to produce a match with the right hand members of the governing equations of that order. The complementary part is set up with the aid of eqn. (6.4). The constants of the complementary

solution for each order are found from the boundary conditions for that order. The total solution is the sum of the particular and complementary solutions of all the orders. This solution procedure will now be applied to the band loading case.

6.4 Solution of Toroidal Shell under Band Load

The analytical solution is discussed with regard to a typical term, q_m, of the Fourier series for the load (6.1). This term is given by

$$q_m = p_m \cos \frac{m\pi\theta}{\theta_o} \tag{6.5}$$

where p_m is a constant specified through eqn. (6.1). In the following, a solution is given for the term q_m up to and including the third order of the MSA series.

6.4.1 Boundary Conditions

The boundary or symmetry conditions satisfied for this typical load term are as follows;

on the curved surface $\xi = \lambda = a/b$

$$(\tau_{\xi\xi})_{mk} = (\tau_{\xi\phi})_{mk} = (\tau_{\xi\theta})_{mk} = 0 \tag{6.6}$$

on the curved surface $\xi = 1.0$

$$(\tau_{\xi\phi})_{mk} = (\tau_{\xi\theta})_{mk} = 0 \tag{6.7}$$

$$(\tau_{\xi\xi})_m = p_m \cos \frac{m\pi\theta}{\theta_o} \tag{6.8}$$

on the symmetric cross-sections

$$\frac{\partial u_{mk}}{\partial \theta} = (\tau_{\phi\theta})_{mk} = (\tau_{\xi\theta})_{mk} = 0 \tag{6.9}$$

where k indicates the order of the solution. In eqn. (6.8), $(\tau_{\xi\xi})_m$ is the sum of different order radial stress components of the load term q_m. In order to satisfy the stress boundary conditions (6.6-6.8) on the shell surfaces, and the symmetry conditions (6.9) on symmetric cross-section surfaces, the functions $F(\theta)$ and $G(\theta)$ must be taken as

$$F(\theta) = \cos\frac{m\pi\theta}{\theta_o}$$
$$G(\theta) = \sin\frac{m\pi\theta}{\theta_o}$$

The contributions from the various orders to the total solution will now be developed, starting from the zeroth order.

6.4.2 The Zeroth Order Solution

The applied band loads are not self-balanced individually. Considering one quadrant there are resultant forces acting in the XOY plane. The stiffness of the shell in the ϕ direction is usually larger than the stiffness in the θ direction. Therefore, the first assumed displacement corresponds to a rigid body motion of the shell cross-section. The zeroth-order displacement solution is thus taken as

$$\begin{aligned} u_{mo} &= a_{mo}\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\ v_{mo} &= -a_{mo}\sin\phi\cos\frac{m\pi\theta}{\theta_o} \\ w_{mo} &= 0 \end{aligned} \tag{6.10}$$

The corresponding zeroth-order stress components are found from eqns. (3.32) as

$$(\tau_{\xi\xi})_{mo} = (\tau_{\phi\phi})_{mo} = (\tau_{\theta\theta})_{mo} = (\tau_{\xi\phi})_{mo} = (\tau_{\xi\theta})_{mo} = (\tau_{\phi\theta})_{mo} = 0 \qquad (6.11)$$

6.4.3 The First Order Solution

For the first order analysis, the right hand side terms of the governing equations are evaluated by substituting the zeroth order displacement functions into eqns. (3.27, 3.30). For this order, evaluation of the right hand members leads to zero values in each case. Thus, the first order governing equations are also homogeneous equations, and only the complementary solution is required. This latter solution may be taken from eqns. (6.4). The first order stress components (3.32) are composed of two parts. The first part corresponds to the current order displacements, and the second part contains the contributions of the previous order displacements. The constants present in the assumed displacement solutions are mostly found to be zero, from the stress boundary conditions. An expectation is the term which is of the same harmonic in the ϕ direction as the second part of the stress components. The non-zero terms of the first order displacement functions are thus

$$\begin{aligned} u_{m1} &= (a_{m1}\xi^{-1} + b_{m1}\xi)\cos\frac{m\pi\theta}{\theta_o} \\ v_{m1} &= 0 \\ w_{m1} &= (e_{m1}\xi^{-1} + f_{m1}\xi)\cos\phi\sin\frac{m\pi\theta}{\theta_o} \end{aligned} \qquad (6.12)$$

The first order stress components are evaluated from eqn. (3.32). By satisfying the stress boundary conditions (6.6, 6.7), one obtains the constants

$$a_{mo} = b_{m1} = \frac{1-2\nu}{1+\nu}\lambda^{-2}a_{m1}$$

$$
\begin{aligned}
a_{m1} &= e_{m1} = 0 \\
f_{m1} &= \frac{1-2\nu}{1+\nu}\frac{m\pi}{\theta_o}\lambda^{-2}a_{m1}
\end{aligned}
\tag{6.13}
$$

Substituting these constants into the assumed solutions, one obtains the zeroth-order displacement and stress fields

$$
\begin{aligned}
u_{m0} &= \frac{1-2\nu}{1+\nu}\lambda^{-2}a_{m1}\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\
v_{m0} &= -\frac{1-2\nu}{1+\nu}\lambda^{-2}a_{m1}\sin\phi\cos\frac{m\pi\theta}{\theta_o} \\
w_{m0} &= 0 \\
(\tau_{\xi\xi})_{mo} &= (\tau_{\phi\phi})_{mo} = (\tau_{\theta\theta})_{mo} = 0 \\
(\tau_{\xi\phi})_{mo} &= (\tau_{\xi\theta})_{mo} = (\tau_{\phi\theta})_{mo} = 0
\end{aligned}
\tag{6.14}
$$

and the first order displacement and stress fields

$$
\begin{aligned}
u_{m1} &= (\xi^{-1} + \frac{1-2\nu}{1+\nu}\lambda^{-2}\xi)a_{m1}\cos\frac{m\pi\theta}{\theta_o} \\
v_{m1} &= 0 \\
w_{m1} &= \frac{1-2\nu}{1+\nu}\frac{m\pi}{\theta_o}\lambda^{-2}a_{m1}\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\xi\xi})_{m1} &= (\lambda^{-2} - \xi^{-2})a_{m1}\cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\phi\phi})_{m1} &= (\lambda^{-2} + \xi^{-2})a_{m1}\cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\theta\theta})_{m1} &= \lambda^{-2}a_{m1}\cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\xi\phi})_{m1} &= (\tau_{\xi\theta})_{m1} = (\tau_{\phi\theta})_{m1} = 0
\end{aligned}
\tag{6.15}
$$

Constant a_{m1} is determined later from the stress boundary condition (6.8).

6.4.4 The Second Order Solution

Substituting eqns. (6.14, 6.15) into eqns. (3.28, 3.30) determines the right hand side members of the second-order governing equations. The second-order governing equations then become

$$\begin{aligned}
\nabla^2 u_{m2} - \frac{u_{m2}}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_{m2}}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_{m2}}{\partial \xi} &= (g_1\xi^{-2} + g_2)\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\
\nabla^2 v_{m2} - \frac{v_{m2}}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_{m2}}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_{m2}}{\partial \phi} &= (g_1\xi^{-2} - g_2)\sin\phi\cos\frac{m\pi\theta}{\theta_o} \\
\nabla^2 w_{m2} &= g_3 \sin\frac{m\pi\theta}{\theta_o}
\end{aligned} \tag{6.16}$$

where

$$\begin{aligned}
g_1 &= \frac{2(1-\nu)}{1-2\nu}a_{m1} \\
g_2 &= -\frac{2\nu}{1+\nu}\left(\frac{m\pi}{\theta_o}\right)^2 a_{m1}\lambda^{-2} \\
g_3 &= \frac{2(1-2\nu)}{1+\nu}\frac{m\pi}{\theta_o}a_{m1}\lambda^{-2}
\end{aligned}$$

The trial function method is used to set up the particular solution, and this is added to the required complementary solution. Thus, the second order displacement functions are taken as

$$\begin{aligned}
u_{m2} &= \{a_{m2} + b_{m2}(1-4\nu)\xi^2 + c_{m2}\xi^{-2} + d_{m2}(3-4\nu)\ln\xi \\
&+ A^*_{m21}\xi^2 + A^*_{m22}\ln\xi\}\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\
v_{m2} &= \{-a_{m2} + b_{m2}(5-4\nu)\xi^2 + c_{m2}\xi^{-2} - d_{m2}(3-4\nu)\ln\xi - d_{m2} \\
&+ B^*_{m21}\xi^2 + B^*_{m22}\ln\xi\}\sin\phi\cos\frac{m\pi\theta}{\theta_o} \\
w_{m2} &= (e_{m2} + f_{m2}\ln\xi + C^*_{m21}\xi^2)\sin\frac{m\pi\theta}{\theta_o}
\end{aligned} \tag{6.17}$$

where $a_{m2}, b_{m2}, c_{m2}, d_{m2}, e_{m2}$ and f_{m2} are the second order free constants. The constants of the particular solution $A^*_{m21}, A^*_{m22}, B^*_{m21}, B^*_{m22}$ and C^*_{m21} are

$$\begin{aligned} A^*_{m21} &= B^*_{m21} = -\frac{\nu(1-2\nu)}{2(1+\nu)}(\frac{m\pi}{\theta_o})^2 a_{m1}\lambda^{-2} \\ A^*_{m22} &= B^*_{m22} = -2(1-\nu)a_{m1} \\ C^*_{m21} &= \frac{1-2\nu}{2(1+\nu)}\frac{m\pi}{\theta_o}a_{m1}\lambda^{-2} \end{aligned} \tag{6.18}$$

The corresponding second order stress components are

$$\begin{aligned} (\tau_{\xi\xi})_{m2} &= [2b_{m2}\xi - 2c_{m2}\xi^{-3} + (3-2\nu)d_{m2}\xi^{-1} - (2-\nu)a_{m1}\xi^{-1}]\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\ (\tau_{\phi\phi})_{m2} &= [6b_{m2}\xi + 2c_{m2}\xi^{-3} - (1-2\nu)d_{m2}\xi^{-1} - \nu a_{m1}\xi^{-1}]\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\ (\tau_{\theta\theta})_{m2} &= [8\nu b_{m2}\xi + 2\nu d_{m2}\xi^{-1} + (1-2\nu)(\frac{m\pi}{\theta_o})^2 + (1-\nu)a_{m1}\xi^{-1}]\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\ (\tau_{\xi\phi})_{m2} &= [2b_{m2}\xi - 2c_{m2}\xi^{-3} - (1-2\nu)d_{m2}\xi^{-1} + (1-\nu)a_{m1}\xi^{-1}]\cos\phi\cos\frac{m\pi\theta}{\theta_o} \\ (\tau_{\xi\theta})_{m2} &= \frac{1}{2}[(f_{m2} - \frac{m\pi}{\theta_o}a_{m1})\xi]\sin\frac{m\pi\theta}{\theta_o} \\ (\tau_{\phi\theta})_{m2} &= 0 \end{aligned} \tag{6.19}$$

The free constants are determined by forcing the stress components $(\tau_{\xi\xi})_{m2}, (\tau_{\xi\phi})_{m2}$ and $(\tau_{\xi\theta})_{m2}$ to satisfy the stress-free boundary conditions on the toroidal shell surfaces. These constants are obtained as

$$\begin{aligned} a_{m2} &= 0 \\ b_{m2} &= -\frac{1}{8(1-\nu)}\frac{1}{1+\lambda^2}a_{m1} \\ c_{m2} &= -\frac{1}{8(1-\nu)}\frac{\lambda^2}{1+\lambda^2}a_{m1} \\ d_{m2} &= \frac{3-2\nu}{4(1-\nu)}a_{m1} \end{aligned} \tag{6.20}$$

$$e_{m2} = 0$$
$$f_{m2} = \frac{m\pi}{\theta_o} a_{m1}$$

6.4.5 The Third Order Solution

The third-order governing equations are more complicated than those for the previous orders. The right hand members contain contributions from the previous three orders. The contributions can be divided into two parts. The first part is independent of ϕ, i. e. a zeroth-harmonic term. The second part is a second harmonic in ϕ for the first two equations, and a first harmonic in ϕ for the third equation. Thus, the third-order governing equations take the form

$$\begin{aligned}
\nabla^2 u_{m3} - \frac{u_{m3}}{\xi^2} - \frac{2}{\xi^2}\frac{\partial v_{m3}}{\partial \phi} + \frac{1}{1-2\nu}\frac{\partial \psi_{m3}}{\partial \xi} = \\
g_4(\xi)\cos\frac{m\pi\theta}{\theta_o} + g_5(\xi)\cos 2\phi \cos\frac{m\pi\theta}{\theta_o} \qquad (6.21)\\
\nabla^2 v_{m3} - \frac{v_{m3}}{\xi^2} + \frac{2}{\xi^2}\frac{\partial u_{m3}}{\partial \phi} + \frac{1}{1-2\nu}\frac{1}{\xi}\frac{\partial \psi_{m3}}{\partial \phi} = g_6(\xi)\sin 2\phi \cos\frac{m\pi\theta}{\theta_o} \\
\nabla^2 w_{m3} = g_7(\xi)\cos\phi \sin\frac{m\pi\theta}{\theta_o}
\end{aligned}$$

where

$$\begin{aligned}
g_4(\xi) &= -4(1-2\nu)b_{m2}\xi - (1-2\nu)d_{m2}\xi^{-1} - 2A^*_{m21}\xi - \frac{1}{2}A^*_{m22}\xi^{-1} \\
&+ \frac{1}{1-2\nu}[4b_{m2}\xi - (3-4\nu)d_{m2}\xi^{-1} - A^*_{m22}\xi^{-1} - \frac{m\pi}{\theta_o}f_{m2}\xi^{-1} - 2\frac{m\pi}{\theta_o}C^*_{m21}\xi] \\
&+ \frac{1-\nu}{1-2\nu}[-a_{m1}\xi^{-1} + b_{m1}\xi + \frac{1}{2(1-\nu)}\frac{m\pi}{\theta_o}f_{m1}\xi] - [\frac{2(1-\nu)}{1-2\nu} + (\frac{m\pi}{\theta_o})^2]a_{m0}\xi \\
&- \{[\frac{1-\nu}{1-2\nu} + (\frac{m\pi}{\theta_o})^2]\xi^{-1} + [\frac{1-\nu}{1+\nu} + \frac{5-8\nu}{2+2\nu}(\frac{m\pi}{\theta_o})^2]\lambda^{-2}\xi\} \\
g_5(\xi) &= 2b_{m2}\xi + 2c_{m2}\xi^{-3} - 2(1-\nu)d_{m2}\xi^{-1} - \frac{1}{2}A^*_{m22}\xi^{-1} - \frac{1}{1-2\nu}[2(3-4\nu)b_{m2}\xi
\end{aligned}$$

$$
\begin{aligned}
&- 2c_{m2}\xi^{-3} + 2A^{*}_{m21}\xi] + \frac{1-\nu}{1-2\nu}[-a_{m1}\xi^{-1} + b_{m1}\xi + \frac{1}{2(1-\nu)}\frac{m\pi}{\theta_o}f_{m1}\xi] \qquad (6.22)\\
&- [\frac{2(1-\nu)}{1-2\nu} + (\frac{m\pi}{\theta_o})^2]a_{m0}\xi - \{\frac{1-\nu}{1-2\nu}\xi^{-1} + [\frac{1-\nu}{1+\nu}\xi + \frac{3-4\nu}{2(1+\nu)}(\frac{m\pi}{\theta_o})^2]\lambda^{-2}\xi\}a_{m1}\\
g_6(\xi) &= -2b_{m2}\xi + 2c_{m2}\xi^{-3} + (1-2\nu)d_{m2}\xi^{-1} + \frac{1}{1-2\nu}[2(3-4\nu)b_{m2}\xi + 2c_{m2}\xi^{-3}\\
&- d_{m2}\xi^{-1} + 2A^{*}_{m21}\xi] + \frac{1}{2}\lambda^{-2}a_{m1}[\frac{2(1-\nu)}{1-2\nu}\lambda^2\xi^{-1} + \frac{2(1-\nu)}{1+\nu}\xi + \frac{1}{1+\nu}(\frac{m\pi}{\theta_o})^2\xi]\\
&+ [\frac{2(1-\nu)}{1-2\nu} + (\frac{m\pi}{\theta_o})^2]a_{m0}\xi + [\frac{1-\nu}{1-2\nu}\xi^{-1} + (\frac{1-\nu}{1+\nu}\xi\lambda^{-2} + \frac{3-4\nu}{2(1+\nu)}(\frac{m\pi}{\theta_o})^2)\lambda^{-2}\xi]a_{m1}\\
g_7(\xi) &= -f_{m2}\xi^{-1} - 2C^{*}_{m21}\xi + \frac{m\pi}{\theta_o}(8b_{m2}\xi + 2d_{m2}\xi^{-1} + \frac{4}{1-2\nu}A^{*}_{m21}\xi\\
&+ \frac{1}{1-2\nu}A^{*}_{m22}\xi^{-1}) + \frac{1+2\nu}{1+\nu}\frac{m\pi}{\theta_o}\lambda^{-2}m1\xi - \frac{2(3-4\nu)}{1-2\nu}(\frac{m\pi}{\theta_o})^2a_{m0}\xi\\
&- \{\frac{3-4\nu}{1-2\nu}\lambda^2\xi^{-1} + \frac{2(1-\nu)}{1+\nu}[1 + (\frac{m\pi}{\theta_o})^2]\xi\}\lambda^{-2}\frac{m\pi}{\theta_o}a_{m1}
\end{aligned}
$$

Contributions from lower harmonic parts usually exceed contributions from higher harmonic parts. Therefore, only the zeroth-harmonic part of the right hand members is considered. Following the procedure used for the previous orders, one obtains the third-order solution for the displacement components as

$$
\begin{aligned}
u_{m3} &= (a_{m3}\xi^{-1} + b_{m3}\xi + A^{*}_{m31}\xi^3 + A^{*}_{m32}\ln\xi)\cos\frac{m\pi\theta}{\theta_o}\\
v_{m3} &= (c_{m3}\xi^{-1} + d_{m3}\xi + B^{*}_{m31}\xi^3 + B^{*}_{m32}\ln\xi)\cos\frac{m\pi\theta}{\theta_o} \qquad (6.23)\\
w_{m3} &= (e_{m3}\xi^{-1} + f_{m3}\xi + C^{*}_{m31}\xi^3 + C^{*}_{m32}\ln\xi)\cos\phi\sin\frac{m\pi\theta}{\theta_o}
\end{aligned}
$$

and for the stress components as

$$
\begin{aligned}
(\tau_{\xi\xi})_{m3} &= (-a_{m3}\xi^{-2} + g_8\xi^2 + \frac{1-\nu}{1-2\nu}A^{*}_{m32}\xi^{-1} + \frac{\nu}{1-2\nu}A^{*}_{m32}\xi^{-1}\ln\xi\\
&+ g_9\ln\xi + g_{10})\cos\frac{m\pi\theta}{\theta_o}\\
(\tau_{\phi\phi})_{m3} &= (a_{m3}\xi^{-2} + g_{11}\xi^2 + \frac{\nu}{1-2\nu}A^{*}_{m32}\xi^{-1} + \frac{1-\nu}{1-2\nu}A^{*}_{m32}\xi^{-1}\ln\xi
\end{aligned}
$$

$$
\begin{aligned}
&- g_9 \ln\xi + g_{10}) \cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\theta\theta})_{m3} &= (g_{12} + \frac{\nu}{1-2\nu} A^*_{m32}\xi^{-1}(1+\ln\xi) - g_{13}\ln\xi + g_{14}\xi^2)\cos\frac{m\pi\theta}{\theta_o} \\
(\tau_{\xi\phi})_{m3} &= 0 \\
(\tau_{\xi\theta})_{m3} &= \frac{1}{2}(-e_{m3}\xi^{-2} + f_{m3} + g_{15}\xi^2 + C^*_{m32}\xi^{-1} - \frac{m\pi}{\theta_o}c_{m2}\xi^{-2} \\
&+ g_{16}\ln\xi + g_{17})\cos\phi\sin\frac{m\pi\theta}{\theta_o} \\
(\tau_{\phi\theta})_{m3} &= \frac{1}{2}(-e_{m3}\xi^{-2} - f_{m3} + g_{18}\xi^2 - C^*_{m32}\xi^{-1}\ln\xi - \frac{m\pi}{\theta_o}c_{m2}\xi^{-2} \\
&+ g_{19}\ln\xi + g_{20})\cos\phi\sin\frac{m\pi\theta}{\theta_o}
\end{aligned}
\tag{6.24}
$$

where

$$
\begin{aligned}
A^*_{m31} &= \frac{a_{m1}}{32(1-\nu)(\lambda^2+\lambda^4)}[2(1-2\nu)^2(1+\lambda^2)\left(\frac{m\pi}{\theta_o}\right)^2 - 4\nu\lambda^2] \\
A^*_{m32} &= -\frac{\nu}{2(1-\nu)}\left(\frac{m\pi}{\theta_o}\right)^2 a_{m1} \\
B^*_{m31} &= B^*_{m32} = 0 \\
C^*_{m31} &= \frac{a_{m1}}{16(1-\nu)(\lambda^2+\lambda^4)}\frac{m\pi}{\theta_o}[4(1-\nu)(1-2\nu)(1+\lambda^2)\left(\frac{m\pi}{\theta_o}\right)^2 \\
&- 2(5-5\nu+\nu^2)\lambda^2 - 4(1-\nu)(2-\nu)] \\
C^*_{m32} &= \frac{3-2\nu}{4(1-\nu)}\frac{m\pi}{\theta_o}a_{m1} \\
a_{m3} &= g_8\lambda^4 + \frac{1-\nu}{1-2\nu}A^*_{m32}\lambda + \frac{\nu}{1-2\nu}A^*_{m32}\lambda\ln\lambda + g_9\lambda^2\ln\lambda + g_{10}\lambda^2 \\
b_{m3} &= c_{m3} = d_{m3} = 0 \\
e_{m3} &= \frac{g_{21}-g_{22}}{1-\lambda^2} \\
f_{m3} &= \frac{g_{21}\lambda^2 - g_{22}}{1-\lambda^2} \\
g_8 &= 3A^*_{m31} + \frac{\nu}{1-2\nu}[4A^*_{m31} - 2b_{m2} + \frac{m\pi}{\theta_o}C^*_{m21} \\
&+ \frac{p_m}{2(1-\lambda^2)\epsilon}\frac{1-2\nu}{1+\nu}(1+\frac{m^2\pi^2}{\theta^2}) + \frac{1}{2}a_{mo}]
\end{aligned}
$$

$$
\begin{aligned}
g_9 &= \frac{\nu}{1-2\nu}[(3-4\nu)d_{m2} + A^*_{m22} + \frac{m\pi}{\theta_o} f_{m2} \\
g_{10} &= \frac{\nu}{1-2\nu}[a_{m2} + \frac{1}{2}d_{m2} + \frac{m\pi}{\theta_o} e_{m2} + \frac{\lambda^2 p_m}{2(1-\lambda^2)\epsilon} \\
g_{11} &= A^*_{m31} + \frac{\nu}{1-2\nu}[4A^*_{m31} - 2b_{m2} + \frac{m\pi}{\theta_o} C^*_{m21} \\
&+ \frac{p_m}{2(1-\lambda^2)\epsilon}\frac{1-2\nu}{1+\nu}(1 + \frac{m^2\pi^2}{\theta^2}) + \frac{1}{2}a_{mo}] \\
g_{12} &= \frac{2\nu}{1-2\nu} b_{m3} + \frac{1-\nu}{1-2\nu}[a_{m2} + \frac{1}{2}d_{m2} + \frac{m\pi}{\theta_o} e_{m2} + \frac{\lambda^2 p_m}{2(1-\lambda^2)\epsilon}] \qquad (6.25) \\
g_{13} &= \frac{1-\nu}{1-2\nu}[(3-4\nu)d_{m2} + A^*_{m22} + \frac{m\pi}{\theta_o} f_{m2}] \\
g_{14} &= \frac{4\nu}{1-2\nu} A^*_{m31} + \frac{1-\nu}{1-2\nu}[-2b_{m2} + \frac{m\pi}{\theta_o} C^*_{m21} \\
&+ \frac{p_m}{2(1-\lambda^2)\epsilon}\frac{1-2\nu}{1+\nu}(1 + \frac{m^2\pi^2}{\theta^2}) + \frac{1}{2}a_{mo}] \\
g_{15} &= 3C^*_{m31} + \frac{m\pi}{\theta_o}[-(1-4\nu)b_{m2} - A^*_{m21} + b_{m1} - \frac{3}{4}a_{mo}] - C^*_{m21} + \frac{3}{4}f_{m1} \\
g_{16} &= \frac{m\pi}{\theta_o}[-(3-4\nu)d_{m2} - A^*_{m22}] - f_{m2} \\
g_{17} &= \frac{m\pi}{\theta_o}(a_{m1} - a_{m2}) - e_{m2} \\
g_{18} &= -C^*_{m31} + \frac{m\pi}{\theta_o}[-(5-4\nu)b_{m2} - B^*_{m21} + \frac{1}{4}a_{mo}] + C^*_{m21} - \frac{1}{4}f_{m2} \\
g_{19} &= \frac{m\pi}{\theta_o}[(3-4\nu)d_{m2} - B^*_{m22} + f_{m2} \\
g_{20} &= \frac{m\pi}{\theta_o}(a_{m2} + d_{m2}) + e_{m2} \\
g_{21} &= \frac{m\pi}{\theta_o} c_{m2} - C^*_{m32} - g_{15} - g_{17} \\
g_{22} &= \frac{m\pi}{\theta_o} c_{m2} - C^*_{m32}\lambda - g_{15}\lambda^4 - g_{16}\lambda^2 \ln\lambda - g_{17}\lambda^2
\end{aligned}
$$

Substituting the solutions up to the third order into eqns. (3.22, 3.23), one obtains the total solution for the typical load term q_m. The last constant a_{m1} is determined by forcing the sum of different order radial stress components $(\tau_{\xi\xi})_m$ to satisfy the

stress boundary condition (6.8)

$$\{(\tau_{\xi\xi})_{m0} + \epsilon(\tau_{\xi\xi})_{m1} + \epsilon^2(\tau_{\xi\xi})_{m2} + \epsilon^3(\tau_{\xi\xi})_{m3}\}_{\xi=1} = p_m \cos\frac{m\pi\theta}{\theta_o} \tag{6.26}$$

Solving this equation, one obtains the constant a_{m1} as

$$a_{m1} = -\frac{p_m}{\epsilon(\lambda^{-2} - 1 + \epsilon^2 g_{23})} \tag{6.27}$$

where

$$\begin{aligned} g_{23} &= (1-\lambda^4)g_8 - \frac{6(1-\nu)}{7(1-2\nu)}(1-\lambda)m^2 + \frac{6\nu}{7(1-2\nu)}m^2\lambda\ln\lambda \\ &- g_9\lambda^2\ln\lambda + (1-\lambda^2)g_{10} \end{aligned} \tag{6.28}$$

By summing the contributions from all the load terms q_m up to M, the total solution to the problem is obtained.

6.5 Numerical Results

Numerical results are presented for the following values of the geometric and material parameters;

$$\begin{aligned} &R/a = 5;\ \lambda = 0.5;\ \theta_o = 90^\circ \\ &\alpha = 8^\circ;\ \nu = 0.3;\ E = 0.206 \times 10^6 MPa \end{aligned} \tag{6.29}$$

This data corresponds to a shell for which the ϵ factor is 0.4. The FORTRAN code for solving this problem is presented in appendix program 3. A check on the convergence of the solution for this ϵ value indicated that satisfactory results were obtained when M was taken as 60 in the Fourier series, and the MSA series was

Table 6.1: The convergence rate of stress components

M	$\sigma_{\phi\phi} - in$	$\sigma_{\theta\theta} - in$	$\sigma_{\phi\phi} - out$	$\sigma_{\theta\theta} - out$
10	-2.5159	-2.1620	-1.7593	-3.0821
20	-2.2311	-1.6140	-1.5121	-1.9031
40	-2.2987	-1.7550	-1.5725	-2.2129
60	-2.3272	-1.8194	-1.5987	-2.3570
80	-2.3384	-1.8451	-1.6091	-2.4148
100	-2.3374	-1.8431	-1.6082	-2.4103
120	-2.3290	-1.8238	-1.6004	-2.3672
300	-2.3219	-1.8076	-1.5939	-2.3309
500	-2.3173	-1.7973	-1.5897	-2.3077

taken to the third order. The maximum stress factors for different M values are presented in table 6.1. It is evident that there is not too much difference of the solution when $M > 60$.

The numerical results obtained by the present method (labelled DBTE) are compared with results obtained from the finite element method (labelled FEM). For the FEM results the MSC NASTRAN software was used. The FEM model contained the upper half of the first quadrant of the shell, and consisted of 200 20-noded solid elements. A 2×10 (thick × circumferential) uniform mesh is made on the cross-section of the toroidal shell, and ten elements with interval $4^o, 4^o, 6^o, 6^o, 8^o, 10^o, 10^o, 12^o, 13^o, 13^o$ are made in longitudinal direction. Stress factors $(\sigma_{ij})_f$ are defined for the six stress components as

$$(\sigma_{ij})_f = \frac{\sigma_{ij}}{q}$$

where $i, j = \xi, \phi, \theta$, σ_{ij} is the actual stress.

A plot of the deformation of the entire shell in the XOY plane is shown in Fig. 6.3. The displacement components are enlarged 5×10^6 times. From this

figure, it is clear that the cross-sections at $\phi = 0, \pi/2, \pi$ and $3\pi/2$ are compressed under the band loads. These cross-sections are moved inward to the axis of the toroidal shell by the horizontal resultant of the band loads. The variation of the radial displacement with θ at $\phi = 0, \pi/2$ and π on the outside surface of the shell is given in Fig. 6.4. The agreement between TE and FEM is better at the top longitudinal line ($\phi = \pi/2$), than at the extrados ($\phi = 0°$) and the intrados ($\phi = \pi$) lines.

Fig. 6.5 shows the variation of the circumferential (SQ) and longitudinal (ST) stress factors vs ϕ, on the outside surface of the shell at $\theta = 0°$. For most cases (e.g. elbow bending, toroidal shell under pressure load), the largest stress factors are located at or close to the intrados. For the present problem, the largest stress factors appear at the extrados. The variation of the circumferential stress factor vs θ, on the outside and inside surfaces, is shown in Figs. 6.6, 6.7. This factor decays rapidly with θ, having significant values over a range of only about 2α from the load. The variation of the longitudinal stress factor vs θ on the outside surface is shown in Fig. 6.8. This variation closely resembles that on the inside surface. The longitudinal stress factor is significant over a range of approximately 3α.

It is observed from Figs. 6.3-6.8 that there is in general good agreement between the TE and FEM results.

The stress components on the inner and outer surfaces are plotted vs θ in Figs. 6.9 - 6.16. On the $\theta = 0°$ cross-section, the shear stress component is zero, and the normal stress component $\sigma_{\phi\phi}$ reaches its maximum value. The maximum value

of the stress component $\sigma_{\theta\theta}$ is located at the point of the edge of the applied band load with $\phi = 0°$. The variation of the stress components at $\phi = 0°, 45°, 90°, 135°$ and $180°$ vs θ on the shell surfaces are presented in Figs. 6.17 - 6.21. The shear stress component $\sigma_{\phi\theta}$ is zero at $\phi = 0°$ and $180°$. The maximum value of the shear stress component is located at the edge of the applied load at the top and bottom points of the shell ($\phi = 90°, 270°$).

Figs. 6.22 - 6.28 present the stress components $\sigma_{\phi\phi}, \sigma_{\theta\theta}$ and $\sigma_{\phi\theta}$ on the surface of $\phi = 0°$ for different values of the parameter ϵ and the ratio of thickness λ. The parameter ϵ varies from 0.1 to 0.4, and the ratio of thickness λ takes the values 0.5 to 0.8. The maximum stress factor of normal stress components $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ with different value of ϵ and λ are presented in Figs. 6.29 - 6.30 for the case of $\alpha = 8°$. The larger the ratio of inside radius over the outside radius is, the higher the stress factor $\sigma_{\phi\phi}$ is. The difference for longitudinal stress component $\sigma_{\theta\theta}$ is much smaller than that for the circumferential stress component.

Conclusions

An analytical solution has been presented to the problem of a thick-walled toroidal shell under band loads, using the displacement-based toroidal elasticity theory. An example concerning a complete toroidal shell subjected to four equally-spaced band loads was considered in detail. Numerical values for displacements and stresses obtained using this solution showed general good agreement with results from the finite element method.

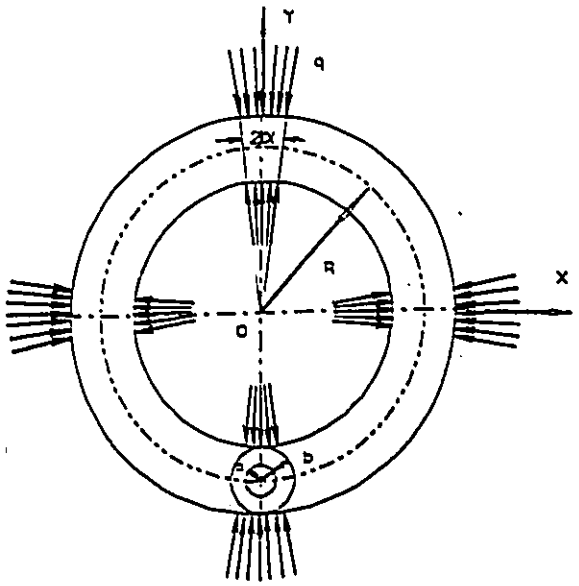

Figure 6.1: Geometry and loading.

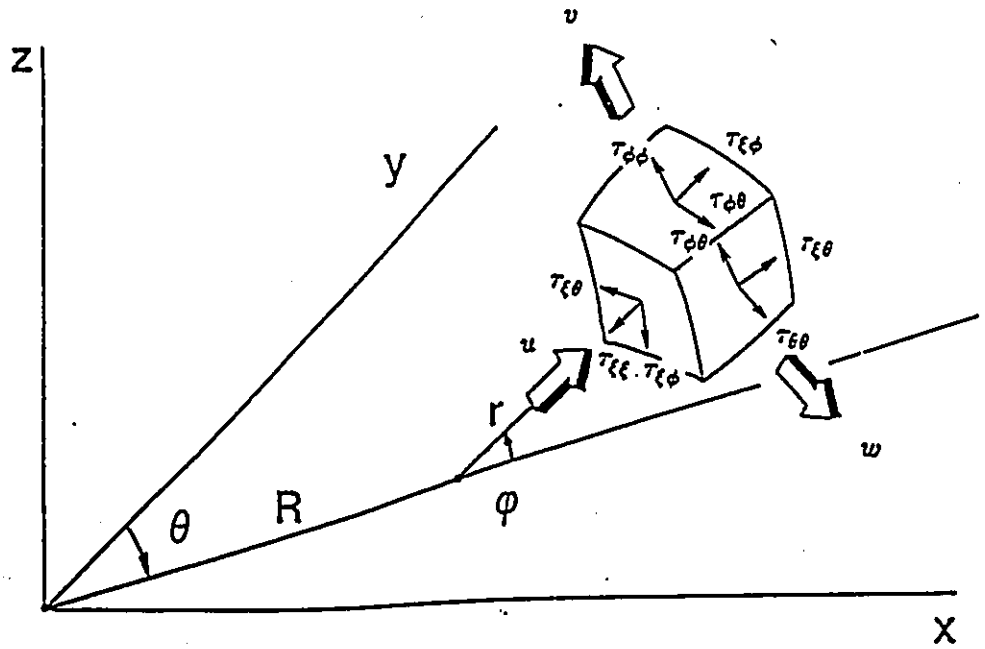

Figure 6.2: Displacement and stress components.

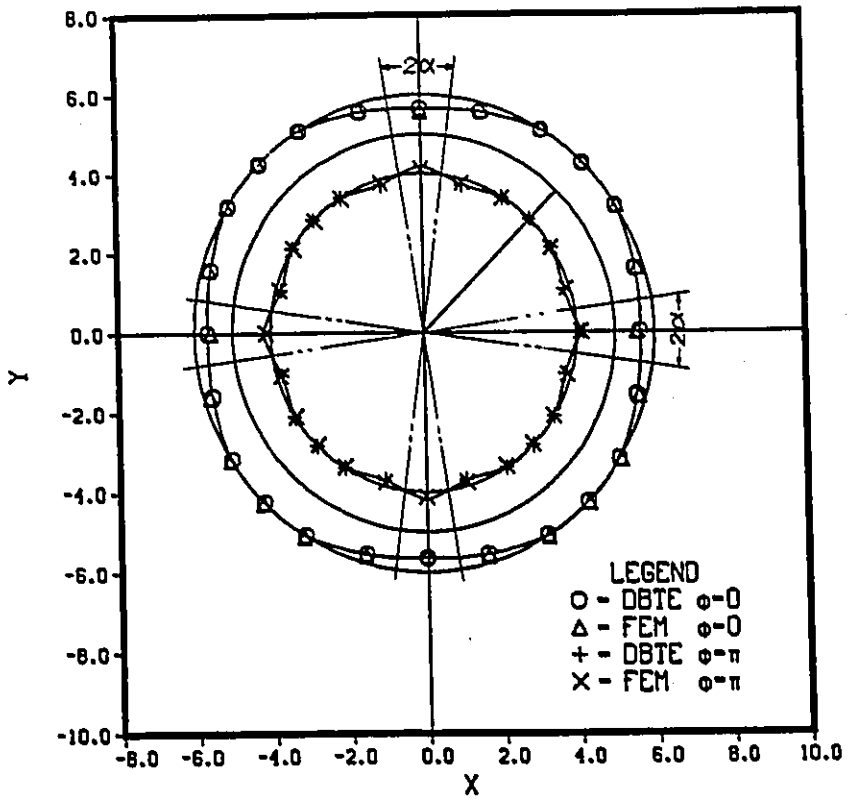

Figure 6.3: Plots of deformed shell in XOY plane.

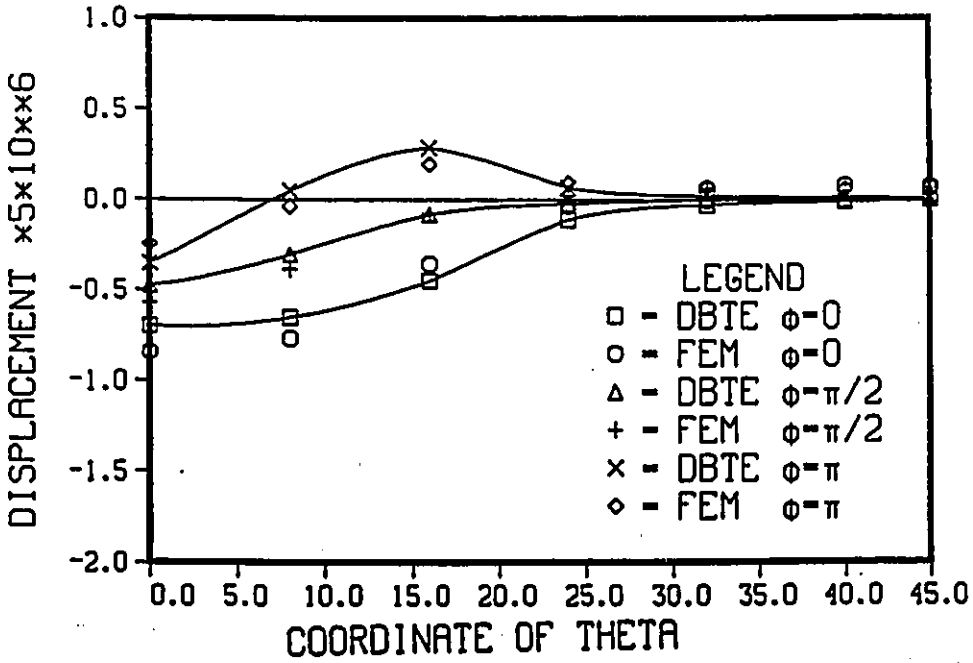

Figure 6.4: Comparison of radial displacement on outside surface.

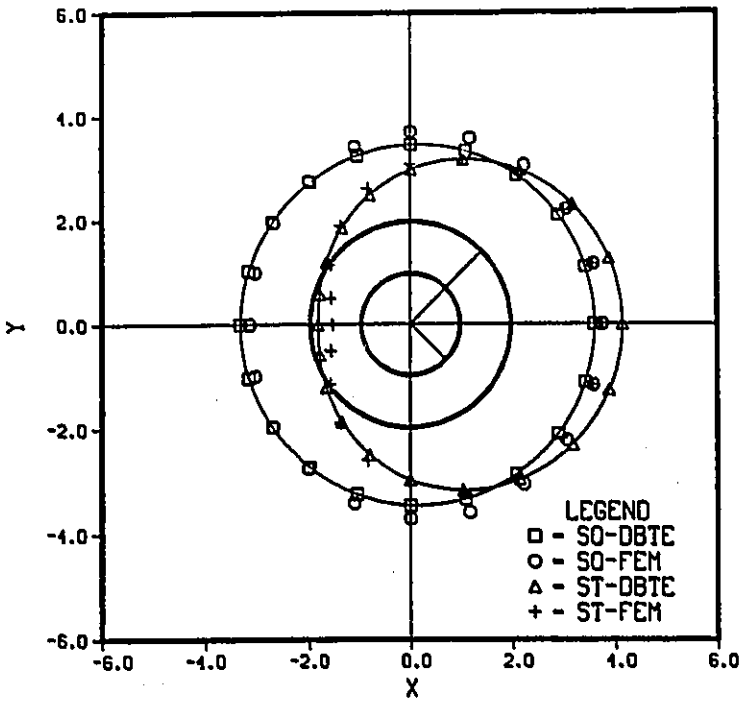

Figure 6.5: Comparison of stress factor $(\sigma_{\theta\theta})_f$ and $(\sigma_{\phi\phi})_f$ on outside surface.

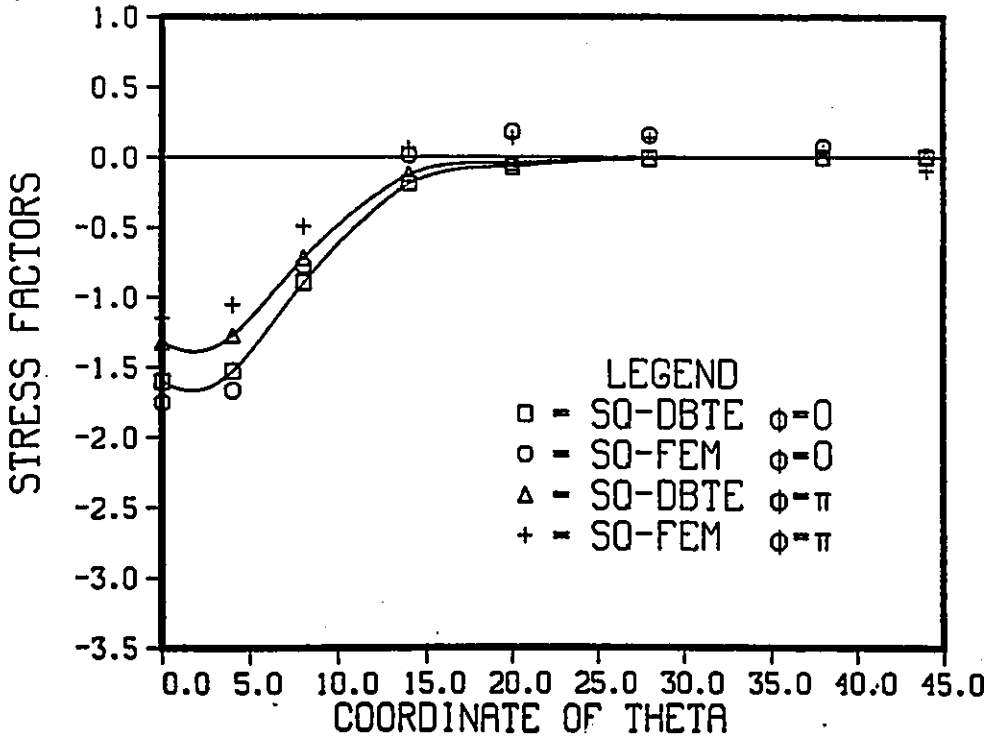

Figure 6.6: Comparison of stress factor $(\sigma_{\phi\phi})_f$ on outside surface.

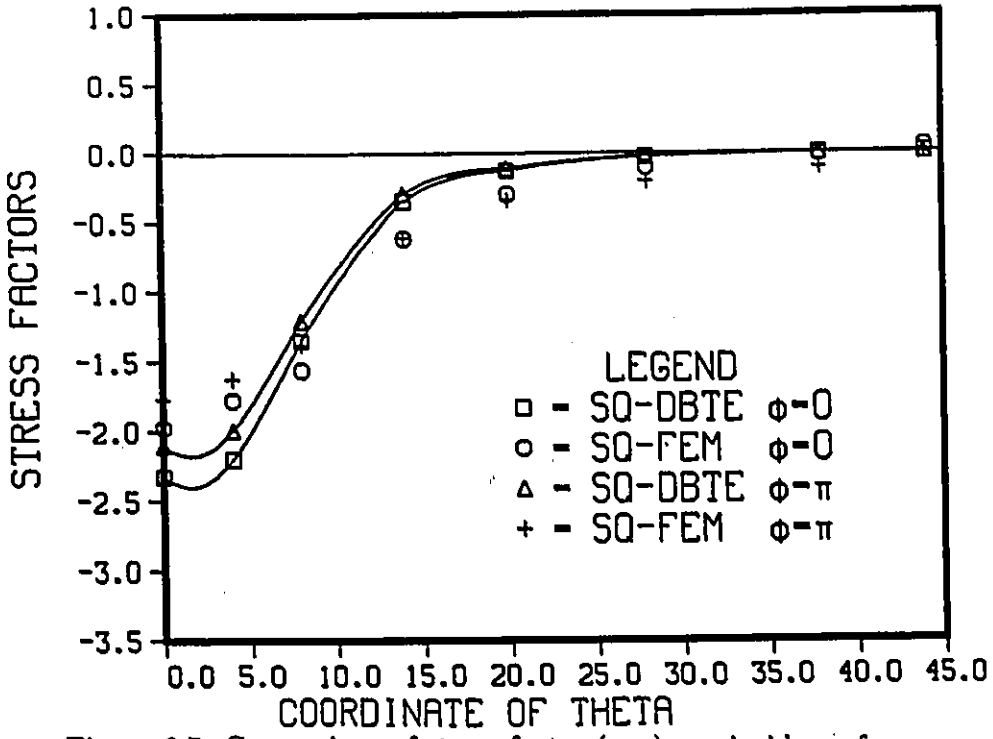

Figure 6.7: Comparison of stress factor $(\sigma_{\phi\phi})_f$ on inside surface.

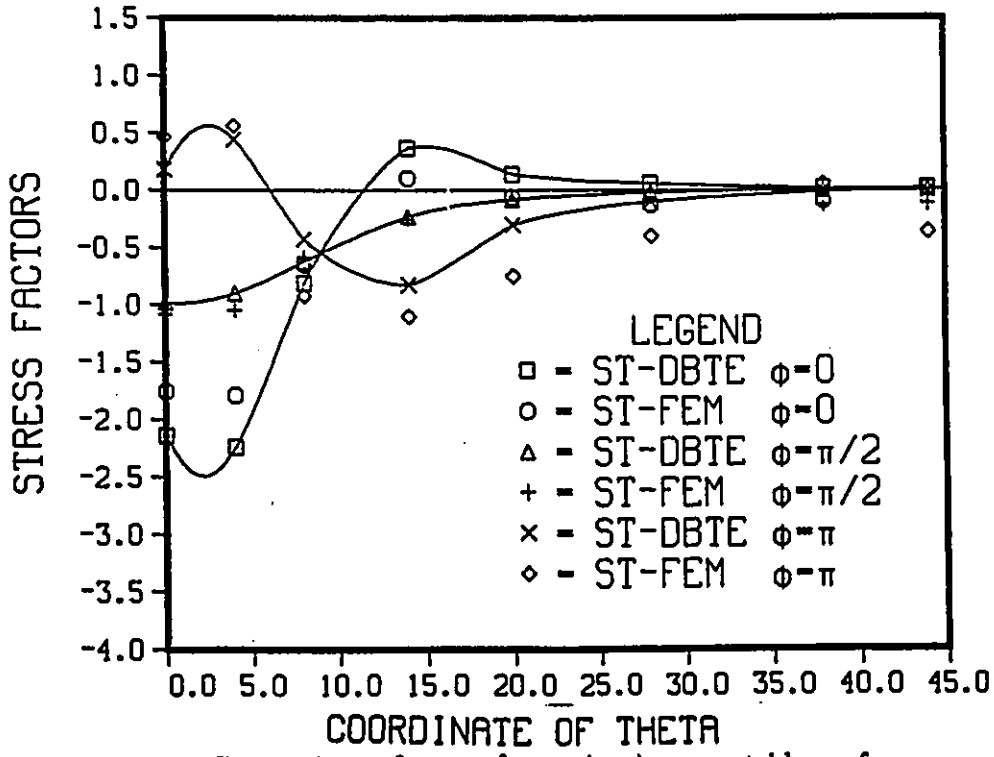

Figure 6.8: Comparison of stress factor $(\sigma_{\theta\theta})_f$ on outside surface.

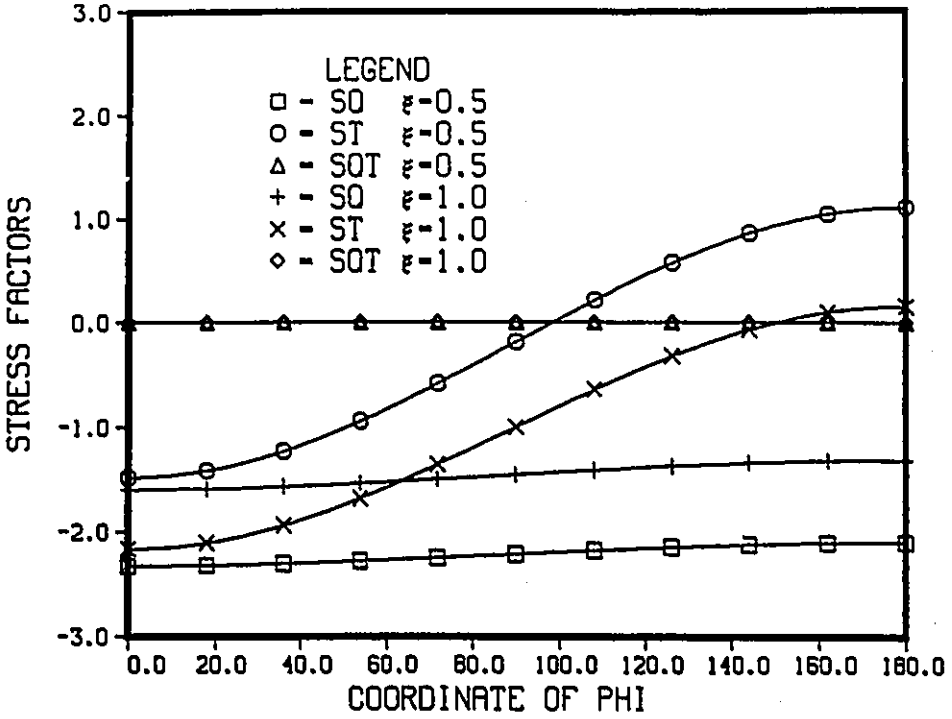

Figure 6.9: Stress factors for θ of 0 degree

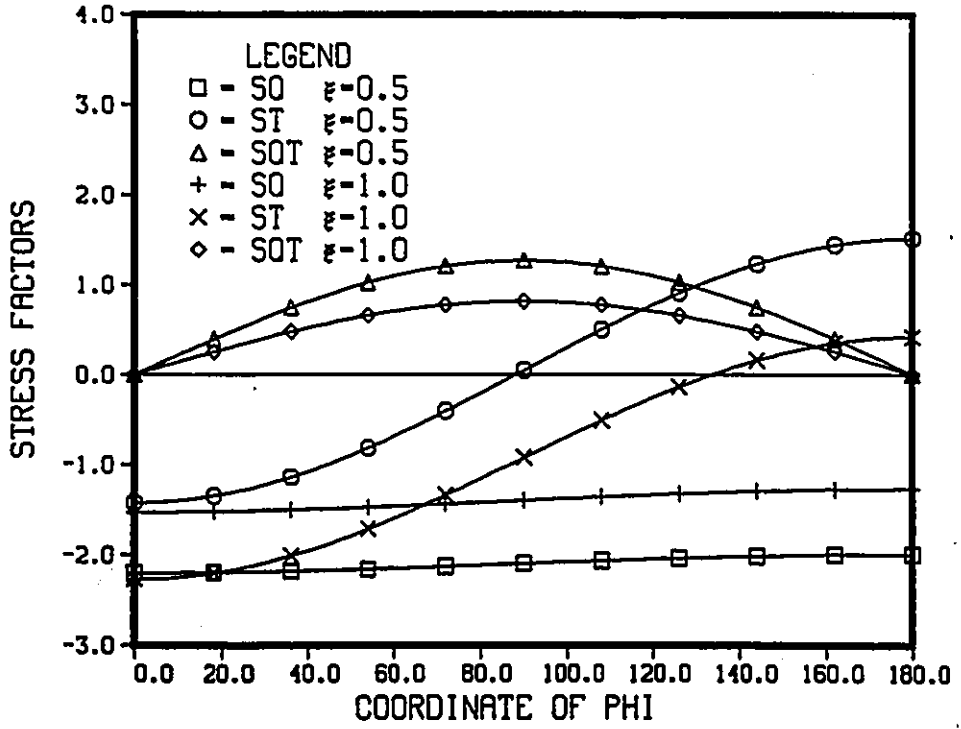

Figure 6.10: Stress factors for θ of 4 degree

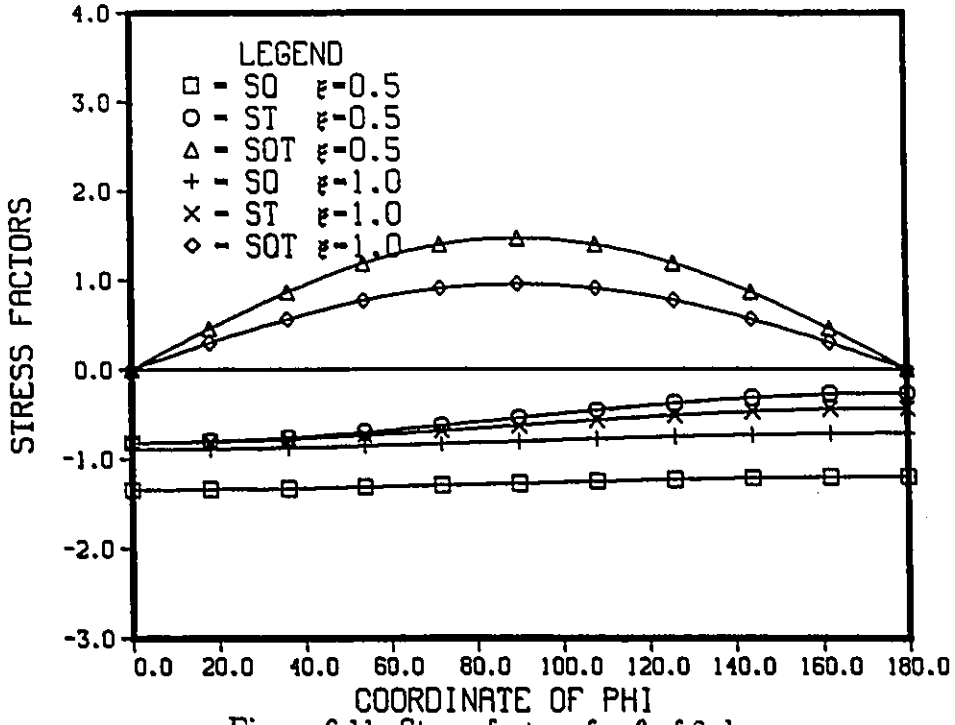

Figure 6.11: Stress factors for θ of 8 degree

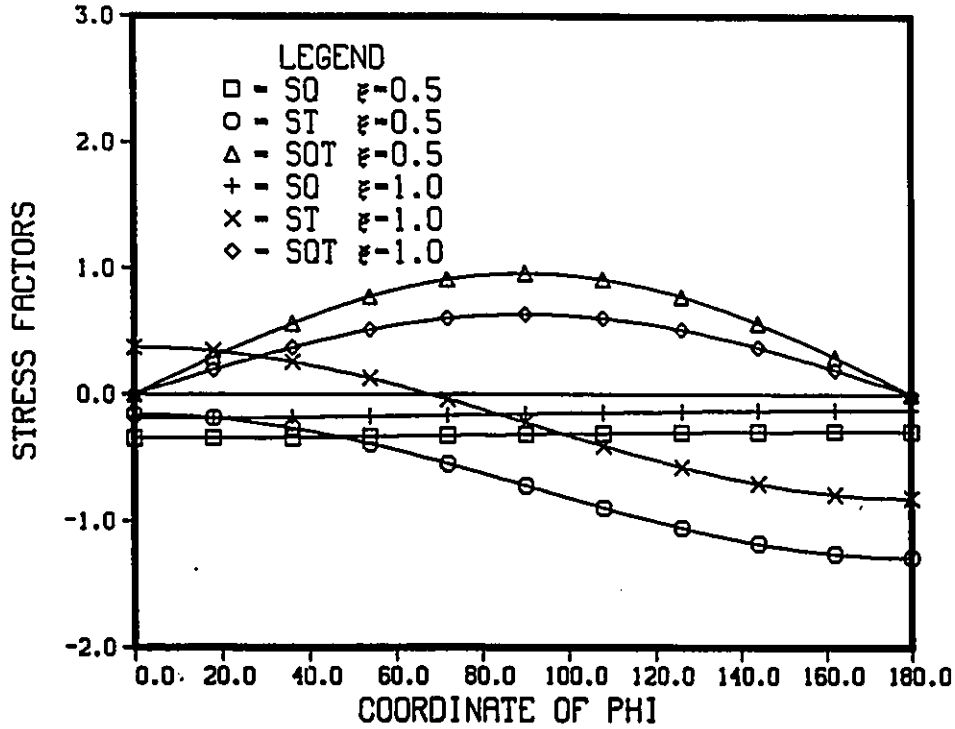

Figure 6.12: Stress factors for θ of 14 degree

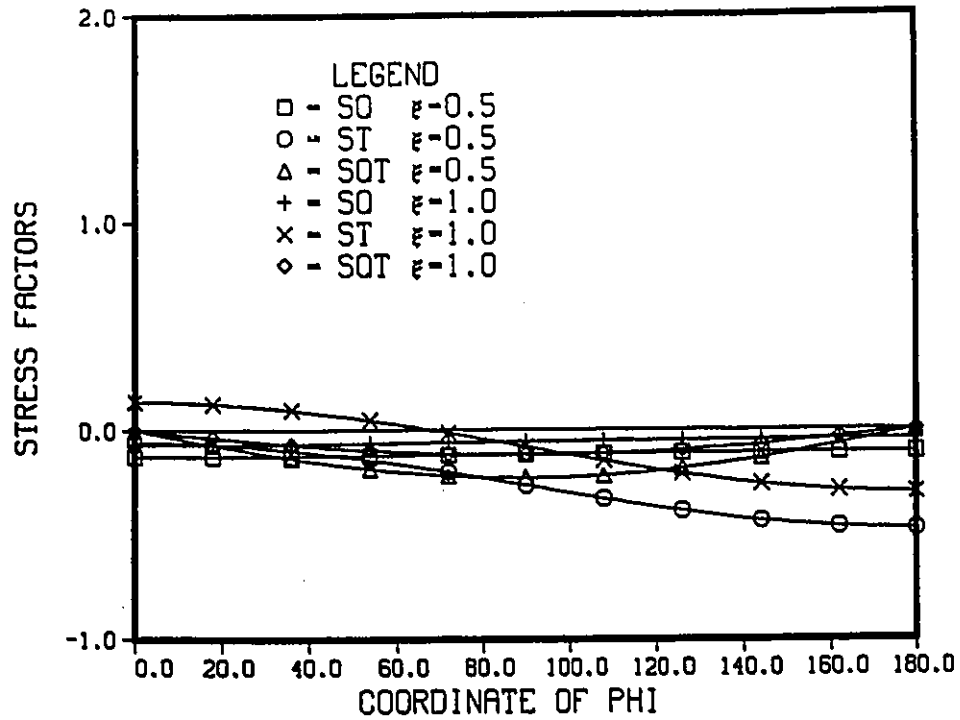

Figure 6.13: Stress factors for θ of 20 degree

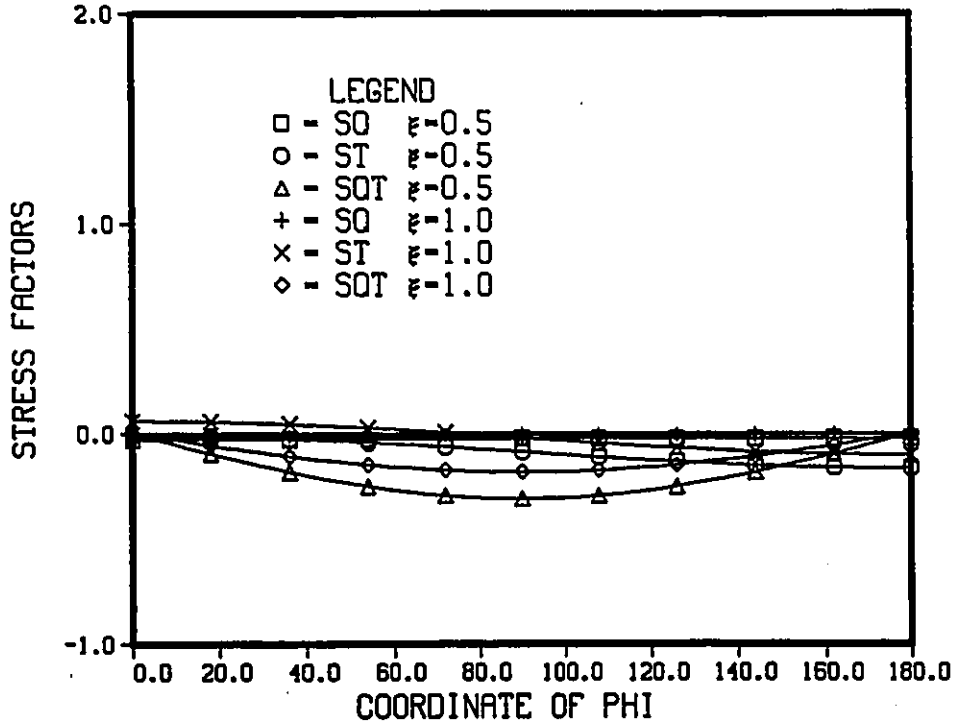

Figure 6.14: Stress factors for θ of 28 degree

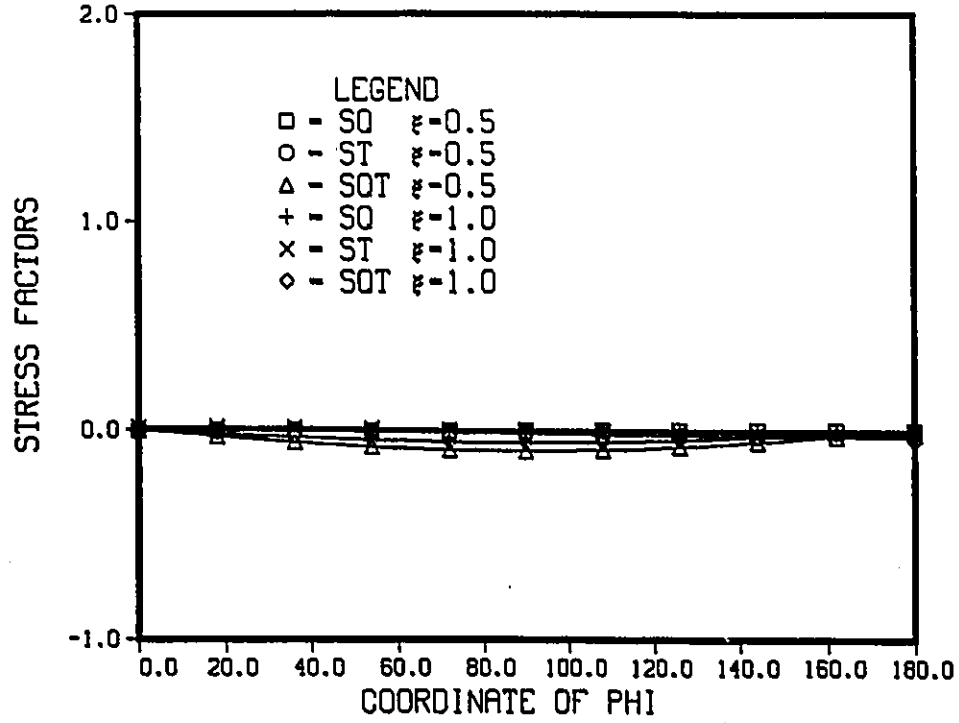

Figure 6.15: Stress factors for θ of 38 degree

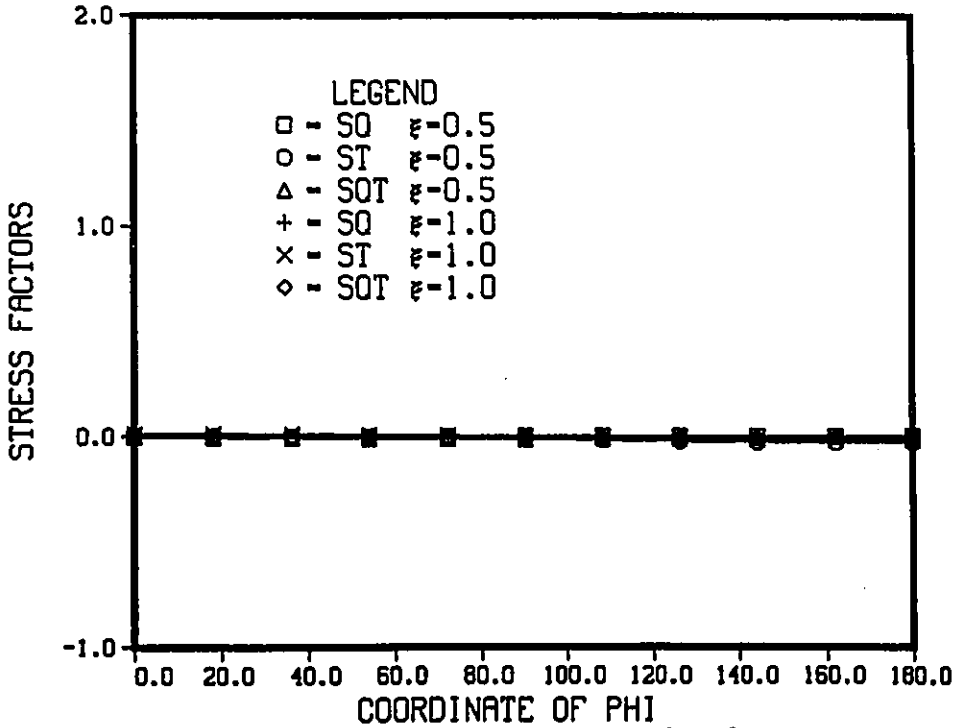

Figure 6.16: Stress factors for θ of 44 degree

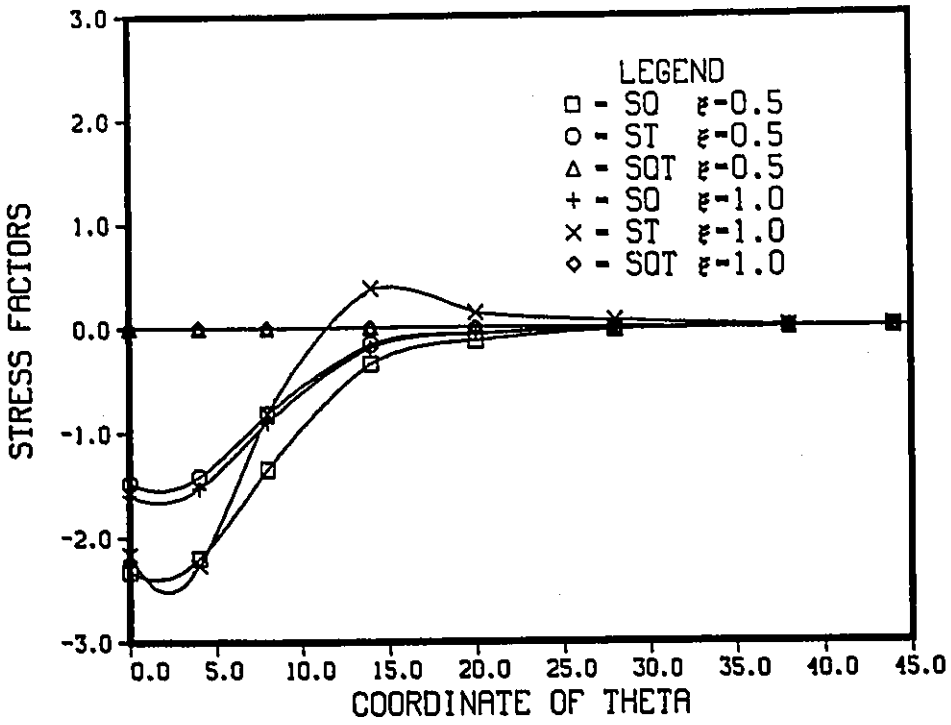

Figure 6.17: Stress factors for ϕ of 0 degree

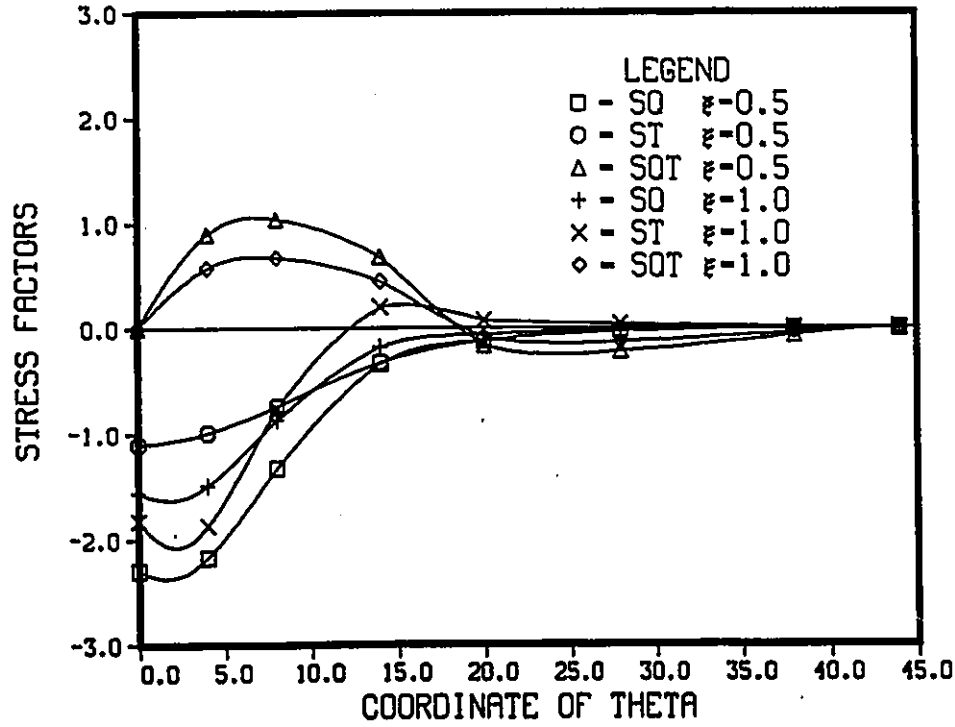

Figure 6.18: Stress factors for ϕ of 45 degree

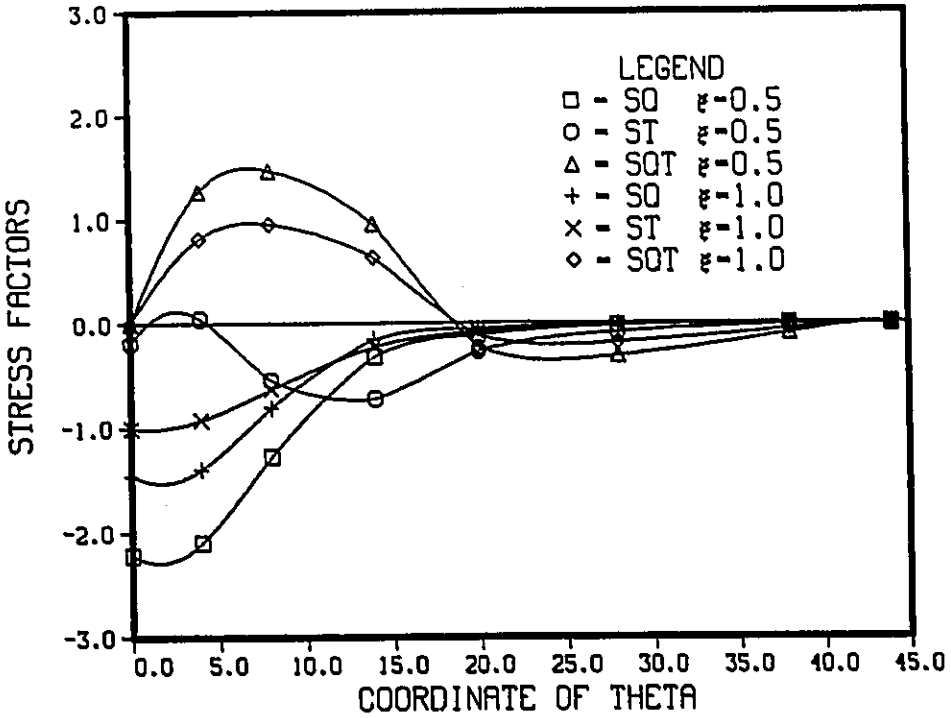

Figure 6.19: Stress factors for ϕ of 90 degree

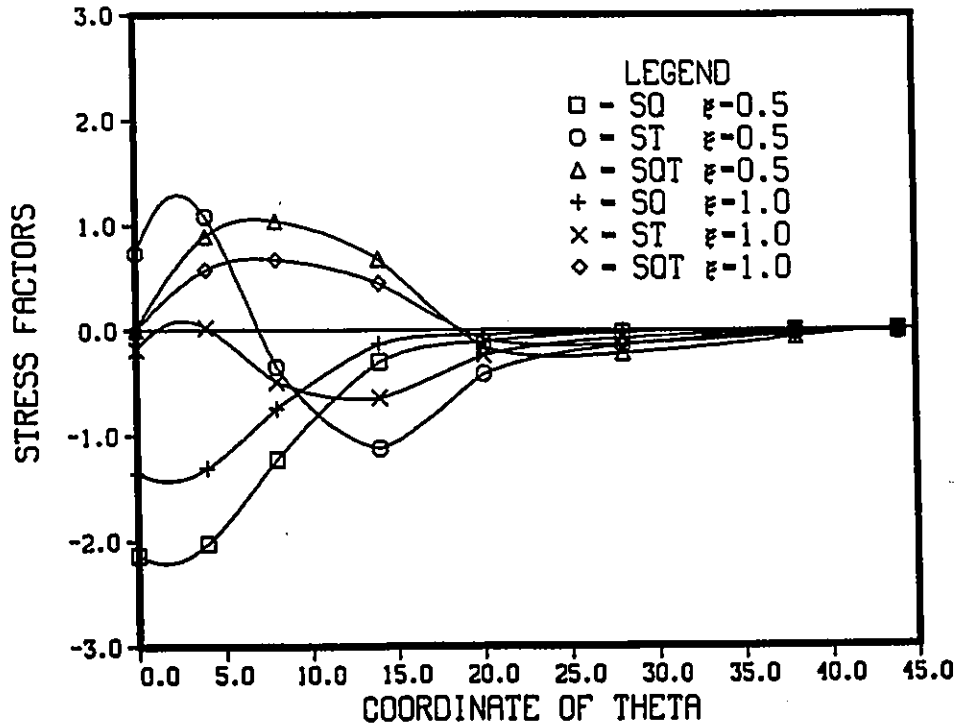

Figure 6.20: Stress factors for ϕ of 135 degree

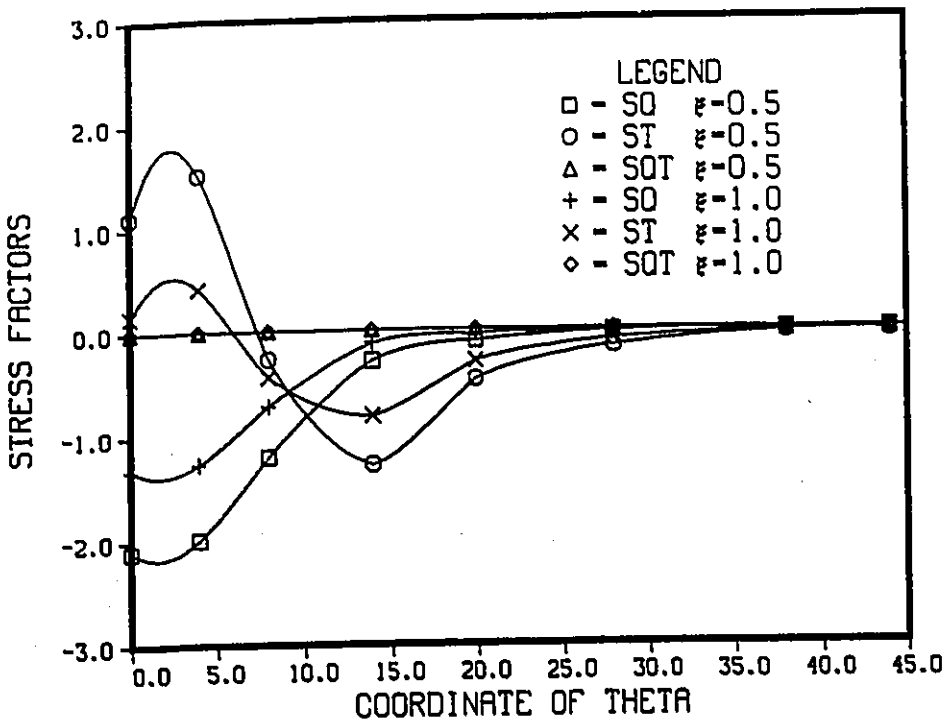

Figure 6.21: Stress factors for ϕ of 180 degree

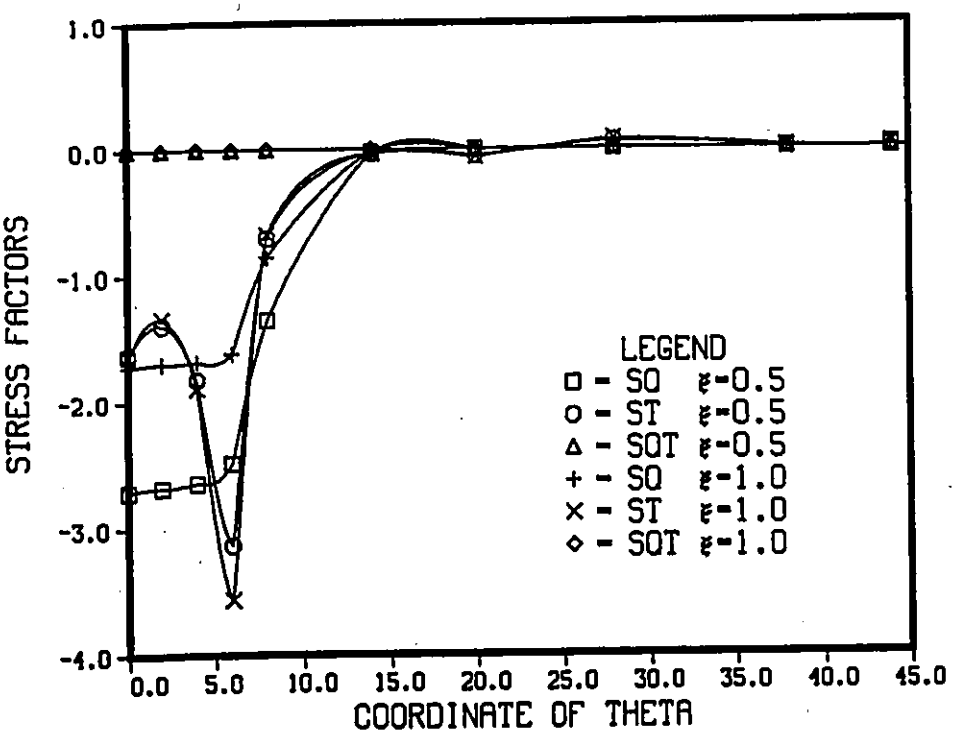

Figure 6.22: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.1$

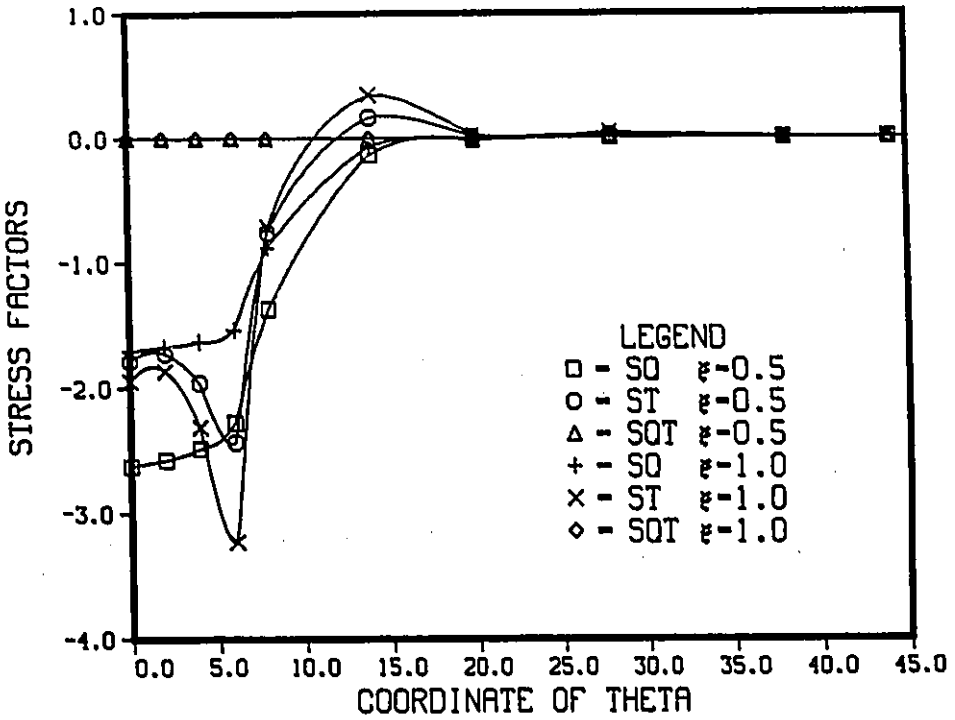

Figure 6.23: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.2$

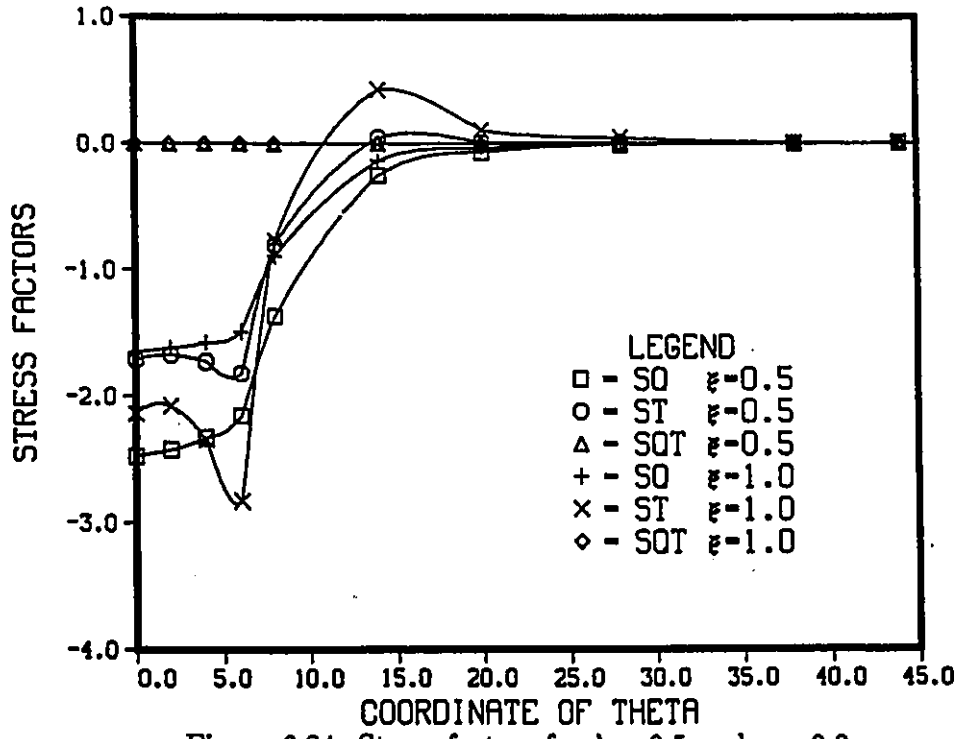

Figure 6.24: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.3$

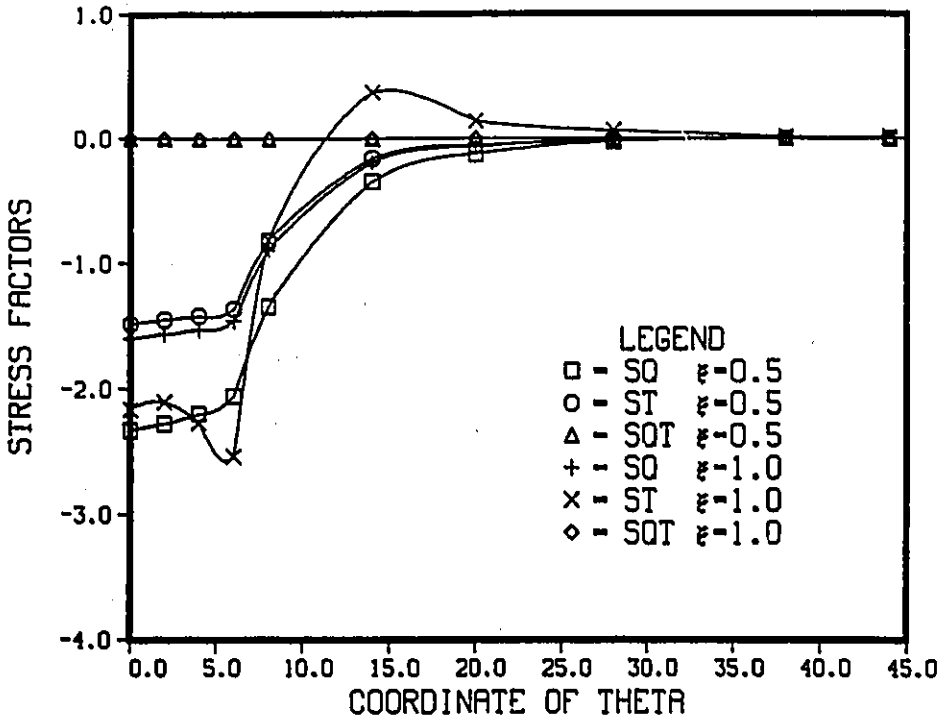

Figure 6.25: Stress factors for $\lambda = 0.5$ and $\epsilon = 0.4$

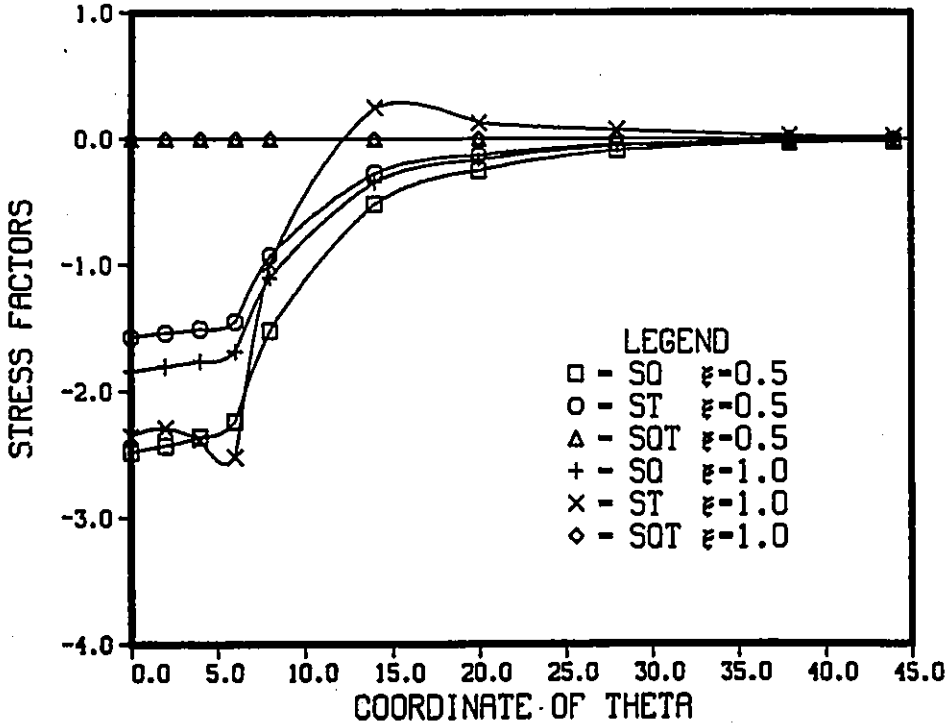

Figure 6.26: Stress factors for $\lambda = 0.6$ and $\epsilon = 0.4$

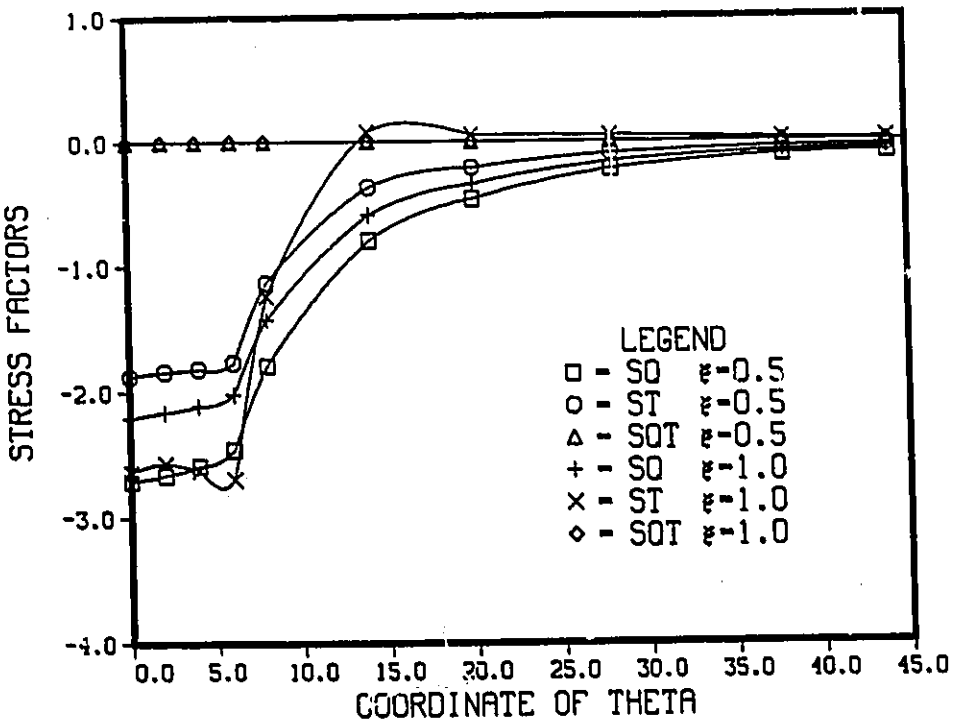

Figure 6.27: Stress factors for $\lambda = 0.7$ and $\epsilon = 0.4$

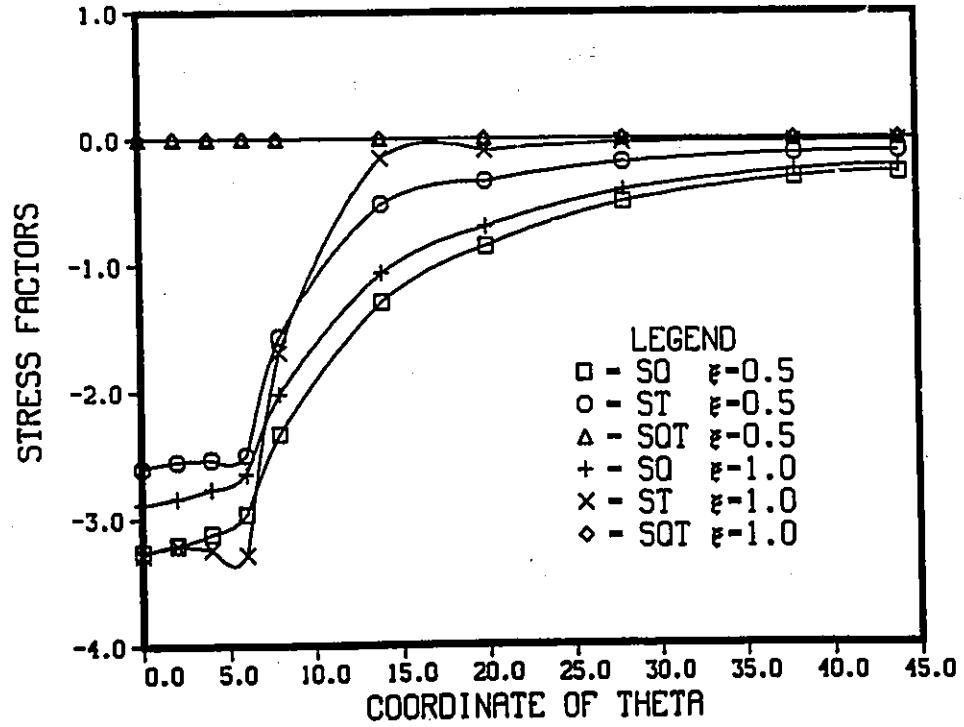

Figure 6.28: Stress factors for $\lambda = 0.8$ and $\epsilon = 0.4$

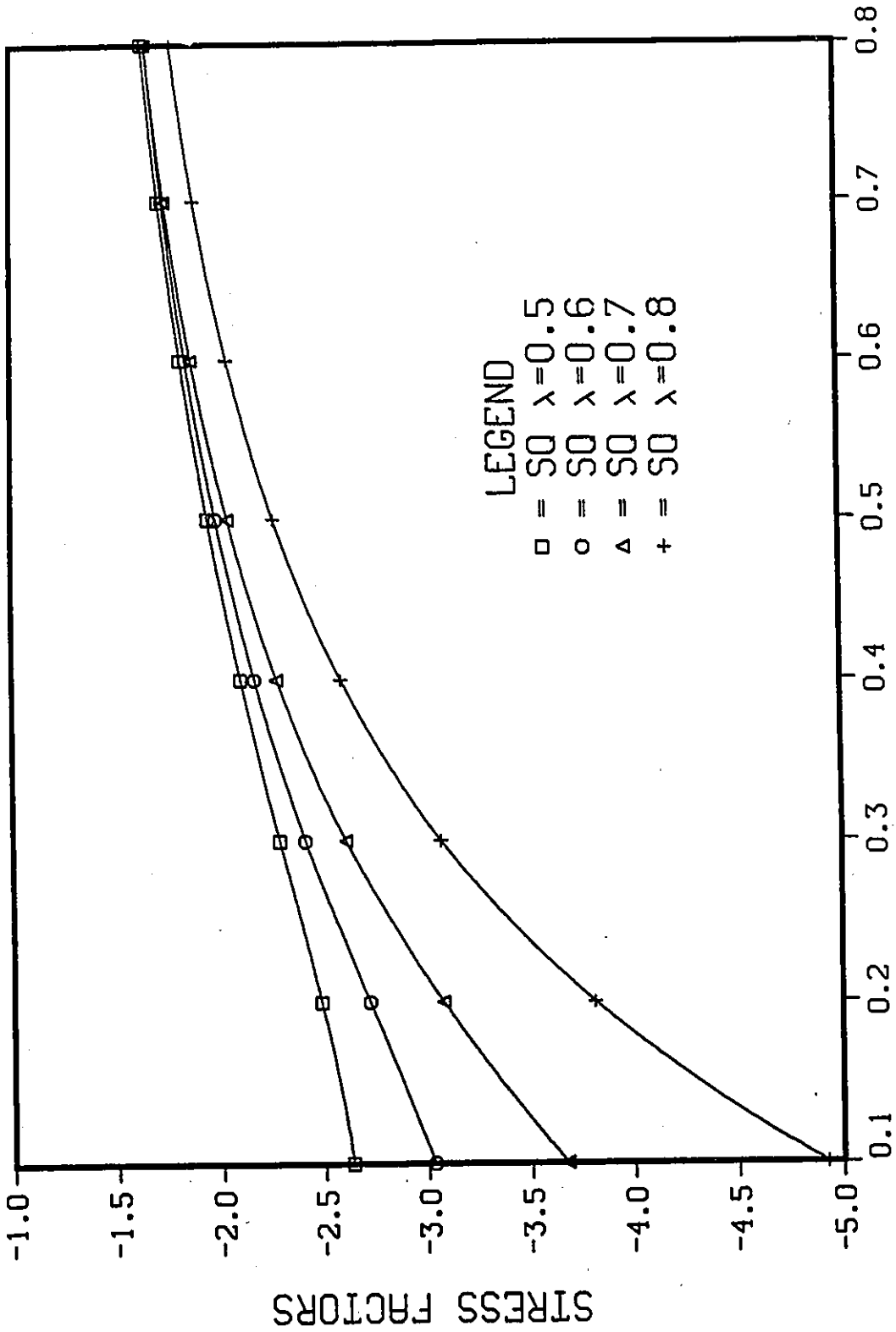

Figure 6.29: Maximum circumferential stress factor vs λ and ϵ

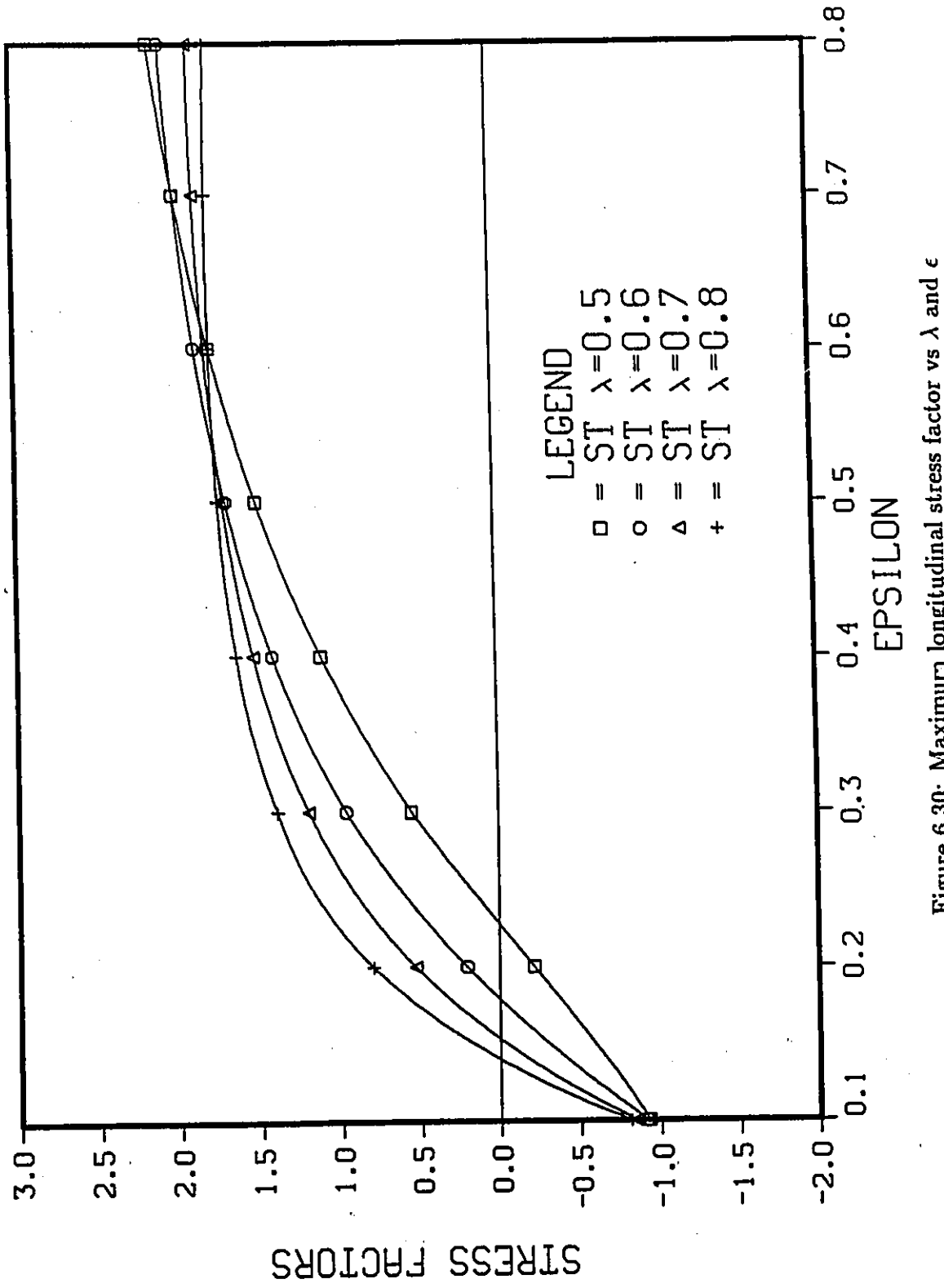

Figure 6.30: Maximum longitudinal stress factor vs λ and ϵ

Chapter 7

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary

Toroidal elasticity is a developing three-dimensional theory which can be used for the elastostatic analysis of thick-walled curved pipes, elbows, and toroidal shells. A displacement based toroidal elasticity has been developed in this thesis. The governing equations (Navier equations) have been set up in the non-classical toroidal coordinates. These governing equations are much more complicated than those represented in Cartesian coordinates. They contain a small parameter $\epsilon = c/R$. The successive approximation method has been employed to simplify the governing equations, and to reconstruct the governing equations into different orders. The left hand sides of the governing equations of different orders have the same form, and the right hand

sides contain only the contribution from displacement functions of previous orders. The main boundary conditions on the shell surfaces are satisfied analytically. The boundary conditions on the end surfaces or edge surfaces are satisfied based on the St. Venant principle, or a point matching technique. Three typical problems are solved by the displacement based toroidal elasticity, as described below.

The first problem, a thick-walled elbow with a fixed end and subjected to twist moment at the other end, has been investigated analytically. Because the central line of the elbow will deform normal to the center plane of the elbow under the applied twist moment, the out-of-plane displacement (normal to the center plane) is the principal displacement component. An out-of-plane displacement model has been assumed. A complementary solution for the general case of out-of-plane deformation problems is obtained by forcing the displacement model to satisfy the zeroth order governing equations in the successive approximation and the main stress boundary conditions on the elbow surfaces. The boundary condition on the end surface has been satisfied globally by the St. Venant principle. Finally, an example of 90° elbow under out-of-plane twist moment is presented. Calculated stresses and displacements have been compared with the stress based toroidal elasticity solution and with data obtained by the boundary element method. Satisfactory agreement has been obtained. It has been concluded that the largest longitudinal stress on the outside surface is located at $\phi = 112.5°, 247.5°$, and the largest circumferential stress is located on the inside surface near $\phi = 45°, 195°, 225°$ and $315°$.

The second problem investigated is a thick-walled axisymmetric toroidal ring subjected to edge loads. This is an axisymmetric problem. The governing equation

and the equations required for successive approximation approach are developed. A displacement model is assumed. By satisfying the governing equations and the main boundary conditions on the ring surfaces, an eigenvalue problem has arisen. The eigenvalues of the ring are classified into three types. There are a triple imaginary eigenvalues, a pair of purely real eigenvalues, and complex eigenvalues appearing in sets of four. The eigenfunctions corresponding to different type of eigenvalues are developed separately. A complex variable solution for an edge load axisymmetric problem has been developed with the aid of homogeneous eigenfunctions. The real and imaginary parts of this complex variable solution are two separate solutions. The governing equation is satisfied separately by the real and imaginary parts. The boundary conditions have been satisfied analytically on the shell surface, and have been satisfied at collocation points by point matching linear least square method on the edge surfaces. A 45° thick-walled toroidal ring subjected to the edge bending moments has been investigated in detail. A FORTRAN program for solving the problem of axisymmetric toroidal ring subjected to edge loads is provided in appendix A. The calculated solution has been compared with a finite element solution, and a generally good solution has been achieved.

The last problemstudied is a thick-walled axisymmetric toroidal shell subjected to non-axisymmetric band loads. The applied band load is expressed in the form of a Fourier Series. Each term of the applied band load is a non-uniformly, continuously distributed load. Because the toroidal shell and applied band load are symmetric about the center plane of the toroidal shell, the center line of the toroidal shell must be kept in the centre plane of the toroidal shell during the deformation

procedure. An in-plane displacement model [2] has been introduced for this problem. By satisfying the boundary conditions on the shell surfaces and the symmetry conditions, the solution for a typical term of the load series is obtained. The data of a thick-walled toroidal shell subjected to four circumferential band loads acting at intervals of 90° have been presented. The calculated stresses and displacements have been compared with data obtained from finite element method. The agreement between these two solutions is satisfactory. The largest stress value is located at the edge of the applied band load on the outside of the toroidal shell.

7.2 Conclusions

The following conclusions can be drawn from the methods, the results and the discussion presented in the previous chapters:

1. Approximate solutions for the thick-walled toroidal shell problems can be obtained analytically by the displacement based toroidal elasticity for many load cases.

2. The governing Navier equations of the toroidal shell developed in the toroidal coordinate system can be solved effectively by the successive approximation method.

3. A satisfactory approximate solution can be obtained even if only the first few orders analysis are considered in the MSA.

4. For the problem with principal deformation in or out of the plane of the toroidal shell, the displacement based toroidal elasticity approach can be simplified by using the corresponding in-plane or out-of-plane deformation model.

5. A satisfactory solution can be obtained even if the boundary conditions on the edge surfaces of the toroidal shell are not satisfied exactly (for example, the boundary conditions are satisfied globally, i.e. only at collocation points).

5. An accurate approximation to the exact solution may be obtained even from a few order of the MSA approach, if the zeroth order solution can be adjusted by some analytic information about the exact solution.

7.3 Recommendations

1. The junction problem of a thick-walled cylindrical vessel with toroispherical head could be solved analytically with the aid of this displacement based toroidal elasticity and a software could be made for practical applications.

2. To complete the displacement based toroidal elasticity, the problem of toroidal shell under thermal loads and dynamic loads should be investigated.

3. The solution obtained from the displacement based toroidal elasticity has only been compared with the solution obtained from the finite element method or the boundary element method. Comparison with experimental investigations would be useful.

4. The boundary conditions on the load applied end surface have been satisfied globally for the problem of 90° elbow under twist moment. In order to satisfy the boundary conditions on the end surface exactly, the deformation model with multi-order harmonic in θ should be introduced.

5. The in-plane and out-of-plane solutions developed in this study can be used together to solve the problem of thick-walled toroidal shell under general load case.

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APPENDIX

Program 1

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C=====================================================================  PRO00010
C  THE 90 DEGREE ELBOW UNDER OUT-OF-PLANE BENDING PROBLEM              PRO00020
C     INPUT DIMENSION DATA AND APPLIED LOAD                           PRO00030
C     RIN:  INSIDE RADIUS OF THE ELBOW CROSS-SECTION;                 PRO00040
C     ROUT: OUTSIDE RADIUS OF THE ELBOW CROSS-SECTION;                PRO00050
C     RTOR: MEAN TOROIDAL RADIUS;                                     PRO00060
C     TAPP: THE APPLIED BENDING MOMENT;                               PRO00070
C  JAN. 30, 1994                                                      PRO00080
C=====================================================================  PRO00090
      REAL LAM,LAM2,LAM4,LAM6,LAM8                                     PRO00100
      DIMENSION COND0(10)                                              PRO00110
C     INPUT DIMENSION DATA AND APPLIED LOAD                            PRO00120
      RIN=1.0                                                          PRO00130
      ROUT=2.0                                                         PRO00140
      RTOR=5.0                                                         PRO00150
      TAPP=5.891*2.0                                                   PRO00160
C---------------------------------------------------------------------  PRO00170
      PI=3.14159265                                                    PRO00180
      E=ROUT/RTOR                                                      PRO00190
      LAM=RIN/ROUT                                                     PRO00200
      LAM2=LAM*LAM                                                     PRO00210
      LAM4=LAM2*LAM2                                                   PRO00220
      LAM6=LAM2*LAM4                                                   PRO00230
      LAM8=LAM4*LAM4                                                   PRO00240
      V=0.3                                                            PRO00250
      V1P1=1.0+V                                                       PRO00260
      V1P4=1.0+4.0*V                                                   PRO00270
      V1M2=1.0-2.0*V                                                   PRO00280
      V3P2=3.0+2.0*V                                                   PRO00290
      V3M2=3.0-2.0*V                                                   PRO00300
      V3M4=3.0-4.0*V                                                   PRO00310
      V5M4=5.0-4.0*V                                                   PRO00320
      E1PV=206000.0/(1.0+V)                                            PRO00330
C---------------------------------------------------------------------  PRO00340
      CONC0=V1P1/4.0/(1.0-LAM2)/(1.0-LAM2)/(1.0-LAM2)                  PRO00350
      COND0(1)=-TAPP*4.0/E/PI/ROUT/ROUT/ROUT/(1.0-LAM4)/(1.0+V)        PRO00360
      N=1                                                              PRO00370
C------CALCULATING CONSTANTS FOR EACH TERM ----------------------      PRO00380
      CBAR=CONC0*COND0(N)                                              PRO00390
      A1=-0.4674*COND0(N)                                              PRO00400
      A2=CONC0*(1.0+LAM2)*(1.0-LAM6)*COND0(N)                          PRO00410
      B2=CONC0*((1.0-LAM6)/3.0-V5M4*(1.0-LAM2)**3/6.0/V3M4)*COND0(N)   PRO00420
      C2=-CONC0*(LAM4-LAM8)*COND0(N)                                   PRO00430
      D2=-CONC0*(LAM2-LAM8)*COND0(N)                                   PRO00440
      F2=-V3P2*(1.0+LAM2)*COND0(N)*N/8.0                               PRO00450
      E2=-V3P2*LAM2*COND0(N)*N/8.0                                     PRO00460
      H1=N*CBAR*(V1P4*(1.0-LAM6)/3.0+(43.-56.*V-32.*V*V                PRO00470
     +   +16.*V*V*V)*(1.-LAM2)**3/12./V1P1/V1M2)                       PRO00480
      H2=N*CBAR*(LAM4-LAM8)                                            PRO00490
      H3=N*CBAR*( -(1.+LAM2)*(1.-LAM6)+V3P2*(1.+LAM2)*(1.-LAM2)**3     PRO00500
     +   /4./V1P1+2.*A1*(1.-LAM2)**3/V1P1/COND0(N) )                   PRO00510
      H4=N*CBAR*(-4.*(1.-V)*(1.-LAM6)*LAM2+ V3P2*(1.-LAM2)**3*LAM2     PRO00520
     +  /4./V1P1 )                                                    PRO00530
      G1=N*CBAR/12.*( (13.-8.*V)*(1.-LAM6)-(11.-16.*V-25.*V*V+8.*V     PRO00540
     +  *V*V)*(1.-LAM2)**3/V1P1/V1M2 )                                PRO00550
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      G2=N*CBAR/2.*(LAM8-LAM4)                                          PRO00560
      G3=N*CBAR/2.*( -(1.+LAM2)*(1.-LAM6)+V3P2*(1.+LAM2)*(1.-LAM2)      PRO00570
     +  **3/4./V1P1+2.*A1*(1.-LAM2)**3/V1P1/COND0(N) )                  PRO00580
      G4=N*CBAR/2.*( (1.-LAM6)/2.-2.*V1M2*(LAM2-LAM8)+V3P2*(1.+LAM2)    PRO00590
     +  *(1.-LAM2)**3/4./V1P1 )                                         PRO00600
      VK=CBAR*( -V1M2*(1.-LAM6)/3.+(1.-V/3.)*(1.-LAM2)**3 )             PRO00610
      VM=CBAR*(LAM4-LAM8)                                               PRO00620
      VN=CBAR*(LAM2-LAM8)                                               PRO00630
      VP=CBAR*( (1.-V)*(1.-LAM6)-(11.-5.*V)*(1.-LAM2)**3/6. )           PRO00640
      G=8.*(LAM4-LAM6)-(1.+LAM2)*(1.-LAM8)                              PRO00650
      B3=( VK*(4.*(LAM4-LAM6)+(1.+LAM2)*(1.-LAM8))-3.*VM*(1-LAM4)       PRO00660
     +  +VN*(4.*V1M2*(LAM2-LAM4)+(1.+LAM2)*(1.-LAM4)) )/8./G            PRO00670
      D3=-( VK*3.*LAM4*(1.-LAM8)+VM*(2.*(LAM4-LAM6)-(1.+LAM2)*          PRO00680
     +  (1.-LAM8))+VN*(2.*(LAM4-LAM8)+V1M2*LAM2*(1.-LAM8)) )            PRO00690
     +  /4./G                                                           PRO00700
      C3=B3+2.*D3+(VK+VM+VN)/8.                                         PRO00710
      A3=4.*B3-D3-(VK+VM-V1M2*VN)/4.                                    PRO00720
      E3=(-H1*LAM6*(1.-LAM)+H2*(LAM-LAM6)+H3*LAM4*(LAM-LAM2)            PRO00730
     +  +H4*LAM2*(LAM-LAM4) )/3./(1.-LAM6)                             PRO00740
      F3=-(H1*(1.-LAM6*LAM)+H2*(1.-LAM)+H3*(1.-LAM4*LAM)                PRO00750
     +  +H4*(1.-LAM2*LAM) )/3./(1.-LAM6)                               PRO00760
      ASTAR1=-V1P1*COND0(N)*( 2.*(1.-V)*(1.-LAM6)-3.*(2.-V)             PRO00770
     +  *(1.-LAM2)**3  )/96./(1.-LAM2)**3                              PRO00780
      BSTAR1=-ASTAR1                                                    PRO00790
      ASTAR2=-(1.-V*V)*COND0(N)*(LAM4-LAM8)/8./(1.-LAM2)**3             PRO00800
      BSTAR2=ASTAR2                                                     PRO00810
      ASTAR3=-V1P1*V3M4*COND0(N)*(LAM2-LAM8)/16./(1.-LAM2)**3           PRO00820
      BSTAR3=ASTAR3*(1.-4.*V)/V3M4                                      PRO00830
      CSTAR1=N*CBAR*( 1.-LAM6 +(21.-28.*V-10.*V*V)*(1.-LAM2)**3         PRO00840
     +  /2./V1P1/V1M2 )/12.                                            PRO00850
      CSTAR2=N*CBAR*(1.-LAM6+V3P2*(1.-LAM2)**3/2./V1P1)*LAM2/4.         PRO00860
C------DO LOOP IN THETA DIRECTION ------------------------------       PRO00870
      DO 100 ITHE=1,7                                                   PRO00880
      THETA=(ITHE-1)*PI/12.0                                            PRO00890
      THE=THETA*180.0/PI                                                PRO00900
C------DO LOOP ON THE INSIDE AND OUTSIDE SURFACE ----------------       PRO00910
      DO 100 I=1,2                                                      PRO00920
      X=LAM+(1.0-LAM)*(I-1)                                             PRO00930
      WRITE(8,190) X,THE                                                PRO00940
  190 FORMAT(/ 5X,'X=',F6.2,5X,'THETA=',F6.2 / 2X,                      PRO00950
     + 'PHI',6X,'SRR',7X,'SFF',7X,'STT',7X,'STF',7X,'SRT',7X,'SFT')     PRO00960
C------DO LOOP IN PHI DIRECTION ---------------------------------      PRO00970
      DO 200 J=1,17                                                     PRO00980
      Q=(J-1)*PI/8.0                                                    PRO00990
      PHI=Q*180.0/PI                                                    PRO01000
      S1=       E*E*(A2-3.0*C2/(X**4)+4.0*D2/X/X                        PRO01010
     +         -V1P1*COND0(N)*X*X/4.0)*COS(N*THETA)*SIN(2.0*Q)          PRO01020
     +         +E*E*E*( 2.*A3*X -4.*B3*X*X*X -4.*C3/(X**5)              PRO01030
     +          +10.*D3/X/X/X +VK*X*X*X +VM/X/X/X +VN*V/X )            PRO01040
     +          *COS(N*THETA)*SIN(3.*Q)                                PRO01050
      S2=       E*E*(-A2+12.0*B2*X*X+3.0*C2/(X**4)                      PRO01060
     +         +V1P1*V1P4*COND0(N)*X*X/4.0/V3M4)*SIN(2.0*Q)            PRO01070
     +         *COS(N*THETA)                                           PRO01080
     +         +E*E*E*( -2.*A3*X +20.*B3*X*X*X +4.*C3/(X**5)           PRO01090
     +           -2.*D3/X/X/X +VK*X*X*X +VM/X/X/X -VN*V/X )            PRO01100
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     +            *COS(N*THETA)*SIN(3.*Q)                                    PRO01110
      S3=        -E*V1P1*COND0(N)*X*SIN(Q)*COS(N*THETA)                      PRO01120
     +          +E*E*(12.0*V*B2*X*X+4.0*V*D2/X/X+V1P1*V3M2/2.0               PRO01130
     +          /V3M4*COND0(N)*X*X)*SIN(2.0*Q)*COS(N*THETA)                  PRO01140
     +          +E*E*E*( 16.*V*B3*X*X*X +8.*V*D3/X/X/X +VP*X*X*X             PRO01150
     +             -(1.-V)*VM/X/X/X -V1P1*VN/X )                             PRO01160
     +             *COS(N*THETA)*SIN(3.*Q)                                   PRO01170
      S4=         E*E*(A2-6.0*E2*X*X+3.0*C2/(X**4)                          PRO01180
     +           -2.0*D2/X/X-V1P1/2.0/V3M4*COND0(N)*X*X)                     PRO01190
     +           *COS(2.0*Q)*COS(N*THETA)                                    PRO01200
     +           +E*E*E*( 2.*A3*X -12.*B3*X*X*X +4.*C3/(X**5)                PRO01210
     +              -6.*D3/X/X/X -(1.-V)*VN/X )                              PRO01220
     +              *COS(N*THETA)*COS(3.*Q)                                  PRO01230
      S5=          E*E*(-V3P2/16.0)*N*(1.0+LAM2-LAM2                        PRO01240
     +           /X/X-X*X)*COND0(N)*SIN(N*THETA)*SIN(Q)                      PRO01250
     +           +E*E*E*( -3.*E3/X/X/X/X +2.*F3*X*X +H1*X*X*X                PRO01260
     +             +H2/X/X/X +H3*X +H4/X )/2.0                               PRO01270
     +             *SIN(N*THETA)*SIN(2.*Q)                                   PRO01280
      S6=          -E*N*COND0(N)*X*SIN(N*THETA)/2.0                          PRO01290
     +           +E*E*(-V3P2/16.0)*N*(1.0+LAM2+LAM2/X/X-3.0*X*X)             PRO01300
     +            *COND0(N)*SIN(N*THETA)*COS(Q)                              PRO01310
     +           -E*E*E*((V3P2*(1.0+LAM2)+3.7392)*X+V3P2*LAM2/X              PRO01320
     +            +(7.0+2.0*V)*X*X*X)*COND0(N)/32.0                          PRO01330
C-------------------------------------------------------------------------   PRO01340
      DU=        E*COND0(N)*(-0.4674+V/2.0*X*X)                             PRO01350
     +             *COS(N*THETA)*SIN(Q)/E1PV                                 PRO01360
     +          +E*E*CONC0*( (1.0+LAM2)*(1.0-LAM6)*X-4.0/3.0*V*(1.0          PRO01370
     +           -LAM6)*X*X*X-(1.0-2.0*V)/3.0*(1.0-LAM2)**3*X*X*X            PRO01380
     +           -(LAM4-LAM8)/X/X/X+4.0*(1.0-V)*(LAM2-LAM8)/X )*             PRO01390
     +           COS(N*THETA)*SIN(2.0*Q)/E1PV                                PRO01400
     +          +E*E*E*( A3*X*X -(1.+4.*V)*B3*X*X*X*X +C3/(X**4)             PRO01410
     +             -V5M4*D3/X/X +ASTAR1*X*X*X*X +ASTAR2/X/X                  PRO01420
     +             +ASTAR3 )*COS(N*THETA)*SIN(3.*Q)/E1PV                     PRO01430
      DV=        COND0(N)*X*COS(N*THETA)/E1PV -E*COND0(N)*(0.4674+           PRO01440
     +            V/2.0*X*X)*COS(N*THETA)*COS(Q)/E1PV                        PRO01450
     +          +E*E*CONC0*( (1.0+LAM2)*(1.0-LAM6)*X-2.0/3.0*V3M2*           PRO01460
     +           (1.0-LAM6)*X*X*X+2.0/3.0*(2.0-V)*(1.0-LAM2)**3*X*X*X        PRO01470
     +           +(LAM4-LAM8)/X/X/X+(2.0-4.0*V)*(LAM2-LAM8)/X )              PRO01480
     +          *COS(N*THETA)*COS(2.0*Q)/E1PV                               PRO01490
     +          +E*E*E*( A3*X*X -(7.-4.*V)*B3*X*X*X*X -C3/(X**4)            PRO01500
     +             -(1.-4.*V)*D3/X/X +BSTAR1*X*X*X*X +BSTAR2/X/X            PRO01510
     +             +BSTAR3 )*COS(N*THETA)*COS(3.*Q)/E1PV                     PRO01520
      DW=        -V3P2*N*COND0(N)*(LAM2/X+(1.0+LAM2)*X-(1.0+2.0*V)           PRO01530
     +           /V3P2*X*X*X)*SIN(N*THETA)*SIN(Q)/8.0/E1PV                   PRO01540
     +           +E*E*E*( E3/X/X/X +F3*X*X*X +CSTAR1*X*X*X*X                 PRO01550
     +              +CSTAR2 )*SIN(N*THETA)*SIN(2.*Q)/E1PV                    PRO01560
      WRITE(8,290) PHI,S1,S2,S3,S4,S5,S6                                     PRO01570
C     WRITE(8,290) PHI,DU,DV,DW                                              PRO01580
  200 CONTINUE                                                               PRO01590
  100 CONTINUE                                                               PRO01600
  290 FORMAT(1X,F5.1,6F10.5)                                                 PRO01610
      END                                                                    PRO01620
C=========================================================================   PRO01630
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Program 2

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C******************** MAIN PROGRAM ************************************PRO00010
C     EIGENFUNCTION SOLUTION FOR THICK-WALLED TOROIDAL RING             PRO00020
C     INPUT THE DIMENSION DATA                                          PRO00030
C     RIT:   INSIDE RADIUS OF THE TOROIDAL RING;                        PRO00040
C     ROT:   OUTSIDE RADIUS OF THE TOROIDAL RING;                       PRO00050
C     RRT:   MEAN TOROIDAL RADIUS;                                      PRO00060
C     POISS: POISSON RATIO;                                             PRO00070
C     EY:    YOUNG'S MODULUS;                                           PRO00080
C     NHIC:  THE NUMBER OF EIGENVALUE SETS TAKEN IN STUDY;              PRO00090
C     THE BOUNDAY CONDITIONS ARE INPUTED IN SUBROUTINE BULLSQ;          PRO00100
C     YANYING ZHU, UNIVERSITY OF OTTAWA   JULY 1992                     PRO00110
C=================================================================     PRO00120
      COMMON/GEOMAL/RIC,ROC,RIT,ROT,ADIM,POISS,EY,NHIC,RRT              PRO00130
      COMMON/CHARAC/ECYLD(40),EVCYLD(40,4),ESPHE(40),EVSPHE(40,4)       PRO00140
      COMMON/ZHU/D(40,4,3),AC(40,3,2),BC(40,3,2),A(50,51)               PRO00150
      COMPLEX*16 ECYLD,EVCYLD,D,AC,BC,A                                 PRO00160
      COMPLEX*16 Z,DCMPLX,W,ZDETC,ZDETS                                 PRO00170
      REAL*8 RIC,ROC,RIS,ROS,ADIM,POISS,EY,ABC1,ABC2                    PRO00180
      REAL*8 X(4,2),U,V,FX(2),DREAL,DIMAG,LAM                           PRO00190
C     INPUT THE DIMENSION DATA                                          PRO00200
      RIT=0.2D0                                                         PRO00210
      ROT=0.4D0                                                         PRO00220
      RRT=0.7D0                                                         PRO00230
      POISS=.3D0                                                        PRO00240
      EY=3.0D7                                                          PRO00250
      LAM=RIT/ROT                                                       PRO00260
      ADIM=.5D0*(RIC+ROC)                                               PRO00270
      RIC=.9D0                                                          PRO00280
      ROC=1.1D0                                                         PRO00290
      NHIC=5                                                            PRO00300
C     SEARCHING THE EIGENVALUES                                         PRO00310
  100 NTALMX=140                                                        PRO00320
      DO 110 LC=1,NHIC                                                  PRO00330
      WRITE(8,116) LC                                                   PRO00340
  116 FORMAT(//,'TOROIDAL ROOT NUMBER',I5)                              PRO00350
      CALL CTART(LC,X)                                                  PRO00360
      LJ=0                                                              PRO00370
      DO 120 LE=1,NTALMX                                                PRO00380
      CALL MULRCD(LJ,X,U,V,FX)                                          PRO00390
      IF(LJ.LE.0) GO TO 121                                             PRO00400
      Z=DCMPLX(U,V)                                                     PRO00410
      W=ZDETC(Z)                                                        PRO00420
      FX(2)=DIMAG(W)                                                    PRO00430
  120 FX(1)=DREAL(W)                                                    PRO00440
  121 U=X(4,1)                                                          PRO00450
      V=X(4,2)                                                          PRO00460
      DO 1001 I=1,2                                                     PRO00470
      DO 1001 J=1,2                                                     PRO00480
      J4=4*(LC-1)+2*(I-1)+J                                             PRO00490
      U=(-1)**(1+J)*X(4,1)                                              PRO00500
      V=(-1)**(1+I)*(-1)**(J+1)*X(4,2)                                  PRO00510
      ECYLD(J4)=DCMPLX(U,V)                                             PRO00520
 1001 CONTINUE                                                          PRO00530
  110 CONTINUE                                                          PRO00540
C     INPUT THE ADJUSTED THIRD GROUP EIGENVALUES                        PRO00550
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      N=4*(NHIC-1)                                                      PRO00560
      DO 1003 I=1,N                                                     PRO00570
      M=I+2                                                             PRO00580
      ECYLD(I)=ECYLD(M)                                                 PRO00590
      IF (I.EQ.1) ECYLD(1)=(2.700956D0,0.0D0)                           PRO00600
      IF (I.EQ.2) ECYLD(2)=(-2.700956D0,0.0D0)                          PRO00610
      Z=ECYLD(I)                                                        PRO00620
      U=DREAL(Z)                                                        PRO00630
      V=DIMAG(Z)                                                        PRO00640
      WRITE(6,9) U,V,I                                                  PRO00650
      WRITE(8,9) U,V,I                                                  PRO00660
    9 FORMAT(//,'EIGNVALUE=',5X,2F18.8,I5)                             PRO00670
 1003 CONTINUE                                                          PRO00680
      CALL ZSOLVC(N,D,AC,BC)                                            PRO00690
      CALL BULLSQ(N,ECYLD,LAM)                                          PRO00700
      CALL STRESS(N,ECYLD,LAM)                                          PRO00710
      CALL DISPLA(N,ECYLD,LAM)                                          PRO00720
      CALL RITE(1)                                                      PRO00730
      STOP                                                              PRO00740
      END                                                               PRO00750
C*********************************************************************** PRO00760
      SUBROUTINE RITE(K)                                                PRO00770
      COMMON/GEOMAL/RIC,ROC,RIT,ROT,ADIM,POISS,EY,NHIC,RRT              PRO00780
      COMMON/CHARAC/ECYLD(40),EVCYLD(40,4),ESPHE(40),EVSPHE(40,4)       PRO00790
      COMPLEX*16 ECYLD,EVCYLD,ESPHE,EVSPHE                              PRO00800
      REAL*8 RIC,ROC,RIS,ROS,ADIM,POISS,EY                              PRO00810
  100 IF(K.NE.1) GO TO 200                                              PRO00820
      WRITE(8,116) NHIC,RIT,ROT,RRT,ADIM,POISS,EY                       PRO00830
  116 FORMAT(//,'GEOMAL DATA'/                                          PRO00840
     1 1X,' NHIC ',6X,'RIT',6X,'ROT',6X,'RRT',5X,                       PRO00850
     2 'ADIM',4X,'POISS',7X,'EY'/1X,I5,5F9.2,F14.2)                     PRO00860
      RETURN                                                            PRO00870
  200 WRITE(8,216)                                                      PRO00880
  216 FORMAT(//,'TOROIDAL EIGENVALUES '/                                PRO00890
     1  1X,'     N',4X,'ALPHAR',4X,'ALPHAI',8X,'BR',8X,'BI',           PRO00900
     2  8X,'CR',8X,'CI')                                               PRO00910
      DO 210 N=1,NHIC                                                   PRO00920
  210 WRITE(8,226) N,ECYLD(N),(EVCYLD(N,L),L=1,4)                       PRO00930
  226 FORMAT(1X,I5,6D10.2/6X,4D10.2)                                    PRO00940
      RETURN                                                            PRO00950
      END                                                               PRO00960
C*********************************************************************** PRO00970
      SUBROUTINE CTART(J,X)                                             PRO00980
C     ROUTINE GIVES TRIAL EIGVAL FOR TOROIDAL SOLUTION                  PRO00990
      COMMON/GEOMAL/RIC,ROC,RIT,ROT,ADIM,POISS,EY,NHIC,RRT              PRO01000
      REAL*8 RIC,ROC,RIS,ROS,ADIM,POISS,EY                              PRO01010
      REAL*8 PIE,CON,H,FACT,AK,XA,DLOG10,DBLE,X(4,2),A(17),B(17)        PRO01020
      PIE=3.14159265358979D0                                            PRO01030
      CON=.1D0                                                          PRO01040
      H=.5D0*(ROC-RIC)/ADIM*3.20D0                                      PRO01050
      FACT=2.D0*H                                                       PRO01060
      IF(J.GT.1) GO TO 200                                              PRO01070
C     CASE FOR FIRST ROOT                                               PRO01080
  100 AK=2.D0                                                           PRO01090
      XA=2.D0*PIE*(AK-.5D0)                                             PRO01100
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      X(2,1)=FACT*(.25D0*XA-DLOG10(XA)/XA)/H                                    PRO01110
      X(2,2)=FACT*(DLOG10(XA))/H                                                PRO01120
      GO TO 300                                                                 PRO01130
C     CASE FOR SUBSEQUENT ROOTS                                                 PRO01140
  200 AK=DBLE(J)                                                                PRO01150
      XA=2.D0*PIE*(AK-.5D0)                                                     PRO01160
      X(2,1)=(.25D0*XA-DLOG10(XA)/XA)/H-6.0D0                                   PRO01170
      IF (J.GT.14) GOTO 211                                                     PRO01180
      X(2,1)=(.25D0*XA-DLOG10(XA)/XA)/H-3.0D0                                   PRO01190
      IF (J.GT.7) GOTO 211                                                      PRO01200
      X(2,1)=(.25D0*XA-DLOG10(XA)/XA)/H                                         PRO01210
  211 X(2,2)=(DLOG10(XA))/H                                                     PRO01220
C     DEFINE COMPLETE X MATRIX                                                  PRO01230
  300 X(1,1)=X(2,1)-CON                                                         PRO01240
      X(3,1)=X(2,1)+CON                                                         PRO01250
      X(1,2)=X(2,2)-CON                                                         PRO01260
      X(3,2)=X(2,2)+CON                                                         PRO01270
      RETURN                                                                    PRO01280
      END                                                                       PRO01290
C-----------------------------------------------------------------------------PRO01300
      COMPLEX FUNCTION ZDETC*16(Z)                                              PRO01310
C     EVALUATE DETERMINANT FOR DETERMINING TOROIDAL EIGVALS                     PRO01320
      COMMON/GEOMAL/RIC,ROC,RIT,ROT,ADIM,POISS,EY,NHIC,RRT                      PRO01330
      COMPLEX*16 Z,ZZ,VV(2),WW(2),V(3,2),W(3,2),ROISZ,ROOSZ                     PRO01340
      COMPLEX*16 ELZZ,ELZU,ELUZ,ELUU,H,B(5,4)                                   PRO01350
      REAL*8 RIC,ROC,RIS,ROS,ADIM,POISS,EY,HH1,HH2,V44,V54,LAM                  PRO01360
      REAL*8 TOL,ARG(2),GAMA,ROIS,ROOS,POZY                                     PRO01370
      V44=4.0D0-4.0D0*0.30D0                                                    PRO01380
      V54=5.0D0-4.0D0*0.30D0                                                    PRO01390
      LAM=RIT/ROT                                                               PRO01400
      H=(0.0D0,1.0D0)                                                           PRO01410
      ZZ=Z*Z                                                                    PRO01420
      B(1,1)=-(ZZ+2.0D0+Z*H)                                                    PRO01430
      B(1,2)=-(ZZ+2.0D0-Z*H)                                                    PRO01440
      B(1,3)=-ZZ+V44+V54*Z*H                                                    PRO01450
      B(1,4)=-ZZ+V44-V54*Z*H                                                    PRO01460
      B(2,1)=Z+ZZ*H                                                             PRO01470
      B(2,2)=Z-ZZ*H                                                             PRO01480
      B(2,3)=-V54*Z-(ZZ-V44)*H                                                  PRO01490
      B(2,4)=-V54*Z+(ZZ-V44)*H                                                  PRO01500
      B(3,1)=-(ZZ+2.0D0+Z*H)*LAM**(Z*H)                                         PRO01510
      B(3,2)=-(ZZ+2.0D0-Z*H)/LAM**(Z*H)                                         PRO01520
      B(3,3)=(ZZ-V44-V54*Z*H)/(-LAM**(2.0D0+Z*H))                               PRO01530
      B(3,4)=(ZZ-V44+V54*Z*H)*(-LAM**(-2.0D0+Z*H))                              PRO01540
      B(4,1)=(Z+ZZ*H)*LAM**(Z*H)                                                PRO01550
      B(4,2)=(Z-ZZ*H)/LAM**(Z*H)                                                PRO01560
      B(4,3)=(-V54*Z-(ZZ-V44)*H)/LAM**(2.0D0+Z*H)                               PRO01570
      B(4,4)=(-V54*Z+(ZZ-V44)*H)/LAM**(2.0D0-Z*H)                               PRO01580
      B(5,1)=B(1,2)*B(2,3)*B(3,4)+B(1,3)*B(2,4)*B(3,2)+B(1,4)*B(2,2)*B(3PRO01590
     #,3)-B(1,4)*B(2,3)*B(3,2)-B(1,3)*B(2,2)*B(3,4)-B(1,2)*B(2,4)*B(3,3)PRO01600
      B(5,2)=B(1,1)*B(2,3)*B(3,4)+B(1,3)*B(2,4)*B(3,1)+B(1,4)*B(2,1)*B(3PRO01610
     #,3)-B(1,4)*B(2,3)*B(3,1)-B(1,3)*B(2,1)*B(3,4)-B(1,1)*B(2,4)*B(3,3)PRO01620
      B(5,3)=B(1,1)*B(2,2)*B(3,4)+B(1,2)*B(2,4)*B(3,1)+B(1,4)*B(2,1)*B(3PRO01630
     #,2)-B(1,4)*B(2,2)*B(3,1)-B(1,2)*B(2,1)*B(3,4)-B(1,1)*B(2,4)*B(3,2)PRO01640
      B(5,4)=B(1,1)*B(2,2)*B(3,3)+B(1,2)*B(2,3)*B(3,1)+B(1,3)*B(2,1)*B(3PRO01650
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     #,2)-B(1,3)*B(2,2)*B(3,1)-B(1,2)*B(2,1)*B(3,3)-B(1,1)*B(2,3)*B(3,2)PRO01660
      ZDETC=-B(4,1)*B(5,1)+B(4,2)*B(5,2)-B(4,3)*B(5,3)+B(4,4)*B(5,4)    PRO01670
      RETURN                                                            PRO01680
      END                                                               PRO01690
C*****************************************************************************PRO01700
      SUBROUTINE ZSOLVC(N,D,AC,BC)                                      PRO01710
C     FIND EIGENVECTOR AND CONSTANTS FOR EIGENFUNCTIONS                 PRO01720
      COMMON/GEOMAL/RIC,ROC,RIT,ROT,ADIM,POISS,EY,NHIC,RRT              PRO01730
      COMMON/CHARAC/ECYLD(40),EVCYLD(40,4),ESPHE(40),EVSPHE(40,4)       PRO01740
      COMPLEX*16 Z,ZZ,W,H,E(4),AC(40,3,2),BC(40,3,2),B(50,51),D(40,4,3) PRO01750
      COMPLEX*16 ZI,ELZU,ELUZ,ELUU,FOURTH,ECYLD,EVCYLD,A(50,51)         PRO01760
      COMPLEX*16 F0,F1,F2,G1,G2,Z1,ZZ1,ZI1,W1,W2,F,G                    PRO01770
      REAL*8 RIC,ROC,RIS,ROS,ADIM,POISS,EY,P,Q,LAM,P1,Q1,ONE            PRO01780
      REAL*8 TOL,ARG(2),GAMA,ROI,ROO,POZY                               PRO01790
      REAL*8 Y,Y11,Y12,Y14,Y23,Y32,Y34,Y44,Y54,Y56,X,PI                 PRO01800
      Y=0.3D0                                                           PRO01810
      Y11=1.0D0-Y                                                       PRO01820
      Y12=1.0D0-2.0D0*Y                                                 PRO01830
      Y14=1.0D0-4.0D0*Y                                                 PRO01840
      Y23=2.0D0-3.0D0*Y                                                 PRO01850
      Y32=3.0D0-2.0D0*Y                                                 PRO01860
      Y34=3.0D0-4.0D0*Y                                                 PRO01870
      Y44=4.0D0-4.0D0*Y                                                 PRO01880
      Y54=5.0D0-4.0D0*Y                                                 PRO01890
      Y56=5.0D0-6.0D0*Y                                                 PRO01900
      LAM=0.5D0                                                         PRO01910
      ONE=1.0D0                                                         PRO01920
      X=EXP(1.0D0)                                                      PRO01930
      PI=ATAN(1.0D0)*4.0D0                                              PRO01940
      H=(0.0D0,1.0D0)                                                   PRO01950
      DO 1111 J=1,N                                                     PRO01960
      Z=ECYLD(J)                                                        PRO01970
      ZZ=Z*Z                                                            PRO01980
      B(1,1)=-(ZZ+2.0D0+Z*H)*ONE**(Z*H)                                 PRO01990
      B(1,2)=-(ZZ+2.0D0-Z*H)/ONE**(Z*H)                                 PRO02000
      B(1,3)=(ZZ-Y44-Y54*Z*H)/(-ONE**(2.0D0+Z*H))                       PRO02010
      B(1,4)=(ZZ-Y44+Y54*Z*H)*(-ONE**(-2.0D0+Z*H))                      PRO02020
      B(2,1)=(Z+ZZ*H)*ONE**(Z*H)                                        PRO02030
      B(2,2)=(Z-ZZ*H)/ONE**(Z*H)                                        PRO02040
      B(2,3)=(-Y54*Z-(ZZ-Y44)*H)/ONE**(2.0D0+Z*H)                       PRO02050
      B(2,4)=(-Y54*Z+(ZZ-Y44)*H)/ONE**(2.0D0-Z*H)                       PRO02060
      B(3,1)=-(ZZ+2.0D0+Z*H)*LAM**(Z*H)                                 PRO02070
      B(3,2)=-(ZZ+2.0D0-Z*H)/LAM**(Z*H)                                 PRO02080
      B(3,3)=(ZZ-Y44-Y54*Z*H)/(-LAM**(2.0D0+Z*H))                       PRO02090
      B(3,4)=(ZZ-Y44+Y54*Z*H)*(-LAM**(-2.0D0+Z*H))                      PRO02100
      B(4,1)=(Z+ZZ*H)*LAM**(Z*H)                                        PRO02110
      B(4,2)=(Z-ZZ*H)/LAM**(Z*H)                                        PRO02120
      B(4,3)=(-Y54*Z-(ZZ-Y44)*H)/LAM**(2.0D0+Z*H)                       PRO02130
      B(4,4)=(-Y54*Z+(ZZ-Y44)*H)/LAM**(2.0D0-Z*H)                       PRO02140
      B(5,1)=B(1,2)*B(2,3)*B(3,4)+B(1,3)*B(2,4)*B(3,2)+B(1,4)*B(2,2)*B(3PRO02150
     #,3)-B(1,4)*B(2,3)*B(3,2)-B(1,3)*B(2,2)*B(3,4)-B(1,2)*B(2,4)*B(3,3)PRO02160
      B(5,2)=B(1,1)*B(2,3)*B(3,4)+B(1,3)*B(2,4)*B(3,1)+B(1,4)*B(2,1)*B(3PRO02170
     #,3)-B(1,4)*B(2,3)*B(3,1)-B(1,3)*B(2,1)*B(3,4)-B(1,1)*B(2,4)*B(3,3)PRO02180
      B(5,3)=B(1,1)*B(2,2)*B(3,4)+B(1,2)*B(2,4)*B(3,1)+B(1,4)*B(2,1)*B(3PRO02190
     #,2)-B(1,4)*B(2,2)*B(3,1)-B(1,2)*B(2,1)*B(3,4)-B(1,1)*B(2,4)*B(3,2)PRO02200
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      B(5,4)=B(1,1)*B(2,2)*B(3,3)+B(1,2)*B(2,3)*B(3,1)+B(1,3)*B(2,1)*B(3PRO02210
     #,2)-B(1,3)*B(2,2)*B(3,1)-B(1,2)*B(2,1)*B(3,3)-B(1,1)*B(2,3)*B(3,2)PRO02220
      W=-B(4,1)*B(5,1)+B(4,2)*B(5,2)-B(4,3)*B(5,3)+B(4,4)*B(5,4)        PRO02230
      D(J,1,1)=-B(5,1)/B(5,4)                                           PRO02240
      D(J,2,1)= B(5,2)/B(5,4)                                           PRO02250
      D(J,3,1)=-B(5,3)/B(5,4)                                           PRO02260
      D(J,4,1)= 1.0D0                                                   PRO02270
      IF (J.GT.2) GOTO 777                                              PRO02280
      D(J,1,1)=-(3.0D0-2.0D0*LAM)*(1.0D0-Z*H)*(4.0D0*Y11-Z*H)           PRO02290
     +     *( LAM**(-Z*H)-(1.0D0-2.0D0*(1.0D0-LAM)*Z*H)*LAM**(Z*H) )    PRO02300
     +     /(Z*H*(1.0D0+Z*H))/((3.0D0-2.0D0*LAM)*LAM**(-Z*H)            PRO02310
     +      -(3.0D0-2.0D0*LAM-2.0D0*(1.0D0-LAM)*Z*H)*LAM**(Z*H) )       PRO02320
      D(J,2,1)=(4.0D0*Y11-Z*H)*(3.0D0-2.0D0*LAM)*LAM**(2.0D0*Z*H)       PRO02330
     +      -(1.0D0+Z*H)*LAM**(2.0D0*Z*H)*D(J,1,1)                     PRO02340
      D(J,3,1)=( Z*H*(1.0D0+Z*H)*D(J,1,1)+Z*H*(1.0D0-Z*H)*D(J,2,1)      PRO02350
     +      +(4.0D0*Y11-Z*H)*(1.0D0-Z*H) )/(4.0D0*Y11+Z*H)             PRO02360
     +      /(1.0D0+Z*H)                                               PRO02370
  777 CONTINUE                                                          PRO02380
 1111 CONTINUE                                                          PRO02390
C-----CALCULATE D1(J,K),D2(J,K)-------------------------------------    PRO02400
      DO 2222 J=1,N                                                     PRO02410
      Z=ECYLD(J)                                                        PRO02420
      ZZ=Z*Z                                                            PRO02430
      ZI=Z*H                                                            PRO02440
      E(1)=((Y54-ZZ)+2.0D0*Y56*ZI)*(ZZ+Y14+4.0D0*Y23*ZI)                PRO02450
     #      -(-2.0D0*Y12*Z+(ZZ-Y14)*H)*(Y44*Z+(ZZ+Y54)*H)               PRO02460
      E(2)=2.0D0*Y11*((ZZ+Y14-4.0D0*Y*ZI)*(ZZ+Y14+4.0D0*Y23*ZI)         PRO02470
     #   -(-2.0D0*Y12*Z+(ZZ-Y14)*H)*(-2.0D0*Y12*Z-(ZZ-Y14)*H))*D(J,1,1)  PRO02480
      E(3)=2.0D0*Y11*((Y54-ZZ+2.0D0*Y56*ZI)*(-2.0D0*Y12*Z+(Y14-ZZ)*H)   PRO02490
     #      -(Y44*Z+(ZZ+Y54)*H)*(ZZ+Y14-4.0D0*Y*ZI))*D(J,1,1)           PRO02500
      AC(J,1,1)=E(2)/E(1)                                               PRO02510
      BC(J,1,1)=E(3)/E(1)                                               PRO02520
      F1=(Y54-ZZ)-2.0D0*Y32*ZI                                          PRO02530
      F2=4.0D0*Y*Z-(ZZ+Y14)*H                                           PRO02540
      G2=ZZ+Y14+4.0D0*Y*ZI                                              PRO02550
      G1=2.0D0*Y32*Z+(-ZZ+Y54)*H                                        PRO02560
      F=2.0D0*Y11*(1.0D0-ZI)*D(J,2,1)                                   PRO02570
      G=2.0D0*Y11*(Z+H)*D(J,2,1)                                        PRO02580
      AC(J,2,1)=F/(F1+F2)                                               PRO02590
      BC(J,2,1)=AC(J,2,1)                                               PRO02600
      F1=(-Y34-ZZ)+2.0D0*Y12*ZI                                         PRO02610
      F2=-Z*Y44+(ZZ-Y34)*H                                              PRO02620
      F=2.0D0*Y11*(ZZ-Y44+Y54*ZI)*D(J,4,1)                              PRO02630
      G1=2.0D0*Y12*Z+(ZZ+Y34)*H                                         PRO02640
      G2=ZZ-Y34+Y44*ZI                                                  PRO02650
      G=2.0D0*Y11*(Y54*Z-(ZZ-Y44)*H)*D(J,4,1)                           PRO02660
      AC(J,3,1)=F/(F1+F2)                                               PRO02670
      BC(J,3,1)=AC(J,3,1)                                               PRO02680
      F1=(Y54-ZZ)+2.0D0*Y32*ZI                                          PRO02690
      F2=4.0D0*Y*Z+(ZZ+Y14)*H                                           PRO02700
      G2=ZZ+Y14-4.0D0*Y*ZI                                              PRO02710
      G1=2.0D0*Y23*Z+(ZZ-Y54)*H                                         PRO02720
      F=(1.0D0+ZI)*D(J,1,1)                                             PRO02730
      G=(Z-H)*D(J,1,1)                                                  PRO02740
      AC(J,1,2)=0.0D0                                                   PRO02750
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      BC(J,1,2)=F/F2                                                    PRO02760
      E(1)=((Y54-ZZ)-2.0D0*Y56*ZI)*(ZZ+Y14-4.0D0*Y23*ZI)                PRO02770
     #     -(Y44*Z-(ZZ-Y54)*H)*(4.0D0*Y*Z-(ZZ-Y14)*H)                   PRO02780
      E(2)=((ZZ+Y14+4.0D0*Y*ZI)*(ZZ+Y14-4.0D0*Y23*ZI)-(-2.0D0*Y12*Z     PRO02790
     #     +(ZZ-Y14)*H)*(4*Y*Z+(-ZZ+Y14)*H))*D(J,2,1)                   PRO02800
      E(3)=((Y54-ZZ-2.0D0*Y56*ZI)*(-2.0D0*Y12*Z+(ZZ-Y14)*H)-(Y44*Z      PRO02810
     #     -(ZZ+Y54)*H)*(ZZ+Y14+4.0D0*Y*ZI))*D(J,2,1)                   PRO02820
      AC(J,2,2)=E(2)/E(1)                                               PRO02830
      BC(J,2,2)=E(3)/E(1)                                               PRO02840
      F1=-Y34-ZZ-2.0D0*Y12*ZI                                           PRO02850
      F2=-Y44*Z-(ZZ-Y34)*H                                              PRO02860
      G2=ZZ-Y34-Y44*ZI                                                  PRO02870
      G1=2.0D0*Y12*Z-(ZZ+Y34)*H                                         PRO02880
      F=(ZZ-Y44-Y54*ZI)*D(J,3,1)                                        PRO02890
      G=(Y54*2+(ZZ-Y44)*H)*D(J,3,1)                                     PRO02900
      AC(J,3,2)=F/(F1+F2)                                               PRO02910
      BC(J,3,2)=AC(J,3,2)                                               PRO02920
C-----------------------------------------------------------------------  PRO02930
      DO 1 I=2,3                                                        PRO02940
      Z=ECYLD(J)                                                        PRO02950
      ZZ=Z*Z                                                            PRO02960
      IF(I.EQ.3) GOTO 2                                                 PRO02970
      B(1,5)=-Y/Y12*(((Y12/Y*(2.0D0+Z*H)+3.0D0+Z*H)*AC(J,1,1)+(Z+H)     PRO02980
     #          *BC(J,1,1)+(1.0D0+Z*H)*D(J,1,1) )*ONE**(1.0D0+Z*H)      PRO02990
     #       +((Y12/Y*(2.0D0-Z*H)+3.0D0-Z*H)*AC(J,2,1)+(Z+H)           PRO03000
     #          *BC(J,2,1)-Y34*D(J,2,1) )*ONE**(1.0D0-Z*H)             PRO03010
     #       +((Y12/Y*Z*H+1.0D0+Z*H)*AC(J,3,1)+(Z+H)*BC(J,3,1)         PRO03020
     #          +(-Y44+Z*H)*D(J,4,1) )*ONE**(-1.0D0+Z*H)  )            PRO03030
      B(2,5)=-(((Z+H)*AC(J,1,1)+(1.0D0+Z*H)*BC(J,1,1))*ONE**(1.0D0+Z*H) PRO03040
     #       +((Z+H)*AC(J,2,1)+(1.0D0-Z*H)*BC(J,2,1))*ONE**(1.0D0-Z*H)  PRO03050
     #       +((Z+H)*AC(J,3,1)-(1.0D0-Z*H)*BC(J,3,1))*ONE**(Z*H-1.0D0)) PRO03060
      B(3,5)=-Y/Y12*(((Y12/Y*(2.0D0+Z*H)+3.0D0+Z*H)*AC(J,1,1)+(Z+H)     PRO03070
     #          *BC(J,1,1)+(1.0D0+Z*H)*D(J,1,1) )*LAM**(1.0D0+Z*H)      PRO03080
     #       +((Y12/Y*(2.0D0-Z*H)+3.0D0-Z*H)*AC(J,2,1)+(Z+H)*BC(J,2,1)  PRO03090
     #                -Y34*D(J,2,1) )*LAM**(1.0D0-Z*H)                  PRO03100
     #       +((Y12/Y*Z*H+1.0D0+Z*H)*AC(J,3,1)+(Z+H)*BC(J,3,1)         PRO03110
     #                +(-Y44+Z*H)*D(J,4,1) )*LAM**(-1.0D0+Z*H)  )       PRO03120
      B(4,5)=-(((Z+H)*AC(J,1,1)+(1.0D0+Z*H)*BC(J,1,1))*LAM**(1.0D0+Z*H) PRO03130
     #       +((Z+H)*AC(J,2,1)+(1.0D0-Z*H)*BC(J,2,1))*LAM**(1.0D0-Z*H)  PRO03140
     #       +((Z+H)*AC(J,3,1)-(1.0D0-Z*H)*BC(J,3,1))*LAM**(Z*H-1.0D0)) PRO03150
      E(1)=B(1,5)                                                       PRO03160
      E(2)=B(2,5)                                                       PRO03170
      E(3)=B(3,5)                                                       PRO03180
      E(4)=B(4,5)                                                       PRO03190
      GOTO 3                                                            PRO03200
    2 B(1,5)=-Y/Y12*(((Y12/Y*(2.0D0+Z*H)+3.0D0+Z*H)*AC(J,1,2)+(Z-H)     PRO03210
     #          *BC(J,1,2)-Y34*D(J,1,1) )*ONE**(1.0D0+Z*H)             PRO03220
     #       +((Y12/Y*(2.0D0-Z*H)+3.0D0-Z*H)*AC(J,2,2)+(Z-H)*BC(J,2,2)  PRO03230
     #          +(1-Z*H)*D(J,2,1) )*ONE**(1.0D0-Z*H)                   PRO03240
     #       +((-Y12/Y+1.0D0-Z*H)*AC(J,3,2)+(Z-H)*BC(J,3,2)            PRO03250
     #          -(Y44+Z*H)*D(J,3,1) )*ONE**(-1.0D0-Z*H) )              PRO03260
      B(2,5)=-(((Z-H)*AC(J,1,2)+(1.0D0+Z*H)*BC(J,1,2))*ONE**(1.0D0+Z*H) PRO03270
     #       +((Z-H)*AC(J,2,2)+(1.0D0-Z*H)*BC(J,2,2))*ONE**(1.0D0-Z*H)  PRO03280
     #      +((Z-H)*AC(J,3,2)-(1.0D0+Z*H)*BC(J,3,2))*ONE**(-1.0D0-Z*H)) PRO03290
      B(3,5)=-Y/Y12*(((Y12/Y*(2.0D0+Z*H)+3.0D0+Z*H)*AC(J,1,2)+(Z-H)     PRO03300
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     #           *BC(J,1,2)-Y34*D(J,1,1) )*LAM**(1.0D0+Z*H)                    PRO03310
     #         +((Y12/Y*(2.0D0-Z*H)+3.0D0-Z*H)*AC(J,2,2)+(Z-H)*BC(J,2,2)       PRO03320
     #           +(1.0D0-Z*H)*D(J,2,1) )*LAM**(1.0D0-Z*H)                      PRO03330
     #         +((-Y12/Y-1.0D0-Z*H)*AC(J,3,2)+(Z-H)*BC(J,3,2)                  PRO03340
     #           -(Y44-Z*H)*D(J,3,1) )*LAM**(-1.0D0-Z*H)  )                    PRO03350
      B(4,5)=-(((Z-H)*AC(J,1,2)+(1.0D0+Z*H)*BC(J,1,2))*LAM**(1.0D0+Z*H)        PRO03360
     #         +((Z-H)*AC(J,2,2)+(1.0D0-Z*H)*BC(J,2,2))*LAM**(1.0D0-Z*H)       PRO03370
     #       +((Z-H)*AC(J,3,2)-(1.0D0+Z*H)*BC(J,3,2))*LAM**(-1.0D0-Z*H))       PRO03380
      E(1)=B(1,5)                                                              PRO03390
      E(2)=B(2,5)                                                              PRO03400
      E(3)=B(3,5)                                                              PRO03410
      E(4)=B(4,5)                                                              PRO03420
    3 CONTINUE                                                                 PRO03430
C-------------------------------------------------------------------           PRO03440
      Z=ECYLD(J)+H*(-1)**I                                                     PRO03450
      ZZ=Z*Z                                                                   PRO03460
      B(1,1)=-(ZZ+2.0D0+Z*H)*ONE**(Z*H)                                        PRO03470
      B(1,2)=-(ZZ+2.0D0-Z*H)/ONE**(Z*H)                                        PRO03480
      B(1,3)=(ZZ-Y44-Y54*Z*H)/(-ONE**(2.0D0+Z*H))                              PRO03490
      B(1,4)=(ZZ-Y44+Y54*Z*H)*(-ONE**(-2.0D0+Z*H))                             PRO03500
      B(2,1)=(Z+ZZ*H)*ONE**(Z*H)                                               PRO03510
      B(2,2)=(Z-ZZ*H)*ONE**(-Z*H)                                              PRO03520
      B(2,3)=(-Y54*Z-(ZZ-Y44)*H)*ONE**(-2.0D0-Z*H)                             PRO03530
      B(2,4)=(-Y54*Z+(ZZ-Y44)*H)*ONE**(-2.0D0+Z*H)                             PRO03540
      B(3,1)=-(ZZ+2.0D0+Z*H)*LAM**(Z*H)                                        PRO03550
      B(3,2)=-(ZZ+2.0D0-Z*H)/LAM**(Z*H)                                        PRO03560
      B(3,3)=(ZZ-Y44-Y54*Z*H)/(-LAM**(2.0D0+Z*H))                              PRO03570
      B(3,4)=(ZZ-Y44+Y54*Z*H)*(-LAM**(-2.0D0+Z*H))                             PRO03580
      B(4,1)=(Z+ZZ*H)*LAM**(Z*H)                                               PRO03590
      B(4,2)=(Z-ZZ*H)*LAM**(-Z*H)                                              PRO03600
      B(4,3)=(-Y54*Z-(ZZ-Y44)*H)*LAM**(-2.0D0-Z*H)                             PRO03610
      B(4,4)=(-Y54*Z+(ZZ-Y44)*H)*LAM**(-2.0D0+Z*H)                             PRO03620
C-------------------------------------------------------------------           PRO03630
      DO 101 KI=1,3                                                            PRO03640
      W=B(KI,4)/B(4,4)                                                         PRO03650
      DO 102 KJ=1,3                                                            PRO03660
  102 A(KI,KJ)=B(KI,KJ)-W*B(4,KJ)                                              PRO03670
  101 A(KI,4)=-B(KI,5)+W*B(4,5)                                                PRO03680
      A(5,1)=A(1,2)*A(2,3)*A(3,4)+A(1,3)*A(2,4)*A(3,2)+A(1,4)*A(2,2)*A(3PRO03690
     #,3)-A(1,4)*A(2,3)*A(3,2)-A(1,3)*A(2,2)*A(3,4)-A(1,2)*A(2,4)*A(3,3)PRO03700
      A(5,2)=A(1,1)*A(2,3)*A(3,4)+A(1,3)*A(2,4)*A(3,1)+A(1,4)*A(2,1)*A(3PRO03710
     #,3)-A(1,4)*A(2,3)*A(3,1)-A(1,3)*A(2,1)*A(3,4)-A(1,1)*A(2,4)*A(3,3)PRO03720
      A(5,3)=A(1,1)*A(2,2)*A(3,4)+A(1,2)*A(2,4)*A(3,1)+A(1,4)*A(2,1)*A(3PRO03730
     #,2)-A(1,4)*A(2,2)*A(3,1)-A(1,2)*A(2,1)*A(3,4)-A(1,1)*A(2,4)*A(3,2)PRO03740
      A(5,4)=A(1,1)*A(2,2)*A(3,3)+A(1,2)*A(2,3)*A(3,1)+A(1,3)*A(2,1)*A(3PRO03750
     #,2)-A(1,3)*A(2,2)*A(3,1)-A(1,2)*A(2,1)*A(3,3)-A(1,1)*A(2,3)*A(3,2)PRO03760
      D(J,1,I)=-A(5,1)/A(5,4)                                                  PRO03770
      D(J,2,I)= A(5,2)/A(5,4)                                                  PRO03780
      D(J,3,I)=-A(5,3)/A(5,4)                                                  PRO03790
      D(J,4,I)=(B(4,5)-B(4,1)*D(J,1,I)-B(4,2)*D(J,2,I)-B(4,3)*D(J,3,I))        PRO03800
     #        /B(4,4)                                                         PRO03810
    1 CONTINUE                                                                 PRO03820
 2222 CONTINUE                                                                 PRO03830
      RETURN                                                                   PRO03840
      END                                                                      PRO03850
```

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C*************************************************************************PRO03860
      SUBROUTINE MULRCD(J,X,A,B,FX)                                       PRO03870
C     ROUTINE TO FIND COMPLEX ROOTS OF A FUNCTION                         PRO03880
      REAL*8 X(4,2),F(4,2),FX(2),A,B,H1,H2,HI1,HI2,DIV,HL1                PRO03890
      REAL*8 HL2,D1,D2,HLF1,HLF2,HLLF1,HLLF2,DF1,DF2,DDF1                 PRO03900
      REAL*8 DDF2,S1,S2,FS1,FS2,SQ1,SQ2,GI1,GI2,UP1,UP2,STR1              PRO03910
      REAL*8 DSQRT,DATAN2,DCOS,DSIN,HH1,HH2,TEST,ZZ,DABS                  PRO03920
      REAL*8 STR2,DWN1,DWN2,GIS1,GIS2,ROOT1,ROOT2,ALPHA,THETA             PRO03930
      J=J+1                                                               PRO03940
      IF(J.GT.4) GO TO 5                                                  PRO03950
      GOTO(1,2,3,4),J                                                     PRO03960
    1 CONTINUE                                                            PRO03970
      A=X(1,1)                                                            PRO03980
      B=X(1,2)                                                            PRO03990
      RETURN                                                              PRO04000
    2 F(1,1)=FX(1)                                                        PRO04010
      F(1,2)=FX(2)                                                        PRO04020
      A=X(2,1)                                                            PRO04030
      B=X(2,2)                                                            PRO04040
      RETURN                                                              PRO04050
    3 F(2,1)=FX(1)                                                        PRO04060
      F(2,2)=FX(2)                                                        PRO04070
      A=X(3,1)                                                            PRO04080
      B=X(3,2)                                                            PRO04090
      RETURN                                                              PRO04100
    4 F(3,1)=FX(1)                                                        PRO04110
      F(3,2)=FX(2)                                                        PRO04120
   13 H1=X(3,1)-X(2,1)                                                    PRO04130
      H2=X(3,2)-X(2,2)                                                    PRO04140
      HI1=X(2,1)-X(1,1)                                                   PRO04150
      HI2=X(2,2)-X(1,2)                                                   PRO04160
      DIV=HI1**2+HI2**2                                                   PRO04170
      HL1=(H1*HI1+H2*HI2)/DIV                                             PRO04180
      HL2=(HI1*H2-H1*HI2)/DIV                                             PRO04190
      D1=1.D0+HL1                                                         PRO04200
      D2=HL2                                                              PRO04210
      HLF1=HL1*F(1,1)-HL2*F(1,2)                                          PRO04220
      HLF2=HL2*F(1,1)+HL1*F(1,2)                                          PRO04230
      HLLF1=HLF1*HL1-HLF2*HL2                                             PRO04240
      HLLF2=HLF1*HL2+HLF2*HL1                                             PRO04250
      DF1=F(2,1)*D1-D2*F(2,2)                                             PRO04260
      DF2=F(2,1)*D2+D1*F(2,2)                                             PRO04270
      DDF1=DF1*D1-DF2*D2                                                  PRO04280
      DDF2=DF2*D1+DF1*D2                                                  PRO04290
      S1=HL1+D1                                                           PRO04300
      S2=HL2+D2                                                           PRO04310
      FS1=S1*F(3,1)-S2*F(3,2)                                             PRO04320
      FS2=S1*F(3,2)+S2*F(3,1)                                             PRO04330
      SQ1=HLF1-DF1+F(3,1)                                                 PRO04340
      SQ2=HLF2-DF2+F(3,2)                                                 PRO04350
      GI1=HLLF1-DDF1+FS1                                                  PRO04360
      GI2=HLLF2-DDF2+FS2                                                  PRO04370
      D1=2.D0*D1                                                          PRO04380
      D2=2.D0*D2                                                          PRO04390
      HL1=2.D0*HL1                                                        PRO04400
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      HL2=2.D0*HL2                                                                PRO04410
      UP1=D2*F(3,2)-D1*F(3,1)                                                     PRO04420
      UP2=-D1*F(3,2)-D2*F(3,1)                                                    PRO04430
      STR1=UP1*SQ1-UP2*SQ2                                                        PRO04440
      STR2=UP1*SQ2+UP2*SQ1                                                        PRO04450
      DWN1=STR1*HL1-STR2*HL2                                                      PRO04460
      DWN2=STR2*HL1+STR1*HL2                                                      PRO04470
      G1S1=GI1*GI1-GI2*GI2                                                        PRO04480
      GIS2=2.D0*GI1*GI2                                                           PRO04490
      SQ1=GIS1+DWN1                                                               PRO04500
      SQ2=GIS2+DWN2                                                               PRO04510
      IF(DABS(SQ1).GT.1.0D-14) GO TO 64                                           PRO04520
   63 ROOT2=DSQRT(DABS(SQ2/2.D0))                                                 PRO04530
      ROOT1=ROOT2                                                                 PRO04540
      IF(SQ2.LT..0D0) ROOT1=-ROOT2                                                PRO04550
      GO TO 65                                                                    PRO04560
   64 ALPHA=DSQRT(DSQRT(SQ1**2+SQ2**2))                                           PRO04570
      THETA=DATAN2(SQ2,SQ1)/2.D0                                                  PRO04580
      ROOT1=ALPHA*DCOS(THETA)                                                     PRO04590
      ROOT2=ALPHA*DSIN(THETA)                                                     PRO04600
   65 DWN1=GI1+ROOT1                                                              PRO04610
      DWN2=GI2+ROOT2                                                              PRO04620
      STR1=GI1-ROOT1                                                              PRO04630
      STR2=GI2-ROOT2                                                              PRO04640
      IF(DWN1*DWN1+DWN2*DWN2.GE.STR1*STR1+STR2*STR2) GO TO 10                     PRO04650
   11 DWN1=STR1                                                                   PRO04660
      DWN2=STR2                                                                   PRO04670
   10 DIV=DWN1**2+DWN2**2                                                         PRO04680
      HL1=(UP1*DWN1+UP2*DWN2)/DIV                                                 PRO04690
      HL2=(UP2*DWN1-UP1*DWN2)/DIV                                                 PRO04700
   23 HH1=HL1*H1-HL2*H2                                                           PRO04710
      HH2=HL1*H2+HL2*H1                                                           PRO04720
      X(4,1)=X(3,1)+HH1                                                           PRO04730
      X(4,2)=X(3,2)+HH2                                                           PRO04740
      A=X(4,1)                                                                    PRO04750
      B=X(4,2)                                                                    PRO04760
      RETURN                                                                      PRO04770
    5 F(4,1)=FX(1)                                                                PRO04780
      F(4,2)=FX(2)                                                                PRO04790
      TEST=(F(4,1)*F(4,1)+F(4,2)*F(4,2))/(F(3,1)*F(3,1)+F(3,2)*F(3,2))            PRO04800
      IF(TEST.LT.10.0) GO TO 27                                                   PRO04810
   28 HL1=.5D0*HL1                                                                PRO04820
      HL2=.5D0*HL2                                                                PRO04830
      GO TO 23                                                                    PRO04840
   27 WRITE(8,88) X(4,1),X(4,2),F(4,1),F(4,2)                                     PRO04850
      WRITE(6,88) X(4,1),X(4,2),F(4,1),F(4,2)                                     PRO04860
   88 FORMAT(1X,4D16.8)                                                           PRO04870
      DO 14 J1=1,3                                                                PRO04880
      X(J1,1)=X(J1+1,1)                                                           PRO04890
      X(J1,2)=X(J1+1,2)                                                           PRO04900
      F(J1,1)=F(J1+1,1)                                                           PRO04910
   14 F(J1,2)=F(J1+1,2)                                                           PRO04920
      ZZ=(HH1*HH1+HH2*HH2)/(X(4,1)*X(4,1)+X(4,2)*X(4,2))                          PRO04930
      IF(ZZ.GT.1.0D-44) GO TO 13                                                  PRO04940
   12 J=0                                                                         PRO04950
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      RETURN                                                                    PRO04960
      END                                                                       PRO04970
C*************************************************************************PRO04980
      REAL FUNCTION DREAL*8 (Z)                                                 PRO04990
C     EXTRACT REAL*8 REAL PART OF DBLE PRE COMPLEX NO                           PRO05000
      COMPLEX*16 Z,ZC,DCONJG,DCMPLX,ZD                                          PRO05010
      REAL*8 OVA,CDABS,SIGN,OVE                                                 PRO05020
      ZC=DCONJG(Z)                                                              PRO05030
      ZC=.5D0*(Z-ZC)                                                            PRO05040
      OVA=CDABS(ZC)                                                             PRO05050
      SIGN=1.D0                                                                 PRO05060
      ZD=DCMPLX(OVA,.0D0)                                                       PRO05070
      ZC=ZD-ZC                                                                  PRO05080
      OVE=CDABS(ZC)                                                             PRO05090
      IF(OVE.GT.1.D-20) SIGN=-1.D0*SIGN                                         PRO05100
      DREAL=SIGN*OVA                                                            PRO05110
      RETURN                                                                    PRO05120
      END                                                                       PRO05130
C*************************************************************************     PRO05140
      REAL FUNCTION DIMAG*8(Z)                                                  PRO05150
C     DIMAG IS DBLE PRE IMAG PART OF COMPLEX*16 Z                               PRO05160
      COMPLEX*16 Z,ZC,DCONJG,DCMPLX,ZD                                          PRO05170
      REAL*8 OVA,CDABS,SIGN,OVE                                                 PRO05180
      ZC=DCONJG(Z)                                                              PRO05190
      ZC=.5D0*(Z-ZC)                                                            PRO05200
      OVA=CDABS(ZC)                                                             PRO05210
      SIGN=1.D0                                                                 PRO05220
      ZD=DCMPLX(.0D0,OVA)                                                       PRO05230
      ZC=ZD-ZC                                                                  PRO05240
      OVE=CDABS(ZC)                                                             PRO05250
      IF(OVE.GT.1.D-20) SIGN=-1.D0*SIGN                                         PRO05260
      DIMAG=SIGN*OVA                                                            PRO05270
      RETURN                                                                    PRO05280
      END                                                                       PRO05290
C*************************************************************************PRO05300
      SUBROUTINE BULLSQ(N,B,LAM)                                                PRO05310
C     DETERMINING THE N FREE CONSTANTS                                          PRO05320
      COMMON/ZHU/D(40,4,3),AC(40,3,2),BC(40,3,2),A(50,51)                       PRO05330
      COMPLEX*16 D,AC,BC,A,B(40),F1,F2,G1,G2,H,ZERO,F0,W                        PRO05340
      COMPLEX*16 AR(160,40),AQ(160,40),Z,ZZ,ZI,WI0,WIQ                          PRO05350
      COMPLEX*16 DCMPLX,F(3),Y,YY,YI,WJ0,WJQ                                    PRO05360
      REAL*8 EPS,U1,U2,U3,U4,DREAL,DIMAG,V,V12,V34,V44,V54,V74                  PRO05370
      REAL*8 EE,SS,TK,TM,TN,T4,CLAM,CLAM1                                       PRO05380
      REAL*8 X,Q,Q0,PRESS,EPSI,C,RRI,RRO,RAB1,RAB2,HL(80)                       PRO05390
      REAL*8 LAM,LAM2,LAM4,LAM6,LAM8,CNOA3,CONB3,CONC3,COND3                    PRO05400
      REAL*8 A1R(160,80),A1Q(320,80),BR(160),BQ(160),B1(320),TOL,XL(80)         PRO05410
      INTEGER IPL(80),IER                                                       PRO05420
      M=N*2                                                                     PRO05430
      MH=M/2                                                                    PRO05440
      Q0=ATAN(1.0D0)                                                            PRO05450
      PRESS=3.0D0                                                               PRO05460
      EE=3.0D7                                                                  PRO05470
      V=0.3D0                                                                   PRO05480
      V12=1.0D0-2.0D0*V                                                         PRO05490
      V34=3.0D0-4.0D0*V                                                         PRO05500
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      V44=4.0D0-4.0D0*V                                                 PRO05510
      V54=5.0D0-4.0D0*V                                                 PRO05520
      V74=7.0D0-4.0D0*V                                                 PRO05530
      LAM2=LAM*LAM                                                      PRO05540
      LAM4=LAM2*LAM2                                                    PRO05550
      LAM6=LAM2*LAM4                                                    PRO05560
      LAM8=LAM4*LAM4                                                    PRO05570
      RRI=0.20625D0                                                     PRO05580
      RRO=0.41250D0                                                     PRO05590
      C=(1.0D0+V)*PRESS*RRI*RRI/(RRO*RRO-RRI*RRI)/EE                    PRO05600
      EPSI=RRO/(1.1D0-RRO)                                              PRO05610
      RAB1=RRI*RRI/(RRO*RRO+RRI*RRI)                                    PRO05620
      RAB2=RRO*RRO/(RRO*RRO+RRI*RRI)                                    PRO05630
      H=(0.0D0,1.0D0)                                                   PRO05640
      ZERO=(0.0D0,0.0D0)                                                PRO05650
      CLAM=(1+V)/4.0D0/(1.0D0-LAM2)/(1.0D0-LAM2)                        PRO05660
      CLAM1=CLAM/(1.0D0-LAM2)                                           PRO05670
      TK=CLAM1*((3.0D0-V)*(1.0D0-LAM2)**3-V12*(1.0D0-LAM6))/3.0D0       PRO05680
      TM=CLAM1*(LAM4-LAM8)                                              PRO05690
      TN=CLAM1*(LAM2-LAM8)                                              PRO05700
      T4=8.0D0*(LAM4-LAM6)-(1.0D0+LAM2)*(1.0D0-LAM8)                    PRO05710
      CONB3=( (4.0D0*(LAM4-LAM6)+(1.0D0+LAM2)*(1.0D0-LAM8))*TK          PRO05720
     #       -3.0D0*(1.0D0-LAM4)*TM + (4.0D0*V12*LAM2+(1.0D0+           PRO05730
     #        LAM2)**2)*(1.0D0-LAM2)*TN )/8.0D0/T4                      PRO05740
      COND3=( -3.0D0*LAM4*(1.0D0-LAM8)*TK -(2.0D0*(LAM4-LAM6)-          PRO05750
     #        (1.0D0+LAM2)*(1.0D0-LAM8))*TM - (2.0D0*(LAM4-LAM8)        PRO05760
     #        +V12*LAM2*(1.0D0-LAM8))*TN )/4.0D0/T4                     PRO05770
      CONA3=4.0D0*CONB3-COND3-(TK+TM-V12*TN)/4.0D0                      PRO05780
      CONC3=CONB3+2.0D0*COND3+(TK+TM+TN)/8.0D0                          PRO05790
      CBB=1./(8.0*(1.-V)*(1.+LAM2))                                     PRO05800
      CLA2=1./(4.0*(1.-V)*(1.+LAM2))                                    PRO05810
      CNU=1./(4.0*(1.-V)*(1.-2.*V))                                     PRO05820
      CONB1=-CBB                                                        PRO05830
      CONC1=LAM2*CBB                                                    PRO05840
      COND1=(3.-2.*V)/(4.*(1.-V))                                       PRO05850
      ASTA1=-2.0*(1.0-V)                                                PRO05860
      BSTA1=-ASTA1                                                      PRO05870
      CONA21=CNU/2.*( 3.*LAM2/(1.+LAM2)+2.*V*(1.+4.*V-4.*V*V)*          PRO05880
     #       LAM2*DLOG(LAM)/(1.-LAM2) )-1./EPSI/EPSI                   PRO05890
      CONB21=( 3.+V*(1.-6.*V)+2.*V*(1.+4.*V-4.*V*V)*LAM2*               PRO05900
     #        DLOG(LAM)/(1.-LAM2) )/8.0/(1.-V)                          PRO05910
      ASTA21=-CBB                                                       PRO05920
      BSTA21=0.0                                                        PRO05930
      G1=-(1.-2.*V)*CBB-CBB*LAM2/2.0+(2.-3.*V+4.*V*V)/(8.0*             PRO05940
     #     (1.0-V)*(1.0-2.0*V) )                                        PRO05950
      G2=-(1.0-2.*V)*CBB*LAM6-CBB*LAM4/2.+(2.-3.*V+4.*V*V)*LAM4         PRO05960
     #     /(8.0*(1.-V)*(1.-2.*V))+LAM4*DLOG(LAM)/4.0                   PRO05970
      G3=CBB*LAM2+1.0/8.0                                               PRO05980
      G4=CBB*LAM4+LAM4/8.0+LAM4*DLOG(LAM)/4.0                           PRO05990
      CONC22=(2.0*G4-2.0*LAM6*G3+(1.+3.*LAM2)*G2-(3.+LAM2)*LAM4*G1)     PRO06000
     #       /(3.0*(1.0-LAM2)**3)                                       PRO06010
      CONA22=(LAM2*G1-G2)/(LAM2-LAM4)-3.0*CONC22/LAM2                   PRO06020
      COND22=(3.0*CONC22*(1.-LAM4)-LAM4*G1+G2)/4./(LAM2-LAM4)           PRO06030
      CONB22=(CONA22+3.*CONC22-2.*COND22-G3)/6.0                        PRO06040
      ASTA22=1.0/24.0/(1.0+LAM2)                                        PRO06050
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      ASTA23=-LAM2*CBB/2.0                                                    PRO06060
      ASTA24=-1.0/4.0                                                         PRO06070
C     FORM THE COMPLEX VECTOR AND MATRICES                                    PRO06080
      DO 801 I=1,2                                                            PRO06090
      Q=(I-1)*Q0                                                              PRO06100
      DO 801 K=1,MH                                                           PRO06110
      X=LAM+(1.0D0-LAM)*(K-1)/(MH-1)                                          PRO06120
      KK=(I-1)*MH+K                                                           PRO06130
C     STRESS COMPONENTS CORRESPONDING TO THE IMAGENARY EIGENVALUES            PRO06140
      AQ(KK,1)=-CLAM*(1.0D0+LAM2)*(1.0D0+LAM2+LAM4-3.0D0*(1.0D0+LAM2)         PRO06150
     #     *X*X+3.0D0*LAM4/X/X/X/X)*(DSIN(2*Q)+H*DCOS(2*(Q)))*EPSI            PRO06160
      AR(KK,1)=-CLAM*((1.D0+LAM2)*(1.0D0+LAM2+LAM4)-(1.0D0+4.0D0*LAM2         PRO06170
     #      +LAM4)*X*X-3.0D0*LAM4*(1.0D0+LAM2)/X/X/X/X+2.0D0*LAM2*(           PRO06180
     #      1.0D0+LAM2+LAM4)/X/X)*(DCOS(2*Q)-H*DSIN(2*(Q)))*(EPSI)            PRO06190
C     INPUT THE BOUNDARY VALUES                                               PRO06200
      BQ(KK)= 2*820.*(1.0D0-X)/(1.0D0-LAM)-800.0                              PRO06210
      BR(KK)=0.0                                                              PRO06220
      IF (I.EQ.1) GOTO 802                                                    PRO06230
      BQ(KK)= 2*950.*(1.0D0-X)/(1.0D0-LAM)-880.                               PRO06240
      BR(KK)=0.0                                                              PRO06250
  802 CONTINUE                                                                PRO06260
C     STRESS COMPONENTS FOR GENERAL EIGENVALUES                               PRO06270
      DO 801 J=1,N                                                            PRO06280
      Z=B(J)                                                                  PRO06290
      ZZ=Z*Z                                                                  PRO06300
      ZI=Z*H                                                                  PRO06310
C     THE NORMAL STRESSES                                                     PRO06320
      F0= (ZZ-2.0D0-3.0D0*ZI)*D(J,1,1)*X**ZI                                  PRO06330
     #   +(ZZ-2.0D0+3.0D0*ZI)*D(J,2,1)*X**(-ZI)                               PRO06340
     #   +(ZZ-V44-V54*ZI)*D(J,3,1)*X**(-2.0D0-ZI)                             PRO06350
     #   +(ZZ-V44+V54*ZI)*D(J,4,1)*X**(-2.0D0+ZI)                             PRO06360
      IF (J.GT.2) GOTO 707                                                    PRO06370
      F0= (ZZ-2.0D0-3.0D0*ZI)*D(J,1,1)*X**ZI                                  PRO06380
     #   +(ZZ-2.0D0+3.0D0*ZI)*D(J,2,1)*X**(-ZI)                               PRO06390
     #   +(ZZ-V44-V54*ZI)*D(J,3,1)*(3.0D0-2.0D0*X)*X**(-ZI)                   PRO06400
     #   +(ZZ-V44+V54*ZI)*D(J,4,1)*(3.0D0-2.0D0*X)*X**(ZI)                    PRO06410
  707 F1= (ZZ-ZI)*D(J,1,2)*X**(ZI-1.0D0)                                      PRO06420
     #   +(ZZ-6.0D0+5.0D0*ZI)*D(J,2,2)*X**(1.0D0-ZI)                          PRO06430
     #   +(ZZ-V34*ZI)*D(J,3,2)*X**(-ZI-1.0D0)                                PRO06440
     #   +(ZZ-2.0D0*V54+V74*ZI)*D(J,4,2)*X**(-3.0D0+ZI)                       PRO06450
      F2= (ZZ-6.0D0-5.0D0*ZI)*D(J,1,3)*X**(1.0D0+ZI)                          PRO06460
     #   +(ZZ+3.0D0*ZI)*D(J,2,3)*X**(-ZI-1.0D0)                               PRO06470
     #   +(ZZ-2.0D0*V54-V54*ZI)*D(J,3,3)*X**(-3.0D0-ZI)                       PRO06480
     #   +(ZZ+V34*ZI)*D(J,4,3)*X**(ZI-1.0D0)                                  PRO06490
      G1=( ( V12/V*(AC(J,1,1)+(Z+H)*BC(J,1,1))+(3.0D0+ZI)*AC(J,1,1)           PRO06500
     #      +(Z+H)*BC(J,1,1)+(1.0D0+ZI)*D(J,1,1) )*X**(1.0D0+ZI)             PRO06510
     #    +( V12/V*(AC(J,2,1)+(Z+H)*BC(J,2,1))+(3.0D0-ZI)*AC(J,2,1)          PRO06520
     #      +(Z+H)*BC(J,2,1)-V34*D(J,2,1) )*X**(1.0D0-ZI)                    PRO06530
     #    +( V12/V*(AC(J,3,1)+(Z+H)*BC(J,3,1))+(1.0D0+ZI)*AC(J,3,1)          PRO06540
     #      +(Z+H)*BC(J,3,1)-(V44-ZI)*D(J,4,1))*X**(ZI-1.0D0) )*V/V12        PRO06550
      G2=( ( V12/V*(AC(J,1,2)+(Z-H)*BC(J,1,2))+(3.0D0+ZI)*AC(J,1,2)           PRO06560
     #      +(Z-H)*BC(J,1,2)-V34*D(J,1,1) )*X**(1.0D0+ZI)                    PRO06570
     #    +( V12/V*(AC(J,2,2)+(Z-H)*BC(J,2,2))+(3.0D0-ZI)*AC(J,2,2)          PRO06580
     #      +(Z-H)*BC(J,2,2)+(1.0D0-ZI)*D(J,2,1) )*X**(1.0D0-ZI)             PRO06590
     #    +( V12/V*(AC(J,3,2)+(Z-H)*BC(J,3,2))+(1.0D0-ZI)*AC(J,3,2)          PRO06600
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     #      +(Z-H)*BC(J,3,2)-(V44-ZI)*D(J,3,1))*X**(-ZI-1.0D0) )*V/V12  PRO06610
      AQ(KK,J+1)=F0*EXP((Q-(1-(-1)**J)*Q0/2)*Z)                         PRO06620
     #  +EPSI*(F1-G1)*EXP((Q-(1-(-1)**J)*Q0/2)*(Z+H))                  PRO06630
     #  +EPSI*(F2-G2)*EXP((Q-(1-(-1)**J)*Q0/2)*(Z-H))                  PRO06640
C     THE SHEAR STRESSES                                               PRO06650
      F0= Z*(1.0D0+ZI)*D(J,1,1)*X**(ZI)                               PRO06660
     #   +Z*(1.0D0-ZI)*D(J,2,1)*X**(-ZI)                              PRO06670
     #   -(V54*Z+(ZZ-V44)*H)*D(J,3,1)*X**(-2.0D0-ZI)                  PRO06680
     #   -(V54*Z-(ZZ-V44)*H)*D(J,4,1)*X**(-2.0D0+ZI)                  PRO06690
      IF (J.GT.2) GOTO 808                                            PRO06700
      F0= Z*(1.0D0-ZI)*D(J,1,1)*X**(ZI)                               PRO06710
     #   +Z*(1.0D0-ZI)*D(J,2,1)*X**(-ZI)                              PRO06720
     #   -(V54*Z+(ZZ-V44)*H)*D(J,3,1)*(3.0D0-2.0D0*X)*X**(-ZI)        PRO06730
     #   -(V54*Z-(ZZ-V44)*H)*D(J,4,1)*(3.0D0-2.0D0*X)*X**(ZI)         PRO06740
  808 F1= Z*(-1.0D0+ZI)*D(J,1,2)*X**(ZI-1.0D0)                        PRO06750
     #   +(3.0D0*Z+(2.0D0-ZZ)*H)*D(J,2,2)*X**(1.0D0-ZI)               PRO06760
     #   -(V34*Z+Z*ZI)*D(J,3,2)*X**(-1.0D0-ZI)                        PRO06770
     #   -(V74*Z-(ZZ-2.0D0*V54)*H)*D(J,4,2)*X**(-3.0D0+ZI)            PRO06780
      G1= ((Z+H)*AC(J,1,1)+(1.0D0+ZI)*BC(J,1,1))*X**(1.0D0+ZI)        PRO06790
     #   +((Z+H)*AC(J,2,1)+(1.0D0-ZI)*BC(J,2,1))*X**(1.0D0-ZI)        PRO06800
     #   +((Z+H)*AC(J,3,1)+(ZI-1.0D0)*BC(J,3,1))*X**(ZI-1.0D0)        PRO06810
      F2=(3.0D0*Z+(ZZ-2.0D0)*H)*D(J,1,3)*X**(1.0D0+ZI)                PRO06820
     #   -Z*(1.0D0+ZI)*D(J,2,3)*X**(-1.0D0-ZI)                        PRO06830
     #   -(V74*Z+(ZZ-2.0D0*V54)*H)*D(J,3,3)*X**(-3.0D0-ZI)            PRO06840
     #   -(V34*Z-Z*ZI)*D(J,4,3)*X**(ZI-1.0D0)                         PRO06850
      G2= ((Z-H)*AC(J,1,2)+(1.0D0+ZI)*BC(J,1,2))*X**(1.0D0+ZI)        PRO06860
     #   +((Z-H)*AC(J,2,2)+(1.0D0-ZI)*BC(J,2,2))*X**(1.0D0-ZI)        PRO06870
     #   +((Z-H)*AC(J,3,2)-(1.0D0+ZI)*BC(J,3,2))*X**(-1.0D0-ZI)       PRO06880
      AR(KK,J+1)=-F0*EXP((Q-(1-(-1)**J)*Q0/2)*Z)                      PRO06890
     #   -EPSI*(F1+G1)*EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))            PRO06900
     #   -EPSI*(F2+G2)*EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))            PRO06910
  801 CONTINUE                                                        PRO06920
      N1=N+1                                                          PRO06930
      NH=N/2                                                          PRO06940
C     DO 703 J=1,N1                                                   PRO06950
C     WRITE(6,29)                                                     PRO06960
C     WRITE(8,29)                                                     PRO06970
C     DO 704 NK=1,N                                                   PRO06980
C     K=NK                                                            PRO06990
C     WRITE(6,19) K,J,AQ(K,J),AR(K,J)                                 PRO07000
C     WRITE(8,19) K,J,AQ(K,J),AR(K,J)                                 PRO07010
C 704 CONTINUE                                                        PRO07020
C  19 FORMAT(1X,I5,2X,I3,4F15.5)                                      PRO07030
C  29 FORMAT(//1X,'POINT',2X,'-N-',5X,'SQ-R',11X,'SQ-I',11X,'SRQ-R',9X, PRO07040
C    +       'SRQ-I')                                                 PRO07050
C 703 CONTINUE                                                        PRO07060
C     REAL FORMATION                                                  PRO07070
      DO 805 I=1,M                                                    PRO07080
      DO 804 J=1,N1                                                   PRO07090
      JA=(J-1)*2                                                      PRO07100
      W=AQ(I,J)                                                       PRO07110
      P=DREAL(W)                                                      PRO07120
      Q=DIMAG(W)                                                      PRO07130
      A1Q(I,JA+1)=P+Q                                                 PRO07140
      A1Q(I,JA+2)=P-Q                                                 PRO07150
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      W=AR(I,J)                                                          PRO07160
      P=DREAL(W)                                                         PRO07170
      Q=DIMAG(W)                                                         PRO07180
      A1Q(M+I,JA+1)=P+Q                                                  PRO07190
  804 A1Q(M+I,JA+2)=P-Q                                                  PRO07200
      W=AQ(I,1)                                                          PRO07210
      P=DREAL(W)                                                         PRO07220
      Q=DIMAG(W)                                                         PRO07230
      A1Q(I,1)=P                                                         PRO07240
      A1Q(I,2)=Q                                                         PRO07250
      W=AR(I,1)                                                          PRO07260
      P=DREAL(W)                                                         PRO07270
      Q=DIMAG(W)                                                         PRO07280
      A1Q(M+I,1)=P                                                       PRO07290
      A1Q(M+I,2)=Q                                                       PRO07300
      B1(I)=BQ(I)                                                        PRO07310
      B1(M+I)=BR(I)                                                      PRO07320
  805 CONTINUE                                                           PRO07330
      N2=2*N1                                                            PRO07340
      M2=2*M                                                             PRO07350
      IA=320                                                             PRO07360
      TOL=0.0                                                            PRO07370
      KBASIS=N2                                                          PRO07380
      CALL LLSQF (A1Q,IA,M2,N2,B1,TOL,KBASIS,XL,HL,IPL,IER)              PRO07390
      DO 2 I=1,N1                                                        PRO07400
      IA1=(I-1)*2                                                        PRO07410
      IA=I-1                                                             PRO07420
      U1=XL(IA1+1)                                                       PRO07430
      U2=XL(IA1+2)                                                       PRO07440
      A(IA,N1)=DCMPLX(U1,U2)                                             PRO07450
      WRITE(8,20) A(IA,N1),IA                                            PRO07460
      WRITE(6,20) A(IA,N1),IA                                            PRO07470
   20 FORMAT(/9X,/'SOLUTION',/2D19.6,I5)                                 PRO07480
    2 CONTINUE                                                           PRO07490
      RETURN                                                             PRO07500
      END                                                                PRO07510
C*********************************************************************** PRO07520
      SUBROUTINE STRESS(N,B,LAM)                                         PRO07530
      COMMON/CHARAC/ECYLD(40),EVCYLD(40,4),ESPHE(40),EVSPHE(40,4)        PRO07540
      COMMON/ZHU/D(40,4,3),AC(40,3,2),BC(40,3,2),A(50,51)                PRO07550
      COMPLEX*16 D,AC,BC,A,F1,F2,G1,G2,H,ZERO,F0,W,W1,B(40),ECYLD        PRO07560
      COMPLEX*16 SR(19,11),SQ(19,11),ST(19,11),SRQ(19,11),Z,ZZ,ZI,WI0    PRO07570
      COMPLEX*16 DCMPLX,F(3),WJ0,WJQ,Z1,ZZ1,ZI1,WIQ,DISU,DISV            PRO07580
      REAL*8 EPS,U1,U2,U3,U4,DREAL,DIMAG,V,V12,V34,V44,V54,V74,CLAM2     PRO07590
      REAL*8 EE,SS,TK,TM,TN,T4,LAM,LAM2,LAM4,CLAM,CLAM1,COND3,LAM6,LAM8 PRO07600
      REAL*8 X,Q,Q0,PRESS,EPSI,C,RRI,RRO,RAB1,RAB2,CONA3,CONB3,CONC3     PRO07610
      M=N+1                                                              PRO07620
      MM=2*N                                                             PRO07630
      MM2=2*MM+1                                                         PRO07640
      Q0=ATAN(1.0D0)                                                     PRO07650
      EE=3.0D7                                                           PRO07660
      V=0.3D0                                                            PRO07670
      V12=1.0D0-2.0D0*V                                                  PRO07680
      V34=3.0D0-4.0D0*V                                                  PRO07690
      V44=4.0D0-4.0D0*V                                                  PRO07700
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      V54=5.0D0-4.0D0*V                                                 PRO07710
      V74=7.0D0-4.0D0*V                                                 PRO07720
      LAM2=LAM*LAM                                                      PRO07730
      LAM4=LAM2*LAM2                                                    PRO07740
      LAM6=LAM2*LAM4                                                    PRO07750
      LAM8=LAM4*LAM4                                                    PRO07760
      RRI=0.20625D0                                                     PRO07770
      RRO=0.41250D0                                                     PRO07780
      EPSI=RRO/(1.1D0-RRO)                                              PRO07790
      H=(0.0D0,1.0D0)                                                   PRO07800
      ZERO=(0.0D0,0.0D0)                                                PRO07810
      CLAM=(1+V)/4.0D0/(1.0D0-LAM2)/(1.0D0-LAM2)                        PRO07820
      CLAM1=CLAM/(1.0D0-LAM2)                                           PRO07830
      CLAM2=(1+V)/8.0D0/(1.0D0-LAM2)/(1.0D0-LAM2)                       PRO07840
C     CALCULATE THE STRESSES                                            PRO07850
      W=A(0,M)                                                          PRO07860
      U1=DREAL(W)                                                       PRO07870
      U2=DIMAG(W)                                                       PRO07880
      DO 130 IP=1,11                                                    PRO07890
      Q=Q0*(IP-1)/10                                                    PRO07900
      QP=(IP-1)*45.0D0/10                                               PRO07910
      DO 130 JP=1,11                                                    PRO07920
      X=LAM+(1.0D0-LAM)*(JP-1)/(10)                                     PRO07930
      SR(IP,JP)=CLAM*((1.0D0+LAM2)*(1.0D0+LAM2+LAM4)+3.0D0*LAM4*(       PRO07940
     #     1.0D0+LAM2)/X/X/X/X-4.0D0*LAM2*(1.0D0+LAM2+LAM4)/X/X-(       PRO07950
     #     1.0D0-LAM2)**2*X*X)*(DSIN(2*Q)*U1                            PRO07960
     #     +H*DCOS(2*(Q))*U2)*EPSI                                      PRO07970
      SQ(IP,JP)=-(1.0D0+LAM2)*(1.0D0+LAM2+LAM4-3.0D0*(1.0D0+LAM2)       PRO07980
     #      *X*X+3.0D0*LAM4/X/X/X/X)*(DSIN(2*Q)*U1                      PRO07990
     #   +     H*DCOS(2*(Q))*U2)*CLAM*EPSI                              PRO08000
      ST(IP,JP)=CLAM2*((2.D0*((1.D0-LAM2)**2.D0+V*(1.D0+LAM2+LAM4)      PRO08010
     #      )*X*X-4.0D0*V*LAM2*(1.0D0+LAM2+LAM4)/X/X)*(DSIN(2*Q)*U1     PRO08020
     #      +H*DCOS(2*(Q))*U2)*EPSI**2.0D0                              PRO08030
     #     - (1.D0+V)*X*(DSIN(Q)*U1+H*DCOS(Q)*U2)*EPSI )                PRO08040
      SRQ(IP,JP)= ((1.D0+LAM2)*(1.0D0+LAM2+LAM4)-(1.0D0+4.0D0*LAM2      PRO08050
     #      +LAM4)*X*X-3.0D0*LAM4*(1.0D0+LAM2)/X/X/X/X+2.0D0*LAM2*(     PRO08060
     #    1.0D0+LAM2+LAM4)/X/X)*(DCOS(2*Q)*U1-H*DSIN(2*(Q))*U2)         PRO08070
     #      *CLAM*(-EPSI)                                               PRO08080
  88  CONTINUE                                                          PRO08090
 130  CONTINUE                                                          PRO08100
                                                                        PRO08110
      DO 303 IP=1,11                                                    PRO08120
      Q=Q0*(IP-1)/10                                                    PRO08130
      QP=(IP-1)*45.0D0/10                                               PRO08140
      DO 300 JP=1,11                                                    PRO08150
      X=LAM+(1.0D0-LAM)*(JP-1)/(10)                                     PRO08160
      DO 299 J=1,N                                                      PRO08170
      Z=B(J)                                                            PRO08180
      ZZ=Z*Z                                                            PRO08190
      ZI=Z*H                                                            PRO08200
      F0=-(ZZ+2.0D0+ZI)*D(J,1,1)*X**ZI                                  PRO08210
     #   -(ZZ+2.0D0-ZI)*D(J,2,1)*X**(-ZI)                               PRO08220
     #   -(ZZ-V44-V54*ZI)*D(J,3,1)*X**(-2.0D0-ZI)                       PRO08230
     #   -(ZZ-V44+V54*ZI)*D(J,4,1)*X**(-2.0D0+ZI)                       PRO08240
      IF (J.GT.2) GOTO 3001                                             PRO08250
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      F0=-(ZZ+2.0D0+ZI)*D(J,1,1)*X**ZI                                          PRO08260
     #    -(ZZ+2.0D0-ZI)*D(J,2,1)*X**(-ZI)                                     PRO08270
     #    -(ZZ-V44-V54*ZI)*D(J,3,1)*(3.0D0-2.0D0*X)*X**(-ZI)                   PRO08280
     #    -(ZZ-V44+V54*ZI)*D(J,4,1)*(3.0D0-2.0D0*X)*X**(ZI)                    PRO08290
 3001 G1=-V/V12*(((V12/V*(2.0D0-Z*H)-3.0D0+Z*H)*AC(J,1,1)+(Z+H)               PRO08300
     #           *BC(J,1,1)+(1.0D0-Z*H)*D(J,1,1) )*X**(1.0D0+Z*H)             PRO08310
     #       +((V12/V*(2.0D0-Z*H)-3.0D0-Z*H)*AC(J,2,1)+(Z+H)*BC(J,2,1)        PRO08320
     #                 -V34*D(J,2,1) )*X**(1.0D0-Z*H)                         PRO08330
     #       +((V12/V*Z*H-1.0D0-Z*H)*AC(J,3,1)+(Z+H)*BC(J,3,1)                PRO08340
     #                 +(-V44-Z*H)*D(J,4,1) )*X**(-1.0D0+Z*H)   )             PRO08350
      G2=-V/V12*(((V12/V*(2.0D0+Z*H)+3.0D0+Z*H)*AC(J,1,2)+(Z-H)               PRO08360
     #           *BC(J,1,2)-V34*D(J,1,1) )*X**(1.0D0+Z*H)                     PRO08370
     #       +((V12/V*(2.0D0-Z*H)-3.0D0-Z*H)*AC(J,2,2)+(Z-H)*BC(J,2,2)        PRO08380
     #          +(1.0D0-Z*H)*D(J,2,1) )*X**(1.0D0-Z*H)                        PRO08390
     #       +((-V12/V+1.0D0-Z*H)*AC(J,3,2)+(Z-H)*BC(J,3,2)                   PRO08400
     #          -(V44+Z*H)*D(J,3,1) )*X**(-1.0D0-Z*H)   )                     PRO08410
      Z1=B(J)+H                                                               PRO08420
      ZZ1=Z1*Z1                                                               PRO08430
      ZI1=Z1*H                                                                PRO08440
      F1=-(ZZ1+2.0D0+ZI1)*D(J,1,2)*X**ZI1                                     PRO08450
     #    -(ZZ1+2.0D0-ZI1)*D(J,2,2)*X**(-ZI1)                                  PRO08460
     #    -(ZZ1-V44-V54*ZI1)*D(J,3,2)*X**(-2.0D0-ZI1)                          PRO08470
     #    -(ZZ1-V44+V54*ZI1)*D(J,4,2)*X**(-2.0D0+ZI1)                          PRO08480
      Z1=B(J)-H                                                               PRO08490
      ZZ1=Z1*Z1                                                               PRO08500
      ZI1=Z1*H                                                                PRO08510
      F2=-(ZZ1+2.0D0+ZI1)*D(J,1,3)*X**ZI1                                     PRO08520
     #    -(ZZ1+2.0D0-ZI1)*D(J,2,3)*X**(-ZI1)                                  PRO08530
     #    -(ZZ1-V44-V54*ZI1)*D(J,3,3)*X**(-2.0D0-ZI1)                          PRO08540
     #    -(ZZ1-V44+V54*ZI1)*D(J,4,3)*X**(-2.0D0+ZI1)                          PRO08550
      SR(IP,JP)=SR(IP,JP)+( F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)               PRO08560
     #    +EPSI*(F1-G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))                   PRO08570
     #    +EPSI*(F2-G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H)) )*A(J,M)          PRO08580
      F0= (ZZ-2.0D0-3.0D0*ZI)*D(J,1,1)*X**ZI                                  PRO08590
     #    +(ZZ-2.0D0+3.0D0*ZI)*D(J,2,1)*X**(-ZI)                               PRO08600
     #    +(ZZ-V44-V54*ZI)*D(J,3,1)*X**(-2.0D0-ZI)                             PRO08610
     #    +(ZZ-V44+V54*ZI)*D(J,4,1)*X**(-2.0D0+ZI)                             PRO08620
      IF (J.GT.2) GOTO 3002                                                   PRO08630
      F0= (ZZ-2.0D0-3.0D0*ZI)*D(J,1,1)*X**ZI                                  PRO08640
     #    +(ZZ-2.0D0+3.0D0*ZI)*D(J,2,1)*X**(-ZI)                               PRO08650
     #    +(ZZ-V44-V54*ZI)*D(J,3,1)*(3.0D0-2.0D0*X)*X**(-ZI)                   PRO08660
     #    +(ZZ-V44+V54*ZI)*D(J,4,1)*(3.0D0-2.0D0*X)*X**(ZI)                    PRO08670
 3002 F1= (ZZ-ZI)*D(J,1,2)*X**(ZI-1.0D0)                                      PRO08680
     #    +(ZZ-6.0D0+5.0D0*ZI)*D(J,2,2)*X**(1.0D0-ZI)                          PRO08690
     #    +(ZZ-V34*ZI)*D(J,3,2)*X**(-ZI-1.0D0)                                PRO08700
     #    +(ZZ-2.0D0*V54+V74*ZI)*D(J,4,2)*X**(-3.0D0+ZI)                       PRO08710
      F2= (ZZ-6.0D0-5.0D0*ZI)*D(J,1,3)*X**(1.0D0+ZI)                          PRO08720
     #    +(ZZ+3.0D0*ZI)*D(J,2,3)*X**(-ZI-1.0D0)                               PRO08730
     #    +(ZZ-2.0D0*V54-V54*ZI)*D(J,3,3)*X**(-3.0D0-ZI)                       PRO08740
     #    +(ZZ+V34*ZI)*D(J,4,3)*X**(ZI-1.0D0)                                  PRO08750
      G1=( ( V12/V*(AC(J,1,1)+(Z+H)*BC(J,1,1))+(3.0D0+ZI)*AC(J,1,1)           PRO08760
     #      +(Z+H)*BC(J,1,1)+(1.0D0+ZI)*D(J,1,1) )*X**(1.0D0+ZI)             PRO08770
     #    +( V12/V*(AC(J,2,1)+(Z+H)*BC(J,2,1))+(3.0D0-ZI)*AC(J,2,1)           PRO08780
     #      +(Z+H)*BC(J,2,1)-V34*D(J,2,1) )*X**(1.0D0-ZI)                     PRO08790
     #    +( V12/V*(AC(J,3,1)+(Z+H)*BC(J,3,1))+(1.0D0+ZI)*AC(J,3,1)           PRO08800
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     #      +(Z+H)*BC(J,3,1)-(V44-ZI)*D(J,4,1))*X**(ZI-1.0D0) )*V/V12   PRO08810
      G2=( ( V12/V*(AC(J,1,2)+(Z-H)*BC(J,1,2))+(3.0D0+ZI)*AC(J,1,2)     PRO08820
     #      +(Z-H)*BC(J,1,2)-V34*D(J,1,1) )*X**(1.0D0+ZI)               PRO08830
     #    +( V12/V*(AC(J,2,2)+(Z-H)*BC(J,2,2))+(3.0D0-ZI)*AC(J,2,2)     PRO08840
     #      +(Z-H)*BC(J,2,2)+(1.0D0-ZI)*D(J,2,1) )*X**(1.0D0-ZI)        PRO08850
     #    +( V12/V*(AC(J,3,2)+(Z-H)*BC(J,3,2))+(1.0D0-ZI)*AC(J,3,2)     PRO08860
     #      +(Z-H)*BC(J,3,2)-(W44+ZI)*D(J,3,1))*X**(-ZI-1.0D0) )*V/V12  PRO08870
      SQ(IP,JP)=SQ(IP,JP)+( F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)         PRO08880
     #           +EPSI*(F1+G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))     PRO08890
     #           +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))     PRO08900
     #            )*A(J,M)                                              PRO08910
      F0=-4*V*((1+ZI)*D(J,1,1)*X**ZI+(1-ZI)*D(J,2,1)*X**(-ZI))          PRO08920
      F1=-4*V*ZI*D(J,1,2)*X**(ZI-1)-4*V*(2-ZI)*D(J,2,2)*X**(1-ZI)       PRO08930
      F2=-4*V*(2+ZI)*D(J,1,3)*X**(1+ZI)+4*V*ZI*D(J,2,3)*X**(-1-ZI)      PRO08940
      G1=( ((3+ZI)*V*AC(J,1,1)+(Z+H)*V*BC(J,1,1)+(1-V)*(1+ZI)*D(J,1,1)) PRO08950
     #   *X**(1+ZI) + ((3-ZI)*V*AC(J,2,1)+(Z+H)*V*BC(J,2,1)-(1-V)*V34   PRO08960
     #   *D(J,2,1))*X**(1-ZI) + ((1+ZI)*V*AC(J,3,1)+(Z+H)*V*BC(J,3,1)+  PRO08970
     #    (1-V)*(ZI-V44)*D(J,4,1))*X**(ZI-1) )/(1-2*V)                  PRO08980
      G2=( ((3+ZI)*V*AC(J,1,2)+(Z-H)*V*BC(J,1,2)-(1-V)*V34*D(J,1,1))    PRO08990
     #   *X**(1+ZI) + ((3-ZI)*V*AC(J,2,2)+(Z-H)*V*BC(J,2,2)+(1-V)*(1-   PRO09000
     #   ZI)*D(J,2,1))*X**(1-ZI) + ((1-ZI)*V*AC(J,3,2)+(Z-H)*V*BC(J,3,  PRO09010
     #   2)-(1-V)*(V44+ZI)*D(J,3,1))*X**(-1-ZI) )/(1-2*V)               PRO09020
      ST(IP,JP)=ST(IP,JP)-( F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)         PRO09030
     #           +EPSI*(F1+G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))     PRO09040
     #           +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))     PRO09050
     #            )*A(J,M)                                              PRO09060
      F0= Z*(1.0D0+ZI)*D(J,1,1)*X**(ZI)                                 PRO09070
     #   +Z*(1.0D0-ZI)*D(J,2,1)*X**(-ZI)                                PRO09080
     #   -(V54*Z+(ZZ-V44)*H)*D(J,3,1)*X**(-2.0D0-ZI)                    PRO09090
     #   -(V54*Z-(ZZ-V44)*H)*D(J,4,1)*X**(-2.0D0+ZI)                    PRO09100
      IF (J.GT.2) GOTO 3003                                             PRO09110
      F0= Z*(1.0D0+ZI)*D(J,1,1)*X**(ZI)                                 PRO09120
     #   +Z*(1.0D0-ZI)*D(J,2,1)*X**(-ZI)                                PRO09130
     #   -(V54*Z+(ZZ-V44)*H)*D(J,3,1)*(3.0D0-2.0D0*X)*X**(-ZI)          PRO09140
     #   -(V54*Z-(ZZ-V44)*H)*D(J,4,1)*(3.0D0-2.0D0*X)*X**(ZI)           PRO09150
3003  F1= Z*(-1.0D0+ZI)*D(J,1,2)*X**(ZI-1.0D0)                          PRO09160
     #   +(3.0D0*Z+(2.0D0-ZZ)*H)*D(J,2,2)*X**(1.0D0-ZI)                 PRO09170
     #   -(V34*Z+Z*ZI)*D(J,3,2)*X**(-1.0D0-ZI)                          PRO09180
     #   -(V74*Z-(ZZ-2.0D0*V54)*H)*D(J,4,2)*X**(-3.0D0+ZI)              PRO09190
      G1= ((Z+H)*AC(J,1,1)+(1.0D0+ZI)*BC(J,1,1))*X**(1.0D0+ZI)          PRO09200
     #   +((Z+H)*AC(J,2,1)+(1.0D0-ZI)*BC(J,2,1))*X**(1.0D0-ZI)          PRO09210
     #   +((Z+H)*AC(J,3,1)+(ZI-1.0D0)*BC(J,3,1))*X**(ZI-1.0D0)          PRO09220
      F2=(3.0D0*Z+(ZZ-2.0D0)*H)*D(J,1,3)*X**(1.0D0+ZI)                  PRO09230
     #   -Z*(1.0D0+ZI)*D(J,2,3)*X**(-1.0D0-ZI)                          PRO09240
     #   -(V74*Z+(ZZ-2.0D0*V54)*H)*D(J,3,3)*X**(-3.0D0-ZI)              PRO09250
     #   -(V34*Z-Z*ZI)*D(J,4,3)*X**(ZI-1.0D0)                           PRO09260
      G2= ((Z-H)*AC(J,1,2)+(1.0D0+ZI)*BC(J,1,2))*X**(1.0D0+ZI)          PRO09270
     #   +((Z-H)*AC(J,2,2)+(1.0D0-ZI)*BC(J,2,2))*X**(1.0D0-ZI)          PRO09280
     #   +((Z-H)*AC(J,3,2)-(1.0D0+ZI)*BC(J,3,2))*X**(-1.0D0-ZI)         PRO09290
      SRQ(IP,JP)=SRQ(IP,JP)-( F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)       PRO09300
     #           +EPSI*(F1+G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))     PRO09310
     #           +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))     PRO09320
     #            )*A(J,M)                                              PRO09330
  299 CONTINUE                                                          PRO09340
      U1=DREAL(SR(IP,JP))                                               PRO09350
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      U2=DIMAG(SR(IP,JP))                                                PRO09360
      T1=(U1+U2)                                                         PRO09370
      U1=DREAL(SQ(IP,JP))                                                PRO09380
      U2=DIMAG(SQ(IP,JP))                                                PRO09390
      T2=(U1+U2)                                                         PRO09400
      U1=DREAL(ST(IP,JP))                                                PRO09410
      U2=DIMAG(ST(IP,JP))                                                PRO09420
      T3=(U1+U2)                                                         PRO09430
      U1=DREAL(SRQ(IP,JP))                                               PRO09440
      U2=DIMAG(SRQ(IP,JP))                                               PRO09450
      T4=(U1+U2)                                                         PRO09460
      WRITE(8,298) X,QP,T1,T2,T3,T4                                      PRO09470
      WRITE(6,298) X,QP,T1,T2,T3,T4                                      PRO09480
  300 CONTINUE                                                           PRO09490
      WRITE(8,298)                                                       PRO09500
      WRITE(6,298)                                                       PRO09510
  303 CONTINUE                                                           PRO09520
                                                                         PRO09530
      DO 402 JPK=1,3                                                     PRO09540
      X=LAM+(1.0D0-LAM)*(JPK-1)/(2.)                                     PRO09550
      JP=1+(JPK-1)*5                                                     PRO09560
      DO 401 IP=1,11                                                     PRO09570
      QP=(IP-1)*45.0D0/10                                                PRO09580
      U1=DREAL(SR(IP,JP))                                                PRO09590
      U2=DIMAG(SR(IP,JP))                                                PRO09600
      T1=(U1+U2)                                                         PRO09610
      U1=DREAL(SQ(IP,JP))                                                PRO09620
      U2=DIMAG(SQ(IP,JP))                                                PRO09630
      T2=(U1+U2)                                                         PRO09640
      U1=DREAL(ST(IP,JP))                                                PRO09650
      U2=DIMAG(ST(IP,JP))                                                PRO09660
      T3=(U1+U2)                                                         PRO09670
      U1=DREAL(SRQ(IP,JP))                                               PRO09680
      U2=DIMAG(SRQ(IP,JP))                                               PRO09690
      T4=(U1+U2)                                                         PRO09700
      WRITE(8,298) X,QP,T1,T2,T3,T4                                      PRO09710
      WRITE(6,298) X,QP,T1,T2,T3,T4                                      PRO09720
  401 CONTINUE                                                           PRO09730
      WRITE(8,298)                                                       PRO09740
      WRITE(6,298)                                                       PRO09750
  402 CONTINUE                                                           PRO09760
  298 FORMAT(1X,2F8.4,4F14.6)                                            PRO09770
      RETURN                                                             PRO09780
      END                                                                PRO09790
C*************************************************************************PRO09800
C* GAUSS METHOD FOR SOLVING LINEAR EQUATIONS                            ***PRO09810
C* A: N*M MATRIX OF COEFFETIONS.                                        ***PRO09820
C* N: THE NUMBER OF EQUATIONS.                                          ***PRO09830
C* M: INLARGED COLUMN; M=N+1.                                           ***PRO09840
C* EPS: ALLOWABLE DIVATION.                                             ***PRO09850
C*************************************************************************PRO09860
      SUBROUTINE GS(A,N,M,EPS)                                           PRO09870
      REAL*8 A(100,101)                                                  PRO09880
      DO 50 K=1,N                                                        PRO09890
      BMAX=0.                                                            PRO09900
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      DO 20 I=K,N                                                       PRO09910
      IF(BMAX-ABS(A(I,K))) 10,20,20                                     PRO09920
   10 BMAX=ABS(A(I,K))                                                  PRO09930
      L=I                                                               PRO09940
   20 CONTINUE                                                          PRO09950
      IF(BMAX.LT.EPS) GOTO 4444                                         PRO09960
      IF (L.EQ.K) GOTO 30                                               PRO09970
      DO 25 J=K,M                                                       PRO09980
      T=A(L,J)                                                          PRO09990
      A(L,J)=A(K,J)                                                     PRO10000
   25 A(K,J)=T                                                          PRO10010
   30 T=1.0/A(K,K)                                                      PRO10020
      K1=K+1                                                            PRO10030
      DO 40 J=K1,M                                                      PRO10040
      A(K,J)=A(K,J)*T                                                   PRO10050
      DO 40 I=K1,N                                                      PRO10060
   40 A(I,J)=A(I,J)-A(I,K)*A(K,J)                                       PRO10070
   50 CONTINUE                                                          PRO10080
      DO 60 IK=2,N                                                      PRO10090
      I=M-IK                                                            PRO10100
      I1=I+1                                                            PRO10110
      DO 60 J=I1,N                                                      PRO10120
   60 A(I,M)=A(I,M)-A(I,J)*A(J,M)                                       PRO10130
 4444 CONTINUE                                                          PRO10140
      RETURN                                                            PRO10150
      END                                                               PRO10160
C*******************************************************************************PRO10170
      SUBROUTINE DISPLA(N,B,LAM)                                        PRO10180
      COMMON/CHARAC/ECYLD(40),EVCYLD(40,4),ESPHE(40),EVSPHE(40,4)       PRO10190
      COMMON/ZHU/D(40,4,3),AC(40,3,2),BC(40,3,2),A(50,51)               PRO10200
      COMPLEX*16 D,AC,BC,A,F1,F2,G1,G2,H,ZERO,F0,W,W1,B(40),ECYLD       PRO10210
      COMPLEX*16 SR(19,11),SQ(19,11),ST(19,11),SRQ(19,11),Z,ZZ,ZI,WI0   PRO10220
      COMPLEX*16 DCMPLX,F(3),WJ0,WJQ,Z1,ZZ1,ZI1,WIQ                     PRO10230
      REAL*8 EPS,U1,U2,U3,U4,DREAL,DIMAG,V,V12,V14,V34,V44,V54,V74       PRO10240
      REAL*8 EE,SS,LAM,LAM2,LAM4,LAM6,LAM8,CLAM,CONSA1                  PRO10250
      REAL*8 X,Q,Q0,EPSI,C,RRI,RRO,CLAM1,CLAM2                          PRO10260
      REAL*8 AST13,AST23,BST13,BST23,A13,B13,C13,D13,A23,B23,C23,D23     PRO10270
      M=N+1                                                             PRO10280
      MM=2*N                                                            PRO10290
      MM2=2*MM+1                                                        PRO10300
      Q0=ATAN(1.0D0)                                                    PRO10310
      EE=3.0D7                                                          PRO10320
      V=0.3D0                                                           PRO10330
      V12=1.0D0-2.0D0*V                                                 PRO10340
      V14=1.0D0-4.0D0*V                                                 PRO10350
      V34=3.0D0-4.0D0*V                                                 PRO10360
      V44=4.0D0-4.0D0*V                                                 PRO10370
      V54=5.0D0-4.0D0*V                                                 PRO10380
      V64=6.0D0-4.0D0*V                                                 PRO10390
      V74=7.0D0-4.0D0*V                                                 PRO10400
      LAM2=LAM*LAM                                                      PRO10410
      LAM4=LAM2*LAM2                                                    PRO10420
      LAM6=LAM4*LAM2                                                    PRO10430
      LAM8=LAM4*LAM4                                                    PRO10440
      RRI=0.2D0                                                         PRO10450
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      RRO=0.4D0                                                                 PRO10460
      C=RRO*(1.0D0+V)/EE                                                        PRO10470
      EPSI=RRO/(1.1D0-RRO)                                                      PRO10480
      H=(0.0D0,1.0D0)                                                           PRO10490
      ZERO=(0.0D0,0.0D0)                                                        PRO10500
C     DETERMINING CONSTANT A_O                                                  PRO10510
      X=(1.0D0+LAM)/2.0D0                                                       PRO10520
      Q=0.0D0                                                                   PRO10530
      W=A(0,M)                                                                  PRO10540
      U1=DREAL(W)                                                               PRO10550
      U2=DIMAG(W)                                                               PRO10560
      CLAM1=(1.0D0+V)/4.0D0/(1.0D0-LAM2)/(1.0D0-LAM2)                           PRO10570
      CLAM2=(1.0D0-V*(1.0D0+V)**2.0D0)/2.0D0/V12/(1.0D0-LAM2)**3.0D0            PRO10580
      AST13=-(1.0D0+V)*U1/12.0D0                                                PRO10590
      BST13= (1.0D0+V)*U1/12.0D0                                                PRO10600
      A13=CLAM1*(1.0D0+LAM2)*(1.0D0+LAM2+LAM4)*U1                               PRO10610
      B13=CLAM1*(1.0D0+LAM2+LAM4+V12*(1.0D0-LAM2)**2.0D0/4.0D0/V)               PRO10620
     # *U1+AST13/4.0D0/V                                                        PRO10630
      C13=-CLAM1*(LAM4+LAM6)*U1                                                 PRO10640
      D13=-CLAM1*(1.0D0+LAM2+LAM4)*LAM2*U1                                      PRO10650
      AST23=-U2/6.0D0                                                           PRO10660
      BST23=-U2/6.0D0                                                           PRO10670
      A23=CLAM2*(1.0D0+LAM2)*(1.0D0-LAM6)*U2                                    PRO10680
      B23=CLAM2*(2.0D0*(1.0D0-LAM6)-(1.0D0-LAM2)**3.0D0)*U2                     PRO10690
      C23=-CLAM2*(LAM4-LAM8)*U2                                                 PRO10700
      D23=-CLAM2*(1.0D0-LAM6)*LAM2*U2                                           PRO10710
      SQ(0,0)= C*( U1/X/EPSI-0.5D0*V*X*X*U1*DCOS(Q)                             PRO10720
     #     + EPSI*(A13*X-V64*B13*X*X*X-C13/X/X/X-D13*(2.0D0                     PRO10730
     #      -4.0D0*V)/X+BST13*X*X*X )*DCOS(2.0D0*Q)                             PRO10740
     #     - U2/EPSI/EPSI*DSIN(Q)+0.5D0*V*(1.0D0+V)*X*X*U2*DSIN(Q)              PRO10750
     #     - EPSI*(A23*X-V64*B23*X*X*X-C23/X/X/X-D23*(2.0D0                     PRO10760
     #      -4.0D0*V)/X+BST23*X*X*X )*DSIN(2.0D0*Q)  )                          PRO10770
                                                                                PRO10780
      DO 300 J=1,N                                                              PRO10790
      Z=B(J)                                                                    PRO10800
      ZZ=Z*Z                                                                    PRO10810
      ZI=Z*H                                                                    PRO10820
      F0=-(-V44*H+Z)*D(J,1,1)*X**(1.0D0+ZI)                                     PRO10830
     #    +( V44*H+Z)*D(J,2,1)*X**(1.0D0-ZI)                                   PRO10840
     #    +(-V44*H+Z)*D(J,3,1)*X**(-1.0D0-ZI)                                  PRO10850
     #    +( V44*H+Z)*D(J,4,1)*X**(-1.0D0+ZI)                                  PRO10860
      IF (J.GT.2) GOTO 565                                                      PRO10870
      F0=-(-V44*H+Z)*D(J,1,1)*X**(1.0D0+ZI)                                     PRO10880
     #    +( V44*H+Z)*D(J,2,1)*X**(1.0D0-ZI)                                   PRO10890
     #    +(-V44*H+Z)*D(J,3,1)*X**(-ZI)*(3.0D0-3.0D0*X+X*X)                    PRO10900
     #    +( V44*H+Z)*D(J,4,1)*X**( ZI)*(3.0D0-3.0D0*X+X*X)                    PRO10910
  565 F1= (-V34*H+Z)*D(J,1,2)*X**ZI                                             PRO10920
     #    -( V14*H+Z)*D(J,2,2)*X**(2.0D0-ZI)                                   PRO10930
     #    -( V34*H+Z)*D(J,3,2)*X**(-ZI)                                        PRO10940
     #    +( V54*H-Z)*D(J,4,2)*X**(-2.0D0+ZI)                                  PRO10950
      F2= (-V54*H+Z)*D(J,1,3)*X**(2.0D0+ZI)                                     PRO10960
     #    +( V34*H+Z)*D(J,2,3)*X**(-ZI)                                        PRO10970
     #    -( V54*H+Z)*D(J,3,3)*X**(-2.0D0-ZI)                                  PRO10980
     #    +( V34*H-Z)*D(J,4,3)*X**( ZI)                                        PRO10990
      G1=BC(J,1,1)*X**(2.0D0+ZI)+BC(J,2,1)*X**(2.0D0-ZI)                        PRO11000
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     #          -BC(J,3,1)*X**(ZI)                                                  PRO11010
     #          -BC(J,3,2)*X**(-ZI)                                                 PRO11020
      G2=BC(J,1,2)*X**(2.0D0+ZI)+BC(J,2,2)*X**(2.0D0-ZI)                          PRO11030
      SQ(0,0)=SQ(0,0)+C*( F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)                     PRO11040
     #   +EPSI*(F1-G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))                       PRO11050
     #   +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H)) )*A(J,M)              PRO11060
  300 CONTINUE                                                                    PRO11070
      W=SQ(0,0)                                                                   PRO11080
      U1=DREAL(W)                                                                 PRO11090
      U2=DIMAG(W)                                                                 PRO11100
      T1=U1+U2                                                                    PRO11110
      CONSA1=-T1*EPSI*EPSI/C                                                      PRO11120
      WRITE(8,299) CONSA1                                                         PRO11130
      WRITE(6,298) CONSA1                                                         PRO11140
  299 FORMAT(1X,'A1=',D14.5)                                                      PRO11150
C     CALCULATING THE DISPLACEMENT COMPONENTS                                     PRO11160
      DO 341 IP=1,19                                                              PRO11170
      QP=(IP-1)*45.0D0/18                                                         PRO11180
      Q=QP*3.1415926/180.0                                                        PRO11190
      DO 340 JP=1,11                                                              PRO11200
      X=LAM+(1.0D0-LAM)*(JP-1)/(10)                                               PRO11210
      W=A(0,M)                                                                    PRO11220
      U1=DREAL(W)                                                                 PRO11230
      U2=DIMAG(W)                                                                 PRO11240
      SR(IP,JP)=-C*(CONSA1/EPSI/EPSI*DSIN(Q)+0.5D0*V*X*X*U1*DSIN(Q)               PRO11250
     #     + EPSI*(A13*X-4.0D0*V*B13*X*X*X+C13/X/X/X-D13*(4.0D0                   PRO11260
     #      -4.0D0*V)/X+AST13*X*X*X )*DSIN(2.0D0*Q)                               PRO11270
     #     + U2/EPSI/EPSI*DCOS(Q)-V*U2/EPSI*X                                     PRO11280
     #     + 0.5D0*V*(1.0D0+V)*X*X*U2*DCOS(Q)                                     PRO11290
     #     + EPSI*(A23*X-4.0D0*V*B23*X*X*X+C23/X/X/X-D23*(4.0D0                   PRO11300
     #      -4.0D0*V)/X+AST23*X*X*X )*DCOS(2.0D0*Q)   )                           PRO11310
      SQ(IP,JP)=C*(CONSA1/EPSI/EPSI*DCOS(Q)+U1/X/EPSI-0.5D0*V*X*X*U1              PRO11320
     #     *DCOS(Q)+EPSI*(A13*X-V64*B13*X*X*X-C13/X/X/X-D13*(2.0D0                PRO11330
     #      -4.0D0*V)/X+BST13*X*X*X )*DCOS(2.0D0*Q)                               PRO11340
     #     - U2/EPSI/EPSI*DSIN(Q)+0.5D0*V*(1.0D0+V)*X*X*U2*DSIN(Q)                PRO11350
     #     - EPSI*(A23*X-V64*B23*X*X*X-C23/X/X/X-D23*(2.0D0                       PRO11360
     #      -4.0D0*V)/X+BST23*X*X*X )*DSIN(2.0D0*Q)   )                           PRO11370
  340 CONTINUE                                                                    PRO11380
  341 CONTINUE                                                                    PRO11390
      DO 350 J=1,N                                                                PRO11400
      Z=B(J)                                                                      PRO11410
      ZZ=Z*Z                                                                      PRO11420
      ZI=Z*H                                                                      PRO11430
      DO 346 IP=1,19                                                              PRO11440
      QP=(IP-1)*45.0D0/18                                                         PRO11450
      Q=QP*3.1415926/180.0                                                        PRO11460
      DO 345 JP=1,11                                                              PRO11470
      X=LAM+(1.0D0-LAM)*(JP-1)/(10)                                               PRO11480
      F0= (-2.0D0*V12+ZI)*D(J,1,1)*X**(1.0D0+ZI)                                  PRO11490
     #   -( 2.0D0*V12+ZI)*D(J,2,1)*X**(1.0D0-ZI)                                  PRO11500
     #   -( V44+ZI)*D(J,3,1)*X**(-1.0D0-ZI)                                       PRO11510
     #   -( V44-ZI)*D(J,4,1)*X**(-1.0D0+ZI)                                       PRO11520
      IF (J.GT.2) GOTO 567                                                        PRO11530
      F0= (-2.0D0*V12+ZI)*D(J,1,1)*X**(1.0D0+ZI)                                  PRO11540
     #   -( 2.0D0*V12+ZI)*D(J,2,1)*X**(1.0D0-ZI)                                  PRO11550
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     #    -( V44+ZI)*D(J,3,1)*X**(-ZI)*(3.0D0-3.0D0*X+X*X)                  PRO11560
     #    -( V44-ZI)*D(J,4,1)*X**( ZI)*(3.0D0-3.0D0*X+X*X)                  PRO11570
   567 F1= (-V34+ZI)*D(J,1,2)*X**ZI                                         PRO11580
     #    -( V14+ZI)*D(J,2,2)*X**(2.0D0-ZI)                                PRO11590
     #    -( V34+ZI)*D(J,3,2)*X**(-ZI)                                     PRO11600
     #    -( V54-ZI)*D(J,4,2)*X**(-2.0D0+ZI)                               PRO11610
      F2= (-V14+ZI)*D(J,1,3)*X**(2.0D0+ZI)                                 PRO11620
     #    -( V34+ZI)*D(J,2,3)*X**(-ZI)                                     PRO11630
     #    -( V54+ZI)*D(J,3,3)*X**(-2.0D0-ZI)                               PRO11640
     #    -( V34-ZI)*D(J,4,3)*X**( ZI)                                     PRO11650
      G1=AC(J,1,1)*X**(2.0D0+ZI)+AC(J,2,1)*X**(2.0D0-ZI)                   PRO11660
     #          +AC(J,3,1)*X**(ZI)                                         PRO11670
      G2=AC(J,1,2)*X**(2.0D0+ZI)+AC(J,2,2)*X**(2.0D0-ZI)                   PRO11680
     #          +AC(J,3,2)*X**(-ZI)                                        PRO11690
      SR(IP,JP)=SR(IP,JP) + C*(                                           PRO11700
     #                  F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)               PRO11710
     #    +EPSI*(F1+G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))              PRO11720
     #    +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))              PRO11730
     #     ) * A(J,M)                                                     PRO11740
                                                                          PRO11750
      F0=-(-V44*H+Z)*D(J,1,1)*X**(1.0D0+ZI)                               PRO11760
     #    +( V44*H+Z)*D(J,2,1)*X**(1.0D0-ZI)                              PRO11770
     #    +(-V44*H+Z)*D(J,3,1)*X**(-1.0D0-ZI)                             PRO11780
     #    +( V44*H+Z)*D(J,4,1)*X**(-1.0D0+ZI)                             PRO11790
      IF (J.GT.2) GOTO 568                                                PRO11800
      F0=-(-V44*H+Z)*D(J,1,1)*X**(1.0D0+ZI)                               PRO11810
     #    +( V44*H+Z)*D(J,2,1)*X**(1.0D0-ZI)                              PRO11820
     #    +(-V44*H+Z)*D(J,3,1)*X**(-ZI)*(3.0D0-3.0D0*X+X*X)               PRO11830
     #    +( V44*H+Z)*D(J,4,1)*X**( ZI)*(3.0D0-3.0D0*X+X*X)               PRO11840
   568 F1= (-V34*H+Z)*D(J,1,2)*X**ZI                                      PRO11850
     #    -( V14*H+Z)*D(J,2,2)*X**(2.0D0-ZI)                              PRO11860
     #    -( V34*H+Z)*D(J,3,2)*X**(-ZI)                                   PRO11870
     #    +( V54*H-Z)*D(J,4,2)*X**(-2.0D0+ZI)                             PRO11880
      F2= (-V54*H+Z)*D(J,1,3)*X**(2.0D0+ZI)                               PRO11890
     #    +( V34*H+Z)*D(J,2,3)*X**(-ZI)                                   PRO11900
     #    -( V54*H+Z)*D(J,3,3)*X**(-2.0D0-ZI)                             PRO11910
     #    +( V34*H-Z)*D(J,4,3)*X**( ZI)                                   PRO11920
      G1=BC(J,1,1)*X**(2.0D0+ZI)+BC(J,2,1)*X**(2.0D0-ZI)                  PRO11930
     #          +BC(J,3,1)*X**(ZI)                                        PRO11940
      G2=BC(J,1,2)*X**(2.0D0+ZI)+BC(J,2,2)*X**(2.0D0-ZI)                  PRO11950
     #          +BC(J,3,2)*X**(-ZI)                                       PRO11960
      SQ(IP,JP)=SQ(IP,JP) + C*(                                           PRO11970
     #                  F0* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*Z)               PRO11980
     #    +EPSI*(F1+G1)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z+H))              PRO11990
     #    +EPSI*(F2+G2)* EXP((Q-(1-(-1)**J)*Q0/2.0D0)*(Z-H))              PRO12000
     #       ) * A(J,M)                                                   PRO12010
    346 CONTINUE                                                          PRO12020
    350 CONTINUE                                                          PRO12030
      DO 402 IP=1,19                                                      PRO12040
      QP=(IP-1)*45.0D0/18                                                 PRO12050
      DO 401 JP=1,11                                                      PRO12060
      X=LAM+(1.0D0-LAM)*(JP-1)/(10)                                       PRO12070
      U1=DREAL(SR(IP,JP))                                                 PRO12080
      U2=DIMAG(SR(IP,JP))                                                 PRO12090
      T1=(U1+U2)                                                          PRO12100
```

```
      U1=DREAL(SQ(IP,JP))                                                    PRO12110
      U2=DIMAG(SQ(IP,JP))                                                    PRO12120
      T2=(U1+U2)                                                             PRO12130
      WRITE(8,298) X,QP,T1,T2                                                PRO12140
      WRITE(6,298) X,QP,T1,T2                                                PRO12150
  401 CONTINUE                                                               PRO12160
      WRITE(8,298)                                                           PRO12170
      WRITE(6,298)                                                           PRO12180
  402 CONTINUE                                                               PRO12190
      DO 501 JP=1,3                                                          PRO12200
      X=LAM+(1.0D0-LAM)*(JP-1)/(2)                                           PRO12210
      DO 502 IP=1,19                                                         PRO12220
      QP=(IP-1)*45.0D0/18                                                    PRO12230
      U1=DREAL(SR(IP,JP))                                                    PRO12240
      U2=DIMAG(SR(IP,JP))                                                    PRO12250
      T1=(U1+U2)                                                             PRO12260
      U1=DREAL(SQ(IP,JP))                                                    PRO12270
      U2=DIMAG(SQ(IP,JP))                                                    PRO12280
      T2=(U1+U2)                                                             PRO12290
      WRITE(8,298) X,QP,T1,T2                                                PRO12300
      WRITE(6,298) X,QP,T1,T2                                                PRO12310
  502 CONTINUE                                                               PRO12320
      WRITE(8,298)                                                           PRO12330
      WRITE(6,298)                                                           PRO12340
  501 CONTINUE                                                               PRO12350
  298 FORMAT(1X,2F8.4,1X,'U=',F19.10,1X,'V=',F19.10,1X,I3)                   PRO12360
      RETURN                                                                 PRO12370
      END                                                                    PRO12380
C*******************************************************************************PRO12390
```

Program 3

```
C******************** MAIN PROGRAM ************************************PRO00010
C     TOROIDAL SHELL SUBJECTED TO OUTSIDE BAND LOADS                   PRO00020
C     INPUT THE DIMENSION DATA AND APPLIED LOAD                        PRO00030
C     RIN:      INSIDE RADIUS OF THE TOROIDAL SHELL;                   PRO00040
C     ROUT:     OUTSIDE RADIUS OF THE TOROIDAL SHELL;                  PRO00050
C     RCENT:    MEAN TOROIDAL RADIUS;                                  PRO00060
C     ALPH:     HALF WIDTH OF THE BAND LOADS;                          PRO00070
C     THE0:     HALF WIDTH OF THE INTERVAL OF THE BAND LOADS;          PRO00080
C     PLOAD:    THE MAGNITUED OF THE APPLIED BAND LOAD;                PRO00090
C     MTERMS:   TERMS TAKEN IN THE LOAD SERIES;                        PRO00100
C     HH9-HH21: CONSTANTS OBTAINED FROM PROGRAM 3;                     PRO00110
C     DATA: JAN. 1994                                                  PRO00120
C-----------------------------------------------------------------------PRO00130
      DIMENSION TRAN(6),STRE(6),QQ(10)                                 PRO00140
C-----------------------------------------------------------------------PRO00150
      DATA QQ/0.0,2.0,4.0,6.0,8.0,14.0,20.0,28.0,38.0,44.0/            PRO00160
      PI=3.1415926                                                     PRO00170
C     INPUT THE DIMENSION DATA AND APPLIED LOAD                        PRO00180
      RIN=0.2                                                          PRO00190
      ROUT=0.4                                                         PRO00200
      RCENT=1.0                                                        PRO00210
      ALPH=PI*8.0/180.0                                                PRO00220
      THE0=PI/2.0                                                      PRO00230
      PLOAD=1.0                                                        PRO00240
      MTERMS=60                                                        PRO00250
      EPS=ROUT/RCENT                                                   PRO00260
C---- CALCULATING MAXIMUM STRESS VS EPSILON -----------------           PRO00270
C     PHI=0.0                                                          PRO00280
C     Q=PHI*180.0/PI                                                   PRO00290
C     THE=0.0                                                          PRO00300
C     THETA=THE*180.0/PI                                               PRO00310
C     DO 100 IE=1,4                                                    PRO00320
C     EPS=0.10*(IE)                                                    PRO00330
C     DO 100 IX=1,2                                                    PRO00340
C     X=0.5*IX                                                         PRO00350
C     WRITE(6,105) X,EPS                                               PRO00360
C     WRITE(8,105) X,EPS                                               PRO00370
C---- CALCULATING STRESS FIELD ON SHELL SURFACE --------------          PRO00380
      DO 100 IT=1,10                                                   PRO00390
      THE=QQ(IT)*PI/180.0                                              PRO00400
      THETA=THE*180.0/PI                                               PRO00410
      DO 100 IX=1,2                                                    PRO00420
      X=0.5*IX                                                         PRO00430
      WRITE(6,101) THETA,X                                             PRO00440
      WRITE(8,101) THETA,X                                             PRO00450
      DO 200 K=1,13                                                    PRO00460
      PHI=(K-1)*(PI/12.0)                                              PRO00470
      Q=PHI*180.0/PI                                                   PRO00480
C-----------------------------------------------------------           PRO00490
      M=0                                                              PRO00500
      PM=2.0*ALPH*PLOAD/THE0                                           PRO00510
      CALL STRESS(X,PHI,THE,M,PM,TRAN,EPS)                             PRO00520
      DO 402 JCOM=1,6                                                  PRO00530
      STRE(JCOM)=TRAN(JCOM)                                            PRO00540
  402 CONTINUE                                                         PRO00550
```

```
      DO 400 MM=1,MTERMS                                                PRO00560
      M=2*MM                                                            PRO00570
      PM=4.0*PLOAD*SIN(M*PI*ALPH/THE0)/M/PI                             PRO00580
      CALL STRESS(X,PHI,THE,M,PM,TRAN,EPS)                              PRO00590
      DO 404 JCOM=1,6                                                   PRO00600
      STRE(JCOM)=STRE(JCOM)+TRAN(JCOM)                                  PRO00610
  404 CONTINUE                                                          PRO00620
  400 CONTINUE                                                          PRO00630
      WRITE(8,210) THETA,(STRE(KP),KP=1,6)                              PRO00640
C     WRITE(8,210) Q,(STRE(KP),KP=1,6)                                  PRO00650
C     WRITE(8,210) EPS,(STRE(KP),KP=1,6)                                PRO00660
  200 CONTINUE                                                          PRO00670
  100 CONTINUE                                                          PRO00680
C-------------------------------------                                  PRO00690
  210 FORMAT(1X,F5.1,6F10.4)                                            PRO00700
  101 FORMAT(/3X,/'SOLUTION FOR',5X,'THETA=',F9.4,5X,'X=',F5.1)         PRO00710
  105 FORMAT(/3X,/'SOLUTION FOR',5X,'X=',F5.1,5X,'EPS=',F6.2)           PRO00720
      STOP                                                              PRO00730
      END                                                               PRO00740
C*********************************************************************  PRO00750
      SUBROUTINE STRESS(X,PHI,THE,M,PM,TRAN,E)                          PRO00760
      DIMENSION G(4,5),TRAN(6)                                          PRO00770
      PI=3.1415926                                                      PRO00780
      V=0.3                                                             PRO00790
      H=0.50                                                            PRO00800
      H2=H*H                                                            PRO00810
      H4=H2*H2                                                          PRO00820
      H6=H4*H2                                                          PRO00830
      THE0=PI/2.0                                                       PRO00840
      PIM=M*PI/THE0                                                     PRO00850
      OPV=1.0+V                                                         PRO00860
      OMV=1.0-V                                                         PRO00870
      OM2V=1.0-2.0*V                                                    PRO00880
      OM3V=1.0-3.0*V                                                    PRO00890
      TM4V=3.0-4.0*V                                                    PRO00900
      DO 2 K=1,6                                                        PRO00910
    2 TRAN(K)=0.0                                                       PRO00920
C-------------------------------------------------------------------    PRO00930
C--- HH1=HH15/A1; HH2=HH16/A1; HH3=HH17/A1; HH4=AST32/A1                PRO00940
      HH1=-0.0428571428+1.371428571*M*M                                 PRO00950
      HH2=0.1071428573+3.0*M*M                                          PRO00960
      HH3=-0.05357142848                                                PRO00970
      HH4=-0.8571428574*M*M                                             PRO00980
      HH5=-HH1*H4-OMV/OM2V*HH4*H-V/OM2V*HH4*H*ALOG(H)-HH2*H2            PRO00990
     #    *ALOG(H)-HH3*H2                                              PRO01000
      HH6=HH1+OMV/OM2V*HH4+HH3                                          PRO01010
      HH7=1.0/H2-1.0                                                    PRO01020
      HH8=HH5+HH6                                                       PRO01030
      A1= -PM / ( E*HH7 + E*E*E*HH8 )                                   PRO01040
C-----THE FOLLOWING CONSTANTS ARE OBTAINED FROM PADLOAD.MAPIN-------    PRO01050
      HH9=(8.861538465*M*M-7.529670334)*M*A1                            PRO01060
      HH10=-2.285714286*M*A1                                            PRO01070
      HH11=2.0*M*A1                                                     PRO01080
      HH12=(5.217582421-0.984615386*M*M)*M*A1                           PRO01090
      HH13=2.285714286*M*A1                                             PRO01100
```

```
      HH14=1.714285715*M*A1                                                  PRO01110
      HH15=(-0.0428571428+1.371428571*M*M)*A1                                PRO01120
      HH16=(0.1071428573+3.0*M*M)*A1                                         PRO01130
      HH17=-0.05357142848*A1                                                 PRO01140
      HH18=(0.0428571428+0.9142857139*M*M)*A1                                PRO01150
      HH19=-0.1249999998*A1                                                  PRO01160
      HH20=(0.2500000003+7.0*M*M)*A1                                         PRO01170
      HH21=(0.371428571+0.685714286*M*M)*A1                                  PRO01180
C-----------------------------------------------------------------           PRO01190
      F1=OM2V*PIM/OMV/H2*A1                                                  PRO01200
      C2=-A1/(8*OPV*(1.0+H2))                                                PRO01210
      F2=PIM*A1                                                              PRO01220
      CST30=-F2/4.0                                                          PRO01230
      F30=-( CST30*(H+H2)+F1*H*(1.+H2)/2.0 )/(1.+H)                          PRO01240
      AST32=-0.8571428574*M*M*A1                                             PRO01250
      CST32=1.714285714*M*A1                                                 PRO01260
      A31=(-0.0346378709-0.961348792*M*M)*A1                                 PRO01270
      VB5=-HH9-CST32+PIM*C2-HH11                                             PRO01280
      VB6=-HH9*H4-CST32*H+PIM*C2-HH10*H2*ALOG(H)-HH11*H2                     PRO01290
      F31=(VB5-VB6)/(1.0-H2)                                                 PRO01300
      E31=(VB5*H2-VB6)/(1.0-H2)                                              PRO01310
C-----STRESS COMPONENTS-----------------------------------------            PRO01320
      TRAN(1)=E*A1*(1.0/H2-1.0/X/X)*COS(PIM*THE)                             PRO01330
     #       +E*E*A1/(4.0*OMV*(1.+H2))*( (1.+H2)/X-X-H2/X/X/X)               PRO01340
     #           *COS(PHI)*COS(PIM*THE)                                     PRO01350
     #       +E**3*(-A31/X/X+HH15*X*X+OMV/OM2V*AST32/X+V/OM2V*              PRO01360
     #         AST32/X*ALOG(X)+HH16*ALOG(X)+HH17)*COS(PIM*THE)              PRO01370
      TRAN(2)=                                                               PRO01380
     #         E*A1*(1.0/H2+1.0/X/X)*COS(PIM*THE)                           PRO01390
     #       -E*E*A1/(4.0*OMV*(1.+H2))*( H2/X/X/X-3.0*X                     PRO01400
     #         -TM4V*(1.0+H2)/X)*COS(PHI)*COS(PIM*THE)                      PRO01410
     #       -E**3*(A31/X/X+HH18*X*X+V/OM2V*AST32/X+OMV/OM2V*               PRO01420
     #         AST32/X*ALOG(X)-HH16*ALOG(X)+HH17)*COS(PIM*THE)              PRO01430
      TRAN(3)=                                                               PRO01440
     #         E*A1/H2*COS(PIM*THE)                                         PRO01450
     #        +E*E*A1/(4.*OMV*(1.+H2))*(-4.*V*X+(4.-2.*V)*(1.+H2)/X         PRO01460
     #          +16.*OMV*OM2V*M*M )*COS(PHI)*COS(PIM*THE)                  PRO01470
     #        +E**3*(HH19+V/OM2V*AST32/X*(1+ALOG(X))+HH20*ALOG(X)          PRO01480
     #          +HH21/X/X)*COS(PIM*THE)                                    PRO01490
      TRAN(4)=E*E*A1/(4.0*OMV*(1.+H2))*( (1.+H2)/X-X-H2/X/X/X)               PRO01500
     #           *COS(PHI)*COS(PIM*THE)                                     PRO01510
      TRAN(5)=E**3*0.5*( F30/X+2*CST30*X+F1*X*X/2.0)                         PRO01520
     #           *SIN(PIM*THE)                                              PRO01530
     #       *E**3*(-E31/X/X+F31+HH9*X*X+CST32/X-PIM*C2/X/X+HH10            PRO01540
     #         *ALOG(X)+HH11)/2.0*COS(PHI)*SIN(PIM*THE)                     PRO01550
      TRAN(6)=E**3*(-E31/X/X-F31+HH12*X*X+HH13*ALOG(X)-CST32/X*              PRO01560
     #         ALOG(X)-PIM*C2/X/X+HH14)/2.*SIN(PHI)*A1*SIN(PIM*THE)         PRO01570
      RETURN                                                                 PRO01580
      END                                                                    PRO01590
C*********************************************************************************PRO01600
```

Program 4

```
C******************** MAIN PROGRAM ************************************PRO00010
#     Padload load solution for toroidal elasticity                    PRO00020
#     Checking the governing equations, boundary condtions, and        PRO00030
#     determining the constants of the particular solution (hh9-hh21). PRO00040
#     a:   inside radius;          b:    outside radius;               PRO00020
#     v:   Poosson ratio;          h:    Ratio of a/b;                 PRO00020
#     t0:  interval of the applied band loads;                         PRO00030
#     Date: Nov. 1993                                                  PRO00050
#*************************************************************************PRO00060
                                                                       PRO00070
#v:=0.3;                                                               PRO00080
#h:=0.8;                                                              PRO00090
pim:=pi*m/t0;                                                          PRO00100
cpv:=1+v:                                                              PRO00110
omv:=1-v:                                                              PRO00120
om2v:=1-2*v:                                                           PRO00130
tm4v:=3-4*v:                                                           PRO00140
cf:=cos(f):                                                            PRO00150
sf:=sin(f):                                                            PRO00160
s2f:=sin(2*f):                                                         PRO00170
ct:=cos(pim*t):                                                        PRO00180
st:=sin(pim*t):                                                        PRO00190
x2:=x*x:                                                               PRO00200
h2:=h*h:                                                               PRO00210
h4:=h2*h2:                                                             PRO00220
h6:=h2*h4:                                                             PRO00230
#a1:=-h*h/(1-h*h);                                                     PRO00240
b1:=a1*om2v/opv/h2;                                                    PRO00250
f12:=2*m*om2v*a1/opv/h2;                                               PRO00260
e11:=0;                                                                PRO00270
a0:=b1;                                                                PRO00280
#                                                                      PRO00290
#zeroth order                                                          PRO00300
uo:= a0*cf*ct;                                                         PRO00310
vo:=-a0*sf*ct;                                                         PRO00320
wo:=0;                                                                 PRO00330
uox:=diff(uo,x):                                                       PRO00340
uof:=diff(uo,f):                                                       PRO00350
uot:=diff(uo,t):                                                       PRO00360
vox:=diff(vo,x):                                                       PRO00370
vof:=diff(vo,f):                                                       PRO00380
vot:=diff(vo,t):                                                       PRO00390
wox:=diff(wo,x):                                                       PRO00400
wof:=diff(wo,f):                                                       PRO00410
wot:=diff(wo,t):                                                       PRO00420
psio:=uox + uo/x + vof/x;                                              PRO00430
psiox:=diff(psio,x):                                                   PRO00440
psiof:=diff(psio,f):                                                   PRO00450
psiot:=diff(psio,t):                                                   PRO00460
dum:=uo*cf - vo*sf + wot:                                              PRO00470
dumm:=normal(dum):                                                     PRO00480
sf2:=sf*sf:                                                            PRO00490
cf2:=cf*cf:                                                            PRO00500
omcd:=1-cf2:                                                           PRO00510
duz:=subs(sf2=omcd,dumm):                                              PRO00520
chio:=normal(duz);
```

```
chicx:=diff(chio,x):                                                    PRO00530
chiof:=diff(chio,f):                                                    PRO00540
chiot:=diff(chio,t):                                                    PRO00550
too:=table():                                                           PRO00560
too[1]:=uox+v*psio/om2v;                                                PRO00570
too[2]:=uo/x+vof/x+v*psio/om2v;                                         PRO00580
too[3]:=v*psio/om2v;                                                    PRO00590
too[4]:=(uof/x + vox - vo/x)/2;                                         PRO00600
too[5]:=wox/2;                                                          PRO00610
too[6]:=wof/x/2;                                                        PRO00620
#rhs evaluation after zeroth order                                      PRO00630
ubo:=cf*uox - sf*(uof/x - vo/x) + chiox/om2v:                           PRO00640
vbo:=cf*vox - sf*(uo/x + vof/x) + chiof/(x*om2v):                       PRO00650
wbo:=cf*wox - sf*wof/x + psiot/om2v:                                    PRO00660
orhs:=table():                                                          PRO00670
orhs[1]:=normal(-ubo);                                                  PRO00680
orhs[2]:=normal(-vbo);                                                  PRO00690
orhs[3]:=normal(-wbo);                                                  PRO00700
#                                                                       PRO00710
#first order                                                            PRO00720
u1:=(a1/x+b1*x)*ct;                                                     PRO00730
v1:=0;                                                                  PRO00740
w1:=f12*x*cf*st;                                                        PRO00750
u1x:=diff(u1,x):                                                        PRO00760
u1f:=diff(u1,f):                                                        PRO00770
u1t:=diff(u1,t):                                                        PRO00780
v1x:=diff(v1,x):                                                        PRO00790
v1f:=diff(v1,f):                                                        PRO00800
v1t:=diff(v1,t):                                                        PRO00810
w1x:=diff(w1,x):                                                        PRO00820
w1f:=diff(w1,f):                                                        PRO00830
w1t:=diff(w1,t):                                                        PRO00840
dum:=u1x + u1/x + v1f/x:                                                PRO00850
psiu:=normal(dum);                                                      PRO00860
psiux:=diff(psiu,x):                                                    PRO00870
psiuf:=diff(psiu,f):                                                    PRO00880
psiut:=diff(psiu,t):                                                    PRO00890
chiu:=u1*cf - v1*sf + w1t;                                              PRO00900
chiux:=diff(chiu,x):                                                    PRO00910
chiuf:=diff(chiu,f):                                                    PRO00920
chiut:=diff(chiu,t):                                                    PRO00930
tu:=table():                                                            PRO00940
tu[1]:=u1x + v*(psiu+chio)/om2v:                                        PRO00950
tu[2]:=u1/x + v1f/x + v*(psiu+chio)/om2v:                               PRO00960
tu[3]:=omv*chio/om2v + v*psiu/om2v:                                     PRO00970
tu[4]:=(u1f/x + v1x - v1/x)/2:                                          PRO00980
tu[5]:=(w1x + uot - wo*cf)/2:                                           PRO00990
tu[6]:=(w1f/x + vot + wo*sf)/2:                                         PRO01000
for j from 1 to 6 do                                                    PRO01010
tu[j]:=normal(tu[j]);                                                   PRO01020
od;                                                                     PRO01030
#right hand side evaluations after first order                          PRO01040
ub1:=cf*u1x - sf*(u1f/x - v1/x) + chiux/om2v:                           PRO01050
uho:=diff(uot,t) - cf*wot - 2*omv*cf*chio/om2v:                         PRO01060
vb1:=cf*v1x - sf*(u1/x+v1f/x) + chiuf/(x*om2v):                         PRO01070
```

```
vho:=diff(vot,t) + sf*wot + 2*omv*sf*chio/om2v:                                  PRO01080
dumd:=cf*w1x - sf*w1f/x + psiut/om2v:                                            PRO01090
wb1:=normal(subs(sf2=cmcd,dumd)):                                                PRO01100
who:=-diff(wot,t) - wo + tm4v*chiot/om2v:                                        PRO01110
urhs:=table():                                                                   PRO01120
urhs[1]:=normal(-ub1 + x*cf*ubo - uho);                                          PRO01130
dumm:=normal(-vb1 + x*cf*vbo - vho):                                             PRO01140
sf2:=sf*sf:                                                                      PRO01150
cf2:=cf*cf:                                                                      PRO01160
omcd:=1-cf2:                                                                     PRO01170
duh:=subs(sf2=omcd,dumm):                                                        PRO01180
urhs[2]:=normal(duh);                                                            PRO01190
urhs[3]:=normal(-wb1 + x*cf*wbo - who);                                          PRO01200
#                                                                                PRO01210
#2nd order solution                                                              PRO01220
om4v:=1-4*v:                                                                     PRO01230
tm4v:=3-4*v:                                                                     PRO01240
fm4v:=5-4*v:                                                                     PRO01250
a2:=0;                                                                           PRO01260
b2:=-a1/(8*omv*(1+h2));                                                          PRO01270
c2:=-h2*b2;                                                                      PRO01280
d2:=(3-2*v)*a1/4/omv;                                                            PRO01290
e2:=0;                                                                           PRO01300
f2:=2*m*a1;                                                                      PRO01310
e21:=0;                                                                          PRO01320
f21:=e11;                                                                        PRO01330
A21:=-v*cm2v*2*m*m*a1/opv/h2;                                                    PRO01340
B21:=A21;                                                                        PRO01350
A22:=-2*omv*a1;                                                                  PRO01360
B22:=-A22;                                                                       PRO01370
C21:=om2v*m*a1/opv/h2;                                                           PRO01380
u2:=(a2+om4v*b2*x*x+c2/x/x+tm4v*d2*ln(x)+A21*x*x+A22*ln(x))*cf*ct;                PRO01390
v2:=(fm4v*b2*x*x-a2+c2/x/x-tm4v*d2*ln(x)+B21*x*x+B22*ln(x)-d2)*sf*ct;             PRO01400
w2:=(e2+f2*ln(x)+C21*x*x)*st;                                                    PRO01410
u2x:=diff(u2,x):                                                                 PRO01420
u2f:=diff(u2,f):                                                                 PRO01430
u2t:=diff(u2,t):                                                                 PRO01440
v2x:=diff(v2,x):                                                                 PRO01450
v2f:=diff(v2,f):                                                                 PRO01460
v2t:=diff(v2,t):                                                                 PRO01470
w2x:=diff(w2,x):                                                                 PRO01480
w2f:=diff(w2,f):                                                                 PRO01490
w2t:=diff(w2,t):                                                                 PRO01500
dum:=u2x + u2/x + v2f/x:                                                         PRO01510
psid:=normal(dum);                                                               PRO01520
psidt:=diff(psid,t):                                                             PRO01530
dum:=u2*cf - v2*sf + w2t:                                                        PRO01540
duz:=subs(sf2=omcd,dum):                                                         PRO01550
chid:=normal(duz);                                                               PRO01560
chidx:=diff(chid,x):                                                             PRO01570
chidf:=diff(chid,f):                                                             PRO01580
chidt:=diff(chid,t):                                                             PRO01590
td:=table():                                                                     PRO01600
td[1]:=u2x + v*(psid+chiu-x*cf*chio)/om2v:                                       PRO01610
td[2]:=u2/x + v2f/x + v*(psid+chiu-x*cf*chio)/om2v:                               PRO01620
```

```
td[3]:=omv*(chiu-x*cf*chio)/om2v + v*psid/om2v:                    PRO01630
td[4]:=(u2f/x + v2x - v2/x)/2:                                     PRO01640
td[5]:=(w2x + u1t - w1*cf-x*cf*(uot-wo*cf))/2:                     PRO01650
td[6]:=(w2f/x + v1t + w1*sf-x*cf*(vot+wo*sf))/2:                   PRO01660
for j from 1 to 6 do                                               PRO01670
td[j]:=normal(td[j]);                                              PRO01680
od;                                                                PRO01690
#right hand side evaluations after order two                       PRO01700
ub2:=cf*u2x - sf*(u2f/x - v2/x) - chidx/om2v:                      PRO01710
uh1:=diff(u1t,t) - cf*w1t - 2*omv*cf*chiu/om2v:                    PRO01720
vb2:=cf*v2x - sf*(u2/x+v2f/x) + chidf/(x*om2v):                    PRO01730
vh1:=diff(v1t,t) + sf*w1t + 2*omv*sf*chiu/om2v:                    PRO01740
wb2:=cf*w2x - sf*w2f/x + psidt/om2v:                               PRO01750
wh1:=-diff(w1t,t) - w1 + tm4v*chiut/om2v:                          PRO01760
ub2:=normal(ub2):                                                  PRO01770
uh1:=normal(uh1):                                                  PRO01780
vb2:=normal(vb2):                                                  PRO01790
vh1:=normal(vh1):                                                  PRO01800
wb2:=normal(wb2):                                                  PRO01810
wh1:=normal(wh1):                                                  PRO01820
drhs:=table():                                                     PRO01830
dumx:=normal(-ub2 + x*cf*ub1 - x2*cf2*ubo - uh1 + 2*x*cf*uho):     PRO01840
dumy:=normal(-vb2 + x*cf*vb1 - x2*cf2*vbo - vh1 + 2*x*cf*vho):     PRO01850
dumz:=normal(-wb2 + x*cf*wb1 - x2*cf2*wbo - wh1 + 2*x*cf*who):     PRO01860
sf2:=sf*sf:                                                        PRO01870
cf2:=cf*cf:                                                        PRO01880
c2f:=cos(2*f):                                                     PRO01890
omcd:=(1-c2f)/2:                                                   PRO01900
dumx:=subs(sf2=omcd,dumx):                                         PRO01910
omcd:=(1+c2f)/2:                                                   PRO01920
dumx:=subs(cf2=omcd,dumx):                                         PRO01930
drhs[1]:=normal(dumx);                                             PRO01940
drhs[2]:=normal(dumy);                                             PRO01950
drhs[3]:=normal(dumz);                                             PRO01960
RHS1:=normal(dumx):                                                PRO01970
RHS2:=normal(dumy):                                                PRO01980
RHS3:=normal(dumz):                                                PRO01990
# the third order particular solution                              PRO02000
uy3:=(A31*x**3+A32*x*ln(x))*ct;                                    PRO02010
vy3:=(B31*x**3+B32*x*ln(x))*ct;                                    PRO02020
wy3:=(C31*x-3+C32*x*ln(x))*st*cf;                                  PRO02030
uy3x:=diff(uy3,x):                                                 PRO02040
uy3f:=diff(uy3,f):                                                 PRO02050
uy3t:=diff(uy3,t):                                                 PRO02060
vy3x:=diff(vy3,x):                                                 PRO02070
vy3f:=diff(vy3,f):                                                 PRO02080
vy3t:=diff(vy3,t):                                                 PRO02090
wy3x:=diff(wy3,x):                                                 PRO02100
wy3f:=diff(wy3,f):                                                 PRO02110
wy3t:=diff(wy3,t):                                                 PRO02120
dum:=uy3x + uy3/x + vy3f/x:                                        PRO02130
psif:=normal(dum);                                                 PRO02140
psifx:=diff(psif,x):                                               PRO02150
psiff:=diff(psif,f):                                               PRO02160
psift:=diff(psif,t):                                               PRO02170
```

```
deltu:=diff(uy3x,x)+uy3x/x+diff(uy3f,f)/x/x:                                    PRO02160
deltv:=diff(vy3x,x)+vy3x/x+diff(vy3f,f)/x/x:                                    PRO02190
deltw:=diff(wy3x,x)+wy3x/x+diff(wy3f,f)/x/x:                                    PRO02200
LHS1:=normal(deltu-uy3/x/x-2*vy3f/x/x+psifx/om2v);                              PRO02210
LHS2:=normal(deltv-vy3/x/x+2*uy3f/x/x+psiff/x/om2v);                            PRO02220
LHS3:=normal(deltw);                                                            PRO02230
#---Comparing both side of the third order governing equations, one----         PRO02240
#--could obtain the following constants-------------------------------          PRO02250
A31:=a1*(2*om2v**2*(1+h2)*pim*pim-4*v*h2)/32/omv/(h2+h4);                       PRO02260
B31:=0.0;                                                                       PRO02270
A32:=-a1*v*pim*pim/2/omv;                                                       PRO02280
B32:=0.0;                                                                       PRO02290
C31:=a1*pim*( 4*omv*om2v*(1+h2)*pim*pim-2*(5-5*v+v*v)-4*omv*(2-v))              PRO02300
     /16/omv/opv/(h2+h4);                                                       PRO02310
C32:=(3-2*v)*pim*a1/4/omv;                                                      PRO02320
cbar:=-a1/h2:                                                                   PRO02330
hh:=3*A33+v/om2v*(6*A33+tm4v*b2-A21+cbar*om2v*(1+4*m*m)/2/opv+a0/2):            PRO02340
hh:=3*C31+2*m*(-om4v*b2-A21+b1-3*a0/4)-C21+3*f12/4:                             PRO02350
hh9:=normal(hh);                                                                PRO02360
hh10:=2*m*(-tm4v*d2-A22)-f2;                                                    PRO02370
hh11:=2*m*(a1-a2)-e2;                                                           PRO02380
hh:=-C31+2*m*(-(5-4*v)*b2-B21+a0/4)+C21-f12/4:                                  PRO02390
hh12:=normal(hh);                                                               PRO02400
hh13:=2*m*(tm4v*d2-B22)+f2;                                                     PRO02410
hh14:=2*m*(a2+d2)+e2;                                                           PRO02420
hh:=3*A31+v/om2v*(4*A31-2*b2+2*m*C21+cbar*om2v/2/opv*(1+4*m*m)+a0/2):           PRO02430
hh15:=normal(hh);                                                               PRO02440
hh16:=v/om2v*(tm4v*d2+A22+2*m*f2);                                              PRO02450
hh17:=v/om2v*(a2+d2/2+2*m*e2+cbar*h2/2);                                        PRO02460
hh:=A31+v/om2v*(4*A31-2*b2+2*m*C21+cbar*om2v/2/opv*(1+4*m*m)+a0/2):             PRO02470
hh18:=normal(hh);                                                               PRO02480
hh19:=2*v/om2v*b3+omv/om2v*(a2+d2/2+2*m*e2+cbar*h2/2);                          PRO02490
hh20:=omv/om2v*(tm4v*d2+A22+2*m*f2);                                            PRO02500
hh:=4*v/om2v*A31+omv/om2v*(-2*b2+2*m*C21+cbar*om2v/2/opv*(1+4*m*m)              PRO02510
     +a0/2):                                                                    PRO02520
hh21:=normal(hh);                                                               PRO02530
rb5:=-hh9-C32+2*m*c2-hh11;                                                      PRO02540
rb6:=-hh9*h4-C32*h+2*m*c2-hh10*h2*ln(h)-hh11*h2;                                PRO02550
rb11:=-hh15*h4-omv/om2v*A32*h-v/om2v*A32*h*ln(h)-hh16*h2*ln(h)-hh17*h2;         PRO02560
rb12:=hh15+omv/om2v*A32+hh17;                                                   PRO02570
rbbb:=rb11+rb12;                                                                PRO02580
a3:=-rb11;                                                                      PRO02590
b3:=0;                                                                          PRO02600
c3:=0;                                                                          PRO02610
d3:=0;                                                                          PRO02620
f3:=(rb5-rb6)/(1-h2);                                                           PRO02630
e3:=-(rb6-rb5*h2)/(1-h2);                                                       PRO02640
#                                                                               PRO02650
#3rd order solution                                                             PRO02660
u3:=(a3/x+b3*x)*ct+uy3:                                                         PRO02670
v3:=(c3/x+d3*x)*ct+vy3:                                                         PRO02680
w3:=(e3/x+f3*x)*st*cf+wy3;                                                      PRO02690
u3x:=diff(u3,x):                                                                PRO02700
u3f:=diff(u3,f):                                                                PRO02710
u3t:=diff(u3,t):                                                                PRO02720
```

```
v3x:=diff(v3,x):                                                          PRO02730
v3f:=diff(v3,f):                                                          PRO02740
v3t:=diff(v3,t):                                                          PRO02750
w3x:=diff(w3,x):                                                          PRO02760
w3f:=diff(w3,f):                                                          PRO02770
w3t:=diff(w3,t):                                                          PRO02780
dum:=u3x + u3/x + v3f/x:                                                  PRO02790
psif:=normal(dum):                                                        PRO02800
psift:=diff(psif,t):                                                      PRO02810
dum:=u3*cf - v3*sf + w3t:                                                 PRO02820
omcd:=(1+c2f)/2:                                                          PRO02830
duz:=subs(cf2=omcd,dum):                                                  PRO02840
chif:=normal(duz):                                                        PRO02850
chifx:=diff(chif,x):                                                      PRO02860
chiff:=diff(chif,f):                                                      PRO02870
chift:=diff(chif,t):                                                      PRO02880
t3:=table():                                                              PRO02890
t3[1]:=u3x + v*(psif+chid-x*cf*chiu+x*x*cf*cf*chio)/om2v:                 PRO02900
t3[1]:=subs(cf2=omcd,t3[1]):                                              PRO02910
t3[2]:=u3/x + v3f/x + v*(psif+chid-x*cf*chiu+x*x*cf*cf*chio)/om2v:         PRO02920
t3[2]:=subs(cf2=omcd,t3[2]):                                              PRO02930
t3[3]:=omv*(chid-x*cf*chiu+x*x*cf*cf*chio)/om2v+v*psif/om2v:              PRO02940
t3[3]:=subs(cf2=omcd,t3[3]):                                              PRO02950
t3[4]:=(u3f/x + v3x - v3/x)/2:                                            PRO02960
t3[5]:=(w3x + u2t-w2*cf-x*cf*(u1t-w1*cf)+(x*cf)-2*(uot-wo*cf))/2:         PRO02970
t3[6]:=(w3f/x+v2t+w2*sf-x*cf*(v1t+w1*sf)+(x*cf)-2*(vot+wo*sf))/2:         PRO02980
#t3[6]:=subs(sf2=omcd,t3[6]):                                             PRO02990
for j from 1 to 6 do                                                      PRO03000
t3[j]:=normal(t3[j]);                                                     PRO03010
od;                                                                       PRO03020
#-------------------------------------------------------------------------PRO03030
quit;                                                                     PRO03040
#*************************************************************************PRO03050
```