On p-adic L-functions arising from Bianchi modular forms

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Abstract

We study and construct p-adic L-functions of Bianchi modular forms, i.e., automorphic forms for GL_2 over quadratic imaginary fields, at *non-ordinary* primes in three different scenarios.

In the first part, we construct signed two-variable p-adic L-functions with bounded coefficients from p-adic L-functions with unbounded coefficients for cuspidal Bianchi modular forms of parallel weight constructed by Williams. This construction extends the works of Pollack, Sprung, and Lei–Loeffler–Zerbes from the elliptic modular forms setting to the Bianchi modular forms setting. Additionally, we extend the results to p-adic L-functions coming from non-parallel weight C-cuspidal Bianchi modular forms constructed by Palacios. We construct logarithmic matrices using Wach modules basis constructed by Berger–Li–Zhu for the decomposition of p-adic L-functions with unbounded coefficients. We use Perrin–Riou's exponential map and the p-adic regulator to prove certain properties of logarithmic matrices.

In the second part, we construct a p-adic Asai L-function, associated to a p-non-ordinary Bianchi modular form, which interpolates special complex L-values of the Asai L-function of that Bianchi modular form. This p-adic L-function has unbounded coefficients. We use modular symbols and some special cohomological elements, called Asai–Eisenstein elements, to construct polynomials. These polynomials satisfy some growth conditions, norm properties, and congruence relations. After taking the limit of these polynomials, we obtain the p-adic Asai L-function with unbounded coefficients. Moreover, we also construct signed p-adic Asai L-functions with bounded coefficients under some assumptions.

In the third part, we construct a two-variable p-adic Asai L-function over the eigenvariety interpolating p-adic Asai L-functions of non-critical small-slope base-change Bianchi modular forms of parallel weight 0. To construct this p-adic L-function, we construct polynomials using a certain overconvergent modular symbol coming from a parallel eigenvariety associated with Bianchi modular forms and Asai–Eisenstein elements over an affinoid in a weight space. Their specialization at the weight $(0, 0)$ Bianchi modular form $\mathcal{F}$ gives the p-adic Asai L function associated to $\mathcal{F}$, which is constructed in the second part.

Dedications

This work is dedicated to my mother, Avanti, and to my late father, Vilas.

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Declaration

The results in this thesis are original, except where explicit reference to the results of others is made. Most of the work is either published or submitted to a mathematical journal.

The results of Chapter 2 and Appendix A are from the article titled *"Signed p-adic L-functions of Bianchi modular forms"*. This article is published in the journal *Research in Number Theory.* The article can be found here: https://doi.org/10.1007/s40993-025-00689-9.

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Chapter 1

Introduction

1.1 Background

A central theme in number theory is the study of special values of complex L-functions, since these values are connected to some important arithmetic data. For example, the Birch and Swinnerton-Dyer conjecture (BSD) predicts that important arithmetic data associated with an elliptic curve over rational numbers, such as its rank and order of the torsion group, are related to properties of its complex L-function at $s = 1$. The use of p-adic L-functions has emerged as one of the most powerful tools to study the special values of L-functions. When such functions can be constructed, they have profound consequences. For example, in their important work, Mazur–Tate–Teitelbaum formulated the p-adic analogue of the BSD conjecture using p-adic L-functions associated with elliptic curves, relating the order of vanishing of p-adic L-functions with certain arithmetic invariants of elliptic curves, similar to those that appear in the classical case. This p-adic version has been proved in a large number of cases. Furthermore, p-adic L-functions play a central role in *Iwasawa main conjectures*, which are one of the important local tools used to prove certain global conjectures, such as the BSD conjecture.

This thesis focuses on p-adic L-functions associated with Bianchi modular forms that are not ordinary at primes p. In particular, we construct two-variable signed p-adic L-functions for Bianchi modular forms extending the works of Pollack, Sprung, and Lei–Loeffler–Zerbes from elliptic modular forms to the setting of Bianchi modular forms. Next, we construct p-adic Asai L-functions for cuspidal Bianchi modular forms that are non-ordinary at p. Furthermore, under some assumptions, we decompose these p-adic Asai L-functions with unbounded coefficients into signed p-adic L-functions with bounded coefficients. Lastly, we construct families of p-adic Asai L-functions associated with small slope cuspidal Bianchi modular forms of weight $(0, 0)$.

1.1.1 L-functions

Let X be an arithmetic object such as a Hecke/Dirichlet character, an elliptic curve, a modular form, or more generally a Galois representation or a Motive. An L-function attached to X is, roughly speaking, a power series

$$L(X,s) := \sum_{n\geqslant 1} \frac{a_n}{n^s},$$

where $s \in \mathbb{C}$ in some suitable right half plane, and the coefficients a_n are related to X. For example, for $X =$ trivial character, we get *the Riemann ζ-function*

$$L(X,s) = \zeta(s) := \sum_{n\geqslant 1} \frac{1}{n^s}.$$

There are deep results and conjectures that relate the special values of L-functions to arithmetic information. For example, for a number field F, the analytic class number formula relates the residue at $s = 1$ of the Dedekind zeta function ζ_F with arithmetic information of F: its class number and regulator. More precisely, the class number formula provides a profound connection between two distinct fields of mathematics.

Elliptic modular forms and their L-functions

Elliptic modular forms are automorphic forms for GL_2 over $\mathbb{Q}$. Let $k \in \mathbb{Z}_{\geqslant 2}$. More precisely, $f : \mathbb{H} \to \mathbb{C}$ is an elliptic modular form of weight k and level $\mathrm{SL}_2(\mathbb{Z})$ if

1. f is holomorphic;
2. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for $\tau \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$;
3. f is holomorphic at ∞.

If f vanishes at the cusp ∞, we say f is a cusp form. One can define ellptic modular forms of level $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, where Γ is any congruence subrgoup. The space of elliptic modular forms of weight k with a given level forms a finite dimensional complex vector space, and the space of cusp forms of weight k is its subspace. There are certain linear operators that act on elliptic modular forms called Hecke operators. See [DS05] for more details.

Suppose $f = \sum_{n\geqslant 1} a_n q^n$ is an Hecke eigenform of level $\Gamma_0(N)$. One can associate an L-function to f:

$$L(f,s) := \sum_{n\geqslant 1} \frac{a_n}{n^s},$$

where $s \in \mathbb{C}$ in some suitable right half-plane. This function admits an Euler product

$$\sum_{n\geqslant 1} \frac{a_n}{n^s} = \prod_{p\nmid N} (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \prod_{p\mid N} (1 - a_p p^{-s})^{-1}.$$

Let E be an elliptic curve of conductor N over $\mathbb{Q}$, and

$$L(E,s) := \prod_{p\nmid N}(1-b_p p^{-s}+p^{1-2s})^{-1}\prod_{p|N}(1-b_p p^{-s})^{-1}.$$

Here, $b_p := p+1-\#\overline{E}(\mathbb{F}_p)$. Then the famous *Modularity theorem* states that

$$L(E,s)=L(f,s),$$

where If f is a newform of weight 2 and level $\Gamma_0(N)\subset \mathrm{SL}_2(\mathbb{Z})$.

The famous Birch–Swinnerton-Dyer conjecture predicts that for an elliptic curve E over $\mathbb{Q}$, one has

$$\mathrm{ord}_{s=1}L(E,s)=\mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q}).$$

The left hand side relies on the analytic continuation of $L(E,s)$, and it holds due to Wiles' (and Taylor–Wiles, Breuil–Conrad–Diamond–Taylor) proof of *the Modularity theorem.*

An important method to connect special L-values with arithmetic information is to construct a p-adic L-function interpolating these values and using Iwasawa theory to relate them to arithmetic invariants. See also [RJW25].

1.1.2 What are p-adic L-functions?

Analytic p-adic L-functions are measures or distributions on p-adic Lie groups, like $\mathbb{Z}_p^\times$, which interpolate special values of certain complex L-functions. One of the earliest examples is the Kubota–Leopoldt p-adic L-function ζ_p^{an}, constructed by Kubota–Leopoldt and Iwasawa. It is a pseudomeasure on $\mathbb{Z}_p^\times$ which interpolates the special values $\zeta(1-k)$ of the complex Riemann ζ-function for all positive integers k. More precisely,

Theorem 1.1.1 (Kubota–Leopoldt, Iwasawa)**.** There exists a (pseudo)measure ζ_p^{an} on $\mathbb{Z}_p^\times$ such that for every $k>0$, we have

$$\int_{\mathbb{Z}_p^\times} x^k\zeta_p^{\mathrm{an}} := \zeta_p^{\mathrm{an}}(x\mapsto x^k)=(1-p^{k-1})\zeta(1-k).$$

Furthermore, let χ be a Dirichlet character of conductor $p^n.n>0$. After consider χ as a locally constant character on $\mathbb{Z}_p^\times$, we have, for all $k>0$,

$$\int_{\mathbb{Z}_p^\times}\chi(x)x^k\zeta_p^{\mathrm{an}}=(1-\chi(p)p^{k-1})L(\chi,1-k),$$

where $L(\chi,s)=\sum_{n\geqslant 1}\frac{\chi(n)}{n^s}$, for $s\in\mathbb{C}$ and $\mathrm{Re}(s)\gg 0$.

Let G be a p-adic Lie group and let $K/\mathbb{Q}_p$ be a finite extension. Let

$$C(G,K) := \mathrm{Hom}_{\mathrm{cts}}(G,K)$$

be the space of all continuous functions from G to K, and

$$M(G,K) := C(G,K)^* = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Hom}_{\mathrm{cts}}(G,K),K)$$

be the K-dual of $C(G,K)$. The space $M(G,K)$ is called *the measure space of* G *over* K. Usually, p-adic measures live in this space.

If $G \cong \mathbb{Z}_p^r$, for some $r \geqslant 1$, one can define the space $C^{la}(G,K)$ of *locally analytic functions* consisiting of $f : G \to K$ such that for all $x \in G$, f can be described as a convergent power series locally around x. Let

$$D(G,K) := C^{la}(G,K)^* = \mathrm{Hom}_{\mathrm{cts}}(C^{la}(G,K),K).$$

The space $D(G,K)$ is called *the space of distributions of* G *over* K, and p-adic distributions live inside this space.

For abelian groups G, we can view p-adic L-functions, i,e, p-adic measures or distributions over G, as power series with coefficients in a p-adic field or its ring of integers. For example, if $G = \mathbb{Z}_p$, then

$$\begin{aligned} M(\mathbb{Z}_p,K) &\xrightarrow{\cong} K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[T]], \\ \mu &\mapsto \int_{\mathbb{Z}_p} (1+T)^x \cdot \mu(x) = \sum_{n \geqslant 0} \left(\int_{\mathbb{Z}_p} \binom{x}{n} \cdot \mu(x) \right) T^n. \end{aligned}$$

This isomorphism is known as *the Amice Transform.*

Similarly,

$$\begin{aligned} D(\mathbb{Z}_p,K) &\xrightarrow{\cong} \mathcal{H}_K(T), \\ \mu &\mapsto \int_{\mathbb{Z}_p} (1+T)^x \cdot \mu(x), \end{aligned}$$

where

$$\mathcal{H}_K(T) := \left\{ f(T) = \sum_{n \geqslant 0} c_n T^n \in K[[T]] : f \text{ converges on the open unit disk in } \mathbb{C}_p \right\}.$$

See [RJW25] and [Col10] for more details about p-adic distributions and measures.

In number theory, we are interested in complex L-functions which have follwowing properties:

1. an Euler product;
2. a meromorphic continuation to the whole complex plane;
3. a functional equation relating s with $1-s$, for $s \in \mathbb{C}$.

If such a complex L-function exists, then it is natural to consider constructing a p-adic L-function that interpolates the special values of the given complex L-function. In [CPR89] and [Coa89], Coates and Perrin-Riou conjectured that for any arithmetic object over $\mathbb{Q}$ (such as modular forms and more generally any motive $\mathcal{M}$ over $\mathbb{Q}$) and for all primes p, there exists a p-adic L-function over $\mathbb{Z}_p^\times$ which interpolates the special/critical L-values of the complex L-function associated with that arithmetic object. For example, when $\mathcal{M} = \mathbb{Q}(\chi)$, where χ is the Dirichlet character of prime power conductor, the corresponding complex L-function is the Dirichlet L-function, and the p-adic L-function is the Kubota–Leopoldt p-adic L-function. We will briefly describe p-adic L-functions associated with elliptic modular forms and Bianchi modular forms in the next subsections.

1.1.3 p-adic L-functions of elliptic modular forms

Iwasawa theory concerns the growth of arithmetic objects, such as Selmer groups, in towers of number fields, such as *cyclotomic fields*. Let us continue with the cyclotmic fields $\mathbb{Q}(\mu_{p^n})$ and the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)$, where $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n \geqslant 1} \mathbb{Q}(\mu_{p^n})$, and μ_{p^n} is the set of p^n roots of unity. Let $\Gamma := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \cong \Delta_{\mathbb{Q}_p} \times \mathbb{Z}_p$, where $\Delta_{\mathbb{Q}_p} \cong (\mathbb{Z}/p\mathbb{Z})^\times$. Let $K/\mathbb{Q}_p$ be a finite extension and $\mathcal{O}_K$ be its ring of integers. Fix a topological generator γ of $\Gamma/\Delta_{\mathbb{Q}_p} \cong \mathbb{Z}_p$. Then the space of K-valued measures on Γ is isomorphic to the Iwasawa alegbra $\Lambda_K(\Gamma) := K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[\Gamma]] \cong K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\Delta_{\mathbb{Q}_p}][[T]]$, where the isomorphism is obtained by sending γ to $(1+T)$. Similarly, the K-valued distribution space over Γ is isomorphic to $\mathcal{H}_K(\Gamma)$, where

$$\mathcal{H}_K(\Gamma) := \left\{ \sum_{n \geqslant 0} c_{n,\sigma} \cdot \sigma \cdot (\gamma - 1)^n : \substack{\sum_{n \geqslant 0} c_{n,\sigma} X^n \in K[[X]] \text{ converges on the open unit disk in } \mathbb{C}_p; \\ \forall \sigma \in \Delta} \right\}. \quad (1.1.1)$$

It is natural to consider one-variable p-adic L-functions as measures or distributions over Γ.

For example, if E is an elliptic curve over $\mathbb{Q}$ and if E is good and ordinary at p, then the p-adic L-function $L_{p,E}$ constructed by Mazur–Swinneton-Dyer lies in $\Lambda_{\mathbb{Q}_p}(\Gamma)$. After representing as a power series in $\mathbb{Q}_p \otimes \mathbb{Z}_p[\Delta_{\mathbb{Q}_p}][[T]]$, it has bounded coefficients. This p-adic L-function interpolates special values of complex Hasse–Weil L-series of E twisted by certain Dirichlet characters. Later, Amice–Velu and Vishik generalized this construction to elliptic modular forms of weight $k \geqslant 2$ and with more general types of reduction.

More precisely, let f be an elliptic modular cusp form of weight $k \geqslant 2$, level $N \geqslant 1$, and nebentypus ϵ_f. Suppose f is a Hecke eigenform and let the T_p Hecke eigenvalue of f be a_p. Consider the Hecke polynomial $X^2 - a_p X + \epsilon_f(p) p^{k-1} = (x - \alpha)(X - \beta)$. Since $\alpha\beta = \epsilon_f(p)(p^{k-1})$ and $\epsilon_f(p)$ is coprime to p, we have $v_p(\alpha), v_p(\beta) \leqslant (k-1)$.. Here v_p is a p-adic valuation such that $v_p(p) = 1$. Let $\lambda \in \{\alpha, \beta\}$, such that $v_p(\lambda) < k - 1$. For such f and λ, by the works of Amice–Velu [AV75] and Vishik [Vis76], one can attach a p-adic L-function $L_p(f, \lambda)$. On the one hand if λ is a p-adic unit, then $L_p(f, \lambda)$ is a p-adic measure. On the other hand, if $v_p(\lambda) > 0$, then $L_p(f, \lambda)$ is a p-adic distribution.

This $L_p(f, \lambda)$ has the following interpolation property:

$$L_p(f, \lambda, \chi^j \omega) := \int_\Gamma \chi^j \omega dL_p(f, \lambda) = \frac{*}{\lambda^r} L(f, \tilde{\omega}^{-1}, j+1), \quad (1.1.2)$$

where

- $*$ is an explicit factor;
- $0 \leqslant j \leqslant k-2$ is an integer;
- ω is a Dirichlet character of conductor $p^r > 1$, and ω factors through $\tilde{\omega} : \mathbb{Z}/p^{r-1}\mathbb{Z} \to \overline{\mathbb{Q}_p}^{\times}$;
- ζ' is a $(p-1)$-th root of unity, and $\zeta_{p^{r-1}}$ is a primitive p^{r-1}th root of unity;
- χ is the p-adic cyclotomic character on Γ such that χ maps γ to a topological generator u of $1+p\mathbb{Z}_p$;
- $L(f, \tilde{\omega}^{-1}, j+1)$ is the value of L-function associated to f and twisted by $\tilde{\omega}^{-1}$.

Suppose f is ordinary at p, i.e., a_p is a p-adic unit. Then one of α or β is also a p-adic unit. Suppose α is a p-adic unit. Then the p-adic L-function $L_p(f, \alpha)$ is a p-adic measure and hence an element of the Iwasawa algebra $\Lambda_K(\Gamma)$. Furthermore, it can be associated with a power series $L_p(f, \alpha, T) \in K[\Delta_{\mathbb{Q}_p}] \otimes_{\mathcal{O}_K} \mathcal{O}_K[[T]]$. Note that one can think of $L_p(f, \alpha, T)$ as a p-adic L-function associated with the p-stabilization f_α of f, such that f_α has weight k, level pN, and $U_p f_\alpha = \alpha f_\alpha$. Additionally, on the algebraic side, one has the Bloch–Kato Selmer group, $\mathrm{Sel}(f)$ over some number field L, which is Λ_K-cotorsion. The Iwasawa main conjecture states that the characteristic ideal of the Pontryagin dual of $\mathrm{Sel}(f)$ is the same as the ideal generated by $L_p(f, \alpha)$ in $\Lambda_K(\Gamma)$. This conjecture holds in many cases due to Kato, Skinner–Urban, and others.

Later, Pollack–Stevens and Bellaïche used *overconvergent modular symbols* and *the Coleman–Mazur eigencurve* to construct p-adic L-functions for elliptic modular forms with the critical slope, i,e, when $v_p(\beta) = k-1$.

1.1.4 Signed p-adic L-functions of elliptic modular forms

Now suppose f is not p-ordinary, i.e., $v_p(a_p) > 0$. Then we know that $0 < v_p(\alpha), v_p(\beta) < k-1$. Therefore, for $? \in \{\alpha, \beta\}$, there exists a p-adic L-function $L_{p,?} := L_p(f, ?)$ which interpolates special values of the complex L-function of f. However, both $L_{p,\alpha}$ and $L_{p,\beta}$ are not *measures* and therefore do not lie in the Iwasawa algebra $\Lambda_K(\Gamma)$. They are *distributions* and can be represented as power series in a much larger algebra $\mathcal{H}_K(\Gamma)$ and have unbounded denominators. Additionally, on the algebraic side, the corresponding Selmer groups are not $\Lambda_K(\Gamma)$-cotorsion. Thus, we do not have *an Iwasawa Main Conjecture* over $\Lambda_K(\Gamma)$ relating $L_{p,?}$ with the appropriate Selmer group.

Remedy

When $a_p = 0$, Pollack in [Pol03] gave a solution to tackle this on the analytic side. Let α_1, α_2 be the roots of $X^2 + \epsilon_f p^{k-1}$. For $i \in \{1, 2\}$, Pollack showed that there exists a decomposition

$$L_{p,\alpha_i} = \log^+_{p,k}(\gamma) L^+_p + \alpha_i \log^-_{p,k}(\gamma) L^-_p, \tag{1.1.3}$$

where $L_p^\pm \in \Lambda_K(\Gamma)$, and $\log_{p,k}^\pm$ are some power series in $\mathcal{H}_{\mathbb{Q}_p}(\Gamma)$ depending only on k. These $\log_{p,k}^\pm$ are related to the p-adic logarithm, e.g., when $k = 2$, we have

$$(\gamma - 1)\log_p^+(\gamma)\log_p^-(\gamma) = \log_p(\gamma) = \sum_{n\geqslant 1} \frac{(-1)^{n+1}(\gamma-1)^n}{n}.$$

Later Sprung (in [Spr12] for $k = 2$ and $a_p \neq 0$) and Lei–Loeffler–Zerbes (in [LLZ10] for $k \geqslant 2$ and $a_p \neq 0$) generalized Pollack's results using a notion of *logarithmic matrices*. Logarithmic matrices are 2×2 matrices with entries in $\mathcal{H}_K(\Gamma)$, with certain growth conditions, and can be thought of as generalizations of Pollack's $\pm$-logarithms $\log_{p,k}^\pm$. For example, in [LLZ10], Lei–Loeffler–Zerbes used p-adic Hodge theory and the theory of Wach modules to prove:

$$\begin{pmatrix} L_{p,\alpha} \\ L_{p,\beta} \end{pmatrix} = M_{\log} \begin{pmatrix} L_p^\flat \\ L_p^\sharp, \end{pmatrix} \tag{1.1.4}$$

where $L_p^{\flat/\sharp} \in \Lambda_K(\Gamma)$ and $M_{\log} \in M_{2,2}(\mathcal{H}_K(\Gamma))$ is a logarithmic matrix.

On the algebraic side, there is a notion of signed Selmer groups due to Kobayashi (for $k = 2$, $a_p = 0$), Lei (for $k \geqslant 2$, $a_p = 0$), Sprung (for $k = 2$, $a_p \neq 0$), and Lei–Loeffler–Zerbes (for $k \geqslant 2$ and $a_p \neq 0$) and signed Iwasawa main conjectures over $\Lambda_K(\Gamma)$ which relate these signed objects with signed p-adic L-functions.

1.1.5 Previous works on Bianchi modular forms

Let F be an imaginary quadratic field. Bianchi modular forms are automorphic forms for GL_2 over F. Like elliptic modular forms, they also satisfy some specific harmonicity and growth conditions. See Definition 2.7.2 for the precise definition.

In [Wil17], Williams constructed a p-adic distribution over the ray class group G_{p^∞} that interpolates the special complex L-values associated with the small-slope Bianchi modular form. In particular, Let $\mathcal{F}$ be a cuspidal Bianchi eigenform over F of weight (k,k) and level $\mathfrak{n}$ such that $(p) \mid \mathfrak{n}$. Let $a_\mathfrak{q}$ denote the $U_\mathfrak{q}$-eigenvalues of $\mathcal{F}$ where $v_p(a_\mathfrak{q}) < (k+1)$ for all $\mathfrak{q} \mid p$. For any ideal $\mathfrak{f}$, we define the operator $U_\mathfrak{f}$ as $U_\mathfrak{f} := \prod_{\mathfrak{p}^n \| \mathfrak{f}} U_\mathfrak{p}^n$. Then Williams constructed a locally analytic distribution $L_{p,\mathcal{F}}$ on the ray class group G_{p^∞} such that for any Hecke character Ξ of infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{f}$, we have

$$L_{p,\mathcal{F}}(\tilde{\Xi}) = (\text{explicit factor})\frac{1}{a_\mathfrak{f}}\frac{L(\mathcal{F},\Xi,1)}{\Omega_\mathcal{F}}, \tag{1.1.5}$$

where $\Omega_\mathcal{F}$ is a complex period. See [Wil17, Theorem 7.4] for more details. Here, the standard complex L-function is defined as

$$L(\mathcal{F},\Xi,s) = \sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_K, \\ (\mathfrak{f},\mathfrak{a})=1}} c(\mathfrak{a},\mathcal{F})\Xi(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-s},$$

where $c(\mathfrak{a},\mathcal{F})$ is the $\mathfrak{a}$-th Fourier coefficient of $\mathcal{F}$ and $s \in \mathbb{C}$ in some suitable right-half plane. In [Pal25], Palacios extended Williams' construction from parallel weight cuspidal Bianchi modular forms to non-parallel weight C-cuspidal Bianchi modular forms.

In [LW20], Loeffler–Williams constructed a p-adic measure $L_p^{\mathrm{As}}(\Psi) \in \mathcal{O}_E[[\mathbb{Z}_p^\times]]$ that interpolates the critical L-values of the Asai L-function attached to a p-ordinary cuspidal Bianchi modular form Ψ of weight (k,k). Here, E is some p-adic field. The twisted Asai L-function of Ψ with a Dirichlet character θ of conductor m is defined by

$$L^{\mathrm{As}}(\Psi,\theta,s) := (*) \cdot \sum_{\substack{n \geqslant 1,\\ (m,n)=1}} c(n\mathcal{O}_F,\Psi)\theta(n)n^{-s},$$

where $(N) = \mathfrak{N} \cap \mathbb{Z}$, $*$ is some explicit factor, and $c(\mathfrak{m},\Psi)$ denotes the Hecke eigenvalue of Ψ at the integral ideal $\mathfrak{m}$. They proved: For any integer $c > 1$ coprime to $6\mathfrak{N}$, there exists a p-adic L-function

$${}_cL_p^{\mathrm{As}}(\Psi) \in \mathcal{O}_E[[\mathbb{Z}_p^\times]]$$

which satisfies the following interpolation property: for any Dirichlet character θ of conductor p^r, and for any integer $0 \leqslant j \leqslant k$, we have

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)d\,{}_cL_p^{\mathrm{As}}(\Psi)(x) = \begin{cases} (*)L^{\mathrm{As}}(\Psi,\overline{\theta},j+1) & \text{if } (-1)^j\theta(-1) = 1, \\ 0 & \text{if } (-1)^j\theta(-1) = -1, \end{cases}$$

where $(*)$ is some non-zero explicit factor. See [LW20, Theorem 7.5] for more details.

Let $\mathcal{F}$ be a cuspidal Bianchi eigenform of weight $(k,k) \in \mathbb{Z}^2_{\geqslant 0}$ and level $\mathfrak{n}$, where $p \mid \mathfrak{n} \subset \mathcal{O}_F$. Assume that $\mathcal{F}$ has a small slope and is non-critical, i.e., the p-adic valuation of the U_p-eigenvalue is less than $k+1$. Let $\phi_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{k,k}(L))$ be the corresponding modular symbol, where $Y_{F,1}(\mathfrak{n})$ is a locally symmetric space, L is a number field, and $V_{k,k}(L)$ is $\mathrm{Sym}^k(L^2)\otimes\mathrm{Sym}^k(L^2)$. Suppose $\mathcal{F}$ is a base change of an elliptic modular form f of weight $k+2$. Let $\mathcal{C}$ denote the Coleman–Mazur eigencurve and BC denote the *base change map.* In [BSW21a], Barrera-Salazar–Williams constructed the base change eigenvariety $\mathcal{E}_{\mathrm{bc}} := \mathrm{BC}(\mathcal{C})$ and the parallel weight eigenvariety $\mathcal{E}_{\mathrm{par}}$, using overconvergent cohomology, such that, $\phi_{\mathcal{F}}$ (and hence $\mathcal{F}$) varies in a 1-dimensional family of overconvergent modular symbols over a curve in the parallel weight space $\mathcal{W}_{\mathrm{par}} \supset \mathcal{W}_{\mathrm{GL}_2(\mathbb{Q})}$. Using the eigenvariety $\mathcal{E}_{\mathrm{bc}} \subset \mathcal{E}_{\mathrm{par}}$, they also constructed a 3-variable p-adic L-function such that it retrieves the two-variable p-adic L-function constructed by Williams in [Wil17] at classical points. Note that their construction of families of p-adic L-functions works for all Bianchi modular forms.

1.2 Main results

Fix an odd prime p. Let $K/\mathbb{Q}_p$ be a p-adic field. For any abelian profinite group G, the space of K-valued locally analytic distributions on G is denoted by $\mathcal{H}_K(G)$. For $r \in \mathbb{R}_{\geqslant 0}$, let $\mathcal{H}_{K,r}(G) \subset \mathcal{H}_K(G)$ denote the space of distributions with growth $O(\log_p^r)$. If $G \cong \mathbb{Z}_p$, then after fixing a topological generator γ of G, one has the following identification:

$$\mathcal{H}_{K,r}(G) = \left\{ \sum_{n\geqslant 0} c_n(\gamma-1)^n : \sum_{n\geqslant 0} c_nT^n \in K[[T]] \text{ and } \sup_n \frac{|c_n|_p}{n^r} < \infty \right\}. \tag{1.2.1}$$

The Iwasawa algebra $\Lambda_K(\mathbb{Z}_p) := K \otimes \mathcal{O}_K[[\mathbb{Z}_p]] \cong K \otimes \mathcal{O}_K[[T]]$ can be identified with $\mathcal{H}_{K,0}(\mathbb{Z}_p)$.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \Gamma_1$, where $\Gamma_1 \cong \mathbb{Z}_p$. Fix a topological generator γ of Γ_1. Let χ be the p-adic cyclotomic character which maps γ to a topological generator of $1 + p\mathbb{Z}_p$. Let $M_{2,2}(R)$ denote the space of 2×2 matrices with coefficients in R.

Fix an imaginary quadratic field F and let $\mathcal{O}_F$ be its ring of integers.

In Chapter 2 and Appendix A, we have extended the construction of signed p-adic L-functions of Pollack, Sprung, and Lei–Loeffler–Zerbes from the setting of elliptic modular forms to the setting of Bianchi modular forms. Assume the fixed prime p splits in the quadratic imaginary F as $p\mathcal{O}_F = \mathfrak{p}\overline{\mathfrak{p}}$, and p does not divide the class number of F. Let $G_{(p)^n}$ denote the ray class group of F modulo p^n. Define $G_{p^\infty} := \varprojlim_n G_{(p)^n}$, the ray class group of F modulo p^∞, which decomposes as $\Delta_F \times \Gamma_{\mathfrak{p}} \times \Gamma_{\overline{\mathfrak{p}}} \cong \Delta_F \times \mathbb{Z}_p \times \mathbb{Z}_p$, where Δ_F is a finite abelian group. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $\varpi_{\mathfrak{q}}$ be a uniformizer of $\mathcal{O}_{F_{\mathfrak{q}}}$, and fix a p-adic valuation $v_p(\cdot)$ such that $v_p(\varpi_{\mathfrak{p}}) = 1$ and $v_p(\varpi_{\overline{\mathfrak{p}}}) = 0$. Fix some topological generators $\gamma_{\mathfrak{p}}$ and $\gamma_{\overline{\mathfrak{p}}}$ for $\Gamma_{\mathfrak{p}}$ and $\Gamma_{\overline{\mathfrak{p}}}$, respectively.

Let $\mathfrak{m} \subset \mathcal{O}_F$ be a nonzero ideal coprime to p, and let $\mathcal{F}$ be a Bianchi modular eigenform of weight $(k, \ell) \in \mathbb{Z}_{\geqslant 0}^2$, level $\mathfrak{m}$, and the central character $\epsilon_{\mathcal{F}}$. Assume that $\mathcal{F}$ vanishes at cusps 0 and ∞. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, suppose $a_{\mathfrak{q}}$ is the $T_{\mathfrak{q}}$-eigenvalue of $\mathcal{F}$, and let $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$ be the roots of Hecke polynomial $X^2 - a_{\mathfrak{q}}X + \epsilon_F(\varpi_{\mathfrak{q}})p$. Note that $\epsilon_{\mathcal{F}}(\varpi_{\mathfrak{p}}) = \varpi_{\mathfrak{p}}^k \cdot \varpi_{\overline{\mathfrak{p}}}^\ell$ and $\epsilon_{\mathcal{F}}(\varpi_{\overline{\mathfrak{p}}}) = \varpi_{\mathfrak{p}}^\ell \cdot \varpi_{\overline{\mathfrak{p}}}^k$, up to p-adic units. Assume $K/\mathbb{Q}_p$ is a finite extension large enough to contain F, and all Hecke eigenvalues of $\mathcal{F}$. Then, for $* \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, the p-stabilization $\mathcal{F}^{*,\dagger}$ is a C-cuspidal Bianchi modular form of weight (k, ℓ) and level $p\mathfrak{m}$. Due to the results of Palacios in [Pal25] (generalizing the results in [Wil17] from cuspidal to C-cuspidal), one can attach a two-variable p-adic distribution $L_{*,\dagger} := L_{p,\mathcal{F}^{*,\dagger}} \in \mathcal{H}_K(G_{p^\infty})$ to $\mathcal{F}^{*,\dagger}$. For simplicity, we will concentrate on the trivial isotypic component of $L_{*,\dagger}$, which we denote by the same symbol. These distributions satisfy:

1. **(Growth condition)** $L_{*,\dagger} \in \mathcal{H}_{K,v_p(*)}(\Gamma_{\mathfrak{p}}) \widehat{\bigotimes}_K \mathcal{H}_{K,v_p(\dagger)}(\Gamma_{\overline{\mathfrak{p}}}) \subset K[[\gamma_{\mathfrak{p}} - 1, \gamma_{\overline{\mathfrak{p}}} - 1]]$;
2. **(Interpolation property)** For any Hecke character Ξ with conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ and of infinity type (q, r) such that $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \in \mathbb{Z}_{\geqslant 1}$, $0 \leqslant q \leqslant k$, and $0 \leqslant r \leqslant \ell$, one has
$$L_{*,\dagger}(\widetilde{\Xi}) = \frac{1}{*^{n_{\mathfrak{p}}}} \frac{1}{\dagger^{n_{\overline{\mathfrak{p}}}}} \,(\text{some explicit factor}) L(\mathcal{F}, \Xi, 1),$$
where $\widetilde{\Xi}$ is a character on G_{p^∞} related to Ξ.

Assume $v_p(a_{\mathfrak{p}}) > \left\lfloor \dfrac{k}{p-2} \right\rfloor$, $v_p(a_{\overline{\mathfrak{p}}}) > \left\lfloor \dfrac{\ell}{p-2} \right\rfloor$, $\alpha_{\mathfrak{p}} \neq \beta_{\mathfrak{p}}$, and $\alpha_{\overline{\mathfrak{p}}} \neq \beta_{\overline{\mathfrak{p}}}$. Under these assumptions, we prove the following theorem about the decomposition of two-variable distributions:

Theorem 1.2.1 (Theorem 2.8.5)**.** There exist two variable power series with bounded coefficients, that is, there exist $L_{\sharp,\sharp}, L_{\sharp,\flat}, L_{\flat,\sharp}, L_{\flat,\flat} \in K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[\gamma_{\mathfrak{p}} - 1, \gamma_{\overline{\mathfrak{p}}} - 1]]$, such that
$$\begin{pmatrix} L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_{\overline{\mathfrak{p}}}^{-1} \underline{M_{\overline{\mathfrak{p}}}^{\ell}} \begin{pmatrix} L_{\sharp,\sharp} & L_{\flat,\sharp} \\ L_{\sharp,\flat} & L_{\flat,\flat} \end{pmatrix} (Q_{\mathfrak{p}}^{-1} \underline{M_{\mathfrak{p}}^{k}})^T,$$

where $\underline{M_{\mathfrak{p}}^{k}} \in M_{2,2}(\mathcal{H}_K(\Gamma_{\mathfrak{p}}))$ and $\underline{M_{\overline{\mathfrak{p}}}^{\ell}} \in M_{2,2}(\mathcal{H}_K(\Gamma_{\overline{\mathfrak{p}}}))$ are modified logarithmic matrices, and for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, $Q_{\mathfrak{q}}$ are certain invertible matrices.

Remark 1.2.2. Note that one can think of Theorem 1.2.1 as a machine to decompose any four 2-variable power series satisfying certain growth and interpolation properties into four 2-variable power series with bounded coefficients, that is, into elements of two-variable Iwasawa algebra $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[T_1, T_2]]$.

To prove Theorem 1.2.1, we adapt the strategy developed by Lei in [Lei14], first factorizing through the variable $\gamma_{\mathfrak{p}}$ and then through $\gamma_{\overline{\mathfrak{p}}}$. Loeffler in [Loe14] proved Theorem1.2.1 in the case where $a_{\mathfrak{p}} = a_{\overline{\mathfrak{p}}} = 0$ and $k = \ell = 0$. Lei in [Lei14] extended this result when $v_p(a_{\mathfrak{p}}), v_p(a_{\overline{\mathfrak{p}}}) > 0$ while assuming $k = \ell = 0$. This work generalizes both results by allowing $k, \ell \geqslant 0$, including the case $k \neq \ell$. Using Wach modules and p-adic Hodge theory, we construct logarithmic matrices $\underline{M_{\mathfrak{p}}^{k}}$ and $\underline{M_{\overline{\mathfrak{p}}}^{\ell}}$ for the decomposition in Theorem 1.2.1. More precisely, we use the explicit Wach module basis described by Berger–Li–Zhu in [BLZ04] to define these matrices. In [LLZ10], Lei–Loeffler–Zerbes constructed logarithmic matrices using Wach modules and crystalline representations related to modular forms. However, in this thesis, due to the use of [BLZ04], logarithmic matrices depend only on the algebraic data $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$, where $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, and are independent of p-adic Hodge theoretic properties of Galois representations associated with Bianchi modular forms, which are mostly conjectural. This allows one to look at logarithmic matrices as algebraic objects.

Furthermore, we prove a decomposition result in one variable. Let $k \in \mathbb{Z}_{\geqslant 2}$, $\alpha, \beta \in \mathcal{O}_K$ such that $\alpha \neq \beta, \alpha\beta = p^{k-1}$, and $v_p(\alpha + \beta) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$. We prove:

Theorem 1.2.3 (Theorem 2.5.5)**.** For $\lambda \in \{\alpha, \beta\}$, let $F_\lambda \in \mathcal{H}_{K,v_p(\lambda)}(\Gamma_1)$ be such that for any Dirichlet character ω of conductor $p^n > 1$ and for any integer $0 \leqslant j \leqslant k-2$, we have $F_\lambda(\chi^j\omega) = \lambda^{-n} \cdot c_{j,\omega}$, where $c_{j,\omega} \in \overline{\mathbb{Q}_p}$ independent of α and β. Then there exist power series $F_\sharp, F_\flat$ with bounded coefficients, i.e., $F_\sharp, F_\flat \in \Lambda_K(\Gamma_1) \cong K \otimes \mathcal{O}_K[[T]]$, and a logarithmic matrix $\underline{M} \in M_{2,2}(\mathcal{H}_K(\Gamma_1))$, such that

$$\begin{pmatrix} F_\alpha \\ F_\beta \end{pmatrix} = Q^{-1}\underline{M} \begin{pmatrix} F_\sharp \\ F_\flat \end{pmatrix}, \tag{1.2.2}$$

where Q is an invertible matrix depending only on α and β, which can be explicitly described.

Regarding the properties of $\underline{M}$, we prove:

Proposition 1.2.4. If $Q^{-1}\underline{M} = [P_{i,j}]_{1 \leqslant i,j \leqslant 2}$, then $P_{1,1}, P_{1,2} \in \mathcal{H}_{K,v_p(\alpha)}(\Gamma_1)$ and $P_{2,1}, P_{2,2} \in \mathcal{H}_{K,v_p(\beta)}(\Gamma_1)$. Moreover, $\det(\underline{M})$ is $O(\log_p^{k-1})$ and $\log_p^{k-1}$ is $O(\det(\underline{M}))$.

To prove Proposition 3.4.2, we use Perrin-Riou's big exponential map, along with the p-adic regulator map and Coleman maps developed by Lei–Loeffler–Zerbes in [LLZ11].

In Chapter 3, we construct p-adic Asai L-functions for p-non-ordinary small-slope cuspidal Bianchi modular forms. Let $\mathcal{F}$ be a cuspidal Bianchi eigenform of weight $(k, k) \in \mathbb{Z}_{\geqslant 0}^2$

and level $\mathfrak{N} \subset \mathcal{O}_F$. From (1.2.1), recall $\mathcal{H}_K(\Gamma) \cong K[\Delta] \otimes \mathcal{H}_K(\Gamma_1)$ to be the space of K-valued p-adic distributions on Γ, and $\mathcal{H}_{K,r}(\Gamma)$ be the space of distributions of with growth $O(\log_p^r)$. For simplicity, we will concentrate on the trivial isotypic component, that is, $\mathcal{H}_K(\Gamma_1)$ and $\mathcal{H}_{K,r}(\Gamma_1)$. Assume that all primes above p in F divide $\mathfrak{n}$. Denote the U_p-eigenvalue of $\mathcal{F}$ by a_p and assume that $\mathcal{F}$ is not ordinary at p and has a small slope, i.e., $0 < v_p(a_p) < k+1$. Let $K/\mathbb{Q}_p$ be a finite extension large enough to contain F and all Hecke eigenvalues of $\mathcal{F}$. We prove:

Theorem 1.2.5. For any integer c coprime to $6\mathfrak{n}$, there exists a $v_p(a_p)$-admissible distribution ${}_cL_p^{\mathrm{As}}(\mathcal{F})$ over Γ_1, i.e., ${}_cL_p^{\mathrm{As}}(\mathcal{F}) \in \mathcal{H}_{K,v_p(a_p)}(\Gamma_1)$, with the following interpolation property: given any integer $0 \leqslant j \leqslant k$, and any Dirichlet character θ of conductor $p^r > 1$, we have

$$
{}_cL_p^{\mathrm{As}}(\mathcal{F})(\chi^j\theta) = \begin{cases} \dfrac{*'}{a_p^r} L^{\mathrm{As}}(\mathcal{F}, \overline{\theta}, j+1) & \text{if } (-1)^j\theta(-1) = 1; \\ 0 & \text{if } (-1)^j\theta(-1) = -1, \end{cases}
$$

where $*'$ is an explicit non-zero factor which depends on c. Under some assumptions on the nebentypus of $\mathcal{F}$, the dependency on c can be removed.

Theorem 1.2.5 is a generalization of [LW20, Theorem 7.5] from the p-ordinary case to the p-non-ordinary case. The power series ${}_cL_p^{\mathrm{As}}(\mathcal{F})$ is constructed using certain polynomials. The idea of constructing polynomials to obtain power series is based on the works of Amice–Velu [AV75], Vishik [Vis76], Perrin-Riou [PR94], and Büyükboduk–Lei [BL21]. We will briefly explain the construction. Let Y be a locally symmetric space of level $\mathfrak{N}$; this is a Bianchi analogue of the modular curve. As Y is a real manifold of real dimension 3, there exists a pairing

$$
\mathrm{H}^1_{\mathrm{c}}(Y, V_{kk}(\mathcal{O}_K)) \times \mathrm{H}^2(Y, V_{kk}^\vee(\mathcal{O}_K)) \to \mathcal{O}_K,
$$

where $V_{kk}(\mathcal{O}_K)$ is a certain weight k coefficient module and $V_{kk}^\vee(\mathcal{O}_K)$ is its $\mathcal{O}_K$-dual. One can extend this pairing to

$$
\mathrm{H}^1_{\mathrm{c}}(Y, V_{kk}(\mathcal{O}_K)) \times \mathrm{H}^2(Y, V_{kk}^\vee(\mathcal{O}_K)) \otimes \mathcal{O}_K[T] \to \mathcal{O}_K[T].
$$

By pairing the appropriate modular symbol associated to $\mathcal{F}$ in $\mathrm{H}^1_{\mathrm{c}}(Y, V_{kk}(\mathcal{O}_K))$ with a suitable element (made up of Loeffler–Williams' *Asai-Eisenstein elements* with the U_p-operator acting on it) in $\mathrm{H}^2(Y, V_{kk}^\vee(\mathcal{O}_K)) \otimes \mathcal{O}_K[T]$, we construct polynomials $P_{r,j}(T) \in K[T]$ of degree $< p^{r-1}$, where $r \in \mathbb{Z}_{\geqslant 1}$ and $0 \leqslant j \leqslant k$ is an integer. See Section 3.6 for the details. They satisfy:

Lemma 1.2.6 (Lemma 3.6.4)**.** For any $r \in \mathbb{Z}_{\geqslant 1}$ and any integer $0 \leqslant j \leqslant k$, we have

1. $\sup \| p^{v_p(a_p)r} P_{r,j}(T) \| < \infty$,
2. $P_{r+1,j}(T) \equiv P_{r,j}(T) \mod ((1+T)^{p^{r-1}} - 1)$,
3. $\sup \left\| p^{(v_p(a_p)-j)r} \sum_{i=0}^{j} (-1)^i \binom{j}{i} P_{r,i}(u^{-i}(1+T)-1) \right\| < \infty$.

Moreover, there exists a polynomial $P_r \in K[T]$ of degree $< (k+1)p^{r-1}$ such that

$$P_r(T) \equiv P_{r,j}(u^{-j}(1+T)-1) \mod (u^{-j}(1+T)^{p^{r-1}}-1).$$

The properties of Loeffler–Williams' Asai-Eisenstein elements, as well as some ideas from [LZ16], are used to prove Lemma 1.2.6. Since $P_{r,j}$ satisfy properties described in Lemma 1.2.6, following [PR94] and [BL21], we define

$$ {}_cL_p^{\mathrm{As}}(\mathcal{F}) := \lim_{r\to\infty} P_r(\gamma-1) \in \mathcal{H}_{K,v_p(a_p)}(\Gamma_1).$$

Moreover, we prove a result on the decomposition of these elements, similar to the works of Pollack, Sprung, and Lei–Loeffler–Zerbes. Assume that p splits in F as $\mathfrak{p}\overline{\mathfrak{p}}$. Let $\mathcal{F}$ be a cuspidal Bianchi eigenform of weight (k,k), level $\mathfrak{m}$, and nebentypus $\epsilon_{\mathcal{F}}$, where $(\mathfrak{m}, p\mathcal{O}_F) = 1$. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $a_{\mathfrak{q}}$ be the $T_{\mathfrak{q}}$-eigenvalue. Assume $v_p(a_{\mathfrak{p}}) > \left\lfloor \frac{k}{p-2} \right\rfloor$ and $v_p(a_{\overline{\mathfrak{p}}}) = 0$. Let α be the root of $X^2 - a_{\overline{\mathfrak{p}}}X + \epsilon_{\mathcal{F}}(p)p^{k+1}$ which is a p-adic unit. Let $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}$ be the roots of $X^2 - a_{\mathfrak{p}}X + \epsilon_{\mathcal{F}}(p)p^{k+1}$ and assume that $\alpha_{\mathfrak{p}} \neq \beta_{\mathfrak{p}}$. Then, for $\bullet \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$, the p-stabilization $\mathcal{F}^{\bullet,\alpha}$ is a cuspidal Bianchi modular form of weight (k,k), level $p\mathfrak{m}$ and with small slope, and therefore one can associate $\mathcal{F}^{\bullet,\alpha}$ with a p-adic distribution ${}_cL_p^{\mathrm{As}}(\mathcal{F}^{\bullet,\alpha}) \in \mathcal{H}_{K,v_p(\bullet)}(\Gamma_1)$. Then, using the methods of Chapter 2, we prove:

Theorem 1.2.7 (Theorem 3.7.3)**.** There exist ${}_cL_p^{\mathrm{As},\flat}, {}_cL_p^{\mathrm{As},\sharp} \in K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[\gamma-1]]$ and a logarithmic matrix $\widetilde{M} \in M_{2,2}(\mathcal{H}_K(\Gamma_1))$ such that

$$\begin{pmatrix} {}_cL_p^{\mathrm{As}}(\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha}) \\ {}_cL_p^{\mathrm{As}}(\mathcal{F}^{\beta_{\mathfrak{p}},\alpha}) \end{pmatrix} = \widetilde{M} \begin{pmatrix} {}_cL_p^{\mathrm{As},\flat} \\ {}_cL_p^{\mathrm{As},\sharp} \end{pmatrix}.$$

In Chapter 4, we construct a two-variable p-adic Asai L-function associated with small slope non-critical cuspidal Bianchi modular forms of weight $(0,0)$. Let f be a cuspidal elliptic modular form of weight 2 and level N, and let $\mathcal{F}$ be its base change. Then, under some assumptions on f, $\mathcal{F}$ is cuspidal Bianchi modular form of weight $(0,0)$ and level $\mathfrak{N}$, where $(N) = \mathfrak{N} \cap \mathbb{Z}$. Assume $p \mid N$ and all prime above p in F divide $\mathfrak{N}$. Furthermore, assume $v_p(a_p) < 1$, where a_p is the U_p-eigenvalue of $\mathcal{F}$. Let $c \in \mathbb{Z}_{>0}$ coprime to $6N$. Let $V = \mathrm{Sp}(\mathcal{T}) \subset \mathcal{E}_{\mathrm{par}}$ be a family passing through $\mathcal{F}$ over $\mathcal{S} = \mathrm{Sp}(\mathfrak{L}) \subset \mathcal{W}_{F,\mathrm{par}}$. We construct:

Theorem 1.2.8 (Theorems 4.6.9, 4.6.15)**.** There exists a two-variable p-adic Asai L-function

$$ {}_c\mathcal{L}_V^{\mathrm{As}} \in \mathcal{T}\widehat{\otimes}\mathfrak{L}\widehat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma),$$

such that the specialization of this power series at the classical point $x_{\mathcal{F}}$ retrieves

$$\mathrm{sp}_{x_{\mathcal{F}}}^{\lambda}({}_c\mathcal{L}_V^{\mathrm{As}}) \doteq {}_cL_p^{\mathrm{As}}(\mathcal{F}) \in \mathcal{H}_{L,v_p(a_p)}(\Gamma).$$

In particular, there exists a point $x_{\mathcal{F}} \in V = \mathrm{Sp}(\mathcal{T})$ that corresponds to $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{N}))$. Let $\mathfrak{m}_{x_{\mathcal{F}}}$ be the maximal ideal corresponding to $\mathcal{F}$ and let $\mathfrak{m}_\lambda$ be the maximal ideal corresponding to $\lambda = (0,0) \in \mathcal{S}(L)$. We naively assume $\mathcal{T}/\mathfrak{m}_{\mathcal{F}} \cong L \cong \mathfrak{L}/\mathfrak{m}_\lambda$. The map $\mathrm{sp}^{\lambda}_{x_{\mathcal{F}}}$ is defined as

$$\mathrm{sp}^{\lambda}_{x_{\mathcal{F}}} : \mathcal{T}\hat{\otimes}\mathfrak{L}\hat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma) \to (T/\mathfrak{m}_{x_{\mathcal{F}}})\hat{\otimes}(\mathfrak{L}/\mathfrak{m}_\lambda)\hat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma) \cong \mathcal{H}_{L,v_p(a_p)}(\Gamma).$$

To construct ${}_{c}\mathcal{L}^{\mathrm{As}}_{V}$, we use the overconvergent cohomology piece $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant 1}$ and construct Asai–Eisenstein elements ${}_{c}\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at}$ over $\mathcal{S}$. Here $\mathcal{D}_{\mathcal{S}}$ is a $\mathcal{S}$-valued distributions on $\mathcal{O}_F \otimes \mathbb{Z}_p$ We pair a certain generator of $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant 1} \otimes \mathcal{T}$ with ${}_{c}\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at}$ to construct polynomials $P^{\mathcal{S}}_{r}$. Here $V = \mathrm{Sp}(\mathcal{T})$, naively, it interpolates the Hecke operators of $\mathcal{F}$. These polynomials satisfy certain norm-compatibility and growth properties. See Theorem 4.6.8 for more details. Since we are dealing with $\mathbb{Q}_p$-Banach spaces, by the methods of by Perrin–Riou in [PR94] and Büyükboduk–Lei in [BL21], we can take limit of $P^{\mathcal{S}}_{r}$ to construct ${}_{c}\mathcal{L}^{\mathrm{As}}_{V}$.

Chapter 2

Signed p-adic L-functions of Bianchi modular forms

2.1 Introduction

The study and construction of p-adic L-functions of arithmetic objects, like modular forms, is one of the central topics in modern number theory. The analytic p-adic L-functions are distributions on p-adic Lie groups like $\mathbb{Z}_p$. For example, let f be an elliptic modular eigenform of weight $k \geqslant 2$, level N, and character ϵ, and let p be a prime such that $p \nmid N$. Let α be a root of the Hecke polynomial $X^2 - a_pX + \epsilon(p)p^{k-1}$ such that $v_p(\alpha) < k-1$, where v_p is the normalized p-adic valuation such that $v_p(p) = 1$, and a_p is the T_p-eigenvalue of f. Then, due to the constructions of Amice-Velu and Vishik (see [AV75], [Vis76]) we can attach to f a p-adic distribution $L_p(f, \alpha)$ of order $v_p(\alpha)$ over $\mathbb{Z}_p^\times$. This $L_p(f, \alpha)$ interpolates the critical values of the complex L-function of f.

We continue with the example of p-adic L-functions of modular forms. When f is good ordinary at p, i.e. $v_p(a_p) = 0$, $L_p(f, \alpha)$ is a bounded measure and hence an element of the Iwasawa algebra $\Lambda_K(\Gamma) \cong K \otimes \mathcal{O}_K[\Delta][[\Gamma_1]]$, where K is some finite extension of $\mathbb{Q}_p$, $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \Delta \times \Gamma_1$, $\Delta \cong (\mathbb{Z}/p\mathbb{Z})^\times$, and $\Gamma_1 \cong \mathbb{Z}_p$. In this setting, the arithmetic is well understood and we have an Iwasawa main conjecture which relates this p-adic L-function with the characteristic ideal of the Selmer group of f (proved in many cases by Kato in [Kat04], Skinner-Urban in [SU14], Wan in [Wan20], etc).

When f is good non-ordinary at p, i.e., a_p is not a p-adic unit, $L_p(f, \alpha)$ is no longer a measure and hence not an element of the Iwasawa algebra. Moreover, it has unbounded denominators and it is an element of a larger algebra known as the distribution algebra $\mathcal{H}_K(\Gamma)$ (see section 2 for the definition). When $a_p = 0$, Pollack in [Pol03] has given a remedy. If α_1, α_2 are the roots of $X^2 + \epsilon(p)p^{k-1}$, Pollack showed that there exists a decomposition

$$L_p(f, \alpha_i) = \log_{p,k}^+ L_p^+ + \alpha_i \log_{p,k}^- L_p^-,$$

where $L_p^\pm \in \Lambda_K(\Gamma)$, for some finite extension K of $\mathbb{Q}_p$, and $\log_{p,k}^\pm$ are some power series in $\mathcal{H}_{\mathbb{Q}_p}(\Gamma_1)$ depending only on k. He also showed that if $k = 2$, then $L_p^\pm$ have integral coefficients, i.e. they lie in $\mathbb{Z}_p[\Delta][[\Gamma]]$. Later Sprung (for $k = 2$) in [Spr12] and Lei–Loeffler–Zerbes (for $k \geqslant 2$) in [LLZ10] have extended the work of Pollack when $a_p \neq 0$ using the method of logarithmic matrices.

Remark 2.1.1. On the algebraic side, we also have notions of signed Selmer groups due to Kobayashi (for $k = 2, a_p = 0$) in [Kob03], Sprung (for $k = 2, v_p(a_p) > 0$) in [Spr12], Lei (for $k \geqslant 2, a_p = 0$) in [Lei11], and Lei–Loeffler–Zerbes (for $k \geqslant 2, v_p(a_p) > 0$) in [LLZ10]. We also have signed Iwasawa main conjectures relating signed p-adic L-functions of non-ordinary modular forms with signed Selmer groups. See [LLZ10] for more details.

In this chapter, we extend the construction of signed p-adic L-functions (due to Pollack, Sprung, and Lei–Loeffler–Zerbes) to the setting of Bianchi modular forms using the two-variable p-adic L-functions constructed by Williams in [Wil17]. Bianchi modular forms are automorphic forms for GL(2) over quadratic imaginary fields. Let K be a quadratic imaginary field. Fix a prime number $p \geqslant 3$, which splits in K as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, and let $k \geqslant 0$ be an integer. Also, assume that p does not divide the class number of K. Let $\mathcal{G}$ be a cuspidal Bianchi eigenform over K of weight (k, k), level $\mathfrak{n}$ such that p divides $\mathfrak{n}$ and $v_p(a_\mathfrak{q}) < (k+1)$, where $a_\mathfrak{q}$ is the $U_\mathfrak{q}$ Hecke eigenvalue for all primes $\mathfrak{q}$ of K which lie above p. Then Williams has constructed a two-variable p-adic L-function $L_p(\mathcal{G})$ (see [Wil17, Theorem 7.4]) of a cuspidal Bianchi modular form $\mathcal{G}$ using overconvergent modular symbols. More precisely, $L_p(\mathcal{G})$ is a locally analytic distribution over the ray class group $\mathrm{Cl}(K, p^\infty) = G_{p^\infty} = \varprojlim_n G_{(p)^n}$, where $G_{(p)^n}$ is the ray class group of K modulo $(p)^n$.

We start with a Bianchi cuspform $\mathcal{F}$ of weight (k, k), level $\mathfrak{m}$ coprime to p, and $\mathcal{F}$ is good non-ordinary at both of the primes above p, that is, $v_p(a_\mathfrak{p}) > 0$ and $v_p(a_{\overline{\mathfrak{p}}}) > 0$, where $a_\mathfrak{p}$ and $a_{\overline{\mathfrak{p}}}$ are $T_\mathfrak{p}$ and $T_{\overline{\mathfrak{p}}}$ Hecke eigenvalues of $\mathcal{F}$ respectively. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, we assume $v_p(a_\mathfrak{q}) > \left\lfloor \frac{k}{p-1} \right\rfloor$. Moreover, let $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ be the roots of Hecke polynomial $X^2 - a_\mathfrak{q} X + p^{k+1}$ which we assume are distinct. Then we get four p-stabilizations of $\mathcal{F}$: $\mathcal{F}^{\alpha_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\alpha_\mathfrak{p}, \beta_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\beta_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}}$ and $\mathcal{F}^{\beta_\mathfrak{p}, \beta_{\overline{\mathfrak{p}}}}$, which are cuspidal Bianchi modular forms of the same weight as $\mathcal{F}$ and level $p\mathfrak{m}$. Thanks to Williams, we can attach a two-variable p-adic L-function to each of the p-stabilizations $L_{*,\dagger} := L_p(\mathcal{F}^{*,\dagger})$, for $* \in \{\alpha_\mathfrak{p}, \beta_\mathfrak{p}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$. The main theorem of this chapter is:

Theorem A (Theorem 2.8.5)**.** There exist two-variable power series with bounded coefficients, that is, there exist $L_{\sharp,\sharp}, L_{\sharp,\flat}, L_{\flat,\sharp}, L_{\flat,\flat} \in \Lambda_E(G_{p^\infty})$ such that

$$\begin{pmatrix} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} & L_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}} & L_{\beta_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_{\overline{\mathfrak{p}}}^{-1} \underline{M_{\overline{\mathfrak{p}}}} \begin{pmatrix} L_{\sharp,\sharp} & L_{\flat,\sharp} \\ L_{\sharp,\flat} & L_{\flat,\flat} \end{pmatrix} (Q_\mathfrak{p}^{-1} \underline{M_\mathfrak{p}})^T,$$

where $\underline{M_\mathfrak{p}}$ and $\underline{M_{\overline{\mathfrak{p}}}}$ are 2×2 logarithmic matrices, $Q_\mathfrak{q} = \begin{pmatrix} \alpha_\mathfrak{q} & -\beta_\mathfrak{q} \\ -p^{k+1} & p^{k+1} \end{pmatrix}$, and $\Lambda_E(G_{p^\infty}) \cong E[\Delta_K] \otimes_{\mathcal{O}_E} \mathcal{O}_E[[T_1, T_2]]$, where Δ_K is a finite abelian group such that $G_{p^\infty} \cong \Delta_K \times \mathbb{Z}_p^2$.

In the Appendix A, we have generalized Theorem A from parallel weight cuspidal Bianchi modular forms to non-parallel C-cuspidal Bianchi modular forms. See Theorem A.2.2.

Remark 2.1.2. In [Loe14], Loeffler proved a special case of Theorem A. He proved a Pollack style $\pm$-decomposition of two-variable unbounded p-adic L-functions attached to Bianchi modular forms of parallel weight 0 with $a_{\mathfrak{q}} = 0$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Even though the result was stated for Bianchi modular forms arising from the base change of weight 2 elliptic modular forms to a quadratic imaginary field, the method works for all Bianchi modular forms as long as $a_{\mathfrak{q}} = 0$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Pollack's distributions $\log_{\mathfrak{p}}^{\pm}, \log_{\overline{\mathfrak{p}}}^{\pm}$ over the ray class group G_{p^∞} for the decomposition were used in Loeffler's proof. See [Loe14, Section 5]. In this chapter, we generalize [Loe14, Corollary 2] to cuspidal Bianchi modular forms of parallel weight $k \geqslant 0$ and $v_p(a_{\mathfrak{q}}) > 0$. We also generalize [Loe14, Proposition 9], [Lei14, Proposition 2.3, Proposition 2.5] from $k = 0$ to $k \geqslant 0$. We use a different approach from [Loe14]. We construct and use Lei–Loeffler–Zerbes style 2×2 logarithmic matrices (which generalize Pollack's $\pm$-logarithms) $\underline{M_{\mathfrak{p}}}$ and $\underline{M_{\overline{\mathfrak{p}}}}$ to decompose p-adic L-functions with unbounded coefficients. To construct these logarithmic matrices, we use p-adic Hodge theoretic tools. For example, one of the key ingredients to construct $\underline{M}$ is the Wach module basis due to Berger-Li-Zhu (in [BLZ04]). The importance of Berger–Li–Zhu's construction is explained briefly in Section 2.1.2.

2.1.1 Logarithmic matrices

In this chapter, we construct logarithmic matrices in the sense of Sprung and Lei–Loeffler–Zerbes. Although we use p-adic Hodge theoretic tools to construct these matrices, one can think of them as purely algebraic elements. More generally, we construct $\underline{M} \in M_{2,2}(\mathcal{H}_E(\Gamma_1))$, where $\mathcal{H}_E(\Gamma_1)$ is the distribution algebra over E, and $M_{2,2}(R)$ is the space of 2×2 matrices with entries in R, having some growth properties. See Section 2.5 for the details.

We do not have much information about p-adic Hodge theoretic properties of the Galois representations of Bianchi modular forms. If the level of the Bianchi modular form is away from p, we expect the corresponding Galois representation to be crystalline at p. Only partial results are known due to Jorza, see [Jor12, Jor13].

We avoid or bypass the use of these conjectural properties of p-adic Galois representations associated with Bianchi modular forms by using Berger–Li–Zhu's construction. We explain it briefly here. Since we have assumed p splits as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in K, the ray class group G_{p^∞} can be decomposed as $\Delta_K \times \Gamma_{\mathfrak{p}} \times \Gamma_{\overline{\mathfrak{p}}}$, where Δ_K is some finite abelian group and $\Gamma_{\mathfrak{p}} \cong \Gamma_{\overline{\mathfrak{p}}} \cong \mathbb{Z}_p$. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, fix a topological generator $\gamma_{\mathfrak{q}}$ of $\Gamma_{\mathfrak{q}}$. For a cuspidal Bianchi modular form $\mathcal{F}$ of level $\mathfrak{m}$ away from p, weight (k, k) and trivial nebentypus, let $a_{\mathfrak{q}}$ be the $T_{\mathfrak{q}}$-eigenvalue of $\mathcal{F}$. We assume $v_p(a_{\mathfrak{q}}) > \left\lfloor \dfrac{k}{p-2} \right\rfloor$. Moreover, let $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$ be the distinct roots of Hecke polynomial $X^2 - a_{\mathfrak{p}}X + p^{k+1}$. Then we construct a logarithmic matrix $\underline{M(\mathfrak{q})} \in M_{2,2}(\Gamma_1)$, and then using *change of variables*, i.e. by changing γ_0 with $\gamma_{\mathfrak{q}}$, we construct another matrix $\underline{M_{\mathfrak{q}}} \in M_{2,2}(\mathcal{H}_E(\Gamma_{\mathfrak{q}}))$ with appropriate properties. See Section 2.8.2 for the construction of $\underline{M_{\mathfrak{q}}}$ and Section 2.6.3 for the details about the change of variable map. Most importantly, the construction of $\underline{M_{\mathfrak{q}}}$ *does not depend* on the p-adic Galois representation of $\mathcal{F}$, but it depends only on the roots of Hecke polynomial $X^2 - a_{\mathfrak{q}}X + p^{k+1}$ and the condition on $v_p(a_{\mathfrak{q}})$. Moreover, in Appendix A, we have constructed logarithmic matrices for

k and ℓ, where k need not be equal to ℓ.

2.1.2 Plan of the chapter

We start with the setup and notations in Section 2.2, which we require throughout the chapter.

In Section 2.3, we recall tools from p-adic Hodge theory and Wach modules. In general, we look at crystalline representations, families of Wach modules and the relation between them due to Berger-Li-Zhu [BLZ04].

In Section 2.4, we recall the exponential map constructed by Perrin-Riou and the p-adic regulator map along with Coleman maps, which were introduced by Lei–Loeffler–Zerbes in [LLZ10] and [LLZ11]. Moreover, we study the relationship between the exponential map and the p-adic regulator map. In this section, we also introduce the logarithm matrix $\underline{M}$, which is an element of $M_{2,2}(\mathcal{H}_E(\Gamma_1))$.

Section 2.5 deals with the factorization of power series in one variable. We first investigate $\underline{M}$ in more depth. Then we prove the following result using $\underline{M}$:

Theorem B (Theorem 2.5.5)**.** Let E be a finite extension of $\mathbb{Q}_p$. Let $\alpha, \beta \in \mathcal{O}_E$ and $k \geqslant 2$ be an integer such that $\alpha\beta = p^{k-1}$. Assume $\alpha \neq \beta$ and $v_p(\alpha+\beta) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$. For each $\lambda \in \{\alpha, \beta\}$, let $F_\lambda \in \mathcal{H}_{E,v_p(\lambda)}(\Gamma)$, such that for any integer $0 \leqslant j \leqslant k-2$, and for any Dirichlet character ω of conductor p^n, we have $F_\lambda(\chi^j\omega) = \lambda^{-n}C_{\omega,j}$, where χ is the p-adic cyclotomic character and $C_{\omega,j} \in \overline{\mathbb{Q}_p}$ that is independent of λ. Then there exist $F_\flat, F_\sharp \in \Lambda_E(\Gamma)$ such that

$$\begin{pmatrix} F_\alpha \\ F_\beta \end{pmatrix} = Q^{-1}\underline{M}\begin{pmatrix} F_\sharp \\ F_\flat \end{pmatrix}.$$

Note that $\underline{M}$ depends on $\alpha+\beta$ and k, and the matrix Q depends on α and β.

Remark 2.1.3. In [BL21, Section 2], the authors proved a similar result as above under the Fontaine-Laffaille condition ($p > k$). In this chapter, we are replacing this condition with a weaker condition $v_p(\alpha+\beta) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$. We also use different methods than the methods used in [BL21]. For example, we obtain properties of $\underline{M}$ using the p-adic regulator $\mathcal{L}_{T_W}$ and Perrin-Riou's exponential map Ω_{T_W}.

In Section 2.6, we develop the two-variable setup and recall definitions of ray class groups, Hecke characters, etc.

Bianchi modular forms and their p-adic L-functions are briefly recalled in Section 2.7.

In the last subsection, we prove the main theorem (Theorem A) of this chapter. We generalize and apply results of [Lei14] in our setting.

2.2 Setup and notations

Fix an odd prime p. Let E be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_E$. Let $\alpha, \beta \in \mathcal{O}_E$ such that $v_p(\alpha+\beta) > 0$, and there exists $v \in \mathcal{O}_E^\times$ and an integer $k \geqslant 2$ such that $\alpha\beta = vp^{k-1}$. We assume that $v^{1/2} \in \mathcal{O}_E^\times$. Denote $a = \alpha + \beta$. We denote $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ by $G_{\mathbb{Q}_p}$.

Assumption 2.2.1. $v_p(a) > m = \left\lfloor \dfrac{k-2}{p-1} \right\rfloor$ and $\alpha \neq \beta$.

We fix a, α, β, and v for the rest of the chapter.

Iwasawa algebras

Let $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$, where μ_{p^n} is the set of all p^n-th roots of unity. Let $\mathbb{Q}_{p,\infty} = \bigcup_{n \geqslant 1} \mathbb{Q}_{p,n}$. Then $\Gamma = \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \cong \Delta \times \mathbb{Z}_p$, where Δ is the torsion group of Γ of order $p-1$. Let Γ_1 be a subgroup of Γ such that $\Gamma_1 \cong \Gamma/\Delta \cong \mathbb{Z}_p$. In other words, Γ_1 is the Galois group of $\mathbb{Q}_{p,\infty}$ over $\mathbb{Q}_{p,1}$. We denote the Iwasawa algebra $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]] \cong \mathcal{O}_E[[\Gamma]]$ over $\mathcal{O}_E$ by $\Lambda_{\mathcal{O}_E}(\Gamma)$. Fix a topological generator γ_0 of Γ_1. Then we can identify $\mathcal{O}_E[[\Gamma_1]]$ with $\mathcal{O}_E[[X]]$ via identification $\gamma_0 \mapsto 1 + X$. This can be extended to $\Lambda_{\mathcal{O}_E}(\Gamma) \cong \mathcal{O}_E[\Delta][[X]]$. We further write $\Lambda_E(\Gamma_1) = E \otimes_{\mathcal{O}_E} \Lambda_{\mathcal{O}_E}(\Gamma_1)$ and $\Lambda_E(\Gamma) = E \otimes_{\mathcal{O}_E} \Lambda_{\mathcal{O}_E}(\Gamma)$. Fix a topological generator u of $1 + p\mathbb{Z}_p$ and let χ be the p-adic cyclotomic character on Γ such that $\chi(\gamma_0) = u$.

Power series rings

Given any power series $F \in E[[X]]$ and $0 < \rho < 1$, we define the sup norm $\|F\|_\rho = \sup_{|z|_p \leqslant \rho} |F(z)|_p$. For any real number $r \geqslant 0$, we define

$$\mathcal{H}_r = \{F \in E[[X]] : \sup\nolimits_{\mathrm{t}}(p^{-tr}\|F\|_{\rho_t}) < \infty\},$$

where $\rho_t = p^{-1/p^{t-1}(p-1)}$ and $t \geqslant 1$ is an integer. Equivalently, we have

$$\mathcal{H}_r = \left\{ F(X) = \sum_{n \geqslant 0} c_n X^n \in E[[X]] : \sup_n \frac{|c_n|_p}{n^r} < \infty \right\}.$$

If $F(X) \in \mathcal{H}_r$, then F is $O(\log_p^r)$, that is

$$\|F\|_\rho = O\left(\| \log_p^r(1+X)) \|_\rho\right)$$

as $\rho \to 1^-$. We write $\mathcal{H}_E = \bigcup_{r \geqslant 0} \mathcal{H}_r$.

We define $\mathcal{H}_{E,r}(\Gamma)$ to be the set of power series $\sum_{n \geqslant 0, \sigma \in \Delta} c_{n,\sigma} \cdot \sigma \cdot (\gamma_0 - 1)^n$, such that $\sum_{n \geqslant 0} c_{n,\sigma} X^n \in \mathcal{H}_r$ for all $\sigma \in \Delta$. In other words, the elements of $\mathcal{H}_{E,r}(\Gamma)$ are the power series in $\gamma_0 - 1$ over $E[\Delta]$ with the growth rate $O(\log_p^r)$. Write $\mathcal{H}_E(\Gamma) = \bigcup_{r \geqslant 0} \mathcal{H}_{E,r}(\Gamma)$. We call $\mathcal{H}_E(\Gamma)$ *the space of distributions on* Γ. We can identify $\mathcal{H}_E(\Gamma)$ with

$$\left\{ F(X) = \sum_{\substack{n \geqslant 0, \\ \sigma \in \Delta}} c_{n,\sigma} \cdot \sigma \cdot X^n \in E[\Delta][[X]] : \sum_{n \geqslant 0} c_{n,\sigma} X^n \text{ converges for all } X \in \mathbb{C}_p \text{ with } |X|_p < 1 \right\}$$

where X corresponds to $\gamma_0 - 1$.

Remark 2.2.2. In [Kat04] and [PR94], the spaces $\mathcal{H}_r$ are defined using $\lim_n n^{-r}|c_n|_p = 0$ instead of $\sup_n(n^{-r}|c_n|_p) < \infty$. For example, if $r = 0$, then by the notation in [Kat04], $\mathcal{H}_0 = E\langle X\rangle$, where $E\langle X\rangle$ is the one-variable Tate algebra. But, in our context, we have the following identification

$$\mathcal{H}_0 = E \otimes_{\mathcal{O}_E} \mathcal{O}_E[[T]].$$

We do not need the stronger notion of Tate algebras in this chapter. See also [Pol03, Lemmas 3.2, 5.2].

Fontaine's rings

Let π be a variable, $\mathbb{A}^+_{\mathbb{Q}_p} = \mathbb{Z}_p[[\pi]]$ and $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{A}^+_{\mathbb{Q}_p}[1/p]$. Let $\mathbb{A}_{\mathbb{Q}_p}$ be the ring of Laurent series $\sum_{i=-\infty}^{+\infty} a_i\pi^i$ such that $a_i \in \mathbb{Z}_p$ and $a_i \to 0$ as $i \to -\infty$. Write $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ for the ring of power series $f(\pi) \in \mathbb{Q}_p[[\pi]]$ such that $f(X)$ converges everywhere in the open unit p-adic disk. We equip $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ with actions of a Frobenius operator φ and Γ by $\varphi : \pi \mapsto (1+\pi)^p - 1$ and $\sigma : \pi \mapsto (1+\pi)^{\chi(\sigma)} - 1$ for all $\sigma \in \Gamma$. We then write $\mathbb{B}^+_{\mathrm{rig},E}$ for the power series ring $E \otimes \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$. We can define a left inverse ψ of φ such that

$$\varphi \circ \psi(f(\pi)) = \frac{1}{p} \sum_{\zeta^p=1} f(\zeta(1+\pi) - 1).$$

Inside $\mathbb{B}^+_{\mathrm{rig},E}$, we have subrings $\mathbb{A}^+_E = \mathcal{O}_E[[\pi]]$ and $\mathbb{B}^+_E = E \otimes \mathbb{A}^+_E$. The actions of φ, ψ, and Γ preserve these subrings. Write $t = \log(1+\pi) \in \mathbb{B}^+_{\mathrm{rig},E}$ and $q = \varphi(\pi)/\pi \in \mathbb{A}^+_E$. Note that $\varphi(t) = pt$ and $\sigma(t) = \chi(\sigma)t$ for all $\sigma \in \Gamma$, since $\log(1+\pi) = \pi \prod_{n\geqslant 1} \frac{\varphi^{n-1}(q)}{p}$.

Mellin transform

We have a $\Lambda_E(\Gamma)$-module isomorphism between $\mathcal{H}_E(\Gamma)$ and $(\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0}$ due to the action of Γ on $\mathbb{B}^+_{\mathrm{rig},E}$, called the Mellin transform. The isomorphism is given by

$$\begin{aligned} \mathfrak{M} : \mathcal{H}_E(\Gamma) &\to (\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0} \\ f(\gamma_0 - 1) &\mapsto f(\gamma_0 - 1) \cdot (1+\pi). \end{aligned}$$

Moreover, $\Lambda_{\mathcal{O}_E}(\Gamma)$ corresponds to $(\mathbb{A}^+_E)^{\psi=0}$ and $\Lambda_{\mathcal{O}_E}(\Gamma_1)$ corresponds to $(1+\pi)\varphi(\mathbb{A}^+_E)$ under $\mathfrak{M}$. Let $\mathcal{H}_E(\Gamma_1) = \{f(\gamma_0 - 1) : f \in \mathcal{H}_E\}$, then $\mathcal{H}_E(\Gamma_1)$ corresponds to $(1+\pi)\varphi(\mathbb{B}^+_{\mathrm{rig},E})$. See [PR01, Section B.2.8] for more details.

2.3 Crystalline representations and Wach modules

In this section, we recall definitions of crystalline representations and Wach modules. Furthermore, we recall the construction of families of Wach modules from [BLZ04]. The primary reference for this section is [BLZ04, Sections 1, 2, and 3].

2.3.1 Crystalline representations

Let $\mathbb{B}_{\mathrm{crys}}$ be Fontaine's crystalline period ring. Recall that we call a $\mathbb{Q}_p$-linear $G_{\mathbb{Q}_p}$-representation V a crystalline representation if V is $\mathbb{B}_{\mathrm{crys}}$-admissible. In other words, V is a crystalline representation if the dimension of the filtered φ-module $\mathbb{D}_{\mathrm{crys}}(V) = (\mathbb{B}_{\mathrm{crys}} \otimes V)^{G_{\mathbb{Q}_p}}$ is $\dim_{\mathbb{Q}_p} V$. For any integer j, we take $\mathbb{Q}_p(j) = \mathbb{Q}_p \cdot e_j$, where $G_{\mathbb{Q}_p}$ acts on e_j via χ^j. We know that $\mathbb{Q}_p(j)$ is a crystalline representation. Then for any crystalline representation V, the representation $V(j) = V(\chi^j) = V \otimes \mathbb{Q}_p(j)$ is again a crystalline representation. Moreover,we have $\mathbb{D}_{\mathrm{crys}}(V(j)) = t^{-j}\mathbb{D}_{\mathrm{crys}}(V) \otimes e_j$. We say a crystalline (or more generally a Hodge–Tate) representation V is positive if its Hodge–Tate weights are negative. Note that we are assuming the p-adic cyclotomic character has Hodge–Tate weight $+1$.

Let E be a finite extension of $\mathbb{Q}_p$. We say that an E-linear $G_{\mathbb{Q}_p}$-representation V is crystalline if and only if the underlying $\mathbb{Q}_p$-linear representation is crystalline. In this case, $\mathbb{D}_{\mathrm{crys}}(V)$ is an E-vector space with E-linear Frobenius and a filtration of E-vector spaces. More precisely, $\mathbb{D}_{\mathrm{crys}}(V)$ is an admissible E-linear filtered φ-module and the functor $V \mapsto \mathbb{D}_{\mathrm{crys}}(V)$ is an equivalence of categories from the category of crystalline E-linear representations to the category of admissible E-linear filtered φ-modules (see [CF00] for more details).

2.3.1.1 Crystalline representations as filtered φ-modules

Let $D_{k,v^{1/2}a}$ be a filtered φ-module given by $D_{k,v^{1/2}a} = Ee_1 \oplus Ee_2$ where:

$$\begin{cases} \varphi(e_1) = p^{k-1}e_2 \\ \varphi(e_2) = -e_1 + (v^{1/2}a)e_2 \end{cases} \quad \text{and} \quad \mathrm{Fil}^i D_{k,v^{1/2}a} = \begin{cases} D_{k,v^{1/2}a} \text{ if } i \leqslant 0 \\ Ee_1 \text{ if } 1 \leqslant i \leqslant k-1 \\ 0 \text{ if } i \geqslant k. \end{cases}$$

Take $e_1' = v^{1/2}e_1$ and $e_2' = e_2$. Thus, e_1', e_2' is another E-basis of $D_{k,v^{1/2}a}$. The matrix of φ with respect to basis e_1', e_2' is

$$\tilde{A}_\varphi = \begin{pmatrix} 0 & -v^{-1/2} \\ v^{1/2}p^{k-1} & v^{-1/2}a \end{pmatrix}.$$

Theorem 2.3.1 (Colmez-Fontaine[CF00], Berger-Li-Zhu[BLZ04])**.** There exists a crystalline E-linear representation $V_{k,v^{1/2}a}$, such that $\mathbb{D}_{\mathrm{crys}}(V^*_{k,v^{1/2}a}) = D_{k,v^{1/2}a}$, where $V^*_{k,v^{1/2}a} = \mathrm{Hom}(V_{k,v^{1/2}a}, E)$.

Proof. See [BLZ04, Section I and Proposition 3.2.4] ∎

From the above theorem, we get

$$\mathbb{D}_{\mathrm{crys}}(V^*_{k,v^{1/2}a}) = Ee_1 \oplus Ee_2 = Ee_1' \oplus Ee_2'.$$

The Hodge–Tate weights of $V_{k,v^{1/2}a}$ are 0 and $k-1$, and thus the Hodge–Tate weights of $V^*_{k,v^{1/2}a}$ are 0 and $1-k$. Let $W = V^*_{k,v^{1/2}a}(\chi^{k-1} \otimes \eta)$, where $\eta : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}_p}^\times$ is an unramified character such that $\eta(\mathrm{Frob}_p) = v^{1/2}$. Therefore, W is a crystalline representation with Hodge–Tate weights 0 and $k-1$.

Let $w_i = e_i' \otimes t^{-(k-1)} e_{k-1} \otimes e_\eta$, for $i = 1, 2$, where e_η is a basis of $\mathbb{Q}_p(\eta)$ and the action of φ on e_η is given by $\varphi(e_\eta) = \eta(\mathrm{Frob}_p^{-1}) e_\eta$. Then w_1, w_2 is a basis of $\mathbb{D}_{\mathrm{crys}}(W)$. The action of φ on w_i can be calculated as

$$\begin{cases} \varphi(w_1) = w_2 \\ \varphi(w_2) = (-1/vp^{k-1}) w_1 + (a/vp^{k-1}) w_2. \end{cases}$$

Thus, the matrix of φ with respect to basis w_1, w_2 is

$$A_\varphi = \begin{pmatrix} 0 & \dfrac{-1}{vp^{k-1}} \\ 1 & \dfrac{a}{vp^{k-1}} \end{pmatrix}.$$

2.3.2 Wach modules

An étale (φ, Γ)-module over $\mathbb{A}_{\mathbb{Q}_p}$ is a free $\mathbb{A}_{\mathbb{Q}_p}$- module M of finite rank, with semilinear action of φ and a continuous action of Γ commuting with each other, such that $\varphi(M)$ generates M as an $\mathbb{A}_{\mathbb{Q}_p}$-module. In [Fon90, A.3.4], Fontaine has constructed a functor $T \mapsto \mathbb{D}(T)$ which associates to every $\mathbb{Z}_p$-linear representation an étale (φ, Γ)-module over $\mathbb{A}_{\mathbb{Q}_p}$. The (φ, Γ)-module $\mathbb{D}(T)$ is defined as $(\mathbb{A} \otimes T)^{H_{\mathbb{Q}_p}}$, where $\mathbb{A}$ is the ring defined in [Fon90] and $H_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p,\infty})$. He also shows that the functor $T \mapsto \mathbb{D}(T)$ is an equivalence of categories. By inverting p, we also get an equivalence of categories between the category of $\mathbb{Q}_p$-linear $G_{\mathbb{Q}_p}$-representations and the category of étale (φ, Γ)-modules over $\mathbb{B}_{\mathbb{Q}_p} = \mathbb{A}_{\mathbb{Q}_p}[p^{-1}]$.

If E is a finite extension of $\mathbb{Q}_p$, we extend the Frobenius and the action of Γ to $E \otimes \mathbb{B}_{\mathbb{Q}_p}$ by E-linearity. We then get an equivalence of categories between $\mathcal{O}_E$-modules (or E-linear $G_{\mathbb{Q}_p}$-representations) and the category of (φ, Γ)-modules over $\mathcal{O}_E \otimes \mathbb{A}_{\mathbb{Q}_p}$ (or over $E \otimes \mathbb{B}_{\mathbb{Q}_p}$), given by $T \mapsto \mathbb{D}(T)$.

In [Ber04], Berger shows that if V is an E-linear $G_{\mathbb{Q}_p}$-representation, then V is crystalline with Hodge–Tate weights in $[a, b]$ if and only if there exists a unique $E \otimes \mathbb{B}^+_{\mathbb{Q}_p}$-module $\mathbb{N}(V) \subset \mathbb{D}(V)$ such that:

1. $\mathbb{N}(V)$ is free of rank $d = \dim_E(V)$ over $E \otimes \mathbb{B}^+_{\mathbb{Q}_p}$;
2. The action of Γ preserves $\mathbb{N}(V)$ and is trivial on $\mathbb{N}(V)/\pi\mathbb{N}(V)$;
3. $\varphi(\pi^b \mathbb{N}(V)) \subset \pi^b \mathbb{N}(V)$ and $\pi^b \mathbb{N}(V)/\varphi^*(\pi^b \mathbb{N}(V))$ is killed by q^{b-a}, where $q = \frac{\varphi(\pi)}{\pi}$ and $\varphi^*(\pi^b(\mathbb{N}(V)))$ is $\mathbb{B}^+_E$-submodule of $\mathbb{N}(V)[\pi^{-1}]$ generated by $\varphi(\pi^b \mathbb{N}(V))$.

Moreover, if V is crystalline and positive, then we can take $b = 0$. In this case, $\mathbb{N}(V)/\pi\mathbb{N}(V)$ is a filtered E-module and there exists an isomorphism $\mathbb{N}(V)/\pi\mathbb{N}(V) \cong \mathbb{D}_{\mathrm{crys}}(V)$ as filtered φ-modules. See [Ber04, Section III.4] for more details.

Let T be a $G_{\mathbb{Q}_p}$-stable lattice in V. Then $\mathbb{N}(T) := \mathbb{D}(T) \cap \mathbb{N}(V)$ is an $\mathcal{O}_E \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-lattice in $\mathbb{N}(V)$. By [Ber04], the functor $T \mapsto \mathbb{N}(T)$ gives a bijection between the $G_{\mathbb{Q}_p}$-stable lattices T in V and the $\mathcal{O}_E \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-lattices in $\mathbb{N}(V)$ which satisfy

1. $\mathbb{N}(T)$ is free of rank $d = \dim_E(V)$ over $\mathcal{O}_E \otimes \mathbb{A}^+_{\mathbb{Q}_p}$;
2. The action of Γ preserves $\mathbb{N}(T)$;
3. $\varphi(\pi^b\mathbb{N}(T)) \subset \pi^b\mathbb{N}(T)$ and $\pi^b\mathbb{N}(T)/\varphi^*(\pi^b\mathbb{N}(T))$ is killed by q^{b-a}, where $\varphi^*(\pi^b(\mathbb{N}(T)))$ is $\mathbb{A}^+_E$-submodule of $\mathbb{N}(T)[\pi^{-1}]$ generated by $\varphi(\pi^b\mathbb{N}(T))$.

The $E \otimes \mathbb{B}^+_{\mathbb{Q}_p}$-module $\mathbb{N}(V) \subset \mathbb{D}(V)$ as well as $\mathcal{O}_E \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-module $\mathbb{N}(T) \subset \mathbb{D}(T)$ are called *Wach modules*.

2.3.2.1 Families of Wach modules

In this section, we recall some results from [BLZ04].

Recall $q = \varphi(\pi)/\pi$. We define λ_+ and λ_- as

$$\lambda_+ = \prod_{n \geqslant 0} \frac{\varphi^{2n+1}(q)}{p}$$

and

$$\lambda_- = \prod_{n \geqslant 0} \frac{\varphi^{2n}(q)}{p}.$$

Lemma 2.3.2. Write $p^m(\lambda_-/\lambda_+)^{k-1} = \sum_{i \geqslant 0} z_i\pi^i$, where $m = \left\lfloor \frac{k-2}{p-1} \right\rfloor$ and define $z = z_0 + z_1\pi + \cdots + z_{k-2}\pi^{k-2}$. Then $z \in \mathbb{Z}_p[[\pi]]$.

Proof. See [BLZ04, Proposition 3.1.1]. ∎

Let Y be a variable. Define a matrix

$$P(Y) = \begin{pmatrix} 0 & -v^{-1/2} \\ v^{1/2}q^{k-1} & Yv^{-1/2}z \end{pmatrix}.$$

Then by [BLZ04, Proposition 3.1.3], for $\gamma \in \Gamma$, there exists a matrix $G_\gamma(Y) \in I_2 + \pi M_{2,2}(\mathbb{Z}_p[[\pi, Y]])$ such that

$$P(Y)\varphi(G_\gamma(Y)) = G_\gamma(Y)\gamma(P(Y)).$$

Note that φ and $\gamma \in \Gamma$ acts trivially on Y.

Lemma 2.3.3. For $\delta = \frac{a}{p^m}$ and $\gamma, \gamma' \in \Gamma$, we have $G_{\gamma\gamma'}(\delta) = G_\gamma(\delta)\gamma'(G_{\gamma'}(\delta))$ and $P(\delta)\varphi(G_\gamma(\delta)) = G_\gamma(\delta)\gamma(P(\delta))$. Therefore, one can use the matrices $P(\delta)$ and $G_\gamma(\delta)$ to define a Wach module $\mathbb{N}_k(\delta)$ over $\mathcal{O}_E[[\pi]]$.

Proof. Define the free $\mathcal{O}_E[[\pi]]$-module of rank 2 with basis n_1, n_2 as: $\mathbb{N}_k(\delta) = \mathcal{O}_E[[\pi]]n_1 \oplus \mathcal{O}_E[[\pi]]n_2$. Endow it with Frobenius φ and an action of $\gamma \in \Gamma$ such that the matrix of φ with respect to the basis n_1, n_2 is $P(\delta) = \begin{pmatrix} 0 & -v^{-1/2} \\ v^{1/2}q^{k-1} & v^{-1/2}\delta \cdot z \end{pmatrix}$ and the matrix of γ is $G_\gamma(\delta)$. See [BLZ04, Proposition 3.2.1] for details. ∎

The above lemma implies $E \otimes \mathbb{N}_k(\delta) = \mathbb{N}(V^*_{k,v^{1/2}a})$, where $\delta = a/p^m$, and $V_{k,v^{1/2}a}$ is the crystalline E-linear representation in Theorem 2.3.1.Here $\mathbb{N}(V^*_{k,v^{1/2}a})$ is the Wach module associated to the crystalline representation $V^*_{k,v^{1/2}a}$. More precisely:

Theorem 2.3.4. The filtered φ-module $E \otimes_{\mathcal{O}_E} (\mathbb{N}_k(\delta)/\pi\mathbb{N}_k(\delta))$ is isomorphic to the φ-module $D_{k,v^{1/2}a}$ defined in the Theorem 2.3.1.

Proof. This can be proved using [BLZ04, Proposition 3.2.4]. ∎

We adapt the above machinery in our setting. Recall that $W = V^*_{k,v^{1/2}a}(\chi^{k-1} \otimes \eta)$. Denote by T_W the $\mathcal{O}_E$-lattice $T(\chi^{k-1} \otimes \eta)$ in W, where T is an $\mathcal{O}_E$-lattice in $V_{k,v^{1/2}a}$ such that $\mathbb{N}_k(\delta) = \mathbb{N}(T^*)$. By an abuse of notation, we write $\mathbb{N}_k(\delta) = \mathbb{N}(T_W) = \mathcal{O}_E[[\pi]]n'_1 \oplus \mathcal{O}_E[[\pi]]n'_2$, where n'_1, n'_2 is a basis after twisting the basis n_1, n_2 of $\mathbb{N}(T)$ with $\chi^{k-1} \otimes \eta$. Then the matrix of φ with respect to $\{n'_1, n'_2\}$ is

$$P = \begin{pmatrix} 0 & -1/vq^{k-1} \\ 1 & \delta \cdot z/vq^{k-1} \end{pmatrix}.$$

Note that $P \equiv A_\varphi \bmod \pi$, since $q \equiv p \bmod \pi$ and $\delta \cdot z \equiv a \bmod \pi$. We fix this $\mathcal{O}_E[[\pi]]$-basis n'_1, n'_2 for $\mathbb{N}(T_W)$ for the rest of the chapter.

2.4 Perrin-Riou's Exponential map, Coleman maps, and the p-adic regulator

We recall definitions of Perrin-Rious's exponential map, the p-adic regulator, and Coleman maps. We also explicitly study these maps and the relationship between them after fixing some basis. The primary reference for this section is [LLZ11].

2.4.1 Iwasawa cohomology and Wach modules

Let V be any crystalline E-linear representation of $G_{\mathbb{Q}_p}$ and let T be an $\mathcal{O}_E$-lattice inside V. The Iwasawa cohomology group $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T)$ is defined by

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T) = \varprojlim H^1(\mathbb{Q}_{p,n}, T),$$

where the inverse limit is taken with respect to the corestriction maps. Then, due to Fontaine (see [CC99, Section II.1]), there exists a canonical $\Lambda_{\mathcal{O}_E}(\Gamma)$-module isomorphism

$$h^1_{\mathrm{Iw}}\colon \mathbb{D}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T),$$

where $\mathbb{D}(T)$ is the (φ, Γ)-module associated to T.

From now on, we consider the p-adic representation $W = V^*_{k,v^{1/2}a}(\chi^{k-1} \otimes \eta)$ and the $\mathcal{O}_E$-lattice T_W in W studied in Section 2.3.2.1 unless mentioned otherwise. Moreover, let $\mathbb{D}_{\mathrm{crys}}(T_W)$ be the image of $\mathbb{N}(T_W)/\pi\mathbb{N}(T_W)$ in $\mathbb{D}_{\mathrm{crys}}(W)$. Then

1. $\mathbb{D}_{\mathrm{crys}}(T_W)$ is filtered φ-module over $\mathcal{O}_E$,
2. $\mathbb{D}_{\mathrm{crys}}(T_W) = \mathcal{O}_E \cdot w_1 \oplus \mathcal{O}_E \cdot w_2$,
3. the matrix of φ with respect to the basis w_1, w_2 is A_φ.

For the representation W, the eigenvalues of the φ are α^{-1}, β^{-1}. From now on we assume that α^{-1} and β^{-1} are not integral powers of the prime p. Since the Hodge–Tate weights of W are 0 and $k-1$, we have the following theorem due to Berger:

Theorem 2.4.1 (Berger, [Ber03, Theorem A.3])**.** For the $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice T_W in W, there exists a $\Lambda_{\mathcal{O}_E}(\Gamma)$-module isomorphism

$$h^1_{\mathrm{Iw},T_W}\colon \mathbb{N}(T_W)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W).$$

Moreover, we can extend this isomorphism from $\Lambda_{\mathcal{O}_E}(\Gamma)$-modules to $\Lambda_E(\Gamma)$-modules

$$h^1_{\mathrm{Iw},W}\colon \mathbb{N}(W)^{\psi=1} \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W),$$

where $\mathbb{N}(W) = E \otimes \mathbb{N}(T_W)$ and $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W) = E \otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T)$.

Remark 2.4.2. For a given p-adic E-linear $G_{\mathbb{Q}_p}$-representation V, the torsion part of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ is given by $V^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\mu_{p^\infty}))}$. But, for our representation W, by [BLZ04, Theorem 4.1.1], $W^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\mu_{p^\infty}))} = 0$, and hence the Iwasawa cohomology $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W)$ is a free $\Lambda_E(\Gamma)$-module of rank 2.

2.4.2 Coleman maps

For the p-adic representation W and the $\mathcal{O}_E$-lattice T_W in W, we deduce $\mathbb{N}(T_W) \subset \varphi^*(\mathbb{N}(T_W))$, since the Hodge–Tate weights of W are non-negative. Here $\varphi^*(\mathbb{N}(T_W))$ is $\mathbb{A}^+_E$-submodule of $\mathbb{N}(T_W)[\pi^{-1}]$ generated by $\varphi(\mathbb{N}(T_W))$ (See [LLZ11, Lemma 1.7]). Hence there exists a well-defined map $1-\varphi\colon \mathbb{N}(T_W) \to \varphi^*(\mathbb{N}(T_W))$ which maps $\mathbb{N}(T_W)^{\psi=1}$ to $(\varphi^*\mathbb{N}(T_W))^{\psi=0}$.

Theorem 2.4.3 (Lei–Loeffler–Zerbes, Berger)**.** The $\Lambda_{\mathcal{O}_E}(\Gamma)$-module $(\varphi^*\mathbb{N}(T_W))^{\psi=0}$ is free of rank 2. Moreover, for any basis v_1, v_2 of $\mathbb{D}_{\mathrm{crys}}(T_W)$, there exists an $\mathcal{O}_E \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-basis n_1, n_2 of $\mathbb{N}(T_W)$ such that $n_i \equiv v_i \bmod \pi$ and $(1+\pi)\varphi(n_1), (1+\pi)\varphi(n_2)$ form a $\Lambda_{\mathcal{O}_E}(\Gamma)$-basis of $(\varphi^*\mathbb{N}(T_W))^{\psi=0}$.

Proof. See[LLZ10, Lemma 3.15] for the proof for any crystalline representation of dimension d. ∎

The above theorem gives an isomorphism of $\Lambda_{\mathcal{O}_E}(\Gamma)$-modules

$$\mathfrak{J}\colon (\varphi^*\mathbb{N}(T_W))^{\psi=0} \to \Lambda_{\mathcal{O}_E}(\Gamma)^{\oplus 2}.$$

Definition 2.4.4 (The Coleman map)**.** We define the Coleman map

$$\underline{\mathrm{Col}} = (\underline{\mathrm{Col}}_i)_{i=1}^2 \colon \mathbb{N}(T_W)^{\psi=1} \to \Lambda_{\mathcal{O}_E}(\Gamma)^{\oplus 2}$$

as the composition $\mathfrak{J} \circ (1-\varphi)$.

This Coleman map $\underline{\mathrm{Col}}$ can be extended to a map from $\mathbb{N}(W)$ to get a $\Lambda_E(\Gamma)$-module homomorphism

$$\underline{\mathrm{Col}}\colon \mathbb{N}(W)^{\psi=1} \to \Lambda_E(\Gamma)^{\oplus 2}.$$

From the above discussion, for the fixed basis n_1', n_2' for $\mathbb{N}(T_W)$ and basis w_1, w_2 for $\mathbb{D}_{\mathrm{crys}}(T_W)$, we get a matrix $\underline{M}$ as follows: The elements $(1+\pi)(\varphi(n_1')), (1+\pi)(\varphi(n_2'))$ form a $\Lambda_{\mathcal{O}_E}(\Gamma)$-basis of $(\varphi^*\mathbb{N}(T_W))^{\psi=0}$. Furthermore the elements $(1+\pi)\otimes w_1, (1+\pi)\otimes w_2$ form a basis of $(\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0} \otimes \mathbb{D}_{\mathrm{crys}}(T_W)$ as a $\mathcal{H}_E(\Gamma)$-module. Since $n_i' \equiv w_i \mod \pi$ for $i = 1, 2$, there exists a unique 2×2 matrix $\underline{M} \in M_{2,2}(\mathcal{H}_E(\Gamma))$ such that

$$\big[(1+\pi)\varphi(n_1') \quad (1+\pi)\varphi(n_2')\big] = \big[(1+\pi)\otimes w_1 \quad (1+\pi)\otimes w_2\big]\,\underline{M}. \tag{2.4.1}$$

That is, $\underline{M}$ is a change of basis matrix for the following homomorphism of $\mathcal{H}_E(\Gamma)$-modules:

$$(\varphi^*\mathbb{N}(T_W))^{\psi=0} \hookrightarrow (\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0} \otimes \mathbb{D}_{\mathrm{crys}}(T_W).$$

Remark 2.4.5. More precisely, $\underline{M} \in M_{2,2}(\mathcal{H}(\Gamma_1))$, since n_i' lie in $(1+\pi)\varphi(\mathbb{N}(T_W)) \subset (1+\pi)\varphi(\mathbb{B}^+_{\mathrm{rig},E}) \otimes \mathbb{D}_{\mathrm{crys}}(T_W)$.

This matrix $\underline{M}$ will play a crucial role in the upcoming sections. More precisely, we will show that $\underline{M}$ is a logarithmic matrix (in the sense of Sprung and Lei–Loeffler–Zerbes) that can be used in the decomposition of power series with unbounded denominators.

2.4.3 Perrin-Riou's Exponental map, the p-adic regulator and the relation between them

Recall that the eigenvalues of φ are not integral powers of p. Under this assumption, we can construct the *Perrin-Riou's exponentia map* as (see [PR94, Section 3.2.3], [Ber03, Section II.5] for the details)

$$\Omega_{T_W,k-1}\colon \mathcal{O}_E \otimes (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{crys}}(T_W) \to \mathcal{H}_E(\Gamma)\otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W).$$

Note that this map is a $\Lambda_{\mathcal{O}_E}(\Gamma)$-module homomorphism. We can extend this to a $\Lambda_E(\Gamma)$-module homomorphism

$$\Omega_{W,k-1} \colon E \otimes (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{crys}}(W) \to \mathcal{H}_E(\Gamma) \otimes \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W).$$

This map interpolates the Bloch–Kato exponential map

$$\exp_{n,j} \colon \mathbb{Q}_{p,n} \otimes \mathbb{D}_{\mathrm{crys}}(W(j)) \to H^1(\mathbb{Q}_{p,n}, W(j)),$$

where j is any integer.

Recall we have fixed a topological generator u of $1 + p\mathbb{Z}_p$ in Section 2.2. For $i \in \mathbb{Z}$, define $\ell_i = \dfrac{\log_p(1+X)}{\log_p(u)} - i$. It is easy to see that $\ell_i \in \mathcal{H}_E(\Gamma)$ for all integers i. Berger gave a description of $\Omega_{W,k-1}$ in the terms of ℓ_i's (see [Ber03, Section II.1, Theorem II.13] for more details) as follows:

$$\Omega_{W,k-1}(z) = (\ell_{k-2} \circ \cdots \circ \ell_0)(1-\varphi)^{-1}(\bar{z}), \tag{2.4.2}$$

where $z \in \mathcal{H}_E(\Gamma) \otimes \mathbb{D}_{\mathrm{crys}}(W)$ and $\bar{z} = (\mathfrak{M} \otimes 1)(z)$.

Definition 2.4.6 (The p-adic regulator)**.** The Perrin-Riou p-adic regulator map $\mathcal{L}_{T_W}$ for the $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice T_W in W is a $\Lambda_{\mathcal{O}_E}(\Gamma)$-homomorphism defined as

$$\mathcal{L}_{T_W} := (\mathfrak{M}^{-1} \otimes 1) \circ (1-\varphi) \circ (h^1_{\mathrm{Iw},T_W})^{-1} \colon H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W) \to \mathcal{H}_E(\Gamma) \otimes \mathbb{D}_{\mathrm{crys}}(T_W).$$

The p-adic regulator $\mathcal{L}_{T_W}$ can be extended to a $\Lambda_E(\Gamma)$-homomorphism as

$$\mathcal{L}_W \colon \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W) \to \mathcal{H}_E(\Gamma) \otimes \mathbb{D}_{\mathrm{crys}}(W).$$

The p-adic regulator map $\mathcal{L}_W$ and the exponential map $\Omega_{W,k-1}$ are related by the following lemma:

Lemma 2.4.7. As maps on $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W)$, we have

$$\mathcal{L}_W = (\mathfrak{M}^{-1} \otimes 1)(\prod_{i=0}^{k-2} \ell_i)(\Omega_{W,k-1})^{-1}.$$

In other words,

$$\Omega_{W,k-1}(\mathcal{L}_W(z)) = (\prod_{i=0}^{k-2} \ell_i)(z), \tag{2.4.3}$$

for all $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, W)$.

Proof. See [LLZ11, Theorem 4.6]. ∎

The following lemma gives a relationship between $\mathcal{L}_{T_W}$ and the Coleman maps.

Lemma 2.4.8 (Lei–Loeffler–Zerbes [LLZ11, Lemma 3.3])**.** For $z \in \mathbb{N}(T_W)^{\psi=1}$, we have

$$(1-\varphi)(z) = \begin{bmatrix}(1+\pi)\otimes w_1 & (1+\pi)\otimes w_2\end{bmatrix} \underline{M}\, \underline{\mathrm{Col}}(z).$$

Thus, we can rewrite $\mathcal{L}_{T_W}$ in terms of $(1+\pi)\otimes w_1, (1+\pi)\otimes w_2$ as

$$\mathcal{L}_{T_W}(z) = \begin{bmatrix}(1+\pi)\otimes w_1 & (1+\pi)\otimes w_2\end{bmatrix} \underline{M} \left(\underline{\mathrm{Col}} \circ (h^1_{\mathrm{Iw},T_W})^{-1}\right)(z),$$

where $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$.

Proof. This can be proved using.

$$(1-\varphi)(z) = \begin{bmatrix}(1+\pi)\varphi(n_1) & (1+\pi)\varphi(n_2)\end{bmatrix} \cdot \underline{\mathrm{Col}}(z),$$

and the definitions of $\underline{M}$ and $\mathcal{L}_{T_W}$. ∎

Now for $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$, $\mathcal{L}_{T_W}(z)$ is an element of $\mathcal{H}_E(\Gamma)\otimes \mathbb{D}_{\mathrm{crys}}(T_W)$. Hence, we can apply any character of Γ to $\mathcal{L}_{T_W}(z)$ to get an element in $E\otimes \mathbb{D}_{\mathrm{crys}}(T_W)$.

Proposition 2.4.9. Let $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$. Then for any integer $0 \leqslant i \leqslant k-2$, and for any Dirichlet character ω of conductor $p^n > 1$, we have

$$(1-\varphi)^{-1}(1-p^{-1}\varphi^{-1})\chi^i(\mathcal{L}_{T_W}(z)\otimes t^i e_{-i}) \in \mathrm{Fil}^0(\mathbb{D}_{\mathrm{crys}}(T_W(-i))), \tag{2.4.4}$$

$$\varphi^{-n}(\chi^i\omega(\mathcal{L}_{T_W}(z)\otimes t^i e_{-i})) \in \mathbb{Q}_{p,n}\otimes \mathrm{Fil}^0\mathbb{D}_{\mathrm{crys}}(T_W(-i)), \tag{2.4.5}$$

where χ is the p-adic cyclotomic character and e_{-i} is the basis of $\mathbb{D}_{\mathrm{crys}}(\mathbb{Z}_p(-i))$.

Proof. We replace V with T_W in [LLZ11, Proposition 4.8] and the result follows. ∎

Lemma 2.4.10. We have

$$\det(\mathcal{L}_W) = \prod_{i=0}^{k-2} \ell_i.$$

Proof. Since the Hodge–Tate weights of W are 0 and $k-1$, we have

$$\dim_E(\mathrm{Fil}^i\mathbb{D}_{\mathrm{crys}}(W)) = 1,$$

for all $-(k-2) \leqslant i \leqslant 0$. Thus, replacing V with W, putting $d=2$ and $n_i = 1$ for all integers $0 \leqslant i \leqslant (k-2)$ in [LLZ11, Corollary 4.7], we get

$$\det(\mathcal{L}_W) = \prod_{i=0}^{k-2}(\ell_i)^{2-1}.$$

∎

2.4.4 The matrices of $\Omega_{W,k-2}, \mathcal{L}_W$ and their connection with $\underline{M}$

For the rest of the chapter, fix an eigenbasis $\mathcal{B}_{\text{eig}} = \{w_\alpha, w_\beta\}$ of φ for $\mathbb{D}_{\text{crys}}(T_W)$, that is, $\mathcal{B}_{\text{eig}}$ is a basis for $\mathbb{D}_{\text{crys}}(T_W)$ and $\varphi(w_\alpha) = (\alpha)^{-1}w_\alpha$ and $\varphi(w_\beta) = (\beta)^{-1}w_\beta$. Thus, $(1+\pi)\otimes w_\alpha, (1+\pi)\otimes w_\beta$ is a basis for $(\mathbb{B}^+_{\text{rig},E})^{\psi=0}\otimes\mathbb{D}_{\text{crys}}(T_W)$. We denote the basis $\{w_1, w_2\}$ for $\mathbb{D}_{\text{crys}}(T_W)$ by $\mathcal{B}$, which we have defined in subsection 3.1.

Recall that the matrix of φ with respect to $\mathcal{B}$ is

$$A_\varphi = \begin{pmatrix} 0 & -1/vp^{k-1} \\ 1 & a/vp^{k-1} \end{pmatrix},$$

and thus we get

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix} = Q^{-1}A_\varphi Q,$$

where $Q = \begin{pmatrix} \alpha & -\beta \\ -vp^{k-1} & vp^{k-1} \end{pmatrix}$.

We know that the $\Lambda_{\mathcal{O}_E}(\Gamma)$-rank of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ is 2, since $\dim_E \mathbb{D}_{\text{crys}}(W) = 2$ and T_W is an $\mathcal{O}_E$-lattice in W. See [PR94, Section 3.2] and [Ber03, Proposition 2.7] for precise details. Thus, we may fix a $\Lambda_{\mathcal{O}_E}(\Gamma)$-basis $\{z_1, z_2\}$ for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$.

2.4.4.1 The matrix of $\Omega_{T_W,k-1}$

Using the expoential map $\Omega_{T_W,k-1}$, we obtain the following equations

$$\Omega_{W,k-1}((1+\pi)\otimes w_\alpha) = a\cdot z_1 + c\cdot z_2,$$
$$\Omega_{W,k-1}((1+\pi)\otimes w_\beta) = b\cdot z_1 + d\cdot z_2,$$

where a, b, c, d are elements in the distribution ring $\mathcal{H}_E(\Gamma)$. In other words, we can write these equations as

$$\begin{bmatrix}\Omega_{W,k-1}((1+\pi)\otimes w_\alpha) & \Omega_{W,k-1}((1+\pi)\otimes w_\beta)\end{bmatrix} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore, with respect to the basis $\{z_1, z_2\}$ for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ and the basis $\mathcal{B}_{\text{eig}}$ for $\mathbb{D}_{\text{crys}}(T_W)$ we can describe the matrix M_Ω of $\Omega_{T_W,k-1}$ as,

$$M_\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4.6}$$

Recall that, if $F \in \mathcal{H}_{E,r}(\Gamma)$, we say F is $O(\log_p^r)$.

Lemma 2.4.11. The elements a, c are $O(\log_p^{v_p(\beta)})$, whereas b, d are $O(\log_p^{v_p(\alpha)})$.

Proof. From [PR94, Section 3.2.4], we note that for any $\mathcal{O}_E$-lattice T in a crystalline representation V, if f is an element of $(\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0} \otimes \mathbb{D}_\nu(T(j))$, where $\mathbb{D}_\nu(T(j))$ is a subspace of $\mathbb{D}_{\mathrm{crys}}(T(j))$ in which φ has slope ν, then $\Omega_{T(j),h+j}(f)$ is $O(\log_p^{h+\nu})$ and $h \in \mathbb{Z}_{>0}$ such that $\mathrm{Fil}^{-h}(\mathbb{D}_{\mathrm{crys}}(T)) = \mathbb{D}_{\mathrm{crys}}(T)$. In other words, $\Omega_{T(j),h+j}(f)$ lies in $\mathcal{H}_{h+\nu}(\Gamma) \otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T(j))$.

For the crystalline representation W and the lattice T_W in W, we know that $\varphi(w_\alpha) = \alpha^{-1} w_\alpha$ and $\varphi(w_\beta) = \beta^{-1} w_\beta$. Thus, $\Omega_{T_W,k-1}((1+\pi) \otimes w_\alpha)$ is $O(\log_p^{(k-1)-v_p(\alpha)}) = O(\log_p^{v_p(\beta)})$, since $v_p(\alpha) + v_p(\beta) = k-1$. Similarly, $\Omega_{T_W,k-1}((1+\pi) \otimes w_\beta)$ is $O(\log_p^{v_p(\alpha)})$.

But $\Omega_{T_W,k-1}((1+\pi) \otimes w_\alpha) = a \cdot z_1 + c \cdot z_2 \in \mathcal{H}_{E,v_p(\beta)}(\Gamma) \otimes \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$. Therefore we conclude that a and c have growth $O(\log_p^{v_p(\beta)})$. In the same manner, b and d have growth $O(\log_p^{v_p(\alpha)})$. ∎

2.4.4.2 The matrix of the p-adic regulator $\mathcal{L}_{T_W}$

After applying the p-adic regulator $\mathcal{L}_{T_W}$ on the $\Lambda_{\mathcal{O}_E}(\Gamma)$-basis z_1, z_2 of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$, we get

$$\begin{aligned} \mathcal{L}_{T_W}(z_1) &= x_1 \cdot w_\alpha + x_3 \cdot w_\beta, \\ \mathcal{L}_{T_W}(z_1) &= x_2 \cdot w_\alpha + x_4 \cdot w_\beta. \end{aligned}$$

We can rewrite these equations as

$$\begin{bmatrix} \mathcal{L}_{T_W}(z_1) & \mathcal{L}_{T_W}(z_2) \end{bmatrix} = \begin{bmatrix} w_\alpha & w_\beta \end{bmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}. \tag{2.4.7}$$

Hence, using the basis $\{z_1, z_2\}$ for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ and the basis $\mathcal{B}_{\mathrm{eig}}$ for $\mathbb{D}_{\mathrm{crys}}(T_W)$, we get a matrix $[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} \in M_{2,2}(\mathcal{H}_E(\Gamma))$ of $\mathcal{L}_{T_W}$ as

$$[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}. \tag{2.4.8}$$

Lemma 2.4.12. We have the equation

$$[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \operatorname{adj} M_\Omega, \tag{2.4.9}$$

where $\operatorname{adj} M_\Omega$ is the adjugate matrix of M_Ω. In particular, x_1, x_2 are $O(\log_p^{v_p(\alpha)})$ and x_3, x_4 are $O(\log_p^{v_p(\beta)})$.

Proof.

From Lemma 2.4.7, we know that

$$\Omega_{W,k-1} \circ \mathcal{L}_W = \left(\prod_{i=0}^{k-2} \ell_i \right).$$

By restricting to the $\mathcal{O}_E$-lattice T_W in W, we have, for any $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$,

$$\Omega_{T_W,k-1}(\mathcal{L}_{T_W}(z)) = \left(\prod_{i=0}^{k-2} \ell_i\right)(z).$$

Thus, by using the $\Lambda_{\mathcal{O}_E}(\Gamma)$-basis $\{z_1, z_2\}$ for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$, we get

$$\Omega_{T_W,k-1}(\mathcal{L}_{T_W}(z_1)) = \left(\prod_{i=0}^{k-2} \ell_i\right)(z_1), \tag{2.4.10}$$

$$\Omega_{T_W,k-1}(\mathcal{L}_{T_W}(z_2)) = \left(\prod_{i=0}^{k-2} \ell_i\right)(z_2). \tag{2.4.11}$$

In matrix form, we can rewrite the equations (3.7.1) and (3.7.2)

$$M_\Omega[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \left(\prod_{i=0}^{k-2} \ell_i\right) I_2.$$

From Lemma 2.4.10, we have $\det(\mathcal{L}_{T_W}) = \prod_{i=0}^{k-2} \ell_i$, hence we have $\det([\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}}) = \prod_{i=0}^{k-2} \ell_i$. Thus

$$M_\Omega[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \det(\mathcal{L}_{T_W}) I_2.$$

Hence,

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = [\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \operatorname{adj} M_\Omega = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{2.4.12}$$

where $\operatorname{adj} M_\Omega$ is the adjugate matrix of the matrix M_Ω. Thus, from Lemma 2.4.11, we get x_1, x_2 have growth $O(\log_p^{v_p(\alpha)})$ and x_3, x_4 have growth $O(\log_p^{v_p(\beta)})$. ∎

We use the basis $\mathcal{B}$ of $\mathbb{D}_{\mathrm{crys}}(T_W)$ and the basis $\{z_1, z_2\}$ for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ to get another matrix $[\mathcal{L}_{T_W}]_{\mathcal{B}}$ such that

$$\begin{bmatrix} \mathcal{L}_{T_W}(z_1) & \mathcal{L}_{T_W}(z_2) \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} [\mathcal{L}_{T_W}]_{\mathcal{B}}.$$

Since $\begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} w_\alpha & w_\beta \end{bmatrix} Q^{-1}$, we have

$$[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = Q^{-1}[\mathcal{L}_{T_W}]_{\mathcal{B}}. \tag{2.4.13}$$

For any non-negative integer n, we write $\omega_n(1+X) = (1+X)^{p^n} - 1$. Let $\Phi_n(1+X) = \omega_n(1+X)/\omega_{n-1}(1+X)$ be the p^n-th cyclotomic polynomial for integers $n > 1$. Recall from Section 2, topological generators γ_0 of Γ_1 and u of $1 + p\mathbb{Z}_p$ such that $\chi(\gamma_0) = u$, where χ is the p-adic cyclotomic character. For any integer $m \geqslant 1$, we define

$$\Phi_{n,m}(1+X) = \prod_{j=0}^{m-1} \Phi_n(u^{-j}(1+X)),$$

$$\begin{aligned}\omega_{n,m}(1+X) &= \prod_{j=0}^{m-1} \omega_n(u^{-j}(1+X)),\\ \delta_m(X) &= \prod_{j=0}^{m-1} (u^{-j}(1+X)-1),\\ \log_{p,m}(1+X) &= \prod_{j=0}^{m-1} \log_p(u^{-j}(1+X)).\end{aligned}$$

Remark 2.4.13. Note that the zeros of $\log_{p,m}(1+X)$, all having simple order, are of the form $u^j\zeta-1$, where ζ is a p^n-th root of unity, for all $n \geqslant 1$ and for all $0 \leqslant j \leqslant m-1$.

Recall from Proposition 3.4.2, for any Dirichlet character ω of conductor $p^n, n > 1$, we have

$$\varphi^{-n}(\chi^i\omega(\mathcal{L}_{T_W}(z) \otimes t^i e_{-i})) \in \mathbb{Q}_{p,n} \otimes \mathrm{Fil}^0\mathbb{D}_{\mathrm{crys}}(T_W(-i)),$$

for any $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ and $0 \leqslant i \leqslant (k-2)$.

Proposition 2.4.14. The second row of the matrix $[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}}$ is divisible by $\Phi_{n-1,k-1}(\gamma_0)$ over $\mathcal{H}_E(\Gamma)$, for all integers $n > 1$.

Proof. The Hodge–Tate weights of the crystalline representation W are 0 and $k-1$. Thus, for $0 \leqslant i \leqslant k-2$, we have

$$\dim_E(\mathrm{Fil}^0\mathbb{D}_{\mathrm{crys}}(W(-i))) = \dim_E(\mathrm{Fil}^{-i}\mathbb{D}_{\mathrm{crys}}(W)) = 1.$$

We know that $\mathrm{Fil}^{-i}\mathbb{D}_{\mathrm{crys}}(T_W)$ is a rank one free $\mathcal{O}_E$-submodule of $\mathbb{D}_{\mathrm{crys}}(T_W)$ generated by w_1 for all $0 \leqslant i \leqslant k-2$. Thus, $\mathrm{Fil}^0\mathbb{D}_{\mathrm{crys}}(T_W(-i))$ is generated by $w_1 \otimes t^i e_{-i}$.

Write

$$\varphi^{-n}(\mathcal{L}_{T_W}(z) \otimes t^i e_{-i}) = F_{1,z} \cdot (w_1 \otimes p^{-ni} t^i e_{-i}) + F_{2,z} \cdot (w_2 \otimes p^{-ni} t^i e_{-i}), \tag{2.4.14}$$

where $F_{1,z}, F_{2,z} \in \mathcal{H}_E(\Gamma)$ and $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$.

Thus, (2.4.5) implies

$$F_{2,z}(\chi^i\omega) = 0, \tag{2.4.15}$$

for all $0 \leqslant i \leqslant k-2$ and for all Dirichlet character ω of conductor p^n, where $n > 1$. Then [LLZ10, Theorem 5.4] implies $\Phi_{n-1,k-1}(\gamma_0)$ divides $F_{2,z}$.

Using the basis z_1, z_2 for $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$ and the basis $\mathcal{B}$ for $\mathbb{D}_{\mathrm{crys}}(T_W)$, we get

$$\begin{bmatrix}\varphi^{-n}(\mathcal{L}_{T_W}(z_1) \otimes t^i e_{-i}) & \varphi^{-n}(\mathcal{L}_{T_W}(z_2) \otimes t^i e_{-i})\end{bmatrix} = \begin{bmatrix} w_1 \otimes t^i e_{-i} & w_2 \otimes t^i e_{-i}\end{bmatrix} \begin{pmatrix} F_{1,z_1} & F_{1,z_2} \\ F_{2,z_1} & F_{2,z_2}\end{pmatrix}.$$

Then the matrix of $\varphi^{-n}\mathcal{L}_{T_W}$ with respect to basis $\mathcal{B}$ is

$$[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}} = \begin{pmatrix} F_{1,z_1} & F_{1,z_2} \\ F_{2,z_1} & F_{2,z_2}\end{pmatrix}.$$

Note that $[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}} \in M_{2,2}(\mathcal{H}(\Gamma))$. Hence, using (2.4.15), we deduce that $\Phi_{n-1,k-1}(\gamma_0)$ divides both F_{2,z_1} and F_{2,z_2} ∎

For $n > 1$, we can write

$$\begin{aligned}\left[\varphi^{-n}(\mathcal{L}_{T_W}(z_1)) \quad \varphi^{-n}(\mathcal{L}_{T_W}(z_2))\right] &= \left[\varphi^{-n}(w_\alpha) \quad \varphi^{-n}(w_\beta)\right][\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}}, \\ &= \left[w_\alpha \quad w_\beta\right]\begin{pmatrix}\alpha^n & 0 \\ 0 & \beta^n\end{pmatrix}[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}}.\end{aligned}$$

Thus, we get a matrix for $\varphi^{-n}\mathcal{L}_{T_W}$ with respect to the eigenbasis $\mathcal{B}_{\mathrm{eig}}$ for $\mathbb{D}_{\mathrm{crys}}(T_W)$ and using (2.4.8), we get

$$[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \begin{pmatrix}\alpha^n & 0 \\ 0 & \beta^n\end{pmatrix}[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \begin{pmatrix}\alpha^n x_1 & \alpha^n x_2 \\ \beta^n x_3 & \beta^n x_4\end{pmatrix}. \tag{2.4.16}$$

2.5 Logarithmic matrix $\underline{M}$ and the factorization of power series in one variable

In this section, we will first explore some properties of $\underline{M}$ which imply that $\underline{M}$ is a logarithmic matrix in the sense of Sprung and Lei–Loeffler–Zerbes. Next, we will use $\underline{M}$ to decompose power series with certain growth conditions into power series with bounded coefficients.

2.5.1 Properties of $\underline{M}$

For any $z \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_W)$, we have

$$\mathcal{L}_{T_W}(z) = \left[(1+\pi)\otimes w_1 \quad (1+\pi)\otimes w_2\right]\underline{M}\begin{pmatrix}\underline{\mathrm{Col}}_1(z) \\ \underline{\mathrm{Col}}_2(z)\end{pmatrix}.$$

In matrix form, we write

$$\left[\mathcal{L}_{T_W}(z_1) \quad \mathcal{L}_{T_W}(z_2)\right] = \left[(1+\pi)\otimes w_1 \quad (1+\pi)\otimes w_2\right]\underline{M}\begin{pmatrix}\underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(z_2)\end{pmatrix}.$$

Thus,

$$[\mathcal{L}_{T_W}]_{\mathcal{B}} = \underline{M}\begin{pmatrix}\underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(z_2)\end{pmatrix}. \tag{2.5.1}$$

Similarly, we have

$$[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = Q^{-1}\underline{M}\begin{pmatrix}\underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(z_2)\end{pmatrix}, \tag{2.5.2}$$

since $[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = Q^{-1}[\mathcal{L}_{T_W}]_{\mathcal{B}}$.

Proposition 2.5.1. The elements in the first row of $Q^{-1}\underline{M}$ are inside $\mathcal{H}_{E,v_p(\alpha)}(\Gamma_1)$, while the elements in the second row are in the $\mathcal{H}_{E,v_p(\beta)}(\Gamma_1)$.

Proof. Recall from Lemma 2.4.12,

$$[\mathcal{L}_{T_W}]_{\mathcal{B}_{\mathrm{eig}}} = \operatorname{adj} M_\Omega = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Therefore, (2.5.2) implies

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = Q^{-1}\underline{M}\begin{pmatrix} \underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(Z_2) \end{pmatrix}.$$

But $\underline{\mathrm{Col}}_i(z_j)$ are $O(1)$ for $i, j \in \{1,2\}$, since they lie in the Iwasawa algebra $\Lambda_E(\Gamma)$. Therefore, after writing $Q^{-1}\underline{M} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$, we get

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}\begin{pmatrix} \underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(Z_2) \end{pmatrix}. \tag{2.5.3}$$

We take isotypic components on both sides with respect to the trivial character of Δ. After writing $[\mathrm{Col}^\Delta]$ for $\begin{pmatrix} \underline{\mathrm{Col}}_1^\Delta(z_1) & \underline{\mathrm{Col}}_1^\Delta(z_2) \\ \underline{\mathrm{Col}}_2^\Delta(z_1) & \underline{\mathrm{Col}}_2^\Delta(Z_2) \end{pmatrix}$, equation (2.5.3) becomes

$$\begin{pmatrix} x_1^\Delta & x_2^\Delta \\ x_3^\Delta & x_4^\Delta \end{pmatrix}\operatorname{adj}[\mathrm{Col}^\Delta] = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}\det([\mathrm{Col}^\Delta]). \tag{2.5.4}$$

The result follows from the Lemma 2.4.12, since $\det([\mathrm{Col}^\Delta])$ is again $O(1)$ and $x_1^\Delta, x_2^\Delta \in \mathcal{H}_{E,v_p(\alpha)}(\Gamma_1)$ and $x_3^\Delta, x_4^\Delta \in \mathcal{H}_{E,v_p(\beta)}(\Gamma_1)$. ∎

Lemma 2.5.2. The second row of $A_\varphi^{-n}\underline{M}$ is divisible by the cyclotomic polynomial $\Phi_{n-1,k-1}(\gamma_0)$ over $\mathcal{H}_E(\Gamma_1)$.

Proof. We know that

$$\left[\mathcal{L}_{T_W}(z_1) \quad \mathcal{L}_{T_W}(z_2)\right] = \left[(1+\pi)\otimes w_1 \quad (1+\pi)\otimes w_2\right]\underline{M}\begin{pmatrix} \underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(z_2) \end{pmatrix}.$$

Let us denote $\begin{pmatrix} \underline{\mathrm{Col}}_1(z_1) & \underline{\mathrm{Col}}_1(z_2) \\ \underline{\mathrm{Col}}_2(z_1) & \underline{\mathrm{Col}}_2(z_2) \end{pmatrix}$ by [Col]. By an abuse of notation, we write φ for $1\otimes\varphi$. Here $1\otimes\varphi$ mean φ does not act on $(1+\pi)$ and acts as φ on w_i.

After applying φ^{-n} on both sides of the above equation, we obtain

$$\begin{aligned} \varphi^{-n}\left(\left[\mathcal{L}_{T_W}(z_1) \quad \mathcal{L}_{T_W}(z_2)\right]\right) &= \left[\varphi^{-n}(\mathcal{L}_{T_W}(z_1)) \quad \varphi^{-n}(\mathcal{L}_{T_W}(z_2))\right], \\ &= \left[(1+\pi)\otimes\varphi^{-n}(w_1) \quad (1+\pi)\otimes\varphi^{-n}(w_2)\right]\underline{M}[\mathrm{Col}], \\ &= \left[(1+\pi)\otimes w_1 \quad (1+\pi)\otimes w_2\right]A_\varphi^{-n}\underline{M}[\mathrm{Col}]. \end{aligned}$$

Therefore, we get

$$[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}} = A_{\varphi}^{-n}\underline{M}[\mathrm{Col}].$$

After rearranging the above equation, we get

$$[\varphi^{-n}\mathcal{L}_{T_W}]_{\mathcal{B}}\,\mathrm{adj}[\mathrm{Col}] = A_{\varphi}^{-n}\underline{M}\,\det([\mathrm{Col}]), \tag{2.5.5}$$

From [LLZ11, Proposition 4.11, Theorem 4.12, Corollary 4.15], we can conclude that $\Phi_{n-1,k-1}(\gamma_0)$ does not divide $\det([\mathrm{Col}])$. Thus, the result follows from Proposition 2.4.14. ∎

Lemma 2.5.3. The determinant of matrix $\underline{M}$ is $\dfrac{\log_{p,k-1}(\gamma_0)}{\delta_{k-1}(\gamma_0-1)}$ upto a unit in $\Lambda_E(\Gamma_1)$.

Proof. This is [LLZ11, Corollary 3.2]. See also [BL21, Lemma 2.7]. ∎

Thus, Proposition 2.5.1, Lemma 2.5.2, and Lemma 2.5.3 imply that the matrix $\underline{M}$ is a logarithmic matrix in the sense of Sprung and Lei–Loeffler–Zerbes.

2.5.2 Factorization using $\underline{M}$

Let F, G be power series in $\mathcal{H}_E(\Gamma)$. We write $F \sim G$, if F is $O(G)$ and G is $O(F)$. Recall that, for power series $F$and G, we say F is $O(G)$ if $\|F\|_\rho = O(\|G\|_\rho)$ as $\rho \to 1^-$.

Lemma 2.5.4. We have $\det(\underline{M}) \sim \log_p^{k-1}$.

Proof. From Lemma 2.5.3, we get $\det(\underline{M}) = *\dfrac{\log_{p,k-1}(\gamma_0)}{\delta_{k-1}(\gamma_0-1)}$, where $*$ is a unit in $\Lambda_E(\Gamma)$. Hence the result follows from the definition of $\log_p^{k-1}(\gamma_0)$ and the fact that $\delta_{k-1}(\gamma_0-1)$ is polynomial and hence $O(1)$. ∎

Theorem 2.5.5. For $\lambda \in \{\alpha, \beta\}$, let $F_\lambda \in \mathcal{H}_{E,v_p(\lambda)}(\Gamma)$, such that for any integer $0 \leqslant j \leqslant k-2$ and for any Dirichlet character ω of conductor p^n we have $F_\lambda(\chi^j\omega) = \lambda^{-n}C_{\omega,j}$, where $C_{\omega,j} \in \overline{\mathbb{Q}_p}$ that is independent of λ. Then, there exist $F_\flat, F_\sharp \in \Lambda_E(\Gamma)$ such that

$$\begin{pmatrix} F_\alpha \\ F_\beta \end{pmatrix} = Q^{-1}\underline{M}\begin{pmatrix} F_\sharp \\ F_\flat \end{pmatrix}. \tag{2.5.6}$$

Proof. From Lemma 2.5.2, we know that the second row of $A_\varphi^{-n}\underline{M}$ is divisible by $\Phi_{n-1,k-1}(\gamma_0)$. Hence we can write

$$A_\varphi^{-n}\underline{M} = \begin{pmatrix} a & b \\ c\cdot\Phi_{n-1,k-1}(\gamma_0) & d\cdot\Phi_{n-1,k-1}(\gamma_0) \end{pmatrix},$$

where a, b, c, d are power series.

Recall, $Q^{-1} = \frac{1}{\det Q}\begin{pmatrix} vp^{k-1} & \beta \\ vp^{k-1} & \alpha \end{pmatrix}$, and $\operatorname{adj} Q^{-1}\underline{M} = \begin{pmatrix} P_4 & -P_2 \\ -P_3 & P_1 \end{pmatrix}$. Note that $\det Q \neq 0$ since $\alpha \neq \beta$.

Thus for every positive integer n, we have

$$\begin{aligned}
Q^{-1}\underline{M} &= Q^{-1}A_{\varphi}^{n}QQ^{-1}A_{\varphi}^{-n}\underline{M}, \\
&= \begin{pmatrix} \frac{1}{\alpha^n} & 0 \\ 0 & \frac{1}{\beta^n} \end{pmatrix} \frac{1}{\det Q} \begin{pmatrix} vp^{k-1} & \beta \\ vp^{k-1} & \alpha \end{pmatrix} \begin{pmatrix} a & b \\ c\cdot\Phi_{n-1,k-1}(\gamma_0) & d\cdot\Phi_{n-1,k-1}(\gamma_0) \end{pmatrix}, \\
&= \frac{1}{\det Q} \begin{pmatrix} \frac{1}{\alpha^n} & 0 \\ 0 & \frac{1}{\beta^n} \end{pmatrix} \begin{pmatrix} a\cdot vp^{k-1} + \beta\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0) & b\cdot vp^{k-1} + \beta\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0) \\ a\cdot vp^{k-1} + \alpha\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0) & b\cdot vp^{k-1} + \alpha\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0) \end{pmatrix}, \\
&= \frac{1}{\det Q} \begin{pmatrix} \frac{1}{\alpha^n}(a\cdot vp^{k-1} + \beta\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0)) & \frac{1}{\alpha^n}(b\cdot vp^{k-1} + \beta\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0)) \\ \frac{1}{\beta^n}(a\cdot vp^{k-1} + \alpha\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0)) & \frac{1}{\beta^n}(b\cdot vp^{k-1} + \alpha\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0)) \end{pmatrix}.
\end{aligned}$$

Hence

$$\operatorname{adj} Q^{-1}\underline{M} = \frac{1}{\det Q} \begin{pmatrix} \dfrac{1}{\beta^n}(b\cdot vp^{k-1} + \alpha\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0)) & \dfrac{-1}{\alpha^n}(b\cdot vp^{k-1} + \beta\cdot d\cdot\Phi_{n-1,k-1}(\gamma_0)) \\ \dfrac{-1}{\beta^n}(a\cdot vp^{k-1} + \alpha\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0)) & \dfrac{1}{\alpha^n}(a\cdot vp^{k-1} + \beta\cdot c\cdot\Phi_{n-1,k-1}(\gamma_0)) \end{pmatrix}. \tag{2.5.7}$$

For integer $0 \leqslant j \leqslant (k-2)$, and Dirichlet character ω of conductor p^n, equation (2.5.7) implies

$$(\operatorname{adj} Q^{-1}\underline{M})(\chi^j\omega) = \begin{pmatrix} \dfrac{1}{\beta^n}* & \dfrac{-1}{\alpha^n}* \\ \dfrac{-1}{\beta^n}*' & \dfrac{1}{\alpha^n}*' \end{pmatrix},$$

where $*, *'$ are in $\overline{\mathbb{Q}_p}$.

Thus, if we write $F_1 = P_4F_\alpha - P_2F_\beta$, then, from (2.5.7), we get

$$\begin{aligned}
F_1(\chi^j\omega) &= P_4(\chi^j\omega)F_\alpha(\chi^j\omega) - P_2(\chi^j\omega)F_\beta(\chi^j\omega), \\
&= \frac{1}{\beta^n}\frac{*C_{\omega,j}}{\alpha^n} - \frac{1}{\alpha^n}\frac{*C_{\omega,j}}{\beta^n}, \\
&= 0.
\end{aligned}$$

Similarly, if we write $F_2 = -P_3F_\alpha + P_1F_\beta$, then $F_2(\chi^j\omega) = 0$.

Hence, for every positive integer n, the zeros of $\Phi_{n-1,k-1}$ are also zeros of F_1 and F_2. In other words, the roots of $\det(Q^{-1}\underline{M}) = (\text{some constant})\dfrac{\log_{p,k-1}(\gamma_0)}{\delta_{k-1}(\gamma_0 - 1)}$ are also the roots of F_1, and F_2. Therefore, $\det Q^{-1}\underline{M}$ divides both F_1, F_2 in $\mathcal{H}_E(\Gamma)$.

Note that P_4F_α is $O(\log_p^{k-1})$, since P_4 is $O(\log_p^{v_p(\beta)})$ and F_α is $O(\log_p^{v_p(\alpha)})$. Similarly, P_2F_β, P_3F_α, and P_1F_β are $O(\log_p^{k-1})$.

Let

$$F_\sharp = \frac{P_4F_\alpha - P_2F_\beta}{\det Q^{-1}\underline{M}} \quad \text{and} \quad F_\flat = \frac{-P_3F_\alpha + P_1F_\beta}{\det Q^{-1}\underline{M}}. \tag{2.5.8}$$

Then $F_\sharp$ and $F_\flat$ have bounded coefficients since the numerators of both of them are $O(\log_p^{k-1})$, denominators of both of them are $\det(Q^{-1}\underline{M})$ and by Lemma 2.5.4 $\det(Q^{-1}\underline{M}) \sim \log_p^{k-1}$. Hence $F_\sharp$ and $F_\flat$ are $O(1)$ (i.e. bounded). Therefore, we can conclude that $F_\sharp$ and $F_\flat$ are in $\Lambda_E(\Gamma)$. This completes the proof. ∎

2.6 Preliminaries about ray class groups, Hecke characters, and the two variable distribution algebras

For the rest of the chapter, we fix a quadratic imaginary field K and a prime $p \geqslant 3$ which splits in K as $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. Let $h_K \geqslant 1$ be the class number of K.

Assumption 2.6.1. For the rest of the chapter, we assume that $p \nmid h_K$.

2.6.1 Ray class groups and ray class fields

Let K_∞ be the unique $\mathbb{Z}_p^2$ extension of K. If $\mathcal{I}$ is an ideal of K, we write $G_{\mathcal{I}}$ for the ray class group K modulo $\mathcal{I}$. We define

$$\mathrm{Cl}(K, p^\infty) = G_{p^\infty} = \varprojlim_n G_{(p)^n}, \; G_{\mathfrak{p}^\infty} = \varprojlim_n G_{\mathfrak{p}^n}, \; G_{\overline{\mathfrak{p}}^\infty} = \varprojlim_n G_{\overline{\mathfrak{p}}^n}.$$

These are the Galois groups of the ray class fields $K(p^\infty), K(\mathfrak{p}^\infty)$ and $K(\overline{\mathfrak{p}}^\infty)$ respectively. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $H(\mathfrak{q}^\infty)$ be the subfield of $G_{\mathfrak{q}^\infty}$ such that $\mathrm{Gal}(H(\mathfrak{q}^\infty)/K) \cong \mathbb{Z}_p$. Note $H(\mathfrak{q}^\infty) \subset K_\infty$.

Remark 2.6.2. We have an isomorphism $G_{p^\infty} \cong \Delta_K \times \mathbb{Z}_p \times \mathbb{Z}_p \cong \Delta \times \overline{\langle\gamma_{\mathfrak{p}}\rangle} \times \overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}$, where Δ_K is a finite abelian group, $\gamma_{\mathfrak{p}}$ and $\gamma_{\overline{\mathfrak{p}}}$ topologically generate $\mathbb{Z}_p$ parts of $G_{\mathfrak{p}^\infty}$ and $G_{\overline{\mathfrak{p}}^\infty}$ respectively. Here by $\overline{\langle x\rangle}$ we mean the topological closure of the cyclic group generated by x.

Remark 2.6.3. By the assumption $p \nmid h_K$, there exists a unique prime in K_∞ above $\mathfrak{p}$ and a unique prime above $\overline{\mathfrak{p}}$. By an abuse of notation, we will also denote by $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ by the unique prime above $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ respectively in K_∞. Therefore, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, $\mathrm{Gal}(H(\mathfrak{q}^\infty)_{\mathfrak{q}}/K_{\mathfrak{q}}) = \overline{\langle\gamma_{\mathfrak{q}}\rangle} \cong \mathrm{Gal}(H(\mathfrak{q}^\infty)/K) \cong \mathbb{Z}_p$.

Since p splits in K, the local field $K_{\mathfrak{q}}$ is isomorphic to $\mathbb{Q}_p$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Thus,$1+\pi_{\mathfrak{q}}\mathcal{O}_{K_{\mathfrak{q}}} \cong 1 + p\mathbb{Z}_p$, where $\pi_{\mathfrak{q}}$ is a uniformizer of $\mathcal{O}_{K_{\mathfrak{q}}}$. Recall the topological generator $u \in 1 + p\mathbb{Z}_p$ such that $\chi(\gamma_0) = u$, where γ_0 generates $\Gamma_1 \cong \mathbb{Z}_p$, and χ is p-adic cyclotomic character. Thus, we may set $u_{\mathfrak{p}} = u_{\overline{\mathfrak{p}}} = u$, where $u_{\mathfrak{q}}$ is a topological generator of $1 + \pi_{\mathfrak{q}}\mathcal{O}_{K_{\mathfrak{q}}}$. From now on, we fix this u.

By local class field theory, there exists a group isomorphism (*Artin map*)

$$\mathrm{Art}_{\mathfrak{q}} : \mathcal{O}_{K_{\mathfrak{q}}}^{\times} \to \mathrm{Gal}(H(\mathfrak{q}^{\infty})_{\mathfrak{q}}/K_{\mathfrak{q}}) \cong \mathrm{Gal}(H(\mathfrak{q}^{\infty})/K),$$

such that

$$\mathrm{Art}_{\mathfrak{q}}(u_{\mathfrak{q}}) = \mathrm{Art}_{\mathfrak{q}}(u) = \gamma_{\mathfrak{q}},$$

where $\gamma_{\mathfrak{q}}$ is a topological generator of $\mathrm{Gal}(H(\mathfrak{q}^{\infty})_{\mathfrak{q}}/K_{\mathfrak{q}})$. By an abuse of notations, let $\gamma_{\mathfrak{q}}$ be a topological generator of $\mathrm{Gal}(H(\mathfrak{q}^{\infty})/K)$ and $\mathrm{Art}_{\mathfrak{q}}(u) = \gamma_{\mathfrak{q}}$.

2.6.2 Hecke characters as the characters on the ray class groups

Let $\mathbb{A}_K^{\times}$ denote the group of ideles of K and write $\mathbb{A}_K^{\times} = \mathbb{A}_{\infty}^{\times} \times \mathbb{A}_f^{\times}$, where $\mathbb{A}_{\infty}^{\times}$ is the infinite part and $\mathbb{A}_f^{\times}$ is the finite part. We can embed $K^{\times}$ into $\mathbb{A}_K^{\times}$ diagonally. Fix embeddings $i_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

Definition 2.6.4 (Hecke characters)**.**

1. A *Hecke character* Ξ of K is a continuous homomorphism $\Xi : \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ that is trivial on $K^{\times}$. In other words, a *Hecke character* of K is a continuous homomorphism $\Xi : K^{\times}\backslash\mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$.
2. We say a Hecke character Ξ is *algebraic* if for each embedding $\kappa : K \hookrightarrow \mathbb{C}$, there exists $n_{\kappa} \in \mathbb{Z}$ such that $\Xi(x) = \prod_{\kappa}(\kappa(x))^{-n_{\kappa}}$ for each x in the connected component of the identity in $K_{\infty}^{\times}$.
3. Let $\Xi : K^{\times}\backslash\mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ be an algebraic Hecke character of K. We say that Ξ has *infinity type* $(q, r) \in \mathbb{Z}^2$ if $\Xi_{\infty}(z) = z^q\overline{z}^r$, where for each place v of K, we let $\Xi_v : K_v^{\times} \to \mathbb{C}^{\times}$ be the v-component of Ξ.
4. The conductor of a Hecke character Ξ is an ideal $\mathfrak{f} := \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$, where the product runs over all primes of K, such that
   - $n_{\mathfrak{p}} = 0$ for almost all primes $\mathfrak{p}$,
   - for finitely many primes $\mathfrak{p}$, $n_{\mathfrak{p}} \in \mathbb{Z}_{>0}$and $\Xi_{\mathfrak{p}}(1 + \mathfrak{p}^{n_{\mathfrak{p}}}) = 1$. Moreover, $n_{\mathfrak{p}}$ is minimal with this property.

From now on, all the Hecke characters mentioned in this chapter are algebraic Hecke characters. We can view Hecke characters as p-adic characters:

Definition 2.6.5 (p-adic avatar of an algebraic Hecke character)**.** Let Ξ be a Hecke character of K of conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ and infinity type (a, b). The p-adic avatar of Ξ is defined as

$$\widehat{\Xi}(x) := x_{\mathfrak{p}}^a x_{\overline{\mathfrak{p}}}^b \cdot i_p i_{\infty}^{-1}(\Xi(x_{fin})),$$

where $x \in \mathbb{A}_K^{\times}$, and for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, $x_{\mathfrak{q}}$ are the $\mathfrak{q}$ componenets of x. See also [Wil17, Section 7.3].

By the class field theory, the correspondence $\Xi \mapsto \hat{\breve{\Xi}}$ establishes a bijection between the set of algebraic Hecke characters of K of conductor dividing (p^∞) and the set of locally algebraic $\overline{\mathbb{Q}_p}$-valued characters of G_{p^∞}. See [Wil17, Theorem 7.3].

Now we combine this p-adic avatar of the Hecke character and the global Artin reciprocity map rec to define a Galois character on the ray class group G_{p^∞}.

Let Ξ be a Hecke character of K of conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ and infinity type (a, b).

Definition 2.6.6. The p-adic Galois character of Ξ is $\tilde{\breve{\Xi}} : G_K^{ab} \to \mathbb{C}_p^\times$ given by

$$\tilde{\breve{\Xi}} = \hat{\breve{\Xi}} \circ \mathrm{rec}^{-1},$$

where $\mathrm{rec} : \mathbb{A}_K^\times/K^\times \to G_K^{\mathrm{ab}}$ is the global Artin reciprocity.

Note that $G_{p^\infty} \subset G_K^{ab}$ and Remark 2.6.3 implies $\mathrm{rec}^{-1}(\gamma_\mathfrak{q}) = \mathrm{Art}_\mathfrak{q}^{-1}(\gamma_\mathfrak{q})$.

By an abuse of notation, let $\tilde{\breve{\Xi}}$ denote $\tilde{\breve{\Xi}}\ |_{G_{p^\infty}}$. Define $\tilde{\breve{\Xi}}_\mathfrak{q} := \tilde{\breve{\Xi}}|_{\overline{\langle\gamma_\mathfrak{q}\rangle}}$. Hence, if Ξ is a Hecke character of the infinity type (a, b) and conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, then

$$\tilde{\breve{\Xi}}_\mathfrak{p}(\gamma_\mathfrak{p}) = u^a\zeta_\mathfrak{p},$$

and

$$\tilde{\breve{\Xi}}_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}}) = u^b\zeta_{\overline{\mathfrak{p}}},$$

where $\zeta_\mathfrak{q}$ is a $(p^{n_\mathfrak{q}-1})$-th root of unity. In other words, for any $\sigma \in \Delta_K$,

$$\tilde{\breve{\Xi}}(\sigma \times \gamma_\mathfrak{p} \times \gamma_{\overline{\mathfrak{p}}}) = \tilde{\breve{\Xi}}_\mathfrak{p}(\gamma_\mathfrak{p}) \cdot \tilde{\breve{\Xi}}_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}}) \times \text{some root of unity.} \tag{2.6.1}$$

From the above discussion, we get

$$\check{\breve{\Xi}} := \tilde{\breve{\Xi}}|_{\overline{\langle\gamma_\mathfrak{p}\rangle}\times\overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}} = \tilde{\breve{\Xi}}_\mathfrak{p} \cdot \tilde{\breve{\Xi}}_{\overline{\mathfrak{p}}}.$$

Moreover, for any Hecke character ω_1 of infinity type $(a, 0)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}$ and Hecke character ω_2 of infinity type $(0, b)$, and conductor $\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, then the product $\widetilde{\omega_1}_\mathfrak{p} \cdot \widetilde{\omega_2}_{\overline{\mathfrak{p}}}$ is a character on $\overline{\langle\gamma_\mathfrak{p}\rangle} \times \overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}$.

2.6.3 Two variable distribution modules

We will extend $\mathcal{H}_E(\Gamma_1)$ from previous sections. Let

$$\mathcal{H}_{\mathbb{C}_p,r}(\Gamma_1) = \{f(\gamma_0 - 1) \colon f(X) = \sum_{n\geqslant 0} a_nX^n \in \mathbb{C}_p[[X]], \sup_n(n^{-r}|a_n|_p) < \infty\}.$$

and

$$\mathcal{H}_{\mathbb{C}_p}(\Gamma_1) = \bigcup_{r\geqslant 0} \mathcal{H}_{\mathbb{C}_p,r}(\Gamma_1).$$

Note that $\mathcal{H}_E(\Gamma_1) = E[[\gamma_0 - 1]] \bigcap \mathcal{H}_{\mathbb{C}_p}(\Gamma_1)$, for any finite field extension E of $\mathbb{Q}_p$.

For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, define a map $\tau_{\mathfrak{q}} = \mathrm{Art}_{\mathfrak{q}} \circ \chi$ between Γ_1 and $\Gamma_{\mathfrak{q}} := \mathrm{Gal}(H(\mathfrak{q}^\infty)/K)$ which sends γ_0 to $\gamma_{\mathfrak{q}}$. This *change of variable map* can be extended to ring isomorphisms

$$\begin{aligned} \tau_{\mathfrak{q}} \colon \mathcal{H}_{\mathbb{C}_p}(\Gamma_1) &\to \mathcal{H}_{\mathbb{C}_p}(\Gamma_{\mathfrak{q}}), \\ f(\gamma_0 - 1) &\mapsto f(\gamma_{\mathfrak{q}} - 1). \end{aligned}$$

Similarly, for any finite field extension E of $\mathbb{Q}_p$, we again have an isomorphism

$$\tau_{\mathfrak{q}} : \mathcal{H}_E(\Gamma_1) \to \mathcal{H}_E(\Gamma_{\mathfrak{q}}).$$

For any finite extension E of $\mathbb{Q}_p$, let

$$\Lambda_E(G_{p^\infty}) = E[\Delta_K] \otimes_{\mathcal{O}_E} \Lambda_{\mathcal{O}_E}(\Gamma_{\mathfrak{p}}) \widehat{\otimes}_{\mathcal{O}_E} \Lambda_{\mathcal{O}_E}(\Gamma_{\overline{\mathfrak{p}}}),$$

where $\Lambda_{\mathcal{O}_E}(\Gamma_{\mathfrak{q}})$ is the Iwasawa algebra $\mathcal{O}_E[[\Gamma_{\mathfrak{q}}]]$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. The two-variable Iwasawa algebra $\Lambda_E(G_{p^\infty})$ is isomorphic to the power series ring

$$E[\Delta_K] \otimes_{\mathcal{O}_E} \mathcal{O}_E[[T_1, T_2]],$$

by identifying $\gamma_{\mathfrak{p}} - 1$ with T_1 and $\gamma_{\overline{\mathfrak{p}}} - 1$ with T_2.

We define the

$$\mathcal{H}_{\mathbb{C}_p,r,s} := \mathcal{H}_{\mathbb{C}_p,r}(\Gamma_{\mathfrak{p}}) \widehat{\otimes}_{\mathbb{C}_p} \mathcal{H}_{\mathbb{C}_p,s}(\Gamma_{\overline{\mathfrak{p}}}),$$

and

$$\mathcal{H}_{E,r,s} := \mathcal{H}_{\mathbb{C}_p,r,s} \cap E[[\gamma_{\mathfrak{p}} - 1, \gamma_{\overline{\mathfrak{p}}} - 1]],$$

for any finite extension E of $\mathbb{Q}_p$. Thus

$$\mathcal{H}_{E,r,s} = \mathcal{H}_{E,r}(\Gamma_{\mathfrak{p}}) \widehat{\otimes}_E \mathcal{H}_{E,s}(\Gamma_{\overline{\mathfrak{p}}}).$$

We also define

$$\mathcal{H}_{\mathbb{C}_p,r,s}(G_{p^\infty}) := \mathbb{C}_p[\Delta_K] \otimes_{\mathbb{C}_p} \mathcal{H}_{\mathbb{C}_p,r,s},$$

and

$$\mathcal{H}_{E,r,s}(G_{p^\infty}) := E[\Delta_K] \otimes_E \mathcal{H}_{E,r,s},$$

where Δ_K is the finite abelian group appearing in the Galois group G_{p^∞}.

Note that

$$\Lambda_E(G_{p^\infty}) = \mathcal{H}_{E,0,0}(G_{p^\infty}),$$

since we can identify $\mathcal{H}_{E,0}(\Gamma_{\mathfrak{q}})$ with $\Lambda_E(\Gamma_{\mathfrak{q}})$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Moreover, $\mathcal{H}_{E,r,s}$ is a $\Lambda_E(\Gamma_{\mathfrak{p}}) \widehat{\otimes}_E \Lambda_E(\Gamma_{\overline{\mathfrak{p}}})$-module, and $\mathcal{H}_{E,r,s}(G_{p^\infty})$ is a $\Lambda_E(G_{p^\infty})$-module.

For $F \in \mathcal{H}_{E,r,s}$, and a Hecke character Ξ of the infinity type (a, b) and conductor $\mathfrak{p}^{n_{\mathfrak{p}}} \overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, define

$$F^{(\widetilde{\Xi}_{\mathfrak{p}})} := F(u^a \zeta_{\mathfrak{p}} - 1, \gamma_{\overline{\mathfrak{p}}} - 1), \tag{2.6.2}$$

$$F^{(\widetilde{\Xi}_{\overline{\mathfrak{p}}})} := F(\gamma_{\mathfrak{p}} - 1, u^b \zeta_{\overline{\mathfrak{p}}} - 1), \tag{2.6.3}$$

where $\zeta_{\mathfrak{q}}$ is a primitive $p^{(n_{\mathfrak{q}} - 1)}$-th root of unity for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$.

Lemma 2.6.7. Let E be a finite extension of $\mathbb{Q}_p$, $F \in \mathcal{H}_{E,r,s}$ be a power series and let Ξ be a Hecke character of K of infinity type (a,b) and conductor $\mathfrak{p}^{n_\mathfrak{p}}.\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$. Then, $F^{(\tilde{\Xi}_\mathfrak{p})} \in \mathcal{H}_{E(\tilde{\Xi}_\mathfrak{p}(\gamma_\mathfrak{p})),s}(\Gamma_{\overline{\mathfrak{p}}})$ and $F^{(\tilde{\Xi}_{\overline{\mathfrak{p}}})} \in \mathcal{H}_{E(\tilde{\Xi}_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}})),r}(\Gamma_\mathfrak{p})$, where $E(\tilde{\Xi}_\mathfrak{q}(\gamma_\mathfrak{q}))$ is finite extension of E by adjoining values of $\tilde{\Xi}_\mathfrak{q}(\gamma_\mathfrak{q})$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$.

Proof. The power series F belongs to the power series ring $\mathcal{H}_{E,r,s}$ which is completed tensor product of $\mathcal{H}_{E,r}(\Gamma_\mathfrak{p})$ and $\mathcal{H}_{E,s}(\Gamma_{\overline{\mathfrak{p}}})$. Thus, if we substitute $\gamma_\mathfrak{p}$ by $u^a\zeta_\mathfrak{p}$, then $F^{(\tilde{\Xi}_\mathfrak{p})}$ will be a power series with growth s and the coefficients of $F^{(\tilde{\Xi}_\mathfrak{p})}$ will be in $E(\tilde{\Xi}_\mathfrak{p}(\gamma_\mathfrak{p}))$.

Similarly, substituting $\gamma_{\overline{\mathfrak{p}}}$ by $u^b\zeta_{\overline{\mathfrak{p}}}$, then $F^{(\tilde{\Xi}_{\overline{\mathfrak{p}}})} \in \mathcal{H}_{E(\tilde{\Xi}_{\overline{\mathfrak{p}}}(\overline{\mathfrak{p}})),s}(\Gamma_\mathfrak{p})$. ∎

Isotypical components. Let $\eta : \Delta_K \to \mathbb{Z}_p^\times$ (or $\eta : \Delta_K \to \overline{\mathbb{Q}_p}^\times$) be a character, where Δ_K is a finite abelian subgroup appearing in G_{p^∞}. Write $e_\eta = \frac{1}{|\Delta_K|}\sum_{\sigma \in \Delta} \eta^{-1}(\sigma)\sigma$. For $* \in \{\mathbb{C}_p, E\}$, if $F \in \mathcal{H}_{*,r,s}(G_{p^\infty})$, write $F^\eta = e_\eta(F)$ for its image in $e_\eta(\mathcal{H}_{*,r,s}(G_{p^\infty})) \cong \mathcal{H}_{*,r,s}$. Note that this isomorphism is a module isomorphism. If $\eta = 1$, we simply write F^{Δ_K} instead of F^η. Note that $F^{\Delta_K} \in \mathcal{H}_{*,r,s}$.

After this point, our integer k will be greater than or equal to 0 rather than 2, since we will be working with Bianchi modular forms which are *cohomological* in nature.

2.7 Cuspidal Bianchi modular forms and their p-adic L-functions

In this section, we will briefly recall the definition of Bianchi modular forms, their L-functions, and the p-adic L-functions constructed by Williams in [Wil17].

2.7.1 Bianchi modular forms

We define Bianchi modular forms adelically. Fix a quadratic imaginary field K. Let σ denote an embedding $K \hookrightarrow \mathbb{C}$, and let c denote the complex conjugate, i.e., $c = \overline{\sigma}$.

Definition 2.7.1 (Level). Let $\widehat{\mathcal{O}_K} := \mathcal{O}_K \otimes \widehat{\mathbb{Z}}$. For any ideal $\mathfrak{m} \in \mathcal{O}_K$, define

1. $\Omega_0(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}_K}) \colon c \in \mathfrak{m}\widehat{\mathcal{O}_K} \right\}$,
2. $\Omega_1(\mathfrak{m}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}_K}) \colon c \in \mathfrak{m}\widehat{\mathcal{O}_K}, a, d \in 1 + \mathfrak{m}\widehat{\mathcal{O}_K} \right\}$.

Fix $k, \ell \in \mathbb{Z}_{\geqslant 0}$. For any ring R, $V_{k+\ell+2}(R)$ denotes the space of homogeneous polynomials over R in two variables of degree $k + \ell + 2$.

Let ε be a Hecke character of infinity type $(-k, -\ell)$ and conductor dividing the level $\mathfrak{m}$. For $x_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Omega_0(\mathfrak{m})$, set $\varepsilon_\mathfrak{m}(x_f) = \varepsilon_\mathfrak{m}(d) = \prod_{\mathfrak{q}|\mathfrak{m}} \varepsilon_\mathfrak{q}(d_\mathfrak{q})$.

Definition 2.7.2 (Bianchi modular forms)**.** We say a function

$$\mathcal{F} : \mathrm{GL}_2(\mathbb{A}_K) \to V_{k+\ell+2}(\mathbb{C})$$

is a cuspform of weight (k, ℓ), level $\mathfrak{m}$ (i.e. level $\Omega_1(\mathfrak{m})$), and central action of ε if it satisfies

1. $\mathcal{F}(zg) = \varepsilon(z)\mathcal{F}(g)$ for $z \in \mathbb{A}_K^\times \cong Z(\mathrm{GL}_2(\mathbb{A}_K^\times))$;
2. For all $u = u_f \cdot u_\infty \in \Omega_0(\mathfrak{m})\mathrm{SU}_2(\mathbb{C})$,

   $$\mathcal{F}(gu) = \varepsilon_{\mathfrak{m}}(u_f)\mathcal{F}(g)\rho_{k+\ell+2}(u_\infty),$$

   where $\rho : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(V_{k+\ell+2}(\mathbb{C}))$ is an irreducible right representation;
3. The function $\mathcal{F}$ is left-invariant under the group $\mathrm{GL}_2(K)$.
4. The function $\mathcal{F}$ is an eigenfunction of Casimir operators ∂_{id} and ∂_c, with eigenvalues $k^2/2 + k$ and $\ell^2/2 + \ell$ respectively. Here $\partial_{\mathrm{id}}/4, \partial_c/4$ are components of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$.
5. We say $\mathcal{F}$ is a *cuspidal* Bianchi modular form if $\mathcal{F}$ satisfies the cuspidal condition for all $g \in \mathrm{GL}_2(\mathbb{A}_K)$, that is, we have

   $$\int_{K\backslash\mathbb{A}_K} \mathcal{F}(ug)\, du = 0,$$

   where we consider $\mathbb{A}_K$ inside $\mathrm{GL}_2(\mathbb{A}_K)$, by the map sending $u \to \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, and du is the Lebesgue measure on $\mathbb{A}_K$.

See [Wil17, Definition 1.2] and [Pal25, Definition 2.1] for the details about the definition of Bianchi modular forms.

Remarks 2.7.3.

1. In a result by Harder, he showed that if $\mathcal{F}$ is a non-zero cuspidal Bianchi modular form of weight (k, ℓ), then $k = \ell$. See [Har87].
2. A Bianchi modular form $\mathcal{F}$ descends to give a collection of h automorphic forms $\mathcal{F}^i : \mathbb{H}_3 \to V_{k+\ell+2}(\mathbb{C})$, where h is the class number of K, and $\mathbb{H}_3 := \mathbb{R}_{\geqslant 0} \times \mathbb{C} \cong \mathrm{GL}_2(\mathbb{C})/[\mathrm{SU}_2(\mathbb{C})Z(\mathrm{GL}_2(\mathbb{C}))]$ is the hyperbolic 3-space.

From now onwards, we will deal with cuspidal Bianchi modular forms, and hence we will fix $k = \ell$ in $\mathbb{Z}_{\geqslant 0}$. We will decompose p-adic L-functions associated to non-parallel weight Bianchi modular forms (of level $\Omega_0(\mathfrak{m})$) in Appendix A.

For a non-trivial cuspform $\mathcal{F}$ of level $\mathfrak{m}$ and weight (k, k), we have the following Fourier expansion:

$$\mathcal{F}\left[\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right] = |t|_K \sum_{a \in K^\times} c(at\mathcal{D}_K, \mathcal{F}) W(at_\infty) \mathrm{e}_K(az), \tag{2.7.1}$$

where $\mathcal{D}_K$ is a generator of the different of K, e_K is an additive character of $K\backslash\mathbb{A}_K$, and W is the Whittaker function. Note that the Fourier coefficients $c(\cdot,\mathcal{F})$ are functions on the fractional ideals of K, and $c(I,\mathcal{F})=0$ if I is not an integral ideal. See [Wil17, Section 1.2], [Pal25, Section 2.3] for the details. For the Whittaker function, see [Pal23, Section 2.3].

Definition 2.7.4 (Twisted L-function of a Bianchi modular form)**.** Let $\mathcal{F}$ be a cuspidal Bianchi modular form of any weight and level $\mathfrak{m}$. Let Ξ be a Hecke character of conductor $\mathfrak{f}$. The twisted L-function of $\mathcal{F}$ is defined as

$$L(\mathcal{F},\Xi,s)=\sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_K,\\ (\mathfrak{f},\mathfrak{a})=1}} c(\mathfrak{a},\mathcal{F})\Xi(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-s},$$

where $c(\mathfrak{a},\mathcal{F})$ is the $\mathfrak{a}$-th Fourier coefficient of $\mathcal{F}$ and $s\in\mathbb{C}$ in some suitable right-half plane.

We also consider the completed L-function

$$\Lambda(\mathcal{F},\Xi,s)=\frac{\Gamma(q+s)\Gamma(r+s)}{(2\pi i)^{q+s}(2\pi i)^{r+s}}L(\mathcal{F},\Xi,s),$$

where (q,r) is the infinity type of Ξ, and $\Gamma(-)$ are the Deligne's Γ-factors.

2.7.2 p-adic L-function of Bianchi modular forms

Let $K/\mathbb{Q}$ be a quadratic imaginary field with class number h_K, discriminant $-D$. Let p be an odd prime splitting in K as $(p)=\mathfrak{p}\overline{\mathfrak{p}}$.

Theorem 2.7.5 (Williams, [Wil17, Theorem 7.4])**.** Let $\mathcal{F}$ be a cuspidal Bianchi eigenform over K of weight (k,k) and level $\mathfrak{n}$ such that $(p)\mid\mathfrak{n}$. Let $a_{\mathfrak{q}}$ denote the $U_{\mathfrak{q}}$-eigenvalues of $\mathcal{F}$ where $v_p(a_{\mathfrak{q}})<(k+1)$ for all $\mathfrak{q}\mid p$. For any ideal $\mathfrak{f}$, we define the operator $U_{\mathfrak{f}}$ as

$$U_{\mathfrak{f}}:=\prod_{\mathfrak{p}^n\|\mathfrak{f}} U_{\mathfrak{p}}^n.$$

Then there exists a locally analytic distribution $L_{p,\mathcal{F}}$ on the ray class group G_{p^∞} such that for any Hecke character Ξ of infinity type $(0,0)\leqslant(a,b)\leqslant(k,k)$ and conductor $\mathfrak{f}$, we have

$$L_{p,\mathcal{F}}(\tilde{\Xi})=(\text{explicit factor})\frac{1}{a_{\mathfrak{f}}}\frac{\Lambda(\mathcal{F},\Xi,1)}{\Omega_{\mathcal{F}}}, \tag{2.7.2}$$

where $a_{\mathfrak{f}}$ is $U_{\mathfrak{f}}$-eigenvalue of $\mathcal{F}$, and $\Omega_{\mathcal{F}}$ is a complex period. Furthermore, the explicit factor is given by

$$\prod_{\mathfrak{p}\mid p} e(\mathfrak{p})\cdot\frac{\Xi(x_{\mathfrak{f}})Dw\tau(\Xi^{-1})}{(-1)^{a+b+2}2\Xi_{\mathfrak{f}}(x_{\mathfrak{f}})},$$

where $-D$ is the discriminant of K, $\tau(\cdot)$ is the Gauss sum, $w=|\mathcal{O}_K^\times|$, and $e_{\mathfrak{p}}$ is described in [Pal23, Equation 3.2].

The distribution $L_{p,\mathcal{F}}$ is $(v_p(a_{\mathfrak{p}}))_{\mathfrak{p}\mid p}$-admissible and is unique with these interpolation and growth properties.

See [Wil17, Definition 6.14] for the definition of admissibility. Naively we say a distribution μ is (h_1, h_2)-admissible if μ is $O(\log_p^{h_1})$ in one variable (corresponding to $\mathfrak{p}$) and is $O(\log_p^{h_2})$ in the other variable (corresponding to $\overline{\mathfrak{p}}$).

Remark 2.7.6. Although we have stated Theorem 2.7.5 for the p split case, Williams proved it without any conditions on p in [Wil17] and for $p = 2$ as well.

Remark 2.7.7. In [Wil17, Section 6.3], Williams has defined the admissibility for locally analytic distributions on $\mathcal{O}_K \otimes \mathbb{Z}_p$. But using methods from [Loe14], we can extend the notion of admissibility for locally analytic distributions on $\mathcal{O}_K \otimes \mathbb{Z}_p$ to locally analytic distributions on ray class group G_{p^∞}. See [Wil17, Section 7.4] for more details.

Remark 2.7.8. For simplicity, assume Δ_F to be trivial and $G_{p^\infty} \cong \Gamma_{\mathfrak{p}} \times \Gamma_{\overline{\mathfrak{p}}} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. For real numbers $r, s \geqslant 0$ and let E be a finite extension of $\mathbb{Q}_p$, we define $D^{(r,s)}(G_{p^\infty}, E)$ to be the set of distributions μ of G_{p^∞} such that for $m, n \in \mathbb{Z}_{\geqslant 0}$ and integers $0 \leqslant i \leqslant \lfloor r \rfloor$ and $0 \leqslant j \leqslant \lfloor s \rfloor$,

$$v_p\left(\int_{\substack{a+p^m\mathbb{Z}_p \\ b+p^n\mathbb{Z}_p}} (x-a)^i (y-b)^j d\mu\right) \geqslant R - (r-i)m - (s-j)n,$$

for some constant $R \in \mathbb{R}$ which only depends on μ. See [Col10, Theorem 2.3.2] for the analogous definitions in the one-variable setting.

Since p splits in the quadratic imaginary field K, we can identify $D^{(u,v)}(G_{p^\infty}, E)$ with $\mathcal{H}_{E,u,v}(G_{p^\infty})$ via Amice transform $\mu \mapsto A_\mu(T_1, T_2) = \displaystyle\int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1+T_1)^x (1+T_2)^y \mu(x, y)$.

Therefore, if p splits in K as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, and $\mathcal{F}$ is a Bianchi modular form which satisfies the conditions of the above theorem, then $L_{p,\mathcal{F}}$ is a $(v_p(a_{\mathfrak{p}}), v_p(a_{\overline{\mathfrak{p}}}))$-admissible locally analytic distribution on G_{p^∞}, that is, by [Wil17] and [Loe14],

$$L_{p,\mathcal{F}} \in D^{(v_p(a_{\mathfrak{p}}), v_p(a_{\overline{\mathfrak{p}}}))}(G_{p^\infty}, F), \tag{2.7.3}$$

where F is a finite extension of $\mathbb{Q}_p$ containing $a_{\mathfrak{p}}$ and $a_{\overline{\mathfrak{p}}}$. By Remark 2.7.8, we can view

$$L_{p,\mathcal{F}} \in \mathcal{H}_{F, v_p(a_{\mathfrak{p}}), v_p(a_{\overline{\mathfrak{p}}})}(G_{p^\infty}).$$

2.8 Factorisation of p-adic L-functions of Bianchi modular forms

In this section, we first define p-stabilizations of cuspidal Bianchi modular forms. Next, we modify our logarithmic matrix $\underline{M}$ and prove the main factorization theorem of two-variable p-adic L-functions. From now we fix the imaginary quadratic field K and all the cuspidal Bianchi modular forms $\mathcal{F}$ we consider are over K. Also, the fixed odd prime p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$.

2.8.1 p-stabilization of Bianchi modular forms

We begin with a cuspidal Bianchi modular form $\mathcal{F}$ of weight (k,k) and level $\mathfrak{m}$ such that $(p\mathcal{O}_K, \mathfrak{m}) = 1$. We further assume $\mathcal{F}$ is a Hecke eigen cuspform and for $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}\}$, we have $T_\mathfrak{q}\mathcal{F} = a_\mathfrak{q}\mathcal{F}$. Note that the norm of $\mathfrak{q}$ is p. Consider the Hecke polynomial $X^2 - a_\mathfrak{q}X + u_\mathfrak{q} \cdot p^{k-1} = (X - \alpha_\mathfrak{q})(X - \beta_\mathfrak{q})$, where $u_\mathfrak{q}$ is some p-adic unit coming from the nebentypus of $\mathcal{F}$.

There are four p-stabilisations $\mathcal{F}^{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}$,and $\mathcal{F}^{\beta_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}$ of level $p\mathfrak{m}$, such that for $* \in \{\alpha_\mathfrak{p}, \beta_\mathfrak{p}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, we have

$$U_\mathfrak{p}(\mathcal{F}^{*,\dagger}) = *\mathcal{F}^{*,\dagger}, \tag{2.8.1}$$

$$U_{\overline{\mathfrak{p}}}(\mathcal{F}^{*,\dagger}) = \dagger\mathcal{F}^{*,\dagger}. \tag{2.8.2}$$

For more details about p-stabilizations, refer to [Pal23, Section 3.3].

Assumption 2.8.1. Throughout the chapter, we assume

1. $\mathcal{F}$ is good non-ordinary at both the primes $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ i.e. $v_p(a_\mathfrak{p})$ and $v_p(a_{\overline{\mathfrak{p}}}) > 0$.
2. $v_p(a_\mathfrak{q}) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$ and $\alpha_\mathfrak{q} \neq \beta_\mathfrak{q}$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$.

Note that for any $x \in \{\alpha_\mathfrak{p}, \beta_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, we know $v_p(x) < k + 1$.

Let E be a finite extension of $\mathbb{Q}_p$ which contains $\alpha_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}, \beta_\mathfrak{p}$ and $\beta_{\overline{\mathfrak{p}}}$, and K.

2.8.2 Modified logarithmic matrices

Recall the ring isomorphism $\tau_\mathfrak{q} : \mathcal{H}_F(\Gamma_1) \to \mathcal{H}_F(\Gamma_\mathfrak{q})$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Consider the change of variable map between matrices induced by $\tau_\mathfrak{q}$:

$$\mathrm{Mat}_\mathfrak{q} : M_{2,2}(\mathcal{H}_F(\Gamma_1)) \to M_{2,2}(\mathcal{H}_F(\Gamma_\mathfrak{q})),$$

such that

$$\mathrm{Mat}_\mathfrak{q}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} \tau_\mathfrak{q}(A) & \tau_\mathfrak{q}(B) \\ \tau_\mathfrak{q}(C) & \tau_\mathfrak{q}(D) \end{pmatrix}.$$

Note that this map is also a ring isomorphism.

Let $\mathcal{F}$ be a Bianchi modular form of level $\mathfrak{m}$ coprime to p and let $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ be the $T_\mathfrak{q}$-eigenvalues, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. From the Sections 2.3, 2.4, and 2.5, we construct a logarithmic matrix $\underline{M(\mathfrak{q})} \in M_{2,2}(\mathcal{H}_E(\Gamma_1))$ using $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$, since $v_p(a_\mathfrak{q}) = v_p(\alpha_\mathfrak{q} + \beta_\mathfrak{q}) > \left\lfloor \dfrac{k}{p-2} \right\rfloor$ and $\alpha_\mathfrak{q}\beta_\mathfrak{q} = u_\mathfrak{q}p^{k+1}$. Let $E/\mathbb{Q}_p$ be a finite extension large enough to contain $\alpha_\mathfrak{q}, \beta_\mathfrak{q}$ and a fixed square root of $u_\mathfrak{q}$. Let

$$A_{\varphi,\mathfrak{q}} = \begin{pmatrix} 0 & \dfrac{-1}{u_\mathfrak{q}p^{k+1}} \\ 1 & \dfrac{a_\mathfrak{q}}{u_\mathfrak{q}p^{k+1}}, \end{pmatrix},$$

$$Q_{\mathfrak{q}} = \begin{pmatrix} \alpha_{\mathfrak{q}} & -\beta_{\mathfrak{q}} \\ -p^{k+1}u_{\mathfrak{q}} & p^{k+1}u_{\mathfrak{q}} \end{pmatrix},$$

$$\underline{M_{\mathfrak{q}}} = \mathrm{Mat}_{\mathfrak{q}}(\underline{M(\mathfrak{q})}).$$

Remark 2.8.2. From Proposition 2.5.1 and Lemmas 2.5.2, 2.5.3, we deduce:

1. If $Q_{\mathfrak{q}}^{-1}(\underline{M(\mathfrak{q})}) = \begin{pmatrix} P_1(\mathfrak{q}) & P_2(\mathfrak{q}) \\ P_3(\mathfrak{q}) & P_4(\mathfrak{q}) \end{pmatrix}$, then $P_1(\mathfrak{q}), P_2(\mathfrak{q}) \in \mathcal{H}_{E,v_p(\alpha_{\mathfrak{q}})}(\Gamma_1)$ and $P_3(\mathfrak{q}), P_4(\mathfrak{q}) \in \mathcal{H}_{E,v_p(\beta_{\mathfrak{q}})}(\Gamma_1)$.
2. The second row of $A_{\varphi,\mathfrak{q}}^{-n}\underline{M(\mathfrak{q})}$ is divisible by $\Phi_{n-1,k+1}(\gamma_0)$ over $\mathcal{H}_E(\Gamma_1)$.
3. The determinant $\det(\underline{M(\mathfrak{q})})$ is $\dfrac{\log_{p,k+1}(\gamma_0)}{\delta_{k+1}(\gamma_0 - 1)}$, up to a unit in $\Lambda_E(\Gamma_1)$.

Remark 2.8.3. Here we get $k+1$ since $k \in \mathbb{Z}_{\geqslant 0}$ and not $k \in \mathbb{Z}_{\geqslant 2}$.

Theorem 2.8.4. For $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ the following are true,

1. The elements in the first row of $Q_{\mathfrak{q}}^{-1}\underline{M_{\mathfrak{q}}}$ are inside $\mathcal{H}_{E,v_p(\alpha_{\mathfrak{q}})}(\Gamma_{\mathfrak{q}})$, while the elements in the second row are in $\mathcal{H}_{E,v_p(\beta_{\mathfrak{q}})}(\Gamma_{\mathfrak{q}})$.
2. The second row of the matrix $A_{\varphi,\mathfrak{q}}^{-n}\underline{M_{\mathfrak{q}}}$ is divisible by the cyclotomic polynomial $\Phi_{n-1,k+1}(\gamma_{\mathfrak{q}})$.
3. The determinant of $\underline{M_{\mathfrak{q}}}$ is $\dfrac{\log_{p,k+1}(\gamma_{\mathfrak{q}})}{\delta_{k+1}(\gamma_{\mathfrak{q}} - 1)}$ upto a unit in $\Lambda_E(\Gamma_{\mathfrak{q}})$.

Proof. First statement follows from Proposition 2.5.1 and the definitions of $Q_{\mathfrak{q}}$ and $\underline{M_{\mathfrak{q}}}$. The second statement follows from Lemma 2.5.2.

For the last statement, from the definitions we have $\underline{M_{\mathfrak{q}}} = \mathrm{Mat}_{\mathfrak{q}}(\underline{M(\mathfrak{q})})$. Thus,

$$\begin{aligned} \det(\underline{M_{\mathfrak{q}}}) &= \det(\mathrm{Mat}_{\mathfrak{q}}(\underline{M(\mathfrak{q})})), \\ &= \tau_{\mathfrak{q}}(\det(\underline{M(\mathfrak{q})})), \end{aligned}$$

since $\tau_{\mathfrak{q}}$ is a ring isomorphism. Hence, the result follows from Lemma 2.5.3. ∎

For the rest of the chapter, we will write

$$Q_{\mathfrak{q}}^{-1}\underline{M_{\mathfrak{q}}} = \begin{pmatrix} P_{1,\mathfrak{q}} & P_{2,\mathfrak{q}} \\ P_{3,\mathfrak{q}} & P_{4,\mathfrak{q}} \end{pmatrix}. \tag{2.8.3}$$

2.8.3 The main theorem and its proof

In this section, we generalize the results of [Lei14] and use them to decompose the two variable p-adic L-functions of Bianchi modular forms.

For the Bianchi cuspform $\mathcal{F}$ of level $\mathfrak{m}$ which is not divisible by p, let $a_{\mathfrak{q}}$ be the $T_{\mathfrak{q}}$-eigenvalue of $\mathcal{F}$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Recall that we have four p-stabilizations $\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}$, and $\mathcal{F}^{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}$ of level $p\mathfrak{m}$, where $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$ are the roots of Hecke polynomial $X^2 - a_{\mathfrak{q}}X + u_{\mathfrak{q}}p^{k-1}$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$.

Therefore, from Theorem 2.7.5 and equation (2.7.3), for $* \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, we have

$$L_{*,\dagger} := L_{p,\mathcal{F}^{*,\dagger}} \in \mathcal{H}_{E,v_p(*),v_p(\dagger)}(G_{p^\infty}),$$

since $v_p(*), v_p(\dagger) < k+1$. Moreover, for any Hecke character Ξ of infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \geqslant 1$, we have

$$L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}(\tilde{\Xi}) = \alpha_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}C_{a,b,\tilde{\Xi}}, \tag{2.8.4}$$

$$L_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}(\tilde{\Xi}) = \alpha_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}C_{a,b,\tilde{\Xi}}, \tag{2.8.5}$$

$$L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}(\tilde{\Xi}) = \beta_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}C_{a,b,\tilde{\Xi}}, \tag{2.8.6}$$

$$L_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}(\tilde{\Xi}) = \beta_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}C_{a,b,\tilde{\Xi}}, \tag{2.8.7}$$

where $C_{a,b,\tilde{\Xi}} \in \overline{\mathbb{Q}_p}$ is a constant independent of $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}$. More precisely, $C_{a,b,\tilde{\Xi}}$ is (explicit factor)$\dfrac{\Lambda(\mathcal{F},\Xi,1)}{\Omega_{\mathcal{F}}}$, since the conductor of Ξ is $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \in \mathbb{Z}_{\geqslant 0}$, we know $L(\mathcal{F},\Xi,1) = L(\mathcal{F}^{*,\dagger},\Xi,1)$ for all p-stabilizations $\mathcal{F}^{*,\dagger}$ of $\mathcal{F}$ and hence $C_{a,b,\tilde{\Xi}}$ is independent of $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}$.

The main theorem is as follows:

Theorem 2.8.5. There exist two variable power series with bounded coefficients, that is, there exist $L_{\sharp,\sharp}, L_{\sharp,\flat}, L_{\flat,\sharp}, L_{\flat,\flat} \in \Lambda_E(G_{p^\infty})$ such that

$$\begin{pmatrix} L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_{\overline{\mathfrak{p}}}^{-1}\underline{M_{\overline{\mathfrak{p}}}}\begin{pmatrix} L_{\sharp,\sharp} & L_{\flat,\sharp} \\ L_{\sharp,\flat} & L_{\flat,\flat} \end{pmatrix}(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})^T. \tag{2.8.8}$$

Remark 2.8.6. Theorem 2.8.5 is analogous to [Lei14, Theorem 2.2] and [BL21, Equation (24)].

We first factorize through the variable $\gamma_{\mathfrak{p}}$ and then through $\gamma_{\overline{\mathfrak{p}}}$. In other words, we will first decompose the matrix $\begin{pmatrix} L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} & L_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix}$ in terms of matrix, say C, and $\underline{M_{\mathfrak{p}}}$. Then we decompose C as a product of matrices $\begin{pmatrix} L_{\sharp,\sharp} & L_{\flat,\sharp} \\ L_{\sharp,\flat} & L_{\flat,\flat} \end{pmatrix}$ and $\underline{M_{\overline{\mathfrak{p}}}}$.

First, we need the following classical result:

Lemma 2.8.7. Let γ be a topological generator of $\mathbb{Z}_p$. Let s, h be non-negative integers and assume $s < h$. If $F \in E[[\gamma - 1]]$ is $O(\log_p^s)$ and vanishes at all characters of type $\chi^i\omega$ for all $0 \leqslant i \leqslant h-1$, where χ is any character which sends γ to another topological generator $u \in 1+p\mathbb{Z}_p$ (for example the cyclotomic character) and ω is any Dirichlet character of conductor $p^n, n \geqslant 1$, then F is identically 0.

Proof. See [AV75, Lemme II.2.5] and [Vis76, Lemma 2.10].

We give a sketch here. We say a power series $G_1 \in \mathcal{H}_E$ is $o(G_2)$ if

$$\|G_1\|_\rho = o(\|G_2\|_\rho)$$

as $\rho \to 1^-$. The power series F is $o(\log_p^h)$, since F is $O(\log_p^s)$ and $s < h$. Suppose F is not 0 and, from the assumption, the zeros of F are of the form $u^j\zeta - 1$, where ζ is a p^n-th root of unity, for all $0 \leqslant j \leqslant h-1$ and $n \geqslant 1$. Then, by Remark 2.4.13, we have

$$F = \log_{p,h}(\gamma) \cdot G,$$

where $G \neq 0$ is another power series. Note that $\log_{p,h}(\gamma)$ is not $o(\log_p^h)$, and therefore F is not $o(\log_p^h)$. This is a contradiction. ∎

Recall that E is a finite extension of $\mathbb{Q}_p$ containing $\alpha_\mathfrak{p}, \beta_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}$ and $\beta_{\overline{\mathfrak{p}}}$. Let $S_\mathfrak{p}$ be the set of all Hecke characters on the ray class group G_{p^∞} with infinity type $(0,0) \leqslant (a,0) \leqslant (k,0)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}$, for $n_\mathfrak{p} > 1$. Similarly, let $S_{\overline{\mathfrak{p}}}$ be the set of all Hecke characters on the ray class group G_{p^∞} with infinity type $(0,0) \leqslant (0,b) \leqslant (0,k)$ and conductor $\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, for $n_{\overline{\mathfrak{p}}} > 1$.

Proposition 2.8.8. There exist $L_{\sharp,\alpha_{\overline{\mathfrak{p}}}}, L_{\flat,\alpha_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E,0,v_p(\alpha_{\overline{\mathfrak{p}}})}(G_{p^\infty})$ and $L_{\sharp,\beta_{\overline{\mathfrak{p}}}}, L_{\flat,\beta_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E,0,v_p(\beta_{\overline{\mathfrak{p}}})}(G_{p^\infty})$ such that

$$\begin{pmatrix} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} & L_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}} & L_{\beta_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = \begin{pmatrix} L_{\sharp,\alpha_{\overline{\mathfrak{p}}}} & L_{\flat,\alpha_{\overline{\mathfrak{p}}}} \\ L_{\sharp,\beta_{\overline{\mathfrak{p}}}} & L_{\flat,\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} (Q_\mathfrak{p}^{-1}\underline{M_\mathfrak{p}})^T.$$

Proof. This is a generalization of [Lei14, Proposition 2.3]. Recall that, for $* \in \{\alpha_\mathfrak{p}, \beta_\mathfrak{p}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, $L_{*,\dagger}$ are locally analytic distributions on the ray class group $G_{p^\infty} \cong \Delta_K \times \overline{\langle\gamma_\mathfrak{p}\rangle} \times \overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}$. For any character $\eta : \Delta_K \to \overline{\mathbb{Z}_p}^\times$, we will prove that for $L^\eta_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}, L^\eta_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E,v_p(\alpha_\mathfrak{p}),v_p(\alpha_{\overline{\mathfrak{p}}})}$, there exist $L^\eta_{\sharp,\alpha_{\overline{\mathfrak{p}}}}, L^\eta_{\flat,\alpha_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E,0,v_p(\alpha_{\overline{\mathfrak{p}}})}$ such that

$$\begin{pmatrix} L^\eta_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \\ L^\eta_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_\mathfrak{p}^{-1}\underline{M_\mathfrak{p}} \begin{pmatrix} L^\eta_{\sharp,\alpha_{\overline{\mathfrak{p}}}} \\ L^\eta_{\flat,\alpha_{\overline{\mathfrak{p}}}} \end{pmatrix}. \tag{2.8.9}$$

and therefore,

$$\begin{pmatrix} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \\ L_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_\mathfrak{p}^{-1}\underline{M_\mathfrak{p}} \begin{pmatrix} L_{\sharp,\alpha_{\overline{\mathfrak{p}}}} \\ L_{\flat,\alpha_{\overline{\mathfrak{p}}}} \end{pmatrix}.$$

Fix $\eta = 1$. The proof for other characters is similar.

Let Ξ be any Hecke character of K of infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, where $n_\mathfrak{p}, n_{\overline{\mathfrak{p}}} \geqslant 1$. Using equations (2.8.4) and (2.8.6), we get

$$L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) = \alpha_\mathfrak{p}^{-n_\mathfrak{p}}(\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{a,b,\widetilde{\Xi}}), \tag{2.8.10}$$

$$L^{\Delta_K}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) = \beta_\mathfrak{p}^{-n_\mathfrak{p}}(\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{a,b,\widetilde{\Xi}}), \tag{2.8.11}$$

where $\check{\Xi} = \widetilde{\Xi}|_{\overline{\langle\gamma_\mathfrak{p}\rangle}\times\overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}} = \widetilde{\Xi}_\mathfrak{p} \cdot \widetilde{\Xi}_{\overline{\mathfrak{p}}}$. Denote $\alpha^{-n_{\overline{\mathfrak{p}}}}C_{a,b,\widetilde{\Xi}}$ by D. Using (2.6.3), we can rewrite equations (2.8.10) and (2.8.11) as

$$L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}_\mathfrak{p}) = \alpha_\mathfrak{p}^{-n_\mathfrak{p}}D, \tag{2.8.12}$$

$$L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}_\mathfrak{p}) = \beta_\mathfrak{p}^{-n_\mathfrak{p}}D. \tag{2.8.13}$$

Note that $L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E',v_p(\alpha_\mathfrak{p})}(\Gamma_\mathfrak{p})$ and $L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E',v_p(\beta_\mathfrak{p})}(\Gamma_\mathfrak{p})$, where $E' = E(\widetilde{\Xi}_{\overline{\mathfrak{p}}}(\gamma_{\overline{\mathfrak{p}}}))$ is a finite field extension of E.

Let

$$G_1 = P_{4,\mathfrak{p}}L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}},$$

and

$$G_2 = -P_{3,\mathfrak{p}}L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K(\widetilde{\Xi}_{\overline{\mathfrak{p}}})}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}},$$

where $Q_\mathfrak{p}^{-1}\underline{M_\mathfrak{p}} = \begin{pmatrix} P_{1,\mathfrak{p}} & P_{2,\mathfrak{p}} \\ P_{3,\mathfrak{p}} & P_{4,\mathfrak{p}} \end{pmatrix} \in M_{2,2}(\mathcal{H}_E(\Gamma_\mathfrak{p}))$.

Then, from Theorem 2.8.4, we get $G_1, G_2 \in \mathcal{H}_{E',k-1}(\Gamma_\mathfrak{p})$, and equations (2.8.12) and (2.8.13) imply $G_1(\widetilde{\Xi}_\mathfrak{p}) = G_2(\widetilde{\Xi}_\mathfrak{p}) = 0$. Hence, from Theorem 2.5.5, we deduce that $\det(Q_\mathfrak{p}^{-1}\underline{M_\mathfrak{p}})$ divides both G_1 and G_2 in $\mathcal{H}_{E',k-1}(\Gamma_\mathfrak{p})$.

Thus,

$$P_{4,\mathfrak{p}}(\check{\Xi})L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) - P_{2,\mathfrak{p}}(\check{\Xi})L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) = -P_{3,\mathfrak{p}}(\check{\Xi})L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) + P_{1,\mathfrak{p}}(\check{\Xi})L^{\Delta_K}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) = 0, \tag{2.8.14}$$

for any Hecke character Ξ of infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$.

Thus, for any Hecke characters $\omega_1 \in S_\mathfrak{p}$, $\omega_2 \in S_{\overline{\mathfrak{p}}}$, (2.8.14) implies

$$(P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}})(\widetilde{\omega_1}_\mathfrak{p}\widetilde{\omega_2}_{\overline{\mathfrak{p}}}) = 0,$$

which we rewrite as

$$(P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_\mathfrak{p})}(\widetilde{\omega_2}_{\overline{\mathfrak{p}}}) = 0. \tag{2.8.15}$$

Similarly.

$$(-P_{3,\mathfrak{p}}L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K}_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_\mathfrak{p})}(\widetilde{\omega_2}_{\overline{\mathfrak{p}}}) = 0. \tag{2.8.16}$$

Hence, the distributions $(P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_{\mathfrak{p}})}$ and $(-P_{3,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_{\mathfrak{p}})}$ vanish at all characters $\omega_2 \in S_{\overline{\mathfrak{p}}}$. Moreover, these two distributions belong to $\mathcal{H}_{E',v_p(\alpha_{\overline{\mathfrak{p}}})}(\Gamma_{\overline{\mathfrak{p}}})$ and $v_p(\alpha_{\overline{\mathfrak{p}}}) < k+1$. Therefore, using Lemma 2.8.7, we get

$$(P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_{\mathfrak{p}})} = 0,$$

$$(-P_{3,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})^{(\widetilde{\omega_1}_{\mathfrak{p}})} = 0.$$

Hence, from the proof of Theorem 2.5.5, we conclude that $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})$ divide $(P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})$ and $(-P_{3,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}})$ over the two-variable distribution algebra $\mathcal{H}_{E,k-1,v_p(\alpha_{\overline{\mathfrak{p}}})}$. More precisely, $\det(Q_{\mathfrak{p}}^{-1}\underline{M})$ divides both distributions in the variable $\gamma_{\mathfrak{p}}-1$ while not touching $\gamma_{\overline{\mathfrak{p}}}-1$.

Write

$$L^{\Delta_K}_{\sharp,\alpha_{\overline{\mathfrak{p}}}} = \frac{P_{4,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}}{\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})},$$

$$L^{\Delta_K}_{\flat,\alpha_{\overline{\mathfrak{p}}}} = \frac{-P_{3,\mathfrak{p}}L^{\Delta_K}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L^{\Delta_K}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}}{\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})}.$$

Then, since $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}) \sim \log_p^{k-1}$, Theorem 2.5.5 implies $L^{\Delta_K}_{\sharp,\alpha_{\overline{\mathfrak{p}}}}$ and $L^{\Delta_K}_{\flat,\alpha_{\overline{\mathfrak{p}}}}$ are $O(1)$ in $\gamma_{\mathfrak{p}}-1$ and hence lie in $\mathcal{H}_{E,0,v_p(\alpha_{\overline{\mathfrak{p}}})}$.

We then write

$$L_{\sharp,\alpha_{\overline{\mathfrak{p}}}} = \frac{P_{4,\mathfrak{p}}L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} - P_{2,\mathfrak{p}}L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}}{\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})},$$

$$L_{\flat,\alpha_{\overline{\mathfrak{p}}}} = \frac{-P_{3,\mathfrak{p}}L_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} + P_{1,\mathfrak{p}}L_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}}{\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})}.$$

Since $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})$ divides each isotypic component of the two distributions in the numerators, $L_{\sharp,\alpha_{\overline{\mathfrak{p}}}}$ and $L_{\flat,\alpha_{\overline{\mathfrak{p}}}}$ are elements in $\mathcal{H}_{E,0,v_p(\alpha_{\overline{\mathfrak{p}}})}(G_{p^\infty})$.

The proof for

$$\begin{pmatrix} L_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \\ L_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}} \begin{pmatrix} L_{\sharp,\beta_{\overline{\mathfrak{p}}}} \\ L_{\flat,\beta_{\overline{\mathfrak{p}}}} \end{pmatrix}.$$

is identical. ∎

Recall $\mathcal{H}_{E,r} := \{f(X) = \sum_{n\geqslant 0} a_nX^n \in E[[X]] \colon \sup_n(n^{-r}|a_n|_p) < \infty\}$. Then, we can identify $\mathcal{H}_{E,r,s} := \mathcal{H}_{E,r}(\Gamma_{\mathfrak{p}})\hat{\otimes}\mathcal{H}_{E,s}(\Gamma_{\overline{\mathfrak{p}}})$ with $\mathcal{H}_{E,r}\hat{\otimes}\mathcal{H}_{E,s}$ by identifying $X = \gamma_{\mathfrak{p}}-1$ and $Y = \gamma_{\overline{\mathfrak{p}}}-1$. We define the operator $\partial_{\mathfrak{p}}$ to be the partial derivative $\dfrac{\partial}{\partial X}$. The next proposition is a generalization of [Lei14, Lemma 2.4].

Proposition 2.8.9. Let Ξ be any character Hecke character of K of the infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_\mathfrak{p}, n_{\overline{\mathfrak{p}}} > 0$. Then, there exist constants $A_{a,b,\Xi}, B_{a,b,\Xi} \in \overline{\mathbb{Q}_p}$ such that

$$\partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} A_{a,b,\widetilde{\Xi}}, \quad \partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} A_{a,b,\widetilde{\Xi}},$$
$$\partial_\mathfrak{p} L_{\beta_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} B_{a,b,\widetilde{\Xi}}, \quad \partial_\mathfrak{p} L_{\beta_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} B_{a,b,\widetilde{\Xi}}.$$

Proof. We will only show that

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}),$$

for any Hecke character Ξ of of the infinity type $(0,0) \leqslant (a,b) \leqslant (k,k)$ and conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_\mathfrak{p}, n_{\overline{\mathfrak{p}}} > 0$.

Fix a Hecke character $\omega_1 \in S_\mathfrak{p}$. Then, for any Hecke character $\omega_2 \in S_{\overline{\mathfrak{p}}}$, (2.8.4) and (2.8.5) imply

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\omega_{1,\mathfrak{p}}}\widetilde{\omega_{2,\overline{\mathfrak{p}}}}) = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\omega_{1,\mathfrak{p}}}\widetilde{\omega_{2,\overline{\mathfrak{p}}}}), \tag{2.8.17}$$

where $L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}, L^{\Delta_K}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}$ are isotypic components of $L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}, L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}$ respectively with respect to the trivial character of Δ_K. Using (2.6.3), we rewrite (2.8.17) as

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\omega_{1,\mathfrak{p}}}) = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\omega_{1,\mathfrak{p}}}). \tag{2.8.18}$$

From Lemma 2.6.7, we know that $L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}, L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}} \in \mathcal{H}_{E',v_p(\alpha_\mathfrak{p})}(\Gamma_\mathfrak{p})$ for some extension E' of E. As $v_p(\alpha_\mathfrak{p}) < k+1$, using Lemma 2.8.7 we have

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}.$$

Hence, their partial derivatives also agree, i.e.

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}} = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L^{\Delta_K(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}. \tag{2.8.19}$$

But, for any power series $F \in \mathcal{H}_{E,v_p(\alpha_\mathfrak{p}),s}$,

$$\partial_\mathfrak{p}(F^{(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})})(\widetilde{\omega_{1,\mathfrak{p}}}) = (\partial_\mathfrak{p} F)(\widetilde{\omega_{1,\mathfrak{p}}}\widetilde{\omega_{2,\overline{\mathfrak{p}}}}),$$

for all Hecke characters $\omega_1 \in S_\mathfrak{p}$. We have this equality since we are partially differentiating with respect to the variable $\gamma_\mathfrak{p} - 1$. More precisely: $F^{(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})} \in \mathcal{H}_{E',v_p(a_\mathfrak{p})}(\Gamma_\mathfrak{p})$. Thus $\partial_\mathfrak{p}(F^{(\widetilde{\omega_{2,\overline{\mathfrak{p}}}})})(\widetilde{\omega_{1,\mathfrak{p}}})$ is first differentiating and then evaluating at the character $\widetilde{\omega_{1,\mathfrak{p}}}$. On the other hand, $\partial_\mathfrak{p}(F)(\widetilde{\omega_{1,\mathfrak{p}}}\widetilde{\omega_{2,\overline{\mathfrak{p}}}})$ is first partially differentiating and then evaluating at the character $\widetilde{\omega_{1,\mathfrak{p}}}\widetilde{\omega_{2,\overline{\mathfrak{p}}}}$.

Thus, for any Hecke character Ξ,

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L^{\Delta_K}_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\check{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L^{\Delta_K}_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\check{\Xi}), \tag{2.8.20}$$

Since equation (2.8.20) is true for any isotypic component, we have

$$\alpha_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{n_{\overline{\mathfrak{p}}}} \partial_\mathfrak{p} L_{\alpha_\mathfrak{p},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}). \tag{2.8.21}$$

∎

Proposition 2.8.10. There exist $L_{\sharp,\sharp}, L_{\sharp,\flat}, L_{\flat,\sharp}, L_{\flat,\flat} \in \Lambda_E(G_{p^\infty})$ such that

$$\begin{pmatrix} L_{\sharp,\alpha_{\overline{\mathfrak{p}}}} & L_{\flat,\alpha_{\overline{\mathfrak{p}}}} \\ L_{\sharp,\beta_{\overline{\mathfrak{p}}}} & L_{\flat,\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = (Q_{\overline{\mathfrak{p}}}^{-1}\underline{M_{\overline{\mathfrak{p}}}}) \begin{pmatrix} L_{\sharp,\sharp} & L_{\flat,\sharp} \\ L_{\sharp,\flat} & L_{\flat,\flat} \end{pmatrix}.$$

Proof. The proof is similar to the proof of [Lei14, Proposition 2.5]. We will prove that

$$\begin{pmatrix} L_{\sharp,\alpha_{\overline{\mathfrak{p}}}} \\ L_{\sharp,\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = (Q_{\overline{\mathfrak{p}}}^{-1}\underline{M_{\overline{\mathfrak{p}}}}) \begin{pmatrix} L_{\sharp,\sharp} \\ L_{\sharp,\flat} \end{pmatrix}.$$

The proof for the other set of power series is similar.

Let $\omega_1 \in S_{\mathfrak{p}}$ and $\omega_2 \in S_{\overline{\mathfrak{p}}}$ and $\Xi = \omega_1.\omega_2$. Recall, from the proof of Proposition 2.8.8, for $* \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, we have

$$L_{\sharp,*} = \frac{P_{4,\mathfrak{p}}L_{\alpha_{\mathfrak{p}},*} - P_{2,\mathfrak{p}}L_{\beta_{\mathfrak{p}},*}}{\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})}. \tag{2.8.22}$$

Thus,

$$L_{\sharp,*}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}) = P_{4,\mathfrak{p}}L_{\alpha_{\mathfrak{p}},*} - P_{2,\mathfrak{p}}L_{\beta_{\mathfrak{p}},*}. \tag{2.8.23}$$

From Theorem 2.8.4, we know that $\det(\underline{M_{\mathfrak{p}}})$ is equal to, upto a unit in $\Lambda_E(\Gamma_{\mathfrak{p}})$, $\dfrac{\log_{p,k-1}(\gamma_{\mathfrak{p}})}{\delta_{k-1}(\gamma_{\mathfrak{p}}-1)}$. Thus, the zeros of of $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})$ are of type $u^j\zeta - 1$, where ζ is a p^n-th root of unity, for all $0 \leqslant j \leqslant k$ and $n \geqslant 1$. Therefore, by the definition of $\widetilde{\omega_{1,\mathfrak{p}}}$, we have $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})(\widetilde{\omega_{1,\mathfrak{p}}}) = 0$. Furthermore, since all the zeros of $\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})$ are simple, we conclude $(\partial_{\mathfrak{p}}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}))(\widetilde{\omega_{1,\mathfrak{p}}}) \neq 0$.

For the rest of the proof, we will use isotypic components corresponding to the trivial character of Δ_K. From (2.8.23), we get,

$$\partial_{\mathfrak{p}}L_{\sharp,*}^{\Delta_K}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}) + L_{\sharp,*}^{\Delta_K}\partial_{\mathfrak{p}}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}) = \partial_{\mathfrak{p}}P_{4,\mathfrak{p}}L_{\alpha_{\mathfrak{p}},*}^{\Delta_K} + P_{4,\mathfrak{p}}\partial_{\mathfrak{p}}L_{\alpha_{\mathfrak{p}},*}^{\Delta_K} - (\partial_{\mathfrak{p}}P_{2,\mathfrak{p}}L_{\beta_{\mathfrak{p}},*}^{\Delta_K} + P_{2,\mathfrak{p}}\partial_{\mathfrak{p}}L_{\beta_{\mathfrak{p}},*}^{\Delta_K}).$$

We evaluate the above equation at $\check{\Xi} = \widetilde{\Xi}|_{\overline{\langle\gamma_{\mathfrak{p}}\rangle}\times\overline{\langle\gamma_{\overline{\mathfrak{p}}}\rangle}}$, where $\Xi = \omega_1\omega_2$ and apply Proposition 2.8.9 together with the equations (2.8.4) to (2.8.7) to get

$$L_{\sharp,*}^{\Delta_K}(\check{\Xi}).(\partial_{\mathfrak{p}}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}))(\check{\Xi}) = (*)^{-n_{\overline{\mathfrak{p}}}} \cdot M_{\tilde{\Xi}}, \tag{2.8.24}$$

where $M_{\tilde{\Xi}}$ is the constant

$$(\partial_{\mathfrak{p}}P_{4,\mathfrak{p}})(\check{\Xi})(\alpha_{\mathfrak{p}}^{-n_{\mathfrak{p}}}C_{a,b,\tilde{\Xi}}) + P_{4,\mathfrak{p}}(\check{\Xi})A_{a,b,\tilde{\Xi}} - (\partial_{\mathfrak{p}}P_{2,\mathfrak{p}})(\check{\Xi})(\beta_{\mathfrak{p}}^{-n_{\mathfrak{p}}}C_{a,b,\tilde{\Xi}}) - P_{2,\mathfrak{p}}(\check{\Xi})B_{a,b,\tilde{\Xi}}.$$

In other words, we have

$$L_{\sharp,\alpha_{\overline{\mathfrak{p}}}}^{\Delta_K}(\check{\Xi}) = \alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}\frac{M_{\tilde{\Xi}}}{(\partial_{\mathfrak{p}}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}))(\check{\Xi})},$$

$$L_{\sharp,\beta_{\overline{\mathfrak{p}}}}^{\Delta_K}(\check{\Xi}) = \beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}}\frac{M_{\tilde{\Xi}}}{(\partial_{\mathfrak{p}}\det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}))(\check{\Xi})}.$$

Since $\Xi = \omega_1\omega_2$, we can rewrite the above equations as

$$L_{\sharp,\alpha_{\overline{\mathfrak{p}}}}^{\Delta_K(\widetilde{\omega_{1,\mathfrak{p}}})}(\widetilde{\omega_{2,\overline{\mathfrak{p}}}}) = \alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \frac{M_{\tilde{\Xi}}}{(\partial_{\mathfrak{p}} \det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}}))(\check{\Xi})},$$

$$L_{\sharp,\beta_{\overline{\mathfrak{p}}}}^{\Delta_K(\widetilde{\omega_{1,\mathfrak{p}}})}(\widetilde{\omega_{2,\overline{\mathfrak{p}}}}) = \beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \frac{M_{\tilde{\Xi}}}{(\partial_{\mathfrak{p}} \det(Q_{\mathfrak{p}}^{-1}\underline{M_{\mathfrak{p}}})).(\check{\Xi})}.$$

Thus, after using Theorem 2.5.5 and the proof of Proposition 2.8.8 (we need to change $\mathfrak{p}$ with $\overline{\mathfrak{p}}$), we deduce the desired result. ∎

Proof of Theorem 2.8.5. Combining the factorizations obtained in Propositions 2.8.8 and 2.8.10, we deduce the result. ∎

Chapter 3

On p-adic Asai L-functions of Bianchi modular forms at non-ordinary primes and their decomposition into bounded p-adic L-functions

3.1 Introduction

Investigating complex L-functions is an important and active area of number theory. More precisely, we are interested in the special values of L-functions, since several conjectures in number theory, including the Bloch–Kato conjecture, describe important arithmetic data in terms of the special (critical) values of L-functions. One of the most important tools for understanding the critical values of complex L-functions is the construction and study of the p-adic avatars of these L-functions, i.e., p-adic L-functions. A one-variable p-adic L-function is a measure or a distribution on $\mathbb{Z}_p^\times$ that interpolates critical values of a given complex L-function. Using the Amice transform, we can associate a p-adic measure with a power series in the Iwasawa algebra $E \otimes \mathcal{O}_E[[T]]$ (or in $E \otimes \mathcal{O}_E[[\mathbb{Z}_p^\times]]$), where E is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_E$ is its ring of integers. Similarly, for $w \in \mathbb{R}_{\geqslant 0}$, a *$w$-tempered* distribution can be associated with a power series in the distribution space $\mathcal{H}_{E,w} \subset E[[T]]$, with unbounded denominators and growth $O(\log_p^w)$. See Section 3.6 for details about $\mathcal{H}_{E,w}$.

In [Coa89] and [CPR89], Coates and Perrin-Riou formulated a conjecture that predicts the existence of p-adic L-functions associated to any motive over $\mathbb{Q}$ having at least one critical L-value. Loeffler and Williams proved this conjecture for the (conjectural) Asai motive attached to a p-ordinary Bianchi eigenform in [LW20]. More precisely, for a p-ordinary cuspidal Bianchi modular form Ψ, they constructed a p-adic measure $L_p^{\mathrm{As}}(\Psi)$ that interpolates the critical L-values of the Asai L-function attached to Ψ. Moreover, their construction is independent of the existence of the motive over $\mathbb{Q}$.

This chapter addresses the next natural step: the construction of a p-adic Asai L-function when Ψ is not ordinary at p. Our construction is also independent of the conjectural Asai motive over $\mathbb{Q}$. We attach a p-adic Asai L-function with unbounded denominators $L_p^{\mathrm{As}}(\Psi)$ to a p-non-ordinary small slope cuspidal Bianchi modular form Ψ such that it interpolates critical values of the Asai L-function associated with Ψ. This construction generalizes the result of Loeffler–Williams [LW20, Theorem 7.5] from the p-ordinary case to the p-non-ordinary case.

Roughly, Loeffler–Williams constructed a p-adic measure $L_p^{\mathrm{As}}(\Psi)$ using Asai–Eisenstein elements only for $j = 0$. Since they work in the p-ordinary setting, by a deep theory of Kings implies that this measure is independent of the twist j. In particular, for a Dirichlet character θ of p-power conductor, by integrating $x^j\theta(x)$ against $L_p^{\mathrm{As}}(\Psi)$ recoveres the special L-value $L^{\mathrm{As}}(\Psi, \overline{\theta}, j + 1)$. See [LW20, Propositions 5.5, 6.3] for more details. We will briefly outline their work in Section 3.4.4. The method of using the only Asai–Eisenstein element for $j = 0$ breaks down in p-non-ordinary setting.

To circumvent this issue, we use the approach taken by Amice–Velu in [AV75] and Višik in [Vis76]: interpolation of twists using polynomials. In this chapter, we obtain the p-adic Asai L-function $L_p^{\mathrm{As}}(\Psi)$ by constructing certain novel family of polynomials $P_{r,j}(T) \in E[T]$ of degree p^{r-1}, where Ψ is a p-non-ordinary weight (k, k) cuspidal Bianchi modular form, $r \in \mathbb{Z}_{\geqslant 1}$ and $0 \leqslant j \leqslant k$. These polynomials are constructed by pairing an appropriate modular symbol associated to Ψ with the *Asai-Eisenstein elements* constructed by Loeffler–Williams in [LW20, Sections 3, 4]. Furthermore, if we evaluate these polynomials at some special elements, we obtain the critical L-values $L^{\mathrm{As}}(\Psi, \overline{\theta}, j + 1)$ with some explicit factor. The polynomials $P_{r,j}$ satisfy certain norm and congruence properties. To prove the congruence properties of polynomials, we state and prove some new congruence results associated to Asai–Eisenstein elements. Then, by applying the techniques developed by Amice–Vélu, Višik, Perrin-Riou, and Büyükboduk–Lei to these polynomials, we construct the p-adic distribution with unbounded coefficients and some growth, that is, an element in the distribution space $\mathcal{H}_{E,v_p(a_p)}(\Gamma)$, where $v_p(a_p)$ is the slope of the 'p-th' Fourier coefficient. We hope that one may use the p-adic distribution $L_p^{\mathrm{As}}(\Psi)$ to formulate and prove the Harron–Pottharst [HP16] style Iwasawa main conjecture over the distribution space, relating the ideal generated by this distribution over the distribution space to a certain Selmer complex. We briefly outline the construction of these polynomials and congruences in Subsection 3.1.2.

We further study this p-adic Asai L-function. In particular, in Section 3.7, we construct Pollack, Sprung, and Lei–Loeffler–Zerbes style signed p-adic Asai L functions using the techniques developed by the author in [Deo26]. These signed p-adic Asai L-functions may be used to formulate signed Iwasawa main conjectures.

3.1.1 The main result

Throughout the chapter, p is an odd rational prime and $k \in \mathbb{Z}_{\geqslant 0}$. Let $F = \mathbb{Q}(\sqrt{-D})/\mathbb{Q}$ be an imaginary quadratic field, $\mathcal{O}_F$ be its ring of integers, and let σ be the complex conjugation map. Let Ψ be a cuspidal Bianchi eigenform over F of level $\mathfrak{N}$ (i.e. $U_{F,1}(\mathfrak{N})$) and weight (k, k). See Definition 3.2.2 for $U_{F,1}(\mathfrak{N})$. The twisted Asai L-function of Ψ with a Dirichlet character θ of conductor m is defined by

$$L^{\mathrm{As}}(\Psi,\theta,s) := L^{(mN)}(\theta^2\epsilon_\Psi|_{(\mathbb{Z}/N\mathbb{Z})^\times}, 2s-2k-2)\cdot \sum_{\substack{n\geqslant 1,\\ (m,n)=1}} c(n\mathcal{O}_F,\Psi)\theta(n)n^{-s},$$

where $(N)=\mathfrak{N}\cap\mathbb{Z}$, $\epsilon_\Psi|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ is the restriction to $(\mathbb{Z}/N\mathbb{Z})^\times$ of the nebentypus ϵ_Ψ of Ψ, $L^{(mN)}(-)$ is the Dirichlet L-function with its Euler factors at the primes dividing mN removed, and $c(\mathfrak{m},\Psi)$ denotes the Hecke eigenvalue of Ψ at the integral ideal $\mathfrak{m}$.

We assume that the level $\mathfrak{N}$ of Ψ is divisible by all the primes above p. Assume that Ψ is p-non-ordinary. More precisely, the U_p Hecke eigenvalue a_p of Ψ is not a p-adic unit, i.e.,$v_p(a_p)>0$, where v_p is a p-adic valuation such that $v_p(p)=1$. We furthermore assume that Ψ has *a small slope* at p, that is, $v_p(a_p)<k+1$. Let $E/\mathbb{Q}_p$ be a finite extension large enough to contain F and all the Hecke eigenvalues of Ψ. We fix some notations for the rest of the chapter. Fix a topological generator u of $1+p\mathbb{Z}_p$. Let $\Gamma=\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)\cong\Delta\times\mathbb{Z}_p$ be the Galois group of cyclotomic extension of $\mathbb{Q}_p$ by adjoining p-power roots to $\mathbb{Q}_p$. Let Γ_1 be a subgroup of Γ such that $\Gamma_1\cong\Gamma/\Delta\cong\mathbb{Z}_p$. Fix a topological generator γ_0 of Γ_1. Let χ be the p-adic cyclotomic character on Γ such that $\chi(\gamma_0)=u$. For any real number $w\geqslant 0$, define the space of *w-admissible distributions* over Γ as

$$\mathcal{H}_{E,w}(\Gamma) := \left\{\sum_{\sigma\in\Delta}\sum_{n\geqslant 0} c_{n,\sigma}\cdot\sigma\cdot(\gamma_0-1)^n : \sup_w \frac{|c_{n,\sigma}|_p}{n^w}<\infty, \forall\sigma\in\Delta\right\}.$$

The main result of this paper is the construction of the p-adic distribution ${}_cL_p^{\mathrm{As}}(\Psi)\in\mathcal{H}_{E,v_p(a_p)}(\Gamma)$ associated with Ψ, which interpolates the critical values of Asai L-function twisted by a Dirichlet character of conductor p^r.

Theorem C (Theorem 3.6.5, Theorem 3.6.7)**.** For any integer $c>1$ coprime to $6\mathfrak{N}$, there exists a $v_p(a_p)$-admissible distribution ${}_cL_p^{\mathrm{As}}(\Psi)$ with the following interpolation property: for any integer $0\leqslant j\leqslant k$ and for any Dirichlet character of conductor $p^r>1$ we have

$${}_cL_p^{\mathrm{As}}(\Psi)(\chi^j\theta) = \begin{cases} \dfrac{*'}{(a_p)^r}L^{\mathrm{As}}(\Psi,\overline{\theta},j+1) & \text{if } (-1)^j\theta(-1)=1,\\ 0 & \text{if } (-1)^j\theta(-1)=-1,\end{cases}$$

where $*'$ is an explicit non-zero factor.

Note that one can get rid of c under some conditions. For more details, see the Subsection 3.6.4.

Remark 3.1.1. By an eigenform, we mean either Ψ is a newform or can be obtained by the *p-stabilization* of some Bianchi newform Ψ' of level $\mathfrak{N}'$ such that $\mathfrak{N}'$ is coprime to p.

Remark 3.1.2. We do not impose any restriction on the fixed odd prime p. Our construction works as long as the p-adic valuation of the U_p-eigenvalue is less than $k+1$.

3.1.2 Outline of the construction of $P_{r,j}$

We briefly outline the construction of the polynomials $P_{r,j}$ of degree p^{r-1} that are used to construct the p-adic L-function ${}_cL_p^{\mathrm{As}}(\Psi)$. For $r \in \mathbb{Z}_{>1}$, any integer $0 \leqslant j \leqslant k$, and $\delta : \Delta \to \mathcal{O}_E^\times$, define,

$$P^{\delta}_{r,j}(T) = \left\langle \phi^*_{\mathcal{F}},\ (U_p)_*^{-r} \frac{1}{A_j} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle, \tag{3.1.1}$$

where

- $\phi^*_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_E))$ is a modular symbol associated to $\mathcal{F}$,
- ${}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))$ is the *Asai-Eisenstein element* constructed by Loeffler–Williams in [LW20],
- $\log_u(t)$ is a unique integer in $[0, p^{r-1})$ related to $t \in (\mathbb{Z}/p^r)^\times$,
- A_j is some combinatorial fudge factor and $a \in \mathcal{O}_F$ such that $a - a^\sigma = \sqrt{-D}$,
- $\langle , \rangle$ denotes the perfect Poincaré duality pairing between $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_E))$ and $\frac{\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))}{(\mathrm{Torsion})}$.

We also define a polynomial when $r = 1$. See Equation (3.6.4). Here $Y^*_{F,1}(\mathfrak{N})$ is a locally symmetric space of a certain level (see 3.2 for the definition). Here $V_{k,k}$ are $T_{k,k}$ are some specific weight (k,k)-coefficient modules. The Asai-Eisenstein element ${}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}$ is related to an Eisenstein series of weight $2k - 2j + 2$.These elements are 'the Betty counterparts' of the known Euler system elements, such as Beilinson–Flach Euler systems. They satisfy certain norm-compatibilities like Beilinson–Flach elements. See [LW20, Theorem 3.13] for more details.

These polynomials satisfy some norm and congruence properties. For example, for $0 \leqslant j \leqslant k$ and $p^r > 0$, $P_{r,j}$ satisfy,

$$\sup_r \left\| p^{(v_p(a_p)-j)r} \sum_{i=0}^{j} (-1)^i \binom{j}{i} P^{\delta}_{r,i}(u^{-i}(1+T)-1) \right\| < \infty. \tag{3.1.2}$$

This property is crucial to construct ${}_cL_p^{\mathrm{As}}(\Psi)$. The more properties of $P_{r,j}$ are described in Lemma 3.6.4. To prove (3.1.2), we first prove the following key lemma, which is a cohomological version of the interpolation of twists (congruence relation):

Lemma A (Lemma 3.5.7)**.** For any $t \in (\mathbb{Z}/p^r)^\times$ and any positive integer j, we have

$$\begin{aligned}&\sum_{i=0}^{j} t^{(j-i)}(a-a^\sigma)^{(j-i)}(j-i)!\mathrm{Res}^{p^{jr}}_{p^r}\,\mathrm{mom}^{(j-i)(j-i)}({}_c\Xi^{[i,i]}_{p^r,\mathfrak{N},ta}) \\ &\in p^{jr}\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{jj}(\mathcal{O}_E)).\end{aligned} \tag{3.1.3}$$

For the definition of the moment map mom, see Subsection 3.4.3.1. The Lemma 3.5.7 is the Betti cohomology analogue of [LZ16, Theorem 3.3.5]. Furthermore, we prove another novel lemma :

Lemma B (Lemma 3.6.1)**.** For any character $\delta : \Delta \to \mathcal{O}_E^\times$, any integer $0 \leqslant j \leqslant k$, and any integer r such that $p^r > 1$, we have

$$\sup_r \left\| p^{-jr} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}\delta(t)(1+T)^{\log_u(t)} \right\| < \infty.$$

To prove Lemmas 3.5.7, 3.6.1, we study locally symmetric spaces $Y_F^*(m, m\mathfrak{N})$ of mixed level, where $m \in \mathbb{Z}$. Similar to the methods of Loeffler–Zerbes in [LZ16] and Loeffler–Williams in [LW20], for each $j \in \mathbb{Z}_{\geqslant 0}$, we construct and study cohomological elements ${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a} \in \mathrm{H}^2(Y_F^*(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_K))$. To construct and study these elements, *moment maps*, *Clebsch–Gordon maps* are used, together with the interplay between them, in a manner similar to those in Kings–Loeffler–Zerbes [KLZ17] and Lei–Loeffler–Zerbes [LLZ18]. Readers are advised to refer to Sections 3.5 and 3.6 for a detailed description of the polynomials, cohomological elements, and congruences.

3.1.3 Signed p-adic Asai L-functions

We give an application of the p-adic Asai L-function ${}_cL_p^{\mathrm{As}}(\Psi)$: the decomposition of p-adic Asai L-functions with unbounded coefficients into signed p-adic Asai L-functions with bounded coefficients. Assume p splits in F as $p\mathcal{O}_F = \mathfrak{p}\bar{\mathfrak{p}}$. See Section 3.7.2 for the setting for the theorem mentioned below. We state a few of them here for the convenience of readers:

1. Ψ is a cuspidal Bianchi eigenform of level $\mathcal{N}$, where $\mathcal{N} \subset \mathcal{O}_F$ is an ideal coprime to p, trivial nebentypus, and weight (k, k). Furthermore, Ψ is $\mathfrak{p}$-non-ordinary and $\bar{\mathfrak{p}}$-ordinary.
2. For $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, let $a_\mathfrak{q}$ be the $T_\mathfrak{q}$-eigenvalue, and $\alpha_\mathfrak{q} \neq \beta_\mathfrak{q}$ be the roots of the polynomial $X^2 - a_\mathfrak{q}X + p^{k+1}$. Assume $v_p(a_\mathfrak{p}) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$.
3. Choose $\alpha_{\bar{\mathfrak{p}}}$ such that $v_p(\alpha_{\bar{\mathfrak{p}}}) = 0$ and for $\bullet \in \{\alpha_\mathfrak{p}, \beta_\mathfrak{p}\}$, let $\tilde{\bullet} = \alpha_{\bar{\mathfrak{p}}}\bullet$. Let $\Psi^{\tilde{\bullet}}$ be p-stabilization of Ψ such that U_p-eigenvalue of $\Psi^{\tilde{\bullet}}$ is $\tilde{\bullet}$.

Since $v_p(\tilde{\alpha}), v_p(\tilde{\beta}) < k+1$, we can attach p-adic distributions ${}_cL_p^{\mathrm{As}}(\mathcal{F}^{\tilde{\alpha}})$ and ${}_cL_p^{\mathrm{As}}(\mathcal{F}^{\tilde{\beta}})$ to $\mathcal{F}^{\tilde{\alpha}}$ and $\mathcal{F}^{\tilde{\beta}}$ respectively using Theorem C. They satisfy certain growth and interpolation properties. Then using the methods in [Deo26], we prove:

Theorem D (Theorem 3.7.3)**.** There exist ${}_cL_p^{\mathrm{As},\sharp}, {}_cL_p^{\mathrm{As},\flat} \in E \otimes \mathcal{O}_E[[\Gamma]] \cong E \otimes \mathcal{O}_E[\Delta][[T]]$ and a logarithmic matrix $\underline{\tilde{M}} \in M_{2,2}(\mathcal{H}_E)$ such that

$$\begin{pmatrix} {}_cL_p^{\mathrm{As}}(\Psi^{\tilde{\alpha}}) \\ \\ {}_cL_p^{\mathrm{As}}(\Psi^{\tilde{\beta}}) \end{pmatrix} = \tilde{Q}^{-1}\underline{\tilde{M}} \begin{pmatrix} {}_cL_p^{\mathrm{As},\sharp} \\ \\ {}_cL_p^{\mathrm{As},\flat} \end{pmatrix}, \tag{3.1.4}$$

where $\tilde{Q} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ -\alpha_{\bar{\mathfrak{p}}}^2 p^{k+1} & -\alpha_{\bar{\mathfrak{p}}}^2 p^{k+1} \end{pmatrix}$, and $\underline{\tilde{M}}$ satisfies properties from Proposition 3.7.2.

Note that by using the methods of [Deo26], we avoid the conjectural p-adic Hodge theoretic properties associated with the Galois representation associated with the Asai motive.

3.1.4 Plan of the chapter

In Section 3.2, we recall the definitions of locally symmetric spaces in which we are interested. We also define Hecke correspondences on these locally symmetric spaces. We define the Asai L-function of Bianchi modular forms in Section 3.3. Moreover, we recall the definitions of Bianchi modular forms and Bianchi modular symbols in this section. In Section 3.4, we describe the weight 2 and higher weight Asai-Eisenstein elements constructed by Loeffler–Williams in [LW20]. The construction of these elements is closely related to the elements constructed in [LLZ18]. One can think of the Asai-Eisenstein elements as a Betti counterpart of known Euler system elements like Beilinson–Flach elements. We do some explicit calculations in Section 3.5 similar to [KLZ17, Theorem 6.2.4] and [LLZ18, Theorem 8.1.4]. We also prove some interpolation and congruence theorems, which are Betti analogues of the theorems in [LZ16, Section 3.3]. In Section 3.6, we prove Theorem C using the interpolation of twists method of Amice–Velu, Perrin-Riou, and Büyükboduk–Lei. In the last section, we first recall the definition and construction of logarithmic matrices briefly. Then we prove Theorem D.

3.2 Locally symmetric spaces

3.2.1 Setup and notations

Let $F/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D$ and denote its ring of integers by $\mathcal{O}_F$. Let $\mathbb{A}_F$ denote the adele ring of F, and the finite adeles are denoted by $\mathbb{A}_F^f$. Define $\widehat{\mathcal{O}_F} := \mathcal{O}_F \otimes \widehat{\mathbb{Z}} = \mathcal{O}_F \otimes \prod_p \mathbb{Z}_p$.

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal such that $\mathfrak{N}$ is divisible by all the primes above p. Throughout this chapter, we assume $\mathfrak{N}$ is small enough such that the locally symmetric space attached to $\mathfrak{N}$ is smooth. Let $N \in \mathbb{Z}$ such that $(N) = \mathbb{Z} \cap \mathfrak{N}$ in $\mathbb{Z}$.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the usual upper-half plane with $\mathrm{GL}_2(\mathbb{R})$-action given by Möbius transformations. Define *the upper-half space* or *hyperbolic 3-space* to be $\mathbb{H}_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R}_{>0}\}$ with $\mathrm{GL}_2(\mathbb{C})$ action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2}{\mid cz+d \mid^2 + \mid c \mid^2 t^2}, \frac{\mid ad-bc \mid t}{\mid cz+d \mid^2 + \mid c \mid^2 t^2} \right).$$

We embed $\mathbb{H} \hookrightarrow \mathbb{H}_3$ via $x + iy \mapsto (x, y)$, which is compatible with the actions of $\mathrm{GL}_2(\mathbb{R})$ on both sides.

3.2.2 Algebraic groups and locally symmetric spaces

The primary references of this section are [LW20] and [LLZ18].

Let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2, \quad G^* = G \times_D \mathbb{G}_m,$$

be the algebraic groups, where where $D := \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ and the map $G \to D$ is determinant.

Definition 3.2.1 (Locally symmetric spaces)**.** For open compact subgroups $U_{\mathbb{Q}} \subset \mathrm{GL}_2(\mathbb{A}^f_{\mathbb{Q}})$, $U \subset G(\mathbb{A}^f_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{A}^f_F)$, and $U^* \subset G^*(\mathbb{A}^f_{\mathbb{Q}}) = \{g \in \mathrm{GL}_2(\mathbb{A}^f_F) \mid \det(g) \in (\mathbb{A}^f_{\mathbb{Q}})^\times\}$, the corresponding locally symmetric spaces are defined as

$$Y_{\mathbb{Q}}(U_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})_+ \backslash [\mathrm{GL}_2(\mathbb{A}^f_{\mathbb{Q}}) \times \mathbb{H}]/U_{\mathbb{Q}},$$

$$Y_F(U) = \mathrm{GL}_2(F) \backslash [\mathrm{GL}_2(\mathbb{A}^f_F) \times \mathbb{H}_3]/U,$$

$$Y^*_F(U^*) = G^*(F)_+ \backslash [G^*(\mathbb{A}^f_{\mathbb{Q}}) \times \mathbb{H}_3]/U^*,$$

where $G^*(F)_+ = \{g \in G^*(F) \colon \det(g) > 0\}$.

3.2.3 Locally symmetric spaces of mixed levels

We define the specific level groups and locally symmetric spaces of a particular interest in this chapter.

Definition 3.2.2 (Congruence subgroups)**.** Let K be either F or $\mathbb{Q}$, and $\mathfrak{m}, \mathfrak{n}$, and $\mathfrak{a}$ be the ideals in $\mathcal{O}_K$. Define

$$U_K(\mathfrak{m}, \mathfrak{n}) := \left\{ B \in \mathrm{GL}_2(\mathcal{O}_K \otimes \widehat{\mathbb{Z}}) \colon B \equiv I_2 \mod \begin{pmatrix} \mathfrak{m} & \mathfrak{m} \\ \mathfrak{n} & \mathfrak{n} \end{pmatrix} \right\},$$

$$U_K(\mathfrak{m}(\mathfrak{a}), \mathfrak{n}) := \left\{ B \in \mathrm{GL}_2(\mathcal{O}_K \otimes \widehat{\mathbb{Z}}) \colon B \equiv I_2 \mod \begin{pmatrix} \mathfrak{m} & \mathfrak{m}\mathfrak{a} \\ \mathfrak{n} & \mathfrak{n} \end{pmatrix} \right\}.$$

We write $Y_K(\mathfrak{m}, \mathfrak{n}) := Y_K(U(\mathfrak{m}, \mathfrak{n}))$ and similarly $Y_K(\mathfrak{m}(\mathfrak{a}), \mathfrak{n})$.

Furthermore, let

$$U^*_F(\mathfrak{m}, \mathfrak{n}) := U_F(\mathfrak{m}, \mathfrak{n}) \cap G^*, \qquad U^*_F(\mathfrak{m}(\mathfrak{a}), \mathfrak{n}) := U_F(\mathfrak{m}(\mathfrak{a}), \mathfrak{n}) \cap G^*.$$

We write $Y^*_F(\mathfrak{m}, \mathfrak{n}) := Y^*_F(U(\mathfrak{m}, \mathfrak{n}))$ and $Y^*_{F,1}(\mathfrak{n}) := Y^*_F((1), \mathfrak{n})$, i.e., $Y^*_F(\mathfrak{m}, \mathfrak{n})$ for $\mathfrak{m} = 1$.

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal and N is an integer such that $(N) = \mathfrak{N} \cap \mathbb{Z}$. We are interested in the following locally symmetric spaces:

1. The usual open modular curve $Y_{\mathbb{Q},1}(N)$ of level $\Gamma_1(N)$. It has only one connected component isomorphic to $\Gamma_1(N)\backslash\mathbb{H}$.
2. Another (mixed level) modular curve which we are interested in is $Y_{\mathbb{Q}}(m, mN)$, for any $m \in \mathbb{Z}_{\geqslant 0}$.
3. The space $Y^*_{F,1}(\mathfrak{N})$ (which will appear so many times later). This space also has a single connected component isomorphic to $\Gamma^*_{F,1}\backslash\mathbb{H}_3$, where
$$\Gamma^*_{F,1} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \colon c \equiv 0, a \equiv d \equiv 1 \mod \mathfrak{N} \right\}.$$
4. We are also interested in (mixed level) $Y^*_F(m, m\mathfrak{N})$, where $m \in \mathbb{Z}_{\geqslant 1}$. The space $Y^*_F(m, m\mathfrak{N})$ is not connected and has connected components indexed by group $(\mathbb{Z}/m\mathbb{Z})^\times$, since the component group of $Y^*_F(m, m\mathfrak{N})$ is $\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q}}/\mathbb{R}_{>0}\det(U^*(m, m\mathfrak{N})) \cong (\mathbb{Z}/m\mathbb{Z})^\times$. The identity component $Y^*_F(m, m\mathfrak{N})^{(1)}$ is isomorphic to $\Gamma^*_F(m, m\mathfrak{N})\backslash\mathbb{H}_3$, where
$$\Gamma^*_F(m, m\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : \begin{matrix} a \equiv 1 \mod m, & b \equiv 0 \mod m \\ c \equiv 0 \mod m\mathfrak{N}, & d \equiv 1 \mod m\mathfrak{N} \end{matrix} \right\}.$$
We will describe some explicit computations related to this space in Section 3.5.
5. The space $Y_{F,1}(\mathfrak{N})$. Since $\det(U_{F,1}(\mathfrak{N})) = \widehat{\mathcal{O}_F}^\times$, $Y_{F,1}(\mathfrak{N})$ has h_F connected components, where h_F is the class number of F. The identity component is isomorphic to $\Gamma_{F,1}\backslash\mathbb{H}_3$, where $\Gamma_{F,1}(\mathfrak{N}) := \mathrm{GL}_2(F) \cap U_{F,1}(\mathfrak{N})$.

Remark 3.2.3. There are natural maps
$$Y_{\mathbb{Q},1}(N) \xrightarrow{\iota} Y^*_{F,1}(\mathfrak{N}) \xrightarrow{j} Y_{F,1}(\mathfrak{N})$$
induced by the natural maps $\mathbb{H} \hookrightarrow \mathbb{H}_3$ and $\mathrm{GL}_2(\mathbb{A}^f_{\mathbb{Q}}) \hookrightarrow G^*(\mathbb{A}^f_{\mathbb{Q}}) \hookrightarrow G^*(\mathbb{A}^f_{\mathbb{Q}})$.

Proposition 3.2.4. If $\mathfrak{N}$ is divisible by some integer $q \geqslant 4$, then $Y^*_{F,1}(\mathfrak{N})$ is a smooth manifold, and
$$\iota \colon Y_{\mathbb{Q},1}(N) \hookrightarrow Y^*_{F,1}(\mathfrak{N})$$
is an injective map and hence a closed immersion. Moreover, let $m \in \mathbb{Z}$ be a positive integer, and if $m\mathfrak{N}$ is divisible some integer $\geqslant 4$, then
$$\iota \colon Y_{\mathbb{Q}}(m, mN) \hookrightarrow Y^*_F(m, m\mathfrak{N})$$
is an injective map and a closed immersion.

Proof. See [LW20, Proposition 2.5]. ∎

Assumption 3.2.5. We will assume the ideal $\mathfrak{N}$ of $\mathcal{O}_F$ is divisible by some integer $q \geqslant 4$ throughout the chapter. Due to this assumption, the space $Y^*_{F,1}(\mathfrak{N})$ will be a smooth manifold and not a non-smooth orbifold.

3.2.4 Hecke correspondences on the locally symmetric spaces

Let $\mathfrak{a}$ be any square-free ideal of $\mathcal{O}_K$. Consider the correspondence diagram

$$\begin{tikzcd}
 & Y_F(1(\mathfrak{a}),\mathfrak{N}) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
Y_{F,1}(\mathfrak{N}) & & Y_{F,1}(\mathfrak{N})
\end{tikzcd}$$

where π_1 is the natural projection map, and π_2 is the map given by the right translation action of $\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathrm{GL}_2(\mathbb{A}_F^f)$, where $\varpi \in \widehat{\mathcal{O}_F}$ is integral adele which generates the ideal $\mathfrak{a}\widehat{\mathcal{O}_F}$. We define Hecke correspondences:

$$(T_\mathfrak{a})_* := (\pi_2)_* \circ (\pi_1)^*,$$
$$(T_\mathfrak{a})^* := (\pi_1)_* \circ (\pi_2)^*$$

as correspondences on $Y_{F,1}(\mathfrak{N})$. When $\mathfrak{a} \mid \mathfrak{N}$, we write $(U_\mathfrak{a})^*$ and $(U_\mathfrak{a})_*$ instead of $(T_\mathfrak{a})^*$ and $(T_\mathfrak{a})_*$ respectively. We can extend these definitions to non-squarefree $\mathfrak{a}$ in the usual way.

Remark 3.2.6. We can use the same construction to define Hecke correspondences on the mixed level locally symmetric space $Y_F(\mathfrak{m},\mathfrak{N})$, but it will not be independent of the choice of generator ϖ of $\mathfrak{a}$, it will depend on the class of $\varpi \mod 1+\mathfrak{m}\widehat{\mathcal{O}_F}$. We will use this when $\mathfrak{a}$ is generated by some $a \in \mathbb{Z}_{>0}$, and in that case we will take $\varpi = a$.

For $m \in \mathbb{Z}_{>0}$, $(T_m)^* = (T_{(m)})^*$. Same for $(T_m)_*, (U_m)^*$, and $(U_m)_*$. From the above remark, it makes sense to define the Hecke operators $(T_m)^*$ and $(T_m)_*$, for $m \in \mathbb{Z}_{>0}$, on the mixed level symmetric spaces $Y_F^*(\mathfrak{m},\mathfrak{N})$. Moreover, $(T_\mathfrak{a})_*$ and $(U_\mathfrak{a})_*$ are the transpose of $(T_\mathfrak{a})^*$ and $(U_\mathfrak{a})^*$ respectively with respect to Poincaré duality.

3.3 Preliminaries related to the Asai L-function of Bianchi modular forms

3.3.1 Bianchi modular forms

We recall definitions of Bianchi modular forms and their Fourier expansions. For an integer $n \geqslant 0$, and a ring R, define $V_n(R)$ to be the space of homogeneous polynomials of degree n in two variables X, Y with coefficients in R. The space $V_n(R)$ can also be described as $\mathrm{Sym}^n(R^2)$. The group of matrices $\mathrm{GL}_2(R)$ acts on $V_n(R)$ both from the right and left, and we denote the corresponding space by $V_n^{(r)}(R)$ or by $V_n^{(\ell)}(R)$, respectively. See [LW20, Section 2A] for the precise definitions.

Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^f)$. Then for any integer $k \geqslant 0$, there is a finite dimensional $\mathbb{C}$-vector space $S_{k,k}(U)$ of *cuspidal Bianchi modular forms* over F of weight (k,k) and level U, which are functions

$$\Psi : \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/U \to V_{2k+2}^{(r)}(\mathbb{C}),$$

satisfying certain transformations under right translation by the group $\mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$, and satisfying certain growth conditions and harmonicity. Fruthermore, the Fourier-Whittaker expansion of Ψ is

$$\Psi\left(\begin{pmatrix} \mathbf{y} & \mathbf{x} \\ 0 & 1 \end{pmatrix}\right) = |\mathbf{y}|_{\mathbb{A}_F} \sum_{\zeta\in F^\times} W_f(\zeta\mathbf{y}_f, \Psi) W_\infty(\zeta\mathbf{y}_\infty) e_F(\zeta\mathbf{x}),$$

where the Kirillov function $W_f(-,\Psi)$ is a locally constant function on $(\mathbb{A}_F^f)^\times$ with support contained in a compact subset of $\mathbb{A}_F^f$, $W_\infty\colon \mathbb{C}^\times \to V_{2k+2}(\mathbb{C})$ be the real analytic function defined in [Wil17, 1.2.1(v)], and $e_F\colon \mathbb{A}_F/F \to \mathbb{C}^\times$ denote the unique continuous character such that its restriction to $F\otimes\mathbb{R}$ is $x_\infty \mapsto e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x_\infty)}$ (Also look at [LW20, Theorem 2.9]).

For $U = U_{F,1}(\mathfrak{N})$, the Kirillov function $W_f(-,\Psi)$ is supported in $\mathcal{D}^{-1}\widehat{\mathcal{O}_F}$, where $\mathcal{D} = (\sqrt{(-D)})$ is the different of F. For an ideal $\mathfrak{m}$ of $\mathcal{O}_F$, we define a coefficient $c(\mathfrak{m},\Psi)$ as the value $W_f(\mathbf{y}_f,\Psi)$ for any $\mathbf{y}_f$ generating the fractional ideal $\mathcal{D}^{-1}\mathfrak{m}\widehat{\mathcal{O}_F}$. The space $S_{k,k}(U_{F,1}(\mathfrak{N}))$ has an action of commuting Hecke operators $(T_\mathfrak{m})^*$ for all ideal $\mathfrak{m}$. Moreover, if Ψ is an eigenvector for all these operators, normalized such that $c(1,\Psi) = 1$, then the $\mathfrak{m}$-th Hecke eigenvalue of Ψ is $c(\mathfrak{m},\Psi)$.

3.3.2 The Asai L-function of a Bianchi modular form

For all $d \in (\mathcal{O}_F/\mathfrak{N})^\times$, the space $S_{k,k}(U_{F,1}(\mathfrak{N}))$ has an action of diamond operators $\langle d\rangle$. On any eigenform Ψ, they act via a character $\epsilon_\Psi\colon (\mathcal{O}_F/\mathfrak{N})^\times \to \mathbb{C}^\times$. Let $\epsilon_\Psi|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ denote the restriction of ϵ_Ψ to $(\mathbb{Z}/N\mathbb{Z})^\times$.

Definition 3.3.1 (Asai L-function of Ψ)**.** Let Ψ be a normalized Hecke eigenform in $S_{k,k}(U_{F,1}(\mathfrak{N}))$ and let θ be a Dirichlet character of conductor m. Define the *Asai L-function* of Ψ by

$$L^{\mathrm{As}}(\Psi,\theta,s) := L^{(mN)}(\theta^2\epsilon_{\Psi,\mathbb{Q}}, 2s-2k-2)\cdot \sum_{\substack{n\geqslant 1,\\ (m,n)=1}} c(n\mathcal{O}_F,\Psi)\theta(n)n^{-s}, \tag{3.3.1}$$

where $(N) = \mathfrak{N}\cap\mathbb{Z}$ and $L^{(mN)}(-,s)$ is the Dirichlet L-function with its Euler factors at the primes dividing mN removed. If θ is trivial, we just write $L^{\mathrm{As}}(\Psi,s)$.

This Asai L-function $L^{\mathrm{As}}(\Psi,\theta,s)$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large (one can take $\mathrm{Re}(s) > k+3$) and has meromorphic continuation for all $s\in\mathbb{C}$. See [LW20, Section 2E] for more details. For s in the half-plane of convergence, $L^{\mathrm{As}}(\Psi,\theta,s)$ can be written as an Euler product

$$L^{\mathrm{As}}(\Psi,\omega,s) = \prod_{\ell\text{ prime}} P_\ell^{\mathrm{As}}(\Psi,\theta,s),$$

where the polynomial $P_\ell^{\mathrm{As}}(\Psi, \theta, s)$ depends only on $\theta(\ell)$ and the Hecke and diamond eigenvalues of Ψ at the primes above ℓ. For simplicity, assume θ and the nebentypus of Ψ are trivial. For primes $\mathfrak{l}$ of F. let $\alpha_{\mathfrak{l}}$ and $\beta_{\mathfrak{l}}$ denote the roots of the polynomial $X^2 - c(\mathfrak{l}, \Psi)X + N(\mathfrak{l})^{k+1}$. Then we have

$$\frac{1}{P_p^{\mathrm{As}}(\Psi, s)} = \begin{cases} (1-\alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1-\alpha_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s})(1-\beta_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1-\beta_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s}) & \text{if } p = \mathfrak{p}\overline{\mathfrak{p}}, \\ (1-\alpha_{\mathfrak{p}}p^{-s})(1-p^{-2s+2})(1-\beta_{\mathfrak{p}}p^{-s}) & \text{if } p = \mathfrak{p}, \\ (1-\alpha_{\mathfrak{p}}^2p^{-s})(1-p^{-s+1})(1-\beta_{\mathfrak{p}}^2p^{-s}) & \text{if } p = \mathfrak{p}^2. \end{cases} \tag{3.3.2}$$

See [Gha99, Section 3] for more details.

Remark 3.3.2. The Asai L-function appearing in Definition 3.3.1 is an "imprimitive" L-function. We can define a "primitive" Asai L-function using automorphic representations attached to Bianchi modular forms. Another way to define the Asai L-function of a Bianchi modular form Ψ is the L-function attached to the tensor induction to $\mathbb{Q}$ of Galois representation associated with Ψ. See [LW20, Section 2F] and [Gha99, Sections 3, 4].

Lemma 3.3.3. [LW20, Lemma 2.11] Let Ψ be a normalized Bianchi eigenform of level $\mathfrak{n}$ coprime to p and let Ψ' be a p-stabilization of Ψ of level $p\mathfrak{n}$. Let $\alpha = c(p\mathcal{O}_F, \Psi')$. Then

$$L^{\mathrm{As}}(\Psi, \theta, s) = L^{\mathrm{As}}(\Psi', \theta, s)$$

for any non-trivial Dirichlet character θ of p-power conductor, and

$$L^{\mathrm{As}}(\Psi', s) = (1-\alpha p^{-s})^{-1} L^{\mathrm{As},(p)}(\Psi, s).$$

3.3.3 Bianchi modular symbols

For any field extension F' of F, we define the left $F'[\mathrm{GL}_2(F)]$-module $V_{kk}(F') := V_k^{(\ell)}(F') \otimes V_k^{(\ell)}(F')^\sigma$. The action of $\mathrm{GL}_2(F)$ is given by: For any $B \in \mathrm{GL}_2(F)$, B acts in the usual way on the first component and via its complex conjugate B^σ on the second component. Due to the action of $\mathrm{GL}_2(F)$, the space $V_{kk}(F')$ gives rise to a local system of F-vector spaces on $Y_{F,1}(\mathfrak{N})$, which we also denote by $V_{kk}(F')$.

Theorem 3.3.4 (Eichler-Shimura-Harder, [LW20, Theorem 2.18])**.** If $\Psi \in S_{kk}(U_{F,1}(\mathfrak{N}))$ is a normalized Hecke eigenform, we have the following isomorphism of 1-dimensional $\mathbb{C}$-vector spaces

$$S_{k,k}(U_{F,1}(\mathfrak{N}))[\Psi] \cong \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathbb{C}))[\Psi] \cong \mathrm{H}^1(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathbb{C}))[\Psi].$$

induced by a canonical Hecke-equivariant injection:

$$S_{k,k}(U_{F,1}(\mathfrak{N})) \hookrightarrow \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathbb{C})).$$

Thus from Theorem 3.3.4, we have $\eta_\Psi \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathbb{C}))$, corresponding to an eigenform $\Psi \in S_{k,k}(U_{F,1}(\mathfrak{N}))$. Fix an eigenform $\Psi \in S_{k,k}(U_{F,1}(\mathfrak{N}))$, and let $F'/\mathbb{Q}$ be a finite extension, large enough, containing F and the Hecke eigenvalues of Ψ, $\mathfrak{P}$ be a prime of F' above p, and $\mathcal{O}_{F',(\mathfrak{P})}$ be the valuation ring of F' at $\mathfrak{P}$. From [LW20, Section 2H], we can regard $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})}))$ as an $\mathcal{O}_{F',(\mathfrak{P})}$-lattice in $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}, V_{kk}(F'))$. Moreover, $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})}))$ is preserved by the action of Hecke operators $(T_{\mathfrak{a}})^*$ and $(U_{\mathfrak{a}})^*$.

Define

$$\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})}))[\Psi] := \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(F'))[\Psi] \cap \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})})).$$

From [LW20, Proposition 2.20], we know that there exists a complex period $\Omega_\Psi \in \mathbb{C}^\times$, such that the quotient $\phi_\Psi := \dfrac{\eta_\Psi}{\Omega_\Psi}$ forms an $\mathcal{O}_{F',(\mathfrak{P})}$-basis of $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})}))[\Psi]$.

Definition 3.3.5. We define

$$\phi^*_\Psi := j^*(\phi_\Psi) \in \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',(\mathfrak{P})}))$$

which is the pullback of ϕ_Ψ under $j : Y^*_{F,1}(\mathfrak{N}) \to Y_{F,1}(\mathfrak{N})$.

This modular symbol ϕ^*_Ψ will be used later to define the p-adic Asai L-function of Ψ.

From now onward, as described in the Introduction, the ideal $\mathfrak{N} \subset \mathcal{O}_F$ is divisible by all the primes above p.

Remark 3.3.6. By an *eigenform* Ψ of weight (k,k) and level $\mathfrak{N}$, i.e., of level $U_{F,1}(\mathfrak{N})$ we mean either Ψ is a *newform* or Ψ is a *p-stabilized p-regular eigenform* in the sense:

- Ψ is an eigenform and for each $\mathfrak{p} \mid p$ in F, $U_{\mathfrak{p}}(\Psi) = c(\mathfrak{p}, \Psi)\Psi$, with $c(\mathfrak{p}, \Psi) \neq 0$,
- there exists an ideal $\mathfrak{M}$ coprime with p and a Bianchi newform $\mathcal{F} \in S_{(k,k)}(U_{F,1}(\mathfrak{M}))$ such that $\mathfrak{N} = \mathfrak{M}\prod_{\mathfrak{p}|p}\mathfrak{p}$ and Ψ is obtained from $\mathcal{F}$ by successive $\mathfrak{p}$-stabilization,
- for each $\mathfrak{p} \mid p$, the roots of $X^2 - c(\mathfrak{p}, \mathcal{F})X + \epsilon_{\mathcal{F}}(\mathfrak{p})N(\mathfrak{p})^{k+1}$ are distinct, where $\epsilon_{\mathcal{F}}$ is the nebentypus of $\mathcal{F}$.

3.4 Asai-Eisenstein elements

In this section, we first recall definitions of modular units and Kato's Siegel units. After that, we recall the definitions and constructions of weight $k = 2$ as well as higher weight $k > 2$ *Asai-Eisenstein elements* from [LW20, Sections 3, 5]. These are compatible families of classes in the Betti cohomology of locally symmetric spaces of Bianchi modular forms. Note that there is no étale cohomology in the Bianchi setting since the Bianchi manifolds, the locally symmetric spaces, are real manifolds of dimension 3 and hence are not algebraic varieties. Loeffler–Williams constructed these elements by pushing forward Kato's Siegel units to the Betti cohomology (under some maps) but using methods similar to the étale cohomology setting from [LLZ18].

3.4.1 Modular units and Siegel units

A *modular unit* on $Y_{\mathbb{Q}}(U)$ is an element of $\mathcal{O}(Y_{\mathbb{Q}}(U))^{\times}$, that is, a regular function on $Y_{\mathbb{Q}}(U)$ with no zeros or poles, where $U \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is an open compact subgroup.

Definition 3.4.1. For $N \geqslant 1$, and $c > 1$ an integer coprime to $6N$, let

$$_{c}g_N \in \mathcal{O}(Y_{\mathbb{Q},1}(N))^{\times}$$

be the Kato's Siegel unit $_{c}g_{0,1/N}$, which is defined in [Kat04].

By an abuse of notation, we use $_{c}g_N$ again for the pullback of this unit to the mixed modular curve $Y_{\mathbb{Q}}(M,N)$, for any $M \geqslant 1$.

Proposition 3.4.2. [Kat04, Section 2.11, Proposition 3.11] The Siegel units are norm compatible, that is, if $N \mid N'$ and N and N' have the same prime divisors, then under the natural projection map $\mathrm{pr} : Y_{\mathbb{Q}}(M,N') \to Y_{\mathbb{Q}}(M,N)$ we have

$$(\mathrm{pr})_*({}_{c}g_{N'}) = {}_{c}g_N.$$

3.4.2 Weight 2 Asai-Eisenstein elements

For a modular unit $u \in \mathcal{O}(Y_{\mathbb{Q},1}(N))^{\times}$, one can associate a Betti realization to u: $C(u) \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N),\mathbb{Z})$. See [LW20, Proposition 3.2].

Definition 3.4.3 (Betti-Eisenstein class)**.** Let $_{c}C_N := C({}_{c}g_N) \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N),\mathbb{Z})$ be the Betti realization of $_{c}g_N$.

Proposition 3.4.2 implies that if $p \mid N$, the classes $_{c}C_{Np^r}$, for $r \geqslant 0$, are compatible under push-forward, and hence defines a class

$$_{c}C_{Np^{\infty}} \in \varprojlim_{r} \mathrm{H}^1(Y_{\mathbb{Q},1}(Np^r),\mathbb{Z}).$$

Let $\mathfrak{N}$ be an ideal in $\mathcal{O}_F$ divisible by some integer $\geqslant 4$ and recall that $Y^*_{F,1}(\mathfrak{N}) = \Gamma^*_{F,1}(\mathfrak{N})\backslash\mathbb{H}_3$.

Definition 3.4.4. Let $m \geqslant 1$ be an integer and $a \in \mathcal{O}_F$. Consider the map

$$\kappa_{a/m} : Y^*_{F,1}(m^2\mathfrak{N}) \to Y^*_{F,1}(\mathfrak{N})$$

given by the left action of $\begin{pmatrix} 1 & a/m \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(F)$ on $\mathbb{H}_3$.

Since $\mathfrak{N}$ is divisible by some integer $\geqslant 4$, we have maps

$$Y_{\mathbb{Q},1}(m^2N) \xrightarrow{i} Y^*_{F,1}(m^2\mathfrak{N}) \xrightarrow{\kappa_{a/m}} Y^*_{F,1}(\mathfrak{N}),$$

where $(N) = \mathbb{Z} \cap \mathfrak{N}$.

Note that there are isomorphisms

$$\mathrm{H}^1(Y_{\mathbb{Q},1}(m^2N),\mathbb{Z}) \cong \mathrm{H}_1^{\mathrm{BM}}(Y_{\mathbb{Q},1}(m^2N),\mathbb{Z}), \qquad \mathrm{H}^2(Y^*_{F,1}(m^2\mathfrak{N}),\mathbb{Z}) \cong \mathrm{H}_1^{\mathrm{BM}}(Y^*_{F,1}(m^2\mathfrak{N}),\mathbb{Z}),$$

where $\mathrm{H}_*^{\mathrm{BM}}$ denotes Borel-Moore homology (homology with non-compact support). See [BM60] for the reference. We define a push-forward map

$$i_* : \mathrm{H}^1(Y_{\mathbb{Q},1}(m^2N),\mathbb{Z}) \to \mathrm{H}^2(Y^*_{F,1}(m^2\mathfrak{N}),\mathbb{Z}),$$

since Borel-Moore homology is covariantly functorial for proper maps.

Definition 3.4.5 (Weight 2 Asai-Eisenstein elements)**.** For $m \geqslant 1$ integer, $a \in \mathcal{O}_F/m\mathcal{O}_F$, and $c > 1$ integer coprime to $6mN$, define

$${}_c\Xi_{m,\mathfrak{N},a} := (\kappa_{a/m})_* \circ (i)_*({}_cC_{m^2N}) \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}),\mathbb{Z}),$$

and

$${}_c\Phi^r_{\mathfrak{N},a} = \sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi_{p^r,\mathfrak{N},at} \otimes [t] \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z}) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times].$$

Remark 3.4.6. We will use another definition for the Asai–Eisenstein element ${}_c\Xi_{m,\mathfrak{N},a}$ later for our patching arguments involving $Y^*_F(m, m\mathfrak{N})$.

Theorem 3.4.7 (Loeffler–Williams)**.**

1. If $\mathfrak{N} \mid \mathfrak{N}'$ are two ideals of $\mathcal{O}_F$ with the same prime factors, then push-forward along the map $Y_{F,1}(\mathfrak{N}') \to Y_{F,1}(\mathfrak{N})$ sends ${}_c\Phi^r_{\mathfrak{N}',a}$ to ${}_c\Phi^r_{\mathfrak{N},a}$ for any valid choices of c, a, r.
2. Let $r \geqslant 1$ be an integer, a be a generator of $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$, and let

   $$\pi_{r+1} : \mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z}) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^{r+1})^\times] \to \mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z}) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]$$

   denote the map which is the identity on the first component and the natural quotient map on the second component. Then we have

   $$\pi_{r+1}({}_c\Phi^{r+1}_{\mathfrak{N},a}) = (U_p)_* \cdot {}_c\Phi^r_{\mathfrak{N},a}, \tag{3.4.1}$$

   where the Hecke operator $(U_p)_*$ acts via its action on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z})$. Similarly, when $r = 0$, we have

   $$\pi_1({}_c\Phi^1_{\mathfrak{N},a}) = ((U_p)_* - 1) \cdot {}_c\Phi^0_{\mathfrak{N},a}.$$

Proof. See [LW20, Lemma 3.12, Theorem 3.13]. ∎

We need the following rephrasing of Theorem 3.4.7 in terms of ${}_c\Xi_{p^r,\mathfrak{N},at}$ elements:

- If $\mathfrak{N}' \mid \mathfrak{N}$, then ${}_c\Xi_{p^r,\mathfrak{N}',at}$ maps to ${}_c\Xi_{p^r,\mathfrak{N},at}$ along the pushforward map $Y^*_F(\mathfrak{N}') \to Y^*_F(\mathfrak{N})$ for all $t \in (\mathbb{Z}/p^r)^\times$.

- For $r > 0$ and for each $t \in (\mathbb{Z}/p^r)^\times$, we have

$$\sum_{\substack{s \in \mathbb{Z}/p^{r+1} \\ s \equiv t \mod p^r}} {}_c\Xi_{p^{r+1},\mathfrak{N},as} = (U_p)_* \, {}_c\Xi_{p^r,\mathfrak{N},at}.$$

- For $r = 0$, we have

$$\sum_{t \in (\mathbb{Z}/p)^\times} {}_c\Xi_{p,\mathfrak{N},at} = ((U_p)_* - 1) \, {}_c\Xi_{1.\mathfrak{N},a}.$$

For more details, see the proof of [LW20, Theorem 3.13].

3.4.2.1 Another description of weight 2 Asai-Eisenstein elements

Let $a \in \mathcal{O}_F$. Consider the composite map

$$Y_{\mathbb{Q}}(m, mN) \overset{\iota}{\hookrightarrow} Y_F^*(m, m\mathfrak{N}) \xrightarrow{u_a} Y_F^*(m, m\mathfrak{N}),$$

where u_a is the action of the matrix $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ on $Y_F^*(m, m\mathfrak{N})$. Note that $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ preserves each component of $Y_F^*(m, m\mathfrak{N})$.

Definition 3.4.8 (Zeta elements)**.** Define ${}_c\mathcal{Z}_{m,\mathfrak{N},a}$ to be the image of ${}_cC_{mN} = C({}_cg_{mN}) \in \mathrm{H}^1(Y_{\mathbb{Q}}(m, mN), \mathbb{Z})$ under the pushforward $(u_a \circ \iota)_*$,i.e.

$${}_c\mathcal{Z}_{m,\mathfrak{N},a} = (u_a \circ \iota)_*({}_cC_{mN}) \in \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathbb{Z}).$$

For $t \in (\mathbb{Z}/m\mathbb{Z})^\times$, let $\mathrm{proj}_t : Y_F^*(m, m\mathfrak{N}) \to Y_F^*(m, m\mathfrak{N})^{(t)}$ be the projection map, and $(\mathrm{proj}_t)_* : \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathbb{Z}) \to \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathbb{Z})^{(t)} := \mathrm{H}^2(Y_F^*(m, m\mathfrak{N})^{(t)}, \mathbb{Z})$ be the projection induced by proj_t.

Let ${}_c\mathcal{Z}_{m,\mathfrak{N},a}(t)$ be the projection of ${}_c\mathcal{Z}_{m,\mathfrak{N},a}$ to the direct summand of $\mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathbb{Z})$ given by the t-th component. In other words,

$${}_c\mathcal{Z}_{m,\mathfrak{N},a}(t) = (\mathrm{proj}_t)_*({}_c\mathcal{Z}_{m,\mathfrak{N},a}),$$

and hence we get

$${}_c\mathcal{Z}_{m,\mathfrak{N},a} = \sum_t {}_c\mathcal{Z}_{m,\mathfrak{N},a}(t).$$

We consider the map

$$s_m : Y_F^*(m, m\mathfrak{N}) \to Y_{F,1}^*(\mathfrak{N})$$

given by the action of $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$. This map corresponds to $(z, t) \mapsto (z/m, t/m)$ on $\mathbb{H}_3$.

Lemma 3.4.9. [LW20, Lemma 4.5] The pushforward of ${}_cC_{m^2N}$ along the map $Y_{\mathbb{Q},1}(m^2N) \to Y_{\mathbb{Q}}(m, mN)$, given by $z \mapsto mz$ on $\mathbb{H}$, is ${}_cC_{mN}$.

Theorem 3.4.10. [LW20, Proposition 4.4] Let $a \in \mathcal{O}_F$ be a generator of $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$. We have

$$(s_{p^r})_*({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) = {}_c\Xi_{p^r,\mathfrak{N},ta},$$

and thus

$${}_c\Phi^r_{\mathfrak{N},a} = \sum_{t \in (\mathbb{Z}/p^r\mathbb{Z})^\times} (s_{p^r})_*({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) \otimes [t].$$

We will use${}_c\mathcal{Z}_{p^r,\mathfrak{N},a}$ for interpolating twists later. We also fix $a \in \mathcal{O}_F$ such that it generates $(\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z}))$.

3.4.3 Higher weight Asai-Eisenstein elements

Let E be a finite extension of $\mathbb{Q}_p$ such that F embeds into E. Let $\mathcal{O}_E$ be its ring of integers. For $k \geqslant 0$ integer, recall $V_k(\mathcal{O}_E) = \mathrm{Sym}^k((\mathcal{O}_E)^2)$, the left $\mathcal{O}_E[\mathrm{GL}_2(\mathbb{Z})]$-module of symmetric polynomials of degree k in 2 variable with coefficients in $\mathcal{O}_E$. Consider $T_k(\mathcal{O}_E) = (V_k(\mathcal{O}_E))^*$ i.e. $T_k(\mathcal{O}_E)$ is the module of symmetric tensors of degree k over $(\mathcal{O}_E)^2$. We then have the $\mathcal{O}_E[\mathrm{GL}_2(\mathcal{O}_F)]$-module $V_{kk}(\mathcal{O}_E) = V_k(\mathcal{O}_E) \otimes (V_k(\mathcal{O}_E))^\sigma$, where $\mathrm{GL}_2(\mathcal{O}_F)$ acts on the first factor via the embedding $\mathcal{O}_F \hookrightarrow \mathcal{O}_E$ and on the second component via its Galois conjugate. Let $T_{kk}(\mathcal{O}_E) = (V_{kk}(\mathcal{O}_E))^*$. The space $T_k(\mathcal{O}_E)$ can be viewed as a local system of $\mathcal{O}_E$-modules on $Y_{\mathbb{Q},1}(N)$ for any integer $N \geqslant 4$. Similarly, $T_{kk}(\mathcal{O}_E)$ gives a local system on $Y^*_F(U)$ and $Y_F(U)$ for sufficiently small U.

3.4.3.1 Moment maps

The linear functional dual to the second basis vector of $\mathcal{O}_E^2$ defines a $\Gamma^*_{F,1}(p^t\mathfrak{N})$-invariant linear functional on $\mathrm{Sym}^k((\mathcal{O}_E/p^t)^2)$ or on $(\mathrm{Sym}^k((\mathcal{O}_E/p^t)^2))^\sigma$ and hence an invariant vector in $T_{kk}(\mathcal{O}_E/p^t)$. This can be seen as a section of the corresponding local system, defining a class

$$e_{F,k,t} \in \mathrm{H}^0(Y^*_{F,1}(p^t\mathfrak{N}), T_{kk}(\mathcal{O}_E/p^t)).$$

Cup-product with $e_{f,k,t}$ defines a *moment map*

$$\mathrm{mom}^{kk} : \varprojlim_t \mathrm{H}^2(Y^*_{F,1}(p^t\mathfrak{N}), \mathbb{Z}) \otimes \mathcal{O}_E \to \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)).$$

This is the Betti cohomology analogue of the moment maps in the étale cohomology of modular curves considered in [KLZ17]. In the next section, we will describe and use $e_{f,k,t}$ in more detail.

By Theorem (3.4.7), the family of classes $({}_c\Phi^r_{\mathfrak{N}p^t,a})_{t\geqslant 0}$ is compatible under pushforward, so it is a valid input to the mom^{kk} after base-extending from $\mathcal{O}_E$ to $\mathcal{O}_E[(\mathbb{Z}/p^r)^\times]$.

Definition 3.4.11. We let ${}_c\Phi^{k,r}_{\mathfrak{N},a}$ be the image of the compatible system $({}_c\Phi^r_{\mathfrak{N}p^t,a})_{t\geqslant 0}$ under mom^{kk}.

More precisely,

$${}_c\Phi^{k,r}_{\mathfrak{N},a} = (\mathrm{mom}^{kk} \otimes \mathrm{Id})(({}_c\Phi^{r}_{\mathfrak{N}p^t,a})_{t \geqslant 0}) \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes \mathcal{O}_E[(\mathbb{Z}/p^r)^\times],$$

and hence, we have

$${}_c\Phi^{k,r}_{\mathfrak{N},a} = \sum_{t \in (\mathbb{Z}/p^r)^\times} \big(\mathrm{mom}^{kk}(({}_c\Xi_{p^r,\mathfrak{N}p^t,a})_{t \geqslant 0})\big) \otimes [t].$$

Note that the action of the Hecke operator $(U_p)_*$ is well defined both on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))$ and on the $\varprojlim_t \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}p^t), \mathbb{Z}_p)$. Moreover, the map mom^{kk} commutes with $(U_p)_*$. Hence, we have the following norm-compatibility relation, for any $k \geqslant 1$,

$$\pi_{r+1}({}_c\Phi^{k,r+1}_{\mathfrak{N},a}) = (U_p)_* \cdot {}_c\Phi^{k,r}_{\mathfrak{N},a}.$$

3.4.3.2 Relation to the weight $2k$ Eisenstein series

From [LW20, Proposition 5.2], for an integer $k \geqslant 0$, we know that there exists a class $\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q}))$, whose image under the comparison map (comparison between the Betti cohomology and the deRham cohomology) is the class of the differential form $-N^k F^{k+2}_{1/N} dw^{\otimes k} d\tau$. Here $F^{k+2}_{1/N}$ is an Eisenstein series of weight $k+2$ that appears in [Kat04]. Via the base-extension, we can consider $\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q}_p))$. This class does not generally lie in the lattice $\mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Z}_p))$. But for any integer $c > 1$ coprime with $2, 3, N$, there exists ${}_c\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_{kj}(\mathbb{Z}_p))$ such that

$${}_c\mathrm{Eis}^k_N = (c^2 - c^{-k}\langle c \rangle)\mathrm{Eis}^k_N$$

holds in $\mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q}_p))$, where $\langle c \rangle$ is the diamond operator acting on $\mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Z}_p))$ (see [Kin15], and [KLZ17] for details). Note that, when $k = j$, we have ${}_c\mathrm{Eis}^0_{mN} = {}_cC_{mN}$.

Definition 3.4.12 (Clebsch-Gordan map)**.** For $j \in \{1, \dots, k\}$We can regard $T_{2k-2j}(\mathcal{O}_E)$ as $\mathrm{SL}_2(\mathbb{Z})$-invariant submodule of the $\mathrm{SL}_2(\mathcal{O}_F)$-module $T_{kk}(\mathcal{O}_E)$, via the *Clebsch-Gordan map*

$$\mathrm{CG}^{[k,k,j]} : T_{2k-2j}(\mathcal{O}_E) \to T_{kk}(\mathcal{O}_E)$$

normalized as is in [KLZ17].

Recall the composition of maps

$$Y_{\mathbb{Q}}(m, mN) \overset{\iota}{\hookrightarrow} Y^*_F(m, m\mathfrak{N}) \xrightarrow{u_a} Y^*_F(m, m\mathfrak{N}).$$

Using this map, we obtain another composition of maps

$$(u_a \circ \iota)_* \circ \mathrm{CG}^{[k,k,kj]} : \mathrm{H}^1(Y_{\mathbb{Q}}(m, mN), T_{2k-2j}(\mathcal{O}_E)) \to \mathrm{H}^2(Y^*_F(m, m\mathfrak{N}), T_{kk}(\mathcal{O}_E)).$$

Definition 3.4.13 (Twisted Asai-Eisenstein element)**.** Let ${}_c\Xi^{k,j}_{m,\mathfrak{N},a} \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))$ be the image of $(u_a \circ \iota)_* \circ \mathrm{CG}^{[k,k,j]}({}_c\mathrm{Eis}^{2k-2j}_{mN})$ under the restrictions to the identity component followed by $(s_m)_*$. Similarly, $\Xi^{k,j}_{m,\mathfrak{N},a}$ is defined, for the analogous element with E-coefficients, using $\mathrm{Eis}^{2k-2j}_{mN}$.

Explicitly, for $t \in (\mathbb{Z}/m\mathbb{Z})^\times$,

$$ {}_c\Xi^{k,j}_{m,\mathfrak{N},at} := (s_m)_* \circ (\mathrm{proj}_t)_* \circ (u_a \circ \iota)_* \mathrm{CG}^{[k,k,j]}({}_c\mathrm{Eis}^{2k-2j-2}_{mN}), $$

where $\mathrm{proj}_t : Y^*_F(m, m\mathfrak{N}) \to Y^*_F(m, m\mathfrak{N})^{(t)}$ is the projection map.

Remark 3.4.14. Note that $\Xi^{k,j}_{p^r,\mathfrak{N},at} \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(E))$, and one has the following equality

$$ \Xi^{k,j}_{p^r,\mathfrak{N},at} = p^{jr} \cdot (\kappa_{at/p^r})_*(\iota_* \mathrm{CG}^{[k,k,j]}(\mathrm{Eis}^{2k-2j}_{p^{2r}N})). $$

This description is convenient to relate this element with special values of the Asai L-function. See [LW20, Lemma 5.4] for the details.

Proposition 3.4.15. [LW20, Proposition 5.5] For any integer $r \geqslant 0$, we have

$$ {}_c\Phi^{k,r}_{\mathfrak{N},a} = \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,0}_{p^r,\mathfrak{N},at} \otimes [t], $$

where the equality takes place in $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes \mathcal{O}_E[(\mathbb{Z}/p^r)^\times]$.

Remark 3.4.16. The action of the Hecke operator $(U_p)_*$ is well defined on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))$ and on $\varprojlim_r \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}p^r), \mathbb{Z}_p)$. Moreover, the maps $\mathrm{mom}^{kk}, \mathrm{mom}^{(k-j)(k-j)}$ commutes with $(U_p)_*$, and see the proof of Theorem 3.5.3 for the relation between $\mathrm{mom}^{(k-j)(k-j)}$ and $\mathrm{CG}^{[k,k,j]}$. Thus, like Theorem 3.4.7, we have the following norm-compatibility: for any integer $r > 1$,and any integer $0 \leqslant j \leqslant k$, we have

$$ \pi_{r+1}\left(\sum_{t \in (\mathbb{Z}/p^{r+1})^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes [t] \right) = (U_p)_* \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes [t]. $$

3.4.4 The p-adic Asai L-function: p-ordinary case

In [LW20], Loeffler–Williams constructed a p-adic measure, that is, a p-adic L-function in the Iwasawa algebra $\Lambda_E(\Gamma)$. They proved:

Theorem. [LW20, Theorem 7.5] For any integer $c > 1$ coprime to $6\mathfrak{N}$, there exists a p-adic L-function

$$ {}_cL_p^{\mathrm{As}}(\Psi) := \left\langle \phi^*_\Psi,\ \varprojlim_r (U_p)_*^{-r} \mathrm{e}_{\mathrm{ord},*}\, {}_c\Phi^{k,r}_{\mathfrak{N},a} \right\rangle \in \mathcal{O}_E[[\mathbb{Z}_p^\times]] $$

which satisfies the following interpolation property: for any Dirichlet character θ of conductor p^r, and for any integer $0 \leqslant j \leqslant k$, we have

$$ \int_{\mathbb{Z}_p^\times} x^j \theta(x) d\,{}_cL_p^{\mathrm{As}}(\Psi)(x) = \begin{cases} (*)L^{\mathrm{As}}(\Psi, \overline{\theta}, j+1) & \text{if } (-1)^j\theta(-1) = 1, \\ 0 & \text{if } (-1)^j\theta(-1) = -1, \end{cases} $$

where $(*)$ is some non-zero explicit factor.

Here and $(e_{\mathrm{ord},*})$ is the Hida's ordinary projector. Note that, roughly, by King's theory of polylogarithms ([Kin15]), ${}_cL_p^{\mathrm{As}}(\Psi)$ is independent of the twist j. Thus, integrating $x^j\theta(x)$ against $d\,{}_cL_p^{\mathrm{As}}(\Psi)$ computes $L^{\mathrm{As}}(\Psi,\overline{\theta},j+1)$. In other words, we have the following equality

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)d\,{}_cL_p^{\mathrm{As}}(\Psi)(x) = \int_{\mathbb{Z}_p^\times} \theta(x)d\,{}_cL_p^{\mathrm{As},j}(\Psi)(x), \tag{3.4.2}$$

where

$${}_cL_p^{\mathrm{As},j}(\Psi) = \varprojlim_r \left\langle \phi_\Psi^*, \quad (U_p)_*^{-r} e_{\mathrm{ord},*} \sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes [t] \right\rangle.$$

3.5 Cyclotomic twists of Asai-Eisenstein elements

In this section, we patch cyclotomic twists of Asai-Eisenstein elements using methods analogous to those in [LZ16]. We first study the hybrid locally symmetric space $Y_F^*(m,m\mathfrak{N})$ in detail. After that following [LLZ14], [LLZ18], and [LW20], we define higher weight zeta elements ${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a}$ which are required for the patching arguments. We also use ${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a}$ to obtain congruences like those in [LZ16] as well as to define ${}_c\Xi^{j,j}_{p^r,\mathfrak{N},at}$.

3.5.1 On mixed level locally symmetric spaces $Y_F^*(m,m\mathfrak{N})$

Recall that the locally symmetric space $Y_F^*(m,m\mathfrak{N})$ is not connected and has $\hat{\mathbb{Z}}^\times/\det(U_F^*(m,m\mathfrak{N}))\cong(\mathbb{Z}/m)^\times$ connected components. Each connected component can be identified with $Y_F^*(m,m\mathfrak{N})^{(1)} = \Gamma_F^*(m,m\mathfrak{N})\backslash\mathbb{H}_3$ by the matrix $\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}$, where

$$\Gamma_F^*(m,m\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : \begin{matrix} a\equiv 1 \mod m, & b\equiv 0 \mod m \\ c\equiv 0 \mod m\mathfrak{N}, & d\equiv 1 \mod m\mathfrak{N}\end{matrix} \right\}$$

and $t\in(\mathbb{Z}/m)^\times$. In other words,

$$Y_F^*(m,m\mathfrak{N})^{(1)} \xrightarrow[\cong]{\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}} Y_F^*(m,m\mathfrak{N})^{(t)}.$$

We have the following commutative diagram for each $t\in(\mathbb{Z}/m)^\times$

$$\begin{tikzcd}
Y_F^*(m,m\mathfrak{N})^{(1)} \arrow[r, "\begin{pmatrix} 1 & -ta \\ 0 & 1\end{pmatrix}"] \arrow[d, "\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}"] & Y_F^*(m,m\mathfrak{N})^{(1)} \arrow[d, "\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}"] \\
Y_F^*(m,m\mathfrak{N})^{(t)} \arrow[r, "\begin{pmatrix} 1 & -a \\ 0 & 1\end{pmatrix}"] & Y_F^*(m,m\mathfrak{N})^{(t)}
\end{tikzcd} \tag{3.5.1}$$

See [LW20, Proof of Proposition 4.4]. If we identify $Y_F^*(m, m\mathfrak{N})^{(t)}$ with $\Gamma_F^*(m, m\mathfrak{N})\backslash\mathbb{H}_3$ via $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, the restriction to this component of the right action of $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ on the adelic symmetric space corresponds to the left action of $\begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix}$ on $\mathbb{H}_3$, i.e, the right action of $\begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix}$.

Thus, the right action by $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ can be decomposed (using the above commutative diagram) as

$$\begin{aligned}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Recall ${}_c\mathcal{Z}_{m,\mathfrak{N},a} = (u_a \circ \iota)({}_cC_{mN}) = \sum_{t\in(\mathbb{Z}/m)^\times} {}_c\mathcal{Z}_{m,\mathfrak{N},a}(t)$, where ${}_c\mathcal{Z}_{m,\mathfrak{N},a}(t) = (\mathrm{proj}_t)_*({}_c\mathcal{Z}_{m,\mathfrak{N},a})$. Moreover, we know

$${}_c\Xi_{p^r,\mathfrak{N},ta} = (s_{p^r})_*({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) = (s_{p^r})_*((\mathrm{proj}_t)_*({}_c\mathcal{Z}_{m,\mathfrak{N},a})).$$

Lets explore the relation between ${}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)$ and ${}_c\Xi_{p^r,\mathfrak{N},ta}$ explicitly. Using the commutative diagram, we can write

$$\begin{aligned}{}_c\mathcal{Z}_{m,\mathfrak{N},a}(t) &= (\mathrm{proj_t})_*({}_c\mathcal{Z}_{m,\mathfrak{N},a}), \\ &= (\mathrm{proj_t})_*((u_a)_*\iota_*({}_cC_{mN})), \\ &= (u_a)_*\left((\mathrm{proj}_t)_*(\iota)_*({}_cC_{mN})\right), \\ &= \left(\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix} \cdot u_{ta} \cdot \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)_* ((\mathrm{proj}_t)_*(\iota)_*({}_cC_{mN})).\end{aligned}$$

If we write $(\mathrm{proj}_t)_*\iota_*({}_cC_{mN}) = \alpha$, then,

$$\begin{aligned}{}_c\mathcal{Z}_{m,\mathfrak{N},a}(t) &= \left(\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{pmatrix} \cdot u_{ta} \cdot \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)_* (\alpha), \\ &= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* (u_{ta})_* \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}_* (\alpha), \\ &= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* (u_{ta})_*(\alpha^{(1)}),\end{aligned}$$

where by superscript (t), we mean the element in $Y_F^*(p^r, p^r\mathfrak{N})^{(t)}$.

Hence,

$$\begin{aligned}
{}_c\Xi_{m,\mathfrak{N},at} &= (s_{p^r})_*({}_c\mathcal{Z}_{m,\mathfrak{N},a}(t)),\\
&= (s_{p^r})_*\left(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_*(\alpha^{(1)})\right),\\
&= \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* \left((s_{p^r})_*(u_{ta})_*(\alpha^{(1)})\right), \text{(because } \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix} \text{ commutes.)}\\
&= ((s_{p^r})_*(u_{ta})_*(\alpha^{(1)}))^{(t)}.
\end{aligned}$$

Since we have defined ${}_c\mathcal{Z}_{m,\mathfrak{N},a} = (u_a)_*\iota_*({}_cC_{mN})$, we get

$$ {}_c\Xi_{m,\mathfrak{N},at} = (s_{p^r})_*(({}_c\mathcal{Z}_{m,\mathfrak{N},ta}(1))(t)).$$

This description will be helpful in the following sections.

3.5.2 Cyclotomic twists calculations

Let $E/\mathbb{Q}_p$ be a finite extension that contains the quadratic imaginary field $F/\mathbb{Q}$ and $\mathcal{O}_F \hookrightarrow \mathcal{O}_E$.

If e_1, e_2 is the standard basis of $(\mathcal{O}_E)^2$, then, for $i \in \{1.2\}$ let $e_{i,r} := e_i \mod p^r$. Then $e_{F,k,r} = e_{2,r}^{[k]} \otimes e_{2,r}^{[k]} \in \mathrm{H}^0(Y^*_{F,1}(\mathfrak{N}p^r), T_{kk}(\mathcal{O}_E/p^r))$ is a section, where $e^{[k]}$ denotes k-th divided power of e. Also, we have chosen e_2 such that

$$(u_a)_*(e_{2,r}^{[k]} \otimes e_{2,r}^{[k]}) = e_{2,r}^{[k]} \otimes e_{2,r}^{[k]}.$$

We observe that

$$(u_a)_*(e_{1,r} \otimes e_{1,r}) = (e_{1,r} - a e_{2,r}) \otimes (e_{1,r} - a^\sigma e_{2,r}).$$

Recall that we have $s_{p^r} : Y^*_F(p^r, p^r\mathfrak{N}) \to Y^*_{F,1}(\mathfrak{N})$ given by the action of $\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}$. By an abuse of notation, write $T_{kk}(\mathcal{O}_E)$ for the local system $\mathcal{O}_E$-modules both on $Y^*_{F,1}$ and Y^*_F. Then let

$$(s_{p^r})_\sharp : T_{kk}(\mathcal{O}_E) \to (s_{p^r})^*(T_{kk}(\mathcal{O}_E))$$

be the map on local systems (sheaves) given by the action of $\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}$ on the representation ($\mathcal{O}_E[\mathrm{GL}_2(\mathcal{O}_F)]$-module) $T_{kk}(\mathcal{O}_E)$. Thus $e_1^{[k]} \otimes e_1^{[k]}$ is in the kernel of $(s_{p^r})_\sharp \mod p^r$. In other words, if we consider the following map

$$(s_{p^r})_\sharp : T_{kk}(\mathcal{O}_E/p^r) \to (s_{p^r})^*(T_{kk}(\mathcal{O}_E/p^r)),$$

then $e_{1,r}^{[k]} \otimes e_{1,r}^{[k]}$ is in the kernel of $(s_{p^r})_\sharp$.

Definition 3.5.1 ($\bmod p^r$ moment maps)**.** We define $\bmod p^r$ moment maps as

$$\begin{aligned} \mathrm{mom}^{kk}_{p^r} : \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}p^r), \mathbb{Z}) \otimes (\mathcal{O}_E/p^r) &\to \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}p^r), T_{kk}(\mathcal{O}_E/p^r)), \\ z &\mapsto z \cup (e^{[k]}_{2,r} \otimes e^{[k]}_{2,r}). \end{aligned}$$

Recall the *Clebsch-Gordan map*: for $j \in \{1, \ldots, k\}$, we have

$$\mathrm{CG}^{[k,k,j]} : T_{2k-2j}(\mathcal{O}_E) \to T_{kk}(\mathcal{O}_E).$$

By putting $k = j$ in $\mathrm{CG}^{[k,k,j]}$, we get

$$\mathrm{CG}^{[j]} := \mathrm{CG}^{[j,j,j]} : (\mathcal{O}_E)^2 \to T_{jj}(\mathcal{O}_E).$$

Let $x, y \in \mathrm{H}^0(Y_{\mathbb{Q}}(p^r, p^r N), \mathbb{Z}_p) \otimes (\mathcal{O}_E/p^r)$ be the sections of order p^r such that $\iota^*(e_{1,r})$ and $\iota^*(e_{2,r})$ agree with the images of sections x, y respectively under the map

$$\mathrm{H}^0(Y_{\mathbb{Q}}(p^r, p^r N), \mathbb{Z}_p) \otimes (\mathcal{O}_E/p^r) \hookrightarrow \mathrm{H}^0(Y_{\mathbb{Q}}(p^r, p^r N), \mathbb{Z}_p) \otimes \iota^*(\mathcal{O}_E/p^r).$$

See [LLZ18, Remark 8.1.2] for a similar situation in the Hilbert modular form setting.

Remark 3.5.2. Modulo p^r, the $\mathrm{CG}^{[j]}$ is defined by the cup-product with

$$\sum_{i=0}^{j} (-1)^i i!(j-i)! e^{[i]}_{1,r} e^{[j-i]}_{2,r} \otimes e^{[j-i]}_{1,r} e^{[i]}_{2,r}.$$

Theorem 3.5.3. For any $t \in (\mathbb{Z}/p^r)^\times$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^r\mathfrak{N}), \mathbb{Z}_p) \otimes (\mathcal{O}_E/p^r) \arrow[r, "\cup(\iota^*(e^{[j]}_{2,r}))^{\otimes 2}"] \arrow[d, "\mathrm{CG}^{[j]}"] & \mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^r N), \iota^*(T_{jj}(\mathcal{O}_E/p^r))) \arrow[d, "t^j j!(a-a^\sigma)^j"] \\
\mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^r N), \iota^*(T_{jj}(\mathcal{O}_E/p^r))) \arrow[d, "\iota_*"] & \mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^r N), \iota^*(T_{jj}(\mathcal{O}_E/p^r))) \arrow[d, "\iota_*"] \\
\mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r)) \arrow[d, "(u_a)_*"] & \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r)) \arrow[d, "(u_a)_*"] \\
\mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r)) \arrow[d, "(s_{p^r})_* \circ (\mathrm{proj}_t)_*"] & \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r)) \arrow[d, "(s_{p^r})_* \circ (\mathrm{proj}_t)_*"] \\
\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r)) \arrow[r, "="] & \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{jj}(\mathcal{O}_E/p^r))
\end{tikzcd}$$

Proof. The proof is similar to the proofs of [KLZ17, Theorem 6.2.4] and [LLZ18, Theorem 8.1.4] in the étale cohomology setting. For the convenience of readers, we will prove this theorem.

We will use a following fact about divided powers:

$$(A+B)^{[m]} = \sum_{k=0}^{m} A^{[k]} B^{[m-k]}.$$

For $z \in \mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^r N), \mathbb{Z}_p) \otimes (\mathcal{O}_E/p^r)$, we have

$$(\mathrm{proj}_t)_*(u_a \circ \iota)_* \mathrm{CG}^{[j]}(z) = (\mathrm{proj}_t)_* \Bigg((u_a \circ \iota)_* \Bigg(z \cup \iota^* \Bigg(\sum_{i=0}^{j} (-1)^i (j-i)! e_{1,r}^{[i]} e_{2,r}^{[j-i]} \otimes e_{1,r}^{[j-i]} e_{2,r}^{[i]} \Bigg)) \Bigg) \Bigg),$$

$$= (u_a)_* \Bigg((\mathrm{proj}_t)_* \Bigg(\iota_*(z) \cup \sum_{i=0}^{j} (-1)^i (j-i)! e_{1,r}^{[i]} e_{2,r}^{[j-i]} \otimes e_{1,r}^{[j-i]} e_{2,r}^{[i]} \Bigg) \Bigg).$$

Now as in the proof of [LW20, Proposition 4.4] and by the calculations involving ${}_c\mathcal{Z}$, if we write $\mathcal{Z}^{[j]}(t) = (\mathrm{proj}_t)_* \left(\iota_*(z) \cup \sum_{i=0}^{j} (-1)^i (j-i)! e_{1,r}^{[i]} e_{2,r}^{[j-i]} \otimes e_{1,r}^{[j-i]} e_{2,r}^{[i]} \right) \in \mathrm{H}^2(Y_F^*(p^r, p^r \mathfrak{N})^{(t)}, T_{jj}(\mathcal{O}_E/p^r))$, then

$$(u_a)_*(\mathcal{Z}^{[j]}(t)) = \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* (u_{ta})_* \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}_* \right) (\mathcal{Z}^{[j]}(t)),$$

$$= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* (u_{ta})_* (\mathcal{Z}^{[j]}(1)). \text{ (naively)}$$

Thus,

$$(u_{ta})_* \left(\iota_*(z)(1) \cup \left(\sum_{i=0}^{j} (-1)^i (j-i)! e_{1,r}^{[i]} e_{2,r}^{[j-i]} \otimes e_{1,r}^{[j-i]} e_{2,r}^{[i]} \right) \right)$$

$$= (u_{ta})_*((\iota_*(z)(1))) \cup \left(\sum_{i=0}^{j} (e_{1,r} - (ta) e_{2,r})^{[i]} e_{2,r}^{[j-i]} \otimes (e_{1,r} - (ta)^\sigma e_{2,r})^{[j-i]} e_{2,r}^{[i]} \right)$$

Therefore,

$$(s_{p^r})_*(u_a)_*(\mathcal{Z}^{[j]}(t))$$

$$= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* (s_{p^r})_* \left((u_{ta})_*(\iota_*(z)(1)) \cup t^j j! (a - a^\sigma)^j (e_{2,r}^{[j]} \otimes e_{2,r}^{[j]}) + \text{ sum involving } e_{1,r} \right),$$

$$= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}_* \left(((s_{p^r})_*(u_{ta})_*(\iota_*(z)(1))) \cup \left(t^j j! (a - a^\sigma)^j (e_{2,r}^{[j]} \otimes e_{2,r}^{[j]}) \right) \right),$$

since $e_{1,r}^{[j]} \otimes e_{1,r}^{[j]}$ is in the kernel of $(s_{p^r})_\sharp$ and $e_{2,r}^{[j]} \otimes e_{2,r}^{[j]}$ is invariant under $(s_{p^r})_\sharp$.

Now, if we chase the diagram on the right-hand side, we get the same equation. ∎

For $0 \leqslant j \leqslant k$ and $Y \in \{Y^*_{F,1}(\mathfrak{N}), Y^*_F(m, m\mathfrak{N})\}$, we define $\mathrm{mom}^{(k-j)(k-j)}$ (similar to mom^{kk})

$$\mathrm{mom}^{(k-j)(k-j)} : \mathrm{H}^2(Y, T_{jj}(\mathcal{O}_E)) \to \mathrm{H}^2(Y, T_{kk}(\mathcal{O}_E)),$$

such that modulo p^r, $\mathrm{mom}^{(k-j)(k-j)}$ is defined by the cup product with the element $e^{[k-j]}_{2,r} \otimes e^{[k-j]}_{2,r}$.

Lemma 3.5.4. For all $0 \leqslant j \leqslant k$, we have

$$\mathrm{mom}^{(k-j)(k-j)}_{p^r} \circ \mathrm{mom}^{jj}_{p^r} = \binom{k}{j}^2 \mathrm{mom}^{kk}_{p^r}.$$

Proof. This is proved in [LLZ18, Lemma 8.2.1]. See also [KLZ17, Lemma 6.3.2]. ∎

Theorem 3.5.5. For $r \geqslant 1$ and any $t \in (\mathbb{Z}/p^r)^\times$, we have, as classes in $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E/p^r))$ (i.e. equality $\mod p^r$),

$$ {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} = t^j(a - a^\sigma)^j j! \binom{k}{j}^2 {}_c\Xi^{k,0}_{p^r,\mathfrak{N},at}.$$

Proof. This is [LW20, Proposition 5.6]. We will prove this for the convenience of the readers. We know that for any $t \in (\mathbb{Z}/p^r)^\times$, by definition,

$${}_c\Xi^{j,j}_{p^r,\mathfrak{N},at} = (s_{p^r})_*(\mathrm{proj}_t)_*(u_a \circ \iota)_*\mathrm{CG}^{[j]}({}_cC_{p^rN}).$$

On the other hand, from [LW20, Proposition 5.5], we have

$${}_c\Xi^{j,0}_{p^r,\mathfrak{N},at} = \mathrm{mom}^{jj}({}_c\Xi_{p^r,\mathfrak{N},at}).$$

Now, modulo p^r, $\mathrm{mom}^{jj}_{p^r}$ is the map defined by the cup product with the element $e^{[j]}_{2,r} \otimes e^{[j]}_{2,r}$. Hence Theorem 3.5.3 implies

$${}_c\Xi^{j,j}_{p^r,\mathfrak{N},at} = t^j(a - a^\sigma)^j j! \, {}_c\Xi^{j,0}_{p^r,\mathfrak{N},at}, \tag{3.5.2}$$

for any $t \in (\mathbb{Z}/p^r)^\times$.

Hence, applying Lemma 3.5.4 to (3.5.2), we get

$$\begin{aligned}\mathrm{mom}^{(k-j)(k-j)}_{p^r}({}_c\Xi^{j,j}_{p^r,\mathfrak{N},at}) &= t^j(a - a^\sigma)^j j! \mathrm{mom}^{(k-j)(k-j)}_{p^r}(\mathrm{mom}^{jj}_{p^r})({}_c\Xi_{p^r,\mathfrak{N},at}),\\ &= t^j(a - a^\sigma)^j j! \binom{k}{j}^2 \mathrm{mom}^{kk}_{p^r}({}_c\Xi_{p^r,\mathfrak{N},at}),\\ &= t^j(a - a^\sigma)^j j! \binom{k}{j}^2 {}_c\Xi^{k,0}_{p^r,\mathfrak{N},at}.\end{aligned}$$

Now using the following commutative diagram (for the local systems on Y_F^* or modules, see [KLZ17, Proposition 5.1.2] for more details),

$$\begin{tikzcd}
(\mathcal{O}_E/p^r)^2 \arrow[r, "\mathrm{CG}^{[j]}"] \arrow[d, "\mathrm{mom}_{p^r}^{2k-2j}"] & T_{jj}(\mathcal{O}_E/p^r) \arrow[d, "\mathrm{mom}_{p^r}^{(k-j)(k-j)}"] \\
T_{2k-2j}(\mathcal{O}_E/p^r) \arrow[r, "\mathrm{CG}^{[k,k,j]}"] & T_{kk}(\mathcal{O}_E/p^r)
\end{tikzcd}$$

we can conclude

$$\mathrm{mom}_{p^r}^{(k-j)(k-j)}({}_c\Xi^{j,j}_{p^r,\mathfrak{N},at}) = {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}.$$

This completes the proof. ∎

3.5.3 Patching arguments

For $j > 0$, like ${}_c\mathcal{Z}_{p^r,\mathfrak{N},a}$, we define

$${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a} = (u_a \circ \iota)_* \mathrm{CG}^{[j]}({}_cC_{p^rN}) \in \mathrm{H}^2(Y_F^*(p^r, p^r\mathfrak{N}), T_{jj}(\mathcal{O}_E)).$$

Therefore, for any $t \in (\mathbb{Z}/p^r)^\times$, we get

$${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a}(t) = (\mathrm{proj}_t)_*(u_a \circ \iota)_* \mathrm{CG}^{[j]}({}_cC_{p^rN}) \in \mathrm{H}^2(Y_F^*(p^r, p^r\mathfrak{N})^{(t)}, T_{jj}(\mathcal{O}_E)), \tag{3.5.3}$$

and

$${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a} = \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},a}(t). \tag{3.5.4}$$

Note that $\mathrm{CG}^{[j]} : (\mathcal{O}_E)^2 \to T_{jj}(\mathcal{O}_E)$ is defined by the cup product with an element $\mathcal{CG}^{[j]}$ such that $\mathcal{CG} \mod p^r$ is $e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}$. Let us denote $e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}$ by $\mathcal{CG}_r$.

Theorem 3.5.6. For $a \in \mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$, $j \in \mathbb{Z}_{>0}$ and any $t \in (\mathbb{Z}/p^r)^\times$, we have the following equality modulo p^{jr}

$$\begin{aligned}
&\sum_{i=0}^{j} t^{(j-i)}(a - a^\sigma)^{(j-i)}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\mathrm{Res}_{p^r}^{p^{jr}}({}_c\mathcal{Z}^{[i]}_{p^r,\mathfrak{N},a}(t)) \\
&= (\mathrm{proj}_t)_* \left(\mathrm{Res}_{p^r}^{p^{jr}}(u_a \circ \iota)_*({}_cC_{p^rN}) \cup (u_{ta})_* \left((t(a-(a)^\sigma)e_{2,jr} \otimes e_{2,jr} + \mathcal{CG}_{jr})^{[j]}\right)\right) \\
&= \mathrm{Res}_{p^r}^{p^{jr}}({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) \cup (u_{ta})_* \left((t(a-(a)^\sigma)e_{2,jr} \otimes e_{2,jr} + \mathcal{CG}_{jr})^{[j]}\right).
\end{aligned}$$

Here, the map $\mathrm{Res}_{p^r}^{p^{jr}}$ is the pullback along (or induced) by natural projection $Y_F^*(p^{jr}, p^{jr}\mathfrak{N}) \to Y_F^*(p^r, p^r\mathfrak{N})$.

Proof. This proof is similar to the proof of [LZ16, Proposition 3.3.4]. But here we are proving it in the setting of the Betty cohomology and for the algebraic group over F rather than

$\mathbb{Q}$. Write Res for $\mathrm{Res}^{p^{jr}}_{p^r}$. From the equation (3.5.1), and calculations involving ${}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)$, we have, modulo p^{jr},

$$\begin{aligned}
&\mathrm{Res}({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) \cup (u_{ta})_* \left((t(a-(a)^\sigma)e_{2,jr}\otimes e_{2,jr}+\mathcal{CG}_{jr})^{[j]}\right)\\
&= \mathrm{Res}\left((\mathrm{proj}_t)_*(u_a\circ\iota)_*({}_cC_{p^rN})\right)) \cup (u_{ta})_* \left((t(a-(a)^\sigma)e_{2,jr}\otimes e_{2,jr}+\mathcal{CG}_{jr})^{[j]}\right),\\
&= \mathrm{Res}\left(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_*((\iota_*\,{}_cC_{p^rN})(1))\right) \cup (u_{ta})_* \left((t(a-(a)^\sigma)e_{2,jr}\otimes e_{2,jr}+\mathcal{CG}_{jr})^{[j]}\right),\\
&= \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_* \left(\mathrm{Res}((\iota_*\,{}_cC_{p^rN})(1)) \cup (t(a-(a)^\sigma)e_{2,jr}\otimes e_{2,jr}+\mathcal{CG}_{jr})^{[j]}\right),\\
&= \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_* \left(\mathrm{Res}((\iota_*\,{}_cC_{p^rN})(1)) \cup \sum_{i=0}^{j}(t(a-a^\sigma)e_{2,jr}\otimes e_{2,jr})^{[j-i]}\mathcal{CG}_{jr}^{[i]}\right),\\
&= \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_* \left(\sum_{i=0}^{j}\mathrm{Res}((\iota_*\,{}_cC_{p^rN})(1)) \cup \mathcal{CG}_{jr}^{[i]} \cup (t(a-a^\sigma)e_{2,jr}\otimes e_{2,jr})^{[j-i]}\right),\\
&= \sum_{i=0}^{j}\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_*(\mathrm{Res}((\iota_*\,{}_cC_{p^rN})(1)) \cup \mathcal{CG}_{jr}^{[i]}) \cup (t(a-a^\sigma)e_{2,jr}\otimes e_{2,jr})^{[j-i]}.
\end{aligned}$$

Note that modulo p^{jr}, $\mathrm{mom}^{(j-i)(j-i)}$ is defined by cup product with the element $e_{2,jr}^{[j-i]}\otimes e_{2,jr}^{[j-i]}$. By linear algebra of symmetric tensors, we have $(t(a-a^\sigma)e_{2,jr}\otimes e_{2,jr})^{[j-i]} = (j-i)!t^{(j-i)}(a-a^\sigma)^{(j-i)}e_{2,jr}^{[j-i]}\otimes e_{2,jr}^{[j-i]}$.

Hence, by the definition of $\mathrm{CG}^{[j]}$ map, we conclude

$$\begin{aligned}
&\sum_{i=0}^{j}\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}_* (u_{ta})_*(\mathrm{Res}((\iota_*\,{}_cC_{p^rN})(1)) \cup \mathcal{CG}_{jr}^{[i]}) \cup (t(a-a^\sigma)e_{2,jr}\otimes e_{2,jr})^{[j-i]},\\
&= \sum_{i=0}^{j} t^{(j-i)}(a-a^\sigma)^{(j-i)}(j-i)!(\mathrm{Res}((\mathrm{proj}_t)_*(u_a)_*(\iota)_*\mathrm{CG}^{[i]}({}_cC_{p^rN})) \cup (e_{2,jr}^{[j-i]}\otimes e_{2,jr}^{[j-i]}),\\
&= \sum_{i=0}^{j} t^{(j-i)}(a-a^\sigma)^{(j-i)}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\mathrm{Res}({}_c\mathcal{Z}^{[i]}_{p^r,\mathfrak{N},a}(t)),
\end{aligned}$$

completing the proof. ∎

Using Theorem 3.5.6, we obtain the following key congruence:

Lemma 3.5.7. For any $t \in (\mathbb{Z}/p^r)^\times$, we have

$$\begin{aligned}&\sum_{i=0}^{j} t^{(j-i)}(a-a^\sigma)^{(j-i)}(j-i)!\mathrm{Res}_{p^r}^{p^{jr}} \mathrm{mom}^{(j-i)(j-i)}({}_c\Xi^{[i,i]}_{p^r,\mathfrak{N},ta}) \\ &\in p^{jr}\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{jj}(\mathcal{O}_E)).\end{aligned} \tag{3.5.5}$$

Proof. The proof is similar to the proof of [LZ16, Theorem 3.3.5].

Recall we have defined ${}_c\Xi^{j,j}_{p^r,\mathfrak{N},at}$ as $(s_{p^r})_*(\mathrm{proj}_t)_*(u_a \circ \iota)_*\mathrm{CG}^{[j]}({}_cC_{p^rN})$. Thus, we have

$${}_c\Xi^{j,j}_{p^r,\mathfrak{N},at} = (s_{p^r})_*({}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},at}(t)),$$

since we have defined ${}_c\mathcal{Z}^{[j]}_{p^r,\mathfrak{N},at}(t) = (\mathrm{proj}_t)_*(u_a \circ \iota)_*\mathrm{CG}^{[j]}({}_cC_{p^rN})$.

From Theorem 3.5.6, $\sum_{i=0}^{j} t^{(j-i)}(a-a^\sigma)^{(j-i)}(j-i)!\mathrm{Res}_{p^r}^{p^{jr}} \mathrm{mom}^{(j-i)(j-i)}({}_c\Xi^{[i,i]}_{p^r,\mathfrak{N},ta})$ modulo p^{jr} boils down to

$$\mathrm{Res}_{p^r}^{p^{jr}}\left((s_{p^r})_*\left({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t) \cup (u_{ta})_*\left((t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + \mathcal{CG}_r)^{[j]}\right)\right)\right).$$

We claim that

$$(s_{p^r})_*(u_{ta})_*\left((t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + \mathcal{CG}_r)^{[j]}\right) = 0.$$

Note that

$$\begin{aligned}(u_{ta})_*(t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + \mathcal{CG}_r) &= t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + (u_{ta})_*(e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}), \\ &= t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + (e_{1,r} - t \cdot a e_{2,r}) \otimes e_{2,r} - e_{2,r} \otimes (e_{1,r} - t \cdot a^\sigma e_{2,r}), \\ &= e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}.\end{aligned}$$

This implies

$$\begin{aligned}(s_{p^r})_*((u_{ta})_*(t(a-a^\sigma)e_{2,r} \otimes e_{2,r} + \mathcal{CG}_r)) &= (s_{p^r})_*(e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}), \\ &= 0,\end{aligned}$$

since $e_{1,r} \otimes e_{2,r} - e_{2,r} \otimes e_{1,r}$ is in $\mathrm{Ker}((s_{p^r})_\sharp) \mod p^r$.

Because this element is killed by $(s_{p^r})_*$ modulo p^r, its j-th tensor power is will be zero after applying $(s_{p^r})_* \mod p^{jr}$ and hence we are done. ∎

In the next section, we will use Theorem 3.5.6 to interpolate (patch) different polynomials to obtain the p-adic distribution.

3.6 Interpolation of twists and construction of the p-adic distribution

Recall $-D$ to be the discriminant of F. We have assumed the level $\mathfrak{N}$ to be divisible by some integer $\geqslant 4$. This is automatic if p is unramified in F and $p \geqslant 5$. ([LW20, Page 1692]). Note that we have taken $a \in \mathcal{O}_F$ such that a generates $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$. Thus take $a = \frac{1}{2}(1+\sqrt{-D})$ if $D \equiv -1 \mod 4$, and $a = \frac{1}{2}\sqrt{-D}$ if $D \equiv 0 \mod 4$. Thus $a - a^{\sigma} = \sqrt{-D}$. Recall $E/\mathbb{Q}_p$ is a finite extension large enough such that F embeds into E and all Hecke eigenvalues are also in E.

3.6.1 Polynomial setup

To construct the distribution, i.e., a power series with unbounded coefficients, we use the technique developed by Perrin-Riou ([PR94, Subsection 1.2]) and by Büyükboduk–Lei ([BL21, Section 2]). To use these methods, we need the language of polynomials, or rather polynomials modulo $\omega_r(X) = (1+X)^{p^r} - 1$.

Write $\mathbb{Z}_p^\times \cong \Delta \times (1 + p\mathbb{Z}_p)$, where Δ is cyclic group of order $p-1$. As mentioned in the introduction, fix topological generator u of $1 + p\mathbb{Z}_p$, i.e., $\overline{\langle u \rangle} = 1 + p\mathbb{Z}_p$. For any integer $r \geqslant 1$, define

$$u_r := u \mod p^r.$$

Then $u_r \in (\mathbb{Z}/p^r)^\times$. Let $\varepsilon : \Delta \to \mathbb{Z}_p^\times$ denote the Teichmüller character. Also, write ε for $\varepsilon : \mathbb{Z}_p^\times \to \Delta \to \mathbb{Z}_p^\times$. Then for any $x \in \mathbb{Z}_p^\times$, $\frac{x}{\varepsilon(x)} \in 1 + p\mathbb{Z}_p$. Now for any $t \in (\mathbb{Z}/p^r)^\times$, let $\tilde{t} \in \mathbb{Z}_p^\times$ be a lift of t. Then there exists a unique integer $0 \leqslant m < p^{r-1}$, such that

$$t \equiv \tilde{t} \equiv \varepsilon(\tilde{t}) u^m \mod p^r.$$

In particular, we get

$$\frac{\tilde{t}}{\varepsilon(\tilde{t})} \equiv u^m \mod p^r$$

and thus $\frac{\tilde{t}}{\varepsilon(\tilde{t})} = u_r^m \in \langle u_r \rangle$, since order of u_r is p^{r-1}. We define $\log_u(t)$ to be a unique integer $0 \leqslant m < p^{r-1}$ such that

$$\log_u(t) := \frac{\tilde{t}}{\varepsilon(\tilde{t})} = u_r^m.$$

Let $\delta : \Delta \to \mathcal{O}_E^\times$ be a group homomorphism. Note that δ is a non-negative integer power of the Teichmüller character ε. We then have a ring homomorphism

$$\begin{aligned} \mathcal{O}_E[(\mathbb{Z}/p^r)^\times] &\to \mathcal{O}_E[T]; \\ [t] &\mapsto \delta(t)(1+T)^{\log_u(t)}. \end{aligned}$$

Note that $\delta(t)$ is an abuse of notation for $\delta(t \mod p)$.

From now onwards, we will use the polynomial setup.

3.6.2 Construction of the p-adic L-function at non-ordinary primes

Define a norm $||\cdot||$ on $E[T]$ as

$$||f|| := \sup_{|z|_p \leqslant 1} |f(z)|_p$$

for any polynomial $f \in E[T]$.

Note that $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes E = \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(E))$ and hence $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))$ is an $\mathcal{O}_E$-lattice in $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(E))$. Hence it defines a norm $|\cdot|$ on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(E))$. By an abuse of notation, we define a norm $||\cdot||$ on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes \mathcal{O}_E[T]$ using $|\cdot|$.

Like ${}_c\Phi^{k,j,r}_{\mathfrak{N},a} = \sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes [t]$ defined in [LW20], we define polynomials, for $\delta \in \Delta^*$:

$$\sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)}$$

lying in $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes \mathcal{O}_E[T]$.

We now use congruences from Lemma 3.5.7 to prove the following congruence theorem.

Lemma 3.6.1. For any character $\delta : \Delta \to \mathcal{O}_E^\times$, any integer $0 \leqslant j \leqslant k$, and $p^r > 0$, we have

$$\sup_r \left|\left| p^{-jr} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}\delta(t)(1+T)^{\log_u(t)} \right|\right| < \infty. \tag{3.6.1}$$

Proof. For simplicity, we assume δ is the trivial character and $r > 1$. The proof for non-trivial δ is similar.

We will use the fact, for $0 \leqslant i \leqslant j \leqslant k$, we have

$$\mathrm{mom}^{(k-j)(k-j)}_{p^r} \mathrm{mom}^{(j-i)(j-i)}_{p^r} = \binom{k-i}{j-i}^2 \mathrm{mom}^{(k-i)(k-i)}_{p^r}.$$

We have the following equality $\mod p^r$:

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(a^\sigma - a)^i i! \binom{k}{i}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)},$$

$$= \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(a^\sigma - a)^i i! \binom{k}{i}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} \mathrm{mom}^{(k-i)(k-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)},$$

$$= \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(a^\sigma - a)^i i! \binom{k}{i}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} \frac{1}{\binom{k-i}{j-i}^2} \mathrm{mom}^{(k-j)(k-j)} \mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)},$$

$$= \mathrm{mom}^{(k-j)(k-j)} \left(\sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(a^\sigma - a)^i i! \binom{k}{i}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} \frac{1}{\binom{k-i}{j-i}^2} \mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)} \right).$$

We can simplify factorials as

$$\binom{j}{i} \times \frac{1}{i!\binom{k}{i}^2} \times \frac{1}{\binom{k-i}{j-i}^2} = \frac{1}{\binom{k}{j}^2}(j-i)!.$$

Thus we get

$$\mathrm{mom}^{(k-j)(k-j)} \left(\sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{1}{(a^\sigma - a)^i i! \binom{k}{i}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} \frac{1}{\binom{k-i}{j-i}^2} \mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)} \right)$$

$$= \mathrm{mom}^{(k-j)(k-j)} \left(\sum_{i=0}^{j} (-1)^i \frac{1}{\binom{k}{j}^2}(j-i)! \frac{1}{(a^\sigma - a)^i} \sum_{t \in (\mathbb{Z}/p^r)^\times} \mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)} \right),$$

$$= \frac{\mathrm{mom}^{(k-j)(k-j)}}{\binom{k}{j}^2} \left(\sum_{i=0}^{j} (j-i)! \frac{1}{(a - a^\sigma)^i} \sum_{t \in (\mathbb{Z}/p^r)^\times} \mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \otimes t^{-i}(1+T)^{\log_u(t)} \right),$$

$$= \frac{\mathrm{mom}^{(k-j)(k-j)}}{\binom{k}{j}^2} \left(\sum_{t \in (\mathbb{Z}/p^r)^\times} \left(\sum_{i=0}^{j} t^{-i}(a - a^\sigma)^{-i}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \right) \otimes (1+T)^{\log_u(t)} \right)$$

From the Lemma 3.5.7, for all $t \in (\mathbb{Z}/p^r)^\times$, we know $\bmod p^{jr}$,

$$\sum_{i=0}^{j} t^{(j-i)}(a - a^\sigma)^{(j-i)}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} = 0.$$

Thus, for any $t \in (\mathbb{Z}/p^r)^\times$,

$$\sum_{i=0}^{j} t^{(-i)}(a - a^\sigma)^{(-i)}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at},$$

$$\begin{aligned} &= \frac{1}{t^j(a - a^\sigma)^j} \sum_{i=0}^{j} t^{(j-i)}(a - a^\sigma)^{(j-i)}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\, {}_c\Xi^{i,i}_{p^r,\mathfrak{N},at} \\ &\in C \cdot p^{jr}\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{jj}(\mathcal{O}_E)), \end{aligned}$$

where C is some positive constant independent of p^r related to $(a^\sigma - a)^j$. Note that if p is unramified in F then $a - a^\sigma$ is a p-adic unit and hence $C = 1$.

Therefore, using the norm $||\cdot||$ on $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))\otimes\mathcal{O}_E[T]$, we deduce

$$\left\|\frac{\mathrm{mom}^{(k-j)(k-j)}}{\binom{k}{j}^2}\left(\sum_{t\in(\mathbb{Z}/p^r)^\times}\left(\sum_{i=0}^{j} t^{-i}(a-a^\sigma)^{-i}(j-i)!\mathrm{mom}^{(j-i)(j-i)}\,{}_c\Xi^{i,i}_{p^r,\mathfrak{N},at}\right)\otimes(1+T)^{\log_u(t)}\right)\right\|$$
$$< C'\cdot p^{jr},$$

for some constant C' independent of r. and hence

$$\sup\left\|p^{-jr}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{1}{(a^\sigma-a)^i i!\binom{k}{i}^2}\sum_{t\in(\mathbb{Z}/p^r)^\times}{}_c\Xi^{k,i}_{p^r,\mathfrak{N},at}\otimes t^{-i}(1+T)^{\log_u(t)}\right\|<\infty. \tag{3.6.2}$$

This completes the proof. ∎

Remark 3.6.2. The equation (3.6.1) can be interpreted as: the sum

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{1}{(a^\sigma-a)^i i!\binom{k}{i}^2}\sum_{t\in(\mathbb{Z}/p^r)^\times}{}_c\Xi^{k,i}_{p^r,\mathfrak{N},at}\otimes t^{-i}(1+T)^{\log_u(t)}$$

lies in $C'\cdot p^{jr}\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))\otimes\mathcal{O}_E[T]$, where C' is some positive constant independent of p^r.

Before going forward, we will introduce some notations related to distributions and power series. We define Iwasawa algebras $\Lambda_E := E\otimes\mathcal{O}_E[[T]]$ and $\Lambda_E(\mathbb{Z}_p^\times) := E\otimes\mathcal{O}_E[[\mathbb{Z}_p^\times]] \cong E\otimes\mathcal{O}_E[\Delta][[T]]$.

For any real number $w\geqslant 0$, let

$$\mathcal{H}_{E,w} := \left\{\sum_{n=0}^{\infty}c_nT^n\in E[[T]] : \sup_n\frac{|c_n|_p}{n^w}<\infty\right\}$$

be the space of w-tempered/admissible distributions. Note that, $\mathcal{H}_{E,w}$ is a Λ_E-module and when $w=0$, $\mathcal{H}_{E,0}=\Lambda_E$. Similarly, we define

$$\mathcal{H}_{E,w}(\mathbb{Z}_p^\times) := \left\{\sum_{\sigma\in\Delta}\sum_{n=0}^{\infty}c_{n,\sigma}\cdot\sigma\cdot T^n\in E[\Delta][[T]] : \sup_n\frac{|c_{n,\sigma}|_p}{n^w}<\infty, \forall\sigma\in\Delta\right\}.$$

Let $\Delta^* = \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}_E^\times)$ be the group of character and let $e_\delta = \frac{1}{|\Delta|}\sum_{d\in\Delta}\delta^{-1}(d)\cdot d\in\mathcal{O}_E[\Delta]$ be the idempotent corresponding to character $\delta\in\Delta^*$. Then

$$\mathcal{H}_{E,w}(\mathbb{Z}_p^\times)\cong\bigoplus_{\delta\in\Delta^*}e_\delta(\mathcal{H}_{E,w}(T)),$$

and $e_\delta(\mathcal{H}_{E,w}(T))\cong\mathcal{H}_{E,w}$ as Λ_E-modules. We say f is $O(\log_p^w)$ whenever $f\in\mathcal{H}_{E,w}(T)$ or $f\in\mathcal{H}_{E,w}(\mathbb{Z}_p^\times)$.

We need the following lemma to construct the distribution.

Lemma 3.6.3. Let $s \geqslant 0$ and $h \geqslant 1$ be integers and $0 \leqslant s < h$. For $0 \leqslant j \leqslant h-1$, let $Q_{r,j}(T) \in E[T]$ be a sequence of polynomials satisfying

1. $\sup ||p^{sr} Q_{r,j}(T)|| < \infty$,
2. $Q_{r+1,j} \equiv Q_{r,j} \mod \omega_{r-1}(T)E[T]$

for all positive integers r. Moreover, suppose that

$$\sup_r \left|\left| p^{(s-j)r} \sum_{i=0}^{j} (-1)^i \binom{j}{i} Q_{r,i}(u^{-i}(1+T)-1) \right|\right| < \infty$$

for all $0 \leqslant j \leqslant h-1$. Then there exists a unique polynomial Q_r of degree $< hp^r$ such that

1. $Q_r(T) \equiv Q_{r,j}(u^{-j}(1+T)-1) \mod \omega_r(u^{-j}(1+T)-1)E[T]$,
2. $\sup ||p^{sr} Q_r(T)|| < \infty$,
3. $Q_{r+1} \equiv Q_r \mod \prod_{i=0}^{h-1} \omega_{r-1}(u^{-i}(1+T)-1)$,

Moreover, the sequence $(Q_r)_r$ converges to a power series Q_∞ such that $Q_\infty \in \mathcal{H}_{E,s}$.

Proof. See [PR94, Lemme 1.2.1, Lemme 1.2.2] and [BL21, Lemma 2.2, Lemma 2.3] for the details. ∎

Now we will construct the distribution using Lemma 3.6.3. Let Ψ be a Bianchi eigenform over F of weight (k,k) and level $\mathfrak{N}$, i.e. of level $U_{F,1}(\mathfrak{N})$. Assume Ψ is p-non ordinary and smalll slope, i.e. $v_p(a_p) > 0$ and $v_p(a_p) < k+1$, where a_p is the U_p-eigenvalue.

Recall from Subsection 3.3.3, that if $F'/\mathbb{Q}$ is a finite extension obtained by adjoining all the Hecke eigenvalues of Ψ to F, and let $\mathfrak{P}$ be a prime above p in E, then we defined

$$\phi_\Psi^* \in \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_{F',\mathfrak{P}})).$$

We enlarge E, if necessary so that we can fix an embedding $F'_{\mathfrak{P}} \hookrightarrow E$. Thus we can consider the modular symbol ϕ_Ψ^* as a a class in $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_E))$ well-defined up to a unit in $\mathcal{O}_E$.

For $\delta \in \Delta^*$ and for integer $r \geqslant 1$, let

$$P^\delta_{r,j}(T) = \left\langle \phi_\Psi^*, (U_p)_*^{-r} \frac{1}{(a^\sigma - a)^j j! \binom{k}{j}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle, \tag{3.6.3}$$

where $\langle , \rangle$ denotes the perfect Poincaré duality pairing

$$\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), V_{kk}(\mathcal{O}_E)) \times \frac{\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E))}{\text{Torsion}} \to \mathcal{O}_E.$$

Note that from Theorem 3.4.7, we have

$$(U_p)_*^{-1} \sum_{t \in (\mathbb{Z}/p)^\times} {}_c\Xi^{k,j}_{p,\mathfrak{N},at} = (1 - p^j (U_p)_*^{-1})\, {}_c\Xi^{k,j}_{1,\mathfrak{N},a}.$$

Thus, when δ is the trivial character and $r = 1$, we get

$$P^{\mathrm{triv}}_{1,j}(T) = \left\langle \phi^*_\Psi, \frac{(1 - p^{-j}(U_p)_*^{-1})}{(a^\sigma - a)^j j! \binom{k}{j}^2}\, {}_c\Xi^{k,j}_{1,\mathfrak{N},a} \right\rangle. \tag{3.6.4}$$

In particular, for $0 \leqslant j \leqslant k$, we get

$$\begin{aligned} P^\delta_{r,j}(T) &= \left\langle ((U_p)^*)^{-r}\phi^*_\Psi, \frac{1}{(a^\sigma - a)^j j! \binom{k}{j}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle, \\ &= \left\langle a_p^{-r}\phi^*_\Psi, \frac{1}{(a^\sigma - a)^j j! \binom{k}{j}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle, \\ &= \frac{a_p^{-r}}{(a^\sigma - a)^j j! \binom{k}{j}^2} \sum_{t \in (\mathbb{Z}/p^r)^\times} \left\langle \phi^*_\Psi, {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \right\rangle \delta(t)(1+T)^{\log_u(t)} \in E[T]. \end{aligned}$$

Moreover, when δ is the trivial character and $r = 1$, we get

$$P^{\mathrm{triv}}_{1,j} = \frac{1}{(a^\sigma - a)^j j! \binom{k}{j}^2} \left(1 - \frac{p^j}{a_p}\right) \langle \phi^*_\Psi, {}_c\Xi^{k,j}_{1,\mathfrak{N},a} \rangle.$$

Let us denote $v_p(a_p)$ by n.

Lemma 3.6.4. For any integer $r \geqslant 1$, any integer $0 \leqslant j \leqslant k$, and any character $\delta \in \Delta^*$, we have

1. $\sup ||p^{nr} P^\delta_{r,j}(T)|| < \infty$,
2. $P^\delta_{r+1,j}(T) \equiv P^\delta_{r,j} \mod \omega_{r-1}(T)$,
3. $\sup_r \left|\left| p^{(n-j)r} \sum_{i=0}^{j} (-1)^i \binom{j}{i} P^\delta_{r,i}(u^{-i}(1+T) - 1) \right|\right| < \infty$.

Proof. Statement (1) follows from the definition of the polynomial $P^\delta_{r,j}(T)$.

Note that from the remark 3.4.16, after converting it in the polynomial setup, we deduce

$$\begin{aligned} &\sum_{t \in (\mathbb{Z}/p^{r+1})^\times} {}_c\Xi^{k,j}_{p^{r+1},\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \\ &\equiv \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \mod \omega_{r-1}(T)(\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), T_{kk}(\mathcal{O}_E)) \otimes \mathcal{O}_E[T]) \end{aligned} \tag{3.6.5}$$

Thus, after pairing both sides of (3.6.5) with the modular symbol ϕ^*_Ψ, we get the second statement.

For the third part, for simplicity assume δ is trivial and integer $r > 1$. We can simplify the expression

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}P^{\delta}_{r,i}(u^{-i}(1+T)-1)$$

$$=\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{a_p^{-r}}{(a^\sigma-a)^i i!\binom{k}{i}^2}\sum_{t\in(\mathbb{Z}/p^r)^\times}\left\langle \phi^*_\Psi, {}_c\Xi^{k,i}_{p^r,\mathfrak{N},at}\right\rangle ((u)^{-i}(1+T))^{\log_u(t)},$$

$$=\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{a_p^{-r}}{(a^\sigma-a)^i i!\binom{k}{i}^2}\sum_{t\in(\mathbb{Z}/p^r)^\times}\left\langle \phi^*_\Psi, {}_c\Xi^{k,i}_{p^r,\mathfrak{N},at}\right\rangle (t)^{-i}(1+T)^{\log_u(t)},$$

$$=\left\langle \phi^*_\Psi, \sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{a_p^{-r}}{(a^\sigma-a)^i i!\binom{k}{i}^2}\sum_{t\in(\mathbb{Z}/p^r)^\times}{}_c\Xi^{k,i}_{p^r,\mathfrak{N},at}\otimes t^{-i}(1+T)^{\log_u(t)}\right\rangle$$

The statement then follows from Lemma 3.6.1 and $n = v_p(a_p)$. For the non-trivial character δ, the proof is the same. ∎

Hence from Lemma 3.6.3 and Lemma 3.6.4, we deduce

Theorem 3.6.5. For any character $\delta \in \Delta^*$, there exists a unique polynomial sequence $P^\delta_r(T) \in E[T]$ of degree $< (k+1)p^{r-1}$ such that

1. $P^\delta_r(T) \equiv P^\delta_{r,j}(u^{-j}(1+T)-1) \mod \omega_{r-1}(u^{-j}(1+T)-1)$,
2. $P^\delta_{r+1}(T) \equiv P^\delta_r(T) \mod \prod_{i=0}^{k}\omega_{r-1}(u^{-i}(1+T)-1)$,
3. $\sup_r ||p^{nr}P^\delta_r|| < \infty$.

Moreover, the sequence P^δ_r converges to ${}_cL_p^{\mathrm{As},\delta}(\Psi) := \lim_{r\to\infty} P^\delta_r \in \mathcal{H}_{E,n}$ and

$${}_cL_p^{\mathrm{As},\delta}(\Psi) \equiv P^\delta_{r,j}(u^{-j}(1+T)-1) \mod \omega_{r-1}(u^{-j}(1+T)-1).$$

Note that, for any real number $w \geqslant 0$, $\mathcal{H}_{E,w}(\mathbb{Z}_p^\times) \cong \mathcal{H}_{E,w}(\Gamma)$ via identification $T \mapsto \gamma_0 - 1$.

Definition 3.6.6 (p-adic Asai distribution)**.** For a p-non-ordinary small slope Bianchi eigenform Ψ of level $\mathfrak{N}$, weight (k,k), and U_p-eigenvalue a_p, define the p-adic distribution attached to Ψ to be

$${}_cL_p^{\mathrm{As}}(\Psi) = \bigoplus_{\delta\in\Delta^*} {}_cL_p^{\mathrm{As},\delta}(\Psi) \in \mathcal{H}_{E,n}(\Gamma),$$

where ${}_cL_p^{\mathrm{As},\delta}(\Psi)$ (after identifying T with $\gamma_0 - 1$) are from Theorem 3.6.5 and $n = v_p(a_p)$ such that $0 < n < k+1$.

Notation. For a Dirichlet character θ of conductor $p^r > 1$ and any integer $0 \leqslant j \leqslant k$, write

$$ {}_cL_p^{\mathrm{As}}(\Psi, \theta, j) := {}_cL_p^{\mathrm{As}}(\Psi)(\chi^j\theta), $$

i.e. evaluating ${}_cL_p^{\mathrm{As}}(\Psi)$ at $\gamma_0 = u^j \cdot \tilde{\zeta} \cdot \zeta_{p^{r-1}}$, where $\tilde{\zeta}$ is a $(p-1)$-th root of unity, corresponding to $\delta \in \Delta^*$ and $\zeta_{p^{r-1}}$is a primitive p^{r-1}-th root of unity.

3.6.3 The interpolation property

We will prove that the p-adic distribution ${}_cL_p^{\mathrm{As}}(\Psi)$ interpolates the critical L-values of the Asai L-function attached to Ψ. We use Section $7C$ and Theorem 7.5 of [LW20] for the interpolation.

Theorem 3.6.7. Let Ψ be a p-non-ordinary small slope Bianchi cusp form of weight (k, k), nebentypus ϵ_Ψ, and level $\mathfrak{N}$, where all primes above p divides $\mathfrak{N}$. Let $a_p = c(p\mathcal{O}_F, \Psi)$ be the U_p-eigenvalue of Ψ which satisfies $0 < v_p(a_p) < k + 1$. For any Dirichlet character θ of conductor p^r and any integer $0 \leqslant j \leqslant k$, ${}_cL_p^{\mathrm{As}}(\Psi)$ satisfies the following interpolation property:

1. If $(-1)^j\theta(-1) = 1$, then

   $$ {}_cL_p^{\mathrm{As}}(\Psi, \theta, j) = \frac{C(c, k, j)G(\theta)}{\Omega_\Psi} \times \mathfrak{e}_p(\Psi, \theta, j) L^{\mathrm{As}}(\Psi, \overline{\theta}, j + 1), $$

   where

   - $\mathfrak{e}_p(\Psi, \theta, j) = \begin{cases} \left(1 - \dfrac{p^j}{a_p}\right) & \text{if } r = 0, \\ \dfrac{1}{a_p^r} & \text{if } r > 0. \end{cases}$
   - $C(c, k, j) := \dfrac{(-1)^{k+1} \cdot p^{jr} j!(\sqrt{-D})}{2 \cdot (2\pi i)^{j+1}} \cdot (c^2 - c^{2j-2k}\epsilon_\Psi(c^{-1})\theta(c)^2)$, where $G(\theta)$ is the Gauss sum associated with θ.

2. If $(-1)^j\theta(-1) = -1$, then

   $$ {}_cL_p^{\mathrm{As}}(\Psi, \theta, j) = 0. $$

Proof. First, we assume $r > 1$ and $\delta \in \Delta^*$. Since $(\mathbb{Z}/p^r)^\times \cong \Delta \times \mathbb{Z}/p^{r-1}$, write $\theta = \delta \cdot \Theta$, where δ is a character on Δ and Θ is a character on $\mathbb{Z}/p^{r-1}$.

For any p^{r-1}-th root ζ, $u^j\zeta$ is a root of polynomial $\omega_{r-1}(u^{-j}\gamma_0 - 1)$.

Now for $\delta \in \Delta^*$, from Theorem 3.6.5, we have

$$ {}_cL_p^{\mathrm{As},\delta}(\Psi) \equiv P_{r,j}^\delta(u^{-j}\gamma_0 - 1) \mod \omega_{r-1}(u^{-j}\gamma_0 - 1) \tag{3.6.6} $$

Hence, to calculate ${}_cL_p^{\mathrm{As}}(\Psi,\theta,j)$ is equivalent to evaluating ${}_cL_p^{\mathrm{As},\delta}(\Psi)$ at $u^j\Theta$, ie, by putting $\gamma_0 = u^j\zeta$, for p^{r-1}-th root of unity ζ.

From equation (3.6.6), it sufficient to calculate $P^{\delta}_{r,j}(\zeta-1)$. Therefore, we get

$$\begin{aligned}
P^{\delta}_{r,j}(\zeta-1) &= \frac{a_p^{-r}}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\sum_{t\in(\mathbb{Z}/p^r)^{\times}}\left\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}\right\rangle \delta(t)(\zeta)^{\log_u(t)},\\
&= \frac{a_p^{-r}}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\sum_{t\in(\mathbb{Z}/p^r)^{\times}}\left\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}\right\rangle \delta(t)\Theta(t),\\
&= \frac{a_p^{-r}}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\sum_{t\in(\mathbb{Z}/p^r)^{\times}}\left\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}\right\rangle \theta(t),
\end{aligned}$$

since $\log_u(t)$ is a unique non-negative integer $< p^{r-1}$ and Θ is a character on $\mathbb{Z}/p^r$. Hence, we have deduced

$${}_cL_p^{\mathrm{As}}(\Psi,\theta,j) = \frac{a_p^{-r}}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\sum_{t\in(\mathbb{Z}/p^r)^{\times}}\left\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}\right\rangle \theta(t).$$

Now, if $r=1$ and δ is non-trivial, following the calculations similar to the above case, we get

$${}_cL_p^{\mathrm{As}}(\Psi,\theta,j) = \frac{a_p^{-1}}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\sum_{t\in(\mathbb{Z}/p)^{\times}}\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{p,\mathfrak{N},at}\rangle \delta(t).$$

Lastly, if $r=1$ and δ is the trivial character, then ${}_cL_p^{\mathrm{As}}(\Psi,j)$ is evaluating ${}_cL_p^{\mathrm{As}}(\Psi)$ at u^j. Therefore, we get

$$\begin{aligned}
{}_cL_p^{\mathrm{As}}(\Psi,j) &= P^{\mathrm{triv}}_{1,j}(u^j-1),\\
&= \frac{1}{(a^{\sigma}-a)^j j!\binom{k}{j}^2}\left(1-\frac{p^j}{a_p}\right)\langle \phi^*_{\Psi}, {}_c\Xi^{k,j}_{1,\mathfrak{N},a}\rangle.
\end{aligned}$$

The theorem then follows from [LW20, Theorem 7.5]. ∎

3.6.4 p-adic distributions without c

Recall Kato's Siegel unit ${}_cg_N \in \mathcal{O}(Y_{\mathbb{Q},1}(N))^{\times}$ from Section 3.4. Note that if $c,d\in\mathbb{Z}_{\geqslant 1}$ coprime to $6N$, then we have

$$(d^2-\langle d\rangle)\,{}_cg_N = (c^2-\langle c\rangle)\,{}_dg_N.$$

Thus, the dependence on c can be removed after extending scalars to $\mathbb{Q}$. More precisely, there is an element $g_N \in \mathcal{O}(Y_{\mathbb{Q},1}(N))^{\times}\otimes\mathbb{Q}$ such that ${}_cg_N = (c^2-\langle c\rangle)\cdot g_N$ for any choice of c. Similarly, for any integer $k>0$, we have

$${}_c\mathrm{Eis}^k_N = (c^2-c^{-k}\langle c\rangle)\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N),T_k(\mathbb{Q}_p)),$$

where ${}_c\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Z}_p))$ appearing in the Definition 3.4.13. See [KLZ17, Theorem 4.4.4] for the details. Following this path, we prove:

Proposition 3.6.8. Let $\epsilon_\Psi : (\mathcal{O}_F/\mathfrak{N})^\times \to \mathcal{O}_E^\times$ be the nebentypus of the Bianchi cusp form Ψ. Assume:

1. the restriction $\epsilon_\Psi|_{(\mathbb{Z}/N)^\times}$ is non-trivial;
2. $\epsilon_\Psi|_{(\mathbb{Z}/N)^\times}$ does not have p-power conductor.

Then there exists a distribution $L_p^{\mathrm{As}}(\Psi) \in \mathcal{H}_{E,v_p(a_p)}(\Gamma)$. such that

$$L_p^{\mathrm{As}}(\Psi) = \frac{1}{\left(c^2 - c^{-2k}\epsilon_\Psi(c^{-1})\left(\gamma_0^{\log_u(c)}\right)^2\right)} \cdot {}_cL_p^{\mathrm{As}}(\Psi)$$

for all valid $c \in \mathbb{Z}_{>1}$, where $\log_u(c)$ is the unique positive integer such that $(uz\tilde{eta}^{\log_u(c)} = c$ for appropriate $(p-1)$-th root of unity.

Before proving this proposition, note that $L_p^{\mathrm{As}}(\Psi)$ satisfies the interpolation property described in Theorem 3.6.7 without the factor involving c. In particular, for any integer $0 \leqslant j \leqslant k$ and for any Dirichlet character θ of conductor p^r, we have

$$L_p^{\mathrm{As}}(\Psi)(u^j\theta) = \frac{C'(k,j)G(\theta)}{\Omega_\Psi} \times \mathfrak{e}_p(\Psi,\theta,j)L^{\mathrm{As}}(\Psi,\overline{\theta},j+1), \tag{3.6.7}$$

where

$$C'(k,j) = \begin{cases} \dfrac{(-1)^{k+1}\cdot p^{jr}j!(\sqrt{-D})}{2\cdot(2\pi i)^{j+1}} & \text{if } (-1)^j\theta(-1) = 1, \\ 0 & \text{if } (-1)^j\theta(-1) = -1, \end{cases}$$

and $\mathfrak{e}_p(\Psi,\theta,j)$ is the same one appearing in Theorem 3.6.7.

Proof of Proposition 3.6.8. We follow [LW20, Proposition 6.7]. Denote $\left(c^2 - c^{-2k}\epsilon_\Psi(c^{-1})\left(\gamma_0^{\log_u(c)}\right)^2\right)$ by υ_c. For any positive integers c, d coprime to $6Np$, consider

$$\upsilon_d \cdot {}_cL_p^{\mathrm{As}}(\Psi).$$

Then

$$\left((\upsilon_d \cdot {}_cL_p^{\mathrm{As}}(\Psi)) - (\upsilon_c \cdot {}_dL_p^{\mathrm{As}}(\Psi))\right)(\chi^j\theta) = 0$$

for any integer $0 \leqslant j \leqslant k$ and for any Dirichlet character θ of conductor p^r. Thus, by the uniqueness of the construction ${}_cL_p^{\mathrm{As}}(\Psi)$, we can conclude

$$(\upsilon_d \cdot {}_cL_p^{\mathrm{As}}(\Psi)) = (\upsilon_c \cdot {}_dL_p^{\mathrm{As}}(\Psi)).$$

In other words, $\upsilon_d \cdot {}_cL_p^{\mathrm{As}}(\Psi)$ is symmetric in c and d.

Now we can choose d such that v_d is a unit in $E \otimes \mathcal{O}_E[[\Gamma]]$, since the conductor of ϵ_Ψ is not a power of p. Note that we get $\epsilon_\Psi(d^{-1})$, since the dual of the diamond operator $\langle d \rangle$ is $\langle d^{-1} \rangle$ under the perfect Poincaré duality.

Therefore, if we define

$$L_p^{\mathrm{As}}(\Psi) := \frac{1}{\left(c^2 - c^{-2k}\epsilon_\Psi(c^{-1})\left(\gamma_0^{\log_u(c)}\right)^2\right)}\, {}_cL_p^{\mathrm{As}}(\Psi), \tag{3.6.8}$$

then $L_p^{\mathrm{As}}(\Psi)$ is independent of c and lies in $\mathcal{H}_{E,v_p(a_p)}(\Gamma)$. ∎

Remark 3.6.9. There is a typo in the equation of $L_p^{\mathrm{As}}(\Psi)$ appearing in [LW20, Proposition 6.7]. There should be $\epsilon_\Psi(c^{-1})$ instead of $\epsilon_\Psi(c)$.

Remark 3.6.10. Even if $\epsilon_\Psi|_{(\mathbb{Z}/N)^\times}$ has p-power conductor, we can still define

$$L_p^{\mathrm{As}}(\Psi) = \frac{1}{\left(c^2 - c^{-2k}\epsilon_\Psi(c^{-1})\left(\gamma_0^{\log_u(c)}\right)^2\right)}\, {}_cL_p^{\mathrm{As}}(\Psi),$$

but this element will not be in the distribution module $\mathcal{H}_{E,v_p(a_p)}(\Gamma)$ but in the fraction field of $\mathcal{H}_{E,v_p(a_p)}(\Gamma)$. This new "distribution" will have poles, i.e., it will be a meromorphic function.

3.7 Signed p-adic Asai L-functions of Bianchi modular forms

This section addresses the factorization of unbounded p-adic L-functions (i.e. p-adic distributions) into bounded signed p-adic L-functions (i.e. p-adic measures) in the spirit of Pollack, Sprung, and Lei–Loeffler–Zerbes. We will apply the machinery of logarithmic matrices, developed by the author in [Deo26], to obtain this factorization.

3.7.1 Wach modules and logarithmic matrix

Define, for any real number $w \geqslant 0$,

$$\mathcal{H}_{E,w}(\Gamma_1) = \left\{ \sum_{n=0}^{\infty} c_n(\gamma_0 - 1)^n : \sup_n \frac{|c_n|_p}{n^w} < \infty \right\},$$

and recall

$$\mathcal{H}_{E,w}(\Gamma) = \left\{ \sum_{\sigma\in\Delta} \sum_{n=0}^{\infty} c_{n,\sigma} \cdot \sigma \cdot (\gamma_0 - 1)^n : \sup_n \frac{|c_{n,\sigma}|_p}{n^w} < \infty, \forall \sigma \in \Delta \right\}.$$

Note that

$$\mathcal{H}_{E,w} \cong \mathcal{H}_{E,w}(\Gamma_1), \tag{3.7.1}$$

$$\mathcal{H}_{E,w}(\mathbb{Z}_p^\times) \cong \mathcal{H}_{E,w}(\Gamma), \tag{3.7.2}$$

where $\mathcal{H}_{E,w}$ and $\mathcal{H}_{E,w}(\mathbb{Z}_p^\times)$ are defined in Section 3.6.2. Also, let $\mathcal{H}_E(\Gamma_1) = \bigcup_{w\geqslant 0}\mathcal{H}_{E,w}(\Gamma_1)$ and $\mathcal{H}_E(\Gamma) = \bigcup_{w\geqslant 0}\mathcal{H}_{E,w}(\Gamma)$. Similar to $\mathcal{H}_{E,w}$, if $f \in \mathcal{H}_{E,w}(\Gamma_1)$, then we say f has growth rate $O(\log_p^w)$.

Let X be a variable and write $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ for the ring of power series $f(X) \in \mathbb{Q}_p[[X]]$ such that f converges everywhere inside the unit p-adic disk. We equip $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ with the actions by the Frobeneius $\varphi : X \mapsto (1+X)^p - 1$ and $\sigma : X \mapsto (1+X)^{\chi(\sigma)} - 1$ for $\sigma \in \Gamma$, where χ is the p-adic cyclotomic character such that $\chi(\gamma_0) = u$. Let $\mathbb{B}^+_{\mathrm{rig},E} = E\otimes\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and inside $\mathbb{B}^+_{\mathrm{rig},E}$ there is a subring $\mathbb{A}^+_E = \mathcal{O}_E[[X]]$ which is also equipped with the actions of φ and Γ. Note that there exists a $\Lambda_E(\Gamma)$-module isomorphism between $(\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0}$ and $\mathcal{H}_E(\Gamma)$ called the Mellin transform, where ψ is a left inverse of φ such that $\varphi\circ\psi(f(X)) = \frac{1}{p}\sum_{\zeta^p=1} f(\zeta(1+X)-1)$. Moreover, we can identify $(\mathbb{A}^+_E)^{\psi=0}$ with $\Lambda_{\mathcal{O}_E}(\Gamma)$. See [PR94, B.2.8] for more details.

Fix $a \in \mathcal{O}_E$ with $v_p(a) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$ and $k \geqslant 0$ be an integer. Let α, β be the distinct roots of polynomial $X^2 - aX + vp^{k+1}$, for some v such that $v^{1/2} \in \mathcal{O}_E^\times$. Then by the methods in [Deo26] (which are based on methods in [BLZ04]), there exists a E-linear crystalline $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-representation V and an $\mathcal{O}_E$-stable lattice T in V such that:

1. there exists an $\mathcal{O}_E$-basis v_1, v_2 of Dieudonné module $\mathbb{D}_{\mathrm{crys}}(T)$ such that the matrix of φ with respect to this basis is
$$A_\varphi = \begin{pmatrix} 0 & -1/vp^{k-1} \\ 1 & a/vp^{k-1} \end{pmatrix}.$$
2. we have a Wach module $\mathbb{N}(T)$ with an $\mathbb{A}^+_E$-basis n_1, n_2 such that, for $i = 1, 2$,
$$n_i \equiv v_i \mod X.$$
See [Ber04], [BLZ04], [LLZ11], and [Deo26] for definitions and more details about Wach modules.
3. $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is a $\Lambda_{\mathcal{O}_E}$-module with basis $(1+\pi)\varphi(n_1), (1+\pi)\varphi(n_1)$, where $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is $\mathbb{A}^+_E$-submodule of $\mathbb{N}(T)[X^{-1}]$ generated by $\varphi(\mathbb{N}(T))$
4. $(\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0}\otimes\mathbb{D}_{\mathrm{crys}}(T)$ is an $\mathcal{H}_E(\Gamma)$-module with basis $(1+\pi)\otimes v_1, (1+\pi)\otimes v_2$.

Definition 3.7.1 (Logarithmic matrix). We define the 2×2 logarithmic matrix $\underline{M} \in M_{2,2}(\mathcal{H}_E(\Gamma))$ as the change of the basis matrix for the following $\mathcal{H}_E(\Gamma)$-module homomorphism:

$$(\varphi^*\mathbb{N}(T))^{\psi=0} \hookrightarrow (\mathbb{B}^+_{\mathrm{rig},E})^{\psi=0}\otimes\mathbb{D}_{\mathrm{crys}}(T),$$

i.e.,

$$\begin{bmatrix}(1+\pi)\varphi(n_1) & (1+\pi)\varphi(n_1)\end{bmatrix} = \begin{bmatrix}(1+\pi)\otimes v_1 & (1+\pi)\otimes v_2\end{bmatrix}\underline{M}. \tag{3.7.3}$$

Note that $\underline{M}$ is unique and in fact $\underline{M} \in M_{2,2}(\mathcal{H}_E(\Gamma_1))$. See [Deo26, Section 4] and [LLZ11] for the details about the construction of $\underline{M}$.

Proposition 3.7.2.

1. The elements in the first row of matrix $Q^{-1}\underline{M}$ have growth $O(\log_p^{v_p(\alpha)})$ and the elements in the second row have growth $O(\log_p^{v_p(\beta)})$, where $Q = \begin{pmatrix} \alpha & -\beta \\ -vp^{k-1} & vp^{k-1} \end{pmatrix}$.
2. The second row of $A_\varphi^{-n}\underline{M}$ is divisible by $\prod_{i=0}^{k} \Phi_{n-1}(u^{-i}\gamma_0 - 1)$ over $\mathcal{H}_E(\Gamma_1)$, where $\Phi_{n-1}(T) = \dfrac{\omega_{n-1}(T)}{\omega_{n-2}(T)}$ is the p^{n-1}-th cyclotomic polynomial.
3. The determinant of $\underline{M}$ is $\dfrac{\log_{p,k+1}(\gamma_0)}{\delta_{k+1}(\gamma_0)}$ up to a unit in $\Lambda_E(\Gamma_1)$. Here $\log_{p,k+1}(\gamma_0) = \prod_{i=0}^{k} \log_p(u^{-i}\gamma_0)$ and $\delta_{k+1}(\gamma_0) = \prod_{i=0}^{k}(u^{-i}\gamma_0 - 1)$.
4. Moreover, $\det(\underline{M})$ is $O(\log_p^{k+1})$ and $\log_p^{k+1}$ is $O(\det(\underline{M}))$.

Proof. See [Deo26, Proposition 5.2, Lemma 5.2, Lemma5.3, Lemma 5.4] for the details. Note that in [Deo26], $k \geqslant 2$. Here, we are taking $k \geqslant 0$, but otherwise, everything is identical. ∎

3.7.2 Decomposition of the distribution into measures

Assume prime p splits in F as $p\mathcal{O}_F = \mathfrak{p}\bar{\mathfrak{p}}$. We enlarge $E/\mathbb{Q}_p$, if necessary, so that it contains F and all Hecke eigenvalues of Ψ.

Let Ψ be a Bianchi cusp form of weight (k, k) and level $\mathcal{N}$ coprime to p.

Furthermore, we assume that:

1. Ψ is $\mathfrak{p}$-non-ordinary and $\bar{\mathfrak{p}}$-ordinary. In other words, for $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, if $a_\mathfrak{q}$ is the $T_\mathfrak{q}$ Hecke eigenvalue, then $v_p(a_\mathfrak{p}) > 0$ and $v_p(a_{\bar{\mathfrak{p}}}) = 0$.
2. $v_p(a_\mathfrak{p}) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$.

For the simplicity of calculations, assume the nebentypus of Ψ is trivial. For $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, let $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ be the roots of Hecke polynomial $X^2 - a_\mathfrak{q}X + p^{k+1}$. We also assume $\alpha_\mathfrak{p} \neq \beta_\mathfrak{p}$. Since we have assumed Ψ is $\bar{\mathfrak{p}}$-non-ordinary, we know $\alpha_{\bar{\mathfrak{p}}}$ a p-adic unit and $v_p(\beta_{\bar{\mathfrak{p}}}) = k + 1$. We also know that $v_p(\alpha_\mathfrak{p}), v_p(\beta_\mathfrak{p}) < k + 1$. Let $\tilde{\alpha} = \alpha_{\bar{\mathfrak{p}}}\alpha_\mathfrak{p}$ and $\tilde{\beta} = \alpha_{\bar{\mathfrak{p}}}\beta_\mathfrak{p}$. Then $v_p(\tilde{\alpha}) = v_p(\alpha_\mathfrak{p})$ and $v_p(\tilde{\beta}) = v_p(\beta_\mathfrak{p})$.

We consider the following two p-stabilizations of Ψ:

$$\Psi^{\tilde{\alpha}} = \Psi^{\alpha_{\bar{\mathfrak{p}}}\alpha_\mathfrak{p}} \text{ and } \Psi^{\tilde{\beta}} = \Psi^{\alpha_{\bar{\mathfrak{p}}}\beta_\mathfrak{p}},$$

such that $\Psi^{\tilde{\alpha}}$ and $\Psi^{\tilde{\beta}}$ are of level $p\mathcal{N}$ and

$$\begin{aligned} U_p(\Psi^{\tilde{\alpha}}) &= \tilde{\alpha} \cdot \Psi^{\tilde{\alpha}}, \\ U_p(\Psi^{\tilde{\beta}}) &= \tilde{\beta} \cdot \Psi^{\tilde{\beta}}. \end{aligned}$$

For more details about the p-stabilizations of Bianchi modular forms, we refer to [Pal23, Section 3.3].

Thus, for $\dagger \in \{\tilde{\alpha}, \tilde{\beta}\}$, from Theorem 3.6.5 and Theorem 3.6.7, we can attach a p-adic distribution ${}_cL_p^{\mathrm{As}}(\Psi^\dagger)$ to $\Psi^\dagger$ such that

- *(Growth property)* ${}_cL_p^{\mathrm{As}}(\Psi^\dagger) \in \mathcal{H}_{E,v_p(\dagger)}(\Gamma)$;
- *(Interpolation)* for any Dirichlet character θ of conductor p^r and any integer $0 \leqslant j \leqslant k$, we have
$$ {}_cL_p^{\mathrm{As}}(\Psi^\dagger, \theta, j) = \frac{*}{\dagger^r} L^{\mathrm{As}}(\Psi^\dagger, \overline{\theta}, j+1),$$
where
$$ * = \begin{cases} \text{some non-zero explicit constant independent of } \dagger & \text{if } (-1)^j\theta(-1) = 1; \\ 0 & \text{if } (-1)^j\theta(-1) = -1. \end{cases}$$

We have the following decomposition theorem:

Theorem 3.7.3. There exist ${}_cL_p^{\mathrm{As},\sharp}, {}_cL_p^{\mathrm{As},\flat} \in E \otimes \mathcal{O}_E[[\Gamma]]$ and a logarithmic matrix $\underline{\tilde{M}} \in M_{2,2}(\mathcal{H}_E(\Gamma))$ such that

$$\begin{pmatrix} {}_cL_p^{\mathrm{As}}(\Psi^{\tilde{\alpha}}) \\ \\ {}_cL_p^{\mathrm{As}}(\Psi^{\tilde{\beta}}) \end{pmatrix} = \tilde{Q}^{-1} \underline{\tilde{M}} \begin{pmatrix} {}_cL_p^{\mathrm{As},\sharp} \\ \\ {}_cL_p^{\mathrm{As},\flat} \end{pmatrix}, \tag{3.7.4}$$

where $\tilde{Q} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ -\alpha_{\overline{\mathfrak{p}}}^2 p^{k+1} & \alpha_{\overline{\mathfrak{p}}}^2 p^{k+1} \end{pmatrix}$, and $\underline{\tilde{M}}$ satisfies properties from Proposition 3.7.2 after replacing α, β with $\tilde{\alpha}, \tilde{\beta}$ respectively.

Proof. Since

$$\begin{aligned} v_p(\tilde{\alpha} + \tilde{\beta}) &= v_p(\alpha_{\overline{\mathfrak{p}}}(\alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}})), \\ &= v_p(\alpha_{\overline{\mathfrak{p}}}) + v_p(a_{\mathfrak{p}}), \\ &= v_p(a_{\overline{\mathfrak{p}}}) > \left\lfloor \frac{k}{p-1} \right\rfloor, \end{aligned}$$

and for $\dagger \in \{\tilde{\alpha}, \tilde{\beta}\}$, ${}_cL_p^{\mathrm{As}}(\Psi^\dagger)$ has growth rate $O(\log_p^{v_p(\dagger)})$ and interpolation property

$$ {}_cL_p^{\mathrm{As}}(\Psi^\dagger, \theta, j) = \frac{c_{\theta,j}}{\dagger^r},$$

where $c_{\theta,j}$ is independent of $\dagger$, the theorem follows from [Deo26, Theorem 5.5]. ∎

We conclude this subsection with some remarks.

Remark 3.7.4. We will explain briefly why we considered the polynomial $X^2 - a_{\mathfrak{q}}X + p^{k+1}$. This is because of the Euler factors of the Asai L-function when prime p splits in F. Recall from the equation (3.3.2), when p splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_F$ and is coprime to the level $\mathcal{N}$, the local L-factor at p of the Asai L-function $L^{\mathrm{As}}(\Psi, s)$ is

$$\frac{1}{P_p^{\mathrm{As}}(\Psi, s)} = (1 - \alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1 - \alpha_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s})(1 - \beta_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1 - \beta_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}p^{-s}),$$

where $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$ are the roots of polynomial $X^2 - a_{\mathfrak{q}}X + p^{k+1}$, for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. For Galois representation theoretic interpretation, see [Gha99, Section 4].

Remark 3.7.5. We can assume Ψ to be $\mathfrak{p}$-ordinary and $\overline{\mathfrak{p}}$-non-ordinary. If we assume that Ψ is non-ordinary at both the primes $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, then we might get into trouble. Note that we are in the finite slope situation, i.e., for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}, v_p(a_{\mathfrak{q}}) < k + 1$. Now if $\alpha_{\mathfrak{q}}, \beta_{\mathfrak{q}}$ are the roots of $X^2 - a_{\mathfrak{q}}X + p^{k+1}$, then we have four p-stabilizations $\Psi^{\bullet,\dagger}$, where $\bullet \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$ and all $\Psi^{\bullet,\dagger}$ are p-non-ordinary. For simplicity, assume $k = 0$. Now suppose we want to decompose the distributions ${}_cL_p^{\mathrm{As}}(\Psi^{\alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}})$ and ${}_cL_p^{\mathrm{As}}(\Psi^{\alpha_{\mathfrak{p}}\beta_{\overline{\mathfrak{p}}}})$ into the linear combination of bounded measures using logarithmic matrices. We know that $0 < v_p(\alpha_{\mathfrak{q}}), v_p(\beta_{\mathfrak{q}}) < 1$. To use logarithmic matrices method, $v_p(\alpha_{\mathfrak{p}}^2\alpha_{\overline{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}})$ should be a positive integer, since from the construction and properties of the logarithmic matrix, we know $\det(\underline{M}) \sim O(\log_p^m)$ where $m = v_p(\alpha_{\mathfrak{p}}^2\alpha_{\overline{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}}) \in \mathbb{Z}_{\geqslant 1}$. But then it might happen that, for example, $v_p(\alpha_{\mathfrak{p}}^2\alpha_{\overline{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}})$ is not an integer, since $v_p(\alpha_{\mathfrak{p}}^2)$ maybe an element in $\mathbb{Q}$ which is not an integer.

3.7.3 Signed p-adic Asai L-function without c

Recall Ψ is a Bianchi eigenform of weight (k, k) and level $\mathcal{N}$ which is coprime to p, such that Ψ is non-ordinary at $\mathfrak{p}$ and ordinary at $\overline{\mathfrak{p}}$. We furthermore assume:

1. The nebentypus $\epsilon_\Psi : (\mathcal{O}_F/\mathcal{N})^\times \to \overline{\mathbb{Q}}^\times$ is non-trivial,
2. the restriction $\epsilon_\Psi|_{(\mathbb{Z}/M)^\times}$ is non-trivial, where $M \in \mathbb{Z}$ such that $(M) = \mathbb{Z} \cap \mathcal{N}$,
3. $\epsilon_\Psi|_{(\mathbb{Z}/M)^\times}$ does not have a p-power conductor.

For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $\alpha_{\mathfrak{q}}$ and $\beta_{\mathfrak{q}}$ be the roots of Hecke polynomial $X^2 - a_{\mathfrak{q}}X + \epsilon_\Psi(\mathfrak{q})p^{k+1}$, where $a_{\mathfrak{q}}$ is $T_{\mathfrak{q}}$-eigenvalue of Ψ. We assume $\alpha_{\mathfrak{q}} \neq \beta_{\mathfrak{q}}$. Like in the previous subsection, let $\tilde{\alpha} = \alpha_{\overline{\mathfrak{p}}}\alpha_{\mathfrak{p}}$ and $\tilde{\beta} = \alpha_{\overline{\mathfrak{p}}}\beta_{\mathfrak{p}}$ and consider two p-stabilizations $\Psi^{\tilde{\alpha}}$ and $\Psi^{\tilde{\beta}}$ of Ψ with the U_p-eigenvalues $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. Note that the nebentypus will not change after the p-stabilization.

Thus, for $\bullet \in \{\tilde{\alpha}, \tilde{\beta}\}$, Proposition 3.6.8 implies there exists a p-adic distribution $L_p^{\mathrm{As}}(\Psi^\bullet) \in \mathcal{H}_{E, v_p(\bullet)}(\Gamma)$ such that

$$L_p^{\mathrm{As}}(\Psi^\bullet) = \frac{1}{\left(c^2 - c^{-2k}\epsilon_\Psi(c^{-1})\left(\gamma_0^{\log_u(c)}\right)^2\right)} \cdot {}_cL_p^{\mathrm{As}}(\Psi^\bullet),$$

where c is a suitable integer coprime to $6pM$, since we have imposed the conditions on the nebentypus ϵ_Ψ. It satisfies the following interpolation property: for any Dirichlet character θ of conductor $p^r > 1$ and any integer $0 \leqslant j \leqslant k$, we have

$$L_p^{\mathrm{As}}(\Psi^\bullet, \theta, j) = \frac{*}{\bullet^r} L^{\mathrm{As}}(\Psi^\bullet, \overline{\theta}, j+1),$$

where

$$* = \begin{cases} \text{some non-zero explicit constant independent of } \bullet \text{ and } c & \text{if } (-1)^j \theta(-1) = 1; \\ 0 & \text{if } (-1)^j \theta(-1) = -1. \end{cases}$$

If we assume $v_p(a_{\mathfrak{p}} + \beta_{\mathfrak{p}}) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$, then we have the following decomposition theorem similar to Theorem 3.7.3:

Theorem 3.7.6. There exist $L_p^{\mathrm{As},\sharp}, L_p^{\mathrm{As},\flat} \in E \otimes \mathcal{O}_E[[\Gamma]]$ and a logarithmic matrix $\underline{\tilde{M}} \in M_{2,2}(\mathcal{H}_E(\Gamma))$ such that

$$\begin{pmatrix} L_p^{\mathrm{As}}(\Psi^{\tilde{\alpha}}) \\ L_p^{\mathrm{As}}(\Psi^{\tilde{\beta}}) \end{pmatrix} = \tilde{Q}^{-1} \underline{\tilde{M}} \begin{pmatrix} L_p^{\mathrm{As},\sharp} \\ L_p^{\mathrm{As},\flat} \end{pmatrix}, \tag{3.7.5}$$

where $\tilde{Q} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ -\alpha_{\bar{\mathfrak{p}}}^2 \epsilon_\Psi(\mathfrak{p}) p^{k+1} & \alpha_{\bar{\mathfrak{p}}}^2 \epsilon_\Psi(\mathfrak{p}) p^{k+1} \end{pmatrix}$, and $\underline{\tilde{M}}$ satisfies properties from Proposition 3.7.2 after replacing α, β with $\tilde{\alpha}, \tilde{\beta}$ respectively. Moreover, for suitable c, we have

$$\begin{pmatrix} L_p^{\mathrm{As},\sharp} \\ L_p^{\mathrm{As},\flat} \end{pmatrix} = \frac{1}{\left(c^2 - c^{-2k} \epsilon_\Psi(c^{-1}) \left(\gamma_0^{\log_u(c)} \right)^2 \right)} \begin{pmatrix} {}_cL_p^{\mathrm{As},\sharp} \\ {}_cL_p^{\mathrm{As},\flat} \end{pmatrix}.$$

Proof. Similar to the proof of Theorem 3.7.3. ∎

Chapter 4

Families of p-adic Asai L-function of Bianchi modular forms

4.1 Introduction

Fix an odd prime p throughout. Let $F/\mathbb{Q}$ be an imaginary quadratic field. Let f be a cuspidal elliptic modular form of weight 2 and level N, and let $\mathcal{F}$ be its base change. Then, under some assumptions on f, $\mathcal{F}$ is cuspidal Bianchi modular form of weight $(0,0)$ and level $\mathfrak{N}$, where $(N) = \mathfrak{N} \cap \mathbb{Z}$. Assume $p \mid N$ and all prime above p in F divide $\mathfrak{N}$. Furthermore, assume $\mathcal{F}$ is small-slope, i.e, $v_p(a_p) < 1$, where a_p is the U_p-eigenvalue of $\mathcal{F}$ and v_p is a p-adic valuation such that $v_p(p) = 1$. Let $c \in \mathbb{Z}_{>0}$ coprime to $6N$.

In this chapter, we construct a two-variable p-adic Asai L-function associated to $\mathcal{F}$. In particular, for $V = \mathrm{Sp}(T) \subset \mathcal{E}_{\mathrm{par}}$ a family passing through $\mathcal{F}$ over $\mathcal{S} = \mathrm{Sp}(\mathfrak{L}) \subset \mathcal{W}_{F,\mathrm{par}}$ we construct:

Theorem E (Theorem 4.6.9)**.** There exists a two-variable p-adic Asai L-function

$${}_c\mathcal{L}_V^{\mathrm{As}} \in T\hat{\otimes}\mathfrak{L}\hat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma),$$

where $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$.

We also recover the p-adic Asai L-function ${}_cL_p^{\mathrm{As}}(\mathcal{F})$ constructed in [Deo25]:

Theorem F (Theorem 4.6.15)**.** We have the following specialization:

$$\mathrm{sp}_{x_{\mathcal{F}}}^{\lambda}({}_c\mathcal{L}_V^{\mathrm{As}}) \doteq {}_cL_p^{\mathrm{As}}(\mathcal{F}) \in \mathcal{H}_{L,v_p(a_p)}(\Gamma).$$

See Section 4.6 for the definitions and details.

We give a brief idea about the construction of the two-variable p-adic Asai L-function. We construct certain polynomials $P_r^{\mathcal{S}} \in T \otimes \mathfrak{L} \otimes L[T]$, using V and constructing *Asai–Eisenstein elements* over $\mathcal{S}$. These polynomials satisfy certain norm-compatibility and growth properties. See Theorem 4.6.8 for the details. Since we are dealing with $\mathbb{Q}_p$-Banach spaces, by the methods described by Perrin–Riou in [PR94] and Büyükboduk–Lei in [BL21], we can take limit of $P_r^{\mathcal{S}}$ to construct ${}_c\mathcal{L}_V^{\mathrm{As}}$.

4.2 Preliminaries and setup

Fix an odd prime p. Let $F/\mathbb{Q}$ be a quadratic imaginary field with the ring of integers denoted by $\mathcal{O}_F$. Let $\mathcal{O}_p = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\varpi_p = \prod_{v|p} \varpi_v$, where v are places above p in F and ϖ_v is the uniformizer of $\mathcal{O}_{F_v}$. Fix a $\mathbb{Z}_p$-module isomorphism $\nu : \mathcal{O}_p \cong \mathbb{Z}_p^2$.

Let $\mathbb{A}_F$ denote the adele ring of F, and the finite adeles are denoted by $\mathbb{A}_F^f$. Define $\widehat{\mathcal{O}_F} := \mathcal{O}_F \otimes \widehat{\mathbb{Z}} = \mathcal{O}_F \otimes \prod_p \mathbb{Z}_p$.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the usual upper-half plane with $\mathrm{GL}_2(\mathbb{R})$-action given by Möbius transformations. Define *upper-half space* or *hyperbolic 3-space* to be $\mathbb{H}_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R}_{>0}\}$ with $\mathrm{GL}_2(\mathbb{C})$ action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2}{| cz+d |^2 + | c |^2 t^2}, \frac{| ad-bc | \, t}{| cz+d |^2 + | c |^2 t^2} \right).$$

We embed $\mathbb{H} \hookrightarrow \mathbb{H}_3$ via $x+iy \mapsto (x,y)$, which is compatible with the actions of $\mathrm{GL}_2(\mathbb{R})$ on both sides.

4.2.1 Algebraic groups setting

Let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2, \quad G^* = G \times_D \mathbb{G}_m,$$

be the algebraic groups, where where $D := \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ and the map $G \to D$ is determinant.

Define T_G to be the torus of G such that $T_G(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathcal{O}_p^\times \right\}$. Furthermore, let B_G be the space of upper triangular matrices, i.e,

$$B_G := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}.$$

Let N_G be the space of unipotent matrices, such that $N(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathcal{O}_p \right\}$. Let $\overline{N}_G$ be the opposite of N, that is, $\overline{N}(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in \mathcal{O}_p \right\}$.

Let I_p denote the following Iwahori group at p:

$$I_p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Z}_p) : c \in \varpi_p \mathcal{O}_p \right\}. \tag{4.2.1}$$

Note that we have the following Iwahori decomposition of I_p

$$I_p = \overline{N}^1(\mathbb{Z}_p) \times T_G(\mathbb{Z}_p) \times N_G(\mathbb{Z}_p),$$

where $\overline{N}^1 = \overline{N}_G(\mathbb{Z}_p) \cap I_p$. We can identify $\overline{N}^1(\mathbb{Z}_p)$ with $\mathcal{O}_p$.

4.2.2 Locally symmetric spaces

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal and N be an integer such that $(N) = \mathfrak{N} \cap \mathbb{Z}$. We are interested in the following locally symmetric spaces:

1. The (open) modular curve $Y_{\mathbb{Q},1}(N)$ of level $\Gamma_1(N)$. We know that $Y_{\mathbb{Q},1}(N)(\mathbb{C}) \cong \Gamma_1(N)\backslash\mathbb{H}$.
2. Another (mixed level) modular curve that we are interested in is
$$Y_{\mathbb{Q}}(m, mN)(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})_+\backslash[\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f) \times \mathbb{H}]/U_{\mathbb{Q}}(m, mN),$$
where $m \in \mathbb{Z}_{\geqslant 0}$, and
$$U_{\mathbb{Q}}(m, mN) := \left\{A \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : A \equiv \mathrm{I}_2 \mod \begin{pmatrix} m & m \\ mN & mN \end{pmatrix}\right\}.$$
3. The space
$$Y_{F,1}(\mathfrak{N}) := \mathrm{GL}_2(F)\backslash[\mathrm{GL}_2(\mathbb{A}_F^f) \times \mathbb{H}_3]/U_{F,1}(\mathfrak{N}),$$
where $U_{F,1}(\mathfrak{N})$ is the space of matrices $A \in \mathrm{GL}_2(\widehat{\mathcal{O}_F})$ such that $A \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{N}$. Moreover, since $\det(U_{F,1}(\mathfrak{N})) = \widehat{\mathcal{O}_F}^\times$, $Y_{F,1}(\mathfrak{N})$ has h_F connected components, where h_F is the class number of F. The identity component is isomorphic to $\Gamma_{F,1}\backslash\mathbb{H}_3$, where
$$\Gamma_{F,1}(\mathfrak{N}) := \mathrm{GL}_2(F) \cap U_{F,1}(\mathfrak{N}).$$
This is the modular curve in the Bianchi setting.
4. The space
$$Y_{F,1}^*(\mathfrak{N}) := G^*(F)_+\backslash[G^*(\mathbb{A}_{\mathbb{Q}}^f) \times \mathbb{H}_3]/U_{F,1}^*(\mathfrak{N}),$$
where $U_{F,1}^*(\mathfrak{N}) := U_{F,1}(\mathfrak{N}) \cap G^*$. This space also has a single connected component isomorphic to $\Gamma_{F,1}^*\backslash\mathbb{H}_3$, where
$$\Gamma_{F,1}^* := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \colon c \equiv 0, a \equiv d \equiv 1 \mod \mathfrak{N}\right\}.$$
5. We are also interested in the mixed level locally symmetric space
$$Y_F^*(m, m\mathfrak{N}) := G^*(F)_+\backslash[G^*(\mathbb{A}_{\mathbb{Q}}^f) \times \mathbb{H}_3]/U_F^*(m, m\mathfrak{N}).$$
Here
$$U_F^*(m, m\mathfrak{N}) := \left\{A \in \mathrm{GL}_2(\widehat{\mathcal{O}_F}) : A \equiv \mathrm{I}_2 \mod \begin{pmatrix} m & m \\ m\mathfrak{N} & m\mathfrak{N} \end{pmatrix}\right\} \cap G^*.$$
Note that $Y_F^*(m, m\mathfrak{N})$ is not connected and has connected components indexed by group $(\mathbb{Z}/m\mathbb{Z})^\times$, since the component group of $Y_F^*(m, m\mathfrak{N})$ is $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{R}_{>0}\det(U^*(m, m\mathfrak{N})) \cong (\mathbb{Z}/m\mathbb{Z})^\times$. The identity component $Y_F^*(m.m\mathfrak{N})^{(1)}$ is isomorphic to $\Gamma_F^*(m.m\mathfrak{N})\backslash\mathbb{H}_3$, where
$$\Gamma_F^*(m.m\mathfrak{N}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : \begin{matrix} a \equiv 1 \mod m, & b \equiv 0 \mod m \\ c \equiv 0 \mod m\mathfrak{N}, & d \equiv 1 \mod m\mathfrak{N} \end{matrix}\right\}.$$

There are natural maps

$$Y_{\mathbb{Q},1}(N) \xrightarrow{\iota} Y^*_{F,1}(\mathfrak{N}) \xrightarrow{j} Y_{F,1}(\mathfrak{N})$$

induced by the natural maps $\mathbb{H} \hookrightarrow \mathbb{H}_3$ and $\mathrm{GL}_2(\mathbb{A}^f_{\mathbb{Q}}) \hookrightarrow G^*(\mathbb{A}^f_{\mathbb{Q}}) \hookrightarrow G^*(\mathbb{A}^f_{\mathbb{Q}})$.

From [LW20, Proposition 2.5], we know that If $\mathfrak{N}$ is divisible by some integer $q \geqslant 4$, then $Y^*_{F,1}(\mathfrak{N})$ is a smooth manifold, and

$$\iota \colon Y_{\mathbb{Q},1}(N) \hookrightarrow Y^*_{F,1}(\mathfrak{N})$$

is a closed immersion. Furthermore, let $m \in \mathbb{Z}$ be a positive integer, and if $m\mathfrak{N}$ is divisible some integer $\geqslant 4$, then

$$\iota \colon Y_{\mathbb{Q}}(m, mN) \hookrightarrow Y^*_F(m, m\mathfrak{N})$$

is an injective map and a closed immersion.

Assumption 4.2.1. Throughout the chapter, we will assume that the ideal $\mathfrak{N}$ of $\mathcal{O}_F$ is divisible by some integer $q \geqslant 4$. Due to this assumption, the space $Y^*_{F,1}(\mathfrak{N})$ will be a smooth manifold and not a non-smooth orbifold.

4.2.3 Elliptic and Bianchi modular forms

We will be restricting ourselves to weight 2 cuspidal elliptic modular forms and weight $(0,0)$ cuspidal Bianchi modular forms.

Let $\mathfrak{N}$ be an ideal in $\mathcal{O}_F$, and let $S_{(0,0)}(U_{F,1}(\mathfrak{N}))$ denote the finite dimensional C-vector space of cuspidal Bianchi form of weight $(0,0)$ and level $\mathfrak{N}$. Bianchi modular forms are automorphic forms over $\mathrm{GL}_2(F)$ satisfying certain automorphy conditions, growth conditions, and harmonicity. See [Wil17, LW20, Deo26, Deo25, Pal23, Pal25] for exact definitions of Bianchi modular forms.

Let $N \in \mathbb{Z}_{\geqslant 1}$ and let $S_2(\Gamma_1(N))$ denote the space of cuspidal elliptic modular forms of weight 2 and level N. For a newform $f \in S_2(\Gamma_1)$, let π_f be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Let $\mathrm{BC}(\pi_f)$ be the base-change of π_f to $\mathrm{GL}_2(\mathbb{A}_F)$. Then $\mathrm{BC}(\pi_f)$ is generated by a normalised new vector $\mathcal{F}$, and $\mathcal{F}$ is a weight $(0,0)$ Bianchi modular form of level $\mathfrak{N}$, where $\mathfrak{N} \subset \mathcal{O}_F$ is an ideal such that $\mathfrak{N} \mid N\mathcal{O}_F$. In other words, the base-change of f to F is $\mathcal{F}$. Furthermore, if f does not have CM by F, then $\mathcal{F} \in S_{(0,0)}(U_{F,q}(\mathfrak{N}))$. See [BSW21a, Section 2.2] for the details.

We will be assuming the following assumptions on f and $\mathcal{F}$:

Assumption 4.2.2 (Conditions on f)**.** Let $p \mid N$ and $f \in S_2(\Gamma_1(N))$, such that:

(A1) f is an eigenform and $U_p f = a_p f$, where a_p is the p-th Fourier coefficient, and assume $a_p \neq 0$;

(A2) f does not have CM by F;

(A3) f is either a newform or is the p-stabilization of a newform g of level prime-to-p;

(A4) If f is the p-stabilization of g, then f is p-regular. In other words, $\alpha_p \neq \beta_p$, where α_p, β_p are the roots of the Hecke polynomial $X^2 - a_p(g)X + \epsilon(p)p$, where $a_p(g)$ is the T_p Hecke eigenvalue of g.

Assumption 4.2.3 (Conditions on $\mathcal{F}$)**.** Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ such that $\mathfrak{p} \mid \mathfrak{N}$, for all $\mathfrak{p} \mid p$ in F. Let $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{N}))$ is a *p-stabilized p-regular eigenform*, i.e., either $\mathcal{F}$ is a *newform* or in the following sense:

(A1) $\mathcal{F}$ is an eigenform and for each $\mathfrak{p} \mid p$ in F, $U_{\mathfrak{p}}(\mathcal{F}) = a_{\mathfrak{p},\mathcal{F}}\mathcal{F}$, with $a_{\mathfrak{p},\mathcal{F}} \neq 0$, where $a_{\mathfrak{p},\mathcal{F}}$ is the '$\mathfrak{p}$-th Fourier coefficient';

(A2) there exists an ideal $\mathfrak{M}$ coprime with p and a Bianchi newform $\mathcal{G} \in S_{(0,0)}(U_{F,1}(\mathfrak{M}))$ such that $\mathfrak{N} = \mathfrak{M}\prod_{\mathfrak{p}|p}\mathfrak{p}$ and $\mathcal{F}$ is obtained from $\mathcal{G}$ by successive $\mathfrak{p}$-stabilization,

(A3) for each $\mathfrak{p} \mid p$, the roots of $X^2 - a_{\mathfrak{p},\mathcal{G}}X + \epsilon_{\mathcal{G}}(\mathfrak{p})N(\mathfrak{p})$ are distinct, where $\epsilon_{\mathcal{G}}$ is the nebentypus of $\mathcal{G}$.

Suppose $\mathcal{F}$ is a Bianchi Hecke eigenform with $U_{\mathfrak{p}}$-eigenvalue $a_{\mathfrak{p},\mathcal{F}}$ that is a base-change of a weight 2 elliptic Hecke cuspform f with U_p-eigenvalue $a_{p,f}$, then we have the following relations:

1. If p splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in F, then $a_{\mathfrak{p},\mathcal{F}} = a_{\overline{\mathfrak{p}},\mathcal{F}} = a_{p,f}$;
2. If p is inert in F, then $a_{p\mathcal{O}_F,\mathcal{F}} = a_{p,f}^2$;
3. If p is ramified as $\mathfrak{p}^2$ in F, then $a_{\mathfrak{p},\mathcal{F}} = a_{p,f}$.

From now on, we are interested in cuspidal Hecke Bianchi eigenforms $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{n}))$ having small slope, i.e.,

$$v_p(a_{p\mathcal{O}_F,\mathcal{F}}) < 1.$$

4.3 Locally analytic functions and distributions

Fix an ideal $\mathfrak{N} \subset \mathcal{O}_F$ such that all primes above p in F divides $\mathfrak{N}$. We first define the weight spaces we need for locally analytic functions, as well as eigenvarieties, later.

Definition 4.3.1 (Three weight spaces)**.**

1. *The Bianchi weight space* is the rigid analytic space $\mathcal{W}_{F,\mathfrak{N}}$ whose L-points, for $L \subset \mathbb{C}_p$ are
$$\mathcal{W}_{F,\mathfrak{N}}(L) = \mathrm{Hom}_{\mathrm{cts}}((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / E(\mathfrak{N}), L^\times),$$
where $E(\mathfrak{N}) := \{\epsilon \in \mathcal{O}_F^\times : \epsilon \equiv 1 \mod \mathfrak{N}\}$.
2. Let $\mathcal{W}_{\mathbb{Q}}$ denote the weight space for $\mathrm{GL}_2/\mathbb{Q}$, that is, the rigid analytic space such that, for $L \subset \mathbb{C}_p$, we have
$$\mathcal{W}_{\mathbb{Q}}(L) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times).$$

3. We define $\mathcal{W}_{F,\mathrm{par}}$ to be the parallel weight line in $\mathcal{W}_{F,\mathfrak{N}}$. More precisely, it is the image of $\mathcal{W}_{\mathbb{Q}}$ in $\mathcal{W}_{F,\mathfrak{N}}$.

Remark 4.3.2. In [Han17] and [Roc26], the weight space $\mathcal{W}_{K,\mathfrak{N}}$ is defined using character on $T_G(\mathbb{Z}_p)$. Our definition matches with theirs after twisting by a power of the norm map. See also [BSW21a, Remark 3.2].

A weight $\lambda \in \mathcal{W}_{K,\mathfrak{N}}(L)$ is called *classical* if it is of the form $\epsilon\lambda^{\mathrm{alg}}$, where ϵ is a finite order character, and $\lambda^{\mathrm{alg}}(z) = z_1^{k_1} z_2^{k_2}$, where $z = (z_1, z_2) \in \mathcal{O}_p^\times \cong (\mathbb{Z}_p^\times)^2$ and $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$.

For an affinoid $\Omega = \mathrm{Sp}(R) \subset \mathcal{W}_{F,\mathfrak{N}}$, there exists a character

$$\kappa_\Omega : (\mathcal{O}_p)^\times \to R^\times,$$

such that for any $\lambda \in \mathcal{W}_{F,\mathfrak{N}}(L)$, the character $\lambda : \mathcal{O}_p^\times \to L^\times$ factors as $\mathcal{O}_p^\times \to R^\times \to L^\times$, where the second map is evaluation at λ. In other words, it is the map $R^\times \to (R/\mathfrak{m}_\lambda)^\times$, where $\mathfrak{m}_\lambda$ is the maximal ideal corresponding to λ.

Definition 4.3.3 (Ring of definition)**.** If R is a $\mathbb{Q}_p$-Banach space, $R_0 \subset R$ is a ring of definition if R_0 is open and bounded.

Examples 4.3.4.

- If $R = \mathbb{Q}_p$, then $R_0 = \mathbb{Z}_p$.
- If $R = \mathbb{Q}_p\langle T\rangle$ the *Tate algebra* in one-variable, then $R_0 = \mathbb{Z}_p\langle T\rangle$.

Given an open affinoid $\Omega = \mathrm{Sp}(R) \in \mathcal{W}_{F,\mathfrak{N}}$, then R is a $\mathbb{Q}_p$-Banach algebra and let R_0 be the ring of definition of R. Using the (spectral) norm associated with R_0, denoted as $|\cdot|_\Omega$, we define

$$r_\Omega := \min\left\{r \in \mathbb{Z}_{\geqslant 0} : |\kappa_\Omega(\gamma_p)|_\Omega < p^{\frac{-1}{p^r(p-1)}}\right\},$$

where γ_p is a topological generator of $\mathcal{O}_p^\times$. See [LW25, Remark 3.2.4] and [Han17, Section 2.2] for more details.

We will consider $\mathcal{W} := \mathcal{W}_{F,\mathrm{par}}$ unless otherwise stated.

4.3.1 Locally analytic functions

Let R be a $\mathbb{Q}_p$-algebra and R_0 be its ring of definition. After identifying $\mathcal{O}_p$ with $\mathbb{Z}_p^2$, for $s \geqslant 0$, we define the space of s-locally analytic R_0-valued functions as

$$\mathbf{A}^{s,\circ}(\mathcal{O}_p, R) := \{f : \mathcal{O}_p \cong \mathbb{Z}_p^2 \to R_0 : z \mapsto f(a + p^s z) \in R_0\langle z_1, z_2\rangle, \forall a \in \mathbb{Z}_p^2\}. \tag{4.3.1}$$

Furthermore, define

$$\mathbf{A}^{s}(\mathcal{O}_p, R) := \mathbf{A}^{s,\circ}(\mathcal{O}_p, R)[1/p]. \tag{4.3.2}$$

More precisely, we have the following identifications

$$\mathbf{A}^{s,\circ}(\mathcal{O}_p, R) = \mathbf{A}^{s,\circ}(\mathcal{O}_p, \mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Z}_p} R_0, \tag{4.3.3}$$

$$\mathbf{A}^s(\mathcal{O}_p, R) = \mathbf{A}^s(\mathcal{O}_p, \mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Z}_p} R, \tag{4.3.4}$$

Note that we can identify $\mathbf{A}^{0,\circ}(\mathcal{O}_p, \mathbb{Q}_p)$ with $\mathbb{Z}_p\langle T_1, T_2\rangle$ and $\mathbf{A}^0(\mathcal{O}_p, \mathbb{Q}_p)$ with $\mathbb{Q}_p\langle T_1, T_2\rangle$. See [Roc26, Section 2.7] and [BH24, Section 5.2] for details.

Now for an affinoid $\Omega \subset \mathcal{W}$, with its tautological character κ_Ω, we extend κ_Ω to $T_G(\mathbb{Z}_p)$ as

$$\begin{aligned}\tilde{\kappa}_\Omega : T_G(\mathbb{Z}_p) &\to R^\times;\\ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} &\mapsto \kappa_\Omega(a).\end{aligned}$$

Definition 4.3.5. For $s \geqslant 0$, define

$$\mathbf{A}_\Omega^{s,\circ} := \{f : I_p \to R^0 : f(gtn) = \tilde{\kappa}_\Omega(t) f(g), \forall g \in I_p, t \in T_G(\mathbb{Z}_p), n \in N_G(\mathbb{Z}_p)\}. \tag{4.3.5}$$

By the Iwahori decomposition of I_p and identifying $\overline{N}^1$ with $\mathcal{O}_p$, we have the following isomorphism:

$$\begin{aligned}\mathbf{A}_\Omega^{s,\circ} &\cong \mathbf{A}^{s,\circ}(\mathcal{O}_p, R),\\ f &\mapsto f|_{\overline{N}^1}.\end{aligned}$$

Furthermore, define

$$\mathbf{A}_\Omega^{s} := \mathbf{A}_\Omega^{s,\circ}[1/p].$$

Both spaces $\mathbf{A}_\Omega^{s,\circ}$ and $\mathbf{A}_\Omega^{s}$ are right I_p-modules with the rule:

$$f|_a(g) = f(ag),$$

for all $a \in I_p$. Moreover, let

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathcal{O}_p) : c \in \varpi_p\mathcal{O}_p, d \in \mathcal{O}_p^\times, ad - bc \neq 0 \right\}. \tag{4.3.6}$$

Note that $I_p \subset \Sigma_0(p)$, and $\Sigma_0(p)$ acts on $\mathbf{A}_\Omega^{s,\circ}, \mathbf{A}_\Omega^{s}$ on right via:

$$f|_b(gtn) = f(b^{-1}gbtn).$$

Remark 4.3.6. In [Roc26] and [BSW21a], the module $\mathbf{A}$ is a left I_p module, and also the definition of $\Sigma_0(p)$ is slightly different.

Note that if $s' > s$, then $\mathbf{A}_\Omega^{s,\circ} \subset \mathbf{A}_\Omega^{s',\circ}$, and this inclusion is compact. The same is true for $\mathbf{A}^{s,\circ}(\mathcal{O}_p, R), \mathbf{A}^s(\mathcal{O}_p, R)$, and $\mathbf{A}_\Omega^s$.

We assume that $\kappa_\Omega(\mathcal{O}_p) \subset (R_0)^\times$, and extend κ_Ω from $\mathcal{O}_p^\times$ to $\mathcal{O}_p$ by defining $\kappa_\Omega(\varpi_p\mathcal{O}_p) = 0$. Also, see [LW25, Lemma 3.2.1].

Lemma 4.3.7. For any $s \geqslant r_\Omega$, $\kappa_\Omega \in \mathbf{A}^{s,\circ}(\mathcal{O}_p, R)$.

Proof. This is essentially [Roc26, Lemma 2.8] and [LZ16, Lemma 4.1.5]. We will sketch a proof here. Write $\mathcal{O}_p^\times$ as (torsion $)\times(1+\varpi_p\mathcal{O}_p)$. It is enough to show that $z \mapsto \kappa_\Omega(1+p^{r_\Omega}z) \in R_0\langle z_1, z_2\rangle$, where $z = (z_1, z_2) \in \mathcal{O}_p \cong \mathbb{Z}_p^2$. Note that, for any $r \in \mathbb{Z}_p^2$,

$$\kappa_\Omega((1+\varpi_p)^r) = \prod_{i=1}^{2}\left(\sum_{n_i \geqslant 0}\binom{r_i}{n_i}(\epsilon T_i)^{n_i}\right),$$

where $v_p(\epsilon) > p^{\overline{p^{r_\Omega}(p-1)}}{}^{-1}$, and $\kappa_\Omega(1+\varpi_p(z)) = (1+\epsilon T_1)(1+\epsilon T_2)$. Here, we are identifying $R_0\langle z_1, z_2\rangle$ with $\mathcal{O}_L[[T_1, T_2]]$ non-canonically, where L is some complete subfield of $\mathbb{C}_p$. Note that $\binom{r_i}{n_i}\epsilon^{n_i}$ are $s = r_\Omega$-locally analytic functions for all n_i. ∎

Lemma 4.3.8. The modules $\mathbf{A}_\Omega^{s,\circ}, \mathbf{A}_\Omega^{s}$ are preserved by the actions of I_p and $\Sigma_0(p)$.

Proof. See [Roc26, Propositions 2.12, 2.18] and [LZ16, Proposition 4.2.5]. ∎

Proposition 4.3.9. For any $s \geqslant r_\Omega$, the map $f_\Omega : gtn \mapsto \widetilde{\kappa}_\Omega(t)$ is an element of $\mathbf{A}_\Omega^{s,\circ}$.

Proof. We know that f_Ω is a maon I_p. From Lemma 4.3.7, we know that $\kappa_\Omega \in \mathbf{A}^{s,\circ}(\mathcal{O}_p, R)$, where $\Omega = \mathrm{Sp}(R) \subset \mathcal{W}$. Hence, we are done. ∎

We will use the map f_Ω later to construct the Asai–Eisenstein elements having coefficients in $\mathbf{A}_\Omega^{s,\circ}$.

Let $L \subset \mathbb{C}_p$, and $\lambda \in \Omega(L) \subset \mathcal{W}(L)$, then define the *specialization map* as:

$$\mathrm{sp}_\lambda : R \to R/\mathfrak{m}_\lambda \cong L.$$

Integrally, we have

$$R_0 \xrightarrow{\mathrm{sp}_\lambda} \mathcal{O}_L.$$

Similar to $\mathbf{A}_\Omega^{s,\circ}, \mathbf{A}_\Omega^{s}$, define

$$\begin{aligned}\mathbf{A}_\lambda^{s,\circ} &:= \{f : I_p \to \mathcal{O}_L : f(gtn) = \widetilde{\lambda}(t)f(g), \forall g \in I_p, t \in T_G(\mathbb{Z}_p), n \in N_G(\mathbb{Z}_p)\},\\ \mathbf{A}_\lambda^{s} &= \mathbf{A}_\lambda^{s,\circ}[1/p],\end{aligned}$$

where $\widetilde{\lambda} : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \lambda(a)$. In other words, for any $s \geqslant 0$, sp_λ induces the following maps, which we denote again by sp_λ:

$$\begin{aligned}\mathrm{sp}_\lambda : \mathbf{A}_\Omega^{s,\circ} &\to \mathbf{A}_\lambda^{s,\circ}\\ \mathrm{sp}_\lambda : \mathbf{A}_\Omega^{s} &\to \mathbf{A}_\lambda^{s}\end{aligned}$$

Lemma 4.3.10. The map sp_λ is $\Sigma_0(p)$-equivariant.

Proof. Follows from the definitions and the fact that $\mathbf{A}_\Omega^{s,\circ}$ is preserved by the action of $\Sigma_0(p)$. ∎

In our case, we are interested in $\lambda = (0,0)$, and hence $\widetilde{\lambda}$ is a trivial character on the torus $T_G(\mathbb{Z}_p)$. Therefore, we can directly write

$$\mathbf{A}^{s,\circ}_{(0,0)} = \mathbf{A}^{s,\circ}(\mathcal{O}_p, L) \text{ and } \mathbf{A}^{s}_{(0,0)} = \mathbf{A}^{s}(\mathcal{O}_p, L). \tag{4.3.7}$$

We can also define

$$\mathcal{A}_\Omega := \lim_{s\to\infty} \mathbf{A}^s_\Omega. \tag{4.3.8}$$

When $\lambda = (0,0)$, the space $\mathcal{A}(L) := \mathcal{A}_{(0,0)}(L)$ is the space of locally analytic functions $f : \mathcal{O}_p \to L$.

4.3.2 Distributions and the integration map

For $\Omega = \mathrm{Sp}(R) \subset \mathcal{W}$ and $s \geqslant 0$, define the space of s-analytic distributions

$$\mathbf{D}^{s,\circ}_\Omega := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^{s,\circ}_\Omega, R_0). \tag{4.3.9}$$

Similarly, define $\mathbf{D}^s_\Omega := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^s_\Omega, R)$. Moreover, we have $\mathbf{D}^s_\Omega = \mathbf{D}^{s,\circ}_\Omega[1/p]$.

Remark 4.3.11. Although we have the identification $\mathbf{A}^{s,\circ}_\Omega \cong \mathbf{A}^{s,\circ}(\mathcal{O}_p, \mathbb{Q}_p)\widehat{\otimes} R_0$, the distribution space $\mathbf{D}^{s,\circ}_\Omega$ is not isomrphic to $\mathbf{D}^{s,\circ}(\mathcal{O}_p, \mathbb{Q}_p)\widehat{\otimes} R_0$. See [Bel12, Remark 3.1] for the details.

The space $\mathbf{D}^{s,\circ}_\Omega$ is a left I_p and $\Sigma_0(p)$-module, and the action is given by

$$g \cdot \mu(f) = \mu(f|_g).$$

Furthermore, define

$$\mathcal{D}_\Omega := \varprojlim_{s\to\infty} \mathbf{D}^s_\Omega. \tag{4.3.10}$$

From [Bel12] and [Han17], one can deduce

$$\mathcal{D}_\Omega \cong \mathcal{D}(\mathcal{O}_p, \mathbb{Q}_p)\widehat{\otimes} R,$$

where $\mathcal{D}(\mathcal{O}_p, \mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(\mathcal{O}_p, \mathbb{Q}_p), \mathbb{Q}_p)$.

For $\lambda \in \Omega(L) \subset \mathcal{W}(L)$, define

$$\mathbf{D}^{s,\circ}_\lambda := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^{s,\circ}_\lambda, \mathcal{O}_L), \tag{4.3.11}$$

$$\mathbf{D}^{s}_\lambda := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^{s}_\lambda, L). \tag{4.3.12}$$

Like in the case of locally analytic functions, we have the following *specialization maps:*

$$\tilde{\mathrm{sp}}_\lambda : \mathbf{D}^{s,\circ}_\Omega \to \mathbf{D}^{s,\circ}_\lambda,$$
$$\tilde{\mathrm{sp}}_\lambda : \mathbf{D}^{s}_\Omega \to \mathbf{D}^{s}_\lambda.$$

When λ corresponds to the weight $(0,0)$, we identify $\mathbf{D}^{s,\circ}_{(0,0)}$ with

$$\mathbf{D}^{s,\circ}(\mathcal{O}_p, L) := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^{s,\circ}(\mathcal{O}_p, \mathbb{Q}_p), \mathcal{O}_L),$$

and $\mathbf{D}^s_{(0,0)}$ with $\mathbf{D}^s(\mathcal{O}_p, L) := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{A}^s(\mathcal{O}_p, \mathbb{Q}_p), L)$.

Definition 4.3.12 (The integration map)**.** Define

$$\begin{aligned}\rho : \mathbf{D}^{s,\circ}(\mathcal{O}_p, L) &\to \mathcal{O}_L^2 \otimes (\mathcal{O}_L^2)^\sigma,\\ \mu &\mapsto \mu(\mathbb{1})(\mathrm{id} \otimes \mathrm{id}^\sigma),\end{aligned}$$

where $\mathbb{1}$ denotes the identity function in $\mathbf{A}^{s,\circ}(\mathcal{O}_p, L)$, and $\mathrm{id} = (1, 1)$, and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ is a non-trivial element of order 2.

Remark 4.3.13. For $k \geqslant 0$, the irreducible L-representation of weight (k, k) of $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ is isomorphic to $\mathrm{Sym}^k(L^2) \otimes \mathrm{Sym}^k(L^2)$. The space $\mathrm{Sym}^k(L^2)$ can be identified with the space of polynomials in $L[X]$ with degree $\leqslant k$. For simplicity, let $G = \mathrm{GL}_2(\mathbb{Q})$, and the integration map is defined as

$$\begin{aligned}\rho_k : \mathcal{D}(L) &\to L[X]^{\leqslant k},\\ \mu &\mapsto \sum_{j=0}^{k} \binom{k}{j} \mu(z^j) X^j.\end{aligned}$$

See [BH24, Definition 5.14] for the details.

Lemma 4.3.14. The specialization map $\tilde{\mathrm{sp}}_\lambda$ and the integration map ρ are $\Sigma_0(p)$-equivariant.

Proof. The map $\tilde{\mathrm{sp}}_\lambda$ is $\Sigma_0(p)$-equivariant since the map $\mathrm{sp}_\lambda : \mathbf{A}_\Omega^{s,\circ} \to \mathbf{A}_\lambda^{s,\circ}$ is $\Sigma_0(p)$-equivariant.

For the integration map ρ, for $g \in \Sigma_0(p)$, we have

$$\begin{aligned}g \cdot (\rho(\mu)) &= g \cdot (\mu(\mathbb{1})(\mathrm{id} \otimes \mathrm{id}^\sigma)),\\ &= \mu(\mathbb{1})(\mathrm{id} \otimes \mathrm{id}^\sigma),\\ &= (\mu|_g(\mathbb{1}))(\mathrm{id} \otimes \mathrm{id}^\sigma),\\ &= \rho(g \cdot \mu),\end{aligned}$$

since the image of ρ lies in $\mathcal{O}_L$, and $\Sigma_0(p)$ acts trivially on $\mathcal{O}_L$. ∎

4.4 Eigenvarieties

We recall some eigenvarieties and some important results from [BSW21a] that we need later to construct the p-adic L-function later.

Note that if $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{N}))$ is an Hecke newform, then by Eichler–Shimura–Harder (see [LW20, Thoerem 2.18]), there exists a modular symobl $\omega_\mathcal{F} \in \mathrm{H}^1_\mathrm{c}(Y_{F,1}(\mathfrak{N}), \mathbb{C})$ such that the following 1-dimensional $\mathbb{C}$-vector spaces are isomorphic:

$$S_{(0,0)}(U_{F,1}(\mathfrak{N}))[\mathcal{F}] \cong \mathrm{H}^1_\mathrm{c}(Y_{F,1}(\mathfrak{N}), \mathbb{C})[\omega_\mathcal{F}].$$

Furthermore, there exists a number field E, large enough to contain F and all Hecke eigenvalues of $\mathcal{F}$, and a complex period $\Omega_\mathcal{F}$, such that

$$\phi_\mathcal{F} := \frac{\omega_\mathcal{F}}{\Omega_\mathcal{F}} \in \mathrm{H}^1_\mathrm{c}(Y_{F,1}(\mathfrak{N}), \mathcal{O}_E).$$

Definition 4.4.1 (Hecke algebra)**.** Let $\mathfrak{H}_{\mathfrak{n},p}$ be the $\mathbb{Z}_p$-algebra generated by $T_\mathfrak{q}, U_\mathfrak{p}$, and $\{\langle v \rangle : v \in \mathrm{CL}(K)\}$.

4.4.1 The Bianchi eigenvariety

Write H_c^* for total cohomology.

Theorem 4.4.2 (Hansen)**.** There exists a separated rigid analytic variety $\mathcal{E}_{\mathfrak{N}}^{\mathrm{bmf}}$, and a morphism $w : \mathcal{E}_{\mathfrak{N}}^{\mathrm{bmf}} \to \mathcal{W}_{K,\mathfrak{N}}$. Moreover, for any finite extension $L/\mathbb{Q}_p$, there exists a bijection between $x \in \mathcal{E}_{\mathfrak{N}}^{\mathrm{bmf}}(L)$ with $w(x) = \lambda = (k,k) \in \mathcal{W}_{F,\mathfrak{N}}(L)$ and eigensystems $\psi_x : \mathfrak{H}_{\mathfrak{N},p} \to L$ of Hecke eigenvalues occurring in $\mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\lambda(L))$.

Here $k \in \mathbb{Z}_{\geqslant 0}$ and $\mathcal{D}_\lambda(L)$ can be identified with $\varprojlim_{s\to\infty} \mathbf{D}_\lambda^s(\mathcal{O}_p, L)$.

We describe the construction more precisely and briefly. A slope-adapted affinoid is a pair (Ω, h), where $\Omega = \mathrm{Sp}(\mathcal{O}(\Omega)) \subset \mathcal{W}_{F,\mathfrak{N}}$ is a two-dimensional affinoid in weight space and $h \in \mathbb{Q}_{\geqslant 0}$ such that there exists a U_p-slope decomposition:

$$\mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega) \cong \mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{\leqslant h} \oplus \mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{>h}$$

stable under the action of $\mathfrak{H}_{\mathfrak{N},p}$, where $\mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{\leqslant h}$ is the space where the p-adic valuation of the U_p-Hecke eigenvalue is $\leqslant h$.

Definition 4.4.3 (Local piece of $\mathcal{E}^{\mathrm{bmf}}$)**.** Let

$$\mathbb{T}_{\Omega,h} := \mathrm{Im}(\mathfrak{H}_{\mathfrak{N},p} \otimes \mathcal{O}(\Omega)) \subset \mathrm{End}(\mathrm{H}_c^*(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{\leqslant h}).$$

Define the local piece of $\mathcal{E}^{\mathrm{bmf}}$

$$\mathcal{E}_{\Omega,h}^{\mathrm{bmf}} := \mathrm{Sp}(\mathbb{T}_{\Omega,h}).$$

The eigenvariety $\mathcal{E}^{\mathrm{bmf}}$ is obtained by gluing the affinoids $\mathcal{E}_{\Omega,h}^{\mathrm{bmf}}$.

Lemma 4.4.4 (Barrera-Salazar–Williams)**.** If $\mathcal{F} \in S_\lambda(U_{F,1}(\mathfrak{N}))$ is a finite-slope cuspidal Bianchi eigenform, then there exists $x \in \mathcal{E}^{\mathrm{bmf}}(L)$ corresponding to $\mathcal{F}$.

Theorem 4.4.5 (Hida, Hansen–Newton)**.** If $\mathcal{F}$ is non-critical and $x_{\mathcal{F}} \in \mathcal{E}^{\mathrm{bmf}}(L)$ corresponds to $\mathcal{F}$, then any irreducible component $\mathcal{I}$ of $\mathcal{E}^{\mathrm{bmf}}$ passing through $x_{\mathcal{F}}$ has dimension 1.

4.4.2 Families in in H_c^1 and the parallel-weight eigenvariety

Bianchi eigenvarieties are constructed using classes that appear only in H_c^2.

Proposition 4.4.6. [BSW21a, Lemma 4.2] Let $x \in \mathcal{E}_{\Omega,h}^{\mathrm{bmf}}$ be a cuspidal classical point. The system of eigenvalues for x occurs in $\mathrm{H}_c^i(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{\leqslant h}$ if and only if $i = 2$.

Moreover,

$$\mathrm{H}_c^0(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\Omega)^{\leqslant h} = 0,$$
$$\mathrm{H}_c^0(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\lambda)^{\leqslant h} = 0.$$

Now in [Wil17], Willimas constructed the p-adic L-functions using overconvergent classes in $\mathrm{H}^1_{\mathrm{c}}$. Note that the classical points in the Bianchi eigenvariety are not Zariski-dense. Moreover, the cuspidal part of the Bianchi eigenvariety is one-dimensional, lying over a two-dimensional weight space. One of the key points of [BSW21a] is to overcome the obstruction. In particular, we can isolate some curves $\mathcal{S}$ in the weight space that allow us to pass from families in $\mathrm{H}^2_{\mathrm{c}}$ to families in $\mathrm{H}^1_{\mathrm{c}}$. Let $x \in \mathcal{E}^{\mathrm{bmf}}_{\Omega,h}(L)$ be any point and let $\mathfrak{m}_x$ be the corresponding maximal ideal in $\mathbb{T}_{\Omega,h}$. Let $\mathcal{P}_x \subset \mathfrak{m}_x$ be a minimal prime of $\mathbb{T}_{\Omega,h}$, and write $\mathcal{P}_\lambda$ for the contraction of $\mathcal{P}_x$ to $\mathcal{O}(\Omega)$.

Definition 4.4.7. Let $\mathcal{S} = \mathrm{Sp}(\mathfrak{L})$, where $\mathfrak{L} = \mathcal{O}(\Omega)/\mathcal{P}_\lambda$.

One can observe that $\mathcal{S} \subset \Omega$ is closed and $\mathcal{S}$ is a rigid curve by Theorem 4.4.5. We say x varies in a family over $\mathcal{S}$ if such a curve $\mathcal{S} \subset \Omega$ arises.

Proposition 4.4.8. [BSW21a, Proposition 4.4] If $x \in \mathcal{E}^{\mathrm{bmf}}_{\Omega,h}(L)$ is a cuspidal classical point that varies in a family over $\mathcal{S}$, then

$$\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h}_{\mathfrak{m}_x} \neq 0,$$

after possibly shrinking $\mathcal{S}$.

We have a maximal ideal $\mathfrak{m}_x \otimes_{\mathcal{O}(\Omega)} \mathfrak{L} \subset \mathfrak{H}_{\mathfrak{N},p} \otimes \mathfrak{L}$, since $\mathfrak{m}_x \in \mathfrak{H}_{\mathfrak{N},p} \otimes \mathcal{O}(\Omega)$. Define

$$\mathbb{T}_{\mathcal{S},h} := \mathrm{Im}(\mathfrak{H}_{\mathfrak{N},p} \otimes \mathfrak{L}) \subset \mathrm{End}_{\mathfrak{L}}(\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h}).$$

The Proposition 4.4.8 implies that the image of $\mathfrak{m}_x \otimes \mathfrak{L} \subset \mathfrak{H}_{\mathfrak{N},p} \otimes \mathfrak{L}$ in $\mathbb{T}_{\mathcal{S},h}$ is a maximal ideal. By abuse of notation, we also denote this image by $\mathfrak{m}_x$. This $\mathfrak{m}_x$ corresponds to a $x \in \mathrm{Sp}(\mathbb{T}_{\mathcal{S},h})$.

If $\mathcal{F}$ is p-regular p-stabilized newform of weight $\lambda = (0,0)$, then the $\mathfrak{H}_{\mathfrak{n},p}$-generalized eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), L)_{(\mathcal{F})}$ is one-dimensional, where $L/\mathbb{Q}_p$ is some finite extension large enough to contain the quadratic imaginary field F and all Hecke eigenvalues of $\mathcal{F}$.

Theorem 4.4.9. [BSW21a, Theorem 4.5] Let $x \in \mathcal{E}^{\mathrm{bmf}}_{\Omega,h}(L)$ correspond to a Bianchi newform $\mathcal{F}$ varying over a curve $\mathrm{Sp}(\mathfrak{L}) = \mathcal{S} \subset \Omega$, and let $w(x) = \lambda$. If $\mathcal{F}$ is p-regular and p-stabilized newform of weight λ and supose $\mathcal{S}$ is smooth at λ, then $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h}_{\mathfrak{m}_x}$ is free of rank 1 over $(\mathbb{T}_{\mathcal{S},h})_{\mathfrak{m}_x}$. Moreover, after replacing $\mathfrak{L}$ with $\mathfrak{L} \otimes_{\mathbb{Q}_p} L$, we get $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h}_{\mathfrak{m}_x}$ is free of rank 1 over $\mathfrak{L}_{\mathfrak{m}_\lambda}$.

Corollary 4.4.10. [BSW21a, Cor. 4.8] After shrinking $\mathcal{S}$, there exists a connected component $V = \mathrm{Sp}(T) \subset \mathrm{Sp}(\mathbb{T}_{\mathcal{S},h})$ containing x such that $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h} \otimes_{\mathbb{T}_{\mathcal{S},h}} T$ is free of rank 1 over T, and T is free of rank 1 over $\mathfrak{L}$. Thus, the weight map $V \to \mathcal{S}$ is étale.

Remark 4.4.11. If we write $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}(L))^{<1}$ for the space of *overconvergent modular symbols* with the slope < 1. Then by [Wil17, Theorem 8.7], we have the following control theorem

$$\rho : \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}(L))^{<1} \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), L)[\phi_{\mathcal{F}}], \tag{4.4.1}$$

where ρ is induced from the integration map $\rho : \mathcal{D}(\mathcal{O}_p, L) \to L$ in Definition 4.3.12. Thus, if $\mathcal{F}$ is non-critical, i.e., the U_p-eigenvalue has the p-adic valuation < 1, then we can lift the modular symbol $\phi_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), L)$ to an overconvergent modular symbol $\Psi_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}(L))^{<1}$.

Using the datum $(\mathcal{W}_{F,\mathrm{par}}, \mathbb{L}, \mathcal{M}, \mathfrak{H}_{\mathfrak{n},p}, \psi_{\mathrm{par}})$, Barrera-Salazar and Williams in [BSW21a] constructed a *parallel-weight eigenvariety* $\mathcal{E}_{\mathrm{par}}$. Here $\mathbb{L}$ is some Fredholm hypersurface, $\mathcal{M}$ is some coherent sheaf related to $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{\leqslant h}$, and ψ_{par} is induced from ψ. They furthermore proved:

Theorem 4.4.12.

1. All irreducible components of $\mathcal{E}_{\mathrm{par}}$ have dimension 1. Moreover, they contain a very Zariski-dense set of classical points.
2. The eigenvariety $\mathcal{E}_{\mathrm{par}}$ is reduced.
3. The inclusion $\mathcal{E}_{\mathrm{par}} \hookrightarrow \mathcal{E}^{\mathrm{bmf}}$ is a closed immersion.

Proof. See [BSW21a, Propositions 5.1, 5.2, Corollary 5.3]. ∎

4.4.3 Coleman–Mazur eigencurve and the base-change eigenvariety

Let $\mathcal{C}$ denote the Coleman–Mazur eigencurve that interpolates elliptic modular forms. Let BC denote the base-change map between eigenvarieties. See [BSW21a, Remark 3.6, Section 5.2] for more details. Let $\mathcal{E}_{\mathrm{bc}}$ denote the image of $\mathrm{BC}(\mathcal{C})$ in $\mathcal{E}_{\mathrm{par}}$.

Recall $\mathfrak{N} \subset \mathcal{O}_F$ is an ideal which is divisible by all primes above p in F and $N \in \mathbb{Z}$ such that $(N) = \mathfrak{N} \cap \mathbb{Z}$ in $\mathbb{Z}$.

Proposition 4.4.13. [BSW21a, Proposition 5.4] Let $f \in S_2(\Gamma_1(N))$ be an eigenform satisfying conditions in Assumptions 4.2.2, and let $x_f \in \mathcal{C}(L)$ be the corresponding point. Suppose f is non-critical, then $\mathcal{C} \to \mathcal{E}_{\mathrm{bc}}$ is locally isomorphic at x_f, and $\mathcal{E}_{\mathrm{bc}}$ is smooth at $\mathrm{BC}(x_f)$.

Note that the above proposition is proved for any weight (k, k) in [BSW21a].

Definition 4.4.14 ($\mathcal{S}$-smooth)**.** A point $x \in \mathcal{E}_{\mathrm{bc}}$ is $\mathcal{S}$-smooth if every irreducible component $\mathcal{I} \subset \mathcal{E}_{\mathrm{par}}$ passing through x is contained in $\mathcal{E}_{\mathrm{bc}}$.

Theorem 4.4.15. If Calegari–Mazur conjecture holds, then every classical base-change point $x_{\mathcal{F}} = \mathrm{BC}(x_f)$ is $\mathcal{S}$-smooth.

Proof. See [BSW21a, Proposition 5.14]. ∎

Remark 4.4.16. There is a conjecture by Calgary and Mazur ([BSW21a, Conjecture 5.13]) which asserts that every ordinary component of $\mathcal{E}_{\mathrm{par}}$ is either twisted base-change or is CM. Non-ordinary CM components do not exist. This conjecture is needed to prove Theorem 4.4.15.

But note that, if $x_{\mathcal{F}}$ arises from the base-change of a non-critical elliptic cusp form x_f, then $x_{\mathcal{F}}$ is $\mathcal{S}$-smooth by Corollary 4.4.10.

4.5 Asai–Eisenstein elements for families

Let $\Omega \subset \mathcal{W} := \mathcal{W}_{F,\mathrm{par}}$ be an affinoid such that $\lambda = (0,0) \in \Omega(L)$, for some finite extension $L/\mathbb{Q}_p$. In this section, we construct Asai–Elements á la Loeffler–Williams over Ω, using methods similar to those used in [Roc26].

Let ${}_cC_N := C({}_cg_N) \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), \mathbb{Z})$ be the Betti realization of ${}_cg_N$, where c is coprime with $6N$ and ${}_cg_N$ is the Kato's zeta element. See [Kat04] for a more detailed account of ${}_cg_N$.

Let $a \in \mathcal{O}_F$, $m \in \mathbb{Z}_{\geqslant 1}$, and $c \in \mathbb{Z}_{\geqslant 1}$ coprime to $6mN$. Consider the composite map

$$Y_{\mathbb{Q}}(m, mN) \overset{\iota}{\hookrightarrow} Y_F^*(m, m\mathfrak{N}) \xrightarrow{u_a} Y_F^*(m, m\mathfrak{N}),$$

where u_a is the action of the matrix $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ on $Y_F^*(m, m\mathfrak{N})$. Note that each component of $Y_F^*(m, m\mathfrak{N})$ is preserved under the action of u_a.

Definition 4.5.1 (Zeta elements). Define ${}_c\mathcal{Z}_{m,\mathfrak{N},a}$ to be the image of ${}_cC_{mN} = C({}_cg_{mN}) \in \mathrm{H}^1(Y_{\mathbb{Q}}(m, mN), \mathbb{Z})$ under the pushforward $(u_a \circ \iota)_*$, i.e.,

$${}_c\mathcal{Z}_{m,\mathfrak{N},a} = (u_a \circ \iota)_*({}_cC_{mN}) \in \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathcal{O}_L).$$

For $t \in (\mathbb{Z}/m\mathbb{Z})^\times$, let $(\mathrm{proj}_t)_* : \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathcal{O}_L) \to \mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathcal{O}_L)^{(t)} := \mathrm{H}^2(Y_F^*(m, m\mathfrak{N})^{(t)}, \mathcal{O}_L)$ be the projection induced by $\mathrm{proj}_t : Y_F^*(m, m\mathfrak{N}) \to Y_F^*(m, m\mathfrak{N})^{(t)}$.

Let ${}_c\mathcal{Z}_{m,\mathfrak{N},a}(t)$ be the projection of ${}_c\mathcal{Z}_{m,\mathfrak{N},a}$ to the direct summand of $\mathrm{H}^2(Y_F^*(m, m\mathfrak{N}), \mathcal{O}_L)$ given by the t-th component. In other words,

$${}_c\mathcal{Z}_{m,\mathfrak{N},a}(t) = (\mathrm{proj}_t)_*({}_c\mathcal{Z}_{m,\mathfrak{N},a}),$$

and hence we get

$${}_c\mathcal{Z}_{m,\mathfrak{N},a} = \sum_t {}_c\mathcal{Z}_{m,\mathfrak{N},a}(t).$$

We consider the map

$$s_m : Y_F^*(m, m\mathfrak{N}) \to Y_{F,1}^*(\mathfrak{N})$$

given by the action of $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$. This map corresponds to $(z,t) \mapsto (z/m, t/m)$ on $\mathbb{H}_3$. See [LW20] and [Deo25] for details.

Let ${}_c\Xi_{m,\mathfrak{N},a} \in \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathcal{O}_L)$ be the *Asai–Eisenstein element* defined in [LW20, Definition 3.9].

Theorem 4.5.2. [LW20, Theorem 3.13, Proposition 4.4] Let $a \in \mathcal{O}_F$ be a generator of $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$. We have

$$(s_{p^r})_*({}_c\mathcal{Z}_{p^r,\mathfrak{N},a}(t)) = {}_c\Xi_{p^r,\mathfrak{N},ta}.$$

Furthermore, we have

$$\sum_{\substack{s \in (\mathbb{Z}/p^{r+1})^\times \\ s \equiv t \mod p^r}} {}_c\Xi_{p^{r+1},\mathfrak{N},as} = (U_p)_* \, {}_c\Xi_{p^r,\mathfrak{N},at}. \tag{4.5.1}$$

Here, the Hecke operator $(U_p)_*$ is defined using the Hecke correspondence on the locally symmetric space $Y^*_{F,1}(\mathfrak{N})$; see [LW20, Section 2C] for more details.

Using Theorem 4.5.2, we redefine ${}_c\Xi_{p^r,\mathfrak{N},at}$ as

$$ {}_c\Xi_{p^r,\mathfrak{N},at} = (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*({}_cC_{p^rN}). \tag{4.5.2}$$

For $\Omega \subset \mathcal{W}$, recall $\mathbf{A}^{s,\circ}_\Omega$ from the Definition 4.3.5. By the action of p-part of group $U^*_F(p^r, p^r\mathfrak{N})$, one can see that the spaces $\mathbf{A}^{s,\circ}_\Omega$, $\mathbf{A}^s_\Omega$, and $\mathcal{A}_\Omega$ form local systems over $Y^*_F(p^r, p^r\mathfrak{N})$, which we denote by the same notations. See [Urb11, Section 1.2] for the definition of local systems. Also, recall $f_\Omega : \overline{n}tn \mapsto \widetilde{\kappa}_\Omega(t)$ that is an element of $\mathbf{A}^{s,\circ}_\Omega$ for $s \geqslant r_\Omega$.

Lemma 4.5.3. The element f_Ω lie in $\mathrm{H}^0(Y^*_F(p^r, p^r\mathfrak{n}), \mathbf{A}^{s,\circ}_\Omega)$.

Proof. See [Roc26, Lemma 3.6]. ∎

Through the pullback along $\iota : Y_\mathbb{Q}(p^r, p^rN) \to Y^*_F(p^r, p^r\mathfrak{N})$, one can regard $i^*(\mathbf{A}^{s,\circ}_\Omega)$ as a local system on $Y_\mathbb{Q}(p^r, p^rN)$. We construct a *zeta element* over Ω as follows: we know that $f_\Omega \mod p^k \in \mathbf{A}^{s,\circ}_\Omega/p^k$ for all $k \in \mathbb{Z}_{\geqslant 0}$. Define

$$ {}_c\mathcal{Z}^\Omega_{p^r,\mathfrak{N},a} := \varprojlim_k \big((u_a \circ \iota)_*({}_cC_{p^rN(p^k)} \cup f_\Omega \mod p^k)\big) \in \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), \mathbf{A}^{s,\circ}_\Omega). \tag{4.5.3}$$

Note that

$$ {}_c\mathcal{Z}^\Omega_{p^r,\mathfrak{N},a} = \sum_t {}_c\mathcal{Z}^\Omega_{p^r,\mathfrak{N},a}(t) \in \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), \mathbf{A}^{s,\circ}_\Omega) = \bigoplus_t \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N})^{(t)}, \mathbf{A}^{s,\circ}_\Omega) \tag{4.5.4}$$

More precisely, we can define a *branching map*, as defined in [Roc26],

$$ \mathrm{br}_\Omega : \varprojlim_k \mathrm{H}^1(Y_\mathbb{Q}(p^rp^k, p^rp^k\mathfrak{N}), \mathbb{Z}_p) \to \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), \mathbf{A}^{s,\circ}_\Omega),$$

such that

$$ \mathrm{br}_\Omega = \varprojlim_k(\mathrm{br}_{\Omega,k}),$$

and $\mathrm{br}_{\Omega,k}(z) = z \cup f_\Omega \mod p^k$.

Definition 4.5.4 (Asai–Eisenstein elements over Ω)**.** For any $r \in \mathbb{Z}_{\geqslant 1}$ and $t \in (\mathbb{Z}/p^r)^\times$, define

$$ {}_c\Xi^\Omega_{p^r,\mathfrak{N},at} := (s_{p^r})_*({}_c\mathcal{Z}^\Omega_{p^r,\mathfrak{N},a}(t)) \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^{s,\circ}_\Omega). \tag{4.5.5}$$

More precisely, we can write

$$ {}_c\Xi^\Omega_{p^r,\mathfrak{N},at} = \varprojlim_k (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*({}_cC_{p^rp^kN} \cup f_\Omega \mod p^k).$$

For $\lambda \in \Omega(L)$, recall the specialization map

$$ \mathrm{sp}_\lambda : \mathbf{A}^{s,\circ}_\Omega \to \mathbf{A}^{s,\circ}_\lambda$$

from Subsection 4.3.1. This induces a map on cohomologies as well:

$$\mathrm{sp}_\lambda : \mathrm{H}^2(Y, \mathbf{A}_\Omega^{s,\circ}) \to \mathrm{H}^2(Y, \mathbf{A}_\lambda^{s,\circ}), \tag{4.5.6}$$

where $Y \in \{Y^*_{F,1}(\mathfrak{N}), Y^*_F(p^r, p^r\mathfrak{N})\}$. For $z^\Omega \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}_\Omega^{s,\circ})$, define

$$\widetilde{z^\lambda} := \mathrm{sp}_\lambda(z^\Omega) \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ}). \tag{4.5.7}$$

Similar to equation (4.5.3), if $\lambda \in \Omega(L)$, we can define elements in $\mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ})$, by taking the cup product with the element $(f_\lambda : \overline{n}tn \mapsto \tilde{\lambda}(t)) \mod p^k$ and then taking limit over k. Here, f_λ is an element of $\mathrm{H}^0(Y^*_{F,1}(p^r, p^r\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ})$. Define

$${}_c\mathcal{Z}^\lambda_{p^r,\mathfrak{N},a} := \varprojlim_k \big((u_a \circ \iota)_*({}_cC_{p^rN(p^k)} \cup f_\lambda \mod p^k)\big) \in \mathrm{H}^2(Y^*_F(p^r, p^r\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ}),$$

and

$${}_c\Xi^\lambda_{p^r,\mathfrak{N},at} := (s_{p^r})_*({}_c\mathcal{Z}^\lambda_{p^r,\mathfrak{N},a}(t)) \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ}).$$

Proposition 4.5.5. For $\lambda \in \Omega(L)$, we have

$$\widetilde{{}_c\Xi}^\lambda_{p^r,\mathfrak{N},at} = {}_c\Xi^\lambda_{p^r,\mathfrak{N},at}. \tag{4.5.8}$$

That is,

$$\mathrm{sp}_\lambda({}_c\Xi^\Omega_{p^r,\mathfrak{N},at}) = {}_c\Xi^\lambda_{p^r,\mathfrak{N},at}.$$

Proof. This is essentially [Roc26, Lemma 3.8]. More precisely, the image of f_Ω under sp_λ is f_λ, since $\lambda \in \Omega(L)$, and λ factors through κ_Ω. ∎

Theorem 4.5.6. We have

$$\sum_{\substack{s \in (\mathbb{Z}/p^{r+1})^\times \\ s \equiv t \mod p^r}} {}_c\Xi^\Omega_{p^{r+1},\mathfrak{N},as} = (U_p)_*\, {}_c\Xi^\Omega_{p^r,\mathfrak{N},at}. \tag{4.5.9}$$

Proof. This follows from the proof of [LW20, Theorem 3.13] and the construction of ${}_c\Xi^\Omega_{p^r,\mathfrak{N},at}$. See also [Loe21, Propositions 4.5.1, 4.5.2] and [Roc26, Theorem 3.10]. ∎

4.6 Construction

Throughout this section, let $f \in S_2(\Gamma_1(N))$ that satisfies the conditions in Assumption 4.2.2. Let $\mathcal{F}$ be the base-change of f, and hence $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{N}))$ that satsifies Assumption 4.2.3. See Remark 4.4.16 if $\mathcal{F}$ is a genuine cuspidal Bianchi modular form, that is, a cuspidal Bianchi modular form that is not the base-change of a cuspidal elliptic modular form.

We also assume that the slope of $\mathcal{F}$ is small at p, i.e.,

$$v_p(a_{p,\mathcal{F}}) < 1, \tag{4.6.1}$$

where $a_{p,\mathcal{F}}$ is the U_p-eigenvalue of $\mathcal{F}$.

From Subsection 4.4.3, we know that there exists a family $V = \mathrm{Sp}(\mathcal{T}) \subset \mathcal{E}_{\mathrm{bc}} \subset \mathcal{E}_{\mathrm{par}}$ over $\mathcal{S} = \mathrm{Sp}(\mathfrak{L}) \subset \mathcal{W} := \mathcal{W}_{K,\mathrm{par}}$. Let $x_{\mathcal{F}} \in V$ be the point corresponding to $\mathcal{F}$. Since we have assumed $\mathcal{F}$ is not critical at p, $\mathcal{E}_{\mathrm{par}}$ is $\mathcal{S}$-smooth at $x_{\mathcal{F}}$. By $\mathcal{S}$-smoothness, V is the unique irreducible component of $\mathrm{Sp}(\mathbb{T}_{\mathcal{S},1}) \subset \mathcal{E}_{\mathrm{par}}$ passing through $x_{\mathcal{F}}$. After shrinking V, if necessary, we can assume V is connected and smooth. Hence, there exists an idempotent e on $\mathbb{T}_{\mathcal{S},1}$ such that $\mathcal{T} = e\mathbb{T}_{\mathcal{S},1}$ is a summand. Furthermore,

$$\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \otimes_{\mathbb{T}_{\mathcal{S},1}} \mathcal{T} = e\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}}).$$

See [BSW21a, Section 6].

Remark 4.6.1. The space $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \otimes_{\mathfrak{L}} \mathcal{T}$ has two $\mathcal{T}$-structures. The space $\mathcal{T}$ acts on $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1}$ via $\mathbb{T}_{\mathcal{S},1}$.

Lemma 4.6.2. [BSW21a, Proposition 6.4] After shrinking $\mathcal{S}$, $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \otimes_{\mathbb{T}_{\mathcal{S},1}} \mathcal{T}$ is free of rank 1 over $\mathcal{T}$.

Definition 4.6.3. Let Φ_V be a $\mathcal{T}$-generator of $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \otimes_{\mathbb{T}_{\mathcal{S},1}} \mathcal{T}$.

For $\lambda \in \mathcal{S}(L)$, the specialization map $\tilde{\mathrm{sp}}_\lambda : \mathcal{D}_{\mathcal{S}} \to \mathcal{D}_\lambda(L)$ induces

$$\tilde{\mathrm{sp}}_\lambda : \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \to \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_\lambda(L))^{<1}. \tag{4.6.2}$$

Proposition 4.6.4. [BSW21a, Proposition 6.7] If $\lambda = (0,0) \in \mathcal{S}(L)$, then $\tilde{\mathrm{sp}}_\lambda(\Phi_V)$ generates the one-dimensional generalized eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}(L))^{<1}$.

Hence, from the Remark 4.4.11 and the Proposition 4.6.4, we can deduce that for $\mathcal{F}$, we have

$$\rho(\tilde{\mathrm{sp}}_\lambda(\Phi_V)) = c_x \phi_{\mathcal{F}}, \tag{4.6.3}$$

under the composition

$$\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \xrightarrow{\tilde{\mathrm{sp}}_\lambda} \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}(L))^{<1} \xrightarrow{\rho} \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), L).$$

Here $c_x \in L^\times$ and $\phi_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{O}_L)$.

Recall the closed immersion $j : Y^*_{F,1}(\mathfrak{N}) \to Y_{F,1}(\mathfrak{N})$. Note that $\mathbf{D}$ forms a local system on $Y_{F,1}(\mathfrak{N})$, and hence also form a local system on $Y^*_{F,1}(\mathfrak{N})$, via the pullback map j^*, where $\mathbf{D} \in \{\mathbf{D}^{s,\circ}_{\mathcal{S}}, \mathbf{D}^s_{\mathcal{S}}, \mathcal{D}_{\mathcal{S}}, \mathbf{D}^{s,\circ}(\mathcal{O}_p, L), \mathbf{D}^s(\mathcal{O}_p, L), \mathcal{D}(L)\}$. We again denote these local systems by $\mathbf{D}$.

Definition 4.6.5. Define

$$\Phi^*_V := j^*(\Phi_V) \in \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1} \otimes_{\mathbb{T}_{\mathcal{S},1}} \mathcal{T}. \tag{4.6.4}$$

Note that we are pulling back the modular symbol part that lies in $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1}$.

Since

$$\Phi_V \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathcal{D}_{\mathcal{S}})^{<1},$$

and $\mathcal{D}_{\mathcal{S}} = \varprojlim_{s} \mathbf{D}^s_{\mathcal{S}}$, we can define Φ^s_V as the projection of Φ_V onto $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathbf{D}^s_{\mathcal{S}})^{<1} \otimes \mathcal{T}$. Note that for all $s \geqslant r_{\mathcal{S}}$,

$$\Phi^s_V \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathbf{D}^s_{\mathcal{S}})^{<1} \otimes \mathcal{T}.$$

Furthermore, since $\mathbf{D}^s_{\mathcal{S}} = \mathbf{D}^{s,\circ}_{\mathcal{S}}[1/p]$, there exists a constant $C > 0$, independent of s, such that

$$\frac{\Phi^s_V}{p^C} \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{N}), \mathbf{D}^{s,\circ}_{\mathcal{S}})^{<1} \otimes \mathcal{T}^\circ, \tag{4.6.5}$$

where T^0 is the ring of definition of T. We denote $\dfrac{\Phi^s_V}{p^C}$ with $\breve{\Phi}^s_V$.

Remark 4.6.6. We will make use of the integrality of the modular symbol part of Φ_V in the next subsection. It is still valid if one keeps $\mathcal{T}$ rather than $\mathcal{T}^\circ$ in 4.6.5.

Fix

$$\breve{\Phi^s}^{*}_{V} := j^*(\breve{\Phi}^s_V) \in \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathbf{D}^{s,\circ}_{\mathcal{S}})^{<1} \otimes \mathcal{T}^\circ. \tag{4.6.6}$$

4.6.1 The polynomials $P^{\mathcal{S}}_r$

We adapt the polynomial setup from [Deo25, Section 6.2]. We will construct polynomials that satisfy certain norm and congruence properties.

Recall that $\mathcal{S} = \mathrm{Sp}(\mathfrak{L}) \subset \mathcal{W}$ and, for all $r \in \mathbb{Z}_{\geqslant 1}$, let

$${}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \in \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^{s,\circ}_{\mathcal{S}}),$$

where $t \in (\mathbb{Z}/p^r)^\times$, $a \in \mathcal{O}_F$ such that a generates $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$. Also, note that all primes above p in F divide $\mathfrak{N}$.

Since $\mathbf{D}^{s,\circ}_{\mathcal{S}}$ is the dual of $\mathbf{A}^{s,\circ}_{\mathcal{S}}$, we have a pairing

$$\mathbf{D}^{s,\circ}_{\mathcal{S}} \otimes \mathbf{A}^{s,\circ}_{\mathcal{S}} \to \mathfrak{L}^\circ. \tag{4.6.7}$$

This pairing induces a perfect pairing (after removing torsion)

$$\langle , \rangle_{\mathcal{S}} : \mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathbf{D}^{s,\circ}_{\mathcal{S}}) \times \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^{s,\circ}_{\mathcal{S}}) \to \mathrm{H}^3(Y^*_{F,1}(\mathfrak{N}), \mathfrak{L}^\circ) \cong \mathfrak{L}^\circ, \tag{4.6.8}$$

since $Y^*_{F,1}(\mathfrak{N})$ is a real manifold of real dimension 3. See also [LW20, Section 3D]. We know that $\mathcal{F}$ varies over V. Let $a_p := a_{p,\mathcal{F}}$, the U_p-eigenvalue of $\mathcal{F}$. We shrink $\mathcal{S}$, if necessary, so that the the slope $v_p(a_p)$ is constant over V. Let a_V be the U_p-eigenvalue of Φ_V. Recall that $v_p(a_p) < 1$.

Write $\mathbb{Z}_p^\times \cong \Delta \times (1 + p\mathbb{Z}_p)$, where Δ is a cyclic group of order $p - 1$. Fix a topological generator u of $1 + p\mathbb{Z}_p$. For any $r \in \mathbb{Z}_{\geqslant 1}$ and $t \in (\mathbb{Z}/p^r)^\times$, define

$$\log_u(t) := \frac{\tilde{t}}{\varepsilon(\tilde{t})},$$

where $\tilde{t} \in \mathbb{Z}_p^\times$ is a lift of t and $\varepsilon : \Delta \to \mathbb{Z}_p^\times$ is the Teichmüller character. Note that $\log_u(t)$ is a unique integer and $0 \leqslant \log_u(t) < p^{r-1}$. Furthermore, let $\delta \in \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}_L^\times)$. We know that δ is a non-negative integral power of ε. See [Deo25, Section 6.1] for more details.

Definition 4.6.7. For $r \in \mathbb{Z}_{\geqslant 1}$ and a non-trivial character $\delta \in \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}_L^\times)$, define

$$P_r^{\mathcal{S},\delta} := \left\langle \widecheck{\Phi^s}^*_V,\ (U_p)_*^{-r} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle_{\mathcal{S}}. \tag{4.6.9}$$

When $r = 1$ and δ is the trivial character, define

$$P_1^{\mathcal{S},\mathrm{triv}} := \left\langle \widecheck{\Phi^s}^*_V,\ (1 - p^{-1}(U_p)_*^{-1})\, {}_c\Xi^{\mathcal{S}}_{1,\mathfrak{N},a} \right\rangle_{\mathcal{S}}. \tag{4.6.10}$$

We simplify the equation (4.6.9) as follows:

$$\begin{aligned}
P_r^{\mathcal{S},\delta} &:= \left\langle \widecheck{\Phi^s}^*_V,\ (U_p)_*^{-r} \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \right\rangle_{\mathcal{S}}, \\
&= \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle \widecheck{\Phi^s}^*_V,\ (U_p)_*^{-r}\, {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \rangle_{\mathcal{S}} \delta(t)(1+T)^{\log_u(t)}, \\
&= \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle ((U_p)^*)^{-r} \widecheck{\Phi^s}^*_V,\ {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \rangle_{\mathcal{S}} \delta(t)(1+T)^{\log_u(t)}, \\
&= \sum_{t \in (\mathbb{Z}/p^r)^\times} \left\langle \frac{1}{a_V^r} \widecheck{\Phi^s}^*_V,\ {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \right\rangle_{\mathcal{S}} \delta(t)(1+T)^{\log_u(t)}, \\
&= \frac{1}{a_V^r} \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle \widecheck{\Phi^s}^*_V,\ {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at} \rangle_{\mathcal{S}}\ \delta(t)(1+T)^{\log_u(t)},
\end{aligned}$$

where $v_p(a_p) = v_p(a_V)$. Thus,

$$P_r^{\mathcal{S},\delta} \in \mathcal{T}^\circ \otimes \mathfrak{L}^\circ \otimes L[T]. \tag{4.6.11}$$

Furthermore, $\mathrm{degree}(P_r^{\mathcal{S},\delta}) < p^{r-1}$.

Note that $\mathfrak{L}$ and T are $\mathbb{Q}_p$-Banach algebras. Let $|\cdot|_{\mathcal{S}}$ denote the norm on the Banach space $\mathfrak{L}$.

Theorem 4.6.8. For any $r \in \mathbb{Z}_{\geqslant 1}$ and any character $\delta : \Delta \to \mathcal{O}_L^\times$, we have

1. $P_{r+1}^{\mathcal{S},\delta} \equiv P_r^{\mathcal{S},\delta} \mod \omega_{r-1}(T)$, where $\omega_n(T) = (1+T)^{p^n} - 1$.
2. $\sup\limits_r \|p^{v_p(a_p)r} P_r^{\mathcal{S},\delta}\| < \infty$, where $\|f\| := \sup_{|z|_p<1} |f(z)|_{\mathcal{S}}$.

Proof. The first statement follows from Theorem 4.5.6. In particular, we have the following congruence relation

$$\begin{aligned}&(U_p)_*^{-(r+1)} \sum_{s \in (\mathbb{Z}/p^{r+1})^\times} {}_c\Xi^{\mathcal{S}}_{p^{r+1},\mathfrak{N},as} \otimes \delta(s)(1+T)^{\log_u(s)} \\ &\equiv (U_p)_*^{-(r)} \sum_{t \in (\mathbb{Z}/p^{r})^\times} {}_c\Xi^{\mathcal{S}}_{p^{r},\mathfrak{N},at} \otimes \delta(t)(1+T)^{\log_u(t)} \quad \bmod \omega_{r-1}(T)\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^{s,\circ}_{\mathcal{S}}) \otimes L[T].\end{aligned}$$

Thus, after pairing both sides with $\breve{\Phi}^{s\,*}_{V}$, we deduce the first statement.

The second statement follows from the definition of $P_r^{\mathcal{S},\delta}$, since the factor $\frac{1}{a_V^r}$ appears in $P_r^{\mathcal{S},\delta}$ and $v_p(a_p) = v_p(a_V)$. ∎

Thus, by [PR94, Lemme 1.2.1, Lemme 1.2.2], [BL21, Lemmas 2.2, 2.3], we deduce:

Theorem 4.6.9. The limit $\lim_{r\to\infty} P_r^{\mathcal{S},\delta}$ is a power series such that

$$ {}_c\mathcal{L}^{\delta}(T) := \lim_{r\to\infty} P_r^{\mathcal{S},\delta} \in T^\circ \widehat{\otimes} \mathfrak{L}^\circ \widehat{\otimes} \mathcal{H}_{L,v_p(a_p)}, \tag{4.6.12}$$

where, for any $m \geqslant 0$,

$$\mathcal{H}_{L,m} := \left\{ \sum_{n \geqslant 1} c_n T^n \in L[[T]] : \sup_n \frac{|c_n|_p}{n^m} < \infty \right\}.$$

Furthermore, we have

$$ {}_c\mathcal{L}^{\delta}(T) \equiv P_r^{\mathcal{S},\delta} \mod \omega_{r-1}(T). \tag{4.6.13}$$

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$. Let γ denote a topological generator of Γ/Δ. Define, for any real number $m \geqslant 0$,

$$\mathcal{H}_{L,m}(\Gamma) := \left\{ \sum_{\sigma\in\Delta} \sum_{n \geqslant 0} c_{n,\sigma} \cdot \sigma \cdot (\gamma - 1)^n : \sup_n \frac{|c_{n,\sigma}|_p}{n^m} < \infty,\ \forall \sigma \in \Delta \right\}.$$

Let e_δ denote the idempotent corresponding to δ, then

$$\mathcal{H}_{L,m}(\Gamma) = \bigoplus_{\delta \in \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}_L^\times)} e_\delta(\mathcal{H}_{L,m}(\gamma - 1)),$$

where each $e_\delta(\mathcal{H}_{L,m}(\gamma - 1))$ is isomorphic to $\mathcal{H}_{L,m}$ after identifying T with $\gamma - 1$.

Remark 4.6.10. Note that we can identify $\mathcal{H}_{L,m}$ with the space of m-admissible distributions $\mathcal{D}_m(\Gamma, L)$ via the Amice transform. See [Col10] for details.

Define ${}_c\mathcal{L}^{\delta}(\gamma - 1)$ by replacing T with $\gamma - 1$ in (4.6.12).

We are now ready to define the *two-variable p-adic Asai L-function* for a small-slope cuspidal Bianchi modular form $\mathcal{F}$ of weight $(0, 0)$.

Definition 4.6.11 (Two-variable p-adic Asai L-function for Bianchi modular forms)**.** Define

$$
{}_c\mathcal{L}_V^{\mathrm{As}} := \bigoplus_{\delta \in \mathrm{Hom}_{\mathrm{cts}}(\Delta, \mathcal{O}_L^\times)} {}_c\mathcal{L}^\delta(\gamma - 1) \in \mathcal{T}^\circ \widehat{\otimes} \mathfrak{L}^\circ \widehat{\otimes} \mathcal{H}_{L, v_p(a_p)}(\Gamma). \tag{4.6.14}
$$

Note that, after idenitifying $\mathcal{H}_{L,v_p(a_p)}$ with $\mathcal{D}_{v_p(a_p)}(\Gamma, L)$, we have

$$
\mathfrak{L}^\circ \widehat{\otimes} \mathcal{D}_{v_p(a_p)}(\Gamma, L) \cong \mathcal{D}_{v_p(a_p)}(\Gamma, \mathfrak{L}),
$$

since both are $\mathbb{Q}_p$-Banach algebras. Thus, one can view

$$
{}_c\mathcal{L}_V^{\mathrm{As}} \in \mathcal{T}^\circ \widehat{\otimes} \mathcal{D}_{v_p(a_p)}(\Gamma, \mathfrak{L}).
$$

We call ${}_c\mathcal{L}_V^{\mathrm{As}}$ a two-variable p-adic L-function, since one variable corresponding to $\mathcal{F}$ varies in $\mathcal{T}^\circ$, and the other variable is the cyclotmic variable, i.e., a finite order character in $\mathcal{D}_{v_p(a_p)}(\Gamma, \mathfrak{L})$.

4.6.2 Specialization at the weight $\lambda = (0, 0)$

Recall that $\mathcal{F}$ is a base-change of a weight 2 cuspidal elliptic modular form, and the weight of $\mathcal{F}$ is $(0,0)$. We have assumed $\mathcal{F}$ has a small slope. In this subsection, we prove that we can recover the p-adic Asai L-function ${}_cL_p^{\mathrm{As}}(\mathcal{F})$ associated with $\mathcal{F}$ constructed in [Deo25], from ${}_cL_V^{\mathrm{As}}$.

Recall from [Deo25], ${}_cL_p^{\mathrm{As}}(\mathcal{F})$ is defined as

$$
\begin{aligned}
{}_cL_p^{\mathrm{As}}(\mathcal{F}) &:= \bigoplus_{\delta \in \Delta^*} {}_cL_p^{\mathrm{As},\delta}(\mathcal{F}) \in \mathcal{H}_{L, v_p(a_p)}(\Gamma), \\
{}_cL_p^{\mathrm{As},\delta}(\mathcal{F}) &= \lim_{r \to \infty} \left(\frac{1}{a_p^r} \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle \phi_{\mathcal{F}}^*, {}_c\Xi_{p^r, \mathfrak{N}, at} \rangle \delta(t) (\gamma)^{\log_u(t)} \right),
\end{aligned}
$$

where $\phi_{\mathcal{F}}^* \in \mathrm{H}_{\mathrm{c}}^1(Y_{F,1}^*(\mathfrak{N}), \mathcal{O}_L)$ is the modular symbol associated to $\mathcal{F}$, and ${}_c\Xi_{p^r,\mathfrak{N},at} \in \mathrm{H}^2(Y_{F,1}^*(\mathfrak{N}), \mathcal{O}_L)$. Let

$$
P_r^\delta := \frac{1}{a_p^r} \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle \phi_{\mathcal{F}}^*, {}_c\Xi_{p^r, \mathfrak{N}, at} \rangle \delta(t) (\gamma)^{\log_u(t)} \in L[\gamma - 1].
$$

We know that, for $\lambda \in \mathcal{S}(L)$,

$$
\begin{aligned}
\tilde{\mathrm{sp}}_\lambda(\widecheck{\Phi^s}^*_V) &= \tilde{\phi}^*{}_\lambda \in \mathrm{H}_{\mathrm{c}}^1(Y_{F,1}^*(\mathfrak{N}), \mathbf{D}_\lambda^{s,\circ}), \\
\mathrm{sp}_\lambda({}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at}) &= \widetilde{{}_c\Xi}^{\lambda}_{p^r,\mathfrak{N},at} \in \mathrm{H}^2(Y_{F,1}^*(\mathfrak{N}), \mathbf{A}_\lambda^{s,\circ}).
\end{aligned}
$$

Recall, for $\lambda = (0,0)$, the integration map

$$
\rho : \mathrm{H}_{\mathrm{c}}^1(Y_{F,1}^*(\mathfrak{N}), \mathcal{D}(\mathcal{O}_L)) \to \mathrm{H}_{\mathrm{c}}^1(Y_{F,1}^*(\mathfrak{N}), \mathcal{O}_L).
$$

Thus, this map induces a dual map

$$
\rho^* : \mathrm{H}^2(Y_{F,1}^*(\mathfrak{N}), \mathcal{O}_L) \to \mathrm{H}^2(Y_{F,1}^*(\mathfrak{N}), \mathcal{A}(\mathcal{O}_L)). \tag{4.6.15}
$$

Lemma 4.6.12. We have

$$\rho^*({}_c\Xi_{p^r,\mathfrak{N},at}) = {}_c\widetilde{\Xi}^{\lambda}_{p^r,\mathfrak{n},at}.$$

Proof. Let $f_\lambda \in \mathcal{A}(\mathcal{O}_L) := \mathcal{A}(\mathcal{O}_p, \mathcal{O}_L)$ denote the identity element, since f_λ map sends $\overline{n}tn$ to $\tilde{\lambda}(t) = z_1^k z_2^k$, and in this case $\lambda = (k,k) = (0,0)$.

Let id^* denote the identity element in the dual of $(\mathcal{O}_L)^2$. Now, for any $\mu \in \mathcal{D}(\mathcal{O}_L)$, we have

$$\begin{aligned}\mu(\rho^*(\mathrm{id}^* \otimes (\mathrm{id}^*)^\sigma)) &= \mathrm{id}^* \otimes (\mathrm{id}^*)^\sigma(\rho(\mu)),\\ &= \mathrm{id}^* \otimes (\mathrm{id}^*)^\sigma(\mu(\mathbb{1})(\mathrm{id} \otimes (\mathrm{id})^\sigma)),\\ &= \mu(\mathbb{1}) = \mu(f_\lambda).\end{aligned}$$

Hence $\rho^*(\mathrm{id}^* \otimes (\mathrm{id}^*)^\sigma) = f_\lambda \in \mathcal{A}(\mathcal{O}_L)$. See also [BH24, Lemma 7.15].

Write ID for $(\mathrm{id}^* \otimes (\mathrm{id}^*)^\sigma)$. Now, we can rewrite ${}_c\Xi_{p^r,\mathfrak{N},at}$ as

$$\varprojlim_k (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*(z_k \cup \mathrm{ID} \mod p^k)$$

where ${}_cC_{p^rN} = (z_k)_k \in \varprojlim_k \mathrm{H}^1(Y_{\mathbb{Q}}(p^rp^k, p^rp^kN), \mathbb{Z}_p) = \mathrm{H}^1(Y_{\mathbb{Q}}(p^r, p^rN), \mathbb{Z}_p)$.

Thus,

$$\begin{aligned}\rho^*({}_c\Xi_{p^r,\mathfrak{N},at}) &= \rho^*(\varprojlim_k (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*(z_k \cup \mathrm{ID} \mod p^k)),\\ &= \varprojlim_k (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*(z_k \cup \rho^*(\mathrm{ID}) \mod p^k),\\ &= \varprojlim_k (s_{p^r})_* \circ (\mathrm{proj}_t)_*(u_a \circ \iota)_*(z_k \cup f_\lambda \mod p^k),\\ &= {}_c\widetilde{\Xi}^{\lambda}_{p^r,\mathfrak{N},at}.\end{aligned}$$

$\blacksquare$

Remark 4.6.13. For any $k \in \mathbb{Z}_{\geqslant 0}$, in the future work, we will prove,

$$\rho^*({}_c\Xi^{k,j}_{p^r,\mathfrak{N},at}) = (*)\,{}_c\widetilde{\Xi}^{\lambda,j}_{p^r,\mathfrak{n},at},$$

where $0 \leqslant j \leqslant k$ and $*$ is some non-zero constant.

For $r \geqslant 1$ and non-trivial δ, define

$$\widetilde{P}^{\delta}_r := \frac{1}{a_p^r} \sum_{t \in (\mathbb{Z}/p^r)^\times} \langle \tilde{\phi}^*_\lambda, {}_c\widetilde{\Xi}^{\lambda}_{p^r,\mathfrak{N},at} \rangle \delta(t) \gamma^{\log_u(t)} \in L[\gamma - 1]. \tag{4.6.16}$$

Here the pairing $\langle , \rangle$ is $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{n}), \mathcal{D}(\mathcal{O}_L)) \times \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathcal{A}(\mathcal{O}_L)) \to \mathcal{O}_L$.

Proposition 4.6.14. As elements in $L[\gamma - 1]$, for $\lambda = (0,0)$, we have

$$\widetilde{P}^{\delta}_r \doteq P^{\delta}_r, \tag{4.6.17}$$

where, by $\doteq$, we mean equal upto a unit in L.

Proof. It is enough to show, for all $r \in \mathbb{Z}_{\geqslant 1}$ and $t \in (\mathbb{Z}/p^r)^\times$,

$$\langle \tilde{\phi}^*_\lambda,\ {}_c\widetilde{\Xi}^{\lambda}_{p^r,\mathfrak{N},at}\rangle \doteq \langle \phi^*_{\mathcal{F}},\ {}_c\Xi_{p^r,\mathfrak{N},at}\rangle. \tag{4.6.18}$$

From equation (4.6.3), we know

$$\rho(\tilde{\phi}^*_\lambda) \doteq \phi^*_{\mathcal{F}}.$$

Hence, from the above equation and Lemma 4.6.12, we get

$$\begin{aligned} \langle \tilde{\phi}^*_\lambda,\ {}_c\widetilde{\Xi}^{\lambda}_{p^r,\mathfrak{N},at}\rangle &= \langle \tilde{\phi}^*_\lambda,\ \rho^*({}_c\Xi_{p^r,\mathfrak{N},at})\rangle, \\ &= \langle \rho(\tilde{\phi}^*_\lambda),\ {}_c\Xi_{p^r,\mathfrak{N},at}\rangle, \\ &\doteq \langle \phi^*_{\mathcal{F}},\ {}_c\Xi_{p^r,\mathfrak{N},at}\rangle, \end{aligned}$$

which concludes the proof. ∎

There exists $x_{\mathcal{F}} \in V = \mathrm{Sp}(\mathcal{T})$ that corresponds to $\mathcal{F} \in S_{(0,0)}(U_{F,1}(\mathfrak{N}))$. Let $\mathfrak{m}_{x_{\mathcal{F}}}$ be the maximal ideal corresponding to $\mathcal{F}$ and let $\mathfrak{m}_\lambda$ be the maximal ideal corresponding to $\lambda = (0,0) \in \mathcal{S}(L)$. We assume $\mathcal{T}/\mathfrak{m}_{\mathcal{F}} \cong L \cong \mathfrak{L}/\mathfrak{m}_\lambda$. Consider the map

$$\mathrm{sp}^\lambda_{x_{\mathcal{F}}} : \mathcal{T}\hat{\otimes}\mathfrak{L}\hat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma) \to (\mathcal{T}/\mathfrak{m}_{x_{\mathcal{F}}})\hat{\otimes}(\mathfrak{L}/\mathfrak{m}_\lambda)\hat{\otimes}\mathcal{H}_{L,v_p(a_p)}(\Gamma) \cong \mathcal{H}_{L,v_p(a_p)}(\Gamma).$$

Theorem 4.6.15. We have

$$\mathrm{sp}^\lambda_{x_{\mathcal{F}}}({}_c\mathcal{L}^{\mathrm{As}}_V) \doteq {}_cL^{\mathrm{As}}_p(\mathcal{F}) \in \mathcal{H}_{L,v_p(a_p)}(\Gamma). \tag{4.6.19}$$

Proof. Let $x := x_{\mathcal{F}}$. We will prove, for all $\delta \in \Delta^*$ and $r \in \mathbb{Z}_{\geqslant 1}$,

$$\overline{\mathrm{sp}}_{x,\lambda}(P^{\mathcal{S},\delta}_r) \doteq P^\delta_r, \tag{4.6.20}$$

where $\overline{\mathrm{sp}}_{x,\lambda} : \mathcal{T}^\circ \otimes \mathfrak{L}^\circ \otimes L[\gamma-1] \to (\mathcal{T}/\mathfrak{m}_x) \otimes (\mathfrak{L}^\circ/\mathfrak{m}_\lambda) \otimes L[\gamma-1] = L[\gamma-1]$.

Suppose equation (4.6.20) is true. Now, from Theorem 4.6.9, we know

$${}_c\mathcal{L}^\delta_V(\gamma-1) \equiv P^{\mathcal{S},\delta}_r \mod \omega_{r-1}(\gamma-1).$$

Therefore,

$$\begin{aligned} \mathrm{sp}_x({}_c\mathcal{L}^\delta_V(\gamma-1)) &\equiv \mathrm{sp}_x(P^{\mathcal{S},\delta}_r \mod \omega_{r-1}(\gamma-1)), \\ &\equiv (\overline{\mathrm{sp}}_x(P^{\mathcal{S},\delta}_r) \mod \omega_{r-1}(\gamma-1), \\ &\equiv P^\delta_r \mod \omega_{r-1}(\gamma-1). \end{aligned}$$

We are working with weak uniform topologies on Banach spaces, and hence

$$\mathrm{sp}_x(\lim_{r\to\infty} P^{\mathcal{S},\delta}_r(\gamma-1)) = \lim_{r\to\infty} \mathrm{sp}_x(P^{\mathcal{S},\delta}_r(\gamma-1)),$$

and hence,

$$\mathrm{sp}_x({}_c\mathcal{L}^\delta_V(\gamma-1)) = \lim_{r\to\infty} P^\delta_r = {}_cL^{\mathrm{As},\delta}_p(\mathcal{F}). \tag{4.6.21}$$

Thus, it only remains to show that equation (4.6.20) holds.

Note that the following diagram commutes (upto some unit in L):

$$\begin{tikzcd}
\big(\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathbf{D}^{s,\circ}_{\mathcal{S}})^{<1} \otimes \mathcal{T}^\circ\big) \arrow[d, "\tilde{\mathrm{sp}}_\lambda"] & \times & \big(\mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^{s,\circ}_{\mathcal{S}})\big) \arrow[r] \arrow[d, "\mathrm{sp}_\lambda"] & \mathcal{T}^\circ \otimes \mathfrak{L}^\circ \arrow[d, "\overline{\mathrm{sp}}_{x,\lambda}"] \\
\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathbf{D}^s_\lambda(\mathcal{O}_L))^{\leqslant h} \arrow[d, "\rho"] & \times & \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathbf{A}^s_\lambda(\mathcal{O}_L)) \arrow[r] & (\mathcal{T}^\circ/\mathfrak{m}_x) \otimes (\mathfrak{L}^\circ/\mathfrak{m}_\lambda) \cong \mathcal{O}_L \arrow[d, "="] \\
\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{N}), \mathcal{O}_L) & \times & \mathrm{H}^2(Y^*_{F,1}(\mathfrak{N}), \mathcal{O}_L) \arrow[r] \arrow[u, "\rho^*"] & \mathcal{O}_L.
\end{tikzcd} \tag{4.6.22}$$

We will explain why the diagram in (4.6.22) commutes. We have

$$\begin{aligned}
\overline{\mathrm{sp}}_{x,\lambda}(\langle \breve{\Phi^s}^*_V, {}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at}\rangle) &= \langle \tilde{\mathrm{sp}}_\lambda(\breve{\Phi^s}^*_V), \mathrm{sp}_\lambda({}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at})\rangle, \\
&= \langle \tilde{\phi}^*_\lambda, {}_c\widetilde{\Xi}^\lambda_{p^r,\mathfrak{N},at}\rangle \\
&= \langle \tilde{\phi}^*_\lambda, \rho^*({}_c\Xi_{p^r,\mathfrak{N},at})\rangle, \\
&= \langle \rho(\tilde{\phi}^*_\lambda), {}_c\Xi_{p^r,\mathfrak{N},at}\rangle, \\
&\doteq \langle \phi^*_{\mathcal{F}}, {}_c\Xi_{p^r,\mathfrak{N},at}\rangle.
\end{aligned}$$

This implies

$$\overline{\mathrm{sp}}_{x,\lambda}(P^{\mathcal{S},\delta}_r) \doteq P^\delta_r.$$

∎

We complete this subsection with a couple of remarks.

Remark 4.6.16. We can remove the dependency on c. We assumed c is coprime to $6N$. Suppose $\epsilon_{\mathcal{F}}$ is the nebentypus of $\mathcal{F}$ and if $\epsilon_{\mathcal{F}}|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ does not have a p-power conductor, then by [Deo25, Proposition 6.8], there exists $L^{\mathrm{As}}_p(\Psi) \in \mathcal{H}_{L,v_p(a_p)}(\Gamma)$ such that

$$L^{\mathrm{As}}_p(\mathcal{F}) = \frac{1}{f_c}\, {}_cL^{\mathrm{As}}_p(\mathcal{F}),$$

where f_c is some non-zero term related to c. We need to construct ${}_c\Xi^{\mathcal{S}}_{p^r,\mathfrak{N},at}$ by using f_c. It will be explored in the future.

Remark 4.6.17. We assumed $\mathcal{F}$ is a base-change of a cuspidal elliptic modular form f of weight 2. If we assume $\mathcal{F}$ is a genuine non-critical cuspidal Bianchi modular form of weight $(0,0)$, i.e., not arising from a cuspidal elliptic modular form, the results in this section are still valid. But in general, classical genuine cuspidal Bianchi modular forms are not always Zariski-dense. It can happen that V only contains a single classical point corresponding to $\mathcal{F}$. See [BSW21a, Remark 6.11] for more details.

Chapter 5

Future works

Artin formalism for signed p-adic L-functions of Bianchi modular forms

Let $F/\mathbb{Q}$ be an imaginary quadratic field, $p \geqslant 3$ be a prime that splits in F, and $k \in \mathbb{Z}_{\geqslant 0}$. Let G_{p^∞} be the ray class group over F modulo p^∞ and Γ be the Galois group of cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Since p splits in F, we can identify Γ with the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of F. Let f be an elliptic cusp form of weight $k+2$ and let $\mathcal{F}$ be a cuspidal Bianchi modular form of weight (k,k) be the base change of f. Let $L_p^{\mathrm{cyc}}(\mathcal{F})$ be the distribution on Γ by projecting Williams' p-adic L-function $L_p(\mathcal{F})$ associated to $\mathcal{F}$ along $G_{p^\infty} \to \Gamma$. Barrera-Salazar–Williams in [BSW21a] proved the following p-adic Artin formalism:

$$L_p^{\mathrm{cyc}}(\mathcal{F}) \doteq L_p(f) \cdot L_p^{\chi_F}(f),$$

where χ_F is the quadratic character associated with F, $L_p^{\chi_F}(f)$ is p-adic L-function associated to the twist of f by χ_F. Recently, for $k=0$, Lei in [Lei24] proved the p-adic Artin formalism for signed p-adic L-functions associated to $\mathcal{F}$. Especially, he proved that if f is p-non-ordinary elliptic modular form of weight 2 and for $\bullet \in \{\flat, \sharp\}$ then

$$L_{\bullet\bullet}^{\mathrm{cyc}}(\mathrm{BC}(f)) = L_\bullet L_\bullet^{\chi_F},$$

where $L_{\bullet,\bullet}, L_\bullet$ are signed p-adic L-functions considered in [Lei14]. We propose to extend this Artin formalism of signed p-adic L-functions of Bianchi modular forms from $k=0$ to $k \geqslant 0$ using the methods developed in [Deo26].

Constructing p-adic Asai L-functions for C-cuspidal Bianchi modular forms

In [Pal25], Palacios constructed p-adic L-functions for C-cuspidal Bianchi modular forms of weight (k,ℓ), where $k,\ell \in \mathbb{Z}_{\geqslant 0}$ and k might not be equal to ℓ. He used *partial modular* *symbols* to construct the distributions. We are interested in investigating and constructing p-adic Asai L-functions for C-cuspidal Bianchi modular forms of non-parallel weights using partial modular symbols, Asai-Eisenstein elements, and the methods from [Deo25].

Families of p-adic Asai L-function for higher weight Bianchi modular forms and the symmetric square p-adic L-function

In Chapter 4, we constructed a two-variable p-adic Asai L-function for a weight $(0,0)$ cuspidal Bianchi modular form arising from a base-change of a weight 2 cuspidal elliptic modular form. The next natural extension that we are pursuing is to construct a two-variable p-adic Asai L-function for a cuspidal Bianchi modular form of weight (k,k), where $k \in \mathbb{Z}_{>0}$. In this case, we have to deal with extra cyclotomic twists $0 \leqslant j \leqslant k$. We need to construct Asai–Eisenstein elements that could incorporate these extra twists.

An application of this two-variable p-adic L-function is to give a new construction of p-adic L-functions related to the symmetric square of an elliptic modular form. More specifically, if $\mathcal{F}$ is a base change of an elliptic modular form f, then we have the following factorization of complex L-functions:

$$L^{\mathrm{As}}(\mathcal{F}, \theta, s) = L(\mathrm{Sym}^2 f, \theta, s) L(\theta \epsilon_f \epsilon_F, s-k-1),$$

where θ is a Dirichlet character of finite order, ϵ_f is the nebentypus of f, and ϵ_F is a quadratic character associated to F. Now if f is a small slope and non-critical Hecke eigenform of weight $k+2$, and $\mathcal{F}$ is a cuspidal Bianchi modular form that is a base change of f, then using the above factorization, and comparing interpolation formulas of ${}_{c}L_p^{\mathrm{As}}(\mathcal{F})$ and the Kubota–Leopoldt p-adic L-function ζ_p^{an}, we can define

$$L_p(\mathrm{Sym}^2 f) := (*) \frac{{}_{c}L_p^{\mathrm{As}}(\mathcal{F})}{\zeta_p^{\mathrm{an}}},$$

where $*$ is some explicit non-zero factor. We will be pursuing this construction in the near future.

We are also interested in constructing the *critical* p-adic Asai L-function, i.e., the p-adic Asai L-function associated to the critical Bianchi modular form. By a critical Bianchi modular form $\mathcal{F}$ of weight (k,k), we mean the slope of $\mathcal{F}$ is k.

Appendix A

Signed p-adic L-functions for non-parallel weight Bianchi modular forms

In this appendix, we extend the results from parallel weight cuspidal Bianchi modular forms to non-parallel weight C-cuspidal Bianchi modular forms. The notion of C-cuspidality is related with the vanishing of the constant term of Fourier expansions of Bianchi modular forms (with level divisible by p) at suitable cusps. For the definitions and proper explanations about *C-cuspidality*, see [Pal25, Section 2]. Note that the space of cuspidal Bianchi modular forms with level at p is a proper subset of the set of C-cuspidal Bianchi modular forms with level at p.

Let us fix some notations first. Fix an odd prime p. Let $K/\mathbb{Q}$ be a quadratic imaginary field and p splits in K as $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. We also assume p does not divide the class number h_K of K. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $K_{\mathfrak{q}} \cong \mathbb{Q}_p$ be the completion of K at the prime $\mathfrak{q}$. Let $\mathcal{O}_{K_{\mathfrak{q}}}$ be the ring of integers of $K_{\mathfrak{q}}$ and let $\varpi_{\mathfrak{q}}$ be its uniformizer. We fix the embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Note that ι_p fixes a p-adic valuation v_p on $\overline{\mathbb{Q}_p}$. Hence, we choose ι_p such that $v_p(\varpi_{\mathfrak{p}}) = 1$ and $v_p(\varpi_{\overline{\mathfrak{p}}}) = 0$. The embeddings ι_∞ and ι_p will give the isomorphism $\iota : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}_p}$ satisfying $\iota \circ \iota_\infty = \iota_p$. Fix non-negative integers k and ℓ.

A.1 p-adic L-functions associated to C-cuspidal Bianchi modular forms

Let $\mathcal{F}$ be a C-cuspidal Bianchi eigenform of weight (k, ℓ) and level $\mathfrak{n}$, where p divides $\mathfrak{n}$. For the Fourier expansion related to any Bianchi modular form, see [Pal25, Section 2.3]. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $a_{\mathfrak{q}}$ be the $U_{\mathfrak{q}}$-eigenvalue of $\mathcal{F}$. Moreover, $\mathcal{F}$ is *small slope*, i.e., $v_p(a_{\mathfrak{p}}) < k + 1$ and $v_p(a_{\overline{\mathfrak{p}}}) < \ell + 1$.

Let $c(\cdot, \mathcal{F})$ denote the Fourier coefficients of $\mathcal{F}$. Like in the cuspidal case, for any Hecke character Ξ with the conductor $\mathfrak{f}$, we define the *L-function of $\mathcal{F}$ twisted by* Ξ:

$$L(\mathcal{F},\Xi,s)=\sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_F,\\(\mathfrak{a},\mathfrak{f})=1}} c(\mathfrak{a},\mathcal{F})\Xi(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-s},$$

where $s\in\mathbb{C}$.

Using Deligne's Γ-factors, we *renormalize* this L-function:

$$\Lambda(\mathcal{F},\Xi,s)=\frac{\Gamma(q+s)\Gamma(r+s)}{(2\pi i)^{q+s}(2\pi i)^{r+s}}L(\mathcal{F},\Xi,s),$$

where (q,r) is the infinity type of Ξ.

The main theorem of [Pal25] is:

Theorem A.1.1. [Pal25, Theorem 4.12] For chosen embeddings ι_∞,ι_p, and ι, there exists a locally analytic distribution $L^\iota_{p,\mathcal{F}}$ on the ray class group G_{p^∞} such that for any Hecke character Ξ of K of conductor $\mathfrak{p}^{n_\mathfrak{p}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ and infinity type $(0,0)\leqslant(q,r)\leqslant(k,\ell)$, we have

$$L^\iota_{p,\mathcal{F}}(\widetilde{\Xi})=(\text{explicit factor})\times\frac{1}{\lambda_\mathfrak{f}}\times\Lambda(\mathcal{F},\Xi,1),\tag{A.1.1}$$

where $\lambda_\mathfrak{f}$ is the $U_\mathfrak{f}=\prod_{\mathfrak{q}|p}U_\mathfrak{q}^{n_\mathfrak{q}}$-eigenvalue of $\mathcal{F}$.
The distribution $L^\iota_{p,\mathcal{F}}$ is $(v_p(a_\mathfrak{q}))_{\mathfrak{q}|p}$-admissible and therefore is unique.

From the Remark 2.7.8, we can see $L^\iota_{p,\mathcal{F}}\in\mathcal{H}_{E,v_p(a_\mathfrak{p}),v_p(a_{\overline{\mathfrak{p}}})}(G_{p^\infty})$, for some finite extension $E/\mathbb{Q}_p$.

A.2 Decomposition of p-adic L-functions of C-cuspidal Bianchi modular form

We first fix a Bianchi modular eigenform $\mathcal{F}$ of weight (k,ℓ) and level $\mathfrak{m}$, where $\mathfrak{m}$ is coprime with p. Furthermore, we assume that $\mathcal{F}$ vanishes at cusps 0 and ∞.

Consider the L-function of $\mathcal{F}$:

$$L(\mathcal{F},s)=\sum_{0\neq\mathfrak{a}\subset\mathcal{O}_K} c(\mathfrak{a},\mathcal{F})\mathrm{N}(\mathfrak{a})^{-s},$$

then the *local Euler factor* at $\mathfrak{q}\in\{\mathfrak{p},\overline{\mathfrak{p}}\}$ is

$$L_\mathfrak{q}(\mathcal{F},s)^{-1}=1-a_\mathfrak{q}p^{-s}+\varepsilon_\mathcal{F}(\varpi_\mathfrak{q})p^{1-2s},\tag{A.2.1}$$

where $a_\mathfrak{q}$ is the $T_\mathfrak{q}$-eigenvalue of $\mathcal{F}$, and $\varepsilon_\mathcal{F}$ is the central character associated to $\mathcal{F}$. Recall that $\varepsilon_\mathcal{F}$ is a Hecke character of the infinity type $(-k,-\ell)$, and conductor coprime with p.

Hence, we will consider the following Hecke polynomials: for the prime ideal $\mathfrak{p}$, we have

$$\begin{aligned} P_{\mathfrak{p}}(X) &:= X^2 - a_{\mathfrak{p}}X + \varepsilon(\varpi_{\mathfrak{p}})p, && \text{(A.2.2)} \\ &= X^2 - a_{\mathfrak{p}}X + \varepsilon_\infty(\varpi_{\mathfrak{p}})^{-1}p, && \text{(A.2.3)} \\ &= X^2 - a_{\mathfrak{p}}X + \varpi_p^k \cdot \varpi_{\overline{\mathfrak{p}}}^\ell \cdot p, && \text{(A.2.4)} \\ &= X^2 - a_{\mathfrak{p}}X + \varpi_{\mathfrak{p}}^{k+1}\varpi_{\overline{\mathfrak{p}}}^{\ell+1}. && \text{(A.2.5)} \end{aligned}$$

Similarly, for prime $\overline{\mathfrak{p}}$, we consider

$$P_{\overline{\mathfrak{p}}}(X) := X^2 - a_{\overline{\mathfrak{p}}}X + \varpi_{\overline{\mathfrak{p}}}^{k+1}\varpi_{\mathfrak{p}}^{\ell+1}. \tag{A.2.6}$$

From now onwards, we assume $\mathcal{F}$ is non-ordinary at both the primes $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, i.e., $v_p(a_{\mathfrak{p}}), v_p(a_{\overline{\mathfrak{p}}}) > 0$. We furthermore assume

1. $v_p(a_{\mathfrak{p}}) > \left\lfloor \dfrac{k}{p-1} \right\rfloor$;
2. $v_p(a_{\overline{\mathfrak{p}}}) > \left\lfloor \dfrac{\ell}{p-1} \right\rfloor$.

For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, let $\alpha_{\mathfrak{q}}$ and $\beta_{\mathfrak{q}}$ be the roots of $P_{\mathfrak{q}}(X)$. Recall that we have a p-adic valuation v_p corresponding to the fixed embedding ι_p such that $v_p(\varpi_{\mathfrak{p}}) = 1$ and $v_p(\varpi_{\overline{\mathfrak{p}}}) = 0$. Hence, the roots $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ satisfy:

$$\begin{aligned} v_p(\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}) &= v_p(\varpi_{\mathfrak{p}}^{k+1}\varpi_{\overline{\mathfrak{p}}}^{\ell+1}), \\ v_p(\alpha_{\mathfrak{p}}) + v_p(\beta_{\mathfrak{p}}) &= v_p(\varpi_{\mathfrak{p}}^{k+1}) + v_p(\varpi_{\overline{\mathfrak{p}}}^{\ell+1}), \\ &= k+1, \end{aligned}$$

and therefore

$$0 < v_p(\alpha_{\mathfrak{p}}), v_p(\beta_{\mathfrak{p}}) < k+1.$$

Similarly,for the roots $\alpha_{\overline{\mathfrak{p}}}$ and $\beta_{\overline{\mathfrak{p}}}$, we have

$$v_p(\alpha_{\overline{\mathfrak{p}}}) + v_p(\beta_{\overline{\mathfrak{p}}}) = \ell + 1,$$

and

$$0 < v_p(\alpha_{\overline{\mathfrak{p}}}), v_p(\beta_{\overline{\mathfrak{p}}}) < \ell + 1.$$

We assume $\alpha_{\mathfrak{q}} \neq \beta_{\mathfrak{q}}$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$.

Like in the cuspidal case, we have four p-stabilizations of $\mathcal{F}$: $\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}$, $\mathcal{F}^{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}$. Note that we have assumed $\mathcal{F}$ vanishes at the cusps 0 and ∞.

Lemma A.2.1. For $* \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$, the p-stabilization $\mathcal{F}^{*,\dagger}$ is a C-cuspidal Bianchi modular form of weight (k, ℓ) and level $p\mathfrak{m}$.

Proof. The proof is similar to the proof of [Pal25, Proposition 5.3]. ∎

Therefore, for $* \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$, the p-stabilizations $\mathcal{F}^{*,\dagger}$ are C-cuspidal, of level $p\mathfrak{m}$, and are of *small slope*, since $v_p(*) < k+1$ and $v_p(\dagger) < \ell + 1$. Hence, by Theorem A.1.1, we can attach a p-adic L-function $L^{\iota}_{*,\dagger} := L^{\iota}_{p,\mathcal{F}^{*,\dagger}} \in \mathcal{H}_{E,v_p(*),v_p(\dagger)}(G_{p^\infty})$ to the C-cuspidal Bianchi modular form $\mathcal{F}^{*,\dagger}$. Furthermore, for any Hecke character Ξ of the infinity type (q,r) such that $0 \leqslant q \leqslant k$ and $0 \leqslant r \leqslant \ell$ of conductor $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$, where $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \in \mathbb{Z}_{>0}$, we have the following interpolation properties:

$$\begin{aligned}
L^{\iota}_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) &= \alpha_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{q,r,\widetilde{\Xi}},\\
L^{\iota}_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) &= \alpha_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{q,r,\widetilde{\Xi}},\\
L^{\iota}_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) &= \beta_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\alpha_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{q,r,\widetilde{\Xi}},\\
L^{\iota}_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}}(\widetilde{\Xi}) &= \beta_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\beta_{\overline{\mathfrak{p}}}^{-n_{\overline{\mathfrak{p}}}} \cdot C_{q,r,\widetilde{\Xi}},
\end{aligned}$$

where $C_{q,r,\widetilde{\Xi}} \in \overline{\mathbb{Q}_p}$ is a constant independent of $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}$. More precisely, $C_{q,r,\widetilde{\Xi}} =$ (some explicit factor) $\times\ \Lambda(\mathcal{F}, \Xi, 1)$, since the conductor of the Hecke character Ξ is $\mathfrak{p}^{n_{\mathfrak{p}}}\overline{\mathfrak{p}}^{n_{\overline{\mathfrak{p}}}}$ with $n_{\mathfrak{p}}, n_{\overline{\mathfrak{p}}} \in \mathbb{Z}_{>0}$, and hence $L(\mathcal{F}, \Xi, 1) = L(\mathcal{F}^{*,\dagger}, \Xi, 1)$ for all $* \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$ and $\dagger \in \{\alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}\}$.

Note that, since $v_p(\varpi_{\mathfrak{p}}) = 1$ and $v_p(\varpi_{\overline{\mathfrak{p}}}) = 0$, we can write $\varpi_{\mathfrak{p}}^{k+1}\varpi^{\ell+1} = p^{k+1} \cdot u_{\mathfrak{p}}$, where $u_{\mathfrak{p}}$ is a suitable unit in some ring of integers of finite extension of $\mathbb{Q}_p$. Similarly, we can write $\varpi_{\mathfrak{p}}^{\ell+1}\varpi_{\overline{\mathfrak{p}}}^{k+1} = p^{\ell+1} \cdot v_{\mathfrak{p}}$. Let $E/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ large enough to contain all Hecke eigenvalues of $\mathcal{F}$, $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}, \beta_{\overline{\mathfrak{p}}}, u_{\mathfrak{p}}^{1/2}$, and $v_{\mathfrak{p}}^{1/2}$.

We construct logarithmic matrices using the methods from Sections 2.3,2.4, and 2.5.

For the prime $\mathfrak{p}$, let

$$A_{\varphi,\mathfrak{p}} = \begin{pmatrix} 0 & \dfrac{-1}{p^{k+1}u_{\mathfrak{p}}} \\ 1 & \dfrac{a_{\mathfrak{p}}}{p^{k+1}u_{\mathfrak{p}}}, \end{pmatrix},$$

$$Q_{\mathfrak{p}} = \begin{pmatrix} \alpha_{\mathfrak{p}} & -\beta_{\mathfrak{p}} \\ -p^{k+1}u_{\mathfrak{p}} & p^{k+1}u_{\mathfrak{p}} \end{pmatrix}.$$

Using this data, we construct a logarithmic matrix $\underline{M^k(\mathfrak{p})} \in M_{2,2}(\mathcal{H}_E(\Gamma_1))$ such that it satisfies:

1. If $Q_{\mathfrak{p}}^{-1}(\underline{M^k(\mathfrak{p})}) = \begin{pmatrix} P_1(\mathfrak{p}) & P_2(\mathfrak{p}) \\ P_3(\mathfrak{p}) & P_4(\mathfrak{p}) \end{pmatrix}$, then $P_1(\mathfrak{p}), P_2(\mathfrak{p}) \in \mathcal{H}_{E,v_p(\alpha_{\mathfrak{p}})}(\Gamma_1)$ and $P_3(\mathfrak{p}), P_4(\mathfrak{p}) \in \mathcal{H}_{E,v_p(\beta_{\mathfrak{p}})}(\Gamma_1)$.
2. The second row of $A_{\varphi,\mathfrak{p}}^{-n}\underline{M^k(\mathfrak{p})}$ is divisible by $\Phi_{n-1,k+1}(\gamma_0)$ over $\mathcal{H}_E(\Gamma_1)$.
3. The determinant $\det(\underline{M^k(\mathfrak{p})})$ is $\dfrac{\log_{p,k+1}(\gamma_0)}{\delta_{k+1}(\gamma_0 - 1)}$, up to a unit in $\Lambda_E(\Gamma_1)$.

Define $\underline{M^k_{\mathfrak{p}}} := \mathrm{Mat}_{\mathfrak{p}}(\underline{M^k(\mathfrak{p})}) \in M_{2,2}(\mathcal{H}_E(\Gamma_{\mathfrak{p}}))$.

For the prime $\overline{\mathfrak{p}}$, let

$$A_{\varphi,\overline{\mathfrak{p}}} = \begin{pmatrix} 0 & \dfrac{-1}{p^{\ell+1}v_{\mathfrak{p}}} \\ 1 & \dfrac{a_{\overline{\mathfrak{p}}}}{p^{\ell+1}v_{\mathfrak{p}}}, \end{pmatrix},$$

$$Q_{\overline{\mathfrak{p}}} = \begin{pmatrix} \alpha_{\overline{\mathfrak{p}}} & -\beta_{\overline{\mathfrak{p}}} \\ -p^{\ell+1}v_{\mathfrak{p}} & p^{\ell+1}v_{\mathfrak{p}} \end{pmatrix}.$$

Using this, we can construct a logarithmic matrix $\underline{M^\ell(\overline{\mathfrak{p}})} \in \mathcal{H}_E(\Gamma_1)$ such that:

1. If $Q_{\overline{\mathfrak{p}}}^{-1}(\underline{M^\ell(\overline{\mathfrak{p}})}) = \begin{pmatrix} P_1(\overline{\mathfrak{p}}) & P_2(\overline{\mathfrak{p}}) \\ P_3(\overline{\mathfrak{p}}) & P_4(\overline{\mathfrak{p}}) \end{pmatrix}$, then $P_1(\overline{\mathfrak{p}}), P_2(\overline{\mathfrak{p}}) \in \mathcal{H}_{E,v_p(\alpha_{\overline{\mathfrak{p}}})}(\Gamma_1)$ and $P_3(\overline{\mathfrak{p}}), P_4(\overline{\mathfrak{p}}) \in \mathcal{H}_{E,v_p(\beta_{\overline{\mathfrak{p}}})}(\Gamma_1)$.
2. The second row of $A_{\varphi,\overline{\mathfrak{p}}}^{-n}\underline{M^\ell(\overline{\mathfrak{p}})}$ is divisible by $\Phi_{n-1,\ell+1}(\gamma_0)$ over $\mathcal{H}_E(\Gamma_1)$.
3. The determinant $\det(\underline{M^\ell(\mathfrak{p})})$ is $\dfrac{\log_{p,\ell-1}(\gamma_0)}{\delta_{\ell+1}(\gamma_0-1)}$, up to a unit in $\Lambda_E(\Gamma_1)$.

Define $\underline{M^\ell_{\overline{\mathfrak{p}}}} := \mathrm{Mat}_{\overline{\mathfrak{p}}}(\underline{M^\ell(\overline{\mathfrak{p}})}) \in \mathcal{H}_E(\Gamma_{\overline{\mathfrak{p}}})$.

Note that we are getting $k+1, \ell+1$ instead of $k-1, \ell-1$ since $k, \ell \in \mathbb{Z}_{\geqslant 0}$.

Hence, by following the same methods used in the proof of Propositions 2.8.8, 2.8.9, 2.8.10, we can conclude:

Theorem A.2.2. There exist $L^\iota_{\sharp,\sharp}, L^\iota_{\sharp,\flat}, L^\iota_{\flat,\sharp}, L^\iota_{\flat,\flat} \in \Lambda_E(G_{p^\infty})$ such that

$$\begin{pmatrix} L^\iota_{\alpha_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} & L^\iota_{\beta_{\mathfrak{p}},\alpha_{\overline{\mathfrak{p}}}} \\ L^\iota_{\alpha_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} & L^\iota_{\beta_{\mathfrak{p}},\beta_{\overline{\mathfrak{p}}}} \end{pmatrix} = Q_{\overline{\mathfrak{p}}}^{-1}\underline{M^\ell_{\overline{\mathfrak{p}}}} \begin{pmatrix} L^\iota_{\sharp,\sharp} & L^\iota_{\flat,\sharp} \\ L^\iota_{\sharp,\flat} & L^\iota_{\flat,\flat} \end{pmatrix} (Q_{\mathfrak{p}}^{-1}\underline{M^k_{\mathfrak{p}}})^T. \tag{A.2.7}$$

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