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(Fixed Point Property)

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COMPUTATIONAL ASPECTS OF ORDER PRESERVING MAPS (*FIXED POINT PROPERTY*)

By

Nazih Chibbani

A thesis submitted to the Faculty of Graduate and Postdoctoral Studies in partial fulfillment of the requirements for the degree of

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Abstract

COMPUTATIONAL ASPECTS OF ORDER PRESERVING MAPS

By Nazih Chibbani

Chairperson of the Supervisory Committee: Professor Nejib Zaguia
Department of Computer Science

The ordered sets field is an important part of the ongoing mathematical, algorithmic and combinatorics research. This young field identifies and addresses many unsolved problems, a considerable group of which were proven (NP). *"Characterizing the finite ordered sets with fixed point property"* is among those open unsolved problems. It was originally introduced and described by Rival (1984); and was later demonstrated to be NP-complete by Williamson. Since then, several researchers worked and published on the subject, describing a variety of algorithms and techniques to address and solve aspects of the problem.

This thesis introduces a computational technique for identifying ordered sets with the **fixed point property** (FPP). The technique consists of generating all possible ordered sets of a given size up to isomorphism and duality, followed by a testing of each ordered set for the FPP. Algorithms for the generation of ordered sets, for isomorphic elimination, and for the FPP testing are presented.

The results obtained from the research were summarized as a list of incidence matrices of all ordered sets with the FPP; irreducible elements free; and retractable elements free up to duality and isomorphism for set sizes ≤ 13 elements. This research expands on the results published by Rutkowski (1989) and Schröder (1993) for set sizes of 10 and 11, respectively, by identifying the set of complying ordered sets of sizes 12 and 13. The results can contribute to future research in the topic, particularly in the area of non-computational techniques, as a comparative data set for validation purposes.

Approved by Professor Nejib Zaguia ______________________________
Chairperson of Supervisory Committee

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Program Authorized
to Offer Degree ______________________________

Date ______________________________

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This work is dedicated to Professor Ivan Rival and to my father Mahmoud. Two people who are with us with their spirits. Professor Rival was my best teacher. He first introduced me to the fascinating world of ordered sets, and he followed the mission when he suggested this topic for my masters degree and supervised my thesis. My father not only taught me life, he also introduced me to the fascinating world of numbers.

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" *I understood that Prof Rival did supervise your thesis and I'm willing to finish the job. Ivan and myself did collaborate for the last twenty years and I feel as a duty to do this.*" Thank you Professor Zaguia for your loyalty, for your help and advice.

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Chapter 1

1 INTRODUCTION

1.1 Ordered set

1.1.1 History and Connection

The mathematical concept of partial orders was proved useful as a unifying tool in such diverse fields as functional analysis, algebra, set theory, combinatorics, and especially **computer science**. Much of the development in the theory of ordered sets arose from its connection to various and diverse fields. However, during the last decade, it has been the combinatorial methods and computational techniques that have particularly stimulated and increased the amount of research in the ordered set.

An ordered set is a set that conforms to the binary relation "≤" such that, the relation "≤" is reflexive, antisymmetric, and transitive. Similar to other like binary relations, order sets can be represented in different instances. Among the various possible schemes utilized to represent ordered sets, the most common representation is that of the upward drawing graph (Figure 1-1). As such, order theory can be seen as a domain with relations to both lattice theory and graph theory. Among the three theories (lattice, graph, and order), order theory is the youngest.

The first ordered sets specialized journal, "Order", was launched in 1984. The journal helped popularize the field by presenting it to a wide audience as fresh field with an abundance of unexplored problems. Several ordered sets problems have been characterized as non-deterministically polynomial (NP) problems. The occurrence of this class of (NP) problems favored the computational methods and algorithmic research, which characterized the domain in the past decade.

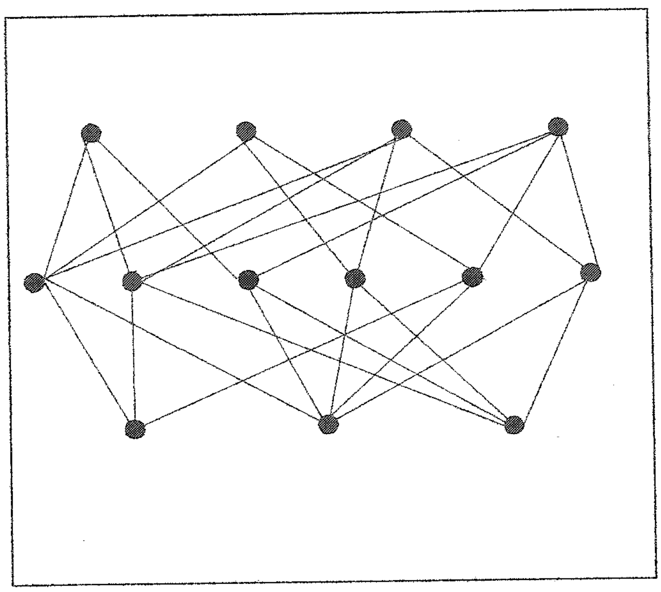

Figure 1-1 Upward drawing of an ordered set

Since the identification and characterization of the finite the FPP problem by Rival (1984), several other research cornerstones were laid down in the foundations and structure of this field.

The recorded history of this problem seems to start in the papers [20] by Knaster in the twenties and then in [32] by Tarski and [9] by Davis in the fifties, where the question is answered for lattices (a lattice is an ordered set in which any finite subset has a supremum and an infimum, it is complete iff every subset has a supremum and an infimum): A lattice has the fixed point property iff it is complete. After that there were papers by Abian and Brown (1971) [1] and Pelczar (1961)[23], which recorded one of today's standard tools: If P is a chain-complete ordered set, f:P →P is order-preserving and there is an element A ∈ P such that f(A) ~ A, then f has a fixed point. The problem for ordered sets seems to have been first raised in the text [6] by Crawley and Dilworth (1973). The next milestone then is Rival's characterization of the fixed point property for finite ordered sets of height 1 in (1976) [27]. Since then, the FPP problem has inspired many possible approaches. Among them are the following

A basic approach is to characterize the fixed point property for certain classes of ordered sets. This is done for example by Fofanova and Rutkowski in(1987) [14] (sets of width 2), by Höft and Höft in (1991)[18] (lexicographic sums), by Rival as mentioned in [27] (sets of height 1), Rutkowski in (1989) [31], Schröder in (1993) [34] (``small'' sets size 10 and 11), Davis and Tarski in (1955) [9] and [38] (for lattices) and possibly most importantly by Roddy in (1994) [30], where the fixed point problem is solved for finite products of finite ordered sets.

Another approach followed by researchers was to prove fixed point theorems that do not necessarily handle an established class of ordered sets, but that provide new insights through possible reductions (Rival in (1976) [27], and by Schröder in (1993) [34] or through the identification of substructures that force the fixed point property (done for example by Baclawski and Björner in (1979) [2], by Edelman (1979) in [12], and by Rutkowski in (1989) [31]).

In 1992, using the language of formal concept analysis, Xia[35] presented the (FPP) and the fixed point free (FPF) as formal concepts of a context from which an algorithm could be derived to determine if a given ordered set has the fixed point property.

1.1.2 Fixed point property

A well-known problem in ordered set theory is the characterization of the finite ordered sets with the fixed point property (FPP) [26].]. An ordered set P is said to have the FPP if and only if:

- for each order preserving self map (OPM) f: P → P, there is at least one fixed point x such that $f(x) = x$;

where f is OPM if and only if: for all x, y in P; if $x \leq y \Rightarrow f(x) \leq f(y)$.

This problem, that was introduced and described by Rival in [27], is the main focus of this thesis. Following Rival's work , further research was carried out and published to describe various algorithms addressing this problem. Special attention was given in this thesis to the results published by Rutkowski (1989) and Schröder (1993): In a paper entitled "The fixed point property for small sets", Rutkowski (1989) identified all ordered sets with the FPP of size ≤ 10, up to isomorphism and duality, containing no irreducible elements. Schröder (1993) introduced the concept of retractable elements, which is a stronger notion than that of irreducible elements. He also characterized all ordered sets with the FPP of size ≤ 11, up to isomorphism and duality, containing no irreducible or retractable elements.

Eventually, both Rutkowski and Schröder results were based on conventional mathematical assumption and proof methods.

1.1.3 Objectives

This thesis is concerned with characterizing all ordered sets with the FPP of size ≤ 13, up to isomorphism and duality, containing no irreducible or retractable elements. The characterization of these ordered sets will be carried out using computational techniques based on reliable algorithms.

1.1.4 Methodology

When searching for the roots of a function $y = f(x)$, a non-numerical approach may produce a closed-form solution leading directly to the roots of the function if they exist. It may also be possible to find the roots using numerical techniques; for discrete functions in the finite domain, a particular numerical method may search for the roots by examining the value of the function at every point in its domain. Then the roots of $f(x)$ may be obtained by evaluating y at every x in the domain.

Similarly, a solution to the problem may be found by generating the set of all ordered sets of size $n \leq 13$, up to isomorphism and duality, then testing every element of this set for the FPP according to a collection of non-trivial criteria.

The computational technique is comprised of three main steps:

1- Generating the set of all ordered sets of size $n \leq 13$

2- Filtering all generated ordered sets for isomorphism and duality.

3- Testing each ordered set for the FPP.

The diagram below (Figure 1-2) represents the data flow through the different phases.

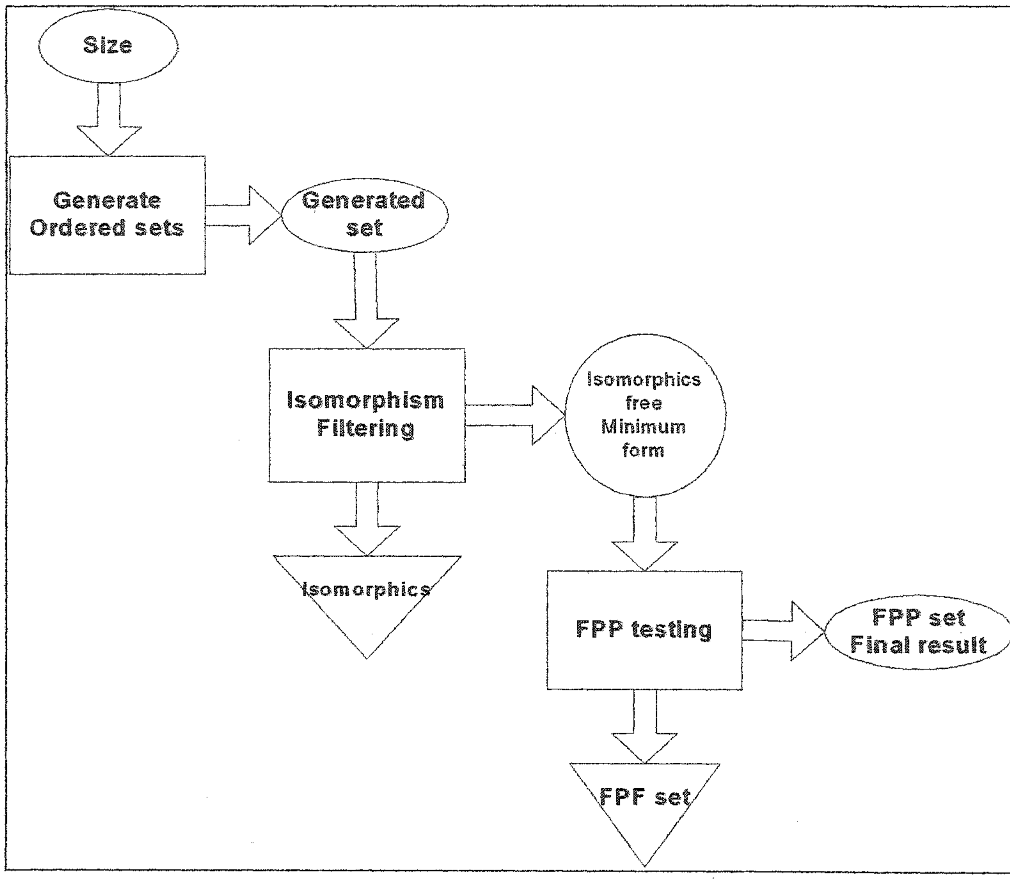

Figure 1-2 Block diagram of the process

1.1.4.1 Generation of Ordered Sets

The generation of ordered sets for this research can be described as a generation of incidence matrices for the binary order relation of ordered sets, which have the three conditions of an ordered set: reflexivity, antisymmetry, and transitivity. A backtracking technique is utilized during the generation process target potential matrices with ordered set properties. Among all of the sets generated, only connected ordered, irreducible element free, retractable element free, and dual free sets are selected

1.1.4.2 Isomorphism

Eventually, the set of generated ordered sets contains isomorphic sets, and the result we aim for is (up to isomorphism); therefore, an elimination process of isomorphic sets should be applied on the result set in order to reduce the set down to one set from each class of isomorphic sets.

The incidence matrices are stored in the resultant set in a vector format, by combining each row, which is a sequence of "0" and "1", in a decimal number, assuming that a row is a binary representation of the decimal number. These vectors are sorted in a lexicographic total order. This key fact contributes enormously to the elimination technique of isomorphic sets.

Meanwhile, in addition to the lexicographical total order property of the resultant set, we define in the same set, an equivalence relation, where two ordered sets are related if they are isomorphic. Thus, all ordered sets in the resultant set can be grouped in disjoint classes, such that, within each class, all elements are isomorphic. Simultaneously, each class is totally ordered; therefore, a selection of only the smallest element in each class provides a set of ordered sets that is isomorphic free, thus filtering out isomorphic.

The method for finding the smallest element in each class, which involves a pair-wise isomorphism comparison for all generated elements, assumes that all the classes are populated and enumerated. This is almost unconceivable; for this very large size of generated resultant. To overcome this obstacle, a computing algorithm is introduced to determine if a generated element is indeed the smallest in its class without comparing it to other elements of the same class.

The algorithm is based on the technique of generating an isomorphic ordered set from a given ordered set, such that the generated set is lexicographically smaller. In fact, the generation of the isomorphic set is possible by relabeling elements in the original ordered set, which implies columns and rows swapping in the incidence matrix, followed by sorting rows as needed. The ordered set is declared the smallest in its class, only when all conceivable permutations attempted to generate a lexicographically smaller set fail. A backtracking technique is applied in order to optimize the number of permutations used. This algorithm does not compare a member set with another; only with itself. The smallest elements found are said to be in their minimal form or, to borrow an analogy from physics, in their stable state.

A detailed description of this algorithm together with illustrative examples is covered in chapter 4.

1.1.4.3 Fixed point property characterization

Fixed point property testing is carried out on the resultant set of smallest element sets subsequent to isomorphism elimination. The testing consists of applying the FPP test on each ordered set in the resultant set.

According to the definition of the FPP, each order preserving self map for a given ordered set, each order preserving self map should have a fixed point. Inversely, an ordered set is fixed point free (FPF) is described as a set with at least one order preserving self map that is fixed point free. As such, the characterization of the FPP can be achieved by searching for a FPF map for the ordered set.

In order to obtain a FPF map for a specific ordered set, if such a map exists, a recursive loop is devised to generate all possible order preserving maps that are potentially fixed point free; which means, for each element A in the ordered set P , we set a range Q for the map f:P → P, such that, elements in the range Q are incomparable with A (x,y ∈P are incomparable iff x $\not\leq$ y and x $\not\geq$ y. It is denoted by x $\nsim$ y). When we find images for all elements A in P; then f:P→P is characterized as fixed point free and consequently, the ordered set P is (FPF). At this point, there is no need for further testing of the remaining self maps in P. However, when f: can not be completely defined; after testing all self maps, P is considered to have the fixed point property (FPP). In addition, we refined the range Q of elements A, by including the property of the order preserving map which is if x≤y ⇒ f(x)≤f(y). Thus, the range of an element x ≤y should be included in the set of comparable elements to z = f(y), since f(x) ≤ f(y). The details of this algorithm, known as the Dynamic Range Technique, are given in Chapter 5 together with appropriate examples.

The final result is a set of vectors with size 10, 11, 12, and 13; that holds all ordered sets with fixed point property (up to isomorphism and duality) irreducible element free , retractable element free.

1.2 Motivation and Objectives

This work is driven by an interest in the behavior of ordered sets and their applications.

The work is particularly interesting because of the special computational techniques required to handle the non-deterministically polynomial (NP) of ordered sets decision problems.

The objectives of the research are:

1. To contribute to a better understanding of the behavior of ordered sets

2. By means of utilizing numerical techniques, to advance the state-of-knowledge of ordered sets by expanding the size of characterized ordered sets with FPP, and documenting the implemented algorithms and techniques.

1.3 Thesis contribution

The results of this research can alleviate and stimulate further efforts leading to enumerating ordered sets with the FPP of size larger than 11 by means of non-computational methods.

Primarily, the thesis introduces algorithms for FPP characterization, for isomorphic sets elimination, and generation of ordered sets of a given size;

A standard representation for isomorphic ordered sets is introduced, together with a technique to convert any ordered set into its standard representation, herein denoted "minimal representation".

An alternative algorithm based on "4-nodes crown retract" is implemented. It represents an efficient technique for the FPP characterization of particular cases of ordered sets of large size.

1.4 Organization

The thesis is organized as follows:

Chapter 1 is an introduction providing an overview of the history of ordered sets and their connection with different fields, and describing the objectives and structure of the research project. An outline of the research methodology is presented, together with the required definitions of the terms comprising the main topic of the research.

Chapter 2 describes the subject problem in details, and the solution strategy and its limitations. The computational approach methodology is presented starting with modeling, followed by algorithm development application and validation. The chapter includes a summary of the ordered sets research background and some related results.

Chapters 3 focuses on the techniques for generating ordered sets of a specific size, ready for FPP testing. It also covers the generation and selection criteria used to limit the generation of potential FPP candidate sets.

Chapter 4 details the concepts, methodology, techniques and algorithms used to eliminate redundancy in the generated ordered sets list by identifying and eliminating isomorphic ordered sets. An algorithm for testing if a given ordered set is in its minimal form is presented.

Chapter 5 presents the implementation of an algorithm used for FPP testing. Examples for explaining the algorithm and validating it, are also presented.

Chapter 6 presents special case techniques that can be applied on particular ordered sets. These techniques are implementations of some technical lemmas.

Chapter 7 presents a summarized account of the research results, followed by the relevant conclusion. An assessment of the success of this effort in meeting the research objectives is presented, together with the appropriate recommendations for future research.

Chapter 2

2 THE PROBLEM CONTEXT

2.1 Introduction

In this chapter, we describe the problem according to the computational approach we selected versus a mathematical technique. A meaningful description of the problem requires at least a brief coverage of the ordered set properties. Likewise, we find it very helpful to present some related published results.

2.2 Ordered Set, Definition

An **ordered set** or a **partially ordered set** or a **poset** P is a pair (P, $\mathscr{R}$) where P is a set, and $\mathscr{R}$ is an order binary relation on P, satisfying the conditions.

- Reflexivity: For all $x \in P$ we have $(x, x) \in \mathscr{R}$.
- Antisymmetry: For all $x, y \in P$. If $(x, y) \in \mathscr{R}$ and $(y, x) \in \mathscr{R}$ then $x = y$.
- Transitivity: For all $x, y, z \in P$. If $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R}$ then $(x, z) \in \mathscr{R}$.

The notations $(x,y) \in \mathscr{R}$, $x \mathscr{R} y$, $x \le y$ in P, and $y \ge x$ in P are used interchangeably. The notation $x < y$ in P means $x \le y$ in P and $x \ne y$.

Distinct elements $x \in P$ and $y \in P$ are **comparable**, when $x \leq y$ or $y \leq x$ in P, we denote by $x \sim y$. Otherwise, we say x and y are **incomparable** and we denote by $x \nsim y$.

Ordered sets can be represented in different instances. We choose to describe the representations that are relevant to our work.

2.2.1 Incidence Matrix

An **incidence matrix** (IM) is an instance of ordered set representation. For an ordered set P or $(P, \mathscr{R})$ with N elements or $|P| = N$. IM is a square matrix of N rows x N columns.

We label elements of P such that $P = \{ a_0, a_1, a_2, \ldots, a_{N-2}, a_{N-1} \}$

Rows and columns in IM are labeled from (0 to N-1) .

$\mathscr{R}$ is the binary relation in $P \Rightarrow \mathscr{R} = \{ (a_i, a_j) \in P^2 \ / \ a_i \leq a_j \}$.

If $(a_i, a_j) \in \mathscr{R} \Rightarrow$ the cell IM(i, j) = 1, otherwise, $(a_i, a_j) \notin \mathscr{R} \Rightarrow$ IM(i, j) = 0.

Eventually, IM is not a unique representation of $(P, \mathscr{R})$. It all depends on how we label the elements in P. However, the partial order in P contributes to grouping P elements into subsets.

By following the partial order in P and quasi sort P elements according to $\mathscr{R}$; IM (incidence matrix) can be transformed into the sum of an upper triangle matrix and a diagonal matrix (Schur)[8] IM = M + D.

If we suppress reflexivity, by treating only the strict order relation, we obtain as the remaining matrix, an **upper triangular matrix.**

a_{10}	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0	
1	0	0	1	1	0	1	1	1	1	1	a_{10}
0	1	0	1	0	1	1	1	1	1	1	a_9
0	0	1	0	1	1	1	1	1	1	1	a_8
0	0	0	1	0	0	1	1	1	1	1	a_7
0	0	0	0	1	0	1	1	1	1	1	a_6
0	0	0	0	0	1	0	0	1	1	0	a_5
0	0	0	0	0	0	1	0	1	0	1	a_4
0	0	0	0	0	0	0	1	0	1	1	a_3
0	0	0	0	0	0	0	0	1	0	0	a_2
0	0	0	0	0	0	0	0	0	1	0	a_1
0	0	0	0	0	0	0	0	0	0	1	a_0

Figure 2-1 Incidence matrix with 4 levels

In addition, if we represent only the covering relation $\prec$ instead of the comparability relation, we obtain the **covering matrix**:

Given any pair $x \in P$ and $y \in P$, we say that x is a **lower cover** of y, or y is an **upper cover** of x , if $x < y$ and there is no $z \in P$ with $x < z < y$, we denote by $x \prec y$.

2.2.2 Upward Drawing

The most common scheme for drawing an ordered set P is a graph G whose vertices V correspond to the elements of P, and whose edges correspond to the covering relations in P. This graph is called the **directed covering graph.**

It is always possible to orient the directed covering graph in such a way that all arrows point upward. When it is done, we erase all arrows and we get what we call an **upward drawing** (Figure 2.2).

Eventually, this upward drawing is not unique. We can draw different instances from the same ordered sets by applying permutation on vertices. All these instances for the same ordered set are **isomorphic**.

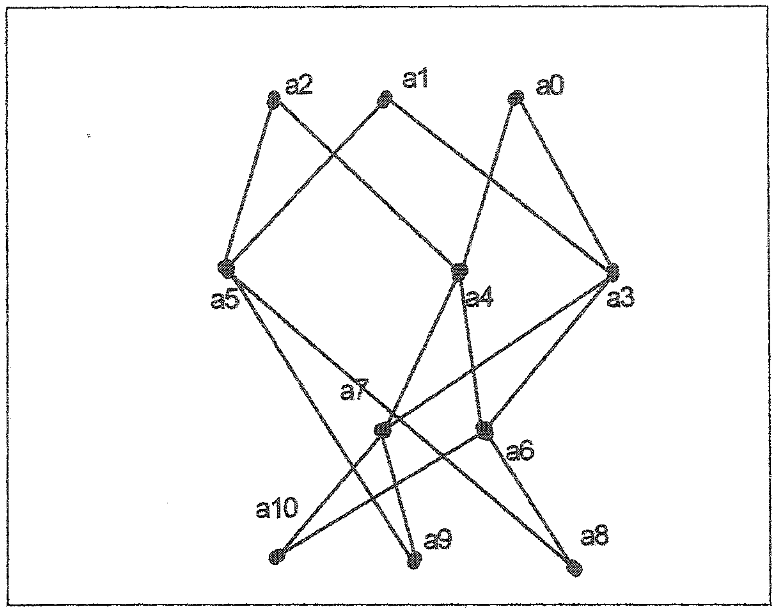

Figure 2-2 Upward drawing graph for Q

There are some specific notations that we need to cover to facilitate the description of the problem. We will use the upward drawing example for a clearer description

2.2.2.1 Up-Set, Down-Set

For **x** ∈ P the **up-set** of **x** is defined and denoted by $(\uparrow \mathbf{x}) = \{\mathbf{y} \in P \,/\, \mathbf{y} \geq \mathbf{x}\}$.
The **down-set** of **x** is defined and denoted by $(\downarrow \mathbf{x}) = \{\mathbf{y} \in P \,/\, \mathbf{y} \leq \mathbf{x}\}$.

For instance, based on the ordered set shown in Figure 2.2, $(\uparrow a_7) = \{ a_0, a_1, a_2, a_3, a_4, a_7 \}$; $(\downarrow a5) = \{ a_5, a_8, a_9 \}$.

2.2.2.2 Maximal Element

Let P be an ordered set. An element **m** $\in$ P is called **maximal** iff there are no elements **a** $\in$ P with **m** < **a**. We denote the set of maximal elements of P by Max(P). e.g. : Max(Q) = $\{ a_0, a_1, a_2 \}$

2.2.2.3 Minimal Element

Let P be an ordered set. An element **m** $\in$ P is called minimal iff there are no elements **a** $\in$ P with **a** < **m**.We denote the set of minimal elements of P by Min(P). e.g. : Min(Q) = $\{ a_8, a_9, a_{10} \}$

2.2.2.4 Upper Bound

Let P be an ordered set. If A $\subseteq$ P, then **u** $\in$ P is called an upper bound of A iff for all **a** $\in$ A we have **u** $\geq$ **a**. e.g.: for A= $\{ a_6, a_7, a_8, a_9 \}$ a_3 is an upper bound.

2.2.2.5 Lower Bound

Let P be an ordered set. If A $\subseteq$ P, then $\ell \in$ P is called a lower bound of A iff for all **a** $\in$ A we have $\ell \leq$ **a**. e.g.: for A= $\{ a_0, a_1, a_2, a_4, a_5 \}$ a_9 is a lower bound.

2.2.2.6 Supremum

Let P be an ordered set and let A $\subseteq$ P. Then the element **u** is called the **lowest upper bound** or **supremum** of A iff **u** is an upper bound of A and for every other upper bound **v** of A, **u**$\leq$**v**. We denote **u** = $\vee$ A or $sup_P(A)$.

e.g. : A= $\{ a_7, a_9, a_{10} \}$ then $sup_Q(A) = a_7$

2.2.2.7 Infimum

Let P be an ordered set and let $A \subseteq P$. Then the element ℓ is called the **greatest lower bound** or **infimum** of A iff ℓ is a lower bound of A and for every other lower bound v of A, $v \leq \ell$. We denote $\ell = \wedge A$ or $\inf_P(A)$.

e.g. : $A=\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ then $\inf_Q(A) = a_8$.

2.2.3 Fixed Point Property / Order preserving map

2.2.3.1 Order-Preserving Map (OPM)

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be ordered sets, and let $f: P \rightarrow Q$ be a map, then f is called an **order preserving map** (OPM) iff for all $x \in P$, and for all $y \in P$ we have:

$$x \leq_P y \Rightarrow f(x) \leq_Q f(y).$$

If P = Q, then $\leq_P = \leq_Q = \leq$ $f: P \rightarrow P$ is called an order preserving self map.

2.2.3.2 Fixed Points Property

Let $(P, \leq)$ be an ordered set, and let $f: P \rightarrow P$ be an order-preserving map.

Then $x \in P$ is called a **fixed point** of f iff $f(x) = x$. If f has no fixed point, f is called **fixed point free.**

P is said to have the **fixed point property** (FPP) iff each order-preserving map (OPM) $f: P \rightarrow P$ has a fixed point. Otherwise, P is fixed point free (FPF)

2.3 *Computational approach*

An ordered set can also be defined as a non-numerical data structure. This non-numerical property favored a modeling approach for a computational methodology. Since the problem is of (NP) nature, the model we conceive should contribute to the efficiency of the computational algorithm. Furthermore, a validation process is applied on the result we obtained before it is considered final. Consequently, the methodology we implemented is divided into three main phases:

a. Modeling

b. Algorithm design

c. Validation case and processing.

2.3.1 Modeling

Choosing the right model depends on a broad understanding of the problem and its ramifications. Thus, we can describe the problem in more details based on the solution strategy we implemented.

a) Generate ordered sets for specific sizes ≤ 13.

b) Eliminate duals and isomorphic sets.

c) Generate and test each order preserving map for (FPF), which consequently leads to (FPP) characterization.

d) Validate the collected results.

Accordingly, a model for an ordered set should satisfy some basic conditions, and matches all phases in the solution strategy. A key element in this model design is to add a numerical dimension to an ordered set, which enormously contributes to the computational process.

In an early stage of this project, we designed a model, which we called " Prime Number Model". In fact, we implemented this model in an algorithm, however, it did not make part of the final implementation and a different model replaced it. We have decided to describe it in this thesis and cover its advantages and disadvantages because of its relation to the final model.

2.3.1.1 Prime number model

We originally introduced the prime numbers model for a single goal: the order preserving maps generation and the fixed point property testing; when the ordered sets generation phase was not the main concern. The following is a mathematical description of the Prime number model for ordered sets.

Let P be an ordered set. For each element $\mathbf{a}_i \in P$, we assign a unique prime number $\mathbf{q}_i$. We consider the mapping:

h: P $\rightarrow$ N where N is the set of positive integers and

$$h(\mathbf{a}_i) = \mathbf{t}_i = \prod_j^{0 \rightarrow N-1} (\mathbf{d}_j) \;/\; (\mathbf{d}_j = \mathbf{q}_i \text{ if } \mathbf{a}_j \leq \mathbf{a}_i) \text{ and } (\mathbf{d}_j = 1 \text{ if } (\mathbf{a}_j \nleq \mathbf{a}_i)\,.$$

Obviously, this mapping h:P $\rightarrow$ T, where T is the image of P by h, is a **bijection.**

Clearly, the divisibility relation S on T (for x and y in T, say xSy if y is divisible by x) is a partial ordering relation equivalent to the ordering relation on P. Therefore, all processing on P is equivalent to processing on T.

2.3.1.2 Model property

The advantage of this model is highlighted in the values of each element image. Each element in T holds all the information we need about its comparability with other elements regardless of the label or index of the element. This property makes permutation generation much simpler. In fact, this model was specifically designed for order preserving maps generation and processing.

Based on the isomorphism property of h: where $a \leq b \Rightarrow h(a) \leq h(b)$. A division of h(a) by h(b), can provide the result whether $h(a) \leq h(b)$ or not, therefore, h: can be easily characterized for (OPM).

The disadvantage of this model appears during the processing of large ordered sets with higher sizes. For instance, an ordered set with 13 elements h(a) = t can reach a value of:

$$2.3.5.7.11.13.17.19.23.29.31.37.41 = t = 304{,}250{,}263{,}527{,}210 \Rightarrow t > 3.\ 10^{14}$$

which is not practical for larger ordered sets.

Moreover, we encountered some difficulties in implementing ordered sets systematic generation and isomorphic sets elimination because of the lack of numerical order needs in this structure. In fact, we can start with any prime number we choose from a bin of prime numbers, as long as it is utilized for the first time. These limitations motivated the design of a different model which we called "Incidence vector model".

2.3.1.3 Incidence Vector Model

The incidence vector model is based, as its name shows, on the ordered set incidence matrix.

Eventually, each element $x_i \in P$ corresponds to a row R_i in the incidence matrix IM. This same row can be formatted as a single binary number, by concatenating 0's and 1's in its cells to compose a single decimal number.

As a result, we get a vector of decimal numbers as images for each element in P. This vector is the range T of P in the codomain $N / T \subset N$.

Notice that $a_i \leq a_j$ in T is equivalent to $a_i \wedge a_j = a_i$ (We denote $a_i \wedge a_j$ as the Boolean logic AND on a_i , a_j . $a_i \vee a_j$ is the Boolean logic OR on a_i, aj)

The incidence vector model is a compatible model (isomorphism).

According to the property of incidence matrix. This matrix is an upper triangular with diagonal = 1.

The first row can only be = 1.

The second row can be 10 or 11

The third row can be 100, 101, 110, or 111

The row n can be any value between 2^r and $(2^{r+1} - 1)$, where r is the row rank.

If we suppress the diagonal then we obtain an incidence matrix with a strict relation <. Any row_r merged value can be between 0 and $(2^r - 1)$.

The element $b_i = f(a_i)$ in T represents the *ith* coordinate of a vector MV or Model vector $b_i = \sum_{j=0}^{N} (2^j . a_{i,j})$.

2.3.1.4 Incidence Vector Model property

It is very easy to obtain this model from the incidence matrix. It is also very simple for generation and comparability. Each coordinate in the incidence vector model, which is a decimal number, holds all needed information about its comparability in the form of binary digits. This model will be the model we utilize through the whole project.

2.3.2 Algorithm design

The corner stone of this project is undoubtedly the algorithm design and implementation. Therefore, we dedicated three different chapters to describe in detail each of the main algorithms: **ordered sets generation**, **isomorphism elimination**, and **(FPP) testing**. The same model commonly links these three algorithms. In fact, in our case, the model choice and algorithm design are mutually related. This model property contributed enormously in the success of these three algorithms. Because of the very large number of possible ordered sets for a specific size, we limited our generation processing to size 14; however, the generation technique does not have a design limitation beside the processing time and CPU resources, but the implementation and coding was limited to size 16. Likewise, the (FPP) characterization was successfully tested for size 16 within a window of a reasonable processing time. On the other hand, the isomorphism elimination implementation was limited to size 14, due excessive possible permutations required for processing. Consequently, these algorithms have limitations, which is eventually expected for a problem of (NP) nature.

2.3.3 Validation

The validation phase is required with this type of project, especially because the result set is unknown. The only hint was the published results of

Rutkowski and Schröder for size 10 and 11. For this reason, we implemented a validation tool set, for testing of all results we obtained, using all possible permutation techniques, which eliminates design errors in the algorithms. These validation algorithms added some limitations on the size, and set the maximum size of the validated results to ≤13, because of the excessive processing needed for all-permutation technique.

2.4 *Related published results (Rutkowski and Schröder)*

We selected the results of Rutkowski in his paper "*The fixed point property for small sets*" and Schröder in his paper "*Fixed point property for 11-Element sets*" as a starting point for our work, and as a validation set for size ≤ 11. Due to the tied connection between this project and these papers, we decided to extract some materials from them, and present them for the benefit of the reader. Both results are up to isomorphism and duality, which are two graph properties that can be defined as:

2.4.1 Duality

The **dual** of an ordered set P on a binary relation $\mathscr{R}$, is denoted by:

P^d and is defined by $P^d = \{(y,x) / (x,y) \in P\}$.

2.4.2 Isomorphism

Let P, Q be ordered sets and let f: P →Q. Then f is called an order-isomorphism iff:

a) f is order preserving

b) f has an inverse f^{-1}.

c) f^{-1} is order-preserving.

The ordered sets P and Q are called order-isomorphic iff there is an isomorphism f:P→Q. The following characterization of isomorphisms reinforces the notion that isomorphic ordered sets can be regarded as "the same".

Proposition 1: Let P, Q be ordered sets. Then f:P→Q is an order-isomorphism iff

a) f is bijective.

b) For all p1, p2 ∈ P we have $p1 \leq p2 \Leftrightarrow f(p1) \leq f(p2)$.

Proposition 2: Let P, Q, R be ordered sets and let f: P→Q and g:Q→R be order-isomorphisms. Then $g \circ f$ is an order-isomorphism.

2.4.3 The fixed point property for small sets (Rutkowski).

Theorem [31]: *There exist exactly eleven (up to isomorphism and duality) ordered sets of size ≤ 10 with the fixed point property and containing no irreducible elements. They are the singleton, and the ten sets given in Fig2.3*

Technical **Lemma 2**[31]: *Let P be a finite ordered set. If P = A ⊕ B then P has the fixed point property (FPP) if and only if A or B has the (FPP)*

For subsets A and B of P, denoted by A < B the situation when x < y for each x ∈ A and y ∈ B. $P = A \oplus B$ if $P = A \cup B$ and $A \neq \emptyset \neq B$ and $A < B$.

2.4.4 Chain

A subset C of an ordered set (P, ≤) is called a **chain** iff ≤ is a total order in C. When C equals P, the ordered set P is called a **chain** as well.

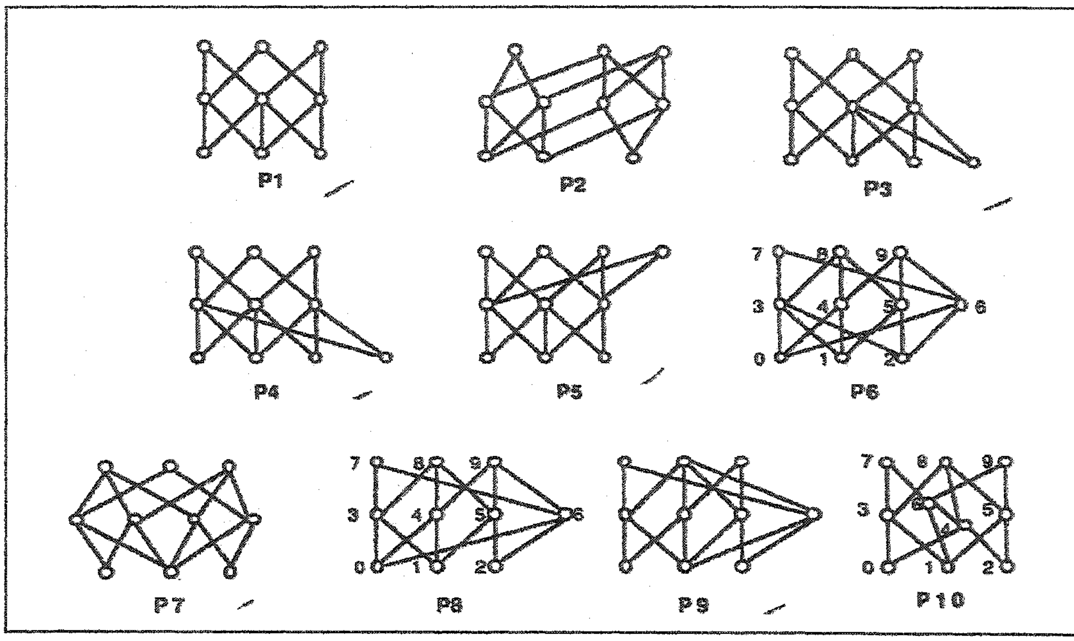

Figure 2-3 Ruthkowski ordered sets with the FPP

2.4.5 Antichain

Let **A** be a subset of an ordered set (P, ≤). **A** is called an **antichain** iff all elements in **A** are non comparable pair-wise.

2.4.6 Length

Let P be an ordered set. The length or the height of P is the length of the longest chain C in P, or max|C| -1.

2.4.7 Crown

Let **n** be an even integer number ≥ 4. An **n-crown** is an ordered set C_n with element set $\{c_1, c_2, \ldots, c_n\}$ such that $c_1 < c_2$, $c_2 > c_3$, $c_3 < c_4, \ldots, c_{n-2} > c_{n-1}$, $c_{n-1} < c_n$, $c_n > c_1$ are the only strict comparabilities.

2.4.8 Fence

Let P be an ordered set. An (n+1)-Fence is an ordered set $P = \{A_0, A_1, A_2,..,A_n\}$ such that $A_0 > A_1$, $A_1 < A_2$, $A_2 > A_3$, $A_3 < A_4$,..$A_{n-1} > A_N$ if N is even, respectively $A_0 < A_1$, $A_1 > A_2$, $A_2 < A_3$, …, $A_{n-1} < A_n$ if n is odd, and such that these are all comparabilities between the points. The length of the fence is n.

2.4.9 Irreducible element

Let P be an ordered set, $x \in P$; x is considered to be **irreducible** in P if it has unique upper cover or unique lower cover. The set of all irreducibles in P is denoted by Ir(P).

Theorem: The removal of irreducible elements preserves the (FPP) as well as the lack of it [27]. This property helped reduce the size of the generated ordered sets; where we can eliminate ordered sets with irreducible elements.

Technical **Lemma** 1[31]: Let P be a finite ordered set. Then

a) Each level of P is Antichain

b) If $x \in P_n$, then there exists a chain $x_0 < x_1 < x_2 < .... < x_n = x$ such that each x_i has rank i.

c) If $x \in P_n$ then $k < n$ then there exists $y \in P_k$ such that $y < x$

Technical **Lemma** 4[31]: Let P be a finite connected ordered set of length 1 with Ir(P)=∅, then either P is singleton or P has no fixed point property.

Technical **Lemma** 5 [31]: Let a <c; a <d; b<c; b<d; a, b ∈Min(P) d ∈Max(P) a ≠ b, d ≠ c. then {a,b,c,d} is a 4-crown and P has a retraction onto this crown (and consequently, it does not have the (FPP)).

Technical Lemma 6 [31]: Assume P to have the following properties:

a) Ir(P) = ∅, L(P)>1, |P|≤10, where L(P) the length of a finite nonempty chain.

b) There exist a ∈Min(P) and b∈Max(P) such that a < b.

Then P does not have the fixed point property.

2.4.10 Fixed point property for 11-Element sets (Schröder)

Theorem: *The ordered sets shown in Figure 2.4 are up to isomorphism and duality all ordered sets with 11 elements that have the fixed point property and no retractable point.*[34].

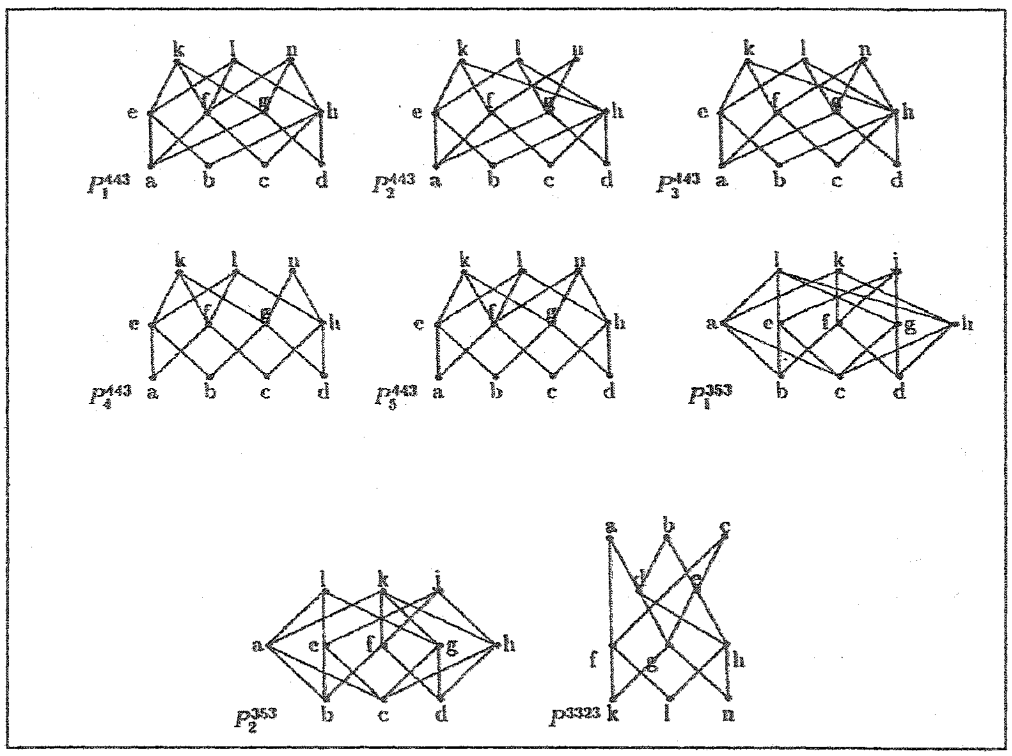

Figure 2-4 Schröder sets FPP size 11

2.4.11 Retracts

A subset Q of an ordered set P is a **retract** of P if there is an order-preserving map (OPM) f: P → Q, which is the identity map on Q. (We use this term up to isomorphism). We call f a retraction map. The class of all (isomorphism types of) retracts of P we denote by $\mathbb{R}(P)$.

2.4.12 Retractable Points

Let P be an ordered set and let $a \in P$ and $b \in P$. a is called **retractable** (to b) iff $a \neq b$ and $(\downarrow a) \setminus \{a\} \subseteq (\downarrow b)$ and $(\uparrow a) \setminus \{a\} \subseteq (\uparrow b)$.[34]

Retractable points have similar behavior to irreducible elements in what is related to the (FPP).

Lemma: Let P be a poset

a) If $a \in P$ is irreducible, then a is retractable to its unique upper, resp., lower cover.

b) Let $a \in P$ be retractable to $b \in P$. Then $a \leq b$ iff b is the unique upper cover of a.

c) Let $a \in P$ be retractable to P. Then r: P →P\{a} defined by r(q):{ q; if q ≠a and b; if q =a, } is a retraction.

d) Let r: P→P\{a} be a retraction. Then a is retractable to r(a).

e) P\{a} is a retract of P iff a is retractable.

Theorem: Let P be an ordered set and let $a \in P$ be retractable to $b \in P$. Then P has the fixed point property iff:

a) P – {a} = P\ {a} has the fixed point property.

b) One of (↑a)\{a} and (↓a)\{a} has the fixed point property.

Proof: "⇒" Assume that P has the fixed point property. a) is a consequence of the previous Lemma part c). To prove b) assume that (↑a)\{a} and (↓a)\{a} don't have the fixed point property.

Let g: (↑a)\{a}→(↑a)\{a} and h: (↓a)\{a}→: (↓a)\{a} be order-preserving maps without fixed points. Define

f(q)= { g(q): if q ∈ (↑a)\{a}; h(q): if q ∈ (↓a)\{a}; a: if q is not comparable to a; b: if q = a }.

f:P→P has no fixed point. And *f*: is order preserving map. Which leads to a contradiction.

" ⇐"Let *f*: P→P be an order preserving map. We will show that *f* has a fixed point. Let r:P→P\{a} be as in the previous Lemma. Then:

r∘(*f*(P\{a}):P\{a}→P\{a} is order preserving and thus has a fixed point q. In case q ≠ b we have that q=r∘(*f*(P\{a}))(q)= *f*(P\{a})(q); and we are done. In case q=b we either have *f*(b)=b or *f*(a)=a. In the first case we are done. In the second case assume that (↑a)\{a} has the fixed point property . Then (↑a)\{a}≠ ∅. Since *f*(b)=a we have that *f*[(↑a)\{a}]⊆*f*[↑b] ⊆(↑a).

In case *f*[(↑a)\{a}]⊆(↑a)\{a} we have by assumption that *f* has a fixed point in (↑a)\{a} and more done. Otherwise there is a q ∈(↑a)\{a} such that *f*(q)=a <q. Now if *f*(a)=a we are done. This leaves the case d= *f*(a)<a. In this case *f* maps (↓d) to itself. Yet ↓d is a retract of P\{a} via s(x)= {x: if x≤d;

d:otherwise} Thus (↓d) has the fixed point property, which implies that *f* has the fixed point in (↓d).

2.5 *Related Theorems , Lemmas, and Properties*

- The problem of deciding whether a finite ordered set is a fixed point free (FPF) was found by Williamson to be NP-Complete [41].
- Every dismantlable ordered set has the fixed point property (Rival)[28].
- Every complete lattice has the fixed point property (Knaster-Tarski) [38].
- Let P be an ordered set with the fixed point property. Then P is connected. [36].
- Let P be a disconnected ordered set. Then P is fixed point free (FPF)

2.5.1 Connectivity

Let P be an ordered set. P is called connected iff for all **a**, **b** $\in$ P there is a fence F $\subseteq$ P with endpoints **a** and **b**. An ordered set that is not connected is called disconnected.

Chapter 3

3 ORDERED SETS GENERATION

3.1 Introduction

The generation of ordered sets is not a trivial problem, especially when the generation is selective. Some published results showed the size of the problem.

The most difficult part in the generation is the elimination of the isomorphic sets. This table below shows a sample of published results (Figure 3-1)

$0 \leq n \leq 6$ Pn = 1; 1; 2; 5; 16; 63; 318;			
P7	2045	1972	J. Wright
P8	16999	1977	S.K. Das
P9	183231	1984	R.H. Mohring
P10	2567284	1990	Culberson & Rawlins
P11	46749427	1990	Culberson & Rawlins
P12	1104891746	1991	Chaunier and Lygeros
P13	33823827452	1992	Chaunier and Lygeros
P14	1338193159771	2000	Heitzig and Reinhold
P15	68275077901156	2002	Brinkmann & McKay
P16	4483130665195087	2002	Brinkmann & McKay

Figure 3-1 numbers of different non-isomorphic orders

In (Figure 3-2) we present the total number of all possible ordered sets.

P1	1
P2	3
P3	19
P4	219
P5	4,231
P6	130,023
P7	6,129,859
P8	431,723,379
P9	44,511,042,511
P10	6,611,065,248,783
P11	1,396,281,677,105,899
P12	414,864,951,055,853,499
P13	171,850,728,381,587,059,351
P14	98,484,324,257,128,207,032,183
P15	77,567,171,020,440,688,353,049,939
P16	83,480,529,785,490,157,813,844,256,579
P17	122,152,541,250,295,322,862,941,281,269,151

Figure 3-2 Total number of ordered sets of size n

This second table shows the magnitude of the generation problem size. Therefore, the generation technique should dynamically eliminate ordered sets that are not potentially candidates to have the (FPP). A necessary condition for an ordered set P to have the (FPP) is the connectivity. This condition eliminates an important part of the whole possible set. We present the global criteria for the generation which is based on the elimination of most of the trivial cases.

3.2 *Generation criteria*

Since each row in the incidence vector model or IV has an upper bound; coordinates in IV can be generated according to this range.

The main objective in generating all ordered sets of a given size; is to characterize which of them have the (FPP).

We will focus only on non-trivial cases and eliminate most of the trivial cases during the generation process.

3.2.1 Infimum and Supremum

Eventually, we should eliminate ordered sets P that have infimum or supremum.(these 2 classes of ordered sets always have (FPP)).

3.2.2 Irreducible elements

In addition, ordered sets with irreducible elements can be eliminated because if **a** is an irreducible element in P. If P\{a} has (FPP) $\Rightarrow$ P has the (FPP); which means, characterizing P of size N falls in the class of P^1 of size N-1 and P^1 should be compared to results of size (N-1), then induces the result to P.

3.2.3 Retractable elements

Retractable elements have similar behavior to irreducible elements. However, the related theorem is slightly different.

Theorem [34]: Let P be an ordered set and let $a \in P$ be a retractable to $b \in P$. Then P has the fixed point property iff

1) $P \setminus \{a\}$ has the fixed point property, and
2) One of $(\uparrow a) \setminus \{a\}$ and $(\downarrow a) \setminus \{a\}$ has the fixed point property.

This theorem also means that, if $P \setminus \{a\}$ (FPF) $\Rightarrow$ P is (FPF)

Proof: Let $Q = P \setminus \{a\}$. Q is (FPF) $\Rightarrow \exists f{:}Q \rightarrow Q \,/\, \forall c \in Q \;\; f(c) \neq c$.

Let g: $P \rightarrow P$ such that, for all $d \in Q \Rightarrow g(d)=f(d)$ and $g(a)=f(b)$ $\Rightarrow$. Clearly g: is order preserving map. g is a (FPF) too and P is (FPF).

If $P \setminus \{a\}$ has (FPP), an extra test needs to be performed on P to satisfy (FPP) condition for P. This test is the condition 2- in the previous theorem.

For these reasons, we limit our ordered set generation to only ordered sets with irreducible elements free and retractable elements free.

3.2.4 Connectivity

Connectivity in ordered sets plays an important role in characterizing (FPP). In fact, all disconnected ordered sets are (FPF).

Proof: Let P be a disconnected ordered set. P can be written as the union of 2 disjoint components V and W such that $V \cap W = \emptyset$ and $V \cup W = P$. Let $\mathbf{x} \in V$ and $\mathbf{y} \in W$, and let **g**: $P \rightarrow P$ where for every $\mathbf{a} \in P$ if $\mathbf{a} \in V$ then $\mathbf{g}(\mathbf{a}) = \mathbf{y}$; otherwise, if $\mathbf{a} \in W$ then $\mathbf{g}(a) = x$. Clearly, **g** OPM and (FPF) .

Therefore, it is not necessary to test any of the disconnected ordered sets for (FPP). However, any generated ordered set has to be tested for connectivity.

A first necessary condition for a graph G connection is that the number of edges ≥ (number of nodes −1). This condition can be easily tested. It is a necessary condition, so it does not mean that when it is true ⇒ G is connected.

Moreover, ordered sets connectivity has some particularity compared to general graph connectivity, which makes connectivity testing easier than known connectivity graph testing. In fact, an ordered set P is an upward graph. Each node $A_i \in P$ belongs to at least one maximal chain C_j . Each maximal chain has an upper bound, which is a maximal node in the ordered set.

This connectivity test technique is based on testing connectivity between maximal chains.

Let P be an ordered set. Let Q = Max(P), set of all maximal elements in P.

For all $\mathbf{m}_i \in Q$, $\mathbf{m}_j \in Q$, consider the two down-sets $M_i = (\downarrow \mathbf{m}_i)$ and $M_j = (\downarrow \mathbf{m}_j)$.

If $\mathbf{M_i} \cap \mathbf{M_j} \neq \varnothing$

In this case , we merge the 2 down-sets in a single component V_{ij} such that

$V_{ij} = (M_i \cup M_j) \neq (\downarrow(\{\mathbf{m}_i, \mathbf{m}_j\}))$

V_{ij} can be denoted as $V(\{\mathbf{m}_i, \mathbf{m}_j\})$

Then we test connectivity with the next maximal element $\mathbf{m}_k$. $M_k = (\downarrow \mathbf{m}_k)$

If $V(\{\mathbf{m}_i, \mathbf{m}_j\}) \cap M_k \neq \varnothing \Rightarrow V(\{\mathbf{m}_i, \mathbf{m}_j\}, \mathbf{m}_k)$ exists, and

$V(\{\mathbf{m}_i, \mathbf{m}_j\}, \mathbf{m}_k) = V(\{\mathbf{m}_i, \mathbf{m}_j\}) \cup M_k) = V(\{\mathbf{m}_i, \mathbf{m}_j, \mathbf{m}_k\})$. We proceed with this test until we cover all maximal elements in Q.

If ($\mathbf{M_i} \cap \mathbf{M_j}$) = $\varnothing \Rightarrow V_{ij}$ does not exists

We skip $\mathbf{m_j}$, and we perform the test on $\mathbf{m_i}$ and the next available maximal element. {$\mathbf{m_i}$, $\mathbf{m_k}$}.

If all maximal elements fail to share elements with ($\downarrow\mathbf{m_i}$), it means that M_i component is not connected to any of the maximal elements, Consequently, P is disconnected.

($\mathbf{M_i} \cap \mathbf{M_k}$) $\neq \varnothing$

We build a new component based on ($\mathbf{m_i}$, $\mathbf{m_k}$).

Eventually, during this loop process, we may drop some of those maximal elements in Q, because of an empty intersection result with the main component V({ $\mathbf{m_1}$, $\mathbf{m_2}$,..., $\mathbf{m_k}$ }). In this case, some extra round are necessary to test if these remaining ($\downarrow\mathbf{m_j}$) are now ready to join a wider component with the condition V({ $\mathbf{m_1}$, $\mathbf{m_2}$,..., $\mathbf{m_k}$ }) $\cap$ ($\downarrow\mathbf{m_j}$) $\neq \varnothing$. In some cases, |Q| -1 iterations are required to complete this test. These extreme cases represent a very small population of the generated ordered sets.

Proposition: P is connected iff V(Q) = P, where Q = Max(P)

Proof: Let P be an ordered set

V(Q) is the subset down-set of Q

- P is connected $\Rightarrow$ V(Q) = P

Assume that V(Q) $\neq$ P and V(Q) $\subset$ P

then, there exists x $\in$ P where x $\notin$ V(Q), and there exists $\mathbf{m_j}$ $\in$Max(P) such that $\mathbf{m_j} \geq$ x

On the other hand, $x \in (\downarrow \mathbf{m}_j)$ and $(\downarrow \mathbf{m}_j) \subset V(Q)$ then $x \in V(Q)$, which leads to a contradiction

$\Rightarrow P \subset V(Q)$ and by definition $V(Q) \subset P \Rightarrow P = V(Q)$.

- If **P = V(Q) ⇒ P connected**

Assume that $P = V(Q)$ and P is disconnected

There exists at least 2 components C and D in P and there exist $\mathbf{m} \in C \cap Q$ and $\mathbf{n} \in D \cap Q$

Assume that there are only 2 components C and D where $Q = Q_C \cup Q_D$ such that Q_C is the set of maximal elements in C and Q_D is the set of maximal elements in D

If we start building V(Q) with only elements from C and we obtain $V(Q_C)$. The next step is to have at least one maximal element **m** from Q_D to joint $V(Q_C)$.

For instance, $V(Q_C, \{\mathbf{m}\}) = V(Q_C) \cup (\downarrow \mathbf{m})$ only if $V(Q_C) \cap \{\downarrow \mathbf{m}\} \neq \varnothing$.

However, if $V(Q_C) \cap \{\downarrow \mathbf{m}\} = \varnothing$, we have to find at least an element $\mathbf{n} \in D$ that can join V(Q) and then if $(\downarrow \mathbf{m}) \cap (\downarrow \mathbf{n}) \neq \varnothing$ **m** can join V(Q).

Obviously, **n** should exist because we know that $\mathbf{m} \in V(Q)$ where $V(Q) = P$. Since **n** is the trigger for **m** to join V(Q) then $V(Qc) \cap \{\downarrow \mathbf{n}\} \neq \varnothing$. As a result, there exist $\mathbf{x} < \mathbf{n}$ and $\mathbf{x} < \mathbf{k}$ such that **k** is a maximal element in C. Therefore, there is a chain between **x** and **k** where $\mathbf{x} \in C$ and there is a chain between **x** and **n** where $\mathbf{n} \in D \Rightarrow \mathbf{x} \in D \Rightarrow C \cap D \neq \varnothing \Rightarrow$ P is connected.

If P has more than 2 components, we apply the same reasoning with a simple difference; we should apply it on all permutations of components, which is longer to prove but the same idea.

At the implementation level, $\cap$ and $\cup$ are Boolean operators $\wedge$ and $\vee$, for columns in the incidence matrix. If we apply the same technique on the minimal elements rather than on the maximal elements, the same Boolean operations $\wedge$ and $\vee$ can be applied on the rows of the incidence matrix, with no needs for columns generation; because, during the generation process, only rows are generated. .

3.2.5 Duality

An important property of the duality is that it conserves the (FPP), therefore, characterizing P and P^d is a redundancy. If P^d testing occurs before P then we can ignore P testing.

All these criteria will be integrated in the next step to make ordered sets generation more efficient for (FPP) characterization.

3.3 Generation algorithm

After describing in detail the incidence vector model IV, in this chapter we exhibit a technique to generate ordered sets of a given size with certain conditions.

The problem of generating ordered sets of size N is reduced to the generation of incidence vector model IV, or to the generation of decimal vectors of dimension N.

We previously mentioned that, the generated vector should have the property of an ordered set or an incidence matrix of an ordered set, in addition to the set of criteria listed below, which guaranties the generation of only non-trivial sets; this means:

a) Reflexive, Antisymmetric and Transitive.

b) Connectivity

c) Irreducible element free

d) Retractable element free

e) No single maximal node or single minimal node supremum and infimum

f) Duality

The property of the incidence matrix as an upper triangular leaves the coordinates of IV with predetermined ranges. Each coordinate in IV is a positive integer $a_i \in [0, 2^i - 1]$.

3.3.1 Basic ordered set property

A simple recursive loop can generate this type of vector. However, a vector V generated by this recursive loop needs to be tested first for ordered set basic definition, followed by the other conditions we listed above.

3.3.1.1 Reflexivity

We can omit the reflexive property by setting the diagonal =0 in the incidence matrix.

3.3.1.2 Antisymmetry

The fact that the incidence matrix (after suppressing the diagonal) is an upper triangular matrix, and rows in V are sorted integer values implies that if

(x C y) then (y $\not\subset$ x) where C is the binary ordered relation. In fact, all cells in the lower triangle are 0. If a cell in the upper triangle =1 then it can automatically find its transpose cell =0.

Even if the incidence matrix is not an upper-triangular matrix, a simple row permutation or node re-labeling can lead to a new instance of the incidence matrix in an upper triangular form.

3.3.1.3 Transitivity

The generated vectors IV are automatically reflexive and antisymmetric, however, the transitivity property needs to be verified.

Let's start with a generated vector IV. The recursive loop generates rows after rows from the upper end of the ordered set to the lower end. Each additional row needs to be tested and compared with the previous rows, for whether it satisfies the transitivity property or not.

During the generation of an ordered set of size N, Let $\mathbf{i}$ be the current generated row where $\mathbf{i} < N$. A transitivity test should be performed between row $\mathbf{i}$ on all previously generated rows from (0 to ($\mathbf{i}$-1)).

Since for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P$, if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{y} \geq \mathbf{z} \Rightarrow \mathbf{x} \geq \mathbf{z}$

Then, when adding a row $\mathbf{i}$, we scan all the previous rows $\mathbf{j}$ where $0 \leq \mathbf{j} < \mathbf{i}$ and $\mathbf{a_j} > \mathbf{a_i}$, which can be seen in the value of row $\mathbf{i}$ as IM($\mathbf{i}$, $\mathbf{j}$)=1

We start with the last element $\mathbf{k}$ where $0 \leq \mathbf{k} \leq \mathbf{j}$ such that $\mathbf{a_k} > \mathbf{a_i}$ This comparability is translated into a cell value in IM where IM($\mathbf{i}$,$\mathbf{k}$) = 1

Let $\mathbf{h}$ be the index of a node $\mathbf{a_h}$ where $0 \leq \mathbf{h} < \mathbf{k}$. If $\mathbf{a_h} > \mathbf{a_k}$ which means that IM($\mathbf{k}$,$\mathbf{h}$) = 1 then $\mathbf{a_h} > \mathbf{a_i}$ should be true, and we should have IM($\mathbf{i}$,$\mathbf{h}$) = 1. Then the cells values of row $\mathbf{i}$ in IM should be adjusted according to the previous rows values, which sets the constraint on the value of the generated row $\mathbf{i}$. The new corrected value is certainly higher than the original generated value for row i in IV. Since IV(i) is a function of (IV(0), IV(1), ...IV(i-1)), while IV(0), IV(1), ..., IV(i-1) haven't been changed , the next value of IV(i) can jump to the adjusted value for transitivity then incremented by one. This dynamic correction reduces the number of cases for testing. After all, the main strategy in the vector generation, is trying to skip non-potential cases, which accelerate the generation by jumping within the recursive loops. We apply the same technique while testing for connectivity necessary condition.

3.3.2 Irreducible element

Ordered sets with irreducible elements should be dropped too. As we mentioned in the generation properties, we should identify and eliminate ordered sets with irreducible elements.

An irreducible element has a row or column in the covering matrix with a single digit or a single edge. As a necessary condition, an incidence matrix should not have a single digit row $\Rightarrow$ we can drop any generated row = 2^i like 1, 2, 4, 8, 16....

From the covering matrix, we should check for any row or column that has a single digit.

3.3.3 Retractable element

A test for retractable element is based on the following definition.

Let P be an ordered set and let a , b $\in$ P. a is called retractable (to b) iff a $\neq$b and: $(\downarrow a) \setminus \{a\} \subseteq (\downarrow b)$ and $(\uparrow a) \setminus \{a\} \subseteq (\uparrow b)$ [34].

Each row in an incidence matrix holds all information we need about its comparability with previous nodes.

In a row(i) of an element a_i when a cell c(i, j) = 1 $\Rightarrow a_i < a_j$. As a result, by applying the definition and comparing rows, we can conclude that for a given element a_i if we can find an element a_j such that $(\uparrow a_i \setminus \{a_i\} \subseteq (\uparrow a_j)$.

In other words, in a row(i) if a cell c(i, k) = 1 $\Rightarrow$ it should be c(j, k) = 1 in row(j) .

We can perform this test by applying a logic AND on 2 coordinates in IV / IV(i) $\wedge$ IV(j).

We also apply the same test on the columns of the incidence matrix or on the row of its transpose in order to keep same technique at the coding level.

If both conditions are true $\Rightarrow$ P has a retractable point and it can be eliminated.

3.3.4 Supremum and Infimum

By setting the second row to 0 after suppressing the diagonal, the ordered set cannot have a single top node. This is also the case for the last row that should not have value with cells all 1's which means for the coordinate (N-1) the value should be $< (2^{N-1} - 1)$.

3.3.5 Duality

The final result of ordered sets with (FPP) should be a list of all ordered sets (up to isomorphism and duality), which means, if P makes it to the final list $\Rightarrow$ P^d should not, and vice-versa.

Duality can be easily be detected during the generation phase. Since the generated vectors occur in a lexicographical order, we can tell which of these 2 sets was generated first by comparing IM with its own dual.

If $IM \leq IM^d \Rightarrow IM$ goes to the final list

If $IM > IM^d \Rightarrow IM$ is ignored because IM^d already part of the list

$IM^d(i, j) = IM(j, N-i)$.

Chapter 4

4 ISOMORPHISM FILTERING

4.1 Introduction

In 1992, Chaunier and Lygeros [5] generated all unlabeled (or isomorphic free) ordered sets of size 13. They actually used an algorithm of essentially the following kind: start with a linear ordering $R \subseteq \mathbf{n} \times \mathbf{n}$, where $\mathbf{n}=\{0,1,,\mathbf{n}-1\}$, then remove pairs (x,y) from R such that $R\backslash\{(x,y)\}$ is still a partial ordering, Now iterate this process and perform a depth first search from top to bottom through the ordered set of all order relation $R^1 \subseteq R$, ordered by set inclusion. In doing so, we get at least one isomorphic instance of each ordered set of size **n**, because any partial ordering is contained in a linear ordering. In order to be able to count the isomorphism classes, the generated order relations have to be compared repeatedly by an isomorphism test. We still need to keep a list of several of the previous relations in the stack. For size 13 this algorithm ran on nine Apollo workstations for six months.

In 2000, Heitzig and Reinhold [15] introduced a new technique, which is called " the orderly algorithm". This algorithm is based on adding elements to an existing ordered set but in a systematic way so that we make sure we get just one for each isomorphism class. Starting with an ordered set of one node, we then generate a tree of n-ordered sets. In this tree the unique predecessor of an ordered set P is its sub-ordered set $P\backslash\{n-1\}$. But we generate only n-ordered sets P that are canonical, which means that every other n-ordered set P^1 that is isomorphic to P is larger than P with respect to a specific linear ordering on the set of all n-ordered sets. As a result of this algorithm is the enumeration of ordered sets of size 14, and isomorphism free.

These 2 different techniques show the difficulty of the isomorphic sets elimination, and its importance in the enumeration of ordered sets.

4.2 Isomorphism

In this chapter we mainly target isomorphism as criteria for redundancy elimination. The final list of ordered sets with (FPP) should not include 2 dual ordered sets or 2 isomorphic sets. Only one of each isomorphic class should be listed in the final result.

The generation technique we use does not check for isomorphism. We utilize a filtering algorithm to select only a single instance from a group of many isomorphic sets. As we explained before, generation of ordered sets is a recursive loop that generates decimals or integers according to certain constraints. These vectors are stored in a table or file in the order of generation. The following is an example of generated vectors of size 12. The generation order is a lexicographic order. Each vector is listed on a line or row. Coordinates are separated by space.

```
1 2 4 8 16 35 69 138 278 605 1179 2255
1 2 4 8 16 35 69 138 278 605 1211 2287
1 2 4 8 16 35 69 138 280 563 1166 2219
1 2 4 8 16 35 69 138 280 599 1310 2397
1 2 4 8 16 35 69 138 281 563 1182 2219
1 2 4 8 16 35 69 138 281 563 1182 2223
1 2 4 8 16 35 69 138 281 597 1182 2255
1 2 4 8 16 35 69 138 281 599 1182 2255
1 2 4 8 16 35 69 138 281 631 1182 2287
1 2 4 8 16 35 69 138 284 563 1101 2151
1 2 4 8 16 35 69 138 284 563 1109 2151
```

4.3 Class construction

Let Z be the set of all generated ordered sets P of size N. The lexicographic order on Z induced by the decimal numerical representation of the generated ordered sets is obviously a total order on Z. For two generated ordered sets P and Q in Z, either P occurs before Q, which means that P < Q, or Q occurs before P which means that Q < P. For instance if

$P = [a_0\ a_1\ a_2\ \ldots\ldots\ a_{N-1}]$ $\quad Q = [\ b_0\ b_1\ b_2\ \ldots.\ b_{N-1}\ ]$; and starting with rank 0 .

If $a_0 < b_0$ then $P < Q$; if $b_0 < a_0$ then $Q < P$; if $a_0 = b_0$ then we should compare a_i and b_i where i can be incremented between 1 and N-1, until we find a couple $(a_i , b_i) \in PxQ$ such that $a_i \neq b_i$.

We denote $\mathscr{R}$ the total ordering relation on Z.

We define an equivalence relation ξon Z , for all $P\in Z$ and for all $Q\in Z$. P $\xi Q \Leftrightarrow$ P and Q are isomorphic.

This construction of two relations in Z such $(Z, \mathscr{R})$ as a total order relation and (Z, ξ) as an equivalence relation leads to a grouping of ordered sets in a systematic way for processing.

Since (Z, ξ) is an equivalence relation this means that there exist m classes in C_i in Z, such that $C_i \cap C_j = \varnothing$ and $\cup_0^{m-1}(C_i) = Z$.

Each class C_p holds a set of ordered sets P_i such that, for all $P_i \neq P_j \in C_p$ P_i and P_j are isomorphic and either $P_i < P_j$ or $P_i > P_j$

In order to achieve our goal of selecting only one ordered set from each class. we need to choose a representative from each class. Since (Z , $\mathscr{R}$) is a total order, then (C_p , $\mathscr{R}$) is a total order too. As a result, there exists an ordered set $P_i \in C_p$ such that for all $P_j \in C_p$ we have $P_i < P_j$

The representative we choose will always be the minimum element P_i in every equivalence class. This approach determines the methodology of isomorphism elimination. In fact, from each class C_p we can keep the minimum ordered set P_{min} and add it to the filtered result.

4.4 Minimum element

The ordered set P_{min} can be obtained without populating the classes. In fact, if we take any element $P_i \in Z$, and run on it a test to determine if we can convert P_i into P_j in such a way that $P_j < P_i$ and P_j and P_i are isomorphic. Then, we can determine whether P_i is in a minimum representation or not.

If P_j exists then P_i is not a minimum element in its class; moreover, P_j does not have to be minimum either.

This means there exists $P_{min} < P_j < P_i$ in a class C_p . As a result, P_i is eliminated from the minimum list and next element is checked. When all possible combinations are tested, and $P_j < P_i$ was not found, P_i is considered as minimum, and it can be added to the minimum list.

We have to mention a key property of graphs. P is a graph and Q is a second graph, created from P by relabeling P nodes. P and Q are isomorphic if and only if the new labeled graph Q coincides with P. [1]

4.5 Algorithm description

Let's review some of the incidence matrix properties, that are related to the model we have chosen. Let P be an ordered set of size N and let $a_i \in P$. After suppressing the diagonal, if row(i) = 0 then a_i is a maximal element. Let IM be an incidence matrix with a suppressed diagonal. We should start by dividing the matrix into levels. These levels correspond to the ordered set levels. The first-top-level L(0) includes all maximal elements, which means all rows equal "0" in the incidence matrix IM. The total of nodes in L(0) is G(0). The first row that has cells different than 0 corresponds to the first node in the following lower level. This second level L(1) includes nodes that have cells in column G(0) as the last cell. The next row G(1) belongs to the next level L(2).

By applying this division process on P, we end up with sub-matrices as it is shown in the following figure 4.1

Starting with level 2, L(1). This sub-matrix has a height of G(1)-G(0) and width of G(0).

This level is a key factor in determining whether P is minimum or not. A property of ordered sets is that maximal element's rows can be swapped between each other without altering the ordered set structure, since the first rows are equal to "0". This is also true for minimal element's columns. However, swapping element's rows or columns in the middle levels can, in most cases impact and alter the structure of the ordered set. Therefore, each rows swapping should be done with corresponding column swapping in order to preserve the order set labeling and structure.

G3			G2		G1			G0			0		
	a_{10}	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0	N	G3
L 3	0	0	0	1	1	1	0	1	1	1	1	a_{10}	
	0	0	0	1	0	1	1	1	1	1	1	a_9	
	0	0	0	0	1	1	1	1	1	1	1	a_8	G2
L2	0	0	0	0	0	1	0	1	1	1	1	a_7	
	0	0	0	0	0	1	0	1	1	1	1	a_6	G1
L1	0	0	0	0	0	0	0	0	1	1	0	a_5	
	0	0	0	0	0	0	0	0	1	0	1	a_4	
	0	0	0	0	0	0	0	0	0	1	1	a_3	G0
L0	0	0	0	0	0	0	0	0	0	0	0	a_2	
	0	0	0	0	0	0	0	0	0	0	0	a_1	
	0	0	0	0	0	0	0	0	0	0	0	a_0	0

Figure 4-1 Levels in the incidence matrix

Now, if any swapping in the first level can make the second level matrix smaller (lexicographically) then P is not minimum.

Node swapping:

For maximal nodes, we start with columns that correspond to rows with only 0's cells. Swapping these columns is like re-labeling nodes.

The next step is to sort the rows in this sub-matrix of level 2 L(1). Sorting rows means swapping nodes in the second level, which results in swapping columns in the second level to match the row swapping. Level 3 or L(2) result should be sorted as well, and we should apply the same routine to the remaining levels.

For a given ordered set P, this described test can be stopped as soon as we encounter a smaller combination than the original matrix; which means that P is not in a minimum representation. Minimum ordered sets have to pass all permutations steps, which means a failure to find a smaller representation. Here's an example how to execute this algorithm.

A swap between $\mathbf{a_0}$ and $\mathbf{a_1}$ converts L(1) into (see Figure 4.2)

a_2	a_0	a_1	L1
1	0	1	a_5
1	1	0	a_4
0	1	1	a_3

Figure 4-2 Sub-matrix Level 2

After sorting we obtain (Figure 4.3)

a_2	a_0	a_1	L1
1	1	0	a_4
1	0	1	a_5
0	1	1	a_3

Figure 4-3 Sub-matrix Level 2 after sorting

Originally L(2) is like Figure 4.4

a_5	a_4	a_3	L2
1	0	1	a_7
1	0	1	a_6

Figure 4-4 Level 3 sub-matrix

After swapping a_4 and a_5 L(2) becomes like Figure 4.5

a_4	a_5	a_3	L2
0	1	1	a_7
0	1	1	a_6

Figure 4-5 Level 3 after swapping 2 columns

Before swapping, the hex value of VI(6) and VI(7) was 2f = (47) in decimal .

After swapping, the hex value of $VI^1(6)$ and $VI^1(7)$ = 1f = (31) in decimal which are smaller than VI(6), VI(7); At the same time, L(0) remains the same. This result means that P is not minimal.

Based on this technique, some cases can be originally eliminated, during the generation phase. As a matter of fact, if the first row in level 2 has 0's between its 1's cells (see figure 4.6), a simple nodes permutation or columns swap can lead to a smaller first row regardless of what the impact is on the remaining levels.

a_3	a_2	a_1	a_0	L1
1	0	1	1	a_7
1	0	0	1	a_6
0	1	1	0	a_5
0	1	1	0	a_4
0	1	0	1	a_3

Figure 4-6 Level 2 in non minimum form

This type of matrix can not even be generated because the first row in L(1) contains 0 between 1's It is obvious that this matrix can be smaller by swapping columns a_1 and a_2 and then we obtain Figure 4.7

a_3	a_1	a_2	a_0	L1
1	1	0	1	a_7
1	0	0	1	a_6
0	1	1	0	a_5
0	1	1	0	a_4
0	0	1	1	a_3

Figure 4-7 Reducing Level 2

However, the smallest combination is represented in Figure 4.8

As a result, the first row should only have adjacent 1's , which means, at the generation level, of IV (Incidence vector) the first row in the second level L(1) can take only $2^j - 1$ as a value where $2 \le j \le (i\text{-}1)$ and i is the rank of the row.

a_3	a_0	a_2	a_1	L1
1	1	1	0	a_7
1	1	0	0	a_6
0	1	0	1	a_3
0	0	1	1	a_5
0	0	1	1	a_4

Figure 4-8 Minimum form of level 2

Moreover, P is an ordered set, IM is its incidence matrix after suppressing diagonal. Let's consider a case where the first row in the second level has k bits of 1's and all of them are adjacent. In IV (incidence vector model) this condition is translated as, the first coordinate after all 0's coordinate is $2^k - 1$.

If any row in level 2 has a total of bits(1) = h where h < k. we can immediately tell that P is not in a minimum form and here is no need for further testing (see figure 4.9)

	a_2	a_1	a_0	L1
1	1	1	0	a_6
1	0	0	1	a_5
0	1	1	1	a_4

Figure 4-9 Non minimum level 2

After swapping a_1 and a_3 and then sorting the rows, we get its minimum form in figure 4.10

Since the generated ordered sets are lexicographically sorted, for a specific IM_i or IV_i, if the minimum test fails at a specific level (L), then the following IV_j

a_3	a_1	a_2	a_0	L1
1	1	1	0	a_6
1	1	0	1	a_4
0	0	1	1	a_5

Figure 4-10 Minimum form level 2

may inherent the same result when the last IV_j has the same rows of IV_i up to the last node in Level(L).

4.6 Algorithm Implementation

Let Z be the set of incidence vector IV_i of size N. where IV_i can be represented as: $IV_i = [\, C_{i0}, C_{i1}, \ldots\ldots, C_{i(N-1)} \,]$ where C_{ij} is a positive integer number; and IM_i is the incidence matrix corresponds to IV_i

The numerical order in N is used to sort $IV_0, IV_1, \ldots, IV_j$ in Z.

We apply on Z the lexicographic order $(N^N, <_N)$.

The condition we applied on the generated vectors determines the pattern of a resulting vector.

- First non zero coordinate should be ($2^R - 1$)
- There is no coordinate with value = 2^b (irreducible elements free)

The second level or level-2 pattern can be as follows (see figure 4.11).

IM					IM^1				
3	2	1	0		3	2	1	0	
0	0	1	1	3	0	0	1	1	3
0	0	1	1	3	0	0	1	1	3
0	1	0	1	5	0	1	0	1	5
1	0	0	1	9	0	1	1	0	6
1	0	1	0	10	1	0	0	1	9
1	1	1	0	14	1	1	1	0	14

Figure 4-11 Comparing Level 2 before and after swapping 2 and 3

The coordinates of the vector VT are positive integers (see figure 4.12). Each cell in VT tells us how many identical rows are equal to a specific number in the rank vector.

A smaller representation can be obtained, when we find a combination that produces vector VT^1 bigger than VT in a lexicographic order sense.

VT	VT^1		Rank					
2	2		0	0	0	1	1	3
1	1		0	0	1	0	1	5
0	1		0	0	1	1	0	6
0	0		0	0	1	1	1	7
1	1		0	1	0	0	1	9
1	0		0	1	0	1	0	10
0	0		0	1	0	1	1	11
0	0		0	1	1	0	0	12
0	0		0	1	1	0	1	13
1	1		0	1	1	1	0	14
0	0		0	1	1	1	1	15
0	0		1	0	0	0	1	17
0	0		1	0	0	1	0	18

Figure 4-12 Template for vector VT

The vector VT^1 does not give an immediate result of its status compared to VT until we apply a sorting on VT^1 rows. This is a possible and feasible method to find smaller combinations. However, it involves the generation of all possible permutations and sorting each of them, unless we find a smaller combination during the generation loop, where we can stop the testing and produce a result.

In our case, this technique is only utilized as a numerical validation process. We improved this technique and we ended up with the following, which we will describe in detail with examples.

4.6.1 Technique description

Let P be an ordered set of size N. IM is the incidence matrix, IV is the incidence vector model $A_0, A_1, A_2, \dots, A_{N-1}$ are nodes in P

Let MA2 be a square matrix of dimension r x r such that r =(G(1) - G(0)) where G(1) – G(0) is the height of the second level matrix of IM (see figure 4-13) .

Let x_{ij} be a cell in MA2 such that $x_{ij}=0$ if $i = j$; and $x_{ij} = x_{ji} = s$, where s is the total number of rows that have comparability to the maximal nodes a_i and a_j and only to these two nodes in Max(P). In another word, the total number of bits for these rows must be two.

MA2				
0	1	2	3	Lev2
0	2	1	1	0
2	0	0	1	1
1	0	0	0	2
1	1	0	0	3

Figure 4-13 MA2 matrix for Level2

We generate a similar cubical matrix 3D for rows with three bits; however, for four bits rows and higher we choose a different technique because of their small population. In some cases, we can use the dual sets, which may have a smaller Max(P).

Now, we need to find a permutation or re-labeling of the first level columns or nodes that may lead to a smaller IV (Incidence vector).

When the first row $R_{G(0)}$ in level2 is (0011_{binary}) or ($3_{decimal}$) this means that the first node $A_{G(0)}$ in level2 is covered by the first 2 nodes in level1 A_0 and A_1 such that $A_{G(0)} \prec A_0$ and $A_{G(0)} \prec A_1$.

The question is raised whether we could find a node $A_i \neq A_{G(0)}$ such that A_i in level 2 where $G(0) < i < G(1)$ and A_i is covered by only two nodes A_j and A_k where $0 \leq j < k < G(0)$ and the couple $(j,k) \neq (0,1) \neq (1,0)$.

If A_i exists, it means that A_j and A_k can take the lead as first two columns in IM. This substitution can help find a smaller permutation if it exists.

The Matrix MA2 helps easily find A_i . In fact, the first step in this search is to find a cell C(j,k) in MA2 such that $(0,1) \neq (j,k) \neq (1,0)$. C(0,1) holds how many nodes in level 2 are only covered by A_0 and A_1.

The next couple of candidate A_j and A_k for the leading columns should have a cell in the vector MA2 such that $MA2(j,k) \geq MA2(0,1)$ and **j** and **k** are indexes where $0 \leq j < k < G(0)$.

When this condition is satisfied, **j** and **k** can be added to a record of new permutations PR where PR(0) =**j** and PR(1) = **k**.

If MA2(j,k) > MA2(0,1) then there is no need to continue the testing, because this permutation, regardless of the remaining nodes position, provides a bigger vector VT^1 , consequently the vector IV is not minimum, therefore, P is not in a minimum representation and can be eliminated.

When MA2(j,k) = MA2(0,1): it means that **j** and **k** are added to the permutation vector, and the next step is to complete the permutation vector with the remaining columns. The selection of the next column CL(2) relies on CL(0), CL(1). It is also related to MA2 and VT.

From the matrix MA2, we need to scan the row(j) for a candidate of column CL(2) which is different than **k**. The first cell ℓ in row(j) we find such that ℓ

is different than k and the value of $MA2(j,\ell) \geq VT(1)$ which represents the total number of rows of value $= (101_{binary})$ or 5_{dec}.

Again, if $MA2(j,\ell) > VT(1)$ then there is no need to continue because the couple (j,ℓ) leads to a smaller combination.

However, when $MA2(j,\ell) = VT(1)$, then ℓ is accepted provisionally, and it has to pass a test for all rows or all coordinates in IV smaller than $(1000)_{binary}$ or $8_{decimal}$ because these rows are related to columns 0, 1, and 2. At this stage PR vector or permutation vector looks like (Figure 4.14):

0	1	2	3	4	5	6	7
j	k	ℓ	∅	∅	∅	∅	∅

Figure 4-14 Permutation list vector

A comparison of $MA2(k,\ell)$ with VT(2) leads to:

If $MA2(k,\ell) > VT(2)$ then the combination is not minimum.

If $MA2(k,\ell) < VT(2)$ then we can completely drop the row ℓ from its current position, and apply a backtracking to the previous state.

If $MA2(k,\ell) = VT(2)$ then we should move to the next row in VT which is the last row where $< 1000_{binary}$ or $8_{decimal}$. In fact, if this row exists it then belongs to a different group because it has 3 bits, then the comparison should be performed between $MA3(j,k,\ell)$ and VT(3).

We can proceed with the same technique for next columns.

When PR is completed, which means that PR reached G(0) elements or coordinates; this also means that the new permutation is neither smaller nor bigger than the original matrix. After sorting rows in VT^1 , the vectors VT and VT^1 should be the same. This vector PR is utilized as the template for the permutation schema for the next levels.

The sorting result represented by the set of vectors PR(D), is applied on the columns of level3 and tested for a smaller combination; where D is the total number of all valid permutations. Moreover, each sorting permutation can generate more than a single column permutation for level3. It depends on whether level2 has identical rows. In fact, identical rows in level2 can generate multiple combinations because when we swap two identical rows in level2, it does not have impact on the structure of the ordered sets, it only affects the lower levels and provides them with the opportunity to minimize the ordered set.

When all permutations fail to find a smaller combination then IV_i is added to the minimum list.

When the first row in level2 IV(G(0)) >3, or IV(G(0)) can be 7, 15, 31..

In this case, all rows in level 2 should have rows with bit-sum ≥ 3 in case IV(G(0)) = 7 ; or a bit-sum ≥4 in case IV(G(0)) = 15 and so on.

In case of IV(G(0)) = 7 we can use the same technique based on MA3 which is a 3D cubical matrix. If IV(G(0)) ≥ 15, which are less frequent cases, we utilize a basic technique to obtain permutations for this category, or we test the dual for a smaller Max(P). We eventually generate all permutations for the maximal nodes and we apply permutation on IM. We execute an immediate test on each of these results for all levels using acceptable permutations that can generate an identical second level.

If we find a smaller combination, we can immediately stop the processing and eliminate this ordered set or save it in the non- minimum set.

Chapter 5

5 FIXED POINT PROPERTY TESTING (FPP)

5.1 Introduction

In the previous chapter, we presented an algorithm that help reduce the list of generated ordered sets of a specific size N up to isomorphism. We use the reduced list in this chapter as an input data for our (FPP) characterization.

Let Z be the reduced list set

$P \in Z$ implies:

- P is an ordered set
- P is irreducible element free
- P is connected
- P is retractable point free
- $P \in Z$ then dual(P) or $P^d \notin Z$
- $P \in Z$ such that if P and Q are isomorphic then $Q \notin Z$
- P does not have supremum or infimum
- P can be either (FPF) or P has the (FPP).

5.2 Preliminaries

5.2.1 Definition 1:

An ordered set P has the fixed point property (FPP) if every order-preserving self-mapping has a fixed point.

5.2.2 Definition 2:

Let (P, $\leq$) be an ordered set and let f: P → P be a map. Then f is called an order preserving self-mapping or isotone map iff for all x , y ∈ P we have:

$x \leq y \Rightarrow f(x) \leq f(y)$

5.2.3 Problem description

Let G = { set of maps f such that f: P → P }

Let M = { set of maps g such that g: P → P is an order-preserving self map}

Based on these definitions:

Let P be a finite ordered set. The characterizing process can be reduced to the finding of an order preserving map g: P → P that has no fixed point.

A simple way to generate f: P → P where $f \in$ G is to run a recursive loop, that covers all possibilities of self-mapping P →P. Eventually, for P of size N there exists N^N different self maps. Each of them has to be tested for (OPM) and the first occurring f:P → P that has fixed point free which characterizes the ordered set P as (FPF). As a result, we can stop the loop at this point.

This technique is extremely time consuming, so we need to reduce the possible cases and limit the loop range to only meaningful maps that are potential candidates for (FPF).

The original loop design assigns for each element in P any element from P as an image. Then the range of the loop varies from 0 → (N-1)

Let f: P → P (OPM) (FPF) . The range of each element $a \in P$ can be limited to $D(\{a\}) \subseteq P\backslash\{a\}=P-\{a\}$.

5.3 Comparable elements:

The range D({a}) can be reduced to only non comparable elements with a .

Proof:

Let f: P → P (FPF) and (OPM).

Let $y \in P$, assume that $f(a) = y$ and $y \neq a$ and $y \sim a$ (comparable)

It means either $y < a$ or $y > a$

The map f is (OPM), which means that:

$$\text{if } y < a \Rightarrow f(y) \leq f(a) \Rightarrow f(y) \leq y$$

P is a finite ordered set. the limit of this inequality leads to:

$f(f(y)) \leq f(y) \Rightarrow f(y^1) \leq y^1 \Rightarrow \ldots \Rightarrow f(y^n) \leq y^n$ such that y^n is a minimal element in $P \Rightarrow f(y^n) = y^n$. This result means that y^n is a fixed point, which led to a contradiction because we assumed that f is (FPF).

If a < y and f is (OPM) and (FPF) then if $f(a) \leq f(y) \Rightarrow y \leq f(y)$. Using the same technique leads to a contradiction too. Consequently, if y ~ a then y $\notin$ D({a}). This property makes the range for each loop smaller and reduces processing. After obtaining each map, we need to test each for (OPM).

5.4 Description of maps generation

This part is described in detail in paragraph 5.6 with examples. In fact, it is based on a recursive loop of depth N where N is the size of the ordered set. Eventually, we generate a map by assigning images from P to all elements in P. This process is achieved by the recursive loop with N^N iterations. It is followed by (OPM) selection.

5.5 Dynamic range

Even with the range reduction of comparable elements the (FPP) test processing was not fast enough, because of the enormous number of maps being generated and tested for OPM. In fact, there is still some room for improvement especially by using the property of (OPM) which helps reduce the range for each element.

Let P be an ordered set, where **a** is an element in P

Let D({**a**}) = {**x** ∈ P such that **x** $\nsim$ **a** } . We choose D({**a**}) as the range for {**a**} for any (OPM).

Let's readjust and reduce dynamically the size of the range D({**a**}) based on the (OPM) conditions.

Let $\mathbf{a}$ be an element in P such that $\mathbf{a} \notin Max(P)$ [Where Max(P) is the set of maximal elements in P.]. There exists an element $\mathbf{b} \in Max(P)$ such that $\mathbf{a} < \mathbf{b}$

Let $M(\mathbf{a}) = (\uparrow \mathbf{a}) \setminus \{\mathbf{a}\}$. Clearly, $M(\mathbf{a}) \neq \varnothing$ because $\mathbf{a} \notin Max(P)$.

For all $\mathbf{x} \in M(\mathbf{a})$ we have $f(\mathbf{a}) \leq f(\mathbf{x})$

Let $C(\{\mathbf{x}\}) = (\downarrow f(\mathbf{x}))$ such that for all $\mathbf{x} \in M(\mathbf{a})$ we have:

$G(M(\mathbf{a})) = \bigcap_{\mathbf{x}} C(\{\mathbf{x}\}) = \bigcap_{\mathbf{x}} (\downarrow f(\mathbf{x}))$ such that $\mathbf{x} \in M(\mathbf{a})$

$f(\mathbf{a})$ always $\in G(M(\mathbf{a}))$.

We previously have $f(\mathbf{a}) \in D(\{\mathbf{a}\})$ set of non-comparable element with $\mathbf{a}$.

Consequently, $f(\mathbf{a}) \in H(\mathbf{a}) = D(\{\mathbf{a}\}) \cap G(M(\mathbf{a}))$

This set $H(\mathbf{a})$ is smaller than $D(\{\mathbf{a}\})$ and it can reduce iterations enormously.

The set $H(\mathbf{a})$ is only available for elements $\mathbf{a} \notin Max(P)$. So for elements $\mathbf{w} \in Max(P)$ we still have to start with $D(\{\mathbf{w}\})$. The moment we get $f(\mathbf{w})$ we can immediately update ranges for next related elements.

In some cases, we only find a single element in $H(\mathbf{x}_i)$ which is very helpful: it means for such a set of images for all elements $> \mathbf{x}_i$, there is only one image $f(\mathbf{x}_i)$ that can be mapped to it, and preserves the order in P (OPM). In some other cases $H(x_i) = \varnothing$ where order can not be preserved for this specific map partially defined by all elements x_0 , x_1, x_2, ..,x_{i-1} .This means that this map has to be abundant and $f(x_{i-1})$, $f(x_{i-2})$,... should take their next possible values.

In the next paragraph, we present the algorithm description with an example of how to adjust the range.

5.6 Case study

Let P be an ordered set of size N

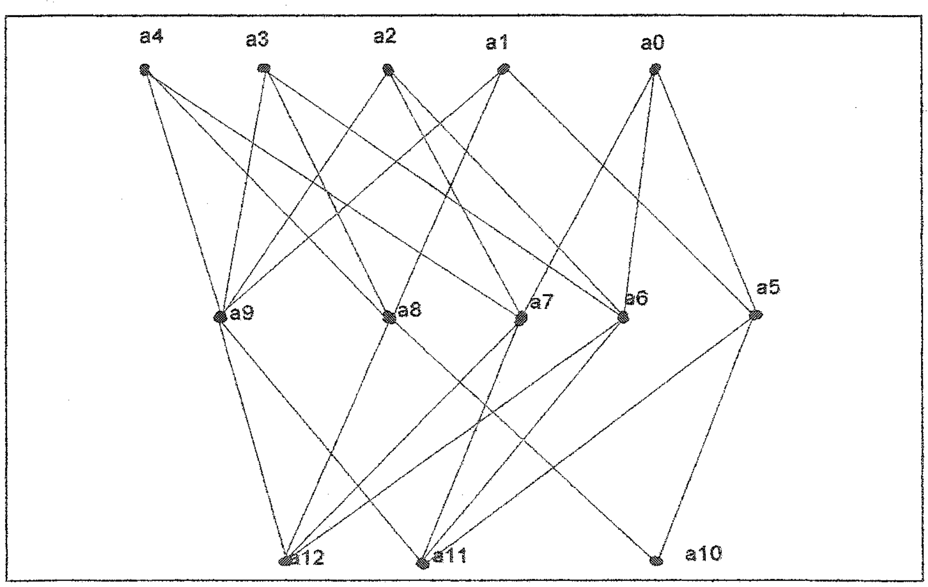

Figure 5-1 Upward drawing ordered set

This ordered set is represented in an upward drawing (Figure 5.1). It is in its minimum form.

The matrix below is the incidence matrix for the same ordered set in figure 5.1 (see figure 5.2).

C12	C11	C10	C9	C8	C7	C6	C5	C4	C3	C2	C1	C0	
1	0	0	1	1	1	1	0	1	1	1	1	1	R12
0	1	0	1	0	1	1	1	1	1	1	1	1	R11
0	0	1	0	1	0	0	1	1	1	0	1	1	R10
0	0	0	1	0	0	0	0	1	1	1	1	0	R9
0	0	0	0	1	0	0	0	1	1	0	1	0	R8
0	0	0	0	0	1	0	0	1	0	1	0	1	R7
0	0	0	0	0	0	1	0	0	1	1	0	1	R6
0	0	0	0	0	0	0	1	0	0	0	1	1	R5
0	0	0	0	0	0	0	0	1	0	0	0	0	R4
0	0	0	0	0	0	0	0	0	1	0	0	0	R3
0	0	0	0	0	0	0	0	0	0	1	0	0	R2
0	0	0	0	0	0	0	0	0	0	0	1	0	R1
0	0	0	0	0	0	0	0	0	0	0	0	1	R0

Figure 5-2 Incidence matrix for case study

These vectors below show a mapping example f:P→ P and a sample of Incidence vector IV with its image f (IV) both are in Hex base Figure 5.3.

P	f(P)		f(IV)		IV
a12	a11		aff		13df
a11	a12		13df		aff
a10	a12		13df		53b
a9	a6		4d		21e
a8	a6		4d		11a
a7	a9		21e		95
a6	a9		21e		4d
a5	a12		13df		23
a4	a2		4		10
a3	a2		4		8
a2	a3		8		4
a1	a0		1		2
a0	a1		2		1

Figure 5-3 OPM sample

The matrix IM is the incidence matrix for P, IV is the incidence vector for P.

This paragraph features an algorithm for (FPP) characterization ordered sets P.

Let $f : P \rightarrow P$ be a map from P to P It is similar to mapping from:

f: D = { IV}→ N^N such that IV is a vector and D a singleton where $D \subset N^N$.

The map f can be represented as a vector of size N where

T = { 1, 2, 4, 8, 10, 23, 4d, 95, 11a, 21e, 53b, aff, 13df }

IV= (1, 2, 4, 8, 10, 23, 4d, 95, 11a, 21e, 53b, aff, 13df)

f(IV) = (2, 1, 8, 4, 4, 13df, 21e, 21e, 4d, 4d, 13df, 13df, aff)

Each coordinate of f(IV) can take any value from T. The modification of f(IV) coordinate is translated by distinct maps from P→ P. In general, if all coordinates are different, we can have up to N^N different maps from f: P → P.

```
void Iteration(int Rank)
{
 Rank++;
 for(I=0;I<N; I++)
  {
    F[Rank]=I;
     {
      if(Rank<N-1)
       Iteration(Rank);
      else
       SaveResult();
     }
  }
  return ;
 };
main()
{
 Poset.Iteration(-1);
};
```

This function **Iteration( )** generates all possible maps from the saved results of vector F[N].

In practice, we don't have to generate all maps. In fact, in the previous paragraphs, we described how we could reduce the range of each node in P, which means reducing the loop iteration in the recursive function.

The loop:

for(I=0;I<N; I++) becomes for(I=0;I<D[S] ; I++) where D[S] is a vector that holds the indexes of all nodes a_J such that $a_S \nsim a_J$

The vector D[S] can be obtained from the incidence matrix.(IM). R_S is a row of rank S in IM

If $a_S < a_J$ the the bit J in R_S is equal to 1.

From (IM) we obtain C_S which is the column of S in IM.

If $a_K < a_S$ then the bit **k** in C_S is equal to 1.

By combining these two vectors C_S and R_S we can obtain a comparability vector for the node a_S we can call it B_S . Combining C_S and R_S is a logic OR on the coordinates of these two vectors. If we represent C_S and R_S as a single binary number, as we already did for IV (incidence vector model) The binary number for $B_S = C_S \vee R_S$ and the vector representation for $B_S(i) = C_S(i) \vee R_S(i)$

From the previous example we have for node (6).

$P = [\, a_{12}\ a_{11}\ a_{10}\ a_9\ a_8\ a_7\ a_6\ a_5\ a_4\ a_3\ a_2\ a_1\ a_0 \,]$

$R_6 = [\,0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\,]$

$C_6 = [\,1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\,]$

$B_6 = [\,1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\,]$

$D_6 = [\,0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\,]$

From B_S we obtain the set of nodes comparable with a_S. Let $D_S = (!\ B_S)$ such that, if $B_S(J) = 1 \Rightarrow D_S = 0$ and if $B_S(J) = 0 \Rightarrow D_S = 1$. (Boolean inverse).

D_S holds all non-comparable elements with a_S such that

$D_S = \{\, x \in P \text{ where } x \nsim a \,\}$.

From the example, we have $D_6 = \{\, a_1, a_4, a_5, a_7, a_8, a_9, a_{10} \,\}$.

Moreover, the process of reducing the range for a_s, suggests that we include the (OPM) property in the compilation of the final range.

From the example, we have $Max(P) = \{\, a_0, a_1, a_2, a_3, a_4 \,\}$

By suppressing diagonals from the incidence matrix, maximal nodes have rows = 0.

By associating the (OPM) property with maximal nodes image assignment, some useful constraints can be added to the remaining node's ranges and reduces them furthermore.

Let $a_S \in P$ such that $a_S \notin Max(P)$ and let $a_K \in (\uparrow a_S) - \{a_S\}$ which means that $a_S < a_K$.

Since $f(a_K) \in P$ then there exists $a_W \in P$ such that $f(a_K) = a_W$

$C_W = (\downarrow a_W)$ is the w_{th} column in the incidence matrix.

From R_S, which is the row of node a_S, we can list all nodes $a_K \in P$ such that $a_K > a_S$.

Therefore, the new range of a_S is:

$H_S = D_S \cap (\downarrow (f(a_K)))$ where $a_K \in R_S - \{a_S\}$. where $R_S = \{ x / x \in P$ and $x > a_S \}$.

Or $H_S = D_S \cap (C_W)$ for all w where $a_W = f(a_K)$ and $a_K \in R_S$.

Example:

$C_6 = \{a_{11}, a_{12}\}$. $R_6 = \{a_0, a_2, a_3\}$

From the same example, we have : $f(a_0) = a_1$; $f(a_2) = a_3$; $f(a_3) = a_2$.

$(\downarrow f(a_0)) = (\downarrow a_1) = C_1 = \{a_1, a_8, a_9, a_{10}, a_{11}, a_{12}\}$

$(\downarrow f(a_2)) = (\downarrow a_3) = C_3 = \{a_3, a_6, a_8, a_9, a_{10}, a_{11}, a_{12}\}$

$(\downarrow f(a_3)) = (\downarrow a_2) = C_2 = \{a_2, a_6, a_7, a_9, a_{11}, a_{12}\}$

$C_1 \cap C_3 \cap C_2 = \{a_9, a_{11}, a_{12}\}$

$D_6 = \{a_1, a_4, a_5, a_7, a_8, a_9, a_{10}\}$ therefore,

$(C_1 \cap C_3 \cap C_2) \cap D_6 = \{ a_9 \}$

In this case, the range of $\{a_6\}$ is a singleton $\{a_9\}$ and only during the recursive loop as long as $f(a_0) = a_1$; $f(a_2) = a_3$; $f(a_3) = a_2$.

This result means $f(a_6) = a_9$.

By following the same strategy, all the ranges can be obtained prior to executing the next step of the loop.

The image of a_6 is a function of $f(a_0)$, $f(a_2)$, $f(a_3)$. For each modification in the images of elements in R_6 the range of a_6 should be updated.

Eventually, if the range for a specific node $\mathbf{a}_S$ is $H_S = \varnothing$, it means that the (OPM) property could not be respected, then we can skip this specific map and generate the next one.

We can only have a decision about a specific ordered set when we complete a whole cycle and obtain images for all elements in P, which is f(P).

When we obtain a set of images. This set f(P) is the definition of a map f:P →P where f is fixed point free (FPF), and (OPM).

So the test can stop and we can conclude that P is (FPF).

Originally, before integrating the (OPM) condition in the dynamic range, an extra test for (OPM) was performed on each generated map.

We implemented this test and we describe it as follows:

5.6.1 Order preserving map (OPM)

We can test for (OPM) by creating a matrix OM(i,j) similar to the incidence matrix IM(i , j), with only elements from f(P) .

The main condition for (OPM) is that rows in OM(i , j) should cover rows in IM (i , j), which means that, for a row(k) if IM(k,ℓ) = 1 then OM(k,ℓ) should be equal to "1" too.

Let OV(i) be the vector that combines rows of OM(i , j) in the same way of incidence vector model construction

For each row k we should have IV(k) < OV(k) and IV(k) $\wedge$ OV(k) = IV(k). which is the Boolean AND of rows IV(k) and OV(k).

5.6.1.1 How to generate OM (incidence matrix for f(P))

Since each row R(k) in f(IV) holds comparability information related to all preceding elements or elements with index < k . Matrix OM can be generated by comparing coordinates in f(IV).

Let $b_x = f(a_x)$ and T_x is the coordinate in row x for the node b_x

Let $b_y = f(a_y)$ and T_y is the coordinate in row y for node b_y .

if $T_x \wedge T_y = T_x$ then $b_x \le b_y$ then OM(x, y) = 1.

We show an example of the OPM retained from the same example we present through this chapter:

C12	C11	C10	C9	C8	C7	C6	C5	C4	C3	C2	C1	C0		$f(P)$
1	0	0	1	1	1	1	0	1	1	1	1	1	R12	a11
0	1	1	1	1	1	1	1	1	1	1	1	1	R11	a12
0	1	1	1	1	1	1	1	1	1	1	1	1	R10	a12
0	0	0	1	1	0	0	0	1	1	1	1	0	R9	a6
0	0	0	1	1	0	0	0	1	1	1	1	0	R8	a6
0	0	0	0	0	1	1	0	1	1	1	0	1	R7	a9
0	0	0	0	0	1	1	0	1	1	1	0	1	R6	a9
0	1	1	1	1	1	1	1	1	1	1	1	1	R5	a12
0	0	0	0	0	0	0	0	1	1	0	0	0	R4	a2
0	0	0	0	0	0	0	0	1	1	0	0	0	R3	a2
0	0	0	0	0	0	0	0	0	0	1	0	0	R2	a3
0	0	0	0	0	0	0	0	0	0	0	1	0	R1	a0
0	0	0	0	0	0	0	0	0	0	0	0	1	R0	a1

Figure 5-4 OPM incidence matrix for $f(P)$

Eventually, failing to find $f(IV)$ means this specific map in progress of generation is not an (OPM), or it has to have fixed point to make it (OPM) (see figure 5.4).

In both cases, when we finish all possible maps without obtaining a single map, then we can consider this specific ordered set having the (FPP) for all its (OPM). And the ordered set P is characterized with the (FPP).

This algorithm was tested up to size (14) with very fast results.

5.7 Special cases

In this project we also implemented different techniques to improve efficiency in characterizing (FPP). These techniques are included in the code, however, the speed impact can be observed only on size ≥ 15. These techniques are presented in the next chapter.

Originally, the code limitation was designed for a size ≤ 16.

Chapter 6

6 SPECIAL CASES TECHNIQUES

6.1 Overview

Some ordered sets have particular properties that can be more compatible when applying more efficient techniques for the (FPP) characterization. In this chapter we cover, two techniques that are especially useful with larger ordered sets than the range we are considering.

These techniques may not give us results and cannot be applied on all cases. However, they are not heuristics. In the case where we conclude that we can not apply these particular cases techniques, a call for the general algorithm is necessary.

These techniques are direct implementations of the following technical lemma:

Lemma: *Let P be a finite ordered set. If $P = A \oplus B$ then P has the (FPP) if and only if A or B has the (FPP).*[31]

$P = A \oplus B$ if $P = A \cup B$ and $A \neq B \neq \emptyset$ and

$A < B$, that is $x < y$ for each $x \in A$ and $y \in B$.

The second technique is based on the retract theorem

Theorem: *Let P be an ordered set. If P has (FPP)* [29], *then every retract of P has the fixed point property.*

We can use a corollary from this theorem

Corollary: *if P has a retract that does not have the (FPP) then P does not have the fixed point property.*

6.2 Implementation of Rutkowski Lemma

As we mentioned before, this technique can only be applied on special cases. When we cannot apply it, we have to refer to the general algorithm.

Let P be an ordered set of size N.

We need to find $A \subset P$ and $B \subset P$ such that $A \oplus B = P$ and $A < B$.

Let's start with $A = \varnothing$ and $B = P$.

Let VB be a vector of size N that holds status of current nodes in B.

In VB originally all coordinates = 1

$VB = [\, 1_0 \;\; 1_1 \;\; 1_2 \;\; 1 \;\; 1 \ldots\ldots 1_{N-1}]$

$\underbrace{\hspace{6cm}}_{N}$

We start adding nodes to A by starting with a_{N-1} and VB(N-1) becomes 0.

A vector VA is a vector that holds data about all nodes y_K such that $y_K > x_i$ and $x_i \in A$.

Originally VA = R(N-1) (row (N-1) or last row) of IM(P) (incidence matrix)

After each step we compare VB with VA

If $VA \wedge VB = VB$ we add this result VB to a vector of result.

We should be careful how we add new nodes. In fact, for each level we have to cover all permutations within the level and not add just consecutive elements. Practically, when we add a node, we should add the whole set of nodes incomparable to it, which ends up to adding at least, the whole level of the node in most cases.

In fact, all processing of VA and VB is a processing of binary numbers V_A and V_B, which are integers in binary base of single dimension.

At the end of the processing, let's assume that we got a set of result with M elements. Each Result R(i) where $0 \leq i < M$ has a coefficient C(i) which is the total number of elements in VB. It is preferable to start processing with C(i) closer to N/2 , which reduces to the minimum the size of A and B.

The first test we should apply on A and B is the connectivity test, because if A is disconnected then A is (FPF) which is the same for B.

We can also eliminate irreducible elements from A and B, if any exists after the cut.

The final subsets A^1 and B^1 can be tested for the (FPP).

We can start with the smaller size set, and apply the general technique we described in the previous chapter. Since A and B are irreducible element free, we can probably tell depending on the size of A and B whether they have (FPP) or not, by comparing these sets with known sets for the (FPP).

Here's the main result of Rutkowski:

Theorem: *There exist exactly eleven (up to isomorphism and duality) ordered sets of size ≤ 10 with the fixed point property and containing no irreducible elements. They are the singleton, and the ten set in Figure 2.7 (listed before).*[31].

This theorem means that any irreducible element free subset ≠ singleton with size ≤ 8 has to be (FPF). For an ordered set P of size 16. If we could find a cut 8 to 8 then P is FPF. For size 9 , there is only 1 set but it has a set of isomorphic sets. A simple way of comparing is to obtain the template set P1 from the Rutkowski result in its minimum form we defined in this project, in addition to the dual of P1.

We convert A and B into their minimum forms too and compare.

Another way to compare the subsets to the previous results would be by listing all isomorphic sets for P1 and then directly comparing with A or B.

Based on Rutkowski result, this comparing technique can work for size 10 as well.

If we want to test for a higher size than 10, an elimination of retractable elements may be needed.

This same Lemma can be implemented in the opposite way.

Let P = A ⊕ B if A is (FPF) and B is (FPF) then P is (FPF).

This case can especially occur, when we get a disconnected subset A or B, In this case we are reducing the size of the original problem and we only need to test one set.

An improvement can be made on this technique in the way we add new elements to the lower set without having to cover all permutation of each level.

This part was not implemented in the code we wrote, but it looks promising and it should improve performance.

6.3 Implementation of the FPF property.

This technique is based on an attempt to find without iteration, for an ordered set P, a (FPF) function f: P → P.

If f exists then f is (FPF) then P is (FPF). If we don't find f , It does not mean that P is (FPF) or P has (FPP), it simply means that P still has to pass the general (FPP) testing.

We try to map P to a subset CR in P where CR is a 4 nodes crown in P.

If we can construct an (OPM) f: P →CR such that f is (FPF) then P is (FPF).

This idea is based on observations we made during the processing of (FPP) testing under the general technique. In fact, several ordered sets we tested with the (FPF) property did have a crown as the range for the (OPM) (FPF) map.

6.3.1 Find the Crown

We first need to find four nodes { A, B, C, D } ∈ P that can construct a crown CR= {A, B, C, D }

We start with two maximal nodes A and B such that {A, B} ⊂ Max(P).

We define G(A) = { x ∈ P such that x ∉ (↓ A) }

We also define G(B) = { x ∈ P / x ∉ (↓B)}.

{ C, D} ⊂ (↓A) ∩ (↓ B) if exists.

G(C) = { x ∈ P / x ∉ ((↓ C) ∪ (↑ C)) or x ≁ C }

G(D) = { x ∈ P / x ∉ ((↓ D) ∪ (↑ D)) or x ≁ D }

We only have one more condition to satisfy which is C ≁ D (C and D incomparable).

Let's say that we find C. Then D should fulfill an extra condition such that, for all x∈ P -{A, B, C, D} there exists y ∈ CR such that x ≁ y.

This condition lets us assign at least one image in CR for each node in (P-CR)

Since the OPM function *f* we are seeking for, is (FPF), then an image *f*(x) for x ∈ P has to be incomparable with x where *f*(x) ≁ x.

This condition constrains the choice of the couple of nodes C, D and it can be written as:

1) A ∈ Max (P).
2) B ∈ Max (P).
3) { C, D } ⊂ (↓A) ∪ (↓B).
4) C ≁ D.
5) G(A) ∪ G(B) ∪ G(C) ∪ G(D) = P.

6) This last condition may guarantee for each element $x \in P$, at least one element y where $x \nsim y$. This element y could be a potential candidate for $f(x)$ value such that $f(x) \nsim x$.

If one of these conditions was not satisfied, then we have to find either two different nodes C and D or restart with two different maximal nodes A and B. We should stop searching after we test all maximal nodes in P.

In fact, it is possible to find a crown that may meet requirements without having the first upper nodes as maximal nodes in P. We did not cover this case in our implementation, because we want a fast result, and this case involves some extra conditions. A general case is more efficient and always guarantees results.

Some extra restraints on CR ={A, B, C, D} need to be verified.

A simple pattern of the combination (P, CR) may easily result in a non-conformal crown for (FPF). In fact, if we find a fence F of (N) shape or its dual (И), such that F overlays on the crown CR. Then the function f cannot be (FPF). This is only a necessary condition.

Let F be a fence of (N) shape where nodes $\{ C, D \} \subset F$ and $F \subset (\downarrow A) \cap (\downarrow B)$.

We have different configurations:

Let $F = \{W, X, Y, Z\}$ such that $C = X$, $D = Z$, $W < C$ and $Y < C$ and $Y < D$

This combination can not exist because Y < A, B, C, and D and Y $\in$ P which is a contradiction with condition 5 that elements in r $\in$P should have at least one element in s $\in$ CR such that r $\nless$ s.(see Figure 6-1 right).

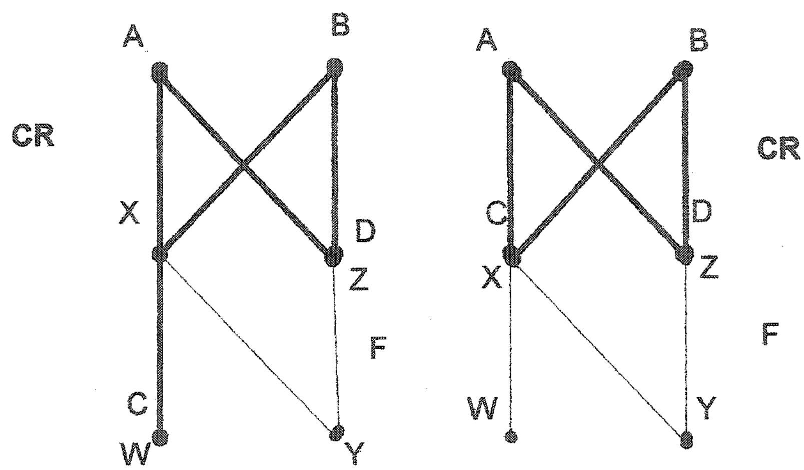

Figure 6-1 2 different combinations with crown and fence

For a fence F as the one in Figure 6-1 left, where W=C, X > C, Y < D and Z = D, we also have {W, X, Y, Z } < { A, B }.

This pattern can not lead to a (FPF) because

{Y} < { D, A, B } $\Rightarrow$ f(Y) = C .

X > C and {X} < { A, B } $\Rightarrow$ f(Y) = D.

However, Y < X $\Rightarrow$ f(Y) < f(X) $\Rightarrow$ C < D $\Rightarrow$ contradiction

Consequently, the crown should not have a (crown, fence) pattern such that the fence F < {A, B} where A and B are two maximal nodes in the crown CR.

If we go one more size with a fence, we obviously get a fence F with shape (**M** or **W**) . We can easily see that if we overlay this fence F such that

F= {U, V, X, Y, Z } under the crown CR such that Z ≡ D; U ≡ C and F < CR, then f(V) = D; f(Y) = C; f(X) > f(Y) = C and f(X) > f(V) = D (Figure 9-2)

However, f(X) ≠ A ≠ B then we can not have CR as crown for a FPF map.

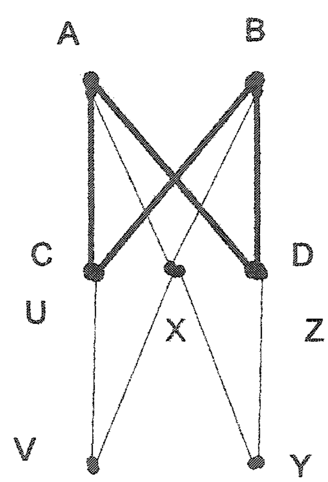

Figure 6-2 Crown with an (M) shape fence.

In general, a necessary condition in order to have a crown{ A, B, C, D } as a range of a map f such that f is (OPM) and (FPF) is that we should not have any fence F that includes {C, D } and F < { A, B }.

Next step is to continue the construction of the (FPF) map f, and covers all elements in P.

Each node S in P can only have its image $f(S)$ in G(S), which is defined before as: $G(S) = \{ X \in P / X \nleftarrow S \}$.

$f(S) \in CR$ as well then $f(S) \in CR \cap G(S)$.

The range of D(S) of S can be at most CR and at least a singleton node from CR, which reduces iteration enormously.

We also use the technique of dynamic range in order to find the images for all elements in P and have the map an (OPM) at the same time.

If we find images for all elements in P, then f is an (OPM) and (FPF) and therefore P is (FPF).

If we don't find images then we can restart iterations on the next possible set of { C, D } which are the minimal elements in the crown CR. If { C, D } does not exist we have to choose the next combination of { A, B} which are the maximal nodes of the crown CR.

Extra iterations are sometimes needed when the range we found for a specific node X is more than a single node; however, this iteration cannot exceed 4, which is the total number of nodes in the crown.

As we mentioned in the beginning of this chapter, these two special cases techniques work side by side with the general technique, and we should have it always as a backup in the structure of our algorithm or at the code level.

Chapter 7

7 CONCLUSION AND FUTURE RESEARCH WORKS

" *Good order is the foundation of all good things* -**Burke**"

7.1 Result evolution

The main objective of this project is to characterize ordered sets based on fixed point property. These results were obtained progressively, depending on the size of the ordered sets. At one point, improvement in the algorithms efficiency was a must. Even with high performance processors, iterations and testing we have to apply on each set is not negligible. Originally, the most processing time was consumed on the final phase, which is the (FPP) testing. An enormous improvement happened once we applied the dynamic range technique.

An example of processing time using a Pentium III Dual processor

For size 13 :

Using technique without dynamic range adjustment > 14 days

Using dynamic range adjustment 157 sec.

This set of results motivates us to work on improving the algorithm.

Size 13 is not a magic number, in fact the numbers of nonisomorphic orders for n-elements ordered set were known up to n =13 [15]. The result we obtained with size 14 cannot be published because of the performance of the validation technique we are using for isomorphism detection, which involves testing of all possibilities, in addition to the huge set of generated result for size 14. Therefore, we limit our validated results to size 13.

For the coding part: we used Borland C++ 5.2 as a programming language on 2 different platforms Windows NT4.0 W/S and Windows 2000 pro Server.

7.2 Results set

Our results are sets of ordered sets P with properties listed in the criteria of generation and testing, that is irreducible element free, retractable point free, non-isomorphic, dual-free, with the fixed point property (FPP), and of size ≤13.

We eventually obtained for size 10, and 11 the same results of Rutkowski and Schröder, but the results for size 12 and 13 have not been published (Figure 7.1).

Size	Ordered sets (FPP) total	
10	4	Ruthkowski [31]
11	8	Schröder[34]
12	107	Computation
13	1209	Computation

Figure 7-1 Main result

We also have the result of total generated ordered sets, the total number of non-isomorphic for each size (Figure 7.2)

Size	ordered sets (total)	Non-isomorphic
11	2206	57
12	35585	568
13	1401850	11173

Figure 7-2 Comparison table

We listed our result as sets of incidence matrices. These incidence matrices are listed in vectors format

Each vector is listed on a line where coordinates are separated by spaces. Each coordinate represents a row in the incidence matrix by converting the coordinate from its hexadecimal representation to a binary base, and then fitting each digit from the binary number in a cell of its specific row.

These vectors are listed in lexicographic order in the appendix

We also list our results up to size 12 by giving all the upward drawing of these ordered sets.

7.3 Observation

The most important observation is related to the upward graphs of all ordered sets with the (FPP). We can see a kind of inherency between ordered sets of different sizes. This part is the core of future research. We started working on it; however its results may not be available with this thesis.

Some observations on the ratio of non-isomorphic ordered sets with the (FPP) relative to the total number of non-isomorphic ordered sets Figure 10.3

Size	Ratio FPP/ total non-isomorphic
11	14 %
12	18.8 %
13	10.8 %

Figure 7-3 Ratio FPP / non-isomorphic

We can also mention that the removal of retractable elements reduced enormously the number of generated ordered sets.

7.4 Other results

Aside from the set of data results we obtained, this project was not only about listing data. In fact, some important results we obtained are the design and implementation of different algorithms or techniques, which can be very useful in ordered sets combinatorics.

7.5 Future works and research opportunities

Our work has introduced several topics of research worth investigating. The following describes possible avenues of research resulting from our work.

1. Investigating the generation of ordered sets (FPP) of size N based on ordered sets of size < N-1. In fact we have some observations related to this topic, we may include it in a future paper.

2. Extending our results to higher sizes by refining even further the computation. This part is especially for ordered sets generation, which is the most time consuming. This can probably done by integrating the isomorphism filtering technique in the generation phase, or the fixed point property algorithm in the generation so the ordered sets we obtained are tailored for (FPP).

3. The original question " characterize ordered sets for (FPP) " is still an unsolved problem.

4. Improving the technique for testing isomorphism based on the minimum representation

7.6 Achieving our objectives

Our first objective, to characterize ordered sets of size ≤ 13 was successfully achieved. Moreover, some new notions we introduced like the minimum representation of ordered sets and the dynamic range technique are part of the original objective, which is to develop an efficient technique to obtain the results. But still, one of the main objectives of this project is to acquire decent knowledge on ordered sets, with deep study of the (FPP) behavior.

7.7 Conclusion

In this thesis, we featured techniques for generating and characterizing ordered sets for the fixed point property. We defined a standard representation of an incidence matrix in a minimum form. Based on the new representation, we implemented an algorithm for isomorphic sets detection.

Moreover, we implemented an algorithm for (FPP) characterization, based on a concept we defined as dynamic range loop.

From the result of the code we implemented we could extend the characterization of ordered sets for (FPP) up to size 13, after obtaining results for size 12 a total of 107 different sets and for size 13 a total of 1209 ordered sets.

We expect these results to be beneficial for further research in (FPP) characterization.

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APPENDIX A

This appendix lists the results for sizes 10, 11, 12 and 13

Ordered sets are listed as vectors of incidence matrix. Each column is a row in an incidence matrix. This representation is in hexadecimal, we need to convert each row in to binary base then fit it in the incidence matrix.

For size 10 we have 4 sets:

1 2 4 b 13 25 46 9b 16f 277
1 2 4 b 13 25 46 af 15f 277
1 2 4 b 13 25 46 bf 16f 277
1 2 4 b 15 26 4f bf 15f 26f

For size 11 we have 8 sets:

1 2 4 8 13 25 49 8e 137 2df 4ef
1 2 4 8 13 25 49 8e 17f 2bf 4df
1 2 4 8 13 25 49 8e 1bf 2df 4ef
1 2 4 8 13 25 4a 8c 137 2df 4ef
1 2 4 8 13 25 4a 8c 17f 2bf 4df
1 2 4 b 13 25 45 86 13f 2cf 4f7
1 2 4 b 13 25 45 86 17f 2af 4d7
1 2 4 b 15 26 5f 9f 17f 2bf 4df

For size 12 we have 107 sets:

1 2 4 8 10 23 4c 95 11a 2ff 5bf 9df
1 2 4 8 13 23 45 89 10e 277 59f 9ef
1 2 4 8 13 23 45 89 10e 2df 53f 9ef
1 2 4 8 13 23 45 89 10e 2df 56f 9bf
1 2 4 8 13 23 45 89 10e 2ff 51f 9ef
1 2 4 8 13 23 45 89 10e 2ff 55f 9af
1 2 4 8 13 23 45 89 10e 2ff 55f 9ef
1 2 4 8 13 23 45 8a 10c 233 5df 9ef
1 2 4 8 13 23 45 8a 10c 257 5bf 9ef
1 2 4 8 13 23 45 8a 10c 277 59f 9ef
1 2 4 8 13 23 45 8a 10c 277 5df 9ef
1 2 4 8 13 23 45 8a 10c 2df 56f 9bf
1 2 4 8 13 23 45 8a 10c 2df 57f 9ef
1 2 4 8 13 23 45 8a 10c 2ff 55f 9af
1 2 4 8 13 23 45 8a 10c 2ff 55f 9ef
1 2 4 8 13 23 45 8a 10c 2ff 5df 9ef
1 2 4 8 13 23 45 8a 10c 35f 5bf 9ef
1 2 4 8 13 23 45 8a 10c 37f 5df 9ef
1 2 4 8 13 25 46 89 10a 237 5df 9ef
1 2 4 8 13 25 46 89 10a 277 59b 9ef
1 2 4 8 13 25 46 89 10a 277 5af 9df
1 2 4 8 13 25 46 89 10a 277 5bf 9ef
1 2 4 8 13 25 46 89 10a 2ad 57f 9df
1 2 4 8 13 25 46 89 10a 2bf 57f 9df
1 2 4 8 13 25 46 89 10a 2bf 57f 9ef
1 2 4 8 13 25 46 89 10a 2df 53f 9ef
1 2 4 8 13 25 46 89 10a 2df 57f 9bf
1 2 4 8 13 25 46 89 10a 2df 57f 9ef

1 2 4 8 13 25 46 89 10a 2ef 57f 9bf

1 2 4 8 13 25 46 89 10a 2ef 57f 9df

1 2 4 8 13 25 46 89 10a 2ff 5bf 9ef

1 2 4 8 13 25 46 89 10a 2ff 5df 9ef

1 2 4 8 13 25 46 89 10e 277 59f 9ef

1 2 4 8 13 25 46 89 10e 277 5bf 9cf

1 2 4 8 13 25 46 89 10e 2df 53f 9ef

1 2 4 8 13 25 46 89 10e 2df 56f 9bf

1 2 4 8 13 25 46 89 10e 2df 57f 9af

1 2 4 8 13 25 46 89 10e 2ff 51f 9ef

1 2 4 8 13 25 46 89 10e 2ff 53f 9df

1 2 4 8 13 25 46 89 10e 2ff 54e 9bf

1 2 4 8 13 25 46 89 10e 2ff 55f 9af

1 2 4 8 13 25 46 89 10e 2ff 55f 9bf

1 2 4 8 13 25 46 89 10e 2ff 59f 9ef

1 2 4 8 13 25 46 89 10e 2ff 5bf 9cf

1 2 4 8 13 25 46 89 14e 277 5df 9ef

1 2 4 8 13 25 46 89 14e 2df 57f 9ef

1 2 4 8 13 25 46 89 14e 2ff 55f 9ef

1 2 4 8 13 25 46 89 14e 2ff 5df 9ef

1 2 4 8 13 25 47 89 10e 277 5bf 9cf

1 2 4 8 13 25 47 89 10e 2df 53f 9ef

1 2 4 8 13 25 47 89 10e 2df 56f 9bf

1 2 4 8 13 25 47 89 10e 2ff 53f 9cf

1 2 4 8 13 25 47 89 10e 2ff 54f 9bf

1 2 4 8 13 25 47 89 10e 2ff 5bf 9cf

1 2 4 8 13 25 47 8a 10c 257 5bf 9ef

1 2 4 8 13 25 47 8a 10c 277 5bf 9cf

1 2 4 8 13 25 47 8a 10c 2bf 55f 9ef

1 2 4 8 13 25 47 8a 10c 2bf 56f 9df

1 2 4 8 13 25 47 8a 10c 2bf 57f 9cf

1 2 4 8 13 25 47 8a 10c 2cf 57f 9bf

1 2 4 8 13 25 47 8a 10c 2df 56f 9bf

1 2 4 8 13 25 47 8a 10c 2df 5bf 9ef

1 2 4 8 13 25 47 8a 10c 2ef 55f 9bf

1 2 4 8 13 25 47 8a 10c 2ef 5bf 9df

1 2 4 8 13 25 47 8a 10c 2ff 5bf 9cf

1 2 4 8 13 25 49 8e 10e 2bf 55f 9ef

1 2 4 8 13 25 49 8e 10e 2bf 57f 9cf

1 2 4 8 13 25 49 8e 10e 2ff 57f 98e

1 2 4 8 13 25 49 8e 10e 2ff 57f 99f

1 2 4 8 13 25 49 8e 10f 2ff 53f 9cf

1 2 4 8 13 25 49 8e 10f 2ff 57f 98f

1 2 4 8 13 25 49 8e 117 2bf 57f 9df

1 2 4 8 13 25 49 8e 117 2ff 57f 99f

1 2 4 8 13 25 49 8e 11f 2ff 57f 99f

1 2 4 8 13 25 49 8e 17f 2bf 4df 8ef

1 2 4 8 13 25 4a 8c 10f 2ff 51f 9ef

1 2 4 8 13 25 4a 8c 10f 2ff 53f 9cf

1 2 4 8 13 25 4a 8c 10f 2ff 56f 99f

1 2 4 8 13 25 4a 8c 117 27f 5bf 9df

1 2 4 8 13 25 4a 8c 117 2bf 57f 9df

1 2 4 8 13 25 4a 8c 117 2ff 537 9df

1 2 4 8 13 25 4a 8c 117 2ff 55f 9bf

1 2 4 8 13 25 4a 8c 117 2ff 57f 99f

1 2 4 8 13 25 4a 8c 117 2ff 57f 9df

1 2 4 8 13 25 4a 8c 117 2ff 5bf 9df

1 2 4 8 13 25 4a 8c 11f 2ff 53f 9df

1 2 4 8 13 25 4a 8c 11f 2ff 57f 99f

1 2 4 8 13 25 4a 8c 137 2ff 57f 9bf

1 2 4 8 13 25 4a 8c 13f 2ff 57f 9bf

1 2 4 8 13 25 4a 8c 16f 2ff 57f 9ef

1 2 4 8 13 25 4a 8d 10e 2df 53f 9ef

1 2 4 8 13 25 4a 8d 10e 2df 56f 9bf

1 2 4 8 13 25 4a 8d 10e 2df 57f 9af

1 2 4 8 13 25 4a 8d 10e 2ef 57f 99f

1 2 4 8 13 25 4a 8d 10e 2ff 57f 98f

1 2 4 8 13 25 4a 8d 10e 2ff 57f 99f

1 2 4 8 13 25 4a 8d 14e 2df 57f 9ef

1 2 4 8 13 25 4a 8d 14e 2ef 57f 9df

1 2 4 8 13 25 4a 8d 14e 2ff 57f 9cf

1 2 4 8 13 25 4a ad 14e 2ff 57f 9ef

1 2 4 8 13 25 4b 8d 10e 2df 56f 9bf

1 2 4 8 13 25 4b 8d 10e 2ff 56f 99f

1 2 4 8 13 25 4b 8e 12d 2ef 57f 9bf

1 2 4 8 13 2d 4e 97 11b 26f 5bf 9df

1 2 4 8 13 2d 4e 97 11b 27f 5bf 9df

1 2 4 b 13 25 46 8f 11b 277 5bf 9df

1 2 4 b 15 26 5f bf 13f 2ff 57f 9bf

For size 13 we have 1209 sets:

1 2 4 8 10 20 47 99 12a 234 5ff aff 137f

1 2 4 8 10 23 43 8c 115 21a 537 aff 13df

1 2 4 8 10 23 43 8c 115 21a 577 abf 13df

1 2 4 8 10 23 43 8c 115 21a 5bf aff 13df

1 2 4 8 10 23 43 8c 115 21a 5ff b3f 13df

1 2 4 8 10 23 45 89 111 21e 4ef b3f 13df

1 2 4 8 10 23 45 89 111 21e 5ff a7f 13bf

1 2 4 8 10 23 45 89 112 21c 467 bbf 13df

1 2 4 8 10 23 45 89 112 21c 4cd b7f 13bf

1 2 4 8 10 23 45 89 112 21c 4ef b3f 13df

1 2 4 8 10 23 45 89 112 21c 4ef b7f 13bf

1 2 4 8 10 23 45 89 112 21c 4ef b7f 13df

1 2 4 8 10 23 45 89 112 21c 577 abf 13df

1 2 4 8 10 23 45 89 112 21c 577 add 13bf

1 2 4 8 10 23 45 89 112 21c 577 aff 139f

1 2 4 8 10 23 45 89 112 21c 577 aff 13bf

1 2 4 8 10 23 45 89 112 21c 577 bbf 13df

1 2 4 8 10 23 45 89 112 21c 5ff a7f 13df

1 2 4 8 10 23 45 89 112 21c 5ff add 133f

1 2 4 8 10 23 45 89 112 21c 5ff add 137f

1 2 4 8 10 23 45 89 112 21c 5ff aff 131e

1 2 4 8 10 23 45 89 112 21c 5ff aff 133f

1 2 4 8 10 23 45 89 112 21c 5ff aff 135f

1 2 4 8 10 23 45 89 112 21c 5ff aff 137f

1 2 4 8 10 23 45 89 112 21c 5ff b3f 13df

1 2 4 8 10 23 45 89 112 21c 5ff b7f 13bf

1 2 4 8 10 23 45 89 112 21c 5ff b7f 13df

1 2 4 8 10 23 45 89 112 21c 67f bbf 13df

1 2 4 8 10 23 45 89 112 21c 6dd b7f 13bf

1 2 4 8 10 23 45 89 112 21c 6ff b5f 13bf

1 2 4 8 10 23 45 89 112 21c 6ff b7f 13bf

1 2 4 8 10 23 45 89 112 21c 77f bbf 13df

1 2 4 8 10 23 45 8a 113 21c 4ef b3f 13df

1 2 4 8 10 23 45 8a 113 21c 4ef b5f 13bf

1 2 4 8 10 23 45 8a 113 21c 533 aff 13df

1 2 4 8 10 23 45 8a 113 21c 557 aff 13bf

1 2 4 8 10 23 45 8a 113 21c 577 abf 13df

1 2 4 8 10 23 45 8a 113 21c 577 aff 139f

1 2 4 8 10 23 45 8a 113 21c 577 aff 13df

1 2 4 8 10 23 45 8a 113 21c 5df aff 137f

1 2 4 8 10 23 45 8a 113 21c 5ff a7f 13df

1 2 4 8 10 23 45 8a 113 21c 5ff aff 135f

1 2 4 8 10 23 45 8a 113 21c 5ff aff 13df

1 2 4 8 10 23 45 8a 113 21c 6ff b3f 13df

1 2 4 8 10 23 45 8a 113 21c 6ff b5f 13bf

1 2 4 8 10 23 45 8a 114 218 467 bbf 13df

1 2 4 8 10 23 45 8a 114 218 4ef b5d 13bf

1 2 4 8 10 23 45 8a 114 218 4ef b7f 13bf

1 2 4 8 10 23 45 8a 114 218 4ef b7f 13df

1 2 4 8 10 23 45 8a 114 218 5ff aff 137f

1 2 4 8 10 23 45 8a 114 218 5ff aff 13df

1 2 4 8 10 23 45 8a 114 219 467 bbf 13df

1 2 4 8 10 23 45 8a 114 219 4ef b5d 13bf

1 2 4 8 10 23 45 8a 114 219 4ef b7f 13bf

1 2 4 8 10 23 45 8a 114 219 4ef bbf 13df

1 2 4 8 10 23 45 8a 114 219 5ff abb 135d

1 2 4 8 10 23 45 8a 114 219 5ff abb 137f

1 2 4 8 10 23 45 8a 114 219 5ff abb 13df

1 2 4 8 10 23 45 8a 114 219 5ff aff 137f

1 2 4 8 10 23 45 8a 114 219 5ff aff 13df

1 2 4 8 10 23 45 8a 114 219 5ff bbf 13df

1 2 4 8 10 23 45 8a 114 219 67f bbf 13df
1 2 4 8 10 23 45 8a 114 219 6df b7f 13bf
1 2 4 8 10 23 45 8a 114 219 6ff b7f 139f
1 2 4 8 10 23 45 8a 116 219 4ef b3f 13df
1 2 4 8 10 23 45 8a 116 219 4ef b5f 13bf
1 2 4 8 10 23 45 8a 116 219 537 aff 13df
1 2 4 8 10 23 45 8a 116 219 577 abb 13df
1 2 4 8 10 23 45 8a 116 219 577 adf 13bf
1 2 4 8 10 23 45 8a 116 219 577 aff 139f
1 2 4 8 10 23 45 8a 116 219 577 aff 13df
1 2 4 8 10 23 45 8a 116 219 59e aff 137f
1 2 4 8 10 23 45 8a 116 219 5bf aff 137f
1 2 4 8 10 23 45 8a 116 219 5bf aff 13df
1 2 4 8 10 23 45 8a 116 219 5df aff 137f
1 2 4 8 10 23 45 8a 116 219 5ff aff 133f
1 2 4 8 10 23 45 8a 116 219 5ff aff 135f
1 2 4 8 10 23 45 8a 116 219 5ff aff 13df
1 2 4 8 10 23 45 8a 116 219 5ff b3f 13df
1 2 4 8 10 23 45 8a 116 219 5ff b5f 13bf
1 2 4 8 10 23 45 8a 11c 233 4ef b7f 13bf
1 2 4 8 10 23 45 8a 11c 233 5ff a77 13bf
1 2 4 8 10 23 45 8a 11c 233 5ff aff 137f
1 2 4 8 10 23 45 8a 11c 233 5ff b7f 13bf
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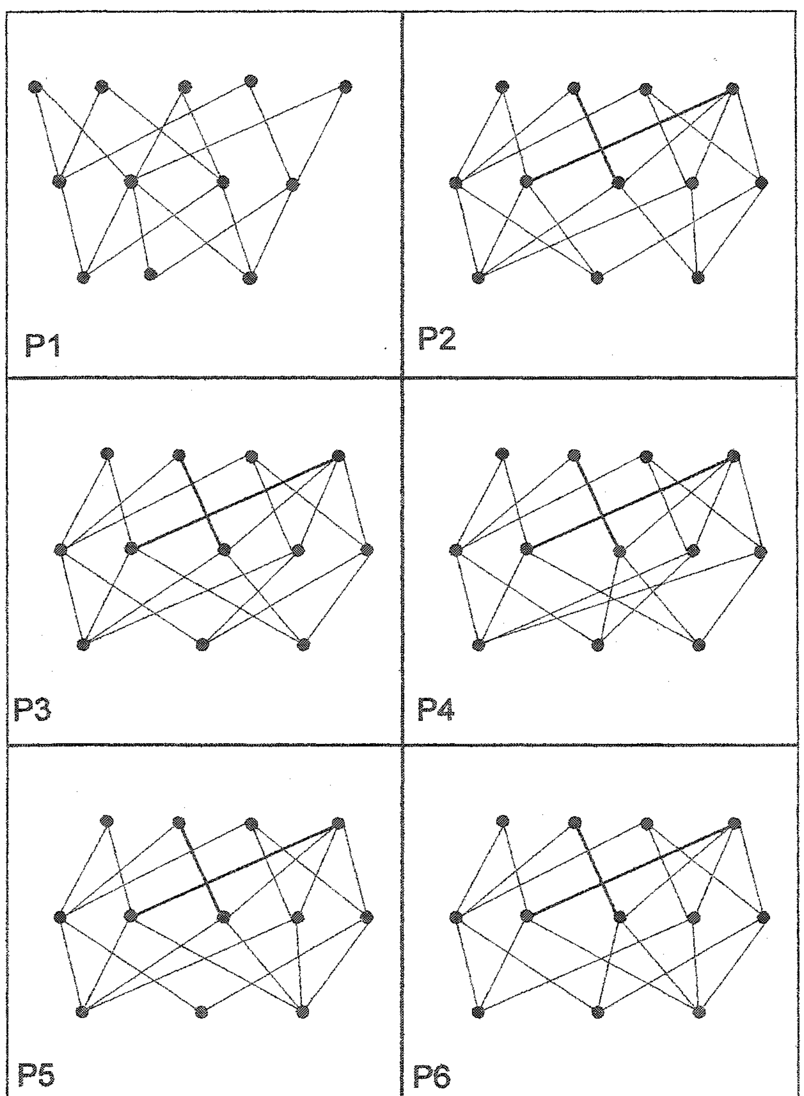
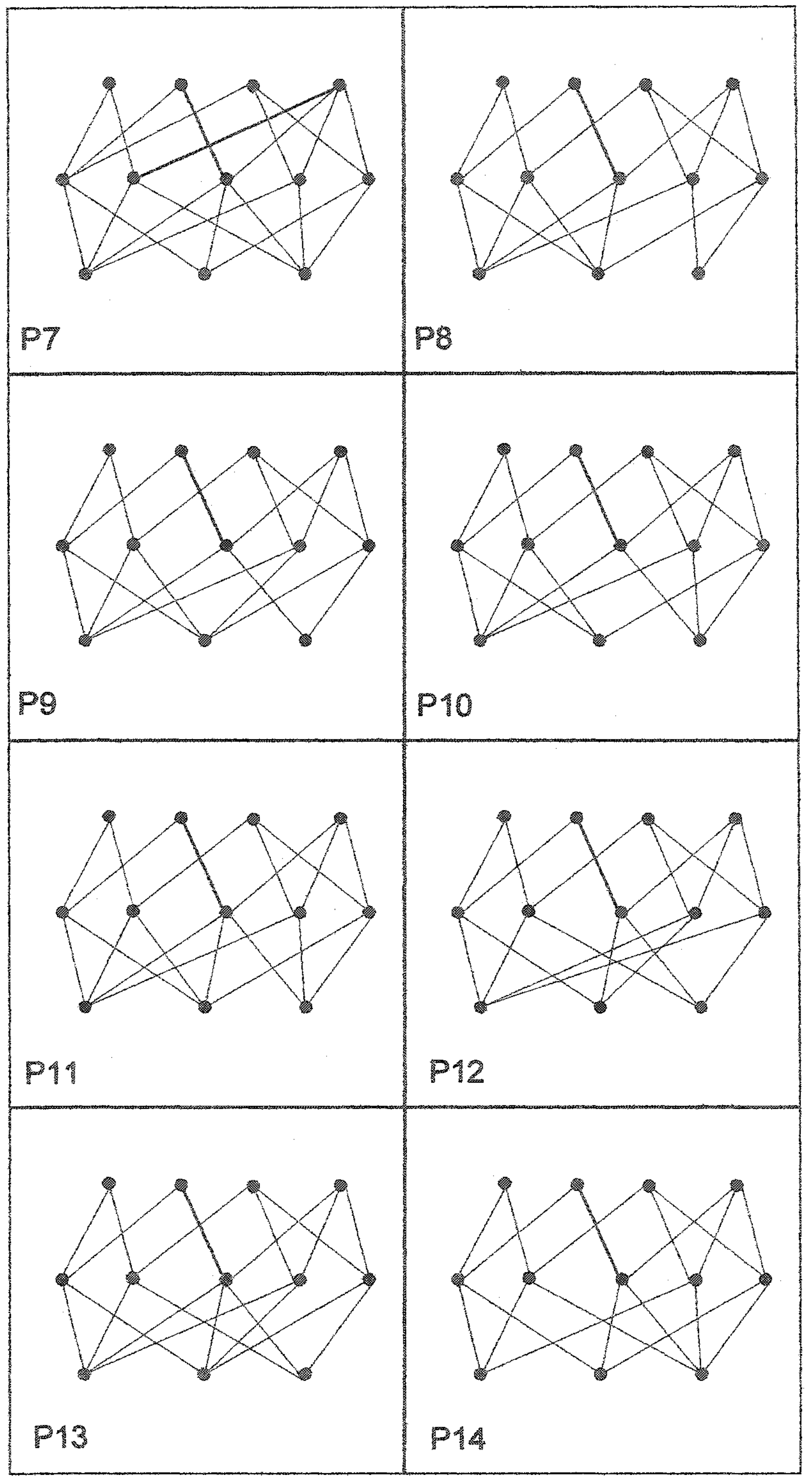
APPENDIX B

This appendix lists Upward drawing for ordered sets (FPP) up to size 12

Size 12:

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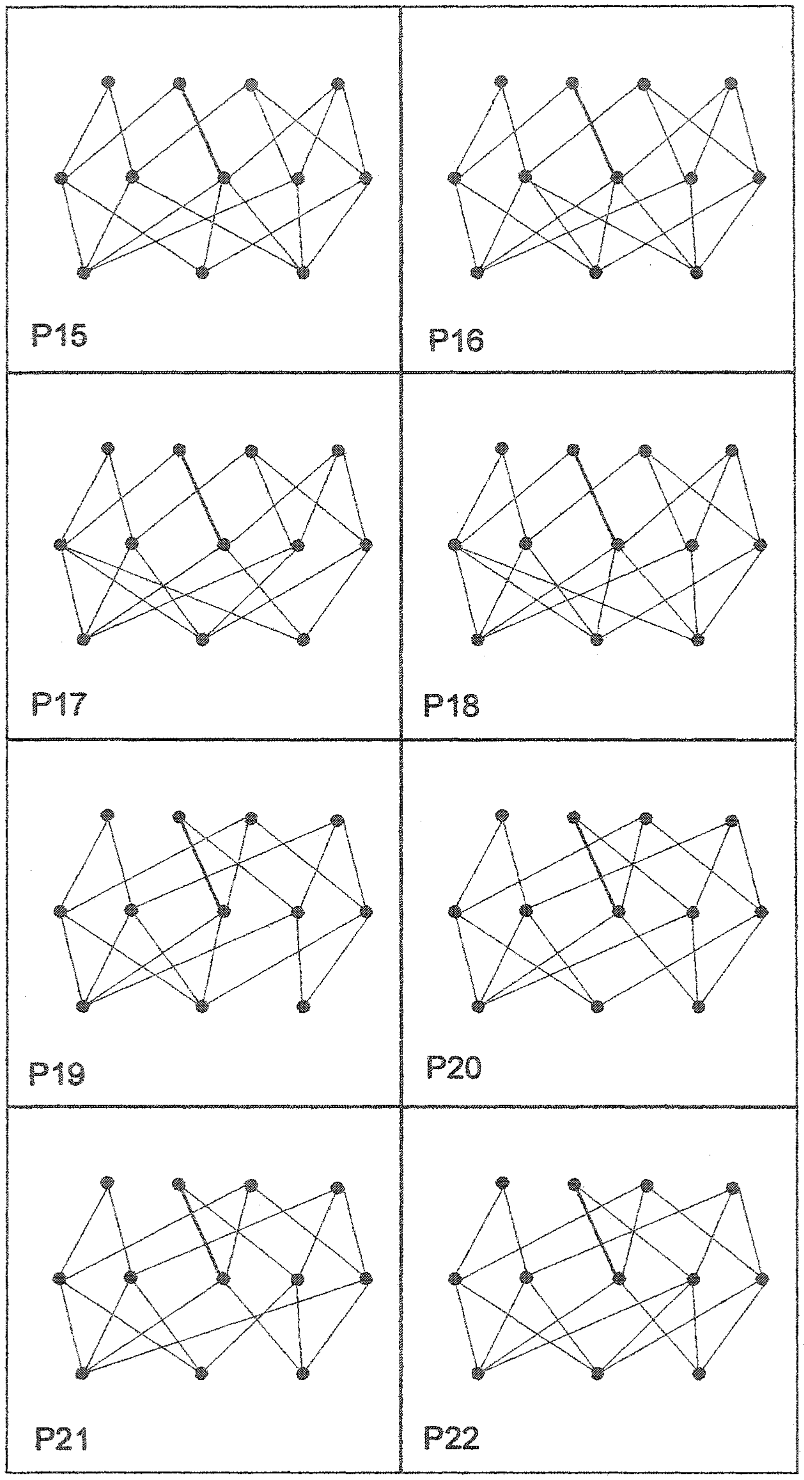
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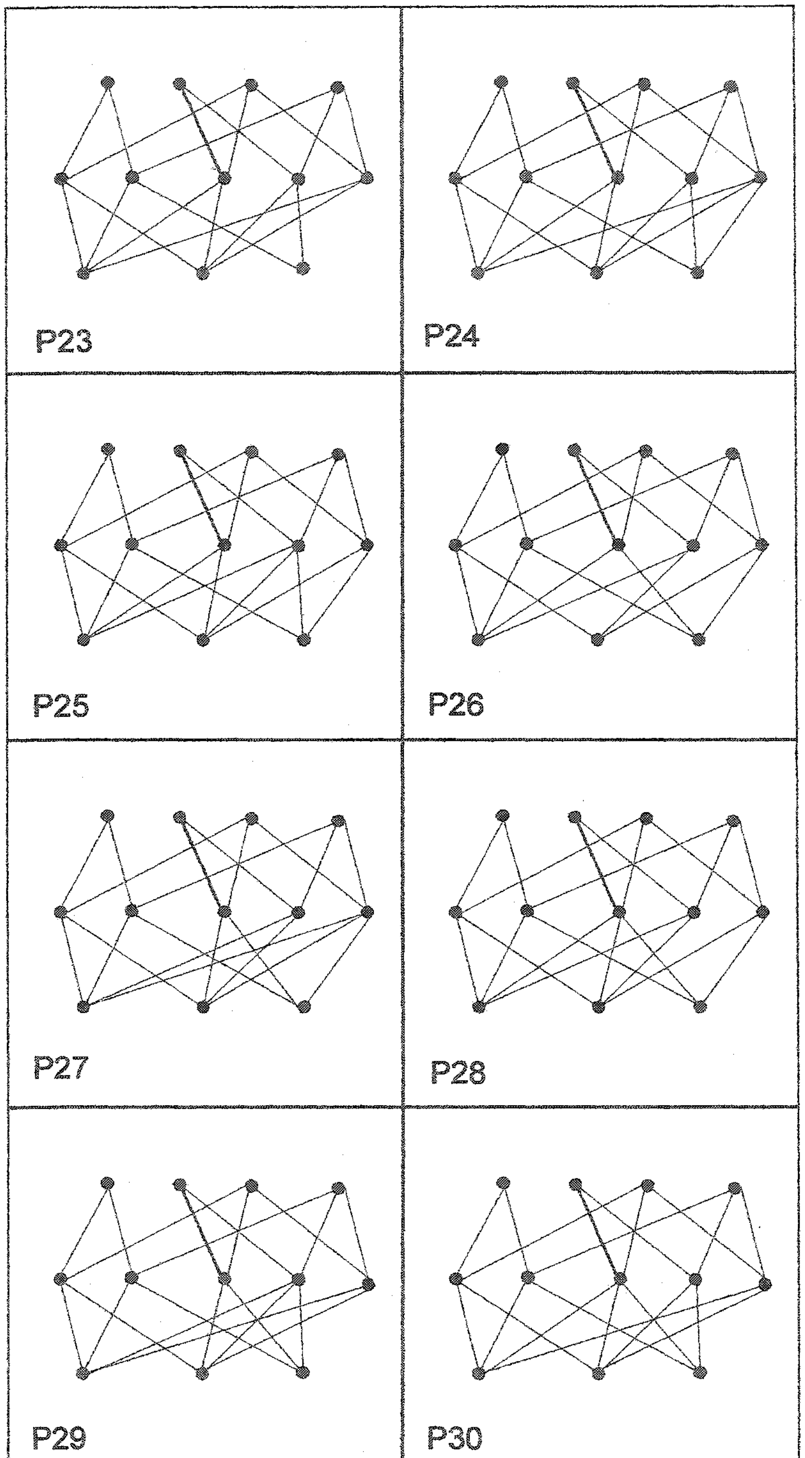
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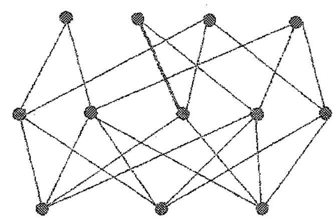
P30



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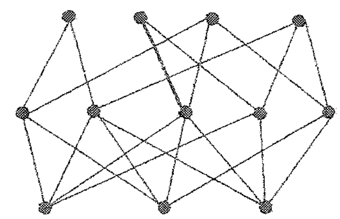

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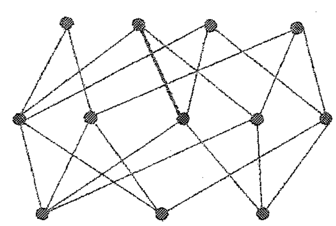

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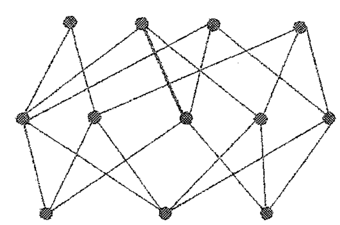

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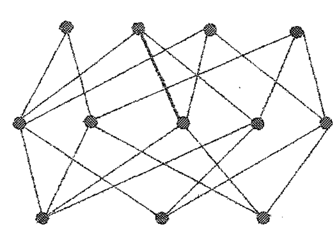

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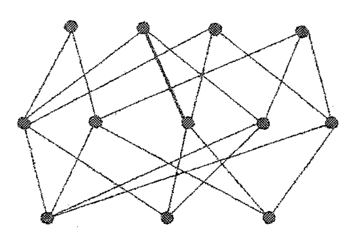

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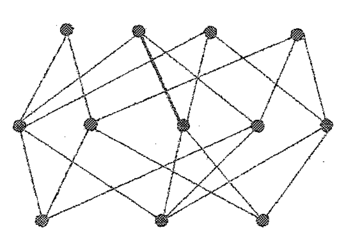

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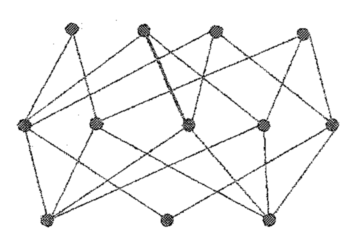

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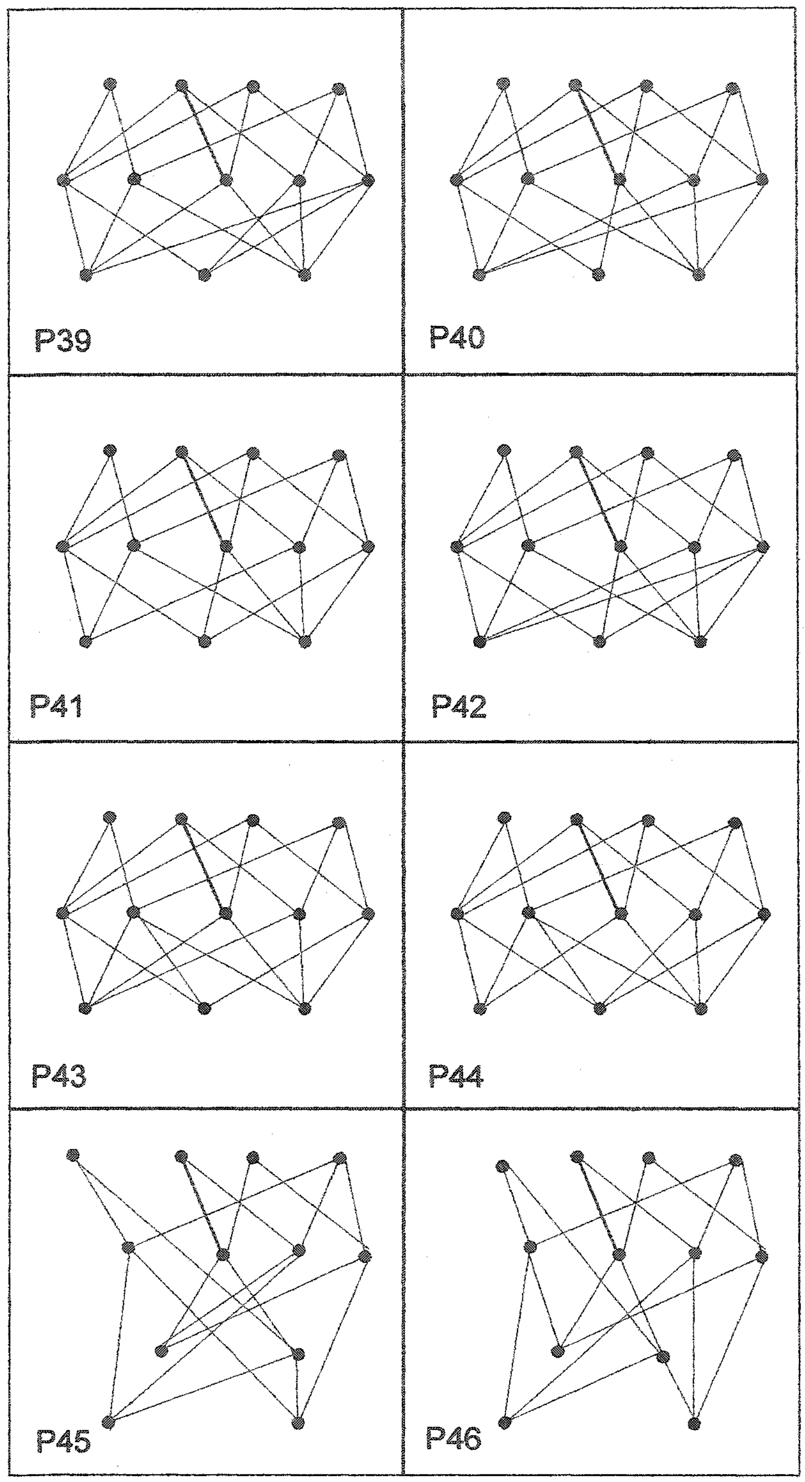
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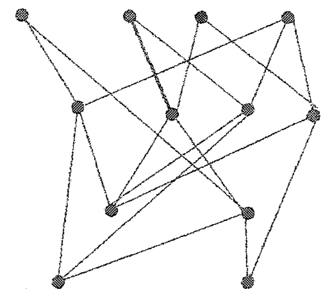
P46



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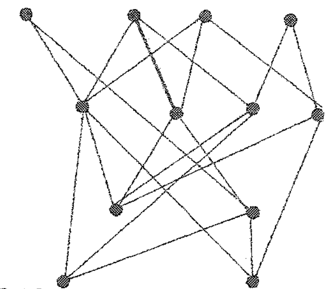

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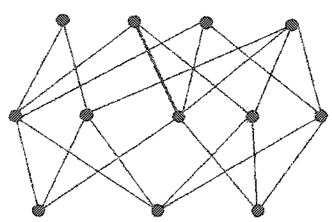

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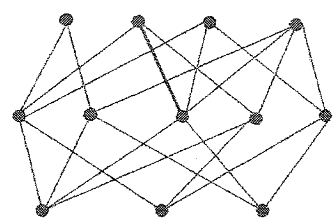

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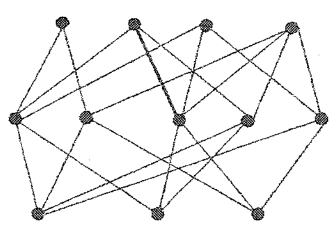

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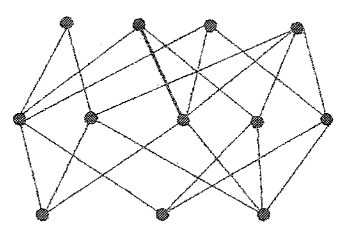

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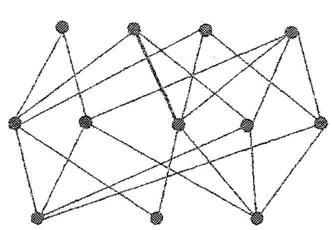

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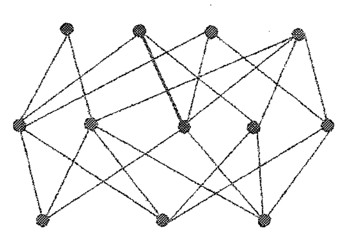

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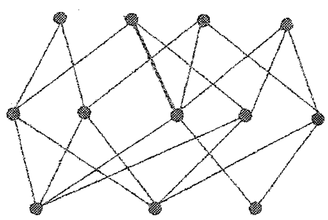

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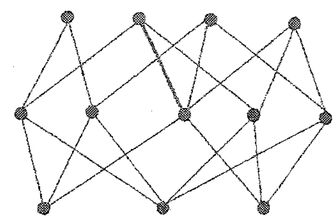

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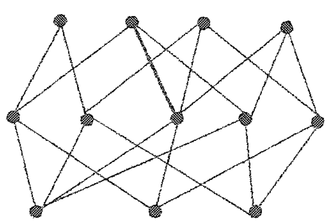

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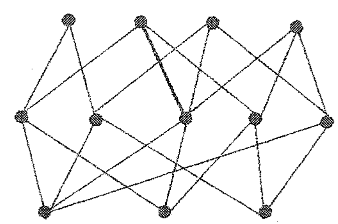

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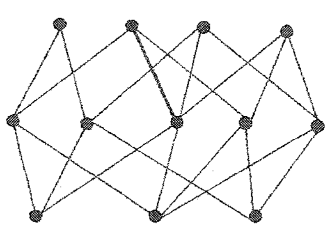

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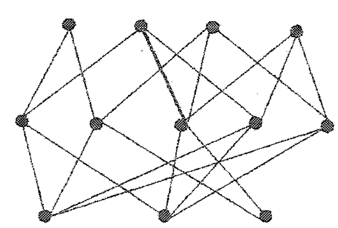

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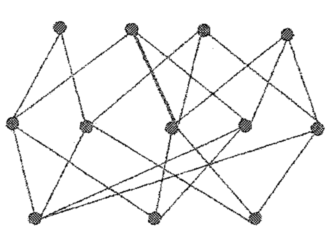

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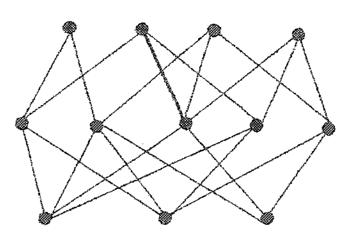

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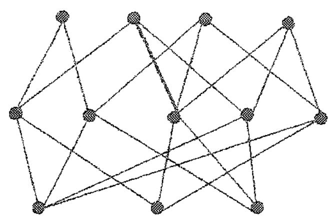
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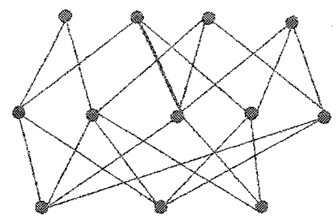
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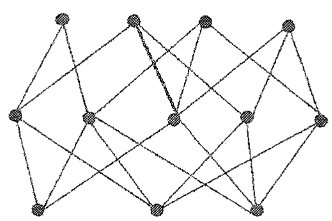
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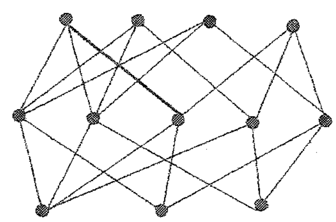
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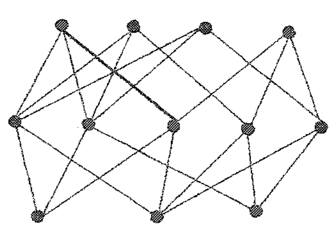
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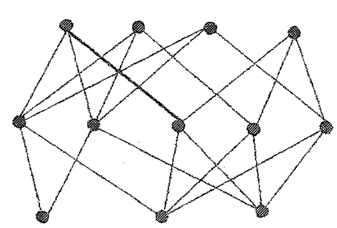
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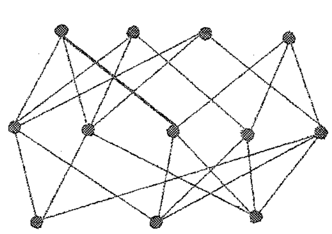
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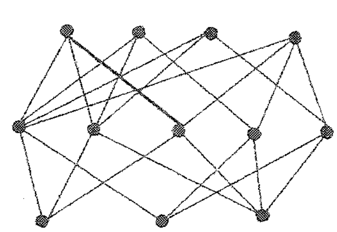
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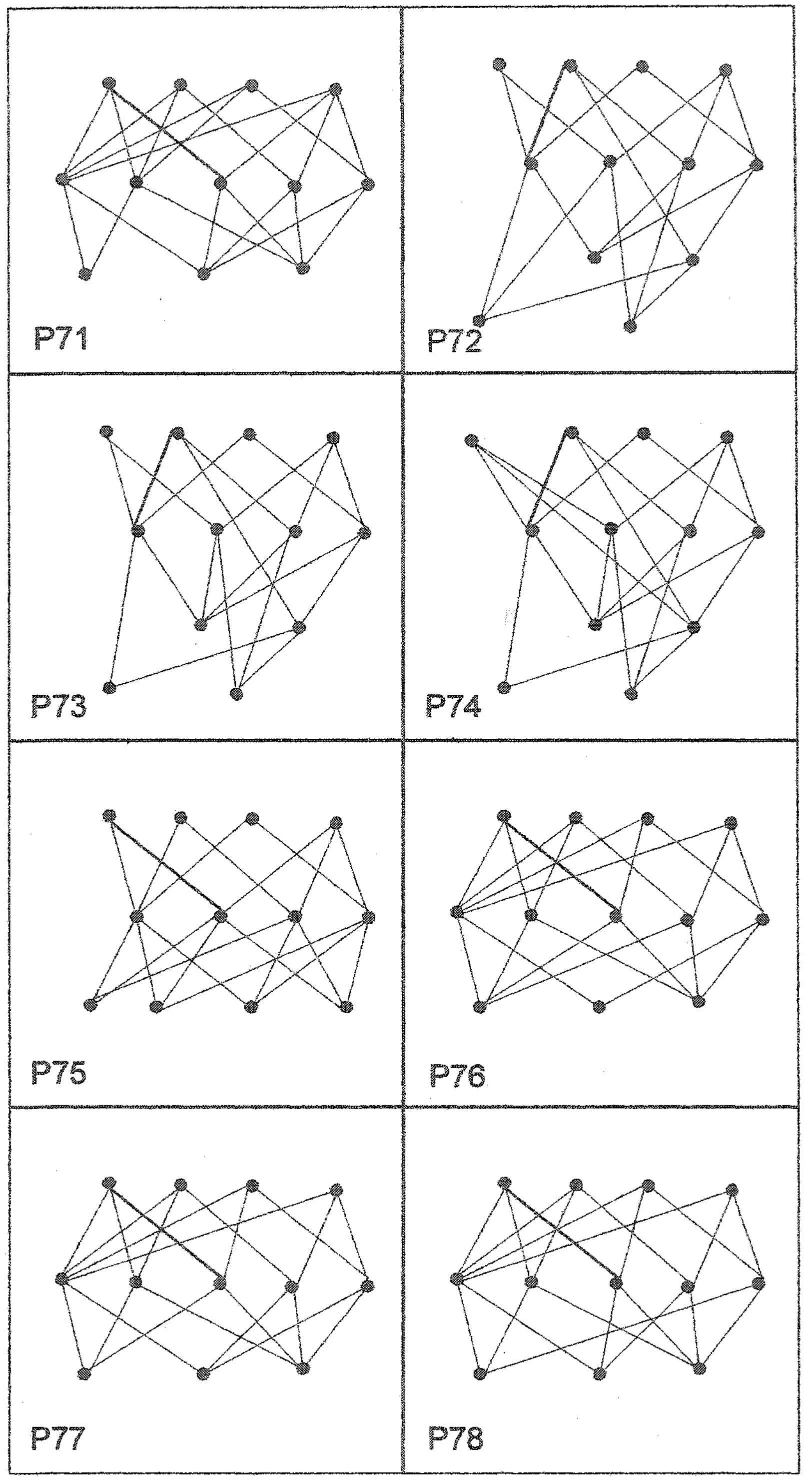
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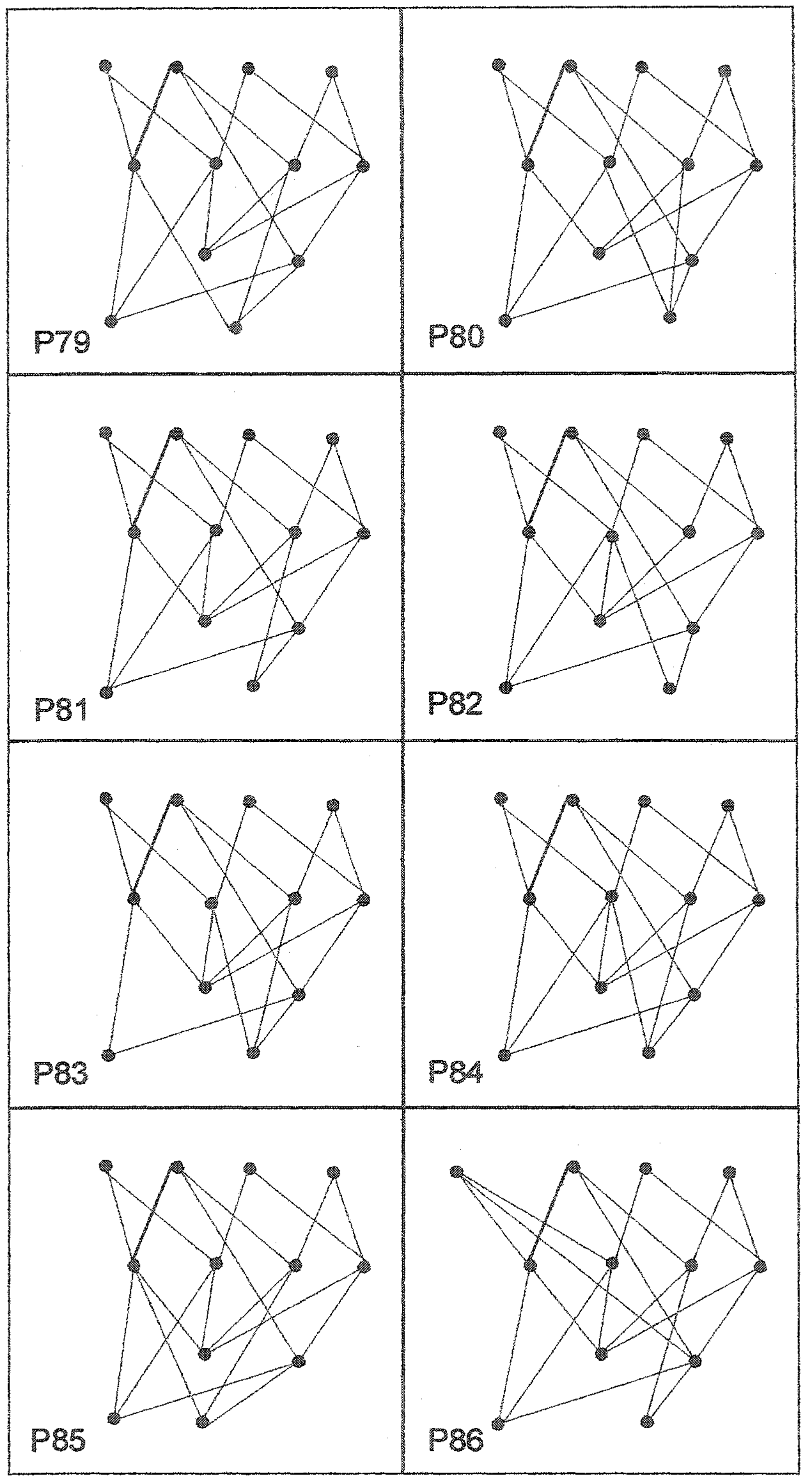
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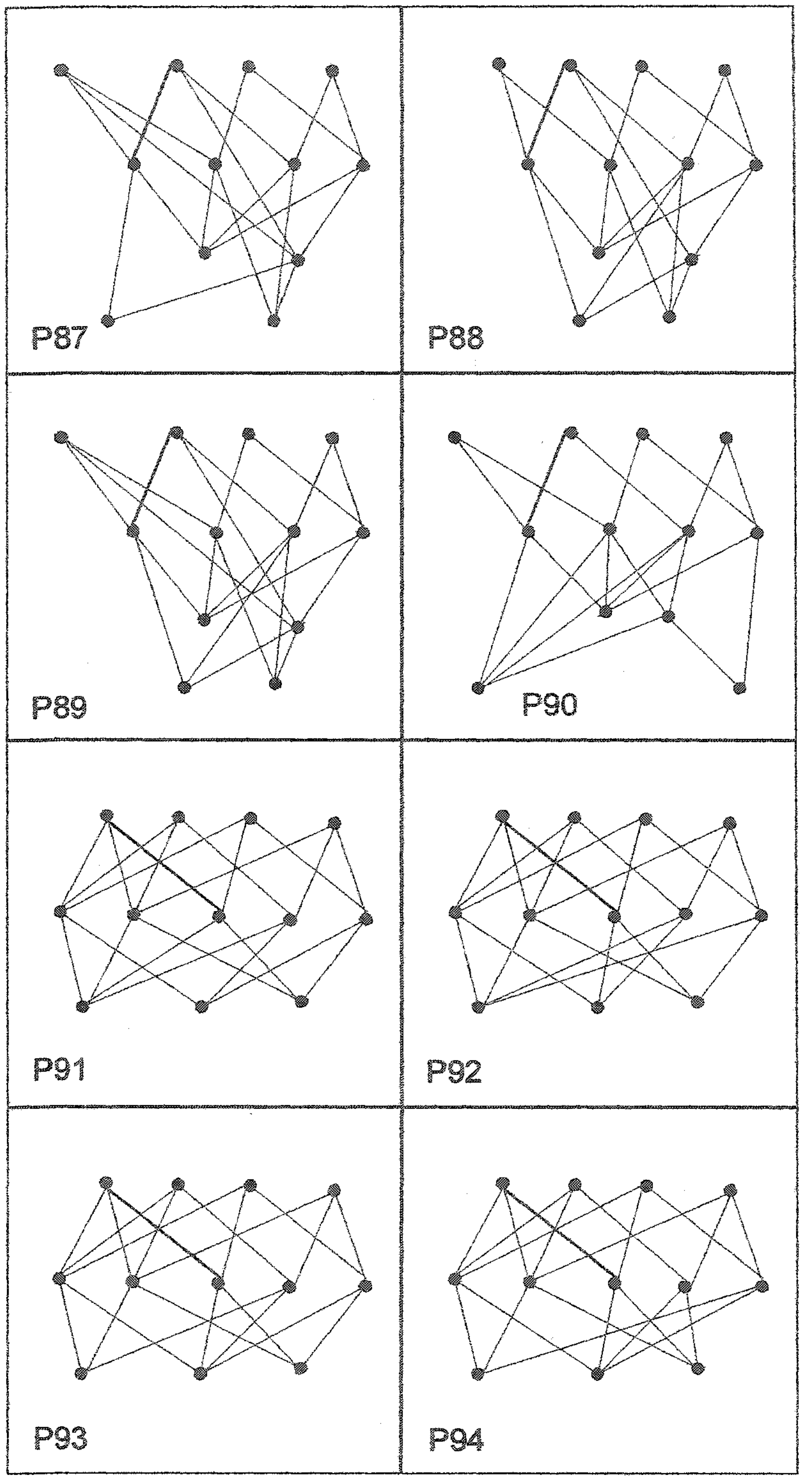
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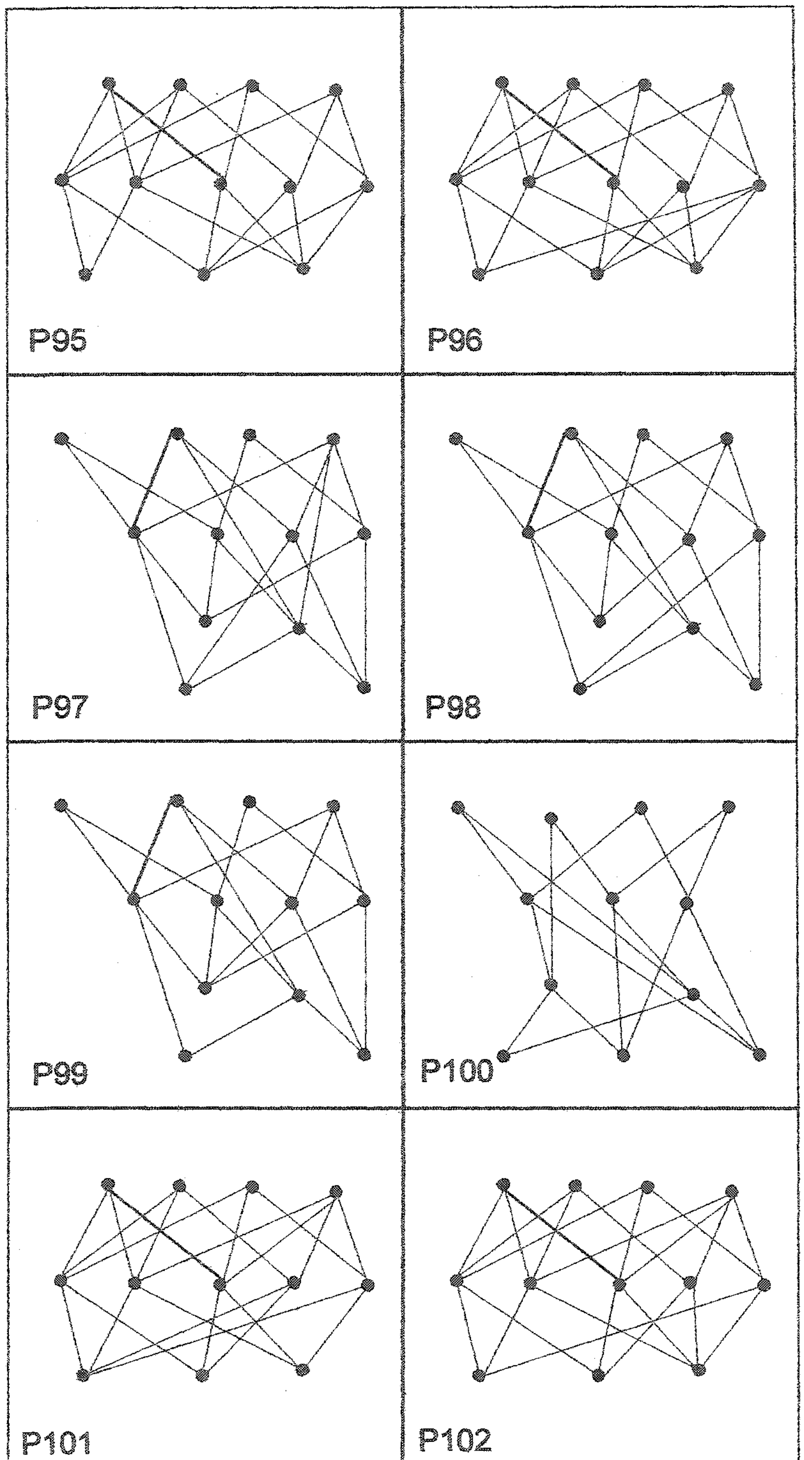
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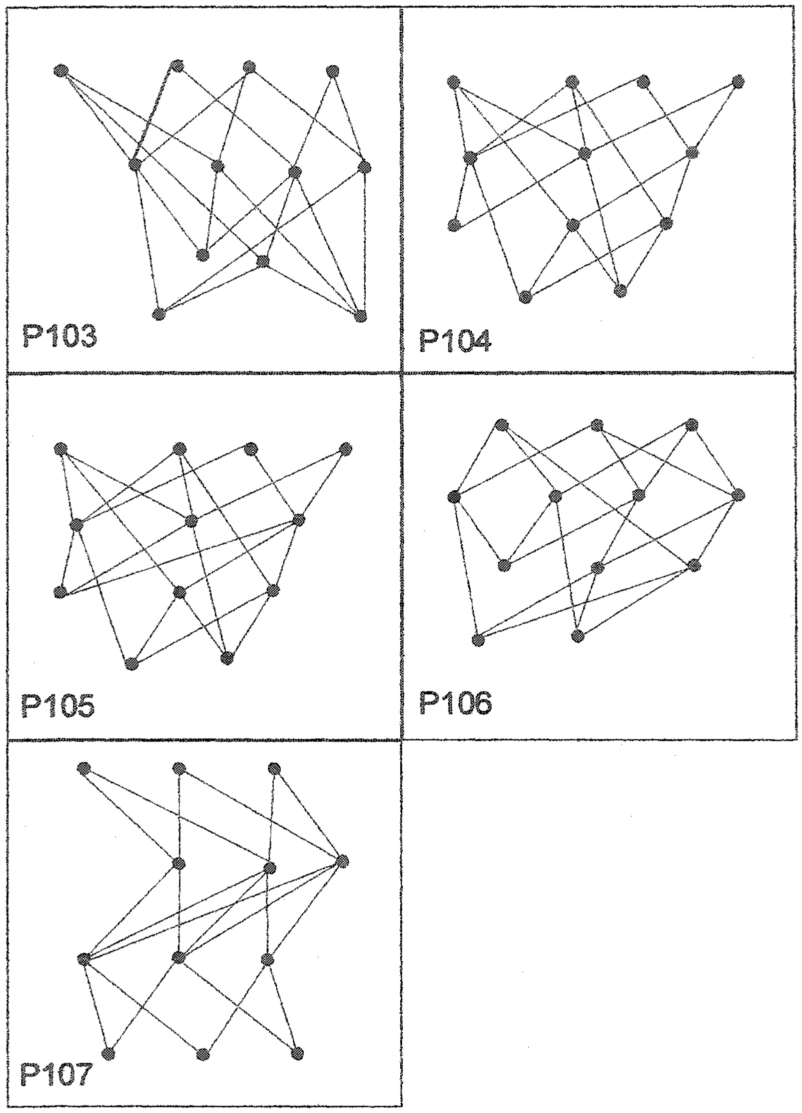
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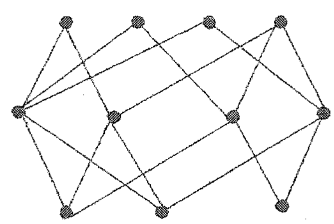
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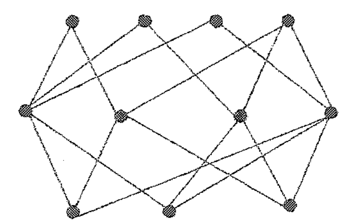
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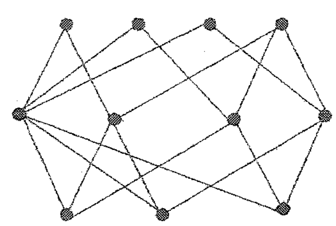
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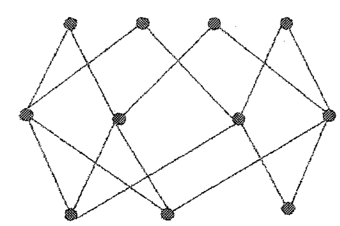
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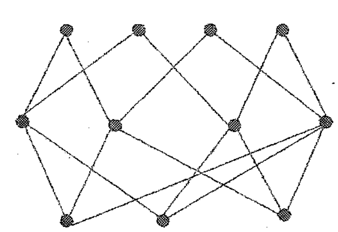
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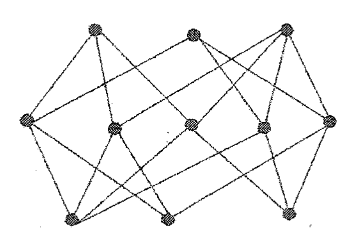
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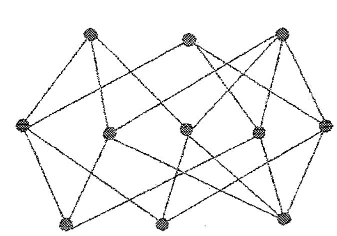
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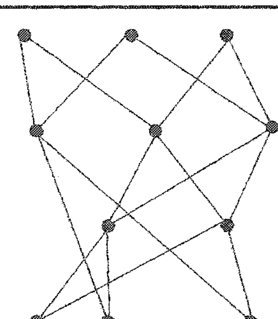
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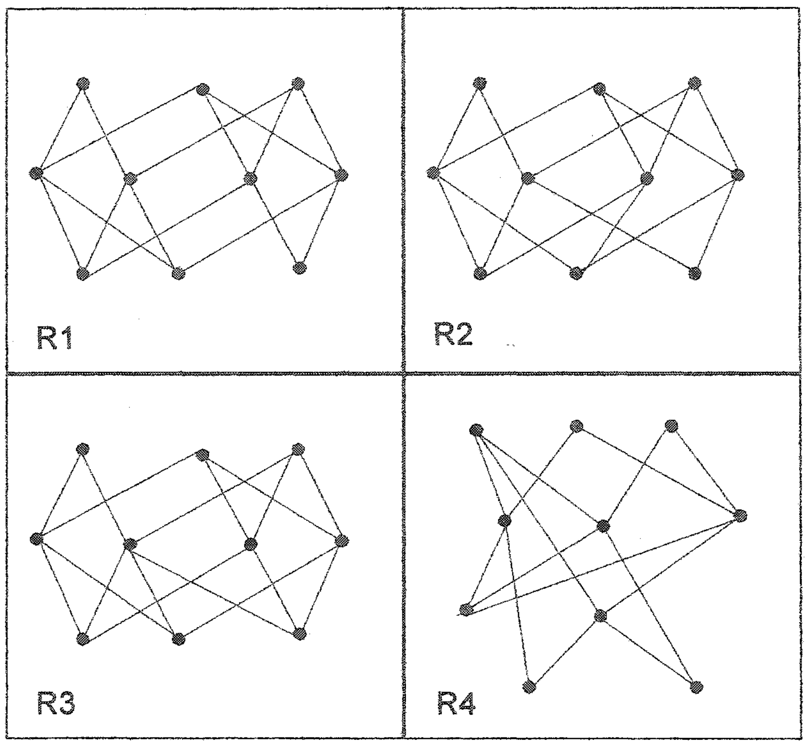

Size 13 Samples: