

A SURVEY OF EGOROV ' S MAXIMUM PRINCIPLE FOR
DISTRIBUTED PARAMETER CONTROL SYSTEMS , AND APPLICATIONS

by

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ABSTRACT

The majority of physical processes in engineering are controllable; consequently, we are interested in obtaining the optimum choice with respect to some criteria.

For the physical processes whose behaviour is described by a system of ordinary differential equations, the solution is given by the L.S. Pontryagin maximum principle [1]. However, there exist a great number of processes described by partial differential equations of various types and with different boundary conditions. These control systems are called distributed parameter control systems.

The present work is divided in two parts:

In the first part, following the original work of A.I. Egorov [2], a review is presented giving the necessary conditions for optimality in a distributed parameter control systems described by parabolic equations.

In the second part, the necessary conditions for optimality are applied to a thermodynamics system and to a biological system.

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INTRODUCTION

Many existing industrial processes and control systems operate under conditions in which their potential working capabilities are not fully exploited. We call a system optimal when it is the best possible, under certain given working conditions.

The main technique for solving the optimal problem for systems described by ordinary differential equations has been based on Pontryagin's Maximum Principle [1], and on the method of dynamic programming due to Bellman [8].

In physical and engineering processes, one often encounters systems whose parameters are distributed in both space and time. The behaviour of these systems is described by partial differential equations, integral equations, integrodifferential equations and sometimes by more general functional equations. Optimal control problems for systems with distributed parameters frequently arise in mechanics, mathematical physics and engineering. In [9], an extensive bibliography may also be found, where different methods are developed which enable one to solve some problems of practical importance. For instance, in [10], the problem of optimal control of temperature distribution in solids is considered.

The difficulty in the formulation of distributed parameter control problems is that, in general consideration, the formulation should be broad enough so as to retain the necessary generality, and on the other hand, it should be narrow enough so as to permit devising effective means for solving the problem. It is far more difficult to develop theories and techniques for distributed parameter optimal control problems than for their lumped equivalents, because of the highly complicated character of the governing equations and because of certain particular features encountered in partial differential equations which are not found in ordinary differential equations.

The present work is divided in two parts:

First, a survey of Egorov's Maximum Principle (Necessary Conditions) for parabolic systems is given.

Next, two applications of Egorov's Maximum Principle in Thermodynamics and Biology are described.

CHAPTER 1

STATEMENT OF THE PROBLEM

1.1 Definitions

Let G be a bounded domain of an n -dimensional Euclidean space E^n .

The elements of E^n are denoted by $x = (x_1, \dots, x_n)$.

We assume that G is bounded by a class $A^{(2)}$ surface.

Next, we give the definition of class $A^{(2)}$ domains [3].

Definition 1.1.1

For $\lambda \in (0, 1]$, we shall say that a function $u(x)$ defined (and continuous) on a set A is Hölder continuous with exponent λ in A , or briefly λ -Hölder continuous in A , if the set

$$\left\{ \frac{|u(x) - u(y)|}{[\rho(x, y)]^\lambda} / (x, y) \in A \otimes A, x \neq y \right\}$$

is bounded from above.

Definition 1.1.2

The supremum of the above set is called the λ -Hölder coefficient of $u(x)$.

Definition 1.1.3

A function $u(x)$ defined on a bounded domain G and possessing there k λ -Hölder continuous derivatives is said to be of class $C^{(k)} [C^{(k, \lambda)}]$ in G .

Obviously $C^{(0)} [C^{(0, \lambda)}]$ will be the class of λ -Hölder continuous functions in G , and $C^{(\infty)} [C^{(\infty, \lambda)}]$ the class of infinitely λ -Hölder differential functions in G .

Now, let G be a domain such that to each $x \in \partial G$ it is possible to associate a hypersphere $\Delta(x)$ with center x , in such a way, that the set $\Delta(x) \cap \partial G$ can be represented by an equation of the form

(a) $\xi_n = \theta(\xi_1, \dots, \xi_{n-1})$,
 with respect to a system of axis ξ_1, \dots, ξ_m , with origin at x , θ being
 a function defined in a suitable region $\Lambda(x)$, of class $C^{(1)}$ there at least,
 and vanishing together with its first derivative at the point x .

Under these hypotheses, there exists a hyperplane tangent to ∂G at x ,
 for every $x \in \partial G$, namely the hyperplane

(b) $\xi_n = 0$

and there exists at x an exterior normal vector η , whose direction cosines
 $X_i(x)$ are continuous functions on ∂G .

Definition 1.1.4

Such a domain G will be said to be of class $A^{(k)} \cap A^{(k, \lambda)}$, if the functions θ
 occurring in (a), are, for every x , of class $C^{(k)} \cap [C^{(k, \lambda)}]$.

Now, for $x \in \Gamma$, let $X(x) = (X_1(x), \dots, X_n(x))$ denote the outer normal
 vector to the boundary Γ , of unit length; then $X_i(x)$, $i = 1, \dots, n$, are
 the direction cosines of the outer normal vector to the boundary Γ .

Next we define the operators L, L_t, M, M_t, P, Q , as follows:

$$(1) \quad (Ly)_i \equiv L_i y = \sum_{v=1}^m \sum_{j,k=1}^n a_{jk}^{iv}(x,t) \frac{\partial^2 y_v}{\partial x_j \partial x_k}, \quad i = 1, \dots, m,$$

where $a_{jk}^{iv}(x,t)$, for $i, n = 1, \dots, m$, $j, k = 1, \dots, n$ are given real
 valued functions of class $C^{(2)}$ in the region $(G+\Gamma) \otimes \mathbb{R}$; the parameter $t \in \mathbb{R}$
 represents time;

We have $(Ly) = ((Ly)_1, \dots, (Ly)_m)$.

Define now the parabolic operator L_t by

$$(2) \quad L_t y = \frac{\partial y}{\partial t} - Ly, \quad \text{for } y \in C^2$$

i. e.
$$(L_t y)_i = \frac{\partial y_i}{\partial t} - \sum_{\nu=1}^m \sum_{j,k=1}^n a_{jk}^{i\nu}(x,t) \frac{\partial^2 y_\nu}{\partial x_j \partial x_k},$$

where $(L_t y) = ((L_t y)_1, \dots, (L_t y)_m)$.

Similarly define:

$$(3) \quad (Mz)_i \equiv M_{i,z} = \sum_{\nu=1}^m \left(\sum_{j,k=1}^n \frac{\partial}{\partial x_j} (a_{jk}^{i\nu} \frac{\partial z_\nu}{\partial x_k}) + \sum_{j=1}^n \frac{\partial}{\partial x_j} (l_j^{i\nu} z_\nu) \right)$$

where

$$i = 1, \dots, m, \quad l_j^{i\nu} = - \sum_{k=1}^n \frac{\partial a_{jk}^{i\nu}}{\partial x_k}, \quad (Mz) = ((Mz)_1, \dots, (Mz)_m) \quad \text{and}$$

$$(3') \quad (M_{it} z) = \frac{\partial z_i}{\partial t} + M_{i,z};$$

$$(4) \quad (Py)_i \equiv P_{i,y} = \sum_{\nu=1}^m (a_{1\nu}^{i\nu}(x,t) \frac{dy_\nu}{dl_{i\nu}} + b_{i\nu}(x,t) y_\nu),$$

where

$$i = 1, \dots, m \quad \text{and} \quad (Py) = ((Py)_1, \dots, (Py)_m).$$

The directions $l^{i\nu}$ are chosen arbitrarily except that $\cos(X, l_{i\nu}) > 0$ and their direction cosines are of class $C^{(1)}$ on Γ ; moreover the functions

$a_{1\nu}^{i\nu}(x,t)$ and $b_{i\nu}(x,t)$ are considered continuously differentiable.

$$(5) \quad (Qz)_i \equiv Q_{i,z} = \sum_{\nu=1}^m (a_{\lambda\nu}^{i\nu}(x,t) \frac{dz}{d\lambda_{i\nu}} + d_{i\nu}(x,t) z_\nu),$$

where

$$i = 1, \dots, m, \quad (Qz) = ((Qz)_1, \dots, (Qz)_m)$$

and the directions $\lambda_{i\nu}$ are chosen independently of $l_{i\nu}$.

We shall study control processes described by the system of parabolic equations:

$$(6) \quad L_t y = f(x, t, y, y_x, u), \quad \text{i.e.}$$

$$(6) \quad L_{it} y \equiv \frac{\partial y_i}{\partial t} - \sum_{v=1}^m \sum_{j,k=1}^n a_{jk}^{iv}(x, t) \frac{\partial^2 y_v}{\partial x_j \partial x_k} = f_i(x, t, y, y_x, u)$$

where:

$i = 1, \dots, m$; $x \in G$; t lies in a closed bounded interval $[0, T]$; the function $f = (f_1, \dots, f_m)$ is considered continuous in t and twice continuously differentiable in all the remaining arguments and the control parameter u takes values in some bounded (open or closed) region U of a p -dimensional Euclidean space.

We assume that the function $y = (y_1, \dots, y_m)$ defined by the system of equations (6), also satisfies the initial condition

$$(7) \quad y(x, 0) = a(x), \quad x \in G$$

where $a(x)$ is a continuous vector valued function.

Further we assume that y satisfies the boundary condition

$$(8) \quad P_i y = \varphi_i(x, t, y, v), \quad x \in \Gamma, \quad t \in [0, T],$$

where the functions φ_i are assumed continuous in t and twice continuously differentiable in all the remaining arguments, and the control parameter v takes values in some bounded (open or closed) region V of a q -dimensional Euclidean space.

The function $\omega(x, t) = (u(x, t), v(x, t))$ will be called an admissible control if all its components are piecewise continuous and $u(x, t)$ and $v(x, t)$ take values in U and V respectively; the class of admissible u will be denoted by U^* and the class of admissible v will be denoted by V^* ; the class of admissible controls ω will be denoted by $\Omega = U^* \otimes V^*$.

In what follows we shall assume that the given functions in (6), (7), (8) satisfy the condition that to each admissible control there corresponds a unique solution of the system (6), (7), (8).

We denote by $E \subset \Omega$ the class of controls ω for which the corresponding solution of (6),(7),(8) satisfies the conditions :

$$(9) \quad \Phi_{\alpha}(x, T, y(x, T)) = 0, \quad \int_G \Psi_{\beta}(x, T, y(x, T), y_x(x, T)) dx = c_{\beta}$$

where:

$\alpha = 1, \dots, j, \quad \beta = 1, \dots, k, \quad \text{with } j+k = m;$

$y(x, T) = (y_1(x, T), \dots, y_m(x, T));$

$\Phi_{\alpha}, \Psi_{\beta}$ are given functions and c_{β} are given constants ;

T is the final time of the process.

We shall say that any $\omega(x, t) \in E$ transfers the system (6) with boundary conditions (8) from the initial state (7) to the final state (9) at the instant T , if the solution of (6),(7),(8) corresponding to $\omega(x, t)$, satisfies (9) at the instant T .

1.2. Completeness of the class of admissible controls

In what follows we shall assume that the class of admissible controls is complete in the sense of A. I. Egorov ([2] , p. 385) so that the optimal problem is not trivial and also it can be solved by the method presented below.

The completeness of the class of admissible controls is defined by the following properties (i), (ii), (iii).

Let $\omega(x, t) = (u(x, t), v(x, t))$ and $\omega_1(x, t) = (u_1(x, t), v_1(x, t))$ be two admissible controls defined in the regions $C = (G+\Gamma) \otimes [0, T]$ and $C_1 = (G+\Gamma) \otimes [0, T_1]$ and transferring the system (6) with boundary conditions (8) from the initial state (7) to the final state (9) at the instants T, T_1 respectively.

Then for an arbitrarily small $\varepsilon > 0$ we can find a control

$\omega_{\varepsilon}(x, t) = (u_{\varepsilon}(x, t), v_{\varepsilon}(x, t))$, defined in the region $C_{\varepsilon} = (G+\Gamma) \otimes [0, T_{\varepsilon}]$ such that

- (i) $\omega_{\varepsilon} \in E$, with final time T_{ε} ;
 (ii) It is defined in the region $C_{\varepsilon} \cap C \cap C_1$ as follows :

$$\omega_{\varepsilon}(x, t) = (u_{\varepsilon}(x, t), v_{\varepsilon}(x, t)) \quad \text{or}$$

$$\omega_{\varepsilon}(x, t) = (u(x, t), v_{\varepsilon}(x, t)) \quad \text{or}$$

$$\omega_{\varepsilon}(x, t) = (u_{\varepsilon}(x, t), v(x, t)) ,$$

where

$$u_{\varepsilon}(x, t) = \begin{cases} u(x, t) & \text{for } (x, t) \in \Lambda_{\varepsilon} \\ u_1(x, t) & \text{for } (x, t) \in C_{\varepsilon} \cap C \cap C_1 - \Lambda_{\varepsilon} \end{cases}$$

and

$$v_{\varepsilon}(x, t) = \begin{cases} v(x, t) & \text{for } (x, t) \in \Lambda_{\varepsilon} \\ v_1(x, t) & \text{for } (x, t) \in C_{\varepsilon} \cap C \cap C_1 - \Lambda_{\varepsilon} \end{cases}$$

and where Λ_{ε} is a closed subset of $C_{\varepsilon} \cap C \cap C_1$

with measure $\mu(\Lambda_{\varepsilon}) = \varepsilon$.

- (iii) The inequality $|T - T_{\varepsilon}| < L\varepsilon$ holds , where the number L is independent of ε .

Now, on the set of admissible controls Ω we define the functional

$$(10) \quad S(\omega) = \sum_{i=1}^m \left(\int_G a_i(x) y_i(x, T) dx + \int_0^T \int_G \beta_i(x, t) y_i(x, t) dx dt + \int_0^T \int_{\Gamma} \gamma_i(x, t) y_i(x, t) d\sigma dt \right)$$

where a_i, β_i, γ_i are given continuous functions and $y_i(x, t)$, $i = 1, \dots, m$, is the solution of the system (6), (7), (8), (9) corresponding to the admissible control ω .

1.3. Formulation of the problem

From among the admissible controls $\omega \in E$, find a control ω^* such that the corresponding solution of (6), (7), (8), (9) realizes the minimum of the functional $S(\omega)$.

The admissible control ω^* and the corresponding solution of the optimal problem being considered will be called the optimal control and the optimal solution, relative to the functional $S(\omega)$.

In the stated problem we observe that the control in the process is effected simultaneously by controls which occur in the system equations (6) and in the boundary conditions (8).

In order to formulate the optimality conditions we introduce the auxiliary functions

$$(11) \quad H(x, t, w, u) = \langle z, f(x, t, y, y_x, u) \rangle \equiv \sum_{i=1}^m z_i f_i(x, t, y, y_x, u)$$

$$(12) \quad h(x, t, p, v) = \langle z, \varphi(x, t, y, v) \rangle \equiv \sum_{i=1}^m z_i \varphi_i(x, t, y, v)$$

where

$$(13) \quad w = (z_1, \dots, z_m; y_1, \dots, y_m; \frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_m}{\partial x_n}), \quad \text{and}$$

$$(14) \quad p = (z_1, \dots, z_m; y_1, \dots, y_m).$$

The auxiliary variable $z(x, t) = (z_1(x, t), \dots, z_m(x, t))$ is defined as the solution of the following system, adjoint to the given.

Let $\omega(x, t)$ be an admissible control and $y = y(x, t)$ the corresponding solution of the system (6), (7), (8), (9).

The adjoint system is the following:

$$(15) \quad M_{it} z = \frac{\partial z_i}{\partial t} = - \frac{\partial H(x, t, w, u)}{\partial y_i} + \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\partial H(x, t, w, u)}{\partial y_{ik}} + \beta_i(x, t) \right)$$

where $y_{ik} = \frac{\partial y_i}{\partial x_k}$ and $(x, t) \in G \otimes [0, T]$,

with initial conditions:

$$(16) \quad z_i(x, T) = - a_i(x) - \sum_{\alpha=1}^j a_\alpha(x) \frac{\partial \varphi_\alpha}{\partial y_i} - \sum_{\beta=1}^k b_\beta \left(\frac{\partial \psi_\beta}{\partial y_i} - \sum_{\nu=1}^n \frac{d}{dx_\nu} \left(\frac{\partial \psi_\beta}{\partial y_{i\nu}} \right) \right),$$

where $x \in G$, the functions $a_i(x)$, $\beta_i(x, t)$ are taken from functional S , the functions φ_α , ψ_β , are taken from conditions (9), while the constants b_β and the functions $a_\alpha(x)$ are as yet undefined;

and with boundary conditions:

$$(17) \quad Q_{iz} = \frac{\partial h(x, t, p, v)}{\partial y_i} + \sum_{k=1}^n \frac{\partial H(x, t, w, u)}{\partial y_{ik}} X_k(x) - \gamma_i(x, t)$$

where $x \in \Gamma$, $i = 1, \dots, m$, the operators Q_i are defined by (5), $X_k(x)$ are the direction cosines of the outer normals to the boundary Γ , and $\gamma_i(x, t)$ are taken from the definition of S .

1.4. Definition of the maximum condition.

Let $\omega(x,t) = (u(x,t), v(x,t)) \in E$ and let $y(x,t), z(x,t)$, be the corresponding solutions of the given system (6), (7), (8), (9), and of its adjoint system (15), (16), (17), respectively.

On the set of admissible controls we define the functionals:

$$(18) \quad J_1(u) = \int_0^T \int_G H(x,t, w(x,t), u) dx dt, \quad \text{for } u \in U^*$$

$$(19) \quad J_2(v) = \int_0^T \int_\Gamma h(x,t, p(x,t), v) d\sigma dt, \quad \text{for } v \in V^* .$$

Definition

We shall say that the admissible control $\omega(x,t) = (u(x,t), v(x,t)) \in E$ satisfies a maximum condition relative to the function $z(x,t)$ if for any other control $\omega'(u'(x,t), v'(x,t)) \in E$, the following inequalities are satisfied:

$$(20) \quad \Delta J_1(u') = \int_D (H(x,t, w(x,t), u') - H(x,t, w(x,t), u(x,t))) dx dt \leq 0 ,$$

$$(21) \quad \Delta J_2(v') = \int_0^T \int_\Gamma (h(x,t, p(x,t), v') - h(x,t, p(x,t), v(x,t))) d\sigma dt \leq 0 ,$$

where $D = G \otimes [0, \tau]$ is the region in which both the controls ω and ω' are defined.

1.4.1 Proposition ([2], p.388)

If we do not impose any restrictions on the form of dependence of the controls on the arguments x and t , namely if as admissible controls we take

all piecewise continuous functions $u(x,t)$ with values in U , (and $v(x,t)$ with values in V), then inequality (20) (and (21) respectively) is equivalent to (20') (and (21') respectively):

$$(20') \quad H(x,t,w(x,t),u(x,t)) \quad ((=)) \quad \sup_{u' \in U^*} H(x,t,w(x,t),u'), \quad (x,t) \in G \otimes [0,T],$$

$$(21') \quad h(x,t,p(x,t),v(x,t)) \quad (=) \quad \sup_{v' \in V^*} h(x,t,p(x,t),v'), \quad (x,t) \in \Gamma \otimes [0,T],$$

where the symbol $((=))$ means equality valid everywhere in the region $C = G \otimes [0,T]$, excepting perhaps points lying on a finite number of n -dimensional surfaces whose $(n+1)$ -dimensional volume equals zero; the symbol $(=)$ is defined analogously except that we take $n-1$ and Γ instead of n and G , respectively.

Proof

The proof is given by contradiction.

Assume that (20') does not follow from (20).

Then: First there exists a control $\hat{u}(x,t) \in U^*$ defined in the region $\hat{C} = G \otimes [0,\hat{T}]$, such that:

$$(22) \quad \int_D (H(x,t,w(x,t),\hat{u}) - H(x,t,w(x,t),u)) dxdt \leq 0,$$

where $D = (G \otimes [0,T]) \cap (G \otimes [0,\hat{T}]) = G \otimes [0,\tau] = C \cap \hat{C}$,

$\tau = \min(T, \hat{T})$ and T is the final time for the optimal control $\omega(x,t)$.

Secondly, there exists $(\hat{x}, \hat{t}) \in D$ such that H is continuous at (\hat{x}, \hat{t}) and

$$(23) \quad H(\hat{x}, \hat{t}, w(\hat{x}, \hat{t}), \hat{u}(\hat{x}, \hat{t})) > H(\hat{x}, \hat{t}, w(\hat{x}, \hat{t}), u(\hat{x}, \hat{t})) .$$

The control $\omega(x,t)$ is piecewise continuous by definition; consequently, the functions $y(x,t), z(x,t)$ corresponding to this control, are continuous; therefore, there exists $\epsilon > 0$ and a set $D_\epsilon \subset D$, containing (\hat{x}, \hat{t}) and having measure

$\mu(D_\varepsilon) = \varepsilon$, such that:

$$(24) \quad H(x, t, \hat{w}(x, t), \hat{u}(x, t)) - H(x, t, w(x, t), u(x, t)) > 0, \quad \text{for every } (x, t) \in D_\varepsilon.$$

Further, construct an auxiliary admissible control $\omega_1(x, t) = (u_1(x, t), v_1(x, t))$, defined in the region $C_1 = G \otimes [0, T_1]$, from which the following conditions are required:

- (i) $\omega_1 \in E$, with final time T_1 .
- (ii) The region C_1 is such that $\mu(D_\varepsilon \cap C_1) > 0$.
- (iii) The control $\omega_1(x, t) = (u_1(x, t), v_1(x, t))$ is defined in the region $D \cap C_1$ as follows:

$$u_1(x, t) = \begin{cases} \hat{u}(x, t), & \text{for } (x, t) \in D_\varepsilon \cap C_1 \\ u(x, t), & \text{for } (x, t) \in (D - D_\varepsilon) \cap C_1 \end{cases}$$

$$v_1(x, t) = v(x, t), \quad \text{for } (x, t) \in D \cap C_1.$$

- (iv) In case that $(C - D) \cap C_1 \neq \emptyset$, define $\omega_1(x, t) = \omega(x, t)$, for every $(x, t) \in (C - D) \cap C_1$.
- (v) Finally, the control $\omega_1(x, t)$ is defined in the region $C_1 - C$ (in case that $C_1 - C \neq \emptyset$), to be any admissible control satisfying condition (i).

Such an admissible control $\omega_1(x, t)$ exists because of the completeness of the class of admissible controls.

It has been assumed that (20) holds; thus for the control ω_1 we shall have:

$$(25) \quad \Delta J_1(u_1) = \int_{C_1 \cap C} (H(x, t, w(x, t), u_1) - H(x, t, w(x, t), u(x, t))) dx dt \leq 0$$

By virtue of (iv), (25) can be written as follows:

$$(26) \quad \Delta J_1(u_1) = \int_{C_1 \cap D} (H(x, t, w(x, t), u_1) - H(x, t, w(x, t), u(x, t))) dx dt \leq 0$$

By virtue of (iii), (26) can be written as:

$$(27) \quad \Delta J_1(u_1) = \int_{D_{\mathcal{E}} \cap C_1} (H(x,t, w(x,t), \hat{u}(x,t)) - H(x,t, w(x,t), u(x,t))) dxdt \leq 0,$$

which contradicts (24); thus it is proved that if (20) holds so does (20').

Next it is proved that if (20') holds so does (20).

Suppose that (20') holds but (20) does not.

Then there exists an admissible control $\omega'(x,t) = (u'(x,t), v'(x,t)) \in E$, satisfying the relation:

$$(28) \quad \Delta J_1(u') = \int_D (H(x,t, w(x,t), u') - H(x,t, w(x,t), u(x,t))) dxdt > 0,$$

where D is the intersection of the domains of definition of the controls ω and ω' i.e. $D = G \otimes [0, T] \cap G \otimes [0, T']$.

Relation (28) implies that there exists $(\hat{x}, \hat{t}) \in D$, such that,

$$(29) \quad H(\hat{x}, \hat{t}, w(\hat{x}, \hat{t}), u'(\hat{x}, \hat{t})) - H(\hat{x}, \hat{t}, w(\hat{x}, \hat{t}), u(\hat{x}, \hat{t})) > 0.$$

Moreover, by the continuity of H it follows that given $\varepsilon > 0$ sufficiently small, there exists a region $D_{\varepsilon} \subset D$, containing (\hat{x}, \hat{t}) , such that, $\mu(D_{\varepsilon}) = \varepsilon$ and the relation :

$$(30) \quad H(x,t, w(x,t), u'(x,t)) - H(x,t, w(x,t), u(x,t)) > 0$$

holds for every $(x,t) \in D_{\varepsilon}$; this contradicts relation (20').

Thus it is proved that if (20') holds so does (20).

The equivalence of relations (21) and (21') can be proved similarly.

CHAPTER 2

NECESSARY CONDITIONS FOR OPTIMALITY

2.1. Formulation of the maximum principle

The maximum principle is stated as follows:

In order that an admissible control $\omega(x,t) = (u(x,t), v(x,t))$, defined on the domain ($x \in G, 0 \leq t \leq T$) and transferring the system (6), with boundary conditions (8), from the state (7) to the state (9), along the trajectory $y(x,t)$, be optimal in the sense of minimizing the functional S (defined by (10)), it is necessary that there exist functions $z_i(x,t)$, $a_\alpha(x)$ and constants b_β , such that:

- (i) The functions $y(x,t)$, $z(x,t)$, $\omega(x,t)$, $a_\alpha(x)$ and the constants b_β form the solution of the given system (6), (7), (8), (9) and of its adjoint system (15), (16), (17).
- (ii) The control $\omega(x,t)$ satisfies the maximum condition relative to the function $z(x,t)$.
- (iii) The condition:

$$(31) \quad \frac{dS}{dt} + \int_G \left(\sum_{\alpha=1}^j a_\alpha(x) \frac{d\phi_\alpha}{dt} + \sum_{\beta=1}^k b_\beta \frac{d\psi_\beta}{dt} \right) dx = 0$$

is satisfied at the final time instant $t = T$.

2.2. Formulas for the increments of functionals J_1, J_2 .

For the proof of the maximum principle it is necessary to compute the increments of the functionals J_1, J_2 , (defined by (18), (19)).

Define, on the set of admissible controls Ω , the auxiliary functional:

$$(32) \quad I(\omega) = I_{db}(\omega) + I_s(\omega) + I_{f_1}(\omega) + I_{f_2}(\omega)$$

where:

$$(33) \quad I_{db}(\omega) = \int_C \left(\sum_{i=1}^m z_i L_{it} y - H(x,t,w(x,t),u) \right) dxdt + \\ + \int_0^T \int_G \left(\sum_{i=1}^m z_i P_i y - h(x,t,p(x,t),v) \right) d\sigma dt,$$

$$(34) \quad I_s(\omega) = S(\omega),$$

$$(35) \quad I_{f_1}(\omega) = \sum_{\alpha=1}^j \int_G a_{\alpha}(x) \bar{\varphi}_{\alpha}(x, T, y(x, T)) dx$$

$$(36) \quad I_{f_2}(\omega) = \sum_{\beta=1}^k b_{\beta} \left(\int_G \psi_{\beta}(x, T, y(x, T), y_x(x, T)) dx - c_{\beta} \right)$$

and where:

$w = (z, y, y_x)$, $p = (z, y)$, $C = G \otimes [0, T]$, the operator P is defined by (4), the functions $\bar{\varphi}_{\alpha}$, ψ_{β} are taken from (9), whereas the functions $y(x, t)$, $z(x, t)$, $a(x)$, the constants b_{β} and the final time T are considered given.

We observe that $I(\omega)$ contains the functionals J_1, J_2 .

Let $\omega \in E$ be an admissible control, and let $y(x, t)$ be the corresponding solution of (6), (7), (8), (9); then (32) reduces to:

$$(37) \quad I(\omega) = S(\omega),$$

and this holds no matter what the functions $z_i(x, t)$, $a_{\alpha}(x)$ and the constants b_{β} are.

Therefore, corresponding to such control-solution pairs, the functional $S(\omega)$ will attain its extremal, whenever $I(\omega)$ does.

For $\omega, \omega_1 \in E$, let

$$(38) \quad \Delta S = S(\omega_1) - S(\omega).$$

If ω is optimal then,

$$(39) \quad S(\omega_1) - S(\omega) \geq 0,$$

and consequently

$$(40) \quad \Delta I = I(\omega_1) - I(\omega_2) \geq 0.$$

In the sequel, the increments of all functionals occurring in inequality (40) will be computed separately.

Let us assume that the optimal control $\omega(x,t) = (u(x,t), v(x,t))$ is defined in the region $C = (G+\Gamma) \otimes [0, T]$, where $\Gamma = \partial G$, and that $\omega_1(x,t) = (u_1(x,t), v_1(x,t))$ is any admissible control in E , defined in the region $C_1 = (G+\Gamma) \otimes [0, T_1]$; it is assumed that T, T_1 are the final times at which the controls ω, ω_1 respectively transfer the system (6) with boundary conditions (8) from the initial state (7) to the final state (9).

Let $D = (G+\Gamma) \otimes [0, \tau]$, be the intersection of C and C_1 , where $\tau = \min(T, T_1)$; consider, for definiteness, that $T \leq T_1$; then $\tau = T$ and $D \doteq C$.

Both the controls $\omega(x,t)$ and $\omega_1(x,t)$ and also their corresponding functions $y^\omega(x,t) = y(x,t,\omega)$, $z^\omega(x,t) = z(x,t,\omega)$ and $y^{\omega_1}(x,t) = y(x,t,\omega_1)$, $z^{\omega_1}(x,t) = z(x,t,\omega_1)$, which form the solution of (6), (7), (8), (9) and (15), (16), (17), are defined in D .

From (6) follows that

$$(41) \quad L_{it} y^\omega(x,t) = f_i(x,t, y^\omega, y_x^\omega, u), \text{ for every } (x,t) \in G \otimes [0, T],$$

$$(42) \quad L_{it} y^{\omega_1}(x,t) = f_i(x,t, y^{\omega_1}, y_x^{\omega_1}, u_1), \text{ for all } (x,t) \in G \otimes [0, T_1].$$

From (11) follows that

$$(43) \quad \frac{\partial H(x,t,w,u)}{\partial z_i} = f_i(x,t, y, y_x, u).$$

Thus (41) and (42) can be written as (44) and (45):

$$(44) \quad L_{it} y^\omega(x, t) = \frac{\partial H(x, t, w^\omega, u)}{\partial z_i}, \quad \text{for all } (x, t) \in G \otimes [0, T]$$

$$(45) \quad L_{it} y^{\omega_1}(x, t) = \frac{\partial H(x, t, w^{\omega_1}, u_1)}{\partial z_i} \quad \text{for all } (x, t) \in G \otimes [0, T_1].$$

From (15) follow (46) and (47):

$$(46) \quad M_{it} z^\omega = - \frac{\partial H(x, t, w^\omega, u)}{\partial y_i} + \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\partial H(x, t, w^\omega, u)}{\partial y_{ik}} \right) + \beta_i(x, t)$$

for all $(x, t) \in G \otimes [0, T]$,

$$(47) \quad M_{it} z^{\omega_1} = - \frac{\partial H(x, t, w^{\omega_1}, u_1)}{\partial y_i} + \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\partial H(x, t, w^{\omega_1}, u_1)}{\partial y_{ik}} \right) + \beta_i(x, t)$$

for all $(x, t) \in G \otimes [0, T_1]$,

where $y_{ik} = \frac{\partial y_i}{\partial x_k}$, and w^ω, w^{ω_1} are the $m(n+2)$ -dimensional vectors:

$$w^\omega = (z^\omega, y^\omega, y_x^\omega) = (z_1^\omega, \dots, z_m^\omega; y_1^\omega, \dots, y_m^\omega; y_{11}^\omega, \dots, y_{mn}^\omega),$$

$$w^{\omega_1} = (z^{\omega_1}, y^{\omega_1}, y_x^{\omega_1}) = (z_1^{\omega_1}, \dots, z_m^{\omega_1}; y_1^{\omega_1}, \dots, y_m^{\omega_1}; y_{11}^{\omega_1}, \dots, y_{mn}^{\omega_1}).$$

Since the operators L_{it}, M_{it} are linear, subtracting (44) from (45), and (46) from (47), we obtain:

$$(48) \quad L_{it} \Delta y = \Delta \frac{\partial H}{\partial z_i}, \quad \text{for all } (x, t) \in D,$$

$$(49) \quad M_{it} \Delta z = - \Delta \frac{\partial H}{\partial y_i} + \sum_{k=1}^n \frac{d}{dx_k} \left(\Delta \frac{\partial H}{\partial y_{ik}} \right), \quad \text{for all } (x, t) \in D,$$

where

$$(50) \quad \Delta y = y^{\omega_1}(x, t) - y^\omega(x, t),$$

$$(51) \quad \Delta z = z^{\omega_1}(x,t) - z^{\omega}(x,t),$$

$$(52) \quad \Delta \frac{\partial H}{\partial w_i} = \frac{\partial H^{\omega_1}}{\partial w_i} - \frac{\partial H^{\omega}}{\partial w_i} = \frac{\partial H(x,t,w(x,t,\omega_1),u_1)}{\partial w_i} - \frac{\partial H(x,t,w(x,t,\omega),u)}{\partial w_i},$$

$$(53) \quad H^{\omega} = H(x,t,w(x,t,\omega),u).$$

Since both functions $y^{\omega_1}(x,t)$, $y^{\omega}(x,t)$ satisfy the initial condition (7), it follows that:

$$(54) \quad \Delta y(x,0) = 0, \text{ for all } x \in G.$$

From (12) follows

$$(55) \quad \frac{\partial h(x,t,p,v)}{\partial z_i} = \varphi_i(x,t,p,v).$$

Thus boundary condition (8) can be written as:

$$(56) \quad P_i y = \frac{\partial h(x,t,p,v)}{\partial z_i}, \text{ for } (x,t) \in \Gamma \otimes [0,T]$$

where p is the $2m$ -dimensional vector $p = (z,y) = (z_1, \dots, z_m; y_1, \dots, y_m)$.

Since $y^{\omega}(x,t)$, $z^{\omega}(x,t)$ and $y^{\omega_1}(x,t)$, $z^{\omega_1}(x,t)$ are the solutions of the systems (6)-(9), and (15)-(17), they satisfy the relations (56) and (17); thus we have the following relations:

$$(57) \quad P_i y^{\omega_1} = \frac{\partial h(x,t,p^{\omega_1},v_1)}{\partial z_i}, \text{ for all } (x,t) \in \Gamma \otimes [0,T_1],$$

$$(58) \quad P_i y^{\omega} = \frac{\partial h(x,t,p^{\omega},v)}{\partial z_i} \text{ for all } (x,t) \in \Gamma \otimes [0,T],$$

$$(59) \quad Q_i z^{\omega_1} = \frac{\partial h(x,t,p^{\omega_1},v_1)}{\partial y_i} + \sum_{k=1}^n \frac{\partial H(x,t,w^{\omega_1},u_1)}{\partial y_{ik}} X_k(x) - \gamma_i(x,t),$$

for all $x \in \Gamma$,

$$(60) \quad Q_i z_i^\omega = \frac{\partial h(x, t, p^\omega, v)}{\partial y_i} + \sum_{k=1}^n \frac{\partial H(x, t, w^\omega, u)}{\partial y_{ik}} X_k(x) - \gamma_i(x, t),$$

for all $x \in \Gamma$.

Since the operators P_i, Q_i are linear, subtracting (58) from (57) we obtain:

$$(61) \quad P_i \Delta y = \Delta \frac{\partial h}{\partial z_i}, \text{ for } (x, t) \in \Gamma \otimes [0, \tau],$$

$$(62) \quad Q_i \Delta z = \Delta \frac{\partial h}{\partial y_i} + \sum_{k=1}^n \Delta \frac{\partial H}{\partial y_{ik}} X_k(x), \text{ for } (x, t) \in \Gamma \otimes [0, \tau],$$

where

$$(63) \quad \Delta \frac{\partial h}{\partial p_i} = \frac{\partial h(x, t, p^{\omega_1}, v_1)}{\partial p_i} - \frac{\partial h(x, t, p^\omega, v)}{\partial p_i}, \text{ where}$$

$$(64) \quad p_i \in \{ z_1, \dots, z_m; y_1, \dots, y_m \}.$$

We compute now the increment of the functional $I_{db}(\omega)$.

$$(65) \quad \begin{aligned} \Delta I_{db}(\omega) = & \int_C \sum_{i=1}^m (z_i^{\omega_1} L_{it} y^{\omega_1} - z_i^\omega L_{it} y^\omega) dx dt - \\ & - \int_C (H(x, t, w^{\omega_1}(x, t), u_1) - H(x, t, w^\omega(x, t), u)) dx dt + \\ & + \int_0^T \int_\Gamma \sum_{i=1}^m (z_i^{\omega_1} P_i y^{\omega_1} - z_i^\omega P_i y^\omega) d\sigma dt - \\ & - \int_0^T \int_\Gamma (h(x, t, p^{\omega_1}(x, t), v_1) - h(x, t, p^\omega(x, t), v)) d\sigma dt + \lambda_1 + \lambda_2, \end{aligned}$$

where

$$(66) \quad \lambda_1 = \int_{C_1 - C} \left(\sum_{i=1}^m z_i^{\omega_1} L_{it} y^{\omega_1} - H(x, t, w^{\omega_1}(x, t), u_1) \right) dx dt, \text{ and}$$

$$(67) \quad \lambda_2 = \int_0^\theta \int_\Gamma \left(\sum_{i=1}^m z_i^{\omega_1} P_i y^{\omega_1} - h(x, t, p^{\omega_1}(x, t), v_1) \right) d\sigma dt,$$

where $\theta = T_1 - T$.

Both λ_1 and λ_2 equal zero, since their integrands are identically zero.

Let us put $z_i^{\omega_1} = z_i^{\omega} + \Delta z_i$ and $y_i^{\omega_1} = y_i^{\omega} + \Delta y_i$; then (65) can be written as:

$$(68) \quad \Delta I_{db}(\omega) = \int_C \left(\sum_{i=1}^m ((z_i^{\omega} + \Delta z_i) L_{it}(y_i^{\omega} + \Delta y) - z_i^{\omega} L_{it} y_i^{\omega}) - \Delta H \right) dxdt + \\ + \int_0^T \int_{\Gamma} \left(\sum_{i=1}^m ((z_i^{\omega} + \Delta z_i) P_i(y_i^{\omega} + \Delta y) - z_i^{\omega} P_i y_i^{\omega}) - \Delta h \right) d\sigma dt.$$

Since the operators L_{it} , P_i are linear, we obtain:

$$(69) \quad \Delta I_{db}(\omega) = \int_C \left(\sum_{i=1}^m (\Delta z_i L_{it} \Delta y + \Delta z_i L_{it} y + z_i L_{it} \Delta y) - \Delta H \right) dxdt + \\ + \int_0^T \int_{\Gamma} \left(\sum_{i=1}^m (\Delta z_i P_i \Delta y + \Delta z_i P_i y + z_i P_i \Delta y) - \Delta h \right) d\sigma dt.$$

For the operators L , M , P , Q , defined by (1), (3), (4), (5), the following formula is valid ([2], p.382):

$$(70) \quad \sum_{i=1}^m \int_G (z_i L_i y - y_i M_i z) dx = \\ = \sum_{i,n=1}^m \sum_{j=1}^n \int_{\Gamma} \left(\sum_{k=1}^n a_{jk}^{iv} \left(z_i \frac{\partial y_j}{\partial x_k} - y_j \frac{\partial z_i}{\partial x_k} \right) + l_j^{iv} y_j z_i \right) X_j(x) d\sigma,$$

This formula can be transformed ([2], p.382) to:

$$(71) \quad \sum_{i=1}^m \int_G (z_i L_i y - y_i M_i z) dx = \sum_{i=1}^m \int_{\Gamma} (z_i P_i y - y_i Q_i z) d\sigma.$$

By use of (71), it can be showed ([2], p.387) that for any twice piecewise continuously differentiable functions $y_i(x, t)$, and $z_i(x, t)$, $i = 1, \dots, m$, the Green-Ostrogradskii formula (72) is valid:

$$(72) \quad \sum_{i=1}^m \int_0^T \int_G (z_i L_{it} y + y_i M_{it} z) dxdt = \sum_{i=1}^m \left(\int_0^T \int_{\Gamma} (y_i Q_i z - z_i P_i y) d\sigma dt + \int_G y_i z_i \Big|_{t=0}^T dx \right)$$

Further, since the functions $\Delta y_i, \Delta z_i$ satisfy (48), (49), with supplementary conditions (54), (61), (62), by applying the Green-Ostrogradskii formula, it can be proved ([2], p.396), that:

$$(73) \quad \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i \Delta y_i d\sigma dt \right) = \\ = \sum_{i=1}^m \left(\int_C \left(\Delta \frac{\partial H}{\partial y_i} \Delta y_i + \sum_{k=1}^n \Delta \frac{\partial H}{\partial y_{ik}} \Delta y_{ik} \right) dx dt + \int_0^T \int_{\Gamma} \Delta \frac{\partial h}{\partial y_i} \Delta y_i d\sigma dt + \int_G \Delta y_i(x, t) \Delta z_i(x, t) dx \right).$$

On the other hand, from (48), and (61) it follows:

$$(74) \quad \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i \Delta y_i d\sigma dt \right) = \\ = \sum_{i=1}^m \left(\int_C \Delta \frac{\partial H}{\partial z_i} \Delta z_i dx dt + \int_0^T \int_{\Gamma} \Delta \frac{\partial h}{\partial z_i} \Delta z_i d\sigma dt \right).$$

Adding by members (73) and (74) we obtain :

$$(75) \quad \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i \Delta y_i d\sigma dt \right) = \\ = \frac{1}{2} \left(\sum_{i=1}^N \int_C \Delta \frac{H}{w_i} \Delta w_i dx dt + \sum_{i=1}^{2m} \int_0^T \int_{\Gamma} \Delta \frac{h}{P_i} \Delta p_i d\sigma dt + \sum_{i=1}^m \int_G \Delta y_i(x, T) \Delta z_i(x, T) dx \right)$$

where $N=m(n+2)$ and the vectors w, p are defined as:

$$w = (w_1, \dots, w_N) = (z_1, \dots, z_m; y_1, \dots, y_m; y_{11}, \dots, y_{mn}); \text{ and}$$

$$p = (p_1, \dots, p_{2m}) = (z_1, \dots, z_m; y_1, \dots, y_m).$$

Next, by virtue of (44) and (56) it follows:

$$(76) \quad \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i \Delta y_i d\sigma dt \right) = \\ = \sum_{i=1}^m \left(\int_C \frac{\partial H}{\partial z_i} \Delta z_i dx dt + \int_0^T \int_{\Gamma} \frac{\partial h}{\partial z_i} \Delta z_i d\sigma dt \right).$$

It can also be proved ([2], p.397 relation (2.39)), that:

$$(77) \quad \sum_{i=1}^m \left(\int_C z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} z_i P_i \Delta y_i d\sigma dt \right) = \\ = \sum_{i=1}^m \left(\int_C \left(\frac{\partial H}{\partial y_i} \Delta y_i + \sum_{k=1}^n \left(\frac{\partial H}{\partial y_{ik}} \Delta y_{ik} \right) \right) dx dt + \int_0^T \int_{\Gamma} \frac{\partial h}{\partial y_i} \Delta y_i d\sigma dt \right) + \Delta \hat{S},$$

where

$$(78) \quad \Delta \hat{S} = \sum_{i=1}^m \left(\int_G z_i(x, T) \Delta y_i(x, T) dx - \int_C z_i(x, t) \Delta y_i(x, t) dx dt - \int_0^T \int_{\Gamma} z_i(x, t) \Delta y_i(x, t) d\sigma dt \right).$$

Adding relations (75), (76), (77) by members, we obtain:

$$(79) \quad \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} \Delta y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i \Delta y_i d\sigma dt \right) + \\ + \sum_{i=1}^m \left(\int_C \Delta z_i L_{it} y_i dx dt + \int_0^T \int_{\Gamma} \Delta z_i P_i y_i d\sigma dt \right) + \\ + \sum_{i=1}^m \left(\int_C \left(\frac{\partial H}{\partial y_i} \Delta y_i + \sum_{k=1}^n \frac{\partial H}{\partial y_{ik}} \Delta y_{ik} \right) dx dt + \int_0^T \int_{\Gamma} \frac{\partial h}{\partial y_i} \Delta y_i d\sigma dt \right) + \Delta \hat{S} = \\ = \int_C \sum_{i=1}^N \left(\frac{\partial H}{\partial w_i} + \frac{1}{2} \Delta \frac{\partial H}{\partial w_i} \right) \Delta w_i dx dt + \int_0^T \int_{\Gamma} \sum_{i=1}^{2m} \left(\frac{\partial h}{\partial p_i} + \frac{1}{2} \Delta \frac{\partial h}{\partial p_i} \right) \Delta p_i d\sigma dt + \\ + \frac{1}{2} \sum_{i=1}^m \int_G \Delta y_i(x, T) \Delta z_i(x, T) dx + \Delta \hat{S}.$$

Hence, by virtue of (79), relation (69) can be written as follows:

$$(80) \quad \Delta I_{db}(\omega) = \int_C \left(\sum_{i=1}^N \left(\frac{\partial H}{\partial w_i} + \frac{1}{2} \Delta \frac{\partial H}{\partial w_i} \right) \Delta w_i - \Delta H \right) dx dt + \\ + \int_0^T \int_{\Gamma} \left(\sum_{i=1}^{2m} \left(\frac{\partial h}{\partial p_i} + \frac{1}{2} \Delta \frac{\partial h}{\partial p_i} \right) \Delta p_i - \Delta H \right) d\sigma dt + \Delta \hat{S} - \eta_2,$$

where

$$(81) \quad \eta_2 = \frac{1}{2} \sum_{i=1}^m \int_G \Delta y_i(x, T) \Delta z_i(x, T) dx.$$

It can be proved ([2], p.p.397-398), by applying Taylor's formula

to the functions $H, h, \frac{\partial H}{\partial w_i}, \frac{\partial h}{\partial w_i}$, and restricting ourselves in the

expansion to second order terms that

$$(82) \quad \Delta I_{db}(\omega) = \Delta \hat{S} - \eta_2 - \int_C (H(x, t, w^{\omega_1}(x, t), u_1) - H(x, t, w^\omega(x, t), u(x, t))) dx dt - \\ - \int_0^T \int_\Gamma (h(x, t, p^{\omega_1}(x, t), v_1) - h(x, t, p^\omega(x, t), v(x, t))) d\sigma dt - \eta_8 - \eta_9,$$

where

$$(83) \quad \eta_8 = \frac{1}{2} \sum_{i=1}^N \int_C \left(\frac{\partial H(x, t, w^{\omega_1}, u_1)}{\partial w_i} - \frac{\partial H(x, t, w^\omega, u)}{\partial w_i} \right) \Delta w_i dx dt + \\ + \frac{1}{2} \sum_{i=1}^{2m} \int_0^T \int_\Gamma \left(\frac{\partial h(x, t, p^{\omega_1}, v_1)}{\partial p_i} - \frac{\partial h(x, t, p^\omega, v)}{\partial p_i} \right) \Delta p_i d\sigma dt,$$

$$(84) \quad \eta_9 = \frac{1}{2} \sum_{i,k=1}^N \int_C \left(\frac{\partial^2 H(x, t, w^{\omega_1}, u_1)}{\partial w_i \partial w_k} - \frac{\partial^2 H(x, t, \hat{w}, u_1)}{\partial w_i \partial w_k} \right) \Delta w_i \Delta w_k dx dt + \\ + \frac{1}{2} \sum_{i,k=1}^{2m} \int_0^T \int_\Gamma \left(\frac{\partial^2 h(x, t, p^{\omega_1}, v_1)}{\partial p_i \partial p_k} - \frac{\partial^2 h(x, t, \hat{p}, v_1)}{\partial p_i \partial p_k} \right) \Delta p_i \Delta p_k d\sigma dt,$$

with $\hat{w}, w' \in [w, w_1]$, and $\hat{p}, p' \in [p, p_1]$.

The increment of the functional $I_s(\omega)$ will have the form ([2], p.398):

$$(85) \quad \Delta I_s(\omega) \equiv \Delta S(\omega) = \sum_{i=1}^m \left(\int_G (a_i(x) \frac{\partial y_i(x, T, \omega)}{\partial T} + \beta_i(x, T) y_i(x, T, \omega)) dx + \right. \\ \left. + \int_\Gamma \gamma_i(x, T) y_i(x, T, \omega) d\sigma \right) \Delta T + \\ + \sum_{i=1}^m \left(\int_G a_i(x) \Delta y_i(x, T, \omega) dx + \int_0^T \int_G \beta_i(x, t) \Delta y_i(x, t, \omega) dx dt + \right. \\ \left. + \int_0^T \int_\Gamma \gamma_i(x, t) \Delta y_i(x, t, \omega) d\sigma dt \right) - \eta_{10},$$

where

$$\begin{aligned}
 (86) \quad \eta_{10} = & \sum_{i=1}^m \int_G (a_i(x) \sum_{k=1}^p \left(\frac{\partial y_i(x, T; \omega)}{\partial T} - \frac{\partial y_i(x, \hat{t}_k, \omega)}{\partial T} \right) \Delta t_k) dx + \\
 & + \sum_{i=1}^m \int_G (\beta_i(x, T) y_i(x, T, \omega) - \beta_i(x, \hat{t}) y_i(x, \hat{t}, \omega_1)) \Delta T dx + \\
 & + \sum_{i=1}^m \Delta T \int_{\Gamma} (\gamma_i(x, T) y_i(x, T, \omega) - \gamma_i(x, \hat{t}) y_i(x, \hat{t}, \omega_1)) d\sigma .
 \end{aligned}$$

The computation of the increments of the functionals $I_{f_1}(\omega)$, and $I_{f_2}(\omega)$, gives the following results ([2], p.p. 393-394) :

$$\begin{aligned}
 (87) \quad \Delta I_{f_1}(\omega) = & \sum_{a=1}^j \int_G a_a(x) (\bar{\phi}_a(x, T_1, y(x, T_1, \omega_1)) - \bar{\phi}_a(x, T, y(x, T, \omega))) dx = \\
 = & \sum_{a=1}^j \int_G a_a(x) \left(\frac{d\bar{\phi}_a(x, T, y(x, T, \omega))}{dT} \Delta T + \sum_{i=1}^m \frac{\bar{\phi}_a(x, T, y(x, T, \omega))}{y_i} \Delta y_i(x, T, \omega) \right) dx - \\
 & - \eta_4 - \eta_5, \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 (88) \quad \Delta I_{f_2}(\omega) = & \sum_{\beta=1}^k b_\beta \int (\psi_\beta(x, T_1, y(x, T_1, \omega_1), y_x(x, T_1, \omega_1)) - \\
 & \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))) dx = \\
 = & \sum_{\beta=1}^k b_\beta \int_G dx \left[\frac{d\psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{dT} \Delta T + \right. \\
 & + \sum_{i=1}^m \left(\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_i} \right) \Delta y_i - \\
 & \left. - \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{dx_k} \right) \Delta y_i(x, T, \omega) \right] - \\
 & - \eta_6 - \eta_7,
 \end{aligned}$$

where

$$(89) \quad \eta_4 = \sum_{a=1}^j \int_G a_a(x) \sum_{i=1}^p \left(\frac{d\hat{\phi}_a(x, T, y(x, T, \omega))}{dT} - \frac{d\hat{\phi}_a(x, \hat{t}_i, y(x, \hat{t}_i, \omega_1))}{dT} \right) \Delta t_i dx,$$

$$(90) \quad \eta_5 = \sum_{a=1}^j \sum_{i=1}^m \int_G a_a(x) \left(\frac{\partial \hat{\phi}_a(x, T, y(x, T, \omega))}{\partial y_i} - \frac{\partial \hat{\phi}_a(x, T, \hat{y})}{\partial y_i} \right) \Delta y(x, T, \omega) dx,$$

$$(91) \quad \eta_6 = \sum_{\beta=1}^k b_\beta \int_G \sum_{i=1}^p \left[\frac{d\psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{dT} - \frac{d\psi_\beta(x, \hat{t}_i, y(x, \hat{t}_i, \omega_1), y_x(x, \hat{t}_i, \omega_1))}{dT} \right] \Delta t_i dx,$$

$$(92) \quad \eta_7 = \sum_{\beta=1}^k b_\beta \int_G \sum_{i=1}^m \left[\left(\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_i} - \frac{\partial \psi_\beta(x, T; \hat{y}, \hat{y}_x)}{\partial y_i} \right) \Delta y(x, T, \omega) + \sum_{k=1}^n \left(\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_{ik}} - \frac{\partial \psi_\beta(x, T, \hat{y}, \hat{y}_x)}{\partial y_{ik}} \right) \Delta y_{ik}(x, T, \omega) \right] dx.$$

Next, adding by members the relations (82), (85), (87), and (88) and taking into account the expression of $\Delta \hat{S}$, given by (78), we have :

$$(93) \quad \begin{aligned} \Delta I(\omega) &= \Delta I_{db}(\omega) + \Delta I_s(\omega) + \Delta I_{f_1}(\omega) + \Delta I_{f_2}(\omega) = \\ &= \sum_{i=1}^m \left(\int_G z_i(x, T) \Delta y_i(x, T, \omega) dx - \int_C \beta_i(x, t) \Delta y_i(x, t, \omega) dx dt \right) - \\ &\quad - \sum_{i=1}^m \int_0^T \int_\Gamma \gamma_i(x, t) \Delta y_i(x, t, \omega) d\sigma dt - \eta_2 \\ &\quad - \int_C (H(x, t, w^{\omega_1}(x, t), u_1) - H(x, t, w^\omega(x, t), u(x, t))) dx dt - \\ &\quad - \int_0^T \int_\Gamma (h(x, t, p^{\omega_1}(x, t), v_1) - h(x, t, p^\omega(x, t), v(x, t))) d\sigma dt - \eta_8 - \eta_9 + \\ &\quad + \sum_{i=1}^m \left[\int_G (a_i(x) \frac{y_i(x, T, \omega)}{T} + \beta_i(x, T) y_i(x, T, \omega)) dx + \right. \\ &\quad \left. + \int_\Gamma \gamma_i(x, T) y_i(x, T, \omega) d\sigma \right] \Delta T + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^m \left[\int_G a_i(x) \Delta y_i(x, T, \omega) dx + \int_0^T \int_G \beta_i(x, t) \Delta y_i(x, t, \omega) dx dt + \right. \\
 & \quad \left. + \int_0^T \int_G \gamma_i(x, t) \Delta y_i(x, t, \omega) d\sigma dt \right] - \eta_{10} + \\
 & + \sum_{\alpha=1}^j \int_G a_\alpha(x) \left(\frac{d\bar{\phi}_\alpha(x, T, y(x, T, \omega))}{dT} \Delta T + \sum_{i=1}^m \frac{\partial \bar{\phi}_\alpha(x, T, y(x, T, \omega))}{\partial y_i} \Delta y_i(x, T, \omega) \right) dx - \\
 & - \eta_4 - \eta_5 + \sum_{\beta=1}^k b_\beta \int_G \frac{d\psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{dT} \Delta T dx + \\
 & + \sum_{\beta=1}^k b_\beta \int_G \sum_{i=1}^m \left[\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_i} \right. \\
 & \quad \left. - \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_{ik}} \right) \right] \Delta y_i - \eta_6 - \eta_7 .
 \end{aligned}$$

In the formula (93) we note the following:

(i) By virtue of (16), it follows that

$$\begin{aligned}
 (94) \quad & \sum_{i=1}^m \int_G z_i(x, T) \Delta y_i(x, T, \omega) dx + \sum_{i=1}^m \int_G a_i(x) \Delta y_i(x, T, \omega) dx + \\
 & \sum_{i=1}^m \sum_{\alpha=1}^j \int_G a_\alpha(x) \frac{\partial \bar{\phi}_\alpha(x, T, y(x, T, \omega))}{\partial y_i} \Delta y_i(x, T, \omega) dx + \\
 & \sum_{i=1}^m b_\beta \int_G \sum_{\beta=1}^k \left[\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_i} \right. \\
 & \quad \left. - \sum_{k=1}^n \frac{d}{dx_k} \left(\frac{\partial \psi_\beta(x, T, y(x, T, \omega), y_x(x, T, \omega))}{\partial y_{ik}} \right) \right] \Delta y_i(x, T, \omega) dx = \\
 & = 0 .
 \end{aligned}$$

(ii) Assuming that condition (31) is imposed as supplementary condition it follows that

$$\begin{aligned}
 (95) \quad & \int_G \left[\sum_{i=1}^m (a_i(x) \frac{\partial y_i(x, T, \omega)}{\partial T} + \beta_i(x, T) y_i(x, T, \omega)) + \right. \\
 & \quad \left. \sum_{\alpha=1}^j a_\alpha(x) \frac{d\bar{\phi}_\alpha(x, T, y(x, T, \omega))}{dT} \right] \Delta T dx +
 \end{aligned}$$

$$\begin{aligned}
 & + \int_G \sum_{\beta=1}^k b_{\beta} \frac{d\psi_{\beta}(x, T, y(x, T, \omega), y_x(x, T, \omega))}{dT} \Delta T dx + \\
 & + \sum_{i=1}^m \int_{\Gamma} \gamma_i(x, T) y_i(x, T, \omega) d\sigma \Delta T = 0
 \end{aligned}$$

Next, by virtue of (94), (95), relation (93) can be written as follows:

$$\begin{aligned}
 (96) \quad \Delta I(\omega) = & - \int_C [H(x, t, w^{\omega_1}(x, t), u_1) - H(x, t, w^{\omega}(x, t), u(x, t))] dx dt - \\
 & - \int_0^T \int_{\Gamma} [h(x, t, p^{\omega_1}(x, t), v_1) - h(x, t, p^{\omega}(x, t), v(x, t))] d\sigma dt - \eta,
 \end{aligned}$$

where

$$(97) \quad \eta = \eta_2 + \sum_{i=4}^{10} \eta_i.$$

Further, from (96) and (40) it follows that

$$\begin{aligned}
 (98) \quad & - \int_C [H(x, t, w^{\omega_1}(x, t), u_1) - H(x, t, w^{\omega}(x, t), u(x, t))] dx dt - \\
 & - \int_0^T \int_{\Gamma} [h(x, t, p^{\omega_1}(x, t), v_1) - h(x, t, p^{\omega}(x, t), v(x, t))] d\sigma dt \geq \eta,
 \end{aligned}$$

or

$$(98') \quad - \Delta J_1(u) - \Delta J_2(v) \geq \eta.$$

Remark.

In order to arrive at relation (98) we need the assumption that condition (31) is imposed in advance. Thus, this relation should not be considered as a necessary condition for optimality.

2.3. Proof of the Maximum Principle

Let the admissible control $\omega(x,t) = (u(x,t), v(x,t))$ defined in the region $C = (G+\Gamma) \otimes [0, T]$, transferring the system (6) with boundary conditions (8), from the initial state (7) to the final state (9) at a time T , be optimal with respect to functional $S(\omega)$, and let $y(x,t,\omega)$ be the corresponding optimal trajectory.

The Maximum Principle is proved by contradiction.

Suppose that the Maximum Principle is false.

Then for any system of functions $z_1(x,t), \dots, z_m(x,t)$, $a_1(x), \dots, a_j(x)$ and constants b_1, \dots, b_k , such that the functions $y(x,t,\omega), \omega(x,t), z(x,t), a(x)$ and the constants b satisfy the system (6), (7), (8), (9), its adjoint system (15), (16), (17), and condition (31), we will have that the control $\omega(x,t)$ does not satisfy the maximum condition relative to the functions $z(x,t)$.

This implies that for every such system of functions and constants we can find an admissible control $\omega_1(x,t) = (u_1(x,t), v_1(x,t))$, defined in the region $C_1 = (G+\Gamma) \otimes [0, T_1]$, transferring the system (6) with boundary conditions (8), from the initial state (7) to the final state (9) at a time T_1 and satisfying at least one of the inequalities:

$$(99) \quad \Delta J_1(u) = \int_0^\tau \int_G [H(x,t, w^\omega(x,t), u_1) - H(x,t, w^\omega(x,t), u)] dx dt > 0,$$

$$(100) \quad \Delta J_2(v) = \int_0^\tau \int_\Gamma [h(x,t, p^\omega(x,t), v_1) - h(x,t, p^\omega(x,t), v)] d\sigma dt > 0,$$

where $\tau = \min(T, T_1)$.

Consider any one such system of functions $z_1(x,t), \dots, z_m(x,t)$, $a_1(x), \dots, a_j(x)$, and constants b_1, \dots, b_k .

Assume, to be specific, that for this system the controls $\omega(x,t)$ and $\omega_1(x,t)$ mentioned above satisfy both the inequality (100) and the inequality (99).

Let $y(x,t,\omega_1)$, $z(x,t,\omega_1)$ denote the solution of (6), (7), (8), (9), (31), and

(15), (16), (17), corresponding to the control $\omega_1(x, t)$, to the functions $a_1(x), \dots, a_j(x)$, and to the constants b_1, \dots, b_k .

From inequality (100) it follows that there exists a point (α, β) in the region $\Lambda = \Gamma \otimes [0, \tau]$, such that:

$$(101) \quad h(\alpha, \beta, p(\alpha, \beta), v_1(\alpha, \beta)) - h(\alpha, \beta, p(\alpha, \beta), v(\alpha, \beta)) > 0 .$$

The functions $h(x, t, p(x, t), v_1(x, t))$ and $h(x, t, p(x, t), v(x, t))$ are piecewise continuous in Λ ; therefore, for an arbitrarily small $\varepsilon_1 > 0$, there exists a $\delta > 0$ and a closed region $\Lambda_{\varepsilon_1} \subset \Gamma \otimes [0, \tau]$, containing (α, β) and having measure $\mu(\Lambda_{\varepsilon_1}) = \varepsilon_1$, such that :

$$(102) \quad h(x, t, p(x, t), v_1(x, t)) - h(x, t, p(x, t), v(x, t)) \geq \delta , \text{ for all } (x, t) \in \Lambda_{\varepsilon_1} .$$

Inequality (102) is obviously satisfied for all $(x, t) \in \Lambda_{\varepsilon} \subset \Lambda_{\varepsilon_1}$.

Consider a certain $\Lambda_{\varepsilon} \subset \Lambda_{\varepsilon_1}$, where $\mu(\Lambda_{\varepsilon}) = \varepsilon$, and $0 < \varepsilon \leq \varepsilon_1$. Construct the auxiliary admissible control $\omega_{\varepsilon}(x, t) = (u_{\varepsilon}(x, t), v_{\varepsilon}(x, t))$ defined on the region $C_{\varepsilon} = (G+\Gamma) \otimes [0, T_{\varepsilon}]$, which transfers the system (6) with boundary conditions (8), from the initial state (7) to the final state (9), at a time T_{ε} , and satisfies the following conditions:

- (i) In the region $C_{\theta} = (G+\Gamma) \otimes [0, \theta]$, where $\theta = \min(T, T_{\varepsilon})$, $\omega_{\varepsilon}(x, t)$ is defined by the formula:

$$u_{\varepsilon}(x, t) = u(x, t) ,$$

$$v_{\varepsilon}(x, t) = \begin{cases} v_1(x, t) , & \text{for } (x, t) \in \Lambda_{\varepsilon} \\ v(x, t) , & \text{for } (x, t) \in C_{\theta} - \Lambda_{\varepsilon} \end{cases}$$

- (ii) In the region $(G+\Gamma) \otimes (\theta, T_{\varepsilon}]$, $\omega_{\varepsilon}(x, t)$ is defined to be any control in E.

- (iii) The inequality $|T - T_\varepsilon| < L\varepsilon$ holds, where the number L is independent of ε .

Such an admissible control $\omega_\varepsilon(x, t)$ exists because of the assumption that the class of admissible controls is complete.

For the controls $\omega(x, t)$, $\omega_\varepsilon(x, t)$, it follows from (18), (19) that:

$$(103) \quad \Delta J_1(u) = J_1(u_\varepsilon) - J_1(u) = 0,$$

$$(104) \quad \begin{aligned} \Delta J_2(v) &= J_2(v_\varepsilon) - J_2(v) = \\ &= \int_0^{\theta} \int_{\Gamma} [h(x, t, p(x, t), v_\varepsilon) - h(x, t, p(x, t), v)] d\sigma dt = \\ &= \int_{\Lambda_\varepsilon} [h(x, t, p(x, t), v_1) - h(x, t, p(x, t), v)] d\Lambda_\varepsilon. \end{aligned}$$

Further, from (102) it follows that:

$$(105) \quad \Delta J_2(v) \geq \delta\mu(\Lambda_\varepsilon) = \delta\varepsilon > 0.$$

Thus

$$(106) \quad \int_0^{\theta} \int_{\Gamma} [h(x, t, p(x, t), v_\varepsilon) - h(x, t, p(x, t), v)] d\sigma dt \geq \delta\varepsilon.$$

It can be proved ([2], p.p. 400-404) that the quantity η defined by (96) is of a higher order of smallness than ε , i.e., $\lim_{\varepsilon \rightarrow 0} \frac{\eta(\varepsilon)}{\varepsilon} = 0$,

and it satisfies the inequality

$$(107) \quad |\eta| \leq F\varepsilon^2, \quad \text{where } F \text{ is constant.}$$

From (106), (107) it follows that:

$$(108) \quad - \int_0^\theta \int_\Gamma [h(x,t,p(x,t),v_\varepsilon) - h(x,t,p(x,t),v)] d\sigma dt - \eta \leq -\delta\varepsilon + F\varepsilon^2.$$

Relation (106) and therefore (108) hold no matter how small ε is .

Therefore ε can be chosen small enough so that $-\delta\varepsilon + F\varepsilon^2$ becomes negative; since (108) still holds it follows that :

$$(109) \quad - \int_0^\theta \int_\Gamma [h(x,t,p(x,t),v_\varepsilon) - h(x,t,p(x,t),v)] d\sigma dt - \eta < 0 .$$

Also, from (99) it follows that:

$$(110) \quad - \int_0^\theta \int_G [H(x,t,w(x,t),u_1) - H(x,t,w(x,t),u)] dx dt < 0 .$$

Adding (109) and (110),

$$(111) \quad - \int_0^\theta \int_G [H(x,t,w(x,t),u_1) - H(x,t,w(x,t),u)] dx dt - \\ - \int_0^\theta \int_\Gamma [h(x,t,p(x,t),v_\varepsilon) - h(x,t,p(x,t),v)] d\sigma dt < \eta .$$

But relation (111) contradicts relation (98) and thus the validity of the Maximum Principle is proved.

CHAPTER 3

APPLICATIONS

3.1. A thermodynamics system (Heat transfer)

3.1.1. Statement of the problem

Let the system be described by the differential equation :

$$(112) \quad \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = u(x,t), \quad (x,t) \in [0,1] \otimes [0,T],$$

where

$x \in G = [0,1]$, is the length variable, $t \in [0,T]$, is the time variable and $y(x,t)$ is a real valued function, defined on $[0,1] \otimes [0,T]$, representing the temperature distribution on a heated bar of length 1.

The initial condition is :

$$(113) \quad y(x,0) = a(x), \quad x \in [0,1],$$

where $a(x)$ is a given continuous real valued function defined on $[0,1]$.

The boundary condition is:

$$(114) \quad \frac{\partial y(x,t)}{\partial x} = y^4(x,t) - v^4(x,t), \quad (x,t) \in \partial G \otimes [0,T].$$

where $v(x,t)$ is the temperature distribution of the furnace.

The final state of the system is determined by the relation:

$$(115) \quad \Phi(x,T,y(x,T)) \equiv y(x,T) - k(x), \quad 0$$

where $k(x)$ is a given continuous real valued function, defined on $[0,1]$, representing the desirable final distribution of temperature along the length 1 of the bar.

The heat distribution on the bar is affected by controls $u(x,t)$, $v(x,t)$, which occur both in equation (112) and in the boundary condition (114).

The class U^* of admissible controls $u(x,t)$ consists of all real valued piecewise continuous functions $u(x,t)$ on $[0,1] \otimes [0,T]$, such that

$$|u(x,t)| \leq 1, \text{ for all } (x,t) \in [0,1] \otimes [0,T].$$

The class V^* of admissible controls $v(x,t)$, is defined similarly.

The class of admissible controls $\omega(x,t) = (u(x,t), v(x,t))$, is $U^* \otimes V^*$.

Our objective is to find a control $\omega^*(x,t) = (u^*(x,t), v^*(x,t)) \in U^* \otimes V^*$, if such a control exists, so that it transfers the system (112) with boundary condition (114), from the initial state (113) to the final state (115), and gives to the functional

$$(116) \quad S(\omega) = \int_{x=0}^1 \alpha_1(x)y(x,T)dx + \int_{t=0}^T \int_{x=0}^1 \beta_1(x,t)y(x,t)dxdt$$

the minimum possible value; the functions $\alpha_1(x)$, $\beta_1(x,t)$ are given real valued continuous functions defined on $[0,1]$ and $[0,1] \otimes [0,T]$ respectively.

3.1.2. Solution of the problem

In order to find the optimal control $\omega^*(x,t)$ we will apply the maximum principle.

From (112) and (1), (2), (3), (3') the following can be written:

$$(117) \quad Ly = \frac{\partial^2 y}{\partial x^2}$$

$$(118) \quad L_t y = \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2}$$

$$(119) \quad Mz = \frac{\partial^2 z}{\partial x^2}$$

$$(120) \quad M_t z = \frac{\partial z}{\partial t} + \frac{\partial^2 z}{\partial x^2}$$

Take $a_1^{iv}(x,t) = 1$, $b_{iv}(x,t) = 0$, and as direction l_{iv} the direction of the $+x$ axis. Thus

$$(121) \quad \frac{dy}{d\lambda_{iv}} = \frac{\partial y}{\partial x}, \quad \text{and from (4):}$$

$$(122) \quad P_y = \frac{\partial y}{\partial x}.$$

Similarly, for $a_{\lambda}^{vi}(x,t) = 1$, $d_{iv}(x,t) = 0$, and taking as direction λ_{iv} , the direction of the $+x$ axis, it follows that:

$$(123) \quad \frac{dz}{d\lambda_{iv}} = \frac{\partial z}{\partial x}.$$

Hence from (15):

$$(124) \quad Q_z = \frac{\partial z}{\partial x}.$$

The equations (112), (113), (114), (115), (116) are special cases of (6), (7), (8), (9), (10) respectively, where

$$\begin{aligned} f(x,t,y,y_x,u) &= u(x,t), \\ \varphi(x,t,y,v) &= y^4(x,t) - v^4(x,t), \\ \bar{\varphi}(x,T,y(x,T)) &= y(x,T) - k(x), \\ \gamma(x,t) &= 0. \end{aligned}$$

Further, from (11), (12), it follows that the Hamiltonians are :

$$(125) \quad H(x,t,w,u) = zu,$$

$$(126) \quad h(x,t,p,v) = z(y^4(x,t) - v^4(x,t)),$$

where

$$(127) \quad w = (z, y, y_x),$$

$$(128) \quad p = (z, y) .$$

From (15), (119) and (125) it follows that the system adjoint to (112) is:

$$(129) \quad \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x^2} = \beta_1(x, t) , \quad (x, t) \in [0, 1] \otimes [0, T] .$$

From (16) and (115) , the initial condition of the adjoint system is:

$$(130) \quad z(x, T) = A(x) , \quad x \in [0, 1] ,$$

where $A(x) = -\alpha_1(x) - a_1(x)$, and where the function $\alpha_1(x)$ is defined in (116) but the function $a_1(x)$ is as yet undefined.

From (17), (124), (125), (126), and since $\gamma(x, t) = 0$, it follows that the boundary condition for the adjoint system is:

$$(131) \quad \frac{\partial z}{\partial x} = 4y^3(x, t)z , \quad (x, t) \in \{0, 1\} \otimes [0, T] .$$

Finally, condition (31) becomes:

$$(132) \quad \int_{x=0}^1 \left[\alpha_1(x) \frac{\partial y(x, T)}{\partial t} + \beta_1(x, T)y(x, T) + a_1(x) \frac{\partial y(x, T)}{\partial t} \right] dx = 0 .$$

According to the Maximum Principle (Necessary conditions for optimality), if there exists an optimal control $\omega^*(x, t) = (u^*(x, t), v^*(x, t)) \in U^* \otimes V^*$ for the stated problem, then there must exist functions $y(x, t), z(x, t), a(x)$, such that these functions together with $\omega^*(x, t)$ satisfy the given system (112), (113), (114), (115) and also its adjoint system (129), (130), (131), and moreover the control $\omega^*(x, t)$ satisfies the maximum condition (20') and (21') relative to the function $z(x, t)$.

From (20'), (21'), it follows that:

$$(133) \quad zu^* \quad (=\) \quad \sup \{ zu : |u| \leq 1 \} , \quad \text{and}$$

$$(134) \quad z(y^4 - v^{*4}) \quad (=\) \quad \sup \{ z(y^4 - v^4) : |v| \leq 1 \}$$

From (133), (134), it follows that:

$$(135) \quad u^*(x,t) = \text{sign} [z(x,t)] , \quad \text{a.e. in } [0,1] \otimes [0,T] ,$$

$$(136) \quad v^*(x,t) = \frac{1 - \text{sign} [z(x,t)]}{2} , \quad \text{a.e. in } [0,1] \otimes [0,T] .$$

Thus if there exists an optimal control $\omega^*(x,t) = (u(x,t), v(x,t))$, then it must be of bang-bang type.

In order to find the expression of the optimal control $\omega^*(x,t)$ as a function of (x,t) , and also the corresponding optimal trajectory, we must proceed as follows:

First we must solve the given system:

$$(112') \quad \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = u(x,t) , \quad (x,t) \in [0,1] \otimes [0,T] ,$$

with initial condition:

$$(113') \quad y(x,0) = a(x) , \quad x \in [0,1] ,$$

with boundary condition:

$$(114') \quad \frac{\partial y}{\partial x} = y^4 - v^4 , \quad (x,t) \in [0,1] \otimes [0,T] ,$$

and final condition:

$$(115') \quad y(x,T) = k(x) , \quad x \in [0,1] .$$

Thus the trajectory $y(x,t)$, functionally dependent on the control ω , can be found.

Next we must solve the adjoint of the given system, i.e.

$$(129') \quad \frac{\partial z}{\partial t} + \frac{\partial^2 z}{\partial x^2} = \beta_1(x,t), \quad (x,t) \in [0,1] \otimes [0,T],$$

with initial condition:

$$(130') \quad z(x,T) = A(x), \quad x \in [0,1],$$

and with boundary condition:

$$(131') \quad \frac{\partial z}{\partial x} = 4y^3(x,t)z, \quad (x,t) \in \{0,1\} \otimes [0,T],$$

where $y(x,t)$ is functionally dependent on ω^* .

The function $a_1(x)$ which is a part of the second member of (130') must be so chosen that the condition,

$$(31') \quad \int_0^1 \left[a_1(x) \frac{\partial y(x,T)}{\partial t} + \beta_1(x,T)y(x,T) + a_1(x) \frac{\partial y(x,T)}{\partial t} \right] dx = 0,$$

is satisfied.

By solving the adjoint system we can find $z(x,t)$ functionally dependent on ω . Next, substituting $z(x,t)$ into (135), (136), we can find respectively the expressions of the optimal controls $u^*(x,t)$ and $v^*(x,t)$.

Finally, putting $u^*(x,t)$ and $v^*(x,t)$, instead of $u(x,t)$ and $v(x,t)$ in relations (112') and (114'), we can find the corresponding optimal trajectory $y^*(x,t)$, by solving the system (112'), (113'), (114'), (115').

Next, an investigation of the method of solution for the system (112), (113), (114), (115), and also for its adjoint system (129), (130), (131), is given.

Let us consider generally the problem:

$$(137) \quad Ly \equiv \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial y}{\partial x_i} + c(x,t)y - \frac{\partial y}{\partial t} = f(x,t),$$

for $(x,t) \in D$;

$$(138) \quad y(x,0) = a(x), \quad x \in B$$

$$(139) \quad \frac{\partial y(x,t)}{\partial \nu(x,t)} = \varphi(x,t,y(x,t),g(t)), \quad (x,t) \in S,$$

where

$\frac{\partial y(x,t)}{\partial \nu(x,t)}$ is the inward conormal derivative,

D is the set $B \otimes (0, \infty)$, S is the lateral boundary of D , and

$f(x,t)$, $\varphi(x,t,u,v)$, $g(t)$ are given functions.

Definition

The operator L defined in (137) is said to be parabolic at (x_0, t_0) , if the matrix $(a_{ij}(x_0, t_0))$ is positive definite, i.e., if for every real vector $\xi = (\xi_1, \dots, \xi_n) \neq 0$, the relation:

$$\sum_{i,j=1}^n a_{ij}(x_0, t_0) \xi_i \xi_j > 0, \text{ is satisfied.}$$

We will use the following

Theorem ([4], p.p. 208-210)

Assume that:

∂B is of class $C^{1+\beta}$ ($0 < \beta < 1$),

L is parabolic in \bar{D} ,

there exist positive constants k_1, k_2 such that $a_{11}(x,t) \geq k_1$, $b_1(x,t) \geq -k_2$,
the coefficients a_{ij}, b_i, c of L are Hölder continuous in \bar{D} and $c \leq 0$,

$\varphi(x,t,u,v)$ is continuous for $(x,t) \in S$, $-\infty < u < \infty$, $-\infty < v < \infty$,

$\varphi(x,t,u,v) \rightarrow \pm\infty$ as $u \rightarrow \pm\infty$, uniformly with respect to (x,t) in \bar{S} and v
in bounded sets,

$\varphi(x,t,u,v)$ is strictly increasing in u and strictly decreasing in v ,

$f(x,t)$ is Hölder continuous in x , uniformly in bounded subsets of \bar{D} ,

$g(t)$ is continuous for $t \in [0, \infty)$,

$a(x)$ is continuous function with compact support in B ,

Under these assumptions there exists a unique solution of (137), (138), (139).

To find this solution we proceed as follows ([4], p.p. 208-210):

Let Z be the Banach space of all functions $p(x,t)$, which are continuous
in \bar{D} with the norm:

$$\| p \| = \sup_D \| p(x,t) \|.$$

For any $R > 0$, define $Z_R = \{ p \in Z : \| p \| \leq R \}$.

For every $p \in Z_R$ define the transformation $w = Tp$ in such a way
that w is the solution of (137), (138), and (139)*, where

$$(139)^* \quad \frac{\partial w}{\partial v} = \varphi(x,t,p(x,t),g(t))$$

The function w is uniquely determined and has the form:

$$(140) \quad w(x,t) = \int_0^t \int_{\partial B} \Gamma(x,t,\xi,\tau) \rho(\xi,\tau) dB_\xi d\tau + G(x,t),$$

where:

dB_ξ is the surface element on ∂B ;

$\Gamma(x, t, \xi, \tau)$ is the fundamental solution of the equation $Ly = 0$,

and the functions $G(x, t)$, $\rho(x, t)$ are defined as follows :

$$(141) \quad G(x, t) = \int_B \Gamma(x, t, \xi, 0) a(\xi) d\xi - \int_0^t \int_B \Gamma(x, t, \xi, \tau) f(\xi, \tau) d\xi d\tau ,$$

and $\rho(x, t)$ is a solution of the integral equation :

$$(142) \quad \rho(x, t) = 2 \int_0^t \int_{\partial B} \frac{\Gamma(x, t, \xi, \tau)}{v(x, t)} \rho(\xi, \tau) dB_\xi d\tau + 2 \frac{G(x, t)}{v(x, t)} - 2\varphi(x, t, \rho(x, t), g(t))$$

In ([4], p.p.208-210) is also proved that T has a fixed point, i.e., that there exists $p \in Z_R$ such that $p = Tp$.

This function $p(x, t)$ is the solution of (137), (138), (139), since, in this case, (139') coincides with (139).

We will apply the previous results to our problem.

Consider the adjoint system:

$$(129''') \quad \frac{\partial z}{\partial t} + \frac{\partial^2 z}{\partial x^2} = \beta_1(x, t) , \quad (x, t) \in [0, 1] \otimes [0, T] ,$$

$$(130''') \quad z(x, T) = A(x) , \quad x \in [0, 1] ,$$

$$(131''') \quad \frac{\partial z}{\partial x} = 4y^3(x, t)z , \quad (x, t) \in [0, 1] \otimes [0, T] .$$

Let us define a new variable s and the functions $\hat{y}(x, s)$, $\hat{\beta}_1(x, s)$, $\hat{z}(x, s)$, as follows:

$$(143) \quad s(t) = T - t , \quad t \in [0, T] ;$$

thus $s(0) = T$, $s(T) = 0$, and for $t \in [0, T]$, we have $s \in [0, T]$.

$$(144) \quad \hat{y}(x, s) = y(x, T-s) = y(x, t) ,$$

$$(145) \quad \hat{\beta}_1(x, s) = \beta_1(x, T-s) = \beta_1(x, t) ,$$

$$(146) \quad \hat{z}(x, s) = z(x, T-s) = z(x, t) .$$

From (146) it follows that :

$$(147) \quad \frac{\partial z}{\partial t} = \frac{\partial \hat{z}}{\partial t} = \frac{\partial \hat{z}}{\partial s} \frac{\partial s}{\partial t} = - \frac{\partial \hat{z}}{\partial s} ,$$

$$(148) \quad \frac{\partial z}{\partial x} = \frac{\partial \hat{z}}{\partial x} ,$$

$$(148') \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 \hat{z}}{\partial x^2} .$$

The adjoint system can be written as follows:

$$(129''') \quad \frac{\partial^2 \hat{z}(x, s)}{\partial x^2} - \frac{\partial \hat{z}(x, s)}{\partial s} = \hat{\beta}_1(x, s) , \quad (x, s) \in [0, 1] \otimes [0, T] ,$$

$$(130''') \quad \hat{z}(x, 0) = A(x) , \quad x \in [0, 1]$$

(because $z(x, T) = z(x, t)|_{t=T} = z(x, T-s)|_{s=0} = \hat{z}(x, s)|_{s=0} = \hat{z}(x, 0)$) ,

$$(131''') \quad \frac{\partial \hat{z}(x, s)}{\partial x} = 4\hat{y}^3(x, s)\hat{z}(x, s)$$

For the above system, all the assumptions of the previous theorem are fulfilled; therefore the system has a unique solution.

To find this solution , we proceed as follows :

Let \hat{Z} be the Banach space of all functions $\hat{z}(x, s)$, which are continuous in $[0, 1] \otimes [0, T]$, with norm :

$$|| \hat{z} || = \sup_{[0, 1] \otimes [0, T]} | \hat{z}(x, s) |$$

For any $R > 0$, define $\hat{Z}_R = \{ \hat{z} \in \hat{Z} : || \hat{z} || \leq R \}$.

For every $\hat{z} \in \hat{Z}_R$, define the transformation $\hat{w} = T\hat{z}$ in such a way that $\hat{w}(x, s)$ is the solution of (129'''), (130'''), and (131''')*, where

$$(131''')^* \quad \frac{\hat{w}(x, s)}{x} = 4\hat{y}^3(x, s)\hat{z}(x, s)$$

The function $\hat{w}(x, s)$ is uniquely determined and has the form :

$$(140') \quad \hat{w}(x, s) = \int_{\eta=0}^s \int_{\partial B} \Gamma(x, s, \xi, \eta) \rho(\xi, \eta) dB_{\xi} d\eta + G(x, s) ,$$

where:

$$B = [0, 1] ;$$

$\Gamma(x, s, \xi, \eta)$ is the fundamental solution of the equation

$$(149) \quad \frac{\partial^2 \hat{z}(x, s)}{\partial x^2} - \frac{\partial \hat{z}(x, s)}{\partial s} = 0 ;$$

and the functions $G(x, s)$, $\rho(x, s)$, as defined as follows:

$$(141') \quad G(x, s) = \int_B \Gamma(x, s, \xi, 0) A(\xi) d\xi - \int_{\eta=0}^s \int_B \Gamma(x, s, \xi, \eta) \hat{\beta}_1(\xi, \eta) d\xi d\eta ,$$

and $\rho(x, s)$ is a solution of the integral equation:

$$(142') \quad \rho(x, s) = 2 \int_{\eta=0}^s \int_{\partial B} \frac{\partial \Gamma(x, s, \xi, \eta)}{\partial x} \rho(\xi, \eta) dB_{\xi} d\eta + 2 \frac{\partial G(x, s)}{\partial x} - 8\hat{y}^3(x, s)\hat{z}(x, s) .$$

We now calculate $\hat{w}(x, s)$.

The fundamental solution of (149) is:

$$(150) \quad \Gamma(x, s, \xi, \eta) = \frac{1}{2\sqrt{\pi(s-\eta)}} \exp\left(-\frac{(x-\xi)^2}{4(s-\eta)}\right) .$$

From (141'), (150), it follows that:

$$(141''') \quad G(x, s) = \frac{1}{2\sqrt{\pi s}} \int_{\xi=0}^1 \exp\left(-\frac{(x-\xi)^2}{4s}\right) A(\xi) d\xi - m \\ - \frac{1}{2\sqrt{\pi}} \int_{\eta=0}^s \int_{\xi=0}^1 \frac{1}{\sqrt{s-\eta}} \exp\left(-\frac{(x-\xi)^2}{4(s-\eta)}\right) \hat{\beta}_1(\xi, \eta) d\xi d\eta .$$

Next, from (142') it follows that:

$$(142''') \quad \rho(x, s) = \frac{1}{2\sqrt{\pi}} \int_{\eta=0}^s \int_{\partial B} \frac{\xi-x}{(s-\eta)^{\frac{3}{2}}} \exp\left(-\frac{(x-\xi)^2}{4(s-\eta)}\right) \rho(\xi, \eta) dB_{\xi} d\eta + \\ + 2 \frac{\partial G(x, s)}{\partial x} - 8 \hat{y}^3(x, s) \hat{z}(x, s) ,$$

where $G(x, s)$ is given by (141''').

From (142''') we should find $\rho(x, s)$ as a functional of a_1, \hat{y}, \hat{z} . Next, we should replace $\rho(x, s)$ in (140') to find the function $\hat{w}(x, s)$ which satisfies (129'''), (130'''), (131''')*

This function $\hat{w}(x, s)$ will be found as a functional of a_1, \hat{y}, \hat{z} .

Now, we know that there exists a function $\hat{z}^*(x, s) \in \hat{Z}_R$, such that $\hat{z}^* = T\hat{z}^* = \hat{w}^*$; this function $\hat{z}^*(x, s)$ will satisfy (129'''), (130''') and (131''') (which now coincides with (131''')*).

Thus, this function $\hat{z}^*(x, s)$ will be the solution of the adjoint system and it will be found from the integral equation:

$$(140''') \quad \hat{z}(x, s) = \int_{\eta=0}^s \int_{\partial B} \frac{1}{2\sqrt{\pi(s-\eta)}} \exp\left(-\frac{(x-\xi)^2}{4(s-\eta)}\right) \rho(\xi, \eta) dB_{\xi} d\eta + G(x, s) ,$$

where $\rho(\xi, \eta)$ is functionally dependent on a_1, \hat{y}, \hat{z} .

Thus, we can find $\hat{z}(x, s)$, and consequently $z(x, t)$, as a functional of a_1, y .

Next, we should solve the initial system (112'), (113'), (114'), (115'), to find the trajectory $y(x, t)$ functionally dependent on the control ω^* .

Substituting then $y(x, t)$, we can have $z(x, t)$ as a functional of a_1, ω^* .

The function $a_1(x)$ must be so chosen, that (132) is satisfied.

Since in relation (132), the function $y(x,t)$, is functionally dependent on ω^* , we can find $a_1(x)$ as a functional of ω^* .

Thus, we can have $z(x,t)$ as a functional of ω^* .

Finally, substituting $z(x,t)$ in (135), (136), we can find respectively the expressions of the optimal controls $u^*(x,t), v^*(x,t)$.

The corresponding optimal trajectory $y^*(x,t)$ can be found by solving the initial system (112), (113), (114); (115), in which $\omega(x,t)$ has been replaced by the optimal control $\omega^*(x,t)$.

3.2. A biological system (Control of sodium transport in toad bladder)

3.2.1. Statement of the problem ([5])

Treating the sodium ions in the cytoplasm as a distributed quadratic source term and modelling the coupling between the diffusional flow and the electrogenic process by an operator in the space-time diffusion model [6,7], the dynamic process of active transport of sodium ions in the epithelial cell of the toad bladder may be formulated [5] as follows :

$$(151) \quad \frac{\partial y(x,t)}{\partial t} = D \frac{\partial^2 y(x,t)}{\partial x^2} + \frac{\partial}{\partial x} (c_1 y(x,t) + c_2 y^2(x,t)) ,$$
$$(x,t) \in [0,1] \otimes [0,\infty)$$

Initial condition:

$$(152) \quad y(x,0) = a(x) , \quad x \in [0,1]$$

The boundary condition is:

$$(153) \quad y(x,t) = \begin{cases} 0 & \text{for } (x,t) \in \{0\} \otimes [0,T] \\ v(t) & \text{for } (x,t) \in \{1\} \otimes [0,T] \end{cases}$$

where:

$y(x,t)$ is the space and time dependent sodium concentration in the cell of width 1;

D is the diffusion coefficient;

c_1, c_2 are reaction constants;

$v(t)$ is the time -dependent boundary condition at the mucosal side of the cell and corresponds to change of the permeability of the membrane.

The function $v(t)$, $t \in [0, T]$, is the control of the process.

It can be shown [5] that in order for the in vivo control operation to be admissible, the control $v(t)$ must satisfy the following inequality:

$$(154) \quad \left| v(t) + \frac{D}{c_2 l} \ln \left(c_1 + 2c_2 \frac{dv(t)}{dt} \right) \right| \leq M,$$

where $M = \frac{D}{c_2 l} \ln \left(\frac{2D}{l} - \frac{c_1}{2c_2} \right)$ is a constant.

However, we consider here a simpler case, where the class V^* of admissible controls consists of all piecewise continuous functions defined on $[0, T]$, and taking values in the control restraint set $V = [0, M]$, i.e.,

$$(155) \quad V^* = \text{P.C.} ([0, T], V),$$

where M is a positive constant.

Let $k(x)$ be a given desirable distribution of the sodium ions concentration along the width of the cell at a time T .

Define on V^* the functional

$$(156) \quad S(v) = \int_{x=0}^1 a_1(x) y(x, T) dx + \int_{t=0}^T \int_{x=0}^1 \beta_1(x, t) y(x, t) dx dt,$$

where:

$y(x, t)$ is the solution of the given system corresponding to v ;

$a_1(x)$, $\beta_1(x, t)$, are given real valued continuous functions with domains $[0, 1]$ and $[0, 1] \otimes [0, \infty)$ respectively.

The functional $S(v)$ represents the "cost" corresponding to the transition of the system from the initial state (152) to the desirable final state:

$$(157) \quad y(x, T) = k(x)$$

at a time T , where $k(x)$ is a given continuous real valued function defined on $[0,1]$.

Our objective is to find a control $v^* \in V^*$, such that it steers the system (151) with boundary conditions (153) from the initial state (152) to the final state (157) and also the functional (156) takes its minimum value for $v = v^*$.

3.2.2. Solution of the problem

We will make use of the Maximum Principle.

From (151) and (1), (2), (3), (3') the following can be written:

$$(158) \quad Ly = D \frac{\partial^2 y}{\partial x^2}$$

$$(159) \quad L_t y = \frac{\partial y}{\partial t} - D \frac{\partial^2 y}{\partial x^2}$$

$$(160) \quad Mz = D \frac{\partial^2 z}{\partial x^2}$$

$$(161) \quad M_t z = \frac{\partial z}{\partial t} + D \frac{\partial^2 z}{\partial x^2}$$

Take $a_{1v}^{iv}(x,t) = 0$, $b_{iv}(x,t) = 1$; then from (4) we have:

$$(162) \quad Py = y.$$

Similarly, for $a_{\lambda}^{vi}(x,t) = 1$, $d_{iv}(x,t) = 0$, and taking as direction λ_{iv} the direction of the $+x$ axis it follows that:

$$(163) \quad \frac{dz}{d\lambda_{iv}} = \frac{\partial z}{\partial x}.$$

Thus, (5) can be written as:

$$(164) \quad Qz = \frac{\partial z}{\partial x}$$

Equation (151) can be written as follows :

$$(151') \quad \frac{\partial y}{\partial t} - D \frac{\partial^2 y}{\partial x^2} = c_1 \frac{\partial y}{\partial x} + 2c_2 y \frac{\partial y}{\partial x}, \quad (x,t) \in [0,1] \otimes [0,\infty).$$

The equations (151'), (152)', (153), (156), (157), are special cases of (6), (7), (8), (9), (10), respectively, where:

$$f(x,t,y,y_x,u) = c_1 \frac{\partial y}{\partial x} + 2c_2 y \frac{\partial y}{\partial x}$$

$$\varphi(x,t,y,v) = \begin{cases} 0 & \text{for } (x,t) \in \{0\} \otimes [0,T] \\ v(t) & \text{for } (x,t) \in \{1\} \otimes [0,T] \end{cases}$$

$$\Phi(x,T,y(x,T)) = y(x,T) - k(x) \quad \text{and}$$

$$\gamma(x,t) = 0.$$

Further, from (11), (12), it follows that the Hamiltonians are:

$$(165) \quad H(x,t,w,u) = z \left(c_1 \frac{\partial y}{\partial x} + 2c_2 y \frac{\partial y}{\partial x} \right),$$

$$(166) \quad h(x,t,p,v) = \begin{cases} 0 & \text{for } (x,t) \in \{0\} \otimes [0,T] \\ z v(t) & \text{for } (x,t) \in \{1\} \otimes [0,T] \end{cases}$$

where $w = (z, y, \frac{\partial y}{\partial x})$, and $p = (z, y)$.

From (15), (161), (165) it follows that the system adjoint to (151) is :

$$(167) \quad \frac{\partial z}{\partial t} + D \frac{\partial z}{\partial x} = z(2c_2 y_x + 2c_2 y + c_1) + \beta_1(x,t).$$

From (157) it follows that,

$$(168) \quad \Phi(x, T, y(x, T)) = y(x, T) - k(x) .$$

From (16), (168), it follows that the initial condition of the adjoint system is:

$$(169) \quad z(x, T) = A(x) \quad , \quad x \in [0, 1] \quad ,$$

where $A(x) = -\alpha_1(x) - a_1(x)$, and where the function $\alpha_1(x)$ is defined in (156) but the function $a_1(x)$ is as yet undefined.

From (17), (164), (165), (166), it follows that the boundary condition for the adjoint system is:

$$(170) \quad \frac{\partial z(x, t)}{\partial x} = z(c_1 + 2c_2 y) \quad , \quad (x, t) \in \{0, 1\} \otimes [0, T] \quad ,$$

Finally, condition (31) becomes :

$$(31'') \quad \int_{x=0}^1 (\alpha_1(x) \frac{\partial y(x, T)}{\partial t} + \beta_1(x, T)y(x, T) + a_1(x) \frac{\partial y(x, T)}{\partial t}) dx = 0 .$$

According to the Maximum Principle (Necessary conditions for optimality), if there exists an optimal control $v^* \in V^*$, for the stated problem, then there must exist functions $y(x, t)$, $z(x, t)$, $\alpha_1(x)$, such that these functions together with $v^*(t)$ satisfy the given system (151), (152), (153), (157), and also its adjoint system (167), (169), (170), and moreover the control $v^*(t)$ satisfies the maximum condition (21') relative to the function $z(x, t)$.

From (21'), it follows that :

$$(171) \quad zv^* (=) \sup \{ .zv : v \in V^* \}$$

From (171), it follows that:

$$(172) \quad v^*(t) = \begin{cases} M & \text{whenever } z > 0 \\ 0 & \text{whenever } z \leq 0 \end{cases} \quad \text{a.e. in } [0, T] .$$

Formula (172) can be written as follows :

$$(173) \quad v^*(t) = \frac{1}{2} M (1 + \text{sign}[z(x,t)]) \quad \text{a.e. in } [0, T] .$$

Thus if there exists an optimal control $v^*(t)$, then it must be of bang-bang type.

In order to find the expression of the optimal control $v^*(t)$ as a function of t , and also the corresponding optimal trajectory $y^*(x,t)$, as a function of (x,t) , we must proceed as in the previous problem (3.1.).

First we must solve the given system:

$$(151') \quad \frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} + \frac{1}{x} (c_1 y + c_2 y^2) , \quad (x,t) \in [0,1] \otimes [0,T]$$

with initial condition:

$$(152') \quad y(x,0) = a(x)$$

with boundary condition:

$$(153') \quad y(x,T) = \begin{cases} 0 & \text{for } (x,t) \in \{0\} \otimes [0,T] \\ v(t) & \text{for } (x,t) \in \{1\} \otimes [0,T] \end{cases}$$

and with final condition:

$$(157') \quad y(x,T) = k(x) .$$

Thus the trajectory $y(x,t)$ functionally dependent on the control v ,

can be found.

Next we must solve the adjoint system:

$$(167') \quad \frac{\partial z}{\partial t} + D \frac{\partial^2 z}{\partial x^2} = z(2c_2 y_x + 2c_2 y + c_1) + \beta_1(x, t),$$

$$(x, t) \in [0, 1] \otimes [0, T],$$

with initial condition :

$$(169') \quad z(x, T) = A(x), \quad x \in [0, 1],$$

and with boundary condition:

$$(170') \quad \frac{\partial z}{\partial x} = z(c_1 + 2c_2 y), \quad (x, t) \in \{0, 1\} \otimes [0, T],$$

where y is functionally dependent on v^* .

The function $a_1(x)$ which is a part of the second member of (169') must be so chosen that the condition:

$$(31''') \quad \int_{x=0}^1 (a_1(x) \frac{\partial y(x, T)}{\partial t} + \beta_1(x, T)y(x, T) + a_1(x) \frac{\partial y(x, T)}{\partial t}) dx = 0$$

is satisfied.

By solving the adjoint system we find $z(x, t)$ functionally dependent on v . Next, substituting $z(x, t)$ in (173) we can find the expression of the optimal control $v^*(t)$.

Finally, substituting $v^*(t)$ instead of $v(t)$ in (153), we can find the expression of the optimal trajectory $y^*(x, t)$, by solving the system (151), (152), (153) and (157).

Next, a description of the method of solution of the given system (151), (152), (153), (157) and also of its adjoint system (167), (169), (170), is given.

Consider the adjoint system:

$$(167'') \quad \frac{\partial z}{\partial t} + D \frac{\partial^2 z}{\partial x^2} = z(2c_2 y_x + 2c_2 y + c_1) + \beta_1(x, t),$$

$$(x, t) \in [0, 1] \otimes [0, T],$$

$$(169'') \quad z(x, T) = A(x), \quad x \in [0, 1],$$

$$(170'') \quad \frac{\partial z}{\partial x} = z(c_1 + 2c_2 y), \quad (x, t) \in \{0, 1\} \otimes [0, T].$$

Let us define a new variable $s(t)$, and the functions $\hat{y}(x, s)$, $\hat{\beta}_1(x, s)$, $\hat{z}(x, s)$ as follows:

$$(174) \quad s(t) = T - t, \quad t \in [0, T];$$

thus $s(0) = T$, $s(T) = 0$, and for $t \in [0, T]$, we have $s \in [0, T]$.

$$(175) \quad \hat{y}(x, s) = y(x, T-s) = y(x, t),$$

$$(176) \quad \hat{\beta}_1(x, s) = \beta_1(x, T-s) = \beta_1(x, t)$$

$$(177) \quad \hat{z}(x, s) = z(x, T-s) = z(x, t).$$

From (174), (175), (177), it follows that:

$$(178) \quad y_x(x, t) = \hat{y}_x(x, s)$$

$$(179) \quad \frac{\partial z}{\partial t} = \frac{\partial \hat{z}}{\partial t} = \frac{\partial \hat{z}}{\partial s} \frac{\partial s}{\partial t} = - \frac{\partial \hat{z}}{\partial s}$$

$$(180) \quad \frac{\partial z}{\partial x} = \frac{\partial \hat{z}}{\partial x}$$

$$(181) \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 \hat{z}}{\partial x^2}$$

Thus the adjoint system can be written as follows:

$$(167''''') \quad D \frac{\partial^2 \hat{z}(x, s)}{\partial x^2} - c \hat{z}(x, s) - \frac{\partial \hat{z}(x, s)}{\partial s} = \hat{\beta}_1(x, s) \quad , \quad (x, s) \in [0, 1] \otimes [0, T] \quad ,$$

$$\text{where } c = 2c_2 \hat{y}_x + 2c_2 \hat{y} + c_1 \quad ,$$

$$(169''''') \quad \hat{z}(x, 0) = A(x) \quad , \quad x \in [0, 1]$$

$$(\text{ because } z(x, T) = z(x, t) \Big|_{t=T} = \hat{z}(x, s) \Big|_{s=0} = \hat{z}(x, 0) \quad ,$$

$$(170''''') \quad \frac{\partial \hat{z}(x, s)}{\partial x} = (c_1 + 2c_2 \hat{y}) \hat{z}(x, s) \quad , \quad (x, s) \in [0, 1] \otimes [0, T] \quad .$$

If we impose the additional assumptions:

$$(182) \quad 2c_2 \hat{y}_x + 2c_2 \hat{y} + c_1 \geq 0 \quad ,$$

$$(183) \quad c_1 + 2c_2 \hat{y} > 0 \quad ,$$

then all the assumptions of the theorem ([4] , p.p. 208-210) , stated in the previous application (p.p.) are fulfilled.

Thus there exists a unique solution of the adjoint system (167''''') , (169''''') , and (170''''') .

To find this solution we proceed ([4] , p.p. 208-210) as follows:

Let \hat{Z} be the Banach space of all functions $z(x, s)$ which are continuous in $[0, 1] \otimes [0, T]$ with norm :

$$\| \hat{z} \| = \sup_{[0, 1] \otimes [0, T]} | \hat{z}(x, s) |$$

For any $R > 0$, define $Z_R = \{ \hat{z} \in \hat{Z} : || \hat{z} || \leq R \}$.

For every $\hat{z} \in \hat{Z}_R$, define the transformation $\hat{w} = T\hat{z}$, in such a way that $\hat{w}(x, s)$ is a solution of (167'''), (169'''), and also satisfies the relation:

$$(170''')^* \quad \frac{\partial \hat{w}(x, s)}{\partial x} = (c_1 + 2c_2 \hat{y}) \hat{z} .$$

The function $\hat{w}(x, s)$ is uniquely determined and has the form:

$$(140''') \quad \hat{w}(x, s) = \int_{\eta=0}^s \int_{\partial B} \Gamma(x, s, \xi, \eta) \rho(\xi, \eta) dB_{\xi} d\eta + G(x, s) ,$$

where:

$$B = [0, 1] ;$$

$\Gamma(x, s, \xi, \eta)$ is the fundamental solution of the equation :

$$(184) \quad D \frac{\partial^2 \hat{z}}{\partial x^2} - c \hat{z} - \frac{\partial \hat{z}}{\partial s} = 0 ;$$

and the functions $G(x, s)$, $\rho(x, s)$, are defined as follows :

$$(141''') \quad G(x, s) = \int_B \Gamma(x, s, \xi, 0) A(\xi) d\xi - \int_{\eta=0}^s \int_B \Gamma(x, s, \xi, \eta) \hat{\theta}_1(\xi, \eta) d\xi d\eta ,$$

and $\rho(x, s)$ is a solution of the integral equation:

$$(142''') \quad \rho(x, s) = 2 \int_{\eta=0}^s \int_{\partial B} \frac{\partial \Gamma(x, s, \xi, \eta)}{\partial x} \rho(\xi, \eta) dB_{\xi} d\eta + 2 \frac{\partial G(x, s)}{\partial x} - 2(c_1 + 2c_2 \hat{y}) \hat{z}(x, s) .$$

We now calculate $\hat{w}(x, s)$.

We find first the fundamental solution of (184).

Following the parametrix method ([4], p.4) we have: the

Inverse (matrix) of D is $\frac{1}{D}$.

Next, we set:

$$(185) \quad \theta(x, \xi) = \frac{1}{D} (x - \xi)^2 .$$

For $s > \eta$, we introduce the function:

$$(186) \quad \Omega(x, s, \xi, \eta) (s-\eta)^{-\frac{1}{2}} \exp\left(-\frac{\theta(x, \xi)}{4(s-\eta)}\right) = (s-\eta)^{-\frac{1}{2}} \exp\left(-\frac{(x-\xi)^2}{4D(s-\eta)}\right)$$

Further we have $C'(x, s) = \frac{1}{2\sqrt{\pi D}}$ and also

$$(187) \quad z_0(x, s, \xi, \eta) = C'(\xi, \eta)\Omega(x, s, \xi, \eta) = \frac{1}{2\sqrt{\pi D(s-\eta)}} \exp\left(-\frac{(x-\xi)^2}{4D(s-\eta)}\right)$$

The function $z_0(x, s, \xi, \eta)$ is a fundamental solution of the equation:

$$(188) \quad D \frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial s} = 0$$

A fundamental solution $\Gamma(x, s, \xi, \eta)$ of equation (184) can be found in the form:

$$(189) \quad \Gamma(x, s, \xi, \eta) = z_0(x, s, \xi, \eta) + \int \int z_0(x, s, \xi', \eta') \Phi(\xi', \eta', \xi, \eta) d\xi' d\eta'$$

where Φ is to be determined by the condition that Γ satisfies equation (184).

Further, after the calculation of $\Gamma(x, s, \xi, \eta)$, we can calculate $G(x, s)$ from the formula (141''').

Next, from (142''') we can find $\rho(x, s)$ functionally dependent on a_1, \hat{y}, \hat{z} .

Replacing then $\rho(x, s)$ in (140'''), we can find the function $\hat{w}(x, s)$, which satisfies (167'''), (169'''), and (170''')*. This function $\hat{w}(x, s)$ will be functionally dependent on a_1, \hat{y}, \hat{z} .

Next, it is known ([4], p.p. 208-210) that there exists a $\hat{z}^* \in \hat{Z}_R$ such that $\hat{z}^* = T \hat{z}^* = \hat{w}^*$; this function $\hat{z}^*(x, s)$ will satisfy (167'''), (169'''), and (170''') (which now coincides with (170''')*).

Thus this function $\hat{z}^*(x, s)$ will be the solution of the adjoint system and it will be found from the integral equation:

$$(140''') \quad z(x, s) = \int_{\eta=0}^s \int_{\partial B} \Gamma(x, s, \xi, \eta) \rho(\xi, \eta) dB_{\xi} d\eta + G(x, s) ,$$

where $\rho(\xi, \eta)$ is functionally dependent on a_1, \hat{y}, \hat{z} .

Thus, we can find $\hat{z}(x, s)$, and consequently $z(x, t)$, functionally dependent on a_1, y .

Next, we should solve the initial system (151), (152), (153), (157), to find the trajectory $y^*(x, t)$ functionally dependent on the control v^* .

Substituting then $y(x, t)$, we can find $z(x, t)$ functionally dependent on a_1, v^* .

The function $a_1(x)$ must be so chosen, that (31''') is satisfied.

Since in (31''') the function $y(x, t)$ is functionally dependent on v^* , we can find $a_1(x)$ as a functional of v^* .

Thus, we can have $z(x, t)$ as a functional of v^* .

Finally, substituting $z(x, t)$ in (173), we can find the expression of the optimal control $v^*(t)$.

The corresponding optimal trajectory $y^*(x, t)$, can be found by solving the initial system (151), (152), (153), (157), in which $v(t)$ has been replaced by the optimal control $v^*(t)$.

CONCLUSION

In chapters 1, 2, a survey of the Egorov's Maximum Principle for distributed parameter control systems, described by parabolic equations has been presented.

The necessary conditions for optimality of the examined system yield a system of equations from which the optimal control may be determined.

In chapter 3, Egorov's Maximum Principle is applied to a Thermodynamics system (heat transfer) and to a Biological system (Control of Sodium Transport in Toad Bladder).

In both applications, the necessary conditions for optimality have converted the optimal control problem to integrodifferential equations the solution of which may yield the optimal control.

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