On r-Quick Convergence in the Law of the Iterated Logarithm for a Class of Dependent Sequences

A Thesis submitted

by

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Contents

Abstract

In this thesis we study Law of the Iterated Logarithm type properties for stationary *-mixing sequences of real random variables $(X_i)_{i=1}^{\infty}$ under certain moment conditions. Essentially we prove that under the assumption that all moments of X_1 exist finitely then for any $r > 0$ we have an r-quick analogue to the functional form of the Law of the Iterated Logarithm. The treatment demonstrates the result directly from the asymptotic properties of the sequence. Several applications along the lines of those given by Strassen of the functional Law of the Iterated Logarithm are also presented.

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0. Introduction In 1964 Strassen published in [17] his well-known functional form of the Law of the Iterated Logarithm (abbreviated LIL) for independent identically distributed (iid) sequences of random variables. Since then there has been considerable interest in extensions and modifications to this LIL. Such extensions involved weakening the conditions placed upon the sequence of random variables, i.e. consideration of weakly dependent sequences, non-stationary sequences or sequences taking values in a space other than the real line. A summary of results of this type may be found in Stout [16, Chapter 5]. All these results share the usual interpretation of convergence and limits of functions.

A new avenue of exploration was opened when, in [18], Strassen introduced the concept of r-quick convergence, a form of convergence which is heavily dependent on the rate at which sequences converge. In the same paper he proved an r-quick analogue to the classical LIL and hypothesised an r-quick analogue for his functional form of the LIL. T.L. Lai in his 1976 paper [13] was able to prove the r-quick functional LIL in the case of iid real random variables. The method of his proof is similar to Strassen's proof of his original functional LIL for iid sequences. The intent of this paper is to extend this latest result to a class of weakly dependent stationary real sequences satisfying certain moment conditions. The proof of this result as in [6], depends heavily on a recent theorem of Ghosh and Babu [8, page 223].

Section 1 of this paper will state certain definitions and concepts used throughout the text. In particular, the concepts of weakly dependent sequences and r-quick convergence are reviewed together with certain related lemmas. Section 2 introduces the main theorem of this paper with appropriate notation. The various definitions and lemmas necessary to the proof of the main theorem are presented in Section 3. The actual proof of the main theorem divides naturally into two parts. Section 4 comprises of the proof of the inclusion portion of the theorem and Section 5 the balance of the proof.

In [17] Strassen gave some applications of his functional LIL and in [13] Lai showed how these applications extended to the r-quick case. Section 6 will affirm the validity of similar applications of our main theorem. Section 7 outlines recent results obtained by Lai [14] in the area of an r-quick Strong Law of Large Numbers. Section 8 consists of some concluding remarks.

1. Definitions Throughout this paper a general knowledge of measure theoretic probability theory is assumed, similar

to that found in Ash [1], or Chung [5]. Let $X = (X_1, X_2, X_3 \ldots)$ be a sequence of real-valued random variables (denoted rrv's) defined on a probability space $(\Omega, \mathfrak{F}, P)$. That is, for each $i = 1,2,3,\ldots$ $X_i : \Omega \to \mathbb{R}$ is measurable with respect to the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$. Let $\mathbb{R}^\infty$ denote the infinite Cartesian product of the real line $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots$ and $\mathcal{B}^\infty$ denote the σ-algebra on $\mathbb{R}^\infty$ generated by all measurable rectangles (all sets of the form $B_1 \times B_2 \times B_3 \times \ldots \times B_n \times \mathbb{R} \times \mathbb{R} \times \ldots$ where $n \in \mathbb{N}$ and $B_i \in \mathcal{B}$ for each $i = 1,2,3\ldots,n$).

Definition. The above sequence of rrv's $X_1, X_2, X_3 \ldots$ is said to be stationary if for any $B \in \mathcal{B}^\infty$ and $j \in \mathbb{N}$

$$P\{\omega \in \Omega : (X_1, X_2, X_3 \ldots) \in B\} = P\{\omega \in \Omega : (X_j, X_{j+1}, X_{j+2}, \ldots) \in B\}$$

Hereafter the set $\{\omega \in \Omega : Y \in B\}$ will be denoted by $[Y \in B]$ for the sake of clarity.

The properties defining a certain class of weakly dependent stochastic sequences are frequently described by a mixing condition which imposes a certain asymptotic independence of the members of the sequence that are sufficiently distant from each other. Three mixing conditions are particularily well known and are listed below.

Definition: (Kesten and O'Brien [10]) let $(X_n : n \in \mathbb{Z})$ or $(X_n : n \in \mathbb{N})$ be a stochastic sequence on a probability space $(\Omega, \mathfrak{F}, P)$. Let $\mathfrak{F}_i^j$ denote the σ-algebra on Ω generated by $(X_i, X_{i+1}, \ldots X_{j-1}, X_j)$ where i may be $-\infty$

and j,∞ . Define the following for $k = 1,2,3...$

(1) $\alpha(k) = \sup(|P(BC) - P(B)P(C)|)$

(2) $\lambda(k) = \sup(|P(C|B) - P(C)|)$

(3) $\rho(k) = \sup(|P(BC)/(P(B)P(C)) - 1|)$.

where in each case the supremum is taken over all $m\in\mathbb{Z}$ (respectively $m\in\{0,1,2,...\}$ and all $B\in\mathcal{F}^{m}_{-\infty}$ $(\mathcal{F}^{m}_{0})$ and $C\in\mathcal{F}^{\infty}_{m+k}$ $(\mathcal{F}^{\infty}_{m+k})$ with $P(B)P(C) > 0$. Then $(X_n : n\in\mathbb{Z}$ or $n\in\mathbb{N})$ is called strong mixing (ϕ-mixing, *-mixing) if $\alpha(k)$ ($\lambda(k)$ or $\rho(k)$ respectively) decreases to 0 as $k\to\infty$. Since $\alpha(k) \le \lambda(k) \le \rho(k)$, it is clear that *-mixing implies ϕ-mixing which in turn implies strong mixing.

Strong mixing and ϕ-mixing are well-known mixing conditions and have been used extensively. A general description of these conditions may be found in [16]. The concept of *-mixing was explored by Blum, Hanson and Koopmans in [3] where it was defined in a slightly weaker fashion than given below. This definition is useful in the sequel.

Definition: Let $(X_n : n\in\mathbb{N})$ be a sequence of rrv's on a probability space $(\Omega, \mathcal{F}, P)$. Define $\mathcal{F}^{j}_{i}$ as before. Then $(X_n : n\in\mathbb{N})$ will be called *-mixing if there exists a positive integer N and a real-valued function $\rho(n)$ defined on the integers $n \ge N$

such that:

(1) ρ is non-increasing with $\lim_{n\to\infty} \rho(n) = 0$ and

(2) if $n \geq N$, $A \in \mathcal{F}_1^m$, $B \in \mathcal{F}_{m+n}^{\infty}$ then $|P(AB) - P(A)P(B)| \leq \rho(n)P(A)P(B)$

In [3], Blum, Hanson and Koopmans give several examples of *mixing sequences. Kesten and O'Brien show in [10] that it is possible to construct *-mixing sequences with $\rho(n)$ going to 0 at any prescribed rate. They also have shown that *-mixing sequences form a proper subset of ϕ-mixing sequences and likewise for ϕ- and strong mixing sequences.

The notion of r-quick limit points and r-quick convergence was first introduced by Strassen in [18], where he proved an r-quick analogue to the classical LIL. Strassen noted the following property of the limsup of a sequence of rrv's θ_n : let $c \in \mathbb{R}$. If the supremum over a null set is taken to be 0, define the random variable T_c by $T_c = \sup(n \in \mathbb{N}: \theta_n > c)$, i.e. the last time θ_n exceeds c. Then the statement "$\limsup_{n\to\infty} \theta_n = x$ almost surely" for some $x \in \mathbb{R}$ can be expressed in terms of T_c in the following manner:

2.1 (a) $T_c < \infty$ a.s. if $c > x$

2.1 (b) $T_c = \infty$ a.s. if $c < x$

This prompted Strassen to make the following definition of an r-quick limsup.

Definition: $\limsup_{n\to\infty} \theta_n = y$ (r-quickly) where $y \in \mathbb{R}$ and $r > 0$ if

2.2 (a) $E(T_c)^r < \infty$ if $c > y$

2.2 (b) $E(T_c)^r = \infty$ if $c < y$

It should be noted that $\limsup_{n\to\infty} \theta_n = y$ (r-quickly) if and only if $y = \sup\ (c \in \mathbb{R} : E(T_c)^r = \infty)$.

In [13] , Lai adds the following.

Definition:

$\limsup_{n\to\infty} \theta_n \le y$ (r-quickly) if 2.2 (a) holds for all $c \in \mathbb{R}$

$\limsup_{n\to\infty} \theta_n \ge y$ (r-quickly) if 2.2 (b) holds for all $c \in \mathbb{R}$

$\limsup_{n\to\infty} \theta_n < \infty$ (r-quickly) if there exists a real constant c for which $E(T_c)^r < \infty$. Otherwise $\limsup_{n\to\infty} \theta_n = \infty$.

Similarily if $E(T_c)^r < \infty$ for all real constants c , then $\limsup_{n\to\infty} \theta_n = -\infty$.

Strassen also defines r-quick limit points in $C[0,1]$ and r-quickly relatively compact sets in $C[0,1]$ (cf. [18] page 319). Stated below are Lai's ([13] page 613) formulation of these concepts.

Definition: For rrv's $(\theta_n : n=1,2,3...)$, θ define

$\theta_n \to 0$ (r-quickly) if $\limsup|\theta_n| = 0$ (r-quickly)

$\theta_n \to \theta$ (r-quickly) if $(\theta_n - \theta) \to 0$ (r-quickly)

Recall: A set S is relatively compact in a metric space M iff the closure of S in M is a compact set of M iff every sequence in S has a convergent subsequence (in M)

Definition: (Lai) Let M be a metric space endowed with its σ-algebra of Borel sets. Let $(\theta_n : n \geq 1)$ be a sequence of random variables taking values in M. Then $(\theta_n : n \geq 1)$ **is said to be** r-quickly relatively norm compact in M if for every $\varepsilon > 0$, there is a finite union $\mathcal{U}$ of ε-spheres in M such that

$$E(\sup(n \in N : \theta_n \notin \mathcal{U}))^r < \infty \quad .$$

An element x of M is called an r-quick limit point of $(\theta_n : n \geq 1)$ in M if for any open neighbourhood V of x, $E(\sup(n \in \mathbb{N} : \theta_n \in V))^r = \infty$.

The following lemmas from Lai [13, page 622] indicate some properties of r-quick limit sets which will be used in Section 6 .

Lemma A: Let (M,d) be a metric space and let $(\theta_n : n \geq 1)$ be a sequence of random variables taking values in M. Then the set K of r-quick limit points of $(\theta_n : n \geq 1)$ is a closed subset of M. Consequently if M is complete and (θ_n) is r-quickly relatively compact in M, then K is a compact subset of M.

Lemma B: Let $(M_1,d_1),(M_2,d_2)$ be metric spaces. Let $\phi : M_1 \to M_2$ be a continuous function. Suppose $(\theta_n : n \geq 1)$ is a sequence of random variables taking values in M_1. Let $L(\theta_n)$ (respectively $L(\phi(\theta_n))$) denote the set of r-quick limit points of $(\theta_n : n \geq 1)$ (respectively $(\phi(\theta_n) : n \geq 1)$. Then $\phi(L(\theta_n)) \subset L(\phi(\theta_n))$. Assume furthermore that the following condition holds: $L(\theta_n)$ is compact and for every $\varepsilon > 0$, $E(\sup(n \in N : \theta_n \notin L_\varepsilon(\theta_n)))^r < \infty$ where $L_\varepsilon(\theta_n)$ denotes the open ε-neighbourhood of $L(\theta_n)$ in M . Then the sequence $(\theta_n : n \geq 1)$ (respectively $(\phi(\theta_n) : n \geq 1)$) is r-quickly relatively compact in M_1 (respectively M_2) and $\phi(L(\theta_n)) = L(\phi(\theta_n))$.

Lemma C: Let $\mathbb{R}$ denote the real line and $(\theta_n : n \geq 1)$ a sequence of rrv's such that $(\theta_n : n \geq 1)$ is r-quickly relatively compact in $\mathbb{R}$. Let $\mathcal{K}$ be the set of all r-quick limit points of the above sequence and let $y = \sup(x : x \in \mathcal{K})$. Then y is finite and $\limsup_{n\to\infty} \theta_n = y$ (r-quickly).

<u>2. Main Theorem</u> Consider a stationary sequence of rrv's $(X_i : i = 1,2,3,\ldots)$ on a probability space $(\Omega, \mathfrak{F}, P)$. Assume that $E X_1 = 0$ and $E X_1^2 < \infty$. Let $S_n = \Sigma_{i=1}^n X_i$ for $n \geq 1$ and define $S_0 = 0$. Let $\| \ \|_c$ denote the supremum norm on the space of all continuous functions $f: [0,1] \to \mathbb{R}$, i.e. $\| f \|_c = \sup(|f(r)| : 0 \leq r \leq 1)$ for $f \in C[0,1]$. Denote by K_s the set of all functions $x \in C[0,1]$ such that x is absolutely continuous , $x(0) = 0$ and $\int_0^1 (dx/dr)^2 dr \leq s^2$. The derivative dx/dr exists almost everywhere with respect to the Lebesgue measure since x is assumed to be absolutely continuous. Define the Hilbert norm for $x \in K_s$ by

$\| x \|_H = \int_0^1 (dx/dr)^2 dr$. If $s = 1$ in the above the set K_s is denoted K_o and is called the Strassen set. It is a well-known fact that K_o is a compact set of $C[0,1]$ (cf. Stout [16], page 282).

Define a family of functions $(S(r), r \geq 0)$ by linearly interpolating $(S_n, n \in \mathbb{N})$. That is, for $r \geq 0$, $r \in \mathbb{R}$ define $S(r)$ by $S(r) = S_{[r]} + (r-[r])(S_{[r]+1} - S_{[r]})$[1] . With all the terms defined as above, it is now possible to proceed with the discussion leading to the main theorem of this paper. In [17] Strassen proved the following functional LIL.

Theorem: (Strassen) Assume the sequence of rrv's $(X_i : i=1,2,3...)$ is iid and $EX_1^2=1$. Let $U_n(r)=(2n\log\log)^{-\frac{1}{2}}S(nr)$ for $0 \leq r \leq 1$ define the sequence $(U_n : n \geq 3)$ of $C[0,1]$ functions. Then with probability 1 $(U_n : n \geq 3)$ is **relatively compact (with respect to $\| \|_c$) and its set of limit points coincides with K_o.**

The proof of the above theorem depends on an "embedding" or "identification" of $(S(r): r \geq 0)$ with a Brownian motion in a particular fashion. In essence the result is first shown for a Brownian motion and the above identification used to extend the result to the iid case. In other settings or under other conditions on $(X_i : i \in N)$ such an embedding ("Invariance") principle may be rather

[1. [ ] denotes the greatest integer function,] difficult to prove. A more elementary proof of the functional LIL in the iid case was given by Chover in [4] . His technique dispenses with the need for looking at Brownian motion and Invariance principles and proves the result directly from the asymptotic properties of the sequence.

The first step in Chover's proof is to show almost sure equicontinuity (uniform in $0 \le r \le 1$) of the family of functions $(U_n(r): n \ge 3)$. This is accomplished using two tools in particular; a technique of looking at convergence along a geometric subsequence (see Feller [7], page 193) and an inequality on the maximum value of partial sums $(S_i i=1,2,3...)$. This equicontinuity result leads immediately to the relative compactness of $(U_n(r): n \ge 3)$. The fact that the set of limit points is K_o follows from the equicontinuity result and consideration of asymptotic distributions (as $n \to \infty$) .

In [18], Strassen showed the following theorem on the r-quick limsup of sample sums.

Theorem: (Strassen) Let $X_1, X_2, X_3 ....$ be a sequence of iid rrv's such that $EX_1 = 0$ and $EX_1^2 = 1$. Let $r > 0$, $\rho > 2(r+1)$ and $E|X_1|^\rho < \infty$. Define S_n for $n \in N$ as before. Then $\limsup_{n\to\infty} (2n\log n)^{-1/2} S_n = r^{1/2}$ (r-quickly).

The similarity of this result to the classical LIL led Strassen to conjecture an r-quick analogue to the functional LIL.

Strassen's Conjecture: With the same assumptions and notation as found in the above theorem, define $\xi_n(t)$ for $0 \le t \le 1$, $n \ge 2$ by: $\xi_n(t) = (2n\log n)^{-1/2} S_n$ at $t = 1/n$, $i = 0,1,2\ldots n$ and $\xi_n(t)$ is linear on $[(i-1)/n, i/n]$, i.e. for $(i-1)/n < t < i/n$ $\xi_n(t) = \xi_n((i-1)/n) + (t-(i-1))(\xi_n(i/n) - \xi_n(\frac{i-1}{n}))$ $i = 0,1,2\ldots n$. Then the sequence $(\xi_n: n \ge 2)$ is r-quickly relatively compact in $C[0,1]$ and the set of its r-quick limit points in $C[0,1]$ is $r^{1/2}K_0$.

Lai, in [13], is able to prove this conjecture under a weaker moment condition than was hypothesised by Strassen. He also succeeds in demonstrating that his condition is the best possible (Theorem 3 of [13]).

Theorem: (Lai) Let $r > 0$ and let $X_1, X_2, X_3 \ldots$ be iid rrv's such that $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2(r+1)} \times (\log^+|X_1|+1)^{-(r+1)} < \infty$. Define ξ_n as above. Then for every $\varepsilon > 0$, letting $\mathcal{U}$ denote the open ε-neighbourhood of $r^{1/2}K_0$, $E(\sup(n\in N: \xi_n \notin \mathcal{U}))^r < \infty$. Thus the sequence $(\xi_n: n \ge 2)$ is r-quickly relatively compact in $C[0,1]$. The set of its r-quick limit points in $C[0,1]$ is $r^{1/2}K_0$.

Lai's proof of the above theorem follows a pattern similar to Strassen's proof of the functional LIL for iid sequences. The demonstration consists of three steps; first the result is shown for real-valued Gaussian sequences (Theorem 4 in [13]), second an r-quick invariance principle is shown for iid sequences (Theorem 5 in [13]) which yields that the partial sums of the sequence behave like Brownian motion; and third, combining the two results to yield the desired theorem. The proof of the r-quick result for Gaussian sequences follows the lines of Chover's proof of the functional LIL ([4], Theorem 1). Lai shows that his Lemma 1 holds, which states a result similar to equicontinuity but imparts additional, and vital, information on the rate at which ξ_n and its approximation approach each other as $n \to \infty$. Using this lemma and the technique of looking along geometric subsequences, Lai proves his Theorem 4 in a fashion roughly similar to the proof of Chover mentioned previously.

The proof of the following LIL for *-mixing sequences is based on the proof of Theorem 4 of Lai [13] and the main Theorem in Deo [6] and thus follows a pattern similar to the proof of the iid LIL given by Chover. The role of the equicontinuity lemma is played by a result similar to Lemma 1 of Lai [13], but adapted to *-mixing sequences and the linear approximations of functions found in Chover [5]. The necessary maximal inequality used is taken from Gikhman and Skorohod [9]. The asymptotic behaviour of $P[|S_n| > a]$ is described using a recent theorem of Ghosh and Babu [8]. This asymptotic result is crucial to the proof of this paper's main theorem, which is presented below.

Definition: For a stationary sequence off rrv's $(X_n: n = 1,2,3\ldots)$ such that $EX_1 = 0$, $EX_1^2 < \infty$ define σ by: $\sigma^2 = EX_1^2 + 2\Sigma_{j=1}^{\infty}\mathrm{Cov}(X_1, X_{1+j})$. For the rest of this paper, we shall assume $\sigma^2 > 0$.

Definition: For a stationary sequence of rrv's as above $(X_n: n = 1,2,3\ldots)$ define $(S_n: n \geq 0)$ and $(\xi_n: n \geq 2)$ as follows: Let $S_n = \Sigma_{i=1}^{n} X_i$ and $S_o = 0$; for $t \in [0,1]$

$\xi_n(t) = (2n\sigma^2 \log n \quad)^{-\frac{1}{2}} S_i$ for $t = i/n$, $i = 0,1,2\ldots n$

$= \xi_n((i-1)/n) + (t-i+1)(\xi_n(i/n) - \xi_n((i-1)/n))$ $(i-1)/n < t < i/n$

$i = 0,1,2\ldots n$

Theorem A: Let $(X_n: n = 1,2,3\ldots)$ be a stationary sequence of rrv's such that $EX_1 = 0$, $EX_1^2 < \infty$ and all the moments of X_1 exist finitely. Assume the sequence is *-mixing with coefficient $\rho(\cdot)$ such that $\Sigma_{n=1}^{\infty}\rho^{1/2}(n) < \infty$. Let $(\xi_n: n \geq 2)$, $(S_n: n \geq 0)$ and σ^2 be as above. Let $r > 0$. Then the sequence $(\xi_n: n \geq 2)$ is r-quickly relatively compact in $C[0,1]$ and the set of its r-quick limit points is $r^{1/2}K_\sigma$.

It is important to note that this result is not derivable from a combination of the previous r-quick analogue to the LIL and one of the many invariance principles for weakly dependent stationary sequences (cf. Philipp and Stout [15]). The best Philipp and Stout achieved under the conditions of Theorem A is the definition of a Brownian motion $Y(t)$ on a richer probability sapce together with $S(n)$ such that

$|S(n) - Y(n)| = O(n^{-5/12})$ almost surely. To derive Theorem A from the corresponding result for Brownian motion, it would have to be established that for any $\varepsilon > 0$:

$E(\sup(n\ \{2,3,4\ldots\}: |S(n) - Y(n)|/(2n\sigma^2 \log n\ )^{1/2} > \varepsilon))^r < \infty$.

Although the Invariance principle yields for almost every $\omega \in \Omega$, $(S(n)-Y(n))/(2n\sigma^2 \log n\ )^{1/2}(\omega) \to 0$, it does not yield a bound on the value of the supremum over n. Thus it does not lead to the finiteness of the expectation of that supremum. In [13], Lai proves an Invariance principle of the appropriate type for iid sequences, but clearly this cannot be applied to a general *-mixing sequence.

3. Some Lemmas

For a sequence $\{X_i : i = 1,2\ldots\}$ as found in Theorem A define σ, $S(t)$ and ξ_n as given for Theorem A under the additional assumption that $0 < \sigma = 1$. Note that for $t \in [0,1]$, $\xi_n(t) = S(nt)/(2n\log n)^{1/2}$, $n = 2,3,4\ldots$.

Definition: Define σ as above and let $\sigma_n^2 = E(\Sigma_{i=1}^n X_i)^2$. In Stout [16, page 310], the following lemma is proved for σ_n and σ.

Lemma: (Stout) As $n \to \infty$ since $\Sigma_{n=1}^{\infty} \rho^{1/2}(n) < \infty$ $\sigma_n^2 = n\sigma^2(1+o(1))$ and $0 \le \sigma^2 < \infty$ □

The following Theorem due to Ghosh and Babu [8] is a remarkably precise statement about the probabilities of moderate deviations of partial sums.

Theorem (Ghosh and Babu) Suppose the stationary process $\{X_n : n \geq 1\}$ is ϕ-mixing. Then for any $c > 0$,

$$P[n^{-1}S_n - \mu > c\sigma(\log n/n)^{1/2}] \sim (2\pi c^2 \log n)^{-\frac{1}{2}} \quad n^{-\frac{1}{2}c^2} \sim 1 - \Phi(c(\log n)^{\frac{1}{2}})$$

and

$$P[|n^{-1}S_n - \mu| > c\sigma(\log n/n)^{1/2}] \sim 2(2\pi c^2 \log n)^{-\frac{1}{2}} \quad n^{-\frac{1}{2}c^2} \sim 2[1 - \Phi(c(\log n)^{\frac{1}{2}})]$$

hold under the conditions

$E|X_1|^{c^2+2+\delta} < \infty$ for some $\delta > 0$

$\Sigma_{n=1}^{\infty} \phi^{1/2}(n) < \infty$ and

$0 \neq \sigma^2 = V(X_1) + 2\Sigma_{j=1}^{\infty} \mathrm{Cov}(X_1, X_{1+j})$

where $\mu = E(X_1)$ □

The following Lemma is adapted from Theorem 2 of Gikhman and Skorohod [9, pg 120] along the lines of Lemma 6 of Deo [6a].

Lemma 1 Let $\{X_i : i = 1,2,3...\}$ be a stationary *-mixing sequence of real-valued random variables with mixing coefficient $\rho(.)$. If $\rho(1) < 1$ and if for some $a \geq 0$ and all $k = 1,2,3...n$

$P[|S_k| \leq a] > \frac{1}{2}$, then

$P[\max\{|S_k| : k = 1,2..n\} > 2a] \leq 2(1-\rho(1))^{-1} P[|S_n| > a]$.

Proof: Define the events A_k, B_k for $k = 1,2...n$ as follows: $A_k = [|S_1| \leq 2a,..,|S_{k-1}| \leq 2a, |S_k| > 2a]$, $B_k = [|S_n - S_k| \leq a]$. Note A_k , $k = 1,2,3..n$ are disjoint events. Hence:

$P[|S_n| > a] \geq P\{\cup_{k=1}^{n} (A_k \cap B_k)\}$

$= \Sigma_{k=1}^{n} P\{A_k \cap B_k\}$ since the A_k are disjoint .

$\geq (1 - \rho(1))\Sigma_{k=1}^{n} P(A_k)P(B_k)$ by *-mixing condition

$\geq \frac{1}{2}(1 - \rho(1))\Sigma_{k=1}^{n} P(A_k)$ by stationarity and the hypothesized condition.

$= \frac{1}{2}(1 - \rho(1))P[\max\{|S_k| : K = 1,2,\ldots n\} > 2a]$ and this clearly leads to the desired inequality. □

Note: In [6a] Deo shows a similar result to the above for ϕ-mixing processes.

The following lemma plays the same role in the proof of theorem A as does the equicontinuity lemma in Chover [4] or Lai's Lemma 1 in [13]. For $f \in C[0,1]$, $m \in N$, define $\pi_m f(t)$ as given below. These functions behave in a similar fashion as the orthogonal representations used by Lai in [13].

$$\pi_m f(t) = f(i/m) \quad \text{if} \quad t = i/m \quad \text{for some } i \in \{0,1,2\ldots m\}$$
$$= f(i/m) + (t-(i/m)(f((i+1)/m)-f(i/m))$$
$$\text{if } i/m < t < (i+1)/m \quad \text{for some } i \in \{0,12\ldots m-1\}$$

Lemma 2: Let $\{X_i, i = 1,2,3\ldots\}$ be a stationary *-mixing sequence of rrv's (with mixing coefficient $\rho(\cdot)$) such that $\Sigma_{i=1}^{\infty}\rho^{\frac{1}{2}}(i) < \infty$. Define S_n, $S(t)$ and σ^2 as previously. Let $a(n) = (2n\sigma^2 \log n)^{\frac{1}{2}}$ $n = 2,3,4\ldots$ and define $\xi_n(t) = S(nt)\cdot a^{-1}(n)$ for $t \in [0,1]$, $n = 2,3,4\ldots$. Then for any given $\varepsilon > 0$, $\gamma > 0$, there exists an $m_o \in N$ such that for all $m \geq m_o$: $P[\| \xi_n - \pi_m \xi_n \|_c > \varepsilon] = o(n^{-\gamma})$.

Proof:

By Stout's Lemma (pg. 16) $\Sigma_{i=1}^{\infty}\rho^{\frac{1}{2}}(i) < \infty$ implies $\sigma < \infty$. Assume $\sigma > 0$, the case $\sigma = 0$ being degenerate, and for the sake of simplicity assume further that $\sigma = 1$.

$$P[\| \xi_n - \pi_m \xi_n \|_c > \varepsilon]$$

$$= P[\max\{\sup\{|\xi_n(t) - \pi_m \xi_n(t)| : t \in](i-1)/m, i/m]\} : i=1,2\ldots m\} > \varepsilon]$$

$$= P[\max\{\sup\{|\xi_n(t) - m((t-(i-1)/m)\xi_n((i-1)/m) + (i/m - t)\xi_n(i/m))| : t \in](i-1)/m, i/m]\} : i=1,2,3\ldots m\} > \varepsilon]$$

$$\leq P[\max\{\sup\{|m(t-(i-1)/m)(\xi_n(t) - \xi_n((i-1)/m))| : t \in](i-1)/m, i/m]\} : i=1,2,3\ldots m\} > \varepsilon/2]$$

$$+ P[\max\{\sup\{|m((i/m)-t)(\xi_n(t) - \xi_n(i/m))| : t \in](i-1)/m, i/m]\} : i=1,2,3\ldots m\} > \varepsilon/2]$$

$$= Q_1 + Q_2$$

$$Q_1 \leq P[\max\{\sup\{|\xi_n(t) - \xi_n((i-1)/m)| : (i-1)/m < t \leq i/m\} : i=1,2,3,\ldots,n\} > \varepsilon/2] \quad \text{since } |t-(i-1)/m| \leq 1/m$$

$$= P[\max\{\sup\{|S(nt) - S(n(i-1)/m)| : (i-1)/m < t \leq i/m\} : i=1,2,3\ldots n\} > \tfrac{1}{2}\varepsilon . a(n)] \tag{3.1}$$

Note that if $\alpha \in [0,1]$ and $s=\alpha k + (1-\alpha)(k+1)$ for some non-negative integer k, then $S(s) = (1-\alpha)S_k + \alpha S_{k+1}$ and

$$|S(s) - S(n(i-1)/m)| \leq |(1-\alpha)S_k - (1-\alpha)S(n(i-1)/m)| + |\alpha S_{k+1} - \alpha S(n(i-1)/m)|$$

$$\leq |S_k - S(n(i-1)/m)| + |S_{k+1} - S(n(i-1)/m)|$$

Thus

$$\max\{\sup\{|S(nt) - S(n(i-1)/m| : (i-1)/m < t \leq i/m\} : 1 = 1,2,3\ldots m\}$$

$$\leq \max\{\max\{|S_j - S(n(i-1)/m)| : n(i-1)/m < j \leq ni/m \; j \in Z\} : i=1,2,3\ldots m\}$$

$$+ \max\{\max\{|S_{j+1} - S(n(i-1)/m)| : n(i-1)/m < j \leq ni/m \; j \in Z\} : i=1,2,3\ldots m\}$$

$\leq 2 \max\{\max\{|S_j - S(n(i-1)/m)| : n(i-1)/m < j \leq (ni/m)+1\ j \in Z\}$

$: i=1,2,3...m\}$

$\leq 2 \max\{|S_j - S(k)| : 0 \leq k \leq j \leq n;\ 0 \leq j - k \leq 1+n/m;\ j \in Z\}$

by increasing the domain of the maximum.

$\leq 4 \max\{|S_j - S_k| : 0 \leq k \leq j \leq n;\ 0 \leq j-k \leq 2+n/m;\ j,k \in Z\}$

by a similar argument as above applied to $S(k)$.

Applying the above result to (3.1) yields

$P[\max\{\sup\{|S(nt) - S(n(i-1)/m)| : (i-1)/m < t < i/m\}$

$: i=1,2,3,..m\} > \frac{1}{2}\varepsilon . a(n)]$

$\leq P[\max\{|S_i - S_j| : 0 \leq j \leq i \leq n;\ 0 \leq i-j \leq 2+n/m;\ i,j \in Z\} > a(n)\varepsilon/8]$

$= P[\max\{\max\{|S_i - S_j| : 0 \leq j \leq i \leq n;\ i-j \leq 2+n/m;\ i \in Z\} : 0 \leq j \leq n;\ j \in Z\}$

$> a(n).\varepsilon/8]$

$\leq P\{U_{j=0}^{n}[\max\{|S_i - S_j| : 0 \leq j \leq i \leq n;\ i-j \leq 2+n/m;\ i,j \in Z\}$

$> a(n).\varepsilon/8]\}$

$\leq \Sigma_{j=o}^{n} P[\max\{|S_{i-j}| : 0 \leq j \leq i \leq n;\ i-j \leq 2+n/m;\ i,j \in Z\}$

$> a(n).\varepsilon/8]$

by stationarity since j is fixed within each set

$\leq \Sigma_{j=o}^{n} P[\max\{|S_k| : 0 \leq k \leq 2+n/m, k \in Z\} > a(n).\varepsilon/8]$

since for $j > n-2-n/m$

$P[\max\{|S_{i-j}| : 0 \leq j \leq i \leq n;\ i-j \leq 2+n/m\} > a]$

$\leq P[\max\{|S_k| : 0 \leq k \leq 2+n/m\} > a]$

$\leq (n+1)P[\max\{|S_k| : 0 \leq k \leq 2+n/m,\ k \in Z\} > a(n).\varepsilon/8]$

So that Lemma 1 may be applied at this point, assume temporarily that $\rho(1) < 1$.

Lemma 1 requires for $k = 1,2,\ldots,[n/m] + 2$ and n sufficiently large that $P[|S_k| \le a(n)\cdot\varepsilon/8] > \frac{1}{2}$. Since $V(S_n) \sim n\sigma^2$ (as stated in Lemma (Stout)), it is easy to see by Chebyshev's inequality that the desired inequalities hold for n sufficiently large, say $n \ge N$. Thus Lemma 1 now yields for $n \ge N$ (under the assumption that $\rho(1) < 1$) that

$(n+1)P[\max\{|S_k|:0\le k\le 2+[n/m],\ k\in Z\} > a(n)\cdot\varepsilon/8]$

$\le 2(n+1):(1-\rho(1))^{-1}P[|S_{2+[n/m]}| > a(n)\cdot\varepsilon/16]$

$= o(n^{-\gamma})$ by applying the theorem of Ghosh and Babu [8] and choosing m large enough. Thus we have shown that $Q_1=o(n^{-\gamma})$. A similar argument shows that $Q_2=o(n^{-\gamma})$. Hence for m large enough $P[\|\xi_n-\pi_m\xi_n\|_c > \varepsilon] = o(n^{-\gamma})$ under the assumption $\rho(1) < 1$. We remove this assumption in the following manner:

Since $\rho(j) \to 0$ as $j \to \infty$, there exists $j_o \ge 1$ such that $\rho(j_o) < 1$. For each $0\le\ell\le j_o-1$ consider the sequence $\{X_{jj_o+\ell}:j = 0,1,2\ldots\}$. Each such sequence is stationary and $*$-mixing as a subsequence of $\{X_i:i = 1,2,3\ldots\}$ and $\rho^{(\ell)}(1) < 1$ by the choice of j_o ($\rho^{(\ell)}$ is the mixing function for $\{X_{jj_o+\ell}:j = 0,1,2\ldots\}$). Thus the desired result holds for the partial sums of these subsequences. It is possible to express S_n as the sum of j_o partial sums of the subsequence $\{X_{jj_o+\ell}:j = 0,1,2..\}$. Applying the result for $\rho(1) < 1$ to each of these partial sums yields the desired conclusion about S_n.

□

Lemma 3 Let $\{X_n : n \geq 1\}$ be a *-mixing stationary sequence with mixing coefficient $\rho(\cdot)$. Let $1 \leq a_1 < b_1 < a_2 < b_2 < \ldots$ be integers. If the event A_n is measurable with respect to $\sigma(X_j : a_n \leq j \leq b_n)$, then for each m

$$[1-\max\{\rho(a_{j+1}-b_j):1\leq j\leq m\}]^m \Pi_{j=1}^m P(A_j)$$
$$\leq P(\cap_{j=1}^m A_j) \leq [1-\max\{\rho(a_{j+1}-b_j):1\leq j\leq m]^m \Pi_{j=1}^m P(A_j)$$ and if η_n is measurable with respect to $\sigma(X_j : a_n \leq j \leq b_n)$, then for real x

$$[1-\max\{\rho(a_{j+1}-b_j):1\leq j\leq m\}]^m P[\Sigma_{j=1}^m \eta_j' \leq x]$$
$$\leq P[\Sigma_{j=1}^m \eta_j \leq n] \leq [1+\max\{\rho(a_{j+1}-b_j):1\leq j\leq m]^m P[\Sigma_{j=1}^m \eta_j' \leq x]$$

where $\eta_1', \eta_2', \eta_3' \ldots$ are independent random variables with η_i' having the same distribution as η_i for each i.

Proof: Clearly the second assertion follows from the first. The proof of the second assertion follows from a simple fact about *-mixing sequences,ie that if A_1 and A_2 are as above

$$[1-\rho(a_2-b_1)]P(A_1)P(A_2) \leq P(A_1 \cap A_2)$$
$$\leq [1+\rho(a_2-b_1)]P(A_1)P(A_2)$$

□

Lemma 4: Let $\{X_{n,i} : n \geq 1\}$, $1 \leq i \leq m$ be m independent copies of a *-mixing stationary sequence $\{X_n : n \geq 1\}$. Let $\theta_1, \theta_2, \ldots, \theta_m$ be real numbers and $Y_n = \Sigma_{i=1}^m \theta_i X_{n,i}$. Then $\{Y_n : n \geq 1\}$ is also a stationary *-mixing sequence. If the mixing coefficient ρ for $\{X_n : n \geq 1\}$ satisfies $\Sigma_{n=1}^\infty \rho^{\frac{1}{2}}(n) < \infty$ then so does the mixing coefficient of

$\{Y_n : n \geq 1\}$. If $\theta_1^2 + \theta_1^2 + \ldots + \theta_m^2 = 1$ then σ^2 (as defined previously) for $\{Y_n : n \geq 1\}$ is the same as that for $\{X_n : n \geq 1\}$.

Proof: This is Lemma 2 in Deo [6]. The proof uses standard approximation arguments by cylinder sets etc. □

As pointed out in [6] this lemma does not appear to be true for ϕ-mixing sequences and this is perhaps the main reason why we limit ourselves to the *-mixing sequences.

Lemma 5 Let $(X_{n,j} : 1 \leq n \leq \infty)$ $1 \leq j \leq k$ be k independent copies of a *-mixing stationary sequence $(Y_n : 1 \leq n < \infty)$ such that $EY_1 = 0$, σ^2 for each of the k copies is the same as σ^2 for $(Y_n : 1 \leq n < \infty)$; take $\sigma^2 = 1$ for simplicity. Then for each $c > 0$

$$\log P[\Sigma_{j=1}^{k} (\Sigma_{i=1}^{n} X_{i,j})^2 > c^2 n \log n] \sim -\tfrac{1}{2} c^2 \log n$$

as sequences in n tending to ∞.

Proof Take $c = 1$ for simplicity. Fix $\delta > 0$. It is possible to find a finite number of unit vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^k$ such that

$$\cap_{j=1}^{m} \{x : \langle u_j, x\rangle \leq 1-\delta\} \subset \{x : ||x||^2 \leq 1\}$$

Hence $P[\Sigma_{j=1}^{k} \{(\Sigma_{i=1}^{n} X_{i,j})^2 / (n \log n)\} > 1]$

$\leq \Sigma_{j=1}^{m} P[\langle u_j, T_n\rangle > 1-\delta]$ where

$$T_n = (\Sigma_{i=1}^{n} X_{i,1}, \Sigma_{i=1}^{n} X_{i,2}, \ldots, \Sigma_{i=1}^{n} X_{i,k}) / (n \log n)^{\frac{1}{2}}$$

But under the given hypotheses, if $a = (a_1, \ldots, a_k)$ is any unit vector in $\mathbb{R}^k$ then by Lemma 4 $\{\Sigma_{j=1}^k a_j X_{n,j} : n \geq 1\}$ is itself a *-mixing stationary sequence with the same $\sigma = 1$. Hence for each j , $1 \leq j \leq m$, by the Ghosh-Babu theorem:

$$\log P[<u_j, T_n> > 1-\delta] \sim -\frac{1}{2}(1-\delta)^2 \log n$$

Hence $(2/\log n) \log P[\Sigma_{j=1}^k \{(\Sigma_{i=1}^n X_{i,j})^2/(n \log n)\} > 1]$ has limsup less than $\limsup (2/\log n) \max\{\log P[<u_j, T_n> > 1-\delta : 1 \leq j \leq m\}$

$$= \limsup\{\max\{(2/\log n) \log P[<u_j, T_n> > 1-\delta] : 1 \leq j \leq m\}\}$$

$= -(1-\delta)^2$ and since δ is arbitrary

≤ -1 .

Now take any unit vector u in $\mathbb{R}^k$, then $\{x : \|x\|^2 \leq 1\} \subset \{x : <u,x> \leq 1\}$ gives

$$P[\Sigma_{j=1}^k \{(\Sigma_{i=1}^n X_{i,j})^2/(n \log n)\} > 1]$$
$$\geq P[<u,T_n> > 1]$$

Hence

$$\liminf (2/\log n) \log P[\Sigma_{j=1}^k \{(\Sigma_{i=1}^n X_{i,j})^2/(n \log n)\} > 1]$$
$$\geq \liminf (2/\log n) \log P[<u,T_n> > 1]$$
$$= -1$$

□

Lemma 6 Let Φ denote the distribution function of the standard normal distribution.

Let $a,b \in \mathbb{R}$, $0 \leq |a| < b$, then

$$\Phi(b) - \Phi(a) \geq (2\pi b^2)^{-\frac{1}{2}} \exp(-a^2/2)\{1-\exp[-(b^2-a^2)/2]\}$$

and if b^2-a^2 is large enough,

$$\Phi(b) - \Phi(a) \geq \frac{1}{2}(2\pi b^2)^{-\frac{1}{2}} \exp(-a^2/2)$$ □

Lemma 7: Let $f(n) \geq 0$ be such that as $n \uparrow \infty$ then $f(n) \uparrow \infty$. Fix $\delta > 0$ and choose any $\gamma > 0$. Then for $\{X_n : n \geq 1\}$ as in Theorem A,

(i) $P[|S(f(n)) - S([f(n)])| > \delta(n \log n)^{\frac{1}{2}}] = o(n^{-\gamma})$

and if $g(n) \geq 2$ is such that $g(n) = o(f(n))$.

(ii) $P[|S(f(n)) - S(f(n) - g(n))| > \delta(n \log n)^{\frac{1}{2}}] = o(n^{-\gamma})$

Proof: Both assertions follow by an application of the Ghosh-Babü Theorem and Chebyshev's inequality. This result will be particularily useful when $f(n) = n$ and $g(n) = n^{\frac{1}{2}}$

□

4. <u>Proof of Theorem A-Inclusion</u> For the sake of simplicity, assume $\sigma = 1$. The first part of the proof of theorem A will demonstrate that the set of r-quick limit points of $(\xi_n: n \geq 2)$ is at most $r^{\frac{1}{2}}K_o$. Let $\varepsilon > 0$, $\delta = 1 + r^{\frac{1}{2}}$ and define $\mathcal{U}_\varepsilon$ as the open ε-neighbourhood of $r^{\frac{1}{2}}K_o$ in $C[0,1]$. Define $L(\varepsilon) = \sup\{n \geq 2: \xi_n \notin (r^{\frac{1}{2}}K_o)_{4\delta\varepsilon} = \mathcal{U}_{4\delta\varepsilon}\}$. $L(\varepsilon)$ is a random variable on $(\Omega, \mathfrak{F}, P)$. If it can be shown that $E(L^r(\varepsilon)) < \infty$ for all $\varepsilon > 0$, then the inclusion follows by the definition of r-quick limit points. Let $\alpha > 1$ and define $\alpha(i)$ for $i = 1,2,3...$ by $\alpha(i) = \alpha^i$. By Lemma 7 for i large, we may use $\alpha(i)$ **instead of** $[\alpha(i)]$ in the sequel.

$$0 \leq E(L^r(\varepsilon)) = \int L^r(\varepsilon) \leq M + \Sigma_{n+1}^{\infty}\alpha(nr)P[L(\varepsilon) \geq \alpha(n)]$$

for some constant $0 \leq M < \infty$. Thus to show $E(L^r(\varepsilon)) < \infty$, it suffices to establish that

$\Sigma_{n=1}^{\infty} \alpha(nr)P[L(\varepsilon) \geq \alpha(n)] < \infty$.

$$\Sigma_{n=1}^{\infty} \alpha(nr)P[L(\varepsilon) \geq \alpha(n)]$$
$$= \Sigma_{n=1}^{\infty} \alpha(nr)P[\xi_j \notin \mathcal{U}_{4\delta\varepsilon} \text{ for some } j \geq \alpha(n)]$$
$$\leq \Sigma_{n=1}^{\infty} \alpha(nr)\, \Sigma_{i=n}^{\infty} P[\xi_{\alpha(i)} \notin \mathcal{U}_{2\varepsilon\delta}]$$
$$+ \Sigma_{n=1}^{\infty} \alpha(nr)\, \Sigma_{i=n}^{\infty} P[\max\{\|\xi_k - \xi_{\alpha(i)}\|_c: \alpha(i) \leq k \leq \alpha(i+1),$$
$$k \text{ an integer}\} \geq 2\varepsilon]$$
$$= T_1 + T_2 \text{ respectively.}$$

We now show that $T_1 < \infty$. From the fact that for any fixed $m \in \mathbb{N}$:

$$[\xi_{\alpha(i)} \notin \mathcal{U}_{2\delta\varepsilon}] \subset \{\xi_{\alpha(i)} \notin \mathcal{U}_{2\delta\varepsilon} \text{ and } \|\pi_m\xi_{\alpha(i)}-\xi_{\alpha(i)}\|_c < \varepsilon]$$
$$\cup\ [\|\pi_m\xi_{\alpha(i)}-\xi_{\alpha(i)}\|_c \geq \varepsilon]\}$$

and if $\|x\|_H \leq (1+\varepsilon)r^{\frac{1}{2}}$ then $x \in (r^{\frac{1}{2}}K_o)_{\delta\varepsilon}$ (since $\|\cdot\|_c < \|\cdot\|_H$ $(1+\varepsilon)^{-1}x \in r^{\frac{1}{2}}K_o$ and hence $\|x-(1+\varepsilon)^{-1}x\|_c \leq \varepsilon(1+\varepsilon)^{-1}\|x\|_H r^{\frac{1}{2}} \leq \delta\varepsilon$), it follows that:

$$T_1 \leq \Sigma_{n=1}^{\infty}\alpha(nr)\ \Sigma_{i=n}^{\infty}\{P[\|\xi_{\alpha(i)} - \pi_m\xi_{\alpha(i)}\|_c \geq \varepsilon]$$
$$+ P[\|\pi_m\xi_{\alpha(i)}\|_H > (1+\varepsilon)r^{\frac{1}{2}}]\}$$
$$\leq \Sigma_{n=1}^{\infty}\alpha(nr)\ \Sigma_{i=n}^{\infty}P[\|\xi_{\alpha(i)} - \pi_m\xi_{\alpha(i)}\|_c \geq \varepsilon]$$
$$+ \Sigma_{n=1}^{\infty}\alpha(nr)\ \Sigma_{i=n}^{\infty} P[\|\pi_m\xi_{\alpha(i)}\|_H > (1+\varepsilon)r^{\frac{1}{2}}] \qquad (4.1)$$

The first double sum in (4.1) above is finite by virtue of Lemma 2 of section 3. Indeed, choose $p > r + 1$ and $m \in N$ sufficiently large so that for some $K \geq 0$

$P[\|\xi_{\alpha(i)} - \pi_m\xi_{\alpha(i)}\|_c > \varepsilon] \leq K\,\alpha(-ip)$. Then

$$\Sigma_{n=1}^{\infty}\alpha(nr)\ \Sigma_{i=n}^{\infty} P[\|\xi_{\alpha(i)} - \pi_m\xi_{\alpha(i)}\|_c \geq \varepsilon]$$
$$\leq \Sigma_{n=1}^{\infty}\alpha(nr)\ \Sigma_{i=n}^{\infty} K\,\alpha(-ip) < \infty \text{ since } r - p < 1.$$

To show that the second double sum in (4.1) converges, consider the following expression for fixed i:

$P[\| \pi_m \xi_{\alpha(i)} \|_H > (1 + \varepsilon) r^{\frac{1}{2}}]$ by definition of $\| \quad \|_H$

$$= P[m \, \Sigma_{j=1}^{m} (\pi_m \xi_{\alpha(i)}(j/m) - \pi_m \xi_{\alpha(i)}((j-1)/m))^2 > (1+\varepsilon)^2 r]$$

$$= P[m \, \Sigma_{j=1}^{m} (S(\alpha(i)\cdot(j/m)) - S(\alpha(i)\cdot((j-1)/m)))^2 > (1+\varepsilon)^2 r \cdot a^2(\alpha(i))]$$

$$\leq P[m \, \Sigma_{j=1}^{m} (\Sigma_{k=1+\alpha(i)\cdot((j-1)/m)}^{\alpha(i)\cdot(j/m)} X_k)^2 > (1+\varepsilon)^2 r \, . \, a^2(\alpha(i))] + K \, \alpha(-ir(1+\varepsilon)^2)$$

for some finite positive constant K by Lemma 7

$$\leq (1+\rho(1))^m \, P[\Sigma_{j=1}^{m} (\Sigma_{k=1}^{\alpha(i)\cdot(1/m)} Y_{k,j})^2 > (1+\varepsilon)^2 r \cdot a^2(\alpha(i))] + K \, \alpha(-ir(1+\varepsilon)^2) \qquad (4.2)$$

By Lemma 3 and stationarity; $\{Y_{k,j} : 1 \leq k \leq \alpha(i)\cdot(1/m)\}$ for $1 \leq j \leq m$ are m independent copies of $\{X_k : 1 \leq k \leq \alpha(i)\cdot(1/m)\}$. Thus expression (4.2) is of order $((1 + \rho(1))^m + K)(\alpha(i))^{-r(1+\varepsilon')}$ for some $0 < \varepsilon' < \varepsilon$ by virtue of Lemma 5.

Hence $\Sigma_{n=1}^{\infty} \alpha(nr) \, \Sigma_{i=n}^{\infty} P[\| \pi_m \xi_{\alpha(i)} \|_H > (1+\varepsilon) r^{\frac{1}{2}}]$ $\leq K(1 + \rho(1))^m \, \Sigma_{n=1}^{\infty} \alpha(nr) \, \alpha\,(-nr(1+\varepsilon')^2) < \infty$ by the properties of geometric sequences and thus it has been established that $T_1 < \infty$.

$$T_2 = \Sigma_{n=1}^{\infty} \alpha(nr) \Sigma_{i=n}^{\infty} P[\max\{\| \xi_k - \xi_{\alpha(i)} \|_c : \alpha(i) \le k \le \alpha(i+1),$$
$$k \text{ an integer}\} \ge 2\varepsilon]$$

Fix $i,k \in N$ such that $\alpha(i) < k \le \alpha(i+1)$; then by the definition of ξ_n and by the triangle inequality introducing $S(\alpha(i)t) \cdot a^{-1}(k)$:

$$\| \xi_k - \xi_{\alpha(i)} \|_c \le \max\{|S(\alpha(i)t)| \cdot |a^{-1}(k) - a^{-1}(\alpha(i))| : 0 \le t \le 1\}$$
$$+ a^{-1}(k)\max\{|S(kt) - S(\alpha(i)t)| : 0 \le t \le 1\}$$

But $\max\{|S(\alpha(i)t)|:0\le t\le 1\} \cdot |a^{-1}(k) - a^{-1}(\alpha(i))| > \varepsilon$

iff $\max\{|S(\alpha(i)t)|:0 \le t \le 1\} > \varepsilon| a^{-1}(k) - a^{-1}(\alpha(i))|^{-1}$

iff $\max\{|S(\alpha(i)t)|:0 \le t \le 1\} > \varepsilon a(\alpha(i))|1-a(\alpha(i))a^{-1}(k)|^{-1}$

and since $\alpha(i) \le k \le \alpha(i+1)$

$|1-a(\alpha(i))a^{-1}(k)|^{-1} \le |1-a(\alpha(i))a^{-1}(\alpha(i+1))|^{-1}$ and as $i \to \infty$ the right-hand expression tends to $|1 - \alpha^{-\frac{1}{2}}|$, and so for i large is $\le (1-\varepsilon)^{-1} |1-\alpha^{-\frac{1}{2}}|$.

Thus $P[\max\{\| \xi_k - \xi_{\alpha(i)} \|_c : \alpha(i) \le k \le \alpha(i+1)\} \ge 2\varepsilon]$

$\le P[\max\{|S(\alpha(i)t)|: 0 \le t \le 1\} > (1-\varepsilon)\varepsilon|\alpha^{-\frac{1}{2}}-1|^{-1}a(\alpha(i))]$

$+ P[\max\{|S(t) - S(s)| : 0 \le s, t \le \alpha(i+1), |t-s| \le (\alpha-1)\alpha(i)\}$
$\ge \varepsilon a(\alpha(i))]$

$= P_1 + P_2$ for a fixed i large.

$$P_1 = P[\max\{|S_j|:j=0,1,\ldots,\alpha(i)\} \geq (1-\varepsilon)\varepsilon|\alpha^{-\frac{1}{2}}-1|^{-1}a(\alpha(i))]$$

by the definition of $S(\cdot)$.

Assume temporarily that $\rho(1) < 1$. Then since $ES_j^2 \sim j\sigma^2$ (see Lemma (Stout) page 16, section 3) and by Chebyshev's Inequality one can obtain the following inequality for any $j \in \{0,1,2 \ldots \alpha(1)\}$:

$$P[|S_j| \leq \tfrac{1}{2} \cdot \varepsilon(1-\varepsilon)|\alpha^{-\frac{1}{2}}-1|^{-1}a(\alpha(i))] \geq \tfrac{1}{2}$$

when i is sufficiently large. Lemma 1 now yields that

$$P[\max\{|S_j|:j=0,1,2\ldots\alpha(i)\} > (1-\varepsilon)\varepsilon|\alpha^{-\frac{1}{2}}-1|^{-1}a(\alpha(i))]$$

$$\leq 2(1-\rho(1))^{-1}P[|S_{\alpha(i)}| > \tfrac{1}{2}(1-\varepsilon)\varepsilon|\alpha^{-\frac{1}{2}}-1|^{-1}a(\alpha(i))]$$

$$\sim 4(1-\rho(1))^{-1}(2\pi c^2\log\alpha(1))^{-\frac{1}{2}}(\alpha(i))^{-\frac{1}{2}c^2}$$ by the Ghosh-Babu theorem where $c=\varepsilon(1-\varepsilon)2^{-\frac{1}{2}}|\alpha^{-\frac{1}{2}}-1|^{-1}$

$$= o(\alpha(i))^{-\frac{1}{2}c^2} \text{ ie. } o(\alpha^{-\frac{1}{4}i\varepsilon^2(1-\varepsilon)^2|\alpha^{-\frac{1}{2}}-1|^{-2}}) \text{ as } i \to \infty .$$

Clearly, by choosing $\alpha > 1$ sufficiently close to 1, the probability above can be taken to be of order $o(\alpha^{-\lambda i})$ for any $\lambda > 0$. The assumption $\rho(1) < 1$ may be removed in a similar fashion to that found in the proof of Lemma 2. Choosing $\lambda = r + 1$ in particular yields

$$\Sigma_{n=1}^{\infty} \alpha(nr)\Sigma_{i=n}^{\infty}P_1 \leq M\Sigma_{n=1}^{\infty}\Sigma_{i=n}^{\infty} \alpha(i-(r+1)) < \infty$$

for M some positive constant.

$$P_2 = P[\max\{|S(t)-S(s)|:0\le s,t\le\alpha(i+1),\ |t-s|\le(\alpha-1)\alpha(i)\} > \varepsilon a(\alpha(i))] .$$

By Lemma 7, to show $\Sigma_{n=1}^{\infty} \alpha(nr)\ \Sigma_{i=n}^{\infty} P_2 < \infty$

it suffices to show $\Sigma_{n=1}^{\infty} \alpha(nr)\ \Sigma_{i=n}^{\infty} P_2' < \infty$

where $P_2' = P[\max\{|S_\ell - S_j|:\ell,j\in\{0,1,\ldots,\alpha(i+1)\},$

$$|\ell-j| \le (\alpha-1)\alpha(i)\} > \tfrac{1}{4}\varepsilon\cdot a(\alpha(i))]$$

$$P_2' \le \Sigma_{\nu=2}^{\alpha/(\alpha-1)} P[\max\{|S_\ell - S_j|:(\nu-2)(\alpha-1)\alpha(i)\le 1,\ell\le\nu(\alpha-1)\alpha(i);\ell,j\in\mathbb{N}\} > \tfrac{1}{4}\varepsilon a(\alpha(i))]$$

where $\alpha > 1$ is chosen so that $\alpha/(\alpha-1)$ is an integer.

$$\le \Sigma_{\nu=2}^{\alpha/(\alpha-1)} P[\max\{|S_\ell - S_{[(\nu-2)(\alpha-1)\alpha(i)]}|:[(\nu-2)(\alpha-1)\alpha(i)] \le \ell \le [\nu(\alpha-1)\alpha(i)],\ \ell\in\mathbb{N}\} > \tfrac{1}{8}\varepsilon a(\alpha(i))]$$

by the triangle inequality

$$= \Sigma_{\nu=2}^{\alpha/(\alpha-1)} P[\max\{|S_\ell|:0\le\ell\le[2(\alpha-1)\alpha(i)]+1,\ \ell\in\mathbb{N}\} > \tfrac{1}{8}\varepsilon a(\alpha(i))]$$

using the stationarity properties of $\{X_n : n \ge 1\}$

$$\le (\alpha/(\alpha-1))P[\max\{|S_\ell|:0\le\ell\le[2(\alpha-1)\alpha(i)]+1,\ \ell\in\mathbb{N}\} > \tfrac{1}{8}\varepsilon a(\alpha(i))]$$

Assume temporarily that $\rho(1) < 1$. Then for i sufficiently large, $1 < \alpha < \frac{3}{2}$ and for any $\ell \in \{0,1,2...[2(\alpha-1)\alpha(i)]$, Chebyshev's inequality yields $P[|S_\ell| \le \frac{1}{16}\varepsilon a(\alpha(i))] \ge \frac{1}{2}$. Lemma 1 and the Ghosh-Babu theorem now show that P_2 is of the order $o(\alpha^{-i(r+1)})$ if $\alpha > 1$ is sufficiently close to 1. Thus $\Sigma_{n=1}^{\infty} \alpha(nr) \Sigma_{i=n}^{\infty} P_2 < \infty$ under the assumption $\rho(1) < 1$. This restriction can be removed in a similar way as before in Lemma 2 . Note that this new requirement placed upon $\alpha > 1$ is compatible with the condition imposed in the P_1 case, i.e. both conditions can be fulfilled simultaneously.

Thus $T_2 \le \Sigma_{n=1}^{\infty}\alpha(nr)\, \Sigma_{i=n}^{\infty} (P_1 + P_2)$

$$\le \Sigma_{n=1}^{\infty} \alpha(nr)\, \Sigma_{i=n}^{\infty} P_1 + \Sigma_{n=1}^{\infty} \alpha(nr)\Sigma_{i=n}^{\infty} P_2 < \infty$$

and hence $E(L^r(\varepsilon)) \le M + T_1 + T_2 < \infty$ for any $\varepsilon > 0$, that is, it has been demonstrated that the sequence $\{\xi_n : n = 1,2,3,...\}$ is r-quickly relatively compact in $C[0,1]$ and, since K_o is a closed set, the set of r-quick limit points of the sequence is contained in $\bigcap_{\varepsilon>0} \mathcal{U}_{4\varepsilon\delta} = r^{\frac{1}{2}}K_o$.

<u>5. Proof of Theorem A-Conclusion</u> Let $h \in r^{\frac{1}{2}}K_o$. We now show that h is an r-quick limit point of $\{\xi_n : n = 2,3,4 ...\}$. Define:

$L^*(\varepsilon) = \sup\{n\geq 1: \|\xi_n - h\|_c \leq \varepsilon(2+\|h\|_c)\}$. To show that that h is an r-quick limit point of the sequence, we must show that $E(L^*(\varepsilon))^r = \infty$.

Let $g_o = (1-\varepsilon)h$, then $\|g_o - h\|_c = \varepsilon\|h\|_c$. Thus to show $E(L^*(\varepsilon))^r = \infty$ it suffices to prove $E(\tilde{L}(\varepsilon))^r = \infty$ where $\tilde{L}(\varepsilon) = \sup\{n\geq 1: \|\xi_n - g_o\|_c \leq 2\varepsilon\}$. Since $E_r (=\{f: f=\pi_n g$ for some $n \in \mathbb{N}$, $g \in r^{\frac{1}{2}}K_o\})$ is dense in $r^{\frac{1}{2}}K_o$, we can assume $g_o \in E_r$, ie for some $k_1 \in \mathbb{N}$ $g \in r^{\frac{1}{2}}K_o$, $g_o = \pi_{k_1} g$. Furthermore $E(\tilde{L}(\varepsilon))^r = \infty$ is implied by

$\Sigma_{n=1}^{\infty} n^{r-1} P[\|\xi_n - g_o\|_c \leq 2\varepsilon] = \infty$ (cf. Ex. 5, pg. 44, Chung [5])

and to demonstrate the latter statement it suffices (by Lemma 2) to show for k an integer multiple of k_1 (so $\pi_k g = \pi_{k_1} g$) that $\Sigma_{n=1}^{\infty} n^{r-1} P[\|\pi_k \xi_n - g_o\|_c \leq \varepsilon] = \infty$.

$\Sigma_{n=1}^{\infty} n^{r-1} P[\|\pi_k \xi_n - g_o\|_c \leq \varepsilon]$ by definition of $\|\ \|_c$

$= \Sigma_{n=1}^{\infty} n^{r-1} P[|\xi_n(i/k) - g(i/k)| \leq \varepsilon,\ i = 1,2,3\ldots k]$

$\geq \Sigma_{n=1}^{\infty} n^{r-1} P[|(\xi_n(i/k) - g(i/k)) - (\xi_n(i/k) - g((i-1)/k))|$

$\leq \varepsilon/k$ for $i=1,2,3,\ldots k]$

$$= \Sigma_{n=1}^{\infty} n^{r-1} P[|\xi_n(i/k) - g(i/k) - b(i)| \le \varepsilon/k$$

$$i = 1,2,3\ldots k] \quad \text{where} \quad b(i) = g(i/k) - g((i-1)/k)$$

$$= \Sigma_{n=1}^{\infty} n^{r-1} P[|S(ni/k) - S(n(i-1)/k) - b(i)\cdot a(n)|$$

$$\le \varepsilon a(n)/k \quad i=1,2,3\ldots k]$$

By Lemma 7, to show the above sum diverges, it suffices to show that the following sum also diverges.

$$\Sigma_{n=1}^{\infty} n^{r-1} P[|S((n-n^{\frac{1}{2}})i/k) - S(n(i-1)/k) - b(i)a(n)|$$

$$\le \varepsilon a(n)/(2k)] = \infty$$

By hypothesis $\rho(n^{\frac{1}{2}}) \to 0$ and the sequence $(X_n)_{n=1}^{\infty}$ is *-mixing. **Hence by Lemma 3 it suffices to prove:**

$$\Sigma_{n=1}^{\infty} n^{r-1} \Pi_{i=1}^{k} P[|S((n-n^{\frac{1}{2}})i/k) - S(n(i-1)/k) - b(i)a(n)|$$

$$\le \varepsilon a(n)/k] = \infty \qquad (5.1)$$

By stationarity and Lemma 7, to show (5.1) it suffices to show:

$$\Sigma_{n=1}^{\infty} n^{r-1} \Pi_{i=1}^{k} P[|S([n/k]) - b(i)a(n)| \le \varepsilon a(n)/k] = \infty$$

Let $i \in \{1,2,3,\ldots k\}$

$$P[|S_{[n/k]} - b(i)a(n)| \le \varepsilon a(n)/k]$$

$$\ge P[(|S_{[n/k]}| + |b(i)|a(n) \le \varepsilon a(n)/k]$$

$$= P[(|b(i)| - \varepsilon/(32k))a(n) \le |S_{[n/k]}|$$

$$\le (|b(i)| + \varepsilon/k)a(n)]$$

$$\ge P[|b(i)|a(n) \le S_{[n/k]} \le (|b(i)| + \varepsilon/k)a(n)]$$

$$= P[S_{[n/k]} \le (|b(i)| + \varepsilon/k)a(n)]$$

$$- P[S_{[n/k]} < |b(i)|a(n)]$$ and for n large

$$\ge P[S_{[n/k]} \le (|b(i)| + \varepsilon/k)(2k[n/k] \log [n/k])^{\frac{1}{2}}]$$

$$- P[S_{[n/k]} \le |b(i)|(1+\zeta)(2k[n.k]\log[n/k])^{\frac{1}{2}}] \qquad (5.2)$$

where $\zeta > 0$ is chosen so that for $i = 1,2...k$

and $|b(i)| \neq 0$: $(1+\zeta)^2|b(i)|^2 < (|b(i)| + \varepsilon/k)^2$

and $r>(1+\xi)^2(r-\varepsilon) = r - \varepsilon''$ for some $\varepsilon'' > 0$. Thus

for n large $1 \le (\log n /\log [n/k])^{\frac{1}{2}} \le (1+\zeta)^{\frac{1}{2}}$ and

$k^{\frac{1}{2}} \le (n/[n/k])^{\frac{1}{2}} \le (1+\zeta)^{\frac{1}{2}}k^{\frac{1}{2}}$. Now by the Ghosh-Babu theorem

the above difference of probabilities is greater than the following difference of Normal values as n tends to infinity for $\gamma > 0$ small enough. Thus expression (5.2) is of order

$$\Phi((|b(i)| + \varepsilon/k)(1-\gamma)(2k \log [n/k])^{\frac{1}{2}})$$

$$- \Phi((|b(i)|(2k)^{\frac{1}{2}})(1+\gamma)(1+\zeta)(\log [n/k])^{\frac{1}{2}}$$

$= \Phi(c(i)) - \Phi(d(i))$ and by the choice of $\zeta > 0$ and

$\gamma > 0$ sufficiently small, as $n \uparrow \infty$

$$c^2(i)-d^2(i) = 2k(\log [n/k])((|b(i)| + \varepsilon/k)^2(1-\gamma)^2$$
$$- (1+\gamma)^2|b(i)|^2(1+\zeta)^2) \uparrow \infty$$

For the rest of this proof let A be a generic constant strictly greater than 0. Hence for large n and by Lemma 6

$$\Phi(c(i))-\Phi(d(i)) \geq \tfrac{1}{2} \cdot c^{-1}(1) \cdot (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}d^2(i)\}$$

$$= \tfrac{1}{2}((|b(i)| + \varepsilon/k)(4\pi k\log[n/k])^{\frac{1}{2}})^{-1}$$
$$\times \exp\{-\tfrac{1}{2}(1+\zeta)^2(1+\gamma)^2b^2(i)\cdot 2 \log[n/k]\}$$

$$\geq A((\log n)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}(1+\zeta)^2(1+\gamma)^2b^2(i)\cdot 2k\log[n/k]\})$$

So $\Pi_{i=1}^{k}P[|S([n/k])-b(i)a(n)| \leq \varepsilon a(n)/k]$

$$\geq A((\log n)^{-\frac{1}{2}}\exp\{-(1+\zeta)^2(1+\gamma)^2k\log[n/k]\cdot\Sigma_{i=1}^{k}b^2(i)\}$$

n^{r-1}x the above product is

$$\geq A((\log n)^{-\frac{1}{2}k}[n/k]^{\{-k(1+\zeta)^2(\Sigma_{i=1}^{k}b^2(i))+r-1\}})$$
$$\geq A((\log n)^{-\frac{1}{2}k}(n)^{\{r-1-k(1+\zeta)^2(\Sigma_{i=1}^{k}b^2(i))\}})$$

But $k\Sigma_{i=1}^{k}b^2(i)=r-\varepsilon'$ $\varepsilon'\geq\varepsilon$ since $g \in (1-\varepsilon)r^{\frac{1}{2}}K_o$

$$\leq r - \varepsilon$$

and hence the above product expression is of the following order:

$$A\{(\log n)^{-\frac{1}{2}k}(n)^{r-1-(r-\varepsilon)(1+\zeta)^2}\}$$
$$\geq A\{(\log n)^{-\frac{1}{2}k}(n)^{r-1-r+\varepsilon''}\}$$

$\geq A(\log n)^{-\frac{1}{2}k}(n)^{-1+\varepsilon''}\}$ and by the integral test:

$\Sigma_{n=1}^{\infty}(\log n)^{-\frac{1}{2}k}n^{-1+\varepsilon''} = \infty$ and hence

$\Sigma_{n=1}^{\infty}n^{r-1}P[||\pi_k\xi_n - g||_c \leq \varepsilon]$

$\geq \Sigma_{n=1}^{\infty}A((\log n)^{-\frac{1}{2}k}n^{-1+\varepsilon''}) = \infty$

Thus it has been established that $E(L^*(\varepsilon))^r = \infty$. By the definition of r-quick limit points, this proves that every $h \in r^{\frac{1}{2}}K_o$ is an r-quick limit point of the sequence $\{\xi_n : n = 2,3,4,..\}$. This completes the proof of the theorem.

6. Applications In [17] Strassen gave some beautiful applications of his functional LIL. Similar applications for the r-quick analogue to the LIL in the iid case were given by Lai in [13]. This section will study similar applications of Theorem A. It is necessary to note here the properties of r-quick limit sets as proven by Lai in [13] and stated in section 1 as Lemmas A, B and C.

In what follows let $X_1, X_2, X_3 \ldots$ be a *-mixing sequence satisfying the requirements of Theorem A (assume further that $\sigma = 1$). Define $r^{\frac{1}{2}}K_o$, S_n and $\xi_n(t)$ for $t \in [0,1]$ as for Theorem A. It is a well-known fact that $r^{\frac{1}{2}}K_o$ is a compact subset of $C[0,1]$. The applications of Theorem A given below are modelled on the examples of Lai [13, page 624] .

Example 1: Define $f: C[0,1] \to \mathbb{R}$ by $f(x) = x(1)$.

Clearly f is continuous and hence by Theorem A and Lemmas B and C, it follows that:

$\limsup_{n\to\infty}(2n\log n)^{-\frac{1}{2}} S_n = \sup\{x(1): x\in r^{\frac{1}{2}}K_o\}$ (r-quickly) and in fact the supremum on the right is attained with $x(t) = r^{\frac{1}{2}}t$, so that the value of that supremum is $r^{\frac{1}{2}}$. Note that this extends Strassen's r-quick LIL to the sequence of Theorem A.

Example 2: Define $\phi: C[0,1] \to \mathbb{R}$ by $\phi(x) = \|x\|_c$. It follows immediately that ϕ is continuous and $\sup\{\phi(x): x\in r^{\frac{1}{2}}K_o\} = r^{\frac{1}{2}}$. Theorem A and Lemmas B and C now imply that:

$$\limsup_{n\to\infty}(2n\log n)^{-\frac{1}{2}} \max\{|S_j|: 1\le j\le n\} = \sup\{\phi(x):x\in r^{\frac{1}{2}}K_o\} = r^{\frac{1}{2}}$$

(r-quickly)

Example 3: Let f be any continuous real-valued function on $[0,1]$ and set $F(t) = \int_t^1 f(s)ds$ for $0\le t\le 1$. Define $\phi: C[0,1] \to \mathbb{R}$ by $\phi(x) = \int_0^1 x(t)f(t)dt$. It can be established that ϕ is continuous and $\sup\{\phi(x):x\in r^{\frac{1}{2}}K_o\} = \{r\int_0^1 F^2(t)dt\}^{\frac{1}{2}}$ (cf. [17, page 219]). Theorem A and Lemmas B and C now imply that:

$$\limsup_{n\to\infty}\int_0^1 f(t)\xi_n(t)dt = (r\int_0^1 F^2(t)dt) \quad \text{(r-quickly)}$$

By an **argument** similar to that found in [13, page 624] and by Example 2 above, it is also possible to show that:

$$\limsup_{n\to\infty}(2n^3\log n)^{-\frac{1}{2}} \Sigma_{i=1}^n f(i/n)S_i = (r\int_0^1 F^2(t)dt) \quad \text{(r-quickly)}$$

Example 4: Let $p \ge 1$ be a real constant. Define the continuous function $\phi: C[0,1] \to \mathbb{R}$ by $\phi(x) = \int_0^1 |x(t)|^p dt$. Using Theorem A and Lemmas B and C in a similar **argument** as that found in [17, pages 220-221], yields that:

$$\limsup_{n\to\infty} n^{-1-(p/2)}(2\log n)^{-p/2}\Sigma_{i=1}^{n}|S_i|^p$$
$$= 2(r/p)^{p/2}(p+2)^{(p-2)/2}\{\int_0^1(1-t^p)^{-\frac{1}{2}}dt\}^{-p} \quad (r\text{-quickly})$$

7. Strong Law It is appropriate at this point to mention the recent results obtained by T. L. Lai in the area of r-quick analogues of the Strong Law of Large Numbers (SLLN). Baum and Katz proved in [2] the following theorem on partial sums S_n of iid real random variables $X_1, X_2, X_3 \ldots$:

Theorem; Let $\alpha > 1/2, p > 0$, $p\alpha > 1$ and assume $EX_1 = 0$ if $\alpha \le 1$, then $E|X_1|^p < \infty$

$$\iff \Sigma_{n=1}^{\infty} n^{p\alpha-2}P[\sup\{j^{-\alpha}|S_j| \ge \varepsilon: j \ge n\}] < \infty \quad \text{for all } \varepsilon > 0$$
$$\iff \Sigma_{n=1}^{\infty} n^{p\alpha-2}P[|S_n| \ge \varepsilon n^{\alpha}] < \infty \quad \text{for some } \varepsilon > 0$$

This result is related to the convergence rate in the Marcinkiewicz-Zygmund SLLN. Considering the concept of r-quick convergence, Lai [14, page 693] states the following r-quick version of the Marcinkiewicz-Zygmund SLLN:

Theorem: Under the above conditions,

$$E|X_1|^p < \infty \iff n^{-\alpha}S_n \to 0 \quad ((p\alpha-1)\text{-quickly})$$
$$\iff n^{-\alpha}X_n \to 0 \quad ((p\alpha-1)\text{-quickly})$$

Lai also shows (cf. [11],[12] and [13]) how this result has useful applications in the area of sequential analysis in statistics. In [14] Lai succeeds in extending the above strong law result to the case where $X_1, X_2, X_3 \ldots\ldots$ is a stationary sequence satisfying certain mixing conditions. In particular, he studied strong and ϕ-mixing sequences and also sequences satisfying the following condition on moderately separated events.

Condition 2.4: There exists $\beta > 1$ and a positive integer m such that as $n \to \infty$,

$$\sup\{P[|X_1| > x, |X_i| > x : i > m]\} = O(P^{\beta}[|X_1| > x]) .$$

Under this condition and if $p > 0$ and $p\alpha > \max\{(\beta p+1), \beta\}/(\beta-1)$ Lai shows that for a stationary sequence $X_1, X_2, X_3 \ldots$, $\alpha > 1$ and

$$E|X_1|^p < \infty$$

$$\Longleftrightarrow n^{-\alpha}S_n \to 0 \qquad ((p\alpha-1)\text{-quickly})$$

$$\Longleftrightarrow n^{-\alpha}X_n \to 0 \qquad ((p\alpha-1)\text{-quickly})$$

These are three of the nine statements proven to be equivalent by Lai in [14]. He also shows that it is possible to loosen the above conditions on α and obtain the same result by requiring the sequence $X_1, X_2, X_3 \ldots$ to be either strong or ϕ-mixing with the mixing coefficient being of a certain order. Applications of these results to renewal theory and first passage times for stationary processes are also given by Lai in [14].

We may make a remark at this point concerning the applicability of Lai's Strong Law result to the case of *-mixing sequence. By definition, if $X_1, X_2, X_3 \ldots$ is a stationary *-mixing sequence there exists an integer m such that for $x \in \mathbb{R}$

and $i \geq m$:

$$P\{[|X_1| > x] \cap [|X_i| > x]\} \leq \{1+\rho(m)\}P[|X_1| > x]P[|X_i| > x]$$
$$\leq \{1+\rho(m)\}P^2[|X_1| > x]$$
$$= O(P^2[|X_1| > x]) \quad \text{as} \quad x \to \infty$$

This clearly implies Condition 2.4 and hence the first part of Lai's theorem 1 holds for all *-mixing stationary sequences. This includes Lai's remarks on iid and m-dependent sequences following his statement of Condition 2.4 ([14,page 694]) .

8. Concluding Remarks Having demonstrated an r-quick analogue to the LIL in the *-mixing case, it is natural to ask whether a further extension to this result exists. In particular, there is a strong possibility that an extension to the ϕ-mixing case can be shown with much the same machinery as used for the *-mixing case. The Ghosh-Babu result is already valid for ϕ-mixing sequences and this paper's Lemma 1 can be shown for ϕ-mixing sequences as remarked previously. These two results would yield in a similar fashion as given in section 3 a ϕ-mixing version of Lemma 2, the "equicontinuity" result. Most of the arguments used in the proof of Theorem A work as well for the ϕ-mixing case, with the exception of Lemmas 3, 4, and 5, where the *-mixing property seems essential. If the difficulties arising in the use of these lemmas can be bypassed, the rest of the proof of Theorem A in the ϕ-mixing case will follow easily.

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