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CONTINUUM MECHANICS AND FINITE ELEMENT NUMERICAL SOLUTIONS
IN GEOTECHNIQUE

by

Amintore Fusco

A thesis
presented to the University of Ottawa
in fulfillment of the
thesis requirement for the degree of
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in
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Amintore Fusco, Ottawa, Canada, 1985.



UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

Alla mia famiglia

"Originality is merely development of the art of one's predecessor, and genuine when a logical development."

T.S. Eliot: introduction to Ezra Pound's selected poems
(1928)

"To make a name for learning
when other roads are barred,
take something very easy
and make it very hard."

Piet Hein: Wide Road (1969)

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PREFACE

A large number of problems in the physical world can be analyzed assuming the medium under investigation as continuous.

Although the basic governing equations for many physical problems have been established more than one century ago, a very small number of practical solutions has been found. The difficulty lies in the fact that the governing field equations are differential equations, most of the time non linear, which in the best case can be solved only for simple boundary conditions.

Engineers, therefore, have always shown a strong tendency to leave aside the Continuum Mechanics Theory and to concentrate rather on the development of simplified approaches to practical problems.

Nowadays, the development of computer facilities and numerical techniques offer the possibility of solving differential equations of great complexity. As a result, a renewed interest in all aspects of the Continuum Mechanics Theory has been noticed in recent years.

Some aspects of the basic assumptions of the Classical Mechanics Theory are today questioned and eventually modified.

In solid mechanics, a number of theoretical and experimental research studies are now trying to establish more realistic Constitutive equations than the classical linear elastic Hooke's law. Researchers in geotechnical engineering are also engaged in the search for a global solution of the consolidation problem which involves the interaction of a fluid and a solid phase.

In line with these developments, this thesis aims to explore and to implement the new possibilities offered by both numerical techniques and recent theoretical achievements tending to analyse a number of geotechnical problems more rigorously.

SUMMARY

The primary objective of this research work was to implement consistently, in a series of finite element programs, the governing field equations for some problems involved in a practical geotechnical analysis.

For this purpose, two different aspects were investigated, namely: flow through a porous medium and elasto-plastic stress analysis in a saturated medium subjected to static load conditions.

This led to the development of four main finite element programs capable of analyzing the following problems:

1. Seepage under steady conditions for confined or unconfined flow in an anisotropic medium;
2. Elasto-plastic analysis of a highly permeable saturated medium;
3. Uncoupled non-linear consolidation in a soil with anisotropic permeability;
4. Elasto-plastic Coupled consolidation in a soil with anisotropic permeability.

Plastic deformation may be described by either linear elastic perfectly plastic models or by a Generalized Cam-clay model. In the first group the failure (yield) surface may be assumed to be represented by either Tresca, von

Mises, Mohr-Coulomb or Drucker-Prager failure criterion. In the Generalized Cam-clay model, the critical state may be represented by either Drucker-Prager or Mohr-Coulomb failure criterion.

In order to achieve these results, an accurate and critical analysis of the main assumptions inherent in the various theories involved was necessary.

The first Chapter deals with the fundamentals of Continuum Mechanics and its extension to Mechanics of a Porous Saturated Medium, or Biphase Medium. The governing equations for steady flow and the two alternative approaches to analyse 3-D consolidation problems, Coupled and Uncoupled Theories, are presented. It is therein also mathematically demonstrated that, within the context of the Uncoupled theory, the governing field equation of consolidation may be derived without necessarily assuming linear elasticity and small deformations. This is exemplified by incorporating the non-linear elastic relationship as proposed by the Cam-clay model in the governing field equation, without any recourse to the hypothesis of small deformations.

The key point in a realistic stress/strain analysis lies in the proper choice of the Constitutive equation. In this respect, Chapter 2 is entirely devoted to a critical review of the fundamental principles of the elastic and plastic theories. In particular, the basic similarity between the Green elastic theory and the incremental theory of

plasticity is highlighted. It is pointed out that the crucial point in both theories is the finding of a potential stress function whose gradient establishes the direction of the relative strain. Then, a correct expression of a reduced form of the gradient vector of an isotropic and quasi-isotropic potential function is derived. The geometrical shape of the elastic potential function for isotropic material is studied, and a possible evaluation of the relative elastic parameters from a conventional triaxial test in large deformation is proposed. As regards the incremental theory of plasticity, it is proved that most of the available expressions for the evaluation of the plastic modulus may be reduced to only two alternative relationships. Following this premise, a number of elasto-plastic models are reexamined in the attempt to identify their potential and limitations. Finally, a generalized version of the two Cam-clay models is proposed, in which the relative plastic potentials have a higher degree of freedom and the Critical state may be also represented by the Mohr-Coulomb failure criterion.

In Chapter 3, the numerical solutions of the field equations discussed in the first Chapter are presented, emphasis being laid upon previous works dealing with algorithms and procedures. It is remarked that current numerical algorithms for the analysis of the unconfined flow are either costly or unable to preserve a well graded

distribution of the elements during the iteration process required to locate the free surface. As regards the numerical implementation of the Coupled and Uncoupled theories of consolidation, they usually present instability in the solution even for the simplest case of linear elastic materials. It appears also that the common procedures for elasto-plastic analysis by Finite Element present several inaccuracies, as for instance in the calculation of the plastic strain gradient and of the yield point. Consequently, the more complex case of the Finite Element implementation of the elasto-plastic coupled consolidation cannot be yet considered numerically solved. In addition, a new expression to calculate the acceleration parameter for a faster convergence in the elasto-plastic analysis by employing the initial stress method is proposed.

In order to overcome the numerical problems illustrated in Chapter 3 new numerical formulations and schemes are proposed. In particular a new algorithm for the analysis of unconfined flow is introduced, along with a more flexible finite element formulation of the Uncoupled and the elasto-(visco)plastic Coupled consolidation theories. A number of practical aspects for a correct numerical implementation of elasto-plastic problems is examined, and an accurate way to evaluate the yielding point for quasi-isotropic functions is proposed. Furthermore, an extrapolating technique to correctly compute the nodal fluid

velocity (or stress) for 8-noded isoparametric element is presented.

Finally, the performance of the various programs herein implemented is illustrated in Chapter 5. The reliability of the programs is supported by a number of examples where analytical solutions are available. The algorithm proposed for the analysis of unconfined flow turned out to be very effective and economical. The present implementation of the Coupled and Uncoupled theories of consolidation appears to be very efficient, having overcome instability in the solutions, and that in an economical way. As far as plastic analysis is concerned, the number of theoretical improvements brought in the numerical implementation allowed to meet completely available numerical expectations; for instance, adopting linear elastic perfectly plastic models, it was possible to prove that the ultimate theoretical values calculated following the Limit Analysis theory could be recovered only by using Tresca or Mohr-Coulomb failure criterion. Moreover, for a semi-infinite medium loaded in plane strain condition by a smooth shallow foundation, the elasto perfectly plastic models predict a "generalized mode of failure", whereas the Generalized Burland model predicts a "localized mode of failure". Furthermore, using the Generalized Burland model in a Coupled elasto-plastic analysis, it was possible for the first time to forecast a delayed failure, an event so often registered in practice.

In conclusion, this research succeeded in providing a fairly efficient and reliable numerical tool for a number of geotechnical problems; nevertheless, as pointed out in Chapter 6, only future experimental works can validate the numerical predictions.

GLOSSARY

Lower case Latin letters:

$\underline{a}$	: directional vector in Prevost's model
$\underline{a}_1, \underline{a}_2, \underline{a}_3$	: vectorial components of the gradient of a stress function
a, b	: Henkel's parameters
a, b, d	: Generalized Burland model coefficients
$\underline{b}, \underline{b}_0$	: current and initial body force vectors
c	: cohesion
c_1, c_2, c_3, c_4	: coefficients
$\underline{c}_F = c^e \underline{\bar{v}}_\sigma F$	
$\underline{c}_G = c^e \underline{\bar{v}}_\sigma G$	
d	: nodal displacement vector
$dI = I dG_0$	: plastic multiplier
$\underline{d}$	: kinematic multiplier
$\underline{e}, \underline{\tilde{e}}$	: homologous vectors to the small and finite deviatoric Lagrange strain tensors respectively
f	: function
$\underline{f}$	: equivalent load vector at the nodes
$\underline{g}$	: equivalent flow vector at the nodes
h, h_c	: size of the yield and bounding surfaces
h, h_0	: current and initial head pressure functions
$\underline{h}, \underline{h}_0$	: current and initial nodal head pressures

i, j, k : indices
 k : hardening parameter
 $l = 1/A$: reciprocal of the plastic modulus
 $l_{ij}, \tilde{l}_{ij}$: small and finite Lagrange strain components
 m : mass
 m_s, m_f : solid and fluid masses
 n : degree of a function, ratio of ellipse's semi-axes
 $\hat{n}_F, \hat{n}_G$: unit gradient to F and G functions
 p : sinks and sources
 $p, p', \tilde{p}$: mean pressures associated to Cauchy stress, Cauchy effective stress tensor and 2nd Piola-Kirchhoff stress tensors

$$p = \frac{I_1}{3} = \frac{\sigma_{kk}}{3}$$

p_e : equivalent preconsolidation pressure
 $q, q', \tilde{q}$: deviatoric invariant associated to Cauchy stress, Cauchy effective stress and 2nd Piola-Kirchhoff stress tensors

$$q = (3 J_2)^{1/2} = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2}$$

r : Generalized Burland model coefficient, reduction factor
 $s, \tilde{s}$: shape functions
 s, s : deviatoric stress components of the Cauchy and 2nd Piola-Kirchhoff stress tensors
 t : time
 $\tilde{t}$: traction force vector
 u : internal specific energy
 $\tilde{u}$: displacement function
 $\bar{v}, \bar{v}_0$: current and initial specific volumes

$\bar{v}_\chi, \bar{v}_\lambda$: specific volume at $p' = 1$. KPa on the swelling and virgin lines
 $\underline{v}, \underline{v}_r, \underline{v}_s, \underline{v}_f$: current, relative, solid and fluid velocity functions
 $\underline{v}^*$: 'smoothed' nodal velocity
 $\underline{v}$: equivalent nodal force in viscoplasticity
 w : internal specific work
 x_v : geometrical height
 $\underline{x}$: spatial coordinate

Upper case Latin letters:

A	: plastic modulus
B	: bulk modulus, B-matrix
BE	: Big Element
B_n	: B-matrix for large displacement
BS	: Bounding Surface
C	: coupling matrix
C^α	: constitutive matrix for state of the material, where α stands for elastic, plastic, elasto-plastic or visco-plastic
CS	: Critical State
CPU	: Central Processing Units
D	: D-matrix in TERCON, damping matrix
DBE	: Dynamic Big Element
E	: Young's modulus
F	: deformation gradient, potential function
FE	: Finite Element
FEM	: Finite Element Method
G	: shear modulus, yield function
H_n	: viscoplastic stiffness submatrix
$I_1, \tilde{I}_1$	: 1st stress invariant of the Cauchy and 2nd Piola-Kirchhoff stress tensors respectively
	$I_1 = \sigma_{kk}$
$J_2, \tilde{J}_2$	: 2nd stress invariants of the deviatoric stress associated to Cauchy and 2nd Piola-Kirchhoff stress tensors respectively

$$J_2 = \frac{1}{2} s_{ij} s_{ij}$$

$J_3, \tilde{J}_3$	: 3rd stress invariant of the deviatoric stress associated to Cauchy and 2nd Piola-Kirchhoff stress tensors respectively ($J_3 = \det (\sigma_{ij})$)
K	: permeability tensor
K^p	: normalized plastic modulus
K^{ep}	: stiffness matrix of the solid phase for state of the material, where a stands for elastic, plastic, elasto-plastic or visco-plastic
$L, \tilde{L}$	: small and finite Lagrange strain tensor
M	: slope of the linear yield (failure) function in p vs. q diagram, smoothing matrix
M_1, M_2	: slopes in the Generalized Burland model, smoothing submatrices
N	: q-intercept of linear yield (failure) function in p vs. q diagram
NCS	: Normally Consolidated Soil
OCS	: Over-consolidated soil
P	: permeability matrix
$\bar{P}$	: stiffness matrix in TERCON
Q	: failure surface
S	: shape matrix
$T, \tilde{T}$	: Cauchy and 2nd Piola-Kirchhoff stress tensors
V, $\tilde{V}$	: current and initial volume
V.	: volume of solid
W	: elastic potential function
$\tilde{X}$	: material coordinates

Greek letters:

- α, β : coordinates of the centre of the Generalized Burland model
- γ_f, γ_s : unit weight of the fluid and of solid respectively
- δ : space diagonal vector
- δ_{ij} : Kronecker delta
- $\tilde{\epsilon}^\alpha, \tilde{\tilde{\epsilon}}^\alpha$: homologous vectors to small and finite Lagrange strain tensors
- $\epsilon_v^\alpha, \tilde{\epsilon}_v^\alpha$: trace of the small and finite Lagrange strain tensor
- $\epsilon_s^\alpha, \tilde{\epsilon}_s^\alpha$: deviatoric invariant associated to the small and finite Lagrange strain tensors
- superscript α stands for elastic (e), plastic (p), elasto-plastic (ep) or viscoplastic (vp)
- $\eta = q/p$: ratio of deviatoric invariant to mean pressure
- $\theta, \tilde{\theta}$: stress invariants, similar to Lode's angle, associated to Cauchy and 2nd Piola-Kirchhoff stress tensors
- $$\theta = \frac{1}{3} \arcsin \left(- \frac{3\sqrt{3}}{2} \frac{J_3}{\tilde{J}_2^{3/2}} \right)$$
- θ : parameter in θ -Wilson time scheme
- λ : Lamé's constant, slope of virgin line in v vs. $\ln p$ space, experimental coefficient in Prevost's model
- μ : distance between the current stress point and its homotetic point on the bounding surface
- ν : Poisson's ratio
- ξ, ξ_c : centroid of the yield and bounding surfaces

ξ, η : local coordinates
 τ, τ^0, τ^e : current, initial and excess pore pressure functions
 τ, τ^0, τ^e : current, initial and excess nodal pore pressures at nodal points
 ρ, ρ_0, ρ_f : current, initial and fluid densities
 $\sigma_{ij}, \sigma'_{ij}, \sigma_{ij}$: total, effective Cauchy and 2nd Piola-Kirchhoff stress tensor components
 $\underline{g}, \underline{g}', \underline{g}$: homologous vectors to the total, effective Cauchy and 2nd Piola-Kirchhoff stress tensors respectively
 ϕ : friction angle
 χ : slope of the swelling line in v vs. $\ln p'$ space
 $\underline{\psi}$: residual force
 ϕ : function
 Ω, Ω_0 : current and initial volumes
 Γ : surface

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Chapter I

CONTINUUM MECHANICS THEORY APPLIED TO GEOTECHNICAL PROBLEMS

1.1 INTRODUCTION

Depending on the scale of investigation, soil can be either considered as a discrete or as a continuum medium.

In general, the scale of investigation of a practical geotechnical engineering problem is so large with reference to a single soil grain, that it is possible to consider the whole soil mass to behave as a continuum medium.

Under the hypothesis of continuity, the geotechnical analysis can be performed, as any other engineering problem involving solid materials, in the frame of the well established Continuum Mechanics Theory.

The only peculiarity arises when soil is saturated with water, or any other fluid. In this case, modern theories try to define an equivalent continuum medium in which, at any infinitesimal material point, both solid and fluid phases are defined.

Following this approach, it is possible to take advantage of the whole Continuum Mechanics Theory, just by extending some of its aspects, in order to account for the interaction between solid and fluid phases.

The fundamental governing field equations which a continuum medium has to respect, in static or dynamic equilibrium with the external applied load, are presented in Section 1.2. The extension of the classical Continuum Mechanics Theory from the monophase to the biphasic medium is summarized in Section 1.3 while, in Section 1.4, some effects of the fluid phase on the solid behaviour is illustrated. With respect to the available consolidation theories, it is proved that the Uncoupled theory does not necessarily require linear elasticity, nor small deformation hypotheses. Finally in Section 1.5, a brief review of the analytical solutions available nowadays for the stress analysis of a saturated medium is reported.

1.2 GOVERNING FIELD EQUATIONS IN A CONTINUOUS SOLID MEDIUM

The Continuum Mechanics theory establishes that a continuum medium, in static or dynamic equilibrium with the applied external forces, should respect a number of field equations.

In Galileo's reference frame it is assumed that mass does not change with time, Eq. 1.1, from which Euler (1757) eventually derived the well known equation of continuity of mass, Eq. 1.2.

$$\frac{dm}{dt} = 0 \quad (1.1)$$

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \operatorname{div} \underline{v} \quad (1.2)$$

where m , ρ and $\underline{v}$ are respectively the mass, the density and the velocity of a material point in a continuum medium.

From Newton's third law of action and reaction, the fundamental laws of equilibrium can be formulated. They can be expressed in the form of Cauchy's equation of motion, (Cauchy 1827), or in the form of the Principle of virtual work as established by Castigliano (1879) and later extended by Signorini (1933).

To be rigorous, Cauchy's equation of motion and the Principle of virtual work should be defined in the reference state, as indicated in Eqs. 1.3 and 1.4 respectively (see for instance Malvern, 1965).

$$\underline{v} \cdot [\underline{\tilde{T}} \cdot \underline{F}^T] + \rho_0 \underline{b}_0 = \rho_0 \frac{d^2 \underline{x}}{dt^2} \quad (1.3)$$

$$\int_{\Omega_0} \underline{\tilde{T}} : \delta \underline{\tilde{L}} \, d\Omega_0 = \int_{\Gamma} \underline{\tilde{t}} \cdot \delta \underline{u} \, d\Gamma - \int_{\Omega} (\underline{\tilde{u}} - \underline{b}) \cdot \delta \underline{u} \, d\Omega \quad (1.4)$$

where ρ_0 and $\underline{b}_0$ are respectively the density and the body force in the initial reference frame, while the tensors $\underline{\tilde{T}}$, $\underline{\tilde{L}}$ and $\underline{F}$ are the 2nd Piola-Kirchhoff stress tensor, the finite

Lagrange strain tensor and the deformation gradient; $\underline{u}$ is instead the displacement vector and $\underline{t}$ is the traction force. Finally, Ω_0 , Ω and Γ are respectively the volume in the original reference frame, the volume and the surface in the current reference frame.

To characterize the material behaviour, it is necessary to define a "Constitutive equation" which establishes the relationship between the applied forces and the induced displacements.

One of the most general forms of Constitutive equation, Eq. 1.5, has been proposed by Truesdell and Noll (1965), but specific forms for describing real materials are still object of intensive studies. Due to its importance and the tremendous development which has taken place in the recent years, Chapter 2 is entirely devoted to some specific forms of constitutive equations.

$$\tilde{T} = f(\tilde{L}) \quad (1.5)$$

It is usually assumed that the material reaches a state of failure whenever the level of the applied stress exceeds a threshold value, which can be described generally as

$$Q(T) = 0 \quad (1.6)$$

where T is the Cauchy stress tensor.

There is no universally accepted definition of failure state. Referring to a material sample subjected to a loading test, a crude definition of failure might be stated as the level of stress for which the sample undergoes intollerable deformations. Roscoe et al. (1958) postulated instead that "a solid material (i.e. soil) at failure may continue to distort, but without further change of specific volume or of (effective) stress q and p ". Such a failure state they also defined as "Critical State Condition".

A vast experimental evidence indicates that a specific form of Eq. 1.6 for most of engineering materials may be expressed as in Eq. 1.7.

$$Q(T) = q + Mp - N \quad (1.7)$$

where M and N are functions of the material parameters c and ϕ , respectively called cohesion and friction angle; q is the deviatoric stress invariant and p the mean pressure of the Cauchy stress tensor.

Several relationships of M and N with c and ϕ have been proposed, Table 1.1 and Figs. 1.1, and their choice is material dependent. Soils at failure apparently follow the Mohr-Coulomb criterion (Parry, 1956 and 1960, Bishop, 1966).

A complete solution of a solid material problem should be able to identify at each point the stress and strain mobilized under the application of a set of external forces, respecting the boundary conditions and the failure criterion.

This may be achieved by solving the system of 15 equations represented by Eqs. 1.3, 1.5 and 1.8, with the 15 unknowns $\tilde{T}$, $\tilde{L}$ and u .

$$\tilde{l}_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \quad (1.8)$$

where $\tilde{l}_{ij}$ are the finite Lagrange strain components.

Unfortunately, because of the mathematical complexity, closed form solutions are rather limited and in most cases restricted to the so-called "Linearized Theory of Elasticity".

In the Linearized Theory of Elasticity the displacements are assumed "sufficiently" small, so that Eqs. 1.3 and 1.8 can be rewritten respectively as Eqs. 1.9 and 1.11 in Table 1.2. Moreover, it is assumed that stress and strain are linearly related, Eq. 1.10 in Table 1.2, and possible states of failure are ignored.

TABLE 1.1

M and N coefficients corresponding to different criteria

(Eq. 1.7)

Type	N	M
Tresca	$\frac{3c}{\sqrt{3} \cos \theta}$	0
von Mises	2c	0
Mohr-Coulomb	$\frac{3c \cos \phi}{\sqrt{3} \cos \theta - \sin \theta \sin \phi}$	$\frac{3 \sin \phi}{\sqrt{3} \cos \theta - \sin \theta \sin \phi}$
Drucker-Prager	$\frac{6c \cos \phi}{3 - \sin \phi}$	$\frac{6 \sin \phi}{3 - \sin \phi}$

TABLE 1.2

Field equations of linearized elasticity (in material coordinates)

$$3 \text{ eqs. of Motion } \quad \frac{\partial \sigma_{ij}}{\partial X_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (1.9)$$

$$6 \text{ Hooke's Law eqs. } \quad \sigma_{ij} = \lambda l_{kk} \delta_{ij} + 2\mu l_{ij} \quad (1.10)$$

$$6 \text{ Geometric eqs. } \quad l_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \quad (1.11)$$

where σ_{ij} and l_{ij} are the components of the Cauchy stress tensor and of the small Lagrange strain tensor; X_j is instead the material coordinate.

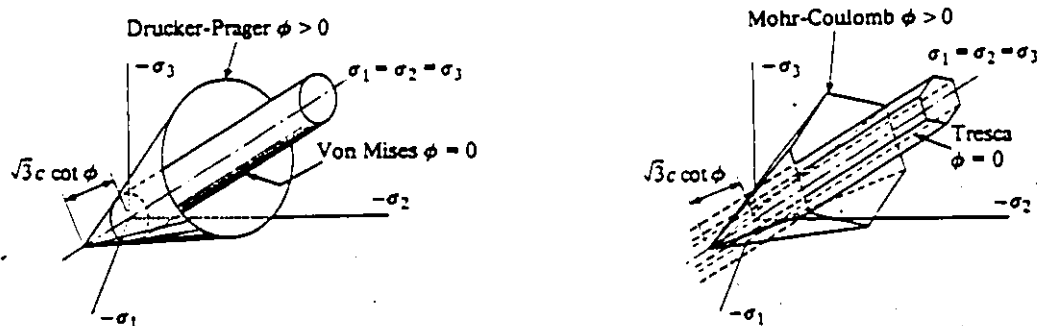


Fig. 1.1a Linear yield surfaces in principal stress space

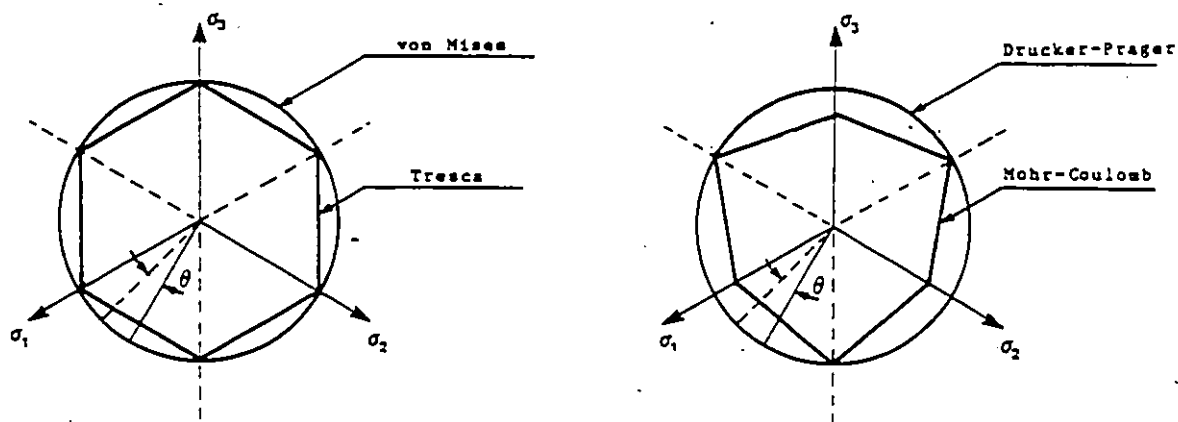


Fig. 1.1b Cross section of a linear yield function onto the octahedral plane

1.3 FUNDAMENTAL LAWS FOR A POROUS SATURATED MATERIAL (BIPHASE MEDIUM)

In case of saturated soils, it is assumed that the two phases, solid and fluid, act in a medium which is continuous for both of them.

Under this hypothesis, all mathematical functions and their derivatives entering the theory are continuous and, as a result, they can be defined at each geometrical point.

In a saturated medium, because of the fluid component, only part of the "total" stress σ_{ij} , in equilibrium with the applied external forces, is carried by the solid phase.

Since fluid cannot support shear, its presence affects only the normal stress component. In mathematical form, this concept can be expressed as in Eq. 1.12, which represents Terzaghi's "Principle of Effective Stress" (after Terzaghi, 1923).

$$\sigma_{ij} = \sigma'_{ij} - \delta_{ij}\pi \quad (1.12)$$

where σ'_{ij} is the effective stress acting on the solid component, π the hydrostatic pressure due to the fluid phase and δ_{ij} is the Kronecker delta. The negative sign has been introduced since π acts in compression.

Always under the hypothesis of continuity of the two phases, the mass of the fluid and of the solid can be defined at any point and their sum represents the global mass of the biphasic medium, Eq. 1.13a.

The equation of conservation of mass can be therefore formulated as in Eq. 1.13b, from which Green and Naghdi (1965) were able to derive the continuity of equation for a biphasic medium, Eq. 1.14.

$$m = m_s + m_f \quad (1.13a)$$

$$\frac{dm}{dt} = \frac{dm_s}{dt} + \frac{dm_f}{dt} = 0 \quad (1.13b)$$

$$\text{div } \underline{v}_r = - \frac{1}{\rho} \frac{d\rho}{dt} \quad (1.14)$$

where m and ρ are the global mass and density of the the biphasic medium respectively, while m_s is the solid mass and m_f the fluid mass.

Green and Naghdi were also able to prove that the relative velocity $\underline{v}_r$, defined as the difference between the velocity of the solid phase $\underline{v}_s$ and that of the fluid phase $\underline{v}_f$, may be expressed as follows:

$$\underline{v}_r = \underline{v}_s - \underline{v}_f = \frac{K}{\gamma_f} \{ \rho_f (\ddot{u} - \ddot{b}) - \nabla_x \pi \} \quad (1.15)$$

where K , the permeability matrix, is a symmetric tensor with 6 independent coefficients while $\ddot{\underline{u}}_f$ and γ_f are the acceleration and the unit weight of the fluid phase respectively.

To prove Eq. 1.15, the aforesaid authors respected Noll's Principle of Material Objectivity (Noll, 1958), Truesdell's Principle of Equipresence (Truesdell, 1951, Truesdell and Toupin, 1960) and the thermodynamic laws.

It is to be noticed that Eq. 1.15 may represent the generalized form of the well known d'Arcy's Law of Permeability, (after d'Arcy, 1856). In fact, neglecting acceleration effects, $\ddot{\underline{u}}_f = 0$, and assuming $\underline{v}_s \ll \underline{v}_f$, Eq. 1.15 reduces to :

$$\underline{v}_f = - \underline{v}_T = - K \text{ grad } \left\{ \frac{1}{g} \underline{b} \cdot \underline{x} + \frac{\pi}{\gamma_f} \right\} = -K \text{ grad } h \quad (1.16)$$

which represents d'Arcy's equation of fluid flow in a porous material, h being a scalar potential function also known as total head.

Under the above conditions and adding the hypothesis of fluid incompressibility, the continuity equation, Eq. 1.14, may be further reduced to

$$\text{div } K \text{ grad } h = \frac{1}{dV_s} \frac{d}{dt} (dV_s) \quad (1.17)$$

since

$$\text{div } \underline{v}_r = \text{div } \underline{v}_s = \frac{1}{dV_s} \frac{d}{dt} (dV_s) \quad (1.18)$$

It has to be observed that the above equation can be derived without any assumption of small displacement since, as established by the continuum mechanics theory, the variation of the density can be related directly to the variation of the actual volume.

Assuming the soil not subjected to any volumetric change, i.e. $d(dV_s) = 0$, Eq. 1.17 takes the form of Laplace's equation describing the regime of steady flow in a porous medium, namely:

$$\text{div } K \text{ grad } h_o = 0 \quad (1.19a)$$

with

$$h_o = \left\{ -\frac{1}{g} \underline{b} \cdot \underline{x} + \frac{\pi^o}{\gamma_f} \right\} \quad (1.19b)$$

where h_o and π^o are respectively the head and the hydrostatic pressure associated to a steady flow situation.

1.4 FLUID EFFECTS IN A STRESS-STRAIN ANALYSIS OF A SOLID MEDIUM

In geotechnical engineering, the inclusion of water may drastically change the safety conditions of a structure. In fact, from Eq. 1.12 the presence of fluid reduces only the mean effective stress p' leaving unaltered the distortional stress component q , Eqs. 1.20.

$$\sigma_{kk} = \sigma'_{kk} - 3\pi \quad \therefore \quad p = p' - \pi \quad (1.20a)$$

$$q = \left(\frac{3}{2} s_{ij} s_{ij} \right)^{1/2} = \left(\frac{3}{2} s'_{ij} s'_{ij} \right)^{1/2} \quad \therefore \quad q = q' \quad (1.20b)$$

Consequently, a stress point T, which in a dry situation would have been considered as safe, Figs. 1.2, may become unsafe in a saturated condition since, due to π , the effective stress violates the failure criterion.

In addition, the fluid component may delay the "exhaustion" of the displacements eventually associated with the application of the additional external forces. In fact, when load is imposed, part of the mean component of the mobilized total stress is equilibrated by the fluid phase. The pressure π may therefore be expressed as the sum of the

initial pressure π^0 and of an induced "excess pore pressure" π^e , Eq. 1.21.

$$\pi = \pi^0 + \pi^e \quad (1.21)$$

With time, π^e tends to dissipate, thereby transferring its contribution to equilibrium to the effective stress, so that additional displacements may take place.

During this physical phenomenon known as the consolidation process, the continuity of mass, Eq. 1.17, may be reduced as follows

$$\text{div } K \text{ grad} \left(\frac{\pi^e}{\gamma_f} \right) = \frac{1}{dV_s} \frac{d}{dt} (dV_s) \quad (1.22)$$

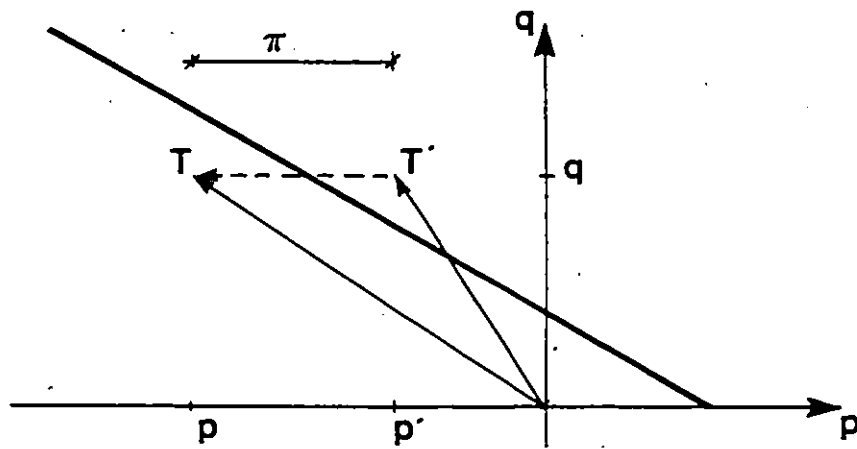


Fig. 1.2 Destabilizing effect of pore pressure

1.5 THEORIES AND CLOSED FORM SOLUTIONS TO CALCULATE PORE PRESSURE

The calculation of pore pressure, therefore, involves two distinct analytical phases, namely:

1. the determination of the initial distribution of pore pressure associated with the preexisting regime of fluid flow, which usually is under steady condition;
2. the calculation of the excess pore pressure, induced by the application of the additional set of external forces, at each stage of the consolidation process.

The first phase requires practically the solution of Laplace's equation. For the mathematical description of the evolution of the excess pore pressure, two theories are available in geotechnical engineering, that is the Uncoupled and the Coupled Theory of Consolidation.

1.5.1 Steady Flow

The mathematical solution of a seepage problem may be obtained from Laplace's equation represented by Eq. 1.19a.

Analytical solutions can be generally achieved only for homogeneous isotropic medium with well defined and simple boundary conditions.

In a number of seepage problems in geotechnical engineering, the flow involves a free surface (unconfined flow). This implies that the boundary conditions cannot be fully defined, since the free surface is not known a priori.

For unconfined flow, the geotechnical engineer has traditionally relied on graphical methods (Casagrande, 1940, Cedergren, 1967).

In the analysis of wells in unconfined flow, groundwater hydrologists have often based their theory on Dupuit's assumptions. Exact analytical methods of handling such problems have been developed, but they are often difficult to apply.

Extensive treatments of these methods are given by Harr (1962), Polubarinova-Kochina (1962), Aravin and Numerov (1965).

1.5.2 Consolidation: Uncoupled Theory

The Uncoupled theory finds its origin in an early postulate by Rendulic (1936), according to which the total stress does not change with time.

This assumption may be generalized in mathematical form as follows:

$$\dot{\sigma}_{ij} = \dot{\sigma}'_{ij} - \delta_{ij} \dot{\pi} = 0 \quad (1.23a)$$

$$\dot{\sigma}'_{ij} = \delta_{ij} \dot{\pi}^e \quad (1.23b)$$

Usually, the Uncoupled consolidation governing equation is derived under the hypothesis of small deformation but, as shown in the following, this assumption is not strictly necessary. In fact, from Eq. 1.23b, the rate the rate of change of the excess pore pressure is directly linked to the rate of change of mean effective stress and consequently it can be related to the solid volumetric change, provided the existence of a direct relationship between the two latter variables, Eq. 1.24.

$$\dot{\pi}^e = \dot{p}' = B \left[\frac{1}{dv_s} \frac{d}{dt} (dv_s) \right] \quad (1.24)$$

where B is a bulk modulus, which is generally a function of the degree of consolidation.

Consequently, by using Eq. 1.24, Eq. 1.22 becomes a parabolic differential equation with the only field variable π^e , Eq. 1.25.

$$\text{div } K \text{ grad } \frac{\pi^e}{\gamma_f} = \frac{1}{B} \dot{\pi}^e \quad (1.25)$$

In the hypothesis of small displacement, the rate of variation of the solid volume is almost equal to the variation of the volumetric strain ϵ^v , Eq. 1.26a. Adding the

hypothesis of linear elasticity, the bulk modulus B may be calculated, depending on the type of problem under investigation, by one of the following expressions (Davis and Poulos, 1968):

$$\frac{1}{dV_s} \frac{d}{dt} dV_s = \dot{\epsilon}_v = \frac{1}{B} \dot{p}' \quad (1.26a)$$

where B for 1-D, plane strain and 3-D is respectively equal to:

$$B = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad (1.26b)$$

$$B = \frac{E}{2(1+\nu)(1-2\nu)} \quad (1.26c)$$

$$B = \frac{E}{3(1-2\nu)} \quad (1.26d)$$

Alternatively, assuming the specific volume $\bar{v}$ to be related to p' as in Eq. 1.27a (Schofield and Wroth, 1968), it follows that B is a non-linear function whose value depends on the current p' and $\bar{v}$, Eqs. 1.27.

$$\bar{v} = \bar{v}_0 - \lambda \ln (-p') \quad (1.27a)$$

$$\frac{d}{dv_s} \frac{d}{dt} (dv_s) = \frac{1}{v} \frac{d\bar{v}}{dt} = - \frac{\lambda}{\bar{v} p'} \frac{dp'}{dt} \quad (1.27b)$$

$$B = - \frac{\bar{v} p'}{\lambda} \quad (1.27c)$$

where λ is a material parameter.

It is to be noticed that in the latter derivation there is no need to assume small displacement in order to relate dv_s to p' .

The reduced form of the equation of continuity of mass, Eq. 1.25, may describe the consolidation process if the initial distribution of the field variable π^e is known.

Henkel (1960) and Henkel and Wade (1966) for instance related the initial excess pore pressure to the variation of total stress due to the application of external load by the following semi-empirical relationship

$$\pi^e = a \Delta p + b \Delta q \quad (1.28)$$

where a and b are material coefficients while Δp and Δq are the total change of mean and distortional stress respectively.

In short, the Uncoupled consolidation theory involves the following steps:

1. calculate the variation of total stress distribution due to the application of the new set of external forces, by solving for instance the system of equations in Table 1.2;
2. estimate the initial excess pore pressure by a semi-empirical expression of the type Eq. 1.28;
3. compute the evolution of the excess pore pressure by solving the parabolic differential equation 1.25;
4. eventually back-calculate the displacement at any given time step once π^e , and consequently σ'_{ij} , have been determined.

Several objections can be made to this procedure.

Firstly, one could question the validity of the assumption stated in Eqs. 1.23, which practically implies that the maximum π^e is mobilized at the time of the load application. Furthermore, in order to calculate π^e , a semi-empirical relationship is needed which may represent another weak point of the approach.

But for now let only the practical computational aspect of the Uncoupled theory be considered.

Although this theory wants to represent a simplified approach to the real consolidation problem, it turns out that its application is in general not simple.

— Closed form solutions of Eq. 1.25 are rather limited to 1-D problems, where π^e is allowed to vary only in one spatial direction. In addition, the medium is assumed to be homogeneous, isotropic, with constant permeability and B coefficient. Some examples of exact closed form solutions have been found by Taylor (1943) while simplified solutions have been proposed by Rendulic (1936) and Terzaghi (1943) always under the aforesaid hypotheses. The solution of the axisymmetric case has been instead proposed by Carslaw and Jaeger (1959).

1.5.3 Consolidation: Coupled Theory

The first attempt to analyze the consolidation problem in a unified and rigorous theoretical procedure was done by Biot (1941a).

In the original Biot's work, the solid part was assumed to obey linear elasticity, the acceleration terms were neglected and the hypothesis of small deformations was advocated. The body weight and the initial pore pressure π^e were not considered since, because of the linearity of the problem, their effect can be superimposed.

Under the above conditions and respecting the Principle of effective stress, the field equations 1.9, 1.10, and 1.11 may be rewritten as Eqs. 1.30, 1.31 and 1.32 in Table 1.3.

Biot correctly identified the use of the continuity equation in order to solve the equilibrium problem which in this case involves the additional unknown π^e .

In the hypotheses of small deformation, uniform and isotropic permeability, Eq. 1.22 can be reduced to

$$K_{ij} \pi_{,ij}^e = \dot{u}_{k,k} \quad (1.29)$$

which, by the application of the Laplace transform, may be proved to take the form of Eq. 1.33, assuming $u_{k,k} = 0$ at $t=0$.

Although the mathematical problem is completely defined, the solution of the above system of equations in Table 1.3 still remains a formidable task. As a result of that, analytical solutions are rather limited and generally confined to flexible footings on homogeneous, isotropic and linear elastic, semi-infinite half space.

Some of those solutions have been published by Biot (1941b, 1941c), Josselin de Jong (1957), Mc Namee and Gibson (1960a, 1960b), Gibson and Mc Namee (1963), and a discussion at length on the validity of this theory has been presented by Schiffman et al. (1969). Solutions for consolidation problems involving finite depth have been proposed by Gibson et al. (1970) and Booker (1973).

A major difference between the Uncoupled and the Coupled theory is the "Mandel-Cryer effect" on the excess pore pressure (Mandel, 1957, Cryer, 1963). Since the excess pore pressure is affected by changes in the mean total stress, it may, at some places within the soil mass, continue to increase for some time after the application of load. This result is in contrast to Rendulic's assumption of constant total stress, and perhaps may explain some delayed failures so often registered in the technical literature.

The existence of the Mandel-Cryer effect is now supported by vast experimental evidence. Interesting for instance is the experimental study performed by Aboshi in 1955.

TABLE 1.3

Field equations of coupled consolidation problem in the
hypothesis of linearized elasticity

3 eqs. of Motion
$$\frac{\partial \sigma'_{ij}}{\partial x_j} - \delta_{ij} \frac{\partial \pi^e}{\partial x_j} = 0 \quad (1.30)$$

6 Hook's Law eqs.
$$\sigma'_{ij} = \lambda l_{kk} \delta_{ij} + 2\mu l_{ij} \quad (1.31)$$

6 Geometrical eqs.
$$l_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.32)$$

1 Continuity eq.
$$l_{ij} * K_{ij} \pi^e_{,ji} = \dot{u}_{k,k} \quad (1.33)$$

16 equations, for 6 effective stresses, 6 strains, 3 displacements and 1 excess pore pressure.

Chapter II

CONSTITUTIVE EQUATIONS

2.1 INTRODUCTION

Three fundamental postulates are assumed to be valid for any constitutive theory of purely mechanical phenomena in a continuous medium. (See Truesdell and Noll (1965), secs. 19 and 26). The three postulates are:

1. the Principle of Determinism for Stress;
2. the Principle of Local Action;
3. the Principle of Material Frame-Indifference.

The application of these principles to the theory of elasticity leads to the conclusion that one of the most general forms of constitutive equations should respect the following form:

$$\tilde{T} = f(\tilde{L}) \quad (2.1)$$

In the hypothesis of small deformation, where $\tilde{T}$ and $\tilde{L}$ reduce respectively to the Cauchy stress tensor T and the linear strain tensor L , Eq. 2.1 assumes the more conventional form of

$$T = f(L)$$

(2.2)

The extension of the functional form of Eq. 2.1 to the case of plastic deformations apparently is not completely rigorous, since the mechanism of slips, which in this case takes place in the material, is not correctly represented (Mandel 1971, 1982). Alternative formulations have been proposed (Dafalias, 1983a and 1983b) but since they are still in process of study and debate, the validity of Eq. 2.1 will be herein assumed even in the range of plastic deformation.

In the recent years, various mathematical models of deformation have been proposed which have led to alternative constitutive equations. Although at a first reading the different models may appear quite different from each other, a more careful study leads to the identification of several common points .

Therefore, this Chapter presents a critical analysis of some of the best known elastic and plastic models and attempts to identify some common features among them.

In the following Sections, the basic assumptions lying behind the classical theory of elasticity and the corresponding geometrical interpretation for the particular case of isotropy are presented and developed. In Section 2.2.1, a possible experimental identification of the Young

and Poisson coefficients from a simple uniaxial test in the case of large deformation is also derived.

A general form of the combined elasto-plastic constitutive equation in large deformation is presented in Section 2.3. In Section 2.4 theoretical considerations on some particular forms of stress functions are developed which will enhance the analysis of the different elasto-plastic models presented in the subsequent Sections 2.5, 2.6 and 2.7. In particular, a new and exact expression of the gradient of a potential function is derived and it is proved that most of the available expressions to calculate the plastic modulus can be reduced, if correct, to only two alternative expressions.

The conclusions of this Chapter are finally drawn in Section 2.8, along with a proposal for a generalized version of the Cam-clay models.

For easier reading and compact formulation, a special tensor transformation is herein introduced, for which 2nd order symmetric tensors are transformed in vectorial form in the following way:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (2.3a)$$

$$\underline{a} = \{a_{ij}\} = \{a_{11}, a_{22}, a_{33}, \sqrt{2} a_{12}, \sqrt{2} a_{13}, \sqrt{3} a_{23}\}^T \quad (2.3b)$$

where the symbol $\{ \}$ indicates the particular nature of the vector.

In the same fashion, the gradient to a certain function, say f , in a 9 dimensional space is transformed into the homologous vector of the type 2.3b, provided the tensor associated to it is symmetric. For instance

$$\text{grad}_{\sigma} f = \left[\frac{\partial f}{\partial \sigma_{ij}} \right] \quad (2.4a)$$

is transformed into

$$\bar{\nabla}_{\sigma} f = \frac{\partial f}{\partial \sigma_{ij}} = \left\{ \frac{\partial f}{\partial \sigma_{11}}, \frac{\partial f}{\partial \sigma_{22}}, \frac{\partial f}{\partial \sigma_{33}}, \sqrt{2} \frac{\partial f}{\partial \sigma_{12}}, \right. \quad (2.4b) \\ \left. \sqrt{2} \frac{\partial f}{\partial \sigma_{13}}, \sqrt{2} \frac{\partial f}{\partial \sigma_{23}} \right\}$$

where $\bar{\nabla}$ represents the gradient vector and the subscript σ the type of space in which the gradient is defined.

It is possible to prove that following this convention, the scalar product of two second order symmetric tensors may be simply transformed into a scalar product of the two homologous vectors; for instance

$$A:B = \underline{a} \cdot \underline{b} = a_{11}b_{11} + a_{22}b_{22} + a_{33}b_{33} + 2a_{12}b_{12} \quad (2.4c) \\ + 2a_{13}b_{13} + 2a_{23}b_{23}$$

where

$$A = [a_{ij}] \quad \therefore \quad \underline{a} = \{a_{ij}\} \quad (2.4d)$$

$$B = [b_{ij}] \quad \therefore \quad \underline{b} = \{b_{ij}\} \quad (2.4e)$$

2.2 ELASTICITY

According to Truesdell and Toupin (1960) a material is called "Green Elastic" or "Hyperelastic" (Green, 1839 and 1841) if there exists an elastic potential function (or strain function), say W , scalar function of the finite Lagrange strain tensor ϵ , whose derivative with respect to a strain determines the corresponding 2nd Piola-Kirchhoff stress component σ . Mathematically this can be expressed as

$$\underline{\tilde{\sigma}} = \overline{\underline{\tilde{v}}}_{\epsilon} W \quad (2.5)$$

This requirement is completely satisfied in the following two fully recoverable thermodynamic processes:

1. Adiabatic and isentropic process where $W = \rho u = \rho w$, being u and w the internal specific energy and the internal specific work respectively;

2. Isothermal deformation with reversible heat conduction where $W = \rho \psi$, being ψ Helmutz's free energy.

When the potential function W is chosen to be a quadratic function of $\tilde{\epsilon}$ (with the condition that W vanishes in the unstrained case), Eq. 2.6a, the material is called "Linear Elastic", since the relationship between stress and strain becomes linear as in Eq. 2.6b.

$$W = \frac{1}{2} \tilde{\epsilon}^T C^e \tilde{\epsilon} \quad (2.6a)$$

$$\tilde{\sigma} = \frac{\partial W}{\partial \tilde{\epsilon}} = C^e \tilde{\epsilon} \quad (2.6b)$$

where C^e is a symmetric tensor containing 21 independent coefficients.

Hence in the current case, due to the linear nature of Eq. 2.6b, W can be also expressed as a strain potential function, that is:

$$W = \frac{1}{2} \tilde{\sigma}^T (C^e)^{-1} \tilde{\sigma} \quad (2.7a)$$

$$\tilde{\varepsilon} = \tilde{\nu}_\sigma W = (C^e)^{-1} \tilde{\sigma} \quad (2.7b)$$

Moreover in the case of adiabatic and isentropic thermodynamic process, the following identity can be proved (appendix A)

$$W = \rho_0 w = \frac{1}{2} \tilde{\sigma} \cdot \tilde{\varepsilon} = \frac{1}{2} (\tilde{p} \tilde{\varepsilon}_v + \tilde{s} \cdot \tilde{e}) \quad (2.7c)$$

where $\tilde{p}$ and $\tilde{s}$ are the trace and the deviatoric component of the 2nd Piola-Kirchhoff stress tensor respectively, while $\tilde{\varepsilon}_v$ and $\tilde{e}$ the trace and the deviatoric component of the finite Lagrange strain tensor.

For "Isotropic Linear Elastic" material, it is well known that elastic matrix C contains only two independent coefficients and the Eq. 2.6b may be expressed as

$$\tilde{\sigma}_{ij} = \frac{E}{\nu+1} (\tilde{\varepsilon}_{ij} + \frac{\nu}{1-2\nu} \tilde{\delta}_{ij} \tilde{\varepsilon}_v) \quad (2.8)$$

In this particular case it is known that the directions of the deviatoric stress $\tilde{s}$ and of the deviatoric strain $\tilde{e}$ coincide, from which it is possible to prove the following interesting relationships (appendix B):

$$W = p_0 \tilde{w} = \frac{1}{2} \tilde{g} \cdot \tilde{\epsilon} = \frac{1}{2} (\tilde{p} \tilde{\epsilon}_v + \tilde{q} \tilde{\epsilon}_s) = \frac{\tilde{p}^2}{2B} + \frac{\tilde{q}^2}{6G} \quad (2.9a)$$

where

$$\tilde{\epsilon}_s = \left(\frac{2}{3} \tilde{e} \cdot \tilde{e} \right)^{1/2} \quad (2.9b)$$

$$B = \frac{E}{3(1-2\nu)} \quad ; \quad G = \frac{E}{2(1+\nu)} \quad (2.9c)$$

it follows

$$\tilde{\epsilon}_v = \frac{\partial W}{\partial \tilde{p}} = \frac{\tilde{p}}{B} \quad (2.10a)$$

$$\tilde{\epsilon}_s = \frac{\partial W}{\partial \tilde{q}} = \frac{\tilde{q}}{3G} \quad (2.10b)$$

From the above relationships, it is possible to deduce that, for isotropic materials, the following mathematical properties hold:

1. the potential function can be alternatively expressed by the scalar product of the 1st and 2nd stress and strain invariants;
2. the potential function may also be mapped onto the stress invariant p vs. q diagram;
3. the deviatoric strain invariant $\tilde{\epsilon}_s$ can be also calculated as the derivative of the potential function with respect to the deviatoric stress invariant $\tilde{q}$.

2.2.1 Experimental Determination of Elastic Constants for Isotropic Materials

For an isotropic linear elastic material the constant values E and ν may be determined from a conventional triaxial test.

In fact, it is possible to prove (appendix C) that in conventional triaxial test the 2nd Piola-Kirchhoff stress $\tilde{\sigma}$ and the finite Lagrange strain $\tilde{\epsilon}$ are simply related to the measurable stress and strain values by the set of Eqs. 2.11.

$$\begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_3 \end{Bmatrix} = \begin{Bmatrix} \frac{\nu}{\nu_0(2\tilde{\epsilon}_1+1)} \sigma_1 \\ (2\tilde{\epsilon}_1 + 1)^{1/2} \sigma_3 \end{Bmatrix} ; \quad \begin{Bmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_3 \end{Bmatrix} = \begin{Bmatrix} \epsilon_1 + \frac{\epsilon_1^2}{2} \\ \epsilon_3 + \frac{\epsilon_3^2}{2} \end{Bmatrix} \quad (2.11)$$

where $\epsilon_1 = \frac{ds_1 - dS_1}{dS_1}$

and dS_i is the initial length of the sample in the i -direction, ds_i the length of the deformed sample in i -direction while σ_i is the applied load.

Therefore in a uniaxial test condition, the values of E and ν can be calculated from the following expressions:

$$\nu = - \frac{\tilde{\epsilon}_3}{\tilde{\epsilon}_1} = - \frac{(\epsilon_3 + \frac{\epsilon_3^2}{2})}{(\epsilon_1 + \frac{\epsilon_1^2}{2})} \quad (2.12a)$$

$$E = \frac{\tilde{\sigma}_1}{\tilde{\epsilon}_1} = \frac{\sigma_1}{\epsilon_1 [\frac{V_0}{V} (1 + \frac{5}{2} \epsilon_1 + 2\epsilon_1^2 + \frac{\epsilon_1^3}{2})]} \quad (2.12b)$$

Thus, Eq. 2.12b establishes a linear relationship only between $\tilde{\sigma}_1$ and $\tilde{\epsilon}_1$, while σ_1 and ϵ_1 are related by a fourth order expression.

In the assumption of small deformation, all the relationships found in this Section are still valid. The only difference lies in the substitution of the 2nd Piola-Kirchhoff stress tensor and the total Lagrange strain tensor by the more familiar Cauchy stress tensor and linear Lagrange strain tensor, respectively.

In particular, because of the hypothesis of small displacement, Eqs. 2-12 may be simplified into Eqs. 2.13. Furthermore, the volume change, which coincides in this case

with the volumetric strain, is linearly related to mean pressure through the bulk modulus B , as Eqs. 2.14 show.

$$v = - \frac{\epsilon_3}{\epsilon_1}, \quad E = \frac{\sigma_1}{\epsilon_1} \quad (2.13)$$

$$\frac{dV - dV_0}{dV_0} = \epsilon_v = \frac{p}{B} \quad (2.14a)$$

where

$$B = \frac{E}{3(1-2v)} \quad (2.14b)$$

Roscoe et al. (1958), extending some earlier suggestions by Terzaghi, apparently found experimentally that the elastic (recoverable) specific volume in soils follows a logarithmic variation with the mean pressure, Eq. 2.15.

Since the rate of change of the specific volume is equal to that of the entire volume, and the latter to the rate of change of volumetric strain in the hypothesis of small deformation, it results that the bulk modulus is not linear, Eq. 2.16.

The type of relationship between p and ϵ^v is clearly shown in Eq. 2.18, obtained by integrating Eq. 2.16, and represented graphically in Figs. 2.1.

$$\bar{v} = \bar{v}_x - \chi \ln(-p) \quad (2.15)$$

$$d\epsilon_v = \frac{1}{dV} d(dV) = \frac{d\bar{v}}{\bar{v}} = - \frac{\chi}{\bar{v}p} dp \quad (2.16)$$

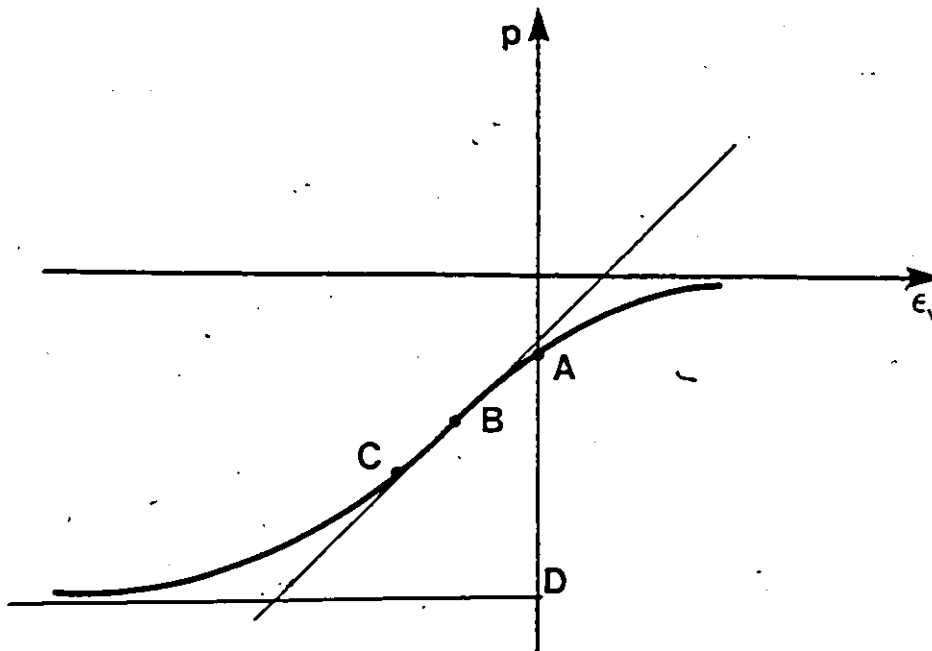
$$B = - \frac{\bar{v} p}{\chi} \quad (2.17a)$$

$$E = - \frac{3(1-2\nu)}{\chi} \bar{p} \bar{v} \quad (2.17b)$$

$$p = - \exp\left\{\frac{\bar{v}}{\chi} [1 - \exp(\epsilon_v - \epsilon_x)]\right\} \quad (2.18)$$

where x is a material parameter while $\bar{v}_x$ and ϵ_x are respectively the specific volume and the volumetric strain for $p = 1$ KPa.

The relation between Young's modulus and x parameter, as in Eq. 2.17b, can be found comparing Eq. 2.17a with Eq. 2.14b. The aforesaid authors apparently assume Poisson's coefficient constant, therefore attributing the non-linear elastic nature of soil only to Young's modulus.



Point	ϵ_v	p
A	0	$-\exp \frac{\bar{v}_y}{x} [1 - \exp(-c_x)]$
B	$c_x \ln \left(\frac{\bar{v}_y}{\bar{v}_x} \right)$	$-\exp \left[\frac{\bar{v}_y}{x} - 1 \right]$
C	c_x	-1
D	0	$-\exp \frac{\bar{v}_y}{x}$

Fig. 2.1 Mean pressure vs. volumetric strain as predicted by Eq. 2.18

2.2.2 The Geometrical Nature of Elastic Potential Functions

The geometrical nature of W can be exactly identified in the linear and isotropic case, (appendix D). There in fact, W , Eq. 2.9, represents an ellipsoid in the principal stress space, Figs. 2.2.

The canonical form of this surface is presented in Eq. 2.19, in which $\bar{\sigma}$, coincides with the spatial diagonal in the principal stress space.

$$\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \bar{\sigma}_3^2 (\nu\sqrt{3} + 1) = 2 E W \quad (2.19)$$

Alternatively, the same surface may be mapped onto the $\tilde{p}$ vs. $\tilde{q}$ diagram, Eqs. 2.20 and Figs. 2.3, where it represents in general an ellipse or, in case $\nu = 0.5$, two parallel lines.

$$\frac{\tilde{p}^2}{a^2} + \frac{\tilde{q}^2}{b^2} = 1 \quad (2.20a)$$

where

$$a^2 = \frac{2 E W}{3(1-2\nu)} = 2 B W \quad (2.20b)$$

$$b^2 = \frac{3 E W}{1 + \nu} = 6 G W \quad (2.20c)$$

In the hypothesis that E varies according to Eq. 2.17a, it can be immediately proved that the potential elastic function W takes the form of Eq. 2.21a which represents an ellipse passing through the origin of p vs. q diagram, Figs. 2.4.



where

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = p \quad (2.21a)$$

$$\frac{a^2}{b^2} = \frac{2 \bar{\nu} W}{3 \chi} \quad (2.21b)$$

$$\frac{a^2}{b^2} = 9 \left(\frac{1-2\nu}{1+\nu} \right) \frac{\bar{\nu} W}{\chi} \quad (2.21c)$$

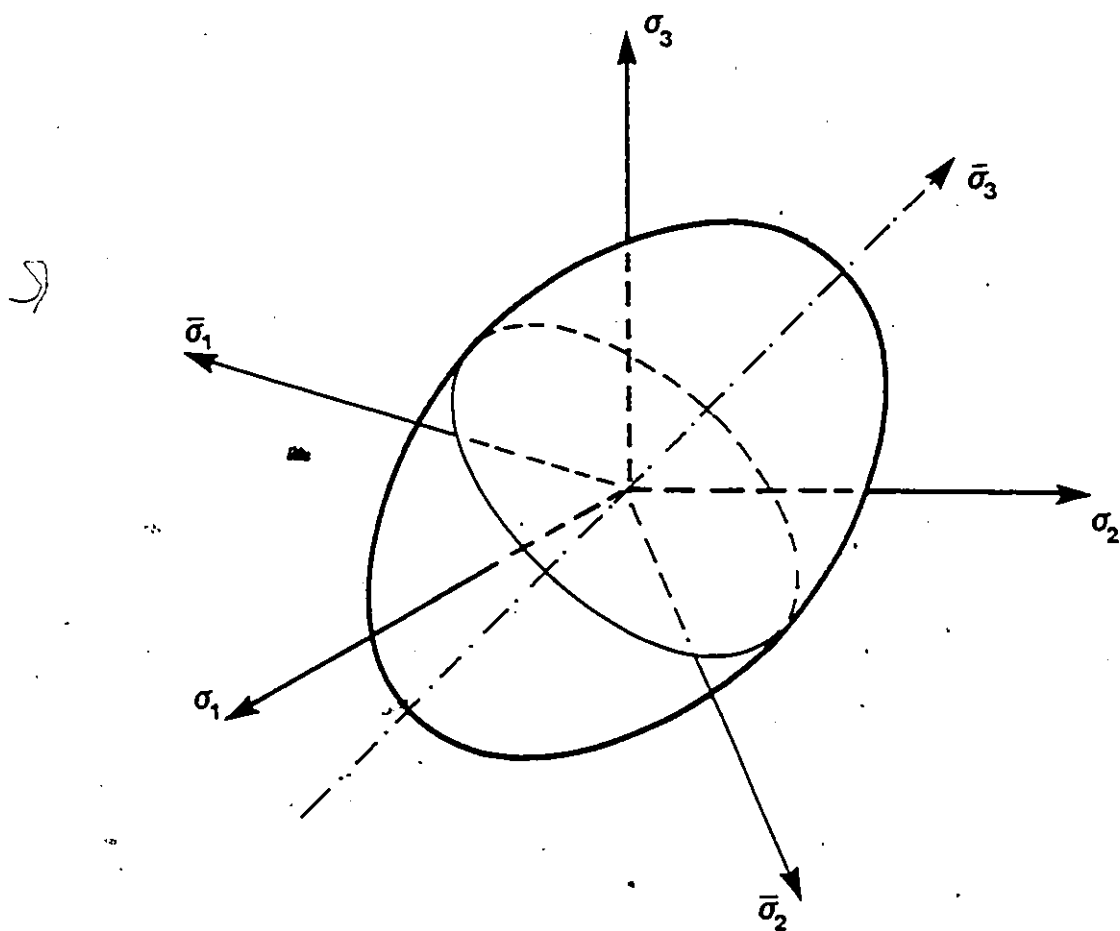


Fig. 2.2 Potential function for isotropic and linear elastic material (eq. 2.19)

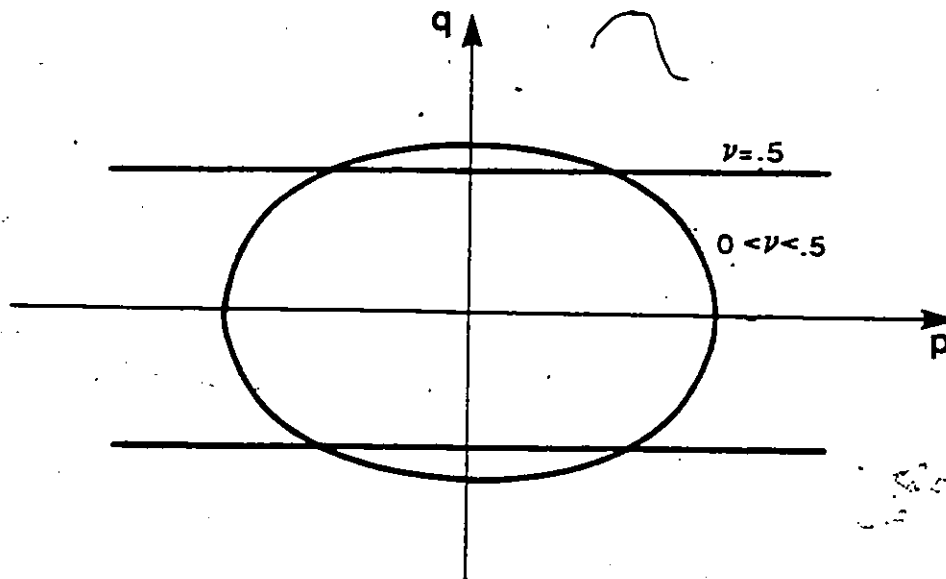


Fig. 2.3 Mapping in p vs. q diagram of a linear elastic and isotropic potential function (eq. 2.20)

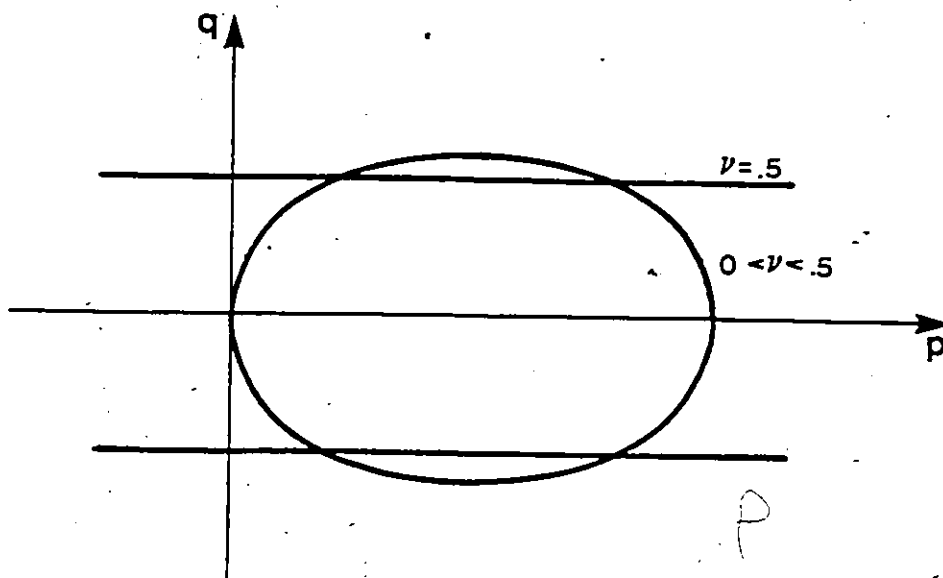


Fig. 2.4 Mapping in p vs. q diagram of a non-linear elastic and isotropic potential function (eq. 2.21)

2.3 PLASTICITY

The previous Section has dealt with the constitutive equation related to a particular process, characterized by a complete recovery to the undeformed configuration upon removal of the applied load.

It is instead known that, in a real material subjected to a cyclic loading process, not the whole deformation is recoverable.

Irreversible deformation, which results from mechanism of slip or from dislocation at the atomic level, and which thereby leads to permanent dimensional changes, is known as "Plastic Deformation".

Such deformation occurs only at the stress intensities, above a certain threshold value, known as "Elastic Limit" or "Yield Stress", determined on a so called "Yield Function" $G(\sigma, k)$.

After the initial yielding, the stress level at which further plastic deformation occurs, is usually assumed to depend on the degree of some hardening parameter k .

Because plastic strain depends upon the entire loading history of the material, plastic stress-strain relationships are very often given in terms of strain increment, the so called "Incremental Theory".

The aforesaid theory assumes the existence of a potential function F whose gradient, in the stress space, determines the direction of the plastic deformation.

The magnitude of the plastic strain is controlled by a scalar parameter $\bar{d}l$ directly related to the total differential of the yield function dG_0 with respect to the stress variables, such that only loading processes may mobilize irreversible deformations.

The first mathematical description of such theory was apparently formulated by Melan (1938) for the case of small deformations.

In order to give more generality to Melan's formulation, it is necessary to respect the Principle of Frame Indifference. Therefore, in a first attempt to extended the definition to the case of total displacement, Melan's formulation is herein proposed in term of Lagrange total strain and 2nd Piola-Kirchhoff stress tensor, as follows:

$$d\tilde{\epsilon}^p = \bar{d}l \bar{V}_{\tilde{\sigma}} F \quad (2.22a)$$

where

$$\bar{d}l = \langle l \, dG_{\tilde{\sigma}} \rangle = \left\langle \frac{dG_{\tilde{\sigma}}}{A} \right\rangle \quad \text{if } dG_{\tilde{\sigma}} > 0 \quad (2.22b)$$

$$\bar{d}l = 0 \quad \text{if } dG_{\tilde{\sigma}} < 0 \quad (2.22c)$$

$$d\tilde{G}_G = \tilde{V}_G G \cdot d\tilde{G} \quad (2.22d)$$

$$A = \frac{1}{2} \quad (2.22e)$$

It is important to notice at this point the close similarity between Eq. 2.22a and Eq. 2.7b which defines the elastic strain.

From the general definition of Eq. 2.22a, after some mathematical manipulation, the value of the plastic deviatoric strain $d\tilde{e}^p$ and of the two plastic strain invariants, $d\tilde{\epsilon}_v^p$ and $d\tilde{\epsilon}_s^p$, may be proved to be equal to the following equations:

$$d\tilde{e}^p = d\tilde{\epsilon}^p - \frac{\delta}{3} d\tilde{\epsilon}_v^p = \overline{dl} \tilde{V}_s F \quad (2.23)$$

$$d\tilde{\epsilon}_v^p = d\tilde{\epsilon}_{kk}^p = \overline{dl} \frac{\partial F}{\partial p} \quad (2.24)$$

$$d\tilde{\epsilon}_s^p = \left(\frac{2}{3} d\tilde{e}^p \cdot d\tilde{e}^p \right)^{1/2} = \frac{2}{3} \overline{dl} |\tilde{V}_s F| \quad (2.25)$$

where the value of $\bar{\nabla}_{\sim} F$ is presented in Table 2.1, and its geometrical interpretation discussed in the next Section.

It is of some interest to observe that if F is an isotropic function, i.e. F function only of the p and q (see Section 2.4), the strain invariant $d\tilde{\epsilon}_s^p$ may be related directly to the $\tilde{q}$ -stress invariant since Eq. 2.25 may be reduced as

$$d\tilde{\epsilon}_s^p = \overline{dl} \frac{\partial F}{\partial \tilde{q}} \quad (2.26)$$

In the technical literature, a few alternative reduced forms of Eq. 2.22a can be found. For instance the plastic strain is sometimes expressed in terms of the unit normal to F and G . Therefore, in the current case of large deformation Eq. 2.22a can be rewritten as follows:

$$d\tilde{\epsilon}^p = \overline{dl} \bar{\nabla}_{\sim} F = \frac{\hat{n}_G \cdot d\tilde{\sigma}}{K^p} n_F \quad (2.27a)$$

where

$$\hat{n}_G = \frac{\bar{\nabla}_{\sim} G}{|\bar{\nabla}_{\sim} G|}, \quad \hat{n}_F = \frac{\bar{\nabla}_{\sim} F}{|\bar{\nabla}_{\sim} F|} \quad (2.27b)$$

$$\frac{1}{K^P} = \frac{|\bar{\nabla}_{\sigma} \tilde{G}| |\bar{\nabla}_{\sigma} \tilde{F}|}{A} \quad (2.27c)$$

where the scalar value K^P is also known as "normalized plastic modulus".

In the common, but nevertheless strong, assumption that the combined elasto-plastic strain can be linearly split into the elastic and plastic strain, Eq. 2.28, the complete elasto-plastic constitutive equation may be proved to take the form of Eq. 2.29.

$$d\tilde{\epsilon} = d\tilde{\epsilon}^e + d\tilde{\epsilon}^p \quad (2.28)$$

$$d\tilde{\sigma} = c^{ep} d\tilde{\epsilon} = [c^e - c^p] d\tilde{\epsilon} \quad (2.29a)$$

where

$$c^p = \frac{\tilde{c}_F^* \tilde{c}_G}{A + \tilde{c}_G^T \cdot \bar{\nabla}_{\sigma} \tilde{F}} \quad (2.29b)$$

$$\tilde{c}_F = c^e \cdot \bar{\nabla}_{\sigma} \tilde{F} ; \quad \tilde{c}_G = c^e \cdot \bar{\nabla}_{\sigma} \tilde{G} \quad (2.29c)$$

where C^e is the elastic constitutive matrix and $*$ is the diac product.

The proof of Eq. 2.29a is practically identical to the one reported in many references (see Yamada et al., 1968, or Zienkiewicz et al., 1969) with the only difference of replacing the variables usually expressed in terms of linear Lagrange strain and Cauchy stress, by the total Lagrange strain and 2nd Piola-Kirchhoff stress.

2.4 GENERALITY ON POTENTIAL FUNCTIONS

Before going into the detail of the specific form of the potential (and/or yield) functions, which are strictly related to the behaviour of the real material, it is felt necessary to investigate some of their mathematical aspects.

In the previous Sections no restriction has been imposed on the general form of the potential functions, W for elasticity and F for plasticity; it has been only assumed that they should be defined in the stress space, Eq. 2.30 in Table 2.1.

In the field of Constitutive equations some forms of potential functions attract particular attention since they allow substantial simplifications.

Herein, it will be defined as "Isotropic functions" the special stress functions which can be equivalently expressed in terms of their stress invariants $\tilde{p}$ and $\tilde{q}$, Eq. 2.31 in Table 2.1. For instance the von Mises or the Drucker-Prager

failure criterion, Table 1.1, as well as the potential functions W for isotropic material, Eqs. 2.9, may be considered of this type.

The geometrical characteristic of Isotropic functions is the complete symmetry about the spatial diagonal in the stress space. They can also be mapped in a $\tilde{p}$ vs. $\tilde{q}$ diagram.

Those particular functions which can be also expressed in terms of the three stress invariants $\tilde{p}$, $\tilde{q}$ and $\tilde{\theta}$, as Eq. 2.32 in Table 2.1, are herein defined as "Quasi-isotropic functions".

Since the angle $\tilde{\theta}$ is defined within a range of 60 degrees, it follows that quasi-isotropic functions present 6 symmetric parts about the spatial diagonal in the stress space. Typical examples of these kinds of functions are given by the Tresca and Mohr-Coulomb failure criteria, Table 1.1.

These functions can still be mapped onto a $\tilde{p}$ vs. $\tilde{q}$ diagram but with the caution to specify the current $\tilde{\theta}$ value.

The gradient of a general stress function, Eq. 2.30, may be decomposed in two vectors $\bar{V}_{\tilde{p}}F$ and $\bar{V}_{\tilde{q}}F$, Table 2.2, orthogonal one to each other. The direction of the vector $\bar{V}_{\tilde{p}}F$ is parallel to the diagonal space and consequently $\bar{V}_{\tilde{q}}F$ lies on the deviatoric plane.

The vector $\bar{V}_{\tilde{q}}F$, whose general expression can be proved to be equal to Eq. b in Table 2.2, for the case of a quasi-isotropic function may be further decomposed into two

vectors $\underline{a}_1c_1$ and $\underline{a}_2c_2$ as in Table 2.3. The direction of $\underline{a}_2$ coincides with that of the current deviatoric stress, while $\underline{a}_1$ contains the derivative of the third deviatoric invariant J_3 with respect to the current deviatoric stress.

It must be emphasized that the last three components of the vector $\underline{a}_2$, as presently derived, are half the value of those usually reported in the technical literature (see for instance Zienkiewicz et al., 1969, Owen and Hinton, 1980).

For isotropic functions, because of the independence on the $\tilde{\theta}$ -value, $\tilde{\underline{v}}_s F$ can still be simplified into Eq. c in Table 2.4. Hence the modulus of $\tilde{\underline{v}}_s F$ may in this case, and only in this case, be expressed in terms of the partial derivative of F with respect to $\tilde{q}$ -invariant, Table 2.4. From this particular property, the plastic strain invariant $d\tilde{\epsilon}_s^p$ may be expressed as in Eq. 2.26.

TABLE 2.1

General forms of stress functions

$$F(\tilde{\sigma}_{ij}, k) = 0 \quad \text{general stress function} \quad (2.30)$$

$$F(\tilde{p}, \tilde{q}, k) = 0 \quad \text{isotropic stress function} \quad (2.31)$$

$$F(\tilde{p}, \tilde{q}, \tilde{\theta}, k) = 0 \quad \text{quasi-isotropic stress function} \quad (2.32)$$

where

$$\tilde{p} = \frac{\tilde{I}_1}{3} = \frac{\tilde{\alpha}_{kk}}{3} \quad (2.33a)$$

$$\tilde{q} = (3 \tilde{J}_2)^{1/2} = \left(\frac{3}{2} \tilde{s}_{ij} \tilde{s}_{ij} \right)^{1/2} \quad (2.33b)$$

$$\tilde{\theta} = \frac{1}{3} \arcsin \left(- \frac{3\sqrt{3}}{2} \frac{\tilde{J}_3}{\tilde{J}_2^{3/2}} \right) \quad (2.33c)$$

$$\tilde{J}_3 = |\tilde{s}_{ij}| \quad (2.33d)$$

TABLE 2.2

Octahedral decomposition of the gradient of a stress
function

$$\tilde{\nabla}_{\tilde{\sigma}} F = \tilde{\nabla}_{\tilde{p}} F + \tilde{\nabla}_{\tilde{s}} F \quad a$$

where

$$\tilde{\nabla}_{\tilde{\sigma}} F = \left\{ \frac{\partial F}{\partial \tilde{\sigma}_{ij}} \right\} \quad b$$

$$\tilde{\nabla}_{\tilde{p}} F = \left\{ \frac{\partial F}{\partial \tilde{p}} \frac{\partial \tilde{p}}{\partial \tilde{\sigma}_{ij}} \right\} = \delta |\tilde{\nabla}_{\tilde{p}} F| = \tilde{a} c_1 \quad c$$

$$c_1 = \frac{1}{3} \frac{\partial F}{\partial \tilde{p}} = \frac{\partial F}{\partial \tilde{I}_1} \quad d$$

$$\tilde{a}_1 = \left\{ \frac{\partial \tilde{I}_1}{\partial \tilde{\sigma}_{ij}} \right\} = \left\{ \delta_{ij} \right\} = \{1, 1, 1, 0, 0, 0\}^T = \tilde{\delta} \quad e$$

$$\tilde{\nabla}_{\tilde{s}} F = \left\{ \frac{\partial F}{\partial \tilde{s}_{ke}} \frac{\partial \tilde{s}_{ke}}{\partial \tilde{\sigma}_{ij}} \right\} = \left\{ \frac{\partial F}{\partial \tilde{s}_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial F}{\partial \tilde{s}_{kk}} \right\} \quad f$$

$$\tilde{\nabla}_{\tilde{p}} F \cdot \tilde{\nabla}_{\tilde{s}} F = 0 \quad g$$

$$|\tilde{\nabla}_{\tilde{\sigma}} F| = (\tilde{\nabla}_{\tilde{\sigma}} F \cdot \tilde{\nabla}_{\tilde{\sigma}} F)^{1/2} = (|\tilde{\nabla}_{\tilde{p}} F|^2 + |\tilde{\nabla}_{\tilde{s}} F|^2)^{1/2} \quad h$$

$$|\bar{\nabla}_{\tilde{p}} F| = (\bar{\nabla}_{\tilde{p}} F \cdot \bar{\nabla}_{\tilde{p}} F)^{1/2} = \frac{1}{\sqrt{3}} \frac{\partial F}{\partial \tilde{p}}$$

1

$$|\bar{\nabla}_{\tilde{s}} F| = (\bar{\nabla}_{\tilde{s}} F \cdot \bar{\nabla}_{\tilde{s}} F)^{1/2} = \left(\frac{\partial F}{\partial \tilde{s}_{ij}} \frac{\partial F}{\partial \tilde{s}_{ij}} - \frac{1}{3} \left(\frac{\partial F}{\partial \tilde{s}_{kk}} \right)^2 \right)^{1/2}$$

2

TABLE 2.3

Octahedral decomposition of the gradient of a quasi-isotropic function

$$\bar{\nabla}_{\tilde{o}} F = \bar{\nabla}_{\tilde{p}} F + \bar{\nabla}_{\tilde{s}} F = a_1 c_1 + a_2 c_2 + a_3 c_3$$

$$\bar{\nabla}_{\tilde{p}} F = a_1 c_1$$

eq c, Table 2.1

$$\bar{\nabla}_{\tilde{s}} F = a_2 c_2 + a_3 c_3$$

where

$$c_2 = \frac{\partial F}{\partial J_2^{1/2}} - \frac{\operatorname{tg} 3\theta}{J_2^{1/2}} \frac{\partial F}{\partial \theta}$$

$$c_3 = - \frac{\sqrt{3}}{2 J_2^{3/2} \cos 3\theta} \frac{\partial F}{\partial \theta}$$

$$a_2 = \left\{ \frac{\partial \tilde{J}_2^{1/2}}{\partial \tilde{s}_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial \tilde{J}_2^{1/2}}{\partial \tilde{s}_{kk}} \right\} = \frac{\tilde{s}}{2 \tilde{J}_2^{1/2}}$$

$$a_3 = \left\{ \frac{\partial \tilde{J}_3}{\partial \tilde{s}_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial \tilde{J}_3}{\partial \tilde{s}_{kk}} \right\} = \left\{ \frac{\partial \tilde{J}_3}{\partial \tilde{s}_{ij}} - \frac{\delta_{ij}}{3} \tilde{J}_2 \right\}$$

$$\left[\frac{\partial J_3}{\partial \tilde{s}_{ij}} \right] = \begin{bmatrix} (\tilde{s}_{22}\tilde{s}_{33}-\tilde{s}_{23}) & (\tilde{s}_{13}\tilde{s}_{23}-\tilde{s}_{33}\tilde{s}_{12}) & (\tilde{s}_{12}\tilde{s}_{23}-\tilde{s}_{22}\tilde{s}_{13}) \\ & (\tilde{s}_{11}\tilde{s}_{33}-\tilde{s}_{13}) & (\tilde{s}_{12}\tilde{s}_{13}-\tilde{s}_{11}\tilde{s}_{23}) \\ \text{symmetric} & & (\tilde{s}_{11}\tilde{s}_{22}-\tilde{s}_{12}) \end{bmatrix} \quad h$$

TABLE 2.4

Octahedral decomposition of the gradient of an isotropic
function

$$\bar{\nabla}_{\tilde{\sigma}} F = \bar{\nabla}_{\tilde{p}} F + \bar{\nabla}_{\tilde{s}} F = \tilde{a}_1 c_1 + \tilde{a}_2 c_2 \quad a$$

$$\bar{\nabla}_{\tilde{p}} F = \tilde{a}_1 c_1 \quad \text{eq. c Table 2.1} \quad b$$

$$\bar{\nabla}_{\tilde{s}} F = \tilde{a}_2 c_2 \quad c$$

where

$$c_2 = \frac{\partial F}{\partial \tilde{J}_2^{1/2}} \quad d$$

$$|\bar{\nabla}_{\tilde{s}} F| = (\bar{\nabla}_{\tilde{s}} F \cdot \bar{\nabla}_{\tilde{s}} F)^{1/2} = \sqrt{\left(\frac{3}{2}\right) \frac{\partial F}{\partial \tilde{q}}} \quad e$$

since

$$\frac{\partial F}{\partial \tilde{\theta}} = 0 \quad f$$

2.5 A BRIEF INTRODUCTION ON PLASTIC MODELS

To calculate the plastic deformation, following Melan's postulate, it is necessary to define the shape and the kinematics (shape and size evolution) in the stress space of both yield and potential functions, and finally to determine the plastic modulus A .

Since the vast majority of the theoretical derivations have been published in terms of Cauchy stress tensor, from this Section to the end of this thesis the further mathematical developments are also reported in the same terms. Consequently, the associated strain is expressed in terms of the small Lagrange strain tensor.

Drucker (1951) established that for a "stable work hardening material" F and G coincide, and the plastic deformation is said to follow an associated flow rule.

Hill (1956) pointed out that the coincidence of F and G has an important mathematical significance, since in this case it is possible to prove the uniqueness of the potential function (von Mises, 1928).

Specific forms of G may be found experimentally, while F might also be determined theoretically, usually by making an hypothesis on the dissipating work associated to the plastic deformation.

When F is theoretically derived, its kinematics is usually implicitly defined, (see for instance Cam-clay models in Section 2.7.1). In general instead the kinematics of F (or G) has to be imposed by introducing specific rules.

If F (or G) is assumed to be represented, as in Eq. 2.34a, by an homogeneous function of order n with separable variables, several kinematic rules have been proposed, Table 2.5.

$$F(\underline{g}, \underline{\xi}, h) = f(\underline{g} - \underline{\xi}) - h^n = 0 \quad (2.34a)$$

where

$$\underline{\xi} = \beta \underline{\delta} + \underline{\alpha} \quad ; \quad \beta = \frac{\xi_{kk}}{3} \quad (2.34b)$$

where $\underline{\xi}$ is the vector defining the centroid of F while the parameter h is a scalar usually related to the size of F . The vector $\underline{\xi}$ can be also decomposed in a hydrostatic component $\beta \underline{\delta}$ along the spatial diagonal and in a deviatoric component $\underline{\alpha}$ lying in the octahedric plane, Eq. 2.34b.

The geometrical meaning of the starting assumption for most of the kinematic rules reported in Table 2.5 is immediate.

Examples of application of the Prager, Ziegler, Philip and Mroz kinematic rules have always implied the additional condition of $dh = 0$.

Mroz's and Prevost's kinematic rules require the existence of a "Bounding Surface" F_c (Mroz, 1967), in which

F may translate and expand but never cross F_c . This condition is essentially ensured by relating the kinematics of F to a certain vector μ which joins the current stress point σ to an homotetic stress point σ_c on F_c . The relationship between σ and σ_c is established by imposing that the two stresses engage the same normal on F and F_c respectively.

As regards Prevost's kinematic rule, it has been herein mathematically verified that it can be applied only under the hypothesis reported in table 2.5 No. 5, and this undoubtedly limits the generality claimed by its author. Furthermore, to calculate $d\xi$ by Prevost's kinematic rule it is also necessary to define the laws of variations of ξ_c and h_c , as presented later in this Chapter.

For a number of models the computation of the plastic modulus A requires that the evolution of G is controlled by a unique "hardening parameter", say k . Hence under this condition, it is possible to prove that A may be only calculated from either Eq. 2.35 or, if G is an isotropic function, from Eq. 2.36.

$$A = \frac{dG}{d\varepsilon_p} \frac{\partial F}{\partial p} = - \frac{\partial G}{\partial k} \frac{dk}{d\varepsilon_p} \frac{\partial F}{\partial p} \quad (2.35)$$

$$A = \frac{dG}{d\varepsilon_s} \frac{\partial F}{\partial q} = - \frac{\partial G}{\partial k} \frac{dk}{d\varepsilon_s} \frac{\partial F}{\partial q} \quad (2.36)$$

The two above equations have been derived from Eq. 2.24 and Eq. 2.26 respectively, and further reduced by the use of the Consistency equation.

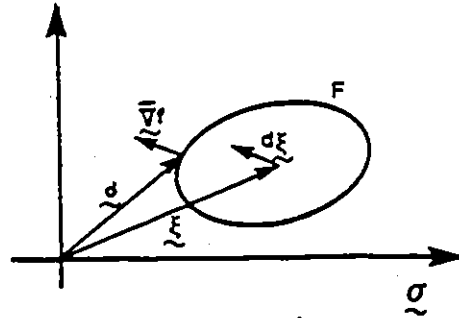
The actual calculation of A obviously requires the existence of a relationship between k and ϵ_V^P or ϵ_S^P . These relationships are, as it is shown in the next Section, determined experimentally, once the exact physical nature of k is established.

TABLE 2.5

Kinematic rules for homogeneous function with separable variable

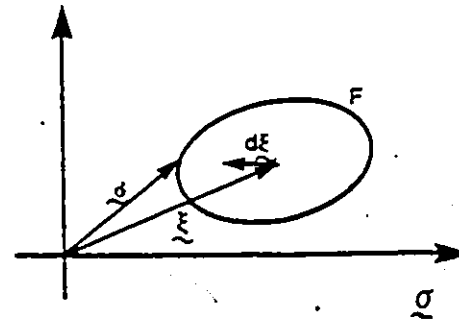
1) Prager

$$d\tilde{\xi} = d\rho \tilde{\nabla}_{\sigma} f = \frac{df_{\sigma} - nh^{n-1}dh}{|\tilde{\nabla}_{\sigma} f|} \cdot n_F$$



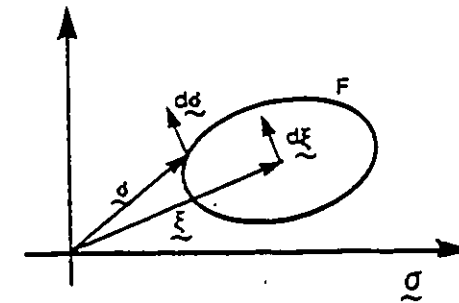
2) Ziegler

$$d\tilde{\xi} = d\rho (\sigma - \xi) = \frac{df_{\sigma} - nh^{n-1}dh}{(\sigma - \xi) \cdot \tilde{\nabla}_{\sigma} f} \cdot (\sigma - \xi)$$



3) Philip

$$d\tilde{\xi} = d\rho d\tilde{g} = \frac{df_{\sigma} - nh^{n-1}dh}{df_{\sigma}} \cdot d\tilde{g}$$



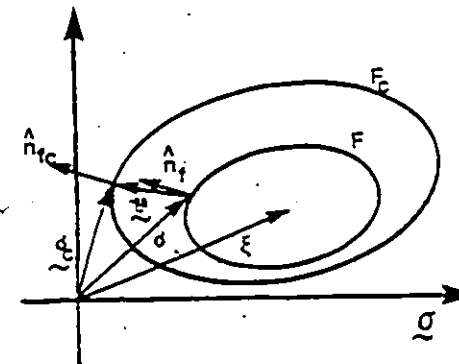
4) Mroz

$$d\tilde{\xi} = d\rho \mu = \frac{df_{\sigma} - nh^{n-1}dh}{\mu \cdot \tilde{\nabla}_{\sigma} f} \cdot \mu$$

where

$$\mu = \sigma - \sigma_c$$

$$\hat{n}_f \equiv \hat{n}_{f_c}$$



5) Prevost

$$d\xi = d\xi_c + a \hat{n}_f + \mu dp$$

where

ξ_c = coordinate of the center of the bounding surface F_c

$$a = n \frac{h_c^{n-1} dh_c}{|\bar{\nabla}_{\sigma} f_c|} - \frac{h^{n-1} dh}{|\bar{\nabla}_{\sigma} f|}$$

$$\mu = \sigma - \sigma_c, \quad \hat{n}_f \equiv \hat{n}_{f_c} = \frac{\bar{\nabla}_{\sigma} f}{|\bar{\nabla}_{\sigma} f|}$$

$$dp = \frac{df_{\sigma} - \bar{\nabla}_{\sigma} f \cdot d\xi_c - n h_c^{n-1} dh_c \frac{|\bar{\nabla}_{\sigma} f|}{|\bar{\nabla}_{\sigma} f_c|}}{\mu \cdot \bar{\nabla}_{\sigma} f}$$

The relationship of a and ρ can be proved only with the following assumption

$$\mu dp = d\sigma - d\sigma_c$$

$$d\sigma = d\xi + \frac{n h^{n-1} dh}{|\bar{\nabla}_{\sigma} f|} \hat{n}_f$$

$$d\sigma_c = d\xi_c + \frac{n h_c^{n-1} dh_c}{|\bar{\nabla}_{\sigma} f_c|} \hat{n}_{f_c}$$

2.6 LINEAR PLASTIC MODELS

It has been found experimentally that for a number of materials the yield function might still be represented by a linear function, Eq. 2.37a or for frictionless materials Eq. 2.37b, similar to the failure function as described by Eq. 1.7.

$$G = q - M(d - p) = 0, \quad M = M(k) \quad (2.37a)$$

$$G = q - N = 0, \quad N = N(k) \quad (2.37b)$$

where d is the common intersection of Q and G with the hydrostatic axis.

As plasticity develops, G expands and eventually it coincides at failure with Q , Figs. 2.5. The law of variation of G is usually assumed to be controlled by a single parameter k .

In the assumption of an isotropic yield function like Eq. 2.37b (von Mises type) and of associated flow rule, the plastic modulus A might be calculated from Eq. 2.36, which in this case simply represents the derivative of q with respect of ϵ_s^p , Eq. 2.38.

Moreover, since the potential function is pressure independent, i.e. from Eq. 2.24 $d\epsilon_V^P = 0$, it follows $d\epsilon_S^P = d\epsilon_a^P$ in a uniaxial test, where ϵ_a^P is the axial plastic strain. The modulus A therefore represents the tangent to the σ_a vs. ϵ_a^P curve in a uniaxial test, Eq. 2.38, where σ_a is the axial stress.

$$A = \frac{dG}{d\epsilon_S^P} = \frac{\sigma}{\partial F} = \frac{dq}{d\epsilon_S^P} = \frac{d\sigma_a}{d\epsilon_a^P} \quad (2.38)$$

Hill (1956) arrived to a similar conclusion but his derivation was based on the postulate that the hardening parameter k is equal to the total plastic work.

It is worthwhile emphasizing once more that the above expression of A can only be applied to von Mises type potential functions.

Some authors postulate that the hardening parameter k is equal to the plastic deviatoric strain, Eq. 2.39a, and consequently A may be calculated as in Eq. 2.39b, originated again from equation 2.36.

$$k = \int d\epsilon_S^P \quad (2.39a)$$

$$A = - \frac{\partial G}{\partial k} \frac{dk}{d\epsilon_s^p} \frac{\partial F}{\partial q} = - \frac{\partial G}{\partial k} \frac{\partial F}{\partial q} \quad (2.39b)$$

where

$$\frac{\partial G}{\partial k} = -(d - p) \frac{\partial M}{\partial k} \quad (2.39c)$$

or

$$\text{or. } \frac{\partial G}{\partial k} = - \frac{\partial N}{\partial k} \quad (2.39d)$$

Usually also in this case the hypothesis of associated rule is advocated and M, or N, is assumed to be related to k by an hyperbolic relationship as follows

$$y = y_0 + \frac{x y_c m_0}{x m_0 + y_c} \quad (2.40)$$

where y might represent M or N and $x=k$. The constants y_0 and m_0 are respectively the values of y and of the slope of the hyperbola at $x=0$. The sum of y_0 and y_c is the limiting value of y for x tending to infinite, Figs. 2.6.

Also in this case, for the mathematical constraints exposed in section 2.4, the calculation of A as in Eq. 2.39 has to be limited to isotropic yield functions.

It is to be noticed that at failure, from either Eq. 2.38 or 2.39a, A results to be equal to zero.

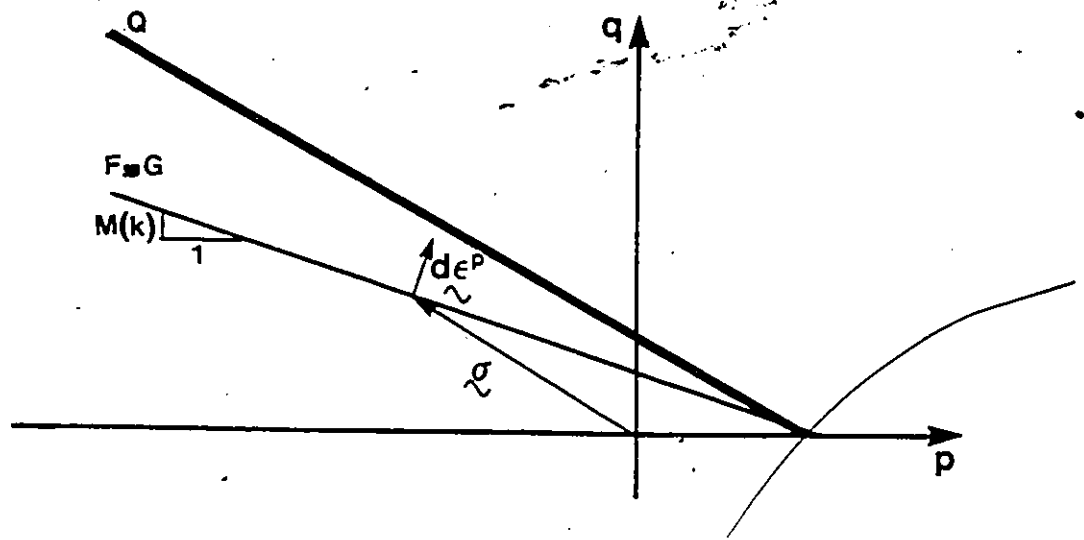


Fig. 2.5 Hardening of a plastic isotropic linear function

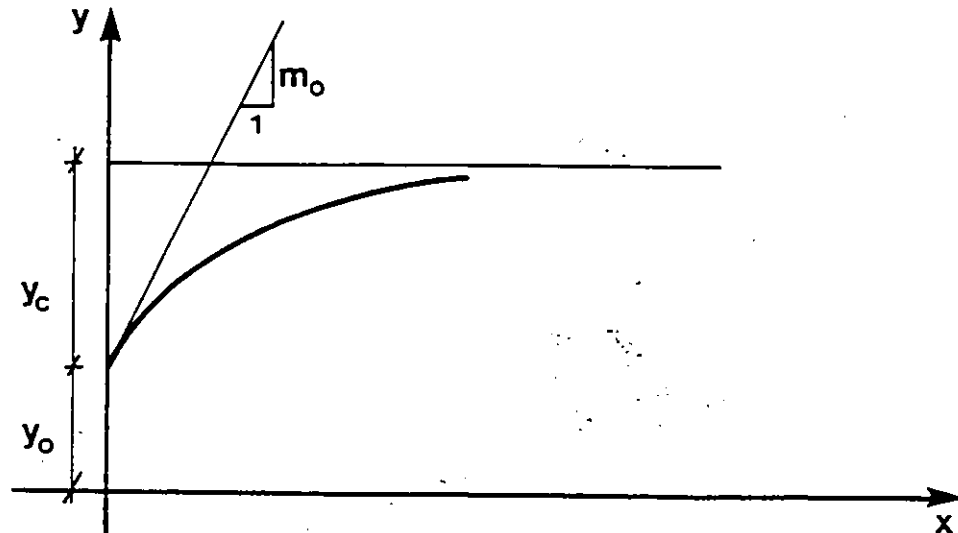


Fig. 2.6 Graphical representation of the hyperbolic relationship eq. 2.40

2.7 CAPPED PLASTIC MODELS

For a number of engineering materials and specially for soils, experimental evidence shows that plasticity takes place even under pure isotropic compression.

The yield and potential surfaces therefore cannot be represented by "linear" functions of the type Eqs. 2.37, but they should rather have a convex shape and cross the hydrostatic axis (space diagonal).

Roscoe et al. (1958) indicate that at failure, or as they called it, "Critical State" (CS), the specific volume in soil tends to a constant value. In the hypothesis of small deformation $d\varepsilon_v = d\bar{v}/\bar{v} = 0$; hence the potential function should present a maximum with respect of p , at the intersection with the failure surface, as indicated by Eq. 2.24.

Apparently Drucker et al. (1957) were the first to suggest that soil might be modelled as an elasto-plastic hardening material. They proposed that successive yield surfaces might resemble extended von Mises' cones with convex end caps. Drucker (1961) discussed this concept in a later paper in which he suggested that the failure surface may not be the yield surface. This point is further emphasized by Drucker (1966) who noted that successive loading surfaces or yields do not approach the failure surface.

Another important feature of soil behaviour can be schematically shown in a q vs. ϵ_a and ϵ_v vs. ϵ_a diagrams, as in Figs. 2.7.

The response of a soil sample tested in a conventional triaxial test may present alternative patterns. It may happen that q and ϵ_v monotonically increase towards certain constant values in correspondence to the state of failure, or CS, of the sample. On the contrary the same soil may present a peak strength after which q falls towards a constant value as ϵ_a increases. The volumetric strain in this case initially contracts and then expands trying to reach a specific value.

There is nowadays enough experimental evidence to indicate that the different responses are related to the previous stress history which the sample has been subjected to.

In the recent years, researchers in geotechnical engineering have widely accepted the existence in the stress space of a so called "Bounding Surface", BS, (Mroz, 1967). The size and the spatial position of BS are related to the first loading history of the soil. The intersection of the BS with the hydrostatic axis is also known as "Equivalent Preconsolidation Pressure", p_e .

If the state of stress still lies on BS, and inside the Critical State Surface (CSS), then the soil is also denoted as "Normally Consolidated Soil", NCS. In this case, further

increase of the applied load will result in an expansion of BS, which follows the stress point.

On the contrary, unloading process leaves unaltered the space position and the size of BS. Soils whose current state of stress is found inside the BS are also indicated as "Over Consolidated Soils", OCS.

The first type of triaxial response previously described is apparently characteristic of a NCS, while the second one is usually associated to an OCS.

On the specific form of the BS there is not yet a concordance of opinion among the researchers. However, with exception of some specific case, there is an evident agreement in identifying the region of BS contained inside the CSS with the yield function associated to the NCS.

The remaining surface of BS, also known as Hvorslev's surface in recognition to the early classic work by Hvorslev (1937), should instead represent the envelope of the peak stress values of an OCS.

The above remarks are further developed in the next Sections.

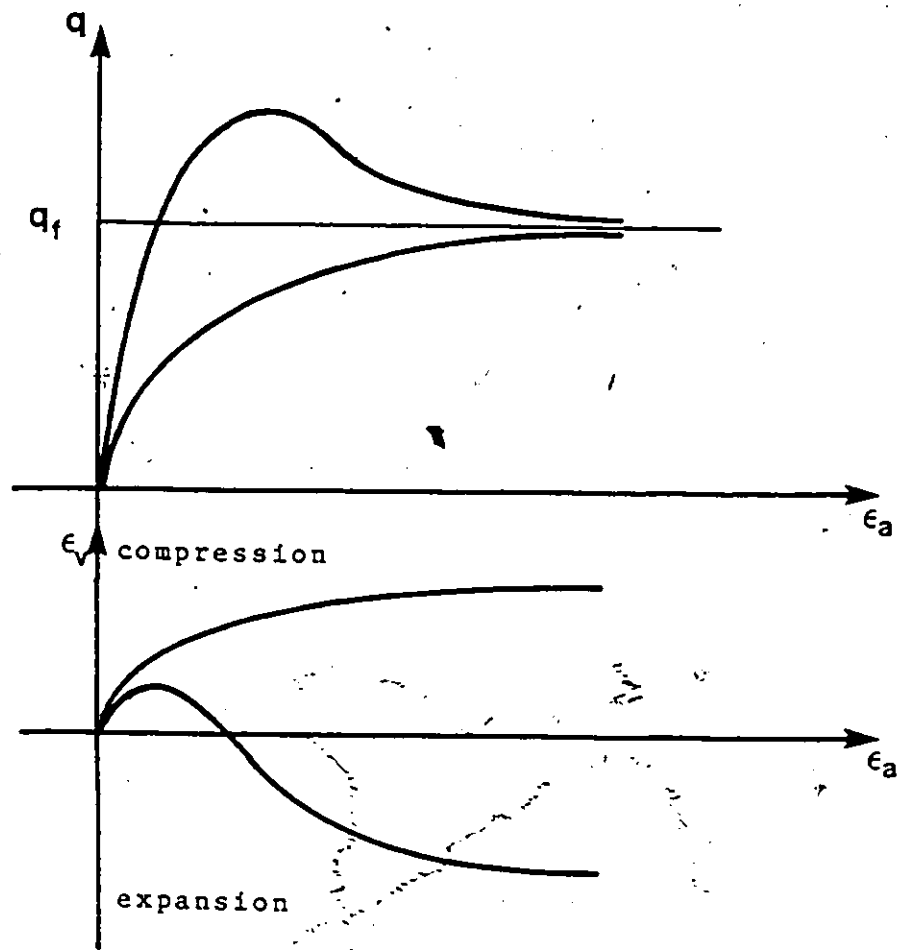


Fig. 2.7 Possible soil response in a conventional triaxial test

2.7.1 Capped Models for Normally Consolidated Soil

Roscoe et al. (1958) published a paper which contained the basis for a number of subsequent hardening models for soils. The paper was concerned primarily with the behaviour of soil in a triaxial test and contained the so-called "State Boundary Surface" (called yield surface in the 1958 paper) and the "Critical State Line" postulates. (The primary credit of this work consists in the attempt to explain the behaviour of soil in a global way, giving new light to the early intuitions of Rendulic (1936), Hvorslev (1937) and Terzaghi (1943).

Roscoe and Poorooshab (1963) utilized the ideas contained in the earlier Roscoe's paper to develop a stress-strain theory for clay which however was not based upon the theory of plasticity.

Calladine (1963) suggested an alternative interpretation of this theory using concepts from strain hardening plasticity.

Subsequently Roscoe et al. (1963) adopted the strain hardening theory of plasticity to formulate a complete stress strain model for normally consolidated or "lightly" over-consolidated clay in the triaxial. This model has since become known as Cam-clay model (Schofield and Wroth, 1968).

Burland (1965) suggested a modified version of the Cam-clay model which was eventually extended to a general three-dimensional stress state by Roscoe and Burland (1968).

The Cam-clay is essentially a strain hardening model whose plastic deformation is assumed to follow an associated flow rule. The potential (and yield) function, Eq. 2.41b, was derived from Taylor's hypothesis (Taylor, 1948) on the internal dissipating work produced by the plastic deformation, Eq. 2.41a.

$$dw^P = p d\epsilon_v^P + q d\epsilon_s^P = M_p d\epsilon_s^P \quad (2.41a)$$

$$F = \frac{q}{M_p} + \ln \frac{p}{p_e} = 0 \quad (2.41b)$$

where p is assumed positive for compression.

In the modified Cam-clay model, Burland (1965 and 1967) proposed a new expression of the dissipating work, Eq. 2.42a, whose integration gives an elliptic yield function in a p vs. q diagram, Eq. 2.42b.

$$dw^P = p d\epsilon_v^P + q d\epsilon_s^P = p [d\epsilon_v^2 + (M d\epsilon_s^P)^2]^{1/2} \quad (2.42a)$$

$$F = p^2 + \frac{q^2}{M^2} - p p_e = 0 \quad (2.42b)$$

where p is assumed positive for compression.

Both surfaces, as shown in Figs. 2.8 and 2.9, have a maximum on the critical state line, implying that the volumetric plastic strain at failure is equal to zero in the hypothesis of small deformation.

Taylor's potential function, though it has the merit to be derived from a clear hypothesis on the plastic work, unfortunately predicts undesirable distortional strain for pure isotropic compression. On the other hand Burland's potential surface, whose starting hypotheses were just briefly discussed by the author, predicts only volumetric strain for isotropic compression in accordance to experimental evidence.

Both surfaces are represented by isotropic functions, and in particular Burland's ellipse is also a homogeneous 2nd order equation with separable variables which can be alternatively expressed as:

$$F = (p + \beta)^2 + \frac{q^2}{n} - h^2 = 0 \quad (2.43a)$$

where

$$P_e = 2\beta = 2h \quad (2.43b).$$

$$n \equiv M$$

(2.43c)

The kinematic rule of both surfaces is implicitly defined since their maximum is always forced to be on the failure surface. In particular the centre of Burland's ellipse, in the general stress space, is forced to lie on the space diagonal, at the distance equal to the size of the major axis of the ellipse.

$$\xi = \delta h \quad (2.44)$$

Mroz et al. (1979) slightly modified Burland's surface in order to allow the ratio of the ellipse semi-axes n to be different from M . This implies mathematically that, for a given n , β and p_e should be respectively equal to

$$\beta = b h = \frac{n}{M} h \quad (2.45a)$$

$$p_e = \beta + h \quad (2.45b)$$

In the Cam-clay models it is postulated that the variation of the plastic specific volume between two next yield surfaces is only dependent on the variation of the p_e and on the current $\bar{v}$. Namely in the hypothesis of small deformation, the rate of volumetric strain may be found from the following expression (Roscoe and Burland, 1968):

$$d\epsilon_v^p = \frac{d\bar{v}^p}{\bar{v}} = \frac{\chi - \lambda}{\bar{v}} \frac{dp_e}{p_e} \quad (2.46)$$

where λ and χ are material constants. Similar relationships have been proposed by Naylor and Pande (1981), Eq. 2.47, and Prevost (1978a), Eq. 2.48, but there the fundamental Cam-clay hypothesis on the plastic volumetric change seems to be lost.

$$d\epsilon_v^p = \chi \frac{dp_e}{p_e} \quad (2.47)$$

$$p_e = \exp(\lambda \epsilon_v^p) \quad \therefore \quad d\epsilon_v^p = \frac{1}{\lambda} \frac{dp_e}{p_e} \quad (2.48)$$

According to the assumptions in Eq. 2.46, the modulus A has been herein derived from Eq. 2.35, with k being

p_e , and Eq. 2.46 used to obtain the following general expression:

$$A = - \frac{\partial G}{\partial p_e} \frac{dp_e}{d\varepsilon_v^p} \frac{\partial F}{\partial p} = \frac{\bar{v} p_e}{\lambda - \chi} \frac{\partial G}{\partial p_e} \frac{\partial F}{\partial p} \quad (2.49)$$

The Cam-clay models, as well as most of the subsequent models, assume the hypothesis of associated flow rule for normally consolidated soils.

For Burland's model and in the hypothesis of associated flow rule, the plastic modulus may also be calculated from the following reduced form of Eq. 2.49:

$$A = \frac{\bar{v} p_e}{\lambda - \chi} \frac{\partial F}{\partial p_e} \frac{\partial F}{\partial p} = - \frac{\bar{v}}{\lambda - \chi} (\bar{v}_\sigma f \cdot g) \frac{\partial F}{\partial p} \quad (2.50a)$$

where

$$f = (p + \beta)^2 + \frac{q^2}{n} \quad (2.50b)$$

$$\bar{v}_\sigma f \cdot g = \frac{\partial F}{\partial p} p + \frac{\partial F}{\partial p} q \quad (2.50c)$$

Poorooshab (1961), after being one of the protagonists in building up the concept of "State Boundary Surface", disagreed completely on the later development which led Roscoe and the group of researchers at Cambridge University to the formulation of the Cam-clay models. He argued that the potential function represents an effective state of soil (namely the constant $\bar{v}$). Poorooshab therefore concluded that the identification of the potential function should be based on experimental work rather than being defined by an hypothetical law on the plastic work dissipation (Poorooshab et al. 1966, and 1967).

Following Poorooshab, yield and potential functions are different and they may be respectively represented by the following equations

$$G = \eta = \frac{q}{p} \quad (2.51)$$

$$F = p \bar{F}(\eta, \bar{v}) = p_e \quad (2.52)$$

where F is a function to be found experimentally and G is a simplified form of the Drucker-Prager function type. Notice that the form of Eq. 2.52 ensures shape similarity to F at any stress level.

Poorooshab also rejected the Cam-clay postulate on the plastic volumetric strain, and instead proposed to calculate the plastic modulus from the deviatoric strain component. Thus following his suggestion, the plastic modulus A can be derived from Eq. 2.36, taking into account Eq. 2.51. This results in:

$$A = \frac{dG}{d\epsilon_s^p} \frac{\sigma}{\partial F} = \frac{d\eta}{d\epsilon_s^p} \frac{\partial F}{\partial q} \quad (2.53)$$

where the relationship between η and ϵ_s^p should be determined experimentally. Incidentally, Nishi and Esashi (1978) proposed that η and ϵ_s^p could be related by an hyperbolic relationship similar to that reported in Eq. 2.40.

Ishihara et al. (1975, 1978) apparently obtained good experimental and numerical agreement by adopting an elasto-plastic model very close to Poorooshab's.

But it is evident that for constant η , including isotropic consolidation, Poorooshab's hardening rule cannot predict any plastic deformation. Furthermore, since A is determined through the use of Eq. 2.36, F and G have to be isotropic functions.

This fact may represent a serious handicap for soils whose strength at failure in compression is different from that in extension.

Lately, there has been an increased interest in trying to model materials which develop plastic anisotropy.

Lerouiel (1977) for instance found experimentally that the curve of constant specific volume has the K_0 -line as axis of symmetry in a p vs. q diagram. The K_0 -line is the stress path that a soil follows when consolidated under a constant ratio of q/p . Similar experimental results were found by Skempton and Sowa (1963) and by Parry and Nadarajah (1973).

Mroz et al. (1979) therefore deduced that also the yield function, namely Burland's ellipse, should not be necessarily symmetric to the hydrostatic axis.

They proposed that "in order to accommodate different observed values of K_0 within the model, we may extend it in a simple way by postulating that consolidation (meaning yield) surface not only expands and translates along p -axis, but also may translate or rotate within the p against q plane."

But it is necessary to remember that the yield surface represented in the p vs. q diagram is only a mapping of the real yield surface defined in the general stress space. This mapping is possible if and only if the surface is an isotropic or a quasi-isotropic function, which implies that it has to be in any case symmetric to the hydrostatic axis. Consequently, it is not clear what type of yield surface in the real stress space g would correspond to a non p -

symmetric yield surface in the p vs. q diagram, as proposed by Mroz.

A more correct approach to this problem may be pursued by defining the yield (and potential) function directly in the real stress space $\underline{g}$ and eventually imposing a non-isotropic condition on it.

Prevost assumed as yield surface a sphere, Eq. 2.54, whose centre is forced to move in a given direction $\underline{a}$, and the size h is related to the distance of the centre from the origin of the reference axes, Eqs. 2.54 .

$$F = \frac{3}{2} (\underline{g} - \underline{\xi})(\underline{g} - \underline{\xi}) - h^2 = 0 \quad (2.54a)$$

$$\underline{\xi} = \underline{a} h \quad (2.54b)$$

$$p_e = \beta + h = \frac{\xi_{kk}}{3} + h = \left(\frac{a_{kk}}{3} + 1 \right) h \quad (2.54c)$$

He also postulated that the plastic volumetric strain is only dependent on p_e , Eq. 2.48, which he defined as the projection of $\underline{\xi}$ on the spatial diagonal plus the size of the

sphere, Eq. 2.54c. Finally Prevost assumed that plastic deformation follows an associated flow rule.

Since the mathematical hypotheses are very similar to Burland's model, the plastic modulus may be derived in a similar fashion. Namely the value of A can be proved to be equal to the following alternative expressions:

$$A = - \frac{\partial F}{\partial p_e} \frac{dp_e}{d\epsilon_v^p} \frac{\partial F}{\partial p} = - \lambda p_e \frac{\partial F}{\partial p_e} \frac{\partial F}{\partial p} = \lambda (\bar{\nabla}_\sigma f \cdot g) \frac{\partial F}{\partial p} \quad (2.55a)$$

where

$$f = \frac{3}{2} (g - \xi)(g - \xi) \quad (2.55b)$$

$$\bar{\nabla}_\sigma f \cdot \sigma = 3(g - \xi) \cdot g \quad (2.55c)$$

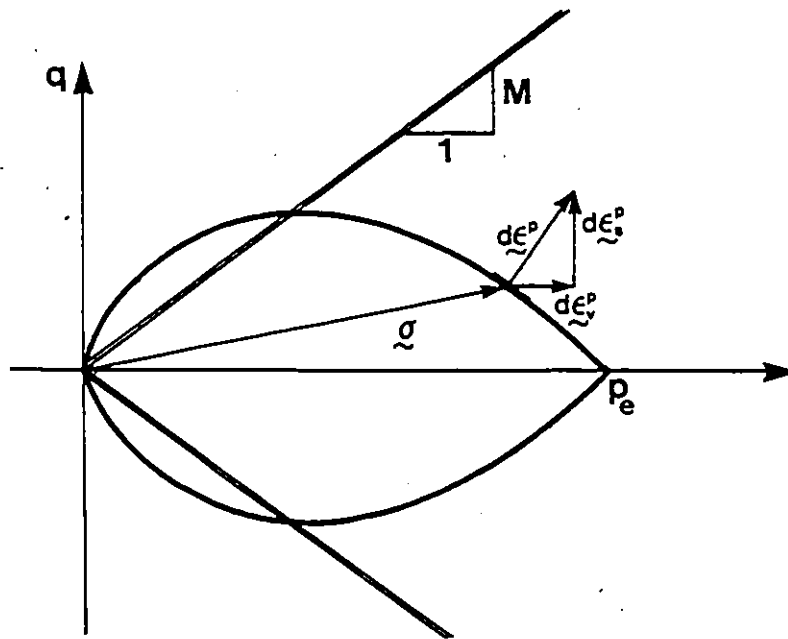


Fig. 2.8 Mapping of Taylor's yield surface in p vs. q diagram

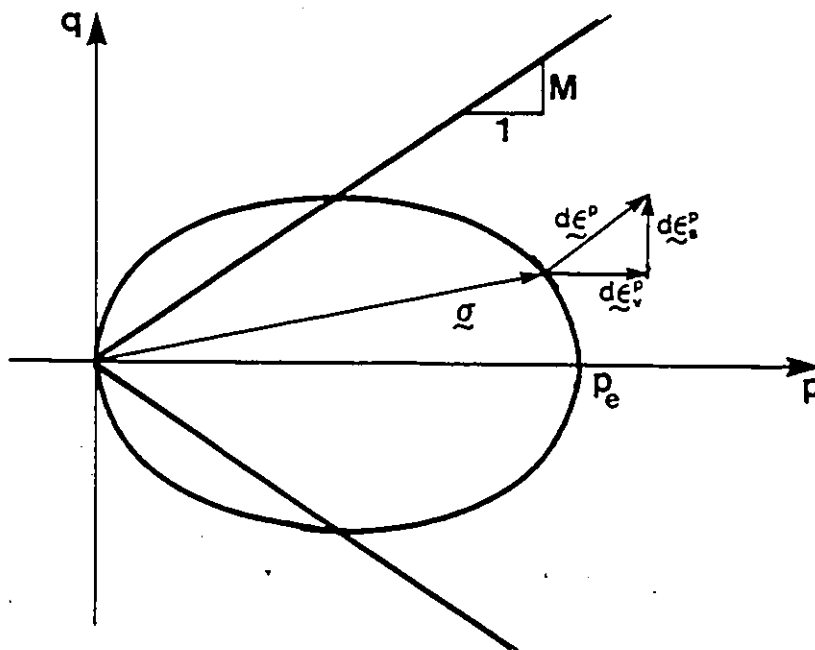


Fig. 2.9 Mapping of Burland's yield surface in p vs. q diagram

2.7.2 Capped Models for Over-consolidated Soil

Naylor and Pande (1981) assumed that the BS is composed of Burland's surface in the NC region and von Mises cone in the Hvorslev region.

For any stress path inside the BS the soil is assumed to respond linear elasticly. Hvorslev's surface is assumed to represent the failure envelope for OCS and the corresponding plastic strain is supposed to follow a non-associated flow rule. The potential function F is in this case given by a paraboloid, Eq. 2.56, having the vertex on the CSS, so that the plastic volumetric strain associated to the critical state condition is zero.

Since the authors postulated that the plastic volumetric deformation follows a type of law as reported in Eq. 2.47, it is believed that the plastic modulus should be calculated by the following Eq. 2.57.

$$G = q + \frac{M_o}{p_e} \left(\frac{p_e}{2} - p \right) = 0 \quad (2.56)$$

$$A = - \frac{\partial G}{\partial p_e} \frac{dp_e}{d\varepsilon_v} \frac{\partial F}{\partial p} = - \frac{p_e}{\chi} \frac{\partial G}{\partial p_e} \frac{\partial F}{\partial p} \quad (2.57)$$

where M_o is a material parameter.

Mroz et al. (1978) proposed that the BS is entirely represented by Burland's ellipse, Eq. 2.43.

Inside the BS plastic deformation may take place. The potential function is represented by a smaller ellipse, Eq. 2.58a, which translates in the p vs. q diagram, with the only condition of not crossing the BS. The size of the ellipse is directly related to the size of the BS, and it is kept constant for any stress path inside the over-consolidated region, Eq. 2.58b.

$$F = (p - \beta)^2 + \frac{(q - \alpha)^2}{n^2} - h^2 = 0 \quad (2.58a)$$

$$h = c h_c \quad (2.58b)$$

where c has a given value.

The kinematic rule is given by the following Eqs. 2.59 and 2.60, and these, according to Mroz, should represent a particular case of the more general rule given in Table 2.5 (no.4).

$$d\xi = \mu dp = \frac{dF}{\mu \cdot \bar{\sigma} f} \quad (2.59)$$

$$\underline{\mu} = \underline{g} - \frac{(\underline{g} - \underline{\xi})}{c} + \underline{\xi}_c \quad (2.60a)$$

where in this case

$$\underline{\sigma} = \{p, q\}^T \quad (2.60b)$$

$$\underline{\xi} = \{\beta, \alpha\}^T, \quad \underline{\xi}_c = \{\beta_c, \alpha_c\}^T \quad (2.60c)$$

and the subscript c specifies the variables which refer to the BS.

The normalized hardening modulus K^P is assumed to vary in accordance to a certain interpolation function (Mroz et al., 1978), dependent on the distance between the current stress point on F_c and the homotetic point on F . Finally the plastic strain is said to follow an associated flow rule. Besides, in Mroz's approach there is always the major problem of identifying the real potential function in the stress space g .

Dafalias et al. (1980, 1982 and 1984) proposed to shape the BS in Hvorslev's region by an hyperboloid which links an ellipsoid at its intersection with the octahedric plane. The BS surface is then completed by Burland's ellipsoid in the NC region.

The authors suggest that inside the BS plastic deformation takes place only for loading stress path. The plastic strain is assumed to have the same direction as the exterior normal of an "image" point on the BS. The image point is defined as the projection of the current stress on the BS from the origin of the stress space. The value of the normalized plastic modulus K^P is related to the distance δ between the stress point and its image, namely

$$K^P = K_b^P + H \frac{\delta}{\delta_o(\sigma, e) - \delta} \quad (2.61)$$

where K_b^P is the value of the normalized plastic modulus at the image point, H is a hardening function and δ_o is a reference distance. The plastic model is finally completed by an assumption of associated flow rule.

Alternatively Oka (1981, 1982 and 1984) assumes the BS to be described by the following Eq. 2.62, which should represent an extension of Taylor's model for the more general case of plastic anisotropy.

$$F = \bar{\eta}_{(o)}^* + M_m^* \ln \frac{p}{p_e} = 0 \quad (2.62a)$$

where

$$\bar{\eta}_{(o)}^* = \left[(\eta_{ij}^* - \eta_{ij(o)}) (\eta_{ij}^* - \eta_{ij(o)}) \right]^{1/2} \quad (2.62b)$$

$$\eta_{ij} = s_{ij}/p \quad (2.62c)$$

Oka suggests that plastic deformation is developed for any stress path inside the BS. The mathematical development of the model is rather complicated because of the nature of its potential function.

However, in order to determine the plastic modulus, the author apparently assumes the plastic deviatoric strain to be given by Eq. 2.63 which instead, as it has been proved previously, is only valid for isotropic function.

$$de_{ij} = d\bar{l} \frac{\partial F}{\partial s_{ij}} \quad (2.63)$$

The use of Oka's model should be therefore restricted only to the case when the BS is simply described by Taylor's equation, Eq. 2.41. Under this hypothesis the potential function for OCS should be represented by the following Eq. 2.64, where the value of M is such as to satisfy the equation at the current stress point.

$$F = \frac{q}{p} + M \ln \frac{p}{p_e} = 0 \quad (2.64a)$$

where

$$\tilde{M} = - \frac{q_n}{p_n \ln \left(\frac{p_n}{p_e} \right)} \quad (2.64b)$$

while q_n and p_n are the stress invariants associated to the current stress and finally p is assumed positive for compression.

Oka, in his general model, postulates that the yield function is given by a cone in the stress space and, in the calculation of the plastic modulus, he assumes the finding of a relationship between the plastic deviatoric strain and deviatoric stress.

Applying Oka's hypotheses to the current simplified case, the plastic modulus should be calculated from the previously reported expression in Eq. 2.53.

Prevost (1978a, 1978b and 1981) also suggests that plasticity takes place in OCS for any loading or unloading stress path. For OCS, Prevost proposes a model which is rather involved mathematically and has been apparently subjected to several modifications by the author himself.

Firstly, Prevost postulates that the BS is completely described by his yield function, Eq. 2.54, which hypothetically should also describe possible state of plastic anisotropy. He also postulates the existence of a number of nesting surfaces, namely spheres, Eq. 2.65, which may translate, expand or contract but never cross the BS nor each other.

For the kinematic rule of a nesting surface, say G_1 , Prevost (1978a) refers back to his general formulation reported in Table 2.5 but with the additional hypotheses reported in Table 2.6. It seems at this point, that the great number of assumptions contained in this kinematic rule are introduced more for mathematical convenience rather than to model the true response of soil.

To complete the kinematic description of G_1 , Prevost (1981) relates the centre and size variation of the next nesting surface G_m to the volumetric strain as follows

$$\xi_m = \bar{\xi}_m \exp(\lambda \varepsilon_v) \quad (2.66a)$$

$$h_m = \bar{h}_m \exp(\lambda \varepsilon_v) \quad (2.66b)$$

where ξ_m and h_m are the initial values of ξ and h respectively and they have to be found experimentally.

Finally, the normalized plastic modulus is assumed to be given by a kind of non-associated flow rule as follows:

$$K^P = K^V + \hat{n}_p K^S \quad (2.67a)$$

$$\hat{n}_p = \frac{1}{|\nabla_{\sigma} G|} \frac{\partial G}{\partial p} = \frac{3}{\sqrt{2}} \frac{(p - s)}{h} \quad (2.67b)$$

$$K^V = \overline{K}^V \left(\frac{p}{p_e} \right)^\eta \quad (2.67c)$$

5

$$K^S = \overline{K}^S \left(\frac{p}{p_e} \right)^\eta \quad (2.67d)$$

where K^V and K^S are the initial values of K^V and K^S for $p = p_e$, while η is a soil constant.

It is evident that the practical application of Prevost's model is very complicated and it involves at least a remarkable work of calibration (Sture, 1981).

TABLE 2.6

Prevost's kinematic rule for over-consolidated and
anisotropic soils

Potential function:

$$G_l = \frac{3}{2} (\sigma - \xi_l)(\sigma - \xi_l) - h_l^2 = 0 \quad 2.65a$$

$$\bar{\nabla}_{\sigma} g_l = 3(\sigma - \xi_l) ; \quad |\bar{\nabla}_{\sigma} g_l| = \sqrt{6} h_l$$

additional hypothesis on the kinematism of the nesting surfaces:

$$dh_e = dh_m ; \quad \bar{\nabla}_{\sigma} g_l d\xi_m = 0$$

$$a = n \left[\frac{h_m^{n-1} dh_m}{|\bar{\nabla}_{\sigma} g_m|} - \frac{h_l^{n-1} dh_l}{|\bar{\nabla}_{\sigma} g_l|} \right] = 0$$

$$dp = \frac{dg_{\sigma}^l - \bar{\nabla}_{\sigma} g_m \cdot d\xi_m - nh_m dh_m \frac{|\bar{\nabla}_{\sigma} g_l|}{|\bar{\nabla}_{\sigma} g_m|}}{\mu \cdot \bar{\nabla}_{\sigma} g_l} = \frac{\bar{\nabla}_{\sigma} g_l \cdot d\xi_l}{\mu \cdot \bar{\nabla}_{\sigma} g_l}$$

$$\mu = \sigma - \sigma_m = \sigma - \xi_m - \frac{h_m}{h_l} (\sigma - \xi_l)$$

kinematic rule:

$$d\tilde{\xi}_l = d\tilde{\xi}_m + a \hat{n}_{g_l} + \mu d\rho = d\tilde{\xi}_m + \frac{\bar{\nabla}_{\sigma l} \cdot d\tilde{\xi}_l}{\mu \cdot \bar{\nabla}_{\sigma l}} \quad 2.65b$$

where the subscript l and m are employed to distinguish the variable related to the current yield surface G_l from those related to a next surface G_m

2.8 CONCLUSION

In spite of the formal similarity between the definition of the elastic and plastic strain, Eqs. 2.7b and 2.22a, the subsequent development of the two theories may appear rather different.

Instead, the crucial point in both theories is the finding of a potential surface and the definition of its evolution in the stress space.

In the classical theory of linear elasticity for isotropic material, the potential function is governed by two parameters, E and ν . On the other hand, the evolution of the potential function for most of the presented plastic models is controlled by a single 'hardening' parameter k .

Perhaps further studies towards an extension of the governing parameters of the plastic potential to two or more parameters may lead to very interesting results.

The study of the possible reduced form of the gradient to the potential function developed in Section 2.4, has allowed a precise mathematical analysis of the different plastic models herein presented.

In particular, it was possible to find that the two available approaches to calculate the modulus A in linear plastic models, Eqs. 2.38 and 2.39b, can be applied only to isotropic potential functions.

However, for elastic perfectly plastic models, where Q always coincides with G , the plastic modulus A should be set

equal to zero, regardless the assumption on k , the hardening parameter. In fact in this case, $d\bar{g}$ cannot leave the failure surface and therefore may be at most tangent to $Q=G$, i.e. orthogonal to the gradient $\bar{V}_0 G$. Hence, the modulus A , from either Eqs. 2.35 or 2.36, is equal to zero, being $dG_0 = \bar{V}_0 G \cdot d\bar{g} = 0$.

The requirement of isotropy on F is also necessary to determine A following Poorooshab's suggestion, Eq. 2.53, while the Cam-clay fundamental postulate on the plastic volumetric strain leads to Eq. 2.49, which usually requires only the additional hypothesis of small deformation.

As regards the specific form of G for NCS, it appears that the two original Cam-clay models still represent a reference point to most of the latest elasto-plastic models.

It must be noticed, however, that both models assume the critical state to be represented by the Drucker-Prager failure criterion having vertex in the origin of the stress space.

There is instead a vast number of evidence (Parry, 1956, Bishop, 1966) which indicates that the Mohr-Coulomb failure criterion, with cohesion generally different than zero, fits the experimental data better.

It is therefore proposed in the following an extension of the two aforesaid models in order to incorporate the Mohr-Coulomb failure criterion in its most general form.

From pure geometrical consideration it is possible to prove that the generalization of Taylor's model, Figs. 2.10, could be given by Eq. 2.68 as reported in Table 2.7. For $c=0$ and $\theta = 30^\circ$ Eq. 2.68 reduces to the original Taylor's model. It is easy to prove that the maximum of Eq. 2.68 with respect of the space diagonal, lies always on the Mohr-Coulomb failure envelope.

Also by pure geometrical means, it is possible to show that a generalization of Burland's model, whose graphical representation in a p vs. q axis for $\theta = 30^\circ$ is shown in Figs. 2.11, is given by Eq. 2.70 as reported in Table 2.7. For $c=0$ and $\theta = 30^\circ$ Eq. 2.70 reduces to Eq. 2.42 which represents the original Burland's model. Also, the maximum of Eq. 2.70 with respect to the hydrostatic axis can be easily verified to lie always on the Mohr-Coulomb failure envelope. In this model the CS2, as indicated in Figs. 2.9, does not represent the critical state line for extension loading condition, but it is merely a mathematical artifice to allow the centre of the ellipse to be shifted from the hydrostatic axis. Also, the variable α has been introduced just for the sake of mathematical completeness: in fact, as said before, the geometrical meaning of $\alpha=0$ with regard to the real potential function in the σ space is not yet clear.

In the assumption of associated flow rule and of the validity of the Cam-clay postulate on the plastic volumetric strain, as summarized by Eq. 2.46, the plastic modulus A of

the two Generalized models, may be proved to be equal to Eqs. 2.69 and 2.71, as reported in Tables 2.7 and 2.8 respectively.

As it can be deduced from section 2.7.2 there is not yet concordance of opinion in modelling the plastic strain for OCS. Apart from the particular shape of Hvorslev's surface, which perhaps is a characteristic of the specific soil under consideration, the major discordance arises in the determination of the potential function associated to the plastic deformation inside BS. Among the different opinions previously presented, Dafalias' proposal to define F as a homotetic surface to BS, seems to be the most appealing. In fact, it has the merit to combine simplicity with generality.

Finally, it seems that more theoretical work is needed for a correct approach to describe plastic deformation in anisotropic soils.

However, besides the interesting theoretical approaches and their mathematical consistency, the validity of these different models has to be verified against the actual response of soils in various boundary value problems. Since closed form solutions are generally unfeasible, this can only be accomplished by correctly implementing the different elasto-plastic models in a finite element program.

Thereafter, the subsequent chapters are entirely devoted to the study of a correct finite element implementation .

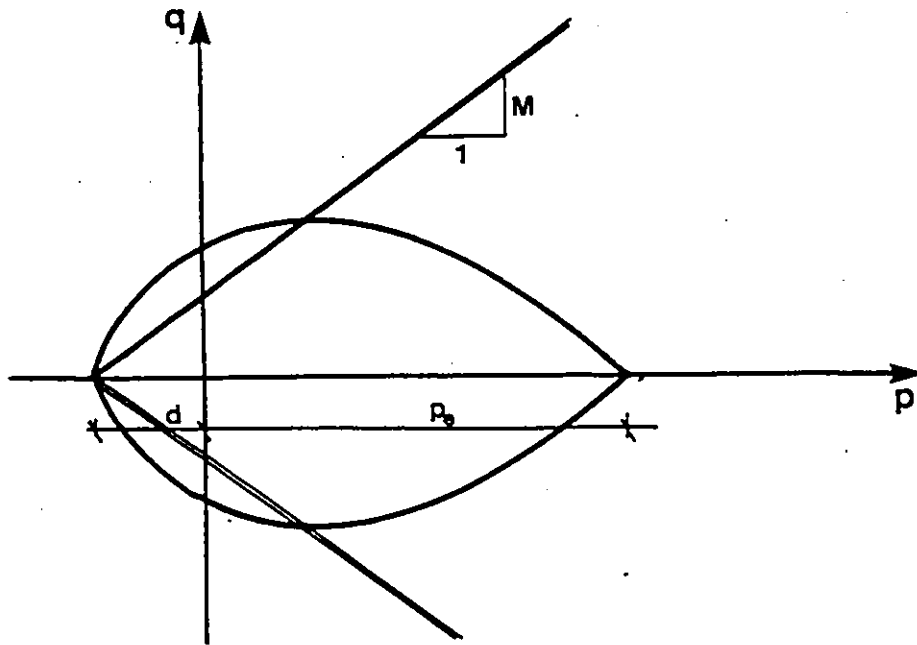


Fig. 2.10 Mapping of the Generalized Taylor yield surface in p vs. q diagram

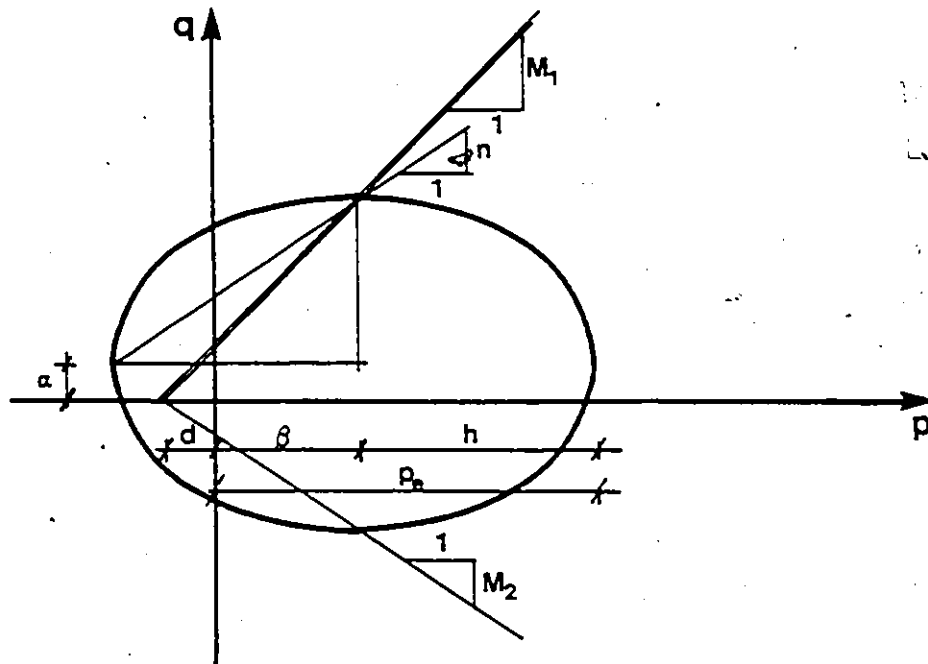


Fig. 2.11 Mapping of the Generalized Burland yield surface in p vs. q diagram

TABLE 2.7

Generalized Taylor model

$$F = \frac{q}{M(\bar{p} - d)} - \ln \frac{d - p}{d + p_e} = 0 \quad 2.68$$

where

$$d = \frac{N}{M} = c \operatorname{ctg} \phi$$

$$M = \frac{3 \sin \phi}{\sqrt{3} \cos \theta - \sin \theta \sin \phi}$$

$$p_e = d + (d - \bar{p}) \exp\left[-\frac{\bar{q}}{M(\bar{p} - d)}\right]$$

plastic modulus

$$A = \frac{\bar{v} p_e}{\lambda - \chi} \frac{\partial F}{\partial p_e} \frac{\partial F}{\partial p} \quad 2.69$$

where

$$\frac{\partial F}{\partial p} = - \frac{M(\bar{p} - d) + \bar{q}}{M(\bar{p} - d)^2}$$

$$\frac{\partial F}{\partial p_e} = \frac{1}{d + p_e}$$

$\bar{p}$, $\bar{q}$ stress invariant associated to the current stress

TABLE 2.8

Generalized Burland model

$$F = (p + \beta)^2 + \frac{(q - \alpha)^2}{n^2} h^2 = 0 \quad 2.70$$

where

$$\alpha = a h; \quad \beta = b h - d$$

$$n = r[2 M_1 - (M_1 - M_2)_{\theta=30^\circ}] ; \quad r = \left(\frac{n}{M_1 + M_2} \right)_{\theta=30^\circ}$$

$$a = n \left(\frac{M_1 - M_2}{M_1 + M_2} \right) ; \quad b = \frac{2n}{M_1 + M_2}$$

$$d = \frac{N_i}{M_i} = c \operatorname{ctg} \phi \quad i = 1, 2$$

$$M_i = \frac{3 \sin \phi_i}{\sqrt{3} \cos \theta - \sin \theta \sin \phi_i} \quad i = 1, 2$$

$$h = \frac{\bar{b} + (\bar{b}^2 - \bar{a}\bar{c})^{1/2}}{\bar{a}} \quad \bar{a} \neq 0$$

$$h = \frac{\bar{c}}{2\bar{b}} \quad a = 0$$

$$\bar{a} = b^2 + \left(\frac{a}{n}\right)^2 - 1$$

$$\bar{b} = \frac{\bar{q} a}{2} - (\bar{p} - d)b$$

$$\bar{c} = \frac{\bar{q}^2}{2} + (\bar{p} - d)$$

$\bar{p}, \bar{q}$ stress invariant associated with the current stress

$$p_e = \beta + h$$

plastic modulus

$$A = \frac{\bar{v}(h-g)}{\lambda-\chi} \frac{\partial F}{\partial h} \frac{\partial F}{\partial p}$$

2.71

where

$$g = \frac{d}{1+b} = \frac{c \cot \phi (M_1 + M_2)}{(M_1 + M_2 + 2n)}$$

$$\frac{\partial F}{\partial p} = 2(p + \beta)$$

$$\frac{\partial F}{\partial p_e} = 2(\bar{a} h - \bar{b})$$

Chapter III

FINITE ELEMENT METHOD APPLIED TO GEOTECHNICAL PROBLEMS

3.1 INTRODUCTION

Seepage, short or/and long term stress analysis are equally essential components of a complete geotechnical study.

An ideal package of Finite Element (FE) programs should therefore include, firstly, the possibility to compute the pore pressure distribution related to the initial flow regime, which is usually under a steady flow condition.

For highly permeable soils, the stress analysis may be performed as in any structural or mechanical problem, with the only peculiarity of taking into account the proper constitutive equation of soils and eventually the buoyant effect of the fluid phase.

For soils having some degree of impermeability, the geotechnical analysis should include the computation of the initial excess pore pressure and its dissipation with time. This can be accomplished by implementing either the Coupled or the Uncoupled theory of consolidation.

In the remainder of this Chapter, after a brief introduction on the Finite Element Method (FEM), a number of previous works concerning the numerical implementation of the aforesaid problems in geotechnique is presented.

3.2 GENERALITY ON THE FINITE ELEMENT METHOD

Although the label "finite element method" first appeared in 1960, when it was used by Clough in a paper on plane elasticity problems, the idea of finite element analysis dates back much further (Polya, 1952 and 1954, Hersch, 1955, Weinberger, 1956).

This method was the result of the development of matrix analysis procedures for structural problems and of the availability of high speed digital computers.

Since then, the method has been greatly expanded and today its application can be seen in any field of engineering science.

Nowadays, the finite element technique is considered as a method which enables to transform systems of integral and differential equations into sets of simple algebraic equations.

In FEM, the closed and bounded subdomain of definition of the solution function, say $f(x, t)$, is subdivided into a "Finite" number of non trivial closed and bounded subdomains called "Elements". A set of points in space, called "Nodal Points" of the system, is used to define the geometry of the element.

The function $f(x, t)$ is then approximated by a piecewise polynomial, Eq. 3.1. The spatial variation of f is described by s , called shape function, while f is the value of $f(x, t)$ at the nodal points.

$$f(\underline{x}, t) = s_1(\underline{x}) f_1(t) = \underline{s} \cdot \underline{f} \quad (3.1)$$

By the use of the interpolating function, Eq. 3.1, and by some mathematical procedures based on variational analysis, the governing field equation of a physical problem is reduced to a simple set of algebraic equations, in which the nodal values $f(t)$ are the unknowns.

Essential for the numerical precision of the solution is the choice of the interpolating function. For 2-D or axisymmetric problems sufficient accuracy can be achieved by using the so called "quadrilateral isoparametric element" (Ergatoudis et al., 1968).

For time dependent problems, $\underline{f}(t)$ is also discretized in time in a form which can be in general written as

$$\underline{f} = \theta \underline{f}_n + (1-\theta) \underline{f}_{n-1} \quad (3.2)$$

where the value of θ depends on the way $f(t)$ is assumed to vary during the interval $dt_n = t_n - t_{n-1}$, Table 3.1. A purely explicit scheme has $\theta = 0$ while a purely implicit scheme has $\theta = 1$. Booker and Small (1975) have shown that any scheme with $\theta > 0.5$ is unconditionally stable.

TABLE 3.1

Different time discretization parameters

function type	θ -value
linear	$\theta = \frac{1}{2}$
parabolic	$\theta = \frac{2}{3}$
logarithmic (Sanduz, 1968)	$\theta = 1 + \frac{1}{\Delta t} - \frac{1}{\ln(1+\Delta t)}$
logarithmic (Hwang et al., 1971)	$\theta = 1 + \frac{1}{\tau_0} - \frac{1}{\ln(1+\tau_0)}$
	$\tau_0 = \frac{\Delta t}{t_{n-1}}$

3.3 SEEPAGE

The basic equation governing the fluid flow or seepage in a porous media is expressed by the Laplace equation in 1.19a.

The corresponding finite element formulations, Eqs. 3.3a, can be obtained from a number of different procedures (Mauersberger, 1965, Huebner, 1975).

$$P \underline{h} = \underline{g} \quad (3.3a)$$

where

$$P = \left[\int_{\Omega} B^T K B d\Omega \right] \quad (3.3b)$$

$$\underline{g} = - \left\{ \int_{\Omega} p s_j d\Omega + \int_{\Gamma} q s_j d\Gamma \right\} \quad (3.3c)$$

$$B = \left[\frac{\partial s_i}{\partial x_j} \right] \quad (3.3d)$$

where p takes into account sources or sinks in the medium and q is the imposed flow on the boundaries.

In a steady confined flow the simple solution of Eqs. 3.3a allows to determine the unknown values of the head pressure h in all nodal points. The velocity can be then calculated from

$$\underline{v} = -K \text{ grad } h = K B \underline{h} \quad (3.4)$$

Theoretical studies had shown that the exact values of $\underline{v}$ can be only calculated on the Gauss integration points of each single element (Barlow, 1976).

For unconfined flow, the analysis becomes more complicated since the position of the free surface is not known a priori.

Taylor and Brown (1967), Finn (1967), Neuman and Witherspoon (1970) solved the aforesaid problem in 2-D following an iterative procedure in which the free surface and the mesh were updated by using empirical laws. At each iteration, a trial free surface was assumed and considered as an impermeable boundary; the convergence was then monitored by verifying that the head pressure, calculated by solving Eqs. 3.3a, was equal to the geometrical height, i.e. $\underline{p} = 0$, along the free surface. This procedure involves the revision of at least a portion of the mesh and of the element stiffness matrices P in that portion at each iteration. Finn and Varoglu (1976) pointed out that the

usual algorithms employed are not able to control the relative size of the elements during iterations. Therefore, they suggested a new approach in which all elements are updated during iterations, in order to keep their relative size constant.

Desai (1976) proposed instead a completely different approach. Here, once Eqs. 3.3a have been solved for given boundary conditions, an approximate location of the free surface is obtained by searching which nodal point satisfies $\pi = 0$. The approximate location is then adjusted or corrected by using the so called "residual approach". Thus, the necessity of revising the mesh is avoided, which can result in considerable saving in computational efforts.

Both methods have advantages and disadvantages but it seems that the former has more generality.

3.4 UNCOUPLED CONSOLIDATION

The parabolic differential equation 1.25 governing the Uncoupled consolidation has essentially the same form of that governing any transient heat conduction problem.

The equivalent FE formulation, Eqs. 3.5a, may be obtained in several different ways, however Galerkin's weighted method appears to be the most immediate.

$$P \ddot{\pi}^e + D \dot{\pi}^e = g \quad (3.5a)$$

$$D = \left[\int_{\Omega} \frac{\gamma_f}{B} s_i s_j d\Omega \right] \quad (3.5b)$$

where P and g are respectively the permeability matrix and the flow vector as defined in Eqs. 3.3b and 3.3c.

Apparently, specific solutions of the Uncoupled consolidation problem by FE are limited and rather restricted to the 1-D case.

Desai and Johnson (1972) solved the 1-D problem for layered soils by employing linear and parabolic 1-D elements. Similarly to Zienkiewicz and Parekh (1970) they adopted the Crank-Nicholson time integration scheme, reducing Eqs. 3.5a to the following form

$$[P + \frac{2}{\Delta t} D] \pi_n^e = g - [P - \frac{2}{\Delta t} D] \pi_{n-1}^e \quad (3.6)$$

They found that the stability and convergence of the solution was highly influenced by the number and type of elements, Figs. 3.1. However, their computational time is very costly, most probably due to the algorithm adopted for the solution of the algebraic system of Eqs. 3.6.

Apparently Gioda and Nova (1982) solved a 2-D Uncoupled problem for homogeneous soils by FE. The time integration scheme adopted is the same as in Desai and Johnson. Unfortunately, no further details could be found either about the type of element employed or about the results obtained.

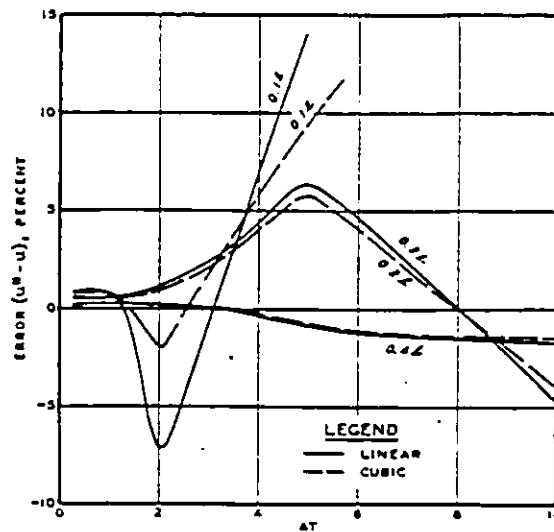


Fig. 3.1a Stability of pore pressure solutions for 21 nodes and $T=0.1$ (after Desai and Johnson, 1972)

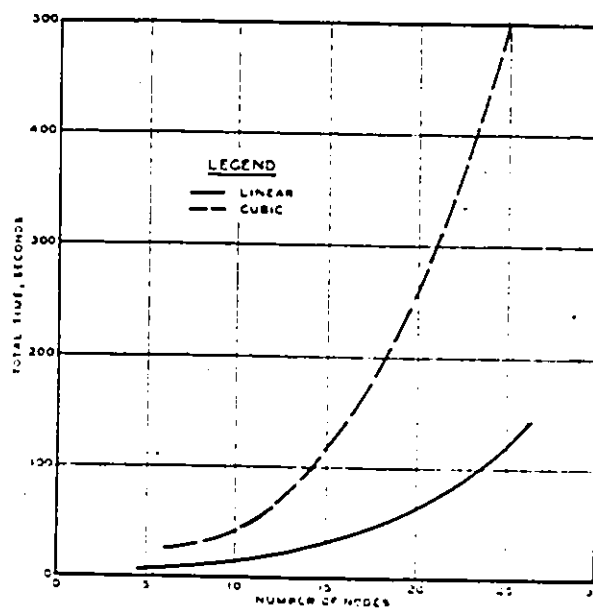


Fig. 3.1b Comparison of total computational times (after Desai and Johnson, 1972)

3.5 EQUILIBRIUM OF A SOLID (DRY) MEDIUM

The state of equilibrium in a solid dry medium may be formulated following three alternative procedures (Desai and Abel, 1972) namely: the displacement (or stiffness) method, the stress (or equilibrium) method and the hybrid method.

Most problems in solid mechanics have been formulated by using the displacement method since it presents several numerical and computational advantages with respect to the other two methods.

In the displacement method, displacement in an element is assumed to be unknown, and the element equations are derived by using the variational procedures based either on the "Principle of Minimum Potential Energy" or on the "Principle of Virtual Work".

The procedures based on the principle of virtual work have apparently gained in the last years more popularity: in fact, it has the advantage of using a concept very familiar to engineers and also contains all the generality required to pursue eventually non-linear analyses.

For dynamic problems the principle of virtual work, Eq. 1.4, is sometimes presented in an extended form as in Eq. 3.7, in order to take into account the dissipation of the internal energy due to viscous propriety of the material.

$$\int_{Q_0} \tilde{\mathbf{g}} \cdot d\tilde{\mathbf{\epsilon}} \, dQ_0 = \int_{\Gamma} \underline{\mathbf{t}} \cdot \delta \underline{\mathbf{u}} \, d\Gamma + \int_Q [\rho(\underline{\mathbf{b}} - \ddot{\underline{\mathbf{u}}}) - c \dot{\underline{\mathbf{u}}}] \delta \underline{\mathbf{u}} \, dQ \quad (3.7)$$

In the displacement method, each of the three displacement components is approximated by an interpolating function of the type Eq. 3.1 so that $\underline{u}$ can be expressed as

$$\underline{u} = S \underline{d} \quad (3.8)$$

where $\underline{d}$ is the displacement at the nodal element, and S , called "Shape Matrix", has the form reported in Table 3.2. Consequently, the virtual Lagrange finite strain can be approximated as

$$\delta \underline{\tilde{\epsilon}} = B_n \delta \underline{d} \quad (3.9)$$

where B_n , known as B-matrix, depends on the current state of deformation and has the form reported in Table 3.2.

The numerical form of the principle of virtual work, Eq. 3.10, can be finally obtained by replacing Eq. 3.8 and Eq. 3.9 into the starting Eq. 3.7.

$$M \ddot{\underline{d}} + D \dot{\underline{d}} + \int_{\Omega_0} B_n^T \underline{\tilde{g}}(t) d\Omega_0 = \underline{f}(t) \quad (3.10a)$$

where

$$M = \int_{\Omega_0} \rho S^T S d\Omega \quad (3.10b)$$

$$D = \int_{\Omega} c S^T S d\Omega \quad (3.10c)$$

$$\underline{f}(t) = \int_{\Gamma} S^T \underline{t} d\Gamma + \int_{\Omega} S^T \underline{b} d\Omega \quad (3.10d)$$

For negligible acceleration and damping parameter $c=0$, Eqs. 3.10 may be further reduced to Eqs. 3.11a or, since equilibrium has also to be respected for any interval of time, they can be expressed in incremental form as in Eqs. 3.11b.

$$\int_{\Omega_0} B_n^T \underline{\tilde{g}}(t) d\Omega_0 = \underline{f}(t) \quad (3.11a)$$

$$\int_{\Omega_0} B_n^T \Delta \underline{\tilde{g}}(t) d\Omega_0 = \Delta \underline{f}(t) \quad (3.11b)$$

In the hypothesis of small deformation the 2nd Piola-Kirchhoff stress tensor $\underline{\tilde{g}}$ is almost equal to the Cauchy stress tensor $\underline{g}$, and B_n becomes a constant matrix, say B , whose components are shown in Table 3.2. Consequently Eq. 3.11b can be reduced to Eq. 3.12.

$$\int_{\Omega} B_n^T \Delta g(\tau) d\Omega = \Delta f(\tau) \quad (3.12)$$

Furthermore, the finite Lagrange strain becomes almost equal to the small Lagrange strain which can be related to the displacement as follows:

$$\underline{\varepsilon} = B \underline{d} \quad (3.13)$$

For the numerical solution of Eqs. 3.12 it is necessary to express the stress in terms of displacement. This can be accomplished by using the proper constitutive equation.

The constitutive equation for elasto-plastic material is represented by Eq. 2.29a, which in the current hypothesis of small deformation may also be expressed as

$$\Delta \underline{g} = C^{ep} \Delta \underline{\varepsilon} = C^{ep} B \Delta \underline{d} \quad (3.14a)$$

where

$$C^{ep} = C^e - C^p = C^e - \frac{C_F^* C_G}{A + C_G^T \cdot \bar{\nabla} F} \quad (3.14b)$$

consequently Eq. 3.12 becomes

$$K^{ep} \Delta \underline{\underline{d}} = \Delta \underline{\underline{f}} \quad (3.15a)$$

where

$$K^{ep} = \int B^T C^{ep} B \, d\Omega \quad (3.15b)$$

Cormeau (1975) proposed to define the rate of the viscoplastic strain as

$$\dot{\underline{\underline{\epsilon}}}^{vp} = \gamma \langle \phi(G) \rangle \bar{\underline{\underline{v}}}_\sigma F \quad (3.16a)$$

where

$$\phi(G) = \left[\exp m \frac{(G - G_o)}{G_o} \right] - 1 \quad (3.16b)$$

or

$$\phi(G) = \frac{(G - G_o)^n}{G_o} \quad (3.16c)$$

while m , n and γ are material parameters.

Zienkiewicz and Cormeau (1973, 1974) found that the corresponding constitutive equation can assume the form of Eq. 3.17a, if the viscoplastic strain rate is discretized according to Eq. 3.2.

$$\Delta \underline{\sigma}_n = C^{vp} (\Delta \underline{\epsilon}_n - \underline{\epsilon}_n^{vp} \Delta t) \quad (3.17a)$$

where

$$C^{vp} = [(C^e)^{-1} + \theta \Delta t H_n]^{-1} \quad (3.17b)$$

$$H_n = \gamma \left[\frac{\partial \phi}{\partial G} \underline{\bar{v}}_G^T * \underline{\bar{v}}_\sigma^F + \phi \underline{\bar{v}}_\sigma^T * \underline{\bar{v}}_\sigma^F \right] \quad (3.17c)$$

Consequently, substituting Eq. 3.17a into Eqs. 3.12, the elasto-viscoplastic problem can be numerically described by the following set of linear equations

$$K^{vp} \Delta \underline{d}_n = \Delta \underline{v}_n \quad (3.18a)$$

where

$$K^{vp} = \int_{\Omega} B^T C^{vp} B \, d\Omega \quad (3.18b)$$

$$\Delta \underline{v}_n = \int_{\Omega} B^T C^{vp} \underline{\epsilon}_n^{vp} \Delta t \, d\Omega + \underline{\Delta f}_n + \underline{\Delta \psi}_n \quad (3.18c)$$

and $\underline{\Delta \psi}_n$ is the residual error in the previous iteration.

Finally, it must be observed that K^{ep} and K^{vp} are symmetric matrices in the hypothesis of associated plastic flow rule (i.e. $G=F$)..

TABLE 3.2

Shape matrix and B-matrices

$$S = \begin{bmatrix} s_i & 0 & 0 \\ 0 & s_i & 0 \\ 0 & 0 & s_i \end{bmatrix} \quad \text{Shape matrix}$$

B-matrix for large deformation

$$B_n = \begin{bmatrix} \frac{\partial s_i}{\partial X} \frac{\partial x}{\partial X} & \frac{\partial s_i}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial s_i}{\partial X} \frac{\partial z}{\partial X} \\ \frac{\partial s_i}{\partial Y} \frac{\partial x}{\partial Y} & \frac{\partial s_i}{\partial Y} \frac{\partial y}{\partial Y} & \frac{\partial s_i}{\partial Y} \frac{\partial z}{\partial Y} \\ \frac{\partial s_i}{\partial Z} \frac{\partial x}{\partial Z} & \frac{\partial s_i}{\partial Z} \frac{\partial y}{\partial Z} & \frac{\partial s_i}{\partial Z} \frac{\partial z}{\partial Z} \\ \left(\frac{\partial s_i}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial x}{\partial X} \frac{\partial s_i}{\partial Y} \right) & \left(\frac{\partial s_i}{\partial X} \frac{\partial y}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial s_i}{\partial Y} \right) & \left(\frac{\partial s_i}{\partial X} \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \frac{\partial s_i}{\partial Y} \right) \\ \left(\frac{\partial s_i}{\partial X} \frac{\partial x}{\partial Z} + \frac{\partial x}{\partial X} \frac{\partial s_i}{\partial Z} \right) & \left(\frac{\partial s_i}{\partial X} \frac{\partial y}{\partial Z} + \frac{\partial y}{\partial X} \frac{\partial s_i}{\partial Z} \right) & \left(\frac{\partial s_i}{\partial X} \frac{\partial z}{\partial Z} + \frac{\partial z}{\partial X} \frac{\partial s_i}{\partial Z} \right) \\ \left(\frac{\partial s_i}{\partial Y} \frac{\partial x}{\partial Z} + \frac{\partial x}{\partial Y} \frac{\partial s_i}{\partial Z} \right) & \left(\frac{\partial s_i}{\partial Y} \frac{\partial y}{\partial Z} + \frac{\partial y}{\partial Y} \frac{\partial s_i}{\partial Z} \right) & \left(\frac{\partial s_i}{\partial Y} \frac{\partial z}{\partial Z} + \frac{\partial z}{\partial Y} \frac{\partial s_i}{\partial Z} \right) \end{bmatrix}$$

B-matrix for small deformation

$$B = \begin{bmatrix} \frac{\partial s_1}{\partial x} & 0 & 0 \\ 0 & \frac{\partial s_1}{\partial y} & 0 \\ 0 & 0 & \frac{\partial s_1}{\partial z} \\ \frac{\partial s_1}{\partial y} & \frac{\partial s_1}{\partial x} & 0 \\ \frac{\partial s_1}{\partial z} & 0 & \frac{\partial s_1}{\partial x} \\ 0 & \frac{\partial s_1}{\partial z} & \frac{\partial s_1}{\partial y} \end{bmatrix}$$

3.5.1 Numerical Algorithms in Elasto-plasticity

In spite of the formal simplicity of the set of Eqs. 3.15a, the numerical solution of an elasto-plastic stress analysis is a very difficult task. In fact, because of the non-linear nature of the problem, a careful choice of the numerical algorithm is required to obtain accurate solutions. Detailed assessments of the solution techniques currently used in elasto-plastic analysis can be found in a wide range of publications (Armen et al., 1972, Oden, 1972, Argyris et al., 1972, Desai and Abel, 1972, Tillerson et al., 1973, Zienkiewicz, 1977).

Because the stiffness matrix of an elasto-plastic problem is stress path dependent and not a unique function of the displacement, the choice of computational techniques is practically restricted to the so-called "Pure Incremental Procedure" and to the "Initial Load Methods".

The "Pure Incremental Procedure" treats the load displacement path as piecewise linear, Figs. 3.2. Solutions obtained following this method, in spite of the high computational cost, do not guarantee accuracy and may, for softening material, not converge.

A more accurate solution may instead be obtained by the "Initial Load Methods" which in addition do not necessarily require any subdivision of the applied load.

The two main classes of Initial Load techniques are known as the "Initial Strain Method" and the "Initial Stress

Method". Although there are many alternative forms in which these methods can be implemented, they are essentially very similar and the differences arise only from the sequence in which certain computations are made.

In the "Initial Strain Method", Figs. 3.3, a first trial solution of the displacement is obtained in the assumption of linear elasticity. The associated variation of the stress field is consequently calculated in the hypothesis of linear elasticity. At this point the method tries to find the plastic strain component, which eventually is associated to the previously calculated stress field, (Desai and Abel, 1972).

The crucial point of this method is the need of using an iterative process whose main step is represented by the Eq. I in Figs. 3.3. It is clear that the Initial Strain Method fails if any of the stress points reaches the failure state, i.e. $A=0$.

The "Initial Stress Method", Figs. 3.4, basically differs from the previous one since it searches for a compatible stress field for a given strain field, rather than viceversa.

The main steps of the Initial Stress Method or ISM are summarized in Table 3.3. A first trial solution $\Delta d_{n(0)}$ is calculated as in step 1, where Δf_n is the current applied load. From the current displacement, the strain $\Delta \epsilon_{n(0)}$ and consequently the compatible stress field $\sigma_{n(i-1)}$ are then

calculated, steps 2-6. The spatial integration of the stress field $\bar{\sigma}_{n(i-1)}$, step 7, gives the equivalent load $\bar{f}_{n(i-1)}$ which is compatible to the current displacement. The residual load $\Delta\psi_{n(i)}$, given as difference between the applied load f_n and $\bar{f}_{n(i-1)}$, step 8, is then balanced by an iterative process as represented by steps 9, 2 - 8.

To compute $\Delta\bar{\sigma}_j$, i.e. the mobilized stress for the current strain increment $\Delta\bar{\epsilon}_j$, step 6, two different procedures are available. Zienkiewicz et al., (1969) proposed to calculate $\Delta\bar{\sigma}_j$ as the equilibrating stress for the mobilized elastic stress $\Delta\bar{\sigma}^e$, Eq. 3.19 in Table 3.4. Herrmann et al. (1983) instead suggested to calculate $\Delta\bar{\sigma}_j$ directly from $\Delta\bar{\epsilon}_j$ by a θ -Wilson scheme as reported in Eq. 3.20 in Table 3.4.

Though the former method may give better accuracy and perhaps reduce the number of steps "m", it has on the other hand the drawback of requiring an estimation of the final elasto-plastic matrix C_j^{ep} .

During the iterative process to balance $\Delta\psi_{n(i-1)}$, the stiffness matrix can be either kept constant and equal to the initial elastic stiffness matrix (Zienkiewicz et al., 1969), or updated as the elasto-plastic tangent stiffness matrix, evaluated at the preexisting stress distribution (Owen and Hinton, 1980), Figs. 3.5.

Following the second procedure, i.e. the "Tangential Stiffness Method" or TSM, the convergence is obtained in relatively few iterations but at each iteration the

computational cost is high since all the stiffness elements should be recalculated.

The first procedure instead, i.e. the "Initial Stiffness Method" or ISM, requires more iterations for convergence but at each iteration its computational cost is less than in TSM since the stiffness is kept constant. Moreover the ISM can be also employed for softening material whereas the TSM can not. In fact, by updating the stiffness matrix for softening material, some of its diagonal terms may become negative bringing the solving process to ill-conditioning.

Nayak and Zienkiewicz (1972a) proposed a procedure to accelerate the convergence for ISM iteration scheme. They practically suggested that an improved value $\Delta d_{n(i)}^*$ of the current displacement $\Delta d_{n(i)}$ may be found as

$$\Delta \tilde{d}_{n(i)}^* = A_{n(i)} \Delta \tilde{d}_{n(i)} \quad (3.21a)$$

where $A_{n(i)}$ is a diagonal matrix whose j-diagonal components are equal to

$$(A_{n(i)})_j = 1 + \left(\frac{\Delta h_{n(i)}}{\Delta d_{n(i)}} \right)_j \quad (3.21b)$$

$$\Delta \tilde{h}_{n(i)} = [(K^e)^{-1} K^p A_{n(i-1)}] \Delta \tilde{d}_{n(i)} \quad (3.21c)$$

and

$$\left(\frac{\Delta h_{n(i)}}{\Delta d_{n(i)}} \right)_j \quad (3.21d)$$

is the ratio of the j -component of $\Delta d_{n(i)}$ and $\Delta h_{n(i)}$.

Thomas (1984) pointed out that the acceleration method as described above "can become unstable and non convergent except in certain special cases (e.g. one dimensional problems). The cause of this is due to the size of the component $\Delta d_{n(i)}$ in the denominator of Eq. 3.21d. If, at any nodal point, these components become small or zero, the corresponding acceleration coefficients will become excessively large and the method will fail. This situation is more likely to arise in multidimensional problems where there is a complex deformation field and stress rotation. Ill-conditioning of this type also occurs with the Aitken δ_2 acceleration method (Jenning, 1977)".

Thomas therefore proposed to reformulate the procedure in term of a single acceleration parameter namely

$$\Delta d_{n(i)}^* = \alpha_{n(i)} \bar{\Delta d}_{n(i)} \quad (3.22a)$$

where

$$\alpha_{n(i)} = \frac{\bar{\Delta d}_{n(i)}^T \{ \bar{\Delta d}_{n(i)} + \bar{\Delta h}_{n(i)} \}}{\bar{\Delta d}_{n(i)}^T \bar{\Delta d}_{n(i)}} \quad (3.22b)$$

$$\alpha_{n(0)} = 1 \quad (3.22c)$$

and apparently

$$\Delta \bar{d}_{\sim n(i)} = \Delta d_{\sim n(i)} - \Delta d_{\sim n(i-1)} \quad (3.22d)$$

$$\Delta \bar{h}_{\sim n(i)} = [(K^e)^{-1} K^p \alpha_{n(i-1)}] \Delta \bar{d}_{\sim n(i)} \quad (3.22e)$$

For completeness it is necessary to say that perhaps Thomas' derivation of the $\alpha_{n(i)}$ parameter may contain an inexactness. In fact, reformulating his idea as reported in appendix E, the improved value $\Delta d_{\sim n(i)}$ should be calculated as follows

$$\Delta d^*_{\sim n(i)} = \alpha_{n(i)} \Delta d_{\sim n(i)} \quad (3.23a)$$

where

$$\alpha_{n(i)} = \frac{\Delta d^T_{\sim n(i)} \{ \Delta d_{\sim n(i)} + \Delta h_{\sim n(i)} \}}{\Delta d^T_{\sim n(i)} \Delta d_{\sim n(i)}} \quad (3.23b)$$

$$\Delta d_{n(i)} = -(K^e)^{-1} \Delta \psi_1 \quad (3.23c)$$

$$\Delta h_{n(i)} = [(K^e)^{-1} K^p \alpha_{n(i-1)}] \Delta d_{n(i)} \quad (3.23d)$$

Another important aspect to consider, in order to correctly implement an elasto-plastic problem, is the calculation of the reduction factor "r", which establishes what portion of the mobilized stress lies inside the yield function, step 5 Table 3.3.

In many published works (Zienkiewicz et al., 1969, Nayak and Zienkiewicz, 1972b, Owen and Hinton, 1980) it is assumed that the "r-factor" follows some sort of linear relationship with the yield function. This can be true only for simple stress paths, such as the conventional triaxial loading path, but in general this assumption may lead to some degree of error.

A correct way to calculate the "r-factor" could be instead pursued by searching for which value of $\sigma_{n-1} + r \Delta \sigma^e$ the yield surface is satisfied, namely: "

$$G((\sigma_{n-1} + r \Delta \sigma^e), k) = 0 \quad (3.24)$$

This approach has been followed by Siriwardane and Desai (1983) who were able to find nice closed form solutions of r-factor for some isotropic yield functions.

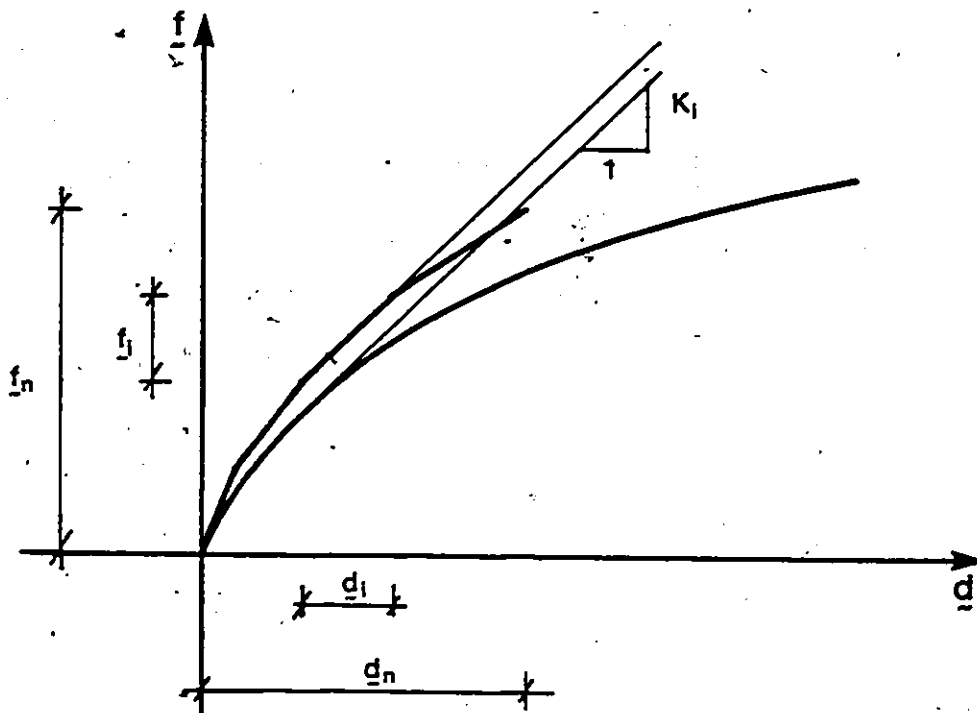
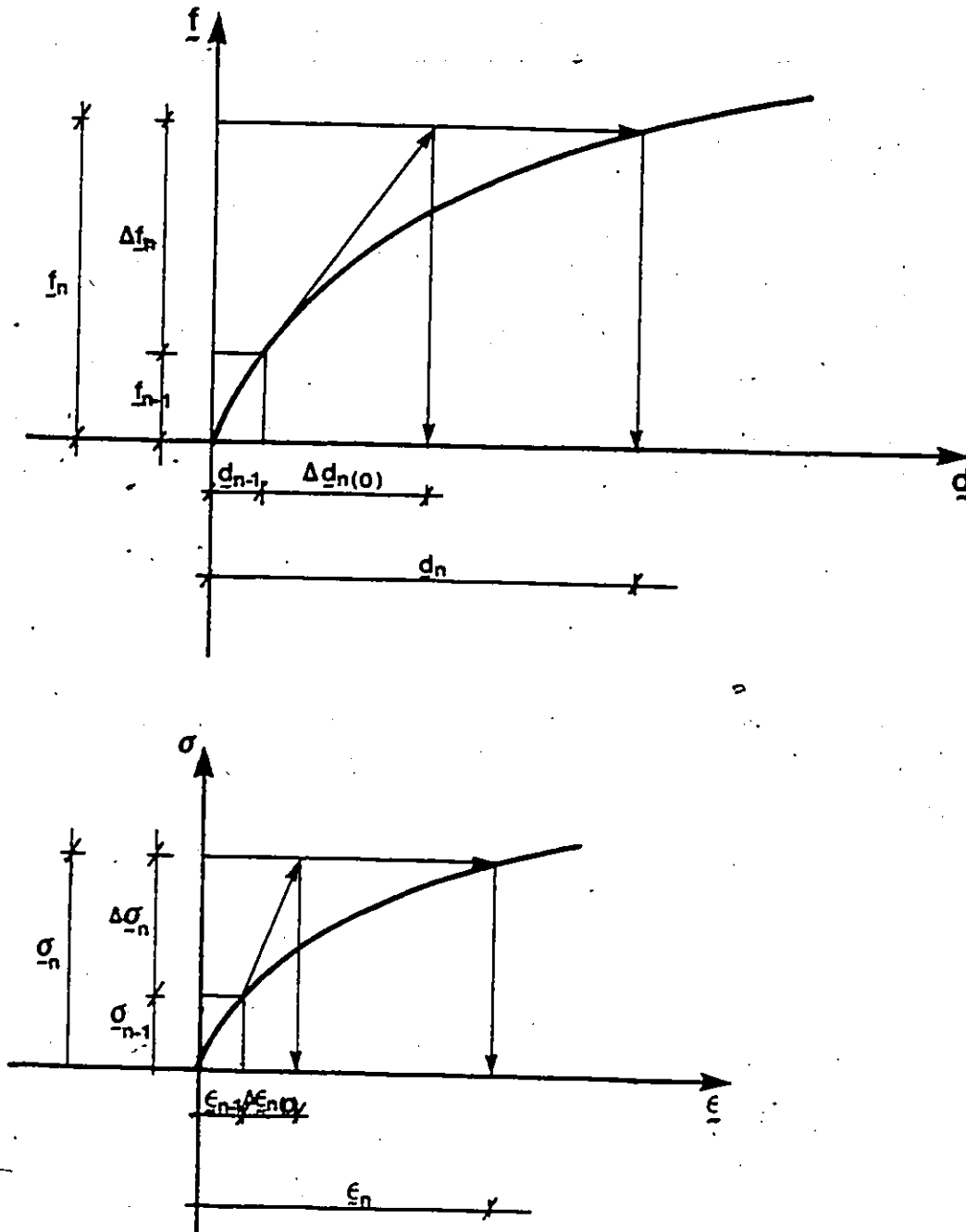


Fig. 3.2 Pure Incremental Procedure



$$\Delta \epsilon_{(i)}^P = \frac{\epsilon_{G(i)}^T [\Delta \epsilon_{(i)} - \Delta \epsilon_{(i-1)}^P]}{A} \quad (1)$$

Fig. 3.3 Initial Strain Method

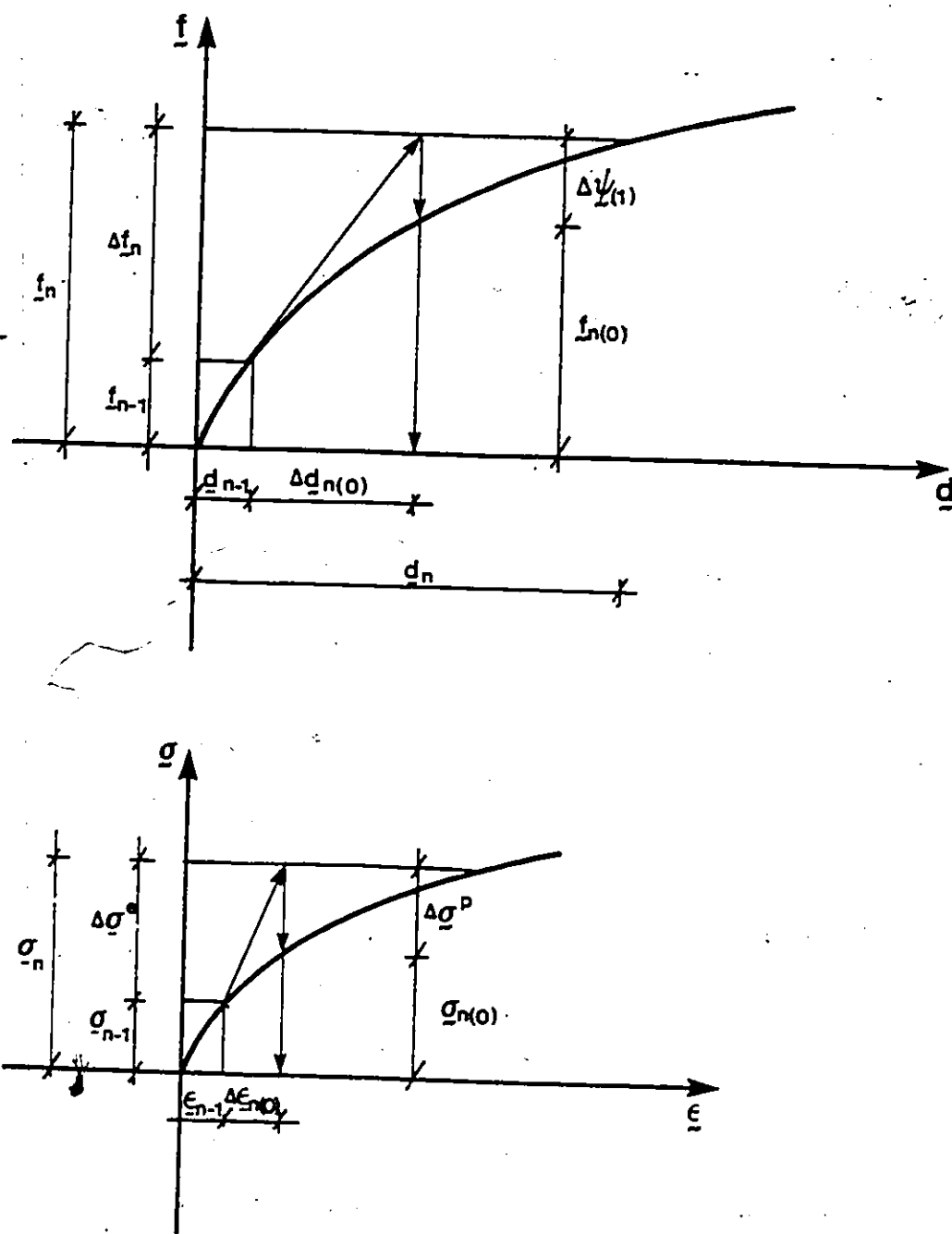


Fig. 3.4 Initial Stress Method

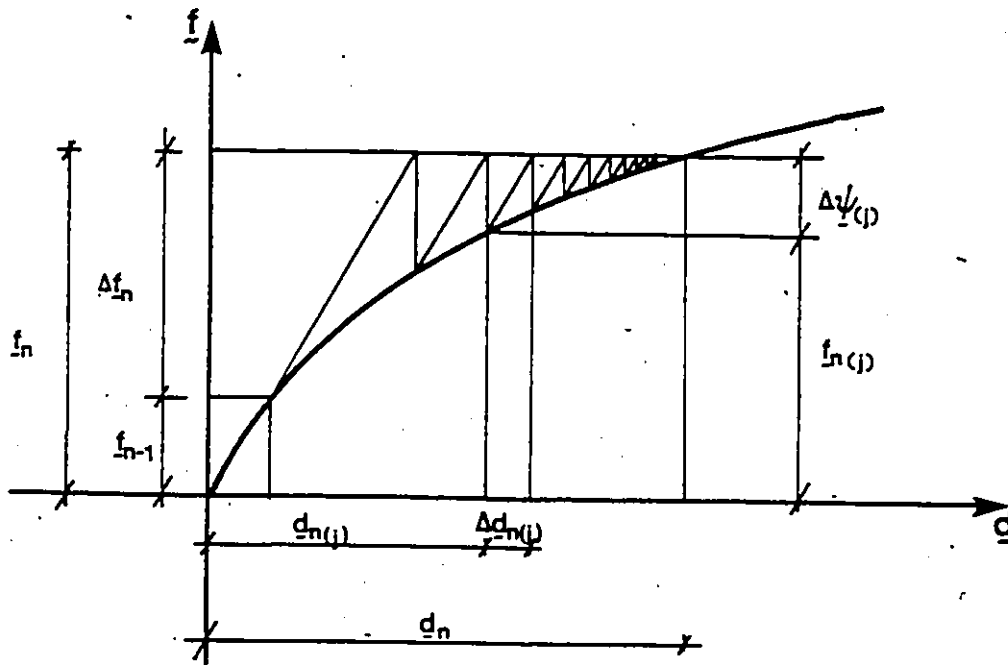


Fig. 3.5a Initial Stress Method with Initial Stiffness

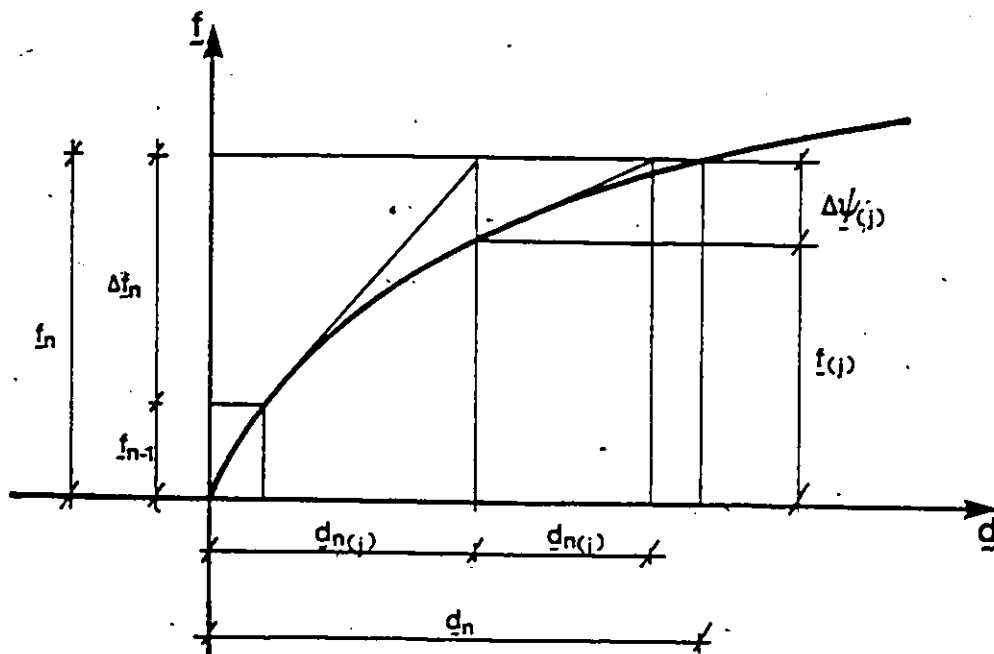


Fig. 3.5b Initial Stress Method with Tangential Stiffness

TABLE 3.3

Summary of steps followed in the Initial
Stress Method

During a particular load increment Δf_n

1. Calculate the increment in displacement by solving $\Delta d_{n(o)} = (K_{n-1})^{-1} \Delta f_n$
2. Compute corresponding strain increment as $\Delta \epsilon_{n(o)} = B \cdot \Delta d_{n(o)}$
3. Calculate the current stress level in the assumption of linear elasticity $\sigma_n = \sigma_{n-1} + C^e \Delta \epsilon_{n(o)}$
 $= \sigma_{n-1} + \Delta \sigma^e$
4. Check if the current stress lies inside the yield surface in which case, σ_n is the actual stress mobilized in the continuum (structure) and is in equilibrium with the current level of applied forces f_n .

If instead σ_n falls outside the yield surface, plasticity develops and find the actual stress mobilized for the current displacement. $d_n = d_{n-1} + \Delta d_n$, in other words proceed to step 5.

5. Differentiate the stress increment contained inside the yield surface from the offset one. $\Delta \sigma^e = r \Delta \sigma^e + (1 - r) \Delta \sigma^e$
 $= r \Delta \sigma^e + s \Delta \sigma^e$

$$\sigma_{n(o)} = \sigma_{n-1} + r \Delta \sigma^e$$

*

r: reduction factor

6. Compute the compatible stress with the current displacement (or strain increment)

$$\bar{\sigma}_{n(i-1)} = \bar{\sigma}_{n(o)} + \sum_{j=1}^m \Delta \bar{\sigma}_j$$

$\Delta \bar{\sigma}_j$ mobilized stress increment defined in Table 3.4.

7. Compute the equivalent load to the $\bar{\sigma}_n$ stress

$$\bar{f}_{n(i-1)} = \int_{\Omega} B^T \bar{\sigma}_{n(i-1)} d\Omega$$

8. Calculate the residual load yet to be balanced

$$\Delta \psi_{n(i)} = f_n - \bar{f}_{n(i-1)}$$

9. Calculate the additional displacement for the residual load

$$\Delta d_{n(i)} = K^{-1} \Delta \psi_{n(i)}$$

Go to Step 2

TABLE 3.4

Computation of the compatible stress for the
current strain increment (step 6 Table 3.3)

Zienkiwicz et al. (1969)

$$\Delta \bar{\sigma}_j = \Delta \bar{\sigma}^e - (d\Gamma \epsilon_F)_j = \Delta \bar{\sigma}^e - \Delta \bar{\sigma}_j^p \quad 3.19a$$

where

$$\Delta \bar{\sigma}^e = \frac{s}{m} \Delta \sigma^e = \frac{s}{m} C^e \Delta \epsilon_n(j) = C^e \Delta \bar{\epsilon}_j \quad 3.19b$$

$$d\bar{l} = \frac{\nabla_{\sigma} G^T \Delta \bar{\sigma}^e}{A + \epsilon_G^T \nabla_{\sigma} F} \quad 3.19c$$

Hermann et al (1983)

$$\Delta \bar{\sigma}_j = [\theta C_j^{ep} + (1-\theta) C_{j-1}^{ep}] \Delta \bar{\epsilon}_j \quad 3.20a$$

where

$$\Delta \bar{\epsilon}_j = \frac{s}{m} \Delta \epsilon_n(j) \quad 3.20b$$

$$C_j^{ep} \text{ should be estimated (see Hermann, 1983) in accordance with the current strain } \epsilon_n(i) \quad 3.20c$$

$$\epsilon_n(i) = \epsilon_{no} + \sum_{j=1}^i \Delta \bar{\epsilon}_j \quad 3.20d$$

$$\epsilon_{no} = \epsilon_{n-1} + r \Delta \epsilon_n \quad 3.20e$$

3.6 EQUILIBRIUM OF A SOLID SATURATED MEDIUM (COUPLED PROBLEM)

The first successful attempt to solve the Coupled consolidation problem for elastic material by FE was made by Sandhu and Wilson in 1969. In that work, the authors applied Gurtin's variational method (Gurtin, 1964, Sandhu, 1968) and found that the associated functional, say Ω_m of the Coupled consolidation problem may be written as in Eq. 3.25 in Table 3.5.

Approximating $\underline{u}$ and $\underline{\pi}^e$ by the appropriate shape function, Eqs. 3.28, and minimizing the obtained functional Ω_m with respect of the nodal variable $\underline{d}$ and $\underline{\pi}^e$, they were able to obtain the two sets of equations, Eqs. 3.26 and 3.27, reported in Table 3.5.

Since each convoluted term may be approximated as in Eq. 3.29b, they were finally able to formulate the Coupled problem by the following two sets of linear algebraic equations

$$\begin{bmatrix} K^e & C \\ C^T & \theta \Delta t P \end{bmatrix} \begin{Bmatrix} \underline{d}_n \\ \underline{\pi}_n \end{Bmatrix} = \begin{Bmatrix} \underline{f}_n \\ \underline{g}_n \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ -C^T & -(1-\theta) \Delta t P \end{bmatrix} \begin{Bmatrix} \underline{d}_{n-1} \\ \underline{\pi}_{n-1} \end{Bmatrix} \quad (3.30a)$$

$$(3.30b)$$

where

$$\underline{g}'_n = \Delta t [\theta \underline{g}_n + (1-\theta) \underline{g}_{n-1}] \quad (3.30c)$$

After Sandhu, Yokoo et al. (1971a and 1971b), Hwang et al. (1971), Valliappan and Lee (1975) analyzed the process of consolidation by always using Gurtin's variational method. Ghaboussi and Wilson (1973) extended the functional Ω_m for compressible fluids.

A different approach was followed by Christian and Bohemer (1970) who simulated the process of consolidation by allowing the solid to undergo deformation with controlled volume change at successive stages. However following Hwang et al. (1971), the method is cumbersome and special problems arise, particularly across the interface between materials with different properties.

Hwang et al. (1972) derived the FE formulation of the Coupled problem in more direct way without using Gurtin's variational method. In fact, they applied Galerkin's method directly to Cauchy's equation of equilibrium and to the continuity of mass, Eqs. 1.30 and 1.33, and eventually were able to obtain the following two sets of equations:

$$K^e \underline{d}(t) + C \underline{\pi}^e(t) = \underline{f}(t) \quad (3.31a)$$

$$C^T \dot{\underline{d}}(t) + P \underline{\pi}^e(t) = \underline{g}(t) \quad (3.31b)$$

where all the matrices and vectors have been already defined in Table 3.6.

The discretization in time of the above equations may be obtained in several ways. Smiths and Hobbs (1976) adopted, for instance, the Crank-Nicholson scheme while Belkeziz and Magnan (1982) employed Garlerkin's method. More flexibility and generality may be achieved by using the θ -Wilson scheme, Eq. 3.2, as proposed for instance by Chang and Duncan (1983). In this case Eqs. 3.31 take the form of the previously reported Eqs. 3.30.

TABLE 3.5

Sandhu's Finite Element derivation of the Coupled elastic
problem

Hypothesis of incompressible fluid and small deformation -

$$\begin{aligned} \Omega(u, \pi) = & \int_{\Omega} \left[\frac{1}{2} \underline{\underline{g}}^T * \underline{\underline{\varepsilon}} - \underline{\underline{\rho}} \underline{\underline{b}}^T * \underline{\underline{u}} + \pi * \underline{\underline{\varepsilon}}_v - \frac{1}{2} \underline{\underline{g}}^* \underline{\underline{v}}_r (\nabla_{\underline{\underline{x}}} \pi + \underline{\underline{\rho}}_f \underline{\underline{b}}) \right] d\Omega \\ & - \int_{\Gamma_1} \underline{\underline{t}} \quad \underline{\underline{u}} \, d\Gamma_2 + \int_{\Gamma_2} \underline{\underline{g}} \, \pi * \underline{\underline{q}} \, d\Gamma_2 \end{aligned} \quad 3.25$$

$$K^e \underline{\underline{d}} + C \underline{\underline{\pi}} = \underline{\underline{f}} \quad 3.26$$

$$C^T \underline{\underline{d}} + \underline{\underline{g}} \otimes P \underline{\underline{\pi}} = \underline{\underline{g}} \otimes \underline{\underline{g}} \quad 3.27$$

where

$$\Pi = \underline{\underline{g}} \cdot \underline{\underline{\pi}} \quad 3.28a$$

$$u = S \cdot \underline{\underline{d}} \quad 3.28b$$

$$K^e = \int_{\Omega} B^T C^e B \, d\Omega \quad \text{elastic stiffness matrix}$$

$$P = \int_{\Omega} \underline{\underline{B}}^T K \underline{\underline{B}} \, d\Omega$$

$$C = \int_{\Omega} (B^T m) * \underline{\underline{S}} \, d\Omega$$

$$m = \{1, 1, 1, 0, 0, 0\}^T$$

B and $\bar{B}$ as in Table 3.2

$\tilde{f}$ and $\tilde{g}$ similar to equations 3.13 and 3.3c

$$\tilde{g} = 1$$

Discretization of the convoluted terms as

$$\tilde{g}^{\oplus} h(t) = \int_0^t h(\tau) \tilde{g}(t-\tau) d\tau = \int_0^t h(\tau) d\tau \quad 3.29a$$

$$\int_{t_{n-1}}^{t_n} h(\tau) d\tau = \Delta t [\theta h_n + (1-\theta) h_{n-1}] \quad 3.29b$$

3.6.1 Time integration

To achieve accuracy in the numerical results, a proper choice of the temporal discretization is extremely important.

A number of different θ -values has been proposed and tested, but still further investigations are necessary for a general positive recommendation.

It appears however, following Sandhu et al. (1977), that good results may be obtained by assuming a logarithmic variation of $\underline{\pi}^e$ with time. In this case θ could be calculated by either one of the two last equations in Table 3.1.

Sandhu observed that "the logarithmic interpolation scheme as proposed by Hwang and al. (1971) has to be considered as an unnecessarily expensive technique". He instead proposed to subdivide the time domain into a number of intervals, as shown for instance in Table 3.6, where Δt is assumed constant. The θ -parameter is then calculated following his earlier suggestion, Table 3.1, and consequently the product $\theta \Delta t$, as well as the global stiffness matrix, become constant values in each time subdomain.

TABLE 3.6

Sandhu's time discretization scheme (after Sandhu et al.
1976)

Steps	Δt	time domain	
10	0.01	0. ,	0.1
10	0.1	0.1 ,	1.1
10	1.0	1.1 ,	11.1
9	10.0	11.1 ,	101.1
10	100.0	101.1 ,	1101.1
8	1000.0	1101.1 ,	9101.1

3.6.2 Choice of Elements

In a Coupled consolidation problem the unknown field variables are the displacement and the excess pore pressure which is essentially a stress.

In the theory of small deformation, stress is proportional to the derivative of the displacement. This implies that the stress is in general described by function one degree lower than that of the displacement.

This theoretical consideration probably led Sandhu (1968) to approximate u and π^e by a parabolic and a linear shape functions respectively. He essentially defined a "composite" triangular element, symbolized as 63, having defined the displacement degree of freedom in 6 nodes, while the pore pressure only at three corner points.

Since then, other types of composite elements have been employed to analyze the Coupled problem, as for instance the "84 isoparametric composite element" in which displacements are defined in all 8 nodes, while the pore pressure is defined only at the four corner points.

Sandhu et al. (1977) presented an interesting comparative analysis on the performance of different types of elements. In that work the authors investigated the performance of composite and standard elements (63, 84, 66, and 88), comparing the numerical results with the exact solution for a 1-D consolidation problem in linear elasticity, Figs. 3.6.

It appears from this investigation that standard elements give better estimate of settlement than the corresponding composite ones. On the other hand the standard elements give errors in pore pressure which do not dissipate in time and are oscillatory in space. Composite elements show also errors in pore pressure but they tend to decrease with time.

This type of errors suggested Cividini et al. (1982, 1983) to apply the so-called "compatible element" to the consolidation problem. Though they were able to obtain some interesting results, this new type of element presents the drawback that the compatibility of the displacement field is respected only in an average sense, i.e. the inter-element compatibility might be violated locally.

An alternative and very appealing approach has been proposed by Reed (1982, 1984) who applied a so called "smoothing technique". Reed's analysis bears on the fact that the pore pressure distribution, obtained by employing standard 8-nodes isoparametric element, gives the exact solution at the 4 Gauss integration points. The correct nodal pore pressure might then be extrapolated from a plane fitting the exact pore pressure values at the Gauss integration points.

Reed chose to fit the exact values by a biquadratic interpolation function and eventually found out that the "smoothed" (correct) nodal pressure π^e is related to the initial (and inexact) nodal pore pressure π^e by the following relation:

$$\tilde{\pi}^e = M \pi^e \quad (3.32a)$$

where

$$M = \frac{1}{6} \begin{bmatrix} 2 & 4 & -2 & 0 & 0 & 0 & -2 & 4 \\ 0 & 4 & 0 & 2 & -1 & 0 & -1 & 2 \\ -2 & 4 & 2 & 4 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & 4 & 2 & 4 & -2 & 0 \\ -1 & 0 & -1 & 2 & 0 & 4 & 0 & 2 \\ -2 & 0 & 0 & 0 & -2 & 4 & 2 & 4 \\ 0 & 2 & -1 & 0 & -1 & 2 & 0 & 4 \end{bmatrix} \quad (3.32b)$$

In this way he could achieve accuracy for both displacement and pore pressure using the conventional 8-nodes isoparametric element.

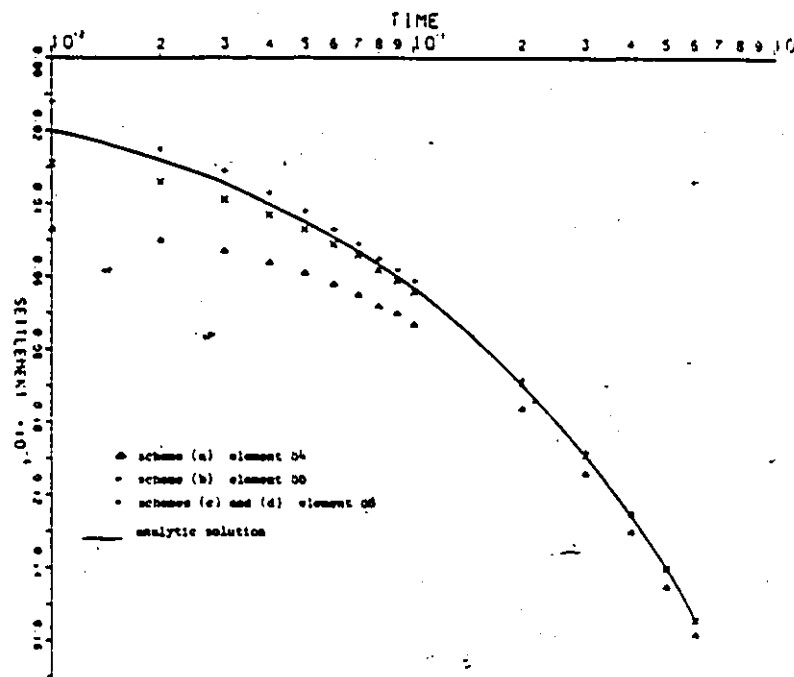


Fig. 3.6a Surface settlement with time for different types of elements (after Reed, 1982)

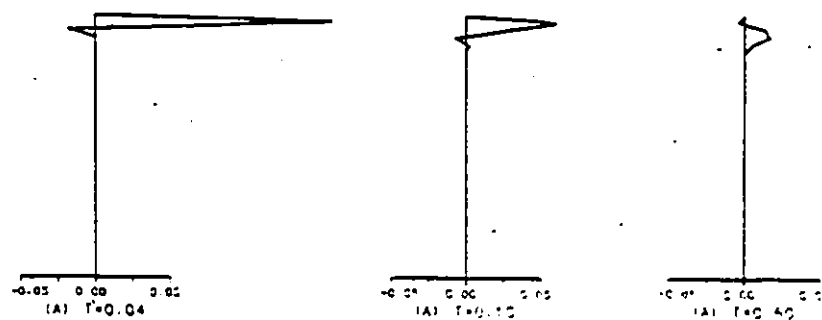


Fig. 3.6b Error in pore pressure employing element 84 (after Reed, 1982)

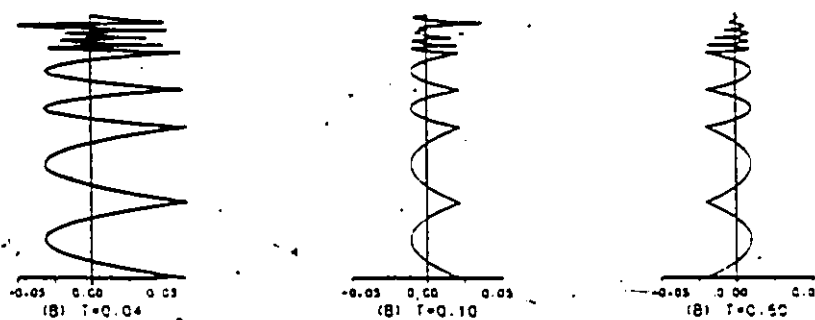


Fig. 3.6c Error in pore pressure employing element 88 (after Reed, 1982)

3.6.3 Non Linear Elastic and Elasto-plastic Numerical Solutions in Coupled Consolidation Problems

Most of the works so far cited have analyzed the Coupled problem assuming a linear elastic material.

The first attempt to take into account the non linear behaviour of soil was apparently made by Lewis et al. (1976). In that work the Young elastic modulus E was assumed to vary following an hyperbolic law as proposed by Duncan and Chang (1970).

Akai and Tamura (1977), Matsui and Abe (1981) introduced a more complete elasto-plastic soil model. In particular the latter authors implemented the Coupled problem assuming the soil to respond according to a modified version of the Cam-clay model. The numerical analysis was carried out by using quadrilateral elements composed of four triangular elements (Wilson, 1965) and the behaviour of soil was assumed to be linear at each time interval.

Siriwardane and Desai (1981), Chang and Duncan (1983) also analyzed the Coupled problem by adopting a modified Cam-clay model. In the first work triangular elements were used, while the more refined 8-nodes isoparametric elements were employed by the latter authors.

Unfortunately none of the above cited works mentions the precautions adopted for reducing the errors in pore pressure which the employed elements have certainly produced. In addition, some of this works do not explain the type of elasto-plastic algorithm used, so that no informations are

available to concretely judge the correctness of the FE
implementation.

Chapter IV

ALTERNATIVE FINITE ELEMENT FORMULATIONS AND ALGORITHMS FOR SOME GEOTECHNICAL PROBLEMS

4.1 INTRODUCTION

The finite element method has so far proved to be a very powerful and versatile tool in solving the different aspects involved in the analysis of different geotechnical problems.

However, as pointed out in the previous Chapter, there are still a number of numerical difficulties to be yet solved. As a matter of fact solving algorithms, stability, accuracy, efficiency and reliability of the FE programs are nowadays an open field of study.

In this research therefore, four basic FE programs have been developed on the basis of new FE formulations and algorithms, namely:

1. SEEP for the seepage analysis of a confined or unconfined flow in a medium with isotropic or anisotropic permeability;
2. TERCON for the analysis of a consolidation problem based on the Uncoupled theory, in a medium with isotropic or anisotropic permeability;
3. ESEPA for an elasto-plastic stress analysis of a solid medium, eventually saturated but highly permeable, under the hypothesis of small deformation;

4. TEPSA for an elasto-plastic stress analysis of a Coupled consolidation problem, in a medium with isotropic or anisotropic permeability, and under the hypothesis of small deformation.

The numerical implementation is presently restricted to 2-D and axisymmetric problems but all program structures are organized for later extension to the more general 3-D case.

In these programs three types of elements may be employed namely 4, 8, or 9-nodes isoparametric elements.

Details of the new FE formulations and of the solving algorithms are presented in the remainder of this Chapter; a description of the common features among the programs and of their possible interaction is presented in the last Section.

4.2 SEEP (SEEPAGE ANALYSIS)

As discussed in Section 3.3, the analysis of a confined flow in a porous medium by FE does not entail any particular problem. In fact, it requires the simple solution of the algebraic system of linear equations 3.3a.

Instead, in an unconfined flow, since the location of the surface is not known a priori, an iterative process is required. Previous FE codes have often incorporated costly and rather involved iterative schemes. An improved algorithm is therefore herein proposed which combines both flexibility and efficiency.

In SEEP, the unconfined flow problem has been approached by introducing a number of so-called "Big Elements", BE, each of them describing a region with the same permeability characteristics. During the numerical computation, each BE is subdivided into smaller elements, in a number which depends on the user's choice.

Initially a completely arbitrary free surface is assumed, as for instance in Figs. 4.1, and the BE defined just below the free surface are identified as "Dynamic Big Elements", DBE.

The numerical solution then proceeds iteratively by considering at each iteration the free surface as an impermeable boundary. Consequently the system of Eqs. 4.41a can be solved for h .

The convergence is controlled by checking if all the nodal points of the DBE lying on the free surface have π equal to zero. If this condition is not met then the nodes of the DBE are moved as showed in Figs. 4.2. In the next iteration all the DBE are again subdivided into a number of smaller sub-elements.

This procedure contains several advantages, one of which is to always assure a good distribution of elements below the free surface, avoiding possible numerical problems. The efficiency of the program is also very high since the updating process is limited only to the coordinates of the DBE nodes and not to the coordinates of each single element.

Furthermore, only the permeability matrices relative to the DBE sub-elements need to be recalculated.

For seepage flow across two vertical layers of soil having different permeability, as for instance in Figs. 4.1, the analysis does not entail any particular difficulty. The discontinuity which the free surface presents on the interface of the two layers may be numerically determined by adequately defining the DBE. The program assumes in this case, for all points lying on the interface BE in Figs. 4.1, the head pressure equal to the current geometrical height.

The outline of the program is provided in Figs. 4.3 and, except for subroutine NODVEL (NODal VELOCITY), it is believed to be self explanatory.

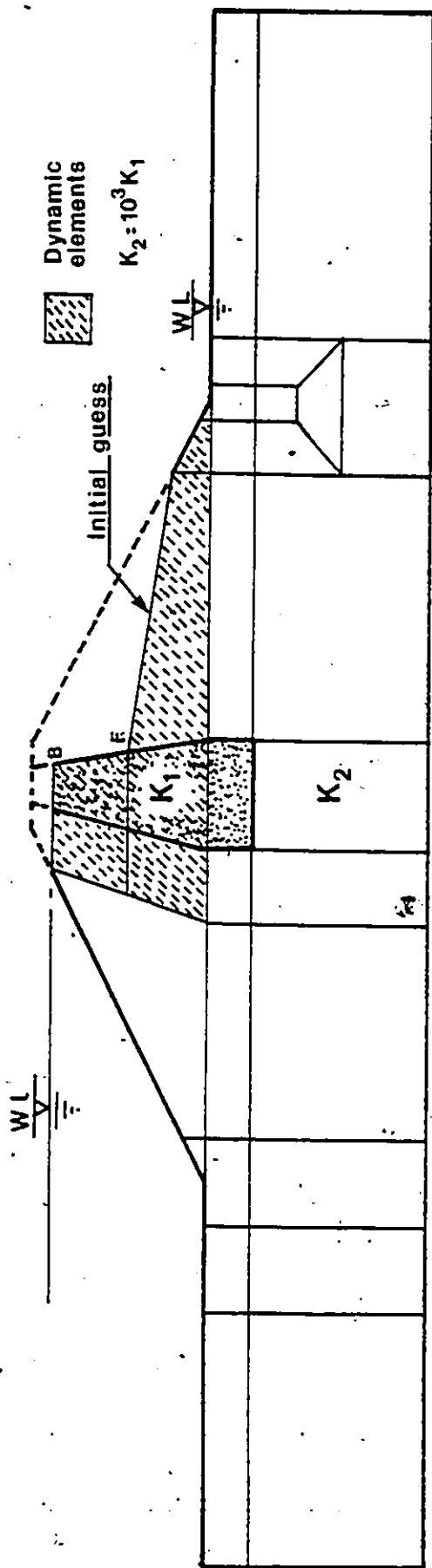


Fig. 4.1 Layout of starting mesh for the seepage analysis in a dam

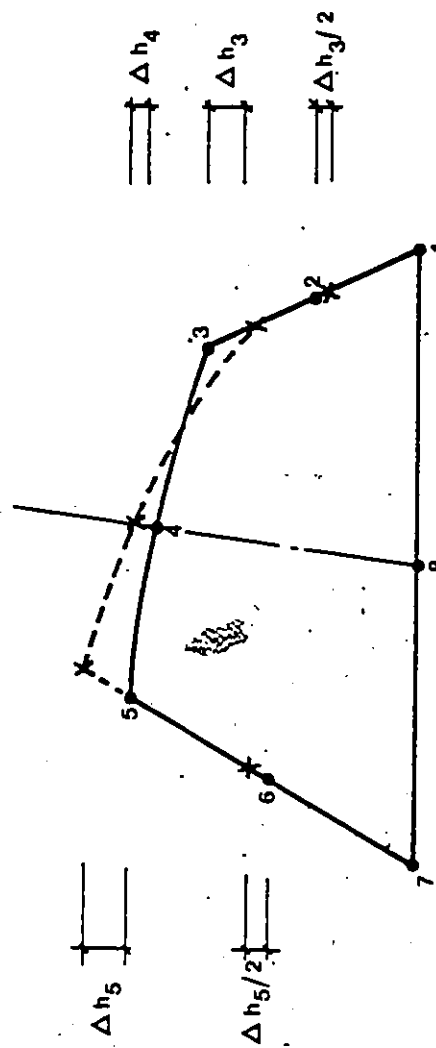


Fig. 4.2 Movement of nodes found on a free surface

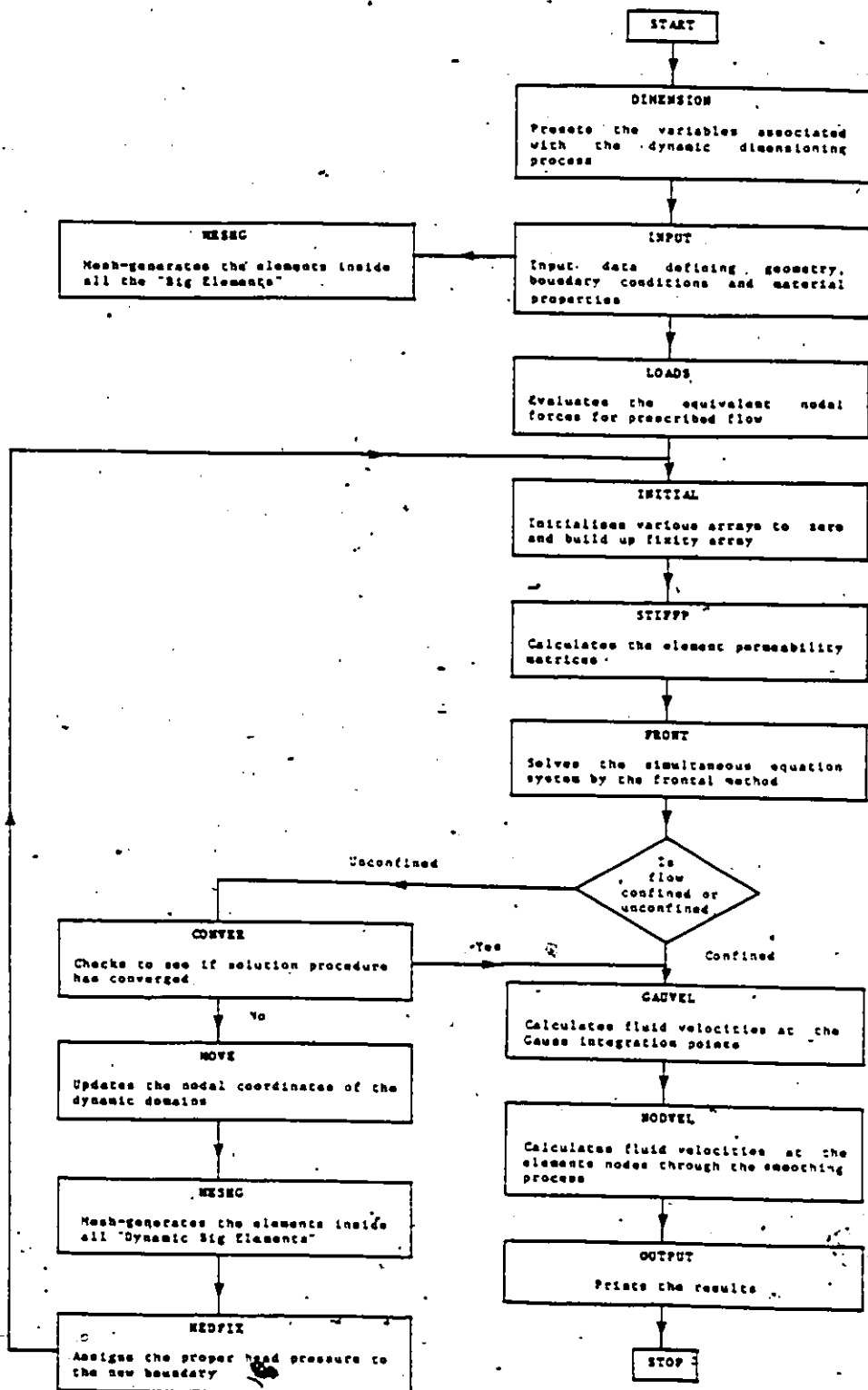


Fig. 4.3 Flowchart SEEP

4.2.1 Velocity at nodes (NODVEL Subroutine)

The calculation of nodal velocity (or of stress at nodes) is of some concern in finite element.

It is known that the velocity as calculated in Eq. 3.4 yields the exact value only at the Gauss integration points (Barlow, 1976). Several techniques have been proposed to estimate the exact value at the nodes (Hinton and Campbell, 1974, Hinton et al., 1975, Herrmann, 1976).

In NODVEL the fluid velocity at nodes is calculated following an extrapolating procedure very closely related to the "smoothing technique" as originally proposed by Hinton and Campbell (1974).

Since in SEEP the total head may be approximated by either a linear or a parabolic function (4-nodes or 8-nodes and 9-nodes isoparametric elements respectively), the resulting velocity field should either be constant or vary linearly within an element. In both cases, the velocity at any point within the element should lie in a plane, and this applies in particular to the exact values at the 4 Gauss integration points.

Like in Reed (1984), the velocity function is approximated by a biquadratic surface of the type

$$v(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta \quad (4.1)$$

where ξ and η are the coordinates in the local reference system of the shape function.

The velocity component $\underline{v}_{(i)}$ at the four Gauss points can therefore be expressed as in Eq. 4.2a from which it is possible to determine the constant $\underline{c}_{(i)}$ of the biquadratic surface, Eq. 4.2b:

$$\underline{v}_{(i)} = M_1 \underline{c}_{(i)} \quad (4.2a)$$

$$\underline{c}_{(i)} = M_1^{-1} \underline{v}_{(i)} \quad (4.2b)$$

where

$$\underline{v}_{(i)} = \{v_j\}_i^T$$

is the i -velocity component at j -Gauss iteration point, while M_1 is a matrix containing the local coordinates of the four Gauss iteration points, Table 4.1.

The values $\underline{v}_{(i)}^*$ of the velocity at node are consequently extrapolated as follows

$$\underline{v}_{(i)}^* = M_2 \underline{c}_{(i)} = M_2 M_1^{-1} \underline{v}_{(i)} \quad (4.3)$$

where

$$\tilde{v}_{(i)}^* = \{v_k^*\}_i^T$$

is the i-velocity component at k-nodal point, while M_1 is a matrix containing the coordinates of the nodes and the local reference system, table 4.1. Finally, the matrix resulting from the product of M_2 and M_1^{-1} is reported also in Table 4.1.

For "m" different elements concurring to the same node, the velocity is calculated as an average of the "m" extrapolated values, namely

$$\bar{\tilde{v}}_{(i)}^* = \frac{\sum_{e=1}^m \tilde{v}_{(i)e}}{m} \quad (4.4)$$

TABLE 4.1

Smoothing matrices to calculate nodal values

$$M_1 = \begin{bmatrix} 1 & -f & -f & f^2 \\ 1 & f & -f & -f^2 \\ 1 & f & f & f^2 \\ 1 & -f & f & -f^2 \end{bmatrix}$$

$$M_2 = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} ; \quad M_2 M_1^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 2e & -1 & 2d \\ a & b & b & a \\ 2d & -1 & 2e & -1 \\ a & a & b & b \\ -1 & 2a & -1 & 2b \\ b & a & a & b \\ 2e & -1 & 2d & -1 \\ b & b & a & a \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

where

$$a = \frac{1+\sqrt{3}}{2} ; \quad b = \frac{1-\sqrt{3}}{2} ; \quad d = \frac{2+\sqrt{3}}{2} ; \quad e = \frac{2-\sqrt{3}}{2}$$

4.3 TERCON (TERZAGHI CONSOLIDATION)

The parabolic differential equation governing the Uncoupled consolidation theory, Eq. 1.25, may be reduced, as already reported, to the following system of linear equations by applying for instance Galerkin's method:

$$P \dot{\pi}^e + D \pi^e = g \quad (4.5)$$

where P and g are respectively the permeability matrix and the equivalent flow vector, as defined in Eqs. 3.3b and 3.3c for the case of seepage. The D matrix is defined in Eq. 3.5b.

Herein, the discretization in time is obtained by the θ -Wilson scheme which, expressed as in Eqs. 4.6, leads to the following system of algebraic equations, Eqs. 4.7.

$$\pi_n^e = \theta \pi_n^e + (1-\theta) \pi_{n-1}^e = \theta \Delta \pi_n^e + \pi_{n-1}^e \quad (4.6a)$$

$$\frac{d\pi_n^e}{dt} = \frac{\Delta \pi_n^e}{\Delta t} \quad (4.6b)$$

$$\overline{P} \Delta \pi_n^e = \Delta g_n^e \quad (4.7a)$$

where

$$\bar{P} = (\theta P + \frac{D}{\Delta t}) \quad (4.7b)$$

$$\Delta g'_n = (\theta \Delta g_n + g_{n-1}) - p \pi_n^e \quad (4.7c)$$

The structure of TERCON follows consequently the FE formulation represented by Eqs. 4.7, Figs. 4.4. The initial value of g is evaluated in subroutine LOADPS, while P and D matrices are calculated in subroutine STIFFP. The time increment and the θ -value are determined in subroutine TMARCH while the complete P matrix and dg vector, Eqs. 4.7b and 4.7c, are built up in subroutine INCREMENT. The variation of excess pore pressure $\Delta \pi_n^e$ is computed in subroutine FRONT and, depending on the user's choice, the flow velocity can be calculated in subroutine VELSEE as

$$\underline{v} = K B \underline{h} = K B (\underline{h}_0 + \frac{\pi_n^e}{\gamma_f}) \quad (4.8a)$$

where

$$\pi_n^e = \sum_{i=1}^n \Delta \pi_i^e \quad (4.8b)$$

and h_0 is the total head corresponding to the preexisting flow condition.

The stage of the consolidation is monitored in subroutine CONVER in which also the necessity of a further iteration is decided.

In the hypothesis of constant bulk modulus and permeability, the matrices D and P are calculated just at the first time increment, while for a more general case they should be updated at each time iteration.

Different choices of time increment Δt_n and θ -values are available and implemented in subroutine TMARCH.

The increment Δt_n can be either kept constant and equal to an input value or calculated as

$$\Delta t_n = t_{n-1} (\exp r - 1) \quad (4.9)$$

where r is an assigned factor. This scheme allowed to identify equally spaced time step in a logarithmic scale.

Alternatively Δt_n may be imposed as constant increment in a certain range of time period, that is

$$\Delta t_n = \frac{\bar{t}_l - \bar{t}_{l-1}}{r} \quad (4.10a)$$

where r is an assigned factor and $\bar{t}_1$ is for instance equal to

$$\bar{t}_\lambda = [10^{-4}, 10^{-3}, \dots, 10^6] \quad (4.10b)$$

The θ -value may either be kept constant to an assigned value or updated according either to Sandhu (1968) or Hwang et al. (1971), Table 3.1.

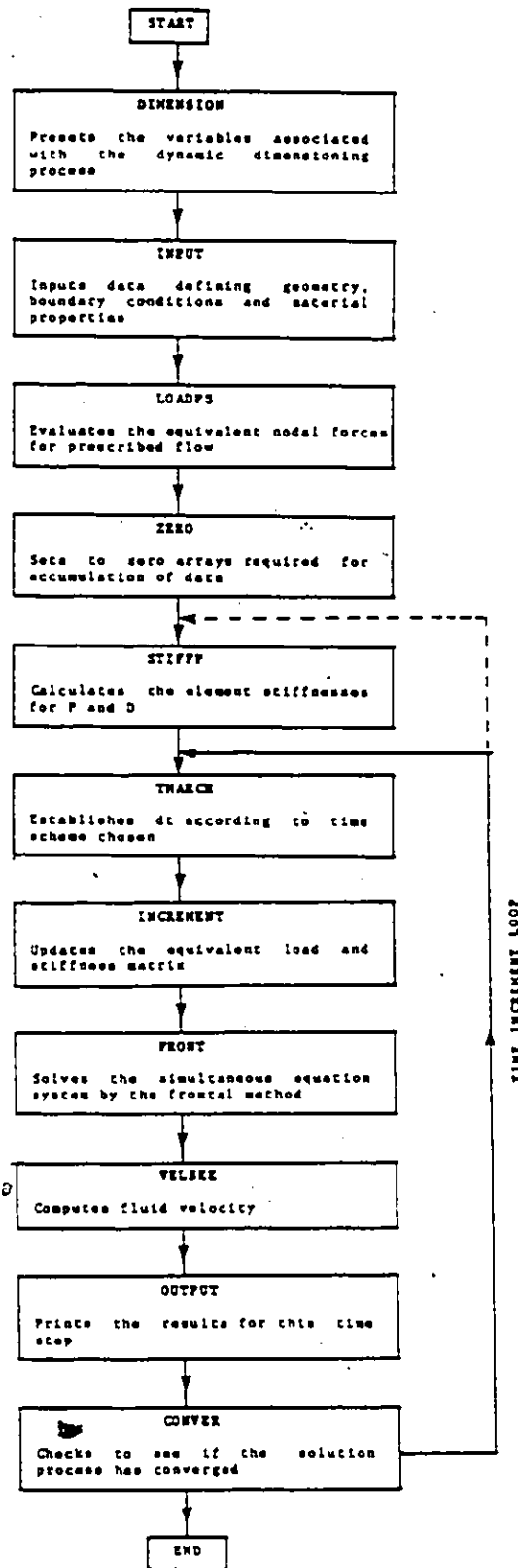


Fig. 4.4 Flowchart TERCON

4.4 STRESS ANALYSIS OF A SATURATED SOLID MEDIUM

In the 3rd chapter, Sandhu's derivation of the FE formulation of the Coupled elastic problem was presented.

Herein, an extension of the Coupled FE formulation to elasto-plastic and elasto-viscoplastic problems, by means of the principle of virtual work, is developed.

As already described in section 3.4, the numerical formulation of the principle of virtual work can be expressed as in Eqs. 3.10. In the hypothesis of small deformation, negligible acceleration $\ddot{\underline{u}}_f = 0$ and damping parameter $c = 0$, the aforesaid equation can be simply reduced to Eq. 4.11.

$$\int_{\Omega} \underline{B}^T \underline{g}(t) d\Omega = \underline{f}(t) \quad (4.11)$$

In a saturated medium $\underline{g}(t)$ is the total stress which, if expressed in terms of effective stress and pore pressure, as in Eq. 4.12b, allows Eq. 4.11 to be rewritten as follows

$$\int_{\Omega} \underline{B}^T \underline{g}'(t) d\Omega - C \underline{\pi}^e(t) = \underline{f}'(t) \quad (4.12a)$$

where

$$\underline{g}(t) = \underline{g}'(t) + \underline{g}'_0 - \underline{\delta}[\underline{\pi}^e(t) + \underline{\pi}^0] \quad (4.12b)$$

$$\underline{\hat{g}} = \{1, 1, 1, 0, 0, 0\}^T = \hat{\underline{m}} \quad (4.12c)$$

$$\pi^e(t) = \underline{\bar{s}}^T \underline{\pi}^e \quad (4.12d)$$

$$C = \int_{\Omega} (B^T \hat{\underline{m}}) * \underline{\bar{s}}^T d\Omega \quad (4.12e)$$

$$\underline{f}'(t) = \underline{f}(t) - \int B^T \underline{g}'_0 d\Omega + C \underline{\pi}^0 \quad (4.12f)$$

while $\underline{g}'_0$ and $\underline{\pi}^0$ are the preexisting effective stress and pore pressure respectively. Finally, $\bar{s}$ is the shape function approximating the excess pore pressure. Also, since the equilibrium has to be respected at any time interval Δt_n , Eqs. 4.12 can be reposed as

$$\int_{\Omega} B^T \Delta \underline{g}'(t) d\Omega - C \Delta \pi^e(t) = \Delta f'(t) \quad (4.13)$$

For elasto-plastic and elasto-viscoplastic materials the stresses are related to the strains as in Eq. 3.14a and 3.17a respectively. Therefore Eq. 4.13 may be alternatively expressed as follows:

Elasto-plastic materials

$$K^{ep} \Delta \underline{d}(t) - C \Delta \underline{\pi}^e(t) = \Delta \underline{f}'(t) \quad (4.14)$$

Elasto-viscoplastic materials

$$K^{vp} \Delta \underline{d}(t) - C \Delta \underline{\pi}^e(t) = \Delta \underline{v}'(t) \quad (4.15a)$$

where

$$\Delta \underline{v}'(t) = \Delta \underline{v}(t) + \Delta \underline{f}'(t) \quad (4.15b)$$

while K^{ep} , K^{vp} and $\Delta \underline{v}(t)$ have been already defined in Eqs. 3.15b, 3.18b and 3.18c respectively.

For highly permeable soils the excess pore pressure dissipates instantaneously, consequently either Eqs. 4.14 or Eqs. 4.15 can be directly solved for the only unknown $\Delta \underline{d}(t)$.

In the more general case $\Delta \underline{\pi}^e \neq 0$ and therefore an additional set of equations is needed to equalize the number

of equations and unknowns. This additional set of equations can be found, following Biot's suggestion, from the equation of continuity of mass for saturated medium, Eq. 1.29, namely

$$\text{div } K \text{ grad } \frac{\pi^e}{v_f} = \frac{d\varepsilon_v}{dt} \quad (4.16)$$

Consistently with the previous derivation, the excess pore pressure π^e and the displacement $\underline{u}$, can be approximated as in Eqs. 4.12e and 3.8 respectively. Finally, applying Galerkin's method, Eq. 4.16 reduces to the following set of equations:

$$-C^T \dot{\underline{d}}(t) - \frac{P}{\gamma_f} \dot{\underline{\pi}}^e(t) = \underline{g}(t) \quad (4.17)$$

where P and $\underline{g}$ are respectively the permeability matrix and the flow vector as defined in Eqs. 3.3b and 3.3c.

Like in TERCON, the discretization on time is obtained by the θ -Wilson scheme. Namely, expressing $\underline{\pi}^e$ and $\dot{\underline{d}}$ as in Eqs. 4.18 and 4.19, it is possible to transform Eqs. 4.17 into the following Eqs. 4.20:

$$\underline{\pi}^e(t) = \theta \Delta \underline{\pi}_{n-1}^e + \underline{\pi}_{n-1}^e \quad (4.18)$$

$$\dot{\tilde{d}} = \frac{\Delta \tilde{d}_n}{\Delta t} \quad (4.19)$$

$$-C^T \Delta \tilde{d}_n - \alpha P \Delta \pi_n^e = \Delta g_n' \quad (4.20a)$$

where

$$\alpha = \frac{\theta \Delta t}{\gamma_f} \quad (4.20b)$$

$$\Delta g_n' = (\theta \Delta g_n + g_{n-1}) + \frac{\Delta t}{\gamma_f} P \pi_{n-1}^e \quad (4.20c)$$

Finally, the complete elasto-plastic and elasto-viscoplastic Coupled FE formulation is obtained from Eq. 4.14, 4.15 and 4.20a, namely:

Elasto-plastic materials

$$\begin{bmatrix} K^{ep} & -C \\ -C^T & -\alpha P \end{bmatrix} \begin{Bmatrix} \Delta \tilde{d}_n \\ \Delta \pi_n^e \end{Bmatrix} = \begin{Bmatrix} \Delta f_n \\ \Delta g_n' \end{Bmatrix} \quad (4.21a)$$

$$(4.21b)$$

Elasto-viscoplastic materials

$$\begin{bmatrix} K^{vp} & -C \\ -C^T & -\alpha P \end{bmatrix} \begin{Bmatrix} \Delta \tilde{d}_n \\ \Delta \tilde{\pi}_n^e \end{Bmatrix} = \begin{Bmatrix} \Delta \tilde{v}_n \\ \Delta \tilde{g}_n' \end{Bmatrix} \quad (4.22a)$$

(4.22b)

In this thesis, the elasto-plastic problems for highly permeable and impermeable soils (Coupled problem) have been implemented.

4.4.1 ESEPA (Effective Stress Elasto-Plastic Analysis)

ESEPA is a program which can analyze, in terms of effective stress, the elasto-plastic deformation of highly permeable soils subjected to a static load. In this case, since the excess pore pressure dissipates instantaneously, the FE problem is limited to the solution of the following reduced form of Eq. 4.21:

$$K^{ep} \Delta \tilde{d} = \Delta \tilde{f}' \quad (4.23)$$

In ESEPA, six different plastic models have been implemented, namely:

1. linear elastic perfectly plastic models in the hypothesis of associated flow rule with failure state described by either von Mises, Tresca, Drucker-Prager or Mohr-Coulomb failure criterion;
2. Generalized Burland's model (Section 2.8) with limiting surface represented by either Drucker-Prager or Mohr-Coulomb failure criterion.

In all plastic models, the elastic component has been assumed to follow the linear Hooke's law for isotropic material, Eq. 2.8.

The solving FE algorithm has been based on the "Initial Stress Method", Section 3.4.1, and at each iteration the stiffness matrix K^{ep} may be either kept constant or updated.

The overall structure of ESEPA is thereby presented in Figs. 4.5.

For saturated soils, the buoyant effect due to the initial pore pressure, along with the preexisting stress distribution, is taken into account in subroutine LOADPS by calculating the equivalent nodal force $f'(t)$ as in Eq. 4.12f.

The initial position of the yield surface at each Gauss integration point is calculated in subroutine PREVOS on the base of the information which the user gives to the program.

The current load increment is calculated in subroutine INCREM, while in subroutine ALGOR it is decided if the stiffness matrix has to be updated. The elasto-plastic

stiffness matrix is computed in subroutine STIFFP according to Eq. 3.15b, and the solution of the system of Eqs. 4.23 is performed in subroutine FRONT.

The real core of ESEPA is represented by subroutine RESIDU which calculates the balanced load for the current displacement field. The algorithm used in subroutine RESIDU is explained in detail in the next section.

The residual forces $\Delta \psi_{n(i)}$ are calculated in subroutine CONVER according to step 8, Table 3.8. The convergence is then monitored by verifying if the norm of the residual forces, calculated as in Eq. 4.24, is less than an imposed tolerance ϵ .

$$\frac{\sum_{i=1}^N (\psi_i)^2}{\sum_{i=1}^N (f_i)^2} * 100 < \epsilon \quad (4.24)$$

where N is the total number of nodes.

Finally, subroutine OUTPUT provides the printout of the displacements, reactions and stresses.

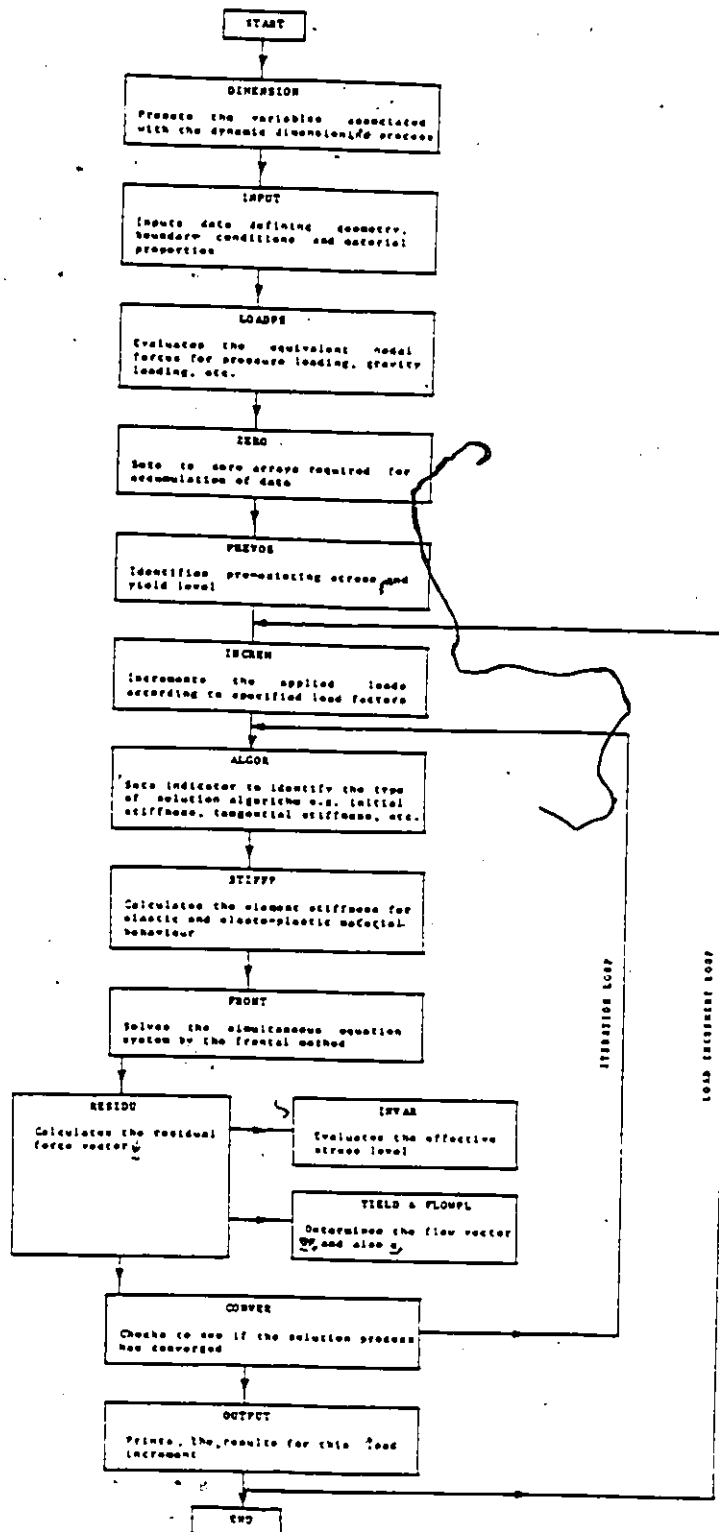


Fig. 4.5 Flowchart ESEPA

4.4.2 Calculation of Unbalanced Load (RESIDU subroutine)

The Initial Stress Method, section 3.4.1, consists in finding iteratively the additional displacements which are used to equilibrate the unbalanced part of the applied load.

The calculation of the balanced load $\underline{f}_n$ for the current displacement $\underline{\Delta d}_n$ is performed in ESEPA by the subroutine RESIDU which is structured as shown in Figs. 4.6.

Two main loops, one on the number of elements and the other on the number of integration points per element, allows to scan the mobilized stress in all Gauss integration points.

At each of these points, the current stress $\underline{g}_n$ is calculated by assuming the material to behave linear elastically for the new displacement increment $\underline{\Delta d}_n$.

In subroutine PLASTIC, the stress $\underline{g}_n$ is checked against the current yield surface. The overall algorithm used in subroutine PLASTIC is shown in Figs. 4.7 where the equivalent pressures, EFFST and SIGYS, are scalar values calculated as in Table 4.2. If $\underline{g}_n$ exceeds the elastic limit, then the intersection of the current stress increment $\underline{\Delta g}^e$ with the yield surface is determined by calculating in subroutine FACTOR the reduction r-factor.

If the current stress is beyond the yield surface, then the actual level of stress $\underline{g}_{n(o)}$ for the current displacement is computed in subroutine ELPLAS as in Eq. 3.19a in Table 3.4.

Finally, the integration of the balanced stress $\sigma_{n(i-1)}$ gives the actual equivalent load which is in equilibrium with the current displacement.

As proved in section 2.8, the plastic modulus A has to be constant and equal to zero for any elastic perfectly plastic model. Instead, in the Generalized Burland model A has to be calculated from Eq. 2.71 in Table 2.8, which depends on the current specific volume and on the location of the yield (potential) function.

It is possible to prove that, since during the equilibrium process the strain increment is considered to be constant, the specific volume can be calculated as follows:

$$\bar{v}_{n(j)} = \bar{v}_{no} + \sum_{i=1}^k \Delta \bar{v} \quad 1 \leq k \leq m \quad (4.25a)$$

where

$$\bar{v}_{no} = 1 + r \Delta \epsilon_{n(j)}^v \bar{v}_{n-1} \quad (4.25b)$$

$$\Delta \bar{v} = \frac{s}{m} \Delta \epsilon_{n(j)}^v \bar{v}_{n(j-1)} \quad (4.25c)$$

$$\Delta \epsilon_{n(j)}^v = (\Delta \epsilon_{11} + \Delta \epsilon_{22} + \Delta \epsilon_{33}) n(j) \quad (4.25d)$$

The location of the yield function is determined in subroutine POTEN through the values of the current stress invariants previously calculated in subroutine INVAR.

The gradient $\bar{\nabla}_\sigma F$ is calculated in subroutines YIELD and FLOW as in Eq. a in Table 2.3, where the constant values c_1 , c_2 and c_3 for each different plastic model are found to be equal to the expressions reported in Table 4.3.

Since the quasi-isotropic models present a sharp corner at $\theta = \pm 30$ degrees, the gradient would result in this case to be indeterminate. To overcome this problem, the sharp corners are "rounded off" by assigning to the gradient vector, for any value of $\theta \leq \pm 29$ degrees, the direction that it would have in the corresponding isotropic case, Table 4.5.

For the elastic perfectly plastic models, the stress point cannot move outside the yield surface; consequently the stress point can only traverse the surface, until both equilibrium conditions and constitutive relation are satisfied. In this case, the only use of the equilibrium equation, step 6 in Table 3.4 is insufficient, since the resulting stress point may drift away from the failure surface (Krieg et al., 1977, Schreyer et al., 1979, Owen and Hinton, 1980).

In ESEPA it has been chosen to relax the equilibrated stress point on the yield surface along its deviatoric component. In this case it is possible to prove that the projection $\bar{g}$ of a stress point g , along its deviatoric component s may be mathematically done as

$$\bar{g} = r s + \hat{m} p \quad (4.26a)$$

where

$$r = \left(\frac{\bar{J}_2}{J_2} \right)^{1/2} \quad (4.26b)$$

and J_2 is the 2nd deviatoric invariant of g while $\bar{J}_2$ is the value of the 2nd deviatoric invariant which satisfies the yield function for p and θ values associated to σ . The actual calculations of the relaxed stress are performed in subroutine BACK.

The r -factor in subroutine FACTOR, is calculated as the scalar value r , $0 \leq r \leq 1$, which satisfies the yield function expressed in term of invariants, namely:

$$G(\bar{I}_1, \bar{J}_2, \bar{J}_3) = 0 \quad (4.27)$$

where the value of the reduced invariants $\bar{I}_1$, $\bar{J}_2$ and $\bar{J}_3$ are found, after some mathematical manipulations, to be equal to the expressions reported in Table 4.4.

For isotropic models, with $\alpha = 0$ in the case of the Generalized Burland model, the r-factor can be directly calculated by solving the second order linear equation reported in Table 4.5. A trial and error procedure on r-factor is instead required for quasi-isotropic yield functions. In subroutine FACTOR the "bisection method" has been employed.

TABLE 4.2

Equivalent yields employed in subroutine PLASTIC

	Yield Function	Equivalent pressure
Tresca	$J_2^{1/2} 2 \cos \theta = 2c$	$2c$
Von Mises	$J_2^{1/2} \sqrt{3} = 2c$	$2c$
Mohr Coulomb	$\frac{I_1 \sin \phi}{3} + J_2^{1/2} (\cos \theta - \frac{\sin \theta \sin \phi}{\sqrt{3}}) = c \cos \phi$	$c \cos \phi$
Drucker-Prager	$I_1 \frac{\sin \phi}{3} + J_2^{1/2} \frac{\sqrt{3}}{2} (1 - \frac{\sin \phi}{3}) = c \cos \phi$	$c \cos \phi$
Generalized Burland	$(\frac{I_1}{3} + \beta)^2 + \frac{(\sqrt{3} J_2^{1/2} - \alpha)^2}{n^2} = h^2$	h

where for the Generalized Burland model with critical state represented by the Mohr Coulomb failure criteria

$$n = t [2 M_1 - (M_1 - M_2)_{\theta=30^\circ}]$$

$$t = \left(\frac{n}{M_1 + M_2} \right)_{\theta=30^\circ}$$

TABLE 4.3

Gradient coefficients

	c_1	c_2	c_3
Tresca	0	$2\cos\theta[1+\operatorname{tg}\theta\operatorname{tg}3\theta]$	$\frac{\sqrt{3}}{J_3} \frac{\sin\theta}{\cos 3\theta}$
Von Mises	0	$\sqrt{3}$	0
Mohr-Coulomb	$\frac{\sin\phi}{3}$	$\cos\theta[(1+\operatorname{tg}\theta\operatorname{tg}3\theta)+\frac{\sin\phi}{\sqrt{3}}(\operatorname{tg}3\theta-\operatorname{tg}\theta)]$	$\frac{\sqrt{3}}{2J_2} \frac{\sin\theta+\cos\theta \sin\phi}{\cos 3\theta}$
Drucker-Prager	$\frac{\sin\phi}{3}$	$\frac{\sqrt{3}}{2} (1-\sin\phi)$	0
Gen. Burland (Mohr-Coulomb)	$\frac{2}{3}(p+\beta)$	$\frac{2\sqrt{3}}{n^2} (q-\alpha)(1+Q)$	$\frac{(q-\alpha)^2}{q^3 n^2} \frac{9}{\cos 3\theta} T$
Gen. Burland (Drucker-Prager)	$\frac{2}{3} (p+\beta)$	$\frac{2\sqrt{3}}{n^2} (q-\alpha)$	0

Gradient coefficients

for $\theta > 29$

	c_1	c_2	c_3
Tresca	0	$\sqrt{3}$	0
Mohr-Coulomb	$\frac{\sin\phi}{3}$	$\frac{\sqrt{3}}{2} (1 \mp \sin\phi)$	0
Gen. Burland	$\frac{2}{3} (p+\beta)$	$\frac{2\sqrt{3}}{\bar{n}^2} (q-\alpha)$	0

where

$$T = \frac{M_1 \theta_1 + M_2 \theta_2}{M_1 + M_2} ; \quad \theta_i = \frac{\sqrt{3} \sin\theta - \cos\theta \sin\phi_i}{\sqrt{3} \cos\theta - \sin\theta \sin\phi_i} ; \quad Q = \left[\frac{(q-\alpha)}{q} \operatorname{tg} 3\theta \right]^T$$

$$M_i = \frac{6 \sin\theta}{3 \mp \sin\theta} ; \quad \bar{n} = \text{in table 2.8}$$

TABLE 4.4

Stress invariant in terms of r-factor

$$\bar{\sigma}_{ij} = \sigma_{ij} + r d\sigma_{ij} \quad \therefore \quad \bar{s}_{ij} = s_{ij} + r ds_{ij}$$

$$\bar{I}_1 = \bar{\sigma}_{ii} = I_1 + r dI_1$$

$$\bar{J}_2 = \frac{1}{2} \bar{s}_{ij} \bar{s}_{ij} = J_2 + 2r Z + r^2 dJ_2$$

$$\bar{J}_3 = \det(\bar{s}_{ij}) = J_3 + r Y + r^2 W + r^3 dJ_3$$

where

$$Y = A + B - C$$

$$A = s_{11}s_{22}ds_{33} + s_{11}s_{33}ds_{22} + s_{22}s_{33}ds_{11}$$

$$B = 2(s_{12}s_{13}ds_{23} + s_{12}s_{23}ds_{13} + s_{13}s_{23}ds_{12})$$

$$C = 2(s_{11}s_{23} + s_{22}s_{13} + s_{33}s_{12}ds_{12}) + (s_{23}^2 ds_{11} + s_{13}^2 ds_{22} + s_{12}^2 ds_{33})$$

$$W = D + E - F$$

$$D = s_{11}ds_{22}ds_{33} + s_{22}ds_{11}ds_{33} + s_{33}ds_{11}ds_{22}$$

$$E = 2(s_{12}ds_{13}ds_{23} + s_{12}ds_{23}ds_{13} + s_{13}ds_{23}ds_{12})$$

$$F = 2(s_{23}ds_{11}ds_{23} + s_{13}ds_{22}ds_{13} + s_{12}ds_{33}ds_{12}) + (s_{11}^2 ds_{23}^2 + s_{22}^2 ds_{13}^2 + s_{23}^2 ds_{12}^2)$$

$$Z = \frac{1}{2} (s_{11}ds_{11} + s_{22}ds_{22} + s_{33}ds_{33}) + s_{12}ds_{12} + s_{13}ds_{13} + s_{23}ds_{23}$$

For 2D problem

$$Z = \frac{1}{2} (s_{11} ds_{11} + s_{22} ds_{22} + s_{33} ds_{33}) + s_{12} ds_{12}$$

$$A = s_{11} s_{22} ds_{33} + s_{11} s_{33} ds_{22} + s_{22} s_{33} ds_{11}$$

$$B = 0$$

$$C = 2s_{33} s_{12} ds_{12} + s_{12}^2 ds_{23}$$

$$D = s_{11} ds_{22} ds_{33} + s_{22} ds_{11} ds_{33} + s_{33} ds_{11} ds_{22}$$

$$E = 0$$

$$F = 2s_{12} ds_{33} ds_{12} + s_{33} ds_{12}^2$$

TABLE 4.5

Calculation of r-factor

$$\bar{p} = \frac{\bar{I}_1}{3} = p + r \, dp ;$$

$$d = \frac{N}{M} = c \, \text{ctg} \, \phi$$

$$\bar{q} = \sqrt{3}(\bar{J}_2^{1/2} = (\bar{J}_2 + 2r \, z + r^2 d J_2)^{1/2}$$

$$\bar{\theta} = \frac{1}{3} \arcsin \left(- \frac{3\sqrt{3}}{2} \frac{\bar{J}_3}{\bar{J}_2^{3/2}} \right)$$

Linear yield functionGeneralized Burland

$$1) \quad F = \bar{q} + M \bar{p} - N = 0$$

$$2) \quad F = (\bar{p} + \beta)^2 + \frac{(\bar{q} - \alpha)^2}{n^2} - h^2 = 0$$

$$3) \quad r^2 A + 2r B + C = 0$$

F in term of r-factor

Type	A	B	C
Linear, $M \neq 0$	$dq^2 - M^2 dp^2$	$3Z - M \, dp(Mp - N)$	$q^2 - (Mp - N)^2$
Burland, $\alpha = 0$	$dp^2 + \frac{dq^2}{n^2}$	$\frac{3Z}{n^2} + dp(p + \beta)$	$(p + \beta)^2 + \frac{q^2}{n^2} - h^2$

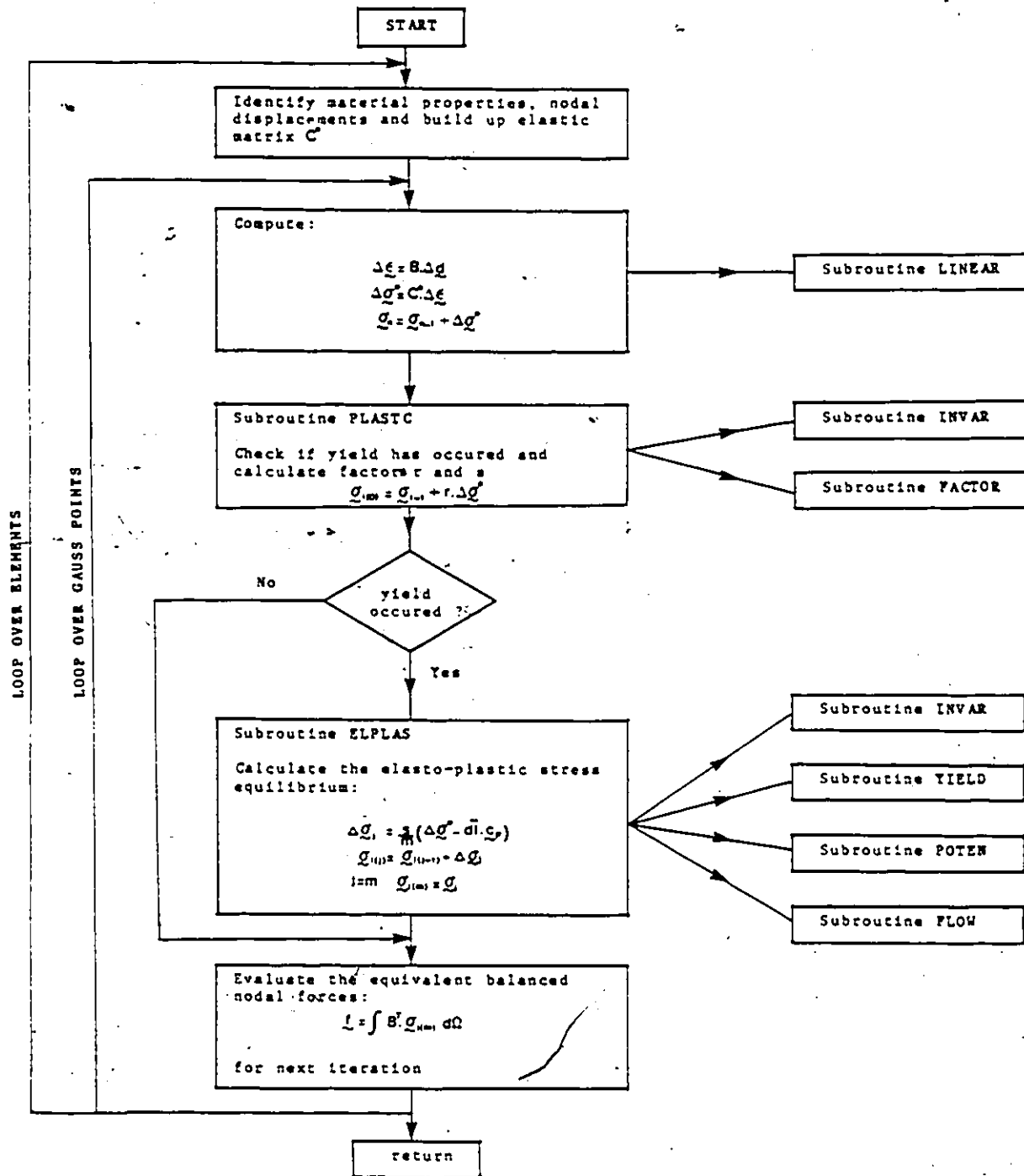


Fig. 4.6 Flowchart RESIDU

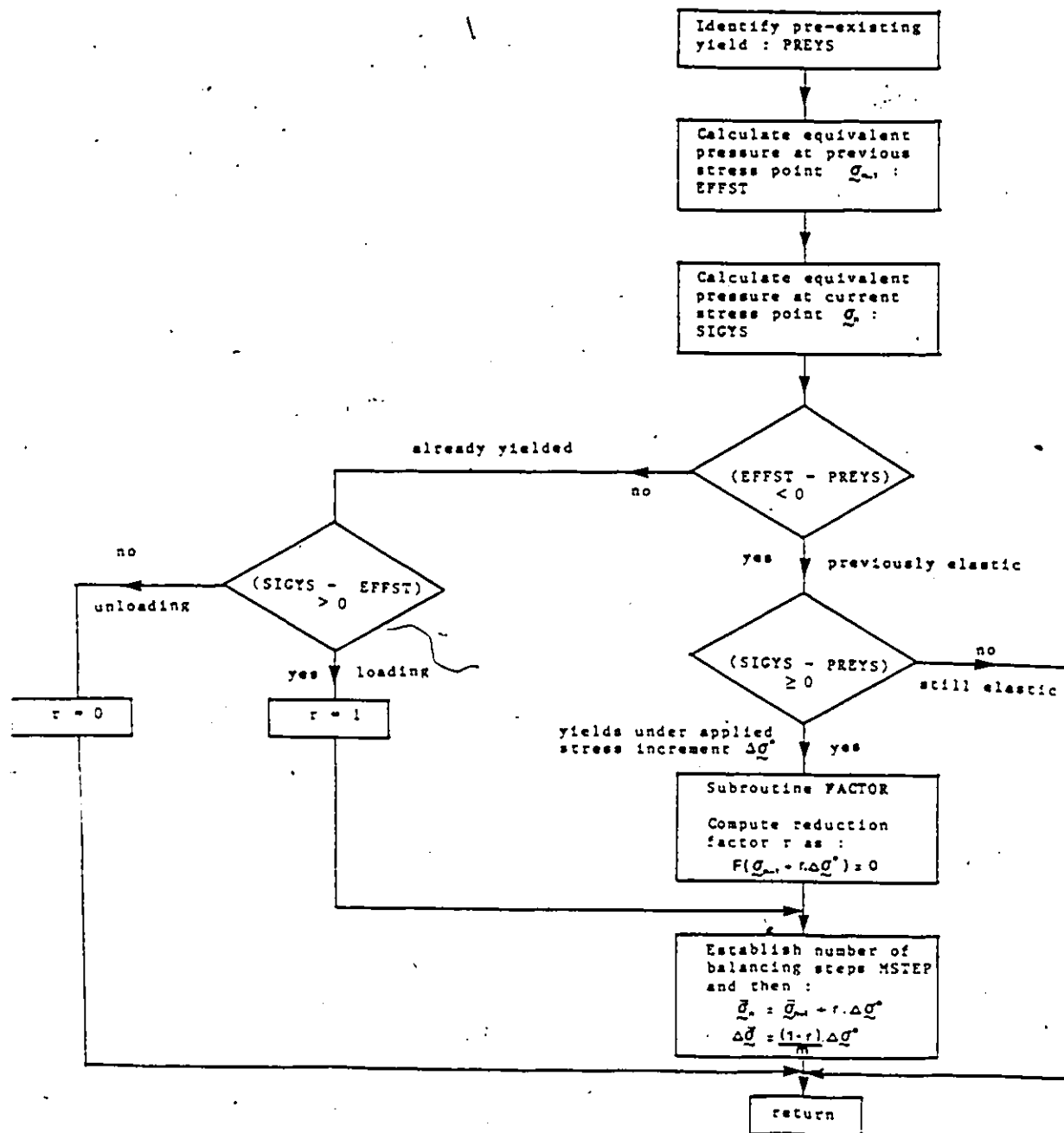


Fig. 4.7 Flowchart PLASTIC

4.4.3 TEPSA (Total Elasto-Plastic Stress Analysis)

TEPSA is the FE implementation of the Coupled problem for elasto-plastic material represented by the set of Eqs. 4.21.

Firstly, it is important to notice that the coefficients of K^{ep} may be even $10 \text{ E}+14$ times bigger than the coefficients of $\theta \text{ At P}$, specially for highly impermeable material and At very small. To avoid ill-conditioning in the solving process, the permeability and the coupled matrices in TEPSA are weighted as follows (Reed, 1984).

$$\begin{bmatrix} K^{ep} & -wC \\ -wC^T & -\alpha w^2 P \end{bmatrix} \begin{bmatrix} \Delta \tilde{u}_n \\ \frac{\Delta \pi_n^e}{w} \end{bmatrix} = \begin{bmatrix} \Delta f_n \\ w \Delta g_n' \end{bmatrix} \quad (4.28a)$$

$$\begin{bmatrix} K^{ep} & -wC \\ -wC^T & -\alpha w^2 P \end{bmatrix} \begin{bmatrix} \Delta \tilde{u}_n \\ \frac{\Delta \pi_n^e}{w} \end{bmatrix} = \begin{bmatrix} \Delta \tilde{f}_n \\ w \Delta g_n' \end{bmatrix} \quad (4.28a)$$

where w is a weighting factor.

For computational reasons, it is far more convenient to consider each node as n -displacements plus 1-pore pressure degree of freedom. This practically implies, for instance in a 2-D problem, to build up the global stiffness matrix and load vector in the following way:

$$\begin{bmatrix} K_{11} & K_{12} & \bar{C}_{11} & \dots & \dots & \dots \\ K_{21} & K_{22} & \bar{C}_{21} & & & \\ \bar{C}_{11} & \bar{C}_{12} & \bar{P}_{11} & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & P_{kk} \end{bmatrix} \begin{Bmatrix} \Delta d_1 \\ \Delta v_1 \\ \Delta \pi_1^e \\ \dots \\ \dots \\ \dots \end{Bmatrix} = \begin{Bmatrix} \Delta f_{x1} \\ \Delta f_{y1} \\ \Delta \bar{g}' \\ \dots \\ \dots \\ \dots \end{Bmatrix} \quad (4.29)$$

where $\bar{C}_{ij} = -w C_{ij}$, $\bar{P}_{ij} = -\alpha w^2 P_{ij}$ and $\bar{\Delta\pi}_j^e = \Delta\pi_j^e / w$

In TEPSA the Coupled problem has been implemented employing the standard 8-nodes isoparametric element and the error in pore pressure is corrected by the "smoothing technique", as proposed by Reed (1984) and reported in section 3.6.2.

TEPSA takes full advantage of ESEPA and in fact it can be considered as an extension of the latter.

The program is structured in three main loops, Figs. 4.8: the first one on load increment, the second on time and the third on the balancing of the residual load due to plastic deformation.

In the program the permeability is assumed to be constant and therefore P and C matrices are calculated only once in subroutine LOADPS, along with the equivalent load vectors $\underline{f}$ and $\underline{g}$, as established by Eqs. 3.3b, 4.12e, 4.12f, and 3.3c respectively.

Like in TERCON, the time marching scheme can be chosen among several different possibilities, section 4.3.

In subroutine CLOAD the equivalent load $\Delta \underline{g}_n$ is calculated as in Eq. 4.20c and then assembled with $\Delta \underline{f}_n$ as reported on the right hand side of Eqs. 4.29.

The stiffness matrix K^{ep} is calculated, as in ESEPA, in subroutine STIFFP and then assembled with C and P matrices, as in Eq. 4.29, in subroutine ASSEM.

The solutions $\Delta \tilde{d}_n$ and $\Delta \pi_n^e$ at each time increment are found in two stages:

The first stage involves the solution of Eqs. 4.28 for the assigned fixity of the problem. The obtained increment of excess pore pressure is then corrected in subroutine SMOOTH by applying the "smoothing technique" as described in Eqs. 3.32.

The second stage requires the finding of the compatible displacement for the corrected value of the excess pore pressure. This is accomplished by re-solving Eqs. 4.28 but this time taking the smoothed pore pressure as additional specified degree of freedom.

Displacements, pore pressures, equivalent flow and forces are sorted from the common array in subroutine SPLIT. In the same subroutine, the effective "equivalent" load carried by the soil skeleton for the current value of pore pressure is computed as follows:

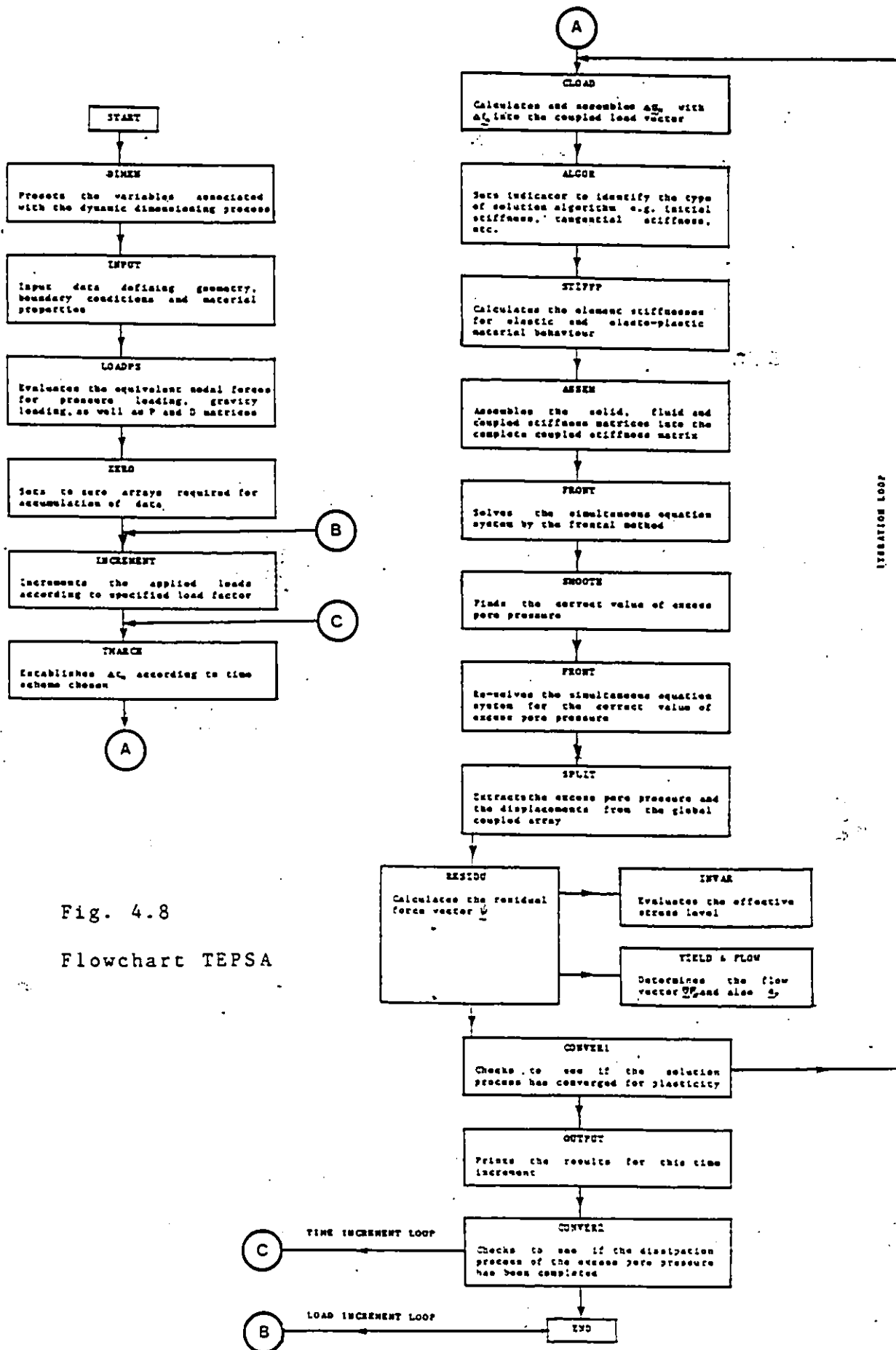
$$\Delta f'_n = \Delta f_n + C \Delta \pi_n^e \quad (4.30)$$

Exactly as in ESEPA, the equivalent load which can be carried by the soil skeleton for to the current displacement is determined in subroutine RESIDU.

The residual load, Eq. 8 Table 3.3, and eventually the need of a next iteration loop for its equilibrium, before

proceeding to another time step, are determined in subroutine CONVER1.

Finally the degree of consolidation is monitored in CONVER2 subroutine.



4.5 CONCLUSIVE REMARKS

In all four FE formulations previously presented, the stiffness matrix is symmetric. This allows the use of the "Frontal Method" (Iron, 1970) in order to solve the system of equations. This technique has proved to be extremely efficient reducing considerably the computational time required by any classical solution scheme. Furthermore, the "Frontal Method" does not need to assemble the global stiffness matrix, which represents a significant saving in computer storage.

From this point of view, it was decided not to implement in ESEPA nor TEPsA the "acceleration method" presented in Section 3.5.1, since it would have required the complete building up of the global stiffness matrix.

The programs have been written in modular form, so that a substantial number of subroutines are common to the four programs. This also allows to verify the reliability of each subroutine since they are tested in completely different contexts.

The FE program PLANET, published by Owen and Hinton (1980), was an important point of reference during the development of these programs. As a matter of fact, a number of PLANET's subroutines have been adopted and all the remaining subroutines developed in this research have been written accordingly. A detailed explanation of the actual implementation of the various programs is reported by Wan

(1985), who gave an invaluable contribution to the writing and testing of those programs.

For an easier input and analysis of the numerous data involved in a FE computation, three peripheral programs have been written, namely: MESHGEN, FEMPLLOT and ELABO.

MESHGEN is a mesh generation program and FEMPLLOT allows the graphic representation of a vast number of results such as principal stresses, displacements, velocities, safety factors as well as the computed mesh. ELABO "elaborates" the output data from ESEPA and TEPSA in order to calculate nodal stresses, principal stress directions, stress invariants, and safety factors. It can also sort the stress distribution along selected sections and the stress history at specific points.

The stresses at the nodes are calculated by the smoothing technique as presented in Section 4.2.1. The safety factor (FS), is herein defined as the ratio of the mobilized τ_{oct} to τ_{oct} at failure, namely:

$$\frac{\bar{q}}{q} = p M + N \quad (4.31a)$$

$$FS = \frac{\bar{\tau}_{oct}}{\tau_{oct}} = \frac{\bar{q}}{q} \quad (4.31b)$$

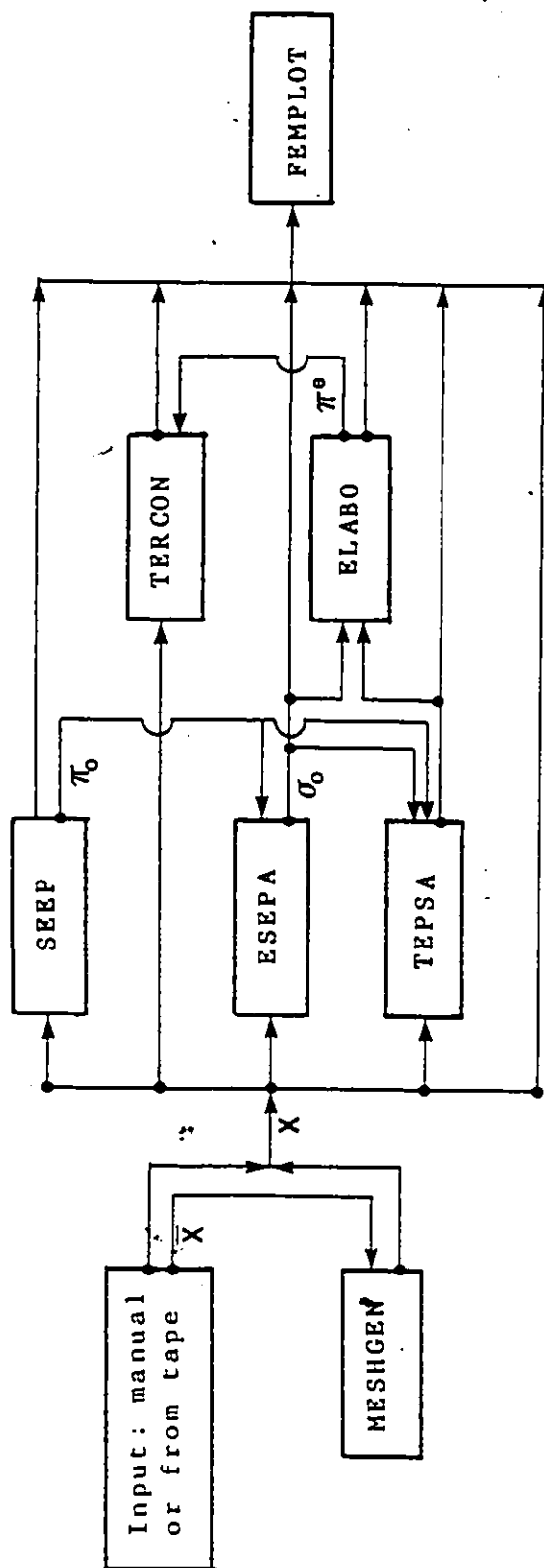
where the parameters M and N are calculated according to Table 1.1 for the current θ while p is the mobilized mean pressure.

The layout illustrating the inter-connections between the programs is shown in Figs. 4.9.

The initial data can be introduced either manually or by tape; alternatively MESHGEN can be used to reduce considerably the calculation (and input) of the geometrical data. The generated mesh can either be checked by sending the data to FEMPLOT, or directly employed to run any of the four main programs.

The output of any of the four main programs may eventually be plotted by FEMPLOT or rather be used to provide the initial values to run of one of the other programs.

For example, the pore pressure calculated by SEEP may be used as initial values for ESEPA or TEPSA. Similarly, the stress distribution calculated by ESEPA may also serve as initial stress in TEPSA. Finally, the calculation of the initial excess pore pressure distribution needed in TERCON may be computed by ELABO, on the basis of the variation of stress eventually calculated by ESEPA.



MESHGEN : MESH GENERATION
 SEEP : SEEPAGE ANALYSIS
 ESEPA : EFFECTIVE STRESS ELASTO-PLASTIC ANALYSIS
 TEPESA : TOTAL ELASTO-PLASTIC STRESS ANALYSIS
 ELABO : ELABORATION NUMERICAL RESULTS
 FEMPLOT : FINITE ELEMENT PLOT

Fig. 4.9 Layout of program inter-connections

Chapter V

NUMERICAL RESULTS

5.1 INTRODUCTION

This Chapter discusses some numerical results obtained from the four FE codes previously described.

The purpose of the following examples is to illustrate two aspects which are thought to be of primary importance: firstly, to verify the efficiency and reliability of the numerical implementations; secondly, to highlight the main differences in material response which the various theories, considered in this research, lead to. For these reasons, a number of cases were studied in order to verify the numerical results with available analytical solutions.

The material parameters, Table 5.1, were arbitrarily chosen and systematically employed in most of the examples.

Basically all programs, except SEEP, were run in the condition of 1-D, axial symmetry and plane strain for semi-infinite layer and semi-infinite medium. The corresponding meshes and boundary assumptions are thereby reported in Figs. 5.1 - 5.4.

Occasionally, and mainly to recover previous numerical results, different cases were considered, and the corresponding conditions are locally reported.

The programs were run on an AMDAHL V8 machine at the University of Ottawa.

TABLE 5.1

Material parameters employed in the different examples

Elastic parameters:

$$E = 10\,000$$

$$\nu = 0$$

Failure parameters:

von Mises and Tresca

$$\text{a) } N = 94.8680, \quad M = 0 \quad \therefore \quad c = 47.4341, \quad \phi = 0$$

$$\text{b) } N = 100, \quad M = 0 \quad \therefore \quad c = 50, \quad \phi = 0$$

Drucker-Prager and Mohr-Coulomb

$$N = 10, \quad M = 1 \quad \therefore \quad c = 4.7434, \quad \phi = 25.3769$$

Generalized Burland Parameters:

$$\lambda = 0.2 \quad \chi = 0.02 \quad \bar{\nu} = 2$$

Permeability

$$K = 4 \times 10^{-6}$$

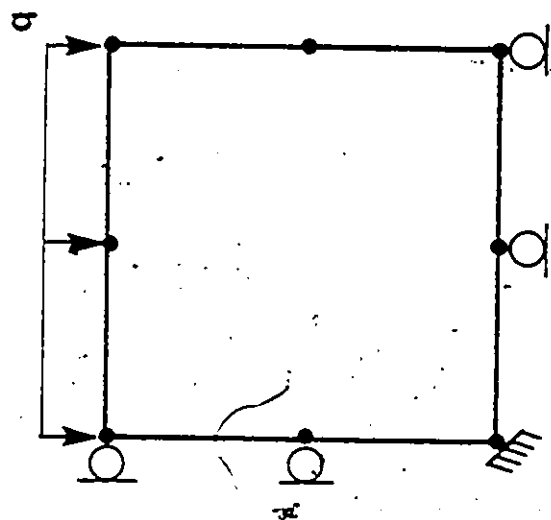


Fig. 5.1 One-element mesh layout

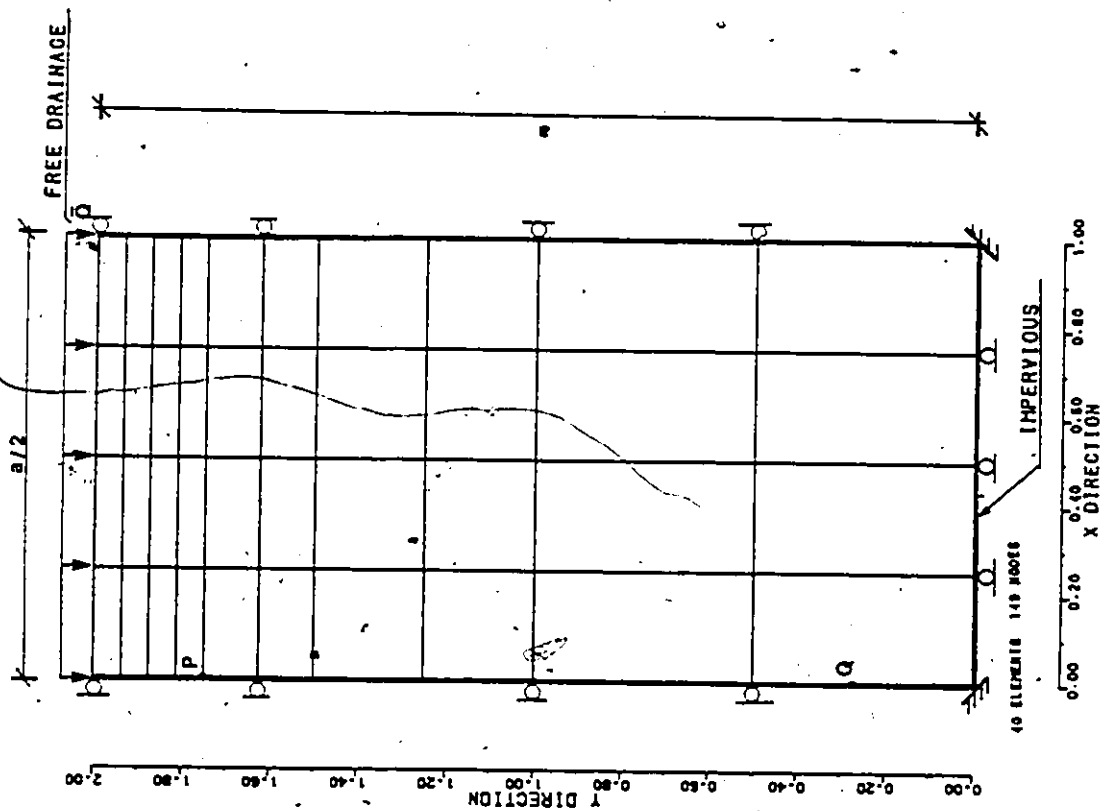


Fig. 5.2a Mesh layout for 1-D conditions

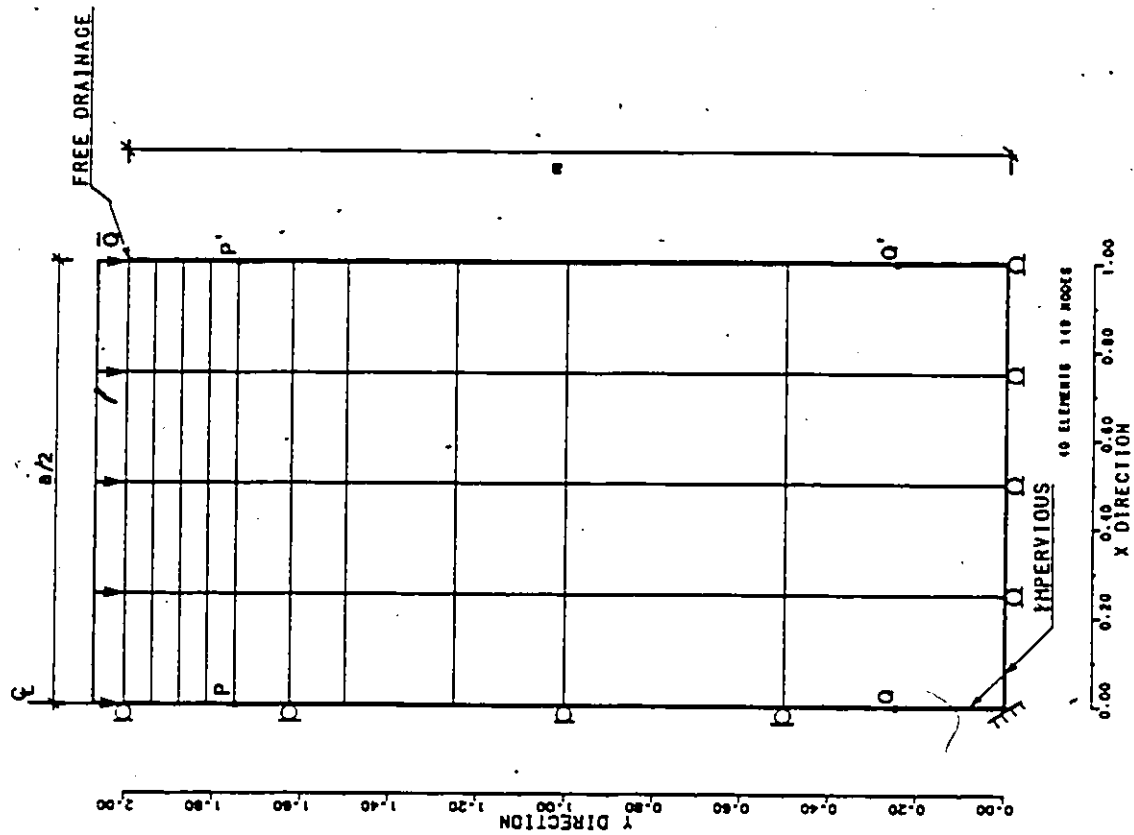


Fig. 5.2c Mesh layout for triaxial conditions

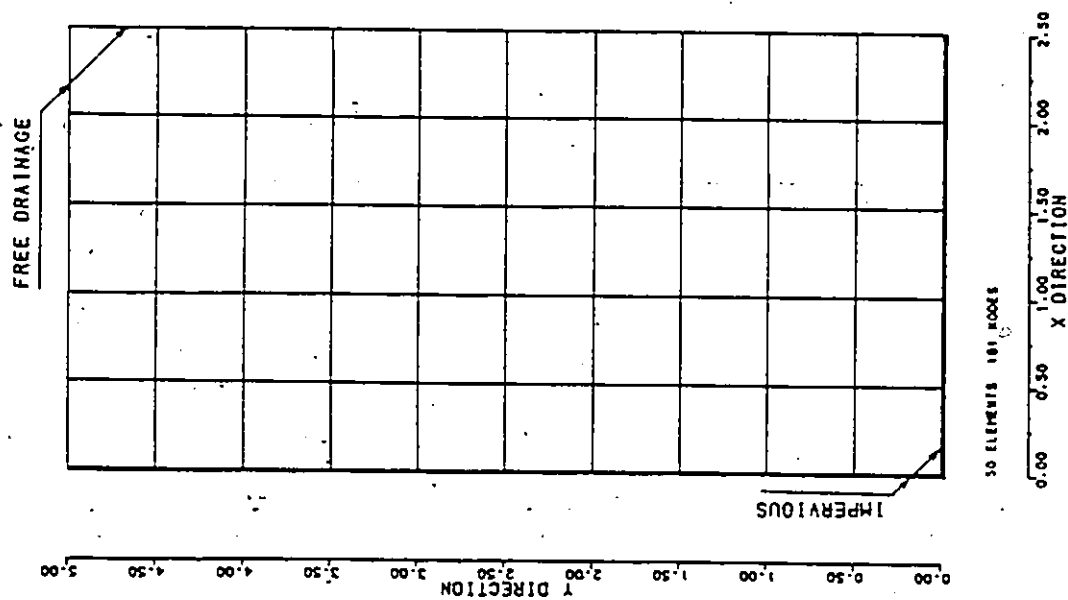


Fig. 5.2b Mesh layout for axisymmetric conditions

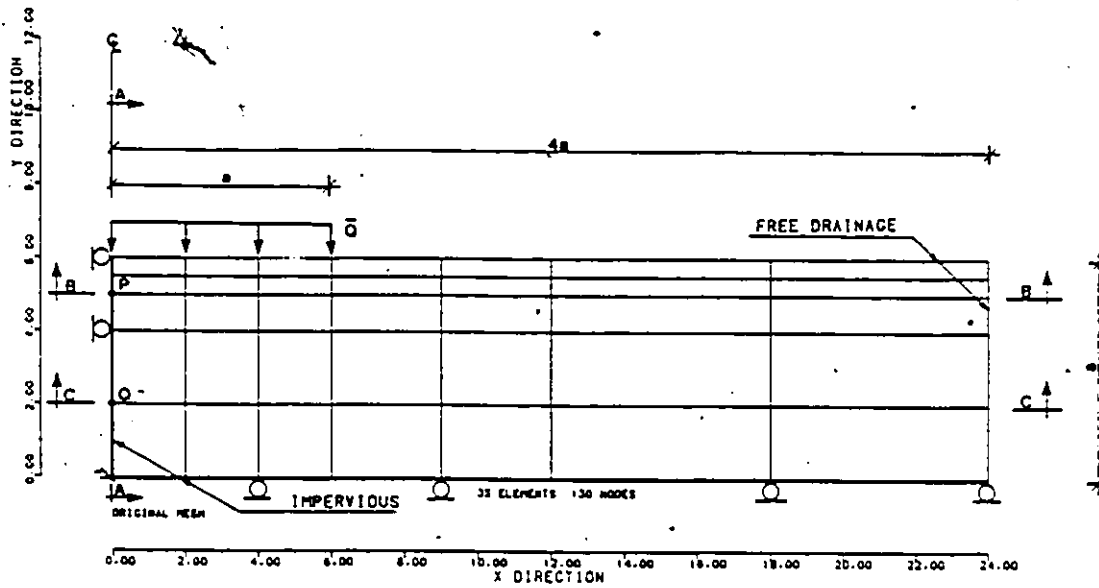


Fig. 5.3 Mesh layout for semi-infinite layer

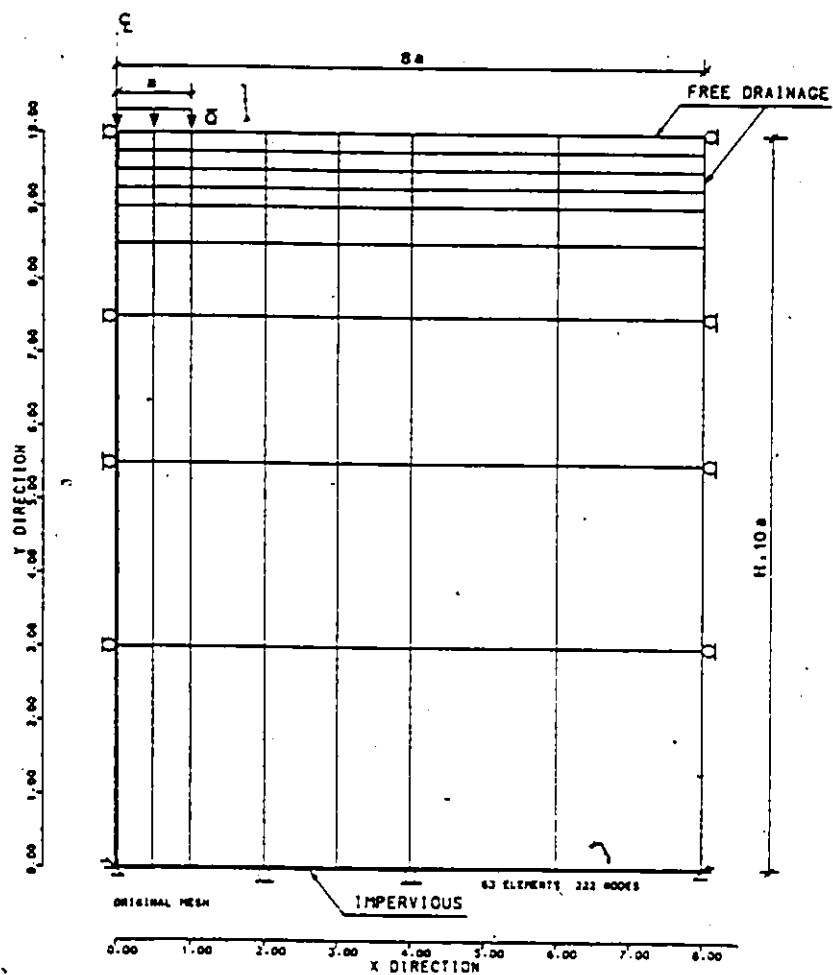


Fig. 5.4 Mesh layout for semi-infinite medium

5.2 SEEP

The solution of confined flow does not entail much difficulty since all boundary conditions are known a priori. In fact, for the analysis of flow below an impermeable wall, Figs. 5.5, the solution required one single iteration with a CPU time of 3.51 sec..

Yet, the main reason to have implemented SEEP was to develop an efficient and reliable algorithm for unconfined flow situations. Unfortunately, analytical solutions are scarce and mostly based on the simplified Dupuit's assumptions; consequently, the choice of examples to compare analytical and numerical results is rather limited.

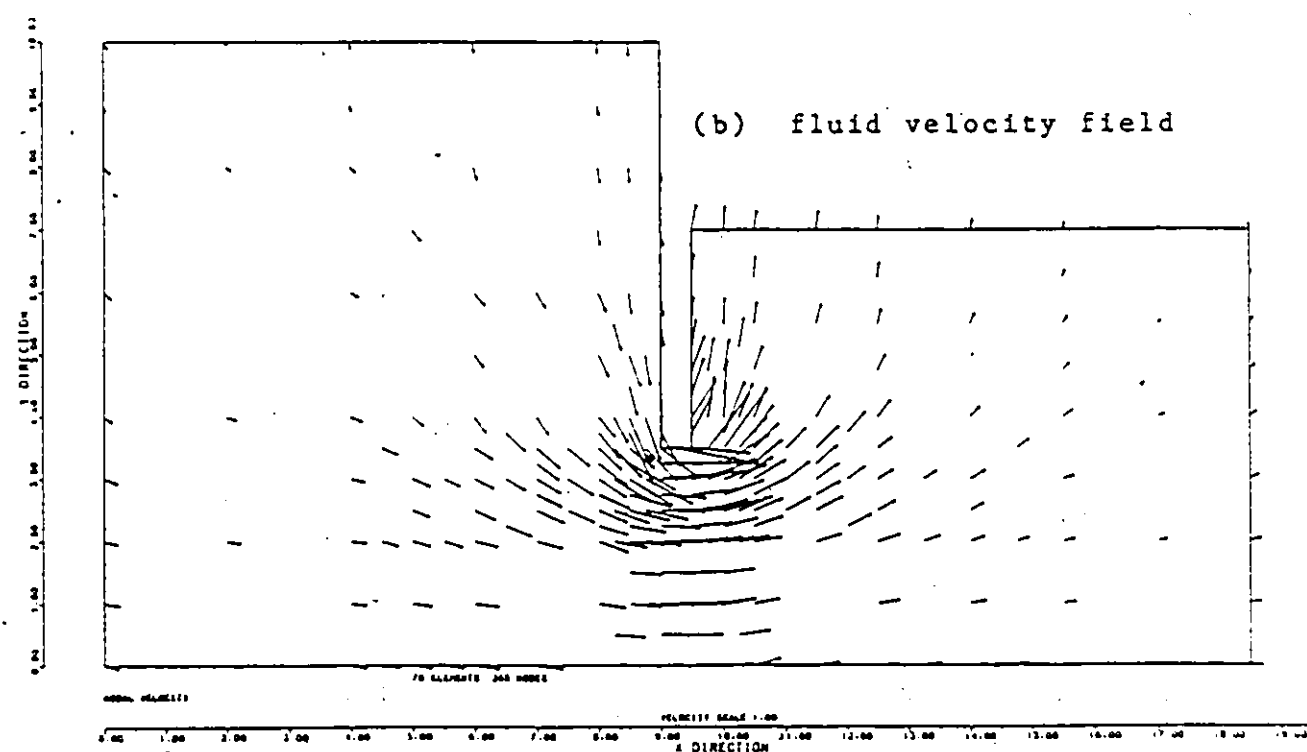
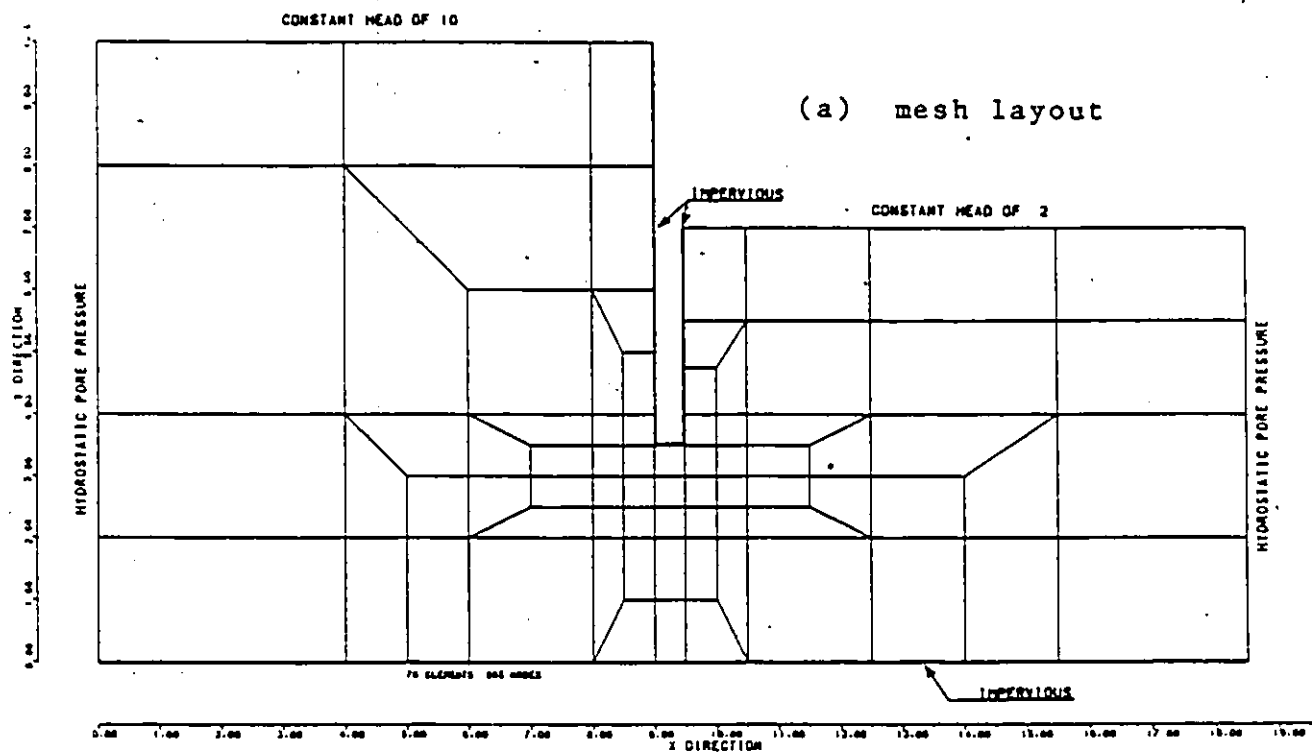
For the case shown in Figs. 5.6, the starting trial free surface was assumed to be horizontal. The program performed 14 iterations, for a total CPU of 3.33 sec., before locating the correct free surface. The exit point downstream was found to be 1.44 m., in excellent agreement with the theoretical expectation (Polubarinova-Kochina, 1962).

The same order of CPU time was necessary to locate the free surface for soils with anisotropic permeability, Figs. 5.7 and 5.8. Incidentally, it is interesting to notice the good distribution of the nodal velocities in Figs. 5.7, obtained by employing the "smoothing technique" as described in Section 4.2. More specific details on the effectiveness of this technique are given later in Section 5.4.2..

SEEP has also confirmed its efficiency in the more complicated case of a dam with a clay core as shown in Figs. 5.9. The initial mesh and boundary conditions were previously described in Section 4.2.

The location of the actual free surface was achieved in 7 iterations. The first iteration, in which all stiffnesses had to be calculated, took 7.34 sec. of CPU time. Each subsequent iteration, which comprised only the updating of the DBE sub-element stiffness, required 3.7 sec. for a total of 29.54 sec. CPU time.

As shown in Figs. 5.9a, the final mesh kept a uniform distribution of elements, which is a prerequisite for reliable numerical results.



Figs. 5.5 Seepage below a sheet pile in a homogeneous soil

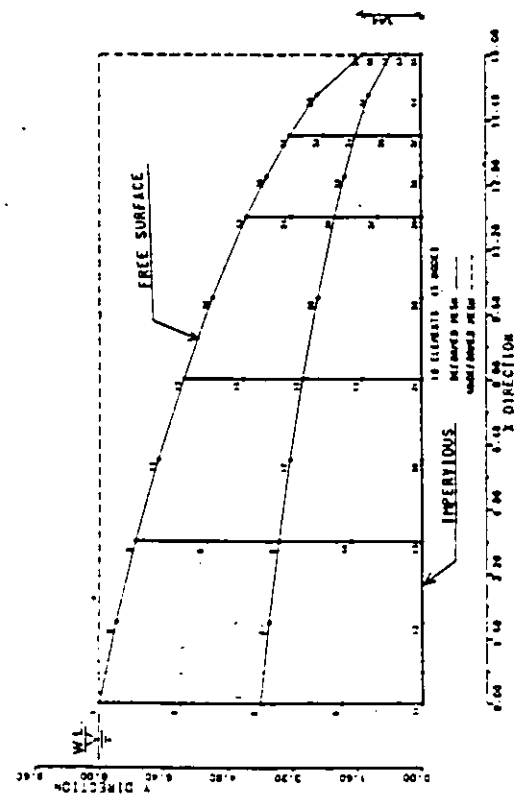


Fig. 5.6 Seepage through an isotropic rectangular dam ($K_x=K_y$): (a) final mesh, (b) fluid velocity field

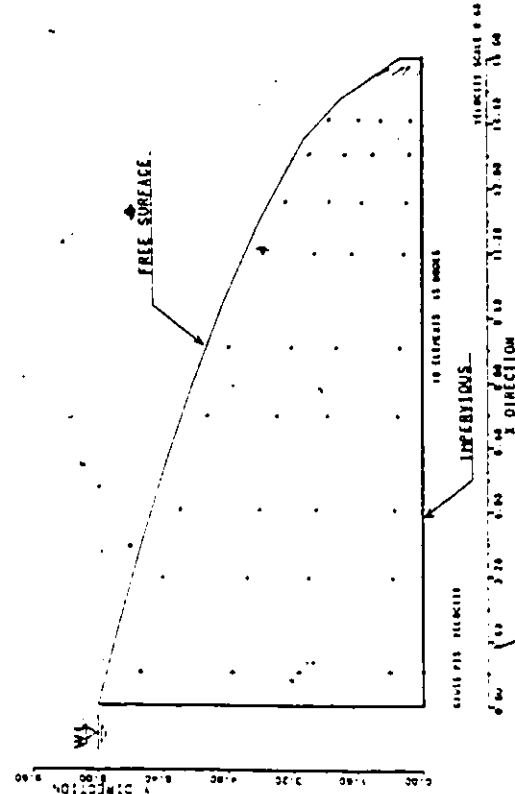
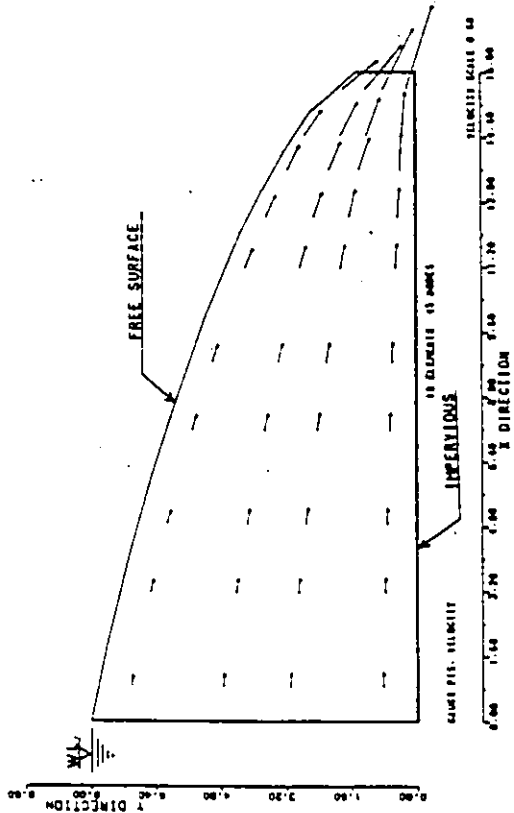


Fig. 5.7 Seepage through an anisotropic rectangular dam ($K_x=10 K_y$): fluid velocity field

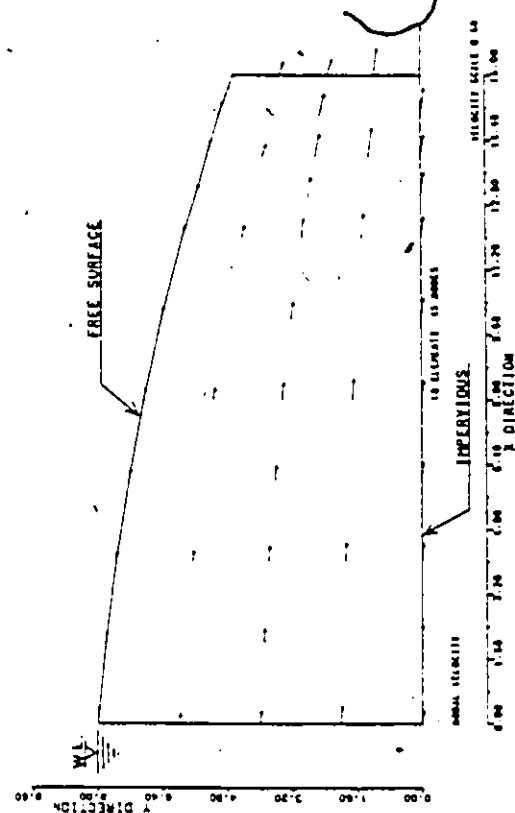


Fig. 5.8 Seepage through an anisotropic rectangular dam ($K_x=0.1 K_y$): fluid velocity field

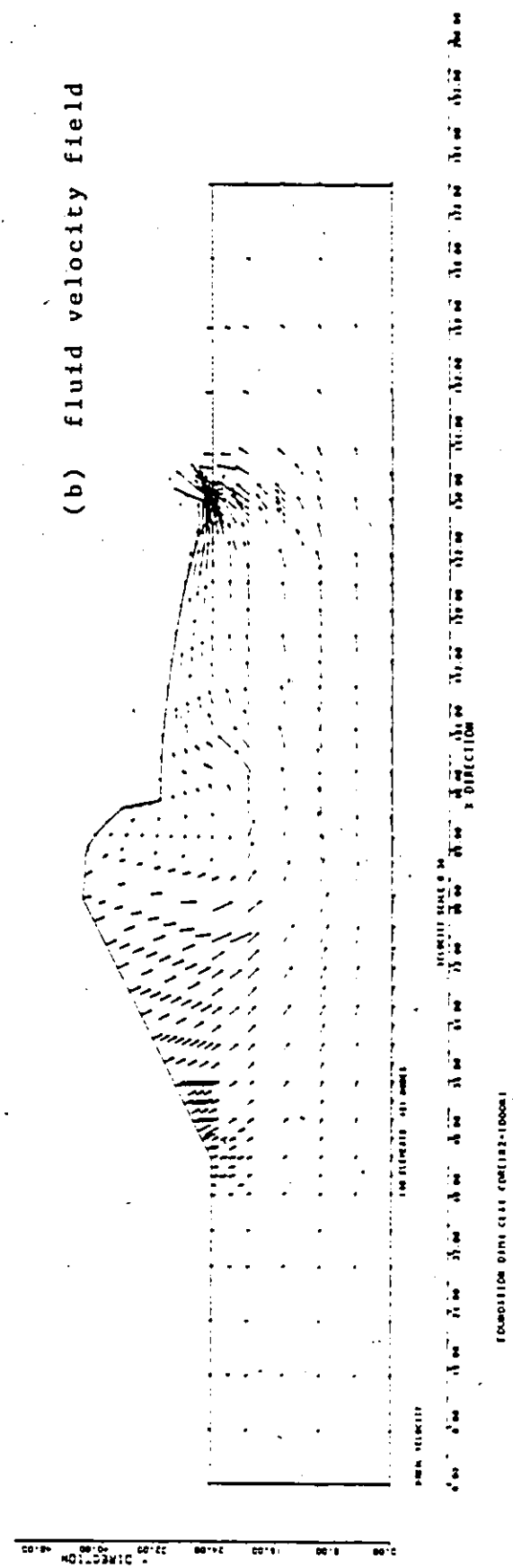
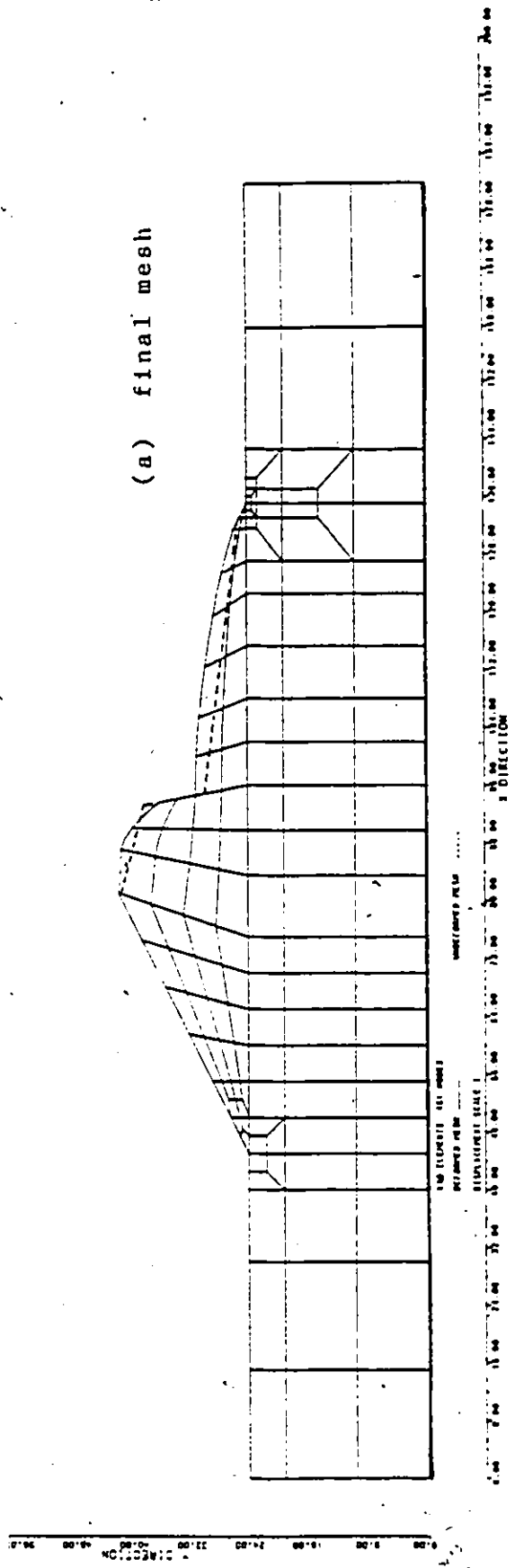


Fig. 5.9 Seepage through a heterogeneous dam

5.3 TERCON

The numerical examples presented in this section were carried out in the assumption of linear elasticity; the corresponding material properties are reported in Table 5.1, while unit specific weight of the fluid was assumed. Consequently, the bulk modulus was determined according to Eqs. 1.26 and it was kept constant during calculation.

Because of the linearity of the problems, the numerical results are generally reported in normalized form. In particular, the time factor T_v is calculated as

$$T_v = \frac{c_v t}{a^2} \quad (5.1a)$$

where

$$c_v = \frac{B K}{\gamma_f} \quad (5.1b)$$

and "a" is a geometrical dimension characteristic of the considered case (see Figs. 5.2, 5.3 and 5.4).

The reliability of TERCON can be verified by comparing the numerical results in 1-D with Taylor's analytical solution (Taylor, 1948), Figs. 5.10. The employed mesh and boundary conditions are shown in Figs. 5.2a, and the agreement can be considered complete.

In accordance with the theoretical expectations, identical results were recovered if the same mesh and boundary conditions were employed to represent a quarter-section of a cylindrical sample. As in the previous case, the initial pore pressure was assumed to be uniformly distributed throughout the sample.

The reliability of TERCON was further confirmed in the axisymmetric case whose external vertical boundary was then assumed to be permeable, Figs. 5.2b. The numerical results showed in Figs. 5.11a matched closely the analytical solution (reported in brackets) proposed by Carslaw and Jaeger (1959). As observed by Smith (1982), the analytical solution, involving rather lengthy expressions of Bessel function took longer computer time than the numerical one. In Figs. 5.11a, b and c the current fluid velocity fields, calculated by setting the head h equal to the excess pore pressure, are presented.

For practical purses, the examples reported in Figs. 5.12 and 5.13 could be quite interesting; they represent the dissipation process that can take place in a soil having a vertical sand drain. In both examples the axis of symmetry of the mesh represented in Figs. 5.2c was assumed to be permeable. The corresponding isochrones, Figs. 5.12b and 5.13b, obviously show that dissipation is much faster when higher horizontal permeability is imposed.

In all the above reported examples, the θ -value was assumed to be constant while the time increments were calculated according to Eq. 4.9. Consequently, the stiffness matrix P in Eq. 4.7 had to be updated at each iteration, which required 0.83 sec. for a total of 38.11 sec. CPU time over 45 iterations. Though this scheme allows to scan the consolidation period in a rather uniform way, it is quite expensive. In fact, solutions performed by employing the second time scheme based on Eqs. 4.10, gave similarly good results but the CPU time was drastically reduced, since the stiffness matrix P had to be updated only at the beginning of each time period.

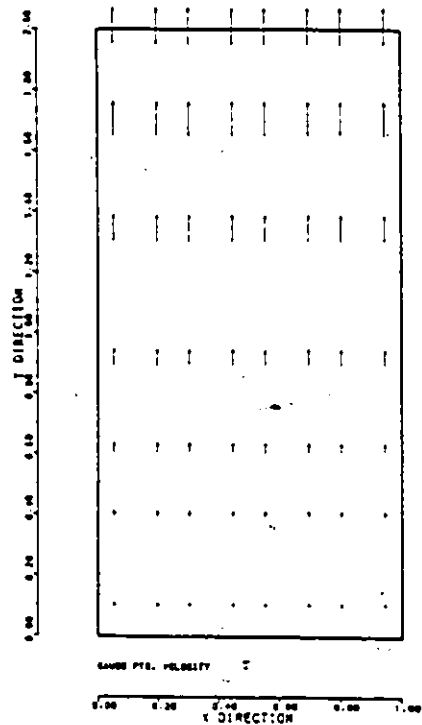
Finally, the consolidation of a semi-infinite layer subjected to a uniform load, $\bar{Q} = 10.$, due to a shallow and smooth foundation was simulated. The corresponding mesh and boundary conditions are shown in Figs. 5.3.

A first solution was calculated at time $t = 10 \text{ E-6}$, followed by a second one at time $t = 10 \text{ E-2}$. From there, the next time increments were calculated as in Eq. 4.10a and θ -value was determined according to Sandhu's suggestion, Table 3.1. To follow the entire dissipation of the excess pore pressure it was necessary to perform 60 iterations for a total CPU time of 37.8 sec..

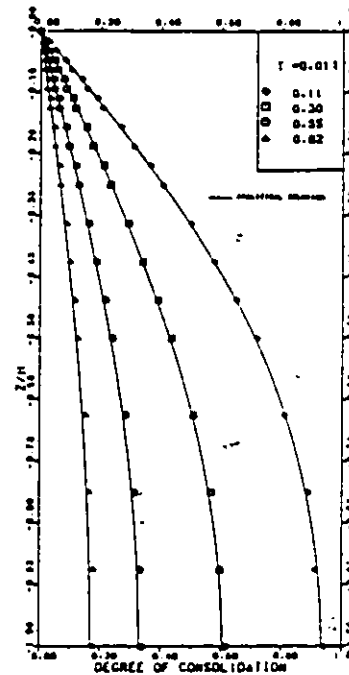
The initial π^e distribution, a prerequisite for the solution of the Uncoupled problem, was assigned equal to the variation of the mean pressure due to the applied load. In

fact, Henkel's parameters "a" and "b" in Eq. 1.28, should be set equal to 1 and 0 respectively under small deformation and isotropic condition. From Figs. 5.14 it can be seen that the excess pore pressure decreases monotonically with time at any point in the medium.

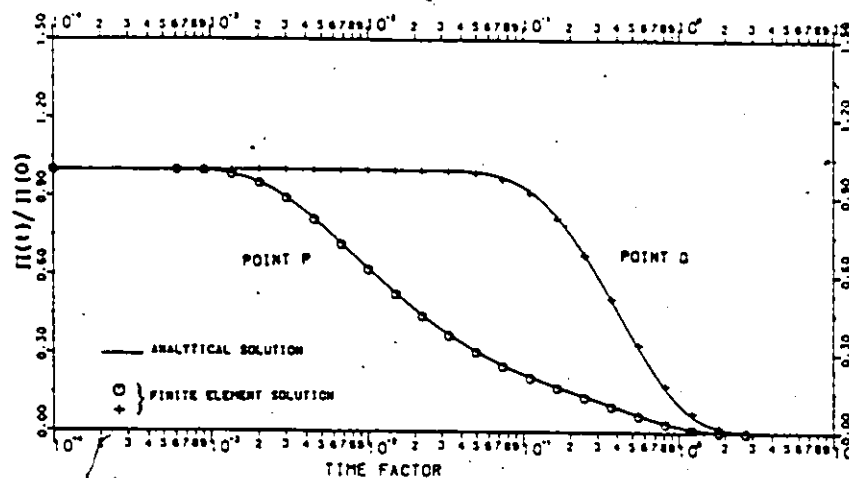
The common engineering practice would solve this problem within the 1-D concept, in the assumption that the π_o^e distribution is constant and equal to the applied load. Accordingly, 90% degree of consolidation in 1-D is obtained at time $t = 763$, which corresponds in a 2-D case to $T_v = 0.42$. From Figs. 5.13 it is instead apparent that the 2-D solution predicts a longer dissipation period.



(a) fluid velocity field

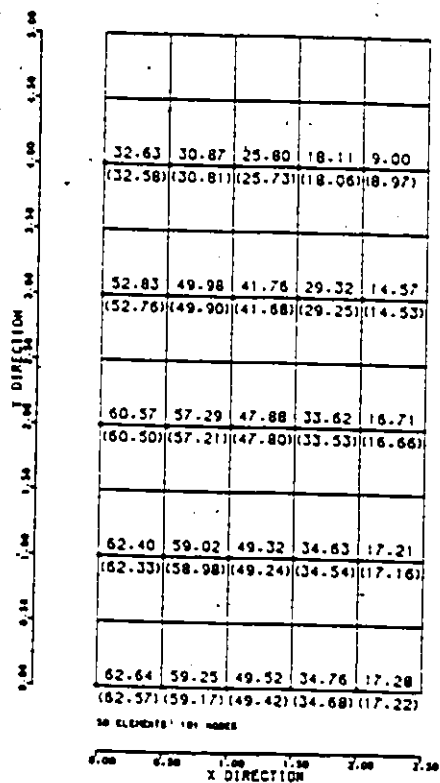


(b) pore pressure isochrones
(comparison with analytical solution)

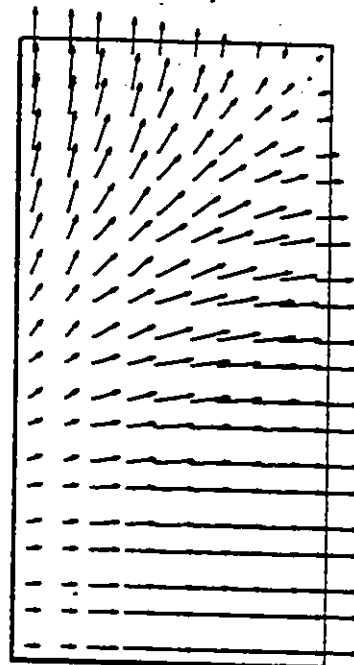


(c) excess pore pressure vs. time
(comparison with analytical solution)

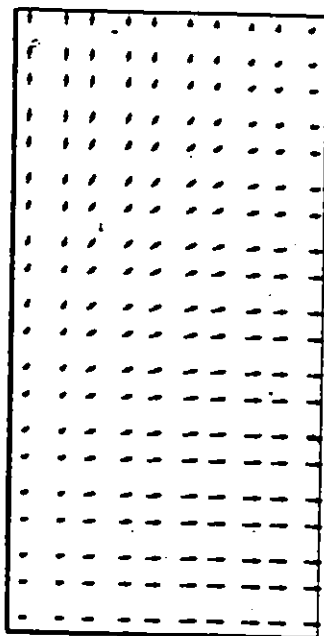
Fig. 5.10 1-D Uncoupled consolidation



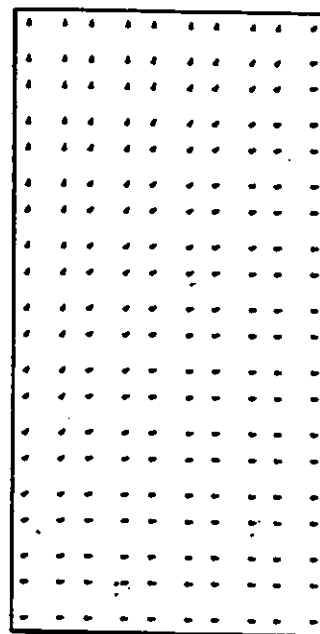
(a) excess pore pressure at $t=1.0$ (comparison with analytical solution)



(b) VELOCITY AT TIME 1.25



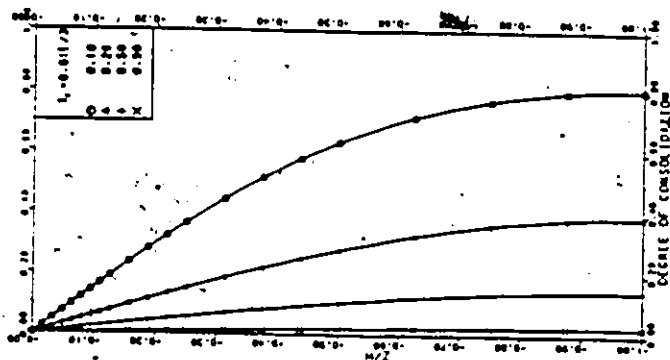
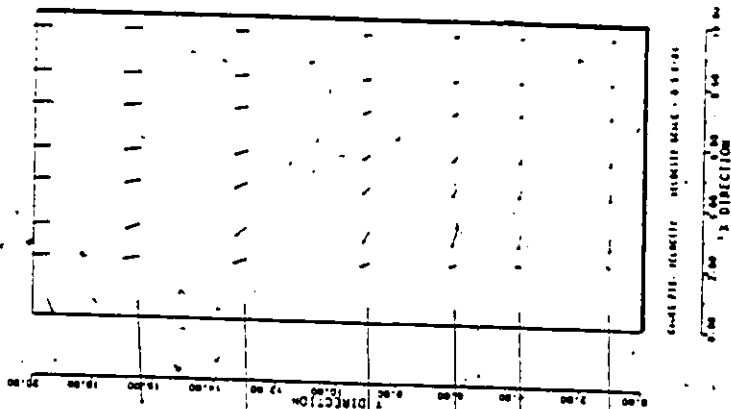
(c) VELOCITY AT TIME 2.50



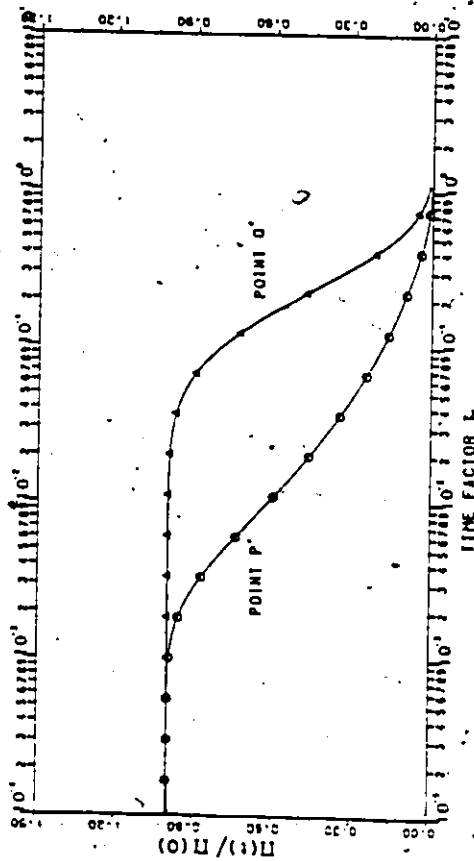
(d) VELOCITY AT TIME 3.75

Fig. 5.11 Uncoupled consolidation in axial symmetry

(a) fluid velocity field
at $T_v=0.1$

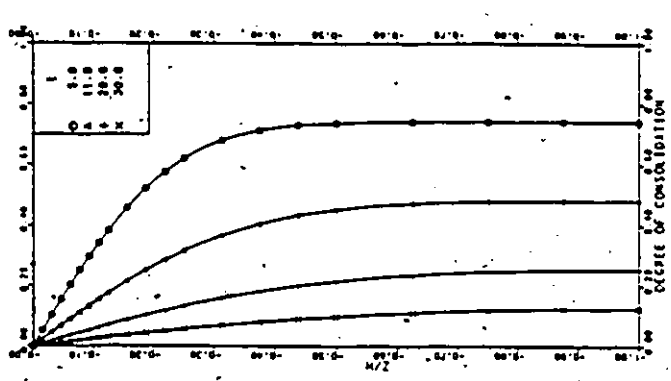
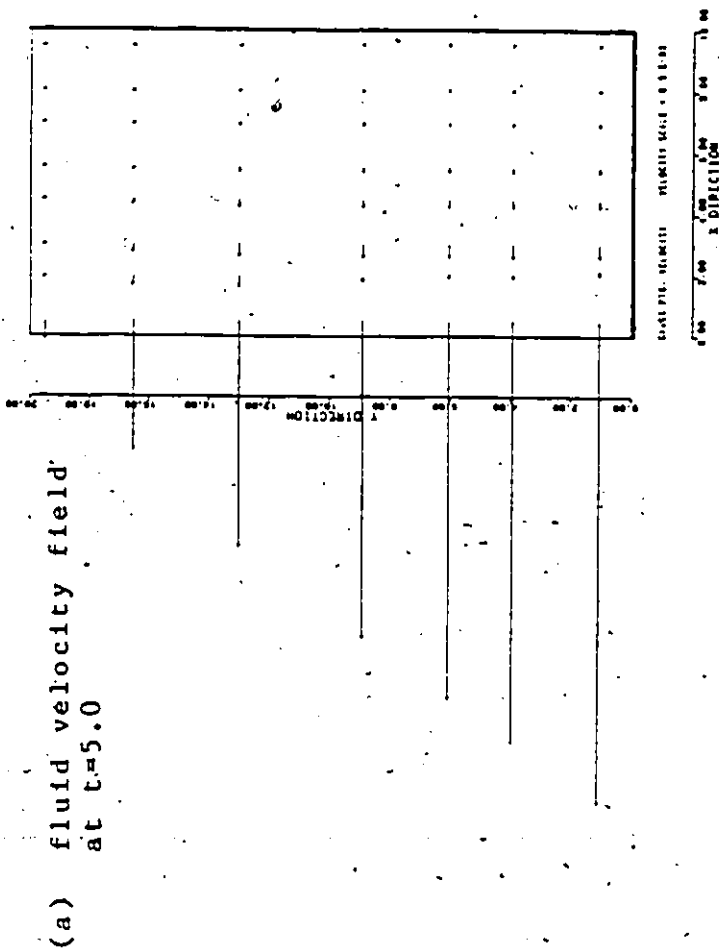


(b) excess pore pressure
isochrones

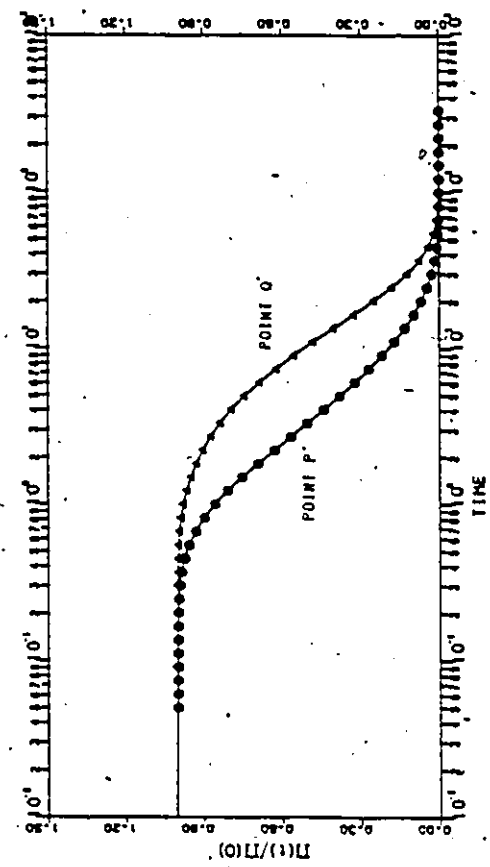


(c) excess pore pressure vs. time

Fig. 5.12 Uncoupled consolidation
in a sand drain ($K_x=K_y$)

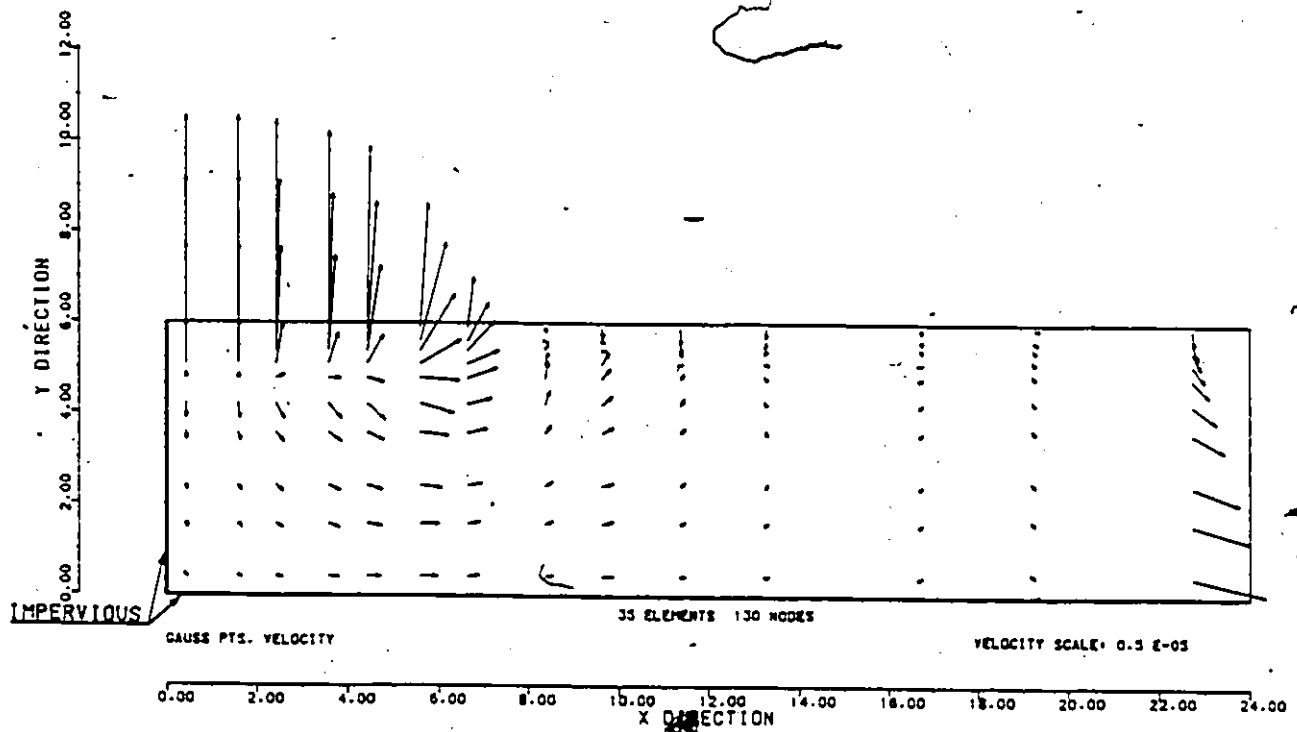


(b) excess pore pressure isochrones



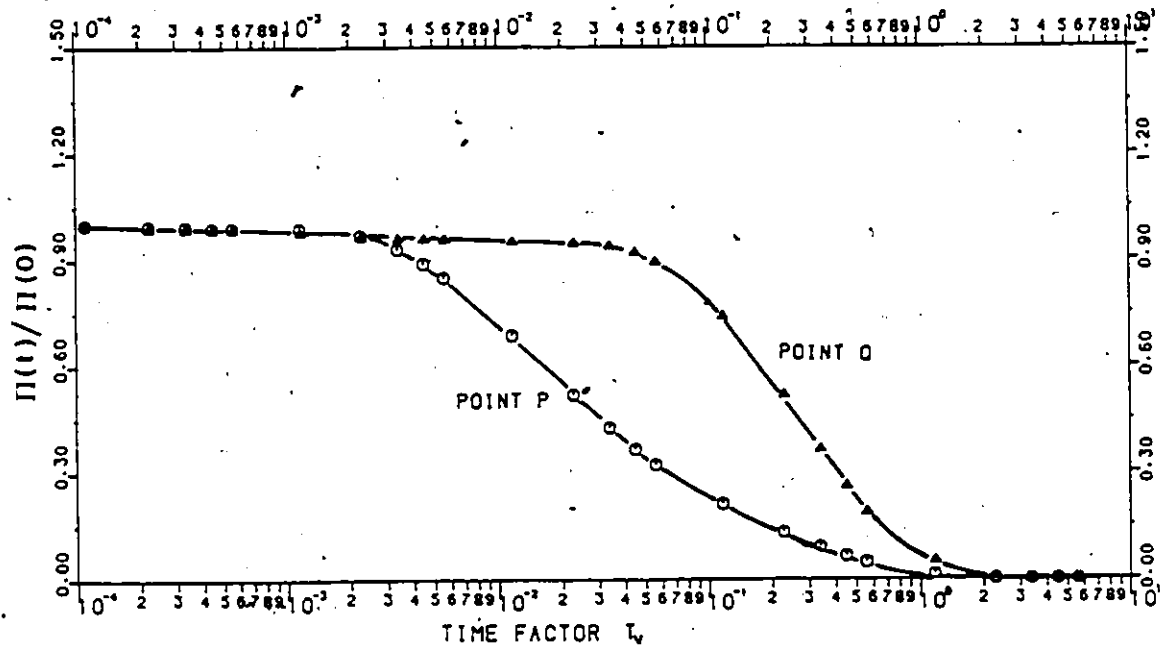
(c) excess pore pressure vs. time

Fig. 5.13 Uncoupled consolidation in sand drain ($K_x=10K_y$)

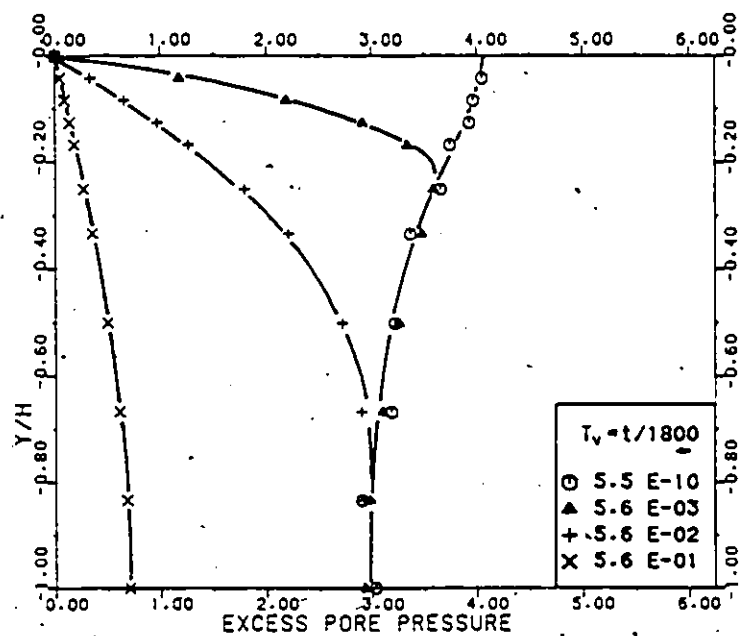


(a) fluid velocity field at $T_v = 5.6 \text{ E-03}$

Fig. 5.14 Uncoupled consolidation in a semi-infinite layer

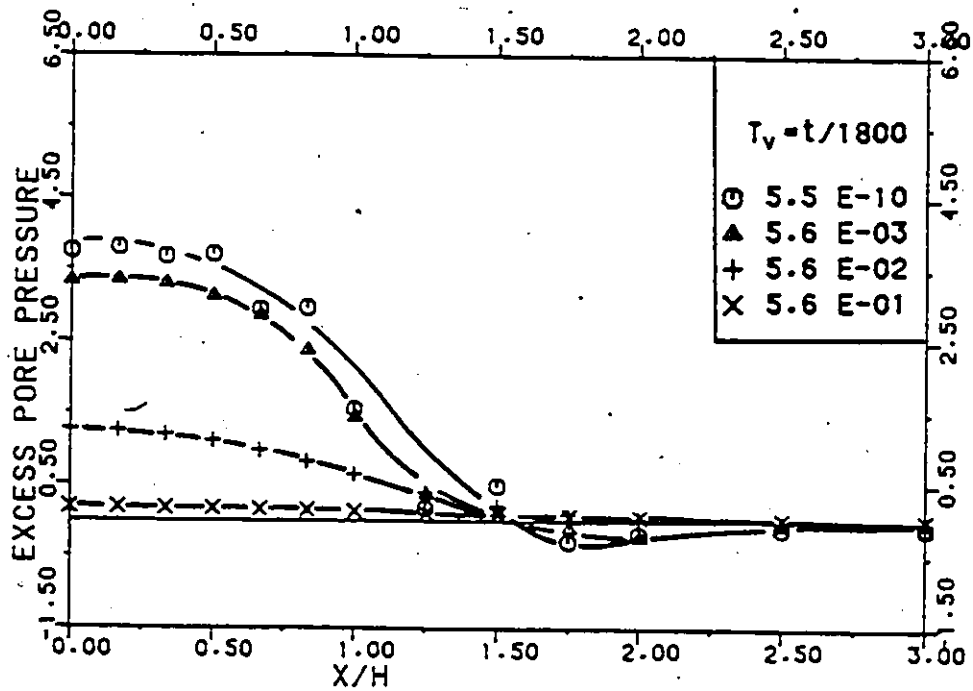


(b) excess pore pressure vs. time

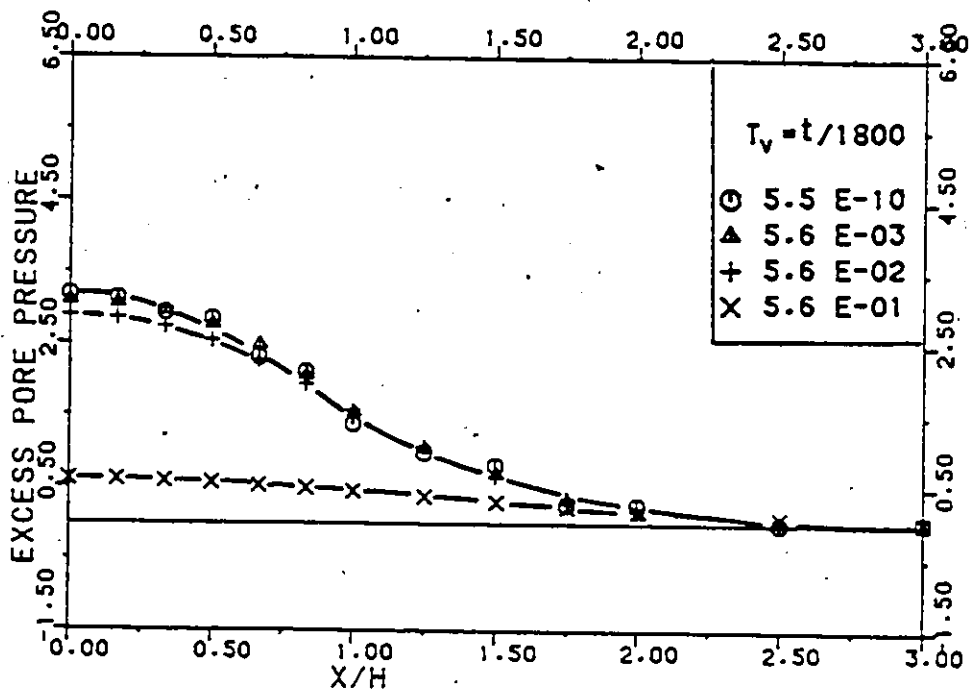


(c) excess pore pressure isochrones at section A-A

Fig. 5.14 Uncoupled consolidation in a semi-infinite layer



(d) excess pore pressure isochrones at section B-B



(e) excess pore pressure isochrones at section C-C

Fig. 5.14 Uncoupled consolidation in a semi-infinite layer

5.4 ESEPA

The performance of ESEPA has been tested in various boundary problems employing all the six implemented elasto-plastic models and some selected cases are presented in the following.

For brevity, materials described by linear elastic perfectly plastic models are herein named sometimes after the type of the failure criterion they are assumed to follow. On the other hand, materials modelled by the Generalized Burland model are also named after the type of failure criterion, but characterized as "Critical".

5.4.1 Numerical Simulation of a Triaxial Test

In order to check the response of the different elasto-plastic models implemented in ESEPA, a preliminary study on the simple triaxial test was conducted.

The numerical simulations were performed on one single element in axisymmetric conditions and the imposed boundaries are shown in Figs. 5.1. The material was considered as weightless and an initial confined pressure of 100 was assigned. The material parameters necessary for the elasto-plastic analysis are reported in Table 5.1 (for Tresca and von Mises, $c = 50$).

The theoretical expectations have been completely satisfied for the four elastic perfectly plastic models. As shown in Figs. 5.15 and 5.16, the axial stress increases

linearly with the imposed displacement, until it reaches a constant value at failure. In the loading path the numerical results obtained for Tresca and Mohr-Coulomb materials coincide with those calculated for von Mises and Drucker-Prager materials respectively. As expected, the level of stress required to reach plasticity under unloading conditions, is lower for Mohr-Coulomb than for Drucker-Prager material.

Figs. 5.15b and 5.16b show a rather unrealistic prediction of plastic volumetric change, specially for Drucker-Prager and Mohr-Coulomb materials.

The same element, Figs. 5.1, was then tested for cycling loading in plane strain conditions. In the loading path, as shown in Figs. 5.17 and 5.18, the numerical results obtained by employing Tresca and Mohr-Coulomb coincide with those computed using von Mises and Drucker-Prager. Imposing further displacements, the induced vertical stress may either still increase (von Mises and Drucker-Prager) or remain constant (Tresca and Mohr-Coulomb).

These distinct post-yield responses herein obtained, are believed to be the basic difference between the failure state described by the two sets of criteria. The difference in responses can be explained by the fact that the limit load in Tresca and Mohr-Coulomb depends also on the θ -value, whereas it is independent in von Mises and Drucker-Prager. In the plane strain examples above reported, θ -value,

remains constant and equal to 30 degrees until the stress point reaches the failure surface. For further displacements, θ starts to vary since the stress in the perpendicular direction to the plane is no longer equal to zero. Though the variation of θ reduces the limit deviatoric invariant, q in Tresca and Mohr-Coulomb, the plastic equilibrium leads to the stabilization of the vertical stress. On the other hand, the variation of θ in von Mises and Drucker-Prager, does not influence the limit value of q and, as a result, the vertical stress keeps increasing.

As simple and obvious as the results may appear, they are the proof of the accuracy that can be achieved by ESEPA. A further confirmation of the reliability of this program is found comparing its results with those obtained by Desai and Siriwardane (1983), Figs. 5.19.

In this example, as in the previous one, an element is loaded in plane strain conditions, until failure occurs. These authors found, in agreement with Davidson and Chen (1974), that plastic deformation is detected around 1000 psi, and failure is reached at 1732 psi. The Drucker-Prager failure criterion was adopted.

The results obtained by ESEPA show instead, in agreement with the theoretical prediction, that plasticity takes place only when the stress point reaches the failure surface, that is at 1732 psi. Depending on the choice of the failure criterion, the stress may either still increase (Drucker-Prager) or remain constant (Mohr-Coulomb).

As for the implementation of the Generalized Burland model, it was possible to recover numerical results previously published, Figs. 5.20.

Herein, the analysis can be extended to over-consolidated clay and represent the critical state also by the Mohr-Coulomb failure criterion.

The results in Figs. 5.21 show that softening can be followed in ESEPA for highly over-consolidated materials, for which it is also possible to predict dilation. In the Generalized Burland model softening is reached when the stress point lies on Hvorslev's surface (Section 2.7) where, since p is smaller than β , the plastic modulus A , calculated as in Eq. 2.71 in Table 2.8, is negative.

The numerical responses in triaxial for normally consolidated soils were not influenced by the choice of the failure criterion in loading condition, whereas different behaviour was predicted upon unloading.

However, further investigations showed that the numerical results are influenced by the type of algorithm employed and by the convergence tolerance ϵ allowed.

In fact, the ISM usually produces numerical results higher than the TSM, but under reduced tolerance $\bar{\epsilon}$, the values tend to coincide with those obtained by TSM, Figs. 5.22. Generally, the same results are obtained irrespective of imposing stress or strain. Incidentally, the results in Figs. 5.20 and 5.21 have been obtained employing the ISM, and imposing a tolerance $\bar{\epsilon} = 1$.

Although it may be concluded that TSM is more reliable than ISM, the former method cannot be always applied for the Cam-clay models, specially when material softens.

Another interesting feature of these numerical experiments was the substantial difference in CPU time registered in the different algorithms employed. In general, the TSM required always much less CPU time than the ISM; also, a big scatter in CPU time was noticed between imposing loads and displacements. For instance, in case of imposed displacements (Figs. 5.22), the CPU time was 2.65 sec. and 2.07 sec. for ISM ($\bar{\epsilon} = 1.$) and TSM ($\bar{\epsilon} = 1.$) respectively; instead when load was applied and TSM ($\bar{\epsilon} = 1.$) adopted, the CPU time was of 2.56 sec..

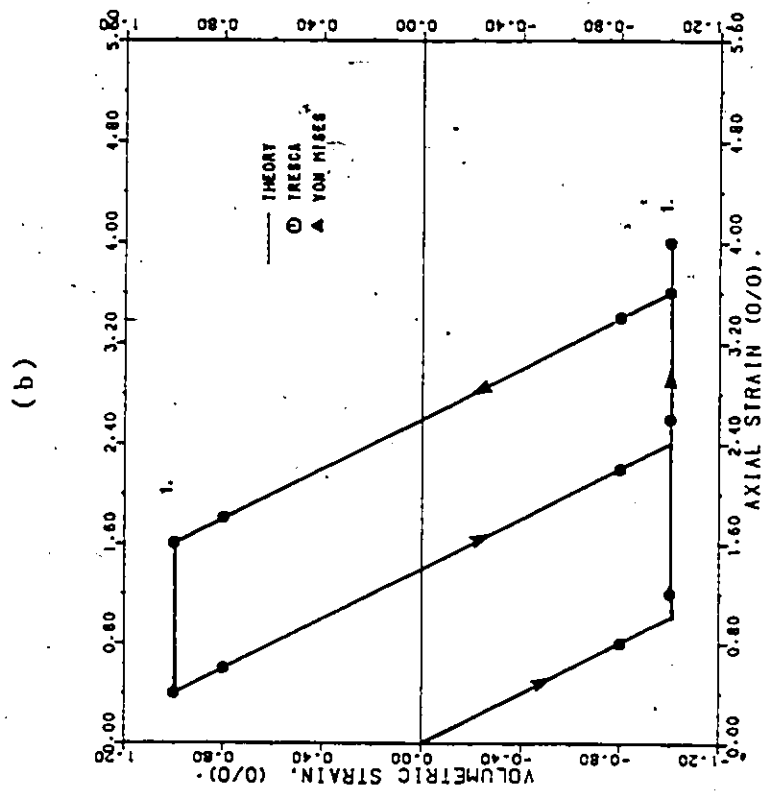
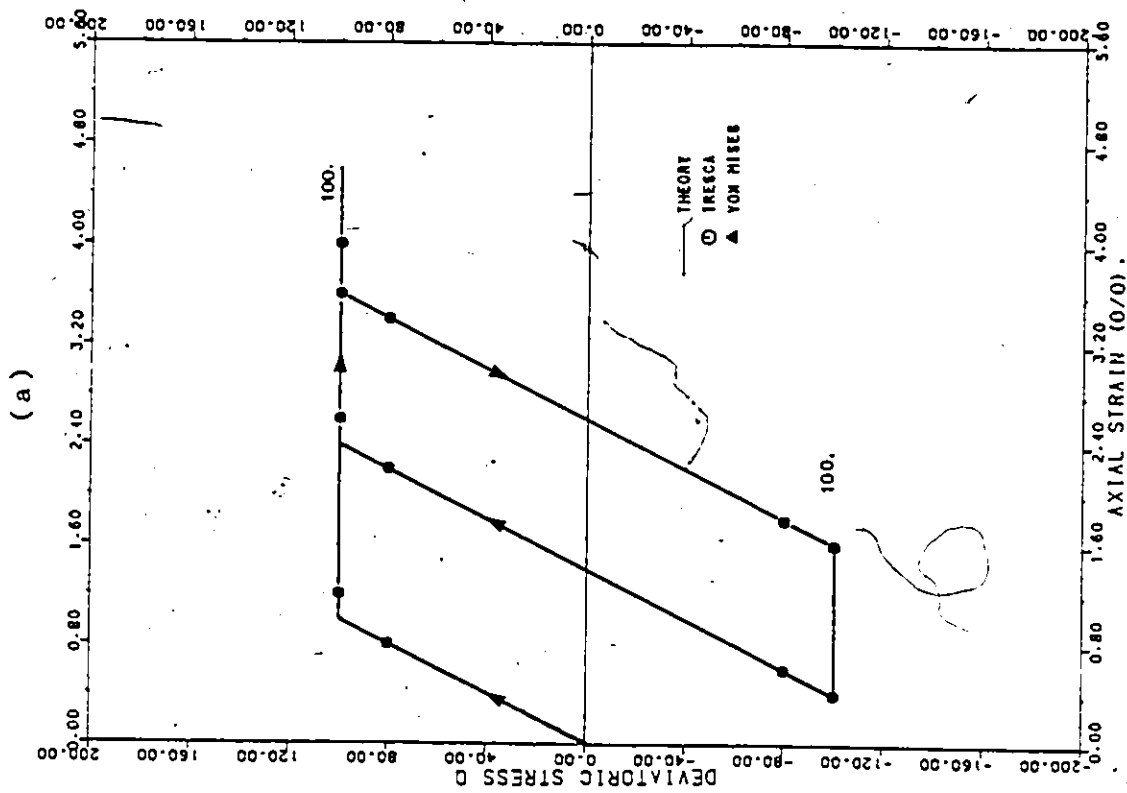


Fig. 5.15 Simulation of a triaxial test: comparison between Tresca and von Mises materials

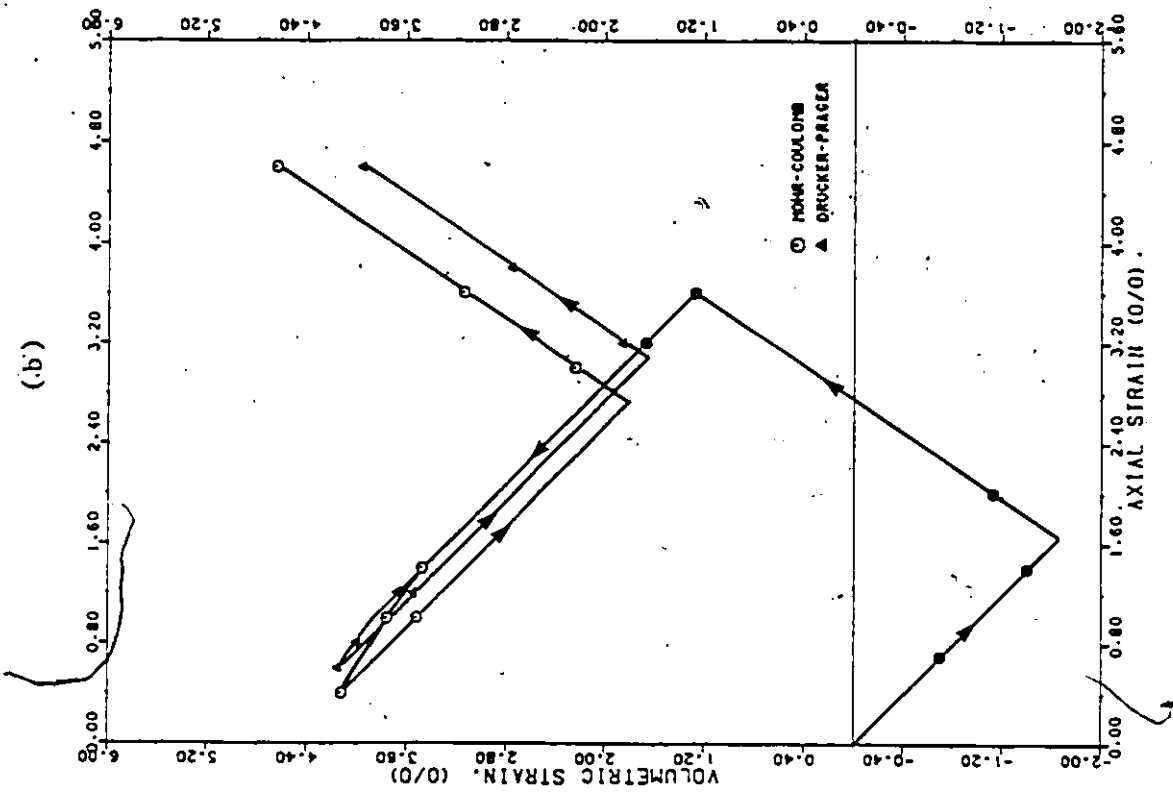
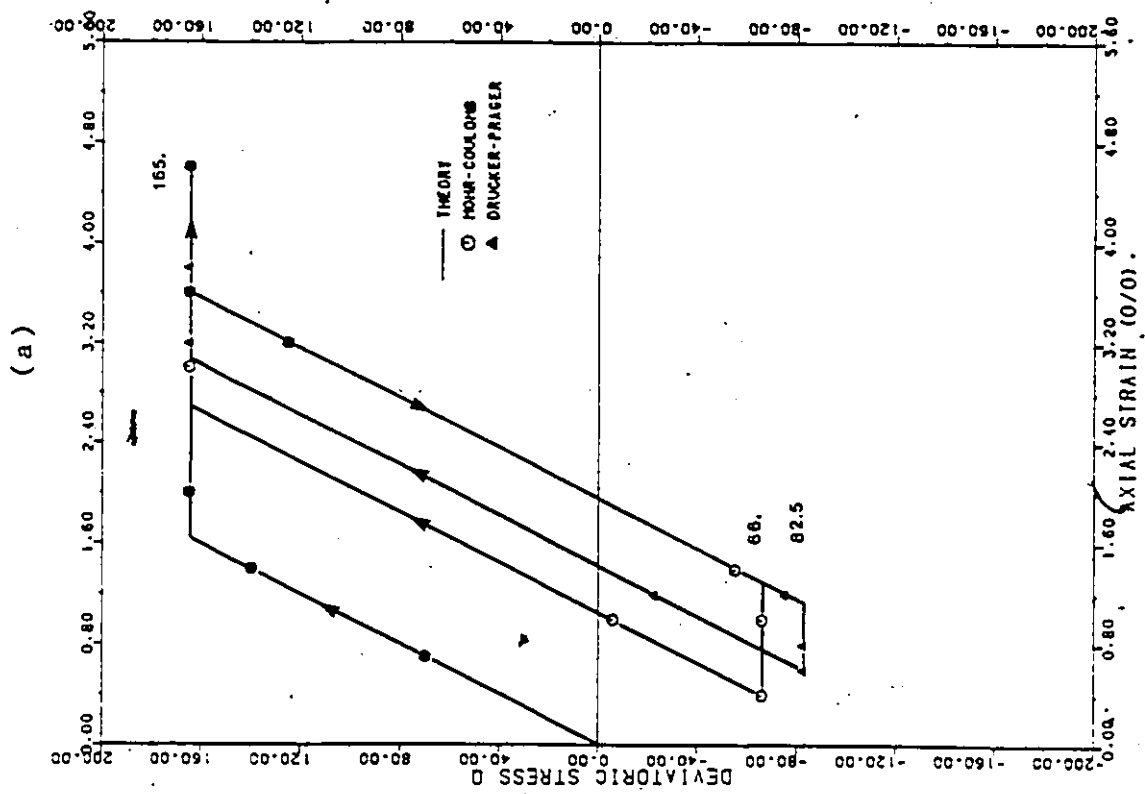


Fig. 5.16 Simulation of a triaxial test: comparison between Mohr-Coulomb and Drucker-Prager materials

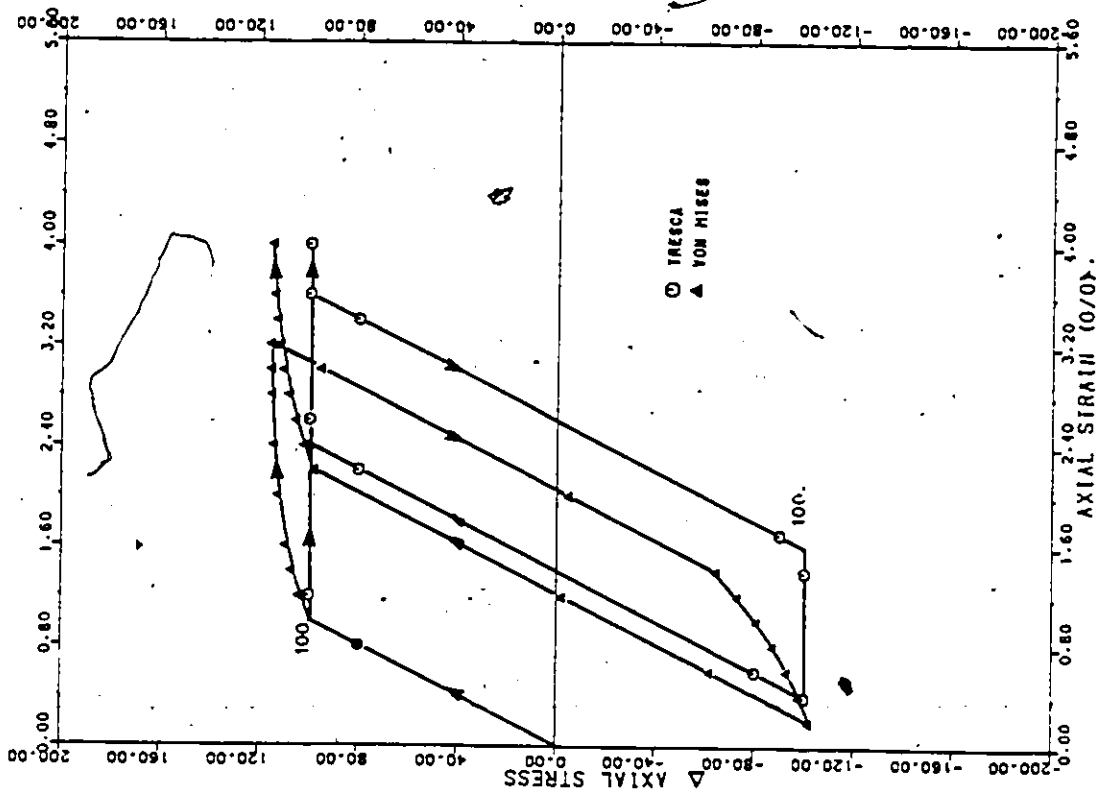


Fig. 5.17 Simple test in plane strain: comparison between Tresca and von Mises materials

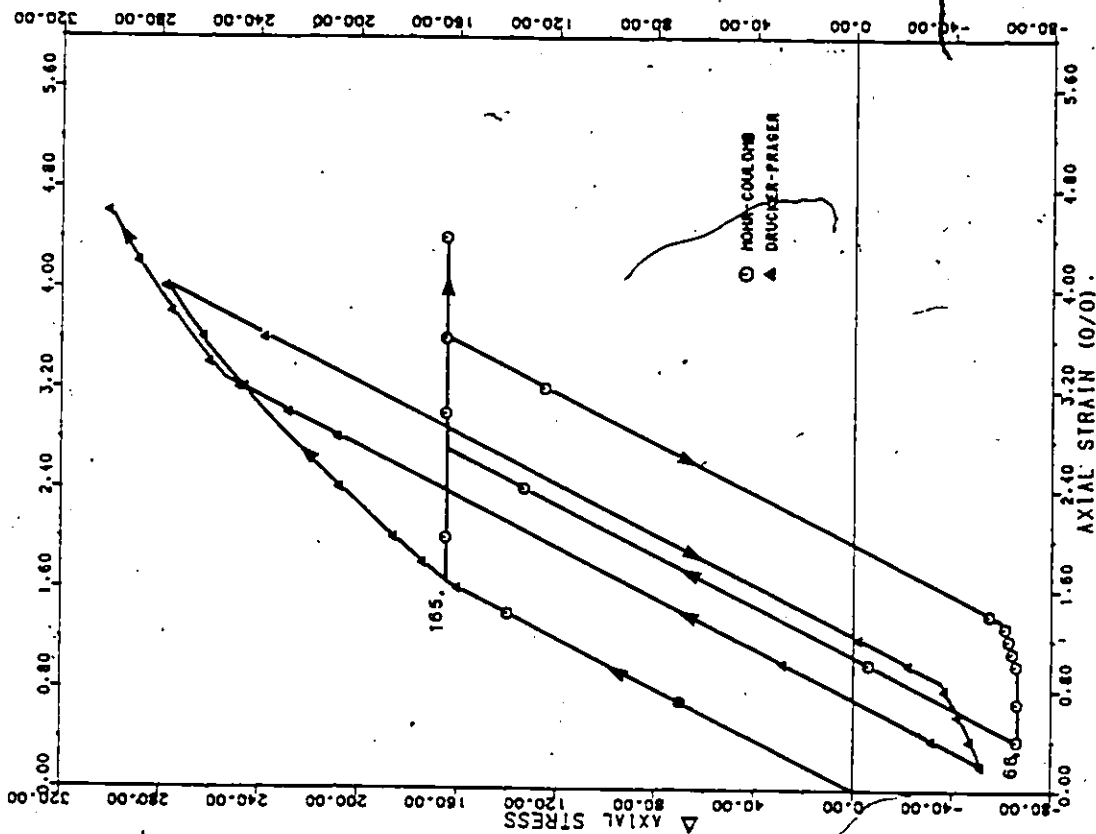


Fig. 5.18 Simple test in plane strain: comparison between Mohr-Coulomb and Drucker-Prager materials

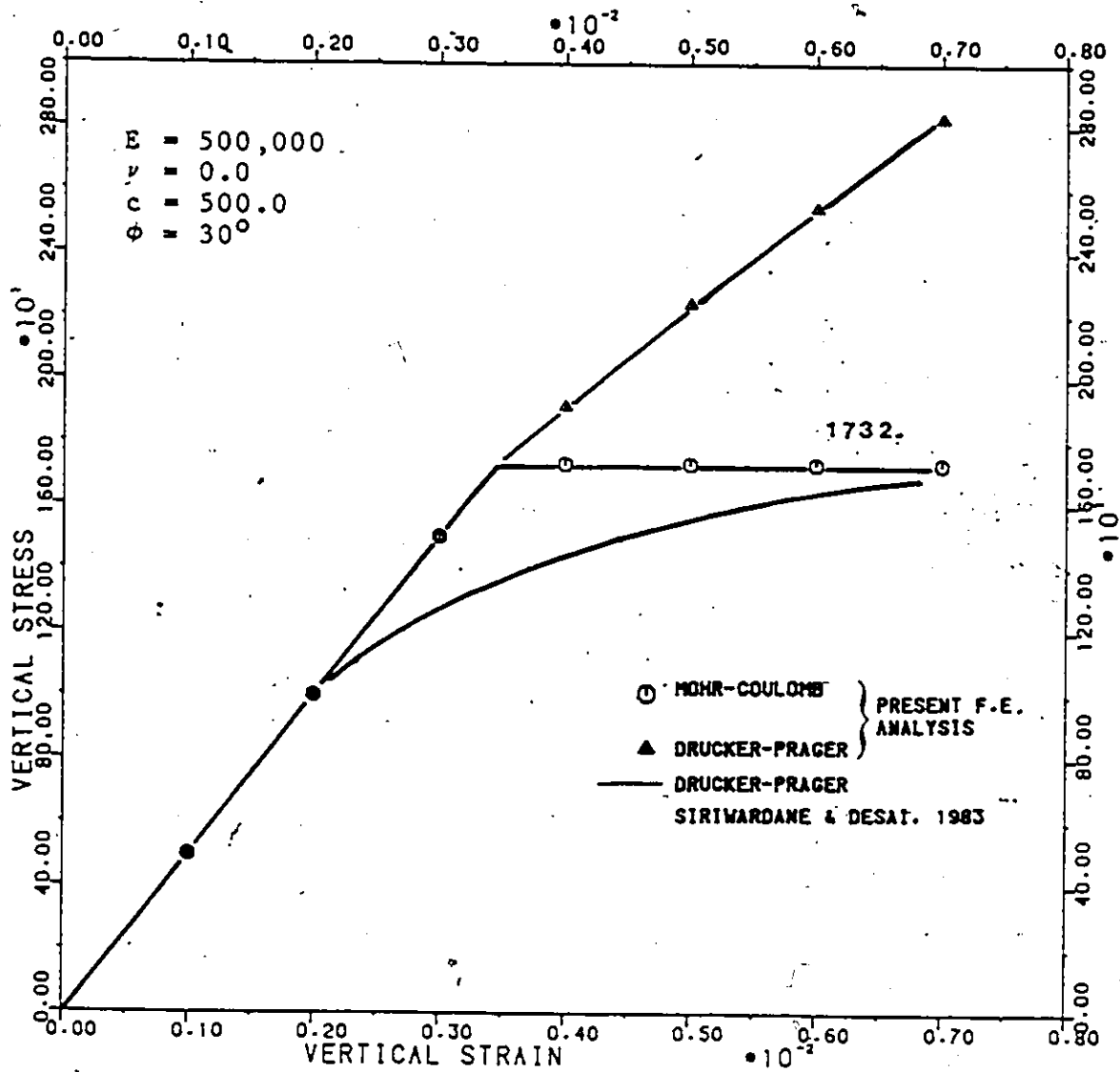


Fig. 5.19 Simple test in plain strain: comparison with numerical results by Siriwardane and Desai (1983)

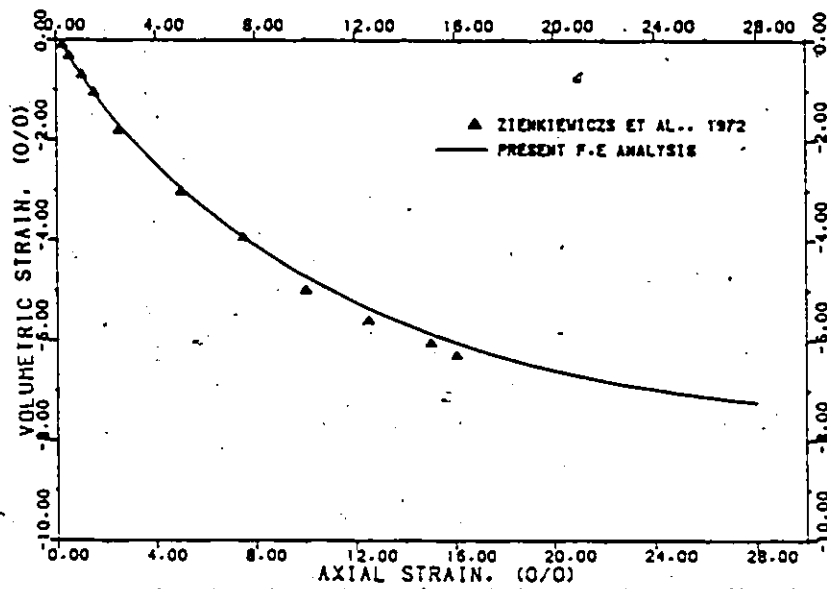
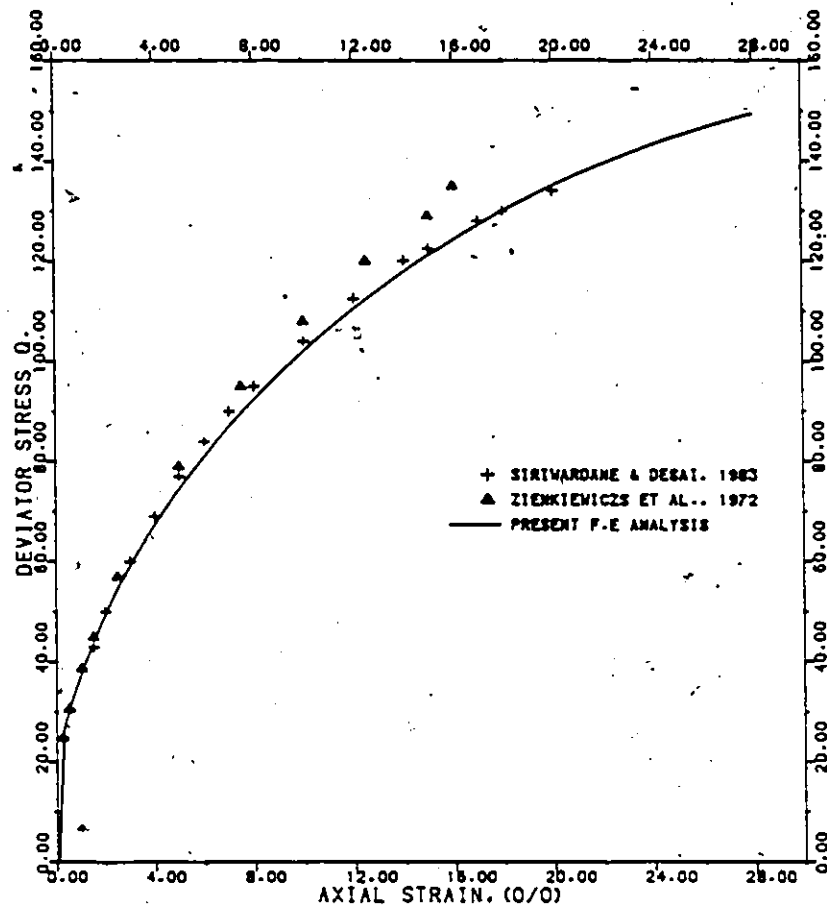


Fig. 5.20 Simulation of a triaxial test: Burland model (comparison with previous numerical results)

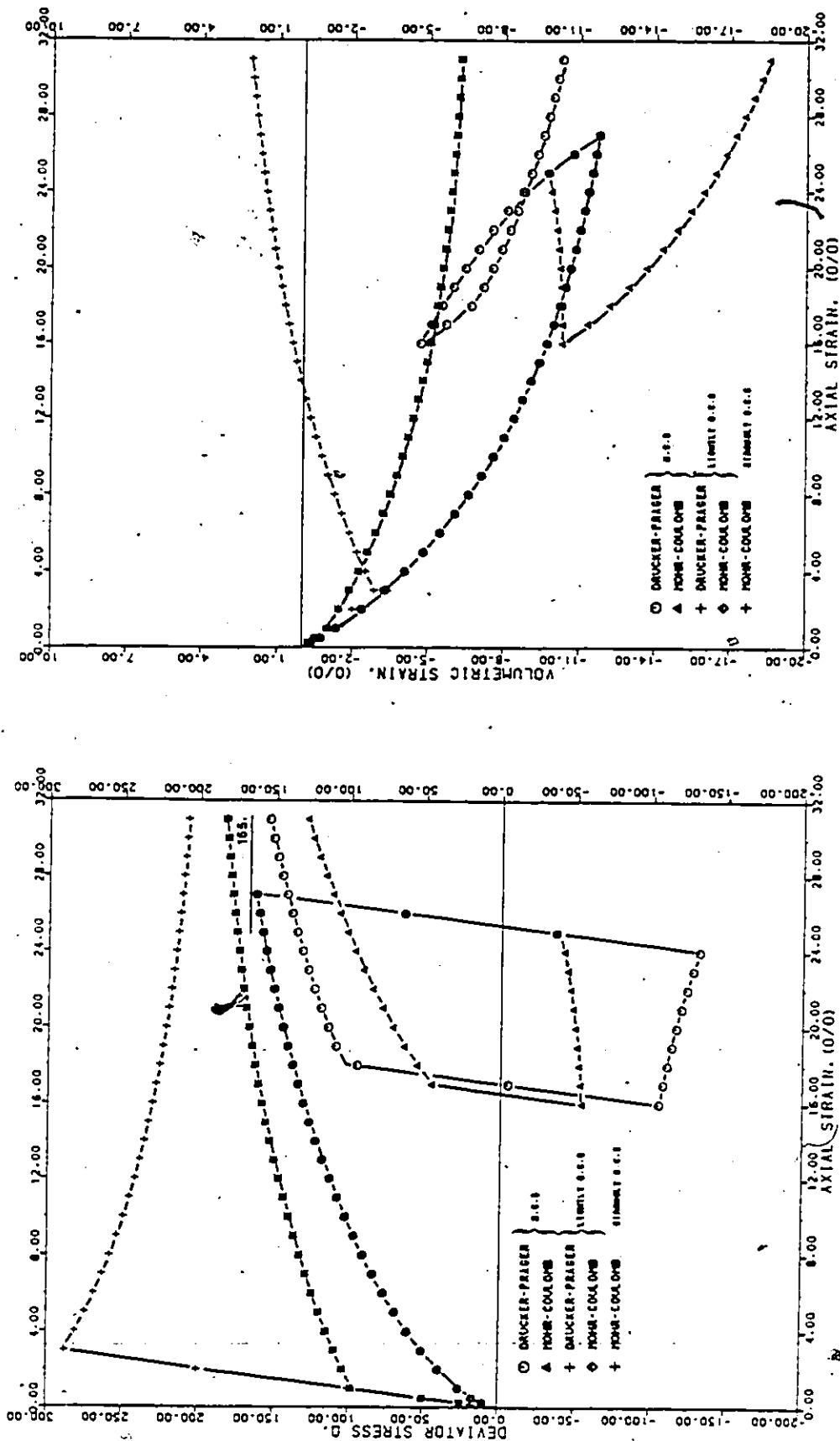


Fig. 5.21 Simulation of a triaxial test: Critical Mohr-Coulomb and Drucker-Prager materials (NCS: $p_e=100$; lightly OCS: $p_e=200$; strongly OCS: $p_e=600$)

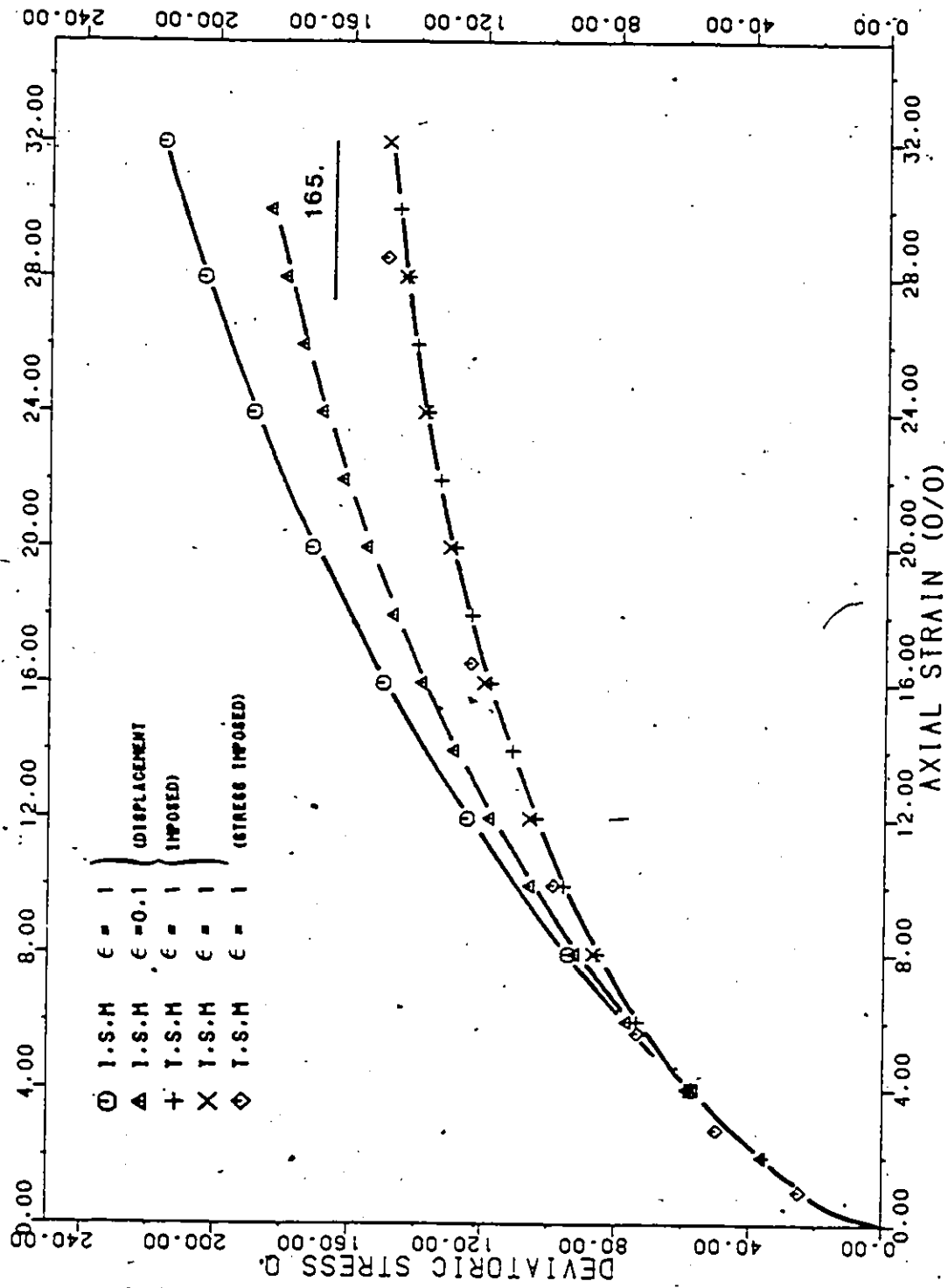


Fig. 5.22 Simulation of a triaxial test: Critical Mohr-Coulomb material (comparison between different algorithms)

5.4.2 Stress Analysis in a Semi-infinite Medium

The numerical test has then been extended to the stress analysis of a semi-infinite medium subjected to a smooth uniform strip loading. The medium was discretized as shown in Figs. 5.4 and the material parameters are, as usual, reported in Table 5.1 (for von Mises and Tresca $c = 47.434$).

A preliminary assessment on the validity of the mesh employed was performed comparing the results obtained in the assumption of linear elasticity with those given by the well known analytical solution (Poulos and Davis, 1974), Figs. 5.23.

This assessment also allowed to verify the efficiency of the smoothing technique described in Section 4.2.1. It was found that the numerical values agree very closely with the analytical ones, and that the smoothed stresses at nodes were as accurate as the stresses at the Gauss integration points. It can be concluded, therefore, that the mesh employed is sufficiently reliable and that the smoothing technique works properly.

The numerical experimentation was then extended to the various elasto-plastic models herein implemented.

The ultimate load for elastic perfectly plastic models with failure surface described by either Tresca or Mohr-Coulomb, Figs. 5.24 and 5.25, coincides with the theoretical values given by the "Limit Analysis Theory" for weightless material (Chen, 1975).

As expected, on the basis of the earlier results obtained for the simple plane strain test, the limit loads calculated for either von Mises or Drucker-Prager material were higher than those obtained by employing the counterparts Tresca or Mohr-Coulomb materials.

Then the weight of the material was taken into account, for the case when the failure is described by either Drucker-Prager or Mohr-Coulomb criterion. This was done by imposing an initial isotropic stress ($\sigma_x = \sigma_y = \sigma_z$, $\tau_{xy} = 0$.) equal to the overburden weight.

In both cases the limit loads were considerably higher than those obtained for weightless materials, Figs. 5.26 and 5.27, but neither of them confirmed the ~~ultimate load~~ indicated by the "Limit Analysis Theory", which is of $\bar{Q} = 376$. In addition, the load-displacement curves for both cases, Figs. 5.28, are very close to each other until failure is reached for Mohr-Coulomb materials.

To confirm the correctness of the present numerical analysis, it was attempted to recover previously published works. Unfortunately not many examples have been found, so that the comparison had to be again restricted to the most recent numerical results published by Desai and Siriwardane (1983). As it can be seen in Figs. 5.29, the load-displacement curve computed by ESEPA diverged significantly from the one calculated by the aforesaid authors for the same case.

Finally the Generalized Burland model with failure described by either Drucker-Prager or Mohr-Coulomb criterion, has been tested. As in the previous examples, an initially isotropic stress distribution equal to the overburden weight was imposed, in order to take into account the weight of the material ($\gamma_s = 20$). The material was considered normally consolidated and its elasto-plastic parameters are reported in Table 5.1.

The numerical results confirmed to be influenced by the choice of the algorithm, whereas the previous elastic perfectly plastic models were not. The load-displacement curve obtained using the ISM always lies above the one calculated by the TSM, Figs. 5.30 and 5.31. Restricting the tolerance $\bar{\epsilon}$, the ISM curve tends towards the TSM curve, but unfortunately without reaching it. On the other hand, the TSM does not allow to follow completely the deformation process, since some points in the medium reached softening behaviour, bringing the solving process to ill-conditioning.

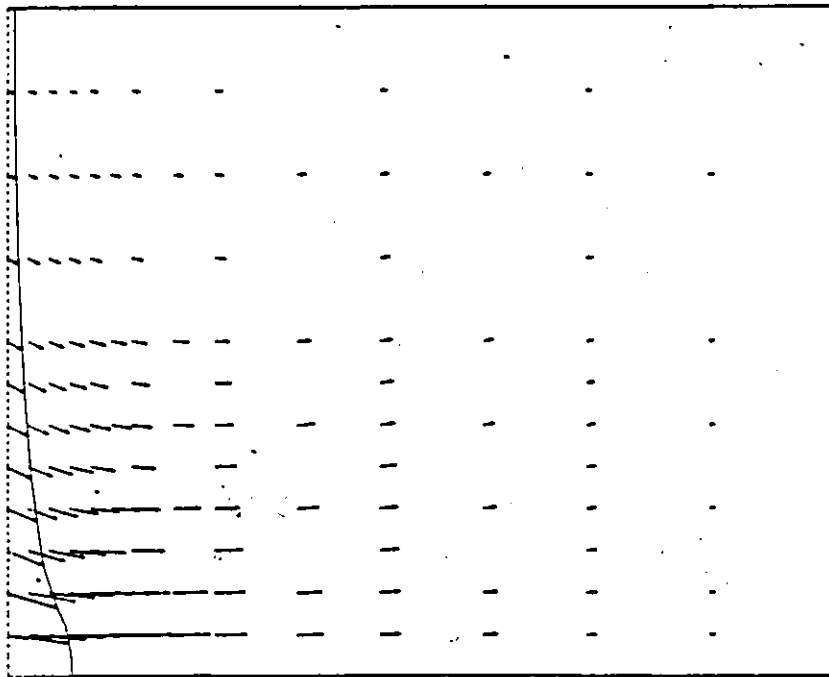
However, the overall response of the model can still be viewed. Firstly, it must be noticed that the order of magnitude of the ultimate loads is by far less than the one calculated by the corresponding elastic perfectly plastic models. This result is most probably due to the fact that the displacement field mobilized by using the Generalized Burland model is rather restricted to a zone beneath the foundation. This can be observed in Figs. 5.32 and 5.33,

which show the displacement fields calculated by the two models, both having as failure surface the one described by Mohr-Coulomb.

From the type of displacement field, it appears that the Generalized Burland model leads to a "localized mode of failure", whereas the elastic perfectly plastic model predicts a "generalized mode of failure".

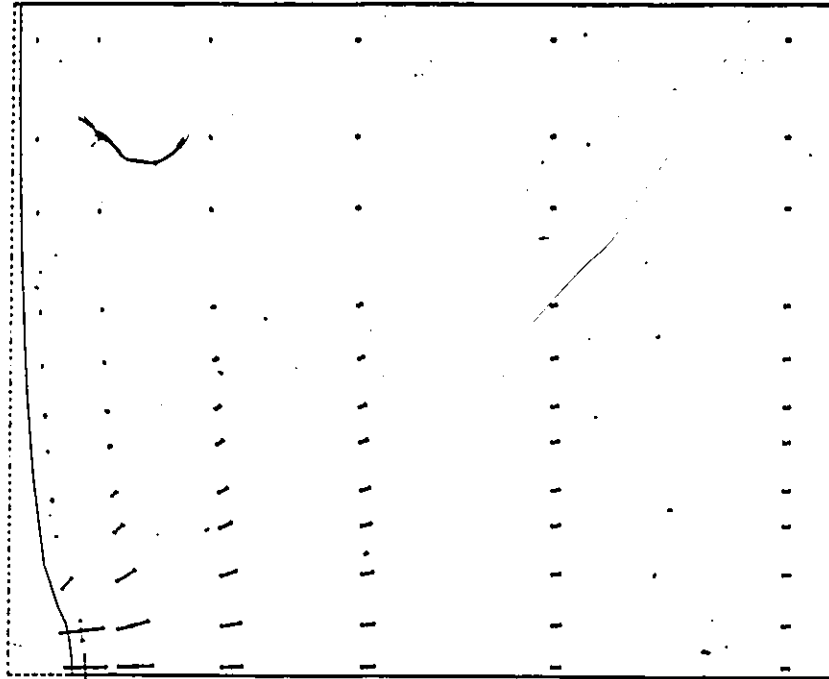
Finally, it is important to report that the numerical analysis performed employing the Generalized Burland model is much more costly than the one computed using an elastic perfectly plastic model. For instance, in the previous case, the CPU time was of 878. sec. when the Generalized Burland model was employed, while it was of only 236. sec. when the same problem was analyzed through the corresponding elastic perfectly plastic model.

Y DIRECTION
0.00 1.00 2.00 3.00 4.00 5.00 6.00 7.00 8.00 9.00 10.00



63 ELEMENTS 222 NODES
DISPLACEMENTS
DISPLACEMENT SCALE 10.
X DIRECTION
0.00 1.00 2.00 3.00 4.00 5.00 6.00 7.00 8.00

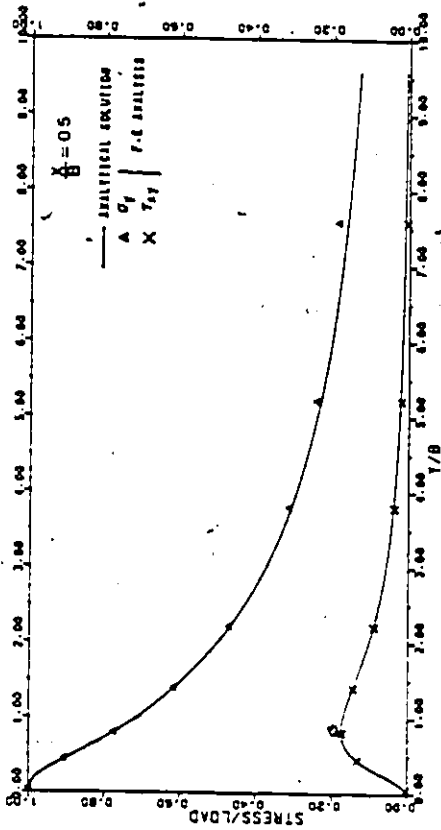
(a) displacement field



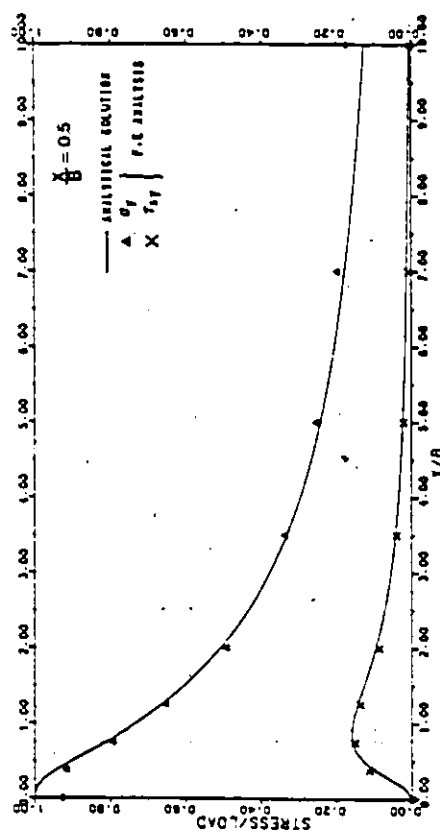
63 ELEMENTS 222 NODES
PRINCIPAL STRESS
UNDEFORMED MEAN
DEFORMED MEAN ———
STRESS SCALE 70.00
X DIRECTION
0.00 1.00 2.00 3.00 4.00 5.00 6.00 7.00 8.00

(b) principal stress field

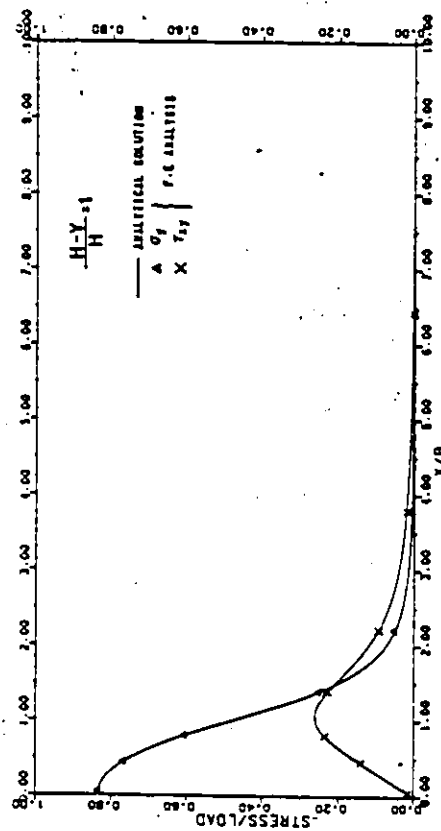
Fig. 5.23 Semi-infinite elastic medium



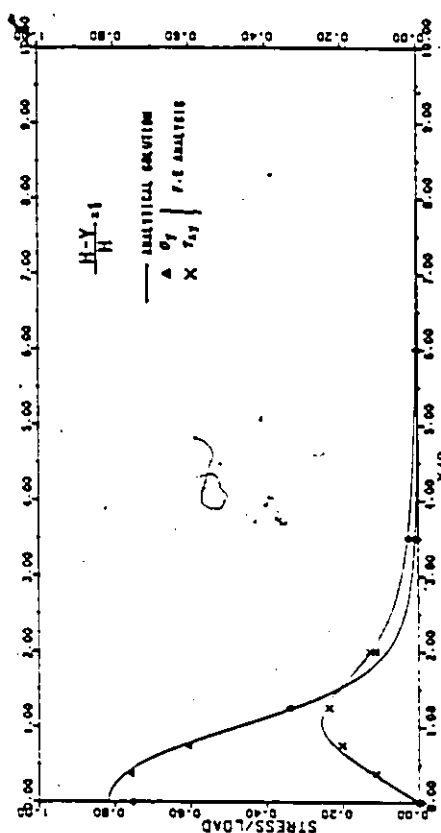
(c) comparison of stresses at Gauss points with analytical solution (vert. section)



(d) comparison of stresses at nodal points with analytical solution (vert. section)



(e) comparison of stresses at Gauss points with analytical solution (horiz. section)



(f) comparison of stresses at nodal points with analytical solution (horiz. section)

Fig. 5.23 Semi-infinite elastic medium

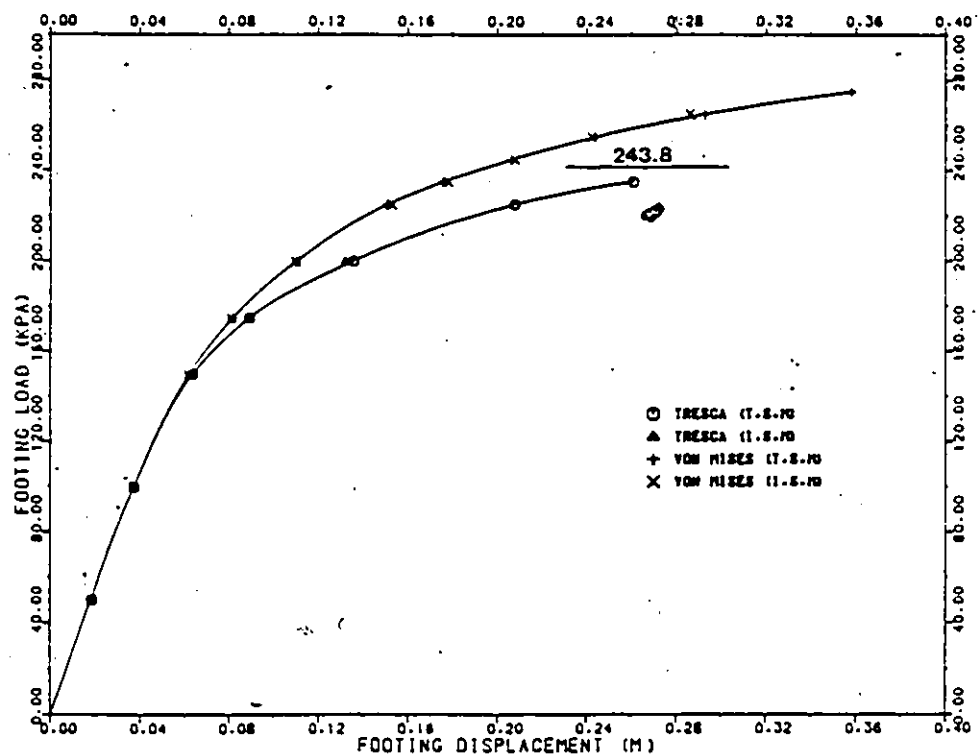


Fig. 5.24 Semi-infinite plastic medium: load displacement curves for weightless Tresca and von Mises materials

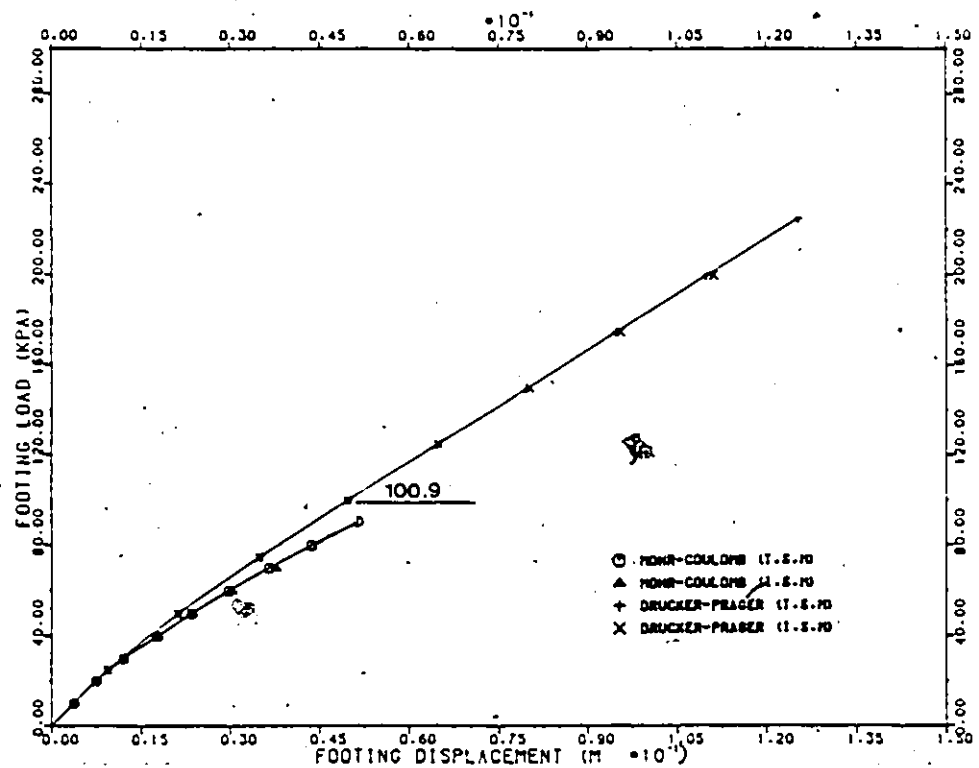


Fig. 5.25 Semi-infinite plastic medium: load displacement curves for weightless Mohr-Coulomb and Drucker-Prager materials

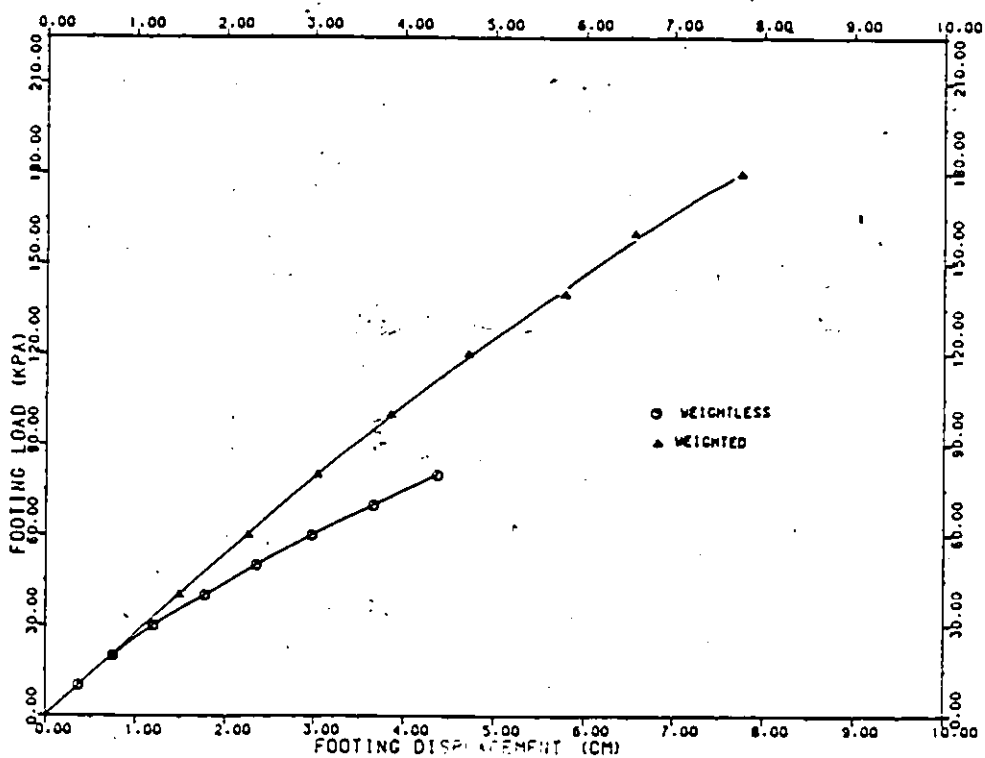


Fig. 5.26 Semi-infinite plastic medium: load displacement curves for weighted and weightless Mohr-Coulomb materials

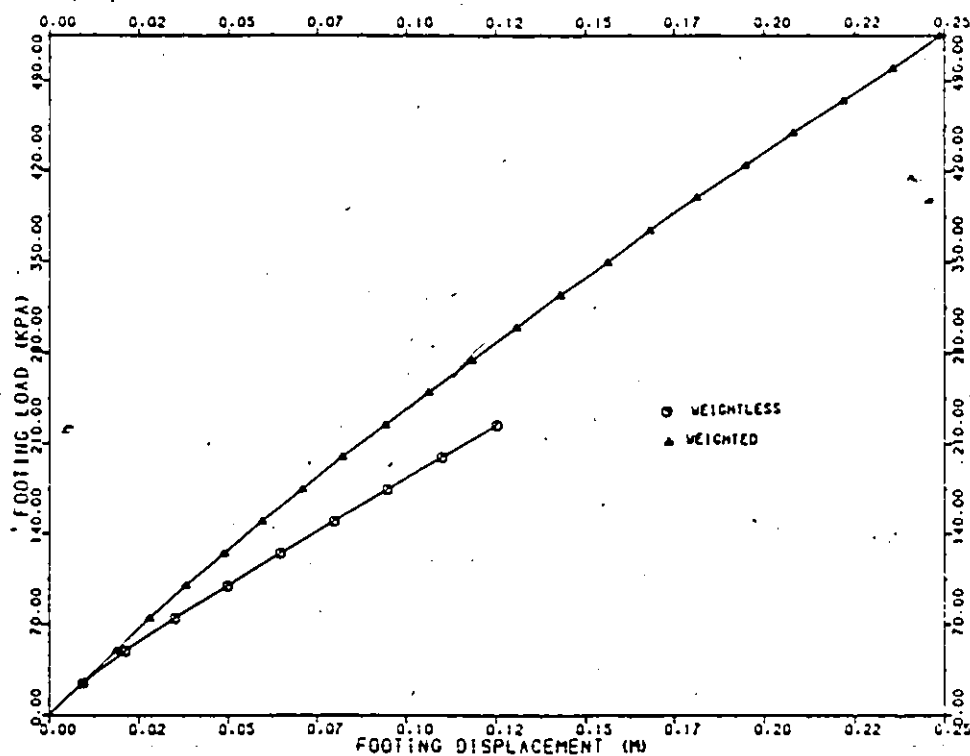


Fig. 5.27 Semi-infinite plastic medium: load displacement curves for weighted and weightless Drucker-Prager materials

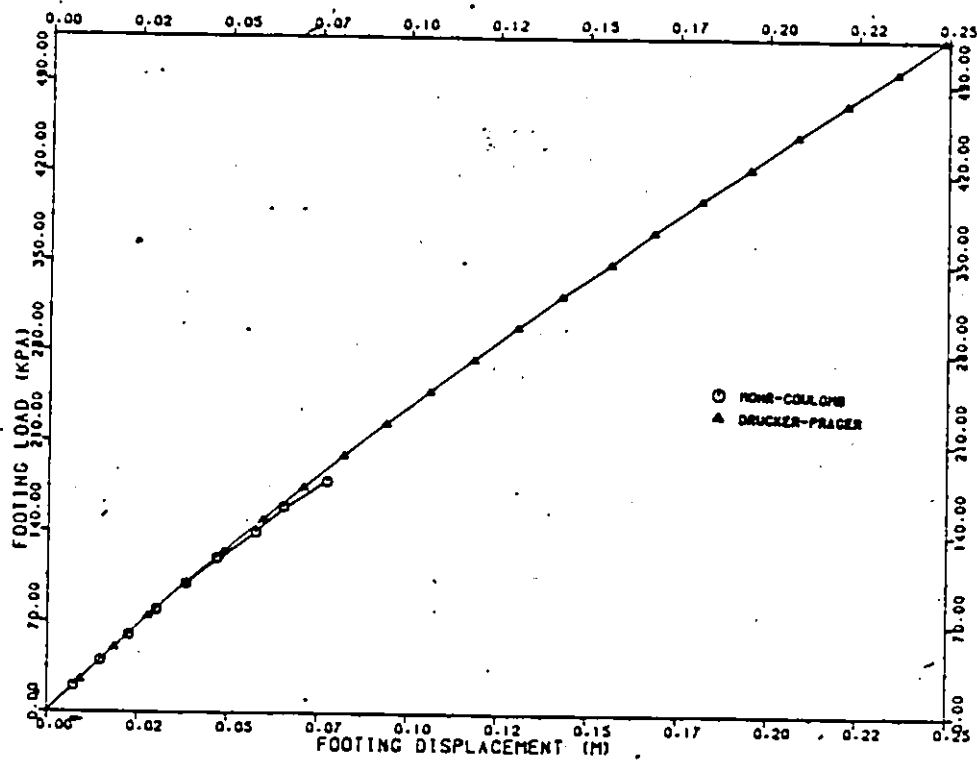
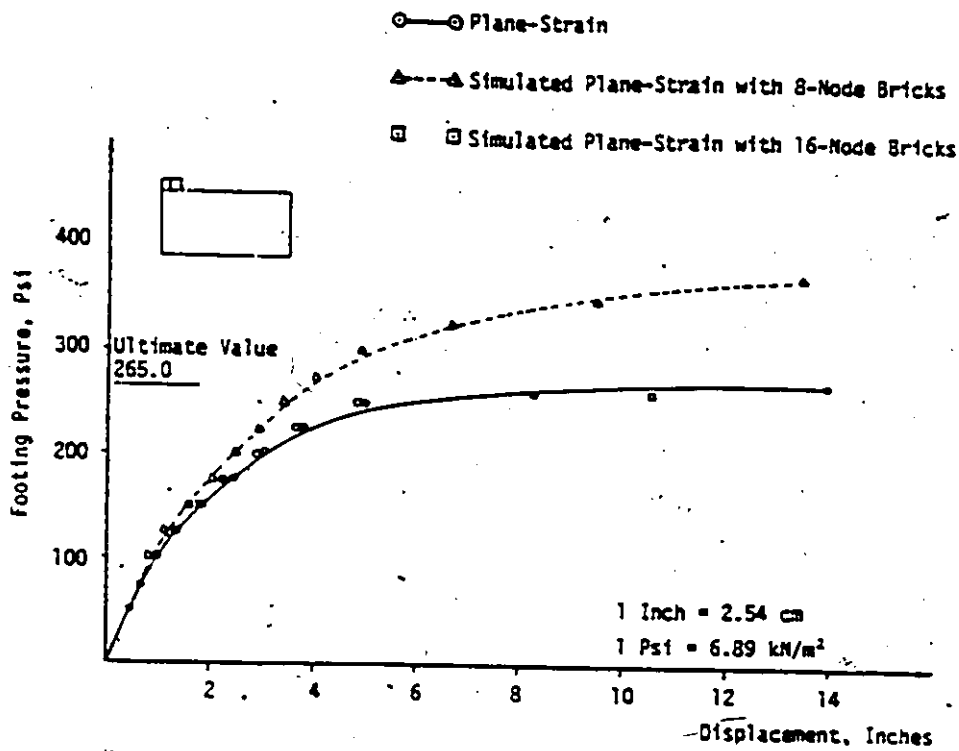
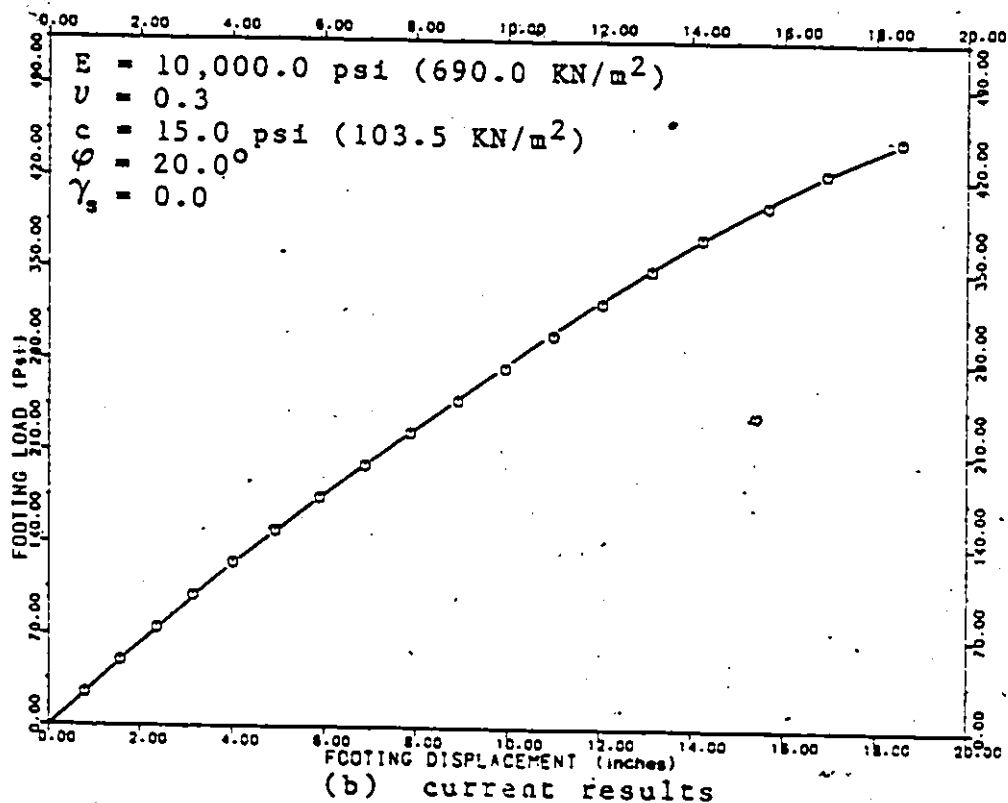


Fig. 5.28 Semi-infinite plastic medium: load displacement curves for weighted Mohr-Coulomb and Drucker-Prager materials



(a) results by Siriwardane and Desai, 1983



(b) current results

Fig. 5.29 Semi-infinite plastic medium: load-displacement curves for Drucker-Prager material (comparison with previous results)

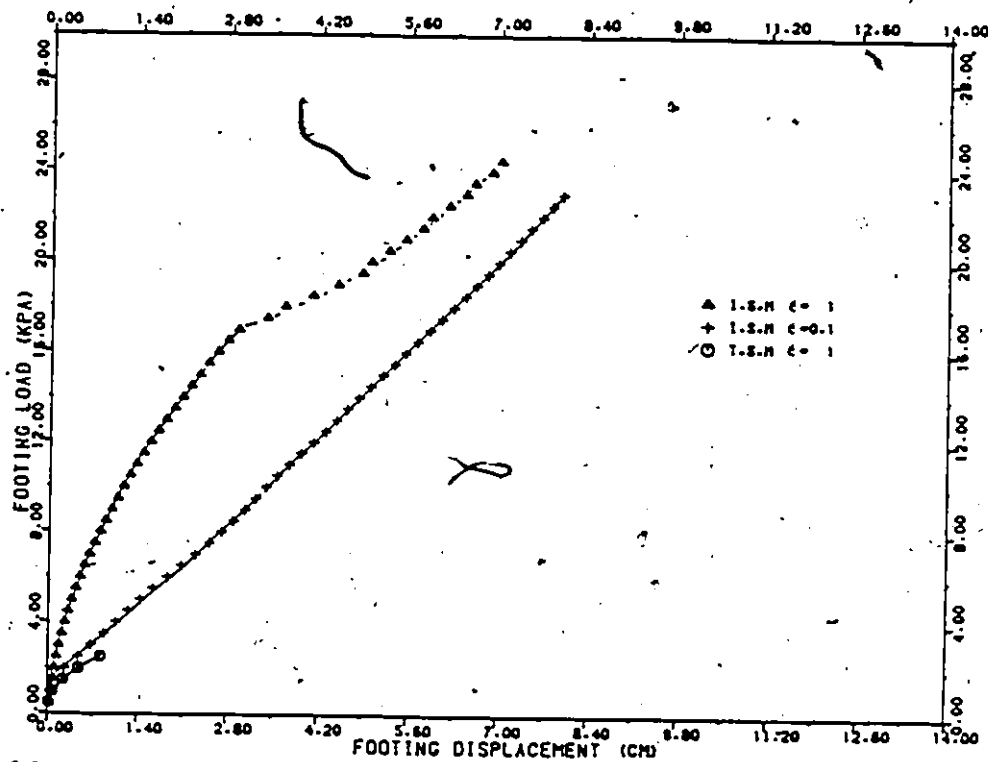


Fig. 5.30 Semi-infinite plastic medium: load displacement curves for Critical Mohr-Coulomb material

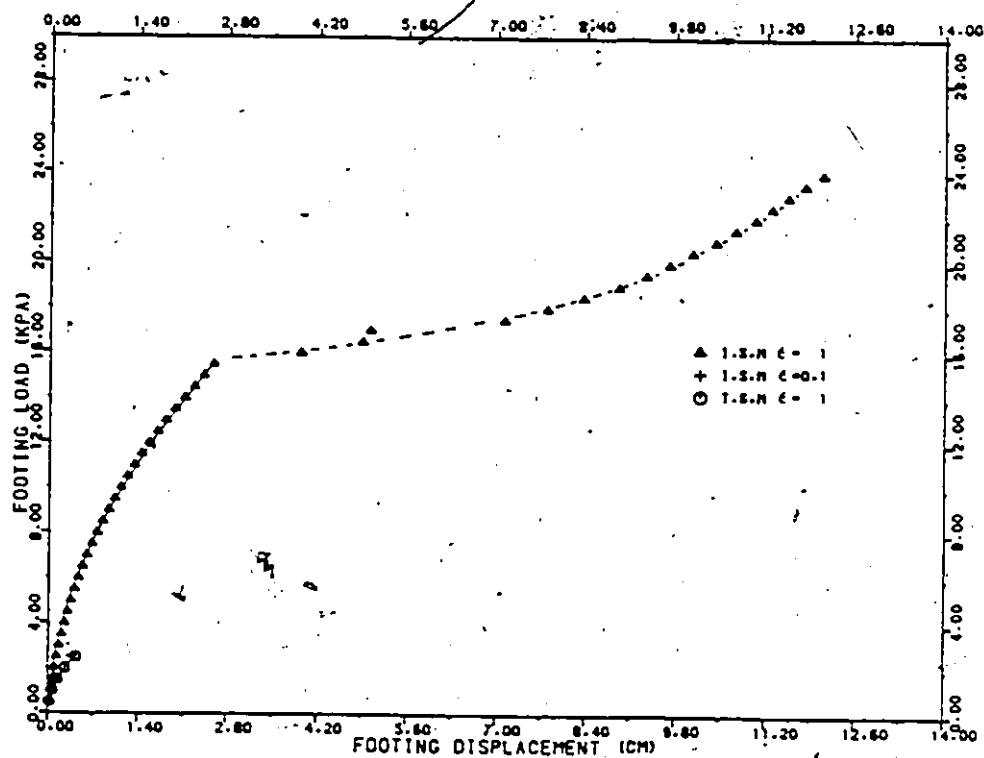


Fig. 5.31 Semi-infinite plastic medium: load displacement curves for Critical Drucker-Prager material

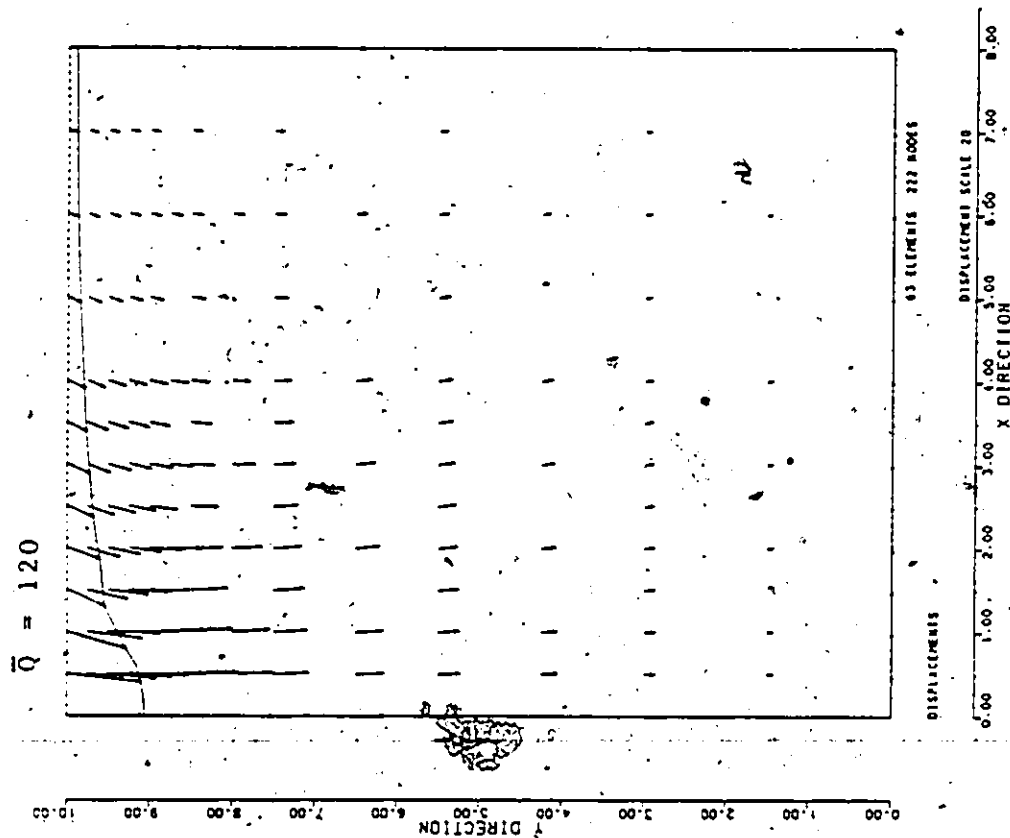
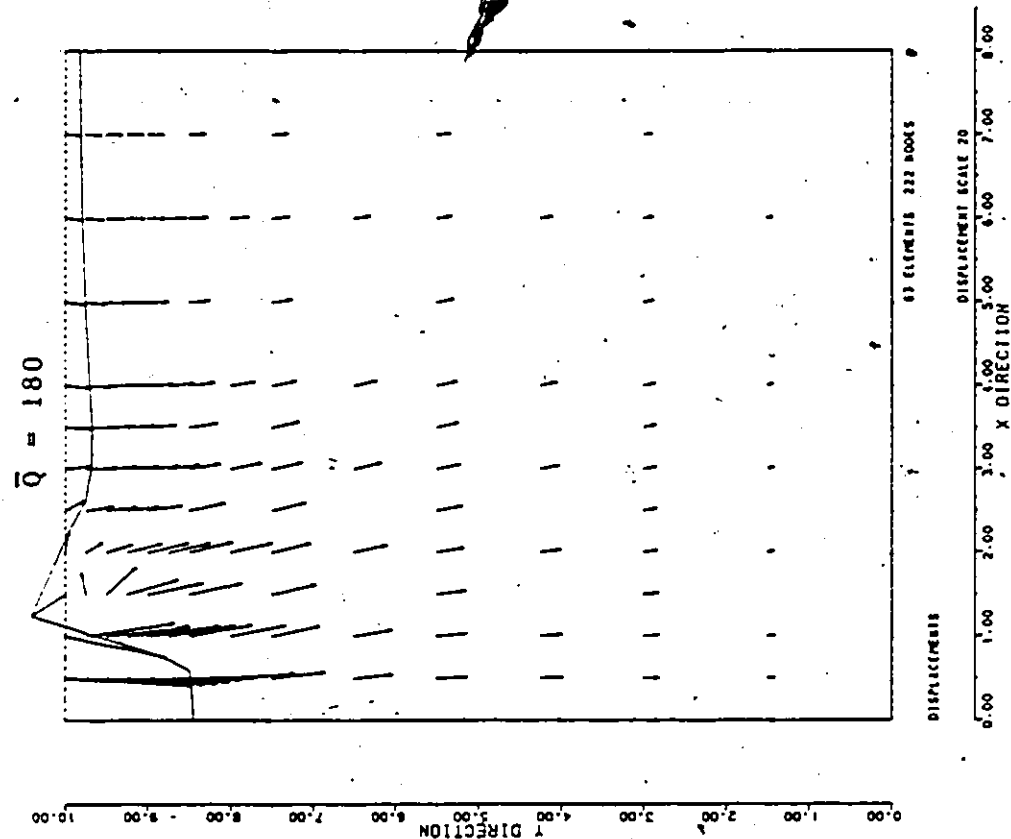


Fig. 5.32 Semi-infinite plastic medium: displacement fields for weighted Mohr-Coulomb material

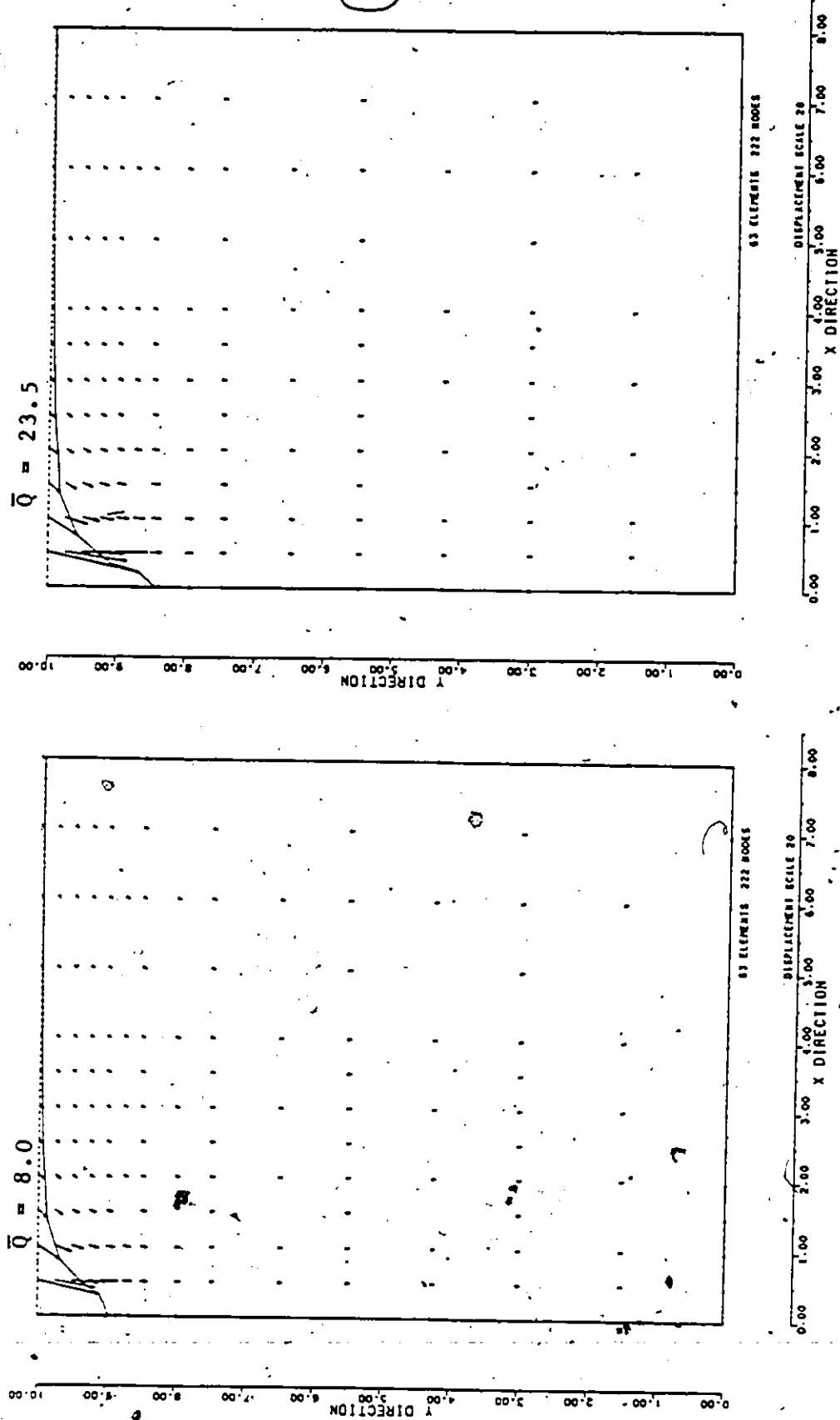


Fig. 5.33 Semi-infinite plastic medium: displacement fields for weighted Critical Mohr-Coulomb material

5.5 TEPSA

This program has been tested in several stages, starting from the 1-D case in the hypothesis of linear elasticity, up to the more complicate problem of semi-infinite medium in elasto-plastic condition.

The elasto-plastic material properties herein adopted are always reported in Table 5.1. The same time scheme was employed throughout all the examples: an initial solution was carried out at time $t = 10 \text{ E-6}$ followed by a second solution at $t = 10 \text{ E-2}$; from there, the next time increments were calculated on the basis of Eq. 4.10a. and θ was determined according to Sandhu's suggestion (Table 3.1).

5.5.1 Elastic Cases

In this Section, because of the linearity of the problems, most of the numerical results are presented in normalized form. In particular, the time factor T_v is calculated as previously described in Section 5.3.

In the 1-D problem the numerical results could be checked with Taylor's closed form solution (Taylor, 1948), since in this case Coupled and Uncoupled theories coincide.

The numerical calculations were carried out using the mesh and boundary conditions reported in Figs. 5.2a; on the top horizontal surface a unit load was applied. The results in Figs. 5.34 show that the smoothing technique has overcome the numerical problems found by Sandhu (1976), and that both displacements and pore pressure are correctly calculated.

Identical numerical results were obtained when the same mesh and boundary conditions, Figs. 5.2a, were assumed to represent a quarter-section of a cylindrical sample. In fact, this case still represents a 1-D problem from both geometrical and hydraulic point of view.

The numerical response in the axial symmetry changed completely when the sample was allowed to deform laterally. In the next two cases, the cylindrical sample was discretized and constrained as reported in Figs. 5.2c, and an initial isotropic pressure of 100 was imposed.

For this case, no analytical solution describing the whole consolidation process is available; nevertheless it can be proved that the initial stresses and excess pore pressure values can be calculated by the following expressions:

$$\Delta p' = 0 \quad \therefore \quad \Delta p = -\pi^e \quad (5.2a)$$

$$\Delta \sigma_r = 0 \quad \therefore \quad \Delta \sigma_r' = \pi^e \quad (5.2b)$$

$$\pi^e = -\frac{\Delta \sigma_z'}{2} = -\frac{\Delta \sigma_z}{3} \quad (5.2c)$$

where σ_a and σ_r are respectively the axial and the radial stresses, while the superscript " ' " is used to distinguish the effective stresses from the total ones. The above relationships can be found from straightforward equilibrium considerations based on the fact that, since the fluid is incompressible, $\Delta p' \approx 0$ at time equal to zero. At infinite time the excess pore pressure is obviously dissipated and the value of the stresses may be calculated by elementary means.

The numerical results in Figs. 5.35 agree completely with the theoretical values calculated from the above relationship (Table 5.2) and, as π^e dissipates, the load is progressively transferred to the effective stress. The values are presented in a normalized form where in particular $\bar{Q}$ is the applied vertical stress ($\bar{Q} = \sigma_a = 135$).

Unlike the prediction of the Uncoupled theory, Figs. 5.10c, π^e may continue to increase in time before starting to decrease, Figs. 5.35c and 5.35d, but overall the consolidation process is faster in the Coupled theory than in the Uncoupled one.

Of some interest is to follow the consolidation process of the same sample when displacements are instead initially imposed. In the case presented in Figs. 5.36 a displacement, equal to the initial one obtained in the previous example ($u = 0.018$), was imposed and kept constant in time. Always with reference to Eqs. 5.2, it is possible to evaluate the

initial stresses and pore pressure values which are reported in Table 5.2.

It is interesting to notice that, since the displacement is kept constant during consolidation, the initial and final effective axial stress must have the same value. It is not surprising, therefore, that the stress evolution inside the sample follows a path which was substantially different from that of the previous case, as clearly shown in Figs. 5.36 and specially in Figs. 5.36g. The normalization is in this case performed with respect to the initial total vertical stress ($\bar{Q} = \sigma_a = 135.$).

The CPU time required to compute the complete consolidation process in the previous two cases was of 156 sec..

The numerical investigation was then extended to the case of semi-infinite layer subjected to a uniform load ($\bar{Q} = 10.$) due to a flexible and smooth shallow foundation. The adopted mesh and boundary conditions are shown in Figs. 5.3. The calculation was extended over 50 iterations, which required a total CPU time of 149 sec..

Unlike the Uncoupled approach, the Coupled theory allows to monitor completely the consolidation process by determining at any time the displacement field, Figs. 5.37a, the fluid velocity field, Figs. 5.37c, the stress field, Figs. 5.37d, and, most important of all, the initial excess pore pressure field without any use of semi-empirical relationship.

The numerical results confirmed also previous findings (Cividini and Rossi, 1983) for which the excess pore pressure, in some place within the medium, continues to increase during the early stage of consolidation, Figs. 5.37f. This confirms once more the Mandel-Cryer effect, which was not detected by the Uncoupled theory, Figs. 5.14.

Finally, Figs. 5.38 shows the numerical results obtained in the case of a semi-infinite medium loaded by a shallow and smooth foundation, ($\bar{Q} = 10.$). The adopted mesh and the boundary condition are reported in Figs. 5.4; the consolidation process was followed over 66 time steps, which required 487 sec. of CPU time.

As shown in Figs. 5.38a, the surface settlement calculated by TEPSA initially deviates by an error of $\pm 2\%$ from the analytical solution proposed by Biot (1941b). When the time factor T_v reaches $10 \text{ E}+2$ the numerical solution converges to the expected theoretical values, ($u(\infty) = 0.37025 \text{ E}-3$), while Biot's solution seems to show no asymptotic bend. It must be considered however, that the author himself pointed out a certain degree of error in his solution (Biot, 1941b).

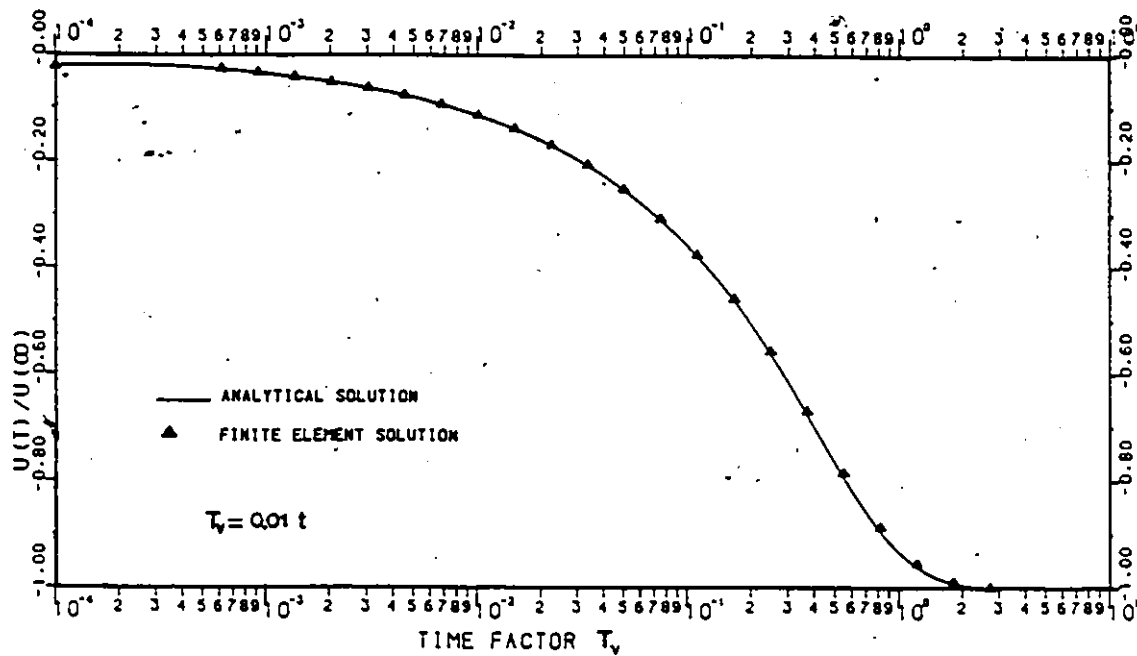
As in the previous case, the excess pore pressure continues to increase during the early stage of consolidation, specially at some points on the centre-line of the foundation, Figs. 5.38d. Consequently, since the variation of the effective stresses may present different

signs, Figs. 5.38d and 5.38e, the induced field of displacement presents alternative patterns during consolidation, Figs. 5.38c.

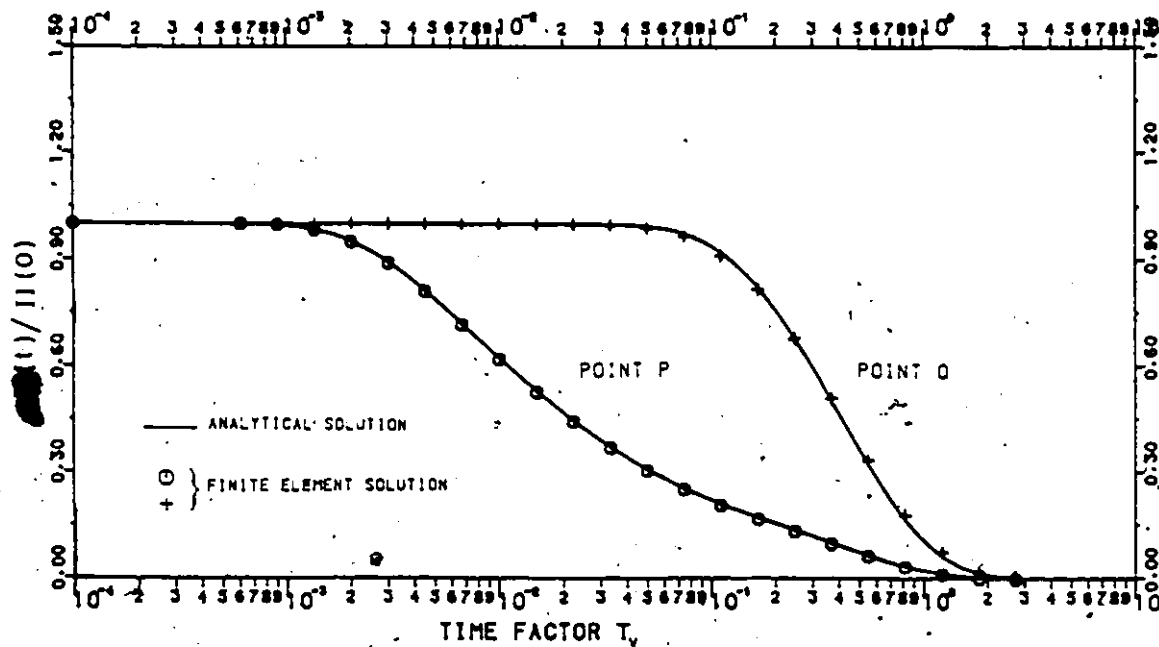
TABLE 5.2

Initial and final values of stresses and excess pore pressure in a triaxial sample

Stress Ratio	Load imposed		Displacement imposed	
	$t = 0$	$t = \infty$	$t = 0$	$t = \infty$
$\frac{\pi^e}{Q}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0
$\frac{\Delta\sigma'_z}{Q}$	$-\frac{2}{3}$	-1	$-\frac{2}{3}$	$-\frac{2}{3}$
$\frac{\Delta\sigma'_r}{Q}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0
$\frac{\Delta p'_e}{\pi_o}$	0	-1	0	$-\frac{2}{3}$
$\frac{\Delta q}{\pi_o}$	3	3	3	2

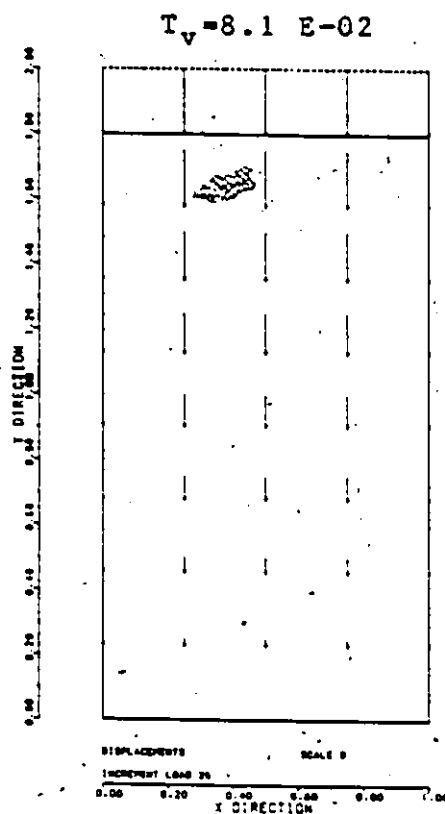
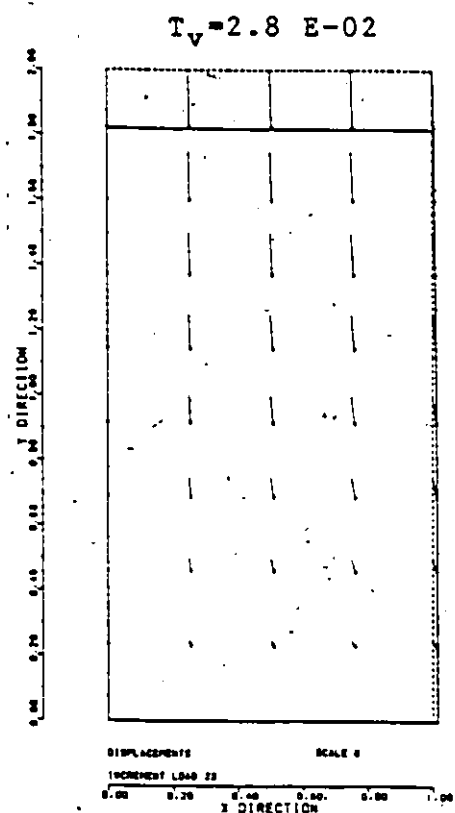
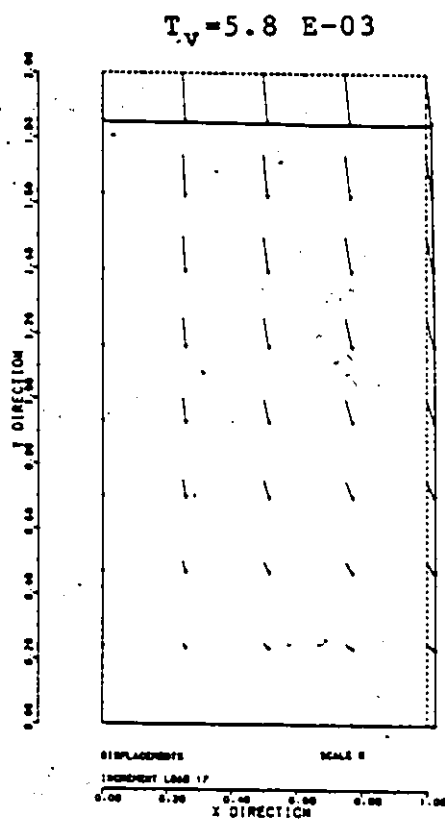
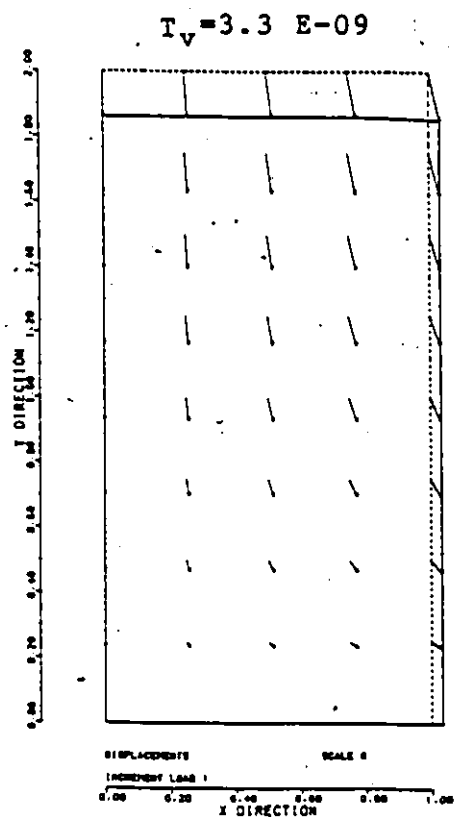


(a) surface settlement vs. time (comparison with analytical solution)



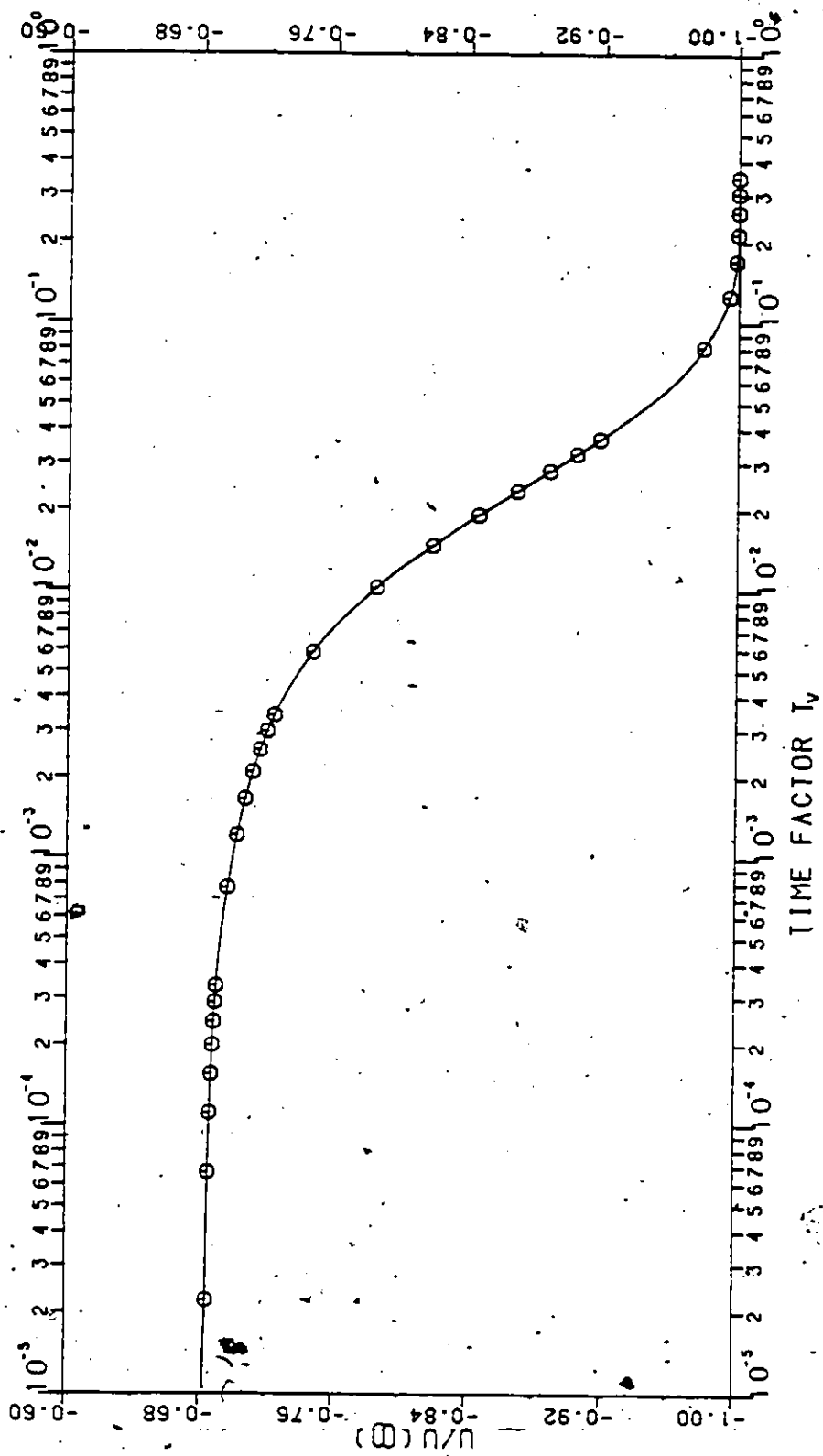
(b) excess pore pressure vs. time (comparison with analytical solution)

Fig. 5.34 1-D Coupled elastic consolidation



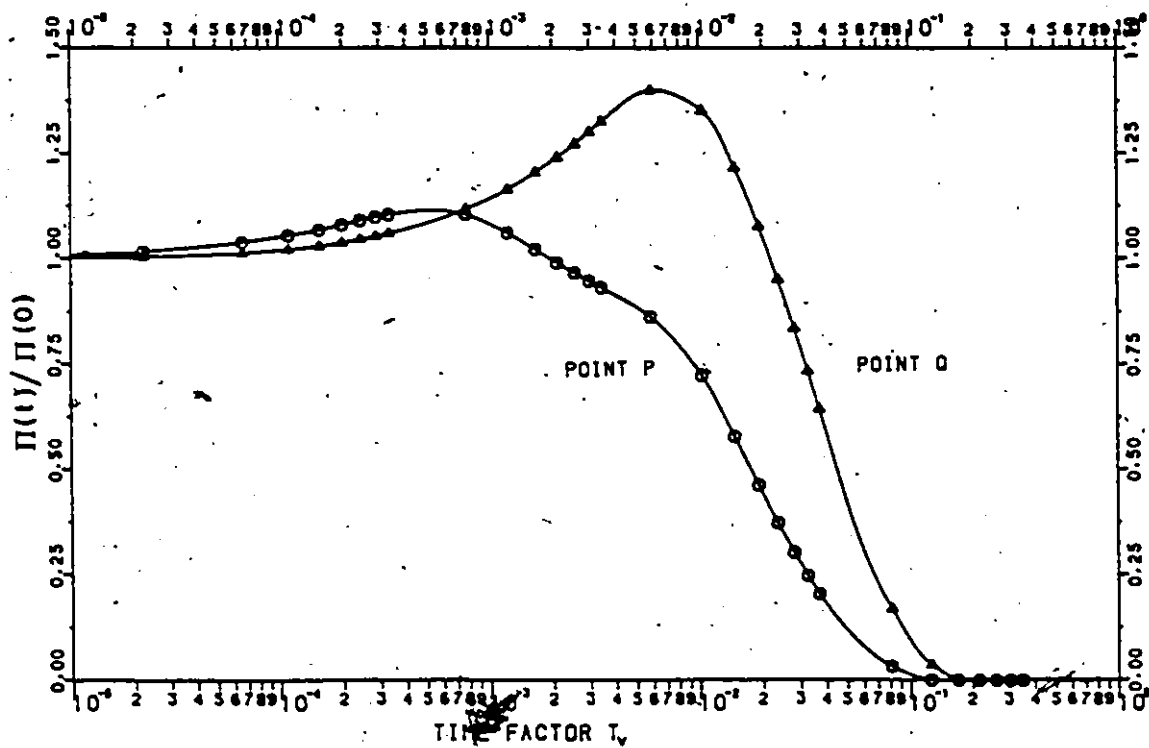
(a) displacement fields at selected times

Fig. 5.35 Triaxial coupled elastic consolidation
(load imposed)

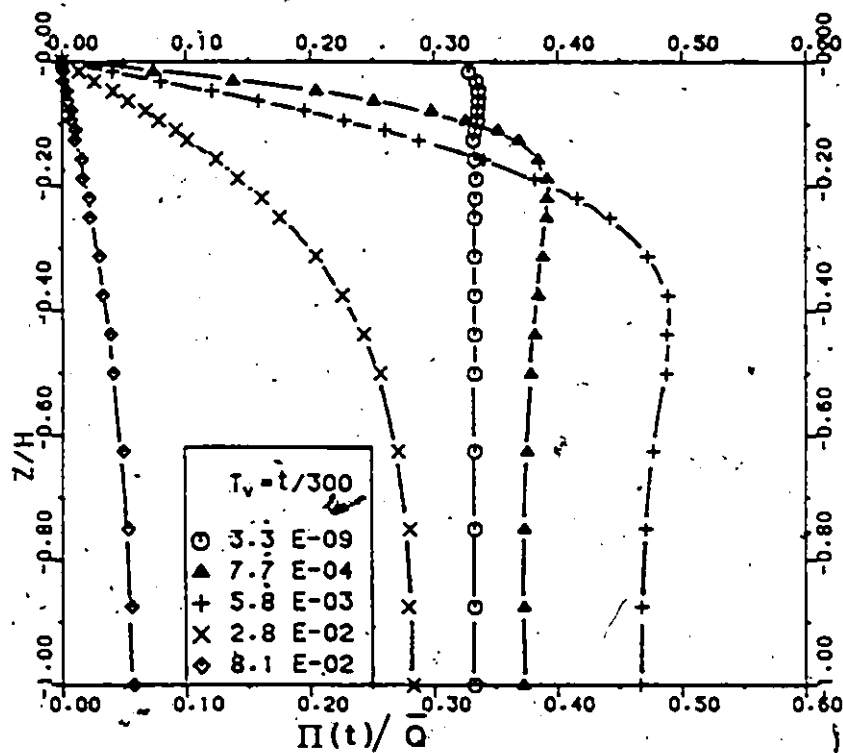


(b) surface settlement vs. time

Fig. 5.35a Triaxial coupled elastic consolidation (load imposed)

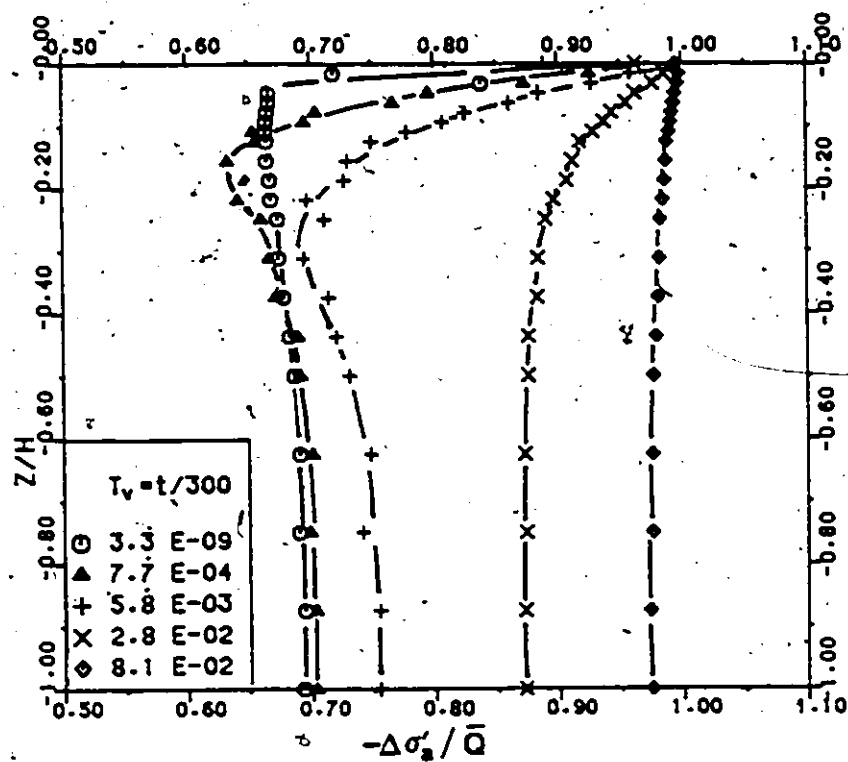


(c) excess pore pressure vs. time

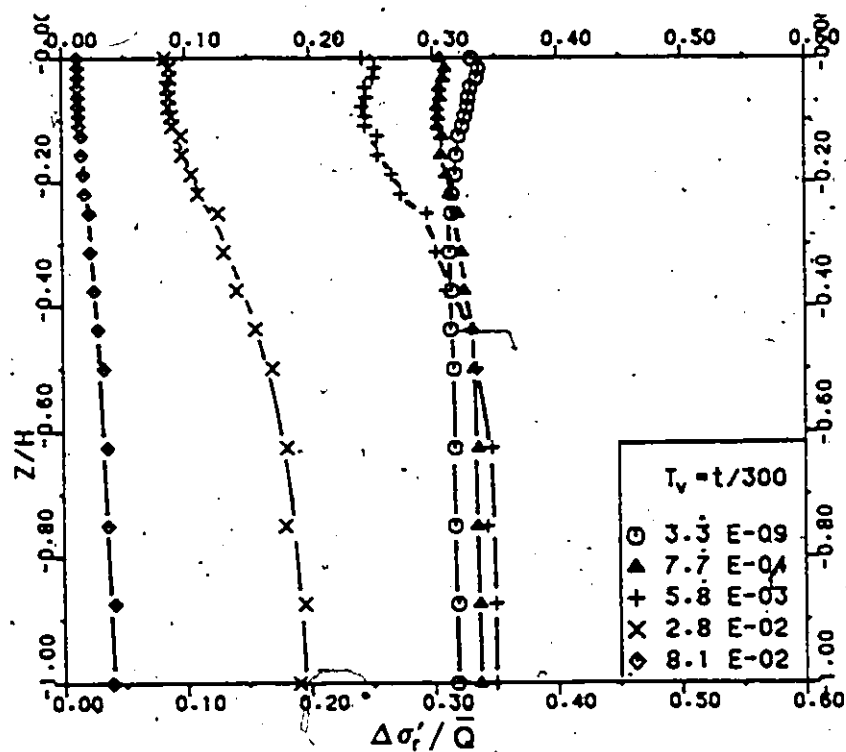


(d) pore pressure isochrones

Fig. 5.35 Triaxial coupled elastic consolidation (load imposed)

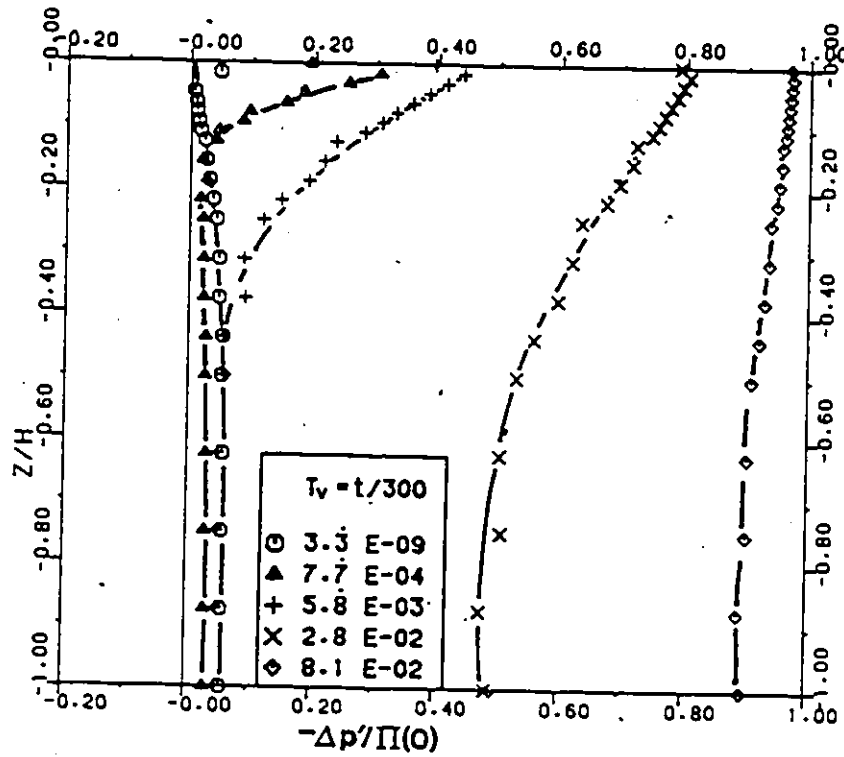


(e) axial effective stress isochrones.

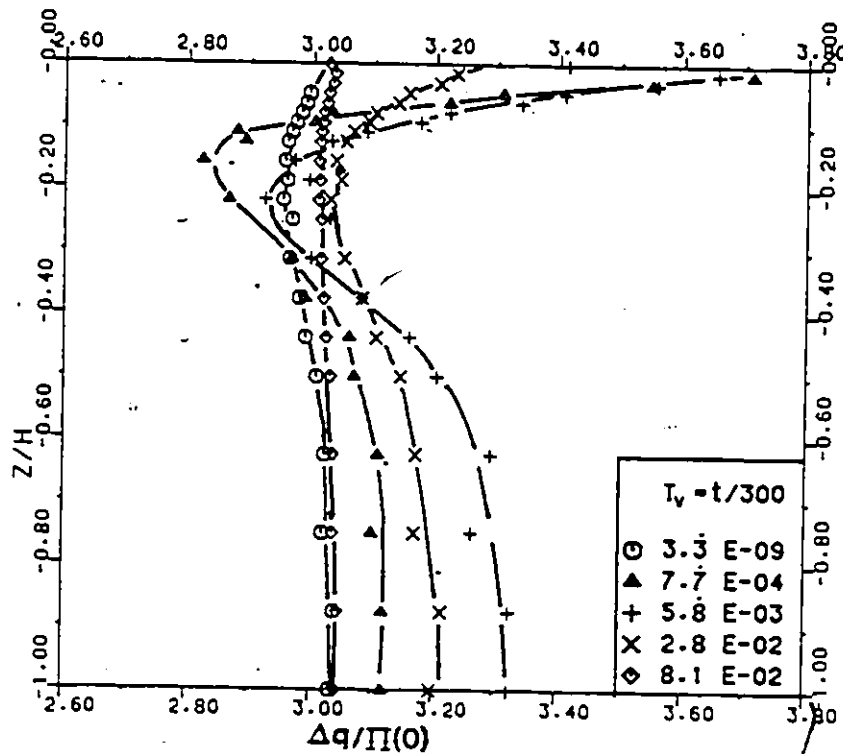


(f) radial effective stress isochrones

Fig. 5.35 Triaxial Coupled elastic consolidation (load imposed)



(g) effective mean pressure isochrones



(h) deviatoric stress isochrones

Fig. 5.35 Triaxial Coupled elastic consolidation (load imposed)

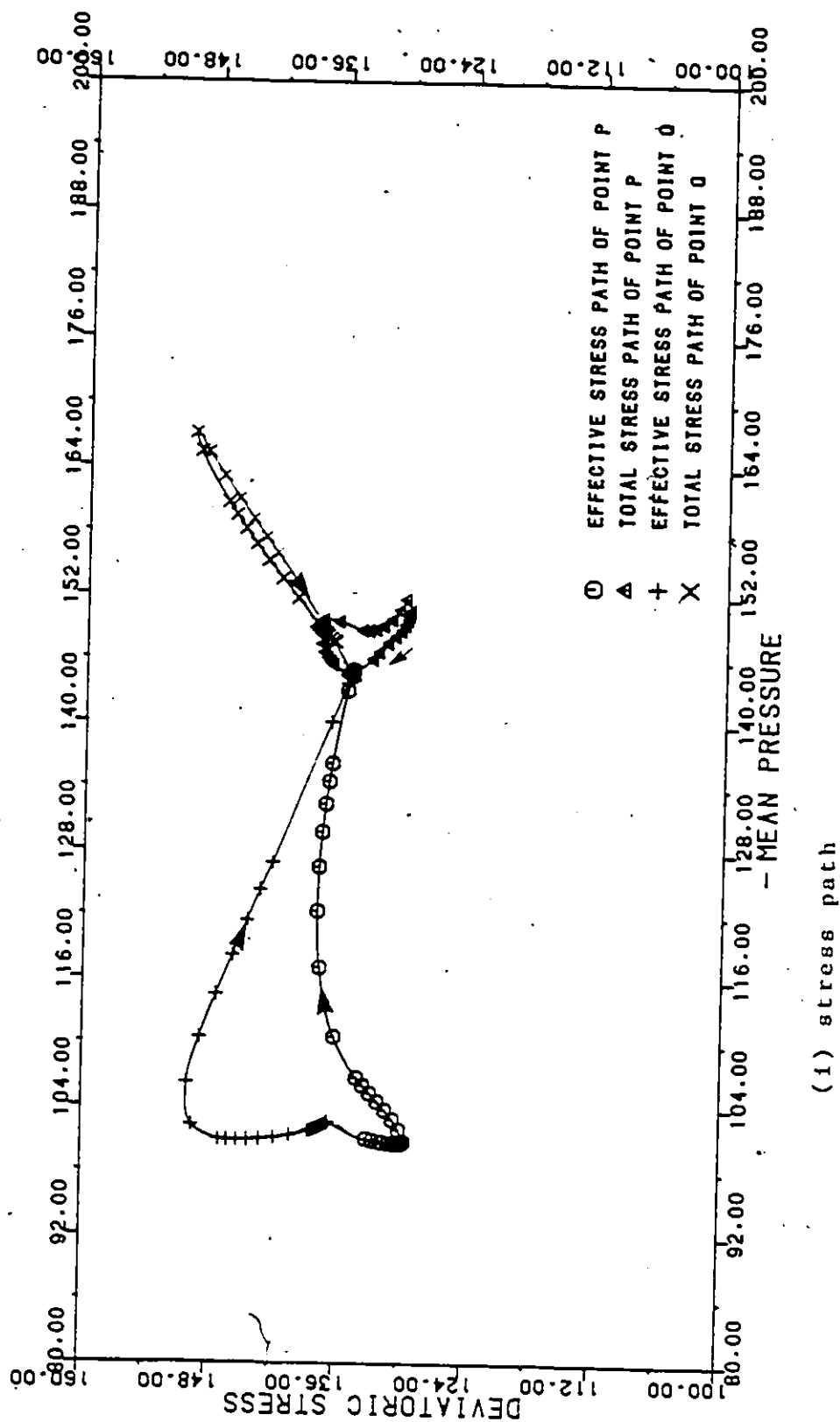
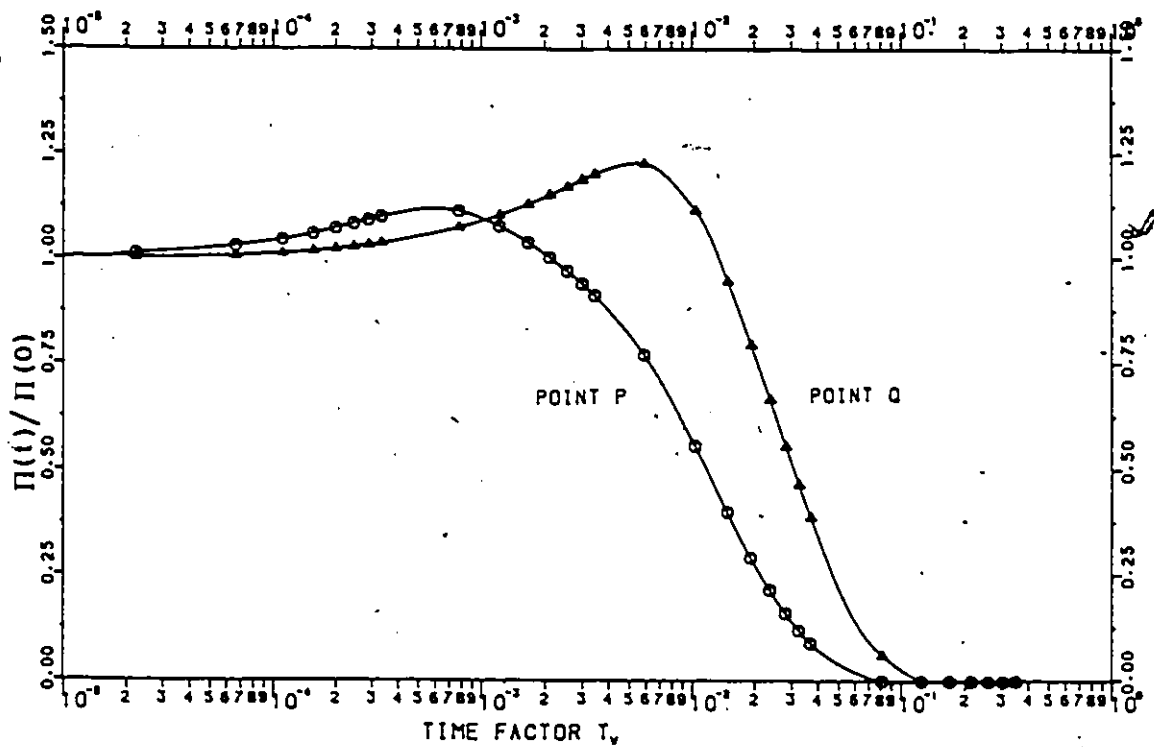
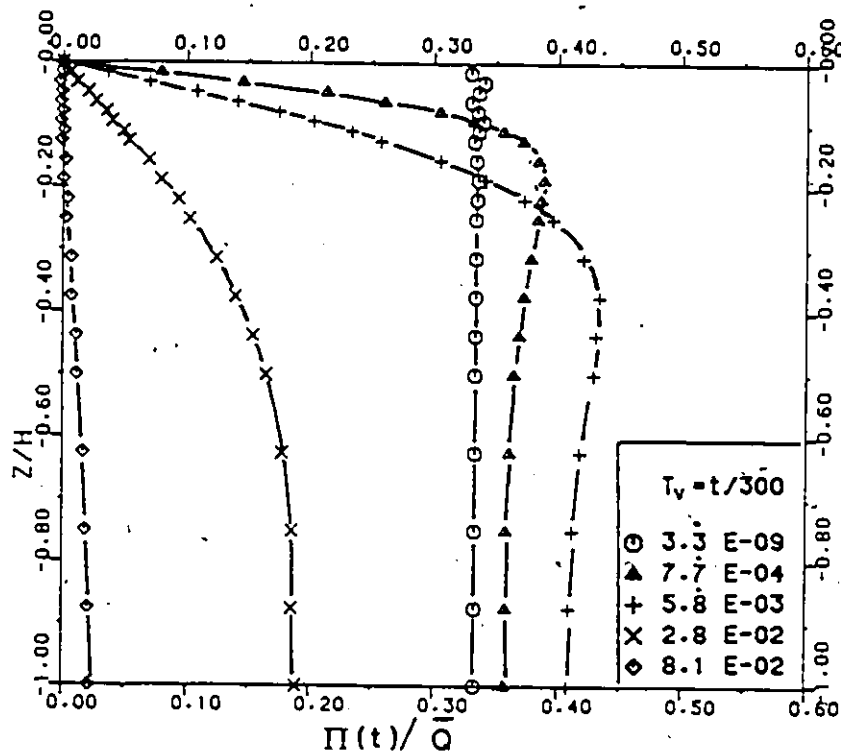


Fig. 5.35 Triaxial Coupled elastic consolidation (load imposed)

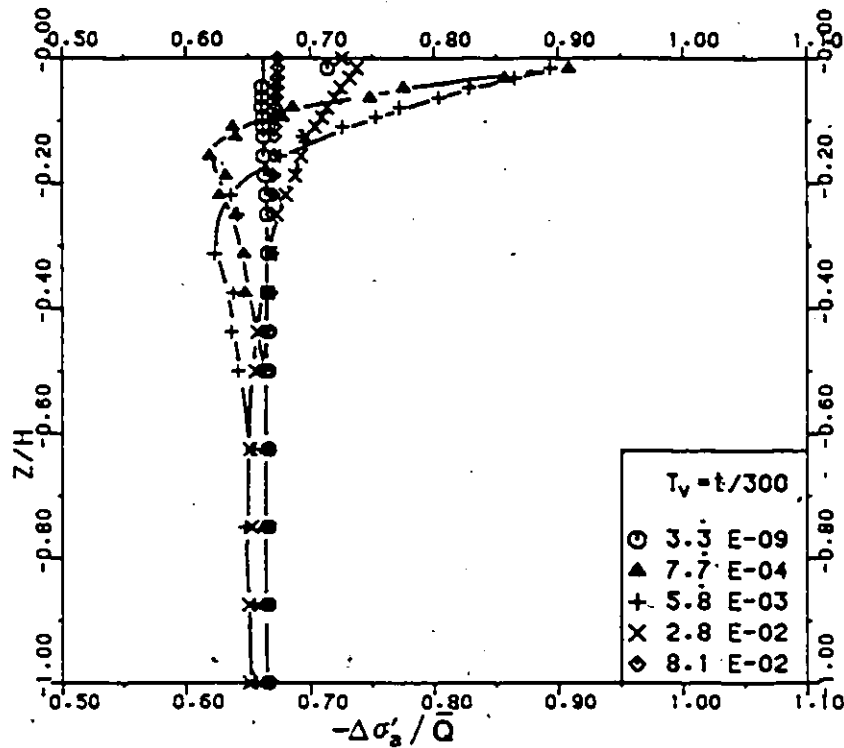


(a) excess pore pressure vs. time

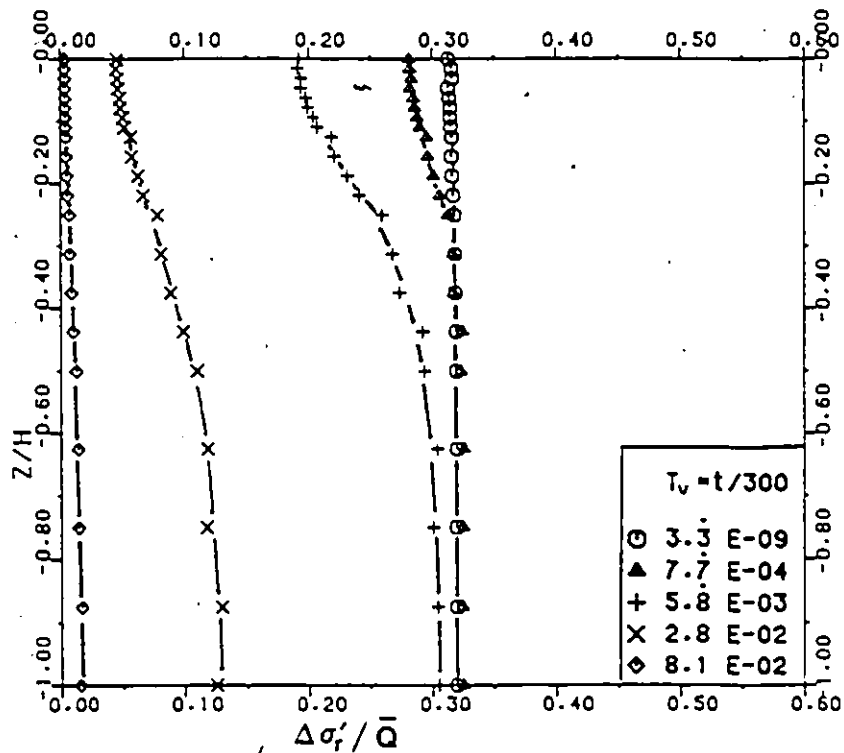


(b) excess pore pressure isochrones

Fig. 5.36 Triaxial Coupled elastic consolidation (displacement imposed)

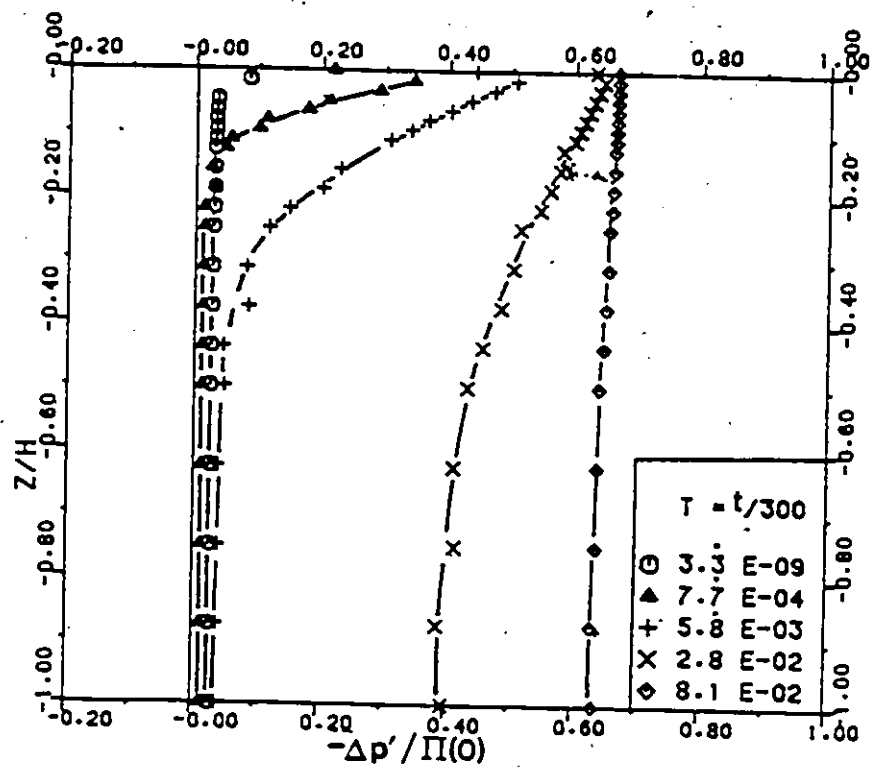


(c) axial effective stress isochrones

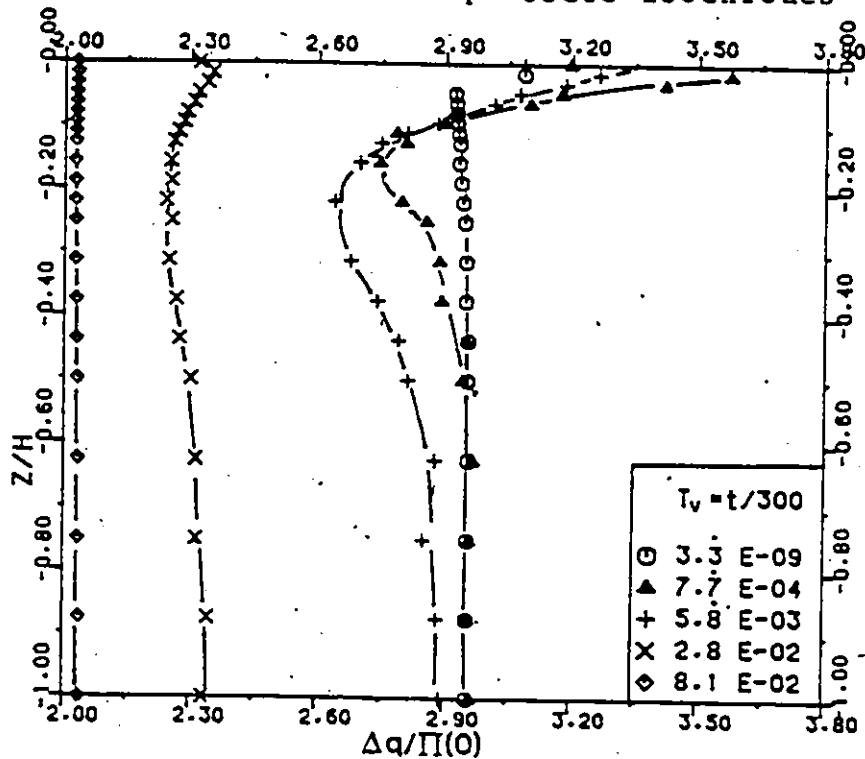


(d) radial effective stress isochrones

Fig. 5.36 Triaxial Coupled elastic consolidation
(displacement imposed)

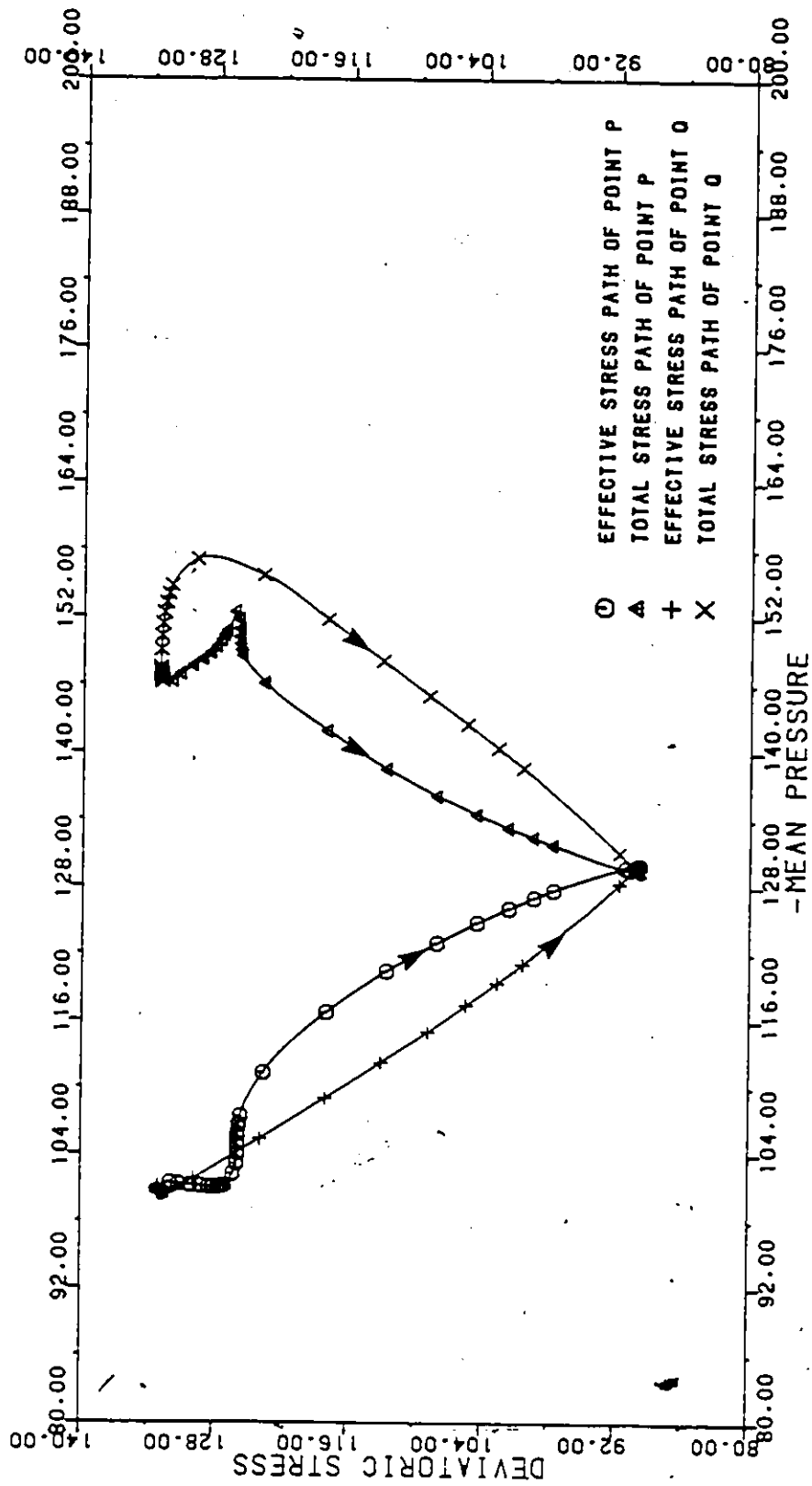


(e) effective mean pressure isochrones



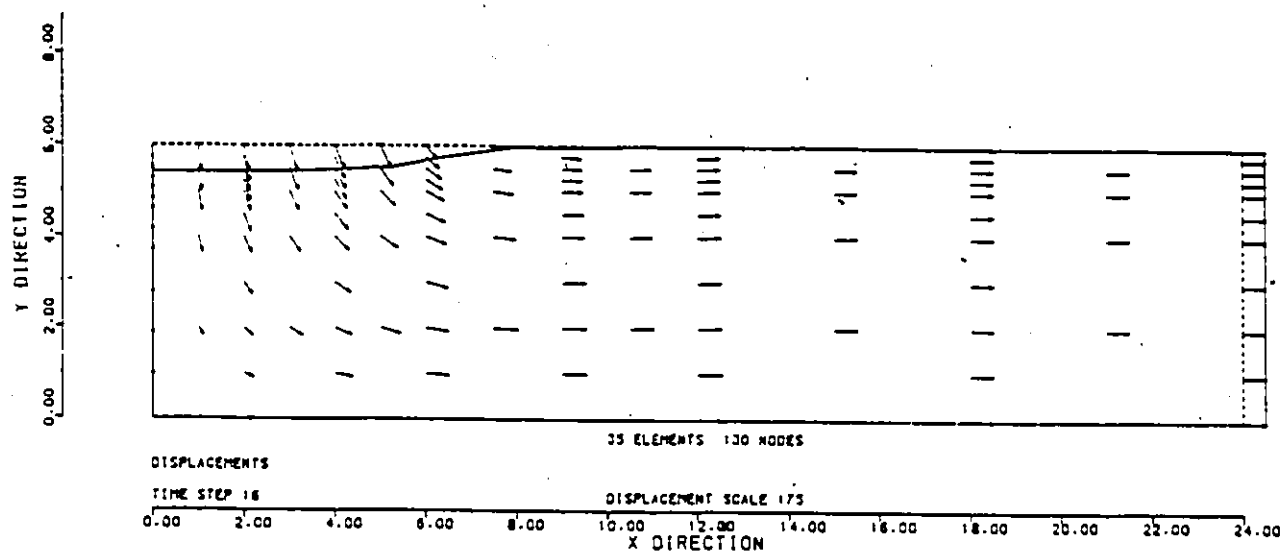
(f) deviatoric stress isochrones

Fig.5.36 Triaxial Coupled elastic consolidation.
(displacement imposed)

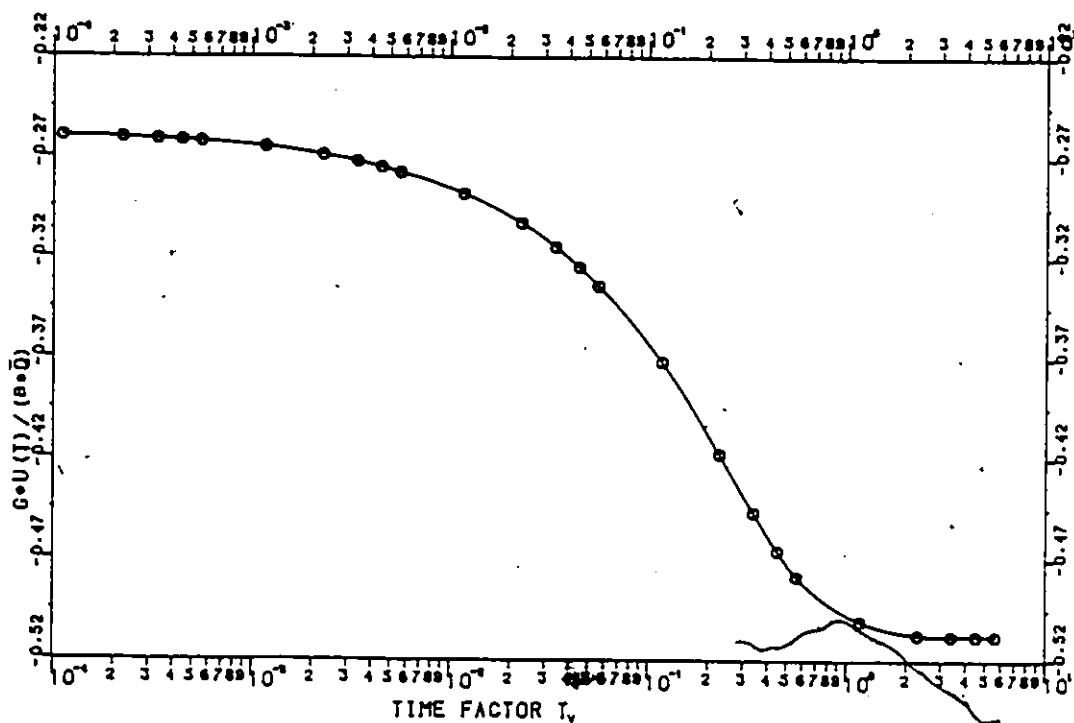


(g) stress path

Fig. 5.36 Triaxial Coupled elastic consolidation
(displacement imposed)

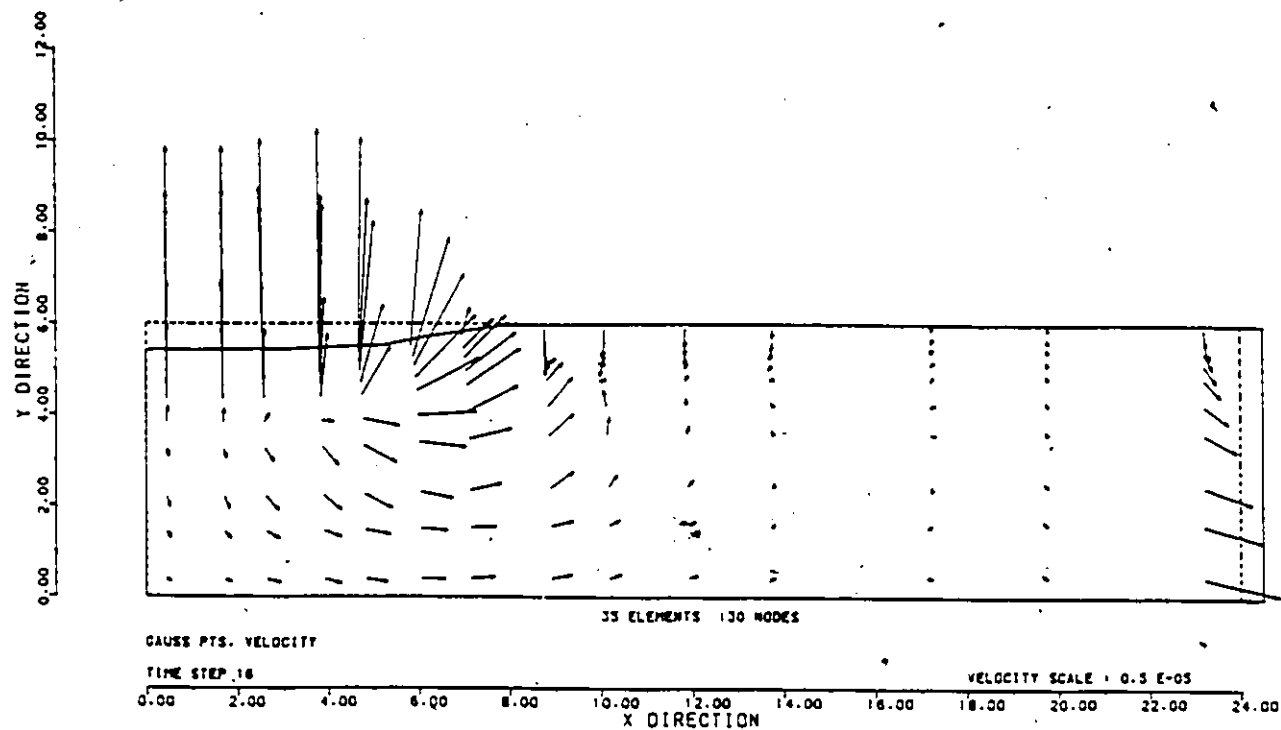


(a) displacement field $T_v = 3.3 \text{ E-04}$

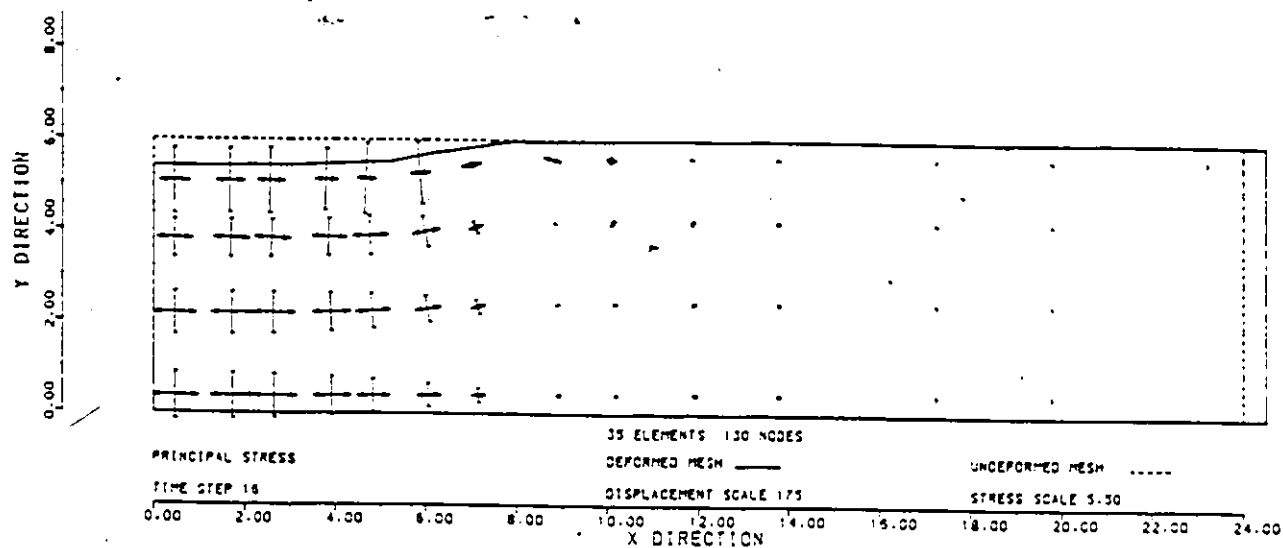


(b) surface settlement vs. time

Fig. 5.37 Coupled elastic consolidation in a semi-infinite layer

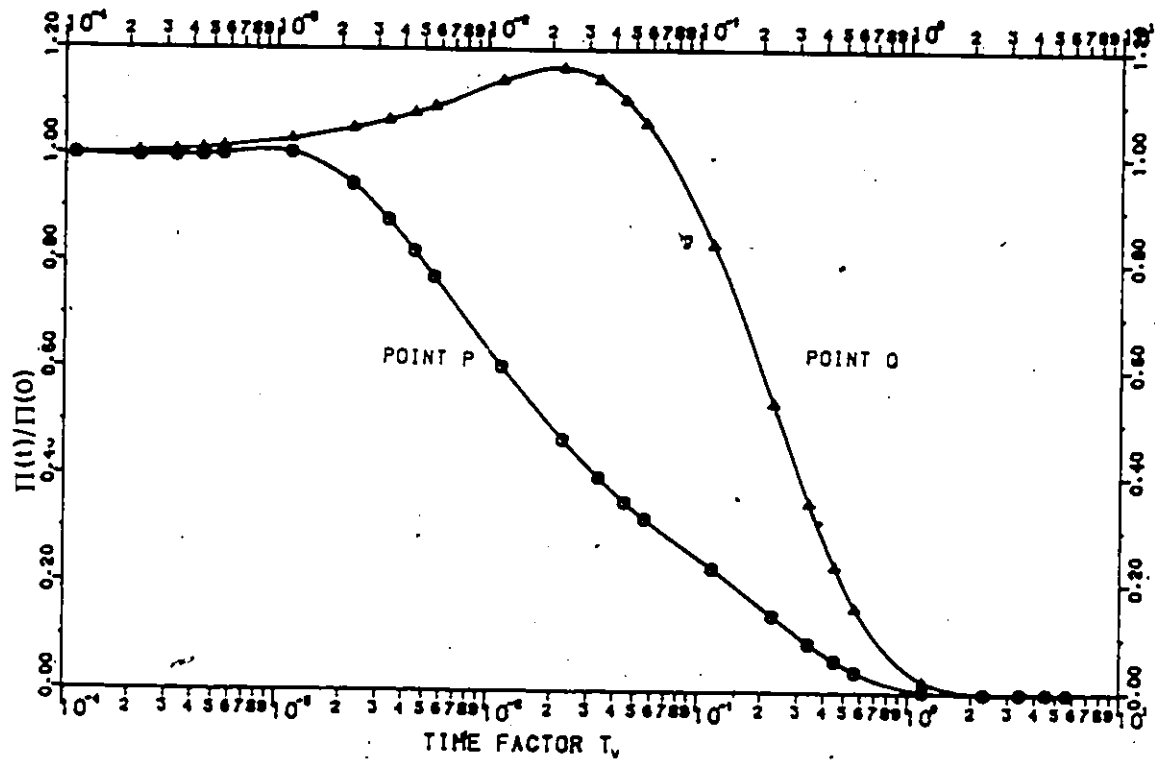


(c) fluid velocity field at $T_v = 3.3 \text{ E-04}$

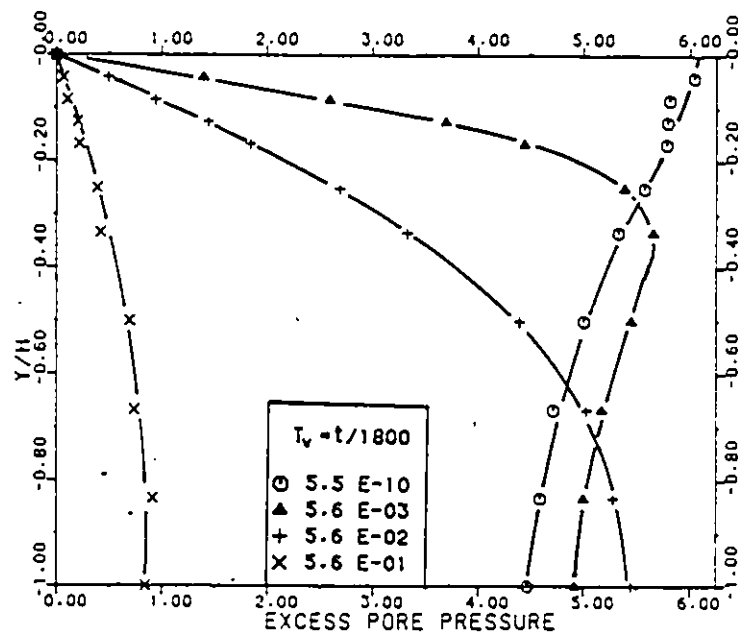


(d) principal effective stress field at $T_v = 3.3 \text{ E-04}$

Fig. 5.37 Coupled elastic consolidation in a semi-infinite layer

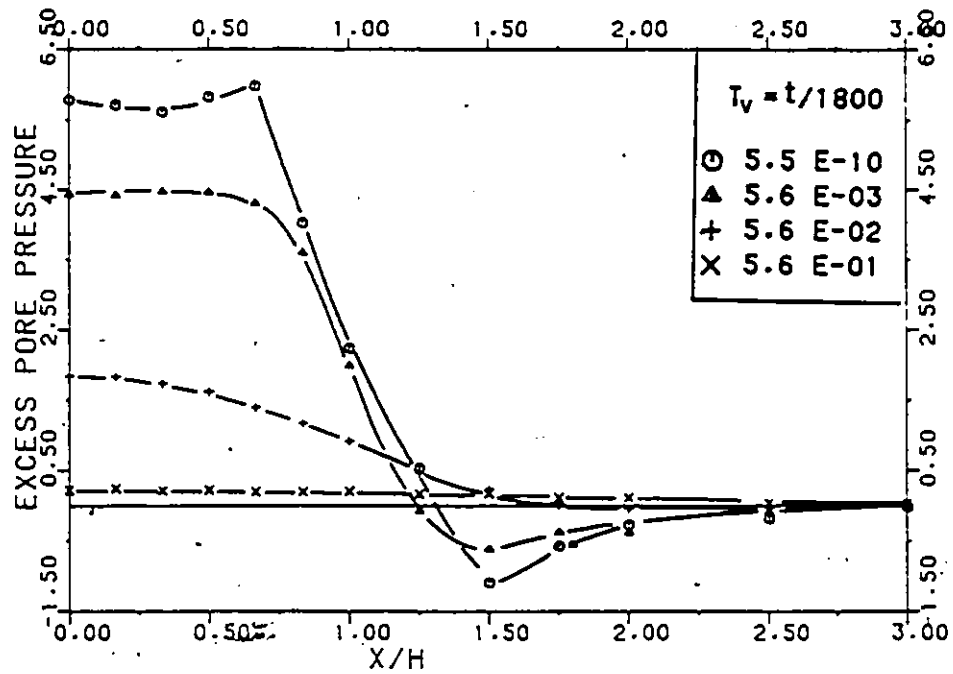


(e) excess pore pressure vs. time

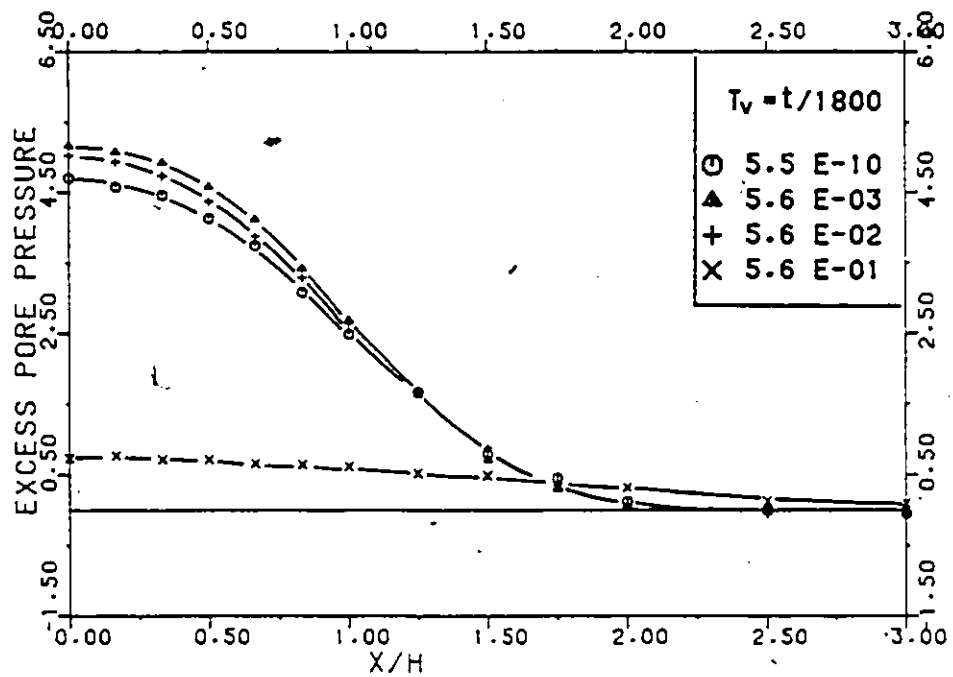


(f) excess pore pressure isochrones at section A-A

Fig. 5.37 Coupled elastic consolidation in a semi-infinite layer

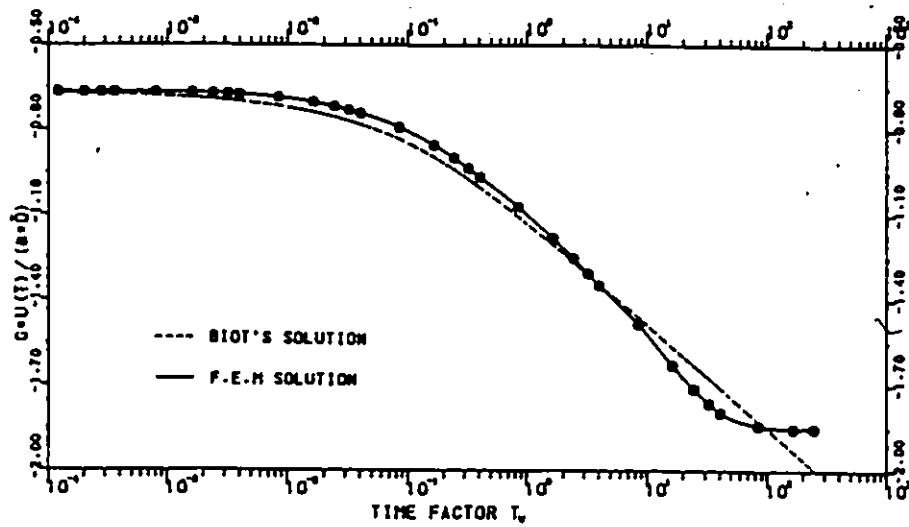


(g) excess pore pressure isochrones at section B-B

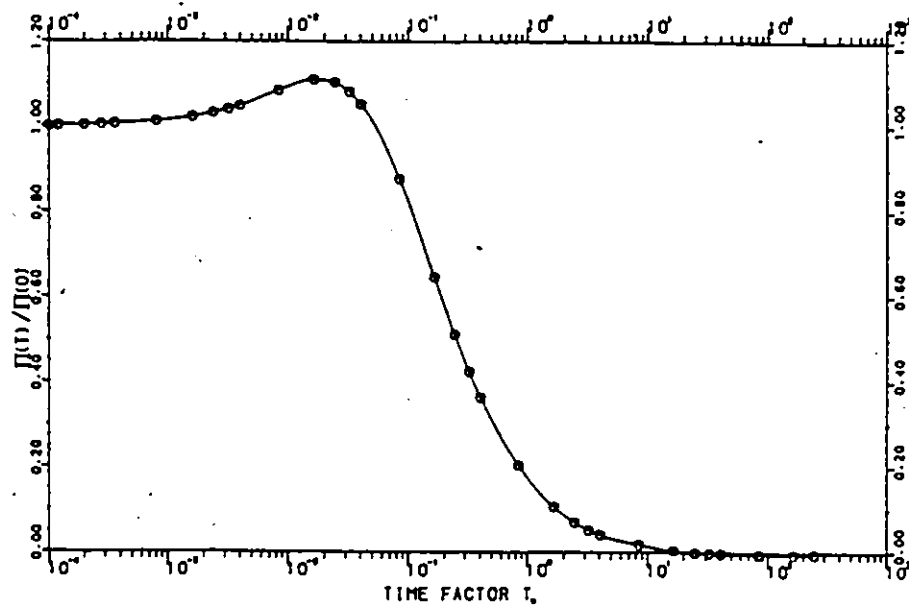


(h) excess pore pressure isochrones at section C-C

Fig 5.37 Coupled elastic consolidation in a semi-infinite layer

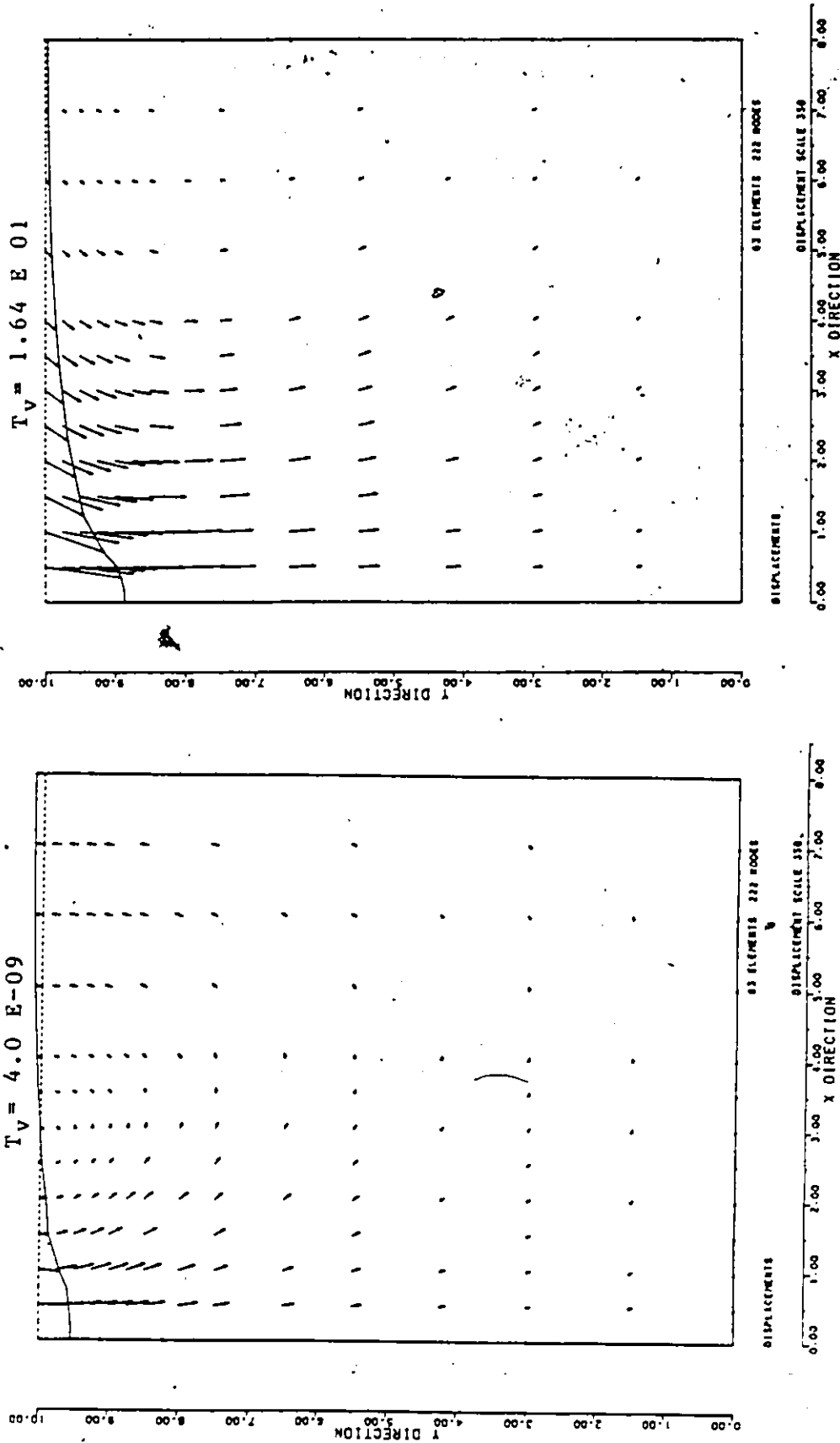


(a) surface settlement vs. time



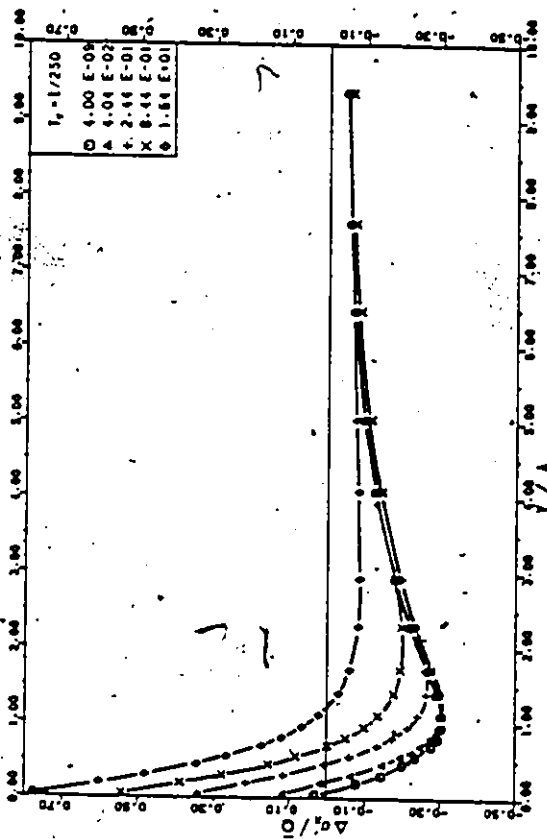
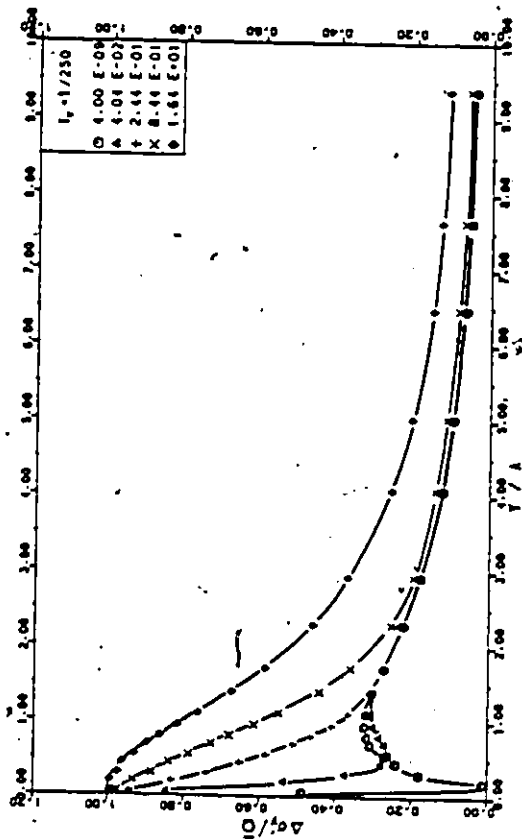
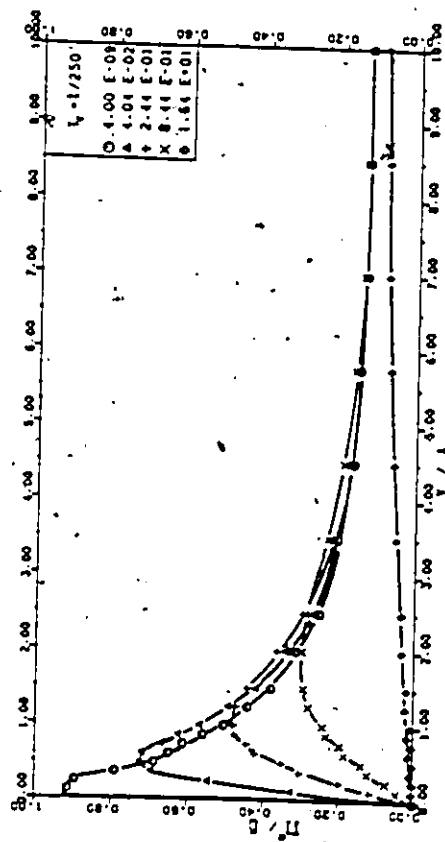
(b) excess pore pressure vs. time

Fig. 5.38 Coupled elastic consolidation in a semi-infinite medium



(c) displacement fields at different times

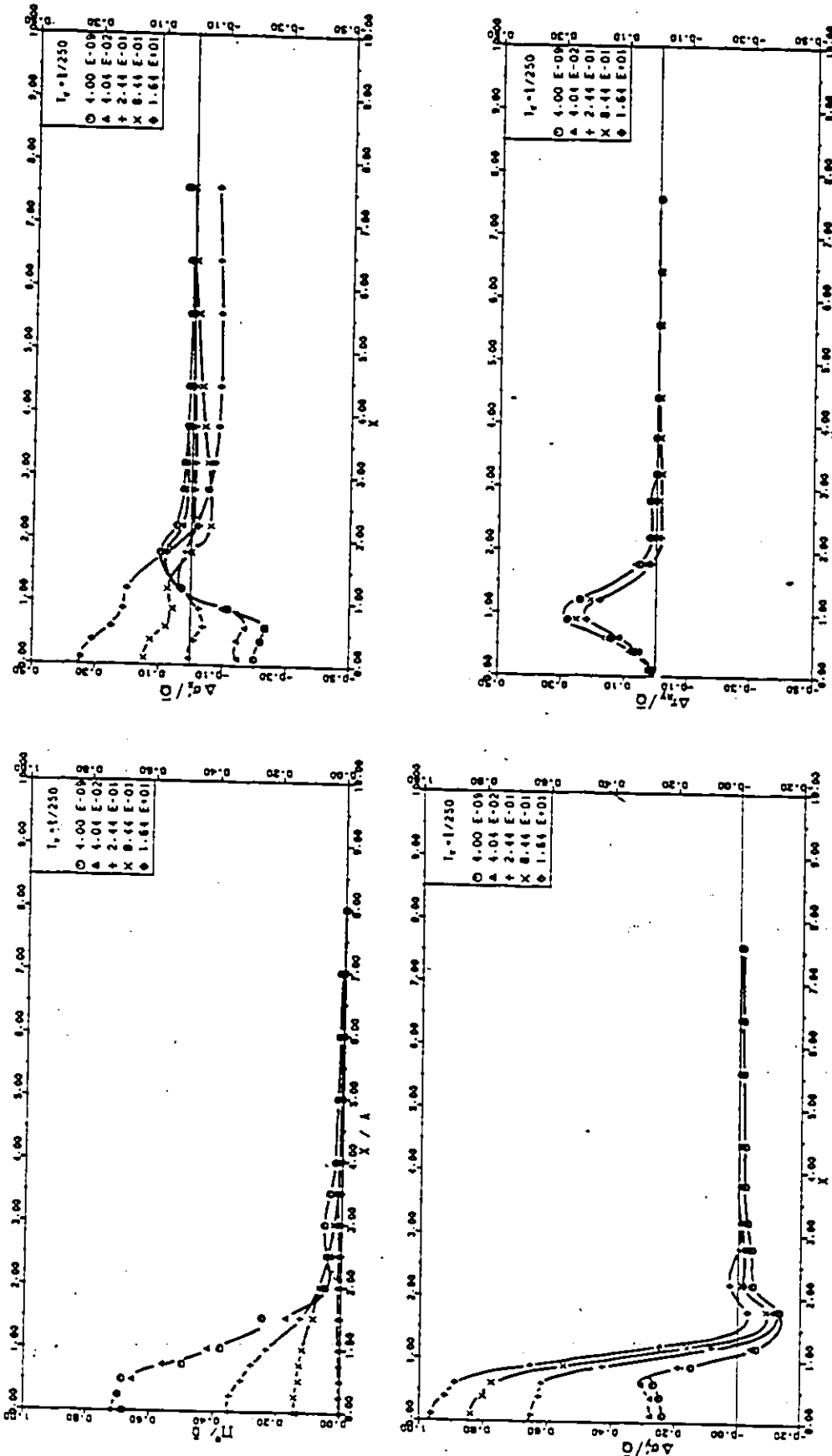
Fig. 5.38 Coupled elastic consolidation in a semi-infinite medium



(d) excess pore pressure and effective stress
isochrones (vert. sect., $X=0.0$)

Fig. 5.38

Coupled elastic consolidation in a
semi-infinite medium



(e) excess pore pressure and effective stress isochrones along the horizontal section at $y/a = 0.5$

Fig. 5.38 Coupled elastic consolidation in a semi-infinite medium

5.5.2 Plastic Cases

Relying on the good performance of TEPSA in elasticity, the investigation has been extended to the elasto-plastic models.

The results herein presented refer only to the elastic perfectly plastic and Generalized Burland model, both having as failure surface the one described by the Mohr-Coulomb criterion (Mohr-Coulomb and Critical Mohr-Coulomb materials respectively).

In order to demonstrate the performance of TEPSA, it was necessary to analyze the behaviour of a cylindrical sample subjected to triaxial conditions. This case in fact, permits the evaluation of the initial stress values in elasto-plasticity by theoretical means.

With reference to Figs. 5.39, the expected numerical responses of both elastic-perfectly plastic and critical-state models are graphically interpreted for the initial time state.

From the starting isotropic stress state, a stress increment leads the total stress point P to stress point A along a line at slope of 3 vertical to 1 horizontal (triaxial line). According to the structure of the program, the first solution is obtained assuming the material as elastic. Consequently, the variation of the mean pressure is solely carried by the fluid phase, bringing the effective stress to point A'. If the effective stress point A' lies

beyond the yield surface, elasto-plastic deformations must occur.

For elasto perfectly plastic materials, the effective stress should be first reduced on the surface at point B' from where the elasto-plastic balancing process takes place. If load is applied to the sample, the effective stress point moves to point D' to satisfy stress equilibrium, while, if displacement is imposed, the stress point moves to a point C', where the elasto-plastic deformations match the imposed displacement.

Similar procedure is carried out when a Critical State model is employed. In this case, the stress point A' is brought down to the yield surface at point P, from where elasto-plastic deformations are allowed to take place. Assuming to have imposed displacement, the effective stress moves to point E' in which the elasto-plastic deformations equalize the imposed displacement.

Interestingly, the initial excess pore pressure in linear perfectly plastic material is smaller than the one expected in linear elastic material. On the contrary, the Generalized Burland model leads to a higher value of excess pore pressure. Moreover, by using the elastic perfectly plastic model, the plastic deformations are exhausted at time $t = 0$ and the subsequent stress path stays inside the yield surface, i.e. in the elastic region. On the other hand, using the Generalized Burland model, elasto-plastic deformations occur during the whole consolidation process.

In the next cases, an initial displacement was imposed to two cylindrical samples of Mohr-Coulomb and Critical Mohr-Coulomb materials respectively. The imposed displacement was the same as in the previous elastic case ($u(0) = 0.018$) and was kept constant in time. The employed mesh and boundary conditions are shown in Figs. 5.2c and the elasto-plastic material parameters are reported in Table 5.1. Furthermore the material was assumed weightless and isotropically consolidated at a pressure of 100. The Critical Mohr-Coulomb material was assumed normally consolidated. The values of the initial effective and total stresses are reported in Figs. 5.39.

The complete stress and pore pressure histories for the linear perfectly plastic and the Generalized Burland 'materials' are shown in Figs. 5.40 and 5.41 respectively. The normalizing stress $\bar{Q}$ is in both cases the initial axial total stress which resulted to be 114 and 52.2 for the elastic perfectly plastic and the Generalized Burland model respectively. The initial excess pore pressure resulted to be 33 and 36.6 in the two cases respectively. Incidentally, in the previous elastic case the initial total axial stress and excess pore pressure were 135 and 45 respectively. It is worth mentioning that the numerical calculations took 551 sec. for the Critical Mohr-Coulomb material, and only 215 sec. for the second type of material.

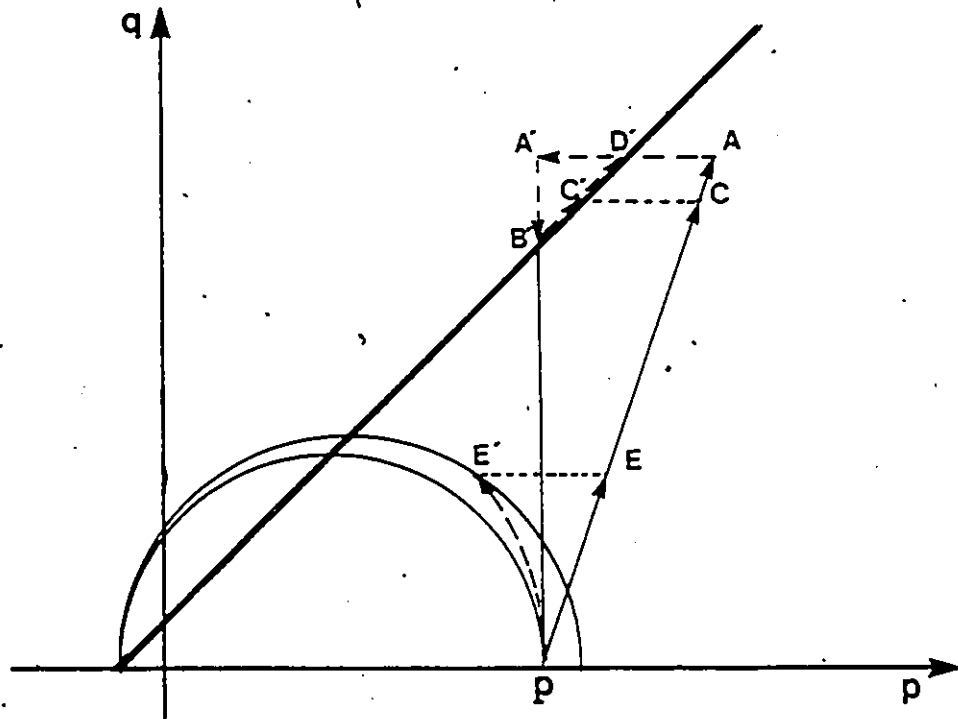
Finally, the numerical experiments were extended to the case of a semi-infinite medium subjected to a uniform load applied by a smooth and shallow foundation. The mesh layout is shown in Figs. 5.4 and the material parameters are reported in Table 5.1. The unit weight of the saturated material was assumed to be 20 and an initial hydrostatic distribution of pore pressure was considered ($\gamma_f = 10$). The weight was therefore taken into account by imposing an initial isotropic stress distribution equal to the effective overburden weight. The Critical Mohr Coulomb material was considered as normally consolidated.

As expected, when deformations are described by the elastic perfectly plastic model, the moment of load application is the most dangerous stage, since this is when much greater plastic deformations occur. In the present case, failure was reached when a load of 105 was applied.

More interesting was the response of the Critical Mohr-Coulomb material, Figs. 5.42. As it can be deduced from the previous paragraphs, the influence of the excess pore pressure brings the initial effective stress point closer to the yield surface, thus restraining the initial amount of plastic deformations. As dissipation proceeds, the effective stress moves further away from the yield surface inducing larger deformations. Meanwhile, as previously noticed in the analogous elastic case, Figs. 5.38b, the excess pore pressure presents its highest rate of dissipation some time

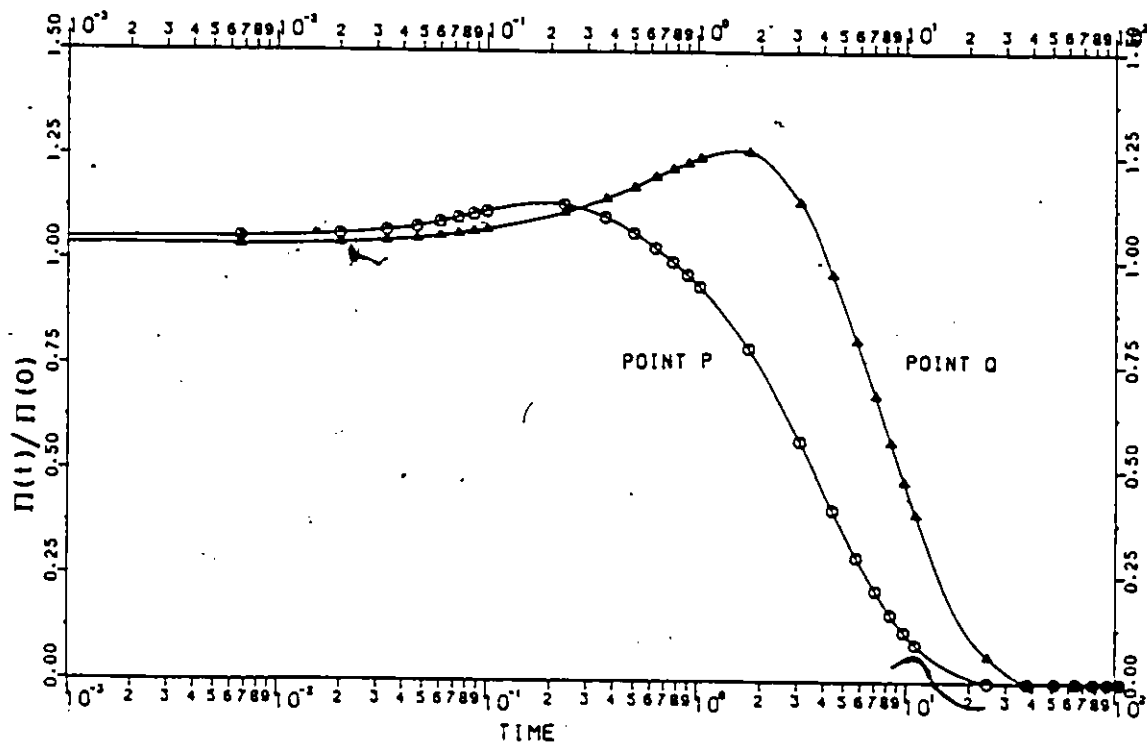
after the load application. From the above considerations, it is possible to explain the delayed failure registered in the present case and thereby shown in Figs. 5.42a.

This model predicted again a "localized mode of failure", Figs. 5.42c. Finally, it is worth mentioning that the numerical analysis required 1539 sec. of CPU time.

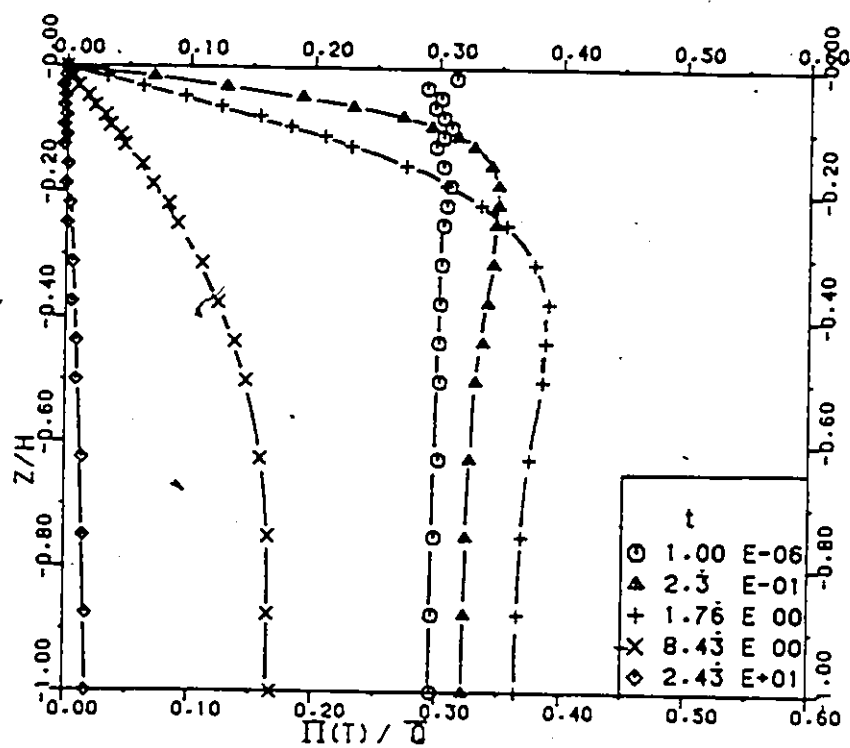


Points	p'	p	q
A		-145	135
A'	-100		135
B'	-100		110
C		-138.3	115
C'	-105		115
D'	-125		135
E'		-116.6	50
E	-80.6		50

Fig. 5.39 Stress equilibrium in coupled elastoplastic consolidation

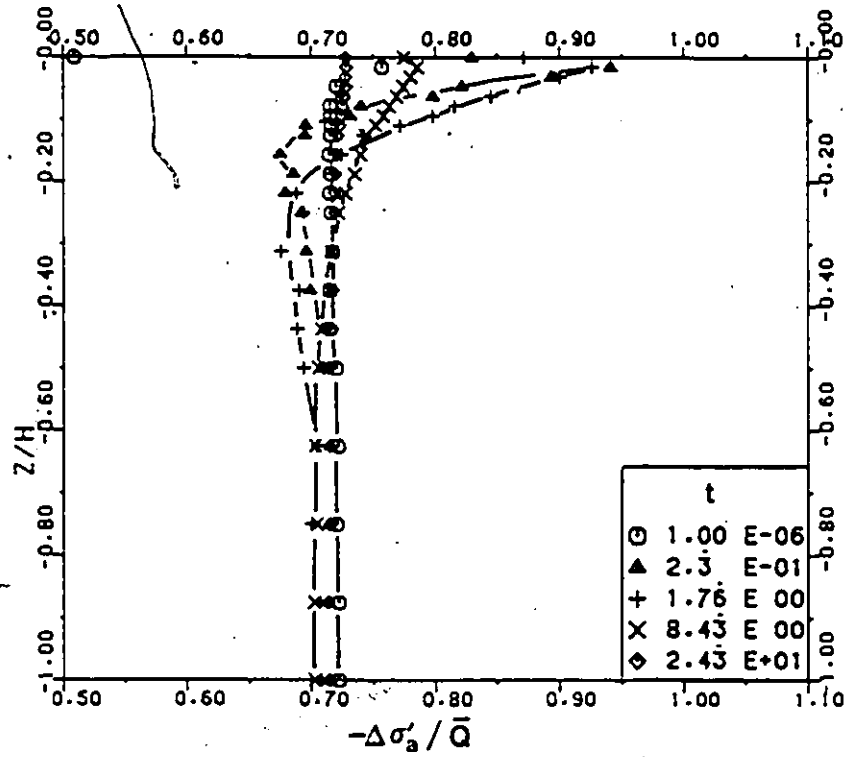


(a) excess pore pressure vs. time

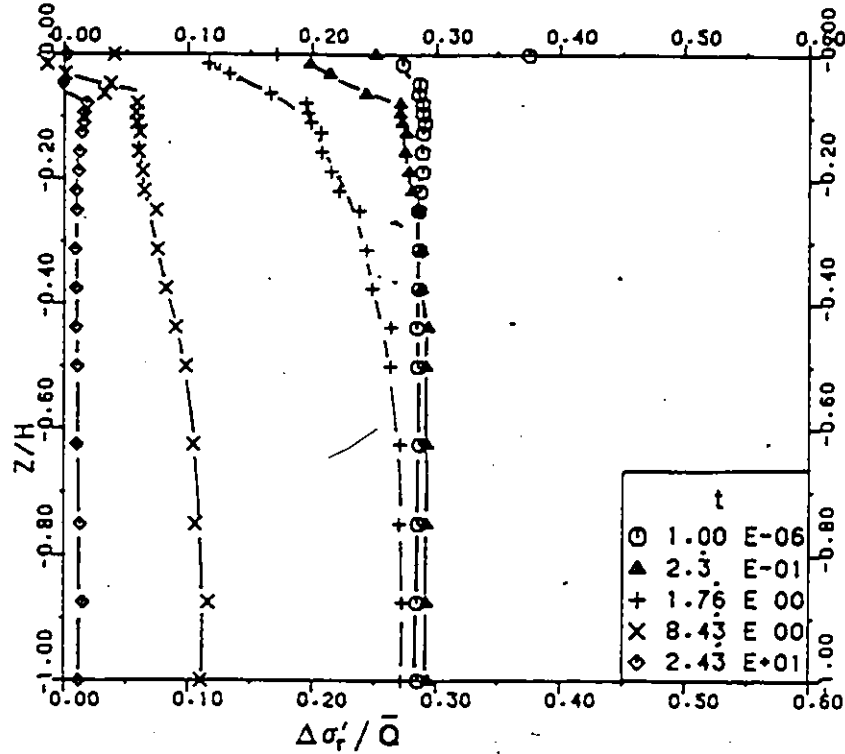


(b) excess pore pressure isochrones

Fig. 5.40 Triaxial Coupled plastic consolidation (Mohr-Coulomb material)

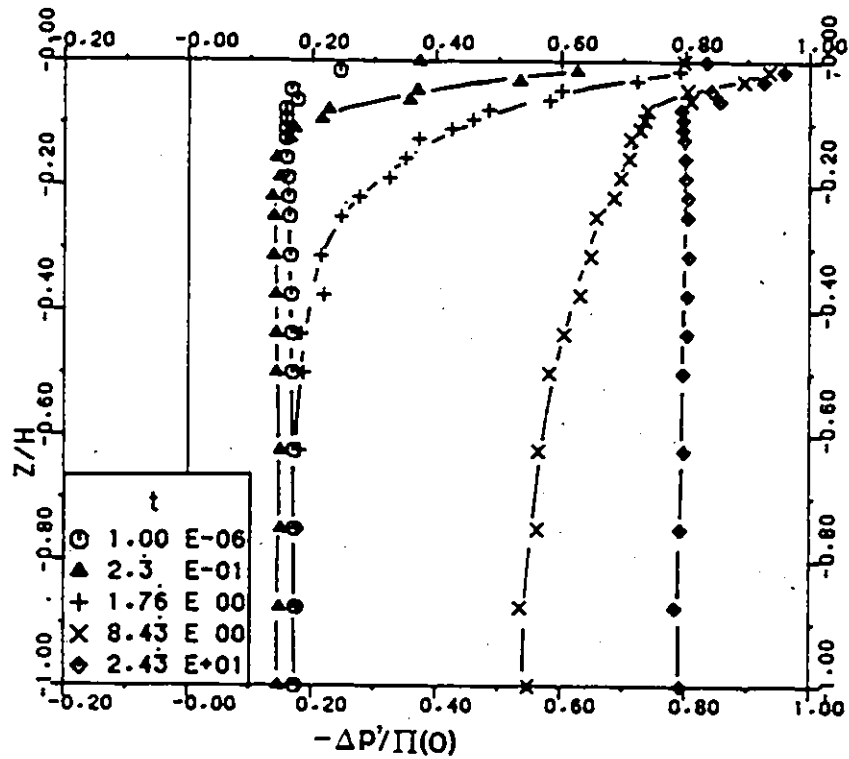


(c) axial effective stress isochrones

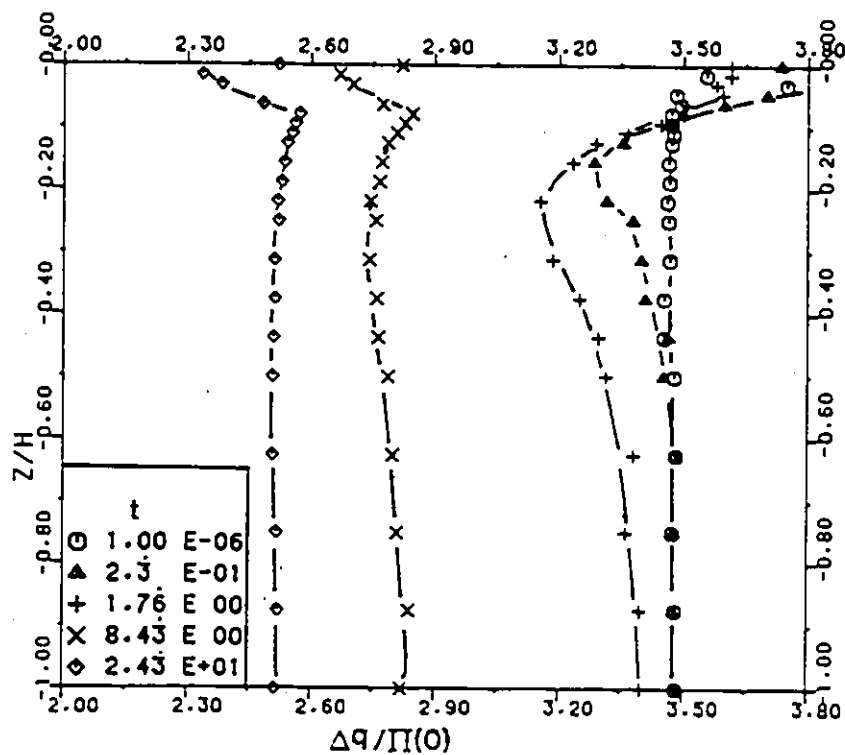


(d) radial effective stress isochrones

Fig. 5.40 Triaxial Coupled plastic consolidation
(Mohr-Coulomb material)

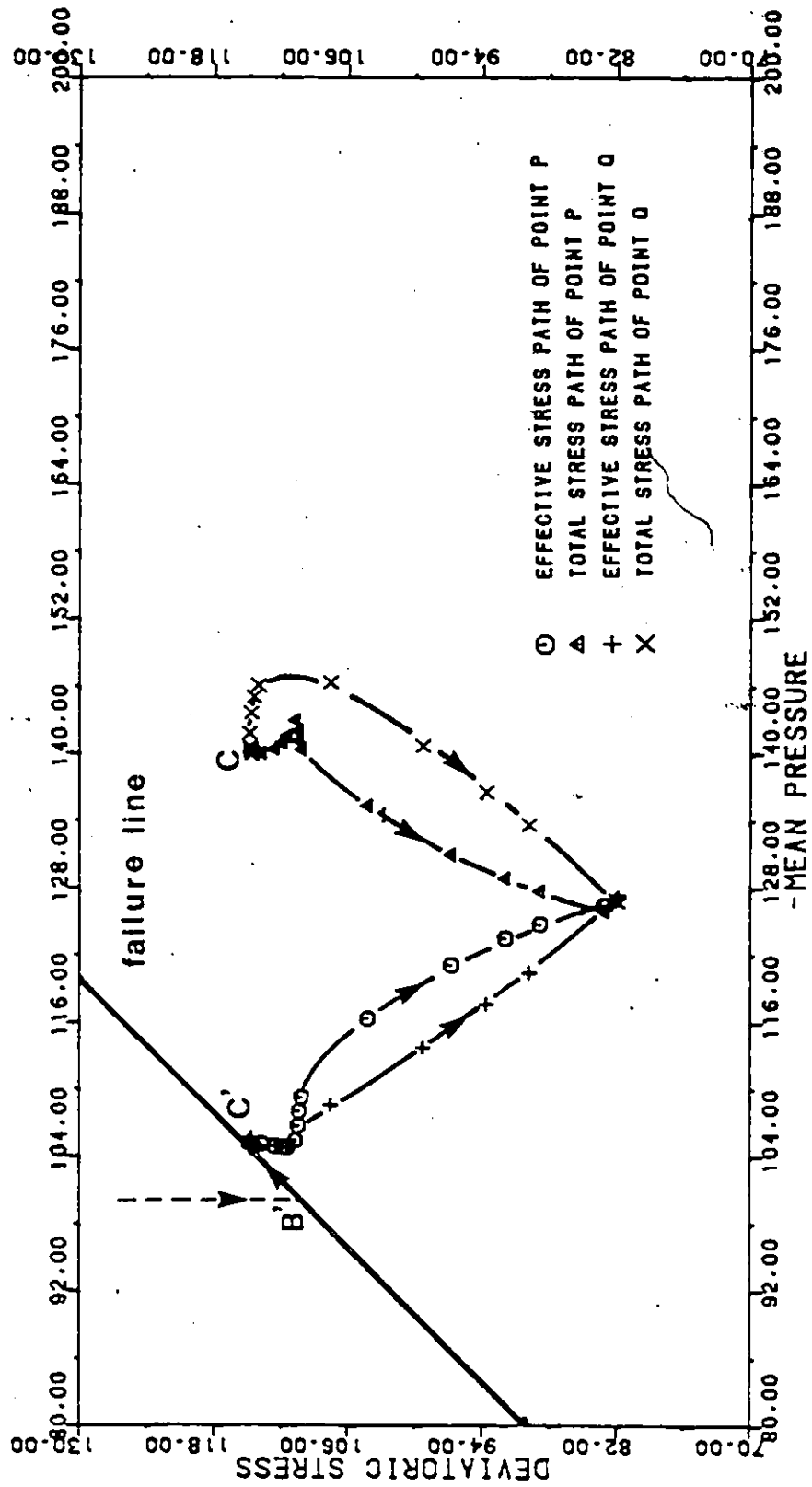


(e) effective mean pressure isochrones



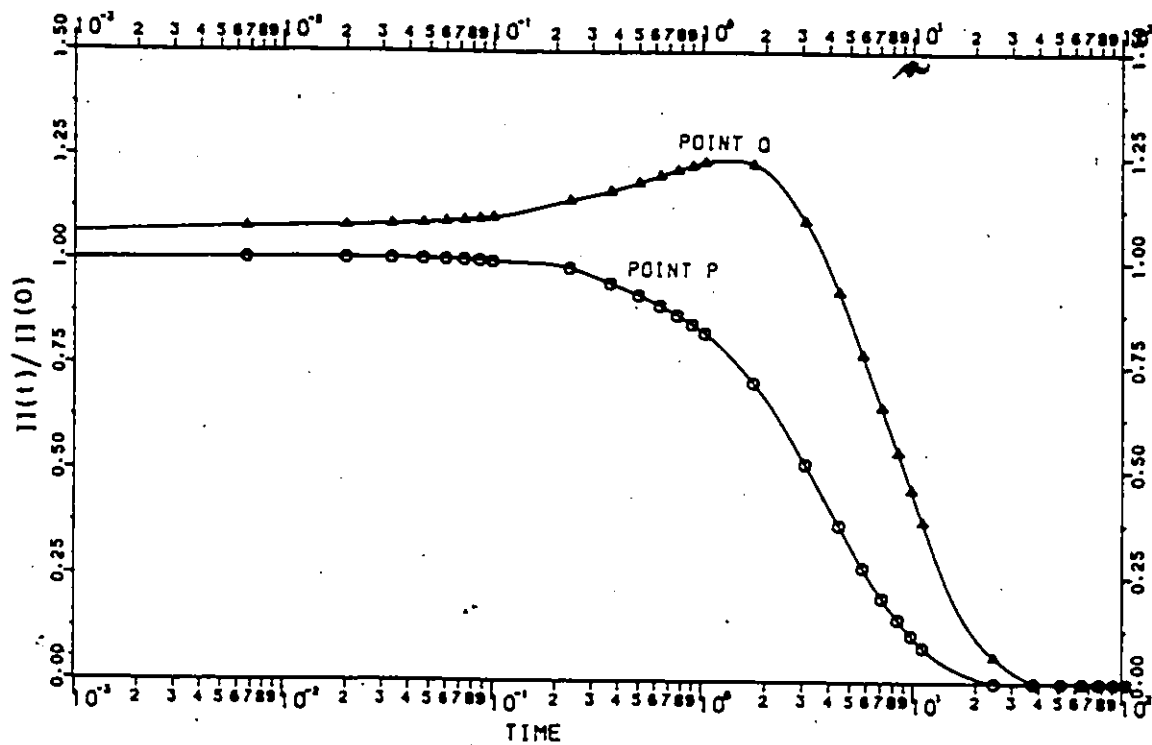
(f) deviatoric stress isochrones

Fig. 5.40 Triaxial Coupled plastic consolidation
(Mohr-Coulomb material)

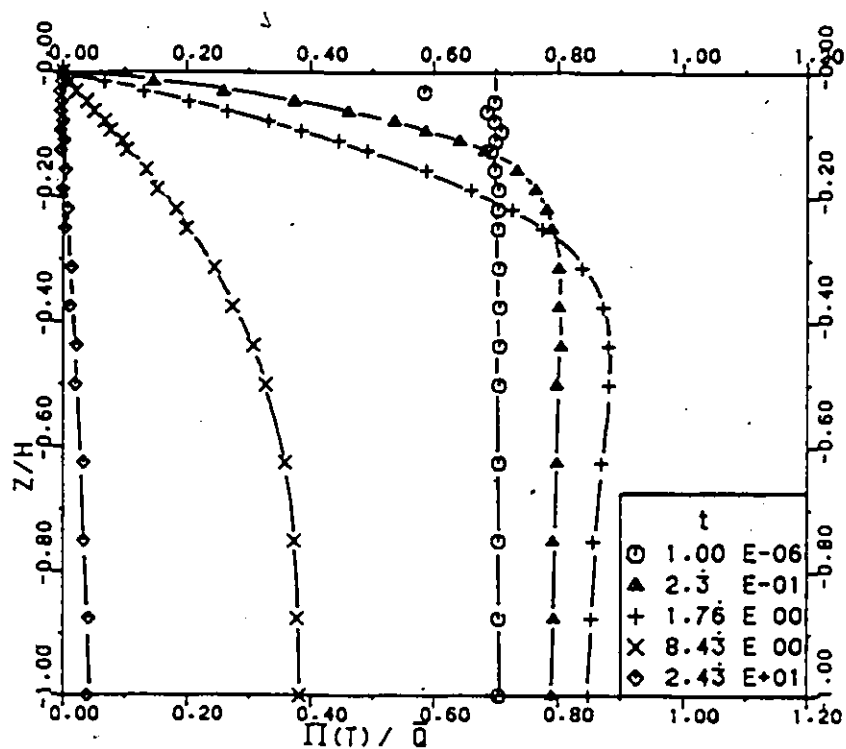


(g) stress path

Fig. 5.40 Triaxial coupled plastic consolidation
(Mohr-Coulomb material)

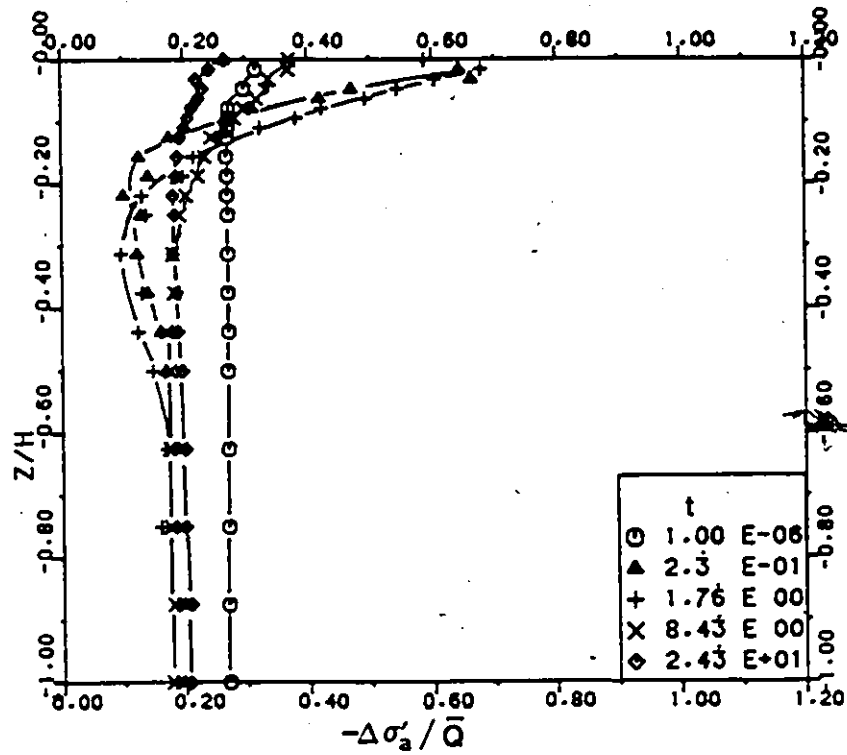


(a) excess pore pressure vs. time

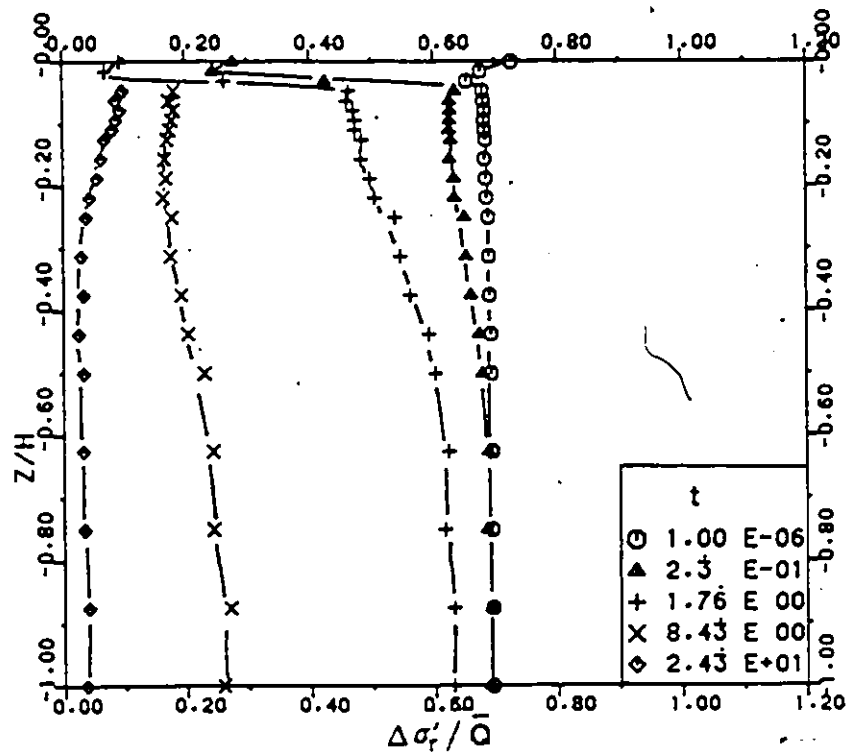


(b) excess pore pressure isochrones

Fig. 5.41 Triaxial Coupled plastic consolidation
(Critical Mohr-Coulomb material)

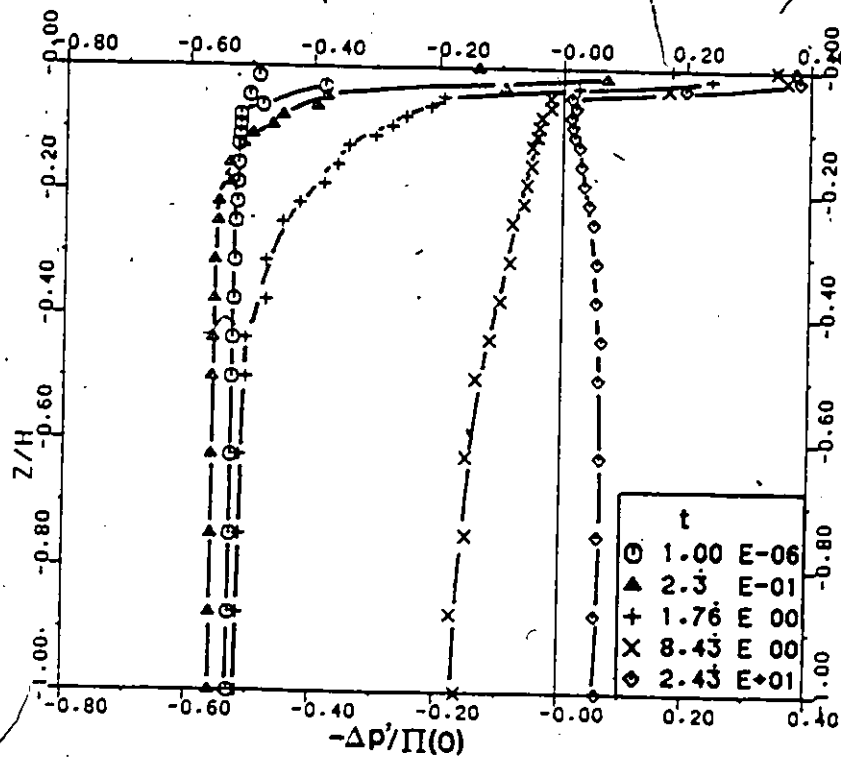


(c) axial effective stress isochrones

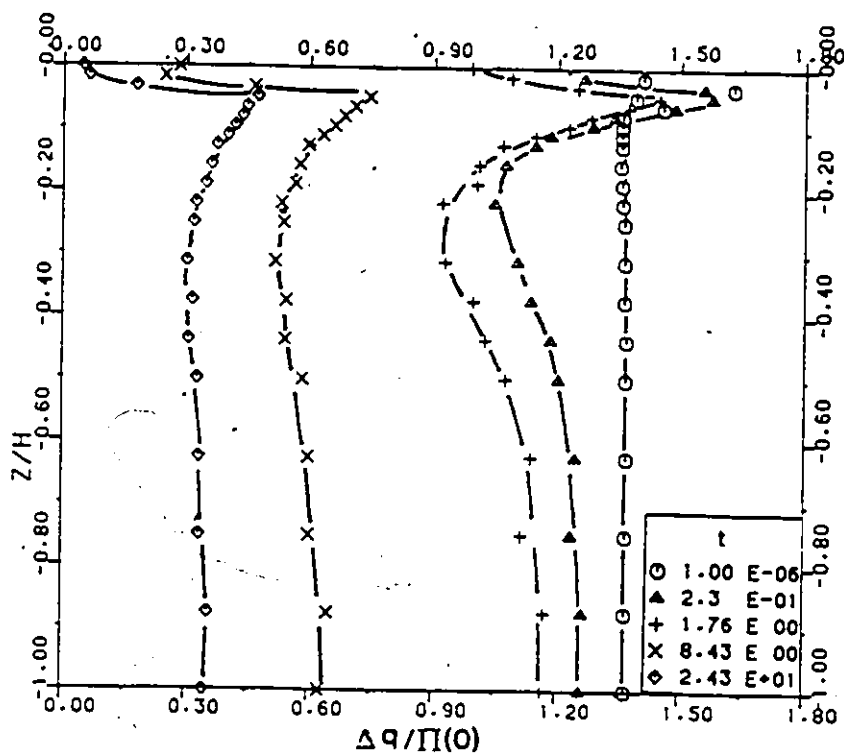


(d) radial effective stress isochrones

Fig. 5.41 Triaxial Coupled plastic consolidation
(Critical Mohr-Coulomb material)

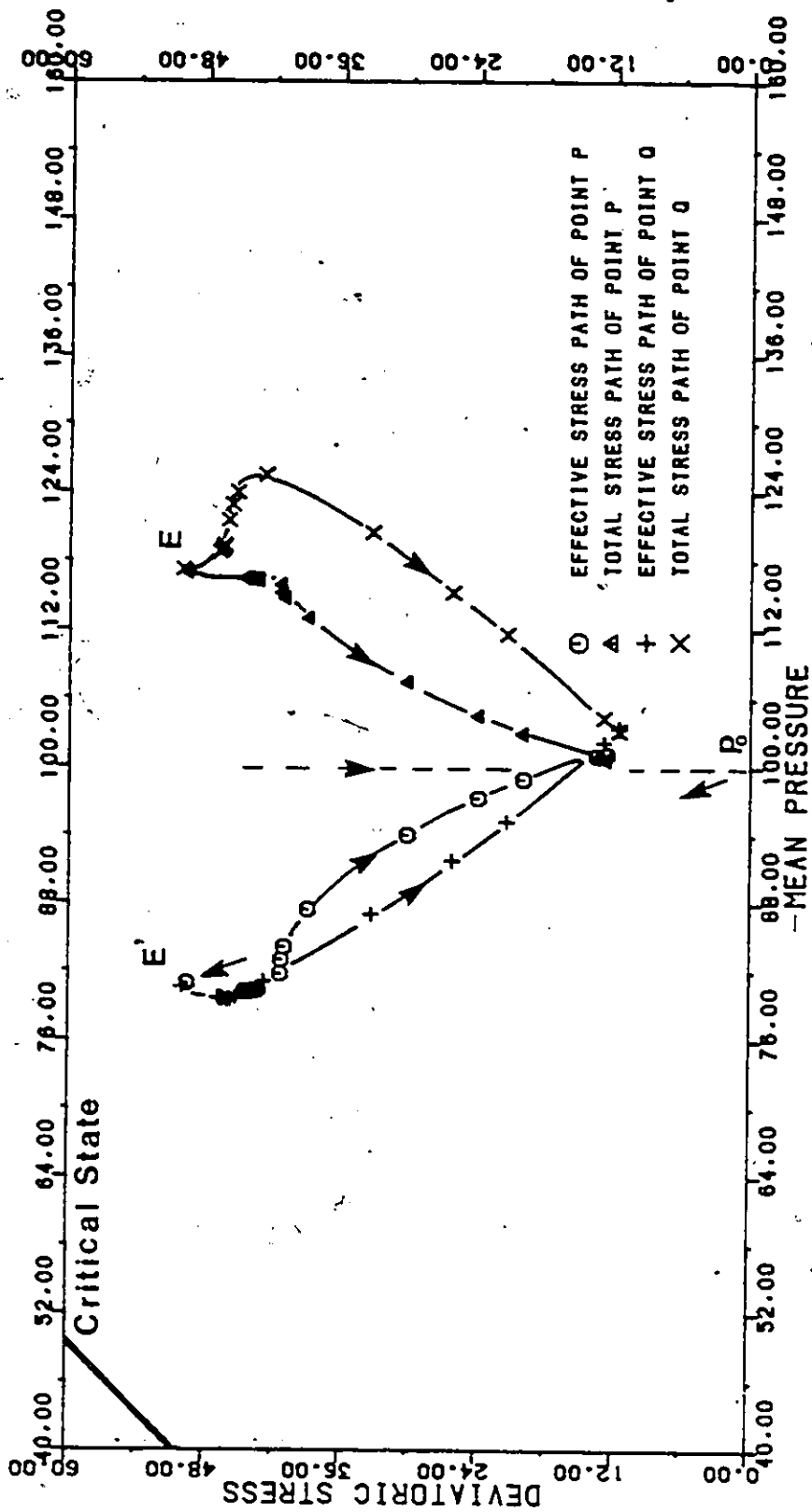


(e) effective mean pressure isochrones



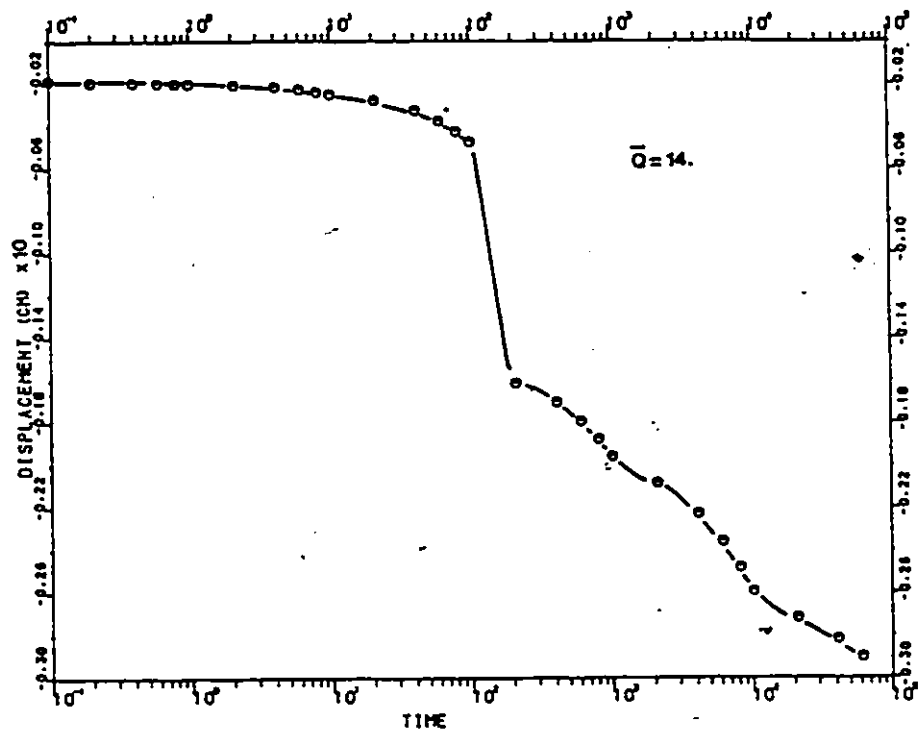
(f) deviatoric stress isochrones

Fig. 5.41 Triaxial Coupled plastic consolidation
(Critical Mohr-Coulomb material)

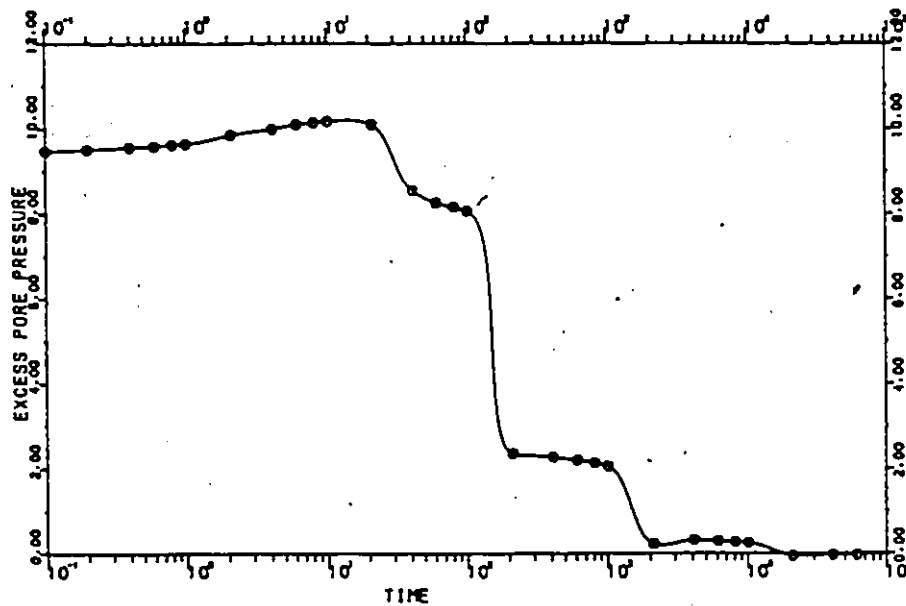


(g) stress path

Fig. 5.41 Triaxial coupled plastic consolidation
(Critical Mohr-Coulomb material)

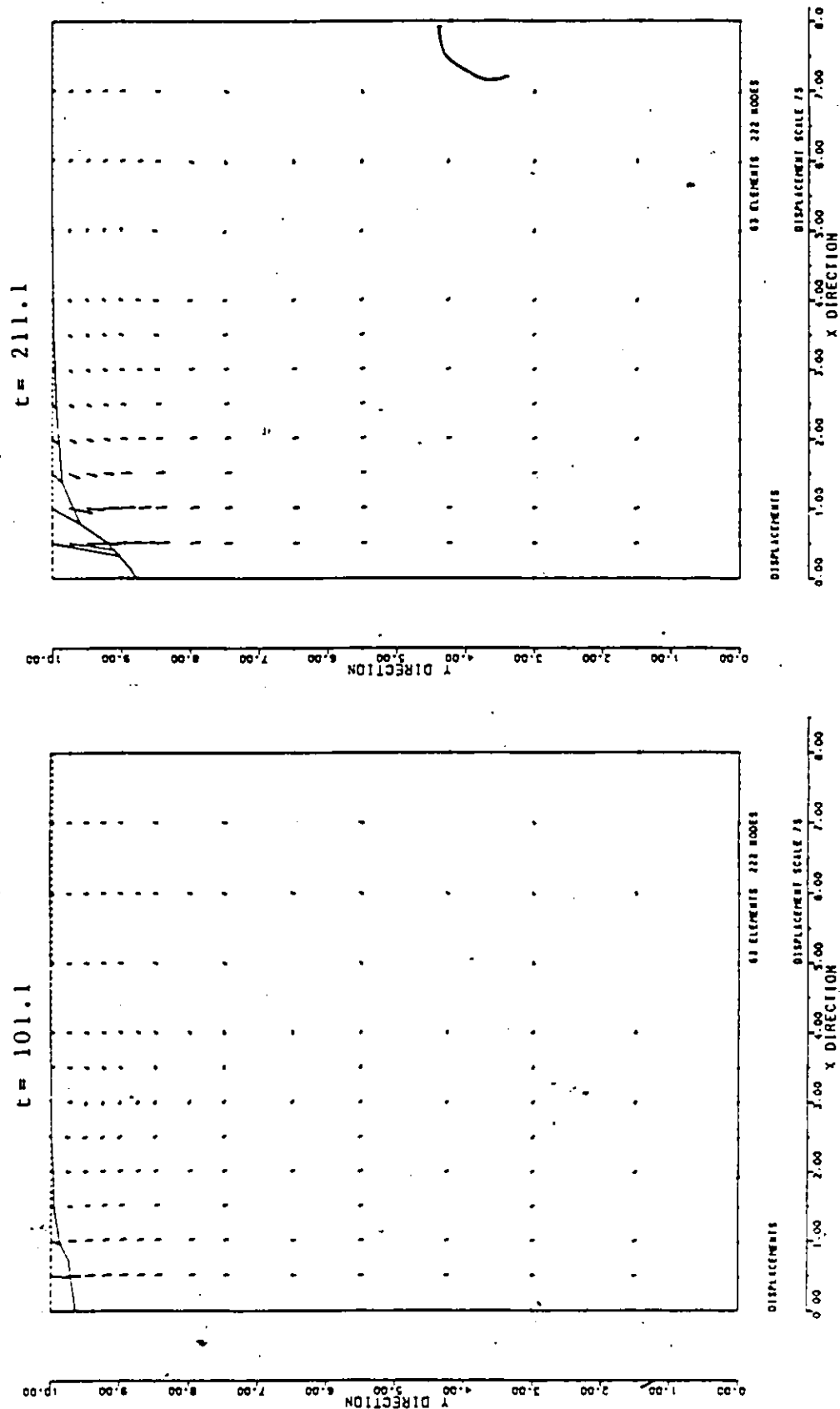


(a) surface settlement vs. time



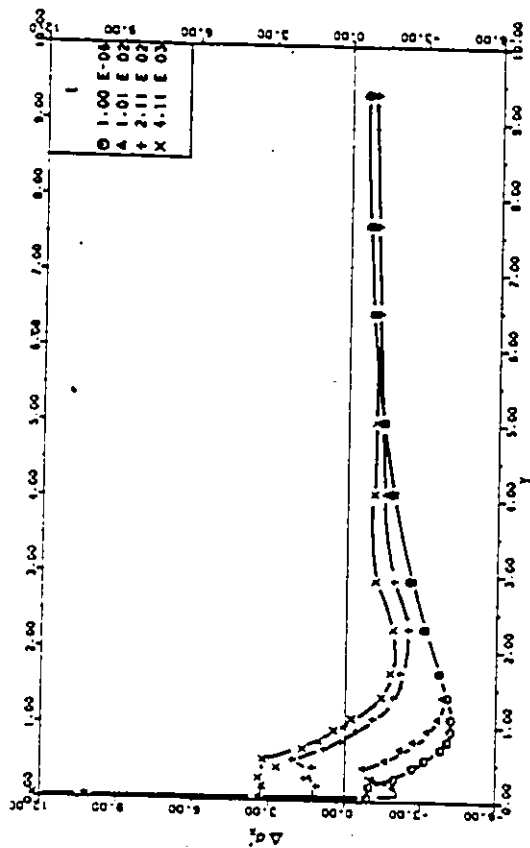
(b) excess pore pressure vs. time

Fig. 5.42 Consolidation in a semi-infinite medium
(Critical Mohr-Coulomb material)



(c) displacement field at different time steps

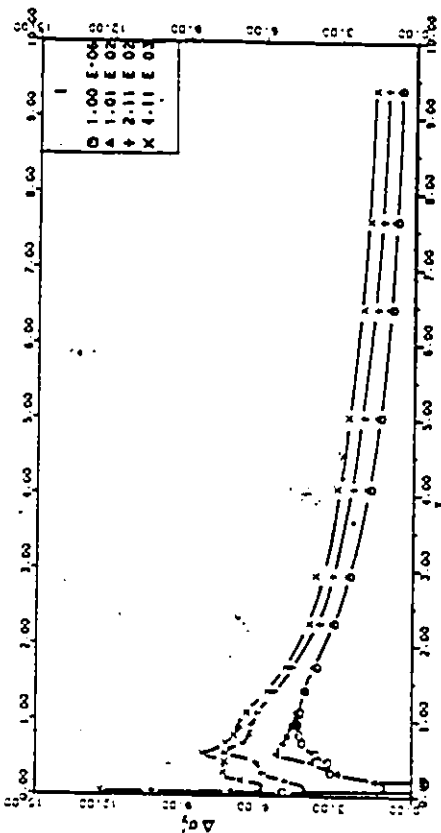
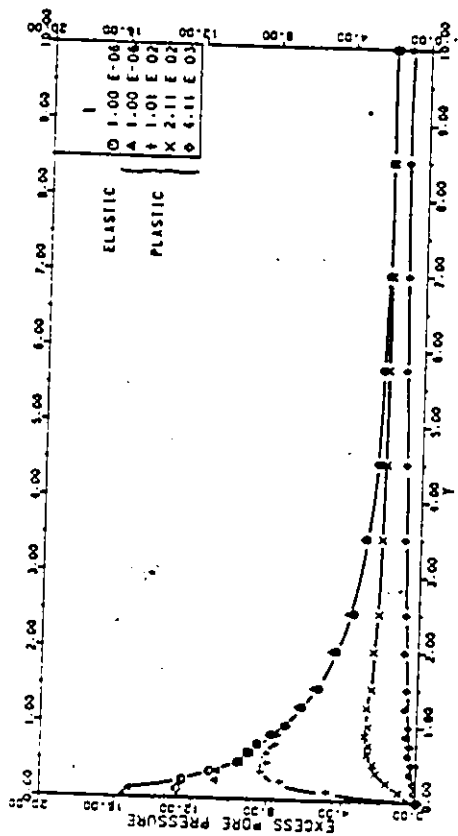
Fig. 5.42 Consolidation in a semi-infinite medium
(Critical Mohr-Coulomb material)

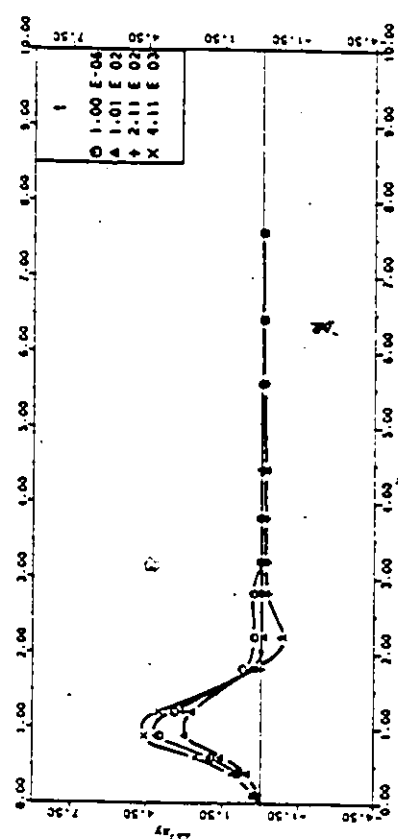
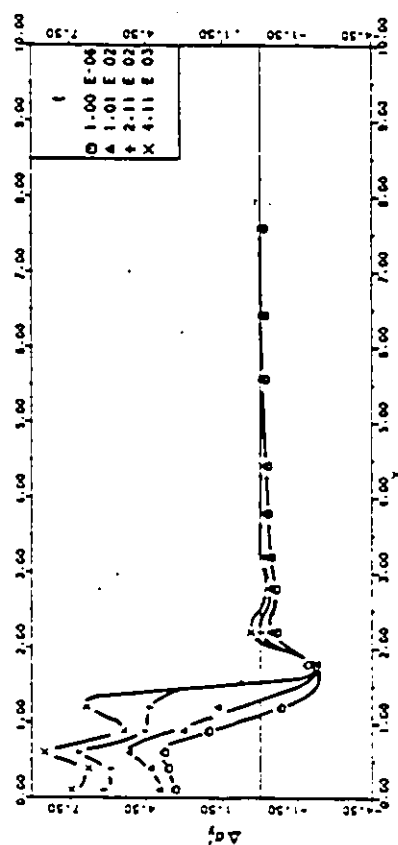
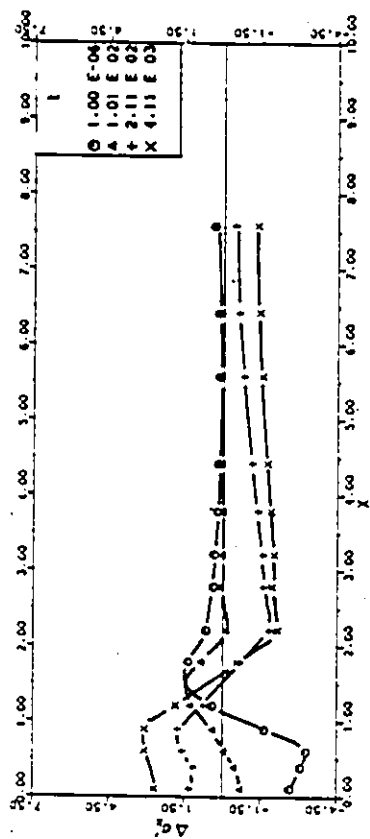
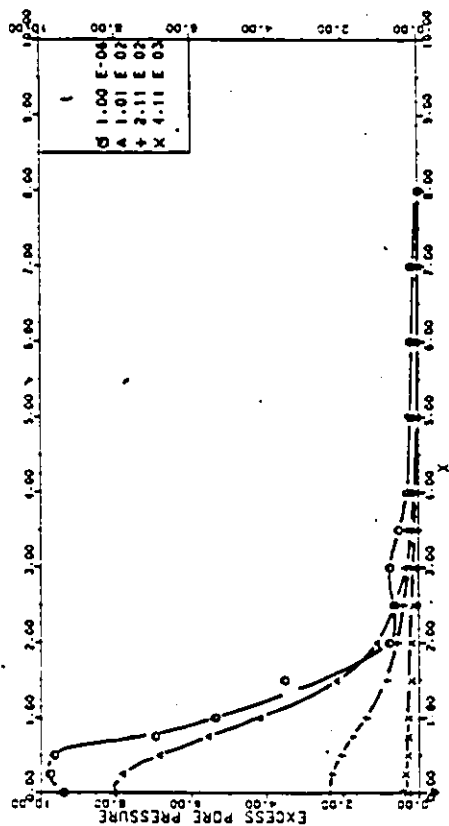


(d) excess pore pressure and effective stress isochrones (vert. sect., $X=0.0$)

Fig. 5.42

Consolidation in a semi-infinite medium
(Critical Mohr-Coulomb material)





(e) excess pore pressure and effective stress isochrones
(horiz. sect., $\gamma=0.5$)

Fig. 5.42 Consolidation in a semi-infinite medium (Critical Mohr-Coulomb material)

5.6 CONCLUSION

- The performance of all four programs herein presented has been very satisfactory, and interesting results have been achieved.

The algorithm on which SEEP is based for the analysis of unconfined flow, may be considered efficient and reliable, and the application of this program to real geotechnical problems should not present any difficulty.

As regards the consolidation problem, the predictions of the Uncoupled theory have proved to be quite different from those of the Coupled theory, even in the simplest case of linear elasticity.

The difference between the two theories is not limited to the Mandel-Cryer effect previously described. The initial values of excess pore pressure calculated as in the example reported in Figs. 5.14, differ substantially from those computed in TEPSA, Figs. 5.37. Perhaps different choices of the coefficients "a" and "b" in Henkel's formula may reduce the gap. If π^e is set equal to the initial value calculated in TEPSA, both theories predict approximately the same time of consolidation, Figs. 5.43, but TERCON required only 37.8 sec. against the 149. sec of TEPSA. However, the coincidence of the consolidation time is restricted to the current case of the Poisson ratio equal to zero. Different choices of Poisson ratio led to some discrepancy between the consolidation times predicted by the two theories.

In a last analysis therefore, TERCON can be very useful to determine economically an order of magnitude of the consolidation time, but it cannot describe correctly the excess pore pressure evolution during consolidation. In fact, the Uncoupled theory can take into account only the hydraulic aspect of the problem, neglecting the interaction of solid and fluid phase, the key point for a correct evaluation of the consolidation process.

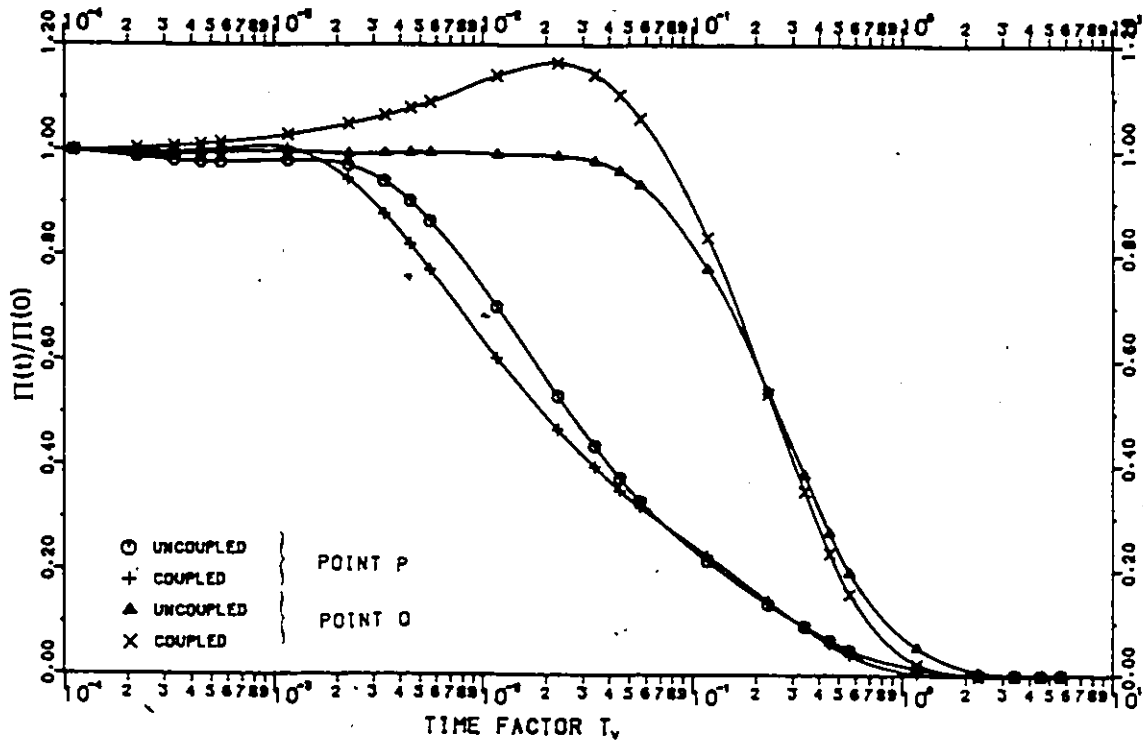
The numerical implementation of the elasto-plastic models in ESEPA produced excellent results in triaxial conditions.

The good performance was then confirmed in the plane strain analysis specially for the linear elastic perfectly plastic models, for which it was possible to recover the ultimate loads predicted by the Limit analysis theory when either Tresca or Mohr-Coulomb failure criterion was employed. Also, it was possible to highlight the different soil responses when states of failure were instead represented by von Mises or Drucker-Prager surface.

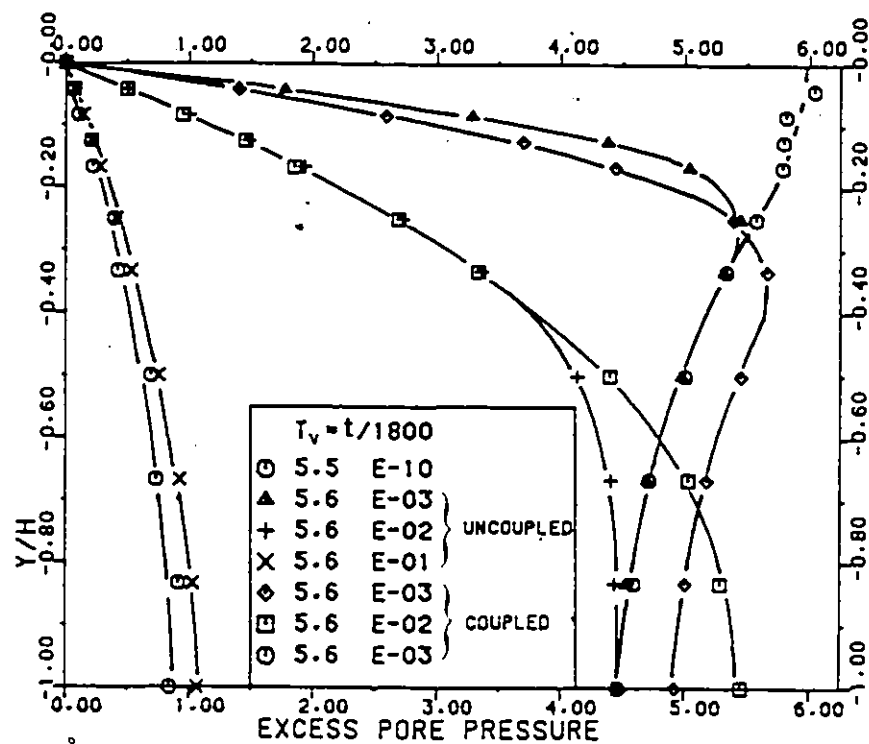
The overall response of the Generalized Burland model for a typical plane strain problem was also viewed, but it is believed that further study towards the improvement of the numerical algorithm is needed. However, it was possible to identify the great difference in numerical response between the perfectly plastic and the Cam-clay models in a general boundary value. As observed in Figs. 5.32 and 5.33, the Cam-clay model does not spread the displacement field as the

perfectly plastic model does. As a result, the former model leads to a much lower ultimate load than the latter, even if failure is described in both cases by the same limiting surface. At this point, only practical experiment can validate the forecast of either plastic models.

Finally TEPSA proved to be also an effective tool to analyze a Coupled consolidation problem in plastic conditions. The axisymmetric case proved that TEPSA is correctly implemented for elasto-plastic analysis, since the numerical results completely satisfied the theoretical expectation. As for the semi-infinite strip load case, the program was also able to highlight the big difference in material response which the two elasto-plastic theories lead to. In particular, it is of extreme interest the delayed failure registered by using the Generalized Burland model.



(a) excess pore pressure vs. time



(b) excess pore pressure isochrones at section A-A

Fig. 5.43 Comparison between Coupled and Uncoupled consolidation theories

Chapter VI

FINAL REMARKS AND CONCLUSIONS

Nowadays, the developments in both theoretical and numerical fields, as well as the tremendous increase in computer power, offer a definite possibility for a more rigorous study of the behaviour of geotechnical structures.

This research work, on the basis of a number of previous experiences and of independent theoretical elaborations, succeeded in developing a set of FE codes for a fairly complete geotechnical analysis.

The algorithm proposed for the analysis of unconfined seepage flow allowed to locate efficiently and economically the free surface, even for a heterogeneous medium with complex geometrical boundary conditions.

As regards the consolidation problem, the new FE formulations and algorithms for both Coupled and Uncoupled theories seem to have overcome instability in the numerical solutions. The comparison between the two theories of consolidation showed that the Coupled theory may describe the dissipation of excess pore pressure more realistically than the Uncoupled theory, being also able to take into proper account the plastic characteristic of soils.

As for plasticity, the numerical implementations herein developed can be considered very satisfactory, having recovered available theoretical predictions. The key point to achieve accuracy in this analysis is undoubtedly represented by the new and correct expressions calculating the plastic gradient and the yield points as derived in Sections 1.2.4 and 4.4.1, respectively. The finite element implementations made possible a comparison soil responses for different boundary conditions, employing the two sets of plastic models considered in this research.

The elastic perfectly plastic models confirmed the theoretical concerns exposed in Chapter 2: in fact, they predicted rather unrealistic volumetric change when employed in triaxial conditions. The Generalized Burland model, instead, could simulate soil behaviour in triaxial more adequately. On the other hand in plane strain conditions, the former model predicts a very low ultimate load compared to the one calculated by the conventional geotechnical formula. However, when employed for the analysis of a consolidation problem, this model forecasts a delayed failure, an event so often registered in engineering practice. Interesting are also the two distinct modes of failure obtained by the two types of models; it appears in fact that the Generalized Burland mode leads to a "localized mode of failure", whereas the elastic perfectly plastic model predicts a "generalized mode of failure".

At this point, only future experimental works can either validate one of the two elasto-plastic models or stimulate further theoretical study for more refined ones. In this regard, the thorough analysis presented in Chapter 2 can turn out to be very useful, since it provides the mathematical frame work for further developments of constitutive equations and helps to avoid the mathematical inconsistency existing in previous models.

Therefore in this research, as well as in most academical works, the term "conclusion" seems to be most inappropriate, since as investigation proceeds, more gateways are opened towards unexplored and exciting possibilities for new developments.

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Appendix A

INTERNAL WORK IN TERMS OF ITS VOLUMETRIC AND DEVIATORIC STRESS/STRAIN COMPONENTS

Expressing stress and strain in terms of their volumetric and deviatoric component, the internal work can be rewritten as

$$\begin{aligned}
 W &= \rho_0 w = \frac{1}{2} \tilde{\sigma} : \tilde{\epsilon} = \frac{1}{2} \tilde{T} : \tilde{L} = \frac{1}{2} \tilde{\sigma}_{ij} \tilde{\epsilon}_{ij} = \\
 &= \frac{1}{2} \left(\tilde{s}_{ij} + \delta_{ij} \frac{\tilde{\sigma}_{kk}}{3} \right) \left(\tilde{e}_{ij} + \delta_{ij} \frac{\tilde{l}_{kk}}{3} \right) = \\
 &= \frac{1}{2} \left(\frac{\tilde{\sigma}_{kk} \tilde{l}_{kk}}{3} + \tilde{s}_{ij} \tilde{e}_{ij} \right) = \frac{1}{2} (\tilde{p} \tilde{\epsilon}_v + \tilde{s} : \tilde{e})
 \end{aligned}$$

Appendix B

ELASTIC ISOTROPIC POTENTIAL FUNCTION IN TERMS OF ITS DEVIATORIC AND VOLUMETRIC STRESS/STRAIN INVARIANTS

For isotropic linear elastic material

$$\bar{\sigma}_{ij} = \frac{E}{1-\nu} \left(\bar{\epsilon}_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \bar{\epsilon}_v \right)$$

consequently

$$\bar{\epsilon}_v = \bar{1}_{kk} = \frac{3(1-2\nu)}{E} \frac{\bar{\sigma}_{kk}}{3} = \frac{\bar{p}}{B}$$

$$\bar{\epsilon}_{ij} = \bar{1}_{ij} - \frac{\delta_{ij}}{3} \bar{1}_{kk} = \frac{1+\nu}{E} \bar{s}_{ij} = \frac{\bar{s}_{ij}}{2G}$$

$$\bar{\epsilon}_s = \left(\frac{2}{3} \bar{\epsilon}_{ij} \bar{\epsilon}_{ij} \right)^{\frac{1}{2}} = \frac{1}{3G} \left(\frac{3}{2} \bar{s}_{ij} \bar{s}_{ij} \right)^{\frac{1}{2}} = \frac{\bar{q}}{3G}$$

Therefore it follows that the expression of the internal work presented in

Appendix A can be further reduced to

$$W = \frac{1}{2} (\bar{p} \bar{\epsilon}_v + \bar{s} \cdot \bar{\epsilon}) = \frac{1}{2} (\bar{p} \bar{\epsilon}_v + \frac{\bar{s} \cdot \bar{s}}{2G})$$

$$= \frac{1}{2} \left(\bar{p} \bar{\epsilon}_v + \frac{\bar{q}^2}{3G} \right) = \frac{1}{2} (\bar{p} \bar{\epsilon}_v + \bar{q} \bar{\epsilon}_s)$$

or rather

$$W = \frac{\bar{p}^2}{2B} + \frac{\bar{q}^2}{3G}$$

Appendix C

STRESS-STRAIN RELATIONSHIPS IN A CONVENTIONAL TRIAXIAL TEST

It is known from Continuum Mechanics that

$$\tilde{T} = J F^{-1} \cdot T \cdot F^{-1}$$

$$F = G^{\frac{1}{2}} = [2\tilde{L} + I]^{\frac{1}{2}}$$

I = identity matrix

G = Green strain tensor

$$\tilde{L} = \left[\frac{1}{2} u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \right] = l_{ij}$$

$$J = \frac{\rho_0}{\rho} = \frac{dV}{dV_0} = (I_3^G)^{\frac{1}{3}}$$

I_3^G = third invariant of G

Furthermore, since

$$\epsilon_i = \frac{ds_i - dS_i}{dS_i} = (2 \tilde{l}_{(ii)} - 1)^{\frac{1}{2}} - 1$$

it follows

$$\tilde{l}_{(ii)} = \epsilon_i + \frac{\epsilon_i^2}{2}$$

where,

dS_i = initial length of the sample

ds_i = deformed length of the sample

i = i -direction (1,2,3)

(ii) - stands to indicate any of these subscripts: 11, 22, 33.

In general, the coordinate $\underline{x}$ of the material point in the deformed configuration may be related to its coordinate $\underline{X}$ in the undeformed configuration as follows

$$\underline{x}(\underline{X}) = \underline{u}(\underline{X}) + \underline{X} = \underline{A} \cdot \underline{X} + \underline{X} = (\underline{A} + \underline{I})\underline{X} = \underline{B}\underline{X}$$

In triaxial conditions, Fig. A3, and under the hypothesis of homogeneity on the material, the following correspondence can be established

$$\underline{x}^{(1)} = \{x_1, 0, 0\}^T$$

$$\underline{x}^{(2)} = \{0, x_3, 0\}^T$$

$$\underline{x}^{(3)} = \{0, 0, x_3\}^T$$

$$\underline{x}^{(1)} = \{x_2, 0, 0\}^T$$

$$\underline{x}^{(2)} = \{0, x_3, 0\}^T$$

$$\underline{x}^{(3)} = \{0, 0, x_3\}^T$$

Consequently B and A have to be diagonal matrices and in particular

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1(x_1) \\ u_2(x_2) \\ u_3(x_3) \end{bmatrix}$$

$$u_{i,j} = 0 \quad \text{for } i \neq j$$

$$\tilde{L}_{(ii)} = u_{i,i} + \frac{u_{i,i}^2}{2}$$

$$\tilde{L}_{(i,j)} = 0$$

Thus, under the hypothesis of isotropy, the direction of the triaxial axis, Fig. C1, coincides with the principal directions of L and therefore of G.

In particular, the 2nd Piola-Kirchhoff stress tensor may be related to the measurable Cauchy stress tensor as

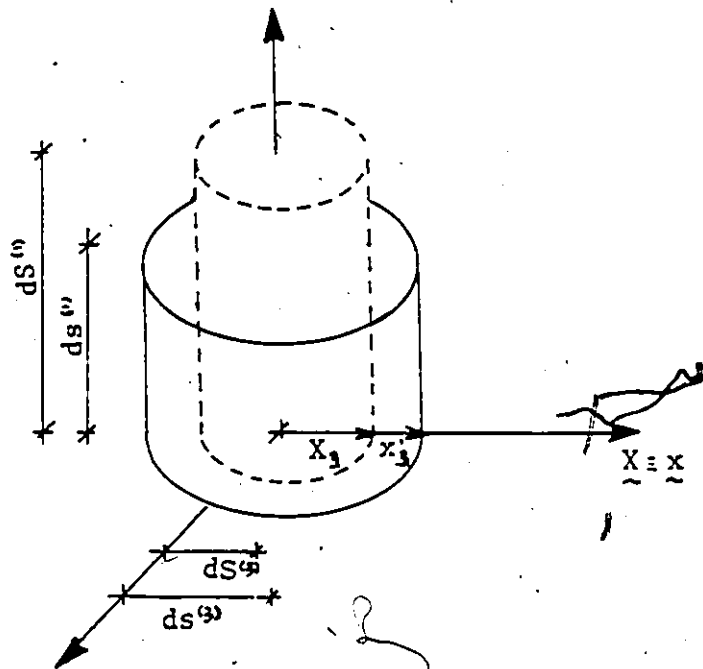
$$\tilde{T} = J F^{-1} T F^{-1} = J \begin{bmatrix} G_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & G_3^{-\frac{1}{2}} & 0 \\ 0 & 0 & G_3^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_3 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} G_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & G_3^{-\frac{1}{2}} & 0 \\ 0 & 0 & G_3^{-\frac{1}{2}} \end{bmatrix}$$

$$J = \frac{dv}{dv_0} = (I_3 G)^{\frac{1}{2}} = G_3 (G_1)^{\frac{1}{2}}$$

$$\begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_3 \end{Bmatrix} = J \begin{Bmatrix} \frac{\sigma_1}{G_1} \\ \frac{\sigma_3}{G_3} \end{Bmatrix} = \begin{Bmatrix} \frac{dv}{dv_0} \frac{\sigma_1}{(2 I_{11} + 1)} \\ (2 \tilde{I}_{11} + 1)^{\frac{1}{2}} \sigma_3 \end{Bmatrix}$$

and of course the finite strain component $l_{(11)}$ may be related to the measurable strain as

$$\begin{Bmatrix} \tilde{l}_{11} \\ \tilde{l}_{33} \end{Bmatrix} = \begin{Bmatrix} \epsilon_1 + \frac{\epsilon_1^2}{2} \\ \epsilon_3 + \frac{\epsilon_3^2}{2} \end{Bmatrix}$$



Appendix D

ISOTROPIC ELASTIC POTENTIAL FUNCTION IN TERMS OF ITS PRINCIPAL STRESS COMPONENTS

$$W = \frac{1}{2} \tilde{\sigma} \cdot \tilde{\epsilon} = \frac{1}{2} \tilde{\sigma} \cdot (C^e)^{-1} \cdot \tilde{\sigma}$$

Since the value of potential functions is frame indifference, the above product can be alternatively written in terms of principal stress. Moreover, in the hypothesis of isotropic material the potential function can be expressed as follows:

$$W = \frac{1}{2E} [(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3) - 2\nu (\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_1\tilde{\sigma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3)]$$

or

$$(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3) - 2\nu(\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_1\tilde{\sigma}_3 + \tilde{\sigma}_2\tilde{\sigma}_3) - 2EW = 0$$

where

$\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ are the principal values of the 2nd Piola-Kirchhoff stress tensor.

From elementary geometry it is possible to prove that the above equation is real ellipsoid whose eigenvalues are respectively

$$\lambda_1 = \lambda_2 = 1 + \nu$$

$$\lambda_3 = 1 - 2\nu$$

It follows then that the canonical form can be written as

$$(1-2\nu)\bar{\sigma}_3^2 + (1+\nu)(\bar{\sigma}_1^2 + \bar{\sigma}_2^2) - 2EW = 0$$

where $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ are the eigen values of $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ and in particular $\bar{\sigma}_3$ has the direction coincident with the direction of the space diagonal in the principal reference system of the 2nd Piola-Kirchhoff stress tensor.

Appendix E

ACCELERATION METHOD: ALTERNATIVE ESTIMATION OF A-COEFFICIENT

In the ISM the iterative process to balance the residual stress $\Delta\psi_{n(i)}$ is based on the recurrent solution of

$$K^e \Delta d_{n(i)} = \Delta\psi_{n(i)} \quad \Delta d_{n(i)} = (K^e)^{-1} \Delta\psi_{n(i)} \quad (1)$$

Let's assume that an improved value of the displacement $\Delta d_{n(i)}^*$ can be found as

$$K^{ep} \Delta d_{n(i)}^* = \Delta\psi_{n(i)}$$

where $K^{ep} = K^e - K^p$ (2)

Premultiplying Eq. 2 by $(K^e)^{-1}$ and employing Eq.1, it is possible to express the improved value $\Delta d_{n(i)}^*$ as follows

$$\Delta d_{n(i)}^* = K^* \Delta d_{n(i)} + \Delta d_{n(i)} \quad (3)$$

$$\text{where } K^* = (K^e)^{-1} K^p$$

Let's also assume that $\Delta d_{n(i)}^*$ is a function of $\Delta d_{n(i)}$, namely

$$\Delta d_{n(i)}^* = A_{n(i)} \Delta d_{n(i)} = [A_{n(i-1)} - \Delta A_{n(i)}] \Delta d_{n(i)} \quad (4)$$

where $A_{n(i)}$, in order to recover Nayak and Zienkiewicz (1972), should be considered as diagonal matrix. Herein instead, shearing Thomas' argument (section 3.5.1), $A_{n(i)}$ is supposed to be a single acceleration parameter.

Equalizing Eq.3 with Eq.4, it follows

$$A_{n(i)} \Delta \tilde{d}_{n(i)} = \Delta \tilde{d}_{n(i)}^p + \Delta \tilde{h}_{n(i)} + \Delta \tilde{\varepsilon}_{n(i)}$$

where

$$\Delta \tilde{h}_{n(i)} = [(K^e)^{-1} K^p A_{n(i-1)}] \Delta \tilde{d}_{n(i)}$$

$$\Delta \tilde{\varepsilon}_{n(i)} = [(K^e)^{-1} K^p \Delta A_{n(i)}] \Delta \tilde{d}_{n(i)}$$

The aim is now to find the value $A_{n(i)}$ such as contained in $\Delta \tilde{\varepsilon}_{n(i)}$.

This can be accomplished by minimizing the modulus of $\Delta \tilde{\varepsilon}_{n(i)}$ with respect of $A_{n(i)}$, namely

$$\frac{\partial}{\partial A_{n(i)}} (\Delta \tilde{\varepsilon}_{n(i)}^T \Delta \tilde{\varepsilon}_{n(i)}) = 0$$

from which it can be proved that

$$A_{n(i)} = \frac{\Delta \tilde{d}_{n(i)}^T \Delta \tilde{d}_{n(i)} - \Delta \tilde{d}_{n(i)}^T \Delta \tilde{h}_{n(i)}}{\Delta \tilde{d}_{n(i)}^T \Delta \tilde{d}_{n(i)}}$$