

CYCLE RANK OF TRANSITION GRAPHS

and the

STAR HEIGHT OF REGULAR EVENTS

by

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Abstract

This thesis is concerned with the (restricted) star height of regular events, as defined by Eggen. Eggen has established the existence of events of arbitrary star height and has shown that the star height of a given event equals the minimum cycle rank of all transition graphs recognizing the event. The problem of establishing the star height of regular events was further investigated by McNaughton and by Dejean and Schützenberger.

In this thesis some new approaches to the star height problem are taken, and the relationship between the star height of a given event and the structure of its reduced state graph is further explored. First a summary of previous results is outlined and some general properties of star height are studied. The notions of confinal sets of trails and simple regular expressions are then introduced. Necessary conditions for the star height of an event to be no less than some integer $k > 0$ are developed and a family of events whose star height equals the rank of the corresponding reduced state graph is exhibited.

Rank-non-increasing transformations on transition graphs are investigated, providing a stronger version of Eggen's theorem. By this version, the star height of a regular event equals the smallest rank of all non-deterministic reduced state graphs representing the event. This result makes possible the introduction of some new concepts, such as the R-projection functions and covering of state graphs, by means of which a number of auxiliary results on star height is obtained. A technique for establishing the star height of an event is developed, as well as necessary conditions for the star height of an event defined by a reset-

free state graph G to be no less than the rank of some subgraph of G . The thesis concludes with a discussion of remaining problems.

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List of Symbols and Notation

Latin Alphabet

A	- finite alphabet
$\mathcal{A}$	- automaton
a	- alphabet letter
B	- set of branches in a digraph
B_L	- set of labelled branches in a transition graph
D	- digraph
D_q	- the derivative (left-quotient) corresponding to state q in a reduced automaton.
E	- regular expression
F	- set of final states in an automaton
G	- transition graph, state graph
$G_0(R)$	- reduced deterministic state graph recognizing R
h	- star height
l	- length of a word
M	- transition function of an automaton
$\mathcal{M}_{\mathcal{A}}$	- syntactic monoid of an automaton $\mathcal{A}$
N	- set of nodes in a graph
m,n	- nodes
P(G)	- set of paths in a transition graph G
p	- path
Q	- set of states in an automaton
q	- state

R	- regular event
r	- cycle rank
S	- strongly connected subgraph
T	- set of trails in a state graph
$T(\mathcal{A})$	- event accepted by automaton $\mathcal{A}$
u, v, w	- words

Greek Alphabet

γ, η, ϕ	- mappings
$h_{\alpha}(E)$	- apparent star height of a regular expression E
ϵ	- null path
λ	- empty word
Λ	- empty event
$\emptyset$	- empty set, empty event

Auxiliary Symbols

$\#N$	- number of elements in set N
$\cup$	- union
$\cap$	- intersection
$-$	- complement
$\cdot$	- concatenation
$*$	- star operator
$\forall$	- for every
$\wedge$	- and
$\vee$	- or
$\in$	- belongs to
$\bar{G}$	- the dual of the graph G

- $\bar{R}$ - the reverse event of R
- $x \setminus R / y$ - two-sided quotient of R by (x, y)
- $x \setminus R, R / x$ - left and right quotients of R by x
- $|E|$ - the event denoted by the regular expression E
- 2^Q - set of all subsets of Q .
- $\tilde{G}_0(R)$ - reduced incomplete automaton accepting R

Notational Conventions

1. Composition of mappings is denoted by juxtaposition ($\gamma\eta(x) = \gamma(\eta(x))$)
2. When no confusion arises, a set S consisting of a single element s will sometimes be denoted by s instead of $\{s\}$.

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Introduction

Since 1956, when regular expressions were first introduced by Kleene [K1], a great deal of research has been centered around them. Their structural properties were investigated and their relation to finite automata has been thoroughly studied. As it turns out, most interesting problems concerning regular expressions arise due to the fact, that there exist infinitely many regular expressions denoting each event, varying in size and structure. Although for some special families of events unique canonical regular expressions have been found ([Br 3], [Yo]), no canonical form for regular expressions is known for the general case. In spite of the fact that methods for checking equivalence of regular expressions are well-known, relatively little is known about the infinite family of all regular expressions denoting a given event. One of the problems requiring the study of this infinite family is the star height problem, namely the problem of determining the star height of a regular event. This problem is of importance because the star height represents a measure of complexity of regular events and the minimum star height would be very desirable in any canonical form for regular expressions.

The notion of star height was first introduced by Eggen [E] who showed that there exist regular events of arbitrary star height. This was done by exhibiting, for each n , events of star height n . A disadvantage in Eggen's examples is the fact that the size of the input alphabet increases with n . Dejean and Schlützenberger [DS] have shown that for each

n there exists an event over the two-letter alphabet that has star height n (it is easy to show that every event over the one-letter alphabet has star height 0 or 1). Eggan has also introduced the notion of (cycle) rank of a digraph and has established a fundamental theorem, by which the star height of a regular event equals the smallest rank of all transition graphs recognizing the event.

Eggan's theorem was used by McNaughton in [Mc 3] for developing graph-theoretical methods for establishing the star height of regular events. It was also proved in that paper, that every regular event of star height n can be mapped by a homomorphism onto an event of star height n over the two-letter alphabet. Pathwise homomorphisms between transition graphs were introduced in [Mc 4]. Such homomorphisms were shown to be rank-non-increasing and were utilized for establishing the existence of an algorithm for determining the star height of pure-group events.

There still remains, however, the difficult open problem of how to determine the star height of a regular event in the general case. In this thesis some new concepts are introduced and some new tools developed for attacking this problem. A number of results on star height of certain families of events is obtained, and some techniques for determining star height are presented.

In Chapters 1, 2 the fundamental definitions of automata, regular expressions, transition graphs and related notions are given and some basic results summarized. The star height problem is introduced

in Chapter 3 and a survey of previous results is outlined. A study of general properties of star height is made in Section 3.3, and some results concerning the changes in star height caused by various operations on events are presented.

In Chapter 4 the notions of trails, cofinal sets of trails in a state graph and simple regular expressions are introduced. The main Lemma is then proved in Section 4.3, relating the apparent star height of a regular expression to the rank of its corresponding set of trails in a given state graph. In Section 4.4 sufficient conditions are found for the star height of an event to be no less than some $k > 0$ and a family of events whose star height equals the rank of the corresponding reduced state graph is exhibited. Section 4.5 deals with the star height of events defined by digraphs. Each such event consists of the set of all paths in a digraph connecting two sets of nodes. A full characterization of the star height of such events is obtained and two questions raised by Eggen in [E] are answered.

Chapter 5 is concerned with rank-non-increasing transformations on transition graphs. In Section 5.1 some elementary modifications on digraphs are introduced and their effect on the rank of the digraph is studied. In Section 5.2 a reduced transition graph is defined and a rank-non-increasing reduction procedure of transition graphs, based on the results of the previous section, is developed. A process of elimination of all empty-word-transitions in a transition graph G is described in Section 5.3. The resulting transition graph is equivalent to G , has no

more nodes than G and no greater rank. It follows that every transition graph can be replaced by an equivalent non-deterministic reduced state graph with no more nodes and of no higher rank. As a result, a stronger version of Eggen's theorem is obtained: The star height of a regular event R equals the smallest rank of all non-deterministic reduced state graphs recognizing R . It follows as a by-product, that the minimum number of nodes in any transition graph recognizing R equals the minimum number of nodes in any non-deterministic state graph recognizing R .

Chapter 6 opens with the definition of the R -projection functions and of covering a state graph by another one, two notions which help relate an arbitrary non-deterministic state graph to the given reduced deterministic state graph recognizing R . These notions, as well as the notion of pathwise homomorphism, are utilized for establishing two fundamental lemmas, by which most results of this chapter are proved. A technique for establishing star height of regular events is developed and some lower bounds to the star height of events recognized by reset-free state graphs are obtained. The star height of any group-free event recognized by a reset-free state graph G is shown to equal the rank of G .

The thesis concludes with a short summary on existing methods for determining star height, and some conjectures concerning the existence of a general algorithm for determining the star height of regular events are discussed. Some other open problems are also mentioned.

Chapter 1

Regular Events and Automata

In this chapter the basic material that is common to all chapters is presented. The reader is referred to [Bo], [Ha] and [Gs] for additional details.

1.1. Regular Expressions and Regular Events

Definition 1.1: Let A be a finite non-empty set of symbols, called letters, and let A^* be the free monoid (i.e. semigroup with identity) generated by A , where the operation is called product or concatenation and is denoted by juxtaposition. The elements of A^* are finite sequences of letters from A and are called words on the alphabet A . The identity element of A^* is the empty word and is denoted by λ . Thus for every $w \in A^*$, $\lambda w = w\lambda = w$. The length of a word $w = a_1 a_2 \dots a_k$, $a_i \in A$ for all $i = 1, \dots, k$, is the number k of letters in it and is denoted by $\ell(w)$. Clearly $\ell(\lambda) = 0$ and for any two words $w_1, w_2 \in A^*$, $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Any subset of A^* is called an event (or a language). The empty event is denoted by ϕ , and $\Lambda = \{\lambda\}$. Let U and V be events; then $U \cup V$, $U \cap V$ and UV are the union, intersection and concatenation (product) respectively of U and V , where

$$UV = \{w \mid w = uv, u \in U, v \in V\}.$$

The difference $U - V$ is defined by:

$$U - V = \{w \mid w \in U, w \notin V\}$$

and the complement of U is defined by $\bar{U} = A^* - U$.

Define for any event U , $U^0 = \Lambda$, $U^1 = U$ and for any $n > 1$, $U^{n+1} = UU^n$. Define the star operator (abbreviated star) on any event U by:

$$U^* = \Lambda \cup U \cup U^2 \cup \dots \cup U^n \cup \dots = \bigcup_{n=0}^{\infty} U^n$$

Clearly for any event U , U^* is a monoid contained in A^* ; in fact, it is the submonoid of A^* generated by U .

For any word $w = a_1 a_2 \dots a_k \in A^*$, the reverse of w is the word $\bar{w} = a_k \dots a_2 a_1$, and for the empty word λ , $\bar{\lambda} = \lambda$. The reverse of an event U is defined by

$$\bar{U} = \{u \mid u = \bar{w} \text{ for some } w \in U\}$$

Definition 1.2: Let $U \subseteq A^*$ and let $x, y \in A^*$. Define the two-sided quotient of U by the pair (x, y) by:

$$x \setminus U / y = \{w \mid xwy \in U\}$$

The left quotient of U by x is $x \setminus U = x \setminus U / \lambda$ and the right quotient of U by y is $U / y = \lambda \setminus U / y$.

Definition 1.3 [K1]: Let A be a finite non-empty alphabet and let λ and ϕ be two distinct symbols not in A . Let $\cup$, $\cap$ and $\cdot$ be associative binary operators and let $*$ and $\bar{}$ be unary operators. A regular expression (on A) is defined inductively as follows:

- (1) $a \in A$, λ and ϕ are regular expressions.
- (2) If P and Q are regular expressions then so are $P \cup Q$, $P \cap Q$, PQ (the dot is usually omitted) P^* and $\bar{P}$.

- (3) Nothing else is a regular expression unless its being so follows from a finite number of applications of Rules (1) and (2).

Definition 1.4: Define a mapping $||$ from the set of all regular expressions (over the alphabet A) into the set of all subsets of A^* inductively as follows:

- (i) For all $a \in A$, $|a| = \{a\}$, $|\lambda| = A$, $|\phi| = \phi$.
- (ii) If P and Q are regular expressions then
$$\begin{aligned}|P \cup Q| &= |P| \cup |Q| \\ |P \cap Q| &= |P| \cap |Q| \\ |PQ| &= |P||Q| \\ |P^*| &= |P|^* \\ |\bar{P}| &= \overline{|P|}\end{aligned}$$

An event R is regular iff there exists a regular expression E such that $|E| = R$. We say that E denotes the regular event R . Thus the class of regular events is the range of the function $||$ just defined.

For example, the regular expression $E = (0 \cup 10^*1)^*$ over the alphabet $A = \{0,1\}$ denotes the set of all words over A with an even number of appearances of the letter 1.

Theorem 1.1: The class of regular events is a Boolean algebra of sets.

Definition 1.5: Regular expressions with no occurrences of the operators $\cap$ and $-$ are called restricted regular expressions. Thus every restricted regular expression is obtained from the letters of the alphabet A , λ and ϕ by a finite number of applications of the operators $\cup$, $.$, $*$ only.

Theorem 1.2 [M Y]: Every regular event can be denoted by a restricted regular expression.

Theorem 1.3: The family of regular events is the smallest family of events containing all finite events and closed under the formation of union, concatenation and star.

In the sequel, by a "regular expression" we shall mean only a restricted regular expression. Regular expressions with the operators $\cap$ and $-$ will be referred to as extended regular expressions.

Notation 1.1: Throughout this paper there will be an ambiguity in notation. When no confusion arises, the brackets $| \quad |$ will be omitted and a regular expression E will be understood as the corresponding event $|E|$. Thus the notation $E = F$, $E \subseteq F$, $w \in E$ for E and F regular expressions and a word w is to be interpreted $|E| = |F|$, $|E| \subseteq |F|$, $w \in |E|$ respectively. For any function f defined on the domain of regular events, $f(E)$ will mean $f(|E|)$, etc. Another ambiguity is that a word on the alphabet and the singleton set of words containing only that word will be represented by the same symbol. There will be no difference in the symbol for a letter of the alphabet and a word of length one. In each case it should be clear from context what meaning is being used at a given time.

Definition 1.6: The reverse of a regular expression E is denoted by E^+ and is defined recursively as follows:

$$a^+ = a \text{ for all } a \in A, \emptyset^+ = \emptyset \text{ and } \lambda^+ = \lambda$$

For all regular expressions E_1, E_2 ,

$$\overleftarrow{E_1 \cup E_2} = \overleftarrow{E_1} \cup \overleftarrow{E_2}$$

$$\overleftarrow{E_1 E_2} = \overleftarrow{E_2} \overleftarrow{E_1}$$

$$\overleftarrow{E_1^*} = (\overleftarrow{E_1})^*$$

Definition 1.7: The unary operator δ is defined for any regular event R by:

$$\delta(R) = \begin{cases} \lambda & \text{if } \lambda \in R \\ \emptyset & \text{otherwise} \end{cases}$$

For a regular expression E , $\delta(E) = \delta(|E|)$.

1.2. Finite Automata

Definition 1.8: A finite non-deterministic automaton (or simply automaton) $\mathcal{A}$ over the (finite) alphabet A is a quadruple $\mathcal{A} = (Q, M, Q_1, F)$, where Q is a finite non-empty set of states, $Q_1 \subseteq Q$ is the set of initial states, $F \subseteq Q$ is the set of final states (or output states) and M is the transition function: $M: Q \times A \rightarrow 2^Q$. If $\#Q_1 = 1$ and for every $q \in Q$, $a \in A$, $\#(M(q, a)) = 1$, then $\mathcal{A}$ is said to be a deterministic automaton. If $\#Q_1 = 1$ and for all $q \in Q$, $a \in A$, $\#(M(q, a)) \leq 1$, then $\mathcal{A}$ is called an incomplete automaton. Thus an incomplete automaton is a special case of a non-deterministic automaton, and a deterministic automaton is a special case of an incomplete automaton.

Usually A is referred to as the input alphabet of the automaton. All automata as well as all events considered from now on will be assumed to be over a fixed alphabet A , unless otherwise specified.

Extend the transition function to subsets of Q . If $Q' \subseteq Q$ then for each $a \in A$,

$$M(Q', a) = \bigcup_{q \in Q'} M(q, a)$$

Next extend M to words $w \in A^*$ as follows: for every $a \in A$, $x \in A^*$, $Q' \subseteq Q$

$$M(Q', xa) = M(M(Q', x), a)$$

and for the empty word λ

$$M(Q', \lambda) = Q'$$

Notation 1.2: When no confusion arises, a singleton subset of Q , namely $Q' = \{q\}$, $q \in Q$, will be denoted by q (with brackets omitted).

Definition 1.9: The automaton $\mathcal{A}$ accepts a word $x \in A^*$ iff $M(Q_1, x) \cap F \neq \emptyset$. The set of all words accepted by $\mathcal{A}$ is called the event accepted by $\mathcal{A}$ and is denoted by $T(\mathcal{A})$ (sometimes the word 'accepted' is replaced by 'recognized' or 'defined').

Theorem 1.4 [K1, RS]: The event accepted by any automaton is regular.

Definition 1.10: Two automata are called equivalent iff they accept the same event.

Definition 1.11: A state q of an automaton $\mathcal{A} = (Q, M, Q_1, F)$ is said to be accessible iff there exist $q_1 \in Q_1$ and $x \in A^*$ such that $q \in M(q_1, x)$. The automaton $\mathcal{A}$ is called connected iff all of its states are accessible. For any automaton $\mathcal{A} = (Q, M, Q_1, F)$, let $\mathcal{A}_c = (Q', M', Q_1, F')$, where $Q' \subseteq Q$ is the set of all accessible states in $\mathcal{A}$, M' is the restriction of M to Q' and $F' = F \cap Q'$. Clearly $\mathcal{A}_c$ is a connected automaton.

Theorem 1.5: For any automaton $\mathcal{A}$, $\mathcal{A}_c$ is equivalent to $\mathcal{A}$.

From now on, all automata considered are assumed to be connected and incomplete, unless otherwise specified. The (single) initial state of the automaton will be denoted by q_1 , unless otherwise specified.

Definition 1.12 [KW]: Let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton and let $q \in Q$. The accepted event of q in $\mathcal{A}$, $A_c^{\mathcal{A}}(q)$, or $A_c(q)$, when $\mathcal{A}$ is understood, is defined by:

$$A_c(q) = \{x \mid x \in A^* \wedge M(q, x) \in F\}$$

Theorem 1.6 [Br 2]: Let $\mathcal{A} = (Q, M, q_1, F)$ be a deterministic automaton and let $T(\mathcal{A}) = R$. Then:

- (a) for every state $q \in Q$ there exists a word $x \in A^*$ such that $A_c(q) = x \setminus R$ (Definition 1.2)

and

- (b) for every $x \in A^*$ there exists at least one state $q \in Q$ such that $x \setminus R = A_c(q)$.

If $\mathcal{A}$ is only an incomplete automaton, then part (a) of Theorem 1.6 still holds but part (b) has to be modified as follows:

- (b) for every $x \in A^*$ such that $x \setminus R \neq \emptyset$, there exists $q \in Q$ such that $x \setminus R = A_c(q)$.

Corollary 1.1: If $R = T(\mathcal{A})$ for some automaton $\mathcal{A}$ then R has a finite number of distinct left quotients.

Definition 1.13: An automaton $\mathcal{A} = (Q, M, q_1, F)$ is called reduced iff for every pair of states $q, q' \in Q$, $A_c(q) \neq A_c(q')$.

Theorem 1.7: For every automaton $\mathcal{A}$, there exists a unique reduced deterministic automaton equivalent to $\mathcal{A}$.

Theorem 1.8 [Br 2]: For every event R with a finite number of distinct left quotients, there can be effectively constructed a reduced deterministic automaton accepting R , provided that it is possible to ascertain, for each $a \in A$ and for each left quotient R_i , which left quotient equals $a \setminus R_i$.

Outline of Proof: Let $R = R_1, R_2, \dots, R_n$ be the set of all distinct left quotients of R . Construct the automaton $\mathcal{A}_0(R) = (\{R_1, \dots, R_n\}, M, R_1, F)$, where

$$F = \{R_i \mid 1 \leq i \leq n, \lambda \in R_i\}$$

and the transition function M is defined by $M(R_i, a) = R_j$, $a \in A$, $1 \leq i, j \leq n$ and $R_j = a \setminus R_i$. One can easily verify that in the automaton $\mathcal{A}_0(R)$, for every state R_i , $1 \leq i \leq n$, $A_c(R_i) = R_i$. Hence $\mathcal{A}_0(R)$ is reduced and accepts R .

Theorem 1.9 [Br 2]: Every regular event has a finite number of distinct left quotients.

Outline of proof: Let E be a regular expression denoting the given event R . Then for each word $x \in A^*$, a regular expression $D_x E$ denoting the left quotient $x \setminus R$, can be obtained from E by recursive application of the following rules:

- (1) For $a \in A$, $D_a a = \lambda$, $D_a b = \emptyset$ for $b = \lambda$, $\emptyset$ or $b \in A$ and $b \neq a$.

For any regular expressions P and Q,

$$(2) \quad D_a(P \cup Q) = D_a P \cup D_a Q$$

$$(3) \quad D_a(PQ) = (D_a P)Q \cup \delta(P)D_a Q$$

$$(4) \quad D_a(P^*) = (D_a P)P^*$$

$$(5) \quad D_\lambda P = P$$

$$(6) \quad D_{ua} P = D_a(D_u P), \quad u \in A^*$$

It can be verified that for any $x \in A^*$ and $R = |E|$, $|D_x E| = x \setminus R$.

The regular expression $D_x E$ obtained from E in the above fashion is called a derivative of E with respect to the word x.

Thus in order to obtain all left quotients of R, one computes the derivatives $D_x E$ for words x taken in a lexicographic order. It is shown in [Br 2] that after a finite number of steps no new derivatives are found, and it follows that R has a finite number of distinct left quotients.

Example 1.1: Let $A = \{0,1\}$ and let $R = 10^*11$. The derivatives of R are:

$$D_\lambda R = R$$

$$D_0 R = \emptyset = D_{0x} R, \quad \text{all } x \in A^*$$

$$D_1 R = 0^*11$$

$$D_{10} R = D_1 R$$

$$D_{11} R = 1$$

$$D_{110} R = D_0 R$$

$$D_{111} R = \lambda$$

$$D_{1110} R = D_0 R$$

$$D_{1111} R = D_0 R$$

Thus no new derivatives of R can be obtained and the set of distinct left quotients of R is: $R_1 = R$, $R_2 = 0 \setminus R = \emptyset$, $R_3 = 1 \setminus R = 0*11$, $R_4 = 11 \setminus R = 1$ and $R_5 = 111 \setminus R = \Lambda$. Using the construction of Theorem 1.8, we obtain the reduced deterministic automaton accepting R : $\mathcal{A}_0(R) = (\{R_1, \dots, R_5\}, M, R_1, F)$ where $F = \{R_5\}$ and M is given by the table in Fig. 1.1 (such a table is called the transition table of the automaton $\mathcal{A}_0(R)$.)

Corollary 1.2: Let $W \subseteq A^*$. The following conditions are equivalent:

- (a) W is regular.
- (b) There exists an automaton $\mathcal{A}$ such that $T(\mathcal{A}) = W$.
- (c) W has a finite number of distinct left quotients.

From Theorem 1.7 and Theorem 1.8 we get:

Corollary 1.3: The automaton $\mathcal{A}_0(R)$ defined in Theorem 1.8 is the unique reduced deterministic automaton accepting the regular event R .

Definition 1.14: Let q be a state in an automaton $\mathcal{A} = (Q, M, q_1, F)$. q is called a dead state (or trap state) iff $A_c(q) = \emptyset$.

Clearly if $\mathcal{A}$ is reduced then there can be at most one dead state $q_\emptyset$ in $\mathcal{A}$, and there is one only if the event $R = T(\mathcal{A})$ has a left-quotient equal to $\emptyset$. One can easily verify that by removing from $\mathcal{A}$ the dead state $q_\emptyset$ and all transitions leading to it and from it, one obtains an incomplete reduced automaton $\tilde{\mathcal{A}}_0(R)$ which is equivalent to $\mathcal{A}$ and has no dead state. Furthermore, for any reduced incomplete automaton $\mathcal{A}' = (Q', M', q_1, F)$ there can be constructed an equivalent reduced deterministic automaton $\mathcal{A} = \mathcal{A}_0(R) = (Q, M, q_1, F)$, where

	0	1
R_1	R_2	R_3
R_2	R_2	R_2
R_3	R_3	R_4
R_4	R_2	R_5
R_5	R_2	R_2

$M(q,a)$

Fig. 1.1

$$Q = \begin{cases} Q' & \text{if } \mathcal{A}' \text{ has a dead state } q_\phi \\ Q' \cup \{q_\phi\}, & \text{where } q_\phi \notin Q' \end{cases}$$

and for all $a \in A$, $q \in Q'$, $M(q, a) = M'(q, a)$ iff $\#(M'(q, a)) = 1$,
 $M(q, a) = q_\phi$ otherwise, and $M(q_\phi, a) = q_\phi$ for all $a \in A$. By Theorem 1.6
 and the above argument, we obtain the following:

Proposition 1.1: For every regular event R there exists a unique reduced
 incomplete automaton $\tilde{\mathcal{A}}_0(R)$ having minimal number of states and accepting
 R . $\tilde{\mathcal{A}}_0(R)$ can be obtained from $\hat{\mathcal{A}}_0(R)$ by removing from the latter the
 dead state together with all transitions leading to and from it.

Notation 1.3: $\tilde{\mathcal{A}}_0(R)$ will be called the minimal incomplete automaton
 accepting the event R . Let q be any state in $\mathcal{A} = \tilde{\mathcal{A}}_0(R)$ (or $\hat{\mathcal{A}}_0(R)$).
 Denote by D_q the corresponding left quotient of R , i.e.

$$D_q = A_c^{\mathcal{A}}(q) = x \setminus R \quad \text{for some } x \in A^*$$

Thus in $\tilde{\mathcal{A}}_0(R)$ ($\hat{\mathcal{A}}_0(R)$) the set $\{D_q \mid q \in Q\}$ coincides with the set
 of all non-empty (possibly empty) distinct left quotients of R .

Definition 1.15: Let $\mathcal{A} = (Q, M, q_1, F)$, where $Q = \{q_1, q_2, \dots, q_n\}$.

Extend further the transition function M as follows:

$M: 2^Q \times 2^{A^*} \rightarrow 2^Q$, where for every $Q' \subseteq Q$, $W \subseteq A^*$,

$$M(Q', W) = \{q \mid q \in Q \wedge q = M(q_i, w) \text{ for some } q_i \in Q', w \in W\}$$

For each i , $1 \leq i \leq n$, define the function $M_i: 2^{A^*} \rightarrow 2^Q$ by:

$$M_i(W) = M(q_i, W), \text{ all } W \subseteq A^*$$

Lemma 1.1: Let $\mathcal{A} = (Q, M, q_1, F)$ and let R, P, Q be events such that $R = PQ$. Then

$$M_i(R) = \bigcup_{q_j \in M_i(P)} M_j(Q), \quad i = 1, 2, \dots, n$$

Proof: Follows immediately from the above definition.

Definition 1.16: A regular expression E of the form $E = P^*$, is called a star expression. An event $W \subseteq A^*$ is a star event iff $W = V^*$ for some event V . V is called the root of W . In [Br 4] it was shown that for every star event W there exists a unique minimal root V_m such that $W = V_m^*$ and V_m is contained in all other roots of W .

Theorem 1.10 [Br 4]: The following equivalent conditions are necessary and sufficient for $W \subseteq A^*$ to be a star event:

(a) $W = W^*$

(b) $W = W^2$

(c) $\lambda \in W$ and for each $u \in A^*$, $\lambda \in u \setminus W$ iff $W \subseteq u \setminus W$

Definition 1.17: Let R be a regular event and let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton accepting R . Define for any subset $B \subseteq Q$,

$$\text{str } B = \{x \mid x \in A^* \wedge M(B, x) \subseteq B\}$$

Proposition 1.2: $\text{str } B$ is a regular star event. Moreover, if $q_1 \in B \subseteq F$ then $\text{str } B \subseteq R$.

Proof: Clearly $\text{str } B = \bigcap_{q \in B} T(\mathcal{A}(q, B))$, where $\mathcal{A}(q, B) = (Q, M, q, B)$.

Hence $\text{str } B$ is regular. Now obviously $\lambda \in \text{str } B$, and for any $x, y \in \text{str } B$, $q \in Q$ we have: $M(q, xy) = M(M(q, x), y) \in B$, which shows

that also $xy \in \text{str } B$. Hence $\text{str } B = (\text{str } B)^2$ and by Theorem 1.10(b) $\text{str } B$ is a star event. If $q_1 \in B \subseteq F$ then $M(q_1, x) \in B \subseteq F$ for all $x \in \text{str } B$ and therefore $\text{str } B \subseteq R$.

Definition 1.18 ([RS], [Me]): Let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton. Define a mapping η from A^* into the set of all transformations of Q into itself as follows: for every $w \in A^*$, $\eta(w) = m_w: Q \rightarrow Q$, where $m_w(q) = M(q, w)$ for all $q \in Q$. One can easily verify that η is a homomorphism from the free monoid A^* onto a finite monoid $\mathcal{M}_{\mathcal{A}}$ of transformations of Q into itself, with the operation of composition of transformations. $\mathcal{M}_{\mathcal{A}}$ is called the syntactic monoid of the automaton $\mathcal{A}$. The syntactic monoid of a regular event R is the syntactic monoid of its reduced automaton $\mathcal{A}_0(R)$.

Example 1.2: Let $A = \{0, 1\}$ and let $R = (0 \cup 1)^* 00 (0 \cup 1)^*$. The transition table of the automaton $\mathcal{A} = \mathcal{A}_0(R)$ is shown in Fig. 1.2, where p is the initial state and $F = \{r\}$ is the set of final states. The syntactic monoid of R consists of the following transformations:

$$i = \begin{pmatrix} p & q & r \\ p & q & r \end{pmatrix} = m_\lambda$$

$$a = \begin{pmatrix} p & q & r \\ q & r & r \end{pmatrix} = m_0 = m_{010}$$

$$b = \begin{pmatrix} p & q & r \\ p & p & r \end{pmatrix} = m_1 = m_{11} = m_{101} = m_{111}$$

$$c = \begin{pmatrix} p & q & r \\ r & r & r \end{pmatrix} = m_{00} = m_{000} = m_{001} = m_{100}$$

$$d = \begin{pmatrix} p & q & r \\ p & r & r \end{pmatrix} = m_{01} = m_{011}$$

$$e = \begin{pmatrix} p & q & r \\ q & q & r \end{pmatrix} = m_{10} = m_{110}$$

	0	1
p	q	p
q	r	p
r	r	r

Fig. 1.2

	i	a	b	c	d	e
i	i	a	b	c	d	e
a	a	c	d	c	c	a
b	b	e	b	c	b	e
c	c	c	c	c	c	c
d	d	a	d	c	d	a
e	e	c	b	c	c	e

Fig. 1.3

The multiplication table of $\mathcal{M}_A$ is shown in Fig. 1.3.

Definition 1.19: A regular event R is group-free [PM] iff its syntactic monoid contains no non-trivial subgroups. The automaton $\mathcal{A}_0(R)$ is group-free iff R is group-free.

Definition 1.20: An automaton $\mathcal{A} = (Q, M, q_1, F)$ is called permutation-free iff for any subset $Q' \subseteq Q$ with at least two elements, and for any word $w \in A^*$, the restriction of m_w to Q' is not a cyclic permutation of Q' onto itself. The regular event R is permutation-free iff $\mathcal{A}_0(R)$ is permutation-free.

Theorem 1.11 [PM]: A regular event R is group-free iff it is permutation-free.

Theorem 1.12 (Schützenberger [Sch], Papert and McNaughton [PM]): The family of group-free events over the alphabet A is a boolean algebra of sets, and a monoid under the operation of concatenation.

Theorem 1.12 is proved by showing that the family of group-free events coincides with the family of all events that can be denoted by an extended regular expression without the star operator, i.e. a regular expression using only the operators $\cup$, $\cap$, $\cdot$ and $^-$.

Definition 1.21: An automaton $\mathcal{A}$ is a pure-group automaton iff $\mathcal{M}_A$ is a group. A regular event R is a pure-group event iff $\mathcal{A}_0(R)$ is a pure-group automaton.

Apparently $\mathcal{A}$ is a pure-group automaton iff all transformations in $\mathcal{M}_A$ are permutations of Q onto itself.

Definition 1.22 [RS]: Let $\mathcal{A} = (Q, N, q_1, G)$ and $\mathcal{B} = (P, M, p_1, F)$ be two deterministic automata. Define their direct product

$$\mathcal{A} \times \mathcal{B} = (Q \times P, N \times M, (q_1, p_1), G \times F)$$

where the transition function $N \times M$ is defined by:

$$(N \times M)((q, p), a) = (N(q, a), M(p, a))$$

for all pairs $(q, p) \in Q \times P$ and $a \in A$.

Theorem 1.13 [RS]: $\mathcal{A} \times \mathcal{B}$ accepts the event $T(\mathcal{A}) \cap T(\mathcal{B})$.

Lemma 1.2: If $\mathcal{A}$ and $\mathcal{B}$ are pure-group automata then so is $\mathcal{A} \times \mathcal{B}$.

Proof: Apparently the syntactic monoid of the direct product of two automata, $\mathcal{M}_{\mathcal{A} \times \mathcal{B}}$, equals the direct product of the two syntactic monoids, $\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{B}}$ [L]. The result now follows immediately.

Theorem 1.14: The family of pure-group events is closed under intersection.

Chapter 2

Digraphs, Transition Graphs and State Graphs

2.1. Finite Digraphs and Cycle Rank

Definition 2.1: A finite directed graph (or simply digraph) is an ordered pair $D = (N, B)$, where N is a finite set of nodes and B , the set of branches of D , is a subset of $N \times N$. An element $(m, n) \in N \times N$ of B will be called a branch from m to n and will be denoted by b_{mn} . We call m the initial node and n the terminal node of the branch b_{mn} .

Definition 2.2: A path p in a digraph $D = (N, B)$ is any finite sequence $b_1 b_2 \dots b_k$ of branches $b_i \in B$, $i = 1, \dots, k$, $k \geq 0$, such that the terminal node of b_i is the initial node of b_{i+1} , $i = 1, 2, \dots, k-1$. p is a path from m to n ($m, n \in N$) if m is the initial node of b_1 and n is the terminal node of b_k . The nodes m and n are the initial and terminal nodes, respectively, of the path p . If $k = 0$ then the path is null. If a non-null path p has the same initial and terminal node, then p is called a loop.

Let p_1 and p_2 be two paths in D such that the terminal node of p_1 is the initial node of p_2 , i.e. $p_1 = b_{n_1 n_2} b_{n_2 n_3} \dots b_{n_k n_{k+1}}$ and $p_2 = b_{n_{k+1} n_{k+2}} \dots b_{n_{\ell} n_{\ell+1}}$, where $\ell \geq k \geq 0$. Then their concatenation is the path $p_1 p_2 = b_{n_1 n_2} b_{n_2 n_3} \dots b_{n_k n_{k+1}} b_{n_{k+1} n_{k+2}} \dots b_{n_{\ell} n_{\ell+1}}$

Definition 2.3: Let $D = (N, B)$ and let $n \in N$. Denote by $B(n)$ the set of all branches in B which contain n as either a terminal node or as an

initial node. For any $N' \subseteq N$ define

$$D - [N'] = (N - N', B - \bigcup_{n \in N'} B(n))$$

thus $D - [N']$ is simply the graph obtained from D by eliminating the nodes of N' together with all branches incident to them.

Definition 2.4: Let $D = (N, B)$ be a digraph and let $n \in N$. Define $\pi^D(n)^\dagger$, the set of predecessors of n in D , by:

$$\pi^D(n) = \{n' \in N \mid (n', n) \in B\}$$

and $\sigma^D(n)^\dagger$, the set of successors of n in D , by

$$\sigma^D(n) = \{n' \in N \mid (n, n') \in B\}$$

Definition 2.5: Two digraphs, $D = (N, B)$ and $D' = (N', B')$ are called isomorphic iff there exists a 1 - 1 mapping ϕ of N onto N' such that $B' = \{(\phi(n), \phi(n')) \mid (n, n') \in B\}$.

Definition 2.6: A digraph $D' = (N', B')$ will be called a subgraph^{††} of $D = (N, B)$ iff $\emptyset \neq N' \subseteq N$ and $D' = D - [N - N']$.

Definition 2.7: A digraph $D = (N, B)$ is strongly connected (s.c.) iff for any pair of nodes, $m, n \in N$, there exists a path in D from m to n .

A section of D is any s.c. subgraph of D which is not properly contained in any other s.c. subgraph of D . It follows that the sections in any digraph D are disjoint, and a node n of D is contained in a section iff there exists in D a non-null loop from n to n (i.e. a loop on n).

† When no confusion arises, we shall omit the superscript D .

†† The word "subgraph" has restricted use here.

Definition 2.8: The (cycle) rank of a digraph (after Eggam [E]):

(a) Let $D = (N, B)$ be a strongly connected digraph. The (cycle) rank, $r(D)$, of D is defined inductively as follows:

(i) $r(D) = 1$ iff there exists a node $n \in N$ such that $D - [n]$ contains no s.c. subgraph.

(ii) $r(D) = k$ iff $r(D)$ is not less than k and for some $n \in N$ all sections of $D - [n]$ have rank at most $k - 1$.

(b) Let $D = (N, B)$ be any digraph. Define $r(D) = 0$ iff D has no sections and

$$r(D) = \max \{r(D') \mid D' \text{ a section of } D\}$$

otherwise.

Thus the rank of a digraph is a measure of the complexity of its loops.

Example 2.1: Let $D = (N, B)$ be the digraph shown in Fig. 2.1. The sections of D are $D_1 = D - [n_3]$ and $D_2 = D - [\{n_1, n_2\}]$. It is easily verified by inspection that $r(D_2) = 1$ and $r(D_1) = 2$. Hence $r(D) = \max \{r(D_1), r(D_2)\} = 2$.

The following lemma is an immediate consequence of the above definition.

Lemma 2.1: Let $D = (N, B)$ and $D' = (N', B')$ be two digraphs such that $N' \subseteq N$ and $B' \subseteq B$. Then $r(D') \leq r(D)$.

Definition 2.9: For any digraph $D = (N, B)$, define $\hat{D}$, the dual of D , to be the digraph $\hat{D} = (N, \hat{B})$, where $\hat{B} = \{(n', n) \in N \times N \mid (n, n') \in B\}$.

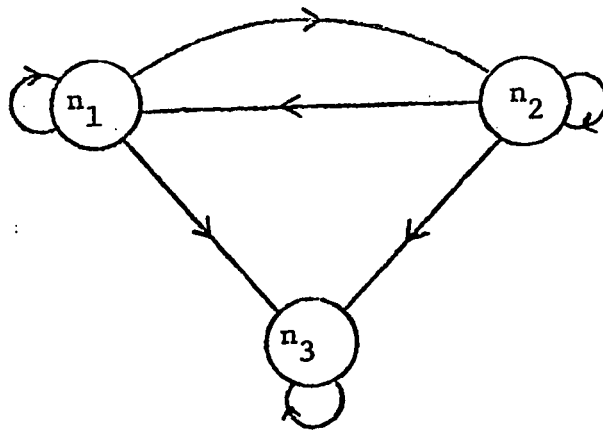


Fig. 2.1

The following lemma is easily verified:

Lemma 2.2: For any digraph D , $r(D) = r(\overleftarrow{D})$, and for every section S of D , $\overleftarrow{S}$ is a section of $\overleftarrow{D}$.

Definition 2.10: Let $D = (N, B)$ be a s.c. digraph. A node $n \in N$ is called a cycle center of D iff $r(D - [n]) < r(D)$. Now let D be any digraph. Then n is a cycle center of D iff n is contained in a section S of D such that $r(S) = r(D)$ and n is a cycle center of S .

The next two propositions are direct consequences of the above definitions.

Proposition 2.1: Let n be a cycle center of a s.c. digraph D . Then $r(D - [n]) = r(D) - 1$.

Proposition 2.2: Let $D = (N, B)$ be a s.c. digraph. Then $r(D) = k \geq 1$ iff (a) there exists a node n in D such that $r(D - [n]) = k - 1$, and (b) for all $m \in N$, $r(D - [m]) \geq k - 1$.

Definition 2.11: Let N be a set of nodes with $\#N = k$. The digraph $D_k = (N, N \times N)$ is called the complete digraph on k nodes.

Proposition 2.3: The complete digraph on $k(>0)$ nodes has rank k .

Proof: Let $k = 1$. Then $D_1 = (\{n\}, \{(n, n)\})$ clearly has rank 1.

The proof proceeds by induction on k : suppose $r(D_{k-1}) = k - 1$ and let $D_k = (N, N \times N)$. Then for any $n \in N$, $D_k - [n] = (N - \{n\}, (N - \{n\}) \times (N - \{n\}))$, which is clearly isomorphic with D_{k-1} . Thus $r(D_k - [n]) = k - 1$ for all $n \in N$ and hence $r(D_k) = k$.

2.2 Transition Graphs and State Graphs

Definition 2.12: Let L be any finite non-void set of symbols, called labels. A finite labelled digraph over L is an ordered pair $G_L = (N, B_L)$, where N is a finite non-void set of nodes and B_L , the set of labelled branches (or simply branches) of G_L , is a subset of $N \times L \times N$. An element $(n_1, a, n_2) \in N \times L \times N$ of B_L is called a branch from n_1 to n_2 labelled by a .

A finite transition graph (or simply transition graph) over a finite alphabet A is a quadruple $G = (N, B_L, N_1, N_2)$, where $G_L = (N, B_L)$ is a finite labelled digraph over the set of labels $L = A \cup \{\lambda\}$, and N_1, N_2 are subsets of N . N_1 is called the set of initial nodes and N_2 - the set of terminal (or final) nodes, of G .

A branch $(n_1, a, n_2) \in N \times A \times N$ in a transition graph G will be called an a-transition (or a transition labelled by a) from n_1 to n_2 in G . A branch $(n_1, \lambda, n_2) \in N \times \{\lambda\} \times N$ will be called a λ -transition from n_1 to n_2 . For any branch $b = (n_1, x, n_2) \in B_L$, $n_1(n_2)$ is the initial node (terminal node) of b . If $(n_1, x, n_2) \in B_L$ then n_2 is an x -successor of n_1 , and n_1 is an x -predecessor of n_2 in G . For each node $n \in N$ in G and for each $x \in A \cup \{\lambda\}$, define $\sigma_x^G(n)^\dagger$ to be the set of all x -successors of n in G , and $\pi_x^G(n)^\dagger$ to be the set of all x -predecessors of n in G .

Convention: All transition graphs, as well as all automata, regular expressions and regular events considered in this text are assumed to be over some fixed finite alphabet A , unless otherwise indicated.

† When no confusion arises, the superscript G will be omitted.

Notation 2.1: In the figures, a terminal node of a transition graph $G = (N, B_L, N_1, N_2)$ is denoted by a double circle, a non-terminal node is indicated by a single circle, a branch $(n_1, x, n_2) \in B_L$ is represented by an arrow from n_1 to n_2 labelled by x and an initial node has a short unlabelled arrow (which is not a branch) pointing to the node.

Example 2.2: The transition graph over the alphabet $A = \{0,1\}$ shown in Fig. 2.2 is $G = (N, B_L, N_1, N_2)$, where $N = \{n_1, n_2, n_3\}$,
 $B_L = \{(n_1, \lambda, n_1), (n_1, 0, n_2), (n_2, 1, n_3), (n_3, \lambda, n_1), (n_3, 0, n_3), (n_3, 0, n_2)\}$,
 $N_1 = \{n_1, n_2\}$ and $N_2 = \{n_2\}$.

Definition 2.13: A path p in a transition graph $G = (N, B_L, N_1, N_2)$ is any finite sequence of branches $b_1 b_2 \dots b_k$, $b_i \in B_L$, $i = 1, \dots, k$, $k \geq 0$, such that the terminal node of b_i is the initial node of b_{i+1} , $i = 1, \dots, k-1$. The number of branches in the path, k , is called the length of p and denoted by $\ell(p)$. If $\ell(p) = 0$ then p is the null path, and is denoted by ϵ . If $m \in N$ is the initial node of b_1 and $n \in N$ is the final node of b_k then p is called a path from m to n , m is called the initial node, and n - the final node, of p . A path from a node n to itself is called a loop on n . A loop of length 1 is called a self-loop. If p is a loop all of whose branches are labelled by λ , then p is a λ -loop.

Definition 2.14: Let p_1 and p_2 be two non-null paths in G such that the final node of p_1 coincides with the initial node of p_2 , i.e.

$p_1 = (n_1, x_1, n_2) \dots (n_k, x_k, n_{k+1})$ and $p_2 = (n_{k+1}, x_{k+1}, n_{k+2}) \dots (n_\ell, x_\ell, n_{\ell+1})$, where $\ell \geq k \geq 0$. Define their concatenation $p_1 \cdot p_2^\dagger$ to be the path $(n_1, x_1, n_2) \dots (n_k, x_k, n_{k+1})(n_{k+1}, x_{k+1}, n_{k+2}) \dots (n_\ell, x_\ell, n_{\ell+1})$.

$\dagger$ Usually the dot will be omitted.

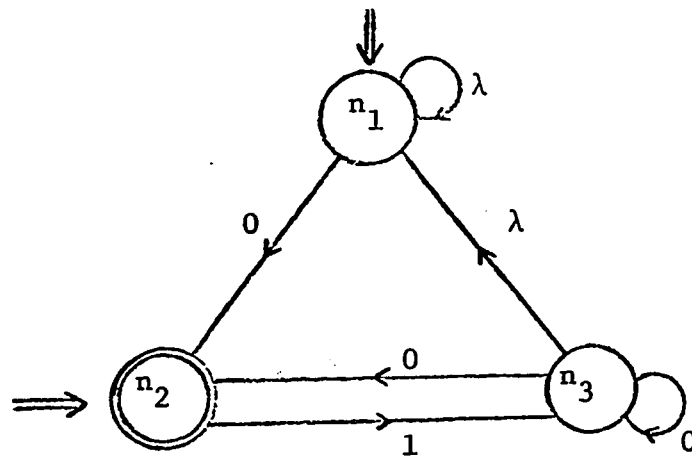


Fig. 2.2

For any path p , define $p.\epsilon = \epsilon.p = p$.

Denote the set of all paths of G by $P(G)$.

Let $P_1, P_2 \subseteq P(G)$. Define $P_1.P_2^\dagger$, the concatenation of P_1 by P_2 , by:

$$P_1.P_2 = \{p \mid p \in P(G) \wedge p = p_1.p_2 \text{ for some } p_1 \in P_1, p_2 \in P_2\}$$

It should be noted that $P_1.P_2$ may be empty even if both P_1 and P_2 are non-empty sets.

Let $p = b_1 b_2 \dots b_k \in P(G)$, $k > 0$. Then any path $p' = b_i b_{i+1} \dots b_j$, $1 \leq i \leq j \leq k$, is called a subpath of p . If $j = k$ then p' is a suffix of p , and if $i = 1$ then p' is a prefix of p .

Definition 2.15: Let $w = a_1 a_2 \dots a_m$ be a word over the alphabet A , and let $p = b_1 b_2 \dots b_k$ be a path in a transition graph $G = (N, B_L, N_1, N_2)$ (over A). We say that w is spelled out by p iff $k \geq m$ and there exists a sequence $1 \leq j_1 < j_2 < \dots < j_m \leq k$ such that for all $i = 1, \dots, m$, b_{j_i} is a branch labelled by a_i , and all other branches b_t , $t \neq j_1, \dots, j_m$, are labelled by λ . The empty word $w = \lambda$ is spelled out by all paths all of whose branches are labelled by λ , and by the null path ϵ .

Definition 2.16: A path p in $G = (N, B_L, N_1, N_2)$ is an admissible path iff the initial node of p belongs to N_1 and the final node of p is in N_2 , i.e. p is a path from an initial node of G to a final node of G .

Definition 2.17: Let $G = (N, B_L, N_1, N_2)$ be a transition graph. Define for every node $n \in N$, the accepted event, $A_c^G(n)^{\dagger\dagger}$, of n in G , as the

† Usually the dot will be omitted.

†† When no confusion arises, the subscript G will be omitted.

set of all words w spelled out by at least one path p from n to a terminal node of G . The preceding event $P_r^G(n)^\dagger$ of node n in G , is the set of all words w spelled out by at least one path from an initial node of G to n .

The event $R(G)$ accepted (recognized, or defined) by the transition graph G is defined by:

$$R(G) = \bigcup_{n \in N_1} A_c^G(n) = \bigcup_{n \in N_2} P_r^G(n)$$

Thus $R(G)$ is the set of all words $w \in A^*$ spelled out by at least one admissible path of G .

Example 2.3: Consider again the transition graph in Fig. 2.2. In this case $R(G) = P_r^G(n_2)$, thus $R(G)$ consists of all words $w \in (0 \cup 1)^*$ spelled out by at least one path from either n_1 or n_2 to n_2 . For instance, $w = 010 \in R(G)$ since w is spelled out by the admissible path $p = (n_1, 0, n_2)(n_2, 1, n_3)(n_3, \lambda, n_1)(n_1, 0, n_2)$.

Remark 2.1: Sometimes sets of paths in a transition graph (or a digraph) will be treated as events over the alphabet of branches B_L (or B resp.). Thus the regular operators $\cup, \cdot, *$ defined for events, will be used for sets of paths as well. For instance, in the last example, the expression $(n_1, \lambda, n_1) * (n_1, 0, n_2)(n_2, 1, n_3)[(n_3, \lambda, n_1)(n_1, \lambda, n_1) * (n_1, 0, n_2) \cup (n_3, 0, n_2)]$ indicates the set of all admissible paths in G spelling out the word 010.

The following proposition is a direct consequence of the above definitions.

[†] When no confusion arises, the superscript G will be omitted.

Proposition 2.4: Let $G = (N, B_L, N_1, N_2)$ and let $(m, \lambda, n) \in B_L$. Then:

$$(1) A_c^G(n) \subseteq A_c^G(m)$$

$$(2) P_r^G(m) \subseteq P_r^G(n)$$

Definition 2.18: A node n in a transition graph G is called admissible [Mc 3] iff there exists in G an admissible path touching n . A branch b in G is admissible iff there is an admissible path in G one of whose branches is b . G is called an all-admissible transition graph iff all branches and nodes of G are admissible.

Clearly, all inadmissible branches and nodes of a transition graph G can be removed from it without modifying the event accepted by G . In the sequel, all transition graphs dealt with will be assumed to be all-admissible.

Definition 2.19: Two transition graphs G and G' are called equivalent iff $R(G) = R(G')$.

Definition 2.20: A transition graph G without any λ -transition (i.e. no branch is labelled by λ) is called a non-deterministic state graph.

A non-deterministic state graph $G = (N, B_L, N_1, N_2)$ with $\#N_1 = 1$ and with the property that for every $n \in N$ and for every $a \in A$, $\#(\sigma_a^G(n)) \leq 1$, is called an incomplete state graph, and if for each $n \in N$, $a \in A$, $\#(\sigma_a^G(n)) = 1$, then G is a deterministic state graph. The nodes of a state graph are usually called states.

Definition 2.21: Let $\mathcal{A} = (Q, M, Q_1, F)$ be a non-deterministic automaton (over alphabet A). Define a non-deterministic state graph, $G_{\mathcal{A}}$, called

the state graph of $\mathcal{A}$, as follows:

$$G_{\mathcal{A}} = (Q, B_L, Q_1, F)$$

where

$$B_L = \{(q, a, q') \in Q \times A \times Q \mid q' \in M(q, a)\}$$

Thus for every transition in $\mathcal{A}$ from a state q to a state q' by input a , there corresponds in $G_{\mathcal{A}}$ a branch (q, a, q') .

Obviously the correspondence between an automaton $\mathcal{A}$ and its state graph $G_{\mathcal{A}}$ is 1 - 1; in fact a state graph can be considered as merely a convenient way of describing a finite automaton. Clearly the automaton is deterministic (incomplete) iff its corresponding state graph is deterministic (incomplete). Moreover, the event $T(\mathcal{A})$ accepted by $\mathcal{A}$ coincides with the event $R(G_{\mathcal{A}})$.

Notation 2.2: If $\mathcal{A} = \mathcal{A}_0(R)$ is the reduced deterministic automaton recognizing R , then $G_{\mathcal{A}}$ will be denoted by $G_0(R)$ and will be called the reduced deterministic state graph recognizing R . In a similar way, the minimal incomplete state graph $\tilde{G}_0(R)$ recognizing R will be defined by:

$$\tilde{G}_0(R) = G_{\tilde{\mathcal{A}}_0(R)} \text{ (Definition 1.14).}$$

Remark: For convenience, $G_0(R)(\tilde{G}_0(R))$ will sometimes be identified with $\mathcal{A}_0(R)(\tilde{\mathcal{A}}_0(R))$, and automaton notation will be used for it rather than graph notation.

Theorem 2.1([RS], [OF]): For every transition graph G there can be constructed a deterministic state graph equivalent to G .

Corollary 2.1: (a) The event recognized by any transition graph is regular. (b) The accepted events and the preceding events of the nodes in a transition graph are regular.

Notation 2.3: Let $\mathcal{A} = (Q, M, q_1, F)$ where $Q = \{q_1, q_2, \dots, q_n\}$ and let $G = G_{\mathcal{A}}$. Denote by $P_i(G)$, $i = 1, \dots, n$, the set of all paths in G with initial node q_i .

Definition 2.22: Let $G = (N, B_L, N_1, N_2)$ be any transition graph. Associate with G a digraph $D_G = (N, B)$, where B is the set of all pairs $(n_1, n_2) \in N \times N$ such that $(n_1, x, n_2) \in B_L$ for some $x \in A \cup \{\lambda\}$. D_G is called the digraph associated with G .

Definition 2.23: The (cycle) rank, $r(G)$, of a transition graph $G = (N, B_L, N_1, N_2)$ is defined to be the rank of the digraph D_G associated with G . The rank $r(\mathcal{A})$ of an automaton $\mathcal{A}$ is the rank of its state graph $G_{\mathcal{A}}$.

Definition 2.24: Let $G = (N, B_L, N_1, N_2)$ and let $N' \subseteq N$. Define

$$G - [N'] = (N - N', B_L - B'_L, N_1 - N', N_2 - N')$$

where

$$B'_L = \{(m, x, n) \mid x \in A \cup \{\lambda\}, m \in N' \text{ or } n \in N'\}$$

Definition 2.25: A transition graph $G' = (N', B'_L, N'_1, N'_2)$ is called a subgraph of $G = (N, B_L, N_1, N_2)$ iff $N' \subseteq N$ and $G' = G - [N - N']$.

Definition 2.26: A transition graph $G = (N, B_L, N_1, N_2)$ is strongly connected (s.c.) iff for every $m, n \in N$ there exists in G a path from m to n . A section of G is any maximal s.c. subgraph of G .

Definition 2.27: Let $G = (N, B_L, N_1, N_2)$ be a transition graph. Define the dual of G , $\overleftarrow{G}$, to be: $\overleftarrow{G} = (N, \overleftarrow{B}_L, N_2, N_1)$, where

$$\overleftarrow{B}_L = \{(m, x, n) \mid (n, x, m) \in B_L\}$$

Clearly $\overleftarrow{\overleftarrow{G}} = G$. The next lemma follows directly from the definitions:

Lemma 2.3: For every transition graph $G = (N, B_L, N_1, N_2)$,

$$(a) \quad r(G) = r(\overleftarrow{G})$$

$$(b) \quad R(G) = \overleftarrow{R(\overleftarrow{G})}$$

$$(c) \quad A_c^G(n) = \overleftarrow{P_r^{\overleftarrow{G}}(n)} \quad , \quad \text{for all } n \in N$$

$$(d) \quad P_r^G(n) = \overleftarrow{A_c^{\overleftarrow{G}}(n)} \quad , \quad \text{for all } n \in N$$

2.3. Homomorphisms of Transition Graphs

The next definitions as well as Theorem 2.2 are due to McNaughton and appear in [Mc 4].

Definition 2.28: A homomorphism from a transition graph $G = (N, B_L, N_1, N_2)$ onto a transition graph $G' = (N', B'_L, N'_1, N'_2)$ is a mapping $\phi: N \cup B_L \xrightarrow{\text{onto}} N' \cup B'_L$, which satisfies the following conditions:

$$(a) \quad \text{for every } n \in N, \phi(n) \in N'.$$

$$(b) \quad \text{for every } (m, x, n) \in B_L, \text{ either}$$

$$\phi(m, x, n) = (\phi(m), x, \phi(n)) \in B'_L, \quad \text{or}$$

$$\phi(m, x, n) = \phi(m) = \phi(n) \in N'$$

$G' = \phi(G)$ is called a homomorphic image of G . The homomorphism ϕ is called full iff for every $(m, x, n) \in B_L$, $\phi(m, x, n) = (\phi(m), x, \phi(n))$ (i.e. no branch of G is mapped onto a node of G'). If ϕ is a full 1-1 homomorphism and if $\phi(N_i) = N'_i$ for $i = 1, 2$, then ϕ is called an isomorphism.

Let ϕ be a homomorphism from G onto G' as above. Extend ϕ to a mapping from $P(G)$ into $P(G')$ as follows: for every path $p = b_1 b_2 \dots b_k$ in G ($b_i \in B_L$), let $\phi(p) = p' = \phi(b_{i_1}) \phi(b_{i_2}) \dots \phi(b_{i_r})$, where $1 \leq i_1 < i_2 < \dots < i_r \leq k$, for each t , $1 \leq t \leq r$, $\phi(b_{i_t})$ is a branch in G' and for all j , $1 \leq j \leq k$ and $j \neq i_t$ for $t = 1, \dots, r$, $\phi(b_j)$ is a node in G' . Clearly $\phi(p)$ is a path in G' . Thus for every path in G there corresponds uniquely a path in G' . The converse, however, is not always true.

Definition 2.29: A homomorphism ϕ from G to G' is a pathwise homomorphism iff for every path p' in G' there exists a path p in G such that $\phi(p) = p'$.

Theorem 2.2 (McNaughton's Pathwise Homomorphism Theorem): If there exists a pathwise homomorphism from G onto G' , then $r(G') \leq r(G)$.

Next we consider two types of homomorphisms of a transition graph G which leave the event accepted by G unchanged.

Theorem 2.3: Let ϕ be a full homomorphism from $G = (N, B_L, N_1, N_2)$ onto $G' = (N', B'_L, N'_1, N'_2)$, such that $\phi(N_i) = N'_i$, $i = 1, 2$, and such that for all $m, n \in N$, $\phi(m) = \phi(n)$ implies $A_c^G(m) = A_c^G(n)$ [alternatively, for all $m, n \in N$, $\phi(m) = \phi(n)$ implies $P_r^G(m) = P_r^G(n)$]. Then G' is equivalent to G .

Proof: Every homomorphism ϕ of the above type can be decomposed into

a product $\phi_k \phi_{k-1} \dots \phi_2 \phi_1$ of full homomorphisms such that each ϕ_i ,

$1 \leq i \leq k$, merges a pair of nodes n_i, n'_i of $G_{i-1} = \phi_{i-1} \phi_{i-2} \dots \phi_1(G)$

(i.e. maps them onto some new node m_i) for which

$A_c^{G_{i-1}}(n_i) = A_c^{G_{i-1}}(n'_i)$, maps all other nodes of G_{i-1} onto themselves

and each branch (n, x, n') of G_{i-1} onto the corresponding branch

$(\phi_i(n), x, \phi_i(n'))$ in G_i . Now it can be easily verified that for every

$i, 1 \leq i \leq k$, and for every node n of G_{i-1} , $A_c^{G_{i-1}}(n) = A_c^{G_i}(\phi_i(n))$. It

follows that G_i is equivalent to G_{i-1} for all $i = 1, \dots, k$ and hence

$G' = \phi(G) = \phi_k \phi_{k-1} \dots \phi_1(G) = G_k$ is equivalent to G .

By replacing in the above proof each accepted event $A_c^{G_i}(n)$ ($i = 1, \dots, k$) by the corresponding preceding event $P_r^{G_i}(n)$, one can obtain a proof for the second part of the theorem.

==

Theorem 2.4: Let ϕ be a pathwise homomorphism from $G = (N, B_L, N_1, N_2)$ onto

$G' = (N', B'_L, N'_1, N'_2)$ such that (1) for each branch $(m, x, n) \in B_L$,

$\phi(m, x, n) \in N'$ implies $x = \lambda$, and (2) for all $n', n'' \in N$, $\phi(n') = \phi(n'')$

implies $n' \in N_i$ iff $n'' \in N_i$, $i = 1, 2$. Then G' is equivalent to G .

Proof: Let w be any word accepted by G . Then there exists a path

$p = (n_1, x_1, n_2) \dots (n_k, x_k, n_{k+1})$ admissible in G and spelling out w .

Now if $x_i \neq \lambda$ then by assumption, $\phi(n_i, x_i, n_{i+1}) = (\phi(n_i), x_i, \phi(n_{i+1}))$.

Thus the path $p' = \phi(p)$ also spells out the same word w . Moreover,

because of condition (2) of the theorem, p' is also an admissible

path in G' and it follows that G' accepts w .

Now let w' be any word spelled out by some admissible path p' in G' . Then since ϕ is a pathwise homomorphism, there exists a path p in G such that $\phi(p) = p'$. By the same argument as above, and condition (2) of the theorem, p spells out the same word w and is an admissible path in G . Hence w' is accepted by G as well. It follows that G and G' are equivalent.

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Chapter 3

The Star Height Problem

3.1. The Star Height of a Regular Event

Definition 3.1: The apparent star height h_α of a regular expression E is defined inductively:

$$h_\alpha(a) = 0 \text{ for } a \in A, \quad h_\alpha(\lambda) = h_\alpha(\phi) = 0$$

$$h_\alpha(E_1 \cup E_2) = \max \{h_\alpha(E_1), h_\alpha(E_2)\}$$

$$h_\alpha(E_1 E_2) = \max \{h_\alpha(E_1), h_\alpha(E_2)\}$$

$$h_\alpha(E^*) = h_\alpha(E) + 1$$

Definition 3.2: The star height $h(R)$ of a regular event R is defined by

$$h(R) = \min \{h_\alpha(E) \mid E \text{ is a regular expression and } |E| = R\}$$

Thus for any regular expression E , $h_\alpha(E)$ is the maximum length of a sequence of stars in the expression E , such that each star is in the scope of the star that follows it. $h(E)$, however, indicates the star height of the event $|E|$ denoted by E (see Notation 1.1) as defined above.

The next proposition is obvious.

Proposition 3.1: Let R be a regular event. Then $h(R) \geq 1$ iff R is infinite.

Example 3.1: Let $E = (10^*1)^*$. Then $h_\alpha(E) = 2$ is the apparent star height of E . However, $h(E) = 1$ because $E_1 = \lambda \cup 1(0 \cup 11)^*1$ is an expression

equivalent to E , i.e. $|E_1| = |E|$.

Example 3.2: Let $E = (0 \cup 10^*1)^*$. Here again $h_\alpha(E) = 2$. Moreover, this event can be shown to be of star height 2 by using Theorem 4.1 of Chapter 4 (the reduced automaton $\mathcal{A}_0(R)$, $R = |E|$, is shown in Fig. 3.1). We present here an outline of a relatively simple algebraic proof to the same effect.[†]

Suppose, by contradiction, that there exists a regular expression E' of apparent star height 1 such that $|E'| = |E|$. Let H_i^* , $i = 1, \dots, n$, $n \geq 1$, be the set of all star expressions appearing in E' . Then the roots H_i (Definition 1.16) of these star expressions must be finite events. Thus let

$$|H_i| = \bigcup_{j=1}^{k_i} w_{ij}, \quad i = 1, \dots, n$$

where each w_{ij} is a word. It can be easily seen that each w_{ij} must contain an even number of 1's (otherwise there would be in $|E'|$ words with an odd number of 1's). Let r be the maximal length of a sequence of consecutive 0's appearing in any w_{ij} , $i = 1, \dots, n$, $j = 1, \dots, k_i$, and let t be the total number of letters 0,1 appearing in E' outside the scope of any star operator. Then a word of the form $u_m = 0^m 10^m 1$, where $m > r$, cannot be a subword of any w_{ij} , nor of any product of words w_{ij} , i.e. u_m , $m > r$, is not a subword of any word in $|H_i^*|$, $i = 1, \dots, n$. It follows that a word $(u_m)^k$, where $m > r$ and $k > n + t$ cannot be denoted by the regular expression E' . But this contradicts the fact, that $(u_m)^k \in |E'|$ for all integers m, k .

[†] Unfortunately, such algebraic proofs can be obtained only for some special cases ($[E]$, $[DS]$) and do not seem to be extensible to general families of events.

3.2. A Survey of Previous Results

The notion of star height of regular events was first introduced by Eggen in [E]. In that paper it is established that there exist regular events of arbitrarily large star height. This result is proved by exhibiting for each integer $k > 0$, an event R_k of star height k . Eggen's proof is purely algebraic and quite cumbersome. Moreover, the number of letters in the alphabet A_k over which R_k is constructed grows together with k . Thus Eggen has raised the question of whether or not there exist events of arbitrarily large star height over the two-letter alphabet.[†] In [DS] Dejean and Schutzenberger have come up with a positive answer to this question, constructing for each $k > 0$, an event F_k over the alphabet $\{0,1\}$ of star height k . F_k ($k = 1, 2, \dots$) is defined to be the event accepted by the automaton $\mathcal{A}_k = (Q, M, q_0, \{q_0\})$, where $Q = \{q_0, q_1, \dots, q_{2^k-1}\}$ and the transition function M is defined by:

$$M(q_i, 0) = q_{i+1 \pmod{2^k}} \quad , \quad i = 0, \dots, 2^k - 1$$

$$M(q_i, 1) = q_{i-1 \pmod{2^k}} \quad , \quad i = 0, \dots, 2^k - 1.$$

Thus F_k is the set of all sequences of 0's and 1's, in which the difference between the number of occurrences of 0 and the number of occurrences of 1 is a multiple of 2^k .

The second part of Eggen's paper introduces the important notion of (cycle) rank of digraphs and the following fundamental theorem is established.

[†] It is not difficult to show, that a regular event over the single-letter alphabet is of star height at most 1.

Theorem 3.1: The star height of a regular event R does not exceed the rank of any transition graph recognizing R .

Theorem 3.1 is proved by constructing, for every transition graph G , a regular expression of apparent star height $r(G)$ denoting the event $R(G)$ recognized by G .

Thus for any regular event R , $r(G_0(R))$ is an upper bound to the star height of R . Unfortunately, there exist many cases in which $h(R) < r(G_0(R))$. For instance, let $R = (0 \cup 1)^*1$ over the alphabet $\{0,1\}$. Then the reduced state graph (shown in Fig. 3.2) has rank 2, whereas R is clearly of star height 1. In the next section it is shown that $r(G_0(R)) - h(R)$ can be arbitrarily large (Corollary 3.4). Thus examining merely the reduced state graph of R does not generally suffice for determining the star height of R . Eggan has shown, however, that the star height of R can be determined by studying the family of all transition graphs recognizing R . This result follows from the next lemma.

Lemma 3.1: For any regular event R there exists a transition graph G recognizing R such that $r(G) = h(R)$.

Lemma 3.1 is proved by constructing, for any given regular expression E , a transition graph G_E with a single initial node and a single terminal node, accepting the event $|E|$ and having rank $r(G_E) = h_\alpha(E)$. The construction is based on the method of Ott and Feinstein [OT], and induction on the shape of the regular expression E is used.

- (1) If $E = \phi$, λ or $E = a$, $a \in A$, then the problem is trivial.

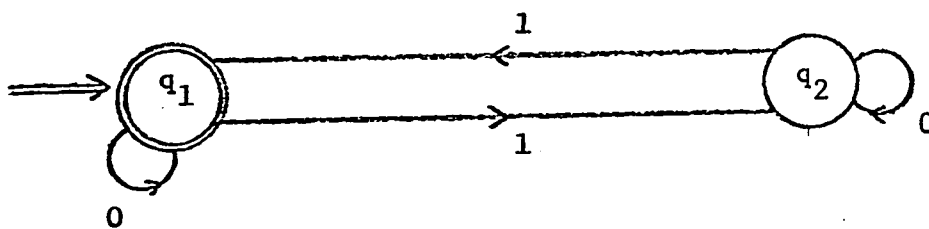


Fig. 3.1

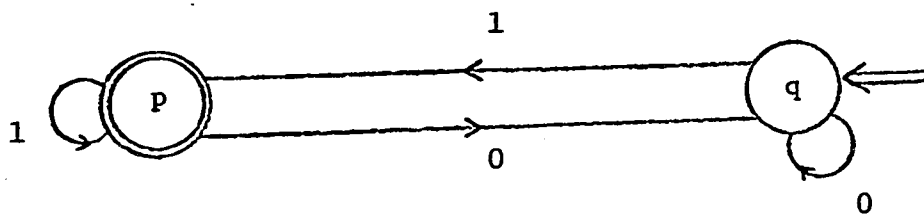


Fig. 3.2

- (2) If there is a transition graph G_E recognizing $|E|$, then the transition graph G_{E^*} recognizing E^* is shown in Fig. 3.3(a).[†]
- (3) If transition graphs G_{E_1} and G_{E_2} recognizing $|E_1|$ and $|E_2|$ resp. have been constructed, the transition graph $G_{E_1 \cup E_2}$ recognizing $|E_1 \cup E_2|$ is obtained as shown in Fig. 3.3.(b).
- (4) The transition graph $G_{E_1 E_2}$ recognizing $|E_1 E_2|$ is constructed from G_{E_1} and G_{E_2} as shown in Fig. 3.3.(c).

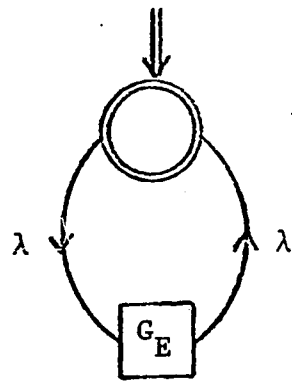
One can see, that the only time a loop is created in G_E is when one encounters a star in E . Hence $r(G_E) = h_\alpha(E)$ and Lemma 3.1 is proved. Combining this lemma with Theorem 3.1, Eggen's Star Height Theorem is achieved.

Eggen's Star Height Theorem: The star height of a regular event R equals $\min_{G \in \mathcal{F}_R} \{r(G)\}$, where $\mathcal{F}_R$ is the family of all transition graphs recognizing R .

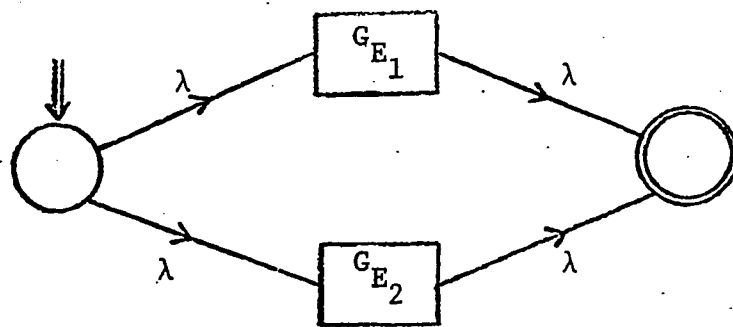
It will be shown in Chapter 5 that a proper subfamily of $\mathcal{F}_R$ is sufficient for determining $h(R)$.

A new technique for establishing the star height of regular events, based on Eggen's Theorem, has been introduced by McNaughton in [Mc 3]. The following fundamental theorem constitutes the essence of the graph-theoretic methods developed in that paper.

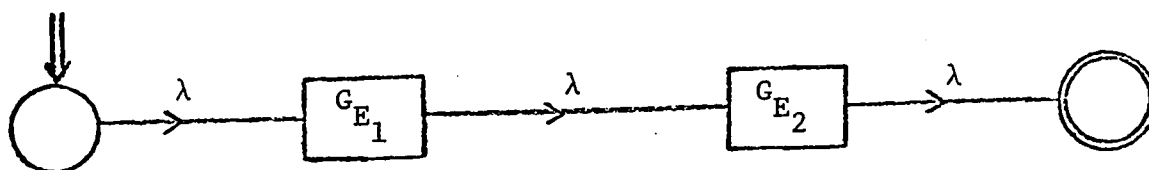
[†] In Fig. 3.3, a λ -transition entering (leaving) the square representing G_E should be interpreted as a λ -transition entering (leaving) the initial (terminal) node of G_E .



(a)



(b)



(c)

Fig. 3.3

Theorem 3.2: If $S_1, \dots, S_m$ are strongly connected subgraphs of a strongly connected subgraph of a transition graph G , and if there is no node common to more than $m - 1$ of the S_i 's then the rank of G is at least one more than the minimum rank of all the S_i 's.

McNaughton's technique will now be illustrated by the following example, which appears in [Mc 3].

Example 3.4: Let $R = (0^*10^*1)^*$ and let $G = (N, B_L, N_1, N_2)$ be an arbitrary transition graph recognizing R . Let N' (N'') be the set of all nodes n in G such that there exists a path to a terminal node from n spelling out a word with an odd number of 1's (an even number of 1's), and any number of 0's interspersed. Clearly N' and N'' are disjoint and so are the corresponding subgraphs $G' = G - [N - N']$ and $G'' = G - [N - N'']$. Now suppose G has k nodes. Then any path starting at a node of N' (N'') and spelling out the word 0^{k+1} must be contained in G' (G'') and must repeat a node. Hence G' (G'') has at least one section. Thus let $A_1, \dots, A_m$ and $B_1, \dots, B_n$ be all sections of G' and G'' respectively. The word $w = (0^{k+1}10^{k+1}1)^{k+1}$ is contained in R . Let p be an admissible path in G spelling out w . p must intersect some A_i ($1 \leq i \leq m$) more than once and between times some B_j ($1 \leq j \leq n$). A_i and B_j are disjoint, each is s.c. and there is a path from each to the other, i.e. both are contained in a s.c. subgraph of G . It follows by Theorem 3.2 that G is of rank ≥ 2 . Since G was arbitrarily chosen, the star height of R cannot be less than 2 by Eggen's Theorem. Hence $h(R) = 2$.

McNaughton's proof techniques are applicable to a great variety of events. The examples of events F_k of star height k constructed in [DS] are re-established in [Mc 3] by means of these methods, as well as a new

way of creating events of arbitrarily large star height, as indicated by the next theorem.

Theorem 3.3: Let R_1 and R_2 be regular events of star height h whose respective alphabets A_1 and A_2 are disjoint, and suppose neither a nor b is in $A_1 \cup A_2$. Then the regular event $R = (aR_1bR_2)^*$ over the alphabet $A_1 \cup A_2 \cup \{a,b\}$ has star height $h + 1$.

Another significant result in [Mc 3] is that for any alphabet $A = \{a_1, \dots, a_n\}$, there exists a homomorphism[†] $\phi: A^* \rightarrow (0 \cup 1)^*$ which preserves star height, i.e. for any $R \subseteq A^*$, $h(\phi(R)) = h(R)$. ϕ is determined by the n equations

$$\phi(a_i) = 10^i 10^{n-i+1} 1, \quad i = 1, \dots, n$$

Thus every event R over the n -letter alphabet can be coded down to an event $\phi(R)$ over the 2-letter alphabet having the same star height.

In the last section of [Mc 3], techniques for reducing the apparent star height of a given regular expression are described. A few isolated principles are obtained (for instance, $(\alpha\beta^*\gamma)^* = \lambda \cup \alpha(\beta \cup \gamma\alpha)^*\gamma$ for all regular events α , β and γ) and a number of illuminating examples are presented, in which such reduction is obtained by looking at the reduced state graph of the event.

Another method for reducing the apparent star height of a given regular expression E is discussed in [Mc 2]. This reduction technique involves a series of transformations on a transition graph G_E representing

[†] An ordinary homomorphism between two free monoids, which is extended to subsets of the monoids in the natural way.

E , leading to an equivalent transition graph G' of smaller rank, from which a regular expression E' of smaller apparent star height can be derived. However, these transformations are carried out on an intuitive basis and are helpful mainly when the regular expression E' is known beforehand. In Chapter 5 of this thesis a similar approach has been adopted, several basic types of rank-non-increasing transformations on transition graphs have been introduced, by means of which any arbitrary transition graph can be converted into an equivalent one with some special properties. The author strongly believes that a more inclusive and powerful set of transformations on transition graphs could be defined, by means of which any arbitrary transition graph recognizing the event R can be modified into an equivalent one of no greater rank, whose number of nodes is bounded by some fixed number k_R , which is a function of the number of states in the reduced automaton $A_0(R)$. This would yield a finite algorithm for finding the star height of any arbitrary regular event R . Such an algorithm, to the best of the author's knowledge, has not yet been established.

There has been found, however, an algorithm for determining the star height of the members of some subfamily of the family of regular events, namely the family of all pure-group events (Definition 1.21). This algorithm is presented in McNaughton's paper [Mc 4]. The algorithm involves constructing a certain finite family of transition graphs, called μ -graphs, for the given pure-group event R . These μ -graphs are derived in a certain manner from the syntactic monoid of R . It is shown that if $h(R) = k$ then there exists a subset of this family of μ -graphs whose

union graph[†] has rank k and recognizes the event R . Thus in order to verify that the star height of R is $\leq i$ ($i = 1, 2, \dots$), one constructs the union graph of all μ -graphs of rank $\leq i$ and checks whether this graph accepts the whole event R .

Another result in [Mc 4] is the following.

Theorem 3.4: If the reduced state graph $G_0(R)$ of a pure-group event R has exactly one terminal node, then the star height of R equals the rank of $G_0(R)$.

This theorem will be obtained in Chapter 4 as a special case of a more general result (Theorem 4.1) which is proved in a totally different way. Note that the class of events F_k of star height k mentioned before [DS] happens to be a special case of Theorem 3.4.

In [D] Dejean proves, that all pure-group events of star height 1 over an alphabet A are of the form:

$$(A^k) * (A^{i_1} \cup A^{i_2} \cup \dots \cup A^{i_n})$$

where for each j , $j = 1, \dots, n$, $0 \leq i_j \leq k - 1$.

3.3 Some General Properties of Star Height

In this section the effect of various operations on the star height of regular events is examined.

We first state some results, due to Brzozowski, showing that star height is preserved by quotients.

[†] By a union graph of several graphs is meant the result of considering these graphs together as a single graph. A node is initial, or terminal, iff it was so before.

Theorem 3.5: Let R be a regular event and let $w \in A^*$. Then $h(w \setminus R) \leq h(R)$. Furthermore, for all integers $n \geq 0$, there exists a word x of length n such that $h(x \setminus R) = h(R)$.

Clearly Theorem 3.5 holds for right quotients as well, and since $x \setminus R / y = (x \setminus R) / y = x \setminus (R / y)$ for all events R and $x, y \in A^*$, we have:

Corollary 3.1: For any regular event R and for any $x, y \in A^*$, $h(x \setminus R / y) \leq h(R)$.

Corollary 3.2: Let R be a regular event and let $G = G_0(R)$. Then if G is strongly connected, $h(x \setminus R) = h(R)$ for all $x \in A^*$.

Next we examine the following problem. Let T_1 and T_2 be regular events and let R be an event obtained from T_1, T_2 by a finite number of applications of the operators $\cup, \cap, \bar{}, \cdot$ and $*$. How is the star height of R related to those of T_1 and T_2 ? - The answer is rather disappointing because in most cases very little connection appears to exist between $h(T_1), h(T_2)$ and $h(R)$. We now proceed to examine specific cases.

Proposition 3.2: For any regular events T_1, T_2 , $h(T_1 \cup T_2) \leq \max \{h(T_1), h(T_2)\}$. Moreover, for every integer $n \geq 0$ there exist events T_1 and T_2 such that $h(T_1) = n$ and $h(T_1 \cup T_2) = 1$.

Proof: The first assertion is trivial. As for the second, simply take any event T_1 of star height n and $T_2 = \overline{T_1}$. Then $h(T_1 \cup T_2) = h(A^*) = 1$.

Proposition 3.3: For any regular events T_1, T_2 , $h(T_1 T_2) \leq \max \{h(T_1), h(T_2)\}$, and for any integer $n \geq 0$ there exist T_1, T_2 such that $h(T_1) = n$ and

$$h(T_1 T_2) = 1.$$

Proof: The first assertion is obvious. As for the second, let T_1 be any event of star height n containing the empty word λ and let $T_2 = A^*$.

$$\text{Then } h(T_1 T_2) = 1.$$

Proposition 3.4: Let T be a regular event. Then $h(T^*) \leq h(T) + 1$ and for any integer $n \geq 0$ there exists an event T of star height n such that $h(T^*) = 1$.

Proof: Let $A = \{0,1\}$ and let $T = 0T_1 \cup 1$, where $h(T_1) = n$ and $\lambda \in T_1$. Now $0 \setminus T = T_1$ showing that $h(T) = h(T_1) = n$ (Theorem 3.5). However, $T^* = (0 \cup 1)^*$ is of star height 1.

Proposition 3.5: For every integer $m \geq 0$, there exists a regular event R such that $h(R) - h(\bar{R}) \geq m$.

Proof: Let $A = \{a_1, a_2, \dots, a_n\}$ be the alphabet, where $n = m + 2 \geq 2$, and let $\mathcal{A}_i = (Q_i, M_i, q_1^i, \{q_1^i\})$, $i = 1, 2, \dots, n$, be the reduced deterministic automaton recognizing the event R_i , consisting of all words over A with an even number of occurrences of the letter a_i (see Example 3.2). Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n = (Q, M, q_1, F)$ (Definition 1.22), where $Q = Q_1 \times \dots \times Q_n$, $q_1 = (q_1^1, \dots, q_1^n)$ and $F = \{q_1\}$. Since each $\mathcal{A}_i$ is a pure-group automaton, so is $\mathcal{A}$ (Theorem 1.14) and by Theorem 3.4, $h(R) = r(G_{\mathcal{A}})$, where R is the event recognized by $\mathcal{A}$. If we show that $r(G_{\mathcal{A}}) \geq n$ then we have $h(R) \geq n$. Furthermore, since $R = R_1 \cap R_2 \cap \dots \cap R_n$, by DeMorgan's law, $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \dots \cup \bar{R}_n$, and since each $\bar{R}_i$ is of star height 2 (again by Theorem 3.4), we get $h(\bar{R}) \leq 2$. Hence $h(R) - h(\bar{R}) \geq n - 2 = m$ as desired.

Thus it remains to show that $r(G_{\mathcal{A}}) \geq n$. We shall prove this by induction on n . For $n = 2$, $\mathcal{A}_1 \times \mathcal{A}_2$ is shown in Fig. 3.4 and apparently has rank 2. Now suppose the assertion holds for $n - 1$ and let $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$. Then for any node $q = (q^1, \dots, q^n)$ of $G_{\mathcal{A}}$, the graph $G_{\mathcal{A}} - [q]$ certainly contains the subgraph $G_{\mathcal{A}} - [N']$, where

$$N' = \{(p_1, \dots, p_n) \mid p_i \in Q_i, i = 1, \dots, n, \text{ and } p_n = q^n\}$$

But since Q_n (as well as the other Q_i 's) consists of only two states, this graph $G_{\mathcal{A}} - [N']$ can be easily seen to be isomorphic with

$G_{\mathcal{A}_1} \times \dots \times G_{\mathcal{A}_{n-1}}$. Hence by the induction hypothesis and Lemma 2.1,

$r(G_{\mathcal{A}} - [q]) \geq r(G_{\mathcal{A}} - [N']) \geq n - 1$ for any state q of $\mathcal{A}$, and therefore $r(G_{\mathcal{A}}) \geq n$. This completes the proof.

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Corollary 3.3: For every integer $n \geq 0$, there exist regular events

$R_1, R_2, \dots, R_n$ such that

$$h(R_1 \cap R_2 \cap \dots \cap R_n) - \max_{1 \leq i \leq n} \{h(R_i)\} \geq n - 2.$$

Corollary 3.4: For any integer $k > 0$ there exists a regular event R such that

$$r(G_0(R)) - h(R) > k$$

Proof: The event $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \dots \cup \bar{R}_n$ defined in the proof of Proposition 3.5 can be easily shown to be accepted by the automaton $\bar{\mathcal{A}}$ obtained from $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ by interchanging the set F of final states with the

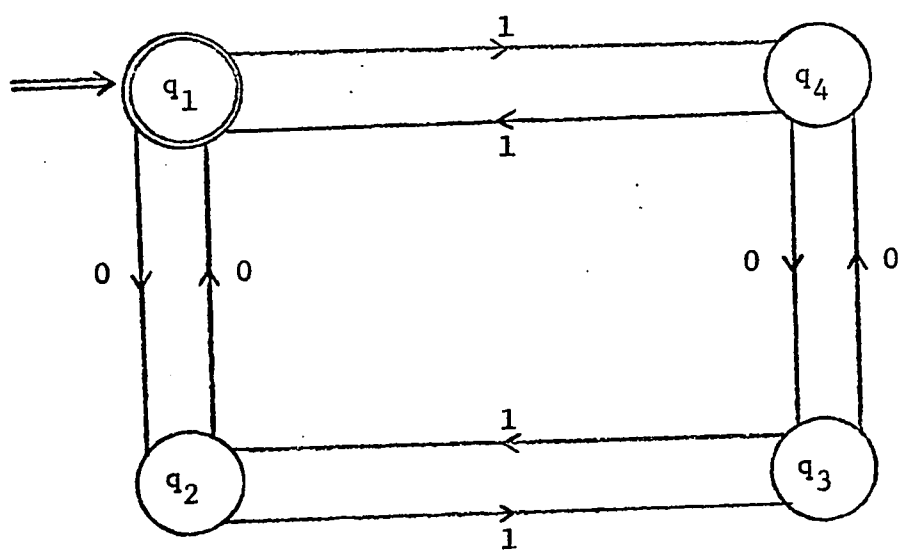


Fig. 3.4.

set $Q = F$. Obviously $\bar{A}$ is also reduced and the result follows.

The next proposition is obvious.

Proposition 3.6: For every regular event R , $h(\bar{R}) = h(R)$.

The next two propositions are due to Brzozowski.

Proposition 3.7: Let R and T be two regular events such that $h(T) = 0$.

Then $h(R) = h(R - T) = h(R \cup T)$.

Proposition 3.8: Let A be an alphabet with at least two letters and let R be a regular event over A . If the automaton $\mathcal{A}_0(R)$ is strongly connected then for any $a \in A$,

$$h(R - aA^*) = h(R).$$

In Proposition 3.7 it was shown that removing from R an event T of star height 0 does not affect star height. The question naturally arises, whether the star height of R can be affected by subtracting from it any subevent $T \subseteq R$ of star height $h(T) < h(R)$.

Proposition 3.9: Let R and T be any regular events such that $T \subseteq R$ and $h(T) < h(R)$. Then $h(R - T) \geq h(R)$.

Proof: If $h(R - T) < h(R)$ then $R = (R - T) \cup T$ would imply $h(R) \leq \max \{h(R - T), h(T)\} < h(R)$.

We conjecture:

Conjecture 3.1:[†] Let R and T be as in Proposition 3.9. Then $h(R - T) = h(R)$.

We have been able to prove only the following special case of

[†] Recently R. McNaughton has disproved this conjecture by a counter example (see Appendix A).

Conjecture 3.1.

Theorem 3.6: Let R be a regular event of star height $k > 1$ such that for any two words $x, y \in A^*$, either $x \setminus R = y \setminus R$ or $x \setminus R \cap y \setminus R = \emptyset$. Then for every subset $T \subseteq R$ such that $h(T) < k$ and $r(G_0(T)) \leq k$, $h(R - T) = h(R)$.

Proof: By Proposition 3.9, $h(R - T) \geq h(R)$. Consider the reduced automaton $\mathcal{A} = \mathcal{A}_0(R) = (Q, M, q_1, F)$. As shown in Chapter 4, Theorem 4.1, $r(\mathcal{A}) = r(\tilde{\mathcal{A}}_0(R)) = k$. Let $\mathcal{A}' = \mathcal{A}_0(T) = (P, N, p_1, G)$ be the reduced deterministic automaton recognizing T , and let $\mathcal{B} = \mathcal{A} \times \mathcal{A}' = (Q \times P, M \times N, (q_1, p_1), F \times G)$ (Definition 1.22). Let $\mathcal{B}'$ be the automaton obtained from $\mathcal{B}$ by changing the set of final states from $F \times G$ to $F \times (P - G)$, and then removing all inaccessible states (Definition 1.11). Let the set of states of $\mathcal{B}'$ be S .

It can be easily verified that the automaton $\mathcal{B}'$ thus obtained accepts the event $R - T$. We claim that $r(\mathcal{B}') \leq k$. To see this, let $(q, p) \in S$ be any state of $\mathcal{B}'$. Then there exists a word w such that $(M \times N)((q_1, p_1), w) = (M(q_1, w), N(p_1, w)) = (q, p)$. Now since $T \subseteq R$, $wx \in T$ implies $wx \in R$ for any word x . Hence all words x taking p to a final state in $\mathcal{A}'$, also take q to a final state in $\mathcal{A}$, i.e. $A_c^{\mathcal{A}'}(p) \subseteq A_c^{\mathcal{A}}(q)$. But since for every two states q', q'' in Q , $A_c^{\mathcal{A}}(q') \cap A_c^{\mathcal{A}}(q'') = \emptyset$ (see Theorem 1.6 and Definition 1.13), it follows that for every state p in $\mathcal{A}'$ such that $A_c^{\mathcal{A}'}(p) \neq \emptyset$, there exists at most one state $q \in Q$ such that $(q, p) \in S$. The only state p in P which may appear in more than one pair (q, p) of S is the dead state $p_\emptyset$, if such a state exists in $\mathcal{A}'$. Thus decompose S into two disjoint subsets: $S = S' \cup S''$, where S' is the set

of all pairs $(q, p) \in S$ such that $p \neq p_\phi$, and S'' is the set of all pairs of the form (q, p_ϕ) appearing in S . We claim:

- (1) There does not exist in $\mathcal{B}'$ any transition from a state of S'' to a state of S' .
- (2) The subgraph $G_{\mathcal{B}'} - [S'']$ of $G_{\mathcal{B}'}$ is isomorphic with some subgraph of $G_{\mathcal{A}'}$.
- (3) The subgraph $G_{\mathcal{B}'} - [S']$ of $G_{\mathcal{B}'}$ is isomorphic with some subgraph of $G_{\mathcal{A}}$.

The first assertion follows from the fact, that for any word x , $N(p_\phi, x) = p_\phi$. As for (2), let $(q, p) \in S'$. Then for any $a \in A$, $(M \times N)((q, p), a) = (M(q, a), N(p, a))$. If $N(p, a) = p_\phi$ then this transition is not contained in $G_{\mathcal{B}'} - [S'']$. Thus suppose $N(p, a) \neq p_\phi$. Define a mapping $\phi: S' \rightarrow P - \{p_\phi\}$ by $\phi(q, p) = p$ for all $(q, p) \in S'$. Then by the above remark, ϕ is a 1 - 1 mapping. Moreover, if for some $(q, p) \in S'$ and $a \in A$, $(M \times N)((q, p), a) = (q', p')$, then by definition, $p' = N(p, a)$. Hence ϕ is an isomorphism of $G_{\mathcal{B}'} - [S'']$ onto some subgraph of $G_{\mathcal{A}'}$ as required.

To show (3), let $Q' = \{q \in Q \mid (q, p_\phi) \in S''\}$. Define a mapping $\psi: S'' \rightarrow Q'$ by $\psi(q, p_\phi) = q$ for all $(q, p_\phi) \in S''$. Obviously ψ is an isomorphism of $G_{\mathcal{B}'} - [S']$ onto the subgraph $G_{\mathcal{A}} - [Q - Q']$, which proves (3).

Now by (1), the set of states of any section in $G_{\mathcal{B}'}$ must be contained in either S' or S'' . Thus:

$$r(G_{\mathcal{B}'}) = \max \{r(G_{\mathcal{B}'} - [S']), r(G_{\mathcal{B}'} - [S''])\} \leq \max \{r(G_{\mathcal{A}}), r(G_{\mathcal{A}'})\} \leq k.$$

Hence by Theorem 3.1, $h(R - T) \leq k$. This completes the proof.

In connection with Conjecture 3.1 we would like to mention another open problem suggested by Brzozowski.

Let R be a regular event of star height ≥ 1 , and let T be a finite event. Is it always true that $h(TR) = h(R) = h(RT)$?

Although it seems rather plausible that this equality always holds, so far we have been unable to give it a formal proof.[†]

Definition 3.3: Let A and B be two finite alphabets. A mapping $\phi: 2^{A^*} \rightarrow 2^{B^*}$ is called a generalized homomorphism iff

- (1) $\phi(\Lambda) = \Lambda$
- (2) for any letter $a \in A$, $\phi(a)$ is a finite event over B .
- (3) for any word $w = a_1 a_2 \dots a_n \in A^*$,
$$\phi(w) = (\phi(a_1))(\phi(a_2)) \dots (\phi(a_n))$$
- (4) for any $W \subseteq A^*$, $\phi(W) = \bigcup_{w \in W} \phi(w)$

An event $R' \subseteq B^*$ is called a homomorphic image of an event $R \subseteq A^*$ iff there exists a generalized homomorphism ϕ from 2^{A^*} into 2^{B^*} such that $R' = \phi(R)$.

The following proposition can be easily verified.

Proposition 3.10: If R' is a homomorphic image of R then $h(R') \leq h(R)$.

[†] R. McNaughton has provided an example of events R, T with the above properties for which this equality does not hold (see Appendix B).

Chapter 4

Star Height of Certain Families of Events

Introduction

In this chapter the relationships between the apparent star height of a given regular expression and the structure of its reduced state graph are investigated.

Sections 4.1 and 4.2 introduce the notions of trails (i.e. state sequences) in a state graph, and simple regular expressions with respect to a given state graph. In section 4.3 the definition of cofinal sets of trails is given, and the Main Lemma, relating the apparent star height of a given regular expression with some properties of its corresponding set of trails in some state graph, is obtained. By means of the Main Lemma, in section 4.4 sufficient conditions are found for the star height of an event R to equal the rank of its reduced state graph $\tilde{G}_0(R)$. Thus families of events of arbitrarily large star height can be easily constructed. Moreover, the rank of $\tilde{G}_0(R)$ is shown to be a lower bound to the star height of certain subsets of R .

Section 4.5 deals with the class of events defined by digraphs. Each member of this class consists of all paths connecting two sets of nodes in some digraph D . It is shown that the star height of each such event R equals the rank of the smallest subgraph of D containing all paths of R . As a by-product, it is shown that every regular event R is a homomorphic image of a group-free event whose star height equals the rank

of $\hat{G}_0(R)$.

4.1. Trails in a State Graph

Throughout this section and the next two sections let G denote an incomplete state graph without a dead state (Definition 1.14) whose corresponding automaton is $\mathcal{A} = (Q, M, q_1, F)$, where $Q = \{q_1, q_2, \dots, q_n\}$.

Definition 4.1: A trail (state sequence) in G is any sequence of states, $t = (q_{i_1}, q_{i_2}, \dots, q_{i_m})$, where $q_{i_j} \in Q$, $m \geq 1$ and for each $1 \leq j \leq m-1$, there exists in G a transition from q_{i_j} to $q_{i_{j+1}}$. q_{i_1} is called the initial node of t and q_{i_m} is the terminal node. A subtrail of t is any trail $t' = (q_{i_k}, q_{i_{k+1}}, \dots, q_{i_\ell})$ where $1 \leq k \leq \ell \leq m$.

Given two trails, $t_1 = (q_{i_1}, q_{i_2}, \dots, q_{i_k})$ and $t_2 = (q_{i_k}, q_{i_{k+1}}, \dots, q_{i_\ell})$, where $1 \leq k \leq \ell$, define their concatenation:

$$t_1 \cdot t_2^\dagger = (q_{i_1}, q_{i_2}, \dots, q_{i_k}, q_{i_{k+1}}, \dots, q_{i_\ell}).$$

Let t be a trail of the form: $(q_{i_1}, q_{i_2}, \dots, q_{i_1})$. Then t is called a loop in G . Define: $t^0 = (q_{i_1})$, $t^1 = t$, and inductively, $t^k = t \cdot t^{k-1}$ for all $k = 1, 2, \dots$. A trail of unit length (q_i) is called a trivial trail or a trivial loop.

Denote the set of all trails in G by $T(G)$.

† The dot will sometimes be omitted.

Let $T, T' \subseteq T(G)$ such that all trails of T have a common terminal node, which also coincides with the common initial node of all trails of T' . Define the concatenation (or product)

$$T \cdot T' = \{t \cdot t' \mid t \in T, t' \in T'\}.$$

Let $L \subseteq T(G)$ be a set of loops in G from state q_i to itself. Define $L^0 = (q_i)$ and by induction: $L^k = L \cdot L^{k-1}$. The star operation is defined by $L^* = \bigcup_{k=0}^{\infty} L^k$.

Denote the set of all trails in $T(G)$ with initial node q_i by $T_i(G)$.

Definition 4.2: For each $i, 1 \leq i \leq n$, define the event

$$W_i(G) = \{w \in A^* \mid \#M(q_i, w) = 1\}$$

Thus $W_i(G)$ is the set of all input words w applicable to the automaton when started at state q_i .

The following properties of $W_i(G)$ can easily be verified:
for each $i, 1 \leq i \leq n$,

1. $W_i(G)$ is regular.
2. $\lambda \in W_i(G)$.
3. If $w \in W_i(G)$ then all prefixes of w also belong to $W_i(G)$.
4. If G is a deterministic state graph then $W_i(G) = A^*$.
5. Let R be the event recognized by G . Then $R \subseteq W_1(G)$.

Definition 4.3: For every $i, 1 \leq i \leq n$, define a mapping $\tau_i: W_i(G) \rightarrow T_i(G)$ as follows: $\tau_i(\lambda) = (q_i)$ and for every non- λ word $w = a_1 a_2 \dots a_k \in W_i(G)$,

$\tau_i(w) = (q_i, q_{j_1}, q_{j_2}, \dots, q_{j_k})$, where $q_{j_r} = M_i(a_1 a_2 \dots a_r)$ for all $1 \leq r \leq k$. Thus τ_i maps each word w of $W_i(G)$ into the trail of states the automaton passes through when started at state q_i and scanning w .

Extend the mapping τ_i to any event $V \subseteq W_i(G)$:

$$\tau_i(V) = \{\tau_i(w) \mid w \in V\} \subseteq T_i(G).$$

The following three lemmas are direct consequences of the above definitions:

Lemma 4.1: The mapping $\tau_i: W_i(G) \rightarrow T_i(G)$ is onto for all $i = 1, \dots, n$.

Lemma 4.2: Let $R \subseteq W_i(G)$ and $R = R_1 \cup R_2$. Then $\tau_i(R) = \tau_i(R_1) \cup \tau_i(R_2)$.

Lemma 4.3: Let $R \subseteq W_i(G)$ and $R = R_1 R_2$ such that $M_i(R_1) = \{q_j\}$ for some $q_j \in Q$. Then $\tau_i(R) = \tau_i(R_1) \cdot \tau_j(R_2)$

4.2. Simple Regular Expressions

Definition 4.4: The string form E_s of a regular expression E is defined inductively as follows:

- (1) If $h_\alpha(E) = 0$ then $E_s = w_1 \cup w_2 \cup \dots \cup w_\ell$
where w_j are words, $j = 1, \dots, \ell$, $\ell \geq 0$.
- (2) If $h_\alpha(E) = k$ then $E_s = F_1 \cup F_2 \cup \dots \cup F_p$, $p > 0$
where each F_i is of the form

$$w_1 H_1^* w_2 H_2^* \dots w_m H_m^* w_{m+1}, \quad m > 0$$

where w_j are words, H_j are in string form and
 $h_\alpha(H_j) \leq k - 1$, $j = 1, \dots, m$.

Thus E_s is formed by a finite number of strings of words and star expressions, that latter having roots which are again made up from a finite number of strings of words and star expressions, and so on. In other words, E and all roots of star expressions appearing in E are represented as a finite union of strings of words and star expressions. The strings F_i , $i = 1, \dots, p$, will be called first level strings of E .

Obviously, because of the distributivity of concatenation over union, every regular expression E may be transformed into its string form E_s .

Example 4.1: Let $E = (1*(1 \cup 0) \cup 2)*(2*1 \cup (0 \cup 2)*(22 \cup 2)1) \cup 00$.
then $E_s = (1*1 \cup 1*0 \cup 2)*2*1 \cup (1*1 \cup 1*0 \cup 2)*(0 \cup 2)*221 \cup (1*1 \cup 1*0 \cup 2)*(0 \cup 2)*21 \cup 00$.

Clearly the transformation of E into E_s does not involve any change in the apparent star height. Thus we have:

Lemma 4.4: For any regular expression E (a) $|E_s| = |E|$
(b) $h_\alpha(E_s) = h_\alpha(E)$

Therefore, as far as the apparent star height is concerned, we can assume without loss of generality that all regular expressions dealt with are in string form. In the sequel, let E always denote a regular expression in string form, unless otherwise specified.

Let E be a regular expression and H^* a star expression appearing in E . H^* may occur in E more than once. In the following, we make a distinction among all appearances of H^* in E , and refer to H^* as a p-star (position-star) in E when a certain appearance of it is considered.

Thus the number of p-stars in E is the number of symbols * appearing in it, whereas the number of distinct star expressions in E may be smaller. For example, if $E = (1*1 \cup 1*0 \cup 2)^*$, the p-stars in E are 1^* , 1^* and E itself.

We shall call the p-stars H_i^* appearing as factors in the first-level strings of E p-stars of level 1 in E. Inductively, a p-star in E is of level k iff it occurs as a factor in one of the strings forming the root of a p-star of level $k - 1$ in E. Thus in Example 4.1 all appearances of $(1*1 \cup 1*0 \cup 2)^*$, $(0 \cup 2)^*$ and 2^* are of level 1 whereas 1^* appears always in level 2.

Definition 4.5: Let E be a regular expression and H^* a p-star of level k in E. Define the predecessor of H^* in E, $\pi_E(H^*)$, by induction on k:

(1) $k = 1$. Then $H^* = H_i^*$ appears as a factor in a string

$w_1 H_i^* w_2 \dots w_i H_i^* \dots w_{n+1}$ of E. Define:

$$\pi_E(H^*) = w_1 H_i^* w_2 \dots w_i H_i^*$$

(2) $k > 1$ and assume the predecessor is defined for all p-stars of level less than k. H^* appears in E within the scope of a p-star of lower level, say H_1^* . Define:

$$\pi_E(H^*) = \pi_E(H_1^*) \cdot \pi_{H_1}(H^*)$$

Define the successor of H^* in E, $\sigma_E(H^*)$, by: $\sigma_E(H^*) = \overleftarrow{\pi_E(H^*)}$

Lemma 4.5: Let G be a state graph recognizing the event R and let E be a regular expression denoting a subset of R. Then for every p-star H^* in E:

- (a) Both events $|\pi_E(H^*)|$ and $|\sigma_E(H^*)|$ are infinite.
- (b) $|\pi_E(H^*) \cdot \sigma_E(H^*)| \subseteq R$
- (c) $|\sigma_E(H^*)| \subseteq \bigcap_{q \in M_1(\pi_E(H^*))} D_q$ (See Notation 1.3)

Proof: (a) Obviously $\pi_E(H^*)$ is not empty and by definition $\pi_E(H^*) = \pi_E(H^*) \cdot H^*$, thus $|\pi_E(H^*)|$ is infinite. The same proof applies to $|\sigma_E(H^*)|$.

(b) Follows directly from the definitions.

(c) By (b), $M_1(\pi_E(H^*) \cdot \sigma_E(H^*))$ is contained in the set F of final states of G. We have:

$$M_1(\pi_E(H^*) \cdot \sigma_E(H^*)) = M(M_1(\pi_E(H^*)), \sigma_E(H^*)) \subseteq F.$$

The last equation shows that for every state q in $M_1(\pi_E(H^*))$, $|\sigma_E(H^*)|$ is contained in the corresponding derivative D_q and hence the result follows.

Definition 4.6: Let G be a state graph and let E be a regular expression.

A p-star H^* of E is called simple with respect to G iff $M_1(\pi_E(H^*))$ is a singleton subset of Q. E is a simple regular expression with respect to G iff $|E| \subseteq W_1(G)$ and every p-star of E is simple w.r.t. G.

Thus if E is simple w.r.t. G then every position [MY] in E corresponds to a single state of G.

Example 4.2: Let G be the state graph in Fig. 4.1. and let

$E = (0^*2 \cup 01)^*0 \cup (0 \cup 2)^*02$. We have $\pi_E((0^*2 \cup 01)^*) = (0^*2 \cup 01)^*$; $\pi_E((0 \cup 2)^*) = (0 \cup 2)^*$ and $\pi_E(0^*) = (0^*2 \cup 01)^*0^*$. By inspection,

$M_1((0^*2 \cup 01)^*) = \{A\}$, $M_1((0 \cup 2)^*) = \{A, B\}$ and $M_1((0^*2 \cup 01)^*0^*) = \{A, B\}$.

Thus $(0^*2 \cup 01)^*$ is a simple star w.r.t. G , while $(0 \cup 2)^*$ and 0^* are not. Hence E is not a simple regular expression w.r.t. G . However, one can easily show that $E' = (00^*2 \cup 01)^*$, for instance, is simple w.r.t. G .

The following lemma can be easily verified:

Lemma 4.6: Let E be a regular expression and let H^* be a p -star in E simple w.r.t. the state graph G , such that $M_1(\pi_E(H^*)) = \{q_r\}$, $q_r \in Q$, and $\pi_E(H^*) \subseteq W_1(G)$. Then:

- (a) $\tau_r(H^*)$ is a set of loops from q_r to itself.
- (b) $\tau_r(H^*) = (\tau_r(H))^*$

Lemma 4.7: Let E be a regular expression simple w.r.t. G and let H^* be a p -star of E such that $M_1(\pi_E(H^*)) = \{q_r\}$, $q_r \in Q$. Then H is a simple regular expression w.r.t. G_r , where G_r is the state graph derived from G by changing the initial state from q_1 to q_r .

Proof: Let H_1^* be any p -star in H . By definition: $\pi_E(H^*) \cdot \pi_H(H_1^*) = \pi_E(H_1^*)$.

Applying the function M_1 we get:

$$\begin{aligned} M_1(\pi_E(H_1^*)) &= M_1(\pi_E(H^*) \cdot \pi_H(H_1^*)) = M(M_1(\pi_E(H^*)), \pi_H(H_1^*)) = M(q_r, \pi_H(H_1^*)) = \\ &= M_r(\pi_H(H_1^*)). \end{aligned}$$

Since E is a simple regular expression w.r.t. G , $M_1(\pi_E(H_1^*))$ is a singleton. Hence so is $M_r(\pi_H(H_1^*))$, which shows that H_1^* is a simple p -star in H w.r.t. G_r . It follows that H is a simple regular expression w.r.t. G_r .

4.3. Cofinal Sets of Trails and Star Height of Simple Regular Expressions

Definition 4.7: Let G be a state graph and G' a s.c. subgraph of G . For every positive integer m define in G an m -trail w.r.t. G' and a trail of exponent m w.r.t. G' by induction on the rank of G' :

- (a) Let $r(G') = 1$. Then a trail in G is an m -trail w.r.t. G' iff it contains a subtrail which is a non-trivial loop in G' .[†] A trail $t \in T(G)$ is of exponent m w.r.t. G' iff $t = t_1 t_2 \dots t_m$, where each t_j ($j = 1, 2, \dots, m$) is an m -trail w.r.t. G' .
- (b) Let $r(G') = k > 1$. Then $t' \in T(G)$ is an m -trail w.r.t. G' iff for every state q in G' there exists a subtrail $(\bar{q}, q_{i_1}, \dots, q_{i_\ell}, q)$ of t' such that $(q_{i_1}, q_{i_2}, \dots, q_{i_\ell})$ is a trail in $G' - [q]$ and is of exponent m w.r.t. some s.c. subgraph of $G' - [q]$ of rank $k - 1$. A trail $t \in T(G)$ is of exponent m w.r.t. G' iff $t = t_1 t_2 \dots t_m$ where each t_j ($j = 1, 2, \dots, m$) is an m -trail w.r.t. G' .

Example 4.3: Let G be the graph in Fig. 4.1. Obviously G has rank 2. Let $m = 3$ and consider the subgraph $G_1 = G - [B]$. Clearly $r(G_1) = 1$ and two of the loops in it are (A,A) and (A,C,A) . Thus every trail containing one of these as subtrail will be a 3-trail w.r.t. G_1 . In particular, (A,A) and (A,C,A) are themselves 3-trails w.r.t. G_1 . A trail of exponent 3 w.r.t. G_1 is any trail which is the product of 3 3-trails; for instance, $(A,A,A,A) = (A,A)(A,A)(A,A)$ or $(C,A,C,A,A,B,B,A,A) = (C,A,C,A)(A,A,B,B,A)(A,A)$.

[†] The definition of an m -trail in the case of $r(G') = 1$ does not depend on m . The name m -trail is introduced here merely for the sake of uniformity of notation.

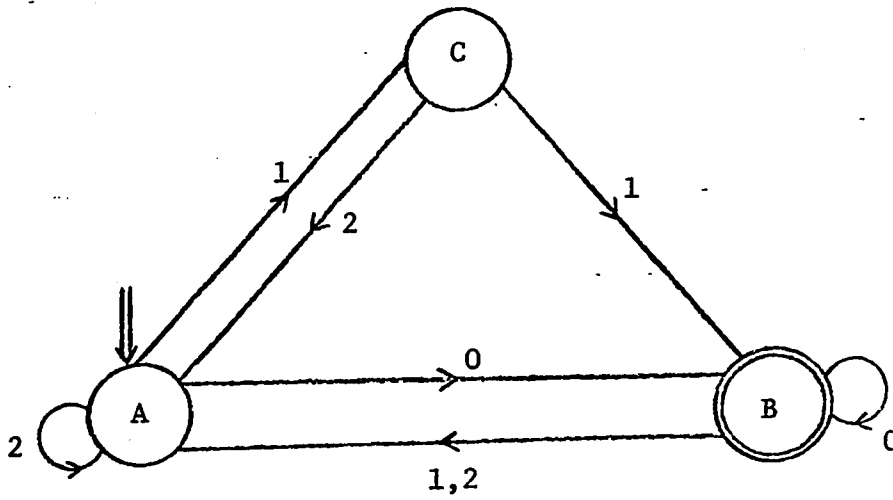


Fig. 4.1

Now consider the s.c. subgraph $G_2 = G - [C]$ which is clearly of rank 2. In order to construct a 3-trail t w.r.t. G_2 we first construct for every state q of G_2 a trail in $G_2 - [q]$ which is of exponent 3 w.r.t. a s.c. subgraph of rank 1 of $G_2 - [q]$. Thus $t_1 = (A,A,A,A)$ is of exponent 3 w.r.t. $G_2 - [B] = G - [\{B,C\}]$, whereas $t_2 = (B,B,B,B)$ is of exponent 3 w.r.t. $G_2 - [A] = G - [\{A,C\}]$. Now, by definition, t_1 must appear in t between two occurrences of B and t_2 must appear in t between two occurrences of A . Thus, for instance, $t = (A,B,B,B,B,A,A,A,A,B)$ is a 3-trail w.r.t. G_2 (in fact, this is the shortest possible one). Another such trail is $t' = (A,B,B,B,B,A,C,B,A,A,A,A,B,A)$. A trail of exponent 3 w.r.t. G_2 is a multiple of 3 3-trails w.r.t. G_2 ; for instance, $(t')^3$ is one such trail, or $[t \cdot (B,A)]^3$ is another one.

The next lemma follows directly from the above definition.

Assume all m -trails as well as all trails of exponent m are w.r.t. some fixed s.c. subgraph of G .

Lemma 4.8: (a) If t is an m -trail then t is also an n -trail for all $n \leq m$.

(b) If t is of exponent m then t is also of exponent n for all $n \leq m$.

(c) If t' is a subtrail of t and t' is an m -trail then t is also an m -trail.

(d) If t' is a subtrail of t and t' is a trail of exponent m then also t is of exponent m .

Lemma 4.9: For every s.c. subgraph G' of a state graph G , and for every positive integer m , there exists in G a trail of exponent m w.r.t. G' .

Proof: Let $r(G') = 1$. Then G' has at least one non-trivial loop and a trail of exponent m w.r.t. G' can be easily constructed. Assume the lemma is true for $r(G') < k$. Let $r(G') = k$. Then for every state q of G' , $G' - [q]$ has rank at least $k - 1$, and by assumption there can be constructed in it a trail t of exponent m w.r.t. some s.c. subgraph of rank $k - 1$. Since G' is s.c., one can find a trail t_1 , from q to the initial node of t , and another one from the terminal node of t back to q , and form their product: $t_1 \cdot t \cdot t_2$. By constructing such trails for all nodes q in G' and then by concatenating them all one after the other (by adding subtrails connecting the terminal node of the one to the initial node of the next one), an m -trail w.r.t. G' is obtained. Hence a trail of exponent m w.r.t. G' can be constructed.

Definition 4.8: A set of trails $T' \subseteq T(G)$ is called cofinal w.r.t. G' iff T' contains trails of exponent m w.r.t. G' for all integers m . A cofinal set of trails in G w.r.t. a s.c. subgraph of rank k will be called a set of trails of rank k in G .

Lemma 4.10: If $\tilde{G} = \tilde{G}_0(R)$ then $\tau_1(R)$ is cofinal w.r.t. every s.c. subgraph G' of $\tilde{G}$.

Proof: Let G' be any s.c. subgraph of $\tilde{G}$. By Lemma 4.9 there exists for every integer m a trail t of exponent m w.r.t. G' . Let the initial node of t be q_i and the terminal node q_j . Then by Lemma 4.1 there exists a word $w \in A^*$ such that $\tau_i(w) = t$. Now let w_1 be a word such that $M_1(w_1) = q_i$ and w_2 a word such that $M_j(w_2)$ is a final state. Then $w_1 \cdot w \cdot w_2 \in R$ and since $\tau_1(w_1 \cdot w \cdot w_2) = \tau_1(w_1) \cdot \tau_i(w) \cdot \tau_j(w_2)$ by

Lemma 4.3, t is a subtrail of $\tau_1(w_1 \cdot w \cdot w_2)$ and therefore the latter is also a trail of exponent m w.r.t. G' (Lemma 4.8). Hence $\tau_1(R)$ contains trails of exponent m w.r.t. G' for every integer m and is cofinal w.r.t. G' .

Lemma 4.11: Let G be a state graph and let E be a regular expression simple w.r.t. G such that $\tau_1(E)$ is of rank $k > 0$ in G . Then there exists a p -star of level 1, H^* , of E such that $\tau_r(H^*)$ is of rank k in G , where r is defined by: $\{q_r\} = M_1(\pi_E(H^*))$.

Proof: By Lemma 4.4 we may assume E is given in string form. Thus

$$E = \bigcup_{j=1}^p F_j \text{ and by Lemma 4.2, } \tau_1(E) = \bigcup_{j=1}^p \tau_1(F_j). \text{ By assumption } \tau_1(E)$$

is cofinal w.r.t. some s.c. subgraph G' of G of rank k and from the definition of a cofinal set it follows that one of the sets $\tau_1(F_j)$ is also cofinal w.r.t. G' . Now F_j is a string of words and stars,

$F_j = w_1 H_1^* w_2 \dots w_\ell H_\ell^* w_{\ell+1}$. By assumption E is simple w.r.t. G . Hence each of the sets: $M_1(w_1 H_1^* \dots w_i)$, $M_1(w_1 H_1^* \dots w_i H_i^*)$, $i = 1, 2, \dots, \ell$, consists of a single state. Let these states be:

$$M_1(w_1 H_1^* \dots w_i) = M_1(w_1 H_1^* \dots w_i H_i^*) = \{q_{j_i}\}. \text{ By Lemma 4.3 we get:}$$

$$\tau_1(F_j) = \tau_1(w_1) \cdot \tau_{j_1}(H_1^*) \cdot \tau_{j_1}(w_2) \cdot \dots \cdot \tau_{j_{\ell-1}}(w_\ell) \cdot \tau_{j_\ell}(H_\ell^*) \cdot \tau_{j_\ell}(w_{\ell+1}).$$

Now $\tau_1(F_j)$ contains trails of exponent m w.r.t. G' for arbitrarily large m , i.e. it contains for any m a trail t which is the multiple of m m -trails w.r.t. G' . Let c be the total length of all words w_i ($i = 1, 2, \dots, \ell + 1$). Then out of the m m -trails in t at least $m - c$ m -trails must be contained in the loops $\tau_{j_i}(H_i^*)$, $i = 1, 2, \dots, \ell$.

Thus for some h , $1 \leq h \leq \ell$, $\tau_{j_h}(H_h^*)$ contains at least $\lceil \frac{m-c}{\ell} \rceil^\dagger = m'$ m -trails w.r.t. G' . But since m grows arbitrarily, so does m' , and recalling that an m -trail is also an m' -trail (Lemma 4.8) we conclude that $\tau_{j_h}(H_h^*)$ contains trails of exponent m' w.r.t. G' for arbitrarily large m' . Hence $\tau_{j_h}(H_h^*)$ is a set of trails of rank k in G , where $\{q_{j_h}\} = \{q_r\} = M_1(\pi_E(H_h^*))$ and H_h^* is the required p -star of level 1 of E .

Main Lemma: Let G be any state graph and let E be a regular expression simple w.r.t. G such that $\tau_1(E)$ has rank $k > 0$ in G . Then $h_\alpha(E) \geq k$.

Proof: The proof will proceed by induction on k .

Let $k = 1$. Then there exists a s.c. subgraph G' of G with $r(G') = 1$ such that $\tau_1(E)$ is cofinal w.r.t. G' . This implies that $|E|$ is infinite and hence $h_\alpha(E) \geq 1$.

Now let $k > 1$ and assume the lemma holds for $k - 1$. Let E be an expression simple w.r.t. G such that $\tau_1(E)$ is of rank k in G . By Lemma 4.11 there exists in E a p -star H^* of level 1 such that $\tau_r(H^*)$ is of rank k , where r is given by: $\{q_r\} = M_1(\pi_E(H^*))$. Clearly $\tau_r(H^*)$ is a set of loops from q_r to itself and by Lemma 4.6, $\tau_r(H^*) = (\tau_r(H))^*$.

Consider the state graph G_r obtained from G by changing the initial state from q_1 to q_r . It follows from Lemma 4.7 that H is a simple regular expression w.r.t. G_r . It will next be shown that $\tau_r(H)$ has rank $k - 1$ in G_r .

So far we have seen that $(\tau_r(H))^*$ has rank k in G . This implies the existence of a s.c. subgraph G' of G which has rank k such

$\dagger [r]$ denotes the largest integer contained in the number r .

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that $(\tau_r(H))^*$ is cofinal w.r.t. G' . Thus for every integer m , $(\tau_r(H))^*$ contains a trail t of the form: $t = t_1 \cdot t_2 \cdot \dots \cdot t_m$, where each t_j is an m -trail w.r.t. G' . It follows from the definition of an m -trail that for every t_j , $1 \leq j \leq m$, there exists a subtrail of the form:

$t' = (q_r, q_{j_1}, q_{j_2}, \dots, q_{j_\ell}, q_r)$ such that $(q_{j_1}, q_{j_2}, \dots, q_{j_\ell})$ is a trail in $G' - [q_r]$ and is of exponent m w.r.t. some s.c. subgraph of rank $k - 1$ of $G' - [q_r]$. Now since $\tau_r(H^*)$ is a set of loops from q_r to itself, so is $\tau_r(H)$, and since $q_{j_i} \neq q_r$ ($i = 1, \dots, \ell$), the above subtrail t' must be contained in a trail of $\tau_r(H)$. Thus $\tau_r(H)$ contains, for arbitrarily large m , a trail of exponent m w.r.t. some s.c. subgraph of rank $k - 1$ of G . But since the number of such subgraphs is finite, $\tau_r(H)$ also contains, for infinitely many m , trails of exponent m w.r.t. one of these subgraphs. Hence $\tau_r(H)$ is a set of trails of rank $k - 1$ in G , and also in G_r . Recalling that q_r is the initial state of G_r and that H is simple w.r.t. G_r , we apply the induction hypothesis to H and deduce that $h_\alpha(H) \geq k - 1$. Hence $h_\alpha(E) \geq k$.

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4.4. Star Height of Events with the Finite Intersection Property

Definition 4.9: Let R be a regular event. R has the finite intersection property (f.i.p.) iff the intersection of every pair of distinct left quotients of R is finite, i.e. for any pair of words x, y , either $x \setminus R = y \setminus R$ or $(x \setminus R) \cap (y \setminus R)$ is finite.

Example 4.4: Let $R = 012^* \cup 2 \cup \lambda$. The left quotients of R (other than R) are:

$$0 \setminus R = 12^*$$

$$1 \setminus R = \emptyset = 00 \setminus R = 02 \setminus R = 20 \setminus R = 21 \setminus R = 22 \setminus R$$

$$2 \setminus R = \lambda$$

$$01 \setminus R = 2^* = 012 \setminus R$$

Thus:

$$R \cap (2 \setminus R) = \lambda = (01 \setminus R) \cap (2 \setminus R)$$

$$R \cap (01 \setminus R) = 2 \cup \lambda$$

and all other intersections of pairs of left quotients are empty.

Hence R has the f.i.p.

Lemma 4.12: Let R be a regular event with the f.i.p. and let $\tilde{G} = \tilde{G}_0(R)$. Then every regular expression E denoting a subset of R is simple w.r.t. $\tilde{G}$.

Proof: Let E be a regular expression denoting a subset of R and let H^* be a p-star in E . Suppose $M_1(\pi_E(H^*))$ contains two distinct states q, q' . Then by Lemma 4.5 $|\sigma_E(H^*)|$ is contained in $D_q \cap D_{q'}$, and it follows that $D_q \cap D_{q'}$ is infinite. But this contradicts the f.i.p. of R . Hence for every p-star H^* of E , $M_1(\pi_E(H^*))$ is a singleton and therefore E is a simple regular expression w.r.t. $\tilde{G}$.

Theorem 4.1: Let R be a regular event with the f.i.p. and let $\tilde{G} = \tilde{G}_0(R)$. Then every regular event $R' \subseteq R$ such that $\tau_1(R')$ has rank k in $\tilde{G}$ is of star height at least k . In particular, the star height of R equals the rank of $\tilde{G}$.

Proof: By Lemma 4.12 every regular expression denoting a subset R' of R is simple w.r.t. $\tilde{G}$. Now, using the Main Lemma, we deduce that in case $\tau_1(R')$ has rank k , every regular expression denoting R' must be of apparent star height at least k . Hence $h(R') \geq k$.

Concerning R itself, Lemma 4.10 implies that $\tau_1(R)$ has rank equal to $r(\tilde{G})$ and therefore $h(R) \geq r(\tilde{G})$. But by Eggen's theorem the star height of R cannot exceed the rank of any state graph recognizing R . Hence $h(R) = r(\tilde{G})$.

We next consider a special case of Theorem 4.1, which was also proved in [Mc 4].

Definition 4.10: Let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton. $\mathcal{A}$ is called reset-free iff for every $a \in A$, $q, q' \in Q$, $q \neq q'$ implies $M(q, a) \neq M(q', a)$ or $M(q, a) = M(q', a) = \phi$. A state graph is reset-free iff its corresponding automaton is reset-free.

Example 4.5: The automaton shown in Fig. 4.2 is reset-free, whereas the one in Fig. 4.3 is not.

Lemma 4.13: Every event recognized by a reset-free automaton with a single output state has the f.i.p.

Proof: Let R be recognized by the reset-free automaton $\mathcal{A} = (Q, M, q_1, F)$, where $F = \{q_f\}$. By Theorem 1.6, for every $q \in Q$, $A_c(q) = x \setminus R$ for some word x , and for every $y \in A^*$, $y \setminus R = A_c(q)$ for some $q \in Q$, or $y \setminus R = \phi$. We now show that for every pair of states $q, q' \in Q$, $A_c(q) \cap A_c(q') = \phi$. To see this, suppose for some word $w = a_1 a_2 \dots a_m$, $w \in A_c(q) \cap A_c(q')$. Then

$M(q,w) = M(q',w) = q_f$. But since $\mathcal{A}$ is reset-free, $M(q,a_1) \neq M(q',a_1)$, $M(q,a_1a_2) \neq M(q',a_1a_2)$ and by induction $M(q,w) \neq M(q',w)$, which is a contradiction. Also λ cannot be included in the intersection of the two accepted events since there is only a single output state. It follows that R has the f.i.p.

==

Theorem 4.2: Let G be a reset-free state graph with a single output state, and let G recognize an infinite event R . Then $h(R) = r(G)$.

Proof: By the proof of the previous lemma, distinct states of G correspond to distinct left quotients of R , and hence G is reduced. Moreover, G cannot have a dead state $q_\emptyset$. For the existence of $q_\emptyset$ implies the existence of two transitions in G , one from a state $q \neq q_\emptyset$ to $q_\emptyset$ and the other from $q_\emptyset$ to itself, both labelled by the same input letter. But this contradicts the fact that G is reset-free. Hence G coincides with the incomplete reduced state graph $\tilde{G}_0(R)$ recognizing R . Also, by the last lemma, R has the f.i.p., and it follows from Theorem 4.1 that $h(R) = r(G)$.

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Clearly every pure-group automaton is also reset-free. Thus Theorem 3.4 (Section 3.2) is a special case of the above theorem.

Example 4.6: Let $\tilde{G} = \tilde{G}_0(R)$ be the state graph shown in Fig. 4.3. It can be easily verified that R has the f.i.p. and hence $h(R) = r(\tilde{G}) = 3$ by Theorem 4.1.

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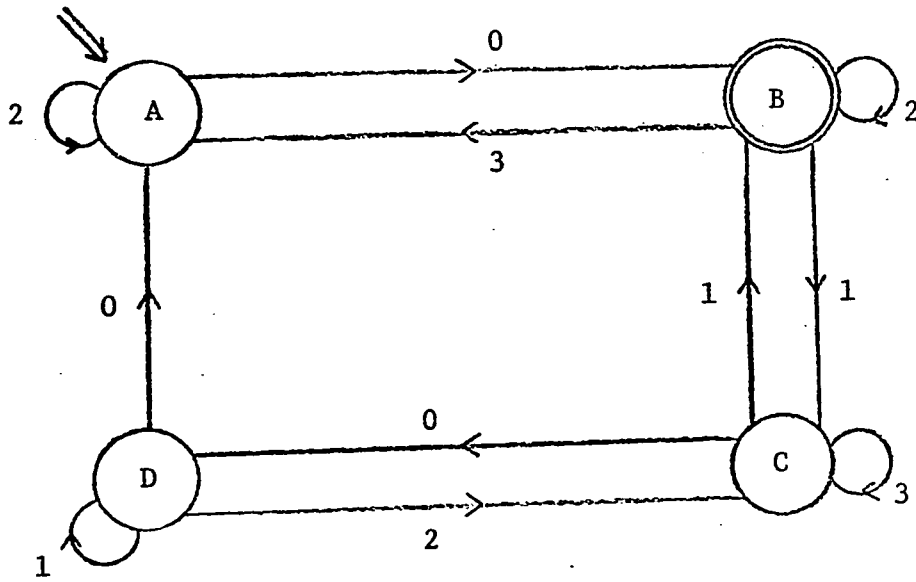


Fig. 4.2

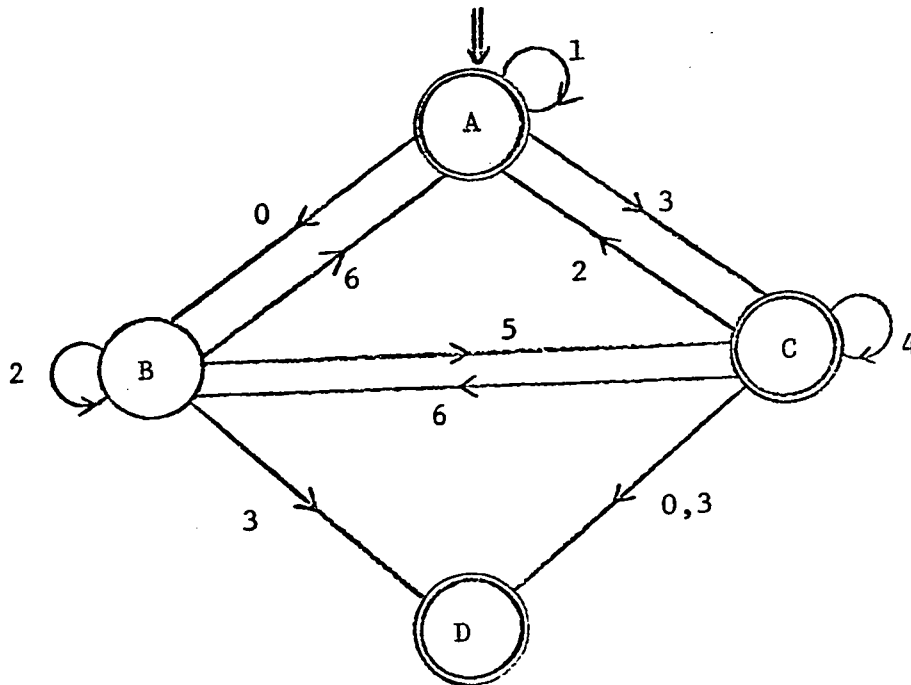


Fig. 4.3

Example 4.7: The event R recognized by the state graph $\tilde{G}$ in Fig. 4.2 has star height $h(R) = r(\tilde{G}) = 3$ by Theorem 4.2.

Example 4.8: Let $R' = (111*02*0 \cup 00)^*$. In order to determine the star-height of R' we first find the quotients of R' :

$$\begin{aligned}
 0 \setminus R' &= 0R' = A \\
 1 \setminus R' &= 1*02*0R' = B \\
 2 \setminus R' &= \phi \\
 00 \setminus R' &= R' = C \\
 01 \setminus R' &= 10 \setminus R' = \phi \\
 11 \setminus R' &= 1*02*0R' = D = 111 \setminus R' \\
 110 \setminus R' &= 2*0R' = E \\
 1100 \setminus R' &= R' \\
 1101 \setminus R' &= \phi \\
 1102 \setminus R' &= 110 \setminus R'
 \end{aligned}$$

The reduced state graph $\tilde{G}' = \tilde{G}_0(R')$ is shown in Fig. 4.4. Clearly $r(\tilde{G}') = 2$ and hence $h(R') \leq 2$. But R' does not have the f.i.p.; in fact, $(0 \setminus R') \cap (110 \setminus R') = 0 \setminus R' = 0R'$, which is infinite. Thus we cannot determine the star height of R' just by observing G' and using Theorem 4.1. However, let us now focus our attention to the state graph $\tilde{G} = \tilde{G}_0(R)$ shown in Fig. 4.5. Obviously $R = (02*0 \cup 1)^*$ and Theorem 4.2 $h(R) = 2$ because $\tilde{G}$ is reset-free with a single output state and $r(\tilde{G}) = 2$. It can be easily verified that $R' \subseteq R$, and furthermore, since R' contains all words $(1^m 02^m 0)^m$ for $m \geq 2$, $\tau_1^G(R')$ contains the set of trails $\{[(A,A)^m(A,B)(B,B)^m(B,A)]^m \mid m \geq 2\}$ which is clearly of rank 2 in G . Hence by Theorem 4.1 and Lemma 4.13 $h(R') \geq r(\tilde{G}) = 2$ and therefore $h(R') = 2$.

Disappointingly, Theorem 4.2 cannot be extended to the case of more than one output state. This is illustrated by the following example:

Example 4.9: Let G be the state graph in Fig. 4.6. Clearly G represents a pure-group automaton and $r(G) = 3$. Yet the event R recognized by G can be denoted by:

$$E = (1 \cup 2 \cup 0(1 \cup 2)^*0)^* \cup (0 \cup 2 \cup 1(0 \cup 2)^*1)^*$$

which is of apparent star height 2. Hence $h(R) < r(G)$.

However, some partial extensions of Theorem 4.2 to reset-free state graphs with more than one terminal node are possible, as will be shown in Chapter 6.

4.5. Star-Height of Events Defined by Digraphs

In [E] Eggen considers a class of events defined by digraphs in the following way: let $D = (N, B)$ be a digraph; then any path $p = b_{n_1 n_2} b_{n_2 n_3} \dots b_{n_{k-1} n_k}$ in D can be considered as a word over the alphabet B , and for any $m, n \in N$, the set of all paths from m to n constitutes an event over B . Clearly such an event is regular; in fact, every digraph can be regarded as an incomplete state graph (without initial and final states specified), where for every branch $(m, n) \in B$, there exists a transition from m to n labelled by b_{mn} . Obviously no two transitions are labelled by the same letter; hence such a state graph is always reset-free.

The following two questions were raised by Eggen in [E]:

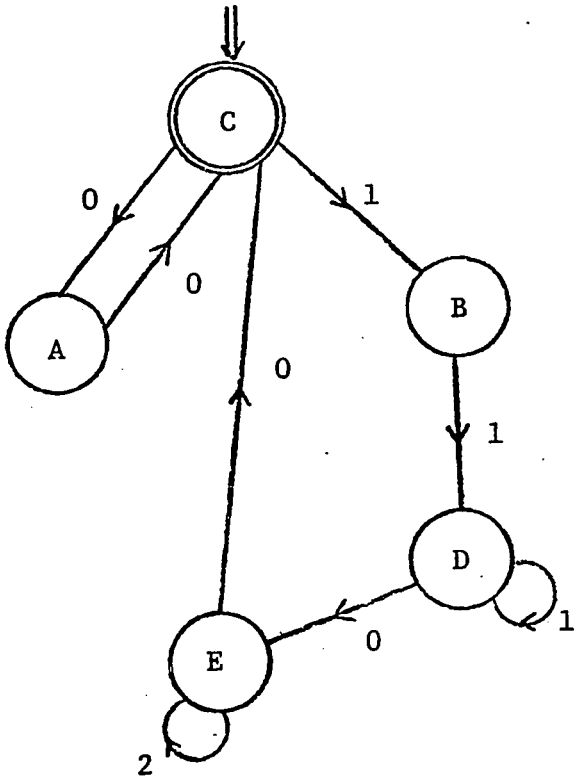


Fig. 4.4

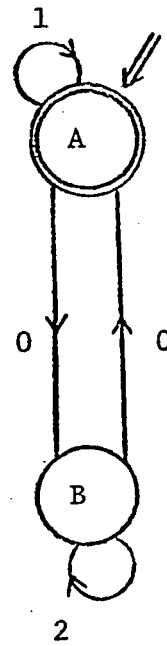


Fig. 4.5

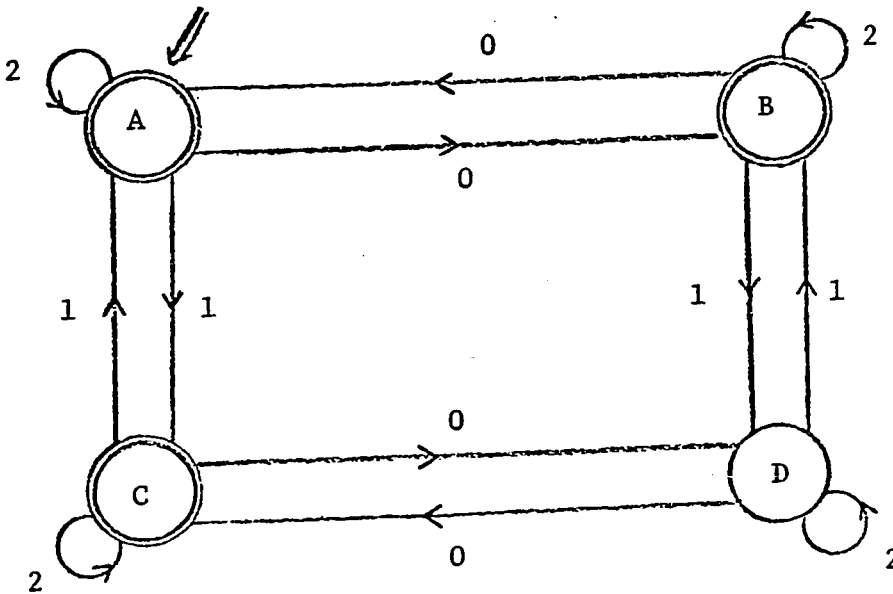


Fig. 4.6

- (1) Is the set of paths from a node to itself in a complete digraph on k nodes of star height k ?
- (2) For a digraph of rank k , do there necessarily exist nodes m and n such that the set of paths from m to n is an event of star height k ?

A positive answer to both questions is provided by the next corollary, which is an immediate consequence of Theorem 4.2.

Corollary 4.1: Let $D = (N, B)$ be a digraph of rank k . Then:

- (a) For any $n \in N$ contained in a section of D of rank k , the set of all paths from n to itself is of star height k .
- (b) For any $m, n \in N$ such that there exists a path from m to n passing through a section of D of rank k , the set of all paths from m to n is of star height k .

Now since the complete digraph on k nodes is strongly connected and has rank k (Proposition 2.3), it follows that for any pair of nodes m, n (not necessarily distinct) in it, the set of all paths from m to n has star height k .

Thus the complete digraph on k nodes can be considered as an incomplete automaton with k states recognizing an event of star height k over an alphabet of k^2 letters. Furthermore, by appropriate labelling of the branches of the complete digraph, one can reduce the number of alphabet letters to k without decreasing the star height of the corresponding event, and moreover, the resulting automaton will be a

pure-group automaton. This is shown in the next corollary.

Corollary 4.2: For every integer k there exists a pure-group automaton with k states recognizing an event of star height k over the k -letter alphabet.

Proof: Let $\mathcal{A}_k = (Q, M, q_0, F)$ be an automaton over the alphabet $A = \{0, 1, \dots, k-1\}$, where $Q = \{q_0, q_1, \dots, q_{k-1}\}$, $F = \{q_0\}$ and the transition function M is defined by: $M(q_i, j) = q_{i+j \pmod k}$, for all $i, j = 0, 1, \dots, k-1$. Clearly $\mathcal{A}_k$ is a pure-group automaton of rank k with a single output state, and Theorem 4.2 yields the result.

For example, the automaton $\mathcal{A}_3$ is as shown in Fig. 4.7.

Definition 4.11: Let $D = (N, B)$ be a digraph and let N_1, N_2 be two non-empty subsets of N . Define $P(N_1, N_2)$ to be the set of all paths in D with initial node in N_1 and terminal node in N_2 . Let $D_{N_1 N_2}$ denote the smallest subgraph of D containing all paths of $P(N_1, N_2)$. Thus $D_{N_1 N_2} = D - [N'']$, where N'' consists of all nodes $n' \in N$ such that no path of $P(N_1, N_2)$ passes through n' .

We now generalize the result of Corollary 4.2 to all events $P(N_1, N_2)$.

Lemma 4.14: Let $D = (N, B)$ and let $n_1 \in N$ and $\emptyset \neq N_2 \subseteq N$. Then

$$h(P(n_1, N_2)) = r(D_{n_1 N_2}).$$

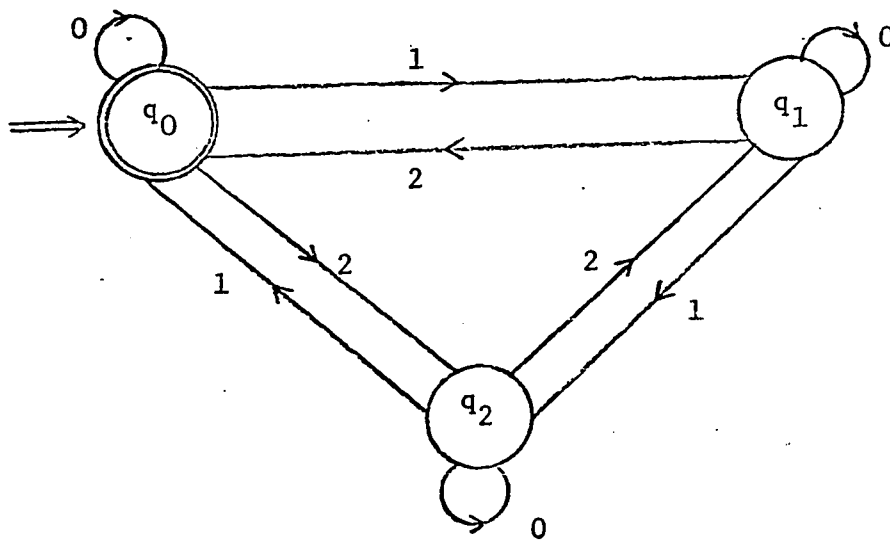


Fig. 4.7

Proof: As has been stated above, any digraph can be regarded as an incomplete state graph, in which the branch from node m to node n is labelled by b_{mn} . Now consider the digraph $D' = D_{n_1 N_2} = (N', B')$ as an incomplete state graph with initial state n_1 and a set of final states N_2 . Clearly the event accepted by D' is $P = P(n_1, N_2)$. Now by Theorem 1.6, the accepted event $A_c(n)$ of any node $n \in N'$ in D' equals one of the left quotients of P , and for every non-empty left quotient $x \setminus P$, $x \in (B')^*$, there exists $n \in N'$ such that $A_c(n) = x \setminus P$. Thus let $m_1, m_2 \in N'$, $m_1 \neq m_2$. Then for $i = 1, 2$, all branches leaving m_i are labelled by letters $b_{m_i n}$, $n \in N'$. Thus two branches, one leaving m_1 and the other leaving m_2 cannot be labelled by the same letter and hence $A_c(m_1) \cap A_c(m_2) \subseteq \Lambda$. This implies that the event P has the f.i.p. Let $\bar{D}' = \bar{D}_{n_1 N_2}$ be the digraph obtained from D' by merging all nodes whose accepted events equal Λ . Apparently $\bar{D}'$ is a reduced state graph without a dead state recognizing P and $r(\bar{D}') = r(D')$. Hence $\bar{D}'$ coincides with $\tilde{G}_0(P)$ and the result then follows from Theorem 4.1.

Lemma 4.15: Let $D = (N, B)$ and let N_1, N_2 be two non-empty subsets of N . Then

$$h(P(N_1, N_2)) = \max \{h(P(n, N_2)) \mid \exists n_1 \in N_1, b_{n_1 n} \in B\}$$

Proof: We first claim that for any $n_1 \in N_1$,

$$b_{n_1 n} \setminus P(N_1, N_2) = \begin{cases} P(n, N_2), & b_{n_1 n} \in B \\ \phi & \text{otherwise} \end{cases}$$

This follows easily from the fact that for all $n \in N$, there can be at most one branch (namely $b_{n_1 n}$) leaving some node of N_1 and labelled by $b_{n_1 n}$. Now by Theorem 3.5,

$$\begin{aligned} h(P(N_1, N_2)) &= \max \{h(b_{n_1 n} \setminus P(N_1, N_2)) \mid n_1 \in N_1, b_{n_1 n} \in B\} = \\ &= \max \{h(P(n, N_2)) \mid n_1 \in N_1, b_{n_1 n} \in B\}. \end{aligned}$$

Theorem 4.3: Let $D = (N, B)$ be a digraph. Then for any two non-empty subsets N_1, N_2 of N , $h(P(N_1, N_2)) = r(D_{N_1 N_2})$.

Proof: By Lemmas 4.15 and 4.14,

$$h(P(N_1, N_2)) = \max \{r(D_{n N_2}) \mid \exists n_1 \in N_1, b_{n_1 n} \in B\}$$

Thus it remains to show that

$$(*) \quad r(D_{N_1 N_2}) = \max \{r(D_{n N_2}) \mid \exists n_1 \in N_1, b_{n_1 n} \in B\}$$

Obviously every graph $D_{n N_2}$ for which $b_{n_1 n} \in B$ for some $n_1 \in N_1$ must be contained in $D_{N_1 N_2}$. Thus $r(D_{n N_2}) \leq r(D_{N_1 N_2})$ for all such graphs $D_{n N_2}$. Moreover, we claim that every section S of $D_{N_1 N_2}$ is contained in at least one such graph $D_{n N_2}$. To see this, let S be any section contained in $D_{N_1 N_2}$. By Definition 4.12, there exists in D a path

$$p = b_{m_1 m_2} b_{m_2 m_3} \dots b_{m_{k-1} m_k} \text{ with } m_1 \in N_1, m_k \in N_2 \text{ and with at least one}$$

node m_i , $1 \leq i \leq k$, in S . Without loss of generality we may assume $i > 1$; for if m_1 is the only node of p contained in S , we may concatenate p on the left by some non-trivial loop ℓ from m_1 to itself (which certainly exists since S is strongly connected), thus obtaining another path ℓp from m_1 to m_2 satisfying the above requirement. Now the path $b_{m_2 m_3} \dots b_{m_{k-1} m_k}$ must be contained in $D_{m_2 N_2}$, and since one of its nodes belongs to the strongly connected subgraph S , there exist in D paths from m_2 to m_k touching any node of S . This implies that the whole section S must be contained in $D_{m_2 N_2}$. But since $b_{m_1 m_2} \in B$ and $m_1 \in N_1$, $D_{m_2 N_2}$ is one of the graphs appearing in (*), and hence

$$r(D_{N_1 N_2}) \leq \max \{r(D_{n N_2}) \mid \exists n_1 \in N_1, b_{n_1 n} \in B\} \text{ and the result follows.}$$

Example 4.10: Let $D' = (N, B)$ be the digraph shown in Fig. 4.8. Let $N_1 = \{A, C, G\}$ and $N_2 = \{B, H\}$. Then $D'_{N_1 N_2} = D'$ and hence $h(P(N_1, N_2)) = r(D') = 3$. Now consider $N_1 = \{A, B\}$, $N_2 = \{A, C\}$; then clearly $D'_{N_1 N_2} = D' - [\{D, E, F, G, H\}]$ and $h(P(N_1, N_2)) = 1$; in fact,

$$P(\{A, B\}, \{A, C\}) = (b_{BA} \cup \lambda)(b_{AB} b_{BA} \cup b_{AC} b_{CB} b_{BA})^*(\lambda \cup b_{AC}).$$

Lemma 4.16: For any digraph $D = (N, B)$ and for any $N_1, N_2 \subseteq N$, $P(N_1, N_2)$ is a group-free event.

Proof: Clearly $P(N_1, N_2) = \bigcup_{n \in N_1} P(n, N_2)$. By Theorem 1.12 every event which is a finite union of group-free events is itself group-free. Thus it suffices to show that each event $P(n, N_2)$ is group-free. Now by

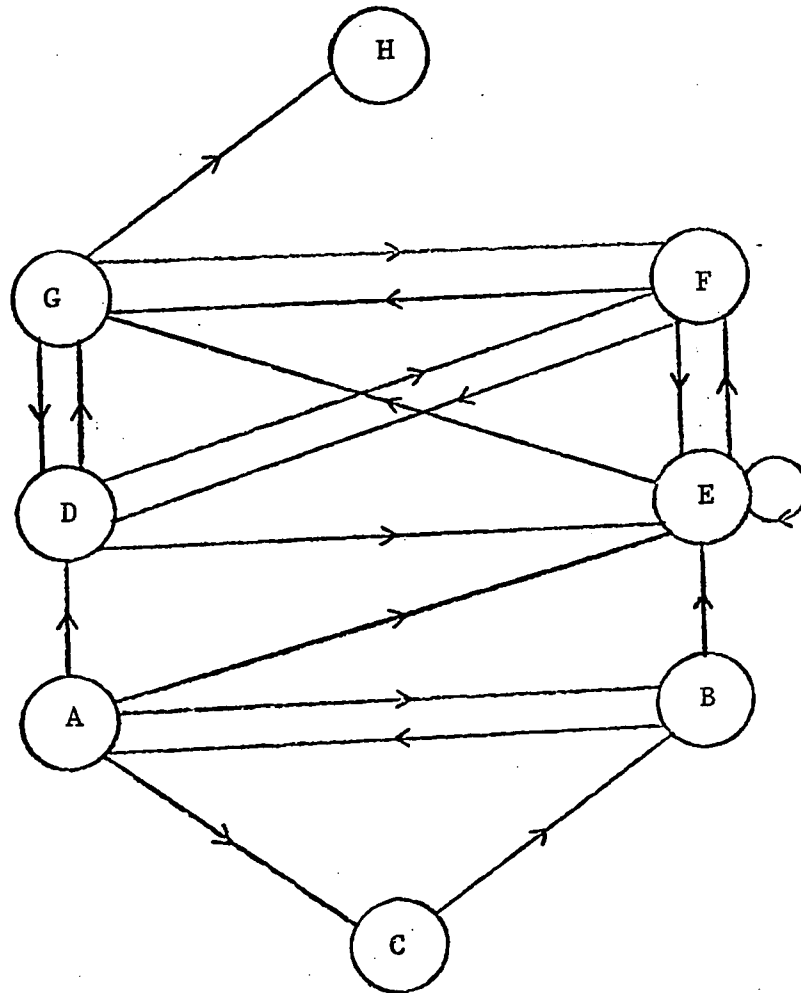


Fig. 4.8

Theorem 1.11 an event R is group-free iff it is permutation-free, i.e., in the reduced state graph for R , no word w induces a non-trivial permutation on any subset of the set of states. Now it has been proved in Lemma 4.14 that for any $n \in N$, $N_2 \subseteq N$, $\bar{D}_{nN_2}$ can be regarded as the reduced (incomplete) state graph recognizing $P(n, N_2)$, with initial state n and the set of final states $\bar{N}_2$. Thus suppose there exists a word $w = b_1 b_2 \dots b_k$ over the alphabet B , and a subset $N' \subseteq N$ with $\#N' \geq 2$ such that w induces a permutation on N' . Then there exist two nodes $n_1, n_2 \in N'$ such that there are in D two branches labelled by b_1 leaving n_1 and n_2 respectively. But this is impossible since no two branches are labelled by the same letter.

An important result of Lemmas 4.14 and 4.16 is the following:

Theorem 4.4: Every regular event R is a homomorphic image of a group-free event R' such that $h(R') = r(\tilde{G}_0(R))$.

Proof: Let $G = \tilde{G}_0(R) = (Q, M, q_1, F)$ and let $D = D_{\tilde{G}} = (Q, B)$ be the digraph corresponding to $\tilde{G}$ (Definition 2.22). Since $\tilde{G}_0(R)$ is the reduced (incomplete) state graph recognizing R , we have $D_{q_1 F} = D$ (Definition 4.12). Hence by Lemma 4.14, $h(P(q_1, F)) = r(D)$ and moreover, by Lemma 4.16, $P(q_1, F)$ is a group-free event. Now define a mapping ϕ from the set of branches B of D to the set of all non-empty subsets of the alphabet A :

$$\phi : B \rightarrow 2^A - \phi$$

$$\forall b_{q_i q_j} \in B, \phi(b_{q_i q_j}) = \{a \in A \mid M(q_i, a) = q_j\}$$

Clearly $\phi(b_{q_i q_j}) \neq \emptyset$ for any $b_{q_i q_j} \in B$. Now extend ϕ to a generalized homomorphism from 2^{B^*} into 2^{A^*} . It is easily seen that $\phi(P(q_1, F)) = R$. Hence $R' = P(q_1, F)$ is the required group-free event.

Chapter 5

Rank Non-Increasing Transformations on Transition Graphs

Introduction:

As was shown in Chapter 4, in some cases the star height of a regular event R can be determined merely by studying the structure of its reduced state graph $\hat{G}_0(R)$. However, as Eggen's Star Height Theorem indicates, in order to determine the star height of R in the general case, one has to examine the vast family of all transition graphs recognizing R . Our aim in this chapter is to show that it suffices to consider only a proper subset of this family, namely the family of all reduced non-deterministic state graphs accepting R (roughly speaking, a transition graph G is reduced if every two accepted events, or preceding events, of G are distinct).

In section 5.1 some elementary modifications of a digraph are introduced and their effect on the rank of the digraph is investigated. These transformations form the foundation for the constructions presented in the rest of the chapter.

In section 5.2 a reduced transition graph is defined and a reduction procedure for an arbitrary transition graph is presented, in which the rank of the graph cannot be increased.

A rank-preserving process of elimination of all λ -transitions in a transition graph is described in section 5.3. This process is based on successive application of basic rank-non-increasing transformations

to the given transition graph G , in which all λ -transitions are eventually replaced by some other non- λ transitions. Thus every transition graph G can be replaced by an equivalent non-deterministic reduced state graph G' , whose rank does not exceed the rank of G . As a result, Eggen's Star Height Theorem can be strengthened as mentioned above. Another consequence is, that for any regular event R , the smallest number of nodes of any transition graph accepting R equals the number of states in a minimal non-deterministic automaton recognizing R [KW].

5.1. Elementary Modifications of Digraphs

In this section two elementary modifications of digraphs are defined and some results concerning the changes in rank caused by these modifications are obtained. These results will be used in the next two paragraphs, when similar modifications of transition graphs will serve as the main tools for proving the end results of this chapter.

Definition 5.1: Let $D = (N, B)$ be a digraph and let $m, n \in N$. Define two digraphs, $D^1(m, n)$ and $D^2(m, n)$, as follows:

$$D^1(m, n) = (N, B \cup \{(m, n') \mid n' \in \sigma(n)\})$$

$$D^2(m, n) = (N, B \cup \{(m', n) \mid m' \in \pi(m)\})$$

Thus $D^1(m, n)$ is obtained from D by adding all branches leaving m and entering successors of n , whereas $D^2(m, n)$ is obtained from D by adding all branches leaving predecessors of m and entering n .

Example 5.1: Let D be the digraph in Fig. 5.1 a. $D^1(m, n)$ and $D^2(m, n)$ are shown in Fig. 5.1 b and Fig. 5.1 c respectively.

Lemma 5.1: $r(D) \leq r(D^i(m, n)) \leq r(D) + 1, i = 1, 2.$

Proof: When a node and all branches touching it are removed, the rank is decreased by at most one.

Lemma 5.2: $D^2(m, n) = [(\overleftarrow{D})^1(n, m)]$

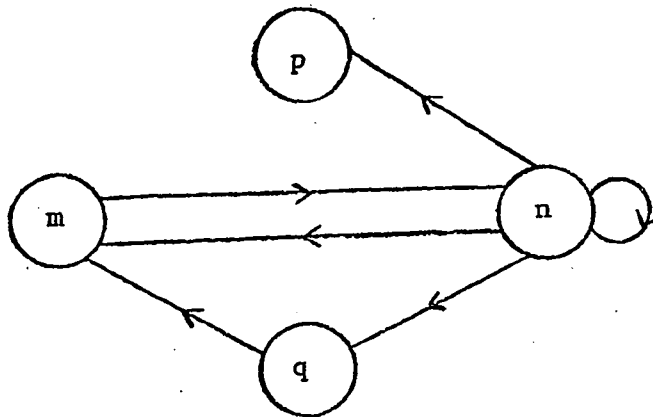
In the following lemmas, let $D = (N, B)$ be a digraph and let $n_1, n_2 \in N.$

Lemma 5.3: (a) If there does not exist any path in D from n_2 to n_1 then all sections of $D^i(n_1, n_2)$ are identical with those of $D, i = 1, 2.$

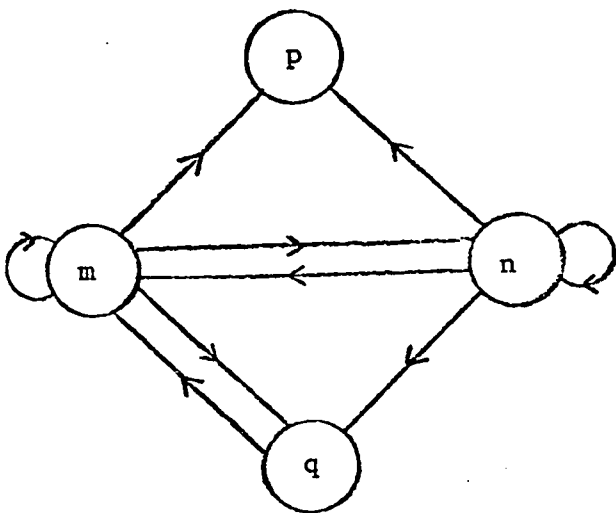
(b) If n_1 and n_2 belong to the same section S of D , then $T_i = S^i(n_1, n_2)$, where T_i is the section containing n_1 and n_2 in $D^i(n_1, n_2)$ ($i = 1, 2$), and all other sections of $D^i(n_1, n_2)$ are identical with those of $D.$

Proof: By definition of $D^1(n_1, n_2)$, the set of branches added to D to obtain $D^1(n_1, n_2)$ is: $B_1 = \{(n_1, n') \mid n' \in \sigma(n_2)\}.$ Since all branches in B_1 are incident to n_1 , none of them can be contained in a section of $D^1(n_1, n_2)$ which does not contain $n_1.$ Hence all sections not containing n_1 remain unchanged when $D^1(n_1, n_2)$ is constructed, and the only section which can possibly be extended is the one containing $n_1.$ Thus let S and T be the sections containing n_1 in D and $D^1(n_1, n_2)$ respectively, or $S = \phi^\dagger (T = \phi)$ if n_1 does not belong to any section of $D (D^1(n_1, n_2)).$

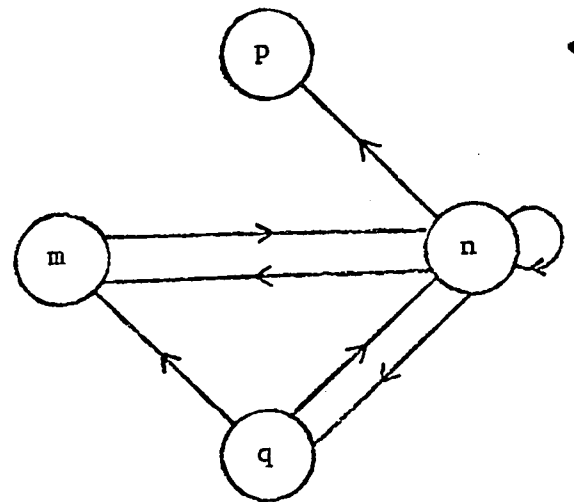
$\dagger \phi$ denotes here the empty digraph with no nodes and no branches.



(a)



(b)



(c)

Fig. 5.1

Clearly $S \subseteq T$. Now suppose $S \neq T$. Then either (i) there exists in T a node n_0 which is not contained in S , or (ii) S and T have the same set of nodes but T has a branch which is not contained in S .

In case (i), since n_0 belongs to T , there exists in T a loop p on n_1 passing through n_0 . Since this loop cannot possibly be contained in S , at least one of its branches must belong to B_1 . Thus let $p = p_1 \cdot p_2$, where $p_2 = b_{n_1 n'} \cdot p'$, and $b_{n_1 n'}$ is the last branch in p contained in B_1 . Since $n' \in \sigma^D(n_2)$, the path $p'' = b_{n_2 n'} \cdot p'$ is a path in D from n_2 to n_1 . But this contradicts our assumption that no such path exists, and hence case (i) is impossible.

As for case (ii), since S and T have the same set of nodes, any branch in T which is not contained in S must clearly belong to B_1 . Thus suppose $b_{n_1 n'} \in B_1$ is contained in T and not in S . Then n' , which is a successor of n_2 in D , is contained in T , and hence in S . But then there is in S a path p from n' to n_1 , and thus the path $b_{n_2 n'} \cdot p$ is a path in D from n_2 to n_1 , which is again a contradiction.

We conclude that $S = T$ and therefore D and $D^1(n_1, n_2)$ have identical sections. The proof for $D^2(n_1, n_2)$ is obtained in a dual way (Lemma 5.2).

(b) Since n_1 and n_2 are in the same section S of D , there exists in D a path $p_{n_1 n_2}$ from n_1 to n_2 .

Starting with $i = 1$, we first show that T_1 and S have a common set of nodes. Obviously every node of S is also contained in T_1 . Thus let n_0 be any node in T_1 . Then there exists in $D^1(n_1, n_2)$ a loop on n_1 passing through n_0 . Replacing in this loop each branch

$b_{n_1 n'}$ of B_1 by the path $p_{n_1 n_2} \cdot b_{n_2 n'}$, we obtain a new loop on n_1 , which also passes through n_0 and is contained in D . Hence n_0 also belongs to S , and T_1 and S have a common set of nodes. Thus T_1 is obtained from S by adding all branches $b_{n_1 n'}$ such that $n' \in \sigma^D(n_2)$ and n' is also contained in S , namely $n' \in \sigma^S(n_2)$. It follows that $T_1 = S^1(n_1, n_2)$. Since all branches added to D in constructing $D^1(n_1, n_2)$ are incident to n_1 , all sections of D other than S remain unchanged in $D^1(n_1, n_2)$.

The proof for $i = 2$ is obtained in a dual way.

Lemma 5.4: If there does not exist in D any path from n_2 to n_1 then

$$r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = r(D).$$

Proof: This is a direct consequence of Lemma 5.3(a) and the definition of the rank.

Lemma 5.5: If n_1 and n_2 belong to the same section S of D and $r(S) < r(D)$ then:

$$r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = r(D^1(n_2, n_1)) = r(D^2(n_2, n_1)) = r(D)$$

Proof: By Lemma 5.3(b), $T_i = S^i(n_1, n_2)$, where T_i is the section of $D^i(n_1, n_2)$ containing n_1 and n_2 ($i = 1, 2$). By Lemma 5.1, $r(T_i) \leq r(S) + 1$

and hence $r(T_i) \leq r(D)$. Since all other sections of $D^i(n_1, n_2)$ are identical with those of D , $r(D^i(n_1, n_2)) = r(D)$. By interchanging n_1 and n_2 we get $r(D^i(n_2, n_1)) = r(D)$, $i = 1, 2$.

Lemma 5.6: If n_1 and n_2 belong to the same section of D and if n_1 is a cycle center of D then:

$$r(D^1(n_1, n_2)) = r(D^2(n_2, n_1)) = r(D).$$

Proof: Let S and T be the sections containing n_1 in D and $D^1(n_1, n_2)$ ($D^2(n_2, n_1)$) respectively. Then by assumption $r(S) = r(D)$ and $r(S - [n_1]) = r(S) - 1$. But by Lemma 5.3, S and T have a common set of nodes, and since $D^1(n_1, n_2)(D^2(n_2, n_1))$ is obtained from D by adding only branches incident to n_1 , $S - [n_1] = T - [n_1]$. Hence $r(T - [n_1]) = r(S) - 1$ and $r(T) = r(S)$. Since, by Lemma 5.3, all other sections of $D^1(n_1, n_2)(D^2(n_2, n_1))$ are identical with those of D , we get $r(D^1(n_1, n_2)) = r(D)(r(D^2(n_2, n_1)) = r(D))$.

Lemma 5.7: Let n_0 be a node of D , $n_0 \neq n_1, n_2$. Then:

$$(D - [n_0])^i(n_1, n_2) = D^i(n_1, n_2) - [n_0], i = 1, 2.$$

Proof: This follows directly from Definition 5.1 and Definition 2.24.

Theorem 5.1: Let $D = (N, B)$ be a digraph and let $n_1, n_2 \in N$. Then at least one of the following conditions must hold:

$$(i) \quad r(D^i(n_1, n_2)) = r(D), \quad i = 1 \text{ and } 2$$

$$(ii) \quad r(D^i(n_2, n_1)) = r(D), \quad i = 1 \text{ and } 2$$

$$(iii) \quad r(D^1(n_1, n_2)) = r(D^2(n_2, n_1)) = r(D).$$

$$(iv) \quad r(D^2(n_1, n_2)) = r(D^1(n_2, n_1)) = r(D).$$

Proof: We consider several cases:

Case (a): There does not exist a path from n_2 to n_1 (from n_1 to n_2).

Then by Lemma 5.4 condition (i) ((ii)) must be satisfied.

Case (b): n_1 and n_2 are both contained in a section S of D and $r(S) < r(D)$. By Lemma 5.5 all the above conditions (i) - (iv) are satisfied.

Case (c): $n_1(n_2)$ is a cycle center of D . Then by Lemma 5.6 condition (iii) ((iv)) holds.

Case (d): n_1 and n_2 are contained in a section S of D , and there exists a cycle center $n_0 \neq n_1, n_2$ of S such that every path in S from n_2 to n_1 (from n_1 to n_2) passes through n_0 .

Let T_i be the section containing n_1 in $D^i(n_1, n_2)$, $i = 1, 2$.

By Lemma 5.3(b), $T_i = S^i(n_1, n_2)$. Now consider the graph $S - [n_0]$.

Clearly $r(S - [n_0]) = r(S) - 1$. By Lemma 5.7 we have:

$$S^i(n_1, n_2) - [n_0] = T_i - [n_0] = (S - [n_0])^i(n_1, n_2).$$

Furthermore, since all paths of S from n_2 to n_1 go through n_0 , in $S - [n_0]$ there is no path from n_2 to n_1 . Applying Lemma 5.4 to $S - [n_0]$ we get:

$$r(S - [n_0])^i(n_1, n_2) = r(S - [n_0]), i = 1, 2$$

and hence

$$r(T_i - [n_0]) = r(S - [n_0]) = r(S) - 1.$$

But then $r(T_i) \leq r(S)$, and since all other sections of $D^i(n_1, n_2)$ are identical with those of D , condition (i) holds.

By interchanging n_1 and n_2 in the above proof, we get condition (ii) for the symmetric case.

Case (e): n_1 and n_2 are contained in the same section S of D such that $r(S) = r(D)$, neither n_1 nor n_2 is a cycle center of S , and for every cycle center n_0 of S there exists in $S - [n_0]$ a loop going through n_1 and n_2 .

The proof is by induction on $r(D)$.

Basis: $r(D) = 1$. Then $r(S) = 1$ and if n_0 is a cycle center of S then $r(S - [n_0]) = 0$. Hence there cannot be any loop in $S - [n_0]$ and the above condition never holds. Thus the theorem is vacuously satisfied.

Induction step: Suppose the theorem holds for all graphs of rank $k - 1$ and let $r(D) = k$. Let n_0 be any cycle center of S ; then $r(S - [n_0]) = k - 1$. Now if $S - [n_0]$ belongs also to Case (e), then by induction hypothesis one of the conditions (i) - (iv) holds for it.

If, on the other hand, the graph $S - [n_0]$ belongs to one of the cases (a) - (d), then it has been proved that one of the conditions (i) - (iv) must be satisfied for it. Now by an argument similar to the

one in the proof for Case (d), with the aid of Lemmas 5.3 and 5.7, it follows that the same condition out of (i) - (iv) must also hold for S . For instance, if condition (i) holds for $S - [n_0]$, then it will also hold for S , i.e. $r(S^i(n_1, n_2)) = r(T_i) = r(S)$, $i = 1, 2$. Thus the rank of the new section T_i replacing S in $D^i(n_1, n_2)$ is also k , and since by Lemma 5.3 the other sections of $D^i(n_1, n_2)$ remained unchanged, (i) will also hold for D . The proof for the other cases (ii) - (iv) is the same.

Example 5.2: Let D be the digraph on Fig. 5.2 (a). The graphs $D^1(n_1, n_2)$, $D^2(n_1, n_2)$, $D^1(n_2, n_1)$ and $D^2(n_2, n_1)$ are shown on Fig. 5.2 (b), (c) (d) and (e) respectively. It is easily seen that $r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = 2$ whereas $r(D^1(n_2, n_1)) = r(D^2(n_2, n_1)) = 1 = r(D)$. Thus condition (ii) is satisfied whereas conditions (i), (iii) and (iv) are not.

Corollary 5.1: Let $D = (N, B)$ and $n_1, n_2 \in N$ such that $(n_1, n_2) \in B$.

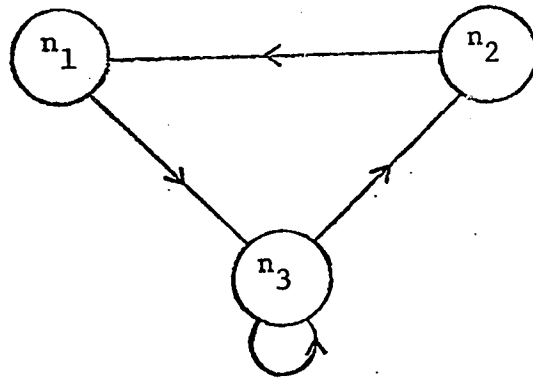
Then either

$$(A) \quad r(D^1(n_1, n_2)) = r(D)$$

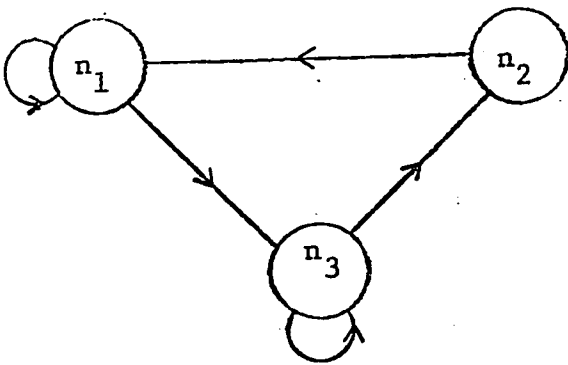
or

$$(B) \quad r(D^2(n_1, n_2)) = r(D)$$

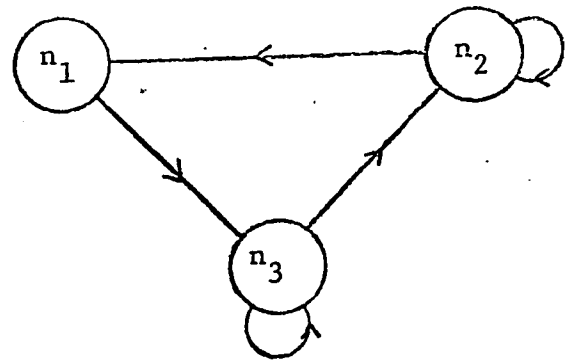
Proof: By Theorem 5.1, at least one of the equalities (i) - (iv) must hold. Out of these, the only one which does not imply (A) or (B) is (ii). Recalling the proof of Theorem 5.1, one can see that,



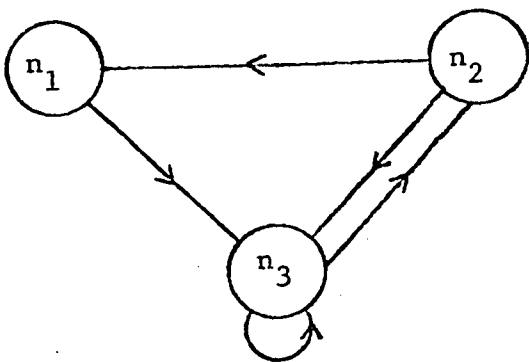
(a)



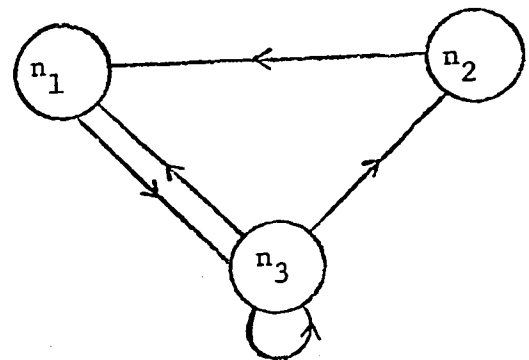
(b)



(c)



(d)



(e)

Fig. 5.2

with the additional assumption $(n_1, n_2) \in B$, one of conditions (i) (iii) or (iv) must be obtained, and hence either (A) or (B) holds.

The second corollary follows immediately from Theorem 5.1:

Corollary 5.2: Let $D = (N, B)$ and $n_1, n_2 \in N$. Then for $i = 1$ and $i = 2$:

$$r(D^i(n_1, n_2)) = r(D) \text{ or } r(D^i(n_2, n_1)) = r(D)$$

5.2. Rank Non-Increasing Reduction of Transition Graphs

Definition 5.2: A transition graph $G = (N, B_L, N_1, N_2)$ is called reduced iff

- (i) $A_c^G(n_1) \neq A_c^G(n_2)$ for all $n_1, n_2 \in N$
- (ii) $P_r^G(n_1) \neq P_r^G(n_2)$ for all $n_1, n_2 \in N$
- (iii) $(n, \lambda, n) \notin B_L$ for all $n \in N$

Note that (i) of the above definition corresponds to the notion of reduced deterministic automaton (Definition 1.13), whereas condition (ii) is always satisfied for deterministic automata, but not necessarily for transition graphs. Condition (iii), not allowing the appearance of self-loops labelled by λ , has been added for convenience. Obviously such self-loops are unnecessary and can be removed from any transition graph without affecting its behaviour.

A reduction procedure for an arbitrary transition graph $G = (N, B_L, N_1, N_2)$ can be easily obtained by applying successively the following basic construction:

Let $K \subseteq N$ be a set of nodes of G , all having the same accepted event (preceding event). Define a mapping ϕ from N onto a set $N' = (N - K) \cup \{n_K\}$, where $n_K \notin N$, by:

$$\phi(n) = n \quad \text{if } n \in N - K$$

$$\phi(n) = n_K \quad \text{if } n \in K$$

Extend ϕ to a full homomorphism (Definition 2.28) from G onto the graph $G' = (N', B'_L, \phi(N_1), \phi(N_2))$, where for every $(n, x, n') \in B_L$, $\phi(n, x, n') = (\phi(n), x, \phi(n'))$ and $B'_L = \phi(B_L)$. By Theorem 2.3, G' is equivalent to G . Thus we may repeat this procedure until a graph G'' is obtained which is equivalent to G and in which every pair of nodes have distinct preceding events and distinct accepted events. Removing all λ -self-loops from G'' yields a reduced transition graph G_R equivalent to G as required.

The only disadvantage in this method is in that the graph G_R

might have greater rank than G . Such a case is shown in the following example.

Example 5.3: Let G be the graph in Fig. 5.3(a). Clearly $r(G) = 1$, and it can be easily verified that $A_c^G(n_1) = A_c^G(n_2)$. Thus merging these two nodes yields the equivalent graph G' (Fig. 5.3(b)) which has rank 2.

Our aim in this section is to develop a reduction procedure of transition graphs, in which the rank of the graph cannot be increased.

Definition 5.3: Let $G = (N, B_L, N_1, N_2)$ be a transition graph and let $m, n \in N$.[†] Define two transition graphs $G^1(m, n)$ and $G^2(m, n)$ as follows:

$$G^i(m, n) = (N, B_L^i, N_1, N_2), \quad i = 1, 2,$$

where

$$B_L^1 = B_L \cup \{(m, x, n') \mid (n, x, n') \in B_L, x \in A \cup \{\lambda\}\}$$

$$B_L^2 = B_L \cup \{(m', x, n) \mid (m', x, m) \in B_L, x \in A \cup \{\lambda\}\}$$

Thus $G^1(m, n)$ is obtained from G by adding, for all $x \in A \cup \{\lambda\}$, x -transitions from m to all nodes n' in $\sigma_x^G(n)$. Similarly, $G^2(m, n)$ is obtained from G by adding, for all $x \in A \cup \{\lambda\}$, x -transitions from all x -predecessors of m to n .

Example 5.4: Let G be the graph on Fig. 5.4(a). $G^1(m, n)$ and $G^2(m, n)$ are shown in Fig. 5.4(b) and (c) respectively.

The next two lemmas follow directly from Definition 5.3.

[†] Throughout the rest of this chapter, whenever two nodes $m, n \in N$ of G are given, it is always assumed that $m \neq n$.

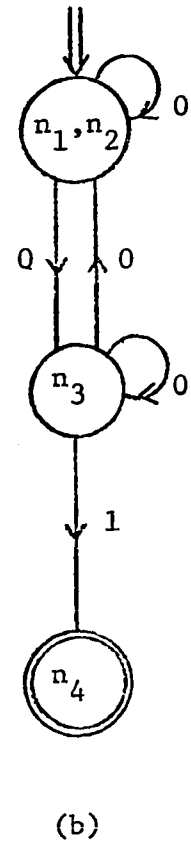
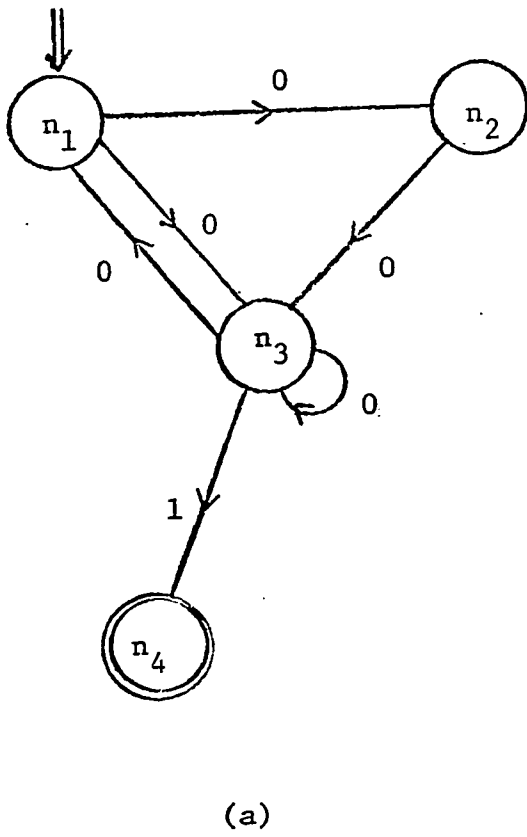
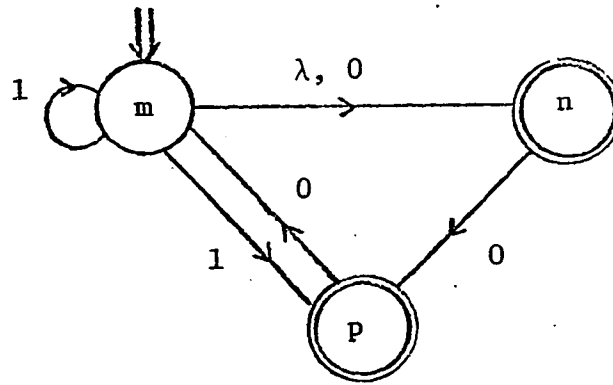
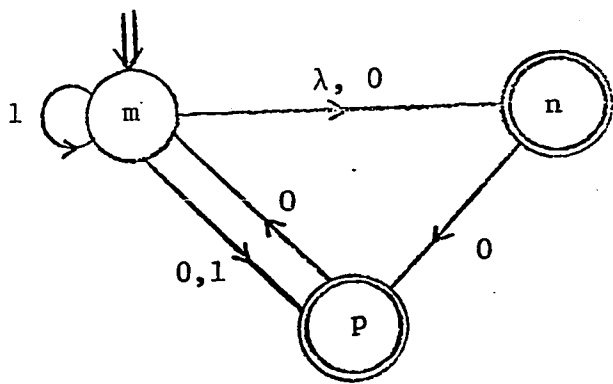


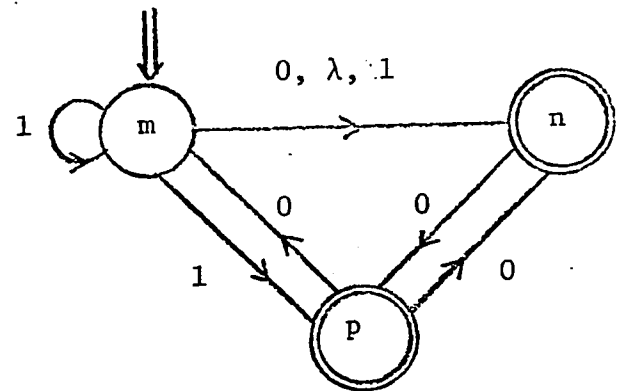
Fig. 5.3



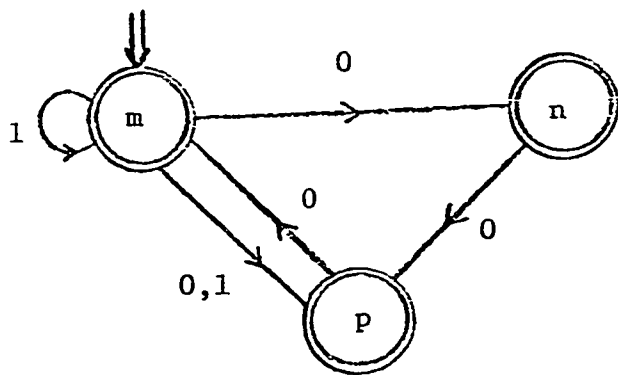
(a)



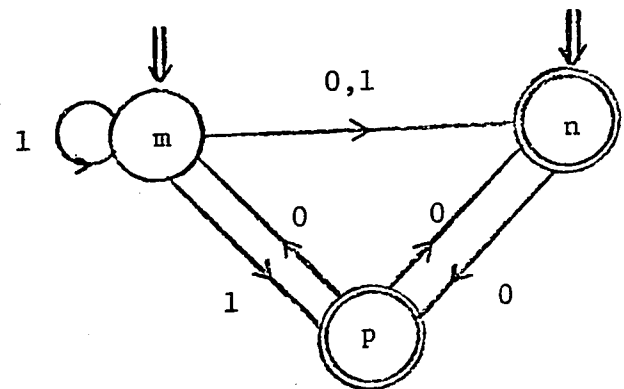
(b)



(c)



(d)



(e)

Fig. 5.4

Lemma 5.8: $\overleftarrow{G^2}(m, n) = (\overleftarrow{G})^1(n, m)$

Lemma 5.9: Let $G = (N, B_L, N_1, N_2)$, $m, n \in N$ and let:

$$\begin{aligned} G^1 &= G^1(m, n) \quad , \quad G^2 = G^2(m, n) \\ G^3 &= (G^1)^1(n, m), \quad G^4 = (G^2)^2(n, m) \\ G^5 &= (G^2)^1(n, m), \quad G^6 = (G^1)^2(n, m) \end{aligned}$$

Then for all $x \in A \cup \{\lambda\}$:

$$(1) \quad \sigma_x^{G^1}(n) \subseteq \sigma_x^{G^1}(m)$$

$$(2) \quad \pi_x^{G^2}(m) \subseteq \pi_x^{G^2}(n)$$

$$(3) \quad \sigma_x^{G^3}(n) = \sigma_x^{G^3}(m)$$

$$(4) \quad \pi_x^{G^4}(n) = \pi_x^{G^4}(m)$$

$$(5) \quad \sigma_x^{G^5}(m) \subseteq \sigma_x^{G^5}(n)$$

$$\pi_x^{G^5}(m) \subseteq \pi_x^{G^5}(n)$$

$$(6) \quad \sigma_x^{G^6}(n) \subseteq \sigma_x^{G^6}(m)$$

$$\pi_x^{G^6}(n) \subseteq \pi_x^{G^6}(m)$$

Lemma 5.10: Let $G = (N, B_L, N_1, N_2)$ and let $m, n \in N$ such that $\sigma_x^G(m) \subseteq \sigma_x^G(n)$ and $\pi_x^G(m) \subseteq \pi_x^G(n)$ for all $x \in A \cup \{\lambda\}$. Then $G - [m]$ is equivalent to G .

Proof: Let p be any admissible path in G spelling out a word w . Replace in p every branch of the form (m', x, m) by (m', x, n) and any branch of the form (m, y, m'') by (n, y, m'') . The new path thus obtained is also admissible and spells out w in $G - [m]$.

Lemma 5.11: Let $G = (N, B_L, N_1, N_2)$ and let $m, n \in N$. Then:

- (a) either $r(G^i(m, n)) = r(G)$ or $r(G^i(n, m)) = r(G)$ for $i = 1, 2$.
- (b) If for some $x \in A \cup \{\lambda\}$, $(n, x, m) \in B_L$ then either $r(G^1(n, m)) = r(G)$ or $r(G^2(n, m)) = r(G)$.

Proof: It can be easily verified that the directed graph $D_{G^i(m, n)}$ associated with $G^i(m, n)$ [Definition 2.22] is $(D_G)^i(m, n)$ [Definition 5.1]. Hence the result follows directly from Corollary 5.2 and Corollary 5.1.

Lemma 5.12: Let $G = (N, B_L, N_1, N_2)$ and let $m, n \in N$ such that

$A_c^G(m) \supseteq A_c^G(n)$ [$P_r^G(m) \subseteq P_r^G(n)$]. Then:

- (a) The graph $G^2 = G^2(m, n)$ [$G^1 = G^1(m, n)$] is equivalent to G .
- (b) $A_c^{G^2}(n') = A_c^G(n')$ [$P_r^{G^1}(n') = P_r^G(n')$] for all nodes $n' \in N$.

Proof: (b) Let $G^2 = (N, B_L^2, N_1, N_2)$ and assume $A_c^G(m) \supseteq A_c^G(n)$.

By definition $B_L^2 \supseteq B_L$, and all branches in $B_L^2 - B_L$ are of the form (m', x, n) , where $x \in A \cup \{\lambda\}$ and $m' \in \pi_x^G(m)$. Let n' be any node of G . Then clearly $A_c^G(n') \subseteq A_c^{G^2}(n')$. Now suppose $w \in A_c^{G^2}(n')$. Then there exists in G^2 a path p from n' to a final node spelling out w .

If p is also a path in G then $w \in A_c^G(n')$. Otherwise p must have branches from $B_L^2 - B_L$. Thus suppose p has $k > 0$ branches from $B_L^2 - B_L$ and let (m', x, n) be the rightmost one, i.e. $p = p' \cdot (m', x, n) \cdot p''$, where $(m', x, n) \in B_L^2 - B_L$ and p'' is a path in G . Let $w = w' x w''$ be the corresponding decomposition of w . Clearly $w'' \in A_c^G(n)$ and hence also $w'' \in A_c^G(m)$; thus there exists in G a path p_1'' from m to a final node spelling out w'' . Now the path $p_1 = p' \cdot (m', x, m) \cdot p_1''$ is also a path in G^2 from n' to a final node spelling out w . Moreover, p_1 has only $k - 1$ branches which are not contained in G . Thus after repeating this procedure $k - 1$ more times we finally obtain a path from n' to a final node, which is contained in G and spells out w . Hence $w \in A_c^G(n')$ and $A_c^G(n') = A_c^{G^2}(n')$.

(a) Since G^2 has the same initial nodes as G , the result follows directly from (b).

The proof for G^1 and $P_r^{G^1}(n')$, $P_r^G(n')$, is obtained by duality.

Lemma 5.13: Let $G = (N, B_L, N_1, N_2)$ and let $\ell = (n_1, \lambda, n_2)(n_2, \lambda, n_3) \dots (n_k, \lambda, n_1)$ be a non-trivial λ -loop in G . Let ϕ be a homomorphism from G onto a graph $\phi(G) = G' = (N', B_L', N_1', N_2')$ defined by:

$$N' = N - \{n_1, n_2, \dots, n_k\} \cup \{m\}$$

where $m \notin N$,

$$\phi(n_1) = \phi(n_2) = \dots = \phi(n_k) = m$$

$$\phi(n) = n, \quad n \neq n_i, \quad i = 1, \dots, k$$

$$\forall i = 1, \dots, k-1, \quad \phi(n_i, \lambda, n_{i+1}) = m$$

$$\phi(n_k, \lambda, n_1) = m$$

and for all other branches $(n, x, n') \in B_L$,

$$\phi(n, x, n') = (\phi(n), x, \phi(n'))$$

and

$$N'_i = \phi(N_i), \quad i = 1, 2$$

Then $r(G') \leq r(G)$, and G' is equivalent to G .

Proof: Let $C = \{n_1, n_2, \dots, n_k\}$. Define two new sets of initial and terminal nodes in G by:

$$N_i'' = \begin{cases} N_i & \text{if } N_i \cap C = \emptyset \\ N_i \cup C & \text{otherwise} \end{cases}$$

$i = 1, 2$, and let $G'' = (N, B_L, N_1'', N_2'')$. One can easily verify that G'' is equivalent to G . Furthermore, the mapping ϕ is a pathwise homomorphism from G'' to G' which satisfies the conditions of Theorem 2.4. To see this, let p' be any path in G' . Replace in p' any branch b'_i entering m by a corresponding branch $b_i \in \phi^{-1}(b'_i)$ in G entering one of the n_j 's, say n_t ($1 \leq t \leq k$). Then replace the succeeding branch b'_{i+1} leaving m by a corresponding branch $b_{i+1} \in \phi^{-1}(b'_{i+1})$ leaving one of the n_j 's, say n_r , preceded by the λ -path from n_t to n_r . In this way one

obtains a path p in G'' such that $\phi(p) = p'$. Hence ϕ is a pathwise homomorphism. By McNaughton's Pathwise Homomorphism Theorem, $r(G') \leq r(G'') = r(G)$, and by Theorem 2.4 G' is equivalent to G'' , and hence to G .

Lemma 5.14: Let $G = (N, B_L, N_1, N_2)$ and let $m, n \in N$ such that:

- (a) $A_c^G(m) \subseteq A_c^G(n) \quad [P_r^G(m) \subseteq P_r^G(n)]$
- (b) $\forall x \in A \cup \{\lambda\}, \pi_x^G(m) \subseteq \pi_x^G(n) \quad [\sigma_x^G(m) \subseteq \sigma_x^G(n)]$
- (c) There does not exist in G a path from n to m spelling out λ [a path from m to n spelling out λ]
- (d) If $m \in N_1$ then also $n \in N_1$ [if $m \in N_2$ then also $n \in N_2$].

Then the graph $G' = G - [m]$ is equivalent to G .

Proof: Obviously the event accepted by G contains the event accepted by G' . Now let w be a word accepted by G . Then there exists in G an admissible path p spelling out w . If p does not touch m then it is also a path in G' and then w is accepted also by G' . Thus suppose p passes through m . Then we have two cases: (i) p starts at m . (ii) p starts at some other node. In Case (i), since $A_c^G(m) \subseteq A_c^G(n)$, we may replace p by some other admissible path starting at n and spelling out w (recall that by (d) if m is an initial node, $m \in N_1$, then so is n). Thus we may assume we have Case (ii). So if p touches m , let

$p = p' \cdot (m', x, m) \cdot (m, y, m'') \cdot p''$, where p' is the maximal prefix of p that does not touch m , $x, y \in A \cup \{\lambda\}$ and $m', m'' \in N$. Let $w = w'xyw''$ be the corresponding decomposition of w . Clearly $yw'' \in A_c^G(m)$ and thus $yw'' \in A_c^G(n)$. Thus there exists in G a path p_1'' from n to a final node spelling out yw'' . Moreover, since by (b) $\pi_x^G(m) \subseteq \pi_x^G(n)$, we have $(m', x, n) \in B_L$. Then the path $p_1 = p' \cdot (m', x, n) \cdot p_1''$ is also an admissible path of G spelling out w . If p_1'' still touches m , we repeat the same procedure once more. Thus let $p_1'' = p_2' \cdot (m_1', x', m) \cdot (m, y', m_1'') \cdot p_2''$, where p_2' is the maximal prefix of p_1'' not touching m . Let the corresponding decomposition of w'' be: $w'' = w_2'x'y'w_2''$. Since by assumption (c) there does not exist in G any path from n to m spelling out λ , the word $w_2'x'$, which is spelled out by $p_2' \cdot (m_1', x', m)$, cannot be λ . Now by the same argument as above, the suffix $(m_1', x', m) \cdot (m, y', m_1'') \cdot p_2''$ can be replaced by a path $(m_1', x', n) \cdot p_3''$, which also leads to a final node and spells out $x'y'w_2''$. Thus we obtain a new path $p_2 = p' \cdot (m', x, n) \cdot p_2' \cdot (m_1', x', n) \cdot p_3''$, in which only the suffix p_3'' may touch m . But this suffix spells out the word $y'w_2''$, which is clearly shorter than the word yw'' spelled out by the corresponding suffix p_1'' of p_1 . This shows that repeating this procedure enough times, we eventually end up with a path p_k ($k \geq 1$) which is admissible in G and spells out w , in which the length of the suffix touching m is 0, i.e. p_k does not touch m at all. Hence p_k is also an admissible path in $G' = G - [m]$ and w is accepted also by G' . This completes the proof of the first part of the theorem. The proof of the second part (with $P_r^G(m) \subseteq P_r^G(n)$, etc.) is obtained in a dual way.

Theorem 5.2: For any transition graph $G = (N, B_L, N_1, N_2)$ there exists an equivalent reduced transition graph $G' = (N', B_L, N'_1, N'_2)$ such that $\#N' \leq \#N$ and $r(G') \leq r(G)$.

Proof: The reduction of G will be achieved by recursive application of the following two basic steps:

Step (A): Let G_i ($i = 0, 1, \dots$) be the graph obtained from G in this process after i basic steps, and let m, n be two nodes of G_i such that $A_c^{G_i}(m) = A_c^{G_i}(n)$. Then by Lemma 5.11 (a) either $r(G_i^2(m, n)) = r(G_i)$ or $r(G_i^2(n, m)) = r(G_i)$. Without loss of generality we may assume the former. By Lemma 5.12, $G_{i2} = G_i^2(m, n)$ is equivalent to G_i and has the same set of accepted events. Thus we may replace G_i by G_{i2} and m and n have equal accepted events also in G_{i2} . Furthermore, by Lemma 5.9 (2), $\pi_x^{G_{i2}}(m) \subseteq \pi_x^{G_{i2}}(n)$ for all $x \in A \cup \{\lambda\}$. Now we have two cases:

Case I: there does not exist any path in G_{i2} from n to m spelling out λ . Then all requirements of Lemma 5.14 are satisfied and hence $G_{i2} - [m]$ is equivalent to G_{i2} . In this case define $G_{i+1} = G_{i2} - [m]$.

Case II: there exists in G_{i2} a path p from n to m spelling out λ . If $\ell(p) \geq 2$, then we have: $p = (n, \lambda, m_1)(m_1, \lambda, m_2) \dots (m_k, \lambda, m)$ for some $k \geq 1$. By Proposition 2.4:

$A_c^{G_{i2}}(n) \supseteq A_c^{G_{i2}}(m_1) \supseteq \dots \supseteq A_c^{G_{i2}}(m_k) \supseteq A_c^{G_{i2}}(m)$, and since m and n have equal accepted events, all the above accepted events are equal. In this case start Step A again, replacing m by m_1 . Thus we may assume that p has

length 1, i.e. there exists in G_{i2} a λ -transition (n, λ, m) . Then if there is also a path from m to n spelling out λ in G_{i2} , then there is a non-trivial λ -loop in G_{i2} . In that case define G_{i+1} to be the graph obtained from G_{i2} by shrinking this λ -loop as indicated in Lemma 5.13.

Thus assume that (n, λ, m) is in G_{i2} but that there does not exist in it any path from m to n spelling out λ . Now by Lemma 5.11 (b), either (1) $r(G_{i2}^1(n, m)) = r(G_{i2})$ or (2) $r(G_{i2}^2(n, m)) = r(G_{i2})$. If (2) is satisfied, we may replace G_{i2} by $G_{i3} = G_{i2}^2(n, m)$, which, by Lemma 5.12 (a), is equivalent to G_{i2} . Now by Lemma 5.12 (b), m and n still have equal accepted events in G_{i3} , and by Lemma 5.9 (2), $\pi_x^{G_{i3}}(n) \subseteq \pi_x^{G_{i3}}(m)$ for all $x \in A \cup \{\lambda\}$. Moreover, if n is an initial node in G_{i3} , we can define also m to be an initial node in G_{i3} without affecting the event accepted by G_{i3} . Now all requirements of Lemma 5.14 are satisfied and hence the graph $G_{i3} - [n]$ is equivalent to G_{i3} . Define in this case $G_{i+1} = G_{i3} - [n]$.

Now suppose (1) is satisfied. Since, by our assumption, the λ -transition (n, λ, m) is in G_{i2} , we have, by Proposition 2.4 (2) (page 28), $P_r^{G_{i2}}(m) \supseteq P_r^{G_{i2}}(n)$.

Thus by Lemma 5.12 (a), the graph $G_{i5} = (G_{i2})^1(n, m)$ is equivalent to G_{i2} and we may replace the latter by the former. Now by Lemma 5.9 (5), $\sigma_x^{G_{i5}}(m) \subseteq \sigma_x^{G_{i5}}(n)$ and $\pi_x^{G_{i5}}(m) \subseteq \pi_x^{G_{i5}}(n)$ and by Lemma 5.10, $G_{i5} - [m]$ is equivalent to G_{i5} . Define in this case $G_{i+1} = G_{i5} - [m]$.

Step (B): Let m, n be two nodes in G_i such that $P_r^{G_i}(m) = P_r^{G_i}(n)$. G_{i+1} is obtained from G_i in a way dual to that defined in Step (A) (Lemma 5.8).

Our reduction procedure is as follows:

- (i) Set $G = G_0$
- (ii) If there exist in G_i ($i = 0, 1, \dots$) any non-trivial λ -loops, shrink them as indicated by Lemma 5.13.
- (iii) If there exist in G_i two nodes m, n such that $A_c^{G_i}(m) = A_c^{G_i}(n)$, apply Step (A) to G_i and return to (ii) with G_{i+1} .
- (iv) If there exist nodes m, n in G_i such that $P_r^{G_i}(m) = P_r^{G_i}(n)$, apply Step (B) to G_i and return to (ii) with G_{i+1} .
- (v) Remove all λ -self-loops from G_i and define $G_i = G'$.

Obviously each graph G_i obtained in the above process is equivalent to G , $r(G_i) \leq r(G)$ and G_i has at least i nodes less than G . Hence the process must terminate after a finite number of steps, yielding the desired reduced graph G' .

5.3. Elimination of λ -transitions

We now use the results of the previous sections to obtain a procedure for replacing all λ -transitions in a given transition graph G by some other non- λ transitions. The resulting graph will be a non-deterministic state graph, equivalent to G and of rank no greater than the

rank of G .

We first present a construction by which a single λ -transition in a transition graph can be replaced by a set of some other transitions (some of which might be again labelled by λ) in such a way that the rank of the graph will not be increased. Such a replacement will be called legitimate.

Definition 5.4: Let $G = (N, B_L, N_1, N_2)$ be a transition graph and let $m, n \in N$ such that $(m, \lambda, n) \in B_L$. Let $G^i(m, n) = (N, B_L^i, N_1, N_2)$, $i = 1, 2$, (Definition 5.3). We now define two more graphs, $G_{mn}(1)$ and $G_{mn}(2)$ as follows:

$$G_{mn}(i) = (N, B_L^i - \{(m, \lambda, n)\}, N_1^i, N_2^i), i = 1, 2$$

where:

$$N_1^1 = N_1$$

$$N_2^2 = N_2$$

$$N_1^2 = \begin{cases} N_1 \cup \{n\} & \text{if } m \in N_1 \\ N_1 & \text{otherwise} \end{cases}$$

$$N_2^1 = \begin{cases} N_2 \cup \{m\} & \text{if } n \in N_2 \\ N_2 & \text{otherwise} \end{cases}$$

Thus $G_{mn}(i)$ is obtained from $G^i(m, n)$ by removing the λ -transition (m, λ, n) and possibly adding m to N_2 or n to N_1 .

Example 5.6: Let G be the graph in Fig. 5.4 (a) (Page 100). Since there exists in G a λ -transition from m to n , $G_{mn(1)}$ and $G_{mn(2)}$ can be constructed, and are shown in Fig. 5.4 (d) and (e) respectively.

Lemma 5.15: For any transition graph G and for any λ -transition (m, λ, n) in G ,

$$(\overleftarrow{G})_{mn(1)} = \overleftarrow{G_{nm(2)}}$$

Proof: This follows directly from Lemma 5.8 and the above definition.

Lemma 5.16: If $G = (N, B_L, N_1, N_2)$ is a reduced transition graph, then there does not exist in G any λ -loop.

Proof: Let $p = (m_1, \lambda, m_2) \dots (m_k, \lambda, m_1)$ be any λ -loop of G . Then by Proposition 2.4, $A_c^G(m_1) = A_c^G(m_2) = \dots = A_c^G(m_k)$ and G cannot be reduced. Also a λ -self-loop cannot exist in G since G is reduced.

Lemma 5.17: Let $G = (N, B_L, N_1, N_2)$ be a reduced transition graph and let $(m, \lambda, n) \in B_L$. Then:

- (1) $G_{mn(1)}$ and $G_{mn(2)}$ are reduced and equivalent to G .
- (2) Either $r(G_{mn(1)}) \leq r(G)$ or $r(G_{mn(2)}) \leq r(G)$.

Proof: The second part follows directly from the above definitions and the application of Corollary 5.1 and Lemma 2.1.

As for (1), let n' be a node of G and let w be any word accepted by n' . Then w is spelled out by a path $p = b_1 b_2 \dots b_k$, leading from n' to a terminal node. Modify p in the following way: for every b_i , $1 \leq i \leq k$, such that $b_i = (m, \lambda, n)$ and $b_{i+1} = (n, x, n'')$ for some $n'' \in N$ and $x \in A \cup \{\lambda\}$, replace $b_i b_{i+1}$ by $b'_i = (m, x, n'')$. Since G is reduced, $b_{i+1} \neq (n, \lambda, n)$; hence $b'_i \neq (m, \lambda, n)$, and it follows from the definition of $G_{mn(1)}$ that b'_i is one of its branches. In addition, if $b_k = (m, \lambda, n)$, then remove b_k from p . Since n must be in this case a terminal node of G , m is a terminal node of $G_{mn(1)}$. Hence the modified path p' obtained from p in this manner is a path in $G_{mn(1)}$ from n' to a terminal node, spelling out the same word w . Hence $A_c^G(n') \subseteq A_c^{G_{mn(1)}}(n')$.

Now let w be a word accepted by n' in $G_{mn(1)}$. Then w is spelled out by a path $p = b_1 b_2 \dots b_k$ in $G_{mn(1)}$ leading from n' to a terminal node. Replace in p every branch $b_i = (m, x, n'')$, which does not belong to G , by $b_i^1 b_i^2 = (m, \lambda, n)(n, x, n'')$, and also in case $b_k = (m', x, m)$ for some $m' \in N$, $x \in A \cup \{\lambda\}$, concatenate p by $b_{k+1} = (m, \lambda, n)$. The new path thus obtained is clearly a path in G from n' to a terminal node of G , spelling out w . Thus we get: $A_c^G(n') = A_c^{G_{mn(1)}}(n')$ for all $n' \in N$, and hence G and $G_{mn(1)}$ have the same set of accepted events. Also, by definition, the set of initial nodes of G is identical with that of $G_{mn(1)}$. Furthermore $G_{mn(1)}$ cannot have any λ -self-loop (n', λ, n') because G does not have any, and if (n', λ, n') were one of the new branches added to G in constructing $G_{mn(1)}$, we would have $n' = m$, $(n, \lambda, m) \in B_L$ and hence $(m, \lambda, n)(n, \lambda, m)$ would be a λ -loop contained in G , contradicting the fact that G is reduced

(Lemma 5.16). All these imply that $G_{mn(1)}$ is equivalent to G and is also reduced.

The proof for $G_{mn(2)}$ is obtained by duality (Lemma 5.15), and in this case we get coincidence of the preceding events of $G_{mn(2)}$ with those of G .

Thus every λ -transition (m, λ, n) in a transition graph G can be replaced by some other transitions so that (after appropriate changes in the sets of initial and final nodes) the resulting graph is equivalent to G and has no higher rank.

Definition 5.5: Let $G = (N, B_L, N_1, N_2)$ and let $(m, \lambda, n) \in B_L$. The construction of either $G_{mn(1)}$ or $G_{mn(2)}$ is called a replacement of (m, λ, n) in G . A replacement is called legitimate iff the corresponding graph $G_{mn(i)}$ has rank no greater than $r(G)$. The sets of branches replacing (m, λ, n) in $G_{mn(1)}$ and $G_{mn(2)}$ are, respectively: $B_1 = \{(m, x, n') \mid (n, x, n') \in B_L\}$ and $B_2 = \{(m', x, n) \mid (m', x, m) \in B_L\}$.

Thus every λ -transition in G has two replacements, at least one of which, by the previous lemma, is legitimate.

Example 5.7: For the replacements of (m, λ, n) shown in Fig. 5.4 (a), (d) and (e) (page 100), $B_1 = \{(m, 0, p)\}$ and $B_2 = \{(p, 0, n), (m, 1, n)\}$. One can easily verify that $r(G_{mn(1)}) = r(G) = 1$ and $r(G_{mn(2)}) = 2 > r(G)$. Hence the replacement of (m, λ, n) by B_1 is legitimate whereas the second replacement, by B_2 , is not.

Throughout the rest of this section let $G = (N, B_L, N_1, N_2)$ be a fixed reduced transition graph.

Definition 5.6: Let $<_G$ be the relation on the set of nodes N defined by: $\forall m, n \in N, m <_G n$ iff there exists in G a non-null path from m to n spelling out the word λ .

Lemma 5.18: $<_G$ is a (strong) partial order on N .

Proof: For any $n \in N, n \not<_G n$ by Lemma 5.16. For any pair of nodes $m, n \in N, m <_G n$ and $n <_G m$ implies the existence of a λ -loop through m and n , which is again impossible by Lemma 5.16. The transitivity follows directly from the definition.

Define $m \leq_G n$ iff $m <_G n$ or $m = n$.

Definition 5.7: The relation $\prec_G$ is defined on the set $N \times \{\lambda\} \times N$ by: $(m_1, \lambda, n_1) \prec_G (m_2, \lambda, n_2)$ iff either $m_1 \leq_G m_2$ and $n_2 <_G n_1$ or $m_1 <_G m_2$ and $n_2 \leq_G n_1$.

Lemma 5.19: $\prec_G$ is a strong partial order on $N \times \{\lambda\} \times N$.

Proof: Follows directly from the above definition and Lemma 5.18.

Example 5.8: Let $G = (N, B_L, N_1, N_2)$ be the transition graph on

Fig. 5.6. The partial order $<_G$ is given by:

$$<_G = \{(p, m), (p, q), (q, m), (q, r), (p, r), (n, m), (n, r), (m, r)\}$$

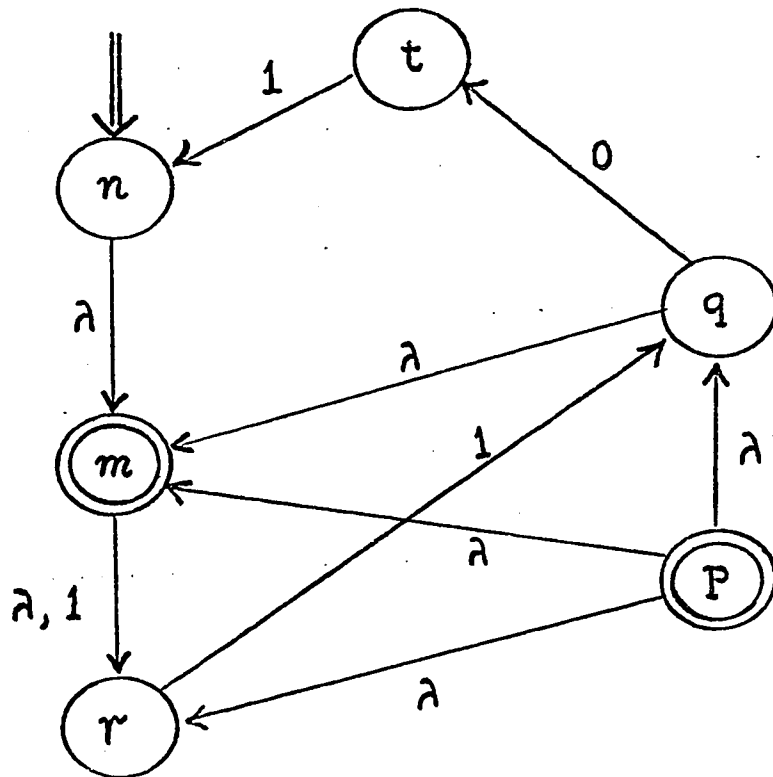


Fig. 5.6.

The restriction of the partial order $\prec_G$ to the set of all λ -transitions of G is:

$$\begin{aligned} & \prec_G / (N \times \{\lambda\} \times N) \cap B_L = \\ & = \{((p, \lambda, r), (p, \lambda, m)), ((p, \lambda, r), (p, \lambda, q)), \\ & \quad ((p, \lambda, r), (m, \lambda, r)), ((p, \lambda, m), (p, \lambda, q)), \\ & \quad ((p, \lambda, m), (q, \lambda, m)), ((p, \lambda, r), (q, \lambda, m))\} \end{aligned}$$

An example of a related pair of λ -transitions, one of which is not contained in G , is: $(p, \lambda, r) \prec_G (p, \lambda, n)$.

Lemma 5.20: Let $(m, \lambda, n) \in B_L$ and let B_i be the set of branches replacing (m, λ, n) in $G_{mn(i)}$, $i = 1, 2$. Then for any (m', λ, n') in B_i , $(m', \lambda, n') \prec_G (m, \lambda, n)$.

Proof: By definition, $m' = m$ if (m', λ, n') is in B_1 and $n' = n$ if this branch is in B_2 . In the former case $(n, \lambda, n') \in B_L$ and hence $n \prec_G n'$, and in the latter case $(m', \lambda, m) \in B_L$ and thus $m' \prec_G m$. In both cases we get $(m', \lambda, n') \prec_G (m, \lambda, n)$.

Lemma 5.21: If G' is a transition graph obtained from G by a finite series of replacements then:

- (a) For any $m', n' \in N$, $m' <_{G'} n'$ implies $m' <_G n'$.
- (b) For any two λ -transitions $b, b' \in N \times \{\lambda\} \times N$, $b <_{G'} b'$ implies $b <_G b'$.

Proof: It suffices to prove (a) and (b) only for the case when G' is obtained from G by a single replacement. The result then follows by induction. Thus assume that $(m, \lambda, n) \in B_L$ and let $G' = G_{mn(i)}$ for $i = 1$ or 2 .

- (a) For $i = 1$: Let $m' <_{G'} n'$ for some $m', n' \in N$. Then there exists in G' a non-null path p from m' to n' spelling out the word λ . If p is also a path in G then clearly $m' <_G n'$. If, however, p is not a path in G , then p contains branches of the replacement of (m, λ, n) , which are of the form (m, λ, n') , where $(n, \lambda, n') \in B_L$. Replacing each such branch (m, λ, n') in p by $(m, \lambda, n)(n, \lambda, n')$, one obtains a new path p' from m' to n' , which is a path in G and spells out λ . Hence $m' <_G n'$ as required. The proof for $i = 2$ follows by duality (Lemma 5.15).

(b) for $i = 1$: Let $b = (m_1, \lambda, n_1)$, $b' = (m_2, \lambda, n_2)$ and suppose $b' \prec_G b$. By definition, either $m_2 \leq_G m_1$ and $n_1 \prec_G n_2$, or $m_2 \prec_G m_1$ and $n_1 \leq_G n_2$. By part (a) of this lemma, we get in the former case $m_2 \leq_G m_1$ and $n_1 \prec_G n_2$ and in the latter case $m_2 \prec_G m_1$ and $n_1 \leq_G n_2$, both of which yield the result.

Theorem 5.3: For any reduced transition graph G , there exists an equivalent reduced non-deterministic state graph G' with the same number of nodes and no greater rank.

Proof: We shall first prove the following assertion:

(**) For any reduced transition graph $G = (N, B_L, N_1, N_2)$ and for any set of λ -transitions $\{b_1, b_2, \dots, b_k\} \subseteq N \times \{\lambda\} \times N$ ($k = 1, 2, \dots$), G can be converted by a series of legitimate replacements into a graph $G^k = (N, B_L^k, N_1^k, N_2^k)$ such that $b_i \notin B_L^k$ for all $i = 1, \dots, k$.

The proof will proceed by induction on k :

Basis: $k = 1$. If $b_1 \notin B_L$ then $G^1 = G$ is the required graph. If $b_1 \in B_L$ then by Lemma 5.16 there exists a legitimate replacement of b_1 . Let the corresponding graph be $G_{mn(i)}$, where $b_1 = (m, \lambda, n)$ and $i = 1$ or 2 . By definition, $G_{mn(i)}$ does not contain b_1 and hence $G^1 = G_{mn(i)}$ is the required graph.

Induction Step: Assume (**) holds for $k - 1$. Let

$\{b_1, b_2, \dots, b_k\} \subseteq N \times \{\lambda\} \times N$ be any set of k λ -transitions.

Since this set is finite and partially ordered by the relation $\prec_G$,

there exists in it an element which is minimal under this partial

ordering, i.e. for some i , $1 \leq i \leq k$, $b_j \not\prec_G b_i$ for all $j = 1, \dots, k$.

Without loss of generality we may assume that b_k is such a minimal

element. Now, applying the induction hypothesis to $\{b_1, \dots, b_{k-1}\}$,

convert G by a finite series of legitimate replacements into a graph G^{k-1} ,

which contains none of the branches $b_1, \dots, b_{k-1}$. If also b_k is not a

branch of G^{k-1} then $G^k = G^{k-1}$ is the required graph. Otherwise, replace

b_k in G^{k-1} by a legitimate replacement, and let B be the set of branches

replacing b_k in this replacement. By Lemma 5.20, $b \prec_{G^{k-1}} b_k$ for any

λ -transition b in B . On the other hand it follows from Lemma 5.21 that

b_k is a minimal element in the set $\{b_1, \dots, b_k\}$ also with respect to

the partial order $\prec_{G^{k-1}}$. We conclude that $B \cap \{b_1, \dots, b_k\} = \emptyset$. Hence

the graph G^k obtained from G^{k-1} by a legitimate replacement of b_k does

not have any of $b_1, \dots, b_k$ among its branches and is therefore the

required graph.

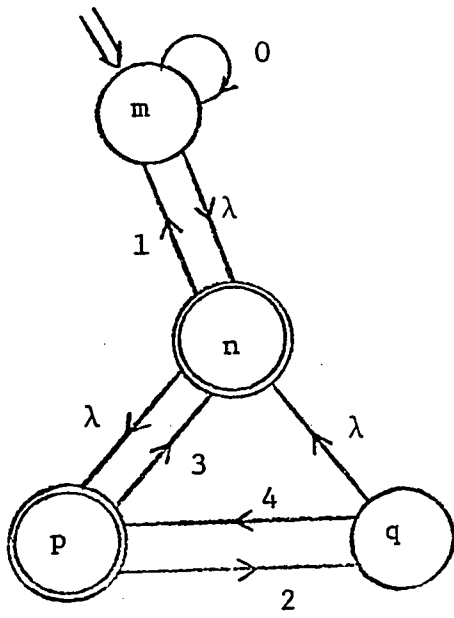
Now apply (**) to $\{b_1, \dots, b_k\} = N \times \{\lambda\} \times N$, where $k = (\#N)^2$. Then the graph $G' = G^k$ does not contain any λ -transitions and is therefore a non-deterministic state graph. Moreover, since G' has been obtained from G by a series of legitimate replacements, Lemma 5.17 implies that G' is reduced, equivalent to G and $r(G') \leq r(G)$ as required.

Example 5.9: Let G be the reduced graph shown in Fig. 5.7 (a).

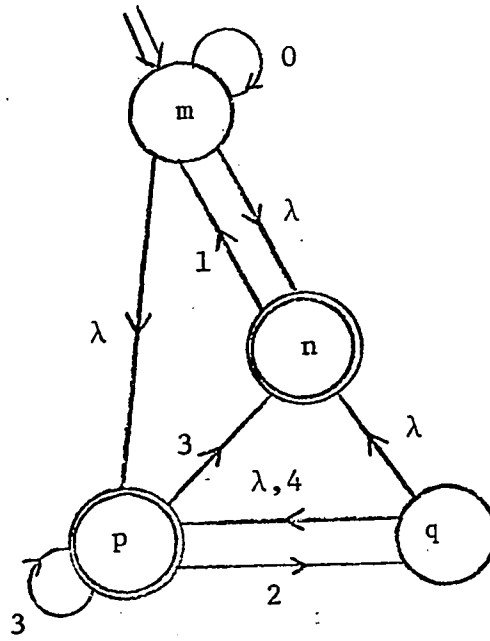
First replace the λ -transition (n, λ, p) by constructing the graph $G^1 = G_{np(2)}$ (Fig. 5.7 (b)). Since $r(G^1) = 2 = r(G)$ this is a legitimate replacement. The set of branches replacing (n, λ, p) in G^1 is $B^1 = \{(m, \lambda, p), (q, \lambda, p), (p, 3, p)\}$, thus G^1 has two new λ -transitions. Go on replacing (q, λ, p) by the single branch $(p, 2, p)$, thus obtaining the graph $G^2 = (G^1)_{qp(2)}$ (Fig. 5.7 (c)). Now replace (m, λ, n) by constructing $G^3 = (G^2)_{mn(1)}$ (Fig. 5.7 (d)), in which m is also a terminal node. Proceed by replacing (m, λ, p) and then (q, λ, n) , obtaining the graphs $G^4 = (G^3)_{mp(2)}$ and $G^5 = (G^4)_{qn(2)}$ (Fig. 5.7 (e), (f)). The graph $G^5 = G'$ is the required non-deterministic state graph equivalent to G .

The following corollary is an immediate result of Theorem 5.2 and Theorem 5.3.

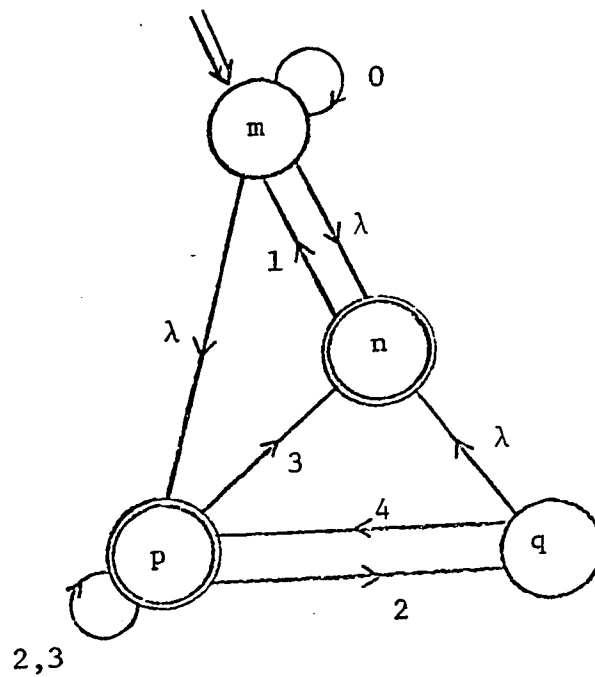
Corollary 5.3: For any transition graph G , there exists an equivalent reduced non-deterministic state graph G' having no more nodes than G and no higher rank.



(a): G

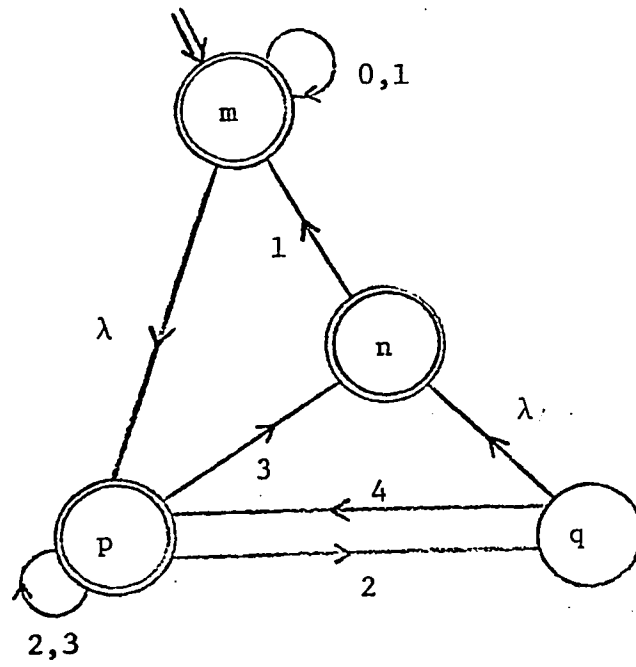


(b): G^1

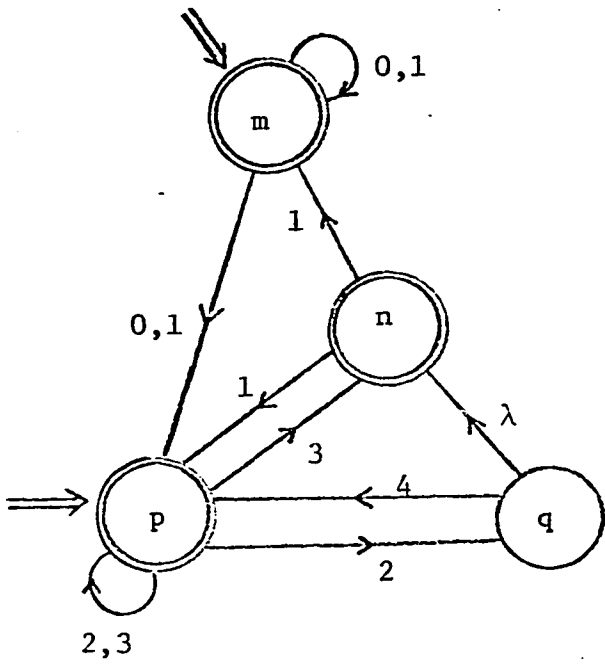


(c): G^2

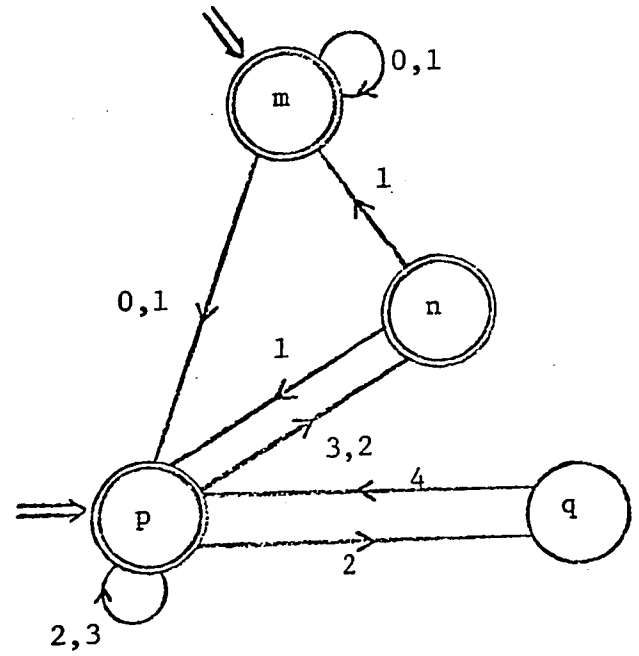
Fig. 5.7



(d): G^3



(e): G^4



(f): $G^5 = G'$

Fig. 5.7

5.4. Applications

An important application of the results obtained in the previous sections is the following stronger version of Eggen's Star Height Theorem.

Theorem 5.4: The star height of a regular event R equals $\min_{G \in \mathcal{F}'_R} \{r(G)\}$,

where $\mathcal{F}'_R$ is the family of all non-deterministic reduced state graphs recognizing R .

We next consider an interesting by-product of Corollary 5.3.

Definition 5.8: For any regular event R , let $M_{nda}(R)$ be the smallest number of states of any non-deterministic automaton recognizing R . Let $M_{tg}(R)$ be the smallest number of nodes of any transition graph accepting R .

An algorithm for finding $M_{nda}(R)$ and some of the corresponding minimum-state non-deterministic automata accepting the event R is discussed in [KW]. As a consequence of Corollary 5.3, the same algorithm applies also for the more general case of finding minimum-state transition graphs recognizing R .

Corollary 5.4: For every regular event R , $M_{nda}(R) = M_{tg}(R)$.

Chapter 6

Further Methods for Establishing Star Height

Introduction: In Chapter 4 the star height of certain families of regular events has been established by relating the rank of the corresponding reduced state graphs to the apparent star height of regular expressions denoting these events. In this chapter a different approach, based on Theorem 5.4, is adopted. Instead of investigating the structure of a regular expression denoting the given event R , we study the rank of a non-deterministic state graph recognizing R . Thus $h(R) = k$ will be proved by showing that any arbitrary non-deterministic state graph accepting R has rank no less than k . This approach is similar to the one employed by McNaughton in [Mc 3,4].

In Section 6.1 two functions, called the R -projection functions, are introduced. These functions, relating the nodes and paths of an arbitrary state graph with sets of nodes and paths in the reduced state graph $\gamma_0(R)$, constitute the foundation for the methods developed in this chapter. By means of these functions, the concept of one state graph covered by another is defined, and in Section 6.2 two fundamental lemmas concerning this concept are proved.

Section 6.3 presents a new proof technique for establishing the star height of a regular event. In some cases only lower bounds to the star height can be obtained but in many cases the exact star height

is found. This technique is illustrated by examples.

In Section 6.4 some results are obtained for the case when $\tilde{G}_0(R)$ is a reset-free state graph. As we have seen in Chapter 4, if $\tilde{G}_0(R)$ is reset-free and has only a single final state, then the star height of R equals the rank of $\tilde{G}_0(R)$. However, this is not always the case when $\tilde{G}_0(R)$ is reset-free with more than one output state. Nevertheless, it is shown that a partial extension of the above result can be achieved if we consider only state graphs $\tilde{G}_0(R)$ which are both reset-free and permutation-free. Some other cases are also considered and sufficient conditions for the star height of R to be no less than the rank of some section of $\tilde{G}_0(R)$ are found.

Throughout this chapter let R be any fixed regular event, let $G = G_0(R) = (\bar{Q}, \bar{M}, q_1, F)$ be the reduced deterministic state graph recognizing R and let $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$ be the reduced incomplete state graph recognizing R . Thus $\bar{Q} = Q \cup \{q_\phi\}$ in case G has a dead state q_ϕ , or $\bar{Q} = Q$ (and hence $\tilde{G} = G$) if there is no dead state in G . In the former case, $M(q, a) = \phi$ iff $\bar{M}(q, a) = q_\phi$ for all $q \in Q, a \in A$ and $M(q, a) = \bar{M}(q, a)$ otherwise. Let $Q = \{q_1, q_2, \dots, q_{m_0}\}$ and assume $m_0 \geq 2$.

In the sequel, let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic all-admissible state graph (not necessarily equivalent to G).

6.1. The R-Projection Functions and Covering of State Graphs

Definition 6.1: Let R be a regular event and let $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$. For any non-deterministic state graph $G' = (N, B_L, N_1, N_2)$ define a function $f_R^{G'}: N \rightarrow 2^Q$ as follows:[†] for each node n of G' ,

$$f_R^{G'}(n) = \{q \mid q \in Q \wedge (P_r^{\tilde{G}}(q) \cap P_r^{G'}(n)) \neq \emptyset\}$$

Thus $f_R^{G'}(n)$ is the set of all states q in $\tilde{G}$ such that $q = M(q_1, w)$ for some word $w \in P_r^{G'}(n)$. Using the function M_1 introduced in Definition 1.15 we obtain:

$$f_R^{G'}(n) = M_1(P_r^{G'}(n))$$

Clearly for all initial nodes $n_1 \in N_1$ of G' , $q_1 \in f_R^{G'}(n_1)$.

Example 6.1: Let $\tilde{G}_0(R)$ and G' be the state graphs shown in Fig. 6.1 and 6.2 respectively. For each node n of G' , the states of $f_R(n)$ are indicated inside the circle representing n .

The function f_R has the following properties.

Lemma 6.1: Let $G' = (N, B_L, N_1, N_2)$ as above. If there exists in G' a path from n to n' ($n, n' \in N$) spelling out the word w , then $M(f_R(n), w) \subseteq f_R(n')$.

Proof: For every $q \in f_R(n)$ there exists a word u such that $M_1(u) = q$ and $u \in P_r^{G'}(n)$. Let $q' = \bar{M}(q, w)$. If $q' = q_\emptyset$ then clearly $M(q, w) = \emptyset \subseteq f_R(n')$. Otherwise $q' = M_1(uw)$ and $uw \in P_r^{G'}(n')$, which implies $q' \in f_R(n')$ as required.

[†] When no confusion arises, the superscript G' will be omitted.

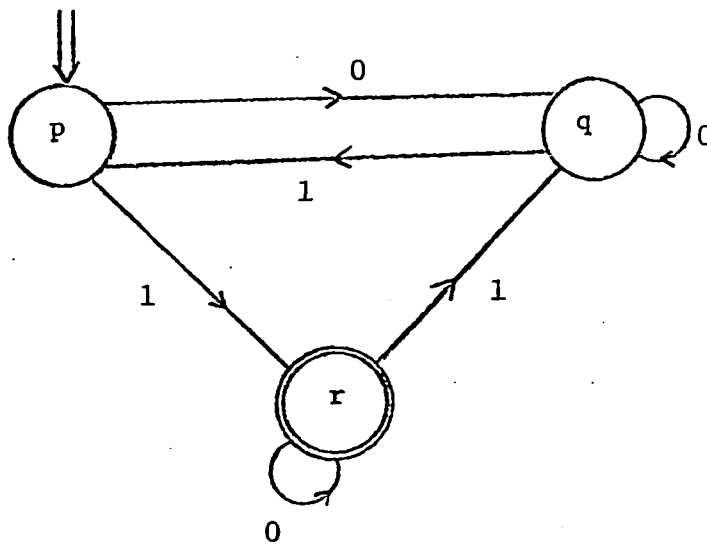


Fig. 6.1

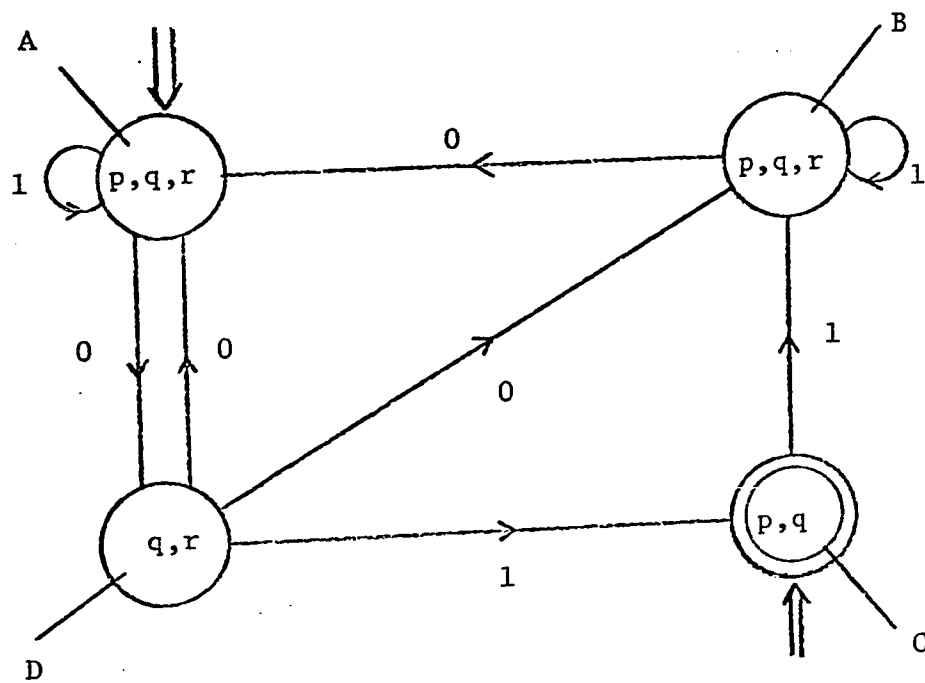


Fig. 6.2

Lemma 6.2: If $G' = (N, B_L, N_1, N_2)$ recognizes R then:

- (a) For any terminal node $n \in N_2$, $f_R(n) \subseteq F$.
- (b) If there exists in G' a path starting at a node m and spelling out the word w , then $M(q, w) \neq \emptyset$ for all states $q \in f_R(m)$. Consequently $f_R(m) = M_1(P_r^{G'}(m)) \neq \emptyset$ for all nodes $m \in N$.

Proof: (a) Let $n \in N_2$ and let $q \in f_R(n)$. Then there exists a word u such that $M_1(u) = q$ and $u \in P_r^{G'}(n)$. The latter implies that u is accepted by G' and hence $M_1(u) = q \in F$.

(b) Suppose $M(q, w) = \emptyset$ for some $q \in f_R(m)$. Then $\bar{M}(q, w) = q_\emptyset$ and also $\bar{M}(q, wv) = q_\emptyset$ for any word v . Let n be the terminal node of the path starting at m and spelling out w , and let $v \in A_c^{G'}(n)$ (such a word v exists since G' is all-admissible by assumption). Let $u \in P_r^{G'}(m) \cap P_r^{\tilde{G}}(q)$; then the word uwv is accepted by G' and hence by $\tilde{G}$. But $M_1(uwv) = M(M_1(u), wv) = M(q, wv) = \emptyset$, a contradiction.

==

Definition 6.2: Let $\tilde{G} = \tilde{G}_0(R)$ and let $G' = (N, B_L, N_1, N_2)$ be any state graph as above. Define a function $g_R^{G'}: P(G') \rightarrow 2^{P(\tilde{G})}$ as follows. For every path p' in G' , starting at a node n' and spelling out a word w , let $g_R^{G'}(p')$ (or $g_R(p')$, when the reference to G' is understood) be the set of all paths p in $\tilde{G}$ spelling out w and starting at nodes contained in $f_R(n')$. Define $g_R(\epsilon) = \epsilon$.

Thus if $p' = (n', a_1, n_2) \dots (n_k, a_k, n_{k+1})$ and

$p = (q^1, c_1, q^2) \dots (q^k, c_k, q^{k+1})$, then $p \in g_R(p')$ iff $k = \ell$, $a_i = c_i$ for all $i = 1, \dots, k$ and $q^1 \in f_R(n)$. Furthermore, by Lemma 6.1 we also have $q^i \in f_R(n_i)$ for all $i = 2, \dots, k+1$.

Clearly $g_R(p')$ is always a finite subset of $P(\hat{G})$; moreover, in case no path in $\hat{G}$ spells out the word w , which is spelled out by p' , $g_R(p') = \emptyset$.

Extend the function g_R to subsets P' of $P(G')$ by:

$$g_R(P') = \bigcup_{p' \in P'} g_R(p')$$

The following lemma is a direct consequence of the above definitions.

Lemma 6.3: Let p' be a path in G' and let $p \in g_R(p')$. If $p = p_1 p_2$ for some $p_1, p_2 \in P(\hat{G})$ then there exist p'_1, p'_2 in $P(G')$ such that $p' = p'_1 p'_2$ and $p_i \in g_R(p'_i)$, $i = 1, 2$.

Thus the functions f_R and g_R map nodes and paths in G' onto the corresponding sets of nodes and paths (resp.) in $\hat{G}_0(R)$; they will be called the R-projection functions.

Before proceeding with the definition of graph covering, we generalize the notion of subgraph as follows.

Definition 6.3: Let $G_1 = (N, B_L, N_1, N_2)$ and $G_2 = (N', B'_L, N'_1, N'_2)$ be two transition graphs. We say that G_2 is a partial graph of G_1 iff $N' \subseteq N$, $B'_L \subseteq B_L$ and $N'_i = N_i \cap N'$, $i = 1, 2$.

Clearly every subgraph of G_1 is also a partial graph of G_1 . Moreover, if G_2 is a partial graph of G_1 then $P(G_2) \subseteq P(G_1)$ and $r(G_2) \leq r(G_1)$.

Definition 6.4: A partial graph G'' of G' covers a partial graph G_1 of $\tilde{G}_0(R)$ iff $P(G_1) \subseteq g_R^{G'}(P(G''))$. In other words, every path p in G_1 is contained in $g_R^{G'}(p')$ for at least one path p' in G'' .

Example 6.2: Let G' and $\tilde{G}$ be the graphs in Fig. 6.1 and 6.2 (page 127) respectively, and let $p' = (D, 0, A)(A, 1, A)(A, 0, D)(D, 1, C)$. Then $g_R(p') = \{p_1, p_2\}$, where

$$p_1 = (q, 0, q)(q, 1, p)(p, 0, q)(q, 1, p)$$

$$p_2 = (r, 0, r)(r, 1, q)(q, 0, q)(q, 1, p)$$

The subgraph $G'' = G' - [\{B, C\}]$ of G' can be seen to cover the subgraph $S = G - [\{p, r\}]$ of G . In fact, every path in $(q, 0, q)^*$ is contained in $g_R(p')$, where

$$p' = [(A, 0, D)(D, 0, A)]^*[(A, 0, D)] \cup \varepsilon \subseteq P(G'').$$

6.2. Two Fundamental Lemmas

Throughout the rest of this chapter it is assumed that the functions f_R and g_R are always those associated with the given non-deterministic state graph G' , and not with any of its partial graphs. Thus the superscript G' will be omitted.

We now proceed with the two main lemmas of this chapter.

Lemma 6.4: Let G' be a non-deterministic state graph recognizing R and let $\tilde{G} = \tilde{G}_0(R)$. Then:

- (a) G' covers $\tilde{G}$
- (b) For every s.c. subgraph S of $\tilde{G}$ there exists a section S' of G' such that S' covers S .

Proof: (a) Let p be any path in $\tilde{G}$. Since $\tilde{G}$ is all-admissible (Definition 2.18), there exists an admissible path p_0 in $\tilde{G}$ containing p as a subpath. Clearly the initial node of p_0 is the initial state q_1 of $\tilde{G}$, and the word w_0 spelled out by p_0 is accepted by $\tilde{G}$, and hence also by G' . Thus there is in G' a path p' from an initial node n_1 to a terminal node spelling out w_0 . Since $q_1 \in f_R(n_1)$ by Lemma 6.1, we get $p_0 \in g_R(p')$. But then there must clearly exist a subpath p'' of p' such that $p \in g_R(p'')$. Hence G' covers $\tilde{G}$.

(b) Let $S_1, S_2, \dots, S_m$ be all sections of G' and assume that for some s.c. subgraph S of $\tilde{G}$, none of the S_i 's covers S . Then there exist paths $p_1, p_2, \dots, p_m$ in S such that $g_R(P(S_i))$ does not contain p_i for all $i = 1, \dots, m$. Since S is strongly connected, one can construct a loop p in S containing all p_i 's ($i = 1, \dots, m$) as subpaths. Now by part (a) of this lemma, there exists for every integer $k > 0$, a path t_k in G' , such that the path $(p)^k$ is contained in $g_R(t_k)$.

Let r be the total number of nodes in G' and let $k > r$. By Lemma 6.3, there exist paths $t'_1, t'_2, \dots, t'_k$ in G' such that $t_k = t'_1 t'_2 \dots t'_k$ and $p \in g_R(t'_i)$ for all $i = 1, 2, \dots, k$. Since $k > r$, at least one of the paths t'_i , $1 \leq i \leq k$, must be contained in a section S_j of G' ($1 \leq j \leq m$). It follows that $p \in g_R(P(S_j))$ and since p_j is a subpath of p , also $p_j \in g_R(P(S_j))$, which contradicts our assumption.

The proof of the next lemma resembles the proof of Theorem 9 in McNaughton's paper [Mc 4].

Lemma 6.5: Let G' be any non-deterministic state graph and let S' be a s.c. subgraph of G' which covers some s.c. subgraph S of $\hat{G}_0(R)$. If for each node n in S' , $f_R(n)$ contains at most one node from S , then $r(S') \geq r(S)$.

Proof: Let $G' = (N, B_L, N_1, N_2)$ and let Q_S be the set of all nodes of S . Let N' be the set of all nodes n in S' for which $f_R(n) \cap Q_S = \emptyset$. Since S' covers S , there exist nodes in S' which are not in N' . Consider the subgraph $\bar{G} = S' - [N'] = (\bar{N}, \bar{B}_L, \bar{N}_1, \bar{N}_2)$ of S' . Let $\bar{B}'_L$ be the set of all branches (n, a, n') in $\bar{G}$, such that for the state q defined by: $\{q\} = f_R(n) \cap Q_S$, $M(q, a)$ is not empty and is also contained in Q_S . By Lemma 6.1, for every branch (n, a, n') in $\bar{B}'_L$, $q' = M(q, a) \in f_R(n')$.

Let $\bar{G}'$ be the partial graph obtained from $\bar{G}$ by removing all branches not in $\bar{B}'_L$, i.e. $\bar{G}' = (\bar{N}, \bar{B}'_L, \bar{N}_1, \bar{N}_2)$. We claim that $\bar{G}'$ also covers S . To see this, let $p = (q^1, a_1, q^2) \dots (q^k, a_k, q^{k+1})$ be any path in S . Since S' covers S , there exists a path $p' = (n_1, a_1, n_2) \dots (n_k, a_k, n_{k+1})$ in S' such that $p \in g_R(p')$. Thus $q^i \in f_R(n_i)$ for all $i = 1, \dots, k+1$ which shows that $\{q_i\} = f_R(n_i) \cap Q_S \neq \emptyset$ and hence $n_i \in \bar{N}$, $i = 1, \dots, k+1$. Moreover, for any branch $b_i = (n_i, a_i, n_{i+1})$ in p' , $M(q_i, a_i) = q_{i+1} \in Q_S$ and hence b_i is in $\bar{B}'_L$. Thus the whole path p' is contained in $\bar{G}'$ and therefore $\bar{G}'$ also covers S .

Define a homomorphism ϕ from $\bar{G}'$ onto S as follows: for every n in $\bar{N}$, $\phi(n) = q$, where $\{q\} = f_R(n) \cap Q_S$. For any branch (n, a, n') in $\bar{B}'_L$, let $\phi((n, a, n')) = (\phi(n), a, \phi(n'))$. To see that the latter is indeed a branch in S , recall that whenever (n, a, n') is in $\bar{B}'_L$, $q' = M(q, a)$ is contained in Q_S , where $q = \phi(n)$, and $q' \in f_R(n')$. But then $q' = \phi(n')$ and hence the above branch is in S .

We now show that ϕ is a pathwise homomorphism from $\bar{G}'$ to S . Extend ϕ to a mapping from $P(\bar{G}')$ to $P(S)$ (as indicated in Definition 2.28) and let $p = (q^1, a_1, q^2) \dots (q^k, a_k, q^{k+1})$ be any path in S . Since $\bar{G}'$ covers S , there exists a path $p' = (n_1, a_1, n_2) \dots (n_k, a_k, n_{k+1})$ in $\bar{G}'$ such that $q^i \in f_R(n_i)$ for all $i = 1, \dots, k+1$. But then $q^i = \phi(n_i)$ for each $i = 1, \dots, k+1$ and consequently $p = \phi(p')$, which shows that ϕ is a pathwise homomorphism. By McNaughton's Pathwise Homomorphism Theorem, $r(S) \leq r(\bar{G}')$ and since $r(\bar{G}') \leq r(S')$ we obtain $r(S) \leq r(S')$ as required.

6.3. A Proof Technique for Establishing Star Height

Definition 6.5: Let $G = G_0(R)$ and $\tilde{G} = \tilde{G}_0(R)$ as above. For any state q_i in $\tilde{G}$, $1 \leq i \leq m_0$, define a mapping $p_i: A^* \rightarrow P_i(G)$ (notation 2.3) as follows:

- (a) $p_i(\lambda) = \epsilon$
- (b) for any non-empty word $w = a_1 a_2 \dots a_k$,
 $p_i(w) = (q_i, a_1, q^1)(q^1, a_2, q^2) \dots (q^{k-1}, a_k, q^k)$,
 where $q^j = M(q_i, a_1 \dots a_j)$ for all j , $1 \leq j \leq k$.

Thus p_i maps every word w to the path in G starting at q_i and spelling out w .

For any event $W \subseteq A^*$, let

$$p_i(W) = \{p_i(w) \mid w \in W\} \subseteq P_i(G).$$

Definition 6.6: For any subset B of the set of states Q of G, define two sets of paths of $\tilde{G}$:

$$L_B = \bigcup_{q_i \in B} p_i(\text{str } B)$$

(see Definition 1.17) and

$$D_B = \bigcup_{q_i \in B} p_i \left(\bigcap_{q_j \in B} D_{q_j} \right)$$

Example 6.3: Let $\tilde{G}$ be the state graph shown in Fig. 6.4 (page 139) and let $B = \{q_1, q_2\}$. Then $\text{str } B = 1^*$ and $D_{q_1} \cap D_{q_2} = 1^*2(0 \cup 1 \cup 2)^*$.

Hence:

$$L_B = [\epsilon \cup (q_1, 1, q_2)][(q_2, 1, q_1)(q_1, 1, q_2)]^*[\epsilon \cup (q_2, 1, q_1)]$$

and

$$D_B = \{[\epsilon \cup (q_1, 1, q_2)][(q_2, 1, q_1)(q_1, 1, q_2)]^*[(q_2, 2, q_3) \cup (q_2, 1, q_1)(q_1, 2, q_3) \cup (q_1, 2, q_3)] \cup (q_3, 0, q_3) \cup (q_3, 1, q_3) \cup (q_3, 2, q_3)]^*\}$$

Definition 6.7: Let S be any subgraph of a state graph G' and let $P' \subseteq P(G')$. We say that P' covers S iff every path p in S is a subpath of at least one path p' in P'.

Example 6.4: Let G' be the graph in Fig. 6.3 and let $S = G - [t]$. Then the set of paths $P' = [((r, 1, r)*(r, 0, q)(q, 0, r))^3]^*(r, 2, t)(t, 0, t)^2$ covers S.

The following lemma is an immediate consequence of the above definitions.

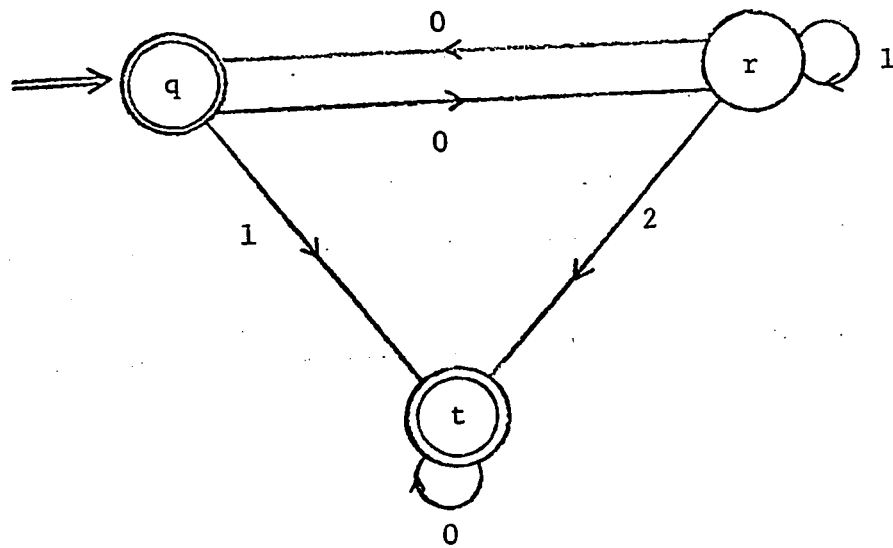


Fig. 6.3

Lemma 6.6: Let G' be any state graph and let S' be a section of G' . Then for every section S of $\tilde{G} = \tilde{G}_0(R)$, S' covers S iff for any set of paths $P' \subseteq P(G')$ covering S' , $g_R(P')$ covers S .

Lemma 6.7: Let G' be any state graph recognizing R and let S' be a section of G' . For any node n in S' , let $L(n)(D(n))$ denote the set of all paths in G' from n to itself (from n to a terminal node of G'). Then $g_R(L(n)) \subseteq L_B$ and $g_R(D(n)) \subseteq D_B$, where $B = f_R(n)$.

Proof: Let $p = (q^1, a_1, q^2) \dots (q^k, a_k, q^{k+1})$ be a path in $g_R(L(n))$.

Let $p' = (n, a_1, n_2)(n_2, a_2, n_3) \dots (n_k, a_k, n)$ be a loop in $L(n)$ such that $p \in g_R(p')$. Then by definition both q^1 and q^{k+1} belong to $B = f_R(n)$. Furthermore, for any state $q_i \in B$, the path in $\tilde{G}$ starting at q_i and spelling out the word $w = a_1 a_2 \dots a_k$ is contained in $g_R(p')$ and hence the terminal node of the path must also be a member of B . This implies that $w \in \text{str } B$, and since $q^1 \in B$ we have $p \in L_B$. Hence

$$g_R(L(n)) \subseteq L_B.$$

The proof for $g_R(D(n))$ is similar. Let p be any path in $g_R(D(n))$, with initial node q^1 , terminal node q^{k+1} , spelling out the word $w = a_1 \dots a_k$. As before, let $p' = (n, a_1, n_2) \dots (n_k, a_k, n_{k+1})$ be a path in G' such that $p \in g_R(p')$ and $p' \in D(n)$. Then $q^1 \in f_R(n) = B$ and $q^{k+1} \in f_R(n_{k+1})$. Since n_{k+1} is a terminal state of G' , by Lemma 6.2 q^{k+1} is a final state of $\tilde{G}$. Moreover, for any state $q_i \in B$, the path in $\tilde{G}$ starting at q_i and spelling out w is in $g_R(p')$ and therefore must terminate at a final state of $\tilde{G}$. This shows that $w \in \bigcap_{q_i \in B} D_{q_i}$ and since $q^1 \in B$, $p \in D_B$ as required.

Theorem 6.1: Let $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$, and let S be a s.c. subgraph of $\tilde{G}$. If for every subset $B \subseteq Q$ with at least two elements, the set L_B does not cover S , then $h(R) \geq r(S)$. The theorem remains true if the set L_B is replaced by D_B .

Proof: We shall prove that, under the above assumptions, every non-deterministic state graph G' recognizing R has rank no less than $r(S)$. The result then follows by Theorem 5.4.

Thus let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic state graph recognizing R . By Lemma 6.4, there exists a section S' of G' covering S . We claim that for any node n of S' , $f_R(n)$ is a singleton.

Suppose, on the contrary, that for some node n in S' , $\#B > 1$, where $B = f_R(n)$. Let $L(n)$ and $D(n)$ be defined as in the last lemma. Apparently any path p' of S' is a subpath of some path in $L(n)$, and since G' is assumed to be an all-admissible state graph, p' is also a subpath of some path in $D(n)$. Thus each of the sets $D(n)$ and $L(n)$ covers S' , and since S' covers S , Lemma 6.6 implies that both sets $g_R(L(n))$ and $g_R(D(n))$ cover S . But by Lemma 6.7 the former set is contained in L_B as well as the latter is contained in D_B . It follows that both L_B and D_B cover S , which contradicts the assumption of the theorem. Hence $f_R(n)$ is a singleton for every node n of S' , and by Lemma 6.5, $r(S') \geq r(S)$. Consequently $r(G') \geq r(S)$, which completes the proof.

Corollary 6.1: Let $\tilde{G} = \tilde{G}_0(R)$ as above and let $S_1, \dots, S_k$ be all sections of $\tilde{G}$ such that $r(S_i) = r(\tilde{G})$, $i = 1, \dots, k$. If for some S_i , $1 \leq i \leq k$, there does not exist a subset $B \subseteq Q$ with at least two elements such that $L_B(D_B)$ covers S_i , then the star height of R equals the rank of $\tilde{G}$. In particular, if for every $B \subseteq Q$, $\#B > 1$, the set $\text{str } B \left(\bigcap_{q \in B} D_q^\dagger \right)$ is finite, then $h(R) = r(\tilde{G})$.

Theorem 6.1 and Corollary 6.1 provide us with a powerful technique for establishing the star height of regular events. This technique will now be illustrated by the following examples.

Example 6.5: Let $\tilde{G} = \tilde{G}_0(R)$ be the state graph shown in Fig. 6.4. One can easily verify that:

$$\begin{aligned} \text{str } \{q_1, q_2, q_3\} &= 1*(\lambda \cup 2(0 \cup 1 \cup 2)*) \\ \text{str } \{q_1, q_2\} &= 1* \\ \text{str } \{q_1, q_3\} &= (0 \cup 11)*(\lambda \cup (2 \cup 12)(0 \cup 1 \cup 2)*) \\ \text{str } \{q_2, q_3\} &= (10*1)*(\lambda \cup (2 \cup 10*2)(0 \cup 1 \cup 2)*) \end{aligned}$$

By inspection, none of the sets L_B , where $B = \{q_1, q_2, q_3\}$, $\{q_1, q_2\}$, $\{q_1, q_3\}$ or $\{q_2, q_3\}$, covers the s.c. subgraph $S = \tilde{G} - [q_3]$ (no path of S containing the branch $(q_2, 3, q_2)$ is included in these sets). Thus by Corollary 6.1, $h(R) = r(S) = 2$.

Example 6.6: Consider the state graph $\tilde{G}$ shown in Fig. 6.5, and let $S = \tilde{G} - [\{q_3, q_4\}]$. One can easily see that for $B = \{q_1, q_2\}$, $\text{str } B = (0 \cup 1)*$ and thus L_B covers S . However, $D_{q_1} \cap D_{q_2} = \emptyset$ by inspection and therefore

† This is the case when R has the finite intersection property, which has also been covered by Theorem 4.1.

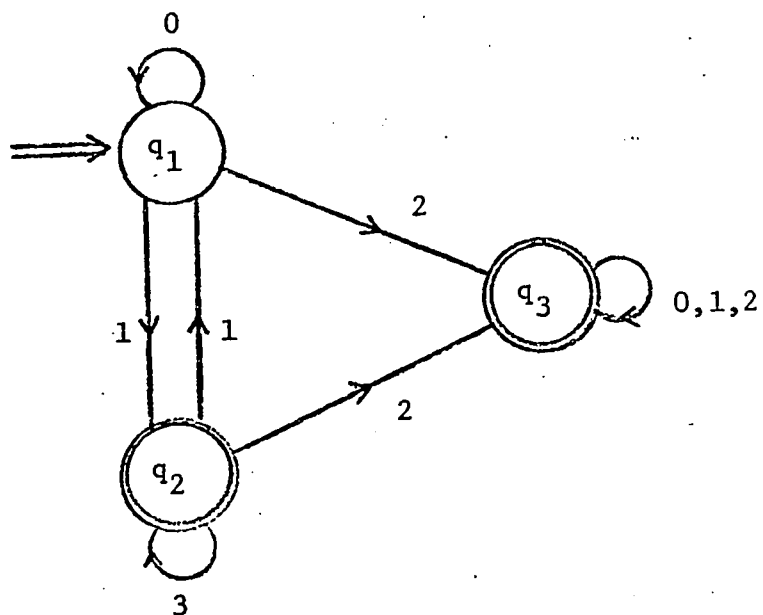


Fig. 6.4

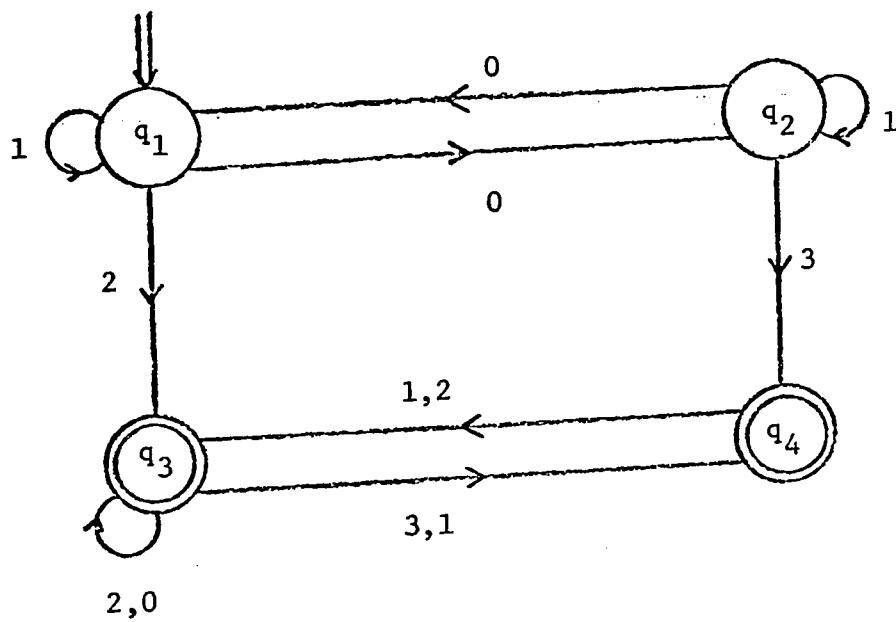


Fig. 6.5

D_B does not cover S . Furthermore, for any other subset $B' \subseteq \{q_1, q_2, q_3, q_4\}$ with $\#B' > 1$, $\bigcap_{q \in B'} D_q$ does not contain any word with the subword 010 and thus $D_{B'}$ cannot cover S . Hence by Corollary 6.1 the event accepted by $\tilde{G}$ has star height 2.

Example 6.7: Let $\tilde{G} = \tilde{G}_0(R)$ be the state graph shown in Fig. 6.6.

By inspection, the section $S = \tilde{G} - [\{q_1, q_5, q_6\}]$ is the only section of rank $r(S) = r(\tilde{G}) = 3$. However, Corollary 6.1 cannot be applied directly to R since both sets L_B and D_B , where $B = \{q_1, q_2\}$, cover S . Thus consider the event $R' = 0 \setminus R / 1$ (Definition 1.2). Since the only branches entering the final states q_5, q_6 and labelled by 1 are $(q_3, 1, q_5)$ and $(q_4, 1, q_6)$, R' is exactly the set of all words spelled out by paths from q_2 to either q_3 or q_4 . Thus the state graph S , with initial state q_2 and final states q_3 and q_4 , recognizes R' , and since it is reduced we have $S = \tilde{G}_0(R')$. Now one can easily show that for any $B \subseteq \{q_2, q_3, q_4\}$, with at least two elements, $\text{str } B = \Lambda$. Hence by Corollary 6.1, $h(R') = 3$. But $h(R') = h(0 \setminus R / 1) \leq h(R)$ by Corollary 3.1, and since $r(\tilde{G}) = 3$ we conclude that $h(R) = 3$.

Consider any set of paths $P' \subseteq P(\tilde{G})$ in $\tilde{G}$ as an event over the alphabet of labelled branches of $\tilde{G}$. Every path p in $\tilde{G}$ represents a word over this alphabet. Thus let P' / p denote the right quotient (Definition 1.2) of P' over the path p , i.e.

$$P' / p = \{p' \in P(G) \mid p'p \in P'\}$$

We now present an extension of Theorem 6.1.

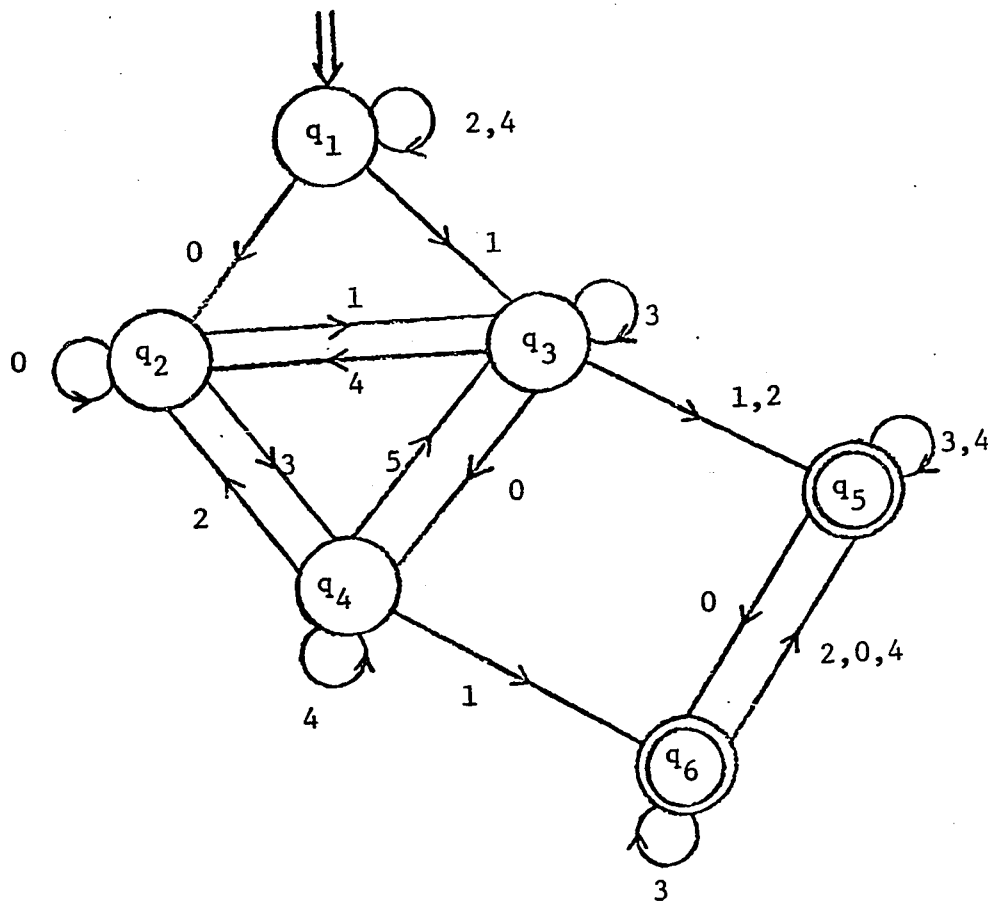


Fig. 6.6

Theorem 6.2: Let $\tilde{G} = \tilde{G}_0(R)$ as above, let S be any section of $\tilde{G}$ and let p be any path leading from a state of S to a final state of $\tilde{G}$. If for no subset $B \subseteq Q$, $\#B > 1$, the set D_B / p covers S , then $h(R) \geq r(S)$.

Proof: Let Q_S be the set of states of S , let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic state graph recognizing R and let $S_1, S_2, \dots, S_m$ be all sections of G' covering S (there exists at least one by Lemma 6.4). If for one of these sections S_i , $\#f_R(n) = 1$ for all nodes n of S_i , then by Lemma 6.5 $r(S_i) \geq r(S)$ and the result follows. Thus suppose on the contrary, that every S_i , $1 \leq i \leq m$, contains a node n_i such that $\#f_R(n_i) > 1$, and let $f_R(n_i) = B_i$. By the assumption of the theorem, D_{B_i} / p does not cover S for all $i = 1, \dots, m$. Then there exist paths p_i in S such that p_i is not the subpath of any path in D_{B_i} / p , $i = 1, \dots, m$. Let $S_{m+1}, \dots, S_{m+h}$ be all the remaining sections of G' ; since none of these covers S , there exist paths $r_1, \dots, r_h$ in S such that $r_i \notin g_R(p(S_{m+i}))$ for all $i = 1, \dots, h$. Thus construct a loop ℓ in S from the initial node q^0 of p to itself, such that each p_i , $i = 1, \dots, m$ and each r_j , $j = 1, \dots, h$, is a subpath of ℓ . Let p_0 be any path in $\tilde{G}$ from q_1 to q^0 . Then for any integer $k > s$, where s denotes the total number of nodes in G' , the path $p' = p_0(\ell)^k p$ is an admissible path in $\tilde{G}$. Thus there exists an admissible path t' in G' spelling out the same word as p' , and since $q_1 \in f_R(n)$ for any $n \in N_1$ we have $p' \in g_R(t')$. By Lemma 6.3, $t' = t_0 t_1 \dots t_k t$ such that $p_0 \in g_R(t_0)$, $\ell \in g_R(t_i)$ for $i = 1, \dots, m$ and $p \in g_R(t)$. Since $k > s$, at least one t_j , $1 \leq j \leq k$, must be contained in some section S' of G' . Now suppose $S' = S_i$ for some i , $1 \leq i \leq m$. Let n be the initial node of t_j and let t'' be a path from n_i to n . Then the path $t^1 = t'' t_j t_{j+1} \dots t_k t$ is a path

from n_i to a terminal node of G' , and hence is contained in $D(n_i)$.

But then, by Lemma 6.7, $g_R(t^1) \in D_{B_i}$ and hence $p''(\ell)^{k-j+1} p \in D_{B_i}$ for

some path $p'' \in g_R(t'')$ terminating at q^0 . It follows that

$p''(\ell)^{k-j+1} \in D_{B_i} / p$ and since p_i is a subpath of ℓ , p_i is a subpath

of the above path, which is contained in D_{B_i} / p . But this contradicts

our assumption.

Thus suppose that $S' = S_{m+i}$ for some $i = 1, \dots, h$. Then

since $\ell \in g_R(t_j)$, the subpath r_i of ℓ is contained in $g_R(t'_j)$ for some

subpath t'_j of t_j . But t'_j is a path in S_{m+i} and this contradicts our

assumption that $r_i \notin g_R(P(S_{m+i}))$.

Hence $\#f_R(n) = 1$ for all nodes of at least one section S_i ,

$1 \leq i \leq m$, and then the result follows by Lemma 6.5 as indicated above.

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Example 6.8: Let $\tilde{G}_0(R)$ be the state graph in Fig. 6.7. One can

verify that $\text{str}\{m, n\} = \text{str}\{m, p\} = \text{str}\{n, p\} = \text{str}\{m, n, p\} = (0 \cup 1)^*$,

and hence for all these sets, L_B covers the section $S = \tilde{G} - [q]$. Also

for any of these sets B , $\bigcap_{q_i \in B} D_{q_i} = (0 \cup 1)^* 2 (0 \cup 2 \cup 3)^*$, and hence D_B

also covers S . However, if we consider the path $p = (n, 3, q)$, then

none of the sets D_B / p , $B \subseteq Q$, $\#B > 1$, covers S . Hence $h(R) = r(S) = 2$.

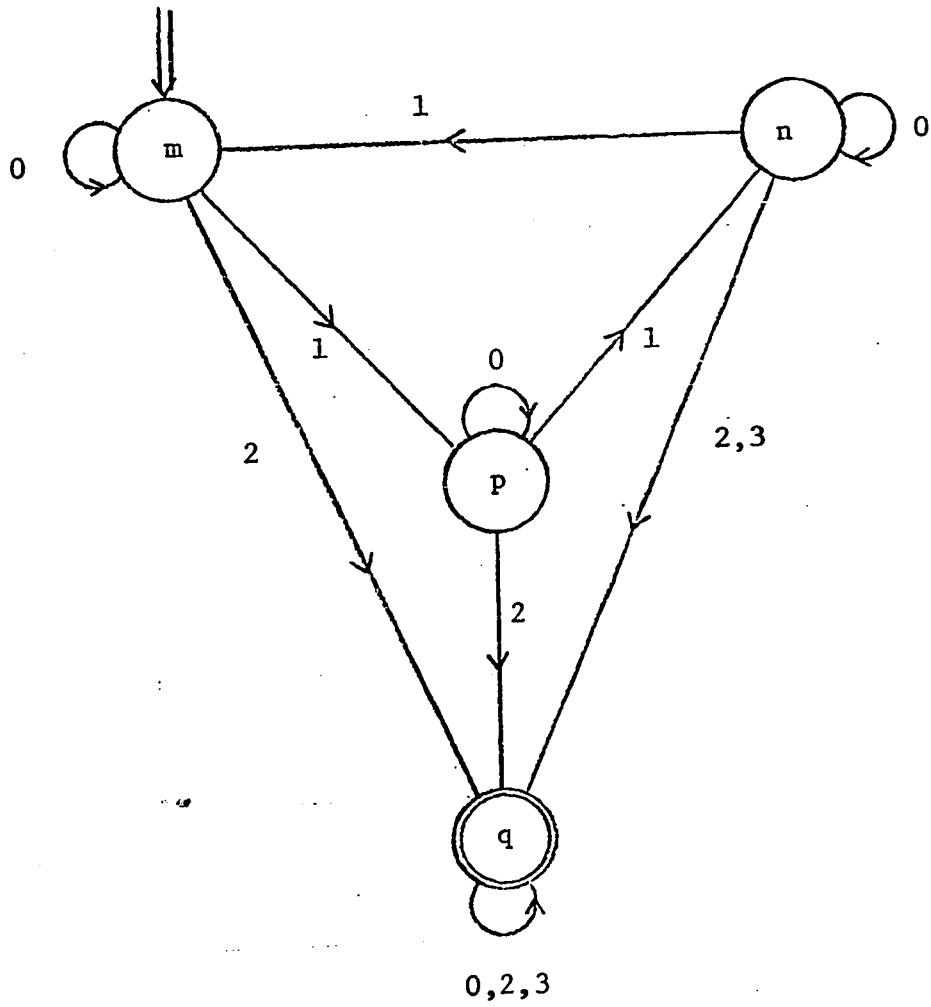


Fig. 6.7

6.4. Results on Reset-Free State Graphs

In this section the state graph $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$ is always assumed to be reset-free (Definition 4.10), unless otherwise specified.

Lemma 6.8: If $\tilde{G} = \tilde{G}_0(R)$ as above and if G' is any non-deterministic state graph recognizing R , then for any pair of nodes n, n' belonging to one section of G' , the following conditions must hold:

$$(a) \quad \#f_R(n) = \#f_R(n').$$

For every section S of $\tilde{G}_0(R)$ with set of states Q_S ,

$$(b) \quad \#(f_R(n) \cap Q_S) = \#(f_R(n') \cap Q_S)$$

$$(c) \quad \text{If } u \text{ is a word spelled out by some path from } n \text{ to } n', \\ \text{then } M((f_R(n) \cap Q_S), u) = f_R(n') \cap Q_S$$

Proof: (a) Since n and n' belong to one section of G' , there exists a word u spelled out by some path p from n to n' , and a word v spelled out by a path p' from n' to n . By Lemma 6.1, $M(f_R(n), u) \subseteq f_R(n')$ and $M(f_R(n'), v) \subseteq f_R(n)$. Furthermore, since $\tilde{G}$ is reset-free and since, by Lemma 6.2 (b), $M(q, u) \neq \emptyset$ for any state q in $f_R(n)$ and $M(q', v) \neq \emptyset$ for any $q' \in f_R(n')$, we have $\#f_R(n) = \#M(f_R(n), u) \leq \#f_R(n') = \#M(f_R(n'), v) \leq \#f_R(n)$, and equality follows.

(c) We first claim that for any subset $B \subseteq f_R(n)$ and for any word w spelled out by a path in G' starting at n , $\#M(B, w) = \#B$. This follows again from Lemma 6.2 (b) and the fact that $\tilde{G}$ is reset-free.

Now let $B = f_R(n) \cap Q_S$ and $B' = f_R(n') \cap Q_S$, and let u and v be words spelled out by paths p and p' as above. By Lemma 6.1, $M(B, u) \subseteq f_R(n')$ and $M(B', v) \subseteq f_R(n)$. If $B' \neq M(B, u)$ then there are two cases: (i) $M(B, u) \subsetneq B'$, or (ii) $M(B, u) \not\subseteq B'$. In case (i) we get: $\#B = \#M(B, u) < \#B' = \#M(B', v)$; it follows that $M(B', v) \not\subseteq B$, which, by exchanging B and B' , u and v , corresponds to case (ii). Thus we may assume that (ii) $M(B, u) \not\subseteq B'$. Then, since $M(B, u) \not\subseteq f_R(n')$, we get $M(B, u) \not\subseteq Q_S$. Thus there exists a state q^1 in B such that $M(q^1, u) \notin Q_S$, and then also $q^2 = M(q^1, uv) \notin Q_S$. Let $q^i = M(q^1, (uv)^{i-1})$, $i = 1, 2, \dots$ (by Lemma 6.2(b), none of the sets $M(q^1, (uv)^{i-1})$ is empty). Then $q^1 \neq q^2$ and we claim that $q^i \neq q^j$ for all $0 < i < j$. Clearly for $j > 1$, $q^j \notin Q_S$ and thus $q^1 \neq q^j$. Now suppose, by contradiction, that k is the smallest integer such that for some $m > k$, $q^k = q^m$. Then $k > 1$ and $q^{k-1} \neq q^{m-1}$; but $M(q^{k-1}, uv) = M(q^{m-1}, uv) = q^k$, which contradicts the fact that $\tilde{G}$ is reset-free. Hence $q^i \neq q^j$ for all $0 < i < j$, which is of course impossible. Consequently $M(B, u) = B'$, which proves (c).

(b) Follows immediately from (c) and the above proof.

Theorem 6.3: If $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$ is reset-free, and if for some state q in a section S of $\tilde{G}$, $[D_q - (\bigcup_{q \neq q' \in Q} D_{q'})] \neq \emptyset$, then

$h(R) \geq r(S)$.

Proof: Let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic state graph recognizing R and assume $r(G') = h(R)$. Let $S_1, \dots, S_m$ be all sections of G' which do not cover S . Then there clearly exist paths $p_1, \dots, p_m$ in S such that $p_i \notin g_R(P(S_i))$ for all $i = 1, \dots, m$. Now let $w_0 \in D_q - (\bigcup_{q \neq q' \in Q} D_{q'})$, and let p_0 be the path in $\tilde{G}$ starting at q , terminating at a final state and spelling out w_0 . Since all paths $p_i, i = 1, \dots, m$, are in the section S , one can construct in S a loop ℓ from q to itself, containing each $p_i, i = 1, \dots, m$, as subpath. Let s be the total number of nodes in G' and let k be any fixed integer greater than s . Since $\tilde{G}$ is all-admissible, there exists in it a path p' from the initial state q_1 to q . Now define the path $p = p'(\ell)^k p_0$. Clearly p is an admissible path in G . Thus the word u spelled out by p is in R , and there must exist in G' an admissible path t spelling out the same word u . Let $n_0(n_t)$ be the initial (terminal) node of t . Then $n_0 \in N_1, n_t \in N_2$ and by Lemma 6.2, $q_1 \in f_R(n_0)$ and $f_R(n_t) \subseteq F$. Hence $p \in g_R(t)$. By Lemma 6.3, there exists a decomposition of t into subpaths: $t = t' t_1 t_2 \dots t_k t_0$ such that $\ell \in g_R(t_i)$ for all $i = 1, \dots, k, p' \in g_R(t')$ and $p_0 \in g_R(t_0)$. But since $k > s$, at least one of the subpaths $t_j (1 \leq j \leq k)$ must be contained in some section of G' . Let this section be S' . Suppose $S' = S_i$ for some $i, 1 \leq i \leq m$. Then, since $\ell \in g_R(t_j)$ and since p_i is a subpath of ℓ , there exists a subpath t'_j of t_j such that $p_i \in g_R(t'_j)$. But this contradicts our assumption that $p_i \notin g_R(P(S_i))$. Hence $S' \neq S_i$ for all $i = 1, \dots, m$ and consequently S' covers S . We claim that for all nodes n of S' , $f_R(n)$ is a singleton.

Suppose this is not so; then by Lemma 6.8 (a) there exists some fixed integer $h \geq 2$ such that for each node n in S' , $f_R(n)$ has cardinality h . Let n_1 be the initial node of the path t_j . Since $l \in g_R(t_j)$ and l starts at q , we have $q \in f_R(n_1)$, and since n_1 is in S' , $\#f_R(n_1) \geq 2$. Let n_k be the terminal node of t_k ; then again $q \in f_R(n_k)$. Furthermore, by Lemma 6.1, $M(f_R(n_1), w) \subseteq f_R(n_k)$, where w is the word spelled out by the path $t_j t_{j+1} \dots t_k$. But since $\tilde{G}$ is reset-free, $f_R(n_1)$ cannot be mapped by any such input w into a set of smaller cardinality (see proof of Lemma 6.8 (c)); thus $\#f_R(n_k) \geq 2$. Then there exists in $f_R(n_k)$ at least one more state $q' \neq q$. Now since p_0 spells out the word w_0 , we have, again by Lemma 6.1, $M(f_R(n_k), w_0) \subseteq f_R(n_t)$. But the latter is contained in F and hence $M(q', w_0) \subseteq M(f_R(n_k), w_0) \subseteq F$. This shows that $w_0 \in D_{q'}$, which contradicts our assumption that $w_0 \notin D_q$, for any $q' \neq q$. We conclude that $f_R(n)$ is a singleton for all nodes n in S' . Combining this with the fact that S' covers S , we get $r(S') \geq r(S)$ by Lemma 6.5. Hence $h(R) = r(G') \geq r(S') \geq r(S)$ as required.

Example 6.9: Let $\tilde{G} = \tilde{G}_0(R)$ be the state graph in Fig. 6.8. One can verify that $\tilde{G}$ is reset-free strongly connected of rank 3. Moreover, the word $w = 21$ belongs to D_{q_3} , but is not contained in D_{q_i} for $i = 1, 2, 4$. Hence by the last theorem and Theorem 3.1, $h(R) = 3$.

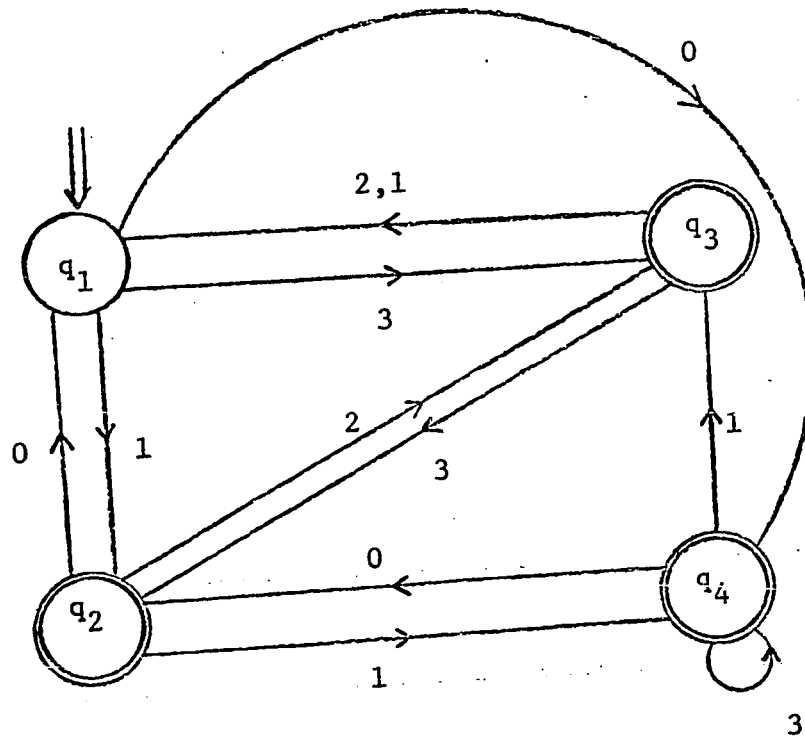


Fig. 6.8

Theorem 6.4: Let $\tilde{G} = \tilde{G}_0(R)$ be reset-free and let S be a section of $\tilde{G}$. If for some state q in S and for some word w_0 , $\bar{M}(q, w_0)$ is in S but $\bar{M}(q', w_0)$ is not in S for all other states q' of S , then the star height of R is no less than $r(S)$.

Proof: Let G' be a non-deterministic state graph recognizing R and assume $h(R) = r(G')$. By Lemma 6.4 there exists a section S' of G' such that S' covers S . We claim that for any node n in S' , $\#(f_R(n) \cap Q_S) = 1$, where Q_S denotes the set of all states of S . By Lemma 6.8 (b) all such sets $f_R(n) \cap Q_S$ (where n is in S') have the same cardinality, say k . Apparently if $k = 0$ then S' cannot cover S . Thus suppose $k \geq 2$. Let p be the path in S starting at q and spelling out w_0 . Then there exists in S' a path p' such that $p \in g_R(p')$. Let $n'(n'')$ be the initial (terminal) node of p' . Then clearly $q \in f_R(n')$ and the sets $B' = f_R(n') \cap Q_S$ and $B'' = f_R(n'') \cap Q_S$ have $k \geq 2$ elements each. Furthermore, by Lemma 6.8 (c), $B'' = \bar{M}(B', w_0)$. But this is impossible since for all states q' in B' except q , $\bar{M}(q', w_0)$ is not contained in Q_S by the assumption of the theorem. Hence $k = 1$ and the result follows immediately from Lemma 6.5.

Example 6.10: Let $\tilde{G} = \tilde{G}_0(R)$ be the state graph shown in Fig. 6.9 and let $S = \tilde{G} - [\{s, t\}]$. One can see that $\bar{M}(p, 2) = r$ is in S whereas $\bar{M}(q, 2) = s$ and $\bar{M}(r, 2) = q_\phi$ are not in S (q_ϕ is not shown in the graph). Hence $h(R) \geq r(S) = 2$ and since $r(\tilde{G}) = 2$, we get $h(R) = 2$.

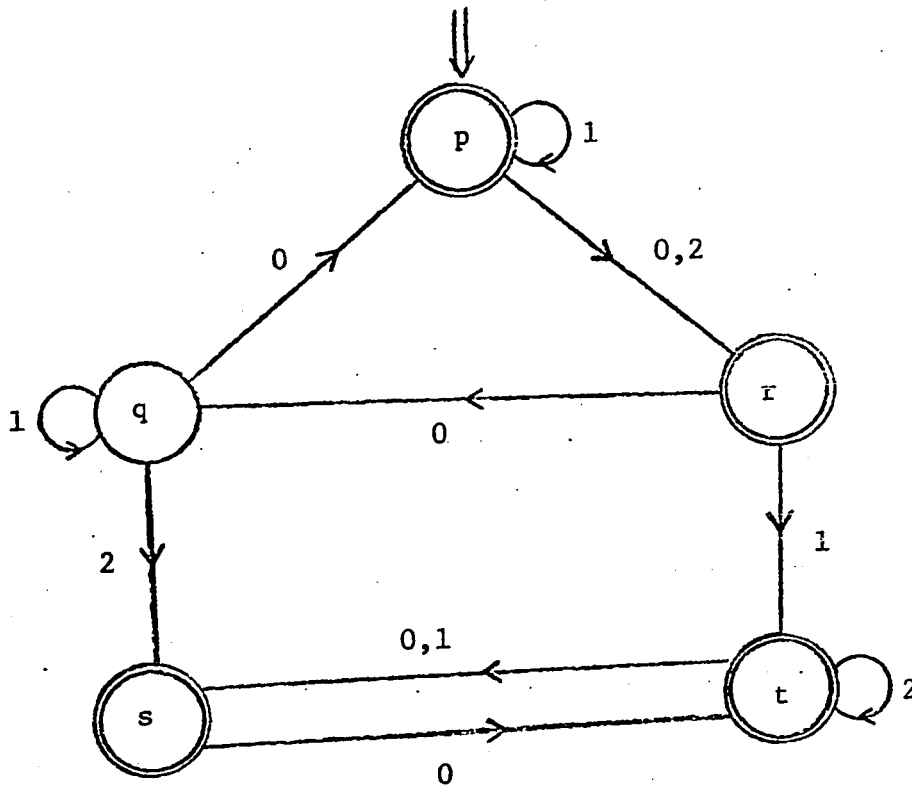


Fig. 6.9

Definition 6.8: Let $\tilde{G} = \tilde{G}_0(R)$, let S be any subgraph of $\tilde{G}$, and let its set of states be Q_S . S is called permutation-free iff for any subset $Q' \subseteq Q_S$ with $\#Q' > 1$ and for any word w , the mapping $m_w: q \rightarrow \bar{M}(q, w)$, $q \in Q'$, is not a cyclic permutation of Q' onto itself (see Definition 1.20).

Lemma 6.9: Let $\tilde{G} = \tilde{G}_0(R)$ be as above and let S be a permutation-free section of $\tilde{G}$. Then there exists a word w_0 and a state q in S such that $\bar{M}(q, w_0)$ is in S whereas $\bar{M}(q', w_0)$ is not in S for all other states q' in S .

Proof: If S consists of a single state then the lemma is trivial. Thus denote the set of states of S by Q_S and suppose $\#Q_S \geq 2$. We first prove the following assertion:

(*) For any two states q_1, q_2 in Q_S , there exists a word w such that either $\bar{M}(q_1, w) \in Q_S$ and $\bar{M}(q_2, w) \notin Q_S$ or $\bar{M}(q_1, w) \notin Q_S$ and $\bar{M}(q_2, w) \in Q_S$.

To prove (*), assume, on the contrary, that for some pair of states q_1, q_2 in S , $\bar{M}(q_1, w) \in Q_S$ iff $\bar{M}(q_2, w) \in Q_S$ for all words w . Let x be a word such that $\bar{M}(q_1, x) = q_2$ (such a word clearly exists since q_1 and q_2 belong to the same section S). By assumption, $q_3 = \bar{M}(q_2, x) = \bar{M}(q_1, x^2)$ is also contained in Q_S , and inductively, $q_i = \bar{M}(q_2, x^{i-2}) = \bar{M}(q_1, x^{i-1}) \in Q_S$ for all $i = 1, 2, \dots$.

Now we claim that for all i, j , $1 \leq i < j < \infty$, $q_i \neq q_j$. For suppose $q_i = q_j$ for some i, j , $i < j$. If $j = i + 1$ then $i > 1$ and $\bar{M}(q_{i-1}, x) = q_i = q_{i+1} = \bar{M}(q_i, x)$ which shows that q_{i-1} and q_i are both mapped by x to q_i . Since we may assume $q_{i-1} \neq q_i$ (by taking i to be the

smallest integer for which $q_i = q_{i+1}$), it follows that x maps two distinct states onto the same state, contrary to the assumption that $\tilde{G}$ is reset-free.

Thus suppose $j - i \geq 2$. Then the word x induces on $\{q_i, q_{i+1}, \dots, q_{j-1}\}$ the non-trivial cyclic permutation:

$$q_i \xrightarrow{x} q_{i+1} \xrightarrow{x} \dots \xrightarrow{x} q_{j-1} \xrightarrow{x} q_i$$

contradicting the fact that S is permutation-free. Thus $q_i \neq q_j$ for all positive integers $i, j, i \neq j$, which is clearly impossible. Hence (*) must hold.

Now let w_1 be a word such that for some pair of states $q_1, q'_1 \in Q_S$, $\bar{M}(q_1, w_1) \in Q_S$ whereas $\bar{M}(q'_1, w_1) \notin Q_S$. Define

$$Q^1 = \{q \mid q \in Q_S \wedge \bar{M}(q, w_1) \in Q_S\}$$

and $Q_1 = \bar{M}(Q^1, w_1)$. Then $0 < \#Q^1 < \#Q_S$ and $\#Q^1 = \#Q_1$ because S is reset-free. If $\#Q_1 > 1$ then there exist two states q_2, q'_2 in Q_1 and a word w_2 such that $\bar{M}(q_2, w_2) \in Q_S$ and $\bar{M}(q'_2, w_2) \notin Q_S$. Define

$$Q^2 = \{q \mid q \in Q_1 \wedge \bar{M}(q, w_2) \in Q_S\}$$

and $Q_2 = \bar{M}(Q^2, w_2)$. Then $0 < \#Q^2 = \#Q_2 < \#Q_1$ as before. By induction. for any $i = 2, 3, \dots$, such that $\#Q_{i-1} > 1$, let w_i be a word such that the set

$$Q^i = \{q \mid q \in Q_{i-1} \wedge \bar{M}(q, w_i) \in Q_S\}$$

is a non-empty proper subset of Q_{i-1} . Then $0 < \#Q^i = \#Q_i < \#Q_{i-1}$, where $Q_i = \bar{M}(Q^i, w_i)$. It follows that for some positive integer k , $\#Q_k = 1$.

Thus let $Q_k = \{q^k\}$, where $q^k \in Q_S$. By definition of the sets Q^i and Q_i , for every $q' \in Q_i$ ($1 < i \leq k$) there exists a state $q'' \in Q_{i-1}$ such that $\bar{M}(q'', w_i) = q'$. It follows by induction, that for every state q' in Q_i as above, there exists a state q^1 in Q_1 such that $\bar{M}(q^1, w_1 w_2 \dots w_i) = q'$. Thus let q be a state in Q_1 such that $\bar{M}(q, w_1 w_2 \dots w_k) = q^k$. Since $\tilde{G}$ is reset-free, for all other states q' in Q_S , $q'' = \bar{M}(q', w_1 w_2 \dots w_k) \neq q^k$. If q'' is contained in Q_S , the definition of the sets Q^i and Q_i implies that q'' should be contained in Q_k , that is, should coincide with q^k . Hence for all states q' of Q_S other than q , $\bar{M}(q', w_1 w_2 \dots w_k)$ is not contained in Q_S . Thus $w_0 = w_1 w_2 \dots w_k$ is the required word and the proof is completed.

Theorem 6.5: If $\tilde{G} = \tilde{G}_0(R)$ is reset-free, then

$$h(R) \geq \max \{r(S) \mid S \text{ a permutation-free section of } \tilde{G}\}$$

Proof: Theorem 6.4 and Lemma 6.9

Corollary 6.2: If $\tilde{G}_0(R)$ is both reset-free and permutation-free then

$$h(R) = r(\tilde{G}_0(R)).$$

Chapter 7

Conclusions and Further Problems

In this thesis the (restricted) star height of regular events has been studied. General properties of star height were investigated as well as some rank-non-increasing transformations on transition graphs, which led to a stronger version of Eggen's Star Height Theorem. The problem of determining the star height of a given event was attacked, both by examining the apparent star height of an arbitrary regular expression denoting the event, and by studying the rank of an arbitrary non-deterministic state graph recognizing the event. The former approach, using the concepts of cofinal sets of trails and simple regular expressions, led to some families of events, whose star height equals the rank of the corresponding reduced deterministic state graphs. The latter approach, utilizing the R-projection functions and the notion of covering of state graphs, provided a more extensive technique for determining star height and proved to be quite powerful.

We now summarize the existing methods for determining the star height of a given regular event R .

- (a) If R is a finite event, $h(R) = 0$. Otherwise $h(R) \geq 1$.
- (b) If R is an event over the single letter alphabet then $h(R) \leq 1$.

- (c) For any event R , $h(R) \leq r(\tilde{G}_0(R))[E]$
- (d) If R is a pure-group event, McNaughton's algorithm [Mc 4] can be used to determine $h(R)$. If, in addition, $\tilde{G}_0(R)$ has a single output state, then $h(R) = r(\tilde{G}_0(R))$.
- (e) If R has the finite intersection property, then $h(R) = r(\tilde{G}_0(R))$ by Theorem 4.1.
- (f) If there exists in $\tilde{G}_0(R)$ a reset-free section S of rank greater than 1, then any of the theorems 6.1, 6.2, 6.5 may be applicable to R .
- (g) If $\tilde{G}_0(R)$ is itself reset-free with more than one final state, Theorems 6.3, 6.4 and Corollary 6.2 should be tried.
- (h) If none of (a) - (g) applies to R , one should try to apply the graph-theoretical technique introduced in [Mc 3].
Alternatively, find an event $R' \supseteq R$ such that Theorem 4.1 can be applied to $\tilde{G}_0(R')$ and R (see Example 4.8). In some special cases, R might belong to the families of events of arbitrarily large star height constructed in either [E] or [Mc 3].

Unfortunately, the above-mentioned techniques for determining the star height of a regular event do not apply to all cases. To the best of our knowledge, the existence of a general algorithm for finding the star height of any given event has not yet been established. We do believe that such an algorithm exists, and would like to present some further conjectures.

Definition 7.1: Let $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$. For any integer $k > 0$, define the subset graph $\tilde{G}_k(R)$ of order k of R as follows:

$$\tilde{G}_k(R) = (N^k, B_L^k, N_1^k, N_2^k)$$

where

$$N^k = \{Q'_{(i)} \mid \emptyset \neq Q' \subseteq Q, 1 \leq i \leq k\}$$

$$B_L^k = \{(Q'_{(i)}, a, Q''_{(j)}) \mid 1 \leq i, j \leq k, a \in A, M(Q', a) \subseteq Q''\}$$

$$N_1^k = \{Q'_{(i)} \mid 1 \leq i \leq k, q_1 \in Q'\}$$

$$N_2^k = \{Q'_{(i)} \mid 1 \leq i \leq k, Q' \subseteq F\}$$

Thus N^k consists of k duplicates of each non-empty subset of Q , and there is a branch labelled by $a \in A$ from any duplicate of a set Q' to any duplicate of a set Q'' iff Q' is mapped by the input a into Q'' . The initial nodes are all duplicates of subsets Q' containing q_1 and the terminal nodes are all duplicates of subsets Q' contained in F .

One can easily verify, that any partial graph of $\tilde{G}_k(R)$ ($k = 1, 2, \dots$) recognizes a subset of R .

Furthermore, if $G' = (N, B_L, N_1, N_2)$ is any non-deterministic state graph recognizing R , then the R -projection function f_R associates with each node n of G' , a subset $f_R(n)$ of Q (which from now on will be called the R -projection set of n). By Lemma 6.1, $M(f_R(n), a) \subseteq f_R(n')$ whenever (n, a, n') is a branch in G' , and by Lemma 6.2, for all initial nodes $n \in N_1$, $f_R(n)$ contains q_1 and for all terminal nodes $n' \in N_2$, $f_R(n') \subseteq F$. It follows that every non-deterministic state graph recognizing R can be

regarded as a partial graph of $\tilde{G}_k(R)$, for a large enough k .

Now consider any state graph $G' = (N, B_L, N_1, N_2)$ recognizing R , and suppose that two nodes n and n' of G' have the same R -projection set, i.e. $f_R(n) = f_R(n')$. One can easily verify that the graph $G'' = (N', B'_L, N'_1, N'_2)$ obtained from G' by a full homomorphism ϕ , which merges n and n' into some new node, say m , and leaves all other nodes fixed, as well as all initial and terminal nodes, is equivalent to G' . Furthermore, for every node n of G' , $f_R^{G'}(n) = f_R^{G''}(\phi(n))$.

By repeating this procedure for all pairs of nodes with equal R -projection sets, one finally obtains an equivalent state graph $\bar{G} = (\bar{N}, \bar{B}_L, \bar{N}_1, \bar{N}_2)$ in which $f_R(n) \neq f_R(n')$ for all $n, n' \in \bar{N}$. However, $\bar{G}$ might have higher rank than G' . Although in most cases the constructions presented in Chapter 5 enable us to obtain from G' an equivalent graph $\bar{G}$ of no greater rank, whose R -projection sets are all distinct, this is not always the case. Two examples will now be presented, one, in which such a construction is possible and another one, in which it is not.

Note that a state graph G' all of whose R -projection sets are distinct, can be regarded as a partial graph of the subset graph $\tilde{G}_1(R)$ of order 1 of R .

Example 7.1: Let $\tilde{G}_0(R)$ and G' be the graphs shown in Fig. 7.1 and Fig. 7.2 respectively. The R -projection sets of the nodes of G' are indicated inside the circles representing the nodes. One can easily verify that G' is equivalent to $\tilde{G}_0(R)$ (in fact both graphs accept the event $(0 \cup 1)^*1$). Moreover, $f_R(n_2) = f_R(n_3) = \{p, q\}$. Thus the full

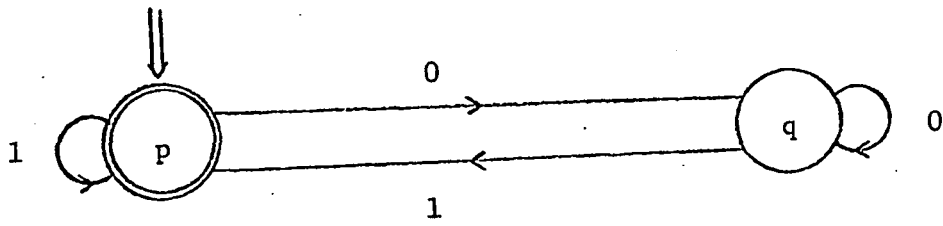


Fig. 7.1

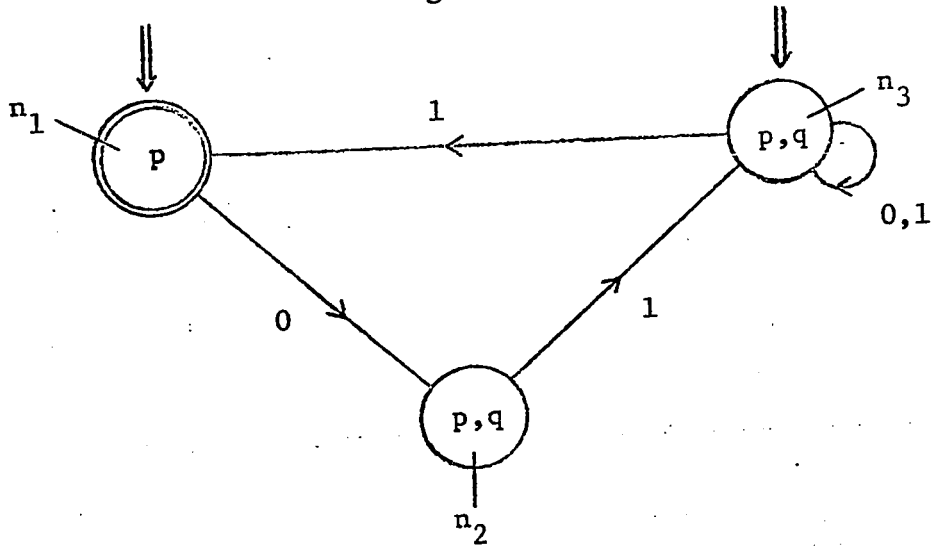


Fig. 7.2

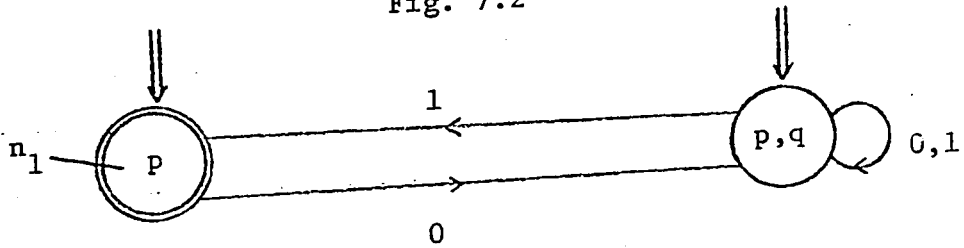


Fig. 7.3

homomorphism ϕ merging n_2 with n_3 can be carried out, yielding the equivalent graph $\bar{G}$ shown in Fig. 7.3. One can see that $r(\bar{G}) = r(G') = 1$ (alternatively, one can use Lemma 5.14 to remove node n_2 from the graph G' without affecting its behaviour. The resulting graph would be $G' - [n_2]$). $\bar{G}$ constitutes a state graph of minimum rank accepting R , in which all R -projection sets are distinct. Thus $\bar{G}$ is a partial graph of $\tilde{G}_1(R)$.

Example 7.2 (farther-fetched after McNaughton's Far-Fetched Example[Mc 3]):

Consider the graphs in Figs. 7.4 and 7.5 respectively. To verify that G' is equivalent to $\tilde{G}(=\tilde{G}_0(R))$, let $w \in R$, and let $p = p_1(w)$ be the admissible path in $\tilde{G}$ spelling out w . There are two cases:

Case (i): p does not contain the subpath $(q, 1, q)^2$. In that case there exists a path p' in $G_1 = G' - [\{n_1, n_2\}]$ such that $p = g_R(p')$. To see this, note that G_1 resembles $\tilde{G}$, except for the missing 1-self-loop on q , which is replaced by another "duplicate" of q and a 1-transition from one duplicate to the other. This modification causes the graph G_1 to be of rank 1, although $\tilde{G}$ has rank 2.

Case (ii): p contains the subpath $(q, 1, q)^2$. In that case one can express p as $p = p_1 \cdot (q, 1, q)^2 \cdot p_2$, where p_2 does not contain $(q, 1, q)^2$ as a subpath. Since the graph $G_2 = G' - [\{n_2, n_3, n_4, n_5, n_6\}]$ obviously covers $\tilde{G}$, one can find a path p'_1 in G_2 such that $p_1 \in g_R(p'_1)$. As for the path $(q, 1, q)^2$, it is certainly contained in $g_R((n_1, 1, n_2)(n_2, 1, n_3))$. Also p_2 , which starts at q and does not contain $(q, 1, q)^2$, is clearly equal to $g_R(p'_2)$ for some path p'_2 in G_1 starting at n_3 . Hence the path $p' = p'_1 \cdot (n_1, 1, n_2)(n_2, 1, n_3) \cdot p'_2$ is an admissible path in G' and

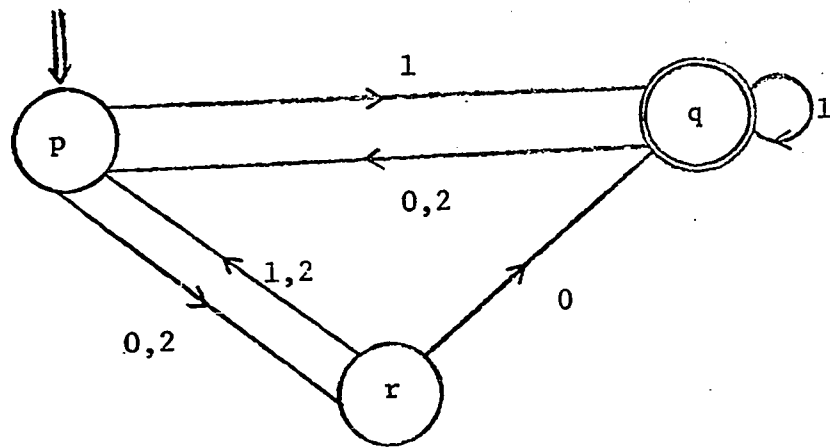


Fig. 7.4

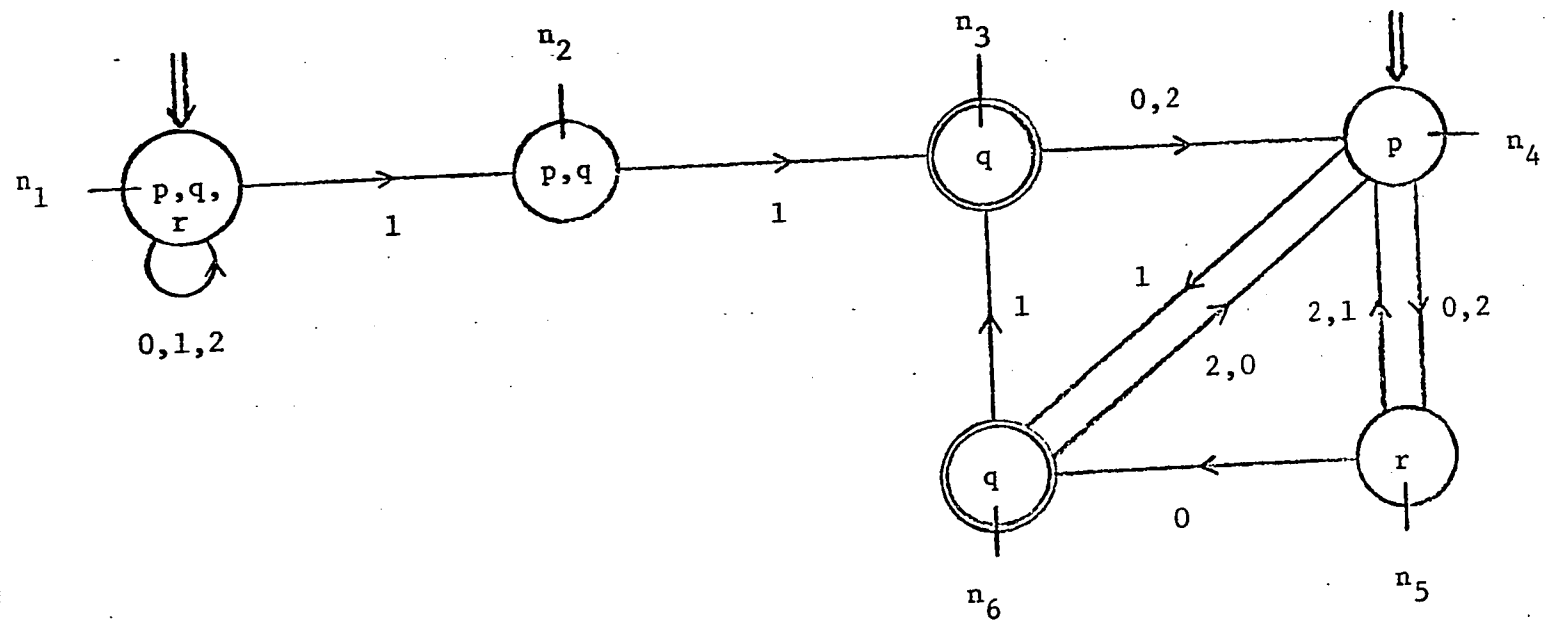


Fig. 7.5

spells out w.

It can be easily verified that G' does not accept any word not in R . Thus G' is equivalent to $\tilde{G}$ and since $r(G') = 1$, we conclude that $h(R) = 1$. Moreover, one can verify (by exhausting all cases) that there does not exist any non-deterministic state graph of rank 1 recognizing R , for which all R -projection sets $f_R(n)$ are distinct. In other words, no partial graph of $\tilde{G}_1(R)$ of rank 1 accepts the whole event R .

Furthermore, for any integer $t > 0$, an example of an event R_t of similar nature can be constructed, such that every non-deterministic state graph of minimum rank recognizing R_t has t duplicates of at least one R_t -projection set $f_{R_t}(n)$, i.e. no partial graph of $\tilde{G}_{t-1}(R_{t-1})$ recognizing R_t has rank $h(R_t)$. However in those examples, the number of states in $\tilde{G}_0(R_t)$ grows together with t . It seems that for a given number of states, say m_0 , in the reduced state graph $\tilde{G}_0(R)$, there exists an upper bound on the number of duplicates of a set $f_R(n)$ necessary for constructing a minimum-rank state graph accepting R . This upper bound should be a function of m_0 alone.

Conjecture 7.1: If $\tilde{G}_0(R)$ has m_0 states, then there exists a partial graph G' of the subset graph $\tilde{G}_k(R)$ of order $k = 2^{m_0} - 1$ such that $r(G') = h(R)$ and G' recognizes R .

As for reset-free state graphs $\tilde{G}_0(R)$, we conjecture:

Conjecture 7.2: If $\tilde{G}_0(R)$ is reset-free then there exists a partial graph of $\tilde{G}_1(R)$ recognizing R of rank $h(R)$.

We have been able to establish Conjecture 7.2 only for reset-free state graphs with no more than two final states.

We would like to mention another interesting open problem concerning regular events, namely the extended star height problem. The apparent star height of an extended regular expression (i.e. using operators $\cup$, $\cap$, $-$, $\cdot$, $*$) is defined in an analogous way to that of a restricted regular expression. The extended star height $h_{\text{ex}}(R)$ of a regular event R is the smallest apparent star height of all extended regular expressions denoting R . Clearly $h_{\text{ex}}(R) \leq h(R)$ for all events R . In many cases $h_{\text{ex}}(R)$ is indeed less than $h(R)$ (take, for instance, $R = (0 \cup 1)^* = \bar{0}$; then $h_{\text{ex}}(R) = 0$ and $h(R) = 1$); in fact, the difference $h(R) - h_{\text{ex}}(R)$ can grow arbitrarily. In [Sch] Schützenberger has proved that $h_{\text{ex}}(R) = 0$ if and only iff R is group-free (see Theorem 1.12). This also establishes the existence of events of extended star height 1. However, to the best of our knowledge, nobody as yet has been able to prove that there exists a regular event of extended star height greater than 1. We believe that there exist events of arbitrarily large extended star height, although other people have made contradictory conjectures.

The star height problem could also be extended to certain families of context-free languages, which can be described by some generalized regular expressions.[†]

[†] See (1) Yntema, M. K.: "A generalized regular notation used to define some families of context-free languages and to obtain their closure properties and containment relations" Ph.D. dissertation, University of Illinois, 1965. (2) Brzozowski, J. A.: "Regular-like expressions for some irregular languages" to appear in the Proceedings of the 1968 Ninth Annual Symposium on Switching and Automata Theory.

Appendix A

The following is a counter-example to Conjecture 3.1 (page 50), suggested by R. McNaughton.

Let $A = \{a_1, a_2, a_3, a_4\}$, let R_1 be the set of all words over A having an even number of a_2 's and an even number of a_3 's, let T be the set of all words over A having an odd number of a_1 's. Let $R = R_1 \cup T$. Note that $R - T$ is the set of all words over A having an even number of a_1 's, an even number of a_2 's and an even number of a_3 's. Now $h(R) = 3$, $h(T) = 2$ and $h(R - T) = 4$.

Appendix B

The following example, due to R. McNaughton, settles the open problem suggested by Brzozowski (Page 53, line 3).

Let $A = \{a_1, a_2, a_3\}$ and let R be the set of all words over A having an even number of at least two of the three letters. Thus, if $E_i = ((A - \{a_i\})^* a_i (A - \{a_i\})^* a_i)^* (A - \{a_i\})^*$, then $R = (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$. Now it can be verified (using the methods introduced in Mc4) that $h(R) = 3$. Let $T = \lambda \cup a_1 \cup a_2 \cup a_3$. Then $TR = \lambda \cup a_1(E_2 \cup E_3) \cup a_2(E_1 \cup E_3) \cup a_3(E_2 \cup E_3)$. Hence $h(TR) \leq 2$. With a little more effort it can be shown that $h(TR) = 2$. Similarly, $h(RT) = 2$.

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ERRATUM

Page 71, lines 10T[†] - 16T should be replaced by:

"If G has a dead state q_ϕ , then clearly $G - [q_\phi]$ is identical with $\tilde{G}_0(R)$. Moreover, since R is infinite, $h(R) \geq 1$ and hence $r(G - [q_\phi]) = r(\tilde{G}_0(R)) \geq 1$. But since q_ϕ cannot be contained in a section of rank > 1 of G , it follows that $r(G - [q_\phi]) = r(G) = r(\tilde{G}_0(R))$. If G has no dead state then G itself coincides with $\tilde{G}_0(R)$. In both cases, since R has the f.i.p. by the last lemma, we get $h(R) = r(\tilde{G}_0(R)) = r(G)$ by Theorem 4.1."

Page 92, line 9B[†] should be replaced by:

"Case (c): n_1 and n_2 are both contained in a section S of D and $n_1(n_2)$ is a cycle center of D . Then by Lemma 5.6 condition ..."

Page 107, line 4B should be replaced by:

"... some $k \geq 1$. We may clearly assume that $m_1 \neq n$. By Proposition 2.4:"

[†] 10T: 10-th line from top

9B: 9-th line from bottom