Isotropy Groups of Quasi-Equational Theories

Jason Parker

Thesis submitted to the Faculty of Science in partial fulfillment of the requirements
for the degree of
Doctorate in Philosophy Mathematics and Statistics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

---

[1] The Ph.D. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

To every small category or Grothendieck topos one may associate its *isotropy group*, which is an algebraic invariant capturing information about the behaviour of automorphisms. In this thesis, we investigate this invariant in the particular context of *quasi-equational theories*, which are multi-sorted equational theories in which operations may be *partially* defined. It is known that every such theory $\mathbb{T}$ has a *classifying topos*, which is a topos that classifies all topos-theoretic models of the theory, and that this classifying topos is in fact equivalent to the covariant presheaf category $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$, with $\mathsf{fp}\mathbb{T}\mathsf{mod}$ being the category of all *finitely presented*, set-based models of $\mathbb{T}$. We then investigate the isotropy group of this classifying topos of $\mathbb{T}$, which will therefore be a presheaf of groups on $\mathsf{fp}\mathbb{T}\mathsf{mod}$, and show that it encodes a notion of *inner automorphism* for the theory. The main technical result of this thesis is a *syntactic* characterization of the isotropy group of a quasi-equational theory, and we illustrate the usefulness of this characterization by applying it to various concrete examples of quasi-equational theories.

Dedications

This work is dedicated to my wife Monica, my sister-in-law Priyanka, and my feline children Mew and Chuck.

Acknowledgements

First and foremost, I wish to thank my advisors Pieter Hofstra and Philip Scott for their invaluable advisement and mentoring throughout my Ph.D career and the writing of this thesis.

I would like to thank all the members of the University of Ottawa Logic group, both past and present, who have given me the opportunity to present my research in the Logic Seminar over the years.

I would like to thank the members of my thesis committee for giving their time to read, critique, and improve this thesis.

I would like to thank my parents for their continual support of my academic journey through the years.

Last but certainly not least, I would like to thank my sister-in-law Priyanka for being the sister that I never had, and my wife Monica for being the best partner in life that I could have ever hoped for.

Contents

List of Symbols

$1^{\mathbb{C}}$	Terminal object of $\mathbb{C}$	6
$\mathcal{B}(\mathbb{T})$	Classifying topos of $\mathbb{T}$	3
$\mathbb{C}^*$	Reduced word strict monoidal category	106
$\mathbb{C}_{\mathbb{T}}$	Syntactic category of $\mathbb{T}$	11
$\mathcal{E}_{\Sigma}$	Completely empty theory over Σ	72
$\mathsf{fp}\mathbb{T}\mathsf{mod}$	Full subcategory of finitely presented $\mathbb{T}$-models	13
$\mathcal{J}_B^M$	Full subcategory of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the sort B	141
$\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$	Abbreviation for $\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{G_{\mathbb{T}} \circ F^M}$	141
$\leftrightarrow_e^*$	Reflexive, symmetric, transitive closure of $\rightarrow_e$	100
$\leftrightarrow_r^*$	Reflexive, symmetric, transitive closure of $\rightarrow_r$	97
Ab	Category of abelian groups and homomorphisms	79
$\mathsf{Aut}(C)$	Group of automorphisms of $C \in \mathbb{C}$	1
Cat	Category of small categories and functors	88
CMon	Category of commutative monoids and homomorphisms	82
$\mathsf{Free}(\mathbb{T})$	Initial model of $\mathbb{T}$	21
Group	Category of groups and group homomorphisms	1
Grpd	Category of small groupoids and functors	93
$\mathsf{Hom}(\mathcal{E}, \mathcal{B}(\mathbb{T}))$	Category of geometric morphisms from $\mathcal{E}$ to $\mathcal{B}(\mathbb{T})$	11
$\mathsf{Inv}(M)$	Group of invertible elements of a monoid M	75
Latt	Category of lattices and lattice homomorphisms	85
$\mathsf{Mod}(\mathbb{T}, \mathbb{C})$	Category of models of $\mathbb{T}$ in $\mathbb{C}$	10
Mon	Category of monoids and monoid homomorphisms	74
$\mathsf{P}\Sigma\mathsf{Str}$	Category of partial Σ-structures	17
Sets	Category of sets and (total) functions	3
$\mathsf{StrMonCat}$	Category of small strict monoidal categories	94
$\mathsf{Sub}(X)$	Subobject poset of X	8
$\mathsf{Term}(\Sigma)$	Class of terms of Σ	4
$\mathsf{Term}^c(\Sigma)$	Set of closed terms of Σ	20
$\mathsf{Term}^c(\Sigma)_A$	Set of closed terms of Σ of sort A	20
Ter	Category of small categories with a terminal object	91
$\mathsf{Tot}(\Sigma)$	Totally defined empty theory over Σ	60

$\mathsf{P}\mathbb{T}\mathsf{mod}$	Full subcategory of $\mathsf{P}\Sigma\mathsf{Str}$ on the models of $\mathbb{T}$	19
ρ_h^A	Signature morphism induced by Σ-morphism h	36
Σ	Signature	4
$\Sigma(M)$	Diagram signature of M	30
$\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$	Extended diagram signature	52
$\Sigma\mathsf{Str}(\mathbb{C})$	Category of interpretations of Σ in $\mathbb{C}$	7
$\Sigma^{\mathcal{J}}$	Functor signature	137
$\Sigma_1 + \Sigma_2$	Disjoint union of signatures Σ_1 and Σ_2	133
Σ_d	Signature extending Σ by new constant d	118
Σ_f	Signature extending Σ by new function symbol f	123
Σ_{Fun}	Set of function symbols of Σ	4
Σ_{Sort}	Set of sorts of Σ	4
$\mathbb{T}(\varphi)$	Abbreviation for $\mathbb{T}(c_1, \ldots, c_n, \varphi)$	66
$\mathbb{T}(M)$	Diagram theory of M	31
$\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$	Extended diagram theory	52
$\mathbb{T}_1 + \mathbb{T}_2$	Union of theories $\mathbb{T}_1$ and $\mathbb{T}_2$	133
$\mathbb{T}_d$	Theory extending $\mathbb{T}$ by new constant d	118
$\mathbb{T}_f$	Theory extending $\mathbb{T}$ by new function symbol f	123
$\mathbb{T}_{\mathsf{Ab}}$	Theory of abelian groups	79
$\mathbb{T}_{\mathsf{Bij}}$	Theory of sets with a bijection	83
$\mathbb{T}_{\mathsf{Cat}}$	Theory of categories	88
$\mathbb{T}_{\mathsf{CMon}}$	Theory of commutative monoids	79
$\mathbb{T}_{\mathsf{Group}}$	Theory of groups	79
$\mathbb{T}_{\mathsf{Grpd}}$	Theory of groupoids	93
$\mathbb{T}_{\mathsf{Inv}}$	Theory of sets with an involution	84
$\mathbb{T}_{\mathsf{Latt}}$	Theory of lattices	85
$\mathbb{T}_{\mathsf{Mon}}$	Theory of monoids	74
$\mathbb{T}_{\mathsf{Str}}$	Theory of strict monoidal categories	94
$\mathbb{T}_{\mathsf{Ter}}$	Theory of categories with a terminal object	91
$\mathbb{T}_{\mathsf{Triv}}$	Trivial single-sorted theory	131
$\mathbb{T}^{\mathcal{J}}$	Functor theory	139
$\widehat{M}$	Canonical diagram structure	30
$\mathcal{Z}_{\mathbb{C}}$	Isotropy group of $\mathbb{C}$	1
$\mathcal{Z}_{\mathbb{T}}^{\mathsf{fp}}$	Covariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}$	64
$f : A \rightharpoondown B$	Partial function from A to B	16
L_n	Free lattice on n generators	85
$M/{\sim}$	Partial quotient Σ-structure	19
$M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$	Initial model of $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$	53
$M\langle \mathsf{x}_A \rangle$	Initial model of $\mathbb{T}(M, \mathsf{x}_A)$	32
$t \downarrow$	Abbreviation for $t = t$	15
t^*	Total function induced by the term t	34

$U_{\mathbb{T}}$	Universal model of $\mathbb{T}$	11
w^e	Normal form of w with respect to $\to_e$	100
w^r	Normal form of w with respect to $\to_r$	97
w^{exp}	Expansion of a word w	100
$W_O^{\mathbb{C}}$	Object words of the strict monoidal category $\mathbb{C}$	95
W_O^r	Set of reduced words in W_O	96
W_O^{sr}	Set of semi-reduced words in W_O	96

Chapter 1

Introduction and Background

1.1 Categorical Background

In this section, we will introduce the necessary *categorical* background and motivation for what follows. We will assume that the reader has familiarity with the basic notions of category theory and topos theory. We will define the *isotropy group* of a small category, and relate this to the notion of the isotropy group of a (Grothendieck) topos.

Given a category $\mathbb{C}$, the assignment

$$C \mapsto \mathsf{Aut}(C),$$

where $\mathsf{Aut}(C)$ is the group of automorphisms of the object C in $\mathbb{C}$, is not generally functorial. More specifically, if $f : D \to C$ happens to be an *isomorphism* in $\mathbb{C}$, then we *can* define a canonical group homomorphism

$$\mathsf{Aut}(C) \to \mathsf{Aut}(D),$$

given by the rule (conjugation by f)

$$\pi \mapsto f^{-1} \circ \pi \circ f$$

for any $\pi \in \mathsf{Aut}(C)$. However, if $f : D \to C$ is *not* an isomorphism, then it is not possible to define such a canonical group homomorphism in general.

To solve this ‘problem’, we introduce the *isotropy group* (functor)

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group}$$

of $\mathbb{C}$. Given $C \in \mathbb{C}$, we set

$$\mathcal{Z}_{\mathbb{C}}(C) := \mathsf{Aut}(\mathbb{C}/C \to \mathbb{C}),$$

the group of natural automorphisms of the forgetful functor $\mathbb{C}/C \to \mathbb{C}$. More concretely, if we let $\mathsf{Cod}(C) := \{f \in \mathbb{C} : \mathsf{cod}(f) = C\}$, then an element $\alpha \in \mathcal{Z}_{\mathbb{C}}(C)$ is a $\mathsf{Cod}(C)$-indexed family of automorphisms

$$\alpha = \Big(\alpha_f : \mathsf{dom}(f) \xrightarrow{\sim} \mathsf{dom}(f)\Big)_{f \in \mathsf{Cod}(C)}$$

with the following naturality property: for any $f' : C'' \to C'$ and $f : C' \to C$ in $\mathbb{C}$ we have

$$f' \circ \alpha_{f \circ f'} = \alpha_f \circ f',$$

as shown in the following commutative diagram:

$$\begin{tikzcd}
C'' \arrow[r, "\alpha_{f \circ f'}"] \arrow[d, "f'"'] & C'' \arrow[d, "f'"] \\
C' \arrow[r, "\alpha_f"'] & C'
\end{tikzcd}$$

The explicit group structure on $\mathcal{Z}_{\mathbb{C}}(C)$ is given as follows: for any $\alpha, \beta \in \mathcal{Z}_{\mathbb{C}}(C)$, we have

$$\alpha \cdot \beta = \Big(\alpha_f \circ \beta_f : \mathsf{dom}(f) \xrightarrow{\sim} \mathsf{dom}(f)\Big)_{f \in \mathsf{Cod}(C)}$$

and

$$\alpha^{-1} = \Big(\alpha_f^{-1} : \mathsf{dom}(f) \xrightarrow{\sim} \mathsf{dom}(f)\Big)_{f \in \mathsf{Cod}(C)},$$

while the unit element of $\mathcal{Z}_{\mathbb{C}}(C)$ is

$$\Big(\mathsf{id}_{\mathsf{dom}(f)} : \mathsf{dom}(f) \xrightarrow{\sim} \mathsf{dom}(f)\Big)_{f \in \mathsf{Cod}(C)}.$$

Now, the assignment $C \mapsto \mathcal{Z}_{\mathbb{C}}(C)$ *is* functorial in C. Explicitly, let $h : D \to C$ be any arrow in $\mathbb{C}$; then

$$\mathcal{Z}_{\mathbb{C}}(h) : \mathcal{Z}_{\mathbb{C}}(C) \to \mathcal{Z}_{\mathbb{C}}(D)$$

is defined by the rule

$$\alpha = \Big(\alpha_f : \mathsf{dom}(f) \xrightarrow{\sim} \mathsf{dom}(f)\Big)_{f \in \mathsf{Cod}(C)} \quad \longmapsto \quad \Big(\alpha_{h \circ g} : \mathsf{dom}(g) \xrightarrow{\sim} \mathsf{dom}(g)\Big)_{g \in \mathsf{Cod}(D)}$$

for any $\alpha \in \mathcal{Z}_{\mathbb{C}}(C)$. Then $\mathcal{Z}_{\mathbb{C}}(h)(\alpha) \in \mathcal{Z}_{\mathbb{C}}(D)$, because if we have arrows $g' : D'' \to D'$ and $g : D' \to D$, then

$$g' \circ \alpha_{h \circ (g \circ g')} = g' \circ \alpha_{(h \circ g) \circ g'} = \alpha_{h \circ g} \circ g',$$

because $\alpha \in \mathcal{Z}_{\mathbb{C}}(C)$. It is then easy to see that $\mathcal{Z}_{\mathbb{C}}(h)$ is a group homomorphism, and that

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group}$$

is functorial.

The functor $\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group}$ is referred to as the *contravariant* isotropy group (functor) of $\mathbb{C}$. One can also consider the *covariant* isotropy group of $\mathbb{C}$, which is the functor

$$\mathcal{Z}_{\mathbb{C}^{op}} : \mathbb{C} = (\mathbb{C}^{op})^{op} \to \mathsf{Group},$$

i.e. the contravariant isotropy group of $\mathbb{C}^{op}$. In fact, in this thesis we will *only* be studying covariant isotropy groups; for further details, see Section 2.1.

We have just shown that every category $\mathbb{C}$ has an isotropy group (functor) $\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group}$, which 'functorializes' the assignment $C \mapsto \mathsf{Aut}(C)$. Since $\mathcal{Z}_{\mathbb{C}}$ is a presheaf of groups on $\mathbb{C}$, it follows that $\mathcal{Z}_{\mathbb{C}}$ is an internal group object in the presheaf category $\mathsf{Sets}^{\mathbb{C}^{op}}$. Thus, every presheaf category $\mathsf{Sets}^{\mathbb{C}^{op}}$ has a canonical internal group object $\mathcal{Z}_{\mathbb{C}}$.

Now, recalling that presheaf categories are particular examples of Grothendieck toposes, the preceding result for presheaf categories has in fact been shown to hold for *all* Grothendieck toposes ([10, 4.3]). Namely, every Grothendieck topos $\mathcal{E}$ has a canonical internal group object $Z_{\mathcal{E}} \in \mathcal{E}$ called the *isotropy group* of $\mathcal{E}$, which *represents* the isotropy group functor $\mathcal{Z}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathsf{Group}$ of $\mathcal{E}$, in the sense that $\mathcal{Z}_{\mathcal{E}}(C) \cong \mathsf{Hom}_{\mathcal{E}}(C, Z_{\mathcal{E}})$ for every object $C \in \mathcal{E}$. This internal group object has the universal property that it acts canonically on every object of the topos (and on itself by conjugation), and every morphism of the topos is equivariant with respect to these actions. In particular, if $\mathcal{E} = \mathsf{Sets}^{\mathbb{C}^{op}}$ for a category $\mathbb{C}$, then the internal isotropy group object $Z_{\mathsf{Sets}^{\mathbb{C}^{op}}} \in \mathsf{Sets}^{\mathbb{C}^{op}}$ is exactly the isotropy group *functor*

$$Z_{\mathsf{Sets}^{\mathbb{C}^{op}}} = \mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group} \hookrightarrow \mathsf{Sets}.$$

The isotropy group of a Grothendieck topos $\mathcal{E}$ has been shown to admit several different descriptions:

- Freyd ([9]) introduced the notion of the *core* of a category which (informally speaking), if it exists, is a monoid in the category that represents the polymorphic unary operations present in the category. Freyd showed that the core of a Grothendieck topos $\mathcal{E}$ *does* exist, and it can then be shown that the isotropy group of $\mathcal{E}$ is the group of invertible elements of the core. Thus, elements of the isotropy group can be interpreted as polymorphic automorphisms in the topos.

  Proof sketch of the claim that the isotropy group of a Grothendieck topos $\mathcal{E}$ is the group of invertible elements of the core: the isotropy group of $\mathcal{E}$ is a representing object for the isotropy group functor $\mathcal{Z}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathsf{Group}$, while the core of $\mathcal{E}$ is a representing object for the 'isotropy monoid' functor $\mathcal{M}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathsf{Mon}$ that sends any $X \in \mathcal{E}$ to the monoid of natural *endomorphisms* of the forgetful functor $\mathcal{E}/X \to \mathcal{E}$. So for any $X \in \mathcal{E}$, we have that $\mathcal{Z}_{\mathcal{E}}(X)$ is the group of invertible elements of the monoid $\mathcal{M}_{\mathcal{E}}(X)$, which then essentially yields the result.

- (For more background on this description, see Section 1.2): For any Grothendieck topos $\mathcal{E}$, there is a geometric theory $\mathbb{T}$ such that $\mathcal{E}$ is the classifying topos $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$. It can then be shown that the isotropy group of $\mathcal{E} = \mathcal{B}(\mathbb{T})$ is the automorphism group of the universal $\mathbb{T}$-model $U_{\mathbb{T}} \in \mathcal{B}(\mathbb{T})$ (for a proof sketch, see [13, 2.2]).

  Spencer Breiner ([5, 4.3.7, 4.3.9]) has also shown that if we represent $\mathcal{E} = \mathcal{B}(\mathbb{T})$ as the topos of sheaves on the (topological) groupoid of $\mathbb{T}$-models, then the isotropy group of $\mathcal{E}$ is the sheaf of groups whose stalk at a $\mathbb{T}$-model M is the group of *definable* automorphisms of M.

As a final introductory remark, let us connect the (contravariant) isotropy group functor $\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathsf{Group}$ of a category $\mathbb{C}$ with the *centre* of $\mathbb{C}$, which is (by definition) the monoid $\mathsf{End}(\mathsf{Id}_{\mathbb{C}})$ of natural endomorphisms of the identity functor $\mathsf{Id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$. Note that if $\mathsf{Aut}(\mathsf{Id}_{\mathbb{C}})$ denotes the group of natural *auto*morphisms of $\mathsf{Id}_{\mathbb{C}}$, then $\mathsf{Aut}(\mathsf{Id}_{\mathbb{C}}) = \mathsf{Inv}(\mathsf{End}(\mathsf{Id}_{\mathbb{C}}))$, the group of *invertible* elements of the monoid $\mathsf{End}(\mathsf{Id}_{\mathbb{C}})$. In Chapter 5, we will refer to $\mathsf{Aut}(\mathsf{Id}_{\mathbb{C}})$ as the *global* isotropy group of the category $\mathbb{C}$. We now remark that if the category $\mathbb{C}$ happens to have a terminal object $1^{\mathbb{C}}$, then it is easy to see that

$$\mathcal{Z}_{\mathbb{C}}\left(1^{\mathbb{C}}\right) = \mathsf{Aut}(\mathsf{Id}_{\mathbb{C}}),$$

i.e. the (contravariant) isotropy group of the terminal object of $\mathbb{C}$ is equal to the global isotropy group of $\mathbb{C}$. Dually, if $\mathbb{C}$ has an initial object $0^{\mathbb{C}}$, then the *covariant* isotropy group of $0^{\mathbb{C}}$ is equal to the global isotropy group of $\mathbb{C}$.

1.2 Logical Background

We will now review some background from categorical logic that we will need in what follows; in particular, the notions of first-order geometric and cartesian theories, and models of such theories. First, we define the notion of a *first-order signature*:

Definition 1.2.1 (Signatures). A *first-order signature* Σ is a triple of sets $\Sigma = (\Sigma_{\mathsf{Sort}}, \Sigma_{\mathsf{Fun}}, \Sigma_{\mathsf{Rel}})$ such that:

- Σ_{Sort} is the set of *sorts* of Σ.
- Σ_{Fun} is the set of *function symbols* of Σ. Each element $f \in \Sigma_{\mathsf{Fun}}$ comes equipped with a tuple of sorts $(A_1, \ldots, A_n, A)$, and we write

$$f : A_1 \times \ldots \times A_n \to A.$$

In case $n = 0$, we write $f : A$.

- Σ_{Rel} is the set of *relation symbols* of Σ. Each element $R \in \Sigma_{\mathsf{Rel}}$ comes equipped with a tuple of sorts $(A_1, \ldots, A_n)$, and we write

$$R : A_1 \times \ldots \times A_n.$$

▮

Next, we define the class of *terms* over a given first-order signature:

Definition 1.2.2 (Terms). Let Σ be a first-order signature. For every sort $A \in \Sigma_{\mathsf{Sort}}$, we assume that we have a countably infinite set V_A of variables of sort A. We now define the class $\mathsf{Term}(\Sigma)$ of *terms* of Σ recursively as follows, while simultaneously defining the *sort* and the set $\mathsf{FV}(t)$ of *free variables* of a term $t \in \mathsf{Term}(\Sigma)$:

- If $A \in \Sigma_{\mathsf{Sort}}$ and $x \in V_A$, then $x \in \mathsf{Term}(\Sigma)$ is of sort A, with $\mathsf{FV}(x) := \{x\}$.
- If $f : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ and $t_1, \ldots, t_n \in \mathsf{Term}(\Sigma)$ with $t_i : A_i$ for each $1 \leq i \leq n$, then $f(t_1, \ldots, t_n) \in \mathsf{Term}(\Sigma)$ is of sort A, and

$$\mathsf{FV}(f(t_1, \ldots, t_n)) := \mathsf{FV}(t_1) \cup \ldots \cup \mathsf{FV}(t_n).$$

  In particular, if c is a constant symbol of sort A, then c is a term of sort A, and $\mathsf{FV}(c) = \emptyset$.

If $t \in \mathsf{Term}(\Sigma)$ and $\mathsf{FV}(t) = \emptyset$, then we will refer to t as a *closed term*. If $t \in \mathsf{Term}(\Sigma)$, then we write $t(x_1, \ldots, x_n)$ to mean that $\mathsf{FV}(t) \subseteq \{x_1, \ldots, x_n\}$. ▮

Now we define the class of *geometric formulas* over a given first-order signature:

Definition 1.2.3 (Geometric Formulas). Let Σ be a first-order signature. We define the class $\mathsf{GeomForm}(\Sigma)$ of *geometric formulas* over Σ recursively as follows, while simultaneously defining the set $\mathsf{FV}(\varphi)$ of *free variables* of a formula $\varphi \in \mathsf{GeomForm}(\Sigma)$:

- If $t_1, t_2 \in \mathsf{Term}(\Sigma)$ are terms of the same sort, then $t_1 = t_2 \in \mathsf{GeomForm}(\Sigma)$, and $\mathsf{FV}(t_1 = t_2) := \mathsf{FV}(t_1) \cup \mathsf{FV}(t_2)$.
- If $R \in \Sigma_{\mathsf{Rel}}$ is a relation symbol with $R : A_1 \times \ldots \times A_n$ and $t_1, \ldots, t_n \in \mathsf{Term}(\Sigma)$ with $t_i : A_i$ for each $1 \leq i \leq n$, then $R(t_1, \ldots, t_n) \in \mathsf{GeomForm}(\Sigma)$, and $\mathsf{FV}(R(t_1, \ldots, t_n)) := \mathsf{FV}(t_1) \cup \ldots \cup \mathsf{FV}(t_n)$.

- $\top, \bot \in \mathsf{GeomForm}(\Sigma)$ (the 'true' and 'false' formulas, respectively), and $\mathsf{FV}(\top) = \mathsf{FV}(\bot) := \emptyset$.
- If $\varphi, \psi \in \mathsf{GeomForm}(\Sigma)$, then
$$\varphi \wedge \psi \in \mathsf{GeomForm}(\Sigma),$$
and $\mathsf{FV}(\varphi \wedge \psi) := \mathsf{FV}(\varphi) \cup \mathsf{FV}(\psi)$.
- If I is a set and $\varphi_i \in \mathsf{GeomForm}(\Sigma)$ is a formula for each $i \in I$ such that $\bigcup_{i \in I} \mathsf{FV}(\varphi_i)$ is finite, then
$$\bigvee_{i \in I} \varphi_i \in \mathsf{GeomForm}(\Sigma),$$
and $\mathsf{FV}(\bigvee_{i \in I} \varphi_i) := \bigcup_{i \in I} \mathsf{FV}(\varphi_i)$.
- If $\psi \in \mathsf{GeomForm}(\Sigma)$ and $x \in V_A$ for some $A \in \Sigma_{\mathsf{Sort}}$, then
$$\exists x \psi \in \mathsf{GeomForm}(\Sigma),$$
and $\mathsf{FV}(\exists x \psi) := \mathsf{FV}(\psi) \setminus \{x\}$.

If $\varphi \in \mathsf{GeomForm}(\Sigma)$ and $\mathsf{FV}(\varphi) = \emptyset$, then we will usually refer to φ as a (geometric) *sentence*. If $\varphi \in \mathsf{GeomForm}(\Sigma)$, then we will write $\varphi(x_1, \ldots, x_n)$ to mean that $\mathsf{FV}(\varphi) \subseteq \{x_1, \ldots, x_n\}$. ∎

Now we define the notions of an *interpretation* of a first-order signature in a category with finite products, and a *homomorphism* between such interpretations:

Definition 1.2.4 (Interpretations). Let Σ be any first-order signature and $\mathbb{C}$ any category with finite products. An *interpretation* M of Σ in $\mathbb{C}$ is given by the following data:

- For any sort $A \in \Sigma_{\mathsf{Sort}}$, an object $M_A \in \mathbb{C}$.
- For any function symbol $f \in \Sigma_{\mathsf{Fun}}$ with $f : A_1 \times \ldots \times A_n \to A$, an arrow
$$f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$$
in $\mathbb{C}$. In particular, if $c : A$ is a constant, then
$$c^M : 1^{\mathbb{C}} \to M_A$$
is a point of M_A (i.e. $1^{\mathbb{C}}$ is the terminal object of $\mathbb{C}$).

- For any relation symbol $R \in \Sigma_{\mathsf{Rel}}$ with $R : A_1 \times \ldots \times A_n$, a subobject

$$R^M \rightarrowtail M_{A_1} \times \ldots \times M_{A_n}.$$

∎

Definition 1.2.5 (Homomorphisms). Let M, M' be interpretations of a first-order signature Σ in a category $\mathbb{C}$ with finite products. A *homomorphism* $H : M \to M'$ is a Σ_{Sort}-indexed collection of morphisms in $\mathbb{C}$

$$H = (H_A : M_A \to M'_A)_{A \in \Sigma_{\mathsf{Sort}}}$$

that commutes with the interpretations of the function and relation symbols of Σ, in the following sense:

- If $f : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ with $n \geq 1$ and $c : A$ is a constant symbol of Σ, then the following diagrams must commute:

$$\begin{tikzcd}
M_{A_1} \times \ldots \times M_{A_n} \arrow[r, "f^M"] \arrow[d, "H_{A_1} \times \ldots \times H_{A_n}"'] & M_A \arrow[d, "H_A"] \\
M'_{A_1} \times \ldots \times M'_{A_n} \arrow[r, "f^{M'}"'] & M'_A
\end{tikzcd}
\qquad
\begin{tikzcd}
1^{\mathbb{C}} \arrow[r, "c^M"] \arrow[dr, "c^{M'}"'] & M_A \arrow[d, "H_A"] \\
 & M'_A
\end{tikzcd}$$

- If $R : A_1 \times \ldots \times A_n$ is a relation symbol of Σ, then there must be a morphism $R^M \to R^{M'}$ making the following diagram commute:

$$\begin{tikzcd}
R^M \arrow[r, tail] \arrow[d] & M_{A_1} \times \ldots \times M_{A_n} \arrow[d, "H_{A_1} \times \ldots \times H_{A_n}"] \\
R^{M'} \arrow[r, tail] & M'_{A_1} \times \ldots \times M'_{A_n}
\end{tikzcd}$$

If $\mathbb{C}$ is a category with finite products, then interpretations of Σ in $\mathbb{C}$ and homomorphisms between them form a category, which we denote by $\Sigma\mathsf{Str}(\mathbb{C})$. Compositions of homomorphisms are defined componentwise, as are the identity homomorphisms. ∎

We can *interpret* the elements of $\mathsf{Term}(\Sigma)$ in any interpretation of Σ, as follows:

Definition 1.2.6 (**Interpretation of Terms**)**.** Let Σ be a first-order signature, let $\mathbb{C}$ be a category with finite products, and let M be an interpretation of Σ in $\mathbb{C}$. For any term $t(x_1, \ldots, x_n) \in \mathsf{Term}(\Sigma)$ with $t : A$ and $x_i : A_i$ for each $1 \leq i \leq n$, we define an arrow

$$t(x_1, \ldots, x_n)^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$$

in $\mathbb{C}$ as follows:

- If $t(x_1, \ldots, x_n) \equiv x_i : A_i$ for some $1 \leq i \leq n$, then

  $$t(x_1, \ldots, x_n)^M := \pi_i : M_{A_1} \times \ldots \times M_{A_n} \to M_{A_i},$$

  the projection onto the ith factor.

- If $t(x_1, \ldots, x_n) \equiv c : A$ for some constant symbol c, then

  $$t(x_1, \ldots, x_n)^M := c^M \circ ! : M_{A_1} \times \ldots \times M_{A_n} \to 1^{\mathbb{C}} \to M_A,$$

  where $! : M_{A_1} \times \ldots \times M_{A_n} \to 1^{\mathbb{C}}$ is the unique arrow from $M_{A_1} \times \ldots \times M_{A_n}$ to the terminal object $1^{\mathbb{C}}$.

- If $t(x_1, \ldots, x_n) \equiv f(t_1, \ldots, t_m) : B$ for some function symbol $f : B_1 \times \ldots \times B_m \to B$ of Σ with $m \geq 1$ and terms $t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n) \in \mathsf{Term}(\Sigma)$ with $t_i : B_i$ for each $1 \leq i \leq m$, then

  $$t(x_1, \ldots, x_n)^M := f^M \circ \left\langle t_1^M, \ldots, t_m^M \right\rangle$$

  $$: M_{A_1} \times \ldots \times M_{A_n} \to M_{B_1} \times \ldots \times M_{B_m} \to M_B.$$

∎

Before we can define how to interpret geometric formulas in interpretations of signatures, we must recall the following types of categories:

Definition 1.2.7. Let $\mathbb{C}$ be a category.

- $\mathbb{C}$ is a *cartesian category* if $\mathbb{C}$ has finite limits.
- $\mathbb{C}$ is a *regular category* if $\mathbb{C}$ is cartesian and every morphism in $\mathbb{C}$ has a unique (up to isomorphism) pullback-stable image factorization (recall that if $f : X \to Y$ is a morphism in $\mathbb{C}$, then an *image factorization* of f is a factorization $f = m \circ e$, where $e : X \twoheadrightarrow \mathsf{Im}(f)$ is a regular epimorphism, and $m : \mathsf{Im}(f) \rightarrowtail Y$ is a monomorphism).

- $\mathbb{C}$ is a *geometric category* if $\mathbb{C}$ is regular and for every object $X \in \mathbb{C}$, the subobject poset $\mathsf{Sub}(X)$ has all small, pullback-stable unions.

∎

We can now *interpret* the elements of $\mathsf{GeomForm}(\Sigma)$ in any interpretation of Σ in any geometric category:

Definition 1.2.8 (**Interpretation of Geometric Formulas**). Let Σ be a first-order signature, let $\mathbb{C}$ be a geometric category, and let M be an interpretation of Σ in $\mathbb{C}$. For any formula $\varphi(x_1, \ldots, x_n) \in \mathsf{GeomForm}(\Sigma)$ with $x_i : A_i$ for each $1 \leq i \leq n$, we define a subobject

$$\varphi(x_1, \ldots, x_n)^M \rightarrowtail M_{A_1} \times \ldots \times M_{A_n}$$

as follows:

- If $\varphi(x_1, \ldots, x_n) \equiv t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ for terms $t_1, t_2 \in \mathsf{Term}(\Sigma)$ of the same sort A, then $\varphi(x_1, \ldots, x_n)^M = (t_1 = t_2)^M$ is defined to be the equalizer of the following pair of arrows:

$$\begin{tikzcd}
(t_1 = t_2)^M \arrow[r, tail] & M_{A_1} \times \ldots \times M_{A_n} \arrow[r, shift left, "t_1^M"] \arrow[r, shift right, "t_2^M"'] & M_A
\end{tikzcd}$$

- If $\varphi(x_1, \ldots, x_n) \equiv R(t_1, \ldots, t_m)$ for some relation symbol $R \in \Sigma_{\mathsf{Rel}}$ with $R : B_1 \times \ldots \times B_m$ and terms $t_1, \ldots t_m \in \mathsf{Term}(\Sigma)$ with $t_i : B_i$ for each $1 \leq i \leq m$, then $\varphi(x_1, \ldots, x_n)^M = R(t_1, \ldots, t_m)^M$ is defined to be the lefthand subobject of the following pullback:

$$\begin{tikzcd}
R(t_1, \ldots, t_m)^M \arrow[r] \arrow[d, tail] & R^M \arrow[d, tail] \\
M_{A_1} \times \ldots \times M_{A_n} \arrow[r, "\langle t_1^M, \ldots, t_m^M \rangle"'] & M_{B_1} \times \ldots \times M_{B_m}
\end{tikzcd}$$

- If $\varphi(x_1, \ldots, x_n) \equiv \top$, then $\varphi(x_1, \ldots, x_n)^M = \top^M$ is defined to be the maximum element of $\mathsf{Sub}(M_{A_1} \times \ldots \times M_{A_n})$.
- If $\varphi(x_1, \ldots, x_n) \equiv \bot$, then $\varphi(x_1, \ldots, x_n)^M = \bot^M$ is defined to be the minimum element of $\mathsf{Sub}(M_{A_1} \times \ldots \times M_{A_n})$.

- If $\varphi(x_1, \ldots, x_n) \equiv \psi_1 \wedge \psi_2$ for some $\psi_1, \psi_2 \in \mathsf{GeomForm}(\Sigma)$, then
$$\varphi(x_1, \ldots, x_n)^M = (\psi_1 \wedge \psi_2)^M$$
is defined to be the intersection of the subobjects
$$\psi_1^M, \psi_2^M \rightarrowtail M_{A_1} \times \ldots \times M_{A_n}.$$
- If $\varphi(x_1, \ldots, x_n) \equiv \bigvee_{i \in I} \varphi_i(x_1, \ldots, x_n)$ for some set I and I-indexed set $\{\varphi_i(x_1, \ldots, x_n) : i \in I\} \subseteq \mathsf{GeomForm}(\Sigma)$, then since the subobject poset $\mathsf{Sub}(M_{A_1} \times \ldots \times M_{A_n})$ has all small unions, we define $\varphi^M = \left(\bigvee_{i \in I} \varphi_i\right)^M$ to be
$$\left(\bigvee_{i \in I} \varphi_i\right)^M := \bigcup_{i \in I} \varphi_i^M \rightarrowtail M_{A_1} \times \ldots \times M_{A_n}.$$
- Finally, let $\varphi(x_1, \ldots, x_n) \equiv \exists x \psi(x_1, \ldots, x_n, x)$ for some $\psi \in \mathsf{GeomForm}(\Sigma)$, and let $x : A$. Since $\mathbb{C}$ is a geometric and hence regular category, we then define $\varphi^M = (\exists x \psi(x_1, \ldots, x_n, x))^M$ to be the image of the following arrow:
$$\psi^M \rightarrowtail M_{A_1} \times \ldots \times M_{A_n} \times M_A \to M_{A_1} \times \ldots \times M_{A_n},$$
with the latter arrow being the projection.

∎

Definition 1.2.9 (**Geometric Sequents**). Let Σ be a first-order signature. By a *geometric sequent* over Σ, we mean an expression of the form $\varphi \vdash^{x_1,\ldots,x_n} \psi$, where $\varphi, \psi \in \mathsf{GeomForm}(\Sigma)$ and $\mathsf{FV}(\varphi), \mathsf{FV}(\psi) \subseteq \{x_1, \ldots, x_n\}$.

If M is an interpretation of Σ in some geometric category $\mathbb{C}$, then we say that M is a *model* of the geometric sequent $\varphi \vdash^{x_1,\ldots,x_n} \psi$ over Σ if
$$\varphi(x_1, \ldots, x_n)^M \subseteq \psi(x_1, \ldots, x_n)^M$$
in the subobject poset $\mathsf{Sub}(M_{A_1} \times \ldots \times M_{A_n})$ (assuming that $x_i : A_i$ for each $1 \leq i \leq n$). ∎

Finally, we can define the notion of a first-order *geometric theory*:

Definition 1.2.10 (Geometric Theories and Models). Let Σ be a first-order signature. A (first-order) *geometric theory* over Σ is a set $\mathbb{T}$ of geometric sequents over Σ, called the (non-logical) *axioms* of $\mathbb{T}$.

If M is an interpretation of Σ in some geometric category $\mathbb{C}$, then we say that M is a *model* of the geometric theory $\mathbb{T}$ if M is a model of every axiom of $\mathbb{T}$.

If $\mathbb{C}$ is a geometric category, we let $\mathsf{Mod}(\mathbb{T}, \mathbb{C})$ be the full subcategory of $\Sigma\mathsf{Str}(\mathbb{C})$ consisting of the models of $\mathbb{T}$. ∎

One can set up a *deduction system* for geometric sequents, wherein certain geometric sequents are designated as *logical axioms*, and there are logical *inference rules* for deriving geometric sequents from other geometric sequents. The actual logical axioms and inference rules for geometric sequents will not concern us in this thesis, so we will not exhibit them. If $\mathbb{T}$ is a geometric theory over Σ and $\varphi \vdash^{\vec{x}} \psi$ is a geometric sequent over Σ (with $\vec{x}$ being an abbreviation for the list of free variables occurring in φ or ψ), then we say that $\mathbb{T}$ *proves* the sequent $\varphi \vdash^{\vec{x}} \psi$, or that this sequent is *provable* in $\mathbb{T}$, if this sequent can be deduced from the (non-logical) axioms of $\mathbb{T}$ by means of the logical axioms and inference rules just mentioned. This deduction system for geometric sequents is then sound and complete for categorical semantics in geometric categories, in the sense that if $\mathbb{T}$ is any geometric theory over Σ and $\varphi \vdash^{\vec{x}} \psi$ is any geometric sequent, then $\mathbb{T}$ proves the sequent $\varphi \vdash^{\vec{x}} \psi$ iff every $\mathbb{T}$-model M in every geometric category models this sequent (cf. e.g. [6, 1.4.15, 1.4.16]).

Having defined geometric theories and models of geometric theories in geometric categories, we now review the notion of a *classifying topos* for a geometric theory. So let Σ be a first-order signature and $\mathbb{T}$ a geometric theory over Σ. If $\mathcal{E}$ is any cocomplete topos (e.g. any Grothendieck topos), then $\mathcal{E}$ is in particular a geometric category (cf. e.g. [4, 3.4.3, 3.4.14]), so it is possible to interpret all geometric formulas over Σ in $\mathcal{E}$. Then it is well-known (cf. e.g. [15, 10.6.1]) that $\mathbb{T}$ has a *classifying topos*, which is a cocomplete topos $\mathcal{B}(\mathbb{T})$ with the following (universal) property: for any cocomplete topos $\mathcal{E}$, there is an equivalence of categories

$$\mathsf{Hom}(\mathcal{E}, \mathcal{B}(\mathbb{T})) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{E})$$

between $\mathsf{Mod}(\mathbb{T}, \mathcal{E})$ and the category of geometric morphisms from $\mathcal{E}$ to $\mathcal{B}(\mathbb{T})$ (natural in $\mathcal{E}$). Specifically, it is known (cf. e.g. [15, 10.6.1]) that $\mathcal{B}(\mathbb{T})$ can be constructed as the category of sheaves on the *syntactic category* $\mathbb{C}_{\mathbb{T}}$ of $\mathbb{T}$, with respect to a certain Grothendieck topology $J_{\mathbb{T}}$ on $\mathbb{C}_{\mathbb{T}}$.

The geometric theory $\mathbb{T}$ also has a *universal model*, which is a model $U_{\mathbb{T}} \in \mathsf{Mod}(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ with the following universal property: for any cocomplete topos $\mathcal{E}$ and any $M \in \mathsf{Mod}(\mathbb{T}, \mathcal{E})$, there is a unique (up to natural isomorphism) geometric morphism

$$c_M : \mathcal{E} \to \mathcal{B}(\mathbb{T})$$

with the property that

$$M \cong c_M^*(U_{\mathbb{T}}),$$

where

$$c_M^* : \mathcal{B}(\mathbb{T}) \to \mathcal{E}$$

is the left exact inverse image functor of the geometric morphism c_M. Briefly: if M is a model of $\mathbb{T}$ in a cocomplete topos $\mathcal{E}$, then M is the inverse image of $U_{\mathbb{T}}$ under a unique geometric morphism $\mathcal{E} \to \mathcal{B}(\mathbb{T})$. The universal $\mathbb{T}$-model $U_{\mathbb{T}}$ then has the following 'minimality' property ([15, 10.7.1]): if $\varphi \vdash^{\vec{x}} \psi$ is any geometric sequent over Σ, then $U_{\mathbb{T}}$ is a model of this sequent iff every $\mathbb{T}$-model M in every cocomplete topos $\mathcal{E}$ is a model of this sequent.

Having given a general overview of first-order geometric theories and classifying toposes of such theories, we now wish to focus more specifically on the subclass of geometric theories that we will be studying for the majority of this thesis: the class of first-order *cartesian* theories (which will turn out to be equivalent to the class of *quasi-equational theories*, to be defined in the next section). To introduce this class of geometric theories, we first need the following definition:

Definition 1.2.11 (**Regular Formulas**)**.** Let Σ be a first-order signature.

- The class $\mathsf{AtomForm}(\Sigma)$ of *atomic formulas* is the subclass of $\mathsf{GeomForm}(\Sigma)$ consisting of the formulas of the form $t_1 = t_2$ and $R(t_1, \dots, t_n)$, where $t_1, t_2 \in \mathsf{Term}(\Sigma)$ are terms of the same sort, $R \in \Sigma_{\mathsf{Rel}}$, and $t_1, \dots, t_n \in \mathsf{Term}(\Sigma)$ are terms of the appropriate sorts.

- If $S \subseteq \mathsf{GeomForm}(\Sigma)$ is a subclass of $\mathsf{GeomForm}(\Sigma)$, then we say that S is *closed under binary conjunction* if $\varphi, \psi \in S$ implies $\varphi \wedge \psi \in S$, and we say that S is *closed under existential quantification* if $\psi \in S$ implies $\exists x \psi \in S$ for any variable x of any sort.

- The class $\mathsf{RegForm}(\Sigma)$ of *regular formulas* over Σ is the smallest subclass of $\mathsf{GeomForm}(\Sigma)$ that contains $\mathsf{AtomForm}(\Sigma)$ and $\top$ and is closed under binary conjunction and existential quantification.

- A geometric sequent $\varphi \vdash^{\vec{x}} \psi$ over Σ is called a *regular sequent* if $\varphi, \psi \in \mathsf{RegForm}(\Sigma)$.

- A geometric theory $\mathbb{T}$ over Σ is called a *regular theory* if all of its axioms are regular sequents.

- Let $\mathbb{T}$ be a regular theory over Σ, and let $\varphi \vdash^{\vec{x}} \psi$ be a regular sequent. Then we say that this regular sequent is *cartesian relative to* $\mathbb{T}$ if the following property holds: for any existential subformula $\exists x \chi(\vec{x}, x)$ of φ or ψ, the following sequent is provable in $\mathbb{T}$:

$$\chi(\vec{x}, x) \wedge \chi(\vec{x}, x') \vdash^{\vec{x}, x, x'} x = x',$$

where x' is a variable distinct from x of the same sort.

Briefly: the regular sequent $\varphi \vdash^{\vec{x}} \psi$ is cartesian relative to $\mathbb{T}$ if every existential quantification occurring in $\varphi \vdash^{\vec{x}} \psi$ is provably unique in $\mathbb{T}$. ∎

We can now define the class of first-order *cartesian theories*:

Definition 1.2.12 (Cartesian Theories). Let Σ be a first-order signature. A geometric theory $\mathbb{T}$ over Σ is a *cartesian theory* if it is a regular theory with the following property: the axioms of $\mathbb{T}$ can be well-ordered in such a way that if $\varphi \vdash^{\vec{x}} \psi$ is any axiom of $\mathbb{T}$ and $\mathbb{T}'$ is the regular theory consisting of all the axioms of $\mathbb{T}$ that precede $\varphi \vdash^{\vec{x}} \psi$ in this well-ordering, then the regular sequent $\varphi \vdash^{\vec{x}} \psi$ is cartesian relative to the regular theory $\mathbb{T}'$.

More briefly: a regular theory $\mathbb{T}$ is cartesian if there is a well-ordering on the axioms of $\mathbb{T}$ such that any existential quantification occurring in an axiom of $\mathbb{T}$ is provably unique in the subtheory of $\mathbb{T}$ consisting of all prior axioms in the well-ordering. ∎

Example 1.2.13 (Examples of Cartesian Theories). Let Σ be a first-order signature.

- A regular theory $\mathbb{T}$ over Σ is called a (multi-sorted) *algebraic theory* over Σ if its axioms are all of the form $\top \vdash^{\vec{x}} t_1(\vec{x}) = t_2(\vec{x})$, with $t_1, t_2 \in \mathsf{Term}(\Sigma)$ of the same sort. Thus, every algebraic theory over Σ is certainly a cartesian theory over Σ.

- The class $\mathsf{HornForm}(\Sigma)$ of *Horn formulas* over Σ is the smallest subclass of $\mathsf{GeomForm}(\Sigma)$ that contains $\mathsf{AtomForm}(\Sigma)$ and $\top$ and is closed under binary conjunction. We say that a geometric sequent $\varphi \vdash^{\vec{x}} \psi$ over Σ is a *Horn sequent* if $\varphi, \psi \in \mathsf{HornForm}(\Sigma)$. Then, we say that a regular theory $\mathbb{T}$ over Σ is a *Horn theory* if every axiom of $\mathbb{T}$ is a Horn sequent. Thus, every Horn theory over Σ is in particular a cartesian theory over Σ.

- It is well-known (cf. e.g. [6, 1.2.7]) that the theory of (small) categories can be presented as a first-order cartesian theory over the signature Σ that has two sorts O, A, three function symbols $\mathsf{id} : O \to A$ and $\mathsf{dom}, \mathsf{cod} : A \to O$, and one relation symbol $\mathsf{Comp} : A \times A \times A$. In particular, provably unique existential quantification is needed to axiomatize the fact that Comp must be a functional relation (i.e. any two composable arrows have exactly one composite). ∎

We will now review the description of the classifying topos of a *cartesian* theory $\mathbb{T}$, which can be presented as a full presheaf topos, rather than as a sheaf topos (as is the case for an arbitrary geometric theory). First, we require the following definition (1.2.14). If $\mathbb{T}$ is a first-order cartesian theory over a first-order signature Σ, then by a $\mathbb{T}$-*cartesian formula* φ over Σ we mean a regular formula φ over Σ with the property that if $\exists x \chi(\vec{x}, x)$ is any existential subformula of φ, the following sequent is provable in $\mathbb{T}$:

$$\chi(\vec{x}, x) \wedge \chi(\vec{x}, x') \vdash^{\vec{x}, x, x'} x = x',$$

where x' is a variable distinct from x of the same sort.

Definition 1.2.14 (**Finitely Presented Models**). Let $\mathbb{T}$ be a first-order cartesian theory over a first-order signature Σ, and let M be a model of $\mathbb{T}$ in the geometric category Sets. We say that M is a *finitely presented* model of $\mathbb{T}$ if it has the following (universal) property: there is a $\mathbb{T}$-cartesian formula $\varphi(x_1, \ldots, x_n)$ over Σ, with $x_i : A_i$ for each $1 \leq i \leq n$, and an n-tuple

$$(a_1, \ldots, a_n) \in \varphi(x_1, \ldots, x_n)^M \subseteq M_{A_1} \times \ldots \times M_{A_n}$$

(the *generators* of M) such that for any $\mathbb{T}$-model N in Sets and any n-tuple $(b_1, \ldots, b_n) \in \varphi(x_1, \ldots, x_n)^N \subseteq N_{A_1} \times \ldots \times N_{A_n}$, there is a unique Σ-homomorphism $f : M \to N$ such that

$$f_{A_i}(a_i) = b_i$$

for each $1 \leq i \leq n$. ∎

For a cartesian theory $\mathbb{T}$, let $\mathsf{fp}\mathbb{T}\mathsf{mod}$ be the full subcategory of $\mathsf{Mod}(\mathbb{T}, \mathsf{Sets})$ on the finitely presented $\mathbb{T}$-models. It is then well-known (cf. e.g. [6, 2.1.21]) that the classifying topos of $\mathbb{T}$ is (equivalent to) the covariant presheaf category $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$. We sketch the proof of this now. Let $\mathcal{E}$ be any cocomplete topos; we must show that there is an equivalence of categories

$$\mathsf{Hom}\left(\mathcal{E}, \mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}\right) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{E}).$$

Let $\mathbb{C}_{\mathbb{T}}$ be the syntactic category of $\mathbb{T}$, which is a cartesian category with the property that $\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathbb{D}) \simeq \mathsf{Mod}(\mathbb{T}, \mathbb{D})$ for any cartesian category $\mathbb{D}$ (where $\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathbb{D})$ is the category of finite-limit-preserving functors from $\mathbb{C}_{\mathbb{T}}$ to $\mathbb{D}$). In particular, since Sets is a cartesian category, we have $\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathsf{Sets}) \simeq \mathsf{Mod}(\mathbb{T}, \mathsf{Sets})$. Then by Gabriel-Ulmer duality ([11], [1]), it follows that the cartesian category $\mathbb{C}_{\mathbb{T}}$ is equivalent to the dual of the full subcategory of finitely presentable objects of $\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathsf{Sets}) \simeq \mathsf{Mod}(\mathbb{T}, \mathsf{Sets})$, which is $\mathsf{fp}\mathbb{T}\mathsf{mod}^{op}$. So $\mathbb{C}_{\mathbb{T}}^{op} \simeq \mathsf{fp}\mathbb{T}\mathsf{mod}$. Next, since $\mathcal{E}$ is a cocomplete topos and hence a cartesian category, we have $\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathcal{E}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{E})$ by the defining property of $\mathbb{C}_{\mathbb{T}}$. By Diaconescu's Theorem ([8]), we also know that

$$\mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathcal{E}) \simeq \mathsf{Hom}\left(\mathcal{E}, \mathsf{Sets}^{\mathbb{C}_{\mathbb{T}}^{op}}\right).$$

Altogether, we obtain

$$\mathsf{Hom}\left(\mathcal{E}, \mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}\right) \simeq \mathsf{Hom}\left(\mathcal{E}, \mathsf{Sets}^{\mathbb{C}_{\mathbb{T}}^{op}}\right) \simeq \mathsf{Lex}(\mathbb{C}_{\mathbb{T}}, \mathcal{E}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{E}),$$

as required. So $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$ is indeed the classifying topos of $\mathbb{T}$.

Definition 1.2.15 (Isotropy Group of Cartesian Theory). Let $\mathbb{T}$ be a first-order cartesian theory, with classifying topos $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$. The *isotropy group* of $\mathbb{T}$ is defined to be the covariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}$ (i.e. the contravariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}^{op}$), or equivalently the (internal) isotropy group object of the covariant presheaf topos $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$. We denote the isotropy group of $\mathbb{T}$ as

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}.$$

∎

The main purpose of this thesis will then be to investigate the isotropy group $\mathcal{Z}_{\mathbb{T}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ of an arbitrary first-order cartesian theory $\mathbb{T}$. In fact, in the next chapter we will investigate the covariant isotropy group of the *full* category $\mathsf{Mod}(\mathbb{T}, \mathsf{Sets})$, i.e. the (internal) isotropy group object of the covariant presheaf topos $\mathsf{Sets}^{\mathsf{Mod}(\mathbb{T},\mathsf{Sets})}$, and we will show that the characterization that we give of *this* isotropy group also applies to the covariant isotropy group of the subcategory $\mathsf{fp}\mathbb{T}\mathsf{mod}$, i.e. to the isotropy group of $\mathbb{T}$.

To facilitate our investigation of the isotropy group of a first-order cartesian theory $\mathbb{T}$, we will actually study a class of logical theories that is 'equivalent' to the class of first-order cartesian theories, namely the class of *quasi-equational theories.* This class of theories is easier to work with than the class of cartesian theories, because the quasi-equational theories do not involve the awkward notion of provably unique existential quantification, and they (unlike cartesian theories) also have explicit free model constructions, which we will need. It is to the definition of quasi-equational theories and the exposition of their needed properties that we now turn.

1.3 Quasi-Equational Theories

The background material in this section follows [19]. We begin with a definition:

Definition 1.3.1 (**Quasi-Equational Theory**). If Σ is a first-order signature in which $\Sigma_{\mathsf{Rel}} = \emptyset$, then we say that Σ is a *relation-free signature*. A *quasi-equational theory* $\mathbb{T}$ is then a set of Horn sequents over a relation-free signature Σ. ▮

In other words, a quasi-equational theory $\mathbb{T}$ is a set of sequents $\varphi \vdash^{\vec{x}} \psi$ over a relation-free signature Σ such that φ, ψ are both finite (possibly empty) conjunctions of equations between Σ-terms. Although quasi-equational theories have thus been defined as certain kinds of geometric theories (over relation-free signatures), the semantics and deduction system of quasi-equational theories will differ from those given for geometric theories in Section 1.2. Namely, function symbols will only need to be *partially* defined in interpretations, and the logical axioms and inference rules of the deduction system for quasi-equational theories (referred to as *partial Horn logic*) will reflect this.

Specifically, one can set up a deduction system of *partial Horn logic* for quasi-equational theories, wherein certain Horn sequents are designated as logical axioms, and there are logical inference rules allowing one to deduce certain Horn sequents from other Horn sequents. We refer the reader to [19] for a list of all the specific logical axioms and inference rules of partial Horn logic. The main novel feature of this deduction system is that equality of terms is *not* assumed to be reflexive, i.e. if $t(\vec{x})$ is a term over a given relation-free signature, then $\top \vdash^{\vec{x}} t(\vec{x}) = t(\vec{x})$ is *not* a logical axiom of partial Horn logic, unless t is a variable. In other words, if we abbreviate the equation $t = t$ by $t \downarrow$ (read: *t is defined*), then unless t is a variable, the sequent $\top \vdash^{\vec{x}} t \downarrow$ is *not* a logical axiom of partial Horn logic. Two specific logical axioms that we will make frequent (implicit) use of are the following:

- (*Strictness of equality axiom*) If $t_1(\vec{x}), t_2(\vec{x})$ are terms of the same sort over a given relation-free signature, then

  $$t_1 = t_2 \vdash^{\vec{x}} t_i \downarrow$$

  is a logical axiom for $i = 1, 2$.

- (*Strictness of functions axiom*) If $f : A_1 \times \ldots \times A_n \to A$ is a function symbol of a given relation-free signature, and $t_i(\vec{x}) : A_i$ is a term over this signature for each $1 \leq i \leq n$, then

  $$f(t_1, \ldots, t_n) \downarrow \vdash^{\vec{x}} t_i \downarrow$$

  is a logical axiom for each $1 \leq i \leq n$. ▮

If $\mathbb{T}$ is a quasi-equational theory over a relation-free signature Σ, and $\varphi \vdash^{\vec{x}} \psi$ is a Horn sequent over Σ, then we say that the sequent $\varphi \vdash^{\vec{x}} \psi$ is *(PHL-)provable* in $\mathbb{T}$ ('PHL' being an acronym for 'partial Horn logic') if there is a finite sequence of Horn sequents whose last member is $\varphi \vdash^{\vec{x}} \psi$, and each member of the sequence is either a logical axiom of partial Horn logic, an axiom of $\mathbb{T}$, or is obtained from previous members of the sequence by an inference rule of partial Horn logic. We also say that $\mathbb{T}$ *proves* the sequent $\varphi \vdash^{\vec{x}} \psi$ (in PHL), or that this sequent is a *(PHL-)theorem* of $\mathbb{T}$. If $\mathbb{T}$ proves a Horn sequent of the form $\top \vdash^{\vec{x}} \varphi$ in PHL, then we usually write this as $\mathbb{T} \vdash^{\vec{x}} \varphi$.

We now review the set-theoretic semantics of partial Horn logic. We recall that if A and B are any sets, then a *partial function* $f : A \rightharpoondown B$ is a total function $f : \mathsf{dom}(f) \to B$, where $\mathsf{dom}(f) \subseteq A$.

Definition 1.3.2 (**Partial Σ-Structure**)**.** Let Σ be a relation-free signature. A (set-based) *partial Σ-structure* M is given by the following data:

1. For every sort $A \in \Sigma$, a set M_A.

2. For every function symbol $f : A_1 \times \ldots \times A_n \to A$ of Σ, a *partial function*

$$f^M : M_{A_1} \times \ldots \times M_{A_n} \rightharpoondown M_A.$$

In case $n = 0$ and $f : A$ is a constant symbol, then $f^M : \{*\} \rightharpoondown M_A$ is a partial function. ∎

For a partial Σ-structure M, a function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ, and $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$, we will sometimes say that f^M *is defined* on $(a_1, \ldots, a_n)$ if $(a_1, \ldots, a_n) \in \mathsf{dom}(f^M)$. Similarly, if $c : A$ is a constant symbol of Σ, we will sometimes say that c^M *is defined* if $* \in \mathsf{dom}(c^M)$.

Definition 1.3.3 (**Σ-Morphism**)**.** Let Σ be a relation-free signature, and let M and N be (set-based) partial Σ-structures. A *Σ-morphism* $h : M \to N$ is a Σ_{Sort}-indexed sequence of *total* functions $h = (h_A : M_A \to N_A)_A$ satisfying the following condition:

- For any function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ and any

$$(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n},$$

if $(a_1, \ldots, a_n) \in \mathsf{dom}(f^M)$, then $(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in \mathsf{dom}(f^N)$ and

$$h_A(f^M(a_1, \ldots, a_n)) = f^N(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in N_A.$$

∎

To improve readability, we will sometimes write the sort indices on Σ-morphisms as superscripts rather than subscripts.

It is easy to verify that the (componentwise) composition of Σ-morphisms is a Σ-morphism, and that the sequence of identity functions $(\mathsf{id}_A : M_A \to M_A)_A$ is a Σ-morphism $\mathsf{id} : M \to M$ that is an identity for composition. So we can form the category $\mathsf{P\Sigma Str}$ of partial Σ-structures and Σ-morphisms.

Before we can define the notion of a (set-based) model of a quasi-equational theory, we must first define the interpretations of terms and Horn formulas in partial (set-based) structures.

Definition 1.3.4 (**Interpretation of Terms in Σ-Structures**)**.** Let Σ be a relation-free signature. Let $t(x_1, \ldots, x_k) : A$ be an element of $\mathsf{Term}(\Sigma)$ with free variables among $x_1 : A_1, \ldots, x_k : A_k$. Let M be a (set-based) partial Σ-structure. We define the *partial* function

$$t(x_1, \ldots, x_k)^M : M_{A_1} \times \ldots \times M_{A_k} \rightharpoonup M_A$$

by induction on t:

- If $t \equiv x_j : A_j$ for some $1 \le j \le k$, then we set

  $$t(x_1, \ldots, x_k)^M := \pi_j : M_{A_1} \times \ldots \times M_{A_k} \to M_{A_j},$$

  the (total) projection onto the j^{th} factor.

- If $t \equiv c$ for some constant $c : A$, then if c^M is defined, we define

  $$t(x_1, \ldots, x_k)^M : M_{A_1} \times \ldots \times M_{A_k} \to M_A$$

  to be the total, constant function with value $c^M(*) \in M_A$, and otherwise we let $t(x_1, \ldots, x_k)^M$ be everywhere undefined (i.e. $\mathsf{dom}\left(t(x_1, \ldots, x_k)^M\right) := \emptyset$).

- If $t \equiv f(t_1, \ldots, t_n) : B$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ with $n \ge 1$ and $t_1, \ldots, t_n \in \mathsf{Term}(\Sigma)$ with $t_i(x_1, \ldots, x_k) : B_i$ for each $1 \le i \le n$, we first set

  $$\begin{aligned} \mathsf{dom}\left(t^M\right) &:= \left\{ \vec{a} \in \bigcap_{1 \le i \le n} \mathsf{dom}\left(t_i^M\right) : \left(t_1^M(\vec{a}), \ldots, t_n^M(\vec{a})\right) \in \mathsf{dom}\left(f^M\right) \right\} \\ &\subseteq M_{A_1} \times \ldots \times M_{A_k}, \end{aligned}$$

and for any $\vec{a} \in \mathsf{dom}\left(t^M\right)$, we set

$$t^M(\vec{a}) := f^M\left(t_1^M(\vec{a}), \ldots, t_n^M(\vec{a})\right) \in M_B,$$

which defines

$$t^M = f(t_1, \ldots, t_n)^M : M_{A_1} \times \ldots \times M_{A_k} \rightharpoondown M_B.$$

∎

Definition 1.3.5 (Interpretation of Horn Formulas in Σ-Structures). Let Σ be a relation-free signature, and let $\varphi(x_1, \ldots, x_n)$ be a Horn formula over Σ whose free variables are among $x_1 : A_1, \ldots, x_n : A_n$. Let M be a (set-based) partial Σ-structure. We define

$$\varphi(x_1, \ldots, x_n)^M \subseteq M_{A_1} \times \ldots \times M_{A_n}$$

by induction on φ:

- If φ is atomic, i.e. if $\varphi \equiv t_1 = t_2$ for some Σ-terms $t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n) : A$ (for some sort A), then
$$\varphi(x_1, \ldots, x_n)^M = (t_1 = t_2)^M := \left\{\vec{a} \in \mathsf{dom}\left(t_1^M\right) \cap \mathsf{dom}\left(t_2^M\right) : t_1^M(\vec{a}) = t_2^M(\vec{a})\right\}.$$
- If $\varphi \equiv \top$, then
$$\top(x_1, \ldots, x_n)^M := M_{A_1} \times \ldots \times M_{A_n}.$$
- If $\varphi \equiv \varphi_1 \wedge \varphi_2$ for Horn formulas $\varphi_1(x_1, \ldots, x_n), \varphi_2(x_1, \ldots, x_n)$ over Σ, then
$$(\varphi_1 \wedge \varphi_2)^M := \varphi_1^M \cap \varphi_2^M \subseteq M_{A_1} \times \ldots \times M_{A_n}.$$

∎

In particular, if $\varphi \equiv t\downarrow$ for some Σ-term $t(x_1, \ldots, x_n) : A$, then

$$\begin{aligned}[t(x_1, \ldots, x_n)\downarrow]^M &= (t = t)^M \\ &= \left\{\vec{a} \in \mathsf{dom}\left(t^M\right) \cap \mathsf{dom}\left(t^M\right) : t^M(\vec{a}) = t^M(\vec{a})\right\} \\ &= \mathsf{dom}\left(t^M\right).\end{aligned}$$

Definition 1.3.6 (**Model of Horn Sequent**)**.** Let Σ be a relation-free signature, let M be a partial Σ-structure, and let $\varphi(\vec{x}), \psi(\vec{x})$ be Horn formulas over Σ. Then we say that M *models* or *satisfies* the Horn sequent $\varphi \vdash^{\vec{x}} \psi$ if $\varphi(x_1, \ldots, x_n)^M \subseteq \psi(x_1, \ldots, x_n)^M$. ▮

Definition 1.3.7 (**Model of Quasi-Equational Theory**)**.** Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ, and let M be a (set-based) partial Σ-structure. Then M is a (set-based) *model* of $\mathbb{T}$ if M satisfies every axiom of $\mathbb{T}$. ▮

For a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ, we now let $\mathsf{P}\mathbb{T}\mathsf{mod}$ be the full subcategory of $\mathsf{P}\Sigma\mathsf{Str}$ on the models of $\mathbb{T}$.

In order to sketch the details of the Initial Model Theorem for quasi-equational theories ([19]), we first require the following definitions.

Definition 1.3.8 (**Partial Congruence**)**.** Let Σ be a relation-free signature and M a partial Σ-structure. For every sort A, let $\sim_A$ be a relation on M_A. Then the Σ_{Sort}-indexed family of relations $(\sim_A)_A$ is a *partial congruence* on M if the following conditions are satisfied:

- For every sort A, the relation $\sim_A$ is an equivalence relation on M_A.
- For every function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ with $n \geq 1$ and all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in M_{A_1} \times \ldots \times M_{A_n}$, if

  $$a_i \sim_{A_i} b_i \text{ for all } 1 \leq i \leq n,$$

  then

  $$(a_1, \ldots, a_n) \in \mathsf{dom}(f^M) \text{ iff } (b_1, \ldots, b_n) \in \mathsf{dom}(f^M)$$

  and

  $$(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathsf{dom}(f^M) \Longrightarrow f^M(a_1, \ldots, a_n) \sim_A f^M(b_1, \ldots, b_n).$$

▮

Definition 1.3.9 (**Partial Quotient Structure**)**.** Let Σ be a relation-free signature and M a partial Σ-structure. Let $\sim\, = (\sim_A)_A$ be a partial congruence on M. We define the *partial quotient* Σ*-structure* $M/{\sim}$ as follows:

- For every sort $A \in \Sigma$, we set
$$(M/{\sim})_A := M_A/{\sim_A},$$
the set of equivalence classes of M_A modulo the equivalence relation $\sim_A$.
- For any constant $c : A$, we define
$$c^{M/\sim} : \{*\} \rightharpoondown M_A/{\sim_A}.$$
If c^M is defined, then we set $\mathsf{dom}\left(c^{M/\sim}\right) := \{*\}$ and
$$c^{M/\sim}(*) := \left[c^M(*)\right]_{\sim_A} \in M_A/{\sim_A},$$
the equivalence class of $c^M(*) \in M_A$ modulo $\sim_A$. Otherwise, we set $\mathsf{dom}\left(c^{M/\sim}\right) := \emptyset$.
- For any function symbol $f : A_1 \times \ldots \times A_n \to A$ with $n \geq 1$, we set
$$\mathsf{dom}\left(f^{M/\sim}\right)$$
$$:= \left\{([a_1], \ldots, [a_n]) \in M_{A_1}/{\sim_{A_1}} \times \ldots \times M_{A_n}/{\sim_{A_n}} : (a_1, \ldots, a_n) \in \mathsf{dom}\left(f^M\right)\right\}.$$
Then for any $([a_1], \ldots, [a_n]) \in \mathsf{dom}\left(f^{M/\sim}\right)$, we set
$$f^{M/\sim}\left([a_1], \ldots, [a_n]\right) := \left[f^M(a_1, \ldots, a_n)\right].$$

Because $\sim$ is a partial congruence on M, it easily follows that $M/{\sim}$ is a well-defined partial Σ-structure. ∎

Now we sketch the details of the Initial Model Theorem from [19] that we will need for our purposes. First, given a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ, we define a specific partial Σ-structure $M^{\mathbb{T}}$.

Definition 1.3.10. Let Σ be a relation-free signature. First, let
$$\mathsf{Term}^c(\Sigma) := \{t \in \mathsf{Term}(\Sigma) : \mathsf{FV}(t) = \emptyset\}$$
be the set of *closed terms* of $\mathsf{Term}(\Sigma)$. For any $A \in \Sigma_{\mathsf{Sort}}$, let
$$\mathsf{Term}^c(\Sigma)_A := \{t \in \mathsf{Term}^c(\Sigma) : t \text{ is of sort } A\}$$
be the set of closed Σ-terms of sort A.

Now let $\mathbb{T}$ be a quasi-equational theory over Σ. We define a partial Σ-structure $M^{\mathbb{T}}$ as follows:

- For any sort $A \in \Sigma$, we set

$$M^{\mathbb{T}}_A := \{t \in \mathsf{Term}^c(\Sigma)_A : \mathbb{T} \vdash t\downarrow\}.$$

- For any constant symbol $c : A$ of Σ, we let $\mathsf{dom}\left(c^{M^{\mathbb{T}}}\right) := \{*\}$ and $c^{M^{\mathbb{T}}}(*) := c \in M^{\mathbb{T}}_A$ if $\mathbb{T} \vdash c\downarrow$, and otherwise $\mathsf{dom}\left(c^{M^{\mathbb{T}}}\right) := \emptyset$.

- For any function symbol $f : A_1 \times \ldots \times A_n \to A$ of Σ with $n \geq 1$, we set

$$\mathsf{dom}\left(f^{M^{\mathbb{T}}}\right) := \{(t_1, \ldots, t_n) \in M^{\mathbb{T}}_{A_1} \times \ldots \times M^{\mathbb{T}}_{A_n} : \mathbb{T} \vdash f(t_1, \ldots, t_n)\downarrow\},$$

  and if $(t_1, \ldots, t_n) \in \mathsf{dom}\left(f^{M^{\mathbb{T}}}\right)$, we set

$$f^{M^{\mathbb{T}}}(t_1, \ldots, t_n) := f(t_1, \ldots, t_n) \in M^{\mathbb{T}}_A.$$

∎

Now we define a partial congruence $\sim^{\mathbb{T}}$ on $M^{\mathbb{T}}$. For any sort $A \in \Sigma$, we set

$$\sim^{\mathbb{T}}_A := \{(t_1, t_2) \in M^{\mathbb{T}}_A \times M^{\mathbb{T}}_A : \mathbb{T} \vdash t_1 = t_2\}.$$

Using the rules of partial Horn logic, it is then straightforward to verify that $\sim^{\mathbb{T}}$ is in fact a partial congruence on $M^{\mathbb{T}}$. We now make the following definition:

Definition 1.3.11. Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ, and let $M^{\mathbb{T}}$ be the partial Σ-structure and $\sim^{\mathbb{T}}$ the partial congruence on $M^{\mathbb{T}}$ just defined. Then, applying Definition 1.3.9, we define the following partial Σ-structure:

$$\mathsf{Free}(\mathbb{T}) := M^{\mathbb{T}}/{\sim^{\mathbb{T}}}.$$

∎

The following theorem is then proven in [19, Theorem 22]:

Theorem 1.3.12 (Initial Model Theorem). *Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ. Then the partial Σ-structure* $\mathsf{Free}(\mathbb{T})$ *is an* initial model *of $\mathbb{T}$, i.e. is an initial object of the category* $\mathsf{P}\mathbb{T}\mathsf{mod}$.

Explicitly, this means that $\mathsf{Free}(\mathbb{T})$ *is a model of $\mathbb{T}$, and that if $N \in$* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, then there is a unique Σ-morphism $h : \mathsf{Free}(\mathbb{T}) \to N$.* ∎

Remark 1.3.13 (Explicit Description of Initial Model). For concreteness and future purposes, we now give the explicit description of $\mathsf{Free}(\mathbb{T})$, for a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ.

- For any sort $A \in \Sigma$, we have

  $$\mathsf{Free}(\mathbb{T})_A := M_A^{\mathbb{T}}/{\sim_A^{\mathbb{T}}} = \{[t] : t \in \mathsf{Term}^c(\Sigma)_A \wedge \mathbb{T} \vdash t\downarrow\},$$

  where $[t]$ is the $\sim_A^{\mathbb{T}}$-congruence class of $t \in M_A^{\mathbb{T}}$ (so for any $s, t \in M_A^{\mathbb{T}}$, we have $[s] = [t]$ iff $\mathbb{T} \vdash s = t$).

- For any constant $c : A$ of Σ, if $\mathbb{T} \vdash c\downarrow$, then $\mathsf{dom}\left(c^{\mathsf{Free}(\mathbb{T})}\right) = \{*\}$ and $c^{\mathsf{Free}(\mathbb{T})}(*) = [c]_A$, and otherwise $\mathsf{dom}\left(c^{\mathsf{Free}(\mathbb{T})}\right) = \emptyset$.

- If $f : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ with $n \geq 1$, then

  $$\mathsf{dom}\left(f^{\mathsf{Free}(\mathbb{T})}\right) = \left\{([t_1], \ldots, [t_n]) \in \prod_{1 \leq i \leq n} \mathsf{Free}(\mathbb{T})_{A_i} : \mathbb{T} \vdash f(t_1, \ldots, t_n)\downarrow\right\},$$

  and for any $([t_1], \ldots, [t_n]) \in \mathsf{dom}\left(f^{\mathsf{Free}(\mathbb{T})}\right)$, we have

  $$f^{\mathsf{Free}(\mathbb{T})}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)].$$

∎

The following notion will be used heavily in what follows:

Definition 1.3.14 (Defined in M). Let Σ be a relation-free signature, let $t \in \mathsf{Term}^c(\Sigma)$ be a closed Σ-term, and let M be a partial Σ-structure. Since $\mathsf{FV}(t) = \emptyset$, we have

$$t^M : \{*\} \rightharpoonup M_A.$$

Then we say that t is *defined in M* if $\mathsf{dom}(t^M) = \{*\}$, and in this case we write $t^M \in M_A$ instead of $t^M(*) \in M_A$.

It is then easily seen that if $f(t_1, \ldots, t_n) \in \mathsf{Term}^c(\Sigma)_A$ for some function symbol $f : A_1 \times \ldots \times A_n \to A$ of Σ and closed terms $t_i \in \mathsf{Term}^c(\Sigma)_{A_i}$, then $f(t_1, \ldots, t_n)$ defined in M implies that (each t_i is defined in M and)

$$f(t_1, \ldots, t_n)^M = f^M(t_1^M, \ldots, t_n^M) \in M_A.$$

∎

We conclude this background section with the following two theorems from [19]. First, if s, t are closed Σ-terms of sort A and M is a partial Σ-structure, we say that $M \models s = t$ if M satisfies the sequent $\top \vdash s = t$. Then we have:

Lemma 1.3.15. *Let Σ be a relation-free signature. If M is a partial Σ-structure and $s, t \in \mathsf{Term}^c(\Sigma)_A$, then $M \models s = t$ iff s and t are defined in M and $s^M = t^M \in M_A$. In particular, $M \models s\downarrow$ iff s is defined in M.*

Proof: First suppose that $M \models s = t$, i.e. that M satisfies the sequent $\top \vdash s = t$. Then $\top^M \subseteq (s = t)^M \subseteq \{*\}$. Since $\top^M = \{*\}$, this implies that $(s = t)^M = \{*\}$, which means that $* \in \mathsf{dom}(s^M), \mathsf{dom}(t^M)$ and $s^M(*) = t^M(*)$. So then s and t are defined in M and $s^M = t^M$, as desired. To prove the converse implication, one just reverses this reasoning. The last statement in the lemma follows because $s\downarrow$ is an abbreviation for $s = s$. ▮

In what follows, we will use the preceding lemma without explicitly referring to it. Now we have ([19, Theorems 23, 24]):

Theorem 1.3.16 (Soundness and Completeness). *Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ.*

- *For any closed terms $s, t \in \mathsf{Term}^c(\Sigma)$ of the same sort, we have*

$$\mathsf{Free}(\mathbb{T}) \models s = t \textit{ iff } \mathbb{T} \vdash s = t.$$

- *For any Horn formulas $\phi(\vec{x}), \psi(\vec{x})$ over Σ, we have:*

the sequent $\phi \vdash^{\vec{x}} \psi$ holds in all (set-based) models of $\mathbb{T}$
iff
$\phi \vdash^{\vec{x}} \psi$ is a PHL-theorem of $\mathbb{T}$.

▮

In what follows, we will frequently refer to (the second part of) Theorem 1.3.16 as the soundness/completeness of partial Horn logic.

Remark 1.3.17 (Deduction Theorem and Theorem on Constants). Two results about partial Horn logic from [19] that we will frequently use are the *deduction theorem* and the *theorem on constants*:

- *Deduction Theorem* [19, Theorem 11]: Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ, and let θ be a Horn *sentence* over Σ. For any Horn formulas φ, ψ over Σ,

  $$\varphi \vdash^{\vec{x}} \psi \text{ is a PHL-theorem of } \mathbb{T} \cup \{\top \vdash \theta\}$$

  iff

  $$\theta \wedge \varphi \vdash^{\vec{x}} \psi \text{ is a PHL-theorem of } \mathbb{T}.$$

- *Theorem on Constants* [19, Theorem 10]: Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ, and let $c \notin \Sigma$ be a constant symbol. For any Horn formulas φ, ψ over Σ,

  $$\varphi \vdash^{\vec{x}} \psi \text{ is a PHL-theorem of } \mathbb{T} \cup \{\top \vdash c\downarrow\}$$

  iff

  $$\varphi[y/c] \vdash^{\vec{x},y} \psi[y/c] \text{ is a PHL-theorem of } \mathbb{T}\text{, for any variable } y \notin \vec{x} \text{ of the same sort as } c.$$

∎

We close this introductory chapter by relating this section to the preceding one. In the previous section, we defined a first-order cartesian theory to be (roughly) a regular theory (over an arbitrary signature) in which every existential quantification is provably unique, and for which the appropriate deduction system is that of geometric logic. In this section, we defined a quasi-equational theory to be a Horn theory over a *relation-free* signature, for which the appropriate deduction system is that of partial Horn logic. Now, it is a result of [19, Theorem 62] that if $\mathbb{T}$ is any first-order cartesian theory over a signature Σ, then there is an 'equivalent' quasi-equational theory $\mathbb{T}'$ over a relation-free signature Σ' with the property that

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Sets}) \simeq \mathsf{P}\mathbb{T}'\mathsf{mod}$$

(and in fact, $\mathbb{T}$ and $\mathbb{T}'$ have equivalent categories of models in any cartesian category whatsoever). Conversely (cf. [19, Section 9]), if $\mathbb{T}$ is a quasi-equational theory over a relation-free signature Σ, then there is an 'equivalent' cartesian theory $\mathbb{T}'$ over a signature Σ' with the property that

$$\mathsf{Mod}(\mathbb{T}', \mathsf{Sets}) \simeq \mathsf{P}\mathbb{T}\mathsf{mod}$$

(with a similar generalization to arbitrary cartesian categories). Therefore, the notions of first-order cartesian theory and quasi-equational theory are essentially equivalent.

Thus, instead of studying the isotropy groups of first-order cartesian theories, we will instead study the isotropy groups of *quasi-equational* theories. More specifically, in the next chapter we will characterize the (internal) isotropy group object of the covariant presheaf topos $\mathsf{Sets}^{\mathsf{P}\mathbb{T}\mathsf{mod}}$, i.e. the covariant isotropy group (functor)

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group},$$

for a given quasi-equational theory $\mathbb{T}$, and we will show that this characterization also applies to the (internal) isotropy group object of $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$, with $\mathsf{fp}\mathbb{T}\mathsf{mod}$ being the full subcategory of $\mathsf{P}\mathbb{T}\mathsf{mod}$ on the *finitely presentable* models (for a suitable notion of finitely presented model of a quasi-equational theory, cf. Definition 2.4.1).

Chapter 2

Isotropy Groups of Quasi-Equational Theories

2.1 Motivation

In this chapter, we will give a syntactic (or proof-theoretic) characterization of the *covariant* isotropy group (functor)

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$$

of an arbitrary quasi-equational theory $\mathbb{T}$. This is the *contravariant* isotropy group of the *dual* category $\mathsf{P}\mathbb{T}\mathsf{mod}^{op}$, so for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, we have that $\mathcal{Z}_{\mathbb{T}}(M)$ is the group of all natural automorphisms of the projection functor

$$\mathsf{P}\mathbb{T}\mathsf{mod}^{op}/M \to \mathsf{P}\mathbb{T}\mathsf{mod}^{op},$$

i.e. of the projection functor

$$M/\mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}.$$

So for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, if $\mathsf{Dom}(M) := \{f \in \mathsf{P}\mathbb{T}\mathsf{mod} : \mathsf{dom}(f) = M\}$, then the elements $\alpha \in \mathcal{Z}_{\mathbb{T}}(M)$ are $\mathsf{Dom}(M)$-indexed families of automorphisms

$$\alpha = \Big(\alpha_f : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\Big)_{f \in \mathsf{Dom}(M)}$$

in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with the following naturality property: for any Σ-morphisms $f : M \to M'$ and $f' : M' \to M''$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$, we have

$$\alpha_{f' \circ f} \circ f' = f' \circ \alpha_f : M' \to M'',$$

as in the following commutative diagram:

$$\begin{tikzcd}
M' \arrow[r, "\alpha_f"] \arrow[d, "f'"'] & M' \arrow[d, "f'"] \\
M'' \arrow[r, "\alpha_{f' \circ f}"'] & M''
\end{tikzcd}$$

Moreover, if $h : M \to N$ is a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, then $\mathcal{Z}_{\mathbb{T}}(h) : \mathcal{Z}_{\mathbb{T}}(M) \to \mathcal{Z}_{\mathbb{T}}(N)$ is the group homomorphism given as follows: for any $\alpha \in \mathcal{Z}_{\mathbb{T}}(M)$ as above,

$$\mathcal{Z}_{\mathbb{T}}(h)(\alpha) = \Big(\alpha_{g \circ h} : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g)\Big)_{g \in \mathsf{Dom}(N)} \in \mathcal{Z}_{\mathbb{T}}(N).$$

In order to motivate our characterization of the isotropy group $\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$, we first review a result proven by George Bergman in [3, Theorems 1, 2]. Bergman considered the covariant isotropy group $\mathcal{Z}_{\mathsf{Group}} : \mathsf{Group} \to \mathsf{Group}$ of the category Group, and proved that if G is any group, then $\mathcal{Z}_{\mathsf{Group}}(G) \cong G$. (Technically, Bergman did not use the terminology of *covariant isotropy*, and instead just referred to $\mathcal{Z}_{\mathsf{Group}}(G)$ as the automorphism group of the forgetful functor $G/\mathsf{Group} \to \mathsf{Group}$). To do this, Bergman showed that if

$$\alpha = (\alpha_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(G)}$$

is a $\mathsf{Dom}(G)$-indexed family of group endomorphisms (not necessarily satisfying the above naturality property), then $\alpha \in \mathcal{Z}_{\mathsf{Group}}(G)$ iff there is a (uniquely determined) element $g \in G$ such that

$$\alpha_f = \mathsf{conj}_{f(g)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)$$

for each $f \in \mathsf{Dom}(G)$, where $\mathsf{conj}_{f(g)}$ is the automorphism of $\mathsf{cod}(f)$ given by conjugation by $f(g)$ (i.e. $\mathsf{conj}_{f(g)}(a) = f(g)af(g)^{-1}$ for $a \in \mathsf{cod}(f)$).

If $g \in G$ has this latter property, then it is easy to verify that $\alpha \in \mathcal{Z}_{\mathsf{Group}}(G)$ (because then each α_f is a group *auto*morphism, and it is easily seen that

$$\mathsf{conj}_{f'(f(g))} \circ f' = f' \circ \mathsf{conj}_{f(g)}$$

for all $f : G \to G'$, $f' : G' \to G''$). To prove the other implication, Bergman reasoned as follows. Suppose that $\alpha \in \mathcal{Z}_{\mathsf{Group}}(G)$. We will then consider the free product group $G\langle x \rangle$, i.e. the free product (or coproduct) of G with the free group on one generator x. Elements of $G\langle x \rangle$ are reduced words over the alphabet $G \cup \{x\}$. There is an obvious inclusion homomorphism $\eta : G \to G\langle x \rangle$, sending each $g \in G$ to the one letter reduced word g. Because $\alpha \in \mathcal{Z}_{\mathsf{Group}}(G)$, we then have a group automorphism

$$\alpha_\eta : G\langle x \rangle \xrightarrow{\sim} G\langle x \rangle$$

with the property that if $f : G\langle x\rangle \to H$ is any group homomorphism, then

$$\alpha_{f\circ\eta} \circ f = f \circ \alpha_\eta : G\langle x\rangle \to H.$$

In particular, if $f : G \to H$ is a group homomorphism and $h \in H$, then by the universal property of $G\langle x\rangle$, there is a unique group homomorphism $f_h : G\langle x\rangle \to H$ such that $f_h \circ \eta = f$ and $f_h(x) = h$, and hence we have

$$\alpha_f \circ f_h = \alpha_{f_h\circ\eta} \circ f_h = f_h \circ \alpha_\eta : G\langle x\rangle \to H,$$

so that

$$\alpha_f(h) = \alpha_f(f_h(x)) = f_h(\alpha_\eta(x)) \in H$$

holds for each $h \in H$.

Now, consider the element

$$\alpha_\eta(x) \in G\langle x\rangle.$$

Bergman showed ([3, Theorem 1]) that $\alpha_\eta(x) \equiv gxg^{-1}$ for some element $g \in G$ (if $g = e$ (the identity element of G), then $\alpha_\eta(x) \equiv x$, because exe^{-1} is not reduced). Now, we show that this element $g \in G$ has the desired property. So let $f : G \to H$ be any group homomorphism with domain G; we must show that

$$\alpha_f = \mathsf{conj}_{f(g)} : H \xrightarrow{\sim} H.$$

So let $h \in H$. Then it is easy to see that the induced group homomorphism $f_h : G\langle x\rangle \to H$ acts as follows: if w is a reduced word over $G \cup \{x\}$, let w_f be the (not necessarily reduced) word over H obtained from w by replacing each occurrence of x by h, and by replacing each occurrence of an element $g' \in G$ by $f(g') \in H$. Then $f_h(w) \in H$ is the element of H obtained by 'evaluating' the word w_f in H. In particular, we have

$$f_h(\alpha_\eta(x)) = f_h(gxg^{-1}) = f(g)hf(g)^{-1} \in H.$$

Thus, we have

$$\alpha_f(h) = f_h(\alpha_\eta(x)) = f(g)hf(g)^{-1} = \mathsf{conj}_{f(g)}(h) \in H$$

for every $h \in H$, so that $\alpha_f = \mathsf{conj}_{f(g)}$, as desired. Finally, such an element $g \in G$ is uniquely determined, for suppose that we also had $g_1 \in G$ with the property that $\alpha_f = \mathsf{conj}_{f(g_1)}$ for every $f \in \mathsf{Dom}(G)$. Then, considering the inclusion morphism $\eta : G \to G\langle x\rangle$, we have

$$gxg^{-1} = \alpha_\eta(x) = \mathsf{conj}_{\eta(g_1)}(x) = \mathsf{conj}_{g_1}(x) = g_1xg_1^{-1},$$

so that the reduced words gxg^{-1} and $g_1xg_1^{-1}$ are identical, which forces $g = g_1$, as desired.

It is easy to verify that the bijective function

$$g \mapsto \Big(\mathsf{conj}_{f(g)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\Big)_{f \in \mathsf{Dom}(G)}$$

$$: G \to \mathcal{Z}_{\mathsf{Group}}(G)$$

is a group homomorphism, so that we indeed have a group isomorphism

$$G \cong \mathcal{Z}_{\mathsf{Group}}(G)$$

for every group G, as claimed. Moreover, these isomorphisms are natural in $G \in \mathsf{Group}$, so that the isotropy group $\mathcal{Z}_{\mathsf{Group}} : \mathsf{Group} \to \mathsf{Group}$ is naturally isomorphic to the identity functor on Group.

Now we will indicate the aspects of the above reasoning that we will generalize in this chapter, in order to give our syntactic characterization of the isotropy group of an arbitrary quasi-equational theory (of which the theory of groups, being a quasi-equational theory $\mathbb{T}$ with $\mathsf{P}\mathbb{T}\mathsf{mod} = \mathsf{Group}$, is an example). First, we observe that the preceding arguments of Bergman also show that for any group G,

$$\mathcal{Z}_{\mathsf{Group}}(G) \cong \left\{gxg^{-1} \in G\langle x \rangle : g \in G\right\},$$

with the righthand set having the following group structure: for any $g_1, g_2 \in G$, we have $(g_1xg_1^{-1})(g_2xg_2^{-1}) = g_1g_2x(g_1g_2)^{-1}$, the unit element is x (the reduction of the non-reduced word exe^{-1}), and for any $g \in G$, we have $(gxg^{-1})^{-1} = g^{-1}xg$.

Next, we claim that the set $\{gxg^{-1} \in G\langle x \rangle : g \in G\}$ is the set of *all and only* those reduced words $w \in G\langle x \rangle$ that satisfy the following properties:

- w is *invertible*, in the sense that there is some reduced word $w^* \in G\langle x \rangle$ such that the reduced word obtained from $w[w^*/x]$ is x, and similarly for $w^*[w/x]$.
- w *commutes generically* with every function symbol of the theory of groups, in the following sense:
  - The reduced word over $G \cup \{x_0, x_1\}$ obtained from $w[x_0/x]w[x_1/x]$ is $w[x_0x_1/x]$.
  - The reduced word over $G \cup \{x\}$ obtained from w^{-1} is $w[x^{-1}/x]$.
  - The element of G obtained by 'evaluating' the word $w[e/x]$ in G is e^G.

Indeed, if $g \in G$, then every reduced word of the form gxg^{-1} easily satisfies the above properties: for the first property, it is easily seen that we can take $(gxg^{-1})^* := g^{-1}xg$. For the second property, the reduced word obtained from $gx_0g^{-1}gx_1g^{-1}$ is clearly $gx_0x_1g^{-1}$, the reduced word obtained from $(gxg^{-1})^{-1} = (g^{-1})^{-1}x^{-1}g^{-1}$ is clearly $gx^{-1}g^{-1}$, and the element of G obtained by evaluating the word geg^{-1} in G is clearly e^G.

Now let $w \in G\langle x\rangle$ be any reduced word satisfying both of the above properties; we must show that $w \equiv gxg^{-1}$ for some $g \in G$. First, w must contain at least one occurrence of x, for otherwise w would not satisfy the invertibility property. Also, w contains at most one, and hence exactly one, occurrence of x, by the following argument of Bergman: if w had at least two occurrences of x, then the second property would fail, because $w[x_0x_1/x]$ would then have an occurrence of x_1 preceding an occurrence of x_0, while it is easy to see that the reduced word obtained from $w[x_0/x]w[x_1/x]$ would have all occurrences of x_1 *following* all occurrences of x_0, and hence these reduced words could not be equal. So w has exactly one occurrence of x, and hence (being reduced) must have one of the following forms:

$$x, gx, xg, g_1xg_2,$$

for some $g, g_1, g_2 \neq e^G \in G$. If $w \equiv x$, then we are done, because x is the reduced word obtained from exe^{-1}. If $w \equiv gx$ for some $g \neq e^G \in G$, then the second property would fail, because the reduced word gx_0gx_1 is not equal to gx_0x_1. Similar reasoning shows that $w \not\equiv xg$, and hence we must have $w \equiv g_1xg_2$ for some $g_1, g_2 \neq e^G \in G$. By the second property, we know that the element of G obtained by evaluating the word $w[e/x] \equiv g_1eg_2$ is e^G, i.e. $e^G = g_1e^Gg_2 = g_1g_2$, so that $g_2 = g_1^{-1}$, and hence $w \equiv g_1xg_2 \equiv g_1xg_1^{-1}$, as desired.

Thus, Bergman essentially proved that for any group G, the isotropy group $\mathcal{Z}_{\mathsf{Group}}(G)$ is isomorphic to the group of all reduced words $w \in G\langle x\rangle$ that are invertible and commute generically with all function symbols of the theory of groups, in the above senses. It is *this* (syntactic) description of the isotropy group of the theory of groups that we will generalize to the isotropy group of *any* arbitrary quasi-equational theory. More specifically, given any model $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ of a *single-sorted* quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ, we will show that $\mathcal{Z}_{\mathbb{T}}(M)$ is isomorphic to the group of all elements of $M\langle x\rangle$ (the coproduct of M with the free $\mathbb{T}$-model on one generator x) that are *invertible* and *commute generically* with all function symbols of Σ (and satisfy one further condition), in a manner similar to the above. This characterization will also be suitably extended to theories over *multi-sorted* signatures.

2.2 The Isotropy Group of a Quasi-Equational Theory

For the remainder of this section, we fix a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ. Our ultimate goal in this section is to give a syntactic characterization of the covariant isotropy group

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}.$$

Our first step towards this goal is to give an explicit description of the process of 'adjoining an indeterminate element' (of a given sort) to a $\mathbb{T}$-model M. More precisely, given $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$, we will give an explicit description/construction of the $\mathbb{T}$-model $M\langle \mathsf{x}_A \rangle$, which will be the coproduct in $\mathsf{P}\mathbb{T}\mathsf{mod}$ of M and the free $\mathbb{T}$-model on one generator x_A of sort A. (cf. the universal property in Proposition 2.2.10).

First, given $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, we need to define new signatures $\Sigma(M), \Sigma(M, \mathsf{x}_A)$ extending Σ and taking into account the elements of M, and new quasi-equational theories $\mathbb{T}(M), \mathbb{T}(M, \mathsf{x}_A)$ extending $\mathbb{T}$ and axiomatizing the specific properties of the model M.

Definition 2.2.1 (**Diagram Signature**)**.** Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. For any $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$, we introduce a new constant symbol $c^M_{B,s} \notin \Sigma_{\mathsf{Fun}}$ of sort B, so that $s \neq s' \in M_B \implies c^M_{B,s} \not\equiv c^M_{B,s'}$. (i.e. these new constants are pairwise distinct).

1. Let $\Sigma(M)$, the *diagram signature of* M, be the relation-free signature defined as follows:

   - $\Sigma(M)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
   - $\Sigma(M)_{\mathsf{Fun}} := \Sigma_{\mathsf{Fun}} \cup \left\{ c^M_{B,s} : B \in \Sigma_{\mathsf{Sort}}, s \in M_B \right\}$.

2. If $A \in \Sigma_{\mathsf{Sort}}$, let $\mathsf{x}_A \notin \Sigma(M)$ be a new constant of sort A, and let $\Sigma(M, \mathsf{x}_A)$ be the signature defined as follows:

   - $\Sigma(M, \mathsf{x}_A)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
   - $\Sigma(M, \mathsf{x}_A)_{\mathsf{Fun}} := \Sigma(M)_{\mathsf{Fun}} \cup \{\mathsf{x}_A\}$. ∎

To increase readability, we will try to omit the sort subscripts and model superscripts from new constants of the diagram signature when no ambiguity can arise (i.e. we will write c_s or c^M_s instead of $c^M_{B,s}$).

If Σ^1 and Σ^2 are relation-free signatures, then we say that $\Sigma^1 \subseteq \Sigma^2$ if $\Sigma^1_{\mathsf{Sort}} \subseteq \Sigma^2_{\mathsf{Sort}}$ and $\Sigma^1_{\mathsf{Fun}} \subseteq \Sigma^2_{\mathsf{Fun}}$. If $\Sigma_1 \subseteq \Sigma_2$ and N is a partial Σ^2-structure, then by $N|_{\Sigma^1}$, or the Σ^1-*reduct* of N, we mean the partial Σ^1-structure obtained from N in the expected way.

If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$, then we clearly have $\Sigma \subseteq \Sigma(M) \subseteq \Sigma(M, \mathsf{x}_A)$.

Remark 2.2.2 (**Canonical Diagram Structure**)**.** If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then M can be expanded to a canonical partial $\Sigma(M)$-structure $\widehat{M}$ such that $\widehat{M}|_\Sigma = M$ in the following way:

- For any sort $B \in \Sigma$, we set $\widehat{M_B} := M_B$.
- For any function symbol $f \in \Sigma$, we set $f^{\widehat{M}} := f^M$.
- For any sort $B \in \Sigma$ and $s \in M_B$, we set $\left(c^M_{B,s}\right)^{\widehat{M}} := s$. ∎

Definition 2.2.3 (**Diagram Theory**)**.** Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$.

1. We define $\mathbb{T}(M)$, the *diagram theory of* M, to be the quasi-equational theory over the diagram signature $\Sigma(M)$ whose axioms are the following Horn sequents over $\Sigma(M)$:
   - All axioms of $\mathbb{T}$.
   - For any sort $B \in \Sigma$ and $s \in M_B$, the axiom $\top \vdash c^M_{B,s} \downarrow$.
   - For any function symbol $f : B_1 \times \ldots \times B_n \to B$ in Σ and any $s_1 \in M_{B_1}, \ldots, s_n \in M_{B_n}$ with $(s_1, \ldots, s_n) \in \mathsf{dom}(f^M)$, the axiom
   $$\top \vdash f\left(c^M_{s_1}, \ldots, c^M_{s_n}\right) = c^M_{f^M(s_1,\ldots,s_n)}.$$
   In particular, if $n = 0$ and $f : B$ is a constant symbol that is defined in M, then $\mathbb{T}(M)$ has the axiom
   $$\top \vdash f = c^M_{f^M}.$$
2. For any sort $A \in \Sigma$, we define $\mathbb{T}(M, \mathsf{x}_A)$ to be the quasi-equational theory over the signature $\Sigma(M, \mathsf{x}_A)$ whose axioms are those of $\mathbb{T}(M)$, together with the axiom $\top \vdash \mathsf{x}_A \downarrow$. ∎

Now we have the following result connecting models of $\mathbb{T}(M)$ to Σ-morphisms with domain M: essentially, every morphism with domain M gives rise to a model of $\mathbb{T}(M)$ on its codomain, and every model of $\mathbb{T}(M)$ induces a morphism from M.

Lemma 2.2.4. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*.*

1. *If* $h : M \to N$ *is any morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, then there is a partial* $\Sigma(M)$*-structure* N^h *such that* $N^h|_\Sigma = N$ *and* $N^h \models \mathbb{T}(M)$*, with the following description:*

- *For any sort* $B \in \Sigma(M)_{\mathsf{Sort}} = \Sigma_{\mathsf{Sort}}$*,*
$$N^h_B := N_B.$$
- *For any function symbol* $f \in \Sigma$*,*
$$f^{N^h} := f^N.$$
- *For any* $A \in \Sigma_{\mathsf{Sort}}$ *and* $a \in M_A$*,*
$$\left(c^M_{A,a}\right)^{N^h} := h_A(a) \in N_A = N^h_A.$$

2. *If* N' *is a partial* $\Sigma(M)$*-structure with* $N' \models \mathbb{T}(M)$*, then there is a unique* Σ*-morphism* $h : M \to N'|_\Sigma$ *such that* $(N'|_\Sigma)^h = N'$*.*

Proof: See Appendix A. ▮

Remark 2.2.5. It is easy to see that the canonical $\Sigma(M)$-structure $\widehat{M}$ is a model of $\mathbb{T}(M)$. ▮

Definition 2.2.6. For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$, we define the partial Σ-structure
$$M\langle \mathsf{x}_A \rangle := \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))|_\Sigma,$$
the Σ-reduct of the $\Sigma(M, \mathsf{x}_A)$-structure $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))$ (which itself is the initial model of $\mathbb{T}(M, \mathsf{x}_A)$).

Since $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))$ is a model of $\mathbb{T}(M, \mathsf{x}_A) \supseteq \mathbb{T}$, it follows that $M\langle \mathsf{x}_A \rangle$ is a model of $\mathbb{T}$. ▮

We will show in Proposition 2.2.10 below that $M\langle \mathsf{x}_A \rangle$ has the universal property previously alluded to, i.e. $M\langle \mathsf{x}_A \rangle$ will be the coproduct in $\mathsf{P}\mathbb{T}\mathsf{mod}$ of M with the initial $\mathbb{T}$-model on one generator x_A of sort A.

Remark 2.2.7. Recall from Remark 1.3.13 that the partial Σ-structure $M\langle \mathsf{x}_A \rangle$ has the following explicit description:

- For any sort $B \in \Sigma$,
$$M\langle \mathsf{x}_A \rangle_B = \left\{ [t] : t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B \wedge \mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow \right\},$$
where $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ is the set of *closed* $\Sigma(M, \mathsf{x}_A)$-terms of sort B, and $[s] = [t]$ iff $\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$.

- For any constant $c : B$ of Σ, if $\mathbb{T}(M, \mathsf{x}_A) \vdash c\downarrow$, then $\mathsf{dom}\left(c^{M\langle \mathsf{x}_A\rangle}\right) = \{*\}$ and
$$c^{M\langle \mathsf{x}_A\rangle}(*) := [c] \in M\langle \mathsf{x}_A\rangle_B,$$
and otherwise $\mathsf{dom}\left(c^{M\langle \mathsf{x}_A\rangle}\right) = \emptyset$.
- If $f : B_1 \times \ldots \times B_n \to B$ is a function symbol in Σ with $n \geq 1$, then
$$\mathsf{dom}\left(f^{M\langle \mathsf{x}_A\rangle}\right) = \left\{([t_1], \ldots, [t_n]) \in \prod_{1\leq i\leq n} M\langle \mathsf{x}_A\rangle_{B_i} : \mathbb{T}(M, \mathsf{x}_A) \vdash f(t_1, \ldots, t_n)\downarrow\right\},$$
and for any $([t_1], \ldots, [t_n]) \in \mathsf{dom}\left(f^{M\langle \mathsf{x}_A\rangle}\right)$, we have
$$f^{M\langle \mathsf{x}_A\rangle}\left([t_1], \ldots, [t_n]\right) = [f(t_1, \ldots, t_n)] \in M\langle \mathsf{x}_A\rangle_B.$$

∎

Before we can state the universal property of $M\langle \mathsf{x}_A\rangle$, we need to define a canonical Σ-morphism $\eta^{M,A} : M \to M\langle \mathsf{x}_A\rangle$ as follows.

Definition 2.2.8. For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$, let
$$\eta^{M,A}_B : M_B \to M\langle \mathsf{x}_A\rangle_B$$
be the (total) function given by the following rule: for any $s \in M_B$,
$$s \mapsto \left[c^M_s\right] \in M\langle \mathsf{x}_A\rangle_B.$$
Since $\mathbb{T}(M, \mathsf{x}_A) \vdash c^M_s\downarrow$ for any $s \in M_B$, this function is well-defined. ∎

If $M, N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $h = (h_A : M_A \to N_A)_{A\in\Sigma}$ is a Σ-morphism, then we say that h is *(sortwise) injective* if each (total) function $h_A : M_A \to N_A$ is injective.

Lemma 2.2.9. *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then the family of total functions*
$$\eta^{M,A} := \left(\eta^{M,A}_C : M_C \to M\langle \mathsf{x}_A\rangle_C\right)_{C\in\Sigma}$$
is a Σ-morphism from M to $M\langle \mathsf{x}_A\rangle$. If $M_A \neq \emptyset$, then $\eta^{M,A}$ is moreover sortwise injective.

Proof: See Appendix A. ∎

We can now state the universal property of the $\mathbb{T}$-model $M\langle \mathsf{x}_A \rangle$. Note that $[\mathsf{x}_A] \in M\langle \mathsf{x}_A \rangle_A$ because $\mathbb{T}(M, \mathsf{x}_A) \vdash \mathsf{x}_A \downarrow$.

Proposition 2.2.10. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$*. For any* $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, any* Σ*-morphism* $h : M \to N$*, and any* $a \in N_A$*, there is a unique* Σ*-morphism*

$$h^a : M\langle \mathsf{x}_A \rangle \to N$$

such that $h^a \circ \eta^{M,A} = h$ *and* $h^a_A([\mathsf{x}_A]) = a \in N_A$*.*

Proof: See Appendix A. ∎

We will now show how to interpret (provably defined) *closed* terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ as *total functions* given by substitution into the indeterminate x_A.

Definition 2.2.11 (Term-Induced Function). Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$. Let $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ be a closed term of some sort $B \in \Sigma$ such that $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$. We define a *total* function

$$t^* : M_A \to M_B$$

as follows.

Let $a \in M_A$. Then $(\widehat{M}, a)$ is a partial $\Sigma(M, \mathsf{x}_A)$-structure that is a model of $\mathbb{T}(M, \mathsf{x}_A)$, because $\widehat{M}$ is a model of $\mathbb{T}(M)$ and $(\widehat{M}, a) \models \mathsf{x}_A \downarrow$. Since $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$, it follows by soundness of partial Horn logic that $(\widehat{M}, a) \models t \downarrow$. So t is defined in $(\widehat{M}, a)$, and we set

$$t^*(a) := t^{(\widehat{M}, a)} \in M_B.$$

∎

Remark 2.2.12. One can also define the total function t^* by induction on the form of $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ as follows:

- If $t \equiv \mathsf{x}_A : A$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and

  $$t^* = \mathsf{x}_A^* : M_A \to M_A$$

  is just the identity function.

- If $t \equiv c_s : B$ for some $s \in M_B$ and $B \in \Sigma_{\mathsf{Sort}}$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and
$$t^* = c_s^* : M_A \to M_B$$
is the constant function on $s \in M_B$.

- If $t \equiv f(t_1, \ldots, t_n) : B$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ and terms $t_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_{B_i}$ for each $1 \leq i \leq n$ and $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t_i \downarrow$ for each $1 \leq i \leq n$ by the rules of partial Horn logic, and hence we have a well-defined total function
$$t_i^* : M_A \to M_{B_i}$$
for each $1 \leq i \leq n$. We then have
$$t^* = f(t_1, \ldots, t_n)^* : M_A \to M_B$$
given by the rule
$$a \mapsto f^M(t_1^*(a), \ldots, t_n^*(a)) \in M_B$$
for $a \in M_A$. ∎

The following lemma now says that provably equal terms induce the same function:

Lemma 2.2.13. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$. Let $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ be of the same sort $B \in \Sigma$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$ (and hence $\mathbb{T}(M, \mathsf{x}_A) \vdash s \downarrow \wedge\, t \downarrow$, by the rules of partial Horn logic). Then*
$$s^* = t^* : M_A \to M_B.$$

Proof: Assume the hypotheses, and let $a \in M_A$. We must show $s^*(a) = t^*(a) \in M_B$, i.e. $s^{(\widehat{M},a)} = t^{(\widehat{M},a)}$. We know that $(\widehat{M}, a)$ is a $\Sigma(M, \mathsf{x}_A)$-structure that is a model of $\mathbb{T}(M, \mathsf{x}_A)$. By soundness of partial Horn logic and the assumption that $\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$, it follows that $(\widehat{M}, a) \models s = t$. Hence, we obtain $s^{(\widehat{M},a)} = t^{(\widehat{M},a)}$, as desired. ∎

We will now show how these term-induced functions can be 'transferred' along Σ-morphisms with domain M. First, we require some additional background definitions on *signature morphisms* from Section 5 of [19].

Definition 2.2.14 (Signature Morphism). Let Σ_1 and Σ_2 be relation-free signatures. A *signature morphism* $\rho : \Sigma_1 \to \Sigma_2$ is given by the following data:

- For any sort B of Σ_1, a sort $\rho(B)$ of Σ_2.

- For any function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ_1, a function symbol $\rho(f) : \rho(B_1) \times \ldots \times \rho(B_n) \to \rho(B)$ of Σ_2. ∎

If $\rho : \Sigma_1 \to \Sigma_2$ is a signature morphism, then any Horn formula φ over Σ_1 has a translation $\rho(\varphi)$ over Σ_2, obtained by applying ρ to the sorts and function symbols occurring in φ. If $\vec{x}$ is a sequence of variables of sorts in Σ_1, then $\rho(\vec{x})$ is the corresponding sequence of variables of sorts in Σ_2.

Definition 2.2.15 (Theory Morphism). Let $\mathbb{T}_1, \mathbb{T}_2$ be quasi-equational theories over respective relation-free signatures Σ_1, Σ_2. Then a signature morphism $\rho : \Sigma_1 \to \Sigma_2$ is a *theory morphism* from $\mathbb{T}_1$ to $\mathbb{T}_2$ if for every axiom $\varphi \vdash^{\vec{x}} \psi$ of $\mathbb{T}_1$, the ρ-translation $\rho(\varphi) \vdash^{\rho(\vec{x})} \rho(\psi)$ is a theorem of $\mathbb{T}_2$. ∎

One can then prove the following lemma by induction on the length of PHL-deductions.

Lemma 2.2.16. *Let $\mathbb{T}_1, \mathbb{T}_2$ be quasi-equational theories over respective relation-free signatures Σ_1, Σ_2. If $\rho : \mathbb{T}_1 \to \mathbb{T}_2$ is a theory morphism and $\varphi \vdash^{\vec{x}} \psi$ is any Horn sequent over Σ_1, then*

$$\varphi \vdash^{\vec{x}} \psi \text{ is a theorem of } \mathbb{T}_1 \text{ implies } \rho(\varphi) \vdash^{\rho(\vec{x})} \rho(\psi) \text{ is a theorem of } \mathbb{T}_2.$$

∎

We now show that any Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ induces a corresponding signature morphism on the diagram signatures:

Definition 2.2.17. Let $h : M \to N$ be a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, and let $A \in \Sigma_{\mathsf{Sort}}$. We define a signature morphism $\rho_h^A : \Sigma(M, \mathsf{x}_A) \to \Sigma(N, \mathsf{x}_A)$ as follows:

- ρ_h^A is the identity on $\Sigma \subseteq \Sigma(M, \mathsf{x}_A)$.
- For any sort $B \in \Sigma$ and $s \in M_B$, we set $\rho_h^A\left(c_s^M\right) := c_{h_B(s)}^N$.
- We set $\rho_h^A(\mathsf{x}_A) := \mathsf{x}_A$. ∎

Lemma 2.2.18. *Let $h : M \to N$ be a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, and let $A \in \Sigma_{\mathsf{Sort}}$. The signature morphism $\rho_h^A : \Sigma(M, \mathsf{x}_A) \to \Sigma(N, \mathsf{x}_A)$ is a theory morphism from $\mathbb{T}(M, \mathsf{x}_A)$ to $\mathbb{T}(N, \mathsf{x}_A)$.*

Proof: See Appendix A. ∎

From Lemmas 2.2.16 and 2.2.18, we then obtain:

Lemma 2.2.19. *Let* $h : M \to N$ *be a* Σ*-morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let* $A, B \in \Sigma_{\mathsf{Sort}}$*.*

- *If* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ *and* $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$*, then* $\mathbb{T}(N, \mathsf{x}_A) \vdash \rho_h^A(t) \downarrow$*.*
- *If* $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ *with* $\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$*, then* $\mathbb{T}(N, \mathsf{x}_A) \vdash \rho_h^A(s) = \rho_h^A(t)$*.* ∎

As our next step, we define for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and any $B \in \Sigma_{\mathsf{Sort}}$ a *partial, one-sided* monoid structure on the set $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$, which will capture the behaviour of (syntactic) *substitution* into the indeterminate x_B. If S is any set, then a *partial, one-sided* monoid structure on S is a triple $(S, \cdot, e)$ with the following properties:

- $\cdot$ is a *partial* binary operation on S, i.e. $\cdot : S \times S \rightharpoondown S$.
- $\cdot$ is *associative*, in the sense that for any $u, v, w \in S$, $u \cdot (v \cdot w)$ is defined iff $(u \cdot v) \cdot w$ is defined, and if both are defined, then they are equal.
- $e \in S$ is a one-sided unit element for $\cdot$, in the sense that $u \cdot e$ is defined for every $u \in S$, and $u \cdot e = u$.

Definition 2.2.20. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$. We define a partial, one-sided monoid structure on the set $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ as follows:

- We set
$$\mathsf{dom}(\cdot) := \{(u, v) \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))^2 \mid v : B\}.$$
  For any $(u, v) \in \mathsf{dom}(\cdot)$, we then set
$$u \cdot v := u[v/\mathsf{x}_B] \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B)).$$
- For the unit, we choose $\mathsf{x}_B \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$. ∎

Remark 2.2.21 (Substitution). In the previous definition we invoked a notion of substitution, which we now explicitly define. If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$ and $v \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ is of sort B, then we define the term $u[v/\mathsf{x}_B] \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ for any $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ in the expected way, as follows:

- If $u \equiv \mathsf{x}_B$, then $u[v/\mathsf{x}_B] := v$.

- If $u \not\equiv \mathsf{x}_B$ is a constant symbol of $\Sigma(M, \mathsf{x}_B)$, then $u[v/\mathsf{x}_B] := u$.
- If $u \equiv f(u_1, \ldots, u_n)$ for some function symbol $f : A_1 \times \ldots \times A_n \to A$ and terms $u_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_{A_i}$ for each $1 \leq i \leq n$, then

$$u[v/\mathsf{x}_B] := f\left(u_1[v/\mathsf{x}_B], \ldots, u_n[v/\mathsf{x}_B]\right).$$

∎

We then clearly have:

Lemma 2.2.22. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$. Then $(\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B)), \cdot, \mathsf{x}_B)$ is a partial, one-sided monoid structure on $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$.* ∎

Remark 2.2.23. If we restrict $\cdot$ to $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$, i.e. to the set of closed $\Sigma(M, \mathsf{x}_B)$-terms of sort B, then

$$\cdot : \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B \times \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B \to \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$$

is *total*, and we obtain a total, two-sided (i.e. ordinary) monoid

$$(\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B, \cdot, \mathsf{x}_B).$$

∎

We will shortly show that the (ordinary) monoid structure on $\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ can be extended to

$$M\langle \mathsf{x}_B \rangle_B = \left\{ [t] : t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B \wedge \mathbb{T}(M, \mathsf{x}_B) \vdash t \downarrow \right\}.$$

Before we can do this, we need the following technical lemma, which essentially says (cf. the third statement) that we can substitute a provably defined term into a provably defined term and still obtain a provably defined term:

Lemma 2.2.24. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$.*

1. *Let $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ with $\mathbb{T}(M, \mathsf{x}_B) \vdash s = t$, and let $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ with $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$. Then*

$$\mathbb{T}(M, \mathsf{x}_B) \vdash s[u/\mathsf{x}_B] = t[u/\mathsf{x}_B].$$

2. *Conversely, if* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$ *and* $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash s = t$, *then*

$$\mathbb{T}(M, \mathsf{x}_B) \vdash u[s/\mathsf{x}_B] = u[t/\mathsf{x}_B].$$

3. *In particular, if* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ *and* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ *and* $\mathbb{T}(M, \mathsf{x}_B) \vdash t \downarrow \wedge u \downarrow$, *then* $\mathbb{T}(M, \mathsf{x}_B) \vdash t[u/\mathsf{x}_B] \downarrow$.

Proof: See Appendix A. ∎

Lemma 2.2.25. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* $B \in \Sigma_{\mathsf{Sort}}$. *The following data give a well-defined (ordinary) monoid structure on* $M\langle \mathsf{x}_B \rangle_B$*:*

- *For any* $[s], [t] \in M\langle \mathsf{x}_B \rangle_B$, *we set*

$$[s] \cdot [t] := [s \cdot t] = [s[t/\mathsf{x}_B]] \in M\langle \mathsf{x}_B \rangle_B.$$

- *The unit is* $[\mathsf{x}_B] \in M\langle \mathsf{x}_B \rangle_B$.

Proof: See Appendix A. ∎

Remark 2.2.26. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $B \in \Sigma_{\mathsf{Sort}}$. We show how the monoid structure on $M\langle \mathsf{x}_B \rangle_B$ is related to the term-induced functions of Definition 2.2.11. Let $\mathsf{Sets}(M_B, M_B)$ be the monoid of (total) endofunctions on the set M_B. Then we have a function

$$^* : M\langle \mathsf{x}_B \rangle_B \to \mathsf{Sets}(M_B, M_B)$$

given by the rule

$$[t]^* := t^* : M_B \to M_B,$$

which is well-defined by Lemma 2.2.13 and the fact that $[t] \in M\langle \mathsf{x}_B \rangle_B$ implies $\mathbb{T}(M, \mathsf{x}_B) \vdash t \downarrow$, so that t^* is a well-defined (total) function by Definition 2.2.11. This function is also a *monoid homomorphism*, because $[s], [t] \in M\langle \mathsf{x}_B \rangle_B$ implies

$$s[t/\mathsf{x}_B]^* = s^* \circ t^* : M_B \to M_B,$$

as we will show below in Lemma 2.2.29. ∎

We now come to our first central definition of this section:

Definition 2.2.27. For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, we define $G_{\mathbb{T}}(M)$ to be the set of all Σ_{Sort}-indexed sequences

$$([s_C])_{C \in \Sigma_{\mathsf{Sort}}}$$

with the following properties:

- For any sort $C \in \Sigma$,

$$[s_C] \in M\langle \mathsf{x}_C \rangle_C = \{[t] : t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C \wedge \mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow\}.$$

- For any Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$, the induced family of total functions

$$\left(\rho_h^C(s_C)^* : N_C \to N_C\right)_{C \in \Sigma}$$

  is a Σ-automorphism of N.

Note that if $[s_C] \in M\langle \mathsf{x}_C \rangle_C$, then $\mathbb{T}(N, \mathsf{x}_C) \vdash \rho_h^C(s_C) \downarrow$ by Lemma 2.2.19, so that $\rho_h^C(s_C)^* : N_C \to N_C$ is indeed a well-defined total function by Definition 2.2.11. ∎

Our first main goal of this section will be to show that $G_{\mathbb{T}}(M)$ carries a group structure, and is in fact isomorphic to the isotropy group $\mathcal{Z}_{\mathbb{T}}(M)$ of M. Before we can accomplish the former goal, we require the following lemmas. The first lemma says that the signature morphism induced by a Σ-morphism interacts properly with substitution.

Lemma 2.2.28. *Let $h : M \to N$ be any Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$. Then for any $C \in \Sigma_{\mathsf{Sort}}$, any $s \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ and $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$,*

$$\rho_h^C(s[t/\mathsf{x}_C]) \equiv \rho_h^C(s)[\rho_h^C(t)/\mathsf{x}_C] \in \mathsf{Term}^c(\Sigma(N, \mathsf{x}_C)).$$

Proof: Straightforward induction on the structure of $s \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$. ∎

The next lemma was mentioned in Remark 2.2.26 and essentially says that substitution interacts properly with term-induced functions:

Lemma 2.2.29. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $B, C \in \Sigma_{\mathsf{Sort}}$, and let $s \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_B$ and $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash s \downarrow \wedge\, t \downarrow$. Then (cf. Definition 2.2.11) we have total functions*

$$s^* : M_C \to M_B \text{ and } t^* : M_C \to M_C,$$

as well as the total function

$$(s \cdot t)^* = (s[t/\mathsf{x}_C])^* : M_C \to M_B$$

(since $\mathbb{T}(M, \mathsf{x}_C) \vdash s[t/\mathsf{x}_C] \downarrow$*, by Lemma 2.2.24). Then*

$$(s \cdot t)^* = s^* \circ t^* : M_C \to M_B.$$

Proof: See Appendix A. ∎

The following technical lemma shows that a term-induced function can also be defined by substituting constants of the diagram signature into the indeterminate of the term, and then evaluating the resulting term in the canonical diagram structure:

Lemma 2.2.30. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, let* $A, B \in \Sigma_{\mathsf{Sort}}$*, and let* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ *with* $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$*. Then (cf. Definition 2.2.11) we have a total function*

$$t^* : M_A \to M_B.$$

Then for any $s \in M_A$*, we have*

$$t^*(s) = t\left[c_s/\mathsf{x}_A\right]^{\widehat{M}} \in M_B,$$

where the righthand side is the interpretation of the closed $\Sigma(M)_B$*-term* $t\left[c_s/\mathsf{x}_A\right]$ *in the* $\Sigma(M)$*-structure* $\widehat{M}$*.*

Proof: Straightforward induction on the structure of terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$. ∎

We will also require the following technical lemma that analyzes the behaviour of the term-induced function obtained by transporting along the canonical morphism $\eta : M \to M\langle \mathsf{x}_A \rangle$.

Lemma 2.2.31. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, let* $A, B \in \Sigma_{\mathsf{Sort}}$*, and let* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ *with* $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$*. Let* $\eta = \eta^{M,A} : M \to M\langle \mathsf{x}_A \rangle$ *be the canonical morphism from Definition 2.2.8. Then*

$$\rho^A_\eta(t) \in \mathsf{Term}^c(\Sigma(M\langle \mathsf{x}_A \rangle, \mathsf{x}_A))_B.$$

1. *Let* $\widehat{M\langle \mathsf{x}_A \rangle}$ *be the* $\Sigma(M\langle \mathsf{x}_A \rangle)$*-structure expanding the* Σ*-structure* $M\langle \mathsf{x}_A \rangle$ *(cf. Remark 2.2.2). Let* $[u] = [u]_A \in M\langle \mathsf{x}_A \rangle_A$ *be arbitrary. Then* $c_{[u]} : A$ *is a constant symbol of* $\Sigma(M\langle \mathsf{x}_A \rangle)$*, and* $\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]$ *is a closed* $\Sigma(M\langle \mathsf{x}_A \rangle)$*-term of sort* B*. Then*

$$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} \in \widehat{M\langle \mathsf{x}_A\rangle}_B = M\langle \mathsf{x}_A\rangle_B \textit{ is defined}$$

and

$$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} = [t[u/\mathsf{x}_A]]_B \in M\langle \mathsf{x}_A\rangle_B.$$

2. *By Lemma 2.2.19, we have* $\mathbb{T}(M\langle \mathsf{x}_A\rangle, \mathsf{x}_A) \vdash \rho^A_\eta(t)\downarrow$*. Then by Definition 2.2.11, we have a total function*

$$\rho^A_\eta(t)^* : M\langle \mathsf{x}_A\rangle_A \to M\langle \mathsf{x}_A\rangle_B.$$

Then for any $[u] \in M\langle \mathsf{x}_A\rangle_A$*, we have*

$$\rho^A_\eta(t)^*([u]) = [t[u/\mathsf{x}_A]] \in M\langle \mathsf{x}_A\rangle_B.$$

Proof: See Appendix A. ∎

The following concept will be required to characterize the isomorphisms between partial Σ-structures, as in Lemma 2.2.33 below.

Definition 2.2.32 (Reflects Definedness). Let $h : M \to N$ be any morphism of partial Σ-structures. We say that h *reflects definedness* if h satisfies the following condition:

- For any function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ and $a_i \in M_{A_i}$ for each $1 \leq i \leq n$,

$$(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in \mathsf{dom}\left(f^N\right) \implies (a_1, \ldots, a_n) \in \mathsf{dom}\left(f^M\right).$$

∎

Lemma 2.2.33. *The isomorphisms of* $\mathsf{P\Sigma Str}$ *are exactly the (sortwise) bijective* Σ*-morphisms that reflect definedness.*

Proof: See Appendix A. ∎

Finally, we will also require the following lemma, which provides a sufficient condition for a family of total functions to be an isomorphism in $\mathsf{P\mathbb{T}mod}$:

Lemma 2.2.34. *Let* $h : M \xrightarrow{\sim} N$ *be an isomorphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let*

$$(k_A : N_A \to M_A)_{A \in \Sigma_{\mathsf{Sort}}}$$

be a Σ_{Sort}*-indexed family of total functions that is sortwise inverse to* $h = (h_A : M_A \to N_A)_A$ *(i.e.* h_A *and* k_A *are mutually inverse for every sort* $A \in \Sigma$*). Then*

$$k := (k_A : N_A \to M_A)_A : N \to M$$

is an isomorphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ *(with inverse* h*).*

Proof: See Appendix A. ∎

We can now prove that the set $G_{\mathbb{T}}(M)$ of Definition 2.2.27 has a group structure, given by substitution:

Proposition 2.2.35. *For any* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, the set* $G_{\mathbb{T}}(M)$ *has a group structure.*

Proof: To define the group multiplication $*$ on $G_{\mathbb{T}}(M)$, let $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M)$. We set

$$([s_C])_C * ([t_C])_C := ([s_C \cdot t_C])_C = ([s_C\,[t_C/\mathsf{x}_C]])_C\,.$$

By Lemma 2.2.25, given $[s_C], [t_C] \in M\langle \mathsf{x}_C \rangle_C$, we know that $[s_C \cdot t_C] \in M\langle \mathsf{x}_C \rangle_C$. Now let $h : M \to N$ be a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. We must show that the family of total functions

$$(\rho_h^C(s_C \cdot t_C)^* : N_C \to N_C)_{C \in \Sigma}$$

is a Σ-automorphism of N. By Lemmas 2.2.28 and 2.2.29, we have for any sort $C \in \Sigma$ that

$$\rho_h^C(s_C \cdot t_C)^* = \left(\rho_h^C(s_C)[\rho_h^C(t_C)/\mathsf{x}_C]\right)^* = \rho_h^C(s_C)^* \circ \rho_h^C(t_C)^* : N_C \to N_C.$$

So the family of functions

$$(\rho_h^C(s_C \cdot t_C)^* : N_C \to N_C)_{C \in \Sigma}$$

is equal to the composition

$$(\rho_h^C(s_C)^* : N_C \to N_C)_{C \in \Sigma} \,\circ\, (\rho_h^C(t_C)^* : N_C \to N_C)_{C \in \Sigma}.$$

But since $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M)$, we know that the two families in the above composition are Σ-automorphisms, and hence their composite is a Σ-automorphism. This proves that $([s_C])_C * ([t_C])_C := ([s_C \cdot t_C])_C \in G_{\mathbb{T}}(M)$. And $*$ is well-defined and associative, because the monoid multiplication on $M\langle \mathsf{x}_C \rangle_C$ is well-defined and associative for every sort $C \in \Sigma$ (by Lemma 2.2.25).

We define the unit element of the group structure on $G_{\mathbb{T}}(M)$ to be $([\mathsf{x}_C])_C$. To see that this is in fact in $G_{\mathbb{T}}(M)$, let $h : M \to N$ be a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. We must show that the family of functions

$$(\rho_h^C(\mathsf{x}_C)^* : N_C \to N_C)_{C \in \Sigma}$$

is a Σ-automorphism of N. For any sort $C \in \Sigma$ we have $\rho_h^C(\mathsf{x}_C) \equiv \mathsf{x}_C$, so we must show that the family

$$(\mathsf{x}_C^* : N_C \to N_C)_{C \in \Sigma}$$

is a Σ-automorphism of N. But for any sort $C \in \Sigma$ we have $\mathsf{x}_C^* = \mathsf{id} : N_C \to N_C$, because for any $c \in N_C$ we have $\mathsf{x}_C^*(c) = \mathsf{x}_C^{(\widehat{N},c)} = c$. So the above family of functions is just the identity Σ-morphism on N, which is a Σ-automorphism. So $([\mathsf{x}_C])_C \in G_{\mathbb{T}}(M)$ and $([\mathsf{x}_C])_C$ is the unit element for $*$, because for every sort $C \in \Sigma$ we have by Lemma 2.2.25 that $[\mathsf{x}_C]$ is the unit element of the monoid structure on $M\langle \mathsf{x}_C \rangle_C$.

Lastly, we show that this monoid structure on $G_{\mathbb{T}}(M)$ admits an inverse operation. So let $([s_C])_C \in G_{\mathbb{T}}(M)$. We define

$$([s_C])_C^{-1} \in G_{\mathbb{T}}(M)$$

as follows. Fix $A \in \Sigma_{\mathsf{Sort}}$. We have the canonical Σ-morphism $\eta = \eta^{M,A} : M \to M\langle \mathsf{x}_A \rangle$. Because $([s_C])_C \in G_{\mathbb{T}}(M)$, it follows that the family of functions

$$(\rho_\eta^C(s_C)^* : M\langle \mathsf{x}_A \rangle_C \to M\langle \mathsf{x}_A \rangle_C)_{C \in \Sigma}$$

is a Σ-automorphism of $M\langle \mathsf{x}_A \rangle$. In particular, the function

$$\rho_\eta^A(s_A)^* : M\langle \mathsf{x}_A \rangle_A \to M\langle \mathsf{x}_A \rangle_A$$

is bijective. Since $[\mathsf{x}_A] \in M\langle \mathsf{x}_A \rangle_A$, it follows that there is a unique element in $M\langle \mathsf{x}_A \rangle_A$, suggestively written $\left[s_A^{-1}\right]$, such that

$$\rho_\eta^A(s_A)^* \left(\left[s_A^{-1}\right]\right) = [\mathsf{x}_A].$$

By Lemma 2.2.31, we then have

$$[\mathsf{x}_A] = \rho_\eta^A(s_A)^* \left(\left[s_A^{-1}\right]\right) = \left[s_A \left[s_A^{-1}/\mathsf{x}_A\right]\right] = [s_A] \cdot \left[s_A^{-1}\right],$$

with the latter multiplication in the monoid structure on $M\langle \mathsf{x}_A \rangle_A$. So $\left[s_A^{-1}\right]$ is a right inverse for $[s_A]$ in the monoid $M\langle \mathsf{x}_A \rangle_A$. To show that $\left[s_A^{-1}\right]$ is also a left inverse for $[s_A]$ in this monoid, we argue as follows. We have

$$\rho_\eta^A(s_A)^* \left(\left[s_A^{-1}\right] \cdot [s_A]\right) = \rho_\eta^A(s_A)^* \left(\left[s_A^{-1}[s_A/\mathsf{x}_A]\right]\right)$$

$$\begin{aligned}
&= \left[s_A \left[s_A^{-1}[s_A/\mathsf{x}_A]/\mathsf{x}_A\right]\right] \\
&= \left[\left(s_A[s_A^{-1}/\mathsf{x}_A]\right)[s_A/\mathsf{x}_A]\right] \\
&= [\mathsf{x}_A[s_A/\mathsf{x}_A]] \\
&= [s_A] \\
&= [s_A[\mathsf{x}_A/\mathsf{x}_A]] \\
&= \rho_\eta^A(s_A)^*([\mathsf{x}_A]).
\end{aligned}$$

The second equality holds by Lemma 2.2.31; the fourth equality holds because

$$\left[s_A\left[s_A^{-1}/\mathsf{x}_A\right]\right] = [\mathsf{x}_A]$$

and the monoid multiplication in $M\langle\mathsf{x}_A\rangle_A$ is well-defined; the last equality holds by Lemma 2.2.31 once again. Since $\rho_\eta^A(s_A)^*$ is injective, it follows that

$$\left[s_A^{-1}\right] \cdot [s_A] = [\mathsf{x}_A],$$

as desired.

Now we set

$$([s_C])_C^{-1} := \left(\left[s_C^{-1}\right]\right)_C.$$

To show that this inverse operation is well-defined, let $([t_C])_C \in G_{\mathbb{T}}(M)$ as well, and suppose that $([s_C])_C = ([t_C])_C$. If $A \in \Sigma_{\mathsf{Sort}}$, we must show that

$$\left[s_A^{-1}\right] = \left[t_A^{-1}\right].$$

But this follows because $\left[s_A^{-1}\right]$ and $\left[t_A^{-1}\right]$ are both left and right inverses of $[s_A] = [t_A]$ in the monoid $M\langle\mathsf{x}_A\rangle_A$ (as shown above), and hence must be equal.

It follows that $\left(\left[s_C^{-1}\right]\right)_C$ is an inverse of $([s_C])_C$, because we have

$$([s_C])_C * \left(\left[s_C^{-1}\right]\right)_C = \left(\left[s_C \cdot s_C^{-1}\right]\right)_C = ([\mathsf{x}_C])_C$$

and similarly $\left(\left[s_C^{-1}\right]\right)_C * ([s_C])_C = ([\mathsf{x}_C])_C$. So it remains to show that $\left(\left[s_C^{-1}\right]\right)_C \in G_{\mathbb{T}}(M)$. Let $h : M \to N$ be an arbitrary Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. We must show that the family of functions

$$\left(\rho_h^C\left(s_C^{-1}\right)^* : N_C \to N_C\right)_{C\in\Sigma}$$

is a Σ-automorphism of N. Since $([s_C])_C \in G_{\mathbb{T}}(M)$, we know that the family of functions

$$(\rho_h^C(s_C)^* : N_C \to N_C)_C$$

is a Σ-automorphism of N. By Lemma 2.2.34, it suffices to show for every sort $A \in \Sigma$ that the functions $\rho_h^A(s_A)^*, \rho_h^A\left(s_A^{-1}\right)^* : N_A \to N_A$ are mutually inverse. We have

$$\begin{aligned}\rho_h^A(s_A)^* \circ \rho_h^A\left(s_A^{-1}\right)^* &= \rho_h^A\left(s_A \cdot s_A^{-1}\right)^* \\ &= \rho_h^A(\mathsf{x}_A)^* \\ &= \mathsf{x}_A^* \\ &= \mathsf{id}.\end{aligned}$$

The first equality follows by Lemmas 2.2.28 and 2.2.29, and the second equality follows by Lemmas 2.2.19 and 2.2.13, using the fact that $\left[s_A \cdot s_A^{-1}\right] = [\mathsf{x}_A]$. That $\rho_h^A\left(s_A^{-1}\right)^* \circ \rho_h^A(s_A)^* = \mathsf{id}$ can be shown similarly. This completes the proof that $(\rho_h^C\left(s_C^{-1}\right)^* : N_C \to N_C)_C$ is a Σ-automorphism of N, and hence that $\left(\left[s_C^{-1}\right]\right)_C \in G_{\mathbb{T}}(M)$. This completes the proof that $G_{\mathbb{T}}(M)$ has a group structure. ∎

We now extend the assignment $M \mapsto G_{\mathbb{T}}(M)$ to a functor

$$G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group},$$

which we will show (in Theorem 2.2.41 below) is naturally isomorphic to the isotropy group $\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$.

Definition 2.2.36. We define a functor $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ as follows:

- On objects, $G_{\mathbb{T}}$ acts by
$$M \mapsto G_{\mathbb{T}}(M).$$
- For morphisms, let $h : M \to M'$ be an arbitrary Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$. We define
$$G_{\mathbb{T}}(h) : G_{\mathbb{T}}(M) \to G_{\mathbb{T}}(M')$$
by
$$([s_C])_C \in G_{\mathbb{T}}(M) \longmapsto \left(\left[\rho_h^C(s_C)\right]\right)_C \in G_{\mathbb{T}}(M').$$

∎

Before we can verify that $G_{\mathbb{T}}$ is a well-defined functor, we require the following lemma, which is easily proved by induction on terms.

Lemma 2.2.37. *Let* $C \in \Sigma_{\mathsf{Sort}}$.

- *If* $h_1 : M \to M'$ *and* $h_2 : M' \to M''$ *are morphisms in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, then for any* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ *we have*
$$\rho^C_{h_2}(\rho^C_{h_1}(u)) \equiv \rho^C_{h_2 \circ h_1}(u) \in \mathsf{Term}^c(\Sigma(M'', \mathsf{x}_C)).$$
- *If* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, then for any* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ *we have*
$$\rho^C_{\mathsf{id}_M}(u) \equiv u.$$

▮

We can now prove:

Proposition 2.2.38. $G_\mathbb{T} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ *is a well-defined functor.*

Proof: See Appendix A. ▮

Before we can prove that $G_\mathbb{T}$ is naturally isomorphic to the isotropy group $\mathcal{Z}_\mathbb{T} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$, we also require the following two technical lemmas, whose proofs may be found in Appendix A.

Lemma 2.2.39. *Let* $h : M \to N$ *be any* Σ*-morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let* $B, C \in \Sigma_{\mathsf{Sort}}$*. Let* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_B$ *with* $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$*. Let* $z \in M_C$*, so that* $(\widehat{M}, z)$ *is a* $\Sigma(M, \mathsf{x}_C)$*-structure and* $(\widehat{N}, h_C(z))$ *is a* $\Sigma(N, \mathsf{x}_C)$*-structure. Then* u *is defined in* $(\widehat{M}, z)$ *and* $\rho^C_h(u)$ *is defined in* $(\widehat{N}, h_C(z))$ *and*
$$h_B\left(u^{(\widehat{M},z)}\right) = \rho^C_h(u)^{(\widehat{N},h_C(z))} \in N_B.$$

▮

Lemma 2.2.40. *Let* $h : M \to N$ *be any morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, let* $A, B \in \Sigma_{\mathsf{Sort}}$*, and let* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ *with* $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$*. Then* $[t] \in M\langle \mathsf{x}_A \rangle_B$ *and* $\rho^A_h(t)^* : N_A \to N_B$ *is a well-defined, total function by Lemma 2.2.19 and Definition 2.2.11.*

If $a \in N_A$*, then by Proposition 2.2.10 there is a unique* Σ*-morphism* $h^a : M\langle \mathsf{x}_A \rangle \to N$ *such that* $h^a \circ \eta^{M,A} = h$ *and* $h^a_A([\mathsf{x}_A]) = a \in N_A$*. Then we have*
$$h^a_B([t]) = \rho^A_h(t)^*(a) \in N_B.$$

▮

We can now prove:

Theorem 2.2.41. *The functor* $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ *of Definition 2.2.36 is naturally isomorphic to the isotropy group* $\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$.

Proof: We construct a natural isomorphism

$$\alpha_{\mathbb{T}} : G_{\mathbb{T}} \overset{\sim}{\Longrightarrow} \mathcal{Z}_{\mathbb{T}}.$$

So let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$; we will define a group isomorphism

$$\alpha_{\mathbb{T}}^{M} : G_{\mathbb{T}}(M) \xrightarrow{\sim} \mathcal{Z}_{\mathbb{T}}(M).$$

Let $([s_C])_C \in G_{\mathbb{T}}(M)$. We define

$$\alpha_{\mathbb{T}}^{M}\left[([s_C])_C\right] \in \mathcal{Z}_{\mathbb{T}}(M).$$

So let $h : M \to N$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. We must define a Σ-automorphism $\alpha_{\mathbb{T}}^{M}\left[([s_C])_C\right]^{h} : N \xrightarrow{\sim} N$ of N. Since $([s_C])_C \in G_{\mathbb{T}}(M)$, we know that the family of total functions

$$(\rho_h^C(s_C)^* : N_C \to N_C)_{C \in \Sigma}$$

is a Σ-automorphism of N. So we set

$$\alpha_{\mathbb{T}}^{M}[([s_C])_C]^{h} := (\rho_h^C(s_C)^* : N_C \to N_C)_{C \in \Sigma}.$$

To verify the naturality condition that elements of $\mathcal{Z}_{\mathbb{T}}(M)$ must satisfy, let $h : M \to N$ and $h' : N \to N'$ be arbitrary morphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}$. We must show that

$$\alpha_{\mathbb{T}}^{M}[([s_C])_C]^{h' \circ h} \circ h' = h' \circ \alpha_{\mathbb{T}}^{M}[([s_C])_C]^{h} : N \to N'.$$

So let $C \in \Sigma_{\mathsf{Sort}}$; we must show that

$$\rho_{h' \circ h}^C(s_C)^* \circ h'_C = h'_C \circ \rho_h^C(s_C)^* : N_C \to N'_C.$$

If $z \in N_C$, then we have

$$\begin{aligned}
\rho_{h' \circ h}^C(s_C)^*(h'_C(z)) &= \rho_{h' \circ h}^C(s_C)^{(\widehat{N'}, h'_C(z))} \\
&= \rho_{h'}^C(\rho_h^C(s_C))^{(\widehat{N'}, h'_C(z))} \\
&= h'_C\left(\rho_h^C(s_C)^{(\widehat{N}, z)}\right) \\
&= h'_C(\rho_h^C(s_C)^*(z)),
\end{aligned}$$

as desired. The second equality holds by Lemma 2.2.37, and the third by Lemma 2.2.39. This proves that $\alpha_{\mathbb{T}}^{M}[([s_C])_C] \in \mathcal{Z}_{\mathbb{T}}(M)$.

Now we show that the function $\alpha_{\mathbb{T}}^M : G_{\mathbb{T}}(M) \to \mathcal{Z}_{\mathbb{T}}(M)$ is a group isomorphism. To show that $\alpha_{\mathbb{T}}^M$ preserves the group multiplication, let $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M)$. We must show that

$$\alpha_{\mathbb{T}}^M[([s_C])_C * ([t_C])_C] = \alpha_{\mathbb{T}}^M[([s_C])_C] * \alpha_{\mathbb{T}}^M[([t_C])_C] \in \mathcal{Z}_{\mathbb{T}}(M).$$

So let $h : M \to N$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. Then we have

$$\begin{aligned}
\alpha_{\mathbb{T}}^M[([s_C])_C * ([t_C])_C]^h &= \alpha_{\mathbb{T}}^M[([s_C \cdot t_C])_C]^h \\
&= (\rho_h^C(s_C \cdot t_C)^* : N_C \to N_C)_{C \in \Sigma} \\
&= ((\rho_h^C(s_C) \cdot \rho_h^C(t_C))^* : N_C \to N_C)_{C \in \Sigma} \\
&= (\rho_h^C(s_C)^* \circ \rho_h^C(t_C)^* : N_C \to N_C)_{C \in \Sigma} \\
&= (\rho_h^C(s_C)^* : N_C \to N_C)_{C \in \Sigma} \circ (\rho_h^C(t_C)^* : N_C \to N_C)_{C \in \Sigma} \\
&= \alpha_{\mathbb{T}}^M[([s_C])_C]^h \circ \alpha_{\mathbb{T}}^M[([t_C])_C]^h \\
&= \alpha_{\mathbb{T}}^M[([s_C])_C] * \alpha_{\mathbb{T}}^M[([t_C])_C]^h,
\end{aligned}$$

as required. The third equality follows by Lemma 2.2.28, and the fourth by Lemma 2.2.29.

To show that $\alpha_{\mathbb{T}}^M$ is injective, let $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M)$ and assume that

$$\alpha_{\mathbb{T}}^M[([s_C])_C] = \alpha_{\mathbb{T}}^M[([t_C])_C];$$

we must show that $([s_C])_C = ([t_C])_C$. So let $B \in \Sigma$ be an arbitrary sort; we must show that $[s_B] = [t_B]$. We have the canonical morphism $\eta : M \to M\langle \mathsf{x}_B \rangle$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$. Then because $\alpha_{\mathbb{T}}^M[([s_C])_C] = \alpha_{\mathbb{T}}^M[([t_C])_C] \in \mathcal{Z}_{\mathbb{T}}(M)$, we obtain equal Σ-automorphisms of $M\langle \mathsf{x}_B \rangle$:

$$\alpha_{\mathbb{T}}^M[([s_C])_C]^\eta = \alpha_{\mathbb{T}}^M[([t_C])_C]^\eta : M\langle \mathsf{x}_B \rangle \xrightarrow{\sim} M\langle \mathsf{x}_B \rangle.$$

So by definition of $\alpha_{\mathbb{T}}^M$, we have

$$(\rho_\eta^C(s_C)^* : M\langle \mathsf{x}_B \rangle_C \to M\langle \mathsf{x}_B \rangle_C)_{C \in \Sigma} = (\rho_\eta^C(t_C)^* : M\langle \mathsf{x}_B \rangle_C \to M\langle \mathsf{x}_B \rangle_C)_{C \in \Sigma}$$

$$: M\langle \mathsf{x}_B \rangle \xrightarrow{\sim} M\langle \mathsf{x}_B \rangle.$$

In particular, we have

$$\rho_\eta^B(s_B)^* = \rho_\eta^B(t_B)^* : M\langle \mathsf{x}_B \rangle_B \to M\langle \mathsf{x}_B \rangle_B.$$

Applying Lemma 2.2.31, we then obtain

$$\begin{aligned}
[s_B] &= [s_B[\mathsf{x}_B/\mathsf{x}_B]] \\
&= \rho_\eta^B(s_B)^*([\mathsf{x}_B])
\end{aligned}$$

$$\begin{aligned} &= \rho_\eta^B(t_B)^*([\mathsf{x}_B]) \\ &= [t_B[\mathsf{x}_B/\mathsf{x}_B]] \\ &= [t_B], \end{aligned}$$

as desired. This proves that $\alpha_{\mathbb{T}}^M$ is injective.

To show that $\alpha_{\mathbb{T}}^M : G_{\mathbb{T}}(M) \to \mathcal{Z}_{\mathbb{T}}(M)$ is surjective, let

$$\beta = \Big(\beta^h : \mathsf{cod}(h) \xrightarrow{\sim} \mathsf{cod}(h)\Big)_{h \in \mathsf{Dom}(M)} \in \mathcal{Z}_{\mathbb{T}}(M).$$

We must show that there is some $([s_C])_C \in G_{\mathbb{T}}(M)$ with

$$\alpha_{\mathbb{T}}^M[([s_C])_C] = \beta.$$

So let $B \in \Sigma_{\mathsf{Sort}}$; we define $[s_B] \in M\langle \mathsf{x}_B \rangle_B$. Consider the canonical morphism $\eta_B : M \to M\langle \mathsf{x}_B \rangle$. Then we have the Σ-automorphism

$$\beta^{\eta_B} : M\langle \mathsf{x}_B \rangle \xrightarrow{\sim} M\langle \mathsf{x}_B \rangle,$$

and hence we have the bijective function

$$\beta_B^{\eta_B} : M\langle \mathsf{x}_B \rangle_B \xrightarrow{\sim} M\langle \mathsf{x}_B \rangle_B.$$

Now we set

$$[s_B] := \beta_B^{\eta_B}([\mathsf{x}_B]) \in M\langle \mathsf{x}_B \rangle_B.$$

We must now show that $([s_C])_C \in G_{\mathbb{T}}(M)$ and $\alpha_{\mathbb{T}}^M[([s_C])_C] = \beta$. We accomplish both goals at once by showing for any morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M that

$$\beta^h = (\rho_h^C(s_C)^* : N_C \to N_C)_C =: \alpha_{\mathbb{T}}^M[([s_C])_C]^h.$$

So let $B \in \Sigma$ be any sort. We must show that

$$\beta_B^h = \rho_h^B(s_B)^* : N_B \to N_B.$$

For any $b \in N_B$, we show that

$$\beta_B^h(b) = \rho_h^B(s_B)^*(b) \in N_B.$$

By Proposition 2.2.10 there is a unique Σ-morphism $h^b : M\langle \mathsf{x}_B \rangle \to N$ with $h^b \circ \eta_B = h$ and $h_B^b([\mathsf{x}_B]) = b \in N_B$. Since $\beta \in \mathcal{Z}_{\mathbb{T}}(M)$, it follows that

$$\beta^h \circ h^b = \beta^{h^b \circ \eta_B} \circ h^b = h^b \circ \beta^{\eta_B} : M\langle \mathsf{x}_B \rangle \to N.$$

In particular, we have

$$h_B^b \circ \beta_B^{\eta_B} = \beta_B^h \circ h_B^b : M\langle \mathsf{x}_B \rangle_B \to N_B.$$

So we obtain

$$\begin{aligned}\beta_B^h(b) &= \beta_B^h(h_B^b([\mathsf{x}_B])) \\ &= h_B^b(\beta_B^{\eta_B}([\mathsf{x}_B])) \\ &= h_B^b([s_B]) \\ &= \rho_h^B(s_B)^*(b),\end{aligned}$$

as desired. The third equality follows by definition of $[s_B]$, while the last equality follows by Lemma 2.2.40. This completes the proof that $\alpha_{\mathbb{T}}^M$ is surjective, and hence a group isomorphism.

To complete the proof of the theorem, we must verify that the collection of group isomorphisms $(\alpha_{\mathbb{T}}^M : G_{\mathbb{T}}(M) \xrightarrow{\sim} \mathcal{Z}_{\mathbb{T}}(M))_M$ is natural in $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. So let $h : M \to N$ be a morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$. We must show that

$$\mathcal{Z}_{\mathbb{T}}(h) \circ \alpha_{\mathbb{T}}^M = \alpha_{\mathbb{T}}^N \circ G_{\mathbb{T}}(h) : G_{\mathbb{T}}(M) \to \mathcal{Z}_{\mathbb{T}}(N).$$

So let $([s_C])_C \in G_{\mathbb{T}}(M)$ and $k : N \to N'$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$. We must show that

$$\mathcal{Z}_{\mathbb{T}}(h)(\alpha_{\mathbb{T}}^M[([s_C])_C])^k = \alpha_{\mathbb{T}}^N(G_{\mathbb{T}}(h)[([s_C])_C])^k : N' \to N'.$$

We have

$$\begin{aligned}\mathcal{Z}_{\mathbb{T}}(h)(\alpha_{\mathbb{T}}^M[([s_C])_C])^k &= \alpha_{\mathbb{T}}^M[([s_C])_C]^{k \circ h} \\ &= (\rho_{k \circ h}^C(s_C)^* : N'_C \to N'_C)_{C \in \Sigma} \\ &= (\rho_k^C(\rho_h^C(s_C))^* : N'_C \to N'_C)_{C \in \Sigma} \\ &= \alpha_{\mathbb{T}}^N[([\rho_h^C(s_C)])_C]^k \\ &= \alpha_{\mathbb{T}}^N(G_{\mathbb{T}}(h)[([s_C])_C])^k,\end{aligned}$$

as required. The first equality follows by definition of $\mathcal{Z}_{\mathbb{T}}(h)$, the second by definition of $\alpha_{\mathbb{T}}^M$, the third by Lemma 2.2.37, the fourth by definition of $\alpha_{\mathbb{T}}^N$, and the last by definition of $G_{\mathbb{T}}(h)$. This completes the proof that

$$\alpha_{\mathbb{T}} : G_{\mathbb{T}} \overset{\sim}{\Longrightarrow} \mathcal{Z}_{\mathbb{T}}$$

is a natural isomorphism. ∎

The following corollary will prove useful in subsequent chapters:

Corollary 2.2.42. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and let*

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$$

be a $\mathsf{Dom}(M)$*-indexed family of* Σ*-endomorphisms in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*. Then* $\pi \in \mathcal{Z}_{\mathbb{T}}(M)$ *iff there is a unique element* $([s_C])_C \in G_{\mathbb{T}}(M)$ *such that*

$$\pi_f = \left(\rho_f^C(s_C)^* : \mathsf{cod}(f)_C \to \mathsf{cod}(f)_C\right)_{C \in \Sigma}$$

for each $f \in \mathsf{Dom}(M)$*.*

Proof: This follows immediately from the definition of the group isomorphism $\alpha_{\mathbb{T}}^M : G_{\mathbb{T}}(M) \xrightarrow{\sim} \mathcal{Z}_{\mathbb{T}}(M)$ in the proof of Theorem 2.2.41. ∎

Let us pause to take stock of what we have shown so far, and what we will show next. We have shown for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ that $\mathcal{Z}_{\mathbb{T}}(M)$ is isomorphic to the group $G_{\mathbb{T}}(M)$ consisting of all sequences $([s_C])_C$ of (congruence classes of) provably defined terms (in the diagram theory of M) whose extensions along any Σ-morphism with domain M induce an automorphism of the codomain. This is essentially an *intermediate* result, which we will use to characterize $\mathcal{Z}_{\mathbb{T}}(M)$ in the way that was described at the end of Section 1.

First, we extend the definitions of $\Sigma(M, \mathsf{x}_A)$ and $\mathbb{T}(M, \mathsf{x}_A)$ to allow for the addition of finitely many indeterminates, rather than just one.

Definition 2.2.43 (Extended Diagram Signature). Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. Given a non-empty, finite sequence of (not necessarily distinct) sorts $A_1, \ldots, A_n \in \Sigma$, let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma(M)$ be pairwise distinct constants with $\mathsf{x}_i : A_i$ for each $1 \leq i \leq n$.

We then define $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$ to be the following relation-free signature:

- $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
- $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)_{\mathsf{Fun}} := \Sigma(M)_{\mathsf{Fun}} \cup \{\mathsf{x}_1 : A_1, \ldots, \mathsf{x}_n : A_n\}$.

Thus, we have $\Sigma(M) \subseteq \Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$. ∎

Definition 2.2.44 (Extended Diagram Theory). For any non-empty, finite sequence of sorts $A_1, \ldots, A_n \in \Sigma$, we define $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$ to be the quasi-equational theory over the relation-free signature $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$ whose axioms are those of $\mathbb{T}(M)$, together with the Horn sequents $\top \vdash \mathsf{x}_i \downarrow$ for each $1 \leq i \leq n$. ∎

Definition 2.2.45. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, and let $A_1, \ldots, A_n \in \Sigma$ be a non-empty, finite sequence of sorts. We then define

$$M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle := \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n))|_{\Sigma},$$

the Σ-reduct of the initial model $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n))$ of $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$.

We may also define a canonical Σ-morphism $\eta^{M,A_1,\ldots,A_n} : M \to M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$ as in Definition 2.2.8. ∎

We can then straightforwardly generalize Proposition 2.2.10 as follows:

Proposition 2.2.46. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and let $A_1, \ldots, A_n \in \Sigma$ be a non-empty, finite sequence of sorts. For any Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ and any $(a_1, \ldots, a_n) \in N_{A_1} \times \ldots \times N_{A_n}$, there is a unique Σ-morphism*

$$h^{a_1,\ldots,a_n} : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle \to N$$

such that $h^{a_1,\ldots,a_n} \circ \eta^{M,A_1,\ldots,A_n} = h$ and $h^{a_1,\ldots,a_n}_{A_i}([\mathsf{x}_i]) = a_i \in N_{A_i}$ for all $1 \leq i \leq n$ (note that $[\mathsf{x}_i] \in M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle_{A_i}$ because $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash \mathsf{x}_i \downarrow$). ∎

We now arrive at the central definition of this section, which formalizes and generalizes the syntactic concepts of 'invertibility' and 'commuting generically' that were discussed in Section 1 in the context of group theory. We also need to add the additional (syntactic) concept of 'reflecting definedness', to capture the fact that isomorphisms of Σ-structures are sortwise bijective Σ-morphisms that reflect definedness (cf. Lemma 2.2.33).

Definition 2.2.47. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $([s_C])_C \in \prod_{C \in \Sigma} M\langle \mathsf{x}_C \rangle_C$.

- Let $f : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ. We say that $([s_C])_C$ *commutes generically with* f if the Horn sequent

  $$f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

  is provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$.

  In particular, if $c : A$ is a constant symbol, then $([s_C])_C$ *commutes generically with* c if the Horn sequent

  $$c \downarrow \vdash s_A[c/\mathsf{x}_A] = c$$

  is provable in $\mathbb{T}(M)$.

- We say that $([s_C])_C$ is *invertible* if for every sort $B \in \Sigma$, there is some $\left[s_B^{-1}\right] \in M\langle \mathsf{x}_B \rangle_B$ such that

  $$\left[s_B\left[s_B^{-1}/\mathsf{x}_B\right]\right] = [\mathsf{x}_B] = \left[s_B^{-1}\left[s_B/\mathsf{x}_B\right]\right] \in M\langle \mathsf{x}_B \rangle_B,$$

  i.e. such that

  $$\mathbb{T}(M, \mathsf{x}_B) \vdash s_B\left[s_B^{-1}/\mathsf{x}_B\right] = \mathsf{x}_B = s_B^{-1}[s_B/\mathsf{x}_B].$$

- We say that $([s_C])_C$ *reflects definedness* if for every function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ with $n \geq 1$, the sequent
$$f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]) \downarrow \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow$$
is provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$. ∎

Remark 2.2.48. Let us see how the above definitions specialize to the theory of groups, which can be expressed as a quasi-equational theory $\mathbb{T}_{\mathsf{Group}}$ over a single-sorted signature Σ_{Group} containing a binary function symbol $\cdot$, a unary function symbol $^{-1}$ (written in infix notation), and a constant symbol e (see Example 2.3.2 below).

So let G be a group, i.e. a model of $\mathbb{T}_{\mathsf{Group}}$, and let $[t] \in G\langle \mathsf{x} \rangle$, so that $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Group}}(G, \mathsf{x}))$ is a closed term of the theory of groups that may contain the indeterminate x and constants from G, and moreover $\mathbb{T}_{\mathsf{Group}}(G, \mathsf{x}) \vdash t \downarrow$ (although this latter condition is redundant, because $\mathbb{T}_{\mathsf{Group}}(G, \mathsf{x})$ actually proves that *all* terms are defined; see Section 1.3 below). It is then a fact of group theory that t is congruent to a reduced multiplicative word over x and constants from G (and their inverses), as mentioned in Section 1.

By the preceding definition, we say that $[t]$ commutes generically with the multiplication symbol $\cdot$ of Σ_{Group} if the sequent
$$\mathsf{x}_1 \cdot \mathsf{x}_2 \downarrow \vdash t[\mathsf{x}_1 \cdot \mathsf{x}_2/\mathsf{x}] = t[\mathsf{x}_1/\mathsf{x}] \cdot t[\mathsf{x}_2/\mathsf{x}]$$
is provable in the extended diagram theory $\mathbb{T}_{\mathsf{Group}}(G, \mathsf{x}_1, \mathsf{x}_2)$. This essentially formalizes the definition of 'commuting generically with $\cdot$' that we gave in Section 1. Similarly, we say that $[t]$ commutes generically with the inverse symbol $^{-1}$ if the sequent
$$\mathsf{x}^{-1} \downarrow \vdash t[\mathsf{x}^{-1}/\mathsf{x}] = t^{-1}$$
is provable in the diagram theory $\mathbb{T}_{\mathsf{Group}}(G, \mathsf{x})$, and we say that $[t]$ commutes generically with the constant symbol e if the sequent
$$e \downarrow \vdash t[e/\mathsf{x}] = e$$
is provable in the diagram theory $\mathbb{T}_{\mathsf{Group}}$, which means that $t^*(e^G) = e^G \in G$. The definition of invertibility given in Definition 2.2.47 also formalizes the notion of invertibility from Section 1. ∎

As we foreshadowed in Section 1, our next goal (cf. Theorem 2.2.53 below) is to prove that $\mathscr{Z}_{\mathbb{T}}(M) \cong G_{\mathbb{T}}(M)$ consists of exactly those sequences $([s_C])_C \in \prod_C M\langle \mathsf{x}_C \rangle_C$ that are invertible, commute generically with all function symbols of Σ, and reflect definedness, in the sense of Definition 2.2.47. To achieve this, we will require the following technical definition:

Definition 2.2.49. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol in Σ, and let $\mathsf{x}_1 : A_1, \ldots, \mathsf{x}_n : A_n \notin \Sigma(M)$ be pairwise distinct constants.

- We define $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f)$ to be the quasi-equational theory

  $$\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \cup \{\top \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow\}$$

  over the signature $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$.

- We define the $\mathbb{T}$-model

  $$M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle := \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f))|_\Sigma.$$

- We define the canonical Σ-morphism

  $$\eta^{M,f} : M \to M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle$$

  as in Definition 2.2.8. ∎

We then have the following universal property, with a proof analogous to that of Proposition 2.2.10, which essentially expresses that $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle$ is the coproduct in $\mathsf{P}\mathbb{T}\mathsf{mod}$ of M with the initial $\mathbb{T}$-model in which f is defined:

Proposition 2.2.50. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol in Σ, and let $\mathsf{x}_1 : A_1, \ldots, \mathsf{x}_n : A_n$ be pairwise distinct constants not in $\Sigma(M)$. For any Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ and any $(a_1, \ldots, a_n) \in N_{A_1} \times \ldots \times N_{A_n}$ such that $(a_1, \ldots, a_n) \in \mathsf{dom}\left(f^N\right)$, there is a unique Σ-morphism*

$$h^{a_1,\ldots,a_n} : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle \to N$$

such that $h^{a_1,\ldots,a_n} \circ \eta^{M,f} = h$ and $h^{a_1,\ldots,a_n}_{A_i}([\mathsf{x}_i]) = a_i \in N_{A_i}$ for all $1 \leq i \leq n$ (note that $[\mathsf{x}_i] \in M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_{A_i}$ because $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f) \vdash \mathsf{x}_i \downarrow$). ∎

Now we prove, in the next two lemmas, that every element of $G_\mathbb{T}(M)$ is invertible, commutes generically with all function symbols of Σ, and reflects definedness.

Lemma 2.2.51. *For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, if $([s_C])_C \in G_\mathbb{T}(M)$, then $([s_C])_C$ is invertible (in the sense of Definition 2.2.47).*

Proof: This is immediate from the definition of the inverse operation in the group $G_\mathbb{T}(M)$. ∎

Lemma 2.2.52. *For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, if $([s_C])_C \in G_{\mathbb{T}}(M)$, then $([s_C])_C$ commutes generically with every function symbol of Σ and reflects definedness.*

Proof: Assume the hypothesis, and let $f : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ. We show that $([s_C])_C$ commutes generically with f and reflects definedness of f. So we must show that the sequents

$$f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

and

$$f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]) \downarrow \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow$$

are provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$. Consider the first sequent. By the deduction theorem of partial Horn logic (cf. Remark 1.3.17), it suffices to show that the sequent

$$\top \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

is provable in the expanded theory $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f)$. Consider the canonical Σ-morphism

$$\eta = \eta^{M,f} : M \to M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle.$$

Since $([s_C])_C \in G_{\mathbb{T}}(M)$, it follows that the induced family of total functions

$$\left(\rho^C_\eta(s_C)^* : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_C \to M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_C\right)_{C \in \Sigma}$$

is a Σ-automorphism of $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle$. We have

$$([\mathsf{x}_1], \ldots, [\mathsf{x}_n]) \in \mathsf{dom}\left(f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}\right),$$

because $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f) \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow$. Then it follows that

$$\left(\rho^{A_1}_\eta(s_{A_1})^*([\mathsf{x}_1]), \ldots, \rho^{A_n}_\eta(s_{A_n})^*([\mathsf{x}_n])\right) \in \mathsf{dom}\left(f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}\right)$$

and

$$\rho^A_\eta(s_A)^*\left(f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}([\mathsf{x}_1], \ldots, [\mathsf{x}_n])\right) = f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}\left(\rho^{A_1}_\eta(s_{A_1})^*([\mathsf{x}_1]), \ldots, \rho^{A_n}_\eta(s_{A_n})^*([\mathsf{x}_n])\right)$$
$$\in M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_A.$$

By a result analogous to Lemma 2.2.31, the following equalities now hold in $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_A$:

$$\begin{aligned}
&s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] \\
&= \rho^A_\eta(s_A)^*\left([f(\mathsf{x}_1, \ldots, \mathsf{x}_n)]\right) \\
&= \rho^A_\eta(s_A)^*\left(f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}([\mathsf{x}_1], \ldots, [\mathsf{x}_n])\right) \\
&= f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}\left(\rho^{A_1}_\eta(s_{A_1})^*([\mathsf{x}_1]), \ldots, \rho^{A_n}_\eta(s_{A_n})^*([\mathsf{x}_n])\right)
\end{aligned}$$

$$= f^{M\langle \mathsf{x}_1,\ldots,\mathsf{x}_n,f\rangle}\left([s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}]],\ldots,[s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]]\right)$$
$$= [f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}],\ldots,s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)].$$

But this means that

$$\top \vdash s_A[f(\mathsf{x}_1,\ldots,\mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}],\ldots,s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

is provable in $\mathbb{T}(M,\mathsf{x}_1,\ldots,\mathsf{x}_n,f)$, as desired.

To show that $\mathbb{T}(M,\mathsf{x}_1,\ldots,\mathsf{x}_n)$ proves the second sequent involving f, it suffices by the deduction theorem (Remark 1.3.17) to show that the sequent

$$\top \vdash f(\mathsf{x}_1,\ldots,\mathsf{x}_n)\downarrow$$

is provable in the expanded theory

$$\mathbb{T}(M,\mathsf{x}_1,\ldots,\mathsf{x}_n)\cup\{\top \vdash f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}],\ldots,s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])\downarrow\}.$$

One then considers the Σ-reduct of the initial model of this theory, and uses a similar argument to the one used for the first sequent involving f (with this initial model playing the role of the $\mathbb{T}$-model $M\langle \mathsf{x}_1,\ldots,\mathsf{x}_n,f\rangle$). ∎

We now show that an element of $\prod_C M\langle \mathsf{x}_C\rangle_C$ belongs to $G_{\mathbb{T}}(M)$ *if and only if* it is invertible, commutes generically with all function symbols, and reflects definedness:

Theorem 2.2.53. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. If $([s_C])_C \in \prod_{C\in\Sigma} M\langle \mathsf{x}_C\rangle_C$, then $([s_C])_C \in G_{\mathbb{T}}(M)$ iff $([s_C])_C$ is invertible, commutes generically with all function symbols of Σ, and reflects definedness (in the sense of Definition 2.2.47).*

Proof: Assume the hypotheses. The 'only if' direction is provided by Lemmas 2.2.51 and 2.2.52. For the 'if' direction, suppose that $([s_C])_C \in \prod_{C\in\Sigma} M\langle \mathsf{x}_C\rangle_C$ is invertible, commutes generically with all function symbols of Σ, and reflects definedness. We must show that $([s_C])_C \in G_{\mathbb{T}}(M)$. So let $h : M \to N$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M. We must show that the family of total functions

$$(\rho_h^C(s_C)^* : N_C \to N_C)_{C\in\Sigma}$$

is a Σ-automorphism of N. By Lemma 2.2.33, it is equivalent to show that this family is a sortwise bijective Σ-endomorphism that reflects definedness.

First, since $([s_C])_C$ is invertible, it follows from the proof that $G_{\mathbb{T}}(M)$ is a group that each function $\rho_h^C(s_C)^* : N_C \to N_C$ is bijective.

To show that $(\rho_h^C(s_C)^* : N_C \to N_C)_{C\in\Sigma}$ is a Σ-morphism, let $f : A_1\times\ldots\times A_n \to A$ be a function symbol of Σ, let $a_i \in N_{A_i}$ for all $1 \le i \le n$, and suppose that

$$(a_1,\ldots,a_n) \in \mathsf{dom}\left(f^N\right).$$

We must show that

$$\left(\rho_h^{A_1}(s_{A_1})^*(a_1), \ldots, \rho_h^{A_n}(s_{A_n})^*(a_n)\right) \in \mathsf{dom}\left(f^N\right)$$

and

$$\rho_h^A(s_A)^*(f^N(a_1, \ldots, a_n)) = f^N\left(\rho_h^{A_1}(s_{A_1})^*(a_1), \ldots, \rho_h^{A_n}(s_{A_n})^*(a_n)\right) \in N_A.$$

By Proposition 2.2.50, there is a unique Σ-morphism

$$h' : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle \to N$$

with $h' \circ \eta^{M,f} = h$ and $h'_{A_i}([\mathsf{x}_i]) = a_i \in N_{A_i}$ for each $1 \leq i \leq n$. Since $([s_C])_C$ commutes generically with f, it follows that the sequent

$$f(\mathsf{x}_1, \ldots, \mathsf{x}_n)\downarrow \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

is provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$, and hence (by the cut rule of partial Horn logic, cf. [19]) that the sequent

$$\top \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])$$

is provable in the expanded theory $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f)$. This implies in particular that $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n, f)$ proves the sequent

$$\top \vdash f(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}])\downarrow,$$

so that

$$\left(\left[s_{A_1}\left[\mathsf{x}_1/\mathsf{x}_{A_1}\right]\right], \ldots, \left[s_{A_n}\left[\mathsf{x}_n/\mathsf{x}_{A_n}\right]\right]\right) \in \mathsf{dom}\left(f^{M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle}\right).$$

Since $h' : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle \to N$ is a Σ-morphism, we then obtain

$$\begin{aligned} &\mathsf{dom}\left(f^N\right) \\ &\ni \left(h'_{A_1}\left(\left[s_{A_1}\left[\mathsf{x}_1/\mathsf{x}_{A_1}\right]\right]\right), \ldots, h'_{A_n}\left(\left[s_{A_n}\left[\mathsf{x}_n/\mathsf{x}_{A_n}\right]\right]\right)\right) \\ &= \left(\rho_h^{A_1}\left(s_{A_1}\right)^*(a_1), \ldots, \rho_h^{A_n}\left(s_{A_n}\right)^*(a_n)\right), \end{aligned}$$

as desired (the equality follows by a lemma analogous to Lemma 2.2.40).

In $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle_A$ we have the equality

$$\left[s_A\left[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A\right]\right] = \left[f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)\right].$$

Since $h' : M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle \to N$ is a Σ-morphism, we then obtain

$$\begin{aligned} &\rho_h^A(s_A)^*(f^N(a_1, \ldots, a_n)) \\ &= h'_A\left(\left[s_A\left[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A\right]\right]\right) \end{aligned}$$

$$
\begin{aligned}
&= h'_A\left(\left[f\left(s_{A_1}\left[\mathsf{x}_1/\mathsf{x}_{A_1}\right],\ldots,s_{A_n}\left[\mathsf{x}_n/\mathsf{x}_{A_n}\right]\right)\right]\right)\\
&= h'_A\left(f^{M\langle \mathsf{x}_1,\ldots,\mathsf{x}_n,f\rangle}\left(\left[s_{A_1}\left[\mathsf{x}_1/\mathsf{x}_{A_1}\right]\right],\ldots,\left[s_{A_n}\left[\mathsf{x}_n/\mathsf{x}_{A_n}\right]\right]\right)\right)\\
&= f^N\left(h'_{A_1}\left(\left[s_{A_1}\left[\mathsf{x}_1/\mathsf{x}_{A_1}\right]\right]\right),\ldots,h'_{A_n}\left(\left[s_{A_n}\left[\mathsf{x}_n/\mathsf{x}_{A_n}\right]\right]\right)\right)\\
&= f^N\left(\rho_h^{A_1}\left(s_{A_1}\right)^*(a_1),\ldots,\rho_h^{A_n}\left(s_{A_n}\right)^*(a_n)\right),
\end{aligned}
$$

with the first and last equalities again holding by a lemma analogous to Lemma 2.2.40. This completes the argument that $(\rho_h^C(s_C)^* : N_C \to N_C)_{C\in\Sigma}$ is a Σ-morphism.

To prove that $(\rho_h^C(s_C)^* : N_C \to N_C)_{C\in\Sigma}$ reflects definedness, one modifies the above argument in the same way as at the end of the proof of Lemma 2.2.52.

We have now shown for each $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ that $(\rho_h^C(s_C)^* : N_C \to N_C)_{C\in\Sigma}$ is a sortwise bijective Σ-endomorphism that reflects definedness, and hence it is a Σ-automorphism of N by Lemma 2.2.33. This finishes the proof that $([s_C])_C \in G_{\mathbb{T}}(M)$, and completes the proof of the theorem. ∎

To summarize the results of this section, we have now shown by Theorems 2.2.41 and 2.2.53 that if $\mathbb{T}$ is any quasi-equational theory over a relation-free signature Σ, then the covariant isotropy group

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$$

is naturally isomorphic to the functor

$$G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$$

that sends any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ to the group of all elements $([s_C])_C \in \prod_{C\in\Sigma} M\langle \mathsf{x}_C\rangle_C$ that are invertible, commute generically with every function symbol of Σ, and reflect definedness (in the sense of Definition 2.2.47). Thus, for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, we have characterized the elements of the isotropy group of M solely in terms of syntactic conditions on the diagram theory $\mathbb{T}(M)$ of M.

We finish this section with a few technical lemmas that will be useful in what follows; their proofs may be found in Appendix A.

Lemma 2.2.54. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$, and suppose that $M_A \neq \emptyset$. For any term $t \in \mathsf{Term}^c(\Sigma(M,\mathsf{x}_A))$ with $\mathbb{T}(M,\mathsf{x}_A) \vdash t\downarrow$ and $t : B$, if x_A does* not *occur in t, then*

$$\mathbb{T}(M,\mathsf{x}_A) \vdash t = c_b$$

for some $b \in M_B$. ∎

Lemma 2.2.55. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and let* $A \in \Sigma_{\mathsf{Sort}}$ *have the property that* $\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$ *for distinct variables* $y_1, y_2 : A$*. Then for every* $a \in M_A$*,*

$$\mathbb{T}(M, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_a.$$

∎

Lemma 2.2.56. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, let* $([s_C])_C \in G_{\mathbb{T}}(M)$*, and let* $A \in \Sigma_{\mathsf{Sort}}$ *have the property that* $M_A \neq \emptyset$ *and* $\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$ *for distinct variables* $y_1, y_2 : A$*. If* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_A$ *and* $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = t$*, then* x_A *occurs in* t*.* ∎

2.3 Totally Defined Theories

We now give a brief discussion and application of the results of the preceding section to the class of *totally defined* quasi-equational theories.

Definition 2.3.1 (**Totally Defined Theories**)**.** Let Σ be a relation-free signature. Let $\mathsf{Tot}(\Sigma)$ be the quasi-equational theory over Σ whose axioms are $\top \vdash^{y_1,\ldots,y_n} f(y_1, \ldots, y_n) \downarrow$ for every function symbol $f : A_1 \times \ldots \times A_n \to A$ of Σ, where $y_1 : A_1, \ldots, y_n : A_n$ are pairwise distinct variables. In particular, if $c : A$ is a constant symbol of Σ, then $\top \vdash c \downarrow$ is an axiom of $\mathsf{Tot}(\Sigma)$.

If $\mathbb{T}$ is a quasi-equational theory over Σ, then we say that $\mathbb{T}$ is *totally defined* if all axioms of $\mathsf{Tot}(\Sigma)$ are axioms of $\mathbb{T}$.

If $\mathbb{T}$ is a totally defined quasi-equational theory over a *single-sorted* relation-free signature Σ whose axioms all have the form $\top \vdash^{\vec{x}} t_1 = t_2$ for $t_1, t_2 \in \mathsf{Term}(\Sigma)$, we say that $\mathbb{T}$ is a *totally defined algebraic theory* (note that every axiom in $\mathsf{Tot}(\Sigma)$ has this form, because $t \downarrow$ is short for $t = t$). ∎

Example 2.3.2. The totally defined (algebraic) theory of groups, over the single-sorted relation-free signature $\Sigma := \{\cdot, ^{-1}, e\}$, has the axioms

$$\top \vdash^{x,y,z} x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$\top \vdash^{x} x \cdot x^{-1} = e = x^{-1} \cdot x,$$

$$\top \vdash x \cdot e = x = e \cdot x,$$

$$\top \vdash^{x,y} x \cdot y \downarrow,$$

$$\top \vdash^{x} x^{-1} \downarrow,$$

$$\top \vdash e \downarrow .$$

Similarly, we have the totally defined (algebraic) theories of monoids, abelian groups, (commutative) rings with unit, etc. ▮

One can easily prove ([19, Section 2]) that if Σ is a relation-free signature and $t(\vec{x}) \in \mathsf{Term}(\Sigma)$ is arbitrary, then $\mathsf{Tot}(\Sigma) \vdash^{\vec{x}} t \downarrow$. In particular, if $\mathbb{T}$ is any totally defined quasi-equational theory over Σ, then $\mathbb{T} \vdash^{\vec{x}} t \downarrow$ for any $t(\vec{x}) \in \mathsf{Term}(\Sigma)$.

Let us now simplify some of the definitions and results of the previous section for totally defined *algebraic* theories, which will comprise many of our examples in the next chapter. So let Σ be any single-sorted relation-free signature, and let $\mathbb{T}$ be any totally defined algebraic theory over Σ. First, if $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $n \geq 1$ and $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma(M)$ are pairwise distinct constants of the unique sort of Σ, then it is trivial to see that $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$ is a totally defined algebraic theory over $\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$, and hence $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash t \downarrow$ for any closed term $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_1, \ldots, \mathsf{x}_n))$. In particular, if $\mathsf{x} \notin \Sigma(M)$ is a constant of the unique sort of Σ, then we have the following simplified description of the $\mathbb{T}$-model $M\langle \mathsf{x} \rangle$:

- $M\langle \mathsf{x} \rangle = \{[t] : t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}))\}$, with $[s] = [t]$ iff $\mathbb{T}(M, \mathsf{x}) \vdash s = t$.
- For any constant c of Σ, $\mathsf{dom}\left(c^{M\langle \mathsf{x} \rangle}\right) = \{*\}$ and
$$c^{M\langle \mathsf{x} \rangle}(*) := [c] \in M\langle \mathsf{x} \rangle.$$
- If f is an n-ary function symbol of Σ with $n \geq 1$, then $\mathsf{dom}\left(f^{M\langle \mathsf{x} \rangle}\right) = M\langle \mathsf{x} \rangle^n$ and
$$f^{M\langle \mathsf{x} \rangle}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)] \in M\langle \mathsf{x} \rangle$$
for any $[t_1], \ldots, [t_n] \in M\langle \mathsf{x} \rangle$. ▮

We can also simplify part of Definition 2.2.47 as follows: if $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $[s] \in M\langle \mathsf{x} \rangle$ and f is an n-ary function symbol of Σ, then $[s]$ *commutes generically with* f if

$$\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash s[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}] = f(s[\mathsf{x}_1/\mathsf{x}], \ldots, s[\mathsf{x}_n/\mathsf{x}]).$$

Also, since $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$ is totally defined, it follows that $[s] \in M\langle \mathsf{x} \rangle$ automatically reflects definedness of f. Hence, we have the following simplification of Theorem 2.2.53 for totally defined algebraic theories:

Corollary 2.3.3. *Let* $\mathbb{T}$ *be a totally defined algebraic theory over a single-sorted relation-free signature* Σ*. Then the isotropy group*

$$\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$$

is naturally isomorphic to the functor

$$G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$$

that sends any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *to the group of all elements* $[s] \in M\langle \mathsf{x} \rangle$ *that are invertible and commute generically with all function symbols of* Σ *(in the above simplified sense).* ∎

2.4 Finitely Presented Models

In the final section of this chapter, we discuss the results of Section 2.2 in the context of *finitely presented* models of a quasi-equational theory.

Definition 2.4.1 (Finitely Presented Models). Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ.

- For any $n \geq 0$ and pairwise distinct constant symbols $c_1, \ldots, c_n \notin \Sigma$ of respective sorts $A_1, \ldots, A_n \in \Sigma_{\mathsf{Sort}}$, let $\Sigma(c_1, \ldots, c_n)$ be the signature defined as follows:
  - $\Sigma(c_1, \ldots, c_n)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
  - $\Sigma(c_1, \ldots, c_n)_{\mathsf{Fun}} := \Sigma_{\mathsf{Fun}} \cup \{c_1, \ldots, c_n\}$.
- For any $n \geq 0$ and pairwise distinct constant symbols $c_1, \ldots, c_n \notin \Sigma$ of sorts belonging to Σ, and any Horn *sentence* φ over the signature $\Sigma(c_1, \ldots, c_n)$, we define

  $$\mathbb{T}(c_1, \ldots, c_n, \varphi)$$

  to be the quasi-equational theory over $\Sigma(c_1, \ldots, c_n)$ whose axioms are as follows:
  - All axioms of $\mathbb{T}$.
  - For each $1 \leq i \leq n$, the axiom $\top \vdash c_i \downarrow$.
  - The axiom $\top \vdash \varphi$.

  If φ is just the sentence $c_1 \downarrow \wedge \ldots \wedge c_n \downarrow$, then we write $\mathbb{T}(c_1, \ldots, c_n)$ instead of $\mathbb{T}(c_1, \ldots, c_n, \varphi)$.

- If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then we say that M is *finitely presented* if there are some $n \geq 0$, some pairwise distinct constant symbols $c_1, \ldots, c_n \notin \Sigma$ of sorts belonging to Σ, and some Horn sentence φ over $\Sigma(c_1, \ldots, c_n)$ such that
$$M \cong \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \varphi))|_\Sigma,$$
i.e. such that M is isomorphic to the Σ-reduct of the initial model of $\mathbb{T}(c_1, \ldots, c_n, \varphi)$.
If $c_1 : A_1, \ldots, c_n : A_n$ and $\varphi \equiv c_1 \downarrow \wedge \ldots \wedge c_n \downarrow$, then we say that M is *free* on n generators of sorts $A_1, \ldots, A_n$. ▮

We can then convert Definition 1.2.14 into a proposition, whose proof may be found in Appendix A:

Proposition 2.4.2. *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then M is finitely presented iff there is a Horn formula $\varphi(x_1, \ldots, x_n)$ over Σ with $x_i : A_i$ for each $1 \leq i \leq n$, and an n-tuple*
$$(a_1, \ldots, a_n) \in \varphi(x_1, \ldots, x_n)^M \subseteq M_{A_1} \times \ldots \times M_{A_n}$$
such that if $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and
$$(b_1, \ldots, b_n) \in \varphi(x_1, \ldots, x_n)^N \subseteq N_{A_1} \times \ldots \times N_{A_n},$$
then there is a unique Σ-morphism $h : M \to N$ with $h_{A_i}(a_i) = b_i$ for all $1 \leq i \leq n$. ▮

Corollary 2.4.3. *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then M is free on n generators of sorts $A_1, \ldots, A_n$ iff there is an n-tuple $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$ such that if $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $(b_1, \ldots, b_n) \in N_{A_1} \times \ldots \times N_{A_n}$, then there is a unique Σ-morphism $h : M \to N$ with $h_{A_i}(a_i) = b_i$ for all $1 \leq i \leq n$.* ▮

We now define $\mathsf{fp}\mathbb{T}\mathsf{mod}$ to be the full subcategory of $\mathsf{P}\mathbb{T}\mathsf{mod}$ on the finitely presented $\mathbb{T}$-models. Before we can show that the main results of Section 2.2 restrict to $\mathsf{fp}\mathbb{T}\mathsf{mod}$, we require the following easy observation, which follows from Proposition 2.4.2 and Propositions 2.2.10 and 2.2.50.

Lemma 2.4.4. *If $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ and $f : B_1 \times \ldots \times B_n \to B$ is a function symbol of Σ and $\mathsf{x}_1 : B_1, \ldots, \mathsf{x}_m : B_m \notin \Sigma(M)$ are pairwise distinct constants, then $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_m \rangle \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ and $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_m, f \rangle \in \mathsf{fp}\mathbb{T}\mathsf{mod}$.* ▮

We now relativize the functors $\mathcal{Z}_{\mathbb{T}}$ and $G_{\mathbb{T}}$ to the subcategory $\mathsf{fp}\mathbb{T}\mathsf{mod}$.

Definition 2.4.5.

- Let $\mathcal{Z}_{\mathbb{T}}^{\mathsf{fp}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ be the covariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}$.
- Let $G_{\mathbb{T}}^{\mathsf{fp}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ be the 'restriction' of $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ to the full subcategory $\mathsf{fp}\mathbb{T}\mathsf{mod}$. In other words, for any $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$, define

  $$G_{\mathbb{T}}^{\mathsf{fp}}(M) \in \mathsf{Group}$$

  to be the group consisting of all elements $([s_C])_C \in \prod_{C \in \Sigma} M\langle \mathsf{x}_C \rangle_C$ with the property that if $h : M \to N$ is any morphism in $\mathsf{fp}\mathbb{T}\mathsf{mod}$, then the induced family of total functions

  $$\left(\rho_h^C(s_C)^* : N_C \to N_C\right)_{C \in \Sigma}$$

  is a Σ-automorphism of N. Define $G_{\mathbb{T}}^{\mathsf{fp}}$ on morphisms in $\mathsf{fp}\mathbb{T}\mathsf{mod}$ in the same way as for $G_{\mathbb{T}}$ (cf. Definition 2.2.36).

Justification. We must verify that $G_{\mathbb{T}}^{\mathsf{fp}}(M)$ is in fact a group (for $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$), and that $G_{\mathbb{T}}^{\mathsf{fp}}$ is a well-defined functor. The group structure of $G_{\mathbb{T}}^{\mathsf{fp}}(M)$ is the same as that of $G_{\mathbb{T}}(M)$ (cf. the proof of Proposition 2.2.35): the definition of the inverse operation in $G_{\mathbb{T}}(M)$ still makes sense for $G_{\mathbb{T}}^{\mathsf{fp}}(M)$, because $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ implies $M\langle \mathsf{x}_A \rangle \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ for any sort $A \in \Sigma$ by Lemma 2.4.4. The proof of Proposition 2.2.38 carries over exactly as it is to show that $G_{\mathbb{T}}^{\mathsf{fp}}$ is functorial. ∎

Theorem 2.4.6. *The functor* $G_{\mathbb{T}}^{\mathsf{fp}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ *is naturally isomorphic to* $\mathcal{Z}_{\mathbb{T}}^{\mathsf{fp}} : \mathsf{fp}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$.

Proof: For any $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$, we define the same isomorphism (relativized to $\mathsf{fp}\mathbb{T}\mathsf{mod}$) as we did in the proof of Theorem 2.2.41. The proof that it is bijective still works in this context because of Lemma 2.4.4. ∎

We also have an analogous version of Corollary 2.2.42 for $\mathsf{fp}\mathbb{T}\mathsf{mod}$, as well as:

Theorem 2.4.7. *Let* $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ *and* $([s_C])_C \in \prod_C M\langle \mathsf{x}_C \rangle_C$. *Then* $([s_C])_C \in G_{\mathbb{T}}^{\mathsf{fp}}(M)$ *iff* $([s_C])_C$ *is invertible, commutes generically with all function symbols of* Σ, *and reflects definedness.*

Proof: The 'only if' direction is still true because Lemmas 2.2.51 and 2.2.52 hold for $M \in \mathsf{fp}\mathbb{T}\mathsf{mod}$, since $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, f\rangle \in \mathsf{fp}\mathbb{T}\mathsf{mod}$ by Lemma 2.4.4. The proof of the 'if' direction is the same as the proof of the 'if' direction in Theorem 2.2.53. ∎

We now wish to show that if M is a finitely presented model of $\mathbb{T}$, then one may somewhat simplify the calculation of the isotropy group $\mathcal{Z}_{\mathbb{T}}(M) \cong G_{\mathbb{T}}(M)$. The elements of $G_{\mathbb{T}}(M)$ are sequences $([s_C])_C \in \prod_C M\langle \mathsf{x}_C\rangle_C$. In particular, if $C \in \Sigma_{\mathsf{Sort}}$, then $s_C \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ is a closed term of sort C with $\mathbb{T}(M, \mathsf{x}_C) \vdash s_C \downarrow$. Hence, s_C may contain the indeterminate x_C, as well as constants from the diagram signature of M. However, if M is *finitely presented*, then we know that M is isomorphic to the (Σ-reduct of the) initial model of $\mathbb{T}(c_1, \ldots, c_n, \varphi)$, for some new constants $c_1, \ldots, c_n \notin \Sigma$ and Horn sentence φ over $\Sigma(c_1, \ldots, c_n)$. Consequently, $s_C \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ may contain constants from $M \cong \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \varphi))$, whose objects are congruence classes of closed terms over $\Sigma(c_1, \ldots, c_n)$. We now want to show that we can regard s_C as being a closed term over $\Sigma(c_1, \ldots, c_n, \mathsf{x}_C)$, rather than over the diagram signature $\Sigma(M, \mathsf{x}_C)$, and we also want to show that we can express the notions of invertibility, commuting generically, and reflecting definedness in terms of the theory $\mathbb{T}(c_1, \ldots, c_n, \varphi)$, rather than in terms of the diagram theory $\mathbb{T}(M)$.

Definition 2.4.8. Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$. Write $\mathbb{T}(\varphi)$ for $\mathbb{T}(c_1, \ldots, c_n, \varphi)$ and M^{φ} for $\mathsf{Free}(\mathbb{T}(\varphi))|_{\Sigma}$. We define

$$G'_{\mathbb{T}}(M^{\varphi})$$

to be the set of all Σ_{Sort}-indexed sequences $([s_C])_C$ with the following properties:

- For any $C \in \Sigma_{\mathsf{Sort}}$,

$$[s_C] \in \{[t] : t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))_C \wedge \mathbb{T}(\varphi, \mathsf{x}_C) \vdash t \downarrow\}.$$

- For any $C \in \Sigma_{\mathsf{Sort}}$, there is some

$$\left[s_C^{-1}\right] \in \{[t] : t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))_C \wedge \mathbb{T}(\varphi, \mathsf{x}_C) \vdash t \downarrow\}$$

  with

$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash s_C\left[s_C^{-1}/\mathsf{x}_C\right] = \mathsf{x}_C = s_C^{-1}[s_C/\mathsf{x}_C].$$

- For any function symbol $f : B_1 \times \ldots \times B_m \to B$ in Σ, $\mathbb{T}(\varphi, \mathsf{x}_1, \ldots, \mathsf{x}_m)$ proves the sequents

$$f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow \vdash s_B[f(\mathsf{x}_1, \ldots, \mathsf{x}_m)/\mathsf{x}_B] = f(s_{B_1}[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, s_{B_m}[\mathsf{x}_m/\mathsf{x}_{B_m}])$$

  and

$$f(s_{B_1}[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, s_{B_m}[\mathsf{x}_m/\mathsf{x}_{B_m}]) \downarrow \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow.$$

∎

Our goal is now to show that the underlying set of $G_{\mathbb{T}}(M^{\varphi})$ is in bijection with the set $G'_{\mathbb{T}}(M^{\varphi})$ (and we will then use this bijection to induce a group structure on $G'_{\mathbb{T}}(M^{\varphi})$, which will then make $G_{\mathbb{T}}(M^{\varphi})$ and $G'_{\mathbb{T}}(M^{\varphi})$ isomorphic as groups). To do this, we require following the definition, which gives translations between the signatures $\Sigma(c_1, \ldots, c_n)$ and $\Sigma(M^{\varphi})$.

Definition 2.4.9. Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$.

- For any $C \in \Sigma_{\mathsf{Sort}}$, we define a signature morphism

  $$\rho_{\varphi}^{C} : \Sigma(c_1, \ldots, c_n, \mathsf{x}_C) \to \Sigma(M^{\varphi}, \mathsf{x}_C)$$

  as follows:

  - ρ_{φ}^{C} is the identity on Σ.
  - For any $1 \leq i \leq n$, we have $[c_i] \in M^{\varphi}_{A_i}$ since $\mathbb{T}(\varphi) \vdash c_i \downarrow$, so we set

    $$\rho_{\varphi}^{C}(c_i) := c_{[c_i]}^{M_{\varphi}} \in \Sigma(M_{\varphi}, \mathsf{x}_C)_{A_i}.$$

  - We set $\rho_{\varphi}^{C}(\mathsf{x}_C) := \mathsf{x}_C$.

- For any $C \in \Sigma_{\mathsf{Sort}}$, we define a sort-preserving function

  $$\sigma_{\varphi}^{C} : \mathsf{Term}^{c}(M^{\varphi}, \mathsf{x}_C)) \to \mathsf{Term}^{c}(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))$$

  as follows. First, for any $B \in \Sigma_{\mathsf{Sort}}$ and

  $$[t] \in M^{\varphi}_{B} = \{[s] : s \in \mathsf{Term}^{c}(\Sigma(c_1, \ldots, c_n))_B \wedge \mathbb{T}(\varphi) \vdash s \downarrow\},$$

  choose a representative $\widehat{[t]} \in \mathsf{Term}^{c}(\Sigma(c_1, \ldots, c_n))_B$.

  - We set $\sigma_{\varphi}^{C}(\mathsf{x}_C) := \mathsf{x}_C$.
  - Let $B \in \Sigma_{\mathsf{Sort}}$ and $[t] \in M^{\varphi}_{B}$. Then we set

    $$\sigma_{\varphi}^{C}\left(c_{[t]}^{M^{\varphi}}\right) := \widehat{[t]},$$

    and this assignment is well-defined.

– If $f : B_1 \times \ldots \times B_m \to B$ is a function symbol of Σ and

$$t_i \in \mathsf{Term}^c(\Sigma(M^\varphi, \mathsf{x}_C))_{B_i}$$

for each $1 \leq i \leq m$, we set

$$\sigma_\varphi^C(f(t_1, \ldots, t_m)) := f(\sigma_\varphi^C(t_1), \ldots, \sigma_\varphi^C(t_m)).$$

∎

The following lemma may be proved by straightforward induction on terms.

Lemma 2.4.10. *Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$.*

1. *For any sort $C \in \Sigma$, the signature morphism ρ_φ^C respects substitution into x_C, i.e. if $s, t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))$ and $t : C$, then*

$$\rho_\varphi^C(s[t/\mathsf{x}_C]) \equiv \rho_\varphi^C(s)[\rho_\varphi^C(t)/\mathsf{x}_C].$$

2. *For any sort $C \in \Sigma$, the signature morphism σ_φ^C respects substitution into x_C, i.e. if $s, t \in \mathsf{Term}^c(\Sigma(M^\varphi, \mathsf{x}_C))$ and $t : C$, then*

$$\sigma_\varphi^C(s[t/\mathsf{x}_C]) \equiv \sigma_\varphi^C(s)[\sigma_\varphi^C(t)/\mathsf{x}_C].$$

∎

The proofs of the following technical lemmas may all be found in Appendix A.

Lemma 2.4.11. *Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants.*

1. *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))_B$ with*

$$(M, a_1, \ldots, a_n) \models t \downarrow$$

for some $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$, then

$$\top \vdash t\,[c_{a_1}/c_1, \ldots, c_{a_n}/c_n] = c_{t^{(M, a_1, \ldots, a_n)}}$$

is provable in $\mathbb{T}(M)$.

2. *If* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* ψ *is a Horn sentence over* $\Sigma(c_1, \ldots, c_n)$ *with*

$$(M, a_1, \ldots, a_n) \models \psi$$

for some $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$, *then*

$$\top \vdash \psi\,[c_{a_1}/c_1, \ldots, c_{a_n}/c_n]$$

is provable in $\mathbb{T}(M)$. ∎

Lemma 2.4.12. *Let* $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ *be pairwise distinct constants and let* φ *be a Horn sentence over* $\Sigma(c_1, \ldots, c_n)$.

1. *For any sort* $C \in \Sigma$, *the signature morphism* ρ_φ^C *is a theory morphism*

$$\rho_\varphi^C : \mathbb{T}(\varphi, \mathsf{x}_C) \to \mathbb{T}(M^\varphi, \mathsf{x}_C).$$

2. *For any sort* $C \in \Sigma$, *any terms* $u, v \in \mathsf{Term}^c(M^\varphi, \mathsf{x}_C))$ *of the same sort and terms* $s, t \in \mathsf{Term}^c(M^\varphi, \mathsf{x}_C))$ *of the same sort, if*

$$u = v \vdash s = t \text{ is provable in } \mathbb{T}(M^\varphi, \mathsf{x}_C),$$

then

$$\sigma_\varphi^C(u) = \sigma_\varphi^C(v) \vdash \sigma_\varphi^C(s) = \sigma_\varphi^C(t) \text{ is provable in } \mathbb{T}(\varphi, \mathsf{x}_C).$$

∎

Lemma 2.4.13. *Let* $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ *be pairwise distinct constants and let* φ *be a Horn sentence over* $\Sigma(c_1, \ldots, c_n)$.

1. *If* $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))$ *is of some sort* $B \in \Sigma$ *and* $\mathbb{T}(\varphi) \vdash t\downarrow$ *and* $C \in \Sigma_{\mathsf{Sort}}$, *then*

$$\mathbb{T}(M^\varphi) \vdash \rho_\varphi^C(t) = c_{[t]}^{M^\varphi}.$$

2. *If* $t \in \mathsf{Term}^c(\Sigma(M^\varphi, \mathsf{x}_C))$ *and* $\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash t\downarrow$, *then*

$$\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash \rho_\varphi^C(\sigma_\varphi^C(t)) = t.$$

3. *If* $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))$ *and* $\mathbb{T}(\varphi, \mathsf{x}_C) \vdash t\downarrow$, *then*

$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash \sigma_\varphi^C(\rho_\varphi^C(t)) = t.$$

∎

We can now prove that the sets $G_{\mathbb{T}}(M^{\varphi})$ and $G'_{\mathbb{T}}(M^{\varphi})$ are in bijective correspondence:

Proposition 2.4.14. *Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$. Then $G_{\mathbb{T}}(M^{\varphi})$ is in bijective correspondence with $G'_{\mathbb{T}}(M^{\varphi})$.*

Proof: We define a bijection

$$G_{\mathbb{T}}(M^{\varphi}) \xrightarrow{\sim} G'_{\mathbb{T}}(M^{\varphi}).$$

So let $([s_C])_C \in G_{\mathbb{T}}(M^{\varphi})$. For each $C \in \Sigma_{\mathsf{Sort}}$, we thus have $s_C \in \mathsf{Term}^c(\Sigma(M_{\varphi}, \mathsf{x}_C))_C$ with $\mathbb{T}(M_{\varphi}, \mathsf{x}_C) \vdash s_C \downarrow$. Then by Lemma 2.4.12, it follows that

$$\sigma_{\varphi}^{C}(s_C) \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))_C$$

and

$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash \sigma_{\varphi}^{C}(s_C) \downarrow .$$

Now we show that $([\sigma_{\varphi}^{C}(s_C)])_C \in G'_{\mathbb{T}}(M^{\varphi})$. Since $([s_C])_C$ is invertible, for each $C \in \Sigma_{\mathsf{Sort}}$ there is some $\left[s_C^{-1}\right] \in M^{\varphi}\langle \mathsf{x}_C \rangle_C$ such that

$$\mathbb{T}(M^{\varphi}, \mathsf{x}_C) \vdash s_C[s_C^{-1}/\mathsf{x}_C] = \mathsf{x}_C = s_C^{-1}[s_C/\mathsf{x}_C].$$

Then we have $[\sigma_{\varphi}^{C}(s_C^{-1})] \in M_C^{\varphi}$ by Lemma 2.4.12 and, using Lemma 2.4.10, the following equations are provable in $\mathbb{T}(\varphi, \mathsf{x}_C)$, as desired:

$$\sigma_{\varphi}^{C}(s_C)\left[\sigma_{\varphi}^{C}\left(s_C^{-1}\right)/\mathsf{x}_C\right] = \sigma_{\varphi}^{C}\left(s_C\left[s_C^{-1}/\mathsf{x}_C\right]\right) = \sigma_{\varphi}^{C}(\mathsf{x}_C) = \mathsf{x}_C = \sigma_{\varphi}^{C}\left(s_C^{-1}\right)\left[\sigma_{\varphi}^{C}(s_C)/\mathsf{x}_C\right].$$

Now let $f : B_1 \times \ldots \times B_m \to B$ be a function symbol of Σ. We must show that $\mathbb{T}(\varphi, \mathsf{x}_1, \ldots, \mathsf{x}_m)$ proves the sequents

$$f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow$$
$$\vdash \sigma_{\varphi}^{B}(s_B)[f(\mathsf{x}_1, \ldots, \mathsf{x}_m)/\mathsf{x}_B] = f\left(\sigma_{\varphi}^{B_1}(s_{B_1})[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, \sigma_{\varphi}^{B_m}(s_{B_m})[\mathsf{x}_m/\mathsf{x}_{B_m}]\right)$$

and

$$f\left(\sigma_{\varphi}^{B_1}(s_{B_1})[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, \sigma_{\varphi}^{B_m}(s_{B_m})[\mathsf{x}_m/\mathsf{x}_{B_m}]\right) \downarrow \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow .$$

Since $([s_C])_C \in G_{\mathbb{T}}(M^{\varphi})$, we know that the following sequents are provable in $\mathbb{T}(M^{\varphi}, \mathsf{x}_1, \ldots, \mathsf{x}_m)$:

$$f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow \vdash s_B[f(\mathsf{x}_1, \ldots, \mathsf{x}_m)/\mathsf{x}_B] = f(s_{B_1}[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, s_{B_m}[\mathsf{x}_m/\mathsf{x}_{B_m}])$$

$$f(s_{B_1}[\mathsf{x}_1/\mathsf{x}_{B_1}], \ldots, s_{B_m}[\mathsf{x}_m/\mathsf{x}_{B_m}]) \downarrow \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_m) \downarrow .$$

By applying (obvious extensions of) Lemmas 2.4.12 and 2.4.10, we then obtain our desired result. This proves that $([\sigma_\varphi^C(s_C)])_C \in G'_{\mathbb{T}}(M^\varphi)$. So we define a function

$$G_{\mathbb{T}}(M^\varphi) \xrightarrow{\sim} G'_{\mathbb{T}}(M^\varphi)$$

by the assignment

$$([s_C])_C \longmapsto ([\sigma_\varphi^C(s_C)])_C,$$

which is well-defined by Lemma 2.4.12.

To show that this function is injective, let $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M^\varphi)$ with

$$([\sigma_\varphi^C(s_C)])_C = ([\sigma_\varphi^C(t_C)])_C;$$

we must show $([s_C])_C = ([t_C])_C$. So let $C \in \Sigma_{\mathsf{Sort}}$. We are given that

$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash \sigma_\varphi^C(s_C) = \sigma_\varphi^C(t_C),$$

and we want to show that

$$\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash s_C = t_C.$$

Since ρ_φ^C is a theory morphism by Lemma 2.4.12, we obtain that

$$\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash \rho_\varphi^C\left(\sigma_\varphi^C(s_C)\right) = \rho_\varphi^C\left(\sigma_\varphi^C(t_C)\right).$$

Then our desired result follows by Lemma 2.4.13.

To show that this function is surjective, let $([t_C])_C \in G'_{\mathbb{T}}(M^\varphi)$. We must show that there is some $([s_C])_C \in G_{\mathbb{T}}(M^\varphi)$ with

$$([\sigma_\varphi^C(s_C)])_C = ([t_C])_C.$$

For any $C \in \Sigma_{\mathsf{Sort}}$, we have $t_C \in \mathsf{Term}(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))_C$ with $\mathbb{T}(\varphi, \mathsf{x}_C) \vdash t_C \downarrow$. Then by Lemma 2.4.12, we have $\rho_\varphi^C(t_C) \in \mathsf{Term}(\Sigma(M^\varphi, \mathsf{x}_C))$ with $\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash \rho_\varphi^C(t_C) \downarrow$. We can now argue as in the first part of the proof that $([t_C])_C \in G'_{\mathbb{T}}(M^\varphi)$ implies $([\rho_\varphi^C(t_C)])_C \in G_{\mathbb{T}}(M^\varphi)$. And for any $C \in \Sigma_{\mathsf{Sort}}$, we have

$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash \sigma_\varphi^C(\rho_\varphi^C(t_C)) = t_C$$

by Lemma 2.4.13, which completes the proof of surjectivity. So we have indeed defined a bijection

$$G_{\mathbb{T}}(M^\varphi) \xrightarrow{\sim} G'_{\mathbb{T}}(M^\varphi).$$

∎

Corollary 2.4.15. *Let* $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ *be pairwise distinct constants and let* φ *be a Horn sentence over* $\Sigma(c_1, \ldots, c_n)$. *If we give* $G'_{\mathbb{T}}(M^{\varphi})$ *the group structure induced by the bijection*

$$G_{\mathbb{T}}(M^{\varphi}) \to G'_{\mathbb{T}}(M^{\varphi})$$

of Proposition 2.4.14, then we have a group isomorphism

$$G_{\mathbb{T}}(M^{\varphi}) \cong G'_{\mathbb{T}}(M^{\varphi}).$$

∎

Remark 2.4.16. It is easy to verify (using Lemmas 2.4.10 and 2.4.13) that the group structure on $G'_{\mathbb{T}}(M^{\varphi})$ is essentially the same as the group structure on $G_{\mathbb{T}}(M^{\varphi})$: the unit element is $([\mathsf{x}_C])_C$, and if $([s_C])_C, ([t_C])_C \in G'_{\mathbb{T}}(M^{\varphi})$, then

$$([s_C])_C \cdot ([t_C])_C = ([s_C[t_C/\mathsf{x}_C])_C$$

and

$$([s_C])_C^{-1} = \left([s_C^{-1}]\right)_C.$$

∎

To conclude this section, we connect the notion of finitely presented model of a *quasi-equational* theory (Definition 2.4.1) with the notion of finitely presented model of a *cartesian* theory (Definition 1.2.14). We have seen at the end of Chapter 1 that the notions of quasi-equational theory and cartesian theory are equivalent. Hence, if $\mathbb{T}$ is any quasi-equational theory with equivalent cartesian theory $\mathbb{T}'$, then $\mathsf{P}\mathbb{T}\mathsf{mod} \simeq \mathsf{Mod}(\mathbb{T}', \mathsf{Sets})$, and thus $\mathsf{fp}\mathbb{T}\mathsf{mod} \simeq \mathsf{fp}\mathbb{T}'\mathsf{mod}$, with $\mathsf{fp}\mathbb{T}\mathsf{mod}$ being the full subcategory of $\mathsf{P}\mathbb{T}\mathsf{mod}$ on the finitely presented models of $\mathbb{T}$ in the sense of Definition 2.4.1, and $\mathsf{fp}\mathbb{T}'\mathsf{mod}$ being the full subcategory of $\mathsf{Mod}(\mathbb{T}', \mathsf{Sets})$ on the finitely presented models of $\mathbb{T}'$ in the sense of Definition 1.2.14. Thus, the covariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}$ is essentially equivalent to the covariant isotropy group of $\mathsf{fp}\mathbb{T}'\mathsf{mod}$, which is the same thing as the internal isotropy group object of the classifying topos $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}'\mathsf{mod}}$ of $\mathbb{T}'$. Thus, when we compute the covariant isotropy group of $\mathsf{fp}\mathbb{T}\mathsf{mod}$, i.e. the internal isotropy group object of the presheaf topos $\mathsf{Sets}^{\mathsf{fp}\mathbb{T}\mathsf{mod}}$, we are indeed (indirectly) computing the isotropy group of the classifying topos of the cartesian theory $\mathbb{T}'$ (as defined in Chapter 1).

Chapter 3

Examples

In this chapter we will apply the results of the preceding chapter to characterize the isotropy groups of many examples of quasi-equational theories. In the first few sections, we will begin by characterizing the isotropy groups of specific (totally defined) algebraic theories, and then we will study some theories with more logical complexity towards the end of the chapter. We will say that a quasi-equational theory $\mathbb{T}$ has *trivial* isotropy (group) if $G_{\mathbb{T}} \cong \mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ is the constant functor on the trivial group, and has *non-trivial* isotropy (group) otherwise.

In many of the examples, the reader will notice that the ability to compute the isotropy group of a theory $\mathbb{T}$ (using the methods developed in Chapter 2) depends heavily on whether there is a solution to the *word problem* for $\mathbb{T}$; more specifically, on whether there is a decidable method for determining whether two closed terms $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ are equal (for $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $C \in \Sigma_{\mathsf{Sort}}$).

3.1 Empty Theories

As our first (easy) example, we will show that any 'empty' quasi-equational theory has trivial isotropy.

Definition 3.1.1 (Empty Theories). Let Σ be any relation-free signature.

- Recall from Definition 2.3.1 that $\mathsf{Tot}(\Sigma)$ is the quasi-equational theory over Σ with axioms $\top \vdash^{\vec{x}} f(\vec{x}) \downarrow$ for each function symbol $f \in \Sigma$. We will refer to $\mathsf{Tot}(\Sigma)$ as the *totally defined empty theory over* Σ.

- If $\mathbb{T}$ is a quasi-equational theory over Σ, then we say that $\mathbb{T}$ is an *empty theory over* Σ if every axiom of $\mathbb{T}$ is an axiom of $\mathsf{Tot}(\Sigma)$. In other words, $\mathbb{T}$ is an empty theory over Σ if every axiom of $\mathbb{T}$ has the form $\top \vdash^{\vec{x}} f(\vec{x}) \downarrow$ for some function symbol $f \in \Sigma$. ▮

The following lemma will be useful in what follows:

Lemma 3.1.2. *Let $\mathbb{T}$ be an arbitrary quasi-equational theory over a relation-free signature Σ.*

- *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $A \in \Sigma_{\mathsf{Sort}}$, and let $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ for some $B \in \Sigma_{\mathsf{Sort}}$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash s \downarrow \wedge\, t \downarrow$. Then*

  $$\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$$

  iff

  $$\rho_h^A(s)^* = \rho_h^A(t)^* : N_A \to N_B$$

  for every Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$.

- *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$ and $y_1, y_2 : A$ are distinct variables, then*

  $$\mathbb{T}(M) \nvdash^{y_1, y_2} y_1 = y_2$$

  iff

  there are a $\mathbb{T}$-model N, a Σ-morphism $h : M \to N$, and distinct elements $a_1 \neq a_2 \in N_A$.

Proof: See Appendix B. ∎

We now require the following technical lemma.

Lemma 3.1.3. *Let Σ be any relation-free signature, let $\mathbb{T}$ be an empty theory over Σ, and let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. For any $C \in \Sigma_{\mathsf{Sort}}$ and $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$,*

$$\mathbb{T}(M, \mathsf{x}_C) \vdash t = \mathsf{x}_C \implies t \equiv \mathsf{x}_C.$$

Proof: See Appendix B. ∎

Proposition 3.1.4. *Let Σ be any relation-free signature, and let $\mathbb{T}$ be an empty theory over Σ. Then $\mathbb{T}$ has trivial isotropy. More precisely, if $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then*

$$G_{\mathbb{T}}(M) = \{([\mathsf{x}_C])_C\} \subseteq \prod_{C \in \Sigma} M\langle \mathsf{x}_C \rangle_C.$$

Proof: The result is almost immediate from Lemma 3.1.3. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $([s_C])_C \in G_{\mathbb{T}}(M)$; we must show that $([s_C])_C = ([\mathsf{x}_C])_C$, i.e. we must show for each $B \in \Sigma_{\mathsf{Sort}}$ that

$$\mathbb{T}(M, \mathsf{x}_B) \vdash s_B = \mathsf{x}_B.$$

Since $([s_C])_C$ is invertible, there is some $\left[s_B^{-1}\right] \in M\langle \mathsf{x}_B \rangle_B$ such that

$$\mathbb{T}(M, \mathsf{x}_B) \vdash s_B[s_B^{-1}/\mathsf{x}_B] = \mathsf{x}_B.$$

By Lemma 3.1.3, it follows that $s_B[s_B^{-1}/\mathsf{x}_B] \equiv \mathsf{x}_B$, which forces $s_B \equiv s_B^{-1} \equiv \mathsf{x}_B$. Since $\mathbb{T}(M, \mathsf{x}_B) \vdash s_B \downarrow$, we then obtain our desired result. ∎

Corollary 3.1.5. *Let Σ be any relation-free signature and $\mathbb{T}$ any empty theory over Σ. If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then $\mathcal{Z}_{\mathbb{T}}(M)$ is the trivial group consisting of only the identity element* $\left(\mathsf{id}_{\mathsf{cod}(f)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(M)} \in \mathcal{Z}_{\mathbb{T}}(M)$. ∎

Corollary 3.1.6. *Let Σ be any relation-free signature and $\mathbb{T}$ an empty theory over Σ. Then $\mathcal{Z}_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ is (naturally isomorphic to) the constant functor on the trivial group.* ∎

An obvious consequence of Proposition 3.1.4 is that if Σ is the single-sorted signature with $\Sigma_{\mathsf{Fun}} := \emptyset$, so that $\mathsf{Tot}(\Sigma)$ is the theory of sets and $\mathsf{P}\mathbb{T}\mathsf{mod}$ is just the usual category Sets of sets and (total) functions, then this theory has trivial isotropy.

3.2 Monoids and Groups

In this section, we will compute the (non-trivial) isotropy groups of the totally defined algebraic theories of monoids and groups. First, we will consider the totally defined theory $\mathbb{T}_{\mathsf{Mon}}$ of monoids. We recall that $\mathbb{T}_{\mathsf{Mon}}$ is the totally defined algebraic theory over the single-sorted signature $\Sigma_{\mathsf{Mon}} := \{\cdot, e\}$ whose axioms are:

$$\top \vdash^{x,y,z} x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$\top \vdash^{x} x \cdot e = x = e \cdot x,$$

$$\top \vdash^{x,y} x \cdot y \downarrow,$$

$$\top \vdash e \downarrow .$$

Since $\cdot$ is associative in any monoid, we will generally not use parentheses when writing iterated products. Also, we have $\mathsf{P}\mathbb{T}_{\mathsf{Mon}}\mathsf{mod} = \mathsf{Mon}$, the usual category of monoids and monoid homomorphisms. If M is any monoid, then for any $n \geq 1$, there is a well-known description of the monoid $M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$, whose needed properties we now record.

Definition 3.2.1 (Multiplicative Words). Let M be any monoid, let $n \geq 1$, and let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma_{\mathsf{Mon}}(M)$ be pairwise distinct constants of the unique sort of Σ_{Mon}.

- A *multiplicative word* over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ is a non-empty finite string over the alphabet $\left\{c^M_m : m \in M\right\} \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$. If $w \equiv w_1 \ldots w_r$ is a multiplicative word over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$, we write $\ell(w) := r$.
- A multiplicative word over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ is said to be *reduced* if either $\ell(w) = 1$, or $\ell(w) \geq 2$ and w does not contain $c^M_{e^M}$ and contains no substring consisting of two consecutive elements of $\{c^M_m : m \in M\}$. ∎

Lemma 3.2.2. *Let M be any monoid, let $n \geq 1$, and let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma(M)$ be pairwise distinct constants of the unique sort of Σ_{Mon}. Note that (because $\mathbb{T}_{\mathsf{Mon}}(M) \vdash c^M_{e^M} = e$)*

$$M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle = \{[w] : w \textit{ is a multiplicative word over } M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}\},$$

with $[w] = [w']$ iff $\mathbb{T}_{\mathsf{Mon}}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash w = w'$.

- *If w is a multiplicative word over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$, then there is a (unique) reduced multiplicative word w^r over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ such that $[w] = [w^r]$.*
- *If w, w' are reduced multiplicative words over $M \cup \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ with $[w] = [w']$, then $w \equiv w'$.* ∎

Lemma 3.2.3. *If M is any monoid and y_1, y_2 are distinct variables of the unique sort of Σ_{Mon}, then $\mathbb{T}_{\mathsf{Mon}}(M) \nvdash^{y_1, y_2} y_1 = y_2$.*

Proof: By Lemma 3.1.2, it suffices to show that there is a monoid N with at least two elements and a monoid homomorphism $h : M \to N$. If M itself already contains at least two elements, then we set $N := M$ and $h := \mathsf{id}_M$. Otherwise, we have $M = \{e^M\}$, and then M can be embedded into any of the many monoids with at least two elements. ∎

If M is any monoid, we say that an element $m \in M$ is *invertible* if there is a (necessarily unique) element $m^{-1} \in M$ with $mm^{-1} = e^M = m^{-1}m$. We write $\mathsf{Inv}(M)$ for the group of all invertible elements of M. We can now characterize the isotropy group of $\mathbb{T}_{\mathsf{Mon}}$.

Proposition 3.2.4. *If M is any monoid, then*

$$G_{\mathbb{T}_{\mathsf{Mon}}}(M) = \{[c_m \mathsf{x} c_{m^{-1}}] \in M\langle \mathsf{x} \rangle : m \in \mathsf{Inv}(M)\}$$

(where we have omitted the superscripts from the object constants of M).

Proof: First, we prove the easier right-to-left inclusion. So let $m \in \mathsf{Inv}(M)$ with inverse $m^{-1} \in M$. We must show that $[c_m \mathsf{x} c_{m^{-1}}] \in G_{\mathbb{T}_{\mathsf{Mon}}}(M)$, i.e. (by Corollary 2.3.3) we must show that $[c_m \mathsf{x} c_{m^{-1}}]$ is invertible and commutes generically with the function symbols $\cdot$ and e of Σ_{Mon}. It is immediate that $[c_m \mathsf{x} c_{m^{-1}}]$ is invertible, because we have $[c_{m^{-1}} \mathsf{x} c_m] \in M\langle \mathsf{x} \rangle$ with

$$[c_m \mathsf{x} c_{m^{-1}} [c_{m^{-1}} \mathsf{x} c_m / \mathsf{x}]] = [c_m c_{m^{-1}} \mathsf{x} c_{m^{-1}} c_m] = [e \mathsf{x} e] = [\mathsf{x}],$$

and similarly $[c_{m^{-1}} \mathsf{x} c_m [c_m \mathsf{x} c_{m^{-1}} / \mathsf{x}]] = [\mathsf{x}]$. To show that $[c_m \mathsf{x} c_{m^{-1}}]$ commutes generically with $\cdot$, we must show that $[c_m \mathsf{x}_1 \mathsf{x}_2 c_{m^{-1}}] = [c_m \mathsf{x}_1 c_{m^{-1}} c_m \mathsf{x}_2 c_{m^{-1}}]$ holds in $M\langle \mathsf{x}_1, \mathsf{x}_2 \rangle$, which is clear. Finally, it is obvious that $[c_m \mathsf{x} c_{m^{-1}}]$ commutes generically with e, because in M we have

$$me^M m^{-1} = mm^{-1} = e^M.$$

This proves that $m \in \mathsf{Inv}(M) \implies [c_m \mathsf{x} c_{m^{-1}}] \in G_{\mathbb{T}_{\mathsf{Mon}}}(M)$.

For the converse inclusion, let w be a multiplicative word over $M \cup \{\mathsf{x}\}$ with $[w] \in G_{\mathbb{T}_{\mathsf{Mon}}}(M) \subseteq M\langle \mathsf{x} \rangle$. By Lemma 3.2.2, we may suppose that w is reduced. The argument now echoes the argument of Bergman ([3, Theorem 1]) given in Section 2.1. First, we show that w contains at most one occurrence of x. Suppose towards a contradiction that w contained at least two occurrences of x. Since $[w]$ commutes generically with $\cdot$, we obtain

$$[w[\mathsf{x}_1 \mathsf{x}_2 / \mathsf{x}]] = [w[\mathsf{x}_1 / \mathsf{x}] w[\mathsf{x}_2 / \mathsf{x}]]$$

in $M\langle \mathsf{x}_1, \mathsf{x}_2 \rangle$. Since w is reduced, it is clear that $w[\mathsf{x}_1 \mathsf{x}_2 / \mathsf{x}]$ is reduced. By Lemma 3.2.2, we have

$$[w[\mathsf{x}_1 \mathsf{x}_2 / \mathsf{x}]] = [w[\mathsf{x}_1 / \mathsf{x}] w[\mathsf{x}_2 / \mathsf{x}]] = [(w[\mathsf{x}_1 / \mathsf{x}] w[\mathsf{x}_2 / \mathsf{x}])^r]$$

and hence

$$w[\mathsf{x}_1 \mathsf{x}_2 / \mathsf{x}] \equiv (w[\mathsf{x}_1 / \mathsf{x}] w[\mathsf{x}_2 / \mathsf{x}])^r.$$

But since w contains at least two occurrences of x, it is clear that $w[\mathsf{x}_1 \mathsf{x}_2 / \mathsf{x}]$ will contain an occurrence of x_2 to the left of an occurrence of x_1, while all occurrences of x_2 will occur to the right of all occurrences of x_1 in $(w[\mathsf{x}_1/\mathsf{x}]w[\mathsf{x}_2/\mathsf{x}])^r$. This is impossible if these words are syntactically identical, so we conclude that w must have at most one occurrence of x.

To show that w must have at least one occurrence of x, it suffices by Lemma 2.2.56 (since $M \neq \emptyset$) to show that $\mathbb{T}_{\mathsf{Mon}}(M) \nvdash^{y_1,y_2} y_1 = y_2$ for distinct variables y_1, y_2 of the unique sort of Σ_{Mon}, but this follows by Lemma 3.2.3.

So w contains exactly one occurrence of x and hence, being reduced, must have one of the following forms:

$$\mathsf{x}, c_m\mathsf{x}, \mathsf{x}c_m, c_{m_1}\mathsf{x}c_{m_2}$$

(with $m, m_1, m_2 \neq e^M$). If $w \equiv \mathsf{x}$, then since $e^M \in \mathsf{Inv}(M)$ with $(e^M)^{-1} = e^M$, we have

$$[w] = [\mathsf{x}] = [e\mathsf{x}e] = [c_{e^M}\mathsf{x}c_{e^M}] = \left[c_{e^M}\mathsf{x}c_{(e^M)^{-1}}\right],$$

as desired. Suppose we had $w \equiv c_m\mathsf{x}$ for some $m \neq e^M \in M$: then since $[w]$ commutes generically with $\cdot$, we would have

$$[c_m\mathsf{x}_1\mathsf{x}_2] = [c_m\mathsf{x}_1c_m\mathsf{x}_2]$$

in $M\langle\mathsf{x}_1, \mathsf{x}_2\rangle$. But since both words are reduced, we would obtain $c_m\mathsf{x}_1\mathsf{x}_2 \equiv c_m\mathsf{x}_1c_m\mathsf{x}_2$ by Lemma 3.2.2, which is clearly false. Similarly, we cannot have $w \equiv \mathsf{x}c_m$ for $m \neq e^M \in M$. The last possibility is that $w \equiv c_{m_1}\mathsf{x}c_{m_2}$ for some $m_1, m_2 \neq e^M \in M$. So $[w] = [c_{m_1}\mathsf{x}c_{m_2}]$, and it just remains to show that m_1 is invertible with $m_1^{-1} = m_2$. Since $[w]$ commutes generically with $\cdot$, we have

$$[c_{m_1}\mathsf{x}_1\mathsf{x}_2c_{m_2}] = [c_{m_1}\mathsf{x}_1c_{m_2}c_{m_1}\mathsf{x}_2c_{m_2}] = [c_{m_1}\mathsf{x}_1c_{m_2m_1}\mathsf{x}_2c_{m_2}]$$

in $M\langle\mathsf{x}_1, \mathsf{x}_2\rangle$. The leftmost word is clearly reduced, and if the rightmost word *were* reduced, then these two clearly syntactically distinct words would be syntactically identical by Lemma 3.2.2. Hence, the rightmost word must be *not* reduced, which implies that $m_2m_1 = e^M$. Also, since $[w]$ commutes generically with e, it follows that in M we have

$$m_1m_2 = m_1e^Mm_2 = e^M.$$

So we have $m_1m_2 = e^M = m_2m_1$, and hence $m_1^{-1} = m_2$, as claimed. This proves that $[w] \in G_{\mathbb{T}_{\mathsf{Mon}}}(M) \implies [w] = [c_m\mathsf{x}c_{m^{-1}}]$ for some $m \in \mathsf{Inv}(M)$, which completes the proof of the proposition. ∎

Corollary 3.2.5. *If M is any monoid and*

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$$

is any $\mathsf{Dom}(M)$*-indexed family of monoid endomorphisms, then* $\pi \in \mathscr{Z}_{\mathbb{T}_{\mathsf{Mon}}}(M)$ *iff there is a (uniquely determined) element* $m \in \mathsf{Inv}(M)$ *such that*

$$\pi_f = \mathsf{conj}_{f(m)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)$$

for every $f \in \mathsf{Dom}(M)$*, where* $\mathsf{conj}_{f(m)}$ *is the monoid automorphism given by conjugation by* $f(m) \in \mathsf{Inv}(\mathsf{cod}(f))$*.*

Proof: By Proposition 3.2.4 and Corollary 2.2.42, it suffices to show for $m \in \mathsf{Inv}(M)$ and $f : M \to N$ a monoid homomorphism that

$$\rho_f \left(c_m \mathsf{x} c_{m^{-1}}\right)^* = \mathsf{conj}_{f(m)} : N \xrightarrow{\sim} N,$$

and that $m_1, m_2 \in \mathsf{Inv}(M)$ and $\left[c_{m_1} \mathsf{x} c_{m_1^{-1}}\right] = \left[c_{m_2} \mathsf{x} c_{m_2^{-1}}\right]$ imply $m_1 = m_2$. First, for any $n \in N$ we have

$$\begin{aligned}\rho_f \left(c_m \mathsf{x} c_{m^{-1}}\right)^* (n) &= \left(c_{f(m)} \mathsf{x} c_{f(m)^{-1}}\right)^* (n)\\ &= \left(c_{f(m)} \mathsf{x} c_{f(m)^{-1}}\right)^{(\widehat{N},n)}\\ &= f(m) n f(m)^{-1}\\ &= \mathsf{conj}_{f(m)}(n),\end{aligned}$$

as desired.

Now suppose $m_1, m_2 \in \mathsf{Inv}(M)$ and $\left[c_{m_1} \mathsf{x} c_{m_1^{-1}}\right] = \left[c_{m_2} \mathsf{x} c_{m_2^{-1}}\right]$. If $m_1 = e^M$, then we have $[\mathsf{x}] = \left[c_{m_1} \mathsf{x} c_{m_1^{-1}}\right] = \left[c_{m_2} \mathsf{x} c_{m_2^{-1}}\right]$, which forces $m_2 = e^M$ by Lemma 3.2.2. Otherwise, if $m_1 \neq e^M$, then for similar reasons we must have $m_2 \neq e^M$, so that the two words are reduced and hence

$$c_{m_1} \mathsf{x} c_{m_1^{-1}} \equiv c_{m_2} \mathsf{x} c_{m_2^{-1}}$$

by Lemma 3.2.2, which forces $c_{m_1} \equiv c_{m_2}$ and hence $m_1 = m_2$. ∎

Let $\mathsf{Inv} : \mathsf{Mon} \to \mathsf{Group}$ be the functor that sends any monoid to its group $\mathsf{Inv}(M)$ of invertible elements, and sends any monoid homomorphism $f : M \to N$ to the group homomorphism $\mathsf{Inv}(f) := f \restriction \mathsf{Inv}(M) : \mathsf{Inv}(M) \to \mathsf{Inv}(N)$. Then we also have:

Corollary 3.2.6. $\mathscr{Z}_{\mathbb{T}_{\mathsf{Mon}}} \cong \mathsf{Inv} : \mathsf{Mon} \to \mathsf{Group}$.

Proof: By Theorem 2.2.41, it suffices to show that there is a natural isomorphism $G_{\mathbb{T}_{\mathsf{Mon}}} \cong \mathsf{Inv} : \mathsf{Mon} \to \mathsf{Group}$. For any monoid M, we define a group isomorphism

$$\beta_M : G_{\mathbb{T}_{\mathsf{Mon}}}(M) \xrightarrow{\sim} \mathsf{Inv}(M)$$

by appealing to Proposition 3.2.4 and setting

$$\beta_M\left([c_m \mathsf{x} c_{m^{-1}}]\right) := m \in \mathsf{Inv}(M)$$

for each $m \in \mathsf{Inv}(M)$. This is well-defined, by the argument given in the proof of Corollary 3.2.5. It is clearly injective, and it is surjective by Proposition 3.2.4. Finally, it is a group homomorphism, because $m_1, m_2 \in \mathsf{Inv}(M)$ implies

$$\left[c_{m_1} \mathsf{x} c_{m_1^{-1}}\right] * \left[c_{m_2} \mathsf{x} c_{m_2^{-1}}\right] = \left[c_{m_1 m_2} \mathsf{x} c_{(m_1 m_2)^{-1}}\right].$$

If we set $\beta := (\beta_M)_{M \in \mathsf{Mon}}$, then β is natural, because for $f : M \to N \in \mathsf{Mon}$ and $m \in \mathsf{Inv}(M)$ we have

$$\begin{aligned}
\mathsf{Inv}(f)\left(\beta_M\left([c_m \mathsf{x} c_{m^{-1}}]\right)\right) &= \mathsf{Inv}(f)(m) \\
&= f(m) \\
&= \beta_N\left(\left[c_{f(m)} \mathsf{x} c_{f(m)^{-1}}\right]\right) \\
&= \beta_N\left(\left[\rho_f\left(c_m \mathsf{x} c_{m^{-1}}\right)\right]\right) \\
&= \beta_N\left(G_{\mathbb{T}_{\mathsf{Mon}}}(f)\left([c_m \mathsf{x} c_{m^{-1}}]\right)\right).
\end{aligned}$$

So $\beta : G_{\mathbb{T}_{\mathsf{Mon}}} \overset{\sim}{\Longrightarrow} \mathsf{Inv}$. ∎

This completes our investigation of the isotropy group of the totally defined algebraic theory of monoids. By making some minor adjustments, the above results carry over almost verbatim to the totally defined theory $\mathbb{T}_{\mathsf{Group}}$ of *groups*, whose axioms were given in Example 2.3.2 (cf. also Bergman's argument from [3, Theorem 1] in Section 2.1):

Proposition 3.2.7. *If G is any group, then*

$$G_{\mathbb{T}_{\mathsf{Group}}}(G) = \left\{[c_g \mathsf{x} c_{g^{-1}}] \in G\langle \mathsf{x} \rangle : g \in G\right\}.$$

Corollary 3.2.8. *If G is any group and*

$$\pi = \left(\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(G)}$$

is any $\mathsf{Dom}(G)$-indexed family of group endomorphisms, then $\pi \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Group}}}(G)$ iff there is a (uniquely determined) element $g \in G$ such that

$$\pi_f = \mathsf{conj}_{f(g)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)$$

for every $f \in \mathsf{Dom}(G)$, where $\mathsf{conj}_{f(g)}$ is the group automorphism given by conjugation by $f(g) \in \mathsf{cod}(f)$.

Corollary 3.2.9. $\mathcal{Z}_{\mathbb{T}_{\mathsf{Group}}} \cong \mathbb{1}_{\mathsf{Group}} : \mathsf{Group} \to \mathsf{Group}$. ∎

3.3 Commutative Monoids and Abelian Groups

In this section, we show that the theory $\mathbb{T}_{\mathsf{CMon}}$ of totally defined *commutative* monoids has trivial isotropy, and that the isotropy group of the theory $\mathbb{T}_{\mathsf{Ab}}$ of totally defined *abelian* groups is (naturally isomorphic to) the constant functor on the two-element group $\mathbb{Z}_2$.

First, we consider the theory $\mathbb{T}_{\mathsf{Ab}}$ over the single-sorted signature $\Sigma_{\mathsf{Ab}} := \{+, -, 0\}$, with binary $+$, unary $-$, and constant symbol 0. The axioms of $\mathbb{T}_{\mathsf{Ab}}$ are those of $\mathbb{T}_{\mathsf{Group}}$ (replacing $\cdot, ^{-1}, e$ respectively by $+, -, 0$) together with the axiom

$$\top \vdash^{x,y} x + y = y + x.$$

Note that $\mathsf{P}\mathbb{T}_{\mathsf{Ab}}\mathsf{mod} = \mathsf{Ab}$, the category of abelian groups and homomorphisms. We now have analogues of Definition 3.2.1 and Lemma 3.2.2:

Definition 3.3.1 (**Reduced Additive Words**). Let G be any abelian group, and let $\mathsf{x} \notin \Sigma_{\mathsf{Ab}}(G)$ be a constant of the unique sort of Σ_{Ab}. A *reduced additive word* over $G \cup \{\mathsf{x}\}$ is an expression of one of the following forms:

- $n\mathsf{x}$, for $n \in \mathbb{Z} \setminus \{0\}$.
- c_g, for $g \in G$.
- $n\mathsf{x} + c_g$, for $n \in \mathbb{Z} \setminus \{0\}$ and $g \neq 0^G \in G$. ∎

Lemma 3.3.2. *Let G be any abelian group, and let $\mathsf{x} \notin \Sigma_{\mathsf{Ab}}(G)$ be a constant of the unique sort of Σ_{Ab}.*

- *If $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Ab}}(G, \mathsf{x}))$, then there is a (unique) reduced additive word t^r over $G \cup \{\mathsf{x}\}$ such that $[t] = [t^r]$ in $G\langle \mathsf{x} \rangle$.*
- *If w, w' are reduced additive words over $G \cup \{\mathsf{x}\}$ with $[w] = [w']$, then $w \equiv w'$.* ∎

Lemma 3.3.3. *If G is any abelian group and y_1, y_2 are distinct variables of the unique sort of Σ_{Ab}, then $\mathbb{T}_{\mathsf{Ab}}(G) \nvdash^{y_1, y_2} y_1 = y_2$.* ∎

Proposition 3.3.4. *If G is any abelian group, then*

$$G_{\mathbb{T}_{\mathsf{Ab}}}(G) = \{[\mathsf{x}], [-\mathsf{x}]\} \subseteq G\langle \mathsf{x} \rangle.$$

Proof: First we show the easy right-to-left inclusion. Of course $[\mathsf{x}] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$. To show that $[-\mathsf{x}] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$, we must show (by Corollary 2.3.3) that $[-\mathsf{x}]$ is invertible and commutes generically with the function symbols $+, -, 0$ of Σ_{Ab}. The inverse of $[-\mathsf{x}]$ is itself, because we clearly have

$$[-\mathsf{x}[-\mathsf{x}/\mathsf{x}]] = [-(-\mathsf{x})] = [\mathsf{x}].$$

That $[-\mathsf{x}]$ commutes generically with $-$ is trivial. That $[-\mathsf{x}]$ commutes generically with $+$ follows because we have

$$[-(\mathsf{x}_1 + \mathsf{x}_2)] = [(-\mathsf{x}_1) + (-\mathsf{x}_2)] \in G\langle \mathsf{x}_1, \mathsf{x}_2 \rangle.$$

And that $[-\mathsf{x}]$ commutes generically with 0 follows because in G we have $-0^G = 0^G$. So $[-\mathsf{x}] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$.

For the converse inclusion, let $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Ab}}(G, \mathsf{x}))$ with $[t] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$, and suppose without loss of generality by Lemma 3.3.2 that t is a reduced additive word over $G \cup \{\mathsf{x}\}$. First, we show that we must have $t \equiv n\mathsf{x}$ for some $n \in \mathbb{Z} \setminus \{0\}$. For suppose otherwise: then (by Definition 3.3.1) either $t \equiv c_g$ for some $g \in G$, or $t \equiv n\mathsf{x} + c_g$ for some $n \in \mathbb{Z} \setminus \{0\}$ and $g \neq 0^G \in G$. Since $[t] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$, the first case is impossible by Lemmas 2.2.56 and 3.3.3 (since $G \neq \emptyset$). So suppose we are in the second case. Since $[t]$ commutes generically with 0, we easily obtain (in G)

$$0^G = n0^G + g = 0^G + g = g,$$

which contradicts the assumption on g. So we must have $[t] = [n\mathsf{x}]$ for some $n \in \mathbb{Z} \setminus \{0\}$. To complete the proof, it thus remains to show that $n \in \{1, -1\}$. Since $[t] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$, there is some $[s] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$ with $[t[s/\mathsf{x}]] = [\mathsf{x}]$. By the argument just used for $[t] \in G_{\mathbb{T}_{\mathsf{Ab}}}(G)$, we know that $[s] = [m\mathsf{x}]$ for some $m \in \mathbb{Z} \setminus \{0\}$. So we have $[\mathsf{x}] = [t[s/\mathsf{x}]] = [n\mathsf{x}[m\mathsf{x}/\mathsf{x}]] = [(nm)\mathsf{x}]$. Since $n, m \neq 0$, we have $nm \neq 0$, and hence $(nm)\mathsf{x}$ is a reduced additive word over $G \cup \{\mathsf{x}\}$. So by Lemma 3.3.2, it follows that $\mathsf{x} \equiv (nm)\mathsf{x}$, so that we must have $nm = 1$, and hence $n \in \{1, -1\}$, as desired. ∎

Corollary 3.3.5. *If G is any abelian group and*

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(G)}$$

is any $\mathsf{Dom}(G)$-indexed family of abelian group endomorphisms, then $\pi \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Ab}}}(G)$ iff either

$$\pi_f = \mathsf{id}_{\mathsf{cod}(f)} \textit{ for each } f \in \mathsf{Dom}(G)$$

or

$$\pi_f = -^{\mathsf{cod}(f)} \textit{ for each } f \in \mathsf{Dom}(G),$$

where $-^{\mathsf{cod}(f)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)$ *is the abelian group automorphism given by* $a \mapsto -a$.
∎

Corollary 3.3.6. $\mathcal{Z}_{\mathbb{T}_{\mathsf{Ab}}} : \mathsf{Ab} \to \mathsf{Group}$ *is (naturally isomorphic to) the constant functor on the two-element group* $\mathbb{Z}_2$. ∎

This completes our investigation of the isotropy group of the totally defined theory of abelian groups. Let us now consider the theory $\mathbb{T}_{\mathsf{CMon}}$ of totally defined commutative monoids over the single-sorted signature $\Sigma_{\mathsf{CMon}} := \{+, 0\}$, with binary $+$ and constant 0. The axioms of $\mathbb{T}_{\mathsf{CMon}}$ are those of $\mathbb{T}_{\mathsf{Mon}}$ (replacing $\cdot, e$ respectively by $+, 0$) together with the axiom $\top \vdash^{x,y} x + y = y + x$. We modify Definition 3.3.1 by stipulating that in the first and third forms, n must be a *positive* integer. Lemmas 3.3.2 and 3.3.3 then hold for $\mathbb{T}_{\mathsf{CMon}}$. By making the appropriate adjustments to the proof of Proposition 3.3.4, we then have:

Proposition 3.3.7. *If M is any commutative monoid, then*

$$G_{\mathbb{T}_{\mathsf{CMon}}}(M) = \{[\mathsf{x}]\} \subseteq M\langle \mathsf{x} \rangle.$$

Corollary 3.3.8. *If M is any commutative monoid, then $\mathcal{Z}_{\mathbb{T}_{\mathsf{CMon}}}(M)$ is the trivial group consisting of only the identity element*

$$\left(\mathsf{id}_{\mathsf{cod}(f)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(M)} \in \mathcal{Z}_{\mathbb{T}_{\mathsf{CMon}}}(M).$$

∎

Since $\mathsf{P}\mathbb{T}_{\mathsf{CMon}}\mathsf{mod} = \mathsf{CMon}$, the category of commutative monoids and homomorphisms, we have:

Corollary 3.3.9. $\mathcal{Z}_{\mathbb{T}_{\mathsf{CMon}}} : \mathsf{CMon} \to \mathsf{Group}$ *is (naturally isomorphic to) the constant functor on the trivial group.* ∎

3.4 Commutative Unital Rings

In this section, we will show that the isotropy group of the totally defined algebraic theory of commutative rings with unit. The theory $\mathbb{T}_{\mathsf{CRing}}$ is the totally defined algebraic theory over the single-sorted signature $\Sigma_{\mathsf{Ring}} := \{+, -, 0, \cdot, 1\}$ (with $+$ binary, $-$ unary, and $0, 1$ constants) whose axioms are those of $\mathbb{T}_{\mathsf{Ab}}$, together with the axioms of $\mathbb{T}_{\mathsf{CMon}}$ (replacing $+, 0$ by $\cdot, 1$), and the distributive axiom

$$\top \vdash^{x,y,z} x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

Then $\mathsf{P}\mathbb{T}_{\mathsf{CRing}}\mathsf{mod} = \mathsf{CRing}$, the usual category of commutative unital rings and ring homomorphisms.

Definition 3.4.1 (Monomials). Let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma_{\mathsf{Ring}}$ be pairwise distinct constants of the unique sort of Σ_{Ring}. A *monomial* over $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ is an expression of the form $\mathsf{x}_1^{a_1} \ldots \mathsf{x}_n^{a_n}$ for integers $a_1, \ldots, a_n \geq 0$. If there is some $1 \leq i \leq n$ with $a_i > 0$, then we say that $\mathsf{x}_1^{a_1} \ldots \mathsf{x}_n^{a_n}$ is a *non-trivial* monomial over $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$. ∎

We have the following well-known description of the elements of $R\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$:

Lemma 3.4.2. *Let R be any commutative unital ring, let $n \geq 1$, and let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma_{\mathsf{Ring}}(R)$ be pairwise distinct constants of the unique sort of Σ_{Ring}.*

- *If $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Ring}}(R, \mathsf{x}_1, \ldots, \mathsf{x}_n))$, then there are $m \geq 1$ and pairwise distinct monomials $t_1, \ldots, t_m$ over $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ and elements $r_1, \ldots, r_m \in R$ with*

$$[t] = [c_{r_1} t_1 + \ldots + c_{r_m} t_m] \in R\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle.$$

- *If $j, k \geq 1$ and $s_1, \ldots, s_j, t_1, \ldots, t_k$ are monomials over $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ (with $s_1, \ldots, s_j$ pairwise distinct and $t_1, \ldots, t_k$ pairwise distinct) and $a_1, \ldots, a_j, b_1, \ldots, b_k \in R$ have the property that*

$$[c_{a_1} s_1 + \ldots + c_{a_j} s_j] = [c_{b_1} t_1 + \ldots + c_{b_k} t_k] \in R\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle,$$

then for any $1 \leq i \leq j$ such that $a_i \neq 0^R$ and s_i is a non-trivial monomial, there must be some $1 \leq i' \leq k$ with $b_{i'} \neq 0^R$ and $s_i \equiv t_{i'}$, and conversely.

∎

Proposition 3.4.3. *If R is any commutative unital ring, then*

$$G_{\mathbb{T}_{\mathsf{CRing}}}(R) = \{[\mathsf{x}]\} \subseteq R\langle \mathsf{x} \rangle.$$

Proof: Let $[t] \in G_{\mathbb{T}_{\mathsf{CRing}}}(R) \subseteq R\langle \mathsf{x} \rangle$, and let us show that we must have $[t] = [\mathsf{x}]$. By Lemma 3.4.2, we have

$$[t] = [c_{a_n}\mathsf{x}^n + \ldots + c_{a_1}\mathsf{x} + c_{a_0}] \in R\langle \mathsf{x} \rangle$$

for some $n \geq 0$ and $a_0, \ldots, a_n \in R$. Since $[t] \in G_{\mathbb{T}_{\mathsf{CRing}}}(R)$, we know that $[t]$ commutes generically with the function symbol $+$ of Σ_{Ring}, which means that

$$\mathbb{T}_{\mathsf{CRing}}(R, \mathsf{x}_1, \mathsf{x}_2) \vdash t[\mathsf{x}_1 + \mathsf{x}_2/\mathsf{x}] = t[\mathsf{x}_1/\mathsf{x}] + t[\mathsf{x}_2/x],$$

so that

$$\begin{aligned} &[c_{a_n}(\mathsf{x}_1 + \mathsf{x}_2)^n + \ldots + c_{a_1}(\mathsf{x}_1 + \mathsf{x}_2) + c_{a_0}] \\ &= [c_{a_n}\mathsf{x}_1^n + c_{a_n}\mathsf{x}_2^n + \ldots + c_{a_1}\mathsf{x}_1 + c_{a_1}\mathsf{x}_2 + c_{a_0+a_0}] \end{aligned}$$

in $R\langle \mathsf{x}_1, \mathsf{x}_2 \rangle$. Now we show for any $1 < i \leq n$ that we must have $a_i = 0^R$. If $i > 1$ and $a_i \neq 0^R$, then (by the binomial theorem for commutative rings) the upper term is congruent to a term of the canonical form described in Lemma 3.4.2 in which the monomial $\mathsf{x}_1^{i-1}\mathsf{x}_2$ occurs with non-zero coefficient from R. By Lemma 3.4.2, it then follows that the lower term must also contain $\mathsf{x}_1^{i-1}\mathsf{x}_2$ with non-zero coefficient from R, which is not the case. So if $1 < i \leq n$, then we must have $a_i = 0^R$. We must also have $a_0 = 0^R$, because otherwise (by Lemma 3.4.2) we would obtain $a_0 = a_0 + a_0$ and hence $a_0 = 0^R$, contrary to assumption.

Thus, we have

$$[t] = [c_a\mathsf{x}] \in R\langle \mathsf{x} \rangle$$

for some $a \in R$. Since $[t] \in G_{\mathbb{T}_{\mathsf{CRing}}}(R)$, we also know that $[t]$ commutes generically with the function symbol 1 of Σ_{Ring}, which entails that

$$\mathbb{T}_{\mathsf{CRing}}(R) \vdash c_a = 1,$$

and hence that

$$[t] = [c_a\mathsf{x}] = [1\mathsf{x}] = [\mathsf{x}] \in R\langle \mathsf{x} \rangle,$$

as desired. This completes the proof. ∎

Corollary 3.4.4. *If R is any commutative unital ring, then $\mathcal{Z}_{\mathbb{T}_{\mathsf{CRing}}}(R)$ is the trivial group consisting of only the identity element*

$$\left(\mathsf{id}_{\mathsf{cod}(f)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(R)} \in \mathcal{Z}_{\mathbb{T}_{\mathsf{CRing}}}(R).$$

∎

Corollary 3.4.5. *$\mathcal{Z}_{\mathbb{T}_{\mathsf{CRing}}} : \mathsf{CRing} \to \mathsf{Group}$ is (naturally isomorphic to) the constant functor on the trivial group.* ∎

3.5 Modules over a Ring

In this section, we will compute the isotropy group of the totally defined algebraic theory of (left) R-modules over a fixed commutative unital ring R. So fix a commutative unital ring R. We define a single-sorted signature $\Sigma_{R\mathsf{mod}}$ to have function symbols $+$ (binary), $-$ (unary), and 0 (constant), together with a unary function symbol f_r for each $r \in R$ (these latter symbols being pairwise distinct). The totally defined algebraic theory $\mathbb{T}_{R\mathsf{mod}}$ over the signature $\Sigma_{R\mathsf{mod}}$ then has the axioms of $\mathbb{T}_{\mathsf{Ab}}$ together with the following axioms for all $r, s \in R$:

- $\top \vdash^{x} f_r(f_s(x)) = f_{rs}(x)$.
- $\top \vdash^{x,y} f_r(x+y) = f_r(x) + f_r(y)$.
- $\top \vdash^{x} f_{r+s}(x) = f_r(x) + f_s(x)$.
- $\top \vdash^{x} f_{1^R}(x) = x$.

From the well-known construction of the free R-module on finitely many generators and the coproduct of R-modules as their direct sum, we then have:

Lemma 3.5.1. *Let M be a left R-module, and let $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma_{R\mathsf{mod}}(M)$ be new constants. For any $t \in \mathsf{Term}^c(\Sigma_{R\mathsf{mod}}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n))$, there are uniquely determined elements $m \in M$ and $r_1, \ldots, r_n \in R$ with*

$$[t] = [c_m + f_{r_1}(\mathsf{x}_1) + \ldots + f_{r_n}(\mathsf{x}_n)] \in M\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle.$$

∎

Let $\mathsf{Unit}(R)$ be the group of units of R (i.e. elements of R that have multiplicative inverses). We now have:

Proposition 3.5.2. *If $M \in R\mathsf{mod}$, then*

$$G_{\mathbb{T}_{R\mathsf{mod}}}(M) = \{[f_r(\mathsf{x})] \in M\langle \mathsf{x} \rangle : r \in \mathsf{Unit}(R)\}.$$

Proof: To prove the right-to-left inclusion, it suffices by Corollary 2.3.3 to show that if $r \in \mathsf{Unit}(R)$, then $[f_r(\mathsf{x})] \in M\langle \mathsf{x} \rangle$ is invertible and commutes generically with the function symbols of $\mathbb{T}_{R\mathsf{mod}}$. Since r is a unit of R, there is some $s \in R$ with $r \cdot^R s = 1^R = s \cdot^R r$. Then $[f_s(\mathsf{x})] \in M\langle \mathsf{x} \rangle$ is the inverse of $[f_r(\mathsf{x})]$, because in $M\langle \mathsf{x} \rangle$ we have

$$[f_r(\mathsf{x})\,[f_s(\mathsf{x})/\mathsf{x}]] = [f_r(f_s(\mathsf{x}))] = [f_{rs}(\mathsf{x})] = [f_{1^R}(\mathsf{x})] = [\mathsf{x}],$$

and conversely. That $[f_r(\mathsf{x})]$ commutes generically with the function symbols of $\Sigma_{R\mathsf{mod}}$ is trivial to verify, using the axioms (and theorems) of $\mathbb{T}_{R\mathsf{mod}}$ (for the function symbols $+, -, 0$) and the commutativity of R (for the function symbols of the form f_s for $s \in R$).

For the less obvious inclusion, let $[t] \in G_{\mathbb{T}_{R\mathsf{mod}}}(M) \subseteq M\langle \mathsf{x} \rangle$, and let us show that there is some $r \in \mathsf{Unit}(R)$ with $[t] = [f_r(\mathsf{x})]$. By Lemma 3.5.1, there are uniquely determined elements $m \in M$ and $r \in R$ with $[t] = [c_m + f_r(\mathsf{x})] \in M\langle \mathsf{x} \rangle$. Since $[t]$ is an element of the isotropy group, it follows that $[t]$ commutes generically with the function symbol $+$, which means that in $M\langle \mathsf{x}_1, \mathsf{x}_2 \rangle$ we will have

$$[c_m + f_r(\mathsf{x}_1) + f_r(\mathsf{x}_2)] = [c_{m+m} + f_r(\mathsf{x}_1) + f_r(\mathsf{x}_2)].$$

By Lemma 3.5.1, it then follows that $m = m + m$, so that $m = 0^M$, and hence

$$[t] = [c_{0^M} + f_r(\mathsf{x})] = [0 + f_r(\mathsf{x})] = [f_r(\mathsf{x})] \in M\langle \mathsf{x} \rangle,$$

so it remains to show that r is a unit of R. Since $[t]$ is an element of the isotropy group, we know that $[t]$ is invertible, and so there is some $[s] \in G_{\mathbb{T}_{R\mathsf{mod}}}(M) \subseteq M\langle \mathsf{x} \rangle$ with

$$[s[t/\mathsf{x}]] = [\mathsf{x}] = [t[s/\mathsf{x}]] \in M\langle \mathsf{x} \rangle.$$

Since $[s]$ is an element of the isotropy group, it follows by the reasoning used thus far for $[t]$ that

$$[s] = [f_{r'}(\mathsf{x})] \in M\langle \mathsf{x} \rangle$$

for some $r' \in R$. Since $[s]$ is the inverse of $[t]$, we then have

$$[f_{1^R}(\mathsf{x})] = [\mathsf{x}] = [t[s/\mathsf{x}]] = [f_r(f_{r'}(\mathsf{x}))] = [f_{rr'}(\mathsf{x})],$$

which implies by Lemma 3.5.1 that $r \cdot^R r' = 1^R$. Similarly, we obtain $r' \cdot^R r = 1^R$, so that $r \in \mathsf{Unit}(R)$, as desired. This completes the proof. ∎

Corollary 3.5.3. *If M is any left R-module and*

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$$

is any $\mathsf{Dom}(M)$-indexed family of R-module endomorphisms, then $\pi \in \mathcal{Z}_{\mathbb{T}_{R\mathsf{mod}}}(M)$ iff there is a (uniquely determined) element $r \in \mathsf{Unit}(R)$ such that

$$\pi_f = f_r^{\mathsf{cod}(f)} : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)$$

for every $f \in \mathsf{Dom}(M)$.

Proof: By Proposition 3.5.2 and Corollary 2.2.42, it remains to show that if $r, s \in \mathsf{Unit}(R)$ and $[f_r(\mathsf{x})] = [f_s(\mathsf{x})] \in M\langle \mathsf{x} \rangle$, then $r = s$, which follows by Lemma 3.5.1. ▮

Corollary 3.5.4. *$\mathcal{Z}_{\mathbb{T}_{R\mathsf{mod}}} : R\mathsf{mod} \to \mathsf{Group}$ is naturally isomorphic to the constant functor on $\mathsf{Unit}(R)$.* ▮

3.6 Sets with a Bijection

We will now compute the isotropy group of a totally defined algebraic theory whose models are sets equipped with a (total) bijection. Specifically, we define the single-sorted signature $\Sigma_{\mathsf{Bij}} := \{f, f^{-1}\}$ with f, f^{-1} unary function symbols, and we define $\mathbb{T}_{\mathsf{Bij}}$ to be the totally defined algebraic theory over Σ_{Bij} with the following axioms:

$$\top \vdash^x f(x) \downarrow \wedge f^{-1}(x) \downarrow,$$

$$\top \vdash^x f(f^{-1}(x)) = x = f^{-1}(f(x)).$$

We will show that the isotropy group of this theory is (naturally isomorphic to) the constant functor on the additive group $\mathbb{Z}$. If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$, then for any $n \in \mathbb{Z}$, we define the term $f^n(t) \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$ in the obvious way.

First, we require the following preparatory lemmas, whose proofs may be found in Appendix B.

Lemma 3.6.1. *Let $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$, and let $\mathsf{x} \notin \Sigma_{\mathsf{Bij}}(M)$ be a constant of the unique sort of Σ_{Bij}. Then for any $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$, either there is some $m \in M$ such that $[t] = [c_m] \in M\langle \mathsf{x} \rangle$, or there is some $n \in \mathbb{Z}$ such that $[t] = [f^n(\mathsf{x})] \in M\langle \mathsf{x} \rangle$.* ▮

Lemma 3.6.2. *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and $\mathsf{x} \notin \Sigma_{\mathsf{Bij}}(M)$ is a constant symbol and for $n, m \in \mathbb{Z}$ we have $[f^n(\mathsf{x})] = [f^m(\mathsf{x})] \in M\langle \mathsf{x} \rangle$, then $n = m$.* ▮

Lemma 3.6.3. *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and y_1, y_2 are distinct variables of the unique sort of Σ_{Bij}, then $\mathbb{T}(M) \nvdash^{y_1, y_2} y_1 = y_2$.* ▮

Proposition 3.6.4. *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$, then*

$$G_{\mathbb{T}_{\mathsf{Bij}}}(M) = \{[f^n(\mathsf{x})] \in M\langle \mathsf{x} \rangle : n \in \mathbb{Z}\}.$$

Proof: To show the right-to-left inclusion, let $n \in \mathbb{Z}$; we must show that $[f^n(\mathsf{x})] \in G_{\mathbb{T}_{\mathsf{Bij}}}(M)$, i.e. (by Corollary 2.3.3) that $[f^n(\mathsf{x})]$ is invertible and commutes generically with the function symbols f, f^{-1}. For invertibility, we have $[f^{-n}(\mathsf{x})] \in M\langle \mathsf{x} \rangle$ and

$$\big[f^n(\mathsf{x})[f^{-n}(\mathsf{x})/\mathsf{x}]\big] = [f^{n-n}(\mathsf{x})] = [f^0(\mathsf{x})] = [\mathsf{x}],$$

and similarly $[f^{-n}(\mathsf{x})[f^n(\mathsf{x})/\mathsf{x}]] = [\mathsf{x}]$. We have that $[f^n(\mathsf{x})]$ commutes generically with f, because in $M\langle \mathsf{x} \rangle$ we have

$$[f^n(\mathsf{x})[f(\mathsf{x})/\mathsf{x}]] = [f^n(f(\mathsf{x}))] = [f^{n+1}(\mathsf{x})] = [f(f^n(\mathsf{x}))],$$

and $[f^n(\mathsf{x})]$ commutes generically with f^{-1} by similar reasoning. Thus, $[f^n(\mathsf{x})] \in G_{\mathbb{T}_{\mathsf{Bij}}}(M)$.

Conversely, suppose that $[t] \in G_{\mathbb{T}_{\mathsf{Bij}}}(M)$. By Lemma 3.6.1, either there is some $m \in M$ with $[t] = [c_m]$, or there is some $n \in \mathbb{Z}$ with $[t] = [f^n(\mathsf{x})]$. The first case is ruled out by Lemmas 2.2.56 and 3.6.3 (since in this case $M \neq \emptyset$), so we are left with the second case, as desired. ∎

Corollary 3.6.5. *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and*

$$\pi = (\pi_h : \mathsf{cod}(h) \to \mathsf{cod}(h))_{h \in \mathsf{Dom}(M)}$$

is any $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$, then $\pi \in \mathscr{Z}_{\mathbb{T}_{\mathsf{Bij}}}(M)$ iff there is a (unique) integer $n \in \mathbb{Z}$ such that

$$\pi_h = \left(f^{\mathsf{cod}(h)}\right)^n : \mathsf{cod}(h) \xrightarrow{\sim} \mathsf{cod}(h)$$

for every $h \in \mathsf{Dom}(M)$.

Proof: By Corollary 2.2.42, Proposition 3.6.4, and Lemma 3.6.2. ∎

Corollary 3.6.6. *$\mathscr{Z}_{\mathbb{T}_{\mathsf{Bij}}} \cong G_{\mathbb{T}_{\mathsf{Bij}}} : \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod} \to \mathsf{Group}$ is naturally isomorphic to the constant functor on the additive group $\mathbb{Z}$.*

Proof: If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$, then (using Proposition 3.6.4) we define a group isomorphism $G_{\mathbb{T}_{\mathsf{Bij}}}(M) \xrightarrow{\sim} \mathbb{Z}$ by $[f^n(\mathsf{x})] \mapsto n$ for each $n \in \mathbb{Z}$. This is well-defined by Lemma 3.6.2, it is clearly injective, it is surjective by Proposition 3.6.4, and it is easily seen to be a group homomorphism, and hence isomorphism. Naturality of these group isomorphisms is trivial. ∎

If we define the single-sorted signature $\Sigma_{\mathsf{Inv}} := \{f\}$ with f unary and define $\mathbb{T}_{\mathsf{Inv}}$ to be the totally defined algebraic theory over Σ_{Inv} with the axioms

$$\top \vdash^{x} f(x) \downarrow$$

and

$$\top \vdash^{x} f(f(x)) = x,$$

then models of $\mathbb{T}_{\mathsf{Inv}}$ are sets with an involution. Then by easy modifications of the arguments given in this section, we obtain:

Proposition 3.6.7.

- *If* $M \in \mathsf{P}\mathbb{T}_{\mathsf{Inv}}\mathsf{mod}$*, then*

$$G_{\mathbb{T}_{\mathsf{Inv}}}(M) = \{[\mathsf{x}], [f(\mathsf{x})]\} \in M\langle \mathsf{x} \rangle.$$

- *If* $M \in \mathsf{P}\mathbb{T}_{\mathsf{Inv}}\mathsf{mod}$ *and* $\pi = (\pi_h : \mathsf{cod}(h) \to \mathsf{cod}(h))_{h \in \mathsf{Dom}(M)}$ *is any* $\mathsf{Dom}(M)$*-indexed family of endomorphisms in* $\mathsf{P}\mathbb{T}_{\mathsf{Inv}}\mathsf{mod}$*, then* $\pi \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Inv}}}(M)$ *iff either* π *is the identity element or*

$$\pi_h = f^{\mathsf{cod}(h)} : \mathsf{cod}(h) \xrightarrow{\sim} \mathsf{cod}(h)$$

  for every $h \in \mathsf{Dom}(M)$.

- $\mathcal{Z}_{\mathbb{T}_{\mathsf{Inv}}} \cong G_{\mathbb{T}_{\mathsf{Inv}}} : \mathsf{P}\mathbb{T}_{\mathsf{Inv}}\mathsf{mod} \to \mathsf{Group}$ *is naturally isomorphic to the constant functor on the group* $\mathbb{Z}_2$. ∎

3.7 Free Lattices

In this section, we will show that the isotropy group of any free lattice on finitely many generators is trivial. We define Σ_{Latt} to be the single-sorted signature with two binary function symbols $\wedge$ and $\vee$. We then define $\mathbb{T}_{\mathsf{Latt}}$ to be the totally defined algebraic theory over Σ_{Latt} with the following axioms:

- $\top \vdash^{x,y} x \wedge y \downarrow$.
- $\top \vdash^{x,y} x \vee y \downarrow$.

- $\top \vdash^{x,y} x \wedge y = y \wedge x$.
- $\top \vdash^{x,y} x \vee y = y \vee x$.
- $\top \vdash^{x,y,z} (x \wedge y) \wedge z = x \wedge (y \wedge z)$.
- $\top \vdash^{x,y,z} (x \vee y) \vee z = x \vee (y \vee z)$.
- $\top \vdash^{x,y} x \wedge (x \vee y) = x$.
- $\top \vdash^{x,y} x \vee (x \wedge y) = x$.

A well-known consequence of these axioms is that $\mathbb{T}_{\mathsf{Latt}}$ proves

$$\top \vdash^{x} x \wedge x = x = x \vee x.$$

The category $\mathsf{P}\mathbb{T}_{\mathsf{Latt}}\mathsf{mod}$ is then just the usual category Latt of lattices and lattice homomorphisms. If L is any lattice and $a, b \in L$, then we will write $a \leq_L b$ for $a \wedge^L b = a$, or equivalently $a \vee^L b = b$, which is known to be a partial order on L (i.e. reflexive, anti-symmetric, transitive).

Let us denote the free lattice on n generators $c_1, \ldots, c_n$ by L_n, i.e.

$$L_n = \mathsf{Free}(\mathbb{T}_{\mathsf{Latt}}(c_1, \ldots, c_n))|_{\Sigma_{\mathsf{Latt}}}.$$

By Corollary 2.4.15, computing $G_{\mathbb{T}_{\mathsf{Latt}}}(L_n)$ is the same as computing $G'_{\mathbb{T}_{\mathsf{Latt}}}(L_n)$, so we will do the latter. Recall that $G'_{\mathbb{T}_{\mathsf{Latt}}}(L_n)$ is the group of elements $[t] \in \mathsf{Free}(\mathbb{T}_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ that are invertible and commute generically with the lattice function symbols.
Now, we will require the following fact about free lattices on finitely many generators, which is due to Whitman ([20], [21]):

Lemma 3.7.1. *Let $n \geq 0$. If $s, t \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ and $[\mathsf{x}] \leq [s] \vee [t]$, then $[\mathsf{x}] \leq [s]$ or $[\mathsf{x}] \leq [t]$.* ▮

Towards proving that the isotropy group of L_n is trivial, we require:

Lemma 3.7.2. *Let $n \geq 0$. If $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ and $[t]$ has a right inverse, i.e. there is some $s \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ with*

$$[t[s/\mathsf{x}]] = [\mathsf{x}] \in \mathsf{Free}(\mathbb{T}_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x})),$$

then

$$[t] = [\mathsf{x}].$$

Proof: We prove this by induction on $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$.

- If $t \equiv \mathsf{x}$, then we trivially obtain the result.
- Suppose $t \equiv c_i$ for some $1 \leq i \leq n$. We claim that $[c_i]$ cannot have a right inverse, which yields the result. If $[c_i]$ *did* have a right inverse, then we would easily obtain $[c_i] = [\mathsf{x}]$, i.e. $\mathbb{T}_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}) \vdash c_i = \mathsf{x}$, and hence $\mathbb{T}_{\mathsf{Latt}} \vdash^{y_1,\ldots,y_n,z} y_i = z$ for pairwise distinct variables $y_1, \ldots, y_n, z$ by the theorem on constants (Remark 1.3.17). This easily implies that every lattice would have at most one element, which is clearly not the case. This proves our claim.
- Suppose $t \equiv t_1 \vee t_2$ for some $t_1, t_2 \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}))$ for which the desired result holds, and suppose that $[t]$ has a right inverse. So there is some $s \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ with

  $$[t_1[s/\mathsf{x}]] \vee [t_2[s/\mathsf{x}]] = [t_1[s/\mathsf{x}] \vee t_2[s/\mathsf{x}]] = [\mathsf{x}] \in \mathsf{Free}(\mathbb{T}_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x})).$$

  From this, we infer

  $$[t_1[s/\mathsf{x}]] \leq [\mathsf{x}] \text{ and } [t_2[s/\mathsf{x}]] \leq [\mathsf{x}]$$

  and

  $$[\mathsf{x}] \leq [t_1[s/\mathsf{x}]] \vee [t_2[s/\mathsf{x}]].$$

  From the latter inequality and Lemma 3.7.1 we obtain

  $$[\mathsf{x}] \leq [t_1[s/\mathsf{x}]] \text{ or } [\mathsf{x}] \leq [t_2[s/\mathsf{x}]].$$

  Since $\leq$ is anti-symmetric, we thus have either $[\mathsf{x}] = [t_1[s/\mathsf{x}]]$ or $[\mathsf{x}] = [t_2[s/\mathsf{x}]]$. Since $\vee$ is commutative, suppose without loss of generality that $[t_1[s/\mathsf{x}]] = [\mathsf{x}]$. Then $[t_1]$ has a right inverse, so by the induction hypothesis it follows that $[t_1] = [\mathsf{x}]$, which entails that $[t_1[s/\mathsf{x}]] = [s]$. Then we have

  $$[\mathsf{x}] = [t_1[s/\mathsf{x}]] \vee [t_2[s/\mathsf{x}]] = [s] \vee [t_2[s/\mathsf{x}]].$$

  So $[s] \leq [\mathsf{x}]$ and $[t_2[s/\mathsf{x}]] \leq [\mathsf{x}]$, and $[\mathsf{x}] \leq [s] \vee [t_2[s/\mathsf{x}]]$. So by Lemma 3.7.1, either $[\mathsf{x}] \leq [s]$ or $[\mathsf{x}] \leq [t_2[s/\mathsf{x}]]$. So either $[\mathsf{x}] = [s]$ or $[\mathsf{x}] = [t_2[s/\mathsf{x}]]$. In the first case, we obtain

  $$[\mathsf{x}] = [t_1[s/\mathsf{x}]] \vee [t_2[s/\mathsf{x}]] = [t_1[\mathsf{x}/\mathsf{x}]] \vee [t_2[\mathsf{x}/\mathsf{x}]] = [t_1] \vee [t_2] = [t_1 \vee t_2] = [t],$$

  as desired. And in the second case, it follows that $[t_2]$ has a right inverse, so by the induction hypothesis, we have $[t_2] = [\mathsf{x}]$. Then we have

  $$[t] = [t_1 \vee t_2] = [t_1] \vee [t_2] = [\mathsf{x}] \vee [\mathsf{x}] = [\mathsf{x}],$$

  as desired (since $\vee$ is idempotent).

- The induction step for $t \equiv t_1 \wedge t_2$ is handled analogously to the previous induction step.

$\blacksquare$

We now easily obtain:

Proposition 3.7.3. *Let $n \geq 0$. Then*

$$G_{\mathbb{T}_{\mathsf{Latt}}}(L_n) = \{[\mathsf{x}]\} \subseteq L_n\langle \mathsf{x} \rangle.$$

Proof: By Corollary 2.4.15, it suffices to show that $G'_{\mathbb{T}_{\mathsf{Latt}}}(L_n) = \{[\mathsf{x}]\}$. So let $[t] \in G'_{\mathbb{T}_{\mathsf{Latt}}}(L_n)$. Then $[t]$ is in particular invertible, so that there is some $s \in \mathsf{Term}^c(\Sigma_{\mathsf{Latt}}(c_1, \ldots, c_n, \mathsf{x}))$ with $[t[s/\mathsf{x}]] = [\mathsf{x}]$. Then by Lemma 3.7.2, it follows that $[t] = [\mathsf{x}]$, as desired. $\blacksquare$

Corollary 3.7.4. *Let $n \geq 0$. Then $\mathscr{Z}_{\mathbb{T}_{\mathsf{Latt}}}(L_n)$ is the trivial group.* $\blacksquare$

One can also use arguments similar to the above, together with the solution of the word problem for finitely presented lattices, to show that the isotropy group of any *finitely presented* lattice is trivial. We omit the details.

3.8 Categories and Groupoids

In this section, we now consider quasi-equational theories with slightly more logical complexity than totally defined algebraic theories. Namely, we will compute the isotropy groups of the quasi-equational theories of categories, groupoids, and categories with a terminal object, which are all two-sorted theories in which not all operations are totally defined. We will in fact show that all of these theories have trivial isotropy. First, we consider the theory of categories.

To define the theory $\mathbb{T}_{\mathsf{Cat}}$ of categories, we first define its signature Σ_{Cat}. We stipulate that Σ_{Cat} has two sorts, the object sort O and the arrow sort A, and four function symbols: $\mathsf{id} : O \to A$, $\mathsf{dom}, \mathsf{cod} : A \to O$, and $\circ : A \times A \to A$ (the latter written in infix notation). We then define $\mathbb{T}_{\mathsf{Cat}}$ to be the quasi-equational theory over Σ_{Cat} whose axioms are the following sequents (throughout, x is a variable of sort O and f, g, h are variables of sort A):

- $\top \vdash^x \mathsf{id}(x) \downarrow$, $\top \vdash^f \mathsf{dom}(f) \downarrow$, and $\top \vdash^f \mathsf{cod}(f) \downarrow$.

- $g \circ f \downarrow \vdash^{f,g} \mathsf{cod}(f) = \mathsf{dom}(g)$ and $\mathsf{cod}(f) = \mathsf{dom}(g) \vdash^{f,g} g \circ f \downarrow$.
- $g \circ f \downarrow \vdash^{f,g} \mathsf{dom}(g \circ f) = \mathsf{dom}(f) \wedge \mathsf{cod}(g \circ f) = \mathsf{cod}(g)$.
- $h \circ (g \circ f) \downarrow \vdash^{f,g,h} h \circ (g \circ f) = (h \circ g) \circ f$.
- $\top \vdash^{x} \mathsf{dom}(\mathsf{id}(x)) = x \wedge \mathsf{cod}(\mathsf{id}(x)) = x$.
- $\top \vdash^{f} f \circ \mathsf{id}(\mathsf{dom}(f)) = f \wedge \mathsf{id}(\mathsf{cod}(f)) \circ f = f$.

Then $\mathsf{P}\mathbb{T}_{\mathsf{Cat}}\mathsf{mod}$ is just the category Cat of all small categories and functors. To show that $G_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C})$ is trivial for any small category $\mathbb{C}$, we require the following preparatory lemmas (the proofs of the first three may be found in Appendix B). If $\mathbb{C}$ is a small category, then $\mathsf{x}_O, \mathsf{x}_A \notin \Sigma_{\mathsf{Cat}}(\mathbb{C})$ will always be distinct constant symbols of sorts O and A respectively.

Lemma 3.8.1. *Let $\mathbb{C}$ be any small category.*

1. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \nvdash \mathsf{x}_O = c_{O,b}$. For any $f \in \mathbb{C}_A$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_{A,f}$.*
2. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = c_{O,b}$ and $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{cod}(\mathsf{x}_A) = c_{O,b}$.*
3. *$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = \mathsf{cod}(\mathsf{x}_A)$.* ∎

Lemma 3.8.2. *If $\mathbb{C}$ is any small category and $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t \downarrow$, then:*

- *If $t : O$, then either $[t] = [\mathsf{x}_O] \in \mathbb{C}\langle \mathsf{x}_O \rangle_O$ or $[t] = [c_{O,b}] \in \mathbb{C}\langle \mathsf{x}_O \rangle_O$ for some $b \in \mathbb{C}_O$.*
- *If $t : A$, then either $[t] = [\mathsf{id}(\mathsf{x}_O)] \in \mathbb{C}\langle \mathsf{x}_O \rangle_A$ or $[t] = [c_{A,f}] \in \mathbb{C}\langle \mathsf{x}_O \rangle_A$ for some $f \in \mathbb{C}_A$.* ∎

Lemma 3.8.3. *If $\mathbb{C}$ is any small category and $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t \downarrow$, then:*

- *If $t : O$, then $[t] = [\mathsf{dom}(\mathsf{x}_A)]$ or $[t] = [\mathsf{cod}(\mathsf{x}_A)]$ or $[t] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$.*
- *If $t : A$, then $[t] = [\mathsf{x}_A]$ or $[t] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$ or $[t] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ or $[t] = [c_{A,f}]$ for some $f \in \mathbb{C}_A$.* ∎

Lemma 3.8.4. *Let $\mathbb{C}$ be any small category.*

1. *For any distinct variables $y_1, y_2 : O$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.*

2. *For any distinct variables $y_1, y_2 : A$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.*

Proof: To prove both claims, it suffices by Lemma 3.1.2 to show that there is a small category $\mathbb{D}$ with at least two objects (and hence at least two arrows) and a functor $F : \mathbb{C} \to \mathbb{D}$. So just let $\mathbb{D}$ be the disjoint union of $\mathbb{C}$ with the discrete category on two objects, and let F be the inclusion functor. ∎

Proposition 3.8.5. *For any small category $\mathbb{C}$,*

$$G_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C}) = \{([\mathsf{x}_O], [\mathsf{x}_A])\} \subseteq \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A.$$

Proof: Let $([s_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C})$; we must show that

$$([s_O], [s_A]) = ([\mathsf{x}_O], [\mathsf{x}_A]).$$

By Lemma 3.8.2, either $[s_O] = [\mathsf{x}_O]$ or $[s_O] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$. But since $([s_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C})$, it follows by Lemmas 2.2.56 and 3.8.4 that we must have $[s_O] = [\mathsf{x}_O]$, as desired.

Now we show $[s_A] = [\mathsf{x}_A]$. By Lemma 3.8.3, we know $[s_A] = [\mathsf{x}_A]$ or $[s_A] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$ or $[s_A] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ or $[s_A] = [c_{A,f}]$ for some $f \in \mathbb{C}_A$. As with $[s_O]$, the last case is impossible. Suppose towards a contradiction that $[s_A] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$. Since $([s_O], [s_A]) = ([\mathsf{x}_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C})$, it commutes generically with the function symbol $\mathsf{cod} : A \to O$, so that

$$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash s_O[\mathsf{cod}(\mathsf{x}_A)/\mathsf{x}_O] = \mathsf{cod}(s_A)$$

(since $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{cod}(\mathsf{x}_A)\downarrow$). Since $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash s_O = \mathsf{x}_O$, we obtain

$$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}_A) \vdash s_O = \mathsf{x}_O$$

and

$$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}_A) \vdash s_O[\mathsf{cod}(\mathsf{x}_A)/\mathsf{x}_O] = \mathsf{cod}(s_A),$$

which together imply

$$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}_A) \vdash \mathsf{cod}(\mathsf{x}_A) = \mathsf{cod}(s_A),$$

and hence (by the theorem on constants in Remark 1.3.17)

$$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{cod}(\mathsf{x}_A) = \mathsf{cod}(s_A),$$

i.e. $[\mathsf{cod}(\mathsf{x}_A)] = [\mathsf{cod}(s_A)] \in \mathbb{C}\langle \mathsf{x}_A \rangle_O$. So, supposing that $[s_A] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$, we would obtain

$$[\mathsf{cod}(\mathsf{x}_A)] = [\mathsf{cod}(s_A)] = [\mathsf{cod}(\mathsf{id}(\mathsf{dom}(\mathsf{x}_A)))] = [\mathsf{dom}(\mathsf{x}_A)],$$

which contradicts Lemma 3.8.1. We reach a similar contradiction if we suppose that $[s_A] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ (instead using the assumption that $([s_O], [s_A])$ commutes generically with dom). So the only remaining possibility is that $[s_A] = [\mathsf{x}_A]$, as desired. ▮

Corollary 3.8.6. *If* $\mathbb{C}$ *is any small category, then* $\mathcal{Z}_{\mathbb{T}_{\mathsf{Cat}}}(\mathbb{C})$ *only consists of the identity element.*

Corollary 3.8.7. $\mathcal{Z}_{\mathbb{T}_{\mathsf{Cat}}} \cong G_{\mathbb{T}_{\mathsf{Cat}}} : \mathsf{Cat} \to \mathsf{Group}$ *is (naturally isomorphic to) the constant functor on the trivial group.* ▮

Richard Garner [12, Proposition 3] has also independently shown that $\mathbb{T}_{\mathsf{Cat}}$ has trivial isotropy by a relatively short *categorical* argument. In general, if one suspects that a theory has *trivial* isotropy, it is usually not too difficult (and sometimes, may even be *easier*) to pursue a purely *categorical* proof of this, rather than one using the logical methods that we have developed so far. However, we should add the caveat that such a proof will likely only be easier if there is an efficient/effective description of the process of freely adjoining an indeterminate element to a model of such a theory (e.g. in the case of the theory of categories, the category obtained from a category $\mathbb{C}$ by freely adjoining an indeterminate object is merely the disjoint union of $\mathbb{C}$ with the terminal category).

Next, we show that the isotropy group of $\mathbb{T}_{\mathsf{Ter}}$, the quasi-equational theory of (small) categories with a chosen terminal object, is also trivial. The signature Σ_{Ter} of this theory extends the signature Σ_{Cat} by adding a new constant $1 : O$ and a new function symbol $! : O \to A$. The quasi-equational theory $\mathbb{T}_{\mathsf{Ter}}$ over the signature Σ_{Ter} then contains all the axioms of $\mathbb{T}_{\mathsf{Cat}}$ together with the following additional axioms (where x is a variable of sort O and f a variable of sort A):

- $\top \vdash 1 \downarrow$.
- $\top \vdash^x\ !(x) \downarrow$.
- $\top \vdash^x \mathsf{dom}(!(x)) = x \wedge \mathsf{cod}(!(x)) = 1$.
- $\mathsf{dom}(f) = x \wedge \mathsf{cod}(f) = 1 \vdash^{x,f} f = !(x)$.

Then $\mathsf{P}\mathbb{T}_{\mathsf{Ter}}\mathsf{mod}$ is the category Ter of small categories with chosen terminal objects, and functors that preserve the terminal objects on the nose.

We now have the following analogues of certain lemmas for $\mathbb{T}_{\mathsf{Cat}}$.

Lemma 3.8.8. *Let $\mathbb{C}$ be a small category with terminal object $1^{\mathbb{C}}$.*

1. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_O) \nvdash \mathsf{x}_O = c_{O,b}$. For any $f \in \mathbb{C}_A$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_{A,f}$.*
2. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = c_{O,b}$ and $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{cod}(\mathsf{x}_A) = c_{O,b}$.*
3. *We have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = \mathsf{cod}(\mathsf{x}_A)$.*

Proof: See Appendix B. ∎

Lemma 3.8.9. *If $\mathbb{C}$ is any small category with terminal object $1^{\mathbb{C}}$ and $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_O))$ with $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_O) \vdash t \downarrow$, then:*

- *If $t : O$, then either $[t] = [\mathsf{x}_O]$ or $[t] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$.*
- *If $t : A$, then either $[t] = [\mathsf{id}(\mathsf{x}_O)]$, or $[t] = [c_{A,f} \circ !(\mathsf{x}_O)]$ for some $f \in \mathbb{C}_A$ with $\mathsf{dom}^{\mathbb{C}}(f) = 1^{\mathbb{C}}$, or $[t] = [c_{A,f}]$ for some $f \in \mathbb{C}_A$.*

Proof: Analogous to the proof of Lemma 3.8.2. ∎

Lemma 3.8.10. *If $\mathbb{C}$ is any small category with terminal object $1^{\mathbb{C}}$ and $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A))$ with $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \vdash t \downarrow$, then:*

- *If $t : O$, then $[t] = [\mathsf{dom}(\mathsf{x}_A)]$ or $[t] = [\mathsf{cod}(\mathsf{x}_A)]$ or $[t] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$.*
- *If $t : A$, then $[t] = [\mathsf{x}_A]$ or $[t] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$ or $[t] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ or $[t] = [c_{A,f} \circ !(\mathsf{dom}(\mathsf{x}_A))]$ for some $f \in \mathbb{C}_A$ with $\mathsf{dom}^{\mathbb{C}}(f) = 1^{\mathbb{C}}$, or $[t] = [c_{A,f} \circ !(\mathsf{cod}(\mathsf{x}_A))]$ for some $f \in \mathbb{C}_A$ with $\mathsf{dom}^{\mathbb{C}}(f) = 1^{\mathbb{C}}$, or $[t] = [c_{A,f}]$ for some $f \in \mathbb{C}_A$.*

Proof: Analogous to the proof of Lemma 3.8.3. ∎

Lemma 3.8.11. *Let $\mathbb{C}$ be any small category with terminal object.*

1. *For any distinct variables $y_1, y_2 : O$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.*

2. *For any distinct variables* $y_1, y_2 : A$, *we have* $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.

Proof: Analogous to the proof of Lemma 3.8.4. ∎

Proposition 3.8.12. *If* $\mathbb{C}$ *is a small category with terminal object* $1^{\mathbb{C}}$, *then*

$$G_{\mathbb{T}_{\mathsf{Ter}}}(\mathbb{C}) = \{([\mathsf{x}_O], [\mathsf{x}_A])\} \subseteq \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A.$$

Proof: Let $([s_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Ter}}}(\mathbb{C})$; we must show that

$$([s_O], [s_A]) = ([\mathsf{x}_O], [\mathsf{x}_A]).$$

By Lemma 3.8.9, either $[s_O] = [\mathsf{x}_O]$ or $[s_O] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$. Since $([s_O], [s_A])$ is an element of isotropy, it follows by Lemmas 2.2.56 and 3.8.11 that we must have $[s_O] = [\mathsf{x}_O]$, as desired.

Now we show $[s_A] = [\mathsf{x}_A] \in \mathbb{C}\langle \mathsf{x}_A \rangle_A$. By Lemma 3.8.10, we have one of the following possibilities:

- $[s_A] = [\mathsf{x}_A]$.
- $[s_A] \in \{[\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))], [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]\}$.
- $[s_A] \in \{[c_{A,f} \circ !(\mathsf{dom}(\mathsf{x}_A))], [c_{A,f} \circ !(\mathsf{cod}(\mathsf{x}_A))]\}$ for some $f \in \mathbb{C}_A$ with $\mathsf{dom}^{\mathbb{C}}(f) = 1^{\mathbb{C}}$.
- $[s_A] = [c_{A,f}]$ for some $f \in \mathbb{C}_A$.

The last case is impossible by Lemmas 2.2.56 and 3.8.11, since $([s_O], [s_A])$ is an element of isotropy. Exactly as in the proof of Proposition 3.8.5, we can also show that the second case is impossible.

Now suppose towards a contradiction that $[s_A] = [c_{A,f} \circ !(\mathsf{dom}(\mathsf{x}_A))]$ for some $f : 1^{\mathbb{C}} \to a \in \mathbb{C}$. As in the proof of Proposition 3.8.5, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{cod}(s_A) = \mathsf{cod}(\mathsf{x}_A)$, i.e. $[\mathsf{cod}(s_A)] = [\mathsf{cod}(\mathsf{x}_A)] \in \mathbb{C}\langle \mathsf{x}_A \rangle_O$. Now in $\mathbb{C}\langle \mathsf{x}_A \rangle_O$ we have

$$[\mathsf{cod}(\mathsf{x}_A)] = [\mathsf{cod}(s_A)] = [\mathsf{cod}\,(c_{A,f} \circ !(\mathsf{dom}(\mathsf{x}_A)))] = [\mathsf{cod}\,(c_{A,f})] = [c_{O,a}]\,,$$

which contradicts Lemma 3.8.8. In the same way, we cannot have $[s_A] = [c_{A,f} \circ !(\mathsf{cod}(\mathsf{x}_A))]$. So the only remaining possibility is that $[s_A] = [\mathsf{x}_A]$, as desired. ∎

Corollary 3.8.13. *If* $\mathbb{C}$ *is any small category with terminal object* $1^{\mathbb{C}}$, *then* $\mathcal{Z}_{\mathbb{T}_{\mathsf{Ter}}}(\mathbb{C})$ *is the trivial group.*

Corollary 3.8.14. $\mathcal{Z}_{\mathbb{T}_{\mathsf{Ter}}} \cong G_{\mathbb{T}_{\mathsf{Ter}}} : \mathsf{Ter} \to \mathsf{Group}$ *is (naturally isomorphic to) the constant functor on the trivial group.* ∎

As the last example of this section, we remark that the isotropy group of $\mathbb{T}_{\mathsf{Grpd}}$, the quasi-equational theory of small groupoids, is also trivial. The signature Σ_{Grpd} extends the signature Σ_{Cat} by adding a new unary function symbol $^{-1} : A \to A$ (written in infix notation). We then define the quasi-equational theory $\mathbb{T}_{\mathsf{Grpd}}$ over the signature Σ_{Grpd} to have all axioms of $\mathbb{T}_{\mathsf{Cat}}$ together with the following additional axioms (where f is a variable of sort A):

- $\top \vdash^{f} f^{-1} \downarrow$.
- $\top \vdash^{f} \mathsf{dom}(f^{-1}) = \mathsf{cod}(f) \wedge \mathsf{cod}(f^{-1}) = \mathsf{dom}(f)$.
- $\top \vdash^{f} f \circ f^{-1} = \mathsf{id}(\mathsf{cod}(f)) \wedge f^{-1} \circ f = \mathsf{id}(\mathsf{dom}(f))$.

Then $\mathsf{P}\mathbb{T}_{\mathsf{Grpd}}\mathsf{mod}$ is just the category Grpd of all small groupoids and functors between them.

To show that the isotropy group of $\mathbb{T}_{\mathsf{Grpd}}$ is trivial, we could argue using the logical methods of Chapter 2, as we did in the last two examples. However, Richard Garner [12, Proposition 3] has independently shown (as for $\mathbb{T}_{\mathsf{Cat}}$) that the isotropy group of $\mathbb{T}_{\mathsf{Grpd}}$ is trivial by using a relatively short categorical argument.

3.9 Strict Monoidal Categories

For our last (and most involved) example of this chapter, we will compute the isotropy group of the quasi-equational theory $\mathbb{T}_{\mathsf{Str}}$ of (small) strict monoidal categories, and will show that it is non-trivial. We define Σ_{Str} to be the signature that extends the signature Σ_{Cat} by adding three new function symbols: $e : O$ (the (object) unit of the strict monoidal structure), $\otimes_O : O \times O \to O$ (the object tensor operation), and $\otimes_A : A \times A \to A$ (the arrow tensor operation). We will write the tensor operations in infix notation. We then define the quasi-equational theory $\mathbb{T}_{\mathsf{Str}}$ of (small) strict monoidal categories to be the quasi-equational theory over the signature Σ_{Str} that extends $\mathbb{T}_{\mathsf{Cat}}$ by adding the following axioms (where x, y, z are variables of sort O, and f, g, f', g', h are variables of sort A):

1. $\top \vdash^{x,y} x \otimes_O y \downarrow$.
2. $\top \vdash^{f,g} f \otimes_A g \downarrow$.

3. $\top \vdash e \downarrow$.
4. $\top \vdash^{f,g} \mathsf{dom}(f \otimes_A g) = \mathsf{dom}(f) \otimes_O \mathsf{dom}(g) \wedge \mathsf{cod}(f \otimes_A g) = \mathsf{cod}(f) \otimes_O \mathsf{cod}(g)$.
5. $\top \vdash^{x,y} \mathsf{id}(x \otimes_O y) = \mathsf{id}(x) \otimes_A \mathsf{id}(y)$.
6. $f' \circ f \downarrow \wedge\, g' \circ g \downarrow \vdash^{f,f',g,g'} (f' \circ f) \otimes_A (g' \circ g) = (f' \otimes_A g') \circ (f \otimes_A g)$.
7. $\top \vdash^{x,y,z} x \otimes_O (y \otimes_O z) = (x \otimes_O y) \otimes_O z$.
8. $\top \vdash^{f,g,h} f \otimes_A (g \otimes_A h) = (f \otimes_A g) \otimes_A h$.
9. $\top \vdash^{x} x \otimes_O e = x \wedge e \otimes_O x = x$.
10. $\top \vdash^{f} f \otimes_A \mathsf{id}(e) = f \wedge \mathsf{id}(e) \otimes_A f = f$.

Because the object and arrow tensor operations are associative, we will omit parentheses around multiple tensor operations when possible. We have $\mathsf{P}\mathbb{T}_{\mathsf{Str}}\mathsf{mod} = \mathsf{StrMonCat}$, the category of (small) strict monoidal categories and strict monoidal functors.

If $\mathbb{C}$ is a small strict monoidal category, then $\mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \notin \Sigma_{\mathsf{Str}}(\mathbb{C})$ will be pairwise distinct constants, the first two of sort O and the last two of sort A. In the next few definitions, we will define the concepts of reduced and semi-reduced *words* over these indeterminates $\mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A$ and the constants of the diagram signature of $\mathbb{C}$.

Definition 3.9.1 (Alphabets). Let $\mathbb{C}$ be a small strict monoidal category. We define the sets of terms, or *alphabets*

$$S_O^{\mathbb{C}} := \{\mathsf{x}_O, \mathsf{x}'_O, \mathsf{dom}(\mathsf{x}_A), \mathsf{cod}(\mathsf{x}_A), \mathsf{dom}(\mathsf{x}'_A), \mathsf{cod}(\mathsf{x}'_A)\} \cup \{c_{O,a} : a \in \mathbb{C}_O\}$$

and

$$S_A^{\mathbb{C}} := \{\mathsf{x}_A, \mathsf{x}'_A, \mathsf{id}(\mathsf{x}_O), \mathsf{id}(\mathsf{x}'_O), \mathsf{id}(\mathsf{dom}(\mathsf{x}_A)), \mathsf{id}(\mathsf{cod}(\mathsf{x}_A)), \mathsf{id}(\mathsf{dom}(\mathsf{x}'_A)), \mathsf{id}(\mathsf{cod}(\mathsf{x}'_A))\}$$
$$\cup \; \{c_{A,f} : f \in \mathbb{C}_A\}\,.$$

Note that

$$S_O^{\mathbb{C}} \subseteq \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_O$$

and

$$S_A \subseteq \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_A.$$

∎

Definition 3.9.2 (Words). Let $\mathbb{C}$ be a small strict monoidal category. If $C \in \{O, A\}$, we define $W_C^{\mathbb{C}}$ to be the smallest set of strings over the alphabet $S_C^{\mathbb{C}} \cup \{\otimes_C\}$ such that:

- $S_C^{\mathbb{C}} \subseteq W_C^{\mathbb{C}}$.
- If $w_1, w_2 \in W_C^{\mathbb{C}}$, then $w_1 \otimes_C w_2 \in W_C^{\mathbb{C}}$.

Informally, $W_C^{\mathbb{C}}$ is the set of all $\otimes_C$-words over the alphabet $S_C^{\mathbb{C}}$. If $\mathbb{C}$ is clear from the context, then we will generally write W_C instead of $W_C^{\mathbb{C}}$.

If $C \in \{O, A\}$ and $w \in W_C$, we define the length $\ell(w)$ of w and the first and last 'letters' $\mathsf{first}(w)$ and $\mathsf{last}(w)$ as follows:

- If $w \in S_C$, then $\ell(w) := 1$ and $\mathsf{first}(w) \equiv \mathsf{last}(w) := w$.
- If $w_1, w_2 \in W_C$, then

$$\ell(w_1 \otimes_C w_2) := \ell(w_1) + \ell(w_2),$$

$$\mathsf{first}(w_1 \otimes_C w_2) := \mathsf{first}(w_1),$$

$$\mathsf{last}(w_1 \otimes_C w_2) := \mathsf{last}(w_2).$$

∎

Definition 3.9.3 (Reduced and Semi-Reduced Words). Let $\mathbb{C}$ be a small strict monoidal category. We say that a word $w \in W_O$ is *reduced* if one of the following holds:

- $\ell(w) = 1$, i.e. $w \in S_O^{\mathbb{C}}$.
- $\ell(w) \geq 2$ and w does not contain $c_{O,e^{\mathbb{C}}}$ and does not contain any substring of the form $c_{O,a} \otimes_O c_{O,b}$ for $a, b \in \mathbb{C}_O$.

We then set

$$W_O^{\mathbb{C},r} := \{w \in W_O^{\mathbb{C}} : w \text{ is reduced}\}.$$

Again, we will usually write W_O^r if $\mathbb{C}$ is clear from the context.

We say that a word $w \in W_O$ is *semi-reduced* if either $\ell(w) = 1$, or $\ell(w) \geq 2$ and w does not contain any substring of the form $c_{O,a} \otimes_O c_{O,b}$ for $a, b \in \mathbb{C}_O$. Then we set

$$W_O^{\mathbb{C},sr} := \{w \in W_O^{\mathbb{C}} : w \text{ is semi-reduced}\}.$$

Thus, a semi-reduced word $w \in W_O$ may contain $c_{O,e^{\mathbb{C}}}$ (even if $\ell(w) > 1$).

Similarly, we say that a word $w \in W_A$ is *reduced* if one of the following holds:

- $\ell(w) = 1$, i.e. $w \in S_A^{\mathbb{C}}$.

- $\ell(w) \geq 2$ and w does not contain $c_{A,\mathsf{id}(e^{\mathbb{C}})}$ and does not contain any substring of the form $c_{A,f} \otimes_A c_{A,g}$ for $f, g \in \mathbb{C}_A$.

We then set
$$W_A^{\mathbb{C},r} := \{w \in W_A^{\mathbb{C}} : w \text{ is reduced}\}.$$

We say that a word $w \in W_A$ is *semi-reduced* if either $\ell(w) = 1$, or $\ell(w) \geq 2$ and w does not contain any substring of the form $c_{A,f} \otimes_A c_{A,g}$ for $f, g \in \mathbb{C}_A$. We then set

$$W_A^{\mathbb{C},sr} := \{w \in W_A^{\mathbb{C}} : w \text{ is semi-reduced}\}.$$

Thus, a semi-reduced word $w \in W_A$ may contain $c_{A,\mathsf{id}(e^{\mathbb{C}})}$ (even if $\ell(w) > 1$). ∎

We now define certain *reduction* or *rewrite* systems on W_O and W_A. Recall that a *reduction system* on a set S is simply a pair $(S, \rightarrow)$, with $\rightarrow$ a binary relation on S.

Definition 3.9.4. Let $\mathbb{C}$ be a small strict monoidal category. We define a reduction system $(W_O^{\mathbb{C}}, \rightarrow_r)$ as follows. For any $w_1, w_2 \in W_O^{\mathbb{C}}$, we stipulate that $w_1 \rightarrow_r w_2$ if either:

there are $a, b \in \mathbb{C}_O$ with $c_{O,a} \otimes_O c_{O,b}$ a subword of w_1, and w_2 is obtained from w_1 by replacing one occurrence of this subword by the constant $c_{O,a \otimes_O^{\mathbb{C}} b}$;

or

$\ell(w_1) \geq 2$ and w_2 is obtained from w_1 by deleting one occurrence of $c_{O,e^{\mathbb{C}}}$ in w_1.

We define a reduction system $(W_A^{\mathbb{C}}, \rightarrow_r)$ analogously. ∎

Recall that a reduction system $(S, \rightarrow)$ is said to be *terminating* if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow \ldots$ in S, and that $(S, \rightarrow)$ is said to be *confluent* if for any $s, s_1, s_2 \in S$, $s \rightarrow^* s_1, s_2$ implies that there is some $s_3 \in S$ with $s_1, s_2 \rightarrow^* s_3$, where $\rightarrow^*$ is the reflexive transitive closure of $\rightarrow$.

Lemma 3.9.5. *For any small strict monoidal category $\mathbb{C}$, the reduction systems $(W_O^{\mathbb{C}}, \rightarrow_r)$ and $(W_A^{\mathbb{C}}, \rightarrow_r)$ are confluent and terminating.*

Proof: We only consider $(W_O^{\mathbb{C}}, \rightarrow_r)$. To show that this reduction system is terminating, we must show that there is no infinite sequence $w_1 \rightarrow_r w_2 \rightarrow_r \ldots$ (with $w_1, w_2, \ldots \in W_O^{\mathbb{C}}$). But if $w \rightarrow_r w'$, then we clearly have $\ell(w') < \ell(w)$, so termination follows.

To show that $(W_O^{\mathbb{C}}, \rightarrow_r)$ is confluent, it suffices by Newman's Lemma ([2], [17]) to show that $(W_O^{\mathbb{C}}, \rightarrow_r)$ is *locally* confluent, since it is terminating. That is, we must show for any $w, w_1, w_2 \in W_O^{\mathbb{C}}$ that if $w \rightarrow_r w_1$ and $w \rightarrow_r w_2$, then there is some $w_3 \in W_O^{\mathbb{C}}$ with $w_1, w_2 \rightarrow_r^* w_3$. We may suppose that w_1 and w_2 are distinct, because otherwise we can clearly take $w_3 := w_1 \equiv w_2$. Then the result follows by an easy case analysis based on the two rules in the definition of $(W_O^{\mathbb{C}}, \rightarrow_r)$. ∎

If $(S, \rightarrow)$ is a reduction system, recall that a *normal form* (with respect to $\rightarrow$) is an element $s \in S$ with the property that there is no $t \in S$ such that $s \rightarrow t$. A normal form (with respect to $\rightarrow$) of an element $x \in S$ is then a normal form $s \in S$ such that $x \rightarrow^* s$. The subsequent corollary now follows from Lemma 3.9.5 by standard arguments from the theory of reduction systems, cf. e.g. [2].

Corollary 3.9.6. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$.

- *If* $w \in W_O^{\mathbb{C}}$, *then* w *has a unique normal form* w^r *with respect to* $\rightarrow_r$.
- *If* $w_1, w_2 \in W_O^{\mathbb{C}}$, *then* $w_1^r \equiv w_2^r$ *iff* $w_1 \leftrightarrow_r^* w_2$ *(with* $\leftrightarrow_r^*$ *being the reflexive, symmetric, transitive closure of* $\rightarrow_r$*).*

Analogous results hold for $(W_A^{\mathbb{C}}, \rightarrow_r)$. ∎

Lemma 3.9.7. *If* $\mathbb{C}$ *is a small strict monoidal category and* $w \in W_O \cup W_A$, *then* w^r *is reduced, and* $w^r \equiv w$ *if* w *is reduced.*

Proof: Let $w \in W_O \cup W_A$. Then w^r is reduced, because if not, then by the definition of 'reduced', there would clearly be some w' with $w^r \rightarrow w'$, contradicting the fact that w^r is a normal form.

Next, we need to show that the normal form of w with respect to $\rightarrow_r$ is itself, if w is reduced, i.e. we need to show that if w is reduced, then there is no $w' \in W_O^{\mathbb{C}}$ with $w \rightarrow_r w'$. But this immediately follows from the definition of $\rightarrow_r$ and the assumption that w is reduced. ∎

Recall that if $\mathbb{C}$ is a small strict monoidal category, then $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$ extends $\mathbb{T}_{\mathsf{Str}}(\mathbb{C})$ by adding the axioms $\{\top \vdash \mathsf{x}_C \downarrow \wedge \mathsf{x}'_C \downarrow : C \in \{O, A\}\}$. Then for any $w \in W_O \cup W_A$, it follows that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w \downarrow,$$

because it is trivial to check that $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s \downarrow$ for every $s \in S_O^{\mathbb{C}} \cup S_A^{\mathbb{C}}$, and the tensor operations are totally defined in $\mathbb{T}_{\mathsf{Str}}$ and hence in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$.

So in the following lemma (and in any subsequent result where we prove that words in $W_O \cup W_A$ have certain properties in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$), we do not need to assume that $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w \downarrow$ for $w \in W_O \cup W_A$.

Lemma 3.9.8. *Let* $\mathbb{C}$ *be any small strict monoidal category. If* $w \in W_O \cup W_A$*, then*

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w = w^r.$$

Proof: If $w_1, w_2 \in W_O$ and $w_1 \to_r w_2$, then it is trivial to check that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w_1 = w_2,$$

which easily yields the result. The proof for W_A is analogous. ∎

In Lemma 3.9.34 below, we will prove (for any $\mathbb{C} \in \mathsf{StrMonCat}$) that any term $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ is provably equal to a word in $W_O \cup W_A$, and hence (by Lemma 3.9.8) to a *reduced* word in $W_O \cup W_A$.

Our next aim is to show that if $w, w' \in W_O \cup W_A$ are reduced words of the same sort with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w = w'$, then $w \equiv w'$. To do this, we will construct a small strict monoidal category $\mathbb{C}^*$ whose set of objects is $W_O^{\mathbb{C},r}$ and whose set of arrows is $W_A^{\mathbb{C},r}$, which will have the property that if $w, w' \in W_O^{\mathbb{C}} \cup W_A^{\mathbb{C}}$ are reduced words of the same sort with $\mathbb{C}^* \models w = w'$, then in fact $w \equiv w'$. We will also show that $\mathbb{C}^* \models \mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$, which will then (by soundness of partial Horn logic) entail our claim.

Towards this goal, we require the following additional definitions. First, given a word $w \in W_O \cup W_A$, we will need (for certain technical reasons, cf. the discussion following Definition 3.9.29) the concept of inserting $c_{O,e^{\mathbb{C}}}$ (if $w : O$) or $c_{A,\mathsf{id}(e^{\mathbb{C}})}$ (if $w : A$) into w 'wherever possible' (so as to still obtain a semi-reduced word).

Definition 3.9.9. Let $\mathbb{C}$ be a small strict monoidal category. We define a reduction system $(W_O^{sr}, \to_e)$ as follows. For any $w_1, w_2 \in W_O^{sr}$, we stipulate that $w_1 \to_e w_2$ if:

- $\mathsf{first}(w_1) \notin \{c_{O,a} : a \in \mathbb{C}_O\}$ and $w_2 \equiv c_{O,e^{\mathbb{C}}} \otimes_O w_1$.
- $\mathsf{last}(w_1) \notin \{c_{O,a} : a \in \mathbb{C}_O\}$ and $w_2 \equiv w_1 \otimes_O c_{O,e^{\mathbb{C}}}$.
- There are $s_1, s_2 \in S_O^{\mathbb{C}} \setminus \{c_{O,a} : a \in \mathbb{C}_O\}$ with $s_1 \otimes_O s_2$ a subword of w_1, and w_2 is obtained from w_1 by replacing one occurrence of this subword by $s_1 \otimes_O c_{O,e^{\mathbb{C}}} \otimes_O s_2$.

We define a reduction system $(W_A^{sr}, \to_e)$ analogously. ∎

So if $w_1 \rightarrow_e w_2$, then w_2 is obtained from w_1 by inserting $c_{O,e^{\mathbb{C}}}$ into w_1 in such a way that the result will still be a semi-reduced word.

Lemma 3.9.10. *For any small strict monoidal category $\mathbb{C}$, the reduction systems $(W_O^{sr}, \rightarrow_e)$ and $(W_A^{sr}, \rightarrow_e)$ are confluent and terminating.*

Proof: We only consider $(W_O^{sr}, \rightarrow_e)$. To show that this reduction system is terminating, we must show that there is no infinite sequence $w_1 \rightarrow_e w_2 \rightarrow_e \dots$ (with $w_1, w_2, \dots \in W_O^{sr}$). To do this, we first define for any $w \in W_O^{sr}$ the concept of a 'hole' in w: a *hole* in w is (an occurrence of) a subword of w of one of the following forms:

- $\mathsf{first}(w)$, if $\mathsf{first}(w) \notin \{c_{O,a} : a \in \mathbb{C}_O\}$.
- $\mathsf{last}(w)$, if $\mathsf{last}(w) \notin \{c_{O,a} : a \in \mathbb{C}_O\}$.
- $s_1 \otimes_O s_2$, with $s_1, s_2 \notin \{c_{O,a} : a \in \mathbb{C}_O\}$.

It is then clear that if $w \in W_O^{sr}$ and $w \rightarrow_e w'$, then w' has strictly fewer holes than w. Since a word can only have finitely many holes, this proves that $\rightarrow_e$ is terminating.

To show that $(W_O^{sr}, \rightarrow_e)$ is confluent, it suffices by Newman's Lemma ([2], [17]) to show that $(W_O^{sr}, \rightarrow_e)$ is *locally* confluent, since it is terminating. That is, we must show for any $w, w_1, w_2 \in W_O^{sr}$ that if $w \rightarrow_e w_1$ and $w \rightarrow_e w_2$, then there is some $w_3 \in W_O^{sr}$ with $w_1, w_2 \rightarrow_e^* w_3$. We may suppose that w_1 and w_2 are distinct, because otherwise we can clearly take $w_3 := w_1 \equiv w_2$. Then the result follows by an easy case analysis based on the three rules in the definition of $(W_O^{sr}, \rightarrow_e)$. ▮

Corollary 3.9.11. *Let $\mathbb{C} \in \mathsf{StrMonCat}$.*

- *If $w \in W_O^{sr}$, then w has a unique normal form w^e with respect to $\rightarrow_e$.*
- *If $w_1, w_2 \in W_O^{sr}$, then $w_1^e \equiv w_2^e$ iff $w_1 \leftrightarrow_e^* w_2$ (with $\leftrightarrow_e^*$ being the reflexive, symmetric, transitive closure of $\rightarrow_e$).*

Analogous results apply to $(W_A^{sr}, \rightarrow_e)$. ▮

The following lemma is proved in the same way as Lemma 3.9.8.

Lemma 3.9.12. *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w \in W_O^{sr} \cup W_A^{sr}$, then*

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w = w^e.$$

▮

We now define the *expansion* w^{exp} of a word as the word obtained by first reducing w and then inserting unit constants wherever possible:

Definition 3.9.13. If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w \in W_O \cup W_A$, then $w^{\mathsf{exp}} := (w^r)^e$. ∎

Combining Lemmas 3.9.8 and 3.9.12, we then have

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w = (w^r)^e =: w^{\mathsf{exp}}$$

for any $w \in W_O \cup W_A$. Also, we have:

Lemma 3.9.14. *If* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $w \in W_O^{sr} \cup W_A^{sr}$*, then*

$$w^r \to_e^* w,$$

and hence $(w^r)^e \equiv w^e$*.*

Proof: If $w \in W_O$ is semi-reduced, then w^r is obtained from w by deleting all occurrences of $c_{O,e^{\mathbb{C}}}$ in w. Using $\to_e$, we can then insert these occurrences of $c_{O,e^{\mathbb{C}}}$ back into w^r to obtain w. Then $(w^r)^e \equiv w^e$ follows by Lemma 3.9.11. ∎

Lemma 3.9.15. *If* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $w \in W_O \cup W_A$*, then*

$$(w^{\mathsf{exp}})^{\mathsf{exp}} \equiv w^{\mathsf{exp}}.$$

Proof: See Appendix B. ∎

Definition 3.9.16. Let $\mathbb{C} \in \mathsf{StrMonCat}$. We define the following functions

$$(-)^{\mathsf{dom}}, (-)^{\mathsf{cod}} : W_A^{\mathbb{C}} \to W_O^{\mathbb{C}},$$

by induction on the structure of $w \in W_A^{\mathbb{C}}$:

- If $w \in S_A^{\mathbb{C}}$, then:
    - If $w \equiv \mathsf{x}_A$, then $w^{\mathsf{dom}} := \mathsf{dom}(\mathsf{x}_A)$ and $w^{\mathsf{cod}} := \mathsf{cod}(\mathsf{x}_A)$, and similarly if $w \equiv \mathsf{x}'_A$.
    - If $w \equiv \mathsf{id}(\mathsf{x}_O)$, then $w^{\mathsf{dom}} \equiv w^{\mathsf{cod}} := \mathsf{x}_O$, and similarly if $w \equiv \mathsf{id}(\mathsf{x}'_O)$.
    - If $w \equiv \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$, then $w^{\mathsf{dom}} \equiv w^{\mathsf{cod}} := \mathsf{dom}(\mathsf{x}_A)$, and similarly if $w \in \{\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)), \mathsf{id}(\mathsf{dom}(\mathsf{x}'_A)), \mathsf{id}(\mathsf{cod}(\mathsf{x}'_A))\}$.

  - If $w \equiv c_{A,f}$ for some $f : a \to b \in \mathbb{C}_A$, then $w^{\mathsf{dom}} := c_{O,a}$ and $w^{\mathsf{cod}} := c_{O,b}$.

- If $w \equiv w_1 \otimes_A w_2$ for some $w_1, w_2 \in W_A^{\mathbb{C}}$, then we set

$$w^{\mathsf{dom}} := w_1^{\mathsf{dom}} \otimes_O w_2^{\mathsf{dom}}$$

  and

$$w^{\mathsf{cod}} := w_1^{\mathsf{cod}} \otimes_O w_2^{\mathsf{cod}}.$$

∎

The following lemma is then easy to prove by induction on the structure of words:

Lemma 3.9.17. *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w \in W_A^{\mathbb{C}}$, then*

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{cod}(w) = w^{\mathsf{cod}} \wedge \mathsf{dom}(w) = w^{\mathsf{dom}}.$$

∎

Lemma 3.9.18. *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w_1, w_2 \in W_A^{\mathbb{C}}$, then*

$$w_1 \to_r^* w_2 \textit{ implies } w_1^{\mathsf{dom}} \to_r^* w_2^{\mathsf{dom}},$$

and similarly for cod.

Proof: It suffices to show that $w_1 \to_r w_2$ implies $w_1^{\mathsf{dom}} \to_r^* w_2^{\mathsf{dom}}$, but this is obvious from the definition of $\to_r$. ∎

Lemma 3.9.19. *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w_1, w_2 \in W_A^{sr}$, then*

$$w_1 \to_e^* w_2 \textit{ implies } w_1^{\mathsf{dom}} \to_e^* w_2^{\mathsf{dom}},$$

and similarly for cod.

Proof: Note first that if w_1, w_2 are semi-reduced, then $w_1^{\mathsf{dom}}, w_2^{\mathsf{dom}}$ are also semi-reduced. It suffices to show that if w_1, w_2 are semi-reduced and $w_1 \to_e w_2$, then $w_1^{\mathsf{dom}} \to_e^* w_2^{\mathsf{dom}}$, but this is obvious from the definition of $\to_e$. ∎

Definition 3.9.20. Let $\mathbb{C} \in \mathsf{StrMonCat}$. We define the following function

$$(-)^{\mathsf{id}} : W_O^{\mathbb{C}} \to W_A^{\mathbb{C}}$$

by induction on the structure of $w \in W_O^{\mathbb{C}}$:

- If $w \in S_O^{\mathbb{C}}$, then:
  - If $w \equiv \mathsf{x}_O$, then $w^{\mathsf{id}} := \mathsf{id}(\mathsf{x}_O)$, and similarly if $w \equiv \mathsf{x}'_O$.
  - If $w \equiv \mathsf{dom}(\mathsf{x}_A)$, then $w^{\mathsf{id}} := \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$, and similarly if
  $$w \in \{\mathsf{cod}(\mathsf{x}_A), \mathsf{dom}(\mathsf{x}'_A), \mathsf{cod}(\mathsf{x}'_A)\}.$$
  - If $w \equiv c_{O,a}$ for some $a \in \mathbb{C}_O$, then $w^{\mathsf{id}} := c_{A,\mathsf{id}^{\mathbb{C}}(a)}$.
- If $w \equiv w_1 \otimes_O w_2$, then
$$w^{\mathsf{id}} := w_1^{\mathsf{id}} \otimes_A w_2^{\mathsf{id}}.$$

∎

The following lemmas are then straightforward (if not trivial) to prove:

Lemma 3.9.21. *For any* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $w \in W_O^{\mathbb{C}}$,

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{id}(w) = w^{\mathsf{id}}.$$

Lemma 3.9.22. *If* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $w \in W_O^{\mathbb{C},r}$, *then* $w^{\mathsf{id}} \in W_A^{\mathbb{C},r}$.

Lemma 3.9.23. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$.

- *If* $w_1, w_2 \in W_O$, *then*
$$w_1 \to_r^* w_2 \text{ implies } w_1^{\mathsf{id}} \to_r^* w_2^{\mathsf{id}}.$$
- *If* $w_1, w_2 \in W_O^{sr}$, *then*
$$w_1 \to_e^* w_2 \text{ implies } w_1^{\mathsf{id}} \to_e^* w_2^{\mathsf{id}}.$$

∎

We now describe how to 'compose' two words $u, v \in W_A$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$.

Definition 3.9.24. Let $\mathbb{C} \in \mathsf{StrMonCat}$ and let $u, v \in S_A^{\mathbb{C}}$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$. We define a 'letter' $u \circ' v \in S_A^{\mathbb{C}}$ as follows. By assumption, one of the following cases must hold:

- If $v \equiv \mathsf{x}_A$, then $v^{\mathsf{cod}} \equiv \mathsf{cod}(\mathsf{x}_A) \equiv u^{\mathsf{dom}}$, so that $u \equiv \mathsf{id}(\mathsf{cod}(\mathsf{x}_A))$. Then we set $u \circ' v := \mathsf{x}_A$. Similarly, if $v \equiv \mathsf{x}'_A$, then we must have $u \equiv \mathsf{id}(\mathsf{cod}(\mathsf{x}'_A))$, and we set $u \circ' v := \mathsf{x}'_A$.
- If $v \equiv \mathsf{id}(\mathsf{x}_O)$, then $v^{\mathsf{cod}} \equiv \mathsf{x}_O \equiv u^{\mathsf{dom}}$, so we must have $u \equiv \mathsf{id}(\mathsf{x}_O)$ as well. Then we set $u \circ' v := \mathsf{id}(\mathsf{x}_O)$. Similarly, if $v \equiv \mathsf{id}(\mathsf{x}'_O)$ then we must have $u \equiv \mathsf{id}(\mathsf{x}'_O)$ as well, and we set $u \circ' v := \mathsf{id}(\mathsf{x}'_O)$.
- If $v \equiv \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$, then $v^{\mathsf{cod}} \equiv \mathsf{dom}(\mathsf{x}_A) \equiv u^{\mathsf{dom}}$, so we must have $u \equiv \mathsf{x}_A$ or $u \equiv \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$. In the first case, we set $u \circ' v := \mathsf{x}_A$, and in the second case we set $u \circ' v := \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$. The definition is analogous if $v \in \{\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)), \mathsf{id}(\mathsf{dom}(\mathsf{x}'_A)), \mathsf{id}(\mathsf{cod}(\mathsf{x}'_A))\}$.
- If $v \equiv c_{A,f}$ for some $f : a \to a' \in \mathbb{C}_A$, then $v^{\mathsf{cod}} \equiv c_{O,a'} \equiv u^{\mathsf{dom}}$. So then we must have $u \equiv c_{A,g}$ for some $g : a' \to a'' \in \mathbb{C}_A$. Then we set $u \circ' v := c_{A,g \circ f}$.

Now let $u, v \in W_A^{\mathbb{C}}$ with $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$. We define a word $u \circ' v \in W_A^{\mathbb{C}}$ as follows. Since $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$, it follows that $\ell(u) = \ell(v) = n$ for some $n \geq 1$. So let $u \equiv u_1 \otimes_A \ldots \otimes_A u_n$ and $v \equiv v_1 \otimes_A \ldots \otimes_A v_n$ with $u_i, v_i \in S_A^{\mathbb{C}}$. By assumption, we have $v_i^{\mathsf{cod}} \equiv u_i^{\mathsf{dom}}$ for all $1 \leq i \leq n$. So we set

$$u \circ' v := (u_1 \circ' v_1) \otimes_A \ldots \otimes_A (u_n \circ' v_n) \in W_A^{\mathbb{C}}.$$

∎

The following lemma is then immediate from the definitions:

Lemma 3.9.25. *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $u, v \in W_A^{\mathbb{C}}$ with $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$, then*

$$(u \circ' v)^{\mathsf{dom}} \equiv v^{\mathsf{dom}}$$

and

$$(u \circ' v)^{\mathsf{cod}} \equiv u^{\mathsf{cod}}.$$

∎

We also have:

Lemma 3.9.26. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $u, v \in W_A^{\mathbb{C}}$ *with* $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$. *Then*

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v \downarrow$$

and

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v = u \circ' v.$$

Proof: See Appendix B. ∎

Now we have the following lemma collecting together several useful facts about the previous definitions. Its proof may be found in Appendix B.

Lemma 3.9.27. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$.

1. *For any* $w \in W_A$, *we have* $((w^r)^{\mathsf{dom}})^r \equiv (w^{\mathsf{dom}})^r$ *and* $((w^r)^{\mathsf{cod}})^r \equiv (w^{\mathsf{cod}})^r$.
2. *For any* $w \in W_A$, *we have* $(w^{\mathsf{exp}})^{\mathsf{cod}} \equiv (w^{\mathsf{cod}})^{\mathsf{exp}}$ *and* $(w^{\mathsf{exp}})^{\mathsf{dom}} \equiv (w^{\mathsf{dom}})^{\mathsf{exp}}$.
3. *For any* $w \in W_O \cup W_A$, *we have* $(w^{\mathsf{exp}})^r \equiv w^r$ *and* $(w^r)^{\mathsf{exp}} \equiv w^{\mathsf{exp}}$.
4. *For any* $u, v \in W_A^{sr}$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, *we have* $(((u \circ' v)^r)^{\mathsf{dom}})^r \equiv (v^{\mathsf{dom}})^r$ *and* $(((u \circ' v)^r)^{\mathsf{cod}})^r \equiv (u^{\mathsf{cod}})^r$.
5. *For any* $u, v \in W_A^{sr}$ *with* $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$,
$$u \circ' v \to_e^* u^e \circ' v^e.$$
6. *For any* $u, v \in W_A^r$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, *we have* $(u \circ' v)^{\mathsf{exp}} \equiv u^e \circ' v^e$.
7. *For any* $u, v \in W_A^r$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, *we have*
$$(u^{\mathsf{exp}} \circ' v^{\mathsf{exp}})^{\mathsf{exp}} \equiv u^{\mathsf{exp}} \circ' v^{\mathsf{exp}} \equiv (u \circ' v)^{\mathsf{exp}}.$$
8. *For any* $s, t, u \in W_A^{sr}$ *with* $s^{\mathsf{dom}} \equiv t^{\mathsf{cod}}$ *and* $t^{\mathsf{dom}} \equiv u^{\mathsf{cod}}$, *we have* $s \circ' (t \circ' u) \equiv (s \circ' t) \circ' u$.
9. *For any* $w \in W_O$, *we have* $(w^{\mathsf{id}})^{\mathsf{dom}} \equiv w \equiv (w^{\mathsf{id}})^{\mathsf{cod}}$.
10. *For any* $w \in W_O$, *we have* $(w^r)^{\mathsf{id}} \equiv (w^{\mathsf{id}})^r$ *and* $(w^{\mathsf{exp}})^{\mathsf{id}} \equiv (w^{\mathsf{id}})^{\mathsf{exp}}$.
11. *For any* $w \in W_A$, *we have* $w \circ' (w^{\mathsf{dom}})^{\mathsf{id}} \equiv w$ *and* $(w^{\mathsf{cod}})^{\mathsf{id}} \circ' w \equiv w$.

12. *For any* $C \in \{O, A\}$ *and any* $u, v \in W_C$, *we have*

$$(u^r \otimes_C v^r)^r \equiv (u \otimes_C v)^r \equiv (u^{\mathsf{exp}} \otimes_C v^{\mathsf{exp}})^r.$$

13. *For any* $s_1, s_2, t_1, t_2 \in W_A^r$ *with* $s_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}}$ *and* $t_1^{\mathsf{dom}} \equiv t_2^{\mathsf{cod}}$, *we have* $(s_1 \circ' s_2) \otimes_A (t_1 \circ' t_2) \equiv (s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2)$.
14. *For any* $w \in W_O$, *we have* $\big(w \otimes_O c_{O,e^{\mathbb{C}}}\big)^r \equiv w^r \equiv \big(c_{O,e^{\mathbb{C}}} \otimes_O w\big)^r$, *and for any* $w \in W_A$, *we have* $\big(w \otimes_A c_{A,\mathsf{id}(e^{\mathbb{C}})}\big)^r \equiv w^r \equiv \big(c_{A,\mathsf{id}(e^{\mathbb{C}})} \otimes_A w\big)^r$.
15. *For any* $C \in \{O, A\}$, *any* $s \in S_C$, *and any* $w \in W_C$, *we have* $(s \otimes_C w)^r \equiv (s \otimes_C w^r)^r$ *and* $(w \otimes_C s)^r \equiv (w^r \otimes_C s)^r$.
16. *For any* $s_1, s_2, t_1, t_2 \in W_A^r$ *with* $s_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}}$ *and* $t_1^{\mathsf{dom}} \equiv t_2^{\mathsf{cod}}$, *we have*

$$(s_1 \otimes_A t_1)^{\mathsf{exp}} \circ' (s_2 \otimes_A t_2)^{\mathsf{exp}} \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^{\mathsf{exp}}.$$

∎

We now define the previously mentioned 'reduced word' strict monoidal category $\mathbb{C}^*$.

Definition 3.9.28 (Reduced Word Strict Monoidal Category). For any $\mathbb{C} \in \mathsf{StrMonCat}$, we define a partial Σ_{Str}-structure (i.e. strict monoidal category structure) $\mathbb{C}^*$ as follows:

- We set
$$\mathbb{C}^*_O := W_O^r$$
and
$$\mathbb{C}^*_A := W_A^r.$$
- We define $\mathsf{id}^* : \mathbb{C}^*_O \to \mathbb{C}^*_A$ (using Lemma 3.9.22) as follows:
$$\mathsf{id}^* := (-)^{\mathsf{id}} : W_O^r \to W_A^r.$$
- We define $\mathsf{dom}^*, \mathsf{cod}^* : \mathbb{C}^*_A \to \mathbb{C}^*_O$ as follows. For any $w \in W_A^r$, we set
$$\mathsf{dom}^*(w) := (w^{\mathsf{dom}})^r \in W_O^r$$
and
$$\mathsf{cod}^*(w) := (w^{\mathsf{cod}})^r \in W_O^r.$$

- We define the partial composition function $\circ^* : \mathbb{C}^*_A \times \mathbb{C}^*_A \rightharpoonup \mathbb{C}^*_A$ as follows. First, we set
$$\mathsf{dom}(\circ^*) := \left\{(s,t) \in W^r_A \times W^r_A : (t^{\mathsf{cod}})^{\mathsf{exp}} \equiv (s^{\mathsf{dom}})^{\mathsf{exp}}\right\}.$$
Now let $(s,t) \in \mathsf{dom}(\circ^*)$, so that $(t^{\mathsf{cod}})^{\mathsf{exp}} \equiv (s^{\mathsf{dom}})^{\mathsf{exp}}$. Then by Lemma 3.9.27.2, we have
$$(t^{\mathsf{exp}})^{\mathsf{cod}} \equiv (t^{\mathsf{cod}})^{\mathsf{exp}} \equiv (s^{\mathsf{dom}})^{\mathsf{exp}} \equiv (s^{\mathsf{exp}})^{\mathsf{dom}},$$
so we set
$$s \circ^* t := (s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r \in W^r_A$$
(in the sense of Definition 3.9.24).
- We define $e^* \in \mathbb{C}^*_O$ as $e^* := c_{O,e^{\mathbb{C}}} \in W^r_O$.
- For $C \in \{O, A\}$, we define $\otimes^*_C : \mathbb{C}^*_C \times \mathbb{C}^*_C \to \mathbb{C}^*_C$ as follows. For any $w_1, w_2 \in W^r_C$, we set
$$w_1 \otimes^*_C w_2 := (w_1 \otimes_C w_2)^r \in W^r_C.$$

▮

Let us try to give some intuition for why we have defined composition in $\mathbb{C}^*$ the way we have. Given how we have defined dom^* and cod^* (which seems quite natural), then in order for $\mathbb{C}^*$ to be a category, the composition operation $\circ^*$ will need to have the property that if $s, t \in \mathbb{C}^*_A = W^r_A$ satisfy $\mathsf{dom}^*(s) = \mathsf{cod}^*(t)$, i.e. $(s^{\mathsf{dom}})^r \equiv (t^{\mathsf{cod}})^r$, then $(s,t) \in \mathsf{dom}(\circ^*)$. However, if $s, t \in W^r_A$ satisfy $(s^{\mathsf{dom}})^r \equiv (t^{\mathsf{cod}})^r$, it may not be obvious how to compose them (using the operation $\circ'$ of Definition 3.9.24) to obtain a reduced word $s \circ^* t \in W^r_A$. For example, let $f : a \to e^{\mathbb{C}}$ and $g : e^{\mathbb{C}} \to b$ be arrows of $\mathbb{C}$ with $f, g \neq \mathsf{id}(e^{\mathbb{C}})$, and consider the reduced words $s := \mathsf{id}(\mathsf{x}_O) \otimes_A c_{A,g}$ and $t := c_{A,f} \otimes_A \mathsf{id}(\mathsf{x}_O)$. Then
$$(s^{\mathsf{dom}})^r \equiv \left(\mathsf{x}_O \otimes_O c_{O,e^{\mathbb{C}}}\right)^r \equiv \mathsf{x}_O \equiv \left(c_{O,e^{\mathbb{C}}} \otimes_O \mathsf{x}_O\right)^r \equiv (t^{\mathsf{cod}})^r,$$
so s and t should be composable in $\mathbb{C}^*$. However, we have
$$s^{\mathsf{dom}} \equiv \mathsf{x}_O \otimes_O c_{O,e^{\mathbb{C}}} \not\equiv c_{O,e^{\mathbb{C}}} \otimes_O \mathsf{x}_O \equiv t^{\mathsf{cod}},$$
so we cannot directly apply Definition 3.9.24 to 'syntactically' compose s and t. However, we *do* have
$$(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv ((s^{\mathsf{dom}})^r)^e \equiv \mathsf{x}^e_O \equiv c_{O,e^{\mathbb{C}}} \otimes_O \mathsf{x}_O \otimes_O c_{O,e^{\mathbb{C}}} \equiv \mathsf{x}^e_O \equiv ((t^{\mathsf{cod}})^r)^e \equiv (t^{\mathsf{cod}})^{\mathsf{exp}},$$
and hence (by Lemma 3.9.27.2) we have $(s^{\mathsf{exp}})^{\mathsf{dom}} \equiv (t^{\mathsf{exp}})^{\mathsf{cod}}$, so that we *can* 'syntactically' compose s^{exp} and t^{exp} using Definition 3.9.24. As a result, we stipulate that

(s,t) should be in $\mathsf{dom}(\circ^*)$ if $(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv (t^{\mathsf{cod}})^{\mathsf{exp}}$, which implies $(s^{\mathsf{exp}})^{\mathsf{dom}} \equiv (t^{\mathsf{exp}})^{\mathsf{cod}}$ by Lemma 3.9.27.2, and then we define $s \circ^* t := (s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r$.

Roughly speaking, two reduced words $s, t \in W_A^r$ may look very different and yet have the property that $\mathsf{dom}^*(s) = \mathsf{cod}^*(t)$, i.e. $(s^{\mathsf{dom}})^r \equiv (t^{\mathsf{cod}})^r$, because s^{dom} and t^{cod} may be 'equal' up to (many) occurrences of $c_{O,e^{\mathbb{C}}}$ in potentially different positions. But once we 'expand' $(s^{\mathsf{dom}})^r$ and $(t^{\mathsf{cod}})^r$ using $\to_e$ to insert $c_{O,e^{\mathbb{C}}}$ wherever possible, the resulting words *will* be identical, and then the resulting expanded versions of s and t will indeed be 'syntactically' composable.

Using primarily Lemma 3.9.27, we now have:

Proposition 3.9.29. *For any $\mathbb{C} \in \mathsf{StrMonCat}$, the partial Σ_{Str}-structure $\mathbb{C}^*$ is a model of $\mathbb{T}_{\mathsf{Str}}$, i.e. $\mathbb{C}^*$ is a strict monoidal category.*

Proof: See Appendix B. ∎

We now define an 'inclusion' strict monoidal functor $i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$.

Definition 3.9.30. Let $\mathbb{C} \in \mathsf{StrMonCat}$. We define a strict monoidal functor

$$i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$$

as follows:

- For any object $a \in \mathbb{C}_O$,

$$i_{\mathbb{C}}(a) := c_{O,a} \in \mathbb{C}^*_O = W^r_O.$$

- For any arrow $f \in \mathbb{C}_A$,

$$i_{\mathbb{C}}(f) := c_{A,f} \in \mathbb{C}^*_A = W^r_A.$$

Justification. For the verification that $i_{\mathbb{C}}$ is a well-defined strict monoidal functor, see Appendix B. ∎

Incidentally, although this will not be used in what follows, we have the following universal property of the reduced word strict monoidal category $\mathbb{C}^*$, whose proof may be found in Appendix B:

Proposition 3.9.31. *Let $\mathbb{C} \in \mathsf{StrMonCat}$. If $\mathbb{D} \in \mathsf{StrMonCat}$ and $F : \mathbb{C} \to \mathbb{D}$ is a strict monoidal functor and $a, a' \in \mathbb{D}_O$ are objects and $f, f' \in \mathbb{D}_A$ are arrows, then there is a unique strict monoidal functor $F^* : \mathbb{C}^* \to \mathbb{D}$ with $F^* \circ i_{\mathbb{C}} = F$ and*

$$F^*(\mathsf{x}_O) = a, F^*(\mathsf{x}'_O) = a', F^*(\mathsf{x}_A) = f, F^*(\mathsf{x}'_A) = f'.$$

Thus,

$$\mathbb{C}^* \cong \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle \in \mathsf{StrMonCat}.$$

∎

For any $\mathbb{C} \in \mathsf{StrMonCat}$, consider the strict monoidal category $\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle$ obtained from $\mathbb{C}$ by freely adjoining 'indeterminate' objects $\mathsf{x}_O, \mathsf{x}'_O$ and 'indeterminate' arrows $\mathsf{x}_A, \mathsf{x}'_A$. We have just constructed a strict monoidal functor $i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$, and we also have $\mathsf{x}_O, \mathsf{x}'_O \in \mathbb{C}^*_O = W^r_O$ and $\mathsf{x}_A, \mathsf{x}'_A \in \mathbb{C}^*_A = W^r_A$. So by the universal property of $\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle$ (cf. Proposition 2.2.10), there is a unique strict monoidal functor

$$i^*_{\mathbb{C}} : \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle \to \mathbb{C}^*$$

such that

$$i^*_{\mathbb{C}} \circ \eta^{\mathbb{C}}_{O,O,A,A} = i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$$

(where $\eta^{\mathbb{C}}_{O,O,A,A} : \mathbb{C} \to \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle$ is the canonical strict monoidal functor) and

$$i^*_{\mathbb{C}}([\mathsf{x}_O]) = \mathsf{x}_O,\ i^*_{\mathbb{C}}([\mathsf{x}'_O]) = \mathsf{x}'_O,\ i^*_{\mathbb{C}}([\mathsf{x}_A]) = \mathsf{x}_A, \text{ and } i^*_{\mathbb{C}}([\mathsf{x}'_A]) = \mathsf{x}'_A.$$

We now have:

Lemma 3.9.32. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $C \in \{O, A\}$. *Then for any* $w \in W_C$ *(so that* $[w]_C \in \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle_C$*) we have*

$$i^*_{\mathbb{C}}([w]_C) = w^r \in \mathbb{C}^*_C = W^r_C.$$

Proof: See Appendix B. ∎

At last, we can show:

Proposition 3.9.33. *If* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $w_1, w_2 \in W^r_O \cup W^r_A$ *are reduced words of the same sort such that* $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w_1 = w_2$, *then* $w_1 \equiv w_2$.

Proof: Let $w_1, w_2 \in W^r_C$ for some $C \in \{O, A\}$, and suppose that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w_1 = w_2.$$

We must show that $w_1 \equiv w_2$. The assumption implies that $[w_1]_C = [w_2]_C$ holds in $\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle_C$. Also, since w_1, w_2 are reduced, we have by Lemma 3.9.7 that $w_1 \equiv w^r_1$ and $w_2 \equiv w^r_2$. Then by Lemma 3.9.32, we obtain

$$w_1 \equiv w^r_1 \equiv i^*_{\mathbb{C}}([w_1]_C) = i^*_{\mathbb{C}}([w_2]_C) \equiv w^r_2 \equiv w_2,$$

as desired. ∎

Proposition 3.9.33 implies that any term $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ is provably equal to *at most one* reduced word in $W_O \cup W_A$. The following lemma (whose proof is deferred to Appendix B) says that any such term is provably equal to *at least one* word in $W_O \cup W_A$, and hence (by Lemma 3.9.8) to *exactly one* reduced word.

Lemma 3.9.34. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$. *If* $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ *with* $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$, *then:*

- *If* $t : O$, *then* $[t] = [w]$ *for some* $w \in W_O^{\mathbb{C}}$.
- *If* $t : A$, *then* $[t] = [w]$ *for some* $w \in W_A^{\mathbb{C}}$. ∎

Combining Lemmas 3.9.8 and 3.9.34 with Proposition 3.9.33, we conclude that any term $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ is provably equal to a *unique* reduced word of the same sort.

Now we finally return to the task of giving an explicit description of the isotropy group of a strict monoidal category $\mathbb{C}$. Our aim is to show that

$$G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}) = \left\{ \begin{array}{l} \left([c_{O,a} \otimes_O \mathsf{x}_O \otimes_O c_{O,a^{-1}}] , \left[c_{A,\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{A,\mathsf{id}(a^{-1})} \right] \right) \in \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A \\ : a \in \mathbb{C}_O \text{ is invertible in the monoid } (\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) \text{ with inverse } a^{-1} \in \mathbb{C}_O. \end{array} \right\}.$$

Toward this goal, we first state the more straightforward right-to-left inclusion, whose proof may be found in Appendix B.

Proposition 3.9.35. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$. *If* $a \in C_O$ *is invertible in the monoid* $(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ *with inverse* $b \in \mathbb{C}_O$, *then*

$$\left([c_{O,a} \otimes_O \mathsf{x}_O \otimes_O c_{O,b}] , \left[c_{A,\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{A,\mathsf{id}(b)} \right] \right) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}).$$

∎

Lemma 3.9.36. *Let* $\mathbb{C} \in \mathsf{StrMonCat}$.

1. *For any distinct variables* $y_1, y_2 : O$, *we have* $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.
2. *For any distinct variables* $y_1, y_2 : A$, *we have* $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \nvdash^{y_1,y_2} y_1 = y_2$.

Proof: By Lemma 3.1.2, it suffices to show that there is a small strict monoidal category $\mathbb{D}$ with at least two objects (and hence at least two arrows) and a strict monoidal functor $F : \mathbb{C} \to \mathbb{D}$. We take $\mathbb{D} := \mathbb{C}^*$ and $F := i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$. Note that $\mathbb{C}^*$ has at least two objects, because (e.g.) $\mathsf{x}_O, \mathsf{x}'_O \in \mathbb{C}^*_O = W^r_O$ and $\mathsf{x}_O \neq \mathsf{x}'_O$. ∎

Now we prove the converse inclusion:

Proposition 3.9.37. *Let $\mathbb{C} \in \mathsf{StrMonCat}$, and let $([s_O], [s_A]) \in \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A$. If $([s_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$, then there is some object $a \in \mathbb{C}_O$ that is invertible in the monoid $(\mathbb{C}_O, \otimes^{\mathbb{C}}_O, e^{\mathbb{C}})$ with inverse $b \in \mathbb{C}_O$ and*

$$([s_O], [s_A]) = ([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}]).$$

Proof: Let $([s_O], [s_A]) \in \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A$ with $([s_O], [s_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$. Since $[s_O] \in \mathbb{C}\langle \mathsf{x}_O \rangle_O$ and $[s_A] \in \mathbb{C}\langle \mathsf{x}_A \rangle_A$, we have that $s_O \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O))_O$ is a term of sort O with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O) \vdash s_O \downarrow$, and similarly $s_A \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A))_A$ is a term of sort A with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A) \vdash s_A \downarrow$. So $s_O, s_A \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ are terms with $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s_O \downarrow \wedge s_A \downarrow$. By Lemmas 3.9.34 and 3.9.8, it follows that there are reduced words $w_O \in W^r_O$ and $w_A \in W^r_A$ with

$$[s_O] = [w_O] \in \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle_O$$

and

$$[s_A] = [w_A] \in \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle_A.$$

Since $([s_O], [s_A]) = ([w_O], [w_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$, it follows that this pair commutes generically with the $\otimes_O$ operation symbol. This means that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O) \vdash w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O] = w_O \otimes_O w_O[\mathsf{x}'_O / \mathsf{x}_O].$$

So this equation is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$ as well. Since w_O is reduced, it is easy to see that $w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O]$ and $w_O[\mathsf{x}'_O / \mathsf{x}_O]$ are reduced as well, so that (by Lemma 3.9.7) we have $w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O]^r \equiv w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O]$ and $w_O[\mathsf{x}'_O / \mathsf{x}_O]^r \equiv w_O[\mathsf{x}'_O / \mathsf{x}_O]$. So by Lemma 3.9.8, we obtain

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O] = (w_O \otimes_O w_O[\mathsf{x}'_O / \mathsf{x}_O])^r.$$

Then by Proposition 3.9.33, it follows that

$$w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O] \equiv (w_O \otimes_O w_O[\mathsf{x}'_O / \mathsf{x}_O])^r. \qquad (\star)$$

Now, suppose towards a contradiction that w_O has at least two occurrences of x_O. Then in $(w_O \otimes_O w_O[\mathsf{x}'_O / \mathsf{x}_O])^r$, all occurrences of x_O will occur to the left of all occurrences of x'_O. But in $w_O[\mathsf{x}_O \otimes_O \mathsf{x}'_O / \mathsf{x}_O]$, there will be an occurrence of x'_O to the left of an occurrence of x_O. But this contradicts the fact $(\star)$ that these words are identical. So it follows that w_O has at most one occurrence of x_O.

By Lemmas 2.2.56 and 3.9.36, it also follows that w_O must have at least one occurrence of x_O, since $([w_O],[w_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ (and $\mathbb{C}_O$ is non-empty, because $e^{\mathbb{C}} \in \mathbb{C}_O$). So w_O has *exactly one* occurrence of x_O.

So (since w_O is reduced), either $w_O \equiv \mathsf{x}_O$, or there are $a, b \neq e^{\mathbb{C}} \in \mathbb{C}_O$ such that $w_O \in \{c_a \otimes_O \mathsf{x}_O, \mathsf{x}_O \otimes_O c_a, c_a \otimes_O \mathsf{x}_O \otimes_O c_b\}$. If $w_O \equiv \mathsf{x}_O$, then we have

$$[s_O] = [w_O] = [\mathsf{x}_O] = [e \otimes_O \mathsf{x}_O \otimes_O e] = [c_{e^{\mathbb{C}}} \otimes_O \mathsf{x}_O \otimes_O c_{e^{\mathbb{C}}}],$$

which is of the desired form (since obviously $e^{\mathbb{C}} \otimes_O^{\mathbb{C}} e^{\mathbb{C}} = e^{\mathbb{C}}$). Otherwise, we reason as follows. If $w_O \equiv c_a \otimes_O \mathsf{x}_O$ for some $a \neq e^{\mathbb{C}} \in \mathbb{C}_O$, then by $(\star)$, we would obtain

$$c_a \otimes_O \mathsf{x}_O \otimes_O \mathsf{x}'_O \equiv (c_a \otimes_O \mathsf{x}_O \otimes_O c_a \otimes_O \mathsf{x}'_O)^r \equiv c_a \otimes_O \mathsf{x}_O \otimes_O c_a \otimes_O \mathsf{x}'_O,$$

which is false. Similarly, we cannot have $w_O \equiv \mathsf{x}_O \otimes_O c_a$ for some $a \neq e^{\mathbb{C}} \in \mathbb{C}_O$. Hence, if $w_O \not\equiv \mathsf{x}_O$, then we must have $w_O \equiv c_a \otimes_O \mathsf{x}_O \otimes_O c_b$ for some $a, b \neq e^{\mathbb{C}} \in \mathbb{C}_O$.

Now we show that $a \otimes_O^{\mathbb{C}} b = e^{\mathbb{C}} = b \otimes_O^{\mathbb{C}} a$. By $(\star)$, we have

$$\begin{aligned} c_a \otimes_O \mathsf{x}_O \otimes_O \mathsf{x}'_O \otimes_O c_b &\equiv (c_a \otimes_O \mathsf{x}_O \otimes_O c_b \otimes_O c_a \otimes_O \mathsf{x}'_O \otimes_O c_b)^r \\ &\equiv (c_a \otimes_O \mathsf{x}_O \otimes_O c_{b \otimes_O^{\mathbb{C}} a} \otimes_O \mathsf{x}'_O \otimes_O c_b)^r. \end{aligned}$$

This implies that we must have $c_{b \otimes_O^{\mathbb{C}} a} \equiv c_{e^{\mathbb{C}}}$ and hence $b \otimes_O^{\mathbb{C}} a = e^{\mathbb{C}}$, as desired.

Since $([s_O],[s_A]) = ([w_O],[w_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$, it follows that $([w_O],[w_A])$ commutes generically with the constant $e : O$, which means that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \vdash w_O[e/\mathsf{x}_O] = e,$$

i.e.

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \vdash c_a \otimes_O e \otimes_O c_b = e.$$

Then we have

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \vdash e = c_a \otimes_O e \otimes_O c_b = c_a \otimes_O c_b = c_{a \otimes_O^{\mathbb{C}} b}.$$

Since the canonical $\Sigma_{\mathsf{Str}}(\mathbb{C})$-structure $\widehat{\mathbb{C}}$ is a model of $\mathbb{T}_{\mathsf{Str}}(\mathbb{C})$, it follows (by soundness of partial Horn logic) that $\widehat{\mathbb{C}} \models c_{a \otimes_O^{\mathbb{C}} b} = e$, which implies that $a \otimes_O^{\mathbb{C}} b = e^{\mathbb{C}}$, as desired.

In summary, we have shown that

$$[s_O] = [c_a \otimes_O \mathsf{x}_O \otimes_O c_b]$$

for some $a \in \mathbb{C}_O$ that is invertible in the monoid $(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ with inverse $b \in \mathbb{C}_O$. Now we consider $[s_A] = [w_A]$, with $w_A \in W_A^r$. As for $[s_O] = [w_O]$ (now using the assumption that $([s_O],[s_A]) = ([w_O],[w_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ commutes generically with the

$\otimes_A$ operation symbol), we can show that $[s_A] = [w_A] = [c_f \otimes_A \mathsf{x}_A \otimes_A c_g]$ for some $f, g \in \mathbb{C}_A$. To complete the proof, we must show that

$$[s_A] = [c_f \otimes_A \mathsf{x}_A \otimes_A c_g] = [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}].$$

Since $([w_O], [w_A]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ commutes with the function symbol $\mathsf{id} : O \to A$, this means that the sequent

$$\top \vdash w_A[\mathsf{id}(\mathsf{x}_O)/\mathsf{x}_A] = \mathsf{id}(w_O)$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O)$ (since $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O) \vdash \mathsf{id}(\mathsf{x}_O) \downarrow$), which implies that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash c_f \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_g = c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)},$$

and hence

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash (c_f \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_g)^r = (c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)})^r$$

by Lemma 3.9.8. Then by Proposition 3.9.33, we obtain

$$(c_f \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_g)^r \equiv (c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)})^r,$$

which easily implies that $c_f \equiv c_{\mathsf{id}(a)}$ and $c_g \equiv c_{\mathsf{id}(b)}$. Hence, we have $f = \mathsf{id}(a)$ and $g = \mathsf{id}(b)$, so that

$$[s_A] = [w_A] = [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}],$$

as desired. This completes the proof of Proposition 3.9.37. ∎

From Propositions 3.9.35 and 3.9.37 we conclude:

Proposition 3.9.38. *If* $\mathbb{C} \in \mathsf{StrMonCat}$, *then*

$$G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}) = \left\{ \begin{array}{l} \left([c_{O,a} \otimes_O \mathsf{x}_O \otimes_O c_{O,a^{-1}}], \left[c_{A,\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{A,\mathsf{id}(a^{-1})}\right]\right) \in \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A \\ : a \in \mathbb{C}_O \textit{ is invertible in the monoid } (\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) \textit{ with inverse } a^{-1} \in \mathbb{C}_O. \end{array} \right\}.$$

∎

We now give a *categorical* description of $\mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ derived from the logical description of $G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$.

Definition 3.9.39 (**Strict Monoidal Inner Automorphisms**)**.** If $\mathbb{D} \in \mathsf{StrMonCat}$ and $d \in \mathbb{D}_O$ is invertible in the monoid $(\mathbb{D}_O, \otimes_O^{\mathbb{D}}, e^{\mathbb{D}})$ with inverse $d^{-1} \in \mathbb{D}_O$, then the strict monoidal automorphism

$$\mathsf{conj}_d : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$$

(with inverse $\mathsf{conj}_{d^{-1}} : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$) is defined as follows:

- For any object $x \in \mathbb{D}_O$,

$$\mathsf{conj}_d(x) := d \otimes_O^{\mathbb{D}} x \otimes_O^{\mathbb{D}} d^{-1}.$$

- For any morphism $f \in \mathbb{D}_A$,

$$\mathsf{conj}_d(f) := \mathsf{id}^{\mathbb{D}}(d) \otimes_A^{\mathbb{D}} f \otimes_A^{\mathbb{D}} \mathsf{id}^{\mathbb{D}}(d^{-1}).$$

We say that a strict monoidal automorphism $F : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ is an *inner automorphism* if there is some $d \in \mathsf{Inv}(\mathbb{D}_O, \otimes_O^{\mathbb{D}}, e^{\mathbb{D}})$ with $F = \mathsf{conj}_d$. ∎

Corollary 3.9.40. *If* $\mathbb{C} \in \mathsf{StrMonCat}$ *and*

$$\pi = (\pi_F : \mathsf{cod}(F) \to \mathsf{cod}(F))_{F \in \mathsf{Dom}(\mathbb{C})}$$

is a $\mathsf{Dom}(\mathbb{C})$*-indexed family of endomorphisms in* $\mathsf{StrMonCat}$*, then* $\pi \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ *iff there is a (uniquely determined) element* $a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ *with*

$$\pi_F = \mathsf{conj}_{F(a)} : \mathsf{cod}(F) \xrightarrow{\sim} \mathsf{cod}(F)$$

for every $F \in \mathsf{Dom}(\mathbb{C})$*.*

Proof: Aside from the uniqueness part, the result easily follows from Corollary 2.2.42 and Proposition 3.9.38. For the uniqueness assertion, let $a, b \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ with

$$[c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}}] = [c_b \otimes_O \mathsf{x}_O \otimes_O c_{b^{-1}}];$$

we must show that $a = b$. By Lemma 3.9.8, we have

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash (c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}})^r = (c_b \otimes_O \mathsf{x}_O \otimes_O c_{b^{-1}})^r .$$

Then by Proposition 3.9.33, we obtain

$$(c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}})^r \equiv (c_b \otimes_O \mathsf{x}_O \otimes_O c_{b^{-1}})^r ,$$

which easily implies that we must have $c_a \equiv c_b$ and hence $a = b$, as desired. ∎

Corollary 3.9.41. *Let* $F : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ *be a strict monoidal automorphism of* $\mathbb{C} \in \mathsf{StrMonCat}$*. Then* F *is an inner automorphism of* $\mathbb{C}$ *iff there is some* $\pi \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ *with* $\pi_{\mathsf{id}_{\mathbb{C}}} = F$*.*

Proof: Suppose first that F is an inner automorphism of $\mathbb{C}$, so that there is some $a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ with $F = \mathsf{conj}_a : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$. For any strict monoidal category $\mathbb{D}$ and strict monoidal functor $G : \mathbb{C} \to \mathbb{D}$, define the strict monoidal automorphism $\pi_G : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ by

$$\pi_G := \mathsf{conj}_{G(a)} : \mathbb{D} \xrightarrow{\sim} \mathbb{D}.$$

Then by Corollary 3.9.40, it follows that the family

$$\pi := \Big(\pi_G : \mathsf{cod}(G) \xrightarrow{\sim} \mathsf{cod}(G)\Big)_{G \in \mathsf{Dom}(\mathbb{C})} \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$$

and

$$\pi_{\mathsf{id}_{\mathbb{C}}} := \mathsf{conj}_{\mathsf{id}_{\mathbb{C}}(a)} = \mathsf{conj}_a = F : \mathbb{C} \xrightarrow{\sim} \mathbb{C},$$

as desired.

Now suppose that there is some $\pi = \Big(\pi_G : \mathsf{cod}(G) \xrightarrow{\sim} \mathsf{cod}(G)\Big)_{G \in \mathsf{Dom}(\mathbb{C})} \in \mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$ with $\pi_{\mathsf{id}_{\mathbb{C}}} = F : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$. Then by Corollary 3.9.40, there is a (uniquely determined) object $a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ with $\pi_G = \mathsf{conj}_{G(a)} : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ for every strict monoidal functor $G : \mathbb{C} \to \mathbb{D}$. Then we have

$$F = \pi_{\mathsf{id}_{\mathbb{C}}} = \mathsf{conj}_{\mathsf{id}_{\mathbb{C}}(a)} = \mathsf{conj}_a : \mathbb{C} \xrightarrow{\sim} \mathbb{C},$$

so that F is an inner automorphism of $\mathbb{C}$, as desired. ∎

For our last corollary, we need the following definition:

Definition 3.9.42. We define the functor $U : \mathsf{StrMonCat} \to \mathsf{Mon}$ as follows:

- For $\mathbb{C} \in \mathsf{StrMonCat}$, we set

$$U(\mathbb{C}) := (\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) \in \mathsf{Mon}.$$

- If $F = (F_O, F_A) : \mathbb{C} \to \mathbb{D}$ is a morphism in $\mathsf{StrMonCat}$, then we set

$$U(F) := F_O : (\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) \to (\mathbb{D}_O, \otimes_O^{\mathbb{D}}, e^{\mathbb{D}}).$$

∎

Corollary 3.9.43. *The isotropy group*

$$\mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}} \cong G_{\mathbb{T}_{\mathsf{Str}}} : \mathsf{StrMonCat} \to \mathsf{Group}$$

is naturally isomorphic to the functor $\mathsf{Inv} \circ U : \mathsf{StrMonCat} \to \mathsf{Group}$.

Proof: We must define a natural isomorphism $\varphi : G_{\mathbb{T}_{\mathsf{Str}}} \overset{\sim}{\Longrightarrow} \mathsf{Inv} \circ U$. So let $\mathbb{C}$ be a (small) strict monoidal category; we define a group isomorphism

$$\varphi_{\mathbb{C}} : G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}) \to \mathsf{Inv}(U(\mathbb{C})) = \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}).$$

Appealing to Proposition 3.9.38, if $a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$, then we define $\varphi_{\mathbb{C}}$ by

$$\left([c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}}], [c_{\mathsf{id}_a} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}_{a^{-1}}}]\right) \mapsto a.$$

The argument given in the proof of Corollary 3.9.40 shows that $\varphi_{\mathbb{C}}$ is well-defined.

To show that $\varphi_{\mathbb{C}}$ is a group homomorphism, it suffices to show for $a, b \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ that

$$\varphi_{\mathbb{C}}\left(\left[c_{a\otimes_O^{\mathbb{C}} b} \otimes_O \mathsf{x}_O \otimes_O c_{(a\otimes_O^{\mathbb{C}} b)^{-1}}\right], \left[c_{\mathsf{id}\left(a\otimes_O^{\mathbb{C}} b\right)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}\left(\left(a\otimes_O^{\mathbb{C}} b\right)^{-1}\right)}\right]\right) = a \otimes_O^{\mathbb{C}} b,$$

which is true by definition. Lastly, it is obvious that $\varphi_{\mathbb{C}}$ is injective, and it is surjective by Proposition 3.9.35. This completes the proof that $\varphi_{\mathbb{C}} : G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}) \xrightarrow{\sim} \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ is a group isomorphism.

To show naturality, let $F = (F_O, F_A) : \mathbb{C} \to \mathbb{D}$ be a morphism in $\mathsf{StrMonCat}$, and let us show

$$\mathsf{Inv}(F_O) \circ \varphi_{\mathbb{C}} = \varphi_{\mathbb{D}} \circ G_{\mathbb{T}_{\mathsf{Str}}}(F) : G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}) \to \mathsf{Inv}(\mathbb{D}_O, \otimes_O^{\mathbb{D}}, e^{\mathbb{D}}).$$

For any $a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$, we have

$$\begin{aligned}
&\mathsf{Inv}(F_O)\left(\varphi_{\mathbb{C}}\left(\left[c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}}\right], \left[c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(a^{-1})}\right]\right)\right) \\
&= \mathsf{Inv}(F_O)(a) \\
&= F_O(a) \\
&= \varphi_{\mathbb{D}}\left(\left[c_{F_O(a)} \otimes_O \mathsf{x}_O \otimes_O c_{F_O(a)^{-1}}\right], \left[c_{\mathsf{id}(F_O(a))} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(F_O(a)^{-1})}\right]\right) \\
&= \varphi_{\mathbb{D}}\left(G_{\mathbb{T}_{\mathsf{Str}}}(F)\left(\left[c_a \otimes_O \mathsf{x}_O \otimes_O c_{a^{-1}}\right], \left[c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(a^{-1})}\right]\right)\right),
\end{aligned}$$

as required. This completes the proof that $\varphi : G_{\mathbb{T}_{\mathsf{Str}}} \overset{\sim}{\Longrightarrow} \mathsf{Inv} \circ U$ is a natural isomorphism. ∎

Let $\mathsf{Disc} : \mathsf{Mon} \to \mathsf{StrMonCat}$ be the functor that sends any monoid M to the discrete strict monoidal category $\mathsf{Disc}(M)$ with $\mathsf{Disc}(M)_O = M$. From Corollary 3.9.43 we then easily obtain:

Corollary 3.9.44. *The isotropy group*

$$\mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}} \cong G_{\mathbb{T}_{\mathsf{Str}}} : \mathsf{StrMonCat} \to \mathsf{Group}$$

is naturally isomorphic to the functor

$$\mathcal{Z}_{\mathbb{T}_{\mathsf{Str}}} \circ \mathsf{Disc} \circ U : \mathsf{StrMonCat} \to \mathsf{Mon} \to \mathsf{StrMonCat} \to \mathsf{Group}.$$

∎

To summarize, we have shown that if $\mathbb{C}$ is any strict monoidal category whose object monoid $(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ contains non-trivial invertible elements, then $\mathbb{C}$ has non-trivial isotropy group isomorphic to $\mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$.

3.10 Other Examples

As the reader can probably tell, many of the examples in this chapter suggest that the isotropy group of a quasi-equational theory encodes a notion of 'conjugation' or 'inner automorphism' for (models of) the theory. In fact, it makes sense to now state the following definition:

Definition 3.10.1 (**Inner Automorphisms**)**.** Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ, and let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. We say that an automorphism $f : M \xrightarrow{\sim} M$ is an *inner automorphism* if one of the following equivalent conditions holds:

- There is an element $\pi \in \mathcal{Z}_{\mathbb{T}}(M)$ with
$$f = \pi_{\mathsf{id}_M} : M \xrightarrow{\sim} M.$$
- There is an element $([s_C])_C \in G_{\mathbb{T}}(M)$ with
$$f = (s_C^* : M_C \to M_C)_C : M \xrightarrow{\sim} M.$$

∎

We now also sketch another quasi-equational theory whose isotropy groups we have (partly) computed, but whose treatment we have chosen *not* to include in this thesis, mainly for reasons of space:

- Racks and Quandles: Racks and quandles are algebraic structures that axiomatize the notion of *conjugation* (without reference to multiplication or inverses). Both theories may be expressed as *algebraic* theories over a single-sorted signature with two binary function symbols $\triangleleft$ and $\triangleleft^{-1}$. The axioms for the theory of racks are as follows:

- $\top \vdash^{x,y,z} x \lhd (y \lhd z) = (x \lhd y) \lhd (x \lhd z)$.
- $\top \vdash^{x,y,z} x \lhd^{-1} (y \lhd^{-1} z) = (x \lhd^{-1} y) \lhd^{-1} (x \lhd^{-1} z)$.
- $\top \vdash^{x,y} (x \lhd y) \lhd^{-1} y = x$.
- $\top \vdash^{x,y} (x \lhd^{-1} y) \lhd y = x$.

The axioms for the theory of quandles are the axioms for the theory of racks together with the following additional axiom:

- $\top \vdash^{x} x \lhd x = x = x \lhd^{-1} x$.

For example, a quandle structure can be specified on (the underlying set of) any group G by setting $g \lhd h := h^{-1}gh$ and $g \lhd^{-1} h := hgh^{-1}$ for any $g, h \in G$. Using the translation of the word problems for free racks and quandles into the word problem for free groups given in [7], we have then shown that the isotropy group of any *free* quandle on n generators is isomorphic to the free group on n generators, while the isotropy group of any *free* rack on n generators is isomorphic to the product of the additive group $\mathbb{Z}$ with the free group on n generators. See also [13, 4.9].

Chapter 4

Closure Properties

In this chapter, we will investigate how modifying the signature and/or axioms of a quasi-equational theory can change the isotropy group of the theory. Specifically, we will examine what happens to the isotropy group of a quasi-equational theory if we add a new function symbol to its signature, and we will compute the isotropy group of a disjoint union of quasi-equational theories in terms of the isotropy groups of the component theories.

4.1 Adding a Constant

For this section, fix a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ.

Definition 4.1.1. Let $d \notin \Sigma$ be a new constant symbol of some sort $A \in \Sigma$.

- We define Σ_d to be the following relation-free signature:
  - $(\Sigma_d)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
  - $(\Sigma_d)_{\mathsf{Fun}} := \Sigma_{\mathsf{Fun}} \cup \{d : A\}$.
- We define $\mathbb{T}_d$ to be the quasi-equational theory over Σ_d whose axioms are those of $\mathbb{T}$ together with the additional axiom

$$\top \vdash d \downarrow .$$

∎

An object of $\mathsf{P}\mathbb{T}_d\mathsf{mod}$ is therefore just a pair (M, d^M) with $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $d^M \in M_A$, while a Σ_d-morphism $h : (M, d^M) \to (N, d^N)$ between $\mathbb{T}_d$-models is a Σ-morphism $h : M \to N$ such that $h_A(d^M) = d^N$. The following result is now not too surprising: the elements of isotropy of a model (M, d^M) of $\mathbb{T}_d$ can be identified with the elements of isotropy of M that preserve d^M:

Proposition 4.1.2. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*. For any* $d^M \in M_A$*,*

$$\mathcal{Z}_{\mathbb{T}_d}(M, d^M) \cong \left\{\pi \in \mathcal{Z}_{\mathbb{T}}(M) : \pi^A_{\mathsf{id}_M}(d^M) = d^M\right\}.$$

Proof: Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $d^M \in M_A$. Note first that the righthand side of the above isomorphism is a subgroup of $\mathcal{Z}_{\mathbb{T}}(M)$, and hence is a group. We define a group isomorphism

$$\varphi^{d^M} : \mathcal{Z}_{\mathbb{T}_d}(M, d^M) \xrightarrow{\sim} \left\{\pi \in \mathcal{Z}_{\mathbb{T}}(M) : \pi^A_{\mathsf{id}_M}(d^M) = d^M\right\}.$$

Let

$$\tau = \Big(\tau_f : \mathsf{cod}(f) \xrightarrow{\sim} \mathsf{cod}(f)\Big)_{f \in \mathsf{Dom}(M, d^M)} \in \mathcal{Z}_{\mathbb{T}_d}(M, d^M).$$

We want to define

$$\varphi^{d^M}(\tau) \in \mathcal{Z}_{\mathbb{T}}(M).$$

To do this, let $F : M \to N$ be any morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M; we must define a Σ-automorphism

$$\varphi^{d^M}(\tau)_F : N \xrightarrow{\sim} N.$$

We have $d^M \in M_A$, and hence $F_A(d^M) \in N_A$, so that $\big(N, F_A(d^M)\big) \in \mathsf{P}\mathbb{T}_d\mathsf{mod}$ and $F : (M, d^M) \to \big(N, F_A(d^M)\big)$ is a Σ_d-morphism. Then

$$\tau_F : \big(N, F_A(d^M)\big) \xrightarrow{\sim} \big(N, F_A(d^M)\big)$$

is a Σ_d-automorphism, and hence in particular a Σ-automorphism of N. So we set

$$\varphi^{d^M}(\tau)_F := \tau_F : N \xrightarrow{\sim} N,$$

which completes the definition of $\varphi^{d^M}(\tau)$. Since $\tau \in \mathcal{Z}_{\mathbb{T}_d}(M, d^M)$, it easily follows that $\varphi^{d^M}(\tau)$ satisfies the naturality condition to be an element of $\mathcal{Z}_{\mathbb{T}}(M)$. Finally, we have

$$\varphi^{d^M}(\tau)^A_{\mathsf{id}_M}(d^M) = \tau^A_{\mathsf{id}_M}(d^M) = d^M,$$

because $\tau_{\mathsf{id}_M} : (M, d^M) \xrightarrow{\sim} (M, d^M)$ is a Σ_d-morphism. So

$$\varphi^{d^M}(\tau) \in \left\{\pi \in \mathcal{Z}_{\mathbb{T}}(M) : \pi^A_{\mathsf{id}_M}(d^M) = d^M\right\}.$$

It is then trivial to verify that φ^{d^M} is an injective group homomorphism.

For surjectivity, let $\pi \in \mathcal{Z}_{\mathbb{T}}(M)$ with $\pi^A_{\mathsf{id}_M}(d^M) = d^M$. We must define $\tau \in \mathcal{Z}_{\mathbb{T}_d}(M, d^M)$ with $\varphi^{d^M}(\tau) = \pi$. So let $F : (M, d^M) \to (N, d^N)$ be any morphism in $\mathsf{P}\mathbb{T}_d\mathsf{mod}$ with domain (M, d^M); we must define a Σ_d-automorphism

$$\tau_F : (N, d^N) \xrightarrow{\sim} (N, d^N).$$

Since F is in particular a Σ-morphism $M \to N$, we have a Σ-automorphism

$$\pi_F : N \xrightarrow{\sim} N.$$

We also have

$$\begin{aligned} d^N &= F_A(d^M) \\ &= F_A(\pi^A_{\mathsf{id}_M}(d^M)) \\ &= (F \circ \pi_{\mathsf{id}_M})_A(d^M) \\ &= (\pi_F \circ F)_A(d^M) \\ &= \pi^A_F(F_A(d^M)) \\ &= \pi^A_F(d^N), \end{aligned}$$

so that $\pi_F : (N, d^N) \xrightarrow{\sim} (N, d^N)$ is also a Σ_d-automorphism (the fourth equality holds because $\pi \in \mathcal{Z}_{\mathbb{T}}(M)$). So we define

$$\tau_F := \pi_F : (N, d^N) \xrightarrow{\sim} (N, d^N).$$

Since $\pi \in \mathcal{Z}_{\mathbb{T}}(M)$, it easily follows that $\tau \in \mathcal{Z}_{\mathbb{T}_d}(M, d^M)$. And we clearly have

$$\varphi^{d^M}(\tau) = \pi,$$

which proves that φ^{d^M} is surjective and thus a group isomorphism. ∎

Corollary 4.1.3. *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. For any $d^M \in M_A$,*

$$G_{\mathbb{T}_d}(M, d^M) \cong \left\{([s_C])_C \in G_{\mathbb{T}}(M) : s^*_A\left(d^M\right) = d^M\right\}.$$

∎

Thus, the isotropy group of a model of $\mathbb{T}_d$ is just (isomorphic to) the subgroup of the isotropy group of the underlying model of $\mathbb{T}$ consisting of all elements of isotropy that preserve the interpretation of the constant d. Recalling the examples of the preceding chapter, we then easily obtain the following corollary:

Corollary 4.1.4.

- *Let $\mathbb{T}$ be the totally defined theory of monoids and let d be a new constant of the unique sort of Σ_{Mon}. Then for any monoid M and $d^M \in M$,*

  $$\mathcal{Z}_{\mathbb{T}_d}(M, d^M) \cong \{m \in \mathsf{Inv}(M) : md^M = d^M m\}.$$

- *Let $\mathbb{T}$ be the totally defined theory of groups and let d be a new constant of the unique sort of Σ_{Group}. Then for any group G and $d^G \in G$,*

$$\mathcal{Z}_{\mathbb{T}_d}(G, d^G) \cong \{g \in G : gd^G = d^G g\}.$$

- *Let $\mathbb{T}$ be the theory of strict monoidal categories.*
  - *If d is a new constant of sort O, then for any $\mathbb{C} \in \mathsf{StrMonCat}$ and $d^{\mathbb{C}} \in \mathbb{C}_O$,*

$$\mathcal{Z}_{\mathbb{T}_d}(\mathbb{C}, d^{\mathbb{C}}) \cong \{a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) : a \otimes_O^{\mathbb{C}} d^{\mathbb{C}} = d^{\mathbb{C}} \otimes_O^{\mathbb{C}} a\}.$$

  - *If d is a new constant of sort A, then for any $\mathbb{C} \in \mathsf{StrMonCat}$ and $d^{\mathbb{C}} \in \mathbb{C}_A$,*

$$\mathcal{Z}_{\mathbb{T}_d}(\mathbb{C}, d^{\mathbb{C}}) \cong \{a \in \mathsf{Inv}(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}}) : \mathsf{id}^{\mathbb{C}}(a) \otimes_A^{\mathbb{C}} d^{\mathbb{C}} = d^{\mathbb{C}} \otimes_A^{\mathbb{C}} \mathsf{id}^{\mathbb{C}}(a)\}.$$

∎

The following corollary is immediate from Proposition 4.1.2.

Corollary 4.1.5. *If $\mathbb{T}$ has trivial isotropy, then so does $\mathbb{T}_d$.* ∎

Recall that a *fixed point* of a total function $f : X \to X$ is an element $x \in X$ with $f(x) = x$. The following corollary is now immediate from Corollary 4.1.3.

Corollary 4.1.6. *$\mathbb{T}_d$ has trivial isotropy*

iff

For any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $([s_C])_C \in G_{\mathbb{T}}(M)$, if $([s_C])_C \neq ([\mathsf{x}_C])_C$, then the total function $s_A^* : M_A \to M_A$ has no fixed points.

∎

Remark 4.1.7.

- For a simple example where $\mathbb{T}$ and $\mathbb{T}_d$ both have non-trivial isotropy, consider the totally defined theory of groups $\mathbb{T}_{\mathsf{Group}}$. Then $\mathbb{T}_{\mathsf{Group}}$ has non-trivial isotropy; in particular, every non-trivial group has non-trivial isotropy group (isomorphic to itself). So if G is any non-trivial group, then by Corollary 4.1.4 it follows that (G, e^G) has non-trivial isotropy as well, so that $(\mathbb{T}_{\mathsf{Group}})_d$ has non-trivial isotropy. Thus, it is not generally the case that if $\mathbb{T}$ has non-trivial isotropy, then $\mathbb{T}_d$ has *trivial* isotropy.

- It is also not generally the case that if $\mathbb{T}$ has non-trivial isotropy, then $\mathbb{T}_d$ will have non-trivial isotropy as well. For a counter-example (due to Simon Henry), let $\mathbb{T}$ be the quasi-equational theory over the signature Σ_{Inv} whose axioms are those of $\mathbb{T}_{\mathsf{Inv}}$ together with the new axiom

  $$f(x) = x \ \vdash^{x,y,z} \ y = z$$

  for pairwise distinct variables x, y, z of the unique sort of Σ_{Inv}. A model of $\mathbb{T}$ is thus either a set with at most one element equipped with the unique endofunction, or else a set with at least two elements equipped with an involution that has no fixed points. If M is any model of $\mathbb{T}$, then by computations extremely similar to those in Section 3.4, we have $\mathcal{Z}_{\mathbb{T}}(M) \cong \mathbb{Z}_2$, so that in particular $\mathbb{T}$ has non-trivial isotropy.

  Now let d be a constant symbol of the unique sort of Σ_{Inv}, and let us show that $\mathbb{T}_d$ has *trivial* isotropy. By Corollary 4.1.6, it suffices to show for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ that if $[f(\mathsf{x})] \neq [\mathsf{x}] \in G_{\mathbb{T}}(M)$, then $f^M : M \to M$ has no fixed points. We will prove the contrapositive: so suppose that f^M has a fixed point, and let us show that $[f(\mathsf{x})] = [\mathsf{x}]$, i.e. that $\mathbb{T}(M, \mathsf{x}) \vdash f(\mathsf{x}) = \mathsf{x}$, i.e. (by Lemma 3.1.2) that if N is any model of $\mathbb{T}$ for which there is a Σ_{Inv}-morphism $h : M \to N$, then $f^N : N \to N$ is the identity involution. Let $m \in M$ be a fixed point of f^M. Since M is a model of $\mathbb{T}$, it follows that (the underlying set of) M is just the singleton $\{m\}$. If $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $h : M \to N$ is a Σ_{Inv}-morphism, then we have $f^N(h(m)) = h(f^M(m)) = h(m)$, so that f^N has a fixed point $h(m)$. But since N is a model of $\mathbb{T}$, it then follows that $N = \{h(m)\}$ is a singleton, so that f^N is the identity involution. This proves that $\mathbb{T}(M, \mathsf{x}) \vdash f(\mathsf{x}) = \mathsf{x}$ if f^M has a fixed point, which shows that $\mathcal{Z}_{\mathbb{T}_d}(M)$ is trivial and hence that $\mathbb{T}_d$ has trivial isotropy, even though $\mathbb{T}$ does not.

- If, instead of defining $\mathbb{T}_d$ to be the theory over Σ_d that extends $\mathbb{T}$ by adding the axiom $\top \vdash d \downarrow$, we simply define $\mathbb{T}_d$ to be the theory $\mathbb{T}$ itself, now regarded as a theory over the signature Σ_d, then a model of $\mathbb{T}_d$ is a pair (M, d^M), with $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $d^M : \{*\} \rightharpoondown M_A$ a *partial* function. It is then trivial to see that the proof of Proposition 4.1.2 carries over to this definition of $\mathbb{T}_d$ essentially unchanged, leading to the following modification of Proposition 4.1.2, and an analogous modification of Corollary 4.1.3:

  If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $d^M : \{*\} \rightharpoondown M_A$, then

  $$\mathcal{Z}_{\mathbb{T}_d}(M, d^M) \cong \left\{\pi \in \mathcal{Z}_{\mathbb{T}}(M) : \pi^A_{\mathsf{id}_M} \circ d^M = d^M\right\}.$$

∎

As shown in the remark, there are examples of quasi-equational theories $\mathbb{T}$ with non-trivial isotropy such that $\mathbb{T}_d$ also has non-trivial isotropy. However, as we will show in the next section, the situation is *not* necessarily the same if we add a totally defined *non-constant* function symbol to a quasi-equational theory; except in some relatively trivial cases, regardless of whether the initial theory has trivial or non-trivial isotropy, part of the isotropy of the resulting theory over the expanded signature will *always* trivialize.

4.2 Adding a Non-Constant Function Symbol

Again, fix a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ.

Definition 4.2.1. Let $f : A_1 \times \ldots \times A_n \to A$ be a function symbol not in Σ, with $n \geq 1$ and $A_1, \ldots, A_n, A \in \Sigma_{\mathsf{Sort}}$.

- We define Σ_f to be the following signature:
    - $(\Sigma_f)_{\mathsf{Sort}} := \Sigma_{\mathsf{Sort}}$.
    - $(\Sigma_f)_{\mathsf{Fun}} : \Sigma_{\mathsf{Fun}} \cup \{f\}$.
- We define $\mathbb{T}_f$ to be the quasi-equational theory over Σ_f whose axioms are those of $\mathbb{T}$ together with the additional axiom

$$\top \vdash^{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \downarrow,$$

where $x_1 : A_1, \ldots, x_n : A_n$ are pairwise distinct variables. ▮

An object of $\mathsf{P}\mathbb{T}_f\mathsf{mod}$ is then just a pair (M, f^M), where $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$ is a *total* function.

We will prove (cf. Proposition 4.2.4) that if $(M, f^M) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$, then 'part' of the isotropy of (M, f^M) is guaranteed to be trivial (provided that $\mathbb{T}$ and/or the sorts $\{A_1, \ldots, A_n, A\}$ involved in f satisfy certain conditions). To motivate this result, let us assume that $\mathbb{T}$ is a quasi-equational theory over a *single-sorted* signature. Then we will prove (cf. Corollary 4.2.5) that if $(M, f^M) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$ and $[t] \in G_{\mathbb{T}_f}(M, f^M)$, then $[t] = [\mathsf{x}]$ (and hence $G_{\mathbb{T}_f}(M, f^M)$ will be trivial). The idea behind this is as follows: if $[t] \in G_{\mathbb{T}_f}(M, f^M)$, then we know that $[t]$ must commute generically with the new function symbol f, which means that $\mathbb{T}_f((M, f^M), \mathsf{x}_1, \ldots, \mathsf{x}_n)$ must prove the equation

$$t[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}] = f(t[\mathsf{x}_1/\mathsf{x}], \ldots, t[\mathsf{x}_n/\mathsf{x}])$$

(since f is totally defined in $\mathbb{T}_f$). But if the *only* axiom governing the behaviour of the new function symbol f in $\mathbb{T}_f$ is an axiom stating that f is totally defined, then it stands to reason that f will not 'interact' with itself or the other function symbols of Σ in any 'non-trivial' way in $\mathbb{T}_f$, and hence the only way for the above equation to be provable in $\mathbb{T}_f((M, f^M), \mathsf{x}_1, \ldots, \mathsf{x}_n)$ is if $[t] = [\mathsf{x}]$.

To prove our main result of this section (Proposition 4.2.4), we first require the following definition and lemmas. Recall that if $(N, f^N) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$ and $\mathsf{x}_1, \ldots, \mathsf{x}_n \notin \Sigma_f(N, f^N)$ are pairwise distinct constants of respective sorts $A_1, \ldots, A_n$, then $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle$ is the coproduct in $\mathsf{P}\mathbb{T}_f\mathsf{mod}$ with (N, f^N) and the initial $\mathbb{T}_f$-model $\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle$ on n generators of sorts $A_1, \ldots, A_n$, and the Σ-reduct $(N, f^N)\langle \mathsf{x}_1, \ldots \mathsf{x}_n\rangle|_\Sigma$ is a model of $\mathbb{T}$.

Definition 4.2.2. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and let $f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$ be a total function. Let $h : (M, f^M) \to (N, f^N)$ be any Σ_f-morphism in $\mathsf{P}\mathbb{T}_f\mathsf{mod}$, and let $a \in N_A$. We define the partial Σ_f-structure $N^a := ((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma, f^{N^a})$ with the total function

$$f^{N^a} : N^a_{A_1} \times \ldots \times N^a_{A_1} = (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_1} \times \ldots \times (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_n}$$
$$\to N^a_A = (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A$$

defined as follows:

- If $([u_1], \ldots, [u_n]) \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_1} \times \ldots \times (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_n}$ and for each $1 \leq i \leq n$ there is some $a_i \in N_{A_i}$ with $[u_i] = \left[c^{(N,f^N)}_{a_i}\right]$, then
$$f^{N^a}([u_1], \ldots, [u_n]) := \left[c^{(N,f^N)}_{f^N(a_1,\ldots,a_n)}\right] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A.$$
- Otherwise (i.e. if there is some $1 \leq i \leq n$ such that $[u_i]$ is *not* in the image of the canonical Σ_f-morphism $\eta : (N, f^N) \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle$), we set
$$f^{N^a}([u_1], \ldots, [u_n]) := \left[c^{(N,f^N)}_{a}\right] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A.$$

Justification. We must verify that f^{N^a} is well-defined. Specifically, let $a_i, b_i \in N_{A_i}$ with $\left[c^{(N,f^N)}_{a_i}\right] = \left[c^{(N,f^N)}_{b_i}\right]$ for each $1 \leq i \leq n$; we must show that

$$\left[c^{(N,f^N)}_{f^N(a_1,\ldots,a_n)}\right] = \left[c^{(N,f^N)}_{f^N(b_1,\ldots,b_n)}\right].$$

Since the canonical morphism $\eta : (N, f^N) \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle$ is (sortwise) injective by Lemma 2.2.9 (because $N_{A_1}, \ldots, N_{A_n}$ are non-empty by hypothesis), the assumption implies that $a_i = b_i$ for each $1 \leq i \leq n$. Then we have $f^N(a_1, \ldots, a_n) = f^N(b_1, \ldots, b_n)$, which yields the desired result. ∎

Lemma 4.2.3. *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and let* $f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$ *be a total function. Let* $h : (M, f^M) \to (N, f^N)$ *be any* Σ_f*-morphism in* $\mathsf{P}\mathbb{T}_f\mathsf{mod}$*, and let* $a \in N_A$*. Then* $N^a \models \mathbb{T}_f$ *and* $\eta : (N, f^N) \to N^a = ((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma, f^{N^a})$ *is a* Σ_f*-morphism.*

Proof: The first claim follows because $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma \models \mathbb{T}$ and f^{N^a} is total. Since $\eta : N \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma$ is a Σ-morphism, it remains to show that it respects the interpretations of f. Since f^N and f^{N^a} are total, we must show that if $(a_1, \ldots, a_n) \in N_{A_1} \times \ldots \times N_{A_n}$, then

$$\eta_A(f^N(a_1, \ldots, a_n)) = f^{N^a}(\eta_{A_1}(a_1), \ldots, \eta_{A_n}(a_n)) \in N^a_A = (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A;$$

but this follows trivially from the definitions of η and f^{N^a}. So $\eta : (N, f^N) \to N^a = ((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma, f^{N^a})$ is a Σ_f-morphism. ∎

Proposition 4.2.4. *Let* $\mathbb{T}$ *be a quasi-equational theory over a relation-free signature* Σ*, and let* $f : A_1 \times \ldots \times A_n \to A$ *be a function symbol with* $n \geq 1$ *and* $A_1, \ldots, A_n, A \in \Sigma_{\mathsf{Sort}}$ *and* $f \notin \Sigma$*. Suppose that at least one of the following conditions holds:*

- $A \in \{A_1, \ldots, A_n\}$.
- *For any* $(N, f^N) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$*, there is some* $1 \leq i \leq n$ *such that* $\mathbb{T}_f(N, f^N)$ *is non-trivial for the sort* A_i *(i.e.* $\mathbb{T}_f(N, f^N) \not\vdash^{y,y'} y = y'$ *for distinct variables* $y, y' : A_i$*).*

Then for any $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *with* $M_{A_1} \times \ldots \times M_{A_n} \neq \emptyset$*, any total function* $f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$*, and any* $([s_C])_C \in G_{\mathbb{T}_f}(M, f^M)$*, we have*

$$[s_A] = [\mathsf{x}_A] \in (M, f^M)\langle \mathsf{x}_A\rangle_A.$$

We also have

$$[s_{A_i}] = [\mathsf{x}_{A_i}] \in (M, f^M)\langle \mathsf{x}_{A_i}\rangle_{A_i}$$

for all $1 \leq i \leq n$*, if we assume in addition that* $M_A \neq \emptyset$ *and that* $\mathbb{T}_f(N, f^N)$ *is non-trivial for the sort* A *for each* $(N, f^N) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$*.*

Proof: Suppose that at least one of the first two stated conditions holds. Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ with $M_{A_1} \times \ldots \times M_{A_n} \neq \emptyset$, let $f^M : M_{A_1} \times \ldots \times M_{A_n} \to M_A$ be a total function, and let $([s_C])_C \in G_{\mathbb{T}_f}(M, f^M)$. We show that $[s_A] = [\mathsf{x}_A]$, i.e. that

$$\mathbb{T}_f\left(\left(M, f^M\right), \mathsf{x}_A\right) \vdash s_A = \mathsf{x}_A.$$

By Lemma 3.1.2, it is equivalent to show for any $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, any total function $f^N : N_{A_1} \times \ldots \times N_{A_n} \to N_A$, any Σ_f-morphism $h : (M, f^M) \to (N, f^N)$, and any $a \in N_A$ that

$$\rho^A_h(s_A)^*(a) = a \in N_A.$$

If $N_A = \{a\}$, then this is trivial, so assume that N_A has at least two elements. Note also that because $M_{A_1} \times \ldots \times M_{A_n} \neq \emptyset$ and there is a Σ-morphism $h : M \to N$, it follows that $N_{A_1} \times \ldots \times N_{A_n} \neq \emptyset$, so that $N_{A_i} \neq \emptyset$ for each $1 \leq i \leq n$.

Given $a \in N_A$, consider the $\mathbb{T}_f$-model $N^a = \big((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma, f^{N^a}\big)$ and the Σ_f-morphism $\eta : \big(N, f^N\big) \to N^a$. Then we have a Σ_f-morphism

$$\eta \circ h : \big(M, f^M\big) \to N^a.$$

Now for each $1 \leq i \leq n$, consider the total function

$$\rho^{A_i}_{\eta\circ h}(s_{A_i})^* : N^a_{A_i} = (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_i} \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_i} = N^a_{A_i}.$$

Since $([s_C])_C \in G_{\mathbb{T}_f}(M, f^M)$, this is a bijection. Since $[\mathsf{x}_i] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_i}$, there is a unique element $[t_i] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_i}$ with

$$\rho^A_{\eta\circ h}(s_{A_i})^*([t_i]) = [\mathsf{x}_i].$$

Since N_A contains at least two elements and one of the stated conditions holds, it follows that there is some $1 \leq i \leq n$ such that $\mathbb{T}_f(N, f^N)$ is non-trivial for the sort A_i. For this i, we now we show that $[\mathsf{x}_i]$ is *not* in the image of $\eta : N \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle|_\Sigma$. If $[\mathsf{x}_i]$ *were* in the image of η, then there would be some $b \in N_{A_i}$ with $[\mathsf{x}_i] = \eta_{A_i}(b) = \left[c^{(N,f^N)}_b\right]$. So we would have $\mathbb{T}_f((N, f^N), \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash \mathsf{x}_i = c^{(N,f^N)}_b$. Then by the theorem on constants (Remark 1.3.17), we would obtain $\mathbb{T}_f(N, f^N) \vdash^{y_1,\ldots,y_n} y_i = c^{(N,f^N)}_b$, with $y_1 : A_1, \ldots, y_n : A_n$ pairwise distinct variables. Since $N_{A_1}, \ldots, N_{A_n} \neq \emptyset$, it follows by an argument used before (cf. e.g. the proof of Lemma 2.2.9) that $\mathbb{T}_f(N, f^N) \vdash^{y_i} y_i = c^{(N,f^N)}_b$, from which it readily follows that $\mathbb{T}_f(N, f^N)$ would be trivial for the sort A_i, contrary to assumption. So $[\mathsf{x}_i]$ is *not* in the image of η.

If $[t_i]$ were in the image of η, then there would again be some $b \in N_{A_i}$ with $[t_i] = \eta_{A_i}(b) = \left[c^{(N,f^N)}_b\right]$. Then in $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_{A_i}$ we would have

$$\begin{aligned}
[\mathsf{x}_i] &= \rho^{A_i}_{\eta\circ h}(s_{A_i})^*([t_i]) \\
&= \rho^{A_i}_{\eta\circ h}(s_{A_i})^*\left(\left[c^{(N,f^N)}_b\right]\right) \\
&= \rho^{A_i}_{\eta}(\rho^{A_i}_h(s_{A_i}))^*\left(\left[c^{(N,f^N)}_b\right]\right) \\
&= \left[\rho^{A_i}_h(s_{A_i})\left[c^{(N,f^N)}_b/\mathsf{x}_i\right]\right]
\end{aligned}$$

$$\in \mathsf{Im}(\eta),$$

contradicting what was just proven. The third equality holds by Lemma 2.2.37, the fourth by Lemma 2.2.31, and the last by Lemma 2.2.54 and the fact that $N_{A_1}, \ldots N_{A_n}$ are all non-empty. So $[t_i]$ is not in the image of η either.

Since $([s_C])_C \in G_{\mathbb{T}_f}(M, f^M)$, it follows that $([s_C])_C$ commutes generically with the function symbol $f : A_1 \times \ldots \times A_n \to A$, which means that

$$\mathbb{T}_f(\left(M, f^M\right), \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f(s_{A_1}[\mathsf{x}_1/\mathsf{x}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}])$$

(since f is totally defined in $\mathbb{T}_f$). By soundness of partial Horn logic and Lemma 2.2.18, this easily entails that

$$\rho^A_{\eta\circ h}(s_A)^* \circ f^{N^a} = f^{N^a} \circ \left\langle \rho^{A_1}_{\eta\circ h}(s_{A_1})^*, \ldots, \rho^{A_n}_{\eta\circ h}(s_{A_n})^* \right\rangle : N^a_{A_1} \times \ldots \times N^a_{A_n} \to N^a_A.$$

We then obtain in $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A$

$$\begin{aligned}
\left[c^{(N,f^N)}_{\rho^A_h(s_A)^*(a)}\right] &= \left[c^{(N,f^N)}_{\left(\rho^A_h(s_A)[c_a/\mathsf{x}_A]\right)^{(\widehat{N,f^N})}}\right] \\
&= \left[\rho^A_h(s_A)\left[c^{(N,f^N)}_a/\mathsf{x}_A\right]\right] \\
&= \rho^A_\eta(\rho^A_h(s_A))^*\left(\left[c^{(N,f^N)}_a\right]\right) \\
&= \rho^A_{\eta\circ h}(s_A)^*\left(\left[c^{(N,f^N)}_a\right]\right) \\
&= \rho^A_{\eta\circ h}(s_A)^*\left(f^{N^a}([t_1], \ldots, [t_n])\right) \\
&= f^{N^a}\left(\rho^{A_1}_{\eta\circ h}(s_{A_1})^*([t_1]), \ldots, \rho^{A_n}_{\eta\circ h}(s_{A_n})^*([t_n])\right) \\
&= f^{N^a}([\mathsf{x}_1], \ldots, [\mathsf{x}_n]) \\
&= \left[c^{(N,f^N)}_a\right].
\end{aligned}$$

The first equality holds by Lemma 2.2.30, the second essentially by Lemma 2.4.11, the third by Lemma 2.2.31, the fourth by Lemma 2.2.37, the fifth by definition of f^{N^a} and the fact that some $[t_i]$ is not in the image of η, the sixth because $([s_C])_C$ commutes generically with f, the seventh by definition of the $[t_i]$, and the last by definition of f^{N^a} and the fact that some $[\mathsf{x}_i]$ is not in the image of η. So we have

$$\eta_A\left(\rho^A_h(s_A)^*(a)\right) = \left[c^{(N,f^N)}_{\rho^A_h(s_A)^*(a)}\right] = \left[c^{(N,f^N)}_a\right] = \eta_A(a) \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle_A.$$

Since η_A is injective by Lemma 2.2.9 (because $N_{A_1}, \ldots, N_{A_n} \neq \emptyset$), this entails that

$$\rho^A_h(s_A)^*(a) = a \in N_A,$$

as desired. This completes the proof that $[s_A] = [\mathsf{x}_A] \in M\langle \mathsf{x}_A \rangle_A$.

Next, we show that $[s_{A_i}] = [\mathsf{x}_{A_i}]$ for each $1 \leq i \leq n$, under the additional assumptions that $M_A \neq \emptyset$ and $\mathbb{T}_f(N, f^N)$ is non-trivial for the sort A for each $(N, f^N) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$. Since $([s_C])_C \in G_{\mathbb{T}_f}(M, f^M)$, we know that $([s_C])_C$ commutes generically with f, which means that the sequent

$$\top \vdash s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)$$

is provable in $\mathbb{T}_f\left(\left(M, f^M\right), \mathsf{x}_1, \ldots, \mathsf{x}_n\right)$ (since f is totally defined in $\mathbb{T}_f$). Since we have now shown that $\mathbb{T}_f((M, f^M), \mathsf{x}_A) \vdash s_A = \mathsf{x}_A$, it follows by an application of the theorem on constants (together with a by now familiar argument using the assumption that $M_A \neq \emptyset$) that

$$\top \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) = f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)$$

is provable in $\mathbb{T}_f\left(\left(M, f^M\right), \mathsf{x}_1, \ldots, \mathsf{x}_n\right)$.

Fix $1 \leq i \leq n$: we show that $[s_{A_i}] = [\mathsf{x}_{A_i}]$, i.e. that $\mathbb{T}_f((M, f^M), \mathsf{x}_{A_i}) \vdash s_{A_i} = \mathsf{x}_{A_i}$. Suppose towards a contradiction that this is false. Then by Lemma 3.1.2, there is some $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, some total function $f^N : N_{A_1} \times \ldots \times N_{A_n} \to N_A$, some Σ_f-morphism $h : (M, f^M) \to (N, f^N)$, and some $a_i \in N_{A_i}$ such that

$$\rho_h^{A_i}(s_{A_i})^*(a_i) \neq a_i \in N_{A_i}.$$

In particular, it follows that N_{A_i} has at least two elements and that $N_A \neq \emptyset$ (since $M_A \neq \emptyset$ and $h_A : M_A \to N_A$ is a total function). Consider the $\mathbb{T}$-model $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle|_\Sigma$, where $\mathsf{x}_A, \mathsf{x}'_A \notin \Sigma_f((N, f^N), \mathsf{x}_1, \ldots, \mathsf{x}_n)$ are distinct constants of sort A. We define a total function

$$f^* : (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_{A_1} \times \ldots \times (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_{A_n}$$

$$\to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A$$

as follows:

- If $([u_1], \ldots, [u_n]) \in \mathsf{dom}\,(f^*)$ and for each $1 \leq j \leq n$ there is some $b_j \in N_{A_j}$ with $[u_j] = \left[c_{b_j}^{(N,f^N)}\right]$, then

$$f^*([u_1], \ldots, [u_n]) := \left[c_{f^N(b_1,\ldots,b_n)}^{(N,f^N)}\right] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A.$$

- If $([u_1], \ldots, [u_n]) \in \mathsf{dom}\,(f^*)$ and $[u_i] = [\mathsf{x}_i]$, then

$$f^*([u_1], \ldots, [u_n]) := [\mathsf{x}_A] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A.$$

- If $([u_1], \ldots, [u_n]) \in \mathsf{dom}\,(f^*)$ and neither of the first two cases holds, then

$$f^*([u_1], \ldots, [u_n]) := [\mathsf{x}'_A] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A.$$

Then f^* is well-defined: the first case in the definition of f^* is well-defined by an argument similar to that used to show that f^{N^a} is well-defined. The first and second cases in the definition of f^* are mutually exclusive, because if $[\mathsf{x}_i] = \left[c_{b_i}^{(N,f^N)}\right]$ for some $b_i \in N_{A_i}$, then we would have

$$\mathbb{T}_f((N, f^N), \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{x}_i = c_{b_i}^{(N,f^N)},$$

which would easily imply that $\mathbb{T}_f(N, f^N) \vdash^{x,y} x = y$ for distinct variables $x, y : A_i$ by the theorem on constants (Remark 1.3.17) and the fact that $N_{A_1}, \ldots, N_{A_n}, N_A \neq \emptyset$ (using a now familiar argument). But this is impossible, because $\widehat{(N, f^N)} \models \mathbb{T}_f(N, f^N)$ and N_{A_i} has at least two elements.

So $((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle|_\Sigma, f^*)$ is a model of $\mathbb{T}_f$. If

$$\eta : (N, f^N) \to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle$$

is the canonical Σ_f-morphism, then an argument like that given for N^a shows that

$$\eta : (N, f^N) \to ((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle|_\Sigma, f^*)$$

is also a Σ_f-morphism. So then

$$\eta \circ h : (M, f^M) \to ((N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle|_\Sigma, f^*)$$

is a Σ_f-morphism. By soundness of partial Horn logic and Lemma 2.2.18, the fact that

$$\top \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) = f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)$$

is provable in $\mathbb{T}_f\left(\left(M, f^M\right), \mathsf{x}_1, \ldots, \mathsf{x}_n\right)$ entails that

$$f^* = f^* \circ \left\langle \rho_{\eta\circ h}^{A_1}(s_{A_1})^*, \ldots, \rho_{\eta\circ h}^{A_n}(s_{A_n})^* \right\rangle \tag{$*$}$$

$$: (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_{A_1} \times \ldots \times (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_{A_n}$$
$$\to (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A.$$

So in $(N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A \rangle_A$ we obtain

$$\begin{aligned}
[\mathsf{x}_A] &= f^*([\mathsf{x}_1], \ldots, [\mathsf{x}_n]) \\
&= f^*\left(\rho_{\eta\circ h}^{A_1}(s_{A_1})^*([\mathsf{x}_1]), \ldots, \rho_{\eta\circ h}^{A_n}(s_{A_n})^*([\mathsf{x}_n])\right) \\
&= f^*\left(\rho_\eta^{A_1}\left(\rho_h^{A_1}(s_{A_1})^*([\mathsf{x}_1])\right), \ldots, \rho_\eta^{A_n}\left(\rho_h^{A_n}(s_{A_n})^*([\mathsf{x}_n])\right)\right)
\end{aligned}$$

$$\begin{aligned}&= f^*\left(\left[\rho_h^{A_1}(s_{A_1})[\mathsf{x}_1/\mathsf{x}_{A_1}]\right], \ldots, \left[\rho_h^{A_n}(s_{A_n})[\mathsf{x}_n/\mathsf{x}_{A_n}]\right]\right)\\ &= [\mathsf{x}'_A];\end{aligned}$$

the first equality holds by definition of f^*, the second by $(*)$, the third by Lemma 2.2.37, and the fourth by Lemma 2.2.31. To justify the last equality, it suffices (by definition of f^*) to show that $\left[\rho_h^{A_i}(s_{A_i})[\mathsf{x}_i/\mathsf{x}_{A_i}]\right] \neq [\mathsf{x}_i]$ and that $\left[\rho_h^{A_i}(s_{A_i})[\mathsf{x}_i/\mathsf{x}_{A_i}]\right] \notin \mathsf{Im}(\eta)$. Since $N_{A_1}, \ldots, N_{A_n}, N_A \neq \emptyset$, the first inequality easily follows from the assumption that

$$\rho_h^{A_i}(s_{A_i})^* : N_{A_i} \to N_{A_i}$$

is not the identity function, and the second easily follows from the assumption that

$$\rho_h^{A_i}(s_{A_i})^* : N_{A_i} \to N_{A_i}$$

is not a constant function. This justifies the above sequence of equalities.

So we have

$$[\mathsf{x}_A] = [\mathsf{x}'_A] \in (N, f^N)\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A\rangle_A,$$

i.e.

$$\mathbb{T}_f((N, f^N), \mathsf{x}_1, \ldots \mathsf{x}_n, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{x}_A = \mathsf{x}'_A.$$

From this and the theorem on constants (Remark 1.3.17) and the fact that $N_{A_1}, \ldots, N_{A_n} \neq \emptyset$ we obtain

$$\mathbb{T}_f((N, f^N), \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{x}_A = \mathsf{x}'_A,$$

and then since $\mathsf{x}_A, \mathsf{x}'_A$ are distinct, we again obtain by the theorem on constants that

$$\mathbb{T}_f(N, f^N) \vdash^{x,y} x = y$$

for distinct variables $x, y : A$. But this contradicts the assumption that $\mathbb{T}_f(N, f^N)$ is non-trivial for the sort A. This contradiction completes the proof that $[s_{A_i}] = [\mathsf{x}_{A_i}]$ for each $1 \leq i \leq n$, as desired. ∎

Corollary 4.2.5. *Let $\mathbb{T}$ be a quasi-equational theory over a* single-sorted *relation-free signature, and let f be a new n-ary function symbol for some $n \geq 1$. Then $\mathbb{T}_f$ has trivial isotropy.*

Proof: The first condition of Proposition 4.2.4 is automatically satisfied, because the signature only has one sort. So if $(M, f^M) \in \mathsf{P}\mathbb{T}_f\mathsf{mod}$ with $M \neq \emptyset$ and $[s] \in G_{\mathbb{T}_f}(M, f^M)$, then Proposition 4.2.4 implies that $[s] = [\mathsf{x}] \in (M, f^M)\langle \mathsf{x}\rangle$, so that $G_{\mathbb{T}_f}(M, f^M)$ is the trivial group. However, it is straightforward to see that since $\mathbb{T}$ is single-sorted, the assumption that $M \neq \emptyset$ is now inessential in the proof of Proposition 4.2.4, which shows that $\mathbb{T}_f$ has trivial isotropy. ∎

Remark 4.2.6.

- If $\mathbb{T}$ and f satisfy neither of the first two conditions in the statement of Proposition 4.2.4, then the (first) conclusion of Proposition 4.2.4 can fail. For example, consider the disjoint union theory $\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}}$ (cf. the next section for more discussion of disjoint unions of theories), where $\mathbb{T}_{\mathsf{Triv}}$ is the quasi-equational theory on the signature with one sort and no function symbols whose only axiom is $\top \vdash^{y_1,y_2} y_1 = y_2$ for distinct variables y_1, y_2 of the unique sort (i.e. the only models of $\mathbb{T}_{\mathsf{Triv}}$ are sets with at most one element). Let $f : X \to Y$ be a new function symbol, with X being the unique sort of $\mathbb{T}_{\mathsf{Triv}}$ and Y the unique sort of $\mathbb{T}_{\mathsf{Group}}$. Then it is easy to see that the first two conditions in Proposition 4.2.4 fail for $\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}}$.

  Now let G be any (non-trivial) group containing an element $g \neq e^G$ that fails to commute with at least one element of G, and consider the model $M := (\{*\}, G, f^M)$ of $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$, where $f^M(*) = e^G$. If $N = (\{*'\}, H, f^N)$ is any model of $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$ for which there is a Σ_f-morphism $h = (h_1, h_2) : M \to N$, then $h_1(*) = *'$ and $h_2 : G \to H$ is a group homomorphism and hence

  $$f^N(*') = f^N(h_1(*)) = h_2(f^M(*)) = h_2(e^G) = e^H.$$

  We now show that the $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$-model M has an element of isotropy whose second component is non-trivial, thus falsifying the (first) conclusion of Proposition 4.2.4. Indeed, let $g \neq e^G \in G$ fail to commute with at least one element of G, and consider

  $$([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}]) \in M\langle \mathsf{x} \rangle_X \times M\langle \mathsf{y} \rangle_Y.$$

  We wish to show that

  $$([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}]) \in Z_{(\mathbb{T}_{\mathsf{Triv}}+\mathbb{T}_{\mathsf{Group}})_f}(M)$$

  and that

  $$[c_g \mathsf{y} c_{g^{-1}}] \neq [\mathsf{y}] \in M\langle \mathsf{y} \rangle_Y.$$

  The second claim easily follows from the assumption that g fails to commute with at least one element of G. For the first claim, it is easily seen that the pair $([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}])$ is invertible and commutes generically with all function symbols of $\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}}$, so it remains to show that it commutes generically with $f : X \to Y$, i.e. we must show that

  $$(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f(M, \mathsf{x}) \vdash c_g f(\mathsf{x}) c_{g^{-1}} = f(\mathsf{x}).$$

  By Lemma 3.1.2, it suffices to show that if $N = (\{*'\}, H, f^N)$ is any model of $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$ for which there is a Σ_f-morphism $h = (h_1, h_2) : M \to N$, then

  $$h_2(g) f^N(*') h_2(g)^{-1} = f^N(*') \in H.$$

But this follows because $f^N(*') = e^H$, as noted above. This proves that $([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}])$ is an element of isotropy of the $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$-model M whose second component is non-trivial, which shows that the (first) conclusion of Proposition 4.2.4 can fail for a theory $\mathbb{T}$ and a new function symbol f that satisfy neither of the first two stated conditions.

- If $\mathbb{T}, f$, satisfy one of the first two conditions in Proposition 4.2.4 but do not satisfy the last (additional) assumption, then the second conclusion of Proposition 4.2.4 can fail. Let $\mathbb{T} := \mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}}$ as in the last point, but now let $f : Y \to X$ (with Y being the unique sort of $\mathbb{T}_{\mathsf{Group}}$ and X the unique sort of $\mathbb{T}_{\mathsf{Triv}}$). Then it is easy to see that the second of the first two conditions in Proposition 4.2.4 is now satisfied, while the last (additional) assumption is *not* satisfied.

  Now let G be a non-abelian group, and consider the model $M := (\{*\}, G, f^M)$ of $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$, where $f^M : G \to \{*\}$ is (obviously) the constant function. Again, let $g \in G$ be an element that fails to commute with at least one element of G, and consider

$$([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}]) \in M\langle \mathsf{x} \rangle_X \times M\langle \mathsf{y} \rangle_Y.$$

  We wish to show that

$$([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}]) \in Z_{(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f}(M)$$

  and that

$$[c_g \mathsf{y} c_{g^{-1}}] \neq [\mathsf{y}] \in M\langle \mathsf{y} \rangle_Y.$$

  The only point at which the argument differs from the argument in the previous point is in showing that $([\mathsf{x}], [c_g \mathsf{y} c_{g^{-1}}])$ commutes generically with $f : Y \to X$, i.e. that

$$(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f(M, \mathsf{y}) \vdash f(\mathsf{y}) = f(c_g \mathsf{y} c_{g^{-1}}).$$

  By Lemma 3.1.2, it suffices to show that if $N = (\{*'\}, H, f^N)$ is any model of $(\mathbb{T}_{\mathsf{Triv}} + \mathbb{T}_{\mathsf{Group}})_f$ for which there is a Σ_f-morphism $h = (h_1, h_2) : M \to N$, then for any $z \in G$ we have

$$f^N(z) = f^N\left(h_2(g) z h_2(g)^{-1}\right) \in \{*'\},$$

  which is clearly true. So M has an element of isotropy whose X-component is trivial and whose Y-component is non-trivial, contrary to the second conclusion of Proposition 4.2.4.

- If, instead of defining $\mathbb{T}_f$ to be the theory over Σ_f that extends $\mathbb{T}$ by adding the axiom $\top \vdash^{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \downarrow$, we simply define $\mathbb{T}_f$ to be the theory $\mathbb{T}$

itself, now regarded as a theory over the signature Σ_f, then a model of $\mathbb{T}_f$ is a pair (M, f^M), with $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $f^M : M_{A_1} \times \ldots \times M_{A_n} \rightharpoonup M_A$ a *partial* function. It is then not difficult to see that the proof of Proposition 4.2.4 carries over to this definition of $\mathbb{T}_f$ essentially unchanged. We just have to modify the first case in the definition of f^{N^a} to take into account that f^N may not be total, and we make a similar adjustment to the definition of f^*. We then have an exact analogue of Proposition 4.2.4 for this alternative definition of the theory $\mathbb{T}_f$. ∎

4.3 Disjoint Union of Theories

In this section, we will compute the isotropy group of a disjoint union of quasi-equational theories in terms of the isotropy groups of the component theories. So let Σ_1 and Σ_2 be disjoint relation-free signatures, and let $\mathbb{T}_1$ and $\mathbb{T}_2$ be quasi-equational theories over the signatures Σ_1 and Σ_2 respectively. We define $\Sigma_1 + \Sigma_2$ to be the (disjoint) union of the signatures Σ_1 and Σ_2, i.e.

$$(\Sigma_1 + \Sigma_2)_{\mathsf{Sort}} := \Sigma_{1_{\mathsf{Sort}}} \cup \Sigma_{2_{\mathsf{Sort}}}$$

and

$$(\Sigma_1 + \Sigma_2)_{\mathsf{Fun}} := \Sigma_{1_{\mathsf{Fun}}} \cup \Sigma_{2_{\mathsf{Fun}}}.$$

We then define $\mathbb{T}_1 + \mathbb{T}_2$ to be the quasi-equational theory over the signature $\Sigma_1 + \Sigma_2$ whose axioms are all those of $\mathbb{T}_1$ and $\mathbb{T}_2$ combined. A model of $\mathbb{T}_1 + \mathbb{T}_2$ is then just a pair (M_1, M_2), where M_1 is a model of $\mathbb{T}_1$ and M_2 is a model of $\mathbb{T}_2$, and a $\Sigma_1 + \Sigma_2$-morphism is just a pair consisting of a Σ_1-morphism and a Σ_2-morphism.

We first require the following technical lemma, relating provability in $\mathbb{T}_1 + \mathbb{T}_2$ to provability in the component theories:

Lemma 4.3.1. *Let $M_1 \in \mathsf{P}\mathbb{T}_1\mathsf{mod}$ and $M_2 \in \mathsf{P}\mathbb{T}_2\mathsf{mod}$. If $i \in \{1, 2\}$ and $C \in \Sigma_{i_{\mathsf{Sort}}}$ and $s, t \in \mathsf{Term}^c(\Sigma_i(M_i, \mathsf{x}_C))$ are of the same sort, then*

$$(\mathbb{T}_1 + \mathbb{T}_2)((M_1, M_2), \mathsf{x}_C) \vdash s = t \iff \mathbb{T}_i(M_i, \mathsf{x}_C) \vdash s = t.$$

Proof: For the 'only if' direction, assume that $(\mathbb{T}_1 + \mathbb{T}_2)((M_1, M_2), \mathsf{x}_C) \vdash s = t$, and suppose that $i = 1$ for concreteness. To prove that $\mathbb{T}_1(M_1, \mathsf{x}_C) \vdash s = t$, it is equivalent by Lemma 3.1.2 to show that if N_1 is any model of $\mathbb{T}_1$ for which there is a Σ_1-morphism $h : M_1 \to N_1$ and an element $c \in N_1^C$, then

$$\rho_h^C(s)^*(c) = \rho_h^C(t)^*(c) \in N_1^A,$$

with $A \in \Sigma_{1_{\mathsf{Sort}}}$ being the sort of s and t.

So let N_1 be any model of $\mathbb{T}_1$ for which there is a Σ_1-morphism $h : M_1 \to N_1$ and an element $c \in N_1^C$. Then we have a $(\Sigma_1 + \Sigma_2)$-morphism $(h, \mathsf{id}_{M_2}) : (M_1, M_2) \to (N_1, M_2)$, with (N_1, M_2) being a model of $\mathbb{T}_1 + \mathbb{T}_2$ and $c \in N_1^C = (N_1, M_2)_C$. From our assumption and Lemma 3.1.2, it then follows that

$$\rho^C_{\left(h,\mathsf{id}_{M_2}\right)}(s)^*(c) = \rho^C_{\left(h,\mathsf{id}_{M_2}\right)}(t)^*(c) \in (N_1, M_2)_A = N_1^A.$$

But since $s, t \in \mathsf{Term}^c(\Sigma_1(M_1, \mathsf{x}_C))$, it is easy to see that

$$\rho^C_{\left(h,\mathsf{id}_{M_2}\right)}(s) = \rho^C_h(s) \text{ and } \rho^C_{\left(h,\mathsf{id}_{M_2}\right)}(t) = \rho^C_h(t),$$

which yields the desired result.

Conversely, if $\mathbb{T}_1(M_1, \mathsf{x}_C) \vdash s = t$, then since (it is easy to see that) $\mathbb{T}_1(M_1, \mathsf{x}_C)$ is a sub-theory of $(\mathbb{T}_1 + \mathbb{T}_2)((M_1, M_2), \mathsf{x}_C)$, the desired result follows. ▮

Proposition 4.3.2. *For any* $M_1 \in \mathsf{P}\mathbb{T}_1\mathsf{mod}$ *and* $M_2 \in \mathsf{P}\mathbb{T}_2\mathsf{mod}$,

$$G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2) \cong G_{\mathbb{T}_1}(M_1) \times G_{\mathbb{T}_2}(M_2).$$

Proof: We define a group isomorphism

$$\varphi^{M_1,M_2} : G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2) \xrightarrow{\sim} G_{\mathbb{T}_1}(M_1) \times G_{\mathbb{T}_2}(M_2).$$

So let

$$([s_C])_{C\in\Sigma_1+\Sigma_2} \in G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2) \subseteq \prod_{C\in\Sigma_1+\Sigma_2} (M_1, M_2)\langle \mathsf{x}_C \rangle_C.$$

So for any $C \in (\Sigma_1 + \Sigma_2)_{\mathsf{Sort}}$, we have $s_C \in \mathsf{Term}^c((\Sigma_1 + \Sigma_2)((M_1, M_2), \mathsf{x}_C))_C$ with $(\mathbb{T}_1 + \mathbb{T}_2)((M_1, M_2), \mathsf{x}_C) \vdash s_C \downarrow$. It is easily seen that if $C \in \Sigma_{i_{\mathsf{Sort}}}$ for $i \in \{1, 2\}$, then $s_C \in \mathsf{Term}^c(\Sigma_i(M_i, \mathsf{x}_C))_C$ because Σ_1 and Σ_2 are disjoint, and thus $\mathbb{T}_i(M_i, \mathsf{x}_C) \vdash s_C \downarrow$ by Lemma 4.3.1. So we have

$$([s_C])_{C\in\Sigma_i} \in \prod_{C\in\Sigma_i} M_i\langle \mathsf{x}_C \rangle_C.$$

It then follows straightforwardly from (a simple generalization of) Lemma 4.3.1 and the assumption that $([s_C])_{C\in\Sigma_1+\Sigma_2} \in G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2)$ that

$$([s_C])_{C\in\Sigma_i} \in G_{\mathbb{T}_i}(M_i)$$

for each $i \in \{1, 2\}$. So we define φ^{M_1,M_2} by the assignment

$$([s_C])_{C\in\Sigma_1+\Sigma_2} \in G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2)$$

$$\longmapsto \langle ([s_C])_{C\in\Sigma_1}, ([s_C])_{C\in\Sigma_2}\rangle \in G_{\mathbb{T}_1}(M_1) \times G_{\mathbb{T}_2}(M_2),$$

which is well-defined and injective by Lemma 4.3.1. It is easy to see that φ^{M_1,M_2} is a group homomorphism, and that it is surjective easily follows from Lemma 4.3.1. So

$$\varphi^{M_1,M_2} : G_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2) \xrightarrow{\sim} G_{\mathbb{T}_1}(M_1) \times G_{\mathbb{T}_2}(M_2)$$

is a group isomorphism. ∎

Corollary 4.3.3. *Let $M_1 \in \mathsf{P}\mathbb{T}_1\mathsf{mod}$ and $M_2 \in \mathsf{P}\mathbb{T}_2\mathsf{mod}$, and let*

$$\pi = \left(\pi_{(f_1,f_2)} : (\mathsf{cod}(f_1), \mathsf{cod}(f_2)) \to (\mathsf{cod}(f_1), \mathsf{cod}(f_2))\right)_{(f_1,f_2)\in\mathsf{Dom}(M_1,M_2)}$$

be a $\mathsf{Dom}(M_1, M_2)$-indexed family of endomorphisms in $\mathsf{P}(\mathbb{T}_1 + \mathbb{T}_2)\mathsf{mod}$. Then $\pi \in \mathcal{Z}_{\mathbb{T}_1+\mathbb{T}_2}(M_1, M_2)$ iff there are (uniquely determined) elements $\pi_1 \in \mathcal{Z}_{\mathbb{T}_1}(M_1)$ and $\pi_2 \in \mathcal{Z}_{\mathbb{T}_2}(M_2)$ such that

$$\pi_{(f_1,f_2)} = \left(\pi_{1_{f_1}}, \pi_{2_{f_2}}\right)$$

for every $(f_1, f_2) \in \mathsf{Dom}(M_1, M_2)$. ∎

Corollary 4.3.4. *If we identify $\mathsf{P}(\mathbb{T}_1 + \mathbb{T}_2)\mathsf{mod}$ with the isomorphic category*

$$\mathsf{P}\mathbb{T}_1\mathsf{mod} \times \mathsf{P}\mathbb{T}_2\mathsf{mod},$$

then

$$\mathcal{Z}_{\mathbb{T}_1+\mathbb{T}_2} \cong \mathcal{Z}_{\mathbb{T}_1} \times \mathcal{Z}_{\mathbb{T}_2} : \mathsf{P}(\mathbb{T}_1 + \mathbb{T}_2)\mathsf{mod} \to \mathsf{Group}.$$

∎

Remark 4.3.5. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are (single-sorted) *algebraic* theories over respective disjoint signatures Σ_1 and Σ_2, one can also define $\mathbb{T}_1 + \mathbb{T}_2$ to be the *algebraic* theory over the *single-sorted* signature Σ obtained by combining Σ_1 and Σ_2 and identifying the unique sort of Σ_1 with the unique sort of Σ_2. We have then shown, using methods from rewriting theory developed in [18], [2] that the isotropy group of any *free* model of $\mathbb{T}_1 + \mathbb{T}_2$ (defined in this way) is *trivial.* For reasons of space and relevance, we have chosen not to include the details. ∎

4.4 Miscellaneous Observations

In the final section of this chapter, we collect together some miscellaneous observations about the isotropy groups of quasi-equational theories, particularly regarding the effects of adding new axioms to a theory (without changing its signature). If $\mathbb{T}$ is a quasi-equational theory over a relation-free signature Σ, then we say that an *extension* of $\mathbb{T}$ is a quasi-equational theory $\mathbb{T}'$ over the same signature Σ such that every axiom of $\mathbb{T}$ is an axiom of $\mathbb{T}'$.

- Adding new axioms to a theory (without changing its signature) can trivialize the isotropy group. More precisely, it is *not* in general the case that if $\mathbb{T}$ is a quasi-equational theory with non-trivial isotropy, then every extension of $\mathbb{T}$ will also have non-trivial isotropy. For an easy example, let $\mathbb{T}$ be any theory with non-trivial isotropy, say the totally defined theory of groups (cf. Proposition 3.2.7). If we extend $\mathbb{T}$ by adding the axiom $\top \vdash^{x,y} x = y$ for distinct variables x, y of the unique sort of Σ_{Group}, then the resulting extension $\mathbb{T}'$ clearly has trivial isotropy.

  For a less frivolous example, let $\mathbb{T}$ be the totally defined theory of monoids, which has non-trivial isotropy (cf. Proposition 3.2.4). If we add the axiom $\top \vdash^{x,y} x \cdot y = y \cdot x$ to $\mathbb{T}$ to obtain the theory $\mathbb{T}'$ of commutative monoids, then $\mathbb{T}'$ has trivial isotropy (cf. Proposition 3.3.7), even though not every model of $\mathbb{T}'$ is a singleton, unlike in the first example.

- Adding new axioms to a theory (without changing its signature) can also create new isotropy. More precisely, it is *not* in general the case that if $\mathbb{T}$ is a quasi-equational theory with trivial isotropy, then every extension of $\mathbb{T}$ will also have trivial isotropy.

  For an easy example, consider the completely empty theory over the signature Σ_{Mon}, which has trivial isotropy (cf. Proposition 3.1.4). Then the totally defined theory of monoids $\mathbb{T}_{\mathsf{Mon}}$ is an extension of this theory which has non-trivial isotropy (cf. Proposition 3.2.4).

  For a less trivial example (starting from a *non-empty* theory with trivial isotropy), consider the theory $\mathbb{T}$ of totally defined semigroups (with a specified element) over the signature Σ_{Mon}, whose axioms are those of $\mathbb{T}_{\mathsf{Mon}}$ minus the axiom $\top \vdash^{x} x \cdot e = x = e \cdot x$. Then by simplifying the arguments given to show that $\mathbb{T}_{\mathsf{Mon}}$ has non-trivial isotropy (cf. Proposition 3.2.4), it is not difficult to see that $\mathbb{T}$ has *trivial* isotropy. But if we extend $\mathbb{T}$ to $\mathbb{T}_{\mathsf{Mon}}$ by adding the axiom $\top \vdash^{x} x \cdot e = x = e \cdot x$, then $\mathbb{T}_{\mathsf{Mon}}$ has non-trivial isotropy.

- If $\mathbb{T}$ is a quasi-equational theory whose free models on finitely many generators all have trivial isotropy, it does not necessarily follow that all finitely presented models of $\mathbb{T}$ will have trivial isotropy. For example, let $\mathbb{T}$ be the theory $\mathbb{T}_{\mathsf{Mon}}$ of totally defined monoids. Since the only invertible element of a free monoid (on finitely many generators) is the identity element, it follows that the isotropy group of any such monoid is trivial, by Proposition 3.2.4. However, since there are clearly finitely presented monoids with non-trivial invertible elements, it is not the case that all finitely presented monoids have trivial isotropy.

Chapter 5

Isotropy Groups of Functor Theories

In this chapter, given a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ and a small indexing category $\mathcal{J}$, we will define a quasi-equational theory $\mathbb{T}^{\mathcal{J}}$ with the property that

$$\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}.$$

We will characterize the isotropy group of $\mathbb{T}^{\mathcal{J}}$ in terms of the isotropy group of $\mathbb{T}$ and the *global* isotropy group of $\mathcal{J}$ (i.e. the group of natural automorphisms of the identity functor on $\mathcal{J}$).

In particular, when $\mathbb{T}$ is the theory of sets, so that $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}} = \mathsf{Sets}^{\mathcal{J}}$, the results of this chapter will allow us to characterize the covariant isotropy groups of presheaf toposes (cf. Corollary 5.3.6), about which nothing was previously known.

In fact, it will follow from the results in this chapter (cf. Corollary 5.3.6 and the remarks immediately thereafter) that if $\mathcal{J}$ is a small category, then the *covariant* isotropy group (functor)

$$\mathcal{Z} : \mathsf{Sets}^{\mathcal{J}} \to \mathsf{Group}$$

of the presheaf topos $\mathsf{Sets}^{\mathcal{J}}$ is *constant* on the so-called *global isotropy group* of $\mathcal{J}$, which is just the group $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ of all natural automorphisms of the identity functor $\mathsf{Id}_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$. This is in dramatic contrast to the *contravariant* isotropy group (functor)

$$\mathcal{Z}' : \left(\mathsf{Sets}^{\mathcal{J}}\right)^{op} \to \mathsf{Group}$$

of $\mathsf{Sets}^{\mathcal{J}}$, which (as shown in [10, 4.12]) is the *representable* presheaf of groups

$$\mathsf{Sets}^{\mathcal{J}}(-, \mathcal{Z}_{\mathcal{J}}) : \left(\mathsf{Sets}^{\mathcal{J}}\right)^{op} \to \mathsf{Group}$$

with representing object

$$\mathcal{Z}_{\mathcal{J}} : \mathcal{J} \to \mathsf{Group} \hookrightarrow \mathsf{Sets},$$

the covariant isotropy group (functor) of $\mathcal{J}$. We will attempt to provide a conceptual motivation for this difference in a future paper based on this chapter.

5.1 Logical Characterization

For the remainder of this section, fix a quasi-equational theory $\mathbb{T}$ over a relation-free signature Σ, and fix a small indexing category $\mathcal{J}$. At a certain point (cf. Proposition 5.1.47) we will need to make an assumption about $\mathbb{T}$, but for now, we can assume that $\mathbb{T}$ is arbitrary.

First, we define a relation-free signature $\Sigma^{\mathcal{J}}$. Since $\mathcal{J}$ is small, we know that its class of objects $\mathcal{J}_O$ is a set, and that for any $i, j \in \mathcal{J}_O$, the class of arrows $\mathcal{J}_A(i, j)$ is a set.

Definition 5.1.1 (**Functor Signature**). We define a relation-free signature $\Sigma^{\mathcal{J}}$ as follows.

- If $i \in \mathcal{J}_O$ and $A \in \Sigma_{\mathsf{Sort}}$, let $A^i \notin \Sigma$ be a new sort, and assume that all of the sorts defined in this way are pairwise distinct. Then we set
$$\Sigma^{\mathcal{J}}_{\mathsf{Sort}} := \{A^i : i \in \mathcal{J}_O, A \in \Sigma_{\mathsf{Sort}}\}.$$
- For any $f : i \to j$ in $\mathcal{J}$ and $A \in \Sigma_{\mathsf{Sort}}$, let
$$\alpha^A_f : A^i \to A^j$$
be a new unary function symbol $\notin \Sigma$, and assume that all of the symbols defined in this way are pairwise distinct.

  For any $i \in \mathcal{J}_O$ and function symbol $g : A_1 \times \ldots \times A_n \to A$ in Σ, let
$$g^i : A^i_1 \times \ldots \times A^i_n \to A^i$$
be a new function symbol $\notin \Sigma$, and assume that all of the symbols defined in this way are pairwise distinct. Then we set
$$\Sigma^{\mathcal{J}}_{\mathsf{Fun}} := \{\alpha^A_f : f \in \mathcal{J}_A, A \in \Sigma_{\mathsf{Sort}}\} \bigcup \{g^i : g \in \Sigma_{\mathsf{Fun}}, i \in \mathcal{J}_O\}.$$

∎

Given a partial $\Sigma^{\mathcal{J}}$-structure, we now show how to derive component Σ-structures from it, indexed by the objects of $\mathcal{J}$.

Definition 5.1.2 (**Component Structures**). Let M be a partial $\Sigma^{\mathcal{J}}$-structure and let $i \in \mathcal{J}_O$. We define a partial Σ-structure M^i as follows:

- For any $A \in \Sigma_{\mathsf{Sort}}$, we set
$$M^i_A := M_{A^i}.$$

- If $g : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ, then $g^i : A_1^i \times \ldots \times A_n^i \to A^i$ is a function symbol of $\Sigma^{\mathcal{J}}$, and we set

$$g^{M^i} := \left(g^i\right)^M : M_{A_1^i} \times \ldots \times M_{A_n^i} = M^i_{A_1} \times \ldots \times M^i_{A_n} \rightharpoondown M^i_A = M_{A^i}.$$

∎

From a morphism of $\Sigma^{\mathcal{J}}$-structures we can also extract morphisms of the component Σ-structures:

Definition 5.1.3 (**Component Morphisms**). Let M, N be partial $\Sigma^{\mathcal{J}}$-structures, let $i \in \mathcal{J}_O$, and let $h : M \to N$ be a $\Sigma^{\mathcal{J}}$-morphism. Then there is a Σ-morphism

$$h^i : M^i \to N^i$$

given by

$$h^i_A := h_{A^i} : M^i_A = M_{A^i} \to N_{A^i} = N^i_A$$

for every $A \in \Sigma_{\mathsf{Sort}}$. ∎

For any object $i \in \mathcal{J}_O$, we now define a signature morphism $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$.

Definition 5.1.4. For any object $i \in \mathcal{J}_O$, we define a signature morphism $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$ as follows:

- For any $A \in \Sigma_{\mathsf{Sort}}$, we set

$$\rho^i(A) := A^i \in \Sigma^{\mathcal{J}}_{\mathsf{Sort}}.$$

- For any function symbol $g : A_1 \times \ldots \times A_n \to A$ in Σ_{Fun}, we set

$$\rho^i(g) := g^i : A_1^i \times \ldots \times A_n^i \to A^i.$$

∎

We now define the quasi-equational theory $\mathbb{T}^{\mathcal{J}}$ that will axiomatize $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$.

Definition 5.1.5 (**Functor Quasi-Equational Theory**). We define $\mathbb{T}^{\mathcal{J}}$ to be the quasi-equational theory over the signature $\Sigma^{\mathcal{J}}$ whose axioms are the following sequents:

1. For any $f : i \to j$ in $\mathcal{J}_A$ and $A \in \Sigma_{\mathsf{Sort}}$, the axiom
$$\top \vdash^{x:A^i} \alpha^A_f(x) \downarrow .$$

2. For any $i \in \mathcal{J}_O$ and $A \in \Sigma_{\mathsf{Sort}}$, the axiom
$$\top \vdash^{x:A^i} \alpha^A_{\mathsf{id}_i}(x) = x.$$

3. For any $f : i \to j$ and $g : j \to k$ in $\mathcal{J}_A$ and $A \in \Sigma_{\mathsf{Sort}}$, the axiom
$$\top \vdash^{x:A^i} \alpha^A_g(\alpha^A_f(x)) = \alpha^A_{g\circ f}(x).$$

4. For any $f : i \to j$ in $\mathcal{J}_A$ and $g : A_1 \times \ldots \times A_n \to A$ in Σ_{Fun}, the axiom
$$g^i(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow \vdash^{\mathsf{x}_1:A^i_1,\ldots,\mathsf{x}_n:A^i_n} \alpha^A_f(g^i(\mathsf{x}_1, \ldots, \mathsf{x}_n)) = g^j(\alpha^{A_1}_f(\mathsf{x}_1), \ldots, \alpha^{A_n}_f(\mathsf{x}_n)).$$

5. For any $i \in \mathcal{J}_O$ and any axiom $\varphi \vdash^{\vec{x}} \psi$ of $\mathbb{T}$, the axiom
$$\rho^i(\varphi) \vdash^{\rho^i(\vec{x})} \rho^i(\psi).$$

∎

Remark 5.1.6. A first easy property of $\mathbb{T}^{\mathcal{J}}$ is that for any object $i \in \mathcal{J}_O$, the signature morphism $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$ is also a theory morphism $\mathbb{T} \to \mathbb{T}^{\mathcal{J}}$, because $\mathbb{T}^{\mathcal{J}}$ includes the axioms in Definition 5.1.5.5. ∎

To begin studying the models of $\mathbb{T}^{\mathcal{J}}$, we first make the following easy observation:

Lemma 5.1.7. *If M is a partial $\Sigma^{\mathcal{J}}$-structure with $M \models \mathbb{T}^{\mathcal{J}}$, then for any object $i \in \mathcal{J}_O$, the partial Σ-structure M^i (cf. Definition 5.1.2) is a model of $\mathbb{T}$.*

Proof: Assume the hypothesis, and let $i \in \mathcal{J}_O$. Since $\rho^i : \mathbb{T} \to \mathbb{T}^{\mathcal{J}}$ is a theory morphism by Remark 5.1.6, it follows by [19, Proposition 28] that $U^i(M)$ is a model of $\mathbb{T}$, where $U^i : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ is the forgetful functor induced by the signature morphism ρ^i. However, it is trivial to observe that $U^i(M) = M^i$, so that M^i is a model of $\mathbb{T}$, as desired. ∎

We now have:

Proposition 5.1.8. *There is an isomorphism of categories*

$$\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}.$$

∎

Before we can start to characterize the isotropy group of $\mathbb{T}^{\mathcal{J}}$, we first require the following purely group-theoretic fact, whose proof is a routine verification.

Lemma 5.1.9. *Let* $F : \mathcal{J} \to \mathsf{Group}$ *be an arbitrary functor, and consider the product group* $\prod_{i \in \mathcal{J}_O} F(i)$*. Then*

$$\left(\prod_{i \in \mathcal{J}_O} F(i)\right)^F := \left\{ (g_i)_{i \in \mathcal{J}_O} \in \prod_{i \in \mathcal{J}_O} F(i) : F(f)(g_j) = g_k \; \forall f : j \to k \in \mathcal{J}_A \right\}$$

is a subgroup of $\prod_{i \in \mathcal{J}_O} F(i)$.

Furthermore, this assignment is the object part of a functor

$$\left(\prod_{i \in \mathcal{J}_O} (-)(i)\right)^{(-)} = \mathsf{lim} : \mathsf{Group}^{\mathcal{J}} \to \mathsf{Group}.$$

∎

We can now begin to characterize the isotropy groups of models of $\mathbb{T}^{\mathcal{J}}$. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. Then by (the proof of) Proposition 5.1.8, there is a corresponding functor $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$. If $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ is the functor from Definition 2.2.36 that is naturally isomorphic to the isotropy group of $\mathbb{T}$ (by Theorem 2.2.41), then we obtain the composite functor

$$G_{\mathbb{T}} \circ F^M : \mathcal{J} \to \mathsf{Group},$$

with

$$(G_{\mathbb{T}} \circ F^M)(i) = G_{\mathbb{T}}(F^M(i)) = G_{\mathbb{T}}(M^i)$$

for every $i \in \mathcal{J}_O$. Then by Lemma 5.1.9, it follows that $\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{G_{\mathbb{T}} \circ F^M}$ is a subgroup of $\prod_i G_{\mathbb{T}}(M^i)$, and hence in particular is a group. Let us denote this subgroup with the cleaner notation $\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$.

Next, we will need to define a certain group $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. Its definition is somewhat subtle/unintuitive, so we ask the reader to bear with us until after we have defined it, at which point we will try to give some intuition/motivation for its definition.

For any $i \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, we say that $\mathbb{T}(M^i)$ is *trivial* for the sort B if $\mathbb{T}(M^i) \vdash^{y,y'} y = y'$ for distinct variables $y, y' : B$. Otherwise, we say that $\mathbb{T}(M^i)$ is *non-trivial* for the sort B.

For any $B \in \Sigma_{\mathsf{Sort}}$, we let $\mathcal{J}_B^M$ be the full subcategory of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the sort B. Let $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$ be the group of natural automorphisms of the identity functor $\mathsf{Id}_{\mathcal{J}_B^M} : \mathcal{J}_B^M \to \mathcal{J}_B^M$.

We will need to consider a certain subgroup of $\prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$, which we will call $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. To define this subgroup, we require the following definition:

Definition 5.1.10 (Degenerate Function Symbols). Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, let $g : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ with $n \geq 1$, and let $i \in \mathcal{J}_O$. Then for any $1 \leq m \leq n$, we say that g^{M^i} is *degenerate in position* m if

$$\mathbb{T}(M^i) \vdash^{y_1,\ldots,y_n,z_m} g(y_1,\ldots,y_n) = g(y_1,\ldots,y_n)[z_m/y_m],$$

where $y_1, \ldots, y_n, z_m$ are pairwise distinct variables of the appropriate sorts.

Otherwise, if $\mathbb{T}(M^i)$ does *not* prove the above equation, we say that g^{M^i} is *non-degenerate in position* m. ∎

Definition 5.1.11. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. We denote an element of $\prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$ by $\psi = (\psi_B)_{B \in \Sigma}$, so that each ψ_B is a natural automorphism of $\mathsf{Id}_{\mathcal{J}_B^M}$, with components $\psi_B(i) : i \xrightarrow{\sim} i$ for $i \in \mathcal{J}_B^M$.

We define

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \subseteq \prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$$

to consist of exactly those elements $\psi \in \prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$ with the following property:

- If $g : A_1 \times \ldots \times A_n \to A$ is any function symbol of Σ with $n \geq 1$, then for any $i \in \mathcal{J}_O$ and $1 \leq m \leq n$ for which g^{M^i} is non-degenerate in position m,

$$\psi_{A_m}(i) = \psi_A(i) : i \xrightarrow{\sim} i.$$

This property is well-defined, in the sense that if g^{M^i} is non-degenerate in position m, then it easily follows that $\mathbb{T}(M^i)$ must be non-trivial for the sorts A and A_m (and hence i must be an object of both $\mathcal{J}_A^M$ and $\mathcal{J}_{A_m}^M$, so that $\psi_A(i)$ and $\psi_{A_m}(i)$ are both well-defined arrows of $\mathcal{J}$).

It is then trivial to verify that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ is indeed a *subgroup* of $\prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$, and hence is a group. ∎

Our ultimate goal in this section will now be to show for any quasi-equational theory $\mathbb{T}$ (satisfying two conditions, cf. Proposition 5.1.47), any small index category $\mathcal{J}$, and any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ that

$$G_{\mathbb{T}^{\mathcal{J}}}(M) \cong \left(\prod_{i \in \mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M,$$

naturally in M. Specifically, we will construct a group isomorphism

$$\beta_M : \left(\prod_{i \in \mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M)$$

for each $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$.

As promised, let us now attempt to give some intuition/motivation for the definition of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$.

First, let us discuss why for each sort $B \in \Sigma$ we needed to consider the full subcategory $\mathcal{J}_B^M$ of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the sort B, rather than just the whole category $\mathcal{J}$. Let $\mathbb{T}$ be the single-sorted algebraic theory of commutative unital rings. It is well-known that the trivial (or zero) ring with underlying set $\{0\}$ has no outgoing ring homomorphism to a ring with more than one element. So (using the same notation for the zero ring as for its underlying set) it follows that $\mathbb{T}(\{0\})$ is trivial for the unique sort of $\mathbb{T}$.

Now let $\mathcal{J}$ any one-object category such that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ is *not* the trivial group (e.g. $\mathcal{J}$ could be the one-object category corresponding to a non-trivial abelian group G, in which case $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ is easily seen to be (isomorphic to) G itself), let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod} = \mathsf{Ring}$ be the constant functor on the trivial ring $\{0\}$, and let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ be the corresponding model of $\mathbb{T}^{\mathcal{J}}$. Since $\mathbb{T}$ is single-sorted, we clearly have $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M = \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, if we ignore the fact that we must consider the full subcategory of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial, which in this case would be the empty subcategory. And since $\mathcal{J}$ has one object and F is a constant functor, we have

$$\left(\prod_{i \in \mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} = G_{\mathbb{T}}(\{0\}) = \{[\mathsf{x}]\},$$

since the isotropy group of $\mathbb{T}$ is trivial, as indicated at the end of Chapter 3. So the group isomorphism to be defined

$$\beta_M : \left(\prod_{i \in \mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M)$$

now simplifies to

$$\beta_M : \{[\mathsf{x}]\} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M).$$

In fact, this isomorphism will be given by the following rule (cf. the proof of Proposition 5.1.20):

$$([\mathsf{x}], \psi) \longmapsto [\alpha_\psi(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

where $\psi : \mathsf{Id}_{\mathcal{J}} \xrightarrow{\sim} \mathsf{Id}_{\mathcal{J}}$ is a natural automorphism of the identity functor on the one-object category $\mathcal{J}$, and hence can be identified with an automorphism of the single object of $\mathcal{J}$ that commutes with every endomorphism of that object. Now, we show that this (desired) group homomorphism β_M will *not* be injective, given that we have defined $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ in terms of the full category $\mathcal{J}$ rather than the full subcategory of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the unique sort of $\mathbb{T}$.

By assumption, we know that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ is not the trivial group, so it contains distinct elements $\psi_1, \psi_2 \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$. To prove that β_M will *not* be injective, we show that

$$\beta_M([\mathsf{x}], \psi_1) = \beta_M([\mathsf{x}], \psi_2) \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

i.e. that

$$[\alpha_{\psi_1}(\mathsf{x})] = [\alpha_{\psi_2}(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{J}}}(M) \subseteq M\langle \mathsf{x} \rangle,$$

i.e. that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}) \vdash \alpha_{\psi_1}(\mathsf{x}) = \alpha_{\psi_2}(\mathsf{x}).$$

To do this, it will clearly suffice to show that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x})$ is trivial for the unique sort of $\mathbb{T}^{\mathcal{J}}$ (note that $\mathbb{T}^{\mathcal{J}}$ is single-sorted, because $\mathbb{T}$ is single-sorted and $\mathcal{J}$ has only one object). And to do *this*, it will suffice by Lemma 3.1.2 and the isomorphism $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ (cf. Proposition 5.1.8) to show that if $\mu : F \to G : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod} = \mathsf{Ring}$ is a natural transformation, then $G(*) \cong \{0\}$, where $*$ is the unique object of $\mathcal{J}$. But if $\mu : F \to G$ is a natural transformation, then $\mu_* : F(*) \to G(*)$ is a ring homomorphism, and then since $F(*) = \{0\}$, it follows that $G(*) \cong \{0\}$, as desired. This shows that if we do *not* restrict to the full subcategory of $\mathcal{J}$ on those objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the unique sort of $\mathbb{T}$, then the group homomorphism β_M may fail to be injective, which we obviously do not want.

This will hopefully help to convince the reader that we need to define $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ to be a subgroup of $\prod_{\mathcal{B} \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}^M_B}\right)$ rather than $\prod_{\mathcal{B} \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}}\right)$ (in the general case where $\mathbb{T}$ may be multi-sorted). Now let us try to motivate why we cannot just define $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ to be the full group $\prod_{\mathcal{B} \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}^M_B}\right)$ in the case where $\mathbb{T}$ is *multi*-sorted. A first vague intuition is that if $\psi = (\psi_B)_{B \in \Sigma} \in \prod_{\mathcal{B} \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}^M_B}\right)$, then we need the distinct ψ_B's to 'interact' properly, if there are (non-degenerate) function symbols in Σ that 'connect' different sorts.

For a more precise intuition, let $\mathcal{J}$ be the one-object category corresponding to the two-element group $\mathbb{Z}_2$, so that $\mathcal{J}$ has just two endomorphisms of its unique object $*$, namely the identity and an involution ψ. Consider the theory $\mathbb{T}$ whose signature has two sorts X and Y and one function symbol $f : X \to Y$, and whose only axiom is

$$\top \vdash^{x:X} f(x) \downarrow .$$

Let us define $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ with $N := (\{a,b\},\{c,d\},f^N)$ for pairwise distinct a,b,c,d with $f^N(a) = c$ and $f^N(b) = d$, and let us define a functor $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ with $F(*) := N$ and $F(\psi) : N \xrightarrow{\sim} N$ the obvious involution homomorphism. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ be the corresponding model of $\mathbb{T}^{\mathcal{J}}$. Since $\mathcal{J}$ has one object and $\mathbb{T}$ is two-sorted, it follows that $\mathbb{T}^{\mathcal{J}}$ is two-sorted, and we also refer to the two sorts of $\mathbb{T}^{\mathcal{J}}$ as X and Y. By Lemma 5.3.2 below, we know that $\mathbb{T}(N)$ is non-trivial for both sorts X and Y and that f^N is non-degenerate in its unique position. So we have $\mathcal{J}_M^X = \mathcal{J}_M^Y = \mathcal{J}$. Since $\mathcal{J}$ has one object and $\mathbb{T}$ (being an empty theory) has trivial isotropy by Proposition 3.1.4, we also have

$$\left(\prod_{i\in\mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} = \{([\mathsf{x}],[\mathsf{y}])\},$$

the trivial group. So the (to be defined) group homomorphism

$$\beta_M : \left(\prod_{i\in\mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M)$$

simplifies to

$$\beta_M : \{([\mathsf{x}],[\mathsf{y}])\} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M).$$

Suppose now that we *ignored* the fact that f^N is non-degenerate (in its unique position) in the definition of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, so that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M = \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$$

and

$$\beta_M : \{([\mathsf{x}],[\mathsf{y}])\} \times (\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})) \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M).$$

For any $\chi, \chi' \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, we will define β_M by

$$(([\mathsf{x}],[\mathsf{y}]),(\chi,\chi')) \longmapsto \left(\left[\alpha_\chi^X(\mathsf{x})\right], \left[\alpha_{\chi'}^Y(\mathsf{y})\right]\right) \in M\langle\mathsf{x}\rangle_X \times M\langle\mathsf{y}\rangle_Y.$$

However, let us now show that if $\chi \neq \chi' \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, then

$$\left(\left[\alpha_\chi^X(\mathsf{x})\right], \left[\alpha_{\chi'}^Y(\mathsf{y})\right]\right) \notin G_{\mathbb{T}^{\mathcal{J}}}(M).$$

In fact, let us show that $\left(\left[\alpha_\chi^X(\mathsf{x})\right], \left[\alpha_{\chi'}^Y(\mathsf{y})\right]\right)$ fails to commute generically with the function symbol $f : X \to Y$, i.e. let us show that

$$\mathbb{T}^{\mathcal{J}}(M,\mathsf{x}) \nvdash \alpha_{\chi'}^Y(f(\mathsf{x})) = f\left(\alpha_\chi^X(\mathsf{x})\right)$$

(note that f is totally defined in $\mathbb{T}$). Let us take χ to be the identity element of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ and χ' to be the other element ψ (the non-identity involution on the unique object $*$), so that we only have to show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}) \nvdash \alpha_{\psi}^{Y}(f(\mathsf{x})) = f(\mathsf{x})$$

(since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}) \vdash \alpha_{\mathsf{id}_*}^{X}(\mathsf{x}) = \mathsf{x}$). By Lemma 3.1.2 and the isomorphism $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$, it suffices to show that there is a functor $G : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ and a natural transformation $\mu : F \to G$ with the property that there is some $a \in G(*)_X$ with

$$G(\psi)_Y \left(f^{G(*)}(a)\right) \neq f^{G(*)}(a) \in G(*)_Y.$$

We take $G := F$ and μ to be the identity natural transformation. Then we have $a \in F(*)_X = N_X = \{a, b\}$ and

$$F(\psi)_Y \left(f^{F(*)}(a)\right) = F(\psi)_Y \left(f^{N}(a)\right) = F(\psi)_Y(c) = d \neq c = f^N(a) \in F(*)_Y = \{c, d\},$$

as desired. So if we do *not* take into account non-degenerate function symbols in the definition of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, then the desired group homomorphism β_M will not have $G_{\mathbb{T}^{\mathcal{J}}}(M)$ as its codomain.

Hopefully we have now given the reader more intuition and motivation for why we needed to define $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ the way we did; briefly, we needed to do so in order to ensure that our desired group isomorphism β_M is injective and has the correct codomain.

Now, towards constructing this group *homo*morphism (cf. Proposition 5.1.20), we require the following technical definitions and lemmas.

Definition 5.1.12. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, $i \in \mathcal{J}_O$, and $C \in \Sigma_{\mathsf{Sort}}$, so that M^i is a partial Σ-structure (cf. Definition 5.1.2). We define a signature morphism

$$\rho_{M^i}^{C} : \Sigma(M^i, \mathsf{x}_C) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i})$$

as follows (where $\mathsf{x}_C \notin \Sigma(M^i)$ and $\mathsf{x}_{C^i} \notin \Sigma^{\mathcal{J}}(M)$ are new constants of sorts C and C^i, respectively):

- On $\Sigma \subseteq \Sigma(M^i, \mathsf{x}_C)$, we stipulate that $\rho_{M^i}^{C}$ agrees with $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$ (cf. Definition 5.1.4).
- If $s \in M_A^i = M_{A^i}$ for some $A \in \Sigma_{\mathsf{Sort}}$, then we set
$$\rho_{M^i}^{C}\left(c_{A,s}^{M^i}\right) := c_{A^i,s}^{M} \in \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}).$$
- We set
$$\rho_{M^i}^{C}(\mathsf{x}_C) := \mathsf{x}_{C^i} \in \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}).$$

∎

Lemma 5.1.13. *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*,* $i \in \mathcal{J}_O$*, and* $C \in \Sigma_{\mathsf{Sort}}$*, the signature morphism* $\rho^C_{M^i}$ *is a theory morphism*

$$\rho^C_{M^i} : \mathbb{T}(M^i, \mathsf{x}_C) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}).$$

Proof: See Appendix C. ∎

The proofs of the next two lemmas are basically trivial from the definitions.

Lemma 5.1.14. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*,* $i \in \mathcal{J}_O$*, and* $C \in \Sigma_{\mathsf{Sort}}$*. For any* $u, v \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))$ *with* $v : C$*, we have*

$$\rho^C_{M^i}(u[v/\mathsf{x}_C]) \equiv \rho^C_{M^i}(u)[\rho^C_{M^i}(v)/\mathsf{x}_{C^i}].$$

Lemma 5.1.15. *Let* $h : M \to N$ *be a* $\Sigma^{\mathcal{J}}$*-morphism in* $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*. Then for any* $B \in \Sigma_{\mathsf{Sort}}$ *and* $k \in \mathcal{J}_O$*, we have an induced signature morphism*

$$\rho^{B^k}_h : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})$$

by Definition 2.2.17. From Definition 5.1.3, we obtain from h *a* Σ*-morphism* $h^k : M^k \to N^k$*, and we also have an induced signature morphism*

$$\rho^B_{h^k} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma(N^k, \mathsf{x}_B)$$

by Definition 2.2.17. Then we have

$$\rho^B_{N^k} \circ \rho^B_{h^k} = \rho^{B^k}_h \circ \rho^B_{M^k} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k}).$$

∎

Definition 5.1.16. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, $f : i \to j \in \mathcal{J}_A$, and $C \in \Sigma_{\mathsf{Sort}}$. We define a signature morphism

$$\sigma^C_f : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^j}) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i})$$

as follows:

- On $\Sigma^{\mathcal{J}}(M)$, we define σ^C_f to be the inclusion into $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i})$.

- We set
$$\sigma_f^C(\mathsf{x}_{C^j}) := \alpha_f^C(\mathsf{x}_{C^i}) : C^j.$$

▮

Since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^j}) \vdash \alpha_f^C(\mathsf{x}_{C^i}) \downarrow$, we then easily obtain:

Lemma 5.1.17. *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*,* $f : i \to j \in \mathcal{J}_A$*, and* $C \in \Sigma_{\mathsf{Sort}}$*, the signature morphism* σ_f^C *is a theory morphism*
$$\sigma_f^C : \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^j}) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}).$$

▮

If $f : i \to j$ is an arrow in $\mathcal{J}$, let us write $f^M := F^M(f) : M^i \to M^j$ (cf. the proof of Proposition 5.1.8). Then we have
$$f^M = F^M(f) = \left(\left(\alpha_f^A\right)^M : M_A^i \to M_A^j \right)_{A \in \Sigma}.$$

Recall that for any $C \in \Sigma_{\mathsf{Sort}}$, the Σ-morphism $f^M : M^i \to M^j$ induces a signature morphism $\rho_{f^M}^C : \Sigma(M^i, \mathsf{x}_C) \to \Sigma(M^j, \mathsf{x}_C)$ by Definition 2.2.17 which is also a theory morphism $\rho_{f^M}^C : \mathbb{T}(M^i, \mathsf{x}_C) \to \mathbb{T}(M^j, \mathsf{x}_C)$ by Lemma 2.2.18.

Definition 5.1.18. For any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, any arrow $f : i \to j$ in $\mathcal{J}$, and any $C \in \Sigma_{\mathsf{Sort}}$, we define a signature morphism $\tau_f^C : \Sigma(M^i, \mathsf{x}_C) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i})$ as
$$\begin{aligned} \tau_f^C &:= \sigma_f^C \circ \rho_{M^j}^C \circ \rho_{f^M}^C \\ &: \Sigma(M^i, \mathsf{x}_C) \to \Sigma(M^j, \mathsf{x}_C) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^j}) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}). \end{aligned}$$

Explicitly, τ_f^C is given by the following data:

- When restricted to $\Sigma \subseteq \Sigma(M^i, \mathsf{x}_C)$, τ_f^C agrees with $\rho^j : \Sigma \to \Sigma^{\mathcal{J}}$.
- For any $s \in M_A^i = M_{A^i}$ (for any $A \in \Sigma_{\mathsf{Sort}}$), we have
$$\begin{aligned} \tau_f^C\left(c_{A,s}^{M^i}\right) &:= \sigma_f^C\left(\rho_{M^j}^C\left(\rho_{f^M}^C\left(c_{A,s}^{M^i}\right)\right)\right) \\ &\equiv \sigma_f^C\left(\rho_{M^j}^C\left(c_{A,f^M(s)}^{M^j}\right)\right) \\ &\equiv \sigma_f^C\left(c_{A^j,f^M(s)}^{M}\right) \\ &\equiv c_{A^j,f^M(s)}^{M} \\ &\equiv c_{A^j,\left(\alpha_f^A\right)^M(s)}^{M}. \end{aligned}$$

- We have

$$\tau_f^C(\mathsf{x}_C) := \sigma_f^C\left(\rho^C_{M^j}\left(\rho^C_{f^M}(\mathsf{x}_C)\right)\right) \equiv \sigma_f^C\left(\rho^C_{M^j}(\mathsf{x}_C)\right) \equiv \sigma_f^C(\mathsf{x}_{C^j}) \equiv \alpha_f^C(\mathsf{x}_{C^i}).$$

∎

We will then need the following technical lemma about the signature morphism τ_f^C:

Lemma 5.1.19. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, let* $f : i \to j$ *be any arrow in* $\mathcal{J}$*, and let* $C \in \Sigma_{\mathsf{Sort}}$*. Then for any term* $u \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))$ *with* $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$ *and* $u : A$*, we have*

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \tau_f^C(u) = \alpha_f^A\left(\rho^C_{M^i}(u)\right).$$

Proof: See Appendix C. ∎

We can now prove:

Proposition 5.1.20. *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, there is a group homomorphism*

$$\beta_M : \left(\prod_{i \in \mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M).$$

Proof: Let $\gamma = (\gamma_i)_i \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$ and $\psi = (\psi_B)_{B \in \Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. We must define

$$\beta_M(\gamma, \psi) \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

with $G_{\mathbb{T}^{\mathcal{J}}}(M)$ being the group of all $\Sigma^{\mathcal{J}}_{\mathsf{Sort}}$-indexed sequences

$$([t_{C^i}])_{i \in \mathcal{J}, C \in \Sigma} \in \prod_{i \in \mathcal{J}, C \in \Sigma} M\langle \mathsf{x}_{C^i} \rangle_{C^i}$$

that are invertible, commute generically with all function symbols of $\Sigma^{\mathcal{J}}$, and reflect definedness. Each $t_{C^i} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))$ is a closed term of sort C^i with $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash t_{C^i} \downarrow$.

So let $i \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$; we define

$$\beta_M(\gamma, \psi)_{C^i} \in M\langle \mathsf{x}_{C^i} \rangle_{C^i}.$$

Since $\gamma_i \in G_{\mathbb{T}}(M^i)$, we know that $\gamma_i^C = [s_C^i] \in M^i\langle \mathsf{x}_C \rangle_C$. So $s_C^i \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))_C$ is a closed term of sort C with $\mathbb{T}(M^i, \mathsf{x}_C) \vdash s_C^i \downarrow$. Then

$$\rho^C_{M^i}(s_C^i) \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))_{C^i}$$

and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \rho^C_{M^i}(s^i_C) \downarrow$, since $\rho^C_{M^i} : \Sigma(M^i, \mathsf{x}_C) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i})$ is a theory morphism from $\mathbb{T}(M^i, \mathsf{x}_C)$ to $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$ by Lemma 5.1.13.

Suppose first that $\mathbb{T}(M^i)$ is *non-trivial* for the sort C. So then $i \in \mathcal{J}^M_C$, and $\psi_C(i) : i \xrightarrow{\sim} i$ is an isomorphism in $\mathcal{J}$. So then $\alpha^C_{\psi_C(i)} : C^i \to C^i$ is a function symbol of $\Sigma^{\mathcal{J}}$, and moreover $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i}) \downarrow$ because $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \mathsf{x}_{C^i} \downarrow$ and $\alpha^C_{\psi_C(i)}$ is provably total in $\mathbb{T}^{\mathcal{J}}$. Then it follows that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \rho^C_{M^i}(s^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right] \downarrow$$

by Lemma 2.2.24.

So then $\left[\rho^C_{M^i}(s^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right]\right] \in M\langle \mathsf{x}_{C^i} \rangle_{C^i}$, and we therefore set

$$\beta_M(\gamma, \psi)_{C^i} := \left[\rho^C_{M^i}(s^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right]\right] \in M\langle \mathsf{x}_{C^i} \rangle_{C^i}.$$

If $\mathbb{T}(M^i)$ is *trivial* for the sort C, then we simply set

$$\beta_M(\gamma, \psi)_{C^i} := [\mathsf{x}_{C^i}] \in M\langle \mathsf{x}_{C^i} \rangle_{C^i}.$$

Before we show that $\beta_M(\gamma, \psi) \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$, we first verify that β_M is well-defined. So let $\gamma = \delta \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$. To show that $\beta_M(\gamma, \psi) = \beta_M(\delta, \psi)$, let $i \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$ be arbitrary; we must show

$$\beta_M(\gamma, \psi)_{C^i} = \beta_M(\delta, \psi)_{C^i}.$$

If $\mathbb{T}(M^i)$ is trivial for the sort C, then we have

$$\beta_M(\gamma, \psi)_{C^i} = [\mathsf{x}_{C^i}] = \beta_M(\delta, \psi)_{C^i},$$

as desired.

So suppose that $\mathbb{T}(M^i)$ is non-trivial for the sort C. Then we must show

$$\left[\rho^C_{M^i}(u^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right]\right] = \left[\rho^C_{M^i}(v^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right]\right] \in M\langle \mathsf{x}_{C^i} \rangle_{C^i},$$

if $\gamma^C_i = [u^i_C]$ and $\delta^C_i = [v^i_C]$. In other words, we must show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \rho^C_{M^i}(u^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right] = \rho^C_{M^i}(v^i_C) \left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right].$$

Since $\gamma = \delta$, we obtain $\gamma_i = \delta_i$ and hence $[u^i_C] = [v^i_C] \in M^i\langle \mathsf{x}_C \rangle$, which means that $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u^i_C = v^i_C$. Then since $\rho^C_{M^i} : \mathbb{T}(M^i, \mathsf{x}_C) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$ is a theory morphism by Lemma 5.1.13, the desired result follows by Lemma 2.2.24. So β_M is indeed well-defined.

Now we prove that $\beta_M(\gamma, \psi) \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, which we do in a series of claims.

Claim 5.1.21. $\beta_M(\gamma, \psi)$ is invertible.

Proof: Let $i \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$. If $\mathbb{T}(M^i)$ is trivial for the sort C, then the result is trivial to verify. So assume that $\mathbb{T}(M^i)$ is non-trivial for the sort C. Since $\gamma_i \in G_{\mathbb{T}}(M^i)$, there is some $[(s^i_C)^{-1}] \in M^i\langle \mathsf{x}_C \rangle_C$ with

$$\left[s^i_C\left[(s^i_C)^{-1}/\mathsf{x}_C\right]\right] = [\mathsf{x}_C] = \left[(s^i_C)^{-1}\left[s^i_C/\mathsf{x}_C\right]\right] \in M^i\langle \mathsf{x}_C \rangle_C,$$

i.e.

$$\mathbb{T}(M^i, \mathsf{x}_C) \vdash s^i_C\left[(s^i_C)^{-1}/\mathsf{x}_C\right] = \mathsf{x}_C = (s^i_C)^{-1}\left[s^i_C/\mathsf{x}_C\right].$$

Now consider $\rho^C_{M^i}\left((s^i_C)^{-1}\right) \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))_{C^i}$: since $\mathbb{T}(M^i, \mathsf{x}_C) \vdash (s^i_C)^{-1} \downarrow$, it follows from Lemma 5.1.13 that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right) \downarrow.$$

Then because $\alpha^C_{\psi_C(i)^{-1}} : C^i \to C^i$ is provably total in $\mathbb{T}^{\mathcal{J}}$, we obtain

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \alpha^C_{\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right) \downarrow,$$

so that

$$\left[\alpha^C_{\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)\right] \in M\langle \mathsf{x}_{C^i} \rangle_{C^i}.$$

So we set

$$\beta_M(\gamma, \psi)^{-1}_{C^i} := \left[\alpha^C_{\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)\right].$$

Then because $\rho^C_{M^i}$ is a theory morphism, the following sequence of equations is provable in $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$, as desired:

$$\begin{aligned}
&\rho^C_{M^i}(s^i_C)\left[\alpha^C_{\psi_C(i)}(\mathsf{x}_{C^i})/\mathsf{x}_{C^i}\right]\left[\alpha^C_{\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)/\mathsf{x}_{C^i}\right] \\
&\equiv \rho^C_{M^i}(s^i_C)\left[\alpha^C_{\psi_C(i)}\left(\alpha^C_{\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)\right)/\mathsf{x}_{C^i}\right] \\
&= \rho^C_{M^i}(s^i_C)\left[\alpha^C_{\psi_C(i)\circ\psi_C(i)^{-1}}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)/\mathsf{x}_{C^i}\right] \\
&= \rho^C_{M^i}(s^i_C)\left[\alpha^C_{\mathsf{id}_i}\left(\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)\right)/\mathsf{x}_{C^i}\right] \\
&= \rho^C_{M^i}(s^i_C)\left[\rho^C_{M^i}\left(\left(s^i_C\right)^{-1}\right)/\mathsf{x}_{C^i}\right] \\
&\equiv \rho^C_{M^i}\left(s^i_C\left[\left(s^i_C\right)^{-1}/\mathsf{x}_C\right]\right) \\
&= \rho^C_{M^i}(\mathsf{x}_C) \\
&= \mathsf{x}_{C^i},
\end{aligned}$$

as desired (the fifth equality follows by Lemma 5.1.14). The other equality is shown similarly. This proves that $\beta_M(\gamma, \psi)$ is invertible. ∎

Claim 5.1.22. $\beta_M(\gamma, \psi)$ commutes generically with all function symbols of $\Sigma^{\mathcal{J}}$.

Proof: First let $i \in \mathcal{J}_O$ and let $g : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ. We must show that $\beta_M(\gamma, \psi)$ commutes generically with the function symbol

$$g^i : A_1^i \times \ldots \times A_n^i \to A^i$$

of $\Sigma^{\mathcal{J}}$. Assume without loss of generality that $\mathbb{T}(M^i)$ is non-trivial for each of the sorts $A_1, \ldots, A_n, A$; if this is *not* the case, then the argument required is a simpler version of the one we are about to give.

We must show that the sequent

$$\begin{aligned} &g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right) \downarrow\ \vdash \left\{\rho^A_{M^i}(s^i_A)\left[\alpha^A_{\psi_A(i)}(\mathsf{x}_{A^i})/\mathsf{x}_{A^i}\right]\right\}\left[g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A^i}\right] \\ &= g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A_1^i}\right)/\mathsf{x}_{A_1^i}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A_n^i}\right]\right)\end{aligned}$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)$ (technically, we need to ensure that the indeterminates on the right side of the equation are pairwise distinct (cf. Definition 2.2.47), but we will ignore this subtlety here and elsewhere in the proof of the proposition to increase readability). Since $\gamma_i \in G_{\mathbb{T}}(M^i)$, we know that the sequent

$$g(\mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \downarrow\ \vdash s^i_A[g(\mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})/\mathsf{x}_A] = g\left(s^i_{A_1}, \ldots, s^i_{A_n}\right) \qquad (*)$$

is provable in the theory $\mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})$. As in Definition 5.1.12 and Lemma 5.1.13, we can define a signature morphism

$$\rho^{\vec{A}}_{M^i} : \Sigma(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \to \Sigma^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)$$

that will be a theory morphism

$$\rho^{\vec{A}}_{M^i} : \mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \to \mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right);$$

on $\Sigma(M^i)$, we define $\rho^{\vec{A}}_{M^i}$ as in Definition 5.1.12, and for any $1 \leq j \leq n$ we set

$$\rho^{\vec{A}}_{M^i}(\mathsf{x}_{A_j}) := \mathsf{x}_{A_j^i}.$$

(Here $\vec{A} = A_1, \ldots, A_n$.) Then it is obvious that for any $1 \leq j \leq n$, the signature morphism $\rho^{\vec{A}}_{M^i}$ agrees with the signature morphism $\rho^{A_j}_{M^i} : \Sigma(M^i, \mathsf{x}_{A_j}) \to \Sigma^{\mathcal{J}}\left(M, \mathsf{x}_{A_j^i}\right)$ when restricted to $\Sigma(M^i, \mathsf{x}_{A_j})$, which implies that $\rho^{\vec{A}}_{M^i}\left(s^i_{A_j}\right) \equiv \rho^{A_j}_{M^i}\left(s^i_{A_j}\right)$ for all $1 \leq j \leq n$. Also (by Lemma 5.1.14), we have

$$\rho^A_{M^i}\left(s^i_A\right)\left[g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A^i}\right] \equiv \rho^{\vec{A}}_{M^i}\left(s^i_A[g(\mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})/\mathsf{x}_A]\right).$$

Now, since $\rho^{\vec{A}}_{M^i} : \mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \to \mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)$ is a theory morphism, it follows that the $\rho^{\vec{A}}_{M^i}$-translation of the aforementioned sequent $(*)$ provable in $\mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})$ will be provable in $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)$. In other words, the following sequent is provable in $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)$:

$$g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right) \downarrow \vdash$$

$$\rho^A_{M^i}(s^i_A)\left[g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A^i}\right] = g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right), \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\right).$$

Now, let us reason in the theory

$$\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right) \cup \left\{\top \vdash g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)\downarrow\right\}$$

(referred to as the 'expanded theory' for the rest of this argument), one of whose theorems is therefore the preceding equation. By substituting $\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right)$ for $\mathsf{x}_{A_1^i}$, …, $\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)$ for $\mathsf{x}_{A_n^i}$, the following equation is then provable in the expanded theory:

$$\begin{aligned}&\rho^A_{M^i}\left(s^i_A\right)\left[g^i\left(\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right), \ldots, \alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)\right)/\mathsf{x}_{A^i}\right]\\ &= g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right)/\mathsf{x}_{A_1^i}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A_n^i}\right]\right).\end{aligned}$$

Since the expanded theory (because of Axiom 5.1.5.4) proves the equation

$$g^i\left(\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right), \ldots, \alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)\right) = \alpha^A_{\psi_A(i)}\left(g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)\right),$$

it follows that the expanded theory proves the equation

$$\begin{aligned}&\rho^A_{M^i}\left(s^i_A\right)\left[\alpha^A_{\psi_A(i)}\left(g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)\right)/\mathsf{x}_{A^i}\right]\\ &= g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right)/\mathsf{x}_{A_1^i}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A_n^i}\right]\right),\end{aligned}$$

i.e. the expanded theory proves the equation

$$\begin{aligned}&\rho^A_{M^i}(s^i_A)\left[\alpha^A_{\psi_A(i)}(\mathsf{x}_{A^i})/\mathsf{x}_{A^i}\right]\left[g^i\left(\mathsf{x}_{A_1^i}, \ldots, \mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A^i}\right]\\ &= g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A_1^i}\right)/\mathsf{x}_{A_1^i}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A_n^i}\right]\right).\end{aligned}$$

So to complete the argument, it remains to show (by the deduction theorem in Remark 1.3.17) that the expanded theory proves the equation

$$g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A_1^i}\right)/\mathsf{x}_{A_1^i}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A_n^i}\right)/\mathsf{x}_{A_n^i}\right]\right)$$

$$= g^i \left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A^i_1}\right)/\mathsf{x}_{A^i_1}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A^i_n}\right)/\mathsf{x}_{A^i_n}\right]\right)$$

(the difference in the two terms being the ψ-subscripts). It suffices to show that for any position $1 \leq m \leq n$, we can 'swap' $\rho^{A_m}_{M^i}\left(s^i_{A_m}\right)\left[\alpha^{A_m}_{\psi_{A_m}(i)}\left(\mathsf{x}_{A^i_m}\right)\right]$ for $\rho^{A_m}_{M^i}\left(s^i_{A_m}\right)\left[\alpha^{A_m}_{\psi_A(i)}\left(\mathsf{x}_{A^i_m}\right)\right]$ within position m in g^i (modulo the expanded theory). If g^{M^i} is degenerate in position m, then this easily follows by the definition of 'degenerate' (cf. Definition 5.1.10): specifically, if $\mathbb{T}(M^i)$ proves the equation in Definition 5.1.10 for g, then it follows from Lemma 5.1.13 that $\mathbb{T}^{\mathcal{J}}(M)$ will prove the corresponding equation for g^i.

Otherwise, if g^{M^i} is *non-degenerate* in position m, then since $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, it follows that $\psi_{A_m}(i) = \psi_A(i) : i \xrightarrow{\sim} i$, which again easily yields the desired result. This completes the proof that $\beta_M(\gamma, \psi)$ commutes generically with the function symbol g^i of $\Sigma^{\mathcal{J}}$.

Now let $B \in \Sigma_{\mathsf{Sort}}$ and let $f : i \to j$ be an arbitrary arrow in $\mathcal{J}$. We must show that $\beta_M(\gamma, \psi)$ commutes generically with the function symbol $\alpha^B_f : B^i \to B^j$ of $\Sigma^{\mathcal{J}}$. Suppose first that $\mathbb{T}(M^i)$ and $\mathbb{T}(M^j)$ are non-trivial for the sort B. Then we must show that the equation

$$\left\{\rho^B_{M^j}(s^j_B)\left[\alpha^B_{\psi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right\}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right] = \alpha^B_f\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right)$$

is provable in $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i})$ (since α^B_f is provably total in $\mathbb{T}^{\mathcal{J}}$). Since $\gamma \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$, we know that

$$G_{\mathbb{T}}\left(F^M(f)\right)(\gamma_i) = \gamma_j,$$

i.e.

$$G_{\mathbb{T}}\left(F^M(f)\right)\left(\left([s^i_C]\right)_{C \in \Sigma}\right) = \left([s^j_C]\right)_{C \in \Sigma}.$$

Recalling our earlier convention that $f^M := F^M(f) : M^i \to M^j$, this equality means that

$$\left([\rho^C_{f^M}(s^i_C)]\right)_{C \in \Sigma} = \left([s^j_C]\right)_{C \in \Sigma} \in G_{\mathbb{T}}(M^j)$$

(cf. Definition 2.2.36). In particular, for our fixed sort B, we have

$$\left[\rho^B_{f^M}(s^i_B)\right] = \left[s^j_B\right] \in M^j\langle \mathsf{x}_B \rangle_B,$$

which means that

$$\mathbb{T}(M^j, \mathsf{x}_B) \vdash \rho^B_{f^M}(s^i_B) = s^j_B.$$

Since $\rho^B_{M^j} : \mathbb{T}(M^j, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j})$ is a theory morphism by Lemma 5.1.13, we then have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \vdash \rho^B_{M^j}\left(\rho^B_{f^M}(s^i_B)\right) = \rho^B_{M^j}(s^j_B).$$

And since $\sigma^B_f : \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i})$ is a theory morphism by Lemma 5.1.17, we obtain

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \sigma^B_f\left(\rho^B_{M^j}\left(\rho^B_{f^M}(s^i_B)\right)\right) = \sigma^B_f\left(\rho^B_{M^j}(s^j_B)\right),$$

i.e. (cf. Definition 5.1.18)

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \tau_f^B(s_B^i) = \sigma_f^B\left(\rho_{M^j}^B(s_B^j)\right).$$

Also, since $\sigma_f^B : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i})$ is the identity except for the fact that $\sigma_f^B(\mathsf{x}_{B^j}) := \alpha_f^B(\mathsf{x}_{B^i})$, it easily follows that

$$\sigma_f^B\left(\rho_{M^j}^B(s_B^j)\right) \equiv \rho_{M^j}^B(s_B^j)\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right].$$

So we have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \tau_f^B(s_B^i) = \rho_{M^j}^B(s_B^j)\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right].$$

Finally, since $\mathbb{T}(M^i, \mathsf{x}_B) \vdash s_B^i \downarrow$, it follows from Lemma 5.1.19 that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \tau_f^B(s_B^i) = \alpha_f^B\left(\rho_{M^i}^B(s_B^i)\right).$$

Combining this equation with the previous one, we then have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \rho_{M^j}^B(s_B^j)\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right] = \alpha_f^B\left(\rho_{M^i}^B(s_B^i)\right).$$

Substituting $\alpha_{\psi_B(i)}^B(\mathsf{x}_{B^i})$ for x_{B^i} and applying Lemma 2.2.24, $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i})$ then proves the equation

$$\rho_{M^j}^B(s_B^j)\left[\alpha_f^B\left(\alpha_{\psi_B(i)}^B(\mathsf{x}_{B^i})\right)/\mathsf{x}_{B^j}\right] = \alpha_f^B\left(\rho_{M^i}^B(s_B^i)\left[\alpha_{\psi_B(i)}^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right).$$

So to complete the argument, it remains to prove that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i})$ proves the equation

$$\rho_{M^j}^B(s_B^j)\left[\alpha_f^B\left(\alpha_{\psi_B(i)}^B(\mathsf{x}_{B^i})\right)/\mathsf{x}_{B^j}\right] = \left\{\rho_{M^j}^B\left(s_B^j\right)\left[\alpha_{\psi_B(j)}^B(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right\}\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right].$$

But the following sequence of equations is provable in $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i})$, as desired:

$$\begin{aligned}
&\left\{\rho_{M^j}^B\left(s_B^j\right)\left[\alpha_{\psi_B(j)}^B(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right\}\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right] \\
&\equiv \rho_{M^j}^B(s_B^j)\left[\alpha_{\psi_B(j)}^B(\alpha_f^B(\mathsf{x}_{B^i}))/\mathsf{x}_{B^j}\right] \\
&= \rho_{M^j}^B(s_B^j)\left[\alpha_{\psi_B(j)\circ f}^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right] \\
&= \rho_{M^j}^B(s_B^j)\left[\alpha_{f\circ\psi_B(i)}^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^j}\right] \\
&= \rho_{M^j}^B(s_B^j)\left[\alpha_f^B(\alpha_{\psi_B(i)}^B(\mathsf{x}_{B^i}))/\mathsf{x}_{B^j}\right].
\end{aligned}$$

The second equality follows by Axiom 5.1.5.3, the third by naturality of $\psi_B \in \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$, and the last by Axiom 5.1.5.3 again.

Now suppose that $\mathbb{T}(M^i)$ is trivial for the sort B, which implies that $\mathbb{T}(M^i, \mathsf{x}_B)$ is also trivial for the sort B. Given the arrow $f : i \to j$, we have the induced Σ-morphism $f^M : M^i \to M^j$, which in turn induces the theory morphism

$$\rho_{f^M}^B : \mathbb{T}(M^i, \mathsf{x}_B) \to \mathbb{T}(M^j, \mathsf{x}_B),$$

the existence of which implies that $\mathbb{T}(M^j, \mathsf{x}_B)$ is also trivial for the sort B. But by Lemma 5.1.13, it then easily follows that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j})$ is trivial for the sort B^j, and hence will prove all equations between terms of this sort, which clearly yields the desired result. And if $\mathbb{T}(M^j)$ is trivial for the sort B, then $\mathbb{T}(M^j, \mathsf{x}_B)$ is trivial for the sort B, which then also yields the desired result, as just explained.

This completes the proof that $\beta_M(\gamma, \psi)$ commutes generically with α^B_f, and completes the proof that $\beta_M(\gamma, \psi)$ commutes generically with all function symbols of $\Sigma^{\mathcal{J}}$.
∎

Claim 5.1.23. $\beta_M(\gamma, \psi)$ reflects definedness.

Proof: Since the function symbols α^B_f are all provably total in $\mathbb{T}^{\mathcal{J}}$, it suffices to just consider the function symbols g^i of $\Sigma^{\mathcal{J}}$, for $i \in \mathcal{J}_O$ and $g : A_1 \times \ldots \times A_n \to A$ in Σ. We will assume without loss of generality that $\mathbb{T}(M^i)$ is non-trivial for the sorts $A_1, \ldots, A_n, A$ (if not, the required argument is simpler).

To show that $\beta_M(\gamma, \psi)$ reflects definedness of g^i (assuming that $n \geq 1$), we must show that the sequent

$$g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A^i_1}\right)/\mathsf{x}_{A^i_1}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A^i_n}\right)/\mathsf{x}_{A^i_n}\right]\right)\downarrow$$

$$\vdash g^i\left(\mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)\downarrow$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)$. Since $\gamma_i \in G_{\mathbb{T}}(M^i)$, we know that γ_i reflects definedness of the function symbol $g \in \Sigma_{\mathsf{Fun}}$, so that the sequent

$$g\left(s^i_{A_1}, \ldots, s^i_{A_n}\right)\downarrow \vdash g(\mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})\downarrow$$

is provable in the theory $\mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n})$. Then since the aforementioned signature morphism $\rho^{\vec{A}}_{M^i} : \Sigma(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \to \Sigma^{\mathcal{J}}\left(M, \mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)$ is a theory morphism $\mathbb{T}(M^i, \mathsf{x}_{A_1}, \ldots, \mathsf{x}_{A_n}) \to \mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)$, it easily follows that $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)$ proves the sequent

$$g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right), \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\right)\downarrow \vdash g^i\left(\mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)\downarrow.$$

By making the appropriate substitutions (and applying Lemma 2.2.24), it then follows that $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{A^i_1}, \ldots, \mathsf{x}_{A^i_n}\right)$ proves the sequent

$$g^i\left(\rho^{A_1}_{M^i}\left(s^i_{A_1}\right)\left[\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A^i_1}\right)/\mathsf{x}_{A^i_1}\right], \ldots, \rho^{A_n}_{M^i}\left(s^i_{A_n}\right)\left[\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A^i_n}\right)/\mathsf{x}_{A^i_n}\right]\right)\downarrow$$

$$\vdash g^i\left(\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A^i_1}\right),\ldots,\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A^i_n}\right)\right)\downarrow.$$

To complete the argument, it now suffices to show that $\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)$ proves the sequent

$$g^i\left(\alpha^{A_1}_{\psi_{A_1}(i)}\left(\mathsf{x}_{A^i_1}\right),\ldots,\alpha^{A_n}_{\psi_{A_n}(i)}\left(\mathsf{x}_{A^i_n}\right)\right)\downarrow\vdash g^i\left(\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)\downarrow.$$

By reasoning similar to that used in the argument that $\beta_M(\gamma,\psi)$ commutes generically with g^i, it is sufficient to prove that $\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)$ proves the sequent

$$g^i\left(\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A^i_1}\right),\ldots,\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A^i_n}\right)\right)\downarrow\vdash g^i\left(\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)\downarrow.$$

Now let us work in the theory

$$\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)\cup\left\{\top\vdash g^i\left(\alpha^{A_1}_{\psi_A(i)}(\mathsf{x}_{A^i_1}),\ldots,\alpha^{A_n}_{\psi_A(i)}(\mathsf{x}_{A^i_n})\right)\downarrow\right\},$$

which we will refer to as the 'expanded theory' for the remainder of the argument. Then (by the deduction theorem in Remark 1.3.17) we need to show that the expanded theory proves the sequent

$$\top\vdash g^i\left(\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)\downarrow.$$

Since $\alpha^A_{\psi_A(i)^{-1}}$ is provably total in $\mathbb{T}^{\mathcal{J}}$, it follows that the expanded theory proves the sequent

$$\top\vdash\alpha^A_{\psi_A(i)^{-1}}\left(g^i\left(\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A^i_1}\right),\ldots,\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A^i_n}\right)\right)\right)\downarrow.$$

Then by Axiom 5.1.5.4, it follows that the expanded theory proves the sequent

$$\top\vdash g^i\left(\alpha^{A_1}_{\psi_A(i)^{-1}}\left(\alpha^{A_1}_{\psi_A(i)}\left(\mathsf{x}_{A^i_1}\right)\right),\ldots,\alpha^{A_n}_{\psi_A(i)^{-1}}\left(\alpha^{A_n}_{\psi_A(i)}\left(\mathsf{x}_{A^i_n}\right)\right)\right)\downarrow.$$

Then by Axioms 5.1.5.3 and 5.1.5.2 we finally obtain that the expanded theory proves the sequent

$$\top\vdash g^i\left(\mathsf{x}_{A^i_1},\ldots,\mathsf{x}_{A^i_n}\right)\downarrow,$$

as required. This proves that $\beta_M(\gamma,\psi)$ reflects definedness. ∎

With the preceding three claims, we have now proved that

$$\beta_M:\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}\times\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M\to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is a well-defined function. To complete the proof of Proposition 5.1.20, we must show that β_M preserves the group multiplication. So let $\gamma = (\gamma_i)_i, \delta = (\delta_i)_i \in (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}}$ and $\psi, \chi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. We must show that

$$\beta_M(\gamma \cdot \delta, \psi \cdot \chi) = \beta_M(\gamma, \psi) \cdot \beta_M(\delta, \chi).$$

So fix $j \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$. Then we must show

$$\beta_M(\gamma \cdot \delta, \psi \cdot \chi)_{B^j} = ((\beta_M(\gamma, \psi) \cdot \beta_M(\delta, \chi))_{B^j}.$$

If $\mathbb{T}(M^j)$ is trivial for the sort B, then the desired result follows immediately from the definition of β_M. So assume otherwise, and let $\gamma_j = \left(\left[s^j_C\right]\right)_C$ and $\delta_j = \left(\left[t^j_C\right]\right)_C$. Also, note that $(\gamma \cdot \delta)_j = \gamma_j \cdot \delta_j = \left(\left[s^j_C \left[t^j_C/\mathsf{x}_C\right]\right]\right)_C$ and $(\psi \cdot \chi)_B(j) = \psi_B(j) \circ \chi_B(j)$. Then we must show that

$$\left[\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\left[\alpha^B_{\psi_B(j)\circ\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right]$$
$$= \left[\rho^B_{M^j}(s^j_B)\left[\alpha^B_{\psi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right] \cdot \left[\rho^B_{M^j}(t^j_B)\left[\alpha^B_{\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right]$$

holds in $M\langle\mathsf{x}_{B^j}\rangle_{B^j}$, i.e. we must show that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j})$ proves the equation

$$\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\left[\alpha^B_{\psi_B(j)\circ\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]$$
$$= \rho^B_{M^j}(s^j_B)\left[\alpha^B_{\psi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right] \cdot \rho^B_{M^j}(t^j_B)\left[\alpha^B_{\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right].$$

First, let $\mathbb{1} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ be the identity element. Then we know that

$$\beta_M(\gamma, \mathbb{1}), \beta_M(\delta, \mathbb{1}), \beta_M(\gamma \cdot \delta, \mathbb{1}) \in G_{\mathbb{T}^{\mathcal{J}}}(M).$$

We have

$$\begin{aligned}
\beta_M(\gamma, \mathbb{1})_{B^j} &= \left[\rho^B_{M^j}\left(s^j_B\right)\left[\alpha^B_{\mathbb{1}_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right] \\
&= \left[\rho^B_{M^j}(s^j_B)\left[\alpha^B_{\mathsf{id}_j}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right]\right] \\
&= \left[\rho^B_{M^j}(s^j_B)[\mathsf{x}_{B^j}/\mathsf{x}_{B^j}]\right] \\
&= \left[\rho^B_{M^j}(s^j_B)\right],
\end{aligned}$$

with the third equality justified by Axiom 5.1.5.2. Similarly, we have

$$\beta_M(\delta, \mathbb{1})_{B^j} = \left[\rho^B_{M^j}\left(t^j_B\right)\right]$$

and

$$\beta_M(\gamma \cdot \delta, \mathbb{1})_{B^j} = \left[\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_{B^j}\right]\right)\right].$$

Since

$$\beta_M(\gamma, \mathbb{1}), \beta_M(\delta, \mathbb{1}), \beta_M(\gamma \cdot \delta, \mathbb{1}) \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

it follows that $\left[\rho^B_{M^j}(s^j_B)\right]$ commutes generically with $\alpha^B_{\psi_B(j)} : B^j \to B^j$, that $\left[\rho^B_{M^j}(t^j_B)\right]$ commutes generically with $\alpha^B_{\chi_B(j)} : B^j \to B^j$, and that $\left[\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\right]$ commutes generically with $\alpha^B_{\psi_B(j)\circ\chi_B(j)} : B^j \to B^j$. This means that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \vdash \rho^B_{M^j}(s^j_B)\left[\alpha^B_{\psi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right] = \alpha^B_{\psi_B(j)}\left(\rho^B_{M^j}(s^j_B)\right),$$

as well as

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \vdash \rho^B_{M^j}(t^j_B)\left[\alpha^B_{\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right] = \alpha^B_{\chi_B(j)}\left(\rho^B_{M^j}(t^j_B)\right)$$

and

$$\begin{aligned}\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j}) \vdash \rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\left[\alpha^B_{\psi_B(j)\circ\chi_B(j)}(\mathsf{x}_{B^j})/\mathsf{x}_{B^j}\right] \\ = \alpha^B_{\psi_B(j)\circ\chi_B(j)}\left(\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\right).\end{aligned}$$

So to complete the argument that β_M preserves group multiplication, it suffices to show that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j})$ proves the equation

$$\alpha^B_{\psi_B(j)\circ\chi_B(j)}\left(\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\right) = \alpha^B_{\psi_B(j)}(\rho^B_{M^j}(s^j_B))\left[\alpha^B_{\chi_B(j)}(\rho^B_{M^j}(t^j_B))/\mathsf{x}_{B^j}\right].$$

But $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^j})$ proves the following sequence of equations, as desired:

$$\begin{aligned}&\alpha^B_{\psi_B(j)}(\rho^B_{M^j}(s^j_B))\left[\alpha^B_{\chi_B(j)}(\rho^B_{M^j}(t^j_B))/\mathsf{x}_{B^j}\right] \\ &\equiv \alpha^B_{\psi_B(j)}\left(\rho^B_{M^j}(s^j_B)\left[\alpha^B_{\chi_B(j)}(\rho^B_{M^j}(t^j_B))/\mathsf{x}_{B^j}\right]\right) \\ &= \alpha^B_{\psi_B(j)}\left(\alpha^B_{\chi_B(j)}\left(\rho^B_{M^j}(s^j_B)\left[\rho^B_{M^j}(t^j_B)/\mathsf{x}_{B^j}\right]\right)\right) \\ &\equiv \alpha^B_{\psi_B(j)}\left(\alpha^B_{\chi_B(j)}\left(\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\right)\right) \\ &= \alpha^B_{\psi_B(j)\circ\chi_B(j)}\left(\rho^B_{M^j}\left(s^j_B\left[t^j_B/\mathsf{x}_B\right]\right)\right).\end{aligned}$$

The second equality follows because $\left[\rho^B_{M^j}(s^j_B)\right]$ commutes generically with $\alpha^B_{\chi_B(j)} : B^j \to B^j$, the third equality follows by Lemma 5.1.14, while the last equality follows by Axiom 5.1.5.3. This completes the argument that β_M preserves the group multiplication.

This finally completes the proof of Proposition 5.1.20. ∎

Our next step is to show that the group homomorphism β_M is bijective. For this purpose, we first require the following definitions.

Definition 5.1.24. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$.

- For any $k \in \mathcal{J}_O$, let

$$\mathsf{Cod}(k) := \{f \in \mathcal{J}_A : \mathsf{cod}(f) = k\}.$$

For any $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, let $\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)$ be the signature obtained from $\Sigma(M^k)$ by adding pairwise distinct new constant symbols $\mathsf{x}^B_f : B$ for every $f \in \mathsf{Cod}(k)$.

- For any $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, let $\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)$ be the quasi-equational theory over the signature $\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)$ obtained from $\mathbb{T}(M^k)$ by adding the axioms $\top \vdash \mathsf{x}^B_f \downarrow$ for every $f \in \mathsf{Cod}(k)$. ∎

Definition 5.1.25 (**α-Restricted Terms**). If $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, then we say that u is *α-restricted* if the only subterms of u of the form $\alpha^C_f(v)$ are those with $C = A$ and $v \equiv \mathsf{x}_{A^i}$ and $\mathsf{dom}(f) = i$.

In other words, $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ is α-restricted if all 'α-subterms' of u have the form $\alpha^A_f(\mathsf{x}_{A^i})$ for some $f \in \mathsf{Dom}(i)$. ∎

Essentially, an α-restricted term is a term in which all of the α-function symbols have been pushed inside 'as far as possible'. In order to prove that every (provably defined) term has an α-restricted equivalent, we require the following lemma:

Lemma 5.1.26. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and let $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ be α-restricted, where $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$. If $u : C^j$ for some $j \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, then for any arrow $f : j \to \mathsf{cod}(f)$ in $\mathcal{J}$, there is an α-restricted term $u^f \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ with $u^f : C^{\mathsf{cod}(f)}$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent*

$$u \downarrow \vdash \alpha^C_f(u) = u^f.$$

Proof: See Appendix C. ∎

With the help of Lemma 5.1.26 we can now show:

Lemma 5.1.27. *If $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, then there is an α-restricted term $u' \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ of the same sort such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent*

$$u \downarrow \vdash u = u'.$$

Proof: See Appendix C. ∎

It is trivial to verify (from the proof of Lemma 5.1.27) that if $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ is already α-restricted, then $u \equiv u'$.

Definition 5.1.28. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, and let

$$\mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$$

be the set of all *α-restricted* terms in $\mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$. We define a map

$$\theta : \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^* \to \bigcup_{k \in \mathcal{J}_O} \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$$

with the property that if $u : C^k$ for $k \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, then $\theta(u) \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$ with $\theta(u) : C$.

We define θ by induction on the structure of $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$:

- For any $f : i \to k$ in $\mathcal{J}$, we set

$$\theta(\mathsf{x}_{A^i}) := \mathsf{x}^A_{\mathsf{id}_i} : A$$

  and

$$\theta(\alpha^A_f(\mathsf{x}_{A^i})) := \mathsf{x}^A_f : A$$

  (note that $\alpha^A_f(\mathsf{x}_{A^i}) : A^k$ and $\mathsf{x}^A_f \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$).

- For any $k \in \mathcal{J}_O$, $C \in \Sigma_{\mathsf{Sort}}$, and $s \in M_{C^k} = M^k_C$, we set

$$\theta\left(c^M_{C^k,s}\right) := c^{M^k}_{C,s}.$$

- For any $k \in \mathcal{J}_O$, any function symbol $g : C_1 \times \ldots \times C_n \to C$ in Σ, and any $u_1, \ldots, u_n \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u_\ell : C^k_\ell$ for all $1 \leq \ell \leq n$, we set

$$\theta(g^k(u_1, \ldots, u_n)) := g(\theta(u_1), \ldots, \theta(u_n)).$$

∎

The idea behind the map θ is that it takes an α-restricted term t and replaces all of the subterms in t of the form $\alpha^A_f(\mathsf{x}_{A^i})$ by constant symbols x^A_f, indexed by the arrows of $\mathcal{J}$. The next result now states that θ preserves provability of equations:

Proposition 5.1.29. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and let $s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ for some $i \in \mathcal{J}_O$ and $A \in \Sigma_{\mathsf{Sort}}$, with $s, t : C^j$ for some $C \in \Sigma_{\mathsf{Sort}}$ and $j \in \mathcal{J}_O$. If*

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash s = t,$$

then

$$\mathbb{T}\left(M^j, \overline{\mathsf{x}^A_{\mathsf{Cod}(j)}}\right) \vdash \theta(s) = \theta(t).$$

Proof: See Appendix C. ∎

We will also need the following technical lemmas, whose proofs may be found in Appendix C.

Lemma 5.1.30. *Let* $\mathbb{T}'$ *be any quasi-equational theory over a relation-free signature* Σ'*, let* $B \in \Sigma'_{\mathsf{Sort}}$*, and let* $\mathcal{C}$ *be a (possibly infinite) set of constants of sort* B *with* $\mathcal{C} \cap \Sigma'_{\mathsf{Fun}} = \emptyset$*. Let* $\Sigma'(\mathcal{C})$ *be the signature with* $\Sigma'(\mathcal{C})_{\mathsf{Sort}} := \Sigma'_{\mathsf{Sort}}$ *and* $\Sigma'(\mathcal{C})_{\mathsf{Fun}} := \Sigma'_{\mathsf{Fun}} \cup \mathcal{C}$*, and let* $\mathbb{T}'(\mathcal{C})$ *be the quasi-equational theory over the signature* $\Sigma'(\mathcal{C})$ *whose axioms are those of* $\mathbb{T}'$ *together with the axioms* $\top \vdash c \downarrow$ *for all* $c \in \mathcal{C}$*.*

Let $s, t \in \mathsf{Term}^c(\Sigma'(\mathcal{C}))$ *be closed terms over* $\Sigma'(\mathcal{C})$ *of the same sort such that at least one of* s *and* t *contains a constant from* $\mathcal{C}$*, and let* $\{c_1, \ldots, c_n\}$ *be the (finite, non-empty) set of all and only those constants of* $\mathcal{C}$ *that occur in either* s *or* t*. Let* $\Sigma'(c_1, \ldots, c_n)$ *and* $\mathbb{T}'(c_1, \ldots, c_n)$ *be the signature and theory defined analogously to* $\Sigma'(\mathcal{C})$ *and* $\mathbb{T}'(\mathcal{C})$*. Then*

$$\mathbb{T}'(\mathcal{C}) \vdash s = t \Longrightarrow \mathbb{T}'(c_1, \ldots, c_n) \vdash s = t.$$

∎

Lemma 5.1.31. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $k \in \mathcal{J}_O$ *and* $B \in \Sigma_{\mathsf{Sort}}$*, and suppose that* $u \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)\right)$ *is of sort* B *and*

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \vdash u = \mathsf{x}^B_{\mathsf{id}_k}.$$

If $\mathbb{T}(M^k)$ *is non-trivial for the sort* B*, then* u *contains at least one occurrence of* $\mathsf{x}^B_{\mathsf{id}_k}$*.*

Proof: See Appendix C. ∎

We will also need the following map θ^*, which essentially takes an α-restricted term t, applies θ to it, and then erases all of the arrow subscripts from the indeterminates of the form x^A_f in $\theta(t)$:

Definition 5.1.32. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$. We define a map

$$\theta^* : \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^* \to \bigcup_{k \in \mathcal{J}_O} \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_A))$$

with the property that if $u : C^k$ for $k \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, then $\theta^*(u) \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_A))$ with $\theta^*(u) : C$. To define θ^*, we first define for each $k \in \mathcal{J}_O$ a signature morphism

$$\lambda_k : \Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \to \Sigma(M^k, \mathsf{x}_A)$$

as follows:

- λ_k is the identity on $\Sigma(M^k)$.
- If $f \in \mathsf{Cod}(k)$, then
$$\lambda_k\left(\mathsf{x}_f^A\right) := \mathsf{x}_A.$$

By a slight abuse of notation, we also denote the induced function on closed terms as

$$\lambda_k : \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}_{\mathsf{Cod}(k)}^A}\right)\right) \to \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_A)).$$

Finally, we set

$$\lambda := \bigcup_{k \in \mathcal{J}_O} \lambda_k : \bigcup_{k \in \mathcal{J}_O} \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}_{\mathsf{Cod}(k)}^A}\right)\right) \to \bigcup_{k \in \mathcal{J}_O} \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_A)),$$

and we then define

$$\theta^* := \lambda \circ \theta : \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^* \to \bigcup_{k \in \mathcal{J}_O} \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_A)),$$

and it is easy to see that θ^* indeed has the stated property. ∎

Before showing that θ^* preserves the provability of a certain restricted kind of sequents, we require the following technical concepts.

Definition 5.1.33 (**i-Local Terms**). Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$.

- If $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, then we say that u is *i-local* if u has the following property:

  For any subterm v of u, there is some sort $C \in \Sigma$ such that $v : C^i$.

  In particular, if u is i-local, then $u : B^i$ for some sort B, and every α-subterm of u has the form $\alpha_f^A(\mathsf{x}_{A^i})$ for some $f : i \to i$.

- Let $f : j \to i$ have codomain i. If $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ is i-local, we define

  $$u[f] \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^j}))^*$$

  (note the change from x_{A^i} to x_{A^j}) to be the term of the same sort defined as follows:

  - If $u \equiv \mathsf{x}_{A^i} : A^i$, then we set

    $$u[f] := \alpha_f^A\left(\mathsf{x}_{A^j}\right) : A^i.$$

  - If $u \equiv \alpha_g^A(\mathsf{x}_{A^i}) : A^i$ for some $g : i \to i$ (since u is i-local), then we set

$$u[f] := \alpha_{g \circ f}^A(\mathsf{x}_{A^j}) : A^i.$$

  - If $u \equiv c_{B^i,s}^M : B^i$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_{B^i}$, then we set

$$u[f] := u : B^i.$$

  - If $u \equiv g^i(u_1, \ldots, u_n) : B^i$ for some function symbol $g : B_1 \times \ldots \times B_n \to B$ in Σ and i-local terms $u_1, \ldots, u_n \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u_\ell : B_\ell^i$ for each $1 \leq \ell \leq n$, then we set

$$u[f] := g^i(u_1[f], \ldots, u_n[f]) : B^i.$$

  In general, $u[f]$ will *not* be the same term as u^f from Lemma 5.1.26.

- If $u \in \mathsf{Term}(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ is i-local and $f : i \to i$, then we say that u *commutes generically with* f if

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \alpha_f^B(u) = u[f]$$

  (assuming that $u : B^i$ for some sort $B \in \Sigma$). ∎

For future reference, we note the following obvious result: if $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ is α-restricted and i-local and $f : i \to i$, then

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$$

$$\implies \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u[f] = u\left[\alpha_f^A(\mathsf{x}_{A^i})/\mathsf{x}_{A^i}\right].$$

We can now prove that θ^* preserves provability of a certain restricted kind of sequents:

Lemma 5.1.34. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. Let $u, s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, with $u : C^i$ and $s, t : D^i$ for some $C, D \in \Sigma_{\mathsf{Sort}}$. Suppose that $u \equiv h^i(u_1, \ldots, u_m)$ for some function symbol $h : C_1 \times \ldots \times C_m \to C$ of Σ and i-local terms $u_1, \ldots, u_m \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u_\ell : C_\ell^i$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u_\ell \downarrow$ for each $1 \leq \ell \leq m$, and assume that u_ℓ commutes generically with each $f : i \to i$ in $\mathcal{J}$.*

If $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

$$u \downarrow \vdash s = t,$$

then $\mathbb{T}(M^i, \mathsf{x}_A)$ proves the sequent

$$\theta^*(u) \downarrow \vdash \theta^*(s) = \theta^*(t).$$

Proof: See Appendix C. ∎

We also have that θ^* preserves the provability of equations:

Lemma 5.1.35. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$. *For any* $s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *with* $s, t : C^k$ *for some* $k \in \mathcal{J}_O$ *and* $C \in \Sigma_{\mathsf{Sort}}$, *if* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ *proves the sequent*

$$\top \vdash s = t,$$

then $\mathbb{T}(M^k, \mathsf{x}_A)$ *proves the sequent*

$$\top \vdash \theta^*(s) = \theta^*(t).$$

Proof: See Appendix C. ∎

The proofs of the following three lemmas may be found in Appendix C.

Lemma 5.1.36. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, *let* $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *be i-local for some* $i \in \mathcal{J}_O$ *and* $A \in \Sigma_{\mathsf{Sort}}$, *and let* $f \in \mathsf{Cod}(i)$. *Then*

$$\theta^*(u) \equiv \theta^*(u[f]) \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_A)).$$

∎

The following lemma says that θ^* interacts properly with substitution, provided that the term being substituted commutes generically with certain arrows of $\mathcal{J}$:

Lemma 5.1.37. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, *let* $u, v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *for some* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$ *with* $v : A^i$, *and suppose that* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u, v \downarrow$. *Suppose also that* u, v *are i-local, and that* v *commutes generically with every arrow* $f : i \to i$ *in* $\mathcal{J}$. *Then*

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(u[v/\mathsf{x}_{A^i}]'\right) = \theta^*(u)[\theta^*(v)/\mathsf{x}_A],$$

where $u[v/\mathsf{x}_{A^i}]'$ *is the* α*-restricted variant of* $u[v/\mathsf{x}_{A^i}]$ *from Lemma 5.1.27.* ∎

We will need the following technical lemma to prove that the group homomorphism β_M is surjective:

Lemma 5.1.38. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$*, and let* $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *be an* α*-restricted,* i*-local term of sort* B^i *for some* $B \in \Sigma_{\mathsf{Sort}}$ *with* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$*. Let* $f : i \to \ell$ *be an arbitrary arrow of* $\mathcal{J}$ *with* $\mathsf{dom}(f) = i$*. Then* $\alpha^B_f(u)$ *has an* α*-restricted variant* $\alpha^B_f(u)'$ *by Lemma 5.1.27, and* $\alpha^B_f(u)' : B^\ell$*, so that* $\theta^*\left(\alpha^B_f(u)'\right) \in \mathsf{Term}^c\left(\Sigma\left(M^\ell, \mathsf{x}_A\right)\right)$*. And* $\theta^*(u) \in \mathsf{Term}^c\left(\Sigma\left(M^i, \mathsf{x}_A\right)\right)$*, so that* $\rho^A_{f^M}(\theta^*(u)) \in \mathsf{Term}^c(\Sigma(M^\ell, \mathsf{x}_A))$*, where* $\rho^A_{f^M} : \mathbb{T}(M^i, \mathsf{x}_A) \to \mathbb{T}(M^\ell, \mathsf{x}_A)$ *is the theory morphism induced by the* Σ*-morphism* $f^M := F^M(f) : M^i \to M^\ell$*. Then*

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \theta^*\left(\alpha^B_f(u)'\right) = \rho^A_{f^M}(\theta^*(u)).$$

∎

We will also require the following technical results regarding the map θ, whose proofs involve straightforward inductions on terms:

Lemma 5.1.39. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$*, and let* $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *be of sort* B^k *for some* $B \in \Sigma_{\mathsf{Sort}}$ *and* $k \in \mathcal{J}_O$*. Fix an arrow* $f : k \to k$ *in* $\mathcal{J}$*. By Lemma 5.1.26, there is a term* $v^f \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *with* $v^f : B^k$*.*

Then $\theta(v), \theta(v^f) \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$*, and for any* $g \in \mathsf{Cod}(k)$*,*

$$\mathsf{x}^A_g \textit{ occurs in } \theta(v) \textit{ iff } \mathsf{x}^A_{f \circ g} \textit{ occurs in } \theta(v^f).$$

Lemma 5.1.40. *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, let* $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k}))$ *for some* $B \in \Sigma_{\mathsf{Sort}}$ *and* $k \in \mathcal{J}_O$*, and suppose that* u *is* α*-restricted and* k*-local. Then it is easy to see that every indeterminate in* $\theta(u) \in \mathsf{Term}\left(\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)\right)$ *has the form* x^B_f *for some arrow* $f : k \to k$ *(since* u *is* k*-local).*

Suppose that the indeterminates occurring in $\theta(u)$ *are* $\mathsf{x}^B_{f_1}, \ldots, \mathsf{x}^B_{f_n}$*, with* $f_1, \ldots, f_n : k \to k$*. Then for any* $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *for some* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$ *with* $v : B^k$*, we know that* $u[v/\mathsf{x}_{B^k}] \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ *has an* α*-restricted variant* $u[v/\mathsf{x}_{B^k}]' \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *(by Lemma 5.1.27). We then have*

$$\theta\left(u[v/\mathsf{x}_{B^k}]'\right) \equiv \theta(u)\left[\theta\left(v^{f_1}\right)/\mathsf{x}^B_{f_1}, \ldots, \theta\left(v^{f_n}\right)/\mathsf{x}^B_{f_n}\right] \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$$

(recall from Lemma 5.1.26 that, for each $1 \leq i \leq n$*,* $v^{f_i} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *is a term of sort* $B^{\mathsf{cod}(f_i)} = B^k$*).* ∎

Finally, we require the following notion of 'α-free variant':

Definition 5.1.41 (**α-Free Variants**)**.** Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. For any $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ that is i-local for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, we define a term

$$u^{-\alpha} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$$

of the same sort, which we will call the *α-free variant* of u:

- If $u \equiv \mathsf{x}_{A^i} : A^i$, then
$$u^{-\alpha} := u : A^i.$$
- If $u \equiv \alpha^A_f(\mathsf{x}_{A^i}) : A^i$ for some arrow $f : i \to i$ (since u is i-local), then
$$u^{-\alpha} := \mathsf{x}_{A^i} : A^i.$$
- If $u \equiv c^M_{B^i,s} : B^i$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_{B^i}$, then
$$u^{-\alpha} := u : B^i.$$
- If $u \equiv g^i(u_1, \ldots, u_n) : B^i$ for some function symbol $g : B_1 \times \ldots \times B_n \to B$ of Σ and i-local terms $u_i \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ of sort B^i_j for each $1 \leq j \leq n$, then
$$u^{-\alpha} := g^i(u_1^{-\alpha}, \ldots, u_n^{-\alpha}) : B^i.$$

∎

Essentially, the α-free variant $u^{-\alpha}$ is obtained from u by 'erasing' all of the α function symbols in u (and since u is i-local, it is possible to do this and obtain a well-defined term of the same sort). We then have the following technical lemma, whose proof is a straightforward induction on terms:

Lemma 5.1.42. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, let $i \in \mathcal{J}_O$, and let $A \in \Sigma_{\mathsf{Sort}}$. For any α-restricted and i-local $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$, we have*

$$\rho^A_{M^i}(\theta^*(u)) \equiv u^{-\alpha},$$

where $\rho^A_{M^i} : \Sigma(M^i, \mathsf{x}_A) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ is the signature morphism from Definition 5.1.12. ∎

We can now finally prove that the group homomorphism $\beta_M : (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$ is injective:

Proposition 5.1.43. *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, the group homomorphism*

$$\beta_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is injective.

Proof: Let $\gamma = (\gamma_i)_i \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$, with $\gamma_i = ([s^i_C])_{C \in \Sigma}$ for each $i \in \mathcal{J}_O$. Also let $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. Suppose that

$$\beta_M(\gamma, \psi) = ([\mathsf{x}_A])_{A \in \Sigma^{\mathcal{J}}},$$

the unit element of the group $G_{\mathbb{T}^{\mathcal{J}}}(M)$. We must show that each γ_i is the unit of the group $G_{\mathbb{T}}(M^i)$, i.e. we must show that

$$\gamma_i = ([\mathsf{x}_C])_C$$

for all $i \in \mathcal{J}_O$, and we must show that $\psi = \mathbb{1}$ is the unit element of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. So fix $i \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, and suppose first that $\mathbb{T}(M^i)$ is non-trivial for the sort B. The hypothesis implies in particular that

$$\beta_M(\gamma, \psi)_{B^i} = [\mathsf{x}_{B^i}],$$

i.e. that

$$\left[\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right] = [\mathsf{x}_{B^i}] \in M\langle \mathsf{x}_{B^i}\rangle_{B^i},$$

which means that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right] = \mathsf{x}_{B^i}.$$

First, we will show that $\psi_B(i) = \mathsf{id}_i$. Note that

$$\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right] \in \mathsf{Term}(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i}))^*$$

(because $\rho^B_{M^i}(s^i_B)$ does not contain any α function symbols). Then by Proposition 5.1.29, we obtain

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash \theta\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right) = \theta(\mathsf{x}_{B^i}).$$

Now, it is trivial to see that the only indeterminate that occurs in $\theta(\rho^B_{M^i}(s^i_B)) \in \mathsf{Term}\left(\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)\right)$ is $\mathsf{x}^B_{\mathsf{id}_i}$. Then since $\rho^B_{M^i}(s^i_B) \in \mathsf{Term}(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i}))$ is also α-restricted and i-local, and $\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]$ is α-restricted, it follows by Lemma 5.1.40 that

$$\theta\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right) \equiv \theta\left(\rho^B_{M^i}(s^i_B)\right)\left[\theta\left(\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})^{\mathsf{id}_i}\right)/\mathsf{x}^B_{\mathsf{id}_i}\right],$$

and hence

$$\theta\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right) \equiv \theta\left(\rho^B_{M^i}(s^i_B)\right)\left[\mathsf{x}^B_{\psi_B(i)}/\mathsf{x}^B_{\mathsf{id}_i}\right],$$

because

$$\theta\left(\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})^{\mathsf{id}_i}\right) \equiv \theta\left(\alpha^B_{\mathsf{id}_i\circ\psi_B(i)}(\mathsf{x}_{B^i})\right) \equiv \theta\left(\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})\right) \equiv \mathsf{x}^B_{\psi_B(i)}.$$

Also, it is easy to see that

$$\theta\left(\rho^B_{M^i}(s^i_B)\right) \equiv s^i_B\left[\mathsf{x}^B_{\mathsf{id}_i}/\mathsf{x}_B\right],$$

and so we obtain

$$\theta\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right) \equiv s^i_B\left[\mathsf{x}^B_{\psi_B(i)}/\mathsf{x}_B\right].$$

Since $\theta(\mathsf{x}_{B^i}) \equiv \mathsf{x}^B_{\mathsf{id}_i}$, from the fact that

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash \theta\left(\rho^B_{M^i}(s^i_B)\left[\alpha^B_{\psi_B(i)}(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right]\right) = \theta(\mathsf{x}_{B^i})$$

we finally deduce that

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash s^i_B\left[\mathsf{x}^B_{\psi_B(i)}/\mathsf{x}_B\right] = \mathsf{x}^B_{\mathsf{id}_i}.$$

Since $\mathbb{T}(M^i)$ is non-trivial for the sort B, it then follows from Lemma 5.1.31 that $\mathsf{x}^B_{\mathsf{id}_i}$ occurs in $s^i_B\left[\mathsf{x}^B_{\psi_B(i)}/\mathsf{x}_{B^i}\right]$, which means that $\mathsf{x}^B_{\psi_B(i)} \equiv \mathsf{x}^B_{\mathsf{id}_i}$ (because $\mathsf{x}^B_{\psi_B(i)}$ is the only indeterminate occurring in $s^i_B\left[\mathsf{x}^B_{\psi_B(i)}/\mathsf{x}_{B^i}\right]$) and hence $\psi_B(i) = \mathsf{id}_i$, as desired. So we may now infer that

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash s^i_B[\mathsf{x}^B_{\mathsf{id}_i}/\mathsf{x}_{B^i}] = \mathsf{x}^B_{\mathsf{id}_i}.$$

Since

$$\lambda_i : \mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \to \mathbb{T}(M^i, \mathsf{x}_B)$$

is a theory morphism by the proof of Lemma 5.1.34, we then obtain

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash \lambda_i\left(s^i_B\left[\mathsf{x}^B_{\mathsf{id}_i}/\mathsf{x}_{B^i}\right]\right) = \lambda_i\left(\mathsf{x}^B_{\mathsf{id}_i}\right).$$

Since λ_i is the identity except on the indeterminates of $\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)$, it then follows that

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash s^i_B = \mathsf{x}_B.$$

This shows that if $\mathbb{T}(M^i)$ is non-trivial for the sort B, then $\gamma^B_i = [s^i_B] = [\mathsf{x}_B]$ and $\psi_B(i) = \mathsf{id}_i$, which implies that ψ_B is the identity natural automorphism of $\mathsf{Id}_{\mathcal{J}^M_B}$, so that ψ is the unit element of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$.

It remains to show that if $\mathbb{T}(M^i)$ is *trivial* for the sort B, then $\gamma_i^B = [s_B^i] = [\mathsf{x}_B]$ in this case as well. But if $\mathbb{T}(M^i)$ is trivial for the sort B, then $\mathbb{T}(M^i, \mathsf{x}_B)$ is trivial for the sort B as well, which implies that

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash s_B^i = \mathsf{x}_B,$$

because $s_B^i, \mathsf{x}_B : B$. This completes the proof that each γ_i is the unit element of $G_{\mathbb{T}}(M^i)$, which completes the proof that β_M is injective. ∎

Since we will need to make two assumptions about $\mathbb{T}$ in order to prove that each β_M is surjective, let us now record what we have proven so far:

Proposition 5.1.44. *Let* $\mathbb{T}$ *be an* arbitrary *quasi-equational theory and* $\mathcal{J}$ *a small index category. Then for any* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, there is an injective group homomorphism*

$$\beta_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M).$$

∎

To prove that each β_M is surjective, we will need to assume that $\mathbb{T}$ satisfies the conditions in the following definitions:

Definition 5.1.45 (Single-Indeterminate Isotropy). Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ. We say that $\mathbb{T}$ has *single-indeterminate isotropy* if $\mathbb{T}$ has the following property:

> For any model $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $([s_C])_{C \in \Sigma} \in G_{\mathbb{T}}(N)$ and $C \in \Sigma_{\mathsf{Sort}}$, there is some $t_C \in \mathsf{Term}^c(\Sigma(N, \mathsf{x}_C))_C$ such that t_C contains *exactly one occurrence* of the indeterminate x_C and $[s_C] = [t_C]$.

∎

In other words, $\mathbb{T}$ has single-indeterminate isotropy if every component of every element of isotropy of every model of $\mathbb{T}$ can be assumed to have exactly one occurrence of the indeterminate. This is not an *overly* restrictive condition, because every theory considered in Chapter 3 has single-indeterminate isotropy, as the reader can easily check. Nonetheless, not every quasi-equational theory satisfies this condition, as we will show later in the chapter (cf. Remark 5.1.58).

Definition 5.1.46 (Single-Sorted Non-Total Operations). Let $\mathbb{T}$ be a quasi-equational theory over a relation-free signature Σ. If $g : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ, then we say that g is *totally defined in* $\mathbb{T}$ if $\mathbb{T}$ proves the sequent

$$\top \vdash^{y_1,\ldots,y_n} g(y_1, \ldots, y_n) \downarrow,$$

where $y_1, \ldots, y_n$ are pairwise distinct variables with $y_i : A_i$ for each $1 \leq i \leq n$. Otherwise, i.e. if $\mathbb{T}$ does *not* prove the above sequent, then we say that g is *not totally defined in* $\mathbb{T}$.

We then say that $\mathbb{T}$ has *single-sorted non-total operations* if $\mathbb{T}$ satisfies the following condition: for any function symbol $g : A_1 \times \ldots \times A_n \to A$ of Σ that is *not* totally defined in $\mathbb{T}$, we have $A_i = A$ for each $1 \leq i \leq n$. ∎

Again, this is not an *overly* restrictive condition, because every theory considered in Chapter 3 has single-sorted non-total operations, as the reader can easily check. In fact, the only theories in Chapter 3 that have function symbols that are *not* totally defined are the theories of categories, groupoids, categories with a terminal object, and strict monoidal categories; but the only function symbol that is *not* totally defined in these theories is the composition operation $\circ : A \times A \to A$, which *does* satisfy the condition at the end of Definition 5.1.46.

Obviously, any quasi-equational theory that is single-sorted and/or totally defined has single-sorted non-total operations (and every theory in Chapter 3 besides the ones just mentioned is single-sorted).

Proposition 5.1.47. *Let $\mathbb{T}$ be a quasi-equational theory with single-indeterminate isotropy and single-sorted non-total operations, and let $\mathcal{J}$ be a small index category. For any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, the group homomorphism $\beta_M : (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$ is surjective.*

Proof: Let $([s_{C^i}])_{i \in \mathcal{J}_O, C \in \Sigma} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$. So for any $i \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, $s_{C^i} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))$ is a closed term of sort C^i with $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash s_{C^i} \downarrow$. Moreover, the $\Sigma^{\mathcal{J}}_{\mathsf{Sort}}$-indexed sequence $([s_{C^i}])_{i,C}$ is invertible, commutes generically with all function symbols of $\Sigma^{\mathcal{J}}$, and reflects definedness. By Lemma 5.1.27, we may assume without loss of generality that for each $i \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, the term $s_{C^i} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))_{C^i}$ is α-restricted, i.e.

$$s_{C^i} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))^*.$$

Then for every $C \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, it follows by Lemma 5.1.35 that $\theta^*(s_{C^i}) \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))_C$ and $\mathbb{T}(M^i, \mathsf{x}_C) \vdash \theta^*(s_{C^i}) \downarrow$. So

$$[\theta^*(s_{C^i})] \in M^i\langle \mathsf{x}_C \rangle_C.$$

Now we define

$$\gamma \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}.$$

For any $i \in \mathcal{J}_O$, we define

$$\gamma_i \in G_{\mathbb{T}}(M^i) \subseteq \prod_C M^i\langle \mathsf{x}_C \rangle_C$$

as follows: for any $C \in \Sigma_{\mathsf{Sort}}$, we set

$$\gamma_i^C := [\theta^*(s_{C^i})] \in M^i\langle \mathsf{x}_C \rangle_C.$$

Our goal is now to show that $\gamma \in (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}}$. We do this via the following series of claims.

Claim 5.1.48. For any $i \in \mathcal{J}_O$, γ_i is invertible.

Proof: Let $B \in \Sigma_{\mathsf{Sort}}$. We must show that there is some

$$[t_B^i] \in M^i\langle \mathsf{x}_B \rangle_B$$

with

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash \theta^*(s_{B^i})[t_B^i/\mathsf{x}_B] = \mathsf{x}_B = t_B^i[\theta^*(s_{B^i})/\mathsf{x}_B].$$

Since $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, there is some

$$\left[s_{B^i}^{-1}\right] \in M\langle \mathsf{x}_{B^i} \rangle_{B^i}$$

with

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right] = \mathsf{x}_{B^i} = s_{B^i}^{-1}[s_{B^i}/\mathsf{x}_{B^i}].$$

Since $s_{B^i}^{-1} \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i}))_{B^i}$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}^{-1} \downarrow$, we may assume without loss of generality that $s_{B^i}^{-1}$ is α-restricted by Lemma 5.1.27. So

$$\theta^*\left(s_{B^i}^{-1}\right) \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_B))$$

has the property that $\mathbb{T}(M^i, \mathsf{x}_B) \vdash \theta^*\left(s_{B^i}^{-1}\right) \downarrow$ by Lemma 5.1.35. Hence, we have

$$\left[\theta^*\left(s_{B^i}^{-1}\right)\right] \in M^i\langle \mathsf{x}_B \rangle_B,$$

so we set

$$[t_B^i] := \left[\theta^*\left(s_{B^i}^{-1}\right)\right].$$

Since $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, it follows that $([s_{C^j}])_{j,C}$ commutes generically with the function symbol $\alpha_f^B : B^i \to B^i$, for any arrow $f : i \to i$ in $\mathcal{J}$. In other words, for any arrow $f : i \to i \in \mathcal{J}$, we have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \alpha_f^B(s_{B^i}) = s_{B^i}[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}] = s_{B^i}[f],$$

the latter equality being provable by the remark after Definition 5.1.33 (since s_{B^i} is i-local, because it is α-restricted and of sort B^i and only contains the indeterminate x_{B^i}). In the same way, we also have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \alpha_f^B\left(s_{B^i}^{-1}\right) = s_{B^i}^{-1}\left[\alpha_f^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^i}\right] = s_{B^i}^{-1}[f]$$

for any arrow $f : i \to i$ in $\mathcal{J}$. Since $s_{B^i}^{-1}$ is i-local, this means that $s_{B^i}^{-1}$ also commutes generically with every $f : i \to i$. Then by Lemma 5.1.37, we obtain

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash \theta^*\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right) = \theta^*(s_{B^i})\left[\theta^*\left(s_{B^i}^{-1}\right)/\mathsf{x}_B\right], \tag{$*$}$$

where $s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'$ is the α-restricted variant of $s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]$ with

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right] = s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'$$

by Lemma 5.1.27. From this latter equation and the defining property of $s_{B^i}^{-1}$, we obtain

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]' = \mathsf{x}_{B^i}.$$

By Lemma 5.1.35, we then obtain

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash \theta^*\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right) = \theta^*(\mathsf{x}_{B^i}) \equiv \mathsf{x}_B.$$

Combining this with $(*)$, we finally have

$$\mathbb{T}(M^i, \mathsf{x}_B) \vdash \theta^*(s_{B^i})\left[\theta^*(s_{B^i}^{-1})/\mathsf{x}_B\right] = \mathsf{x}_B,$$

as desired. The converse equality is proven analogously. This completes the proof that γ_i is invertible. ∎

Claim 5.1.49. γ_i commutes generically with all function symbols of Σ.

Proof: Let $g : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ. We must show that the sequent

$$g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n}) \downarrow \vdash \theta^*(s_{B^i})\left[g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n})/\mathsf{x}_B\right] = g\left(\theta^*\left(s_{B_1^i}\right), \ldots, \theta^*\left(s_{B_n^i}\right)\right)$$

is provable in the theory $\mathbb{T}(M^i, \mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n})$ (as in the proof of Proposition 5.1.20, we technically need to ensure that the indeterminates on the right side of the above equation are pairwise distinct (cf. Definition 2.2.47), but we will ignore this subtlety here and elsewhere in the proof of the proposition to increase readability). Since

$([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, we know that $([s_{C^j}])_{j,C}$ commutes generically with the function symbol $g^i : B^i_1 \times \ldots B^i_n \to B^i$ of $\Sigma^{\mathcal{J}}$, which means that the sequent

$$g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \downarrow \vdash s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right] = g^i\left(s_{B^i_1}, \ldots, s_{B^i_n}\right)$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)$. Since the terms $s_{B^i_1}, \ldots, s_{B^i_n}$ are all α-restricted, it easily follows that the terms $g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)$ and $g^i\left(s_{B^i_1}, \ldots, s_{B^i_n}\right)$ are α-restricted. By a simple extension of Lemma 5.1.27, there is an α-restricted variant $s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right]'$ of $s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right]$ such that the sequent

$$g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \downarrow \vdash s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right] = s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right]'$$

is provable in $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)$. For each $1 \leq m \leq n$, the indeterminate $\mathsf{x}_{B^i_m}$ is clearly i-local, we have $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \vdash \mathsf{x}_{B^i_m} \downarrow$, and for each arrow $f : i \to i$ in $\mathcal{J}$ we have

$$\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \vdash \alpha^{B_m}_f\left(\mathsf{x}_{B^i_m}\right) = \mathsf{x}_{B^i_m}[f],$$

which means that $\mathsf{x}_{B^i_m}$ commutes generically with f. Then by a simple extension of Lemma 5.1.34 and the assumption that

$$g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \downarrow \vdash s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right] = g^i\left(s_{B^i_1}, \ldots, s_{B^i_n}\right)$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M, \mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)$, we obtain that

$$g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n}) \downarrow \vdash \theta^*\left(s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right]'\right) = g\left(\theta^*\left(s_{B^i_1}\right), \ldots, \theta^*\left(s_{B^i_n}\right)\right)$$

is provable in the theory $\mathbb{T}(M^i, \mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n})$. Now, we will be done if we can show that $\mathbb{T}(M^i, \mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n})$ proves the sequent

$$g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n}) \downarrow \vdash \theta^*\left(s_{B^i}\left[g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)/\mathsf{x}_{B^i}\right]'\right) = \theta^*(s_{B^i})\left[g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n})/\mathsf{x}_B\right].$$

Since

$$\theta^*\left(g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)\right) \equiv g(\mathsf{x}_{B_1}, \ldots, \mathsf{x}_{B_n}),$$

it suffices by a simple extension of Lemma 5.1.37 to show that $g^i(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n})$ commutes generically with every $f : i \to i$ in $\mathcal{J}$, i.e. it suffices to show that the sequent

$$g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right) \downarrow \vdash \alpha^B_f\left(g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)\right) = g^i\left(\mathsf{x}_{B^i_1}, \ldots, \mathsf{x}_{B^i_n}\right)[f],$$

i.e. the sequent

$$g^i\left(\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)\downarrow\vdash\alpha_f^B\left(g^i\left(\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)\right)=g^i\left(\alpha_f^{B_1}\left(\mathsf{x}_{B_1^i}\right),\ldots,\alpha_f^B\left(\mathsf{x}_{B_n^i}\right)\right)$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)$ for each $f : i \to i$ in $\mathcal{J}$. But this is true by Axiom 5.1.5.4. This completes the argument that γ_i commutes generically with all function symbols of Σ. ∎

Claim 5.1.50. γ_i reflects definedness.

Proof: Let the function symbol $g \in \Sigma$ be as above (with $n \geq 1$). We must show that the sequent

$$g\left(\theta^*\left(s_{B_1^i}\right),\ldots,\theta^*\left(s_{B_n^i}\right)\right)\downarrow\vdash g(\mathsf{x}_{B_1},\ldots,\mathsf{x}_{B_n})\downarrow$$

is provable in the theory $\mathbb{T}(M^i,\mathsf{x}_{B_1},\ldots,\mathsf{x}_{B_n})$. Since $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, we know that $([s_{C^j}])_{j,C}$ reflects definedness, which implies that the sequent

$$g^i\left(s_{B_1^i},\ldots,s_{B_n^i}\right)\downarrow\vdash g^i\left(\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)\downarrow$$

is provable in the theory $\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)$. As remarked above, the terms in the latter sequent are both α-restricted. For each $1 \leq m \leq n$, the term $s_{B_m^i}$ is i-local and satisfies $\mathbb{T}^{\mathcal{J}}\left(M,\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)\vdash s_{B_m^i}\downarrow$. The term $s_{B_m^i}$ also commutes generically with every $f : i \to i$ in $\mathcal{J}$, because $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$ and thus commutes generically with the function symbol $\alpha_f^{B_m} : B_m^i \to B_m^i$. So by a simple extension of Lemma 5.1.34, it follows that $\mathbb{T}(M^i,\mathsf{x}_{B_1},\ldots,\mathsf{x}_{B_n})$ proves the sequent

$$\theta^*\left(g^i\left(s_{B_1^i},\ldots,s_{B_n^i}\right)\right)\downarrow\vdash\theta^*\left(g^i\left(\mathsf{x}_{B_1^i},\ldots,\mathsf{x}_{B_n^i}\right)\right)\downarrow.$$

But (recalling the definition of θ^*) this is the desired sequent. ∎

So $\gamma \in \prod_i G_{\mathbb{T}}(M^i)$ by the previous three claims, and now we must show that $\gamma \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$. To show this, let $f : i \to k$ be an arbitrary morphism of $\mathcal{J}$. We must show that

$$G_{\mathbb{T}}(f^M)(\gamma_i)=\gamma_k$$

(recall that $f^M := F^M(f) : M^i \to M^k$). Unravelling the definitions, this means that we must show for any $B \in \Sigma_{\mathsf{Sort}}$ that

$$\left[\rho_{f^M}^B\left(\theta^*(s_{B^i})\right)\right]=\left[\theta^*(s_{B^k})\right]$$

holds in $M^k\langle \mathsf{x}_B \rangle_B$, i.e. that

$$\mathbb{T}(M^k, \mathsf{x}_B) \vdash \rho^B_{f^M}(\theta^*(s_{B^i})) = \theta^*(s_{B^k}),$$

where $\rho^B_{f^M} : \mathbb{T}(M^i, \mathsf{x}_B) \to \mathbb{T}(M^k, \mathsf{x}_B)$ is the theory morphism induced by the Σ-morphism $f^M : M^i \to M^k$ by Definition 2.2.17.

Since $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, we know that $([s_{C^j}])_{j,C}$ commutes generically with the function symbol $\alpha^B_f : B^i \to B^k$ of $\Sigma^{\mathcal{J}}$, which means that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right] = \alpha^B_f(s_{B^i})$$

(since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \alpha^B_f(\mathsf{x}_{B^i}) \downarrow$). By Lemma 5.1.27, there are α-restricted variants $s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]', \alpha^B_f(s_{B^i})' \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i}))$ of these terms. So we have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]' = \alpha^B_f(s_{B^i})'.$$

Then by Lemma 5.1.35 we obtain

$$\mathbb{T}(M^k, \mathsf{x}_B) \vdash \theta^*\left(s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right) = \theta^*\left(\alpha^B_f(s_{B^i})'\right),$$

since both of the arguments of θ^* are of sort B^k. By Lemma 5.1.36, since s_{B^k} is k-local and $f \in \mathsf{Cod}(k)$, we have

$$\theta^*(s_{B^k}) \equiv \theta^*(s_{B^k}[f]).$$

We also have (by the observation following Definition 5.1.33)

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^k}[f] = s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right],$$

and hence

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^k}[f] = s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'.$$

Then by Lemma 5.1.35 we deduce

$$\mathbb{T}(M^k, \mathsf{x}_B) \vdash \theta^*(s_{B^k}) \equiv \theta^*(s_{B^k}[f]) = \theta^*\left(s_{B^k}\left[\alpha^B_f(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right) = \theta^*\left(\alpha^B_f(s_{B^i})'\right).$$

So to obtain our desired result, it suffices to show that

$$\mathbb{T}(M^k, \mathsf{x}_B) \vdash \theta^*\left(\alpha^B_f(s_{B^i})'\right) = \rho^B_{f^M}(\theta^*(s_{B^i}));$$

but this is true by Lemma 5.1.38, given that s_{B^i} is α-restricted and i-local and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i} \downarrow$. This completes the proof that $\gamma \in (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}}$.

To complete the proof that β_M is surjective, we must now construct an element $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ and then show that $\beta_M(\gamma, \psi) = ([s_{C^j}])_{j,C}$. So for every sort $B \in \Sigma$, we must construct a natural automorphism $\psi_B : \mathsf{Id}_{\mathcal{J}_B^M} \to \mathsf{Id}_{\mathcal{J}_B^M}$. Let i be any object of $\mathcal{J}_B^M$. Then by definition of $\mathcal{J}_B^M$, it follows that the theory $\mathbb{T}(M^i)$ is non-trivial for the sort B. We now wish to define an arrow (which will turn out to be an isomorphism) $\psi_B(i) : i \to i$.

We have shown that $\gamma_i := ([\theta^*(s_{C^i})])_{C \in \Sigma} \in G_{\mathbb{T}}(M^i)$. Then because $\mathbb{T}$ has single-indeterminate isotropy, we can assume without loss of generality that $\theta^*(s_{B^i}) \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_B))_B$ has exactly one occurrence of the indeterminate x_B. From this, it then follows that $s_{B^i} \in \mathsf{Term}(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^i}))_{B^i}$ has exactly one occurrence of the indeterminate x_{B^i} (because distinct occurrences of x_{B^i} in s_{B^i} correspond to distinct occurrences of x_B in $\theta^*(s_{B^i})$). More precisely, because $\theta^*(s_{B^i})$ has exactly one occurrence of x_B, it follows that $\rho^B_{M^i}(\theta^*(s_{B^i}))$ has exactly one occurrence of x_{B^i}. But by Lemma 5.1.42 we know that $\rho^B_{M^i}(\theta^*(s_{B^i})) \equiv s_{B^i}^{-\alpha}$, and so the α-free variant $s_{B^i}^{-\alpha}$ of s_{B^i} has exactly one occurrence of x_{B^i}, which implies that s_{B^i} has exactly one occurrence of x_{B^i}.

Now consider $\theta(s_{B^i}) \in \mathsf{Term}^c\left(\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)\right)$. Since s_{B^i} has exactly one occurrence of x_{B^i}, it follows that $\theta(s_{B^i})$ has exactly one indeterminate from $\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)$, and moreover, the subscript of this indeterminate will be an arrow from i to i. Then we define $\psi_B(i) : i \to i$ to be this arrow. In other words, we define $\psi_B(i) : i \to i$ so that $\mathsf{x}^B_{\psi_B(i)}$ is the unique indeterminate occurring in $\theta(s_{B^i})$.

Claim 5.1.51. $\psi_B(i) : i \to i$ is an isomorphism.

Proof: From the proof that $\gamma_i := ([\theta^*(s_{C^i})])_{C \in \Sigma} \in G_{\mathbb{T}}(M^i)$, it follows that $\gamma_i^{-1} = \left(\left[\theta^*\left(s_{C^i}^{-1}\right)\right]\right)_{C \in \Sigma} \in G_{\mathbb{T}}(M^i)$. Then, as for s_{B^i}, it follows that $s_{B^i}^{-1}$ has exactly one occurrence of the indeterminate x_{B^i}, and so we define $\psi_B(i)^{-1} : i \to i$ from $\theta\left(s_{B^i}^{-1}\right)$ in the same way that we defined $\psi_B(i)$ from $\theta(s_{B^i})$. Now we need to verify that $\psi_B(i)$ and $\psi_B(i)^{-1}$ are in fact mutually inverse. First, we know that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right] = \mathsf{x}_{B^i} = s_{B^i}^{-1}[s_{B^i}/\mathsf{x}_{B^i}].$$

By Lemma 5.1.27, there is an α-restricted variant $s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'$ of $s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]$, so we obtain

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]' = \mathsf{x}_{B^i}.$$

Then by Proposition 5.1.29, we have

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash \theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right) = \theta(\mathsf{x}_{B^i}) \equiv \mathsf{x}^B_{\mathsf{id}_i}.$$

By Lemma 5.1.40, since the unique indeterminate that $\theta(s_{B^i}) \in \mathsf{Term}^c\left(\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)\right)$ contains is $\mathsf{x}^B_{\psi_B(i)}$, it follows that

$$\theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right) \equiv \theta(s_{B^i})\left[\theta\left(\left(s_{B^i}^{-1}\right)^{\psi_B(i)}\right)/\mathsf{x}^B_{\psi_B(i)}\right].$$

So then the unique indeterminate that occurs in $\theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right)$ will be the unique indeterminate that occurs in $\theta\left(\left(s_{B^i}^{-1}\right)^{\psi_B(i)}\right)$. But since the unique indeterminate that occurs in $\theta\left(s_{B^i}^{-1}\right)$ is $\mathsf{x}^B_{\psi_B(i)^{-1}}$, it follows by Lemma 5.1.39 that the unique indeterminate that occurs in $\theta\left(\left(s_{B^i}^{-1}\right)^{\psi_B(i)}\right)$ is $\mathsf{x}^B_{\psi_B(i)\circ\psi_B(i)^{-1}}$. In summary, the unique indeterminate that occurs in $\theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right)$ is $\mathsf{x}^B_{\psi_B(i)\circ\psi_B(i)^{-1}}$.

Now, since $\mathbb{T}(M^i)$ is non-trivial for the sort B and

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash \theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right) = \mathsf{x}^B_{\mathsf{id}_i},$$

it follows by Lemma 5.1.31 that $\mathsf{x}^B_{\mathsf{id}_i}$ occurs in $\theta\left(s_{B^i}\left[s_{B^i}^{-1}/\mathsf{x}_{B^i}\right]'\right)$. But then we must have $\mathsf{x}^B_{\psi_B(i)\circ\psi_B(i)^{-1}} \equiv \mathsf{x}^B_{\mathsf{id}_i}$, which forces $\psi_B(i)\circ\psi_B(i)^{-1} = \mathsf{id}_i$, since distinct arrows with codomain i correspond to distinct indeterminates in $\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)$. The proof that $\psi_B(i)^{-1} \circ \psi_B(i) = \mathsf{id}_i$ is analogous. This completes the proof that $\psi_B(i) : i \to i$ is an isomorphism. ∎

Claim 5.1.52. ψ_B is a natural automorphism of $\mathsf{Id}_{\mathcal{J}_B^M}$.

Proof: Let $i, k \in \mathcal{J}_B^M$, which means that the theories $\mathbb{T}(M^i)$ and $\mathbb{T}(M^k)$ are both non-trivial for the sort B. And let $h : i \to k$ be an arbitrary arrow in $\mathcal{J}$. We must show that

$$h \circ \psi_B(i) = \psi_B(k) \circ h : i \to k.$$

We know that $\alpha_h^B : B^i \to B^k$ is a function symbol of $\Sigma^{\mathcal{J}}$, so because $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, it follows that $([s_{C^j}])_{j,C}$ commutes generically with this function symbol, which means that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \alpha_h^B(s_{B^i}) = s_{B^k}\left[\alpha_h^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right].$$

By Lemma 5.1.27, the righthand term in the above equation has an α-restricted variant $s_{B^k}\left[\alpha_h^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'$, and by Lemma 5.1.26, since $s_{B^i} : B^i$ is α-restricted and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}\downarrow$, we know that $s_{B^i}^h : B^k$ is an α-restricted term with

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash \alpha_h^B(s_{B^i}) = s_{B^i}^h.$$

Altogether, we then have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^i}) \vdash s_{B^i}^h = s_{B^k}\left[\alpha_h^B(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]',$$

with both terms α-restricted. By Proposition 5.1.29, we then obtain

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \vdash \theta(s^h_{B^i}) = \theta\left(s_{B^k}\left[\alpha^B_h(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right).$$

Since the unique indeterminate that occurs in $\theta(s_{B^i})$ is $\mathsf{x}^B_{\psi_B(i)}$, it follows by Lemma 5.1.39 that the unique indeterminate that occurs in $\theta(s^h_{B^i})$ is $\mathsf{x}^B_{h\circ\psi_B(i)}$. Also, we know by Lemma 5.1.40 that

$$\theta\left(s_{B^k}\left[\alpha^B_h(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right) \equiv \theta(s_{B^k})\left[\theta\left(\alpha^B_h(\mathsf{x}_{B^i})^{\psi_B(k)}\right)/\mathsf{x}^B_{\psi_B(k)}\right],$$

since $\mathsf{x}^B_{\psi_B(k)}$ is the unique indeterminate that occurs in $\theta(s_{B^k})$. But (by the proof of Lemma 5.1.26) we have

$$\theta\left(\alpha^B_h(\mathsf{x}_{B^i})^{\psi_B(k)}\right) \equiv \theta\left(\alpha^B_{\psi_B(k)\circ h}(\mathsf{x}_{B^i})\right) \equiv \mathsf{x}^B_{\psi_B(k)\circ h},$$

which means that the unique indeterminate that occurs in $\theta\left(s_{B^k}\left[\alpha^B_h(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right)$ is $\mathsf{x}^B_{\psi_B(k)\circ h}$.

Now, suppose towards a contradiction that

$$h \circ \psi_B(i) \neq \psi_B(k) \circ h.$$

Then we would have

$$\mathsf{x}^B_{h\circ\psi_B(i)} \not\equiv \mathsf{x}^B_{\psi_B(k)\circ h},$$

since distinct arrows with codomain k correspond to distinct indeterminates in $\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)$. By the preceding discussion, it would then follow that in the equation

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \vdash \theta(s^h_{B^i}) = \theta\left(s_{B^k}\left[\alpha^B_h(\mathsf{x}_{B^i})/\mathsf{x}_{B^k}\right]'\right),$$

i.e.

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \vdash \theta(s^h_{B^i}) = \theta(s_{B^k})\left[\mathsf{x}^B_{\psi_B(k)\circ h}/\mathsf{x}^B_{\psi_B(k)}\right],$$

the two terms have no indeterminates in common. From the previous line, we can infer

$$\mathbb{T}\left(M^k, \mathsf{x}^B_{h\circ\psi_B(i)}, \mathsf{x}^B_{\psi_B(k)\circ h}\right) \vdash \theta(s^h_{B^i}) = \theta(s_{B^k})\left[\mathsf{x}^B_{\psi_B(k)\circ h}/\mathsf{x}^B_{\psi_B(k)}\right]$$

by Lemma 5.1.30. Now let y, y' be distinct variables of sort B. Then by the theorem on constants, we may conclude

$$\mathbb{T}(M^k) \vdash^{y,y'} \theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right] = \theta(s_{B^k})\left[y'/\mathsf{x}^B_{\psi_B(k)}\right], \qquad (*)$$

since the indeterminates $\mathsf{x}^B_{h\circ\psi_B(i)}, \mathsf{x}^B_{\psi_B(k)\circ h} : B$ are distinct. Now, we showed during the proof that $\psi_B(k)$ is an isomorphism that

$$\theta\left(s_{B^k}\left[s^{-1}_{B^k}/\mathsf{x}_{B^k}\right]'\right) \equiv \theta(s_{B^k})\left[\theta\left(\left(s^{-1}_{B^k}\right)^{\psi_B(k)}\right)/\mathsf{x}^B_{\psi_B(k)}\right].$$

Hence, by substituting $\theta\left(\left(s^{-1}_{B^k}\right)^{\psi_B(k)}\right)$ for y' in $(*)$, we obtain

$$\mathbb{T}\left(M^k, \mathsf{x}^B_{\mathsf{id}_k}\right) \vdash^y \theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right] = \theta(s_{B^k})\left[\theta\left(\left(s^{-1}_{B^k}\right)^{\psi_B(k)}\right)/\mathsf{x}^B_{\psi_B(k)}\right] \equiv \theta\left(s_{B^k}\left[s^{-1}_{B^k}/\mathsf{x}_{B^k}\right]'\right).$$

But we also know (from the same proof) that

$$\mathbb{T}\left(M^k, \mathsf{x}^B_{\mathsf{id}_k}\right) \vdash \theta\left(s_{B^k}\left[s^{-1}_{B^k}/\mathsf{x}_{B^k}\right]'\right) = \mathsf{x}^B_{\mathsf{id}_k},$$

so we finally obtain

$$\mathbb{T}\left(M^k, \mathsf{x}^B_{\mathsf{id}_k}\right) \vdash^y \theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right] = \mathsf{x}^B_{\mathsf{id}_k}.$$

Since $\mathsf{x}^B_{\mathsf{id}_k}$ does not appear in $\theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right]$, it then follows from the theorem on constants (Remark 1.3.17) that if $y' : B$ is a variable distinct from y, then

$$\mathbb{T}(M^k) \vdash^{y,y'} \theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right] = y',$$

and y' does not appear in $\theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right]$. But then if $y'' : B$ is a variable distinct from both y and y', we also obtain

$$\mathbb{T}(M^k) \vdash^{y,y''} \theta(s^h_{B^i})\left[y/\mathsf{x}^B_{h\circ\psi_B(i)}\right] = y''.$$

Finally, we deduce that

$$\mathbb{T}(M^k) \vdash^{y',y''} y' = y'',$$

which contradicts the assumption that $\mathbb{T}(M^k)$ is non-trivial for the sort B. So our assumption that

$$h \circ \psi_B(i) \neq \psi_B(k) \circ h$$

is incorrect, which implies that

$$h \circ \psi_B(i) = \psi_B(k) \circ h,$$

as desired. This completes the argument that ψ_B is a natural automorphism of $\mathsf{Id}_{\mathcal{J}^M_B}$. ▮

To complete the proof that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, we must also verify:

Claim 5.1.53. If $g : B_1 \times \ldots \times B_n \to B$ is any function symbol of Σ with $n \geq 1$, then for any $i \in \mathcal{J}_O$ and $1 \leq m \leq n$ such that g^{M^i} is non-degenerate in position m,

$$\psi_{B_m}(i) = \psi_B(i) : i \to i.$$

Proof: For simplicity, we will let $g : A \times B \to C$ be a binary function symbol of Σ with $A \neq B$, and let $i \in \mathcal{J}_O$. Suppose that g^{M^i} is non-degenerate in position 1 (the argument for position 2 being analogous), which means that

$$\mathbb{T}(M^i) \nvdash^{y_1, y_1', y_2} g(y_1, y_2) = g(y_1', y_2)$$

for pairwise distinct variables $y_1, y_1' : A, y_2 : B$. Then (as remarked in the definition of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$) this implies that $\mathbb{T}(M^i)$ is non-trivial for the sort C, so that $i \in \mathcal{J}_C^M$ and hence $\psi_A(i), \psi_C(i) : i \to i$ are defined. We must now show that

$$\psi_A(i) = \psi_C(i) : i \to i.$$

If g is *not* totally defined in $\mathbb{T}$, then the assumption that $\mathbb{T}$ has single-sorted non-total operations implies that $A = B = C$, which obviously entails the desired result. So suppose that g *is* totally defined in $\mathbb{T}$. If the sorts A and C are identical, then the desired result trivially follows, so suppose that $A \neq C$. Suppose towards a contradiction that

$$\psi_A(i) \neq \psi_C(i) : i \to i.$$

Since $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, it follows that $([s_{C^j}])_{j,C}$ commutes generically with the function symbol $g^i : A^i \times B^i \to C^i$ of $\Sigma^{\mathcal{J}}$. Since g is totally defined in $\mathbb{T}$, this entails that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}, \mathsf{x}_{B^i}) \vdash g^i(s_{A^i}, s_{B^i}) = s_{C^i}\left[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}\right].$$

Let $s_{C^i}\left[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}\right]'$ be the α-restricted variant of $s_{C^i}\left[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}\right]$ from (a simple extension of) Lemma 5.1.27. Then by (simple extensions of) Lemma 5.1.27 and Proposition 5.1.29 and the definition of θ we have

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash g\left(\theta(s_{A^i}), \theta(s_{B^i})\right) = \theta\left(s_{C^i}[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}]'\right).$$

By a simple extension of Lemma 5.1.40, we have

$$\theta\left(s_{C^i}\left[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}\right]'\right) \equiv \theta(s_{C^i})\left[\theta\left(g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})^{\psi_C(i)}\right)/\mathsf{x}^C_{\psi_C(i)}\right].$$

We also have (cf. a simple extension of Lemma 5.1.26)

$$\begin{aligned}
\theta\left(g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})^{\psi_C(i)}\right) &\equiv \theta\left(g^i\left(\mathsf{x}_{A^i}^{\psi_C(i)}, \mathsf{x}_{B^i}^{\psi_C(i)}\right)\right) \\
&\equiv \theta\left(g^i\left(\alpha^A_{\psi_C(i)}(\mathsf{x}_{A^i}), \alpha^B_{\psi_C(i)}(\mathsf{x}_{B^i})\right)\right) \\
&\equiv g\left(\theta\left(\alpha^A_{\psi_C(i)}(\mathsf{x}_{A^i}), \theta(\alpha^B_{\psi_C(i)}(\mathsf{x}_{B^i})\right)\right) \\
&\equiv g\left(\mathsf{x}^A_{\psi_C(i)}, \mathsf{x}^B_{\psi_C(i)}\right).
\end{aligned}$$

From this, we obtain

$$\theta\left(s_{C^i}\left[g^i(\mathsf{x}_{A^i}, \mathsf{x}_{B^i})/\mathsf{x}_{C^i}\right]'\right) \equiv \theta(s_{C^i})\left[g\left(\mathsf{x}^A_{\psi_C(i)}, \mathsf{x}^B_{\psi_C(i)}\right)/\mathsf{x}^C_{\psi_C(i)}\right].$$

Finally, we have

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash g\left(\theta(s_{A^i}), \theta(s_{B^i})\right) = \theta(s_{C^i})\left[g\left(\mathsf{x}^A_{\psi_C(i)}, \mathsf{x}^B_{\psi_C(i)}\right)/\mathsf{x}^C_{\psi_C(i)}\right].$$

Now, we know that $\mathsf{x}^A_{\psi_A(i)}$ is the unique indeterminate occurring in $\theta(s_{A^i})$, and that $\mathsf{x}^C_{\psi_C(i)}$ is the unique indeterminate occurring in $\theta(s_{C^i})$. Because of our assumption that $\psi_A(i) \neq \psi_C(i)$, it follows that $\mathsf{x}^A_{\psi_A(i)} \not\equiv \mathsf{x}^A_{\psi_C(i)} \in \Sigma\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}\right)$. For the remainder of the argument, we will assume that $\mathbb{T}(M^i)$ is non-trivial for the sort B as well, so that $i \in \mathcal{J}^M_B$ and $\psi_B(i) : i \to i$ is defined, and we will also assume that $\psi_B(i) = \psi_C(i)$ (so that $\mathsf{x}^B_{\psi_B(i)} \equiv \mathsf{x}^B_{\psi_C(i)}$). Without these assumptions, the required argument is a simpler version of the one we are about to give.

So, in the above equation, we can substitute $\theta\left(\left(s^{-1}_{A^i}\right)^{\psi_A(i)}\right)$ for $\mathsf{x}^A_{\psi_A(i)}$ in the lefthand term, and $\theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)$ for $\mathsf{x}^B_{\psi_B(i)} \equiv \mathsf{x}^B_{\psi_C(i)}$ in both terms, and we obtain

$$\begin{aligned}\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash g\left(\theta(s_{A^i})\left[\theta\left(\left(s^{-1}_{A^i}\right)^{\psi_A(i)}\right)/\mathsf{x}^A_{\psi_A(i)}\right], \theta(s_{B^i})\left[\theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)/\mathsf{x}^B_{\psi_B(i)}\right]\right) \\ = \theta(s_{C^i})\left[g\left(\mathsf{x}^A_{\psi_C(i)}, \theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)\right)/\mathsf{x}^C_{\psi_C(i)}\right]\end{aligned}$$

(note that $A \neq B$ implies $\mathsf{x}^A_{\psi_A(i)} \not\equiv \mathsf{x}^B_{\psi_B(i)}$). Earlier in the proof of the proposition, we saw that

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}\right) \vdash \theta(s_{A^i})\left[\theta\left(\left(s^{-1}_{A^i}\right)^{\psi_A(i)}\right)/\mathsf{x}^A_{\psi_A(i)}\right] = \mathsf{x}^A_{\mathsf{id}_i}$$

and

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash \theta(s_{B^i})\left[\theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)/\mathsf{x}^B_{\psi_B(i)}\right] = \mathsf{x}^B_{\mathsf{id}_i},$$

so we obtain

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right) \vdash g\left(\mathsf{x}^A_{\mathsf{id}_i}, \mathsf{x}^B_{\mathsf{id}_i}\right) = \theta(s_{C^i})\left[g\left(\mathsf{x}^A_{\psi_C(i)}, \theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)\right)/\mathsf{x}^C_{\psi_C(i)}\right].$$

We can also repeat the above argument to show that

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}, {\mathsf{x}^A_{\mathsf{id}_i}}'\right) \vdash g\left({\mathsf{x}^A_{\mathsf{id}_i}}', \mathsf{x}^B_{\mathsf{id}_i}\right) = \theta(s_{C^i})\left[g\left(\mathsf{x}^A_{\psi_C(i)}, \theta\left(\left(s^{-1}_{B^i}\right)^{\psi_B(i)}\right)\right)/\mathsf{x}^C_{\psi_C(i)}\right],$$

where ${\mathsf{x}^A_{\mathsf{id}_i}}' \notin \Sigma\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)$ is a new constant of sort A. So then we obtain

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}, {\mathsf{x}^A_{\mathsf{id}_i}}'\right) \vdash g\left(\mathsf{x}^A_{\mathsf{id}_i}, \mathsf{x}^B_{\mathsf{id}_i}\right) = g\left({\mathsf{x}^A_{\mathsf{id}_i}}', \mathsf{x}^B_{\mathsf{id}_i}\right).$$

By (a slight variation of) Lemma 5.1.30, it then follows that

$$\mathbb{T}\left(M^i, \mathsf{x}^A_{\mathsf{id}_i}, \mathsf{x}^B_{\mathsf{id}_i}, {\mathsf{x}^A_{\mathsf{id}_i}}'\right) \vdash g\left(\mathsf{x}^A_{\mathsf{id}_i}, \mathsf{x}^B_{\mathsf{id}_i}\right) = g\left({\mathsf{x}^A_{\mathsf{id}_i}}', \mathsf{x}^B_{\mathsf{id}_i}\right).$$

By the theorem on constants again, it follows that if $y, y' : A$ are distinct variables and $z : B$ is a variable, then

$$\mathbb{T}(M^i) \vdash^{y,y',z} g(y,z) = g(y',z),$$

which contradicts the assumption that g^{M^i} is non-degenerate in position 1. This contradiction implies that we must have

$$\psi_A(i) = \psi_C(i) : i \to i$$

after all. ∎

Hence, we may finally conclude that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, and therefore

$$(\gamma, \psi) \in \left(\prod_i G_{\mathbb{T}}(M^i) \right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M.$$

To complete the proof that β_M is surjective, we must now show that

$$\beta_M(\gamma, \psi) = ([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M).$$

So let $B \in \Sigma_{\mathsf{Sort}}$ and $k \in \mathcal{J}_O$ be arbitrary: we must show

$$\beta_M(\gamma, \psi)_{B^k} = [s_{B^k}] \in M\langle \mathsf{x}_{B^k} \rangle_{B^k}.$$

First suppose that $\mathbb{T}(M^k)$ is trivial for the sort B. Then $\beta_M(\gamma, \psi)_{B^k} = [\mathsf{x}_{B^k}]$, so we must show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash \mathsf{x}_{B^k} = s_{B^k}.$$

By hypothesis, we know that $\mathbb{T}(M^k) \vdash^{y,y'} y = y'$ for distinct variables $y, y' : B$. Then by Lemma 5.1.13, we have that

$$\rho^B_{M^k} : \mathbb{T}(M^k, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k})$$

is a theory morphism, which implies that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash^{z,z'} z = z'$$

for distinct variables $z, z' : B^k$. Since $\mathsf{x}_{B^k}, s_{B^k} : B^k$, this yields the desired result.

Now suppose that $\mathbb{T}(M^k)$ is non-trivial for the sort B. Unravelling the definitions, we must then show that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash \rho^B_{M^k}\left(\theta^*(s_{B^k})\right)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right] = s_{B^k}.$$

Since s_{B^k} is k-local, it follows by Lemma 5.1.42 that

$$\rho^B_{M^k}(\theta^*(s_{B^k})) \equiv s^{-\alpha}_{B^k},$$

so that we must show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash s^{-\alpha}_{B^k}\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right] = s_{B^k}.$$

We have assumed that s_{B^k} has a unique occurrence of the indeterminate x_{B^k}. Either this occurrence is just x_{B^k} (without being the argument of any α function symbol), or (as easily follows from the definition of $\psi_B(k)$) it has the form $\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})$. In the first case, we had set $\psi_B(k) := \mathsf{id}_k$, and it is then easy to see that $s^{-\alpha}_{B^k} \equiv s_{B^k}$, so we must show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash s_{B^k}\left[\alpha^B_{\mathsf{id}_k}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right] = s_{B^k}.$$

But this follows because

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash \alpha^B_{\mathsf{id}_k}(\mathsf{x}_{B^k}) = \mathsf{x}_{B^k}.$$

Now suppose that the unique occurrence of x_{B^k} in s_{B^k} has the form $\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})$. Then it is easy to see that

$$s^{-\alpha}_{B^k}\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right] \equiv s_{B^k},$$

from which the desired result follows, because $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash s_{B^k} \downarrow$. This completes the proof that

$$\beta_M(\gamma, \psi) = ([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

which proves that β_M is surjective. ∎

Before we can conclude that the family of isomorphisms $(\beta_M)_{M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}}$ is natural (assuming that $\mathbb{T}$ has single-indeterminate isotropy and single-sorted non-trivial operations), we must first make explicit the functoriality of the assignment

$$M \mapsto \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \in \mathsf{Group}.$$

First, we show that the assignment

$$M \mapsto \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$$

is functorial.

Definition 5.1.54. We define a functor

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}$$

as follows:

- For any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, we set
$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)}(M) := \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M} \in \mathsf{Group}.$$
- For any $\Sigma^{\mathcal{J}}$-morphism $h : M \to N \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, we define a group homomorphism
$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{h} : \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M} \to \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{N}$$
by
$$(\psi_B)_{B\in\Sigma} \mapsto (\psi_B^h)_{B\in\Sigma},$$
with
$$\psi_B^h(i) := \psi_B(i) : i \xrightarrow{\sim} i$$
for each $i \in \mathcal{J}_B^N$.

Justification. For the verification that this is a well-defined functor, see Appendix C. ∎

Definition 5.1.55. We define a functor

$$G^*_{\mathbb{T}^{\mathcal{J}}} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}$$

as follows. From Lemma 5.1.9, we have a functor

$$\left(\prod_i G_{\mathbb{T}} \circ F^{(-)}\right)^{G_{\mathbb{T}} \circ F^{(-)}} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}^{\mathcal{J}} \to \mathsf{Group}.$$

From Definition 5.1.54, we also have a functor

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}.$$

If $\times : \mathsf{Group} \times \mathsf{Group} \to \mathsf{Group}$ is the product group functor, then we set

$$G^*_{\mathbb{T}^{\mathcal{J}}} := \times \circ \left\langle \left(\prod_i G_{\mathbb{T}} \circ F^{(-)}\right)^{G_{\mathbb{T}} \circ F^{(-)}}, \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} \right\rangle$$

$$: \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group} \times \mathsf{Group} \to \mathsf{Group}.$$

∎

Theorem 5.1.56. *Let $\mathbb{T}$ be a quasi-equational theory with single-indeterminate isotropy and single-sorted non-total operations, and let $\mathcal{J}$ be a small index category. Then there is a natural isomorphism*

$$\beta : G^*_{\mathbb{T}^\mathcal{J}} \xrightarrow{\sim} G_{\mathbb{T}^\mathcal{J}}$$
$$: \mathsf{P}\mathbb{T}^\mathcal{J}\mathsf{mod} \to \mathsf{Group}.$$

Proof: For any $M \in \mathsf{P}\mathbb{T}^\mathcal{J}\mathsf{mod}$, we have defined in Proposition 5.1.20 a group homomorphism

$$\beta_M : G^*_{\mathbb{T}^\mathcal{J}}(M) = \left(\prod_i G_\mathbb{T} \circ F^M\right)^{G_\mathbb{T} \circ F^M} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^M = \left(\prod_i G_\mathbb{T}(M^i)\right)^\mathcal{J} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^M$$
$$\to G_{\mathbb{T}^\mathcal{J}}(M),$$

which is injective by Proposition 5.1.43 and surjective by Proposition 5.1.47, provided that $\mathbb{T}$ has single-indeterminate isotropy and single-sorted non-total operations. Thus, β_M is a group isomorphism. So it just remains to verify that the family $(\beta_M)_{M \in \mathsf{P}\mathbb{T}^\mathcal{J}\mathsf{mod}}$ is natural in M.

For this purpose, let $h : M \to N$ be an arbitrary $\Sigma^\mathcal{J}$-morphism in $\mathsf{P}\mathbb{T}^\mathcal{J}\mathsf{mod}$. We must show that

$$G_{\mathbb{T}^\mathcal{J}}(h) \circ \beta_M = \beta_N \circ \left(\prod_i G_\mathbb{T}(h^i))^\mathcal{J} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^h\right)$$
$$: \left(\prod_i G_\mathbb{T}(M^i)\right)^\mathcal{J} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^M \to G_{\mathbb{T}^\mathcal{J}}(N).$$

Towards this end, let $(\gamma, \psi) \in \left(\prod_i G_\mathbb{T}(M^i)\right)^\mathcal{J} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^M$ be arbitrary, and let $B \in \Sigma_\mathsf{Sort}$ and $k \in \mathcal{J}_O$. We must show that

$$(G_{\mathbb{T}^\mathcal{J}}(h) \circ \beta_M)(\gamma, \psi)_{B^k} = \left(\beta_N \circ \left(\prod_i G_\mathbb{T}(h^i))^\mathcal{J} \times \mathsf{Aut}(\mathsf{Id}_\mathcal{J})^h\right)\right)(\gamma, \psi)_{B^k}$$
$$\in N\langle \mathsf{x}_{B^k} \rangle_{B^k}.$$

We have $\gamma_k \in G_\mathbb{T}(M^k)$, so let $\gamma_k^B = \left[s_B^k\right] \in M^k\langle \mathsf{x}_B \rangle_B$, and let $\psi = (\psi_C)_{C \in \Sigma}$. First suppose that $\mathbb{T}(N^k)$ is non-trivial for the sort B. Then because there is a theory morphism $\rho_{h^k} : \mathbb{T}(M^k) \to \mathbb{T}(N^k)$, it easily follows that $\mathbb{T}(M^k)$ is also non-trivial for the sort B. So then $k \in \mathcal{J}_B^M$ and $k \in \mathcal{J}_B^N$.

Unravelling the definitions, the element on the left side of the above equation is then

$$\left[\rho_h^{B^k}\left(\rho^B_{M^k}(s^k_B)[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}]\right)\right] \in N\langle \mathsf{x}_{B^k} \rangle_{B^k},$$

where $\rho^B_{M^k} : \mathbb{T}(M^k, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k})$ is the theory morphism of Lemma 5.1.13, and $\rho^{B^k}_h : \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is the theory morphism induced by the $\Sigma^{\mathcal{J}}$-morphism $h : M \to N$. Conversely (because $\mathbb{T}(N^k)$ is non-trivial for the sort B), the element on the right side of the above equation is

$$\left[\rho^B_{N^k}(\rho^B_{h^k}(s^k_B))\left[\alpha^B_{\psi^h_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right] \in N\langle \mathsf{x}_{B^k}\rangle_{B^k},$$

where $\rho^B_{h^k} : \mathbb{T}(M^k, \mathsf{x}_B) \to \mathbb{T}(N^k, \mathsf{x}_B)$ is the theory morphism induced by the Σ-morphism $h^k : M^k \to N^k$, and $\rho^B_{N^k} : \mathbb{T}(N^k, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is the theory morphism of Lemma 5.1.13. So we have to prove that these two congruence classes are equal, i.e. we have to prove that

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \rho^{B^k}_h\left(\rho^B_{M^k}(s^k_B)[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}]\right) = \rho^B_{N^k}(\rho^B_{h^k}(s^k_B))\left[\alpha^B_{\psi^h_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right].$$

For our first simplification, we know by definition of $\psi^h_B : \mathsf{Id}_{\mathcal{J}^N_B} \xrightarrow{\sim} \mathsf{Id}_{\mathcal{J}^N_B}$ that $\psi^h_B(k) = \psi_B(k)$, so we are reduced to showing that

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \rho^{B^k}_h\left(\rho^B_{M^k}(s^k_B)[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}]\right) = \rho^B_{N^k}(\rho^B_{h^k}(s^k_B))[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}].$$

Then by Lemma 2.2.28 we have

$$\begin{aligned}\rho^{B^k}_h\left(\rho^B_{M^k}(s^k_B)[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}]\right) &\equiv \rho^{B^k}_h(\rho^B_{M^k}(s^k_B))[\rho^{B^k}_h(\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k}))/\mathsf{x}_{B^k}]\\ &\equiv \rho^{B^k}_h(\rho^B_{M^k}(s^k_B))[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}],\end{aligned}$$

since $\rho^{B^k}_h(\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})) \equiv \alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})$. So we must now show

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \rho^{B^k}_h(\rho^B_{M^k}(s^k_B))[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}] = \rho^B_{N^k}(\rho^B_{h^k}(s^k_B))[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}].$$

To show this, it clearly suffices to show

$$\rho^{B^k}_h(\rho^B_{M^k}(s^k_B)) \equiv \rho^B_{N^k}(\rho^B_{h^k}(s^k_B)) \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})),$$

which follows by Lemma 5.1.15.

Now suppose that $\mathbb{T}(N^k)$ is *trivial* for the sort B. Then it trivially follows that

$$\begin{aligned}(G_{\mathbb{T}^{\mathcal{J}}}(h) \circ \beta_M)(\gamma, \psi)_{B^k} &= \left(\beta_N \circ \left(\prod_i G_{\mathbb{T}}(h^i))^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^h\right)\right)(\gamma, \psi)_{B^k}\\ &\in N\langle \mathsf{x}_{B^k}\rangle_{B^k},\end{aligned}$$

because we infer from the hypothesis and the existence of the theory morphism $\rho^B_{N^k} : \mathbb{T}(N^k, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ that $\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is trivial for the sort B^k, which implies that

$$N\langle \mathsf{x}_{B^k}\rangle_{B^k} = \left\{[t] : t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k}))_{B^k} \wedge \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash t\downarrow\right\}$$

is a singleton (since for any $[s], [t] \in N\langle \mathsf{x}_{B^k} \rangle_{B^k}$, we have $[s] = [t]$ iff $\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash s = t$).

This completes the proof that the family $(\beta_M)_{M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}}$ is natural in M, so that we have indeed constructed a natural isomorphism

$$\beta : G^*_{\mathbb{T}^{\mathcal{J}}} \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}.$$

∎

Finally, appealing to Lemma 5.1.9, we have the following corollary:

Corollary 5.1.57. *Let $\mathbb{T}$ be a quasi-equational theory with single-indeterminate isotropy and single-sorted non-total operations, and let $\mathcal{J}$ be a small index category. Then there is a natural isomorphism*

$$\beta : \times \circ \left\langle \mathsf{lim}\left(G_{\mathbb{T}} \circ F^{(-)}\right), \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} \right\rangle \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}$$

$$: \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}.$$

So for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ with corresponding functor $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, we have

$$G_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathsf{lim}\left(G_{\mathbb{T}} \circ F^M\right) \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M.$$

∎

Remark 5.1.58. We needed the assumption that $\mathbb{T}$ has single-indeterminate isotropy in order to show in Proposition 5.1.47 that the injective group homomorphism

$$\beta_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is also *surjective* for each small category $\mathcal{J}$ and $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. We now show that there *do* exist theories without single-indeterminate isotropy, which have models M for which β_M is *not* surjective.

Let Σ be a signature with one sort X and one binary function symbol $*$, and let $\mathbb{T}$ be the totally defined algebraic theory over Σ with the following axioms:

- $\top \vdash^{x,y} x * y \downarrow$.
- $\top \vdash^{x,y} (y * y) * (y * x) = x$.
- $\top \vdash^{\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3,\mathsf{x}_4} (\mathsf{x}_1 * \mathsf{x}_2) * (\mathsf{x}_3 * \mathsf{x}_4) = (\mathsf{x}_1 * \mathsf{x}_3) * (\mathsf{x}_2 * \mathsf{x}_4)$.

If we define $M := (\{0,1\}, *^M)$ by

$$0 * 0 = 1,$$

$$0 * 1 = 0,$$

$$1 * 0 = 1,$$

$$1 * 1 = 0,$$

then one can easily verify that $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$.

We now show that $\mathbb{T}$ does *not* have single-indeterminate isotropy. Consider the initial model $\mathsf{Free}(\mathbb{T}) \in \mathsf{P}\mathbb{T}\mathsf{mod}$. We have

$$\mathsf{x} * \mathsf{x} \in \mathsf{Term}^c(\Sigma(\mathsf{Free}(\mathbb{T}), \mathsf{x}))$$

with

$$\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}) \vdash \mathsf{x} * \mathsf{x} \downarrow,$$

and hence

$$[\mathsf{x} * \mathsf{x}] \in \mathsf{Free}(\mathbb{T})\langle \mathsf{x} \rangle.$$

Now we show that $[\mathsf{x} * \mathsf{x}] \in G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))$. First, this element is invertible, because its inverse is itself, since $\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x})$ proves the equations

$$(\mathsf{x} * \mathsf{x})[\mathsf{x} * \mathsf{x}/\mathsf{x}] = (\mathsf{x} * \mathsf{x}) * (\mathsf{x} * \mathsf{x}) = \mathsf{x}.$$

Next, this element commutes generically with the function symbol $*$, because $\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}_1, \mathsf{x}_2)$ proves the equations

$$\begin{aligned}(\mathsf{x} * \mathsf{x})[\mathsf{x}_1 * \mathsf{x}_2/\mathsf{x}] &\equiv (\mathsf{x}_1 * \mathsf{x}_2) * (\mathsf{x}_1 * \mathsf{x}_2) \\ &= (\mathsf{x}_1 * \mathsf{x}_1) * (\mathsf{x}_2 * \mathsf{x}_2) \\ &\equiv (\mathsf{x} * \mathsf{x})[\mathsf{x}_1/\mathsf{x}] * (\mathsf{x} * \mathsf{x})[\mathsf{x}_2/\mathsf{x}].\end{aligned}$$

That $[\mathsf{x} * \mathsf{x}]$ reflects definedness of $*$ is trivial, since $*$ is totally defined in $\mathbb{T}$. This proves that $[\mathsf{x} * \mathsf{x}] \in G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))$.

To complete the proof that $\mathbb{T}$ does *not* have single-indeterminate isotropy, it now remains to show that there is no $t \in \mathsf{Term}^c(\Sigma(\mathsf{Free}(\mathbb{T}), \mathsf{x}))$ such that t contains exactly one occurrence of x and $[\mathsf{x} * \mathsf{x}] = [t]$. It is easy to see that if $t \in \mathsf{Term}^c(\Sigma(\mathsf{Free}(\mathbb{T}), \mathsf{x}))$ has exactly one occurrence of x, then t must be x itself, because (the underlying set of) $\mathsf{Free}(\mathbb{T})$ is empty, because $\mathsf{Term}(\Sigma)$ has no closed terms. To show $[\mathsf{x} * \mathsf{x}] \neq [\mathsf{x}]$, i.e. $\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}) \nvdash \mathsf{x} * \mathsf{x} = \mathsf{x}$, it is equivalent by Lemma 3.1.2 to show that there is some $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and some Σ-morphism $h : \mathsf{Free}(\mathbb{T}) \to N$ and some element $a \in N$ with $a *^N a \neq a$. If we consider the $\mathbb{T}$-model $M = (\{0,1\}, *^M)$ defined above, then $0 \in M$ with $0 *^M 0 \neq 0$, and (by initiality of $\mathsf{Free}(\mathbb{T})$), there is a Σ-morphism $\mathsf{Free}(\mathbb{T}) \to M$. So $[\mathsf{x} * \mathsf{x}] \neq [\mathsf{x}]$, and thus $\mathbb{T}$ does not have single-indeterminate isotropy.

Given that $\mathbb{T}$ does *not* have single-indeterminate isotropy, we now exhibit a small category $\mathcal{J}$ and a model $N \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ for which β_N is *not* surjective.

Let M be any commutative monoid containing elements m_1, m_2, m_3 with m_2 invertible and $m_1 \neq m_2$ and $m_1 m_3 = m_1 m_2^{-1} = m_2 m_3$ (e.g. M could be the multiplicative commutative monoid $(\{0,1\}, \cdot, 1)$ with $m_1 = m_3 = 0$ and $m_2 = 1$). Let $\mathcal{B}(M)$ be the one-object category corresponding to M (whose arrows are labelled by the objects of M, with composition given by monoid multiplication in M). Let $F : \mathcal{B}(M) \to \mathsf{P}\mathbb{T}\mathsf{mod}$ be the constant functor on $\mathsf{Free}(\mathbb{T}) \in \mathsf{P}\mathbb{T}\mathsf{mod}$. Since $\mathcal{B}(M)$ has only one object and $\mathbb{T}$ is single-sorted, it follows that $\mathbb{T}^{\mathcal{B}(M)}$ is single-sorted, so we identify the unique sort of $\mathbb{T}^{\mathcal{B}(M)}$ with the unique sort of $\mathbb{T}$.

Let $M^F \in \mathsf{P}\mathbb{T}^{\mathcal{B}(M)}\mathsf{mod}$ be the model of $\mathbb{T}^{\mathcal{B}(M)}$ corresponding to the functor $F : \mathcal{B}(M) \to \mathsf{P}\mathbb{T}\mathsf{mod}$. Then

$$\beta_{M^F} : G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))^{\mathcal{B}(M)} \times \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right)^{M^F} \to G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$$

is an injective group homomorphism. Since (the underlying set of) $\mathsf{Free}(\mathbb{T})$ is empty, it easily follows that $G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))^{\mathcal{B}(M)} = G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))$, and since $\mathbb{T}(\mathsf{Free}(\mathbb{T}))$ is non-trivial for its unique sort (using Lemma 3.1.2, because there is a model of $\mathbb{T}$ with at least two elements, as shown above), it easily follows that $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right)^{M^F} = \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right)$. So we have

$$\beta_{M^F} : G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T})) \times \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right) \to G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right).$$

Also, it is easy to see that the group $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right)$ is (isomorphic to) the group of invertible elements of the centre of M (and hence of M, because M is commutative).

We will now exhibit an element of $G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$ that is *not* in the image of β_{M^F}. Consider the closed term $\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}) \in \mathsf{Term}^c\left(\Sigma^{\mathcal{B}(M)}\left(M^F, \mathsf{x}\right)\right)$; we will show that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$. First we show that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]$ is invertible. We show that we can take $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]^{-1} := \left[\alpha_{m_3}(\mathsf{x}) * \alpha_{m_2^{-1}}(\mathsf{x})\right]$. Using the axioms of $\mathbb{T}^{\mathcal{B}(M)}$ and the assumed equalities $m_1 m_3 = m_1 m_2^{-1} = m_2 m_3$ and $m_2 m_2^{-1} = e^M$, we have

$$\begin{aligned}
&[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]\left[\alpha_{m_3}(\mathsf{x}) * \alpha_{m_2^{-1}}(\mathsf{x})/\mathsf{x}\right] \\
&= \left[\alpha_{m_1}\left(\alpha_{m_3}(\mathsf{x}) * \alpha_{m_2^{-1}}(\mathsf{x})\right) * \alpha_{m_2}\left(\alpha_{m_3}(\mathsf{x}) * \alpha_{m_2^{-1}}(\mathsf{x})\right)\right] \\
&= \left[\left(\alpha_{m_1}\left(\alpha_{m_3}(\mathsf{x})\right) * \alpha_{m_1}\left(\alpha_{m_2^{-1}}(\mathsf{x})\right)\right) * \left(\alpha_{m_2}\left(\alpha_{m_3}(\mathsf{x})\right) * \alpha_{m_2}\left(\alpha_{m_2^{-1}}(\mathsf{x})\right)\right)\right] \\
&= \left[\left(\alpha_{m_1 m_3}(\mathsf{x}) * \alpha_{m_1 m_2^{-1}}(\mathsf{x})\right) * \left(\alpha_{m_2 m_3}(\mathsf{x}) * \alpha_{m_2 m_2^{-1}}(\mathsf{x})\right)\right] \\
&= \left[\left(\alpha_{m_1 m_3}(\mathsf{x}) * \alpha_{m_1 m_3}(\mathsf{x})\right) * \left(\alpha_{m_1 m_3}(\mathsf{x}) * \alpha_{e^M}(\mathsf{x})\right)\right] \\
&= \left[\left(\alpha_{m_1 m_3}(\mathsf{x}) * \alpha_{m_1 m_3}(\mathsf{x})\right) * \left(\alpha_{m_1 m_3}(\mathsf{x}) * \mathsf{x}\right)\right] \\
&= [\mathsf{x}],
\end{aligned}$$

and the other required equality can be shown similarly (using also the commutativity of the monoid M).

Now we show that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]$ commutes generically with the function symbol $*$ of $\mathbb{T}$. Using the axioms of $\mathbb{T}^{\mathcal{B}(M)}$, we have

$$\begin{aligned}
&[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]\,[\mathsf{x}_1 * \mathsf{x}_2/\mathsf{x}] \\
&= [\alpha_{m_1}(\mathsf{x}_1 * \mathsf{x}_2) * \alpha_{m_2}(\mathsf{x}_1 * \mathsf{x}_2)] \\
&= [(\alpha_{m_1}(\mathsf{x}_1) * \alpha_{m_1}(\mathsf{x}_2)) * (\alpha_{m_2}(\mathsf{x}_1) * \alpha_{m_2}(\mathsf{x}_2))] \\
&= [(\alpha_{m_1}(\mathsf{x}_1) * \alpha_{m_2}(\mathsf{x}_1)) * (\alpha_{m_1}(\mathsf{x}_2) * \alpha_{m_2}(\mathsf{x}_2))] \\
&= [(\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}))[\mathsf{x}_1/\mathsf{x}] * (\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}))[\mathsf{x}_2/\mathsf{x}]]\,,
\end{aligned}$$

as required.

Since $\mathbb{T}$ and hence $\mathbb{T}^{\mathcal{B}(M)}$ is a totally defined theory, it remains to show that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]$ commutes with each function symbol of $\mathbb{T}^{\mathcal{B}(M)}$ of the form α_m for $m \in M$. Using the axioms of $\mathbb{T}^{\mathcal{B}(M)}$ and the commutativity of M, we have

$$\begin{aligned}
&[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})]\,[\alpha_m(\mathsf{x})/\mathsf{x}] \\
&= [\alpha_{m_1}(\alpha_m(\mathsf{x})) * \alpha_{m_2}(\alpha_m(\mathsf{x}))] \\
&= [\alpha_{m_1 m}(\mathsf{x}) * \alpha_{m_2 m}(\mathsf{x})] \\
&= [\alpha_{m m_1}(\mathsf{x}) * \alpha_{m m_2}(\mathsf{x})] \\
&= [\alpha_m(\alpha_{m_1}(\mathsf{x})) * \alpha_m(\alpha_{m_2}(\mathsf{x}))] \\
&= [\alpha_m\,(\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}))]\,,
\end{aligned}$$

as required. This completes the proof that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$.

To show that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$ is *not* in the image of β_{M^F}, suppose towards a contradiction that we *did* have some $[t] \in G_{\mathbb{T}}(\mathsf{Free}(\mathbb{T}))$ and $m \in \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right) = \mathsf{Inv}(M)$ with

$$[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})] = \beta_{M^F}([t], m) = [t\,[\alpha_m(\mathsf{x})/\mathsf{x}]] \in G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$$

(since $\mathbb{T}(\mathsf{Free}(\mathbb{T}))$ is non-trivial for its unique sort, cf. the definition of β_{M^F} in Proposition 5.1.20), so that

$$\mathbb{T}^{\mathcal{B}(M)}\left(M^F, \mathsf{x}\right) \vdash \alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}) = t\,[\alpha_m(\mathsf{x})/\mathsf{x}]\,.$$

Since $\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x}), t\,[\alpha_m(\mathsf{x})/\mathsf{x}] \in \mathsf{Term}^c\left(\Sigma^{\mathcal{B}(M)}\left(M^F, \mathsf{x}\right)\right)$ are clearly α-restricted, it follows by Proposition 5.1.29 and Lemma 5.1.30 and the definition of θ that

$$\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}_{m_1}, \mathsf{x}_{m_2}, \mathsf{x}_m) \vdash \mathsf{x}_{m_1} * \mathsf{x}_{m_2} = t[\mathsf{x}_m/\mathsf{x}].$$

By Lemma 2.2.24 and Lemma 5.1.30, we may substitute x for all of $\mathsf{x}_{m_1}, \mathsf{x}_{m_2}, \mathsf{x}_m$ and obtain

$$\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}) \vdash \mathsf{x} * \mathsf{x} = t.$$

So from

$$\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}_{m_1}, \mathsf{x}_{m_2}, \mathsf{x}_m) \vdash \mathsf{x}_{m_1} * \mathsf{x}_{m_2} = t[\mathsf{x}_m/\mathsf{x}],$$

we then (easily) obtain

$$\mathbb{T}(\mathsf{Free}(\mathbb{T}), \mathsf{x}_{m_1}, \mathsf{x}_{m_2}, \mathsf{x}_m) \vdash \mathsf{x}_{m_1} * \mathsf{x}_{m_2} = \mathsf{x}_m * \mathsf{x}_m.$$

However, since $m_1 \neq m_2$ (so that $\mathsf{x}_{m_1} \not\equiv \mathsf{x}_{m_2}$), this is easily shown to be false by appealing to Lemma 3.1.2. Specifically, if $m \neq m_1, m_2$, then the above equation is not provable because in the aforementioned model M of $\mathbb{T}$ we have $M \not\models x * y = z * z$ for distinct variables x, y, z, and if $m = m_1$, then the above equation is not provable because $M \not\models x * y = x * x$. If $m = m_2$, then to show that the above equation is not provable, we must show that $\mathbb{T} \nvdash^{x,y} x * y = y * y$ for distinct variables x, y (notice that this general equation *does* hold in the aforementioned model M of $\mathbb{T}$). Consider the following Σ-structure $N := (\{0, 1, 2\}, *^N)$ with

$$0 *^N 0 = 1 *^N 0 = 2 *^N 0 = 0,$$

$$0 *^N 1 = 1 *^N 2 = 2 *^N 2 = 1,$$

$$1 *^N 1 = 2 *^N 1 = 0 *^N 2 = 2.$$

According to the Mace4 model searcher [16], this is a model of $\mathbb{T}$ (one could also verify this by hand), and we have $N \not\models x * y = y * y$, since (e.g.) we have $0 *^N 1 = 1 \neq 2 = 1 *^N 1$.

This completes the proof that $[\alpha_{m_1}(\mathsf{x}) * \alpha_{m_2}(\mathsf{x})] \in G_{\mathbb{T}^{\mathcal{B}(M)}}\left(M^F\right)$ is *not* in the image of β_{M^F}, which shows that Proposition 5.1.47 does *not* hold in general for theories *without* single-indeterminate isotropy.

∎

Remark 5.1.59. We also needed the assumption that $\mathbb{T}$ has single-sorted non-total operations in order to show in Proposition 5.1.47 that the injective group homomorphism

$$\beta_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is *surjective* for each small category $\mathcal{J}$ and $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. Specifically, given $([s_{C^j}])_{j,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$ for $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, we constructed $(\gamma, \psi) \in (\prod_i G_{\mathbb{T}}(M^i))^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ with $\beta_M(\gamma, \psi) = ([s_{C^j}])_{j,C}$, and we used the assumption that $\mathbb{T}$ has single-sorted non-total operations to show that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$; more specifically, we used this assumption to show that if $g : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ with $n \geq 1$ and $i \in \mathcal{J}_O$ and g^{M^i} is non-degenerate in position $1 \leq m \leq n$, then $\psi_{A_m}(i) = \psi_A(i) : i \xrightarrow{\sim} i$.

We now show that there is a theory $\mathbb{T}$ without single-sorted non-total operations for which there is a small category $\mathcal{J}$ and a model M of $\mathbb{T}^{\mathcal{J}}$ such that β_M is *not* surjective. Let $\mathbb{T}$ be the theory over the signature Σ with two sorts X and Y and one unary function symbol $f : X \to Y$ whose only axiom is

$$f(x) \downarrow \vdash^{x,y,y'} y = y'$$

for pairwise distinct variables $x : X$ and $y, y' : Y$. A model of $\mathbb{T}$ therefore consists of a pair of sets A, B and a partial function $g : A \to B$ with the property that if $\mathsf{dom}(g) \neq \emptyset$, then $|B| \leq 1$. Then $\mathbb{T}$ does *not* have single-sorted non-total operations, because $X \neq Y$ and $\mathbb{T}$ does *not* prove the sequent $\top \vdash^{x:X} f(x) \downarrow$ (by soundness of partial Horn logic), since there is obviously a model M of $\mathbb{T}$ in which f^M is not total.

Let $\mathcal{J}$ be the one-object category corresponding to a non-trivial abelian group, so that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ is non-trivial. Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ be the constant functor on $\mathsf{Free}(\mathbb{T})$, which is the Σ-structure consisting of two empty sets and the empty function. Let $M^F \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ be the corresponding model of $\mathbb{T}^{\mathcal{J}}$. It is not difficult to see that the model $\mathsf{Free}(\mathbb{T})$ has trivial isotropy and that $\mathbb{T}(\mathsf{Free}(\mathbb{T})) = \mathbb{T}$ is non-trivial for the sorts X and Y (identifying the two sorts of $\mathbb{T}^{\mathcal{J}}$ with the two sorts of $\mathbb{T}$), so that

$$\beta_{M^F} : \{([\mathsf{x}], [\mathsf{y}])\} \times (\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}))^{M^F} \to G_{\mathbb{T}^{\mathcal{J}}}\left(M^F\right).$$

Moreover, since $f^{\mathsf{Free}(\mathbb{T})} : \emptyset \to \emptyset$ is non-degenerate in its unique position (since $\mathbb{T}(\mathsf{Free}(\mathbb{T})) = \mathbb{T}$ does not prove the sequent $\top \vdash^{x,x'} f(x) = f(x')$ for distinct variables $x, x' : X$ by soundness of partial Horn logic, since there is obviously a model M of $\mathbb{T}$ in which $|M_X| \geq 2$ and f^M is completely undefined, hence not constant), it follows that

$$(\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}))^{M^F} = \Delta\left(\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})\right),$$

the diagonal relation on $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$. So then

$$\beta_{M^F} : \{([\mathsf{x}], [\mathsf{y}])\} \times \Delta\left(\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})\right) \to G_{\mathbb{T}^{\mathcal{J}}}\left(M^F\right)$$

is given by the rule

$$(([\mathsf{x}], [\mathsf{y}]), (\psi, \psi)) \mapsto \left(\left[\alpha^X_\psi(\mathsf{x})\right], \left[\alpha^Y_\psi(\mathsf{y})\right]\right) \in G_{\mathbb{T}^{\mathcal{J}}}\left(M^F\right)$$

for each $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$.

Since we assumed $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ to be non-trivial, let $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ be non-trivial, and let us show that

$$\left([\mathsf{x}], \left[\alpha^Y_\psi(\mathsf{y})\right]\right) \in G_{\mathbb{T}^{\mathcal{J}}}\left(M^F\right).$$

It is easy to verify that $\left([\mathsf{x}], \left[\alpha^Y_\psi(\mathsf{y})\right]\right)$ is invertible, with inverse $\left([\mathsf{x}], \left[\alpha^Y_{\psi^{-1}}(\mathsf{y})\right]\right)$. It is also easy to verify that $\left([\mathsf{x}], \left[\alpha^Y_\psi(\mathsf{y})\right]\right)$ commutes generically with every function symbol of $\mathbb{T}^{\mathcal{J}}$ of the form α_h^X or α_h^Y for h an endomorphism of the unique object of $\mathcal{J}$, because all endomorphisms of this object commute with each other (since $\mathcal{J}$ corresponds to an abelian group). We must lastly show that $\left([\mathsf{x}], \left[\alpha_\psi^Y(\mathsf{y})\right]\right)$ commutes generically with and reflects definedness of the function symbol $f : X \to Y$. For the first claim, we must show that $\mathbb{T}^{\mathcal{J}}(M^F, \mathsf{x})$ proves the sequent

$$f(\mathsf{x}) \downarrow \vdash \alpha_\psi^Y(f(\mathsf{x})) = f(\mathsf{x}),$$

but this follows easily from the unique axiom of $\mathbb{T}$. For the second claim, we must show that $\mathbb{T}^{\mathcal{J}}(M^F, \mathsf{x})$ proves the sequent

$$f(\mathsf{x}) \downarrow \vdash f(\mathsf{x}) \downarrow,$$

which is just a logical axiom of partial Horn logic. This proves that $\left([\mathsf{x}], \left[\alpha_\psi^Y(\mathsf{y})\right]\right) \in G_{\mathbb{T}^{\mathcal{J}}}\left(M^F\right)$.

We now show that $\left([\mathsf{x}], \left[\alpha_\psi^Y(\mathsf{y})\right]\right)$ is *not* in the image of β_{M^F}. If it *were* in the image of β_{M^F}, then there would be some $h \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ with

$$\left([\mathsf{x}], \left[\alpha_\psi^Y(\mathsf{y})\right]\right) = \beta_{M^F}\left(([\mathsf{x}], [\mathsf{y}]), (h, h)\right) = \left(\left[\alpha_h^X(\mathsf{x})\right], \left[\alpha_h^Y(\mathsf{y})\right]\right),$$

which entails that

$$\mathbb{T}^{\mathcal{J}}(M^F, \mathsf{x}) \vdash \alpha_h^X(\mathsf{x}) = \mathsf{x}$$

and

$$\mathbb{T}^{\mathcal{J}}(M^F, \mathsf{y}) \vdash \alpha_\psi^Y(\mathsf{y}) = \alpha_h^Y(\mathsf{y}).$$

From these two facts we will deduce that ψ must be the identity element of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, contrary to assumption. Since $\alpha_h^X(\mathsf{x})$ and x are clearly α-restricted, it follows from Proposition 5.1.29 and Lemma 5.1.30 and the definition of θ that

$$\mathbb{T}\left(\mathsf{Free}(\mathbb{T}), \mathsf{x}_h^X, \mathsf{x}_{\mathsf{id}}^X\right) \vdash \mathsf{x}_h^X = \mathsf{x}_{\mathsf{id}}^X,$$

where id is the identity arrow on the unique object of $\mathcal{J}$. Since $\mathbb{T}(\mathsf{Free}(\mathbb{T})) = \mathbb{T}$ is non-trivial for the sort X (as previously indicated), it then follows by Lemma 5.1.31 that we must have $\mathsf{x}_h^X \equiv \mathsf{x}_{\mathsf{id}}^X$, which implies $h = \mathsf{id}$. So we then obtain

$$\mathbb{T}^{\mathcal{J}}(M^F, \mathsf{y}) \vdash \alpha_\psi^Y(\mathsf{y}) = \alpha_h^Y(\mathsf{y}) = \alpha_{\mathsf{id}}^Y(\mathsf{y}) = \mathsf{y}.$$

In an exactly similar way (now using the fact that $\mathbb{T}(\mathsf{Free}(\mathbb{T})) = \mathbb{T}$ is non-trivial for the sort Y), we then obtain $\psi = \mathsf{id}$, contrary to assumption. This contradiction shows that $\left([\mathsf{x}], \left[\alpha_\psi^Y(\mathsf{y})\right]\right)$ is *not* in the image of β_{M^F}, so that β_{M^F} is *not* surjective, and hence Proposition 5.1.47 does *not* hold in general for theories $\mathbb{T}$ that have multi-sorted operations which are *not* totally defined. ∎

5.2 Categorical Characterization

In this section, we will deduce a categorical characterization of the isotropy group $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}$ of $\mathbb{T}^{\mathcal{J}}$ from the logical characterization of $G_{\mathbb{T}^{\mathcal{J}}}$ given in the first section. Unless otherwise stated, $\mathbb{T}$ is an arbitrary quasi-equational theory and $\mathcal{J}$ is an arbitrary small index category.

Definition 5.2.1. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, with component models $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ for all $i \in \mathcal{J}_O$. Let

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$$

be a $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. For each $i \in \mathcal{J}_O$, let

$$\phi^i = \left(\phi^i_g : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g)\right)_{g \in \mathsf{Dom}(M^i)} \in \mathcal{Z}_{\mathbb{T}}(M^i).$$

Finally, let $\psi = (\psi_B)_{B \in \Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$.

We say that π is *determined by* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ and the family $(\phi^i)_{i \in \mathcal{J}_O} \in \prod_{i \in \mathcal{J}_O} \mathcal{Z}_{\mathbb{T}}(M^i)$ if the following holds for any morphism $f : M \to N$ in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ with domain M, any $B \in \Sigma_{\mathsf{Sort}}$, and any $k \in \mathcal{J}_O$:

- If $k \notin \mathcal{J}^M_B$, then
$$\pi_f^{B^k} = \mathsf{id} : N_{B^k} \to N_{B^k}.$$
- If $k \in \mathcal{J}^M_B$, then
$$\pi_f^{B^k} = \left(\phi^k_{f^k \circ F^M(\psi_B(k))}\right)_B \circ F^N(\psi_B(k))_B :$$
$$(N^k_B =)\, N_{B^k} \to N_{B^k}\, (= N^k_B),$$
where $F^N : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ is the functor corresponding to $N \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, and $f^k : M^k \to N^k$ is the Σ-morphism induced by the $\Sigma^{\mathcal{J}}$-morphism $f : M \to N$ (cf. Definition 5.1.3). ∎

Definition 5.2.2. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, with component models $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ for all $i \in \mathcal{J}_O$. For each $i \in \mathcal{J}_O$, let

$$\phi^i = \left(\phi^i_g : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g)\right)_{g \in \mathsf{Dom}(M^i)} \in \mathcal{Z}_{\mathbb{T}}(M^i).$$

Let $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ be the functor corresponding to $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$.

We say that the family $(\phi^i)_{i \in \mathcal{J}_O} \in \prod_{i \in \mathcal{J}_O} \mathcal{Z}_{\mathbb{T}}(M^i)$ is *compatible* if for every arrow $f : i \to j$ in $\mathcal{J}$ and every $g \in \mathsf{Dom}(M^j)$ we have

$$\phi^j_g = \phi^i_{g \circ F^M(f)} : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g).$$

In other words, we say that $(\phi^i)_{i\in\mathcal{J}_O}$ is compatible if

$$(\phi^i)_{i\in\mathcal{J}_O} \in \mathsf{lim}(\mathcal{Z}_{\mathbb{T}} \circ F^M) \in \mathsf{Group}.$$

∎

We will prove below (cf. Lemma 5.2.4) that if $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ and $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i)$ is *compatible*, then there is a unique $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ that is determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$. For this purpose, we will require the following lemma, which can be proved by a straightforward induction on terms.

Lemma 5.2.3. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, $k \in \mathcal{J}_O$, and $B \in \Sigma_{\mathsf{Sort}}$. If $u \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_B))$ with $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$ and $u : C$, then*

$$\rho^B_{M^k}(u)^* = u^* : M^k_B = M_{B^k} \to M_{C^k} = M^k_C.$$

∎

Now we have the following lemma, whose proof may be found in Appendix C.

Lemma 5.2.4. *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, with component models $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ for all $i \in \mathcal{J}_O$, suppose that $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i)$ is compatible, and suppose that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. Then there is a unique $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$.* ∎

For our first theorem of this section, we will also require the following lemma, whose proof may again be found in Appendix C.

Lemma 5.2.5. *Let $f : M \to N$ be a $\Sigma^{\mathcal{J}}$-morphism in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, and let $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ be the functor corresponding to M. Also let $k \in \mathcal{J}_O$ and let $g : k \to k$ be an arbitrary arrow in $\mathcal{J}$, and let $B \in \Sigma_{\mathsf{Sort}}$. Finally, let $u \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_B))$ with $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$ and $u : C$. Then*

$$\alpha^C_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u)\right)\right)^* = \rho^B_{f^k\circ F^M(g)}(u)^* \circ \left(\alpha^B_g\right)^N$$
$$: N_{B^k} \to N_{C^k},$$

where $\rho^B_{f^k\circ F^M(g)} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma(N^k, \mathsf{x}_B)$ is the signature morphism induced by the Σ-morphism $f^k \circ F^M(g) : M^k \to N^k$, and $\rho^B_{M^k} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k})$ is the signature morphism of Definition 5.1.12, and $\rho^{B^k}_f : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is the signature morphism induced by $f : M \to N$. ∎

We can now give a more explicit characterization of $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}$, based on the characterization of $G_{\mathbb{T}^{\mathcal{J}}}$ given in the last section.

Theorem 5.2.6. *Let $\mathbb{T}$ be a quasi-equational theory with single-indeterminate isotropy and single-sorted non-total operations, and let $\mathcal{J}$ be a small index category. Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, and let*

$$\pi = \left(\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(M)}$$

be a $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. Then $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ iff there is a (uniquely determined) compatible family $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i)$ and a (uniquely determined) element $\psi = (\psi_B)_{B\in\Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ such that π is determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$.

Proof: Let $\pi = \left(\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(M)}$ be a $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. Suppose first that $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$. By the isomorphism $G_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ given in (the proof of) Theorem 2.2.41, this element of $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ corresponds to the following element of $G_{\mathbb{T}^{\mathcal{J}}}(M)$:

$$\left(\pi_{\eta_{C^i}}\left([\mathsf{x}_{C^i}]\right)\right)_{C^i\in\Sigma^{\mathcal{J}}} \in \prod_{C^i\in\Sigma^{\mathcal{J}}} M\langle \mathsf{x}_{C^i}\rangle_{C^i},$$

where $\eta_{C^i} : M \to M\langle \mathsf{x}_{C^i}\rangle$ is the canonical $\Sigma^{\mathcal{J}}$-morphism. By Theorem 5.1.56, we then know that there is a unique element $(\gamma^i)_{i\in\mathcal{J}} \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$ and a unique element $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ such that

$$\beta_M(\gamma,\psi) = \left(\pi_{\eta_{C^i}}\left([\mathsf{x}_{C^i}]\right)\right)_{C^i\in\Sigma^{\mathcal{J}}}.$$

Specifically, if $\gamma^k_B = [s^k_B] \in M^k\langle \mathsf{x}_B\rangle_B$ for each $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, then for all such k, B, if $k \notin \mathcal{J}^M_B$, then

$$\pi_{\eta_{B^k}}\left([\mathsf{x}_{B^k}]\right) = \beta_M(\gamma,\psi)_{B^k} = [\mathsf{x}_{B^k}] \in M\langle \mathsf{x}_{B^k}\rangle_{B^k},$$

and otherwise

$$\pi_{\eta_{B^k}}\left([\mathsf{x}_{B^k}]\right) = \beta_M(\gamma,\psi)_{B^k} = \left[\rho^B_{M^k}(s^k_B)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right] \in M\langle \mathsf{x}_{B^k}\rangle_{B^k}.$$

Now, for each $i \in \mathcal{J}_O$, we define $\phi^i \in \mathcal{Z}_{\mathbb{T}}(M^i)$ to be the element that corresponds to $\gamma^i \in G_{\mathbb{T}}(M^i)$ via the isomorphism $\mathcal{Z}_{\mathbb{T}}(M^i) \cong G_{\mathbb{T}}(M^i)$.

We now show that the family $(\phi^i)_{i\in\mathcal{J}}$ is compatible. So let $f : i \to j$ be an arbitrary arrow in $\mathcal{J}$, and let $g : M^j \to N$ be an arbitrary Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M^j. We must show

$$\phi^j_g = \phi^i_{g\circ F^M(f)} : N \overset{\sim}{\to} N.$$

By definition of ϕ^i and ϕ^j (cf. the proof of Theorem 2.2.41), we must show

$$\left(\rho_g^C \left(s_C^j\right)^* : N_C \to N_C\right)_C = \left(\rho_{g\circ F^M(f)}^C \left(s_C^i\right)^* : N_C \to N_C\right)_C : N \xrightarrow{\sim} N,$$

where $\rho_g^C : \Sigma(M^j, \mathsf{x}_C) \to \Sigma(N, \mathsf{x}_C)$ and $\rho_{g\circ F^M(f)}^C : \Sigma(M^i, \mathsf{x}_C) \to \Sigma(N, \mathsf{x}_C)$ are the signature morphisms induced by the Σ-morphisms $g : M^j \to N$ and $g \circ F^M(f) : M^i \to N$. So let $B \in \Sigma$ be any sort; we must show that

$$\rho_g^B(s_B^j)^* = \rho_{g\circ F^M(f)}^B(s_B^i)^* : N_B \to N_B.$$

By Lemma 2.2.13, it suffices to show that

$$\mathbb{T}(N, \mathsf{x}_B) \vdash \rho_g^B(s_B^j) = \rho_{g\circ F^M(f)}^B(s_B^i).$$

Since $(\gamma^i)_{i\in\mathcal{J}} \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$, it follows that

$$G_{\mathbb{T}}(F^M(f))(\gamma^i) = \gamma^j,$$

which implies in particular that

$$\left[\rho_{F^M(f)}^B(s_B^i)\right] = \left[s_B^j\right] \in M^j\langle \mathsf{x}_B\rangle_B,$$

where $\rho_{F^M(f)}^B : \Sigma(M^i, \mathsf{x}_B) \to \Sigma(M^j, \mathsf{x}_B)$ is the signature morphism induced by the Σ-morphism $F^M(f) : M^i \to M^j$. In other words, we have

$$\mathbb{T}(M^j, \mathsf{x}_B) \vdash \rho_{F^M(f)}^B(s_B^i) = s_B^j.$$

Since $\rho_g^B : \Sigma(M^j, \mathsf{x}_B) \to \Sigma(N, \mathsf{x}_B)$ is a theory morphism $\mathbb{T}(M^j, \mathsf{x}_B) \to \mathbb{T}(N, \mathsf{x}_B)$, we then obtain

$$\mathbb{T}(N, \mathsf{x}_B) \vdash \rho_g^B\left(\rho_{F^M(f)}^B(s_B^i)\right) = \rho_g^B(s_B^j).$$

But by Lemma 2.2.37, we have

$$\rho_g^B\left(\rho_{F^M(f)}^B(s_B^i)\right) \equiv \rho_{g\circ F^M(f)}^B(s_B^i),$$

which yields the desired result. This proves that $(\phi^i)_{i\in\mathcal{J}}$ is compatible.

Next, we show that π is determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$. So let $f : M \to N$ in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, let $k \in \mathcal{J}_O$, and let $B \in \Sigma_{\mathsf{Sort}}$. First, let

$$\pi_{\eta_{B^k}}([\mathsf{x}_{B^k}]) =: [t_{B^k}] \in M\langle \mathsf{x}_{B^k}\rangle_{B^k}.$$

Then by the proof of Theorem 2.2.41, we have

$$\pi_f^{B^k} = \rho_f^{B^k}(t_{B^k})^* : N_{B^k} \to N_{B^k},$$

where $\rho_f^{B^k} : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is the signature morphism induced by the $\Sigma^{\mathcal{J}}$-morphism $f : M \to N$.

Suppose first that $k \notin \mathcal{J}_B^M$. Then $\mathbb{T}(M^k)$ is trivial for the sort B, and we must show

$$\pi_f^{B^k} = \rho_f^{B^k}(t_{B^k})^* = \mathsf{id} : N_{B^k} \to N_{B^k}.$$

Since $\mathbb{T}(M^k)$ is trivial for the sort B, it easily follows (by remarks made earlier in the chapter) that $\mathbb{T}^{\mathcal{J}}(M)$ is trivial for the sort B^k, and then because of the $\Sigma^{\mathcal{J}}$-morphism $f : M \to N$, it follows that $\mathbb{T}^{\mathcal{J}}(N)$ is trivial for the sort B^k. Since N (more precisely, the expansion of N to its canonical $\Sigma^{\mathcal{J}}(N)$-structure) is a model of $\mathbb{T}^{\mathcal{J}}(N)$, it then follows that N_{B^k} contains at most one element, which yields the desired conclusion.

Now suppose that $k \in \mathcal{J}_B^M$. Then we must show that

$$\pi_f^{B^k} = \left(\phi^k_{f^k \circ F^M(\psi_B(k))}\right)_B \circ F^N(\psi_B(k))_B : N_{B^k} \to N_{B^k},$$

i.e.

$$\rho_f^{B^k}(t_{B^k})^* = \left(\phi^k_{f^k \circ F^M(\psi_B(k))}\right)_B \circ F^N(\psi_B(k))_B : N_{B^k} \to N_{B^k}.$$

By definition of ϕ^k (and the proof of Theorem 2.2.41) we have

$$\left(\phi^k_{f^k \circ F^M(\psi_B(k))}\right)_B = \rho^B_{f^k \circ F^M(\psi_B(k))}(s_B^k)^* : N_B^k \to N_B^k,$$

and so we must show

$$\rho_f^{B^k}(t_{B^k})^* = \rho^B_{f^k \circ F^M(\psi_B(k))}(s_B^k)^* \circ F^N(\psi_B(k))_B : N_{B^k} \to N_{B^k}.$$

Also, since

$$[t_{B^k}] = \pi_{\eta_{B^k}}([\mathsf{x}_{B^k}]) = \left[\rho^B_{M^k}(s_B^k)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right],$$

we have

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash t_{B^k} = \rho^B_{M^k}(s_B^k)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right].$$

Now, we know that

$$\beta_M(\gamma, \mathbb{1}) \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

where $\mathbb{1}$ is the unit element of the group $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. In particular, we have

$$\begin{aligned}
\beta_M(\gamma, \mathbb{1})_{B^k} &= \left[\rho^B_{M^k}(s_B^k)\left[\alpha^B_{\mathbb{1}_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right] \\
&= \left[\rho^B_{M^k}(s_B^k)\left[\alpha^B_{\mathsf{id}_k}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right] \\
&= \left[\rho^B_{M^k}(s_B^k)\left[\mathsf{x}_{B^k}/\mathsf{x}_{B^k}\right]\right] \\
&= \left[\rho^B_{M^k}(s_B^k)\right] \\
&\in M\langle \mathsf{x}_{B^k}\rangle_{B^k},
\end{aligned}$$

and since $\beta_M(\gamma, \mathbb{1})$ commutes generically with the function symbol $\alpha^B_{\psi_B(k)} : B^k \to B^k$, it follows that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \vdash \rho^B_{M^k}(s^k_B)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right] = \alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right).$$

Then because $\rho^{B^k}_f : \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ is a theory morphism, we obtain

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \rho^{B^k}_f\left(\rho^B_{M^k}(s^k_B)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right) = \rho^{B^k}_f\left(\alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right)\right).$$

Finally, we obtain

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \rho^{B^k}_f(t_{B^k}) = \rho^{B^k}_f\left(\alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right)\right).$$

By Lemma 2.2.13, we then deduce that

$$\rho^{B^k}_f(t_{B^k})^* = \rho^{B^k}_f\left(\alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right)\right)^* : N_{B^k} \to N_{B^k},$$

so we must show

$$\rho^{B^k}_f\left(\alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right)\right)^* = \rho^B_{f^k \circ F^M(\psi_B(k))}(s^k_B)^* \circ F^N(\psi_B(k))_B : N_{B^k} \to N_{B^k}.$$

Finally, we have

$$F^N(\psi_B(k))_B := \left(\alpha^B_{\psi_B(k)}\right)^N,$$

so we are ultimately reduced to showing that

$$\rho^{B^k}_f\left(\alpha^B_{\psi_B(k)}\left(\rho^B_{M^k}(s^k_B)\right)\right)^* = \rho^B_{f^k \circ F^M(\psi_B(k))}(s^k_B)^* \circ \left(\alpha^B_{\psi_B(k)}\right)^N$$
$$: N_{B^k} \to N_{B^k},$$

but this is true by Lemma 5.2.5 above (given that $\mathbb{T}(M^k, \mathsf{x}_B) \vdash s^k_B \downarrow$ and that $\rho^{B^k}_f$ commutes with $\alpha^B_{\psi_B(k)}$).

This completes the argument that π is determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$, which completes the proof of the 'only if' direction of Theorem 5.2.6.

To prove the 'if' direction of Theorem 5.2.6, now only assuming that $\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$ is a $\mathsf{Dom}(M)$-indexed family of *endo*morphisms in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, suppose that there is some compatible family $(\phi^i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} \mathscr{Z}_{\mathbb{T}}(M^i)$ and an element $\psi = (\psi_B)_{B \in \Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ such that π is determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$. We must show that $\pi \in \mathscr{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$.

By Lemma 5.2.4, there *does* exist an element $\pi' \in \mathscr{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ that is determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$. Since π is also determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$, it easily follows that $\pi = \pi'$, so that $\pi \in \mathscr{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$, as desired. This completes the proof of Theorem 5.2.6. ∎

We will now rephrase Theorem 5.2.6 in terms of the isomorphism $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. First, we rephrase all of the preceding definitions in terms of $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$.

Definition 5.2.7. Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, and let

$$\pi = (\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu))_{\mu \in \mathsf{Dom}(F)}$$

be a $\mathsf{Dom}(F)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. For each $i \in \mathcal{J}_O$, let

$$\phi^i = \left(\phi^i_g : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g)\right)_{g \in \mathsf{Dom}(F(i))} \in \mathcal{Z}_{\mathbb{T}}(F(i)).$$

Finally, let $\psi = (\psi_B)_{B \in \Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M^F}$ (where $M^F \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ is the model of $\mathbb{T}^{\mathcal{J}}$ corresponding to the functor F).

We say that π is *determined by* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M^F}$ and the family $(\phi^i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} \mathcal{Z}_{\mathbb{T}}(F(i))$ if the following holds for any morphism $\mu : F \to G$ in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$, any $B \in \Sigma_{\mathsf{Sort}}$, and any $k \in \mathcal{J}_O$:

- If $k \notin \mathcal{J}_B^{M^F}$, then
$$\pi_\mu(k)_B = \mathsf{id} : G(k)_B \to G(k)_B.$$
- If $k \in \mathcal{J}_B^{M^F}$, then
$$\begin{aligned}\pi_\mu(k)_B = \left(\phi^k_{\mu(k) \circ F(\psi_B(k))}\right)_B \circ G(\psi_B(k))_B \\ : G(k)_B \to G(k)_B\end{aligned}$$
(where $\mu(k) : F(k) \to G(k)$ is a Σ-morphism). ∎

Definition 5.2.8. Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$. For each $i \in \mathcal{J}_O$, let

$$\phi^i = \left(\phi^i_g : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g)\right)_{g \in \mathsf{Dom}(F(i))} \in \mathcal{Z}_{\mathbb{T}}(F(i)).$$

We say that the family $(\phi^i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} \mathcal{Z}_{\mathbb{T}}(F(i))$ is *compatible* if for every arrow $f : i \to j$ in $\mathcal{J}$ and every $g \in \mathsf{Dom}(F(j))$ we have

$$\phi^j_g = \phi^i_{g \circ F(f)} : \mathsf{cod}(g) \xrightarrow{\sim} \mathsf{cod}(g).$$

In other words, we say that $(\phi^i)_{i \in \mathcal{J}}$ is compatible if

$$(\phi^i)_{i \in \mathcal{J}} \in \mathsf{lim}(\mathcal{Z}_{\mathbb{T}} \circ F) \in \mathsf{Group}.$$

∎

We now have the following version of Theorem 5.2.6 in terms of $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$:

Corollary 5.2.9. *Let $\mathbb{T}$ be a quasi-equational theory with single-indeterminate isotropy and single-sorted non-total operations, and let $\mathcal{J}$ be a small index category. Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, and let*

$$\pi = (\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu))_{\mu \in \mathsf{Dom}(F)}$$

be a $\mathsf{Dom}(F)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. Then $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}}(F)$ iff there is a (uniquely determined) compatible family $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(F(i))$ and a (uniquely determined) element $\psi = (\psi_B)_{B\in\Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M^F}$ such that π is determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$. ∎

Before we give some specific applications of the general results proven so far in this chapter, we will extract an important consequence of Theorem 5.1.56 that does *not* rely on the assumptions that $\mathbb{T}$ has single-indeterminate isotropy and single-sorted non-total operations, but which only applies to index categories $\mathcal{J}$ satisfying a certain strict condition. Namely, we have the following consequence of (the proof of) Theorem 5.1.56:

Corollary 5.2.10. *Let $\mathbb{T}$ be an* arbitrary *quasi-equational theory, and let $\mathcal{J}$ be a small index category. If $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $\mathcal{J}_B^M$ has only* trivial *endomorphisms for each $B \in \Sigma_{\mathsf{Sort}}$, then*

$$\beta_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is a group isomorphism, and

$$G_{\mathbb{T}^{\mathcal{J}}}(M) \cong \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}.$$

Proof: We know from Propositions 5.1.20 and 5.1.43 that if $\mathbb{T}$ is an *arbitrary* quasi-equational theory and $\mathcal{J}$ an arbitrary small index category, then β_M is an *injective* group homomorphism for each $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. We only needed the assumptions that $\mathbb{T}$ has single-indeterminate isotropy and single-sorted non-total operations in order to prove that β_M is *surjective*. In the proof of Proposition 5.1.47, given $([s_{C^i}])_{i,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, we constructed a pair

$$(\gamma, \psi) \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$$

with $\beta_M(\gamma, \psi) = ([s_{C^i}])_{i,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, and we only used the assumption that $\mathbb{T}$ has single-indeterminate isotropy to show that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ can be defined from $([s_{C^i}])_{i,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$. More specifically, given any sort $B \in \Sigma$, in order to define $\psi_B \in \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$, we used the assumption that $\mathbb{T}$ has single-indeterminate isotropy to conclude (for each $i \in \mathcal{J}_B^M$) that s_{B^i} contains exactly one occurrence of the indeterminate x_{B^i}, and then we defined $\psi_B(i) : i \xrightarrow{\sim} i$ to be the subscript arrow of the unique indeterminate occurring in $\theta(s_{B^i}) \in \mathsf{Term}^c\left(\Sigma\left(M^i, \overline{\mathsf{x}^B_{\mathsf{Cod}(i)}}\right)\right)$. Then we used the assumption that $\mathbb{T}$ has single-sorted non-total operations in order to prove that $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$.

However, if we know for each sort $B \in \Sigma$ that $\mathcal{J}_B^M$ has only *trivial* endomorphisms, then even if $\mathbb{T}$ does *not* have single-indeterminate isotropy, so that s_{B^i} may have *multiple* occurrences of x_{B^i} for certain $i \in \mathcal{J}_B^M$, we know that the only indeterminate occurring in $\theta(s_{B^i})$ will be $\mathsf{x}_{\mathsf{id}_i}$, even if it has multiple occurrences in $\theta(s_{B^i})$ (since $\mathsf{id}_i : i \to i$ is the *only* endomorphism of $i \in \mathcal{J}_B^M$). Hence, if we set ψ_B to be the unit element of $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$ for each sort $B \in \Sigma$, then $\psi = (\psi_B)_{B\in\Sigma}$ will be the unit element of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, and we will still have $\beta_M(\gamma, \psi) = ([s_{C^i}])_{i,C} \in G_{\mathbb{T}^{\mathcal{J}}}(M)$, so that β_M will still be surjective and hence a group isomorphism.

The last claim of the corollary now easily follows, because if $\mathcal{J}_B^M$ has only trivial endomorphisms for every sort B, then certainly $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right)$ is the trivial group for every sort B, which easily implies that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ is the trivial group, so that

$$\left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \cong \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}.$$

In fact, if we define

$$\beta'_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \to G_{\mathbb{T}^{\mathcal{J}}}(M)$$

by

$$\beta'_M(\gamma) := \beta_M(\gamma, \mathbb{1}) \in G_{\mathbb{T}^{\mathcal{J}}}(M)$$

for every $\gamma \in \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$ (where $\mathbb{1} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ is the unit element), then

$$\beta'_M : \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M)$$

is a group isomorphism. ∎

From Corollary 5.2.10 and (the proof of) Theorem 5.1.56, we easily deduce the following corollary:

Corollary 5.2.11. *Let $\mathbb{T}$ be an* arbitrary *quasi-equational theory and $\mathcal{J}$ a small index category with only* trivial *endomorphisms. Then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, there is a group isomorphism*

$$\beta'_M : \mathsf{lim}\left(G_{\mathbb{T}} \circ F^M\right) = \left(\prod_i G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}(M)$$

given by

$$\beta'_M(\gamma) := \beta_M(\gamma, \mathbb{1}) \in G_{\mathbb{T}^{\mathcal{J}}}(M),$$

and these group isomorphisms form a natural automorphism

$$\beta' := (\beta'_M)_M : \mathsf{lim}\left(G_{\mathbb{T}} \circ F^{(-)}\right) = \left(\prod_i G_{\mathbb{T}} \circ F^{(-)}\right)^{G_{\mathbb{T}} \circ F^{(-)}} \xrightarrow{\sim} G_{\mathbb{T}^{\mathcal{J}}}$$

$$: \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}.$$

∎

We may now deduce the following categorical versions of the previous corollary:

Corollary 5.2.12. *Let $\mathbb{T}$ be an* arbitrary *quasi-equational theory and $\mathcal{J}$ a small index category with only* trivial *endomorphisms.*

- *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and let*

  $$\pi = \left(\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f)\right)_{f \in \mathsf{Dom}(M)}$$

  be a $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. Then $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ iff there is a (uniquely determined) compatible family $(\phi^i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i)$ such that π is determined by $(\phi^i)_{i \in \mathcal{J}}$, in the sense that

  $$\pi_f^{B^k} = \left(\phi^k_{f^k}\right)_B : N_{B^k} \to N_{B^k}$$

  for every morphism $f : M \to N$ in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and every $B^k \in \Sigma^{\mathcal{J}}_{\mathsf{Sort}}$.

- *Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, and let*

  $$\pi = \left(\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu)\right)_{\mu \in \mathsf{Dom}(F)}$$

  be a $\mathsf{Dom}(F)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. Then $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}}(F)$ iff there is a (uniquely determined) compatible family $(\phi^i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} \mathcal{Z}_{\mathbb{T}}(F(i))$ such that π is determined by $(\phi^i)_{i \in \mathcal{J}}$, in the sense that

  $$\pi_\mu(k)_B = \left(\phi^k_{\mu(k)}\right)_B : G(k)_B \to G(k)_B$$

  for every morphism $\mu : F \to G$ in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ and every $B^k \in \Sigma^{\mathcal{J}}_{\mathsf{Sort}}$. ∎

5.3 Applications

First, for an *arbitrary* quasi-equational theory $\mathbb{T}$, we deduce characterizations of the isotropy group of $\mathbb{T}^{\mathcal{J}}$ for certain specific, commonly occurring index categories $\mathcal{J}$ with only *trivial* endomorphisms.

Corollary 5.3.1. *Let $\mathbb{T}$ be an* arbitrary *quasi-equational theory. Note that in each of the following examples, the index category $\mathcal{J}$ has only* trivial *endomorphisms.*

- *Let $\mathcal{J}$ be any small* discrete *category (i.e. $\mathcal{J}$ has no arrows other than identities). Then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, i.e. any collection $(M^i)_{i\in\mathcal{J}}$ of $\mathbb{T}$-models, we have*
  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i).$$
  In particular, if $\mathcal{J}$ is a discrete category on two objects i and j, then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, i.e. any pair (M^i, M^j) of $\mathbb{T}$-models, we have
  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathcal{Z}_{\mathbb{T}}(M^i) \times \mathcal{Z}_{\mathbb{T}}(M^j).$$
- *Let $\mathcal{J}$ be the category with two objects i, j and two parallel morphisms $f, g : i \to j$. Then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, i.e. any quadruple (M^i, M^j, f^M, g^M) consisting of $\mathbb{T}$-models M^i, M^j and Σ-morphisms $f^M, g^M : M^i \to M^j$, we have*
  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathsf{Eq}\left(\mathcal{Z}_{\mathbb{T}}(f^M), \mathcal{Z}_{\mathbb{T}}(g^M)\right)$$
  $$= \left\{\gamma^i \in \mathcal{Z}_{\mathbb{T}}(M^i) : \mathcal{Z}_{\mathbb{T}}(f^M)(\gamma^i) = \mathcal{Z}_{\mathbb{T}}(g^M)(\gamma^i)\right\},$$
  where $\mathsf{Eq}\left(\mathcal{Z}_{\mathbb{T}}(f^M), \mathcal{Z}_{\mathbb{T}}(g^M)\right)$ is the equalizer in Group of the group homomorphisms $\mathcal{Z}_{\mathbb{T}}(f^M), \mathcal{Z}_{\mathbb{T}}(g^M) : \mathcal{Z}_{\mathbb{T}}(M^i) \to \mathcal{Z}_{\mathbb{T}}(M^j)$.
- *Let $\mathcal{J}$ be the category with three objects i, j, k and two morphisms $f : i \to k$ and $g : j \to k$. Then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, i.e. any quintuple $(M^i, M^j, M^k, f^M, g^M)$ consisting of $\mathbb{T}$-models M^i, M^j, M^k and Σ-morphisms $f^M : M^i \to M^k$ and $g^M : M^j \to M^k$, we have*
  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathcal{Z}_{\mathbb{T}}(M^i) \times_{\mathcal{Z}_{\mathbb{T}}(M^k)} \mathcal{Z}_{\mathbb{T}}(M^j)$$
  $$= \left\{(\gamma^i, \gamma^j) \in \mathcal{Z}_{\mathbb{T}}(M^i) \times \mathcal{Z}_{\mathbb{T}}(M^j) : \mathcal{Z}_{\mathbb{T}}(f^M)(\gamma^i) = \mathcal{Z}_{\mathbb{T}}(g^M)(\gamma^j)\right\},$$
  where $\mathcal{Z}_{\mathbb{T}}(M^i) \times_{\mathcal{Z}_{\mathbb{T}}(M^k)} \mathcal{Z}_{\mathbb{T}}(M^j)$ is the pullback in Group of the pair $\mathcal{Z}_{\mathbb{T}}(f^M) : M^i \to M^k, \mathcal{Z}_{\mathbb{T}}(g^M) : M^j \to M^k$ of group homomorphisms. ∎

Now, we provide applications of the results of the preceding sections for quasi-equational theories $\mathbb{T}$ with single-indeterminate isotropy and single-sorted non-total operations, for which we *can* consider index categories with non-trivial endomorphisms. Let us note that the following quasi-equational theories *do* have single-indeterminate isotropy and single-sorted non-total operations, as one can easily verify by an inspection of Chapter 3:

- Any quasi-equational theory with single-sorted non-total operations that has trivial isotropy, which includes the following theories: any empty theory with single-sorted non-total operations (e.g. the theory of sets), the theory of commutative monoids, the theories of categories and groupoids, and the theory of categories with a terminal object.
- The theories of monoids and groups, and the theory of abelian groups.
- The theory of sets with a bijection or involution.
- The theory of strict monoidal categories.

First, let us consider the class of empty theories with single-sorted non-total operations. We first require the following technical lemma (which does *not* rely on the assumption of $\mathbb{T}$ having single-sorted non-total operations).

Lemma 5.3.2. *Let Σ be an arbitrary relation-free signature, and let $\mathbb{T}$ be an empty theory over Σ. If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then for any $B \in \Sigma_{\mathsf{Sort}}$, $\mathbb{T}(M)$ is non-trivial for the sort B. If $g : B_1 \times \ldots \times B_n \to B$ is any function symbol of Σ, then for any $1 \leq m \leq n$, g^M is non-degenerate in position m.*

Proof: See Appendix C. ∎

Now let $\mathcal{J}$ be an arbitrary small index category, and let $\mathbb{T}$ be an empty theory over Σ with single-sorted non-total operations. We conclude from Lemma 5.3.2 that if $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, then $\mathcal{J}_B^M = \mathcal{J}$ for each $B \in \Sigma_{\mathsf{Sort}}$, because if $i \in \mathcal{J}_O$, then $\mathbb{T}(M^i)$ is non-trivial for the sort B, since $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$. So then

$$\prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}_B^M}\right) = \prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}).$$

Next, if $i \in \mathcal{J}_O$ and $g : A_1 \times \ldots \times A_n \to A$ is any function symbol of Σ, then since $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$, it follows by Lemma 5.3.2 that g^{M^i} is non-degenerate in every position $1 \leq m \leq n$. So for any $\psi = (\psi_B)_{B \in \Sigma} \in \prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, we have $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$ iff for every function symbol $g : A_1 \times \ldots \times A_n \to A$ in Σ and every $1 \leq m \leq n$ we have $\psi_A = \psi_{A_m} : \mathsf{Id}_{\mathcal{J}} \xrightarrow{\sim} \mathsf{Id}_{\mathcal{J}}$. In fact, let $\mathcal{G}(\Sigma)$ be the (undirected) graph defined as follows: the nodes are the sorts of Σ, and for any distinct sorts $A \neq B \in \Sigma$, there is an edge between A and B iff there is some function symbol $g : C_1 \times \ldots \times C_n \to C$ with $A, B \in \{C_1, \ldots, C_n, C\}$. Then it easily follows from the observation just made that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \cong \prod_{\mathsf{Comp}(\mathcal{G}(\Sigma))} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}),$$

where $\mathsf{Comp}(\mathcal{G}(\Sigma))$ is the set of connected components of the graph $\mathcal{G}(\Sigma)$.

Finally, we know by Proposition 3.1.4 that $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ has trivial isotropy for every $i \in \mathcal{J}_O$, so that $\mathcal{Z}_{\mathbb{T}}(M^i)$ is the trivial group. Hence, we obtain:

Corollary 5.3.3. *Let Σ be any relation-free signature and $\mathbb{T}$ any empty theory over Σ with single-sorted non-total operations, and let $\mathcal{J}$ be any small index category. Then for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ we have*

$$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \prod_{\mathsf{Comp}(\mathcal{G}(\Sigma))} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$$

(naturally in M).

In particular, if Σ is the signature with only one sort and no function symbols, then $\mathsf{Tot}(\Sigma)$ is the theory of sets (and $\mathcal{G}(\Sigma)$ is the trivial graph), and for any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ we have

$$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$$

(naturally in M). ∎

We can also give the following more categorical version of Corollary 5.3.3:

Corollary 5.3.4. *Let Σ be any relation-free signature, let $\mathbb{T}$ be any empty theory with single-sorted non-total operations, and let $\mathcal{J}$ be any small index category. Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, and let*

$$\pi = (\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu))_{\mu \in \mathsf{Dom}(F)}$$

be any $\mathsf{Dom}(F)$-indexed family of endomorphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. Then $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}}(F)$ iff there is a (uniquely determined) element $\psi \in \prod_{\mathsf{Comp}(\mathcal{G}(\Sigma))} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ such that π is determined by ψ, in the sense that

$$\pi_\mu(k)_B = G(\psi_U(k))_B : G(k)_B \xrightarrow{\sim} G(k)_B$$

for every morphism $\mu : F \to G$ in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ and every sort $B^k \in \Sigma^{\mathcal{J}}$, where $U \in \mathsf{Comp}(\mathcal{G}(\Sigma))$ is the connected component of $\mathcal{G}(\Sigma)$ containing B.

In particular, if Σ has only one sort and no function symbols, then $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}}(F)$ iff there is a (uniquely determined) element $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ such that π is determined by ψ, in the sense that

$$\pi_\mu(k) = G(\psi(k)) : G(k) \xrightarrow{\sim} G(k)$$

for every morphism $\mu : F \to G$ in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}} = \mathsf{Sets}^{\mathcal{J}}$ and every object $k \in \mathcal{J}_O$. ∎

Now we consider non-empty *single-sorted* quasi-equational theories with single-indeterminate isotropy (note that any such theory automatically has single-sorted non-total operations). For any such theory $\mathbb{T}$, let us refer to its sort as 'X'. If M is any model of such a theory, then we will just say that $\mathbb{T}(M)$ is *non-trivial* if it is non-trivial for this unique sort X. Then (by Lemma 3.1.2) $\mathbb{T}(M)$ is trivial iff there is no Σ-morphism from M to a $\mathbb{T}$-model N with $|N| \geq 2$. Such a situation is quite rare (although examples exist: for example, the zero ring (as a model of the theory of rings with unit) has trivial diagram theory, because there is no ring homomorphism from the zero ring to any non-zero ring with unit). Hence, to simplify the presentation of the upcoming results, we will generally assume that if $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$, then $F(i) \in \mathsf{P}\mathbb{T}\mathsf{mod}$ is non-trivial for each $i \in \mathcal{J}_O$.

Now let $\mathcal{J}$ be any small index category with $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ (where $\mathbb{T}$ is a single-sorted quasi-equational theory). Assume also that $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ is non-trivial for each $i \in \mathcal{J}_O$. If we set $\mathcal{J}^M := \mathcal{J}^M_X$, then we therefore have $\mathcal{J}^M = \mathcal{J}$. So $\prod_{B \in \Sigma_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{J}^M_B}\right)$ then becomes just $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$. Also, since Σ has only one sort, it follows that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) = \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$. Hence, we obtain the following simplification of earlier results:

Corollary 5.3.5. *Let Σ be any* single-sorted *relation-free signature and $\mathbb{T}$ any quasi-equational theory over Σ with single-indeterminate isotropy, and let $\mathcal{J}$ be any small index category.*

- *For any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ such that $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ is non-trivial for all $i \in \mathcal{J}_O$, we have*

  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \left(\prod_i \mathcal{Z}_{\mathbb{T}}(M^i)\right)^{\mathcal{J}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$$

  (naturally in M).

- *Let $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ with $F(i) \in \mathsf{P}\mathbb{T}\mathsf{mod}$ non-trivial for all $i \in \mathcal{J}_O$, and let*

  $$\pi = \left(\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu)\right)_{\mu \in \mathsf{Dom}(F)}$$

be any $\mathsf{Dom}(F)$*-indexed family of endomorphisms in* $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. *Then* $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}^{\mathcal{J}}}(F)$ *iff there is a (uniquely determined) compatible family* $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(F(i))$ *and a (uniquely determined) element* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ *such that* π *is determined by* $(\phi^i)_{i\in\mathcal{J}}$ *and* ψ*, in the sense that*

$$\pi_\mu(k) = \phi^k_{\mu(k)\circ F(\psi(k))} \circ G(\psi(k)) : G(k) \xrightarrow{\sim} G(k)$$

for every morphism $\mu : F \to G$ *in* $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ *and every* $k \in \mathcal{J}_O$. ∎

We now have the following specialization of Corollary 5.3.5 to single-sorted quasi-equational theories with *trivial* isotropy.

Corollary 5.3.6. *Let* Σ *be any single-sorted relation-free signature and* $\mathbb{T}$ *any quasi-equational theory over* Σ *with* trivial *isotropy, and let* $\mathcal{J}$ *be any small index category.*

- *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *such that* $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *is non-trivial for all* $i \in \mathcal{J}_O$*, we have*

  $$\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$$

  (naturally in M*).*

- *Let* $F : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ *with* $F(i) \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *non-trivial for all* $i \in \mathcal{J}_O$*, and let*

  $$\pi = (\pi_\mu : \mathsf{cod}(\mu) \to \mathsf{cod}(\mu))_{\mu\in\mathsf{Dom}(F)}$$

  be any $\mathsf{Dom}(F)$*-indexed family of endomorphisms in* $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. *Then* $\pi \in \mathcal{Z}_{\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}^{\mathcal{J}}}(F)$ *iff there is a (uniquely determined) element* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ *such that* π *is determined by* ψ*, in the sense that*

  $$\pi_\mu(k) = G(\psi(k)) : G(k) \xrightarrow{\sim} G(k)$$

  for every morphism $\mu : F \to G$ *in* $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ *and every* $k \in \mathcal{J}_O$. ∎

Since the theory of sets (i.e. the completely empty theory over the single-sorted signature with no function symbols) has trivial isotropy by Proposition 3.1.4, and every model of this theory (i.e. every set) has non-trivial diagram theory (by Lemma 5.3.2), Corollary 5.3.6 provides a characterization of the covariant isotropy group of a presheaf topos, as was promised at the beginning of the chapter.

Given a group G, we can also compute the covariant isotropy group of the category of G-sets. Let $\mathcal{B}(G)$ be the one-object category corresponding to the group G. Then the category of G-sets is just the functor category $\mathsf{Sets}^{\mathcal{B}(G)}$. Also, it is easy to see that the global isotropy group $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(G)}\right)$ of $\mathcal{B}(G)$ is (isomorphic to) the centre $Z(G)$ of G. So from Corollary 5.3.6 we obtain:

Corollary 5.3.7. *Let G be a group with corresponding one-object category $\mathcal{B}(G)$, so that $\mathsf{Sets}^{\mathcal{B}(G)}$ is the category of G-sets. For any G-set $M \in \mathsf{Sets}^{\mathcal{B}(G)}$, we have*

$$\mathcal{Z}_{\mathsf{Sets}^{\mathcal{B}(G)}}(M) \cong Z(G).$$

More concretely, let

$$\pi = (\pi_f : \mathsf{cod}(f) \to \mathsf{cod}(f))_{f \in \mathsf{Dom}(M)}$$

be any $\mathsf{Dom}(M)$-indexed family of endomorphisms in $\mathsf{Sets}^{\mathcal{B}(G)}$. Then $\pi \in \mathcal{Z}_{\mathsf{Sets}^{\mathcal{B}(G)}}(M)$ iff there is a (uniquely determined) element $g \in Z(G)$ such that π is determined by g, in the sense that

$$\pi_f : N \xrightarrow{\sim} N$$

is given by

$$\pi_f(n) = gn \in N \qquad (n \in N)$$

for every morphism $f : M \to N$ in $\mathsf{Sets}^{\mathcal{B}(G)}$. ∎

More generally, if M is any monoid with corresponding one-object category $\mathcal{B}(M)$, then the category of M-sets is just the functor category $\mathsf{Sets}^{\mathcal{B}(M)}$, and the global isotropy group $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{B}(M)}\right)$ of $\mathcal{B}(M)$ is (isomorphic to) the subgroup $\mathsf{Inv}(Z(M))$ of the invertible elements of the centre $Z(M)$ of M. So we have a corresponding analogue of Corollary 5.3.7 for M-sets.

As our final application of the preceding results of this chapter, we will calculate the global isotropy group $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}})$ of a product of small categories $\mathcal{J} \times \mathcal{K}$ in terms of the global isotropy groups $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}), \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$ of the factor categories $\mathcal{J}, \mathcal{K}$. First, we require a preparatory lemma:

Lemma 5.3.8. *Let $\mathcal{J}, \mathcal{K}$ be any small categories. Let Σ be the signature containing just one sort X and no function symbols, and let $\mathcal{E}_\Sigma$ be the empty theory over this signature (i.e. the theory of sets). Let $M \in \mathsf{P}(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}\mathsf{mod}$ be arbitrary. Then for any object $k \in \mathcal{K}_O$, the theory $\mathcal{E}_\Sigma^{\mathcal{J}}(M^k)$ is non-trivial for every sort in $\Sigma^{\mathcal{J}}$. Moreover, if $g \in \Sigma^{\mathcal{J}}$ is any function symbol, then g^{M^k} is non-degenerate in every position.*

Finally, we have

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \prod_{\mathsf{Comp}(\mathcal{K})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}}),$$

where $\mathsf{Comp}(\mathcal{J}), \mathsf{Comp}(\mathcal{K})$ are respectively the sets of connected components of the categories $\mathcal{J}, \mathcal{K}$.

Proof: First, note that we have (by Proposition 5.1.8 and general category theory)

$$\begin{aligned}\mathsf{P}(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}\mathsf{mod} &\cong (\mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}}\mathsf{mod})^{\mathcal{K}} \\ &\cong (\mathsf{P}\mathcal{E}_\Sigma\mathsf{mod}^{\mathcal{J}})^{\mathcal{K}} \\ &= (\mathsf{Sets}^{\mathcal{J}})^{\mathcal{K}} \\ &\cong \mathsf{Sets}^{\mathcal{J}\times\mathcal{K}}.\end{aligned}$$

Then the first two claims easily follow from the fact that if $F : \mathcal{J} \times \mathcal{K} \to \mathsf{Sets}$ is any functor, then there is a natural transformation from F to a functor $F' : \mathcal{J}\times\mathcal{K} \to \mathsf{Sets}$ such that $F'(j,k) \in \mathsf{Sets}$ has at least two elements for each $(j,k) \in \mathcal{J} \times \mathcal{K}$, and such that for each arrow $(f,g) : (j_1,k_1) \to (j_2,k_2)$ in $\mathcal{J} \times \mathcal{K}$, $F'(f,g) : F'(j_1,k_1) \to F'(j_2,k_2)$ is not a constant function.

For the last claim, fix an arbitrary $M \in \mathsf{P}(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}\mathsf{mod}$ and an arbitrary $k \in \mathcal{K}_O$; then $M^k \in \mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}}\mathsf{mod}$. We first calculate $\mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}}}(M^k)$. Since $\mathcal{E}_\Sigma$ (the theory of sets) has trivial isotropy (cf. Proposition 3.1.4) and $(M^k)^i \in \mathsf{P}\mathcal{E}_\Sigma\mathsf{mod} = \mathsf{Sets}$ has non-trivial diagram theory for all $i \in \mathcal{J}_O$ (by Lemma 5.3.2), it follows by Corollary 5.3.6 that

$$\mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}}}(M^k) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}).$$

Now, by the first claim, we know for each $k \in \mathcal{K}_O$ that the theory $\mathcal{E}_\Sigma^{\mathcal{J}}(M^k)$ is non-trivial for every sort of $\Sigma^{\mathcal{J}}$. So for each $j \in \mathcal{J}_O$, we have that $\mathcal{E}_\Sigma^{\mathcal{J}}(M^k)$ is non-trivial for the sort X^j for every $k \in \mathcal{K}_O$, and hence $\mathcal{K}^M_{X^j} = \mathcal{K}$, so that $\mathsf{Aut}\left(\mathsf{Id}_{\mathcal{K}^M_{X^j}}\right) = \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$. Hence, we have

$$\prod_{B\in\Sigma^{\mathcal{J}}_{\mathsf{Sort}}} \mathsf{Aut}\left(\mathsf{Id}_{\mathcal{K}^M_B}\right) = \prod_{j\in\mathcal{J}_O} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$$

(since $\Sigma^{\mathcal{J}}_{\mathsf{Sort}} = \{X^j : j \in \mathcal{J}_O\}$). Next, if $\psi = (\psi_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$, then since for every arrow $f : i \to \ell$ in $\mathcal{J}$ and every $k \in \mathcal{K}_O$ we know (by the second claim) that the function symbol $\alpha_f^{M^k}$ of $\Sigma^{\mathcal{J}}$ is non-degenerate in its only position, it follows that $\psi = (\psi_j)_{j\in\mathcal{J}} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})^M$ iff $\psi_i = \psi_\ell$ for every pair of objects $i, \ell \in \mathcal{J}_O$ for which there is an arrow from i to ℓ. So we deduce that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})^M \cong \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}}),$$

where $\mathsf{Comp}(\mathcal{J})$ is the set of connected components of $\mathcal{J}$.

Now, we know by Theorem 5.1.56 that

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \left(\prod_{k\in\mathcal{K}} \mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}}}(M^k)\right)^{\mathcal{K}} \times \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})^M.$$

By the foregoing, we then obtain

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \left(\prod_{k\in\mathcal{K}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})\right)^{\mathcal{K}} \times \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}}).$$

However, because

$$\mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}}} : \mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}$$

is the constant functor on $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ (by Proposition 3.1.4 and Corollary 5.3.6), it then follows that if $\psi = (\psi_k)_{k\in\mathcal{K}} \in \prod_{k\in\mathcal{K}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, then $\psi \in \left(\prod_{k\in\mathcal{K}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})\right)^{\mathcal{K}}$ iff $\psi_k = \psi_{k'}$ for any objects k, k' in the same connected component of $\mathcal{K}$. So then we obtain

$$\left(\prod_{k\in\mathcal{K}} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})\right)^{\mathcal{K}} \cong \prod_{\mathsf{Comp}(\mathcal{K})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}),$$

where $\mathsf{Comp}(\mathcal{K})$ is the set of connected components of $\mathcal{K}$. Altogether, we then have

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \prod_{\mathsf{Comp}(\mathcal{K})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}}),$$

as desired. ∎

If $\mathcal{J}$ is any small category, we now define $\mathcal{Z}(\mathcal{J}) := \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$, the global isotropy group of $\mathcal{J}$. Then we have the following result:

Corollary 5.3.9. *For any small categories $\mathcal{J}$ and $\mathcal{K}$, we have*

$$\mathcal{Z}(\mathcal{J} \times \mathcal{K}) \cong \prod_{\mathsf{Comp}(\mathcal{K})} \mathcal{Z}(\mathcal{J}) \times \prod_{\mathsf{Comp}(\mathcal{J})} \mathcal{Z}(\mathcal{K}),$$

where $\mathsf{Comp}(\mathcal{J}), \mathsf{Comp}(\mathcal{K})$ are respectively the sets of connected components of $\mathcal{J}, \mathcal{K}$. In particular, if $\mathcal{J}$ and $\mathcal{K}$ are both connected, then

$$\mathcal{Z}(\mathcal{J} \times \mathcal{K}) \cong \mathcal{Z}(\mathcal{J}) \times \mathcal{Z}(\mathcal{K}).$$

Concretely: if $\pi = (\pi_{j,k} : (j,k) \to (j,k))_{(j,k)\in\mathcal{J}\times\mathcal{K}}$ is a family of endomorphisms in $\mathcal{J} \times \mathcal{K}$, then $\pi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}})$ iff there are (uniquely determined) elements $\psi \in \prod_{\mathsf{Comp}(\mathcal{K})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ and $\chi \in \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$ such that π is determined *by ψ and χ, in the sense that if $U \in \mathsf{Comp}(\mathcal{J})$ is the connected component of $j \in \mathcal{J}$ and $V \in \mathsf{Comp}(\mathcal{K})$ is the connected component of $k \in \mathcal{K}$, then*

$$\pi_{j,k} = (\psi_V(j), \chi_U(k)) : (j,k) \xrightarrow{\sim} (j,k).$$

In particular, if $\mathcal{J}$ *and* $\mathcal{K}$ *are connected, then* $\pi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}})$ *iff there are (uniquely determined) elements* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ *and* $\chi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}})$ *such that* π *is* determined by ψ *and* χ, *in the sense that*

$$\pi_{j,k} = (\psi(j), \chi(k)) : (j,k) \xrightarrow{\sim} (j,k)$$

for all $j \in \mathcal{J}, k \in \mathcal{K}$.

Proof: Applying the definitions, we must show that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}}) \cong \prod_{\mathsf{Comp}(\mathcal{K})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \times \prod_{\mathsf{Comp}(\mathcal{J})} \mathsf{Aut}(\mathsf{Id}_{\mathcal{K}}).$$

Let Σ be the signature consisting of just one sort and no function symbols. Fix an arbitrary $M \in \mathsf{P}(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}\mathsf{mod}$. Then by Lemma 5.3.8, it suffices to show

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}}).$$

First, note that we have (by Proposition 5.1.8 and general category theory)

$$\begin{aligned}
\mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}\times\mathcal{K}}\mathsf{mod} &\cong \mathsf{P}\mathcal{E}_\Sigma\mathsf{mod}^{\mathcal{J}\times\mathcal{K}} \\
&= \mathsf{Sets}^{\mathcal{J}\times\mathcal{K}} \\
&\cong (\mathsf{Sets}^{\mathcal{J}})^{\mathcal{K}} \\
&= (\mathsf{P}\mathcal{E}_\Sigma\mathsf{mod}^{\mathcal{J}})^{\mathcal{K}} \\
&\cong (\mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}}\mathsf{mod})^{\mathcal{K}} \\
&\cong \mathsf{P}(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}\mathsf{mod}.
\end{aligned}$$

Let $M^* \in \mathsf{P}\mathcal{E}_\Sigma^{\mathcal{J}\times\mathcal{K}}\mathsf{mod}$ correspond to M via the above sequence of isomorphisms. Then we have

$$\mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}\times\mathcal{K}}}(M^*) \cong \mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \in \mathsf{Group}.$$

Since $\mathcal{E}_\Sigma$ has trivial isotropy (by Proposition 3.1.4) and every set clearly has non-trivial diagram theory, it follows by Corollary 5.3.6 that

$$\mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}\times\mathcal{K}}}(M^*) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}}).$$

So we obtain

$$\mathcal{Z}_{(\mathcal{E}_\Sigma^{\mathcal{J}})^{\mathcal{K}}}(M) \cong \mathcal{Z}_{\mathcal{E}_\Sigma^{\mathcal{J}\times\mathcal{K}}}(M^*) \cong \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}\times\mathcal{K}}),$$

as desired. ∎

Chapter 6

Further Directions

In this closing chapter, we will discuss some open questions relating to this thesis that could be pursued in further research. For other open questions not discussed here, see [13, Section 5].

- One very general open question concerns whether it is possible to devise conditions on the *axioms* of a quasi-equational theory to guarantee that a theory will have either trivial or non-trivial isotropy. In other words, one might wonder whether there is a certain condition P on the axioms of a quasi-equational theory with the property that a quasi-equational theory $\mathbb{T}$ satisfies condition P iff $\mathbb{T}$ has trivial isotropy, and similarly for non-trivial isotropy. To illustrate the potential difficulties involved in devising such a condition, consider the algebraic theory $\mathbb{T}$ of (totally defined) monoids. As we mentioned in the last section of Chapter 4, the finitely generated free monoids all have trivial isotropy, since no free monoid has any non-trivial invertible elements. However, since there *are* (non-free) finitely presented monoids with non-trivial invertible elements, it follows that there are finitely presented monoids with *non-trivial* isotropy. This suggests that the problem of determining whether a theory has trivial isotropy from just its signature and axioms may be algorithmically undecidable in general, because the equations that hold in a free monoid are just those that can be deduced from the monoid axioms.

  Furthermore, while the theory of monoids has non-trivial isotropy, the theory of *commutative* monoids has *trivial* isotropy. However, it is not obvious why simply adding the axiom $\top \vdash^{x,y} x \cdot y = y \cdot x$ to the theory of monoids would cause the isotropy of the resulting theory to become trivial. This again suggests that it is very difficult to determine from the axioms of a theory *alone* whether that theory will have trivial or non-trivial isotropy.

  In general, one might suspect that in order for a theory to have non-trivial isotropy, the theory must encode some notion of 'invertibility'; however, this vaguely stated condition is certainly not *sufficient* for a theory to have non-trivial isotropy. For example, the theories of groupoids and commutative monoids encode notions of 'invertibility' but have *trivial* isotropy. It is still an open question whether a theory *must* encode some notion of 'invertibility' in order to have *non-trivial* isotropy, because all of the example theories with non-trivial isotropy that we have considered *do* encode some notion of 'invertibility' (e.g. monoids, (abelian) groups, sets with a bijection/involution, strict monoidal categories, (certain) functor categories etc.).

- Given quasi-equational theories $\mathbb{T}_1$ and $\mathbb{T}_2$ over disjoint signatures Σ_1 and Σ_2, one can form a quasi-equational theory $\mathbb{T}$ that axiomatizes models of $\mathbb{T}_1$ in the cartesian category $\mathsf{P}\mathbb{T}_2\mathsf{mod}$. For example, in Chapter 3 we considered the theory $\mathbb{T}_{\mathsf{Str}}$ of strict monoidal categories, which axiomatizes models of $\mathbb{T}_{\mathsf{Mon}}$ in $\mathsf{P}\mathbb{T}_{\mathsf{Cat}}\mathsf{mod} = \mathsf{Cat}$ (since a strict monoidal category can be regarded as a monoid object in Cat). There was certainly some overlap between the arguments used to characterize the isotropy group of $\mathbb{T}_{\mathsf{Str}}$, and the arguments used to characterize the isotropy groups of $\mathbb{T}_{\mathsf{Mon}}$ and $\mathbb{T}_{\mathsf{Cat}}$. So it would be interesting to try to characterize the isotropy group of the theory $\mathbb{T}$ (axiomatizing models of $\mathbb{T}_1$ in $\mathsf{P}\mathbb{T}_2\mathsf{mod}$) in terms of the isotropy groups of $\mathbb{T}_1$ and $\mathbb{T}_2$.

- In Chapter 5, we introduced the notion of a quasi-equational theory $\mathbb{T}$ having single-indeterminate isotropy and single-sorted non-total operations, in order to completely characterize the isotropy group of $\mathbb{T}^{\mathcal{J}}$ (for any small index category $\mathcal{J}$). First, it would be interesting to try to determine what categorical significance or interpretation the notion of single-indeterminate isotropy has (if any), because its definition is expressed completely in terms of logic.

  Second, it would obviously be interesting to try to extend/modify the results of Chapter 5 in order to completely characterize the isotropy group of $\mathbb{T}^{\mathcal{J}}$ for a given *arbitrary* quasi-equational theory $\mathbb{T}$ and arbitrary small index category $\mathcal{J}$.

- Of course, one could also extend the work in this thesis by computing the isotropy groups of specific quasi-equational theories that we did not consider here.

- Finally, one could step beyond the work in this thesis by trying to characterize and compute the isotropy groups of theories with more logical complexity considered in Chapter 1, e.g. regular theories, coherent theories (which extend regular theories by allowing for finite disjunction), and full geometric theories. However, these theories generally do not have initial model constructions, which we relied on heavily to characterize the isotropy groups of quasi-equational (i.e. cartesian) theories, so the methods developed in the present thesis do not (obviously) apply to such classes of theories.

Appendix A

Chapter 2 Proofs

Lemma (2.2.4)**.** *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$.*

1. *If $h : M \to N$ is any morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, then there is a partial $\Sigma(M)$-structure N^h such that $N^h|_\Sigma = N$ and $N^h \models \mathbb{T}(M)$, with the following description:*

    - *For any sort $B \in \Sigma(M)_{\mathsf{Sort}} = \Sigma_{\mathsf{Sort}}$,*

    $$N^h_B := N_B.$$

    - *For any function symbol $f \in \Sigma$,*

    $$f^{N^h} := f^N.$$

    - *For any $A \in \Sigma_{\mathsf{Sort}}$ and $a \in M_A$,*

    $$\left(c^M_{A,a}\right)^{N^h} := h_A(a) \in N_A = N^h_A.$$

2. *If N' is a partial $\Sigma(M)$-structure with $N' \models \mathbb{T}(M)$, then there is a unique Σ-morphism $h : M \to N'|_\Sigma$ such that $(N'|_\Sigma)^h = N'$.*

Proof: To prove (1), let $h : M \to N$ be any Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$. If N^h is the partial $\Sigma(M)$-structure described in the statement of (1), then it is obvious that $N^h|_\Sigma = N$, so it remains to show that $N^h \models \mathbb{T}(M)$. Since $N^h|_\Sigma = N$ and $N \models \mathbb{T}$, it follows that $N^h \models \mathbb{T}$. So it remains to show that N^h satisfies the axioms of $\mathbb{T}(M) \setminus \mathbb{T}$.

- Let $A \in \Sigma_{\mathsf{Sort}}$ and $a \in M_A$. We must show that N^h satisfies the axiom $\top \vdash c^M_{A,a} \downarrow$ of $\mathbb{T}(M)$, i.e. we must show that $c^M_{A,a}$ is defined in N^h; but this is true by definition of N^h.

- Let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol of Σ, and let $a_1 \in M_{A_1}, \ldots, a_n \in M_{A_n}$, $a \in M_A$ with $(a_1, \ldots, a_n) \in \mathsf{dom}(f^M)$ and $f^M(a_1, \ldots, a_n) = a$. We must show that N^h satisfies the axiom $\top \vdash f\left(c^M_{a_1}, \ldots, c^M_{a_n}\right) = c^M_a$ of $\mathbb{T}(M)$, i.e. we must show that
$$\left(\left(c^M_{a_1}\right)^{N^h}, \ldots, \left(c^M_{a_n}\right)^{N^h}\right) \in \mathsf{dom}\left(f^{N^h}\right)$$
and
$$f^{N^h}\left(\left(c^M_{a_1}\right)^{N^h}, \ldots, \left(c^M_{a_n}\right)^{N^h}\right) = \left(c^M_a\right)^{N^h}.$$
The first claim easily follows by definition of N^h and the fact that h is a Σ-morphism. For the second claim, we have
$$\begin{aligned} f^{N^h}\left(\left(c^M_{a_1}\right)^{N^h}, \ldots, \left(c^M_{a_n}\right)^{N^h}\right) &= f^N\left(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)\right) \\ &= h_A(f^M(a_1, \ldots, a_n)) \\ &= h_A(a) \\ &= \left(c^M_a\right)^{N^h}, \end{aligned}$$
as desired.

This completes the proof that $N^h \models \mathbb{T}(M)$, which proves (1).

To prove (2), let N' be a $\Sigma(M)$-structure with $N' \models \mathbb{T}(M)$. We must show that there is a unique Σ-morphism $h : M \to N'|_\Sigma$ with $(N'|_\Sigma)^h = N'$. Let $A \in \Sigma$ be any sort; we must define a total function $h_A : M_A \to (N'|_\Sigma)_A = N'_A$. So let $a \in M_A$. Then $c^M_{A,a}$ is a constant symbol of $\Sigma(M)$, and since $N' \models \mathbb{T}(M)$, it follows that $N' \models c^M_{A,a} \downarrow$. So we set
$$h_A(a) := \left(c^M_{A,a}\right)^{N'} \in N'_A.$$
Now we show that $h : M \to N'|_\Sigma$ is a Σ-morphism. So let $f : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ, and let $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$ be an element of $\mathsf{dom}(f^M)$. We must show that
$$(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in \mathsf{dom}\left(f^{N'}\right)$$
and
$$h_A(f^M(a_1, \ldots, a_n)) = f^{N'}\left(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)\right) \in N'_A.$$
Since $N' \models \mathbb{T}(M)$, we have $N' \models f\left(c^M_{a_1}, \ldots, c^M_{a_n}\right) = c^M_{f^M(a_1, \ldots, a_n)}$, i.e.
$$f^{N'}\left(\left(c^M_{a_1}\right)^{N'}, \ldots, \left(c^M_{a_n}\right)^{N'}\right) = \left(c^M_{f^M(a_1, \ldots, a_n)}\right)^{N'}.$$

This implies that

$$(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) = \left(\left(c^M_{a_1}\right)^{N'}, \ldots, \left(c^M_{a_n}\right)^{N'}\right) \in \mathsf{dom}\left(f^{N'}\right)$$

and

$$\begin{aligned} f^{N'}(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) &= f^{N'}\left(\left(c^M_{a_1}\right)^{N'}, \ldots, \left(c^M_{a_n}\right)^{N'}\right) \\ &= \left(c^M_{f^M(a_1,\ldots,a_n)}\right)^{N'} \\ &= h_A(f^M(a_1, \ldots, a_n)), \end{aligned}$$

as desired. This proves that $h : M \to N'|_\Sigma$ is a Σ-morphism. Also, it is obvious from the definition of h that $(N'|_\Sigma)^h = N'$, as desired.

Lastly, we must prove that $h : M \to N'|_\Sigma$ is the unique Σ-morphism with $(N'|_\Sigma)^h = N'$. So let $k : M \to N'|_\Sigma$ be any Σ-morphism with $(N'|_\Sigma)^k = N'$; we must show that $h = k$. So let $A \in \Sigma_{\mathsf{Sort}}$ and $a \in M_A$; then

$$h_A(a) = \left(c^M_a\right)^{N'} = \left(c^M_a\right)^{(N'|_\Sigma)^k} = k_A(a),$$

as desired (the last equality follows by definition of $(N'|_\Sigma)^k$). This completes the proof of (2). ∎

Lemma (2.2.9)**.** *If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then the family of total functions*

$$\eta^{M,A} := \left(\eta^{M,A}_C : M_C \to M\langle \mathsf{x}_A\rangle_C\right)_{C\in\Sigma}$$

is a Σ-morphism from M to $M\langle \mathsf{x}_A\rangle$. If $M_A \neq \emptyset$, then $\eta^{M,A}$ is moreover sortwise injective.

Proof: Let $f : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ, let $b_i \in M_{B_i}$ for all $1 \leq i \leq n$, and suppose that $(b_1, \ldots, b_n) \in \mathsf{dom}(f^M)$. We must show that

$$\left(\eta^{M,A}_{B_1}(b_1), \ldots, \eta^{M,A}_{B_n}(b_n)\right) = ([c_{b_1}], \ldots, [c_{b_n}]) \in \mathsf{dom}\left(f^{M\langle \mathsf{x}_A\rangle}\right),$$

and that

$$\eta^{M,A}_B(f^M(b_1, \ldots, b_n)) = f^{M\langle \mathsf{x}_A\rangle}([c_{b_1}], \ldots, [c_{b_n}]) \in M\langle \mathsf{x}_A\rangle_B.$$

To prove the first claim, we must show that

$$\mathbb{T}(M, \mathsf{x}_A) \vdash f(c_{b_1}, \ldots, c_{b_n}) \downarrow.$$

But this follows by the rules of partial Horn logic and the fact that

$$\mathbb{T}(M, \mathsf{x}_A) \vdash f\left(c_{b_1}, \ldots, c_{b_n}\right) = c_{f^M(b_1,\ldots,b_n)}.$$

For the second claim, we have

$$\begin{aligned} \eta_B^{M,A}(f^M(b_1, \ldots, b_n)) &:= \left[c_{f^M(b_1,\ldots,b_n)}\right]_B \\ &= \left[f\left(c_{b_1}, \ldots, c_{b_n}\right)\right]_B \\ &= f^{M\langle \mathsf{x}_A\rangle}\left(\left[c_{b_1}\right], \ldots, \left[c_{b_n}\right]\right), \end{aligned}$$

as desired. The second equality holds because $\mathbb{T}(M, \mathsf{x}_A)$ proves the above equation. This completes the proof that $\eta^{M,A} : M \to M\langle \mathsf{x}_A\rangle$ is a Σ-morphism.

To show that $\eta^{M,A}$ is sortwise injective under the assumption that $M_A \neq \emptyset$, let $C \in \Sigma_{\mathsf{Sort}}$, and let us show that $\eta_C^{M,A} = \eta_C$ is injective. So let $a, b \in M_C$, and suppose that

$$[c_a] = \eta_C(a) = \eta_C(b) = [c_b] \in M\langle \mathsf{x}_A\rangle_C.$$

Then $\mathbb{T}(M, \mathsf{x}_A) \vdash c_a = c_b$, and hence $\mathbb{T}(M)$ proves the sequent $\top \vdash^{y:A} c_a = c_b$ by the theorem on constants (Remark 1.3.17), where $y : A$ is a variable. Since $M_A \neq \emptyset$, there is some $s \in M_A$, and hence $\mathbb{T}(M, \mathsf{x}_A)$ proves the sequent $\top \vdash c_s \downarrow$. By the partial term substitution rule of partial Horn logic (see [19]), it follows from $\top \vdash^{y:A} c_a = c_b$ being provable in $\mathbb{T}(M, \mathsf{x}_A)$ that the sequent $c_s \downarrow \vdash c_a = c_b$ is provable in $\mathbb{T}(M, \mathsf{x}_A)$. By the cut rule for partial Horn logic, it then follows that the sequent $\top \vdash c_a = c_b$ is provable in $\mathbb{T}(M, \mathsf{x}_A)$. Since $\widehat{M} \models \mathbb{T}(M)$, we then obtain $\widehat{M} \models c_a = c_b$ (by soundness of partial Horn logic) and hence $a = c_a^{\widehat{M}} = c_b^{\widehat{M}} = b$, as desired. ∎

Proposition (2.2.10). *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$. For any $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, any Σ-morphism $h : M \to N$, and any $a \in N_A$, there is a unique Σ-morphism*

$$h^a : M\langle \mathsf{x}_A\rangle \to N$$

such that $h^a \circ \eta^{M,A} = h$ and $h_A^a([\mathsf{x}_A]) = a \in N_A$.

Proof: Let $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $h : M \to N$ be any Σ-morphism, and let $a \in N_A$. By definition, we have $M\langle \mathsf{x}_A\rangle := \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))|_\Sigma$, the Σ-reduct of the $\Sigma(M, \mathsf{x}_A)$-structure $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))$. From Chapter 1, we also know that $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))$ is the initial model of $\mathbb{T}(M, \mathsf{x}_A)$ in the category of all partial $\Sigma(M, \mathsf{x}_A)$-structures. By Lemma 2.2.4, we associate to N and h a $\Sigma(M)$-structure N^h such that $N^h|_\Sigma = N$ and $N^h \models \mathbb{T}(M)$. So $a \in N_A = N_A^h$, and hence (N^h, a) is a $\Sigma(M, \mathsf{x}_A)$-structure. Moreover, we have $(N^h, a) \models \mathbb{T}(M, \mathsf{x}_A)$ because $N^h \models \mathbb{T}(M)$ and $(N^h, a) \models \mathsf{x}_A \downarrow$. So there is a unique $\Sigma(M, \mathsf{x}_A)$-morphism $k : \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A)) \to (N^h, a)$.

Now we define $h^a := k$. This makes sense, because $\Sigma(M, \mathsf{x}_A)_{\mathsf{Sort}} = \Sigma_{\mathsf{Sort}}$. So for any sort $B \in \Sigma$, we have

$$h^a_B := k_B : M\langle \mathsf{x}_A \rangle_B = \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))_B \to (N^h, a)_B = N_B.$$

Since k is a $\Sigma(M, \mathsf{x}_A)$-morphism and $\Sigma \subseteq \Sigma(M, \mathsf{x}_A)$, it follows that h^a is a Σ-morphism. Also, because k preserves the interpretation of the constant x_A (which is defined in both $\mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A))$ and (N^h, a)), it follows that

$$h^a_A([\mathsf{x}_A]) = k_A([\mathsf{x}_A]) = k_A\left(\mathsf{x}_A^{\mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A))}\right) = \mathsf{x}_A^{(N^h,a)} = a,$$

as desired. To show that $h^a \circ \eta^{M,A} = h$, let B be any sort, and let $b \in M_B$. We must show that

$$h^a_B\left(\eta^{M,A}_B(b)\right) = h_B(b) \in N_B.$$

We have

$$\begin{aligned} h^a_B\left(\eta^{M,A}_B(b)\right) &= h^a_B\left([c_b]\right) \\ &= k_B\left([c_b]\right) \\ &= k_B\left(\left(c^M_b\right)^{\mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A))}\right) \\ &= \left(c^M_b\right)^{(N^h,a)} \\ &= \left(c^M_b\right)^{N^h} \\ &= h_B(b), \end{aligned}$$

as desired. The fourth equality follows because $k : \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A)) \to (N^h, a)$ is a $\Sigma(M)$-morphism, and the last equality follows by definition of the $\Sigma(M)$-structure N^h.

Lastly, we must show that $h^a := k$ is the unique Σ-morphism $M\langle \mathsf{x}_A \rangle \to N$ with the desired properties. So let $j : M\langle \mathsf{x}_A \rangle \to N$ be any Σ-morphism with $j \circ \eta^{M,A} = h$ and $j_A([\mathsf{x}_A]_A) = a \in N_A$. We must show that $j = h^a$, i.e. that $j = k$. By the uniqueness of k, it suffices to show that j is also a $\Sigma(M, \mathsf{x}_A)$-morphism $j : \mathsf{Free}(\mathbb{T}(M, \mathsf{x}_A)) \to (N^h, a)$. We already know that j is a Σ-morphism. Also, we have that j preserves the interpretation of the constant x_A, since

$$j_A\left(\mathsf{x}_A^{\mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A))}\right) = j_A([\mathsf{x}_A]) = a = \mathsf{x}_A^{(N^h,a)}.$$

So it remains to show for any $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$ that

$$j_B\left(\left(c^M_s\right)^{\mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A))}\right) = \left(c^M_s\right)^{(N^h,a)}.$$

We have

$$\begin{aligned} j_B\left(\left(c_s^M\right)^{\mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A))}\right) &= j_B\left([c_s]\right) \\ &= j_B\left(\eta_B^{M,A}(s)\right) \\ &= h_B(s) \\ &= \left(c_s^M\right)^{N^h} \\ &= \left(c_s^M\right)^{(N^h,a)}, \end{aligned}$$

as desired. The penultimate equality holds by definition of the $\Sigma(M)$-structure N^h. This finishes the proof that $j : \mathsf{Free}(\mathbb{T}(M,\mathsf{x}_A)) \to (N^h, a)$ is a $\Sigma(M,\mathsf{x}_A)$-morphism, and proves that $j = k = h^a$. ∎

Lemma (2.2.18)**.** *Let $h : M \to N$ be a Σ-morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let $A \in \Sigma_{\mathsf{Sort}}$. The signature morphism $\rho_h^A : \Sigma(M,\mathsf{x}_A) \to \Sigma(N,\mathsf{x}_A)$ is a theory morphism from $\mathbb{T}(M,\mathsf{x}_A)$ to $\mathbb{T}(N,\mathsf{x}_A)$.*

Proof: If $\varphi \vdash^{\vec{x}} \psi$ is an axiom of $\mathbb{T}$, then its ρ_h^A-translation is just itself, because ρ_h^A is the identity on Σ. So then $\rho_h^A(\varphi) \vdash^{\rho_h^A(\vec{x})} \rho_h^A(\psi) \equiv \varphi \vdash^{\vec{x}} \psi$ is an axiom, and hence a theorem, of $\mathbb{T} \subseteq \mathbb{T}(N,\mathsf{x}_A)$.

If $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$, then the ρ_h^A-translation of the $\mathbb{T}(M,\mathsf{x}_A)$-axiom $\top \vdash c_s^M \downarrow$ is $\top \vdash c_{h_B(s)}^N \downarrow$. But this is an axiom and hence theorem of $\mathbb{T}(N,\mathsf{x}_A)$, since $h_B(s) \in N_B$.

Now let $f : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ with $s_i \in M_{B_i}$ for all $1 \leq i \leq n$, and suppose that $(s_1, \ldots, s_n) \in \mathsf{dom}(f^M)$. Then

$$\top \vdash f\left(c_{s_1}^M, \ldots, c_{s_n}^M\right) = c_{f^M(s_1,\ldots,s_n)}^M$$

is an axiom of $\mathbb{T}(M,\mathsf{x}_A)$, whose ρ_h^A-translation is

$$\top \vdash f\left(c_{h_{B_1}(s_1)}^N, \ldots, c_{h_{B_n}(s_n)}^N\right) = c_{h_B(f^M(s_1,\ldots,s_n))}^N.$$

Since $h : M \to N$ is a Σ-morphism, it follows that

$$h_B(f^M(s_1, \ldots, s_n)) = f^N(h_{B_1}(s_1), \ldots, h_{B_n}(s_n)).$$

But

$$\top \vdash f\left(c_{h_{B_1}(s_1)}^N, \ldots, c_{h_{B_n}(s_n)}^N\right) = c_{f^N(h_{B_1}(s_1),\ldots,h_{B_n}(s_n))}^N$$

is an axiom and hence theorem of $\mathbb{T}(N,\mathsf{x}_A)$, as required. Finally, the ρ_h^A-translation of the $\mathbb{T}(M,\mathsf{x}_A)$-axiom $\top \vdash \mathsf{x}_A \downarrow$ is just itself, which is also an axiom and hence theorem of $\mathbb{T}(N,\mathsf{x}_A)$. This shows that the ρ_h^A-translation of every $\mathbb{T}(M,\mathsf{x}_A)$-axiom is a $\mathbb{T}(N,\mathsf{x}_A)$-theorem, which completes the proof. ∎

Lemma (2.2.24)**.** *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* $B \in \Sigma_{\mathsf{Sort}}$.

1. *Let* $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash s = t$*, and let* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$*. Then*

$$\mathbb{T}(M, \mathsf{x}_B) \vdash s[u/\mathsf{x}_B] = t[u/\mathsf{x}_B].$$

2. *Conversely, if* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$ *and* $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ *with* $\mathbb{T}(M, \mathsf{x}_B) \vdash s = t$*, then*

$$\mathbb{T}(M, \mathsf{x}_B) \vdash u[s/\mathsf{x}_B] = u[t/\mathsf{x}_B].$$

3. *In particular, if* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ *and* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ *and* $\mathbb{T}(M, \mathsf{x}_B) \vdash t \downarrow \wedge u \downarrow$*, then* $\mathbb{T}(M, \mathsf{x}_B) \vdash t[u/\mathsf{x}_B] \downarrow$.

Proof: To prove (1), let $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))$ with $\mathbb{T}(M, \mathsf{x}_B) \vdash s = t$, and let $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B$ with $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$. Since

$$\mathbb{T}(M, \mathsf{x}_B) := \mathbb{T}(M) \cup \{\top \vdash \mathsf{x}_B \downarrow\},$$

it follows by the theorem on constants (Remark 1.3.17) that

$$\mathbb{T}(M) \vdash^{y:B} s[y/\mathsf{x}_B] = t[y/\mathsf{x}_B],$$

where $y : B$ is a variable. For the same reason, the assumption that $\mathbb{T}(M, \mathsf{x}_B) \vdash u \downarrow$, i.e. that $\mathbb{T}(M, \mathsf{x}_B) \vdash u = u$, implies that

$$\mathbb{T}(M) \vdash^{y:B} u[y/\mathsf{x}_B] \downarrow .$$

Then by the partial term substitution rule of partial Horn logic, we obtain

$$\mathbb{T}(M) \vdash^{y:B} s\left[u[y/\mathsf{x}_B]/\mathsf{x}_B\right] = t\left[u[y/\mathsf{x}_B]/\mathsf{x}_B\right].$$

Then by the theorem on constants again, we obtain

$$\mathbb{T}(M, \mathsf{x}_B) \vdash s[u/\mathsf{x}_B] = t[u/\mathsf{x}_B],$$

as desired. The proof of (2) is analogous. ∎

Lemma (2.2.25)**.** *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* $B \in \Sigma_{\mathsf{Sort}}$*. The following data give a well-defined (ordinary) monoid structure on* $M\langle \mathsf{x}_B \rangle_B$*:*

- *For any* $[s], [t] \in M\langle \mathsf{x}_B \rangle_B$*, we set*

$$[s] \cdot [t] := [s \cdot t] = [s[t/\mathsf{x}_B]] \in M\langle \mathsf{x}_B \rangle_B.$$

- *The unit is* $[\mathsf{x}_B] \in M\langle \mathsf{x}_B \rangle_B$.

Proof: First, we must show that the definition of the monoid multiplication on $M\langle \mathsf{x}_B \rangle_B$ is well-defined. If $[s], [t] \in M\langle \mathsf{x}_B \rangle_B$, then we know that $\mathbb{T}(M, \mathsf{x}_B) \vdash s \downarrow \wedge\, t \downarrow$. Then by Lemma 2.2.24, it follows that $\mathbb{T}(M, \mathsf{x}_B) \vdash s[t/\mathsf{x}_B] \downarrow$, so that

$$[s] \cdot [t] := [s \cdot t] = [s[t/\mathsf{x}_B]] \in M\langle \mathsf{x}_B \rangle_B,$$

as required.

Also, if $[s], [s'], [t], [t'] \in M\langle \mathsf{x}_B \rangle_B$ with $[s] = [s']$ and $[t] = [t']$, then we have $\mathbb{T}(M, \mathsf{x}_B) \vdash s = s'$ and $\mathbb{T}(M, \mathsf{x}_B) \vdash t = t'$. Since $\mathbb{T}(M, \mathsf{x}_B) \vdash t \downarrow$, it follows by Lemma 2.2.24 that $\mathbb{T}(M, \mathsf{x}_B) \vdash s[t/\mathsf{x}_B] = s'[t/\mathsf{x}_B]$. And then since $\mathbb{T}(M, \mathsf{x}_B) \vdash t = t'$, we obtain that $\mathbb{T}(M, \mathsf{x}_B) \vdash s'[t/\mathsf{x}_B] = s'[t'/\mathsf{x}_B]$ by Lemma 2.2.24 again, so that $\mathbb{T}(M, \mathsf{x}_B) \vdash s[t/\mathsf{x}_B] = s'[t'/\mathsf{x}_B]$, i.e. $\mathbb{T}(M, \mathsf{x}_B) \vdash s \cdot t = s' \cdot t'$. So then

$$[s] \cdot [t] = [s \cdot t] = [s' \cdot t'] = [s'] \cdot [t'],$$

as required.

So the monoid structure on $M\langle \mathsf{x}_B \rangle_B$ is well-defined, and the fact that $(M\langle \mathsf{x}_B \rangle, \cdot, [\mathsf{x}_B])$ *is* a monoid follows easily from the fact (cf. Remark 2.2.23) that $(\mathsf{Term}^c(\Sigma(M, \mathsf{x}_B))_B, \cdot, \mathsf{x}_B)$ is a monoid. ▮

Lemma (2.2.29)**.** *Let* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, let* $B, C \in \Sigma_{\mathsf{Sort}}$*, and let* $s \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_B$ *and* $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ *with* $\mathbb{T}(M, \mathsf{x}_C) \vdash s \downarrow \wedge\, t \downarrow$*. Then (cf. Definition 2.2.11) we have total functions*

$$s^* : M_C \to M_B \text{ and } t^* : M_C \to M_C,$$

as well as the total function

$$(s \cdot t)^* = (s[t/\mathsf{x}_C])^* : M_C \to M_B$$

(since $\mathbb{T}(M, \mathsf{x}_C) \vdash s[t/\mathsf{x}_C] \downarrow$*, by Lemma 2.2.24). Then*

$$(s \cdot t)^* = s^* \circ t^* : M_C \to M_B.$$

Proof: Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $C \in \Sigma_{\mathsf{Sort}}$. Fix an arbitrary $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow$. We prove the desired claim by induction on terms $s \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash s \downarrow$.

- If $s \equiv \mathsf{x}_C : C$, then $\mathbb{T}(M, \mathsf{x}_C) \vdash \mathsf{x}_C \downarrow$, and for any $c \in M_C$ we have

$$(s \cdot t)^*(c) = (\mathsf{x}_C \cdot t)^*(c) = t^*(c) = \mathsf{x}_C^{(\widehat{M}, t^*(c))} = \mathsf{x}_C^*(t^*(c)) = s^*(t^*(c)).$$

- If $s : B$ is any other constant symbol of $\Sigma(M, \mathsf{x}_C)$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash s \downarrow$, then for any $c \in M_C$ we have

  $$(s \cdot t)^*(c) = s^*(c) = s^{(\widehat{M},c)} = s^{\widehat{M}} = s^{(\widehat{M},t^*(c))} = s^*(t^*(c));$$

  the third and fourth equalities hold because $s \not\equiv \mathsf{x}_C$.

- Suppose that $s \equiv f(s_1, \ldots, s_n)$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ with $n \geq 1$ and terms $s_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_{B_i}$ for all $1 \leq i \leq n$, and suppose that $\mathbb{T}(M, \mathsf{x}_C) \vdash f(s_1, \ldots, s_n) \downarrow$. Then by the rules of partial Horn logic, it follows that $\mathbb{T}(M, \mathsf{x}_C) \vdash s_i \downarrow$ for all $1 \leq i \leq n$. So for each $1 \leq i \leq n$ we have a total function

  $$s_i^* : M_C \to M_{B_i}$$

  with the property that

  $$(s_i \cdot t)^* = s_i^* \circ t^* : M_C \to M_{B_i}$$

  by the induction hypothesis. Then for any $c \in M_C$ we have

  $$\begin{aligned} (s \cdot t)^*(c) &= (f(s_1, \ldots, s_n) \cdot t)^*(c) \\ &= f(s_1 \cdot t, \ldots, s_n \cdot t)^*(c) \\ &= f(s_1 \cdot t, \ldots, s_n \cdot t)^{(\widehat{M},c)} \\ &= f^{(\widehat{M},c)}\left((s_1 \cdot t)^{(\widehat{M},c)}, \ldots, (s_n \cdot t)^{(\widehat{M},c)}\right) \\ &= f^M((s_1 \cdot t)^*(c), \ldots, (s_n \cdot t)^*(c)) \\ &= f^M(s_1^*(t^*(c)), \ldots, s_n^*(t^*(c))) \\ &= f^{(\widehat{M},t^*(c))}\left(s_1^{(\widehat{M},t^*(c))}, \ldots, s_n^{(\widehat{M},t^*(c))}\right) \\ &= f(s_1, \ldots, s_n)^{(\widehat{M},t^*(c))} \\ &= f(s_1, \ldots, s_n)^*(t^*(c)) \\ &= s^*(t^*(c)). \end{aligned}$$

∎

Lemma (2.2.31). *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $A, B \in \Sigma_{\mathsf{Sort}}$, and let $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$. Let $\eta = \eta^{M,A} : M \to M\langle \mathsf{x}_A \rangle$ be the canonical morphism from Definition 2.2.8. Then*

$$\rho_\eta^A(t) \in \mathsf{Term}^c(\Sigma(M\langle \mathsf{x}_A \rangle, \mathsf{x}_A))_B.$$

1. *Let $\widehat{M\langle \mathsf{x}_A\rangle}$ be the $\Sigma(M\langle \mathsf{x}_A\rangle)$-structure expanding the Σ-structure $M\langle \mathsf{x}_A\rangle$ (cf. Remark 2.2.2). Let $[u] = [u] \in M\langle \mathsf{x}_A\rangle_A$ be arbitrary. Then $c_{[u]} : A$ is a constant symbol of $\Sigma(M\langle \mathsf{x}_A\rangle)$, and $\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]$ is a closed $\Sigma(M\langle \mathsf{x}_A\rangle)$-term of sort B. Then*

   $$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} \in \widehat{M\langle \mathsf{x}_A\rangle}_B = M\langle \mathsf{x}_A\rangle_B \text{ is defined}$$

   and

   $$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} = [t[u/\mathsf{x}_A]] \in M\langle \mathsf{x}_A\rangle_B.$$

2. *By Lemma 2.2.19, we have $\mathbb{T}(M\langle \mathsf{x}_A\rangle, \mathsf{x}_A) \vdash \rho^A_\eta(t)\downarrow$. Then by Definition 2.2.11, we have a total function*

   $$\rho^A_\eta(t)^* : M\langle \mathsf{x}_A\rangle_A \to M\langle \mathsf{x}_A\rangle_B.$$

   Then for any $[u]_A \in M\langle \mathsf{x}_A\rangle_A$, we have

   $$\rho^A_\eta(t)^*([u]) = [t[u/\mathsf{x}_A]] \in M\langle \mathsf{x}_A\rangle_B.$$

Proof: To prove (1), we first show that (under the given hypotheses)

$$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} \in \widehat{M\langle \mathsf{x}_A\rangle}_B = M\langle \mathsf{x}_A\rangle_B$$

is defined. By the hypotheses and Lemma 2.2.19, we have

$$\mathbb{T}(M\langle \mathsf{x}_A\rangle, \mathsf{x}_A) \vdash \rho^A_\eta(t)\downarrow .$$

By definition of $\mathbb{T}(M\langle \mathsf{x}_A\rangle)$, we also know that $\mathbb{T}(M\langle \mathsf{x}_A\rangle) \vdash c_{[u]}\downarrow$, and thus

$$\mathbb{T}(M\langle \mathsf{x}_A\rangle, \mathsf{x}_A) \vdash c_{[u]}\downarrow .$$

Then by Lemma 2.2.24, it follows that

$$\mathbb{T}(M\langle \mathsf{x}_A\rangle, \mathsf{x}_A) \vdash \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]\downarrow,$$

and hence (by the theorem on constants in Remark 1.3.17) $\mathbb{T}(M\langle \mathsf{x}_A\rangle)$ proves the sequent

$$\top \vdash^{y:A} \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]\downarrow,$$

with $y : A$ a variable. By the partial term substitution rule for partial Horn logic, we then have that $\mathbb{T}(M\langle \mathsf{x}_A\rangle)$ proves the sequent $c_{[u]}\downarrow \vdash \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]\downarrow$, and hence by the cut rule for partial Horn logic, we obtain that $\mathbb{T}(M\langle \mathsf{x}_A\rangle)$ proves the sequent

$$\top \vdash \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]\downarrow .$$

Now $\widehat{M\langle \mathsf{x}_A\rangle}$ is a model of $\mathbb{T}(M\langle \mathsf{x}_A\rangle)$, and hence by soundness of partial Horn logic it follows that

$$\widehat{M\langle \mathsf{x}_A\rangle} \models \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]\downarrow,$$

so that

$$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} \in \widehat{M\langle \mathsf{x}_A\rangle}_B = M\langle \mathsf{x}_A\rangle_B$$

is defined, as desired.

Now, fixing $[u] \in M\langle \mathsf{x}_A\rangle_A$, we prove by induction on terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t\downarrow$ that

$$\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} = [t[u/\mathsf{x}_A]].$$

- If $t \equiv \mathsf{x}_A$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash \mathsf{x}_A\downarrow$ and

$$\begin{aligned}\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} &= \rho^A_\eta(\mathsf{x}_A)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= \mathsf{x}_A\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= \left(c^{M\langle \mathsf{x}_A\rangle}_{[u]}\right)^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= [u]\\ &= [\mathsf{x}_A[u/\mathsf{x}_A]]\\ &= [t[u/\mathsf{x}_A]].\end{aligned}$$

- If $t \equiv c^M_s$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash c^M_s\downarrow$ and

$$\begin{aligned}\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} &= \rho^A_\eta\left(c^M_s\right)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= c^{M\langle \mathsf{x}_A\rangle}_{\eta_B(s)}\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= \left(c^{M\langle \mathsf{x}_A\rangle}_{B,\eta_B(s)}\right)^{\widehat{M\langle \mathsf{x}_A\rangle}}\\ &= \eta_B(s)\\ &= \left[c^M_s\right]\\ &= \left[c^M_s[u/\mathsf{x}_A]\right]\\ &= [t[u/\mathsf{x}_A]].\end{aligned}$$

- Suppose $t \equiv f(t_1, \ldots, t_n)$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ and some terms $t_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_{B_i}$ for all $1 \leq i \leq n$, and suppose that $\mathbb{T}(M, \mathsf{x}_A) \vdash f(t_1, \ldots, t_n)\downarrow$. Then by the rules of partial Horn logic, we have that $\mathbb{T}(M, \mathsf{x}_A) \vdash t_i\downarrow$ for all $1 \leq i \leq n$, and hence

$$\rho^A_\eta(t_i)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} = [t_i[u/\mathsf{x}_A]] \in \widehat{M\langle \mathsf{x}_A\rangle}_{B_i} = M\langle \mathsf{x}_A\rangle_{B_i}$$

for each $1 \leq i \leq n$ by the induction hypothesis. Then we have

$$\begin{aligned}
&\rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\
&= \rho^A_\eta(f(t_1,\ldots,t_n))\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\\
&= f\left(\rho^A_\eta(t_1)\left[c_{[u]}/\mathsf{x}_A\right],\ldots,\rho^A_\eta(t_n)\left[c_{[u]}/\mathsf{x}_A\right]\right)^{\widehat{M\langle \mathsf{x}_A\rangle}}\\
&= f^{\widehat{M\langle \mathsf{x}_A\rangle}}\left(\rho^A_\eta(t_1)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}},\ldots,\rho^A_\eta(t_n)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}}\right)\\
&= f^{M\langle \mathsf{x}_A\rangle}\left([t_1[u/\mathsf{x}_A]],\ldots,[t_n[u/\mathsf{x}_A]]\right)\\
&= [f\left(t_1[u/\mathsf{x}_A],\ldots,t_n[u/\mathsf{x}_A]\right)]\\
&= [f(t_1,\ldots,t_n)[u/\mathsf{x}_A]]\\
&= [t[u/\mathsf{x}_A]],
\end{aligned}$$

as desired (the fifth equality follows by definition of $f^{M\langle \mathsf{x}_A\rangle}$, cf. Remark 2.2.7).

To prove (2), we combine part (1) and Lemma 2.2.30. For any $[u] \in M\langle \mathsf{x}_A\rangle_A$, we have

$$\rho^A_\eta(t)^*([u]) = \rho^A_\eta(t)\left[c_{[u]}/\mathsf{x}_A\right]^{\widehat{M\langle \mathsf{x}_A\rangle}} = [t[u/\mathsf{x}_A]],$$

as desired. ∎

Lemma (2.2.33)**.** *The isomorphisms of* $\mathsf{P\Sigma Str}$ *are exactly the (sortwise) bijective* Σ*-morphisms that reflect definedness.*

Proof: First let $h = (h_A : M_A \to N_A)_{A\in\Sigma} : M \xrightarrow{\sim} N$ be any isomorphism in $\mathsf{P\Sigma Str}$. We must show that h is a sortwise bijective Σ-morphism that reflects definedness. Let $h^{-1} = (h_A^{-1} : N_A \to M_A)_{A\in\Sigma} : N \xrightarrow{\sim} M$ be the inverse of h. Since $h \circ h^{-1} = \mathsf{id}_N$ and $h^{-1} \circ h = \mathsf{id}_M$, it clearly follows that h is a sortwise bijective Σ-morphism. To show that h reflects definedness, let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol of Σ, let $(a_1,\ldots,a_n) \in M_{A_1} \times \ldots \times M_{A_n}$, and assume that $(h_{A_1}(a_1),\ldots,h_{A_n}(a_n)) \in \mathsf{dom}\left(f^N\right)$. We must show that $(a_1,\ldots,a_n) \in \mathsf{dom}\left(f^M\right)$. But we know that $h^{-1} : N \to M$ is a Σ-morphism, so the assumption implies that

$$\left(h^{-1}_{A_1}(h_{A_1}(a_1)),\ldots,h^{-1}_{A_n}(h_{A_n}(a_n))\right) = (a_1,\ldots,a_n) \in \mathsf{dom}\left(f^M\right),$$

as desired.

Now let $h = (h_A : M_A \to N_A)_{A\in\Sigma} : M \to N$ be a sortwise bijective Σ-morphism that reflects definedness; we must show that h is an isomorphism of P-Σ-Str, i.e. we must show that there is a Σ-morphism $h^{-1} : N \to M$ such that $h \circ h^{-1} = \mathsf{id}_N$ and

$h^{-1} \circ h = \mathsf{id}_M$. To define h^{-1}, let $A \in \Sigma$ be any sort. Since $h_A : M_A \to N_A$ is a bijection, we define $h_A^{-1} : N_A \to M_A$ to be the inverse function of h_A. Then we clearly have $h \circ h^{-1} = \mathsf{id}_N$ and $h^{-1} \circ h = \mathsf{id}_M$, so it remains to show that h^{-1} is a Σ-morphism $N \to M$.

To prove this, let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol of Σ, let $(a_1, \ldots, a_n) \in N_{A_1} \times \ldots \times N_{A_n}$, and suppose that $(a_1, \ldots, a_n) \in \mathsf{dom}\left(f^N\right)$. We must show that

$$\left(h_{A_1}^{-1}(a_1), \ldots, h_{A_n}^{-1}(a_n)\right) \in \mathsf{dom}\left(f^M\right)$$

and

$$h_A^{-1}\left(f^N(a_1, \ldots, a_n)\right) = f^M\left(h_{A_1}^{-1}(a_1), \ldots, h_{A_n}^{-1}(a_n)\right) \in M_A.$$

Since

$$(a_1, \ldots, a_n) = \left(h_{A_1}\left(h_{A_1}^{-1}(a_1)\right), \ldots, h_{A_n}\left(h_{A_n}^{-1}(a_n)\right)\right) \in \mathsf{dom}\left(f^N\right)$$

and h reflects definedness, it follows that $\left(h_{A_1}^{-1}(a_1), \ldots, h_{A_n}^{-1}(a_n)\right) \in \mathsf{dom}\left(f^M\right)$, as desired. Then because h is a Σ-morphism, we have

$$\begin{aligned} h_A^{-1}\left(f^N(a_1, \ldots, a_n)\right) &= h_A^{-1}\left(f^N\left(h_{A_1}\left(h_{A_1}^{-1}(a_1)\right), \ldots, h_{A_n}\left(h_{A_n}^{-1}(a_n)\right)\right)\right) \\ &= h_A^{-1}\left(h_A\left(f^M\left(h_{A_1}^{-1}(a_1), \ldots, h_{A_n}^{-1}(a_n)\right)\right)\right) \\ &= f^M\left(h_{A_1}^{-1}(a_1), \ldots, h_{A_n}^{-1}(a_n)\right), \end{aligned}$$

as desired. ∎

Lemma (2.2.34). *Let $h : M \xrightarrow{\sim} N$ be an isomorphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let*

$$(k_A : N_A \to M_A)_{A \in \Sigma_{\mathsf{Sort}}}$$

be a Σ_{Sort}-indexed family of total functions that is sortwise inverse to $h = (h_A : M_A \to N_A)_A$ (i.e. h_A and k_A are mutually inverse for every sort $A \in \Sigma$). Then

$$k := (k_A : N_A \to M_A)_A : N \to M$$

is an isomorphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ *(with inverse h).*

Proof: It suffices to show that the family of total functions $k := (k_A : N_A \to M_A)_A : N \to M$ is a Σ-morphism, because then it will be an isomorphism by virtue of having h as its inverse. So let $f : A_1 \times \ldots \times A_n \to A$ be any function symbol of Σ, let $a_i \in N_{A_i}$ for each $1 \leq i \leq n$, and suppose that $(a_1, \ldots, a_n) \in \mathsf{dom}(f^N)$. We must show that $(k_{A_1}(a_1), \ldots, k_{A_n}(a_n)) \in \mathsf{dom}(f^M)$ and

$$k_A(f^N(a_1, \ldots, a_n)) = f^M(k_{A_1}(a_1), \ldots, k_{A_n}(a_n)) \in M_A.$$

Since $h : M \xrightarrow{\sim} N$ is an isomorphism, we know by Lemma 2.2.33 that h reflects definedness. So the assumption that

$$(a_1, \ldots, a_n) = (h_{A_1}(k_{A_1}(a_1)), \ldots, h_{A_n}(k_{A_n}(a_n))) \in \mathsf{dom}\left(f^N\right)$$

implies that

$$(k_{A_1}(a_1), \ldots, k_{A_n}(a_n)) \in \mathsf{dom}\left(f^M\right),$$

as desired. We then have

$$\begin{aligned} f^M(k_{A_1}(a_1), \ldots, k_{A_n}(a_n)) &= k_A(h_A(f^M(k_{A_1}(a_1), \ldots, k_{A_n}(a_n)))) \\ &= k_A(f^N(h_{A_1}(k_{A_1}(a_1)), \ldots, h_{A_n}(k_{A_n}(a_n)))) \\ &= k_A(f^N(a_1, \ldots, a_n)), \end{aligned}$$

as desired. ∎

Proposition (2.2.38). $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ *is a well-defined functor.*

Proof: Let $h : M \to M'$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$. We must show that

$$G_{\mathbb{T}}(h) : G_{\mathbb{T}}(M) \to G_{\mathbb{T}}(M')$$

is a well-defined group homomorphism. If $([s_C])_C \in G_{\mathbb{T}}(M)$, then $\mathbb{T}(M, \mathsf{x}_C) \vdash s_C \downarrow$ for every sort $C \in \Sigma$, and hence $\mathbb{T}(M', \mathsf{x}_C) \vdash \rho_h^C(s_C) \downarrow$ for every sort $C \in \Sigma$ by Lemma 2.2.19. So

$$\left([\rho_h^C(s_C)]\right)_{C\in\Sigma} \in \prod_{C\in\Sigma} M'\langle \mathsf{x}_C\rangle_C,$$

as required. Also, it easily follows by Lemma 2.2.19 that $G_{\mathbb{T}}(h)$ is well-defined.

To see that $\left([\rho_h^C(s_C)]\right)_{C\in\Sigma} \in G_{\mathbb{T}}(M')$, let $h' : M' \to M''$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$ with domain M'. We must show that the family of total functions

$$\left(\rho_{h'}^C(\rho_h^C(s_C))^* : M''_C \to M''_C\right)_{C\in\Sigma}$$

is a Σ-automorphism of M''. Since $([s_C])_C \in G_{\mathbb{T}}(M)$ and $h' \circ h : M \to M''$ is a Σ-morphism, we know that the family of total functions

$$\left(\rho_{h'\circ h}^C(s_C)^* : M''_C \to M''_C\right)_{C\in\Sigma}$$

is a Σ-automorphism of M''. But by Lemma 2.2.37, we know that $\rho_{h'}^C(\rho_h^C(s_C)) \equiv \rho_{h'\circ h}^C(s_C)$ for every sort $C \in \Sigma$. So we then obtain

$$\left(\rho_{h'}^C(\rho_h^C(s_C))^* : M''_C \to M''_C\right)_{C\in\Sigma} = \left(\rho_{h'\circ h}^C(s_C)^* : M''_C \to M''_C\right)_{C\in\Sigma},$$

so that the lefthand family is indeed a Σ-automorphism of M'', as required.

To show that $G_{\mathbb{T}}(h)$ preserves group multiplication, let $([s_C])_C, ([t_C])_C \in G_{\mathbb{T}}(M)$. Then we have

$$\begin{aligned} G_{\mathbb{T}}(h)\left[([s_C])_C * ([t_C])_C\right] &= G_{\mathbb{T}}(h)\left[([s_C \cdot t_C])_C\right] \\ &= G_{\mathbb{T}}(h)\left[([s_C\,[t_C/\mathsf{x}_C]])_C\right] \\ &= \left(\left[\rho_h^C(s_C[t_C/\mathsf{x}_C])\right]\right)_C \\ &= \left(\left[\rho_h^C(s_C)\,\left[\rho_h^C(t_C)/\mathsf{x}_C\right]\right]\right)_C \\ &= \left(\left[\rho_h^C(s_C)\cdot\rho_h^C(t_C)\right]\right)_C \\ &= \left(\left[\rho_h^C(s_C)\right]\right)_C * \left(\left[\rho_h^C(t_C)\right]\right)_C \\ &= G_{\mathbb{T}}(h)\left[([s_C])_C\right] * G_{\mathbb{T}}(h)\left[([t_C])_C\right], \end{aligned}$$

as desired; the fourth equality follows by Lemma 2.2.28. This completes the proof that $G_{\mathbb{T}}(h)$ is a well-defined group homomorphism.

Now we show that $G_{\mathbb{T}}$ is functorial. So let $h : M \to M', h' : M' \to M''$ be arbitrary morphisms in $\mathsf{P}\mathbb{T}\mathsf{mod}$; we show that

$$G_{\mathbb{T}}(h' \circ h) = G_{\mathbb{T}}(h') \circ G_{\mathbb{T}}(h) : G_{\mathbb{T}}(M) \to G_{\mathbb{T}}(M'').$$

If $([s_C])_C \in G_{\mathbb{T}}(M)$, then we have

$$\begin{aligned} G_{\mathbb{T}}(h')\left(G_{\mathbb{T}}(h)\left[([s_C])_C\right]\right) &= G_{\mathbb{T}}(h')\left[\left(\left[\rho_h^C(s_C)\right]\right)_C\right] \\ &= \left(\left[\rho_{h'}^C(\rho_h^C(s_C))\right]\right)_C \\ &= \left(\left[\rho_{h'\circ h}^C(s_C)\right]\right)_C \\ &= G_{\mathbb{T}}(h' \circ h)\left[([s_C])_C\right], \end{aligned}$$

with the third equality being justified by Lemma 2.2.37.

Now let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$; we must show that

$$G_{\mathbb{T}}(\mathsf{id}_M) = \mathsf{id} : G_{\mathbb{T}}(M) \to G_{\mathbb{T}}(M).$$

For any $([s_C])_C \in G_{\mathbb{T}}(M)$, we have

$$G_{\mathbb{T}}(\mathsf{id}_M)\left[([s_C])_C\right] = \left(\left[\rho_{\mathsf{id}_M}^C(s_C)\right]\right)_C = ([s_C])_C,$$

as desired (the second equality being justified by Lemma 2.2.37).

This completes the proof that $G_{\mathbb{T}} : \mathsf{P}\mathbb{T}\mathsf{mod} \to \mathsf{Group}$ is a well-defined functor. ∎

Lemma (2.2.39)**.** *Let* $h : M \to N$ *be any* Σ*-morphism in* $\mathsf{P}\mathbb{T}\mathsf{mod}$*, and let* $B, C \in \Sigma_{\mathsf{Sort}}$*. Let* $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_B$ *with* $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$*. Let* $z \in M_C$*, so that* $(\widehat{M}, z)$ *is a* $\Sigma(M, \mathsf{x}_C)$*-structure and* $(\widehat{N}, h_C(z))$ *is a* $\Sigma(N, \mathsf{x}_C)$*-structure. Then* u *is defined in* $(\widehat{M}, z)$ *and* $\rho_h^C(u)$ *is defined in* $(\widehat{N}, h_C(z))$ *and*

$$h_B\left(u^{(\widehat{M},z)}\right) = \rho_h^C(u)^{(\widehat{N},h_C(z))} \in N_B.$$

Proof: First, note that $(\widehat{M}, z) \models \mathbb{T}(M, \mathsf{x}_C)$ and hence $(\widehat{M}, z) \models u \downarrow$ by soundness of partial Horn logic and the assumption that $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$. So it follows that u is defined in $(\widehat{M}, z)$. We also have $(\widehat{N}, h_C(z)) \models \mathbb{T}(N, \mathsf{x}_C)$ and $\mathbb{T}(N, \mathsf{x}_C) \vdash \rho_h^C(u) \downarrow$ by Lemma 2.2.19 and the assumption that $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$. So by soundness of partial Horn logic, we obtain that $(\widehat{N}, h_C(z)) \models \rho_h^C(u) \downarrow$, so that $\rho_h^C(u)$ is defined in $(\widehat{N}, h_C(z))$.

Fixing $z \in M_C$, we now prove by induction on $u \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$ that

$$h_B\left(u^{(\widehat{M},z)}\right) = \rho_h^C(u)^{(\widehat{N},h_C(z))} \in N_B,$$

if $u : B$.

- If $u \equiv \mathsf{x}_C : C$, then $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$ and $\rho_h^C(\mathsf{x}_C) \equiv \mathsf{x}_C$, so

  $$h_C\left(\mathsf{x}_C^{(\widehat{M},z)}\right) = h_C(z) = \mathsf{x}_C^{(\widehat{N},h_C(z))} = \rho_h^C(\mathsf{x}_C)^{(\widehat{N},h_C(z))}.$$

- If $u \equiv c_s^M : B$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$, then $\mathbb{T}(M, \mathsf{x}_C) \vdash u \downarrow$ and $\rho_h^C\left(c_s^M\right) \equiv c_{h_B(s)}^N$ and

  $$h_B\left(\left(c_s^M\right)^{(\widehat{M},z)}\right) = h_B(s) = \left(c_{h_B(s)}^N\right)^{(\widehat{N},h_C(z))} = \rho_h^C\left(c_s^M\right)^{(\widehat{N},h_C(z))}.$$

- Suppose that $u \equiv f(u_1, \ldots, u_n)$ for some function symbol $f : A_1 \times \ldots \times A_n \to A$ of Σ and terms $u_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_{A_i}$ for all $1 \leq i \leq n$, and suppose that $\mathbb{T}(M, \mathsf{x}_C) \vdash f(u_1, \ldots, u_n) \downarrow$. Then by the rules of partial Horn logic, it follows that $\mathbb{T}(M, \mathsf{x}_C) \vdash u_i \downarrow$ for all $1 \leq i \leq n$. So by the induction hypothesis, we have

  $$h_{A_i}\left(u_i^{(\widehat{M},z)}\right) = \rho_h^C(u_i)^{(\widehat{N},h_C(z))}$$

  for each $1 \leq i \leq n$. Then we have

  $$\begin{aligned} h_A\left(f(u_1, \ldots, u_n)^{(\widehat{M},z)}\right) &= h_A\left(f^{(\widehat{M},z)}\left(u_1^{(\widehat{M},z)}, \ldots, u_n^{(\widehat{M},z)}\right)\right) \\ &= h_A\left(f^M\left(u_1^{(\widehat{M},z)}, \ldots, u_n^{(\widehat{M},z)}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= f^N\left(h_{A_1}\left(u_1^{(\widehat{M},z)}\right),\ldots,h_{A_n}\left(u_n^{(\widehat{M},z)}\right)\right)\\
&= f^N\left(\rho_h^C(u_1)^{(\widehat{N},h_C(z))},\ldots,\rho_h^C(u_n)^{(\widehat{N},h_C(z))}\right)\\
&= f^{(\widehat{N},h_C(z))}\left(\rho_h^C(u_1)^{(\widehat{N},h_C(z))},\ldots,\rho_h^C(u_n)^{(\widehat{N},h_C(z))}\right)\\
&= f(\rho_h^C(u_1),\ldots,\rho_h^C(u_n))^{(\widehat{N},h_C(z))}\\
&= \rho_h^C(f(u_1,\ldots,u_n))^{(\widehat{N},h_C(z))},
\end{aligned}
$$

as desired.

▮

Lemma (2.2.40)**.** *Let $h : M \to N$ be any morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, let $A, B \in \Sigma_{\mathsf{Sort}}$, and let $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$. Then $[t] \in M\langle \mathsf{x}_A\rangle_B$ and $\rho_h^A(t)^* : N_A \to N_B$ is a well-defined, total function by Lemma 2.2.19 and Definition 2.2.11.*

If $a \in N_A$, then by Proposition 2.2.10 there is a unique Σ-morphism $h^a : M\langle \mathsf{x}_A\rangle \to N$ such that $h^a \circ \eta^{M,A} = h$ and $h^a_A([\mathsf{x}_A]) = a \in N_A$. Then we have

$$h^a_B([t]) = \rho_h^A(t)^*(a) \in N_B.$$

Proof: Fix $a \in N_A$. We prove the desired equation by induction on terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$.

- If $t \equiv \mathsf{x}_A : A$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and $\rho_h^A(t) \equiv \mathsf{x}_A$. We then have

$$h^a_A([\mathsf{x}_A]) = a = \mathsf{x}_A^{(\widehat{N},a)} = \rho_h^A(\mathsf{x}_A)^{(\widehat{N},a)} = \rho_h^A(\mathsf{x}_A)^*(a).$$

- If $t \equiv c_b^M$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $s \in M_B$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and $\rho_h^A\left(c_s^M\right) \equiv c^N_{h_B(s)}$. Then we have

$$
\begin{aligned}
h^a_B\left(\left[c_s^M\right]\right) &= h^a_B(\eta_B^{M,A}(s))\\
&= h_B(s)\\
&= \left(c^N_{h_B(s)}\right)^{(\widehat{N},a)}\\
&= \rho_h^A\left(c_s^M\right)^{(\widehat{N},a)}\\
&= \rho_h^A\left(c_s^M\right)^*(a).
\end{aligned}
$$

- Let $t \equiv f(t_1, \ldots, t_n)$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ and terms $t_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_{B_i}$ for each $1 \leq i \leq n$, and suppose that

$\mathbb{T}(M, \mathsf{x}_A) \vdash f(t_1, \ldots, t_n) \downarrow$. Then by the rules of partial Horn logic, it follows that $\mathbb{T}(M, \mathsf{x}_A) \vdash t_i \downarrow$ for each $1 \leq i \leq n$. So by the induction hypothesis, we have

$$h^a_{B_i}([t_i]) = \rho^A_h(t_i)^*(a) \in N_{B_i}$$

for each $1 \leq i \leq n$. Also, we have $\rho^A_h(f(t_1, \ldots, t_n)) \equiv f(\rho^A_h(t_1), \ldots, \rho^A_h(t_n))$. So we obtain

$$\begin{aligned} h^a_B([f(t_1, \ldots, t_n)]) &= h^a_B\left(f^{M\langle \mathsf{x}_A \rangle}([t_1], \ldots, [t_n])\right) \\ &= f^N(h^a_{B_1}([t_1]), \ldots, h^a_{B_n}([t_n])) \\ &= f^N\left(\rho^A_h(t_1)^*(a), \ldots, \rho^A_h(t_n)^*(a)\right) \\ &= f^{(\widehat{N},a)}\left(\rho^A_h(t_1)^{(\widehat{N},a)}, \ldots, \rho^A_h(t_n)^{(\widehat{N},a)}\right) \\ &= f(\rho^A_h(t_1), \ldots, \rho^A_h(t_n))^{(\widehat{N},a)} \\ &= f(\rho^A_h(t_1), \ldots, \rho^A_h(t_n))^*(a) \\ &= \rho^A_h(f(t_1, \ldots, t_n))^*(a), \end{aligned}$$

as desired (the second equality holds because $h^a : M\langle \mathsf{x}_A \rangle \to N$ is a Σ-morphism, and the third equality holds by the induction hypothesis).

∎

Lemma (2.2.54). *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$, and suppose that $M_A \neq \emptyset$. For any term $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and $t : B$, if x_A does* not *occur in t, then*

$$\mathbb{T}(M, \mathsf{x}_A) \vdash t = c_b$$

for some $b \in M_B$.

Proof: We prove this by induction on terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and x_A not occurring in t. If $t \equiv c_b$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $b \in M_B$, then $\mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow$ and the result clearly holds. Since $t \not\equiv \mathsf{x}_A$, this completes the base case.

For the induction step, suppose that $t \equiv f(t_1, \ldots, t_n)$ for some function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ and terms $t_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_{B_i}$ for all $1 \leq i \leq n$, and suppose that $\mathbb{T}(M, \mathsf{x}_A) \vdash f(t_1, \ldots, t_n) \downarrow$ and that x_A does not occur in t. By the rules of partial Horn logic, it follows for each $1 \leq i \leq n$ that $\mathbb{T}(M, \mathsf{x}_A) \vdash t_i \downarrow$ and that x_A does not occur in t_i. So by the induction hypothesis, for each $1 \leq i \leq n$ there is some $b_i \in M_{B_i}$ such that $\mathbb{T}(M, \mathsf{x}_A) \vdash t_i = c_{b_i}$. Since $\mathbb{T}(M, \mathsf{x}_A) \vdash f(t_1, \ldots, t_n) \downarrow$, it follows by the rules of partial Horn logic that $\mathbb{T}(M, \mathsf{x}_A) \vdash f\left(c_{b_1}, \ldots, c_{b_n}\right)\downarrow$. Then by the theorem on constants (Remark 1.3.17), it follows that $\mathbb{T}(M)$ proves the sequent

$$\top \vdash^{y:A} f\left(c_{b_1}, \ldots, c_{b_n}\right)\downarrow,$$

with $y : A$ a variable. Since $M_A \neq \emptyset$, it follows by an argument used before (cf. e.g. the proof of Lemma 2.2.9) that $\mathbb{T}(M)$ also proves the sequent $\top \vdash f\left(c_{b_1}, \ldots, c_{b_n}\right)\downarrow$. Since $\widehat{M} \models \mathbb{T}(M)$, we obtain $\widehat{M} \models f\left(c_{b_1}, \ldots, c_{b_n}\right)\downarrow$, which entails that $(b_1, \ldots, b_n) \in \mathsf{dom}\left(f^M\right)$. Suppose that $f^M(b_1, \ldots, b_n) = b \in M_B$. Then $\mathbb{T}(M) \vdash f\left(c_{b_1}, \ldots, c_{b_n}\right) = c_b$, and hence $\mathbb{T}(M, \mathsf{x}_A)$ proves this sequent as well. In conclusion, we obtain

$$\mathbb{T}(M, \mathsf{x}_A) \vdash t \equiv f(t_1, \ldots, t_n) = f\left(c_{b_1}, \ldots, c_{b_n}\right) = c_b$$

for some $b \in M_B$, as desired. ∎

Lemma (2.2.55)**.** *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and let $A \in \Sigma_{\mathsf{Sort}}$ have the property that $\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$ for distinct variables $y_1, y_2 : A$. Then for every $a \in M_A$,*

$$\mathbb{T}(M, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_a.$$

Proof: Let M and A satisfy the hypotheses, and let $a \in M_A$. Suppose towards a contradiction that $\mathbb{T}(M, \mathsf{x}_A) \vdash \mathsf{x}_A = c_a$. Then, letting $y_1, y_2 : A$ be distinct variables of sort A, we have by the theorem on constants (Remark 1.3.17) that $\mathbb{T}(M) \vdash^{y_1} y_1 = c_a$ and $\mathbb{T}(M) \vdash^{y_2} y_2 = c_a$, from which we obtain $\mathbb{T}(M) \vdash^{y_1,y_2} y_1 = y_2$, contrary to hypothesis. ∎

Lemma (2.2.56)**.** *Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $([s_C])_C \in G_{\mathbb{T}}(M)$, and let $A \in \Sigma_{\mathsf{Sort}}$ have the property that $M_A \neq \emptyset$ and $\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$ for distinct variables $y_1, y_2 : A$. If $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_A$ and $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = t$, then x_A occurs in t.*

Proof: Assume the hypotheses, and let $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_A$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = t$. If x_A did *not* occur in t, then by Lemma 2.2.54 (and the fact that $\mathbb{T}(M, \mathsf{x}_A) \vdash t\downarrow$, since $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = t$) it would follow that there is some $a \in M_A$ such that $\mathbb{T}(M, \mathsf{x}_A) \vdash t = c_a$, and hence $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = c_a$. Since $([s_C])_C \in G_{\mathbb{T}}(M)$, we know that $([s_C])_C$ is invertible, and so there is some $\left[s_A^{-1}\right] \in M\langle \mathsf{x}_A \rangle_A$ such that $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A[s_A^{-1}/\mathsf{x}_A] = \mathsf{x}_A$. Since $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A = c_a$ and $c_a[s_A^{-1}/\mathsf{x}_A] \equiv c_a$, we then obtain $\mathbb{T}(M, \mathsf{x}_A) \vdash c_a = \mathsf{x}_A$. But this contradicts Lemma 2.2.55, given the assumption on A. So x_A must occur in t, as desired. ∎

Proposition (2.4.2). *If* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$*, then* M *is finitely presented iff there is a Horn formula* $\varphi(x_1, \ldots, x_n)$ *over* Σ *with* $x_i : A_i$ *for each* $1 \leq i \leq n$*, and an* n*-tuple*

$$(a_1, \ldots, a_n) \in \varphi(x_1, \ldots, x_n)^M \subseteq M_{A_1} \times \ldots \times M_{A_n}$$

such that if $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and*

$$(b_1, \ldots, b_n) \in \varphi(x_1, \ldots, x_n)^N \subseteq N_{A_1} \times \ldots \times N_{A_n},$$

then there is a unique Σ*-morphism* $h : M \to N$ *with* $h_{A_i}(a_i) = b_i$ *for all* $1 \leq i \leq n$*.*

Proof: If M is finitely presented, then there are some $n \geq 0$, some pairwise distinct constant symbols $c_1, \ldots, c_n \notin \Sigma$ of respective sorts $A_1, \ldots, A_n \in \Sigma$, and some Horn sentence ψ over $\Sigma(c_1, \ldots, c_n)$ such that

$$M \cong \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))|_\Sigma.$$

Then we take $\varphi(x_1, \ldots, x_n) := \psi[x_1/c_1, \ldots, x_n/c_n]$, and we take

$$(a_1, \ldots, a_n) := ([c_1], \ldots, [c_n])$$

$$\in \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))_{A_1} \times \ldots \times \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))_{A_n}$$

(recall that $\mathbb{T}(c_1, \ldots, c_n, \psi) \vdash c_i \downarrow$ for each $1 \leq i \leq n$).

Now let $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ with $(b_1, \ldots, b_n) \in \varphi(x_1, \ldots, x_n)^N$. Then $(N, b_1, \ldots, b_n)$ is a $\Sigma(c_1, \ldots, c_n)$-structure with $(N, b_1, \ldots, b_n) \models \mathbb{T}(c_1, \ldots, c_n, \psi)$. Since $\mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))$ is the initial model of $\mathbb{T}(c_1, \ldots, c_n, \psi)$, there is a unique $\Sigma(c_1, \ldots, c_n)$-morphism

$$h : \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi)) \to (N, b_1, \ldots, b_n),$$

which will clearly be a Σ-morphism

$$h : \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))|_\Sigma \to N$$

with the property that $h_{A_i}([c_i]) = b_i$ for each $1 \leq i \leq n$. If we precompose h with the isomorphism $M \xrightarrow{\sim} \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))|_\Sigma$, then we obtain our desired (unique) Σ-morphism $M \to N$.

Conversely, suppose that there is a Horn formula $\varphi(x_1, \ldots, x_n)$ over Σ with $x_i : A_i$ for each $1 \leq i \leq n$, and an n-tuple $(a_1, \ldots, a_n) \in \varphi(x_1, \ldots, x_n)^M$ with the stated universal property, and set $\psi := \varphi[c_1/x_1, \ldots, c_n/x_n]$, a Horn sentence over $\Sigma(c_1, \ldots, c_n)$. Then $(M, a_1, \ldots, a_n)$ is a $\Sigma(c_1, \ldots, c_n)$-structure with the same universal property as $\mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))$, which implies that $(M, a_1, \ldots, a_n) \cong \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))$, and hence $M \cong \mathsf{Free}(\mathbb{T}(c_1, \ldots, c_n, \psi))|_\Sigma$, so that M is indeed finitely presented. ∎

Lemma (2.4.11)**.** *Let* $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ *be pairwise distinct constants.*

1. *If* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))_B$ *with*

$$(M, a_1, \ldots, a_n) \models t \downarrow$$

for some $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$*, then*

$$\top \vdash t\,[c_{a_1}/c_1, \ldots, c_{a_n}/c_n] = c_{t^{(M,a_1,\ldots,a_n)}}$$

is provable in $\mathbb{T}(M)$*.*

2. *If* $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *and* ψ *is a Horn sentence over* $\Sigma(c_1, \ldots, c_n)$ *with*

$$(M, a_1, \ldots, a_n) \models \psi$$

for some $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$*, then*

$$\top \vdash \psi\,[c_{a_1}/c_1, \ldots, c_{a_n}/c_n]$$

is provable in $\mathbb{T}(M)$*.*

Proof: We prove (1) by induction on $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))$.

- If $t \equiv c_i$ for some $1 \leq i \leq n$, then for $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$ we have $t^{(M,a_1,\ldots,a_n)} = a_i$ and the desired result easily follows, since $\mathbb{T}(M) \vdash c^M_{A_i,a_i} \downarrow$.
- Suppose $t \equiv f(t_1, \ldots, t_m)$ for some function symbol $f : B_1 \times \ldots \times B_m \to B$ of Σ and $t_i \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))_{B_i}$ for each $1 \leq i \leq m$. Let $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$ with

$$(M, a_1, \ldots, a_n) \models f(t_1, \ldots, t_m) \downarrow .$$

Then

$$(M, a_1, \ldots, a_n) \models t_i \downarrow$$

for each $1 \leq i \leq m$ and

$$\left(t_1^{(M,a_1,\ldots,a_n)}, \ldots, t_m^{(M,a_1,\ldots,a_n)}\right) \in \mathsf{dom}\left(f^M\right) .$$

The induction hypothesis and the definition of $\mathbb{T}(M)$ then easily yield the desired result.

We prove (2) by induction on the structure of the Horn sentence ψ.

- Suppose ψ has the form $s = t$ for some closed $s, t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))$ of the same sort B, and let $(a_1, \ldots, a_n) \in M_{A_1} \times \ldots \times M_{A_n}$ with

  $$(M, a_1, \ldots, a_n) \models s = t.$$

  Then

  $$(M, a_1, \ldots, a_n) \models s \downarrow \wedge\, t \downarrow$$

  and

  $$s^{(M,a_1,\ldots,a_n)} = t^{(M,a_1,\ldots,a_n)} \in M_B.$$

  By part (1), the following equations are then provable in $\mathbb{T}(M)$, as desired:

  $$s\left[c_{a_1}/c_1, \ldots, c_{a_n}/c_n\right] = c_{s^{(M,a_1,\ldots,a_n)}} = c_{t^{(M,a_1,\ldots,a_n)}} = t\left[c_{a_1}/c_1, \ldots, c_{a_n}/c_n\right].$$

- If $\psi \equiv \top$, then we trivially obtain the result.

- If $\psi \equiv \psi_1 \wedge \psi_2$ for Horn sentences ψ_1, ψ_2 over $\Sigma(c_1, \ldots, c_n)$, then the result follows easily from the induction hypothesis.

∎

Lemma (2.4.12)**.** *Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$.*

1. *For any sort $C \in \Sigma$, the signature morphism ρ^C_φ is a theory morphism*

   $$\rho^C_\varphi : \mathbb{T}(\varphi, \mathsf{x}_C) \to \mathbb{T}(M^\varphi, \mathsf{x}_C).$$

2. *For any sort $C \in \Sigma$, any terms $u, v \in \mathsf{Term}^c(M^\varphi, \mathsf{x}_C))$ of the same sort and terms $s, t \in \mathsf{Term}^c(M^\varphi, \mathsf{x}_C))$ of the same sort, if*

   $$u = v \vdash s = t \textit{ is provable in } \mathbb{T}(M^\varphi, \mathsf{x}_C),$$

   then

   $$\sigma^C_\varphi(u) = \sigma^C_\varphi(v) \vdash \sigma^C_\varphi(s) = \sigma^C_\varphi(t) \textit{ is provable in } \mathbb{T}(\varphi, \mathsf{x}_C).$$

Proof: To prove (1), we must show that the ρ^C_φ-translation of any axiom of $\mathbb{T}(\varphi, \mathsf{x}_C)$ is a theorem of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$. By definition of ρ^C_φ, the ρ^C_φ-translation of any axiom of $\mathbb{T}$ is itself, and hence is an axiom of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$. Similarly, the ρ^C_φ-translation of the axiom $\top \vdash \mathsf{x}_C \downarrow$ is just itself, which is also an axiom of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$. For any $1 \leq i \leq n$, the ρ^C_φ-translation of the axiom $\top \vdash c_i \downarrow$ of $\mathbb{T}(\varphi, \mathsf{x}_C)$ is $\top \vdash c^{M^\varphi}_{[c_i]} \downarrow$, which is an axiom of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$. Finally, we must show that the ρ^C_φ-translation of the axiom $\top \vdash \varphi$ of $\mathbb{T}(\varphi)$ is a theorem of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$. The ρ^C_φ-translation of this axiom is

$$\top \vdash \varphi \left[c^{M^\varphi}_{[c_1]}/c_1, \ldots, c^{M^\varphi}_{[c_n]}/c_n\right].$$

So the result follows from Lemma 2.4.11 because

$$(M^\varphi, [c_1], \ldots, [c_n]) \models \varphi,$$

i.e. because

$$\mathsf{Free}(\mathbb{T}(\varphi)) \models \varphi.$$

This proves that ρ^C_φ is a theory morphism, which proves (1).

To prove (2), we show that the σ^C_φ-translation of any axiom of $\mathbb{T}(M^\varphi, \mathsf{x}_C)$ is provable in $\mathbb{T}(\varphi, \mathsf{x}_C)$. It is clear that the σ^C_φ-translation of any axiom of $\mathbb{T}$ is just itself, and similarly for the axiom $\top \vdash \mathsf{x}_C \downarrow$, and these are axioms of $\mathbb{T}(\varphi, \mathsf{x}_C)$. Now let $B \in \Sigma_{\mathsf{Sort}}$ and $[t] \in M^\varphi_B$; we must show that the σ^C_φ-translation of the axiom $\top \vdash c^{M^\varphi}_{[t]} \downarrow$ is provable in $\mathbb{T}(\varphi, \mathsf{x}_C)$, i.e. we must show that the sequent

$$\top \vdash \widehat{[t]} \downarrow$$

is provable in $\mathbb{T}(\varphi, \mathsf{x}_C)$. But if $[t] \in M^\varphi_B$, then this means that $\mathbb{T}(\varphi) \vdash t \downarrow$, and since $\mathbb{T}(\varphi) \vdash t = \widehat{[t]}$, this yields the desired result.

Lastly, let $f : B_1 \times \ldots \times B_m \to B$ be a function symbol of Σ, let $[t_i] \in M^\varphi_{B_i}$ for each $1 \leq i \leq m$, and suppose that $([t_1], \ldots, [t_m]) \in \mathsf{dom}\left(f^{M^\varphi}\right)$. We must show that $\mathbb{T}(\varphi, \mathsf{x}_C)$ proves the σ^C_φ-translation of the axiom

$$\top \vdash f\left(c^{M^\varphi}_{[t_1]}, \ldots, c^{M^\varphi}_{[t_m]}\right) = c^{M^\varphi}_{[f(t_1,\ldots,t_m)]},$$

which is

$$\top \vdash f\left(\widehat{[t_1]}, \ldots, \widehat{[t_m]}\right) = [f(\widehat{t_1, \ldots}, t_m)].$$

But this follows because $\mathbb{T}(\varphi) \vdash f(t_1, \ldots, t_m) \downarrow$ (since $([t_1], \ldots, [t_m]) \in \mathsf{dom}\left(f^{M^\varphi}\right)$) and $\mathbb{T}(\varphi) \vdash t_i = \widehat{[t_i]}$ for each $1 \leq i \leq m$ and

$$\mathbb{T}(\varphi) \vdash f(t_1, \ldots, t_m) = [f(\widehat{t_1, \ldots}, t_m)].$$

This completes the proof of (2). ∎

Lemma (2.4.13). *Let $c_1 : A_1, \ldots, c_n : A_n \notin \Sigma$ be pairwise distinct constants and let φ be a Horn sentence over $\Sigma(c_1, \ldots, c_n)$.*

1. *If* $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))$ *is of some sort* $B \in \Sigma$ *and* $\mathbb{T}(\varphi) \vdash t \downarrow$ *and* $C \in \Sigma_{\mathsf{Sort}}$, *then*
$$\mathbb{T}(M^\varphi) \vdash \rho^C_\varphi(t) = c^{M^\varphi}_{[t]}.$$
2. *If* $t \in \mathsf{Term}^c(\Sigma(M^\varphi, \mathsf{x}_C))$ *and* $\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash t \downarrow$, *then*
$$\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash \rho^C_\varphi(\sigma^C_\varphi(t)) = t.$$
3. *If* $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))$ *and* $\mathbb{T}(\varphi, \mathsf{x}_C) \vdash t \downarrow$, *then*
$$\mathbb{T}(\varphi, \mathsf{x}_C) \vdash \sigma^C_\varphi(\rho^C_\varphi(t)) = t.$$

Proof: We prove (1) by induction on terms $t \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))$ with $\mathbb{T}(\varphi) \vdash t \downarrow$. Let $C \in \Sigma_{\mathsf{Sort}}$.

- Suppose $t \equiv c_i$ for some $1 \leq i \leq n$. Then $\mathbb{T}(\varphi) \vdash c_i \downarrow$ and
$$\rho^C_\varphi(c_i) \equiv c^{M^\varphi}_{[c_i]},$$
which yields the result.
- Suppose $t \equiv f(t_1, \ldots, t_m)$ for some function symbol $f : B_1 \times \ldots \times B_m \to B$ of Σ and $t_i \in \mathsf{Term}^c(\Sigma(c_1, \ldots, c_n))_{B_i}$ for each $1 \leq i \leq m$. If $\mathbb{T}(\varphi) \vdash f(t_1, \ldots, t_m) \downarrow$, then $\mathbb{T}(\varphi) \vdash t_i \downarrow$ for each $1 \leq i \leq m$, so by the induction hypothesis we have
$$\mathbb{T}(M^\varphi) \vdash \rho^C_\varphi(t_i) = c^{M^\varphi}_{[t_i]}$$
for each $1 \leq i \leq m$. Then $\mathbb{T}(M^\varphi)$ proves the following sequence of equations, as desired:
$$\rho^C_\varphi(f(t_1, \ldots, t_m)) = f\left(\rho^C_\varphi(t_1), \ldots, \rho^C_\varphi(t_m)\right) = f\left(c^{M^\varphi}_{[t_1]}, \ldots, c^{M^\varphi}_{[t_m]}\right) = c^{M^\varphi}_{[f(t_1,\ldots,t_m)]}.$$
The last equality holds because $([t_1], \ldots, [t_m]) \in \mathsf{dom}\left(f^{M^\varphi}\right)$ and $f^{M^\varphi}([t_1], \ldots, [t_m]) = [f(t_1, \ldots, t_m)]$, since $\mathbb{T}(\varphi) \vdash f(t_1, \ldots, t_m) \downarrow$.

The proof of (2) is by induction on terms $t \in \mathsf{Term}^c(\Sigma(M^\varphi, \mathsf{x}_C))$ with $\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash t \downarrow$. The only 'non-trivial' case is when $t \equiv c^{M^\varphi}_{[s]}$ for some $B \in \Sigma_{\mathsf{Sort}}$ and $[s] \in M^\varphi_B$. Then
$$\rho^C_\varphi\left(\sigma^C_\varphi\left(c^{M^\varphi}_{[s]}\right)\right) \equiv \rho^C_\varphi\left(\widehat{[s]}\right).$$

Since $\mathbb{T}(\varphi) \vdash s = \widehat{[s]}$, we obtain
$$\mathbb{T}(M^\varphi, \mathsf{x}_C) \vdash \rho^C_\varphi\left(\widehat{[s]}\right) = \rho^C_\varphi(s)$$

by Lemma 2.4.12. Since $[s] \in M^{\varphi}_{B}$ implies $s \in \mathsf{Term}^{c}(\Sigma(c_1, \ldots, c_n))_B$ and $\mathbb{T}(\varphi) \vdash s \downarrow$, we then obtain

$$\mathbb{T}(M^{\varphi}, \mathsf{x}_C) \vdash \rho^{C}_{\varphi}(s) = c^{M^{\varphi}}_{[s]}$$

by (1), as desired.

The proof of (3) is also by induction on terms $t \in \mathsf{Term}^{c}(\Sigma(c_1, \ldots, c_n, \mathsf{x}_C))$ with $\mathbb{T}(\varphi, \mathsf{x}_C) \vdash t \downarrow$, the only 'non-trivial' case being $t \equiv c_i$ for some $1 \leq i \leq n$, in which case

$$\sigma^{C}_{\varphi}\left(\rho^{C}_{\varphi}(c_i)\right) \equiv \sigma^{C}_{\varphi}\left(c^{M^{\varphi}}_{[c_i]}\right) \equiv \widehat{[c_i]},$$

and we have

$$\mathbb{T}(\varphi) \vdash \widehat{[c_i]} = c_i,$$

as desired. ∎

Appendix B

Chapter 3 Proofs

Lemma (3.1.2)**.** Let $\mathbb{T}$ be an arbitrary quasi-equational theory over a relation-free signature Σ.

- Let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $A \in \Sigma_{\mathsf{Sort}}$, and let $s, t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_A))_B$ for some $B \in \Sigma_{\mathsf{Sort}}$ with $\mathbb{T}(M, \mathsf{x}_A) \vdash s\downarrow \wedge\, t\downarrow$. Then

  $$\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$$

  iff

  $$\rho_h^A(s)^* = \rho_h^A(t)^* : N_A \to N_B$$

  for every Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$.

- If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $A \in \Sigma_{\mathsf{Sort}}$ and $y_1, y_2 : A$ are distinct variables, then

  $$\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$$

  iff

  there are a $\mathbb{T}$-model N, a Σ-morphism $h : M \to N$, and distinct elements $a_1 \neq a_2 \in N_A$.

Proof: Assume the hypotheses of the first claim. First, we note by Lemma 2.2.19 and Definition 2.2.11 that if $\mathbb{T}(M, \mathsf{x}_A) \vdash s\downarrow \wedge\, t\downarrow$ and $h : M \to N$ is a Σ-morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, then $\rho_h^A(s)^*, \rho_h^A(t)^* : N_A \to N_B$ are indeed well-defined total functions.

Now, if $\mathbb{T}(M, \mathsf{x}_A) \vdash s = t$, then for any Σ-morphism $h : M \to N$ in $\mathsf{P}\mathbb{T}\mathsf{mod}$ we obtain $\mathbb{T}(N, \mathsf{x}_A) \vdash \rho_h^A(s) = \rho_h^A(t)$ by Lemma 2.2.19, so that $\rho_h^A(s)^* = \rho_h^A(t)^*$ by Lemma 2.2.13.

Conversely, suppose that $\mathbb{T}(M, \mathsf{x}_A) \nvdash s = t$, which means that $[s] \neq [t] \in M\langle \mathsf{x}_A \rangle_B$. Consider the canonical Σ-morphism $\eta : M \to M\langle \mathsf{x}_A \rangle$. By Lemma 2.2.31, we then have

$$\rho_\eta^A(s)^*([\mathsf{x}_A]) = [s[\mathsf{x}_A/\mathsf{x}_A]] = [s] \neq [t] = [t[\mathsf{x}_A/\mathsf{x}_A]] = \rho_\eta^A(t)^*([\mathsf{x}_A]),$$

so that $\rho^A_\eta(s)^* \neq \rho^A_\eta(t)^* : M\langle \mathsf{x}_A\rangle_A \to M\langle \mathsf{x}_A\rangle_B$, as desired.

To prove the second claim, suppose first that $\mathbb{T}(M) \nvdash^{y_1,y_2} y_1 = y_2$. Then by the completeness of partial Horn logic, it follows that there is a model N of $\mathbb{T}(M)$ with $N \not\models^{y_1,y_2} y_1 = y_2$, which means that there must be distinct elements $a_1 \neq a_2 \in N_A$. By Lemma 2.2.4, there is also a Σ-morphism $h : M \to N|_\Sigma$, and $N|_\Sigma$ is a model of $\mathbb{T}$. Conversely, if $\mathbb{T}(M) \vdash^{y_1,y_2} y_1 = y_2$, then by soundness of partial Horn logic it follows that every model N of $\mathbb{T}(M)$ satisfies $y_1 = y_2$. Now if $h : M \to N$ is any morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}$, then by Lemma 2.2.4 there is a $\Sigma(M)$-structure N^h with $N^h|_\Sigma = N$ and $N^h \models \mathbb{T}(M)$. So we have $N^h \models^{y_1,y_2} y_1 = y_2$, which implies that $a_1 = a_2$ for all $a_1, a_2 \in N^h_A = N_A$, as desired. ∎

Lemma (3.1.3)**.** *Let Σ be any relation-free signature, let $\mathbb{T}$ be an empty theory over Σ, and let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$. For any $C \in \Sigma_{\mathsf{Sort}}$ and $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$,*

$$\mathbb{T}(M, \mathsf{x}_C) \vdash t = \mathsf{x}_C \implies t \equiv \mathsf{x}_C.$$

Proof: If $\mathbb{T}(M, \mathsf{x}_C) \nvdash t \downarrow$, then we trivially obtain the result, because $\mathbb{T}(M, \mathsf{x}_C) \vdash t = \mathsf{x}_C$ implies $\mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow$ by the rules of partial Horn logic. So it remains to prove the result for terms $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow$.

We will prove the contrapositive. So let $t \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_C$ with $\mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow$ and $t \not\equiv \mathsf{x}_C$, and we will show that $\mathbb{T}(M, \mathsf{x}_C) \nvdash t = \mathsf{x}_C$. By Lemma 3.1.2 (since $\mathbb{T}(M, \mathsf{x}_C) \vdash t \downarrow \wedge\, \mathsf{x}_C \downarrow$), it will suffice to construct a $\mathbb{T}$-model N and a Σ-morphism $h : M \to N$ with $a \in N_C$ and

$$\rho^C_h(t)^*(a) \neq \rho^C_h(\mathsf{x}_C)^*(a) = \mathsf{x}^*_C(a) = a \in N_C.$$

Since $t \not\equiv \mathsf{x}_C$, there are two possible cases:

- Suppose $t \equiv c_s$ for some $s \in M_C$. We define a partial Σ-structure N as follows. Let $a_0, b_0 \notin \bigcup_{B \in \Sigma} M_B$. Then for every $B \in \Sigma_{\mathsf{Sort}}$, we set

  $$N_B := M_B \cup \{a_0, b_0\},$$

  and for every function symbol $f : B_1 \times \ldots \times B_n \to B$ of Σ, we set $\mathsf{dom}\left(f^N\right) := N_{B_1} \times \ldots \times N_{B_n}$ and

  $$f^N \restriction \mathsf{dom}\left(f^M\right) := f^M$$

  and

  $$f^N(b_1, \ldots, b_n) := b_0 \in N_B$$

  for any $(b_1, \ldots, b_n) \in N_{B_1} \times \ldots \times N_{B_n} \setminus \mathsf{dom}\left(f^M\right)$. It is clear that N is a model of $\mathbb{T}$, and that if $h_B : M_B \to N_B$ is the inclusion map for each $B \in \Sigma_{\mathsf{Sort}}$, then

$h := (h_B : M_B \to N_B)_{B \in \Sigma}$ is a Σ-morphism $M \to N$. With $a_0 \in N_C$, we then have

$$\rho_h^C(t)^*(a_0) = \left(c_{h_C(s)}\right)^*(a_0) = \left(c_{h_C(s)}\right)^{(\widehat{N}, a_0)} = h_C(s) = s \neq a_0,$$

since $s \in M_C$ but $a_0 \notin M_C$.

- Suppose $t \equiv f(t_1, \dots, t_n)$ for some function symbol $f : C_1 \times \ldots \times C_n \to C$ of Σ and $t_i \in \mathsf{Term}^c(\Sigma(M, \mathsf{x}_C))_{C_i}$ for each $1 \leq i \leq n$, and consider the $\mathbb{T}$-model N and Σ-morphism $h : M \to N$ from the previous case. With $a_0 \in N_C$, we have

  $$\rho_h^C(f(t_1, \dots, t_n))^*(a_0) = f^N(\rho_h^C(t_1)^*(a_0), \dots, \rho_h^C(t_n)^*(a_0)) \neq a_0,$$

  since $\mathsf{Im}\left(f^N\right) \subseteq M_C \cup \{b_0\}$, but $a_0 \notin M_C \cup \{b_0\}$.

∎

Lemma (3.6.1)**.** *Let $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$, and let $\mathsf{x} \notin \Sigma_{\mathsf{Bij}}(M)$ be a constant of the unique sort of Σ_{Bij}. Then for any $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$, either there is some $m \in M$ such that $[t] = [c_m] \in M\langle \mathsf{x} \rangle$, or there is some $n \in \mathbb{Z}$ such that $[t] = [f^n(\mathsf{x})] \in M\langle \mathsf{x} \rangle$.*

Proof: We prove this by induction on $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$.

- If $t \equiv \mathsf{x}$, then we have $[t] = [\mathsf{x}] = [f^0(\mathsf{x})]$.
- If $t \equiv c_m$ for some $m \in M$, then we clearly have $[t] = [c_m]$.
- Suppose $t \equiv f(s)$ for some $s \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$. By the induction hypothesis, either there is some $m \in M$ such that $[s] = [c_m]$, or there is some $n \in \mathbb{Z}$ such that $[s] = [f^n(\mathsf{x})]$. In the first case, we have

  $$[t] = [f(s)] = [f(c_m)] = \left[c_{f^M(m)}\right]$$

  with $f^M(m) \in M$, as desired. In the other case, we have

  $$[t] = [f(s)] = [f(f^n(\mathsf{x}))] = [f^{n+1}(\mathsf{x})]$$

  (regardless of whether $n \geq 0$ or $n < 0$).
- If $t \equiv f^{-1}(s)$ for some $s \in \mathsf{Term}^c(\Sigma_{\mathsf{Bij}}(M, \mathsf{x}))$, then the reasoning is similar to that in the previous case.

∎

Lemma (3.6.2)**.** *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and $\mathsf{x} \notin \Sigma_{\mathsf{Bij}}(M)$ is a constant symbol and for $n, m \in \mathbb{Z}$ we have $[f^n(\mathsf{x})] = [f^m(\mathsf{x})] \in M\langle\mathsf{x}\rangle$, then $n = m$.*

Proof: We will prove the contrapositive. So let $n, m \in \mathbb{Z}$ with $n \neq m$. We must show that $[f^n(\mathsf{x})] \neq [f^m(\mathsf{x})] \in M\langle\mathsf{x}\rangle$, i.e. we must show that $\mathbb{T}_{\mathsf{Bij}}(M, \mathsf{x}) \nvdash f^n(\mathsf{x}) = f^m(\mathsf{x})$. By Lemma 3.1.2, it clearly suffices to construct $N \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ with a Σ_{Bij}-morphism $h : M \to N$ and some $a \in N$ with $\left(f^N\right)^n(a) \neq \left(f^N\right)^m(a)$. Assume without loss of generality that $M \cap \mathbb{Z} = \emptyset$, and set $N := M \cup \mathbb{Z}$, with $f^N \restriction M := f^M$ and $(f^{-1})^N \restriction M := (f^{-1})^M$ and $f^N(n) := n + 1$ and $(f^{-1})^N(n) := n - 1$ for each $n \in \mathbb{Z}$. Then clearly N is a model of $\mathbb{T}_{\mathsf{Bij}}$ and the inclusion map gives a Σ_{Bij}-morphism $M \to N$. Then $0 \in N$ and we clearly have $\left(f^N\right)^n(0) = n \neq m = \left(f^N\right)^m(0)$, as desired. ∎

Lemma (3.6.3)**.** *If $M \in \mathsf{P}\mathbb{T}_{\mathsf{Bij}}\mathsf{mod}$ and y_1, y_2 are distinct variables of the unique sort of Σ_{Bij}, then $\mathbb{T}(M) \nvdash^{y_1, y_2} y_1 = y_2$.*

Proof: By Lemma 3.1.2, it is equivalent to show that there is a $\mathbb{T}_{\mathsf{Bij}}$-model N with at least two elements and a Σ_{Bij}-morphism $h : M \to N$. If M already has at least two elements, then we take $N := M$ and $h := \mathsf{id}_M$. Otherwise, M has at most one element. If we take $a \neq b \notin M$ and define $N := \{a, b\}$ with $f^N = (f^{-1})^N$ the identity function, then N is a model of $\mathbb{T}_{\mathsf{Bij}}$ and there is clearly a Σ_{Bij}-morphism $M \to N$. ∎

Lemma (3.8.1)**.** *Let $\mathbb{C}$ be any small category.*

1. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \nvdash \mathsf{x}_O = c_{O,b}$. For any $f \in \mathbb{C}_A$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_{A,f}$.*
2. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = c_{O,b}$ and $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{cod}(\mathsf{x}_A) = c_{O,b}$.*
3. *$\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = \mathsf{cod}(\mathsf{x}_A)$.*

Proof: For the first claim in (1), let $b \in \mathbb{C}_O$ be arbitrary. We must show $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \nvdash \mathsf{x}_O = c_b$. By Lemma 3.1.2, it suffices to show that there is a small category $\mathbb{D}$, a functor $F : \mathbb{C} \to \mathbb{D}$, and an object $d \in \mathbb{D}_O$ with $F(b) \neq d$. Let $\mathbb{C}^*$ be the disjoint union of $\mathbb{C}$ with the terminal category on the object $*$. Let $F : \mathbb{C} \to \mathbb{C}^*$ be the inclusion functor. Then we have $* \in \mathbb{C}^*$ with $F(b) = b \neq *$, as desired.

The proof of the second claim in (1) is similar, except that instead of the disjoint union of $\mathbb{C}$ with the terminal category, we use the disjoint union of $\mathbb{C}$ with the category that has just two distinct objects and one arrow between them. This latter construction can also be used to prove claims (2) and (3). ▮

Lemma (3.8.2)**.** *If* $\mathbb{C}$ *is any small category and* $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))$ *with* $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t\downarrow$*, then:*

- *If* $t : O$*, then either* $[t] = [\mathsf{x}_O] \in \mathbb{C}\langle \mathsf{x}_O \rangle_O$ *or* $[t] = [c_{O,b}] \in \mathbb{C}\langle \mathsf{x}_O \rangle_O$ *for some* $b \in \mathbb{C}_O$*.*
- *If* $t : A$*, then either* $[t] = [\mathsf{id}(\mathsf{x}_O)] \in \mathbb{C}\langle \mathsf{x}_O \rangle_A$ *or* $[t] = [c_{A,f}] \in \mathbb{C}\langle \mathsf{x}_O \rangle_A$ *for some* $f \in \mathbb{C}_A$*.*

Proof: We prove the claims by induction on $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t\downarrow$.

- If $t \equiv \mathsf{x}_O : O$ or $t \equiv c_{O,b} : O$ for some $b \in \mathbb{C}_O$ (in which cases $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t\downarrow$), then the desired result obviously holds. Similarly if $t \equiv c_{A,f} : A$ for some $f \in \mathbb{C}_A$.
- Suppose $t \equiv \mathsf{dom}(t') : O$ for some $t' \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))_A$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash \mathsf{dom}(t')\downarrow$ (which implies $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash t'\downarrow$ by partial Horn logic). By the induction hypothesis, we have $[t'] = [\mathsf{id}(\mathsf{x}_O)]$ or $[t'] = [c_{A,f}]$ for some $f : a \to b \in \mathbb{C}$. In the first case, we obtain $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}(\mathsf{id}(\mathsf{x}_O))] = [\mathsf{x}_O]$, as desired. In the second case, we obtain $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}\,(c_{A,f})] = [c_{O,a}]$, as desired. The reasoning for cod is similar.
- Suppose $t \equiv \mathsf{id}(t') : A$ for some $t' \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))_O$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash \mathsf{id}(t')\downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash t'\downarrow$). By the induction hypothesis, we have $[t'] = [\mathsf{x}_O]$ or $[t'] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$. In the first case, we have $[t] = [\mathsf{id}(t')] = [\mathsf{id}(\mathsf{x}_O)]$, and in the second case we have $[t] = [\mathsf{id}(t')] = [\mathsf{id}\,(c_{O,b})] = \left[c_{A,\mathsf{id}^{\mathbb{C}}(b)}\right]$, as desired.
- Suppose $t \equiv t_1 \circ t_2 : A$ for some $t_1, t_2 \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O))_A$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash t_1 \circ t_2\downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O) \vdash t_i\downarrow$ for $i = 1, 2$). For $i = 1, 2$, we have by the induction hypothesis that $[t_i] = [\mathsf{id}(\mathsf{x}_O)]$ or $[t_i] = [c_{A,f_i}]$ for some $f_i \in \mathbb{C}_A$. If $[t_1] = [t_2] = [\mathsf{id}(\mathsf{x}_O)]$, then we have $[t] = [t_1 \circ t_2] = [\mathsf{id}(\mathsf{x}_O) \circ \mathsf{id}(\mathsf{x}_O)] = [\mathsf{id}(\mathsf{x}_O)]$, as desired. Conversely, if we have $[t_1] = [c_{A,f_1}]$ and $[t_2] = [c_{A,f_2}]$, then $\mathsf{dom}^{\mathbb{C}}(f_2) = \mathsf{cod}^{\mathbb{C}}(f_1)$ (since $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t_1 \circ t_2\downarrow$), and we have $[t] = [t_1 \circ t_2] = [c_{A,f_2} \circ c_{A,f_1}] = \left[c_{A,f_2 \circ^{\mathbb{C}} f_1}\right]$, as desired.

  To conclude, we show that it is not possible that $[t_1] = [\mathsf{id}(\mathsf{x}_O)]$ and $[t_2] = [c_{A,f_2}]$, and conversely. By assumption, we have $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash t_1 \circ t_2\downarrow$, which implies that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_O) \vdash \mathsf{dom}(t_1) = \mathsf{cod}(t_2)$, i.e. $[\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)]$ holds in $\mathbb{C}\langle \mathsf{x}_O \rangle_O$. So if we had $[t_1] = [\mathsf{id}(\mathsf{x}_O)]$ and $[t_2] = [c_{A,f_2}]$, then in $\mathbb{C}\langle \mathsf{x}_O \rangle_O$ we would have

$$[\mathsf{x}_O] = [\mathsf{dom}(\mathsf{id}(\mathsf{x}_O))] = [\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)] = [\mathsf{cod}\,(c_{A,f_2})] = \left[c_{O,\mathsf{cod}^{\mathbb{C}}(f_2)}\right],$$

which contradicts (1) of Lemma 3.8.1. Similarly, we cannot have $[t_2] = [\mathsf{id}(\mathsf{x}_O)]$ and $[t_1] = [c_{A,f_1}]$.

$\blacksquare$

Lemma (3.8.3)**.** *If* $\mathbb{C}$ *is any small category and* $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))$ *with* $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t \downarrow$*, then:*

- *If* $t : O$*, then* $[t] = [\mathsf{dom}(\mathsf{x}_A)]$ *or* $[t] = [\mathsf{cod}(\mathsf{x}_A)]$ *or* $[t] = [c_{O,b}]$ *for some* $b \in \mathbb{C}_O$*.*
- *If* $t : A$*, then* $[t] = [\mathsf{x}_A]$ *or* $[t] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$ *or* $[t] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ *or* $[t] = [c_{A,f}]$ *for some* $f \in \mathbb{C}_A$*.*

Proof: We prove this by induction on terms $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t \downarrow$.

- If $t \equiv c_{O,b} : O$ for some $b \in \mathbb{C}_O$ or $t \equiv c_{A,f} : A$ for some $f \in \mathbb{C}_A$, then the desired result clearly holds. Similarly if $t \equiv \mathsf{x}_A : A$.
- Suppose that $t \equiv \mathsf{dom}(t') : O$ for some $t' \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))_A$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{dom}(t') \downarrow$ (so that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t' \downarrow$). By the induction hypothesis, we then have one of the following cases:
  - If $[t'] = [\mathsf{x}_A]$, then we have $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}(\mathsf{x}_A)]$, as desired.
  - If $[t'] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$, then we have $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}(\mathsf{id}(\mathsf{dom}(\mathsf{x}_A)))] = [\mathsf{dom}(\mathsf{x}_A)]$, as desired.
  - If $[t'] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$, then we have $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}(\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)))] = [\mathsf{cod}(\mathsf{x}_A)]$, as desired.
  - If $[t'] = [c_{A,f}]$ for some $f : a \to b \in \mathbb{C}$, then we have $[t] = [\mathsf{dom}(t')] = [\mathsf{dom}\,(c_{A,f})] = [c_{O,a}]$, as desired.

  The reasoning for $t \equiv \mathsf{cod}(t')$ is analogous.
- Suppose that $t \equiv \mathsf{id}(t') : A$ for some $t' \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))_O$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{id}(t') \downarrow$ (so that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t' \downarrow$). By the induction hypothesis, we then have one of the following cases:

  - If $[t'] = [\mathsf{dom}(\mathsf{x}_A)]$, then we have $[t] = [\mathsf{id}(t')] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$, as desired.
  - If $[t'] = [\mathsf{cod}(\mathsf{x}_A)]$, then we have $[t] = [\mathsf{id}(t')] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$, as desired.
  - If $[t'] = [c_{O,b}]$ for some $b \in \mathbb{C}_O$, then we have $[t] = [\mathsf{id}(t')] = [\mathsf{id}\,(c_{O,b})] = \left[c_{A,\mathsf{id}^{\mathbb{C}}(b)}\right]$, as desired.

- Suppose that $t \equiv t_1 \circ t_2 : A$ for some $t_1, t_2 \in \mathsf{Term}^c(\Sigma_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A))_A$ with $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t_1 \circ t_2 \downarrow$ (so that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t_i \downarrow$ for $i = 1, 2$). Then $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{dom}(t_1) = \mathsf{cod}(t_2)$, so $[\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)] \in \mathbb{C}\langle \mathsf{x}_A \rangle_O$. By the induction hypothesis, we then have one of the following cases:

  - Suppose $[t_2] = [\mathsf{x}_A]$, so that $[\mathsf{cod}(t_2)] = [\mathsf{cod}(\mathsf{x}_A)]$. Since $[\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)]$, it follows by Lemma 3.8.1 and the induction hypothesis for t_1 that we must have $[t_1] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$. Then we have

    $$[t] = [t_1 \circ t_2] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)) \circ \mathsf{x}_A] = [\mathsf{x}_A],$$

    as desired.
  - Suppose $[t_2] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$, so that $[\mathsf{cod}(t_2)] = [\mathsf{cod}(\mathsf{id}(\mathsf{dom}(\mathsf{x}_A)))] = [\mathsf{dom}(\mathsf{x}_A)]$. Since $[\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)]$, it follows by Lemma 3.8.1 and the induction hypothesis for t_1 that we must have $[t_1] = [\mathsf{x}_A]$ or $[t_1] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$. In the first case we obtain $[t] = [t_1 \circ t_2] = [\mathsf{x}_A \circ \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))] = [\mathsf{x}_A]$, and in the second case we obtain $[t] = [t_1 \circ t_2] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A)) \circ \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))] = [\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))]$, as desired.
  - Suppose $[t_2] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$, so that $[\mathsf{cod}(t_2)] = [\mathsf{cod}(\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)))] = [\mathsf{cod}(\mathsf{x}_A)]$. As in the first case, we must then have $[t_1] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))]$ as well, so that

    $$[t] = [t_1 \circ t_2] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A)) \circ \mathsf{id}(\mathsf{cod}(\mathsf{x}_A))] = [\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))],$$

    as desired.
  - Suppose $[t_2] = [c_{A,f_2}]$ for some $f_2 \in \mathbb{C}_A$. Then $[\mathsf{cod}(t_2)] = [\mathsf{cod}\,(c_{A,f_2})] = \left[c_{O,\mathsf{cod}^{\mathbb{C}}(f_2)}\right]$. Since $[\mathsf{dom}(t_1)] = [\mathsf{cod}(t_2)]$, it follows by Lemma 3.8.1 and the induction hypothesis for t_1 that we must have $[t_1] = [c_{A,f_1}]$ for some $f_1 \in \mathbb{C}_A$. Since $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash t_1 \circ t_2 \downarrow$, it follows that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}, \mathsf{x}_A) \vdash c_{A,f_2} \circ c_{A,f_1} \downarrow$, and hence (by the theorem on constants) that $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}) \vdash c_{A,f_2} \circ c_{A,f_1} \downarrow$. Since $\widehat{\mathbb{C}} \models \mathbb{T}_{\mathsf{Cat}}(\mathbb{C})$, it then follows (by soundness of partial Horn logic) that $\widehat{\mathbb{C}} \models c_{A,f_2} \circ c_{A,f_1} \downarrow$, which implies that $f_2 \circ^C f_1$ is defined in $\mathbb{C}$. So then $\mathbb{T}_{\mathsf{Cat}}(\mathbb{C}) \vdash c_{A,f_2} \circ c_{A,f_1} = c_{f_2 \circ^{\mathbb{C}} f_1}$, from which we infer that

    $$[t] = [t_1 \circ t_2] = [c_{A,f_2} \circ c_{A,f_1}] = \left[c_{f_2 \circ^{\mathbb{C}} f_1}\right],$$

    as desired.

∎

Lemma (3.8.8)**.** *Let $\mathbb{C}$ be a small category with terminal object $1^{\mathbb{C}}$.*

1. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_O) \nvdash \mathsf{x}_O = c_{O,b}$. For any $f \in \mathbb{C}_A$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{x}_A = c_{A,f}$.*
2. *For any $b \in \mathbb{C}_O$, we have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = c_{O,b}$ and $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{cod}(\mathsf{x}_A) = c_{O,b}$.*
3. *We have $\mathbb{T}_{\mathsf{Ter}}(\mathbb{C}, \mathsf{x}_A) \nvdash \mathsf{dom}(\mathsf{x}_A) = \mathsf{cod}(\mathsf{x}_A)$.*

Proof: Fix $b \in \mathbb{C}_O$ and $f \in \mathbb{C}_A$. We prove (1) and (2) for b and (1) for f all at once. By Lemma 3.1.2, it suffices to find a small category $\mathbb{D}$ with terminal object $1^{\mathbb{D}}$, a functor $F : \mathbb{C} \to \mathbb{D}$ with $F(1^{\mathbb{C}}) = 1^{\mathbb{D}}$, an object $d \in \mathbb{D}_O$ with $F(b) \neq d$, an arrow $g \in \mathbb{D}_A$ with $F(f) \neq g$, and an arrow $h : d_1 \to d_2 \in \mathbb{D}_A$ with $F(b) \neq d_1$ and $F(b) \neq d_2$.

Suppose first that $\mathbb{C}$ has another object b' besides b. Then $\mathbb{C}$ also has at least two distinct arrows, namely $\mathsf{id}^{\mathbb{C}}(b) \neq \mathsf{id}^{\mathbb{C}}(b')$. So we set $\mathbb{D} := \mathbb{C}$ and $F := \mathsf{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, we set $d := b'$, we set g to be an arrow in $\mathbb{C}$ distinct from f, and we set $h := \mathsf{id}^{\mathbb{C}}(b')$. Also, since $\mathbb{C}$ has a terminal object and at least two distinct objects, it follows that $\mathbb{C}$ must contain distinct objects b_1, b_2 and an arrow $f : b_1 \to b_2$. Then by Lemma 3.1.2, we obtain (3).

Now suppose that $\mathbb{C}$ has only one object. Since $\mathbb{C}$ has a terminal object, it follows that $\mathbb{C}$ is the terminal category. By embedding $\mathbb{C}$ into the category with just two distinct objects and one arrow between them, it then easily follows by Lemma 3.1.2 that (1), (2), and (3) hold for $\mathbb{C}$. ∎

Lemma (3.9.15)**.** *If $\mathbb{C} \in \mathsf{StrMonCat}$ and $w \in W_O \cup W_A$, then*

$$(w^{\mathsf{exp}})^{\mathsf{exp}} \equiv w^{\mathsf{exp}}.$$

Proof: We show the claim for $w \in W_O$. First, we claim that if $v \in W_O$ is reduced, then $(v^e)^r \equiv v$. But we clearly have $v^e \to_r^* v$, so we obtain $(v^e) \leftrightarrow_r^* v$. Then by Corollary 3.9.6 we obtain $(v^e)^r \equiv v^r \equiv v$ (by Lemma 3.9.7, since v is reduced).

Now if $w \in W_O$ is arbitrary, we have

$$(w^{\mathsf{exp}})^{\mathsf{exp}} \equiv (((w^r)^e)^r)^e \equiv (w^r)^e \equiv w^{\mathsf{exp}},$$

since w^r is reduced (by Lemma 3.9.7). ∎

Lemma (3.9.26)**.** *Let* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $u, v \in W_A^{\mathbb{C}}$ *with* $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$*. Then*

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v \downarrow$$

and

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v = u \circ' v.$$

Proof: First let $u, v \in S_A^{\mathbb{C}}$ with $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$. Then by Lemma 3.9.17, we have

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{cod}(v) = v^{\mathsf{cod}} \equiv u^{\mathsf{dom}} = \mathsf{dom}(u),$$

so that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v \downarrow .$$

By checking the four base cases in Definition 3.9.24, it is also easy to see that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v = u \circ' v.$$

Now suppose that $u, v \in W_A^{\mathbb{C}}$ with $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$. Then as in Definition 3.9.24, we infer that $u \equiv u_1 \otimes_A \ldots \otimes_A u_n$ and $v \equiv v_1 \otimes_A \ldots \otimes_A v_n$ for some $n \geq 1$ and $u_i, v_i \in S_A$ with $v_i^{\mathsf{cod}} \equiv u_i^{\mathsf{dom}}$ for all $1 \leq i \leq n$. By Lemma 3.9.17, we again obtain

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{cod}(v) = v^{\mathsf{cod}} \equiv u^{\mathsf{dom}} = \mathsf{dom}(u),$$

so that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u \circ v \downarrow .$$

By the first part of the proof, we also know that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u_i \circ v_i \downarrow$$

and

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash u_i \circ v_i = u_i \circ' v_i$$

for all $1 \leq i \leq n$. So then $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A)$ proves the following sequence of equalities, as desired:

$$\begin{aligned} u \circ v &\equiv (u_1 \otimes_A \ldots \otimes_A u_n) \circ (v_1 \otimes_A \ldots \otimes_A v_n) \\ &= (u_1 \circ v_1) \otimes_A \ldots \otimes_A (u_n \circ v_n) \\ &= (u_1 \circ' v_1) \otimes_A \ldots \otimes_A (u_n \circ' v_n) \\ &=: (u_1 \otimes_A \ldots \otimes_A u_n) \circ' (v_1 \otimes_A \ldots \otimes_A v_n) \\ &\equiv u \circ' v. \end{aligned}$$

∎

Lemma (3.9.27)**.** *Let* $\mathbb{C} \in \mathsf{StrMonCat}$.

1. *For any* $w \in W_A$*, we have* $((w^r)^{\mathsf{dom}})^r \equiv (w^{\mathsf{dom}})^r$ *and* $((w^r)^{\mathsf{cod}})^r \equiv (w^{\mathsf{cod}})^r$.
2. *For any* $w \in W_A$*, we have* $(w^{\mathsf{exp}})^{\mathsf{cod}} \equiv (w^{\mathsf{cod}})^{\mathsf{exp}}$ *and* $(w^{\mathsf{exp}})^{\mathsf{dom}} \equiv (w^{\mathsf{dom}})^{\mathsf{exp}}$.
3. *For any* $w \in W_O \cup W_A$*, we have* $(w^{\mathsf{exp}})^r \equiv w^r$ *and* $(w^r)^{\mathsf{exp}} \equiv w^{\mathsf{exp}}$.
4. *For any* $u, v \in W_A^{sr}$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$*, we have* $(((u \circ' v)^r)^{\mathsf{dom}})^r \equiv (v^{\mathsf{dom}})^r$ *and* $(((u \circ' v)^r)^{\mathsf{cod}})^r \equiv (u^{\mathsf{cod}})^r$.
5. *For any* $u, v \in W_A^{sr}$ *with* $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$,
$$u \circ' v \to_e^* u^e \circ' v^e.$$
6. *For any* $u, v \in W_A^r$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$*, we have* $(u \circ' v)^{\mathsf{exp}} \equiv u^e \circ' v^e$.
7. *For any* $u, v \in W_A^r$ *with* $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$*, we have*
$$(u^{\mathsf{exp}} \circ' v^{\mathsf{exp}})^{\mathsf{exp}} \equiv u^{\mathsf{exp}} \circ' v^{\mathsf{exp}} \equiv (u \circ' v)^{\mathsf{exp}}.$$
8. *For any* $s, t, u \in W_A^{sr}$ *with* $s^{\mathsf{dom}} \equiv t^{\mathsf{cod}}$ *and* $t^{\mathsf{dom}} \equiv u^{\mathsf{cod}}$*, we have* $s \circ' (t \circ' u) \equiv (s \circ' t) \circ' u$.
9. *For any* $w \in W_O$*, we have* $(w^{\mathsf{id}})^{\mathsf{dom}} \equiv w \equiv (w^{\mathsf{id}})^{\mathsf{cod}}$.
10. *For any* $w \in W_O$*, we have* $(w^r)^{\mathsf{id}} \equiv (w^{\mathsf{id}})^r$ *and* $(w^{\mathsf{exp}})^{\mathsf{id}} \equiv (w^{\mathsf{id}})^{\mathsf{exp}}$.
11. *For any* $w \in W_A$*, we have* $w \circ' (w^{\mathsf{dom}})^{\mathsf{id}} \equiv w$ *and* $(w^{\mathsf{cod}})^{\mathsf{id}} \circ' w \equiv w$.
12. *For any* $C \in \{O, A\}$ *and any* $u, v \in W_C$*, we have*
$$(u^r \otimes_C v^r)^r \equiv (u \otimes_C v)^r \equiv (u^{\mathsf{exp}} \otimes_C v^{\mathsf{exp}})^r.$$
13. *For any* $s_1, s_2, t_1, t_2 \in W_A^r$ *with* $s_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}}$ *and* $t_1^{\mathsf{dom}} \equiv t_2^{\mathsf{cod}}$*, we have* $(s_1 \circ' s_2) \otimes_A (t_1 \circ' t_2) \equiv (s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2)$.
14. *For any* $w \in W_O$*, we have* $\big(w \otimes_O c_{O,e^{\mathbb{C}}}\big)^r \equiv w^r \equiv \big(c_{O,e^{\mathbb{C}}} \otimes_O w\big)^r$*, and for any* $w \in W_A$*, we have* $\big(w \otimes_A c_{A,\mathsf{id}(e^{\mathbb{C}})}\big)^r \equiv w^r \equiv \big(c_{A,\mathsf{id}(e^{\mathbb{C}})} \otimes_A w\big)^r$.
15. *For any* $C \in \{O, A\}$*, any* $s \in S_C$*, and any* $w \in W_C$*, we have* $(s \otimes_C w)^r \equiv (s \otimes_C w^r)^r$ *and* $(w \otimes_C s)^r \equiv (w^r \otimes_C s)^r$.

16. *For any* $s_1, s_2, t_1, t_2 \in W_A^r$ *with* $s_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}}$ *and* $t_1^{\mathsf{dom}} \equiv t_2^{\mathsf{cod}}$, *we have*

$$(s_1 \otimes_A t_1)^{\mathsf{exp}} \circ' (s_2 \otimes_A t_2)^{\mathsf{exp}} \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^{\mathsf{exp}}.$$

Proof:

- For (1), we only prove the claim for dom. So we must prove for any $w \in W_A$ that $((w^r)^{\mathsf{dom}})^r \equiv (w^{\mathsf{dom}})^r$. By Corollary 3.9.6, it is equivalent to show that

$$(w^r)^{\mathsf{dom}} \leftrightarrow_r^* w^{\mathsf{dom}}.$$

  But we have $w \to_r^* w^r$, and hence $w^{\mathsf{dom}} \to_r^* (w^r)^{\mathsf{dom}}$ by Lemma 3.9.18, as desired.

- For (2), we only prove the claim for dom. So we must show for any $w \in W_A$ that $(w^{\mathsf{exp}})^{\mathsf{dom}} \equiv (w^{\mathsf{dom}})^{\mathsf{exp}}$, i.e. that

$$((w^r)^e)^{\mathsf{dom}} \equiv ((w^{\mathsf{dom}})^r)^e.$$

  By part (1), it suffices to show

$$((w^r)^e)^{\mathsf{dom}} \equiv (((w^r)^{\mathsf{dom}})^r)^e.$$

  And to show *this*, it suffices to show for any *reduced* $v \in W_A^r$ that

$$(v^e)^{\mathsf{dom}} \equiv (v^{\mathsf{dom}})^{\mathsf{exp}},$$

  since w^r is reduced. So let $v \in W_A^r$ be reduced, and let us show

$$(v^e)^{\mathsf{dom}} \equiv (v^{\mathsf{dom}})^{\mathsf{exp}} \equiv ((v^{\mathsf{dom}})^r)^e.$$

  So we must show that the normal form of $(v^{\mathsf{dom}})^r$ with respect to $\to_e$ is $(v^e)^{\mathsf{dom}}$. Since $v \to_e^* v^e$, it then follows by Lemma 3.9.19 that $v^{\mathsf{dom}} \to_e^* (v^e)^{\mathsf{dom}}$. Since $(v^{\mathsf{dom}})^r \to_e^* v^{\mathsf{dom}}$ by Lemma 3.9.14 (because v^{dom} is semi-reduced if v is reduced), we finally obtain $(v^{\mathsf{dom}})^r \to_e^* (v^e)^{\mathsf{dom}}$, which yields the result.

- To prove (3), we first note that the second claim is trivial: we have

$$(w^r)^{\mathsf{exp}} \equiv ((w^r)^r)^e \equiv (w^r)^e \equiv w^{\mathsf{exp}},$$

  since w^r is reduced and hence $(w^r)^r \equiv w^r$ by Lemma 3.9.7.

  For the first claim in (3), let $w \in W_O$; we show that $(w^{\mathsf{exp}})^r \equiv w^r$ (the proof for $w \in W_A$ is analogous). So we must show $((w^r)^e)^r \equiv w^r$. To show this, it suffices to show that if $v \in W_O$ is *reduced*, then $(v^e)^r \equiv v$. Because then for arbitrary $w \in W_O$ we would obtain the desired claim, since w^r is reduced. But we proved this during the proof of Lemma 3.9.15.

- Claim (4) follows from Lemma 3.9.25 and (1).

- To prove (5), we must show for any $u, v \in W_A^{sr}$ with $v^{\mathsf{cod}} \equiv u^{\mathsf{dom}}$ that

  $$u \circ' v \to_e^* u^e \circ' v^e.$$

  First, we show that the right-hand composition is well-defined, i.e. we show that $(u^e)^{\mathsf{dom}} \equiv (v^e)^{\mathsf{cod}}$. We have

  $$\begin{aligned} (u^e)^{\mathsf{dom}} &\equiv ((u^r)^e)^{\mathsf{dom}} && \text{(by Lemma 3.9.14)} \\ &\equiv (u^{\mathsf{exp}})^{\mathsf{dom}} \\ &\equiv (u^{\mathsf{dom}})^{\mathsf{exp}} && \text{(by part (2))} \\ &\equiv (v^{\mathsf{cod}})^{\mathsf{exp}} && \text{(by assumption)} \\ &\equiv (v^{\mathsf{exp}})^{\mathsf{cod}} && \text{(by part (2))} \\ &\equiv ((v^r)^e)^{\mathsf{cod}} \\ &\equiv (v^e)^{\mathsf{cod}} && \text{(by Lemma 3.9.14),} \end{aligned}$$

  as required. It is now intuitively clear that the desired result holds, since $u^e \circ' v^e$ can be obtained from $u \circ' v$ by inserting occurrences of $c_{A,\mathsf{id}(e^{\mathbb{C}})}$.

- To prove (6), we first note that if $u, v \in W_A^r$ have the property that $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, then $u^e \circ' v^e$ is actually defined, as shown in the proof of (5). Now, we must show for all $u, v \in W_A^r$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$ that

  $$(u \circ' v)^{\mathsf{exp}} \equiv u^e \circ' v^e,$$

  i.e.

  $$((u \circ' v)^r)^e \equiv u^e \circ' v^e.$$

  So we want to show that the normal form of $(u \circ' v)^r$ with respect to $\to_e$ is $u^e \circ' v^e$. Since u, v are reduced, it easily follows that $u \circ' v$ is semi-reduced. So by Lemma 3.9.14 we obtain $(u \circ' v)^r \to_e^* u \circ' v$. We also have $u \circ' v \to_e^* u^e \circ' v^e$ by part (5). So we finally have $(u \circ' v)^r \to_e^* u^e \circ' v^e$, which yields the result.

- To prove (7), let $u, v \in W_A^r$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$. First, we note that $u^{\mathsf{exp}} \circ' v^{\mathsf{exp}} \in W_A$ is defined, because $(u^{\mathsf{exp}})^{\mathsf{dom}} \equiv (v^{\mathsf{exp}})^{\mathsf{cod}}$ by the assumption and part (2). Now, we must show

  $$(u^{\mathsf{exp}} \circ' v^{\mathsf{exp}})^{\mathsf{exp}} \equiv u^{\mathsf{exp}} \circ' v^{\mathsf{exp}} \equiv (u \circ' v)^{\mathsf{exp}}.$$

  Since u, v are reduced, it follows by Lemma 3.9.7 that $u^r \equiv u$ and $v^r \equiv v$. So then we have $u^{\mathsf{exp}} \equiv (u^r)^e \equiv u^e$ and similarly $v^{\mathsf{exp}} \equiv v^e$. So the right-hand $\equiv$ is just part (6). For the left-hand $\equiv$, we have

  $$(u^{\mathsf{exp}} \circ' v^{\mathsf{exp}})^{\mathsf{exp}} \equiv (u^e \circ' v^e)^{\mathsf{exp}}$$

$$
\begin{aligned}
&\equiv ((u \circ' v)^{\mathsf{exp}})^{\mathsf{exp}} && \text{(by part (6))}\\
&\equiv (u \circ' v)^{\mathsf{exp}} && \text{(by Lemma 3.9.15)}\\
&\equiv u^{\mathsf{exp}} \circ' v^{\mathsf{exp}} && \text{(by the right-hand } \equiv\text{)},
\end{aligned}
$$

as desired.

- The proof of (8) is almost trivial. The proof of (9) is by an easy induction on $\ell(w)$.

- For (10), we first prove that if $w \in W_O$, then $(w^r)^{\mathsf{id}} \equiv (w^{\mathsf{id}})^r$. So we want to prove that the normal form of w^{id} with respect to $\to_r$ is $(w^r)^{\mathsf{id}}$. It suffices by Corollary 3.9.6 to prove that $w^{\mathsf{id}} \to_r^* (w^r)^{\mathsf{id}}$. But we have $w \to_r^* w^r$ and hence $w^{\mathsf{id}} \to_r^* (w^r)^{\mathsf{id}}$ by Lemma 3.9.23, as desired.

  Now we show for any $w \in W_O$ that $(w^{\mathsf{exp}})^{\mathsf{id}} \equiv (w^{\mathsf{id}})^{\mathsf{exp}}$, i.e. that $((w^r)^e)^{\mathsf{id}} \equiv ((w^{\mathsf{id}})^r)^e$. By what we just proved, it will suffice to show for any $w \in W_O$ that

  $$((w^r)^e)^{\mathsf{id}} \equiv ((w^r)^{\mathsf{id}})^e.$$

  Since w^r is reduced, it will then suffice to show for any *reduced* $v \in W_O^r$ that

  $$(v^e)^{\mathsf{id}} \equiv (v^{\mathsf{id}})^e.$$

  So we want to show that the normal form of v^{id} with respect to $\to_e$ is $(v^e)^{\mathsf{id}}$. But we have $v \to_e^* v^e$ and hence $v^{\mathsf{id}} \to_e^* (v^e)^{\mathsf{id}}$ by Lemma 3.9.23, which yields the result.

- The proof of (11) is by an easy induction on $\ell(w)$.

- For (12), we only show the claim for W_O. So we must show for all $u, v \in W_O$ that

  $$(u^r \otimes_O v^r)^r \equiv (u \otimes_O v)^r \equiv (u^{\mathsf{exp}} \otimes_O v^{\mathsf{exp}})^r,$$

  i.e.

  $$(u^r \otimes_O v^r)^r \equiv (u \otimes_O v)^r \equiv ((u^r)^e \otimes_O (v^r)^e)^r.$$

  It will suffice to show that the first $\equiv$ holds for all $u, v \in W_O$. Because then we obtain the second $\equiv$ as follows:

  $$
  \begin{aligned}
  ((u^r)^e \otimes_O (v^r)^e)^r &\equiv (((u^r)^e)^r \otimes_O ((v^r)^e)^r)^r && \text{(by the first } \equiv \text{ applied to } (u^r)^e, (v^r)^e)\\
  &\equiv ((u^{\mathsf{exp}})^r \otimes_O (v^{\mathsf{exp}})^r)^r\\
  &\equiv (u^r \otimes_O v^r)^r && \text{(by part (3))}\\
  &\equiv (u \otimes_O v)^r && \text{(by the first } \equiv \text{ applied to } u, v).
  \end{aligned}
  $$

  So we must show

  $$(u \otimes_O v)^r \equiv (u^r \otimes_O v^r)^r$$

for all $u, v \in W_O$. By Corollary 3.9.6, it suffices to show that

$$u \otimes_O v \leftrightarrow_r^* u^r \otimes_O v^r.$$

But we have $u \to_r^* u^r$ and $v \to_r^* v^r$, so it easily follows that $u \otimes_O v \to_r^* u^r \otimes_O v^r$, as desired.

- The proof of (13) is by an easy application of the definitions, while the proofs of (14) and (15) are straightforward applications of Corollary 3.9.6.
- To prove (16), let $s_1, s_2, t_1, t_2 \in W_A^r$ satisfy the hypotheses. First, we show that both sides of the equality are defined. For the left side, we have

$$\begin{aligned} ((s_1 \otimes_A t_1)^{\mathsf{exp}})^{\mathsf{dom}} &\equiv ((s_1 \otimes_A t_1)^{\mathsf{dom}})^{\mathsf{exp}} && \text{(by part (2))} \\ &\equiv (s_1^{\mathsf{dom}} \otimes_O t_1^{\mathsf{dom}})^{\mathsf{exp}} \\ &\equiv (s_2^{\mathsf{cod}} \otimes_O t_2^{\mathsf{cod}})^{\mathsf{exp}} && \text{(by hypothesis)} \\ &\equiv ((s_2 \otimes_A t_2)^{\mathsf{cod}})^{\mathsf{exp}} \\ &\equiv ((s_2 \otimes_A t_2)^{\mathsf{exp}})^{\mathsf{cod}} && \text{(by part (2))}, \end{aligned}$$

as required. And for the right side, we have

$$(s_1 \otimes_A t_1)^{\mathsf{dom}} \equiv s_1^{\mathsf{dom}} \otimes_O t_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}} \otimes_O t_2^{\mathsf{cod}} \equiv (s_2 \otimes_A t_2)^{\mathsf{cod}}.$$

Now we must show

$$(s_1 \otimes_A t_1)^{\mathsf{exp}} \circ' (s_2 \otimes_A t_2)^{\mathsf{exp}} \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^{\mathsf{exp}},$$

i.e.

$$((s_1 \otimes_A t_1)^r)^e \circ' ((s_2 \otimes_A t_2)^r)^e \equiv \left(((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r\right)^e.$$

So we must show that the normal form of $((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r$ with respect to $\to_e$ is $((s_1 \otimes_A t_1)^r)^e \circ' ((s_2 \otimes_A t_2)^r)^e$. There are two cases to consider:

  - Suppose that $s_1 \otimes_A t_1$ is reduced. Then at least one of $\mathsf{last}(s_1), \mathsf{first}(t_1)$ is *not* an element of $\{c_{A,f} : f \in \mathbb{C}_A\}$. Since $s_1^{\mathsf{dom}} \equiv s_2^{\mathsf{cod}}$ and $t_1^{\mathsf{dom}} \equiv t_2^{\mathsf{cod}}$, it then easily follows that the same is true for $\mathsf{last}(s_2), \mathsf{first}(t_2)$. Then since s_2, t_2 are reduced, it follows that $s_2 \otimes_A t_2$ is reduced as well. So then $(s_1 \otimes_A t_1)^r \equiv s_1 \otimes_A t_1$ and $(s_2 \otimes_A t_2)^r \equiv s_2 \otimes_A t_2$ by Lemma 3.9.7, and it also follows that $(s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2)$ is semi-reduced. So by Lemma 3.9.14, we have

$$((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r \to_e^* (s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2).$$

Since $s_1 \otimes_A t_1 \to_e^* (s_1 \otimes_A t_1)^e$ and $s_2 \otimes_A t_2 \to_e^* (s_2 \otimes_A t_2)^e$, we then obtain by part (5)

$$(s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2) \to_e^* (s_1 \otimes_A t_1)^e \circ' (s_2 \otimes_A t_2)^e.$$

Altogether, we have

$$((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r \to_e^* (s_1 \otimes_A t_1)^e \circ' (s_2 \otimes_A t_2)^e,$$

which yields the result.

- Suppose that $s_1 \otimes_A t_1$ is *not* reduced. Since s_1, t_1 *are* reduced, this must be because

  $$\mathsf{last}(s_1), \mathsf{first}(t_1) \in \{c_{A,f} : f \in \mathbb{C}_A\}.$$

  By reasoning used in the previous case, it then follows that $\mathsf{last}(s_2), \mathsf{first}(t_2) \in \{c_{A,f} : f \in \mathbb{C}_A\}$, so that $s_2 \otimes_A t_2$ also fails to be reduced. Let

  $$s_1 \equiv \widehat{s_1} \otimes_A c_{A,f_1},$$
  $$t_1 \equiv c_{A,g_1} \otimes_A \widehat{t_1},$$
  $$s_2 \equiv \widehat{s_2} \otimes_A c_{A,f_2},$$
  $$t_2 \equiv c_{A,g_2} \otimes_A \widehat{t_2},$$

  with $\widehat{s_1}, \widehat{t_1}, \widehat{s_2}, \widehat{t_2}$ (possibly empty) reduced words in W_A such that $\widehat{s_1}^{\mathsf{dom}} \equiv \widehat{s_2}^{\mathsf{cod}}$ and $\widehat{t_1}^{\mathsf{dom}} \equiv \widehat{t_2}^{\mathsf{cod}}$, and $\mathsf{dom}^{\mathbb{C}}(f_1) = \mathsf{cod}^{\mathbb{C}}(f_2)$ and $\mathsf{dom}^{\mathbb{C}}(g_1) = \mathsf{cod}^{\mathbb{C}}(g_2)$. Then

  $$\begin{aligned}&((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r \\ \equiv &\left(\left(\widehat{s_1} \otimes_A c_{A,f_1} \otimes_A c_{A,g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2} \otimes_A c_{A,g_2} \otimes_A \widehat{t_2}\right)\right)^r.\end{aligned}$$

  We also have

  $$(s_1 \otimes_A t_1)^r \equiv \left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right)^r$$

  and

  $$(s_2 \otimes_A t_2)^r \equiv \left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right)^r.$$

  Since

  $$\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}$$

  and

  $$\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}$$

  are semi-reduced, it follows from Lemma 3.9.14 that

  $$((s_1 \otimes_A t_1)^r)^e \equiv \left(\left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right)^r\right)^e \equiv \left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right)^e$$

and

$$((s_2 \otimes_A t_2)^r)^e \equiv \left(\left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right)^r\right)^e \equiv \left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right)^e .$$

In the proof of (6), we showed that if u, v are semi-reduced and $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, then the normal form of $(u \circ' v)^r$ with respect to $\to_e$ is $u^e \circ' v^e$. So it now remains to show that

$$\left(\left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right)\right)^r$$
$$\equiv \left(\left(\widehat{s_1} \otimes_A c_{A,f_1} \otimes_A c_{A,g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2} \otimes_A c_{A,g_2} \otimes_A \widehat{t_2}\right)\right)^r .$$

By Lemma 3.9.6, it suffices to show that

$$\left(\widehat{s_1} \otimes_A c_{A,f_1} \otimes_A c_{A,g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2} \otimes_A c_{A,g_2} \otimes_A \widehat{t_2}\right)$$
$$\to_r^* \left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right).$$

Well, we have

$$\begin{aligned}
&\left(\widehat{s_1} \otimes_A c_{A,f_1} \otimes_A c_{A,g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2} \otimes_A c_{A,g_2} \otimes_A \widehat{t_2}\right) \\
&\equiv \left(\widehat{s_1} \circ' \widehat{s_2}\right) \otimes_A \left(c_{A,f_1} \circ' c_{A,f_2}\right) \otimes_A \left(c_{A,g_1} \circ' c_{A,g_2}\right) \otimes_A \left(\widehat{t_1} \circ' \widehat{t_2}\right) \\
&\equiv \left(\widehat{s_1} \circ' \widehat{s_2}\right) \otimes_A c_{A,f_1 \circ f_2} \otimes_A c_{A,g_1 \circ g_2} \otimes_A \left(\widehat{t_1} \circ' \widehat{t_2}\right) \\
&\to_r \left(\widehat{s_1} \circ' \widehat{s_2}\right) \otimes_A c_{A,(f_1 \circ f_2) \otimes_A^{\mathbb{C}} (g_1 \circ g_2)} \otimes_A \left(\widehat{t_1} \circ' \widehat{t_2}\right) \\
&\equiv \left(\widehat{s_1} \circ' \widehat{s_2}\right) \otimes_A c_{A,(f_1 \otimes_A^{\mathbb{C}} g_1) \circ (f_2 \otimes_A^{\mathbb{C}} g_2)} \otimes_A \left(\widehat{t_1} \circ' \widehat{t_2}\right) \\
&\equiv \left(\widehat{s_1} \circ' \widehat{s_2}\right) \otimes_A \left(c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \circ' c_{A,f_2 \otimes_A^{\mathbb{C}} \circ g_2}\right) \otimes_A \left(\widehat{t_1} \circ' \widehat{t_2}\right) \\
&\equiv \left(\widehat{s_1} \otimes_A c_{A,f_1 \otimes_A^{\mathbb{C}} g_1} \otimes_A \widehat{t_1}\right) \circ' \left(\widehat{s_2} \otimes_A c_{A,f_2 \otimes_A^{\mathbb{C}} g_2} \otimes_A \widehat{t_2}\right),
\end{aligned}$$

as desired.

This completes the proof of (16).

∎

Proposition (3.9.29)**.** *For any $\mathbb{C} \in \mathsf{StrMonCat}$, the partial Σ_{Str}-structure $\mathbb{C}^*$ is a model of $\mathbb{T}_{\mathsf{Str}}$, i.e. $\mathbb{C}^*$ is a strict monoidal category.*

Proof: We begin by showing that $\mathbb{C}^*$ is a category. First, the functions $\mathsf{id}^* : \mathbb{C}_O^* \to \mathbb{C}_A^*$ and $\mathsf{dom}^*, \mathsf{cod}^* : \mathbb{C}_A^* \to \mathbb{C}_O^*$ are all total, as required.

Now let $s, t \in \mathbb{C}^*_A = W^r_A$: we must show that

$$(s,t) \in \mathsf{dom}(\circ^*) \text{ iff } \mathsf{dom}^*(s) = \mathsf{cod}^*(t).$$

If $(s,t) \in \mathsf{dom}(\circ^*)$, then this means that $(t^{\mathsf{cod}})^{\mathsf{exp}} \equiv (s^{\mathsf{dom}})^{\mathsf{exp}}$. Then we have

$$\begin{aligned}
\mathsf{dom}^*(s) &= (s^{\mathsf{dom}})^r && \text{(by definition)}\\
&\equiv ((s^{\mathsf{dom}})^{\mathsf{exp}})^r && \text{(by Lemma 3.9.27.3)}\\
&\equiv ((t^{\mathsf{cod}})^{\mathsf{exp}})^r && \text{(by assumption)}\\
&\equiv (t^{\mathsf{cod}})^r && \text{(by Lemma 3.9.27.3 again)}\\
&= \mathsf{cod}^*(t), && \text{(by definition)}
\end{aligned}$$

as desired. And if $\mathsf{dom}^*(s) = \mathsf{cod}^*(t)$, then this means $(s^{\mathsf{dom}})^r \equiv (t^{\mathsf{cod}})^r$, which implies

$$\begin{aligned}
(s^{\mathsf{dom}})^{\mathsf{exp}} &\equiv ((s^{\mathsf{dom}})^r)^{\mathsf{exp}} && \text{(by Lemma 3.9.27.3)}\\
&\equiv ((t^{\mathsf{cod}})^r)^{\mathsf{exp}} && \text{(by assumption)}\\
&\equiv (t^{\mathsf{cod}})^{\mathsf{exp}}, && \text{(by Lemma 3.9.27.3 again)}
\end{aligned}$$

which means that $(s,t) \in \mathsf{dom}(\circ^*)$, as desired.

Next, let $s, t \in W^r_A$ with $(s,t) \in \mathsf{dom}(\circ^*)$. We must show that

$$\mathsf{dom}^*(s \circ^* t) = \mathsf{dom}^*(t) \text{ and } \mathsf{cod}^*(s \circ^* t) = \mathsf{cod}^*(s).$$

Since both claims are proven analogously, we only consider the case for dom. Unravelling the definitions, we must show that

$$(((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^r \equiv (t^{\mathsf{dom}})^r.$$

We have

$$\begin{aligned}
(((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^r &\equiv ((t^{\mathsf{exp}})^{\mathsf{dom}})^r && \text{(by Lemma 3.9.27.4)}\\
&\equiv ((t^{\mathsf{dom}})^{\mathsf{exp}})^r && \text{(by Lemma 3.9.27.2)}\\
&\equiv (t^{\mathsf{dom}})^r, && \text{(by Lemma 3.9.27.3)}
\end{aligned}$$

as required.

Now we must show that composition is associative. So let $s, t, u \in W^r_A$ and suppose that $(t,u) \in \mathsf{dom}(\circ^*)$ and $(s, t \circ^* u) \in \mathsf{dom}(\circ^*)$. We must then show that $(s,t) \in \mathsf{dom}(\circ^*)$ and $(s \circ^* t, u) \in \mathsf{dom}(\circ^*)$ and that

$$s \circ^* (t \circ^* u) = (s \circ^* t) \circ^* u.$$

That $(t,u) \in \mathsf{dom}(\circ^*)$ means that

$$(t^{\mathsf{dom}})^{\mathsf{exp}} \equiv (u^{\mathsf{cod}})^{\mathsf{exp}},$$

and that $(s, t \circ^* u) \in \mathsf{dom}(\circ^*)$ means that

$$(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv ((t \circ^* u)^{\mathsf{cod}})^{\mathsf{exp}} \equiv (((t^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r)^{\mathsf{cod}})^{\mathsf{exp}}.$$

To show $(s, t) \in \mathsf{dom}(\circ^*)$ means to show $(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv (t^{\mathsf{cod}})^{\mathsf{exp}}$. First, we have

$$\begin{aligned}
(s^{\mathsf{dom}})^r &\equiv ((s^{\mathsf{dom}})^{\mathsf{exp}})^r && \text{(by Lemma 3.9.27.3)}\\
&\equiv ((((t^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r)^{\mathsf{cod}})^{\mathsf{exp}})^r && \text{(by assumption)}\\
&\equiv (((t^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r)^{\mathsf{cod}})^r && \text{(by Lemma 3.9.27.3)}\\
&\equiv ((t^{\mathsf{exp}})^{\mathsf{cod}})^r && \text{(by Lemma 3.9.27.4)}\\
&\equiv ((t^{\mathsf{cod}})^{\mathsf{exp}})^r && \text{(by Lemma 3.9.27.2)}\\
&\equiv (t^{\mathsf{cod}})^r. && \text{(by Lemma 3.9.27.3)}
\end{aligned}$$

So then (using Lemma 3.9.27.3) we obtain

$$(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv ((s^{\mathsf{dom}})^r)^{\mathsf{exp}} \equiv ((t^{\mathsf{cod}})^r)^{\mathsf{exp}} \equiv (t^{\mathsf{cod}})^{\mathsf{exp}},$$

as desired.

To show that $(s \circ^* t, u) \in \mathsf{dom}(\circ^*)$, we must show that

$$(u^{\mathsf{cod}})^{\mathsf{exp}} \equiv ((s \circ^* t)^{\mathsf{dom}})^{\mathsf{exp}} \equiv (((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^{\mathsf{exp}}.$$

Since $(u^{\mathsf{cod}})^{\mathsf{exp}} \equiv (t^{\mathsf{dom}})^{\mathsf{exp}}$ by assumption, it suffices to show

$$(t^{\mathsf{dom}})^{\mathsf{exp}} \equiv (((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^{\mathsf{exp}}.$$

First, we have

$$\begin{aligned}
(((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^r &\equiv ((t^{\mathsf{exp}})^{\mathsf{dom}})^r && \text{(by Lemma 3.9.27.4)}\\
&\equiv ((t^{\mathsf{dom}})^{\mathsf{exp}})^r && \text{(by Lemma 3.9.27.2)}\\
&\equiv (t^{\mathsf{dom}})^r, && \text{(by Lemma 3.9.27.3)}
\end{aligned}$$

so that we obtain (using Lemma 3.9.27.3)

$$(t^{\mathsf{dom}})^{\mathsf{exp}} \equiv ((t^{\mathsf{dom}})^r)^{\mathsf{exp}} \equiv ((((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^r)^{\mathsf{exp}} \equiv (((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{dom}})^{\mathsf{exp}},$$

as desired.

To complete the proof that $\circ^*$ is associative, we must now show that

$$s \circ^* (t \circ^* u) = (s \circ^* t) \circ^* u.$$

Unravelling the definitions, we must show that

$$(s^{\mathsf{exp}} \circ' ((t^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r)^{\mathsf{exp}})^r \equiv (((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r)^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r,$$

i.e. (by Lemma 3.9.27.3)

$$(s^{\mathsf{exp}} \circ' (t^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^{\mathsf{exp}})^r \equiv ((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^{\mathsf{exp}} \circ' u^{\mathsf{exp}})^r,$$

i.e. (by Lemma 3.9.27.7)

$$(s^{\mathsf{exp}} \circ' (t^{\mathsf{exp}} \circ' u^{\mathsf{exp}}))^r \equiv ((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}}) \circ' u^{\mathsf{exp}})^r.$$

To show *this*, it of course suffices to show

$$s^{\mathsf{exp}} \circ' (t^{\mathsf{exp}} \circ' u^{\mathsf{exp}}) \equiv (s^{\mathsf{exp}} \circ' t^{\mathsf{exp}}) \circ' u^{\mathsf{exp}}.$$

But this follows from Lemma 3.9.27.8.

Next, we must show for any $w \in \mathbb{C}^*_O = W^r_O$ that

$$\mathsf{dom}^*(\mathsf{id}^*(w)) = w \text{ and } \mathsf{cod}^*(\mathsf{id}^*(w)) = w.$$

As usual, we only show the claim for dom. By the definitions, we must show

$$((w^{\mathsf{id}})^{\mathsf{dom}})^r \equiv w.$$

But since $w \in W^r_O$, we have by Lemma 3.9.27.9 and Lemma 3.9.7 that

$$((w^{\mathsf{id}})^{\mathsf{dom}})^r \equiv w^r \equiv w,$$

as desired.

To complete the proof that $\mathbb{C}^*$ is a category, we must show for any $s \in \mathbb{C}^*_A = W^r_A$ that

$$s \circ^* \mathsf{id}^*(\mathsf{dom}^*(s)) = s \text{ and } \mathsf{id}^*(\mathsf{cod}^*(s)) \circ^* s = s.$$

As usual, we only show the claim for dom. Unravelling the definitions, we must show that

$$(s^{\mathsf{exp}} \circ' (((s^{\mathsf{dom}})^r)^{\mathsf{id}})^{\mathsf{exp}})^r \equiv s.$$

By Lemma 3.9.27.10, this simplifies to showing

$$(s^{\mathsf{exp}} \circ' (((s^{\mathsf{dom}})^{\mathsf{id}})^r)^{\mathsf{exp}})^r \equiv s,$$

and then by Lemma 3.9.27.3, this simplifies to showing

$$(s^{\mathsf{exp}} \circ' ((s^{\mathsf{dom}})^{\mathsf{id}})^{\mathsf{exp}})^r \equiv s.$$

By Lemma 3.9.27.10 again, it suffices to show

$$(s^{\mathsf{exp}} \circ' ((s^{\mathsf{dom}})^{\mathsf{exp}})^{\mathsf{id}})^r \equiv s,$$

and then by Lemma 3.9.27.2, it suffices to show

$$(s^{\mathsf{exp}} \circ' ((s^{\mathsf{exp}})^{\mathsf{dom}})^{\mathsf{id}})^r \equiv s.$$

But since $s \in W_A^r$, we apply Lemma 3.9.27.11 and Lemma 3.9.27.3 and Lemma 3.9.7 to obtain

$$(s^{\mathsf{exp}} \circ' ((s^{\mathsf{exp}})^{\mathsf{dom}})^{\mathsf{id}})^r \equiv (s^{\mathsf{exp}})^r \equiv s^r \equiv s,$$

as desired. This completes the proof that $\mathbb{C}^*$ is a category.

Now we show that $\mathbb{C}^*$ is a *strict monoidal* category. First, we have that $e^* := c_{e^{\mathbb{C}}} \in \mathbb{C}_O^* = W_O^r$ is defined, and that $\otimes_O^* : \mathbb{C}_O^* \times \mathbb{C}_O^* \to \mathbb{C}_O^*$ and $\otimes_A^* : \mathbb{C}_A^* \times \mathbb{C}_A^* \to \mathbb{C}_A^*$ are totally defined operations, as required.

Now we must show that $\otimes_A^*$ commutes with dom^* and cod^*. So let $s, t \in \mathbb{C}_A^* = W_A^r$: we must show that

$$\mathsf{dom}^*(s \otimes_A^* t) = \mathsf{dom}^*(s) \otimes_O^* \mathsf{dom}^*(t) \text{ and } \mathsf{cod}^*(s \otimes_A^* t) = \mathsf{cod}^*(s) \otimes_O^* \mathsf{cod}^*(t).$$

As usual, we only treat the case for dom. Unravelling the definitions, we must show that

$$(((s \otimes_A t)^r)^{\mathsf{dom}})^r \equiv ((s^{\mathsf{dom}})^r \otimes_O (t^{\mathsf{dom}})^r)^r.$$

By Lemma 3.9.27.12, it then suffices to show

$$(((s \otimes_A t)^r)^{\mathsf{dom}})^r \equiv (s^{\mathsf{dom}} \otimes_O t^{\mathsf{dom}})^r.$$

But by Lemma 3.9.27.1 and the definition of the function $(-)^{\mathsf{dom}}$, we obtain

$$(((s \otimes_A t)^r)^{\mathsf{dom}})^r \equiv ((s \otimes_A t)^{\mathsf{dom}})^r \equiv (s^{\mathsf{dom}} \otimes_O t^{\mathsf{dom}})^r,$$

as desired.

Next, we must show that id^* commutes with $\otimes_O^*$. So let $s, t \in \mathbb{C}_O^* = W_O^r$: we must show

$$\mathsf{id}^*(s \otimes_O^* t) = \mathsf{id}^*(s) \otimes_A^* \mathsf{id}^*(t).$$

Unravelling the definitions, we must show

$$((s \otimes_O t)^r)^{\mathsf{id}} \equiv (s^{\mathsf{id}} \otimes_A t^{\mathsf{id}})^r.$$

By Lemma 3.9.27.10, it suffices to show

$$((s \otimes_O t)^{\mathsf{id}})^r \equiv (s^{\mathsf{id}} \otimes_A t^{\mathsf{id}})^r.$$

And to show this, it suffices to show

$$(s \otimes_O t)^{\mathsf{id}} \equiv s^{\mathsf{id}} \otimes_A t^{\mathsf{id}}.$$

But this is true by definition of the function $(-)^{\mathsf{id}} : W_O \to W_A$.

Now we must show that $\circ^*$ and $\otimes_A^*$ commute with each other. So let $s_1, s_2, t_1, t_2 \in \mathbb{C}_A^* = W_A^r$ with $(s_1, s_2) \in \mathsf{dom}(\circ^*)$ and $(t_1, t_2) \in \mathsf{dom}(\circ^*)$. We must show that

$$(s_1 \circ^* s_2) \otimes_A^* (t_1 \circ^* t_2) = (s_1 \otimes_A^* t_1) \circ^* (s_2 \otimes_A^* t_2).$$

Unravelling the definitions, we must show that

$$((s_1^{\mathsf{exp}} \circ' s_2^{\mathsf{exp}})^r \otimes_A (t_1^{\mathsf{exp}} \circ' t_2^{\mathsf{exp}})^r)^r \equiv (((s_1 \otimes_A t_1)^r)^{\mathsf{exp}} \circ' ((s_2 \otimes_A t_2)^r)^{\mathsf{exp}})^r.$$

By Lemma 3.9.27.12 applied to the left-hand side and Lemma 3.9.27.3 applied to the right-hand side, it then suffices to show

$$((s_1^{\mathsf{exp}} \circ' s_2^{\mathsf{exp}}) \otimes_A (t_1^{\mathsf{exp}} \circ' t_2^{\mathsf{exp}}))^r \equiv ((s_1 \otimes_A t_1)^{\mathsf{exp}} \circ' (s_2 \otimes_A t_2)^{\mathsf{exp}})^r.$$

By Lemma 3.9.27.16 and Lemma 3.9.27.3 applied to the right-hand side, it then suffices to show

$$((s_1^{\mathsf{exp}} \circ' s_2^{\mathsf{exp}}) \otimes_A (t_1^{\mathsf{exp}} \circ' t_2^{\mathsf{exp}}))^r \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r.$$

By Lemma 3.9.27.7 applied to the left-hand side, it then suffices to show

$$((s_1 \circ' s_2)^{\mathsf{exp}} \otimes_A (t_1 \circ' t_2)^{\mathsf{exp}})^r \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r.$$

By Lemma 3.9.27.12 applied to the left-hand side, it then suffices to show

$$((s_1 \circ' s_2) \otimes_A (t_1 \circ' t_2))^r \equiv ((s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2))^r.$$

Finally, it suffices to show

$$(s_1 \circ' s_2) \otimes_A (t_1 \circ' t_2) \equiv (s_1 \otimes_A t_1) \circ' (s_2 \otimes_A t_2).$$

But this is true by Lemma 3.9.27.13.

Now we must show that $\otimes_O^*$ and $\otimes_A^*$ are associative operations on $\mathbb{C}_O^*$ and $\mathbb{C}_A^*$ respectively. Let $C \in \{O, A\}$, and let $s, t, u \in \mathbb{C}_C^* = W_C^r$. We must show that

$$s \otimes_C^* (t \otimes_C^* u) = (s \otimes_C^* t) \otimes_C^* u.$$

Unravelling the definitions, we must show that

$$(s \otimes_C (t \otimes_C u)^r)^r \equiv ((s \otimes_C t)^r \otimes_C u)^r.$$

Since s and u are reduced, we have $s^r \equiv s$ and $u^r \equiv u$ by Lemma 3.9.7, so it is equivalent to show

$$(s^r \otimes_C (t \otimes_C u)^r)^r \equiv ((s \otimes_C t)^r \otimes_C u^r)^r.$$

Then by Lemma 3.9.27.12, it is equivalent to show

$$(s \otimes_C (t \otimes_C u))^r \equiv ((s \otimes_C t) \otimes_C u)^r.$$

But this is true, because we have

$$s \otimes_C (t \otimes_C u) \equiv (s \otimes_C t) \otimes_C u.$$

To complete the proof, we must show that $e^* := c_{e^{\mathbb{C}}} \in \mathbb{C}^*_O = W^r_O$ is a unit for $\otimes^*_O$, and that $\mathsf{id}^*(e^*) = \mathsf{id}^*(c_{e^{\mathbb{C}}}) = c_{\mathsf{id}_{e^{\mathbb{C}}}} \in \mathbb{C}^*_A = W^r_A$ is a unit for $\otimes^*_A$. For the object tensor, we must show for any $w \in \mathbb{C}^*_O = W^r_O$ that

$$w \otimes^*_O e^* = w = e^* \otimes^*_O w,$$

i.e.

$$(w \otimes_O c_{e^{\mathbb{C}}})^r \equiv w \equiv (c_{e^{\mathbb{C}}} \otimes_O w)^r.$$

But this follows by Lemma 3.9.27.14, since $w \equiv w^r$ (by Lemma 3.9.7, since w is reduced). The proof for the arrow tensor is analogous.

This finally completes the proof that $\mathbb{C}^*$ is a strict monoidal category. ∎

Definition (3.9.30)**.** Let $\mathbb{C} \in \mathsf{StrMonCat}$. We define a strict monoidal functor

$$i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^*$$

as follows:

- For any object $a \in \mathbb{C}_O$,

$$i_{\mathbb{C}}(a) := c_{O,a} \in \mathbb{C}^*_O = W^r_O.$$

- For any arrow $f \in \mathbb{C}_A$,

$$i_{\mathbb{C}}(f) := c_{A,f} \in \mathbb{C}^*_A = W^r_A.$$

Justification. First we show that i is functorial. So let $f \in \mathbb{C}_A$ be arbitrary: we must show that

$$\mathsf{dom}^*(i_{\mathbb{C}}(f)) = i_{\mathbb{C}}(\mathsf{dom}^{\mathbb{C}}(f)) \text{ and } \mathsf{cod}^*(i_{\mathbb{C}}(f)) = i_{\mathbb{C}}(\mathsf{cod}^{\mathbb{C}}(f)).$$

For the first claim, we have

$$\mathsf{dom}^*(i_{\mathbb{C}}(f)) = \mathsf{dom}^*(c_{A,f}) = \left((c_{A,f})^{\mathsf{dom}}\right)^r \equiv \left(c_{O,\mathsf{dom}^{\mathbb{C}}(f)}\right)^r \equiv c_{O,\mathsf{dom}^{\mathbb{C}}(f)} = i_{\mathbb{C}}(\mathsf{dom}^{\mathbb{C}}(f)),$$

as desired. The proof of the second claim is analogous.

Now let $f : a \to a', g : a' \to a'' \in \mathbb{C}_A$. We must show

$$i_{\mathbb{C}}(g \circ^{\mathbb{C}} f) = i_{\mathbb{C}}(g) \circ^{*} i_{\mathbb{C}}(f).$$

We have

$$\begin{aligned} i_{\mathbb{C}}(g \circ^{\mathbb{C}} f) &= c_{A, g \circ^{\mathbb{C}} f} \\ &\equiv \left(c_{A, g \circ^{\mathbb{C}} f}\right)^{r} \\ &\equiv \left(c_{A,g} \circ' c_{A,f}\right)^{r} \\ &\equiv \left((c_{A,g})^{\mathsf{exp}} \circ' (c_{A,f})^{\mathsf{exp}}\right)^{r} \\ &\equiv \left(i_{\mathbb{C}}(g)^{\mathsf{exp}} \circ' i_{\mathbb{C}}(f)^{\mathsf{exp}}\right)^{r} \\ &= i_{\mathbb{C}}(g) \circ^{*} i_{\mathbb{C}}(f), \end{aligned}$$

as desired.

Finally, let $a \in \mathbb{C}_O$: we must show

$$i_{\mathbb{C}}(\mathsf{id}^{\mathbb{C}}(a)) = \mathsf{id}^{*}(i_{\mathbb{C}}(a)).$$

We have

$$i_{\mathbb{C}}(\mathsf{id}^{\mathbb{C}}(a)) = c_{A, \mathsf{id}^{\mathbb{C}}(a)} \equiv (c_{O,a})^{\mathsf{id}} = \mathsf{id}^{*}\,(c_{O,a}) = \mathsf{id}^{*}(i_{\mathbb{C}}(a)),$$

as desired. This completes the proof that i is functorial.

To prove that $i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^{*}$ is a *strict monoidal* functor, we first show that $i_{\mathbb{C}}$ preserves the tensor operations. So let $C \in \{O, A\}$ and $x, y \in \mathbb{C}_C$. Then we have

$$i_{\mathbb{C}}(x \otimes_C^{\mathbb{C}} y) = c_{x \otimes_C^{\mathbb{C}} y} \equiv (c_x \otimes_C c_y)^{r} = c_x \otimes_C^{*} c_y = i_{\mathbb{C}}(x) \otimes_C^{*} i_{\mathbb{C}}(y),$$

as required. Finally, we have that $i_{\mathbb{C}}$ preserves the object tensor unit, since

$$i_{\mathbb{C}}(e^{\mathbb{C}}) = c_{O, e^{\mathbb{C}}} = e^{*}.$$

This completes the proof that $i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^{*}$ is a strict monoidal functor. ∎

Proposition (3.9.31)**.** Let $\mathbb{C} \in \mathsf{StrMonCat}$. If $\mathbb{D} \in \mathsf{StrMonCat}$ and $F : \mathbb{C} \to \mathbb{D}$ is a strict monoidal functor and $a, a' \in \mathbb{D}_O$ are objects and $f, f' \in \mathbb{D}_A$ are arrows, then there is a unique strict monoidal functor $F^{*} : \mathbb{C}^{*} \to \mathbb{D}$ with $F^{*} \circ i_{\mathbb{C}} = F$ and

$$F^{*}(\mathsf{x}_O) = a, F^{*}(\mathsf{x}'_O) = a', F^{*}(\mathsf{x}_A) = f, F^{*}(\mathsf{x}'_A) = f'.$$

Thus,

$$\mathbb{C}^{*} \cong \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle \in \mathsf{StrMonCat}.$$

Proof: We define a strict monoidal functor $F^* : \mathbb{C}^* \to \mathbb{D}$ as follows. First, we define F^* on $S_O^{\mathbb{C}}$ and $S_A^{\mathbb{C}}$. On $S_O^{\mathbb{C}}$, we set:

$$F^*(\mathsf{x}_O) := a, F^*(\mathsf{x}'_O) := a', F^*(\mathsf{dom}(\mathsf{x}_A)) := \mathsf{dom}^{\mathbb{D}}(f), F^*(\mathsf{cod}(\mathsf{x}_A)) := \mathsf{cod}^{\mathbb{D}}(f),$$

$$F^*(\mathsf{dom}(\mathsf{x}'_A)) := \mathsf{dom}^{\mathbb{D}}(f'), F^*(\mathsf{cod}(\mathsf{x}'_A)) := \mathsf{cod}^{\mathbb{D}}(f'),$$

and

$$F^*(c_{O,b}) := F(b)$$

for any $b \in \mathbb{C}_O$. We define F^* on $S_A^{\mathbb{C}}$ in an analogous fashion.

For any $s_1 \otimes_O \ldots \otimes_O s_n \in W_O$ and $t_1 \otimes_A \ldots \otimes_A t_m \in W_A$, we then set

$$F^*(s_1 \otimes_O \ldots \otimes_O s_n) := F^*(s_1) \otimes_O^{\mathbb{D}} \ldots \otimes_O^{\mathbb{D}} F^*(s_n) \in \mathbb{D}_O$$

and

$$F^*(t_1 \otimes_A \ldots \otimes_A t_m) := F^*(t_1) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} F^*(t_m) \in \mathbb{D}_A.$$

These definitions obviously restrict to $W_O^r = \mathbb{C}_O^*$ and $W_A^r = \mathbb{C}_A^*$, which gives the definition of F^* on objects and arrows of $\mathbb{C}^*$.

Before we show that F^* is a strict monoidal functor, we first show that if $w_1, w_2 \in W_O$, then $w_1 \to_r w_2$ implies $F^*(w_1) = F^*(w_2)$, and if $w_1, w_2 \in W_O^{sr}$, then $w_1 \to_e w_2$ implies $F^*(w_1) = F^*(w_2)$, and similarly for W_A and W_A^{sr}. But this is obvious from the definitions of $F^*, \to_r$, and $\to_e$, and the fact that $\mathbb{D}$ is a strict monoidal category and $F : \mathbb{C} \to \mathbb{D}$ is a strict monoidal functor. Thus, for any $w \in W_O \cup W_A$ we have $F^*(w) = F^*(w^r)$, for any $w \in W_O^{sr} \cup W_A^{sr}$ we have $F^*(w) = F^*(w^e)$, and hence for any $w \in W_O \cup W_A$ we have $F^*(w^{\mathsf{exp}}) = F^*((w^r)^e) = F^*(w)$.

Now we show that F^* is a strict monoidal functor from $\mathbb{C}^*$ to $\mathbb{D}$. To show that F^* is functorial, we must first show for any $w \in \mathbb{C}_A^* = W_A^r$ that

$$\mathsf{dom}^{\mathbb{D}}(F^*(w)) = F^*(\mathsf{dom}^*(w)) \text{ and } \mathsf{cod}^{\mathbb{D}}(F^*(w)) = F^*(\mathsf{cod}^*(w)).$$

We only consider dom. So we must show that $\mathsf{dom}^{\mathbb{D}}(F^*(w)) = F^*(\mathsf{dom}^*(w)) = F^*\left((w^{\mathsf{dom}})^r\right)$. Since $F^*((w^{\mathsf{dom}})^r) = F^*(w^{\mathsf{dom}})$, it suffices to show $\mathsf{dom}^{\mathbb{D}}(F^*(w)) = F^*(w^{\mathsf{dom}})$, which easily follows from the fact that $\mathsf{dom}^{\mathbb{D}}(F^*(s)) = F^*(s^{\mathsf{dom}})$ for any $s \in S_A^{\mathbb{C}}$, which itself follows from the definition of F^*.

Before we show that F^* preserves composition, we first show that if $u, v \in W_A$ and $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, then

$$F^*(u \circ' v) = F^*(u) \circ^{\mathbb{D}} F^*(v).$$

It is clear that this is true for $u, v \in S_A^{\mathbb{C}}$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$ (by definition of F^* and functoriality of F). For the general case, let $u, v \in W_A$ with $u^{\mathsf{dom}} \equiv v^{\mathsf{cod}}$, so that $u \equiv s_1 \otimes_A \ldots \otimes_A s_n$ and $v \equiv t_1 \otimes_A \ldots \otimes_A t_n$ for some $n \geq 1$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in S_A^{\mathbb{C}}$ with $s_i^{\mathsf{dom}} \equiv t_i^{\mathsf{cod}}$ for each $1 \leq i \leq n$. Then we have

$$F^*(u \circ' v) = F^*((s_1 \otimes_A \ldots \otimes_A s_n) \circ' (t_1 \otimes_A \ldots \otimes_A t_n))$$

$$\begin{aligned}
&= F^*((s_1 \circ' t_1) \otimes_A \ldots \otimes_A (s_n \circ' t_n)) \\
&= F^*(s_1 \circ' t_1) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} F^*(s_n \circ' t_n) \\
&= (F^*(s_1) \circ^{\mathbb{D}} F^*(t_1)) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} (F^*(s_n) \circ^{\mathbb{D}} F^*(t_n)) \\
&= (F^*(s_1) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} F^*(s_n)) \circ^{\mathbb{D}} (F^*(t_1) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} F^*(t_n)) \\
&= F^*(s_1 \otimes_A \ldots \otimes_A s_n) \circ^{\mathbb{D}} F^*(t_1 \otimes_A \ldots \otimes_A t_n) \\
&= F^*(u) \circ^{\mathbb{D}} F^*(v),
\end{aligned}$$

as desired; the second equality follows by definition of $\circ'$, the third by definition of F^*, the fourth by the case $n = 1$, the fifth because $\mathbb{D}$ is a strict monoidal category, and the sixth by definition of F^*.

To show that F^* preserves composition, let $s, t \in W_A^r = \mathbb{C}_A^*$ with $\mathsf{dom}^*(s) = \mathsf{cod}^*(t)$, i.e. $(s^{\mathsf{dom}})^{\mathsf{exp}} \equiv (s^{\mathsf{cod}})^{\mathsf{exp}}$, i.e. $((s^{\mathsf{dom}})^r)^e \equiv ((t^{\mathsf{cod}})^r)^e$. We must show

$$F^*(s \circ^* t) = F^*(s) \circ^{\mathbb{D}} F^*(t),$$

i.e.

$$F^*((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r) = F^*(s) \circ^{\mathbb{D}} F^*(t).$$

First, we have $F^*((s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})^r) = F^*(s^{\mathsf{exp}} \circ' t^{\mathsf{exp}})$. By what we just showed, it now suffices to prove

$$F^*(s^{\mathsf{exp}}) \circ^{\mathbb{D}} F^*(t^{\mathsf{exp}}) = F^*(s) \circ^{\mathbb{D}} F^*(t);$$

but this follows because (as remarked earlier in the proof) we have $F^*(s^{\mathsf{exp}}) = F^*(s)$ and $F^*(t^{\mathsf{exp}}) = F^*(t)$, as desired.

To show that F^* preserves identity arrows, we must show for any $w \in W_O^r$ that

$$F^*(\mathsf{id}^*(w)) = \mathsf{id}^{\mathbb{D}}(F^*(w)),$$

i.e.

$$F^*(w^{\mathsf{id}}) = \mathsf{id}^{\mathbb{D}}(F^*(w)).$$

That this is true for any $w \in S_O^{\mathbb{C}}$ is obvious by definition of F^* and the assumption that F is functorial. If $w = s_1 \otimes_O \ldots \otimes_O s_n$ for $s_1, \ldots, s_n \in S_O^{\mathbb{C}}$, then we have

$$\begin{aligned}
F^*(w^{\mathsf{id}}) &= F^*((s_1 \otimes_O \ldots \otimes_O s_n)^{\mathsf{id}}) \\
&= F^*(s_1^{\mathsf{id}} \otimes_A \ldots \otimes_A s_n^{\mathsf{id}}) \\
&= F^*(s_1^{\mathsf{id}}) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} F^*(s_n^{\mathsf{id}}) \\
&= \mathsf{id}^{\mathbb{D}}(F^*(s_1)) \otimes_A^{\mathbb{D}} \ldots \otimes_A^{\mathbb{D}} \mathsf{id}^{\mathbb{D}}(F^*(s_n)) \\
&= \mathsf{id}^{\mathbb{D}}(F^*(s_1) \otimes_O^{\mathbb{D}} \ldots \otimes_O^{\mathbb{D}} F^*(s_n)) \\
&= \mathsf{id}^{\mathbb{D}}(F^*(s_1 \otimes_O \ldots \otimes_O s_n)) \\
&= \mathsf{id}^{\mathbb{D}}(F^*(w)),
\end{aligned}$$

as desired; the second equality follows by definition of $(-)^{\mathsf{id}}$, the third by definition of F^*, the fourth by the base case, the fifth because $\mathbb{D}$ is a strict monoidal category, and the sixth by definition of F^*. This completes the proof that $F^* : \mathbb{C}^* \to \mathbb{D}$ is functorial.

Now we show that F^* is a *strict monoidal* functor. To show that F^* preserves the object tensor (the argument for the arrow tensor being identical), let $w_1, w_2 \in W_O^r$, and let us show

$$F^*(w_1 \otimes_O^* w_2) = F^*(w_1) \otimes_O^{\mathbb{D}} F^*(w_2).$$

We have

$$F^*(w_1 \otimes_O^* w_2) = F^*((w_1 \otimes_O w_2)^r) = F^*(w_1 \otimes_O w_2) = F^*(w_1) \otimes_O^{\mathbb{D}} F^*(w_2),$$

as desired. Lastly, by definition of F^* and the fact that F is a strict monoidal functor, we have

$$F^*\left(e^{\mathbb{C}^*}\right) = F^*(c_{O,e^{\mathbb{C}}}) = F(e^{\mathbb{C}}) = e^{\mathbb{D}},$$

so that F^* preserves the unit of the object tensor. This completes the proof that $F^* : \mathbb{C}^* \to \mathbb{D}$ is a strict monoidal functor.

It is true by definition that F^* maps $\mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A$ to the required objects and arrows of $\mathbb{D}$. We also have $F^* \circ i_{\mathbb{C}} = F$, because for any object $a \in \mathbb{C}_O$ we have

$$F^*(i_{\mathbb{C}}(a)) = F^*(c_{O,a}) = F(a),$$

and similarly for arrows of $\mathbb{C}$.

Finally, we show that F^* is the *unique* strict monoidal functor $\mathbb{C}^* \to \mathbb{D}$ with the stated properties. So let $G : \mathbb{C}^* \to \mathbb{D}$ be a strict monoidal functor that maps $\mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A$ to the stated objects and arrows and also satisfies $G \circ i_{\mathbb{C}} = F : \mathbb{C} \to \mathbb{D}$. We must show that $F^* = G : \mathbb{C}^* \to \mathbb{D}$. So we must show for any $w \in W_O^r$ that $F^*(w) = G(w)$ (the argument for W_A^r being analogous). If $w \in S_O^{\mathbb{C}}$ and $w \equiv \mathsf{x}_O$ or $w \equiv \mathsf{x}'_O$, then this is true. If $w \equiv \mathsf{dom}(\mathsf{x}_A)$, then we have

$$\begin{aligned}
G(\mathsf{dom}(\mathsf{x}_A)) &= G(\mathsf{dom}(\mathsf{x}_A)^r) \\
&= G((\mathsf{x}_A^{\mathsf{dom}})^r) \\
&= G(\mathsf{dom}^*(\mathsf{x}_A)) \\
&= \mathsf{dom}^{\mathbb{D}}(G(\mathsf{x}_A)) \\
&= \mathsf{dom}^{\mathbb{D}}(f) \\
&= F^*(\mathsf{dom}(\mathsf{x}_A)),
\end{aligned}$$

as desired; similar calculations work for cod and x'_A. If $a \in \mathbb{C}_O$, then we have

$$G(c_{O,a}) = G(i_{\mathbb{C}}(a)) = F(a) = F^*(c_{O,a}),$$

as desired. So $F^*(w) = G(w)$ for all $w \in S_O^{\mathbb{C}}$. In the general case, if $w \equiv s_1 \otimes_O \ldots \otimes_O s_n$ for $s_1, \ldots, s_n \in S_O^{\mathbb{C}}$, then we have

$$\begin{aligned} G(w) &= G(w^r) \\ &= G((s_1 \otimes_O \ldots \otimes_O s_n)^r) \\ &= G(s_1 \otimes_O^* \ldots \otimes_O^* s_n) \\ &= G(s_1) \otimes_O^{\mathbb{D}} \ldots \otimes_O^{\mathbb{D}} G(s_n) \\ &= F^*(s_1) \otimes_O^{\mathbb{D}} \ldots \otimes_O^{\mathbb{D}} F^*(s_n) \\ &= F^*(s_1 \otimes_O \ldots \otimes_O s_n) \\ &= F^*(w), \end{aligned}$$

as desired; the first equality follows by Lemma 3.9.7 (since w is reduced), the third by definition of $\otimes_O^*$ in $\mathbb{C}^*$, the fourth because G is a strict monoidal functor, the fifth by the base case, and the sixth by definition of F^*. This completes the proof that $G = F^*$, which proves that F^* is the *unique* strict monoidal functor $\mathbb{C}^* \to \mathbb{D}$ with the stated properties.

The last statement of the Proposition follows because $\mathbb{C}^*$ now has the universal property of $\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle$ (cf. Proposition 2.2.10). ▮

Lemma (3.9.32)**.** *Let* $\mathbb{C} \in \mathsf{StrMonCat}$ *and* $C \in \{O, A\}$*. Then for any* $w \in W_C$ *(so that* $[w]_C \in \mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A \rangle_C$*) we have*

$$i_{\mathbb{C}}^*([w]_C) = w^r \in \mathbb{C}_C^* = W_C^r.$$

Proof: We show the claim for $C = O$ (the proof for $C = A$ is analogous). It suffices to show for *reduced* $w \in W_O$ that

$$i_{\mathbb{C}}^*([w]) = w \in W_O^r.$$

Because then for arbitrary $w \in W_O$, since w^r is reduced and $[w] = [w^r]$ by Lemma 3.9.8, we obtain

$$i_{\mathbb{C}}^*([w]) = i_{\mathbb{C}}^*([w^r]) = w^r,$$

as desired.

We prove this by induction on the length of $w \in W_O^r$.

- If $w \equiv \mathsf{x}_O$, then we have

$$i_{\mathbb{C}}^*([w]) = i_{\mathbb{C}}^*([\mathsf{x}_O]) \equiv \mathsf{x}_O \equiv w,$$

  as desired, and similarly if $w \equiv \mathsf{x}'_O$.

- If $w \equiv \mathsf{dom}(\mathsf{x}_A)$, then we have

  $$\begin{aligned} i^*_{\mathbb{C}}([w]) &= i^*_{\mathbb{C}}\left([\mathsf{dom}(\mathsf{x}_A)]\right) \\ &= i^*_{\mathbb{C}}\left(\mathsf{dom}^{\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A\rangle}([\mathsf{x}_A])\right) \\ &= \mathsf{dom}^*(i^*_{\mathbb{C}}([\mathsf{x}_A])) \\ &= \mathsf{dom}^*(\mathsf{x}_A) \\ &= \left(\mathsf{x}_A^{\mathsf{dom}}\right)^r \\ &\equiv \mathsf{dom}(\mathsf{x}_A)^r \\ &\equiv \mathsf{dom}(\mathsf{x}_A) \\ &\equiv w, \end{aligned}$$

  as desired. The reasoning for cod and x'_A is analogous.

- If $w \equiv c_{O,a}$ for some $a \in \mathbb{C}_O$, then we have

  $$i^*_{\mathbb{C}}([w]) = i^*_{\mathbb{C}}\left([c_{O,a}]\right) = i^*_{\mathbb{C}}\left(\eta^{\mathbb{C}}_{O,O,A,A}(a)\right) = i_{\mathbb{C}}(a) = c_{O,a} \equiv w,$$

  as desired.

- Suppose that $w \equiv s_1 \otimes_O \ldots \otimes_O s_{n+1} \in W^r_O$ for some $n \geq 1$. Then $s_1 \otimes_O \ldots \otimes_O s_n$ is reduced, so by the induction hypothesis we have

  $$i^*_{\mathbb{C}}([s_1 \otimes_O \ldots \otimes_O s_n]) \equiv s_1 \otimes_O \ldots \otimes_O s_n.$$

  Then we have

  $$\begin{aligned} i^*_{\mathbb{C}}([w]) &= i^*_{\mathbb{C}}([s_1 \otimes_O \ldots \otimes_O s_n \otimes_O s_{n+1}]) \\ &= i^*_{\mathbb{C}}\left([s_1 \otimes_O \ldots \otimes_O s_n] \otimes_O^{\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A\rangle} [s_{n+1}]\right) \\ &= i^*_{\mathbb{C}}([s_1 \otimes_O \ldots \otimes_O s_n]) \otimes^*_O i^*_{\mathbb{C}}([s_{n+1}]) \\ &= (s_1 \otimes_O \ldots \otimes_O s_n) \otimes^*_O s_{n+1} \\ &= (s_1 \otimes_O \ldots \otimes_O s_n \otimes_O s_{n+1})^r \\ &\equiv s_1 \otimes_O \ldots \otimes_O s_{n+1} \\ &\equiv w, \end{aligned}$$

  as desired, where the penultimate equality holds by Lemma 3.9.7 because $s_1 \otimes_O \ldots \otimes_O s_{n+1}$ is assumed to be reduced.

∎

Lemma (3.9.34)**.** *Let* $\mathbb{C} \in \mathsf{StrMonCat}$*. If* $t \in \mathsf{Term}^c(\Sigma_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ *with* $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$*, then:*

- *If* $t : O$*, then* $[t] = [w]$ *for some* $w \in W^{\mathbb{C}}_O$*.*
- *If* $t : A$*, then* $[t] = [w]$ *for some* $w \in W^{\mathbb{C}}_A$*.*

Proof: We prove the lemma by induction on terms $t \in \mathsf{Term}^c(\Sigma(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$.

- If
$$t \in \{\mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}_A\} \cup \{c_{O,a} : a \in \mathbb{C}_O\} \cup \{c_{A,f} : f \in \mathbb{C}_A\},$$
then $t \in W_O \cup W_A$ and $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$, so that we have $[t] = [t]$.

- If $t \equiv e : O$, then $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ and
$$[t] = [e] = [c_{O,e^{\mathbb{C}}}],$$
with $c_{O,e^{\mathbb{C}}} \in S_O \subseteq W_O$.

- Suppose $t \equiv \mathsf{dom}(s) : O$ for some $s \in \mathsf{Term}^c(\Sigma(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_A$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s \downarrow$ as well). Then by the induction hypothesis, there is some word $s_1 \otimes_A \ldots \otimes_A s_n \in W_A$ with
$$[s] = [s_1 \otimes_A \ldots \otimes_A s_n].$$
Then we have
$$\begin{aligned} [t] &= [\mathsf{dom}(s)] \\ &= [\mathsf{dom}(s_1 \otimes_A \ldots \otimes_A s_n)] \\ &= [\mathsf{dom}(s_1) \otimes_O \ldots \otimes_O \mathsf{dom}(s_n)] \\ &= [s_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O s_n^{\mathsf{dom}}], \qquad \text{(by Lemma 3.9.17)} \end{aligned}$$
with $s_i^{\mathsf{dom}} \in S_O$ for all $1 \leq i \leq n$. The reasoning for cod is analogous.

- Suppose $t \equiv \mathsf{id}(s) : A$ for some $s \in \mathsf{Term}^c(\Sigma(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_O$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s \downarrow$ as well). Then by the induction hypothesis, there is some word $s_1 \otimes_O \ldots \otimes_O s_n \in W_O$ with
$$[s] = [s_1 \otimes_O \ldots \otimes_O s_n].$$
Then we have
$$[t] = [\mathsf{id}(s)]$$

$$\begin{aligned}
&= [\mathsf{id}(s_1 \otimes_O \ldots \otimes_O s_n)] \\
&= [\mathsf{id}(s_1) \otimes_A \ldots \otimes_A \mathsf{id}(s_n)] \\
&= [s_1^{\mathsf{id}} \otimes_A \ldots \otimes_A s_n^{\mathsf{id}}], && \text{(by Lemma 3.9.21)}
\end{aligned}$$

with $s_i^{\mathsf{id}} \in S_A$ for all $1 \leq i \leq n$.

- Suppose $t \equiv s_1 \circ s_2 : A$ for some $s_1, s_2 \in \mathsf{Term}^c(\Sigma(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_A$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s_i \downarrow$ for $i = 1, 2$ as well). Then by the induction hypothesis, there are $n, m \geq 1$ and $u_1, \ldots, u_n, v_1, \ldots, v_m \in S_A$ with
$$[s_1] = [u_1 \otimes_A \ldots \otimes_A u_n]$$
and
$$[s_2] = [v_1 \otimes_A \ldots \otimes_A v_m].$$
By Lemma 3.9.8, we may assume without loss of generality that $u_1 \otimes_A \ldots \otimes_A u_n$ and $v_1 \otimes_A \ldots \otimes_A v_m$ are reduced. Then the assumption that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash s_1 \circ s_2 \downarrow$ implies that
$$\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash \mathsf{cod}(s_2) = \mathsf{dom}(s_1),$$
i.e.
$$[\mathsf{cod}(s_2)] = [\mathsf{dom}(s_1)].$$
So then we obtain
$$\begin{aligned}
[v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}}] &= [\mathsf{cod}(v_1) \otimes_O \ldots \otimes_O \mathsf{cod}(v_m)] && \text{(by Lemma 3.9.17)} \\
&= [\mathsf{cod}(v_1 \otimes_A \ldots \otimes_A v_m)] \\
&= [\mathsf{cod}(s_2)] \\
&= [\mathsf{dom}(s_1)] \\
&= [\mathsf{dom}(u_1 \otimes_A \ldots \otimes_A u_n)] \\
&= [\mathsf{dom}(u_1) \otimes_O \ldots \otimes_O \mathsf{dom}(u_n)] \\
&= [u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}}]. && \text{(by Lemma 3.9.17)}
\end{aligned}$$
Then by Lemma 3.9.8, we obtain
$$\begin{aligned}
[(v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}})^r] &= [v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}}] \\
&= [u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}}] \\
&= [(u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}})^r],
\end{aligned}$$
i.e. we have
$$\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash (v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}})^r = (u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}})^r.$$

Then by Proposition 3.9.33, we obtain

$$(v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}})^r \equiv (u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}})^r.$$

Then we have

$$((v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}})^r)^{\mathsf{exp}} \equiv ((u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}})^r)^{\mathsf{exp}},$$

and thus

$$(v_1^{\mathsf{cod}} \otimes_O \ldots \otimes_O v_m^{\mathsf{cod}})^{\mathsf{exp}} \equiv (u_1^{\mathsf{dom}} \otimes_O \ldots \otimes_O u_n^{\mathsf{dom}})^{\mathsf{exp}}$$

by Lemma 3.9.27.3. Then by Lemma 3.9.27.2, we obtain

$$((v_1 \otimes_A \ldots \otimes_A v_m)^{\mathsf{exp}})^{\mathsf{cod}} \equiv ((u_1 \otimes_A \ldots \otimes_A u_n)^{\mathsf{exp}})^{\mathsf{dom}}.$$

So we have

$$(u_1 \otimes_A \ldots \otimes_A u_n)^{\mathsf{exp}} \circ' (v_1 \otimes_A \ldots \otimes_A v_m)^{\mathsf{exp}} \in W_A.$$

Finally, we show that

$$[t] = [s_1 \circ s_2] = [(u_1 \otimes_A \ldots \otimes_A u_n)^{\mathsf{exp}} \circ' (v_1 \otimes_A \ldots \otimes_A v_m)^{\mathsf{exp}}].$$

We have

$$\begin{aligned}
[t] &= [s_1 \circ s_2] \\
&= [(u_1 \otimes_A \ldots \otimes_A u_n) \circ (v_1 \otimes_A \ldots \otimes_A v_m)] \\
&= [(u_1 \otimes_A \ldots \otimes_A u_n)^{\mathsf{exp}} \circ (v_1 \otimes_A \ldots \otimes_A v_m)^{\mathsf{exp}}] \quad &&\text{(by the remark after Lemma 3.9.12)} \\
&= [(u_1 \otimes_A \ldots \otimes_A u_n)^{\mathsf{exp}} \circ' (v_1 \otimes_A \ldots \otimes_A v_m)^{\mathsf{exp}}], \quad &&\text{(by Lemma 3.9.26)}
\end{aligned}$$

as desired.

- Suppose $t \equiv t_1 \otimes_O t_2 : O$ for some $t_1, t_2 \in \mathsf{Term}^c(\Sigma(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A))_O$ with $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t \downarrow$ (so that $\mathbb{T}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O, \mathsf{x}_A, \mathsf{x}'_A) \vdash t_i \downarrow$ for $i = 1, 2$ as well). By the induction hypothesis, we have that

  $$[t_i] = [w_i]$$

  for some $w_i \in W_O$ for $i = 1, 2$. Then we have

  $$[t] = [t_1 \otimes_O t_2] = [w_1 \otimes_O w_2],$$

  with $w_1 \otimes_O w_2 \in W_O$, as desired. The reasoning for $\otimes_A$ is analogous.

∎

Proposition (3.9.35)**.** *Let* $\mathbb{C} \in \mathsf{StrMonCat}$*. If* $a \in C_O$ *is invertible in the monoid* $(\mathbb{C}_O, \otimes_O^{\mathbb{C}}, e^{\mathbb{C}})$ *with inverse* $b \in \mathbb{C}_O$*, then*

$$\left([c_{O,a} \otimes_O \mathsf{x}_O \otimes_O c_{O,b}]\,, \left[c_{A,\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{A,\mathsf{id}(b)}\right]\right) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C}).$$

Proof: We must show that this pair is invertible, commutes generically with all function symbols of Σ_{Str}, and reflects definedness of all function symbols of Σ_{Str}. First, we show that this pair is invertible, with inverse

$$([c_b \otimes_O \mathsf{x}_O \otimes_O c_a], [c_{\mathsf{id}(b)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(a)}]) \in \mathbb{C}\langle \mathsf{x}_O \rangle_O \times \mathbb{C}\langle \mathsf{x}_A \rangle_A.$$

We have

$$\begin{aligned}
[(c_a \otimes_O \mathsf{x}_O \otimes_O c_b)[c_b \otimes_O \mathsf{x}_O \otimes_O c_a/\mathsf{x}_O]] &= [c_a \otimes_O c_b \otimes_O \mathsf{x}_O \otimes_O c_a \otimes_O c_b] \\
&= [c_{a \otimes_O^{\mathbb{C}} b} \otimes_O \mathsf{x}_O \otimes_O c_{a \otimes_O^{\mathbb{C}} b}] \\
&= [c_{e^{\mathbb{C}}} \otimes_O \mathsf{x}_O \otimes_O c_{e^{\mathbb{C}}}] \\
&= [e \otimes_O \mathsf{x}_O \otimes_O e] \\
&= [\mathsf{x}_O],
\end{aligned}$$

as desired. The converse equality is shown in the exact same way. The other two equalities have similar proofs, using the facts $\mathsf{id}(a) \otimes_A^{\mathbb{C}} \mathsf{id}(b) = \mathsf{id}(a \otimes_O^{\mathbb{C}} b) = \mathsf{id}(e^{\mathbb{C}}) = \mathsf{id}(b) \otimes_A^{\mathbb{C}} \mathsf{id}(a)$. So the pair $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}])$ is invertible.

Now we must show that $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}])$ commutes generically with all function symbols of Σ_{Str}. First, we show that this pair commutes generically with $\mathsf{id} : O \to A$. Then we must show that the sequent

$$\top \vdash \mathsf{id}(c_a \otimes_O \mathsf{x}_O \otimes_O c_b) = c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)}$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O)$ (since $\mathbb{T}_{\mathsf{Str}}(C, \mathsf{x}_O) \vdash \mathsf{id}(\mathsf{x}_O) \downarrow$), i.e. it suffices to show that the equality

$$[\mathsf{id}(c_a \otimes_O \mathsf{x}_O \otimes_O c_b)] = [c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)}]$$

holds in $\mathbb{C}\langle \mathsf{x}_O \rangle_A$. But we have

$$[\mathsf{id}(c_a \otimes_O \mathsf{x}_O \otimes_O c_b)] = [\mathsf{id}(c_a) \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A \mathsf{id}(c_b)] = [c_{\mathsf{id}(a)} \otimes_A \mathsf{id}(\mathsf{x}_O) \otimes_A c_{\mathsf{id}(b)}],$$

as desired.

Now we show that the pair in question commutes generically with $\mathsf{dom} : A \to O$ (the proof for cod is similar). Since $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{dom}(\mathsf{x}_A) \downarrow$, it suffices to show that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A) \vdash \mathsf{dom}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) = c_a \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_b,$$

i.e. we must show that the equality

$$[\mathsf{dom}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)})] = [c_a \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_b]$$

holds in $\mathbb{C}\langle \mathsf{x}_A \rangle_O$. But we have

$$[\mathsf{dom}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)})] = [\mathsf{dom}(c_{\mathsf{id}(a)}) \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O \mathsf{dom}(c_{\mathsf{id}(b)})] = [c_a \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_b],$$

as desired.

Now we show that the pair in question commutes generically with $\circ : A \times A \to A$. This means that we must show that the sequent

$$\mathsf{x}_A \circ \mathsf{x}'_A \downarrow \vdash (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) \circ (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)}) = c_{\mathsf{id}(a)} \otimes_A (\mathsf{x}_A \circ \mathsf{x}'_A) \otimes_A c_{\mathsf{id}(b)}$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)$. By the deduction theorem for partial Horn logic (Remark 1.3.17), it suffices to show that

$$\top \vdash (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) \circ (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)}) = c_{\mathsf{id}(a)} \otimes_A (\mathsf{x}_A \circ \mathsf{x}'_A) \otimes_A c_{\mathsf{id}(b)}$$

is provable in the theory $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A) \cup \{\top \vdash \mathsf{x}_A \circ \mathsf{x}'_A \downarrow\}$. In this theory, the following sequence of equalities is provable:

$$\begin{aligned}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) \circ (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)}) &= (c_{\mathsf{id}(a)} \circ c_{\mathsf{id}(a)}) \otimes_A (\mathsf{x}_A \circ \mathsf{x}'_A) \otimes_A (c_{\mathsf{id}(b)} \circ c_{\mathsf{id}(b)}) \\ &= c_{\mathsf{id}(a) \circ \mathsf{id}(a)} \otimes_A (\mathsf{x}_A \circ \mathsf{x}'_A) \otimes_A c_{\mathsf{id}(b) \circ \mathsf{id}(b)} \\ &= c_{\mathsf{id}(a)} \otimes_A (\mathsf{x}_A \circ \mathsf{x}'_A) \otimes_A c_{\mathsf{id}(b)},\end{aligned}$$

as desired.

Next, we show that the pair in question commutes generically with $\otimes_O : O \times O \to O$ and $\otimes_A : A \times A \to A$. For $\otimes_O$, we must show that the sequent

$$\top \vdash (c_a \otimes_O \mathsf{x}_O \otimes_O c_b) \otimes_O (c_a \otimes_O \mathsf{x}'_O \otimes_O c_b) = c_a \otimes_O (\mathsf{x}_O \otimes_O \mathsf{x}'_O) \otimes_O c_b$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O)$ (since $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_O, \mathsf{x}'_O) \vdash \mathsf{x}_O \otimes_O \mathsf{x}'_O \downarrow$), i.e. we must show that

$$[(c_a \otimes_O \mathsf{x}_O \otimes_O c_b) \otimes_O (c_a \otimes_O \mathsf{x}'_O \otimes_O c_b)] = [c_a \otimes_O (\mathsf{x}_O \otimes_O \mathsf{x}'_O) \otimes_O c_b]$$

holds in $\mathbb{C}\langle \mathsf{x}_O, \mathsf{x}'_O \rangle_O$. We have

$$\begin{aligned}[(c_a \otimes_O \mathsf{x}_O \otimes_O c_b) \otimes_O (c_a \otimes_O \mathsf{x}'_O \otimes_O c_b)] &= [c_a \otimes_O \mathsf{x}_O \otimes_O c_b \otimes_O c_a \otimes_O \mathsf{x}'_O \otimes_O c_b] \\ &= [c_a \otimes_O \mathsf{x}_O \otimes_O c_{b \otimes^{\mathbb{C}}_O a} \otimes_O \mathsf{x}'_O \otimes_O c_b] \\ &= [c_a \otimes_O \mathsf{x}_O \otimes_O c_{e^{\mathbb{C}}} \otimes_O \mathsf{x}'_O \otimes_O c_b] \\ &= [c_a \otimes_O \mathsf{x}_O \otimes_O e \otimes_O \mathsf{x}'_O \otimes_O c_b] \\ &= [c_a \otimes_O \mathsf{x}_O \otimes_O \mathsf{x}'_O \otimes_O c_b] \\ &= [c_a \otimes_O (\mathsf{x}_O \otimes_O \mathsf{x}'_O) \otimes_O c_b],\end{aligned}$$

as required. The proof for $\otimes_A$ is analogous.

Lastly, we must show that the pair in question commutes generically with $e : O$. So we must show that the sequent

$$\top \vdash c_a \otimes_O e \otimes_O c_b = e$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C})$ (since $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}) \vdash e\downarrow$). But $\mathbb{T}_{\mathsf{Str}}(\mathbb{C})$ proves the following equations, as required:

$$c_a \otimes_O e \otimes_O c_b = c_a \otimes_O c_b = c_{a \otimes^{\mathbb{C}}_O b} = c_{e^{\mathbb{C}}} = e.$$

This completes the proof that the pair $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}])$ commutes generically with all function symbols of Σ_{Str}.

Lastly, we must show that the pair $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}])$ reflects definedness of all (non-constant) function symbols of Σ_{Str}. Since $\circ : A \times A \to A$ is the only function symbol of Σ_{Str} that is not totally defined in $\mathbb{T}_{\mathsf{Str}}$, it suffices to show that the pair in question reflects definedness of $\circ$. So we must show that the sequent

$$(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) \circ (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)})\downarrow \vdash \mathsf{x}_A \circ \mathsf{x}'_A\downarrow$$

is provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)$. By the deduction theorem for partial Horn logic (Remark 1.3.17), it suffices to show that the sequent

$$\top \vdash \mathsf{x}_A \circ \mathsf{x}'_A\downarrow$$

is provable in the theory

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A) \cup \{\top \vdash (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}) \circ (c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)})\downarrow\}.$$

Denote this theory by $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)^+$. Then it suffices to show that

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)^+ \vdash \mathsf{cod}(\mathsf{x}'_A) = \mathsf{dom}(\mathsf{x}_A).$$

Well, we have

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)^+ \vdash \mathsf{cod}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}'_A \otimes_A c_{\mathsf{id}(b)}) = \mathsf{dom}(c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}),$$

and hence

$$\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)^+ \vdash c_a \otimes_O \mathsf{cod}(\mathsf{x}'_A) \otimes_O c_b = c_a \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_b.$$

Then the following equalities are provable in $\mathbb{T}_{\mathsf{Str}}(\mathbb{C}, \mathsf{x}_A, \mathsf{x}'_A)^+$, which completes the argument:

$$\begin{aligned} \mathsf{cod}(\mathsf{x}'_A) &= e \otimes_O \mathsf{cod}(\mathsf{x}'_A) \otimes_O e \\ &= c_{e^{\mathbb{C}}} \otimes_O \mathsf{cod}(\mathsf{x}'_A) \otimes_O c_{e^{\mathbb{C}}} \end{aligned}$$

$$
\begin{aligned}
&= c_{b \otimes^{\mathbb{C}}_O a} \otimes_O \mathsf{cod}(\mathsf{x}'_A) \otimes_O c_{b \otimes^{\mathbb{C}}_O a} \\
&= c_b \otimes_O c_a \otimes_O \mathsf{cod}(\mathsf{x}'_A) \otimes_O c_b \otimes_O c_a \\
&= c_b \otimes_O c_a \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_b \otimes_O c_a \\
&= c_{e^{\mathbb{C}}} \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O c_{e^{\mathbb{C}}} \\
&= e \otimes_O \mathsf{dom}(\mathsf{x}_A) \otimes_O e \\
&= \mathsf{dom}(\mathsf{x}_A).
\end{aligned}
$$

This completes the proof that the pair $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}])$ reflects definedness of all (non-constant) function symbols of Σ_{Str}, which in turn completes the proof that $([c_a \otimes_O \mathsf{x}_O \otimes_O c_b], [c_{\mathsf{id}(a)} \otimes_A \mathsf{x}_A \otimes_A c_{\mathsf{id}(b)}]) \in G_{\mathbb{T}_{\mathsf{Str}}}(\mathbb{C})$. ∎

Appendix C

Chapter 5 Proofs

Proposition (5.1.8)**.** *There is an isomorphism of categories*

$$\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}.$$

Proof: We define mutually inverse functors

$$F^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$$

and

$$M^{(-)} : \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}} \to \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}.$$

First, we define $F^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$. So let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$; we define a corresponding functor

$$F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}.$$

For any object $i \in \mathcal{J}_O$, we set

$$F^M(i) := M^i,$$

which is a partial Σ-structure that is a model of $\mathbb{T}$ by Lemma 5.1.7. Now let $f : i \to j$ in $\mathcal{J}$; we must define a Σ-morphism

$$F^M(f) : M^i \to M^j.$$

For any $A \in \Sigma_{\mathsf{Sort}}$, we have a partial function

$$(\alpha_f^A)^M : M_{A^i} \rightharpoonup M_{A^j},$$

i.e.

$$(\alpha_f^A)^M : M_A^i \rightharpoonup M_A^j.$$

Since M is a model of $\mathbb{T}^{\mathcal{J}}$, it follows by the axioms in Definition 5.1.5.1 that each such partial function is in fact *total*. So we set

$$F^M(f) := \left(\left(\alpha_f^A\right)^M\right)_{A\in\Sigma} : M^i \to M^j.$$

It then follows from the axioms in Definition 5.1.5.4 and the assumption that $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ that $F^M(f)$ is indeed a Σ-morphism, which completes the definition of the functor $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$.

To show that F^M preserves identities, let $i \in \mathcal{J}_O$; we must show that $F^M(\mathsf{id}_i)$ is the identity Σ-morphism on M^i. To show this, we must show that $F^M(\mathsf{id}_i)_A := (\alpha^A_{\mathsf{id}_i})^M : M^i_A \to M^i_A$ is the identity function on $M^i_A = M_{A^i}$. But this follows from Definition 5.1.5.2 and the assumption that $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. The fact that F^M preserves composition follows similarly, using the axioms in Definition 5.1.5.3. This completes the proof that $F^M : \mathcal{J} \to \mathbb{T}$-mod is functorial.

To complete the definition of the functor $F^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$, let $h : M \to N$ be a $\Sigma^{\mathcal{J}}$-morphism in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$; we must define a natural transformation

$$F^h : F^M \Longrightarrow F^N.$$

So let $i \in \mathcal{J}_O$; we must define a Σ-morphism

$$F^h_i : F^M(i) = M^i \to N^i = F^N(i).$$

Appealing to Definition 5.1.3, we set

$$F^h_i := h^i : M^i \to N^i.$$

To verify naturality of F^h, let $f : i \to j$ be an arrow in $\mathcal{J}$; we must show

$$F^N(f) \circ F^h_i = F^h_j \circ F^M(f) : M^i \to N^j,$$

i.e.

$$\left(\alpha^A_f\right)^N \circ h_{A^i} = h_{A^j} \circ \left(\alpha^A_f\right)^M$$

for each $A \in \Sigma_{\mathsf{Sort}}$, which follows because h is a $\Sigma^{\mathcal{J}}$-morphism. So $F^h : F^M \Longrightarrow F^N$ is indeed a natural transformation.

This completes the definition of the functor $F^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$; it is trivial to see that $F^{(-)}$ is functorial.

Now we define the functor $M^{(-)} : \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}} \to \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$. So let $F : \mathcal{J} \to \mathbb{T}$-mod be an arbitrary functor; we construct a corresponding partial $\Sigma^{\mathcal{J}}$-structure M^F that will be a model of $\mathbb{T}^{\mathcal{J}}$. For any object $i \in \mathcal{J}_O$, we know that $F(i)$ is a partial Σ-structure that is a model of $\mathbb{T}$, and for any arrow $f : i \to j$ in $\mathcal{J}$, we know that $F(f) : F(i) \to F(j)$ is a Σ-morphism. To define the $\Sigma^{\mathcal{J}}$-structure M^F, let $i \in \mathcal{J}_O$ and $A \in \Sigma_{\mathsf{Sort}}$. We set

$$M^F_{A^i} := F(i)_A.$$

For any arrow $f : i \to j$ in $\mathcal{J}$ and $A \in \Sigma_{\mathsf{Sort}}$, we set

$$\left(\alpha^A_f\right)^{M^F} := F(f)_A : M^F_{A^i} = F(i)_A \rightharpoonup F(j)_A = M^F_{A_j}.$$

And for any $i \in \mathcal{J}_O$ and function symbol $g : A_1 \times \ldots \times A_n \to A$ in Σ, we set

$$(g^i)^{M^F} := g^{F(i)} : M^F_{A^i_1} \times \ldots \times M^F_{A^i_n} = F(i)_{A_1} \times \ldots \times F(i)_{A_n} \rightharpoonup F(i)_A = M^F_A.$$

This completes the definition of the partial $\Sigma^{\mathcal{J}}$-structure M^F.

Now we show that $M^F \models \mathbb{T}^{\mathcal{J}}$. Axiom 1 in Definition 5.1.5 is satisfied, because each $\left(\alpha^A_f\right)^{M^F} := F(f)_A$ is a total function, since $F(f)$ is a Σ-morphism. Since F is functorial, it also easily follows that M^F satisfies Axioms 2 and 3. Axiom 4 is also satisfied, because $F(f) : F(i) \to F(j)$ is a Σ-morphism for any arrow $f : i \to j$ in $\mathcal{J}$.

Finally, we show that M^F satisfies Axiom 5. So fix $i \in \mathcal{J}_O$ and let $\varphi \vdash^{\vec{x}} \psi$ be an axiom of $\mathbb{T}$. We must show that M^F satisfies the corresponding $\mathbb{T}^{\mathcal{J}}$-axiom $\rho^i(\varphi) \vdash^{\rho^i(\vec{x})} \rho^i(\psi)$. In other words, we must show that

$$\rho^i(\varphi)^{M^F} \subseteq \rho^i(\psi)^{M^F}.$$

Since $F(i) \models \mathbb{T}$, we know that

$$\varphi^{F(i)} \subseteq \psi^{F(i)}.$$

Also, it is trivial to observe that $F(i) = \left(M^F\right)^i$ (cf. Definition 5.1.2), and moreover that $\left(M^F\right)^i = U^i\left(M^F\right)$, where U^i is the obvious forgetful functor from $\mathsf{P}\Sigma^{\mathcal{J}}\mathsf{Str}$ to $\mathsf{P}\Sigma\mathsf{Str}$ induced by the signature morphism $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$. Then by [19, Lemma 27], we obtain

$$\rho^i(\varphi)^{M^F} = \varphi^{F(i)} \subseteq \psi^{F(i)} = \rho^i(\psi)^{M^F},$$

as desired. This completes the proof that $M^F \models \mathbb{T}^{\mathcal{J}}$, i.e. $M^F \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$.

To define $M^{(-)}$ on morphisms, let $\eta : F \to G$ be a morphism in $\mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}}$; we must define a $\Sigma^{\mathcal{J}}$-morphism

$$M^\eta : M^F \to M^G.$$

For any $i \in \mathcal{J}_O$ and $A \in \Sigma_{\mathsf{Sort}}$, we set

$$M^\eta_{A^i} := \eta^A_i : M^F_{A_i} = F(i)_A \to G(i)_A = M^G_{A^i}.$$

To verify that M^η is a $\Sigma^{\mathcal{J}}$-morphism, first let $A \in \Sigma_{\mathsf{Sort}}$ and let $f : i \to j$ be an arrow in $\mathcal{J}$. We must show that

$$M^\eta_{A^j} \circ (\alpha^A_f)^{M^F} = (\alpha^A_f)^{M^G} \circ M^\eta_{A^i}$$

(since α^A_f is total in the $\mathbb{T}^{\mathcal{J}}$-models M^F, M^G), i.e.

$$\eta^A_j \circ F(f)_A = G(f)_A \circ \eta^A_i,$$

which is true by naturality of $\eta : F \to G$.

Now let $i \in \mathcal{J}_O$, let $g : A_1 \times \ldots \times A_n \to A$ be a function symbol of Σ, and let $(a_1, \ldots, a_n) \in \mathsf{dom}\left(\left(g^i\right)^{M^F}\right) = \mathsf{dom}\left(g^{F(i)}\right)$. We must show that

$$\left(M^\eta_{A^i_1}(a_1), \ldots, M^\eta_{A^i_n}(a_n)\right) \in \mathsf{dom}\left(\left(g^i\right)^{M^G}\right) = \mathsf{dom}\left(g^{G(i)}\right)$$

and

$$M^\eta_{A^i}\left(\left(g^i\right)^{M^F}(a_1, \ldots, a_n)\right) = \left(g^i\right)^{M^G}\left(M^\eta_{A^i_1}(a_1), \ldots, M^\eta_{A^i_n}(a_n)\right).$$

Both claims follow from the definition of M^η and the fact that $\eta_i : F(i) \to G(i)$ is a Σ-morphism. This completes the proof that $M^\eta : M^F \to M^G$ is a $\Sigma^{\mathcal{J}}$-morphism.

This completes the definition of the functor $M^{(-)} : \mathsf{P}\mathbb{T}\mathsf{mod}^{\mathcal{J}} \to \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$; it is trivial to see that $M^{(-)}$ is functorial.

Finally, it is straightforward to see that the functors $F^{(-)}$ and $M^{(-)}$ are mutually inverse, which completes the proof of Proposition 5.1.8. ▮

Lemma (5.1.13)**.** *For any* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, $i \in \mathcal{J}_O$, *and* $C \in \Sigma_{\mathsf{Sort}}$, *the signature morphism* $\rho^C_{M^i}$ *is a theory morphism*

$$\rho^C_{M^i} : \mathbb{T}(M^i, \mathsf{x}_C) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}).$$

Proof: We must show that the $\rho^C_{M^i}$-translation of any axiom of $\mathbb{T}(M^i, \mathsf{x}_C)$ is provable in $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$:

- First, we know that the $\rho^C_{M^i}$-translation of any axiom of $\mathbb{T}$ (considered as an axiom of $\mathbb{T}(M^i, \mathsf{x}_C)$) is an axiom of $\mathbb{T}^{\mathcal{J}}$ and hence of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$, because $\rho^C_{M^i}$ restricted to $\Sigma \subseteq \Sigma(M^i, \mathsf{x}_C)$ is equal to $\rho^i : \Sigma \to \Sigma^{\mathcal{J}}$, and $\rho^i : \mathbb{T} \to \mathbb{T}^{\mathcal{J}}$ is a theory morphism by Remark 5.1.6.

- If $s \in M^i_A = M_{A^i}$ for some $A \in \Sigma_{\mathsf{Sort}}$, then $\mathbb{T}(M^i, \mathsf{x}_C)$ has the axiom $\top \vdash c^{M^i}_{A,s} \downarrow$. The $\rho^C_{M^i}$-translation of this axiom is $\top \vdash c^M_{A^i,s} \downarrow$, which is an axiom of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$.

- If $g : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ and $s_1, \ldots, s_n \in M^i_{A_1}, \ldots, M^i_{A_n}$ with $(s_1, \ldots, s_n) \in \mathsf{dom}\left(g^{M^i}\right)$, then $\mathbb{T}(M^i, \mathsf{x}_C)$ has the axiom

  $$\top \vdash g\left(c^{M^i}_{A_1,s_1}, \ldots, c^{M^i}_{A_n,s_n}\right) = c^{M^i}_{A,g^{M^i}(s_1,\ldots,s_n)}.$$

  The $\rho^C_{M^i}$-translation of this axiom is

  $$\top \vdash g^i\left(c^M_{A^i_1,s_1}, \ldots, c^M_{A^i_n,s_n}\right) = c^M_{A^i,g^{M^i}(s_1,\ldots,s_n)}.$$

  But this is an axiom of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$, because $g^{M^i} = (g^i)^M$.

- Lastly, $\mathbb{T}(M^i, \mathsf{x}_C)$ has the axiom $\top \vdash \mathsf{x}_C \downarrow$, whose $\rho^C_{M^i}$-translation is $\top \vdash \mathsf{x}_{C^i} \downarrow$, which is an axiom of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$.

∎

Lemma (5.1.19)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, let* $f : i \to j$ *be any arrow in* $\mathcal{J}$*, and let* $C \in \Sigma_{\mathsf{Sort}}$*. Then for any term* $u \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))$ *with* $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$ *and* $u : A$*, we have*

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \tau^C_f(u) = \alpha^A_f \left(\rho^C_{M^i}(u)\right).$$

Proof: First note that the above equation 'type-checks', because $u \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))_A$ implies $\tau^C_f(u) : \tau^C_f(A) \equiv \rho^j(A) \equiv A^j$, and we also have $\rho^C_{M^i}(u) \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{C^i}))$ and $\rho^C_{M^i}(u) : \rho^C_{M^i}(A) \equiv \rho^i(A) \equiv A^i$, and hence $\alpha^A_f(\rho^C_{M^i}(u)) : A^j$, since $\alpha^A_f : A^i \to A^j$.

Now we prove the claim by induction on terms $u \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))$ with $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$:

- If $u \equiv \mathsf{x}_C : C$, then $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$ and

  $$\tau^C_f(u) \equiv \tau^C_f(\mathsf{x}_C) \equiv \alpha^C_f(\mathsf{x}_{C^i}) \equiv \alpha^C_f \left(\rho^C_{M^i}(\mathsf{x}_C)\right) \equiv \alpha^C_f \left(\rho^C_{M^i}(u)\right),$$

  which entails the desired conclusion.

- If $u \equiv c^{M^i}_{A,s} : A$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $s \in M^i_A = M_{A^i}$ (so that $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$), then we have

  $$\tau^C_f(u) \equiv \tau^C_f \left(c^{M^i}_{A,s}\right) \equiv c^M_{A^j, \left(\alpha^A_f\right)^M(s)}$$

  (cf. Definition 5.1.18), as well as

  $$\alpha^A_f \left(\rho^C_{M^i}(u)\right) \equiv \alpha^A_f \left(\rho^C_{M^i}\left(c^{M^i}_{A,s}\right)\right) \equiv \alpha^A_f \left(c^M_{A^i,s}\right),$$

  and (by definition of $\mathbb{T}^{\mathcal{J}}(M)$) we indeed have

  $$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \alpha^A_f \left(c^M_{A^i,s}\right) = c^M_{A^j, \left(\alpha^A_f\right)^M(s)},$$

  as desired.

- For the induction step, suppose $u \equiv g(u_1, \ldots, u_n) : A$ for some function symbol $g : A_1 \times \ldots \times A_n \to A$ in Σ and terms $u_1, \ldots, u_n \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_C))$ of the appropriate sorts, and suppose that $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \downarrow$. Then (by the rules of partial Horn logic) it follows that $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u_\ell \downarrow$ for all $1 \leq \ell \leq n$. So by the induction hypothesis, we obtain

  $$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \tau^C_f(u_\ell) = \alpha^{A_\ell}_f \left(\rho^C_{M^i}(u_\ell)\right)$$

for each $1 \leq \ell \leq n$. Now we must show

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \tau_f^C(g(u_1, \ldots, u_n)) = \alpha_f^A\left(\rho_{M^i}^C(g(u_1, \ldots, u_n))\right),$$

i.e.

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash g^j\left(\tau_f^C(u_1), \ldots, \tau_f^C(u_n)\right) = \alpha_f^A\left(g^i\left(\rho_{M^i}^C(u_1), \ldots, \rho_{M^i}^C(u_n)\right)\right).$$

Since $\mathbb{T}(M^i, \mathsf{x}_C) \vdash u \equiv g(u_1, \ldots, u_n) \downarrow$ and $\rho_{M^i}^C : \mathbb{T}(M^i, \mathsf{x}_C) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i})$ is a theory morphism by Lemma 5.1.13, it follows that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \rho_{M^i}^C(g(u_1, \ldots, u_n)) \equiv g^i\left(\rho_{M^i}^C(u_1), \ldots, \rho_{M^i}^C(u_n)\right) \downarrow.$$

Then by Axiom 5.1.5.4 we obtain

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{C^i}) \vdash \alpha_f^A\left(g^i\left(\rho_{M^i}^C(u_1), \ldots, \rho_{M^i}^C(u_n)\right)\right) = g^j\left(\alpha_f^{A_1}\left(\rho_{M^i}^C(u_1)\right), \ldots, \alpha_f^{A_n}\left(\rho_{M^i}^C(u_n)\right)\right),$$

which yields the desired result by the induction hypothesis.

∎

Lemma (5.1.26)**.** *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and let $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ be α-restricted, where $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$. If $u : C^j$ for some $j \in \mathcal{J}_O$ and $C \in \Sigma_{\mathsf{Sort}}$, then for any arrow $f : j \to \mathsf{cod}(f)$ in $\mathcal{J}$, there is an α-restricted term $u^f \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ with $u^f : C^{\mathsf{cod}(f)}$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent*

$$u \downarrow \vdash \alpha_f^C(u) = u^f.$$

Proof: We prove this by induction on α-restricted terms $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$.

- If $u \equiv \mathsf{x}_{A^i} : A^i$, then u is clearly α-restricted. Let $f : i \to k$ be any morphism in $\mathcal{J}$ with domain i. Then we set

$$u^f \equiv \mathsf{x}_{A^i}^f := \alpha_f^A(\mathsf{x}_{A^i}) : A^k.$$

Then $u^f \equiv \alpha_f^A(\mathsf{x}_{A^i})$ is α-restricted and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

$$x_{A_i} \downarrow \vdash \alpha_f^A(x_{A^i}) = \alpha_f^A(\mathsf{x}_{A^i}) \equiv x_{A^i}^f,$$

since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \mathsf{x}_{A^i} \downarrow$ and α_f^A is provably total in $\mathbb{T}^{\mathcal{J}}$.

- Let $u \equiv c^M_{C^j,s} : C^j$ for some $j \in \mathcal{J}_O$, $C \in \Sigma_{\mathsf{Sort}}$, and $s \in M_{C^j}$. Then u is α-restricted. Now let $f : j \to k$ be any morphism in $\mathcal{J}$ with domain j. Then we have $\left(\alpha^C_f\right)^M(s) \in M_{C^k}$, so we set
$$u^f \equiv \left(c^M_{C^j,s}\right)^f := c^M_{C^k,\left(\alpha^C_f\right)^M(s)} : C^k.$$
Then u^f is α-restricted and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ does prove the sequent
$$c^M_{C^j,s} \downarrow \vdash \alpha^C_f\left(c^M_{C^j,s}\right) = c^M_{C^k,\left(\alpha^C_f\right)^M(s)} \equiv \left(c^M_{C^j,s}\right)^f,$$
as desired.

- Let $j \in \mathcal{J}_O$ and let $g : C_1 \times \ldots \times C_n \to C$ be a function symbol of Σ and suppose that $u \equiv g^j(u_1, \ldots, u_n) : C^j$ for terms $u_\ell \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))_{C^j_\ell}$ for all $1 \leq \ell \leq n$. Suppose that $u \equiv g^j(u_1, \ldots, u_n)$ is α-restricted. Then for each $1 \leq \ell \leq n$ it easily follows that u_ℓ is α-restricted. Now let $f : j \to k$ be any morphism in $\mathcal{J}$ with domain j. By the induction hypothesis, for every $1 \leq \ell \leq n$ there is an α-restricted term $u^f_\ell \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ with $u^f_\ell : C^k_\ell$ such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent
$$u_\ell \downarrow \vdash \alpha^{C_\ell}_f(u_\ell) = u^f_\ell.$$
Then we set
$$u^f \equiv g^j(u_1, \ldots, u_n)^f := g^k\left(u^f_1, \ldots, u^f_n\right) : C^k.$$
Since $u^f_1, \ldots, u^f_n$ are α-restricted, it follows that $u^f \equiv g^k\left(u^f_1, \ldots, u^f_n\right)$ is α-restricted.

  Now we wish to show that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent
$$g^j(u_1, \ldots, u_n) \downarrow \vdash \alpha^C_f(g^j(u_1, \ldots, u_n)) = g^k\left(u^f_1, \ldots, u^f_n\right).$$
By the deduction theorem of partial Horn logic (cf. Remark 1.3.17), it suffices to show that the expanded theory
$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \cup \{\top \vdash g^j(u_1, \ldots, u_n) \downarrow\}$$
proves the sequent
$$\top \vdash \alpha^C_f(g^j(u_1, \ldots, u_n)) = g^k\left(u^f_1, \ldots, u^f_n\right).$$
By the rules of partial Horn logic, the expanded theory proves $\top \vdash u_\ell \downarrow$ for each $1 \leq \ell \leq n$. So by the induction hypothesis and the cut rule, the expanded theory proves
$$\top \vdash \alpha^{C_\ell}_f(u_\ell) = u^f_\ell$$

for each $1 \leq \ell \leq n$. It also follows by Axiom 5.1.5.4 that the expanded theory proves

$$\top \vdash \alpha_f^C(g^j(u_1, \ldots, u_n)) = g^k\left(\alpha_f^{C_1}(u_1), \ldots, \alpha_f^{C_n}(u_n)\right).$$

So we obtain that the expanded theory proves

$$\top \vdash \alpha_f^C(g^j(u_1, \ldots, u_n)) = g^k\left(\alpha_f^{C_1}(u_1), \ldots, \alpha_f^{C_n}(u_n)\right) = g^k\left(u_1^f, \ldots, u_n^f\right),$$

as required.

- Suppose that $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ is an α-restricted α-term. Then it must be the case that $u \equiv \alpha_g^A(\mathsf{x}_{A^i}) : A^\ell$ for some $g : i \to \ell$ in $\mathcal{J}$. Now let $h : \ell \to \ell'$ be any morphism in $\mathcal{J}$ with domain ℓ. We set

  $$u^h \equiv \alpha_g^A(\mathsf{x}_{A^i})^h := \alpha_{h \circ g}^A(\mathsf{x}_{A^i}) : A^{\ell'}.$$

  Then u^h is α-restricted, and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

  $$\alpha_g^A(\mathsf{x}_{A^i}) \downarrow \vdash \alpha_h^A(\alpha_g^A(\mathsf{x}_{A^i})) = \alpha_{h \circ g}^A(\mathsf{x}_{A^i}) \equiv \alpha_g^A(\mathsf{x}_{A^i})^h,$$

  with the middle equality justified by Axiom 5.1.5.3.

∎

Lemma (5.1.27)**.** *If $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$, then there is an α-restricted term $u' \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ of the same sort such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent*

$$u \downarrow \vdash u = u'.$$

Proof: We prove the Lemma by induction on terms $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$.

- If $u \equiv \mathsf{x}_{A^i}$ or $u \equiv c_{C^j,s}^M$ for some $j \in \mathcal{J}_O, C \in \Sigma_{\mathsf{Sort}}$, and $s \in M_{C^j}$, then we set $u' := u$, and the desired properties are clearly satisfied.
- Suppose that $j \in \mathcal{J}_O$ and that $g : C_1 \times \ldots \times C_n \to C$ is a function symbol of Σ, and let $u \equiv g^j(u_1, \ldots, u_n) : C^j$ for some terms $u_1, \ldots, u_n \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ with $u_\ell : C_\ell^j$ for each $1 \leq \ell \leq n$. By the induction hypothesis, for each $1 \leq \ell \leq n$ there is an α-restricted term $u'_\ell \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ of sort C_ℓ^j such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

  $$u_\ell \downarrow \vdash u_\ell = u'_\ell.$$

Then we set

$$u' \equiv g^j(u_1, \ldots, u_n)' := g^j(u_1', \ldots, u_n') : C^j,$$

which is α-restricted. It then easily follows from the induction hypothesis and the rules of partial Horn logic (including the deduction theorem from Remark 1.3.17) that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

$$g^j(u_1, \ldots, u_n) \downarrow \vdash g^j(u_1, \ldots, u_n) = g^j(u_1', \ldots, u_n').$$

- Suppose that $C \in \Sigma_{\mathsf{Sort}}$ and that $f : j \to \ell$ is a morphism of $\mathcal{J}$, and that $u \equiv \alpha_f^C(v) : C^\ell$ for some term $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ of sort C^j. Then by the induction hypothesis, there is an α-restricted term $v' \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ of sort C^j such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

  $$v \downarrow \vdash v = v'.$$

  By Lemma 5.1.26, there is also an α-restricted term $(v')^f \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))$ such that $(v')^f : C^\ell$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

  $$v' \downarrow \vdash \alpha_f^C(v') = (v')^f.$$

  We then set

  $$u' \equiv \alpha_f^C(v)' := (v')^f.$$

  Then (as mentioned) $(v')^f$ is α-restricted.

  Now we want to show that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

  $$\alpha_f^C(v) \downarrow \vdash \alpha_f^C(v) = (v')^f.$$

  By the deduction theorem for partial Horn logic (cf. Remark 1.3.17), it suffices to prove that the expanded theory

  $$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \cup \{\top \vdash \alpha_f^C(v) \downarrow\}$$

  proves the sequent

  $$\top \vdash \alpha_f^C(v) = (v')^f.$$

  By the rules of partial Horn logic, the expanded theory proves the sequent $\top \vdash v \downarrow$. Then by the cut rule and the definition of v', the expanded theory proves the sequent $\top \vdash v = v'$ and thus the sequent $\top \vdash v' \downarrow$. By the cut rule and the definition of $(v')^f$, the expanded theory then proves the sequent $\top \vdash \alpha_f^C(v') = (v')^f$. From this and the fact that the expanded theory proves the sequent $\top \vdash v = v'$, we obtain our result.

∎

Proposition (5.1.29)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and let* $s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *for some* $i \in \mathcal{J}_O$ *and* $A \in \Sigma_{\mathsf{Sort}}$*, with* $s, t : C^j$ *for some* $C \in \Sigma_{\mathsf{Sort}}$ *and* $j \in \mathcal{J}_O$*. If*

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash s = t,$$

then

$$\mathbb{T}\left(M^j, \overline{\mathsf{x}^A_{\mathsf{Cod}(j)}}\right) \vdash \theta(s) = \theta(t).$$

Proof: First, recall for $k \in \mathcal{J}_O$ that $\left(M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle, \left(\left[x^A_f\right]\right)_{f \in \mathsf{Cod}(k)}\right)$ is the initial model of $\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$ in the category of all partial $\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$-structures (the reader will be reminded of its explicit description during the course of the proof). It has the following universal property (cf. Proposition 2.2.10): if $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then for any tuple of elements $(n_f)_{f \in \mathsf{Cod}(k)} \in \prod_{f \in \mathsf{Cod}(k)} N_A$ and any Σ-morphism $h : M^k \to N$, there is a unique Σ-morphism $\widehat{h} : M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle \to N$ with the property that

$$\widehat{h} \circ \eta^{M^k} = h : M^k \to N$$

and

$$\widehat{h}_A\left(\left[\mathsf{x}^A_f\right]\right) = n_f \in N_A$$

for each $f \in \mathsf{Cod}(k)$ (where $\eta^{M^k} : M^k \to M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle$ is the canonical Σ-morphism).

We now begin the proof of the proposition by first constructing a special partial $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i})$-structure $\mathfrak{M}$ that will be a model of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$.

For any $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, we set

$$\mathfrak{M}_{B^k} := M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle_B = \left\{[u] : u \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)_B \wedge \mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \vdash u\downarrow\right\}.$$

Now let $f : k \to \ell$ be any morphism of $\mathcal{J}$, and let $B \in \Sigma_{\mathsf{Sort}}$: we must define

$$\left(\alpha^B_f\right)^{\mathfrak{M}} : \mathfrak{M}_{B^k} = M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle_B \rightharpoonup M^\ell\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}}\right\rangle_B = \mathfrak{M}_{B^\ell}.$$

We know that $f^M := F^M(f) : M^k \to M^\ell$ is a Σ-morphism. Then by the universal property of $M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle$, there is a unique Σ-morphism

$$\widehat{f^M} : M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle \to M^\ell\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}}\right\rangle$$

such that

$$\widehat{f^M} \circ \eta^{M^k} = \eta^{M^\ell} \circ f^M : M^k \to M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle$$

and

$$\widehat{f^M_A}\left(\left[\mathsf{x}^A_g\right]\right) = \left[\mathsf{x}^A_{f \circ g}\right] \in M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle_A$$

for every $g \in \mathsf{Cod}(k)$, where $\eta^{M^k} : M^k \to M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$ and $\eta^{M^\ell} : M^\ell \to M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle$ are the canonical Σ-morphisms. So we set

$$\left(\alpha^B_f\right)^{\mathfrak{M}} := \widehat{f^M_B} : \mathfrak{M}_{B^k} = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B \rightharpoondown M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle_B = \mathfrak{M}_{B^\ell}.$$

Now let $k \in \mathcal{J}_O$ and let $g : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ: we must define

$$\left(g^k\right)^{\mathfrak{M}} : \mathfrak{M}_{B^k_1} \times \ldots \times \mathfrak{M}_{B^k_n} \rightharpoondown \mathfrak{M}_{B^k},$$

i.e.

$$\left(g^k\right)^{\mathfrak{M}} : M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_1} \times \ldots \times M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_n} \rightharpoondown M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B .$$

We know that $M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$ is a Σ-structure, so we simply set

$$\left(g^k\right)^{\mathfrak{M}} := g^{M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle}.$$

Now let $k \in \mathcal{J}_O$, $B \in \Sigma_{\mathsf{Sort}}$, and $b \in M_{B^k} = M^k_B$; we define

$$\left(c^M_{B^k,b}\right)^{\mathfrak{M}} \in \mathfrak{M}_{B^k} = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B .$$

We set

$$\left(c^M_{B^k,b}\right)^{\mathfrak{M}} := \left[c^{M^k}_{B,b}\right] = \eta^{M^k}_B(b) \in M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B .$$

Lastly, we set

$$\mathsf{x}^{\mathfrak{M}}_{A^i} := \left[\mathsf{x}^A_{\mathsf{id}_i}\right] \in \mathfrak{M}_{A^i} = M^i \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}} \right\rangle_A .$$

This completes the definition of the partial $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i})$-structure $\mathfrak{M}$.

Observe (cf. Definition 5.1.2) that for any $k \in \mathcal{J}_O$ we have

$$\mathfrak{M}^k = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle .$$

Now we show that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$. First, we show that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}$. To show that $\mathfrak{M}$ satisfies Axiom 5.1.5.1, let $f : k \to \ell$ be any morphism in $\mathcal{J}$, and let $B \in \Sigma_{\mathsf{Sort}}$. We must show that $\left(\alpha^B_f\right)^{\mathfrak{M}} : \mathfrak{M}_{B^k} \to \mathfrak{M}_{B^\ell}$ is a total function. But this follows because $\left(\alpha^B_f\right)^{\mathfrak{M}} := \widehat{f^M_B}$ and $\widehat{f^M}$ is a Σ-morphism.

To show that $\mathfrak{M}$ satisfies Axiom 5.1.5.2, let $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$. We must show that $\left(\alpha^B_{\mathsf{id}_k}\right)^{\mathfrak{M}} := \left(\widehat{\mathsf{id}^M_k}\right)_B : \mathfrak{M}_{B^k} \to \mathfrak{M}_{B^k}$ is the identity function. Since $F^M : \mathcal{J} \to \mathbb{T}$-mod is functorial, we know that $F^M(\mathsf{id}_k) = \mathsf{id}^M_k : M^k \to M^k$ is the identity Σ-morphism, which then implies that $\widehat{\mathsf{id}^M_k} : M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle \to M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$ is the identity Σ-morphism, which yields the desired result. The fact that $\left(\alpha^B_g\right)^{\mathfrak{M}} \circ \left(\alpha^B_f\right)^{\mathfrak{M}} = \left(\alpha^B_{g \circ f}\right)^{\mathfrak{M}}$ for any composable $f, g \in \mathcal{J}_A$ is shown similarly, so that $\mathfrak{M}$ satisfies Axiom 5.1.5.3.

To show that $\mathfrak{M}$ satisfies Axiom 5.1.5.4, it suffices to show that for any arrow $f : k \to \ell$ in $\mathcal{J}$, the collection of total functions

$$\left(\left(\alpha^B_f\right)^{\mathfrak{M}} : \mathfrak{M}_{B^k} = \mathfrak{M}^k_B \to \mathfrak{M}_{B^\ell} = \mathfrak{M}^\ell_B\right)_{B \in \Sigma}$$

is a Σ-morphism $\mathfrak{M}^k \to \mathfrak{M}^\ell$, i.e. $M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle \to M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle$. But by definition of $\mathfrak{M}$, we know that this collection of total functions is equal to the collection

$$\left(\widehat{f^M_B}\right)_{B \in \Sigma} = \widehat{f^M} : M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle \to M^\ell \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(\ell)}} \right\rangle,$$

which is a Σ-morphism.

Finally, we must show that $\mathfrak{M}$ satisfies Axiom 5.1.5.5. To do this, it suffices to show for any $k \in \mathcal{J}_O$ that $\mathfrak{M}^k$ is a model of $\mathbb{T}$. But we have $\mathfrak{M}^k = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$, and $M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$ is a model of $\mathbb{T}$, since M^k is a model of $\mathbb{T}$ (by Lemma 5.1.7, since $M \models \mathbb{T}^{\mathcal{J}}$). This completes the proof that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}$.

Now we must show that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M)$. First, for any $k \in \mathcal{J}_O$, any $B \in \Sigma_{\mathsf{Sort}}$, and any $b \in M_{B^k} = M^k_B$, we must show that $\mathfrak{M}$ satisfies the $\mathbb{T}^{\mathcal{J}}(M)$-axiom

$$\top \vdash c^M_{B^k,b} \downarrow .$$

But this is true, because $\left(c^M_{B^k,b}\right)^{\mathfrak{M}} \in \mathfrak{M}_{B^k}$ is defined, by construction of $\mathfrak{M}$.

Now let $k \in \mathcal{J}_O$, let $g : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ, and let $b_1 \in M_{B^k_1}, \ldots, b_n \in M_{B^k_n}$ with $(b_1, \ldots, b_n) \in \mathsf{dom}\left((g^k)^M\right)$ and $(g^k)^M(b_1, \ldots, b_n) = b \in M_{B^k}$. We must show that $\mathfrak{M}$ satisfies the $\mathbb{T}^{\mathcal{J}}(M)$-axiom

$$\top \vdash g^k\left(c^M_{B^k_1,b_1}, \ldots, c^M_{B^k_n,b_n}\right) = c^M_{B^k,b}.$$

We have

$$\begin{aligned} (g^k)^{\mathfrak{M}}\left(\left(c^M_{B^k_1,b_1}\right)^{\mathfrak{M}}, \ldots, \left(c^M_{B^k_n,b_n}\right)^{\mathfrak{M}}\right) &= g^{M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle}\left(\left[c^{M^k}_{B_1,b_1}\right], \ldots, \left[c^{M^k}_{B_n,b_n}\right]\right) \\ &= \left[g\left(c^{M^k}_{B_1,b_1}, \ldots, c^{M^k}_{B_n,b_n}\right)\right] \end{aligned}$$

$$\begin{aligned}
&= \left[c^{M^k}_{B,g^{M^k}(b_1,\ldots,b_n)}\right] \\
&= \left[c^{M^k}_{B,(g^k)^M(b_1,\ldots,b_n)}\right] \\
&= \left[c^{M^k}_{B,b}\right] \\
&= \left(c^M_{B^k,b}\right)^{\mathfrak{M}},
\end{aligned}$$

as desired.

Next, let $f : k \to \ell$ be any morphism of $\mathcal{J}$ and let $B \in \Sigma_{\mathsf{Sort}}$, so that $\alpha^B_f : B^k \to B^\ell$. Suppose that $s_1 \in M_{B^k} = M^k_B$ and $\left(\alpha^B_f\right)^M(s_1) = s_2 \in M_{B^\ell} = M^\ell_B$. We must show that $\mathfrak{M}$ satisfies the $\mathbb{T}^{\mathcal{J}}(M)$-axiom

$$\top \vdash \alpha^B_f\left(c^M_{B^k,s_1}\right) = c^M_{B^\ell,s_2}.$$

We have

$$\begin{aligned}
\left(\alpha^B_f\right)^{\mathfrak{M}}\left(\left(c^M_{B^k,s_1}\right)^{\mathfrak{M}}\right) &= \widehat{f^M_B}\left(\left[c^{M^k}_{B,s_1}\right]\right) \\
&= \widehat{f^M_B}\left(\eta^{M^k}_B(s_1)\right) \\
&= \eta^{M^\ell}_B\left(f^M_B(s_1)\right) \\
&= \eta^{M^\ell}_B\left(\left(\alpha^B_f\right)^M(s_1)\right) \\
&= \eta^{M^\ell}_B(s_2) \\
&= \left[c^{M^\ell}_{B,s_2}\right] \\
&= \left(c^M_{B^\ell,s_2}\right)^{\mathfrak{M}},
\end{aligned}$$

as desired. This completes the proof that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M)$.

Finally, we have that $\mathfrak{M}$ satisfies the sequent $\top \vdash \mathsf{x}_{A^i}\downarrow$, since $\mathsf{x}^{\mathfrak{M}}_{A^i} = \left[\mathsf{x}^A_{\mathsf{id}_i}\right] \in \mathfrak{M}_{A^i} = M^i\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}\right\rangle_A$ is defined. This completes the proof that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$.

Now we prove the following claim, where we write $\mathfrak{M}^k_+ := \left(\mathfrak{M}^k, \left(\left[\mathsf{x}^A_f\right]\right)_{f\in\mathsf{Cod}(k)}\right)$ for $k \in \mathcal{J}_O$, so that

$$\mathfrak{M}^k_+ = \left(\mathfrak{M}^k, \left(\left[\mathsf{x}^A_f\right]\right)_{f\in\mathsf{Cod}(k)}\right) = \left(M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle, \left(\left[\mathsf{x}^A_f\right]\right)_{f\in\mathsf{Cod}(k)}\right)$$

is a $\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$-structure that is the initial model of $\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$. Recall that if $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ is of sort B^k for some $B \in \Sigma_{\mathsf{Sort}}$ and $k \in \mathcal{J}_O$, then $\theta(v) \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)\right)$ is a term of sort B.

Claim. If $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ is of sort B^k for some $B \in \Sigma_{\mathsf{Sort}}$ and $k \in \mathcal{J}_O$, then $v^{\mathfrak{M}} \in \mathfrak{M}_{B^k}$ is defined iff $\theta(v)^{\mathfrak{M}^k_+} \in \left(\mathfrak{M}^k_+\right)_B = \mathfrak{M}^k_B = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B$ is defined, and if both are defined, then

$$v^{\mathfrak{M}} = [\theta(v)] = \theta(v)^{\mathfrak{M}^k_+} \in \mathfrak{M}_{B^k} = M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_B .$$

Proof: This is proved by induction on terms $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$:

- If $v \equiv \mathsf{x}_{A^i}$, then both
$$v^{\mathfrak{M}} = \mathsf{x}^{\mathfrak{M}}_{A^i} = \left[\mathsf{x}^A_{\mathsf{id}_i}\right]$$
and
$$\theta(v)^{\mathfrak{M}^i_+} = \theta(\mathsf{x}_{A^i})^{\mathfrak{M}^i_+} = \left(\mathsf{x}^A_{\mathsf{id}_i}\right)^{\mathfrak{M}^i_+} = \left[\mathsf{x}^A_{\mathsf{id}_i}\right]$$
are defined, and (as indicated) we have
$$v^{\mathfrak{M}} = [\theta(v)] = \theta(v)^{\mathfrak{M}^i_+}.$$

- If $v \equiv c^M_{B^k,b} : B^k$ for some $k \in \mathcal{J}_O$, $B \in \Sigma_{\mathsf{Sort}}$, and $b \in M_{B^k} = M^k_B$, then both
$$v^{\mathfrak{M}} = \left(c^M_{B^k,b}\right)^{\mathfrak{M}} = \left[c^{M^k}_{B,b}\right]$$
and
$$\theta(v)^{\mathfrak{M}^k_+} = \theta\left(c^M_{B^k,b}\right)^{\mathfrak{M}^k_+} = \left(c^{M^k}_{B,b}\right)^{\mathfrak{M}^k_+} = \left(c^{M^k}_{B,b}\right)^{M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle} = \left[c^{M^k}_{B,b}\right]$$
are defined, and (as indicated) we have
$$v^{\mathfrak{M}} = [\theta(v)] = \theta(v)^{\mathfrak{M}^k_+}.$$

- If $v \equiv \alpha^A_f(\mathsf{x}_{A^i}) : A^k$ for some $f : i \to k \in \mathcal{J}$, then both
$$v^{\mathfrak{M}} = \alpha^A_f(\mathsf{x}_{A^i})^{\mathfrak{M}} = \left(\alpha^A_f\right)^{\mathfrak{M}}\left(\mathsf{x}^{\mathfrak{M}}_{A^i}\right) = \widehat{f^M_A}\left(\left[\mathsf{x}^A_{\mathsf{id}_i}\right]\right) = \left[\mathsf{x}^A_{f\circ\mathsf{id}_i}\right] = \left[\mathsf{x}^A_f\right]$$
and
$$\theta(v)^{\mathfrak{M}^k_+} = \theta\left(\alpha^A_f(\mathsf{x}_{A^i})\right)^{\mathfrak{M}^k_+} = \left(\mathsf{x}^A_f\right)^{\mathfrak{M}^k_+} = \left[\mathsf{x}^A_f\right]$$
are defined, and (as indicated) we have
$$v^{\mathfrak{M}} = [\theta(v)] = \theta(v)^{\mathfrak{M}^k_+}.$$

- Suppose that $v \equiv g^k(v_1, \ldots, v_n)$ for some $k \in \mathcal{J}_O$ and function symbol $g : B_1 \times \ldots \times B_n \to B$ in Σ and terms $v_1, \ldots, v_n \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $v_\ell : B^k_\ell$ for each $1 \leq \ell \leq n$. If $v^{\mathfrak{M}} = g^k(v_1, \ldots, v_n)^{\mathfrak{M}} \in \mathfrak{M}_{B^k}$ is defined, then $v_1^{\mathfrak{M}} \in \mathfrak{M}_{B_1^k}, \ldots, v_n^{\mathfrak{M}} \in \mathfrak{M}_{B_n^k}$ are all defined, so by the induction hypothesis it follows that $\theta(v_1)^{\mathfrak{M}^k_+} \in M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_1}, \ldots, \theta(v_n)^{\mathfrak{M}^k_+} \in M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_n}$ are all defined and moreover

  $$v_\ell^{\mathfrak{M}} = [\theta(v_\ell)] = \theta(v_\ell)^{\mathfrak{M}^k_+} \in M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_\ell}$$

  for each $1 \leq \ell \leq n$. Since $g^k(v_1, \ldots, v_n)^{\mathfrak{M}} \in \mathfrak{M}_{B^k}$ is defined, it follows that

  $$(v_1^{\mathfrak{M}}, \ldots, v_n^{\mathfrak{M}}) \in \mathsf{dom}\left(\left(g^k\right)^{\mathfrak{M}}\right) = \mathsf{dom}\left(g^{M^k\left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle}\right),$$

  i.e.

  $$([\theta(v_1)], \ldots, [\theta(v_n)]) \in \mathsf{dom}\left(g^{M^k\left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle}\right),$$

  which means by definition of $M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle$ that

  $$\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \vdash g(\theta(v_1), \ldots, \theta(v_n)) \downarrow,$$

  i.e.

  $$\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \vdash \theta(v) \downarrow.$$

  Since $\mathfrak{M}^k_+$ is a model (in fact, the initial model) of $\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$, it follows by soundness of partial Horn logic that

  $$\mathfrak{M}^k_+ \models \theta(v) \downarrow,$$

  so that $\theta(v)^{\mathfrak{M}^k_+}$ is defined, as desired. To show that $\theta(v)^{\mathfrak{M}^k_+}$ being defined implies that $v^{\mathfrak{M}}$ is defined, one essentially reverses the above reasoning (using [19, Theorem 23] - the fact that provability of a closed equation in $\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right)$ coincides with its satisfaction in the initial model $\mathfrak{M}^k_+$).

  Finally, if both $v^{\mathfrak{M}}$ and $\theta(v)^{\mathfrak{M}^k_+}$ are defined, then by the induction hypothesis we obtain that

  $$v_\ell^{\mathfrak{M}} = [\theta(v_\ell)] = \theta(v_\ell)^{\mathfrak{M}^k_+} \in M^k \left\langle \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}} \right\rangle_{B_\ell}$$

  for each $1 \leq \ell \leq n$, which yields

  $$v^{\mathfrak{M}} = g^k(v_1, \ldots, v_n)^{\mathfrak{M}}$$

$$
\begin{aligned}
&= \left(g^k\right)^{\mathfrak{M}} \left(v_1^{\mathfrak{M}}, \ldots, v_n^{\mathfrak{M}}\right) \\
&= g^{M^k\left\langle\overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right\rangle}([\theta(v_1)], \ldots, [\theta(v_n)]) \\
&= [g(\theta(v_1), \ldots, \theta(v_n))] \\
&= [\theta(g^k(v_1, \ldots, v_n))] \\
&= [\theta(v)],
\end{aligned}
$$

as well as

$$
\begin{aligned}
\theta(v)^{\mathfrak{M}^k_+} &= \theta(g^k(v_1, \ldots, v_n))^{\mathfrak{M}^k_+} \\
&= g(\theta(v_1), \ldots, \theta(v_n))^{\mathfrak{M}^k_+} \\
&= g^{\mathfrak{M}^k_+}\left(\theta(v_1)^{\mathfrak{M}^k_+}, \ldots, \theta(v_n)^{\mathfrak{M}^k_+}\right) \\
&= (g^k)^{\mathfrak{M}} \left(v_1^{\mathfrak{M}}, \ldots, v_n^{\mathfrak{M}}\right) \\
&= g^k(v_1, \ldots, v_n)^{\mathfrak{M}} \\
&= v^{\mathfrak{M}},
\end{aligned}
$$

as desired.

∎

We at last prove the actual statement of the proposition. So let $s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $s, t : C^j$ for some $C \in \Sigma_{\mathsf{Sort}}$ and $j \in \mathcal{J}_O$, and suppose that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash s = t$. Since $\mathfrak{M}^j_+$ is the initial model of $\mathbb{T}\left(M^j, \overline{\mathsf{x}^A_{\mathsf{Cod}(j)}}\right)$, then to show that $\mathbb{T}\left(M^j, \overline{\mathsf{x}^A_{\mathsf{Cod}(j)}}\right) \vdash \theta(s) = \theta(t)$, it suffices by [19, Theorem 23] to show that $\mathfrak{M}^j_+$ satisfies the sequent $\top \vdash \theta(s) = \theta(t)$. Since $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$, it follows by hypothesis and the soundness of partial Horn logic that $\mathfrak{M} \models s = t$. Then $s^{\mathfrak{M}}, t^{\mathfrak{M}}$ are defined and equal, which implies by the Claim that $\theta(s)^{\mathfrak{M}^j_+}, \theta(t)^{\mathfrak{M}^j_+}$ are defined and equal. So $\mathfrak{M}^j_+$ satisfies the desired equation, which completes the proof of Proposition 5.1.29. ∎

Lemma (5.1.30)**.** *Let $\mathbb{T}'$ be any quasi-equational theory over a relation-free signature Σ', let $B \in \Sigma'_{\mathsf{Sort}}$, and let $\mathcal{C}$ be a (possibly infinite) set of constants of sort B with $\mathcal{C} \cap \Sigma'_{\mathsf{Fun}} = \emptyset$. Let $\Sigma'(\mathcal{C})$ be the signature with $\Sigma'(\mathcal{C})_{\mathsf{Sort}} := \Sigma'_{\mathsf{Sort}}$ and $\Sigma'(\mathcal{C})_{\mathsf{Fun}} := \Sigma'_{\mathsf{Fun}} \cup \mathcal{C}$, and let $\mathbb{T}'(\mathcal{C})$ be the quasi-equational theory over the signature $\Sigma'(\mathcal{C})$ whose axioms are those of $\mathbb{T}'$ together with the axioms $\top \vdash c\downarrow$ for all $c \in \mathcal{C}$.*

Let $s, t \in \mathsf{Term}^c(\Sigma'(\mathcal{C}))$ be closed terms over $\Sigma'(\mathcal{C})$ of the same sort such that at least one of s and t contains a constant from $\mathcal{C}$, and let $\{c_1, \ldots, c_n\}$ be the (finite, non-empty) set of all and only those constants of $\mathcal{C}$ that occur in either s or t. Let

$\Sigma'(c_1, \ldots, c_n)$ and $\mathbb{T}'(c_1, \ldots, c_n)$ be the signature and theory defined analogously to $\Sigma'(\mathcal{C})$ and $\mathbb{T}'(\mathcal{C})$. Then

$$\mathbb{T}'(\mathcal{C}) \vdash s = t \Longrightarrow \mathbb{T}'(c_1, \ldots, c_n) \vdash s = t.$$

Proof: Suppose that $\mathbb{T}'(\mathcal{C}) \vdash s = t$. To prove that $\mathbb{T}'(c_1, \ldots, c_n) \vdash s = t$, it suffices by completeness of partial Horn logic to show that any $\Sigma'(c_1, \ldots, c_n)$-structure M with $M \models \mathbb{T}'(c_1, \ldots, c_n)$ satisfies $M \models s = t$. And to show this, it suffices by the hypothesis and soundness of partial Horn logic to show that any $\Sigma'(c_1, \ldots, c_n)$-structure M with $M \models \mathbb{T}'(c_1, \ldots, c_n)$ can be expanded to a $\Sigma'(\mathcal{C})$-structure M' with $M' \models \mathbb{T}'(\mathcal{C})$. So we must set $M'|_{\Sigma'(c_1,\ldots,c_n)} := M$, and for any constant $c \in \mathcal{C} \setminus \{c_1, \ldots, c_n\}$, we set $c^{M'} := c_1^M \in M_B = M'_B$ (which we can do, because $\{c_1, \ldots, c_n\}$ is non-empty by assumption). Then we clearly have $M' \models \mathbb{T}'(\mathcal{C})$, which completes the proof. ▮

Lemma (5.1.31)**.** *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ and $k \in \mathcal{J}_O$ and $B \in \Sigma_{\mathsf{Sort}}$, and suppose that $u \in \mathsf{Term}^c\left(\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)\right)$ is of sort B and*

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \vdash u = \mathsf{x}^B_{\mathsf{id}_k}.$$

If $\mathbb{T}(M^k)$ is non-trivial for the sort B, then u contains at least one occurrence of $\mathsf{x}^B_{\mathsf{id}_k}$.

Proof: Assume all of the hypotheses, but suppose towards a contradiction that u does *not* contain any occurrence of $\mathsf{x}^B_{\mathsf{id}_k}$. Let

$$\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- := \Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \setminus \left\{\mathsf{x}^B_{\mathsf{id}_k}\right\},$$

and let

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- := \mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right) \setminus \left\{\top \vdash \mathsf{x}^B_{\mathsf{id}_k} \downarrow\right\},$$

a quasi-equational theory over the signature $\Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^-$. Since u does not contain $\mathsf{x}^B_{\mathsf{id}_k}$, it follows by the theorem on constants (Remark 1.3.17) that if $y, y' : B$ are distinct variables, then

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- \vdash^{y:B} u = y$$

and

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- \vdash^{y':B} u = y',$$

so that

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- \vdash^{y,y':B} y = y'.$$

Now let $\mathsf{x}, \mathsf{x}' : B$ be distinct constant symbols with $\mathsf{x}, \mathsf{x}' \notin \Sigma\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^-$. By the theorem on constants again, we then obtain

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^B_{\mathsf{Cod}(k)}}\right)^- \cup \{\top \vdash \mathsf{x}\downarrow \wedge\, \mathsf{x}'\downarrow\} \vdash \mathsf{x} = \mathsf{x}'.$$

By Lemma 5.1.30, it then follows that

$$\mathbb{T}(M^k) \cup \{\top \vdash \mathsf{x}\downarrow \wedge\, \mathsf{x}'\downarrow\} \vdash \mathsf{x} = \mathsf{x}'.$$

By yet another application of the theorem on constants, it finally follows that

$$\mathbb{T}(M^k) \vdash^{y,y'} y = y',$$

with $y, y' : B$ distinct variables, contrary to the supposition that $\mathbb{T}(M^k)$ is non-trivial for the sort B. So u must contain at least one occurrence of $\mathsf{x}^B_{\mathsf{id}_k}$, as desired. ∎

Lemma (5.1.34)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*. Let* $u, s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *for some* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$*, with* $u : C^i$ *and* $s, t : D^i$ *for some* $C, D \in \Sigma_{\mathsf{Sort}}$*. Suppose that* $u \equiv h^i(u_1, \ldots, u_m)$ *for some function symbol* $h : C_1 \times \ldots \times C_m \to C$ *of* Σ *and* i*-local terms* $u_1, \ldots, u_m \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *with* $u_\ell : C^i_\ell$ *and* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u_\ell\downarrow$ *for each* $1 \le \ell \le m$*, and assume that* u_ℓ *commutes generically with each* $f : i \to i$ *in* $\mathcal{J}$*.*

If $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ *proves the sequent*

$$u\downarrow \vdash s = t,$$

then $\mathbb{T}(M^i, \mathsf{x}_A)$ *proves the sequent*

$$\theta^*(u)\downarrow \vdash \theta^*(s) = \theta^*(t).$$

Proof: First, it is trivial to see that the signature morphism

$$\lambda_i : \Sigma\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}\right) \to \Sigma(M^i, \mathsf{x}_A)$$

from Definition 5.1.32 is in fact a theory morphism

$$\lambda_i : \mathbb{T}\left(M^i, \overline{\mathsf{x}^A_{\mathsf{Cod}(i)}}\right) \to \mathbb{T}(M^i, \mathsf{x}_A).$$

Now let $u, s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u : C^i$ and $s, t : D^i$ for some $C, D \in \Sigma_{\mathsf{Sort}}$. We will not need to assume anything about u until near the end of the proof.

Let $\ell \in \mathcal{J}_O$, and let $\mathcal{J}_A(i, \ell)$ be the set of arrows in $\mathcal{J}$ from i to ℓ. For any such arrow $f : i \to \ell$, we have by Lemma 5.1.26 an α-restricted term $u^f \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u^f : C^\ell$ such that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the sequent

$$u\downarrow \vdash \alpha^C_f(u) = u^f$$

and $\theta^*\left(u^f\right) \in \mathsf{Term}^c(\Sigma(M^\ell, \mathsf{x}_A))$.

For any $\ell \in \mathcal{J}_O$, consider the quasi-equational theory

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \cup \left\{\top \vdash \theta^*\left(u^f\right)\downarrow : f \in \mathcal{J}_A(i,\ell)\right\}$$

over the signature $\Sigma(M^\ell, \mathsf{x}_A)$, and let us denote the Σ-reduct of the initial model of this theory by

$$M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle.$$

For any $B \in \Sigma_{\mathsf{Sort}}$,

$$M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B = \left\{\begin{array}{l}[v] : v \in \mathsf{Term}^c(\Sigma(M^\ell, \mathsf{x}_A))_B \text{ and} \\ \mathbb{T}(M^\ell, \mathsf{x}_A) \cup \left\{\top \vdash \theta^*\left(u^f\right)\downarrow : f \in \mathcal{J}_A(i,\ell)\right\} \vdash v\downarrow\end{array}\right\}.$$

These models have the following universal property, which can be easily derived from the initiality in the definition of $M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle$. Recall that if $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ and $h : M^\ell \to N$ is a Σ-morphism, then N^h is the $\Sigma(M^\ell)$-structure from Lemma 2.2.4 with $N^h|_\Sigma = N$ and $N^h \models \mathbb{T}(M^\ell)$.

Claim. Let $\ell \in \mathcal{J}_O$. If $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then for any Σ-morphism $h : M^\ell \to N$ and any element $n_A \in N_A$ such that

$$\left(N^h, n_A\right) \models \theta^*(u^g)\downarrow$$

for each $g \in \mathcal{J}_A(i,\ell)$, there is a unique Σ-morphism

$$\widehat{h} : M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle \to N$$

with the property that

$$\widehat{h} \circ \eta^{M^\ell} = h : M^\ell \to N$$

and

$$\widehat{h}_A\left([\mathsf{x}_A]\right) = n_A \in N_A$$

(where $\eta^{M^\ell} : M^\ell \to M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle$ is the canonical Σ-morphism). ∎

As in the proof of Proposition 5.1.29, we now define a $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i})$-structure $\mathfrak{M}$ that will be a model of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \cup \{\top \vdash u\downarrow\}$. For any $B \in \Sigma_{\mathsf{Sort}}$ and $\ell \in \mathcal{J}_O$ we set

$$\mathfrak{M}_{B^\ell} := M^\ell\left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B.$$

Now let $g : \ell \to \ell'$ be any morphism of $\mathcal{J}$ and let $B \in \Sigma_{\mathsf{Sort}}$: we must define

$$\left(\alpha_g^B\right)^{\mathfrak{M}} : \mathfrak{M}_{B^\ell} \rightharpoondown \mathfrak{M}_{B^{\ell'}}.$$

We know that $g^M := F^M(g) : M^\ell \to M^{\ell'}$ is a Σ-morphism, so we have a Σ-morphism

$$\eta^{M^{\ell'}} \circ g^M : M^\ell \to M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle .$$

We also have

$$[\mathsf{x}_A] \in M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle_A .$$

It is then tedious but not difficult to show that

$$\left(M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle^{\eta^{M^{\ell'}} \circ g^M}, [\mathsf{x}_A] \right) \models \theta^*(u^h) \downarrow$$

for each $h \in \mathcal{J}_A(i,\ell)$, since the interpretation of $\theta^*(u^h)$ in this $\Sigma(M^\ell, \mathsf{x}_A)$-structure will be

$$\left[\theta^* \left(u^{g\circ h}\right)\right] \in M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle_C ,$$

because

$$\mathbb{T}\left(M^{\ell'}, \mathsf{x}_A\right) \cup \left\{\top \vdash \theta^* \left(u^f\right) \downarrow : f \in \mathcal{J}_A(i,\ell')\right\} \vdash \theta^* \left(u^{g\circ h}\right) \downarrow,$$

as $g \circ h \in \mathcal{J}_A(i,\ell')$. Then by the universal property of $M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle$ in the above Claim, there is a unique Σ-morphism

$$\widehat{g^M} : M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle \to M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle$$

such that

$$\widehat{g^M} \circ \eta^{M^\ell} = \eta^{M^{\ell'}} \circ g^M : M^\ell \to M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle$$

and

$$\widehat{g^M_A} \left([\mathsf{x}_A]\right) = [\mathsf{x}_A] \in M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle_A .$$

So we set

$$\left(\alpha_g^B\right)^{\mathfrak{M}} := \widehat{g^M_B} : \mathfrak{M}_{B^\ell} = M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle_B \rightharpoonup M^{\ell'} \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell'} \right\rangle_B = \mathfrak{M}_{B^{\ell'}} .$$

Now let $\ell \in \mathcal{J}_O$ and let $g : B_1 \times \ldots \times B_n \to B$ be a function symbol of Σ: we must define

$$\left(g^\ell\right)^{\mathfrak{M}} : \mathfrak{M}_{B_1^\ell} \times \ldots \times \mathfrak{M}_{B_n^\ell} \rightharpoonup \mathfrak{M}_{B^\ell},$$

i.e.

$$\left(g^\ell\right)^{\mathfrak{M}} : M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle_{B_1} \times \ldots \times M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle_{B_n}$$

$$\rightharpoonup M^\ell \left\langle \mathsf{x}_A, \theta^* \left(u^f\right)_{f:i\to\ell} \right\rangle_B .$$

We know that $M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle$ is a Σ-structure, so we simply set

$$\left(g^\ell\right)^{\mathfrak{M}} := g^{M^\ell\left\langle \mathsf{x}_A,\theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle}.$$

Now let $\ell \in \mathcal{J}_O$, $B \in \Sigma_{\mathsf{Sort}}$, and $b \in M_{B^\ell} = M^\ell_B$; we define

$$\left(c^M_{B^\ell,b}\right)^{\mathfrak{M}} \in \mathfrak{M}_{B^\ell} = M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B.$$

We set

$$\left(c^M_{B^\ell,b}\right)^{\mathfrak{M}} := \left[c^{M^\ell}_{B,b}\right] = \eta^{M^\ell}_B(b) \in M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B.$$

Lastly, we set

$$\mathsf{x}^{\mathfrak{M}}_{A^i} := [\mathsf{x}_A] \in \mathfrak{M}_{A^i} = M^i \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to i}\right\rangle_A.$$

This completes the definition of the partial $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i})$-structure $\mathfrak{M}$.

Observe (cf. Definition 5.1.2) that for any $\ell \in \mathcal{J}_O$ we have

$$\mathfrak{M}^\ell = M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle.$$

The verification that $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i})$ is now essentially identical to the analogous verification in the proof of Proposition 5.1.29.

We now have the following claim, whose proof is almost identical to the proof of the analogous claim in the proof of Proposition 5.1.29. Recall that if $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i}))^*$ is of sort B^ℓ for some $B \in \Sigma_{\mathsf{Sort}}$ and $\ell \in \mathcal{J}_O$, then $\theta^*(v) \in \mathsf{Term}^c(\Sigma(M^\ell, \mathsf{x}_A))_B$ and hence can be interpreted in the $\Sigma(M^\ell, \mathsf{x}_A)$-structure

$$\mathfrak{M}^\ell_+ := \left(\mathfrak{M}^\ell, [x_A]\right) = \left(M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle, [x_A]\right).$$

Claim. If $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i}))^*$ is of sort B^ℓ for some $B \in \Sigma_{\mathsf{Sort}}$ and $\ell \in \mathcal{J}_O$, then $v^{\mathfrak{M}} \in \mathfrak{M}_{B^\ell}$ is defined iff $\theta^*(v)^{\mathfrak{M}^\ell_+} \in \mathfrak{M}^\ell_B = M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B$ is defined, and if both are defined, then

$$v^{\mathfrak{M}} = [\theta^*(v)] = \theta^*(v)^{\mathfrak{M}^\ell_+} \in \mathfrak{M}_{B^\ell} = M^\ell \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to\ell}\right\rangle_B.$$

∎

Finally, we show that $\mathfrak{M} \models u\downarrow$, i.e. that $u^{\mathfrak{M}}$ is defined in $\mathfrak{M}_{C^i}$ (recall that $u : C^i$). By the preceding Claim, it is equivalent to show that $\theta^*(u)^{\mathfrak{M}^i_+} \in \mathfrak{M}^i_C = M^i \left\langle \mathsf{x}_A, \theta^*\left(u^f\right)_{f:i\to i}\right\rangle_C$ is defined. Since $\mathfrak{M}^i_+$ is a model (in fact, the initial model) of $\mathbb{T}(M^i, \mathsf{x}_A) \cup \left\{\top \vdash \theta^*\left(u^f\right) \downarrow : f \in \mathcal{J}_A(i,i)\right\}$, it suffices by soundness of partial Horn logic to show that this theory proves the sequent $\top \vdash \theta^*(u) \downarrow$. But this is true because $\theta^*(u) \equiv \theta^*\left(u^{\mathsf{id}^i}\right)$ (since $u \equiv u^{\mathsf{id}^i}$) and this theory (by definition) does indeed prove the sequent $\top \vdash \theta^*\left(u^{\mathsf{id}^i}\right) \downarrow$. Thus, we have proven that $\mathfrak{M}$ is a $\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i})$-structure that is a model of $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \cup \{\top \vdash u \downarrow\}$.

Now let us finally prove the actual statement of the lemma. We initially assumed that $u, s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i}))^*$ with $u : C^i$ and $s, t : D^i$ for some $C, D \in \Sigma_{\mathsf{Sort}}$. Suppose in addition that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i})$ proves the sequent $u \downarrow \vdash s = t$. Before showing that $\mathbb{T}(M^i, \mathsf{x}_A)$ proves the sequent $\theta^*(u) \downarrow \vdash \theta^*(s) = \theta^*(t)$, we first show that the sequent $\top \vdash \theta^*(s) = \theta^*(t)$ is a theorem of

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \left\{\top \vdash \theta^*\left(u^f\right) \downarrow : f \in \mathcal{J}_A(i,i)\right\}.$$

Since $\mathfrak{M}^i_+$ is the initial model of this theory, it suffices by [19, Theorem 23] to show that $\mathfrak{M}^i_+$ satisfies the sequent $\top \vdash \theta^*(s) = \theta^*(t)$. Since $\mathfrak{M} \models \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i})$, it follows by hypothesis and the soundness of partial Horn logic that $\mathfrak{M}$ satisfies the sequent $u \downarrow \vdash s = t$. Since also $\mathfrak{M} \models u \downarrow$, it then follows that $\mathfrak{M} \models s = t$. So $s^{\mathfrak{M}}, t^{\mathfrak{M}}$ are defined and equal, which (by the preceding Claim) implies that $\theta^*(s)^{\mathfrak{M}^i_+}, \theta^*(t)^{\mathfrak{M}^i_+}$ are defined and equal, so that $\mathfrak{M}^i_+$ models the sequent $\top \vdash \theta^*(s) = \theta^*(t)$, as desired. Thus, the sequent $\top \vdash \theta^*(s) = \theta^*(t)$ is indeed a theorem of

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \left\{\top \vdash \theta^*\left(u^f\right) \downarrow : f \in \mathcal{J}_A(i,i)\right\}.$$

Next, we wish to show that the sequent $\top \vdash \theta^*(s) = \theta^*(t)$ is a theorem of

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \{\top \vdash \theta^*(u) \downarrow\}.$$

By what we just showed, it suffices to prove that if $f \in \mathcal{J}_A(i,i)$, then $\top \vdash \theta^*\left(u^f\right) \downarrow$ is a theorem of

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \{\top \vdash \theta^*(u) \downarrow\}.$$

We now invoke the additional assumptions on u made in the statement of the lemma: namely, we assume that $u \equiv h^i(u_1, \ldots, u_m)$ for some function symbol $h : C_1 \times \ldots \times C_m \to C$ of Σ and i-local terms $u_1, \ldots, u_m \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i}))^*$ with $u_\ell : C_\ell^i$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash u_\ell \downarrow$ for each $1 \leq \ell \leq m$, and we also assume that u_ℓ commutes generically with each $f : i \to i$ in $\mathcal{J}$. The latter assumption (cf. Definition 5.1.33) means that if $f : i \to i$, then

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash \alpha_f^C(u_\ell) = u_\ell[f]$$

for each $1 \leq \ell \leq m$. Since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash \alpha_f^C(u_\ell) = u_\ell^f$ (by Lemma 5.1.26), we then obtain that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A_i}) \vdash u_\ell^f = u_\ell[f]$$

for each $1 \leq \ell \leq m$, with both $u_\ell^f, u_\ell[f] \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A_i}))^*$. By Proposition 5.1.29, we then obtain

$$\mathbb{T}\left(M^i, \overline{\mathsf{x}_{\mathsf{Cod}(i)}^A}\right) \vdash \theta\left(u_\ell^f\right) = \theta(u_\ell[f]).$$

Since

$$\lambda_i : \mathbb{T}\left(M^i, \overline{\mathsf{x}_{\mathsf{Cod}(i)}^A}\right) \to \mathbb{T}(M^i, \mathsf{x}_A)$$

is a theory morphism, we then obtain

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \lambda_i\left(\theta\left(u_\ell^f\right)\right) = \lambda_i\left(\theta\left(u_\ell[f]\right)\right),$$

i.e.

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(u_\ell^f\right) = \theta^*(u_\ell[f]).$$

By Lemma 5.1.36 (since u_ℓ is i-local) we have $\theta^*(u_\ell[f]) \equiv \theta^*(u_\ell)$, and thus

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(u_\ell^f\right) = \theta^*(u_\ell).$$

Altogether, we want to show for $f : i \to i$ that the sequent $\top \vdash \theta^*\left(u^f\right) \downarrow$, i.e. the sequent $\top \vdash h\left(\theta^*\left(u_1^f\right), \ldots, \theta^*\left(u_m^f\right)\right) \downarrow$, is provable in the theory

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \{\top \vdash \theta^*(u) \downarrow\},$$

i.e. the theory

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \{\top \vdash h(\theta^*(u_1), \ldots, \theta^*(u_m)) \downarrow\}.$$

But this follows from the just mentioned fact that

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(u_\ell^f\right) = \theta^*(u_\ell)$$

for each $1 \leq \ell \leq m$.

This proves that the sequent $\top \vdash \theta^*(s) = \theta^*(t)$ is a theorem of

$$\mathbb{T}(M^i, \mathsf{x}_A) \cup \{\top \vdash \theta^*(u) \downarrow\}.$$

By the deduction theorem for partial Horn logic (cf. Remark 1.3.17), it then follows that $\mathbb{T}(M^i, \mathsf{x}_A)$ proves the sequent

$$\theta^*(u) \downarrow \vdash \theta^*(s) = \theta^*(t),$$

which completes the proof of Lemma 5.1.34. ∎

Lemma (5.1.35)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$*. For any* $s, t \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *with* $s, t : C^k$ *for some* $k \in \mathcal{J}_O$ *and* $C \in \Sigma_{\mathsf{Sort}}$*, if* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ *proves the sequent*

$$\top \vdash s = t,$$

then $\mathbb{T}(M^k, \mathsf{x}_A)$ *proves the sequent*

$$\top \vdash \theta^*(s) = \theta^*(t).$$

Proof: Suppose that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash s = t$. By Proposition 5.1.29, it then follows that

$$\mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \vdash \theta(s) = \theta(t),$$

since $s, t : C^k$. Since

$$\lambda_k : \mathbb{T}\left(M^k, \overline{\mathsf{x}^A_{\mathsf{Cod}(k)}}\right) \to \mathbb{T}(M^k, \mathsf{x}_A)$$

is a theory morphism by the proof of Lemma 5.1.34, it then follows that

$$\mathbb{T}(M, \mathsf{x}_A) \vdash \theta^*(s) \equiv \lambda_k(\theta(s)) = \lambda_k(\theta(t)) \equiv \theta^*(t),$$

as desired. ∎

Lemma (5.1.36)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, let* $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *be* i*-local for some* $i \in \mathcal{J}_O$ *and* $A \in \Sigma_{\mathsf{Sort}}$*, and let* $f \in \mathsf{Cod}(i)$*. Then*

$$\theta^*(u) \equiv \theta^*(u[f]) \in \mathsf{Term}^c(\Sigma(M^i, \mathsf{x}_A)).$$

Proof: Let $f : j \to i$. We prove the lemma by induction on i-local terms $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$:

- If $u \equiv \mathsf{x}_{A^i} : A^i$, then $u[f] \equiv \alpha^A_f(\mathsf{x}_{A^j})$. Then we have

  $$\theta^*(\mathsf{x}_{A^i}) \equiv \lambda_i(\theta(\mathsf{x}_{A^i})) \equiv \lambda_i\left(\mathsf{x}^A_{\mathsf{id}_i}\right) \equiv \mathsf{x}_A \equiv \lambda_i\left(\mathsf{x}^A_f\right) \equiv \lambda_i(\theta(\alpha^A_f(\mathsf{x}_{A^j}))) \equiv \theta^*(u[f]).$$

- Suppose $u \equiv \alpha^A_g(\mathsf{x}_{A^i}) : A^i$ for some arrow $g : i \to i$ in $\mathcal{J}$. Then $u[f] \equiv \alpha^A_{g\circ f}(\mathsf{x}_{A^j})$. We then have

  $$\begin{aligned} \theta^*(u) &\equiv \theta^*(\alpha^A_g(\mathsf{x}_{A^i})) \\ &\equiv \lambda_i(\theta(\alpha^A_g(\mathsf{x}_{A^i}))) \\ &\equiv \lambda_i\left(\mathsf{x}^A_g\right) \\ &\equiv \mathsf{x}_A \\ &\equiv \lambda_i\left(\mathsf{x}^A_{g\circ f}\right) \\ &\equiv \lambda_i(\theta(\alpha^A_{g\circ f}(\mathsf{x}_{A^j}))) \\ &\equiv \theta^*(u[f]). \end{aligned}$$

- If $u \equiv c^M_{B^i,s} : B^i$ for some sort $B \in \Sigma$ and $s \in M_{B^i}$, then $u[f] \equiv u \equiv c^M_{B^i,s}$. So then the desired result obviously holds.
- The induction step for function symbols in Σ is straightforward.

∎

Lemma (5.1.37)**.** *Let $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, let $u, v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ for some $A \in \Sigma_{\mathsf{Sort}}$ and $i \in \mathcal{J}_O$ with $v : A^i$, and suppose that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u, v \downarrow$. Suppose also that u, v are i-local, and that v commutes generically with every arrow $f : i \to i$ in $\mathcal{J}$. Then*

$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(u[v/\mathsf{x}_{A^i}]'\right) = \theta^*(u)[\theta^*(v)/\mathsf{x}_A],$$

where $u[v/\mathsf{x}_{A^i}]'$ is the α-restricted variant of $u[v/\mathsf{x}_{A^i}]$ from Lemma 5.1.27.

Proof: Fix $v \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with the properties that $v : A^i$ and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash v \downarrow$ and v is i-local, and assume that v commutes generically with every $f : i \to i$, which means (cf. Definition 5.1.33) that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \alpha^A_f(v) = v[f].$$

We will prove the desired claim by induction on α-restricted, i-local terms $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$:

- If $u \equiv \mathsf{x}_{A^i} : A^i$, then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$ and $u[v/\mathsf{x}_{A^i}] \equiv v$. Since $\theta^*(u) \equiv \theta^*(\mathsf{x}_{A^i}) \equiv \mathsf{x}_A$, we must show

  $$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*(v') = \theta^*(v).$$

  By Lemma 5.1.27 and the assumption that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash v \downarrow$, we know that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash v = v'$. Then the desired result follows by Lemma 5.1.35.

- Suppose $u \equiv \alpha^A_g(\mathsf{x}_{A^i})$ for some $g : i \to i \in \mathcal{J}_A$ (since u is i-local). Then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$, and we have $u[v/\mathsf{x}_{A^i}] \equiv \alpha^A_g(v)$. By Lemma 5.1.27 and the assumption that v commutes generically with g, we have

  $$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \alpha^A_g(v)' = \alpha^A_g(v) = v[g].$$

  Since

  $$\theta^*(u) \equiv \theta^*(\alpha^A_g(\mathsf{x}_{A^i})) \equiv \lambda_i(\theta(\alpha^A_g(\mathsf{x}_{A^i}))) \equiv \lambda_i(\mathsf{x}^A_g) \equiv \mathsf{x}_A,$$

  it now suffices (by Lemma 5.1.35) to show

  $$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*(v[g]) = \theta^*(v).$$

  But this follows by Lemma 5.1.36 (since v is i-local) and the fact that $\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*(v) \downarrow$ (which itself follows from Lemma 5.1.35 and the assumption that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash v \downarrow$).

- Suppose $u \equiv c^M_{D^i,s}$ for some $D \in \Sigma_{\mathsf{Sort}}$ and $s \in M_{D^i}$. Then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$, and we have $u[v/\mathsf{x}_{A^i}] \equiv u \equiv c^M_{D^i,s}$, as well as $u[v/\mathsf{x}_{A^i}]' \equiv \left(c^M_{D^i,s}\right)' \equiv c^M_{D^i,s}$ by the proof of Lemma 5.1.27. Also, we have $\theta^*(u) \equiv \theta^*\left(c^M_{D^i,s}\right) \equiv c^{M^i}_{D,s}$. So our goal is now to show
$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash \theta^*\left(c^M_{D^i,s}\right) = \theta^*\left(c^M_{D^i,s}\right),$$
i.e.
$$\mathbb{T}(M^i, \mathsf{x}_A) \vdash c^{M^i}_{D,s} = c^{M^i}_{D,s},$$
which is true, because $\mathbb{T}(M^i, \mathsf{x}_A) \vdash c^{M^i}_{D,s} \downarrow$.

- The induction step for function symbols in Σ is straightforward.

∎

Lemma (5.1.38)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *and* $A \in \Sigma_{\mathsf{Sort}}$ *and* $i \in \mathcal{J}_O$*, and let* $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ *be an* α*-restricted,* i*-local term of sort* B^i *for some* $B \in \Sigma_{\mathsf{Sort}}$ *with* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$*. Let* $f : i \to \ell$ *be an arbitrary arrow of* $\mathcal{J}$ *with* $\mathsf{dom}(f) = i$*. Then* $\alpha^B_f(u)$ *has an* α*-restricted variant* $\alpha^B_f(u)'$ *by Lemma 5.1.27, and* $\alpha^B_f(u)' : B^\ell$*, so that* $\theta^*\left(\alpha^B_f(u)'\right) \in \mathsf{Term}^c\left(\Sigma\left(M^\ell, \mathsf{x}_A\right)\right)$*. And* $\theta^*(u) \in \mathsf{Term}^c\left(\Sigma\left(M^i, \mathsf{x}_A\right)\right)$*, so that* $\rho^A_{f^M}(\theta^*(u)) \in \mathsf{Term}^c(\Sigma(M^\ell, \mathsf{x}_A))$*, where* $\rho^A_{f^M} : \mathbb{T}(M^i, \mathsf{x}_A) \to \mathbb{T}(M^\ell, \mathsf{x}_A)$ *is the theory morphism induced by the* Σ*-morphism* $f^M := F^M(f) : M^i \to M^\ell$*. Then*
$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \theta^*\left(\alpha^B_f(u)'\right) = \rho^A_{f^M}(\theta^*(u)).$$

Proof: We prove the lemma by induction on i-local terms $u \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$.

- If $u \equiv \mathsf{x}_{A^i} : A^i$, then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$ and
$$\rho^A_{f^M}(\theta^*(u)) \equiv \rho^A_{f^M}(\theta^*(\mathsf{x}_{A^i})) \equiv \rho^A_{f^M}(\mathsf{x}_A) \equiv \mathsf{x}_A.$$
By the proof of Lemma 5.1.27, we also have
$$\theta^*(\alpha^A_f(u)') \equiv \theta^*(\alpha^A_f(\mathsf{x}_{A^i})') \equiv \theta^*(\alpha^A_f(\mathsf{x}_{A^i})) \equiv \mathsf{x}_A.$$
So we must prove
$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \mathsf{x}_A = \mathsf{x}_A,$$
which is true because $\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \mathsf{x}_A \downarrow$.

- Suppose $u \equiv c^M_{B^i,s} : B^i$ for some sort $B \in \Sigma$ and $s \in M_{B^i} = M^i_B$. Then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$, and we have

  $$\rho^A_{f^M}(\theta^*(u)) \equiv \rho^A_{f^M}\left(\theta^*\left(c^M_{B^i,s}\right)\right) \equiv \rho^A_{f^M}\left(c^{M^i}_{B,s}\right) \equiv c^{M^\ell}_{B,f^M(s)} \equiv c^{M^\ell}_{B,\left(\alpha^B_f\right)^M(s)}.$$

  By the proofs of Lemmas 5.1.26 and 5.1.27, we also have

  $$\begin{aligned}\theta^*\left(\alpha^B_f(u)'\right) &\equiv \theta^*\left(\alpha^B_f\left(c^M_{B^i,s}\right)'\right)\\ &\equiv \theta^*\left(\left(\left(c^M_{B^i,s}\right)'\right)^f\right)\\ &\equiv \theta^*\left(\left(c^M_{B^i,s}\right)^f\right)\\ &\equiv \theta^*\left(c^M_{B^\ell,\left(\alpha^B_f\right)^M(s)}\right)\\ &\equiv c^{M^\ell}_{B,\left(\alpha^B_f\right)^M(s)}.\end{aligned}$$

  So we must show that

  $$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash c^{M^\ell}_{B,(\alpha^B_f)^M(s)} = c^{M^\ell}_{B,(\alpha^B_f)^M(s)},$$

  but this is true because $\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash c^{M^\ell}_{B,(\alpha^B_f)^M(s)} \downarrow$.

- Suppose $u \equiv \alpha^A_g(\mathsf{x}_{A^i}) : A^i$ for some $g : i \to i$ in $\mathcal{J}$ (because u is i-local). Then $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$, and we have

  $$\rho^A_{f^M}(\theta^*(u)) \equiv \rho^A_{f^M}(\theta^*(\alpha^A_g(\mathsf{x}_{A^i}))) \equiv \rho^A_{f^M}(\mathsf{x}_A) \equiv \mathsf{x}_A,$$

  as well as

  $$\begin{aligned}\theta^*\left(\alpha^A_f(u)'\right) &\equiv \theta^*\left(\alpha^A_f\left(\alpha^A_g(\mathsf{x}_{A^i})\right)'\right)\\ &\equiv \theta^*\left(\left(\alpha^A_g(\mathsf{x}_{A^i})'\right)^f\right)\\ &\equiv \theta^*\left(\alpha^A_g(\mathsf{x}_{A^i})^f\right)\\ &\equiv \theta^*\left(\alpha^A_{f\circ g}(\mathsf{x}_{A^i})\right)\\ &\equiv \mathsf{x}_A.\end{aligned}$$

  So we must prove $\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \mathsf{x}_A = \mathsf{x}_A$, which is true because $\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \mathsf{x}_A \downarrow$.

- Suppose $u \equiv g^i(u_1, \ldots, u_n) : B^i$ for some $g : B_1 \times \ldots \times B_n \to B$ in Σ and i-local terms $u_1, \ldots, u_n \in \mathsf{Term}^c(\Sigma^{\mathcal{J}}(M, \mathsf{x}_{A^i}))^*$ with $u_j : B^i_j$ for each $1 \leq j \leq n$, and suppose that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \downarrow$. Then by the rules of partial Horn logic, it follows that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u_j \downarrow$ for each $1 \leq j \leq n$. So by the induction hypothesis, we obtain for each $1 \leq j \leq n$ that

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \theta^* \left(\alpha_f^{B_j}(u_j)' \right) = \rho^A_{f^M}(\theta^*(u_j)).$$

Since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash u \equiv g^i(u_1, \ldots, u_n) \downarrow$, it follows by Axiom 5.1.5.4 that

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \alpha_f^B(u) \equiv \alpha_f^B(g^i(u_1, \ldots, u_n)) = g^\ell \left(\alpha_f^{B_1}(u_1), \ldots, \alpha_f^{B_n}(u_n) \right).$$

Then since

$$\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i}) \vdash \alpha_f^B(u) = \alpha_f^B(u)'$$

and $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{A^i})$ proves the following sequence of equations

$$\begin{aligned} g^\ell \left(\alpha_f^{B_1}(u_1), \ldots, \alpha_f^{B_n}(u_n) \right) &= g^\ell \left(\alpha_f^{B_1}(u_1), \ldots, \alpha_f^{B_n}(u_n) \right)' \\ &\equiv g^\ell \left(\alpha_f^{B_1}(u_1)', \ldots, \alpha_f^{B_n}(u_n)' \right) \end{aligned}$$

by Lemma 5.1.27, it follows by Lemma 5.1.35 that

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \theta^* \left(\alpha_f^B(u)' \right) = \theta^* \left(g^\ell \left(\alpha_f^{B_1}(u_1)', \ldots, \alpha_f^{B_n}(u_n)' \right) \right),$$

i.e.

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash \theta^* \left(\alpha_f^B(u)' \right) = g \left(\theta^* \left(\alpha_f^{B_1}(u_1)' \right), \ldots, \theta^* \left(\alpha_f^{B_n}(u_n)' \right) \right).$$

So to obtain the desired result, it suffices to show that

$$\mathbb{T}(M^\ell, \mathsf{x}_A) \vdash g \left(\theta^* \left(\alpha_f^{B_1}(u_1)' \right), \ldots, \theta^* \left(\alpha_f^{B_n}(u_n)' \right) \right) = \rho^A_{f^M} \left(\theta^*(g^i(u_1, \ldots, u_n)) \right).$$

But the following sequence of equations is provable in $\mathbb{T}(M^\ell, \mathsf{x}_A)$, as desired:

$$\begin{aligned} g \left(\theta^* \left(\alpha_f^{B_1}(u_1)' \right), \ldots, \theta^* \left(\alpha_f^{B_n}(u_n)' \right) \right) &= g \left(\rho^A_{f^M}(\theta^*(u_1)), \ldots, \rho^A_{f^M}(\theta^*(u_n)) \right) \\ &\equiv \rho^A_{f^M} \left(g \left(\theta^*(u_1), \ldots, \theta^*(u_n) \right) \right) \\ &\equiv \rho^A_{f^M} \left(\theta^*(g^i(u_1, \ldots, u_n)) \right) \end{aligned}$$

(where the first equality follows by the induction hypothesis).

∎

Definition (5.1.54)**.** We define a functor

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}$$

as follows:

- For any $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, we set

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)}(M) := \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M} \in \mathsf{Group}.$$

- For any $\Sigma^{\mathcal{J}}$-morphism $h : M \to N \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, we define a group homomorphism

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{h} : \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M} \to \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{N}$$

  by

$$(\psi_B)_{B\in\Sigma} \mapsto (\psi_B^h)_{B\in\Sigma},$$

  with

$$\psi_B^h(i) := \psi_B(i) : i \xrightarrow{\sim} i$$

  for each $i \in \mathcal{J}_B^N$.

Justification. We must show that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)}$ is a well-defined functor. Let $h : M \to N$ be a morphism in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, and let $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{M}$. Then $\psi = (\psi_B)_{B\in\Sigma}$, with $\psi_B : \mathsf{Id}_{\mathcal{J}_B^M} \to \mathsf{Id}_{\mathcal{J}_B^M}$ a natural automorphism of the identity functor on $\mathcal{J}_B^M$, which is the full subcategory of $\mathcal{J}$ consisting of all objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(M^i)$ is non-trivial for the sort B.

Now let $B \in \Sigma$ be an arbitrary sort; we show that the natural automorphism $\psi_B^h : \mathsf{Id}_{\mathcal{J}_B^N} \to \mathsf{Id}_{\mathcal{J}_B^N}$ of the identity functor on $\mathcal{J}_B^N$ is well-defined, where $\mathcal{J}_B^N$ is the full subcategory of $\mathcal{J}$ consisting of all objects $i \in \mathcal{J}_O$ for which $\mathbb{T}(N^i)$ is non-trivial for the sort B. So let $i \in \mathcal{J}_B^N$, so that $\mathbb{T}(N^i)$ is non-trivial for the sort B. Since $h^i : M^i \to N^i$ is a Σ-morphism, it follows that we have a theory morphism

$$\rho_{h^i} : \mathbb{T}(M^i) \to \mathbb{T}(N^i)$$

by Definition 2.2.17. If $\mathbb{T}(M^i)$ were trivial for the sort B, then we would have

$$\mathbb{T}(M^i) \vdash^{y,y'} y = y'$$

for distinct variables $y, y' : B$. Then the existence of this theory morphism would imply that

$$\mathbb{T}(N^i) \vdash^{y,y'} y = y',$$

so that $\mathbb{T}(N^i)$ would be trivial for the sort B, contrary to assumption. Thus, it must be that $\mathbb{T}(M^i)$ is non-trivial for the sort B, so that $i \in \mathcal{J}_B^M$, and hence $\psi_B(i) : i \xrightarrow{\sim} i$ is defined. So our definition

$$\psi_B^h(i) := \psi_B(i) : i \xrightarrow{\sim} i$$

makes sense.

We now show that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^h(\psi) := (\psi^h_B)_{B\in\Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^N.$$

If $B \in \Sigma_{\mathsf{Sort}}$, then since ψ_B is a natural automorphism of $\mathsf{Id}_{\mathcal{J}^M_B}$, it easily follows that ψ^h_B is a natural automorphism of $\mathsf{Id}_{\mathcal{J}^N_B}$. Now let $g : B_1 \times \ldots \times B_n \to B$ be any function symbol of Σ with $n \geq 1$, and let $i \in \mathcal{J}_O$ and $1 \leq m \leq n$ with g^{N^i} non-degenerate in position m. We must show that

$$\psi^h_{B_m}(i) = \psi^h_B(i) : i \to i.$$

As before, we can easily show that g^{N^i} being non-degenerate in position m implies that g^{M^i} is also non-degenerate in position m, and hence because $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, we obtain

$$\psi^h_{B_m}(i) = \psi_{B_m}(i) = \psi_B(i) = \psi^h_B(i) : i \to i,$$

as required. This completes the argument that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^h(\psi) := (\psi^h_B)_{B\in\Sigma} \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^N.$$

Also, it easily follows from the definitions that

$$\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^h : \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M \to \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^N$$

is a group homomorphism, and that $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^{(-)} : \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Group}$ is functorial. ∎

Lemma (5.2.4)**.** *Let* $M \in \mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$*, with component models* $M^i \in \mathsf{P}\mathbb{T}\mathsf{mod}$ *for all* $i \in \mathcal{J}_O$*, suppose that* $(\phi^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} \mathcal{Z}_{\mathbb{T}}(M^i)$ *is compatible, and suppose that* $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$*. Then there is a unique* $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ *determined by* ψ *and* $(\phi^i)_{i\in\mathcal{J}}$*.*

Proof: Assume the hypotheses. Then we have $\phi^i \in \mathcal{Z}_{\mathbb{T}}(M^i)$ for each $i \in \mathcal{J}_O$. Let $\gamma^i = ([s^i_C])_{C\in\Sigma}$ be the element of $G_{\mathbb{T}}(M^i)$ that corresponds to ϕ^i under the isomorphism $\mathcal{Z}_{\mathbb{T}}(M^i) \cong G_{\mathbb{T}}(M^i)$ (cf. Theorem 2.2.41). Then we have $(\gamma^i)_{i\in\mathcal{J}} \in \prod_{i\in\mathcal{J}} G_{\mathbb{T}}(M^i)$. Now we show that $(\gamma^i)_{i\in\mathcal{J}} \in \left(\prod_{i\in\mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$. So let $f : j \to k$ be any arrow in $\mathcal{J}$; we must show that

$$G_{\mathbb{T}}(f^M)(\gamma^j) = \gamma^k,$$

where $f^M = F^M(f) : M^j \to M^k$. Unravelling the definitions, we must show for any sort $B \in \Sigma$ that

$$\left[\rho^B_{f^M}(s^j_B)\right] = \left[s^k_B\right] \in M^k\langle \mathsf{x}_B\rangle_B,$$

where $\rho^B_{f^M} : \Sigma(M^j, \mathsf{x}_B) \to \Sigma(M^k, \mathsf{x}_B)$ is the signature morphism induced by the Σ-morphism f^M. By the explicit description of the isomorphisms $\mathscr{Z}_{\mathbb{T}}(M^j) \cong G_{\mathbb{T}}(M^j)$ and $\mathscr{Z}_{\mathbb{T}}(M^k) \cong G_{\mathbb{T}}(M^k)$ (cf. the proof of Theorem 2.2.41), we have

$$\left[s^j_B\right] := \left(\phi^j_{\eta^j_B}\right)_B ([\mathsf{x}_B]) \in M^j\langle \mathsf{x}_B\rangle_B$$

and

$$\left[s^k_B\right] := \left(\phi^k_{\eta^k_B}\right)_B ([\mathsf{x}_B]) \in M^k\langle \mathsf{x}_B\rangle_B,$$

where $\eta^j_B : M^j \to M^j\langle \mathsf{x}_B\rangle$ and $\eta^k_B : M^k \to M^k\langle \mathsf{x}_B\rangle$ are the canonical Σ-morphisms. Let

$$f^M\langle \mathsf{x}_B\rangle : M^j\langle \mathsf{x}_B\rangle \to M^k\langle \mathsf{x}_B\rangle$$

be the unique Σ-morphism with the properties

$$f^M\langle \mathsf{x}_B\rangle \circ \eta^j_B = \eta^k_B \circ f^M : M^j \to M^k\langle \mathsf{x}_B\rangle$$

and

$$f^M\langle \mathsf{x}_B\rangle_B([\mathsf{x}_B]) = [\mathsf{x}_B] \in M^k\langle \mathsf{x}_B\rangle_B$$

(cf. Proposition 2.2.10). Then for any $u \in \mathsf{Term}^c(\Sigma(M^j, \mathsf{x}_B))$ of sort B with $\mathbb{T}(M^j, \mathsf{x}_B) \vdash u\downarrow$, it is easy to see that

$$f^M\langle \mathsf{x}_B\rangle_B([u]) = \left[\rho^B_{f^M}(u)\right] \in M^k\langle \mathsf{x}_B\rangle_B.$$

Since our goal is to show

$$\left[\rho^B_{f^M}(s^j_B)\right] = \left[s^k_B\right] \in M^k\langle \mathsf{x}_B\rangle_B,$$

it is now equivalent to show

$$f^M\langle \mathsf{x}_B\rangle_B\left(\left(\phi^j_{\eta^j_B}\right)_B ([\mathsf{x}_B])\right) = \left(\phi^k_{\eta^k_B}\right)_B ([\mathsf{x}_B]) \in M^k\langle \mathsf{x}_B\rangle_B,$$

i.e.

$$\left(f^M\langle \mathsf{x}_B\rangle \circ \phi^j_{\eta^j_B}\right)_B ([\mathsf{x}_B]) = \left(\phi^k_{\eta^k_B}\right)_B ([\mathsf{x}_B]).$$

We have

$$\begin{aligned} f^M\langle \mathsf{x}_B\rangle \circ \phi^j_{\eta^j_B} &= \phi^j_{f^M\langle \mathsf{x}_B\rangle \circ \eta^j_B} \circ f^M\langle \mathsf{x}_B\rangle \\ &= \phi^j_{\eta^k_B \circ f^M} \circ f^M\langle \mathsf{x}_B\rangle \\ &= \phi^j_{\eta^k_B \circ F^M(f)} \circ f^M\langle \mathsf{x}_B\rangle \\ &= \phi^k_{\eta^k_B} \circ f^M\langle \mathsf{x}_B\rangle; \end{aligned}$$

the first equality holds because $\phi^j \in \mathcal{Z}_{\mathbb{T}}(M^j)$, the second equality holds by definition of $f^M\langle \mathsf{x}_B\rangle$, the third equality holds by definition of f^M, and the last equality holds because the family $(\phi^i)_{i\in\mathcal{J}}$ is compatible. Then we obtain

$$\begin{aligned}\left(f^M\langle \mathsf{x}_B\rangle \circ \phi^j_{\eta^j_B}\right)_B([\mathsf{x}_B]) &= \left(\phi^k_{\eta^k_B}\circ f^M\langle\mathsf{x}_B\rangle\right)_B([\mathsf{x}_B]) \\ &= \left(\phi^k_{\eta^k_B}\right)_B\left(f^M\langle\mathsf{x}_B\rangle_B\left([\mathsf{x}_B]\right)\right)\\ &= \left(\phi^k_{\eta^k_B}\right)_B([\mathsf{x}_B]),\end{aligned}$$

as desired. This completes the argument that $(\gamma^i)_{i\in\mathcal{J}} \in \left(\prod_{i\in\mathcal{J}} G_{\mathbb{T}}(M^i)\right)^{\mathcal{J}}$.

Consequently, since $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})^M$, we obtain

$$\beta_M\left((\gamma^i)_{i\in\mathcal{J}},\psi\right)\in G_{\mathbb{T}^{\mathcal{J}}}(M).$$

Now, let $\pi \in \mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ correspond to $\beta_M\left((\gamma^i)_{i\in\mathcal{J}},\psi\right)$ under the isomorphism $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)\cong G_{\mathbb{T}^{\mathcal{J}}}(M)$. We will show that π is determined by ψ and $(\phi^i)_{i\in\mathcal{J}}$. So let $f: M\to N$ be an arbitrary morphism in $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ with domain M, and let $B^k\in\Sigma^{\mathcal{J}}$ be any sort. If $k\notin \mathcal{J}^M_B$, then the desired result is trivial to verify. So assume that $k\in\mathcal{J}^M_B$. Then we must show

$$\pi^{B^k}_f = \left(\phi^k_{f^k\circ F^M(\psi_B(k))}\right)_B \circ F^N(\psi_B(k))_B : N_{B^k}\to N_{B^k}.$$

Since $(\phi^i)_{i\in\mathcal{J}}$ is compatible, it is equivalent to show

$$\pi^{B^k}_f = \left(\phi^k_{f^k}\right)_B\circ F^N(\psi_B(k))_B : N_{B^k}\to N_{B^k}.$$

By the proof of Theorem 2.2.41 and the definition of β_M, we have

$$\pi^{B^k}_f = \rho^{B^k}_f\left(\rho^B_{M^k}(s^k_B)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]\right)^* = \rho^{B^k}_f\left(\rho^B_{M^k}(s^k_B)\right)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]^*$$

$$: N_{B^k}\to N_{B^k}$$

(the second equality holds by Lemma 2.2.28 and since $\rho^{B^k}_f$ fixes $\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})$). Since $\mathbb{T}^{\mathcal{J}}(N,\mathsf{x}_{B^k})\vdash \alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})\downarrow$, it follows that

$$\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})^* = \left(\alpha^B_{\psi_B(k)}\right)^N : N_{B^k}\to N_{B^k}$$

is a well-defined total function. Then by Lemma 2.2.29, we obtain

$$\begin{aligned}\pi^{B^k}_f &= \rho^{B^k}_f\left(\rho^B_{M^k}(s^k_B)\right)\left[\alpha^B_{\psi_B(k)}(\mathsf{x}_{B^k})/\mathsf{x}_{B^k}\right]^* \\ &= \rho^{B^k}_f\left(\rho^B_{M^k}(s^k_B)\right)^*\circ\left(\alpha^B_{\psi_B(k)}\right)^N \\ &= \rho^B_{N^k}\left(\rho^B_{f^k}(s^k_B)\right)^*\circ\left(\alpha^B_{\psi_B(k)}\right)^N,\end{aligned}$$

with the last equality justified by Lemma 5.1.15. So now, we are reduced to showing that

$$\rho^B_{N^k}\left(\rho^B_{f^k}(s^k_B)\right)^* \circ \left(\alpha^B_{\psi_B(k)}\right)^N = \left(\phi^k_{f^k}\right)_B \circ F^N(\psi_B(k))_B.$$

However, since

$$F^N(\psi_B(k))_B := \left(\alpha^B_{\psi_B(k)}\right)^N$$

is a bijection (since $\psi_B(k) : k \xrightarrow{\sim} k$ is an isomorphism in $\mathcal{J}$), it is equivalent to prove that

$$\rho^B_{N^k}\left(\rho^B_{f^k}(s^k_B)\right)^* = \left(\phi^k_{f^k}\right)_B : N_{B^k} \to N_{B^k}.$$

Then by Lemma 5.2.3, it is equivalent to prove

$$\rho^B_{f^k}(s^k_B)^* = \left(\phi^k_{f^k}\right)_B : N_{B^k} \to N_{B^k}.$$

However, if $\delta : G_{\mathbb{T}}(M^k) \xrightarrow{\sim} \mathcal{Z}_{\mathbb{T}}(M^k)$ is the isomorphism from (the proof of) Theorem 2.2.41, then (given that $\gamma^k = \left(\left[s^k_C\right]\right)_{C \in \Sigma}$) we have

$$\rho^B_{f^k}(s^k_B)^* = \left(\delta\left(\gamma^k\right)_{f^k}\right)_B = \left(\delta\left(\delta^{-1}\left(\phi^k\right)\right)_{f^k}\right)_B = \left(\phi^k_{f^k}\right)_B,$$

as desired. This completes the proof that π is determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$. Finally, it is trivial to observe (cf. Definition 5.2.1) that π is the *only* element of $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}}(M)$ determined by ψ and $(\phi^i)_{i \in \mathcal{J}}$, which completes the proof. ∎

Lemma (5.2.5)**.** *Let* $f : M \to N$ *be a* $\Sigma^{\mathcal{J}}$*-morphism in* $\mathsf{P}\mathbb{T}^{\mathcal{J}}\mathsf{mod}$, *and let* $F^M : \mathcal{J} \to \mathsf{P}\mathbb{T}\mathsf{mod}$ *be the functor corresponding to* M. *Also let* $k \in \mathcal{J}_O$ *and let* $g : k \to k$ *be an arbitrary arrow in* $\mathcal{J}$, *and let* $B \in \Sigma_{\mathsf{Sort}}$. *Finally, let* $u \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_B))$ *with* $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$ *and* $u : C$. *Then*

$$\alpha^C_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u)\right)\right)^* = \rho^B_{f^k \circ F^M(g)}(u)^* \circ \left(\alpha^B_g\right)^N$$
$$: N_{B^k} \to N_{C^k},$$

where $\rho^B_{f^k \circ F^M(g)} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma(N^k, \mathsf{x}_B)$ *is the signature morphism induced by the* Σ*-morphism* $f^k \circ F^M(g) : M^k \to N^k$, *and* $\rho^B_{M^k} : \Sigma(M^k, \mathsf{x}_B) \to \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k})$ *is the signature morphism of Definition 5.1.12, and* $\rho^{B^k}_f : \Sigma^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ *is the signature morphism induced by* $f : M \to N$.

Proof: First, we verify that the above equation makes sense under the given hypotheses. Since $\alpha^B_g : B^k \to B^k$, we have $(\alpha^B_g)^N : N_{B^k} \to N_{B^k}$ (which is a total function because $N \models \mathbb{T}^{\mathcal{J}}$). Since $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$ and $\rho^B_{f^k \circ F^M(g)} : \mathbb{T}(M^k, \mathsf{x}_B) \to \mathbb{T}(N^k, \mathsf{x}_B)$ is a theory morphism by Lemma 2.2.18, it follows that $\mathbb{T}(N^k, \mathsf{x}_B) \vdash \rho^B_{f^k \circ F^M(g)}(u)\downarrow$, so that $\rho^{B}_{f^k\circ F^M(g)}(u)^* : N_{B^k} = N^k_B \to N^k_C = N_{C^k}$ is a well-defined total function. Similar remarks show that the left side of the above equation is also a well-defined total function.

Now, we prove the above equation by induction on terms $u \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_B))$ with $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$:

- Suppose $u \equiv \mathsf{x}_B : B$. Then $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$. We have

  $$\alpha^B_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u)\right)\right) \equiv \alpha^B_g\left(\rho^{B^k}_f(\mathsf{x}_{B^k})\right) \equiv \alpha^B_g(\mathsf{x}_{B^k}) : B^k,$$

  along with

  $$\rho^B_{f^k\circ F^M(g)}(u) \equiv \mathsf{x}_B : B.$$

  Since

  $$\mathsf{x}^*_B = \mathsf{id} : N^k_B \to N^k_B,$$

  we must show

  $$\alpha^B_g(\mathsf{x}_{B^k})^* = \left(\alpha^B_g\right)^N : N_{B^k} \to N_{B^k}.$$

  But for any $d \in N_{B^k}$, we have

  $$\alpha^B_g(\mathsf{x}_{B^k})^*(d) := \alpha^B_g(\mathsf{x}_{B^k})^{(\widehat{N},d)} = (\alpha^B_g)^{(\widehat{N},d)}\left(\mathsf{x}^{(\widehat{N},d)}_{B^k}\right) = \left(\alpha^B_g\right)^N(d),$$

  as desired (here $\widehat{N}$ is the canonical expansion of N to a $\Sigma^{\mathcal{J}}(N)$-structure, and $(\widehat{N}, d)$ is a $\Sigma^{\mathcal{J}}(N, \mathsf{x}_{B^k})$-structure).

- Suppose $u \equiv c^{M^k}_{D,s} : D$ for some sort $D \in \Sigma$ and $s \in M^k_D = M_{D^k}$. Then $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u\downarrow$, and we have

  $$\alpha^D_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u)\right)\right) \equiv \alpha^D_g\left(\rho^{B^k}_f\left(c^M_{D^k,s}\right)\right) \equiv \alpha^D_g\left(c^N_{D^k,f(s)}\right) : D^k,$$

  while

  $$\rho^B_{f^k\circ F^M(g)}(u) \equiv c^{N^k}_{D,f^k(F^M(g)(s))} : D.$$

  So we must show

  $$\alpha^D_g\left(c^N_{D^k,f(s)}\right)^* = \left(c^{N^k}_{D,f^k(F^M(g)(s))}\right)^* \circ \left(\alpha^B_g\right)^N : N_{B^k} \to N_{D^k},$$

  i.e.

  $$\alpha^D_g\left(c^N_{D^k,f(s)}\right)^* = \left(c^{N^k}_{D,f^k(F^M(g)(s))}\right)^* : N_{B^k} \to N_{D^k}$$

  (since $\left(c^{N^k}_{D,f^k(F^M(g)(s))}\right)^* : N^k_B \to N^k_D$ is a constant function). To show this, it is equivalent to prove that

  $$\left(\alpha^D_g\right)^N\left(f_{D^k}(s)\right) = f^k_D\left(F^M(g)_D(s)\right) \in N_{D^k}.$$

But

$$F^M(g)_D := \left(\alpha_g^D\right)^M : M_D^k = M_{D^k} \to M_{D^k} = M_D^k,$$

so the result follows because $f_D^k = f_{D^k}$ and $f : M \to N$ is a $\Sigma^{\mathcal{J}}$-morphism.

- Suppose that $h : C_1 \times \ldots \times C_n \to C$ is a function symbol of Σ, suppose that $u_i \in \mathsf{Term}^c(\Sigma(M^k, \mathsf{x}_B))$ is a term of sort C_i for each $1 \leq i \leq n$, and suppose that $u \equiv h(u_1, \ldots, u_n) : C$. Finally, assume that $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u \downarrow$, which implies (by the rules of partial Horn logic) that $\mathbb{T}(M^k, \mathsf{x}_B) \vdash u_i \downarrow$ for all $1 \leq i \leq n$. So by the induction hypothesis, for each $1 \leq i \leq n$ we have

$$\alpha_g^{C_i}\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_i)\right)\right)^* = \rho^B_{f^k \circ F^M(g)}(u_i)^* \circ \left(\alpha_g^B\right)^N$$
$$: N_{B^k} \to N_{C_i^k}.$$

Now, we have

$$\begin{aligned}\alpha_g^C\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u)\right)\right) &\equiv \alpha_g^C\left(\rho_f^{B^k}\left(\rho_{M^k}^B\left(h\left(u_1, \ldots, u_n\right)\right)\right)\right)\\ &\equiv \alpha_g^C\left(\rho_f^{B^k}\left(h^k\left(\rho_{M^k}^B(u_1), \ldots, \rho_{M^k}^B(u_n)\right)\right)\right)\\ &\equiv \alpha_g^C\left(h^k\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right), \ldots, \rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right)\right).\end{aligned}$$

Since $\mathbb{T}(M^k, \mathsf{x}_B) \vdash h(u_1, \ldots, u_n) \downarrow$, the existence of the theory morphisms $\rho_{M^k}^B : \mathbb{T}(M^k, \mathsf{x}_B) \to \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k})$ and $\rho_f^{B^k} : \mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_{B^k}) \to \mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k})$ implies that

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash h^k\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right), \ldots, \rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right) \downarrow.$$

Then by Axiom 5.1.5.4, we obtain

$$\mathbb{T}^{\mathcal{J}}(N, \mathsf{x}_{B^k}) \vdash \alpha_g^C\left(h^k\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right), \ldots, \rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right)\right)$$
$$= h^k\left(\alpha_g^{C_1}\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right)\right), \ldots, \alpha_g^{C_n}\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right)\right).$$

So by Lemma 2.2.13, we obtain

$$\begin{aligned}&\alpha_g^C\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u)\right)\right)^*\\ &= \alpha_g^C\left(h^k\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right), \ldots, \rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right)\right)^*\\ &= h^k\left(\alpha_g^{C_1}\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_1)\right)\right), \ldots, \alpha_g^{C_n}\left(\rho_f^{B^k}\left(\rho_{M^k}^B(u_n)\right)\right)\right)^*\\ &: N_{B^k} \to N_{C^k}.\end{aligned}$$

Also, we have

$$\rho^B_{f^k \circ F^M(g)}(u) \equiv \rho^B_{f^k \circ F^M(g)}\left(h(u_1, \ldots, u_n)\right) \equiv h\left(\rho^B_{f^k \circ F^M(g)}(u_1), \ldots, \rho^B_{f^k \circ F^M(g)}(u_n)\right),$$

and hence

$$\rho^B_{f^k \circ F^M(g)}(u)^* = h\left(\rho^B_{f^k \circ F^M(g)}(u_1), \ldots, \rho^B_{f^k \circ F^M(g)}(u_n)\right)^* : N^k_B \to N^k_C.$$

Altogether, we must show

$$h^k\left(\alpha^{C_1}_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u_1)\right)\right), \ldots, \alpha^{C_n}_g\left(\rho^{B^k}_f\left(\rho^B_{M^k}(u_n)\right)\right)\right)^*$$

$$= h\left(\rho^B_{f^k \circ F^M(g)}(u_1), \ldots, \rho^B_{f^k \circ F^M(g)}(u_n)\right)^* \circ \left(\alpha^B_g\right)^N : N_{B^k} \to N_{C^k},$$

which now easily follows by the induction hypothesis (and the fact that $h^{N^k} = (h^k)^N$).

∎

Lemma (5.3.2)**.** *Let Σ be an arbitrary relation-free signature, and let $\mathbb{T}$ be an empty theory over Σ. If $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, then for any $B \in \Sigma_{\mathsf{Sort}}$, $\mathbb{T}(M)$ is non-trivial for the sort B. If $g : B_1 \times \ldots \times B_n \to B$ is any function symbol of Σ, then for any $1 \leq m \leq n$, g^M is non-degenerate in position m.*

Proof: Let $\mathbb{T}$ be an empty theory over Σ, let $M \in \mathsf{P}\mathbb{T}\mathsf{mod}$, let $B \in \Sigma$ be any sort, let $g : A_1 \times \ldots \times A_n \to A$ be any function symbol of Σ, and let $1 \leq m \leq n$. We show that $\mathbb{T}(M)$ is non-trivial for the sort B and that g^M is non-degenerate in position m, i.e. we show that

$$\mathbb{T}(M) \not\vdash^{y, y' : B} y = y'$$

for distinct variables $y, y' : B$ and that

$$\mathbb{T}(M) \not\vdash^{y_1, \ldots, y_n, z_m} g(y_1, \ldots, y_n) = g(y_1, \ldots, y_n)[z_m / y_m]$$

for pairwise distinct variables $y_1, \ldots, y_n, z_m$ of the correct sorts. We accomplish both goals simultaneously by appealing to Lemma 3.1.2 and showing that there is some $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$ such that N_B contains at least two elements, and there are $a_1 \in N_{A_1}, \ldots, a_m \neq b_m \in N_{A_m}, \ldots, a_n \in N_{A_n}$ such that

$$(a_1, \ldots, a_m, \ldots, a_n), (a_1, \ldots, b_m, \ldots, a_n) \in \mathsf{dom}(g^N)$$

and

$$g^N(a_1, \ldots, a_m, \ldots, a_n) \neq g^N(a_1, \ldots, b_m, \ldots, a_n) \in N_A,$$

and there is a Σ-morphism $h : M \to N$.

We define a partial Σ-structure N as follows. For any sort $A \in \Sigma$, let $N_A := M_A \cup \{0, 1\}$, where $0, 1 \notin \bigcup_C M_C$ are distinct. For any function symbol $f : C_1 \times \ldots \times C_r \to C$ of Σ other than g, we set f^N to be the total function defined by

$$f^N \restriction \mathsf{dom}(f^M) := f^M$$

and

$$f^N(\mathsf{x}_1, \ldots, \mathsf{x}_r) := 0 \in N_C$$

for any $(\mathsf{x}_1, \ldots, \mathsf{x}_r) \in N_{C_1} \times \ldots \times N_{C_r} \setminus \mathsf{dom}(f^M)$. We then set g^N to be the total function defined by

$$g^N \restriction \mathsf{dom}(g^M) := g^M$$

and

$$g^N(0, \ldots, 0) := 0 \in N_A$$

and

$$g^N(\mathsf{x}_1, \ldots, \mathsf{x}_n) := 1 \in N_A$$

otherwise. It is clear that $N \in \mathsf{P}\mathbb{T}\mathsf{mod}$, that N_B contains at least two elements, and that the desired condition on g^N is satisfied. If we then set $h : M \to N$ to the inclusion morphism, then h is clearly a Σ-morphism and the proof is complete. ∎

Bibliography

[1] J. Adamek, H-E Porst. Algebraic theories of quasivarieties. Journal of Algebra 208, 379-398, 1998.

[2] F. Baader, T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1999.

[3] G. Bergman. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. Publicacions Matematiques 56, 91-126, 2012.

[4] F. Borceux. *Handbook of Categorical Algebra 3: Categories of Sheaves.* Cambridge University Press, 1994.

[5] S. Breiner. Scheme representation for first-order logic. PhD thesis, Carnegie Mellon University, 2016.

[6] O. Caramello. *Theories, Sites, Toposes.* Oxford University Press, 2017.

[7] P. Dehornoy. Some aspects of the SD-world. Preprint, 2017. Available at https://arxiv.org/abs/1711.09792.

[8] R. Diaconescu. Change of base for toposes with generators. Journal of Pure and Applied Algebra 6(3), 191-218, 1975.

[9] P. J. Freyd. Core Algebra Revisited. Theoretical Computer Science 374(1-3), 193-200, 2007.

[10] J. Funk, P. Hofstra, B. Steinberg. Isotropy and crossed toposes. Theory and Applications of Categories 26, 660-709, 2012.

[11] P. Gabriel, F. Ulmer. *Lokal Praesentierbare Kategorien.* Springer Lecture Notes in Mathematics 221, 1971.

[12] R. Garner. Inner automorphisms of groupoids. Preprint, 2019. Available at https://arxiv.org/abs/1907.10378.

[13] P. Hofstra, J. Parker, P. Scott. Isotropy of Algebraic Theories. Electronic Notes in Theoretical Computer Science 341, 201-217, 2018.

[14] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Clarendon Press, 2002.

[15] S. Mac Lane, I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, 1992.

[16] W. McCune. Prover9 and Mace4. http://www.cs.unm.edu/ mccune/Prover9. 2005-2010.

[17] M.H.A. Newman. On theories with a combinatorial definition of "equivalence". Annals of Mathematics, 43(2), 223-243, 1942.

[18] T. Nipkow. Combining Matching Algorithms: The Regular Case. Journal of Symbolic Computation 12, 633-653, 1991.

[19] E. Palmgren, S.J. Vickers. Partial Horn logic and cartesian categories. Annals of Pure and Applied Logic 145, 314-353, 2007.

[20] P. M. Whitman. Free Lattices. Annals of Mathematics 42, 325-329, 1941.

[21] P. M. Whitman. Free Lattices II. Annals of Mathematics 43, 104-115, 1941.