A STUDY IN THE FOURIER-HOLMGREN NORM FOR PSEUDO-DIFFERENTIAL OPERATORS

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CONTENTS

ABSTRACT

The purpose of this thesis is to study the Fourier-Holmgren norm for pseudo-differential operators. We shall first study the algebraic properties of the Fourier-Maximum class of pseudo-differential operators as developed by Friedrichs [1].

We shall then develop the Fourier-Holmgren class of pseudo-differential operators, show that this class satisfies certain algebraic inequalities that are analogous to those of the Fourier-Maximum class and also prove that this class forms an algebra.

Next we shall prove that the sharp form of Garding's inequality [4] still holds for a slightly restricted class of Fourier-Holmgren pseudo-differential operators.

Finally, we shall examine the class of co-kernels that obey the Γ_{FH} condition.

Note

Throughout this thesis if we refer to an equation in the same section we give only the number of the equation, whereas, if we refer to an equation in another section we give the section number and then the equation number.

{e.g. 1.2(24) refers to equation (24) in section 1.2}

INTRODUCTION

Pseudo-differential operators are a generalization of differential operators. The study of pseudo-differential operators was developed by Calderón, Zygmund, Friedrichs, Lax, Nirenberg, Kohn, Kumano-go, Hörmander and many others. Friedrichs' lecture notes [1] on the Fourier-Maximum class of pseudo-differential operators were used extensively in the development of this thesis and will be referred to often. These authors developed an algebra for a variety of singular integral operators or pseudo-differential operators.

Hörmander [12] proved a sharp form of Gårding's inequality for pseudo-differential operators acting on scalar-valued functions. Lax and Nirenberg [4], by a somewhat different technique, extended this inequality to Kohn and Nirenberg's [3] operators acting on vector-valued functions and Friedrichs [1] used a 'mollifier' in the proof he gave for pseudo-differential operators of the Fourier-Maximum type. Also Vaillancourt [10] gave a "simple" proof for the sharp form of Gårding's inequality for the Fourier-Maximum class of pseudo-differential operators. We shall summarize the notations adopted in sections 0.1 and 0.2.

§0.1 We shall write $x=(x_1,\ldots,x_n)$ for a point in R^n and $\xi=(\xi_1,\ldots,\xi_n)$ for its dual. If $\rho=(\rho_1,\ldots,\rho_n)$ is an n-tuple of non-negative integers, then we shall write

$|\rho| = \rho_1+\rho_2+\ldots+\rho_n$,

$x^\rho = x_1^{\rho_1} \ldots x_n^{\rho_n}$, $\xi^\rho = \xi_1^{\rho_1} \ldots \xi_n^{\rho_n}$,

$\partial_x^\rho = (\frac{\partial}{\partial x_1})^{\rho_1} \ldots (\frac{\partial}{\partial x_n})^{\rho_n}$, $\partial_\xi^\rho = (\frac{\partial}{\partial \xi_1})^{\rho_1} \ldots (\frac{\partial}{\partial \xi_n})^{\rho_n}$,

$D_x = (D_{x_1}, \ldots, D_{x_n})$, $D_x^\rho = (-i)^{|\rho|}(\frac{\partial}{\partial x})^\rho$.

We shall denote the real scalar product $x\xi = x\cdot\xi = \sum_j x_j \xi_j$ and for brevity we shall write

$g_{ix} = -i\partial_x g$, $g_\xi = \partial_\xi g$, $g_{ix\cdot\xi} = \sum_j -i\partial_{x_j}\partial_{\xi_j} g$,

$g_{ix}\cdot y = \sum_j -i\partial_{x_i} g y_j$, $g_\xi\cdot y = \sum_j g_{\xi_j} y_j$, etc.

The operators used in this thesis will be defined with the aid of Fourier transformation, namely u(x) is the Fourier transform of $\omega(\xi)$,

$$u(x) = \hat{\omega}(x) = (2\pi)^{-n}\int e^{ix\xi}\omega(\xi)\,d\xi, \qquad (1)$$

and $\omega(\xi)$ is the inverse Fourier transform of u(x),

$$\omega(\xi) = \int e^{-ix\xi}u(x)\,dx. \qquad (2)$$

Using the operator D_x^ρ on formula (1), we get

$$D_x^\rho u(x) = (2\pi)^{-n}\int \xi^\rho e^{ix\xi}\omega(\xi)\,d\xi. \qquad (3)$$

Hence the differential operator

$$G = \sum_{|\rho|\le r} g_\rho(X) D_x^\rho, \qquad (4)$$

where X is the operator of multiplication by x, may be represented by

$$Gu(x) = (2\pi)^{-n}\int e^{ix\xi}g(x;\xi)\omega(\xi)\,d\xi. \qquad (5)$$

In equation (5) the matrix function,

$$g(x;\xi) = \sum_{|\rho|\le r} g_\rho(x)\xi^\rho$$

is called the symbol of G.

The operator G acts on vector functions $u(x)$, where $x \epsilon R^n$. These functions form a Hilbert space under the L^2-norm

$$||u||^2 = \int |u(x)|^2 dx \, . \qquad (6)$$

For any real number s, the norm $||u||_s$ is defined by means of the inverse Fourier transform ω of u:

$$||u||_s^2 = ||\omega||_s^2 = (2\pi)^{-n} \int \langle\xi\rangle^{2s} |\omega(\xi)|^2 d\xi \qquad (7)$$

where

$$\langle\xi\rangle^2 = 1 + |\xi|^2 .$$

By Parseval's relation, $||u|| = ||u||_0$. The class of functions with finite s-norm forms a Hibert space, denoted by H^s and $H^0 = L^2$.

§0.2 A basic starting point in the development of Friedrich's class of pseudo-differential operators is the definition of a space of 'co-kernels' denoted by G_0. Any measurable function $\gamma(\chi;\xi)$ is said to be a co-kernel in G_0 if

(i) $\sup_\xi |\gamma(\chi;\xi)| < \infty$, for all χ, (8)

(ii) $(2\pi)^{-n} \int \sup_\xi |\gamma(\chi;\xi)| d\chi < \infty$. (9)

In view of (9) above, the r-norm of γ, $||\gamma||_r$, is defined by

$$||\gamma||_r = (2\pi)^{-n} \int \sup_\xi |\gamma(\chi;\xi)| \langle\xi\rangle^r d\chi \, . \qquad (10)$$

This norm is referred to as the "Fourier-Maximum" r-norm often abbreviated as "FM" r-norm.

G_0 is then used to generate a space of symbols denoted by $\hat{G}_0$, where each symbol $g(x;\xi)$ in $\hat{G}_0$ is the Fourier transform of some $\gamma(\chi;\xi)$ in G_0; i.e.

$$g(x;\xi) = (2\pi)^{-n}\int e^{ix\chi}\gamma(\chi;\xi)\,d\chi . \tag{11}$$

The spaces G_0 and $\hat{G}_0$ actually arise in the definition of the pseudo-differential operator $G=g(X;D_x)$ associated with the symbol $g(x;\xi)$. In defining the action of the operator on a function $u(x)$, we set

$$Gu(x) = (2\pi)^{-n}\int e^{ix\xi}[\Gamma\omega(\xi)]\,d\xi \tag{12}$$

where

$$\Gamma\omega(\xi) = (2\pi)^{-n}\int\gamma(\xi-\xi';\xi')\omega(\xi')\,d\xi' .$$

Throughout this thesis we will be dealing primarily with pseudo-differential operators of the Fourier-Holmgren (we abbreviate this by FH) type, where the FH r-norm is defined as follows:

$$||\gamma||_{r,FH}=\sup_{\substack{\xi' \\ \xi_1}}(2\pi)^{-n}\int|\gamma(\xi-\xi';\xi')|\langle\xi'\rangle^{r}{}^{d\xi}_{d\xi'} \tag{13}$$

We shall write $||\gamma||_{FH}$ for $||\gamma||_{0,FH}$.

Friedrichs [1] assigns to the pseudo-differential operator the Fourier-Maximum norms

$$||G||_r = ||\Gamma||_r = ||g||_r = ||\gamma||_r . \tag{14}$$

Similarly, we assign to the FH pseudo-differential the FH norms

$$||G||_{r,FH} = ||\Gamma||_{r,FH} = ||g||_{r,FH} = ||\gamma||_{r,FH} . \tag{15}$$

The co-kernel $\gamma(\chi;\xi)$ is of FM or FH order r, $\gamma \epsilon O(r)$ if

$$\gamma(\chi;\xi)<\xi>^{r}$$

is bounded in the FM or FH norm; i.e. $\gamma(\chi;\xi)$ is bounded in the FM or FH r-norm. Since $||G||_{FH} \leq ||G||_{FM}$ we see that FM operators are FH operators.

In Chapter 1, we shall define pseudo-differential operators of the Fourier-Maximum class, summarize the notations and outline the algebraic properties of pseudo-differential operators that have been used by Friedrichs [1]. We shall further show that the class of scalar FM symbols form a commutative Banach algebra under the product of symbols, i.e. the convolution product of co-kernels. The FM operators form a non-commutative Banach algebra under the operator product.

Chapter 2 will be devoted to pseudo-differential operators of the FH class and we shall develop an operator algebra for this class. Specifically we shall show that this class forms a non-commutative Banach algebra under the operator product. Note that in extending the FM class to the FH class, we loose the convolution product.

Chapter 3 will be concerned with showing that the ordinary form of Gårding's inequality for pseudo-differential operators proved by Lax and Nirenberg [4] for

Kohn and Nirenberg's [3] class of pseudo-differential operators and by Friedrichs [1] for pseudo-differential operators of FM type also holds for pseudo-differential operators of the FH type which are also bounded in a slightly stronger norm.

In Chapter 4, we shall examine the significance of the extra condition,

$$(\Gamma_{FH}) \quad \lim_{\chi' \to 0} ||\gamma^{\chi'} - \gamma||_{FH} = 0 \tag{16}$$

where

$$\gamma^{\chi'}(\xi-\xi';\xi') = \gamma(\xi-\xi'+\chi';\xi'). \tag{17}$$

Friedrichs [1] used the extra condition

$$(\Gamma) \quad \lim_{\chi' \to 0} ||\gamma^{\chi'} - \gamma||_{FM} = 0 \tag{18}$$

to prove an inversion theorem for symbols in $\hat{G}_0$. Sormani [9] examined the significance of the extra condition (18) in regard to how restrictive it was on elements in $\hat{G}_0$ and how necessary it was as a condition in an inversion theorem. We shall then investigate the properties of a subset, E_0^{FH}, of G_0^{FH} that obey the extra condition (16). The set of co-kernels that obey (16) will be seen to form a non-commutative Banach sub-algebra.

The following remark on the non-commutativity of differential and pseudo-differential operators will be used in subsequent chapters.

Remark 0.1 Differential operators with variable coefficients or pseudo-differential operators in general do not commute.

CHAPTER I

Fourier-Maximum Pseudo-Differential Operators

In this chapter we shall define L^2-continuous pseudo-differential operators of the Fourier-Maximum class, summarize the notations and outline the algebraic properties of pseudo-differential operators that have been used by Friedrichs [1]. In the next chapter we shall define L^2-continuous pseudo-differential operators of a larger class, the Fourier-Holmgren class, and develop a more restrictive algebra for this class.

§1.1 Pseudo-differential operators are a generalization of differential operators. This will be seen by using Fourier transformation to define a differential operator. Let $\hat{\omega}$ be the Fourier transform (FT) of ω in $C_0^\infty(R^n)$,

$$u(x) = \hat{\omega}(x) = \int e^{ix\xi}\omega(\xi)\,\bar{d}\xi \qquad (1)$$

where the constant $(2\pi)^{-n}$ has been absorbed in $\bar{d}\xi$, namely

$$\bar{d}\xi = (2\pi)^{-n}d\xi = (2\pi)^{-n}d\xi_1\ldots d\xi_n,$$

and $\omega(\xi)$ be the inverse Fourier transform (IFT) of $\hat{\omega}(x)$, i.e.

$$\omega(\xi) = \check{u}(\xi) = \int e^{-i\xi x}u(x)\,dx, \qquad (2)$$

both integrations being over R^n.

Consider the differential operator

$$G = \sum_{|\rho|\le r} g_\rho(X)D_x^\rho .$$

Since $D_x^\rho u(x) = \int e^{ix\xi}\xi^\rho \omega(\xi)\, d\xi$, then the differential operator G can be represented in the following way,

$$Gu(x) = \int e^{ix\xi} \sum_{|\rho|\le r} g_\rho(x)\xi^\rho\, \omega(\xi)\, d\xi$$

$$\doteq \int e^{ix\xi} g(x;\xi)\omega(\xi)\, d\xi, \qquad (3)$$

where the function $g(x;\xi) = \sum_{|\rho|\le r} g_\rho(x)\xi^\rho$ is called the symbol of the differential operator. Pseudo-differential operators may be defined by (3) with symbols $g(x;\xi)$ which are not necessarily polynomials in ξ.

Any measurable function $\gamma(\chi;\xi')$ is said to be the co-kernel of the pseudo-differential operator G if

(i) $\sup_{\xi'}|\gamma(\chi;\xi')| < \infty$,

(ii) $\int \sup_{\xi'}|\gamma(\chi;\xi')|\, d\chi < \infty$.

The latter integral will then be taken as the norm of γ, i.e.

$$||\gamma|| = \int \sup_{\xi'}|\gamma(\chi;\xi')|\, d\chi. \qquad (4)$$

This norm is referred to as the "Fourier-Maximum" norm or "FM" norm.

The space of all functions $\gamma(\chi;\xi)$ for which this norm is finite will be denoted by G_0. This space is complete with respect to the FM norm [9]. G_0 is then used to generate a space of symbols, denoted by $\hat{G}_0$, where each symbol is the Fourier transform of $\gamma(\chi;\xi)\in G_0$; thus

$$g(x;\xi) = \int e^{ix\xi}\gamma(\chi;\xi)\, d\chi. \qquad (5)$$

It should be mentioned that convolutions of co-kernels with respect to the χ-variable in G_0 go over to multiplication of symbols in the Fourier transform space $\hat{G}_0$. The spaces G_0 and $\hat{G}_0$ arise in the definition of the pseudo-differential operator $G = g(X;D_x)$ associated with the symbol $g(x;\xi)$. In defining the action of G on a function $u(x)$, we set

$$G\,u(x) = \int e^{ix\xi}[\Gamma\omega(\xi)]\bar{d}\xi \,, \qquad (6)$$

where

$$\Gamma\omega(\xi) = \int\gamma(\xi-\xi';\xi')\omega(\xi')\bar{d}\xi'$$

and

$$u(x) = \int e^{ix\xi}\omega(\xi)\bar{d}\xi.$$

Friedrichs [1] extended the class $\hat{G}_0$ to the class of all functions of the form

$$g(x;\xi) = g_0(x;\xi)+\gamma_\infty(\xi), \qquad (7)$$

where $g_0 \epsilon \hat{G}_0$ while $\gamma_\infty(\xi)$ is any bounded and measurable function. This results from the fact that the symbol $g_0(\chi;\xi)\epsilon\hat{G}_0$ belongs to a restricted class of functions which tend to zero as $|x|\to\infty$. The class of functions γ_∞ will be denoted by $\hat{G}_\infty$ and that of the functions $g_0+\gamma_\infty$ by $\hat{G}$.

At the same time, Friedrichs [1] introduced another space of co-kernels, G_∞, whose elements are of the form $\not{\delta}(\chi)\gamma_\infty(\xi)$, where $\not{\delta}(\chi) = (2\pi)^n\delta(\chi)$ is a modified delta function such that

$$\int\not{\delta}(\chi)\bar{d}\chi = 1 \qquad (8)$$

and where $\gamma_\infty(\xi)$ is a bounded measurable function. The two spaces, G_0 and G_∞, are then combined to form a single space of co-kernels denoted by G and consisting of elements of the form,

$$\gamma(\chi;\xi) = \gamma_\infty(\chi;\xi) + \phi(\chi)\gamma_\infty(\xi) \,. \tag{9}$$

The norm assigned to G is the extension of the FM norm for G_0, i.e.

$$|\gamma(\chi;\xi)| = |\gamma_0(\chi;\xi)| + \phi(\chi)|\gamma_\infty(\xi)| ,$$

and $$\sup_\xi|\gamma(\chi;\xi)| = \sup_{\xi'}|\gamma_0(\chi;\xi)| + \phi(\chi)\sup_\xi|\gamma_\infty(\xi)| ,$$

so that

$$||\gamma|| = ||\gamma_0|| + \sup_\xi|\gamma_\infty(\xi)| \,. \tag{10}$$

We now show that FM operators are bounded in L^2.

L^2-continuity theorem for FM: The inequality $||\Gamma\omega||^2_{L^2} \le ||\Gamma||^2_{L^2}||\omega||^2_{L^2}$ holds if $||\Gamma||_{L^2}=||\gamma||_{L^2}$.

Proof: To prove this theorem we use the estimate

$$\left|\int\gamma(\xi-\xi';\xi')\omega(\xi')\,\text{đ}\xi'\right| \le \int\sup_{\xi''}|\gamma(\xi-\xi';\xi'')||\omega(\xi')|\,\text{đ}\xi'$$

and the inequality

$$\int\left|\int\rho(\xi-\xi')\omega(\xi')\,\text{đ}\xi'\right|^2\text{đ}\xi \le \left[\int|\rho(\chi)|\,\text{đ}\chi\right]^2 \int|\omega(\xi)|^2\,\text{đ}\xi \,.$$

Hence,

$$||\Gamma\omega||^2_{L^2} = \int\left|\int\gamma(\xi-\xi';\xi')\omega(\xi')\,\text{đ}\xi'\right|^2 d\xi \le \left[\int\sup_{\xi'}|\gamma(\chi;\xi')|\,\text{đ}\chi\right]^2 \int|\omega(\xi')|^2\,\text{đ}\xi' \le ||\Gamma||^2_{L^2}||\omega||^2_{L^2} \,.$$

§1.2 In this section we shall discuss some of the algebraic properties of the FM class of pseudo-differential operators.

Definition 1.1 Let G_1 and G_2 be two pseudo-differential operators with co-kernels $\gamma_1(\xi-\xi';\xi')$ and $\gamma_2(\xi-\xi';\xi')$ respectively. The addition G_1+G_2 is defined as:

$$(G_1+G_2)u(x) = \int e^{ix\xi}(\Gamma_1+\Gamma_2)\omega(\xi)\,\text{đ}\xi$$

$$= \int e^{ix\xi}[\int(\gamma_1(\xi-\xi';\xi')+\gamma_2(\xi-\xi';\xi'))\omega(\xi')\,\text{đ}\xi']\,\text{đ}\xi. \quad (1)$$

The operator product G_2G_1 is defined as:

$$G_2G_1 \equiv G_{21} = g_2(X;D_x)g_1(X;D_x), \quad (2)$$

and the symbol product g_2g_1 is defined as:

$$g_2g_1(x;\xi) = g_2(x;\xi)g_1(x;\xi). \quad (3)$$

From the definition (2) we remark that

$$G_2G_1u(x)=\int e^{ix\xi}[\int\gamma_2(\xi-\xi'';\xi'')[\int\gamma_1(\xi''-\xi';\xi')\omega(\xi')\,\text{đ}\xi']\,\text{đ}\xi'']\,\text{đ}\xi \quad (4)$$

with $u=\hat{\omega}\in L^2$. By Fubini's theorem [7], the order of integration can be interchanged; thus

$$G_2G_1u(x)=\int e^{ix\xi}[[\int\int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\text{đ}\xi'']\omega(\xi')\,\text{đ}\xi']\,\text{đ}\xi. \quad (5)$$

We see that the operator G_2G_1 is associated with the co-kernel,

$$\gamma_{21}(\xi-\xi';\xi') = \int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\text{đ}\xi''. \quad (6)$$

With $\chi=\xi-\xi'$ and $\chi'=\xi''-\xi'$, we have

$$|\gamma_{21}(\chi;\xi')| \le \int\sup_{\xi''}|\gamma_2(\chi-\chi';\xi'')|\sup_{\xi'}|\gamma_1(\chi';\xi')|\,\text{đ}\chi'$$

and

$$\int\sup_{\xi'}|\gamma_{21}(\chi;\xi')|d\chi \le \int\int\sup_{\xi''}|\gamma(\chi-\chi';\xi'')|\,\text{đ}\chi\sup_{\xi'}|\gamma(\chi';\xi')|\,\text{đ}\chi'$$

$$= \int\sup_{\xi''}|\gamma(\chi'';\xi'')|\,\text{đ}\chi''\int\sup_{\xi'}|\gamma(\chi';\xi')|\,\text{đ}\chi'.$$

This implies that γ_{21} is an FM co-kernel and its norm satisfies the following inequality,

$$||\gamma_{21}|| \le ||\gamma_2||\cdot||\gamma_1||.$$

Since the IFT of the product $g_2(x;\xi)g_1(x;\xi)$ is given by the convolution

$$(\gamma_2 * \gamma_1)(\chi;\xi') = \int \gamma_2(\chi-\chi';\xi')\gamma_1(\chi';\xi')\,\text{đ}\chi' \tag{8}$$

we have

$$\sup_{\xi'}|(\gamma_2 * \gamma_1)(\chi;\xi')| \leq \int \sup_{\xi''}|\gamma_2(\chi-\chi';\xi'')|\sup_{\xi'}|\gamma_1(\chi';\xi')|\,\text{đ}\chi'$$

and

$$\int \sup_{\xi'}|(\gamma_2 * \gamma_1)(\chi;\xi')|\,\text{đ}\chi \leq \int\int \sup_{\xi''}|\gamma_2(\chi-\chi';\xi'')\,\text{đ}\chi \sup_{\xi'}|\gamma_1(\chi';\xi')|\,\text{đ}\chi'$$

or

$$||\gamma_2 * \gamma_1|| \leq ||\gamma_2||\cdot||\gamma_1||$$

which implies that $(\gamma_2 * \gamma_1)(\chi;\xi')$ defined by (8) belongs to the class, G, of FM co-kernels and hence the product of symbols in $\hat{G}$ is a symbol in $\hat{G}$. The operator associated with this product symbol $g_2 g_1$, i.e. the convolution product, is denoted by

$$G_2 * G_1 = g_2 g_1(X;D_x). \tag{9}$$

<u>Theorem 1.1</u> Under the convolution product, the space of FM scalar co-kernels forms a commutative Banach algebra.

<u>Proof</u> (1) G is a linear space. (2) the convolution product,

$$(\gamma_1 * \gamma_2)(\chi;\xi) = \int \gamma_1(\chi-\chi';\xi)\gamma_2(\chi';\xi)\,\text{đ}\chi'$$

satisfies the four axioms of a commutative algebra:

(i) $(\gamma_1 * \gamma_2) * \gamma_3 = \gamma_1 * (\gamma_2 * \gamma_3)$.

This follows from

$$\begin{aligned}[(\gamma_1 * \gamma_2) * \gamma_3](\chi;\xi) &= \int (\gamma_1 * \gamma_2)(\chi-\chi';\xi)\gamma_3(\chi';\xi)\,\text{đ}\chi' \\ &= \int\int \gamma_1(\chi-\chi'-\chi'';\xi)\gamma_2(\chi'';\xi)\gamma_3(\chi';\xi)\,\text{đ}\chi''\text{đ}\chi' \\ &= \int\int \gamma_1(\chi-\chi''';\xi)\gamma_2(\chi'''-\chi';\xi)\gamma_3(\chi';\xi)\,\text{đ}\chi'''\text{đ}\chi'\end{aligned}$$

$$= \int \gamma_1(\chi-\chi''';\xi)\left[\int \gamma_2(\chi'''-\chi';\xi)\gamma_3(\chi';\xi)\,đ\chi'\right]đ\chi'''$$
$$= [\gamma_1*(\gamma_2*\gamma_3)](\chi;\xi).$$

Note that the order of integration can be interchanged by Fubini's theorem.

(ii) commutativity

$\gamma_1*\gamma_2 = \gamma_2*\gamma_1$, if γ_1 and γ_2 are scalar functions.

This follows from

$$(\gamma_1*\gamma_2)(\chi;\xi) = \int \gamma_1(\chi-\chi';\xi)\gamma_2(\chi';\xi)\,đ\chi'$$
$$= \int \gamma_1(\chi'';\xi)\gamma_2(\chi-\chi'';\xi)\,đ\chi''$$
$$= (\gamma_2*\gamma_1)(\chi;\xi) .$$

(iii) $\alpha\beta(\gamma_1*\gamma_2) = (\alpha\gamma_1)*(\beta\gamma_2)$.

(iv) $\gamma_1*(\gamma_2+\gamma_3) = \gamma_1*\gamma_2+\gamma_1*\gamma_3$.

(3) $\mathcal{G}$ is a Banach algebra because it is complete with respect to the FM norm [9] and because

$$||\gamma_2*\gamma_1|| \leq ||\gamma_2||\cdot||\gamma_1|| .$$

This completes the proof.

Since the matrix product is non-commutative, matrix symbols form a non-commutative algebra. However, for applications, it suffices that the determinant of the symbol be in $\hat{\mathcal{G}}_0$; hence the above theorem applies.

In assigning the operator $G = g(X;D_x)$ to a polynomial symbol $g(x;\xi) = \sum_{|\rho|\leq r} g_\rho(x)\xi^\rho$, Friedrichs [1] chose to apply D_x^ρ first and then multiplication by $g_\rho(x)$. By reversing the order of applying D and X, Friedrichs [1] denoted by G^R the 'reversed' operator

$$G^R = \sum_{|\rho|\le r} D_x^\rho g_\rho(X) = g^R(X;D_x) . \qquad (10)$$

The superscript R stands for "reversor". Since the IFT of a(x)u(x) is given by the convolution

$$\int \alpha(\xi-\xi')\omega(\xi')đ\xi' ,$$

the IFT of $\beta(D_x)a(X)u(x)$ is

$$\beta(\xi)\int \alpha(\xi-\xi')\omega(\xi')đ\xi' . \qquad (11)$$

The co-kernel of this operator, $\beta(\xi)\alpha(\xi-\xi')$, can be written in the form

$$\gamma^R(\xi-\xi';\xi') = \alpha(\xi-\xi')\beta(\xi-\xi'+\xi') . \qquad (12)$$

In terms of the co-kernel

$$\gamma(\chi;\xi') = \alpha(\chi)\beta(\xi')$$

of the operator G, the "reversed" co-kernel $\gamma^R(\xi-\xi';\xi')$ can be written as

$$\gamma^R(\chi;\xi') = \gamma(\chi;\chi+\xi') . \qquad (13)$$

In general we associate the reversed operator G^R with the co-kernel $\gamma^R(\chi;\xi') = \gamma(\chi;\chi+\xi')$ if $\gamma(\chi;\xi')$ is the co-kernel of the operator G.

Since

$$\sup_{\xi'}|\gamma^R(\chi;\xi')| = \sup_{\xi'}|\gamma(\chi;\chi+\xi')| = \sup_{\xi''}|\gamma(\chi;\xi'')|$$

we have

$$||\gamma^R|| = ||\gamma|| , \qquad (14)$$

which implies that γ^R is of FM type.

We shall now give the Reversor inequalities and the product - difference inequality which hold for the algebra of pseudo-differential operators of FM type, and which we shall extend to FH pseudo-differential operators in Chapter II.

Reversor inequality I_1^1: If G is of order 1 and $G_{ix\cdot\xi}$ of order 0,

$$||G^R-G|| \le ||G_{ix\cdot\xi}||. \qquad (15)$$

Note that the superscript 1 stands for order 1 and the subscript 1 stands for one term in the expansion.

Proof. Since the operator G^R has its co-kernel $\gamma^R(\chi;\xi) = \gamma(\chi;\chi+\xi)$, the difference of the co-kernels γ^R and γ can be written as

$$\gamma^R(\chi;\xi)-\gamma(\chi;\xi) = \int_0^1 \frac{d}{d\alpha}\gamma(\chi;\xi+\alpha\chi)\,d\alpha \qquad (16)$$

$$= \int_0^1 \sum_{j=1}^{m} \chi_j\gamma_{\xi_j}(\chi;\xi+\alpha\chi)\,d\alpha$$

$$= \int_0^1 \gamma_{ix\cdot\xi}(\chi;\xi+\alpha\chi)\,d\alpha .$$

Noticing that

$\sup_\xi|\gamma^R(\chi;\xi)-\gamma(\chi;\xi)| \le \sup_{\xi'}|\gamma_{ix\cdot\xi}(\chi;\xi')|$

and integrating with respect to χ,

we have

$$||\gamma^R-\gamma|| \le ||\gamma_{ix\cdot\xi}|| .$$

Hence we obtain inequality (15).

Reversor inequality I_1^0: If G and G_x are of order 0 and $G_{ix\cdot\xi}$ of order -1,

$$||G^R-G||_1 \le ||G_{ix\cdot\xi}||_1+2||G_x|| . \qquad (17)$$

Proof. In order to derive inequality (17), we use an idea due to Kumano-go [11] to write

$$[\gamma^R(\chi;\xi)-\gamma(\chi;\xi)]<\xi>=[\gamma(\chi;\xi+\chi)-\gamma(\chi;\xi)]<\xi>$$

$$=\int_0^1\frac{d}{d\alpha}[\gamma(\chi;\xi+\alpha\chi)<\xi+\alpha\chi>]d\alpha-\gamma(\chi;\xi+\chi)[<\xi+\chi>-<\xi>] \qquad (18)$$

$$=\int_0^1[\chi\cdot\gamma_\xi(\chi;\xi+\alpha\chi)<\xi+\alpha\chi>+\gamma(\chi;\xi+\alpha\chi)\chi\cdot(\xi+\alpha\chi)<\xi+\alpha\chi>^{-1}d\alpha-$$

$$\gamma(\chi;\xi+\chi)[<\xi+\chi>-<\xi>][\frac{<\xi+\chi>+<\xi>}{<\xi+\chi>+<\xi>}] \qquad (19)$$

$$=\int_0^1\gamma_{ix\cdot\xi}(\chi;\alpha\chi)<\xi+\alpha\chi>d\alpha+\int_0^1\gamma_x\cdot(\chi;\xi+\alpha\chi)\cdot(\xi+\alpha\chi)<\xi+\alpha\chi>^{-1}d\alpha$$

$$-\gamma(\chi;\xi+\chi)\chi\cdot(\xi+\chi+\xi)[<\xi+\chi>+<\xi>]^{-1} . \qquad (20)$$

We then have

$$|\gamma^R(\chi;\xi)-\gamma(\chi;\xi)|<\xi> \le \int_0^1|\gamma_{ix\cdot\xi}(\chi+\xi+\alpha\chi)|<\xi+\alpha\chi>d\alpha$$

$$+\int_0^1|\gamma_x(\chi;\xi+\alpha\chi)|d\alpha+|\gamma_x(\chi;\xi+\chi)| \qquad (21)$$

$$\le\sup_{\xi'}|\gamma_{ix\cdot\xi}(\chi;\xi')|<\xi'>+2\sup_{\xi'}|\gamma_x(\chi;\xi')| . \qquad (22)$$

We now integrate inequality (22) with respect to χ to obtain

$$||\gamma^R-\gamma||_1 \le ||\gamma_{ix\cdot\xi}||_1 + 2||\gamma_x|| \qquad (23)$$

from which inequality (17) follows.

The notion of reversal can be extended to a function of more than one variable. Consider a function

$$\overset{2}{g} = \overset{2}{g}(x_2;\xi_2|x_1;\xi_1) \qquad (24)$$

of two pairs of variables $(\vec{x}_1;\vec{\xi}_1)$ and $(\vec{x}_2;\vec{\xi}_2)$ each of which is an n-vector, $\vec{x}_1=\{x_1,\ldots,x_n\},\ldots,\vec{\xi}_2=\{\xi_1,\ldots,\xi_n\}$. By taking the IFT of (24) with respect to each x variable, we can introduce the function $\overset{2}{\gamma}(\chi_2;\xi_2|\chi_1;\xi_1)$ assuming

$$||\overset{2}{\gamma}|| = \iint\sup_{\xi_1\xi_2}|\overset{2}{\gamma}(\chi_2;\xi_2|\chi_1;\xi_1)\,đ\chi_1 đ\chi_2$$

is finite. The functions $\overset{2}{\gamma}$ and $\overset{2}{g}$ will be called the "double" FM co-kernel and "double" FM symbol respectively. The double co-kernel $\overset{2}{\gamma}$ generates the operator $\overset{2}{\Gamma}$ by the relation

$$\overset{2}{\Gamma}\omega(\xi) = \iint \overset{2}{\gamma}(\xi-\xi_2;\xi_2|\xi_2-\xi_1;\xi_1)\omega(\xi_1)\text{đ}\xi_1\text{đ}\xi_2 . \tag{25}$$

The operator $\overset{2}{G}$ corresponding to this operator $\overset{2}{\Gamma}$, by taking FT, will be assigned to the symbol $\overset{2}{g}$ and denoted by

$$\overset{2}{G} = \overset{2}{g}(X;D_x|X;D_x) . \tag{26}$$

In order to estimate the difference

$$G_2{*}G_1-G_2G_1 \tag{27}$$

we note that the product G_2G_1 and the convolution product $G_2{*}G_1$ are generated by the symbols

$$g_{21}(x_2;\xi_2|x_1;\xi_1) = g_2(x_2;\xi_2)g_1(x_1;\xi_1) \tag{28}$$

$$g_2g_1(x_2;\xi_2|x_1;\xi_1) = g_2(x_2;\xi_1)g_1(x_1;\xi_1) \tag{29}$$

respectively, so that the multiple symbol of the product difference operator is

$$g_2(x_2;\xi_1)g_1(x_1;\xi_1)-g_2(x_2;\xi_2)g_1(x_1;\xi_1) . \tag{30}$$

Going to the co-kernels $\gamma_2{*}\gamma_1$ and γ_{21}, we obtain

$$\begin{aligned}
&(\gamma_2{*}\gamma_1)(\chi;\xi')-\gamma_{21}(\chi;\xi') \\
&=\int[\gamma_2(\chi-\chi';\xi')\gamma_1(\chi';\xi')-\gamma_2(\chi-\chi';\xi'+\chi')\gamma_1(\chi';\xi')]\text{đ}\chi' && (31) \\
&=\int[\gamma_2(\xi-\xi'';\xi')-\gamma_2(\xi-\xi'';\xi'')]\gamma_1(\xi''-\xi';\xi')\text{đ}\xi'' \\
&=\int(-1)\int_0^1\frac{d}{d\alpha}[\gamma_2(\xi-\xi'';\xi'+\alpha(\xi''-\xi'))d\alpha]\gamma_1(\xi''-\xi';\xi')\text{đ}\xi'' \\
&=-\iint_0^1\sum_{\mu=1}^{n}\gamma_{2,\xi_\mu}(\xi-\xi'';\xi'+\alpha(\xi''-\xi'))\gamma_{1,ix_\mu}(\xi''-\xi';\xi')d\alpha\,\text{đ}\xi'' && (32) \\
&=-\iint_0^1\gamma_{2,\xi}\cdot(\xi-\xi'';\xi'+\alpha(\xi''-\xi'))d\alpha\cdot\gamma_{1,ix}(\xi''-\xi';\xi')\text{đ}\xi'' .
\end{aligned}$$

Assuming that g_1 and g_2 are of order zero and that the derivatives $g_{2,\xi}$ and $g_{1,ix}$ are of orders -1 and 0 respectively, we observe that the difference $G_2{*}G_1-G_2G_1$ is of order $(0,-1)$ and hence of order -1 when regarded as an operator associated with a simple symbol. Since the difference $G_2{*}G_1-G_2G_1$ possesses a "leading" difference $-g_{2,\xi}g_{1,ix}$, then by virtue of inequality (17), we have that

$$||G_2{*}G_1-G_2G_1||_1 \le ||G_{2,\xi}||_1||G_{1,x}||+2||G_{2,x}||\,||G_{1,x}||. \tag{33}$$

Symbols of order $\frac{1}{2}$ will play a special role for us, therefore we shall now derive the inequalities involving such symbols. We shall first prove the following inequality

$I_1^{\frac{1}{2}}$: If g and g_x are of order $\frac{1}{2}$ and $g_{ix\cdot\xi}$ is of order $-\frac{1}{2}$,

$$||g^R-g||_{\frac{1}{2}} \le ||g_{ix\cdot\xi}||_{\frac{1}{2}} + \frac{3}{2}||g_x||_{-\frac{1}{2}}. \tag{34}$$

Proof. By writing

$$[\gamma(\chi;\xi+\chi)-\gamma(\chi;\xi)]<\xi>^{\frac{1}{2}}$$
$$=[\gamma(\chi;\xi+\chi)<\xi+\chi>^{\frac{1}{2}}-\gamma(\chi;\xi)<\xi>^{\frac{1}{2}}]-\gamma(\chi;\xi+\chi)[<\xi+\chi>^{\frac{1}{2}}-<\xi>^{\frac{1}{2}}] \tag{35}$$

and expressing the difference $[<\xi+\chi>^{\frac{1}{2}}-<\xi>^{\frac{1}{2}}]$ as

$$\begin{aligned}<\xi+\chi>^{\frac{1}{2}}-<\xi>^{\frac{1}{2}} &= \frac{<\xi+\chi>^{\frac{1}{2}}+<\xi>^{\frac{1}{2}}}{<\xi+\chi>^{\frac{1}{2}}+<\xi>^{\frac{1}{2}}}\,<\xi+\chi>^{\frac{1}{2}}-<\xi>^{\frac{1}{2}} \qquad (36)\\ &= [<\xi+\chi>^{\frac{1}{2}}+<\xi>^{\frac{1}{2}}]^{-1}[<\xi+\chi>-<\xi>]\\ &= [<\xi+\chi>^{\frac{1}{2}}+<\xi>^{\frac{1}{2}}]^{-1}[<\xi+\chi>+<\xi>]^{-1}(\xi+\chi+\xi)\cdot\chi\end{aligned}$$

we have

$$|<\xi+\chi>^{\frac{1}{2}}-<\xi>^{\frac{1}{2}}| \le [<\xi+\chi>^{\frac{1}{2}}+<\xi>^{\frac{1}{2}}]^{-1}|\chi| \le <\xi+\chi>^{-\frac{1}{2}}|\chi|. \tag{37}$$

We now write the first term of the right hand side of equation (35) as

$$\gamma(\chi;\xi+\chi)\langle\xi+\chi\rangle^{\frac{1}{2}}-\gamma(\chi;\xi)\langle\xi\rangle^{\frac{1}{2}}$$
$$=\int_0^1\frac{d}{d\alpha}[\gamma(\chi;\xi+\alpha\chi)\langle\xi+\alpha\chi\rangle^{\frac{1}{2}}]d\alpha$$
$$=\int_0^1[\chi\cdot\gamma_{\cdot\xi}(\chi;\xi+\alpha\chi)\langle\xi+\alpha\chi\rangle^{\frac{1}{2}}+\tfrac{1}{2}\chi\cdot\check{\gamma}(\chi;\xi+\alpha\chi)\cdot(\xi+\alpha\chi)\langle\xi+\alpha\chi\rangle^{-\frac{1}{2}}$$
$$\langle\xi+\alpha\chi\rangle^{-1}]d\alpha\,.$$

Hence,

$$|\gamma(\chi;\xi+\chi)-\gamma(\chi;\xi)|\langle\xi\rangle^{\frac{1}{2}}\le\sup_{\xi'}|\gamma_{ix\cdot\xi}(\chi;\xi')|\langle\xi'\rangle^{\frac{1}{2}}$$
$$+\tfrac{1}{2}\sup_{\xi'}|\gamma_x(\chi;\xi')|\langle\xi'\rangle^{-\frac{1}{2}}+\sup_{\xi'}|\gamma_x(\chi;\xi')|\langle\xi'\rangle^{-\frac{1}{2}}$$

and integrating with respect to χ, we have

$$||\gamma^R-\gamma||_{\frac{1}{2}}\le||\gamma_{ix\cdot\xi}||_{\frac{1}{2}}+\frac{3}{2}||\gamma_x||_{-\frac{1}{2}}$$

from which inequality (34) follows.

Another inequality involving the order ½ refers to the 'double' norm of G, which we define as

$$||G||_{\beta,\alpha}=||\langle D_x\rangle^{\beta}G\langle D_x\rangle^{\alpha}|| \qquad (38)$$

for an operator G of order $(\alpha+\beta)$. It may also be associated with the multiple symbol

$$\langle\xi_2\rangle^{\beta}g(x_1;\xi_1)\langle\xi_1\rangle^{\alpha}.$$

Using the fact that

$$|\langle\xi+\chi\rangle^{\frac{1}{2}}-\langle\xi\rangle^{\frac{1}{2}}|\le\langle\xi+\chi\rangle^{-\frac{1}{2}}|\chi|$$

we obtain, with $\beta=\frac{1}{2}$,

$$|[\langle\xi_2\rangle^{\frac{1}{2}}\gamma(\xi_2-\xi_1;\xi_1)\langle\xi_1\rangle^{\alpha}-\gamma(\xi_2-\xi_1;\xi_1)\langle\xi_1\rangle^{\alpha+\frac{1}{2}}]| \qquad (39)$$
$$=|\gamma(\xi_2-\xi_1;\xi_1)\langle\xi_1\rangle^{\alpha}[\langle\xi_2\rangle^{\frac{1}{2}}-\langle\xi_1\rangle^{\frac{1}{2}}]|$$
$$\le|\gamma(\xi_2-\xi_1;\xi_1)|\,\frac{\langle\xi_1\rangle^{\alpha}}{\langle\xi_1\rangle^{\frac{1}{2}}}\,|\xi_2-\xi_1|$$

and hence the inequality

$I(\frac{1}{2},\alpha)$: If G and G_x are of order $-(\alpha+\frac{1}{2})$,

$$\left|\, ||G||_{\frac{1}{2},\alpha} - ||G||_{\alpha+\frac{1}{2}} \right| \leq ||G_x||_{\alpha-\frac{1}{2}} . \tag{40}$$

For $\alpha=\frac{1}{2}$, we have

$$I(\tfrac{1}{2},\tfrac{1}{2}): \quad \left|\, ||G||_{\frac{1}{2},\frac{1}{2}} - ||G||_{1} \right| \leq ||G_x|| \tag{41}$$

for an operator of order -1.

Note that inequality (41) is true for the reversed operator G^R of G.

CHAPTER II

Fourier-Holmgren Pseudo-Differential Operators

A natural question arises as to whether or not one could enlarge the class of FM pseudo-differential operators in ~~such~~ a way that they remain L^2-continuous and that at least the operators, if not their symbols, form an algebra.

We first consider the Fourier-Holmgren (FH) norm

$$||\gamma||_{FH}=\max\{\sup_{\xi'}\int|\gamma(\xi-\xi';\xi')|\,\not{d}\xi,\sup_{\xi}\int|\gamma(\xi-\xi';\xi')|\,\not{d}\xi'\} \quad (1)$$
$$=\max\{||\gamma||_+,||\gamma||_-\}<\infty$$

defined by Friedrichs [1] for the co-kernel $\gamma(\chi;\xi)$.

A basic starting point in the development of Friedrichs' [1] class of pseudo-differential operators was the definition of the space of co-kernels denoted by G. To this end, we denote the space of co-kernels with finite FH norm by G^{FH} and that of the symbols $\hat{G}^{FH}$. We shall derive for the class of FH pseudo-differential operators, inequalities analogous to those we derived for the FM pseudo-differential operators.

We show that an operator with finite FH norm is L^2-continuous, i.e. the inequality $||\Gamma\omega||^2_{L^2} \le ||\Gamma||_+||\Gamma||_-||\omega||^2_{L^2}$ holds if $||\Gamma||_+=||\gamma||_+<\infty$ and $||\Gamma||_-=||\gamma||_-<\infty$.

Proof.

$$||\Gamma\omega||^2_{L^2}=\int\left|\int\gamma(\xi-\xi';\xi')\omega(\xi')\,\not{d}\xi'\right|^2\not{d}\xi$$

$$\leq\int[\int|\gamma(\xi-\xi';\xi')|^{\frac{1}{2}}|\gamma(\xi-\xi';\xi')|^{\frac{1}{2}}|\omega(\xi')|\not{d}\xi']^2\not{d}\xi$$
$$\leq\int[\int|\gamma(\xi-\xi';\xi')|\not{d}\xi'][\int|\gamma(\xi-\xi';\xi')||\omega(\xi')|^2\not{d}\xi']\not{d}\xi$$
$$\leq[\sup_\xi\int|\gamma(\xi-\xi';\xi')|\not{d}\xi']\int[\int|\gamma(\xi-\xi';\xi')||\omega(\xi')|^2\not{d}\xi']\not{d}\xi$$
$$\leq[\sup_\xi\int|\gamma(\xi-\xi';\xi')|\not{d}\xi']\int\not{d}\xi'[\sup_{\xi'}\int|\gamma(\xi-\xi';\xi')|\not{d}\xi]|\omega(\xi')|^2$$
$$\leq[\sup_\xi\int|\gamma(\xi-\xi';\xi')|\not{d}\xi'][\sup_{\xi'}\int|\gamma(\xi-\xi';\xi')|\not{d}\xi]\int|\omega(\xi')|^2\not{d}\xi'$$
$$=||\gamma||_-||\gamma||_+||\omega||^2_{L^2}.$$

The boundedness of $\gamma(\chi;\xi)$ in the FH norm does not imply the boundedness of the reversed co-kernel $\gamma^R(\chi;\xi)$ nor of the intermediary co-kernels $\gamma(\chi;\xi+\alpha\chi)$, $0\leq\alpha\leq1$, which occur in the reversor inequalities. To extend the L^2-continuity theorem to the reversed co-kernels, we remark that

$$||\gamma^R||=\max\{||\gamma^R||_+,||\gamma^R||_-\}$$
$$=\max\{\sup_{\xi'}\int|\gamma(\xi-\xi';\xi')|\not{d}\xi,\sup_\xi\int|\gamma(\xi-\xi';\xi)|\not{d}\xi'\}$$
$$\equiv\max\{||\gamma||_\sim,||\gamma||_+\}.$$

Hence, G^R is L^2-continuous if $||\gamma||_\sim$ and $||\gamma||_+$ are bounded.

For the reversor inequalities, we shall require that $\gamma_\alpha=\gamma(\chi;\xi'+\alpha\chi)$ $0\leq\alpha\leq1$, be in G^{FH}. Hence G_α is L^2-bounded if $||\gamma_\alpha||_+$ and $||\gamma||_-$ are bounded. Note that $\gamma_0=\gamma$ and $\gamma_1=\gamma^R$.

In $R^1\times R^1$, γ_α is in G^{FH}, $0\leq\alpha\leq1$, if

$$\max_L\int_L|\gamma|ds<\infty,$$

where L is any line with slope $|m|\geq1$ in the (ξ,ξ')-plane.

In particular, for γ,

$$||\gamma||_+ = \max_{\text{vertical lines}} \int |\gamma| ds,$$

$$||\gamma||_- = \max_{\text{lines with slope}=-1} \int |\gamma| ds$$

and

$$||\gamma||_\sim = \max_{\text{lines with slope}=1} \int |\gamma| ds.$$

We give a simple example of an FH co-kernel which is not an FM co-kernel. Consider the simple function of a real valuable

$$\psi(\sigma) = \begin{cases} 1 & \text{for } -1 \le \sigma \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\gamma(\chi;\xi') = \psi(\chi + \tfrac{1}{2}\xi').$$

Clearly,

$$||\gamma||_{FH} = 2,$$

$$||\gamma||_{FM} = \infty,$$

<u>Definition 2.1.</u> Let G_1 and G_2 be two pseudo-differential operators, acting on L^2 functions, with co-kernels $\gamma_1(\xi-\xi';\xi')$ and $\gamma_2(\xi-\xi';\xi')$ respectively. The sum G_1+G_2 is defined as follows:

$$(G_1+G_2)u(x) = \int e^{ix\xi}[\Gamma_1+\Gamma_2]\omega(\xi){\mathchar'26\mkern-12mu d}\xi$$

$$= \int e^{ix\xi}[\int\{\gamma_1(\xi-\xi';\xi')+\gamma_2(\xi-\xi';\xi')\}\omega(\xi'){\mathchar'26\mkern-12mu d}\xi']{\mathchar'26\mkern-12mu d}\xi$$

and the operator product G_2G_1 is defined as:

$$G_2G_1u(x) = \int e^{ix\xi}[\int\gamma_2(\xi-\xi'';\xi'')[\int\gamma_1(\xi''-\xi';\xi')\omega(\xi'){\mathchar'26\mkern-12mu d}\xi']{\mathchar'26\mkern-12mu d}\xi'']{\mathchar'26\mkern-12mu d}\xi. \quad (3)$$

<u>Remark 2.1</u> Clearly, the operator G_2G_1 is non-commutative.

Algebra Theorem 2.1. The class of pseudo-differential operators with finite FH norm forms a non-commutative algebra under the operator product.

Proof. It is clear that G_1+G_2 belongs to FH. To prove that $G_2G_1 \in FH$, we interchange the order of integration in ξ' and ξ'' in the definition (3) of G_2G_1, to be justified below. We thus obtain

$$G_2G_1u(x) = \int e^{ix\xi}\Big[\int\Big[\int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\bar{d}\xi''\Big]\omega(\xi')\,\bar{d}\xi'\Big]\bar{d}\xi. \quad (4)$$

We see that the product operator corresponds to the co-kernel

$$\gamma_{21} = \int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\bar{d}\xi'' \quad (5)$$

and we have

$$\begin{aligned}||\gamma_{21}||_+ &= \sup_{\xi'}\int\Big|\int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\bar{d}\xi''\Big|\bar{d}\xi \quad (6)\\ &\le \sup_{\xi'}\int\int|\gamma_2(\chi;\xi'')|\,d\chi\,|\gamma_1(\xi''-\xi';\xi')|\,\bar{d}\xi''\\ &\le \sup_{\xi',\xi''}\int|\gamma_2(\chi;\xi'')|\,\bar{d}\chi\int|\gamma_1(\chi';\xi')|\,\bar{d}\chi'\\ &\le \sup_{\xi''}\int|\gamma_2(\chi;\xi'')|\,\bar{d}\chi\,\sup_{\xi'}\int|\gamma_1(\chi';\xi')|\,\bar{d}\chi'\\ &= ||\gamma_2||_+||\gamma_1||_+ ,\end{aligned}$$

$$\begin{aligned}||\gamma_{21}||_- &= \sup_{\xi}\int\Big|\int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\bar{d}\xi''\Big|\bar{d}\xi' \quad (7)\\ &\le \sup_{\xi}\int\sup_{\xi'''}\int|\gamma_2(\xi-\xi'';\xi'')|\,\bar{d}\xi''|\gamma_1(\xi'''-\xi';\xi')|\,\bar{d}\xi'\\ &\le ||\gamma_2||_-||\gamma_1||_- \ .\end{aligned}$$

This justifies the change in the order of integration by Fubini's theorem [7] and shows that γ_{21} is an FH co-kernel. By remark 2.1, the algebra is non-commutative. Thus the theorem is proved.

In order to prove that the space G^{FH} is complete with respect to the FH norm, we shall use the following theorems [5].

<u>Theorem I</u>. If a sequence of measurable non-negative functions $\{\phi_k(x)\}_{k=1}^{\infty}$ is such that $\sum_{k=1}^{\infty} \int \phi_k(x)dx<\infty$, then $\sum_{k=1}^{\infty} \phi_k(x)$ converges to a finite limit almost everywhere.

<u>Theorem II</u>. If a sequence of measurable non-negative functions $\{\psi_k(x)\}_{k=1}^{\infty}$ converges to the function $\psi(x)$ almost everywhere then

$$\int \psi(x)dx \le \lim\inf \int \psi_k(x)dx .$$

<u>Theorem 2.2</u>. The space G^{FH} is complete with respect to the FH norm.

<u>Proof</u>. Let $\{\gamma_n(\chi;\xi')\}_{n=1}^{\infty}$ be a Cauchy sequence of elements $\gamma_n \epsilon G^{FH-}$. Given $\epsilon>0$, there exists $N=N(\epsilon)$ such that

$$\|\gamma_n-\gamma_m\|_- = \sup_{\xi} \int |\gamma_n(\xi-\xi';\xi')-\gamma_m(\xi-\xi';\xi')| \not{d}\xi' \tag{8}$$
$$< \epsilon$$

for $m,n>N$. For each number k, we can choose an n_k such that

$$\|\gamma_n-\gamma_m\|_- < \frac{1}{2^k} \tag{9}$$

if $m,n>n_k$ and $n_1<n_2<n_3<\ldots$.

Hence we have

$$\|\gamma_{n_{k+1}}-\gamma_{n_k}\|_- < \frac{1}{2^k} , \tag{10}$$

which implies that

$$\sum_{k=1}^{\infty} \|\gamma_{n_{k+1}}-\gamma_{n_k}\|_- < \sum_{k=1}^{\infty} \frac{1}{2^k} . \tag{11}$$

Hence the series

$$\sum_{k=1}^{\infty} \sup_{\xi} \int |\gamma_{n_{k+1}}(\xi-\xi';\xi')-\gamma_{n_k}(\xi-\xi';\xi')| \not{d}\xi'$$
$$= \sum_{k=1}^{\infty} \|\gamma_{n_{k+1}}-\gamma_{n_k}\|_- < \infty \tag{12}$$

which implies that the series

$$\sum_{k=1}^{\infty} \int |\gamma_{n_{k+1}}(\xi-\xi';\xi') - \gamma_{n_k}(\xi-\xi';\xi')|\,d\xi' \tag{13}$$

converges for almost all values of ξ. By Theorem I, we see that

$$\sum_{k=1}^{\infty} |\gamma_{n_{k+1}}(\xi-\xi';\xi') - \gamma_{n_k}(\xi-\xi';\xi')| \tag{14}$$

converges for almost all values of ξ and ξ'. It follows that the series

$$\gamma_{n_1}(\xi-\xi';\xi') + \sum_{k=1}^{\infty} [\gamma_{n_{k+1}}(\xi-\xi';\xi') - \gamma_{n_k}(\xi-\xi';\xi')] \tag{15}$$

converges almost everywhere to a finite limit γ as a function of ξ and ξ'. Hence, there exists a function $\gamma(\xi-\xi';\xi')$ defined almost everywhere such that

$$\lim_{k\to\infty} \gamma_{n_k}(\xi-\xi';\xi') = \gamma(\xi-\xi';\xi') \tag{16}$$

for almost all values of ξ and ξ'.

Next, we show that $\gamma(\xi-\xi';\xi') \in G^{FH}$. For fixed j we have almost everywhere

$$\lim_{k\to\infty} |\gamma_{n_k}(\xi-\xi';\xi') - \gamma_{n_j}(\xi-\xi';\xi')|$$
$$= |\gamma(\xi-\xi';\xi') - \gamma_{n_j}(\xi-\xi';\xi')|. \tag{17}$$

Hence by Theorem II, we have

$$\int |\gamma(\xi-\xi';\xi') - \gamma_{n_j}(\xi-\xi';\xi')|\,d\xi'$$
$$\le \liminf_{k\to\infty} \int |\gamma_{n_k}(\xi-\xi';\xi') - \gamma_{n_j}(\xi-\xi';\xi')|\,d\xi'. \tag{18}$$

Now

$$\|\gamma_{n_k} - \gamma_{n_j}\|_{-} \le \|\gamma_{n_j} - \gamma_{n_{j+1}}\|_{-} + \|\gamma_{n_{j+1}} - \gamma_{n_{j+2}}\|_{-} + \ldots + \|\gamma_{n_{k-1}} - \gamma_{n_k}\|_{-} \tag{19}$$
$$\le \frac{1}{2^j} + \frac{1}{2^{j+1}} + \ldots = \frac{1}{2^{j-1}},$$

which implies that

$$\liminf_{k\to\infty} \int |\gamma_{n_k}(\xi-\xi';\xi') - \gamma_{n_j}(\xi-\xi';\xi')|\,d\xi' < \infty. \tag{20}$$

This means that $\sup_{\xi}\int|\gamma(\xi-\xi';\xi')-\gamma_{n_j}(\xi-\xi';\xi')|d\xi'<\infty$, so that $\gamma(\xi-\xi';\xi')-\gamma_{n_j}(\xi-\xi';\xi')\in G^{FH-}$. Since

$$\gamma(\xi-\xi';\xi')=\gamma(\xi-\xi';\xi')-\gamma_{n_j}(\xi-\xi';\xi')+\gamma_{n_j}(\xi-\xi';\xi') \qquad (21)$$

and each of these is in G^{FH-}, their sum, $\gamma(\xi-\xi';\xi')$ must be in G^{FH-}. Hence

$$\lim_{j\to\infty}||\gamma-\gamma_{n_j}||_-=0, \qquad (22)$$

i.e. $\gamma_n(\xi-\xi';\xi')$ converges with respect to the FH- norm to a function $\gamma(\xi-\xi';\xi')\in G^{FH-}$.

Similarly, we can show that G^{FH+} is complete. This proves theorem 2.2.

Corollary 2.3. The space G^{FH} is a non-commutative Banach algebra.

Proof. (i) G^{FH} is a linear space. (2) For each pair of elements γ_1 and $\gamma_2\in G^{FH}$, there is a unique product,

$$\gamma_2\gamma_1=\int\gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')d\xi''$$

such that

$$(\gamma_3\gamma_2)\gamma_1=\gamma_3(\gamma_2\gamma_1).$$

This follows from

$$[(\gamma_3\gamma_2)\gamma_1](\xi-\xi';\xi')=\int[\gamma_3(\xi-\xi''';\xi''')\gamma_2(\xi'''-\xi'';\xi'')d\xi''']\gamma_1(\xi''-\xi';\xi')d\xi''.$$

By Fubini's theorem [7], the order of integration can be interchanged so that

$$\int[\int\gamma_3(\xi-\xi''';\xi''')\gamma_2(\xi'''-\xi'';\xi'')d\xi''']\gamma_1(\xi''-\xi';\xi')d\xi''$$
$$=\int\gamma_3(\xi-\xi''';\xi''')[\int\gamma_2(\xi'''-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')d\xi'']d\xi'''$$
$$=\gamma_3(\gamma_2\gamma_1).$$

(ii) $\alpha\beta(\gamma_2\gamma_1)=(\alpha\gamma_2)(\beta\gamma_1)$, where α and β are complex numbers ,

(iii) $\gamma_3(\gamma_2+\gamma_1) = \gamma_3\gamma_2+\gamma_3\gamma_1$.

Hence, G^{FH} is a non-commutative algebra. It is also a non-commutative Banach algebra because it is complete with respect to the FH norm and because

$$||\gamma_2\gamma_1||_{FH} \leq ||\gamma_2||_{FH}||\gamma_1||_{FH} .$$

We shall now prove three reversor inequalities which hold for the algebra of pseudo-differential operators of the FH type.

Reversor inequality I_1^1: If G is of order 1 and $G_{ix\cdot\xi}$ of order 0,

$$||G^R-G||_{L^2} \leq \sup_{0\leq\alpha\leq 1} ||G_{ix\cdot\xi,\alpha}||_{L^2} \leq ||G_{ix\cdot\xi,\alpha}||_{FH} . \tag{38}$$

Proof. Since the operator G^R has its co-kernel $\gamma^R(\chi;\xi)=\gamma(\chi;\chi+\xi)$, the difference of the co-kernels γ^R and γ can be written as

$$\gamma^R(\chi;\xi)-\gamma(\chi;\xi) = \int_0^1 \frac{d}{d\alpha}\gamma(\chi;\xi+\alpha\chi)\,d\alpha \tag{39}$$

$$= \int_0^1 \gamma_{ix\cdot\xi}(\chi;\xi+\alpha\chi)\,d\alpha. \tag{40}$$

By the L^2 continuity theorem for FH with $\gamma_\alpha(\chi;\xi') = \gamma(\chi;\xi'+\alpha\chi)$, we have

$$||\Gamma_\alpha\omega||^2_{L^2} \leq \int\left|\int\gamma(\xi-\xi';\xi'+\alpha(\xi-\xi'))\omega(\xi'){\mathchar'26\mkern-12mu d}\xi'\right|^2{\mathchar'26\mkern-12mu d}\xi \tag{41}$$

$$\leq ||\gamma_\alpha||_+||\gamma_\alpha||_-||\omega||^2_{L^2}$$

whence inequality (38) follows.

Reversor inequality I_1^0: If G and G_x are of order 0 and $g_{ix\cdot\xi}$ of order -1,

$$||G^R-G||_{H^1} \leq \max_{0\leq\alpha\leq 1} ||G_{ix\cdot\xi,\alpha}||_{1,FH}+2\max_{0\leq\alpha\leq 1}||G_{x,\alpha}||_{FH} \tag{42}$$

where $H^1 = \{f\epsilon L^2, \nabla f\epsilon L^2\}$.

Proof. Using inequality 1.2(21) and substituting

$$h_\alpha = \gamma_{ix\cdot\xi}(\chi;\xi+\alpha\chi)<\xi+\alpha\chi>, \qquad (43)$$

$$k_\alpha = \gamma_x(\chi;\xi+\alpha\chi), \qquad (44)$$

$$k_1 = \gamma_x(\chi;\xi+\chi), \qquad (45)$$

we have

$$\begin{aligned} ||(\Gamma^R-\Gamma)<\xi>\omega||_{L^2} &\le ||\int_0^1 (h_\alpha\omega+k_\alpha\omega+k_1\omega)\,d\alpha||_{L^2} \qquad (46)\\ &\le \int_0^1 (||h_\alpha\omega||_{L^2}+||k_\alpha\omega||_{L^2}+||k_1\omega||_{L^2})\,d\alpha \\ &\le \max_{0\le\alpha\le 1} (||h_\alpha||_{L^2}||\omega||_{L^2}+||k_\alpha||_{L^2}||\omega||_{L^2}+||k_1||_{L^2}||\omega||_{L^2}) \\ &\le \max_{0\le\alpha\le 1} (||h_\alpha||_{FH}||\omega||_{L^2}+||k_\alpha||_{FH}||\omega||_{L^2}+||k_1||_{FH}||\omega||_{L^2}) \\ &\le \max_{0\le\alpha\le 1} (||h_\alpha||_{FH}+2||k_\alpha||_{FH})||\omega||_{L^2} \\ &\le \max_{0\le\alpha\le 1} (||\gamma_{ix\cdot\xi,\alpha}||_{1,FH}+2||\gamma_{x,\alpha}||_{FH})||\omega||_{L^2}\,. \end{aligned}$$

Hence we obtain the desired result.

<u>Reversor inequality</u> $I(\frac{1}{2},\alpha)$: If G is of order $-(\alpha+\frac{1}{2})$

$$|||G||_{H^{\frac{1}{2}},\alpha}-||G||_{H^{\alpha+\frac{1}{2}}}| \le ||G_x||_{\alpha-\frac{1}{2},FH}\,. \qquad (47)$$

<u>Proof</u>. By using inequality 1.2 (37), we write

$$\begin{aligned} &|[<\xi_2>^{\frac{1}{2}}\gamma(\xi_2-\xi_1;\xi_1)<\xi_1>^{\alpha}-\gamma(\xi_2-\xi_1;\xi_1)<\xi_1>^{\alpha+\frac{1}{2}}]| \qquad (48)\\ &=|\gamma(\xi_2-\xi_1;\xi_1)<\xi_1>^{\alpha}[<\xi_2>^{\frac{1}{2}}-<\xi_1>^{\frac{1}{2}}]| \\ &=|\gamma(\xi_2-\xi_1;\xi_1)|\frac{<\xi_1>^{\alpha}}{<\xi_1>^{\frac{1}{2}}}\,|\xi_2-\xi_1| \\ &=|\gamma_{ix}(\xi_2-\xi_1;\xi_1)|<\xi_1>^{\alpha-\frac{1}{2}}\,. \end{aligned}$$

Then

$$\begin{aligned} &\int|[<\xi_2>^{\frac{1}{2}}\gamma(\xi_2-\xi_1;\xi_1)<\xi_1>^{\alpha}-\gamma(\xi_2-\xi_1;\xi_1)<\xi_1>^{\alpha+\frac{1}{2}}]|\begin{matrix}\not{d}\xi_2\\ \not{d}\xi_1\end{matrix} \\ &\le\max\{\sup_{\substack{\xi_1\\ \xi_2}}\int|\gamma_{ix}(\xi_2-\xi_1;\xi_1)|<\xi>^{\alpha-\frac{1}{2}}\begin{matrix}\not{d}\xi_2\\ \not{d}\xi_1\end{matrix}\} \qquad (49)\\ &=||\gamma_{ix}||_{\alpha-\frac{1}{2},FH} \end{aligned}$$

and hence the desired inequality follows:

$$\left|\,||G||_{H^{\frac{1}{2},\alpha}}-||G||_{H^{\alpha+\frac{1}{2}}}\right| \le ||G_x||_{\alpha-\frac{1}{2},FH}. \tag{50}$$

We shall use inequality (47) for the case $\alpha=\frac{1}{2}$ in which it reads

$$I(\tfrac{1}{2},\tfrac{1}{2}):\ \left|\,||G||_{H^{\frac{1}{2},\frac{1}{2}}}-||G||_{H^{1}}\right| \le ||G_x||_{FH} \tag{51}$$

for an operator of order -1.

Now we shall prove inequality (51) for the reversed operator G^R of G. Since

$$\int\{\langle\xi_2\rangle^{\alpha}\gamma(\xi_2-\xi_1;\xi_2)\langle\xi_1\rangle^{\frac{1}{2}}-\gamma(\xi_2-\xi_1;\xi_2)\langle\xi_2\rangle^{\alpha+\frac{1}{2}}\Big|\begin{matrix}đ\xi_2\\ đ\xi_1\end{matrix}$$

$$\le\sup_{\substack{\xi_1\\ \xi_2}}\int|\gamma(\xi_2-\xi_1;\xi_2)|\langle\xi_2\rangle^{\alpha}|\langle\xi_1\rangle^{\frac{1}{2}}-\langle\xi_2\rangle^{\frac{1}{2}}\Big|\begin{matrix}đ\xi_2\\ đ\xi_1\end{matrix}$$

$$=\max\{\sup_{\substack{\xi_1\\ \xi_2}}\int|\gamma_{ix}(\xi_2-\xi_1;\xi_2)|\langle\xi_2\rangle^{\alpha-\frac{1}{2}}\,]\begin{matrix}đ\xi_2\\ đ\xi_1\end{matrix}\}$$

$$=||\gamma_{ix}||_{\alpha-\frac{1}{2},0;FH}.$$

Hence

$$\left|\,||G^R||_{H^{\alpha,\frac{1}{2}}}-||G^R||_{H^{\alpha+\frac{1}{2},0}}\right| \le ||G^R_x||_{\alpha-\frac{1}{2},0;FH}. \tag{52}$$

Taking $\alpha=\frac{1}{2}$, we have

$$\left|\,||G^R||_{H^{\frac{1}{2},\frac{1}{2}}}-||G^R||_{H^{1,0}}\right| \le ||G^R_x||_{FH}. \tag{53}$$

Chapters 3 and 4 will be concerned solely with pseudo-differential operators of the FH type. For this reason, we mention the following facts. Since

$$g(x;\xi')=\int e^{ix\chi}\gamma(\chi;\xi')\,đ\chi \tag{54}$$

then

$$||g||_{FM}=\int\sup_{\xi'}|\gamma(\chi;\xi')|\,đ\chi \ge ||g||_{FH}. \tag{55}$$

This inequality implies that an FM symbol is an FH symbol. Moreover, the product $g_2(x;\xi')g_1(x,\xi')$ of an FM symbol g_2 and an FH symbol g_1 is an FH symbol. In fact since

$$g_2(x;\xi')g_1(x;\xi')=\int e^{ix\chi}(\gamma_2*\gamma_1)(\chi;\xi')\not{d}\chi \tag{56}$$

$$=\int\int e^{ix(\chi-\chi')}\gamma_2(\chi-\chi';\xi')\not{d}\chi e^{ix\chi'}\gamma_1(\chi';\xi')\not{d}\chi'$$

we have

$$||g_2 g_1||_{FH} = ||\gamma_2*\gamma_1||_{FH}$$

$$=\max\{\sup_{\xi'}\int\left|\int\gamma_2(\xi-\xi'';\xi')\gamma_1(\xi''-\xi';\xi')\not{d}\xi''\right|\not{d}\xi,$$

$$\sup_{\xi}\int\left|\int\gamma_2(\xi-\xi'';\xi')\gamma_1(\xi''-\xi';\xi')\not{d}\xi''\right|\not{d}\xi'\} \tag{57}$$

$$=\max\{A_+,A_-\},$$

where

$$A_+\le\int\sup_{\xi'''}|\gamma_2(\xi-\xi'';\xi''')|\not{d}\xi\sup_{\xi^{iv}}\int|\gamma_1(\xi''-\xi';\xi^{iv})|\not{d}\xi''$$

$$\le||\gamma_2||_{FM}||\gamma_1||_+$$

and

$$A_-\le\sup_{\xi}\int\sup_{\xi'''}|\gamma_2(\xi-\xi'';\xi''')|\sup_{\xi^{iv}}\int|\gamma_1(\xi^{iv}-\xi';\xi')|\not{d}\xi''\not{d}\xi'$$

$$\le||\gamma_2||_{FM}||\gamma_1||_-$$

Hence

$$||\gamma_2*\gamma_1||_{FH}\le||\gamma_2||_{FM}||\gamma_1||_{FH}.$$

The operator associated with $g_2(x;\xi')g_1(x;\xi')$ is the convolution product operator G_2*G_1, and from equality (56), we have

$$||G_2*G_1||_{FH}\le||G_2||_{FM}||G_1||_{FH}. \tag{58}$$

Consider the simple function of a real variable (given earlier in this chapter):

$$\psi(\sigma)=\{\begin{matrix}\pi \text{ for } 1\le\sigma\le 1\\ 0 \text{ otherwise}\end{matrix}$$

and let

$$\begin{aligned}\gamma(\chi;\xi') &= \psi(\chi+\tfrac{1}{2}\xi') \\ &= \psi([\xi-\xi']+\tfrac{1}{2}\xi').\end{aligned}$$

Then

$$\gamma*\gamma(\xi-\xi';\xi') = \int\psi(\xi-\xi''+\tfrac{1}{2}\xi')\psi(\xi''-\xi'+\tfrac{1}{2}\xi')\,\text{đ}\xi'',$$

and

$$\begin{aligned}\|\gamma*\gamma\|_{-} &= \sup_{\xi}\iint\psi(\xi-\xi''+\tfrac{1}{2}\xi')\psi(\xi''-\tfrac{1}{2}\xi')\,\text{đ}\xi''\,\text{đ}\xi' \\ &= \infty .\end{aligned}$$

This shows that the convolution of two FH co-kernels is not in general an FH co-kernel.

The product G_2G_1 of an FM operator G_2 and an FH operator G_1 is an FH operator, since from

$$G_2G_1u(x)=\int e^{ix\xi}\left[\int\gamma_2(\xi-\xi'';\xi'')\left[\int\gamma_1(\xi''-\xi';\xi')\omega(\xi')\,\text{đ}\xi'\right]\text{đ}\xi''\right]\text{đ}\xi, \qquad (59)$$

we have

$$\|G_2G_1\|_{FH} \le \|G_2\|_{FM}\|G_1\|_{FH}. \qquad (60)$$

We even have from the Algebra Theorem 2.1, the sharper inequality

$$\|G_2G_1\|_{FH} \le \|G_2\|_{FH}\|G_1\|_{FH}. \qquad (61)$$

CHAPTER III

Gårding's Inequality and the Sharp Form of Gårding's Inequality

In this chapter we shall show that a sharp form of Gårding's inequality for pseudo-differential operators as proved by Lax and Nirenberg [4] also holds for pseudo-differential operators of the FH type. We begin by proving the following theorem relating to double symbols, from which follows Gårding's inequality in its ordinary form.

Theorem 3.1. A pseudo-differential operator G with non-negative FH symbol g, with g and g_x of order 0 and g_ξ of order -1, differs from a non-negative FH operator $\tilde{G}$ by the sum of two operators, one of arbitrarily small norm and one which is of order -1 as well as order $-\frac{1}{2},-\frac{1}{2}$. That is to say, for every $\epsilon>0$ there is an H_ϵ and T_ϵ with $||H_\epsilon||_{FH}<\epsilon$ and $||T_\epsilon||_{1,FH}<\infty$, as well as $||T_\epsilon||_{\frac{1}{2},\frac{1}{2};FH}<\infty$, such that

$$(u,(G+H_\epsilon+T_\epsilon)u)\geq 0. \tag{1}$$

Proof. To prove this theorem, we first use convolution operators J_ρ acting on [illegible]$x;\xi)$, which are defined by

$$J_\rho g(x;\xi) = \int g(x;\zeta)\,j_\rho(\xi-\zeta)\,\not{d}\zeta \tag{2}$$
$$= g_\rho(x;\xi).$$

The kernel $j_\rho(\sigma)$ should belong to C^∞, be supported in $|\sigma|\leq\rho$, and have integral 1,

$$\int j_\rho(\sigma)\,\not{d}\sigma=1. \tag{3}$$

To construct such J_ρ, we choose a smooth even function $q_1^2(\sigma)$ with support in the unit ball, $|\sigma|\leq 1$ and having integral 1, i.e.

$$\int q_1^2(\sigma)\not{d}\sigma=1\ . \tag{4}$$

We set $q_\rho(\sigma)=\rho^{-n/2}q_1(\rho^{-1}\sigma)$ and take

$$j_\rho(\sigma)=q_\rho^2(\sigma)\ .$$

By writing

$$J_\rho\gamma(\chi;\xi)-\gamma(\chi;\xi)=\int[\gamma(\chi;\xi-\sigma)-\gamma(\chi;\xi)]j_\rho(\sigma)\not{d}\sigma, \tag{5}$$

we have

$$\begin{aligned}[J_\rho\gamma(\chi;\xi)-\gamma(\chi;\xi)]<\xi>&=\int[\int_0^1\frac{d}{d\alpha}[\gamma(\chi;\xi-\alpha\sigma)<\xi-\alpha\sigma>]d\alpha-\gamma(\chi;\xi-\sigma)[<\xi-\sigma>\\ &\qquad-<\xi>]j_\rho(\sigma)\not{d}\sigma \qquad (6)\\ &=\int[\int_0^1-\sigma\cdot\gamma_{\cdot\xi}(\chi;\xi-\alpha\sigma)<\xi-\alpha\sigma>-\gamma(\chi;\xi-\alpha\sigma)\sigma\cdot(\xi-\alpha\sigma)<\xi-\alpha\sigma>^{-1}d\alpha\\ &\quad-\gamma(\chi;\xi-\alpha\sigma)[<\xi-\sigma>-<\xi>][\frac{<\xi-\sigma>+<\xi>}{<\xi-\sigma>+<\xi>}]]j_\rho(\sigma)\not{d}\sigma\ .\end{aligned}$$

Hence we obtain

$$\begin{aligned}[J_\rho\gamma(\chi;\xi)-\gamma(\chi;\xi)]<\xi>&=-\int[\int_0^1\sigma\cdot\gamma_{\cdot\xi}(\chi;\xi-\alpha\sigma)<\xi-\alpha\sigma>d\alpha\\ &\quad+\int\gamma(\chi;\xi-\alpha\sigma)\sigma\cdot(\xi-\alpha\sigma)<\xi-\alpha\sigma>^{-1}d\alpha+\gamma(\chi;\xi-\alpha\sigma)\sigma\cdot(-\xi+\sigma-\xi)\\ &\quad[<\xi-\sigma>+<\xi>]^{-1}]j(\sigma)\not{d}\sigma\ .\end{aligned} \tag{7}$$

Noting that

$$\frac{|\sigma\cdot(\xi-\alpha\sigma)|}{<\xi-\alpha\sigma>}\le\frac{|\sigma||\xi-\alpha\sigma|}{<\xi-\alpha\sigma>}\le|\sigma|\le\rho \tag{8}$$

and

$$\frac{|\sigma\cdot(-\xi+\sigma-\xi)|}{<\xi-\sigma>+<\xi>}\le\frac{|\sigma|[|\xi|+|\sigma-\xi|]}{<\xi-\sigma>+<\xi>}\le|\sigma|\le\rho, \tag{9}$$

the FH norm of the first term in the right hand side of (7) is

$$\begin{aligned}&\int[\sup_{\substack{\xi\\ \xi'}}|\int[\int_0^1\gamma_{\xi\cdot}(\xi-\xi';\xi'-\alpha\sigma)<\xi'-\alpha\sigma>\cdot\sigma d\alpha]j_\rho(\sigma)\not{d}\sigma|]^{\not{d}\xi'}_{\not{d}\xi} \qquad (10)\\ &\le\rho\int_0^1 d\alpha\sup_{\substack{\xi\\ \xi'}}\int|\gamma_\xi(\xi-\xi';\xi'-\alpha\sigma)|<\xi'-\alpha\sigma>^{\not{d}\xi'}_{\not{d}\xi}\int j_\rho(\sigma)\not{d}\sigma\\ &\le\rho||\gamma_\xi||_{1,FH}\ .\end{aligned}$$

Using inequality (8), the FH norm of the second term in the right hand side of (7) is

$$\sup_{\substack{\xi \\ \xi'}} \int \Big| \int \Big[\int_0^1 \gamma(\xi-\xi';\xi'-\alpha\sigma)\sigma\cdot(\xi'-\alpha\sigma)<\xi'-\alpha\sigma>^{-1} d\alpha \Big] j_\rho(\sigma)\,\text{đ}\sigma \Big|_{\text{đ}\xi}^{\text{đ}\xi'} \qquad (11)$$

$$\leq \rho \int_0^1 d\alpha \sup_{\substack{\xi \\ \xi'}} \int \big|\gamma(\xi-\xi';\xi'-\alpha\sigma)\big|_{\text{đ}\xi}^{\text{đ}\xi'} \int j_\rho(\sigma)\,\text{đ}\sigma$$

$$\leq \rho \| \gamma \|_{FH}.$$

Also, using inequality (9), the FH norm of the third term in the right hand side of (7) is,

$$\sup_{\substack{\xi \\ \xi'}} \int \Big| \int [\gamma(\xi-\xi';\xi'-\alpha\sigma)\sigma\cdot(-\xi'+\sigma-\xi')[<\xi'-\sigma>+<\xi'>]^{-1}] j_\rho(\sigma)\,\text{đ}\sigma]_{\text{đ}\xi}^{\text{đ}\xi'} \qquad (12)$$

$$\leq \rho \sup_{\substack{\xi \\ \xi'}} \int \big|\gamma(\xi-\xi';\xi'-\alpha\sigma)\big|_{\text{đ}\xi}^{\text{đ}\xi'} \int j_\rho(\sigma)\,\text{đ}\sigma$$

$$\leq \rho \| \gamma \|_{FH}.$$

Hence

$$\| G_\rho - G \|_{H^1} \leq \rho\{ \| g_\xi \|_{1,FH} + 2 \| g \|_{FH} \}. \qquad (13)$$

Using inequality 2.1 (51) and noting that

$$\| g_\rho \|_{FH} \leq \| g \|_{FH} \quad \text{and} \quad \| g_{\rho,x} \|_{FH} \leq \| g_x \|_{FH},$$

we have

$$\| G_\rho - G \|_{H^{\frac{1}{2},\frac{1}{2}}} \leq \rho\{ \| g_\xi \|_{1,FH} + 2 \| g \|_{FH} \} + 2 \| g_x \|_{FH}. \qquad (14)$$

Now the simple symbol g_ρ can be rearranged as the double symbol

$$\tilde{g}_\rho(\xi_2|x_1;\xi_1) = \int q_\rho(\xi_2-\zeta) g(x_1;\zeta) q_\rho(\zeta-\xi_1)\,\text{đ}\zeta \qquad (15)$$

which can be shown to possess the following properties:

(i) the symbol $\tilde{g}_\rho$ is a hermitian self-adjoint kernel for each value of the x-variable:

$$\tilde{g}^H_\rho(\xi_2|x_0;\xi_1) = \tilde{g}_\rho(\xi_1|x_0;\xi_2), \qquad (16)$$

(ii) for all values of x_0 the symbol g is a non-negative kernel; i.e. the inequality

$$\iint \overline{\omega}(\xi_2)\tilde{g}_\rho(\xi_2|x_0;\xi_1)\omega(\xi_1)\not{d}\xi_1\not{d}\xi_2 \geq 0 \tag{17}$$

holds for all values of x_0 and all functions ω in L^2. Property (i) is obtained from the fact that $g^H=g$ implies $\tilde{g}_\rho^{HR}=\tilde{g}_\rho$. Property (ii) results from the following

$$\iint \overline{\omega}(\xi_2)\tilde{g}_\rho(\xi_2|x_0;\xi_1)\omega(\xi_1)\not{d}\xi_1\not{d}\xi_2$$

$$= \iint \overline{\omega}(\xi_2)[\int q_\rho(\xi_2-\zeta)g(x_0;\zeta)q_\rho(\zeta-\xi_1)\not{d}\zeta]\omega(\xi_1)\not{d}\xi_1\not{d}\xi_2$$

$$= \int[\int[q_\rho(\xi_2-\zeta)\overline{\omega}(\xi_2)\not{d}\xi_2]g(x_0;\zeta)\int[q_\rho(\zeta-\xi_1)\omega(\xi_1)\not{d}\omega_1]]\not{d}\zeta$$

$$\geq 0.$$

Hence, $\tilde{G}_\rho=\tilde{g}_\rho(D_x|X;D_x)$ is a non-negative operator by Lemma 3.1.

By the Mean Value Theorem, we have

$$|q_\rho(\xi_2-\zeta)-q_\rho(\xi_1-\zeta)| \leq \sup_\sigma|q_\rho'(\sigma)||X| \tag{18}$$

$$= \rho^{-n/2-1}\sup_\sigma|q_1'(\rho^{-1}\sigma)||\chi|.$$

We may then derive the estimate

$$||\tilde{G}_\rho-G_\rho||_{L^2} \leq C\rho^{-n/2-1}||G_x||_{FH} \tag{19}$$

in which the constant C depends only on the choice of the function $q_1(\sigma)$ with support $|\sigma|\leq 1$. We now choose $\rho^{\frac{n}{2}+1} \geq C^{-1}||G_x||_{FH}$ and set $\tilde{G}_\rho-G_\rho=H_\epsilon$ so that $||H_\epsilon||_{FH}<\epsilon$. Also, $T_\epsilon=G_\rho-G$ satisfies $||T_\epsilon||_{-1,FH}<\infty$ and $||T_\epsilon||_{\frac{1}{2},\frac{1}{2};FH}<\infty$. Consequently, $G+H_\epsilon+T_\epsilon=\tilde{G}_\rho$ satisfies inequality (1),

$$(u,(G+H_\epsilon+T_\epsilon)u) \geq 0,$$

and we obtain the desired result.

Since $(u,T\ u)\leq||T_\epsilon||_{\frac{1}{2},\frac{1}{2};FH}||u||^2_{-\frac{1}{2}}$, inequality (1) implies the inequality

$$(u,Gu)+\epsilon\|u\|^2+C_\epsilon\|u\|^2_{-\frac{1}{2}} \geq 0 \tag{20}$$

with $C_\epsilon \geq \|T_\epsilon\|_{\frac{1}{2},\frac{1}{2};FH}$. Specifically we may take $C_\epsilon=\epsilon^{-1}C$. Inequality (20) is Gårding's inequality in its ordinary form.

Under slightly more restrictive circumstances the above statement holds with $H_\epsilon=0$, so that the inequality (20) holds without the term $\epsilon\|u\|^2$. This inequality is referred to as Gårding's inequality in a sharp form.

Theorem 3.2. Suppose the symbol $g(x;\xi)$ is a hermitian and non-negative square matrix; then it admits a hermitian self-reversed non-negative rearrangement $\overset{2}{g}(\xi_2|x_1;\xi_1)$ which generates a self-adjoint non-negative operator $\overset{2}{G}$. The difference $\overset{2}{G}-\mathrm{Sy}G$ of the symmetric part, $\mathrm{Sy}G=\frac{1}{2}G+\frac{1}{2}G^{H}$ from $\overset{2}{G}$ is of the order $(-\frac{1}{2},-\frac{1}{2})$, that is to say, the inequality

$$\|\overset{2}{G}-\mathrm{Sy}G\|_{H^{\frac{1}{2}},\frac{1}{2}} \leq C_0 \tag{21}$$

holds with an appropriate constant C_0. This inequality implies the inequality

$$|(u,(\overset{2}{G}-\mathrm{Sy}G)u)| \leq C_0\|u\|^2_{-\frac{1}{2}} \tag{22}$$

and, consequently, by virtue of $(u,\overset{2}{G}u) \geq 0$, the inequality

$$\mathrm{Re}(u,Gu)+C_0\|u\|^2_{-\frac{1}{2}} \geq 0 . \tag{23}$$

The differentiability conditions imposed on the symbol g are that $g, g_x, g_{xx} \in O(0)$ and $g_{\xi\xi} \in O(-2)$ in the norm $\|\ \|_{FH+}$ which is slightly stronger than the FH norm i.e.

$$\|\gamma\|_{FH+}=\max\{\max_{0\leq\alpha\leq1} \sup_{\substack{\xi\\ \xi'\\ |\sigma|\leq 1}} \int|\gamma(\xi-\xi';\xi'+\alpha(\xi-\xi')-\langle\xi'+\alpha(\xi-\xi')\rangle^{\frac{1}{2}}\sigma)|\genfrac{}{}{0pt}{}{\not{d}\xi'}{\not{d}\xi}\}. \tag{24}$$

Proof. We choose a smooth even function $q^2(\sigma)$ with support in the unit ball, $|\sigma|\leq 1$ and having integral 1,

$$\int q^2(\sigma)\,\text{đ}\sigma = 1\,. \tag{25}$$

In order to construct an approximation to the symbol g we shall use $q^2(\sigma)$ to mollify g in such a way as to obtain the mollified symbol

$$\overset{1}{g}(x;\xi) = \int g(x;\xi-\langle\xi\rangle^{\frac{1}{2}}\sigma)\,q^2(\sigma)\,\text{đ}\sigma\,. \tag{26}$$

It is convenient to rewrite the right hand side of equation (26) by using the change of variable

$$\zeta = \xi-\langle\xi\rangle^{\frac{1}{2}}\sigma \tag{27}$$

and the notation

$$\phi(\xi;\zeta) = \langle\xi\rangle^{-n/4}q(\langle\xi\rangle^{-\frac{1}{2}}[\xi-\zeta]), \tag{28}$$

so that the symbol $\overset{1}{g}(x;\xi)$ becomes

$$\overset{1}{g}(x;\xi) = \int g(x;\zeta)\,\phi^2(\xi;\zeta)\,\text{đ}\zeta\,. \tag{29}$$

If we denote by $\overset{1}{G}$ the pseudo-differential operator with symbol $\overset{1}{g}$, the adjoint $\overset{1}{G}{}^*$ has the symbol

$$\overset{1}{g}{}^{HR}(x;\xi) = \int g^H(x_1;\zeta)\,\phi^2(\xi_2;\zeta)\,\text{đ}\zeta\,. \tag{30}$$

By the assumption that the symbol $g(x;\xi)$ is hermitian,

$$g^H(x;\zeta) = g(x;\zeta), \tag{31}$$

it follows that

$$\begin{aligned}\overset{1}{g}{}^{HR}(x;\xi) &= \int g(x_1;\zeta)\,\phi^2(\xi_2;\zeta)\,\text{đ}\zeta \\ &= \overset{1}{g}(x_1;\xi_2)\,.\end{aligned} \tag{32}$$

Finally, we define the double symbol by

$$\overset{2}{g}(\xi_2|x_1;\xi_1) = \int\phi(\xi_2;\zeta)\,g(x_1;\zeta)\,\phi(\xi_1;\zeta)\,\text{đ}\zeta. \tag{33}$$

One can easily see that $\overset{2}{G}$ is symmetric, i.e. $\overset{2}{G}{}^*=\overset{2}{G}$, since $\overset{2}{g}$ is hermitian and self-reversed:

$$\overset{2}{g}(\xi_2|x_1;\xi_1) = \overset{2}{g}(\xi_1|x_1;\xi_2)\,. \tag{34}$$

Denote by g^R and $\overset{1}{g}{}^R$ the symbols of the adjoints G^* and $\overset{1}{G}{}^*$. In order to facilitate a systematic proof of the sharp form of Gårding's inequality, we shall divide the proof into four lemmas.

Lemma 3.1. (Friedrichs [1])

$$\overset{2}{G} \geq 0, \tag{35}$$

Proof. The symbol $\overset{2}{g}(\xi_2|x_0;\xi_1)$ is a non-negative kernel for each value x_0 of x,

$$\iint \overline{\omega}(\xi_2) e^{-i\xi_2 x_0} \overset{2}{g}(\xi_2|x_0;\xi_1) e^{i\xi_1 x_0} \omega(\xi_1) \bar{d}\xi_1 \bar{d}\xi_2 \tag{36}$$

$$= \iiint \overline{\omega}(\xi_2) e^{-i\xi_2 x_0} \phi(\xi_2;\zeta) g(x_0;\zeta) \phi(\xi_1;\zeta) e^{i\xi_0 x_0} \omega(\xi_1) \bar{d}\xi \bar{d}\xi_1 \bar{d}\xi_2 \geq 0.$$

To conclude the proof we integrate with respect to x_0 and ζ; hence inequality (35) results in the form

$$(\omega, \overset{2}{\Gamma}\omega) = \iint \overline{\omega}(\xi_2) \overset{2}{\gamma}(\xi_2|\xi_2-\xi_1;\xi_1) \omega(\xi_1) \bar{d}\xi_1 \bar{d}\xi_2 \geq 0. \tag{37}$$

Lemma 3.2.

$$||\overset{1}{G}-G||_{H^1} \leq C_A ||G_{\xi\xi}||_{2,FH+} \tag{38}$$

and

$$||\overset{1}{G}{}^R-G^R||_{H^{1,0}} \leq C_A ||G_{\xi\xi}||_{2,FH+} \tag{39}$$

Proof. We use the fact that $\int q^2(\sigma)\bar{d}\sigma=1$ to write the difference $\overset{1}{g}-g$ in the form

$$\overset{1}{g}(x;\xi)-g(x;\xi) = \int [g(x;\xi-\langle\xi\rangle^{\frac{1}{2}}\sigma)-g(x;\xi)]q^2(\sigma)\bar{d}\sigma, \tag{40}$$

and expand g in the integral in a Taylor series in $\langle\xi\rangle^{\frac{1}{2}}\sigma$ with remainder of second order in integral form

$$\overset{1}{g}-g = -\int g_{\xi}(x;\xi)\cdot\sigma q^2(\sigma)\bar{d}\sigma\langle\xi\rangle^{\frac{1}{2}}$$
$$+\iint_0^1 (1-\mu) g_{\xi\cdot\xi}(x;\xi-\mu\langle\xi\rangle^{\frac{1}{2}})\cdot\sigma\cdot\sigma q^2(\sigma)\,d\mu\,\bar{d}\sigma\langle\xi\rangle. \tag{41}$$

The first term involving the factor $\langle\xi\rangle^{\frac{1}{2}}$ is zero since

$q^2(\sigma)$ is even. Now multiplying equation (41) by $\langle\xi\rangle$ and going to co-kernels, we can rewrite the difference $(\overset{1}{\gamma}-\gamma)\langle\xi\rangle$ as follows

$$[\overset{1}{\gamma}(\chi;\xi)-\gamma(\chi;\xi)]\langle\xi\rangle=\int\int_0^1 (1-\mu)\gamma_{\xi\cdot\xi\cdot}(\chi;\xi-\mu\langle\xi\rangle^{\frac{1}{2}}\sigma)\langle\xi-\mu\langle\xi\rangle^{\frac{1}{2}}\sigma\rangle^2$$
$$\langle\xi-\mu\langle\xi\rangle^{\frac{1}{2}}\sigma\rangle^{-2}\langle\xi\rangle^2\cdot\sigma\cdot\sigma q^2(\sigma)\,d\mu d\sigma . \qquad (42)$$

Since $q(\sigma)$ has support in the unit ball $|\sigma|\le 1$, we may use the inequality

$$\int\sigma\cdot\sigma q^2(\sigma)\,\bar{d}\sigma\le 1 \qquad (43)$$

and the constant

$$C_A = \sup_{\xi,|\mu\sigma|\le 1} \langle\xi-\mu\langle\xi\rangle^{\frac{1}{2}}\sigma\rangle^{-2}\langle\xi\rangle^2 \qquad (44)$$

to obtain the estimate

$$|\overset{1}{\gamma}(\xi-\xi';\xi')-\gamma(\xi-\xi';\xi')|\langle\xi'\rangle\le C_A\int_0^1 (1-\mu)\,|\gamma_{\xi\xi}(\xi-\xi';\xi'-\mu\langle\xi'\rangle^{\frac{1}{2}}\sigma)|\cdot$$
$$\langle\xi'-\mu\langle\xi'\rangle^{\frac{1}{2}}\sigma\rangle^2 d\mu , \qquad (45)$$

Hence

$$||\overset{1}{\gamma}-\gamma||_{H^1}\le C_A\sup_{0\le\mu\le 1}\sup_{\substack{\xi\\ \xi'}}\int\int_0^1 (1-\mu)\,|\gamma_{\xi\xi}(\xi-\xi';\xi'-\mu\langle\xi'\rangle^{\frac{1}{2}}\sigma)|$$
$$\langle\xi'-\mu\langle\xi'\rangle^{\frac{1}{2}}\sigma\rangle^2 d\mu \;\substack{\bar{d}\xi' \\ \bar{d}\xi'} \qquad (46)$$

$$\le C_A||\gamma_{\xi\xi}||_{2,FH+} . \qquad (47)$$

This inequality is the required result. Similarly, inequality (39) follows.

<u>Lemma 3.3</u>.

$$||\overset{2}{G}-Sy\overset{1}{G}||_{H^{\frac{1}{2},\frac{1}{2}}} \le C_B||G_{xx}||_{FH+} . \qquad (48)$$

Proof. We note that the operators $\overset{1}{G}$ and $\overset{1}{G}^R$ are associated with the contracted multiple symbols $\overset{2}{g}(\xi_1|x_1;\xi_1)$ and $\overset{2}{g}(\xi_2|x_1;\xi_2)$ and hence with the co-kernels $\overset{2}{\gamma}(\xi_1|\xi_2-\xi_1;\xi_1)$ and $\overset{2}{\gamma}(\xi_2|\xi_2-\xi_1;\xi_2)$. This can be seen from equations (29) and (33) which we rewrite as

$$\overset{1}{g}(x;\xi) = \int g(x;\zeta)q^2(\langle\xi\rangle^{-\frac{1}{2}}(\xi-\zeta))\,đ\zeta\langle\xi\rangle^{-n/2} \qquad (49)$$

and

$$\overset{2}{g}(\xi_2|x_1;\xi_1) = \langle\xi_2\rangle^{-n/4}\int q(\langle\xi_2\rangle^{-\frac{1}{2}}(\xi_2-\zeta))g(x_1;\zeta)$$
$$q(\langle\xi_1\rangle^{\frac{1}{2}}(\zeta-\xi_1))\,đ\zeta\langle\xi_1\rangle^{-n/4}. \qquad (50)$$

Recalling that $\overset{2}{\gamma}(\xi_2|\xi_2-\xi_1;\xi_1)$ is self-reversed,

$$\overset{2}{\gamma}(\xi_2|\xi_2-\xi_1;\xi_1) = \overset{2}{\gamma}(\xi_1|\xi_2-\xi_1;\xi_2), \qquad (51)$$

we see that $\overset{1}{G}+\overset{1}{G}^R-2\overset{2}{G}$ is associated with the co-kernel

$$\overset{2}{\gamma}(\xi|\chi;\xi)+\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)-2\overset{2}{\gamma}(\xi+\chi|\chi;\xi)$$
$$= \overset{2}{\gamma}(\xi|\chi;\xi)-\overset{2}{\gamma}(\xi+\chi|\chi;\xi)+\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)-\overset{2}{\gamma}(\xi|\chi;\xi+\chi), \qquad (52)$$

where $\xi=\xi_1$ and $\chi=\xi_2-\xi_1$.

We multiply equation (52) by $\langle\xi+\chi\rangle^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}}$, then add, subtract and group terms to obtain finite Taylor expansions in χ involving only $\overset{2}{\gamma}$ and derivatives of $\overset{2}{\gamma}$:

$$\langle\xi+\chi\rangle^{\frac{1}{2}}\{\overset{2}{\gamma}(\xi|\chi;\xi)+\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)-2\overset{2}{\gamma}(\xi+\chi|\chi;\xi)\}\langle\xi\rangle^{\frac{1}{2}}$$
$$= \langle\xi+\chi\rangle^{\frac{1}{2}}\{\overset{2}{\gamma}(\xi|\chi;\xi)-\overset{2}{\gamma}(\xi+\chi|\chi;\xi)+\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)$$
$$-\overset{2}{\gamma}(\xi|\chi;\xi+\chi)\}\langle\xi\rangle^{\frac{1}{2}}$$
$$= \{[\overset{2}{\gamma}(\xi|\chi;\xi)\langle\xi\rangle^{\frac{1}{2}}-\overset{2}{\gamma}(\xi+\chi|\chi;\xi)\langle\xi+\chi\rangle^{\frac{1}{2}}]\langle\xi\rangle^{\frac{1}{2}}$$
$$+\overset{2}{\gamma}(\xi|\chi;\xi)[\langle\xi+\chi\rangle^{\frac{1}{2}}-\langle\xi\rangle^{\frac{1}{2}}]\langle\xi\rangle^{\frac{1}{2}}\}$$
$$+\{[\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)\langle\xi+\chi\rangle^{\frac{1}{2}}-\overset{2}{\gamma}(\xi|\chi;\xi+\chi)\langle\xi\rangle^{\frac{1}{2}}]\langle\xi+\chi\rangle^{\frac{1}{2}}$$
$$+\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)[\langle\xi\rangle^{\frac{1}{2}}-\langle\xi+\chi\rangle^{\frac{1}{2}}]\langle\xi+\chi\rangle^{\frac{1}{2}}\} \qquad (53)$$
$$= \{\langle\xi+\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)\langle\xi+\chi\rangle^{\frac{1}{2}}-\langle\xi+\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi+\chi|\chi;\xi)\langle\xi\rangle^{\frac{1}{2}}$$
$$-\langle\xi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi|\chi;\xi+\chi)\langle\xi+\chi\rangle^{\frac{1}{2}}+\langle\xi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi|\chi;\xi)\langle\xi\rangle^{\frac{1}{2}}\}$$
$$-\langle\xi+\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi+\chi|\chi;\xi+\chi)\{\langle\xi+\chi\rangle^{\frac{1}{2}}-\langle\xi\rangle^{\frac{1}{2}}\}$$
$$+\langle\xi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi|\chi;\xi)\{\langle\xi+\chi\rangle^{\frac{1}{2}}-\langle\xi\rangle^{\frac{1}{2}}\} \qquad (54)$$
$$= \int_0^1\int_0^1 \frac{d}{d\nu}\frac{d}{d\mu}\langle\xi+\mu\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi+\xi\chi|\chi;\xi+\nu\chi)\langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\mu\, d\nu$$
$$-\int_0^1\frac{d}{d\mu}\langle\xi+\mu\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}(\xi+\mu\chi|\chi;\xi+\nu\chi)\,d\mu\int_0^1\frac{d}{d\nu}\langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\nu\,. \qquad (55)$$

In order that an estimate for the FH norm of this expression be obtained in terms of the norm of $\overset{2}{\gamma}$ and the derivatives of $\overset{2}{\gamma}$, we note that

$$\overset{2}{\gamma}(\xi|\chi;\xi)\ \frac{d}{d\mu}\ \langle\xi+\mu\chi\rangle^{\frac{1}{2}} = \tfrac{1}{2}\overset{2}{\gamma}_{ix\cdot}(\xi|\chi;\xi)\langle\xi+\mu\chi\rangle^{-3/2}\cdot(\xi+\mu\chi) \qquad (56)$$

where the last factor is bounded by 1, i.e.

$$|\langle\xi+\mu\chi\rangle^{-3/2}(\xi+\mu\chi)| \le 1\ . \qquad (57)$$

Hence performing the differentiations with respect to μ and ν, we have

$$\int_0^1\int_0^1 [2(\tfrac{1}{2})\frac{(\xi+\mu\chi)\cdot}{\langle\xi+\mu\chi\rangle^{3/2}}\ \overset{2}{\gamma}_{\cdot ix\cdot ix}(\xi+\mu\chi|\chi;\xi+\nu\chi)(\tfrac{1}{2})\frac{\cdot(\xi+\nu\chi)}{\langle\xi+\nu\chi\rangle^{3/2}}$$

$$+\ \tfrac{1}{2}\frac{(\xi+\mu\chi)\cdot}{\langle\xi+\mu\chi\rangle^{3/2}}\ \overset{2}{\gamma}_{\cdot ix(ix\cdot\xi_2)}(\xi+\mu\chi|\chi;\xi+\nu\chi)\langle\xi+\nu\chi\rangle^{\frac{1}{2}}$$

$$+\ \tfrac{1}{2}\langle\xi+\mu\chi\rangle^{\frac{1}{2}}\overset{2}{\gamma}_{(ix\cdot\xi_1)ix\cdot}(\xi+\mu\chi|\chi;\xi+\nu\chi)\frac{(\xi+\nu\chi)}{\langle\xi+\nu\chi\rangle^{3/2}}$$

$$+\ \langle\xi+\mu\chi\rangle^{\frac{1}{2}}\ \gamma_{(ix\cdot\xi_2)(ix\cdot\xi_1)}(\xi+\mu\chi|\chi;\xi+\nu\chi)\langle\xi+\nu\chi\rangle^{\frac{1}{2}}]\,d\mu\,d\nu$$

$$-[\int_0^1\ \tfrac{1}{2}\frac{(\xi+\mu\chi)\cdot}{\langle\xi+\mu\chi\rangle^{3/2}}\ \gamma_{\cdot ix}(\xi+\mu\chi|\chi;\xi+\nu\chi)\,d\mu\ \int_0^1\frac{d}{d\nu}\ \langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\nu$$

$$+\int_0^1\langle\xi+\mu\chi\rangle^{\frac{1}{2}}\ \frac{d}{d\mu}\ \overset{2}{\gamma}(\xi+\mu\chi|\chi;\xi+\nu\chi)\int_0^1\ \frac{d}{d\nu}\ \langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\nu\ . \qquad (58)$$

We find that

$$\langle\xi+\chi\rangle^{\frac{1}{2}}|\overset{1}{\gamma}+\overset{1}{\gamma}^{R}-2\overset{2}{\gamma}|\langle\xi\rangle^{\frac{1}{2}} \le \tfrac{1}{2}\int_0^1\int_0^1|\overset{2}{\gamma}_{xx}(\xi+\mu\chi|\chi;\xi+\nu\chi)|\,d\mu\,d\nu$$

$$+\ \int_0^1\int_0^1\langle\xi+\mu\chi\rangle^{\frac{1}{2}}|\overset{2}{\gamma}_{(ix\cdot\xi_2)(ix\cdot\xi_1)}(\xi+\mu\chi;\chi;\xi+\nu\chi)|\langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\mu\,d\nu$$

$$+\ \int_0^1\int_0^1\langle\xi+\mu\chi\rangle^{\frac{1}{2}}|\overset{2}{\gamma}_{(ix\cdot\xi_1)ix}(\xi+\mu\chi|\chi;\xi+\nu\chi)|\,d\mu\,d\nu$$

$$+\ \int_0^1\int_0^1|\overset{2}{\gamma}_{ix(ix\cdot\xi_2)}(\xi+\mu\chi|\chi;\xi+\nu\chi)|\langle\xi+\nu\chi\rangle^{\frac{1}{2}}d\mu\,d\nu\ . \qquad (59)$$

Before we bound the L^2 norm of the four terms in the right hand side of inequality (59) by the FH norm of the simple co-kernel $\gamma(\chi;\xi)$ and derivatives of γ, we obtain such a bound for $\overset{2}{\Gamma}\omega$,

$$||\overset{2}{\Gamma}\omega||_{L^2} \le ||\gamma||_{+}||\omega||_{L^2} \ . \tag{60}$$

This estimate is obtained by a repeated use of Schwarz's inequality:

$$||\overset{2}{\Gamma}\omega||^2_{L^2} = \int|\int[\phi(\xi_2;\zeta)\gamma(\xi_2-\xi_1;\zeta)\phi(\xi_1;\zeta){\not d}\zeta]\omega(\xi_1){\not d}\xi_1|^2{\not d}\xi_2 \tag{61}$$

$$\le \int[\int[\int\phi^2(\xi_2;\zeta)|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta]^{\frac{1}{2}}[\int\phi^2(\xi_1;\zeta)$$

$$|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta]^{\frac{1}{2}}|\omega(\xi_1)|{\not d}\xi_1]^2{\not d}\xi_2 \tag{62}$$

$$\le \int[\int\int\phi^2(\xi_2;\zeta)|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta{\not d}\xi_1][\int\int\phi^2(\xi_1;\zeta)$$

$$|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta|\omega(\xi_1)|^2{\not d}\xi_1]{\not d}\xi_2 \tag{63}$$

$$\le [\sup_{\xi_3}\int\int\phi^2(\xi_3;\zeta)|\gamma(\xi_3-\xi_1;\zeta)|{\not d}\zeta{\not d}\xi_1]$$

$$[\sup_{\xi_4}\int\int\phi^2(\xi_4;\zeta)|\gamma(\xi_2-\xi_4;\zeta)|{\not d}\zeta{\not d}\xi_2]\int|\omega(\xi_1)|^2{\not d}\xi_1 \tag{64}$$

$$= [\sup_{\xi_3}\int\int q^2(\eta)|\gamma(\xi_3-\xi_1;\xi_3-\langle\xi_3\rangle^{\frac{1}{2}}\eta)|{\not d}\eta d\xi_1]$$

$$[\sup_{\xi_4}\int\int q^2(\eta)|\gamma(\xi_2-\xi_4;\xi_4-\langle\xi_4\rangle^{\frac{1}{2}}\eta)|{\not d}\eta{\not d}\xi_2]$$

$$||\omega||^2_{L^2} \tag{65}$$

$$\le ||\gamma||^2_{+}||\omega||^2_{L^2} \le ||\gamma||^2_{FH}||\omega||^2_{L^2} \ . \tag{66}$$

We now bound the first term of (59). Writing γ for $\frac{1}{2}\gamma_{xx}$ we have:

$$\int|\int\int_0^1\int_0^1 \overset{2}{\gamma}(\xi_2+\mu(\xi_2-\xi_1)|\xi_2-\xi_1;\xi_1+\nu(\xi_2-\xi_1))d\mu d\nu \ \omega(\xi_1){\not d}\xi_1|^2{\not d}\xi_2$$

$$\le \int\{\int_0^1\int_0^1\int[\int\phi^2(\xi_1+\mu(\xi_2-\xi_1);\zeta)|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta]^{\frac{1}{2}}$$

$$[\int\phi^2(\xi_1+\nu(\xi_2-\xi_1;\zeta)|\gamma(\xi_2-\xi_1;\zeta)|{\not d}\zeta]^{\frac{1}{2}}$$

$$|\omega(\xi_1)|{\not d}\xi_1 \ d\mu d\nu\}^2{\not d}\xi_2 \tag{67}$$

$$\leq \int\{\int_0^1\int_0^1[\int\int\phi^2(\xi_1+\mu(\xi_2-\xi_1);\zeta)\,|\gamma(\xi_2-\xi_1;\zeta)|\,\not{d}\zeta\,\not{d}\xi_1]^{\frac{1}{2}}$$

$$[\int\int\phi^2(\xi_1+\nu(\xi_2-\xi_1);\zeta)\,|\gamma(\xi_2-\xi_1;\zeta)|\,\not{d}\zeta\,|\omega(\xi_1)|^2\not{d}\xi_1]^{\frac{1}{2}}d\mu d\nu\}^2\not{d}\xi_2 \qquad (68)$$

$$\leq \int\{\int_0^1\int_0^1\int\int\phi^2(\xi_1+\mu(\xi_2-\xi_1);\zeta)\,|\gamma(\xi_2-\xi_1;\zeta)|\,\not{d}\zeta\,\not{d}\xi_1]\,d\mu d\nu\}$$

$$\{\int_0^1\int_0^1\int\int\phi^2(\xi_1+\nu(\xi_2-\xi_1);\zeta)\,|\gamma(\xi_2-\xi_1;\zeta)|\,\not{d}\zeta\,|\omega(\xi_1)|^2\not{d}\xi_1 d\mu d\nu\}\not{d}\xi_2 \qquad (69)$$

$$\leq \{\int_0^1\sup_{\xi_3}\int\int q^2(\eta)\,|\gamma(\xi_3-\xi_1;\xi_1+\mu(\xi_3-\xi_1)$$

$$-<\xi_1+\mu(\xi_3-\xi_1)>^{\frac{1}{2}}\eta)|\,\not{d}\eta\,\not{d}\xi_1 d\mu\}\ .$$

$$\{\int_0^1\sup_{\xi_4}\int\int q^2(\eta)\,|\gamma(\xi_2-\xi_4;\xi_4+\nu(\xi_2-\xi_4)$$

$$-<\xi_4+\nu(\xi_2-\xi_4)>^{\frac{1}{2}}\eta)|\,\not{d}\eta\,\not{d}\xi_2 d\nu\}\cdot\int|\omega(\xi_1)|^2\not{d}\xi_1 \qquad (70)$$

$$= ||\gamma||^2_{FH+}||\omega||^2_{L^2} \qquad (71)$$

Hence we have the FH bound

$$\tfrac{1}{2}||\gamma_{xx}||_{FH+}$$

for the first term on the right hand side of (59).

We now estimate the second term on the right hand side of (59) in a similar way

$$\int|\int\int_0^1\int_0^1<\xi_1+\mu(\xi_2-\xi_1)>^{\frac{1}{2}}\int\phi_{\xi}(\xi_1+\mu(\xi_2-\xi_1);\zeta)\gamma_{\cdot ixix\cdot}(\xi_2-\xi_1;\zeta)$$

$$\phi_{\xi}(\xi_1+\nu(\xi_2-\xi_1);\zeta)\,\not{d}\zeta<\xi_1+\nu(\xi_2-\xi_1)>^{\frac{1}{2}}d\mu d\nu\ \omega(\xi_1)\,\not{d}\xi_1|^2\not{d}\xi_2$$

$$\leq \{\int_0^1\sup_{\xi_3}\int\int<\xi_1+\mu(\xi_3-\xi_1)>\phi^2_{\xi}(\xi_1+\mu(\xi_3-\xi_1);\zeta)$$

$$|\gamma_{xx}(\xi_3-\xi_1;\zeta)|\,\not{d}\zeta\,\not{d}\xi_1 d\mu\}\cdot$$

$$\{\int_0^1\sup_{\xi_4}\int\int<\xi_4+\nu(\xi_2-\xi_4)>\phi^2_{\xi}(\xi_4+\nu(\xi_2-\xi_4);\zeta)$$

$$|\gamma_{xx}(\xi_2-\xi_4;\zeta)|\,\not{d}\zeta\,\not{d}\xi_2 d\nu\}\cdot\int|\omega(\xi_1)|^2\not{d}\xi_1\ . \qquad (72)$$

We compute ϕ_{ξ},

$$\partial_\xi \phi(\xi,\eta) = <\xi>^{-n/4}\{-\tfrac{m}{4}<\xi>^{-2}\xi q(\eta)+<\xi>^{-\frac{1}{2}}q_\eta(\eta) - \tfrac{1}{2}<\xi>^{-2}\xi q_{\eta\cdot}(\eta)\cdot\eta \tag{73}$$

where $\eta = <\xi>^{-\frac{1}{2}}[\xi-\zeta]$.

We notice that the worst term in this expression is the second one, $<\xi>^{-\frac{1}{2}}q_\eta$, involving $<\xi>^{-\frac{1}{2}}$ while the first term involves only $<\xi>^{-2}$; the same is true for the third one since $q_{\eta\cdot}(\eta)\cdot\eta$ is a smooth function with bounded support. Hence, $<\xi>^{\frac{1}{2}}\partial_\xi\phi(\xi;\zeta)$ is bounded in ξ and has σ-support in the unit ball.

Accordingly, there exists a constant C_ϕ such that

$$\sup_\xi \int [<\xi>^{\frac{1}{2}}\partial_\xi \phi(\xi,\eta)]^2 \not{d}\eta = C_\phi^2 . \tag{74}$$

Using equation (74), we find that inequality (72) becomes

$$C_\phi^4||\gamma_{xx}||^2_{FH+}||\omega||^2_{L^2} . \tag{75}$$

Hence, we have the FH bound

$$C_\phi^2||\gamma_{xx}||_{FH+} \tag{76}$$

Similarly,

$$\int\left|\iint_0^1\int_0^1 <\xi_1+\mu(\xi_2-\xi_1)>^{\frac{1}{2}}\int\phi_\xi(\xi_1+\mu(\xi_2-\xi_1);\zeta)\gamma_{ix\cdot ix}(\xi_2-\xi_1;\zeta) \phi(\xi_1+\nu(\xi_2-\xi_1);\zeta)\not{d}\zeta<\xi_1+\nu(\xi_2-\xi_1)>^{\frac{1}{2}}d\mu d\nu\ \omega(\xi_1)\not{d}\xi_1\right|^2 d\xi_2 \tag{77}$$

$$\le C_\phi^2||\gamma_{xx}||_{FH+}||\omega||_{L^2} ,$$

$$\left|\left|\int_0^1\int_0^1\gamma_{ix(ix\cdot\xi_2)}(\xi+\mu\chi|\chi;\xi+\nu\chi)<\xi+\nu\chi>^{\frac{1}{2}}d\mu d\nu\right|\right|_{L^2} \le C_\phi||\gamma_{xx}||_{FH+} \tag{78}$$

Combining these estimates, we find

$$||\overset{1}{\gamma}+\overset{1}{\gamma}{}^R-2\overset{2}{\gamma}||_{H^{\frac{1}{2},\frac{1}{2}}} \le [C_\phi^2+2C_\phi+\tfrac{1}{2}]||\gamma_{xx}||_{FH+} \tag{79}$$

$$\le C_B||\gamma_{xx}||_{FH+} , \tag{80}$$

where

$C_B = \frac{1}{2}C_\phi^2 + C_\phi + \frac{1}{4}$,

from which inequality (48) follows.

<u>Lemma 3.4</u>

$$||Sy(G-\overset{1}{G})||_{H^{\frac{1}{2},\frac{1}{2}}} \le \tfrac{1}{2}||G-\overset{1}{G}||_{H^1} + \tfrac{1}{2}||G^R-\overset{1}{G}{}^R||_{H^{1,0}} + 2||G_x||_{FH+}. \qquad (81)$$

<u>Proof</u>. We shall use the inequalities 2(51) and 2(52),

$$I(\tfrac{1}{2},\tfrac{1}{2}): \quad \Big|\,||g||_{H^{\frac{1}{2},\frac{1}{2}}} - ||g||_{H^1}\Big| \le ||g_x||_{FH} , \qquad (82)$$

$$I^R(\tfrac{1}{2},\tfrac{1}{2}): \quad \Big|\,||g^R||_{H^{\frac{1}{2},\frac{1}{2}}} - ||g^R||_{H^{1,0}}\Big| \le ||g_x^R||_{FH} \le ||g_x||_{FH+}. \qquad (83)$$

We apply these inequalities to the symbol $g-\overset{1}{g}$ in place of g, observing that

$$\underset{\xi}{\sup_{\xi'}}\int|\overset{1}{\gamma}(\chi;\xi')|^{\text{đ}\xi}_{\text{đ}\xi'} \le \underset{\xi}{\sup_{\xi'}}\iint[\,|\gamma(\xi-\xi';\xi'-\sigma\langle\xi'\rangle^{\frac{1}{2}})|\; q^2(\sigma)\,\text{đ}\sigma]^{\text{đ}\xi}_{\text{đ}\xi'} \qquad (84)$$

$$\le \sup_{\substack{\xi' \\ \xi \\ |\sigma|\le 1}}\int|\gamma(\xi-\xi';\xi'-\langle\xi\rangle^{\frac{1}{2}}\sigma)|^{\text{đ}\xi'}_{\text{đ}\xi} \qquad (85)$$

$$\le ||\gamma||_{FH+} . \qquad (86)$$

Similarly, $||\overset{1}{\gamma}_x||_{FH} \le ||\gamma_x||_{FH+}$. Consequently we have

$$||(g-\overset{1}{g})_x||_{FH} \le 2||g_x||_{FH+} . \qquad (87)$$

Hence the lemma is proved.

Combining these lemmas we obtain

$$||SyG-G^2||_{H^{\frac{1}{2},\frac{1}{2}}} \le C_A||G_{\xi\xi}||_{2,FH+} + C_B||G_{xx}||_{FH+} + 2||G_x||_{FH+} , \qquad (88)$$

from which the statement of Theorem 3.2 follows.

CHAPTER IV

Properties of the space E_0^{FH}

In this chapter we shall examine the significance of the extra condition (Γ'_{FH}):

$$\lim_{\chi'\to 0}\Big[\sup_{\substack{\xi'\\ \xi}}\int\left|\gamma(\xi-\xi'+\chi';\xi')-\gamma(\xi-\xi';\xi')\right|\begin{matrix} d\xi \\ \\ d\xi'\end{matrix}\Big] = 0, \qquad (1)$$

in regard to how restrictive it is on elements of G_0^{FH}. For simplicity we consider only the FH norm defined by (1) or 2.1(1).

Sormani [9] has given examples of functions in G_0 that do not obey the FM(Γ) condition

$$\lim_{\chi'\to 0}\Big[\int\sup_{\xi}\left|\gamma(\chi+\chi';\xi)-\gamma(\chi;\xi)\right|d\chi\Big] = 0, \qquad (2)$$

thus proving that (Γ) is a true restriction on G_0. The subset of co-kernels in G_0 that obeyed (Γ) formed a Banach sub-algebra and was denoted by E_0. The space E_0 and hence the condition (Γ) was characterized by the following approximation theorem: a co-kernel $\gamma(\chi;\xi')$ is an element of E_0 if and only if γ can be approximated arbitrarily closely with respect to the FM norm by functions in G_0 of the form

$$\sum_{i=1}^{N}\alpha_i(\chi)\beta_i(\xi), \qquad (3)$$

where each $\alpha_i(\chi)$ is absolutely integrable and each $\beta_i(\xi)$ is bounded and measurable.

In this section we shall examine the properties of the subset of co-kernels E_0^{FH} in G_0^{FH} that obey the extra condition (Γ_{FH}). They will be seen to form a non-commutative Banach sub-algebra under the operator product. Let E_0^{FH} be the subset of G_0^{FH} that consists of all functions $\gamma(\chi;\xi)\epsilon G_0^{FH}$ that obey the extra conditions (Γ_{FH}). Thus, E_0^{FH} consists of all bounded and measurable functions $\gamma(\xi-\xi';\xi')$ which satisfies:

(i) $\sup_{\substack{\xi' \\ \xi}} \int |\gamma(\xi-\xi';\xi')| \Big|^{\not{d}\xi}_{\not{d}\xi'} = ||\gamma||_{FH} < \infty$,

(ii) $\lim_{\chi\to 0} ||\gamma^{\chi'}-\gamma|| = 0$, where $\gamma^{\chi'} = \gamma(\xi-\xi'+\chi';\xi)$.

The first fact concerning E_0^{FH} that should be established is that E_0^{FH} is a proper subset of G_0^{FH}. This can be accomplished with an example of a function in G_0^{FH} that do not obey the (Γ_{FH}) condition. Consider the following function of a real variable,

$$\psi(\sigma) = \begin{cases} \pi & \text{for } -1\leq\sigma\leq 1 \\ 0 & \text{otherwise} \end{cases},$$

and let

$$\gamma(\chi;\xi') = \langle\xi'\rangle\psi(\langle\xi'\rangle[\chi+\tfrac{1}{2}\xi']).$$

Clearly, γ is in G_0^{FH} since

$$||\gamma||_+ = 1$$

and

$$||\gamma||_- < 4.$$

Since the χ- support of $\gamma(\chi;\xi')$ is of width $\langle\xi\rangle^{-1}$, it follows that for fixed $\chi'\neq 0$, the supports of $\gamma(\chi+\chi';\xi')$ and $\gamma(\chi;\xi')$ are disjoint for $|\xi'|$ sufficiently large; namely $\langle\xi'\rangle > 2\,|\chi'|^{-1}$.

Hence

$$||\gamma^{\chi'}-\gamma||_{+} = 2$$

does not tend to zero as $\chi'\to 0$.

Let us investigate some of the properties of the subspace E_0^{FH}.

Lemma 4.1 If $\gamma_1 \epsilon E_0^{FH}$ and $\gamma_2 \epsilon E_0^{FH}$, then $\lambda\gamma_1+\mu\gamma_2 \epsilon E_0^{FH}$ for any complex numbers λ and μ.

Lemma 4.2 If $\gamma_1 \epsilon G_0^{FH}$ and $\gamma_2 \epsilon E_0$, then $\gamma_1 * \gamma_2 \epsilon E_0^{FH}$.

Proof. By inequality 2(58)

$$\gamma_1 * \gamma_2 \epsilon G_0^{FH}.$$

Now,

$$||(\gamma_1\ \gamma_2)^{\chi'}-\gamma_1\ \gamma_2||_{-} = \sup_{\xi}\int\left|\int[\gamma_1(\xi-\xi''-\chi';\xi')-\gamma_1(\xi-\xi'';\xi'] .\ \gamma_2(\xi''-\xi';\xi')\not{d}\xi''\right|\not{d}\xi'$$

$$=\sup_{\xi}\int\left|\int\gamma_1(\xi-\xi''-\xi';\xi')[\gamma_2(\xi''-\chi';\xi')- \gamma_2(\xi'';\xi')]\not{d}\xi''\right|\not{d}\xi'$$

$$\leq\int\sup_{\xi}\int|\gamma_1(\xi-\xi''-\xi';\xi')|\not{d}\xi' \sup_{\xi'''}|\gamma_2(\xi''-\chi';\xi''')- \gamma_2(\xi'';\xi''')|\not{d}\xi''$$

$$=||\gamma_1||_{-}||\gamma_2^{\chi'}-\gamma_2||_{FM}$$

tends to zero as $\chi'\to 0$.

Similarly,

$$||\gamma_1 * \gamma_2^{\chi'}-\gamma_1 * \gamma_2||_{+} \leq ||\gamma_1||_{+}||\gamma_2^{\chi'}-\gamma_2||_{FM}$$

tends to zero as $\chi'\to 0$.

Theorem 4.1 If $\gamma_1(\chi;\xi)\epsilon G_0^{FH}$ and $\gamma_2(\chi;\xi)\epsilon E_0^{FH}$ are two co-kernels corresponding to the operators G_1 and G_2, then the function

$$\gamma_{21} = \int \gamma_2(\xi-\xi'';\xi'')\gamma_1(\xi''-\xi';\xi')\,\not{d}\xi'' ,$$

is a co-kernel in E_0^{FH} that corresponds to the operator product G_2G_1.

<u>Proof</u>. This follows from the Algebra Theorem 2.1 and from the fact that

$$\gamma_{21}^{\chi'} = \gamma_2^{\chi'}\gamma_1 ,$$

namely,

$$\begin{aligned} \|\gamma_{21}^{\chi'}-\gamma_{21}\|_{FH} &= \|(\gamma_2^{\chi'}-\gamma_2)\gamma_1\|_{FH} , \\ &\le \|\gamma_2^{\chi'}-\gamma_2\|_{FH}\|\gamma_1\|_{FH} . \end{aligned}$$

tends to zero as $\chi'\to 0$. Thus $\gamma_{21}\in E_0^{FH}$.

Since

$$\gamma_{21}^{\chi'} = \gamma_2^{\chi'}\gamma_1 = \gamma_2\gamma_1^{\chi'} ,$$

the above theorem states that E_0^{FH} is a two sided ideal in G_0^{FH}.

<u>Lemma 4.3</u> The space E_0^{FH} is closed under the operator product in G_0^{FH}.

<u>Proof</u>. Let $\gamma\in G_0^{FH}$ be the limit of a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of elements in E_0^{FH}. Then

$$\|\gamma^{\chi'}-\gamma\|_{\pm} \le \|\gamma^{\chi'}-\gamma_n^{\chi'}\|_{\pm} + \|\gamma_n^{\chi'}-\gamma_n\|_{\pm} + \|\gamma_n-\gamma\|_{\pm} . \qquad (20)$$

The first and the third terms can be made arbitrarily small by taking n large enough and then the second term can be made arbitrarily small by taking $|\chi'|$ small enough. This proves the lemma.

<u>Lemma 4.4</u> The space E_0^{FH} is a non-commutative Banach algebra.

<u>Proof</u>. Since E_0^{FH} is a closed subspace of a non-commutative Banach algebra G_0^{FH}, it follows that E_0^{FH} is a Banach space which, by Remark 2.1, is non-commutative.

It is an open question how to characterize a subset of symbols in $\hat{G}^{FH}$ which admits inverses in $\hat{G}^{FH}$. Because of the loss of the convolution product in G^{FH} and the non-commutativity of the operator product, the Gelfand theory used by Sormani [9] or the methods used by Friedrichs [1] for FM symbols do not carry over to the FH case.

BIBLIOGRAPHY

[1] FRIEDRICHS, K.O. Pseudo-Differential Operators, An Introduction. Lecture Notes, Revised Edition, with the assistance of Rémi Vaillancourt. Courant Inst. Math. Sci. New York University, 1970.

[2] HÖRMANDER, L. Linear Partial Differential Operators, Springer-Verlag Inc. New York 1969.

[3] KOHN, J.J. & NIRENBERG, L. An algebra of pseudo-differential operators. Comm. Pure Appl. Math. Vol. 18 (1965), pp. 269-305.

[4] LAX, P.D. & NIRENBERG, L. On stability for difference schemes; a sharp form of Gårding's inequality. Comm. Pure Appl. Math. Vol. 19 (1966), pp. 473-492.

[5] NATHANSON, I.P. Theory of Functions of a Real Variable. Vol. 1, (translated by Leo F. Boron), Ungar, New York, 1955.

[6] NIRENBERG, L. Pseudo-Differential Operators. Global Analysis, (Proc. Sym. Pure Math. Vol. 16, A.M.S., Providence, R.I., 1970), pp. 149-167.

[7] ROYDEN, H.L. Real Analysis. The MacMillan Company. Collier-MacMillan Ltd. London, 1968.

[8] RUBINSTEIN, Zalman. A Course in Ordinary and Partial Differential Equations. Academic Press. New York, 1969.

[9] SORMANI, M.J. Inversion of Symbols Associated with Fourier Maximum Pseudo-Differential Operators. Thesis. New York University, 1971.

[10] VAILLANCOURT, Rémi. A Simple Proof of LAX-NIRENBERG Theorem. Comm. Pure Appl. Math., Vol. 23 (1970), pp. 151-163.

[11] KUMANO-GO, H. Pseudo-differential operators and the uniqueness of the Cauchy problem. Comm. Pure Appl. Math. Vol. 22 (1969), pp. 73-129.

[12] HÖRMANDER, L. Pseudo-differential operators and non-elliptic boundary value problems. Annals. Math. Vol. 83 (1966), pp. 269-305.