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INFORMATION THEORETIC ASPECTS OF THE RELIABILITY OF BINARY SWITCHING NETWORKS

by

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Submitted to the Department of Electrical Engineering in partial fulfilment of the requirements for the degree of Master of Science

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of Electrical Engineering

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Chairman of the examining committee

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ABSTRACT

The subject of this thesis is the analysis of binary switching networks.

Many schemes for improving the performance of an unreliable binary switching element (gate) have been advanced. Only those relying on the combination of many unreliable gates into a more reliable network - performing the original function - are here considered. Some of the well-known schemes are analysed and reduced to a common format.

This particular format was selected because it facilitated the application of information theoretic concepts to reliability improvement.

Information theoretic concepts, applied under certain restrictions, resulted in the development of a lower bound, LB, on the unreliability (probability of error) of binary switching networks.

ACKNOWLEDGEMENTS

The author wishes to thank Professor G.S. Glinski, Chairman of the Department of Electrical Engineering, who supervised this research project.

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1. Introduction

A theoretically perfect switching element is, in practice, invariably imperfect or "noisy". A standard method of reducing the probability of error, at this switching element's output, is to replace this single faulty element by a network of faulty elements. Moore and Shannon [12], von Neumann [15], et al. [1, 14], have advanced different schemes using different mathematical approaches to achieve this reduction in error. By utilizing a common language to describe these schemes, using this language to compare a single element with an equivalent net, and then employing the concepts of entropy and information originating in information theory, it is hoped to gain further insight into the reliability of switching functions.

Although attention will be concentrated on the <u>two-input, single-output, binary switching element</u>, hereafter called a <u>gate</u>, much of the following will be equally applicable to the many-input, many-output, binary case. The extension to this more general case will be indicated as it occurs.

2. Probabilistic Properties of Perfect Gates

A perfect gate may be characterized by a function f, such that

$$f:\ A \times B \qquad C, \qquad (2.1)$$

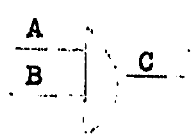
where

$$A = B = C = \quad 0, 1 \quad ,$$

and A, B are input spaces,

C is the output space.

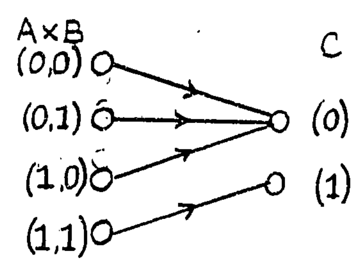
It may be noted that this definition of f is also the definition of a Boolean function. A common tabular representation of such a function is the "Canonical Sum of Fundamental Products" [11] table. This table is constructed by listing the elements of the domain A x B, by considering each element as a binary number, and then arranging these numbers in ascending order. To each element of A x B, the function f then assigns a single element of the range C. An equivalent "transition diagram" [13] is obtained by graphically displaying the elements of both domain and range and indicating relations by connecting lines (See Fig. 1, b) and c), for an illustration).



a)

A x B	C
(0,0)	(0)
(0,1)	(0)
(1,0)	(0)
(1,1)	(1)

b)



c)

3 Equivalent Representations of the Boolean "AND" Function:
a) Gate Symbol, b) Tabular Form, c) Transition Diagram.

Fig. 1

Although, to this point, all definitions have been "deterministic" rather than "probabilistic", a more general case is attained, if the "transition diagram", Fig. 1c), is reconsidered in probabilistic terms as follows:

1) A conditional probability may be defined as the probability of occurrence of a specific output event, given a specific input event. A line between a specific input and output, therefore, indicates a conditional probability of 1. The absence of a line, accordingly, indicates a conditional probability of 0.

In other words, given the occurrence of an input event (i, j), the presence of a line to an output (k) indicates that $p\left\{(k) / (i, j)\right\} = 1$. The absence of a line to k, consequently indicates $p\left\{(k) / (i, j)\right\} = 0$.

2) To each input event (i, j) assign a probability $p\left\{(i, j)\right\}$, which, for convenience, will be called r_{ij}.

These conventions lead logically to the representation of the transition diagram by the following matrix formula:

$$Q = RP, \tag{2.2}$$

where $R = \left[r_{00}, r_{01}, r_{10}, r_{11} \right]$ is the input probability distribution vector,

P is a matrix whose elements are of the form $p\left\{(k) / (i, j)\right\}$, where (k) designates the column and (i, j) the row. For convenient tabulation, both the row and column designators may be considered to be binary numbers, starting at zero. It can be seen that P characterizes the gate.

$Q = \left[q_0, q_1 \right]$ is the output probability distribution vector, with q_k the probability of occurrence of the output event (k).

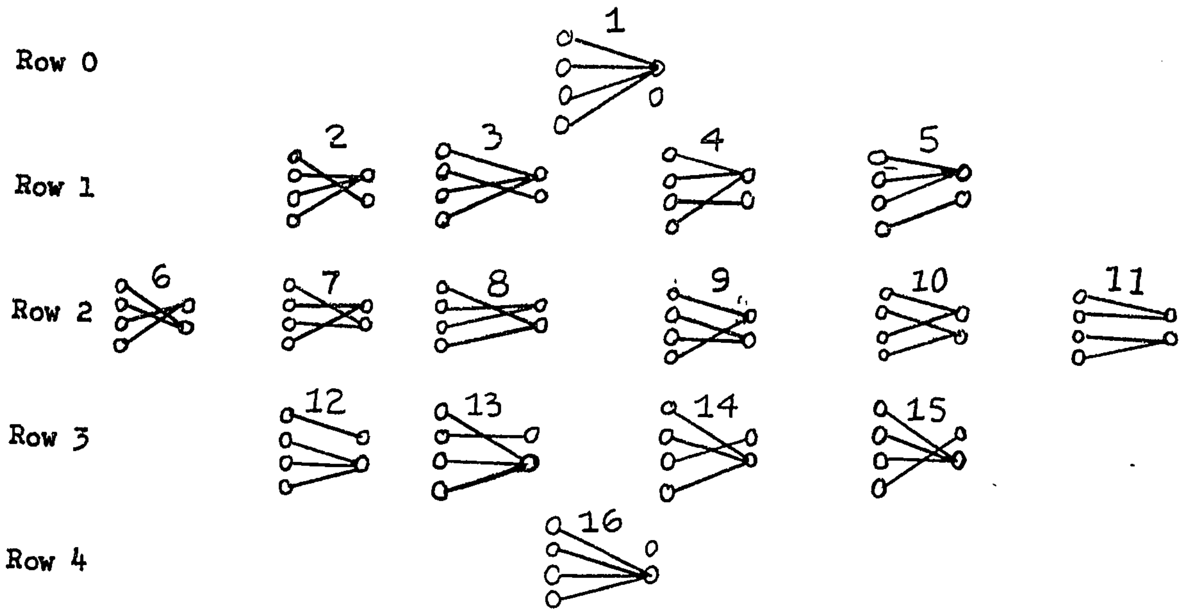
Take, as an example of P, the P of an "AND" gate. It can be written as

$$P = \begin{bmatrix} p\left\{(0) / (0,0)\right\} & p\left\{(1) / (0,0)\right\} \\ p\left\{(0) / (0,1)\right\} & p\left\{(1) / (0,1)\right\} \\ p\left\{(0) / (1,0)\right\} & p\left\{(1) / (1,0)\right\} \\ p\left\{(0) / (1,1)\right\} & p\left\{(1) / (1,1)\right\} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the elements of P, for the sake of conciseness, will henceforth be called $p_{(ij)(k)}$ with (k) and (ij) as above. Note also, that for any row of P, $p_{(ij)(0)} + p_{(ij)(1)} = 1$.

It should be noted that (2.2) is, by definition, applicable to the m-input, n-output switching network. In this case, R would be a row vector of 2^m elements, Q a row vector of 2^n elements, and P a $2^m \times 2^n$ matrix.

To conclude this section, the 16 possible gates will be listed for later use. The reason for the particular classification of Fig. 2) [4], will also emerge later. For the moment, it is sufficient to note that the gates are arranged in rows with each gate in a row having the same number of lines terminating in an output of 1. For example, all gates in row 3 have 2 lines terminating in a 1.



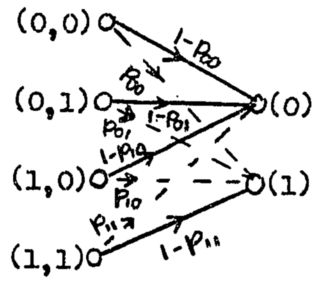
Classification of Gates.

Fig. 2.

3. Probabilistic Properties of Imperfect Gates

A perfect gate may be said to give the "correct" output when any input (i,j) is applied. An imperfect or noisy gate, given input (i,j) will be said to give rise to the "incorrect" output with a probability p_{ij}.

(2.2), therefore, still applies, but the 1's and 0's of P are replaced, respectively, by the appropriate $1 - p_{ij}$'s and p_{ij}'s (see fig. 3 for an illustration). In other words, $P_{(ij)(k)}$ is now either p_{ij} or $1-p_{ij}$. For example, in Fig. 3), $P_{(01)(0)}$ is $1-p_{01}$.



a)

$$Q = RP$$

$$\text{where } P = \begin{bmatrix} 1-p_{00} & p_{00} \\ 1-p_{01} & p_{01} \\ 1-p_{10} & p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix}$$

b)

a) Transition Diagram and, b) Matrix Formula for a "Noisy" AND Gate.

Fig. 3.

It should be noted that the probability of error, P(e), of an imperfect element is given by

$$P(e) = \sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}p_{ij}$$

$$= R\Phi^t,$$

where $\Phi = \left[p_{00}, p_{01}, p_{10}, p_{11} \right]$

and superscript "t" indicates the transpose.

4. Networks of Perfect Gates

In all gate reliability improvement schemes that will be studied, a single faulty gate will be replaced by a network. For purposes of comparison with the original gate, this network must be reduced to an equivalent gate. In order to develop a standard method of reduction, attention will be concentrated on a simple type of network, the cascade. The cascade may be defined as a configuration of gates where the last or output gate is designated the n^{th} rank, the gate or gates directly feeding the n^{th} rank are designated the $n-1^{th}$ rank, and this process is continued till the source, which feeds the first rank, is reached. Since, by construction, there is no feedback, and any rank may only be fed by elements of the previous rank, time may be neglected, and combinatorial methods may be applied. The reduction method, however, must take cognizance of the fact that the output of a gate in the k^{th} rank depends, in general, on the outputs of gates in all $k-1$ previous ranks.

To develop this reduction method, first consider a cascade of perfect gates. All gates will be designated by the superscript rs, where r indicates the rank and s the row, starting from the top. The object, now, is to find a matrix characterization for the cascade. For this cascade to replace a single gate, the cascade's characteristic matrix must be identical with the single gate's matrix. In other words, if the cascade may be reduced to an equivalent gate characterized by P_{EQUIV}, then (2.2) may be written as

$$Q = RP_{EQUIV}. \tag{4.1}$$

Since, in the two cases, R is unchanged, and Q must be unchanged in order for the cascade to be equivalent to the single gate, therefore, P_{EQUIV} must equal P.

The reduction method details will be developed in the next section which treats the more general case of cascades composed of noisy gates.

It is possible, however, to point out an interesting aspect of cascades of noiseless gates by consideration of Fig. 4. The single "AND" gate of Fig. 4a) is replaced by the simple, 3-gate cascade of Fig. 4b), which also performs the Boolean "AND" function. Although the cascade leaves the input-output relation of the simple gate unchanged, the input probability distribution to gate 21, an "AND" gate, is no longer R. In other words, if we assume that source R has 2 statistically independent outputs, A and B, such that

$$r_{00} = p(A=0,\ B=0) = p(A=0)\ p(B=0)$$

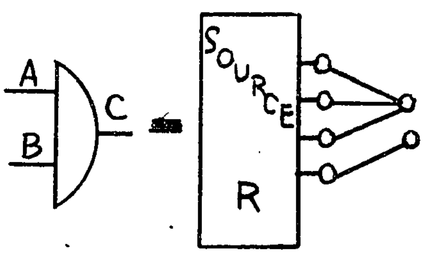
$$r_{01} = p(A=0,\ B=1) = p(A=0)\ p(B=1)$$

$$r_{10} = p(A=1,\ B=0) = p(A=1)\ p(B=0)$$

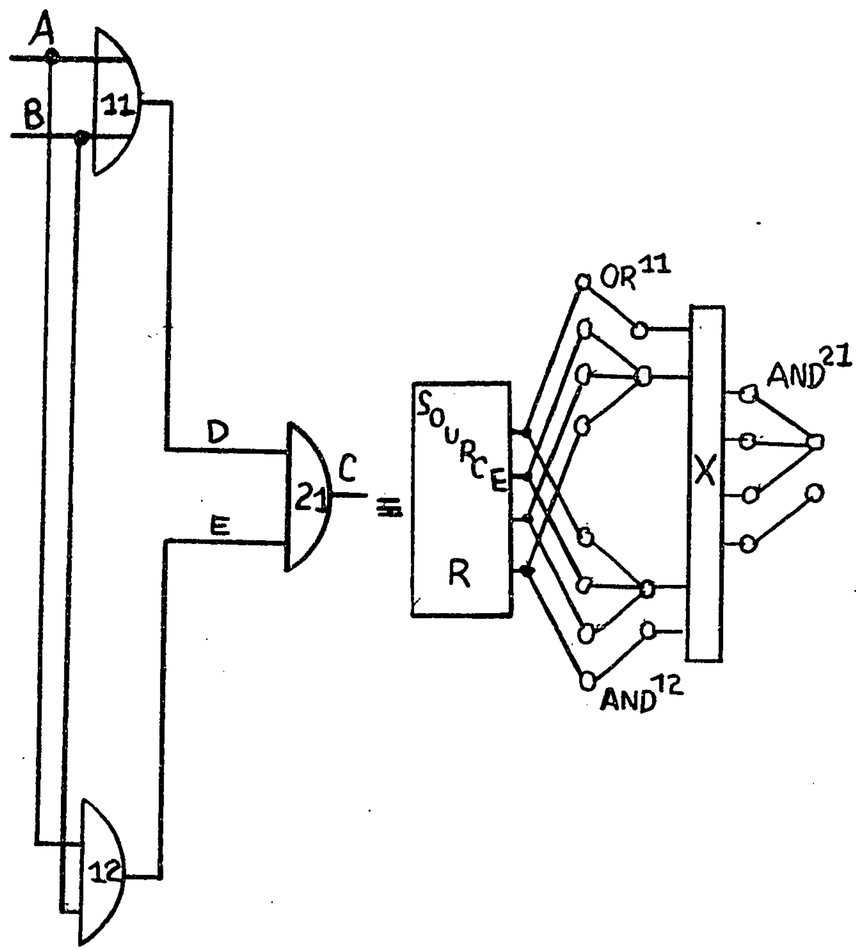
$$r_{11} = p(A=1,\ B=1) = p(A=1)\ p(B=1)$$

then both the single gate and the cascade have an output $Q = \boxed{r_{00} + r_{01} + r_{10},\ r_{11}}$. Although the "AND" gate of Fig. 4a) has an input distribution of R, the "AND" gate (gate 21) of Fig. 4b) has an input distribution of $R^{21} = \boxed{r_{00},\ 0,\ r_{01} + r_{10},\ r_{11}}$. To sum up, the cascade of perfect gates in Fig. 4b), may be considered to be a means of changing the input probability distribution to an "AND" gate from statistical independence to a particular statistical dependence [7].

In passing, note that in the schematic of the cascade in Fig. 4b), a box with a cross product sign is placed between the first and second ranks. This symbolism is used since, from a set theory standpoint, the set of inputs to gate 21 is the cross product



a) AND Gate



b) An AND Gate Cascade Equivalent

Fig 4)

of the 2 previous output sets (the output sets of gates 11 and 12).

5. Networks of Imperfect Gates

A network reduction method, the Z-matrix method, will be developed in this section. Each gate of the network is characterized by its own P which corresponds to its own conditional probability of error structure. The Z-matrix method will relate, transform and reduce these P's to the P_{EQUIV} of (4.1). The superscript rs on P_{EQUIV} will indicate that the network up to, and including gate rs has been reduced to a P_{EQUIV}.

In order that the P_{EQUIV} may be obtained, a step-by-step reduction, starting at rank 1 and working to the right, must be carried out. As the reduction proceeds to the k^{th} rank, a matrix characterizing the effect of the previous k-1 ranks, is required. This matrix, premultiplying the P matrix of a k^{th} rank gate, will result in a P_{EQUIV}. We will call this matrix the Z matrix.

To illustrate the method of obtaining this Z matrix and hence P_{EQUIV}, consider a noisy network of the type shown in Fig. 4b):

1) Gate 11 is represented by P^{11} and $Q^{11} = RP^{11}$ (5.2)

2) Gate 12 is represented by P^{12} and $Q^{12} = RP^{12}$ (5.3)

3) Gate 21 is represented by P^{21} and $Q^{21} = R^{21}P^{21}$. (5.4)

The problem is to rewrite this last relation in the form

$$Q^{21} = RZ^{21}P^{21} = RP^{21}_{EQUIV} \tag{5.5}$$

where $Z^{21}P^{21}$ is, then the required P^{21}_{EQUIV} and Z^{21} is the Z matrix sought.

It can be seen that Z^{21} is a 4x4 matrix, since, by definition, any R is a 1x4 matrix.

The elements of Z^{21}, and their significance, are derived as follows:

1) Since the element $p^{11}_{(ij)(k)}$ of P^{11} indicates the probability of output (k) occurring, <u>source</u> input (i,j) having occurred,

2) Since the element $p^{12}_{(ij)(l)}$ of P^{12} indicates the probability of output (l) occurring, <u>source</u> input (i,j) having occurred,

3) Then the element $z^{21}_{(ij)(kl)}$ of Z^{21} indicates the probability of (k,l) occurring at the input to 21, <u>source</u> input (i,j) having occurred.

i.e. $z^{21}_{(ij)(kl)} = f(p^{11}_{(ij)(k)}, p^{12}_{(ij)(l)})$ where the function is determined by the statistical relations between the elements of P^{11} and P^{12}. These elements, of course, characterize the conditional probabilities of error of gates 11 and 12. As an example, consider the "Noisy" case of Fig. 4b).

$$P^{11} = \begin{bmatrix} 1-p^{11}_{00} & p^{11}_{00} \\ p^{11}_{01} & 1-p^{11}_{01} \\ p^{11}_{10} & 1-p^{11}_{10} \\ p^{11}_{11} & 1-p^{11}_{11} \end{bmatrix} \qquad P^{12} = \begin{bmatrix} 1-p^{12}_{00} & p^{12}_{00} \\ 1-p^{12}_{01} & p^{12}_{01} \\ 1-p^{12}_{10} & p^{12}_{10} \\ p^{12}_{11} & 1-p^{12}_{11} \end{bmatrix}$$

In terms of previous definitions, choose two elements, say,

$p^{11}_{(01)(0)} = p^{11}_{01}$, and $p^{12}_{(01)(1)} = p^{12}_{01}$, to be the elements of our sample calculation. Assume that these errors occur with statistical independence. $z^{21}_{(01)(01)}$ is then a simple product of these elements, i.e.

$$z^{21}_{(01)(01)} = p^{11}_{(01)(0)} \cdot \quad p^{12}_{(01)(1)} = p^{11}_{01} \cdot p^{12}_{01},$$

and Z^{21} =

$$\begin{bmatrix} (1-p^{11}_{00})\cdot(1-p^{12}_{00}) & (1-p^{11}_{00})\cdot\ p^{12}_{00} & p^{11}_{00}\cdot(1-p^{12}_{00}) & p^{11}_{00}\cdot\ p^{12}_{00} \\ p^{11}_{01}\cdot(1-p^{12}_{01}) & p^{11}_{01}\cdot\ p^{12}_{01} & (1-p^{11}_{01})\cdot(1-p^{12}_{01}) & (1-p^{11}_{01})\cdot\ p^{12}_{01} \\ p^{11}_{10}\cdot(1-p^{12}_{10}) & p^{11}_{10}\cdot\ p^{12}_{10} & (1-p^{11}_{10})\cdot(1-p^{12}_{10}) & (1-p^{11}_{10})\cdot\ p^{12}_{10} \\ p^{11}_{11}\cdot\ p^{12}_{11} & p^{11}_{11}\cdot(1-p^{12}_{11}) & (1-p^{11}_{11})\cdot\ p^{12}_{11} & (1-p^{11}_{11})\cdot(1-p^{12}_{11}) \end{bmatrix}.$$

Z is a matrix representing the effect of all gates prior to the last one. Each component of the source, R, is split into 4 parts by the row of the Z matrix with the same row index. Each column index designates the input of the last gate on which the source component acts.

6. Moore and Shannon Scheme for Increased Reliability

Moore and Shannon outlined a method for reliability improvement of networks in which certain general properties of relays were specified by certain parameters. Iteration techniques based on these parameters were developed and the resulting larger networks were made, accordingly, more reliable [5] [12].

The basic element, the relay, has the following properties:

1) Given an input signal of "0" (i.e. relay coil un-energized) the output will be "0" (i.e. contact open) with a probability

1-c and the output will be "1" (i.e. contact closed), with a probability c.

2) Given an input "1" (i.e. coil energized), the output will be "1" with probability a, and "0" with probability 1-a.

It can be seen that this characterization of a relay as a 1-input, 1-output noisy switching function is consistent with the definitions of section 3.

Consider now 2 perfect relays, the first with input space $A = \{0,1\}$, and the second with input space $B = \{0,1\}$, where the events of A and B are statistically independent, i.e.

$$\text{if } p(A=1) = r_A \text{ and therefore } p(A=0) = 1-r_A,$$

$$\text{and if } p(B=1) = r_B \text{ and therefore } p(B=0) = 1-r_B,$$

then the two relays together may be considered to have a source input space AxB, where

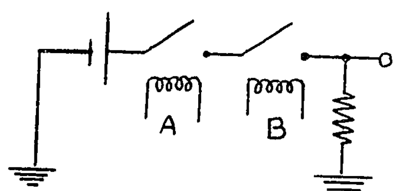
$$\begin{aligned} p(0,0) &= (1-r_A)(1-r_B) = r_{00}, \\ p(0,1) &= (1-r_A)(r_B) = r_{01}, \\ p(1,0) &= (r_A)(1-r_B) = r_{10}, \\ p(1,1) &= (r_A)(r_B) = r_{11}. \end{aligned}$$

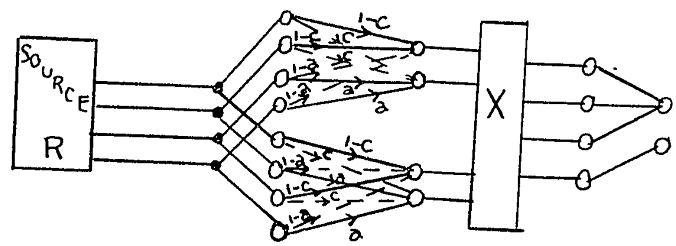
Using this source AxB, the relays may now be considered to be 2-input, 1-output switching functions. If in the 1-input characterization, the first relay was represented by $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, in the 2-input characterization it is represented by $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. Similarly, if the second relay was represented by $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it is now represented by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the manner of section 3, the appropriate p_{ij}'s are inserted to give the noisy relay representation.

If the two relays are "ANDED" together, the "ANDING" function is a result of the geometry of the network. If, as in Moore and Shannon, errors are assumed to occur only in the relay contacts, the "AND" gate is, accordingly, noiseless (see Fig. 5).



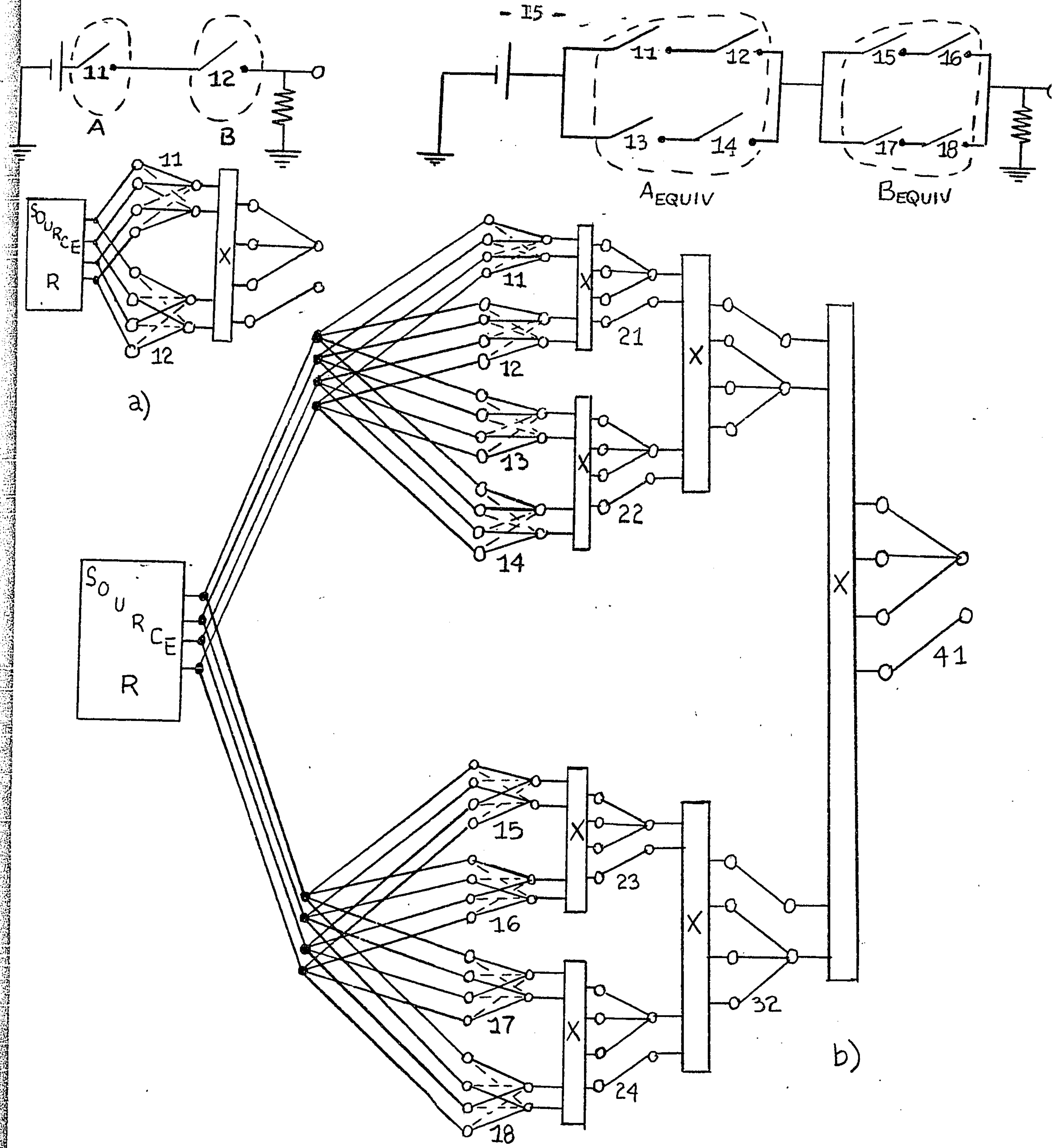


a) Relay Representation b) Transition Diagram Representation. Noisy "AND" Function with Relays.

Fig. 5.

It is apparent that no matter how complex the geometry of the network, if only one source (i.e. 2 inputs, or in functional terms, two arguments) is used, then the equivalent transition diagram will have noisy gates in the first rank only. This, in fact, is characteristic of the Moore and Shannon method.

As an illustration of the Moore and Shannon method, the following example will be worked out in detail. This is done in detail in order to fix the Z-matrix method firmly in mind, to provide a basis for comparison with other reliability increasing methods, and to provide a basis for later entropy applications.



a) Original AND   b) Equivalent Cascaded AND

A Moore and Shannon Iteration of an Original "AND" Net

Fig 6)

EXAMPLE

1) The network of Fig. 5a) is iterated by replacing each relay by 4 relays as indicated in Fig. 6b). As before, it is assumed that the error occurrences in the gates are statistically independent. It is also assumed that the relay coils are in parallel.

2) The original network's $P_{EQUIV} = Z^{21}P^{21}$ is P_{EQUIV} =

$$\begin{bmatrix} (1-p^{11}_{00})(1-p^{12}_{00}) & (1-p^{11}_{00})\ (p^{12}_{00}) & (p^{11}_{00})(1-p^{12}_{00}) & (p^{11}_{00})\ (p^{12}_{00}) \\ (1-p^{11}_{01})\ (p^{12}_{01}) & (1-p^{11}_{01})(1-p^{12}_{01}) & (p^{11}_{01})\ (p^{12}_{01}) & (p^{11}_{01})(1-p^{12}_{01}) \\ (p^{11}_{10})(1-p^{12}_{10}) & (p^{11}_{10})\ (p^{12}_{10}) & (1-p^{11}_{10})(1-p^{12}_{10}) & (1-p^{11}_{10})\ (p^{12}_{10}) \\ (p^{11}_{11})\ (p^{12}_{11}) & (p^{11}_{11})(1-p^{12}_{01}) & (1-p^{11}_{11})\ (p^{12}_{11}) & (1-p^{11}_{11})(1-p^{12}_{11}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where in the Z^{21} matrix, the following identifications are made

$$p^{11}_{00} = p^{11}_{01} = c \qquad p^{11}_{10} = p^{11}_{11} = 1-a$$

$$p^{12}_{00} = p^{12}_{10} = c \qquad p^{12}_{01} = p^{12}_{11} = 1-a,$$

i.e.

$$P_{EQUIV} = \begin{bmatrix} 1-c^2 & c^2 \\ 1-ca & ca \\ 1-ca & ca \\ 1-a^2 & a^2 \end{bmatrix} \equiv \begin{bmatrix} 1-p_{00} & p_{00} \\ 1-p_{01} & p_{01} \\ 1-p_{10} & p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix}.$$

and, therefore, the equivalent matrix has the following conditional probabilities:

$$p_{00} = c^2$$

$$p_{01} = ca$$

$$p_{10} = ca$$

$$p_{11} = 1-a^2$$

3) In the equivalent cascaded "AND"

$$p_{00}^{11} = p_{00}^{12} = p_{00}^{13} = p_{00}^{14} = c \qquad p_{10}^{11} = p_{10}^{12} = p_{10}^{13} = p_{10}^{14} = 1-a$$

$$p_{01}^{11} = p_{01}^{12} = p_{01}^{13} = p_{01}^{14} = c \qquad p_{11}^{11} = p_{11}^{12} = p_{11}^{13} = p_{11}^{14} = 1-a$$

$$p_{00}^{15} = p_{00}^{16} = p_{00}^{17} = p_{00}^{18} = c \qquad p_{01}^{15} = p_{01}^{16} = p_{01}^{17} = p_{01}^{18} = 1-a$$

$$p_{10}^{15} = p_{10}^{16} = p_{10}^{17} = p_{10}^{18} = c \qquad p_{11}^{15} = p_{11}^{16} = p_{11}^{17} = p_{11}^{18} = 1-a$$

By construction

$$Z^{21}P^{21} = Z^{22}P^{22}$$

$$Z^{23}P^{23} = Z^{24}P^{24}$$

$$Z^{21}P^{21} = Z^{22}P^{22} =$$

$$\begin{bmatrix} (1-c)^2 & (1-c)c & c(1-c) & c^2 \\ (1-c)^2 & (1-c)c & c(1-c) & c^2 \\ (1-a)^2 & (1-a)a & a(1-a) & a^2 \\ (1-a)^2 & (1-a)a & a(1-a) & a^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-c^2 & c^2 \\ 1-c^2 & c^2 \\ 1-a^2 & a^2 \\ 1-a^2 & a^2 \end{bmatrix} \equiv \begin{bmatrix} 1-p_{00} & p_{00} \\ 1-p_{01} & p_{01} \\ p_{10} & 1-p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix} \equiv P^{21}_{EQUIV} = P^{22}_{EQUIV}.$$

i.e.

$$p^{21}_{00} = p^{21}_{01} = c^2 \qquad p^{21}_{10} = p^{21}_{11} = 1-a^2$$

$$p^{22}_{00} = p^{22}_{01} = c^2 \qquad p^{22}_{10} = p^{22}_{11} = 1-a^2.$$

In a similar fashion it can be shown that

$$P^{23}_{EQUIV} = P^{24}_{EQUIV} = \begin{bmatrix} 1-c^2 & c^2 \\ 1-a^2 & a^2 \\ 1-c^2 & c^2 \\ 1-a^2 & a^2 \end{bmatrix} \equiv \begin{bmatrix} 1-p_{00} & p_{00} \\ p_{01} & 1-p_{01} \\ 1-p_{10} & p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix}$$

i.e.

$$p^{23}_{00} = p^{23}_{10} = c^2 \qquad p^{23}_{10} = p^{23}_{01} = 1-a^2$$

$$p^{24}_{00} = p^{24}_{10} = c^2 \qquad p^{24}_{01} = p^{24}_{11} = 1-a^2 .$$

Repetition of this procedure for gates of rank 3, yields:

$$Z^{31}P^{31} = \begin{bmatrix} (1-c^2)^2 & (1-c^2)^2 & c^2(1-c^2) & c^4 \\ (1-c^2)^2 & (1-c^2)c^2 & c^2(1-c^2) & c^4 \\ (1-a^2)^2 & (1-a^2)a^2 & a^2(1-a^2) & a^4 \\ (1-a^2)^2 & (1-a^2)a^2 & a^2(1-a^2) & a^4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(2c^2-c^4) & 2c^2-c^4 \\ 1-(2c^2-c^4) & 2c^2-c^4 \\ (1-a^2)^2 & 1-(1-a^2)^2 \\ (1-a^2)^2 & 1-(1-a^2)^2 \end{bmatrix} \equiv P^{31}_{EQUIV}$$

i.e. $p^{31}_{00} = p^{31}_{01} = 2c^2-c^4 \qquad p^{31}_{10} = p^{31}_{11} = (1-a^2)^2$

and $Z^{32}P^{32} =$

$$\begin{bmatrix} (1-c^2)^2 & (1-c^2)^2 & c^2(1-c^2) & c^4 \\ (1-a^2)^2 & (1-a^2)a^2 & a^2(1-a^2) & a^4 \\ (1-c^2)^2 & (1-c^2)c^2 & c^2(1-c^2) & c^4 \\ (1-a^2)^2 & (1-a^2)a^2 & a^2(1-a^2) & a^4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(2c^2-c^4) & 2c^2-c^4 \\ (1-a^2)^2 & 1-(1-a^2)^2 \\ (1-(2c^2-c^4) & 2c^2-c^4 \\ (1-a^2)^2 & 1-(1-a^2)^2 \end{bmatrix} \equiv P^{32}_{EQUIV},$$

i.e. $p^{32}_{00} = p^{32}_{10} = 2c^2-c^4 \qquad p^{32}_{01} = p^{32}_{11} = (1-a^2)^2.$

Finally $Z^{41}P^{41} =$

$$\begin{bmatrix} (1-c^2)^4 & (1-c^2)^2(2c^2-c^4) & (2c^2-c^4)(1-c^2)^2 & (2c^2-c^4)^2 \\ (1-c^2)^2(1-a^2)^2 & (1-c^2)^2(2a^2-a^4) & (2c^2-c^4)(1-a^2)^2 & (2c^2-c^4)(2a^2-a^4) \\ (1-a^2)^2(1-c^2)^2 & (1-a^2)^2(2c^2-c^4) & (2a^2-a^4)(1-c^2)^2 & (2a^2-a^4)(2c^2-c^4) \\ (1-a^2)^4 & (1-a^2)^2(2a^2-a^4) & (2a^2-a^4)(1-a^2)^2 & (2a^2-a^4)^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1-(2c^2-c^4)^2 & (2c^2-c^4)^2 \\ 1-(2c^2-c^4)(2a^2-a^4) & (2c^2-c^4)(2a^2-a^4) \\ 1-(2c^2-c^4)(2a^2-a^4) & (2c^2-c^4)(2a^2-a^4) \\ 1-(2a^2-a^4)^2 & (2a^2-a^4)^4 \end{bmatrix} \equiv \begin{bmatrix} 1-p_{00} & p_{00} \\ 1-p_{01} & p_{01} \\ 1-p_{10} & p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix} = P_{EQUIV}.$$

Identifying the p_{ij} of the equivalent cascaded "AND" leads to

$$p_{00} = (2c^2-c^4)^2$$

$$p_{01} = (2c^2-c^4)(2a^2-a^4)$$

$$p_{10} = (2c^2-c^4)(2a^2-a^4)$$

$$p_{11} = 1-(2a^2-a^4)^2$$

4) The calculations are completed by comparing the p_{ij}'s of the original with the equivalent cascade.

It can be seen that the c of the original is replaced by $2c^2-c^4$ in the equivalent cascade, and that the a of the original is replaced by $2a^2-a^4$ in the equivalent cascade. This means that the relay network replacing the original relay network has p_{ij}'s less than the original relay's p_{ij}'s if $c<.618$ and $a>.618$. Under these conditions, therefore, the original A and B relays have been improved. If, in addition, ca of the original is more than the equivalent network's ca, then $P(e)_{EQUIV} < P(e)_{ORIGINAL}$ for all R.

While the approach presented here (in contrast with Moore and Shannon's original approach) is cumbersome and only indicates <u>whether</u>

probability of error has been improved, not <u>how</u> to improve this reliability, it does provide a language. This language is easily adaptable to an information theoretic approach developed in subsequent sections. In general, other methods of increasing reliability can easily be represented in terms of this language. In the next two sections, where they will be treated more fully, we will observe that:

1) Von Neumann's scheme can be shown to have p_{ij}'s $\neq 0$ in all ranks except the last.
2) Recursive triangular networks - although the errors are defined in terms of probabilities of the gates carrying out not the design function but some other functions - can be transformed into p_{ij} representation, with some or all p_{ij}'s $\neq 0$.

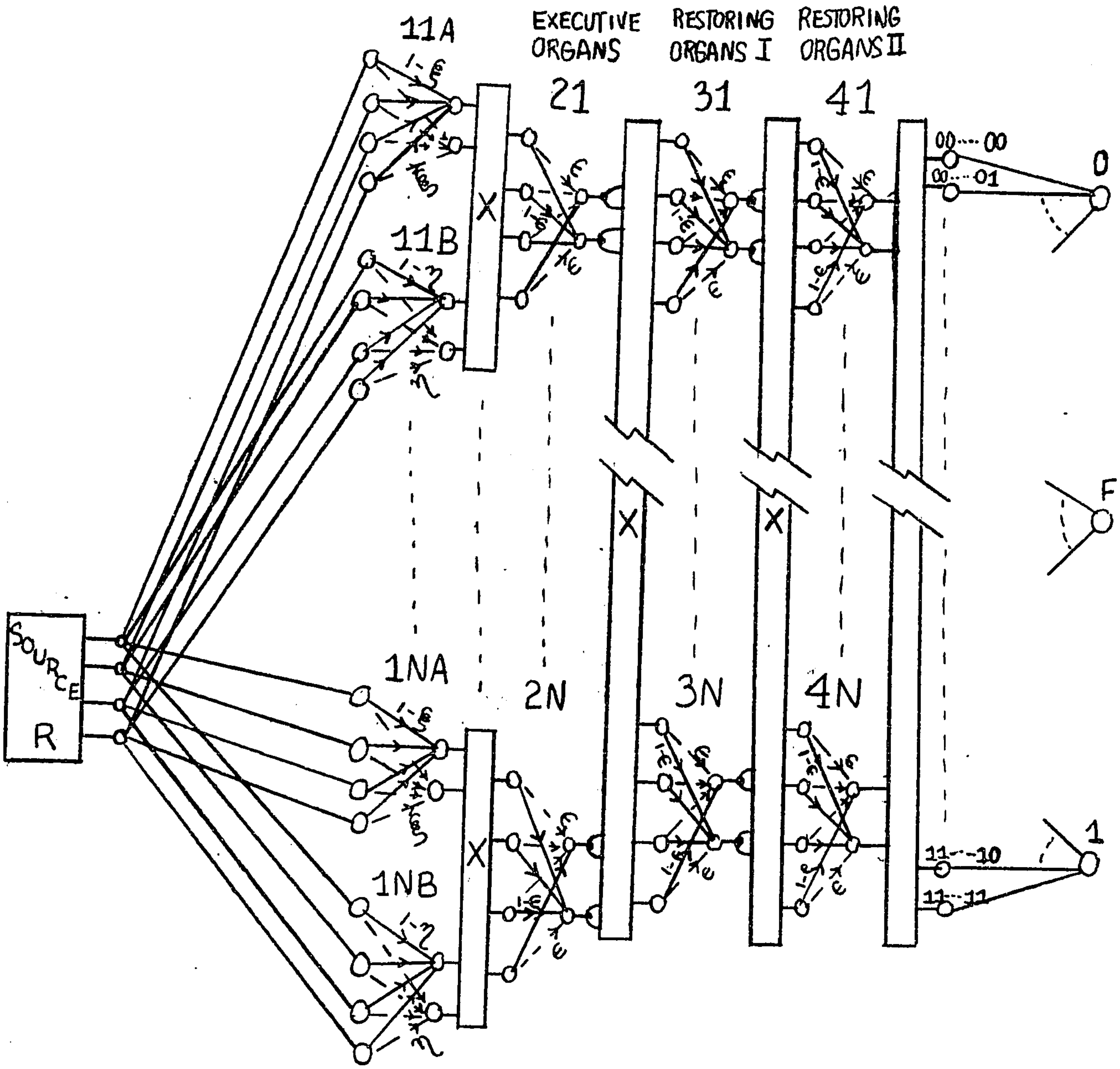
7. <u>von Neumann Scheme for Increased Reliability</u> [15]

Using the format developed in sections 5 and 6, the von Neumann scheme for increasing reliability of a Sheffer stroke organ (gate 15 of Fig. 2) is represented graphically in Fig. 7. Statistical independence of error occurrence is again assumed.

The first rank of gates represents the N lines of bundle A, ξ N of which are stimulated, and the N lines of bundle B, η N of which are stimulated.

From section 5, it follows that,

$$P^{21}_{EQUIV} = Z^{21}P^{21} \qquad (5.5)$$



von Neumann Iteration of an Original Sheffer Stroke Organ

Fig. 7)

i.e.

$$P^{2i}_{EQUIV} = \begin{bmatrix} (1-\xi)(1-\eta) & (1-\xi)(\eta) & (\xi)(1-\eta) & (\xi)(\eta) \\ (1-\xi)(1-\eta) & (1-\xi)(\eta) & (\xi)(1-\eta) & (\xi)(\eta) \\ (1-\xi)(1-\eta) & (1-\xi)(\eta) & (\xi)(1-\eta) & (\xi)(\eta) \\ (1-\xi)(1-\eta) & (1-\xi)(\eta) & (\xi)(1-\eta) & (\xi)(\eta) \end{bmatrix} \begin{bmatrix} \epsilon & 1-\epsilon \\ \epsilon & 1-\epsilon \\ \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{bmatrix} \qquad (7.1)$$

where $i = 1,2,\ldots,N$.

It should be noted that in von Neumann's exposition, $0 \leq \xi, \eta \leq 1$. The stipulation that $p_{ij} \leq 0.5$, would seem to be inconsistent with the range of ξ and η. This apparent inconsistency is resolved if we associate p^{1iA}_{ij} with either ξ, or $1-\xi$, whichever is less than 0.5. Similar considerations apply to p^{1iB}_{ij}. This has the effect of making the function of the 1st rank gates dependent on the values of ξ and η. The form of P^{2i}_{EQUIV}, however, is always

$$P^{2i}_{EQUIV} = \begin{bmatrix} 1-\mathcal{J} & \mathcal{J} \\ 1-\mathcal{J} & \mathcal{J} \\ 1-\mathcal{J} & \mathcal{J} \\ 1-\mathcal{J} & \mathcal{J} \end{bmatrix} \qquad (7.2)$$

where $\mathcal{J} = (1-\epsilon)(1-\xi\eta) + (\epsilon)(\xi\eta)$.

Each of the N lines emanating from the executive organs is split into two and the resulting 2N lines are randomly connected to the restoring organs 1. With a sufficiently large N, this has the effect of normalizing the error input to each executive organ around the mean $\mathcal{J}$ with a very small dispersion from the mean. A large N, by the central limit theorem, also makes the elements of the P^{2i} statistically independent. We can now, accordingly, write

$$P^{3i}_{EQUIV} = \begin{bmatrix} 1-\omega & \omega \\ 1-\omega & \omega \\ 1-\omega & \omega \\ 1-\omega & \omega \end{bmatrix} \tag{7.4}$$

where

$$\omega = (1-\epsilon)(1-\mathcal{S}^2) + (\epsilon)(\mathcal{S}^2). \tag{7.5}$$

Repeating the process for the next rank of gates results in

$$P^{4i}_{EQUIV} = \begin{bmatrix} 1-\psi & \psi \\ 1-\psi & \psi \\ 1-\psi & \psi \\ 1-\psi & \psi \end{bmatrix} \tag{7.6}$$

where

$$\psi = (1-\epsilon)(1-\omega^2) + (\epsilon)(\omega^2).$$

The final stage consists of setting a fiduciary level Δ, such that; if ΔN or less of the N outputs of rank 4 fire (i.e. are 1 in our terminology) the system is said not to fire (i.e. to be 0); if $(1-\Delta)N$ or more of the lines fire, the system is said to fire; if neither of these conditions hold the system is said to be in error.

The final stage may be represented by a perfect N-line-input, 1-line-output, binary input, ternary output switching function, which in terms of the graphical model of section 5 would have the equation

$$P_{EQUIV} = \begin{bmatrix} (1-\psi)^N, & (1-\psi)^{N-1}(\psi),\ldots, & (1-\psi)(\psi)^{N-1}, & (\psi)^N \\ (1-\psi)^N, & (1-\psi)^{N-1}(\psi),\ldots, & (1-\psi)(\psi)^{N-1}, & (\psi)^N \\ (1-\psi)^N, & (1-\psi)^{N-1}(\psi),\ldots, & (1-\psi)(\psi)^{N-1}, & (\psi)^N \\ (1-\psi)^N, & (1-\psi)^{N-1}(\psi),\ldots, & (1-\psi)(\psi)^{N-1}, & (\psi)^N \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \cdot & \cdot & \cdot \\ 0 & 1 & 0 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \text{"0"}\text{"F"}\text{"1"} \end{array} \qquad (7.8)$$

In explanation of (7.8): 1) The inputs to the last switching function may be considered to be binary numbers N digits long (i.e. there are 2^N such inputs). 2) If there are ΔN or less 1's in the input the output is a "0". 3) If there are $(1-\Delta)N$ or more 1's in this number the output is a 1. 4) If the number of 1's in the input number lies in between, the output is in error (F).

It can be seen that

$$P_{EQUIV} = \begin{bmatrix} \sum_{j=0}^{\Delta N}\binom{N}{j}(\psi)^j(1-\omega)^{N-j} & \sum_{j=\Delta N+1}^{(1-\Delta)N-1}\binom{N}{j}(\psi)^j(1-\omega)^{N-j} & \sum_{j=(1-\Delta)N}^{N}\binom{N}{j}(\psi)^j(1-\psi)^{N-j} \\ '' & '' & '' \\ '' & '' & '' \\ '' & '' & '' \end{bmatrix} \qquad (7.9)$$

It can be seen that the system does behave as a Sheffer stroke, i.e. 1) If system operates perfectly ($\epsilon = 0$) then:

a) If $\xi = \eta = 1$ i.e. input to system is always 11, then output is 0

b) If $\xi = 0, \eta = 1$ " " " " " " 01, " " " 1

c) If $\xi = 1, \eta = 0$ " " " " " " 10, " " " 1

d) If $\xi = 0, \eta = 0$ " " " " " " 00, " " " 1 .

2) If system operates with small Sheffer stroke organ error ($\epsilon \simeq 0$) and argument error is kept within fiduciary levels, then:

a) If $\xi, \eta \geqslant (1-\Delta)$ then, $\psi \simeq 0$ and the output is most likely 0

b) If $\xi \leqslant \Delta, \eta \geqslant (1-\Delta)$ " $\psi \simeq 1$ " " " " " " 1

c) If $\xi \geqslant (1-\Delta), \eta \leqslant \Delta$ " $\psi \simeq 1$ " " " " " " 1

d) If $\xi, \eta \leqslant \Delta$ " $\psi \simeq 1$ " " " " " " 1 .

8. "Triangular Switching Network" Scheme [2] for Increased Reliability.

In order to improve the reliability of a single gate, a method of iteration (or recursion) is developed and the resulting cascade is called a "Triangular Switching Network". The original gate is replaced by a homogeneous triangular network. A homogeneous network is one constructed of identical gates. Triangular means that the network is a cascade with 2 gates in the first rank and one in the second.

Each new gate is, in turn, replaced by a homogeneous triangular network. This process is repeated a requisite N times till the desired triangular switching network is achieved. It should be noted that the original single gate ($=3^0$) is replaced by 3 gates ($=3^1$) in 2 ranks (2^1). The next, or second, iteration results in 9 gates ($=3^2$) in 4 ranks ($=2^2$). After N iterations there are a total of 3^N

gates in 2^N ranks.

The noise, or error, in a gate is viewed from a different standpoint than the one characterized by conditional probabilities of error. It is assumed that a gate may perform switching functions other than its design function with stated probabilities. Certain physical considerations lead this analysis to the restriction of possible switching functions to the 6 positive functions (a positive function is a Boolean function which, in the sum of products form, contains no negated variable). These 6 functions are called, F, C, A, B, D, and T. In terms of our classification of gates in Fig. 2, they are respectively gates 1, 5, 6, 7, 12, and 16. Any gate is assumed to perform these functions with the respective probabilities of F_o, C_o, A_o, B_o, D_o, and T_o. These probabilities are assumed statistically independent.

Although this error formulation is different, it has, obviously, an equivalent conditional probability of error representation. The relations between the two forms can be developed as follows:

1) $F_o + C_o + A_o + B_o + D_o + T_o = 1$
2) Assume an input distribution R, where as usual $\sum_{i,j} r_{ij} = 1$
3) Given an input ij, the gate, depending on which of the 6 functions it is performing, will give an output of either 0 or 1. Group the terms giving a zero

output in one category, those giving a one output in another.

i.e.

$$
\begin{array}{l|l}
P(\text{Output}=0) = r_{00}F_0+r_{01}F_0+r_{10}F_0+r_{11}F_0 & P(\text{output}=1)= \\
\quad + r_{00}C_0+r_{01}C_0+r_{10}C_0 & \qquad\qquad +r_{11}C_0 \\
\quad + r_{00}A_0+r_{01}A_0 & \qquad +r_{10}A_0+r_{11}A_0 \\
\quad + r_{00}B_0 \qquad +r_{10}B_0 & +r_{01}B_0 \qquad +r_{11}B_0 \\
\quad + r_{00}D_0 & +r_{01}D_0+r_{10}D_0+r_{11}D_0 \\
 & +r_{00}T_0+r_{01}T_0+r_{10}T_0+r_{11}T_0 \\
\hline
\quad = r_{00}(1-T_0)+r_{01}(F_0+C_0+A_0) & =r_{00}(T_0)+r_{01}(T_0+D_0+B_0) \\
\quad + r_{10}(F_0+C_0+B_0)+r_{11}(F_0) & +r_{10}(T_0+D_0+A_0)+r_{11}(1-F_0)
\end{array}
$$

4) Putting the above in the matrix form of the conditional probability of error representation leads to

$$
P = \begin{bmatrix} 1-T_0 & T_0 \\ F_0+C_0+A_0 & T_0+D_0+B_0 \\ F_0+C_0+B_0 & T_0+D_0+A_0 \\ F_0 & 1-F_0 \end{bmatrix}
$$

5) The p_{ij}'s may, now, be identified when the design function is specified. If, for example, the gate is designed to be an AND gate (C), then,

$$P = \begin{bmatrix} 1-p_{00} & p_{00} \\ 1-p_{01} & p_{01} \\ 1-p_{10} & p_{10} \\ p_{11} & 1-p_{11} \end{bmatrix}, \quad \text{and}$$

$p_{00}=T_0$, $p_{01}=T_0+D_0+B_0$, $p_{10}=T_0+D_0+A_0$, and $p_{11}=F_0$.

Having established the relations between the 2 approaches, let us revert to the analysis [2] under discussion. The iteration is developed in terms of the 6 functions, i.e. if the single gate probabilities are F_0, C_0, A_0, B_0, D_0, and T_0, then the first iteration yields, F_1, C_1, A_1, B_1, D_1, and T_1 the respective probabilities of the network functioning as F, C, A, B, D and T. The network probabilities are, of course, in terms of the gate probabilities.

It is possible, by the methods of section 5, to arrive at the same recursion formulas obtained in this analysis. First assuming that because of physical symmetry, $A_0=B_0$, it can be seen that

$$P_{EQUIV} = \begin{bmatrix} (1-T_0)^2 & (1-T_0)\,T_0 & (T_0(1-T_0) & T_0^{\,2} \\ (F_0+C_0+A_0)^2 & (F_0+C_0+A_0)(T_0+D_0+A_0) & (T_0+D_0+A_0)(F_0+C_0+A_0) & (T_0+D_0+A_0) \\ '' & '' & '' & '' \\ F_0^{\,2} & F_0(1-F_0) & (1-F_0)F_0 & (1-F_0)^2 \end{bmatrix} \begin{bmatrix} 1-T_0 & T_0 \\ F_0+C_0+A_0 & T_0+D_0+A_0 \\ '' & '' \\ F_0 & 1-F_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-T_1 & T_1 \\ F_1+C_1+A_1 & T_1+D_1+A_1 \\ \text{''} & \text{''} \\ F_1 & T_1 \end{bmatrix}, \tag{8.2}$$

where

$$\begin{aligned} T_1 &= T_0\left\{(1-T_0)^2 + 2(1-T_0)(T_0+D_0+A_0) + T_0(1-F_0)\right\} \\ &= T_0\left\{1+2D_0+2D_0+2A_0+T_0(-T_0+1-F_0-2D_0-2A_0)\right\} \\ &= T_0\left\{1+2D_0+2A_0+T_0(C_0-D_0)\right\}, \end{aligned}$$

and similarly,

$$\begin{aligned} F_1 &= F_0\left\{1+2C_0+2A_0+F_0(D_0-C_0)\right\} \\ A_1 &= A_0\left\{2A_0+A_0(C_0+D_0)+4C_0D_0+2C_0T_0+2D_0F_0\right\} \\ D_1 &= D_0\left\{2A_0+2A_0{}^2+2C_0T_0+D_0(2+C_0-2T_0) - D_0{}^2\right\} \\ C_1 &= C_0\left\{2A_0+2A_0{}^2+2D_0F_0+C_0(2+D_0-2F_0) - C_0{}^2\right\}. \end{aligned}$$

It can be seen that for N iterations the recursion formulas above remain of the same form but the subscripts are all enlarged by N. As an example

$$F_{N+1} = F_N\left\{1+2C_N+2A_N+F_N(D_N-C_N)\right\}.$$

This analysis yielded several interesting results, among which are the following:

1) If $F_0+T_0 \neq 0$, if $(C_0+D_0) \neq 0$, then $\lim\limits_{N\to\infty}(C_N+D_N) = 1$

2) Given $F_0=T_0=0$, if $(C_0+D_0) \neq 0$, then $\lim\limits_{N\to\infty}(C_N+D_N) = 1.$

In particular, if $C_0 > D_0$, then the network converges to C,

" $D_0 > C_0$, " " " " " D,

" $C_0 = D_0$, then $\lim_{N\to\infty} C_N = \lim_{N\to\infty} D_N = 1/2$.

3) If $C_0+A_0+B_0$ $(=C_0+2A_0) = 1$, then if the desired network function is C, an iteration improves the network.

4) If $C_0+F_0=1$, then if the desired network function is C, an iteration degrades the network.

Having shown that various reliability improvement schemes can, without great difficulty, be represented by the Z matrix format, we now consider the application of entropy to this format.

9. Entropy and Information

The mathematical concept of entropy is a useful way of showing the relation of an input set $X = \left\{x_1, x_2, \ldots, x_n\right\}$, where each event x_k has a stationary probability of occurrence $p(x_k)$, to an output set $Y = \left\{y_1, y_2; y_m\right\}$, each of whose events y_i has a stationary probability of occurrence $p(y_i)$. The two sets are related by a set of probabilities of joint occurrences $p(x_k, y_i)$. In all cases to be considered, the statistical relations $p(x_k, y_i) = p(x_k/y_i)p(y_i) = p(y_i/x_k)p(x_k)$, will, of course, hold.

The entropy function is a convex function (see appendix II) of the type $-\sum_j Z_j \text{ Log } Z_j$ (where Log Z_j will be understood to designate $\text{Log}_2 Z_j$ throughout).

There are many excellent texts [6],[9],[10],[13], which may be consulted for a detailed analysis of entropy. To relate entropy to gates, however, it is sufficient, for our purposes, to state only the basic equation, i.e.

$$I(X;Y) = H(X) - H(X/Y) = H(Y) - H(Y/X) \qquad (9.1)$$

where $I(X;Y)$, the transinformation, or mutual information, is defined as $I(X;Y) = \sum_X\sum_Y p(x_k,y_i)\ \mathrm{Log}\ \dfrac{p(x_k/y_i)}{p(x_k)}$,

$H(X)$, the entropy, or average self information of X, is defined as $H(X) = -\sum_X p(x_k)\ \mathrm{Log}\ p(x_k)$,

$H(Y)$, the entropy of Y, is defined as

$$H(Y) = -\sum_Y p(y_i)\ \mathrm{Log}\ p(y_i),$$

$H(X/Y)$, is, therefore, defined as

$$H(X/Y) = -\sum_X\sum_Y p(x_k,\ y_i)\ \mathrm{Log}\ p(x_k/y_i), \text{ and}$$

$H(Y/X)$, is, accordingly, defined as

$$H(Y/X) = -\sum_X\sum_Y p(x_k,\ y_i)\ \mathrm{Log}\ p(y_i/x_k).$$

10. Information Theoretic Properties of Perfect Gates

To apply the foregoing entropy definitions to the gate characterizations of section 2 requires the terminology of section 2 be given its equivalent name. A x B becomes X, C becomes Y, and (2.1) becomes

$$f:\ X \longrightarrow Y \qquad (10.1).$$

For a perfect gate, the function performed is a mapping (i.e. for every x_k there is one and only one y_i). The conditional probabilities $p(y_i/x_k)$ are, therefore, equal to either 0 or 1. This means, by definition, that $H(Y/X) = 0$. This leads to equation (9.1) having the form

$$I(X;Y) = H(X) - H(X/Y) = H(Y) - 0. \tag{10.2}$$

(10.2) may be interpreted as follows: The average amount of information at the input $H(X)$ is not all available at the output since the average amount of information there is $H(Y)$, i.e. an average amount of information $H(X/Y)$ called the equivocation (measuring the average confusion about X, knowing Y) is lost in the process.

The terms of (10.2) are computed as follows:

$$H(X) = -R \text{ Log } R^t, \tag{10.3}$$

$$H(Y) = -Q \text{ Log } Q^t, \tag{10.4}$$

Where superscript "t" indicates the matrix transpose, and R and Q are defined before in section 2.

11. Information Theoretic Properties of Noisy Gates

The introduction of noise into the gate leads to $H(Y/X) \neq 0$, by definition. This average amount of information $H(Y/X)$ is called the noise or error entropy and measures the average confusion about Y, knowing X. i.e. knowing X, Y can no longer be determined with certainty.

(9.1) may be rewritten as:

$$H(X) + H(Y/X) = H(Y) + H(X/Y) = H(X,Y), \tag{11.1}$$

where the terms are computed as follows:

1) $H(X)$ is unchanged from the perfect case of (10.3),
2) $H(Y/X) = RK^t$, (11.2)

where

$$K = \left[H(p_{00}),\ H(p_{01}),\ H(p_{10}),\ H(p_{11}) \right] \qquad (11.3)$$

and $$H(p_{ij}) = -\left\{ p_{ij} \operatorname{Log} p_{ij} + (1-p_{ij}) \operatorname{Log}(1-p_{ij}) \right\}. \qquad (11.4)$$

Note that a probability, such as p_{ij}, appearing in brackets after H will indicate an entropy of the form of (11.3).

3) H(Y) retains the form of (10.4), but, of course, the value of Q will be changed.

12. Information Theoretic Properties of Gate Networks

When a noisy gate is replaced by an equivalent gate cascade, then the input source, R, is unchanged, hence H(X) is unchanged, but the reliability is improved if

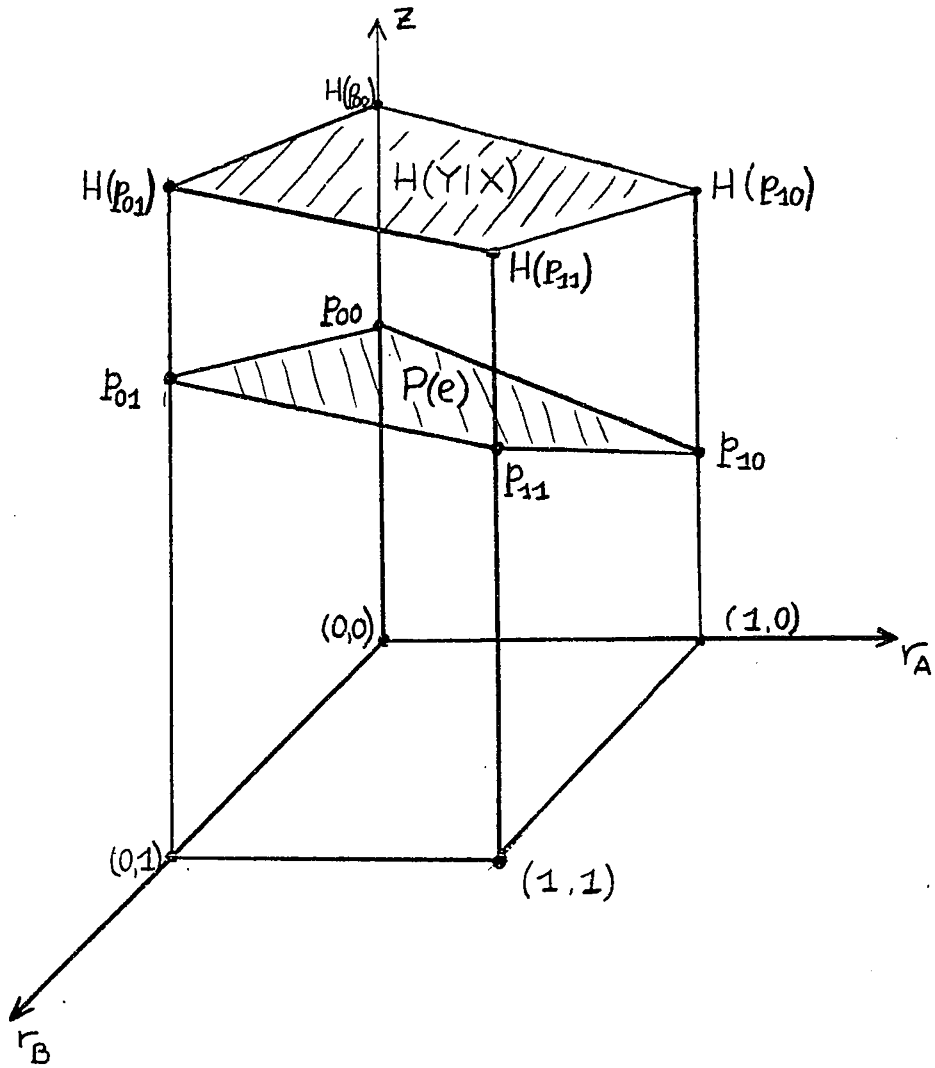
$$P(e)_{ORIGINAL} = \sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij} p_{ij(ORIGINAL)} > P(e)_{EQUIV} = \sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij} p_{ij(EQUIV)}. \qquad (12.1)$$

It could be profitable to relate P(e) to an entropy term. H(Y/X), under certain restrictions, is such an entropy. A justification, for the previous statement, is developed on the following 6 steps:

1) Attention will be restricted to sources whose 2 outputs, A and B, are statistically independent,

i.e. $R = \left[(1-r_A)(1-r_B),\ (1-r_A)r_B,\ r_A(1-r_B),\ r_A r_B \right]$.

This restriction is much less a constraint on the general case than it would appear to be on first sight. Many cases of sources with statistically dependent outputs may, as in section 4, be represented by a statistically independent source followed by a rank, or ranks, of perfect gates. These ranks of perfect gates may be considered as additions to the front end of the existing cascade. Analytic methods,



Graphical Representation of H(Y/X) and P(e)

Fig. 8

developed in subsequent sections, will be equally applicable to the new cascade.

2) $P(e) = \sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}p_{ij}$, and therefore, from above,

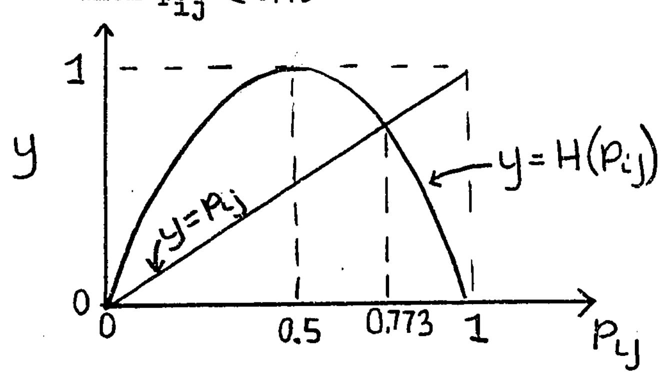
$$P(e) = (1-r_A)(1-r_B)p_{00}+(1-r_A)r_Bp_{01}+r_A(1-r_B)p_{10}+r_Ar_Bp_{11}. \tag{12.2}$$

3) Consider r_A and r_B to be the x and y axes of a 3-dimensional cartesian coordinate system. Along any fixed r_A, $Z = P(e)$ is a linear function of r_B (i.e. a line). Along any fixed r_B, $Z = P(e)$ is a linear function of r_A (i.e. a line). $Z = P(e)$ is a surface which at the four corners under consideration -(0,0), (0,1), (1,0), (1,1) - is p_{00}, p_{01}, p_{10}, and p_{11} respectively (see Fig. 8).

4) From Fig. 9), the relation between p_{ij} and $H(p_{ij})$, it is apparent that

$$\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}H(p_{ij}) = H(Y/X) \geqslant \sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}p_{ij} = P(e) \tag{12.3}$$

when $p_{ij} < .773$



Relation between $H(p_{ij})$ and p_{ij}

Fig. 9

5) Restricting p_{ij} to the domain $0 \leq p_{ij} \leq 0.5$, both p_{ij} and $H(p_{ij})$ are monotonically increasing with p_{ij}, and therefore, the surface $z = H(Y/X)$ has its minimum at the same corner as the surface $z = P(e)$.

6) With restriction $0 \leq p_{ij} \leq 0.5$, $P(e)_{ORIG.} \geq P(e)_{EQUIV}$ for all R, if and only if

$$H(Y/X)_{ORIG} \geqslant H(Y/X)_{EQUIV}. \qquad (12.4)$$

The stipulation that p_{ij} be less than, or equal to, 0.5 is not unreasonable. A gate is customarily designed to perform its design function over 50% of the time, regardless of input. If the hardware doesn't conform to this specification, the design function may be renamed. As an example, consider an "OR" gate with $p_{11} = 0.8$ and all other $p_{ij} \leq 0.5$. This gate performs as an "EXCLUSIVE OR" gate within specifications.

Since P(e) has been related to H(Y/X), the following final sections will be devoted to obtaining the $H(Y/X)_{EQUIV}$ of a network by considering the entropies of the component gates, and to obtaining a lower bound for $H(Y/X)_{EQUIV}$.

13. Application of Information Theory to Switching Networks

To recapitulate, a format for the representation of gate networks and some mathematical relations from information theory applicable to gates have thus far been developed. How can these, most usefully, be applied?

The most general approach is that of Cowan and Winograd [3].

Their argument, in simplified form, is as follows:

1) In developing information theory, Shannon considered the case of communication through a simple noisy channel. Given a message input set X and a simple noisy channel, fully specified by the $p(y_j/x_i)$'s, he defined a channel capacity C. This $C = \max_X I(X;Y)$, is obtained by varying the symbol transmission probabilities of the input set. C is an upper bound on the average amount of information that can be provided by each received symbol about the corresponding transmitted signal. Shannon showed that if $H(X) \leq C$, then there exists at least one code such that a suitably encoded message may be transmitted over the channel and recovered with an arbitrarily small frequency of identification error. Coding may be considered to be the replacement of a set of symbols by another, usually larger, set according to a fixed scheme. In other words, given a fixed channel with capacity C, and a message ensemble X, such that $H(X) \leq C$, a coder may be found to match the message sequences to the noisy channel so that a decoder at the other end of the channel will decode the received messages with an arbitrarily small probability of error.

Smallness of error is obtained at the expense of complexity at the coder and decoder and time delay in decoding.

2) Cowan and Winograd, in an analogous manner, define a computation capacity C^* for a module (a complex switching network) and show that, under certain restrictions, an arbitrarily reliable network, composed of modules all having computation capacity C^*, may be designed. These results are obtained by considering the coder-channel-

decoder trilogy to be incorporated in every module. Every increase in network reliability entails more modules and more complex modules.

While this approach is of prime importance in formulating a general theory of reliable computation, it is possible that information theory concepts may also be used in smaller and more specific areas of reliable computation, such as obtaining $H(Y/X)_{EQUIV}$, or a lower bound on it.

Why a lower bound on $H(Y/X)_{EQUIV}$? In section 12 it was noted that $H(Y/X)_{EQUIV}$ directly reflects P(e). By finding such a lower bound, therefore, a lower limit is set on P(e).

In order to attempt a reduction in the arbitrariness of the choice, the gate network must be re-examined. A network of gates may be considered to be a computation channel. Unlike the simple communication channel, the computation channel may be considered to be not fixed, but variable, depending on the exact configuration of the constituent gates. Synthesis of a more reliable gate network, accordingly, may be considered to be the synthesis of a better computation channel.

Whereas in the simple communication case, the fixed characteristics of a channel result is a single statistical figure of merit C, which is then used in obtaining an optimal code, in a computation channel we may reasonably consider that only the constituent gates have fixed characteristics. The figure of merit of the network synthesized from these gates would depend on the configuration of the gates and vary with the varying arrangement of these gates. By analogy with the communication case, then, a likely approach is to define some figure(s) of merit for the individual gates; obtain an optimal figure for the equivalent gate (the synthesized computation channel); finally, it is to be hoped that in the process of obtaining this optimal figure there will be indication of how to arrange (code) the gates in order to achieve this optimum.

In section 5, it was seen that a gate network may be explicitly reduced to an equivalent gate by what was designated the Z-matrix approach. It can, therefore, be seen that once a gate network is designed, its theoretical performance can be directly verified by use of Z-matrix techniques. No statistical (entropy) techniques need be used. Statistical techniques, however, may be useful, given particular gates, in giving lower bounds on network error towards which coding may aim. It may also conceivably be used to gain better insight into all types of switching networks, sequential (with feedback) as well as combinatorial.

Starting, then, with an individual gate, it can be seen that the matrix K (its characterization in the entropy field) is not a substantial economy over the probability matrix P. Any attempt to develop an exact analogue of the Z-matrix method in the entropy field would lead to a method nearly as laborious as the Z-matrix. Even if the attempt were successful, the result would be a mere duplication, giving no additional insights into networks.

If the figure(s) of merit characterizing a gate are to be simpler (or at least different) than the K matrix, a re-examination of the Z-matrix method is called for. These new figure(s) must, after all, be at least based on the Z-matrix.

The crucial fact to note is that the sum of the 4 elements of any row of the Z matrix is 1. Taking as a point of departure the equation

$$Z^{rs}P^{rs} = P^{rs}_{EQUIV} \tag{13.1}$$

which characterizes the Z-matrix method, this reduction method may be reinterpreted in the following terms:

1. Each gate, rs, of a network takes 4 input distributions (the Z^{rs} matrix) and transforms them into 4 output distributions (the P^{rs}_{EQUIV} matrix).

2. The output of gate rs and another gate, say rt, then feed into a gate of the next rank, say (r+1)u. In other words, P^{rs}_{EQUIV} and P^{rt}_{EQUIV} are related to each other and together form the new Z matrix, $Z^{(r+1)u}$, which feeds gate (r+1)u.

3. The process is repeated till the last gate is reached. The last gate then transforms its Z into the required P_{EQUIV} of the network.

To sum up, the Z-matrix method, instead of being regarded as the repeated multiplication of conditional probability matrices, may be regarded as the repeated transformation of input distributions into output distributions by a cascade of gates.

It must be noted that at every step of the Z-matrix procedure the output distribution may, in terms of entropy, be represented by the appropriate K of (11.3). Since $H(Y/X) = RK^t$ (11.2), it can immediately be seen that in order to reduce H(Y/X) it is sufficient to reduce all 4 elements of K^t. If, in turn, a lower bound is found on each element of K^t; the least of these then chosen and transmitted through the net, a lower bound will have been established on the smallest element of K^t_{EQUIV}, hence on K^t_{EQUIV} and hence on $H(Y/X)_{EQUIV}$.

Such a lower bound will be developed in the next section.

14. A Lower Bound on $H(Y/X)_{EQUIV}$

There is no loss in generality if in developing the required lower bound, the terminology of section 2 is used. i.e.,

$$\begin{bmatrix} r_{00} & r_{01} & r_{10} & r_{11} \end{bmatrix} \begin{bmatrix} p_{(00)(0)} & p_{(00)(1)} \\ p_{(01)(0)} & p_{(01)(1)} \\ p_{(10)(0)} & p_{(10)(1)} \\ p_{(11)(0)} & p_{(11)(1)} \end{bmatrix} = \begin{bmatrix} q_0 & q_1 \end{bmatrix} \tag{2.2}$$

or RP = Q, where R now represents a row kl of Z^{rs}, P represents the gate, and Q represents a row kl of P^{rs}_{EQUIV}.

By Theorem 2 of Appendix I, it can be seen that

$$\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij} p_{(ij)(0)} \text{ Log } p_{(ij)(0)} \geqslant q_0 \text{Log} q_0 \tag{14.1}$$

and that

$$\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij} p_{(ij)(1)} \text{ Log } p_{(ij)(1)} \geqslant q_1 \text{Log} q_1 \tag{14.2}$$

Adding (14.1) and (14.2) leads to

$$\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij} H(p_{ij}) \leqslant H(q_1) \tag{14.3}$$

If the left-hand side of (14.3) is defined as α, it can be seen that α is a lower bound on $H(q_1)$. It is, however, unsatisfactory for at least two reasons:

1. It is composed of a mixture of entropy and probability terms.

2. If P is noiseless, i.e. all $p_{ij} = 0$, then $\alpha = 0$.

Stated another way, α is far from being a greatest lower bound which would, of course, be most desirable. In the Moore and Shannon case, for example, α would equal zero after the first rank. It would accordingly be of little use.

In order to develop a lower bound based on an approach such as the one embodied in (14.3) we will make the 3 following simplifying assumptions:

1. Let $q_1 \leqslant 0.5$. This is done purely for convenience and consistency. Whether $q_1 \leqslant 0.5$ or $q_0 \leqslant 0.5$ is irrelevant in the same sense that this is a problem in coding and not of lower bounds on entropies (cf. step 6 of section 12, for an analoguous case).

2. In any gate there is both a maximum and a minimum p_{ij} (though they may be equal). Call these respectively p_{max} and p_{min}.

3. For a perfect gate (all $p_{ij} = 0$), let q_{1p} designate q_1 and let Δq_1 be the difference between the actual q_1 and the perfect q_1 (i.e. q_{1p}). In other words $q_1 \overset{\Delta}{=} q_{1p} + \Delta q_1$. (14.4)

On the basis of assumption 2, we can define a lower bound on Δq_1 as $\Delta q_{1min} \stackrel{\Delta}{=} (1-q_{1p})p_{min} - q_{1p}p_{max}$. (14.5)

It now follows that if we define q_{1min} as

$$q_{1min} \stackrel{\Delta}{=} q_{1p} + q_{1min}$$

$$= q_{1p}(1-p_{min}-p_{max}) + p_{min} \quad (14.6)$$

then, by definition, $q_{1min} \leqslant q_1$ (14.7)

and hence $H(q_{1min}) \leqslant H(q_1)$. (14.8)

$H(q_{1min})$ is, accordingly, another lower bound on $H(q_1)$.

As a final step, a lower bound on $H(q_1)$, call it H(LB), must be found. H(LB) should satisfy the following properties:

1. H(LB) should be less than $H(q_{1min})$. This is still a lower bound on $H(q_1)$ and is done so that the p_{ij}'s on which H(LB) will be based will be reduced from 4 to 2 (p_{min} and p_{max}) for each gate.

2. H(LB) must be greater than $H(p_{min})$. This is evident from (14.3). It follows that because $\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}H(p_{min}) = H(p_{min})$ and because $\sum_{i=0}^{1} \sum_{j=0}^{1} r_{ij}H(p_{ij}) = \alpha$, therefore, $H(p_{min}) \leqslant \alpha$. (14.9)

This is, in entropy terms, the well-known fact that the error out of a gate is at least as great as the minimum error in the gate.

3. H(LB) should increase with increasing p_{min}.

4. When the p_{ij}'s all equal zero, H(LB) should equal $H(q_{1p})$.

$$H(LB) = H(p_{min}) + H(q_{1p}) \frac{H(p_{min})}{H(p_{min}+p_{max})} \quad (14.10)$$

satisfies the above 4 properties under the following qualifications.

1. $H(LB) \leqslant H(q_{1min})$ only under the empirically derived

restriction that $p_{min} \leqslant 0.02$. This area of small error is, however, the most commonly occurring and hence most interesting, as has been noted by Zemanek [16].

A justification may be attempted in the following steps:

a. With a given p_{min}, it may be noted that

$$H(LB) \xrightarrow[p_{max} \rightarrow p_{min}]{} \text{Maximum, while } H(q_{1min}) \text{ increases only slightly.}$$

b. We are, therefore, interested in the limits on p_{min}, such that $H(q_{1p}) \dfrac{H(p_{min})}{H(p_{min}+p_{max})} \leqslant H(q_{1min}) - H(p_{min})$,

or, by a. above $$H(q_{1p}) \frac{H(p_{min})}{H(2p_{min})} \quad H(q_{1p}[1-2p_{min}]+p_{min}) - H(p_{min}) \qquad (14.11).$$

c. Since $q_{1p} \leqslant 0.5$ by construction, therefore, $2q_{1p}p_{min} < p_{min}$, and

hence $$H(q_{1p}) \leqslant H(q_{1min}) \qquad (14.12).$$

d. This satisfies the prerequisites of Theorem 2, Appendix 2, and we may accordingly impose a relaxed upper bound on $\dfrac{H(p_{min})}{H(2p_{min})}$ of

$$\frac{H(p_{min})}{H(2p_{min})} \lesssim \frac{(1-2p_{min})q_{1p}}{q_{1p}} \qquad (14.13)$$

This inequality is easily solved graphically ($p_{min} \lesssim 0.1$) and indicates an upper bound on p_{min}, which is determined to be about 0.02 (See Table 1).

15 Conclusion

Converting H(LB) into LB by means of entropy tables results in a lower bound on the probability of error out of any specific gate. A problem remains in compounding two H(LB)'s to obtain the $H(q_{1p})$

of the next gate. In the case of Moore and Shannon, and von Neumann where errors are made statistically independent, a solution in the entropy field is to approximate $H(LB_1 \cdot LB_2)$, the minimum $H(q_{1p})$, by $H(LB_1)H(LB_2)$, (cf. Theorem 1, Appendix 2).

This problem of compounding q_{1p} in the case of statistically dependent errors and, ultimately, in multi-input, multi-output sequential circuits is a matter for further inquiry. So, unfortunately, is the answer to the overriding problem, i.e., are there coding methods inherent in some form of H(LB)?

In the meantime, H(LB) may be used to indicate a lower bound on error in a network. This may be done directly in some simple cases and with conversion from entropy to probability between gates in the case of more complex cascades.

To conclude:

1. Some well-known gate-reliability-improvement methods have been presented in a single format (the Z matrix method) which facilitates the application of entropy concepts.

2. A lower bound, H (LB), was synthesized. When converted back into probability, it gave a lower bound on the conditional probabilities of error of the gate equivalent to a network composed of specified gates.

3. This lower bound, while not a greatest lower bound, does, at least, make the derivation of a greatest lower bound a possibility.

4. In the meanwhile, it is to be hoped that even H(LB) may be of use in further analysis of the more complex networks mentioned - i.e. sequential circuits with feedback, having component gates with non-statistically independent probabilities, and possibly many non-binary inputs and outputs.

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TABLE 1

Some Comparisons of $H(LB)$ and $H(q_{1min})$

	q_{1p}	$H(q_{1p})$	$H(q_{1min})$	$H(LB)$
$p_{min} = .03 \quad H(p_{min}) = .194$				
$p_{max} = .03 \quad p_{min}+p_{max} = .06$	.01	.081	.2395	.242
$H(p_{min}+p_{max}) = .327$	.05	.291	.395	.367
$\dfrac{H(p_{min})}{H(p_{min}+p_{max})} = .594$	.40	.971	.9785	.771

	q_{1p}	$H(q_{1p})$	$H(q_{1min})$	$H(LB)$
$p_{min} = .02 \quad H(p_{min}) = .141$				
$p_{max} = .02 \quad p_{min}+p_{max} = .04$	.01	.081	.192	.188
$H(p_{min}+p_{max}) = .242$	.05	.291	.358	.311
$\dfrac{H(p_{min})}{H(p_{min}+p_{max})} = .583$	.40	.971	.961	.711

	q_{1p}	$H(q_{1p})$	$H(q_{1min})$	$H(LB)$
$p_{min} = .02 \quad H(p_{min}) = .141$				
$p_{max} = .08 \quad p_{min}+p_{max} = .100$	.01	.081	.189	.165
$H(p_{min}+p_{max}) = .469$	.05	.291	.347	.225
$\dfrac{H(p_{min})}{H(p_{min}+p_{max})} = .301$	.40	.971	.958	.433

	q_{1p}	$H(q_{1p})$	$H(q_{1min})$	$H(LB)$
$p_{min} = .01 \quad H(p_{min}) = .081$				
$p_{max} = .02 \quad p_{min}+p_{max} = .02$	.01	.081	.140	.128
$H(p_{min}+p_{max}) = .141$	.05	.291	.323	.249
$\dfrac{H(p_{min})}{H(p_{min}+p_{max})} = .575$	.40	.971	.972	.640

TABLE 1 (Cont)

$p_{min} = .001$ $H(p_{min}) = .011$	q_{1p}	$H(q_{1p})$	$H(q_{1min})$	$H(LB)$
$p_{max} = .001$ $p_{min}+p_{max} = .002$	.001	.011	.021	.016
$H(p_{min}+p_{max}) = .021$	.01	.081	.087	.066
$\frac{H(p_{min})}{H(p_{min}+p_{max})} = .549$				

Appendix I

Convex Functions: Some Properties [8] [9] [13]

Definition: $f(x)$ is convex in (a,b) if $\frac{d^2f(x)}{dx^2} \geqslant 0$ in (a,b).

Theorem 1: If $f(x)$ is convex in (a,b), then for every x_1, x_2 in (a,b) such that $x_1 < x_2$, $\alpha f(x_1) + (1-\alpha)f(x_2) \geqslant f(\alpha x_1 + [1-\alpha]\, x_2)$ for $0 \leqslant \alpha \leqslant 1$.

Proof: Define an x_α such that $x_1 \leqslant x_\alpha \leqslant x_2$.

It follows that $0 \leqslant \frac{x_\alpha - x_1}{x_2 - x_1} \leqslant 1$, accordingly, define $1-\alpha$, and α as follows:

$$\frac{x_\alpha - x_1}{x_2 - x_1} \equiv 1-\alpha, \text{ and } \frac{x_2 - \alpha_\alpha}{x_2 - x_1} \equiv \alpha.$$

Now define $q(x_\alpha)$ as the straight line joining $f(x_1)$ to $f(x_2)$

i.e. $$q(x_\alpha) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (x_\alpha - x_1). \qquad \text{(I1)}$$

By definition of convexity, $\frac{df(x)}{dx}$ increases monotonically with increasing x, and hence $f(x)$ may be represented by an upward opening curve such as that of Fig. A. It may, hence, be added that $q(x_\alpha) \geqslant f(x_\alpha)$. (I2)

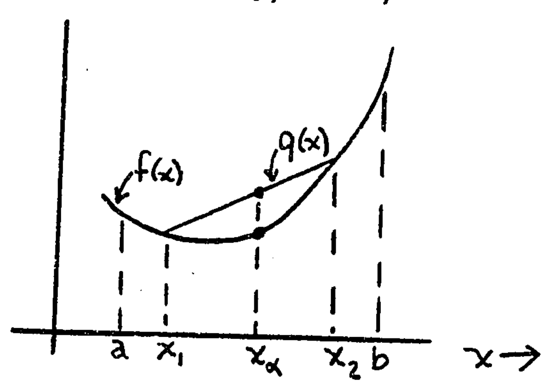

Fig. A

But, from (I1),

$$q(x_\alpha) = f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}\ (x_\alpha - x_1),\ \text{and rearranging}$$

the terms $q(x_\alpha) = \frac{x_2-x_\alpha}{x_2-x_1}\ f(x_1)\ +\ \frac{x_\alpha-x_1}{x_2-x_1}\ f(x_2)$. (I3)

Substituting the definitions of α and $1-\alpha$ leads to $q(x_\alpha) = \alpha f(x_1) + (1-\alpha)f(x_2)$.
(I4)

Note, that by decomposition,

$$x_\alpha = \frac{x_2-x_\alpha}{x_2-x_1}\ x_1 + \frac{x_\alpha-x_1}{x_2-x_1}\ x_2, \tag{I5}$$

and, again, by definition of α and $1-\alpha$,

$$x_\alpha = \alpha x_1 + \left[1-\alpha\right] x_2, \tag{I6}$$

and hence $f(x_\alpha) = f(\alpha x_1 + \left[1-\alpha\right] x_2)$. (I7)

Substituting (I4) and (I7) in (I2), obtain

$$\alpha f(x_1) + (1-\alpha)f(x_2) \geqslant f(\alpha x_1 + 1-\alpha\ x_2)_{\text{Q.E.D.}} \tag{I8}$$

Theorem 2: If $\sum_{i=1}^{n} q_i = 1$ and $q_i \geqslant 0$, and if $Q(x)$ is convex, then

$$\sum_{i=1}^{n} q_i Q(x_i) \geqslant Q\left(\sum_{i=1}^{n} q_i x_i\right).$$

Proof: Assume that the theorem is true for $\sum_{i=1}^{n-1}$, and prove it true for $\sum_{i=1}^{n}$ by induction.

$$Q\Big(\sum_{i=1}^{n} q_i x_i\Big) = Q\Big(q_1x_1 + \big[q_2 + \ldots + q_r\big]\,\frac{q_2x_2 + \ldots + q_nx_n}{\big[q_2 + \ldots + q_n\big]}\Big)$$

$$\leqslant q_1 Q(x_1) + \big[q_1 + \ldots + q_n\big]\, Q\left(\frac{q_2x_2 + \ldots + q_nx_n}{\big[q_2 + \ldots + q_n\big]}\right), \qquad \text{(I9)}$$

by Theorem 1.

But, by the assumption above,

$$Q\left(\frac{q_2x_2 + \ldots + q_nx_n}{q_n + \ldots + q_n}\right) \leqslant \frac{q_2}{\big[q_2 + \ldots + q_n\big]}\, Q(x_2) + \ldots + \frac{q_n}{\big[q_2 + \ldots + q_n\big]}\, Q(x_n), \qquad \text{(I10)}$$

i.e., substituting (I10) in (I9) yields

$$Q\Big(\sum_{i=1}^{n} q_i x_i\Big) \leqslant \sum_{i=1}^{n} q_i Q(x_i). \qquad \text{(I11)}$$

But the assumption is true for n-1=2 (a restatement of Theorem 1), hence the theorem is true for all n.

Note: $\mathrm{Ln}\ x \leqslant x-1$, (I12)

and, hence, $\mathrm{Log}\ x\ (= \mathrm{Ln}\ x\ \mathrm{Log}e) \leqslant (x-1)\ \mathrm{Log}e$. (I13)

This relation (I12) may be derived as follows:

1. Call Ln x, $f(x)$.
2. Define a tangent to $f(x)$ at $x=x_0$ as $q(x)$.
3. Since $-f(x)$ is convex, $q(x) \geqslant f(x)$, for any x. (I14)
4. Let $x_0=1$, then $q(x)=x-1$, and substituting for $f(x)$ and $q(x)$ in (I14),

$\mathrm{Ln}\ x \leqslant x-1$. (See Fig. B)

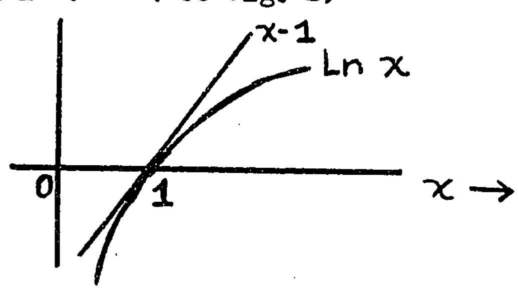

Fig. B

Appendix II

Some Properties of H(x)

Given, in this appendix, that $0 \leqslant b \leqslant a \leqslant 0.5$, it follows:

Lemma 1. (i) $a \leqslant -(1-a)\mathrm{Log}(1-a)$

(ii) $-(1-a)\mathrm{Log}(1-a) \leqslant -a\mathrm{Log}a$

(iii) $-a\mathrm{Log}a \leqslant H(a)$

Proof:

(1) The function xLogx is convex; hence, by Theorem 1 of Appendix 1, for any x between x = .5 and x = 1, the function is less than the line joining these two points. The equation of this line is x-1.

In other words, for $0.5 \leqslant x \leqslant 1$, $x\mathrm{Log}x \leqslant x-1$. (II1)

Let x = 1-a, and substitute in (II1),

i.e. $a \leqslant -(1-a)\mathrm{Log}(1-a)$.

(11) By reference to Figure C, note the following:

a. In the range 0 to 1, the convex function -Logx has a slope of minimum magnitude at x = 1, hence so has the function Logx in this range.

b. $\mathrm{Log}x = \frac{\mathrm{Ln}\ x}{\mathrm{Ln}\ e} = 1.44\ \mathrm{Ln}\ x,$ (II2)

hence, the slope at x = 1 is $1.44\ d\ \frac{\mathrm{Ln}\ x}{dx}\Big|_{x\ =\ 1} = 1.44.$ (II3)

Through Log 1 and Log(1-a) draw a straight line, and call it y_2. The slope of y_2 is, by definition, greater than y_1.

Observe that

$$\frac{a}{1-a} = \frac{-y_2 \text{ at } x=1-a}{-y_2 \text{ at } x=a} \quad . \qquad \text{(II4)}$$

Now, by convexity of $-\text{Log}x$,

$$-y_2 \text{ at } x = a \leqslant -\text{Log}a, \tag{II5}$$

and, by construction,

$$-y_2 \text{ at } x = 1-a = -\text{Log}(1-a), \tag{II6}$$

hence, substituting (II5) and (II6) in (II4) yields

$$\frac{a}{1-a} \geqslant \frac{-\text{Log}(1-a)}{-\text{Log}a}, \tag{II7}$$

or $\qquad -(1-a)\text{Log}(1-a) \leqslant -a\text{Log}a.$

(111) Since H(a) is by definition

$$H(a) = -\left[a\text{Log}a + (1-a)\text{Log}(1-a)\right],$$

and both right-hand terms are greater than zero, it follows that

$$-a\text{Log}a \leqslant H(a).$$

Theorem 1: (1) $ab \leqslant bH(a)$

(11) $bH(a) \leqslant aH(b)$

(111) $aH(b) \leqslant H(ab)$

(IV) $H(ab) \leqslant H(a)H(b)$.

Proof: (1) By lemma 1, $a \leqslant H(a)$, hence, $ab \leqslant bH(a)$.

(11) An equivalent statement is $\frac{b}{a} \leqslant \frac{H(b)}{H(a)}$. (II8)

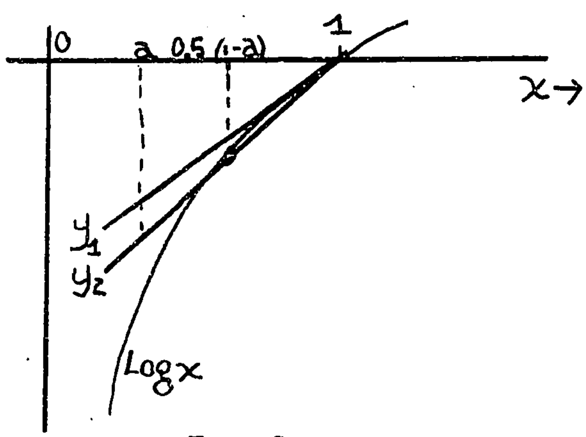

Fig. C

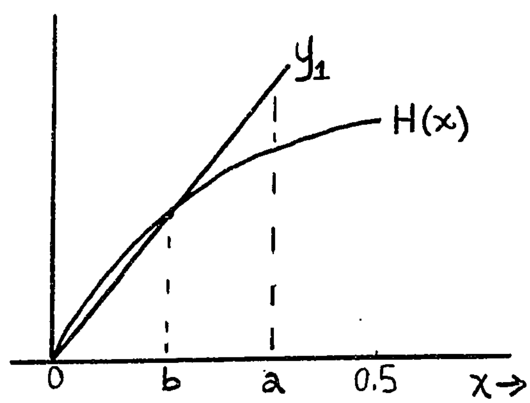

Fig. D

It can be seen (by reference to Figure D) that if a line joining the origin and H(x) at x = b, call it the function y_1, is constructed, then,

$$\frac{b}{a} = \frac{y_1 \text{ at } b}{y_1 \text{ at } a}. \qquad \text{(II9)}$$

But, by construction,

$$y_1 \text{ at } b = H(b) \qquad \text{(II10)}$$

and, by convexity,

$$y_1 \text{ at } a \geqslant H(a). \qquad \text{(II11)}$$

Substituting (II10) and (II11) in (II9) yields $\frac{b}{a} \leqslant \frac{H(b)}{H(a)}$.

(111) By (11) above, $\frac{H(ab)}{H(b)} \geqslant \frac{ab}{b} = a,$ (II12)

hence, $aH(b) \leqslant H(ab)$.

(1V) Make the following definitions:

$$-(1-x)\text{Log}(1-x) \equiv x + \Delta x', \qquad \text{(II13)}$$

and $$-x\text{Log}x \equiv x + \Delta x. \qquad \text{(II14)}$$

By lemma 1, $\Delta x \geqslant \Delta x' \geqslant 0$. (II15)

H(a)H(b) may now be rewritten as

$$\begin{aligned} H(a)H(b) &= \left[(a+\Delta a) + (a+\Delta a')\right]\left[(b+\Delta b) + (b+\Delta b')\right] \\ &= \left[2a+\Delta a+\Delta a'\right]\left[2b+\Delta b+\Delta b'\right] \\ &= 4ab+2b\Delta a+2a\Delta b+\Sigma, \end{aligned} \qquad \text{(II16)}$$

where $\Sigma = 2a\Delta b'+2b\Delta a'+\Delta a\Delta b+\Delta ab'+\Delta a'\Delta b+\Delta a'\Delta b'$, and therefore $\Sigma \geqslant 0$.

$$\begin{aligned} \text{Now, } -ab\text{Log}ab &= -\left[ab\text{Log}a + ab\text{Log}b\right] \\ &= b(a+\Delta a) + a(b+\Delta b) \\ &= 2ab + a\Delta b + b\Delta a. \end{aligned} \qquad \text{(II17)}$$

Note that $-ab\text{Log}ab \geqslant -(1-ab)\text{Log}(1-ab)$ by lemma 1. (II18)

It follows that $H(ab) = -\left[ab\text{Log}ab + (1-ab)\text{Log}(1-ab)\right]$

$$\leqslant -2ab\text{Log}ab$$

$$= 4ab + 2b\Delta a + 2a\Delta b. \tag{II19}$$

Comparing (II19) with (II16), it can be seen that even an upper bound on H(ab) is less than H(a)H(b), i.e.,

$$H(ab) \leqslant H(a)H(b).$$

Theorem 2:

Given $0 \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 0.5$

$$\frac{H(a)-H(c)}{H(b)-H(d)} \leqslant \frac{a-c}{b-d}$$

Proof: Since -H(x) is convex, the slope of H(x) decreases with x. By the Mean Value Theorem the slope of the line joining H(a) and H(b), has a slope less that of the tangent at b. Similarly, the slope of the line joining H(c) and H(d) is greater than that of the tangent at c. But, by convexity, H'(x) at c$\geqslant$H'(x) at b, hence,

$$\frac{H(a)-H(b)}{a-b} \leqslant \frac{H(c)-H(d)}{c-d} \quad \text{(See Fig. E)} \tag{II20}$$

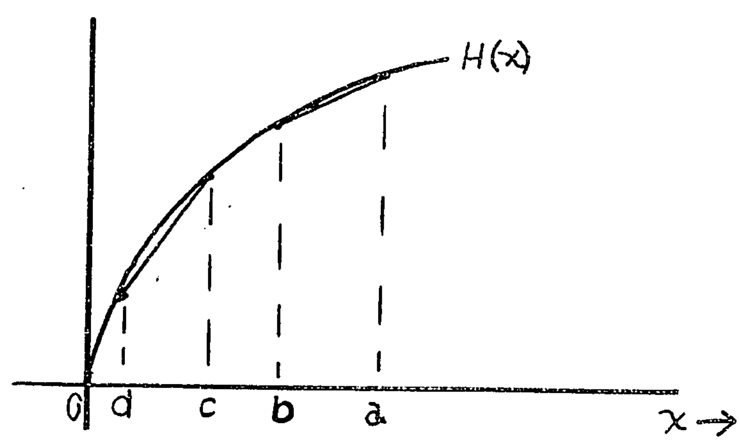

Fig. E

Consider now, $\dfrac{H(x)-H(d)}{x-d}$, where $d \leqslant x \leqslant a$, (II21)

and $\dfrac{H(a)-H(x)}{a-x}$. (II22)

Observe that $$(II21) \geqslant \frac{H(a)-H(d)}{a-d}, \text{ and}$$
$$(II22) \leqslant \frac{H(a)-H(d)}{a-d} .$$

Hence, substituting x = c in (II21) and x = b in (II22) leads to

$$\frac{H(a)-H(c)}{H(b)-H(d)} \leqslant \frac{a-c}{b-d} .$$

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