An Investigation of Algebraic Bounds for Graphs

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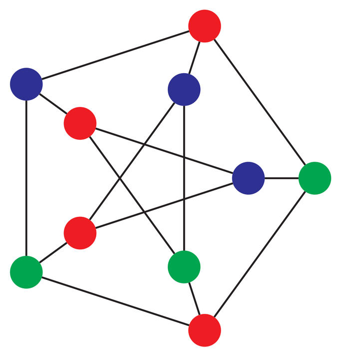
Introduction

Graphs can be used to represent many different practical problems including scheduling problems, data systems and communication networks. Two important parameters of a graph include the independence number and the chromatic number denoted α and χ respectively.

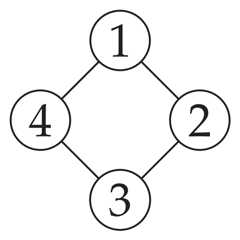


$\chi = 3$ and $\alpha = 4$

In general these parameters are difficult to compute and no efficient algorithm exists for finding them. However we can find bounds on these parameters.

Algebraic Bounds

One way to represent a graph is by its adjacency matrix and one can find bounds on α and χ from the eigenvalues of this matrix.



$$\rightarrow A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Two such bounds on α include the ratio bound and the inertia bound.

Theorem 1. *For a graph G with n vertices and adjacency matrix, A, if $n^+(A)$ and $n^-(A)$ represent the number of positive and negative eigenvalues of A and τ is the least eigenvalue of A then:*

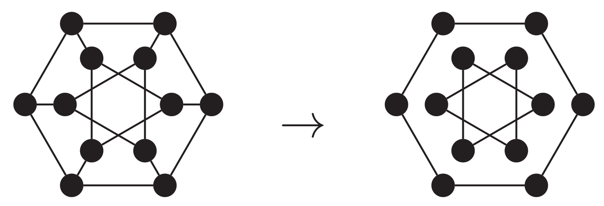
Inertia Bound: $\alpha(G) \leq \min\{n-n^+(A), n-n^-(A)\}$

Ratio Bound (if G is k-regular): $\alpha(G) \leq \dfrac{n}{1+\frac{k}{-\tau}}$

Investigation

It is not known when these bounds are tight. Moreover these bounds can be difficult to apply if the eigenvalues are difficult to compute. By deleting edges, the bounds are easier to compute and still give an estimate on α.

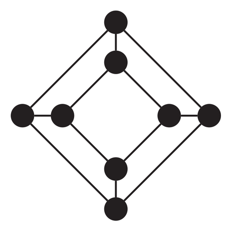
In this project the ratio bound and the inertia bound are applied to the families of cycles and generalized Petersen graphs in an attempt to characterize how well these bounds behave for these particular graph families.



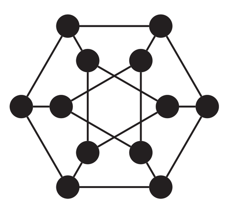
Applying the Bounds to Generalized Petersen Graphs

Let C_n denote the cyclic graph on n vertices, $C(n,k)$ denote the graph on n vertices with a skip of k and $GP(n,k)$ denote the generalized Petersen graph.

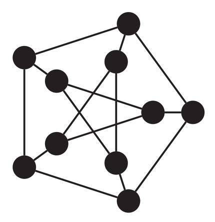
Applying the inertia and ratio bounds directly to some of the generalized Petersen graphs (using eigenvalues computed with MATLAB) gave the following results:



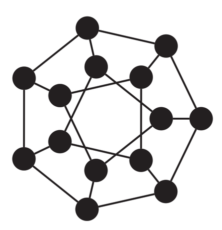
Inertia	Ratio	α
4	4	4



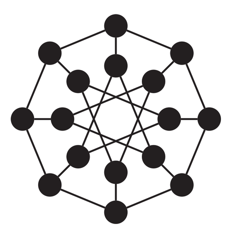
Inertia	Ratio	α
7	5.12	4



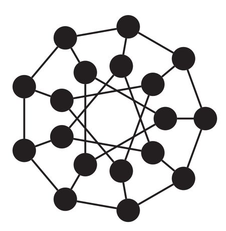
Inertia	Ratio	α
4	4	4



Inertia	Ratio	α
6	6.12	5



Inertia	Ratio	α
8	8	8



Inertia	Ratio	α
8	8.24	7

The bounds appear to give good estimates on α. For the graphs studied, the inertia bound was inexact in all cases when $n \in 3\mathbb{N}$ and $k \in 2\mathbb{N}$ except for the Petersen graph. For the ratio bound no observable pattern could be deduced which explained the cases when the bound was inexact.

References

[1] Norman Biggs *Algebraic Graph Theory*, Cambridge University Press, 1996.

[2] Chris Godsil *Interesting Graphs and their Colourings*, Unpublished Notes, 2004.

[3] Michael W. Newman *Independent Sets and Eigenspaces*, Ph.D Thesis, 2004.

Applying the Bounds to Cycles

For C_n, the eigenvalues and the independence numbers are well known:

Proposition 2. *For $n \geq 3$, the eigenvalues for C_n are $\lambda_r = 2\cos(2\pi r/n)$ where $0 \leq r \leq n-1$. The independence number for C_n is $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$*

Therefore for C_n, the least eigenvalue is $2\cos(\frac{2\pi}{n}\lfloor\frac{n}{2}\rfloor)$. Using Proposition 2, the following lemma was proven.

Lemma 3. *For C_n, the inertia bound yields:*

$$\alpha(C_n) \leq \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even and not a multiple of 4} \\ \frac{n}{2}+1 & \text{if } n \text{ is a multiple of 4} \end{cases}$$

The ratio bound gives:

$$\alpha(C_n) \leq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n}{1+1/\cos(\pi/n)} & \text{if } n \text{ is odd} \end{cases}$$

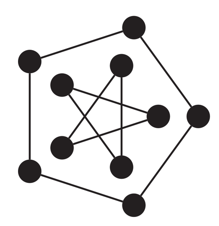
For odd $n \geq 3$ it can be shown that $\frac{n-1}{2} \leq \frac{n}{1+1/\cos(\pi/n)} < \frac{n}{2}$ with equality in the left-most inequality only when $n = 3$. Hence the ratio gives the exact answer whenever n is even or equal to 3. In the other cases, equality is still given however only after rounding.

For the inner graphs $C(n,k)$, it was noticed that the structure of these graphs is connected to the subgroups of the cyclic group of order n:

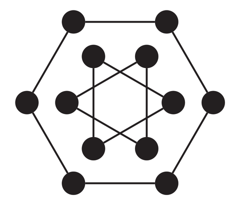
Lemma 4. *Let $g = \gcd\{n,k\}$. Then the graph $C(n,k)$ is a disjoint union of g cycles $C_{\frac{n}{g}}$.*

It follows that removing the connecting edges from the inner and outer graphs of $GP(n,k)$ divides the graph into disjoint cycles. In this case the bounds are easy to compute but the best possible result is that $\alpha(GP(n,k)) \leq \alpha(C_n) + g \times \alpha(C_{\frac{n}{g}})$. Applying either of the bounds will not always give this ideal case however it is known when the errors will occur. Therefore they can always be eliminated either by rounding or by subtraction so as to always give the ideal answer.

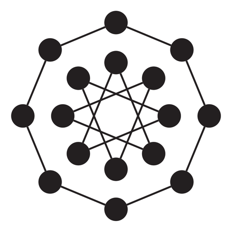
For the graphs studied, it appears that $\alpha(C_n) + g \times \alpha(C_{\frac{n}{g}})$ is a fairly good estimate on $\alpha(GP(n,k))$. However exactly when equality is given is not yet fully understood. Notice that in removing the edges from $GP(9,3)$ the bounds give a better estimate on α than if we apply them to the original graph.



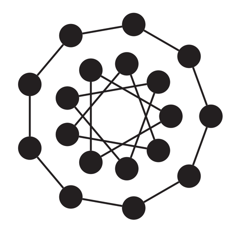
$\alpha(\cup C_i)$	$\alpha(GP(5,2))$
4	4



$\alpha(\cup C_i)$	$\alpha(GP(6,2))$
5	4



$\alpha(\cup C_i)$	$\alpha(GP(8,3))$
8	8



$\alpha(\cup C_i)$	$\alpha(GP(9,3))$
7	7

Conclusion

For cyclic graphs, the inertia bound was shown to be tight in all cases except when $n \in 4\mathbb{N}$. The ratio bound was shown to be tight in all cases upon rounding. Specifically, equality was shown to be present in the ratio bound whenever n is even or equal to 3.

For the generalized Petersen graphs, both the inertia and ratio bounds were generally witnessed to be tight. From the examples completed it appears that strict inequality is observed in the inertia bound whenever $n \in 3\mathbb{N}$ or $k \in 2\mathbb{N}$. No pattern could be seen for when the ratio bound is inexact. More investigation is needed to find and confirm patterns.

Further research involves attempting to find a generalized expression for the ratio and inertia bounds for the $GP(n,k)$ graphs with knowledge of an expression for their eigenvalues.

Acknowledgements

I would like to thank my supervisor for his guidance, patience, and many interesting talks. I would also like to thank the Undergraduate Research Opportunity Program for providing me with such a great opportunity.