

Representation theory of compact inverse semigroups

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Abstract

W. D. Munn proved that a finite dimensional representation of an inverse semigroup is equivalent to a \star -representation if and only if it is bounded. The first goal of this thesis will be to give new analytic proof that every finite dimensional representation of a compact inverse semigroup is equivalent to a \star -representation.

The second goal is to parameterize all finite dimensional irreducible representations of a compact inverse semigroup in terms of maximal subgroups and order theoretic properties of the idempotent set. As a consequence, we obtain a new and simpler proof of the following theorem of Shneperman: a compact inverse semigroup has enough finite dimensional irreducible representations to separate points if and only if its idempotent set is totally disconnected.

Our last theorem is the following: every norm continuous irreducible $*$ -representation of a compact inverse semigroup on a Hilbert space is finite dimensional.

Résumé

W.D. Munn a démontré que une représentation de dimension finie d'un inverse demi-groupe est équivalente à la \star -représentation si et seulement si la représentation est bornée. Notre premier but dans cette thèse est de donner une démonstration analytique. On va prouver que toute représentation de dimension finie est équivalente à une \star -représentation.

Le deuxième but est de donner une paramétrisation de toutes les représentations irréductibles de dimension finies d'un inverse demi-groupe compact, en utilisant les sous groupes maximaux et les propriétés de l'ordre défini sur l'ensemble des idempotents. Par conséquent, on obtient une preuve simple et nouvelle du théorème suivant: un inverse demi-groupe compact a suffisamment de représentations irréductibles de dimension finie pour séparer les points si et seulement si l'ensemble des idempotent est totalement déconnecté.

Notre dernier théorème est le suivant: toute \star -représentation irréductible avec la norme continues d'un inverse demi-groupe compact sur un espace de Hilbert est de dimension finie.

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Dedication

This thesis is dedicated to my parents, all my family, and all my friends.

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Introduction

Inverse semigroups provide an abstract characterization of partial one-to-one transformations from a set to itself. A *partial transformation* of a set X is a function from A to B , where A and B are subsets of X . The set of all partial bijections form an inverse monoid denoted by $I(X)$.

$$f \circ f^{-1} = 1_{\text{range}(f)} \quad f^{-1} \circ f = 1_{\text{dom}(f)}.$$

Thus an inverse monoid S is a monoid such that for all $x \in S$ there exists a unique x^* such that $x^*xx^* = x^*$ and $xx^*x = x$. Mark Lawson in his book [11] (Inverse Semigroups: The Theory of Partial Symmetries) gives a very good introduction to the subject.

A pseudogroup is an example of an inverse semigroup. Pseudogroups were defined to develop a Galois theory for differential equations. Roughly speaking a *pseudogroup* is just a collection of partial homeomorphisms between the open subsets of a topological space which is closed under composition and inversion.

We can define a restricted product between the elements of an inverse semigroup as follows: $s \cdot t$ exists if and only if $s^*s = tt^*$. An inverse semigroup S with its restricted product form a groupoid; we are going to denote this groupoid by G_S . The category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors. An inductive groupoid G is a groupoid with a partial order defined on G with the following axioms:

- $x \leq y$ implies $x^{-1} \leq y^{-1}$.

- If $x \leq y$ and $u \leq v$ then $xu \leq yv$ whenever the product is defined.
- If $e \leq d(x)$, then there exists a unique element $(x|e)$ such that $(x|e) \leq x$ and $d(x|e) = e$.
- The set of identities forms a meet semilattice.

The following is an example of tiling inverse semigroup. Let Γ be a tiling of \mathbb{R}^n . A *doubly-pointed pattern* is a triple (p_2, P, p_1) where p_1 and p_2 are two distinguished tiles from the pattern P . We say $(p_2, P, p_1) \sim (q_2, Q, q_1)$ if and only if there exists a translation τ of \mathbb{R}^n such that $\tau(P) = Q$, $\tau(p_1) = q_1$ and $\tau(p_2) = q_2$. Let $S(\Gamma)$ be the set of all doubly pointed pattern classes and zero with the following binary operation $[p_2, P, p_1][q_2, Q, q_1] = [\tau_1(p_2), \tau_1(P) \cup \tau_2(Q), \tau_2(q_1)]$ where τ_1 and τ_2 are translations of \mathbb{R}^n such that $\tau_1(P)$ and $\tau_2(Q)$ are patterns and $\tau_1(p_1) = \tau_2(q_2)$. If τ_1 and τ_2 do not exist, then the product defined to be 0. Tiling inverse semigroups are examples of 0- E -unitary inverse semigroup. From such semigroup we can construct a topological groupoid and then C^* -algebra to study a particle moving in a solid which is modeled by tilings. The theory of tiling semigroup was first defined by Johannes Kellendonk in [10].

In this thesis, we want to develop a theory of compact inverse semigroups. A compact inverse monoid is an inverse monoid with a compact Hausdorff topology such that the multiplication and inversion maps are continuous. The following are some examples of compact inverse semigroups.

- Let G be a compact group. A rook matrix over G is an $n \times n$ matrix over $G \cup \{0\}$ with at most one nonzero entry in each column and row. The semigroup $R_n(G)$ of rook matrices over G is a compact inverse monoid.
- The product of the semilattice $([0, 1], \min)$ and the circle group \mathbb{T} (a compact Clifford inverse monoid).

- A semidirect product $E \rtimes G$ of a compact semilattice and a compact group G .

$$(e, g) \star (e', g') = (e(g \cdot e'), gg') \text{ and the inverse } (e, g)^* = (g^{-1} \cdot e, g^{-1}).$$

- Let L/K be an algebraic field extension. Then the set of all isomorphisms between subfields of L which contain K and fix the elements of K pointwise form an inverse Galois monoid.
- Let G be a profinite group. A subset X in G is a coset if and only if $XX^{-1}X = X$ and $X^{-1}XX^{-1} = X^{-1}$. The set $K(G) = \{X \mid X \text{ is a closed coset}\}$ is an inverse monoid with respect to the following product $X \cdot Y =$ the smallest closed coset containing X and Y . In fact if $G = \varprojlim G_i$ then $K(G) = \varprojlim K(G_i)$ where G_i is finite.
- Let G be a compact group acting continuously on a compact Hausdorff space X . The set of all nonempty closed subsets of X denoted by $K(X)$ is a compact space with Vietoris topology. We can extend this action to a continuous one on $K(X)$. The semidirect product $K(X) \rtimes G$ is a compact inverse semigroup, where $K(X)$ is made a semilattice via union.
- In the same setting as the previous example, the semigroup of all partial homeomorphisms of X between open subsets which extend to the acting of an element of G is a compact inverse monoid.
- Our main example is the following: Let G be a compact group acting continuously on a compact space X . Set $K(X)$ to be the set of all non-empty closed subsets of X . Take a quotient of the semidirect product of $K(X) \rtimes G$ by equivalence relation $(A, g) \equiv (B, h)$ if and only if $A = B$, $g^{-1}A = h^{-1}B$ and the mapping $g^{-1}(X \setminus A) \rightarrow (X \setminus A)$ given by $a \mapsto ga$ is the same as the map $h^{-1}(X \setminus B) \rightarrow (X \setminus B)$ given by $b \mapsto gb$ see the example 1.6.2 (2) for more details.

W. D. Munn proved in his paper [12] that a finite dimensional representation of an inverse semigroup is equivalent to a \star -representation if and only if it is bounded. We will give in this thesis a new analytic proof, which is a new proof of an old and more general result of Munn which states that is every finite dimensional representation of a compact inverse semigroup is equivalent to a \star -representation. In fact, suppose that we have a finite dimensional representation π of a compact inverse semigroup S . Since $\pi(S)$ has only finitely many idempotents, we can construct a Haar system on the underling groupoid of the inverse semigroup $\pi(S)$. Using the latter system and the Möbius function we will be able to show that we can turn this representation into a \star -representation. The Möbius function plays crucial role in semigroup representation theory see [24] and [25].

In chapter three, we prove our main result (new): we will parameterize all finite dimensional irreducible representations of a compact inverse semigroup in terms of maximal subgroups and order theoretic proprieties of the idempotent set. As a consequence, we obtain a new and simpler proof of the following theorem of Shneperman [21]: a compact inverse semigroup has enough finite dimensional irreducible representations to separate points if and only if its idempotent set is totally disconnected. The representation theory of discrete inverse semigroups was developed in many papers by Steinberg in [26] and Exel [3]. The third chapter of Clifford and Preston's book [2] (representation theory of finite semigroup) inspired our approach.

We end by proving that any norm continuous irreducible \ast -representation of a compact inverse semigroup on a Hilbert space is finite dimensional.

Chapter 1

Preliminaries

1.1 Inverse semigroup theory

In this chapter, we will give some basic results and facts about the theory of inverse semigroups. Inverse semigroups were first defined independently by Wagner in 1952 [27] and Gordon Preston in 1954 [14]. We are interested in inverse semigroups because they give a nice abstract characterization of partial symmetries; see [11].

1.1.1 Inverse semigroups

A semigroup S is a set together with an associative binary operation on S . An element $e \in S$ is called an *idempotent* if $ee = e$. We denote the set of idempotents by $E(S)$. If a semigroup S contains an identity 1 (i.e., $1s = s1 = s$ for all $s \in S$), then S is called a monoid.

Definition 1.1.1 *Let S be a semigroup. Then S is an inverse semigroup if and only if, for every element $s \in S$, there exists unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.*

The following theorem yields an alternative approach to inverse semigroups.

Theorem 1.1.2 *Let S be a semigroup. Then S is an inverse semigroup if and only if*

1. *S is regular. This means that for every element $a \in S$ there is an element $b \in S$, called an inverse of a , satisfying $a = aba$ and $b = bab$.*
2. *The idempotents of S commute.*

In particular the idempotents of an inverse semigroup form a subsemigroup. A group can be identified as a very special inverse semigroup as the following proposition shows.

Proposition 1.1.3 *An inverse semigroup is a group if and only if it has a unique idempotent element.*

Example 1.1.4 Let I_X be the set of all the partial injective functions from X to itself. Every partial injective function has a unique partial inversion. I_X with the natural way to compose partial functions is an inverse monoid. The identity here is 1_X and the elements 1_A where $A \subset X$ are the idempotent elements that are not identities.

Now, we will give some properties of inverse semigroups that can be found in [11] (Proposition 1.4.1 and Lemma 1.4.2).

Proposition 1.1.5 *Let S be an inverse semigroup.*

1. *For any $s \in S$, both s^*s and ss^* are idempotents and $s(s^*s) = s$ and $(ss^*)s = s$.*
2. *$(s^*)^* = s$ for all $s \in S$.*
3. *For any idempotent $e \in S$ and any $s \in S$, the element s^*es is an idempotent.*
4. *If e is an idempotent in S , then $e^* = e$.*
5. *$(s_1 \dots s_n)^* = s_n^* \dots s_1^*$ for all $s_1, \dots, s_n \in S$ with $n \geq 2$.*

6. For every idempotent e and element s , there exists an idempotent f such that $es = sf$.

7. For every idempotent e and element s , there exists an idempotent f such that $se = fs$.

1.1.2 The natural partial order

We define the *natural partial order* on arbitrary inverse semigroup S . Let $s, t \in S$. Define

$$s \leq t \iff s = te$$

for some idempotent e .

For example, the partial order on I_X (the set of all partial bijections between X and itself) is via restriction. The natural partial order on a group is equality. Indeed, let f, h be in I_X such that $f \leq h \iff h \circ 1_{\text{dom}(f)} = f$ where $f^* \circ f = 1_{\text{dom}(f)}$ is the domain of a function f .

The following results are in [11] (Lemma 1.4.6, Proposition 1.4.7 and Proposition 1.4.8)

Lemma 1.1.6 *Let S be an inverse semigroup. Then the following are equivalent:*

1. $s \leq t$.
2. $s = ft$ for some idempotent f .
3. $s^* \leq t^*$.
4. $s = ss^*t$.
5. $s = ts^*s$.

Definition 1.1.7 *A non-empty subset I of a partially ordered set (P, \leq) is an ideal; if the following conditions hold:*

1. For every x in I , $y \leq x$ implies that y is in I . (I is a lower set or downset.)
2. For every x, y in I , there is some element z in I , such that $x \leq z$ and $y \leq z$. (I is a directed set.)

The dual notion to that of ideal is filter.

Definition 1.1.8 A non-empty subset F of a partially ordered set (P, \leq) is a filter if the following conditions hold:

1. For every x in F , $y \geq x$ implies that y is in F . (F is an upper set.)
2. For every x, y in F , there is some element z in F , such that $x \geq z$ and $y \geq z$. (F is a codirected set.)

In the presence of a semilattice structure, ideals and filters admit simpler descriptions.

Proposition 1.1.9 Let E be a poset.

1. If E is a join semilattice with identity, then a subset $I \subseteq E$ is an ideal if and only if it is a lower set and closed under finite joins.
2. If E is a meet semilattice with identity, then a subset $I \subseteq E$ is a filter if and only if it is an upper set and closed under finite meets.

Definition 1.1.10 Let E be a poset.

1. The smallest ideal that contains a given element p is a principal ideal. The principal ideal p^\downarrow generated by p is the set $p^\downarrow = \{x \in E \mid x \leq p\}$.
2. Similarly, we can define a principal filter as a smallest filter that contains an element p . The principal filter p^\uparrow generated by p is the set $p^\uparrow = \{x \in E \mid x \geq p\}$.

Proposition 1.1.11 Let S be an inverse semigroup.

1. The relation \leq is a partial order on S .
2. For idempotents $e, f \in S$, we have that $e \leq f \iff e = ef = fe$.
3. If $s \leq t$ and $u \leq v$, then $su \leq tv$.
4. If $s \leq t$, then $s^*s \leq t^*t$ and $ss^* \leq tt^*$.
5. $E(S)$ is a lower set of S .

Notice that inverse semigroup homomorphisms preserve order.

Proposition 1.1.12 *Let S be any semigroup. Define a relation \leq on $E(S)$ by*

$$e \leq f \iff e = ef = fe.$$

Then \leq is a partial order on $E(S)$. If S is an inverse semigroup, then $(E(S), \leq)$ is a meet semilattice.

Now, similar to what is done in the group case, Cayley's theorem, which states that every group G is isomorphic to a subgroup of the symmetric group on G , we have the following theorem for inverse semigroups.

Theorem 1.1.13 (The Wagner-Preston representation theorem) *Let S be an inverse semigroup. Then there is a set X and an injective homomorphism $\theta : S \rightarrow I_X$.*

One can take $X = S$.

1.2 Groupoids and Green's relations

1.2.1 Groupoids

To give the definition of groupoid, we will start by defining a category.

Definition 1.2.1 A category \mathcal{G} consists of two sets:

- The class of objects denoted by \mathcal{G}^0 .
- The class of arrows denoted by \mathcal{G}^1 .
- Two mappings from the set of arrows to the set of objects, called source and target, denoted respectively by d and r . One writes an arrow $f : A \rightarrow B$ such that $d(f) = A$ and $r(f) = B$.
- The arrows f and g are composable maps if $r(f) = d(g)$ (we write $\exists gf$). The composition of $f : A \rightarrow B$ and $g : B \rightarrow C$ is written as $g \circ f$ or gf .
- There is a map id which assigns to each object A in \mathcal{G}^0 an identity arrow $id_A : A \rightarrow A$. Such that these two axioms hold:
- If $f : A \rightarrow B$ an arrow then $f id_A = f$ and $id_B f = f$.
- If $f : A \rightarrow B, g : B \rightarrow C$ and $h : C \rightarrow D$ are arrows in \mathcal{G} then $(hg)f = h(gf)$.

A category is small if the arrows and objects are sets. In this thesis, all the categories are assumed small.

Definition 1.2.2 A groupoid is a category \mathcal{G} such that for every element $x \in \mathcal{G}$ there exists x^{-1} such that $xx^{-1} = r(x)$ and $x^{-1}x = d(x)$.

Remark 1.2.3 We can prove that x^{-1} is unique. We can also view a groupoid as a set with a partially defined multiplication for which the usual proprieties of a group hold whenever they make sense.

Notation 1.2.4 Let \mathcal{G} be a groupoid.

- \mathcal{G}^0 the set of identities;

- Let e, f be identities.
 - $\mathcal{G}^e = \{x \in \mathcal{G} \mid r(x) = e\} = r^{-1}(\{e\})$,
 - $\mathcal{G}_e = \{x \in \mathcal{G} \mid d(x) = e\} = d^{-1}(\{e\})$,
 - $\mathcal{G}_f^e = \{x \in \mathcal{G} \mid r(x) = e \text{ and } d(x) = f\} = r^{-1}(\{e\}) \cap d^{-1}(\{f\})$.
- $\mathcal{G}^2 = \{(x, y) \in \mathcal{G} \times \mathcal{G} \text{ such that } r(y) = d(x)\}$
- Let X be a topological space. Then $C(X)$ is the ring of continuous complex-valued functions on X .

Definition 1.2.5 A *topological groupoid* consist of a groupoid \mathcal{G} and a topology τ compatible with the groupoid structure, i.e., the maps

$$m : \mathcal{G}^2 \longrightarrow \mathcal{G} \quad (x, y) \longrightarrow xy \text{ and } i : \mathcal{G} \longrightarrow \mathcal{G} \quad x \longrightarrow x^{-1} \text{ are continuous}$$

Proposition 1.2.6 *Let \mathcal{G} be a topological groupoid. Then*

1. *The map $i : x \longrightarrow x^{-1}$ is a homeomorphism;*
2. *The maps r and d are continuous.*
3. *If \mathcal{G} is Hausdorff, \mathcal{G}^0 is closed in \mathcal{G} ;*
4. *If \mathcal{G}^0 is Hausdorff, \mathcal{G}^2 is closed in $\mathcal{G} \times \mathcal{G}$;*
5. *\mathcal{G}^0 is both a subspace of \mathcal{G} and a quotient of \mathcal{G} (by the map r or d); the induced topology and the quotient topology coincide.*

Here, we will give an example of compact groupoid inside $M_n(\mathbb{C})$. But before that let us define linear isometry and partial isometry.

Definition 1.2.7 *Given two normed vector spaces V and W , a linear isometry is a linear map $f : V \longrightarrow W$ that preserves the norms:*

$$\|f(v)\| = \|v\|$$

for all v in V . Linear isometries are distance-preserving maps in the above sense. They are global isometries if and only if they are surjective.

Definition 1.2.8 A partial isometry W is a linear map between Hilbert spaces H and K such that the restriction of W to the orthogonal complement of its kernel is an isometry. We call the orthogonal complement of the kernel of W the initial subspace of W , and the range of W is called the final subspace of W .

For example, Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Then A is a partial isometry with initial subspace $\{0\} \oplus \mathbb{C}$ and final subspace $\mathbb{C} \oplus \{0\}$.

Example 1.2.9 The set of all partial isometries in $M_n(\mathbb{C})$ forms a compact groupoid

with a partial product defined as follows: $A \cdot B = \begin{cases} AB & \text{if } A^*A = BB^* \\ \text{undefined} & \text{else.} \end{cases}$

Definition 1.2.10 A connected component of a groupoid \mathcal{G} is a maximal subgroupoid $\mathcal{C} \subseteq \mathcal{G}$ such that for every elements $e \neq f \in \mathcal{C}^0$ there exists $s \in \mathcal{C}$ with $ss^{-1} = e$ and $s^{-1}s = f$.

Remark 1.2.11 Notice that the connected component is in the algebraic sense, not the topological sense.

In an inverse semigroup, we can define a restricted product between elements as follows:

$$\exists s \cdot t \iff ss^* = t^*t$$

Proposition 1.2.12 Every inverse semigroup S is a groupoid with respect to its restricted product. (We call this groupoid the underlying groupoid \mathcal{G}_S of S .) Moreover, $\mathcal{G}_S^0 = E(S)$ and the inverse of an element $s \in \mathcal{G}_S^0$ is the element $s^* \in S$.

The following definition is a variant of the definition of *left Haar system* for locally compact groupoids. But instead of locally compact groupoids, we use compact groupoids, and so we want to consider probability measures.

Definition 1.2.13 *A (normalized) left Haar system for a compact groupoid is a family $\{\mu^e\}_{e \in \mathcal{G}^0}$ of regular (probability) Borel measures on each compact Hausdorff space \mathcal{G}^e such that the following three conditions are satisfied:*

1. *The support of each μ^e is the whole \mathcal{G}^e .*
2. *For any $f \in C(\mathcal{G})$ the function,*

$$\begin{aligned} \mathcal{G}^0 &\longrightarrow \mathbb{C} \\ e &\longmapsto \int_{\mathcal{G}^e} f(x) d\mu^e(x) \end{aligned}$$

is continuous; (Continuity condition)

3. *For any $x \in \mathcal{G}$ and $f \in C(\mathcal{G})$,*

$$\int_{\mathcal{G}^{d(x)}} f(xz) d\mu^{d(x)}(z) = \int_{\mathcal{G}^{r(x)}} f(y) d\mu^{r(x)}(y).$$

(Invariance condition)

In the literature, many definitions of left Haar system have been given (see Seda in [18] and [19] and Westman [28] and [29]). The definition that we mentioned above was adopted by Renault in [15]. A left Haar system in a groupoid \mathcal{G} allows us to construct the regular representation of \mathcal{G} . Moreover, the resulting C^* -algebras of \mathcal{G} have been studied by Renault in [15]; the continuity condition was crucial. The definition given above for left Haar system was adopted by Seda in [20]. This definition is the best possible and the continuity condition cannot be replaced by anything weaker.

Haar systems do not exist in general. It is of course the continuity condition which can fail. The existence of a left Haar system requires some topological conditions.

The following counterexample was given by Seda [20].

Counterexample 1.2.14 Let $X = [0, 1]$ denote the closed unit interval in the x -axis and let T denote the unit circle in the yz -plane in \mathbb{R}^3 . Then $M = [0, 1] \times T$ is a compact inverse monoid if we consider the interval $[0, 1]$ as a semilattice with \min and T as a group in the usual way; M is an example of Clifford inverse monoid see [11]. Let G be the subinverse monoid $\{(x, t) \in M \mid x > \frac{1}{2} \Rightarrow t = 1\}$. Then the underlying topological groupoid of G has no left Haar system since the image of a small open ball at $(\frac{1}{2}, -1)$ by r or d is not open. By the Proposition below 1.2.15, G does not have a left Haar system.

The following Proposition is given in Paterson's book [13].

Proposition 1.2.15 *Let G be a locally compact (compact) groupoid with a left Haar system. Then the maps r, d are open maps from G to G^0 .*

In this thesis, we do not need the continuity condition. More work has been done by Peter Hahn in the second countable case [6].

1.2.2 Green's relations on inverse semigroups

Green's relations constitute a fundamental tool in semigroup theory: they characterize the elements of a semigroup in terms of the principal ideals [5].

Definition 1.2.16 *Let S be an inverse semigroup.*

1. $s \mathcal{R} t$ if and only if $ss^* = tt^*$ if and only if $sS = tS$.
2. $s \mathcal{L} t$ if and only if $s^*s = t^*t$ if and only if $Ss = St$.
3. Put $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. This is an equivalence relation. Elements which are \mathcal{H} -related begin and end at the same idempotent of the underlying groupoid.

4. $s \mathcal{D} t$ if and only if s and t are in the same connected component of underlying groupoid; i.e., there exists $a \in S$ such that $t \mathcal{L} a \mathcal{R} s$ (equivalently, there exists $a \in S$ such that $t \mathcal{R} a \mathcal{L} s$)
5. $s \mathcal{J} t$ if and only if $SsS = StS$ if and only if $s \mathcal{D} s' \leq t$ and $t \mathcal{D} t' \leq s$ for some $s', t' \in S$. (For the proof see Lawson's book [11] page 88.)
6. $s \leq_{\mathcal{J}} t$ if and only if $SsS \subseteq StS$.

1.3 Compact semigroups

Definition 1.3.1 We define a compact semigroup to be a semigroup S equipped with a compact (Hausdorff) topology such that the map $S \times S \rightarrow S$ given by $(x, y) \mapsto xy$ is continuous.

Our next Definitions 1.3.2 and 1.3.6 and also Propositions 1.3.3 and 1.3.5 can be found in Rhodes and Steinberg's book [16] and Hofmann's book [8]. They show that Green's relations are closed in a compact semigroup, and prove the stability of Green's relations. Then we can deduce that the relations \mathcal{J} and \mathcal{D} coincide in compact semigroups.

Definition 1.3.2 (Closed equivalence relation) An equivalence relation R on a topological space X is said to be closed if R is a closed subspace of $X \times X$.

Proposition 1.3.3 Let S be a compact semigroup. Then each of Green's relations is closed.

Next, it is proved in [16] that compact semigroups are stable.

Definition 1.3.4 A semigroup S is said to be stable if

$$sx \mathcal{J} s \iff sx \mathcal{R} s \quad \text{and also} \quad xs \mathcal{J} s \iff xs \mathcal{L} s$$

where s and x are in S .

Proposition 1.3.5 *In a stable semigroup, Green's relations \mathcal{J} and \mathcal{D} coincide.*

A proof of the following can be found in [16].

Proposition 1.3.6 *Compact semigroups are stable and so Green's relations \mathcal{J} and \mathcal{D} coincide in a compact semigroup.*

Lemma 1.3.7 *Let S be a compact inverse semigroup. If $e \leq f$ and $e \mathcal{J} f$ then $e = f$.*

Proof: Observe that $e \leq f$ implies that $e = fe = ef$. Then $f \mathcal{J} e = fe$ and so by the stability (Proposition 1.3.6), $f \mathcal{R} fe = e \Rightarrow f = ex$ for some $x \in S$. So $e = ef = eex = ex = f$. ■

We will mostly be interested in compact inverse monoids.

1.4 Algebraic lattices and domains

The order structure of a compact inverse monoid will play a key role in this thesis.

Definition 1.4.1 *Let L be a complete lattice. Then $k \in L$ is a compact element if and only if for every subset X of L such that $k \leq \bigvee X$ there exists $F \subseteq X$ finite such that $k \leq \bigvee F$. We denote the set of compact elements by $\mathcal{K}(L)$*

Remark 1.4.2 The set $\mathcal{K}(L)$ is closed under finite joins.

Proposition 1.4.3 *Let L be a complete lattice. Then $k \in L$ is a compact element if and only if whenever D is a directed subset of L such that $k \leq \bigvee D$, then $k \leq d$ for some $d \in D$.*

Proof: Suppose that whenever D is a directed subset of L such that $k \leq \bigvee D$, then $k \leq d$ for some $d \in D$. Let X be any subset of L such that $k \leq \bigvee X$. Let \mathcal{F} be the set of all finite joins of elements of X . Then $\bigvee \mathcal{F} = \bigvee X$, so $k \leq \bigvee \mathcal{F}$. But \mathcal{F} is directed so $k \leq d$ for some $d \in \mathcal{F}$, but $d = \bigvee F$ for some $F \subseteq X$.

Now suppose that k is a compact element. Then for any directed set D such that $k \leq \bigvee D$ there exist finite subset F of D such that $k \leq \bigvee F$. Thus there exists an element $d \in D$ such that $\bigvee F \leq d$ because D is directed. Thus $k \leq \bigvee F \leq d$ for some $d \in D$. ■

Motivated by this proposition, we can define in general a compact element in a poset as follows:

Definition 1.4.4 *Let P be a poset. Then $x \in P$ is a compact element if and only if whenever D is a directed subset of P such that $x \leq \bigvee D$ (and so in particular $\bigvee D$ exists) then $x \leq d$ for some $d \in D$.*

For a poset P in general, we continue to use $\mathcal{K}(P)$ for the compact elements of P .

Definition 1.4.5 *A complete lattice L is called an algebraic lattice if each of its elements can be written as a join of compact elements.*

Example 1.4.6

- Let X be a set. $(2^X, \cup, \cap)$ is an algebraic lattice, and $\mathcal{K}(2^X)$ is the set of finite subsets of X .
- Let G be a group, the set L of normal subgroups of G with \subseteq is an algebraic lattice; $\mathcal{K}(L) = \{\langle x_1, x_2, \dots, x_n \rangle^G \mid x_1, x_2, \dots, x_n \in G\}$

Definition 1.4.7 *A closure operator on a poset P is a map $c : P \rightarrow P$ such that*

1. *c is order preserving*

2. $x \leq c(x)$ for every $x \in P$

3. $c^2 = c$.

Moreover a closure operator is called continuous if it preserves directed joins, i.e., if D is directed $c(\bigvee D) = \bigvee c(D)$

Lemma 1.4.8 *L is an algebraic lattice if and only if there exists a continuous closure operator $c : 2^X \rightarrow 2^X$ with $c(2^X) \simeq L$ for some set X . Moreover, $\mathcal{K}(L)$ is the image of finite subsets of 2^X under c .*

Proof: Let us denote by $Id(\mathcal{K}(L))$ the set of all ideals of $\mathcal{K}(L)$. Suppose that L is an algebraic lattice. Then

$$c : 2^{\mathcal{K}(L)} \longrightarrow 2^{\mathcal{K}(L)}$$

$$A \mapsto \bigcap_{\substack{I \in Id(\mathcal{K}(L)) \\ A \subseteq I}} I$$

is a continuous closure operator. Indeed, we prove

$$\bigcap_{\substack{I \in Id(\mathcal{K}(L)) \\ A \subseteq I}} I = \{b \in \mathcal{K}(L) \mid \exists a_1, a_2, \dots, a_n \in A \text{ with } b \leq a_1 \vee a_2 \vee \dots \vee a_n\}$$

We need to prove the isomorphism: $c(2^{\mathcal{K}(L)}) = Id(\mathcal{K}(L)) \simeq L$. Let us define the two maps.

$$\alpha : Id(\mathcal{K}(L)) \longrightarrow L \qquad \gamma : L \longrightarrow Id(\mathcal{K}(L))$$

$$I \mapsto \bigvee I \qquad \ell \mapsto \ell^\perp \cap \mathcal{K}(L)$$

The details are completely straightforward. ■

Example 1.4.9 Let G be a group and $L = \text{sub}(G)$ be the lattice of subgroups. Then

$$c : 2^G \longrightarrow 2^G$$

$$X \mapsto \langle X \rangle$$

is a continuous closure operator, moreover $\mathcal{K}(L)$ is equal to the set of all finitely generated subgroups.

Lemma 1.4.10 *Let E be a join semilattice with identity. Then $Id(E)$ is an algebraic lattice and*

$$Id(E) \simeq \text{hom}_\vee(E, 2)$$

where $2 = \{0, 1\}$ is a join semilattice with order $0 \leq 1$ and $\varphi \in \text{hom}_\vee(E, 2)$ means that $\varphi(0) = 0$ and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. Here $\text{hom}_\vee(E, 2)$ is a lattice with reverse of the pointwise ordering.

Proof: Define

$$\alpha : Id(E) \longrightarrow \text{hom}_\vee(E, 2)$$

$$I \mapsto \chi_{I^c}$$

where I^c is the complement of I in $Id(E)$ and $\chi_{I^c}(x) = \begin{cases} 1 & \text{if } x \notin I \\ 0 & \text{if } x \in I. \end{cases}$

Conversely, define

$$\beta : \text{hom}_\vee(E, 2) \longrightarrow Id(E)$$

$$\phi \mapsto \ker(\phi) = \phi^{-1}(0).$$

One checks that α and β are inverses. ■

Corollary 1.4.11 *Let L be an algebraic lattice. Then*

$$L \simeq Id(\mathcal{K}(L)) \simeq \text{hom}_\vee(\mathcal{K}(L), 2).$$

Definition 1.4.12 *Let (S, \leq) be a poset. Then S is an algebraic domain if and only if*

1. S admits directed joins.
2. $\forall x \in S, x^\downarrow \cap \mathcal{K}(S)$ is directed.
3. $x = \bigvee (x^\downarrow \cap \mathcal{K}(S))$.

Algebraic lattices are algebraic domains.

1.5 Compact semilattices

In this section we assume all semilattices are meet semilattices with identity unless we specify they are join semilattices with identity. The following well known results about partially ordered spaces can be found in [4] and in Johnstone's book [9].

Lemma 1.5.1 *Let X be a topological space equipped with a partial order \leq . The following conditions are equivalent:*

1. The set $\{(x, y) \mid x \leq y\}$ is closed in $X \times X$.
2. Given any two points x, y of X with $x \not\leq y$, we can find disjoint (not necessarily open) neighbourhoods U, V of x, y such that U is an upper set and V is lower set.

A topological poset satisfying the conditions of the Lemma 1.5.1 will be called *order-Hausdorff* or a *pospace*. Clearly, any order-Hausdorff space is Hausdorff. The following Proposition and Corollary are taken from Peter T. Johnstone's book [9] page 271.

Lemma 1.5.2 *Let X be pospace. If K is a compact subset of X . Then K^\downarrow is closed.*

In particular, for every $x \in X$ where X is pospace the sets x^\uparrow and y^\downarrow are closed in X .

Corollary 1.5.3 *Let X be a compact pospace, and $x, y \in X$ with $x \not\leq y$. Then there exist disjoint open neighbourhoods U, V of x, y respectively such that U is upper and V is lower.*

Theorem 1.5.4 *Let X be a compact pospace. Then the sets of the form $U \cap V$, where U is an open upper set and V is an open lower set, form a base for the topology on X .*

Theorem 1.5.5 (Monotone Convergence Theorem) *Let X be a compact pospace. Then*

1. *Any directed set $D \subseteq X$, considered as a net in X , converges to a (unique) limit, which is a least upper bound for D in X , (i.e., $\lim D = \bigvee D$).*
2. *Similarly, any codirected $D \subseteq X$ converges to its greatest lower bound in X , (i.e., $\lim D = \bigwedge D$).*

Proof: We will prove (1); (2) is dual. Suppose x is a limit point of $D = \{x_\alpha\}$. We fix α_0 . Then $x_\alpha \geq x_{\alpha_0}$ for all α large enough. Since $x_{\alpha_0}^\uparrow$ is closed, thus $x \geq x_{\alpha_0}$. Since α_0 is arbitrary therefore $x \geq x_\alpha$ for all α . Suppose y is also an upper bound for D . Then y^\downarrow is closed and contains the net $D = \{x_\alpha\}$. Hence $x \in y^\downarrow$, that is $x \leq y$. Therefore, $x = \bigvee \{x_\alpha\}$, the least upper bound. Since any limit point is equal to the join of the net D , therefore there is a unique limit point. But X is compact, this implies that the net converges to $x = \lim D = \bigvee D$. ■

Definition 1.5.6 *A complete lattice is called meet continuous if*

$$x \wedge \bigvee D = \bigvee_{d \in D} (x \wedge d)$$

for any directed set D .

Not all complete lattices are meet continuous.

Counterexample 1.5.7 Let $L =]0, 1[^2 \cup \{(0, 0)\} \cup \{(1, 1)\}$ be the subset of the square $[0, 1]^2$. Then L is complete lattice but is not meet continuous. Consider $D = \{\frac{1}{3}\} \times]0, 1[$ with $\sup D = (1, 1)$, if $x = (\frac{2}{3}, \frac{1}{2})$, then we have $x \sup D = x$, but $\sup xD = (\frac{1}{3}, \frac{1}{2}) \neq x$.

Proposition 1.5.8 *If E is a compact semilattice, then E is a compact pospace.*

Proof: Let E be a compact semilattice. Then the set $\{(a, b) \mid a \leq b\}$ is the equalizer of the multiplication and the projection to the first coordinate. ■

From this we shall obtain that E is a complete lattice.

Corollary 1.5.9 *Let E be a compact semilattice with identity. Then E is a complete lattice which is meet continuous.*

Proof: Since E is a semilattice with identity therefore E has an empty meet. The pairwise meet is the product and any codirected set has a meet (by the monotone convergence theorem). Thus E has all meets. Since the multiplication is continuous therefore E is meet continuous using the monotone convergence again. ■

The following theorem is taken from [9] page 274.

Theorem 1.5.10 *Let E be a compact semilattice. Then a downset is closed if and only if it is closed under directed joins.*

Proposition 1.5.11 *Let E be a compact semilattice and $e \in E$. Then $e \in \mathcal{K}(E)$ if and only if e^\uparrow is clopen if and only if e^\uparrow is open.*

Proof: Suppose that e^\uparrow is clopen and let D be a directed set such that $e \leq \bigvee D = \lim D$. So $\lim D \in e^\uparrow$ which is open, so there exists $d \in D$ such that $d \in e^\uparrow$, thus $e \leq d$. Therefore e is compact.

Conversely, e^\uparrow is closed by the Lemma 1.5.2. To prove the complement of e^\uparrow is closed, we first observe that lower sets are closed if and only if they are closed under directed joins. So let $D \subseteq E \setminus e^\uparrow = e^\downarrow$ be a directed set, and suppose that $\bigvee D \notin e^\downarrow$; then $e \leq \bigvee D$ which implies by compactness of e that there exists $d \in D$ such that $e \leq d$, contradiction. ■

Notation 1.5.12

- Let E be a topological semilattice with identity. Then $\text{hom}_{\text{Top}}(E, \{0, 1\})$ is the set of all continuous monoid homomorphisms where $\{0, 1\}$ is a semilattice with the discrete topology.
- Let E be a complete semilattice. Then $\text{hom}_{\text{Lat}}(E, \{0, 1\})$ is the set of all mapping which preserve all meets and directed joins.

The following theorem is taken from “A Compendium of Continuous Lattices” [4].

Theorem 1.5.13 *Let E be a compact semilattice with identity. Then*

$$\text{hom}_{\text{Top}}(E, \{0, 1\}) \cong \text{hom}_{\text{Lat}}(E, \{0, 1\})$$

and they are join semilattices with respect to the reverse of the pointwise ordering. (i.e, pointwise ordering means $f \leq g \iff f(x) \leq g(x)$ for all x).

Let us denote $\widehat{E} = \text{hom}_{\text{Top}}(E, \{0, 1\}) = \text{hom}_{\text{Lat}}(E, \{0, 1\})$. The following is well known:

Theorem 1.5.14 $\widehat{E} \simeq \mathcal{K}(E)$.

Proof: If $\varphi : E \longrightarrow \{0, 1\}$, then $\bigwedge \varphi^{-1}(1) \in \mathcal{K}(E)$. And if $e \in \mathcal{K}(E)$, then $\chi_{e^\uparrow} \in \text{hom}_{\text{Top}}(E, \{0, 1\})$. ■

Definition 1.5.15 *A profinite semilattice is a topological semilattice that is isomorphic to the inverse limit of an inverse system of discrete finite semilattices.*

The following theorem is proved in [9].

Theorem 1.5.16 *A compact semilattice E is totally disconnected if and only if it is profinite.*

As a corollary, one has Hofmann's duality for 0-dimensional semilattices [7].

Corollary 1.5.17 *Let E be a compact semilattice. Then \widehat{E} separates the points of E (i.e., if $e, f \in E$ there exists $\varphi \in \widehat{E}$ such that $\varphi(e) \neq \varphi(f)$) if and only if E is an algebraic lattice if and only if E is totally disconnected.*

1.6 Compact inverse monoids

In this section, we will introduce some facts about compact inverse monoids.

Definition 1.6.1 *A compact inverse monoid (semigroup) is an inverse monoid (semigroup) S with a compact Hausdorff topology such that the multiplication and inversion are continuous.*

1.6.1 Examples of compact inverse semigroups

First we provide trivial examples.

1. Compact groups.
2. Compact semilattices such as $[0, 1]$ equipped with the min operation..
3. Closed subsemigroups of $(T \cup \{0\})^X$ with the product topology where $T = \{z \in \mathbb{C} \mid |z| = 1\}$.

Here we will give more examples of compact inverse semigroups.

Example 1.6.2

1. Let G be a compact group, let E be a compact lattice, and suppose that $G \times E \rightarrow E$ is a continuous action. Define $S = E \rtimes G = E \times G$ with the following product:

$$(e, g)(f, h) = (e \cdot (gf), gh)$$

and the inverse

$$(e, g)^* = (g^{-1}e, g^{-1}).$$

Thus the semidirect product $S = E \rtimes G$ is a compact inverse semigroup. Define a congruence \equiv on S as follows:

$$(e, g) \equiv (f, h) \iff e = f \text{ and } g^{-1}e = h^{-1}f.$$

We claim \equiv is closed. Suppose $(e, g) \not\equiv (f, h)$.

Case 1: If $e \neq f$, then there exist U, V disjoint neighbourhoods of e, f respectively such that $(U \times G) \times (V \times G)$ is a neighbourhood of $((e, g), (f, h))$ which does not intersect the congruence.

Case 2: Suppose that $g^{-1}e \neq h^{-1}f$. Let W, Z be disjoint neighbourhoods of $g^{-1}e, h^{-1}f$ respectively. Then there exist neighbourhoods U of (e, g) and V of (f, h) such that if $(e', g') \in U$ then $(g')^{-1}e' \in W$ and for (f', h') in V then $(h')^{-1}f' \in Z$. Then $U \times V$ is a neighbourhood of $((e, g), (f, h))$ such that $U \times V$ misses the congruence.

2. Let $X \neq \emptyset$ be a compact space. Denote by $K(X)$ the set of nonempty closed subsets of X , that is,

$$K(X) := \{\emptyset \neq Y \subseteq X \mid Y \text{ is closed}\}$$

Define the Vietoris topology on $K(X)$ as follows. Let $\gamma = \{U_1, U_2, \dots, U_n\}$ be a finite family of open sets. Let us denote by Γ the set of all finite families of open sets.

A basis for the topology consists of

$$\mathcal{N}(\gamma) := \{Y \in K(X) \mid Y \subseteq \bigcup_{U_i \in \gamma} U_i \text{ and } Y \cap U_i \neq \emptyset \text{ for all } i\}$$

It is known that $K(X)$ is a compact Hausdorff space and the union is continuous in this topology and so $(K(X), \cup)$ is a compact semilattice see [9].

Let us verify that

$$\begin{aligned} \alpha : K(X) \times K(X) &\longrightarrow K(X) \\ (A, B) &\mapsto A \cup B \end{aligned}$$

is continuous. Indeed, let $A, B \in K(X)$ and $V(\gamma)$ be an open neighbourhood of $A \cup B$ where $\gamma = \{U_1, U_2, \dots, U_n\}$. Put $\gamma_1 = \{U \in \gamma \mid A \cap U \neq \emptyset\}$ $\gamma_2 = \{U \in \gamma \mid U \cap B \neq \emptyset\}$. Therefore, $V(\gamma_1)$ is an open neighbourhood of A and $V(\gamma_2)$ is an open neighbourhood of B . Moreover, $\alpha(V(\gamma_1) \times V(\gamma_2)) \subseteq V(\gamma_1 \cup \gamma_2)$. Indeed, let $(C, D) \in V(\gamma_1) \times V(\gamma_2)$. Thus $C \subseteq \bigcup_{U \in \gamma_1} U$ and $D \subseteq \bigcup_{U \in \gamma_2} U$. So $C \cup D \subseteq \bigcup \gamma$. Moreover, if $U \in \gamma$ then $U \in \gamma_i$ for some $i \in \{1, 2\}$. So either $C \cap U$ or $D \cap U$ is not empty. Thus the map α is continuous.

Consider $\mathcal{B} = \{\mathcal{N}(\gamma) \mid \gamma \in \Gamma\}$ the basis of the Vietoris topology on $K(X)$.

Remark 1.6.3 The mapping

$$\begin{aligned} j : X &\longrightarrow K(X) \\ x &\mapsto \{x\} \end{aligned}$$

defines a topological embedding of X in $K(X)$.

Proof: Clearly j is one-to-one. The mapping j is open. If $U \subseteq X$ is open, put $\gamma = \{U\}$. Then $j(U) = j(X) \cap V(\gamma)$. ■

Now, we want to prove some basic properties about the Vietoris topology. Let

$f : X \longrightarrow Y$ be a continuous map. Define $f_* : K(X) \longrightarrow K(Y)$. Let $\gamma = \{U_1, \dots, U_n\}$ where U_1, \dots, U_n are open subsets in Y . $N(\gamma) = \{Z \subseteq Y \mid Z \subseteq \bigcup_{U_i \in \gamma} U_i \text{ and } Z \cap U_i \neq \emptyset \text{ for all } i\}$. Let $f^{-1}(\gamma) = \{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$. Then $f_*^{-1}(N(\gamma)) = N(f^{-1}(\gamma))$.

Indeed, if $A \in f_*^{-1}(N(\gamma))$ then $f(A) \subseteq \bigcup_{U_i \in \gamma} U_i$ and $f(A) \cap U_i \neq \emptyset$ for all i . Then $A \subseteq f^{-1} \circ f(A) \subseteq f^{-1} \left(\bigcup_{U_i \in \gamma} U_i \right) = \bigcup_{U_i \in \gamma} f^{-1}(U_i)$. Since $f(A) \cap U_i \neq \emptyset$, there exists $a \in A$ such that $f(a) \in U_i$. Thus $a \in A \cap f^{-1}(U_i)$. Therefore $A \in N(f^{-1}(\gamma))$.

Suppose $A \in N(f^{-1}(\gamma))$. Then $A \subseteq \left(\bigcup_{U_i \in \gamma} f^{-1}(U_i) \right)$ and $A \cap f^{-1}(U_i) \neq \emptyset$ for all i . So $f(A) \subseteq f \left(\bigcup_{U_i \in \gamma} f^{-1}(U_i) \right) = \bigcup_{U_i \in \gamma} f \circ f^{-1}(U_i) \subseteq \bigcup_{U_i \in \gamma} U_i$. $f(A) \cap U_i \supseteq f(A \cap f^{-1}(U_i)) \neq \emptyset$, since $A \cap f^{-1}(U_i) \neq \emptyset$. So $f(A) \in N(\gamma)$. Therefore $A \in f_*^{-1}(N(\gamma))$.

Therefore K is a functor from compact Hausdorff spaces to compact Hausdorff semilattices.

Lemma 1.6.4 *If X, Y are compact spaces then the map*

$$\begin{aligned} \varphi : K(X) \times K(Y) &\longrightarrow K(X \times Y) \\ (A, B) &\mapsto A \times B \end{aligned}$$

is continuous.

Proof: Let $A \times B \in N(\gamma)$, where $\gamma = \{U_1, \dots, U_n\}$, be a neighbourhood of $A \times B$. Then $\bigcup_{U_i \in \gamma} U_i = \bigcup_{\alpha \in I} (V_\alpha \times W_\alpha)$ where V_α and W_α are open in X and Y respectively. There exist $\alpha_1, \dots, \alpha_m$ such that $A \times B \subseteq \bigcup_{i=1}^m (V_{\alpha_i} \times W_{\alpha_i})$ by compactness. Let $V = \bigcup_{i=1}^m V_{\alpha_i}$ and $W = \bigcup_{i=1}^m W_{\alpha_i}$. Then $A \subseteq V$ and $B \subseteq W$ where V and W are open. Moreover, $V \times W \subseteq \bigcup_{U_i \in \gamma} U_i$ and $(A \times B) \cap U_i \neq \emptyset$ for all i . Choose $(a_i, b_i) \in (A \times B) \cap U_i$. Then there exist L_i and M_i open in X and Y respectively such that $a_i \in L_i \subseteq V$ and $b_i \in M_i \subseteq W$ and $L_i \times M_i \subseteq U_i$.

The claim is that if $\gamma_1 = \{V, L_1, \dots, L_m\}$ and $\gamma_2 = \{W, M_1, \dots, M_m\}$ then $N(\gamma_1) \times N(\gamma_2)$ contains (A, B) and maps into $N(\gamma)$. Indeed, $A \subseteq V$ then $A \cap V \neq \emptyset$ and $a_i \in A \cap L_i$ for all i . So $A \in N(\gamma_1)$. Similarly $B \in N(\gamma_2)$. If $(C, D) \in N(\gamma_1) \times N(\gamma_2)$ then $C \times D \subseteq V \times W \subseteq \bigcup_{U_i \in \gamma} U_i$ and $(C \times D) \cap (L_i \times M_i) \neq \emptyset$ and $L_i \times M_i \subseteq U_i$. So $(C \times D) \cap U_i \neq \emptyset$ implies $C \times D \in N(\gamma)$. Thus φ is continuous. ■

Lemma 1.6.5 *Let G be a topological group and X a topological space. Every continuous action of G on X is extendible to an action of G on $K(X)$.*

Proof: Let $f : G \times X \longrightarrow X$ be the action. We have the following composition of continuous maps

$$G \times K(X) \xrightarrow{j \times 1} K(G) \times K(X) \xrightarrow{\varphi} K(G \times X) \xrightarrow{K(f)} K(X).$$

The composition sends (g, A) to $gA = \{ga \mid a \in A\}$. ■

We let $K'(X) = K(X) \cup \{\emptyset\}$ where \emptyset is an isolated point then $K'(X)$ is a compact semilattice. Then G acts continuously on $K(X)$ by lemma 1.6.5 and hence on $K'(X)$ where $g\emptyset = \emptyset$ for all $g \in G$. Thus we can form the compact inverse monoid $K'(X) \rtimes G$. Define

$$\begin{aligned} \varphi : K'(X) \rtimes G &\longrightarrow I_X \\ (C, g) &\mapsto 1_{(X \setminus C)} \circ g \end{aligned}$$

where we view g as a permutation of X . So $1_{(X \setminus C)} \circ g : g^{-1}(X \setminus C) \longrightarrow X \setminus C$ maps x to gx . The image $I(G, X)$ of φ consists of all restriction of the elements of G to the open subsets of X (or all homeomorphisms between open subsets of X that are restrictions of actions of elements of G). We endow $I(G, X)$ with

the quotient topology from φ . To show $I(G, X)$ is a compact inverse monoid, we just need to check the congruence defined below is closed.

$$(C, g) \equiv (D, h) \iff \varphi(C, g) = \varphi(D, h).$$

Indeed, assume that $(C, g) \not\equiv (D, h)$. We find neighbourhoods A and B of (C, g) and (D, h) respectively such that $A \times B$ does not intersect \equiv . Set $U = X \setminus C$ and $V = X \setminus D$.

Case 1: If $C \neq D$. Take disjoint neighbourhoods N_1 and N_2 of C, D in $K'(X)$. Then $A = N_1 \times G$ and $B = N_2 \times G$ do the job.

Case 2: If $g^{-1}C \neq h^{-1}D$. Let N_1, N_2 be a disjoint open neighbourhoods of $g^{-1}C, h^{-1}D$ in $K'(X)$. Then there exist neighbourhoods A, B of $(C, g), (D, h)$ respectively such that the action takes A into N_1 and B into N_2 . Then A and B work.

Case 3: $C = D$ and $g^{-1}C = h^{-1}D$ but there exists $x \in g^{-1}U = h^{-1}V$ such that $gx \neq hx$. Choose disjoint neighbourhoods N_1, N_2 of gx, hx respectively. There exist neighbourhoods M_1, M_2 of g and h respectively such that $M_i x \subseteq N_i$ by continuity of the action. The set

$$W = \{Y \in K'(X) \mid x \notin Y\} = N(\gamma) \cup \{\emptyset\}$$

where $\gamma = \{X - \{x\}\}$ is open in $K'(X)$. Since $x \in g^{-1}U$ implies that $x \notin g^{-1}C$ and similarly $x \notin h^{-1}D$, there exist (by the continuity of the action of G on $K'(X)$) neighbourhoods L_1, L_2 of g, h respectively and R_1, R_2 of $C = D$ in $K'(X)$ such that $g' \in L_1, C' \in R_1$ that implies that $(g')^{-1}C' \in W$ and $h' \in L_2, D' \in R_2$ implies that $(h')^{-1}D' \in W$. Let $A = R_1 \times (L_1 \cap M_1)$ and $B = R_2 \times (L_2 \cap M_2)$. So $(C, g) \in A$ and $(D, h) \in B$. If $(C', g') \in A$, then

- (a) $x \notin (g')^{-1}C'$ implies that $x \in (g')^{-1}(X \setminus C')$
- (b) $g'x \in N_1$ since $g' \in M_1$.

If $(D', h') \in B$, then

- (a) $x \notin (h')^{-1}D'$ implies that $x \in (h')^{-1}(X \setminus D')$
- (b) $h'x \in N_2$.

Thus $\varphi(C', g') \neq \varphi(D', h')$.

Remark 1.6.6 For a compact space X , $K(X)$ is algebraic if and only if X admits a basis consisting of compact open (clopen) sets (that is, X is totally disconnected).

3. $K'(\{0, 1\}^\omega) \rtimes \text{Isom}(\{0, 1\}^\omega)$ is a compact inverse monoid (actually this is profinite). So is $I(\text{Isom}(\{0, 1\}^\omega), \{0, 1\}^\omega)$.
4. Let G be compact group; $K'(G) \rtimes G$ is a compact inverse monoid.
5. Let G be a compact group. A *rook matrix* over G is an $n \times n$ matrix over $G \cup \{0\}$ with at most one nonzero entry in each column and row. The semigroup $R_n(G)$ of rook matrices over G is a compact inverse monoid. Rook monoids are defined later in the definition 3.2.1. We will show in the proposition 3.2.3 that the rook matrices $R_n(G)$ form a compact inverse monoid. A profinite inverse monoid is an inverse limit of finite inverse monoids; it is thus a totally disconnected compact inverse monoid.
6. The Galois inverse monoid is an example of compact inverse monoid. Similarly to what is done for Galois groups, the Galois theory of field extensions of infinite degree gives rise naturally to Galois inverse monoids that are profinite. Specifically, if K/k is a field extension, we consider the inverse monoid $\text{InvGal}(K/k)$ consisting of all field isomorphisms from $\sigma : L \rightarrow L'$ where L and L' are subextensions of K , and also σ keeps all elements of k fixed.

This inverse monoid is a profinite inverse monoid with respect to the Krull topology. A basis is given by sets of the form $B(\sigma)$ where $\sigma : L \rightarrow L'$ is an isomorphism with $[L : k]$ and $[L' : k]$ finite and $B(\sigma) = \{\tau \in \text{InvGal}(K/k) \mid \tau \geq \sigma\}$. This definition is equivalent to the following definition:

This inverse monoid is the inverse limit of the finite inverse semigroups $\text{InvGal}(F/k)$, where F ranges over all intermediate fields such that F/k is a finite extension.

The resulting collection of open sets forms a topology on $\text{InvGal}(K/k)$ called the Krull topology, and $\text{InvGal}(K/k)$ is a profinite inverse monoid under the Krull topology. Note that $\text{Gal}(K/k)$ is the group of units of $\text{InvGal}(K/k)$.

7. Let G be a profinite group.

A subset X in G is a coset iff $XX^{-1}X = X$ and $X^{-1}XX^{-1} = X^{-1}$.

The set $\mathbf{K}(G) = \{X \mid X \text{ is a closed coset}\}$ is an inverse monoid with respect to the following product $X \cdot Y =$ the smallest closed coset containing X and Y . In fact if $G = \varprojlim G_i$ then $\mathbf{K}(G) = \varprojlim \mathbf{K}(G_i)$ where G_i is finite.

8. Every compact inverse semigroup with finitely many idempotents is a closed subsemigroup of a finite direct product inverse semigroups of the form $R_n(G)$ with G a compact group via the Schützenberger representation.
9. Every compact inverse semigroup with totally disconnected semilattice of idempotents is a closed subsemigroup of direct products of inverse semigroups of the form $R_n(G)$ with G a compact group. This will follow from results that we prove later.

1.6.2 Properties of compact inverse semigroups

Proposition 1.6.7 *If S is a compact semigroup which is algebraically an inverse semigroup, then the inverse map $S \rightarrow S$*

$$s \mapsto s^*$$

is continuous, i.e., S is a compact inverse semigroup.

Proof: Let $\{x_\alpha\}$ be a net in S which converges to $x \in S$. We need to prove that x_α^* converges to x^* . Let $\{x_{\alpha_j}^*\}$ be a convergent subnet, say it converges to s . Then $xsx = \lim x_{\alpha_j} x_{\alpha_j}^* x_{\alpha_j} = \lim x_{\alpha_j} = x$. Similarly, $sxs = s$ and thus $s = x^*$. So any convergent subnet of $\{x_\alpha^*\}$ converges to x^* . Therefore, by compactness $x_\alpha^* \rightarrow x^*$. ■

In a manner similar to that of Proposition 1.5.8, we can prove the following more general result.

Proposition 1.6.8 *If S is a compact inverse monoid, then (S, \leq) is a compact pospace. Also $E(S)$ is a compact semilattice (with identity).*

Proof: Let s, t be in S . We have $s \leq t \iff ss^*t = s$. Therefore the set $\{(s, t) \in S \times S \mid s \leq t\}$ is the equalizer of the projection to the first coordinate and the continuous map $S \times S \rightarrow S$

$$(s, t) \mapsto ss^*t.$$

Thus the relation \leq is closed. Observe that the two maps

$$r : S \rightarrow S$$

$$d : S \rightarrow S$$

$$s \mapsto ss^*$$

$$s \mapsto s^*s$$

are continuous and also $E(S) = r(S) = d(S)$, thus $E(S)$ is compact. ■

In the following proposition we show that if a \mathcal{J} -class contains a compact idempotent element then it follows that all its idempotents are compact.

Proposition 1.6.9 *Let S be a compact inverse monoid and $e, f \in E = E(S)$. If $e \in \mathcal{K}(E)$ such that $f \mathcal{J} e$ then $f \in \mathcal{K}(E)$.*

Proof: Since S is compact, then $e \mathcal{J} f$ implies $e \mathcal{D} f$. Therefore, there exists $s \in S$ such that $s^*s = e$ and $ss^* = f$, and so $f = ses^*$. Now let D be a directed set in $E(S)$ such that $f = ses^* \leq \bigvee D$. Then $e = s^*ses^*s = s^*fs \leq s^*(\bigvee D)s = s^*\lim Ds = \lim s^*Ds = \bigvee s^*Ds$. Since s^*Ds is directed, there exists $d \in D$ such that $e \leq s^*ds$ and so $f = ses^* \leq ss^*ds s^* \leq d$. Thus $f \in \mathcal{K}(E)$. \blacksquare

We define a \mathcal{J} -class J to be order compact if it contains a compact element.

Notation 1.6.10 Let S be a compact inverse semigroup and let J be a \mathcal{J} -class.

- $J^\dagger = \{s \in S \mid SJS \not\subseteq SsS\} = \{s \in S \mid J \not\leq_{\mathcal{J}} s\}$ is an ideal in S and is closed if and only if J is order compact, in which case it is clopen. (by Theorem 1.6.11 below).
- $J^\uparrow = \{s \in S \mid SJS \subseteq SsS\}$.
- Define $J^0 = SJS/(J^\dagger \cap SJS)$; this is in bijection with $J \cup \{0\}$.
- Moreover, S/J^\dagger is in bijection with $J^\uparrow \cup \{0\}$.

Theorem 1.6.11 *Let S be a compact inverse monoid. Let J be a \mathcal{J} -class. Then*

J^\dagger is clopen if and only if J contains a compact idempotent e .

Proof: Let e be a compact idempotent element in J . First we prove J^\dagger is closed. Let $\{s_\alpha\}_{\alpha \in D} \subseteq J^\dagger$ converge to s for some directed set D . Therefore for every $\alpha \in D$ we have $e \leq_{\mathcal{J}} s_\alpha$ thus $e = h_\alpha s_\alpha t_\alpha$ for some $h_\alpha, t_\alpha \in S$. Without loss of generality, we can assume that h_α and t_α converge to h and t by passing to the subnet. Then $e = hst$ which implies that $e \leq_{\mathcal{J}} s$, therefore J^\dagger is closed.

Now we will prove that J^\uparrow is open. Observe that $J^\uparrow = \{s \in S \mid e \leq_{\mathcal{J}} s\} = \{s \in S \mid e \leq_{\mathcal{J}} ss^*\} = r^{-1}(E(J^\uparrow))$. So it suffices to prove that $E(J^\uparrow)$ is open in $E(S)$. Then using the fact that the map $r : S \rightarrow E(S)$ is a continuous map, we deduce that J^\uparrow is open. Now $e \in SfS$ if and only if there exists $u \in E(S)$ such that $e \mathcal{D} u \leq f$; thus

$$\begin{aligned} E(J^\uparrow) &= \{f \in E(S) \mid \exists u \in E(S) \ e \mathcal{D} u \leq f\} \\ &= \bigcup_{\substack{u \in E(S) \\ u \mathcal{D} e}} \{f \in E(S) \mid u \leq f\} \\ &= \bigcup_{\substack{u \in E(S) \\ u \mathcal{D} e}} u^\uparrow. \end{aligned}$$

Since $e \mathcal{D} u$, Proposition 1.6.9 implies that u is a compact element. By Proposition 1.5.11, u^\uparrow is open in $E(S)$. Thus $E(J^\uparrow)$ is open in $E(S)$.

Conversely, suppose that the complement of J^\uparrow in S is closed and let $e \in E(J)$. We will prove that the complement of e^\uparrow in $E(S)$ is closed, therefore e is compact.

Let $(x_d)_{d \in D}$ be a net in the complement of e^\uparrow in $E(S)$ converging to x . Suppose that $x \in e^\uparrow$. Then $e \leq x$, which implies that $e = ex = xe$. But $ex_d < e$ since $x_d \notin e^\uparrow$. Thus $ex_d \leq_{\mathcal{J}} e$. By Lemma Lemma 1.3.7, we cannot have $ex_d \mathcal{J} e$. Therefore, $ex_d <_{\mathcal{J}} e$ which implies that $ex_d \notin J^\uparrow$. But (ex_d) converges to $ex = e$, so using the fact that the complement of J^\uparrow is closed implies that e is in the complement of J^\uparrow which is a contradiction. \blacksquare

The following proposition is a consequence of the Monotone Convergence Theorem.

Proposition 1.6.12 *Let S be a compact inverse monoid. If $D \subseteq S$ is directed, then $\bigvee D$ exists and $\lim D = \bigvee D$ (viewed as a net). Moreover, $s \cdot \bigvee D = \bigvee (s \cdot D)$ and $(\bigvee D) \cdot s = \bigvee (D \cdot s)$.*

Proposition 1.6.13 *Let S be a compact inverse semigroup. Then we have the following properties:*

1. *All the \mathcal{J} -classes are closed.*
2. *Let J be a \mathcal{J} -class. Then $E(J) = J \cap E(S)$ is closed hence compact.*
3. *Let $e \in E(S)$ be a compact element. Then $E(J_e) = \{f \in E(S) \mid SeS = Sfs\}$ is finite.*
4. *Let e be an idempotent element in a \mathcal{J} -class J_e . Then $eSe \cap J_e = G_e^e$ where $G_e^e = \{s \in S \mid ss^* = e = s^*s\}$ is the maximal subgroup at e .*

Proof:

1. Let J be a \mathcal{J} -class; the claim is that J is closed. Indeed, let $(s_{\lambda \in D}) \subseteq J$ a net converging to s , where D is a directed set. Let $j \in J$. Then $Ss_{\lambda}S = SjS$, therefore there exists $u_{\lambda}, v_{\lambda}, u'_{\lambda}$ and v'_{λ} in S such that $u_{\lambda}s_{\lambda}v_{\lambda} = j$ and $u'_{\lambda}jv'_{\lambda} = s_{\lambda}$. By compactness and subnet convergence we can assume that (u_{λ}) converges to u , (u'_{λ}) converges to u' , (v_{λ}) converges to v and (v'_{λ}) converges to v' . So, $usv = \lim u_{\lambda}s_{\lambda}v_{\lambda} = j$ and $u'jv' = \lim u'_{\lambda}jv'_{\lambda} = \lim s_{\lambda} = s$ which implies that $SsS = SjS$, thus $s \in J$.
2. J and $E(S)$ are closed so consequently $E(J)$ is compact.
3. We proved in the Proposition 1.6.9 that all the elements of $E(J_e)$ are compact. We have $E(J_e) \subseteq \bigcup_{e \in E(J_e)} e^{\uparrow}$. Thus $E(J_e) \subseteq e_1^{\uparrow} \cup e_2^{\uparrow} \cup \dots \cup e_n^{\uparrow}$ for some $n \in \mathbb{N}$. Let $f \in E(J_e)$. Then $f \geq e_i$ for some $i \in \{1, 2, \dots, n\}$ which implies $f = e_i$ because the idempotents of any \mathcal{J} class are not comparable by Lemma 1.3.7.
4. Let $x \in eSe$ and $x \mathcal{J} e$. Then $ex = x \mathcal{J} e$. By the stability of the compact inverse semigroup S , we conclude that $x = ex \mathcal{R} e$. Trivially, $G_e^e \subseteq eSe \cap J_e$. Similarly $x \mathcal{L} e$. Therefore $x \mathcal{H} e$. Thus $x \in G_e^e$.

The rest of the proposition shows that the property of an element of S being compact is controlled by the idempotents. ■

Proposition 1.6.14 *Let S be a compact inverse monoid, then $x \in \mathcal{K}(S)$ if and only if $xx^* \in \mathcal{K}(E(S))$, if and only if $x^*x \in \mathcal{K}(E(S))$.*

Proof: We will prove the first if and only if statement; the second is trivial. Suppose that x is compact. Let $D \subseteq E(S)$ be a directed set such that $xx^* \leq \bigvee D$. Then $xx^*x \leq (\bigvee D)x = \bigvee Dx$ implies there exists $d \in D$ such that $x \leq dx \Rightarrow xx^* \leq dxx^* \leq d$. Thus xx^* is compact.

Conversely, let $D \subseteq S$ be a directed set such that $x \leq \bigvee D$. This means that $x \leq \bigvee_{d \in D} d$. Then $r(x) \leq r(\bigvee_{d \in D} d) = \bigvee_{d \in D} r(d)$ by the continuity of r . Hence there exists $d_1 \in D$ such that $xx^* \leq r(d_1) = d_1d_1^*$. Then $x = xx^*x \leq d_1d_1^*(\bigvee D) = d_1$, since $d_1 \leq \bigvee D$ implies $d_1d_1^*(\bigvee D) = d_1$. Thus x is compact. ■

It follows that $\mathcal{K}(S)$ is a full subgroupoid of the underlying groupoid \mathcal{G}_S . It also follows that an idempotent is compact in S if and only if it is compact in $E(S)$.

Theorem 1.6.15 *Let S be a compact inverse monoid. Then the following conditions are equivalent:*

1. (S, \leq) is an algebraic domain,
2. $E(S)$ is an algebraic lattice,
3. $E(S)$ is a compact totally disconnected semilattice.

Proof: Since $E(S)$ is a compact semilattice, it is a complete lattice. Suppose that (S, \leq) is an algebraic domain. Let $e \in E(S)$. Then $e = \bigvee (e^\downarrow \cap \mathcal{K}(S)) = \bigvee (e^\downarrow \cap \mathcal{K}(E(S)))$, thus $E(S)$ is an algebraic lattice. Now suppose that $E(S)$ is

algebraic and prove that (S, \leq) is an algebraic domain. We have to prove the following conditions hold

1. S admits directed joins.
2. For all $s \in S$ the set $s^\downarrow \cap \mathcal{K}(S)$ is directed.
3. $s = \bigvee (s^\downarrow \cap \mathcal{K}(S))$ for all $s \in S$.

By the compactness of S , the first condition is true. Let $x_1, x_2 \in s^\downarrow \cap \mathcal{K}(S)$ which implies that $x_1 \leq s$, $x_2 \leq s$ and $x_1, x_2 \in \mathcal{K}(S)$. Thus by the Proposition 1.6.14 $x_1x_1^*, x_2x_2^* \in \mathcal{K}(E(S))$. Put $a = x_1x_1^* \vee x_2x_2^* \in \mathcal{K}(E(S))$. Note that $a \leq r(s) = ss^*$. The element as is compact because $r(as) = a$ which is compact. And also $as \geq x_1x_1^*s = x_1$ and $as \geq x_2x_2^*s = x_2$. Thus the second condition holds. We have $\bigvee (s^\downarrow \cap \mathcal{K}(S)) \leq s$. Observe that $ss^* \in E(S)$, thus there exists a directed set $C \subseteq \mathcal{K}(E(S))$ such that $\bigvee_{c \in C} c = ss^*$ (and so in particular, each $c \leq ss^*$). We have

$$\bigvee_{c \in C} cs = \left(\bigvee_{c \in C} c \right) s = ss^*s = s.$$

Also $cs \in s^\downarrow \cap \mathcal{K}(S)$ because $(cs)(cs)^* = c$. Therefore $\bigvee (s^\downarrow \cap \mathcal{K}(S)) = s$.

The equivalence of (2) and (3) is the Corollary 1.5.17. ■

1.7 Compact groups

This section contains a summary of some of the necessary facts about the representation theory of compact groups. Recall that $C(G)$ is the space of continuous functions $G \rightarrow \mathbb{C}$.

Definition 1.7.1 *Let G be a compact group. A Borel measure μ on G is called left invariant if $\mu(gB) = \mu(B)$ for all $g \in G$ and $B \subset G$ a Borel set. It is called a Haar measure if it is a regular Borel measure with support G .*

Theorem 1.7.2 (Existence and Uniqueness Theorem of Haar measure) *For each compact group G there exists a Haar measure which is unique up to a positive scalar multiple. In particular there is a unique Haar probability measure called a normalized Haar measure.*

Remark 1.7.3

- In this thesis Haar measure will always mean normalized Haar measure.
- Haar measure is also right invariant for compact groups.

Let \mathcal{H} be a complex Banach space, we denote by $Gl(\mathcal{H})$ the group of continuous isomorphisms of \mathcal{H} onto itself. A *representation* π of a compact group G in \mathcal{H} is a homomorphism

$$\pi : G \longrightarrow Gl(\mathcal{H})$$

for which all maps

$$\begin{aligned} G &\longrightarrow \mathcal{H} \\ g &\mapsto \pi(g)v \end{aligned}$$

are continuous for each $v \in \mathcal{H}$. Equivalently, $\pi : G \longrightarrow Gl(\mathcal{H})$ is continuous in the strong operator topology.

A representation π of a group G in \mathcal{H} defines an action

$$\begin{aligned} \pi : G \times \mathcal{H} &\longrightarrow \mathcal{H} \\ (g, v) &\mapsto \pi(g)v \end{aligned}$$

It can be proved that this action is *jointly continuous*, see [17].

A unitary operator is a bounded linear operator $U : \mathcal{H} \rightarrow \mathcal{H}$ on a Hilbert space \mathcal{H} satisfying $U^*U = UU^* = I$ where U^* is the adjoint of U , and $I : \mathcal{H} \rightarrow \mathcal{H}$ is the identity operator. This property is equivalent to the following:

1. The range of U is dense, and
2. U preserves the inner product $\langle - | - \rangle$ on the Hilbert space, i.e., for all vectors x and y in the Hilbert space,

$$\langle Ux | Uy \rangle = \langle x | y \rangle.$$

To see this, notice that U preserves the inner product implies U is an isometry (thus, a bounded linear operator). The fact that U has dense range ensures it has a bounded inverse U^{-1} . It is clear that $U^{-1} = U^*$.

We say that the representation π is *unitary* when \mathcal{H} is a Hilbert space and each operator $\pi(g)$ is unitary operator (thus each $\pi(g)$ must be *isometric and surjective*). Thus π is unitary when \mathcal{H} is a Hilbert space and

$$\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1}) \quad (g \in G)$$

The representation π of G is said to be irreducible when \mathcal{H} and $\{0\}$ are distinct and are the only two invariant subspaces under all operators $\pi(g)$ where $g \in G$.

Proposition 1.7.4 *Let π be a unitary representation of G in the Hilbert space \mathcal{H} . If \mathcal{H}_1 is an invariant subspace of \mathcal{H} (with respect to all operators $\pi(g)$, $g \in G$), then the orthogonal $\mathcal{H}_2 = \mathcal{H}_1^\perp$ of \mathcal{H}_1 is also invariant.*

Proof: Let $v \in \mathcal{H}_2$, then all $\pi(g)v$ are also orthogonal to \mathcal{H}_1 . Indeed, $\forall x \in \mathcal{H}_1$ we have $\langle x | \pi(g)v \rangle = \langle \pi(g)^*x | v \rangle = \langle \pi(g^{-1})x | v \rangle = 0$. ■

Chapter 2

Complete reducibility

In this chapter, we will prove our first result about the representation theory of compact inverse monoids (complete reducibility). We will show that any finite dimensional representation of a compact inverse semigroup is completely reducible. This can also be deduced from a result of Munn. We denote by $M_n(\mathbb{C})$ the semigroup of $n \times n$ matrices over \mathbb{C} . Our first observation is that any semilattice in $M_n(\mathbb{C})$ has at most 2^n elements. We are going to prove this statement in steps.

Step 1: Any idempotent matrix is diagonalizable. Indeed, let A be any idempotent matrix. Thus $A^2 = A$, which implies that $A(A - I_n) = 0$. Therefore, the minimal polynomial $P(X)$ of A divides the polynomial $X(X - 1)$ and hence the only possible cases for $P(X)$ are:

- $P(X) = X$, so $A = 0$
- $P(X) = X - 1$ implies that $A = I_n$
- $P(X) = X(X - 1)$.

Thus A is similar to diagonal matrix because the minimal polynomial is a product of distinct linear factors.

Step 2: We will prove that, if $S \subseteq M_n(\mathbb{C})$ is a commutative semigroup and A is a matrix in $M_n(\mathbb{C})$ commuting with each element of S , then each eigenspace of A is S -invariant.

Proof: Let \mathcal{H}_λ be an eigenspace of the matrix A . Let B be a matrix in S . We will prove that $B \cdot \mathcal{H}_\lambda \subseteq \mathcal{H}_\lambda$. Let x be an element of \mathcal{H}_λ .

$$A(Bx) = B(Ax) = B(\lambda x) = \lambda Bx$$

Thus $Bx \in \mathcal{H}_\lambda$ ■

Step 3: We will show that any commutative semigroup S of diagonalizable matrices in $M_n(\mathbb{C})$ is conjugate to a semigroup of diagonal matrices (i.e., is simultaneously diagonalizable).

Proof: When $n = 1$, all the matrices are scalar, so there is nothing to prove. Suppose that the assumption is true for all $k \leq n$. The case where all the matrices in S are scalar is trivial. Let A be in S such that A is not scalar. Then A has at least two distinct eigenvalues, hence two distinct eigenspaces. By Step 2, the eigenspaces are invariant subspaces of \mathbb{C}^n with respect to all matrices in the semigroup S . Thus the eigenspaces of A are S -invariant, so by induction, all the restrictions are simultaneously diagonalizable. As the sum of the eigenspaces of A is \mathbb{C}^n the result follows. ■

Step 4: The diagonal idempotent matrices only have $\{0, 1\}$ entries; there are thus 2^n of them.

Step 5: Conclude that any semilattice in $M_n(\mathbb{C})$ has at most 2^n elements. We proved in Step 1 that any idempotent matrix is diagonalizable and by Step 3 any commutative semigroup of diagonalizable matrices is simultaneously diagonalizable. Therefore any semilattice in $M_n(\mathbb{C})$ is conjugate to a semilattice of diagonal matrices

and thus the semilattice has at most 2^n elements.

Remark 2.0.5 Consider the interval $[0, 1]$ as a semilattice with respect to \min . This semilattice has no non-trivial continuous representations in finite dimensions. Suppose that we have a continuous representation in finite dimension $\varphi : [0, 1] \rightarrow M_n(\mathbb{C})$. Then $\varphi([0, 1])$ is connected and finite. Thus it has one element.

2.1 Haar system for groupoids with finitely many identities

Definition 2.1.1 A compact groupoid is a topological groupoid which is compact and Hausdorff as a topological space.

Let \mathcal{G} be a compact groupoid with finitely many identities. We can define a (left) Haar system on such a groupoid as follows: Let e be in \mathcal{G}^0 and let us denote by C_e the connected component of e and $C_e^0 \subseteq C_e$ the set of identities in C_e . Now for each $e \neq f \in C_e^0$ choose $g_{e,f} \in C_e$ such that $r(g_{e,f}) = f$ and $d(g_{e,f}) = e$. Usually, when $e = f$ we choose $g_{e,e} = e$.

Define a measure on Borel subsets B of \mathcal{G}^e as follows

$$\eta^e(B) = \frac{1}{|C_e^0|} \sum_{f \in C_e^0} \nu_e(B_f^e g_{e,f}) \quad (2.1.1)$$

where ν_e is the (normalized) Haar measure on the compact group \mathcal{G}_e^e and $B \cap \mathcal{G}_f^e = B_f^e$.

Remark 2.1.2

1. $r^{-1}(\{e\}) = \bigsqcup_{f \in C_e^0} \mathcal{G}_f^e$ and $\mathcal{G}_f^e = r^{-1}(\{e\}) \cap d^{-1}(\{f\})$.
2. Since $\mathcal{G}_f^e = r^{-1}(\{e\}) \cap d^{-1}(\{f\})$ therefore \mathcal{G}_f^e is clopen, so $B_f^e g_{e,f}$ is a Borel subset of \mathcal{G}_e^e .

Lemma 2.1.3 *Let \mathcal{G} be a compact groupoid. If $g_{e,f} : e \rightarrow f$ and ν_e and ν_f are Haar measures associated to the isotropy groups \mathcal{G}_e^e and \mathcal{G}_f^f respectively, then*

1. $\nu_e(C) = \nu_f(g_{e,f}Cg_{e,f}^{-1})$ where C is a Borel subset of \mathcal{G}_e^e .
2. $\nu_e(Cg_{e,f}) = \nu_f(g_{e,f}C)$ where C is a Borel subset of \mathcal{G}_f^f .

Proof:

1. We need to show that $\nu_f(g_{e,f}(\cdot)g_{e,f}^{-1})$ is a normalized Haar measure on the group \mathcal{G}_e^e . Let C be a Borel subset of \mathcal{G}_e^e . If $h \in \mathcal{G}_e^e$ then $\nu_f(g_{e,f}hg_{e,f}^{-1}Cg_{e,f}^{-1}) = \nu_f(g_{e,f}hg_{e,f}^{-1} \cdot g_{e,f}Cg_{e,f}^{-1}) = \nu_f(g_{e,f}Cg_{e,f}^{-1})$ establishing left invariance. Trivially $\nu_f(g_{e,f}\mathcal{G}_e^eg_{e,f}^{-1}) = \nu_f(\mathcal{G}_f^f) = 1$. By the uniqueness of Haar measure, we have $\nu_e(C) = \nu_f(g_{e,f}Cg_{e,f}^{-1})$ for all Borel sets $C \subseteq \mathcal{G}_e^e$.
2. Since $C \subseteq \mathcal{G}_f^f$, then $Cg_{e,f} \subseteq \mathcal{G}_e^e$, therefore $\nu_e(Cg_{e,f}) = \nu_f(g_{e,f}Cg_{e,f}g_{e,f}^{-1}) = \nu_f(g_{e,f}C)$ by (1).

■

The following Lemma is used to show that the measure η^e defined on \mathcal{G}_f^e is independent of the choices of the $g_{e,f}$.

Lemma 2.1.4 *If B is a Borel subset of \mathcal{G}_f^e and $g_{e,f}, h_{e,f} : e \rightarrow f$ then*

$$\nu_e(Bg_{e,f}) = \nu_e(Bh_{e,f}).$$

Proof: By Lemma 2.1.3, $\nu_e(Bg_{e,f}) = \nu_f(g_{e,f}B) = \nu_f(h_{e,f}g_{e,f}^{-1}(g_{e,f}B)) = \nu_f(h_{e,f}B) = \nu_e(Bh_{e,f})$. ■

We can deduce the following Corollary showing that the measure (2.1.1) well-defined (that is, it is independent of the choices made).

Corollary 2.1.5 *Let \mathcal{G} be a compact groupoid with finitely many identities. Then the measure η^e defined in (2.1.1) is independent of the choice of $g_{e,f}$. That is, if $h_{e,f} \in C_e$ are such that $r(h_{e,f}) = f$ and $d(h_{e,f}) = e$ then*

$$\eta^e(B) = \frac{1}{|C_e^0|} \sum_{f \in C_e^0} \nu_e(B_f^e h_{e,f}).$$

Now we can define a measure on all Borel subsets of \mathcal{G}^e as follows:

$$\eta^e(B) = \frac{1}{|C_e^0|} \sum_{f \in C_e^0} \nu_f(g_{e,f} B_f^e) \quad (2.1.2)$$

Equality results from Lemma 2.1.3 and Corollary 2.1.5.

Proposition 2.1.6 *Let \mathcal{G} be a compact groupoid with finitely many identities. Then the family of measures $\{\eta^e\}_{e \in \mathcal{G}^0}$ defined in (2.1.2) is a normalized left Haar system on the groupoid \mathcal{G} .*

Proof: We verify the three properties.

1. The support of the measure η^e is by definition:

$$\text{Supp}(\eta^e) = \{x \in \mathcal{G}^e \mid \forall N_x \text{ neighbourhood of } x, \eta^e(N_x) > 0\}.$$

First suppose that $\nu_e(N_e) = 0$ for some neighbourhood of e in \mathcal{G}^e . Since the group \mathcal{G}_e^e is compact, there exist g_1, g_2, \dots, g_n elements of \mathcal{G}_e^e such that $\mathcal{G}_e^e = \bigcup_{i=1}^{n_e} g_i N_e$. Therefore $\nu_e(\mathcal{G}_e^e) \leq \sum_{i=1}^{n_e} \nu_e(g_i N_e) = 0$, since $\nu_e(g_i N_e) = \nu_e(N_e)$ where ν_e is the Haar measure on the group \mathcal{G}_e^e . This is a contradiction.

In general, let N_x be neighbourhood of some element x in \mathcal{G}^e . Therefore $r(x) = e$.

$$\eta^e(N_x) = \frac{1}{|C_e^0|} \sum_{f \in C_e^0} \nu_e((N_x \cap \mathcal{G}_f^e) g_{e,f})$$

then $\nu_e((N_x \cap \mathcal{G}_f^e) g_{e,f}) > 0$ because $(N_x \cap \mathcal{G}_f^e) g_{e,f}$ is a neighbourhood of e . Therefore $\eta^e(N_x) > 0$.

2. For any $f \in C(\mathcal{G})$, the function $f^0 : \mathcal{G}^0 \rightarrow \mathbb{C}$, where $f^0(e) = \int_{\mathcal{G}^e} f \, d\eta^e$ belongs to $C(\mathcal{G}^0)$, because \mathcal{G}^0 is finite.
3. Let $B \subseteq \mathcal{G}^{d(x)}$ be a Borel set. Then we will show that $\eta^{d(x)}(B) = \eta^{r(x)}(xB)$. First, we will prove the following:

$$\nu_{d(x)} \left(B_f^{d(x)} g_{d(x),f} \right) = \nu_{r(x)} \left((xB)_f^{r(x)} g_{r(x),f} \right).$$

From Lemma 2.1.3 and Corollary 2.1.5 we can deduce that

$$\nu_{d(x)} \left(B_f^{d(x)} g_{d(x),f} \right) = \nu_{r(x)} \left(xB_f^{d(x)} g_{d(x),f} x^{-1} \right).$$

Now, $xB_f^{d(x)} = (xB)_f^{r(x)}$ and put $h_{r(x),f} = g_{d(x),f} x^{-1}$, thus

$$\begin{aligned} \nu_{d(x)} \left(B_f^{d(x)} g_{d(x),f} \right) &= \nu_{r(x)} \left((xB)_f^{r(x)} g_{d(x),f} x^{-1} \right) \\ &= \nu_{r(x)} \left((xB)_f^{r(x)} h_{r(x),f} \right) \\ &= \nu_{r(x)} \left((xB)_f^{r(x)} g_{r(x),f} \right) \text{ using Lemma 2.1.3.} \end{aligned}$$

Moreover, we have $|C_{d(x)}^0| = |C_{r(x)}^0|$ because x connects these two identities. So,

$$\begin{aligned} \eta^{d(x)}(B) &= \frac{1}{|C_{d(x)}^0|} \sum_{f \in C_{d(x)}^0} \nu_{d(x)} \left(B_f^{d(x)} g_{d(x),f} \right) \\ &= \frac{1}{|C_{r(x)}^0|} \sum_{f \in C_{r(x)}^0} \nu_{r(x)} \left((xB)_f^{r(x)} g_{r(x),f} \right) \\ &= \eta^{r(x)}(xB) \end{aligned}$$

■

Dually, we can define a right Haar system in any groupoid. Before giving some proprieties of Haar systems, we give the following definition.

Definition 2.1.7 *A left Haar system $\{\lambda^e\}_{e \in \mathcal{G}^0}$ in a groupoid \mathcal{G} is right invariant if for every $B \subseteq \mathcal{G}_f^e$ Borel and $x \in \mathcal{G}^f$ we have $\lambda^e(B) = \lambda^e(Bx)$.*

Proposition 2.1.8 *Let \mathcal{G} be a compact groupoid with finitely many identities which admits $\{\lambda^e\}_{e \in \mathcal{G}^0}$ as left Haar system and let $g_{e,f} \in \mathcal{G}_e^f$.*

1. *Let e, f be in C_e^0 , then $\mathcal{G}_e^e \cong \mathcal{G}_f^f$ via the map $x \mapsto g_{e,f}xg_{e,f}^{-1}$. And also*

$$\nu_e(B) = \frac{\lambda^e(B)}{\lambda^e(\mathcal{G}_e^e)} = \frac{\lambda^f(g_{e,f}(B)g_{e,f}^{-1})}{\lambda^f(\mathcal{G}_f^f)} = \nu_f(g_{e,f}(B)g_{e,f}^{-1})$$

where B is a Borel subset of \mathcal{G}_e^e and also ν_e and ν_f are the normalized Haar measures on \mathcal{G}_e^e and \mathcal{G}_f^f respectively. Moreover, if $\lambda^e(\mathcal{G}_e^e) = \lambda^f(\mathcal{G}_f^f)$, then $\lambda_{|\mathcal{G}_e^e}^e(-) = \lambda_{|\mathcal{G}_f^f}^f(g_{e,f}(-)g_{e,f}^{-1})$ and if B is Borel subset of \mathcal{G}_f^f then $\lambda^e(Bg_{e,f}) = \lambda^f(g_{e,f}B)$.

2. *If $x \in \mathcal{G}$ then the left invariance condition is equivalent to $\lambda^{r(x)}(B) = \lambda^{d(x)}(x^{-1}B)$ where $B \subseteq \mathcal{G}^{r(x)}$.*
3. *The system $\{\lambda_e\}_{e \in \mathcal{G}^0}$ is a right Haar system for the groupoid \mathcal{G} , where $\lambda_e(B) = \lambda^e(B^{-1})$ for every B a Borel subset of \mathcal{G}_e .*
4. *A normalized $\{\lambda^e\}_{e \in \mathcal{G}^0}$ left Haar system is a right invariant if and only if $\lambda^e(\mathcal{G}_e^e) = \frac{1}{|C_e^0|}$ for all $e \in \mathcal{G}^0$.*

Proof:

1. It is a general fact that if $\alpha : G \rightarrow H$ is continuous homomorphism of compact groups, then $\lambda_H = \lambda_G \circ \alpha^{-1}$ where λ_G and λ_H are a Haar measure on G and H respectively. Therefore $\nu_f(g_{e,f}(-)g_{e,f}^{-1}) = \nu_e$. Thus we need to prove that

$$\beta(-) = \frac{\lambda^f(g_{e,f}(-)g_{e,f}^{-1})}{\lambda^f(\mathcal{G}_f^f)}$$

is Haar measure on \mathcal{G}_e^e . Indeed, let $x \in \mathcal{G}_e^e$ and let B be a Borel subset of \mathcal{G}_e^e . Then $\lambda^f(g_{e,f}xBg_{e,f}^{-1}) = \lambda^e(Bg_{e,f}^{-1})$ by applying a left invariance of the Haar system. But $\lambda^e(Bg_{e,f}^{-1}) = \lambda^f(g_{e,f}Bg_{e,f}^{-1})$ by left invariance of the system. Thus

β is a left invariant measure. Clearly $\beta(\mathcal{G}_e^e) = 1$. And also the support of the measure β is \mathcal{G}_e^e . Indeed, let U be an open subset of \mathcal{G}_e^e then

$$\beta(U) = \frac{\lambda^f(g_{e,f}Ug_{e,f}^{-1})}{\lambda^f(\mathcal{G}_f^f)} > 0$$

since $g_{e,f}Ug_{e,f}^{-1}$ is an open subset of \mathcal{G}_f^f and also \mathcal{G}_f^f is an open subset of \mathcal{G}^f . For the last equality let $B \subseteq \mathcal{G}_f^f$, then

$$\lambda^e(Bg_{e,f}) = \nu^e(Bg_{e,f})\lambda^e(\mathcal{G}_e^e) = \nu^f(g_{e,f}(Bg_{e,f})g_{e,f}^{-1})\lambda^f(\mathcal{G}_f^f) = \lambda^f(g_{e,f}B).$$

2. It is trivial.

3. Let $x \in \mathcal{G}$ then

$$\begin{aligned} \lambda_{r(x)}(B_{r(x)}) &= \lambda^{r(x)}(B_{r(x)}^{-1}) \\ &= \lambda^{d(x)}(x^{-1}B_{r(x)}^{-1}) \\ &= \lambda^{d(x)}((B_{r(x)}x)^{-1}) \\ &= \lambda_{d(x)}(B_{r(x)}x). \end{aligned}$$

4. Suppose that $\{\lambda^e\}_{e \in \mathcal{G}^0}$ is right invariant. Then for any choice of $g_{e,f} \in \mathcal{G}_e^f$ we have

$$\begin{aligned} 1 &= \lambda^e(\mathcal{G}_e^e) \\ &= \sum_{f \in C_e^0} \lambda^e(\mathcal{G}_f^e g_{e,f}) \\ &= \sum_{f \in C_e^0} \lambda^e(\mathcal{G}_e^e) \\ &= |C_e^0| \lambda^e(\mathcal{G}_e^e). \end{aligned}$$

Therefore $\lambda^e(\mathcal{G}_e^e) = \frac{1}{|C_e^0|}$.

Conversely, suppose that $\lambda^e(\mathcal{G}_e^e) = \frac{1}{|C_e^0|}$ for all $e \in \mathcal{G}^0$. Therefore $\lambda^f(\mathcal{G}_f^f) =$

$\lambda^e(\mathcal{G}_e^e)$ for every $e, f \in C_e^0$. Suppose that B is Borel subset of \mathcal{G}_f^e . Then we have

$$\lambda^e(Bg_{e,f}) = \lambda^f(g_{e,f}B) = \lambda^e(B)$$

using (1). Therefore $\{\lambda^e\}_{e \in \mathcal{G}^0}$ is right invariant. ■

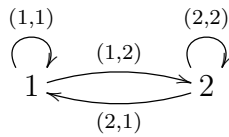
Similar to the construction of the left Haar system defined in (2.1.2), we can form a right Haar system on a compact groupoid with finitely many identities as follows:

$$\eta_e(B) := \frac{1}{|C_e^0|} \sum_{e \in C_e^0} \nu_e(h_{f,e}B_e^f) = \frac{1}{|C_e^0|} \sum_{e \in C_e^0} \nu_f(B_e^f h_{f,e}) \quad (2.1.3)$$

Remark 2.1.9

1. The two constructions are related (i.e., $\eta_e(B) = \eta^e(B^{-1})$ for every B a Borel set in G_e).
2. Notice that $\eta^e(\mathcal{G}_e^e) = \eta^f(\mathcal{G}_f^f)$ for every $e, f \in C_e^0$, which means that $\{\eta^e\}_{e \in \mathcal{G}^0}$ is right invariant.

In general, a left Haar system need not be right invariant. Indeed, let \mathcal{G} be the following groupoid.



Define $\lambda^1 = \frac{1}{4}\delta_{(1,1)} + \frac{3}{4}\delta_{(1,2)}$ and $\lambda^2 = \frac{1}{4}\delta_{(2,1)} + \frac{3}{4}\delta_{(2,2)}$. Then $\{\lambda^1; \lambda^2\}$ is a (normalized) left Haar system but is not right invariant.

Remark 2.1.10 The space of normalized left Haar systems is in bijection with the product $\prod_{\substack{C \text{ connected} \\ \text{component}}} \mathcal{P}(C^0)$ where $\mathcal{P}(X)$ is the set of regular Borel measures on X .

Theorem 2.1.11 *In a compact groupoid with finitely many identities, there exists a unique normalized left Haar system that is also right invariant, namely the measure defined in (2.1.2).*

Proof: Let $B \subseteq \mathcal{G}^e \subseteq \bigsqcup_{f \in C_e^0} \mathcal{G}_f^e$ be Borel, where $e \in \mathcal{G}^0$, assume $\{\lambda^e\}_{e \in \mathcal{G}^0}$ is a left Haar system which is right invariant. Then

$$\begin{aligned}
 \lambda^e(B) &= \sum_{f \in C_e^0} \lambda^e(B_f^e g_{e,f}) && \text{by right invariance of } \{\lambda^e\}_{e \in \mathcal{G}^0}. \\
 &= \left[\sum_{f \in C_e^0} \nu_e(B_f^e g_{e,f}) \right] \lambda^e(\mathcal{G}_e^e) && \text{because } B_f^e g_{e,f} \subseteq \mathcal{G}_e^e \text{ and Proposition 2.1.8(1).} \\
 &= \eta^e(B) |C_e^0| \lambda^e(\mathcal{G}_e^e). \\
 &= \eta^e(B) && \text{by Proposition 2.1.8(4).}
 \end{aligned}$$

■

Dually, the normalized right Haar system that is also left invariant measure is unique. (that is, the measure defined in (2.1.3) is the unique right normalized Haar system that is also left invariant.)

2.2 Solomon's Theorem

In this section, all the theorems are taken from the book, "Incidence Algebras" [23]. Let E be a finite semilattice and K any commutative ring with unit. Define the *incidence algebra* $I(E, K) = \{f : E \times E \rightarrow K \mid f(x, y) = 0 \text{ if } x \not\leq y\}$, with operations given by

$$\begin{aligned}
 (f + g)(x, y) &= f(x, y) + g(x, y), \\
 (f \cdot g)(x, y) &= \sum_{x \leq z \leq y} f(x, z)g(z, y) \\
 (rf)(x, y) &= r \cdot f(x, y)
 \end{aligned}$$

for $f, g \in I(E, K)$ with $r \in K$ and $x, y, z \in E$. Then $I(E, K)$ is a K -algebra.

The function $\delta \in I(E, K)$ is given by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{otherwise.} \end{cases}$$

One can prove that δ is the identity of the K -algebra $I(E, K)$.

Next, let $\chi \in I(E, K)$ be defined by

$$\chi(x, y) = \begin{cases} 1 & \text{if } x < y; \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exists k such that $\chi^k = 0$ namely, if k is larger than the length of the longest strict chain in E . Let $\zeta = \delta + \chi$ which we call the *zeta function* of E . Note that

$$\zeta^{-1} = \delta - \chi + \chi^2 - \chi^3 + \dots$$

which is well defined since χ is nilpotent. Normally ζ^{-1} is denoted by μ and called the *Möbius function* of E .

Notation 2.2.1

1. We denote by K^E the set of all functions $E \rightarrow K$ with pointwise multiplication and addition.
2. If $f \in E$, then $\delta_f : E \rightarrow K$ where

$$\delta_f(e) = \begin{cases} 1 & \text{if } e = f; \\ 0 & \text{if } e \neq f. \end{cases}$$

Theorem 2.2.2 [Solomon]: *Let E be finite semilattice. Then for any commutative ring with unit K , one has $KE \cong K^E$ where KE is the semigroup algebra of E over K . More explicitly*

$$\psi : KE \rightarrow K^E$$

$$e \mapsto \chi_{e\downarrow} = \sum_{f \leq e} \delta_f$$

is an isomorphism with inverse given by

$$\begin{aligned} \psi^{-1} : K^E &\longrightarrow KE \\ \delta_e &\mapsto \sum_{f \leq e} f\mu(f, e). \end{aligned}$$

For more details, see Solomon [22].

Corollary 2.2.3 *Let E be finite semilattice. If $\pi : E \longrightarrow M_n(\mathbb{C})$ is a representation of E . Then $\pi' : E \longrightarrow M_n(\mathbb{C})$ given by $\pi'(e) = \sum_{f \leq e} \pi(f)\mu(f, e)$ satisfies*

$$\pi'(e)\pi'(\bar{e}) = \begin{cases} \pi'(e) & \text{if } e = \bar{e} \\ 0 & \text{if } e \neq \bar{e} \end{cases}$$

i.e., $\{\pi'(e) \mid e \in E\}$ is an orthogonal set of idempotents.

Proof: Let $\pi : E \longrightarrow M_n(\mathbb{C})$ be a representation of E . We can extend this representation to $\tilde{\pi} : \mathbb{C}E \longrightarrow M_n(\mathbb{C})$ by sending $\sum a_e e \mapsto \sum a_e \pi(e)$. Then

$$\pi' : E \longrightarrow \mathbb{C}^E \xrightarrow{\psi^{-1}} \mathbb{C}E \xrightarrow{\tilde{\pi}} M_n(\mathbb{C})$$

by sending

$$e \mapsto \delta_e \mapsto \sum_{f \leq e} f\mu(f, e) \mapsto \sum_{f \leq e} \pi(f)\mu(f, e).$$

We have

$$\delta_e \delta_{\bar{e}} = \begin{cases} \delta_e & \text{if } e = \bar{e} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\pi'(e)\pi'(\bar{e}) = \tilde{\pi}(\psi^{-1}(\delta_e))\tilde{\pi}(\psi^{-1}(\delta_{\bar{e}})) = \tilde{\pi}(\psi^{-1}(\delta_e \delta_{\bar{e}})) = \begin{cases} \pi'(e) & \text{if } e = \bar{e} \\ 0 & \text{otherwise.} \end{cases}$ ■

2.3 Complete reducibility

The definition that we will give for a representation of compact groupoid is a special case of the more general way of defining a representation of compact and locally compact groupoids; see Massoud [1]. There, Massoud defined a representation using a bundle of Hilbert spaces; he showed that irreducible representations have finite dimensional fibres. He proved the Schur's lemma, the Gelfand-Raikov theorem and the Peter-Weyl theorem in this context.

Definition 2.3.1 *A representation π of a compact groupoid \mathcal{G} on a Banach space \mathcal{H} is a map $\pi : \mathcal{G} \rightarrow B(\mathcal{H})$ such that*

$$\pi(gh) = \begin{cases} \pi(g)\pi(h) & \text{if } d(g) = r(h); \\ 0 & \text{otherwise.} \end{cases}$$

and for which all maps

$$\begin{aligned} \mathcal{G} &\longrightarrow \mathcal{H} \\ g &\longmapsto \pi(g)v \end{aligned}$$

are continuous with $v \in \mathcal{H}$.

Equivalently, $\pi : \mathcal{G} \rightarrow B(\mathcal{H})$ is continuous in the strong operator topology.

The representation π is *non-degenerate* if $\pi(g)v = 0$ for some $v \in V$ and all $g \in \mathcal{G}$ then $v = 0$ (i.e., $\text{Span}(\pi(\mathcal{G})v) \neq \{0\}$ for all $0 \neq v \in V$) which is equivalent to $\pi(e)v = 0$ for all $e \in \mathcal{G}^0$ implies that $v = 0$.

A representation π satisfying $\pi(g)^* = \pi(g^{-1})$ is called a **-representation*.

Lemma 2.3.2 *Let \mathcal{G} be a compact groupoid. If $\pi : \mathcal{G} \rightarrow M_n(\mathbb{C}) \cong \text{End}_n(V)$ is a non-degenerate representation, then:*

1. *The set $U = \{e \in \mathcal{G}^0 \mid \pi(e) \neq 0\}$ is finite.*

2. The image $\pi(U)$ consists of orthogonal idempotents summing to 1.

Proof: If $e \neq f$ then $\pi(e)\pi(f) = 0$, therefore $\pi(e)$ and $\pi(f)$ are idempotent and orthogonal. But $M_n(\mathbb{C})$ does not have any infinite orthogonal sets of idempotents. So U is finite. If $v \in V = \mathbb{C}^n$ the claim is that

$$\sum_{e \in U} \pi(e)v = v.$$

Let $w = v - \sum_{e \in U} \pi(e)v$. Then if $f \in \mathcal{G}^0$ either $f \notin U$ and so $\pi(f)w = 0 \cdot w = 0$ or $f \in U$ and $\pi(f)w = \pi(f)v - \pi(f)v = 0$ using orthogonality. So $\pi(f)w = 0$ for all $f \in \mathcal{G}^0$, which implies $w = 0$ by non-degeneracy. So $\sum_{e \in U} \pi(e) = 1$. ■

Retaining the notation of the Lemma above we have:

Corollary 2.3.3 *Let \mathcal{G} be a compact groupoid. Let $\pi : \mathcal{G} \rightarrow M_n(\mathbb{C}) \cong \text{End}_n(V)$ be a finite dimensional non-degenerate representation of \mathcal{G} , where V is a complex vector space of dimension n . Then:*

1. The space $V = \mathbb{C}^n \cong \bigoplus_{e \in U} \pi(e)V$.

2. Let $V_e = \pi(e)V$ and $g : e \rightarrow f$. Then

(a) $\pi(g)(V_e) = V_f$.

(b) if $x \neq e$ then $\pi(g)(V_x) = 0$. In particular $\ker(\pi(g)) = \bigoplus_{e \neq x \in U} V_x$.

3. For every $h \in \mathcal{G}$ we have:

$$\pi(h^{-1}) : V_{r(h)} \rightarrow V_{d(h)} \text{ is inverse to } \pi(h) : V_{d(h)} \rightarrow V_{r(h)}.$$

Proof: The first part follows from lemma 2.3.2. We have $\pi(e) = \pi(g^{-1})\pi(g)$ and $\pi(g) = \pi(g)\pi(e)$, therefore $\ker \pi(g) = \ker \pi(e) = \bigoplus_{e \neq x \in U} V_x$ using the fact that the idempotents are orthogonal. Now $\pi(f) = \pi(g)\pi(g^{-1})$ and $\pi(g) = \pi(f)\pi(g)$ which implies $\text{Im}(\pi(g)) = \text{Im}(\pi(f)) = V_f$. Therefore $\pi(g)V_e \subseteq \text{Im}(\pi(g)) = V_f$. Now,

if $v \in V_f$, then $v = \pi(f)v = \pi(g)(\pi(g^{-1})v)$ and by the above for g^{-1} we have $\text{Im}(\pi(g^{-1})) = \text{Im}(\pi(e)) = V_e$. Thus $\pi(g)V_e = V_f$. Finally, $\pi(h^{-1}h) = \pi(d(h))$, $\pi(hh^{-1}) = \pi(r(h))$ and if $e \in \mathcal{G}^0$ then $\pi(e)|_{V_e} = 1_{V_e}$. Thus 3 holds. \blacksquare

Proposition 2.3.4 *Let \mathcal{G} be a compact groupoid. Let $\pi : \mathcal{G} \rightarrow M_n(\mathbb{C}) \cong \text{End}_n(V)$ be a finite dimensional non-degenerate representation of \mathcal{G} , where V is a complex vector space of dimension n . Let $U = \{e \in \mathcal{G}^0 \mid \pi(e) \neq 0\}$. Then, for each $e \in U$, there exists a positive definite hermitian form ϕ_e on V_e for which every $g : e \rightarrow f$ the map $\pi(g) : V_e \rightarrow V_f$ is an isometry, consequently, the sum $\bigoplus_{e \in U} \phi_e$ is a positive definite hermitian form on V such that π is a $*$ -representation.*

Proof: We may assume that the groupoid has finitely many idempotents by Lemma 2.3.2. Indeed, if $g \in \mathcal{G}$ then $\pi(g) = 0$ implies $\pi(g^{-1}g) = 0$. So $\mathcal{G}' = \{g \in \mathcal{G} \mid \pi(g) \neq 0\} = d^{-1}(U)$ is compact with finitely many idempotents and it suffices to consider $\pi|_{\mathcal{G}'}$. Suppose that V is equipped with some positive definite hermitian form $\langle -, - \rangle$. Since the maps $g \rightarrow \pi(g^{-1})v$ are continuous, the maps

$$g \rightarrow \langle \pi(g^{-1})v \mid \pi(g^{-1})w \rangle \quad v, w \in V$$

will be also continuous. The left Haar system enables us to define the form $\phi_e(v, w)$ (for w and v in V_e) via

$$\phi_e(v, w) = \int_{\mathcal{G}^e} \langle \pi(g^{-1})v \mid \pi(g^{-1})w \rangle d\lambda^e$$

It is clear that ϕ_e defines a positive definite Hermitian form on V_e , since $\langle - \mid - \rangle$ is positive definite and integration of a positive function is positive. And for all $e \in U$, (V_e, ϕ_e) are complex vector spaces with positive definite Hermitian forms ϕ_e , and moreover the $\phi = \bigoplus_{e \in U} \phi_e$ is a positive definite form on $V = \bigoplus_{e \in U} V_e$ making the collection of subspaces $\{V_e\}$ orthogonal. Finally, invariance of ϕ comes from the Haar

system. Let $v = \sum_{e \in \mathcal{G}^0} v_e$, $w = \sum_{e \in \mathcal{G}^0} w_e$ with $v_e, w_e \in V_e$ and let $h : x \rightarrow y$. We need to show that $\phi(\pi(h)v, w) = \phi(v, \pi(h^{-1})w)$ But

$$\begin{aligned}
\phi(\pi(h)v, w) &= \phi(\pi(h)v_x, w) = \phi(\pi(h)v_x, w_y) \\
&= \int_{\mathcal{G}^y} \langle \pi(y^{-1})\pi(h)v_x \mid \pi(y^{-1})w_y \rangle d\lambda^y \\
&= \int_{\mathcal{G}^y} \langle \pi((h^{-1}g)^{-1})v_x \mid \pi((h^{-1}g)^{-1})\pi(h^{-1})w_y \rangle d\lambda^y \\
&= \int_{\mathcal{G}^x} \langle \pi(z^{-1})v_x \mid \pi(z^{-1})\pi(h^{-1})w_y \rangle d\lambda^x \quad \text{by invariance} \\
&= \phi(v_x, \pi(h^{-1})w_y) \\
&= \phi(v, \pi(h^{-1})w_y) \\
&= \phi(v, \pi(h^{-1})w).
\end{aligned}$$

So π is equivalent to a *-representation. ■

Let \mathcal{H} be a complex Banach space. Then a *representation* π of a topological inverse semigroup S in \mathcal{H} is a homomorphism

$$\pi : S \rightarrow B(\mathcal{H})$$

for which all maps

$$\begin{aligned}
S &\rightarrow \mathcal{H} \\
s &\mapsto \pi(s)v
\end{aligned}$$

are continuous with $v \in \mathcal{H}$.

Equivalently, $\pi : S \rightarrow B(\mathcal{H})$ is continuous in the strong operator topology.

We say that the representation π is *-representation when \mathcal{H} is a Hilbert space and

$$\pi(s)^* = \pi(s^*) \quad (s \in S)$$

In this case $\pi(s)$ is a partial isometry.

We say that a representation is *non-degenerate* if $\pi(e)v = 0$ for all $e \in E(S)$ implies $v = 0$. Or equivalently $\pi(s)v = 0$ for all $s \in S$ implies $v = 0$.

Definition 2.3.5 Let $\pi : S \longrightarrow M_n(\mathbb{C})$ be a non-degenerate representation of an inverse semigroup S with finitely many idempotents. Define

$$\begin{aligned} \pi' : \mathcal{G}_S &\longrightarrow M_n(\mathbb{C}) \\ s &\mapsto \sum_{e \leq s^{-1}s} \pi(se)\mu(e, s^{-1}s). \end{aligned}$$

Lemma 2.3.6 Let π' be the map defined in the Definition 2.3.5. Then

$$\pi'(s) = \sum_{e \leq ss^{-1}} \pi(es)\mu(e, ss^{-1}).$$

Proof: Using the following order isomorphism,

$$\begin{aligned} (ss^{-1})^\downarrow &\longrightarrow (s^{-1}s)^\downarrow \\ e &\mapsto s^{-1}es \end{aligned}$$

$$\begin{aligned} \pi'(s) &= \sum_{e \leq s^{-1}s} \pi(se)\mu(e, s^{-1}s) \\ &= \sum_{f \leq ss^{-1}} \pi(ss^{-1}fs)\mu(f, ss^{-1}) \\ &= \sum_{f \leq ss^{-1}} \pi(fs)\mu(f, ss^{-1}) \end{aligned}$$

■

Proposition 2.3.7 If $\pi : S \longrightarrow M_n(\mathbb{C})$ is a non-degenerate representation of an inverse semigroup S with finitely many idempotents, and let π' be as in Definition 2.3.5. Then π' is a non-degenerate representation and

$$\pi(s) = \sum_{e \leq s^{-1}s} \pi'(se) = \sum_{f \leq ss^{-1}} \pi'(fs) \quad (2.3.1)$$

Proof: The equation (2.3.1) is true by the Möbius inversion theorem [23] page 34. Let us prove that π' is a non-degenerate representation.

First, we will prove this useful result: $\pi'(s) = \pi(s)\pi'(s^{-1}s) = \pi'(ss^{-1})\pi(s)$.

$$\begin{aligned}
\pi'(s) &= \sum_{e \leq s^{-1}s} \pi(se)\mu(e, s^{-1}s) \\
&= \sum_{e \leq s^{-1}s} \pi(s)\pi(s^{-1}se)\mu(e, s^{-1}s) \\
&= \pi(s) \sum_{e \leq s^{-1}s} \pi(s^{-1}se)\mu(e, s^{-1}s) \\
&= \pi(s)\pi'(s^{-1}s).
\end{aligned}$$

Dually, we can prove the other equality.

By Solomon's theorem $\pi'_{|\mathcal{G}_S^0}$ is a representation of \mathcal{G}_S^0 . Let t and s be two elements of the groupoid \mathcal{G}_S such that the product ts is defined which means that $ss^{-1} = t^{-1}t$, then

$$\begin{aligned}
\pi'(t)\pi'(s) &= \pi(t)\pi'(t^{-1}t)\pi'(ss^{-1})\pi(s) \\
&= \pi(t)\pi'(t^{-1}t)\pi(s) && \text{by Solomon's theorem} \\
&= \pi'(tt^{-1})\pi(t)\pi(s) \\
&= \pi'(tt^{-1})\pi(ts) \\
&= \pi'(tt^{-1}tt^{-1})\pi(ts) \\
&= \pi'(tss^{-1}t^{-1})\pi(ts) \\
&= \pi'(ts).
\end{aligned}$$

If $t^{-1}t \neq ss^{-1}$ then $\pi'(t)\pi'(s) = \pi(t)\pi'(t^{-1}t)\pi'(ss^{-1})\pi(s) = 0$, again by Solomon's theorem.

Let $s_\alpha \rightarrow s$. Then $s_\alpha^{-1}s_\alpha = s^{-1}s$ for α large enough since $|E(S)| < \infty$. So without loss of generality we may assume that $s_\alpha^{-1}s_\alpha = s^{-1}s$ for all α . Then

$$\pi'(s_\alpha)v = \sum_{e \leq s^{-1}s} \pi(s_\alpha e)\mu(e, s^{-1}s)v \rightarrow \sum_{e \leq s^{-1}s} \pi(se)\mu(e, s^{-1}s)v = \pi'(s)v$$

Now we need to prove the non-degeneracy of π' . Suppose that $\pi'(g)v = 0$ for all $g \in \mathcal{G}_S$ and some $v \in V = \mathbb{C}^n$. Therefore $\pi(s)v = 0$ for all $s \in S$ by (2.3.1) which

implies that $v = 0$. ■

Proposition 2.3.8 *Let $\pi : S \longrightarrow M_n(\mathbb{C})$ be a non-degenerate representation of an inverse semigroup S with finitely many idempotents. and let π' be as in the Definition 2.3.5. Then*

π is a $$ -representation if and only if π' is a $*$ -representation*

i.e., $\pi(s)^ = \pi(s^{-1}) \iff \pi'(s)^* = \pi'(s^{-1})$ for all s*

Proof: Suppose that π' is a $*$ -representation. Then $\pi(s^{-1}) = \sum_{e \leq ss^{-1}} \pi'(s^{-1}e) =$

$$\sum_{e \leq ss^{-1}} \pi'((es)^{-1}) = \sum_{e \leq ss^{-1}} \pi'(es)^* = \pi(s)^*.$$

Now suppose that π is a $*$ -representation, then

$$\begin{aligned} \pi'(s^{-1}) &= \sum_{e \leq ss^{-1}} \pi(s^{-1}e)\mu(e, ss^{-1}) \\ &= \sum_{e \leq ss^{-1}} \pi((es)^{-1})\mu(e, ss^{-1}) \\ &= \sum_{e \leq ss^{-1}} \pi(es)^*\mu(e, ss^{-1}) \\ &= \pi'(s)^*. \end{aligned}$$

■

Now, we have reached the main theorem.

Theorem 2.3.9 *If S is a compact inverse semigroup and $\pi : S \longrightarrow M_n(\mathbb{C})$ is non-degenerate representation then π is equivalent to a $*$ -representation.*

Proof: Put $T = \pi(S)$, so T is an inverse semigroup with finitely many idempotents because $T \subset M_n(\mathbb{C})$. Thus without loss of generality, we may assume that S has finitely many idempotents and π is an inclusion by replacing S by T . We conclude

π' is equivalent to a $*$ -representation by Proposition 2.3.4 and so π is equivalent to $*$ -representation by Proposition 2.3.8 ■

A finite-dimensional representation of a compact inverse monoid S is a continuous homomorphism of S into the monoid of $n \times n$ complex matrices for some n . From the preceding theorem, we deduce the following.

Corollary 2.3.10 *Every finite-dimensional representation of a compact inverse semi-group is completely reducible (i.e., direct sum of irreducible ones.)*

This could be deduced as special case of Munn's paper [12] but our proof is more direct and analytic and avoids Schur's result.

Remark 2.3.11 If $\pi : S \rightarrow B(\mathcal{H})$ is a non-degenerate representation of S on a Hilbert space, it is equivalent to a $*$ -representation if $|\pi(E(S))| < \infty$ by an adaptation of the preceding argument.

Chapter 3

Irreducible representations of compact inverse monoids

In this chapter, we are going to prove our new result which is the main result of the thesis.

3.1 A parametrization of the irreducible representations

In this section, we will prove one direction of our main theorem 3.2.24. Starting with irreducible representation of a compact inverse semigroup and finding the maximal subgroups G at e which parameterize the irreducible representation of the compact inverse semigroup.

Definition 3.1.1 *Let S be semigroup. Define $Idc(S)$ to be the set of all closed ideals in S .*

Theorem 3.1.2 *Let S be a compact inverse monoid and let*

$$\varphi : S \longrightarrow M_n(\mathbb{C})$$

be an irreducible representation of S . Define $D := \{I \in \text{Idc}(S) \mid \varphi(I) \neq 0\}$. Then

1. D has a minimal element I .
2. If I' is any ideal (not necessarily closed) such that $\varphi(I') \neq 0$, then $I \subseteq I'$, that is, I is the unique minimal with $\varphi(I) \neq 0$.
3. $I = SeS$ for some $e \in E(S)$.
4. e is minimal in $E(S)$ with $\varphi(e) \neq 0$.
5. The \mathcal{J} -class J_e is order compact.
6. $\varphi(J_e^\dagger) = 0$, where $J_e^\dagger = \{s \in S \mid SeS \not\subseteq SsS\}$
7. If $V = \mathbb{C}^n$, then $g \mapsto \varphi(g)|_{\varphi(e)V}$ is an irreducible representation of the maximal subgroup at e .

Proof:

1. Let $\{I_\alpha\}_{\alpha \in J}$ be a chain of closed ideals in D with respect to reverse inclusion. Let $K = \varphi^{-1}(E(\varphi(S)) \setminus \{0\})$. Thus K is closed, because $E(\varphi(S)) \setminus \{0\}$ is a finite point set in a Hausdorff space, and is thus closed.

For every $\alpha \in J$ there exists $x \in I_\alpha$ such that $\varphi(x) \neq 0$. Now $I_\alpha \cap K \neq \emptyset$, since $xx^* \in I_\alpha \cap K$ because $\varphi(x) \neq 0 \iff \varphi(xx^*) \neq 0$. Since every finite intersection of these sets $\bigcap_{\alpha \in J' \text{ finite}} I_\alpha \cap K \neq \emptyset$ and the fact that S is compact then $\mathcal{I} := \bigcap_{\alpha \in J} I_\alpha \cap K \neq \emptyset$, and so $\bigcap_{\alpha \in J} I_\alpha \in D$. Zorn's Lemma implies that D has minimal elements.

2. Let I be a minimal ideal in D and I' be any ideal such that $\varphi(I') \neq 0$. Take $V = \mathbb{C}^n$. Then $\text{Span}(\varphi(I')V) \neq 0$ and is S -invariant, since I' is an ideal, thus $\text{Span}(\varphi(I')V) = V$. Therefore $\text{Span}(\varphi(I \cdot I')V) = \text{Span}(\varphi(I)V) \neq 0$. So $\varphi(I \cdot I') \neq 0$ and $I \cdot I' \subseteq I$. There exists $s \in I \cdot I'$ with $\varphi(s) \neq 0$ hence $SsS \subseteq$

$I \cdot I' \subseteq I$, but SsS is closed, therefore $SsS = I$. Thus $I = I \cdot I' \subseteq I \cap I' \subseteq I'$.
So $I \subseteq I'$.

3. Let $s \in I$ such that $\varphi(s) \neq 0$, then $SsS = Sss^*S = Ss^*sS \subseteq I$ and $\varphi(SsS) \neq 0$, so $SsS = Sss^*S = Ss^*sS = I$ by minimality of I . Let $e = ss^*$.
4. Suppose there exists $f \in E(S)$ such that $\varphi(f) \neq 0$ and $f \leq e$. Minimality of SeS implies that $SfS = SeS$ and hence $f = e$ by Lemma 1.3.7.
5. To show that e is compact, it suffices to prove that e^\uparrow is clopen in $E(S)$. To do that we show $e^\uparrow = \varphi^{-1}(\varphi(e)^\uparrow) \cap E(S)$ where $\varphi(e)^\uparrow$ is taken inside of $\varphi(E(S))$. Then since $\varphi(E(S))$ is finite it follows that $\varphi^{-1}(\varphi(e)^\uparrow)$ is clopen in S . Thus we need to prove the equality $e^\uparrow = \varphi^{-1}(\varphi(e)^\uparrow) \cap E(S)$. For that, let $f \in e^\uparrow$ which implies that $e \leq f$ thus $\varphi(e) \leq \varphi(f)$ therefore $\varphi(f) \in \varphi(e)^\uparrow$. Conversely, suppose that $f \in \varphi^{-1}(\varphi(e)^\uparrow) \cap E(S)$. Then $\varphi(fe) = \varphi(f)\varphi(e) = \varphi(e) \neq 0$ and we know that $fe \leq e$, which implies that $fe = e$ by minimality of e . Thus $f \in e^\uparrow$.
6. We have that J_e^\dagger is an ideal. Suppose that $\varphi(J_e^\dagger) \neq 0$ which implies that $SeS = I \subseteq J_e^\dagger$ by (2), a contradiction because $e \notin J_e^\dagger$. Therefore $\varphi(J_e^\dagger) = 0$.
7. Let us denote the maximal subgroup at e by G_e^e . Let

$$\begin{aligned} \varphi : G_e^e &\longrightarrow \text{End}(\varphi(e)V) \\ g &\longmapsto \varphi(g)|_{\varphi(e)V} \end{aligned}$$

be the restriction of φ to the maximal subgroup G_e^e . We claim this restriction is irreducible. Suppose that there exists an invariant subspace $0 \neq W \subseteq \varphi(e)V$. So $\text{Span}(\varphi(S)W) = V$ because φ is irreducible, thus $\text{Span}(\varphi(Se)W) = V$, hence $\varphi(e)V = \text{Span}(\varphi(eSe)W)$. In a compact inverse semigroup we have $eSe \cap J_e = G_e^e$ using the Proposition 1.6.13 (4). So by minimality of SeS , we get

that $\varphi(eSe \setminus G_e^e) = 0$, so $\varphi(eSe) = \varphi(G_e^e)$, therefore $\varphi(e)V = \text{Span}(\varphi(eSe)W) = \text{Span}(\varphi(G_e^e)W) = W$.

■

3.2 Construction of irreducible representations from group representations

In this section, we will prove the other direction of our main theorem 3.2.24. Given a compact inverse semigroup S , let e be a compact element; take the maximal subgroup G_e at e . From an irreducible representation of G_e we will construct an irreducible representation of the compact inverse semigroup S .

Definition 3.2.1 *Let G be a group. A rook matrix over G is an $n \times n$ matrix over $G \cup \{0\}$ with at most one nonzero entry in each column and row. Rook matrices are also called monomial matrices or weighted permutation matrices.*

Notation 3.2.2

- We will denote the set of all rook matrices over G by $R_n(G)$.
- $CM_n(G)$, column monomial matrices; is the set of all $n \times n$ matrices over $G \cup \{0\}$ with at most one nonzero entry in each column.
- $RM_n(G)$, row monomial matrices; is the set of all $n \times n$ matrices over $G \cup \{0\}$ with at most one nonzero entry in each row.

Proposition 3.2.3 *The semigroup $R_n(G)$, consisting of $n \times n$ rook matrices over a compact group G together with matrix multiplication, is a compact inverse monoid.*

Proof: The inverse of any matrix $M = (a_{i,j})$ is given by $M^* = (a_{j,i}^*)$ where

$$a^* = \begin{cases} a^{-1} & \text{if } a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $MM^*M = M$. Indeed, $(MM^*M)_{i,j} = \sum_{k,l} a_{i,k} a_{l,k}^* a_{l,j}$ There is at most one term i_0 so that $a_{i,i_0} \neq 0$. Then we need $k = i_0, l = i$ for the above to not be 0 since we have rook matrices. Thus

$$(MM^*M)_{i,j} = \begin{cases} a_{i,i_0} a_{i_0,i}^* a_{i_0,j} & \text{if } i_0 \text{ exists} \\ 0 & \text{else} \end{cases}$$

Now we need $j = i_0$ to not have $a_{i,j} = 0$. Therefore

$$(MM^*M)_{i,j} = \begin{cases} a_{i,j} a_{i,j}^* a_{i,j} = a_{i,j} & \text{if } i_0 = j \\ 0 & \text{else} \end{cases}$$

So $MM^*M = M$ Similarly $M^*MM^* = M^*$.

To define a topology, give $G \cup \{0\}$ the topology of G with $\{0\}$ as an isolated point. This is a compact inverse monoid. Now $R_n(G) \subseteq (G \cup \{0\})^{n^2}$ with product topology.

Clearly $R_n(G)$ is closed subspace of $(G \cup \{0\})^{n^2}$ and multiplication is continuous. ■

Example 3.2.4 In the inverse semigroup $R_3(G)$, let

$$M = \begin{pmatrix} 0 & g & 0 \\ h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where g, h are in G

So the inverse of the above matrix is

$$M^* = \begin{pmatrix} 0 & h^{-1} & 0 \\ g^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark 3.2.5 The elements of $R_n(G)$ are known as rook matrices because the locations of the nonzero elements correspond to positions of a set of mutually non-attacking rooks on a checkerboard of size n .

Example 3.2.6 Let G be a group and let I be a set. Then the *Brandt inverse semigroup* $B_I(G)$ is defined as follows:

$$B_I(G) = \{(i, g, j) \mid i, j \in I, g \in G\} \cup \{0\}.$$

The product is given by

$$(i, g, j)(k, g', \ell) = \begin{cases} (i, gg', \ell) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

and 0 is a multiplicative zero.

If the set I is finite, we are going to denote the Brandt inverse semigroup by $B_n(G)$ where $n = |I|$. We can see the elements of $B_n(G)$ as $n \times n$ rook matrices. Indeed, define first the matrix

$$E_{i,j} = [e_{l,k}]_{n \times n}$$

such that

$$e_{l,k} = \begin{cases} 1 & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

The map

$$(i, g, j) \longleftrightarrow gE_{i,j}$$

is an isomorphism of $B_n(G)$ with a subsemigroup of $R_n(G)$. Observe that if G is compact, then $B_n(G)$ is closed subsemigroup of $R_n(G)$

Let S be a topological semigroup, then a *left translation* of S is a continuous map $\lambda : S \rightarrow S$ satisfying the following condition: $\lambda(st) = \lambda(s)t$. We will denote the set of all left translations of S by $LT(S)$. *Right translation* ρ is defined dually, i.e., it is a continuous map $\rho : S \rightarrow S$ such that $\rho(st) = s\rho(t)$. The set of all right translations is denoted by $RT(S)$. A left translation λ and a right translation ρ are said to be *linked* if $s(\lambda(t)) = (\rho(s))t$, for all $s, t \in S$. The collection of all pairs $(\lambda; \rho)$ of linked left and right translations is a semigroup $\Omega(S)$ called the translational hull of S . Multiplication is given by

$$(\lambda_1; \rho_1)(\lambda_2; \rho_2) = (\lambda_1\lambda_2; \rho_2\rho_1).$$

Example 3.2.7 For example, if $s \in S$ define a left translation $\lambda_s(t) = st$ and a right translation $\rho_s(t) = ts$. These two translations are linked because of the associativity of the semigroup.

Remark 3.2.8 One can consider $LT(S)$ and $RT(S)$ with the compact open topology and hence $\Omega(S)$ is a topological semigroup (in fact a topological monoid). However for the case of interest in this thesis we define a topology more directly.

Lemma 3.2.9 $LT(B_n(G))$ is isomorphic to $CM_n(G)$.

Proof: Let $M \in CM_n(G)$ then define

$$\begin{aligned} \lambda_M : B_n(G) &\longrightarrow B_n(G) \\ A &\longmapsto \lambda_M(A) = MA. \end{aligned}$$

This map is well defined. Indeed, $MA \in B_n(G)$. If $A = 0$ then $MA = 0 \in B_n(G)$. Suppose that $A = [a_{i,j}]_{n \times n} \neq 0$, therefore there exists unique $a_{i_0, j_0} \neq 0$ and all $a_{i,j} = 0$ for $(i, j) \neq (i_0, j_0)$. Suppose that $M = [b_{i,j}]$. If $MA \notin B_n(G)$, then there exists $i_1 \neq i_2$ such that $b_{i_1, i_0} a_{i_0, j_0} \neq 0$ and $b_{i_2, i_0} a_{i_0, j_0} \neq 0$, which implies that $b_{i_1, i_0} \neq 0$ and $b_{i_2, i_0} \neq 0$

this is a contradiction, because $M \in CM_n(G)$ which means that M has at most one nonzero entry in each column.

λ_M is left translation since the multiplication of matrices is associative.

Conversely, suppose that $\lambda : B_n(G) \longrightarrow B_n(G)$ is a left translation. First note that $\lambda(0) = \lambda(0 \cdot 0) = \lambda(0) \cdot 0 = 0$. Then

$$\lambda(E_{i,j}) = \lambda(E_{i,i}E_{i,j}) = \lambda(E_{i,i})E_{i,j} = g_i E_{k,\ell} E_{i,j}. \quad (*)$$

where g_i is an element in G and $\lambda(E_{i,i}) = g_i E_{k,\ell}$.

- case 1: $\lambda(E_{i,i}) = 0$, therefore $\lambda(E_{i,j}) = 0$ for all $j \in \{1, 2, \dots, n\}$.
- case 2: $\ell \neq i$ implies $\lambda(E_{i,j}) = 0$ for all $j \in \{1, 2, \dots, n\}$.
- case 3: $\ell = i$ implies that $\lambda(E_{i,j}) = g_i E_{k,j}$ for all $j \in \{1, 2, \dots, n\}$.

Now define a partial function $\sigma : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$ by

$$\sigma(i) = \begin{cases} k & \text{if } \lambda(E_{i,i}) = g_i E_{k,\ell}, g_i \in G \\ \text{undefined} & \text{if } \lambda(E_{i,i}) = 0 \end{cases}$$

Note if $\lambda(E_{i,i}) \neq 0$, then (*) implies that $\ell = i$, thus $\lambda(E_{i,i}) = g_i E_{\sigma(i),i}$. Put

$$M = \sum_{k \in \text{dom}(\sigma)} g_k E_{\sigma(k),k} \in CM_n(G)$$

Claim: $\lambda(gE_{i,j}) = M(gE_{i,j})$.

$$\begin{aligned} M(gE_{i,j}) &= \sum_{k \in \text{dom}(\sigma)} (g_k E_{\sigma(k),k})(gE_{i,j}) = \begin{cases} g_i g E_{\sigma(i),i} E_{i,j} & \text{if } i \in \text{dom}(\sigma) \\ 0 & \text{if } i \notin \text{dom}(\sigma). \end{cases} \\ &= \begin{cases} g_i g E_{\sigma(i),j} & \text{if } i \in \text{dom}(\sigma) \\ 0 & \text{if } i \notin \text{dom}(\sigma) \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} \lambda(gE_{i,j}) = \lambda(E_{i,i}(gE_{i,j})) = \lambda(E_{i,i})(gE_{i,j}) &= \begin{cases} g_i g E_{\sigma(i),i} E_{i,j} & \text{if } i \in \text{dom}(\sigma) \\ 0 & \text{if } i \notin \text{dom}(\sigma) \end{cases} \\ &= \begin{cases} g_i g E_{\sigma(i),j} & \text{if } i \in \text{dom}(\sigma) \\ 0 & \text{if } i \notin \text{dom}(\sigma) \end{cases} \end{aligned}$$

Therefore, $\lambda = \lambda_M$.

We have to show that $M \neq M'$ then $\lambda_M \neq \lambda_{M'}$. Indeed, suppose that $M = [b_{i,j}]$ and $M' = [b'_{i,j}]$. Then there exists i_0, j_0 such that $b_{i_0, j_0} \neq b'_{i_0, j_0}$ then $[ME_{j_0, i_0}]_{i_0, i_0} = b_{i_0, j_0} \neq b'_{i_0, j_0} = [M'E_{j_0, i_0}]_{i_0, i_0}$. Thus $ME_{j_0, i_0} \neq M'E_{j_0, i_0} \Rightarrow \lambda_M(E_{j_0, i_0}) \neq \lambda_{M'}(E_{j_0, i_0}) \Rightarrow \lambda_M \neq \lambda_{M'}$ ■

Dually, we can prove the following lemma.

Lemma 3.2.10 *$RT(B_n(G))$ is isomorphic to $RM_n(G)$.*

Now we characterize $\Omega(B_n(G))$.

Proposition 3.2.11 *If $M \in CM_n(G)$ and $N \in RM_n(G)$, then*

(M, N) are linked if and only if $M = N$ and hence they are rook matrices.

Proof: Suppose that $M = [m_{i,j}]_{n \times n}$ and $N = [n_{i,j}]_{n \times n}$ are linked. Thus $E_{i,j}(ME_{k,\ell}) = (E_{i,j}N)E_{k,\ell}$ for all $E_{i,j}$ and $E_{k,\ell}$ in $B_n(G)$. Therefore $E_{i,j}(ME_{k,\ell}) = m_{j,k}E_{i,\ell} = (E_{i,j}N)E_{k,\ell} = n_{j,k}E_{i,\ell}$. So $m_{j,k} = n_{j,k}$ for all j, k . Thus $M = N$. The other direction is trivial. ■

Corollary 3.2.12 $\Omega(B_n(G))$ is isomorphic to $R_n(G)$. And if $(\lambda; \rho) \in \Omega(B_n(G))$, then associated rook matrix is equal to

$$\sum_{\{j|\lambda(E_{j,j})\neq 0\}} \lambda(E_{j,j}) = \sum_{j \in \text{dom}(\sigma)} g_j E_{\sigma(j),j}$$

where $g_j \in G$ and σ is a partial bijection.

Thus if G is compact, we topologize $\Omega(B_n(G))$ to be homeomorphic to $R_n(G)$.

Definition 3.2.13 Let S be a compact inverse semigroup and I closed ideal of S . Define $S/I = S/\equiv_I$ where

$$a \equiv_I b \Leftrightarrow a = b \quad \text{or} \quad a, b \in I$$

Note that \equiv_I is a closed congruence on S .

Remark 3.2.14

1. S/I is a compact inverse semigroup with the quotient topology.
2. S/I is in bijection with $(S \setminus I) \cup \{0\}$ where the class of I corresponds to 0.

Multiplication is defined by

$$s \cdot s' = \begin{cases} ss' & ss' \notin I \\ 0 & \text{elsewhere.} \end{cases}$$

3. The map

$$\begin{aligned} \pi : S &\longrightarrow S/I \\ s &\mapsto [s] \end{aligned}$$

is a continuous onto homomorphism.

Let T be a semigroup and I_0 be an ideal of T . Then there is a homomorphism

$$\begin{aligned} \omega_{I_0} : T &\longrightarrow \Omega(I_0) \\ t &\mapsto (\lambda_t|_{I_0}, \rho_t|_{I_0}) \end{aligned}$$

Proposition 3.2.15 *Let S be a compact inverse semigroup and J be an order compact \mathcal{J} -class. Then $J^0 = J \cup \{0\}$ is a compact inverse semigroup and 0 is isolated.*

Proof: We have J^\dagger is clopen when J is order compact, which implies $J^\dagger \cap SJS$ is closed in SJS . Therefore J^0 is a compact inverse semigroup. The 0 is the class of $J^\dagger \cap SJS$ which is clopen. Thus $\{0\}$ is clopen. ■

We recall here that order compact \mathcal{J} -classes have finitely many idempotents.

Proposition 3.2.16 *Let J be an order compact \mathcal{J} -class, and let G be the maximal group at some idempotent e in J , and denote by n the number of idempotents in J . Then $B_n(G) \cong J^0$ as topological inverse semigroups.*

Proof: Since J is order compact, $|E(J)| < \infty$. Choose for each $f \in J^0$ an idempotent element, an element $p_f \in J$ such that $p_f : e \rightarrow f$, that is $p_f^* p_f = e$ and $p_f p_f^* = f$. Define $\alpha : J^0 \rightarrow B_n(G)$ by $\alpha(0) = 0$ and $\alpha(h) = (hh^*, p_{hh^}^* h p_{hh^}, h^* h)$.

$$\begin{array}{ccc} e & \xrightarrow{p_{h^*h}} & h^*h \\ & \searrow_{p_{hh^}} & \downarrow h \\ & & hh^* \end{array}$$

The map α is continuous because the maps $h \mapsto hh^*$, $h \mapsto h^*h$ and $f \mapsto p_f$ are continuous. The map α is an isomorphism. First, we prove that it is an homomorphism. Let h and k be in J . If $hk = 0$, we have $h^*h \neq k^*k$, which implies $\alpha(h)\alpha(k) = (hh^*, p_{hh^}^* h p_{hh^}, h^* h)(kk^*, p_{kk^}^* k p_{kk^}, k^* k) = 0 = \alpha(hk)$. If $hk \neq 0$, we have $h^*h = k^*k$, which implies

$$\alpha(hk) = (hkk^*h^*, p_{hkk^*h^}^* h k p_{hkk^*h^}, k^* h^* h k)$$

$$= (hh^*, p_{hh^*}^* hkp_{k^*k}, k^*k)$$

On the other hand,

$$\begin{aligned} \alpha(h)\alpha(k) &= (hh^*, p_{hh^*}^* hp_{h^*h}, h^*h)(kk^*, p_{kk^*}^* hp_{k^*k}, k^*k) \\ &= (hh^*, p_{hh^*}^* hp_{h^*h} p_{kk^*}^* kp_{k^*k}, k^*k) \\ &= (hh^*, p_{hh^*}^* hp_{h^*h} p_{h^*h}^* kp_{k^*k}, k^*k) \\ &= (hh^*, p_{hh^*}^* hh^* hkp_{k^*k}, k^*k) \\ &= (hh^*, p_{hh^*}^* hkp_{k^*k}, k^*k) \end{aligned}$$

We define the inverse as $\alpha^{-1}(f', gf) = p_{f'} g p_f^*$. ■

We continue to fix J an order compact \mathcal{J} -class. Put $n = |E(J)|$ and let G be a maximal subgroup around an idempotent element of J . So, now take S/J^\dagger . It is a semigroup with unique minimal ideal J^0 . Thus S/J^\dagger acts by left translations on J^0 via left multiplication and also on the right by right multiplication. These translations are linked by the associativity of the semigroup, giving a homomorphism $\omega : S/J^\dagger \rightarrow \Omega(J^0)$. So we have the following commutative diagram:

$$\begin{array}{ccc} J^0 & \xrightarrow{\alpha} & B_n(G) \\ \swarrow & & \searrow \\ S/J^\dagger & \xrightarrow{\omega} & \Omega(J^0) \cong R_n(G) \\ \uparrow \pi & \nearrow \varphi_J & \\ S & & \end{array}$$

where $\alpha : J^0 \rightarrow B_n(G)$ is the continuous isomorphism from Proposition 3.2.16. (Commutativity of the diagram is essentially in the proof of step 1 of Theorem 3.2.20 below.) Then by the Corollary 3.2.12, we can conclude that

$$\varphi_J : S \rightarrow R_n(G)$$

$$s \mapsto \sum_{\{e_j \in E(J) \mid se_j \in J\}} \alpha(s \cdot e_j)$$

is a homomorphism where α is as defined in the proof of the Proposition 3.2.16.

Lemma 3.2.17 *The map φ_J defined above is continuous.*

Proof: Define $I(s) = \{e_j \in E(J) \mid se_j \in J\}$ for $s \in S$. Let s_α be a net in S converging to s . By continuity of α and multiplication it suffices to prove that $I(s_\alpha) = I(s)$ for α large enough. Fix $e_j \in E(J)$. If $se_j \notin J$ then $se_j \in J^\dagger$. But J^\dagger is open. So $s_\alpha e_j \in J^\dagger$ for α large enough, which implies that $s_\alpha e_j \notin J$ for α large enough. So $e_j \notin I(s)$ implies that $e_j \notin I(s_\alpha)$ for α large enough.

Suppose now that $e_j \in I(s)$. Then $se_j \in J \subseteq J^\dagger$ which is open. Therefore $s_\alpha e_j \in J^\dagger$ for α large enough. But $s_\alpha e_j \leq_{\mathcal{J}} e_j$ which implies that $s_\alpha e_j \in J$ for α large enough. Therefore $e_j \in I(s)$ implies that $e_j \in I(s_\alpha)$ for α large enough. Since $|E(J)| < \infty$, thus $I(s) = I(s_\alpha)$ for α large enough. \blacksquare

Definition 3.2.18 *The tensor product or Kronecker product of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ is defined as follows*

$$A \otimes B = (Ab_{i,j}) = \begin{bmatrix} Ab_{11} & Ab_{12} & \dots & Ab_{1,n} \\ \vdots & \vdots & & \vdots \\ Ab_{n,1} & Ab_{n,2} & \dots & Ab_{n,n} \end{bmatrix}$$

Retaining the notations as above we have the following. Let ρ be a representation of G . Thus $\rho : G \rightarrow M_m(\mathbb{C})$. Define $\widehat{\rho} : R_n(G) \rightarrow M_{nm}(\mathbb{C})$ by the following:

If $A = (a_{i,j}) \in R_n(G)$, then $\widehat{\rho}(A) = (\rho(a_{i,j}))$ where $\rho(0) = 0$ (the $m \times m$ zero matrix).

Note that $\widehat{\rho}(gE_{i,j}) = \rho(g) \otimes E_{i,j}$. Clearly the representation $\widehat{\rho}$ is continuous. Observe

that if two representations are equivalent $\rho \sim \phi$ then $\widehat{\rho} \sim \widehat{\phi}$. Indeed, suppose that $\rho \sim \phi$ then there exists a matrix $T \in M_m(\mathbb{C})$ such that $T\rho(g)T^{-1} = \phi(g)$ where $g \in G$. Therefore for all $A \in R_n(G)$ $(T \otimes I_n)\widehat{\rho}(A)(T \otimes I_n)^{-1} = \widehat{\phi}(A)$.

The following theorem is called the Burnside theorem. We are going to use it to prove the theorem 3.2.20.

Theorem 3.2.19 *Let G be a group and $\varphi : G \rightarrow M_n(\mathbb{C})$ be a representation. Then φ is an irreducible representation if and only if $\text{Span}(\varphi(G)) = M_n(\mathbb{C})$.*

Recall that J is an order compact \mathcal{J} -class and G_J is maximal subgroup.

Theorem 3.2.20 *If $\rho : G \rightarrow M_m(\mathbb{C})$ is an irreducible representation, then*

$$\widehat{\rho} \circ \varphi_J : S \rightarrow M_{nm}(\mathbb{C})$$

is an irreducible representation.

Proof:

Step 1: We will show that $B_n(G) = \varphi_J(J) \cup \{0\}$. Indeed, let $h \in J$ then

$$\varphi_J(h) = \sum_{\{e_j \in E(J) \mid he_j \in J\}} \alpha(h \cdot e_j).$$

Since idempotents in a \mathcal{J} -class are incomparable and $(he_j)^*(he_j) = e_j h^* h e_j \leq h^* h$, it follows that $\{e_j \in E(J) \mid he_j \in J\} = \{h^* h\}$. Therefore $\varphi_J(h) = \alpha(h)$ which is a bijection of J^0 with $B_n(G)$.

Step 2: We will prove that $\widehat{\rho}|_{B_n(G)}$ is an irreducible representation. Indeed, we have $\widehat{\rho}(gE_{i,j}) = \rho(g) \otimes E_{i,j}$. Moreover

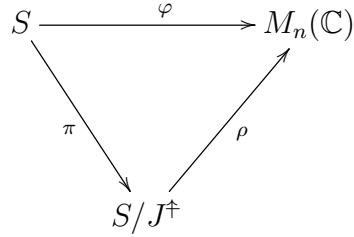
$$\begin{aligned} \text{Span}_{\substack{g \in G \\ i,j \in \{1,2,\dots,n\}}} (\rho(g) \otimes E_{i,j}) &= \text{Span}_{i,j \in \{1,2,\dots,n\}} \{ \text{Span}_{g \in G} \rho(g) \otimes E_{i,j} \} \\ &= \text{Span}_{i,j \in \{1,2,\dots,n\}} (M_n(\mathbb{C}) \otimes E_{i,j}) \quad \text{by the Burnside thm for } G. \\ &= M_{nm}(\mathbb{C}) \end{aligned}$$

Again by the Burnside Theorem $\widehat{\rho}|_{B_n(G)}$ is an irreducible representation.

Step 3: We conclude that no nontrivial subspace of \mathbb{C}^{nm} is invariant under the action of elements of J . Thus the representation $\widehat{\rho} \circ \varphi_J$ is irreducible. ■

Now suppose that we have an irreducible representation of a compact inverse monoid $\varphi : S \rightarrow M_k(\mathbb{C})$. Let e be a minimal idempotent that does not map to 0 (using Theorem 3.1.2 the idempotent e exists and is unique up to \mathcal{J} -equivalence) and set $J = J_e$.

Our goal is to show that φ is constructed as in Theorem 3.2.20 for a unique compact \mathcal{J} -class J and representation ρ of the maximal subgroup G . The following diagram commutes by Theorem 3.1.2 (6).



From the diagram it follows that ρ determines φ . We first need to prove that $\rho|_J$ uniquely determines ρ .

Let us prove the following proposition.

Proposition 3.2.21 *Let S be a semigroup and $I \subseteq S$ an ideal. Suppose that $\varphi : S \rightarrow M_n(\mathbb{C})$ is an irreducible representation of S such that $\varphi(I) \neq 0$. Then $\varphi|_I$ is irreducible and uniquely determines φ .*

Proof: We have $\text{Span}(\varphi(I))$ is an ideal in $\text{Span}(\varphi(S))$. But $\text{Span}(\varphi(I)) \subseteq \text{Span}(\varphi(S)) = M_n(\mathbb{C})$ by Burnside theorem. Since $M_n(\mathbb{C})$ is simple, we have $\text{Span}(\varphi(I)) = M_n(\mathbb{C})$. Thus $I_n \in \text{Span}(\varphi(I))$ which implies that $I_n = \sum_{i \in F} c_i \varphi(x_i)$ where $c_i \in \mathbb{C}$, $x_i \in I$ and F is finite set of indices. If $s \in S$ then $\varphi(s) = \varphi(s)I_n = \sum_{i \in F} c_i \varphi(sx_i)$. Thus $\varphi|_I$ is uniquely determines φ . ■

Since $\varphi(J^\dagger) = 0$ then $J^0 = J \cup \{0\}$ is an ideal in S/J^\dagger . We have $\rho(e) \neq 0$ which implies that $\rho(J^0) \neq 0$. Thus $\rho|_{J^0}$ is irreducible and determines ρ uniquely and therefore φ as well.

Let G be a maximal subgroup around an idempotent element of J . It remains to prove $\rho|_G$ is irreducible and determines $\rho|_J$.

Proposition 3.2.22 *Let G be a compact group. Let $\rho : B_n(G) \longrightarrow M_k(\mathbb{C})$ be an irreducible representation of $B_n(G)$ such that $\rho(0) = 0$. Then*

1. *The representation*

$$\begin{aligned} \bar{\rho} : G &\longrightarrow \text{End}(\rho(E_{1,1})\mathbb{C}^k) = \rho(E_{1,1})M_k(\mathbb{C})\rho(E_{1,1}) \\ g &\mapsto \rho(gE_{1,1}) \end{aligned}$$

is irreducible.

2. *If ρ_1 and ρ_2 are two irreducible representation of $B_n(G)$ such that $\rho_1 \sim \rho_2$. Then $\bar{\rho}_1 \sim \bar{\rho}_2$.*

Proof:

1. We have

$$\begin{aligned} \rho(E_{1,1})M_k(\mathbb{C})\rho(E_{1,1}) &= \rho(E_{1,1})\text{Span}(\rho(B_n(G)))\rho(E_{1,1}) \\ &= \text{Span}(\rho(E_{1,1})\rho(B_n(G))\rho(E_{1,1})) \\ &= \text{Span}(\bar{\rho}(G)) \end{aligned}$$

by Burnside theorem $\bar{\rho}$ is irreducible.

2. Suppose that $\rho_0 \sim \rho_1$, this implies that there exists $T \in M_k(\mathbb{C})$ such that $T\rho_0(gE_{i,j})T^{-1} = \rho_1(gE_{i,j})$. Thus $\bar{\rho}_1(g) = \rho_1(gE_{1,1}) = T\rho_0(gE_{1,1})T^{-1} = T\bar{\rho}_0(g)T^{-1}$

■

With the same notation, we have the following.

Proposition 3.2.23 *Let G be a compact group. There is a bijection between the collections of representations of G and of $B_n(G)$ (sending 0 to 0) preserving direct sums and hence irreducibility, i.e., ρ is determined by $\bar{\rho}$ up to equivalence.*

Proof: Let $\rho : B_n(G) \rightarrow M_k(\mathbb{C})$ with $\rho(0) = 0$ and non-degenerate. By an argument analogous to Lemma 2.3.2 for groupoids, we have $I_k = \sum_{i=1}^n \rho(E_{i,i})$. Let $V = \mathbb{C}^k$ and $V_i = \rho(E_{i,i})V$. Then $V = V_1 \oplus V_2 \dots \oplus V_n$ as the $\rho(E_{i,i})$ are orthogonal. We have

$$E_{i,i}E_{j,j} = \begin{cases} E_{i,i} & i = j \\ 0 & \text{else.} \end{cases}$$

Note that $\dim(V_1) = \dim(V_2) = \dots = \dim(V_n)$, since $\rho(E_{j,i}) : V_i \rightarrow V_j$ with inverse $\rho(E_{i,j}) : V_j \rightarrow V_i$ since

$$E_{i,j}E_{j,i} = E_{i,i}, \quad E_{j,i}E_{i,j} = E_{j,j}.$$

Let $m = \dim(V_1)$. Then $k = nm$. Choose a basis B for V_1 . Then $\rho(E_{j,1})B = B_j$ is basis for V_j and $\rho(E_{1,1})B = B$. So

$$\mathcal{B} = B \cup \rho(E_{2,1})B \cup \dots \cup \rho(E_{n,1})B = B \cup B_2 \cup \dots \cup B_n$$

is a basis for V .

Define $\widehat{\rho}$ to be the extension of $\bar{\rho}$ to $B_n(G)$ where we use the basis B for $V_1 = \rho(E_{1,1})V$ to make $\bar{\rho}$ a matrix representation. i.e,

$$\begin{aligned} \widehat{\rho} : B_n(G) &\rightarrow M_m(\mathbb{C}) \\ gE_{i,j} &\mapsto \bar{\rho}(g) \otimes E_{i,j} \end{aligned}$$

The claim is $\rho \sim \widehat{\rho}$. We show that $\widehat{\rho}$ is ρ in the basis \mathcal{B} . We have $\rho(gE_{i,j})$ in \mathcal{B} basis is $\rho(gE_{i,j})(V_k) = 0$ if $j \neq k$ since $E_{i,j}E_{k,k} = 0$. Also $\rho(gE_{i,j})(V_j) = V_i$. A typical element of $\rho(E_{j,1})B$ is $\rho(E_{j,1})v$ where $v \in B$.

$$\rho(gE_{i,j})\rho(E_{j,1})v = \rho(E_{i,1})\rho(gE_{1,1})\rho(E_{1,j})\rho(E_{j,1})(v)$$

$$\begin{aligned}
 &= \rho(E_{i,1})\rho(gE_{1,1})(v) \text{ because } \rho(E_{1,j})\rho(E_{j,1})(v) = \rho(E_{1,1})v = v \\
 &= \rho(E_{i,1})\bar{\rho}(g)v.
 \end{aligned}$$

Therefore

$$\rho(gE_{i,j}) = \begin{pmatrix} & & \rho(E_{j,1})B & & \\ & 0 & \dots & 0 & \dots & 0 \\ & \vdots & & & & \vdots \\ & 0 & \dots & \bar{\rho}(g) & \dots & 0 \\ & \vdots & & & & \vdots \\ & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \rho(E_{i,1})B$$

Thus $\rho(gE_{i,j}) = \bar{\rho}(g) \otimes E_{i,j}$. ■

Fix a maximal subgroup G_J for every \mathcal{J} -class.

Theorem 3.2.24 (Main theorem (new)) *Let S be a compact inverse monoid. There is a bijection between finite dimensional irreducible representations of S and pairs (J, φ) where J is an order compact \mathcal{J} -class and φ is an irreducible representation of the maximal subgroup of J .*

Proof: In the Theorem 3.1.2 we proved that for every finite dimensional irreducible representation of a compact inverse semigroup φ there exists a minimum compact element e . Therefore we have a unique \mathcal{J} -class J_e such that the restriction of φ to the maximal subgroup G_{J_e} is irreducible. Moreover given a \mathcal{J} -class J_e and φ an irreducible representation of maximal subgroup G_{J_e} we were able to construct a unique (up to equivalence) finite dimensional representation irreducible of the compact inverse semigroup S . ■

Example 3.2.25 Let $S = R_n(G)$ and $I = B_n(G)$. Let $\varphi : G \rightarrow M_k(\mathbb{C})$ then define

$$\begin{aligned}\widehat{\varphi} : B_n(G) &\longrightarrow M_{kn}(\mathbb{C}) \\ gE_{i,j} &\mapsto \widehat{\varphi}(gE_{i,j}) = \varphi(g) \otimes E_{i,j} \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) = M_{kn}(\mathbb{C}).\end{aligned}$$

Note

$$\widehat{\varphi}(E_{1,1}) + \dots + \widehat{\varphi}(E_{n,n}) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & \dots & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dots & I_k \end{pmatrix} = I_{kn}$$

By Proposition 3.2.21 $\widehat{\varphi}$ has exactly one extension to $R_n(G)$. Then the extension is given by

$$[g_{i,j}] \mapsto [\varphi(g_{i,j})]$$

where $\varphi(0) = 0$.

3.3 The radical of a compact inverse monoid.

Let S be a compact inverse semigroup and s, t two elements in S . We define the radical of S to be the congruence $[\text{rad}(S)]$ given by $s \equiv t \pmod{\text{rad}(S)}$ if and only if $\varphi(s) = \varphi(t)$ for any finite dimensional irreducible representation φ .

Theorem 3.3.1 (New) *Let S be a compact inverse semigroup and s, t two elements in S . Then $s \equiv t \pmod{\text{rad}(S)}$ if and only if $s^\perp \cap \mathcal{K}(S) = t^\perp \cap \mathcal{K}(S)$.*

Proof: Suppose that $s^\perp \cap \mathcal{K}(S) = t^\perp \cap \mathcal{K}(S)$ for some s and t in S . Let π be a finite dimensional irreducible representation of S . Therefore there exist J , an order compact \mathcal{J} -class, and $\varphi : G \rightarrow M_d(\mathbb{C})$ is an irreducible representation of the maximal subgroup G of J . The set $E(J) = \{e_1, \dots, e_m\}$ of idempotents of J is finite. Assume G is the maximal subgroup at e_1 . Then $\pi : S \rightarrow M_{dm}(\mathbb{C})$ up to equivalence is defined

as follows, choose $p_i \in J$ so that $p_i^*p_i = e_1$, $p_i p_i^* = e_i$. Let $s \in S$. If $se_i \in J$, define $\sigma_s(i) = j$ where $e_j = se_i s^*$. Then

$$\pi(s) = \sum_{\{i|se_i \in J\}} \varphi(p_{\sigma_s(i)}^* s p_i) \otimes E_{\sigma_s(i)i}. \quad (*)$$

If we index the rows and columns by $E(J)$ and write P_{e_i} for P_i , then the equation (*) applied to s and t becomes.

$$\begin{aligned} \pi(s) &= \sum_{\{x \in J | x \leq s\}} \varphi(p_{xx^*}^* x p_{x^*x}) \otimes E_{xx^*, x^*x} \\ &= \sum_{\{x \in J | x \leq t\}} \varphi(p_{xx^*}^* x p_{x^*x}) \otimes E_{xx^*, x^*x} \\ &= \pi(t) \end{aligned}$$

since $J \subset \mathcal{K}(S)$.

Now suppose that $s^\downarrow \cap \mathcal{K}(S) \neq t^\downarrow \cap \mathcal{K}(S)$. Then without loss of generality there exists $k \leq t$ a compact element and $k \notin s^\downarrow \cap \mathcal{K}(S)$. The \mathcal{J} -class J of k is compact. Let $E(J) = \{e_1, e_2, \dots, e_m\}$ and retain the notation above. We have two cases:

Case 1: Suppose that there exists $k \neq h \in s^\downarrow \cap \mathcal{K}(S)$ such that $h^*h = k^*k = e$ and $hh^* = kk^* = f$.

$$e \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} f$$

By the Peter-Weyl theorem there exists an irreducible representation φ of the maximal subgroup G such that $\varphi(p_{\sigma_h(i)}^* h p_i) \neq \varphi(p_{\sigma_k(i)}^* k p_i)$. Thus we can separate s and t by the finite dimensional irreducible representation π of S defined as in (*).

Case 2: Suppose that there exists no $h \in s^\downarrow \cap \mathcal{K}(S)$ such that $h^*h = k^*k = e$ and $hh^* = kk^* = f$. Let φ be the trivial representation of G then

$$\pi(s) = \sum_{\{x \in J | x \leq s\}} I_m \otimes E_{xx^*, x^*x}$$

and

$$\pi(t) = \sum_{\{x \in J | x \leq t\}} I_m \otimes E_{xx^*, x^*x}$$

and these are different because for s we have the term $I_n \otimes E_{f,e}$ but for t there is no term $I_n \otimes E_{f,e}$. ■

Theorem 3.3.2 *Let S be a compact inverse semigroup. Then the following are equivalent.*

1. S has enough finite dimensional representations to separate points.
2. $[\text{rad}(S)]$ is the trivial congruence.
3. S is an algebraic domain.
4. $E(S)$ is an algebraic lattice.
5. $E(S)$ is totally disconnected.

Proof: We proved in the Theorem 1.6.15 the equivalence between (3), (4) and (5). Now, suppose that S has enough representations finite dimensional to separate points. By the compactness of S the first condition holds (S admits directed joins). And also the second condition holds ($\forall s \in S \quad s^\downarrow \cap \mathcal{K}(S)$ is directed). Now, Let $s \in S$. Put $t = \bigvee s^\downarrow \cap \mathcal{K}(S)$. Suppose that $t < s$, therefore $(t^\downarrow \cap \mathcal{K}(S)) \subset (s^\downarrow \cap \mathcal{K}(S))$. But $t = \bigvee s^\downarrow \cap \mathcal{K}(S)$ thus $(s^\downarrow \cap \mathcal{K}(S)) \subset (t^\downarrow \cap \mathcal{K}(S))$, which implies that $(s^\downarrow \cap \mathcal{K}(S)) = (t^\downarrow \cap \mathcal{K}(S))$. Therefore $s = t$. ■

The equivalence of (1) and (5) is Shneperman's Theorem see [21].

Corollary 3.3.3 (Shneperman's Theorem) *A compact inverse monoid S has enough finite dimensional irreducible representations to separate points if and only if $E(S)$ is totally disconnected.*

3.4 Norm continuous irreducible $*$ -representations are finite dimensional

In this section, we will prove that any norm continuous irreducible $*$ -representation of a compact inverse semigroup on a Hilbert space is finite dimensional. This result is similar to the following theorem for compact groups see [17].

Theorem 3.4.1 *Any irreducible representation of a compact group is finite dimensional.*

The key idea, as in the finite dimensional case, is to find a primitive idempotent.

Definition 3.4.2 *If S is an inverse semigroup, then a primitive idempotent is a minimal nonzero idempotent.*

Lemma 3.4.3 *Let S be a compact inverse semigroup and suppose that e is a primitive idempotent. Then e is compact.*

Proof: Suppose $e \leq \bigvee_{d \in D} d$ where D is a directed set of idempotents. Since multiplication commutes with directed joins in a compact semilattice, we obtain $e \leq \bigvee_{d \in D} ed$. But $ed \leq e$ so by primitivity $ed = e$ or $ed = 0$. If $ed = 0$ all $d \in D$, we would get $e \leq 0$, a contradiction. So $ed = e$ some $d \in D$, that is, $e \leq d$ some $d \in D$. ■

Lemma 3.4.4 *Let e be an idempotent bounded operator on a Hilbert space \mathcal{H} . Then $Im(e)$ is weak*-closed.*

Proof: Suppose $\{v_\alpha\}_{\alpha \in D}$ is a net converging to v in the weak* topology with the v_α in $Im(e)$. So $ev_\alpha = v_\alpha$ for all n . Let w be any vector in \mathcal{H} . Then $\langle ev | w \rangle = \langle v | e^*w \rangle = \lim \langle v_\alpha | e^*w \rangle = \lim \langle ev_\alpha | w \rangle = \lim \langle v_\alpha | w \rangle = \langle v | w \rangle$.

Thus $ev = v$. ■

Proposition 3.4.5 *Let $f : E \rightarrow B(\mathcal{H})$ be a norm continuous $*$ -representation of a compact semilattice on a Hilbert space. Then $f(E)$ contains a minimal nonzero idempotent.*

Proof: If $0 \notin f(E)$, then the minimum of $f(E)$ will do. Otherwise, we use Zorn's lemma. Let D be a descending chain of nonzero idempotents of $f(E)$. Then if we view D as a net, it converges to its meet e . We need to show that $e \neq 0$.

Let A be the norm-closed subalgebra of $B(\mathcal{H})$ spanned by $f(E)$. Then A is a commutative C^* -algebra and so is isomorphic to $C_c(X)$ where X is the spectrum of A . Now a projection on X must be the characteristic function of a compact open set and the ordering translates into inclusion. So a descending chain D of nonzero idempotents in $f(E)$ corresponds in $C_c(X)$ to the characteristic functions of a descending chain of non-empty compact open subspaces $\{U_\alpha\}_{\alpha \in D}$. Since convergence in $C_c(X)$ is uniform convergence, it follows that e is the characteristic function of the non-empty subset $\bigcap_{\alpha \in D} U_\alpha$. This proves that meet of each descending chain of nonzero idempotents of $f(E)$ is nonzero. ■

Theorem 3.4.6 *Every norm continuous irreducible $*$ -representation of a compact inverse semigroup on a Hilbert space is finite dimensional.*

Proof: Let $f : S \rightarrow B(\mathcal{H})$ be a norm continuous irreducible $*$ -representation. Then $f(E(S))$ is nonzero and compact. By the above proposition, $f(E(S))$ contains a minimal nonzero idempotent e and this idempotent is compact. Now the same argument as the finite dimensional case shows that we obtain an irreducible representation $G_e \rightarrow B(e\mathcal{H})$ (where G_e is maximal subgroup at e) by restricting. By the

theorem for groups 3.4.1, it follows that $e\mathcal{H}$ is finite dimensional. If J is the \mathcal{J} -class of e , then the same argument as the finite dimensional case shows that $\bigoplus_{f \in E(J)} f\mathcal{H}$, which is a finite dimensional (since $E(J)$ is finite) and hence closed, is invariant under S . Thus $\mathcal{H} = \bigoplus_{f \in E(J)} f\mathcal{H}$ is finite dimensional. ■

Conclusion

The existence of a left regular representation is one of the main goals of my future work. First, we will define a new measure on compact inverse semigroups in order to define a left regular representation. Let S be a locally compact inverse semigroup. A P-V-invariant measure on S is a Borel σ measure on S with global support such that $\sigma(sB) = \sigma(B)$ for every Borel subset of s^*sS . The existence of this measure relies on the existence of a Haar system (in the sense of Seda [19]). Then define a left translation by

$$\begin{aligned}\pi(s) : L^2(S, \sigma) &\longrightarrow L^2(S, \sigma) \\ f(x) &\mapsto f(s^*x)\chi_{s^*sS}\end{aligned}$$

This agrees with the usual definition when S is discrete and σ is counting measure.

Some other ideas for future work.

- Let S be a compact inverse monoid with totally disconnected idempotent set. $G_S = \mathcal{K}(S)$ is the set of order compact elements which is a topological groupoid. We want to show

$$\text{Rep}(S) \simeq \text{Rep}(G_S).$$

- The existence of P-V-invariant measure of a locally compact inverse monoid.
- Defining characters of a compact inverse semigroup and generalize theorems from group case to inverse semigroup.

-
- Give a parametrization of finite dimensional irreducible representations of locally compact inverse semigroups.
 - Prove or disprove: every irreducible $*$ -representation of compact inverse semigroup is finite-dimensional.
 - Prove the P -theorem for discrete inverse semigroup.
 - Given an action of a semigroup T on an inverse semigroup S , show that $C^*(S \rtimes T) \cong C^*(S) \rtimes T$.
 - Recover the topology on S from $G_S = \mathcal{K}(S)$.

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