SPECTRAL ASYMPTOTICS
FOR
POLAR VECTOR STURM-LIOUVILLE PROBLEMS

by

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To my parents

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Abstract

We study the spectrum of the vector boundary value problem

$$y'' + \lambda R(x)y = 0$$

$$y(a) = y(b) = 0,$$

where $R(x)$ is a real continuous n by n symmetric matrix function and, in particular, discuss the existence of two infinite sequences of eigenvalues. We also present two asymptotic formulae for the distribution functions of the positive and negative eigenvalues when $R(x)$ belongs to two different classes of matrices.

1. Introduction

For the scalar boundary value problem

$$y'' + \lambda r(x)y = 0 \qquad \left[\,'' = \frac{d^2}{dx^2}\right] \tag{1.1}$$
$$y(a) = y(b) = 0$$

there exists positive and negative sequences of eigenvalues which have the limit points $+\infty$ and $-\infty$ when the continuous function $r(x)$ changes sign in (a,b), (cf., [6]). In fact the mere condition that $r(x) > 0$ and $r(x) < 0$ on two intervals of positive Lebesgue measure in (a,b) guarantees the above existence result, (cf., [6]). Under a similar hypothesis, it will be proved in Chapter 3 that for the vector boundary value problem

$$y'' + \lambda R(x)y = 0 \tag{1.2}$$

$$y(a) = y(b) = 0 \tag{1.3}$$

where $R(x)$ is a real continuous n by n symmetric matrix, there exist positive and negative sequences of eigenvalues with limit points $+\infty$ and $-\infty$. Our proof is based on a paper of two-parameter eigenvalue problems by Paul Binding and Patrick J. Browne, [1]. In Chapter 3 we consider the two-parameter vector boundary value problem

$$-y'' + \lambda_1 R(x)y + \lambda_2 Iy = 0 \tag{1.4}$$

with boundary conditions *(1.3)* and then we give a detailed study of the geometric behaviour of the functions $\lambda_2^i(\lambda_1)$ for $i = 0,1,2,\ldots$, in the $\lambda_1\lambda_2$-plane.

By a *solution* y of (1.2) [(1.4)] we mean a vector valued function such that y, y' are absolutely continuous on [a,b], $y'' \in \mathcal{L}_2^{(n)}[a,b]$ and which satisfies the differential equation (1.2) [(1.4)].

The other major result that we present in this work is as follows: In [5] I.C. Gohberg and M.G. Krein show that the number of eigenvalues $n(s)$ in the closed interval [0,s] for the problem (1.2) with the boundary conditions (1.3) satisfies the asymptotic formula

$$\lim_{s\to\infty} \frac{n(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R(x))dx \tag{1.5}$$

when R(x) is of *positive type*. Here $\Lambda_{\frac{1}{2}}(R(x))$ is the sum of the square roots of the moduli of all the eigenvalues of the square matrix R(x) and n(s) is called the *distribution function* of eigenvalues. In Chapter 4 we extend this result to two different classes of matrices and show that when R(x) "changes sign" in (a,b) the distribution function of the positive eigenvalues $n_+(s)$ of the problem (1.2) and (1.3) satisfies the asymptotic formula

$$\lim_{s\to\infty} \frac{n_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R_+(x))dx \; . \tag{1.6}$$

Here $R_+(x) = \frac{1}{2}[|R(x)| + R(x)]$. This asymptotic formula (1.6) agrees with the result (1.5) of I.C. Gohberg and M.G. Krein since $R(x) = R_+(x)$ and all the eigenvalues of the boundary value problem are positive when R(x) is non negative definite on (a,b). A formula similar to (1.6) holds for the negative eigenvalues of the same problem. The distribution function of the negative eigenvalues $n_-(s)$ of the problem (1.2) and (1.3) satisfies the asymptotic formula

$$\lim_{s\to\infty} \frac{n_-(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R_-(x))dx,$$

where $R_-(x) = \frac{1}{2}[|R(x)| - R(x)]$.

It is interesting to observe in this context, that the distribution function $N_+(s)$ of the positive eigenvalues of the scalar boundary value problem *(1.1)* satisfies the asymptotic formula

$$\lim_{s\to\infty} \frac{N_+(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sqrt{r_+(x)}\, dx,$$

where $r_+(x) = \max(0,r(x))$. This is a direct result from [7].

2. Definitions and preliminary results

2.1. Matrix notation

Ordinary matrix notation is used throughout; in particular, matrices of one column are called *vectors*. For a vector $y = (y_\alpha)$, $(\alpha = 1,\dots,n)$ the norm $||y||_{R_n}$ is given by $(y_1^2 + y_2^2 + \dots + y_n^2)^{\frac{1}{2}}$, R_n is the usual linear vector space of ordered n-tuples of real numbers. Only real valued matrices are considered. The n by n identity matrix is denoted by I_n, or merely I when there is no ambiguity, while 0 is the zero matrix regardless of dimension. For A an m by n matrix the n by m matrix obtained by interchanging rows and columns of A is denoted by A^t, and called the *transpose* of A. A square matrix $A = A_{\alpha\beta}$, $(\alpha,\beta = 1,\dots,n)$ is said to be symmetric if $A = A^t$. The real symmetric n by n matrix A is said to be positive (non negative) definite if $y^tAy > 0$ $(y^tAy \geq 0)$ for all nonzero vectors y belonging to R_n. If A is an n by n matrix, then the supremum of $\{||Ay||_{R_n} : y \in R_n, ||y||_{R_n} \leq 1\}$ is called the *norm of A* and denoted by $\nu[A]$. Moreover for a given symmetric matrix the symbol $|A|$ *will denote the non negative definite symmetric square root matrix of* A^2, and A_+, A_- will denote the corresponding $A_+ = \frac{1}{2}[|A|+A]$, $A_- = \frac{1}{2}[|A|-A]$. It is true (see, e.g., Riesz-Nagy [11], Section 108), that A_+ and A_- are non negative definite symmetric matrices such that $A_+ \geq A$ and $A \geq -A_-$. The notation $A \geq B$ $(A > B)$ is used to signify that A and B are symmetric matrices of the same dimensions, and A-B is a non negative (positive) definite square matrix. A *matrix function* $A(t) = A_{\alpha\beta}(t)$, $(\alpha = 1,\dots,m,$ $\beta = 1,\dots,n)$ is called *continuous, integrable*, etc., when each element $A_{\alpha\beta}$ possesses the specified property. If A(t) is (Lebesgue) integrable on [a,b], - hereafter this compact interval is denoted by Q, -then $\int_Q A(t)dt$ denotes the matrix of integrals of the respective elements of $A(t)$.

We denote by $\mathcal{C}(Q)$ the set of all n by n matrix functions that are continuous on Q.

$\mathcal{L}_2(Q)$ is the Hilbert space consisting of all equivalence classes of real valued measurable functions $f(x)$ on Q that satisfy

$$||f||^2_{\mathcal{L}_2(Q)} = \int_Q |f(x)|^2 dx < \infty .$$

The inner product in $\mathcal{L}_2(Q)$ is defined by

$$(f,g)_{\mathcal{L}_2(Q)} = \int_Q f(x)g(x)dx \qquad (f,g \in \mathcal{L}_2(Q)).$$

$\mathcal{L}_2^{(n)}(Q)$ is the Hilbert space consisting of all n dimensional vector functions $f(x) = (f_\alpha(x))$, $(\alpha = 1,\ldots,n)$, $x \in Q$, with real valued measurable coordinates $f_\alpha(x)$ $(\alpha = 1,\ldots,n)$ such that

$$||f||^2_{\mathcal{L}_2^{(n)}(Q)} = \int_Q \sum_{\alpha=1}^{n} |f_\alpha(x)|^2 dx < \infty .$$

The inner product in $\mathcal{L}_2^n(Q)$ is defined in a natural way by the formula

$$(f,g)_{\mathcal{L}_2^{(n)}(Q)} = \int_Q g^t(x)f(x)dx = \int_Q \sum_{\alpha=1}^{n} g_\alpha(x)f_\alpha(x)dx \qquad (f,g \in \mathcal{L}_2^{(n)}(Q)).$$

The Hilbert space $\mathcal{L}_2^{(n)}(Q)$ is considered throughout this work and therefore it is of particular importance. We will write $\mathcal{L}_2^{(n)}(Q) = H$ for simplicity.

$\mathcal{L}_2^{(nxn)}(QxQ)$ is the Hilbert space consisting of matrix functions $A(x,t) = A_{\alpha\beta}(x,t)$, $(\alpha,\beta = 1,\ldots,n)$ with elements measurable on QxQ such that

$$||A||^2_{\mathcal{L}_2^{(nxn)}(QxQ)} = \int_Q \int_Q \sum_{\alpha,\beta=1}^{n} |A_{\alpha\beta}(x,t)|^2 dxdt < \infty ,$$

in which the inner product of two elements A(x,t) and B(x,t) is defined by the formula

$$(A,B)_{\mathcal{L}_2^{(n\times n)}(Q\times Q)} \doteq \int_Q \int_Q \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(x,t)B_{\alpha\beta}(x,t)dxdt.$$

2.2. A Sturm-Liouville operator

In this section we consider a vector Sturm-Liouville operator T defined by $Ty = -y''$ on H. T is not defined on the whole space H, but only on the set $\mathcal{D}(T)$, where

$$\mathcal{D}(T) = \{y \in H\colon y,\ y' \text{ are absolutely continuous on } [a,b],\ y'' \in H \text{ and } y(a) = y(b) = 0\}.$$

Some known results for $\mathcal{L}_2(Q)$ can be extended to show that H is separable and $\mathcal{D}(T)$ is dense in H. Hence T is said to be densely defined on H.

Here, our objective is to examine some properties of the operator T. But we need some definitions before we start this investigation.

Definition 2.2.1. Let T be a densely defined linear operator on a Hilbert space H. Let D(T*) be the set of $v \in H$ for which there is an $z \in H$ with $(Tu,v) = (u,z)$ for all u belong to the domain D(T) of T. For each such $v \in D(T^*)$, we define $T^*v = z$. T* is called the *adjoint* of T.

Definition 2.2.2. A densely defined operator T on a Hilbert space H is called *symmetric* if $(Tu,v) = (u,Tv)$ for all $u,v \in D(T)$.

Definition 2.2.3. T is called *self-adjoint* if T is symmetric and $D(T) = D(T^*)$.

Now the following theorem is an easy result, (cf., [11], section 119).

Theorem 2.2.4. The Sturm-Liouville operator $Ty = -y''$ is self adjoint on H.

Definition 2.2.5. A symmetric operator T for which there exists a real constant α such that $(Ty,y) \geq \alpha(y,y)$ for all $y \in D(T)$ is said to be *bounded below*. In particular if $(Ty,y) > 0$ for all $y \in D(T)$ then T is said to be *positive*.

Next we prove the

Theorem 2.2.6. The Sturm-Liouville operator $Ty = -y''$ is positive on H.

Proof.
$$(Ty,y)_H = \int_a^b (-y''(t),y(t))dt$$
$$= -(y^t)'y\Big|_a^b + \int_a^b (y'(t),y'(t))dt$$
$$= \int_a^b (y'(t),y'(t))dt$$
$$> 0 \qquad \text{for all } y \in \mathcal{D}(T),\ y \not\equiv 0.$$

For, $\int_a^b (y'(t),y'(t))dt = 0$ implies that $y'(t) = 0$ a.e. and hence $y(t) = \text{constant}$. Thus $y(t) \equiv 0$ since $y(a) = 0$.

Definition 2.2.7. The *resolvent set* of T is the set $\rho(T)$ of all λ such that the range of $\lambda I-T$ is dense in the Banach space X and $\lambda I-T$ has a continuous inverse. For $\lambda \in \rho(T)$ the operator $(\lambda I-T)^{-1}$ is called the *resolvent operator*. The *spectrum* of T is the set $\sigma(T)$ of all scalar values not in $\rho(T)$.

Definition 2.2.8. Let X and Y be two Banach spaces and let $T:X \to Y$ be a linear operator. T is said to be compact if T(S) lies in a compact subset of Y, where $S = \{x \in X: ||x|| \leq 1\}$.

Definition 2.2.9. Let T be a linear operator on a Hilbert space H. Then T has a *compact resolvent* if there is a λ_0 in the resolvent $\rho(T)$ for which $(\lambda_0 I-T)^{-1}$ is compact.

We need a few more results before the next theorem.

Corollary 2.2.10. $\lambda = 0$ is not an eigenvalue of the operator $Ty = -y''$.

Proof. If $\lambda = 0$ is an eigenvalue, then $Ty = 0$ and hence $(Ty,y) = 0$. But this implies $y \equiv 0$ from theorem 2.2.6.

Corollary 2.2.11. T is an invertible operator.

Proof. Since $Ty = 0$ implies $y = 0$ the corollary is proved.

Theorem 2.2.12. T has a compact resolvent.

Since $\lambda = 0$ is not an eigenvalue of T it suffices to show that T^{-1} is a compact operator. But in order to do this we need to construct a *matrix Green's function*.

2.3. A matrix Green's function

It is assumed that a matrix Green's function possesses similar properties as in the scalar case([2],Chapter 5.14.1). Therefore a matrix Green's function $G(x,\xi)$ is a function of x and a parameter ξ, $(a < \xi < b)$, satisfying three conditions.

(i) $G(x,\xi)$, considered as a function of x, satisfies the differential equation

$$Y'' = 0 \qquad (a)$$

at all points of the interval (a,b) except at the point $x = \xi$.

(ii) $G(x,\xi)$ satisfies (as a function of x) both boundary conditions

$$Y(a) = Y(b) = 0 \qquad (b).$$

(iii) $G(x,\xi)$ is everywhere continuous for $a \leq x \leq b$. But its derivative $G_x(x,\xi)$ with respect to x is continuous only for $a \leq x < \xi$ and $\xi < x \leq b$, and at $x = \xi$

$$G_x(\xi+0,\xi) - G_x(\xi-0,\xi) = -I \qquad (c).$$

A basis for the solution space of equation *(a)* is $\{I, xI\}$. Therefore, ([14],page 130),

$$G(x,\xi) = \begin{cases} A(\xi) + xB(\xi) & a \leq x < \xi \\ C(\xi) + xD(\xi) & \xi < x \leq b. \end{cases} \qquad (d)$$

Here $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are four unknown n by n matrix functions of ξ, which can be determined from the boundary conditions *(b)*, the continuity condition $G(\xi+0,\xi) = G(\xi-0,\xi)$ and *(c)*.

Applying these three conditions in *(d)* we find

$$A(\xi) + aB(\xi) = 0$$

$$C(\xi) + bD(\xi) = 0$$

$$A(\xi) + \xi B(\xi) - C(\xi) - \xi D(\xi) = 0$$

$$D(\xi) - B(\xi) = -I.$$

Solving the above four matrix equations for A, B, C and D we have

$$A(\xi) = -a\frac{(b-\xi)}{(b-a)} I$$

$$B(\xi) = \frac{(b-\xi)}{(b-a)} I$$

$$C(\xi) = b\frac{(\xi-a)}{(b-a)} I$$

$$D(\xi) = -\frac{(\xi-a)}{(b-a)} I.$$

Therefore

$$G(x,\xi) = \begin{cases} \dfrac{(b-\xi)(x-a)}{(b-a)} I \equiv g(x,\xi)I & a \leq x < \xi \\ \dfrac{(b-x)(\xi-a)}{(b-a)} I \equiv g(x,\xi)I & \xi < x \leq b. \end{cases}$$

Lemma 2.3.1. The integral operator K defined by

$$(Ky)(\xi) = \int_a^b G(x,\xi)y(x)dx \qquad \text{for } y \in H$$

is the inverse operator of T.

Proof. Let $z(\xi) = (Ky)(\xi) = \int_a^b G(x,\xi)y(x)dx$

$$= \int_a^\xi \frac{(b-\xi)(x-a)}{(b-a)} y(x)dx + \int_\xi^b \frac{(b-x)(\xi-a)}{(b-a)} y(x)dx.$$

So that $z(a) = z(b) = 0$.

Therefore $z = Ky$ satisfies the boundary conditions. By successive differentiation we see that

$$z'(\xi) = -\frac{1}{(b-a)}\int_a^\xi (x-a)y(x)dx + \frac{1}{(b-a)}\int_\xi^b (b-x)y(x)dx$$

$$z''(\xi) = -\frac{(\xi-a)}{(b-a)} y(\xi) - \frac{(b-\xi)}{(b-a)} y(\xi)$$

$$= -\frac{(\xi-a+b-\xi)}{(b-a)} y(\xi)$$

$$= -y(\xi).$$

So that $TKy = y$.

Moreover $(KTy)(\xi) = \int_a^b G(x,\xi)(Ty)(x)dx$

$$= -\int_a^b g(x,\xi)y''(x)dx$$

$$= -\int_a^{\xi-0} g(x,\xi)y''(x)dx - \int_{\xi-0}^{\xi+0} g(x,\xi)y''(x)dx - \int_{\xi-0}^{\xi+0} g(x,\xi)y''(x)dx.$$

Now

$$\int_a^{\xi-0} g(x,\xi)y''(x)dx = g(x,\xi)y'(x)\Big|_a^{\xi-0} - \int_a^{\xi-0} g_x(x,\xi)y'(x)dx$$

$$= g(x,\xi)y'(x)\Big|_a^{\xi-0} - g_x(x,\xi)y(x)\Big|_a^{\xi-0} + \int_a^{\xi-0} g_{xx}(x,\xi)y(x)dx$$

$$= g(\xi-0,\xi)y'(\xi-0) - g_x(\xi-0,\xi)y(\xi-0) + \int_a^{\xi-0} g_{xx}(x,\xi)y(x)dx,$$

using the properties of $G(x,\xi)$.

Similarly

$$\int_{\xi+0}^{b} g(x,\xi)y''(x)dx = -g(\xi+0,\xi)y'(\xi+0) + g_x(\xi+0,\xi)y(\xi+0)$$

$$+ \int_{\xi+0}^{b} g_{xx}(x,\xi)y(x)dx,$$

and

$$\int_{\xi-0}^{\xi+0} g(x,\xi)y''(x)dx = g(x,\xi)y'(x)\Big|_{\xi-0}^{\xi+0} - g_x(x,\xi)y(x)\Big|_{\xi-0}^{\xi+0}$$

$$+ \int_{\xi-0}^{\xi+0} g_{xx}(x,\xi)y(x)dx.$$

Therefore

$$(KTy)(\xi) = - \int_a^b g_{xx}(x,\xi)y(x)dx$$

$$= -\left(\int_a^b g_{xx}(x,\xi)y_1(x)dx,\dots, \int_a^b g_{xx}(x,\xi)y_n(x)dx\right)^t$$

$$= (y_1(\xi),\dots,y_n(\xi))^t, \quad \text{(see Appendix B)},$$

$$= y(\xi).$$

Thus KTy = y and K is the inverse operator of T.

The proof of Theorem 2.2.12 is immediate from the following lemma.

Lemma 2.3.2. The integral operator defined by

$$(T^{-1}y)(\xi) = \int_a^b G(x,\xi)y(x)dx \qquad \text{for } y \in H$$

is compact.

Proof. $G(x,\xi) = g(x,\xi)I$ belongs to $\mathcal{L}_2^{(nxn)}(QxQ)$ since

$$\int_a^b \int_a^b \sum_{\beta=1}^n \sum_{\alpha=1}^n g^2(x,\xi)\delta_{\alpha\beta}dx\, d\xi = n \int_a^b \int_a^b g^2(x,\xi)dxd\xi$$

$$= n \frac{(b-a)^2}{90} .$$

Hence G is *Hilbert-Schmidt* and so the lemma follows by Theorem A.1.

3. Properties of the spectrum

3.1. A discrete spectrum

In this chapter attention will be made to the spectrum of the vector boundary value problem

$$\begin{aligned} Ty &= \lambda R(x)y \\ y(a) &= y(b) = 0 \end{aligned} \tag{3.1}$$

under the following assumptions on $R(x)$.

H1 $R(x) = R_{\alpha\beta}(x)$, $(\alpha, \beta = 1,\ldots,n)$, is a real symmetric matrix belonging to $\mathcal{L}(Q)$.

$H2_0$ There exist at least two nonempty intervals in (a,b) such that $R(x) > 0$ on one interval and $R(x) < 0$ on the other.

Generally a Sturm-Liouville boundary value problem is equivalent to a Fredholm integral equation of the second kind which incorporates the appropriate boundary conditions and some properties of the spectrum of differential operator can be studied more effectively through the corresponding integral operator.

Lemma 3.1.1. The boundary value problem *(3.1)* is equivalent to the integral equation

$$y(x) = \lambda \int_a^b G(\xi,x)R(\xi)y(\xi)d\xi, \tag{3.2}$$

which is a homogeneous Fredholm equation of the second kind.

Proof. From Lemma 2.3.1. $y = KTy$ and then $y = K(\lambda Ry)$ since $Ty = \lambda Ry$. i.e., $y = \lambda KRy$. Now the equation *(3.2)* follows from the definition of the integral operator K.

Lemma 3.1.2. The integral operator M defined by

$$(My)(x) = \int_a^b G(t,x)R(t)y(t)dt \qquad \text{for } y \in H$$

is compact.

Proof. It is enough to show that the matrix kernel $M(t,x) = G(t,x)R(t)$ is Hilbert-Schmidt or belongs to $\mathcal{L}_2^{(nxn)}(QxQ)$. Then the lemma follows from Theorem A.1.

Let $M_{\alpha\beta}(t) = g(t,x)R_{\alpha\beta}(t)$ for $\alpha,\beta = 1,\dots,n$. Then

$$\begin{aligned}
\int_a^b\int_a^b \sum_{\alpha,\beta=1}^n |M_{\alpha\beta}(t,x)|^2 dtdx &= \int_a^b\int_a^b \sum_{\alpha,\beta=1}^n |g(t,x)R_{\alpha\beta}(t)|^2 dtdx \\
&\leq \int_a^b\int_a^b \sum_{\alpha,\beta=1}^n |g(t,x)|^2 |R_{\alpha\beta}(t)|^2 dtdx \\
&= \int_a^b \sum_{\alpha,\beta=1}^n \int_a^b |g(t,x)|^2 |R_{\alpha\beta}(t)|^2 dtdx \\
&\leq \int_a^b \sum_{\alpha,\beta=1}^n \max_{t\in Q} |R_{\alpha\beta}(t)|^2 \int_a^b |g(t,x)|^2 dt \; dx \\
&\leq \int_a^b n^2 \max_{\alpha,\beta} \max_{t\in Q} |R_{\alpha\beta}(t)|^2 \int_a^b |g(t,x)|^2 dtdx \\
&= n^2 \max_{\alpha,\beta} \max_{t\in Q} |R_{\alpha\beta}(t)|^2 \int_a^b\int_a^b |g(t,x)|^2 dtdx \\
&< \infty
\end{aligned}$$

since $\int_a^b \int_a^b |g(t,x)|^2 dtdx = \frac{(b-a)^2}{90}$ and $\max_{\alpha,\beta} \max_{t \in Q} |R_{\alpha\beta}(t)|^2$ is finite as the $R_{\alpha\beta}$ are continuous on Q.

The integral equation *(3.2)* can be written as $My = \mu y$, where $\mu = \frac{1}{\lambda}$.

As we mentioned before, the objective in this chapter is to study the spectrum of the problem *(3.1)*. So we now present the first theorem in this context.

Theorem 3.1.3. The spectrum of the boundary value problem *(3.1)* is a discrete set having no limit points except perhaps $\lambda = \pm\infty$. Further, any nonzero λ belonging to the spectrum is an eigenvalue of finite multiplicity.

Proof. This follows from the Riesz-Schauder theorem (see, Appendix A) since M is a compact operator.

Theorem 3.1.4. All the eigenvalues of the boundary value problem *(3.1)* are real.

Proof. Assume the nonzero complex number $\lambda = \alpha + i\beta$ (α,β real) is an eigenvalue of *(3.1)* and y ($\not\equiv 0$) is the corresponding eigenvector. For this vector y,

$$y'' = -(\alpha + i\beta)Ry \quad \text{and} \quad y(a) = y(b) = 0.$$

Then $\overline{y}'' = -\overline{(\alpha + i\beta)}R\overline{y} = -(\alpha - i\beta)R\overline{y}$. Taking the inner product with y in H,

$$(y,\overline{y}'') = -(\alpha - i\beta)(y,R\overline{y}). \qquad (*)$$

Now

$$(y,\overline{y}'') = \int_a^b (\overline{y^t(x)})''y(x)dx$$

$$= (\overline{y^t(x)})'y(x)\Big|_a^b - \int_a^b (\overline{y^t(x)})'y'(x)dx$$

$$= -(y',y') \text{ and this is real.}$$

Equating the imaginary parts of (*),

$$0 = \beta(y, R\bar{y})$$

since $(y, R\bar{y})$ is real. Therefore either $\beta = 0$ or $(y, R\bar{y}) = 0$. But $(y, R\bar{y}) = 0$ implies $(y, \bar{y}'') = -(y', \bar{y}') = 0$ from *(*)*. Then $y'(x) = 0$ a.e. and so $y(x)$ = constant after the integration. Thus $y \equiv 0$ since $y(a) = 0$. Hence $\beta = 0$ and so λ is real.

3.2. Existence theorem

A proof of the existence of an infinite sequence of eigenvalues for a Sturm-Liouville boundary value problem is very important in the theory of ordinary differential equations. In the scalar case this is proved ([6],Chapter 10.61) by using Sturm's oscillation theorems. But it is not possible to extend the same method to the vector boundary value problem *(3.1)*. Thus some other method has to be developed to solve this problem.

A result of two parameter eigenvalue problems by Paul Binding and Patrick J. Browne, [1], is the major motivation for our study in this section.

Consider the self adjoint eigenvalue problem

$$\omega(\underline{\lambda})y = 0 \qquad \text{for } y \in H,$$

where

$$\omega(\underline{\lambda}) = T + \lambda_1 R(x) + \lambda_2 I$$

with $Ty = -y''$, $\mathcal{D}(T)$ is as in section 2.2 and $R(x)$ satisfies *H1*, *H2*$_o$. $\underline{\lambda} = (\lambda_1, \lambda_2)$ is a vector.

We notice that T and R(x) are self adjoint operators on the separable Hilbert space H, T is bounded below and has compact resolvent. Therefore results in [1] are applicable for the above eigenvalue problem $\omega(\underline{\lambda})y = 0$.

Define

$$U = \{u \in H: ||u||_H = 1\},$$

a *vector* $\underline{v}(u) = ((Ru,u),1)$ for $u \in U$, and sets K, C^- by

$$K = \{\underline{\lambda}: \underline{v}(u).\underline{\lambda} \geq 0 \text{ for all } u \in U\},$$

$$C^- = \{\underline{\lambda}: \underline{v}(u).\underline{\lambda} > \alpha \text{ for some } \alpha > 0, \text{ for all } u \in U\},$$

Also define sets P^i, Z^i, N^i as those sets of $\underline{\lambda}$ for which $\rho^i(\underline{\lambda})$ -- the ith eigenvalue of $\omega(\underline{\lambda})$ according to multiplicity -- is positive, zero or negative respectively, $(i = 0,1,2,\ldots)$. Here,

$$\rho^i(\underline{\lambda}) = \max\{\min\{(\omega(\underline{\lambda})u,u) \mid u \in U \cap \mathcal{D}(T), (u,y_i)=0\} \mid y_j \in H, 1 \leq j \leq i\}.$$

We also define

$$\sigma^i(\underline{\lambda}) = \sup_{E} \inf_{u \in E^\perp \cap U} (s(\underline{\lambda})u,u), \qquad i = 0,1,2,\ldots,$$

for $\underline{\lambda} \in R^2$. Here E denotes an i-dimensional subspace of H and $s(\underline{\lambda}) = \lambda_1 R(x) + \lambda_2 I$.

<u>Lemma 3.2.1</u>. The cone K contains the nonempty cone K' given by

$$K' = \{\underline{\lambda} \in R^2: \lambda_1 \geq 0, \lambda_2 \geq \sup_{x \in Q} \nu[R_-(x)]\lambda_1;$$

$$\lambda_1 \leq 0, \lambda_2 \geq -\sup_{x \in Q} \nu[R_+(x)]\lambda_1\}.$$

<u>Lemma 3.2.2</u>. $C^- \neq \emptyset$.

<u>Lemma 3.2.3</u>. (i) If $\sigma^i(\underline{\lambda}) > 0$ then there exists a number $\alpha_* \geq 0$ so that $\alpha\underline{\lambda} \in P^i$ for all $\alpha > \alpha_*$.

(ii) If $\sigma^i(\underline{\lambda}) < 0$ then there exists a number $\beta_* \geq 0$ so that $\beta\underline{\lambda} \in N^i$ for all $\beta > \beta_*$.

<u>Lemma 3.2.4</u>. $N^i \cap \Lambda_j \neq \emptyset$ for $j = 1,2,3,4$ and $i = 0,1,2,\ldots$ where Λ_j is the interior of the jth quadrant of the $\underline{\lambda}$ plane.

<u>Lemma 3.2.5</u>. For each integer $i \geq 0$ there is a unique continuous function $\lambda_2^i: R \to R$ such that $\rho^i(\lambda_1, \lambda_2^i(\lambda_1)) = 0$. Moreover there is a vector $y^i \in U \cap \mathcal{D}(T)$ which satisfies $\omega(\lambda_1, \lambda_2^i(\lambda_1))y^i = 0$ and Z^i is the graph of λ_2^i and is a continuous curve in the $\underline{\lambda}$-plane.

The graph of Z^0 (Fig. 1) can be drawn by using the following lemma.

<u>Lemma 3.2.6</u>. (i) $P^0 \neq \emptyset$

(ii) P^0 is convex

(iii) $Z^0 \neq \emptyset$

(iv) $Z^0 = \partial P^0 = \partial N^0$

(v) $P^0 \supseteq K$.

Also we have for higher order eigenvalues of $\omega(\underline{\lambda})$

<u>Lemma 3.2.7</u>. (i) $Z^i \neq \emptyset$

(ii) $Z^i = \partial P^i \cap \partial N^i$

(iii) $P^{i-1} \subseteq P^i$, $N^{i+1} \subseteq N^i$

(iv) Z^i form a countable sequence of curves with no finite accumulation.

(v) Each Z^i crosses each ray $R_+\underline{\lambda}$ at most once.

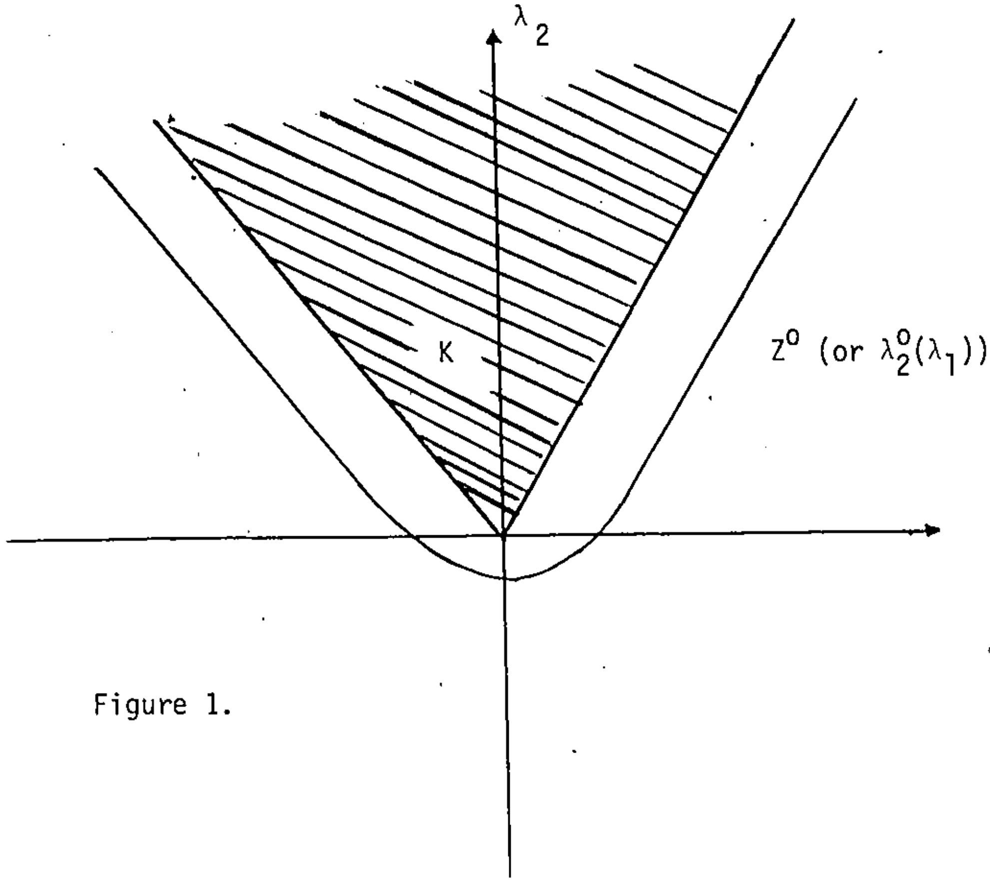

Figure 1.

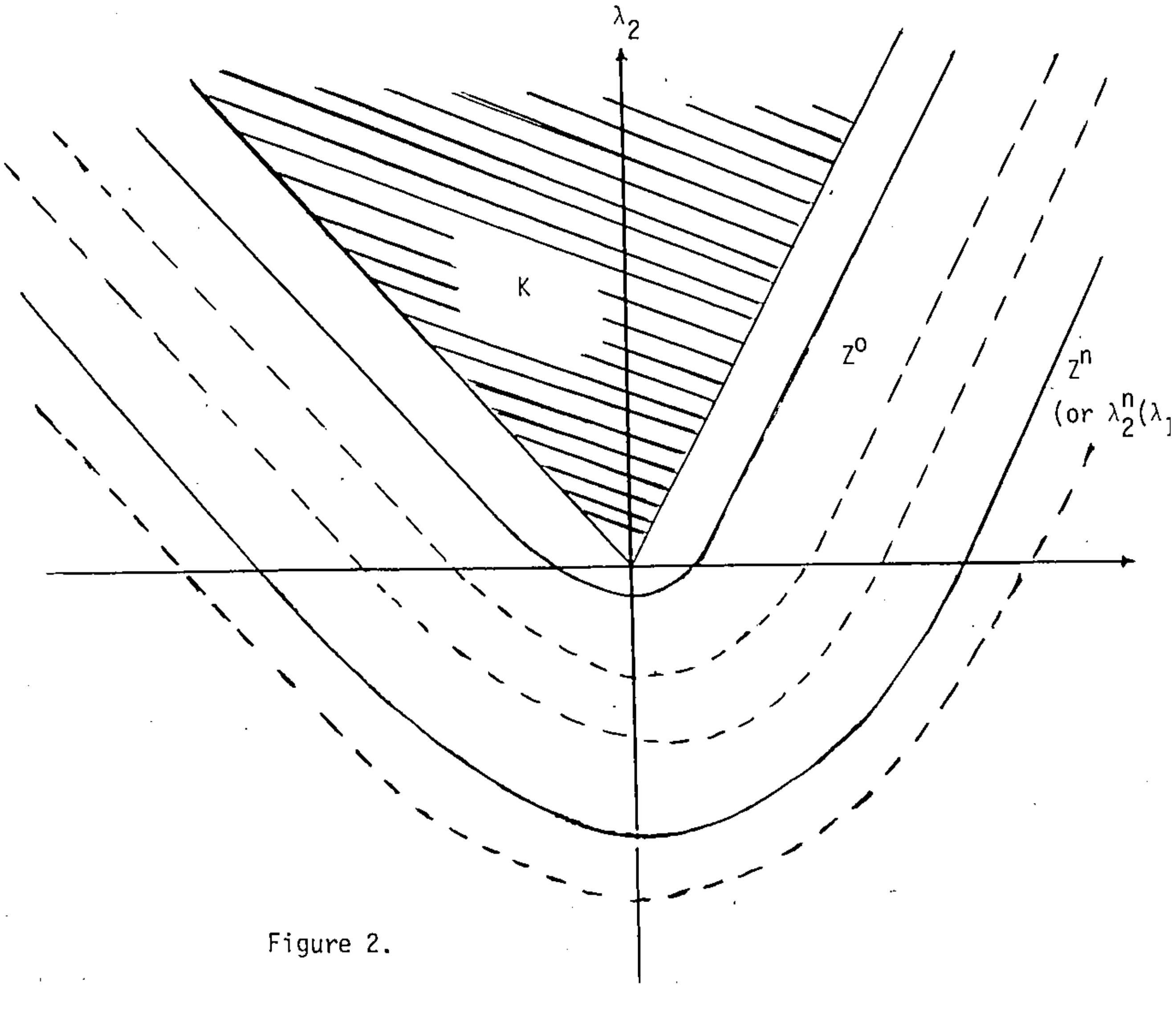

Figure 2.

The results in lemma 3.2.7. are illustrated in figure 2. It shows that each λ_2^i vanishes precisely two times for positive and negative values of λ_1.

Thus in the eigenvalue problem

$$-y'' + \lambda_1 R(x)y + \lambda_2 Iy = 0,$$

$\lambda_2 = 0$ for a countable number of positive and negative values of λ_1, with no finite accumulation point.

Hence we proved the following existence theorem.

<u>Theorem 3.2.8</u>. *When* R(x) *satisfies H1 and* $H2_0$ *the Sturm-Liouville vector boundary value problem (3.1) has two countable sequences of real eigenvalues whose only limit points are* $+\infty$ *and* $-\infty$.

<u>Proof of lemma 3.2.1</u>.

$$\underline{v}(u).\underline{\lambda} = \lambda_1(Ru,u) + \lambda_2$$

$$= \lambda_1 \int_a^b u^t Rudx + \lambda_2 \int_a^b u^t udx.$$

Now when $\lambda_1 \geq 0$ and $\lambda_2 \geq \sup_{x \in Q} \nu[R\_(x)]\lambda_1$

$$\underline{v}(u).\underline{\lambda} \geq \lambda_1 \int_a^b u^t Rudx + \sup_{x \in Q} \nu[R\_(x)]\lambda_1 \int_a^b u^t udx$$

$$= \lambda_1 \int_a^b u^t [R(x) + \sup_{x \in Q} \nu[R\_(x)]I]udx$$

$$\geq 0 \quad \text{for all } u \in U, \text{ since } R(x) + \sup_{x \in Q} \nu[R\_(x)]I \geq 0$$

on Q.

Now consider those $\lambda_1 \leq 0$ for which $\lambda_2 \geq -\sup_{x \in Q} \nu[R_+(x)]\lambda_1$. Then

$$\underline{v}(u).\underline{\lambda} \geq \lambda_1 \int_a^b u^t[R(x) - \sup_{x \in Q} \nu[R_+(x)]I]udx$$

$$\geq 0 \quad \text{for all } u \in U, \text{ since } R(x) - \sup_{x \in Q} \nu[R_+(x)]I \leq 0$$

on Q.

We also notice that $\sup_{x \in Q} \nu[R_+(x)] > 0$ and $\sup_{x \in Q} \nu[R_-(x)] > 0$. For, $\sup_{x \in Q} \nu[R_+(x)] = 0$ implies $\nu[R_+(x)] = 0$ for $x \in Q$ and then $R_+(x) = 0$ a.e. This contradicts the assumption $H2_o$. A similar argument is valid for the $\sup_{x \in Q} \nu[R_-(x)]$.

Proof of Lemma 3.2.2.

Consider $\underline{\lambda} = (0,1)$. Then $\underline{v}(u).\underline{\lambda} = 0. \int_a^b u^t Rudx + 1 = 1$ for all $u \in U$. Thus $C^- \neq \emptyset$.

When $R(x) < 0$ on an interval E of the real line we can find a compact interval $E^o \subseteq E$ such that $R(x) \leq -\delta I$ on E^o for some $\delta > 0$. For, $R(x) < 0$ implies all eigenvalues $\omega_i(x)$, $(i = 1,\ldots,n)$, of $R(x)$ are negative on E and given interval E there exist a compact interval E^o such that $E^o \subseteq E$. Now the continuous functions $\omega_i(x)$ achieve their maximum values on compact interval E^o and $\max_{x \in E^o} \omega_i(x) < 0$ for each i. Find some $\delta > 0$ such that $\max_i \max_{x \in E^o} \omega_i(x) \leq -\delta < 0$. Then $\max_i \max_{x \in E^o} \omega_i(x) + \delta \leq 0$ so that $\omega_i(x) + \delta \leq 0$ for each i on E^o. Thus $R(x) + \delta I \leq 0$ on E^o since $\omega_i(x) + \delta$, $(i = 1,\ldots,n)$, are the eigenvalues of $R(x) + \delta I$.

Remark. It can also be shown that for a symmetric matrix $R(x)$ which is positive definite on the compact interval E^o there is a positive constant κ such that $\kappa I \leq R(x)$ for $x \in E^o$, (see, e.g.,[10], problem F.1.6) .

Proof of Lemma 3.2.3. ([1], Theorem 5.4).

Proof of emma 3.2.4.

Here we prove $N^i \cap \Lambda_1 \neq \emptyset$. Similar proofs can be given for $N^i \cap \Lambda_j \neq \emptyset$ when $j = 2,3$ and 4.

From Lemma 3.2.3. (ii) $N^i \cap \Lambda_1 \neq \emptyset$ if $\sigma^i(\underline{\lambda}) < 0$ for some $\underline{\lambda} \in \Lambda_1$. So it suffices to give a $\underline{\lambda} \in \Lambda_1$ such that $\sigma^i(\underline{\lambda}) < 0$.

Assuming $u_j(\underline{\lambda};x)$ denote the eigenvectors of $\sigma^j(\underline{\lambda})$ for $j = 1,2,\ldots,i-1$ we can rewrite the definition of $\sigma^i(\underline{\lambda})$ as

$$\sigma^i(\underline{\lambda}) = \inf_{\substack{u \in U \\ (u,u_j(\underline{\lambda};x))=0 \\ j=1,2,\ldots,i-1}} (s(\underline{\lambda})u,u)$$

Suppose $\sigma^i(\underline{\lambda}) \geq 0$ for all $\underline{\lambda} \in \Lambda_1$. Then $(s(\underline{\lambda})u,u) \geq 0$ for all $\underline{\lambda} \in \Lambda_1$ and for all $u \in U$ such that $(u,u_j(\underline{\lambda};x)) = 0$, $j = 1,2,\ldots,i-1$.

Let $u_j(\underline{\lambda};x) = (u_{j1}(\underline{\lambda};x),\ldots,u_{jn}(\underline{\lambda};x))^t$ for $j = 1,\ldots,i-1$ and $R(x) < 0$ on the interval $E \subseteq Q$. Then there exists some compact interval $E^o \subseteq E$ such that $R(x) \leq -\delta I$ on E^o for some $\delta > 0$.

$u_j \in H$ implies $u_{jk} \in \mathcal{L}_2(Q)$ and then $u_{jk} \in \mathcal{L}_2(E^o)$ for $j = 1,\ldots,i-1$ and $k = 1,\ldots,n$. Now first consider the set of functions $u_{11}(\underline{\lambda};x)$, $u_{21}(\underline{\lambda};x),\ldots,$ $u_{i-11}(x)$, i.e. first components of the vectors $u_j(\underline{\lambda};x)$ for $j = 1,\ldots,i-1$, on $\mathcal{L}_2(E^o)$. Using the Gramm-Schmidt orthogonalization process find $h_1(\underline{\lambda};x) \in \mathcal{L}_2(E^o)$ so that $(h_i,u_{j1})_{\mathcal{L}_2(E^o)} = 0$ for

$j = 1,2,\ldots,i-1$. Here $(h_1,u_{j1})_{\mathcal{L}_2(E^o)} = \int_{E^o} h_1(\underline{\lambda};x)u_{j1}(\underline{\lambda};x)dx$. Similarly find $h_k(\underline{\lambda};x)$ so that $(h_k,u_{jk})_{\mathcal{L}_2(E^o)} = 0$ for each $k = 2,\ldots,n$ and $j = 1,\ldots,i-1$.

We normalize $h(\underline{\lambda};x) = (h_1(\underline{\lambda};x),\ldots,h_n(\underline{\lambda};x))^t$ in $\mathcal{L}_2(E^o)$ so that $(h,h)_{\mathcal{L}_2(E^o)} = 1$.

Now define

$$g_k(\underline{\lambda};x) = \begin{cases} h_k(\underline{\lambda};x) & x \in E^o \\ 0 & \text{elsewhere} \end{cases}$$

Then $g(\underline{\lambda};x) = (g_1(\underline{\lambda};x),\ldots,g_n(\underline{\lambda};x))^t \in U$ since

$$\begin{aligned}(g,g)_H &= \int_a^b g^t(\underline{\lambda};x)g(\underline{\lambda};x)dx \\ &= \int_{E^o} h^t(\underline{\lambda};x)h(\underline{\lambda};x)dx \\ &= (h,h)_{\mathcal{L}_2(E^o)} = 1.\end{aligned}$$

Further

$$\begin{aligned}(g,u_j) &= \int_a^b g^t u_j \\ &= \int_a^b \sum_{k=1}^n g_k u_{jk} = \sum_{k=1}^n \int_a^b g_k u_{jk} = \sum_{k=1}^n \int_{E^o} h_k u_{jk} \\ &= \sum_{k=1}^n (h_k,u_{jk})_{\mathcal{L}_2(E^o)} \\ &= 0 \qquad \text{for each } j = 1,\ldots,i-1.\end{aligned}$$

Now fix $\underline{\lambda} = (1, \delta/2)$. Then $\underline{\lambda} \in \Lambda_1$.

$$(s(\underline{\lambda})g,g) = (R(x)g,g) + \delta/2$$

$$= \int_a^b g^t(x)R(x)g(x)dx + \delta/2$$

But $$\int_a^b g^tRg = \int_{E^o} h^tRh$$

$$\leq -\delta \int_{E^o} h^th$$

$$= -\delta\ , \quad \text{since } R(x) \leq -\delta I \text{ on } E^o.$$

Thus $(s(\underline{\lambda})g,g) \leq -\delta + \delta/2 = -\delta/2 < 0$ for $g \in U$. This contradicts the fact that $(s(\underline{\lambda})u,u) \geq 0$ for all $u \in U$ and therefore $\sigma^i(\underline{\lambda}) < 0$ for some $\underline{\lambda} \in \Lambda_1$.

Proof of Lemma 3.2.5. ([1], Section 6).

Proof of Lemma 3.2.6. (i) ([1], Theorem 3.4.(b)),
(ii) ([1], Lemma 3.1),
(iii) ([1], Corollary 3.5.),
(iv) ([1], Theorem 3.6.),
(v) ([1], Lemma 4.1.).

Proof of Lemma 3.2.7. (i) ([1], Theorem 5.5.),
(ii), (iii), (iv) and (v) ([1], Lemma 6.1.).

4. An asymptotic formula for the distribution function of the spectrum

In this chapter we obtain an asymptotic formula for the distribution function $n_+(s)$ of positive eigenvalues of the boundary value problem *(3.1)* when $R(x)$ belongs to two different classes of matrices. Indeed, as we mentioned before, our results extend those of I.C. Gohberg and M.G. Krein [5].

Distribution function of eigenvalues for the problem *(3.1)* is denoted by $n(s)$. Then $n(s) = n_+(s) + n_-(s)$, where $n_+(s)$ is the number of eigenvalues in $(0,s]$ while $n_-(s)$ is the number of eigenvalues in $[-s,0)$. Since we are interested here only in $n_+(s)$ and henceforth $n_+(s)$ will be denoted by $n(s)$ for simplicity.

Lemma 4.1. *When* $R(x)$ *is a diagonal matrix,* λ_0 *is an eigenvalue of the boundary value problem (3.1) if and only if it is an eigenvalue of some scalar boundary value problem*

$$\begin{aligned} y_i'' + \lambda^i R_{ii}(x) y_i &= 0 \\ y_i(a) = y_i(b) &= 0 \end{aligned} \qquad (4.1)_i$$

$i = 1,\ldots,n$. *i.e.,* $\lambda_0 = \lambda^i$ *for some* i.

Proof. Suppose λ_0 is an eigenvalue of the problem *(3.1)*. Then the eigenvector y is nonzero and thus

$$\begin{aligned} y_i'' + \lambda_0 R_{ii}(x) y_i &= 0 \\ y_i(a) = y_i(b) &= 0 \end{aligned}$$

for some i, so that *(4.1)*$_i$ has a non-trivial solution with $\lambda_0 = \lambda^i$. Thus λ_0 is an eigenvalue of the scalar problem *(4.1)*$_i$.

For the converse assume $\lambda^i = \lambda_0$ is an eigenvalue of the scalar problem *(4.1)*$_i$ for some i. Then

$$y_i'' + \lambda_0 R_{ii}(x) y_i = 0$$

$$y_i(a) = y_i(b) = 0.$$

Setting $y(x) = (0 \;\; 0 \; \ldots \; \underset{\text{ith place}}{y_i(x)} \; \ldots \; 0)^t$ we see that

$$y'' + \lambda_0 R(x) y = 0$$

$$y(a) = y(b) = 0.$$

Thus λ_0 is an eigenvalue of *(3.1)*.

The following corollary about the spectrum is now immediate.

<u>Corollary 4.2</u>. *The spectrum of (3.1) is* $\bigcup_{i=1}^{n} \bigcup_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \lambda_m^i$, *where* $\{\lambda_m^i\}_{m=1}^{\infty}$ *is the sequence of positive eigenvalues and* $\{\lambda_{-m}^i\}_{m=1}^{\infty}$ *is the sequence of negative eigenvalues of the problem (4.1)*$_i$.

These countable sets of positive and negative eigenvalues are now arranged so that each is repeated a number of times equal to its multiplicity and each in order of non decreasing numerical value. We denote these two new sequences by

$$\mu_1^+, \mu_2^+, \mu_3^+, \ldots$$

and

$$\mu_1^-, \mu_2^-, \mu_3^-, \ldots \quad .$$

This is said to be the *complete sequence* (see Gohberg and Krein [5]) of eigenvalues for the problem *(3.1)*. It is easy to see that

$$\bigcup_{m=1}^{\infty} \mu_m^+ = \bigcup_{i=1}^{n} \bigcup_{m=1}^{\infty} \lambda_m^i \quad \text{and} \quad \bigcup_{m=1}^{\infty} \mu_m^- = \bigcup_{i=1}^{n} \bigcup_{m=1}^{\infty} \lambda_{-m}^i .$$

Let us denote the distribution function for the positive eigenvalues of the scalar problem $(4.1)_i$ by $N^i(s)$ and $\sum_{i=1}^{n} N^i(s)$ by $N(s)$.

In view of the above relation between $\{\mu_m^+\}_{m=1}^{\infty}$ and $\{\lambda_m^i\}_{\substack{i=1\\m=1}}^{\substack{i=n\\m=\infty}}$ we have the following lemma.

Lemma 4.3. $N(s) = n(s)$.

Lemma 4.4. *When* $R_{ii}(x)$ *changes sign finitely many times in the interval* Q

$$\lim_{s\to\infty} \frac{N^i(s)}{\sqrt{s}} = \frac{1}{\pi} \int_a^b \sqrt{R_{ii}^+(x)}\, dx,$$

where $R_{ii}^+(x) = \max(0, R_{ii}(x))$.

Proof. Consider the scalar boundary value problem

$$y_i'' + \mu R_{ii}(x) y_i = 0$$

$$y_i(a) = y_i(b) = 0.$$

It is known (see [7]) that this problem has a sequence of positive eigenvalues $\{\mu_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} \frac{m^2}{\mu_m} = \frac{K_i^2}{\pi^2}$, where $K_i = \int_a^b \sqrt{R_{ii}^+(x)}\, dx$.

Now consider some arbitrary sequence $\{x_m\}_{m=1}^{\infty}$ so that $x_m \to \infty$ as $m \to \infty$. Then we can find some integer $k(m)$ and a constant M such that

$$0 < \mu_{k(m)-1} \leq x_m \leq \mu_{k(m)} \quad \text{for} \quad m > M. \tag{*}$$

This implies that

$$N^i(\mu_{k(m)-1}) \leq N^i(x_m) \leq N^i(\mu_{k(m)})$$

or
$$k(m)-1 \leq N^i(x_m) \leq k(m)\ . \tag{**}$$

Combining *(*)* and *(**)*, we have

$$\frac{k(m)-1}{\mu_{k(m)}^{\frac{1}{2}}} \leq \frac{N^i(x_m)}{x_m^{\frac{1}{2}}} \leq \frac{k(m)}{\mu_{k(m)-1}^{\frac{1}{2}}} \qquad \text{for } m > M.$$

But $\lim\limits_{m\to\infty} \frac{k(m)-1}{\mu_{k(m)}^{\frac{1}{2}}} = \frac{K_i}{\pi}$ and $\lim\limits_{m\to\infty} \frac{k(m)}{\mu_{k(m)-1}^{\frac{1}{2}}} = \frac{K_i}{\pi}$ from [7]. Thus $\lim\limits_{m\to\infty} \frac{N^i(x_m)}{x_m^{\frac{1}{2}}} = \frac{K_i}{\pi}$.

Since $\{x_m\}_{m=1}$ is arbitrary, this result is true for every sequence $\{x_m\}_{m=1}$ such that $x_m \to \infty$ as $m \to \infty$. Then by definition $\lim\limits_{s\to\infty} \frac{N^i(s)}{\sqrt{s}} = \frac{K_i}{\pi}$.

Now we are ready to present a major result.

Theorem 4.5. *When* R(x) *is diagonal and satisfies H1 and H2$_1$ the distribution function* n(s) *of positive eigenvalues of the problem (3.1) satisfies the asymptotic formula*

$$\lim_{s\to\infty} \frac{n(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R_+(x))dx \tag{4.2}$$

A *similar formula holds for the negative eigenvalues.* (*see p. 3*).

Here

H2₁. For each $R_{ii}(x)$ there exists a decomposition $(a,b) = \bigcup_{j=1}^{N_i}(a_j^{(i)}, b_j^{(i)})$ such that $R_{ii}(x) > 0$ on $(a_k^{(i)}, b_k^{(i)})$ and $R_{ii}(x) < 0$ on $(a_\ell^{(i)}, b_\ell^{(i)})$ for some two integers k and ℓ, $1 \le k,\ell \le N_i$.

<u>Proof of Theorem 4.5</u>. From the Lemma 4.4 for each i, (i = 1,2,...,n),

$\lim_{s\to\infty} \frac{N^i(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \sqrt{R_{ii}^+(x)}\, dx$. Thus

$$\lim_{s\to\infty} \sum_{i=1}^{n} \frac{N^i(s)}{\sqrt{s}} = \sum_{i=1}^{n} \frac{1}{\pi}\int_a^b \sqrt{R_{ii}^+(x)}\, dx$$

$$= \frac{1}{\pi}\int_a^b \sum_{i=1}^{n} \sqrt{R_{ii}^+(x)}\, dx.$$

On the other hand

$$\lim_{s\to\infty} \sum_{i=1}^{n} \frac{N^i(s)}{\sqrt{s}} = \lim_{s\to\infty} \frac{N(s)}{\sqrt{s}}$$

$$= \lim_{s\to\infty} \frac{n(s)}{\sqrt{s}}.$$

Since $R(x)$ is diagonal, $R_{ii}^+(x)$ are the eigenvalues of the matrix $R_+(x)$.

Therefore $\Lambda_{\frac{1}{2}}(R_+(x)) = \sum_{i=1}^{n} \sqrt{R_{ii}^+(x)}$ and the theorem is proved.

We introduce a special class of matrices called *functionally commutative* matrices. An n by n matrix $A(t)$ for $t \in E$, where E is some nonempty set of real values, is said to be functionally commutative in case

$A(t)A(s) - A(s)A(t) = 0$ for all $t,s \in E$ (see Freedman [3]).

Corollary 4.6. *When* $R(x)$ *is functionally commutative and satisfies H1 and* $H2_1$ *the distribution function* $n(s)$ *of positive eigenvalues of the problem (3.1) satisfies the asymptotic formula (4.2).*

Proof. Since $R(x)$ is symmetric and functionally commutative, we have $R(x) = PJ(x)P^{-1}$ with a constant matrix P and a diagonal matrix $J(x)$, (see, [3], theorem 8) . Now make the transformation $\tilde{y} = P^{-1}y$ in *(3.1)*. Then *(3.1)* is equivalent to

$$\tilde{y}'' + \lambda J(x)\tilde{y} = 0$$

$$\tilde{y}(a) = \tilde{y}(b) = 0 ,$$

so that the distribution function $n(s)$ satisfies

$$\lim_{s \to \infty} \frac{n(s)}{\sqrt{s}} = \frac{1}{\pi} \int_a^b \Lambda_{\frac{1}{2}}(J_+(x))$$

by Theorem 4.5. Since $R_+(x) = PJ_+(x)P^{-1}$ the eigenvalues of $R_+(x)$ and $J_+(x)$ are equal and nonnegative. Thus $\Lambda_{\frac{1}{2}}(J_+(x)) = \Lambda_{\frac{1}{2}}(R_+(x))$.

Remark. Since every constant square matrix is functionally commutative the corollary 4.6 is true for any n by n constant symmetric matrix R.

In the remaining part of this thesis, we obtain the same asymptotic formula with $R(x)$ belonging to another class of matrices. We consider all n by n matrices which have the property $H2_2$. Here

H2$_2$. For a piecewise continuous symmetric matrix function R(x) such that both $R_+(x)$ and $R_-(x)$ are not identically 0, the square root matrix $|R|$ of R^2 is positive definite on the compact interval Q.

Once again we study the spectral properties of two parameter eigenvalue problems. The self adjoint eigenvalue problem

$$\omega_+(\underline{\lambda})y = 0, \; y \in H$$

is considered with $\omega_+(\underline{\lambda}) = T + \lambda_1 R_+(x) + \lambda_2 R_-(x)$. We refer to [1] and section 3.2 for notation used hereafter.

The cone $K = \{\underline{\lambda}: \underline{v}(u).\underline{\lambda} \geq 0$ for all $u \in U\}$ is the first quadrant of $\lambda_1\lambda_2$-plane and $C^- = \{\underline{\lambda}: \underline{v}(u).\underline{\lambda} > \alpha$ for some $\alpha > 0$ for all $u \in U\}$ is nonempty when the assumption *H2*$_2$ is made on R(x). For, the vector $\underline{\lambda} = (1,1) \in C^-$ since

$$\underline{v}(u).\underline{\lambda} = \int_a^b u^t R_+(x) u dx + \int_a^b u^t R_-(x) u dx$$

$$= \int_a^b u^t |R| u dx$$

$$\geq k \qquad \text{for all } u \in U,$$

where $k > 0$ is such that $|R| \geq kI$ on the compact interval Q (see, remark, section 3.2).

Thus the lemmas 3.2.5, 3.2.6 and 3.2.7 are true also in this case. The curves of the continuous functions λ_2^i are shown in the figure 3.

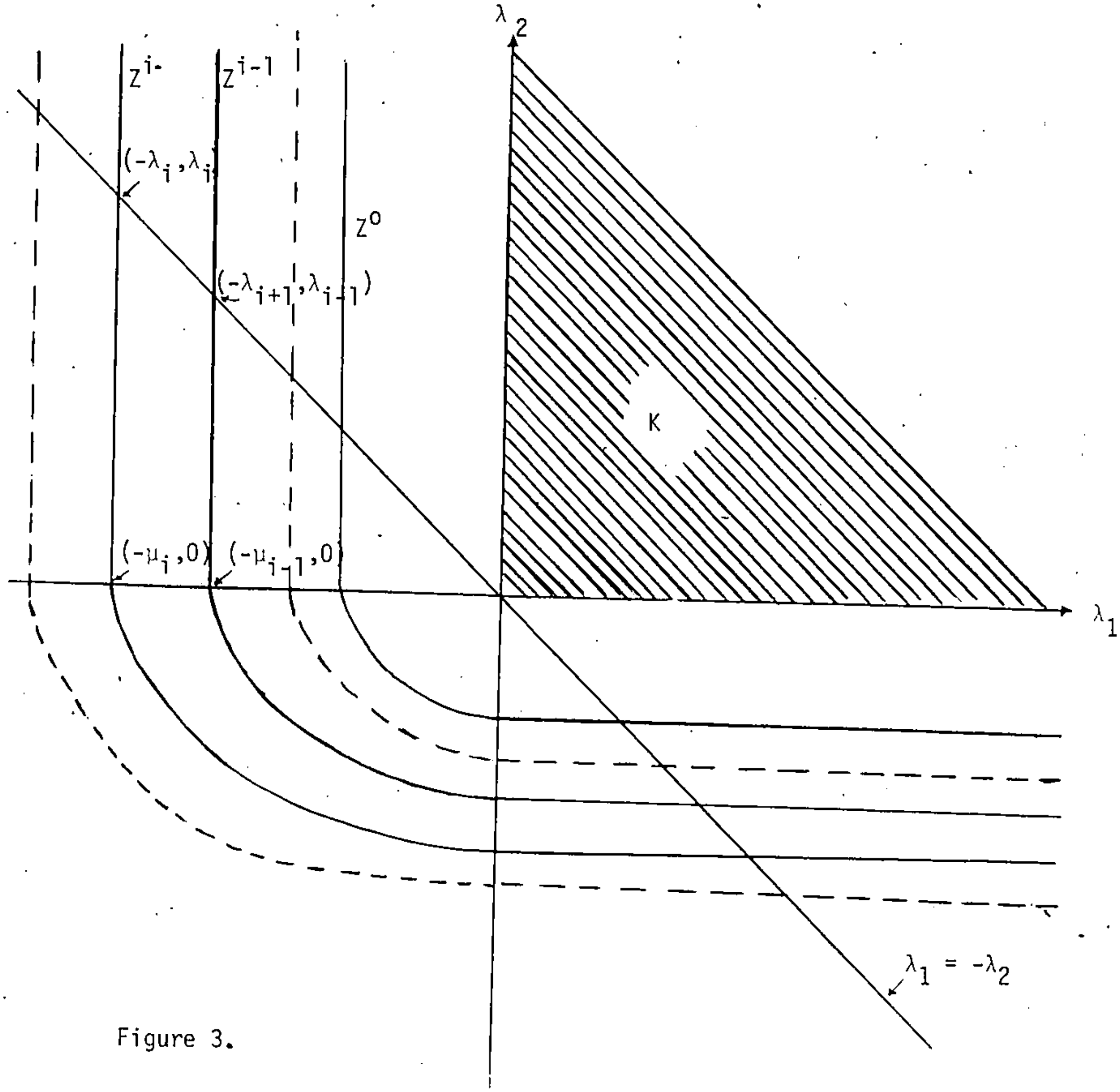

Figure 3.

In view of figure 3 we state and prove the following lemma.

Lemma 4.7. *The ith eigenvalue of the problem (3.1) is given by the intersection point of curve* Z^i *and line* $\lambda_1 = -\lambda_2$ *while the ith eigenvalue of the problem*

$$y'' + \mu R_+(x)y = 0$$
$$y(a) = y(b) = 0 \tag{4.3}$$

is given by the intersection point of Z^i *and* $\lambda_2 = 0$.

Proof. Let the intersection point of Z^i and $\lambda_1 = -\lambda_2$ be $(-\lambda_i, \lambda_i)$ for $\lambda_i > 0$. Then $\rho^i(-\lambda_i, \lambda_i) = 0$ and $\omega_+(-\lambda_i, \lambda_i)y = 0$ for some $y \in U \cap \mathcal{D}(T)$.

i.e.

$$\begin{aligned} Ty - \lambda_i R_+(x)y + \lambda_i R_-(x)y &= -y'' - \lambda_i[R_+(x) - R_-(x)]y \\ &= -y'' - \lambda_i R(x)y \\ &= 0 \end{aligned}$$

and $y(a) = y(b) = 0$. Thus λ_i is an eigenvalue of the problem *(3.1)*.

If there is an eigenvalue γ such that $\lambda_i < \gamma < \lambda_{i-1}$ of *(3.1)* then $y'' + \gamma R(x)y = 0$ for some $y \in U \cap \mathcal{D}(T)$. i.e., $-y'' - \gamma R_+(x)y + \gamma R_-(x)y = 0$ or $Ty - \gamma R_+(x)y + \gamma R_-(x)y = 0$. Thus $\omega_+(-\gamma, \gamma)y = 0$. But this implies zero is an eigenvalue of $\omega_+(-\gamma, \gamma)$ and $(-\gamma, \gamma)$ belongs to some Z^k. Hence there must be a curve which passes through this point. Since there is no curve Z^k between Z^i and Z^{i-1} there cannot be an eigenvalue γ between λ_i and λ_{i-1}. Thus λ_i is the ith positive eigenvalue of the problem *(3.1)*.

Similarly it can be shown that μ_i is the ith eigenvalue for the problem *(4.3)* and μ_i is given by the intersection point of Z^i and λ_1 axis.

Corollary 4.8. $\lim_{m\to\infty} \lambda_m = \mu_m$.

Proof. Since the Z^i all asymptotically resemble the two boundary rays of ∂K, (see [1]), i.e., λ_1 and λ_2 axes in this case, we see that $\lambda_m \sim \mu_m$ as $m \to \infty$.

The following corollary is now an immediate result concerning $\tilde{n}(s)$ -- the distribution function of eigenvalues of the problem *(4.3)*.

Corollary 4.9. $\lim_{s\to\infty} \frac{\tilde{n}(s)}{\sqrt{s}} \doteq \lim_{s\to\infty} \frac{n(s)}{\sqrt{s}}$, *provided this limit exists.*

Lemma 4.10. $\lim_{s\to\infty} \frac{\tilde{n}(s)}{\sqrt{s}} = \frac{1}{\pi} \int_a^b \Lambda_{\frac{1}{2}}(R_+(x))dx.$

Proof. Proof is in 3 steps.

Step 1. Consider the boundary value problem

$$\begin{aligned} y'' + \mu R_\varepsilon(x)y &= 0 \\ y(a) = y(b) &= 0 \end{aligned} \tag{4.4}$$

where $R(x) = R_+(x) + \varepsilon I$ for some $\varepsilon > 0$. Let $\mu_n(\varepsilon)$ denote the eigenvalues and $n_\varepsilon(s)$ denotes the distribution functions of eigenvalues of the problem *(4.4)*. $R_\varepsilon(x)$ is a symmetric positive definite matrix in the interval Q and hence by Sturm-Liouville theory, $\mu_n(\varepsilon)$ is real and $\mu_n(\varepsilon) > 0$ for every n. Moreover if $\varepsilon_m \downarrow 0$ as $m \to \infty$, $\mu_n(\varepsilon_m) \le \mu_n(\varepsilon_{m+1})$ since $R_{\varepsilon_m}(x) \ge R_{\varepsilon_{m+1}}(x)$ (see Appendix C). Thus since $R_\varepsilon(x) > R_+(x)$, we have $\mu_n(\varepsilon) \le \mu_n$. Therefore

$\lim_{m\to\infty} \mu_n(\varepsilon_m)$ exists for each n. Moreover since

$$\mu_n(\varepsilon_m) = \max_{E} \min_{y \in E \cap U} \frac{\int_a^b |y'(x)|^2 dx}{\int_a^b y^t R_+(x) y + \varepsilon_m}$$

by the minmax principle

$$\lim_{m\to\infty} \mu_n(\varepsilon_m) = \max_{E} \min_{y \in E \cap U} \frac{\int_a^b |y'(x)|^2 dx}{\int_a^b y^t R_+(x) y dx}$$

$$= \mu_n$$

i.e., $\mu_n(\varepsilon) = \mu_n$ as $\varepsilon \to 0^+$ for each n. But this implies that $n_\varepsilon(s) = \tilde{n}(s)$ as $\varepsilon \to 0^+$ for $s \in (0,\infty)$.

Step 2. Let $\omega_j(x)$ $(j = 1,\ldots,n)$ be the eigenvalues of the matrix $R_+(x)$ Then $\omega_j(x) \geq 0$ on Q for each j. Let $f_m(x) = \sqrt{\omega_j(x)+\varepsilon_m}$ then $\lim_{m\to\infty} f_m(x) = \sqrt{\omega_j(x)}$ for $x \in Q$. Moreover $|f_m(x)| = \sqrt{\omega_j(x)+\varepsilon_m} \leq \sqrt{\omega_j(x)+\varepsilon_1}$ and $\sqrt{\omega_j(x)+\varepsilon_1}$ is integrable since $\omega_j(x)$ is integrable (see [9]) for each j. Then $\lim_{m\to\infty} \int_a^b f_m(x)dx = \lim_{m\to\infty} \int_a^b \sqrt{\omega_j(x)+\varepsilon_m}\, dx$

$$= \int_a^b \sqrt{\omega_j(x)}\, dx \qquad \text{by Lebesgue's dominated}$$

convergence theorem for each j. Therefore

$$\lim_{m\to\infty} \int_a^b \sum_{j=1}^n \sqrt{\omega_j(x)+\varepsilon_m}\, dx = \int_a^b \sum_{j=1}^n \sqrt{\omega_j(x)}\, dx$$

or

$$\lim_{\varepsilon\to 0^+} \int_a^b \Lambda_{\frac{1}{2}}(R_\varepsilon(x))dx = \int_a^b \Lambda_{\frac{1}{2}}(R_+(x))dx \quad \text{since}$$

$\omega_j(x) + \varepsilon$, $(j = 1,\ldots,n)$ are the eigenvalues of $R_\varepsilon(x)$.

Step 3. From I.C. Gohberg and M.G. Krein, ([5], Chapter VI.7), we have

$$\lim_{s\to\infty} \frac{n_\varepsilon(s)}{\sqrt{s}} = \frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R_\varepsilon(x))dx \qquad \text{for each } \varepsilon > 0.$$

From this there follows the lemma.

We now present the final theorem.

Theorem 4.11. *When* R(x) *satisfies* $H2_2$ *the distribution function n(s) of positive eigenvalues of the problem (3.1) satisifes the asymptotic formula (4.2).*

Proof. From Lemma 4.10 $\lim_{s\to\infty} \frac{\tilde{n}(s)}{\sqrt{s}}$ exists and equals to $\frac{1}{\pi}\int_a^b \Lambda_{\frac{1}{2}}(R_+(x))dx$.

Now using corollary 4.9, we have $\lim_{s\to\infty} \frac{n(s)}{\sqrt{s}} = \lim_{s\to\infty} \frac{\tilde{n}(s)}{\sqrt{s}}$ and hence we have proved the theorem.

Remark. In the results of I.C. Gohberg and M.G. Krein [5], there appears some additional factor 4. But it must be corrected as 1.

Appendix A

Compact operators

The following two theorems are in connection with compact operators on a Hilbert space:

Theorem A.1. ([4], III.9)

The operator A defined by

$$(Af)(x) = \int_Q A(x,t)f(t)dt \qquad \text{for } f\in \mathcal{L}_2^{(n)}(Q)$$

is compact if $A(x,t)\in \mathcal{L}_2^{(nxn)}(QxQ)$.

Theorem A.2. (Riesz-Schauder theorem -- [8], Theorem VI.15)

Let A be a compact operator on $\mathcal{L}_2^{(n)}(Q)$, then $\sigma(A)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity.

Appendix B

Some properties of Green's functions

For the Green's function

$$g(x,t) = \begin{cases} (b-t)(x-a)/(b-a) & a \leq x < t \\ (b-x)(t-a)/(b-a) & t < x \leq b \end{cases}$$

$$-g_{xx}(x,t) = \delta(x-t).$$

Where $\delta(x-t)$ is the *Dirac delta function* and has the following properties (cf. [13], Chapter 1.1)

$$\delta(x-t) = \begin{cases} 0 & x \neq t \\ \infty & x = t \end{cases}$$

$$\int_a^b \delta(x-t)dx = \begin{cases} 0 & \text{if } t \in (a,b) \\ 1 & \text{if } t \in (a,b) \end{cases}$$

If $u(x)$ is a continuous function

$$\int_a^b u(x)\delta(x-t)dx = \begin{cases} 0 & \text{if } t \in (a,b) \\ u(t) & \text{if } t \in (a,b) \end{cases}$$

Thus for a continuous function $u(x)$

$$-\int_a^b g_{xx}(x,t)u(x)dx = u(t).$$

Direct integration shows that

$$\int_a^b \int_a^b |g(x,t)|^2 dxdt = \frac{(b-a)^2}{90}.$$

Appendix C

The Courant minimax principle

For the Sturm-Liouville boundary value problem *(4.4)* the quotient defined by

$$\mathcal{R}_{\varepsilon}(y) = \frac{\int_a^b |y'(x)|^2 dx}{\int_a^b y^t R_{\varepsilon}(x) y dx} \qquad \text{for } y \in H$$

is called the *Rayleigh quotient.*

Let $C(u_1,u_2,\ldots,u_{n-1})$ denote the *minimum value of the Rayleigh quotient* $\mathcal{R}_{\varepsilon}(y)$ *subject to the n-1 constraints*

$$\int_a^b u_i^t(x) y(x) dx = 0 \qquad \text{for } i = 1,2,\ldots,n-1,$$

where $u_1,u_2,\ldots,u_{n-1}$ are any given n-1 vector functions from the U. Then *the nth eigenvalue* $\mu_n(\varepsilon)$ *of (4.4) is equal to the maximum value of the expression* $C(u_1,u_2,\ldots,u_{n-1})$ *over all possible vector functions* $u_1,u_2,\ldots,u_{n-1}$ *in the* U. This result is often referred to as *Courant's minimax principle* ([2], Chapter 6.1.4).

Let
$$\mathcal{R}_{\varepsilon_m}(y) = \frac{\int_a^b |y'(x)|^2 dx}{\int_a^b y^t R_{\varepsilon_m}(x) y dx}$$

and
$$\mathcal{R}_{\varepsilon_{m+1}}(y) = \frac{\int_a^b |y'(x)|^2 dx}{\int_a^b y^t R_{\varepsilon_{m+1}}(x) y dx} .$$

Then $\mu_n(\varepsilon_m) \le \mu_n(\varepsilon_{m+1})$ for all n = 1,2,3... if the Rayleigh quotients satisfy the condition

$$\mathcal{R}_{\varepsilon_m}(y) \le \mathcal{R}_{\varepsilon_{m+1}}(y) \qquad \text{for all vector functions } y \in U,$$

([12], Chapter 7.6).

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