ABSTRACT

The problem to be considered in this work is the simultaneous development of velocity and temperature profiles for a Newtonian fluid, in laminar flow in the entrance region, of a circular tube. The physical properties, i.e. thermal conductivity and viscosity are assumed to be a linear functions of temperature. A variational formulation based on the concept of local potential is first applied to transform the two fundamental coupled nonlinear partial differential equations into two tractable ordinary differential equations. The equations are then solved by the analog/hybrid computer. Consequently, momentum boundary layer thickness, thermal boundary layer thickness, local and average Nusselt numbers, and local and average friction factors are determined for the flow. The study has yielded information concerning flow performance as well as information on the sensitivity of thermal conductivity and viscosity to the Nusselt number and friction factor. A S/360 CSMP (Continuous System Modelling Program) approach in solving the problem is also discussed. Wherever possible, the accuracy of this study is demonstrated by comparing the calculated results with the corresponding available solutions in the literature. Close agreement has been obtained in all the cases. The results show that the variable properties have substantial influence on the flow performance.

ACKNOWLEDGMENT

The author is greatly indebted to Dr. S. C. Cheng who originated this project and gave him the opportunity to work on it. Dr. Cheng's inspiration, constructive criticism and generous consultation were of immeasurable importance in carrying the work to completion.

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NOMENCLATURE

A	viscosity coefficient
A	amplifier
ADC	analog-to-digital converters
B	conductivity coefficient
c	potentiometer coefficient
C	comparator
C_p	specific heat at constant pressure
d	diameter of the tube
D/A	digital-to-analog (switch)
e_o	output analog signal
$e_1, e_2, \ldots e_n$	input analog signals
E	local potential
f	local friction factor
$\bar{f}$	average friction factor
h	heat transfer coefficient
HG	high gain
IC	initial condition mode
κ	thermal conductivity
K	amplification
ℓ	entrance length
L, L1	logic signals
Nu	local Nusselt number
$\overline{Nu}$	average Nusselt number

OP	operate mode
p	pressure
P	potentiometer, set by servometer
Pr_o	Prandtl number in the inlet section, $\frac{\mu_o C_p}{\kappa_o}$
q	heat flux
r, z	cylindrical coordinates
r_o	radius of the tube
R	relay
Re_o	diameter Reynold number, $\frac{\rho u_o d}{\mu_o}$
s	surface
S	switch
SL	senseline
t	tolerance for the Verify mode
t	time
T	temperature
u	axial velocity
v	radial velocity
w	substituted variable for Δ^{*2}
x	substituted variable for Y^{*3}
y	radial coordinate measured from wall, $(r_o - r)$
Y*	ratio between the thermal and momentum boundary layers, $\frac{\Delta_t}{\Delta}$

$\dot{Y}^*$	derivative of Y^* with respect to z^*

Greek Symbols

ρ	density
δ	variational notation
μ	dynamic viscosity
Δ	momentum boundary layer thickness
Δ_t	thermal boundary layer thickness
θ	dimensionless temperature variable, $\frac{T - T_o}{T_\omega - T_o}$
α	thermal diffusivity, $\frac{\kappa}{\rho C_p}$
ν	kinematic viscosity
τ	shear stress

Subscripts

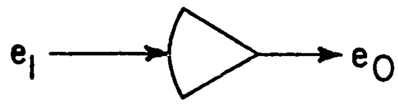
c	condition in the frictionless core of the tube
o	condition at the entrance of the tube
r, z	derivative with respect to r and z, respectively
ω	wall property

Superscripts

*	dimensionless quantity
o	stationary state

"PROGRAMMER SYMBOLS"

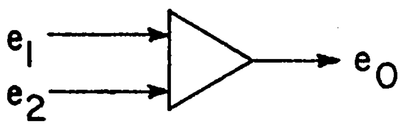
1.



High gain dc amplifier

$e_o = -Ke_1 \quad (K > 10^8)$

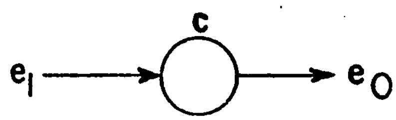
2.



Summer - inverter

$e_o = -(e_1 + e_2)$

3.

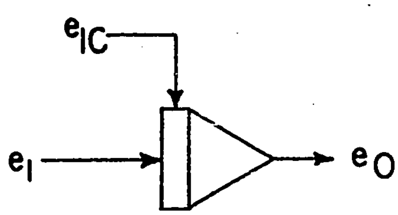


Grounded Potentiometer

$e_o = c\ e_1$

$0 \leq c \leq 1$

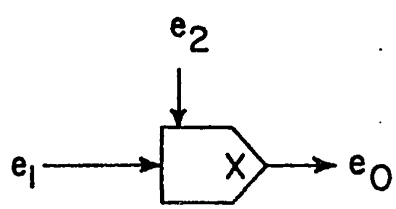
4.



Integrator

$e_o = -\int e_1\, dt - e_{IC}$

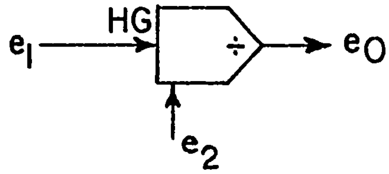
5.



Multiplier

$e_o = -e_1 e_2$

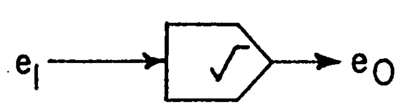
6.



Divider

$$e_o = -\frac{e_1}{e_2}$$

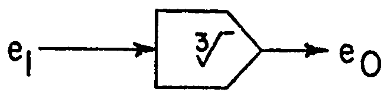
7.



Square root

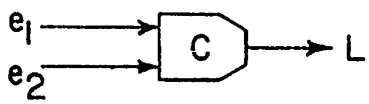
$e_o = -\sqrt{e_1}$

8.



$e_o = -\sqrt[3]{e_1}$

9.

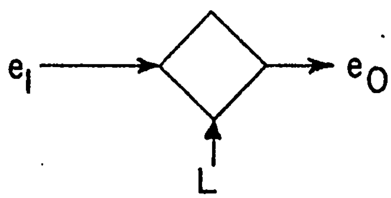


Comparator

When $e_1 + e_2 \geq 0$, L is TRUE

When $e_1 + e_2 < 0$, L is FALSE

10.

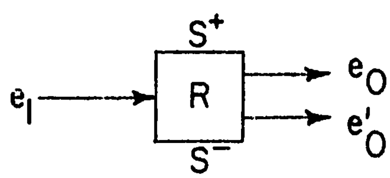


D/A Switch

When L is TRUE, e_1 and e_o are connected

When L is FALSE, e_1 is grounded

11.

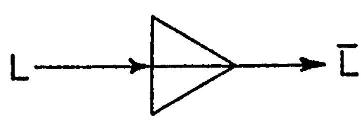


Relay or Functional Switch

When S^+ is TRUE, e_1 and e_o are connected

When S^- is TRUE, e_1 and e_o^1 are connected

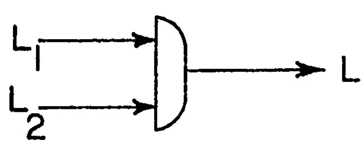
12.



Logic Inverter

When L is TRUE, $\bar{L}$ is FALSE and vice versa

13.



AND GATE

When L_1 and L_2 are TRUE, L is TRUE,
otherwise L is FALSE

CHAPTER 1

INTRODUCTION

A duct of circular cross section has wide engineering applications and its fully developed flow cases have been the subject of many analytical and experimental studies. In contrast very few studies on laminar flow and heat transfer in the entrance region of a circular tube can be found in the literature. However, the situation existing in a tube entry is significant in many engineering problems which involve tubes and ducts with small length to diameter ratios. Such is the case in the inlet passages of jet engines, short tube heat exchangers, tubes of radiators, intercoolers and passageways in wind tunnels.

When a fluid flows steadily through a tube, the flow pattern in the region immediately downstream of the entrance to the tube depends in great degree on the distance from the tube entrance. After this distance has become sufficiently great, the variation in flow pattern vanishes, the velocity profile remains unaltered, and the friction coefficient is independent of the distance from the inlet. In addition to this, if a fluid flowing adiabatically into a duct or channel, enters a region having a wall temperature different from that of the fluid, thermal entrance effects are also present.

This temperature distribution takes various forms depending on the boundary conditions and past history of the fluid.

Unfortunately, most of the work conducted so far assumed constant properties for cases having $Pr_0 < 1$. In this study, the effects of variable viscosity and thermal conductivity on the laminar flow in the entrance region of a circular tube with $Pr_0 > 1$ are investigated by assuming a constant wall temperature.

The study involves the derivation of two coupled differential equations by applying the variational formulation to the basic equations of the flow. These equations are then solved by the analog/hybrid computer and by CSMP (Continuous System Modelling Program). The various quantities to be determined include momentum and thermal boundary layer thicknesses, local and average Nusselt numbers, and local and average friction factors. Wherever possible, comparison is made between the results obtained from this work and those available from the literature, i.e. constant properties case.

CHAPTER 2

LITERATURE SURVEY

Until recently, analytical studies of laminar forced-convection flow in circular and noncircular ducts have been concerned primarily with three groups of problems.

The first group dealt with the development of the velocity profile in the entrance region of a duct with no heat transfer. The fluid is assumed to enter the duct at a uniform velocity. As the fluid moves down the duct, a boundary layer begins forming at the entrance and grows on the wall surface due to the fact that the fluid immediately adjacent to the wall has a velocity of zero. The fully developed velocity profile exists when the two edges of the boundary layer coincide.

The second group was concerned with the flow in which the velocity profile of the fluid is already fully developed but enters a section of duct having a wall temperature different from the temperature of the entering fluid. This is the problem of a purely thermal entrance region.

Finally, the third group was concerned with the so-called fully developed problems in which the heat transfer and friction parameters do not change along the length of the duct. The results can be applied to long ducts. For obvious reasons, the nature of laminar flow and heat transfer under the fully developed condition has been studied extensively by many investigators over the last few decades.

Only recently has some attention been given to the case of simultaneous development of velocity and temperature profiles in the entrance region of circular ducts. Understanding of the flow characteristics and heat transfer in this region is of practical importance as these configurations are the most widely occurring. In technical applications, one is often more concerned with the development of the hydrodynamic and thermal boundary layers together, rather than merely the thermal boundary layer alone.

All of the published solutions for laminar flow heat transfer in a tube are based on the idealization of either a fully established velocity profile, or a uniform velocity profile. When oils are the primary fluid, assumption of a fully established parabolic velocity profile, even though both velocity and temperature are uniform at the tube entrance, does not lead to a significant error because the velocity profile is established much more rapidly than the temperature profile. In contrast, for very low Prandtl number fluids, such as the liquid metals, the temperature profile develops much faster than the velocity profile.

However, when fluids with intermediate Prandtl numbers are used, no such flow idealization yields an adequate description of the velocity distribution that exists within a circular tube.

One of the earliest analytical studies on the combined entry length problem of laminar flow and heat transfer in a circular tube appears to be that of Kays (1)≠. He obtained a numerical solution for Prandtl number of 0.71, employing Langhaar's velocity profiles (2). In this case, the radial component of velocity was neglected. He reported the relationship for the local and mean Nusselt numbers for the three different boundary conditions.

Ulrichson and Schmitz (3) refined the work of Kays by including the radial component of velocity in the entrance region. The values of the axial component of velocity were taken from the work of Langhaar (2) and those of radial components of velocity were obtained from the equation of continuity and Langhaar profiles. The main aim of Ulrichson and Schmitz was to study the effect of this refinement and

≠ Numbers in the parenthesis designate the references at the end of the thesis.

their calculations had shown a significant decrease in the calculated local Nusselt number in the entrance region from that obtained by Kays.

R. Manohar(4) solved the nonlinear equations of laminar flow of a viscous incompressible fluid in the entrance region of a circular tube by a numerical method to obtain the velocity of flow in this region. This velocity distribution was used in solving the energy equation numerically to obtain temperature profiles under constant wall temperature and also under constant heat flux at the wall. The local Nusselt number was calculated and compared with those given by other researchers for Prandtl number of 0.7.

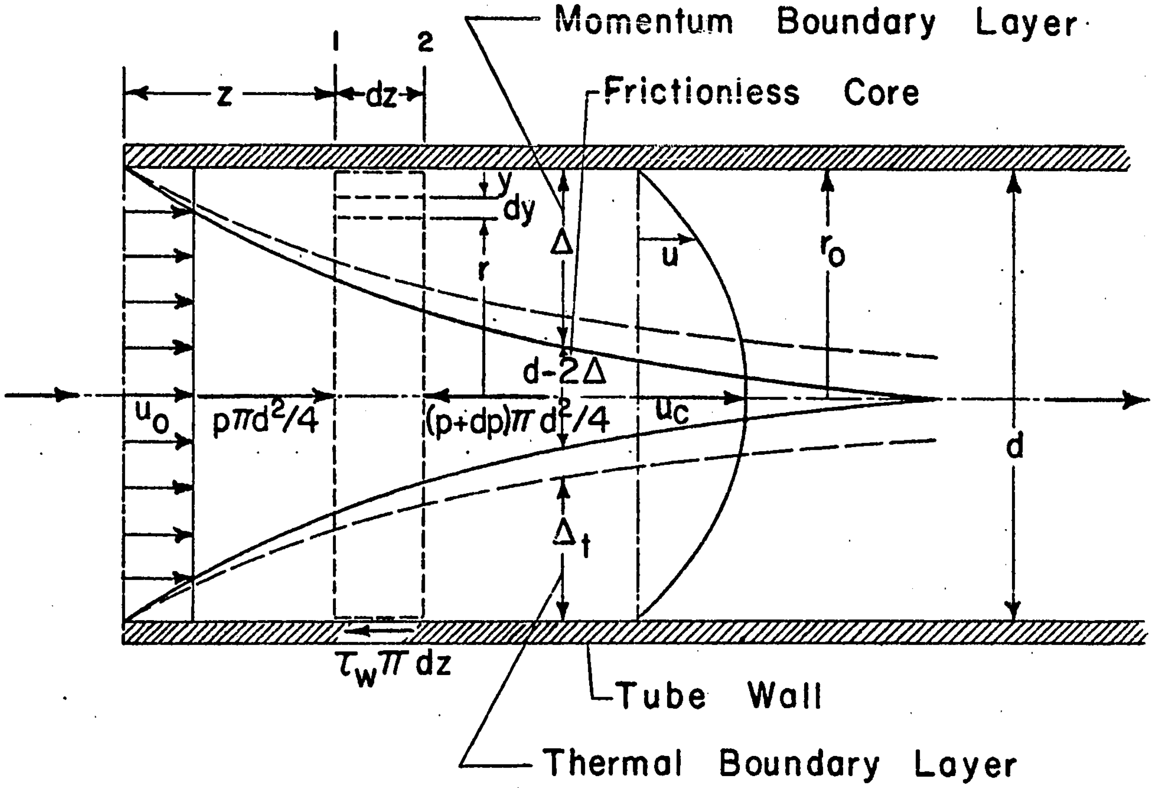


FIG. I SCHEMATIC DIAGRAM OF THE PROBLEM

CHAPTER 3

ANALYTICAL STUDIES

3.1 PHYSICAL MODEL

The physical model selected for a study of the development of the velocity and temperature profiles in the entrance region of a circular tube is the following:

1. A steady laminar flow exists at all points along the tube.

2. The flow is of an incompressible fluid and the fluid properties are constant except for the viscosity and the thermal conductivity which are linear functions of temperature.

3. The velocity and temperature profiles are uniform across the entrance section, and the hydrodynamic and thermal boundary layer thicknesses are zero at the entrance section.

4. The wall temperature is uniform, which differs from the entering fluid temperature.

5. The flow in the region outside the boundary layer is a potential flow.

6. There exists velocity and thermal boundary layers

of definite thickness.

7. Natural convection effects are not considered.

8. The usual boundary layer conceptions

$$\frac{\partial u}{\partial y} >> \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial v}{\partial y}$$

$$\frac{\partial p}{\partial z} \simeq \frac{dp}{dz} \quad \text{and}$$

$$\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial z}$$

will be applied and also, it is assumed that the viscous dissipation is negligible as compared with heat conduction.

The effect of viscosity in the entrance region of a circular tube is confined primarily to the fluid layers near the wall, although in actual practice small effects exist everywhere in the flow. It will be assumed in this analysis that viscosity plays a role only in a definite region adjacent to the wall, called the velocity boundary layer Δ. The development of the boundary layer in the laminar entry region of a tube is complicated by the fact that, as the layer thickness increases, the frictionless core of the flow outside the boundary layer undergoes a contraction in cross sectional area and is, therefore, accelerated. The increasing core velocity produces a pressure gradient along the pipe and this in turn affects the growth of the boundary layer. Thus there is a definite interaction between the boundary layer flow and the potential flow.

The principal effects of heat transfer will be felt by the fluid layers close to the walls, although small effects will exist everywhere. The assumption is made that the effects of the heat transfer play a role only in a definite region adjacent to the wall called the thermal boundary layer Δ_t. The fluid outside the thermal boundary layer will be uninfluenced by the heat transfer and will therefore have a uniform temperature identical to the value at the entrance of a tube. In this analysis, only the case where the momentum boundary layer is greater than the thermal boundary layer is considered; hence, the region of interest of heat transfer is between the duct entrance and the point where the momentum boundary layers meet.

3.2 BASIC EQUATIONS

In cylindrical coordinates, let r denote the radial direction while z denotes the axial coordinate. The velocities corresponding to these coordinates are v and u respectively. For the case of incompressible laminar fluid flow, the equations of conservations of mass, momentum and energy[5] with variable physical properties for the cylindrical coordinates could be expressed in the following form:

Continuity:

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0 \tag{1}$$

Momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu(T)r\frac{\partial u}{\partial r}\right) - \frac{\partial p}{\partial z} \tag{2}$$

Energy:

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial z} + v\frac{\partial T}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\kappa(T)r\frac{\partial T}{\partial r}\right) \tag{3}$$

For the potential flow, using the expression,

$$u_c\frac{\partial u_c}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{4}$$

Eq. (2) can be written as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu(T)r\frac{\partial u}{\partial r}\right) + \rho u_c\frac{\partial u_c}{\partial z} \tag{5}$$

Although the problem under study is a steady-state case, one must nevertheless retain the time-dependent character of the equations when using a variational technique as can be seen clearly by the following section.

3.3 VARIATIONAL FORMULATION OF PROBLEM

The solution of the basic Eqs. (1), (3), (5), in the developing region is essentially one of finding the velocity and temperature distribution. However, even though these equations can be greatly simplified, they are still extremely difficult if not impossible to solve because of the non-linearities involved in the equations. The variational technique can, however, be used to transform these equations into a more tractable form.

An important formulation of the variational principle in thermoscience was derived by Glansdorff et al[6] based on minimum entropy production. However, the formulation is only applicable to a rather narrow class of systems. The restrictions are of such nature that most problems of engineering interest are excluded. Later, Glansdorff and Prigogine[7] removed the restrictions by modifying the formulation using the concept of local potential-generalized entropy production.

In order to construct a local potential for the problem for use in the variational method, a technique used in references [8,9,10] is followed. Upon multiplying Eq. (1) by $-\left(\frac{\rho}{2}\right)\frac{\partial (v)^2}{\partial t}$ Eq. (5) by $\frac{\partial u}{\partial t}$ and Eq. (3) by $\frac{\partial T}{\partial t}$ and adding the resultant expressions, one obtains

$$\Psi = -\rho\left(\frac{\partial u}{\partial t}\right)^2 - \rho C_p\left(\frac{\partial T}{\partial r}\right)^2$$

$$= \rho u \frac{\partial u}{\partial t}\frac{\partial u}{\partial z} + \rho v \frac{\partial u}{\partial t}\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\mu(T) r \frac{\partial u}{\partial r}\right)\right)$$

$$-\frac{\rho}{2}\frac{\partial v^2}{\partial t}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv)\right) + \rho C_p u \frac{\partial T}{\partial t}\frac{\partial T}{\partial z}$$

$$+ \rho C_p v \frac{\partial T}{\partial t}\frac{\partial T}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\kappa(T)\ r \frac{\partial T}{\partial t}\frac{\partial T}{\partial r}\right)$$

$$+ \frac{\kappa}{2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial r}\right)^2 - \rho \frac{\partial u}{\partial t} u_c \frac{\partial u_c}{\partial z} \leq 0 \qquad (6)$$

The arguments of the variational techniques require the specification of function ϕ as follows:

$$\phi = \iint_s \Psi r\ dr\ dz \leq 0 \qquad (7)$$

where s is the area of interest in the z - r plane which is bounded by a curve enclosing s. The integrand Ψ can be rearranged in the form

$$\Psi = \frac{\partial}{\partial z}\left(\rho u^2 \frac{\partial u}{\partial t}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r\ u\ v \frac{\partial u}{\partial t}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\mu(T)\ r \frac{\partial u}{\partial r}\frac{\partial u}{\partial t}\right)$$

$$- \rho u \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv)\right) - \rho u^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right)$$

$$- \rho u v \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + \frac{\mu(T)}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right)^2$$

$$- \frac{\rho}{2}\left(\frac{\partial v^2}{\partial t}\right)\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv)\right) + \rho C_p u \frac{\partial T}{\partial z}\frac{\partial T}{\partial t}$$

$$+ \rho C_p v \frac{\partial T}{\partial r}\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(\kappa(T) \frac{\partial T}{\partial t}\frac{\partial T}{\partial r}\right)$$

$$+ \frac{\kappa(T)}{2}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial r}\right)^2 - \rho \frac{\partial u}{\partial t} u_c \frac{\partial u_c}{\partial z} \leq 0 \qquad (8)$$

Combining Eqs. (7), (8) and using Gauss theorem, one obtains

$$\phi = \iint_s\left(- \rho u^2 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right) - \rho u v \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + \frac{\mu(T)}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right)^2\right.$$

$$- \frac{\rho}{2} \frac{\partial u^2}{\partial t} \left(\frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (rv) \right) - \frac{\rho}{2} \frac{\partial v^2}{\partial t} \left(\frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (rv) \right)$$

$$+ \rho C_p u \frac{\partial T}{\partial z} \frac{\partial T}{\partial t} + \rho C_p v \frac{\partial T}{\partial r} \frac{\partial T}{\partial t} + \frac{\kappa(T)}{2} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial r} \right)^2$$

$$- \rho \frac{\partial u}{\partial t} u_c \frac{\partial u_c}{\partial z} \Bigg) r \, dr \, dz$$

$$+ \int_c \left(\rho u^2 \frac{\partial u}{\partial t} r \, dr - \frac{1}{r} \rho uv \left(r \frac{\partial u}{\partial t} \right) r \, dz + \frac{\mu(T)}{r} \left(r \frac{\partial u}{\partial r} \frac{\partial u}{\partial t} \right) r \, dz \right.$$

$$\left. + \frac{1}{r} \left(r\kappa(T) \frac{\partial T}{\partial t} \frac{\partial T}{\partial r} \right) r \, dz \right) \leq 0 \qquad (9)$$

Near the stationary state, the concept of local potential gives

$$\phi = \frac{\partial}{\partial t} \iint_s \left(- \rho u^{o2} \frac{\partial u}{\partial z} - \rho u^o v^o \frac{\partial u}{\partial r} + \frac{\mu^o(T)}{2} \left(\frac{\partial u}{\partial r}\right)^2 \right.$$

$$- \frac{\rho}{2} \left(u^2 + v^2 \right) \left(\frac{\partial u^o}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (rv^o) \right) + \rho C_p u^o T \frac{\partial T^o}{\partial z}$$

$$\left. + \rho C_p v^o T \frac{\partial T^o}{\partial r} + \frac{\kappa^o(T)}{2} \left(\frac{\partial T}{\partial r} \right)^2 - \rho u_c^o \frac{\partial u_c^o}{\partial z} u \right) r \, dr \, dz$$

$$+ \frac{\partial}{\partial t} \int_c \left(\rho u^{o2} ur \, dr - \rho u^o v^o ur \, dz + \mu^o(T) \frac{\partial u^o}{\partial r} urdz \right.$$

$$\left. + \kappa^o(T) \frac{\partial T^o}{\partial r} rdz \right) \leq 0 \qquad (10)$$

Therefore the local potential is

$$E = \iint_S \Bigg(- \rho u^{o^2} \frac{\partial u}{\partial z} - \rho u^o v^o \frac{\partial u}{\partial r} + \frac{\mu^o(T)}{2} \left(\frac{\partial u}{\partial r}\right)^2$$

$$- \frac{\rho}{2} (u^2 + v^2) \left(\frac{\partial u^o}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r v^o)\right)$$

$$+ \rho C_p u^o T \frac{\partial T^o}{\partial z} + \rho C_p v^o T \frac{\partial T^o}{\partial r} + \frac{\kappa^o(T)}{2} \left(\frac{\partial T}{\partial r}\right)^2$$

$$- \rho u_c^o \frac{\partial u_c^o}{\partial z} u \Bigg) r \, dr \, dz$$

$$+ \int_C \Bigg(\rho u^{o^2} u r dr - \rho u^o v^o u r dz + \mu^o(T) \frac{\partial u^o}{\partial r} u r dz$$

$$+ \kappa^o(T) T \frac{\partial T^o}{\partial r} r dz \Bigg) \qquad (11)$$

with subsidiary conditions

$$u^o = u$$
$$v^o = v$$
$$T^o = T$$

The line integral portion of Eq. (11) can be simplified by using the boundary conditions. In this study, the area of interest is bounded by the lines $z = 0$, $z = \ell$, $r = r_o$, $r = r_o - \Delta$.

The boundary conditions for the problem are

$$u = 0 \quad \text{at} \quad r = r_o$$

$$v = 0 \quad \text{at} \quad r = r_o$$

$$u = u_c \quad \text{at} \quad r = r_o - \Delta$$

$$u = u_o \quad \text{at} \quad z = 0$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad r = r_o - \Delta$$

and

$$T = T_\omega \quad \text{at} \quad r = r_o$$

$$T = T_o \quad \text{at} \quad r = r_o - \Delta_t$$

$$T = T_o \quad \text{at} \quad z = 0$$

Therefore, the contribution stemming from the line integral is

$$E_{line} = \int_{r_o}^{r_o - \Delta} (\rho u^{o^2} ur)_{z=\ell} dr + \int_{r_o-\Delta}^{r_o} (\rho u^{o^2} ur)_{z=0} dr$$

$$+ \int_{\ell}^{0} (-\rho u^o v^o ur)_{r=r_o-\Delta} dz + \int_{0}^{\ell} (-\rho u^o v^o ur)_{r=r_o} dz$$

$$+ \int_{\ell}^{0} \left(\mu^o(T) \frac{\partial u^o}{\partial r} ur \right)_{r=r_o-\Delta} dz + \int_{0}^{\ell} \left(\mu^o(T) \frac{\partial u^o}{\partial r} ur \right)_{r=r_o} dz$$

$$+ \int_{\ell}^{0} \left(\kappa^o(T) T \frac{\partial T^o}{\partial r} r \right)_{r=r_o-\Delta} dz + \int_{0}^{\ell} \left(\kappa^o(T) \frac{\partial T^o}{\partial r} r \right)_{r=r_o} dz \qquad (12)$$

Imposing the boundary conditions, one obtains

$$E_{line} = \int_{r_o}^{r_o - \Delta} \left\{ (\rho u^{o^2} ur)_{z=\ell} - (\rho u^{o^2} ur)_{z=0} \right\} dr$$

$$+ \int_0^{\ell} \left\{ (\rho u^o v^o u)_{r=r_o-\Delta} + \kappa_\omega^o \left\{ \frac{\partial T^o}{\partial r} \right\}_{r=r_o} T_\omega \right\} dz \qquad (13)$$

By using Eqs. (1) and (13), Eq. (11) can be reduced to

$$E = \iint_S \Bigg(- \rho u^{o^2} \frac{\partial u}{\partial z} - \rho u^o v^o \frac{\partial u}{\partial r} + \frac{\mu^o(T)}{2} \left\{ \frac{\partial u}{\partial r} \right\}^2 + \rho C_p u^o T \frac{\partial T^o}{\partial z}$$

$$+ \rho C_p v^o T \frac{\partial T^o}{\partial r} + \frac{\kappa^o(T)}{2} \left\{ \frac{\partial T}{\partial r} \right\}^2 - \rho u_c^o \frac{\partial u_c^o}{\partial z} u \Bigg) r\, dr\, dz$$

$$+ \int_{r_o}^{r_o - \Delta} \left\{ (\rho u^{o^2} ur)_{z=\ell} - (\rho u^{o^2} ur)_{z=0} \right\} dr \qquad (14)$$

Or

$$E = \iint_S F\,(r,\, z,\, u,\, u_z,\, u_r,\, T,\, T_z,\, T_r)\; r\, dr\, dz$$

$$+ \int_{r_o}^{r_o - \Delta} \left\{ (\rho u^{o^2} ur)_{z=\ell} - (\rho u^{o^2} ur)_{z=0} \right\} dr \qquad (14a)$$

Where

$$F = - \rho u^{o^2} \frac{\partial u}{\partial z} - \rho u^o v^o \frac{\partial u}{\partial r} + \frac{\mu^o(T)}{2} \left\{ \frac{\partial u}{\partial r} \right\}^2 + \rho C_p u^o T \frac{\partial T^o}{\partial z}$$

$$+ \rho C_p v^o T \frac{\partial T}{\partial r} + \frac{\kappa^o(T)}{2} \left\{ \frac{\partial T}{\partial r} \right\}^2 - \rho u_c^o \frac{\partial u_c^o}{\partial z} u \qquad (14b)$$

In order to prove that Eq. (14) is the local potential of the problem, one takes the variation of local potential E (Eq. (14a)) with respect to u and T, which gives

$$\left(\frac{\delta E}{\delta u}\right)_{u^o} = 0 \qquad (15)$$

$$\left(\frac{\delta E}{\delta T}\right)_{T^o} = 0 \qquad (16)$$

Eq. (15) can be written as

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial F}{\partial u_r}\right) = 0 \qquad (17)$$

$$-\rho u_c \frac{\partial u_c^o}{\partial z} - \frac{\partial}{\partial z}\left(-\rho u^{o^2}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(-\rho u^o r v^o + \mu^o(T) r \frac{\partial u}{\partial r}\right) = 0 \qquad (18)$$

Using the subsidiary conditions

$u^o = u$

$v^o = v$, Eq. (18) becomes

$$-\rho u_c \frac{\partial u_c}{\partial z} + \frac{\partial}{\partial z}\left(\rho u^2\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho u r v\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\mu(T) r \frac{\partial u}{\partial r}\right) = 0$$

After rearranging the terms, the above equations gives

$$\rho u \frac{\partial u}{\partial z} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left\{\mu(T)\frac{\partial u}{\partial r}\right\} + \rho u_c \frac{\partial u_c}{\partial z}$$

which is the momentum boundary layer equation in the z direction.

Similarly from Eq. (16) it can be shown as

$$\frac{\partial F}{\partial T} - \frac{\partial}{\partial z}\left\{\frac{\partial F}{\partial T_z}\right\} - \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial F}{\partial T_r}\right\} = 0$$

i.e.

$$\rho C_p\left\{u\frac{\partial T}{\partial z} + v\frac{\partial T}{\partial r}\right\} = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\kappa(T)\frac{\partial T}{\partial r}\right\}$$

which is the energy boundary layer equation. Thus it has been shown that Eq. (14) is the local potential of the problem.

To determine the momentum and thermal boundary layers, the method of partial integration is used and the following velocity and temperature distributions are assumed:

Velocity Profile:

$$\frac{u}{u_c} = -\left\{\frac{r_o - r}{\Delta}\right\}^2 + 2\left\{\frac{r_o - r}{\Delta}\right\} \quad \text{for } r_o \geq r \geq r_o - \Delta \qquad (19)$$

Temperature Profile:

$$\frac{T - T_\omega}{T_o - T_\omega} = \frac{3}{2}\frac{r_o - r}{\Delta_t} - \frac{1}{2}\left\{\frac{r_o - r}{\Delta_t}\right\}^3 \text{ for } r_o \geq r \geq r_o - \Delta_t > r_o - \Delta \qquad (20)$$

with the following boundary conditions

At $r = r_o$ $u = 0$

$T = T_\omega$

At $r = r_o - \Delta$ $u = u_c$

$\frac{\partial u}{\partial r} = 0$

At $r = r_o - \Delta_t$ $T = T_o$

$\frac{\partial T}{\partial r} = 0$

In order to determine the relation between the core and entrance velocities, mass balance may be applied to a portion of the entrance section between z = 0 (the inlet) and an arbitrary cross section z to give

$$\rho u_o \pi r_o^2 = \int_0^{r_o-\Delta} \rho u 2\pi r dr + \int_{r_o-\Delta}^{r_o} \rho u 2\pi r dr \tag{21}$$

Substituting Eq. (19) in above, it yields

$$\frac{u_c}{u_o} = \frac{1}{1 - \frac{2}{3}\frac{\Delta}{r_o} + \frac{1}{6}\frac{\Delta^2}{r_o^2}} \tag{22}$$

For simplicity, the viscosity and thermal conductivity are chosen as linear functions of temperature.

i.e.

$$\frac{\mu}{\mu_o} = 1 + A\theta \tag{23}$$

$$\frac{\kappa}{\kappa_o} = 1 + B\theta \tag{24}$$

Where

$$\theta = \frac{T - T_o}{T_\omega - T_o} \tag{25}$$

In the above expressions, a positive A and negative B indicate cooling of the fluid, while a negative A and a positive B indicate heating of the fluid.

Substituting these expressions for velocity, temperature, viscosity and conductivity into Eq. (14), the following two coupled equations are obtained. (The rather tedious calculations involved in making this step are given in the Appendix A).

$$\left\{ \left\{\frac{21}{10} + \frac{1239}{640} B\right\} \left\{\frac{1}{Pr_o}\right\} - \left\{\frac{3}{10}\frac{B}{Pr_o}\right\} Y^*\Delta^* \right\} \left\{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right\}$$

$$+ \left\{-\frac{21}{128} Y^{*2}\dot{Y}^* + \frac{1}{20} Y^{*3}\dot{Y}^*\Delta^{*2} + \frac{1}{10} Y^{*3}\dot{Y}^*\Delta^{*3} - \frac{21}{256} Y^{*3}\Delta^*\dot{\Delta}^*\right\}$$

$$- \left\{\frac{\frac{2}{3}\dot{\Delta}^* - \frac{1}{3}\Delta^*\dot{\Delta}^*}{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}}\right\} \frac{21}{256} Y^{*3}\Delta^{*2} = 0 \tag{26}$$

$$\frac{2}{3} - A\left\{\Delta^*\left\{\frac{2}{5}Y^{*2} - \frac{1}{2}Y^{*3}\right\} - \frac{3}{2}Y^* + \frac{6}{5}Y^{*2} - \frac{1}{3}Y^{*3}\right\}$$

$$+ \frac{1}{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}}\left\{\frac{49}{5040}\Delta^{*2}\dot{\Delta}^* - \frac{17}{1260}\Delta^*\dot{\Delta}^*\right\}$$

$$+ \frac{\frac{2}{3}\dot{\Delta}^* - \frac{1}{3}\Delta^*\dot{\Delta}^*}{\left\{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right\}^2}\left\{\frac{187}{10080}\Delta^{*3} - \frac{13}{252}\Delta^{*2}\right\} = 0 \qquad (27)$$

Where the non-dimensionalized quantities are defined as

$$Y^* = \frac{\Delta_t}{\Delta}$$

$$z^* = \frac{z}{dRe_o}$$

$$\Delta^* = \frac{\Delta}{r_o}$$

$$\Delta_t^* = \frac{\Delta_t}{r_o}$$

$$\dot{\Delta}^* = 2\,\dot{\Delta}\,Re_o$$

$$\dot{Y}^* = d\,\dot{Y}\,Re_o$$

By substituting

$$w = \Delta^{*2}$$

$$x = Y^{*3}$$

which gives

$$\dot{w} = 2\,\Delta^*\,\dot{\Delta}^*$$

and

$$\dot{x} = 3\,Y^{*2}\,\dot{Y}^*$$

Eqs. (26) and (27) reduce to

$$\dot{x}w\left(D1 - D2\sqrt[3]{x} - D3\sqrt{w}\sqrt[3]{x}\right)$$

$$= \frac{1}{Pr_o}\left(C1 - C2\sqrt{w}\sqrt[3]{x}\right)\left\{1 - \frac{2}{3}\sqrt{w} + \frac{1}{6}w\right\}$$

$$- \dot{w}x\left\{D4 + \frac{C3\sqrt{w} - C4\,w}{1 - \frac{2}{3}\sqrt{w} + \frac{1}{6}w}\right\} \qquad (28)$$

$$E6 + A\left\{\frac{3}{2}\sqrt[3]{x} - \frac{6}{5}x^{2/3} + \frac{1}{3}x - \frac{2}{5}\sqrt{w}\,x^{2/3} + \frac{1}{2}\sqrt{w}\,x\right\}$$

$$= \dot{w}\left(\frac{E5 - E4\sqrt{w}}{1 - \frac{2}{3}\sqrt{w} + \frac{1}{6}w} + \frac{E1\sqrt{w} - E2\,w + E3\,w\sqrt{w}}{\left\{1 - \frac{2}{3}\sqrt{w} + \frac{1}{6}w\right\}^2}\right) \qquad (29)$$

where

$$C1 = \frac{21}{10} + \frac{1239}{640}B$$

$$C2 = \frac{3}{10}B$$

$$C3 = \frac{7}{256}$$

$$C4 = \frac{7}{512}$$

$$D1 = \frac{7}{128}$$

$$D2 = \frac{1}{60}$$

$$D3 = \frac{1}{30}$$

$$D4 = \frac{21}{512}$$

$$E1 = \frac{13}{756}$$

$$E2 = 0.014778$$

$$E3 = \frac{187}{60480}$$

$$E4 = \frac{49}{10080}$$

$$E5 = \frac{17}{2520}$$

$$E6 = \frac{2}{3}$$

with

$$\Delta^* = \sqrt{W} \qquad (30)$$

$$\Delta_t^* = \sqrt{W}\ \sqrt[3]{X} \qquad (31)$$

The range of solution is from $0 \leq \Delta^* \leq 1$.

3.4 NUSSELT NUMBER

Once momentum and thermal boundary layer thicknesses are known, the local and average Nusselt numbers can be found easily.

An expression for the local Nusselt number, Nu, is obtained by equating the heat transfer rate per unit area and time, q, as determined by the difference between the wall temperature and the mean fluid temperature with q as determined by the temperature gradient at the wall. The former expression for q is

$$q = h\ (T_\omega - T_m) \tag{32}$$

The latter expression for q is

$$q = \kappa_\omega \left\{\frac{\partial T}{\partial r}\right\}_{r=r_o}$$

or

$$q = -\frac{3}{2}\frac{\kappa_\omega}{\Delta_t}(T_o - T_\omega) \tag{33}$$

Therefore, by equating Eqs. (32) and (33) and by solving for Nu, the following result is obtained

$$h\ (T_\omega - T_m) = -\frac{3}{2}\frac{\kappa_\omega}{\Delta_t}(T_o - T_\omega)$$

or

$$h\,\theta_m = \frac{3}{2}\frac{\kappa_\omega}{\Delta_t}$$

where

$$\theta_m = \frac{T_m - T_\omega}{T_o - T_\omega}$$

or

$$\frac{2\,h\,r_o}{\kappa_o} = \frac{3(1+B)}{\theta_m\,\Delta_t^*}$$

Therefore

$$Nu = \frac{3(1+B)}{\theta_m\,\Delta_t^*} \qquad (34)$$

To calculate the mean fluid temperature, T_m and hence θ_m, the following procedure is adopted

$$T_m = \frac{\int_o^{r_o} T\,u\,r\,dr}{\int_o^{r_o} u\,r\,dr} \qquad (35)$$

$$T_m - T_\omega = \frac{\int_o^{r_o}\left\{T_o - T_\omega\right\}\left\{\frac{3}{2}\frac{r_o - r}{\Delta_t} - \frac{1}{2}\left\{\frac{r_o - r}{\Delta_t}\right\}^3\right\} u_c\left\{-\left\{\frac{r_o - r}{\Delta}\right\}^2 + 2\,\frac{r_o - r}{\Delta}\right\} rdr}{\int_o^{r_o} u_c\left\{-\left\{\frac{r_o - r}{\Delta}\right\}^2 + 2\,\frac{r_o - r}{\Delta}\right\} rdr}$$

After integration and introducing the non-dimensional quantities one obtains

$$\theta_m = \frac{T_m - T_\omega}{T_o - T_\omega} = \frac{\left\{\frac{1}{2} - \frac{3}{140}\frac{\Delta_t^{*4}}{\Delta^{*2}} + \frac{1}{12}\frac{\Delta_t^{*3}}{\Delta^{*2}} + \frac{1}{24}\frac{\Delta_t^{*3}}{\Delta^{*2}} - \frac{1}{5}\frac{\Delta_t^{*2}}{\Delta^{*}} + \frac{1}{12}\Delta^{*2} - \frac{1}{3}\Delta^{*}\right\}}{\left\{\frac{1}{2} + \frac{1}{12}\Delta^{*2} - \frac{1}{3}\Delta^{*}\right\}} \tag{36}$$

The mean Nusselt number $\overline{Nu}$ is the local Nusselt number averaged over the tube length. Hence,

$$\overline{Nu} = \frac{3(1 + B)}{z^*} \int_o^{z^*} \frac{dz^*}{\theta_m \; \Delta_t^*} \tag{37}$$

3.5 FRICTION FACTOR

The local friction coefficient is defined as

$$f = \frac{-2 \mu_\omega \left.\frac{\partial u}{\partial r}\right|_{r=r_o}}{\rho \, u_o^2} \tag{38}$$

or

$$f = \frac{4 \mu_\omega u_c}{\Delta \rho u_o^2}$$

Using Eq. (22) and rearranging the terms, one obtains

$$fRe_o = \frac{8(1 + A)}{\Delta^*\left\{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right\}} \tag{39}$$

The average friction coefficient is found from the following expression

$$\bar{f}Re_o = \frac{1}{z^*}\int_o^{z^*} f\ Re_o\ dz^* \tag{40}$$

or

$$\bar{f}Re_o = \frac{8(1 + A)}{z^*}\int_o^{z^*} \frac{1}{\Delta^*\left\{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right\}}\ dz^* \tag{41}$$

While choosing viscosity coefficient A and thermal conductivity coefficient B, it must be kept in mind that, in general, for incompressible fluids, the thermal conductivity is slightly dependent on temperature and the viscosity is always decreasing with temperature.

The computer solutions of Eqs. (28) and (29) are discussed in the next chapter. The calculations and plotting of the quantities defined by Eqs. (30), (31), (34), (35), (36), (37), (39), and (40) are also discussed in the next chapter.

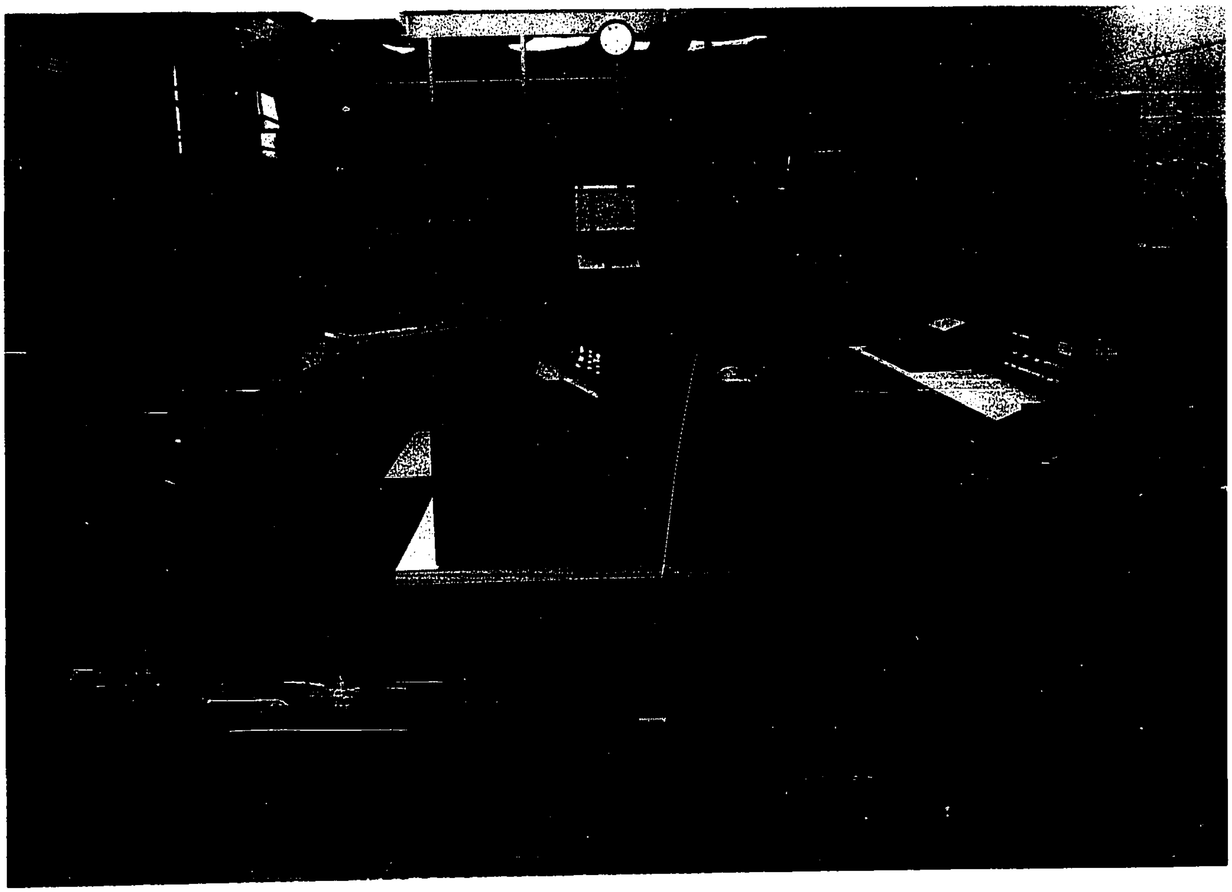


FIG. 2 EAI-690 HYBRID COMPUTER

CHAPTER 4

MECHANICS OF COMPUTER SOLUTIONS

Once the complete definition of the problem, based on the model chosen for study, has been achieved, the next step in the study is consideration as to how the equations are to be solved.

The Eqs. (28) and (29) will be solved first, on an analog/hybrid computer and secondly on a digital computer, using a digital-analog simulator. It would be worthwhile to give a brief description of the computers and the techniques used before the equations are solved.

4.1 THE ANALOG/HYBRID COMPUTER SOLUTION OF THE PROBLEM

As its name suggests, the Hybrid Computer derives its existence from the union of dissimilar elements. These heterogeneous parts are a digital computer and an analog computer. In order to function, however, the system has a third essential component; namely, the interface hardware which permits the effective communication between the two computing domains.

In order to acquire insight into the computing role of the hybrid computer, it is of value to examine briefly some of the characteristics of the two constituent computing machines.

4.1.1 THE ANALOG COMPUTER[11]

Some of the noteworthy features of the analog computer are:

1. Dependent variables within the machine treated in continuous form.

2. Accuracy limited by the quality of the computer components, and rarely better than 0.01%.

3. Parallel operation with all computational elements operating simultaneously.

4. Ability to perform efficiently such operations as multiplications, division, addition, integration, and non-linear function generations; on the other hand, very limited ability to make logical decisions, store numerical data, and handle nonnumeric information.

5. Programming techniques which consist largely of substituting analog computing elements for corresponding elements in a physical system under simulation.

6. Provision to experiment by adjusting coefficient settings on the computer, thereby gaining direct insight into the system operation.

4.1.2 THE DIGITAL COMPUTER[11]

The following attributes are commonly associated with digital computer system:

1. Handling of dependent variables, and indeed all data within the computer, in discretized form.

2. Serial operation, i.e., only one or a limited number of operations being carried out at one time.

3. Accuracy relatively independent of the quality of system components and determined primarily by the number of bits contained in memory registers and by the specific numerical technique selected for a specific problem.

4. Solution times relatively long and determined by the complexity of a problem.

5. Ability to reduce errors inherent in the computer solution by increasing the length of time required to obtain the solution on the computer.

6. Ability to perform a limited number of arithmetic operations including particularly addition and multiplication; more complex operations such as integration and differentiation must be performed by approximate technique.

7. Facility for memorizing numerical and nonnumerical data indefinitely.

8. Facility to perform logic operations and decisions utilizing numerical as well as nonnumeric data.

4.2 THE HYBRID COMPUTER[11~16]

Hybrid computer techniques inevitably represent an effort to combine in one computer system, some of the characteristics normally associated with analog systems and some of the characteristics associated with digital computer systems. It should be appreciated, however, that the extent to which this objective is realized in practice is intimately dependent on the nature of the problem under consideration. For the problems within an appropriate class, the hybrid computer provides a powerful, and indeed indispensable, computational aid.

In any genuine hybrid problem, the computational task is divided between the two computers. This division should always be made in a manner which takes the most advantage of the computing power inherent in the system. The following are, at present, the chief motivations for interconnecting digital and analog computers:

1. To combine the speed of an analog computer with the

accuracy of digital computer.

2. To permit the use of system hardware in a digital simulation.

3. To increase the flexibility of an analog simulation by using digital memory and control.

4. To increase the speed of a digital computation by utilizing analog subroutines.

5. To permit the processing of incoming data which are partially discrete and partially continuous.

Besides, the various areas of application which are well suited to hybrid computer investigation are:

1. Simulation of complex dynamic systems.

2. System optimization.

3. Random processes.

4. Solution of partial differential equations.

At the present time, virtually all hybrid computers employ so-called "stand alone" analog and digital computers which can function independently as well as in the interconnected mode. The ways in which analog and digital computer installations can be used together can be classified into two broad categories:

1. Unilateral operation in which information flows across the interface between the analog and the digital sections in only one direction.

2. Bilateral operation in which the flow across this interface is in both directions.

The computer used for the solution of the problem at hand is an EAI-690 hybrid computer, located in the Analysis Laboratory, National Research Council, Ottawa, and is shown in Fig.(2). The two coupled Eqs. (28) and (29) are solved on analog computer while all other computations are performed by the digital computer; hence the hybrid system is used in an unilateral fashion.

4.3 THE ANALOG PROGRAM

The magnitude and time scaled circuit of the problem at hand is shown on Figs. (3), (3-a) and (3-b). A discussion of the standard procedure used can be found in references[17,18]. $t=1000z^*$ represents the relationship between the computer independent variable t and the problem independent variable z^*.

The solution of thermal boundary layer equation (Eq.(29)) involves a division whose result is initially infinity. To prevent this requires use of logic hardware which disconnects the division circuit in the neighbourhood of $z^* = 0$. This is achieved by disconnecting the numerator input to the divider, during the interval $(0, \Delta_o^*)$, with the D/A switch S064, which is controlled through the comparator C 099. The output logic signal L1 of C 099 changes from FALSE to TRUE at $\Delta^* = \Delta_o^*$, thereby reconnecting the numerator input to the divider. During the interval $(0, \Delta_o^*)$ the output of amplifier A105 must remain at zero in order to satisfy the physical model of the problem (i.e., a small unheated length is assumed). This is achieved by disconnecting the input to integrator A105 by D/A switch S069 during the interval $(0, \Delta_o^*)$ which is also being controlled by the logic signal L1 from the comparator C 101. The value of the unheated length z_o^*, corresponding to $\Delta_o^* = 0.001$ is found to be 1.0×10^{-8}. In this unheated region $0 \leq z^* \leq z_o^*$, $\Delta_t^* = 0$ (i.e., $x = 0$).

A negative or positive value of A is achieved by the use of relay R19 which switches between two alternate circuits. When the logic input to relay R19 is TRUE, the value of A is positive and for FALSE logic input, A is negative. Here the position of relay is controlled manually by push buttons, on the control panel, which supply the appropriate logic input signal. During an automated run, the relay can also be set or reset by the digital computer.

The region of interest for the variable w is from zero to its maximum value of one, (i.e., $0 \leq \Delta^* \leq 1$) and also variable x must never be greater than one because of the assumptions used when deriving the equations (i.e., $0 \leq Y^* \leq 1$). To prevent any of these variables from becoming greater than one, a logic circuit consisting of comparators C004 and C049, a synthesized OR gate and the ORH (OVER RIDE HOLD) input is made. When either w or x becomes greater than one, their respective comparators output change from FALSE to TRUE, with which the input to ORH changes from FALSE to TRUE, and the analog computer is put in the HOLD mode. This freezes the solution of problem at that point and thus prevents the computer from overloading.

Amplifier A030 as shown in Fig. (3-a) is used to produce a time sweep which is required when plotting results and computing the average value of Nusselt number and friction factor.

4.4 UTILIZATION OF HYTRAN OPERATIONS INTERPRETER[19]

Hytran Operations Interpreter (HOI) language has been created specifically for the preparation, setup, and execution of analog or hybrid computations. The process of computation using HOI takes the form of a dialogue between the operator and the system. Operations are initialized by loading the system program into memory. The system accepts the input from cards, paper tape, cartridge tape, magnetic tape disk, or the teletype keyboard.

An important function of HOI is to provide a method of performing a digital computer check of the analog program. The check-out procedure is a two phase operation; the first being checking out the analog circuit diagram and then, once its validity has been established, checking out the wiring on the actual analog computer.

The first phase of this procedure is an entirely digital operation and requires no interaction with an actual computer. This phase of the check-out is conducted in the OFF-LINE mode.

In order to check the actual computer wiring, it is obviously necessary to interact with the analog computer. This is achieved by entering the ON-LINE execution mode.

In order to expedite editing and documenting of the HOI program, all information required for setup and check-out is written, using standard "Part and Step" numbers that correspond to the component being set or checked by that statement.

4.4.1 OFF-LINE CHECK

Block diagram of the OFF-LINE check is shown in Table (2).

The purpose of the OFF-LINE check is to insure that the analog computer connection diagram,where the components are assigned to perform the appropriate mathematical functions, truly represents the original set of equations given in the problem statement. In the off-line mode, the variables names which refer to analog components, are treated as digital variables. The whole HOI program is executed as a conventional digital program and the computed values are stored in core. When the computer is set in the VERIFY mode as shown in (5) of the block diagram, all subsequent equations are modified to compare the value of the quantity defined on the left side of the calculated value from the expression on the right. If the absolute value of the difference exceeds the specified tolerance, an error message will result. The error message will indicate the part and step number of the statement, name and quantity of the left side, value as computed, and the value read from the variable list. When the OFF-LINE check is completed with no error message, then the analog computer diagram and the HOI program which represents the diagram are in good agreement.

4.4.2 ON-LINE CHECK

Block diagram of the ON-LINE check is shown in Table (3).

The initialization and execution of the basic ON-LINE setup and check-out of the analog program operation is performed by the statements in Part 2 of HOI program. The actual operation of the analog computing components is checked against the program as specified by the computer diagram or the original problem statement. The on-line/verify mode isolates any faults in either the patching or defective analog computer component. When on-line check is completed with no error messages, then the analog computer has been patched correctly according to the analog diagram. When many nonlinear components are used in a circuit loop as in the present problem, error messages may still be produced due to a loss of accuracy in these loops.

4.5 AUTOMATED PROGRAM OPERATION[20,21]

To eliminate manual interference of the operation of the problem, the program can be automated. Though this program could be written in HOI, Fortran is chosen here because it performs at relatively high speed and can allocate more core area when storing data.

This step involves the writing of a Fortran program which controls the analog program, reads results from the analog computer through the interface, and then calculates the local and average values of the Nusselt numbers and friction factors by a digital computer. The program written to execute the present problem is shown on Table (4). A block diagram of the first part of the program is shown on Table 5.

4.6 CONTINUOUS SYSTEM MODELLING PROGRAM (CSMP)

The IBM S/360 Continuous System Modelling Program utilizes an application oriented, input-language, designed for system-simulation work. The program provides an application-oriented language that allows these problems to be prepared directly and simply from either a block diagram representation or a set of ordinary differential equations. The program includes a basic set of functional blocks with which the components of a continuous system may be represented, and accepts application-oriented statements for defining the connections between these functional blocks. S/360 CSMP also accepts FORTRAN statements, thereby allowing the user to readily handle nonlinear and time variant problems of considerable complexity.

4.6.1 STRUCTURE OF THE MODEL

The procedural technique for using the CSMP program involves the use of three operational segments known as Initial, Dynamic and Terminal which describe the computations to be performed before, during, and after each simulation run.

INITIAL

This segment is intended exclusively for computation of initial condition values and those parameters that the user prefers to express in terms of more basic parameters. Computations are made once and only at the start of the simulation run. The use of this segment is optional and depends on the problem housekeeping needs of the program user.

DYNAMIC

The Dynamic segment is normally the most extensive in the model. It includes the complete description of the system dynamics, together with any other computations desired during the run. Functionally, the Dynamic segment is analogous to the block diagram representation or to the ordinary differential equation representation of system dynamics. Data statements can be intermixed in this segment as automatic sorting takes place during program translation. This segment is also called a parallel processing segment.

TERMINAL

The Terminal segment is used for those computations desired after completion of each run and it begins when TIME = FINTIM and the execution is procedural rather than parallel. The use of Terminal segment is optional.

All the aspects regarding the use of S/360 CSMP in obtaining a solution of a problem are described in the literature[22,23].

4.6.2 S/360 CSMP SOLUTION OF THE PROBLEM

A S/360 CSMP program to solve Eqs. (28) and (29) is rather straightforward to implement. Basically, the only functional blocks required are two integrators, a comparator and a switching function. On Table (6), a program to obtain momentum and thermal boundary layer thicknesses, local Nusselt number and local friction factor for different combinations of values of viscosity coefficient A and conductivity coefficient B, for Prandtl number 5 is given. All the values of the constants that are required for solving the equations are calculated in the Initial segment. The statement representing the differential equations are located in the Dynamic segment.

To prevent division by zero at initial conditions, the COMPARATOR function L = COMPAR (P, 0.001) – 0.5, which determines the switching point, and the INSW function XDOT = INSW (L, 0.0, X1/WW) are provided. When the value of P becomes greater than 0.001, the comparator output becomes HIGH and the value of XDOT is determined from the relation XDOT = X1/WW; otherwise XDOT is equal to zero. These two functional blocks serve a similar function as the D/A switches and comparator in the analog computer circuit. In order to compare the S/360 CSMP results with the ones obtained from hybrid computer, the same switching point as taken in the hybrid computer solution is used. An executional statement FINISH is used to terminate the run when any one of the variable w and x becomes equal to one. Typical CSMP output plots are shown in Figs. (19 ~ 46).

CHAPTER 5

DISCUSSIONS AND RESULTS

5.1 HEAT TRANSFER AND FLUID DYNAMICS RESULTS

A number of runs were obtained for the combination of various parameters. A comparison for constant properties case (i.e. A=0.0, B=0.0) between the results obtained by the variational method, CSMP and those obtained by the integral method and the one available in the literature is shown in Figs. (47), (48) and (49). The figures show a very close agreement in all the cases. Table (7) shows the comparison of hydrodynamic entrance length obtained from the present analysis with the others available in the literature.

For the case of variable properties, there are no suitable solutions available in literature for comparison. However, the results from this study agree qualitatively with the work for flow through a pipe by Yang[26], flow over a flat plate by Su[27], and flow between parallel plates by Dumouchel[28].

From the results shown graphically on Figs. (4~49), the following observations can be made for the flow in the entrance region of a circular tube:

I $\quad$ For constant Pr_o and B.

1. The entrance length, based on momentum boundary layer thickness, increases with decreasing viscosity coefficient A.

2. There is a small increase in the local Nusselt number for all z*, as the viscosity coefficient A decreases. A similar behaviour is observed for the average Nusselt number but its value is higher than the local value for all z*. This is due to the fact that positive A indicates cooling of liquid in the flow, thus corresponding increase in viscosity near the wall slows down the flow, results in a lower heat transfer relative to the constant properties case.

3. There is a large decrease in the value of the local friction factor for all z* as the viscosity coefficient A decreases. A similar behaviour is observed for the average friction factor but its value is always larger than the local value for all z* due to the similar reason as above.

II $\quad$ For constant Pr_o and A.

The local and average Nusselt numbers increase with the increase in B while the effect on the local and average friction factor is negligible for all z*.

III For constant A and B.

The effect on the local and average friction factors with the change of Prandtl number is negligible for all z* while the local and average values of Nusselt numbers increase as the Prandtl number increases for all z*.

The variational method used in this analysis provides a means of determining effects of variable physical properties on the flow of the fluid in the entrance region of a circular tube. The results from this study indicate that the effects of thermal conductivity and viscosity variation on the laminar flow of fluid in the circular tube are slight to moderate for heat transfer and more severe for wall friction. To engineers accustomed to working with constant property analysis, the results can be surprising and they show that the constant property idealization may lead to either dangerous or conservative design, depending on the application.

It is believed that the results can be improved by assuming a modified velocity and temperature profiles and using more realistic expressions of viscosity and thermal conductivity.

5.2 USE OF COMPUTER TECHNIQUE

Among the methods used for the solution of the present problems, it is clear that CSMP provides the easiest method for solving ordinary nonlinear differential equations. For a person with little knowledge of Fortran and analog computers, it can be learned quickly and it prevents the user from going into the details of numerical methods involved for solving nonlinear differential equations..

On the other hand, learning the techniques and use of analog/hybrid computer requires much more time, but its advantage over CSMP becomes more evident when one simulates a physical system involving various parameters instead of just solving differential equations. With the use of display unit, one can immediately study the effects of changing the problem parameters and can have a greater feel of the system under simulation.

In conclusion, the S/360 CSMP is best suited for problems with small number of parameters, whereas analog/hybrid solution is best suited for solving complex simulation problems involving large number of parameters.

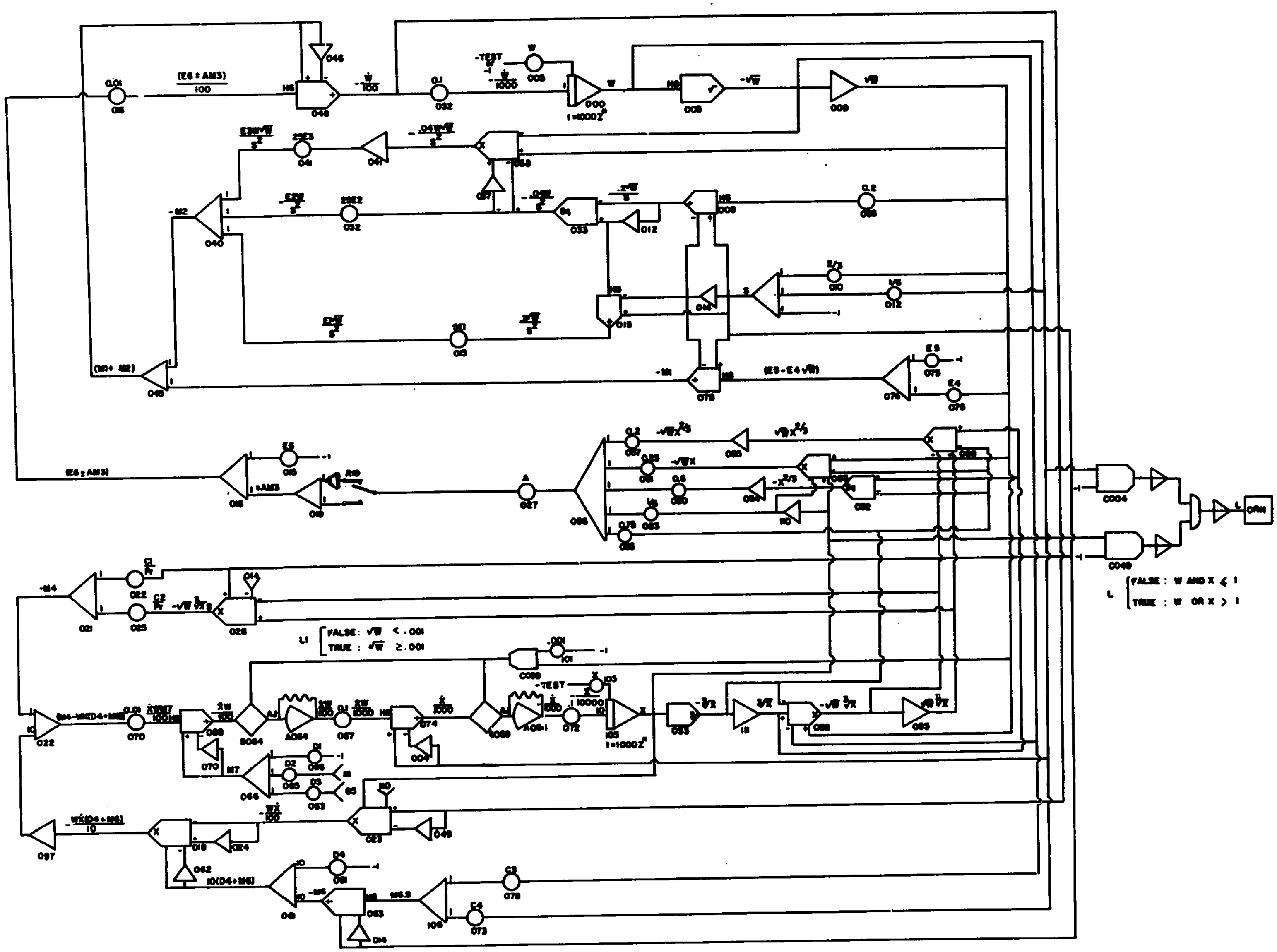


FIG. 3 SCALED ANALOG COMPUTER CIRCUIT DIAGRAM WITH ASSOCIATED AMPLIFIERS

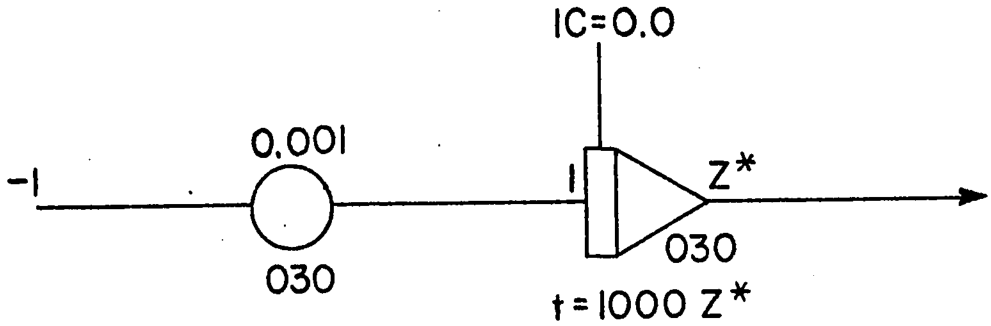


FIG. 3-a

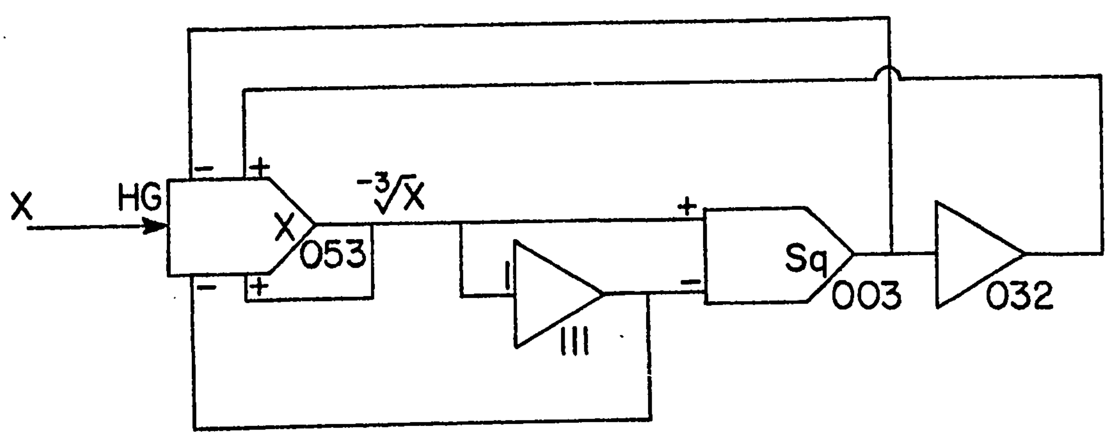


FIG. 3-b CUBE ROOT CIRCUIT

TABLE 1. THE HOI PROGRAM

```
1.010        "APPLICATION OF A HYBRID COMPUTER TECHNIQUE"
1.020        "AND VARIATIONAL METHOD TO SOLVE THE FLOW IN"
1.030        "THE ENTRANCE REGION OF A CIRCULAR TUBE WITH"
1.040        "VARIABLE PHYSICAL PROPERTIES"
1.050        "OFF LINE CHECK":
1.060        @NA, XMODE: NORMAL:
1.070        "ESTABLISH VARIABLES" 12: 13;
1.080        "COMPUTE THEORETICAL VALUES"
1.090        21; 22; 23; 31;
1.100        "OFF LINE CHECK OF DESIGN"
1.110        0.00001, VERIFY; 32; 33;
1.120        "OFF LINE CHECK COMPLETE":
1.130       2.

2.010        "ON LINE CHECK":
2.020        @WA, XMODE: NORMAL:
2.030        "SET UP CONSOLE"680, CONSOLE: 1, USE: 1, CONSOLE;
2.040        "SET POTS"@SP, MODE: 21: @PC, MODE:
2.050        "CHECK POTS" 0.0002, VERIFY: 21: "POTS OK? : HALT:
2.060        "VERIFY CONNECTIONS"0.0005, VERIFY; @ST, MODE:
2.070        31; 32; 33;
2.080        0.005, VERIFY;
2.090        "GO FOR STATIC CHECK"22; 23;
2.100        "STATIC CHECK COMPLETE"
2.110        NORMAL; HALT;
2.120       3.

3.010        "RUN PHASE":
3.020        @WA, XMODE: NORMAL;
3.030        PR =2.5
3.040        @SP, M; 21.025; 21.022: @PC, M; HALT;

12.001       "PARAMETERS"
12.002        PR =2.5
12.003        A =0.0
12.004        B =0.0
12.005        C1=2.1+(1239.0/640.0)*B
12.006        C2=0.3*B
12.007        C3=7.0/256.0
12.008        C4=7.0/512.0
12.009        D1=7.0/128.0
12.010        D2=1.0/60.0
12.011        D3=1.0/30.0
12.012        D4=21.0/512.0
12.013        E1=13.0/756.0
12.014        E2=0.014778
12.015        E3=187.0/60480.0
12.016        E4=49.0/10080.0
12.017        E5=17.0/2520.0
12.018        E6=2.0/3.0
12.019        "INITIAL CONDITIONS"
12.020        W=0.25
12.021        X=0.027
```

```
13.001      "DIFFERENTIAL EQUATIONS"
13.002      P=SQR(W)
13.003      Q=X↑(1.0/3.0), Q1=Q↑2
13.004      R=W*P
13.005      R1=P*Q
13.006      S=1.0-(2.0/3.0)*P+(1.0/6.0)*W
13.007      S1=S↑2
13.008      M1=(E5-E4*P)/S
13.009      M2=(E1*P-E2*W+E3*R)/S1
13.010      M3=1.5*Q-1.2*Q1+(1.0/3.0)*X-0.4*P*Q1+0.5*P*X
13.011      M4=((C1-C2*R1)*S)/PR
13.012      M6=(C3*P-C4*W)/S
13.013      M7=D1-D2*Q-D3*R1
13.014      WDOT=(E6+A*M3)/(M1+M2)
13.015      XDOT=(M4-WDOT*X*(D4+M6))/(M7*W)
13.016      P<0.001? XDOT=0.0

21.001      "POT SETTINGS"
21.003      1C003=W
21.010      1C010=2.0/3.0
21.012      1C012=1.0/6.0
21.013      1C013=5.0*E1
21.015      1C015=E6
21.016      1C016=0.01
21.022      1C022=C1/PR
21.025      1C025=C2/PR
21.027      1C027=A
21.030      1C030=0.001
21.032      1C032=25.0*E2
21.035      1C035=0.2
21.041      1C041=25.0*E3
21.052      1C052=0.1
21.055      1C055=0.15
21.057      1C057=0.4
21.061      1C061=D4
21.063      1C063=D3
21.065      1C065=D2
21.066      1C066=D1
21.067      1C067=0.1
21.070      1C070=0.01
21.072      1C072=0.1
21.073      1C073=C4
21.075      1C075=E5
21.076      1C076=E4
21.078      1C078=C3
21.080      1C080=0.12
21.081      1C081=0.5
21.083      1C083=1.0/3.0
21.101      1C101=0.001
21.105      1C105=X

22.001      "DERIVATIVES THEORETICAL INPUTS"
22.002      1D000=-WDOT/1000
22.003      1D105=-XDOT/1000
```

```
23.001        "THEORETICAL AMPLIFIERS OUTPUT"
23.002        1A000=W
23.003        1A003=-P
23.004        1A004=-W
23.009        1A009=P
23.010        1A010=S
23.012        1A012=0.2*P/S
23.014        1A014=-S
23.015        1A015=0.2*P/S1
23.016        1A016=WDOT*(M1+M2)
23.018        1A018=-WDOT*X*(D4+M6)/10
23.020        1A020=-A*M3
23.021        1A021=-M4
23.022        1A022=XDOT*W*M7
23.023        1A023=-X*WDOT/100
23.024        1A024=X*WDOT/100
23.028        1A028=-R1*S
23.032        1A032=(-1.0)*X↑(2.0/3.0)
23.033        1A033=-0.04*W/S1
23.034        1A034=X↑(2.0/3.0)
23.037        1A037=0.04*W/S1
23.038        1A038=-0.04*R/S1
23.040        1A040=-M2
23.041        1A041=0.04*R/S1
23.043        1A043=-R1
23.045        1A045=M1+M2
23.046        1A046=-M1-M2
23.048        1A048=-WDOT/100
23.049        1A049=WDOT/100
23.053        1A053=-Q
23.055        1A055=P*Q↑2
23.056        1A056=M3
23.058        1A058=-0.2*P/S
23.061        1A061=10.0*(D4+M6)
23.062        1A062=-10.0*(D4+M6)
23.063        1A063=-M6
23.064        1A064=(XDOT*W/100)*((SGN(P-0.001)+1)/2)
23.066        1A066=M7
23.068        1A068=-XDOT*W/100
23.069        1A069=-(XDOT/1000)*((SGN(P-0.001)+1)/2)
23.070        1A070=-M7
23.074        1A074=XDOT/1000
23.076        1A076=E5-E4*P
23.078        1A078=-M1
23.083        1A083=-P*X
23.085        1A085=R1
23.088        1A088=-P*Q↑2
23.097        1A097=WDOT*X*(D4+M6)/10
23.105        1A105=X
23.106        1A106=M6*S
23.110        1A110=-X
23.111        1A111=Q
```

```
31.001          "POT OUTPUT"
31.003          1P003=1C003*(-1)
31.010          1P010=1C010*1A009
31.012          1P012=1C012*1A004
31.013          1P013=1C013*1A015
31.015          1P015=1C015*(-1)
31.016          1P016=1C016*1A016
31.022          1P022=1C022*1A010
31.025          1P025=1C025*1A028
31.027          1P027=1C027*1A056
31.032          1P032=1C032*1A033
31.035          1P035=1C035*1A009
31.041          1P041=1C041*1A041
31.052          1P052=1C052*1A048
31.055          1P055=1C055*1A053
31.057          1P057=1C057*1A055
31.061          1P061=1C061*(-1)
31.063          1P063=1C063*1A085
31.065          1P065=1C065*1A111
31.066          1P066=1C066*(-1)
31.067          1P067=1C067*1A064
31.070          1P070=1C070*1A022
31.072          1P072=1C072*1A069
31.073          1P073=1C073*1A000
31.075          1P075=1C075*(-1)
31.076          1P076=1C076*1A009
31.078          1P078=1C078*1A003
31.080          1P080=1C080*1A034
31.081          1P081=1C081*1A083
31.083          1P083=1C083*1A110
31.101          1P101=1C101*(-1)
31.105          1P105=1C105*(-1)

32.001          "DERIVATIVE CONNECTIONS"
32.002          1D000=1P052
32.105          1D105=10*1P072

33.001          "AMPLIFIER CONNECTIONS"
33.002          1A000=-1P003
33.003          1A003=-1*(1A000↑.5)
33.004          1A004=-1A000
33.009          1A009=-1A003
33.010          1A010=1.0-1P010-1P012
33.012          1A012=-1A058
33.014          1A014=-1A010
33.015          1A015=1A012/1A010
33.016          1A016=-1P015-1A020
33.018          1A018=-1A024*1A061
33.020          1A020=-1P027
33.021          1A021=-1P025-1P022
33.022          1A022=-1A021-10*1A097
33.023          1A023=-1A105*1A049
33.024          1A024=-1A023
33.028          1A028=-1A085*1A010
```

```
33.032        1A032=-1*(1A111↑2)
33.033        1A033=-1*(1A012↑2)
33.037        1A037=-1A033
33.038        1A038=-1A037*1A009
33.040        1A040=-1P041-1P032-1P013
33.041        1A041=-1A038
33.043        1A043=-1A111*1A009
33.045        1A045=-1A040-1A078
33.046        1A046=-1A045
33.048        1A048=-1P016/1A045
33.049        1A049=-1A048
33.053        1A053=-1*(1A105↑(1/3))
33.055        1A055=-1A088
33.056        1A056=-10*(1P055+1P080)-1P057-1P083-1P081
33.058        1A058=-1P035/1A010
33.061        1A061=-10*(1P061+1A063)
33.062        1A062=-1A061
33.063        1A063=-1A106/1A010
33.064        1A064=-(1A068/2+1A068*((SGN(P-0.001)/2)))
33.066        1A066=-1P066-1P065-1P063
33.068        1A068=-1P070/1A066
33.069        1A069=-(1A074/2+1A074*((SGN(P-0.001)/2)))
33.070        1A070=-1A066
33.074        1A074=1P067/1A000
33.076        1A076=-1P075-1P076
33.078        1A078=-1A076/1A010
33.083        1A083=-1A009*1A105
33.085        1A085=-1A043
33.088        1A088=-1A085*1A111
33.097        1A097=-1A018
33.105        1A105=-1P105
33.106        1A106=-1P073-1P078
33.110        1A110=-1A105
33.111        1A111=-1A053
```

←

TABLE 2. EXECUTION OF OFF-LINE CHECK

(1) Set HOI in NO-ANALOG and NORMAL modes.

↓

(2) Define values for analog program constants, parameters, initial conditions and values in equations by executing parts 12 and 13.

↓

(3) Compute potentiometer coefficients by executing Part 21.

↓

(4) By executing Parts 22, 23 and 31, compute static values for all analog components from the original equations using the values defined in (2) and (3).

↓

(5) Set HOI in VERIFY mode with tolerance of t = 0.00001.

↓

(6) By executing Parts 32 and 33, determine for each component its expected output based on the calculated static check outputs of components which are specified as inputs to this component on the analog interconnection diagram. Then compare this expected output to the calculated theoretical output from (4).

TABLE 3. EXECUTION OF ON-LINE CHECK

Set HOI in WITH ANALOG and NORMAL modes.

↓

Set Analog computer in the SET POT mode and set potentiometer coefficient by executing Part 21.

↓

Set analog computer in POT. COEFFICIENT mode and HOI in verify mode with tolerance of t = 0.00002.

↓

Verify actual potentiometer setting against desired coefficient by executing Part 21.

↓

Set analog computer in the STATIC TEST mode and HOI in VERIFY mode with tolerance of t = 0.0005.

↓

By executing Parts 31, 32 and 33, compare each component actual output with the calculated value based upon the actual outputs of components which are specified as inputs to this component on the analog interconnection diagram.

↓

Set HOI in VERIFY mode with tolerance of t = 0.005.

↓

By executing Parts 22 and 23, compare each component actual output with the theoretical output.

TABLE 4. THE FORTRAN PROGRAM

```
      DIMENSION CH(3),ZZ(600),TBL(600),RR(600),THM(600)
      REAL MBL(600)
      LOGICAL ISL
      LOGICAL LSL
C     INIALIZE LINKAGE
      CALL QSHYIN(IERR,680)
21    TYPE 1
      ACCEPT 2,A,B
1     FORMAT(30H ENTER VALUES OF A,B',/)
2     FORMAT(2F10.7)
C     SET ANALOG AND LOGIC MODES
      CALL QSSECN(IERR)
      CALL QSRUN(IERR)
      CALL QWPR(4HP027,A,IERR)
C     START SOLUTION AND SAMPLING
      CALL QSIC(IERR)
      CALL QSDLY(100)
      CALL QSOP(IERR)
      CALL QRSLL(1,ISL,IERR)
3     CALL QRSLL(1,ISL,IERR)
      IF(.NOT.ISL)GO TO 3
      ZZ(1)=0.0
      N=1
C     READ ANALOG TO DIGITAL CONVERTERS
4     CALL QRBADR(CH,1,3,IERR)
      N=N+1
      ZZ(N)=CH(1)
      MBL(N)=CH(2)
      TBL(N)=CH(3)
C     TEST FOR MBL=1
      CALL QRCPL(4,LSL,IERR)
      IF(LSL)GO TO 6
C     TEST FOR TIMING ERROR
      CALL QRSLL(1,ISL,IERR)
      IF(ISL)GO TO 22
C     NEXT PULSE
5     CALL QRSLL(1,ISL,IERR)
      IF(.NOT.ISL)GO TO 5
      GO TO 4
C     PRINTING OF MBL AND TBL
6     DO 7 J=2,N
      Z=ZZ(J)
      R=MBL(J)
7     WRITE(6,8)Z,R
8     FORMAT(2F12.4)
C     PRINT TBL VS Z
      DO 9 J=2,N
      Z=ZZ(J)
      R=TBL(J)
9     WRITE(6,10)Z,R
10    FORMAT(2F12.4)
C     PRINTING OF LOCAL AND AVERAGE NUSSELT NUMBER
      J=2
      Z=ZZ(J)
```

```
      THM(J)=(0.5-(3.0/140.0)*(TBL(J)**4)/(MBL(J)**2)+(1.0/12.0)
     C*(TBL(J)**3)/MBL(J)+(1.0/24.0)*(TBL(J)**3)/(MBL(J)**2)
     C-(1.0/5.0)*(TBL(J)**2)/MBL(J)+(1.0/12.0)*(MBL(J)**2)
     C-(1.0/3.0)*MBL(J))/(0.5+(1.0/12.0)*(MBL(J)**2)-(1.0/3.0)*MBL(J))
```

PAGE 2

```
      R=(3.0*(1.0+B))/(THM(J)*TBL(J))
      RR(J)=R
      WRITE(6,11)Z,R,THM(J)
 11   FORMAT(3F12.4)
C     LOCAL NUSSELT NUMBER VS Z
      DO 12 J=3,N
      Z=ZZ(J)
      THM(J)=(0.5-(3.0/140.0)*(TBL(J)**4)/(MBL(J)**2)+(1.0/12.0)
     C*(TBL(J)**3)/MBL(J)+(1.0/24.0)*(TBL(J)**3)/(MBL(J)**2)
     C-(1.0/5.0)*(TBL(J)**2)/MBL(J)+(1.0/12.0)*(MBL(J)**2)
     C-(1.0/3.0)*MBL(J))/(0.5+(1.0/12.0)*(MBL(J)**2)-(1.0/3.0)*MBL(J))
      R=(3.0*(1.0+B))/(THM(J)*TBL(J))
      RR(J)=R
 12   WRITE(6,13)Z,R,THM(J)
 13   FORMAT(3F12.4)
C     INTEGRATE BY TRAPEZOIDAL RULE
      TREPEZ=0.0
      K=N-1
      DO 14 J=2,K
      Z=ZZ(J+1)
      TREPEZ=TREPEZ+0.5*(RR(J)+RR(J+1))*(ZZ(J+1)-ZZ(J))
      P=ZZ(J+1)-ZZ(2)
      R=TREPEZ/P
 14   WRITE(6,15)Z,R,TREPEZ
 15   FORMAT(3F12.4)
C     PRINTING OF LOCAL AND AVERAGE FRICTION FACTOR*REYNOLD NUMBER
      J=2
      Z=ZZ(J)
      R=8.0*(1.0+A)/(MBL(J)*(1.0-(2.0/3.0)*MBL(J)+(1.0/6.0)*MBL(J)*
     CMBL(J)))
      RR(J)=R
      WRITE(6,16)Z,R
 16   FORMAT(2F12.4)
C     PLOT LOCAL FRICTION FACTOR*REYNOLD NUMBER VS Z
      DO 17 J=3,N
      Z=ZZ(J)
      R=8.0*(1.0+A)/(MBL(J)*(1.0-(2.0/3.0)*MBL(J)+(1.0/6.0)*MBL(J)*
     CMBL(J)))
      RR(J)=R
 17   WRITE(6,18)Z,R
 18   FORMAT(2F12.4)
C     INTEGRATE BY TRAPEZOIDAL RULE
      TREP=0.0
      K=N-1
      DO 19 J=2,K
```

```
      Z=ZZ(J+1)
      TREP=TREP+0.5*(RR(J)+RR(J+1))*(ZZ(J+1)-ZZ(J))
      Q=ZZ(J+1)-ZZ(2)
      R=TREP/Q
19    WRITE(6,20)Z,R,TREP
20    FORMAT(3F12.4)
      GO TO 21
22    WRITE(6,23)
23    FORMAT(20H TIMING ERROR')
      END
```

TABLE 5. BLOCK DIAGRAM OF FORTRAN PROGRAM

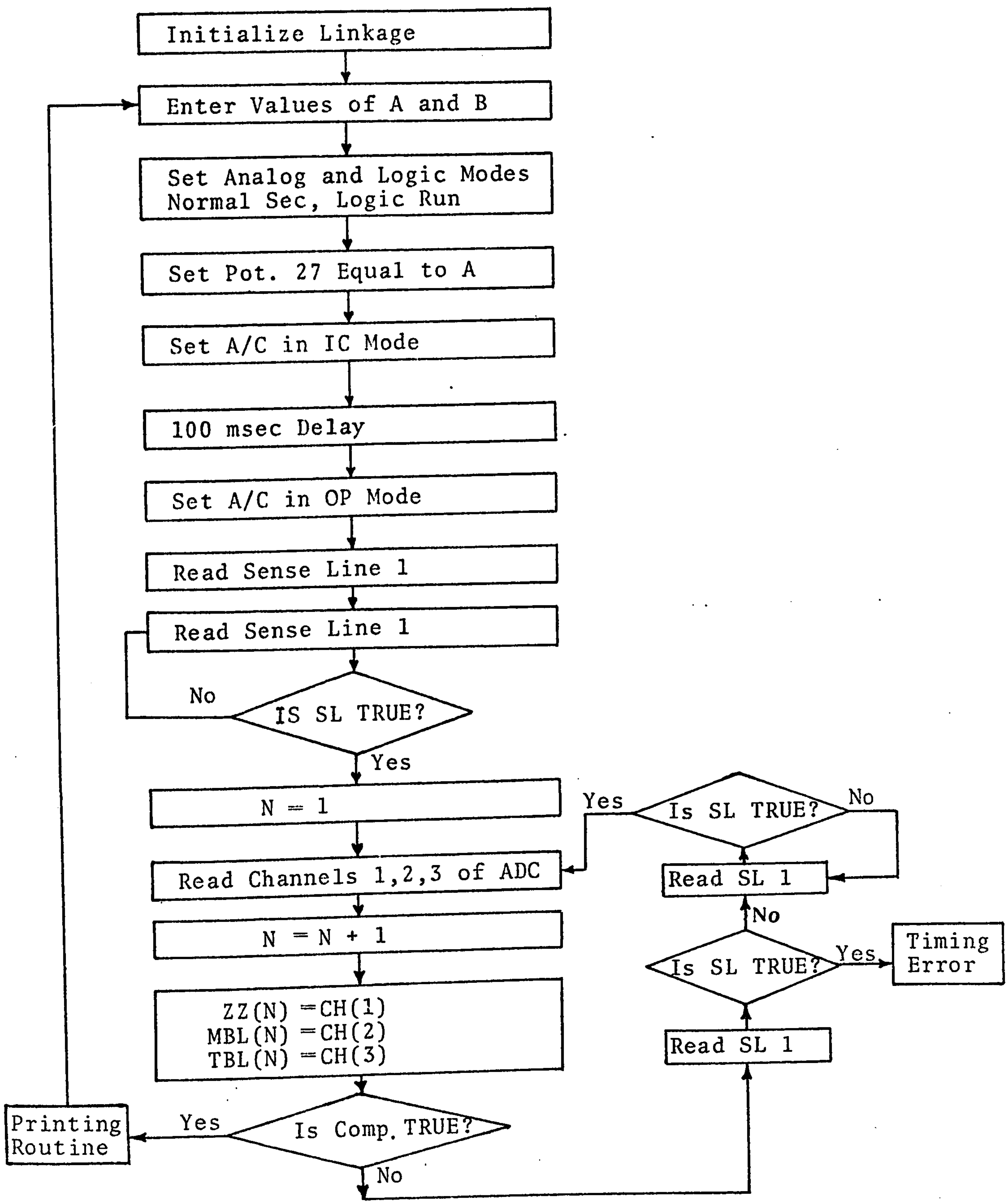


TABLE 6. THE S/360 CSMP PROGRAM

```
****CONTINUOUS SYSTEM MODELING PROGRAM****

***PROBLEM INPUT STATEMENTS***

TITLE        HEAT TRANSFER IN THE ENTRANCE REGION OF A TUBE
RENAME       TIME=Z
PARAM        PP=5.0, A=0.0, B=0.0
INITIAL
             C1=2.1+(1239.0/640.0)*B
             C2=0.3*B
             C3=7.0/256.0
             C4=7.0/512.0
             D1=7.0/128.0
             D2=1.0/60.0
             D3=1.0/30.0
             D4=21.0/512.0
             E1=13.0/756.0
             E2=0.014778
             E3=187.0/60480.0
             E4=49.0/10080.0
             E5=17.0/2520.0
             E6=2.0/3.0
DYNAMIC
             P=W**0.5
             Q=X**(1.0/3.0)
             Q1=Q**2
             R=W*P
             R1=P*Q
             S=1.0-(2.0/3.0)*P+(1.0/6.0)*W
             S1=S**2
             M1=(E5-E4*P)/S
             M2=(E1*P-E2*W+E3*R)/S1
             M3=1.5*Q-1.2*Q1+(1.0/3.0)*X-0.4*P*Q1+0.5*P*X
             WDOT=(E6+A*M3)/(M1+M2)
             W=INTGRL(0.0,WDOT)
             L=COMPAR(P,0.001)-0.5
             M4=((C1-C2*R1)*S)/PP
             M6=(C3*P-C4*W)/S
             M7=D1-D2*Q-D3*R1
             X1=(M4-WDOT*X*(D4+M6))/M7
             WW=INSW(L,1.0,W)
             XDOT=INSW(L,0.0,X1/WW)
             X=INTGRL(0.0,XDOT)
             PP=INSW(L,1.0,P)
             THN=INSW(L,1.0,(0.5-(3.0/140.0)*(R1**4)/(PP**2)+(1.0/12.0)   ...
             *(R1**3)/PP+(1.0/24.0)*(R1**3)/(PP**2)-(1.0/5.0)*(R1**2)/    ...
             PP+(1.0/12.0)*(PP**2)-(1.0/3.0)*PP)/(0.5+(1.0/12.0)*(PP      ...
             **2)-(1.0/3.0)*PP))
             L1=COMPAR(R1,0.001)-0.5
```

```
          RR=INSW(L1,1.0,R1)
          NUL=INSW(L1,60.0,3.0*(1.+B)/(THM*RR))
          FREN=INSW(L,60.0,8.0*(1.0+A)/(PP*(1.-(2./3.)*PP+(1./6.)     ...
          *(PP**2))))
TIMER     FINTIM=0.06,DELT=0.00001,OUTDEL=0.001
FINISH    W=1.0,X=1.0
METHOD    RKSFX
PRTPLT    P
LABEL     MOMENTUM BOUNDARY LAYER
PRTPLT    R1
LABEL     THERMAL BOUNDARY LAYER
PRTPLT    NUL
LABEL     LOCAL NUSSELT NUMBER
PRTPLT    FREN
LABEL     LOCAL FRICTION FACTOR*REYNOLD NUMBER
END
STOP
```

TABLE 7. HYDRODYNAMIC ENTRANCE LENGTH

	Theory	z*
1.	Schiller	0.0288
2.	Shapiro, Siegel, Kline[24]	
i)	Modified Cubic Profile	0.0300
ii)	Modified Pohlhausen Profile	0.0296
3.	Present Study	
i)	Variational Method (analog/hybrid)	0.0316
ii)	Integral Method[25]	0.0290
iii)	CSMP	0.0299

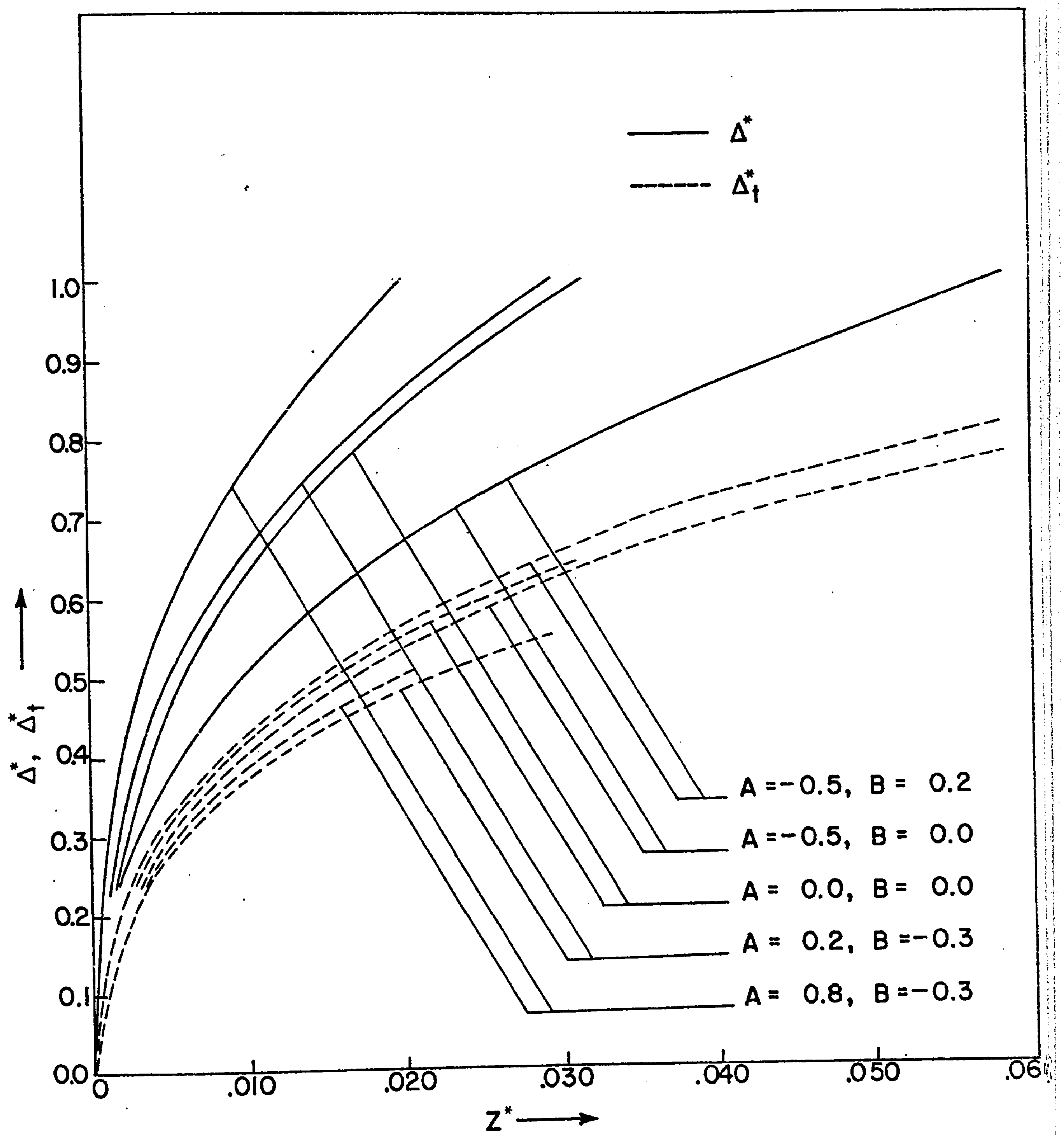


FIG. 4 MOMENTUM AND THERMAL BOUNDARY LAYER THICKNESS FOR $Pr_o = 2.5$

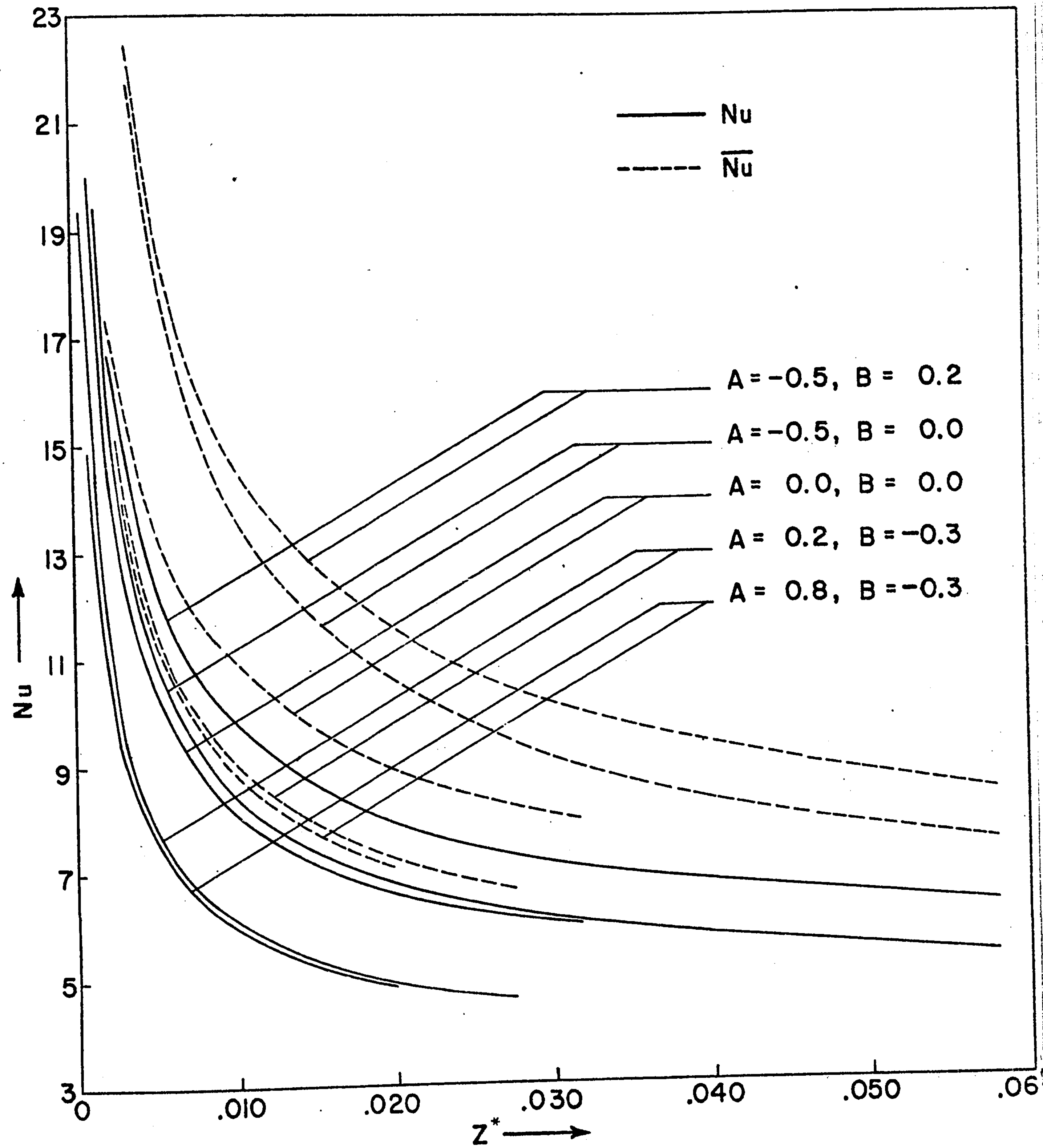


FIG. 5 LOCAL AND AVERAGE NUSSELT NUMBER FOR Pr_0 = 2.5

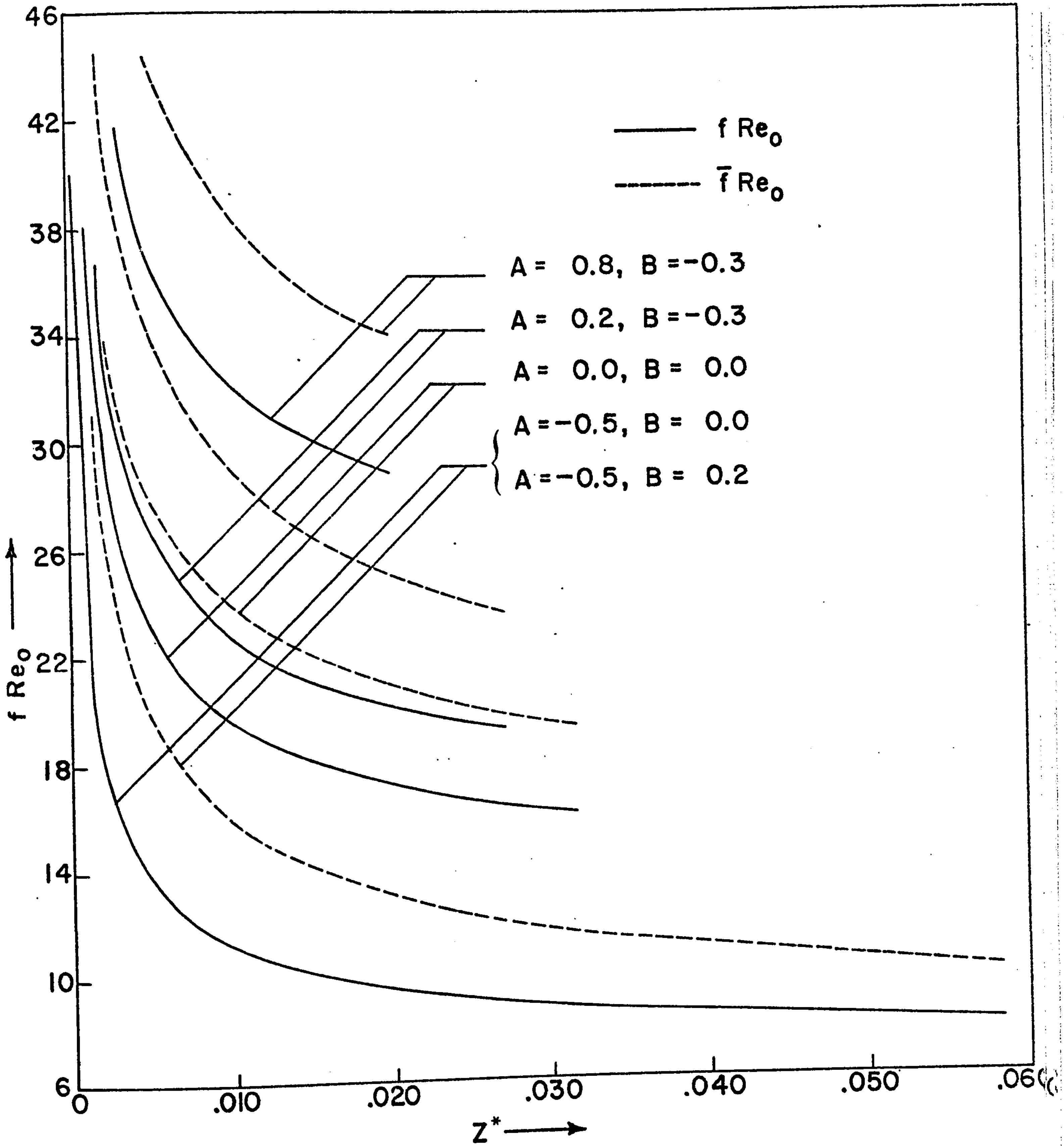


FIG. 6 LOCAL AND AVERAGE FRICTION FACTOR FOR Pr_0 = 2.5

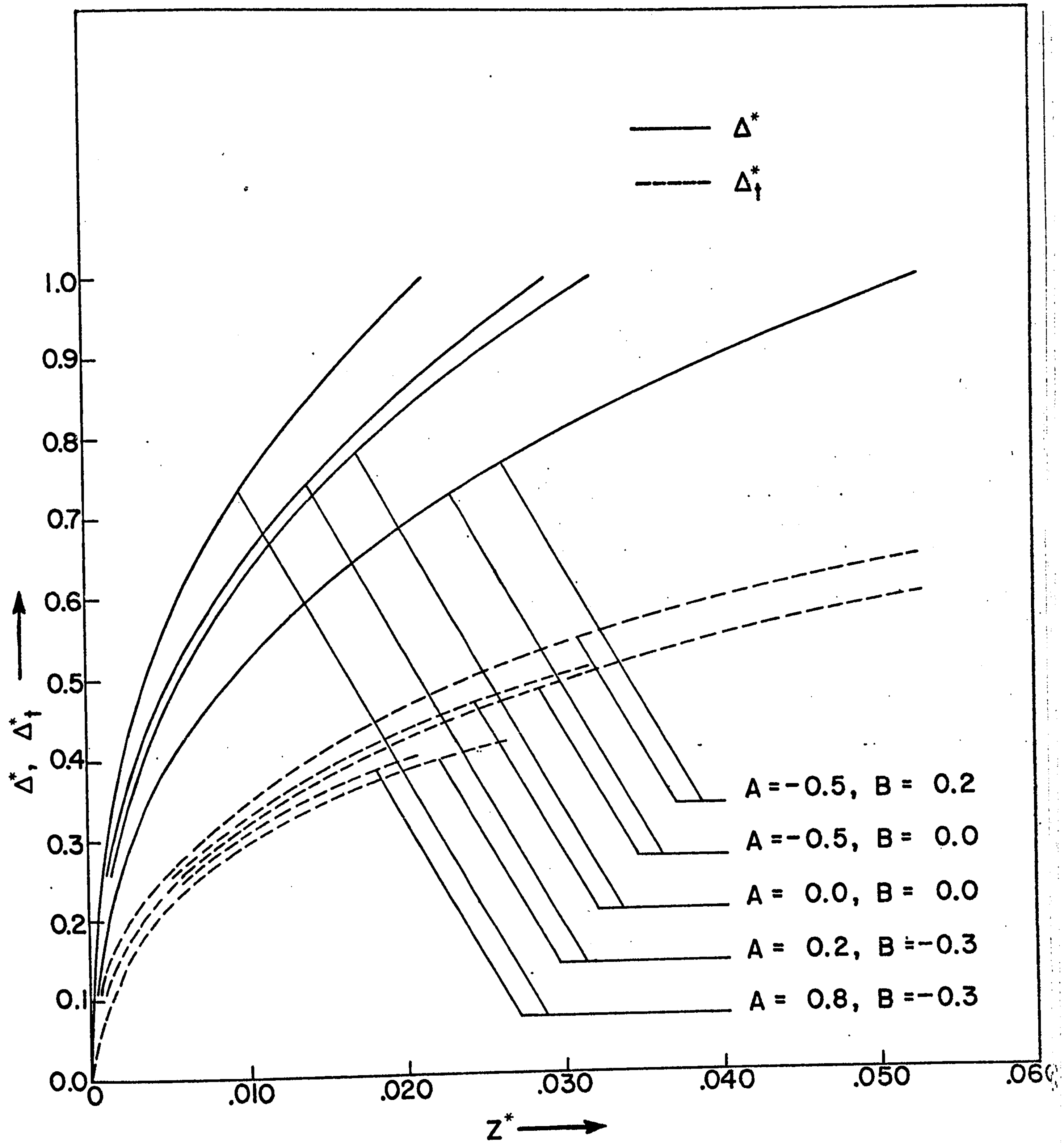


FIG. 7 MOMENTUM AND THERMAL BOUNDARY LAYER THICKNESS FOR $Pr_0 = 5.0$

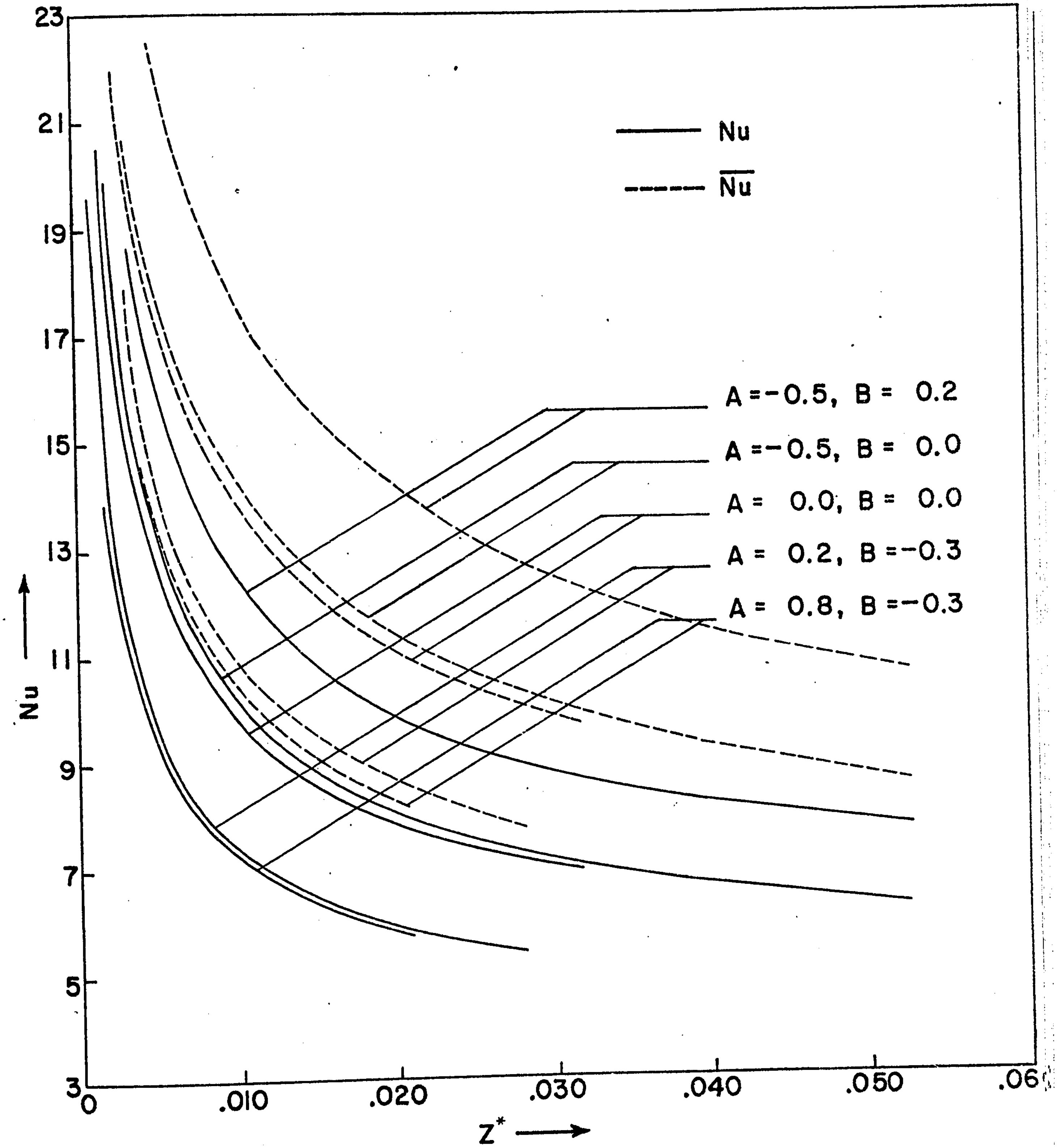


FIG. 8 LOCAL AND AVERAGE NUSSELT NUMBER FOR Pr_o = 5.0

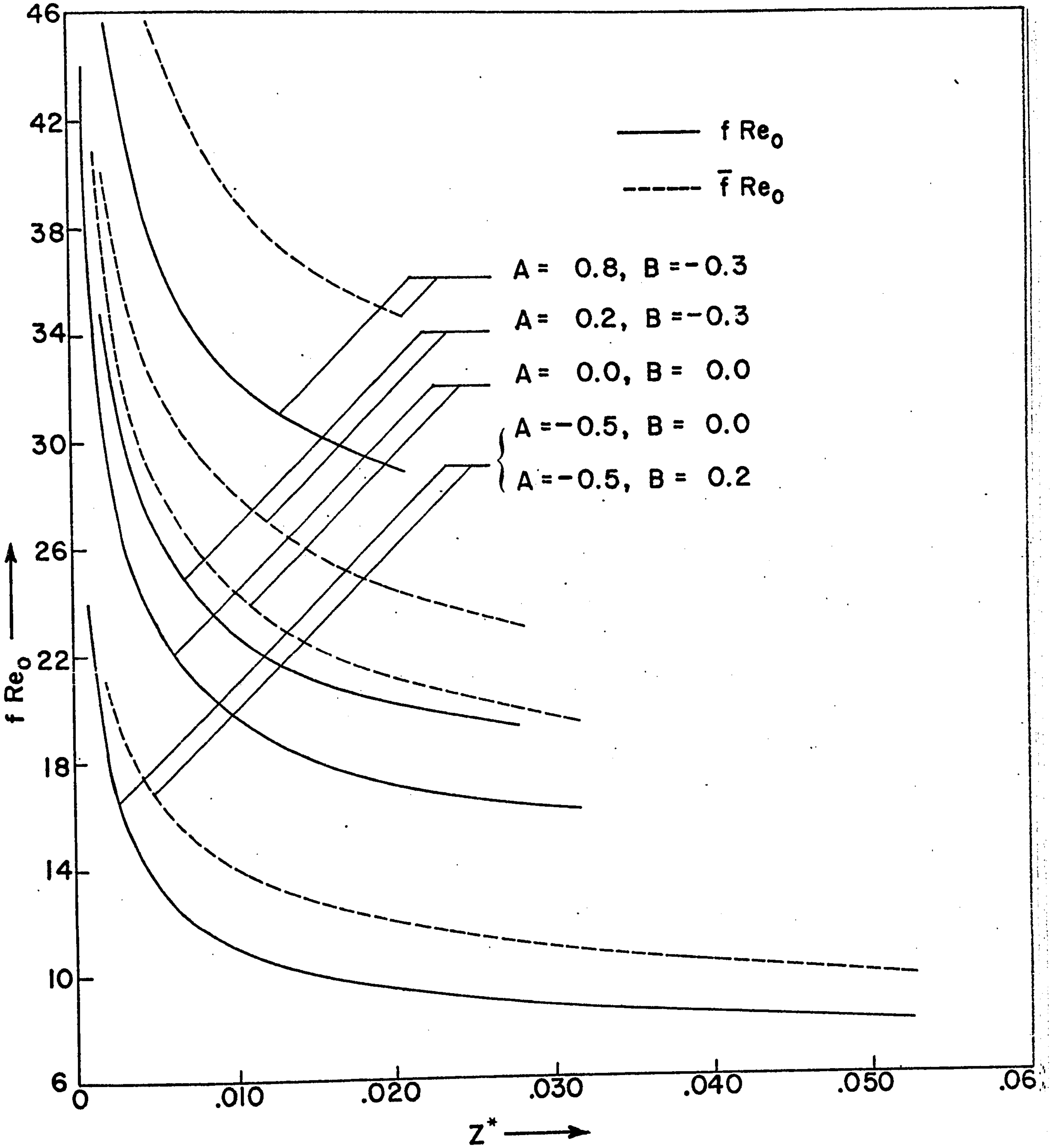


FIG. 9 LOCAL AND AVERAGE FRICTION FACTOR FOR $Pr_0 = 5.0$

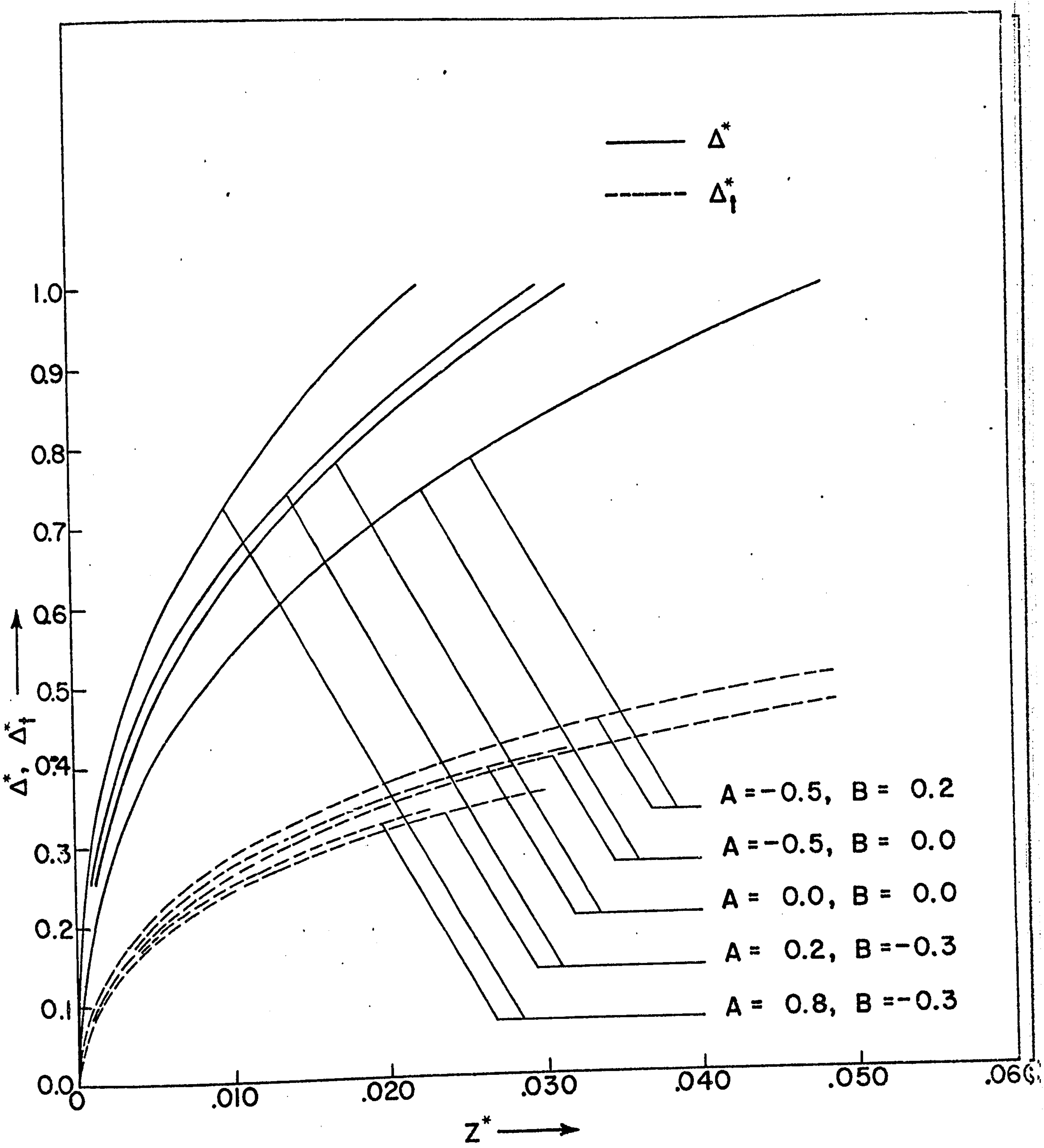


FIG. 10 MOMENTUM AND THERMAL BOUNDARY LAYER THICKNESS FOR $Pr_0 = 10.0$

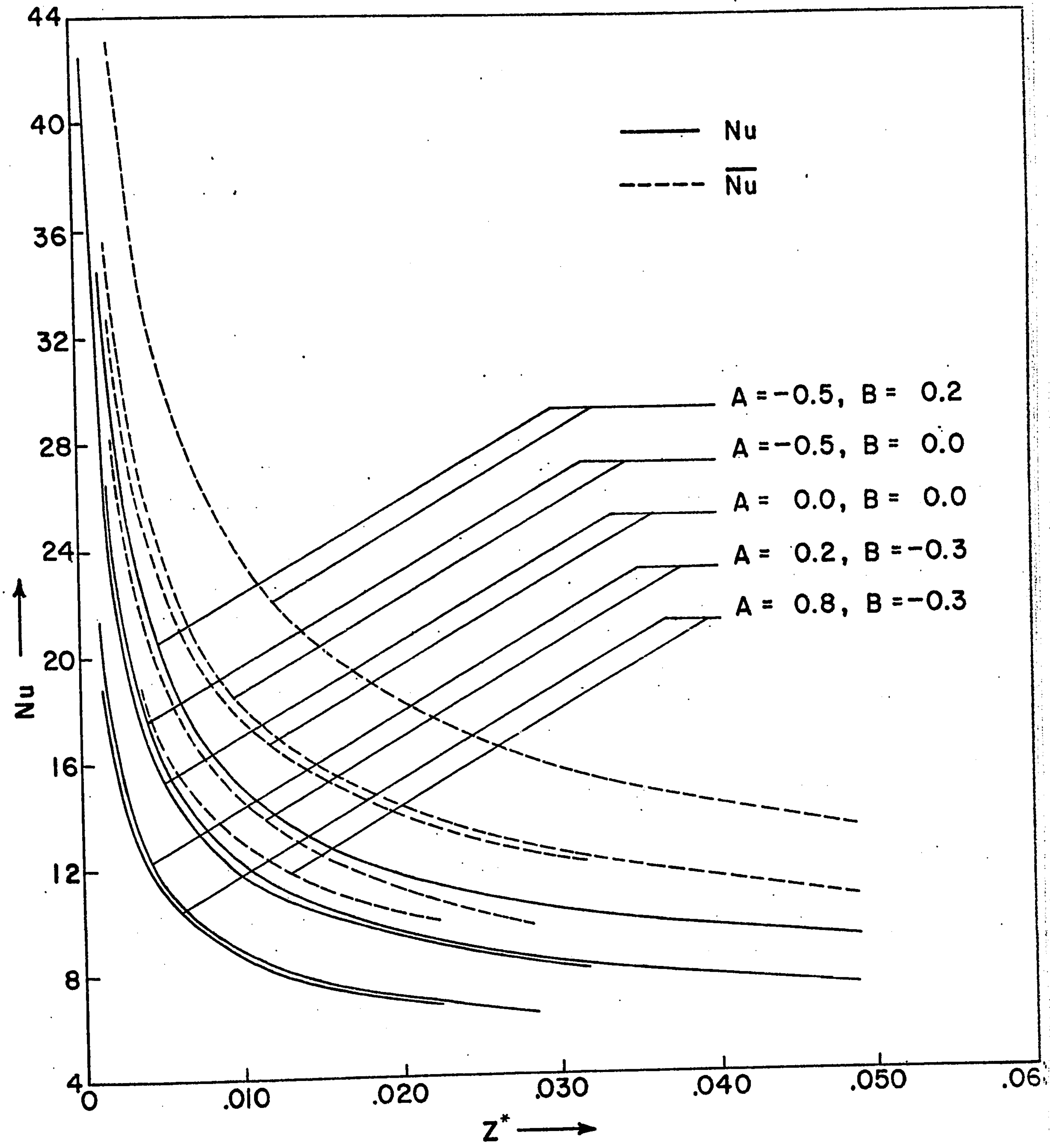


FIG. 11 LOCAL AND AVERAGE NUSSELT NUMBER FOR Pr_0 = 10.0

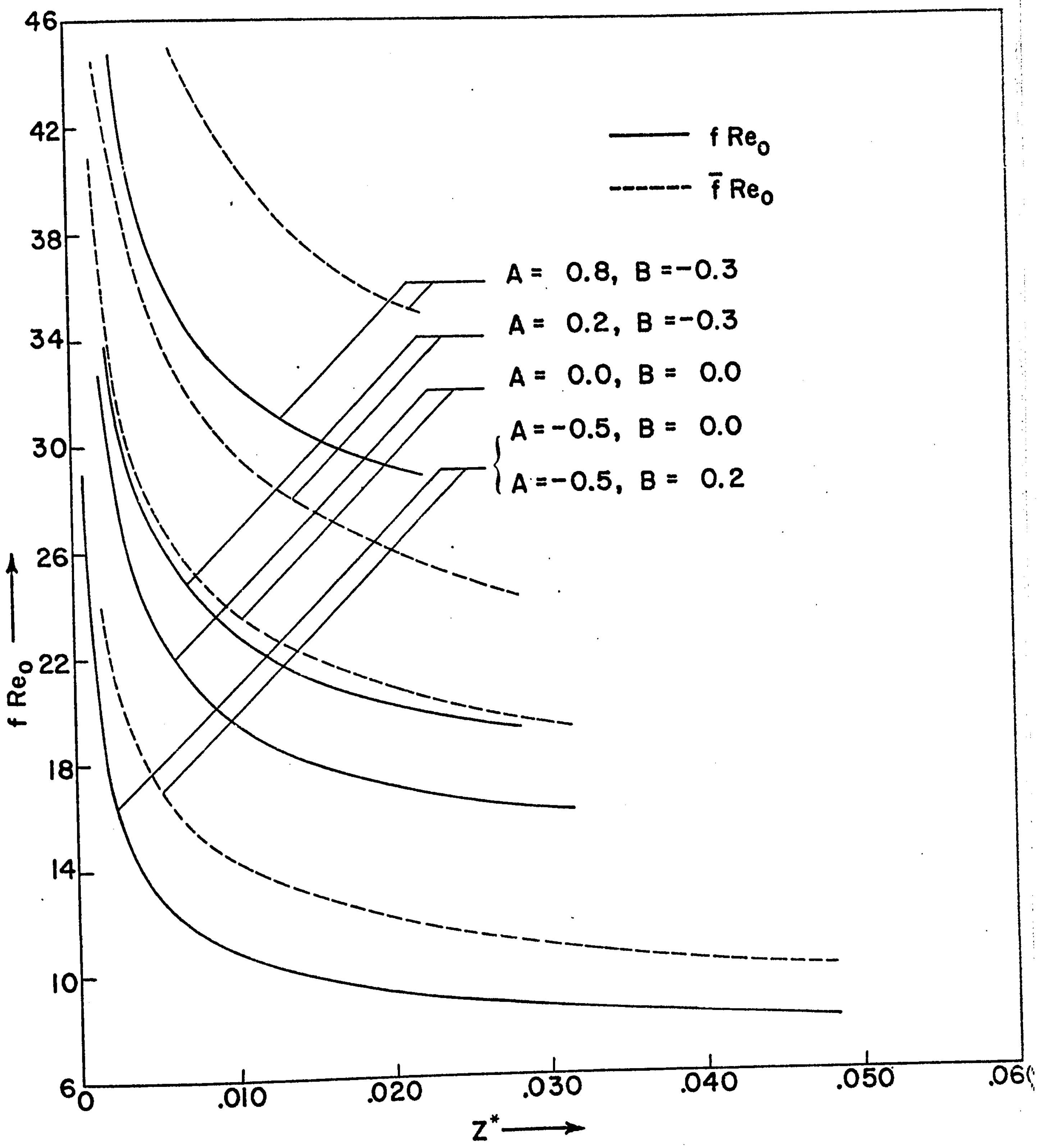


FIG. 12 LOCAL AND AVERAGE FRICTION FACTOR FOR Pr_0 = 10.0

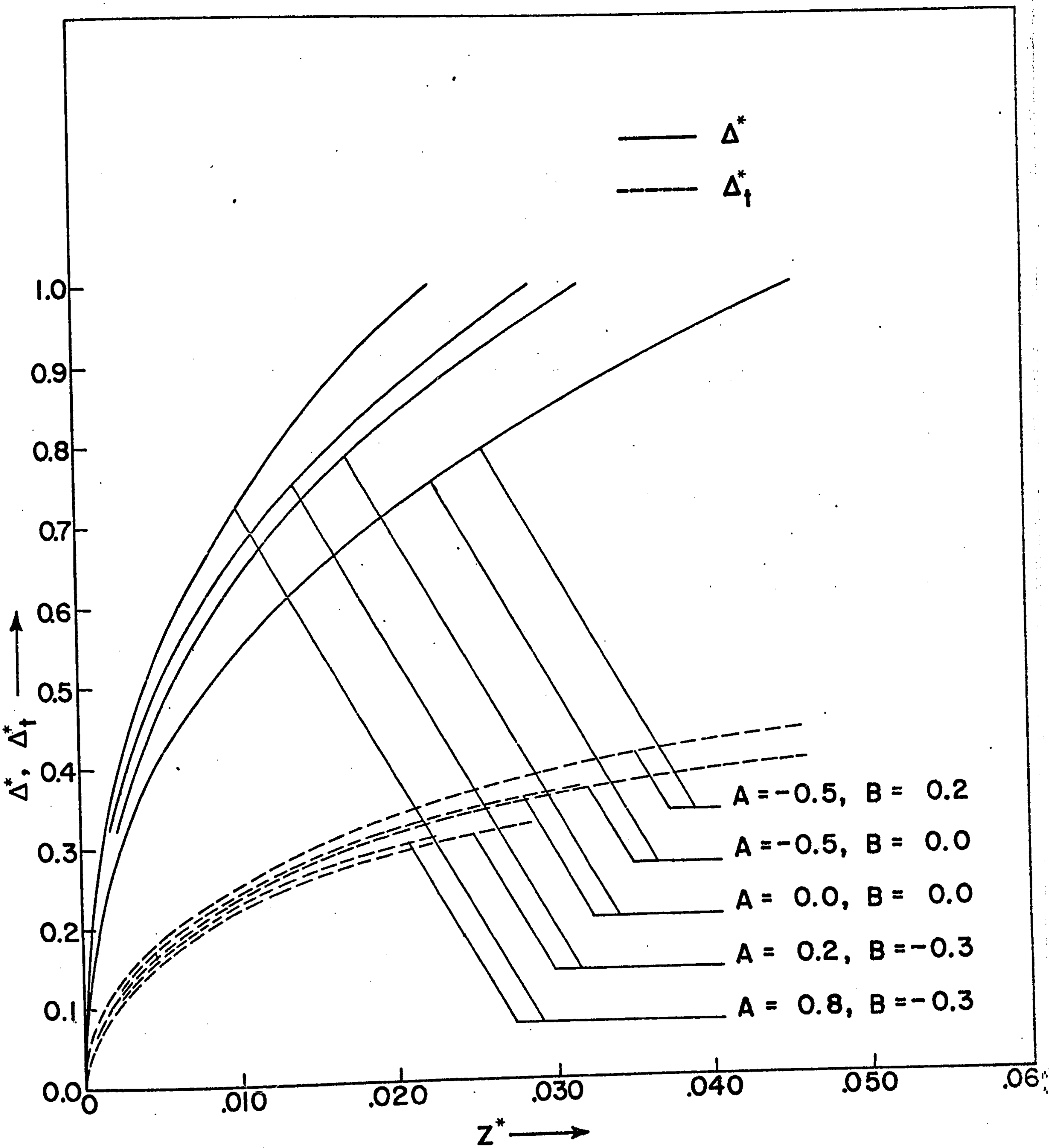


FIG. 13 MOMENTUM AND THERMAL BOUNDARY LAYER THICKNESS FOR Pr_0 = 15.0

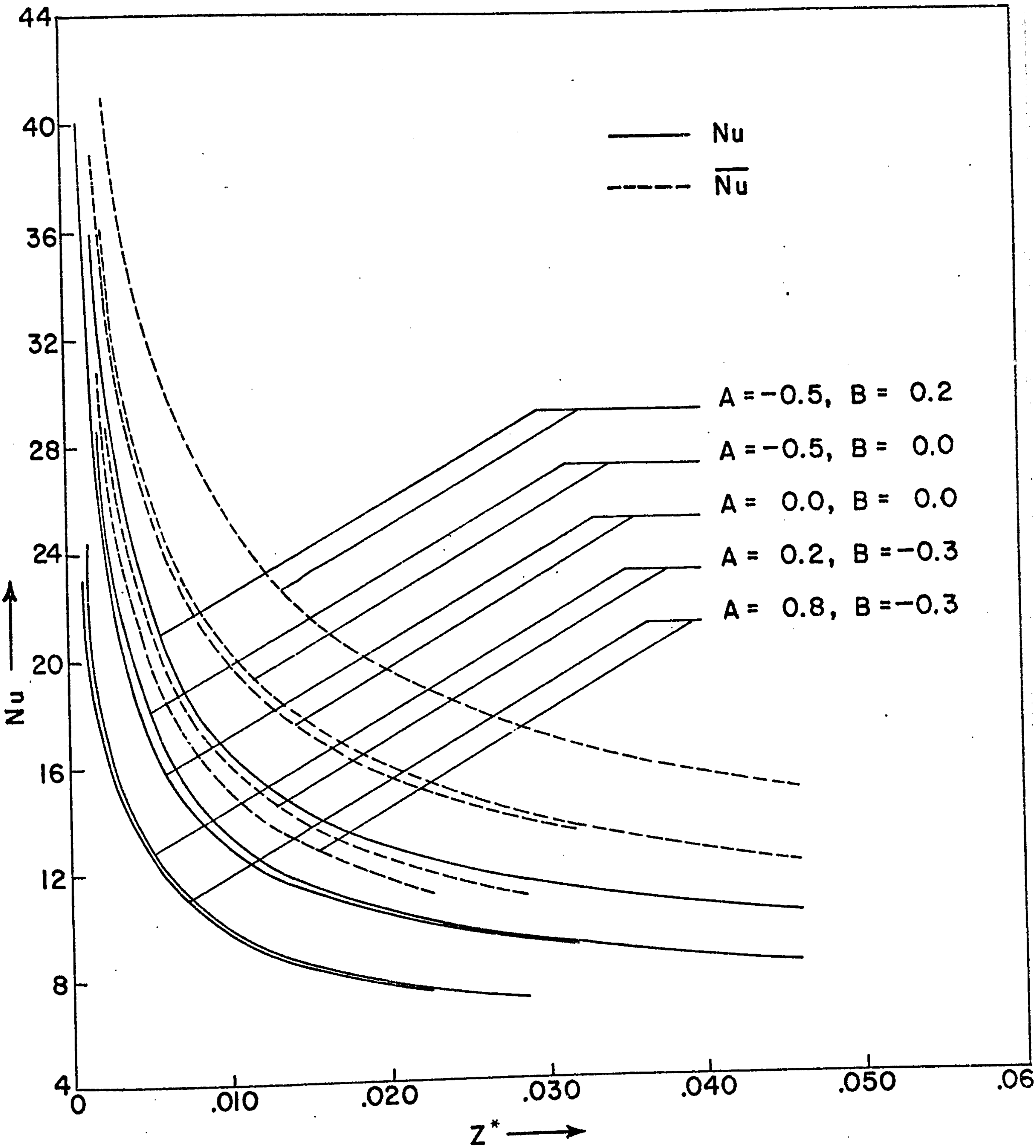


FIG. 14 LOCAL AND AVERAGE NUSSELT NUMBER FOR Pr_0 = 15.0

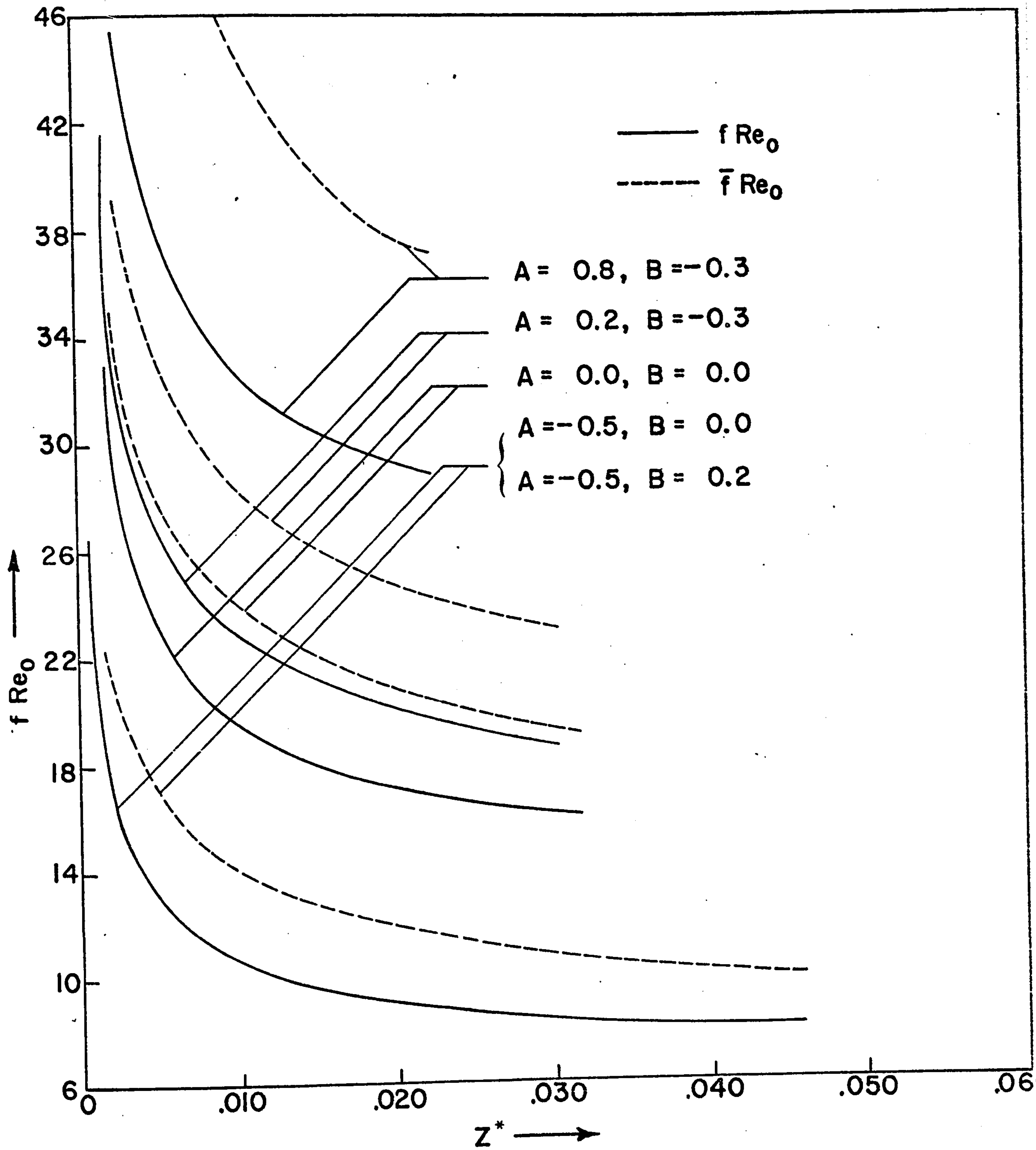


FIG. 15 LOCAL AND AVERAGE FRICTION FACTOR FOR Pr_o = 15.0

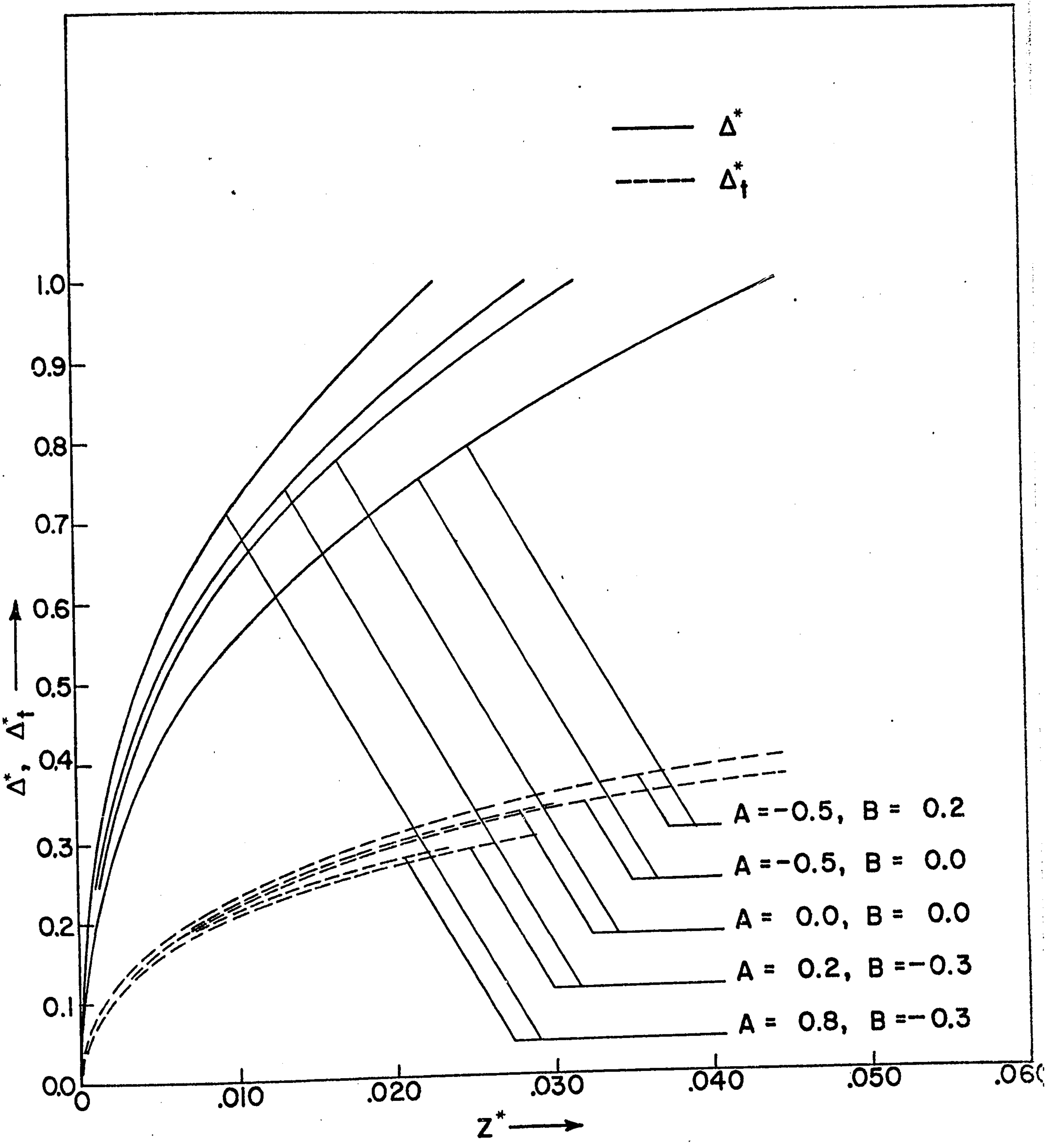


FIG. 16 MOMENTUM AND THERMAL BOUNDARY LAYER THICKNESS FOR Pr_0 = 20.0

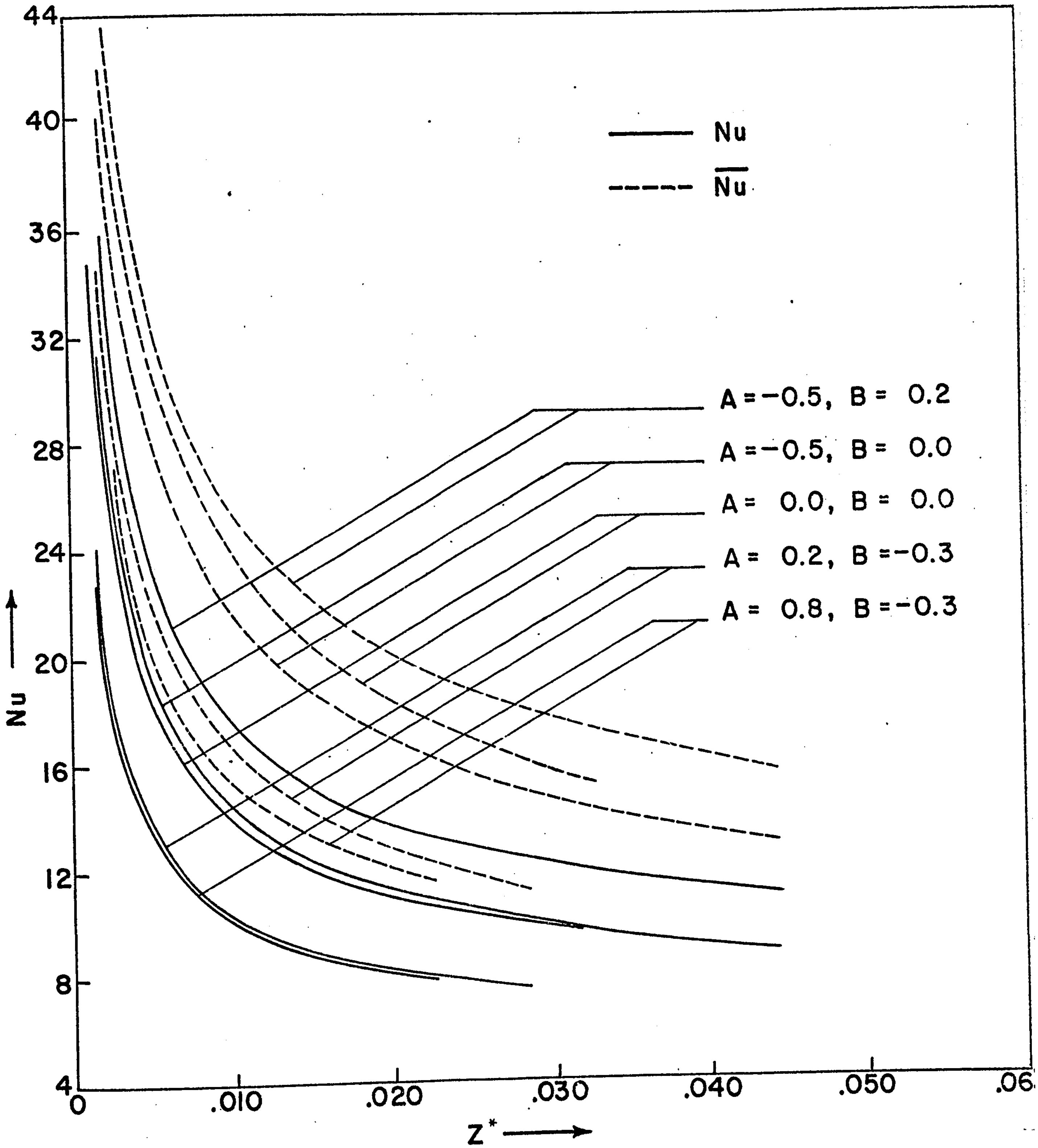


FIG. 17 LOCAL AND AVERAGE NUSSELT NUMBER FOR Pr_0 = 20.0

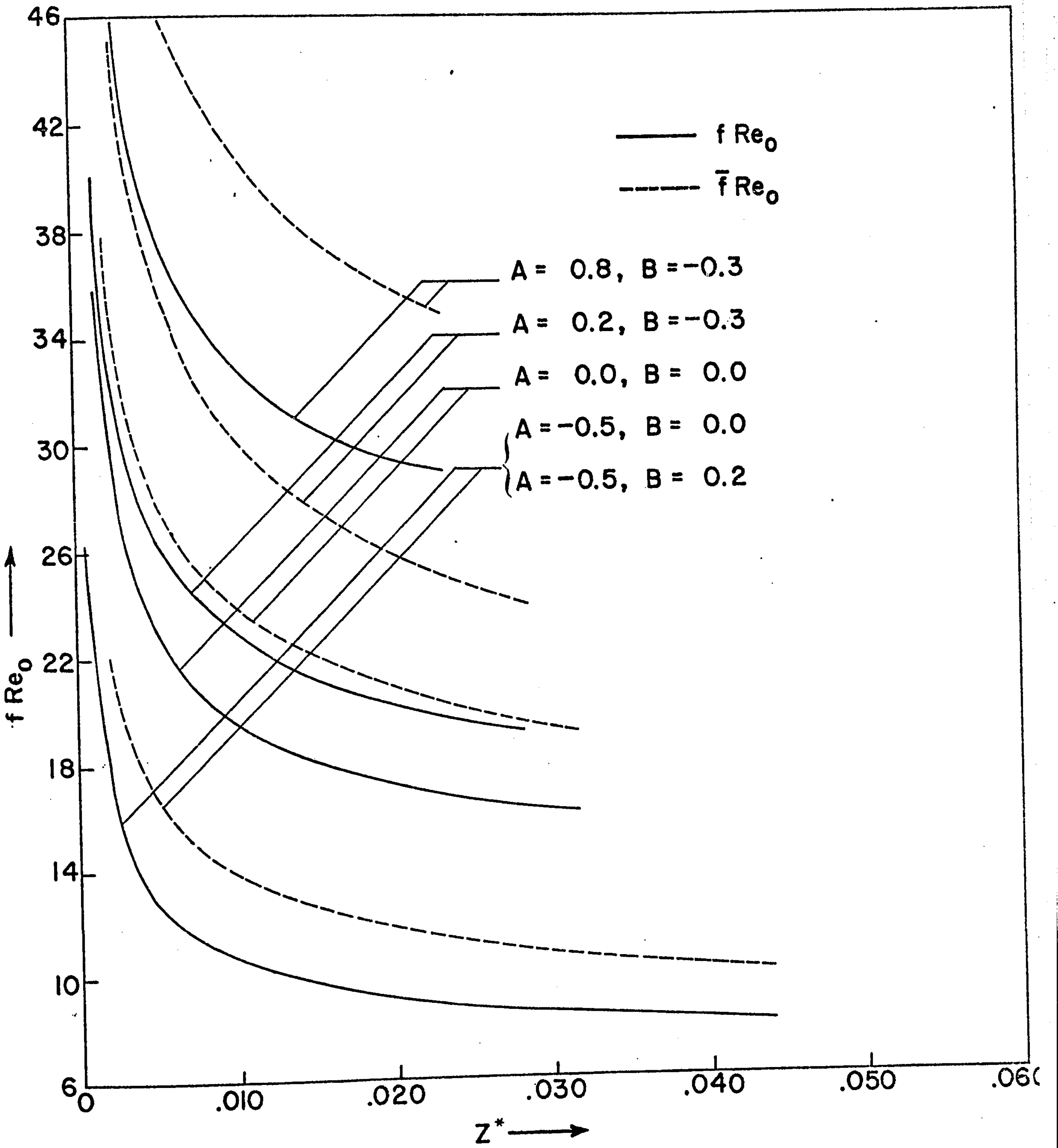


FIG. 18 LOCAL AND AVERAGE FRICTION FACTOR FOR $Pr_o = 20.0$

MOMENTUM BOUNDARY LAYER PAGE 1

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0001E 00

Z	P
0.0	0.0
1.0000E-03	2.5972E-01
2.0000E-03	3.4754E-01
3.0000E-03	4.1318E-01
4.0000E-03	4.6341E-01
5.0000E-03	5.0301E-01
6.0000E-03	5.4024E-01
7.0000E-03	5.7380E-01
8.0000E-03	6.0427E-01
9.0000E-03	6.3234E-01
1.0000E-02	6.5845E-01
1.1000E-02	6.8291E-01
1.2000E-02	7.0598E-01
1.3000E-02	7.2784E-01
1.4000E-02	7.4865E-01
1.5000E-02	7.6854E-01
1.6000E-02	7.8761E-01
1.7000E-02	8.0594E-01
1.8000E-02	8.2361E-01
1.9000E-02	8.4069E-01
2.0000E-02	8.5723E-01
2.1000E-02	8.7327E-01
2.2000E-02	8.8886E-01
2.3000E-02	9.0403E-01
2.4000E-02	9.1882E-01
2.5000E-02	9.3326E-01
2.6000E-02	9.4737E-01
2.7000E-02	9.6117E-01
2.8000E-02	9.7470E-01
2.9000E-02	9.8796E-01
2.9930E-02	1.0001E 00

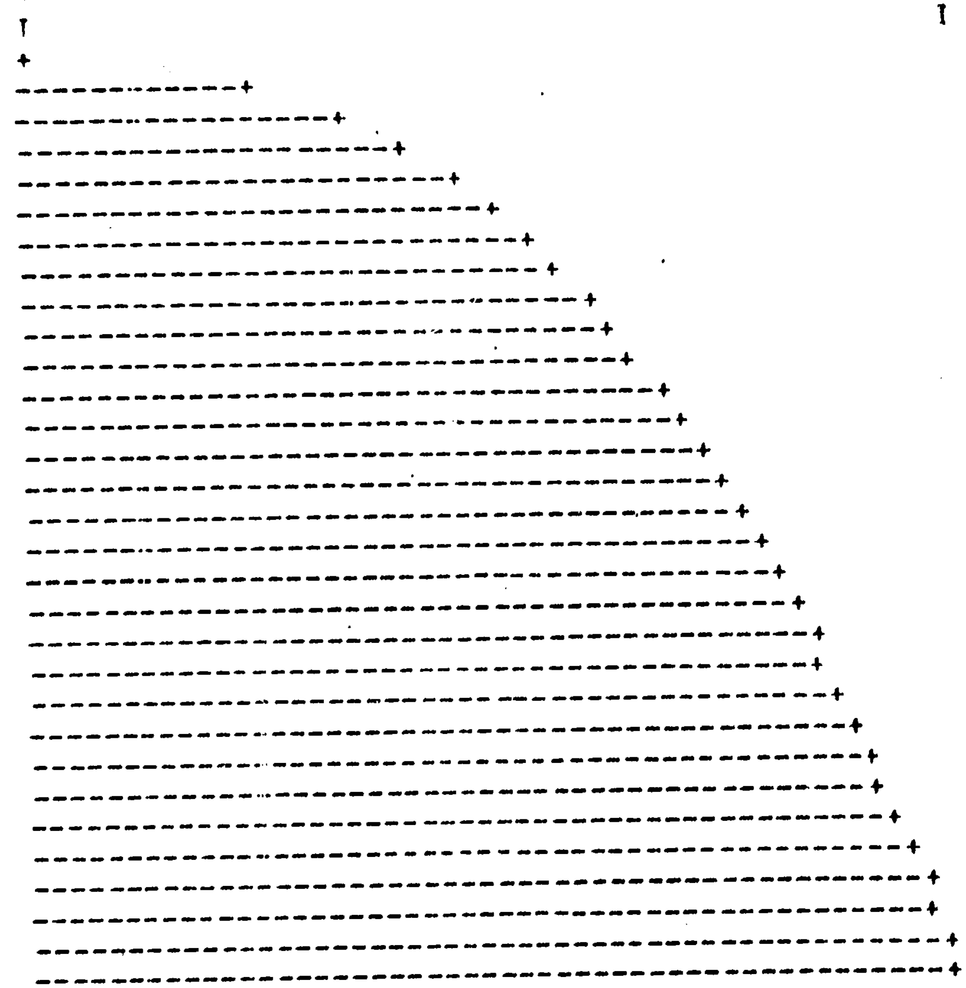


FIG. 19 MOMENTUM BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE (Pr_o = 5.0)

THERMAL BOUNDARY LAYER　　PAGE 1

R1 VERSUS Z

MINIMUM 0.0　　MAXIMUM 5.0209E-01

Z	R1
0.0	0.0
1.0000E-03	1.2200E-01
2.0000E-03	1.6951E-01
3.0000E-03	2.0307E-01
4.0000E-03	2.2987E-01
5.0000E-03	2.5250E-01
6.0000E-03	2.7226E-01
7.0000E-03	2.8990E-01
8.0000E-03	3.0588E-01
9.0000E-03	3.2055E-01
1.0000E-02	3.3412E-01
1.1000E-02	3.4677E-01
1.2000E-02	3.5864E-01
1.3000E-02	3.6982E-01
1.4000E-02	3.8041E-01
1.5000E-02	3.9047E-01
1.6000E-02	4.0005E-01
1.7000E-02	4.0922E-01
1.8000E-02	4.1800E-01
1.9000E-02	4.2643E-01
2.0000E-02	4.3454E-01
2.1000E-02	4.4235E-01
2.2000E-02	4.4990E-01
2.3000E-02	4.5719E-01
2.4000E-02	4.6425E-01
2.5000E-02	4.7110E-01
2.6000E-02	4.7774E-01
2.7000E-02	4.8419E-01
2.8000E-02	4.9046E-01
2.9000E-02	4.9656E-01
2.9930E-02	5.0209E-01

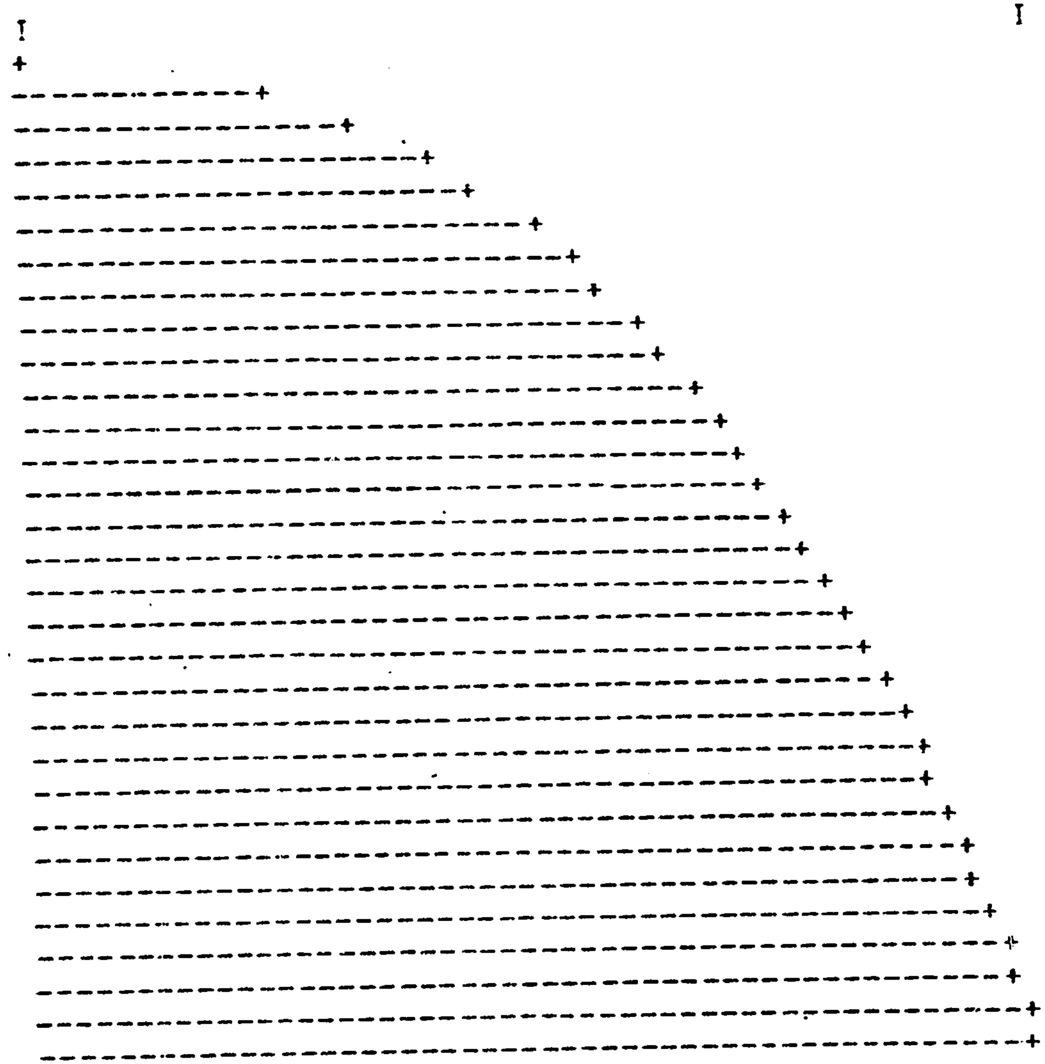


FIG. 20 THERMAL BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE ($Pr_0 = 5.0$)

```
LOCAL NUSSELT NUMBER                                                                  PAGE    1

                                 MINIMUM                 NUL     VERSUS Z                MAXIMUM
                                6.9788E 00                                              6.0000E 01
 Z                   NUL         I                                                           I
0.0                6.0000E 01   -----------------------------------------------------------+
1.0000E-03         2.5181E 01   -------------------+
2.0000E-03         1.8341E 01   -----------+
3.0000E-03         1.5451E 01   --------+
4.0000E-03         1.3755E 01   -------+
5.0000E-03         1.2608E 01   ------+
6.0000E-03         1.1766E 01   -----+
7.0000E-03         1.1113E 01   ----+
8.0000E-03         1.0589E 01   ----+
9.0000E-03         1.0155E 01   ---+
1.0000E-02         9.7895E 00   ---+
1.1000E-02         9.4754E 00   ---+
1.2000E-02         9.2019E 00   ---+
1.3000E-02         8.9611E 00   --+
1.4000E-02         8.7470E 00   --+
1.5000E-02         8.5550E 00   --+
1.6000E-02         8.3816E 00   --+
1.7000E-02         8.2241E 00   --+
1.8000E-02         8.0802E 00   --+
1.9000E-02         7.9480E 00   +
2.0000E-02         7.8262E 00   +
2.1000E-02         7.7134E 00   +
2.2000E-02         7.6086E 00   +
2.3000E-02         7.5109E 00   +
2.4000E-02         7.4195E 00   +
2.5000E-02         7.3329E 00   +
2.6000E-02         7.2535E 00   +
2.7000E-02         7.1777E 00   +
2.8000E-02         7.1062E 00   +
2.9000E-02         7.0385E 00   +
2.9930E-02         6.9788E 00   +
```

FIG. 21 LOCAL NUSSELT NUMBER FOR CONSTANT PROPERTIES CASE ($Pr_0 = 5.0$)

```
LOCAL FRICTION FACTOR*REYNOLD NUMBER                                                      PAGE    1

                                     MINIMUM               FREN   VERSUS Z                    MAXIMUM
                                    1.6000E 01                                               8.5098E 01
 Z                   FREN             I                                                          I
0.0               6.0000E 01       ----------------------------------+
1.0000E-03        3.6753E 01       -----------------+
2.0000E-03        2.9195E 01       ----------+
3.0000E-03        2.5847E 01       --------+
4.0000E-03        2.3855E 01       ------+
5.0000E-03        2.2501E 01       -----+
6.0000E-03        2.1506E 01       ----+
7.0000E-03        2.0727E 01       ----+
8.0000E-03        2.0120E 01       ---+
9.0000E-03        1.9612E 01       ---+
1.0000E-02        1.9165E 01       ---+
1.1000E-02        1.8820E 01       ---+
1.2000E-02        1.8503E 01       --+
1.3000E-02        1.8226E 01       --+
1.4000E-02        1.7980E 01       --+
1.5000E-02        1.7761E 01       --+
1.6000E-02        1.7564E 01       --+
1.7000E-02        1.7385E 01       --+
1.8000E-02        1.7223E 01        +
1.9000E-02        1.7074E 01        +
2.0000E-02        1.6938E 01        +
2.1000E-02        1.6812E 01        +
2.2000E-02        1.6695E 01        +
2.3000E-02        1.6586E 01        +
2.4000E-02        1.6485E 01        +
2.5000E-02        1.6391E 01        +
2.6000E-02        1.6302E 01        +
2.7000E-02        1.6218E 01        +
2.8000E-02        1.6140E 01        +
2.9000E-02        1.6065E 01        +
2.9930E-02        1.6000E 01        +
```

FIG. 22 LOCAL FRICTION FACTOR FOR CONSTANT PROPERTIES CASE (Pr_o = 5.0)

MOMENTUM BOUNDARY LAYER PAGE 1

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0000E 00

Z	P
0.0	0.0
2.0000E-03	2.9201E-01
4.0000E-03	3.8380E-01
6.0000E-03	4.4973E-01
8.0000E-03	5.0281E-01
1.0000E-02	5.4795E-01
1.2000E-02	5.8762E-01
1.4000E-02	6.2326E-01
1.6000E-02	6.5577E-01
1.8000E-02	6.8579E-01
2.0000E-02	7.1378E-01
2.2000E-02	7.4005E-01
2.4000E-02	7.6489E-01
2.6000E-02	7.8847E-01
2.8000E-02	8.1097E-01
3.0000E-02	8.3252E-01
3.2000E-02	8.5323E-01
3.4000E-02	8.7319E-01
3.6000E-02	8.9248E-01
3.8000E-02	9.1116E-01
4.0000E-02	9.2930E-01
4.2000E-02	9.4695E-01
4.4000E-02	9.6414E-01
4.6000E-02	9.8092E-01
4.8000E-02	9.9733E-01
4.8330E-02	1.0000E 00

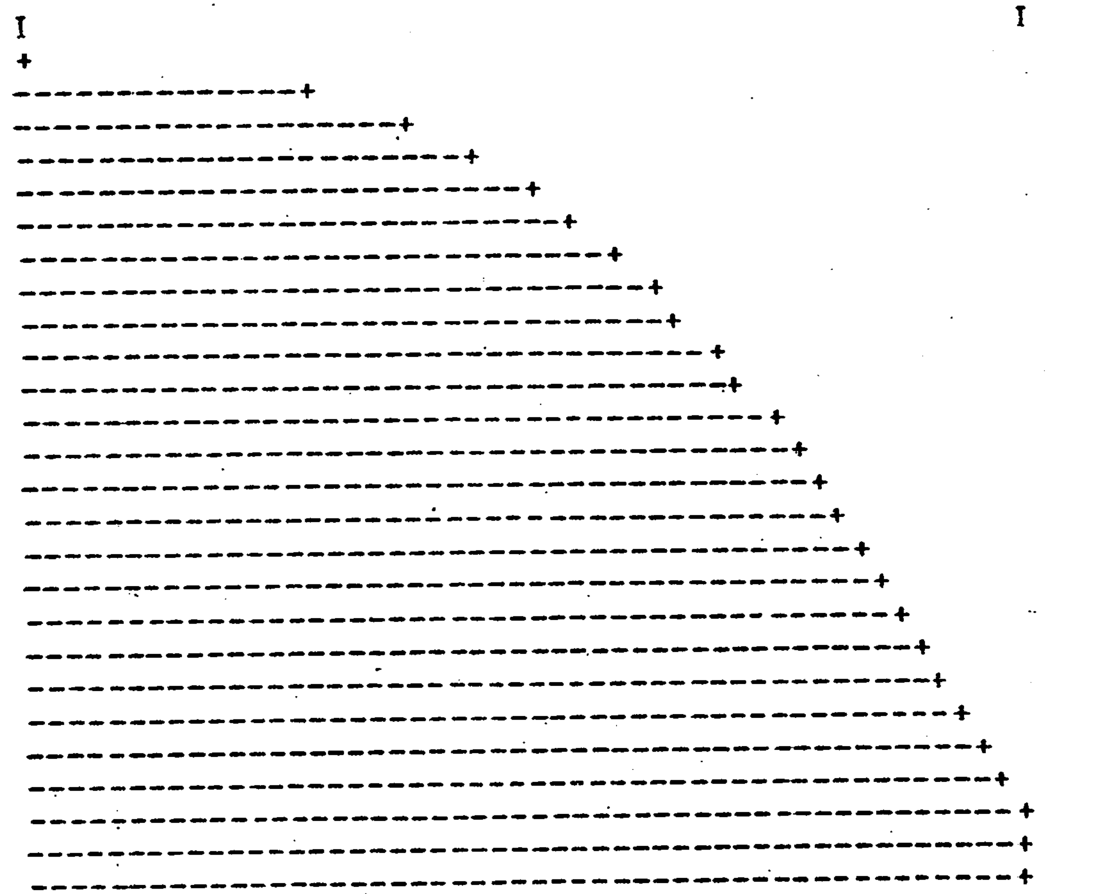


FIG. 23 MOMENTUM BOUNDARY LAYER THICKNESS FOR

$Pr_0 = 5.0$, $A = -0.5$, $B = 0.0$

THERMAL BOUNDARY LAYER PAGE 1

MINIMUM 0.0 R1 VERSUS Z MAXIMUM 5.8829E-01

Z	R1
0.0	0.0
2.0000E-03	1.6277E-01
4.0000E-03	2.2168E-01
6.0000E-03	2.6357E-01
8.0000E-03	2.9706E-01
1.0000E-02	3.2535E-01
1.2000E-02	3.5003E-01
1.4000E-02	3.7204E-01
1.6000E-02	3.9197E-01
1.8000E-02	4.1022E-01
2.0000E-02	4.2708E-01
2.2000E-02	4.4278E-01
2.4000E-02	4.5748E-01
2.6000E-02	4.7132E-01
2.8000E-02	4.8439E-01
3.0000E-02	4.9679E-01
3.2000E-02	5.0859E-01
3.4000E-02	5.1984E-01
3.6000E-02	5.3060E-01
3.8000E-02	5.4091E-01
4.0000E-02	5.5081E-01
4.2000E-02	5.6034E-01
4.4000E-02	5.6951E-01
4.6000E-02	5.7836E-01
4.8000E-02	5.8690E-01
4.8330E-02	5.8829E-01

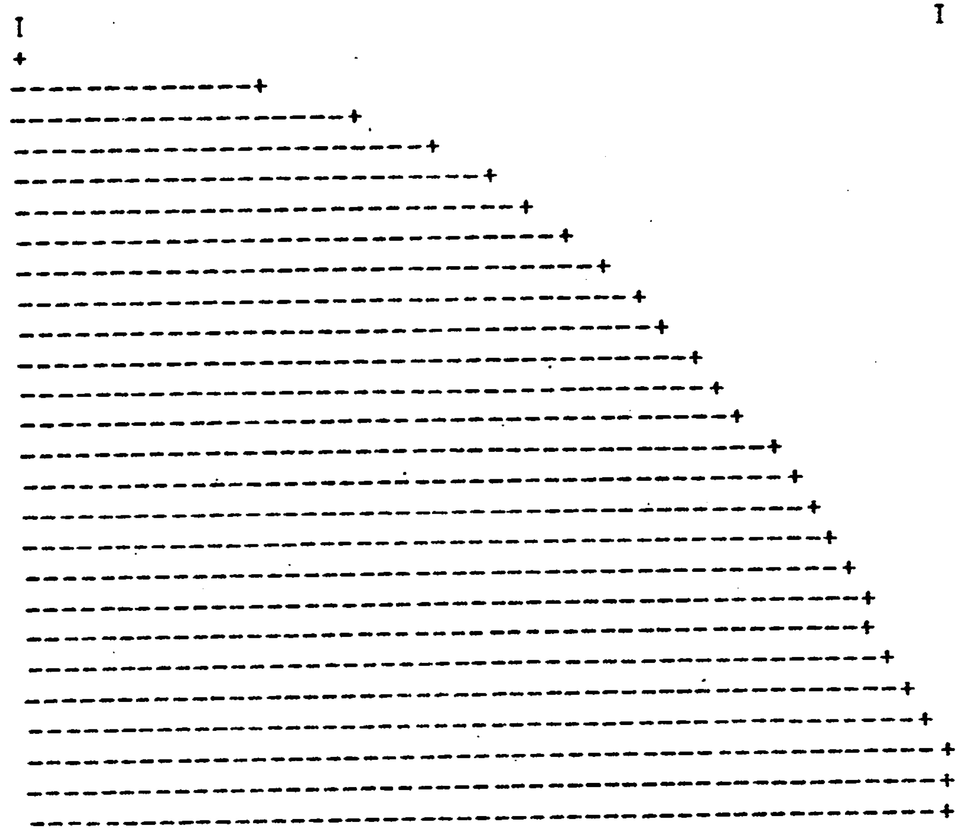


FIG. 24 THERMAL BOUNDARY LAYER THICKNESS FOR

$Pr_o = 5.0, \quad A = -0.5, \quad B = 0.0$

```
LOCAL NUSSELT NUMBER                                                                        PAGE    1

                                   MINIMUM                NUL     VERSUS Z                      MAXIMUM
                                  6.2597E 00                                                   5.0000E 01
  Z                 NUL           I                                                                I
0.0             6.0000E 01        -----------------------------------------------------------------+
2.0000E-03      1.9131E 01        ------------+
4.0000E-03      1.4296E 01        --------+
6.0000E-03      1.2187E 01        -----+
8.0000E-03      1.0937E 01        ----+
1.0000E-02      1.0088E 01        ---+
1.2000E-02      9.4624E 00        --+
1.4000E-02      8.9781E 00        --+
1.6000E-02      8.5890E 00        --+
1.8000E-02      8.2678E 00        -+
2.0000E-02      7.9972E 00        -+
2.2000E-02      7.7652E 00        -+
2.4000E-02      7.5637E 00        -+
2.6000E-02      7.3867E 00        -+
2.8000E-02      7.2297E 00        +
3.0000E-02      7.0893E 00        +
3.2000E-02      6.9629E 00        +
3.4000E-02      6.8484E 00        +
3.6000E-02      6.7440E 00        +
3.8000E-02      6.6485E 00        +
4.0000E-02      6.5607E 00        +
4.2000E-02      6.4797E 00        +
4.4000E-02      6.4047E 00        +
4.6000E-02      6.3350E 00        +
4.8000E-02      6.2700E 00        +
4.8330E-02      6.2597E 00        +
```

FIG. 25 LOCAL NUSSELT NUMBER FOR $Pr_o = 5.0$, $A = -0.5$, $B = 0.0$

```
LOCAL FRICTION FACTOR*REYNOLD NUMBER                                               PAGE    1

                            MINIMUM                FREN    VERSUS Z                  MAXIMUM
                          8.0000E 00                                               6.0000E 01
 Z                FREN      I                                                           I
0.0           6.0000E 01    ------------------------------------------------------------+
2.0000E-03    1.6715E 01    ---------+
4.0000E-03    1.3558E 01    ------+
6.0000E-03    1.2119E 01    ---+
8.0000E-03    1.1253E 01    ---+
1.0000E-02    1.0661E 01    --+
1.2000E-02    1.0224E 01    --+
1.4000E-02    9.8853E 00    -+
1.6000E-02    9.6135E 00    -+
1.8000E-02    9.3895E 00    -+
2.0000E-02    9.2010E 00    -+
2.2000E-02    9.0398E 00    +
2.4000E-02    8.9001E 00    +
2.6000E-02    8.7775E 00    +
2.8000E-02    8.6690E 00    +
3.0000E-02    8.5721E 00    +
3.2000E-02    8.4850E 00    +
3.4000E-02    8.4061E 00    +
3.6000E-02    8.3343E 00    +
3.8000E-02    8.2685E 00    +
4.0000E-02    8.2081E 00    +
4.2000E-02    8.1522E 00    +
4.4000E-02    8.1004E 00    +
4.6000E-02    8.0522E 00    +
4.8000E-02    8.0072E 00    +
4.8330E-02    8.0000E 00    +
```

FIG. 26 LOCAL FRICTION FACTOR FOR $Pr_0 = 5.0$, $A = -0.5$, $B = 0.0$

MOMENTUM BOUNDARY LAYER

PAGE 1

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0001E 00

Z	P
0.0	0.0
2.0000E-03	2.8979E-01
4.0000E-03	3.8079E-01
6.0000E-03	4.4619E-01
8.0000E-03	4.9885E-01
1.0000E-02	5.4364E-01
1.2000E-02	5.8300E-01
1.4000E-02	6.1835E-01
1.6000E-02	6.5061E-01
1.8000E-02	6.8040E-01
2.0000E-02	7.0816E-01
2.2000E-02	7.3422E-01
2.4000E-02	7.5885E-01
2.6000E-02	7.8224E-01
2.8000E-02	8.0455E-01
3.0000E-02	8.2592E-01
3.2000E-02	8.4645E-01
3.4000E-02	8.6624E-01
3.6000E-02	8.8535E-01
3.8000E-02	9.0387E-01
4.0000E-02	9.2184E-01
4.2000E-02	9.3932E-01
4.4000E-02	9.5636E-01
4.6000E-02	9.7298E-01
4.8000E-02	9.8922E-01
4.9360E-02	1.0001E 00

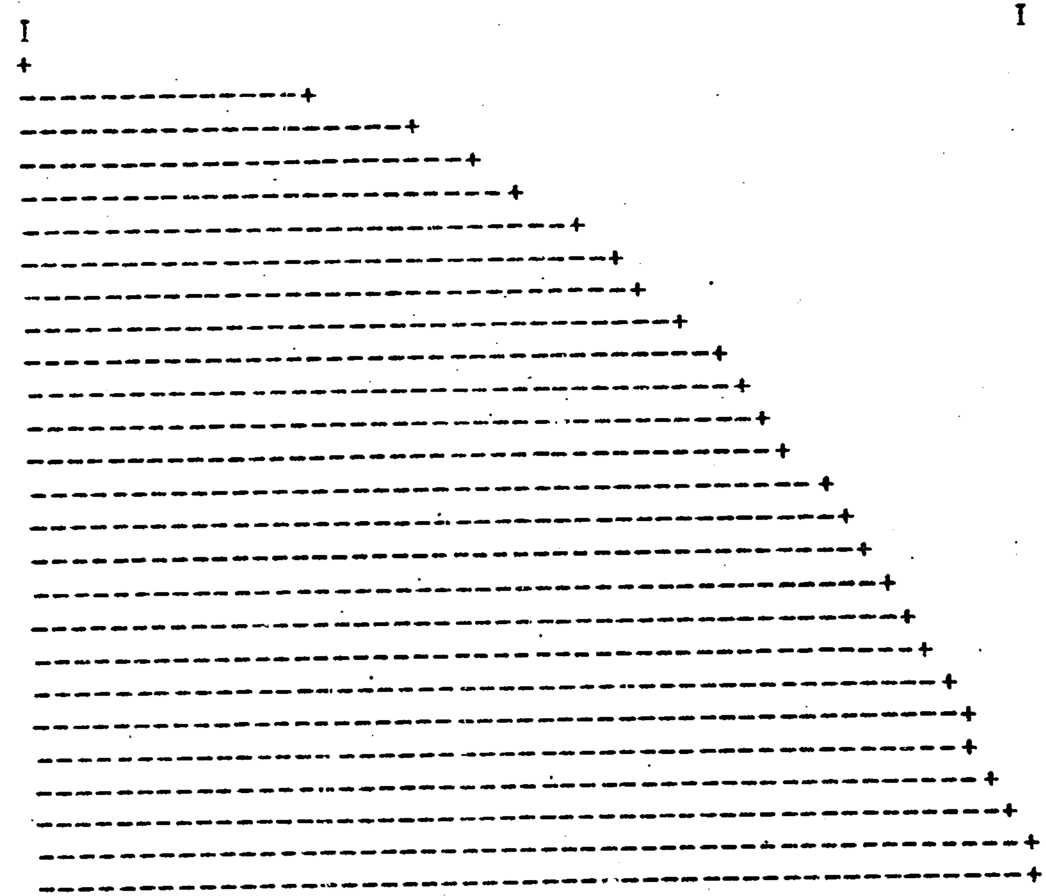


FIG. 27 MOMENTUM BOUNDARY LAYER THICKNESS FOR

$Pr_0 = 5.0$, $A = -0.5$, $B = 0.2$

THERMAL BOUNDARY LAYER — PAGE 1

MINIMUM 0.0 — R1 VERSUS Z — MAXIMUM 6.2378E-01

Z	R1
0.0	0.0
2.0000E-03	1.7200E-01
4.0000E-03	2.3413E-01
6.0000E-03	2.7829E-01
8.0000E-03	3.1358E-01
1.0000E-02	3.4337E-01
1.2000E-02	3.6936E-01
1.4000E-02	3.9252E-01
1.6000E-02	4.1348E-01
1.8000E-02	4.3267E-01
2.0000E-02	4.5040E-01
2.2000E-02	4.6690E-01
2.4000E-02	4.8234E-01
2.6000E-02	4.9686E-01
2.8000E-02	5.1058E-01
3.0000E-02	5.2359E-01
3.2000E-02	5.3595E-01
3.4000E-02	5.4775E-01
3.6000E-02	5.5902E-01
3.8000E-02	5.6981E-01
4.0000E-02	5.8017E-01
4.2000E-02	5.9013E-01
4.4000E-02	5.9972E-01
4.6000E-02	6.0896E-01
4.8000E-02	6.1789E-01
4.9360E-02	6.2378E-01

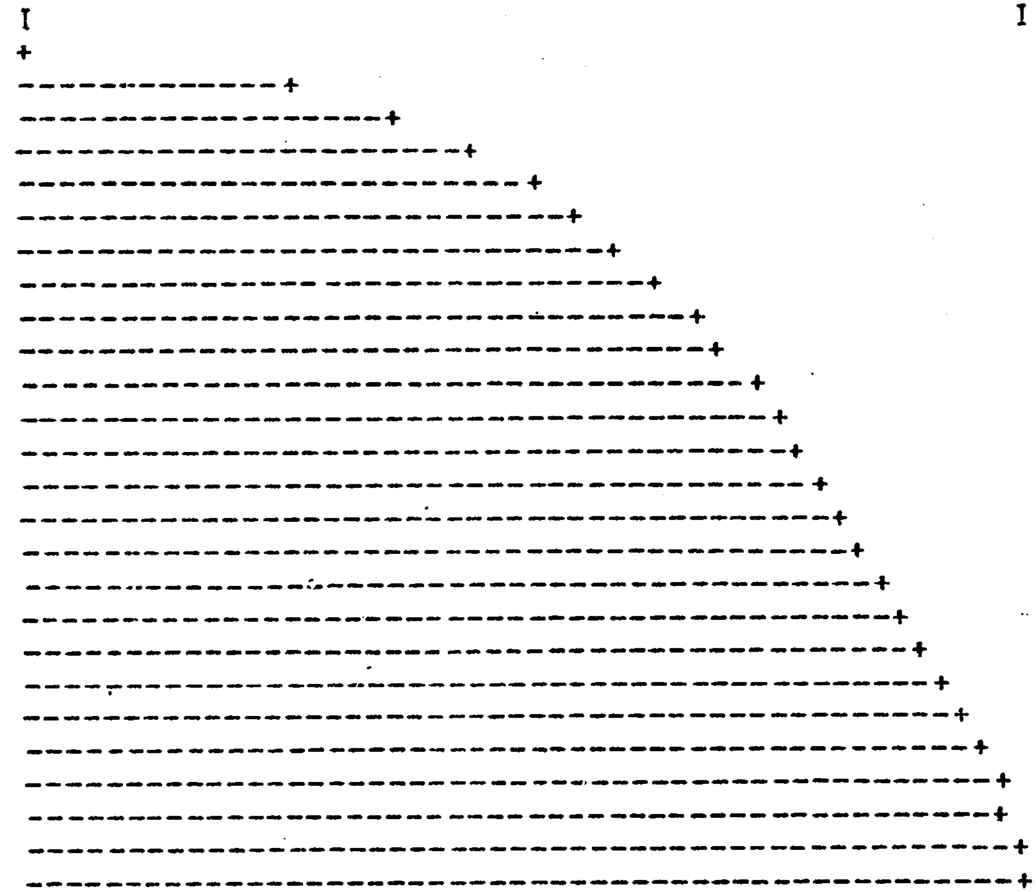


FIG. 28 THERMAL BOUNDARY LAYER THICKNESS FOR

$Pr_0 = 5.0$, $A = -0.5$, $B = 0.2$

```
LOCAL NUSSELT NUMBER                                                              PAGE    1

                                   MINIMUM                 NUL     VERSUS Z                      MAXIMUM
                                  7.2403E 00                                                   6.0000E 01
 Z                    NUL            I                                                              I
0.0                 6.0000E 01      ------------------------------------------------------+
2.0000E-03          2.1815E 01      --------------+
4.0000E-03          1.6341E 01      ---------+
6.0000E-03          1.3954E 01      -------+
8.0000E-03          1.2540E 01      ------+
1.0000E-02          1.1579E 01      -----+
1.2000E-02          1.0873E 01      ----+
1.4000E-02          1.0326E 01      ---+
1.6000E-02          9.8972E 00      ---+
1.8000E-02          9.5253E 00      ---+
2.0000E-02          9.2205E 00      --+
2.2000E-02          8.9596E 00      --+
2.4000E-02          8.7331E 00      --+
2.6000E-02          8.5344E 00      --+
2.8000E-02          8.3582E 00      --+
3.0000E-02          8.2009E 00      +
3.2000E-02          8.0593E 00      +
3.4000E-02          7.9312E 00      +
3.6000E-02          7.8146E 00      +
3.8000E-02          7.7080E 00      +
4.0000E-02          7.6101E 00      +
4.2000E-02          7.5198E 00      +
4.4000E-02          7.4363E 00      +
4.6000E-02          7.3588E 00      +
4.8000E-02          7.2866E 00      +
4.9360E-02          7.2403E 00      +
```

FIG. 29 LOCAL NUSSELT NUMBER FOR $Pr_0 = 5.0$, $A = -0.5$, $B = 0.2$

LOCAL FRICTION FACTOR*REYNOLD NUMBER PAGE 1

MINIMUM 7.9998E 00 FREN VERSUS Z MAXIMUM 6.0000E 01

Z	FREN	I ... I
0.0	6.0000E 01	--+
2.0000E-03	1.6817E 01	---------+
4.0000E-03	1.3637E 01	------+
6.0000E-03	1.2185E 01	-----+
8.0000E-03	1.1311E 01	---+
1.0000E-02	1.0713E 01	--+
1.2000E-02	1.0271E 01	--+
1.4000E-02	9.9292E 00	-+
1.6000E-02	9.6545E 00	-+
1.8000E-02	9.4280E 00	-+
2.0000E-02	9.2374E 00	-+
2.2000E-02	9.0744E 00	-+
2.4000E-02	8.9330E 00	+
2.6000E-02	8.8090E 00	+
2.8000E-02	8.6992E 00	+
3.0000E-02	8.6011E 00	+
3.2000E-02	8.5129E 00	+
3.4000E-02	8.4330E 00	+
3.6000E-02	8.3603E 00	+
3.8000E-02	8.2938E 00	+
4.0000E-02	8.2325E 00	+
4.2000E-02	8.1760E 00	+
4.4000E-02	8.1236E 00	+
4.6000E-02	8.0748E 00	+
4.8000E-02	8.0292E 00	+
4.9360E-02	7.9958E 00	+

FIG. 30 LOCAL FRICTION FACTOR FOR $Pr_0 = 5.0$, $A = -0.5$, $B = 0.2$

MOMENTUM BOUNDARY LAYER PAGE 1

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0000E 00

Z	P
0.0	0.0
1.0000E-03	2.7108E-01
2.0000E-03	3.6379E-01
3.0000E-03	4.2961E-01
4.0000E-03	4.8228E-01
5.0000E-03	5.2687E-01
6.0000E-03	5.6594E-01
7.0000E-03	6.0092E-01
8.0000E-03	6.3277E-01
9.0000E-03	6.6211E-01
1.0000E-02	6.8939E-01
1.1000E-02	7.1496E-01
1.2000E-02	7.3908E-01
1.3000E-02	7.6193E-01
1.4000E-02	7.8370E-01
1.5000E-02	8.0450E-01
1.6000E-02	8.2445E-01
1.7000E-02	8.4364E-01
1.8000E-02	8.6216E-01
1.9000E-02	8.8005E-01
2.0000E-02	8.9739E-01
2.1000E-02	9.1422E-01
2.2000E-02	9.3059E-01
2.3000E-02	9.4653E-01
2.4000E-02	9.6208E-01
2.5000E-02	9.7727E-01
2.6000E-02	9.9213E-01
2.6540E-02	1.0000E 00

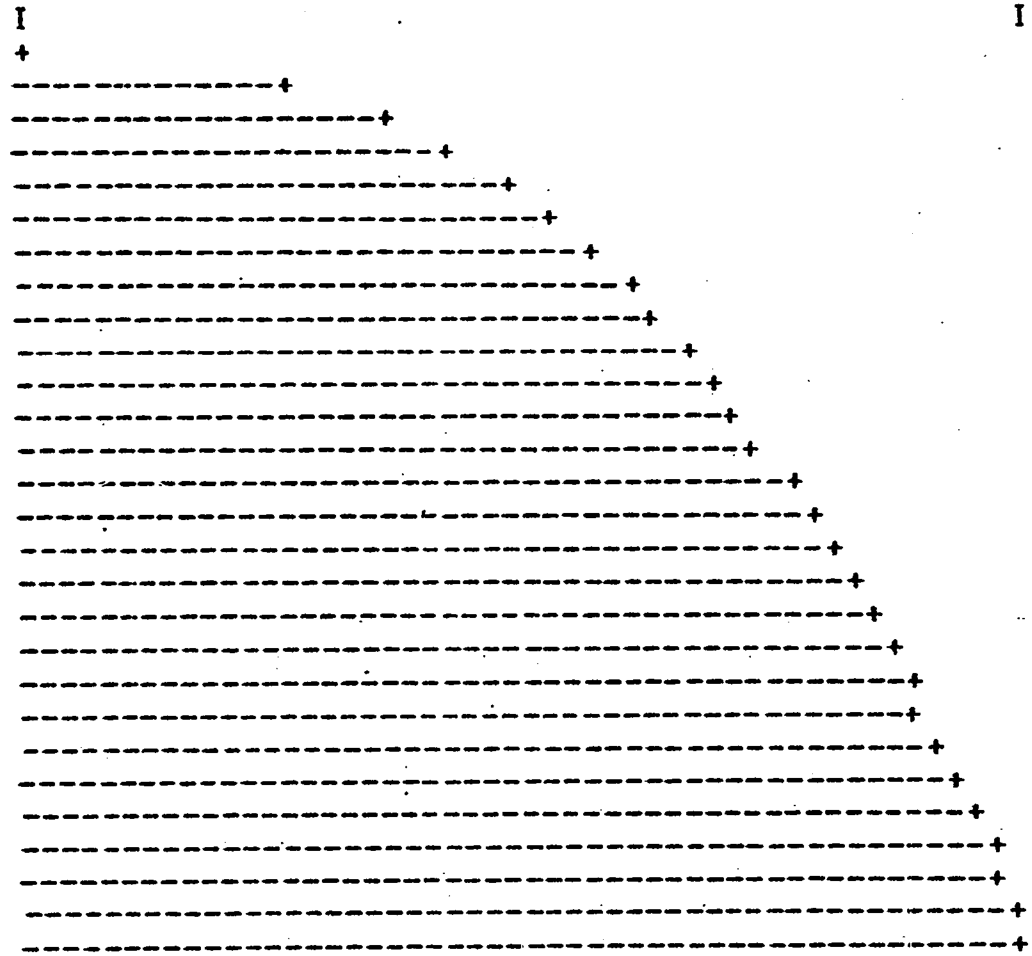


FIG. 31 MOMENTUM BOUNDARY LAYER THICKNESS FOR

$Pr_o = 5.0, \quad A = 0.2, \quad B = -0.3$

THERMAL BOUNDARY LAYER

PAGE 1

MINIMUM 0.0 — R1 VERSUS Z — MAXIMUM 4.3610E-01

Z	R1
0.0	0.0
1.0000E-03	1.1066E-01
2.0000E-03	1.5382E-01
3.0000E-03	1.8424E-01
4.0000E-03	2.0851E-01
5.0000E-03	2.2899E-01
6.0000E-03	2.4687E-01
7.0000E-03	2.6283E-01
8.0000E-03	2.7729E-01
9.0000E-03	2.9055E-01
1.0000E-02	3.0282E-01
1.1000E-02	3.1426E-01
1.2000E-02	3.2500E-01
1.3000E-02	3.3511E-01
1.4000E-02	3.4469E-01
1.5000E-02	3.5380E-01
1.6000E-02	3.6248E-01
1.7000E-02	3.7077E-01
1.8000E-02	3.7872E-01
1.9000E-02	3.8636E-01
2.0000E-02	3.9371E-01
2.1000E-02	4.0080E-01
2.2000E-02	4.0764E-01
2.3000E-02	4.1426E-01
2.4000E-02	4.2067E-01
2.5000E-02	4.2688E-01
2.6000E-02	4.3291E-01
2.6540E-02	4.3610E-01

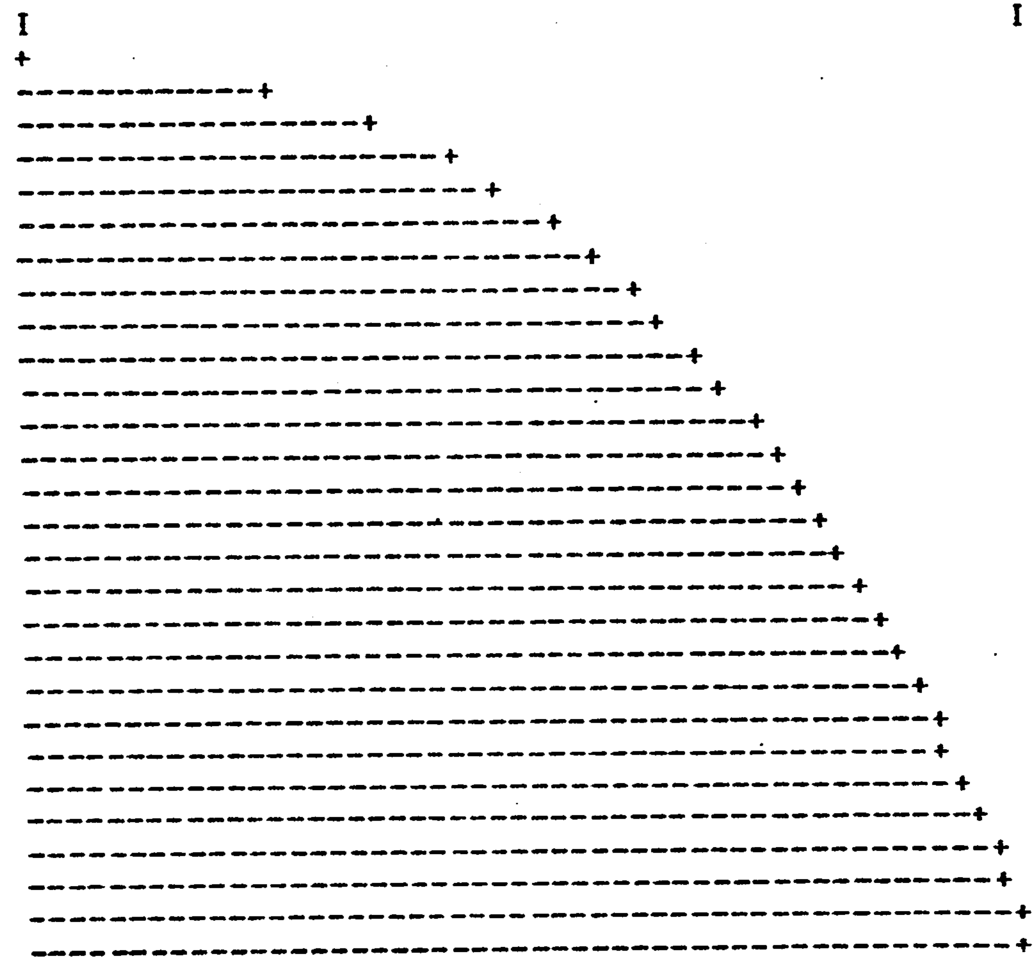


FIG. 32 THERMAL BOUNDARY LAYER THICKNESS FOR

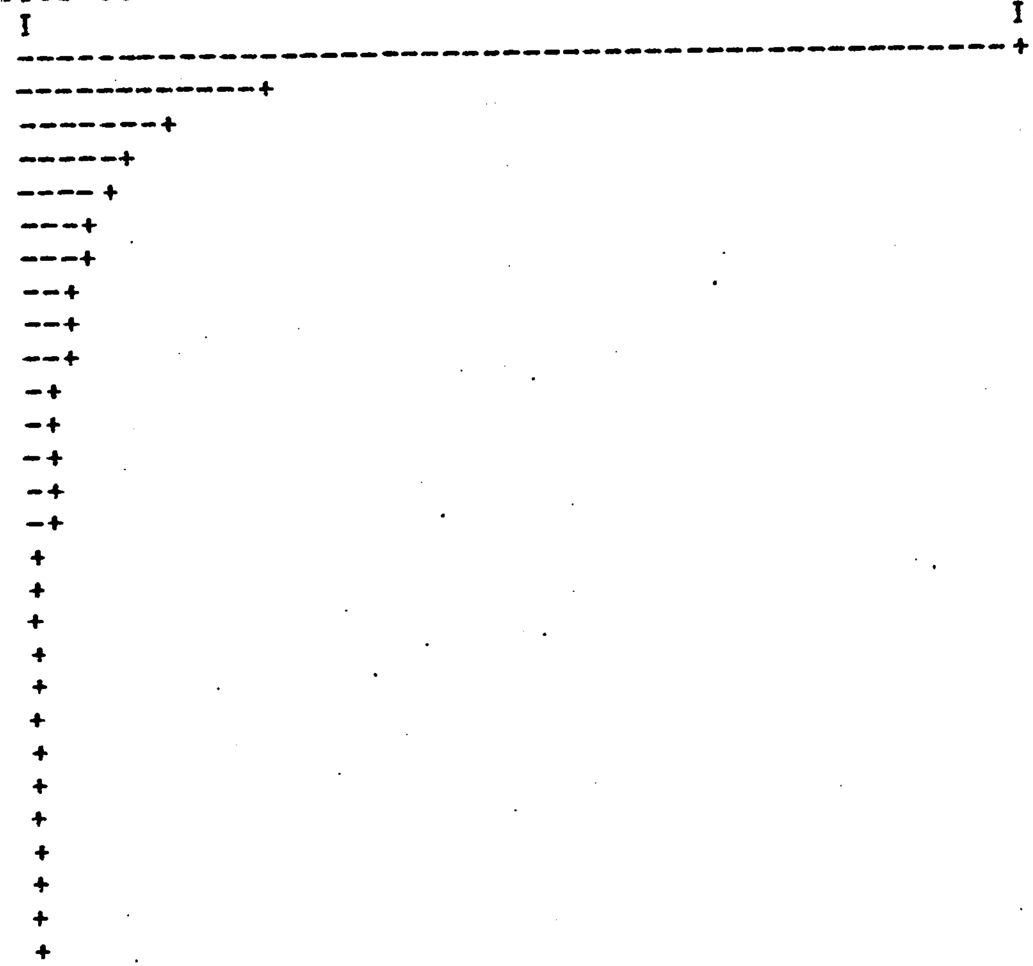
$Pr_0 = 5.0$, $A = 0.2$, $B = -0.3$

LOCAL NUSSELT NUMBER

PAGE 1

NUL VERSUS Z

MINIMUM 5.4336E 00 — MAXIMUM 6.0000E 01



Z	NUL
0.0	6.0000E 01
1.0000E-03	1.9344E 01
2.0000E-03	1.4054E 01
3.0000E-03	1.1821E 01
4.0000E-03	1.0512E 01
5.0000E-03	9.6253E 00
6.0000E-03	8.9740E 00
7.0000E-03	8.4693E 00
8.0000E-03	8.0634E 00
9.0000E-03	7.7276E 00
1.0000E-02	7.4440E 00
1.1000E-02	7.2003E 00
1.2000E-02	6.9879E 00
1.3000E-02	6.8008E 00
1.4000E-02	6.6342E 00
1.5000E-02	6.4847E 00
1.6000E-02	6.3496E 00
1.7000E-02	6.2267E 00
1.8000E-02	6.1144E 00
1.9000E-02	6.0111E 00
2.0000E-02	5.9158E 00
2.1000E-02	5.8275E 00
2.2000E-02	5.7453E 00
2.3000E-02	5.6687E 00
2.4000E-02	5.5969E 00
2.5000E-02	5.5296E 00
2.6000E-02	5.4663E 00
2.6540E-02	5.4336E 00

FIG. 33 LOCAL NUSSELT NUMBER FOR $Pr_o = 5.0$, $A = 0.2$, $B = -0.3$

LOCAL FRICTION FACTOR*REYNOLD NUMBER

PAGE 1

FREN VERSUS Z

MINIMUM 1.9200E 01 — MAXIMUM 1.0212E 02

Z	FREN	I
0.0	6.0000E 01	--------------------------+
1.0000E-03	4.2589E 01	---------------+
2.0000E-03	3.3852E 01	---------+
3.0000E-03	3.0020E 01	-------+
4.0000E-03	2.7753E 01	------+
5.0000E-03	2.6216E 01	-----+
6.0000E-03	2.5090E 01	---+
7.0000E-03	2.4221E 01	---+
8.0000E-03	2.3526E 01	--+
9.0000E-03	2.2954E 01	--+
1.0000E-02	2.2474E 01	-+
1.1000E-02	2.2064E 01	-+
1.2000E-02	2.1709E 01	-+
1.3000E-02	2.1399E 01	-+
1.4000E-02	2.1124E 01	-+
1.5000E-02	2.0879E 01	-+
1.6000E-02	2.0658E 01	+
1.7000E-02	2.0459E 01	+
1.8000E-02	2.0278E 01	+
1.9000E-02	2.0112E 01	+
2.0000E-02	1.9960E 01	+
2.1000E-02	1.9819E 01	+
2.2000E-02	1.9689E 01	+
2.3000E-02	1.9568E 01	+
2.4000E-02	1.9456E 01	+
2.5000E-02	1.9350E 01	+
2.6000E-02	1.9251E 01	+
2.6540E-02	1.9200E 01	+

FIG. 34 LOCAL FRICTION FACTOR FOR $Pr_0 = 5.0$, $A = 0.2$, $B = -0.3$

MOMENTUM BOUNDARY LAYER

PAGE 1

MINIMUM 0.0 — P VERSUS Z — MAXIMUM 1.0001E 00

Z	P
0.0	0.0
1.0000E-03	3.0000E-01
2.0000E-03	4.0428E-01
3.0000E-03	4.7761E-01
4.0000E-03	5.3602E-01
5.0000E-03	5.8539E-01
6.0000E-03	6.2856E-01
7.0000E-03	6.6721E-01
8.0000E-03	7.0238E-01
9.0000E-03	7.3478E-01
1.0000E-02	7.6493E-01
1.1000E-02	7.9320E-01
1.2000E-02	8.1989E-01
1.3000E-02	8.4521E-01
1.4000E-02	8.6935E-01
1.5000E-02	8.9245E-01
1.6000E-02	9.1464E-01
1.7000E-02	9.3602E-01
1.8000E-02	9.5669E-01
1.9000E-02	9.7670E-01
2.0000E-02	9.9613E-01
2.0210E-02	1.0001E 00

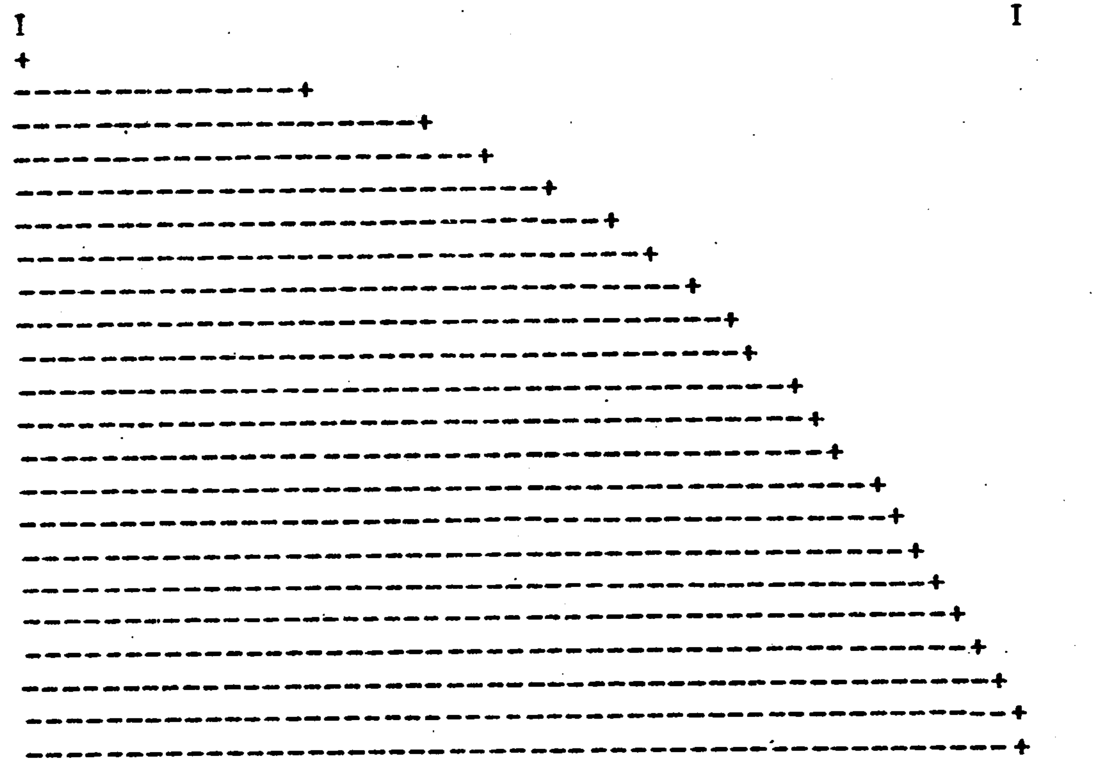


FIG. 35 MOMENTUM BOUNDARY LAYER THICKNESS FOR

$Pr_o = 5.0$, $A = 0.8$, $B = -0.3$

THERMAL BOUNDARY LAYER

PAGE 1

MINIMUM 0.0 R1 VERSUS Z MAXIMUM 3.9792E-01

Z	R1
0.0	0.0
1.0000E-03	1.1359E-01
2.0000E-03	1.5747E-01
3.0000E-03	1.8819E-01
4.0000E-03	2.1258E-01
5.0000E-03	2.3309E-01
6.0000E-03	2.5096E-01
7.0000E-03	2.6686E-01
8.0000E-03	2.8125E-01
9.0000E-03	2.9443E-01
1.0000E-02	3.0661E-01
1.1000E-02	3.1796E-01
1.2000E-02	3.2859E-01
1.3000E-02	3.3860E-01
1.4000E-02	3.4807E-01
1.5000E-02	3.5706E-01
1.6000E-02	3.6563E-01
1.7000E-02	3.7382E-01
1.8000E-02	3.8166E-01
1.9000E-02	3.8919E-01
2.0000E-02	3.9643E-01
2.0210E-02	3.9792E-01

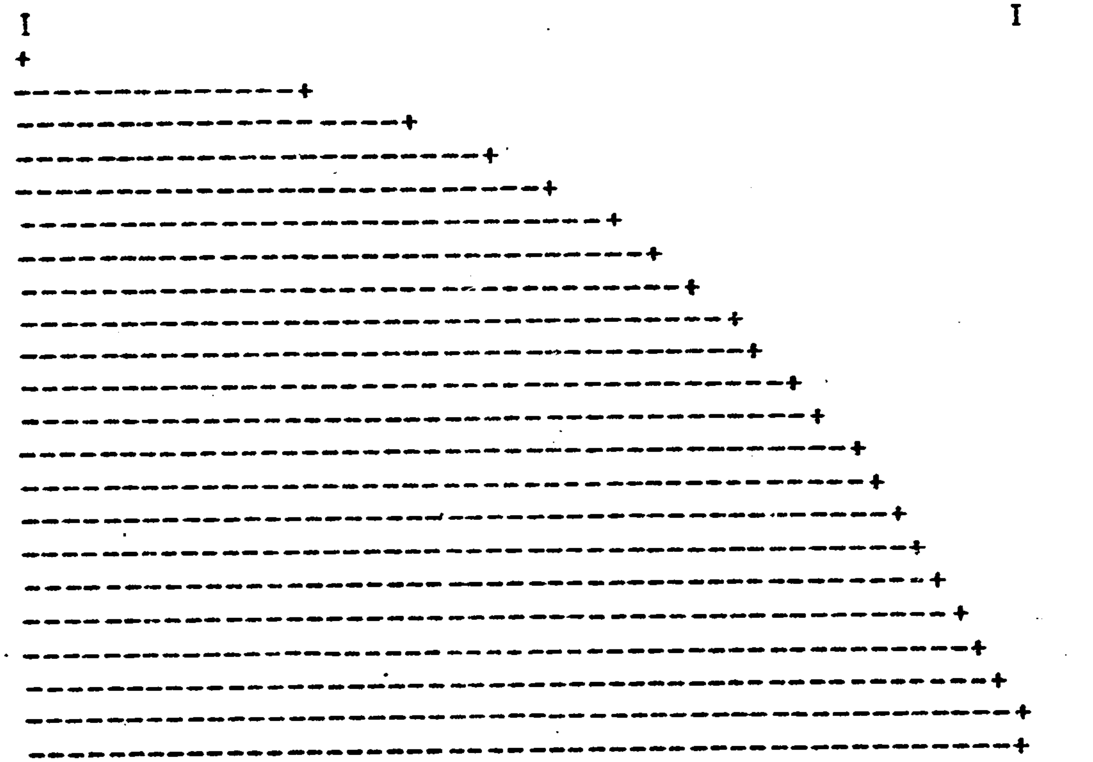


FIG. 36 THERMAL BOUNDARY LAYER THICKNESS FOR $Pr_0 = 5.0$, $A = 0.8$, $B = -0.3$

```
LOCAL NUSSELT NUMBER                                                              PAGE    1

                                MINIMUM                NUL    VERSUS Z                  MAXIMUM
                               5.8464E 00                                              6.0000E 01
  Z                  NUL          I                                                        I
0.0               6.0000E 01   -----------------------------------------------------------+
1.0000E-03        1.8836E 01   ------------+
2.0000E-03        1.3717E 01   --------+
3.0000E-03        1.1563E 01   ------+
4.0000E-03        1.03C0E 01   ----+
5.0000E-03        9.4456E 00   ---+
6.0000E-03        8.8176E 00   --+
7.0000E-03        8.33C9E 00   --+
8.0000E-03        7.9393F 00   -+
9.0000E-03        7.615?F 00   -+
1.0000E-02        7.3416F 00   -+
1.1000E-02        7.1064F 00   -+
1.2000F-02        6.9013E 00   +
1.3000F-02        6.72C5F 00   +
1.4000E-02        6.5596F 00   +
1.5000E-02        6.4152F 00   +
1.6000F-02        6.2846E C0   +
1.7000F-02        6.1658F 00   +
1.8000F-02        6.0571F 00   +
1.9000E-02        5.9571E 00   +
2.0000F-02        5.8649F 00   +
2.0210F-02        5.8464F 00   +
```

FIG. 37 LOCAL NUSSELT NUMBER FOR $Pr_0 = 5.0$, $A = 0.8$, $B = -0.3$

```
LOCAL FRICTION FACTOR*REYNOLD NUMBER                                              PAGE   1

                         MINIMUM                FREN    VERSUS Z                  MAXIMUM
                        2.8799E 01                                               1.5318E 02
   Z              FREN       I                                                      I
0.0            6.0000E 01   -------------+
1.0000E-03     5.8895E 01   -------------+
2.0000E-03     4.7008E 01   --------+
3.0000E-03     4.1898E 01   -----+
4.0000E-03     3.8903E 01   ----+
5.0000E-03     3.6888E 01   ---+
6.0000E-03     3.5419E 01   --+
7.0000E-03     3.4291E 01   --+
8.0000E-03     3.3392E 01   -+
9.0000E-03     3.2656E 01   -+
1.0000E-02     3.2039E 01   -+
1.1000E-02     3.1514E 01   -+
1.2000E-02     3.1061E 01   +
1.3000E-02     3.0665E 01   +
1.4000E-02     3.0315E 01   +
1.5000E-02     3.0004E 01   +
1.6000E-02     2.9724E 01   +
1.7000E-02     2.9471E 01   +
1.8000E-02     2.9241E 01   +
1.9000E-02     2.9031E 01   +
2.0000E-02     2.8837E 01   +
2.0210E-02     2.8799E 01   +
```

FIG. 38 LOCAL FRICTION FACTOR FOR $Pr_0 = 5.0$, $A = 0.8$, $B = -0.3$

MOMENTUM BOUNDARY LAYER

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0001E 00

Z	P
0.0	0.0
1.0000E-03	2.5972E-01
2.0000E-03	3.4754E-01
3.0000E-03	4.1018E-01
4.0000E-03	4.6041E-01
5.0000E-03	5.0301E-01
6.0000E-03	5.4034E-01
7.0000E-03	5.7380E-01
8.0000E-03	6.0427E-01
9.0000E-03	6.3234E-01
1.0000E-02	6.5845E-01
1.1000E-02	6.8291E-01
1.2000E-02	7.0598E-01
1.3000E-02	7.2784E-01
1.4000E-02	7.4865E-01
1.5000E-02	7.6854E-01
1.6000E-02	7.8761E-01
1.7000E-02	8.0594E-01
1.8000E-02	8.2361E-01
1.9000E-02	8.4069E-01
2.0000E-02	8.5723E-01
2.1000E-02	8.7327E-01
2.2000E-02	8.8886E-01
2.3000E-02	9.0403E-01
2.4000E-02	9.1882E-01
2.5000E-02	9.3326E-01
2.6000E-02	9.4737E-01
2.7000E-02	9.6117E-01
2.8000E-02	9.7470E-01
2.9000E-02	9.8796E-01
2.9930E-02	1.0001E 00

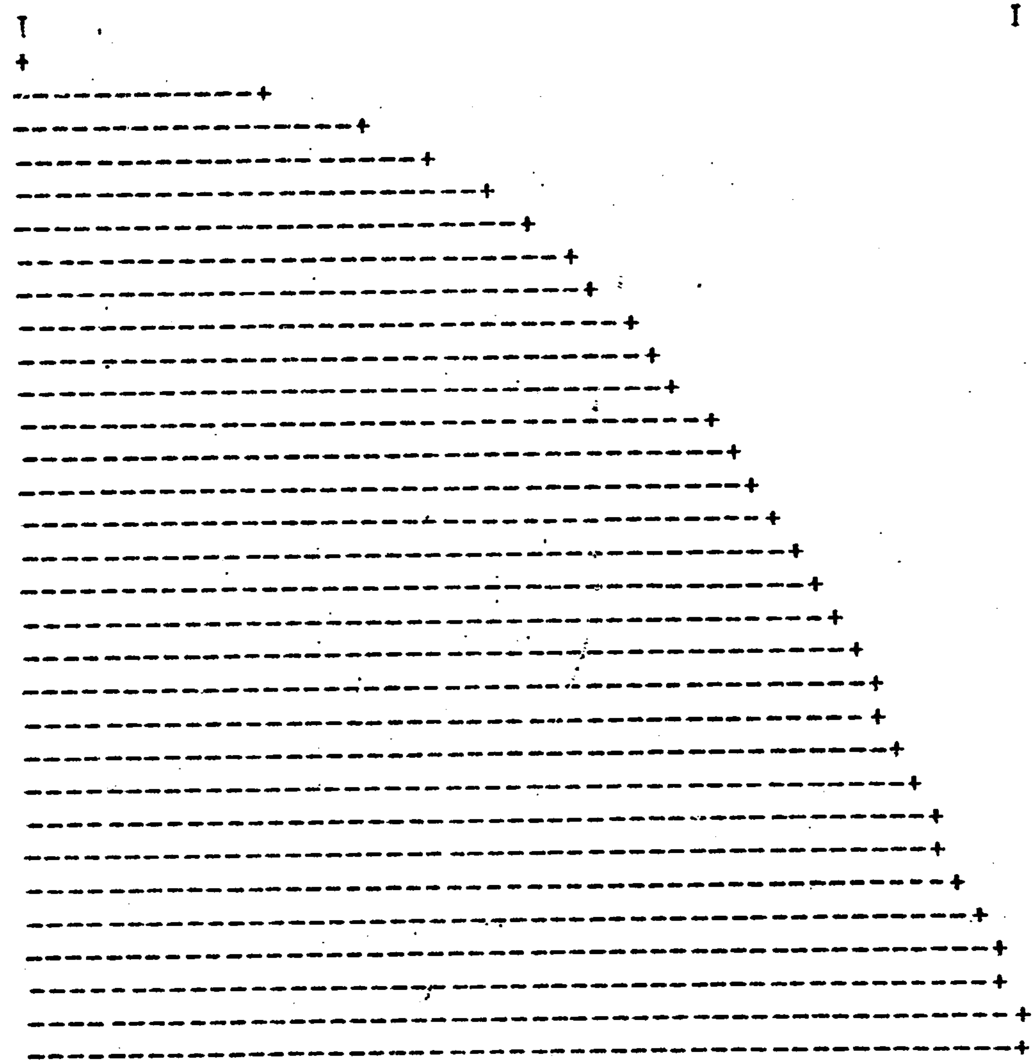


FIG. 39 MOMENTUM BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE ($Pr_0 = 2.5$)

THERMAL BOUNDARY LAYER PAGE 1

MINIMUM 0.0 R1 VERSUS Z MAXIMUM 6.3157E-01

Z	R1
0.0	0.0
1.0000E-03	1.5466E-01
2.0000E-03	2.1459E-01
3.0000E-03	2.5690E-01
4.0000E-03	2.9066E-01
5.0000E-03	3.1917E-01
6.0000E-03	3.4405E-01
7.0000E-03	3.6625E-01
8.0000E-03	3.8626E-01
9.0000E-03	4.0480E-01
1.0000E-02	4.2186E-01
1.1000E-02	4.3776E-01
1.2000E-02	4.5266E-01
1.3000E-02	4.6669E-01
1.4000E-02	4.7997E-01
1.5000E-02	4.9257E-01
1.6000E-02	5.0458E-01
1.7000E-02	5.1604E-01
1.8000E-02	5.2702E-01
1.9000E-02	5.3755E-01
2.0000E-02	5.4767E-01
2.1000E-02	5.5742E-01
2.2000E-02	5.6682E-01
2.3000E-02	5.7590E-01
2.4000E-02	5.8468E-01
2.5000E-02	5.9318E-01
2.6000E-02	6.0142E-01
2.7000E-02	6.0942E-01
2.8000E-02	6.1718E-01
2.9000E-02	6.2473E-01
2.9930E-02	6.3157E-01

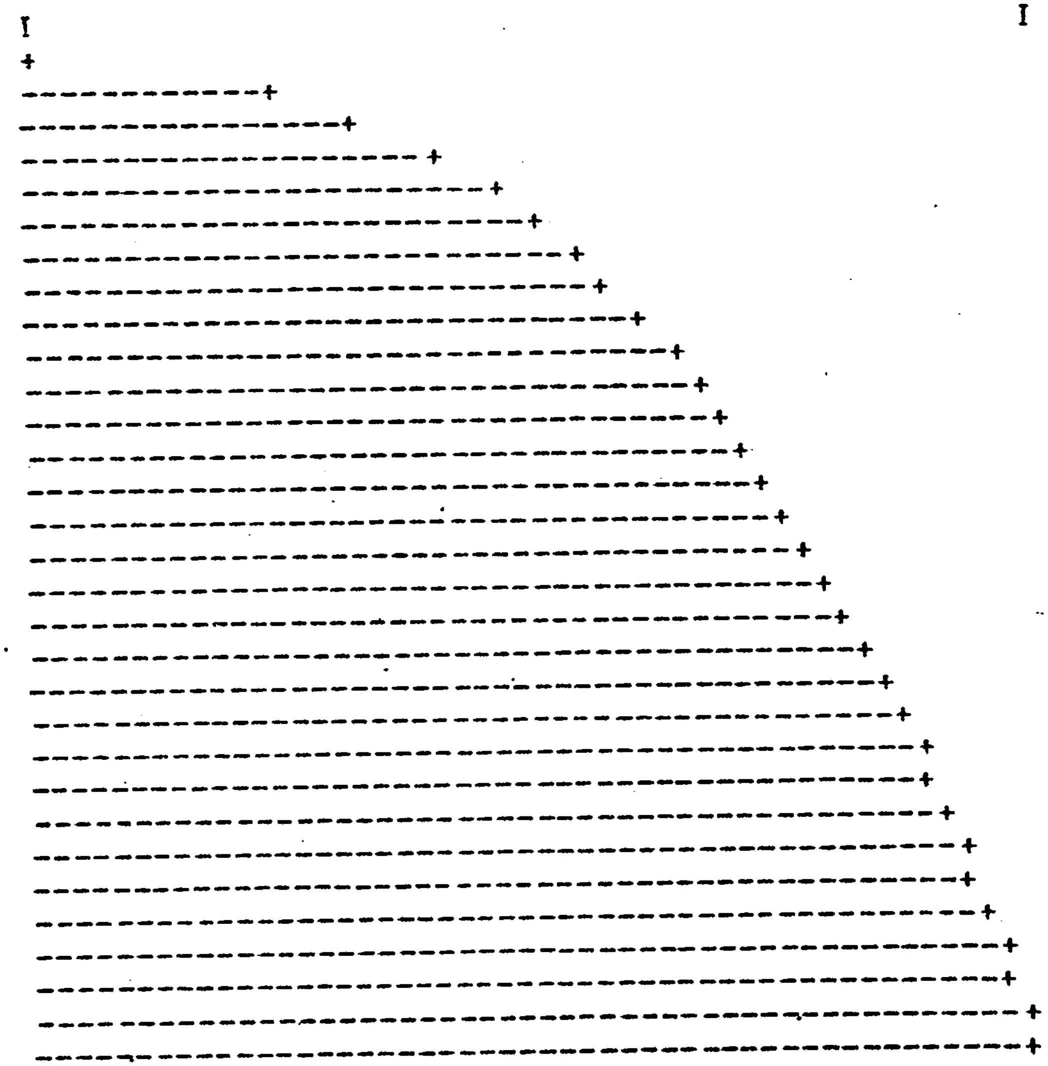


FIG. 40 THERMAL BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE ($Pr_0 = 2.5$)

LOCAL NUSSELT NUMBER PAGE 1

MINIMUM 5.9883E 00 NUL VERSUS Z MAXIMUM 6.0000E 01

Z	NUL
0.0	6.0000E 01
1.0000E-03	2.0124E 01
2.0000E-03	1.4770E 01
3.0000E-03	1.2510E 01
4.0000E-03	1.1186E 01
5.0000E-03	1.0292E 01
6.0000E-03	9.6365E 00
7.0000E-03	9.1301E 00
8.0000E-03	8.7241E 00
9.0000E-03	8.3893E 00
1.0000E-02	8.1075E 00
1.1000E-02	7.8661E 00
1.2000E-02	7.6565E 00
1.3000E-02	7.4724E 00
1.4000E-02	7.3092E 00
1.5000E-02	7.1633E 00
1.6000E-02	7.0319E 00
1.7000E-02	6.9128E 00
1.8000E-02	6.8043E 00
1.9000E-02	6.7050E 00
2.0000E-02	6.6137E 00
2.1000E-02	6.5295E 00
2.2000E-02	6.4514E 00
2.3000E-02	6.3789E 00
2.4000E-02	6.3113E 00
2.5000E-02	6.2481E 00
2.6000E-02	6.1889E 00
2.7000E-02	6.1333E 00
2.8000E-02	6.0810E 00
2.9000E-02	6.0317E 00
2.9930E-02	5.9883E 00

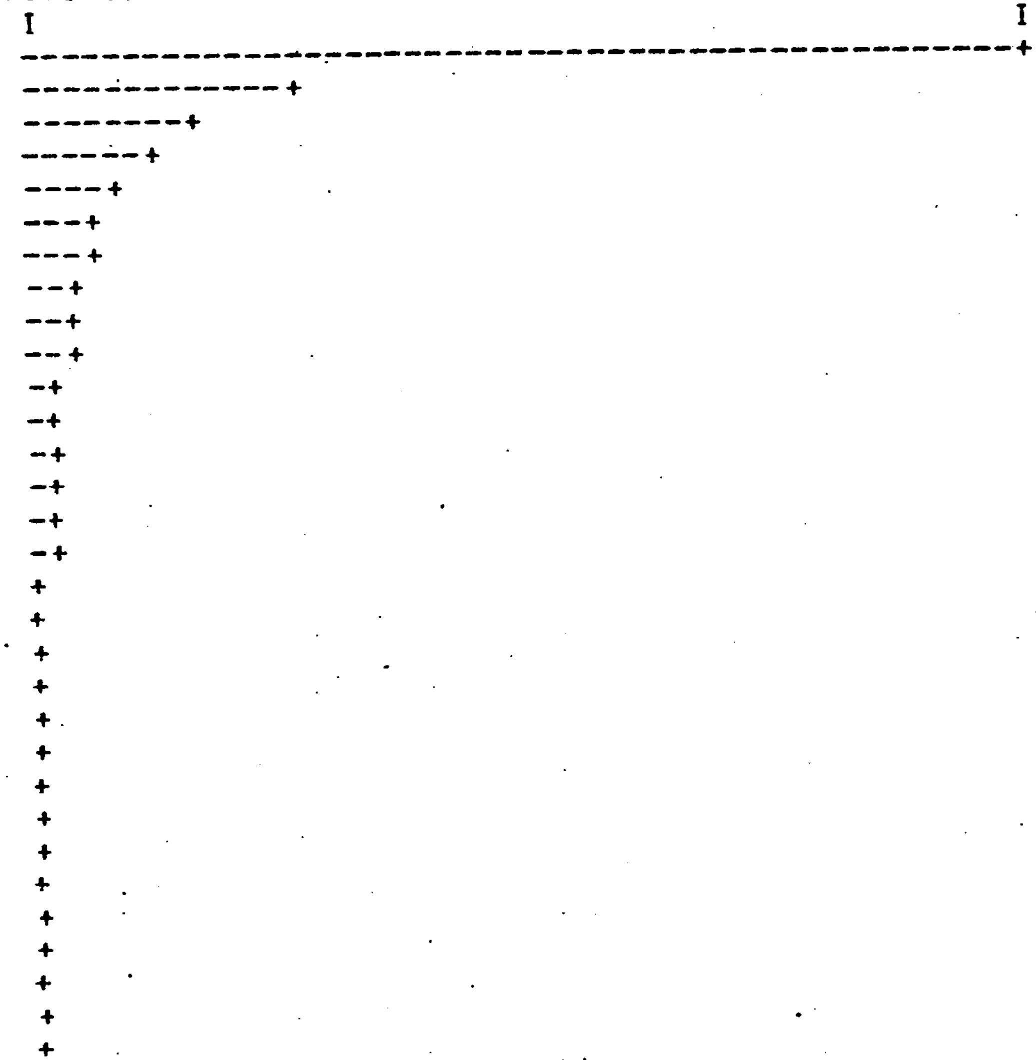


FIG. 41 LOCAL NUSSELT NUMBER FOR CONSTANT PROPERTIES CASE (Pr_0 = 2.5)

```
LOCAL FRICTION FACTOR*REYNOLD NUMBER                                          PAGE   1

                                MINIMUM          FREN   VERSUS Z              MAXIMUM
                               1.6000E 01                                   8.5098E 01
 Z                 FREN          I                                              I
0.0              6.0000E 01     -----------------------------------+
1.0000E-03       3.6753E 01     -----------------+
2.0000E-03       2.9195E 01     -----------+
3.0000E-03       2.5847E 01     --------+
4.0000E-03       2.3855E 01     ------+
5.0000E-03       2.2501E 01     -----+
6.0000E-03       2.1506E 01     ----+
7.0000E-03       2.0737E 01     ----+
8.0000E-03       2.0120E 01     ---+
9.0000E-03       1.9612E 01     ---+
1.0000E-02       1.9185E 01     ---+
1.1000E-02       1.8820E 01     ---+
1.2000E-02       1.8503E 01     --+
1.3000E-02       1.8226E 01     --+
1.4000E-02       1.7980E 01     --+
1.5000E-02       1.7761E 01     --+
1.6000E-02       1.7564E 01     --+
1.7000E-02       1.7385E 01     --+
1.8000E-02       1.7223E 01     +
1.9000E-02       1.7074E 01     +
2.0000E-02       1.6938E 01     +
2.1000E-02       1.6812E 01     +
2.2000E-02       1.6695E 01     +
2.3000E-02       1.6586E 01     +
2.4000E-02       1.6485E 01     +
2.5000E-02       1.6391E 01     +
2.6000E-02       1.6302E 01     +
2.7000E-02       1.6218E 01     +
2.8000E-02       1.6140E 01     +
2.9000E-02       1.6065E 01     +
2.9930E-02       1.6000E 01     +
```

FIG. 42 LOCAL FRICTION FACTOR FOR CONSTANT PROPERTIES CASE (Pr_o = 2.5)

MOMENTUM BOUNDARY LAYER PAGE 1

MINIMUM 0.0 P VERSUS Z MAXIMUM 1.0001E 00

Z	P
0.0	0.0
1.0000E-03	2.5972E-01
2.0000E-03	3.4754E-01
3.0000E-03	4.1018E-01
4.0000E-03	4.6041E-01
5.0000E-03	5.0301E-01
6.0000E-03	5.4034E-01
7.0000E-03	5.7380E-01
8.0000E-03	6.0427E-01
9.0000E-03	6.3234E-01
1.0000E-02	6.5845E-01
1.1000E-02	6.8291E-01
1.2000E-02	7.0598E-01
1.3000E-02	7.2784E-01
1.4000E-02	7.4865E-01
1.5000E-02	7.6854E-01
1.6000E-02	7.8761E-01
1.7000E-02	8.0594E-01
1.8000E-02	8.2361E-01
1.9000E-02	8.4069E-01
2.0000E-02	8.5723E-01
2.1000E-02	8.7327E-01
2.2000E-02	8.8886E-01
2.3000E-02	9.0403E-01
2.4000E-02	9.1882E-01
2.5000E-02	9.3326E-01
2.6000E-02	9.4737E-01
2.7000E-02	9.6117E-01
2.8000E-02	9.7470E-01
2.9000E-02	9.8796E-01
2.9930E-02	1.0001E 00

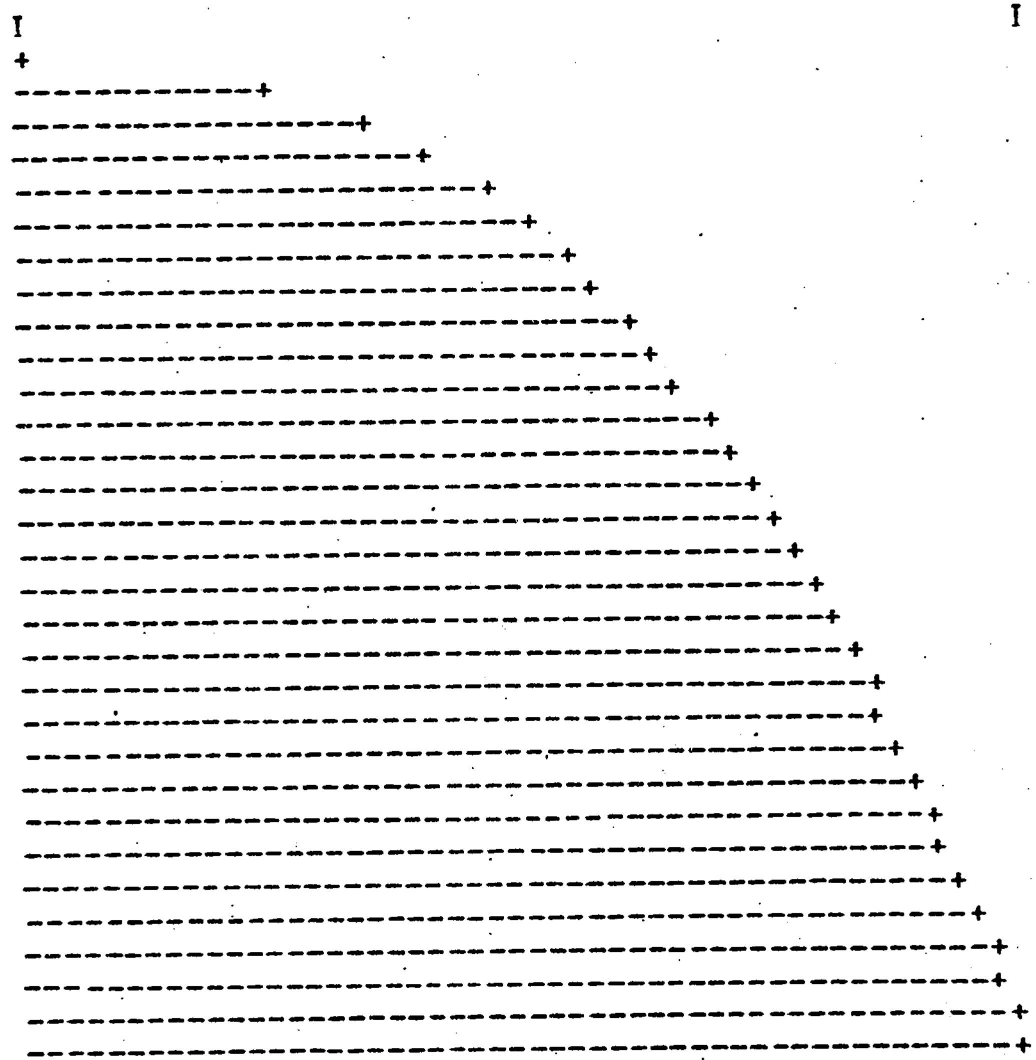


FIG. 43 MOMENTUM BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE (Pr_0 = 20.0)

THERMAL BOUNDARY LAYER PAGE 1

MINIMUM 0.0 R1 VERSUS Z MAXIMUM 3.1664E-01

Z	R1
0.0	0.0
1.0000E-03	7.6195E-02
2.0000E-03	1.0606E-01
3.0000E-03	1.2717E-01
4.0000E-03	1.4404E-01
5.0000E-03	1.5829E-01
6.0000E-03	1.7075E-01
7.0000E-03	1.8186E-01
8.0000E-03	1.9195E-01
9.0000E-03	2.0120E-01
1.0000E-02	2.0977E-01
1.1000E-02	2.1776E-01
1.2000E-02	2.2527E-01
1.3000E-02	2.3234E-01
1.4000E-02	2.3905E-01
1.5000E-02	2.4542E-01
1.6000E-02	2.5150E-01
1.7000E-02	2.5732E-01
1.8000E-02	2.6289E-01
1.9000E-02	2.6825E-01
2.0000E-02	2.7341E-01
2.1000E-02	2.7839E-01
2.2000E-02	2.8320E-01
2.3000E-02	2.8786E-01
2.4000E-02	2.9237E-01
2.5000E-02	2.9675E-01
2.6000E-02	3.0100E-01
2.7000E-02	3.0514E-01
2.8000E-02	3.0916E-01
2.9000E-02	3.1308E-01
2.9930E-02	3.1664E-01

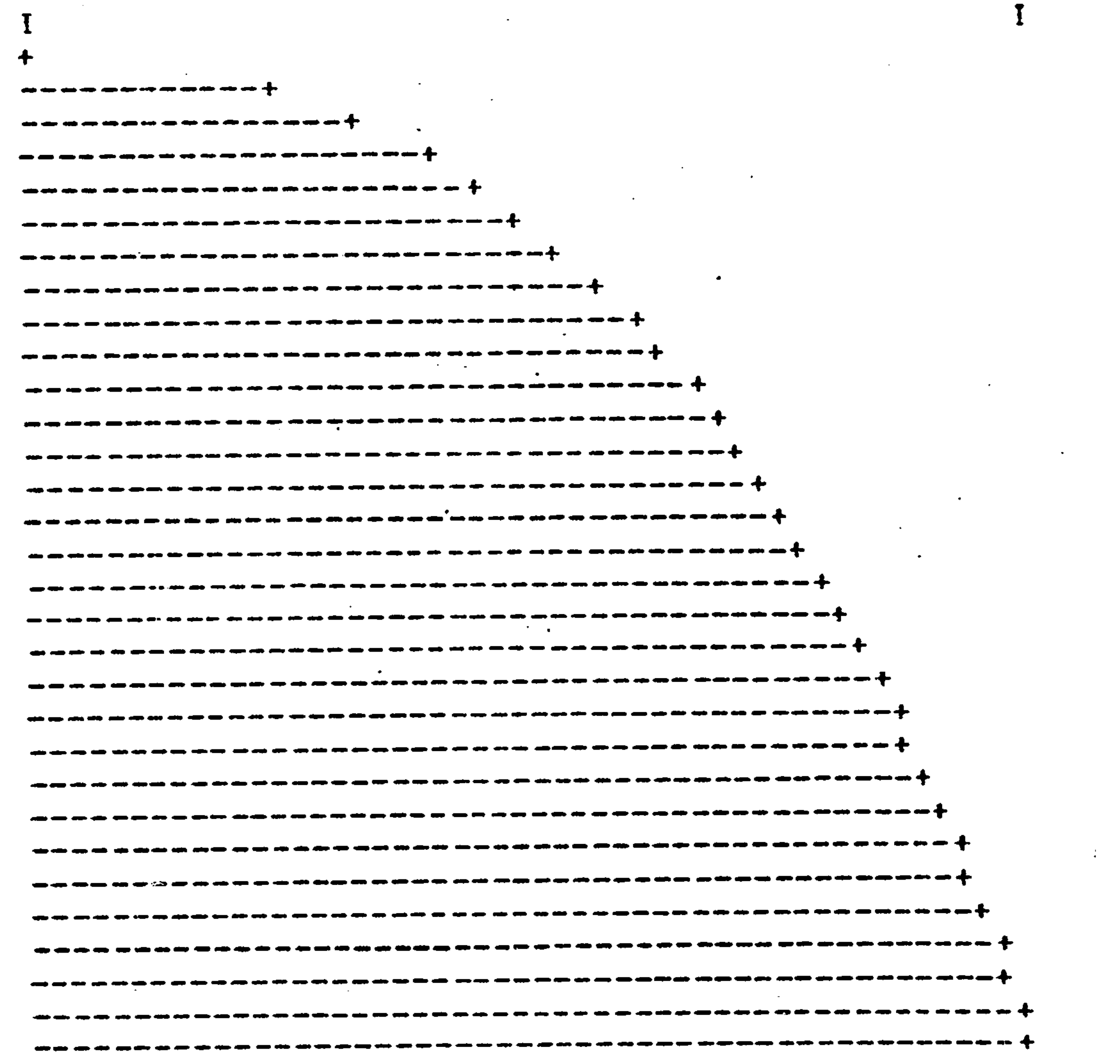


FIG. 44 THERMAL BOUNDARY LAYER THICKNESS FOR CONSTANT PROPERTIES CASE (Pr_o = 20.0)

```
LOCAL NUSSELT NUMBER                                                                  PAGE    1

                                 MINIMUM                   NUL    VERSUS  Z                   MAXIMUM
                                1.0135E 01                                                   5.0000E 01
 Z                     NUL      I                                                                 I
0.0                6.0000E 01   --------------------------------------------------------------+
1.0000E-03         6.0000E 01   --------------------------------------------------------------+
2.0000E-03         2.8708E 01   --------------------+
3.0000E-03         2.4035E 01   ---------------+
4.0000E-03         2.1291E 01   -------------+
5.0000E-03         1.9431E 01   -----------+
6.0000E-03         1.8062E 01   ---------+
7.0000E-03         1.7000E 01   --------+
8.0000E-03         1.6144E 01   --------+
9.0000E-03         1.5436E 01   -------+
1.0000E-02         1.4836E 01   -----+
1.1000E-02         1.4320E 01   -----+
1.2000E-02         1.3869E 01   ----+
1.3000E-02         1.3472E 01   ----+
1.4000E-02         1.3117E 01   ---+
1.5000E-02         1.2799E 01   ---+
1.6000E-02         1.2510E 01   ---+
1.7000E-02         1.2247E 01   ---+
1.8000E-02         1.2007E 01   --+
1.9000E-02         1.1785E 01   --+
2.0000E-02         1.1580E 01   --+
2.1000E-02         1.1390E 01   --+
2.2000E-02         1.1213E 01   --+
2.3000E-02         1.1047E 01   +
2.4000E-02         1.0892E 01   +
2.5000E-02         1.0746E 01   +
2.6000E-02         1.0608E 01   +
2.7000E-02         1.0478E 01   +
2.8000E-02         1.0355E 01   +
2.9000E-02         1.0238E 01   +
2.9930E-02         1.0135E 01   +
```

FIG. 45 LOCAL NUSSELT NUMBER FOR CONSTANT PROPERTIES CASE (Pr_o = 20.0)

```
LOCAL FRICTION FACTOR*REYNOLD NUMBER                                                      PAGE    1

                              MINIMUM                    FREN    VERSUS Z                MAXIMUM
                            1.6000E 01                                                  8.5098E 01
  Z                  FREN          I                                                         I
0.0                 6.0000E 01    ---------------------------------+
1.0000E-03          3.6753E 01    ----------------+
2.0000E-03          2.9195E 01    ----------+
3.0000E-03          2.5847E 01    --------+
4.0000E-03          2.3855E 01    ------+
5.0000E-03          2.2591E 01    -----+
6.0000E-03          2.1506E 01    ---+
7.0000E-03          2.0737E 01    ---+
8.0000E-03          2.0120E 01    --+
9.0000E-03          1.9612E 01    --+
1.0000E-02          1.9185E 01    --+
1.1000E-02          1.8820E 01    --+
1.2000E-02          1.8503E 01    -+
1.3000E-02          1.8226E 01    -+
1.4000E-02          1.7980E 01    -+
1.5000E-02          1.7761E 01    -+
1.6000E-02          1.7564E 01    -+
1.7000E-02          1.7385E 01    -+
1.8000E-02          1.7223E 01    +
1.9000E-02          1.7074E 01    +
2.0000E-02          1.6938E 01    +
2.1000E-02          1.6812E 01    +
2.2000E-02          1.6695E 01    +
2.3000E-02          1.6586E 01    +
2.4000E-02          1.6485E 01    +
2.5000E-02          1.6391E 01    +
2.6000E-02          1.6302E 01    +
2.7000E-02          1.6218E 01    +
2.8000E-02          1.6140E 01    +
2.9000E-02          1.6065E 01    +
2.9930E-02          1.6000E 01    +
```

FIG. 46   LOCAL FRICTION FACTOR FOR CONSTANT PROPERTIES CASE (Pr_0 = 20.0)

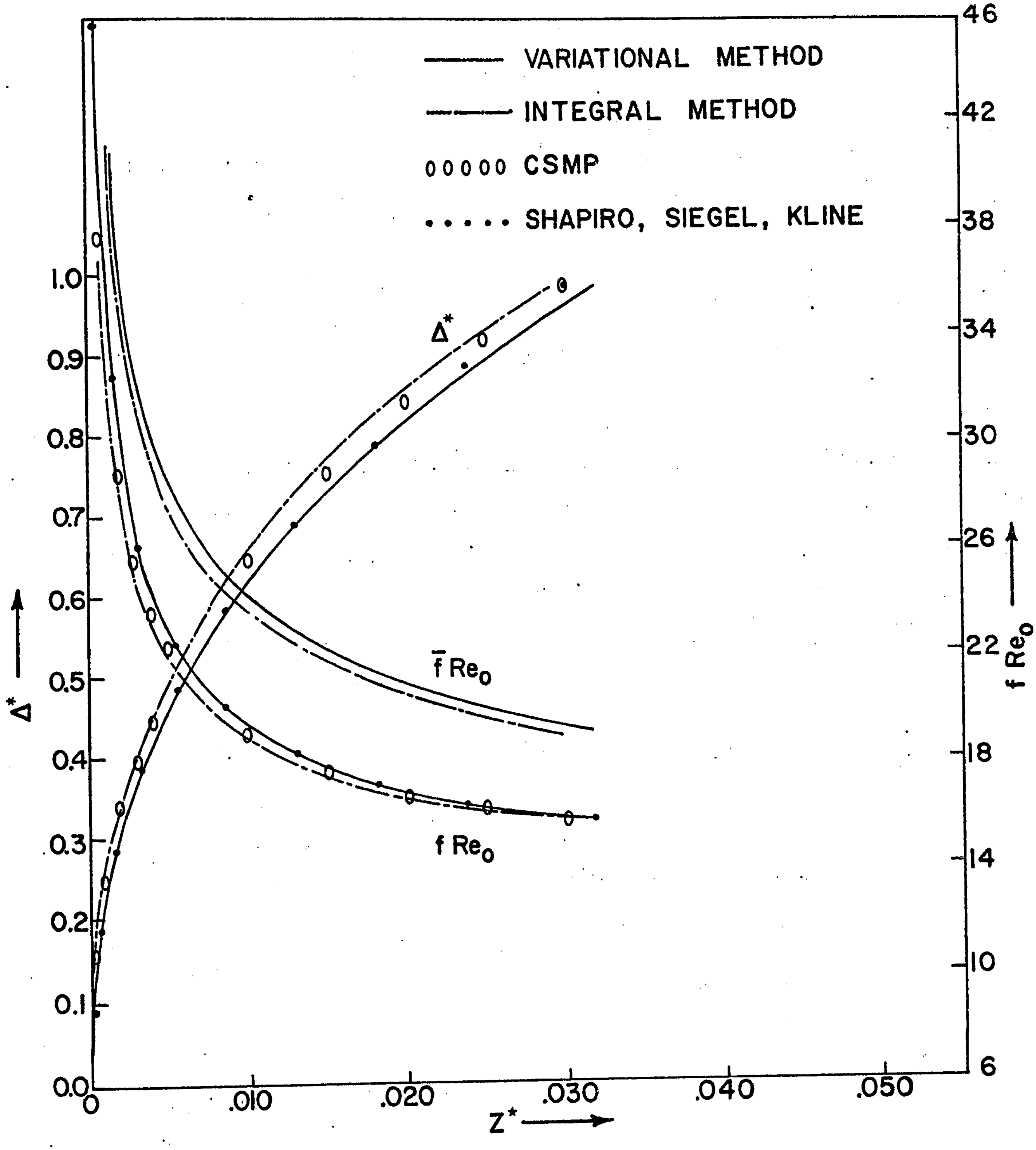


FIG. 47 COMPARISON OF MOMENTUM BOUNDARY LAYER THICKNESS, LOCAL AND AVERAGE FRICTION FACTOR FOR CONSTANT PROPERTIES CASE

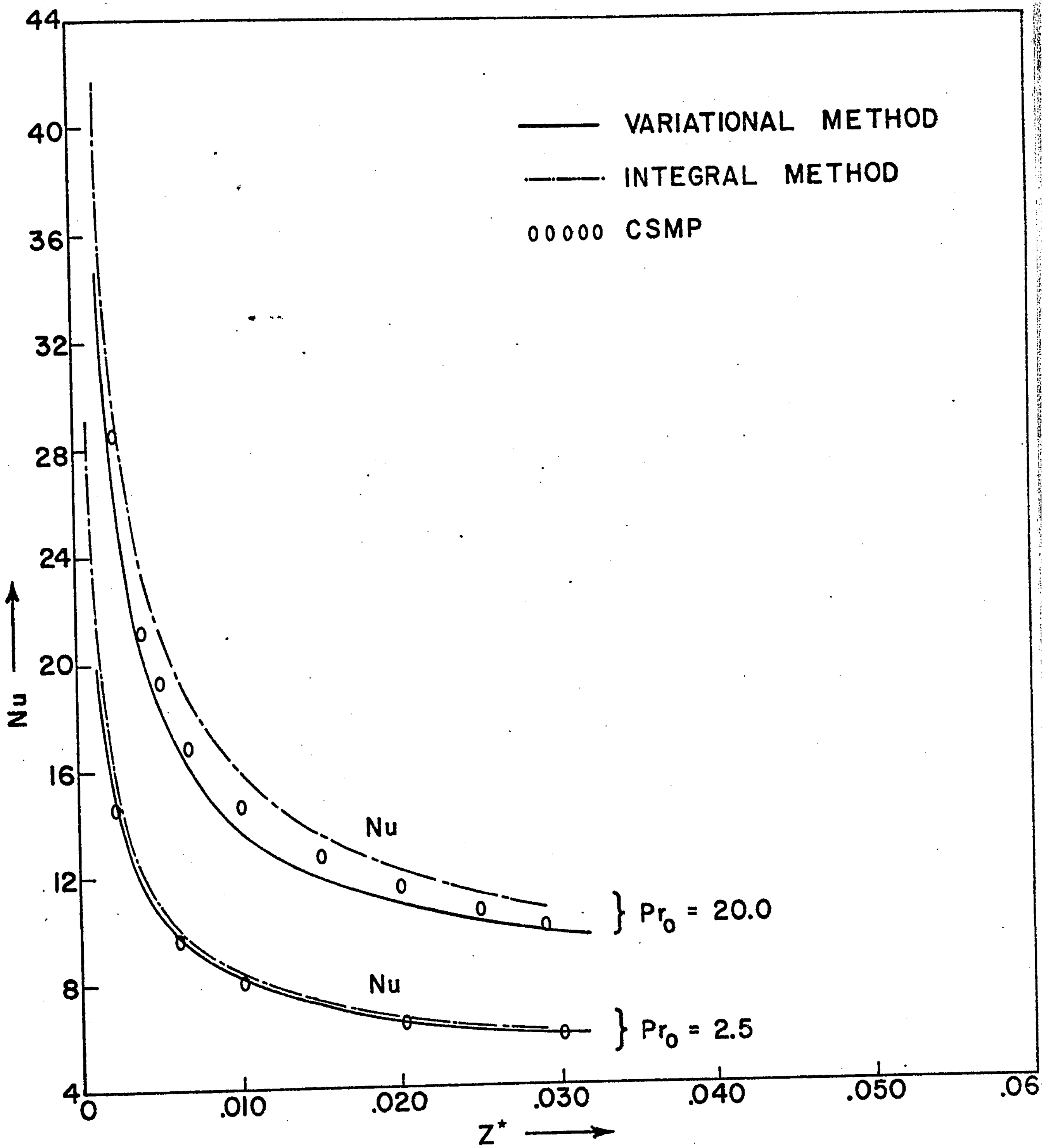


FIG. 48 COMPARISON OF LOCAL NUSSELT NUMBER FOR CONSTANT PROPERTIES CASE (Pr_0 = 2.5, 20.0)

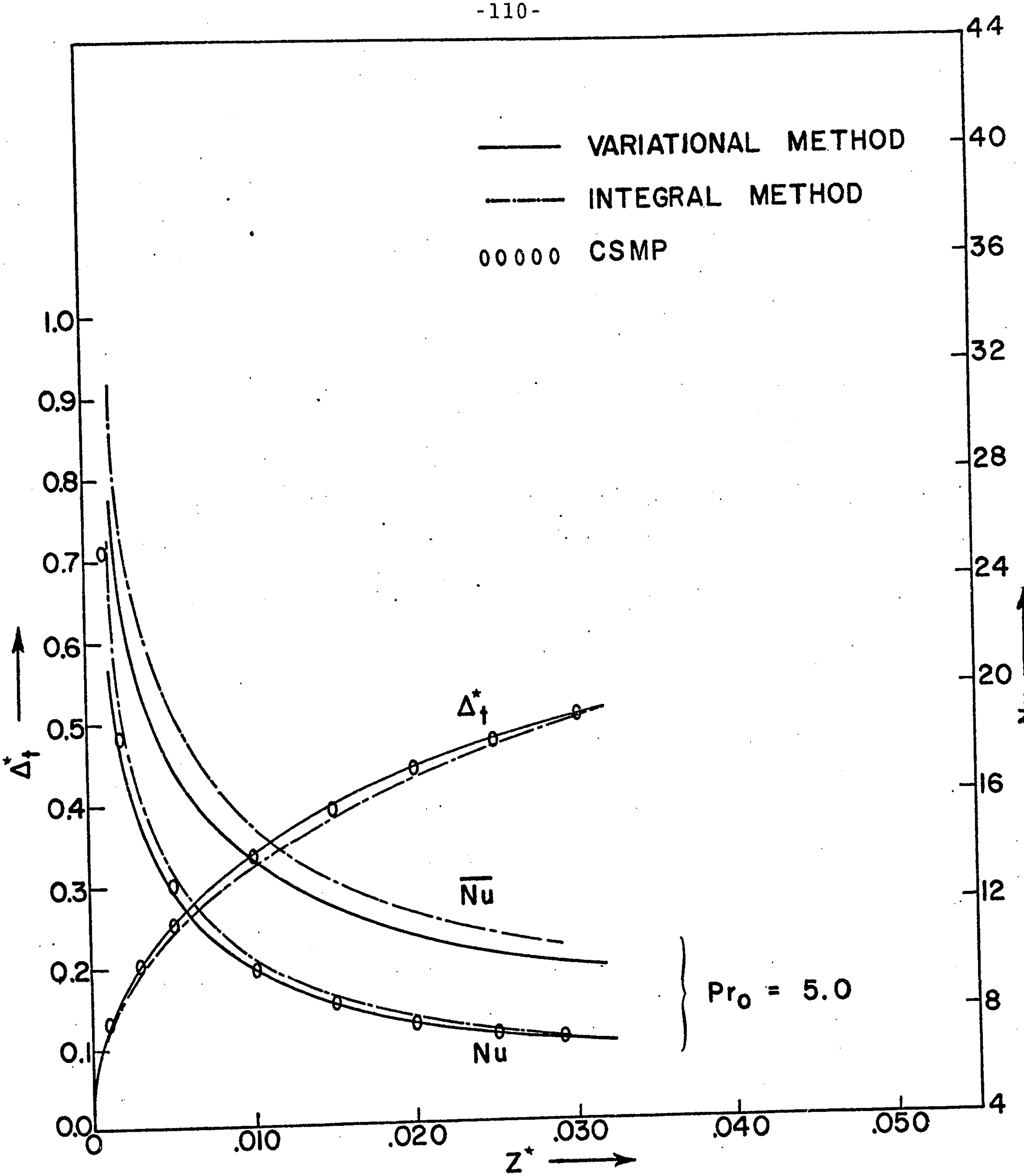


FIG. 49 COMPARISON OF THERMAL BOUNDARY LAYER THICKNESS, LOCAL AND AVERAGE NUSSELT No. FOR CONSTANT PROPERTIES CASE (Pr_0 = 5.0)

APPENDIX A

DERIVATION OF COUPLED EQUATIONS WITH VARIATIONAL TECHNIQUE

Rewriting the Eq. (14) and dropping the last term in the line integral which is not subject to variation

$$E = \iint \left(\frac{\kappa^o(T)}{2} \left(\frac{\partial T}{\partial r}\right)^2 + \frac{\mu^o(T)}{2} \left(\frac{\partial u}{\partial r}\right)^2 + \rho C_p u^o T \frac{\partial T^o}{\partial z} + \rho C_p v^o T \frac{\partial T^o}{\partial r} - \rho u^{o^2} \frac{\partial u}{\partial z} - \rho u^o v^o \frac{\partial u}{\partial r} - \rho u_c^o \frac{\partial u_c^o}{\partial z} u \right) r\,dr\,dz + \int_{r_o}^{r_o-\Delta} (\rho u^{o^2} ur)\,dr \Big|_{z=\ell} \qquad \text{.....(A-1)}$$

Since

$$\frac{u}{u_c} = -\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}$$

it follows that

$$\frac{\partial u}{\partial z} = 2u_c\dot{\Delta}\left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2}\right\} + \frac{\partial u_c}{\partial z}\left\{-\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}\right\}$$

and

$$\frac{\partial u}{\partial r} = 2u_c\left(\frac{r_o-r}{\Delta^2} - \frac{1}{\Delta}\right)$$

From the continuity equation

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0$$

$$v = -\frac{1}{r}\int_{r_o}^{r} r\,\frac{\partial u}{\partial z}\,dr$$

$$v = \frac{1}{r}\Bigg(2\,u_c\dot{\Delta}\left\{-\frac{1}{\Delta^3}\left\{\frac{(r_o-r)^4}{4} - \frac{r_o}{3}(r_o-r)^3\right\}\right.$$

$$\left. -\frac{1}{\Delta^2}\left\{-\frac{(r_o-r)^3}{3} + \frac{r_o}{2}(r_o-r)^2\right\}\right\}$$

$$+\frac{\partial u_c}{\partial z}\left\{\frac{1}{\Delta^2}\left\{\frac{(r_o-r)^4}{4} - \frac{r_o}{3}(r_o-r)^3\right\}\right.$$

$$\left. +\frac{2}{\Delta}\left\{-\frac{(r_o-r)^3}{3} + \frac{r_o}{2}(r_o-r)^2\right\}\right\}\Bigg)$$

Also it is assumed

$$\frac{T-T_\omega}{T_o-T_\omega} = \frac{3}{2}\,\frac{r_o-r}{\Delta_t} - \frac{1}{2}\,\frac{(r_o-r)^3}{\Delta_t^3}$$

and so

$$T = (T_o-T_\omega)\left\{\frac{3}{2}\,\frac{r_o-r}{\Delta_t} - \frac{1}{2}\,\frac{(r_o-r)^3}{\Delta_t^3}\right\} + T_\omega$$

$$\frac{\partial T}{\partial r} = \frac{3}{2}(T_o-T_\omega)\left\{-\frac{1}{\Delta_t} + \frac{(r_o-r)^2}{\Delta_t^3}\right\}$$

and

$$\frac{\partial T}{\partial z} = (T_o-T_\omega)\left\{-\frac{3}{2}\,\frac{(r_o-r)}{\Delta_t^2}\,\dot{\Delta}_t + \frac{3}{2}\,\frac{(r_o-r)^3}{\Delta_t^4}\,\dot{\Delta}_t\right\}$$

Also

$$\mu(T) = \mu_o(1 + A\theta)$$

$$\kappa(T) = \kappa_o(1 + B\theta)$$

Substituting all the above expressions into Eq. (A-1) one obtains

$$E = \iint \Bigg(\frac{1}{2} \left\{ \kappa_o - \kappa_o B \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t^o} - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^{o\,3}} - 1 \right\} \right\}$$

$$(T_o - T_\omega)^2 \frac{9}{4} \left\{ -\frac{1}{\Delta_t} + \frac{(r_o - r)^2}{\Delta_t^3} \right\}^2$$

$$+ \frac{1}{2} \left\{ \mu_o - \mu_o A \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t^o} - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^{o\,3}} - 1 \right\} \right\} \; 4 \; u_c^2 \left\{ \frac{r_o - r}{\Delta^2} - \frac{1}{\Delta} \right\}^2$$

$$+ \rho C_p u_c \left\{ -\frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\} \left\{ (T_o - T_\omega) \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t} \right. \right.$$

$$\left. \left. - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^3} \right\} + T_\omega \right\} \; \frac{3}{2} (T_o - T_\omega) \left\{ -\frac{r_o - r}{\Delta_t^{o\,2}} \dot{\Delta}_t^o + \frac{(r_o - r)^3}{\Delta_t^{o\,4}} \dot{\Delta}_t^o \right\}$$

$$+ \rho C_p \frac{1}{r} \left\{ 2 \; u_c \; \dot{\Delta}^o \left\{ -\frac{1}{\Delta^{o\,3}} \left\{ \frac{(r_o - r)^4}{4} - \frac{r_o}{3} (r_o - r)^3 \right\} \right. \right.$$

$$\left. \left. \left. - \frac{1}{\Delta^{o\,2}} \left\{ -\frac{(r_o - r)^3}{3} + \frac{r_o}{2} (r_o - r)^2 \right\} \right\} \right\} \right\}$$

$$\left\{ (T_o - T_\omega) \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t} - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^3} \right\} + T_\omega \right\}$$

$$\frac{3}{2} (T_o - T_\omega) \left\{ - \frac{1}{\Delta_t^o} + \frac{(r_o - r)^2}{\Delta_t^{o\,3}} \right\}$$

$$+ \rho C_p \frac{\partial u_c}{\partial z} \frac{1}{r} \left\{ \frac{1}{\Delta^{o\,2}} \left\{ \frac{(r_o - r)^4}{4} - \frac{r_o}{3} (r_o - r)^3 \right\} \right.$$

$$\left. + \frac{2}{\Delta^o} \left\{ - \frac{(r_o - r)^3}{3} + \frac{r_o}{2} (r_o - r)^2 \right\} \right\}$$

$$\left\{ (T_o - T_\omega) \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t} - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^3} + T_\omega \right\} \frac{3}{2} (T_o - T_\omega) \right.$$

$$\left. \left\{ - \frac{1}{\Delta_t^o} + \frac{(r_o - r)^2}{\Delta_t^{o\,3}} \right\} \right\}$$

$$- \rho u_c^2 \left\{ - \frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\}^2 2 u_c \dot{\Delta} \left\{ \frac{r_o - r}{\Delta^3}^2 - \frac{r_o - r}{\Delta^2} \right\}$$

$$- \rho u_c^2 \frac{\partial u_c}{\partial z} \left\{ - \frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\}^2 \left\{ - \frac{(r_o - r)^2}{\Delta^2} + 2 \frac{r_o - r}{\Delta} \right\}$$

$$- \rho u_c \left\{ - \frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\} \frac{1}{r} \left\{ 2 u_c \dot{\Delta}^o \left\{ - \frac{1}{\Delta^{o\,3}} \right. \right.$$

$$\left\{\frac{(r_o-r)^4}{4} - \frac{r_o}{3}(r_o-r)^3\right\}$$

$$- \frac{1}{\Delta^{o2}}\left\{-\frac{(r_o-r)^3}{3} + \frac{r_o}{2}(r_o-r)^2\right\}\Bigg\}\Bigg\} \quad 2\,u_c\left\{\frac{r_o-r}{\Delta^2} - \frac{1}{\Delta}\right\}$$

$$- \rho u_c\left\{-\frac{(r_o-r)^2}{\Delta^{o2}} + 2\,\frac{r_o-r}{\Delta^o}\right\} \frac{1}{r}\frac{\partial u_c}{\partial z}\left\{\frac{1}{\Delta^{o2}}\left\{\frac{(r_o-r)^4}{4}\right.\right.$$

$$\left. - \frac{r_o}{3}(r_o-r)^3\right\}$$

$$+ \frac{2}{\Delta^o}\left\{-\frac{(r_o-r)^3}{3} + \frac{r_o}{2}(r_o-r)^2\right\}\Bigg\} \quad 2\,u_c\left\{\frac{r_o-r}{\Delta^2} - \frac{1}{\Delta}\right\}$$

$$\left. - \rho u_c^o \frac{\partial u_c^o}{\partial z} u_c\left\{-\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}\right\}\right) r\,dr\,dz$$

$$+ \int_{r_o}^{r_o-\Delta} \rho u_c^3\left\{-\frac{(r_o-r)^2}{\Delta^{o2}} + 2\,\frac{r_o-r}{\Delta^o}\right\}\left\{-\frac{(r_o-r)^2}{\Delta^2}\right.$$

$$\left. + 2\,\frac{r_o-r}{\Delta}\right\}^2 r\,dr \qquad \text{.....(A-2)}$$

Taking variation of E (Eq. (A-2)) with respect to Δ_t, one obtains

$$\delta E = \int_o^{\ell}\int_{r_o}^{r_o-\Delta_t}\left(\frac{9}{4}\left\{\kappa_o - \kappa_o B\left\{\frac{3}{2}\,\frac{r_o-r}{\Delta_t^o} - \frac{1}{2}\,\frac{(r_o-r)^3}{\Delta_t^{o3}} - 1\right\}\right\}\right. \quad (T_o-T_\omega)^2$$

$$\left\{-\frac{1}{\Delta_t}+\frac{(r_o-r)^2}{\Delta_t^3}\right\}\left\{\frac{1}{\Delta_t^2}-3\frac{(r_o-r)^2}{\Delta_t^4}\right\}$$

$$+\frac{9}{4}\rho C_p u_c\left\{-\frac{(r_o-r)^2}{\Delta^{o^2}}+2\frac{r_o-r}{\Delta^o}\right\}(T_o-T_\omega)^2$$

$$\left\{-\frac{r_o-r}{\Delta_t^{o^2}}\dot{\Delta}_t^o+\frac{(r_o-r)^3}{\Delta_t^{o^4}}\dot{\Delta}_t^o\right\}\left\{-\frac{r_o-r}{\Delta_t^2}+\frac{(r_o-r)^3}{\Delta_t^4}\right\}$$

$$+\rho C_p\frac{1}{r}\left\{2u_c\dot{\Delta}^o\left\{-\frac{1}{\Delta^{o^3}}\left\{\frac{(r_o-r)^4}{4}-\frac{r_o}{3}(r_o-r)^3\right\}\right.\right.$$

$$\left.\left.\left.-\frac{1}{\Delta^{o^2}}\left\{-\frac{(r_o-r)^3}{3}+\frac{r_o}{2}(r_o-r)^2\right\}\right\}\right\}$$

$$\frac{9}{4}(T_o-T_\omega)^2\left\{-\frac{1}{\Delta_t^o}+\frac{(r_o-r)^2}{\Delta_t^{o^3}}\right\}\left\{-\frac{r_o-r}{\Delta_t^2}+\frac{(r_o-r)^3}{\Delta_t^4}\right\}$$

$$+\rho C_p\frac{\partial u_c}{\partial z}\frac{1}{r}\left\{\frac{1}{\Delta^{o^2}}\left\{\frac{(r_o-r)^4}{4}-\frac{r_o}{3}(r_o-r)^3\right\}\right.$$

$$\left.+\frac{2}{\Delta^o}\left\{-\frac{(r_o-r)^3}{3}+\frac{r_o}{2}(r_o-r)^2\right\}\right\}$$

$$\left.\frac{9}{4}(T_o-T_\omega)\left\{-\frac{1}{\Delta_t^o}+\frac{(r_o-r)^2}{\Delta_t^{o^3}}\right\}\left\{-\frac{r_o-r}{\Delta_t^2}+\frac{(r_o-r)^3}{\Delta_t^4}\right\}\right)\delta\Delta_t\, r\,dr\,dz$$

.....(A-3)

There is no longer any need to distinguish between the varied Δ_t and unvaried Δ_t^o versions of thermal boundary layer thickness. Dropping the superscripts and integrating between the limits r_o to $r_o-\Delta_t$, one obtains

$$\delta E = \int_o^{\ell} \Bigg(\kappa_o (T_o - T_\omega)^2 \left\{ \left\{ \frac{3}{5} + \frac{177}{320} B \right\} \frac{r_o}{\Delta_t^2} - \frac{3}{35} \frac{B}{\Delta_t} \right\}$$

$$+ \rho C_p \, u_c \dot{\Delta}_t \, (T_o - T_\omega)^2 \left\{ - \frac{3}{80} \frac{\Delta_t^2}{\Delta^2} + \frac{4}{35} \frac{\Delta_t}{\Delta} + \frac{2}{35} r_o \frac{\Delta_t}{\Delta^2} - \frac{3}{16} \frac{r_o}{\Delta} \right\}$$

$$+ \rho C_p \, u_c \dot{\Delta} \, (T_o - T_\omega)^2 \left\{ \frac{3}{160} \frac{\Delta_t^3}{\Delta^3} - \frac{4}{105} \frac{\Delta_t^2}{\Delta^2} - \frac{4}{105} r_o \frac{\Delta_t^2}{\Delta^3} \right.$$

$$\left. + \frac{3}{32} r_o \frac{\Delta_t}{\Delta^2} \right\}$$

$$- \rho C_p \frac{\partial u_c}{\partial z} (T_o - T_\omega)^2 \left\{ \frac{3}{320} \frac{\Delta_t^3}{\Delta^2} - \frac{4}{105} \frac{\Delta_t^2}{\Delta} - \frac{2}{105} r_o \frac{\Delta_t^2}{\Delta^2} \right.$$

$$\left. + \frac{3}{32} r_o \frac{\Delta_t}{\Delta} \right\} \Bigg) \delta\Delta_t \, dz \qquad \ldots\ldots \text{(A-4)}$$

Since δE must vanish for all $\delta\Delta_t$, therefore

$$\frac{\kappa_o}{\rho C_p} \left\{ \frac{3}{5} + \frac{177}{320} B \right\} r_o - \frac{3}{35} B \frac{\kappa_o}{\rho C_p} \Delta_t$$

$$+ u_c \left\{ - \frac{3}{80} \frac{\Delta_t^4}{\Delta^2} \dot{\Delta}_t + \frac{4}{35} \frac{\Delta_t^3}{\Delta} \dot{\Delta}_t + \frac{2}{35} r_o \frac{\Delta_t^3}{\Delta^2} \dot{\Delta}_t - \frac{3}{16} r_o \frac{\Delta_t^2}{\Delta} \dot{\Delta}_t \right.$$

$$+ \frac{1}{15} Y^{*4} \Delta^{*2} \dot{\Delta}^{*} - \frac{21}{1280} Y^{*5} \Delta^{*2} \dot{\Delta}^{*} \Big\}$$

$$- \left\{ \frac{\frac{2}{3}\dot{\Delta}^{*} - \frac{1}{3}\Delta^{*}\dot{\Delta}^{*}}{1 - \frac{2}{3}\Delta^{*} + \frac{1}{6}\Delta^{*2}} \right\} \left\{ \frac{21}{256} Y^{*3}\Delta^{*2} - \frac{1}{60} Y^{*4}\Delta^{*2} - \frac{1}{30} Y^{*4}\Delta^{*3} \right.$$

$$\left. + \frac{21}{2560} Y^{*5}\Delta^{*3} \right\} = 0 \qquad .....(A-6)$$

Similarly, taking the variation of E (Eq. (A-2)) with respect to Δ, one obtains

$$\delta E = \int_0^{\ell}\int_{r_o}^{r_o-\Delta} \Bigg(4\ u_c^2 \left\{ \mu_o - \mu_o A \left\{ \frac{3}{2}\frac{r_o-r}{\Delta_t^o} - \frac{1}{2}\frac{(r_o-r)^3}{\Delta_t^{o\,3}} - 1 \right\}\right\}$$

$$\left\{ \frac{r_o-r}{\Delta^2} - \frac{1}{\Delta} \right\} \left\{ -2\frac{r_o-r}{\Delta^3} + \frac{1}{\Delta^2} \right\} \delta\Delta$$

$$- 2\ \rho u_c^3 \left\{ -\frac{(r_o-r)^2}{\Delta^{o\,2}} + 2\frac{r_o-r}{\Delta^o} \right\}^2 \left\{ \left\{ -3\frac{(r_o-r)^2}{\Delta^4} \right.\right.$$

$$\left. + 2\frac{r_o-r}{\Delta^3} \right\} \dot{\Delta}\,\delta\Delta + \left\{ \frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2} \right\} \delta\dot{\Delta} \Bigg\}$$

$$- 2\ \rho u_c^2 \frac{\partial u_c}{\partial z} \left\{ -\frac{(r_o-r)^2}{\Delta^{o\,2}} + 2\frac{r_o-r}{\Delta^o} \right\}^2$$

$$\left\{ \frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2} \right\} \delta\Delta$$

$$+ \frac{1}{15} Y^{*4} \Delta^{*2} \dot{\Delta}^{*} - \frac{21}{1280} Y^{*5} \Delta^{*2} \dot{\Delta}^{*} \Big\}$$

$$- \left\{ \frac{\frac{2}{3} \dot{\Delta}^{*} - \frac{1}{3} \Delta^{*} \dot{\Delta}^{*}}{1 - \frac{2}{3} \Delta^{*} + \frac{1}{6} \Delta^{*2}} \right\} \left\{ \frac{21}{256} Y^{*3} \Delta^{*2} - \frac{1}{60} Y^{*4} \Delta^{*2} - \frac{1}{30} Y^{*4} \Delta^{*3} \right.$$

$$\left. + \frac{21}{2560} Y^{*5} \Delta^{*3} \right\} = 0 \qquad \text{.....(A-6)}$$

Similarly, taking the variation of E (Eq. (A-2)) with respect to Δ, one obtains

$$\delta E = \int_{o}^{\ell} \int_{r_o}^{r_o - \Delta} \Bigg(4 u_c^2 \left\{ \mu_o - \mu_o A \left\{ \frac{3}{2} \frac{r_o - r}{\Delta_t^o} - \frac{1}{2} \frac{(r_o - r)^3}{\Delta_t^{o\,3}} - 1 \right\} \right\}$$

$$\left\{ \frac{r_o - r}{\Delta^2} - \frac{1}{\Delta} \right\} \left\{ -2 \frac{r_o - r}{\Delta^3} + \frac{1}{\Delta^2} \right\} \delta\Delta$$

$$- 2 \rho u_c^3 \left\{ - \frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\}^2 \left\{ \left\{ -3 \frac{(r_o - r)^2}{\Delta^4} \right. \right.$$

$$\left. \left. + 2 \frac{r_o - r}{\Delta^3} \right\} \dot{\Delta}\, \delta\Delta + \left\{ \frac{(r_o - r)^2}{\Delta^3} - \frac{r_o - r}{\Delta^2} \right\} \delta\dot{\Delta} \right\}$$

$$- 2 \rho u_c^2 \frac{\partial u_c}{\partial z} \left\{ - \frac{(r_o - r)^2}{\Delta^{o\,2}} + 2 \frac{r_o - r}{\Delta^o} \right\}^2$$

$$\left\{ \frac{(r_o - r)^2}{\Delta^3} - \frac{r_o - r}{\Delta^2} \right\} \delta\Delta$$

$$- 4 \rho u_c^3 \dot{\Delta}^o \left\{- \frac{(r_o-r)^2}{\Delta^{o2}} + 2 \frac{r_o-r}{\Delta^o}\right\} \frac{1}{r} \left\{- \frac{1}{\Delta^{o3}} \left\{\frac{(r_o-r)^4}{4}\right.\right.$$

$$\left.\left. - \frac{r_o}{3} (r_o-r)^3\right\} - \frac{1}{\Delta^{o2}} \left\{- \frac{(r_o-r)^3}{3} + \frac{r_o}{2} (r_o-r)^2\right\}\right\}$$

$$\left\{- 2 \frac{r_o-r}{\Delta^3} + \frac{1}{\Delta^2}\right\} \delta\Delta$$

$$- 2 \rho u_c^2 \frac{\partial u_c}{\partial z} \frac{1}{r} \left\{- \frac{(r_o-r)^2}{\Delta^{o2}} + \frac{2(r_o-r)}{\Delta^o}\right\}\left\{ \frac{1}{\Delta^{o2}} \left\{ \frac{(r_o-r)^4}{4}\right.\right.$$

$$\left.\left. - \frac{r_o}{3} (r_o-r)^3\right\} + \frac{2}{\Delta^o} \left\{- \frac{(r_o-r)^3}{3} + \frac{r_o}{2} (r_o-r)^2\right\}\right\}$$

$$\left\{- 2 \frac{(r_o-r)}{\Delta^3} + \frac{1}{\Delta^2}\right\} \delta\Delta$$

$$\left. - 2 \rho u_c^o \frac{\partial u_c^o}{\partial z} u_c \left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2}\right\} \delta\Delta \right) r \, dr \, dz$$

$$+ \int_{r_o}^{r_o-\Delta} 2 \rho u_c^3 \left\{- \frac{(r_o-r)^2}{\Delta^{o2}} + 2 \frac{r_o-r}{\Delta^o}\right\}^2$$

$$\left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{(r_o-r)}{\Delta^2}\right\} \delta\Delta \, r \, dr \qquad \text{.....(A-7)}$$

As before, there is no longer any need to distinguish between the varied Δ and unvaried Δ^{o} versions of the momentum boundary layer thickness. By using the fact that $\delta\dot{\Delta} = \frac{d(\delta\Delta)}{dz}$, the term involving $\delta\dot{\Delta}$ on the right hand side of Eq. (A-7) can be written as

$$I = -2\int_{o}^{\ell}\int_{r_o}^{r_o-\Delta} \rho u_c^3 \left\{-\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}\right\}^2 \left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2}\right\} r\,dr\,dz\,\frac{d}{dz}(\delta\Delta) \qquad \text{.....(A-8)}$$

Integration by parts yields

$$\begin{aligned} I = &-2\int_{r_o}^{r_o-\Delta} \rho u_c^3 \left\{-\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}\right\}^2 \left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2}\right\} \delta\Delta\, r\,dr \\ &+ 2\int_{o}^{\ell}\int_{r_o}^{r_o-\Delta} \frac{d}{dz}\left\{\rho u_c^3 \left\{-\frac{(r_o-r)^2}{\Delta^2} + 2\,\frac{r_o-r}{\Delta}\right\}^2 \left\{\frac{(r_o-r)^2}{\Delta^3} - \frac{r_o-r}{\Delta^2}\right\}\right\} \delta\Delta\, r\,dr\,dz \end{aligned} \qquad \text{.....(A-9)}$$

The first term cancels the contribution of the line integral of Eq. (A-7).

Finally, dropping the superscript and integrating from r_o to $r_o - \Delta$, Eq. (A-7) becomes

$$\delta E = \int_0^{\ell} \left(\left(\frac{2}{3} \mu_o \, u_c^2 \, \frac{r_o}{\Delta^2} - \mu_o u_c^2 A \right\} \frac{1}{\Delta} \left\{ \frac{2}{5} \frac{\Delta_t^2}{\Delta^2} - \frac{1}{2} \frac{\Delta_t^3}{\Delta^3} + \frac{6}{35} \frac{\Delta_t^4}{\Delta^4} \right\} . \right.$$

$$+ \frac{r_o}{\Delta^2} \left\{ - \frac{3}{2} \frac{\Delta_t}{\Delta} + \frac{6}{5} \frac{\Delta_t^2}{\Delta^2} - \frac{1}{3} \frac{\Delta_t^3}{\Delta^3} \right\} \Bigg\}$$

$$+ \rho u_c^3 \, \dot{\Delta} \left\{ \frac{11}{105} - \frac{4}{21} \frac{r_o}{\Delta} \right\} + \rho u_c^2 \, \frac{\partial u_c}{\partial z} \left\{ - \frac{47}{140} \Delta + \frac{19}{35} r_o \right\}$$

$$- \rho u_c^2 \, \frac{\partial u_c}{\partial z} \left\{ - \frac{47}{420} \Delta + \frac{19}{105} r_o \right\} + \rho u_c^3 \, \dot{\Delta} \left\{ - \frac{83}{1260} + \frac{43}{315} \frac{r_o}{\Delta} \right\}$$

$$\left. + \rho u_c^2 \, \frac{\partial u_c}{\partial z} \left\{ \frac{331}{2520} \Delta - \frac{74}{315} r_o \right\} - \rho u_c^2 \, \frac{\partial u_c}{\partial z} \left\{ - \frac{\Delta}{6} + \frac{r_o}{3} \right\} \right) \delta\Delta \; dz$$

.....(A-10)

Since δE must vanish for all $\delta\Delta$, therefore

$$\frac{2}{3} r_o - A \left\} \Delta \left\{ \frac{2}{5} \frac{\Delta_t^2}{\Delta^2} - \frac{1}{2} \frac{\Delta_t^3}{\Delta^3} + \frac{6}{35} \frac{\Delta_t^4}{\Delta^4} \right\} + r_o \left\{ - \frac{3}{2} \frac{\Delta_t}{\Delta} + \frac{6}{5} \frac{\Delta_t^2}{\Delta^2} \right. \right.$$

$$\left. \left. - \frac{1}{3} \frac{\Delta_t^3}{\Delta^3} \right\} \right\}$$

$$+ \frac{u_c}{\nu} \left\{ \frac{49}{1260} \Delta^2 \, \dot{\Delta} - \frac{17}{315} r_o \, \Delta \, \dot{\Delta} \right\} + \frac{1}{\nu} \frac{\partial u_c}{\partial z} \left\{ \frac{187}{2520} \Delta^3 - \frac{13}{63} \Delta^2 \right\} = 0$$

.....(A-11)

Introducing the non-dimensional quantities and substituting for u_c and $\frac{\partial u_c}{\partial z}$, one obtains

$$\frac{2}{3} - A\left[\Delta^*\left[\frac{2}{5} Y^{*2} - \frac{1}{2} Y^{*3} + \frac{6}{35} Y^{*4}\right] - \frac{3}{2} Y^* + \frac{6}{5} Y^{*2} - \frac{1}{3} Y^{*3}\right]$$

$$+ \frac{1}{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}}\left[\frac{49}{5040}\Delta^{*2}\dot{\Delta}^* - \frac{17}{1260}\Delta^*\dot{\Delta}^*\right]$$

$$+ \frac{\frac{2}{3}\dot{\Delta}^* - \frac{1}{3}\Delta^*\dot{\Delta}^*}{\left[1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right]^2}\left[\frac{187}{10080}\Delta^{*3} - \frac{13}{252}\Delta^{*2}\right] = 0 \quad \text{.....(A-12)}$$

According to assumption of $\Delta_t < \Delta$, Eqs. (A-6) and (A-12) can be simplified by neglecting the higher order terms Y^{*4} and Y^{*5} and giving:

$$\left[\left[\frac{21}{10} + \frac{1239}{640} B\right] \frac{1}{Pr_o} - \frac{3}{10}\frac{B}{Pr_o} Y^* \Delta^*\right]\left[1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right]$$

$$+ \left[-\frac{21}{128} Y^{*2}\dot{Y}^*\Delta^{*2} + \frac{1}{20} Y^*\dot{Y}^*\Delta^{*2} + \frac{1}{10} Y^{*3}\dot{Y}^*\Delta^{*3} - \frac{21}{256} Y^{*3}\Delta^*\dot{\Delta}^*\right]$$

$$-\left[\frac{\frac{2}{3}\dot{\Delta}^* - \frac{1}{3}\Delta^*\dot{\Delta}^*}{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}}\right]\frac{21}{256} Y^{*3}\Delta^{*2} = 0 \quad \text{.....(A-13)}$$

$$\frac{2}{3} - A\left[\Delta^* \left[\frac{2}{5}Y^{*2} - \frac{1}{2}Y^{*3}\right] - \frac{3}{2}Y^* + \frac{6}{5}Y^{*2} - \frac{1}{3}Y^{*3}\right]$$

$$+ \frac{1}{1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}}\left[\frac{49}{5040}\Delta^{*2}\dot{\Delta}^* - \frac{17}{1260}\Delta^*\dot{\Delta}^*\right]$$

$$+ \frac{\frac{2}{3}\dot{\Delta}^* - \frac{1}{3}\Delta^*\dot{\Delta}^*}{\left[1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right]^2}\left[\frac{187}{10080}\Delta^{*3} - \frac{13}{252}\Delta^{*2}\right] = 0 \qquad \text{.....(A-14)}$$

APPENDIX B

THE INTEGRAL METHOD FOR THE FLOW WITH CONSTANT PROPERTIES

The problem of laminar flow heat transfer in a circular tube with simultaneously developing velocity and temperature profiles has been studied by many investigators (1, 3, 4, 29). All of them used various numerical methods to solve the momentum and energy differential equations which are lengthy and require a large computational time. The integral method is very simple to use and within the knowledge of the author, no one has used the integral method for solving the laminar flow heat transfer problem in a circular duct. The method yields a simple expression for predicting the momentum and thermal boundary layer thicknesses in a circular tube.

MOMENTUM INTEGRAL EQUATION

Momentum equation in momentum integral form can be written by considering a control volume as shown in Fig. (1). If the net force on the control volume is equated to the outgoing flux of momentum, one obtains the equation:

$$- dp\, \pi \frac{d^2}{4} - \tau_\omega \pi\, d\, dz = d \int_0^\Delta \pi \rho u^2 (d-2y)\, dy + d\left[\rho \frac{\pi}{4} (d-2\Delta)^2 u_c^2\right]$$

.....(B-1)

By the use of Eq. (4) and using $\tau_\omega = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0}$, the Eq. (B-1) reduces to the form

$$u_c \frac{\partial u_c}{\partial z} - 2\frac{\nu}{r_o}\left[\frac{\partial u}{\partial y}\right]_{y=0} = \frac{2}{r_o^2}\frac{d}{dz}\left[u_c^2\int_0^{\Delta}\frac{u^2}{u_c^2}(r_o-y)\,dy\right]$$

$$+\frac{d}{dz}\left[\left[1-\frac{\Delta}{r_o}\right]^2 u_c^2\right] \qquad \text{(B-2)}$$

All of the terms in this equation can then be evaluated by the use of the Eq. (19), which after rearranging the terms gives

$$-16z^* = \int_0^{\Delta^*}\left[\frac{P_1(1-2P_2-2P_3)}{P_4^2}+\frac{P_5+P_6}{P_4}\right]\Delta^*\,d\Delta^* \qquad \text{.....(B-3)}$$

where

$$P_1 = -\frac{2}{3}+\frac{1}{3}\Delta^*$$

$$P_2 = \frac{16}{15}\Delta^* - \frac{11}{15}\Delta^{*2}$$

$$P_3 = 1 - 2\Delta^* + \Delta^{*2}$$

$$P_4 = 1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}$$

$$P_5 = \frac{16}{15} - \frac{22}{15}\Delta^*$$

$$P_6 = -2 + 2\Delta^*$$

Eq. (B-3) is a transcendental equation in Δ^* and can be solved by using any one of the iterative schemes available. Interval Halving Technique was used in the present study to solve Eq. (B-3) for Δ^*.

ENERGY INTEGRAL EQUATION

The heat flow equation for the thermal boundary layer in a tube can be obtained from Fig. (1). Energy flows into the differential annulus by convection and by conduction through plane 1 and leaves through plane 2. In addition to the energy, there is a radial flow of energy (heat) from the tube surface by conduction and a radial flow by convection at Δ_t. No radial heat flow due to temperature gradient takes place at Δ_t because the temperature gradient is, by definition of thermal boundary layer, zero at the edge of the thermal boundary layer. Neglecting the axial conduction term, and equating the heat energy entering the annulus to heat leaving gives

$$2 \pi r_o q_o dz = d\left[\int_o^{\Delta_t} C_p \ t \ \rho \ u \ 2\pi \ (r_o - y) \ dy\right] - C_p \ t_o \ d\left[\int_o^{\Delta_t} \rho \ u \ 2 \pi \ (r_o - y) \ dy\right] \qquad \text{.....(B-4)}$$

which after algebraic manipulation reduces to

$$z^* = \frac{Pr_o}{2} \frac{\Delta_t^{*3}}{\Delta^*} \frac{1}{\left[1 - \frac{2}{3}\Delta^* + \frac{1}{6}\Delta^{*2}\right]} \left[\frac{1}{15} - \frac{1}{72}\frac{\Delta_t^*}{\Delta^*} - \frac{1}{36}\Delta_t^* + \frac{1}{140}\frac{\Delta_t^{*2}}{\Delta^*}\right] \qquad \text{.....(B-5)}$$

Eq. (B-5) is the energy integral equation which is used to calculate Δ_t^* for known value of Δ^* from Eq. (B-3) by any iterative method. Newton-Raphson method was used in the present study to solve the above equation.

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