OPTIMAL CONTROL OF AN ECONOMIC SYSTEM MODELLED ON THE BASIS OF THE CYCLICAL GROWTH THEORY

by

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TO MY PARENTS

AND MY WIFE

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ABSTRACT

This thesis presents an economic model based on the Cyclical Growth Theory developed by Keynes-Phillips-Bergstrom (KPB) (1,2). The KPB model has been modified by labeling certain economic factors as the control variables. By introducing realistic constraints on the control variables and a suitable performance criterion and applying Pontryagin's Maximum Principle (13), optimal control laws have been derived. Numerical results are obtained using the Davidon-Fletcher-Powell method (3,6,11) with Fibonacci Search technique (7,8,11,16).

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INTRODUCTION

The economic standing of a country can be measured by several economic standards. Among the most important considered are the standard of living, per capita consumption, fluctuation of the price level and employment level. Thus it is important to develop a realistic economic model incorporating all these factors, so that it can be used for the purpose of prediction and regulation of national economic trends.

This thesis presents a national economic model based on the Cyclical Growth Theory developed by Keynes, Phillips and Bergstrom (2). The model appears to be more realistic than those used in the past (4, 15) because of its inclusion of intrinsic feedback mechanism that exists between the relevant economic variables.

However the original KPB Model is not suitable for direct application of optimal control theory. Therefore the KPB model has been modified by labeling certain economic factors such as "Marginal Propensity to Save", "Employment Ratio" and "Price Factor as a variable positive multiple of average unit-production-cost" as the control variables and introducing realistic constraints on these variables. The optimal control laws have been derived by applying Pontryagin's Maximum Principle (13). The optimality problem is then reduced to a two-point boundary value problem (TPBVP). Numerical results are obtained using the Davidon-Fletcher-Powell method with Fibonacci Search technique.

In Chapter I a brief review of the fundamental results of Pontryagin et al. (13) are presented for reference.

In Chapter II, an economic model based on the Cyclical

Growth Theory developed by Keynes-Phillips -Bergstrom (2) is reviewed. This model is then modified suitably to incorporate variables that could be considered as control parameters. In the last section of this chapter two types of optimal control problems are formulated and by application of Pontryagin's Maximum Principle and transversality condition (13) optimal control policies are derived.

In Chapter III iterative method developed by Davidon-Fletcher-Powell (3, 6, 11) is used along with Fibonacci Search technique (7, 8, 11, 16) to solve the two-point boundary value problems arising from the optimality conditions given in Chapter II. A complete flow chart showing the detailed computer program used in solving the problem along with numerical results are presented.

A brief discussion of the results is presented in the conclusion section.

CHAPTER I

A BRIEF REVIEW OF PONTRYAGIN'S MAXIMUM PRINCIPLE

INTRODUCTION

In this section we will state without proof the Maximum Principle and the Transversality Condition of Pontryagin et al (13). These results are used in Chapter II to derive necessary conditions for optimality of an economic system.

(1.1) BASIC ASSUMPTIONS :

It is assumed that the control system is described by a system of differential equations :

$$\frac{d}{dt} x^i = f^i(x, u)$$

$$i = 1, 2, \ldots, n \qquad (1.1.1)$$

In vector form :

$$\frac{d}{dt} x = f(x, u) . \qquad (1.1.2)$$

where $f(x, u) \triangleq (f^1, \ldots, f^n)$,

$x = (x_1, x_2, \ldots, x_n)$ is the state vector, and $u = (u_1, u_2, \ldots, u_r)$ is the control vector. The functions f^i are defined for $x \in E^n$ and for $u \in U$ where U is the control region, which may be any set in some n-dimensional Euclidean space E^n. It is also assumed that $\frac{\partial}{\partial x_j} f^i(x_1, x_2, \ldots, x_n, u)$; i, j = 1, 2, ..., n is continuous on $E^n \times U$.

In practical problems the control region U is usually a closed bounded set in E^n representing constraints on control efforts.

(1.2) FORMULATION OF THE OPTIMAL CONTROL PROBLEM

Let U denote the set of all bounded measurable functions defined on $I = [t_o, t_1]$ to E^n such that for each $u \in U$, $u(t) \in U$ a.e. . The set U is the class of admissible controls.

The functional to be minimized is $J = \int_{t_o}^{t_1} f^o(x,u)dt$,

where it is assumed that $f^o(x,u)$ satisfies the same conditions as the functions $f^i(x,u), i = 1, 2, \ldots, n$.

Define a new phase co-ordinate $x^o = f^o(x,u)$. By adjoining the new co-ordinate, we obtain the (n+1) dimensional phase space E^{n+1}. Then equation (1.1.2) takes the form

$$\frac{d}{dt}\bar{x} = \bar{f}(x,u) \qquad \text{where } \bar{x} = (x^o, x_1, x_2, \ldots, x_n) \tag{1.2.1}$$

and,

$$\bar{f}(x,u) = (f^o(x,u), f^1(x,u), \ldots, f^n(x,u)) .$$

(1.3) STATEMENT OF THE FUNDAMENTAL PROBLEM

In the (n+1) dimensional phase space E^{n+1} the point $\bar{x}_o = (o, x_o)$ and the line $\pi \triangleq \{(x^o, x_1) : x^o \in E\}$ is given. The line π is parallel to the x^o axis, passing through the point $(0, x_1)$. Among the admissible controls $u = u(t)$, having the property that the corresponding solution $\bar{x}(t)$ of equation (1.2.1) with initial conditions $\bar{x}(t_o) = \bar{x}_o$ intersects π, find one whose point of intersection with π has the smallest co-ordinate x^o .

The solution to this problem is given by the Maximum Principle of Pontryagin. These results are given below.

(1.4) THE MAXIMUM PRINCIPLE

We consider the following function H (known as the Hamiltonian) :

$$H = (\psi, f(x,u)) = \sum_{\alpha=0}^{n} \psi_\alpha f^\alpha(x,u)$$

where $\psi = (\psi_0, \psi_1, \ldots, \psi_n)$ is an auxiliary vector function, satisfying the differential equation ,

$$\frac{d}{dt}\psi_i = -\sum_{\alpha=0}^{n} \frac{\partial}{\partial x_i} f^\alpha(x,u)\,\psi_\alpha \qquad (1.4.1)$$

The above systems (1.2.1) and (1.4.1) may be rewritten with the aid of this function H in the form of the following systems of coupled differential equations :

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial \psi_i} \qquad i = 0, 1, \ldots, n \qquad (1.4.2)$$

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad i = 0, 1, \ldots, n \qquad (1.4.3)$$

With this preparation we present the Maximum Principle of Pontryagin in the following theorem.

THEOREM 1.4.1. (Maximum Principle Pontryagin et al, (13) ; Th 8, pp 81) .

Let u(t), $t_0 \leq t \leq t_1$, be an admissible control such that the corresponding trajectory $\overline{x}(t)$ (1.4.2) which starts at the point

$\bar{x}_o$ at time t_o is defined on the interval $t_o \le t \le t_1$, and passes at time t_1 through a point on the line π. In order that u(t) and x(t) be optimal it is necessary that there exist a nonzero absolutely continuous vector function $\psi(t) = (\psi_o(t), \psi_1(t), \ldots, \psi_n(t))$ corresponding to the functions u(t) and x(t) (1.4.3) such that :

(1) The function $H(\psi(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$ almost everywhere in the interval $t_o \le t \le t_1$,

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) \qquad (1.4.4)$$

where $M(\psi, x) = \max_{u \in U} H(\psi, x, u)$.

(2) at the terminal time t_1 the relations

$$\psi_o(t_1) \le 0, \quad M(\psi(t_1), x(t_1)) = 0 \qquad (1.4.5)$$

are satisfied. Furthermore if $\psi(t)$, $x(t)$, and $u(t)$ satisfy systems (1.4.2), (1.4.3) and condition (1), the time functions $\psi_o(t)$ and $M(\psi(t), x(t))$ are constant. Thus (1.4.5) may be verified at any time t, $t_o \le t \le t_1$ and not just at t_1.

For the time optimal case, $f^o(x,u) \equiv 1$, and we consider the Hamiltonian function,

$H = \psi_o + \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x,u)$, where ψ is a n-dimensional vector (i.e. $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$). Here we consider n-dimensional phase space only. Since ψ_o is constant the Hamiltonian system becomes :

$$\frac{d}{dt} x_i = \frac{\partial H}{\partial \psi_i} \qquad i = 1, 2, \ldots, n \qquad (1.4.6)$$

$$\frac{d}{dt} \psi_i = -\frac{\partial H}{\partial x^i} \qquad i = 1, 2, \ldots, n \qquad (1.4.7)$$

The necessary condition for time-optimality is given below .

THEOREM 1.4.2 (Pontryagin et al (13) ; Th 2, pp 20-21)

Let $u(t)$, $t_o \leq t \leq t_1$, be an admissible control which transfers the phase point from x_o to x_1, and let $x(t)$, $t \in [t_o, t_1]$ be the corresponding trajectory (1.4.6) so that $x(t_o) = x_o$, $x(t_1) = x_1$. In order that $u(t)$ and $x(t)$ be time-optimal it is necessary that there exist a nonzero, continuous vector function $\psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t))$ corresponding to $u(t)$ and $x(t)$ such that :

(1) for all t, $t_o \leq t \leq t_1$, the function $H(\psi(t), x(t), u(t))$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$:

i.e $H(\psi(t), x(t), u(t)) = M(\psi(t), x(t))$; $\qquad (1.4.8)$

(2). at the terminal t_1 the relation

$$M(\psi(t_1)\ x(t_1)) \geq 0 \qquad (1.4.9)$$

is satisfied. Furthermore if $\psi(t)$, $x(t)$, and $u(t)$ satisfy systems (1.4.6) , (1.4.7) and condition (1), the time function $M(\psi(t), x(t))$ is constant. Thus (1.4.9) may be verified at any time $t \in [t_o, t_1]$ and not just at t_1.

(1.5) THE TRANSVERSALITY CONDITION.

Let s_o and s_1 be a pair of fixed (smooth) manifolds in R^n and suppose $x_o \in s_o$ and $x_1 \in s_1$ be certain points and T_o and T_1 the tangent planes of s_o and s_1 which pass through these points. The planes T_o and T_1 are in E^n and have dimensions r_o and r_1 respectively. Let $u(t)$, $x(t)$, $t_o \leq t \leq t_1$, be the solution of the optimal control problem with fixed end points x_o and x_1 and let $\psi(t)$ be a vector whose existence is assured by theorem 1.4.1. The vector $\psi(t)$ satisfies the transversality condition at the right-hand end points of the trajectory $x(t)$ (i.e $x(t_1)$), if the vector $\psi(t_1) = (\psi_1(t_1), \psi_2(t_1), \ldots, \psi_n(t_1))$ is orthogonal to T_1.

The transversality condition signifies that $(\psi(t_1), \theta) = 0$ for every vector $\theta = (\theta^1, \theta^2, \ldots, \theta^n)$ belonging (or parallel) to T_1. Similar transversality condition holds true at the left hand end point, that is $(\psi(t_o), \theta) = 0$ for all $\theta \in T_o$.

THEOREM 1.5.1 (Transversality condition)

Let $u(t)$, $t_o \leq t \leq t_1$, be an admissible control which transfers the phase point from some position $x_o \in s_o$ to the position $x_1 \in s_1$, and let $x(t)$ be the corresponding trajectory (starting at the point $x_o = (0, x_o)$). In order that $u(t)$ and $x(t)$ yield the solution of the optimal control problem with variable end-points, it is necessary that there exist a nonzero continuous vector function $\psi(t)$, which satisfies the conditions of Theorem 1.4.1 and the transversality conditions at both the end-points of the trajectory $x(t)$.

CHAPTER II

OPTIMAL CONTROL OF AN ECONOMIC SYSTEM MODELLED ON THE BASIS OF THE CYCLICAL GROWTH THEORY

INTRODUCTION

In this Chapter a national economic model based on the Cyclical Growth Theory developed by Keynes, Phillips and Bergstrom (1, 2) is presented. Due to inclusion of intrinsic feedback mechanisms that exist between the relevant economic variables, this model appears to be more realistic than those used in the past (4, 11). However the original KPB model is not suitable for direct application of optimal control theory. Therefore the model here has been modified by labeling the "marginal propensity to save", "employment ratio" and "the price level as a (variable) multiple of unit - production - cost" as the control variables.

(2. 1) THE ECONOMIC MODEL

In this section we present the economic model based on the Cyclical Growth Theory (2, p 53).

The model is described by the following system of non-linear differential equations :

$$
S_o \left\{
\begin{aligned}
\frac{dC}{dt} &= \alpha C \log \left\{ \frac{(1-s) Y}{C} \right\} \\
\frac{dK}{dt} &= \beta K \log \left\{ \frac{PY - WL_e}{(1+g) RPK} \right\} \\
\frac{dY}{dt} &= \gamma \left\{ C + \beta K \log \left(\frac{PY - WL_e}{(1+g) RPK} \right) - Y \right\} \\
\frac{dW}{dt} &= \lambda W \log \left(\frac{L_e}{L_s} \right) + aW \\
\frac{dR}{dt} &= \delta R \log \left(\frac{M_d}{M_s} \right) \\
\frac{dM_s}{dt} &= \theta M_s \log \left(\frac{L_s}{L_e} \right) \\
M_d &= \Gamma Y^{\mu} R^{-\nu} \\
P &= B W L_e / Y
\end{aligned}
\right.
$$

where α, β, γ, λ, δ, θ, Γ, μ, B, g, a and ν are positive constants. The state of the system (at any time) is described by the vector $x \triangleq (C, K, Y, W, R, M_s)$ where $C \triangleq$ per capita consumption, $K \triangleq$ amount of fixed capital, $Y \triangleq$ gross national production (GNP), $W \triangleq$ wage rate, $R \triangleq$ interest rate, $M_s \triangleq$ supply of money, $M_d \triangleq$ real demand for money and $P \triangleq$ Price level.

(2.2) MODEL OF ECONOMIC SYSTEM INCLUDING CONTROLS

As stated earlier, the original KPB model is not suitable for direct application of optimal control theory. Therefore it has been modified by labeling "Marginal Propensity to Save", Employment Ratio and Price Index as a "variable mutliple of unit-production cost" as the control variables.

It appears reasonable to assume that the economic system described above can be controlled through appropriate choice of such variables as s - "the Marginal Propensity to Save", $\frac{L_e}{L_s}$ - "the employment ratio" and P - " the price level". Since choice of saving factor s determines the proportional investment through the relation,

$$I = sY \tag{2.2.1}$$

we will choose s, denoted as u_1, as one of the control variables. The second control variable will be chosen as the employment ratio,

$$u_2 = \frac{L_e}{L_s} \tag{2.2.2}$$

where L_s is the actual supply of labor proportional to the population growth and L_e is the labour employed. Furthermore since price is related to unit production cost ($\frac{WL_e}{Y}$) we will consider the factor

B(system equation S_0), denoted as u_3, also as one of the other control variables. In this situation,

$$P \triangleq u_3 \left(\frac{WL_e}{Y} \right) \qquad (2.2.3)$$

with $u_3 \geq 1$.

THE CONTROL WITH CONSTRAINTS

The control vector $u \triangleq (u_1, u_2, u_3)$ has natural constraints. Obviously $u_1 \geq 0$ and to allow for positive investment it is required that $u_1 < 1$. Thus $0 < u_1 \leq \omega_1$ for any preassigned positive number $\omega_1 < 1$. For the control variable u_2 it is clear that $0 \leq u_2 \leq 1$. However to limit unemployment to 5%, for example, u_2 can be chosen to satisfy the inequality $\omega_2 \leq u_2 \leq 1$ with $\omega_2 \geq .95$. Similarly to limit profit margins and fluctuation of price levels it may be decided to put an upper limit on u_3, for example $\omega_3' \leq u_3 \leq \omega_3$, where $\omega_3' \geq 1$ and ω_3 can be equal or less than 2.

THE SYSTEM MODEL WITH CONTROLS

The economic model in the modified form incorporating the control variables (2.2.1), (2.2.2) and (2.2.3) as described above is given by the following system of nonlinear differential equations :

$$S \begin{cases} \dfrac{dC}{dt} = \alpha C \log \left\{ \dfrac{(1-u_1)\,Y}{C} \right\} & \triangleq f^1 \\[2ex] \dfrac{dK}{dt} = \beta K \log \left\{ \dfrac{(u_3-1)\;Y}{(1+g)\,R\,u_3 K} \right\} & \triangleq f^2 \\[2ex] \dfrac{dY}{dt} = \gamma \left\{ C + \beta K \log \left(\dfrac{(u_3-1)\;Y}{(1+g)\,R\,u_3 K} \right) - Y \right\} & \triangleq f^3 \\[2ex] \dfrac{dW}{dt} = \lambda W \log (u_2) + aW & \triangleq f^4 \\[2ex] \dfrac{dR}{dt} = \delta R \log \left(\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}} \right) & \triangleq f^5 \\[2ex] \dfrac{dM_s}{dt} = \theta M_s \log \left(\dfrac{1}{u_2} \right) & \triangleq f^6 \end{cases}$$

In the following section two kinds of optimal control problem are formulated, the fixed time optimal control problem and the time optimal control problem.

(2.3) FIXED TIME OPTIMAL CONTROL PROBLEM

The prosperity of a country can be roughly measured by the standard of living and hence by its per capita consumption level. Thus a desirable terminal state of the economic system S may be described by the manifold,

$$M_1 \triangleq \{ C(t_1) = C_1,\ K(t_1) = K_1, Y(t_1) = Y_1, W(t_1) > 0, R(t_1) > 0, M_s(t_1) > 0 \} \subset E^6,$$

Where t_1 is the prescribed terminal time, and C_1, K_1 and Y_1 are the desired targets.

Now the problem is : subject to the dynamic constraint S and assuming, for simplicity, that u_2, u_3 are fixed, find a control policy u_1 over the plan period $[0, t_1]$, that will transfer the system S from the present state $x_0 \triangleq (C(o), K(o), Y(o), W(o), R(o), M_s(o))$ to the desired manifold M_1 and minimize the cost functional,

$$J(u) = \int_o^{t_1} \left[\frac{Wu_2L_s}{Y} (u_3 - 1) \right] dt \qquad (2.3.1),$$

where $\left(\frac{Wu_2L_s}{Y} \right)$ is the cost per unit production and the integrand $\left(u_3 \frac{Wu_2 L_s}{Y} - \frac{Wu_2 L_s}{Y} \right)$ is the excess revenue over the cost of production.

Applying Pontryagin's Maximum Principle and the transversality condition (theorem 1.4.1 pp 5 - 7 , theorem 1.5.1 ,p 8), the Hamiltonian function $\bar{H}$ is given by :

$$H = -\left[\frac{W u_2 L_s}{Y}(u_3 - 1)\right] + \left\{\alpha C \log\left(\frac{(1-u_1)Y}{C}\right)\right\}\psi_1$$

$$+ \left\{\beta K \log\left(\frac{(u_3-1)Y}{(1+g)R\, u_3 K}\right)\right\}\psi_2 + \left\{\gamma\left[C + \beta K \log\left(\frac{(u_3-1)}{(1+g)R}\right.\right.\right.$$

$$\left.\left.\left.\frac{Y}{u_3 K}\right) - Y\right]\right\}\psi_3 + \left\{\lambda W \log(u_2) + a W\right\}\psi_4$$

$$+ \left\{\delta R \log\left(\frac{\Gamma Y^{\mu}}{M_s R^{\nu}}\right)\right\}\psi_5 + \left\{\theta M_s \log\left(\frac{1}{u_2}\right)\right\}\psi_6$$

Maximizing the Hamiltonian H we obtain the form of the control variables.

It is easily verified that the control :

$$u_1 = 1 - \mathrm{Exp}\,(b_1 \log(1 - \omega_1)) \qquad (2.3.2)$$

where $b_1 = \frac{1}{2}(1 - \mathrm{Sign}(\alpha\,\psi_1 C))$ and $\psi = (\psi_1, \psi_2, \ldots \psi_6)$ is the solution of the adjoint system S' .

$$S' \begin{cases} \dfrac{d\psi_1}{dt} = \alpha\,\psi_1\left[1 - \log\left(\dfrac{(1-u_1)Y}{C}\right)\right] - \gamma\,\psi_3 \\[2ex] \dfrac{d\psi_2}{dt} = \beta(\psi_2 + \gamma\psi_3)\left[1 - \log\left(\dfrac{(u_3-1)Y}{(1+g)R\,u_3 K}\right)\right] \\[2ex] \dfrac{d\psi_3}{dt} = \gamma\psi_3 - \dfrac{\alpha C\psi_1}{Y} - \beta K\left(\dfrac{\psi_2 + \gamma\psi_3}{Y}\right) - \mu\dfrac{\delta R\psi_5}{Y} - \dfrac{W u_2 L_s}{Y^2}(u_3 - 1) \\[2ex] \dfrac{d\psi_4}{dt} = -\lambda\,\psi_4 \log(u_2) - a\psi_4 + \dfrac{u_2 L_s}{Y}(u_3 - 1) \\[2ex] \dfrac{d\psi_5}{dt} = \beta K\left(\dfrac{\psi_2 + \gamma\psi_3}{R}\right) - \delta\,\psi_5 \log\left(\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}}\right) + \delta\nu\psi_5 \\[2ex] \dfrac{d\psi_6}{dt} = \dfrac{\delta R\,\psi_5}{M_s} + \alpha\psi_6 \log(u_2) \end{cases}$$

By applying the transversality condition we obtain the end conditions that must be satisfied by the adjoint system S'. These conditions are $\psi_1(t_1) = Z_1$, $\psi_2(t_1) = Z_2$, $\psi_3(t_1) = Z_3$, $\psi_4(t_1) = 0$, $\psi_5(t_1) = 0$ and $\psi_6(t_1) = 0$.

Where the vector $Z \triangleq (Z_1\ Z_2,\ Z_3,\ 0, 0, 0)$ is not identically zero.

Substitution of the expression (2.3.2) for the control u_1 into the system S and the adjoint system S' gives rise to the following two-point boundary value problem (TPBVP) :

$$
S \begin{cases}
\dfrac{dC}{dt} = \alpha C \{\log [1 - (1 - \mathrm{Exp} (\log \dfrac{(1-\omega_1)}{2} (1 - \mathrm{Sign} (\alpha \psi_1 C))))] \\
\qquad + \log (\dfrac{Y}{C}) \} \\
\dfrac{dK}{dt} = \beta K \log \{ \dfrac{(u_3 - 1) Y}{(1+g) R u_3 K} \} \\
\dfrac{dY}{dt} = \gamma \{ C + \beta K \log (\dfrac{(u_3 - 1) Y}{(1+g) R u_3 K}) - Y \} \\
\dfrac{dW}{dt} = \lambda W \log (u_2) + aW \\
\dfrac{dR}{dt} = \delta R \log (\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}}) \\
\dfrac{dM_s}{dt} = \theta M_s \log (\dfrac{1}{u_2})
\end{cases}
$$

with the boundary conditions :

$$C(0) = C_o,\ K(0) = K_o,\ Y(0) = Y_o,\ W(0) = W_o,\ R(0) = R_o,\ M_s(0) = M_{s_o}$$

$$C(t_1) = C_1,\ K(t_1) = K_1,\quad Y(t_1) = Y_1$$

and ,

$$S' \begin{cases} \dfrac{d\psi_1}{dt} = a\psi_1 \{1 - [\log(1-(1-\mathrm{Exp}(\dfrac{\log(1-\omega_1)}{2}(1-\mathrm{Sign}(a\psi_1 C))))) \\ \qquad + \log(\dfrac{Y}{C})]\} - \gamma\psi_3 \\ \dfrac{d\psi_2}{dt} = \beta(\psi_2+\gamma\psi_3)(1-\log(\dfrac{(u_3-1)Y}{(1+g)Ru_3K})) \\ \dfrac{d\psi_3}{dt} = \gamma\psi_3 - \dfrac{aC\psi_1}{Y} - \beta K(\dfrac{\psi_2+\gamma\psi_3}{Y}) + \mu\dfrac{\delta R\psi_5}{Y} - \dfrac{Wu_2L_s}{Y^2}(u_3-1) \\ \dfrac{d\psi_4}{dt} = -\lambda\psi_4\log(u_2) - a\psi_4 + \dfrac{u_2L_s}{Y}(u_3-1) \\ \dfrac{d\psi_5}{dt} = \beta K(\dfrac{\psi_2+\gamma\psi_3}{R}) - \delta\psi_5\log(\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}}) + \delta\nu\psi_5 \\ \dfrac{d\psi_6}{dt} = \dfrac{\delta R\psi_5}{M_s} + \theta\psi_6\log(u_2) \end{cases}$$

With boundary conditions :

$$\psi_4(t_1) = 0, \quad \psi_5(t_1) = 0, \quad \psi_6(t_1) = 0$$

The optimality problem now reduces to solving the above TPBVP. It is clear that we have six initial conditions and six terminal conditions. Clearly this problem can not be solved by usual methods of integration of differential equations. The problem is solved by using the Davidon-Fletcher-Powell method along with the Fibonacci Search technique which is presented in Chapter III.

(2.4) TIME OPTIMAL CONTROL PROBLEM

It is often required to achieve a specified economic target in the shortest possible time. This is known as the time optimal control problem as discussed in this section.

Let the desired terminal state of the economic system S, described by the manifold,

$$M_1 \triangleq \{ C(t^*) = C_1,\ K(t^*) = K_1,\ Y(t^*) = Y_1,\ W(t^*) > 0,$$
$$R(t^*) = R_1,\ M_s(t^*) > 0 \} \subset E^6$$

be given.

The problem is: subject to the dynamic constraint S, choose policies of "Marginal Propensity to Save", "employment ratio" and "price index" considered as the control vector $u \triangleq (u_1, u_2, u_3)$ that will transfer the system S from the initial state $x_o \triangleq (C_o, K_o, Y_o, W_o, R_o, M_{s_o})$ to a desired state $x_1 \in M_1 \triangleq \{C(t^*) = C_1, K(t^*) = K_1, Y(t^*) = Y_1, W(t^*) > 0, R(t^*) = R_1, M_s(t^*) > 0\}$ in the shortest possible time. That is minimize the cost functional, $J = \int_o^{t^*} 1\, dt$.

Again applying Pontryagin's Maximum Principle and the transversality condition we obtain the necessary conditions of optimality.

The Hamiltonian H is given by:

$$H = -1 + \{\alpha C \log(\frac{(1-u_1)Y}{C})\}\psi_1 + \{\beta K \log(\frac{(u_3-1)Y}{(1+g)Ru_3K})\}\psi_2$$
$$+ \{\gamma[C + \beta K \log(\frac{(u_3-1)Y}{(1+g)Ru_3K}) - Y]\}\psi_3 + \{\lambda W \log(u_2) + aW\}\psi_4$$
$$+ \{\delta R \log(\frac{\Gamma Y^{\mu}}{M_s R^{\nu}})\}\psi_5 + \{\theta M_s \log(\frac{1}{u_2})\}\psi_6$$

Maximizing the Hamiltonian with respect to the control vector (u_1, u_2, u_3) we obtain the form of the controls. These are given below :

$$u_1 = 1 - \text{Exp}\,(b_1 \log(1-\omega_1)) \qquad (2.4.1)$$

$$u_2 = \text{Exp}\,(b_2 \log(\omega_2)) \qquad (2.4.2)$$

and,

$$u_3 = \frac{1}{1 - \text{Exp}\{ b'_3 \log(1 - \frac{1}{\omega'_3}) + b_3 \log(1 - \frac{1}{\omega_3})\}} \qquad (2.4.3)$$

where,

$$b_1 = \frac{1}{2}\{1 - \text{Sign}\,(\alpha\,\psi_1 C)\},$$

$$b_2 = \frac{1}{2}\{1 - \text{Sign}\,(\lambda\psi_4 W - \theta\,\psi_6 M_s)\},$$

$$b'_3 = \frac{1}{2}\{1 - \text{Sign}\,(\beta\,\psi_2 K + \gamma\beta\,\psi_3 K)\},$$

$$b_3 = \frac{1}{2}\{1 + \text{Sign}\,(\beta\,\psi_2 K + \gamma\beta\,\psi_3 K)\}$$

and $\psi = (\psi_1, \psi_2, \ldots, \psi_6)$ is the solution of the adjoint system S",

$$S'' \begin{cases} \dfrac{d\psi_1}{dt} = \alpha\psi_1 \left[1 - \log\left(\dfrac{(1-u_1)\,Y}{C}\right)\right] - \gamma\,\psi_3 \\[2ex] \dfrac{d\psi_2}{dt} = \beta(\psi_2 + \gamma\psi_3)\left\{1 - \log\left(\dfrac{(u_3-1)\,Y}{(1+g)\,R\,u_3\,K}\right)\right\} \\[2ex] \dfrac{d\psi_3}{dt} = \gamma\psi_3 - \dfrac{\alpha\psi_1 C}{Y} + \beta K\left(\dfrac{\psi_2 + \gamma\psi_3}{Y}\right) - \dfrac{\mu\,\delta\,R\,\psi_5}{Y} \\[2ex] \dfrac{d\psi_4}{dt} = -\lambda\,\psi_4 \log(u_2) - a\,\psi_4 \\[2ex] \dfrac{d\psi_5}{dt} = \beta K\left(\dfrac{\psi_2 + \gamma\psi_3}{R}\right) - \delta\,\psi_5 \log\left(\dfrac{\Gamma\,Y^{\mu}}{M_s R^{\nu}}\right) + \delta\,\nu\,\psi_5 \\[2ex] \dfrac{d\psi_6}{dt} = \dfrac{\delta\,R\,\psi_5}{M_s} + \theta\,\psi_6 \log(u_2) \end{cases}$$

Again by applying the transversality condition it is found that the adjoint system S" must satisfy the end conditions:

$\psi_1(t^*) = Z_1, \ \psi_2(t^*) = Z_2, \ \psi_3(t^*) = Z_3, \ \psi_4(t^*) = 0,$

$\psi_5(t^*) = Z_5, \ \psi_6(t^*) = 0.$ Where the vector $Z \underline{\Delta} (Z_1, Z_2, Z_3, 0, Z_5, 0)$ is not identically zero.

Substitution of the expressions (2.4.1) (2.4.2) and (2.4.3) for the controls u_1, u_2 and u_3 into the system S and the adjoint S" gives rise to the following two-point boundary value problem (TPBVP):

$$
S \begin{cases}
\dfrac{dC}{dt} = \alpha C \{ \log [1 - (1 - \text{Exp}(\dfrac{\log(1-\omega_1)}{2}(1 - \text{sign}(\alpha\psi_1 C)))] + \log(\dfrac{Y}{C}) \} \\
\dfrac{dK}{dt} = \beta K \{ \log [1 - (1 - \text{Exp}(\dfrac{\log(1-\frac{1}{\omega_3})}{2}(1 - \text{Sign}(\beta\psi_2 K + \gamma\beta\psi_3 K)) \\
\qquad + \dfrac{\log(1-\frac{1}{\omega_3})}{2}(1 + \text{Sign}(\beta\psi_2 K + \gamma\beta\psi_3 K))))] + \log(\dfrac{Y}{(1+g)RK}) \} \\
\dfrac{dY}{dt} = \gamma \{ C + \dfrac{dK}{dt} - Y \} \\
\dfrac{dW}{dt} = \lambda W \log [\text{Exp}(\dfrac{\log(\omega_2)}{2}(1 - \text{Sign}(\lambda\psi_4 W - \theta\psi_6 M_s)))] + aW \\
\dfrac{dR}{dt} = \delta R \log(\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}}) \\
\dfrac{dM_s}{dt} = -\theta M_s \log [\text{Exp}(\dfrac{\log(\omega_2)}{2}(1 - \text{Sign}(\lambda\psi_4 W - \theta\psi_6 M_s)))]
\end{cases}
$$

with the boundary conditions:

$C(0) = C_o, \ K(0) = K_o, \ Y(0) = Y_o, \ W(\theta) = W_o, \ R(0) = R_o, \ M_s(0) = M_{s_o}$

$C(t^*) = C_1, \ K(t^*) = K_1, \ Y(t^*) = Y_1, \ R(t^*) = R_1,$

and,

$$
S'' \begin{cases}
\dfrac{d\psi_1}{dt} = \alpha\psi_1 \{1 - [\log(1 - (1 - \text{Exp}(\dfrac{\log(1-\omega_1)}{2}(1 - \text{Sign}(\alpha\psi_1 C)))) \\
\quad) + \log(\dfrac{Y}{C})]\} - \gamma\psi_3 \\[2ex]
\dfrac{d\psi_2}{dt} = \beta(\psi_2 + \gamma\psi_3)\{1 - \log(1-(1-\text{Exp}(\dfrac{\log(1-\frac{1}{\omega_3})}{2}(1 - \text{Sign} \\
\quad (\beta\psi_2 K + \gamma\beta\psi_3 K)) + \dfrac{\log(1-\frac{1}{\omega_3})}{2}(1 + \text{Sign}(\beta\psi_2 K + \gamma\beta\psi_3 K) \\
\quad)))) + \log(\dfrac{Y}{(1+g)RK})\} \\[2ex]
\dfrac{d\psi_3}{dt} = \gamma\psi_3 - \dfrac{\alpha\psi_1 C}{Y} - \beta K(\dfrac{\psi_2 + \gamma\psi_3}{Y}) - \dfrac{\mu\delta R\psi_5}{Y} \\[2ex]
\dfrac{d\psi_4}{dt} = -\lambda\psi_4 \log[\text{Exp}(\dfrac{\log(\omega_2)}{2}(1 - \text{Sign}(\lambda\psi_4 W - \theta\psi_6 M_s)))] \\
\quad - a\psi_4 \\[2ex]
\dfrac{d\psi_5}{dt} = \beta K(\dfrac{\psi_2 + \gamma\psi_3}{R}) - \delta\psi_5 \log(\dfrac{\Gamma Y^{\mu}}{M_s R^{\nu}}) + \delta\nu\psi_5 \\[2ex]
\dfrac{d\psi_6}{dt} = \dfrac{\delta R\psi_5}{M_s} + \theta\psi_6 \log[\text{Exp}(\dfrac{\log(\omega_2)}{2}(1 - \text{Sign}(\lambda\psi_4 W - \\
\quad \theta\psi_6 M_s)))]
\end{cases}
$$

with the boundary conditions : $\psi_4(t^*) = 0$, $\psi_6(t^*) = 0$.

This problem is also solved using the same method and technique as discussed before.

CHAPTER III

AN ITERATIVE METHOD AND COMPUTATIONAL RESULTS

INTRODUCTION

For the fixed time problem consisting of the two-point boundary value problem (pp 14-15), the iterative method consists of choosing an initial value for the vector $\psi(o) = \psi_o$ and integrating the differential equations (systems S and S') forward yielding a solution $x(t, \psi_o)$, $\psi(t, \psi_o)$ on $0 \le t \le T$ as a function of ψ_o. The desired terminal conditions are achieved by minimizing a terminal error function through an iterative procedure. This procedure is designed to find ψ_o^* that yields a minimum to the terminal error criterion to be given in the iterative steps. For the time optimal control problem (pp. 18-19), in addition to choosing ψ_o, it is also necessary to choose a terminal time T^o as the initial guess. Finally the desired terminal condition is achieved by minimizing a terminal error criterion by appropriate choice of T^o and ψ_o. This is done through an iterative procedure as given below.

(3.1) THE ITERATION STEPS

The iterative scheme consists of two loops one being the major and the other an inner loop. We denote by K the K th stage of iteration in the main loop and by N the N th step of iteration in the inner loop.

STEP 1.

SET $K = 0 \quad t = 0$

STEP 2

READ $\frac{dx_i}{dt} = f_i$, $\frac{d\psi_i}{dt} = -f'_{x_i}\psi$

$x_i(o) = x_{i_o}$ $\quad i = 1, 2, \ldots, n$

$x_i(t^*) = x_i^*$ $\quad i = 1, 2, \ldots, M$

$\psi_i(t^*) = \psi_i^*$ $\quad i = M+1, \ldots, n$

STEP 3 INITIAL GUESS

$\psi_i^K(o) = \psi_i^K \quad , \quad t^K \quad i = 1, 2, \ldots n.$

STEP 4 INTEGRATE FORWARD $(0 — t^K)$

$\frac{dx_i}{dt} = f_i \quad , \quad \frac{d\psi_i}{dt} = -f'_{x_i}\psi \quad ; \quad i = 1, 2, \ldots, n$

STEP 5

EVALUATE

$$E(\psi^K, t^K) = \sqrt{\sum_{i=1}^{M} [x_i^* - x_i(\psi^K, t^K)]^2 + \sum_{i=M+1}^{n} [\psi_i^* - \psi_i(\psi^K, t^K)]^2}$$

$(0 \le M \le n)$

STEP 6 CHECK

$$E(\psi^K, t^K) \le \epsilon \text{ SOME PREASSIGNED NUMBER } \epsilon > 0,$$

IF TRUE GO TO STEP 31, OTHERWISE GO TO STEP 7.

STEP 7 SET $K = K + 1$.

STEP 8 EVALUATE

$$S_i^{K-1} = \frac{\partial E}{\partial \psi_i} \simeq \frac{E(\psi_1^{K-1}, \ldots, \psi_i^{K-1} + \epsilon', \ldots, \psi_n^{K-1}; t^{K-1}) - E(\psi_1^{K-1}, \ldots, \psi_i^{K-1}, \ldots, \psi_n^{K-1}; t^{K-1})}{\epsilon'}$$

$$(S^{K-1} = (S_1^{K-1}, S_2^{K-1}, \ldots, S_n^{K-1})) \qquad i = 1, 2, \ldots, n$$

$$S_t^{K-1} = \left[\sum_{i=1}^{n} \frac{\partial E}{\partial x_i}(x(\psi_i^{K-1}, t^{K-1})) \cdot \dot{x}_i + \sum_{i=1}^{n} \frac{\partial E}{\partial \psi_i}(x(\psi_i^{K-1}, t^{K-1})) \cdot \dot{\psi}_i\right]_{t=t^{K-1}}$$

STEP 9 EVALUATE

$$L_1^N = F_P = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{P+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{P+1}\right]$$

$$P = 3, 4, \ldots, 20 \qquad L_o^N = 0$$

STEP 10 SET

$$N = 1$$

STEP 11 DETERMINE

$$a_1^N = \frac{F_{P-2}}{F_P}(L_1^N - L_o^N) + L_o^N$$

$$a_2^N = \frac{F_{P-1}}{F_P}(L_1^N - L_o^N) + L_o^N$$

STEP 12 EVALUATE

$$\hat{E}(a_1^N) = \hat{E}(\psi^{K-1} + a_1^N S^{K-1},\ t^{K-1} + a_1^N S_t^{K-1})$$

$$\hat{E}(a_2^N) = \hat{E}(\psi^{K-1} + a_2^N S^{K-1},\ t^{K-1} + a_2^N S_t^{K-1})$$

STEP 13 COMPARE

$\hat{E}(a_1^N) > \hat{E}(a_2^N)$, IF TRUE GO TO STEP 14,

OTHERWISE GO TO STEP 18

STEP 14 SET

$a_1^{N+1} = a_2^N \quad L_o^{N+1} = a_1^N$ and $L_1^{N+1} = L_1^N$

STEP 15 CHECK

$N < (P-1)$, IF TRUE GO TO STEP 16,

OTHERWISE $N = P-1$, SET $N = N+1$ GO TO STEP 26

STEP 16 SET

$N = N+1$ GO TO STEP 17

STEP 17 EVALUATE

$$a_2^N = \frac{F_{P-N}}{F_{P+1-N}} (L_1^N - L_o^N) + L_o^N$$

GO TO STEP 12.

STEP 18 SET

$L_1^{N+1} = a_2^N, \quad L_o^{N+1} = L_o^N, \; a_2^{N+1} = a_1^N$

STEP 19 CHECK

$N < (P-1)$ IF TRUE GO TO STEP 20, OTHERWISE

$N = P-1$, SET $N = N+1$ GO TO STEP 22.

STEP 20 SET

$N = N+1$ GO TO STEP 21

STEP 21 EVALUATE

$$a_1^N = \frac{F_{P-1-N}}{F_{P+1-N}} (L_1^N - L_o^N) + L_o^N$$

GO TO STEP 12.

STEP 22 EVALUATE

$$\hat{E}(a_2^N) = \hat{E}(\psi^{K-1} + a_2^N S^{K-1}, \; t^{K-1} + a_2^N S_t^{K-1})$$

$$\hat{E}(a_2^N + \hat{\epsilon}) = \hat{E}(\psi^{K-1} + (a_2^N + \hat{\epsilon}) S^{K-1}, \; t^{K-1} + (a_2^N + \hat{\epsilon}) S_t^{K-1})$$

STEP 23 COMPARE

$\hat{E}(a_2^N) > \hat{E}(a_2^N + \hat{\epsilon})$ IF TRUE GO TO STEP 24,

OTHERWISE GO TO STEP 25.

STEP 24 EVALUATE

$$a^{K-1} = \frac{1}{2}\left(a_2^N + L_2^N\right)$$ GO TO STEP 30

STEP 25 EVALUATE

$$a^{K-1} = \frac{1}{2}\left(L_o^N + a_2^N\right)$$ GO TO STEP 30

STEP 26 EVALUATE

$$\hat{E}\left(a_1^N\right) = \hat{E}\left(\psi^{K-1} + a_1^N S^{K-1},\ t^{K-1} + a_1^N S_t^{K-1}\right)$$

$$\hat{E}\left(a_1^N + \hat{\epsilon}\right) = \hat{E}\left(\psi^{K-1} + (a_1^N + \hat{\epsilon})S^{K-1},\ t^{K-1} + (a_1^N + \hat{\epsilon})S_t^{K-1}\right)$$

STEP 27 COMPARE

$$\hat{E}\left(a_1^N\right) > \hat{E}\left(a_1^N + \hat{\epsilon}\right)$$ IF TRUE GO TO STEP 28,

OTHERWISE GO TO STEP 29

STEP 28 EVALUATE

$$a^{K-1} = \frac{1}{2}\left(a_1^N + L_1^N\right)$$ GO TO STEP 30

STEP 29 EVALUATE

$$a^{K-1} = \frac{1}{2}\left(L_o^N + a_1^N\right)$$ GO TO STEP 30

STEP 30 DETERMINE

$$\psi^K = \psi^{K-1} + a^{K-1}S^{K-1}, \quad t^K = t^{K-1} + a^{K-1}S_t^{K-1}$$

GO TO STEP 4

STEP 31 ITERATIVE PROCESS IS COMPLETED

PRINT

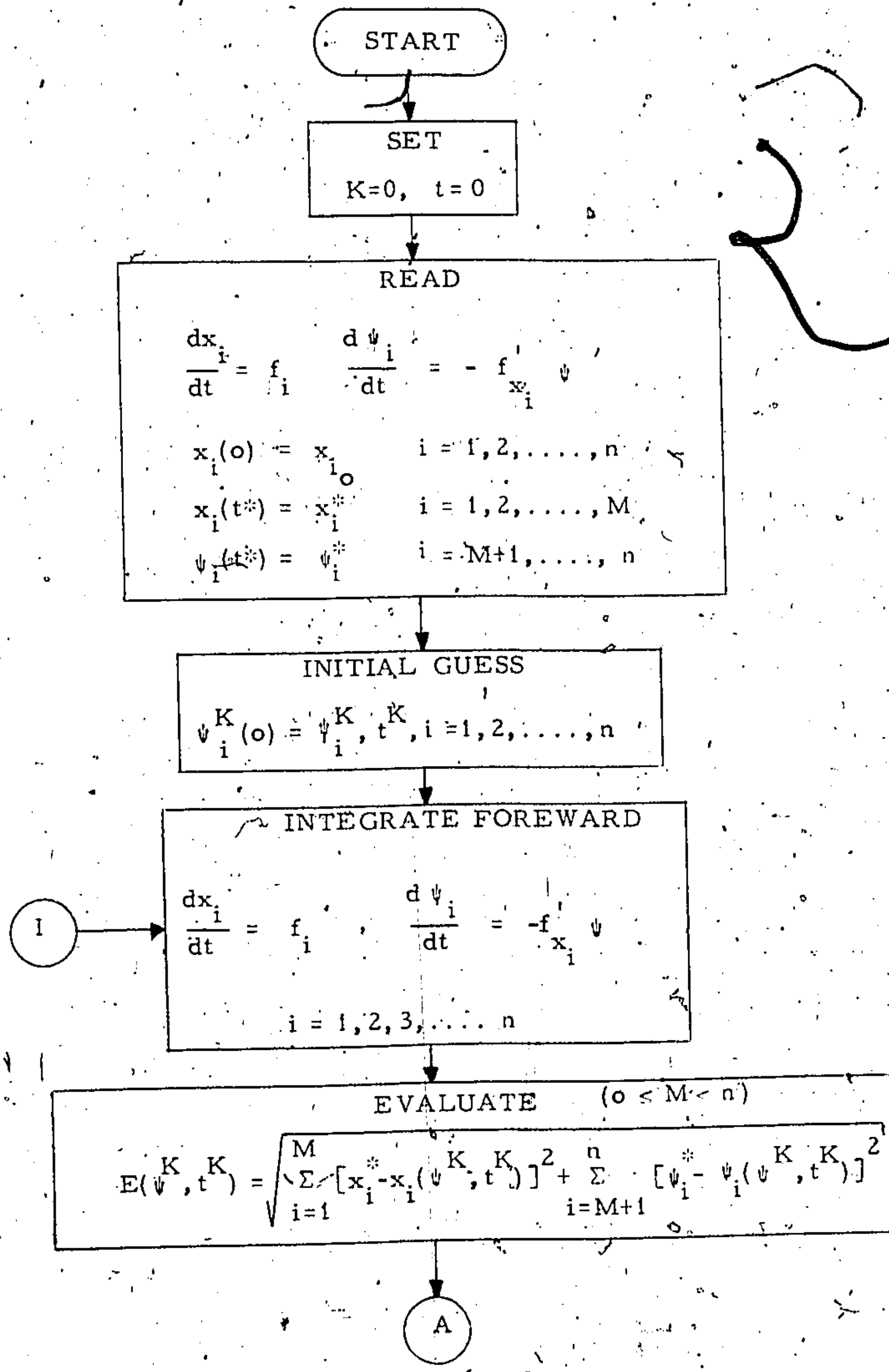
$$x^{*}(t) \quad , \quad t \in [0, t^{K}]$$

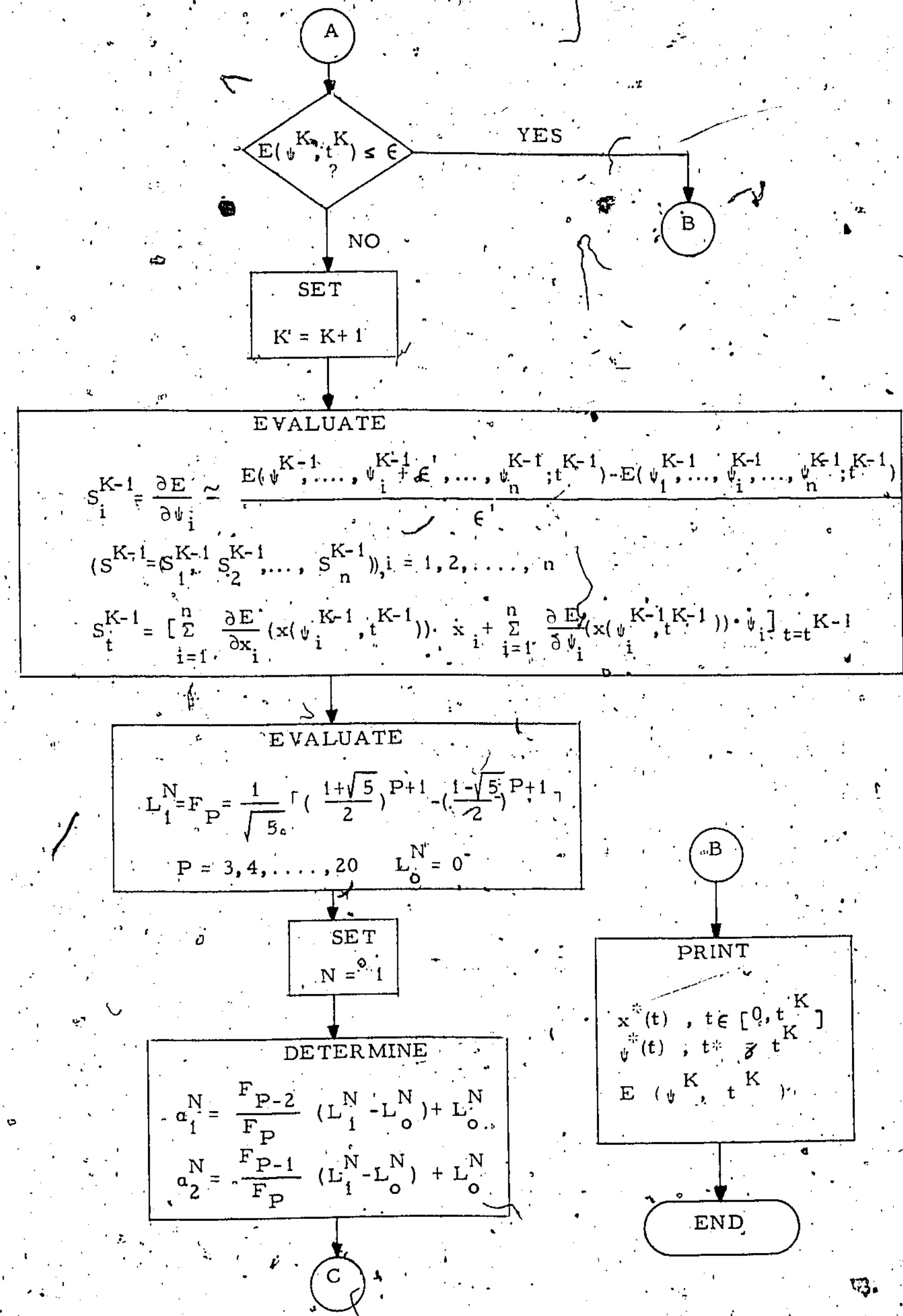
$$\psi^{*}(t) \quad ,$$

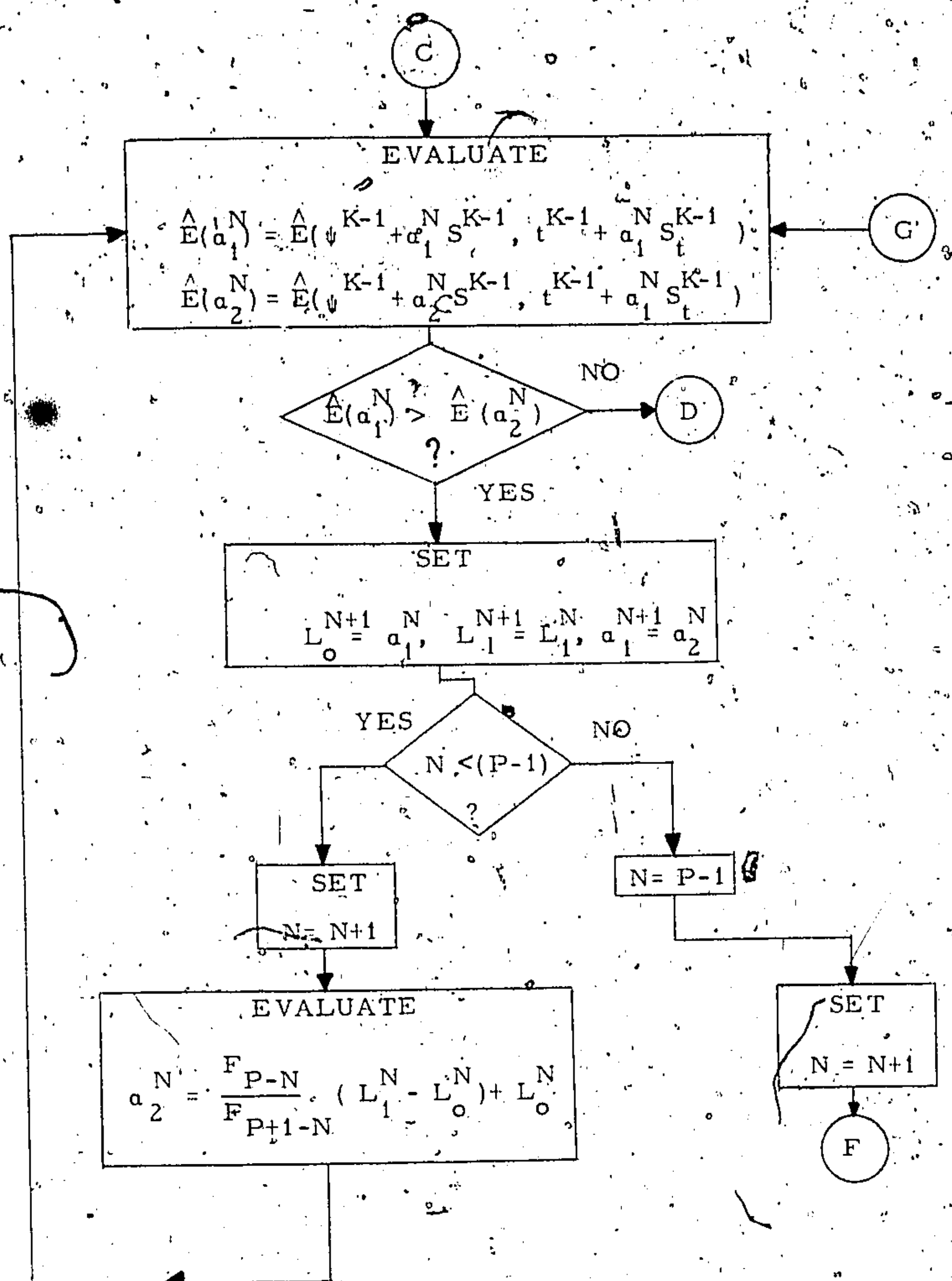
$$t^{*} = t^{K} \quad , \quad E(\psi^{K}, t^{K}).$$

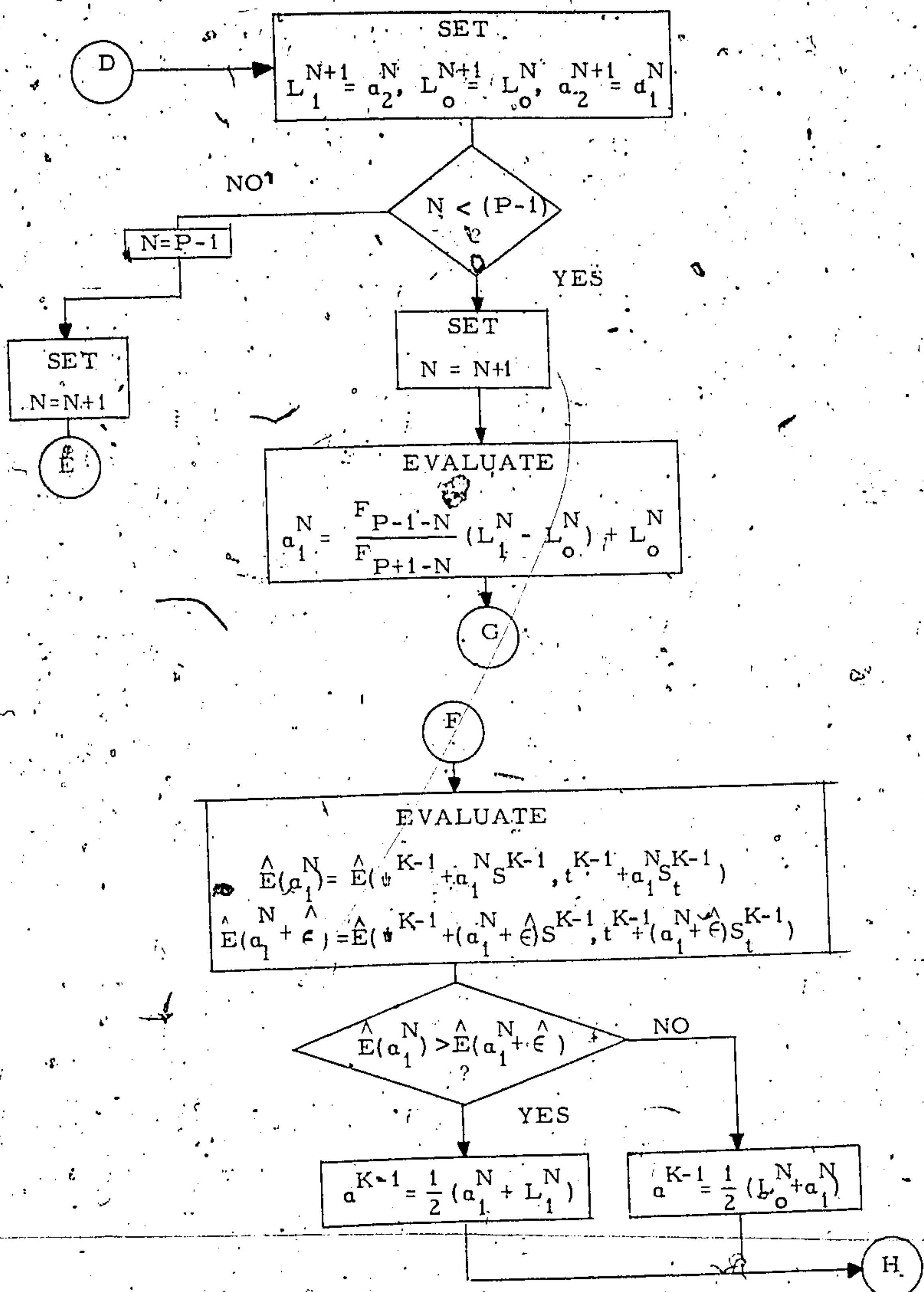
END

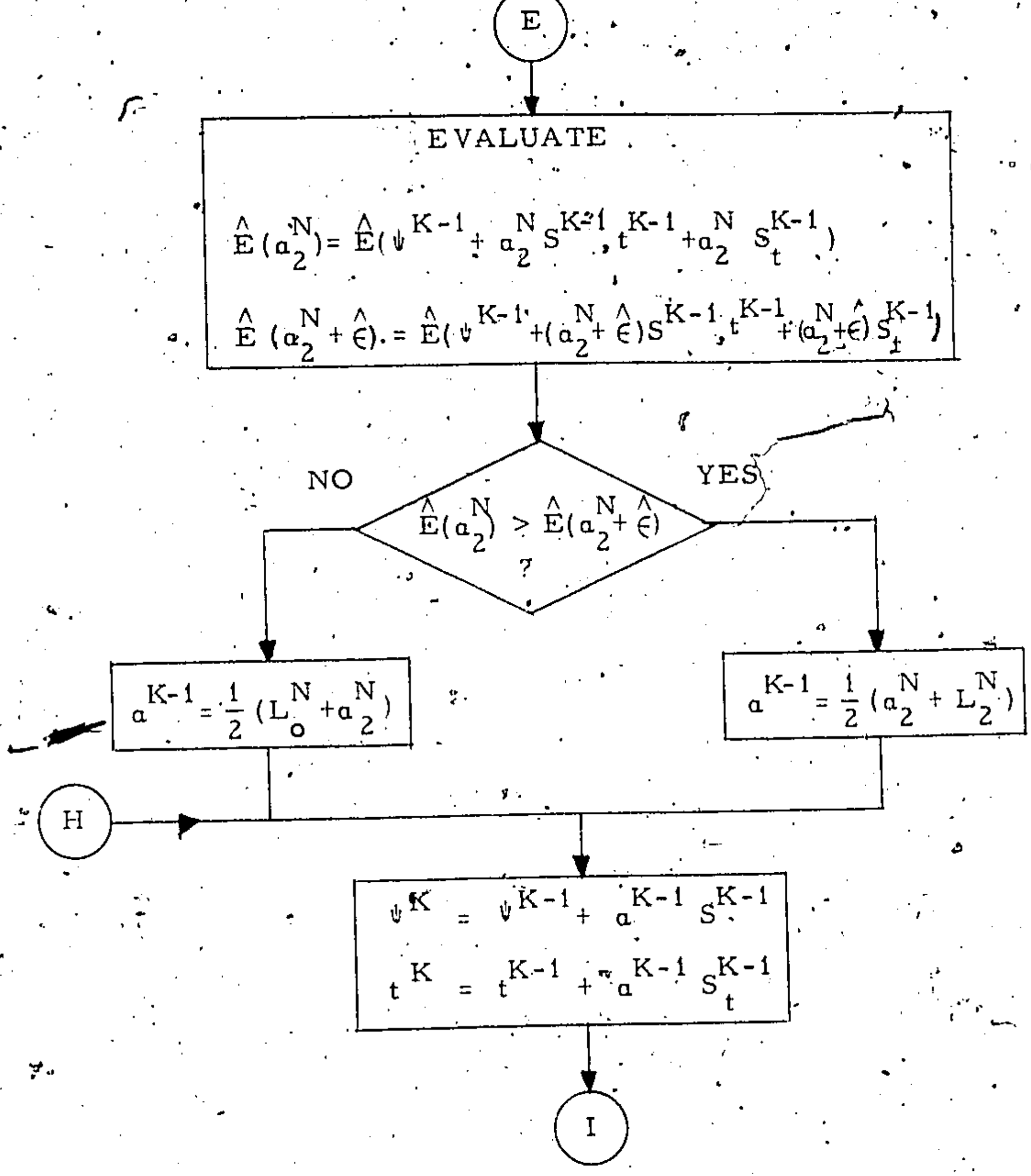
(3.2) THE FLOW CHART




A
$E(\psi^K, t^K) \le \epsilon$?
YES
NO
B
SET
$K = K+1$
EVALUATE
$S_i^{K-1} = \frac{\partial E}{\partial \psi_i} \simeq \frac{E(\psi^{K-1}, \ldots, \psi_i^{K-1} + \epsilon', \ldots, \psi_n^{K-1}; t^{K-1}) - E(\psi_1^{K-1}, \ldots, \psi_i^{K-1}, \ldots, \psi_n^{K-1}; t^{K-1})}{\epsilon'}$
$(S^{K-1} = (S_1^{K-1}, S_2^{K-1}, \ldots, S_n^{K-1})), i = 1, 2, \ldots, n$
$S_t^{K-1} = [\sum_{i=1}^{n} \frac{\partial E}{\partial x_i}(x(\psi_i^{K-1}, t^{K-1})) \cdot \dot{x}_i + \sum_{i=1}^{n} \frac{\partial E}{\partial \psi_i}(x(\psi_i^{K-1}, t^{K-1})) \cdot \dot{\psi}_i]_{t=t^{K-1}}$
EVALUATE
$L_1^N = F_P = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{P+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{P+1} \right]$
$P = 3, 4, \ldots, 20 \quad L_o^N = 0$
SET
$N = 1$
DETERMINE
$a_1^N = \frac{F_{P-2}}{F_P} (L_1^N - L_o^N) + L_o^N$
$a_2^N = \frac{F_{P-1}}{F_P} (L_1^N - L_o^N) + L_o^N$
C
B
PRINT
$x^*(t)$, $t \in [0, t^K]$
$\psi^*(t)$; $t^* \geq t^K$
$E(\psi^K, t^K)$
END


C
EVALUATE
$\hat{E}(a_1^N) = \hat{E}(\psi^{K-1} + a_1^N S^{K-1},\ t^{K-1} + a_1^N S_t^{K-1})$
$\hat{E}(a_2^N) = \hat{E}(\psi^{K-1} + a_2^N S^{K-1},\ t^{K-1} + a_1^N S_t^{K-1})$
G
$\hat{E}(a_1^N) > \hat{E}(a_2^N)$?
NO
D
YES
SET
$L_o^{N+1} = a_1^N,\ L_1^{N+1} = L_1^N,\ a_1^{N+1} = a_2^N$
YES
NO
$N < (P-1)$?
SET
N = N+1
N= P-1
EVALUATE
$a_2^N = \frac{F_{P-N}}{F_{P+1-N}} (L_1^N - L_o^N) + L_o^N$
SET
N = N+1
F


D
SET
$L_1^{N+1} = a_2^N,\ L_o^{N+1} = L_o^N,\ a_2^{N+1} = a_1^N$
$N < (P-1)$?
NO
N=P-1
SET
N=N+1
E
YES
SET
N = N+1
EVALUATE
$a_1^N = \frac{F_{P-1-N}}{F_{P+1-N}}(L_1^N - L_o^N) + L_o^N$
G
F
EVALUATE
$\hat{E}(a_1^N) = \hat{E}(\psi^{K-1} + a_1^N S^{K-1}, t^{K-1} + a_1^N S_t^{K-1})$
$\hat{E}(a_1^N + \hat{\epsilon}) = \hat{E}(\psi^{K-1} + (a_1^N + \hat{\epsilon})S^{K-1}, t^{K-1} + (a_1^N + \hat{\epsilon})S_t^{K-1})$
$\hat{E}(a_1^N) > \hat{E}(a_1^N + \hat{\epsilon})$?
NO
YES
$a^{K-1} = \frac{1}{2}(a_1^N + L_1^N)$
$a^{K-1} = \frac{1}{2}(L_o^N + a_1^N)$
H


E
EVALUATE
Ê(a₂ᴺ) = Ê(ψ^(K-1) + a₂ᴺ S^(K-1), t^(K-1) + a₂ᴺ S_t^(K-1))
Ê(a₂ᴺ + ε̂) = Ê(ψ^(K-1) + (a₂ᴺ + ε̂) S^(K-1), t^(K-1) + (a₂ᴺ + ε̂) S_t^(K-1))
NO
YES
Ê(a₂ᴺ) > Ê(a₂ᴺ + ε̂) ?
a^(K-1) = ½ (L₀ᴺ + a₂ᴺ)
a^(K-1) = ½ (a₂ᴺ + L₂ᴺ)
H
ψ^K = ψ^(K-1) + a^(K-1) S^(K-1)
t^K = t^(K-1) + a^(K-1) S_t^(K-1)
I

(3. 3) COMPUTATIONAL RESULTS

In this section computational results of fixed time optimal control problems and time optimal control problems solved by using Davidon-Fletcher-Powell method (3, 6, 11) along with Fibonacci Search technique (7, 8, 11, 16) are presented.

The numerical values given to the coefficients of the following problems are :

$\alpha = 0.1 \times 10^{-1}$, $\beta = 0.1$, $\gamma = 0.1$, $\lambda = 0.1 \times 10^{-1}$, $\delta = 0.1 \times 10^{-1}$

$\nu = 1.0$, $\theta = 0.1$, $g = 0.4$, $\Gamma = 0.1 \times 10^{-3}$, $a = 0.4 \times 10^{-1}$,

$L_s = 0.1 \times 10^{3}$, $\mu = 1.0$.

P. 3. 3. 1 : THE FIXED TIME OPTIMAL CONTROL PROBLEM

EXAMPLE 1

Initial state : $x_o \triangleq \{ C_o = 0.4,\ K_o = 0.1,\ Y_o = 0.43,\ W_o = 1.0,\ R_o = 0.21 \times 10^{-1},\ M_{s_o} = 1.0 \}$

Desired final state : $x_1 \in M_1 \triangleq \{ C_1 = 0.401,\ K_1 = 0.5,\ Y_1 = 0.45,\ W_1 > 0,\ R_1 > 0,\ M_{s_1} > 0 \}$

The control constraint : $\omega_1 = .02$ ($u_2 = .95$, $u_3 = 1.5$ assumed fixed)

Time : [0, 5]

$\psi^o \triangleq (\psi_1^o, \psi_2^o, \ldots, \psi_6^o)$	$E(\psi^o)$	No. of Iterations	ψ^o(end conditions after 6 iterations)	$E(\psi^o)$ (after 6 iterations)
0.1×10^{-2}			-0.4015×10^{3}	
0.2×10^{-2}			-0.9974×10^{2}	
0.1×10^{-2}	6.190789×10^{5}	6	0.1745	1.735281×10^{-5}
0.1×10^{-1}			0.2463×10^{-3}	
0.3×10^{-1}			-0.1745×10^{-2}	
0.1×10^{-3}			-0.1781×10^{-3}	

The Cost : 5.996782×10^{2}

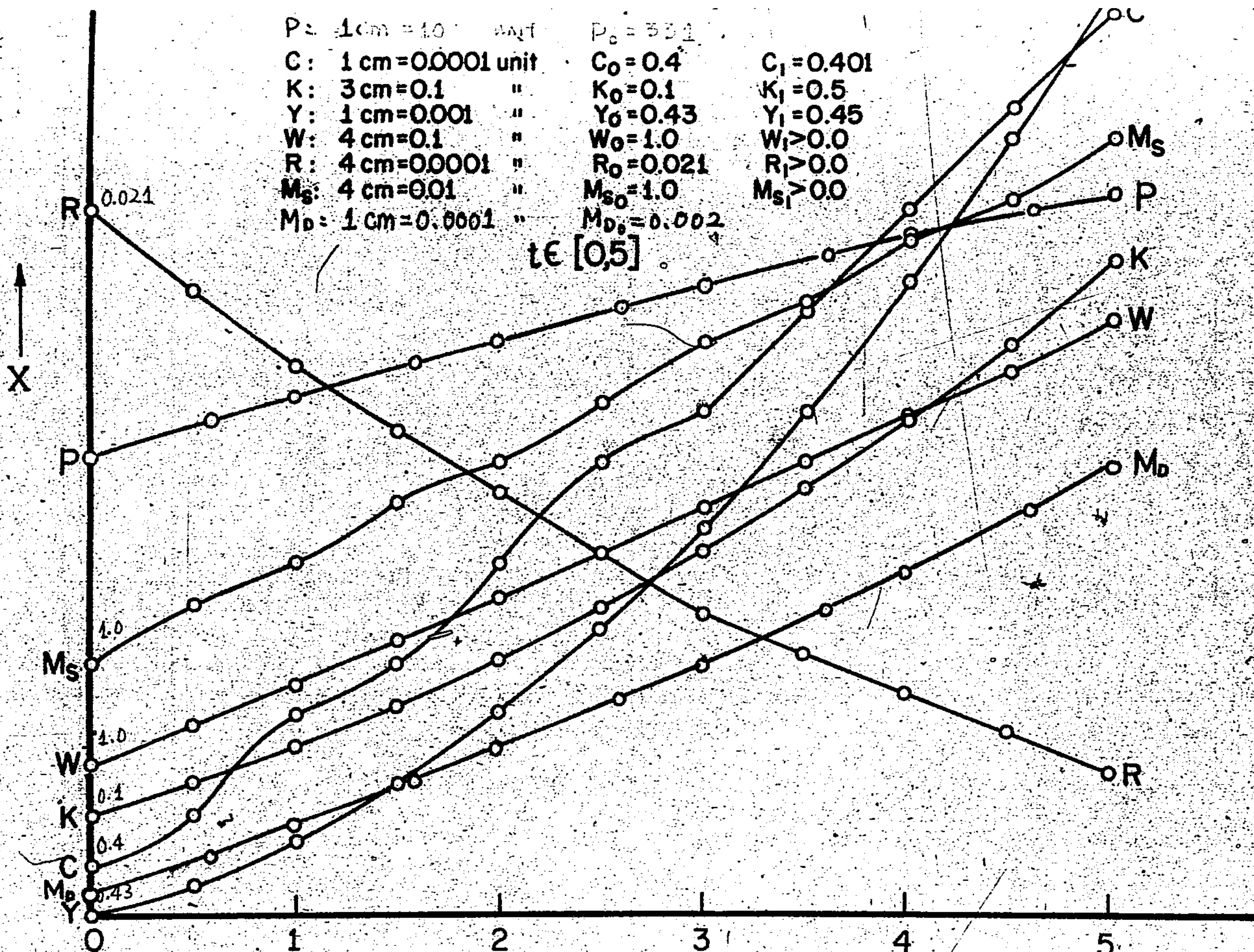

C : 1 cm = 0.0001 unit C0 = 0.4 C1 = 0.401
K : 3 cm = 0.1 " K0 = 0.1 K1 = 0.5
Y : 1 cm = 0.001 " Y0 = 0.43 Y1 = 0.45
W : 4 cm = 0.1 " W0 = 1.0 W1 > 0.0
R : 4 cm = 0.0001 " R0 = 0.021 R1 > 0.0
Ms : 4 cm = 0.01 " Ms0 = 1.0 Ms1 > 0.0
MD : 1 cm = 0.0001 " MD0 = 0.002
t ∈ [0,5]
X
t
R
P
Ms
W
K
C
MD
Y
0.021
1.0
1.0
0.1
0.4
0.43
0
1
2
3
4
5

FIG 1.1

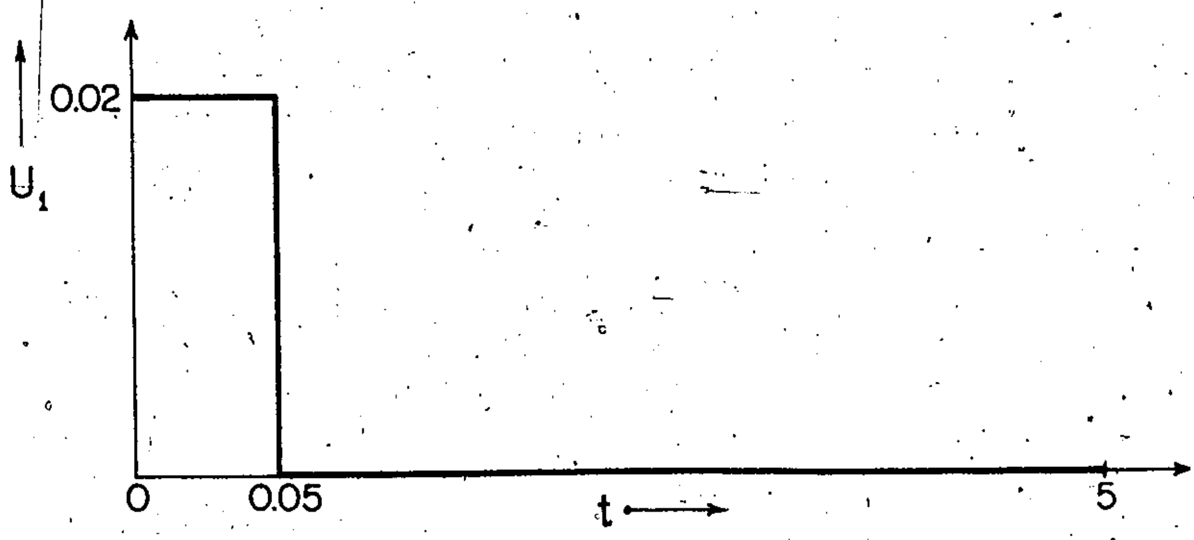


FIG 1.2

EXAMPLE 2

In this example, 30% marginal propensity to save, 3% unemployment and $L_s = L_o e^{bt}$ where $L_o = 0.1 \times 10^3$ and $b = 0.2$ are considered.

Inital state : $x_o \underline{\Delta} \{C_o = 0.4,\ K_o = 0.1,\ Y_o = 0.43,\ W_o = 1.0,$

$R_o = 0.21 \times 10^{-1},\ M_{s_o} = 1.0\}$

Desired final state: $x_1 \in M_1 \underline{\Delta} \{C_1 = 0.401,\ K_1 = 0.5,\ Y_1 = 0.45,\ W_1 > 0,$

$R_1 > 0,\ M_{s_1} > 0\}$

The control constraint : $\omega_1 = 0.3$ ($u_2 = 0.97$, $u_3 = 1.5$ assumed fixed)

Time : $[0, 5]$

$\psi^o \underline{\Delta} (\psi_1^o, \psi_2^o, \ldots, \psi_6^o)$	$E(\psi^o)$	No. of iterations	ψ^o(end conditions after 9 iterations)	$E(\psi^o)$ (after 9 iterations)
0.1×10^{-2}			-0.6904×10^{3}	
0.2×10^{-2}			-0.1463×10^{3}	
0.1×10^{-2}	1.683797×10^{6}	9	0.2913×10^{4}	6.723392×10^{-6}
0.1×10^{-1}			-0.6898×10^{-10}	
0.3×10^{-1}			-0.1570×10^{-9}	
0.1×10^{-3}			0.2491×10^{-10}	

The cost : 0.1065553×10^{4}

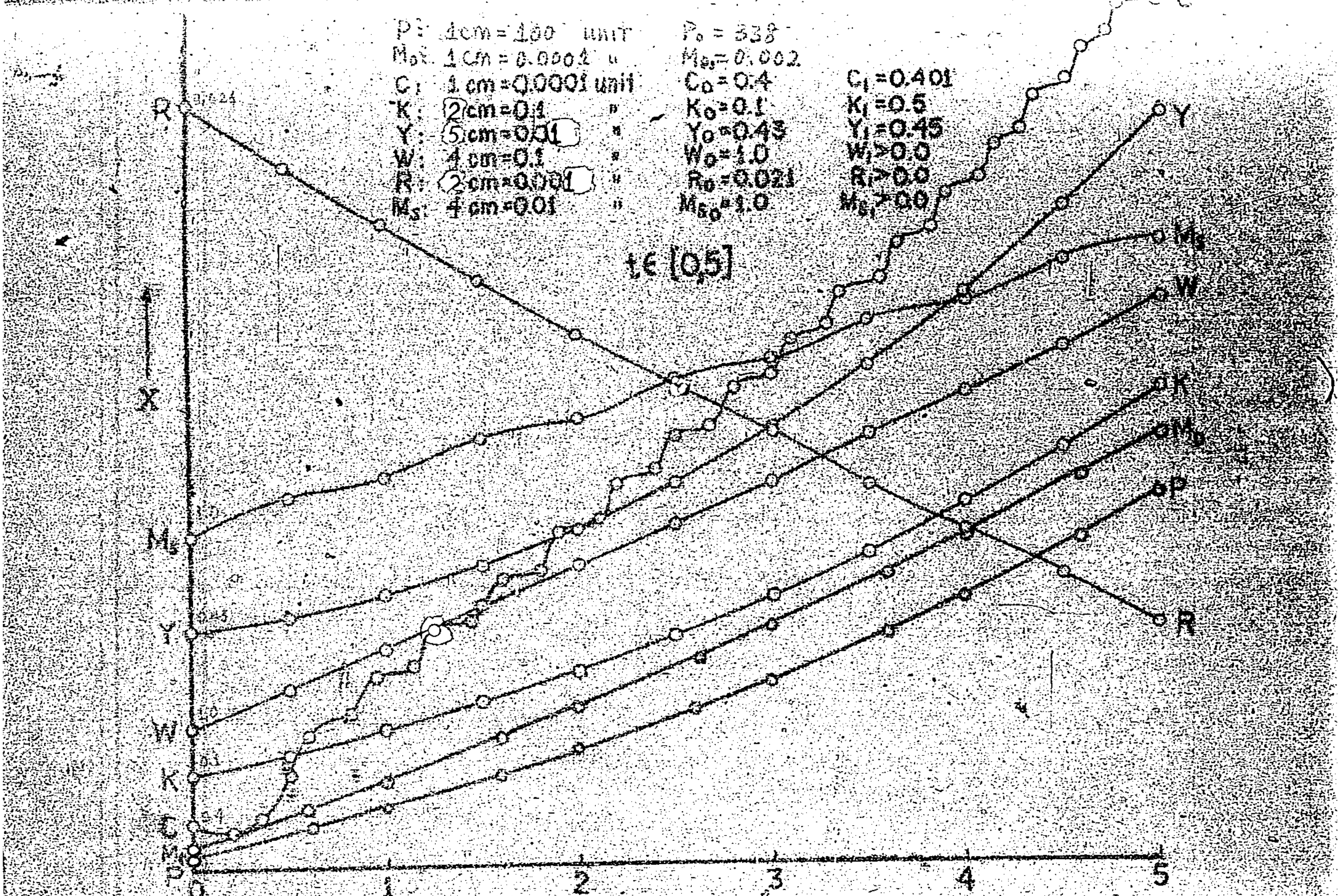

P: 1cm = 100 unit
M_o: 1cm = 0.0001 "
C: 1 cm = 0.0001 unit
K: 2 cm = 0.1 "
Y: 5 cm = 0.01 "
W: 4 cm = 0.1 "
R: 2 cm = 0.001 "
M_s: 4 cm = 0.01 "
P_o = 338
M_{o_o} = 0.002
C_o = 0.4
K_o = 0.1
Y_o = 0.43
W_o = 1.0
R_o = 0.021
M_{s_o} = 1.0
C_1 = 0.401
K_1 = 0.5
Y_1 = 0.45
W_1 > 0.0
R_1 > 0.0
M_{s_1} > 0.0
t ∈ [0,5]
X
t
0
1
2
3
4
5
R
M_s
Y
W
K
C
M_o
P

FIG 2.1

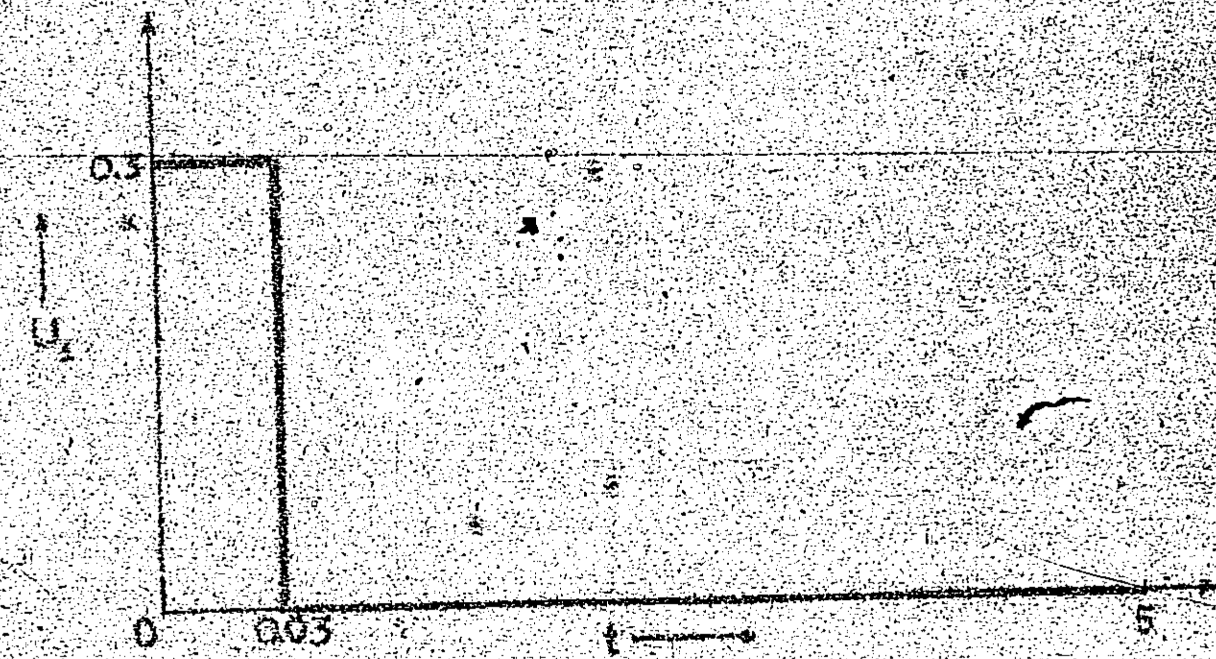


FIG 2.2

P 3.3.2 : THE TIME OPTIMAL CONTROL PROBLEM

EXAPLE 1

Initial state : $x_o \triangleq \{C_o = 0.4,\ K_o = 0.1,\ Y_o = 0.43,\ W_o = 1.0,$

$R_o = 0.21 \times 10^{-1},\ M_{s_o} = 1.0\}$

Desired final state : $x_1 \in M_1 \triangleq \{C_1 = 0.402,\ K_1 = 0.52,\ Y_1 = 0.45$

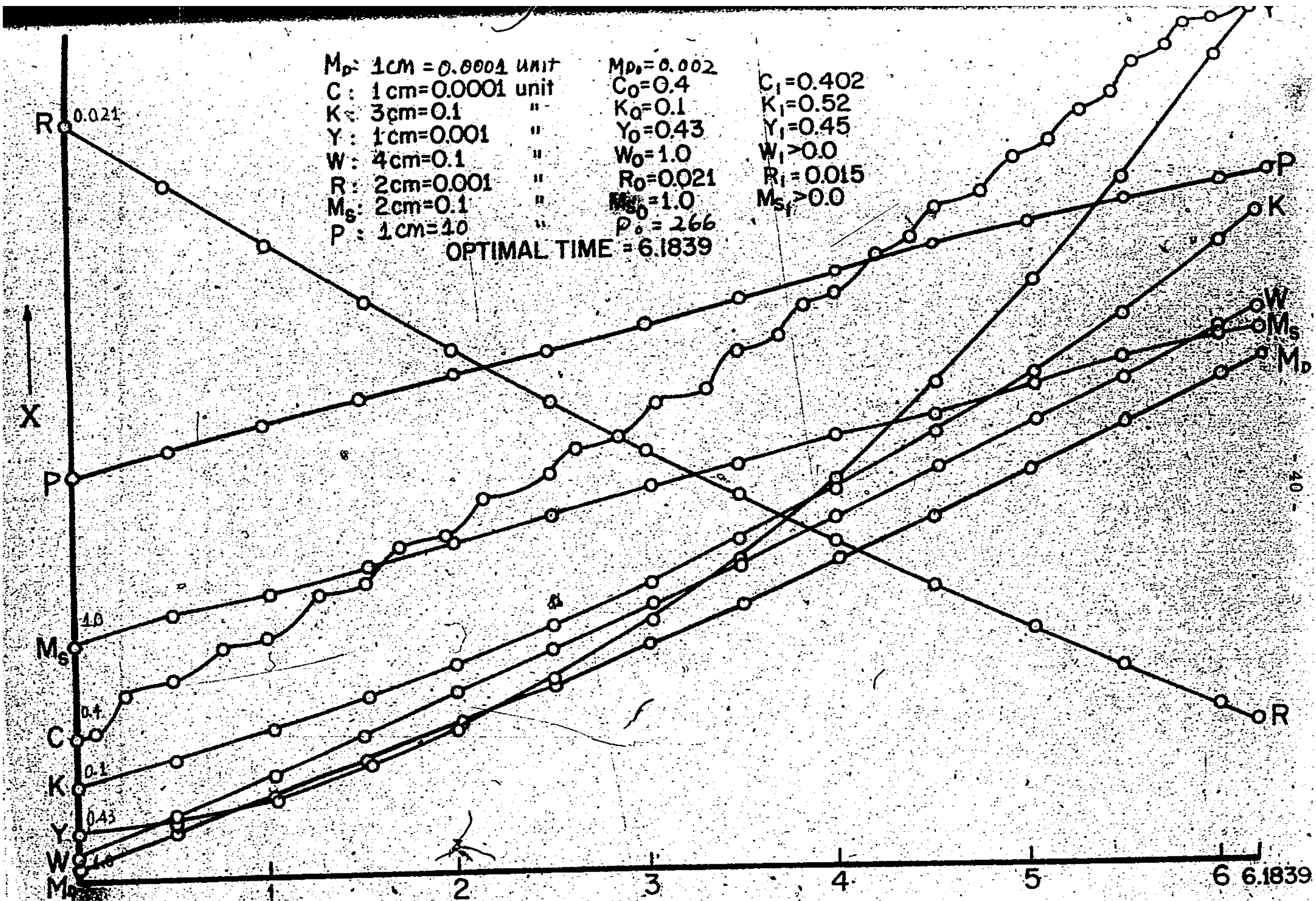
$W_1 > 0,\ R_1 = 0.014,\ M_{s_1} > 0\}$

The control constraints : $\omega_1 = .02$, $\omega_2 = .95$, $\omega_3 = 1.2$

and $\omega_3 = 1.5$.

$\psi^o \triangleq (\psi_1^o, \psi_2^o, \dots \psi_6^o)$	$E(\psi^o)$	No. of Iterations	ψ^o(end conditions after 5 iterations)	$E(\psi^o)$ (after 5 iterations)
0.71			-0.6297×10^{-1}	
0.29			-0.4607×10^{-1}	
2.2	0.3767508	5	0.7128	1.157586×10^{-3}
-0.62×10^{-1}			0.8101×10^{-2}	
0.73×10^{3}			0.1157×10^{4}	
-2.0			0.3419×10^{-2}	

The optimal time : 6.183914



M_D: 1cm = 0.0001 unit
C : 1cm=0.0001 unit
K : 3cm=0.1 "
Y : 1cm=0.001 "
W : 4cm=0.1 "
R : 2cm=0.001 "
M_S: 2cm=0.1 "
P : 1cm=10 "
M_{D_0} = 0.002
C_0=0.4
K_0=0.1
Y_0=0.43
W_0=1.0
R_0=0.021
M_{S_0}=1.0
P_0 = 266
C_1=0.402
K_1=0.52
Y_1=0.45
W_1>0.0
R_1=0.015
M_{S_1}>0.0
OPTIMAL TIME = 6.1839
X
t
R
P
M_S
C
K
Y
W
M_D
0.021
1.0
0.4
0.1
0.43
1
2
3
4
5
6
6.1839

FIG 1.1

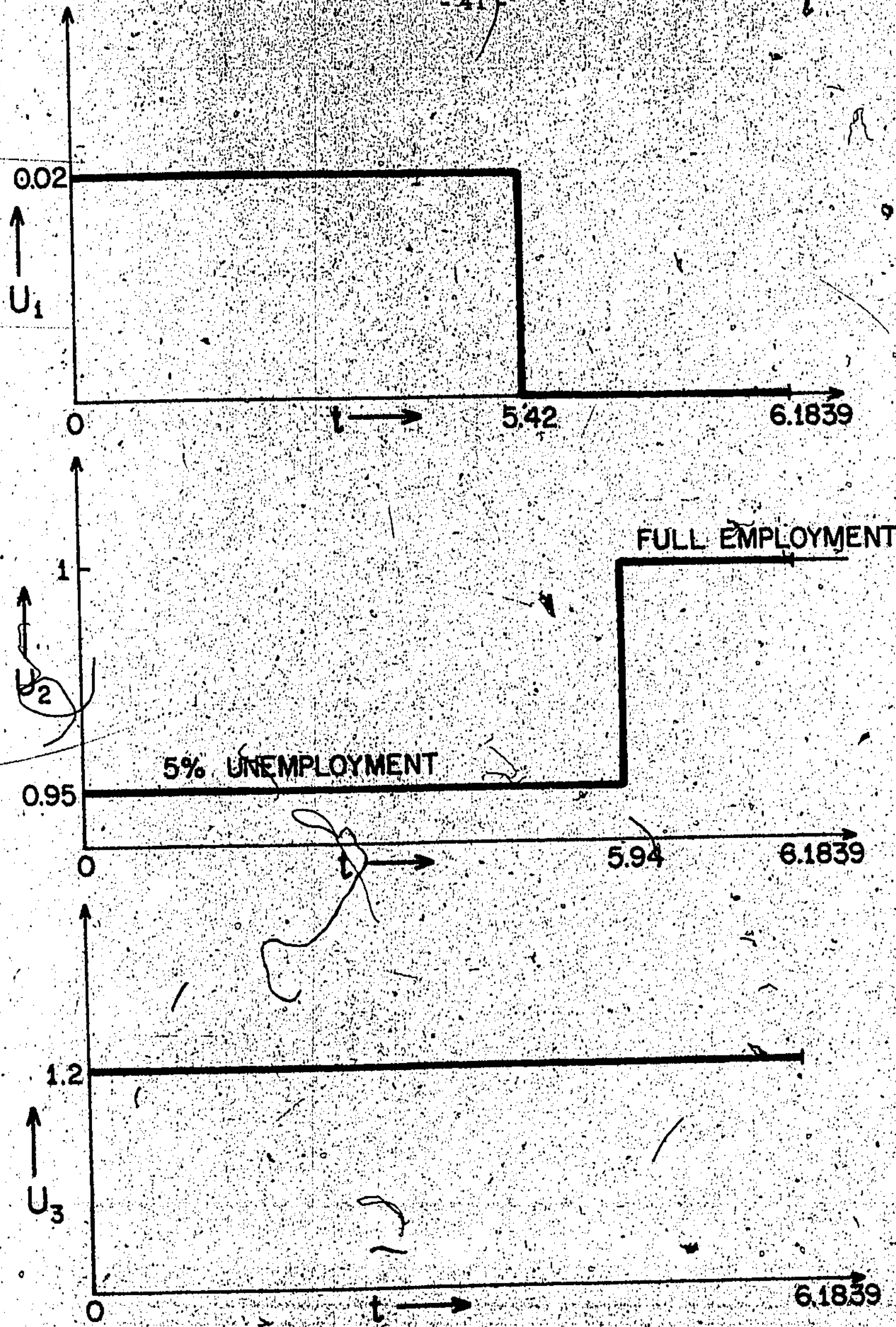

0.02
U_1
0
t
5.42
6.1839
FULL EMPLOYMENT
1
U_2
5% UNEMPLOYMENT
0.95
0
t
5.94
6.1839
1.2
U_3
0
t
6.1839

FIG 1.2

EXAMPLE 2.

In this example, 30% marginal propensity to save, 3% unemployment and $L_s = L_o e^{bt}$ where $L_o = 0.1 \times 10^3$ and $b = 0.2$ are considered.

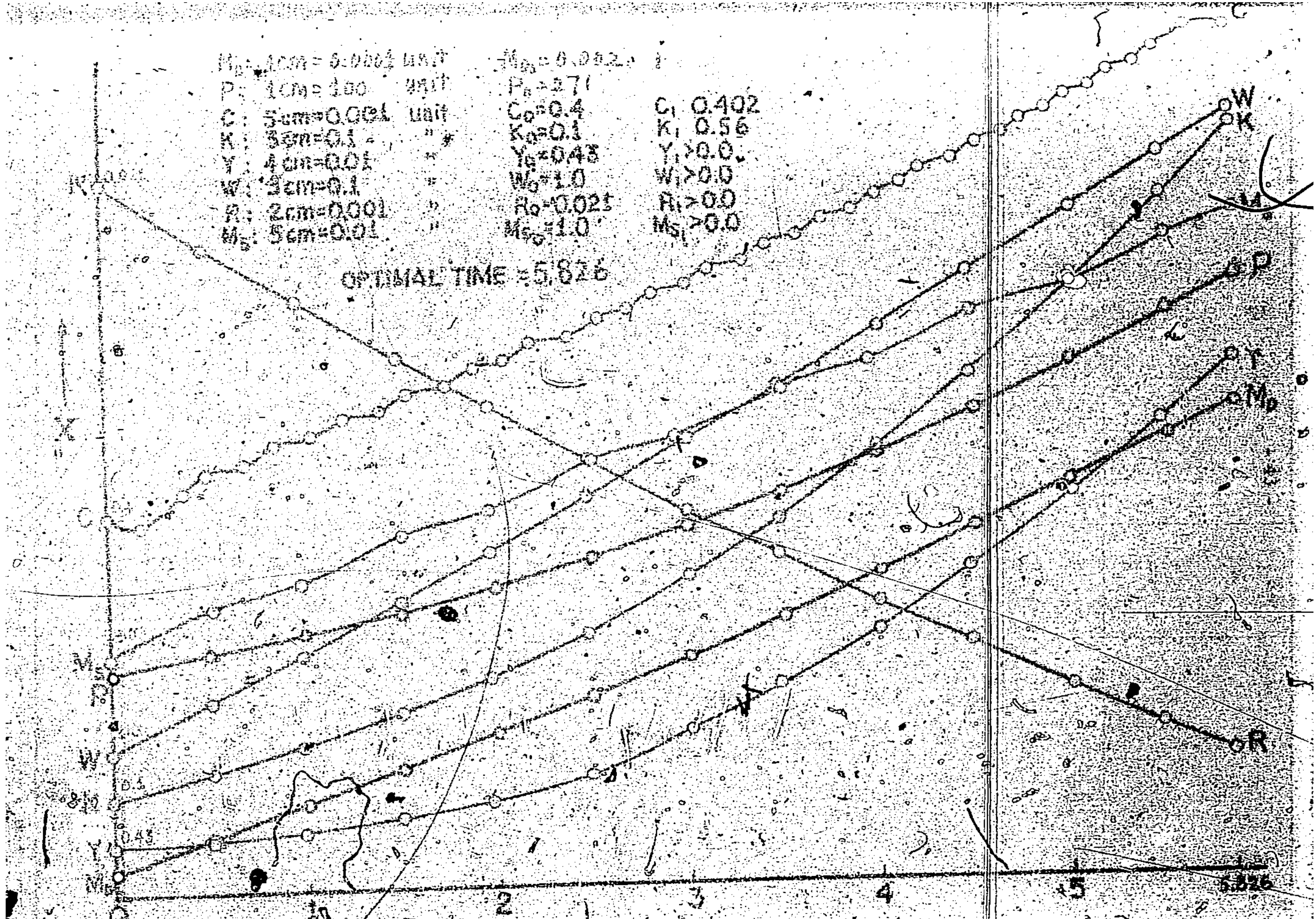
Initial state : $x_o \underline{\Delta} \{ C_o = 0.4,\ K_o = 0.1,\ Y_o = 0.43,\ W_o = 1.0,$
$R_o = 0.21 \times 10^{-1},\ M_{s_o} = 1.0 \}$

Desired final state : $x_1 \in M_1 \underline{\Delta} \{ C_1 = 0.402,\ K_1 = 0.56,\ Y_1 > 0,\ W_1 > 0,$
$R_1 > 0,\ M_{s_1} > 0 \}$

The constrol constraints : $\omega_1 = 0.3,\ \omega_2 = 0.97,\ \omega_3' = 1.2$ and
$\omega_3 = 1.5$

$\psi^o \underline{\Delta} (\psi_1^o, \psi_2^o, \ldots, \psi_6^o)$	$E(\psi^o)$	No. of iterations	ψ^o (end conditions after 12 iterations)	$E(\psi^o)$ (after 12 iterations)
0.722×10^6			0.8238×10^2	
0.5×10^{-1}			-0.1656	
0.1×10^{-3}	2.914052×10^{10}	12	-0.2107×10^{-2}	3.41502×10^{-5}
0.1×10^{-3}			0.4235×10^{-2}	
0.3×10^{-2}			-0.2238×10^{-2}	
0.1×10^{-3}			0.2322×10^{-2}	

The optimal time : 5.826061



P: 1cm=100 unit
C: 5cm=0.001 unit
K: 3cm=0.1
Y: 4cm=0.01
W: 3cm=0.1
R: 2cm=0.001
M_s: 5cm=0.01
P_0=271
C_0=0.4
K_0=0.1
Y_0=0.43
W_0=1.0
R_0=0.021
M_{s0}=1.0
C_1 0.402
K_1 0.56
Y_1>0.0
W_1>0.0
R_1>0.0
M_{s1}>0.0
OPTIMAL TIME =5.826
W
K
P
Y
M_p
R
2
3
4
5
5.826

Fig 2.1

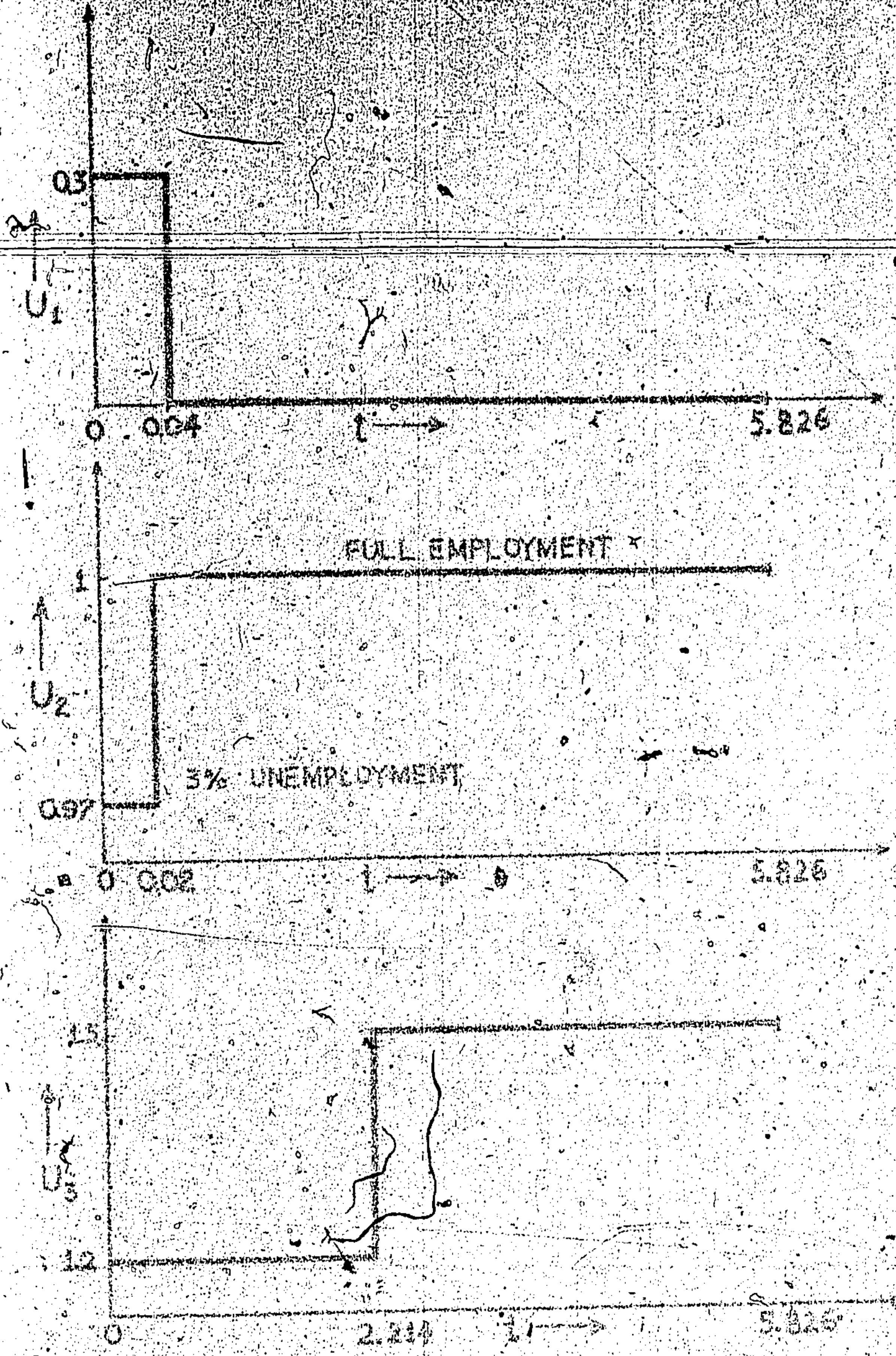

0.3
U_1
0
0.04
t →
5.826
FULL EMPLOYMENT
1
U_2
3% UNEMPLOYMENT
0.97
0
0.02
t →
5.826
1.5
U_3
1.2
0
t →
5.826

FIG 2.2

DISCUSSIONS AND CONCLUSION

An economic model based on the Cyclical Growth Theory developed by Keynes-Phillips-Bergstrom (KPB) (1, 2) with the inclusion of "Marginal Propensity to Save", "Employment Ratio" and "Price Factor" as the Control variables has been presented.

In the problems of optimization of national economic system two types of problem arise naturally.

Fixed plan period : For a fixed plan period the objective is to achieve a specified economic goal and at the same time minimize excessive rise of profit . This is achieved by optimal choice of an investment policy over the plan period under the constraints of fixed employment ratio and price index (unit cost of production). This is a fixed time problem and has been considered in Chapter 3 examples 1, 2 pp 33 - 38 .

Variable plan period : In the case of developing countries, the initial standard of living (eg. per capita consumption, wage rate etc) may be below subsistance level or near subsistance level. In such a situation it is desirable to raise the standard of living in as short a time as possible within the limit of available resources. This is to be achieved by appropriate choice of investment rate, employment level and price level with appropriate constraints. This gives rise to a standard time optimal control problem.

Applying Pontryagin's Maximum Principle (13), optimal control laws have been derived. The optimality problem is then reduced to a two-point boundary value problem and has been solved by using Davidon - Fletcher - Powell method (3, 6, 11) along with Fibonacci Search technique (7, 8, 11, 16).

The fixed time result indicates a positive investment throughout the plan period under the constraints of fixed employment ratio and price index. The desired terminal state is achieved even when the upper limit for the "Marginal Propensity to Save" is fixed at 30%.

The result of time optimal problem indicates that the desired terminal state is achieved with a combination of unemployment initially, full employment at the later stages and positive investment throughout.

For future suggestion, it would be interesting to consider further advanced multi-sector models which include foreign trades, fiscal policy and technical progress etc. Also the computational method has to be further improved to save heavy computational cost.

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