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THE BORSUK-ULAM THEOREM AND SOME OF ITS APPLICATIONS IN ANALYSIS

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by

G.K.R. RAO

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INTRODUCTION

This thesis deals with a proof of the Borsuk-Ulam Theorem and treats some applications of the Borsuk-Ulam Theorem in Analysis. We may express the content of the theorem in question goes as follows: Every continuous function mapping the unit sphere of an $(n+1)$-dimensional real Euclidean space into an n-dimensional real Euclidean space sends at least one pair of antipodal, i.e., diametrically opposite, points to the same. - An illustration of this theorem in meteorology would be to consider the function which assigns to each point on the surface of the earth an ordered pair of real numbers such that the first entry indicates the temperature and the second entry denotes the atmospheric pressure at that point. Assuming of course that this function is continuous and that the earth is a perfect sphere, the Borsuk-Ulam Theorem asserts that there is at least one pair of diametrically opposite points on the surface of the earth at which the pressure-temperature reading of the atmosphere are the same at any given time.

In Chapter I we present an "ad hoc" proof of the Borsuk-Ulam Theorem leaning on A.W. Tucker's work [11]. In our treatment we endeavour to give a reasonably self-contained presentation consistent with our purpose.

We use the Borsuk-Ulam Theorem in Chapter II to discuss the notion of opening of two linear subsets of a Banach space. Here our presentation differs from that encountered in the literature [7] on the subject in as much as we avoid the use of the intricate concept of regular closure to prove the crucial result that the opening of two linear subspaces in a Banach space equals the opening of their orthogonal complements in the dual Banach space. Moreover we take up the notion of deficiency index of a symmetric operator in a Hilbert space in this context.

In Chapter III we commence with a proof of Brouwer's Fixed Point Theorem based on the Borsuk-Ulam Theorem, and we use the former to derive Schauder's Fixed Point Theorem. We close with an illustration of the analytic significance of Schauder's Fixed Point Theorem.

CHAPTER I

BORSUK-ULAM THEOREM

We begin with some definitions. In the sequel the word mapping will be understood to mean a continuous mapping.

Let R^{n+1} be the (n+1)-dimensional Euclidean space, so that each point p of R^{n+1} corresponds to (n+1) real co-ordinates $(x_1, \dots, x_{n+1})$; the distance between two points $p=(x_1, \dots, x_{n+1})$ and $q=(y_1, \dots, y_{n+1})$ being given by

$$d(p,q) = \{\sum_{i=1}^{n+1} (x_i - y_i)^2\}^{1/2} .$$

Thus R^{n+1} is a real (n+1)-dimensional vector space (over R^1) and is a metric space as well.

DEFINITION 1. Points $p_0, p_1, \dots, p_{k+1}$ in R^{n+1} are said to be *affinely independent* if the vectors $(p_1 - p_0), \dots, (p_{k+1} - p_0)$, are linearly independent in R^{n+1}.

It is clear that any permutation of the set $\{p_0, p_1, \dots, p_{k+1}\}$ of affinely independent points is also affinely independent. Clearly $k \leq n+1$.

Next we introduce the notion of a geometric simplex.

DEFINITION 2. Let $M = \{p_1, p_2, \dots, p_{k+1}\}$, where

$k \leq n+1$ denote a set of affinely independent points of R^{n+1}. Then the convex closure of the set M, i.e., the intersection of all closed convex sets containing M, is called a k - d i m e n s i o n a l s i m p l e x or a k - s i m p l e x with v e r t i c e s $p_1, p_2, \ldots, p_{k+1}$. We denote the simplex by $\sigma^k\langle p_1, p_2, \ldots, p_{k+1}\rangle$.

One shows *) that:

If $p_1, p_2, \ldots, p_{k+1}$ are affinely independent, then every point x of the simplex $\sigma^k\langle p_1, p_2, \ldots, p_{k+1}\rangle$ can be uniquely expressed in the form

$$x = \alpha_1 p_1 + \ldots + \alpha_{k+1} p_{k+1},$$

where $\alpha_i \geq 0$ $(i=1, 2, \ldots, k+1)$ and $\sum_{i=1}^{k+1} \alpha_i = 1$.

The α_i, $i=1, \ldots, k+1$ are then called the b a r y c e n t r i c c o - o r d i n a t e s of x and we denote the point x in the simplex σ^k as the point $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$.

DEFINITION 3. Given a k-dimensional simplex σ^k, we can choose any $s+1$ vertices of $\sigma^k (s \leq k)$ and determine

*) See reference [1], page 412, of the Bibliography.

an s-dimensional simplex σ^s. (These $s+1$ points are clearly affinely independent). The simplex σ^s is then called s-dimensional f a c e of the given k-dimensional simplex, or s - f a c e of σ^k.

The zero-dimensional faces of σ^k are its $(k+1)$ vertices.

DEFINITION 4. Let $\sigma^k = \sigma^k \langle p_1, p_2, \ldots, p_{k+1} \rangle$ be a k-simplex in R^{n+1}. Then the point $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ where $\alpha_i = \frac{1}{k+1}$ for $i=1, 2, \ldots, k+1$, is referred to as the b a r y c e n t r e of the simplex σ^k, and denoted by $\mathbf{b}\sigma^k$.

REMARK: In particular, if p is a vertex (i.e., a single point), then bp, the barycentre of p as a simplex, is p itself.

We shall use the simplex $\sigma^{k-1}\langle p_1, p_2, \ldots, \hat{p}_j, \ldots, p_{k+1} \rangle$ to denote that $(k-1)$-face of $\sigma^k \langle p_1, p_2, \ldots, p_j, \ldots, p_{k+1} \rangle$ which is obtained by deleting the vertex p_j from the collection of vertices $p_1, \ldots, p_{k+1}$.

Next we introduce the notion of barycentric subdivision of a simplex.

DEFINITION 5. Let $\sigma^k = \sigma^k\langle p_1, p_2, \ldots, p_{k+1}\rangle$ be a k-simplex in R^{n+1} $(k \leq n+1)$ and $\sigma_1, \sigma_2, \ldots, \sigma_{\ell+1}$ be an arbitrary collection of faces of σ^k such that σ_i is face of σ_{i+1}, $(i=1, 2, \ldots, \ell)$. Then the corresponding barycentres $b\sigma_1, b\sigma_2, \ldots, b\sigma_{\ell+1}$ are affinely independent and hence generate a ℓ-dimensional simplex $\sigma^\ell\langle b\sigma_1, b\sigma_2, \ldots, b\sigma_{\ell+1}\rangle$. The collection of all such simplexes which can be obtained from the given simplex σ^k in the manner described is called the b a r y c e n t r i c s u b d i v i s i o n of σ^k.

REMARK: We note the well-known fact that on a finite-dimensional space all norms are equivalent. It proves to be convenient for us to continue the discussion in terms of the Banach space of all ordered (n+1)-tuples of real numbers $x = (x_1, x_2, \ldots, x_{n+1})$, which we denote by E^{n+1}, and for which the norm $||x||$ of any x is defined by

$$||x|| = \sum_{i=1}^{n+1} |x_i| .$$

Accordingly in Definitions 1 to 5, we may replace R^{n+1} by E^{n+1}.

Now the subset Σ^n of E^{n+1} consisting of all points $p = (x_1, x_2, \ldots, x_{n+1})$, whose coordinates satisfy the equality

$$|x_1| + |x_2| + \dots + |x_n| = 1 , \qquad (A)$$

is the boundary of the convex region obtained if the equality in Equation (A) is replaced by the inequality $\leq$. Hence Σ^n is an n-sphere in E^{n+1} .

We describe below certain symbols used frequently in the sequel.

<u>n-sphere</u> $\Sigma^n = \{p \in E^{n+1}: ||p|| = 1\}$.

<u>Unit ball</u> $K^{n+1} = \{p \in E^{n+1}: ||p|| \leq 1\}$.

<u>Upper hemisphere</u> $\Sigma^n_+ = \{p=(x_1, \dots , x_{n+1}) \in \Sigma^n: x_{n+1} \geq 0\}$.

<u>Lower hemisphere</u> Σ^n_- is defined analogously.

<u>Equator of Σ^n</u> $= \Sigma^{n-1} = \Sigma^n_+ \cap \Sigma^n_-$.

We shall denote by $e_i (i=1, 2, \dots , n+1)$ the point of E^{n+1} with co-ordinates $(\delta^i_1, \delta^i_2, \dots , \delta^i_i, \dots , \delta^i_{n+1})$, where δ^i_j equals 1 if i=j and equals 0 otherwise.

Then:

K^{n+1} is the convex closure of the points

$$e_1, e_2, \dots , e_{n+1}; -e_1, -e_2, \dots , -e_{n+1} ,$$

i.e.,

$$K^{n+1} = \text{conv}[e_1, e_2, \dots , e_{n+1}; -e_1, -e_2, \dots, -e_{n+1}].$$

Note that $\{e_1, e_2, \dots , e_{n+1}\}$ form a linearly independent set of vectors of E^{n+1} .

DEFINITION 6. TRIANGULATION OF THE SPHERE.

A finite collection $T = \{\sigma_j^i\}$ of the simplexes in E^{n+1} will be referred to as a t r i a n g u l a t i o n o f t h e s p h e r e Σ^n if the following conditions are satisfied:

(i) The n-dimensional simplexes of the collection T constitute a covering of Σ^n in the sense,

$$\bigcup_{\substack{j \\ \sigma_j^n \in T}} \sigma_j^n = \Sigma^n \quad ,$$

(ii) If $\sigma \in T$ and σ' is a face of σ, then $\sigma' \in T$, and

(iii) For any two simplexes $\sigma_1, \sigma_2 \in T$, either

$\sigma_1 \cap \sigma_2 = \square$, the null-set, or

$\sigma_1 \cap \sigma_2 = \sigma_3$, where σ_3 is non-null, belongs to T, and is a face of both σ_1 and σ_2.

DEFINITION 7. Let T be a triangulation of Σ^n. Then (as defined in Definition 5) we can barycentrically subdivide every simplex of T and obtain a collection $T^{(1)}$ of simplexes, which again constitute a triangulation of Σ^n. The triangulation $T^{(1)}$ is said to be the b a r y c e n t r i c s u b d i v i - s i o n of the triangulation T.

DEFINITION 8. Let $T^{(0)}=T$ be a triangulation of Σ^n and define inductively $T^{(m)}$ as the barycentric subdivision of $T^{(m-1)}$. The triangulation $T^{(m)}$ of Σ^n will be called the barycentric subdivision of order m of T, or briefly a subdivision of T.

DEFINITION 9. By the mesh of a triangulation T of Σ^n we mean the supremum of the diameters of the simplexes of T, i.e.,

$$\text{mesh } (T) = \sup_{\sigma \in T} d(\sigma) \quad .$$

Obviously

$$\text{mesh } (T) = \sup_{\sigma^n \in T} d(\sigma^n) \; .$$

THEOREM 1.*) Let T be a triangulation of the Σ^n. Then

$$\text{mesh } (T^{(m)}) \leq \left(\frac{n}{n+1}\right)^m \text{mesh}(T) \; .$$

(The simplexes of $T^{(m)}$ can thus be made arbitrarily small by taking m sufficiently large).

Earlier we referred to a triangulation of Σ^n. For our convenience, we shall introduce a standard triangulation Σ^n, which we shall henceforward refer to as the basic triangulation of Σ^n.

*) For a proof, see [4], or Hilton-Wylie, "Homology Theory", p.24.

DEFINITION 10. First of all if $e_i (i=1, 2, \ldots, n+1)$ denotes the point of E^{n+1} with coordinates $(\delta_1^i, \delta_2^i, \ldots, \delta_i^i, \ldots \delta_{n+1}^i)$ then by e_{-i} we shall mean the point $-e_i$ $(i=1, 2, \ldots, n+1)$. Let us consider any simplex of the form

$$\sigma^k = \sigma^k \langle e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}} \rangle$$

$$i_j = \pm 1, \pm 2, \ldots, \pm (n+1)$$

where $|i_1| < |i_2| < \ldots < |i_{k+1}|$. (1)

The collection $T_0 = \{\sigma^k\}$ $(k=1, 2, \ldots n)$ of all such simplexes constitutes a triangulation of the sphere Σ^n and will be referred to as the *basic triangulation* of the sphere Σ^n.

N.B. Condition (1) above ensures the affine independence of the set $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}}\}$. It also assigns a definite orientation to the simplex in the sense that it introduces an ordering of the vertices. We shall make the following agreement:

IN THE SEQUEL WE SHALL DENOTE BY T AN ITERATED BARYCENTRIC SUBDIVISION OF THE BASIC TRIANGULATION T_0.

DEFINITION 11. Simplicial Vertex mapping. Let T_1 and T_2 be any two triangulations of Σ^n. A mapping f which assigns to every vertex of T_1 a vertex of T_2 is called a (T_1, T_2) - simplicial vertex mapping provided the following condition is satisfied:

If $p_1, p_2, \ldots, p_{n+1}$ are vertices of an n-simplex of T_1, then $f(p_1), f(p_2), \ldots, f(p_{n+1})$ are vertices of some simplex of T_2 of possibly lower dimension than n. If two or more distinct vertices of $\sigma \in T_1$ are mapped by f into a single vertex, then this simplex σ is said to be degenerate under the mapping f, or simply degenerate; otherwise a simplex of T_1 is said to be non-degenerate under f. (2)

(In this definition the word mapping does not signify a continuous mapping).

DEFINITION 12. If T_1 and T_2 be two triangulations of Σ^n. A mapping $f: \Sigma^n \to \Sigma^n$ is said to be a (T_1, T_2) -simplicial map if

(i) f, as a mapping on the set of vertices of T_1 satisfies condition (2) of Def. 19, i.e., f is (T_1, T_2) -simplicial vertex mapping.

(ii) f is linear in terms of the barycentric co-ordinates. More precisely, if $x \in \sigma^n\langle p_1, p_2, \ldots, p_{n+1}\rangle$ and $x = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_{n+1} p_{n+1}$, then $f(x) = \alpha_1 f(p_1) + \alpha_2 f(p_2) + \ldots + \alpha_{n+1} f(p_{n+1})$.

THEOREM 2. A simplicial mapping is determined by its behaviour on vertices. Precisely, if a mapping f on the vertices of a triangulation T_1 of Σ^n is a (T_1, T_2)-simplicial vertex mapping, then it can be extended uniquely to a (T_1, T_2)-simplicial mapping of Σ^n into itself.

Proof: Extend f linearly in each simplex of T_1, i.e., for $x \in \sigma^n\langle p_1, \ldots, p_{n+1}\rangle$, $x = \sum_{i=1}^{n+1} \alpha_i p_i$, $\alpha_i \geq 0$, with $\sum_{i=1}^{n+1} \alpha_i = 1$, we define a mapping F by

$$F(x) = \sum_{i=1}^{n+1} \alpha_i f(p_i) . \qquad (3)$$

F is, obviously, continuous.

DEFINITION 13. The mapping α: $\Sigma^n \to \Sigma^n$, defined by,

$$\alpha(x) = -x \quad \text{for all} \quad x \in \Sigma^n ,$$

is called the antipodal map of Σ^n onto itself. Two elements of any sort (e.g., points, simplexes, sets, etc.) corresponding under α will be called antipodal.

With the understanding about notations used, it is clear that α, both on T_0 and T, is (T_0, T_0) - and (T,T)-simplicial mapping of Σ^n into itself.

DEFINITION 14. Any mapping $f: \Sigma^n \to \Sigma^n$ of the sphere Σ^n into itself is said to be an odd mapping or an antipodal mapping if f maps antipodal pair of points into antipodal pair of points i.e.,

$$f(-x) = -f(x) \ \forall \ x \in \Sigma^n .$$

Remark: Let S^n denote the Euclidean n-sphere i.e., the set of all points $\{(x_1, x_2, \ldots, x_{n+1})\}$ such that

$$x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 .$$

A homeomorphism between S^n and Σ^n is immediately established by central projection from the origin. Under this homeomorphism antipodal points on Σ^n correspond to antipodal points of S^n. Thus there is no loss of generality resulting from the particular representation of the n-sphere that we have chosen. (See the remark following Def. 5 in this chapter).

In the sequel let T be a subdivision of the basic triangulation T_0 of Σ^n and f be an odd (T, T_0)-simplicial mapping of Σ^n into itself. Then f sends antipodal vertices of T into antipodal vertices of T_0 and hence antipodal simplexes of T into antipodal simplexes of T_0.

In what follows we shall find it convenient to classify simplexes of the iterated triangulation T. This classification is partly indicated in the two definitions below.

DEFINITION 15. Let f be an odd (T,T_0)-simplicial mapping of Σ^n into itself. A k-simplex $s^k=s^k\langle p_1, p_2, \dots p_{k+1}\rangle$ of the triangulation T is said to be of the "p l u s k i n d" if its image $f(s^k)$ under f is a k-simplex σ^k of T_0, i.e.,

$$\sigma^k=\sigma^k\langle e_{i_1}, e_{i_2}, \dots , e_{i_k}\rangle , i_j = \pm 1, \pm 2, \dots \pm(n+1),$$

$$|i_1| < |i_2| < \dots < |i_{k+1}| ,$$

and satisfies in addition the following two properties: (i) adjacent vertices e_{i_j} and $e_{i_{j+1}}$ have opposite signed subscripts $(j=1, 2, \dots , n)$, (ii) the first vertex e_{i_1} has positive subscript i_1.

We shall denote by T_k^+ the set of all k-simplexes of "plus kind" in the triangulation $T(k=0, 1, \dots , n)$. Thus in particular, T_n^+ consists of all n-simplexes of T which are mapped by f onto the simplex:

$$\sigma^n\langle e_1, e_{-2}, e_3, \dots , e_{(-1)^n(n+1)}\rangle .$$

DEFINITION 16. A k-simplex s^k of the triangulation T is said to be of the "m i n u s k i n d", if its antipodal image $\alpha(s^k)$ is a simplex of the "plus kind", i.e., $f(\alpha(s^k))$ is a k-simplex $\sigma^k=\sigma^k\langle e_{i_1}, e_{i_2}, \dots ,e_{i_{k+1}}\rangle$, where

$$|i_1| < |i_2| < \dots < |i_{k+1}|, (i_j= \pm 1, \pm 2, \dots, \pm (n+1))$$

<u>and</u> (i) e_{i_j} <u>and</u> $e_{i_{j+1}}$ <u>have opposite signed subscripts,</u>

<u>and</u> (ii) e_{i_1} <u>has positive subscript</u>.

We shall denote by T_k^- $(k=0, 1, \ldots, n)$ the set of all k-simplexes of "minus kind."

Note that T_n^- consists of all n-simplexes of T which are mapped by f onto the simplex

$$\sigma^n \langle e_{-1}, e_2, e_{-3}, \ldots, e_{(-1)^{n+1}(n+1)} \rangle .$$

We remark, in passing this stage, that there are other possibilities.

(I) For a $s^k \in T$, $f(s^k)$ is a k-simplex, but $s^k \notin T_k^+$ and $s^k \notin T_k^-$.

(II) For an $s^k \in T$, $f(s^k)$ is a j-simplex, where $0 \leq j < k$.

Definitions 15, 16 and remarks (I) and (II) above describe the said classification.

In the sequel, we shall use the following notations, with regard to any arbitrary set $A \subset \Sigma^n$: $\ell_k^+(A)$ = the total number of k-simplexes s^k of "plus kind" in T such that $s^k \subset A$.

N_1 = the set of n-simplexes $s^n \in T_n^+$ such that $s^n \subset \Sigma_+^n$

N_2 = the set of n-simplexes $s^n \in T_n^-$ such that $s^n \subset \Sigma_+^n$

N_3 = the set of n-simplexes $s^n \in T$ such that $s^n \subset \Sigma_+^n$ and $s^n \notin T_n^+$ and $s^n \notin T_n^-$.

LEMMA 1[*]. <u>Let</u> s^n <u>be an arbitrary n-simplex of the triangulation</u> T <u>of</u> Σ^n.

(i) <u>If</u> $s^n \in T_n^+$ <u>or</u> $s^n \in T_n^-$, <u>then</u> $\ell_{n-1}^+(s^n) = 1$

(ii) <u>If</u> $s^n \notin T_n^+$ <u>and</u> $s^n \notin T_n^-$, <u>then</u> $\ell_{n-1}^+(s^n) \equiv 0 (\mathrm{mod}.2)$.

Proof: (i) $\sigma^n\langle e_1, e_{-2}, e_3, \dots , e_{(-1)^n(n+1)}\rangle$ is clearly the only n-simplex in T_0, where the first vertex has positive subscript, and consecutive vertices e_j, $e_{j+1}(j=1, 2, \dots, n)$ have oppositedly signed subscripts. Let this simplex be the image under f of the n-simplex $s^n\langle p_1, p_2, \dots , p_{n+1}\rangle$, where $f(p_1)=e_1$, $f(p_2)=e_{-2}$, ..., $f(p_{n+1}) = e_{(-1)^n(n+1)}$. (Note that f is an odd (T,T_0)-simplicial mapping). The only $(n-1)$-simplex of plus kind contained in s^n is obtained by deleting the last vertex p_{n+1}. In this case

$$f(s^{n-1}\langle p_1, \dots , p_n\rangle) = \sigma^{n-1}\langle e_1, e_{-2}, \dots e_{(-1)^{n-1}n}\rangle.$$

Note that $\sigma^{n-1}\langle e_1, e_{-2}, \dots , e_{(-1)^{n-1}n}\rangle$ is the only $(n-1)$-simplex for which first vertex has positive subscript, and

*) See for instance Reference [3], p. 95.

consecutive vertices have oppositedly signed subscripts, and which is contained in $\sigma^n\langle e_1, e_{-2}, \ldots, e_{(-1)^n(n+1)}\rangle$.

Thus, in this case $\iota^+_{n-1}(s^n) = 1$.

Now let $s^n \in T^-_n$. Then

$$f(s^n) = \sigma^n\langle e_{-1}, e_2, e_{-3}, \ldots , e_{(-1)^{n+1}(n+1)}\rangle$$

where on the right hand side we have an n-simplex with

(i) subscript of the first vertex is negative

(ii) subscripts of consecutive vertices e_j, e_{j+1} have opposite signs.

The simplex on the right hand side is the only n-simplex of this variety in T_0, and it has only one (n-1)-simplex of the same type contained in it viz. $\sigma^{n-1}\langle e_{-1}, e_2, e_{-3}, \ldots , e_{(-1)^n n}\rangle$.

If $s^n = s^n\langle p_1, p_2, \ldots , p_{n+1}\rangle$ where $f(p_1) = e_{-1}$, $f(p_2) = e_2$, $f(p_3) = e_{-3}$, ..., $f(p_{n+1}) = e_{(-1)^{n+1}(n+1)}$, then by deleting the first vertex p_1 we get the only (n-1)-simplex of "plus kind"contained in s^n.

Thus again $\iota^+_{n-1}(s^n) = 1$.

The proof of the first part is complete.

Proof of second part:

Let $s^n \notin T^+_n$ and $s^n \notin T^-_n$. A number of different cases arise.

(A) Let $f(s^n)$ be an n-simplex.

i.e., $f(s^n) = \sigma^n\langle e_{i_1}, e_{i_2}, \ldots, e_{i_{n+1}}\rangle$, $i_j = \pm 1, \pm 2, \ldots, \pm(n+1)$

with $|i_1| < |i_2| < \ldots < |i_{n+1}|$, where σ^n is not an n-simplex of the type considered in the proof of the first part. We may write as usual

$$s^n = s^n\langle p_1, p_2, \ldots, p_{n+1}\rangle$$

where $f(p_j) = e_{i_j}$, $j=1, \ldots, n+1$.

Firstly if in $\sigma^n\langle e_{i_1}, e_{i_2}, \ldots, e_{i_{n+1}}\rangle$, the subscript of the first vertex is positive and subscripts of adjacent vertices are opposite in sign, except that the subscripts of a certain pair of adjacent vertices, say i_j, i_{j+1} have the same sign. (4)

Then both the simplexes of dimension (n-1), namely,

$$\left.\begin{array}{l} \sigma^{n-1}\langle e_{i_1}, e_{i_2}, \ldots, \hat{e}_{i_j}, \ldots, e_{i_{n+1}}\rangle \text{ and} \\ \sigma^{n-1}\langle e_{i_1}, e_{i_2}, \ldots, \hat{e}_{i_{j+1}}, \ldots, e_{i_{n+1}}\rangle \end{array}\right\} \quad (5)$$

are the images under f of simplexes of "plus kind" viz.

$$s^{n-1}\langle p_1, p_2, \ldots, \hat{p}_j \ldots, p_{n+1}\rangle$$

and

$$s^{n-1}\langle p_1, p_2, \ldots, \hat{p}_{j+1}, \ldots, p_{n+1}\rangle$$

resp., and thus $\ell^+_{n-1}(s^n) = 2 \equiv 0 \pmod 2$.

If in addition to the pair of vertices e_{i_j}, $e_{i_{j+1}}$ another pair e_{i_r}, $e_{i_{r+1}}$ $(i_j \neq i_r)$ also have subscripts of the same sign, then it is obvious that we cannot have (n-1)-simplexes of type (5) contained in σ^n, and thus $\ell^{+}_{n-1}(s^n) = 0$.

Secondly if in statement (4) above we have that the subscript of the first vertex e_{i_1}, viz. i_1, is negative, and other conditions unchanged, then it is clear that $\ell^{+}_{n-1}(s^n) = 0$.

Now if σ^n is neither of the type considered in (4), or of the one just above, nor of the two types considered in proving part (i), then

$$\ell^{+}_{n-1}(s^n) = 0 .$$

(B) Let $f(s^n)$ be a (n-1)-simplex σ^{n-1}, say,

$$s^n = s^n\langle p_1, p_2, \dots , p_{n+1}\rangle \quad \text{and}$$

$$\sigma^{n-1} = \sigma^{n-1}\langle e_{i_1}, e_{i_2}, \dots , e_{i_n}\rangle .$$

In this case, some pair of distinct vertices, say p_i, $p_j (i \neq j)$ are mapped into one vertex, e.g., e_{i_j}. Clearly, if σ^{n-1} has vertices the subscripts of which alternate in sign, with the first subscript being assumed positive, then there are two (n-1)-simplexes of "plus kind" contained in s^n, viz.,

$$s^{n-1}\langle p_1, p_2, \dots, p_i, \dots, \hat{p}_j, \dots, p_{n+1}\rangle \text{ and}$$

$$s^{n-1}\langle p_1, p_2, \dots, \hat{p}_i, \dots, p_j, \dots, p_{n+1}\rangle .$$

Thus $\iota^+_{n-1}(s^n) = 2 \equiv 0 \pmod 2$.

On the other hand if σ^{n-1} has the sign of subscript of the first vertex negative and signs of adjacent vertices e_{i_r}, $e_{i_{r+1}}$ $(r=1, 2, \dots , n-1)$ opposite to each other then clearly $\iota^+_{n-1}(s^n) = 0$, for in the ordered enumeration(i.e., by $|i_1| < |i_2| < \dots < |i_n|$),

$$i_1, i_2, \dots , i_n$$

there are (n-1)-changes of sign and we commence with a negative sign, and it is not possible to have (n-1)-changes in sign after commencing with a positive sign, keeping the relative ordering unaltered.

It is now obvious that if σ^{n-1} is an n-simplex which is not of either type considered above then

$$\iota^+_{n-1}(s^n) = 0.$$

The proof is completed by noting finally that if $f(s^n)$ is a j-simplex where $j < n-1$, then $\iota^+_{n-1}(s^n) = 0$.

THEOREM 3*). <u>Let</u> T <u>be a subdivision of the basic triangulation</u> T_0 <u>of the sphere</u> Σ^n <u>and</u> f <u>an odd</u> (T,T_0)-<u>simplicial mapping of</u> Σ^n <u>into itself. Then there exists</u>

*) See [3], p. 95.

<u>an n-simplex of</u> T_0 <u>which is the image under</u> f <u>of an odd number of n-simplexes of</u> T.

Proof: It will be shown that the simplex

$$\sigma^n\langle e_1, e_{-2}, e_3, e_{-4}, \ldots, e_{(-1)^n(n+1)}\rangle$$

is the image under f of an odd number of n-simplexes of T. As n-simplexes of "plus kind" can only be mapped by f into σ^n, it is enough to establish that

$$\iota_n^+ (\Sigma^n) \equiv 1 \pmod 2 . \qquad \text{(i)}$$

We proceed to prove by induction, and note that for n=0 the equation (i) is true since the sphere Σ^0 consists of just two points Σ_+^0 and Σ_-^0. Assume that (i) is true for (n-1) i.e.,

$$\iota_{n-1}^+ (\Sigma^{n-1}) \equiv 1 \pmod 2 . \qquad \text{(ii)}$$

It is required to prove (i) for n.

To do this we enumerate the (n-1)-simplexes $s^{n-1} \in T$ of the "plus kind" lying on the upper hemisphere Σ_+^n counting each simplex s^{n-1} as many times as it occurs as a face of a simplex $s^n \subset \Sigma_+^n$ of the triangulation T.

Since each simplex $s^{n-1} \subset \Sigma_+^n$ is either a face of two simplexes $s_1^n, s_2^n \subset \Sigma_+^n$ or else is a face of just one, in which case $s^{n-1} \subset \Sigma^{n-1}$, we conclude that the parity of the number of simplexes we have counted is equal to the parity of the number $\iota_{n-1}^+ (\Sigma^{n-1})$; two integers a and b have the same parity if $a-b$ is even, and otherwise have different parity.

On the other hand, the parity of the number of simplexes we have counted is equal to the parity of the number

$$\sum_{s^n \in N_1} \ell^+_{n-1}(s^n) + \sum_{s^n \in N_2} \ell^+_{n-1}(s^n) + \sum_{s^n \in N_3} \ell^+_{n-1}(s^n) \quad ,$$

where the symbols N_1, N_2, N_3 have been described earlier.

By Lemma 1, we have $\sum_{s^n \in N_3} \ell^+_{n-1}(s^n) \equiv 0 \pmod 2$.

Hence

$$\sum_{s^n \in N_1} \ell^+_{n-1}(s^n) + \sum_{s^n \in N_2} \ell^+_{n-1}(s^n) \equiv \ell^+_{n-1}(\Sigma^{n-1}) \pmod 2 \qquad \text{(iii)}$$

Now let us consider all n-simplexes s^n of "plus kind" in triangulation T, contained in the upper hemisphere Σ^n_+. By an earlier reasoning (see Lemma 1, part (i)), each such s^n contains one and only one (n-1)-simplex s^{n-1} of "plus kind" as its face. Thus it follows that there can be as many (n-1)-simplexes of "plus kind" $\subset \Sigma_+^n$ as there are simplexes of latter type in Σ_+^n. (All the concerned simplexes belong to T). Hence

$$\sum_{s^n \in N_1} \ell_{n-1}^{+}(s^n) = \ell_n^{+}(\Sigma_+^n) .$$

Similarly,

$$\sum_{s^n \in N_2} \ell_{n-1}^{+}(s^n) = \ell_n^{-}(\Sigma_+^n) .$$

Hence we get from (iii) that

$$\ell_n^{+}(\Sigma_+^n) + \ell_n^{-}(\Sigma_+^n) \equiv \ell_{n-1}^{+}(\Sigma^{n-1}) \pmod 2 \qquad \text{(iv)}.$$

Since the number $\ell_n^{-}(\Sigma_n^+)$ of n-simplex of the "minus kind" in the upper hemisphere Σ_+^n is equal to the number $\ell_n^{+}(\Sigma_-^n)$ of n-simplexes of "plus kind" in the lower hemisphere Σ_-^n, we get from (iv) the result

$$\ell_n^{+}(\Sigma_+^n) + \ell_n^{+}(\Sigma_-^n) \equiv \ell_{n-1}^{+}(\Sigma^{n-1}) \pmod 2. \qquad \text{(v)}.$$

But

$$\ell_n^{+}(\Sigma_+^n) + \ell_n^{+}(\Sigma_-^n) = \ell_n^{+}(\Sigma^n) \quad \text{as} \quad \Sigma^n = \Sigma_+^n \cup \Sigma_-^n ,$$

and we supposed that $\ell_{n-1}^{+}(\Sigma^{n-1}) \equiv 1 \pmod 2$ (Induction hypothesis). Hence we get

$$\ell_n^{+}(\Sigma^n) \equiv 1 \pmod 2$$

This completes the proof.

Remark:*) In fact every n-simplex of the basic triangulation T_0 can be shown to have the property mentioned in the lemma.

At this stage it may be emphasized that Theorem 3 constitutes the key result in the proof of Borsuk-Ulam Theorem.

We also need the

Lemma 2: Let $\Phi = \{M_1, M_2, \dots , M_n\}$ be a family of non-empty closed subsets $M_i (i=1, \dots , n)$, of a compact metric space X. Then there exists a real number $\varepsilon > 0$ (the so-called Lebesgue number of Φ) such that, whenever $A \supset X$, $d(A) < \varepsilon$ and $A \cap M_{i_j} \neq \square$ for $j=1, \dots , k$, then

$$M_{i_1} \cap M_{i_2} \cap \dots \cap M_{i_k} \neq \square .$$

We do not prove**) this lemma.

LEMMA 3***). The intersection of (n+1) non-empty closed sets $M_1, M_2, \dots , M_{n+1}$ on the sphere Σ^n is not empty if the (n+1) sets and their antipodal sets together cover Σ^n, but no one of the sets contains a pair of antipodal points.

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*) See [4], [11]; **in [4] consult p. 135.**

**) For a proof, see [1], page 234.

***) See [11]; [4], **p. 136.**

Proof: Let α denote the antipodal map. We suppose that Σ^n is covered by the $(n+1)$ sets $M_i(i=1,\dots,n+1)$ and the antipodal sets $\alpha(M_i)$ $(i=1, 2, \dots, n+1)$. We order the covering as follows:

$$\left.\begin{array}{l} M_1, M_{-1}; M_{-2}, M_2; M_3, M_{-3}; M_{-4}, M_4; \dots ; M_{(-1)^n(n+1)}, \\ M_{(-1)^{n+1}(n+1)}, \text{ where } \alpha(M_i)=M_{-i}, \ i=1, 2, \dots, n+1 . \end{array}\right\} \quad (6)$$

Let us denote by ε the Lebesgue number of the covering (6) and T a barycentric subdivision of the basic triangulation T_0, satisfying the condition mesh$(T) < \varepsilon$. (Note that we may have to take an iterated barycentric subdivision of T_0 for this purpose).

Take a vertex p of T. With the covering ordered as in (7), let M_i be the first set of the covering which contains $p(i= \pm 1, \pm 2, \dots \pm (n+1))$.

Set $f(p) = e_{(-1)^{i+1}i}$, where f is a map.

This function f defines a vertex mapping from T to T_0 in such a way that antipodal vertices of T are mapped into antipodal vertices of T_0 and adjacent vertices of T (i.e., joined by an edge) are mapped into non-antipodal points. [Note that no one of the sets $M_i(i=1, 2, \dots, n+1)$ contains a pair of antipodal points (hypothesis)].

Then by Theorem 2, f can be extended to a unique (T,T_0)-simplicial antipodal (because f is an antipodal (T, T_0)-vertex mapping) mapping $\bar{f}$ of Σ^n into itself.

Now by Theorem 3, the simplex $\sigma^n\langle e_1, e_{-2}, e_3, \ldots, e_{(-1)^n(n+1)}\rangle$ of T_0 is the image under $\bar{f}$ of at least one n-simplex s^n of T. This simplex has at least one vertex in each of the sets,

$$M_1, M_2, \ldots, M_{n+1};$$

(indeed, let $s^n = s^n\langle p_1, p_2, \ldots, p_{n+1}\rangle$ and let none of the vertices $p_i (i=1, 2, \ldots, n+1)$ belong to some particular set M_j (j fixed) among the sets $M_1, M_2, \ldots, M_{n+1}$. Then $p_i \notin M_j$ for all $i=1, \ldots, n+1$. Hence $f(p_i) \neq e_{(-1)^{j+1}j}$ for any $i=1, \ldots, n+1$. But $e_{(-1)^{j+1}j}$ is a vertex of $\sigma^n\langle e_1, e_{-2}, e_3, \ldots, e_{(-1)^n(n+1)}\rangle$ which is $f(s^n)$ and thus some vertex of s^n must belong to M_j. This contradicts the earlier assertion

$$p_i \notin M_j \quad \forall \quad i=1, 2, \ldots, n+1).$$

Thus $s^n \cap M_i \neq \square \quad i=1, 2, \ldots, n+1$. By virtue of Lemma 2, the intersection of these (n+1) sets is not empty since the diameter of s^n is less than the Lebesgue number of the covering.

The proof is over.

THEOREM 4. (Lusternik-Schnirelman-Borsuk-Theorem). <ins>If the sphere</ins> Σ^n <ins>is covered by</ins> (n+1) <ins>non-empty closed sets, then at least one of them contains a pair of antipodal points</ins>.

Proof: Let $M_1, M_2, \ldots, M_{n+1}$ be the $(n+1)$ closed sets which cover Σ^n. Then the $(n+1)$ antipodal $\alpha(M_i)$ $(i=1, \ldots, n+1)$, also constitute a covering of Σ^n.

Let us suppose that none of the M_i contains a pair of antipodal points. Then the sets } (7)

$$M_1, \alpha(M_1); M_2, \alpha(M_2); \ldots ; M_{n+1}, \alpha(M_{n+1})$$

satisfy the conditions of Lemma 3. In this event the $(n+1)$ sets $M_1, M_2, \ldots, M_{n+1}$ have some point in common:

$$M_1 \cap M_2 \cap \ldots \cap M_{n+1} \neq \square \ .$$

Let

$$p \in M_1 \cap M_2 \cap \ldots \cap M_{n+1} \neq \square \ .$$

Therefore $p \in M_i$ for $i=1, 2, \ldots, n+1$. As $\bigcup_{i=1}^{n+1} \alpha(M_i) = \Sigma^n$,

so $p \in \alpha(M_j)$ for some j.

Thus the point $p \in M_j$ and also in $\alpha(M_j)$. $p \in \alpha(M_j)$ implies $\alpha(p) \in M_j$. Thus $p, \alpha(p)$ are both in M_j.

Thus the set M_j contains a pair of antipodal points. This contradicts (7), and the contradiction establishes the theorem.

LEMMA 4: <u>If</u> n <u>non-empty closed sets and their antipodal sets together cover</u> Σ^n, <u>then at least one of the sets contains a pair of antipodal points</u>.

Proof:*) Let the n closed sets $M_1, M_2, \ldots, M_n$ together with their n antipodal images cover Σ^n.

Suppose also that none of these sets contain a pair of antipodal points. Adjoin $M_{n+1} = \alpha(M_n)$, $\alpha(M_{n+1})=M_n$ to get a covering by (n+1) pairs of antipodal closed sets.

Then by Lemma 3, it follows that

$$M_1 \cap M_2 \cap M_3 \cap \ldots \cap M_n \cap \alpha(M_n) \neq \Box .$$

Hence $M_n \cap \alpha(M_n) \neq \Box$, which is impossible since M_n and $\alpha(M_n)$ are disjoint or else contain a pair of antipodal points.

The proof is completed.

THEOREM 5. (Borsuk - Ulam Theorem):
Any continuous mapping of Σ^n into E^n maps at least one pair of antipodal points into the same point.

Proof: Suppose there is a map $f: \Sigma^n \to E^n$ such that whenever $x \in \Sigma^n$, $f(-x) \neq f(x)$.

We shall write $f=(f_1, f_2, \ldots, f_n)$, where $f_i(x)$ is the i^{th} co-ordinate of the point $f(x)$, $(i=1, 2, \ldots, n)$.

Now Σ^n is compact. Then it follows that

$$\beta = \inf_{x \in \Sigma^n} \max_{1 \leq i \leq n} |f_i(x) - f_i(-x)| > 0 \tag{8}$$

*) cf [4] p. 138 (Th. 21.3).

Let

$$M_i = \{x \in \Sigma^n \text{ such that } (f_i(x) - f_i(-x)) \geq \frac{\beta}{2}\}, i=1, 2, \ldots, n$$

so that

$$\alpha(M_i)=\{x \in \Sigma^n \text{ such that } (f_i(x) - f_i(-x)) \leq -\frac{\beta}{2}\}, i=1,2, \ldots, n.$$

Then $M_1, M_2, \ldots, M_n$ are closed sets in Σ^n, none of which contain a pair of antipodal points [for if x and $-x$ where in M_i, then $f_i(x) - f_i(-x) \geq \frac{\beta}{2}$ and $f_i(-x) - f_i(x) \geq \frac{\beta}{2}$ simultaneously and $\beta > 0$. This is not possible] and

$$\Sigma^n = \bigcup_{i=1}^{n} (M_i \cup \alpha(M_i)) \quad .$$

The latter fact follows because given any $x \in \Sigma^n$, we have $f(x) \neq f(-x)$ (by supposition). Then by (8) it follows that $x \in M_i$ for some i or in $\alpha(M_j)$ for some j.

Thus n closed sets $M_i(i=1, 2, \ldots, n)$ and their n antipodal sets $\alpha(M_i)$ $(i=1, 2, \ldots, n)$ cover Σ^n and none of these $2n$ sets contain a pair of antipodal points. This contradicts Lemma 4.

Hence the theorem.

THEOREM 6: <u>There is no continuous mapping of</u> Σ^n <u>into</u> Σ^{n-1}, <u>which maps antipodal points of</u> Σ^n <u>into antipodal points of</u> Σ^{n-1}.

Proof: Let us assume the contrary: there is a continuous map $f: \Sigma^n \to \Sigma^{n-1}$ with $f(x) = -f(-x)$ for

$x \in \Sigma^n$. This can be considered as a mapping of Σ^n into E^n. We have $f(x) \neq f(-x)$, which is in contradiction to the Borsuk-Ulam Theorem.

In the discussion so far we have shown that (i) Lemma 4 implies Theorem 4,(ii) Theorem 4 implies Theorem 5 and (iii) Theorem 5 implies Theorem 6. In fact we could show that (iv) Theorem 6 implies Theorem 5 and (v)*) Theorem 5 implies Theorem 4. We show (iv). Thus Theorems 4, 5 and 6 are equivalent. In this context we remark that in Chapter 2 we use the Borsuk-Ulam Theorem under the formulation of Theorem 6. (See also remark before Theorem 8). We sketch a proof of assertion (iv): Firstly, it is obvious that Lemma 4 implies Theorem 6. Indeed, if there was an antipodal map $f\colon \Sigma^n \to \Sigma^{n-1}$, then put $f=(f_1, \ldots, f_n)$. The 2n sets $M_i, \alpha(M_i)$, $i=1, \ldots, n$, where $M_i = \{x \in \Sigma^n$ such that $f_i(x) \geq \frac{1}{n}\}$, form a covering of Σ^n and none of these sets contain a pair of antipodal points. This contradicts Lemma 4. Hence a map f as supposed does not exist.

Next, assume that there is a mapping $g\colon \Sigma^n \to E^n$ such that $g(-x) \neq g(x)$ for all $x \in \Sigma^n$. Define $f\colon \Sigma^n \to \Sigma^{n-1}$ by

$$f(x) = \frac{g(-x)-g(x)}{|g(-x)-g(x)|} \quad \text{for all } x \in \Sigma^n .$$

Then f is an antipodal map, contradicting Theorem 6.

*) See [1], page 350.

The following theorem is used in Chapter III in connection with a proof of Brouwer's Theorem. (See Lemma 1 in Chapter III).

First we need the

DEFINITION 14. <u>A continuous mapping</u> $f: X \to A$ <u>of a topological space</u> X <u>on a subset</u> A <u>is called a r e t r a c - t i o n of</u> X <u>on</u> A <u>if</u> f/A <u>is the identity map</u>. A <u>is then called a r e t r a c t of</u> X.

THEOREM 7. <u>There is no continuous mapping</u> $f: K^{n+1} \to \Sigma^n$ <u>of the ball</u> K^{n+1} <u>onto its boundary</u> Σ^n <u>such that</u> f/Σ^n <u>is odd</u>.

Proof. If possible, let $f: K^{n+1} \to \Sigma^n$ be such a mapping. Put $f = (f_1, f_2, \ldots , f_{n+1})$ where $f_i: K^{n+1} \to R$. Then we have

$$\sum_{i=1}^{n+1} |f_i(x)| = 1, \quad x \in K^{n+1} ,$$

$$f_i(x) = -f_i(-x), \; x \in \Sigma^n, \; i=1, \ldots , n+1.$$

Let r be a retraction

$$r: \; E^{n+2} \to K^{n+1} , \text{ and}$$

put for each $x \in \Sigma^{n+1}$, $(i=1, 2, \ldots , n+1)$,

$$g_i(x) = \begin{cases} f_i(r(x)) & \text{if } x \in \Sigma_+^{n+1} \\ -f_i(r(-x)) & \text{if } x \in \Sigma_-^{n+1} . \end{cases} \tag{9}$$

By virtue of (9), the functions

$$g_i: \; \Sigma^{n+1} \to R$$

are continuous and satisfy

$$\sum_{i=1}^{n+1} |g_i(x)| = 1 \quad \text{for all } x \in \Sigma^{n+1}$$

We also have $g_i(x) = -g_i(-x)$ for $x \in \Sigma^{n+1}$.

Hence the mapping g defined by

$$g=(g_1, g_2, \ldots, g_{n+1})$$

is a continuous mapping of Σ^{n+1} into Σ^n, which **is an** antipodal **map**. But this contradicts Theorem 6.

Therefore no such map as supposed exists. This completes the proof.

Now Theorem 7 implies the Borsuks' Antipodal Theorem. Though we do not use this latter theorem in later chapters, we remark that it implies*) each one of the equivalent statements in Theorems 4, 5 and 6.

Theorem 8. (Borsuks' Antipodal Theorem). Let f: $\Sigma^n \to \Sigma^n$ be an odd mapping of Σ^n into itself. Then f is not homotopic to a constant map.

Proof. Let $I = [0,1]$, the closed unit interval of the real line. If f is homotopic to a constant map, then there exists a mapping $h: \Sigma^n \times I \to \Sigma^n$, with $h(x,0) = f(x)$, $h(x,1) = c_0$, a constant, for each $x \in \Sigma^n$. Setting $y=tx$, where $x \in \Sigma^n$ and $0 < t \leq 1$, so that $y \in K^{n+1} - \{0\}$, and defining a mapping $\bar{f}: K^{n+1} \to \Sigma^n$ by $\bar{f}(y) = h(x, 1-t)$, $\bar{f}(0)=c_0$, $(x \in \Sigma^n)$, we note that $\bar{f}/\Sigma^n=f$ is odd, thus contradicting Theorem 7.

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*) See references [1], [6], [11]; **in [1] consult p. 349.**

CHAPTER II

OPENING OF TWO LINEAR SUBSETS OF A BANACH SPACE

In the sequel we shall consider Banach spaces over the real or complex numbers. We commence with some definitions.

DEFINITION 1. Let E be a Banach space. A subset $M \subset E$ is called a g e n e r a t o r of E if the linear hull of M is dense in E, i.e.,

$$\overline{[M]} = E,$$

where the square brackets denote the linear hull of the set they enclose and the bar put above signifies the closure of the set relative to the norm topology of E.

We shall denote the cardinal number of the set M by α_M.

DEFINITION 2. The smallest of the cardinal numbers α_M, where M is an arbitrary set generating E, is called the d i m e n s i o n of the space E and is denoted by $\dim E$.

DEFINITION 3. Let M be a subset of a normed linear space E and x an element of E. The number $d(x,M)$, called the d i s t a n c e of the element x from the set M, stands for

$$\inf_{y \in M} \| x-y \| .$$

DEFINITION 4. Let E be a Banach space and M a subset of E. A subset A of M is called an *α - grating* of the set M if for any $x, y \in A$, we have

$$|| x-y || \geq \alpha .$$

DEFINITION 5. If an α-grating A of a set M is not a proper subset of any other α-grating of M, then it is called a *maximal* α-grating of the set M.

Thus if A is a maximal α-grating of the set M, then for each $z \in M$

$$d(z,A) = \inf_{x \in A} || x-z || < \alpha .$$

By transfinite process every α-grating can be extended up to a maximal one; hence, in particular follows the existence of a maximal α-grating.

LEMMA 1: Let X be a normed linear space and M a closed linear subspace in X. Suppose $M \neq X$. Then there exists, for any $\epsilon > 0$ with $0 < \epsilon < 1$, an x_ϵ in X such that

$$|| x_\epsilon || = 1 \quad \text{and} \quad d(x_\epsilon,M) \geq 1-\epsilon .$$

NOTE: The element x_ϵ is therefore "nearly orthogonal" to M.

Proof: Let $y \in X-M$. Since M is closed

$$d(y,M) = \inf_{m \in M} || y-m || = \alpha > 0 .$$

Thus there exists an element m_ε in M such that

$$|| \, y-m_\varepsilon \, || \leq \alpha\left(1 + \frac{\varepsilon}{1-\varepsilon}\right) \quad (\text{Note: } 0 < \varepsilon < 1) .$$

Clearly the vector $x_\varepsilon = \dfrac{y-m_\varepsilon}{||y-m_\varepsilon||}$ has unit norm

and

$$|| \, x_\varepsilon - m \, || = || \, y - m_\varepsilon || ^{-1} \, || \, y - m_\varepsilon - || \, y - m_\varepsilon || \, m \, ||$$

$$\geq || \, y - m_\varepsilon || ^{-1} \, \alpha \geq \left(\frac{1}{1-\varepsilon}\right)^{-1} = 1-\varepsilon .$$

This completes the proof.

Theorem 1 below relates the dimension of an arbitrary Banach space E with the cardinality of each maximal α-grating of the unit ball in E, where α is any real number in the open interval $(0,1)$.

THEOREM 1.*) Let E be a Banach space with dim E infinite. Then the cardinality of each maximal α-grating A of the unit ball ($|| \, x \, || \leq 1$) for $0 < \alpha < 1$ coincides with dim E.

Proof: Let A be a certain maximal α-grating of the unit ball $|| \, x \, || \leq 1$. Let it be assumed that $\alpha_A <$ dim E. Then the set L of all linear combinations with rational coefficients of elements of A is not dense in E.. In this case $\overline{L}$ is a proper subset of E. Then by Lemma 1 above it follows that for any $\varepsilon > 0$ $(0 < \varepsilon < 1)$ we can find an element x in E having unit

*) For Theorems 1 to 5 see reference [7] in the Bibliography; the relevant locations are pp. 188 - 190 and pp. 227 - 230.

norm, i.e., $|| x || = 1$, such that

$$d(x,L) \geq 1-\varepsilon .$$

Select an $\varepsilon < 1-\alpha$. Then $d(x,L) > \alpha$. We now note that $x \in E$ and $L \supset A$, where A is an α-grating. This implies that x is in an α-grating which contains A. Thus we arrive at a contradiction to the supposition that A is a maximal α-grating. Hence $\alpha_A \not< \dim E$.

Now from the definition of an α-grating it follows that neighbourhoods of radius $\frac{\alpha}{2}$ of elements of the α-grating do not intersect. But as in each of these neighbourhoods there is at least one element of any set dense in E, we have that

$$\alpha_A \not> \dim E.$$

Therefore $\alpha_A = \dim E$. The proof is complete.

REMARK: The assertion of the theorem is clearly valid if in its statement we replace the unit ball $|| x || \leq 1$ by the unit sphere $|| x || = 1$.

We note further: if E_1 is any subspace of the Banach space E, then $\dim E_1 \leq \dim E$. For if A_1 is some maximal α-grating $(0 < \alpha < 1)$ of the unit ball $K_1 \subset E_1$ and α_{A_1} is the cardinality of A_1, then by Theorem 1

$$\dim E_1 = \alpha_{A_1} .$$

Denoting by A a maximal α-grating of the unit ball in E which is an extension of A_1 and letting α_A be the cardinal number of A, then dim E = α_A . It is obvious however that $\alpha_{A_1} \leq \alpha_A$ and hence

$$\dim E_1 \leq \dim E .$$

DEFINITION 6. Let L_1 and L_2 be two linear subspaces of a Banach space E. The o p e n i n g of L_1 and L_2 in E is defined to be the number $\theta(L_1, L_2)$ satisfying the equality

$$\theta(L_1, L_2) = \max \left\{ \sup_{\substack{x \in L_1 \\ ||x||=1}} d(x, L_2), \quad \sup_{\substack{y \in L_2 \\ ||y||=1}} d(y, L_1) \right\}$$

From the definition we have two immediate properties of the opening:

firstly, $\quad 0 \leq \theta(L_1, L_2) = \theta(L_2, L_1) \leq 1,$ and

secondly $\quad \theta(\overline{L}_1, \overline{L}_2) = \theta(L_1, L_2)$.

THEOREM 2. Let L_1, L_2 be two linear subspaces of a Banach space E and let $\theta(L_1, L_2) = \alpha < 1$. If either dim L_1 or dim L_2 is a finite number, then

$$\dim L_1 = \dim L_2 .$$

N.B. By dim L we understand dim $\overline{L}$.

Proof: First we note that the statement in the theorem is equivalent to the assertion: If dim $L_1 = n$, dim $L_2 > n$, then $\theta(L_1, L_2) = 1$.

It is clear from def. 6 that for establishing $\theta(L_1, L_2) = 1$, it is sufficient to show that in L_2 there is a vector y of unit norm ($\| y \| = 1$) orthogonal to L_1, i.e.,

$$d(y, L_1) = 1 .$$

Without loss of generality we may suppose that $\dim L_2 = n+1$. Let us designate by E_1 the linear hull of L_1 and L_2, i.e., $E_1 = [L_1 + L_2]$. For a start let us assume that in E_1 the sphere $\| z \| = 1$ ($z \in E_1$) is strictly convex, i.e., does not contain segments. Then for each $z \in E_1$ there will exist only one $x \in L_1$ for which $d(z, L_1) = \| z-x \|$.

It is easy to see that the "projecting" operator Φ defined by $x = \Phi(z)$ ($z \in E_1$) is continuous and has the property

$$\Phi(-z) = -\Phi(z) .$$

The orthogonality of z to L_1 implies:

$$d(z, L_1) = \| z-\Phi(z) \| = \| z \| \qquad (1)$$

But by the uniqueness of a projection, (1) leads to $\Phi(z) = 0$.

If we now assume that on the unit sphere S_2 ($\| y \| = 1$) of the space L_2 the operator Φ is everywhere different from zero, then by virtue of the compactness of S_2, we may assert that the operator Ψ

defined by

$$\Psi(y) = \frac{\Phi(y)}{|| \Phi(y) ||}$$

is continuous on S_2 .

But this operator maps continuously the n-dimensional sphere S_2 into the (n-1)-dimensional sphere $S_1(||x||=1)$ of the space L_1 in such a way that any antipodal pair of points is transformed into an antipodal pair of points, i.e.,

$$\Psi(-y) = -\Psi(y)$$

and this is impossible in view of Borsuk-Ulam theorem.

Indeed, the sphere S_2 of the complex space L_2 can be mapped homeomorphically with preservation of central symmetry into a 2n-dimensional sphere S_2' of ordinary Euclidean space. Then the sphere S_1 can be mapped homeomorphically into a 2(n-1)-dimensional sphere S_1' which is a proper subset of S_2' . After this, the mapping $\Psi(y)$ induces in a natural manner an odd mapping of the sphere S_2' into its subset S_1' . However S_1' does not coincide with S_2' , and by the Borsuk-Ulam Theorem, a continuous map of a sphere into a sphere of lower dimension collapses some pair of antipodal points.

Therefore the theorem is proved by assuming that in the space E_1, the unit sphere $|| z || = 1$ is strictly convex. We now consider the general case where the condition of strict convexity is waived.

This case can however be reduced to the one already considered by proving that for any $\epsilon > 0$, it is always possible to construct in E_1 a new norm $\|z\|_o$ such that,

$$\|z\| \leq \|z\|_o \leq (1 + \epsilon)\|z\| \quad (z \in E_1), \qquad (2)$$

and so that the new unit sphere is strictly convex, i.e., for every two vectors $z_1, z_2 \in E_1$ with different directions, we have

$$\|z_1 + z_2\|_o < \|z_1\|_o + \|z_2\|_o . \qquad (3)$$

Then

$$\theta_o(L_1, L_2) = \max \left\{ \sup_{\substack{x \in L_1 \\ \|x\|_o = 1}} d_o(x, L_2), \quad \sup_{\substack{y \in L_2 \\ \|y\|_o = 1}} d_o(y, L_1) \right\}$$

where d_o is the metric induced by the norm $\| \ \|_o$.

Hence it follows that

$$\theta_o(L_1, L_2) \leq (1 + \epsilon)\, \theta\, (L_1, L_2) \quad .$$

In the above $\theta_o(L_1, L_2)$ is the opening of L_1 and L_2 in E_1 relative to the norm $\| \ \|_o$ and $\theta(L_1, L_2)$ is the opening relative to the norm $\| \ \|$.

Now, because of (3) (strict convexity) and by what has been established earlier, it follows that $\theta_o(L_1, L_2) = 1$ and on account of the arbitrariness of $\epsilon > 0$, we have that $\theta(L_1, L_2) = 1$ as well.

Finally we must show how such a norm $\| \ \|_o$ is realized. Let $\| \ \|_2$ be any norm in E_1 with respect to which the sphere $\| z \|_2 = 1$ of E_1 is strictly convex; e.g., it may be defined by setting

$$\| \xi_1 e_1 + \dots + \xi_m e_m \|_2 = \sqrt{\{\xi_1^2 + \dots + \xi_m^2\}} ,$$

where $\{e_1, \dots , e_m\}$ is some basis in E_1.

Let K (a strictly positive number) denote the maximum of $\| z \|_2$ on the unit sphere $\| z \| = 1$. Then we have that

$$\| z \|_2 \leq K \| z \| \text{ for all } z \text{ in } E_1 . \qquad (4)$$

Since the norm $\| z \|_2$ satisfies strict convexity, i.e., for every two vectors $z_1, z_2 \in E_1$ with different directions

$$\| z_1 + z_2 \|_2 < \| z_1 \|_2 + \| z_2 \|_2 , \qquad (5)$$

then for any $\delta > 0$ the same condition is also satisfied by the norm $\| \ \|_o$ defined by

$$\| z \|_o = \| z \| + \delta \| z \|_2 . \qquad (6)$$

Indeed, from (6), if z_1, z_2 have different directions, we get

$$\| z_1 + z_2 \|_o = \| z_1 + z_2 \| + \delta \| z_1 + z_2 \|_2$$

$$< \| z_1 \| + \| z_2 \| + \delta (\| z_1 \|_2 + \| z_2 \|_2).$$

This is so because $\| z_1 + z_2 \| \leq \| z_1 \| + \| z_2 \|$ and $\| z_1 + z_2 \|_2 < \| z_1 \|_2 + \| z_2 \|_2$ for such z_1, z_2 by (5).

Thus $\| z_1 + z_2 \|_o < \| z_1 \|_o + \| z_2 \|_o$, so that the sphere $\| z \|_o = 1$ is strictly convex in E_1.

For the norm $\| \ \|_o$, the inequality (2) is satisfied by setting $\delta = \frac{\epsilon}{K}$; whence

$$\| z \|_o = \| z \| + \frac{\epsilon}{K} \| z \|_2 \leq \| z \| + \frac{\epsilon}{K} K \| z \| = (1+\epsilon)\|z\|.$$

Consequently, by the arbitrariness of $\delta > 0$, the required construction has been realized.

This completes the proof of the theorem.

THEOREM 3. Let L_1 and L_2 be two linear subspaces of a Banach space and let $\theta(L_1, L_2) < \frac{1}{2}$. Then $\dim L_1 = \dim L_2$.

Proof: In view of Theorem 2, it is only necessary to examine the case when $\dim L_1$ and $\dim L_2$ are not finite.

Let $\theta(L_1, L_2) = \frac{1}{2} - b$, where $0 < b < \frac{1}{2}$.

Construct in unit ball K_1 of the linear subspace L_1 a maximal α-grating A for $\alpha = 1 - \frac{b}{2}$.

By hypothesis, for each element $x \in A$ ($\subset L_1$) there is an element $y_x \in L_2$ such that

$$\| y_x - x \| < \frac{1}{2} - \frac{b}{2}$$

This follows from the assumption $\theta(L_1, L_2) = \frac{1}{2} - b$, i.e.,

$$\max\left\{ \sup_{\substack{x \in L_1 \\ \|x\|=1}} d(x, L_2),\ \sup_{\substack{y \in L_2 \\ \|y\|=1}} d(y, L_1) \right\} = \frac{1}{2} - b < \frac{1}{2} - \frac{b}{2}.$$

If $x_1, x_2 \in A$, then $|| x_1 - x_2 || \geq 1 - \frac{b}{2}$. So

$$|| y_{x_1} - y_{x_2} || \geq || x_1 - x_2 || - || x_1 - y_{x_1} || - || x_2 - y_{x_2} ||$$
$$> \frac{b}{2} .$$

From this it follows that the elements y_x corresponding to elements x of the maximal $(1 - \frac{b}{2})$ - grating A of the ball K_1 are contained in some $\frac{b}{2}$ - grating of the ball of radius $\frac{3}{2}$ of the linear set L_2.

This is so because

$$|| y_x || \leq || y_x - x || + || x || < \frac{1}{2} - \frac{b}{2} + 1 < \frac{3}{2}$$

Consequently, in view of Theorem 1

$$\dim L_1 \leq \dim L_2 \text{ holds.}$$

Interchanging the roles of L_1 and L_2, we arrive at the assertion of the theorem.

DEFINITION 7. Let L be a linear subspace of a Banach space E and E^* the dual space of E. The subspace $L^{\perp} \subset E^*$ consisting of all functionals vanishing on every element of L is called the *orthogonal complement* of L in E^*.

Clearly, $L^{\perp}$ is a closed subspace in E^*. With the results developed till now, we proceed to study certain facts about openings of subspaces in the dual space E^*, these subspaces being orthogonal complements of corresponding linear subspaces in the space E. A question that suggests itself spontaneously is: If L_1, L_2 are two linear subspaces in a Banach space E and if $L_1^\perp$, $L_2^\perp$ are their orthogonal complements in E^*, resp., will $\theta(L_1^\perp, L_2^\perp) = \theta(L_1, L_2)$ hold? It turns out that the answer is in the affirmative and this will be established in Theorem 4.

LEMMA 2: Let L be a linear subspace in a normed linear space E and let d denote the distance from x to L i.e.,

$$d = \inf_{y \in L} \| x - y \|$$

(Note that the infimum need not be attained for any $y \in L$). If f is a linear functional vanishing on L, then

$$| f(x) | \leq d \| f \|$$

Furthermore there is a linear functional f_o in $L^\perp$ for which the equality holds.

REMARK: The lemma expresses the duality of two variational problems; the lemma asserts that

$$\inf_{y \in L} \| x - y \| = \max_{\substack{f \in L^\perp \\ \| f \| = 1}} | f(x) | ,$$

where the maximum is actually attained for some f: On the left we have a minimizing problem, on the right a maximizing one.

Proof of Lemma 2: For any y in L, we have

$$| f(x) | = | f(x-y) | \leq \| f \| \ \| x-y \| ;$$

this is so because f vanishes on L.

Since $\| x-y \|$ can be made as close to d as we wish, the inequality in the lemma follows. To construct a linear functional f for which the equality holds, we must first define it on the space spanned by x and L and then extend it to on E by use of the Hahn-Banach theorem.

Any vector in this space is of the form $ax + y$, $y \in L$ and we define the linear functional by

$$f(ax - y) = ad ;$$

certainly $f(x) = d$. The norm of f on this space is

$$\| f \| = \sup_{a,y} \frac{|a|d}{\|ax-y\|} = \sup_{o \neq a,y} \frac{d}{\|x - \frac{y}{|a|}\|} = \frac{d}{d} = 1$$

and so any extension $\tilde{f}$ to the entire space, with norm 1 has the desired properties.

LEMMA 3: L is a linear subspace of a normed linear space E. For any f in E*, of distance d from $L^{\perp}$ we have

$$d = \min_{g \in L^{\perp}} \| f-g \| = \sup_{x \in L} \frac{|f(x)|}{\| x \|} ,$$

where the minimum is actually attained by some g_o in $L^{\perp}$.

Proof: Let $f \in E^*$ and g by any functional from $L^{\perp}$; thus $g(x) = 0$ for $x \in L$. Then for x in L we have

$$|f(x)| = |(f-g)x| \leq \|f-g\| \, \|x\| \quad \text{for any } g \text{ in } L^{\perp} .$$

Thence for x in L,

$$|f(x)| \leq \inf_{g \in L^{\perp}} \| f-g \| \, \| x \| ,$$

i.e.,

$$| f(x) | \leq d \, || x || . \qquad (I)$$

From (I) we have

$$\sup_{\substack{||x||=1 \\ x \in L}} | f(x) | \leq d .$$

To complete the proof of the lemma we need only to construct a $g_o \in L^{\perp}$ so that

$$d \leq || f - g_o || = \sup_{x \in L} \frac{|f(x)|}{|| x ||}$$

But by the Hahn-Banach Theorem, there exists a linear functional $\tilde{f}$ in E^* agreeing with f on L and with $|| \tilde{f} || = \sup_{\substack{x \in L \\ ||x||=1}} | f(x) |$, so that we may take $g_o = f - \tilde{f}$

Lemmas 2 and 3 are equivalent to the following Lemmas 2' and 3', respectively.

LEMMA 2': Let L be a closed linear subspace of a normed linear space E. Then $(E/L)^*$ is isometrically isormorphic to $L^{\perp}$.

LEMMA 3': L^* is isometrically isormorphic to $E^*/L^{\perp}$, where L is any linear subspace of the normed linear space E.

THEOREM 4. Let L_1 and L_2 be two linear subspaces of a Banach space E and let $L_1^{\perp}$ and $L_2^{\perp}$ be their orthogonal complements in E^*. Then

$$\theta(L_1, L_2) = \theta(L_1^{\perp}, L_2^{\perp}) .$$

Proof: By Lemma 2, for any $y \in E$

$$d(y,L) = \sup_{f \in L^{\perp},\ ||f||=1} | f(y) | \quad . \qquad \text{(II)}$$

Using (II) we can write

$$\theta(L_1, L_2) = \sup \{| g(x) |, | f(y) |\} ,$$

where the supremum is to be taken over all x in L_1, y in L_2 f in $L_1^{\perp}$, g in $L_2^{\perp}$ with $|| x ||=|| y ||=|| f ||=|| g ||=1.$

By Lemma 3 ,

$$d(f, L^{\perp}) = \sup_{\substack{x \in L \\ ||x||=1}} | f(x) | \; .$$

This implies that

$$\theta(L_1^{\perp}, L_2^{\perp}) = \sup \{| f(y) |, | g(x) |\} ,$$

where the supremum is again taken over all $x \in L_1$, $y \in L_2$ $f \in L_1^{\perp}$, $g \in L_2^{\perp}$ such that $||x|| = ||y|| = ||f|| = ||g|| = 1.$

Hence $\theta(L_1, L_2) = \theta(L_1^{\perp}, L_2^{\perp})$.

The proof is complete.

THEOREM 5. <u>Let</u> L_1 <u>and</u> L_2 <u>be two linear subspaces of a Banach space</u> E <u>and let</u> $\theta(L_1, L_2) < \frac{1}{2}$. <u>Then</u>

$$\dim L_1^{\perp} = \dim L_2^{\perp} ,$$

<u>where</u> $L_1^{\perp}$, $L_2^{\perp}$ <u>denote the orthogonal complements</u> of L_1, L_2 <u>respectively in</u> E^*.

Proof: By Theorem 3, if $\theta(L_1, L_2) < \frac{1}{2}$, then dim L_1 = dim L_2. From Theorem 4, $\theta(L_1, L_2) = \theta(L_1^\perp, L_2^\perp)$. Now $\theta(L_1, L_2) < \frac{1}{2}$ implies: $\theta(L_1^\perp, L_2^\perp) < \frac{1}{2}$; and so by Theorem 4, it follows that dim $L_1^\perp$ = dim $L_2^\perp$. The proof is complete.

Let A be a linear (i.e. an additive and homogeneous) operator in a Banach space. We denote its domain of definition by $D(A)$ and the set of all its values, i.e., range by $R(A)$.

DEFINITION 8. A point λ_o of the complex plane is called a p o i n t o f r e g u l a r t y p e for the operator A, if there exists some positive number k_{λ_o} such that

$$||(A - \lambda_o I)\, x\, || \geq k_{\lambda_o}\, ||\, x\, ||$$

for all x in $D(A)$, where by I we understand the identity operator.

THEOREM 6. Points of regular type of an operator A form an open set.

Proof: If λ_o is a point of regular type for the operator A, then for λ satisfying $|\lambda-\lambda_o| < k_{\lambda_o}$, we have

$$\begin{aligned} ||(A - \lambda I)x|| &= ||\, \{A - \lambda_o I - (\lambda - \lambda_o)I\}\, x\, || \\ &\geq ||\, (A - \lambda_o I)\, x\, || - |\lambda - \lambda_o|\, ||\, x\, || \\ &\geq k_\lambda\, ||\, x\, || \end{aligned}$$

where $k_\lambda = k_{\lambda_o} - |\lambda - \lambda_o|$.

Indeed λ is also a point of regular type for the operator A.

We shall refer to this set of all points of regular type as the domain of regularity of the operator.

THEOREM 7.*) Let G be a non-empty open connected set of the complex plane consisting of points of regular type for the operator A. Then the dimension of the orthogonal complement N_λ in E^* to $R(A - \lambda I) \subset E$, is the same for all λ in G.

Proof: To prove the theorem we shall show that for each point λ_o in G we can find a neighbourhood W such that for all λ in W,

$$\dim N_\lambda = \dim N_{\lambda_o} .$$

Let W be a neighbourhood of point λ_o of radius $\frac{1}{4} k_{\lambda_o}$; then for all λ in W

$$\|(A - \lambda I) x\| \geq \|(A - \lambda_o I) x\| - |\lambda - \lambda_o| \, \|x\|$$

$$\geq \frac{3}{4} k_{\lambda_o} \|x\|, \qquad (x \in D(A)) \quad \text{(III)},$$

and

$$\|(A - \lambda I) x - (A - \lambda_o I) x\| = |\lambda - \lambda_o| \, \|x\| < \frac{k_{\lambda_o}}{4} \|x\|$$

$$< \frac{1}{3} \|(A - \lambda I) x\| \qquad \text{(IV)}$$

It follows from (III) that

$$\theta[R(A - \lambda I), R(A - \lambda_o I)] \leq \frac{1}{3} . \qquad (\lambda \in W) .$$

*) See [2] , p. 98.

In view of Theorem 5, dim N_{λ} = dim N_{λ_o} .

The proof is complete.

In the remainder of this chapter we shall discuss an application of the foregoing theorem to the study of linear symmetric operators in a complex Hilbert space.*)

DEFINITION 9. A linear operator A in a Hilbert space H with domain of definition $D(A)$ and range $R(A)$ is called H e r m i t i a n if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad (x, y \in D(A)) .$$

and is called s y m m e t r i c if it is Hermitian and $D(A)$ is dense in H.

DEFINITION 10. Let H_1 and H_2 be Hilbert spaces and A an arbitrary operator from H_1 into H_2 whose domain of definition $D(A)$ is dense in H_1. It can occur that for some y in H_2 there is a vector z in H_1 such that for all $x \in D(A)$ we have

$$\langle Ax, y\rangle = \langle x, z\rangle .$$

(This is the case by a theorem of F. Riesz if and only if $\langle Ax, y\rangle$ is a bounded linear form in $D(A)$).

We denote by D^* the set of all $y \in H_2$ for which such a z in H_1 exists.

We now define a linear operator A^* from H_2 to H_1 whose domain is $D(A^*) = D^*$ by the relation

$$A^*y = z .$$

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*) See [2], p. 99.

The mapping A^* is called the *a d j o i n t* of A. A^* is well defined on account of the assumption $\overline{D(A)} = H_1$.

DEFINITION 11. An operator A in a Hilbert space H is called *c l o s e d* if $x_n \in D(A)$, $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} Ax_n = y$ imply

$$x \in D(A) \quad \text{and} \quad Ax = y.$$

It immediately follows that A^*, the adjoint of A, is a closed linear transformation.

Indeed, let $(y_n)_{n=1}^{\infty}$ be a sequence of vectors from $D(A^*)$ such that

$$y_n \to y \quad \text{and} \quad z_n = A^*y_n \to z .$$

For each n,

$$\langle Ax, y_n\rangle = \langle x, A^*y_n\rangle = \langle x, z_n\rangle .$$

Hence

$$\lim_{n\to\infty} \langle Ax, y_n\rangle = \lim_{n\to\infty} \langle x, z_n\rangle$$

Using the continuity of the inner product,we have

$$\langle Ax, y\rangle = \langle x, z\rangle \quad (\text{for all } x \text{ in } D(A)) .$$

Consequently, $y \in D(A^*)$ and $A^*y = z$. Thus A^* is closed.

DEFINITION 12. An operator A in a Hilbert space H is said to be *s e l f - a d j o i n t* if it is symmetric and $A = A^*$.

It is clear that a self-adjoint operator A is necessarily closed.

We note that: The upper and lower complex half planes*) are connected subsets of the domain of regularity of an arbitrary symmetric operator.

Indeed, if A is a symmetric operator and $\lambda = x+iy$ where $y \neq 0$, then

$$||(A - \lambda I)f||^2 = ||(A - xI)f||^2 + y^2||f||^2 \geq y^2|| f ||^2 \text{ for}$$

each f in $D(A)$.

Hence the upper half complex plane $\{z | z= x + iy, y > 0\}$ and the lower half complex plane $\{z | z= x + iy, y < 0 \}$ are connected subsets of the domain of regularity of an arbitrary symmetric operator.

DEFINITION 13. Any point λ of regular type will be called a r e g u l a r p o i n t for the operator A if the set $R(A - \lambda I)$ coincides with the entire space H.

LEMMA 4: If A is a self-adjoint operator in a Hilbert space H, then

$$R(A \pm iI) = H .$$

Proof: A is self-adjoint. Then for x in $D(A)$ we have

$$||(A \pm iI)\, x||^2 = || Ax||^2 + || x ||^2 .$$

Thus $Ax \pm ix = 0$ if and only if $x = 0$. (i)

Also $R(A \pm iI)$ is dense in H. If y is orthogonal to $R(A \pm iI)$, then

*) The half planes are assumed to be open, i.e., the real line is excluded.

$$0 = \langle y, Ax + ix \rangle = \langle y, Ax \rangle - \langle iy, x \rangle ,$$

Thus $y \in D(A^*)$ and $Ay = iy$. But we have just seen that this can be only if $y = 0$.

To see that $R(A + iI) = H$, take an $h \in H$. Since $R(A + iI)$ is dense in H, there is a sequence $(h_n)_{n=1}^{\infty}$ such that

$$h_n = Ax_n + ix_n \to h \quad . \tag{ii}$$

Then by (i)

$$\| h_n - h_m \|^2 = \| A(x_n - x_m) + i\,(x_n - x_m) \|^2$$

$$= \| A(x_n - x_m) \|^2 + \| x_n - x_m \|^2 .$$

Thus $(x_n)_{n=1}^{\infty}$ and $(Ax_n)_{n=1}^{\infty}$ converge to certain vectors x and z. Since A is closed, $x \in D(A)$ and $z = Ax$. By (ii) $h = Ax + ix \in R(A + iI)$. Thus $R(A + iI) = H$.

In the same way we obtain $R(A - iI) = H$.

LEMMA 5: _If_ A _is a symmetric operator in a Hilbert space_ H _and_ $R(A) = H$, _then_ A _is self-adjoint_.

Proof: It is easy to see that A^* extends A. Therefore, if we show that $D(A^*) \subset D(A)$ the lemma will follow.

Let $y \in D(A^*)$ and let $A^*y = z \in H$. Since $R(A) = H$ there must be some w in $D(A)$ such that

$$z = Aw .$$

For any x in $D(A)$, we now have

$$\langle Ax, y \rangle = \langle x, A^*y \rangle = \langle x, Aw \rangle .$$

Since w is in $D(A)$ and $A < A^*$, we must have

$$Aw = A^*w \ ;$$

therefore $\langle Ax, y-w \rangle = 0$ (for all x in $D(A)$) .
Since A is an onto mapping it follows that $y-w = 0$ or $y = w$ in $D(A)$; consequently,

$$D(A) \supset D(A^*)$$

which establishes the theorem.

Theorem 7 lends meaning to the following definition.

DEFINITION 14. The dimension of the space N_λ for $\lambda \in D$ will be called the d e f i c i e n c y n u m b e r of the operator A in the connected region D.

As remarked earlier, the upper and lower halves of the complex plane are connected regions for the domain of regularity of a symmetric operator A.

Thus symmetric operators can have no more than two distinct deficiency numbers. If one of the points of the real axis in the complex plane is of regular type for a symmetric operator A, then the deficiency numbers of A are equal.

DEFINITION 15. The pair (m, n) where m is the deficiency number of the symmetric operator A in the upper half plane and n is the deficiency number in the lower half plane, is called the d e f i c i e n c y i n d e x of A.

REMARK: The deficiency index of a symmetric operator A is $(0, 0)$ if and only if A is self-adjoint.

The verification of this remark is immediate from lemmas 4 and 5.

CHAPTER III

SCHAUDERS FIXED POINT THEOREM

In this chapter we first prove Brouwer's fixed point theorem, and with the help of the latter we prove Schauder's fixed point theorem.

Theorem 7 of Chapter I implies:

LEMMA 1. Σ^n is not a retract of K^{n+1}.

DEFINITION 1. *Let* f *be a mapping of a topological space* X *into itself,*

$$f: X \to X .$$

Then a point $x \in X$ *is called a fixed point of the mapping* f *if* $x = f(x)$.

We say that the topological space X *has a fixed point property if for every mapping* $f: X \to X$ *of* X *into itself there exists a fixed point of* f.

THEOREM 1. (Brouwer's fixed point theorem). *Every mapping* $f: K^{n+1} \to K^{n+1}$ *has at least one fixed point.*

Proof: Suppose that x and $f(x)$ are always distinct points. Join $f(x)$ to x by segment and produce this segment beyond x until it meets the boundary Σ^n in the point $r(x)$. The correspondence $x \to r(x)$ is a retraction

$$r: K^{n+1} \to \Sigma^n ,$$

contradicting the lemma above. Hence the supposition is incorrect.

The proof is complete.

REMARK: Clearly, Brouwer's fixed-point theorem is valid in any space homeomorphic to K^{n+1}.

We now discuss some matters of use later in this chapter.

DEFINITION 2. Let E be a Banach space and $(x_n)_{n=1}^{\infty}$ be a sequence of elements of E. Then the series $\sum_{n=1}^{\infty} x_n$ is said to c o n v e r g e to the sum $x \in E$ if the partial sums $u_n = \sum_{r=1}^{n} x_r$ converge to x i.e. for each $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$||x - \sum_{r=1}^{n} x_r|| < \varepsilon \quad \text{for all } n \geq N(\varepsilon) .$$

The series $\sum_{n=1}^{\infty} x_n$ is said to be a b s o l u t e l y c o n v e r g e n t if the numerical series $\sum_{n=1}^{\infty} ||x_n||$ is convergent.

PROPOSITION 1: Let E be a Banach space and $(x_n)_{n=1}^{\infty}$ a sequence of elements in E. Let $\sum_{n=1}^{\infty} x_n$ be be absolutely convergent. Then $\sum_{n=1}^{\infty} x_n$ is convergent.

Proof: By hypothesis, given $\epsilon > 0$ there exists $N > 0$ such that whenever $q > p > N$ (q, p positive integers)

$$\sum_{r=p}^{q} ||x_r|| < \epsilon \qquad (1)$$

Now if $y_i = \sum_{r=1}^{i} x_n$, then

$$||y_q - y_p|| = ||\sum_{r=p+1}^{q} x_r|| \leq \sum_{r=p+1}^{q} ||x_r|| < \epsilon$$

by (1). So $(y_i)_{i=1}^{\infty}$ is a Cauchy sequence in E and from the completeness of E, it follows that $\sum_{n=1}^{\infty} x_n$ converges. The proof is complete.

PROPOSITION 2: <u>Let</u> $(x_n)_{n=1}^{\infty}$ <u>be a sequence of points of a bounded closed convex subset</u> X <u>of a Banach space</u> E. <u>Then for each sequence</u> $(\alpha_n)_{n=1}^{\infty}$ <u>of non-negative reals with</u> $\sum_{n=1}^{\infty} \alpha_n = 1$, <u>the series</u> $\sum_{n=1}^{\infty} \alpha_n x_n$ <u>is convergent and its sum belongs to</u> X.

Proof: By hypothesis, for each $\epsilon > 0$ $(0 < \epsilon < 1)$ there exists $N > 0$ such that

$$\sum_{i=n}^{\infty} \alpha_i < \epsilon \quad \text{for all} \quad n \geq N .$$

Putting

$$u_r = \sum_{i=1}^{r} \alpha_i \, || \, x_i \, ||$$

then for $m > n > N$ we have

$$u_m - u_n = \sum_{i=n+1}^{m} \alpha_i || x_i || < \epsilon K$$

where K is an upper bound for $||x||$, as x runs through X. Clearly $(u_r)_{r=1}^{\infty}$ is a Cauchy sequence of real numbers and so is convergent. Thus $\sum_{i=1}^{\infty} \alpha_i x_i$ is absolutely convergent, and by Proposition 1, also convergent. Thus the sum of the series $\sum_{i=1}^{\infty} \alpha_i x_i$ belongs to E.

To show that this sum belongs to X, put

$$y_n = \sum_{i=1}^{n} \beta_i^{(n)} x_i \text{ where } \beta_i^{(n)} = \frac{\alpha_i}{\sum_{k=1}^{n} \alpha_k} > \alpha_i .$$

Clearly $y_n \in X$, as X is convex. Now,

$$||\sum_{k=1}^{n} \alpha_k x_k - y_n|| = ||\sum_{k=1}^{n} \alpha_k x_k - \sum_{k=1}^{n} \beta_k^{(n)} x_k||$$

$$= ||\sum_{k=1}^{n} (\alpha_k - \beta_k^{(n)}) x_k|| \leq \sum_{k=1}^{n} |\beta_k^{(n)} - \alpha_k| \; ||x_k|| .$$

Also, $$\sum_{k=1}^{n} (\beta_k^{(n)} - \alpha_k) = \sum_{k=1}^{n} \left(\frac{\alpha_k}{\sum_{i=1}^{n} \alpha_i} - \alpha_k \right)$$

$$= \frac{1}{(\sum_{i=1}^{n} \alpha_i)} \left\{ \sum_{k=1}^{n} (\alpha_k - \alpha_k \sum_{i=1}^{n} \alpha_i) \right\}$$

$$= 1 - \sum_{i=1}^{n} \alpha_i$$

Thus the difference $(\sum_{k=1}^{n} \alpha_k x_k - y_n)$ converges to zero, and the fact that X is closed now ensures that the sum of the series $\sum_{i=1}^{\infty} \alpha_i x_i$ belongs to X.

This completes the proof.

DEFINITION 3. Let X be a metric space and E a Banach space. A mapping F of X into E, i.e.,

$$F: X \to E,$$

is called a c o m p a c t m a p p i n g if $\overline{F(X)}$ is compact.

DEFINITION 4. A sequence $(f_n)_{n=1}^{\infty}$ of mappings of a topological space X into a metric space (Y,d) is said to be u n i f o r m l y c o n v e r g e n t (or to c o n v e r g e u n i f o r m l y) to the mapping f of X into Y if for every $\varepsilon > 0$ there exists a number $n(\varepsilon)$ such that for all $x \in X$

$$d(f_k(x), f(x)) < \varepsilon \text{ whenever } k > n(\varepsilon).$$

THEOREM 2. X is a metric space and E a Banach space. The limit F of a uniformly convergent sequence $(F_n)_{n=1}^{\infty}$ of compact mappings $F_n: X \to E$ (n=1, 2, ...), is a compact mapping of X into E.

Proof: On account of Hausdorff's theorem, which asserts that a metric space is compact if and only if it is complete and totally bounded, it is sufficient to show that F(X) is totally bounded in E, that is, for arbitrary $\varepsilon > 0$

there exists a finite number of points $y_1, y_2, \dots, y_k$ in E such that

$$F(X) \subset \bigcup_{i=1}^{k} N(y_i, \varepsilon) .$$

By hypothesis, there exists an integer k such that

$$||F(x) - F_k(x)|| < \frac{\varepsilon}{2} \text{ for all } x \text{ in } X .$$

F_k being compact, there exists for $F_k(X)$ an $\frac{\varepsilon}{2}$ - net $N = \{y_1, y_2, \dots, y_k\}$, i.e., for arbitrary $x \in X$ we have

$$||F_k(x) - y_i|| < \frac{\varepsilon}{2} \text{ for some } i; \text{ hence}$$

$$||F(x) - y_i|| \leq ||F(x) - F_k(x)|| + ||F_k(x) - y_i|| < \varepsilon .$$

This implies that

$$F(X) \subset \bigcup_{i=1}^{k} N(y_i, \varepsilon) .$$

The proof is complete.

REMARK: It is clear that for a bounded metric space the notion of compact mapping coincides with that of completely continuous mapping: A mapping $F: X \to E$, is said to be completely continuous if F is compact (see Def. 3) on every bounded set of X.

DEFINITION 5. X _is a metric space and_ E _a Banach space. A compact mapping_ $F: X \to E$ _is said to be_ _f i n i t e d i m e n s i o n a l_ _if_ F _sends all of_ X _into some finite dimensional subspace_ E^n _of_ E.

From Theorem 2 it follows that the limit of a uniformly convergent sequence of finite dimensional mappings is compact. It turns out conversely, that every compact mapping can be uniformly approximated by finite-dimensional ones.

THEOREM 3.*) Let X be a metric space and E a Banach space. A mapping $F: X \to E$ is a compact mapping on X if and only if for every $\epsilon > 0$ there exists a finite dimensional mapping $F_\epsilon : X \to E_n$ such that

$$||F(x) - F_\epsilon(x)|| < \epsilon \quad \text{for all } x \in X .$$

Proof: The sufficiency follows from Theorem 2.

The necessity is proved below:

Suppose that $F: X \to E$ is a compact mapping, i.e. $\overline{F(X)}$ is compact in E. By Hausdorff's theorem, for a given $\epsilon > 0$ we can find a finite subset $y_1, y_2, \ldots, y_k$ of E such that every point of the compact set $\overline{F(X)}$ has a distance less than ϵ from at least one of the y_i, i.e. for every $x \in X$, $||y_i - F(X)|| < \epsilon$ for some i.

Let E^n be the finite dimensional subspace of E spanned by the points $y_1, y_2, \ldots, y_k$.

For each $i = 1, 2, \ldots, k$, put

$$u_i(x) = \max \{0, \epsilon - ||F(x) - y_i||\} \qquad (x \in X).$$

Obviously u_i $(i=1, 2, \ldots, k)$ are continuous real-valued functions on X. Since for each $x \in X$ there is an integer

*) Theorem 3 and its corollary are direct generalizations of the well-known results in Hilbert space situation. In this connection the interested reader may consult chapter viii of Berberian, S. K. Introduction to Hilbert space, Oxford Univ. Press, 1961.

i such that $u_i(x) > 0$, hence

$$\sum_{i=1}^{k} u_i(x) \neq 0 \quad \text{for} \quad x \in X,$$

and we can define

$$\lambda_i(x) = \frac{u_i(x)}{\sum_{j=1}^{k} u_j(x)} \quad , \; x \in X \; .$$

It follows from the definition of $u_i(x)$, that for $i=1, 2, \ldots, k$, $\lambda_i(x)$ is continuous on X and

$$\lambda_i(x) = 0 \quad \text{if and only if} \quad ||F(x) - y_i|| \geq \epsilon \qquad (2) .$$

Moreover, for each x in X,

$$\sum_{i=1}^{k} \lambda_i(x) = 1. \qquad (3)$$

Put

$$F_\varepsilon(x) = \sum_{j=1}^{k} \lambda_i(x)\, y_i \quad .$$

Since $F_\varepsilon(x)$ is a linear combination of the vectors y_i in E^n it is clear that F_ϵ is a finite dimension mapping

$$F_\varepsilon : \; X \to E_n \quad ;$$

on account of (2) and (3) there holds for each $x \varepsilon X$

$$||F(x) - F_\varepsilon(x)|| = || \sum_{i=1}^{k} \lambda_i(x) \, (F(x) - y_i)||$$

$$\leq \sum_{i=1}^{k} \lambda_i(x)||F(x) - y_i||$$

$$< \quad \varepsilon \; .$$

The proof is complete.

COROLLARY. *A mapping* $F: X \to E$ *is compact if and only if* F *can be represented on* X *as a uniformly convergent series*

$$F(x) = \sum_{n=0}^{\infty} F_n(x)$$

where the mappings $F_n: X \to E^n$ $(n=0, 1, \dots)$ *are finite dimensional and such that*

$$||F_n(x)|| \leq \frac{1}{2^n} \text{ for all } x \in X \text{ and } n=1, 2, \dots$$

Proof: Sufficiency part follows from Theorem 2.

Necessity: For $\epsilon = \frac{1}{2^{n+2}}$ there exists, by Theorem 3, a finite dimensional mapping $G_n: X \to E$ such that

$$||G_n(x) - F(x)|| \leq \frac{1}{2^{n+2}} .$$

Putting

$$F_o(x) = G_o(x) \text{ and } F_n(x) = G_n(x) - G_{n-1}(x)$$

for $n=1, 2, \dots$, we obtain a sequence of finite dimensional mappings. Since

$$\sum_{i=1}^{n} F_i(x) = G_n(x) ,$$

then $\sum_{i=1}^{\infty} F_i(x)$ is convergent to $F(x)$.

Moreover,

$$||F_n(x)|| = ||G_n(x) - G_{n-1}(x)|| \leq ||G_n(x) - F(x)||+||F(x)-G_{n-1}(x)||$$

$$\leq \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} < \frac{1}{2^n} ,$$

is continuous on $E - X$, and

$$\sum_{n=1}^{\infty} \alpha_n(x) = \frac{1}{\lambda(x)} \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n(x) = 1 .$$

Now we define a mapping α of $E - X$ into X by setting

$$\alpha(x) = \sum_{n=1}^{\infty} \alpha_n(x) y_n .$$

X is convex, $y_n \in X$, $\alpha_n(x) \geq 0$ and $\sum_{n=1}^{\infty} \alpha_n(x) = 1$.

Hence by proposition 2 of this chapter, it follows that $\alpha(x)$ is in X.

The continuity of α follows at once from the fact that the series

$$\alpha(x) = \frac{1}{\lambda(x)} \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n(x) y_n$$

is absolutely and uniformly convergent. (Use proposition 1 of this chapter and the theorem: The limit of an uniformly convergent sequence of continuous mappings of a metric space X into a metric space Y is a continuous mapping of X into Y). Finally we define

$$r(x) = \begin{cases} \alpha(x) & \text{for } x \text{ in } E - X \\ x & \text{for } x \text{ in } X \end{cases} .$$

r is continuous on X and $E - X$; to prove continuity on E, it suffices to prove that for every sequence $(x_n)_{n=1}^{\infty}$, $x_n \in E - X$ and $\lim_{n\to\infty} x_n = x_0$ in X, the sequence $\alpha(x_n)$ is convergent to x_0 .

Let $\varepsilon > 0$ be given and x be an arbitrary point of the sequence (x_n) such that $||x - x_0|| < \varepsilon$. Denote by (y_{n_k}) a subsequence of (y_n) such that $||y_{n_k} - x|| < 3\varepsilon$. By the definition of $\lambda_n(x)$, for the remaining n, $\lambda_n(x) = 0$; hence

$$\alpha(x) = \sum_{k=1}^{\infty} \alpha_{n_k}(x)\, y_{n_k} .$$

By proposition 2 of this chapter, $\alpha(x) \in \text{Conv}\,\{y_{n_k}\}$, hence $||\alpha(x)-x|| < \varepsilon$ and

$$||\alpha(x)-x_0|| \leq ||\alpha(x)-x|| + ||x - x_0|| < 2\varepsilon + \varepsilon = 3\varepsilon,$$

which completes the proof.

With the help of Brouwer's fixed point theorem we shall now establish a special case of the Schauder's fixed point theorem.

THEOREM 5. <u>Let</u> $F\colon\ E \to E$ <u>be a compact mapping of a Banach space</u> E <u>into itself. Then</u> F <u>has at least one fixed point.</u>

Proof: By Theorem 3 there exists for $k=1, 2, \ldots$ a finite dimensional mapping $F_k\colon E \to E_{n(k)}$ such that

$$||F(x) - F_k(x)|| \leq \frac{1}{k} \quad \text{for} \quad x \in E . \qquad (4)$$

The restricted mapping $F_k^* = F_k / E_{n(k)}$ is a compact mapping of a finite dimensional space $E_{n(k)}$ into itself, and hence by virtue of Brouwer's fixed point theorem, has a fixed point x_k in $E_{n(k)}$.

Thus we have a sequence of points (x_k) such that

$$x_k = F_k(x_k) \tag{5}$$

From (4) and (5) we obtain

$$||F(x_k) - x_k|| \leq \frac{1}{k}, \; k=1, 2, \ldots \tag{6}$$

Since F is a compact mapping we may suppose, without loss of generality, that the sequence $F(x_k)$ is convergent, i.e.,

$$\lim_{k\to\infty} F(x_k) = x . \tag{7}$$

From this, taking into account (6) we infer that $\lim\limits_{k\to\infty} x_k = x$, and hence by the continuity of F we have

$$\lim_{k\to\infty} F(x_k) = F(x) .$$

From this and (7), we obtain $F(x) = x$, which completes the proof.

DEFINITION 6. <u>Let</u> X <u>be a subset of</u> E, <u>a Banach space. We shall say that the set</u> X <u>has the f i x e d p o i n t p r o p e r t y i n t h e n a r r o w s e n s e , if for every compact mapping</u> $F: X \to X$ <u>there exists a fixed point of</u> F.

THEOREM 6.*) <u>The fixed-point property in the narrow sense is invariant under retraction mappings i.e. if</u> X <u>has the fixed point property in the narrow sense, then so does every retract of</u> X.

REMARK: The simplest example of a set having the fixed point property in the narrow sense, is the whole Banach space. This was proved in Theorem 5.

Proof of Theorem 6: Assume that X has fixed point property in the narrow sense i.e. for every compact mapping $F: X \to X$ there exists a fixed point of F, and let $r: X \supset A$ be an arbitrary retraction of X onto A. It suffices to prove that A has the fixed-point property in the narrow sense. For this, let

$$f: A \to A$$

be an arbitrarily given compact mapping. Consider the composed mapping

$$g = h \circ f \circ r : X \to X ,$$

where $h: A \subset X$ stands for the inclusion map. Clearly, g is a compact mapping. Since X has the fixed point property, g must have a fixed point i.e. a point x in X such that $g(x) = x$.

Since $g(x) = f(r(x))$, this implies that the fixed point x of g must be in A. Hence

$$r(x) = x \text{ and } x = g(x) = f(r(x)) = f(x) .$$

This proves that x is a fixed point of f, and hence A has the fixed point property.

*) See [5] page 20.

THEOREM 7.*) (SCHAUDER'S FIXED POINT THEOREM). A closed convex set X in a Banach space E has the fixed point property in the narrow sense.

Proof: Let $F: X \to X$ be an arbitrary mapping of X into itself. We denote by X^* the convex closure of the range of the mapping F i.e. the set Conv(F(X)). Consider the restriction F^* of F to X^*, i.e., $F^* = F/X^*$. The set X^* is bounded, closed, separable and convex; hence by Theorem 4, X^* is a retract of X; then by Theorem 6, we conclude that X^* has the fixed point property in the narrow sense, i.e., the mapping $F^*: X^* \to X$ and therefore the mapping F has a fixed point. This completes the proof.

The question arises whether a non-compact closed and convex set X in an infinite dimensional Banach space has the fixed point property. The answer is in the negative.

PROPOSITION 3:**) There exists a mapping $\varphi: K \to K$ of the closed unit ball K of the Hilbert space ℓ_2 into itself which has no fixed point.

Proof: Let $x = (x_1, x_2, \ldots)$ be an arbitrary point of the closed unit ball K in the Hilbert space ℓ_2, i.e.,

$|| x || = \{ \sum_{i=1}^{\infty} x_i^2 \}^{1/2} \leq 1$, and let ψ be a mapping

*) See [3], [10]; in [3] consult p. 124.
**) See [8], p. 270.

of K into itself defined by the formula:

$$\psi(x) = (0, x_1, x_2, \ldots) \qquad (8) .$$

Obviously ψ does not have fixed points on the sphere $S = \{x: ||x|| = 1\}$ (because the origin $(0, 0, \ldots)$ is the only fixed point of ψ).

Let $\overline{x} = (1, 0, \ldots)$ and define for $x \in K$:

$$\varphi(x) = \frac{1}{2}(1 - ||x||)\,\overline{x} + \psi(x) \qquad (9) .$$

Obviously φ is a mapping of K into itself. We shall prove that φ does not have a fixed point. Assume the contrary that $x^* = \varphi(x^*)$ for some x^* in K. Then we must have

$$x^* - \psi(x^*) = \frac{1}{2}(1 - ||x^*||)\,\overline{x} \qquad (10) .$$

Let $x^* = (x_1^*, x_2^*, \ldots)$; from (10) it follows that $x^* \neq 0$, and $||x^*|| < 1$. From (9) and (10) we have

$$x_1^* = \frac{1}{2}(1 - \sum_{i=1}^{\infty} x_1^*), \; x_1^* = x_2^* = x_3^* = \ldots .$$

Since this is incompatible with the fact that $\sum_{i=1}^{\infty} x_i^* < \infty$ the mapping φ cannot have a fixed point. This completes the proof.

This also shows that it is impossible to prove a fixed point theorem in a Banach space which is valid for arbitrary continuous mappings.

In fact it has been proved that the unit ball in a Banach space E has the fixed point property if and only if E is finite dimensional.(See [6], p. 362).

Finally, we look at an application of Theorem 7.

THEOREM 8.*) Let T be a compact mapping of a Banach space E into itself. Let $\lambda_1 \in [0,1]$. Then either there is an $x \in E$ such that,

$$x = \lambda_1 T(x) \ ,$$

or the set $\{x | x \in E \text{ such that } x = \lambda T(x),\ 0 < \lambda < 1\}$ is not bounded.

Proof: Let $U = \{x|\ ||x|| \leq 1\}$

$$nU = \{y | y = nx \text{ where } x \in U\},$$

where n is a positive integer.

Suppose there is a $\lambda_0 \in (0, 1)$ such that the equation

$$x = \lambda_0 T(x)$$

does not have a solution. We prove: given positive integer n there is an x_n such that

$$x_n = \mu_n \lambda_0 T(x_n) \ ,$$

where $\mu_n \in (0,1)$ and $||x_n|| = n$. Define the mapping R_n: $E \to E$ as follows:

$$\left.\begin{aligned} R_n(x) &= \lambda_0 T(x) \quad \text{for } x \text{ such that } \lambda_0 T(x) \in nU \\ &= \frac{n\lambda_0 T(x)}{||\lambda_0 T(x)||} \quad \text{for } x \text{ such that } \lambda_0 T(x) \in E - nU. \end{aligned}\right\}$$

Then the restriction R_n/nU is continuous. If $x \in nU$, then if $\lambda_0 T(x) \in nU$,

$$R_n(x) = \lambda_0 T(x) \in nU.$$

If $\lambda_0 T(x) \in E - nU$, then

*) See [9], p. 415.

$$||R_n(x)|| = \frac{n}{||\lambda_0 T(x)||} \cdot ||\lambda_0 T(x)|| = n .$$

Thus if $x \in nU$, then $R_n(x) \in nU$. Since T is compact, then R_n is compact. Since nU is a bounded closed convex set, then by Schauder's fixed-point theorem there is an x_n in nU such that

$$R_n(x_n) = x_n .$$

Now suppose that $\lambda_0 T(x_n) \in nU$. Then $\lambda_0 T(x_n) = R_n(x_n) = x_n$, which contradicts the property of λ_0. Therefore $\lambda_0 T(x_n) \in E - nU$, and

$$R_n(x_n) = \frac{n}{||\lambda_0 T(x_n)||} \lambda_0 T(x_n) = x_n$$

and $||x_n|| = n$. Also $||\lambda_0 T(x_n)|| > n$. Then $\frac{n}{||\lambda_0 T(x_n)||} = \mu_n$ where $0 < \mu_n < 1$.

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