# A CHARACTERIZATION OF ALMOST CONTINUOUS FUNCTIONS

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by

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## INTRODUCTION

The theory of functions of a real variable consists in part of definitions of the properties of functions which map the real line into itself. Included among these properties are the usual well known properties such as continuity, integrability, differentiability etc. as well as properties not usually possessed by the functions met with in the other branches of Science.

The theory is also concerned with the various characterizations of each of these properties and with the relationships that exist among these properties. Each characterization of a property gives a criterion for determining whether a function possesses this property.

The theory is further concerned with the classification of functions according to the properties they possess. The criteria obtained from the characterization of the properties are used in the classification of functions.

The class of continuous functions is perhaps the most important class in Pure and Applied Mathematics. In this thesis we are interested in the study of the generalisations of continuity. One is generally interested in knowing what happens if a function satisfies some weaker condition, than the conditions of continuity.

In the literature there are many 'weakened' forms of continuity e.g. upper and lower semi-continuity, approximate continuity and many others. Here in this thesis we discuss a weakened form of continuity namely almost continuity of a real function $f(x)$, and show that the classes of almost continuous, approximately continuous, and continuous functions form a descending chain.

The thesis is divided into four chapters. The first chapter deals with the definitions of the concepts used in the development of this work. In the second chapter we give some definitions of continuity and show how the definition of semi-continuity can be obtained by weakening the condition in the definition of continuity. The third chapter deals with another important generalisation, namely approximate continuity, which was first introduced by Denjoy in 1915. In the fourth chapter we discuss a new generalisation of continuity namely almost continuity, and prove a characterization theorem for the same. Finally, we show the relation between approximate continuous and almost continuous functions.

## Chapter I

## DEFINITIONS

In this chapter we define all the notions, used in the development of this thesis.

Definition 1.1: - Let R be the set of all real numbers, and let $a, b \in R$ with $a < b$, then any set $I = \{x \in R | a < x < b\}$ is known as an open interval.

Definition 1.2: - A set G R is said to be open, if for each $p \in G$ there is an open interval I such that $p \in I \subset G$.

Definition 1.3: - Let $x \in R$. A subset N of R is known as a neighbourhood of x, if there is an open set G such that $x \in G \subset N$.

Definition 1.4: - Let O be an open set in R and $x \in O$. The set $\{O - \{x\}\}$ is known as a partial or deleted neighbourhood of x.

Definition 1.5: - Let S be any set in R. The complement of S relative to R, denoted by C(S), consists of all real numbers $x \in R$ such that $x \notin S$.

Definition 1.6: - The set $C = \{x \in R | a \leq x \leq b\}$ where $a, b \in R$ with $a < b$, is said to be a closed interval.

Definition 1.7: - A set $P \subset R$ is said to be a closed set if C(P) is an open set.

Definition 1.8: - Let Y be a subset of R. A subset U' of Y is said to be open relative to Y, if and only if $U' = U \cap Y$, for some open subset U of R.

Definition 1.9: - Let X be a subset of R. A subset C' of X is said to be closed relative to X if and only if $C' = C \cap X$, where C is some closed subset of R.

Definition 1.10: - Let G be any subset of R, then a point $x_o$ is called a <u>limit point</u> of G, if every deleted neighbourhood of $x_o$ contains at least one point of the set G.

Definition 1.11: - Let P be any set, the set of all limit points of P is said to be a <u>derived set</u> and is denoted by $P'$.

Definition 1.12: - Let $S \subset R$ be any set, then $S \cup S'$ is known as the <u>closure</u> of the set S, and is denoted by $\bar{S}$.

Definition 1.13: - If $x \in E$ and x is not a limit point of E, then x is known as an <u>isolated</u> point of E.

Let f(x) be a real function defined by $f : R \to R$ where for each $x \in R$ $f(x) = y$, $y \in R$.

Definition 1.14: - Let $f : R \to R$. The subset $\Gamma_f \subset R \times R$, which consists of all ordered pairs of the form $(x, f(x))$, is called the <u>graph</u> of $f : R \to R$.

<u>Remark</u>: - Let A and B be sets contained in R. Given a set $\Gamma$ of $A \times B$ there is a function $f : A \to B$ such that $\Gamma$ is the graph of $f : A \to B$, if for each $x \in A$, there is one and only one element of the form $(x,y) \in \Gamma$. The function f is defined by $y = f(x)$.

Definition 1.15: - A set, which is either finite or has a one to one correspondence between its elements and the set of all positive integers, is called a <u>countable</u> set.

Definition 1.16: - A function f such that the domain of f is the set of positive integers is called a <u>sequence</u>. We may indicate such a sequence by writing $x_1\ x_2\ x_3 \ldots x_n \ldots$, where $x_n = f(n)$ and n is a positive integer.

Definition 1.17: - A sequence $\{a_n\}$ of real numbers is said to <u>converge</u> to a real number L as n tends to infinity if and only if for each $\varepsilon > 0$ there exists an integer $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$. L is called the limit of the sequence.

Definition 1.18: - If $A \subset R$ and there exists a $k \in R$ such that for all $a \in A$, $a \leq k$ then k is called an <u>upper bound</u> of A.

Definition 1.19: - If no member of A is greater than k, and at least one member of A is greater than $k - \varepsilon$ where $\varepsilon > 0$ and arbitrary, then k is known as the <u>least upper bound</u> of A, and we write Sup (A) = k. (Supremum)

Definition 1.20: - If $S \subset R$ and there exists an $m \in R$ such that for every $s \in S$, $s \geq m$, then m is called a <u>lower bound</u> of S.

Definition 1.21: - If no member of S is less than m and at least one member of the set S is less than $m + \varepsilon$, where $\varepsilon > 0$ and arbitrary, then m is known as the greatest lower bound of the set S, and we write Inf (S) = m. (Infimum)

Definition 1.22: - The <u>limit superior</u> of a sequence $\{x_n\}$ of real numbers is the right most limit point of the sequence. It is designated $\lim_{n\to\infty} \text{Sup}\ x_n$ or simply $\lim \text{Sup}\ x_n$.

The <u>limit inferior</u> of a sequence $\{x_n\}$ of real numbers is the left most limit point of the sequence, and is designated $\lim_{n\to\infty} \text{Inf}\ x_n$ or simply $\lim \text{Inf}\ x_n$.

Definition 1.23: - Let a real-valued function $y = f(x)$ be defined on R. Then the <u>limit inferior</u> of $f(x)$ at $x = a$, denoted by $\liminf_{x \to a} f(x)$, is the greatest lower bound of the limits inferior of $\{f(x_n)\}$ over all sequences $\{x_n\}$ which have the limit a.

Definition 1.24: - The <u>limit superior</u> of $f(x)$ at $x = a$, denoted by $\limsup_{x \to a} f(x)$, is the least upper bound of the $\limsup \{f(x_n)\}$ for all sequences $\{x_n\}$ converging to a.

Definition 1.25: - Let X be any set. A family $\mathcal{R}$ of subsets of X is said to be a <u>ring</u> of sets if the following conditions are satisfied.

(a) For each $A, B \in \mathcal{R}$, $A \cup B \in \mathcal{R}$

(b) For each $A, B \in \mathcal{R}$, $A - B \in \mathcal{R}$

Definition 1.26: - Let $\mathcal{R}$ be a ring of sets and $R^+ = \{x \in R \mid 0 \leq x \leq +\infty\}$ where R is the set of real numbers. Then a mapping of $\mathcal{R}$ into $R^+$ is called a <u>set-function</u>.

Definition 1.27: - A set function $\mu$ is said to be <u>additive</u> if for each $A, B \in \mathcal{R}$ with $A \cap B = \phi$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition 1.28: - A set-function $\mu$ is said to be <u>countably additive</u> if, for each sequence $\{A_n\}_{(n \geq 1)}$ of subsets of $\mathcal{R}$ with $A_n \cap A_m = \phi$ for $n \neq m$, $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Definition 1.29: - A set function $\mu$ defined on $\mathcal{R}$ is said to be a <u>measure</u> if the following conditions are satisfied.

(a) $0 \leq \mu(A) \leq +\infty$

(b) $\mu(\phi) = 0$

(c) $\mu$ is countably additive.

Definition 1.30: - Let R denote the real line and A a subset of R.

Let $\lambda(A) = \inf \sum_{n=1}^{\infty} (b_n - a_n)$ for all sequences of pair-wise disjoint open intervals $(a_n, b_n)$ such that $A \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$. Then the set function $\lambda$ is an <u>outer measure</u> on the ring of all subsets of R.

Definition 1.31: - Let R denote the real line and $\mathcal{R}$ the ring of all subsets of R. A subset A of R is said to be $\lambda$-measurable if for each $T \in \mathcal{R}$,

$\lambda(T) = \lambda(T \cap A) + \lambda(T \cap C(A))$.

Definition 1.32: - The outer measure $\lambda$ is a measure on the ring of all $\lambda$-measurable subsets of R and is known as the <u>Lebesque measure</u>.

## Chapter II

In this chapter, $f(x)$ denotes a real function defined by $f : R \to R$ where R denotes the set of all real numbers, such that for each $x \varepsilon R$, $f(x) = y$, $y \varepsilon R$. We first give the definition of continuity at a point in several forms.

Definition 2.1: - If $\xi$ is a real number, then $f(x)$ is said to be <u>continuous</u> at $\xi$ if for a given $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that $|f(x) - f(\xi)| < \varepsilon$ whenever $|x - \xi| < \delta$.

Definition 2.2: - $f(x)$ is said to be <u>continuous</u> at $\xi$ if for every convergent sequence $\{x_n\}$ whose limit is $\xi$, $\{f(x_n)\}$ converges and $\lim_{n\to\infty} f(x_n) = f(\xi)$.

Definition 2.3: - $f(x)$ is said to be <u>continuous</u> at $\xi$ if for each $\varepsilon > 0$ there is a neighbourhood N of $\xi$, such that for every $x \varepsilon N$, $|f(x) - f(\xi)| < \varepsilon$.

Definition 2.4: - $f(x)$ is said to be <u>continuous</u> at $\xi$ if for every open set G containing $f(\xi)$ there is an open set H containing $\xi$ such that for every $x \varepsilon H$, $f(x) \varepsilon G$.

These four definitions are equivalent. It is easy to see that definitions 2.1, 2.3, and 2.4 are equivalent. We only prove the equivalence of definitions 2.1 and 2.2.

Proposition 2.1: - Definitions 2.1 and 2.2 are equivalent.

Proof: - Suppose that the function $f(x)$ is continuous at $\xi \varepsilon R$ according to definitions 2.1. Let $\{x_n\}$ be a convergent sequence whose limit is $\xi$. Let $\varepsilon > 0$ be given and suppose $\delta > 0$ is such that if $|x - \xi| < \delta$ then $|f(x) - f(\xi)| < \varepsilon$. There is an N such that if $n > N$ then $|x_n - \xi| < \delta$, and so $|f(x_n) - f(\xi)| < \varepsilon$. Hence $\lim f(x_n) = f(\xi)$, and so f(x) is continuous at $\xi$ according to definition 2.2.

Suppose that f(x) is not continuous at $\xi$ according to definition 2.1. Then there is a $k > 0$ such that for every $\delta > 0$ there is an x with $|x - \xi| < \delta$ and $|f(x) - f(\xi)| \geq k$. Now there is an $x_1$ such that $|x_1 - \xi| < 1$ and $|f(x_1) - f(\xi)| \geq k$. Then there is an $x_2$ such that $|x_2 - \xi| < \min [|x_1 - \xi|, \frac{1}{2}]$ and $|f(x_2) - f(\xi)| \geq k$. Continuing in this way, we obtain a sequence $\{x_n\}$ which converges to $\xi$, such that $|f(x_n) - f(\xi)| \geq k$ for every n. Hence, f(x) is not continuous at $\xi$ according to definition 2.2. It follows that if f(x) is continuous at $\xi$ according to definition 2.2, then it is continuous at $\xi$ according to definition 2.1.

Definition 2.5: - f(x) is said to be <u>continuous</u> on R if it is continuous at every point of R.

Now we define the continuity of a function f(x) relative to a set $E \subset R$ as follows.

Definition 2.6: - A function f(x) defined on the real line R is said to be continuous at $\alpha \in R$ relative to a set $E \subset R$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in E$ and

$| x - \alpha | < \delta$ then $| f(x) - f(\alpha) | < \varepsilon$.

Definition 2.7: - $f(x)$ is said to be continuous on R relative to E if it is continuous at every point of R relative to the set E.

We now prove the following theorem, in order to illustrate the relative concepts involved.

<u>Theorem 2.1</u>: - A function $f(x)$, defined on a set $M \subset R$, is continuous on M relative to M if and only if, for every real number p, the sets $E_1 = \{x \mid f(x) > p\}$ and $E_2 = \{x \mid f(x) < p\}$ are open relative to M.

Proof: - Suppose $f(x)$ is continuous on M relative to M. Let $\alpha \in M$ be such that $f(\alpha) > p$. Then there is an $\varepsilon > 0$ such that $f(\alpha) - \varepsilon > p$. Since $f(x)$ is continuous at $\alpha$ relative to M, there is a $\delta > 0$ such that if $x \in M$ and $| x - \alpha | < \delta$ then $| f(x) - f(\alpha) | < \varepsilon$. Hence there is a neighbourhood N of $\alpha$ such that for every $x \in N \cap M$, $f(x) > f(\alpha) - \varepsilon > p$. Thus the set $E_1$ is open relative to M. Similarly it can be shown that $E_2$ is open relative to M.

Conversely, suppose that $E_1$ and $E_2$ are open relative to M, for every p. Let $\alpha \in M$, and let $\varepsilon > 0$. Then since $E_1 = \{x \mid f(x) > f(\alpha) - \varepsilon\}$ is open relative to M, there is a neighbourhood $N_1$ of $\alpha$ such that for each $x \in N_1 \cap M$, $f(x) > f(\alpha) - \varepsilon$. Moreover, since $E_2 = \{x \mid f(x) < f(\alpha) + \varepsilon\}$ is open relative to M, there is a neighbourhood $N_2$ of $\alpha$ such that for each $x \in N_2 \cap M$,

$f(x) < f(\alpha) + \varepsilon$. Let $N = N_1 \cap N_2$. Clearly N is a neighbourhood of $\alpha$, and for each $x \varepsilon N \cap M$, $f(\alpha) - \varepsilon < f(x) < f(\alpha) + \varepsilon$ i.e. $| f(x) - f(\alpha) | < \varepsilon$. Hence $f(x)$ is continuous on M relative to M.

Corollary: - $f(x)$ is continuous on M relative to M if and only if, for every k the sets $E_1 = \{x \mid f(x) \geq k\}$ and $E_2 = \{x \mid f(x) \leq k\}$ are closed relative to M.

An important generalization of continuity will now be obtained by dropping part of the demand imposed upon a function by the definition of continuity. The condition of continuity of a function $f(x)$, at a point $\alpha$, namely that, if $\varepsilon > 0$ be any given number, an open interval $(\alpha - \delta, \alpha + \delta)$ exists such that, for any point $x \varepsilon (\alpha - \delta, \alpha + \delta)$, $|f(x) - f(\alpha)| < \varepsilon$, this inequality can be divided into two separate conditions, namely $f(x) < f(\alpha) + \varepsilon$ and $f(x) > f(\alpha) - \varepsilon$. It is quite possible that, at a point x, one of these conditions may be satisfied and not the other. This consideration gives rise to the definition of a <u>notion</u> called semi-continuity.

Definition 2.8: - A function $f(x)$ is said to be <u>upper semi-continuous</u> at $\alpha \varepsilon R$, if corresponding to every arbitrarily chosen positive number $\varepsilon$, there exists a $\delta > 0$ such that for all $x \varepsilon (\alpha - \delta, \alpha + \delta)$, $f(x) < f(\alpha) + \varepsilon$.

Similarly if an open neighbourhood of the point $\alpha$ can be determined for each $\varepsilon > 0$ such that $f(x) > f(\alpha) - \varepsilon$, then $f(x)$ is said to be lower semi-continuous at $\alpha$.

**Example: -** Define a function f(x) as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ c > 0 & \text{if } x = 0 \end{cases}$$

It is easily seen that f(x) is not continuous, but f(x) is upper semi-continuous.

Similarly

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -c & \text{if } x = 0 \quad c > 0 \end{cases}$$

Then f(x) is lower semi-continuous at x = 0, but not continuous.

## Chapter III

In this chapter we shall discuss approximate continuity of a real function $f(x)$. The notion was first introduced by 'Denjoy' in Bulletin de la société mathématique de France Vol. XLIII (1915) p. 165. The concept of approximate continuity depends upon the notion of metric density. We shall introduce the concept of "interval function", and "relative measure" in order to define metric density.

Definition 3.1: - Let $I$ be any open or closed interval. An interval function $\mu(I)$ is a correspondence which associates a real number with every interval $I$.

Definition 3.2: - An interval function $\mu(I)$ is said to be non decreasing if $I \subset F$ then $\mu(I) \leq \mu(F)$ and non increasing if whenever $I \subset F$ then $\mu(I) \geq \mu(F)$.

Definition 3.3: - An interval function $\mu(I)$ is said to be bounded, if there is a real number $M > 0$ such that, for every $I$, $| \mu(I) | \leq M$.

Here we shall only consider the bounded interval functions. Every bounded interval function has two associated point functions namely upper and lower point functions denoted by $\overline{\mu}(x)$ and $\underline{\mu}(x)$ respectively. We shall explain the notation used in the definition of point functions.

For example:

$\sup \{\mu(I) \mid x \in I, \lambda(I) < 1/n\}$ signifies the least upper bound of the set of values of $\mu(I)$ for all $I$, containing $x$, and that have Lebesgue measure less than $1/n$, $n > 0$ integer, where $\lambda$ denotes the Lebesgue measure.

Now consider:

$$\overline{\mu}_n(x) = \sup\{\mu(I) \mid x \in I,\ \lambda(I) < 1/n\},$$

and

$$\underline{\mu}_n(x) = \inf\{\mu(I) | x \in I,\ \lambda(I) < 1/n\} \text{ for all integer } n > 0$$

The sequence $\{\overline{\mu}_n(x)\}$ is a non increasing bounded sequence, hence by the fundamental theorem of bounded sequences it converges. To show that $\{\overline{\mu}_n(x)\}$ is a non increasing sequence, we note that the set of intervals I, containing x, of measure less than 1/n + 1 is a subset of those whose measure is less than 1/n so that

$$\overline{\mu}_{n+1}(x) = \sup\{\mu(I) \mid x \in I,\ \lambda(I) < 1/n + 1\} \leq$$

$$\sup\{\mu(I) \mid x \in I,\ \lambda(I) < 1/n\} = \overline{\mu}_n(x)$$

Similarly it can be shown that $\{\underline{\mu}_n(x)\}$ is a non decreasing bounded sequence, and hence converges.

Definition 3.4: - If $\mu(I)$ is a bounded interval function, then its upper and lower point functions are given respectively by,

$$\overline{\mu}(x) = \lim_{n\to\infty} \overline{\mu}_n(x) = \lim_{n\to\infty} [\sup\{\mu(I)|x \in I,\ \lambda(I) < 1/n\}]$$

$$\underline{\mu}(x) = \lim_{n\to\infty} \underline{\mu}_n(x) = \lim_{n\to\infty} [\inf\{\mu(I)|x \in I,\ \lambda(I) < 1/n\}]$$

Now we define the concept of relative measure of a set S in an interval I.

Definition 3.5: - If S is a $\lambda$-measurable set and I is any open interval, the <u>relative measure</u> of S in I is given by the number

$$\frac{\lambda(S \cap I)}{\lambda(I)} .$$

Remark: - We observe that the relative measure is an interval function such that

$$0 \le \frac{\lambda(S \cap I)}{\lambda(I)} \le 1.$$

If the relative measure of S in I is designated by $\mu(I)$, then the associated point functions namely $\overline{\mu}(x)$, and $\underline{\mu}(x)$, are respectively called the upper and lower metric density of S at x.

We give these definitions in detail. Let S be a $\lambda$-measurable set and x a real number. For every positive integer n, let

$$\overline{\mu}_n(x) = \sup \{\frac{\lambda(S \cap I)}{\lambda(I)} \mid x \in I, \lambda(I) < 1/n\},$$

and

$$\underline{\mu}_n(x) = \inf \{\frac{\lambda(S \cap I)}{\lambda(I)} \mid x \in I, \lambda(I) < 1/n\};$$

and let

$$\overline{\mu}(x) = \lim_{n \to \infty} \overline{\mu}_n(x),$$

and

$$\underline{\mu}(x) = \lim_{n \to \infty} \underline{\mu}_n(x).$$

Definition 3.6: - The numbers $\overline{\mu}(x)$ and $\underline{\mu}(x)$ are called the upper metric density and lower metric density of S at x respectively. If $\overline{\mu}(x) = \underline{\mu}(x)$ the metric density of S at x is said to exist and the number $\mu(x) = \overline{\mu}(x) = \underline{\mu}(x)$ is called the metric density of S at x. [5]

The metric density of a set need not exist as is shown by the following example.

Example 1: - Let S be a closed interval $[\frac{1}{2}, 1]$. For every positive integer n, the relative measure of S in the interval $(\frac{1}{2} - \frac{1}{n^2}, \frac{1}{2} + \frac{1}{n})$ is given by

$$\frac{\lambda\{[1/2, 1] \cap (1/2 - 1/n^2, 1/2 + 1/n)\}}{\lambda(1/2 - 1/n^2, 1/2 + 1/n)} =$$

$$\frac{\lambda[1/2, 1/2 + 1/n)}{\lambda(1/2 - 1/n^2, 1/2 + 1/n)} = \frac{n}{n+1}.$$

The relative measure of S in the interval $(1/2 - 1/n, 1/2 + 1/n^2)$ is given by

$$\frac{\lambda\{[1/2, 1] \cap (1/2 - 1/n, 1/2 + 1/n^2)\}}{\lambda(1/2 - 1/n, 1/2 + 1/n^2)} = \frac{1}{n+1}$$

It follows that the upper metric density of S at 1/2 is equal to 1, and the lower metric density of S at 1/2 is equal to 0. Hence the metric density of S at 1/2 does not exist.

**Example 2: -** Let $S = \bigcup_{n=1}^{\infty} (1/2 + 1/n, 1/2 + 1/n + 1/n^2)$. Now it is easily seen that the relative measure of S is positive in every open interval containing the point 1/2, and the metric density of S at 1/2 exists and is equal to 0.

**Proposition 3.1: -** If S is any $\lambda$-measurable set and the metric density of S exists at a point x, then the metric density of C(S) exists at x and the sum of the two metric densities is equal to 1.

**Proof: -** Let I be any open interval containing the point x since S is measurable,

$\lambda(S \cap I) + \lambda(C(S) \cap I) = \lambda(I)$ so that

$$\frac{\lambda(S \cap I)}{\lambda(I)} + \frac{\lambda(C(S) \cap I)}{\lambda(I)} = 1$$

The proposition is now immediate consequence of the definition.

**Proposition 3.2:** – Every open set S has metric density unity at a point $x \in S$.

**Proof:** – Since S is an open set, for every $x \in S$ there exists an open interval I such that $x \in I \subset S$. Now the relative measure of S in I is given by,

$\frac{\lambda(S \cap I)}{\lambda(I)} = \frac{\lambda(I)}{\lambda(I)} = 1.$ Hence the result follows by taking the sup and then limit.

**Definition 3.7:** – A set G is said to be metrically dense at a point p, [p may or may not belong to G] if every interval containing p, contains a subset of G which has the measure greater than zero.

**Remark:** – We observe that the point x at which a set G is metrically dense must belong to $G'$.

Now we define the approximate continuity of a real function $f(x)$ as follows.

**Definition 3.8:** – A real function $f(x)$ is said to be approximately continuous at $\alpha \in R$ if for every $\varepsilon > 0$ the set $E = \{x \mid x \in R, \mid f(x) - f(\alpha) \mid < \varepsilon\}$ has metric density unity at $\alpha$. [5]

**Proposition 3.3:** – A continuous function $f(x)$ is approximately continuous, but the converse is not true.

**Proof:** – Let $f(x)$ be continuous at $\alpha \in R$. Then for a given $\varepsilon > 0$ the set $N = \{x \mid x \in R, \mid f(x) - f(\alpha) \mid < \varepsilon\}$ is open, and $\alpha \in N$.

Hence the metric density of N at $\alpha$ is unity by the previous proposition.

For the converse, consider the following example. For each positive integer n, consider the intervals of the form $(\frac{1}{n+1}, \frac{1}{n}) \subset [0,1]$. Now from each such interval remove a subinterval $S_n$ of the length

$$\frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n^2(n+1)}, \quad n = 1, 2, 3 \dots\dots\dots$$

Let S denote the union of all the intervals so removed, and let

$$f(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \in C(S) \end{cases}$$

By symmetry we can define the function in the interval $[-1,0]$. It is easily seen that f(x) is not continuous at x = 0, we show that f(x) is approximately continuous at x = 0.

Let $\delta > 0$ be any real number such that $\delta < 1$, and let $I_\delta = (-\delta, \delta)$ be a neighbourhood of 0. There exists a positive integer m, such that $1/m \leq \delta$, and for all $n \geq m$

$$\bigcup_{n=m}^{\infty} S_n \subset I_\delta$$

Now the relative measure of the set C(S) in $I_\delta$ is given by the number,

$$\frac{\lambda(I_\delta \cap C(S))}{\lambda(I)}.$$

Clearly $\limsup_{\delta\to 0} \frac{\lambda(I_\delta \cap C(S)}{\lambda(I_\delta)} = \limsup_{\delta\to 0} \frac{\lambda(I_\delta - \bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)}$

$= \limsup_{\delta\to 0} \frac{\lambda(I_\delta) - \lambda(\bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)}$, since $I_\delta$ and $\bigcup_{n=m}^{\infty} S_n$ are λ-measurable, and $\bigcup_{n=m}^{\infty} S_n \subset I_\delta$

$= \limsup_{\delta\to 0} \left(1 - \frac{\lambda(\bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)}\right)$

But $\lambda(\bigcup_{n=m}^{\infty} S_n) = 2 \sum_{n=m}^{\infty} \frac{1}{n^2(n+1)}$, since $S_n$'s are disjoint.

So $\frac{\lambda(\bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)} = \frac{1}{\delta} \sum_{n=m}^{\infty} \frac{1}{n^2(n+1)}$.

Since $\frac{1}{\delta} \leq m$, $\frac{\lambda(\bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)} \leq m \sum_{n=m}^{\infty} \frac{1}{n^2(n+1)}$

$$\leq \sum_{n=m}^{\infty} \frac{m}{n} \cdot \frac{1}{n(n+1)}$$

$$\leq \sum_{n=m}^{\infty} \frac{1}{n(n+1)}$$

Now as $\delta \to 0$, $m \to \infty$ therefore $\lim \frac{\lambda(\bigcup_{n=m}^{\infty} S_n)}{\lambda(I_\delta)} = 0$

Hence $\limsup_{\delta\to 0} \frac{\lambda(I_\delta \cap C(S))}{\lambda(I_\delta)} = 1$. Similarly it can be shown that the $\liminf_{\delta\to 0} \frac{\lambda(I_\delta \cap C(S))}{\lambda(I_\delta)} = 1$. Thus the metric density of the set $C(S)$ at 0 exists and is equal to unity. Now given $\varepsilon > 0$, the set

$N = \{ x \varepsilon [-1, 1] \mid f(x) < \varepsilon \} \supset C(S)$.

Therefore the metric density of N is unity at 0.

This proves that $f(x)$ is approximately continuous at 0.

Theorem 3.1: - The necessary and sufficient condition that a function $f(x)$ should be approximately continuous at the point $\alpha$ is that it should be continuous, at $\alpha$, relatively to a set of points G which has the metric density unity at $\alpha$. [5]

Proof: Sufficiency: - Suppose $f(x)$ is continuous at $\alpha$ with respect to G which has metric density unity at the point $\alpha$. Then for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \varepsilon (\alpha - \delta, \alpha + \delta) \cap G$, $| f(x) - f(\alpha) | < \varepsilon$. If $\delta' < \delta$ then $I_{\delta'} = (\alpha - \delta', \alpha + \delta')$ is an open neighbourhood of $\alpha$. Now the relative measure of $(\alpha - \delta, \alpha + \delta) \cap G$ in $I_{\delta'}$ is given by,

$$\frac{\lambda(I_{\delta'} \cap (\alpha - \delta, \alpha + \delta) \cap G)}{\lambda(I_{\delta'})} = \frac{\lambda(I_{\delta'} \cap G)}{\lambda(I_{\delta'})} .$$

Since G has metric density unity at $\alpha$,

$$\limsup_{\delta' \to 0} \frac{\lambda(I_{\delta'} \cap G)}{\lambda(I_{\delta'})} = \liminf_{\delta' \to 0} \frac{\lambda(I_{\delta'} \cap G)}{\lambda(I_{\delta'})} = 1.$$

Hence the set $(\alpha - \delta, \alpha + \delta) \cap G$ has metric density unity at $\alpha$. This proves that $f(x)$ is approximately continuous.

Necessity: - Suppose that for each $\varepsilon > 0$ the set

$G_\varepsilon(\alpha) = \{x \mid x \varepsilon R, \ | f(x) - f(\alpha) | < \varepsilon \}$ has metric density unity at $\alpha$. Let $\{\varepsilon_n\}$ be a sequence of decreasing values of $\varepsilon$ that converges to zero. Now for each $\varepsilon_n$ a number $\delta_n < \delta_{n-1}$ can be determined such that

$\lambda\{G_{\varepsilon_n}(\alpha)\cap(\alpha - \delta_n, \alpha + \delta_n)\} > 2\delta_n(1 - \varepsilon_n)$, because f(x) is approximately continuous at $\alpha$. The numbers $\delta_n$ may, if necessary, be so altered that $\frac{\delta_{n+1}}{\delta_n} < \frac{1}{\varepsilon_n}$, for every value of n.

Let $F_n =$

$$\{G_{\varepsilon_n}(\alpha)\cap(\alpha + \varepsilon_n\delta_{n+1}, \alpha + \delta_n)\}\cup\{G_{\varepsilon_n}(\alpha)\cap(\alpha - \delta_n, \alpha - \varepsilon_n\delta_{n+1})\},$$

then $F_{n+m} \subset G_{\varepsilon_{n+m}}(\alpha) \subset G_{\varepsilon_n}(\alpha)$ for all $m \geq 1$.

Clearly $G_{\varepsilon_n}(\alpha) = F_n \cup (G_{\varepsilon_n}(\alpha)\cap I_{\delta_{n+1}\varepsilon_n})$, where

$I_{\delta_{n+1}\varepsilon_n} = (\alpha - \delta_{n+1}\varepsilon_n, \alpha + \delta_{n+1}\varepsilon_n)$.

Therefore

$$\lambda(G_{\varepsilon_n}(\alpha)) \leq \lambda(F_n) + \lambda(G_{\varepsilon_n}(\alpha)\cap I_{\delta_{n+1}\varepsilon_n}),$$

or

$$\lambda(F_n) \geq \lambda(G_{\varepsilon_n}(\alpha)) - \lambda(G_{\varepsilon_n}(\alpha)\cap I_{\delta_{n+1}\varepsilon_n}).$$

or

$$> 2\delta_n(1 - \varepsilon_n) - 2\delta_{n+1}\varepsilon_n$$

or

$$> 2\delta_n(1 - \varepsilon_n) - 2\delta_n\varepsilon_n \text{ since } \delta_{n+1} < \delta_n$$

$$> 2\delta_n(1 - 2\varepsilon_n).$$

Thus $\lambda(F_n) > 2\delta_n(1 - 2\varepsilon_n)$.

Let $G = \bigcup_{n=1}^{\infty} F_n$, then the set G has metric density unity at $\alpha$, since $\lambda(F_n) > 2\delta_n(1 - 2\varepsilon_n)$. Let $\{x_m\}_{m=1}^{\infty}$ be a monotonic sequence converging to $\alpha$, and such that all of them are points of G. Clearly each $x_m$ belongs to $\bigcup_{n=1}^{\infty} F_n$ for some n. Let that n be denoted by $n_m$. We then have $| f(x_m) - f(\alpha) | < \varepsilon_{n_m}$; now as $m \to \infty$, $n_m \to \infty$ and $\varepsilon_{n_m} \to 0$ therefore the sequence $\{f(x_m)\}$ converges to $f(\alpha)$. Hence $f(x)$ is continuous relative to G; also G has metric density unity at $\alpha$.

Chapter IV

In this chapter along with the properties of certain sets, the concept of almost continuity will be described and a characterization theorem for the same will be proved.

Definition 4.1: - A set S is said to be dense in a set M if $\overline{S} \supset M$.

Definition 4.2: - A set E is said to be non-dense in R, if for each open interval $I \subset R$ there is a subinterval $F \subset I$ such that $F \cap E = \phi$.

Definition 4.3: - A set S is said to be of the first category if it is a countable union of non-dense sets.

Definition 4.4: - A set M, which is not of the first category is known as a set of the second category.

Definition 4.5: - A set S which is complementary to a set of the first category is known as a residual set.

The distinction between sets of the first and second categories was first introduced by Baire, in Annali de math. Vol. VII (1899) P. 65.

Examples: -

1) Every finite set is a non-dense set .

2) Let $P_1, P_2, P_3, \ldots P_n, \ldots$ be a countable set of points in (a,b). The finite sets $(P_1)$, $(P_1, P_2, P_3)$ $\ldots (P_1, P_2, \ldots P_n), \ldots$, are non-dense, and form a countable union. Hence form a set of the first category. Clearly the remaining points of (a,b) form a residual set.

3) The set of all real numbers, is a set of the second category.

Now we will discuss the existence of a non-dense set, and a set of the first category with the help of a binary relation $\mathcal{R}$ between open sets and points on the real line. [1]

Definition 4.6: - Let I be any open set and x be a point on the real line, then the symbol $I \mathcal{R} x$ shall mean that the open set I has a relation $\mathcal{R}$ to the point x.

Definition 4.7: - The relation $\mathcal{R}$ is said to be closed, if the relationships $I \mathcal{R} x_n$ and $\lim_{n\to\infty} x_n \to x$ imply $I \mathcal{R} x$.

Lemma 4.1: - If $\mathcal{R}$ is a closed relation, then the set

$$M = \left\{ x \;\middle|\; \begin{array}{l} \text{(a) } N \mathcal{R} x \text{ for each neighbourhood } N \text{ of } x, \text{ and} \\ \quad \text{a partial neighbourhood } N_p \text{ of } x \text{ exists} \\ \quad \text{such that} \\ \text{(b) } N_p \not{\mathcal{R}}\, x \quad (\text{i.e. } N_p \mathcal{R}\, x \text{ is false.}) \end{array} \right.$$

is a non-dense set.

Proof: - Since $\mathcal{R}$ is a closed relation and $N_p \not{\mathcal{R}}\, x$, there exists no sequence of points $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $N_p \mathcal{R} x_n$ for every n. Hence a neighbourhood N of x exists such that $N_p \not{\mathcal{R}}\, y$ for each $y \in N$. In particular $N_p \not{\mathcal{R}}\, Z$ for each $Z \in N_p \cap N$. Since $N_p$ is a neighbourhood of Z, it follows that no $Z \in M$. Thus every neighbourhood of a point $x \in M$ contains an open set which is completely free from the points of M. Hence M is a non-dense set.

Now let $y = f(x)$ be any given real function defined for every point $x \in R$. We define a relation $\mathcal{R}_{r_1, r_2}$ as follows, where $r_1 < r_2$ are real numbers.

Definition 4.8: - If $x \in R$, and $I \subset R$ is any open set, then $I \, R_{r_1,r_2} \, x$ if any only if there exists $\{x_n\}_{n=1}^{\infty} \in I$, such that $\lim x_n = x$, $\lim f(x_n)$ exists, and $r_1 \leq \lim f(x_n) \leq r_2$

Lemma 4.2: - The relation $R_{r_1,r_2}$ is a closed relation

Proof: - Suppose that $\lim x^n = x$ and $I \, R_{r_1,r_2} \, x^n$ for each n. Then for every n, there exists a sequence $\{x_m^n\}$, $m = 1, 2 \ldots$ of points of I such that $\lim_{m\to\infty} x_m^n = x^n$, $\lim f(x_m^n)$ exists and $r_1 \leq \lim f(x_m^n) \leq r_2$.

Now in view of the preceding relations, we may select a sequence $\{y_n\}_{n=1}^{\infty}$ from $\{x_m^n\}$ such that $\lim_{n\to\infty} y_n = x$ and $r_1 \leq \lim \inf f(y_n) \leq \lim \sup f(y_n) \leq r_2$.

From $\{y_n\}$ we can select a subsequence $\{x_n\}$ such that $\lim f(x_n)$ exists. Then $\lim_{n\to\infty} x_n = x$ and $r_1 \leq \lim f(x_n) \leq r_2$

Lemma 4.3: - If $R_{r_1,r_2}$ is a closed relation, then for every real function $f(x)$, and for every pair of real numbers $r_1 < r_2$ the set

$$M_{r_1,r_2} = \left\{ x \in R \;\middle|\; \begin{array}{l} (1)\ N \, R_{r_1,r_2} \, x, \text{ for every neighbourhood } N \text{ of } x \\ \text{and} \\ (2)\ N_p \, \not R_{r_1,r_2} \, x, \text{ for some partial neighbourhood } N_p \text{ of } x \end{array} \right.$$

is a non-dense set, and $M = \bigcup_{r_1,r_2 \in Q} M_{r_1,r_2}$ is a set of the first category.

Proof: - By Lemma 4.1, $M_{r_1, r_2}$ is a non-dense set. Since $r_1$, $r_2$ vary over the rationals, M is equal to the countable union of non-dense sets. Hence M is a set of the first category.

Now we define the concept of almost continuity of a real function f(x) and relate its properties in terms of the sets discussed above.

Definition 4.9: - A real function f(x) is said to be <u>almost continuous</u> at $\alpha \in R$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that the set of points x, for which $|f(x) - f(\alpha)| < \varepsilon$ forms a dense set in $(\alpha - \delta, \alpha + \delta)$. Symbolically we can write as $\overline{f^{-1}(f(\alpha) - \varepsilon , f(\alpha) + \varepsilon)} \supset (\alpha - \delta, \alpha + \delta)$

Definition 4.10: - A function f(x) is said to be almost continuous on R if it is almost continuous at each point of R.

Proposition 4.1: - Every continuous function is almost continuous but the converse is not true.

Proof: - Let f(x) be continuous at $\alpha \in R$

$\Rightarrow$ for a given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$| f(x) - f(\alpha) | < \varepsilon$ for $| x - \alpha | < \delta$

$\Rightarrow (f(\alpha) - \varepsilon, f(\alpha) + \varepsilon) \supset f(\alpha - \delta, \alpha + \delta)$

$\Rightarrow f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon) \supset (\alpha - \delta, \alpha + \delta)$

$\Rightarrow \overline{f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)} \supset (\alpha - \delta, \alpha + \delta)$

Hence f(x) is almost continuous at $\alpha \in R$.

For the converse consider the following example. Define f(x) as follows:

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Now for each $\varepsilon > 0$, $(1 - \varepsilon, 1 + \varepsilon)$ is a neighbourhood of 1, consider $f^{-1}(1 - \varepsilon, 1 + \varepsilon)$, if $\varepsilon < 1$ then $f^{-1}(1 - \varepsilon, 1 + \varepsilon)$ is the set of all rationals. Hence

$$\overline{f^{-1}(1 - \varepsilon, 1 + \varepsilon)} \supset R.$$

Now if $\varepsilon > 1$ then $f^{-1}(1 - \varepsilon, 1 + \varepsilon)$ is the whole real line and $\overline{R} \supset R$. Hence $f(x)$ is almost continuous at $x \varepsilon R$. Clearly $f(x)$ is not continuous.

Lemma 4.4: - A set M of isolated points is a set of the first category.

Proof: - Since each $p \varepsilon M$ is an isolated point, no p can be a limit point of elements in M. Hence there exists rationals $r_p$, $s_p$ such that $r_p < p < s_p$, and $(r_p, s_p)$ contains no other elements of M.

Therefore we have a one to one correspondence between the points p and a subset of rationals. Hence $M = \bigcup_{p \varepsilon M} p$ is a countable union of non-dense sets. Therefore M is a set of the first category.

Theorem 4.1: - The set of points x on the real line at which a real function f(x) is almost continuous forms a residual set. On the other hand, given a residual set S a function f(x) exists which is almost continuous on the points of S. [1]

Proof: - Consider the graph of the function f(x). The points of the graph of f(x), are either isolated points or not.

The isolated points of the graph form a countable set, therefore the points p on the real line whose corresponding points of the graph are isolated points and f is not almost continuous at p, are also isolated points hence form a countable set. Therefore by the previous Lemma the set of points, at which f is not almost continuous, and the corresponding points on the graph are isolated points, is a set of the first category. Consequently it is sufficient to prove that the points $p \in R$ for which $P' = (p, f(p))$ is a limit point of the graph of $f(x)$, but $f(x)$ is not almost continuous at p, form a set of the first category.

Since $f(x)$ is not almost continuous at p, for some $\varepsilon > 0$ and for every $\delta > 0$ the set $M = \{q \mid q \varepsilon (p - \delta, p + \delta),$ $| f(q) - f(p) | < \varepsilon \}$ is not dense in $(p - \delta, p + \delta)$. Let us denote $(p - \delta, p + \delta)$ by N, a neighbourhood of p. Since M is not dense in N, an open interval $I \subset N$ exists, for every neighbourhood of p, such that I is completely free from points of M and $| f(q) - f(p) | \geq \varepsilon$ for all $q \varepsilon I$. Since $\delta$ is arbitrary a partial neighbourhood $N_p$ of p exists such that $| f(p) - f(q) | \geq \varepsilon$ holds for all $q \varepsilon N_p$. Now let $r_1$ be a rational number between $f(p) - \varepsilon$ and $f(p)$, and $r_2$ a rational number between $f(p)$ and $f(p) + \varepsilon$. Then it follows from the inequality $| f(q) - f(p) | \geq \varepsilon$ for all $q \varepsilon N_p$ that the set $E = \{ (q\ f(q)) \mid q \varepsilon N_p \}$ which is a subset of the graph of f, has no point (p, y) as a limit with $r_1 \leq y \leq r_2$. Hence $N_p \not R_{r_1, r_2} p$ (i.e. $N_p R_{r_1, r_2} p$ is false). On the other hand, P' is a limit point of the graph, hence it is a limit point of every neighbourhood N', where N is a neighbourhood of P. Therefore N' has (p, z), with $r_1 \leq z \leq r_2$ as a limit point. Hence by Lemma 4.3, p is a point of non-dense set $M_{r_1 r_2}$ associated with the closed relation $R_{r_1, r_2}$. Since $r_1$, $r_2$ vary over rationals, $M = \bigcup_{r_1, r_2 \in Q} M_{r_1, r_2}$ is a set of the first category. Hence the points p at which f(x) is not almost continuous belongs to a set of the first category. Therefore the points at which f(x) is almost continuous form a residual set.

For the converse, let E be the set of the first category complementary to the given residual set S. Then we can write $E = \bigcup_n E_n$ where each $E_n$ is a non-dense set, and $E_k \cap E_m = \phi \quad k \neq m$.

Now define f(x) as follows:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \epsilon E_n \\ 2 & \text{if } x \epsilon S. \end{cases}$$

Now it is easy to see that f(x) is almost continuous at S.

Theorem 4.2: - A function $f(x)$ is almost continuous at a point $\alpha \in R$ if and only if it is continuous with respect to a set G which is dense in a neighbourhood $U_\alpha$ of $\alpha$.

Proof: - Let $f(x)$ be continuous at $\alpha$ with respect to G which is dense in some neighbourhood of $\alpha$, say $U_\alpha$.

Since $f(x)$ is continuous with respect to G, for a given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all

$x \varepsilon (\alpha - \delta, \alpha + \delta) \cap G$,

$f(x) \varepsilon (f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)$,

or $(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon) \supset f\{(\alpha - \delta, \alpha + \delta) \cap G\}$

$$\Rightarrow f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon) \supset (\alpha - \delta, \alpha + \delta) \cap G$$

$$\Rightarrow f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon) \supset (\alpha - \delta, \alpha + \delta) \cap G \cap U_\alpha.$$

$$\Rightarrow \overline{f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)} \supset \overline{(\alpha - \delta, \alpha + \delta) \cap G \cap U_\alpha}$$

Since G is dense in $U_\alpha$, it follows that

$\overline{G \cap (\alpha - \delta, \alpha + \delta) \cap U_\alpha} \supset (\alpha - \delta, \alpha + \delta) \cap U_\alpha$.

Therefore $\overline{f^{-1}(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)} \supset (\alpha - \delta, \alpha + \delta) \cap U_\alpha$ which is a neighbourhood of $\alpha$. Hence $f(x)$ is almost continuous at $\alpha$.

Next suppose that $f(x)$ is almost continuous at $\alpha \varepsilon R$. Consider the set $G_n(\alpha) = \{x \varepsilon R \mid |f(x) - f(\alpha)| < 1/n\}$ for each positive integer n. Now for each integer $n > 0$ there exists a $\delta_n > 0$ such that $\overline{G_n(\alpha)} \supset (\alpha - \delta_n, \alpha + \delta_n)$, because $f(x)$ is almost continuous at $\alpha$. We can choose $\{\delta_n\}_{n=1}^{\infty}$ to be a decreasing sequence of positive numbers converging to zero, such that for any positive integer $m$, $(\alpha - \delta_m, \alpha + \delta_m) \subset (\alpha - \delta_n, \alpha + \delta_n)$, for $m > n$.

Let $E_n(\alpha) = G_n(\alpha) \cap \{(\alpha + \delta_{n+1}, \alpha + \delta_n) \cup (\alpha - \delta_n, \alpha - \delta_{n+1})\}$.

Since $G_n(\alpha)$ is dense in $(\alpha - \delta_n, \alpha + \delta_n)$, so is $E_n(\alpha)$ in $(\alpha + \delta_{n+1}, \alpha + \delta_n) \cup (\alpha - \delta_n, \alpha - \delta_{n+1})$ for each $n$.

Hence $G = \bigcup_{n \geq 1} E_n$ is dense in

$$\bigcup_{n \geq 1} (\alpha + \delta_{n+1}, \alpha + \delta_n) \cup (\alpha - \delta_n, \alpha - \delta_{n+1})$$

$= (\alpha, \alpha + \delta_1) \cup (\alpha - \delta_1, \alpha)$. From this it follows that $G$ is dense in $(\alpha - \delta_1, \alpha + \delta_1)$ which is a neighbourhood of $\alpha$.

Now let $\{x_m\}$ be an increasing or decreasing sequence in $G$ such that $x_m \to \alpha$. Since $G = \bigcup_{n \geq 1} E_n$ and $E_n$'s are decreasing, there exists a positive integer $m_n$ such that $x_m \in E_{m_n}(\alpha)$. If $m$ increases so does $m_n$ as is quite clear. Thus $x_m \in G_{m_n}(\alpha)$ by the definition of $E_n$'s$(\alpha)$, i.e. $| f(x_m) - f(\alpha) | < \frac{1}{m_n}$. This shows that $f(x_m) \to f(\alpha)$ as $m \to \infty$. This completes the proof.

Now in the following theorem we show that the class of almost continuous functions is larger than the class of approximately continuous functions.

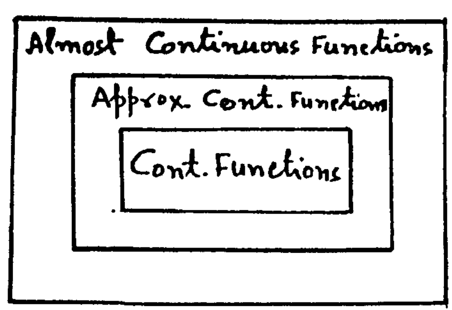
**Theorem 4.3: -** If $f(x)$ is approximately continuous then $f(x)$ is almost continuous but the converse is not true.

**Proof: -** Since $f(x)$ is approximately continuous at $\alpha \in R$ this implies that $f(x)$ is continuous at $\alpha \in R$ with respect to a set G which has a metric density unity at the point $\alpha$. We have to show that the set G is dense in some neighbourhood of $\alpha$. If not, then for each $\delta > 0, I_\delta \cap G = \{\alpha\}$ where $I_\delta$ is a neighbourhood of $\alpha$. This implies that $\lambda(I_\delta \cap G) = 0$ which is a contradiction, because $f(x)$ is approximately continuous at $\alpha$. Hence there exists a $\delta$ such that G is dense in $I_\delta$. Therefore $f(x)$ is almost continuous at $\alpha \in R$. For the converse, consider the following example:

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Clearly $f(x)$ is almost continuous with respect to rationals, but $f(x)$ is not approximately continuous, because the set of rationals has a measure zero.

Thus combining the results of Proposition 3.3, 4.1 and above theorem we observe that, the following is true.



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