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MODULAR ELLIPTIC SURFACES

By
Rachid Ejouamai
January 2006

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
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Master of Science in Mathematics¹

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INTRODUCTION

In arithmetic geometry, one of the important themes is to prove that certain geometric objects have an arithmetic nature. This can lead to many important results about the geometric object itself as well as to intrinsic arithmetic results. This phenomenon can also be found when one proves that certain algebraic objects are topological in nature, or geometric. For instance when a general group is an algebraic group or a Lie group, or when it is the fundamental group or the monodromy group of a geometric object, then the structure of the group becomes more apparent and very rich.

A more relevant example can be illustrated in the theory of elliptic curves. On one hand, an elliptic curve over the complex numbers is topologically a complex torus, that is the quotient of the complex field by a rank two lattice. This can lead to determining all the invariants of the curve, as well as the standard analytic objects attached to the elliptic curve. On the other hand, the same elliptic curve over the rational numbers is modular, in the sense that it is parameterized by the quotient of the upper half of the complex plane by a discrete group (of arithmetic type). This has another interpretation in terms of L-functions and modular forms. The modularity of elliptic curves, which used to be called the Taniyama-Shimura conjecture, was proven a decade ago by Wiles and as a consequence, Fermat's last theorem was proven.

It is natural to try extending these facts to objects of higher dimension; in particular, it is natural to ask whether the same phenomenon can exist for algebraic surfaces or for three-dimensional algebraic varieties. In fact, any similar considerations for these higher dimensional varieties have strong implications in algebraic geometry and in physics.

The main focus in this thesis is on elliptic surfaces. These are algebraic surfaces

with an elliptic fibration over a curve. The notion of modular elliptic surfaces was introduced by Shioda in the 1970s and it involves discrete groups of arithmetic nature similar to the ones used to define modular elliptic curves. However, the construction of a modular elliptic surface is more of a topological nature similar to the fact that an elliptic curve is topologically a torus. Yet, the ingredients for this construction are essentially arithmetic. The full arithmetic nature of the elliptic surfaces would involve a deep understanding of the zeta functions or the L-functions in terms of automorphic forms attached to the discrete groups defining the modular elliptic surfaces. Because of the complexity of the structure, the arithmetic nature in terms of L-functions is left to a second stage, and we deal only with the topological modularity, which is by itself very rich and still is an unexplored topic of research. A natural question to ask is: when is an elliptic surface modular? There is a theorem of Nori which states some sufficient conditions for an elliptic surface to be modular, and this is what we will try to explain in this work.

This thesis is organized as follows: In the first chapter, we introduce the cohomology of sheaves and their applications to complex manifolds and to algebraic surfaces. In particular, we define the numerical invariants of an algebraic surface such as the Betti numbers, the Hodge numbers and the Picard number. In the second chapter, we introduce the notion of an elliptic surface, and we characterize the singular fibers of these surfaces. We recall the notion of the Mordell-Weil group and the Neron-Severi group for such surfaces, and we determine their numerical invariants. The third chapter deals with modular elliptic surfaces. We explain their construction and we determine their singular fibers in terms of the base curves as well as their numerical invariants. We also study some new examples of modular elliptic surfaces. In the last chapter, we explain Nori's theorem about the modularity of an elliptic surface. We will provide a proof, which is essentially the same as Nori's proof, but without the heavy technical tools used by Nori, namely, we will not use the notion of generalized modular elliptic surfaces, which were introduced, to carry out the proof. We will however prove an intermediate result, which we will establish directly in a way that is inspired from Nori's proof.

It remains to study the modular elliptic surfaces from the number-theoretic point of view. Many attempts have been made in this regard by many authors, but only on particular surfaces, and a generic treatment of the subject has yet to appear.

Chapter 1

Algebraic surfaces

The purpose of this chapter is to introduce algebraic surfaces together with their invariants. This requires the introduction of cohomology of manifolds with coefficients in a sheaf, and algebraic curves over the field of complex numbers. We will define all the necessary ingredients which are used later.

1.1 Cohomology

1.1.1 Sheaves

We start this section with the classical definition of a sheaf for which we need the notion of a presheaf. Let R be a fixed ring, a presheaf \mathcal{F} of R -modules over a topological space X is:

1. An assignment to each open set U of X an R -module $\mathcal{F}(U)$.
2. A collection of R -linear maps $r_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ whenever $V \subseteq U$ such that:
 - (a) $r_U^U = I_{\mathcal{F}(U)}$.
 - (b) If $W \subseteq V \subseteq U$ then $r_W^U = r_W^V \circ r_V^U$.

A morphism of presheaves $h : \mathcal{F} \rightarrow \mathcal{C}$ is collection of R -linear maps $h(U) : \mathcal{F}(U) \rightarrow \mathcal{C}(U)$ such that the following diagram commutes whenever $V \subseteq U$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{h(U)} & \mathcal{C}(U) \\ r_V^U \downarrow & & \downarrow \rho_V^U \\ \mathcal{F}(V) & \xrightarrow{h(V)} & \mathcal{C}(V) \end{array}$$

A presheaf \mathcal{F} is called a sheaf if for every open subset U of X and every open covering $\{U_\alpha\}_{\alpha \in A}$ of U we have:

1. If $s, t \in \mathcal{F}(U)$ and $r_{U_\alpha}^U(s) = r_{U_\alpha}^U(t)$ for all $\alpha \in A$ then $s = t$.
2. If $s_\alpha \in \mathcal{F}(U_\alpha)$ and if for $U_\alpha \cap U_\beta \neq \emptyset$ we have $r_{U_\alpha \cap U_\beta}^{U_\alpha}(s_\alpha) = r_{U_\alpha \cap U_\beta}^{U_\beta}(s_\beta)$ for all α and β , then there exists an $s \in \mathcal{F}(U)$ such that $r_{U_\alpha}^U(s) = s_\alpha$, for all α .

A morphism of sheaves \mathcal{F} and \mathcal{C} is a morphism of \mathcal{F} and \mathcal{C} as presheaves.

Examples.

1. Let X a topological space, and G an abelian group. The assignment $U \mapsto G$ for each open subset U of X , defines a sheaf over X called the constant sheaf associated with G where the restriction homomorphisms r_V^U are all equal to I_G ; the identity map of G .
2. Let X a topological space. For each open subset U of X define $\mathcal{C}(U)$ to be the set of continuous real valued functions on U , then the assignment $U \mapsto \mathcal{C}(U)$ defines a sheaf of \mathbb{R} -algebras called the sheaf of real valued continuous functions on X where the restriction homomorphisms are the usual restrictions.

More elaborate examples of sheaves will be given in later sections.

Let \mathcal{F} be a sheaf of R -modules over a topological space X , and let $x \in X$. We consider $v(x) = \{U \text{ open set of } X \text{ such that } x \in U\}$. It can be shown that $v(x)$ is a

directed set, and it is easy to see that the family $\{\mathcal{F}(U), r_U^V\}_{U \in v(x)}$ is a directed system of R -modules. The stalk of the sheaf \mathcal{F} at x is by definition the direct limit $\varinjlim \mathcal{F}(U)$ of the directed system $\{\mathcal{F}(U), r_U^V\}_{U \in v(x)}$. It is also an R -module and we denote it by \mathcal{F}_x . A morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{C}$ induces an R -linear map f_x on the stalks: $f_x : \mathcal{F}_x \rightarrow \mathcal{C}_x$, for each $x \in X$.

1.1.2 Cohomology with coefficients in a sheaf

A sequence of sheaves $\mathcal{F}^1 \rightarrow \mathcal{F}^2 \rightarrow \dots \rightarrow \mathcal{F}^m \rightarrow \dots$ is said to be exact if the induced sequences; $\mathcal{F}_x^1 \rightarrow \mathcal{F}_x^2 \rightarrow \dots \rightarrow \mathcal{F}_x^m \rightarrow \dots$ are exact for all $x \in X$.

A resolution of a sheaf \mathcal{F} is an exact sequence of the form

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^1 \rightarrow \dots \rightarrow \mathcal{F}^m \rightarrow \dots$$

Let \mathcal{F} be a sheaf over X and S a closed subset of X . If $v(S) = \{U \text{ open subset of } X \mid S \subseteq U\}$, then $v(S)$ is a directed set and the family $\{\mathcal{F}(U), r_U^V\}_{U \in v(S)}$ is a directed system. Let $\mathcal{F}(S)$ the direct limit of the directed system $\{\mathcal{F}(U), r_U^V\}_{U \in v(S)}$. A sheaf \mathcal{F} is said to be soft if for any closed subset S of X the natural map $\mathcal{F}(X) \rightarrow \mathcal{F}(S)$ is surjective.

A *soft* resolution of a sheaf \mathcal{F} is a resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^1 \rightarrow \dots \rightarrow \mathcal{F}^m \rightarrow \dots$$

such that each \mathcal{F}^i is soft.

Theorem 1.1.1. (*[10] Page 56*)

For any sheaf \mathcal{F} over a topological space X , there exists a soft resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^1 \rightarrow \dots \rightarrow \mathcal{F}^m \rightarrow \dots$$

Let \mathcal{F} be a sheaf over a topological space X , and consider a soft resolution of \mathcal{F}

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow \dots \rightarrow \mathcal{C}^m \rightarrow \dots,$$

one can prove that the induced sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{C}^0(X) \longrightarrow \mathcal{C}^1(X) \longrightarrow \dots \longrightarrow \mathcal{C}^m(X) \longrightarrow \dots$$

is a cochain complex, in the sense that $Im(\mathcal{C}^{q-1} \longrightarrow \mathcal{C}^q) \subseteq Ker(\mathcal{C}^q \longrightarrow \mathcal{C}^{q+1})$. (see [10], Page 57).

We define the q^{th} cohomology group of X with coefficients in \mathcal{F} to be

$$H^q(X, \mathcal{F}) = \frac{Ker(\mathcal{C}^q \longrightarrow \mathcal{C}^{q+1})}{Im(\mathcal{C}^{q-1} \longrightarrow \mathcal{C}^q)} \quad \text{if } q \geq 1, \quad \text{and } H^0(X, \mathcal{F}) = \mathcal{F}(X).$$

One can prove by standard homological arguments that $H^q(X, \mathcal{F})$ does not depend on the particular soft resolution of \mathcal{F} .

1.2 Complex manifolds

1.2.1 Holomorphic Atlas

Let X be a Hausdorff topological space with a countable basis. A holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on X consists of the following:

1. A covering $X = \bigcup_{\alpha \in A} U_\alpha$ with the U_α open subsets of X .
2. Homeomorphisms $\varphi_\alpha : U_\alpha \longrightarrow D_\alpha$ with D_α open subsets of \mathbb{C}^n , $n \geq 1$

such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphic map of open subsets of \mathbb{C}^n .

A complex manifold is a Hausdorff topological space with a countable basis which is endowed with a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. If the φ_α 's map into \mathbb{C}^n for a fixed positive integer n , then X is said to be of dimension n . The elements $(U_\alpha, \varphi_\alpha)$

of the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ are called charts. For $x \in X$ a chart $(U_\alpha, \varphi_\alpha)$ is called a local coordinate at x if $x \in U_\alpha$ and $\varphi_\alpha(x) = 0$. If D is an open subset of \mathbb{C}^n , we denote by $\mathcal{O}(D)$ the ring of complex-valued holomorphic functions on D .

Let X be a complex manifold with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Let U be an open subset of X , then a holomorphic function on U is a complex-valued function f such that for all $\alpha \in A$ with $U_\alpha \cap U \neq \emptyset$, we have $f \circ \varphi_\alpha^{-1} \in \mathcal{O}(\varphi_\alpha(U_\alpha \cap U))$. We denote the ring of holomorphic functions on U by $\mathcal{O}_X(U)$, the assignment $U \rightarrow \mathcal{O}_X(U)$ defines a sheaf \mathcal{O}_X on X called the sheaf of holomorphic functions.

1.2.2 Morphisms

Let X, Y be two complex manifolds. A map $\varphi : X \rightarrow Y$ is said to be a morphism (or holomorphic) if it is continuous and for any open subset V of Y , if $f \in \mathcal{O}_Y(V)$ then $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$. Moreover, φ is said to be an isomorphism if it is bijective and φ^{-1} is a morphism, and it is said to be an embedding if φ is an isomorphism from X into its image in Y .

An important example of a complex manifold is the complex projective space. For $n \geq 1$, we consider the following equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $x \sim y$ if there exists $\lambda \in \mathbb{C}^*$ such that $y = \lambda x$. The quotient $\mathbb{C}^{n+1} \setminus \{0\} / \sim$ is called the complex projective space of dimension n , and it is denoted by $\mathbb{P}^n(\mathbb{C})$ or simply \mathbb{P}^n . For $x = (x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, we denote its equivalence class by $[x_0, \dots, x_n]$. The space \mathbb{P}^n has the quotient topology by means of the natural surjection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$$

$$(x_0, \dots, x_n) \longmapsto [x_0, \dots, x_n].$$

Therefore, \mathbb{P}^n is a Hausdorff space with a countable basis and π is continuous. If S^{n+1} denotes the unit sphere of \mathbb{C}^{n+1} , then $\mathbb{P}^n = \pi(S^{n+1})$ is compact.

For $\alpha = 0, 1, \dots, n$, let $U_\alpha = \{[x_0, \dots, x_n] \in \mathbb{P}^n \mid x_\alpha \neq 0\}$. Then U_α is an open subset of \mathbb{P}^n and $\mathbb{P}^n = \bigcup_{\alpha=0}^n U_\alpha$. If we define $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$, $\varphi_\alpha([x_0, \dots, x_n]) = (\frac{x_0}{x_\alpha}, \dots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \dots, \frac{x_n}{x_\alpha})$.

Then one can show that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=0}^n$ is a holomorphic atlas on \mathbb{P}^n , so that \mathbb{P}^n has the structure of a compact complex manifold.

A compact complex manifold X which admits an embedding into \mathbb{P}^n for some n is called a projective algebraic manifold. An algebraic curve is a projective algebraic manifold of dimension 1 which is also referred to as a Riemann surface. An algebraic surface is a projective algebraic manifold of dimension 2.

1.2.3 The sheaf of differential forms

Let X be a complex manifold of dimension n . For $x \in X$ let $\mathcal{O}_{X,x}$ be the \mathbb{C} -algebra of germs of holomorphic functions at x . The elements of $\mathcal{O}_{X,x}$ are by definition the equivalence classes of pairs (f, U) , where U is an open subset of X containing x and $f \in \mathcal{O}_X(U)$, and where the equivalence relation is $(f, U) \sim (g, V)$ if $f|_{U \cap V} = g|_{U \cap V}$. The \mathbb{C} -algebra $\mathcal{O}_{X,x}$ is also referred to as the stalk of the sheaf \mathcal{O}_X at x .

A derivation of the \mathbb{C} -algebra $\mathcal{O}_{X,x}$ is a linear form $D : \mathcal{O}_{X,x} \rightarrow \mathbb{C}$, such that $D(f \cdot g) = g(x) \cdot D(f) + f(x) \cdot D(g)$. Let $T_x(X)$ be the set of all derivations, it is a subspace of the dual space $\mathcal{O}_{X,x}^*$, and we call it the tangent space of X at x . If X has dimension n then $\dim_{\mathbb{C}} T_x(X) = n$ for all $x \in X$.

The cotangent space of X at x is by definition the dual space $T_x(X)^*$ of the tangent space $T_x(X)$. For any integer $p \geq 0$ define the p -exterior space $\Lambda^p T_x(X)^*$ of $T_x(X)^*$. We have $\Lambda^p T_x(X)^* = 0$ for all $p > \dim_{\mathbb{C}} T_x(X)^*$.

Let $\Lambda^p T(X)^*$ denotes the disjoint union $\bigcup_{x \in X} \Lambda^p T_x(X)^*$, and consider the map $\pi : \Lambda^p T(X)^* \rightarrow X$ given by $\pi(\xi) = x$ if $\xi \in \Lambda^p T_x(X)^*$. In other words $\pi^{-1}(x) = \Lambda^p T_x(X)^*$. One can prove that $\Lambda^p T(X)^*$ has a canonical structure of a complex manifold such that the map π is holomorphic.

Let U be an open subset of X , a differential form ω of degree p on U is a holomorphic map $\omega : U \rightarrow \Lambda^p T(X)^*$ such that $\pi \circ \omega = I_U$, where I_U is the identity map of U . Let $\Omega_X^p(U)$ denote the set of all differential forms on U of degree p . It has the structure of an $\mathcal{O}_X(U)$ -module, and the assignment $U \rightarrow \Omega_X^p(U)$ defines a sheaf Ω_X^p over X called the sheaf of differential forms of degree p on X .

1.2.4 The Hodge decomposition

If X is a compact complex manifold, consider the q^{th} -cohomology group $H^q(X, \Omega_X^p)$ with coefficients in the sheaf Ω_X^p , and $H^r(X, \mathbb{C})$ the r^{th} -cohomology group associated with the constant sheaf \mathbb{C} . Then one can prove that $H^q(X, \Omega_X^p)$ and $H^r(X, \mathbb{C})$ are finite dimensional \mathbb{C} -vector spaces.

Define

$$h^{q,p} = \dim_{\mathbb{C}} H^q(X, \Omega_X^p) \quad , \quad b_r = \dim_{\mathbb{C}} H^r(X, \mathbb{C}).$$

The numbers $h^{q,p}$ are called the Hodge numbers, and the b_r are called the Betti numbers.

Theorem 1.2.1. *(The Hodge decomposition theorem, [10] Page 189) Let X be an algebraic complex manifold, then there exists a direct sum decomposition:*

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^q(X, \Omega_X^p)$$

In particular,

$$b_r = \sum_{p+q=r} h^{q,p}. \tag{1}$$

Theorem 1.2.2. *([10] Corollary 2.8 Page 179) Let X be an algebraic complex manifold of complex dimension n , then:*

$$b_r = b_{2n-r}, \quad r = 0, \dots, 2n \tag{2}$$

$$h^{p,q} = h^{q,p} = h^{n-p, n-q}, \quad p, q = 0, \dots, n. \tag{3}$$

1.3 Algebraic curves

1.3.1 Ramification and degree

A Riemann surface is a one-dimensional connected complex manifold. An algebraic curve is a compact Riemann surface. The following proposition describes morphisms (holomorphic maps) between compact Riemann surfaces.

Proposition 1.3.1. [9] *Let C and C' be two algebraic curves, and let $f : C \rightarrow C'$ be a morphism. Then f is either constant or surjective.*

If $f : C \rightarrow C'$ is a nonconstant morphism, then C is called a covering of C' . If $z_0 \in C$ and $y_0 = f(z_0) \in C'$ and u and t are respectively local coordinates at z_0 and y_0 , then there exists a neighborhood V of z_0 such that for every $z \in V$, we have

$$t(f(z)) = a_e u(z)^e + a_{e+1} u(z)^{e+1} + \dots, \quad a_e \neq 0.$$

The integer e is independent of the choice of u and t and $e \geq 1$. It is called the ramification index of f at z_0 and is denoted by $e_f(z_0)$ or simply e_{z_0} . We say that z_0 is ramified if $e_{z_0} > 1$ and it is unramified if $e_{z_0} = 1$.

Theorem 1.3.2. [9] *Let $f : C \rightarrow C'$ be a nonconstant morphism of algebraic curves. Then:*

1. *There are only finitely many ramified points.*
2. *If $y \in C'$, then $f^{-1}(y)$ is a finite set and the integer $n = \sum_{x \in f^{-1}(y)} e_x$ does not depend on $y \in C'$; we call it the degree of f and we denote it $\deg(f)$.*
3. *The map f is an isomorphism if and only if $\deg(f) = 1$.*
4. *If $g : C' \rightarrow C''$ is a nonconstant morphism of algebraic curves and $h = g \circ f$ then*

$$\deg(h) = \deg(g) \cdot \deg(f) \quad , \quad e_h(z) = e_f(z) \cdot e_g(f(z)) \quad \text{for } z \in C.$$

1.3.2 The Hurwitz formula

The genus of an algebraic curve C is by definition the integer

$$g = h^{0,1} = \dim_{\mathbb{C}} H^0(C, \Omega_C),$$

where $H^0(C, \Omega_C)$ is the 0^{th} cohomology group of C with coefficients in the sheaf Ω_C of differential forms on C . The Euler characteristic χ of the curve C is by definition the integer

$$\chi = 2 - 2g.$$

A rational algebraic curve is an algebraic curve of genus zero, or equivalently, an algebraic curve with Euler characteristic 2. One can prove that an algebraic curve C is rational if and only if it is isomorphic to \mathbb{P}^1 .

Theorem 1.3.3. (*Hurwitz formula*) [9]. *Let $f : C \rightarrow C'$ be a morphism of algebraic curves, if g and g' are respectively the genera of C and C' , then:*

$$2g - 2 = (2g' - 2) \cdot \deg(f) + \sum_{x \in C} (e_x - 1), \quad (4)$$

where e_x is the ramification index of f at x .

Notice that the sum $\sum_{x \in C} (e_x - 1)$ makes sense, since there are only finitely many ramified points.

1.4 Elliptic curves

1.4.1 The modular group

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ denote the complex upper half-plane. The modular group is defined by

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

It acts on $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by linear fractional transformations as follows:

$$\sigma \cdot z = \frac{az + b}{cz + d}, \text{ if } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ and } z \in \overline{\mathbb{H}}.$$

Two elements z and z' are equivalent if there exists $\sigma \in \text{SL}_2(\mathbb{Z})$ such that $\sigma \cdot z = z'$. We note that $\mathbb{Q} \cup \{\infty\}$ is stable under this action, and that σ and $-\sigma$ act in the same way. In order to get an effective action, we consider the quotient $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z}) / \{\pm I\}$.

A nonidentity element $\sigma \in \text{PSL}_2(\mathbb{Z})$ is called an elliptic element if it has one fixed point in \mathbb{H} or, equivalently, $|\text{tr}\sigma| < 2$, i.e. $\text{tr}\sigma \in \{-1, 0, 1\}$. It is called a parabolic element if σ has only one fixed point in $\mathbb{Q} \cup \{\infty\}$ or, equivalently, $\text{tr}\sigma \in \{-2, 2\}$.

A point $z \in \mathbb{H}$ is called an elliptic point if it is fixed by some elliptic element. A point $r \in \mathbb{Q} \cup \{\infty\}$ is called a cusp if it is fixed by some parabolic element.

Let $z \in \overline{\mathbb{H}}$, such that z is a cusp or an elliptic point. We denote the stabilizer of z in $\mathrm{PSL}_2(\mathbb{Z})$ by $\mathrm{PSL}_2(\mathbb{Z})_z$. For example, ∞ is a cusp and $\mathrm{PSL}_2(\mathbb{Z})_\infty$ is the infinite cyclic group generated by $T \cdot z = z + 1$ corresponding to the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The point i is an elliptic point and $\mathrm{PSL}_2(\mathbb{Z})_i = \{I, S\}$ where $S \cdot z = -1/z$ corresponding to the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The point $\rho = e^{\frac{2i\pi}{3}}$ is an elliptic point and $\mathrm{PSL}_2(\mathbb{Z})_\rho = \{I, ST, (ST)^2\}$.

One can prove the following:

1. Every cusp r is equivalent to the point ∞ , and $\mathrm{PSL}_2(\mathbb{Z})_r$ is conjugate to $\mathrm{PSL}_2(\mathbb{Z})_\infty = \langle T \rangle$ in $\mathrm{PSL}_2(\mathbb{Z})$.
2. Every elliptic point z is equivalent to either i or ρ . Moreover $\mathrm{PSL}_2(\mathbb{Z})_z$ is conjugate to $\mathrm{PSL}_2(\mathbb{Z})_i = \{I, S\}$ when z is equivalent to i , and it is conjugate to $\mathrm{PSL}_2(\mathbb{Z})_\rho = \{I, ST, (ST)^2\}$ when z is equivalent to ρ . (see [4])

The quotient $\mathrm{PSL}_2(\mathbb{Z}) \backslash \overline{\mathbb{H}}$ is called a modular curve. One can prove that this modular curve has a natural structure of an algebraic curve and it is rational, i.e. isomorphic to \mathbb{P}^1 .

We say a function f is modular if f is meromorphic on $\overline{\mathbb{H}}$ and verifies the relation $f(z) = f(\sigma.z)$ for all $\sigma \in \mathrm{PSL}_2(\mathbb{Z})$. It is clear that a modular function yields a well defined meromorphic function on the modular curve $\mathrm{PSL}_2(\mathbb{Z}) \backslash \overline{\mathbb{H}}$.

If $k \geq 1$ be an integer, we say that a function f is a modular form of weight $2k$ if f is holomorphic on $\overline{\mathbb{H}}$, and for every $z \in \mathbb{H}$ and every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. we have

$$f(\sigma.z) = (cz + d)^{-2k} f(z).$$

1.4.2 Lattices

A lattice Λ in \mathbb{C} is a subgroup generated by two complex numbers linearly independent over \mathbb{R} . Thus $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. We can order ω_1, ω_2 so that $\text{Im}(\omega_1/\omega_2) > 0$. Let $M = \{(\omega_1, \omega_2) \in \mathbb{C}^2 \mid \text{Im}(\omega_1/\omega_2) > 0\}$. The group $\text{SL}_2(\mathbb{Z})$ acts on M by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2).$$

It is easy to prove that two pairs (ω_1, ω_2) and (ω'_1, ω'_2) in M generate the same lattice if and only if they are equivalent under the action of $\text{SL}_2(\mathbb{Z})$. Thus if \mathfrak{R} denotes the set of all lattices in \mathbb{C} , the map $\text{SL}_2(\mathbb{Z}) \backslash M \rightarrow \mathfrak{R}$ defined by $[(\omega_1, \omega_2)] \mapsto \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a bijection. Now, consider the actions of \mathbb{C}^* on M and on \mathfrak{R} defined by the rules: $z \cdot (\omega_1, \omega_2) = (z \cdot \omega_1, z \cdot \omega_2)$ and $z \cdot \Lambda = \{z \cdot \lambda \mid \lambda \in \Lambda\}$. The map

$$M/\mathbb{C}^* \rightarrow \mathbb{H}, \quad [(\omega_1, \omega_2)] \mapsto \frac{\omega_1}{\omega_2}$$

is a bijection and the action of $\text{SL}_2(\mathbb{Z})$ on M is by linear fractional transformations. Therefore we have the following bijections

$$\mathfrak{R}/\mathbb{C}^* \rightarrow \text{SL}_2(\mathbb{Z}) \backslash M/\mathbb{C}^* \rightarrow \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$$

$$\mathbb{Z}\omega \oplus \mathbb{Z} \mapsto (\omega, 1) \mapsto \omega.$$

1.4.3 Elliptic curves

Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice, then the quotient \mathbb{C}/Λ is a torus and has a natural structure of a compact Riemann surface, so that the natural surjection $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$ is a holomorphic map. The torus \mathbb{C}/Λ is called the elliptic curve associated with the lattice Λ . Two elliptic curves \mathbb{C}/Λ and \mathbb{C}/Λ' are isomorphic if and only if there exists $\alpha \in \mathbb{C}^*$ such that $\Lambda' = \alpha\Lambda$. Therefore $\mathfrak{R}/\mathbb{C}^*$ is the set of isomorphism classes of elliptic curves, and every elliptic curve is isomorphic to $\mathbb{C}/\mathbb{Z}\omega \oplus \mathbb{Z}$ for some $\omega \in \mathbb{H}$.

Let $\Lambda(\omega) = \mathbb{Z}\omega \oplus \mathbb{Z}$ be a lattice. A $\Lambda(\omega)$ -elliptic function is a meromorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f(z + \tau) = f(z)$ for all $z \in \mathbb{C}$ and all $\tau \in \Lambda(\omega)$. An elliptic

function defines a function on the elliptic curve $\mathbb{C}/\mathbb{Z}\omega \oplus \mathbb{Z}$.

The Weierstrass \wp -function relative to a lattice $\Lambda(\omega)$, $\omega \in \mathbb{H}$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\tau \in \Lambda(\omega)} \left(\frac{1}{(z - \tau)^2} - \frac{1}{\tau^2} \right).$$

One can prove that $\wp(z)$ converges absolutely and uniformly on every compact subset of $\mathbb{C} \setminus \Lambda(\omega)$. Moreover $\wp(z)$ is a $\Lambda(\omega)$ -elliptic function.

The Eisenstein series of weight $2k$ relative to $\Lambda(\omega)$ is defined by

$$G_{2k}(\omega) = G_{2k}(\Lambda(\omega)) = \sum_{\tau \in \Lambda(\omega), \tau \neq 0} \frac{1}{\tau^{2k}} = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus (0,0)} \frac{1}{(n\omega + m)^{2k}}.$$

The series G_{2k} converges absolutely, and the function $\omega \mapsto G_{2k}(\omega)$ is a modular form of weight $2k$. We have the following result:

Theorem 1.4.1. (*[9], Theorem 3.5*) *For every $z \in \mathbb{C}$, $z \notin \Lambda(\omega)$ we have:*

$$\wp'(z)^2 = 4\wp(z)^3 - g_4(\omega)\wp(z) - g_6(\omega),$$

where $g_4 = 60G_4$ and $g_6 = 140G_6$.

The discriminant of the elliptic curve $\mathbb{C}/\Lambda(\omega)$ is defined by

$$\Delta(\omega) = g_4(\omega)^3 - 27g_6(\omega)^2.$$

The function $\omega \mapsto \Delta(\omega)$ is a modular form of weight 12.

We also define the j -invariant of the elliptic curve $\mathbb{C}/\Lambda(\omega)$ by

$$j(\omega) = 1728 \frac{g_2^3(\omega)}{\Delta(\omega)}$$

and the function $\omega \mapsto j(\omega)$ is a modular function. We have the following result:

Theorem 1.4.2. (*[4], Proposition 5, Chapter VII*) *The j -function is holomorphic on \mathbb{H} and has a simple pole at infinity. Moreover, it induces an isomorphism of compact Riemann surfaces*

$$j : PSL_2(\mathbb{Z}) \setminus \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1.$$

1.5 Algebraic surfaces

1.5.1 Divisors

Let X be an algebraic surface i.e. a compact complex manifold of dimension 2 which admits an embedding into \mathbb{P}^n for some positive integer n . An algebraic curve C , such that $C \subseteq X$ is called a *prime divisor* on X . A *divisor* D on X is a formal linear combination $D = \sum n_i C_i$ with $n_i \in \mathbb{Z}$ and C_i are prime divisors on X , the C_i are called the prime components of D . The set of all divisors form a group called the *group of divisors* of X and it is denoted by $Div(X)$; it is the free abelian group generated by the prime divisors. The support of a divisor $D = \sum n_i C_i$ is the union $\bigcup_i C_i$ and it is denoted by $Supp(D)$.

Let $\mathbb{C}(X)$ denotes the field of meromorphic functions on X . For $f \in \mathbb{C}(X)^*$ consider the sets:

$$A_0(f) = \{C \text{ a curve on } X \mid f|_C = 0\}, \quad A_\infty(f) = \{C \text{ a curve on } X \mid f|_C = \infty\}.$$

For $C \in A_0(f)$, let $v_C(f)$ denotes the order of vanishing of f at C , and for $C \in A_\infty(f)$ we define the order of vanishing of f at C to be $v_C(f) = -v_C(1/f)$. One can prove that there are only finitely many curves C in X such that $v_C(f) \neq 0$. Therefore the formal sum $\sum_{C \in A_0(f) \cup A_\infty(f)} v_C(f) \cdot C$ makes sense; we call it the divisor associated with f and we denote it by (f) .

As an example, we consider $f : \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ defined by

$$f([x_0 : x_1 : x_2]) = \frac{(x_2 x_1 - x_0^2)^3}{(x_0 x_1 - x_2^2)^2 (x_0 - x_1)^2}.$$

Let C_1, C_2 and C_3 be the curves on \mathbb{P}^2 defined by

$$C_1 : x_2 x_1 - x_0^2 = 0, \quad C_2 : x_0 x_1 - x_2^2 = 0, \quad C_3 : x_0 - x_1 = 0.$$

We have $A_0(f) = \{C_1\}$ and $A_\infty(f) = \{C_2, C_3\}$. Moreover, $v_{C_1}(f) = 3, v_{C_2}(f) = -2$ and $v_{C_3}(f) = -2$. Therefore, the divisor associated with f is

$$(f) = 3C_1 - 2C_2 - 2C_3.$$

We note also that for $f_1, f_2 \in \mathbb{C}(X)^*$ we have: $(f_1 \cdot f_2) = (f_1) + (f_2)$ and $(1/f_1) = -(f_1)$.

A divisor $D \in \text{Div}(X)$ is called principal if it has the form (f) for some $f \in \mathbb{C}(X)^*$. The set of principal divisors form a subgroup of $\text{Div}(X)$ and it is denoted by $P_r(X)$. We say that two divisors $D_1, D_2 \in \text{Div}(X)$ are linearly equivalent, and we write $D_1 \sim D_2$, if there exists $f \in \mathbb{C}(X)^*$ such that $D_1 - D_2 = (f)$. The group of equivalence classes, $\text{Div}(X)/\sim = \text{Div}(X)/P_r(X)$, is a group called the Picard group of X and it is denoted $\text{Pic}(X)$.

1.5.2 Intersection numbers of divisors

Let X be an algebraic surface and C_1, C_2 be curves on X such that $C_1 \cap C_2$ consists of a finite number of points. We define the *intersection number* of C_1 and C_2 to be the number of points in $C_1 \cap C_2$ counted with their multiplicities. This number is denoted by $C_1 \cdot C_2$.

Now let $D = \sum n_i C_i$ and $\Delta = \sum m_j \Gamma_j$ be two divisors such that $\text{Supp}(D) \cap \text{Supp}(\Delta)$ is finite, that is, $C_i \cap \Gamma_j$ is finite for every i, j . We define the *intersection number* of D and Δ by

$$D \cdot \Delta = \sum n_i m_j C_i \cdot \Gamma_j.$$

If for $D = \sum n_i C_i$ and $\Delta = \sum m_j \Gamma_j$, $\text{Supp}(D) \cap \text{Supp}(\Delta)$ is not finite, then one can find a divisor Δ' such that $\Delta \sim \Delta'$ and $\text{Supp}(D) \cap \text{Supp}(\Delta')$ is finite, and thus we define the *intersection number* of D and Δ to be $D \cdot \Delta = D \cdot \Delta'$. More generally, we have the following:

Theorem 1.5.1. ([5], pages 151 and 152) *One can define an intersection pairing $\text{Div}(X) \times \text{Div}(X) \rightarrow \mathbb{Z}$, written $(D_1, D_2) \mapsto D_1 \cdot D_2$ with the following properties:*

1. $D_1 \cdot D_2$ is bilinear and symmetric.
2. If $D_1 \sim D_2$ then $D_1 \cdot D = D_2 \cdot D$ for all $D \in \text{Div}(X)$.

The second assertion of the theorem shows that the intersection number depends only on the linear equivalence classes of the divisors, and therefore we have a well defined pairing $\text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbb{Z}$ which we also call the intersection number.

A curve C is called a -1 curve or an *exceptional curve of the second kind* if its self intersection C^2 is equal to -1 .

Two divisors D_1 and D_2 in $Div(X)$ are said to be *numerically equivalent* if $D_1 \cdot D = D_2 \cdot D$ for all $D \in Div(X)$; we write $D_1 \equiv D_2$ if they are numerically equivalent. By the previous theorem, if $D_1 \sim D_2$ then $D_1 \equiv D_2$. For each $\Delta \in Div(X)$, consider the group homomorphism $\varphi_\Delta : Div(X) \rightarrow \mathbb{Z}$ defined by $\varphi_\Delta(D) = \Delta \cdot D$. By definition of the numerical equivalence, $D_1 \equiv D_2$ is equivalent $\Delta \cdot (D_1 - D_2) = 0$ for all $\Delta \in Div(X)$ which is equivalent to

$$D_1 - D_2 \in K := \bigcap_{\Delta \in Div(X)} Ker(\varphi_\Delta).$$

The group

$(Div(X)/\equiv) = Div(X)/K$ is called the *Neron-Severi group* of X , and it is denoted by $NS(X)$. It is a finitely generated torsion free abelian group and we denote its rank by ρ . The number ρ is called the *Picard number* of X (see [5]).

1.5.3 The canonical class

Let Ω_X^2 be the $\mathbb{C}(X)$ -vector space of meromorphic differential forms of degree 2. Then Ω_X^2 has dimension 1 over $\mathbb{C}(X)$ since X has dimension 2. Let $\omega \in \Omega_X^2, \omega \neq 0$ then any element ω' has the form $\omega' = f \cdot \omega$.

As in the case of functions, one can associate to ω a divisor (ω) . For $\omega' \in \Omega_X^2, \omega' \neq 0$ there exists $f \in \mathbb{C}(X), f \neq 0$, such that $\omega' = f \cdot \omega$. Moreover $(\omega') = (f \cdot \omega) = (f) + (\omega)$, so $(\omega') - (\omega) = (f)$. Therefore, for all nonzero differential forms ω' and ω in Ω_X^2 the divisors (ω) and (ω') are linearly equivalent, and hence they define the same element in $Pic(X)$. This element is called the *canonical class* of X and it is denoted by K_X .

1.5.4 The numerical invariants of an algebraic surface

Let b_k , $0 \leq k \leq 4$ be the Betti numbers of X . We have seen in Theorem 1.2.2 the following:

$$b_0 = b_4 = 1, \quad b_1 = b_3. \quad (5)$$

The Euler characteristic of X is defined by

$$e(X) = \sum_{i=0}^4 (-1)^i b_i = 2 - 2b_1 + b_2. \quad (6)$$

Let $h^{p,q}$, $p, q = 0, 1, 2$, be the Hodge numbers. By Theorem 1.2.2 we have:

$$h^{p,q} = h^{q,p} = h^{2-p,2-q}. \quad (7)$$

Furthermore, we have

$$b_1 = h^{0,1} + h^{1,0} = 2h^{0,1}, \quad (8)$$

and

$$b_2 = h^{0,2} + h^{1,1} + h^{2,0} = 2h^{2,0} + h^{1,1}. \quad (9)$$

The integer $h^{0,1}$ is called the *irregularity* of X and it is denoted by q . The integer $h^{0,2}$ is called the *geometric genus* of X and it is denoted by P_g . We have

$$b_1 = 2q, \quad b_2 = 2P_g + h^{1,1}. \quad (10)$$

We also define the *arithmetic genus* of X to be

$$P_a = 1 - q + P_g. \quad (11)$$

Let K_X be the canonical class of X , and K_X^2 its self intersection number, then we have

Theorem 1.5.2. (*Noether formula, [5]*)

$$K_X^2 + e(X) = 12P_a. \quad (12)$$

Finally, let $NS(X)$ be the Neron-severi group of X , then one can prove that the \mathbb{C} -vector space $NS(X) \otimes_{\mathbb{Z}} \mathbb{C}$ has a natural embedding into the \mathbb{C} -vector space $H^1(X, \Omega_X^1)$ ([5]). Therefore, we have the inequality

$$\rho \leq h^{1,1}. \quad (13)$$

Chapter 2

Elliptic Surfaces

In this chapter, we provide a short exposition of the general theory of elliptic surfaces due to Kodaira, as well as the major theorems in the theory.

2.1 Elliptic fibration

2.1.1 Definitions and Examples

Definition 2.1.1. *Let C be an algebraic curve over \mathbb{C} . An elliptic surface over C consists of the following data:*

1. *A projective surface X over \mathbb{C} .*
2. *A morphism $\pi : X \rightarrow C$ such that for all but finitely many points $t \in C$, the fiber C_t is an algebraic curve of genus 1.*
3. *A section to π , by which we mean a morphism $\sigma_0 : C \rightarrow X$ such that $\pi \circ \sigma_0 = 1_C$.*

Conditions 2 and 3 imply that for almost all points $t \in C$, the pair $(C_t, \sigma_0(t))$ is an elliptic curve over \mathbb{C} . Also, condition 2 implies that there exists a finite set $\Sigma \subsetneq C$

such that for all t in Σ , $\pi^{-1}(t)$ is not an elliptic curve, we will suppose that such a set is always non-empty.

Definition 2.1.2. *An elliptic surface $\pi : X \rightarrow C$ is called relatively minimal if for any t in C , the fiber C_t is not an exceptional curve of the second kind, i.e. $C_t^2 \neq -1$.*

From now on, elliptic surfaces will be assumed to be relatively minimal.

The standard example is given as follows: Let C be an algebraic curve over \mathbb{C} , and let $\mathbb{C}(C)$ be its field of meromorphic functions. Consider the family of curves in \mathbb{P}^2

$$C_t : y^2 = x^3 + f(t)x + g(t), \quad t \in C. \quad (14)$$

For each $t \in C$ such that $f(t)$ and $g(t)$ are defined and $\Delta(t) = 4f(t)^3 + 27g(t)^2 \neq 0$, the curve C_t is an elliptic curve. Define the algebraic surface X by

$$X = \{([x : y : z], t) \in \mathbb{P}^2 \times C \mid y^2z = x^3 + f(t)xz^2 + g(t)z^3\}.$$

Since X is a subvariety of $\mathbb{P}^2 \times C$, projection onto the second factor defines a morphism

$$\pi : X \rightarrow C$$

$$([x : y : z], t) \mapsto t.$$

We claim that the surface X together with the map π onto C , is an elliptic surface. In fact we need to check condition 2 and 3 of the definition. For 2 define

$$\Sigma = \{t \in C \mid f(t) = \infty \text{ or } g(t) = \infty \text{ or } \Delta(t) = 0\}.$$

In other words,

$$\Sigma = \{\text{poles of } f\} \cup \{\text{poles of } g\} \cup \{\text{zeros of } \Delta\}.$$

Since C is an algebraic curve and $f, g, \Delta \in \mathbb{C}(C)$, Σ is a finite set, and for $t \in C \setminus \Sigma$ the cubic

$$C_t = \pi^{-1}(t) = \{(x, y) \mid y^2 = x^3 + f(t)x + g(t)\}$$

is non singular.

For condition 3, we define the map

$$\begin{aligned}\sigma_0 : C &\longrightarrow X \\ t &\longmapsto ([0 : 1 : 0], t).\end{aligned}$$

Then σ_0 is a morphism and we have $\pi \circ \sigma_0 = 1_C$.

2.2 Singular fibers

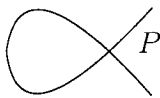
2.2.1 The list of singular fibers

Definition 2.2.1. *Let $\pi : X \longrightarrow C$ be an elliptic surface, and Σ the finite set in C such that for all t in Σ the fiber C_t is not an elliptic curve.*

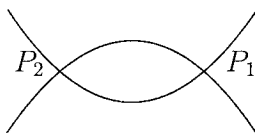
The fibers C_t , $t \in \Sigma$, are called the singular fibers or the exceptional fibers of the elliptic surface $\pi : X \longrightarrow C$.

In the list below, we provide the names and descriptions of singular fibers. It turns out that the list we provide is complete for an elliptic surface.

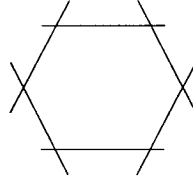
Type I_1 is a rational curve Θ_0 with one double point P .



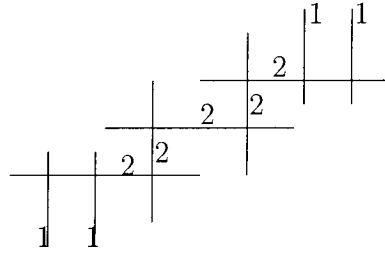
Type I_2 is a divisor of the form $\Theta_0 + \Theta_1$, where Θ_0 and Θ_1 are rational curves with $\Theta_0 \cdot \Theta_1 = P_1 + P_2$



Type I_b is a divisor of the form $\Theta_0 + \Theta_1 + \dots + \Theta_{b-1}$, where $b \geq 3$ and the Θ_i are rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \dots = \Theta_{b-1} \cdot \Theta_0 = 1$



Type I_b^* is a divisor of the form $\Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2(\Theta_4 + \Theta_5 + \dots + \Theta_{b+4})$ where $b \geq 0$ and the Θ_i are rational curves with $\Theta_0 \cdot \Theta_4 = \Theta_1 \cdot \Theta_4 = \Theta_2 \cdot \Theta_{4+b} = \Theta_3 \cdot \Theta_{4+b} = \Theta_4 \cdot \Theta_5 = \Theta_5 \cdot \Theta_6 = \dots = \Theta_{3+b} \cdot \Theta_{4+b} = 1$

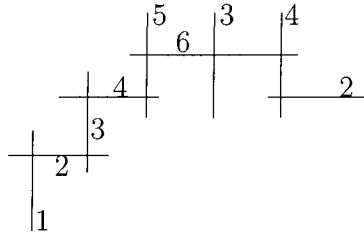


Type II is a rational curve Θ_0 with one cusp.

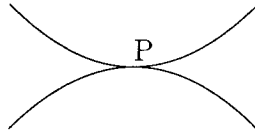


Type II^* is a divisor of the form $\Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 5\Theta_4 + 6\Theta_5 + 4\Theta_6 + 3\Theta_7 + 2\Theta_8$, where the Θ_i are rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_3 = \Theta_3 \cdot \Theta_4 = \Theta_4 \cdot \Theta_5 =$

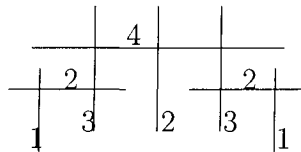
$$\Theta_5 \cdot \Theta_6 = \Theta_5 \cdot \Theta_7 = \Theta_6 \cdot \Theta_8 = 1$$



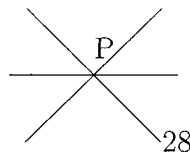
Type III is a divisor of the form $\Theta_0 + \Theta_1$, where Θ_0 and Θ_1 are rational curves with $\Theta_0 \cdot \Theta_1 = 2P$



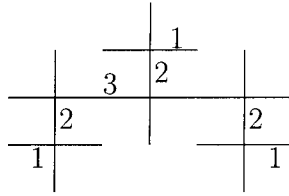
Type III* is a divisor of the form $\Theta_0 + 2\Theta_1 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7$, where the Θ_i are rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_3 = \Theta_3 \cdot \Theta_4 = \Theta_3 \cdot \Theta_5 = \Theta_4 \cdot \Theta_6 = \Theta_6 \cdot \Theta_7 = 1$



Type IV is a divisor of the form $\Theta_0 + \Theta_1 + \Theta_2$, where the Θ_i are rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_0 = P$



Type IV* is a divisor of the form $\Theta_0 + 2\Theta_1 + 3\Theta_2 + 2\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6$, where the Θ_i are rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_2 \cdot \Theta_3 = \Theta_2 \cdot \Theta_4 = \Theta_3 \cdot \Theta_5 = \Theta_4 \cdot \Theta_6 = 1$



2.2.2 The classification theorem for singular fibers

We have

Theorem 2.2.2. ([2], Theorem 6.2) *The singular fibers of an elliptic surface are of the types listed above.*

Proposition 2.2.3. ([2] III, Page 14 Table II) *The Euler characteristic of singular fibers are listed in the table below*

Type of C_v	I_b	I_b^*	II	II*	III	III*	IV	IV*
$e(C_v)$	b	$b + 6$	2	10	3	9	4	8

Table 1

In particular we observe that:

$$e(C_v) = \begin{cases} m_v & \text{if } C_v \text{ is of type } I_b, \\ m_v + 1 & \text{otherwise.} \end{cases}$$

Where m_v is the number of prime components of C_v .

2.3 The Mordell-Weil group and the Neron-Severi group

2.3.1 The Mordell-Weil group

If $\pi : X \rightarrow C$ is an elliptic surface with section: $\sigma_0 : C \rightarrow X$, we set

$$S(X_C) = \{\sigma : C \rightarrow X \text{ a morphism} \mid \pi \circ \sigma = 1_C\}.$$

Let Σ denote the finite set such that for all t in $C \setminus \Sigma$ the fiber C_t is an elliptic curve. We define on $S(X_C)$ the structure of an abelian group with identity element σ_0 as follows

$$(\sigma + \tau)(t) = \sigma(t) \boxplus_t \tau(t) \quad \forall t \in C \setminus \Sigma$$

and

$$(-\sigma)(t) = \boxminus_t \sigma(t) \quad \forall t \in C \setminus \Sigma,$$

where \boxplus_t and \boxminus_t denote the addition and the inverse in the elliptic curve $(C_t, \sigma_0(t))$. The maps $(\sigma, \tau) \mapsto (\sigma + \tau)$ and $\sigma \mapsto (-\sigma)$ can be extended to C in a unique way. Now, if σ, τ and $\lambda \in S(X_C)$, then $(\sigma + \tau) + \lambda$ and $\sigma + (\tau + \lambda)$ coincide on $C \setminus \Sigma$, and so they coincide everywhere. Also, the maps $\sigma + \tau$ and $\tau + \sigma$ coincide on $C \setminus \Sigma$ and thus they coincide everywhere.

Definition 2.3.1. *The group $S(X_C)$ is called the group of sections or the Mordell-Weil group of the elliptic surface $\pi : X \rightarrow C$. The elements of $S(X_C)$ are called the sections of $\pi : X \rightarrow C$. For $\sigma \in S(X_C)$, the curve $\sigma(C) \subseteq X$ is also called a section.*

In what follows we state the Mordell-Weil theorem on elliptic surfaces over \mathbb{C} .

Theorem 2.3.2. *([8], Theorem 6.1) The group $S(X_C)$ is a finitely generated abelian group. More precisely,*

$$S(X_C) \cong \mathbb{Z}^r \oplus S(X_C)_{tor}, \quad r \in \mathbb{N},$$

and

$$S(X_C)_{tor} \cong \mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2}, \quad e_2 | e_1 \quad \text{and} \quad e_2 \geq 1.$$

The integer r is called the *Mordell-Weil rank* of the elliptic surface $\pi : X \longrightarrow C$.

2.3.2 The Neron-Severi group

Since

$$S(X_C) \cong \mathbb{Z}^r \oplus S(X_C)_{tor}, \quad r \in \mathbb{N},$$

and

$$S(X_C)_{tor} \cong \mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2}, \quad e_2 | e_1 \text{ and } e_2 \geq 1.$$

the group $S(X_C)$ has r generators $\sigma_1, \sigma_2, \dots, \sigma_r$ of infinite order and at most two generators τ_1, τ_2 of order e_1, e_2 .

For $\sigma \in S(X_C)$ we denote the image curve $\sigma(C)$ in X by (σ) , we put

$$D_k = (\sigma_k) - (\sigma_0), \quad 1 \leq k \leq r,$$

and

$$D'_h = (\tau_h) - (\sigma_0), \quad h = 1, 2.$$

Let Σ denote the finite set of points v of C for which $C_v = \pi^{-1}(v)$ is a singular fiber. For each $v \in \Sigma$, we denote by θ_{v_i} ($0 \leq i \leq m_v - 1$) the prime components of the divisor C_v , m_v being the number of prime components. We take θ_{v_0} to be the unique component of C_v containing $\sigma_0(v)$. Then we have

$$C_v = \theta_{v_0} + \sum_{i \geq 1} \mu_{vi} \cdot \theta_{v_i}, \quad \mu_{vi} \geq 1.$$

Let A_v denote the square matrix of size $(m_v - 1)$ whose (i, j) -coefficient is $(\theta_{v_i} \theta_{v_j})$, $i, j \geq 1$, where (DD') denotes the intersection number of the divisors D and D' on X .

Finally we take and fix a non-singular fiber C_{u_0} , $u_0 \notin \Sigma$.

With this notations we state :

Theorem 2.3.3. ([7], Theorem 1.1) *The Neron-severi group $NS(X)$ of the elliptic surface $\pi : X \longrightarrow C$, is generated by the following divisors :*

$$C_{u_0}, \quad \theta_{v_i} \quad (1 \leq i \leq m_v - 1, v \in \Sigma)$$

$$(\sigma_0), \quad D_k \quad 1 \leq k \leq r \quad \text{and} \quad D'_h \quad (h = 1, 2).$$

The fundamental relations among them are given by (at most) two relations :

$$e_1 D'_1 \equiv e_1 (D'_1(\sigma_0)).C_{u_0} + e_1 \sum_{v \in \Sigma} (\theta_{v,1}, \dots, \theta_{v,m_v-1}) A_v^{-1} \begin{pmatrix} (D'_1 \theta_{v1}) \\ \vdots \\ (D'_1 \theta_{v,m_v-1}) \end{pmatrix}$$

and

$$e_2 D'_2 \equiv e_2 (D'_2(\sigma_0)).C_{u_0} + e_2 \sum_{v \in \Sigma} (\theta_{v,1}, \dots, \theta_{v,m_v-1}) A_v^{-1} \begin{pmatrix} (D'_2 \theta_{v1}) \\ \vdots \\ (D'_2 \theta_{v,m_v-1}) \end{pmatrix}$$

where e_1, e_2 are the orders of τ_1, τ_2 and \equiv denotes the numerical equivalence.

As a consequence, we have the following corollary

Corollary 2.3.4. (The Shioda-Tate Formulae) ([7], Corollary 1.5) The Picard number ρ of an elliptic surface is given by the formula:

$$\rho = r + 2 + \sum_{v \in \Sigma} (m_v - 1) \quad (15)$$

where m_v is the number of prime components of the singular fiber C_v , and r is the Mordell-Weil rank of the group of sections $S(X_C)$.

Proof. Let

$$p = r + 4 + \sum_{v \in \Sigma} (m_v - 1)$$

Theorem 2.3.3 shows that $NS(X)$ has p generators with two relations among them. Therefore the corollary can be proved simply by :

$$(\text{rank}) = (\text{the number of generators}) - (\text{the number of relations}).$$

□

2.4 Invariants of an elliptic surface

2.4.1 The homological invariant

Let $\pi : X \rightarrow C$ be an elliptic surface. In this section we suppose that the set Σ contains all $t \in C$ for which the fiber C_t is singular and may contain also some points with nonsingular fibers.

Take a point $O \in C \setminus \Sigma$ and consider the fundamental group of the Riemann surface $C \setminus \Sigma$, $\pi_1(C \setminus \Sigma, O)$. Let $\beta \in \pi_1(C \setminus \Sigma, O)$ which we represent by a closed path, that is $\beta : [0, 1] \rightarrow C \setminus \Sigma$ starting and ending at O . Let $H_1(C_O, \mathbb{Z})$ be the singular homology group of C_O with coefficients in \mathbb{Z} . We fix a basis $\{\gamma_{10}, \gamma_{20}\}$ of $H_1(C_O, \mathbb{Z})$ and choose $\gamma_1(s), \gamma_2(s)$ to be a basis of $H_1(C_{\beta(s)}, \mathbb{Z})$ which depend continuously on $s \in [0, 1]$, such that $\gamma_1(0) = \gamma_{10}$ and $\gamma_2(0) = \gamma_{20}$. Then on the fiber C_O , we have

$$\begin{cases} \gamma_1(1) \sim a_\beta \gamma_{10} + b_\beta \gamma_{20} \\ \gamma_2(1) \sim c_\beta \gamma_{10} + d_\beta \gamma_{20} \end{cases}$$

where the symbol \sim denotes the homology equivalence on C_0 and $a_\beta, b_\beta, c_\beta$, and d_β are integers such that $a_\beta d_\beta - b_\beta c_\beta = 1$. We write $(S_\beta) = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then (S_β) depends only on the homotopy class of β on $C \setminus \Sigma$ relative to the point O . Thus $\beta \mapsto (S_\beta)$ defines a representation of the fundamental group $\pi_1(C \setminus \Sigma, O)$. We denote this representation by

$$\rho : \pi_1(C \setminus \Sigma, O) \rightarrow \mathrm{SL}_2(\mathbb{Z})$$

$$\beta \mapsto (S_\beta) = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}$$

Definition 2.4.1. *The map ρ is called the homological invariant of the elliptic surface $\pi : X \rightarrow C$.*

2.4.2 The functional invariant

For any t in $C \setminus \Sigma$, the elliptic curve $(C_t, \sigma_o(t))$ can be presented as a complex torus $\mathbb{C}/\omega(t).\mathbb{Z} \oplus \mathbb{Z}$ where $\omega(t) \in \mathbb{H}$. The complex number $\omega(t)$ is unique up to the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ in \mathbb{H} . Therefore, the multi-valued function $\omega : C \setminus \Sigma \longrightarrow \mathbb{H}$ gives us a well defined holomorphic function

$$\bar{\omega} : C \setminus \Sigma \longrightarrow \mathrm{PSL}_2(\mathbb{Z}) \backslash \overline{\mathbb{H}}.$$

If we compose with the j -modular map $j : \mathrm{PSL}_2(\mathbb{Z}) \backslash \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1$ we obtain a holomorphic function

$$\tilde{J} : C \setminus \Sigma \longrightarrow \mathbb{P}^1.$$

For any t in $C \setminus \Sigma$, the quantity $\tilde{J}(t)$ is just the j -invariant of the elliptic curve $(C_t, \sigma_o(t))$. The function \tilde{J} can be extended to a unique morphism; $J : C \longrightarrow \mathbb{P}^1$ since C is an algebraic curve and Σ is a finite set. Obviously the poles of J lie in the finite set Σ . Finally by a suitable choice of the finite set Σ , we may assume that $J(t) \neq 0, 1, \infty$ for $t \in C \setminus \Sigma$.

Definition 2.4.2. *The morphism $J : C \longrightarrow \mathbb{P}^1$ is called the functional invariant of the elliptic surface $\pi : X \longrightarrow C$.*

There are several results on the relationships between the homological invariant, the functional invariant and the singular fibers which will be summarized in a table below. Let us first fix some notation.

Let $\rho : \pi_1(C \setminus \Sigma, O) \longrightarrow \mathrm{SL}_2(\mathbb{Z})$ be the the homological invariant. For every point t in C consider β_t , the loop element in $\pi_1(C \setminus \Sigma)$, which goes around t once with positive orientation. Let $[\rho(\beta_t)]$ to denote the conjugacy class of $\rho(\beta_t)$ in $\mathrm{SL}_2(\mathbb{Z})$, and e_t the ramification index of the functional invariant J at t .

With this terminology we have

Theorem 2.4.3. *([2], Page 604, Table I) We have the following table:*

C_t	$[\rho(\beta_t)]$	e_t	Behavior of J at t
I_0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$e_t \equiv 0[3]$	<i>Regular</i>
I_0^*	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	or $e_t \equiv 0[2]$	
I_b	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$e_t = b$	<i>pole of order b</i>
I_b^*	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$		
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$e_t \equiv 1[3]$	$J(t) = 0$
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$		
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$e_t \equiv 2[3]$	$J(t) = 0$
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$		
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$e_t \equiv 1[2]$	$J(t) = 1$
III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		

Table 2

2.5 The construction of elliptic surfaces

2.5.1 The period map

Let C be an algebraic curve over \mathbb{C} , and $J : C \longrightarrow \mathbb{P}^1$ be a non-constant morphism, and set $\Sigma = J^{-1}(\{0, 1, \infty\})$. Clearly, Σ is a finite set and $J : C \setminus \Sigma \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a holomorphic map. Now we consider the universal covering $P : U \longrightarrow C \setminus \Sigma$ of

$C \setminus \Sigma$, and the map $j : \overline{\mathbb{H}} \setminus j^{-1}(\{0, 1, \infty\}) \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$, where j is the modular function and $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. One can prove that there exists a unique holomorphic map $\omega : U \longrightarrow \overline{\mathbb{H}} \setminus j^{-1}(\{0, 1, \infty\})$ such that $j \circ \omega = J \circ P$ (see [2] section 8), that is, we have the following commutative diagram;

$$\begin{array}{ccc} U & \xrightarrow{\omega} & \overline{\mathbb{H}} \setminus j^{-1}(\{0, 1, \infty\}) \\ P \downarrow & & \downarrow j \\ C \setminus \Sigma & \xrightarrow{J} & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array}$$

Definition 2.5.1. *The map ω is called the period map associated with the pair (C, J) .*

Proposition 2.5.2. *([2], page 579) Fix a point $O \in C \setminus \Sigma$. Let ω be the period map of $J : C \longrightarrow \mathbb{P}^1$. Then there exists a representation of the fundamental group in $PSL_2(\mathbb{Z})$, $\bar{\rho} : \pi_1(C \setminus \Sigma, O) \longrightarrow PSL_2(\mathbb{Z})$, such that for all β in $\pi_1(C \setminus \Sigma, O)$ and all u in U we have*

$$\omega(\beta \cdot u) = \bar{\rho}(\beta) \cdot \omega(u).$$

Proof. Since for all β in $\pi_1(C \setminus \Sigma, O)$, we have $P(\beta \cdot u) = P(u)$, (P is the covering map and $\beta \cdot u$ is the natural action of $\pi_1(C \setminus \Sigma, O)$ in the covering space U we get $J(P(\beta \cdot u)) = J(P(u))$. Since $j \circ \omega = J \circ P$ we get $j(\omega(\beta \cdot u)) = j(\omega(u))$. Therefore, there exists σ_β in $PSL_2(\mathbb{Z})$ such that $\omega(\beta \cdot u) = \sigma_\beta \cdot \omega(u)$, and σ_β is uniquely determined by β since $PSL_2(\mathbb{Z})$ acts freely on $\overline{\mathbb{H}} \setminus j^{-1}(\{0, 1, \infty\})$, we take $\bar{\rho}(\beta) = \sigma_\beta$. Now $\omega((\beta \cdot \gamma) \cdot u) = \sigma_\beta \cdot \omega(\gamma \cdot u) = \sigma_\beta \cdot \sigma_\gamma \cdot \omega(u)$ shows that $\bar{\rho}$ is a group homomorphism. \square

Proposition 2.5.3. *([3] Proposition 2.2) The group $\Gamma := \bar{\rho}(\pi_1(C \setminus \Sigma, O))$ has finite index in $PSL_2(\mathbb{Z})$. Moreover, there exists a morphism $\bar{\omega} : C \longrightarrow \Gamma \backslash \overline{\mathbb{H}}$, such that the*

following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\bar{\omega}} & \Gamma \backslash \overline{\mathbb{H}} \\ & \searrow J & \downarrow J_\Gamma \\ & & \mathbb{P}^1 \end{array}$$

where J_Γ is the natural morphism defined in chapter 3 Page 45. In addition, if J is only ramified over $0, 1$ and ∞ with $e_a = 1$ or 3 when $J(a) = 0$ and $e_a = 1$ or 2 when $J(a) = 1$, then $\bar{\omega}$ is an isomorphism, in which the curve C can be viewed as a modular curve in this case.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\omega} & \overline{\mathbb{H}} \setminus j^{-1}(\{0, 1, \infty\}) \\ P \downarrow & & \downarrow j \\ C \setminus \Sigma & \xrightarrow{J} & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array}$$

For a fixed t in $C \setminus \Sigma$ and a fixed u_t in $P^{-1}(t)$ we have

$$P^{-1}(t) = \pi_1(C \setminus \Sigma).u_t = \{\beta.u_t | \beta \in \pi_1(C \setminus \Sigma)\}$$

and

$$\omega(P^{-1}(t)) = \{\omega(\beta.u_t) | \beta \in \pi_1(C \setminus \Sigma)\}.$$

By Proposition 2.4.1, we have

$$\omega(P^{-1}(t)) = \{\bar{\rho}(\beta).\omega(u_t) | \beta \in \pi_1(C \setminus \Sigma)\}.$$

Let $\Gamma = \bar{\rho}(\pi_1(C \setminus \Sigma))$, then $\omega(P^{-1}(t)) = \{\gamma.\omega(u_t) | \gamma \in \Gamma\} = \Gamma.\omega(u_t)$. Therefore we have a well defined holomorphic map;

$$\begin{aligned} \bar{\omega} : C \setminus \Sigma &\longrightarrow \Gamma \backslash \overline{\mathbb{H}} \\ t &\longmapsto \Gamma.\omega(u_t) \end{aligned}$$

The map $\bar{\omega}$ can be extended in a canonical manner to a morphism; $\bar{\omega} : C \longrightarrow \Gamma/\overline{\mathbb{H}}$ since C is an algebraic curve, and we denote the extension also by $\bar{\omega}$. Let t in $C \setminus \Sigma$, we have $J_\Gamma(\bar{\omega}(t)) = J_\Gamma(\Gamma.\omega(u_t)) = j(\mathrm{PSL}_2(\mathbb{Z}).\omega(u_t)) = j(\omega(t))$. By the above diagram we have $j(\omega(t)) = J(P(u_t)) = J(t)$ meaning that $J_\Gamma \circ \bar{\omega}$ and J coincide in $C \setminus \Sigma$ and hence they coincide in C since Σ is a finite set. Moreover, we have $\deg(J) = \deg(J_\Gamma).\deg(\bar{\omega})$. Since $\deg(J)$ is finite by assumption, it follows that $\deg(J_\Gamma)$ is also finite and therefore Γ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$. (Notice that $\deg(J_\Gamma) = [\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma]$, see Theorem 3.1.1 in chapter 3.) \square

2.5.2 The explicit construction

The main theorem is the following.

Theorem 2.5.4. (*[2], section 8, Page 580*)

Let $J : C \longrightarrow \mathbb{P}^1$ be a morphism of finite degree, $\Sigma = J^{-1}\{0, 1, \infty\}$ and

$$\bar{\rho} : \pi_1(C \setminus \Sigma, O) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$$

its monodromy representation introduced in Proposition 2.5.2 .

Then for any map $\rho : \pi_1(C \setminus \Sigma, O) \longrightarrow \mathrm{SL}_2(\mathbb{Z})$ such that the diagram:

$$\begin{array}{ccc} & & \mathrm{SL}_2(\mathbb{Z}) \\ & \nearrow \rho & \downarrow s \\ \pi_1(C \setminus \Sigma, O) & \xrightarrow{\bar{\rho}} & \mathrm{PSL}_2(\mathbb{Z}) \end{array}$$

is commutative, there exists a unique elliptic surface $\pi : X \longrightarrow C$, such that:

1. The functional invariant of $\pi : X \longrightarrow C$ is J .
2. The homological invariant of $\pi : X \longrightarrow C$ is ρ .
3. The singular fibers of $\pi : X \longrightarrow C$ lie over the finite set Σ .
4. The elliptic surface; $\pi : X \longrightarrow C$ is relatively minimal.

Proof. (Sketch). Set $\Sigma = J^{-1}\{0, 1, \infty\}$ and let $P : U \longrightarrow C \setminus \Sigma$ be the universal covering of $C \setminus \Sigma$ and $\rho : \pi_1(C \setminus \Sigma) \longrightarrow \mathrm{SL}_2(\mathbb{Z})$ be a lift of the map $\bar{\rho} : \pi_1(C \setminus \Sigma) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$. For β in $\pi_1(C \setminus \Sigma)$ set

$$\rho(\beta) = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We define the group $V = \{(\beta, n_1, n_2) | \beta \in \pi_1(C \setminus \Sigma), n_1, n_2 \in \mathbb{Z}\}$ where the product is given by

$$(\beta, n_1, n_2) \cdot (\gamma, m_1, m_2) = (\beta \cdot \gamma, a_\gamma \cdot n_1 + c_\gamma \cdot n_2 + m_1, b_\gamma \cdot n_1 + d_\gamma \cdot n_2 + m_2).$$

The unit of V is $(1, 0, 0)$ and the inverse is given by:

$$(\beta, n_1, n_2)^{-1} = (\beta^{-1}, c_\beta \cdot n_2 - d_\beta \cdot n_1, b_\beta \cdot n_1 - a_\beta \cdot n_2).$$

The group V acts on $U \times \mathbb{C}$ by:

$$(\beta, n_1, n_2) \cdot (u, z) = \left(\beta \cdot u, \frac{n_1 \cdot \omega(u) + n_2 + z}{c_\beta \cdot \omega(u) + d_\beta} \right).$$

It is easy to prove that the action is well defined and that if $v \in V$, $v \neq 1$ then v has no fixed point in $U \times \mathbb{C}$. Consequently the quotient; $X' = (U \times \mathbb{C})/V$ is a non-singular analytic surface. We denote by $[u, z]$ the orbit $V \cdot (u, z)$ in X' and we define the map

$$\begin{aligned} \pi' : X' &\longrightarrow C \setminus \Sigma \\ [u, z] &\longmapsto P(u), \end{aligned}$$

where P is the covering map $P : U \longrightarrow C \setminus \Sigma$. One can verify that π' is well-defined and holomorphic. Now let $t \in C \setminus \Sigma$ and choose $u_t \in U$ such that $P(u_t) = t$, then $\pi'^{-1}(t) = \{[\beta \cdot u_t, z] | \beta \in \pi_1(C \setminus \Sigma), z \in \mathbb{C}\} = \{[u_t, z] \in X' | z \in \mathbb{C}\}$. The last equality follows from the equivalence between (u_t, z) and $(\beta \cdot u_t, (\beta, n_1, n_2) \cdot (u_t, z))$. The fiber $\pi'^{-1}(t)$ is obviously an abelian group since $[u_t, z] + [u_t, z'] = [u_t, z + z']$.

Let

$$\begin{aligned} h_t : \pi'^{-1}(t) &\longrightarrow \mathbb{C}/\omega(u_t) \cdot \mathbb{Z} \oplus \mathbb{Z} \\ [u_t, z] &\longmapsto [z]. \end{aligned}$$

It is easy to check that h_t is an isomorphism of groups, and one can check that it is a biholomorphic map. Therefore for any t in $C \setminus \Sigma$ the fiber $\pi'^{-1}(t)$ is an elliptic curve. A section to π' is simply given by

$$\begin{aligned} \sigma'_0 : C \setminus \Sigma &\longrightarrow X' \\ t &\longmapsto [u_t, o] \end{aligned}$$

It can be shown ([2], section 8) that there exists a unique algebraic surface X and maps $\pi : X \longrightarrow C$, $\sigma_0 : C \longrightarrow X$ such that

1. $X' \subseteq X$
2. $\pi : X \longrightarrow C$ is a relatively minimal elliptic surface with section $\sigma_0 : C \longrightarrow X$
3. the restriction of π to X' is the map $\pi' : X' \longrightarrow C \setminus \Sigma$, and the restriction of σ_0 to $C \setminus \Sigma$ is the map $\sigma'_0 : C \setminus \Sigma \longrightarrow X'$.

In other words X is obtained from X' by adding the singular fibers which lie over Σ . Finally by definition of J and ρ , it is clear that they are respectively the functional and the homological invariants of $\pi : X \longrightarrow C$. □

2.6 The numerical invariants of an elliptic surface

For the numerical invariants of an elliptic surface, the major theorems are

Theorem 2.6.1. (*[2], III, page 14*) *Let $\pi : X \rightarrow C$ be an elliptic surface then we have the following:*

1. *The canonical class, K_X of X satisfies $K_X^2 = 0$.*
2. *The irregularity q of X is equal to the genus g of the curve C .*

Theorem 2.6.2. (*[2], III Theorem 12.2*) *Let $\pi : X \rightarrow C$ be an elliptic surface and denote by μ the degree of the functional invariant J . Then the Euler characteristic $e(X)$ of X is given by :*

$$e(X) = \sum_{v \in \Sigma} e(C_v) =$$

$$\mu + 6 \cdot \sum_{b \geq 0} \nu(I_b^*) + 2\nu(II) + 10\nu(II^*) + 3\nu(III) + 9\nu(III^*) + 4\nu(IV) + 8\nu(IV^*)$$

where $\nu(T)$ is the number of singular fibers of type T .

As a consequence we have the following:

Corollary 2.6.3. *The numerical invariants of an elliptic surface satisfy:*

$$b_1 = 2g \tag{16}$$

$$b_2 = e(X) + 4g - 2 \tag{17}$$

$$P_a = 1/12 e(X) \tag{18}$$

$$P_g = 1/12 e(X) + g - 1 \tag{19}$$

$$h^{1,1} = 5/6 e(X) + 2g. \tag{20}$$

Where b_1 and b_2 are the Betti numbers, and P_a and P_g are the arithmetic and the geometric genera.

Proof. By (10) (section 1.5.4) we have $b_1 = 2q$, and by Theorem 2.6.1 we have $q = g$, so this proves (16). Now (17) follows from (6 section 1.5.4), and (18) is a consequence of Noether formula (Theorem 1.5.2) combined with the equality $K_X^2 = 0$ (Theorem 2.6.1). Finally (19) and (20) follow from (10) and (11) (section 1.5.4). □

The previous corollary shows that to compute the numerical invariants of an elliptic surface it suffices to compute $e(X)$ and g . Moreover, $e(X)$ is related to the singular fibers of X (by Theorem 2.6.2). We recall also the Shioda-Tate formula given in section 2.3 namely the Picard number ρ and the Mordell-Weil rank r are related by the following formula:

$$\rho = r + 2 + \sum_{v \in \Sigma} (m_v - 1),$$

where m_v is the number of irreducible components of the singular fiber C_v .

Theorem 2.6.4. ([7]) *Let t_1 denotes the number of singular fibers of type I_b , $b \geq 1$ (in other words, $t_1 = \sum_{b \geq 1} \nu(I_b)$), and t the total number of singular fibers. Put $l = 4g + 2t - t_1 - 4$. Then we have the following :*

1. $l = b_2 - \rho + r$
2. $r \leq l - 2P_g$
3. $\nu(I_0^*) + \nu(II) + \nu(III) + \nu(IV) \leq l - 2P_g$

Proof. We shall prove 1 and 2 of the theorem. For 3 see ([7], Proposition 2.8). By Theorem 2.6.2 we have $e(X) = \sum_{v \in \Sigma} e(C_v)$ and we know by proposition 2.2.3 that

$$e(C_v) = \begin{cases} m_v & \text{if } C_v \text{ is of type } I_b, \\ m_v + 1 & \text{otherwise.} \end{cases}$$

Then if we denote by Σ_1 the subset of Σ such that C_v is of type I_b , and by Σ' its

complement in Σ , we get

$$\begin{aligned}
e(X) &= \sum_{v \in \Sigma_1} m_v + \sum_{v \in \Sigma'} (m_v + 1) \\
&= \sum_{v \in \Sigma_1} (m_v - 1) + t_1 + \sum_{v \in \Sigma'} (m_v - 1) + 2(t - t_1) \\
&= \sum_{v \in \Sigma} (m_v - 1) + 2t - t_1.
\end{aligned}$$

Therefore, using the Shioda-Tate formula, we get

$$e(X) = \rho - r - 2 + 2t - t_1.$$

Since $b_2 = e(X) + 4g - 2$, we have the desired formula. This proves the first assertion.

Now, since $b_2 = h^{1,1} + 2P_g$ we have $r = l - 2P_g + (\rho - h^{1,1})$ and we know that $h^{1,1} \geq \rho$, therefore $r \leq l - 2P_g$, and this proves the second assertion.

□

Chapter 3

Modular Elliptic Surfaces

In this chapter we will give an exposition of the theory of modular elliptic surfaces due to Shioda. These are elliptic surfaces over modular curves and their geometry is encoded in the modular group associated to the modular curve. Their properties are simpler than those of general elliptic surfaces. For instance, the Mordell-Weil rank for a modular elliptic surface vanishes, and the only singular fibers are of type I_b , I_b^* or IV^* .

3.1 Modular elliptic surfaces

3.1.1 Modular Curves

Throughout this section we assume that Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Let $P\Gamma = s(\Gamma)$, where $s : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{PSL}_2(\mathbb{Z})$ is the natural surjection.

A point $z \in \overline{\mathbb{H}}$ is called a cusp (resp. an elliptic) point for $P\Gamma$ if it is fixed by a parabolic (resp. an elliptic) element $\gamma \in P\Gamma$ and $\gamma \neq I$.

Let $r \in \mathbb{Q} \cup \{\infty\}$ be a cusp for $P\Gamma$. The stabilizer $P\Gamma_r$ of r in $P\Gamma$, is a subgroup of finite index of the stabilizer $\mathrm{PSL}_2(\mathbb{Z})_r$ of r in $\mathrm{PSL}_2(\mathbb{Z})$. The index of $P\Gamma_r$ in $\mathrm{PSL}_2(\mathbb{Z})_r$ is called the cusp width of r .

If $z \in \mathbb{H}$ is an elliptic point for $P\Gamma$, then $P\Gamma_z$ is a subgroup of $PSL_2(\mathbb{Z})_z$. Meanwhile, $PSL_2(\mathbb{Z})_z$ is conjugate to either $\langle S \rangle = \{I, S\}$ or $\langle ST \rangle = \{I, ST, (ST)^2\}$ in $PSL_2(\mathbb{Z})$, hence $PSL_2(\mathbb{Z})_z$ has order 2 or 3. Since $P\Gamma_z$ is non trivial then $P\Gamma_z = PSL_2(\mathbb{Z})_z$. Two $P\Gamma$ -cusp (resp. $P\Gamma$ -elliptic) points z and z' are $P\Gamma$ -equivalent if there exists $\gamma \in P\Gamma$ such that $\gamma \cdot z = z'$.

An element α in the quotient $P\Gamma \backslash \overline{\mathbb{H}}$ is said to be a cusp (resp. an elliptic point) if one (therefore all) of its representatives in $\overline{\mathbb{H}}$ is a $P\Gamma$ -cusp (resp. a $P\Gamma$ -elliptic) point. The quotient $P\Gamma \backslash \overline{\mathbb{H}}$ is called the modular curve associated with Γ . We have the following result:

Theorem 3.1.1. (*[6], pages 19 and 20*) *We have*

1. *The quotient $P\Gamma \backslash \overline{\mathbb{H}}$ is a compact Riemann surface.*
2. *The natural map*

$$\begin{aligned} f_\Gamma : P\Gamma \backslash \overline{\mathbb{H}} &\longrightarrow PSL_2(\mathbb{Z}) \backslash \overline{\mathbb{H}} \\ P\Gamma \cdot z &\longmapsto PSL_2(\mathbb{Z}) \cdot z \end{aligned}$$

is a holomorphic map, and $\deg f_\Gamma = [PSL_2(\mathbb{Z}) : P\Gamma]$.

3. *Let $p \in PSL_2(\mathbb{Z}) \backslash \overline{\mathbb{H}}$, and let $\alpha \in f_\Gamma^{-1}(p)$. One can choose $z \in \overline{\mathbb{H}}$ such that $\alpha = P\Gamma \cdot z$, and $p = PSL_2(\mathbb{Z}) \cdot z$. Then the ramification index of f_Γ at α is given by*

$$e_{f_\Gamma}(\alpha) = [PSL_2(\mathbb{Z})_z : P\Gamma_z].$$

Recall that the classical modular function j introduced in the first chapter defines an isomorphism between $PSL_2(\mathbb{Z}) \backslash \overline{\mathbb{H}}$ and \mathbb{P}^1 . By composing f_Γ with j we get a holomorphic function $J_\Gamma = j \circ f_\Gamma : P\Gamma \backslash \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1$, and $\deg(J_\Gamma) = \deg(f_\Gamma) = [PSL_2(\mathbb{Z}) : P\Gamma]$.

Let $\alpha \in P\Gamma \backslash \overline{\mathbb{H}}$. We know that $e_{J_\Gamma}(\alpha) = e_{f_\Gamma}(\alpha) \cdot e_j(f_\Gamma(\alpha))$, and since j is non-ramified, we have $e_{J_\Gamma}(\alpha) = e_{f_\Gamma}(\alpha)$. Let z be a representative of α in $\overline{\mathbb{H}}$ mod $P\Gamma$. By the above theorem, we have

$$e_{J_\Gamma}(\alpha) = [PSL_2(\mathbb{Z})_z : P\Gamma_z].$$

There are three cases:

1. If z is neither a $\mathrm{PSL}_2(\mathbb{Z})$ -cusp nor a $\mathrm{PSL}_2(\mathbb{Z})$ -elliptic point then $\mathrm{PSL}_2(\mathbb{Z})_z = \{I\}$, and hence $e_{J_\Gamma}(\alpha) = 1$.
2. If z is a cusp then $J_\Gamma(\alpha) = j(z) = \infty$ and therefore α is a pole of J_Γ and $e_{J_\Gamma}(\alpha) = \mathrm{ord}_{J_\Gamma}(\alpha) = [\mathrm{PSL}_2(\mathbb{Z})_z : P\Gamma_z]$.
3. If z is a $\mathrm{PSL}_2(\mathbb{Z})$ -elliptic point then $J_\Gamma(\alpha) = j(z) = 1$ if $z \sim i$ and $J_\Gamma(\alpha) = j(z) = 0$ if $z \sim \rho$. Here we have two subcases:
 - (a) If z is a $P\Gamma$ -elliptic point then $P\Gamma_z = \mathrm{PSL}_2(\mathbb{Z})_z$ so $e_{J_\Gamma}(\alpha) = 1$.
 - (b) If z is not a $P\Gamma$ -elliptic point then $P\Gamma_z = \{I_d\}$ so $e_{J_\Gamma}(\alpha) = 2$ if $z \sim i$ and $e_{J_\Gamma}(\alpha) = 3$ if $z \sim \rho$.

We have proved the following result:

Proposition 3.1.2. *The map $J_\Gamma : P\Gamma \setminus \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1$ is only ramified over $0, 1$ and ∞ , and we have*

1. *If $J_\Gamma(\alpha) = \infty$ then α is a cusp point and $e_{J_\Gamma}(\alpha) = \mathrm{ord}_{J_\Gamma}(\alpha)$*
2. *If $J_\Gamma(\alpha) = 0$ then $e_{J_\Gamma}(\alpha) = 1$ if α is an elliptic point, and $e_{J_\Gamma}(\alpha) = 3$ otherwise.*
3. *If $J_\Gamma(\alpha) = 1$ then $e_{J_\Gamma}(\alpha) = 1$ if α is an elliptic point, and $e_{J_\Gamma}(\alpha) = 2$ otherwise.*

The following theorem provides a formula for computing the genus of the modular curve $P\Gamma \setminus \overline{\mathbb{H}}$. We denote by μ the index of $P\Gamma$ in $\mathrm{PSL}_2(\mathbb{Z})$, s_1, s_2 the number of elliptic points in $P\Gamma \setminus \overline{\mathbb{H}}$ of order 2 and 3 respectively, and t' the number of cusps in $P\Gamma \setminus \overline{\mathbb{H}}$.

Theorem 3.1.3. *([6], Proposition 1.40) The genus g of $P\Gamma \setminus \overline{\mathbb{H}}$ is given by the following formula:*

$$g = 1 + \mu/12 - t'/2 - s_2/4 - s_3/3. \quad (21)$$

Proof. Since the genus of \mathbb{P}^1 is 0, the Hurwitz formula for the map $J_\Gamma : P\Gamma \setminus \overline{\mathbb{H}} \rightarrow \mathbb{P}^1$ yields

$$2g - 2 = -2\deg(J_\Gamma) + \sum_{\alpha \in P\Gamma \setminus \overline{\mathbb{H}}} (e_{J_\Gamma}(\alpha) - 1),$$

and since J_Γ is ramified over 0, 1 and ∞ only, we have

$$2g - 2 = -2\mu + \sum_{\alpha \in J_\Gamma^{-1}(0)} (e_\alpha - 1) + \sum_{\alpha \in J_\Gamma^{-1}(1)} (e_\alpha - 1) + \sum_{\alpha \in J_\Gamma^{-1}(\infty)} (e_\alpha - 1). \quad (22)$$

where $\mu = \deg(J_\Gamma)$ and $e_\alpha = e_{J_\Gamma}(\alpha)$. Let $n = |J_\Gamma^{-1}(0)|$. We know that $\mu = \sum_{\alpha \in J_\Gamma^{-1}(0)} e_\alpha$ and therefore

$$\sum_{\alpha \in J_\Gamma^{-1}(0)} (e_\alpha - 1) = \mu - n. \quad (23)$$

Let ϵ_3 be the subset of $J_\Gamma^{-1}(0)$, consisting of elliptic points of order 3, by our notation $|\epsilon_3| = s_3$ and by the above proposition, $e_\alpha = 1$ when $\alpha \in \epsilon_3$ and $e_\alpha = 3$ when $\alpha \in J_\Gamma^{-1}(0) \setminus \epsilon_3$. Therefore

$$\sum_{\alpha \in J_\Gamma^{-1}(0)} (e_\alpha - 1) = \sum_{\alpha \in J_\Gamma^{-1}(0) \setminus \epsilon_3} (e_\alpha - 1) = 2(n - s_3).$$

If we compare this with (23), we have $n = \mu/3 + 2s_3/3$, and by substituting this expression for n in (23), we get

$$\sum_{\alpha \in J_\Gamma^{-1}(0)} (e_\alpha - 1) = 2(\mu - s_3)/3.$$

Similarly, we have

$$\sum_{\alpha \in J_\Gamma^{-1}(1)} (e_\alpha - 1) = (\mu - s_2)/2 \quad \text{and} \quad \sum_{\alpha \in J_\Gamma^{-1}(\infty)} (e_\alpha - 1) = \mu - t'.$$

Taking these into formula (22) proves the theorem. \square

Throughout this chapter, we will assume that the group Γ acts effectively on $\overline{\mathbb{H}}$. In other words, $-I \notin \Gamma$.

By this assumption the canonical surjection $s : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$ induces an isomorphism between Γ and $P\Gamma$. Since there are no elliptic elements in $\mathrm{SL}_2(\mathbb{Z})$ of order 2, the curve $P\Gamma \backslash \overline{\mathbb{H}}$ has no elliptic points of order 2 (i.e. $s_2 = 0$).

For an elliptic point $\alpha \in P\Gamma \backslash \overline{\mathbb{H}}$, the stabilizer $P\Gamma_z$ of a representative z of α in \mathbb{H} has order 3, and it is conjugate to $\langle ST \rangle$ in $\mathrm{PSL}_2(\mathbb{Z})$. Let Γ_z be the stabilizer of z in Γ , then Γ_z is conjugate to either $\langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ or $\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \rangle$ in $\mathrm{SL}_2(\mathbb{Z})$, (since Γ_z has order 3 and $-I \notin \Gamma_z$).

Similarly if $\alpha \in P\Gamma \backslash \overline{\mathbb{H}}$ is a cusp and if $r \in \mathbb{Q} \cup \{\infty\}$ is a representative of α , then Γ_r is conjugate to either $\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \rangle$ or $\langle \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} \rangle$ in $\mathrm{SL}_2(\mathbb{Z})$, where $b = \mathrm{ord}_{J_\Gamma}(\alpha) \geq 1$.

We call α a cusp of the first (resp. second) kind if Γ_r is conjugate to $\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \rangle$ (resp. $\langle \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} \rangle$) in $\mathrm{SL}_2(\mathbb{Z})$. Equivalently, if σ_r is the generator of Γ_r , then it is easy to see that α is of the first kind if $\mathrm{tr}(\sigma_r) = 2$ and it is of the second kind if $\mathrm{tr}(\sigma_r) = -2$.

3.1.2 Construction of modular elliptic surfaces

In the following, we provide a procedure of constructing modular elliptic surfaces. Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ such that $-I \notin \Gamma$. Let $J_\Gamma : P\Gamma \backslash \overline{\mathbb{H}} \longrightarrow \mathbb{P}^1$ be the natural map defined in the previous section, $\Sigma = J_\Gamma^{-1}\{0, 1, \infty\}$ and $\bar{\rho} : \pi_1(C_\Gamma \backslash \Sigma) \longrightarrow \mathrm{PSL}_2(\mathbb{Z})$ be the representation given in Proposition 2.5.2. If $\Lambda = \bar{\rho}(\pi_1(C_\Gamma \backslash \Sigma))$, then by proposition 2.5.3 there exists an isomorphism $\bar{\omega} : P\Gamma \backslash \overline{\mathbb{H}} \longrightarrow$

$\Lambda \backslash \overline{\mathbb{H}}$ such that the following diagram commutes

$$\begin{array}{ccc} P\Gamma \backslash \overline{\mathbb{H}} & \xrightarrow{\bar{\omega}} & \Lambda \backslash \overline{\mathbb{H}} \\ & \searrow J_\Gamma & \downarrow J_\Lambda \\ & & \mathbb{P}^1 \end{array}$$

It follows that the subgroups $P\Gamma$ and Λ are conjugate in $\mathrm{PSL}_2(\mathbb{Z})$, hence there exists $g \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\Lambda = g^{-1} \cdot P\Gamma \cdot g$. By composing with the map

$$\begin{aligned} \Lambda &\longrightarrow P\Gamma \\ \lambda &\longmapsto g \cdot \lambda \cdot g^{-1} \end{aligned}$$

we may assume that $\bar{\rho}(\pi_1(C_\Gamma \backslash \Sigma)) = P\Gamma$. Now the isomorphism $s : \Gamma \longrightarrow P\Gamma$ gives us a natural lift ; $\rho = s^{-1} \circ \bar{\rho} : \pi_1(C_\Gamma \backslash \Sigma) \longrightarrow \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and the following diagram is obviously commutative

$$\begin{array}{ccc} & & \Gamma \\ & \nearrow \rho & \downarrow s \\ \pi_1(C \backslash \Sigma) & \xrightarrow{\bar{\rho}} & P\Gamma \end{array}$$

Following Kodaira (Theorem 2.5.4), there exists a unique elliptic surface over $P\Gamma \backslash \overline{\mathbb{H}}$ with functional invariant J_Γ and homological invariant ρ . This elliptic surface is called the modular elliptic surface associated with the subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, and we denote it by X_Γ .

3.2 Singular fibers, Numerical invariants and the Mordell-Weil group.

3.2.1 Singular fibers

The following theorem enumerates the possible singularities for a modular elliptic surface.

Theorem 3.2.1. ([7], Proposition 4.2). Let $X_\Gamma \rightarrow C_\Gamma$, where $C_\Gamma = P\Gamma \backslash \overline{\mathbb{H}}$, be a modular elliptic surface. Then the only possible singular fibers are of types I_b, I_b^* or IV^* which lie over the cusps and the elliptic points of C_Γ . Let t_1 (resp. t_2) the number of cusps of the first (resp. second) kind and s the number of elliptic points then

$$\sum_{b \geq 1} \nu(I_b) = t_1, \quad \sum_{b \geq 1} \nu(I_b^*) = t_2, \quad \nu(IV^*) = s.$$

Proof. This proof uses frequently the table of Theorem 2.4.3. First we prove that the fibers C_v such that v is a cusp or an elliptic point are singular.

If v is an elliptic point in $P\Gamma \backslash \overline{\mathbb{H}}$, by Proposition 3.1.2 we have $J_\Gamma(v) = 0$ and $e_v = 1$. Hence, by table 2.4.3, C_v is a singular fiber of one of the types II or IV^* . In the case that C_v is of type II we know, by table 2.4.3, that $[\rho(\beta_v)] = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Therefore there exists $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $\rho(\beta_v) = g^{-1} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot g$. It follows that $\rho(\beta_v)^3 = -I$ and since $\rho(\beta_v) \in \Gamma$ we have $-I \in \Gamma$, which is a contradiction. Therefore C_v is of type IV^* .

If v is a cusp of the first kind then $[\rho(\beta_v)] = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, so C_v is a singular fiber of type I_b , while if it is of the second kind then $[\rho(\beta_v)] = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$, so C_v is a singular fiber of type I_b^* .

Conversely, let v be a point in $\Gamma \backslash \overline{\mathbb{H}}$ such that C_v is a singular fiber. Then we have five cases:

1. C_v of type I_0^* : In this case $[\rho(\beta_v)] = -I \in \Gamma$ which is impossible.
2. $C_v \in \{III, III^*\}$: We get $[\rho(\beta_v)] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, in which case $\rho(\beta_v)^2 = -I \in \Gamma$ which is impossible.
3. $C_v \in \{II^*, IV\}$: we have $e_v \equiv 2[3]$ and $J(v) = 0$. This is a contradiction since, by Proposition 3.1.2, $e_v = 1$ or 3 when $J_\Gamma(v) = 0$.

4. $C_v \in \{II, IV^*\}$: In this case C_v cannot be II , otherwise $-I \in \Gamma$ and hence C_v is of type IV^* in this case $J_\Gamma(v) = 0$ and $e_v \equiv 1[3]$, and by Proposition 3.1.2 v is an elliptic point.
5. $C_v \in \{I_b, I_b^*\}$: we have $J_\Gamma(v) = \infty$ and by Proposition 3.1.2, v is a cusp. If C_v is of type I_b then $[\rho(\beta_v)] = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ in which case v is a cusp of the first kind, and if it is of type I_b^* , then $[\rho(\beta_v)] = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$ in which case v is a cusp of the second kind.

This completes the proof. □

The table of Theorem 2.4.3 reduces to the following:

C_t	$[\rho(\beta_t)]$	e_t	Behavior of J at t
I_0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$e_t \equiv 0[3]$ or $e_t \equiv 0[2]$	Regular
I_b	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$e_t = b$	pole of order b
I_b^*	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$		
IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$e_t = 1$	$J(t) = 0$

Table 3

3.2.2 Numerical invariants

Recall that the genus of the curve $P\Gamma \setminus \overline{\mathbb{H}}$ is given by the formula:

$$g = 1 + \mu/12 + (t_1 + t_2)/2 + s/3 \quad (24)$$

where $\mu = \deg J_\Gamma = [P\Gamma : \text{PSL}_2(\mathbb{Z})]$ and t_1, t_2 are respectively the number of cusps of the first and second kind, and s denotes the number of elliptic points. By Theorem

2.6.2, we have

$$e(X) = \mu + 6 \sum_{b \geq 1} \nu(I_b^*) + 8\nu(IV^*),$$

and by Theorem 3.3.1, we have

$$e(X) = \mu + 6t_2 + 8s. \tag{25}$$

We have also seen the formulae (Corollary 2.6.3):

$$\begin{aligned} b_1 &= 2g \\ b_2 &= e(X) + 4g - 2 \\ P_a &= 1/12 e(X) \\ P_g &= 1/12 e(X) + g - 1 \\ h^{1,1} &= 5/6 e(X) + 2g. \end{aligned}$$

Therefore, we can compute all the numerical invariants of a modular elliptic surface X_Γ if we know the characteristics t_1 , t_2 , s and μ of the group Γ .

Theorem 3.2.2. (*[7],*) *Let X_Γ be a modular elliptic surface, then the Mordell-Weil rank r vanishes and the Picard number ρ is equal to $h^{1,1}$.*

Proof. By Theorem 2.6.4 we have $l = b_2 - \rho + r$ and $r \leq l - 2P_g$, where $l = 4g + 2t - t_1 - 4$ and t being the total number of singular fibers (the number of elliptic points and cusps in this case) and t_1 is the number of singular fibers of type I_b , $b \geq 1$ (the number of cusps of the first kind). To prove that $r = 0$ it suffices to prove that $l = 2P_g$.

By Corollary 2.6.3, we have $2P_g = 1/6 e(X) + 2g - 2$, but $e(X) = \mu + 6t_2 + 8s$. Therefore, $2P_g = \mu/6 + t_2 + 4s/3 + 2g - 2$. Using the genus formula, we have $\mu/6 = 2s/3 + t_1 + t_2 + 2g - 2$, and hence $2P_g = 4g + 2t - t_1 - 4 = l$ which shows that $r = 0$. Now, replacing r and l in the formula $l = b_2 - \rho + r$ yields $2P_g = b_2 - \rho$, and since $2P_g = b_2 - h^{1,1}$, we have $\rho = h^{1,1}$. \square

As a consequence of the above theorem we have $r = 0$.

Therefore by Theorem 2.3.2, the Mordell-Weil group of a modular elliptic surface X_Γ is a finite group of the form

$$\mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \quad e_2 | e_1 \quad \text{and} \quad e_2 \geq 1$$

In the following we give a more precise result on the Mordell-Weil group.

Theorem 3.2.3. (*[7], Theorem 5.2*) *Let X_Γ be the elliptic modular surface attached to Γ , and $S(X_\Gamma)$ the Mordell-Weil group of X_Γ .*

1. *If Γ has torsion (i.e. $s > 0$) then $S(X_\Gamma)$ is either trivial or a cyclic group of order 3.*
2. *If Γ has a cusp of the second kind ($t_2 > 0$) then $S(X_\Gamma)$ is either trivial or isomorphic to one of the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, \mathbb{Z}_2 or \mathbb{Z}_4 .*
3. *If Γ is torsion free and all cusps are of the first kind, then $S(X_\Gamma)$ is isomorphic to a subgroup of $\mathbb{Z}_n \oplus \mathbb{Z}_n$, where n denotes the least common multiple of the b_i (here the singular fibers of $S(X_\Gamma)$ are of types I_{b_i} , $1 \leq i \leq t_1$)*

3.2.3 A particular case of modular elliptic surfaces

In the following we will see that modular elliptic surfaces associated with torsion free subgroups of $\mathrm{SL}_2(\mathbb{Z})$, can be realized in a more concrete way. In fact, when Γ is torsion free the modular curve $P\Gamma \setminus \overline{\mathbb{H}}$ has no elliptic points. If we denote by Σ the finite set of cusps of $P\Gamma \setminus \overline{\mathbb{H}}$, then $(P\Gamma \setminus \overline{\mathbb{H}}) \setminus \Sigma = P\Gamma \setminus \mathbb{H}$. Since Γ acts freely on \mathbb{H} , the natural map

$$\begin{aligned} P : \mathbb{H} &\longrightarrow P\Gamma \setminus \mathbb{H} \\ h &\longmapsto \Gamma h \end{aligned}$$

is a covering of $(P\Gamma \setminus \overline{\mathbb{H}}) \setminus \Sigma = P\Gamma \setminus \mathbb{H}$. Since \mathbb{H} is simply connected, \mathbb{H} is the universal covering of $P\Gamma \setminus \mathbb{H}$. The fundamental group $\pi_1(P\Gamma \setminus \mathbb{H})$ of $P\Gamma \setminus \mathbb{H}$ is isomorphic to the group of automorphisms of \mathbb{H} which commute with P , but the latter is just Γ .

Therefore we can identify $\pi_1(P\Gamma \backslash \mathbb{H})$ with Γ . Now the period map ω defined in section 2.5.1 is just the identity map of \mathbb{H} .

The group V introduced in the proof of Theorem 2.5.4 is simply given by

$$V = \{(\gamma, n_1, n_2) | \gamma \in \Gamma, n_1, n_2 \in \mathbb{Z}\},$$

and its action on $\mathbb{H} \times \mathbb{C}$ takes the form :

$$(\gamma, n_1, n_2)(h, z) = \left(\gamma.h, \frac{n_1.h + n_2 + z}{c_\gamma.h + d_\gamma} \right),$$

where $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$.

Take the quotient $(\mathbb{H} \times \mathbb{C})/V$, and consider the map

$$\begin{aligned} \pi' : (\mathbb{H} \times \mathbb{C})/V &\longrightarrow P\Gamma \backslash \overline{\mathbb{H}} \\ [h, z] &\longmapsto P(h) = \Gamma h \end{aligned}$$

It is well-defined and holomorphic, and $\pi'^{-1}(\Gamma h) = \{[h, z] : z \in \mathbb{C}\}$ is naturally an abelian group. The map

$$\begin{aligned} \delta_h : \pi'^{-1}(\Gamma h) &\longrightarrow \mathbb{C}/h\mathbb{Z} \oplus \mathbb{Z} \\ [h, z] &\longmapsto [z] \end{aligned}$$

is biholomorphic and is an isomorphism of groups, hence $\pi'^{-1}(\Gamma.h)$ is an elliptic curve. A section to π' is given by :

$$\begin{aligned} \sigma'_0 : P\Gamma \backslash \overline{\mathbb{H}} &\longrightarrow (\mathbb{H} \times \mathbb{C})/V \\ \Gamma.h &\longmapsto [h, 0]. \end{aligned}$$

Finally one can complete the analytic surface $(\mathbb{H} \times \mathbb{C})/V$ by adding the singular fibers which lie over the finite set of cusps of $P\Gamma \backslash \overline{\mathbb{H}}$, and this gives us an algebraic surface X_Γ together with a map $\pi : X_\Gamma \longrightarrow P\Gamma \backslash \overline{\mathbb{H}}$ such that the restriction of π to $(\mathbb{H} \times \mathbb{C})/V$ is π' . The singular fibers of X_Γ will be of the type I_b or I_{b^*} , $b \geq 1$. Let $S(X_\Gamma)$ the Mordell-Weil group of X_Γ . A section $\sigma \in S(X_\Gamma)$ is uniquely determined by its restriction; $\sigma' : P\Gamma \backslash \overline{\mathbb{H}} \longrightarrow (\mathbb{H} \times \mathbb{C})/V$. The following proposition will be very useful for the computation of $S(X_\Gamma)$ for some examples.

Proposition 3.2.4. (*[7], page 39*) Let Γ be a torsion free subgroup of $SL_2(\mathbb{Z})$ of finite index. If $\sigma : P\Gamma \backslash \overline{\mathbb{H}} \rightarrow X_\Gamma$ is a section and σ' is its restriction to $P\Gamma \backslash \mathbb{H}$, then σ' has the form, $\sigma'(\Gamma.h) = [h, a_1h + a_2]$ where $a_1, a_2 \in \mathbb{Q}$ with the property : $(a_1, a_2).(\gamma - I) \in \mathbb{Z} \oplus \mathbb{Z}$ for all $\gamma \in \Gamma$.

Proof. The section σ' acts on cosets by $\sigma'(\Gamma.h) = [h, \zeta(h)]$, where $\zeta(h) : \mathbb{H} \rightarrow \mathbb{C}$ is a holomorphic function. Since obviously $\sigma'(\Gamma(\gamma.h)) = \sigma'(\Gamma.h)$, then $(h, \zeta(h)) \sim (\gamma.h, \zeta(\gamma.h)) \pmod{V}$. That is, there exists $(\lambda, n_1, n_2) \in V$ such that

$$(\gamma.h, \zeta(\gamma.h)) = (\lambda, n_1, n_2).(h, \zeta(\gamma.h)) = (\lambda.h, \frac{n_1.h + n_2 + \zeta(h)}{c_\lambda.h + d_\lambda}).$$

It follows that $\gamma.h = \lambda.h$. In other words, $\lambda^{-1}.\gamma$ fixes h . Since Γ is torsion-free, it has no fixed points on \mathbb{H} for nonidentity elements. Therefore $\lambda^{-1}.\gamma = I$, that is, $\lambda = \gamma$. Consequently:

$$\zeta(\gamma.h) = \frac{n_1.h + n_2 + \zeta(h)}{c_\gamma.h + d_\gamma} \quad \text{for all } \gamma \in \Gamma. \quad (26)$$

We know that $S(X_\Gamma)$ is finite and hence σ has finite order $m \geq 1$. Therefore, for all $h \in \mathbb{H}$, $\sigma'(\Gamma.h)$ is such that $m.\sigma'(\Gamma.h) = \sigma'_0(\Gamma.h)$ in the elliptic curve $(C_h, \sigma_0(h))$, that is, $[h, m.\zeta(h)] = [h, 0]$. Since $(C_h, \sigma_0(h))$ is identified with $\mathbb{C}/h.\mathbb{Z} \oplus \mathbb{Z}$, $[h, m.\zeta(h)]$ corresponds to $[m.\zeta(h)]$ and $[h, 0]$ corresponds to $[0]$. We conclude that $m.\zeta(h)$ and 0 are identified under the action of $h.\mathbb{Z} \oplus \mathbb{Z}$ in \mathbb{C} . Thus, there exist $p, q \in \mathbb{Z}$ such that $m.\zeta(h) = p.h + q$ for all $h \in \mathbb{H}$, i.e.

$$\zeta(h) = (p/m).h + q/m \quad \text{for all } h \in \mathbb{H}. \quad (27)$$

It remains to show that $(p/m, q/m).(\gamma - I) \in \mathbb{Z} \oplus \mathbb{Z}$. We use (26) and (27) as follows:

$$\zeta(\gamma.h) = (p/m).(\gamma.h) + q/m = \frac{n_1.h + n_2 + \zeta(h)}{c_\gamma.h + d_\gamma} = \frac{n_1.h + n_2 + (p/m).h + q/m}{c_\gamma.h + d_\gamma}.$$

Therefore

$$(p/m).\frac{a_\gamma.h + b_\gamma}{c_\gamma.h + d_\gamma} + q/m = \frac{n_1.h + n_2 + (p/m).h + q/m}{c_\gamma.h + d_\gamma}.$$

Hence

$$(c_\gamma.h + (d_\gamma - 1)).(q/m) + ((a_\gamma - 1).h + b_\gamma).(p/m) = n_1.h + n_2 \quad \text{for all } h \in \mathbb{H}$$

This last equation can be written as:

$(p/m, q/m) \cdot \begin{pmatrix} a_\gamma - 1 & b_\gamma \\ c_\gamma & d_\gamma - 1 \end{pmatrix} \cdot \begin{pmatrix} h \\ 1 \end{pmatrix} = (n_1, n_2) \cdot \begin{pmatrix} h \\ 1 \end{pmatrix}$ for all $h \in \mathbb{H}$. This concludes the proof. \square

3.3 Examples

3.3.1 Example 1

The following example is due to Shioda ([7]). We define the principal congruence group of level n by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{n}, b \equiv c \equiv 0 \pmod{n} \right\}$$

The group $\Gamma(n)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $-I \notin \Gamma(n)$ for $n \geq 3$. Reduction mod n induces a group homomorphism $\varphi_n : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}_n)$. Clearly $\mathrm{Ker}\varphi_n = \Gamma(n)$. Therefore $\Gamma(n)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We denote the modular curve $P\Gamma(n) \backslash \overline{\mathbb{H}}$ by $C(n)$.

Theorem 3.3.1. ([6], Pages 20, 21, 22 and 23) *Let $n \geq 3$, then*

1. *The group $\Gamma(n)$ has no elliptic elements.*
2. *The cusps of $\Gamma(n)$ are all of the first kind. Moreover, if $r \in \mathbb{Q} \cup \{\infty\}$ is a cusp, then $\Gamma(n)_r$ is conjugate to $\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \rangle$ in $\mathrm{SL}_2(\mathbb{Z})$.*
3. *The map φ_n is surjective and the index of $P\Gamma(n)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is*

$$\mu(n) = \frac{n^3}{2} \prod_{p|n} (1 - p^{-2}). \quad (28)$$

4. *The number of cusps in $C(n)$ is*

$$t(n) = \mu(n)/n = \frac{n^2}{2} \prod_{p|n} (1 - p^{-2}). \quad (29)$$

Proof. Suppose that $\Gamma(n)$ has an elliptic element σ . Then σ is conjugate to one of the following elements in $\mathrm{SL}_2(\mathbb{Z})$:

$$\pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $\Gamma(n)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, then $\Gamma(n)$ must contain one of these which is impossible when $n \geq 3$. This proves 1 To show 2, let $r \in \mathbb{Q} \cup \{\infty\}$ be a cusp. Write $r = p/q$ with $\gcd(p, q) = 1$ and consider

$$\sigma_r = \begin{pmatrix} -npq + 1 & np^2 \\ -nq^2 & -npq + 1 \end{pmatrix}.$$

Then $\Gamma(n)_r = \langle \sigma_r \rangle$. Now, if $u, v \in \mathbb{Z}$ are such that $pu - qv = 1$, then $\lambda = \begin{pmatrix} u & -v \\ -q & p \end{pmatrix}$ sends r to ∞ and $\lambda \cdot \sigma_r \cdot \lambda^{-1} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Therefore $\Gamma(n)_r$ is conjugate to $\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \rangle$ in $\mathrm{SL}_2(\mathbb{Z})$ and hence α is a cusp of the first kind. This proves 2. The formulae in 3 and 4 are standard see ([6], page 22). We note that $\mu(n) = n \cdot t(n)$. \square

Now let $X(n)$ be the modular elliptic surface associated with $\Gamma(n)$. Since $C(n)$ has no elliptic points and all cusps are of the first kind and their stabilizers are conjugate to $\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \rangle$ in $\mathrm{SL}_2(\mathbb{Z})$, the only singular fibers of $X(n)$ are of type I_n , and their number is $\mu(n)/n$.

To compute the numerical invariants of $X(n)$ it suffices to compute the genus $g(n)$ of $C(n)$ and the Euler characteristic $e(X(n))$ of $X(n)$. We have

$$g_n = 1 + \frac{\mu(n)}{12} - \frac{t(n)}{2} = 1 + \left(\frac{n-6}{12n}\right)\mu(n) \quad (30)$$

and

$$e(X(n)) = \mu(n). \quad (31)$$

Hence by Corollary 2.6.3 we can compute all of the numerical invariants of $X(n)$.

Theorem 3.3.2. ([7], Theorem 5.5) Let $S(X(n))$ be the Mordell-Weil group of $X(n)$, then

$$S(X(n)) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

Proof. Since $\Gamma(n)$ is torsion free then by Theorem 3.2.3, $S(X(n))$ is isomorphic to a subgroup of $\mathbb{Z}_n \oplus \mathbb{Z}_n$, and so it suffice to prove that $|S(X(n))| = n^2$. By Proposition 3.2.4 for any $\sigma \in S(X(n))$ the restriction of σ to $P\Gamma(n) \setminus \mathbb{H}$ is given by $\sigma'([z]) = [z, r_1 z + r_2]$ with $r_1, r_2 \in \mathbb{Q}$ satisfying $(r_1, r_2)(\gamma - I) \in \mathbb{Z} \oplus \mathbb{Z}$ for all $\gamma \in \Gamma(n)$. In particular for $\gamma_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, we have $(r_1, r_2) \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \in \mathbb{Z} \oplus \mathbb{Z}$ and hence $r_1 = m_1/n$, $m_1 \in \mathbb{Z}$. Similarly, if we consider $\gamma_2 = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ we have $r_2 = m_2/n$, $m_2 \in \mathbb{Z}$. Therefore,

$$\sigma'([z]) = [z, \frac{m_1}{n}z + \frac{m_2}{n}], \quad m_1, m_2 \in \mathbb{Z}.$$

Now if $m'_1, m'_2 \in \mathbb{Z}$ such that $m'_1 \equiv m_1 \pmod{n}$ and $m'_2 \equiv m_2 \pmod{n}$, then we can write $m'_1 = nk_1 + m_1$ and $m'_2 = nk_2 + m_2$. If consider $(I, k_1, k_2) \in V_{\Gamma(n)}$, then

$$(I, k_1, k_2)(z, \frac{m_1}{n}z + \frac{m_2}{n}) = (z, \frac{m'_1}{n}z + \frac{m'_2}{n}),$$

and therefore

$$[z, \frac{m_1}{n}z + \frac{m_2}{n}] = [z, \frac{m'_1}{n}z + \frac{m'_2}{n}]$$

and so the couples $m = (m_1, m_2)$ and $m' = (m'_1, m'_2)$ define the same section. If we set $\sigma'_m([z]) = [z, \frac{m_1}{n}z + \frac{m_2}{n}]$ the set of sections is

$$S(X(n)) = \{\sigma_m | m \in \{0, 1, \dots, n-1\} \times \{0, 1, \dots, n-1\}\}.$$

If $\sigma_m = \sigma_{m'}$, then for all $[z] \in P\Gamma(n) \setminus \mathbb{H}$, we have

$$[z, \frac{m_1}{n}z + \frac{m_2}{n}] = [z, \frac{m'_1}{n}z + \frac{m'_2}{n}].$$

In other words, there exists $(\gamma, k_1, k_2) \in V_{\Gamma(n)}$ such that

$$(\gamma, k_1, k_2)(z, \frac{m_1}{n}z + \frac{m_2}{n}) = (z, \frac{m'_1}{n}z + \frac{m'_2}{n}),$$

and hence $\gamma.z = z$. Since $\Gamma(n)$ acts without fixed points in \mathbb{H} , we have $\gamma = I$. Therefore,

$$\frac{m'_1 z + m'_2}{n} = \frac{m_1 z + m_2}{n} + k_1 z + k_2.$$

It follows that

$$(m'_1 - (m_1 + nk_1))z + (m'_2 - (m_2 + nk_2)) = 0 \quad \forall z \in \mathbb{H},$$

hence $m'_1 = nk_1 + m_1$ and $m'_2 = nk_2 + m_2$. Since $m_1, m_2, m'_1, m'_2 \in \{0, 1, \dots, n-1\}$ then $m_1 = m'_1$ and $m_2 = m'_2$, hence $|S(X(n))| = n^2$. \square

3.3.2 Example 2

Define the congruence group $\Gamma_1(n)$

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{n}, c \equiv 0 \pmod{n} \right\}.$$

Then $\Gamma_1(n)$ is a subgroup $\mathrm{SL}_2(\mathbb{Z})$ and $-I \notin \Gamma_1(n)$ for $n \geq 3$. Obviously we have $\Gamma(n) \subseteq \Gamma_1(n)$ for all n . Let $C_1(n)$ denotes the modular curve $P\Gamma_1(n) \setminus \overline{\mathbb{H}}$.

Theorem 3.3.3. ([1] Lemmas 7.10 and 7.11) Let $n \geq 3$.

1. The modular curve $C_1(n)$ has no elliptic points for $n \geq 4$ and $C_1(3)$ has one elliptic point.
2. $C_1(n)$ has only cusps of the first kind for $n \neq 4$ and $C_1(4)$ has two cusps of the first kind and one cusp of the second kind. Moreover, if for $n \neq 4$, $\alpha \in C_1(n)$ is a cusp and $r \in \mathbb{Q} \cup \{\infty\}$ is a representative of α . Then $\Gamma_1(n)_r$ is conjugate to $\begin{pmatrix} 1 & \frac{n}{d} \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$, where $r = p/q$, $\mathrm{gcd}(p, q) = 1$ and $d = \mathrm{gcd}(q, n)$.
3. The index of $P\Gamma_1(n)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is

$$\mu(n) = \frac{n^2}{2} \prod_{p|n} (1 - p^{-2}). \quad (32)$$

4. The number of cusps in $C_1(n)$ is

$$t(n) = \frac{1}{2} \sum_{d|n} \varphi(d) \varphi\left(\frac{n}{d}\right), \quad n \neq 4, \quad (33)$$

and $t(4) = 3$, where φ is the Euler function.

Proof. For 1. and 4. see ([1] Lemmas 7.10 and 7.11). As for 2., let $\alpha \in C_1(n)$ $n \neq 4$ be a cusp and $r \in \mathbb{Q} \cup \{\infty\}$ be a representative of α . Put $r = p/q$ with $\gcd(p, q) = 1$ and $d = \gcd(n, q)$. Consider

$$\sigma_r = \begin{pmatrix} -np_d^q + 1 & \frac{n}{d}p^2 \\ -n\frac{q^2}{d} & np_d^q + 1 \end{pmatrix}.$$

Then $\Gamma_1(n)_r = \langle \sigma_r \rangle$. Now, let $u, v \in \mathbb{Z}$ such that $pu - qv = 1$, then $\lambda = \begin{pmatrix} p & v \\ q & u \end{pmatrix}$

lies in $\mathrm{SL}_2(\mathbb{Z})$ and it sends ∞ to r , moreover $\lambda^{-1}\sigma_r\lambda = \begin{pmatrix} 1 & \frac{n}{d} \\ 0 & 1 \end{pmatrix}$ hence $\Gamma_1(n)_r$ is

conjugate to $\begin{pmatrix} 1 & \frac{n}{d} \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$.

For $n = 4$ we verify that $\infty, 0$ and $\frac{1}{2}$ are pairwise non equivalent under the action of $\Gamma_1(4)$ and since by 4) we have $t(4) = 3$ then $\{\infty, 0, \frac{1}{2}\}$ is a complete set of representatives of cusps of $C_1(4)$. We can also verify that

$$\Gamma_1(4)_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

$$\Gamma_1(4)_0 = \left\langle \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right\rangle$$

which is conjugate to

$$\left\langle \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \right\rangle$$

in $\mathrm{SL}_2(\mathbb{Z})$ by the transformation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$\Gamma_1(4)_{\frac{1}{2}} = \left\langle \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \right\rangle$$

which is conjugate to

$$\left\langle \left(\begin{array}{cc} -1 & -1 \\ 0 & -1 \end{array} \right) \right\rangle$$

by the transformation $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$. Therefore $[0], [\infty]$ are cusps of the first kind and $[\frac{1}{2}]$ is a cusp of the second kind.

As for 3., we have the index formula

$$[\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma_1(n)] = \frac{[\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma(n)]}{[P\Gamma_1(n) : P\Gamma(n)]}.$$

Since by Theorem 3.3.1

$$[\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma(n)] = \frac{n^3}{2} \prod_{p|n} (1 - p^2),$$

then, to determine $[\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma_1(n)]$, it suffices to compute $[P\Gamma_1(n) : P\Gamma(n)]$. Recall the epimorphism $\varphi_n : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}_n)$ for which $\mathrm{Ker}\varphi_n = \Gamma(n)$. Moreover,

$$\varphi_n(\Gamma_1(n)) = \left\{ \begin{pmatrix} \bar{1} & \bar{m} \\ \bar{0} & \bar{1} \end{pmatrix} \mid \bar{m} \in \mathbb{Z}_n \right\},$$

and hence $|\varphi_n(\Gamma_1(n))| = n$ which yields $[P\Gamma_1(n) : P\Gamma(n)] = n$. Therefore,

$$[\mathrm{PSL}_2(\mathbb{Z}) : P\Gamma_1(n)] = \frac{n^2}{2} \prod_{p|n} (1 - p^{-2}).$$

□

For $n \geq 5$, consider the modular elliptic surface $X_1(n)$ associated with $\Gamma_1(n)$. The modular curve $C_1(n)$ has no elliptic points and all cusps are of the first kind. More precisely, if $\alpha \in C_1(n)$ is a cusp and $r \in \mathbb{Q} \cup \{\infty\}$ write $r = p/q$, $\mathrm{gcd}(p, q) = 1$ and let $d = \mathrm{gcd}(n, q)$, then by the previous theorem $\Gamma_1(n)_r$ is conjugate to

$$\left\langle \left(\begin{array}{cc} 1 & \frac{n}{d} \\ 0 & 1 \end{array} \right) \right\rangle$$

in $\mathrm{SL}_2(\mathbb{Z})$ and hence the singular fiber over α is of type $I_{\frac{n}{d}}$. Therefore the singular fibers of $X_1(n)$ are of types I_d , $d|n$, and their number is the number of cusps which is given by the formula

$$t(n) = \frac{1}{2} \sum_{d|n} (\varphi(d)\varphi(\frac{n}{d})).$$

To compute the numerical invariants of $X_1(n)$, it suffices to compute the genus g_n of $C_1(n)$ and the Euler characteristic $e(X_1(n))$ of $X_1(n)$. We have

$$g_n = 1 + \frac{\mu(n)}{12} - \frac{t(n)}{2} = 1 + \frac{n^2}{24} \prod_{p|n} (1 - p^{-2}) - \frac{1}{4} \sum_{d|n} (\varphi(d)\varphi(\frac{n}{d})) \quad (34)$$

and

$$e(X_1(n)) = \mu(n) = \frac{n^2}{2} \prod_{p|n} (1 - p^{-2}). \quad (35)$$

As for the Mordell-Weil group $S(X_1(n))$ of $X_1(n)$ for $n \geq 5$, we have the following:

Theorem 3.3.4. *For $n \geq 5$, the group $S(X_1(n))$ is isomorphic to \mathbb{Z}_n .*

Proof. Since $\Gamma_1(n)$ is torsion free then by Proposition 3.4.1 we have: if $\sigma \in S(X_1(n))$ then its restriction σ' to $P\Gamma_1(n) \setminus \mathbb{H}$, has the form $\sigma'(\Gamma.h) = [h, a_1h + a_2]$ where $a_1, a_2 \in \mathbb{Q}$ with the property : $(a_1, a_2) \cdot (\gamma - I) \in \mathbb{Z} \oplus \mathbb{Z}$ for all $\gamma \in \Gamma_1(n)$. Taking $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ yields $a_1 \in \mathbb{Z}$, and taking $\gamma_1 = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ yields $a_2 = \frac{m}{n}$, $m \in \mathbb{Z}$, and therefore $\sigma'([z]) = [z, m_1z + \frac{m_2}{n}]$ with $m_1, m_2 \in \mathbb{Z}$. Since $(z, m_1z + \frac{m_2}{n})$ is equivalent to $(z, \frac{m_2}{n})$ by means of $(I, m_1, 0) \in V_{\Gamma_1(n)}$ we have $\sigma'([z]) = [z, m_2/n]$, $m_2 \in \mathbb{Z}$. As in the case of $\Gamma(n)$, one can easily prove that $m \equiv m'[n]$ if and only if $[z, \frac{m}{n}] = [z, \frac{m'}{n}]$. Therefore, $S(X_1(n)) = \{\sigma_m \mid 0 \leq m \leq n - 1\}$ with $\sigma'_m([z]) = [z, m/n]$ for all $z \in \mathbb{H}$.

Finally, the map

$$\begin{aligned} \chi : \mathbb{Z}_n &\longrightarrow S(X_1(n)) \\ \bar{m} &\longmapsto \sigma_m \end{aligned}$$

is clearly a group isomorphism. □

3.3.3 Example 3

Define the Hecke congruence subgroup $\Gamma_0(n)$ by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid n \mid c \right\}.$$

Unlike $\Gamma(n)$ and $\Gamma_1(n)$, we have $-I \in \Gamma_0(n)$ for every $n \geq 1$, and so we cannot associate to it a modular elliptic surface as discussed in this chapter. However, we will show that for a class of integers n , we will find at least one subgroup Γ of $\Gamma_0(n)$ such that $-I \notin \Gamma$ and $\Gamma_0(n) = \{\pm I\} \cdot \Gamma$, i.e. $P\Gamma = P\Gamma_0(n)$. In other words $P\Gamma_0(n)$ has a lift Γ in $\mathrm{SL}_2(\mathbb{Z})$ not containing $-I$, and hence we can associate to Γ a modular elliptic surface X_Γ which will have $C_0(n) = P\Gamma_0(n) \backslash \overline{\mathbb{H}}$ as its base curve.

Theorem 3.3.5. (*[6], Proposition 1.43*) *Let $C_0(n)$ be the modular curve associated with the group $P\Gamma_0(n)$ and $J_n : C_0(n) \rightarrow \mathbb{P}^1$ be the natural map, then the following holds*

1. *The degree of J_n is given by*

$$\mu(n) = n \cdot \prod_{p \mid n} (1 + p^{-1}). \quad (36)$$

2. *The number of elliptic points of order 2 is given by*

$$s_2(n) = \begin{cases} 0, & \text{if } n \text{ is divisible by } 4; \\ \prod_{p \mid n} (1 + (\frac{-1}{p})), & \text{otherwise.} \end{cases} \quad (37)$$

3. *The number of elliptic points of order 3 is given by*

$$s_3(n) = \begin{cases} 0, & \text{if } n \text{ is divisible by } 9; \\ \prod_{p \mid n} (1 + (\frac{-3}{p})), & \text{otherwise.} \end{cases} \quad (38)$$

4. *The number of cusps is given by*

$$t'(n) = \sum_{d \mid n} \varphi\left(d, \frac{n}{d}\right). \quad (39)$$

Where φ is the Euler's function, and $\left(\frac{\cdot}{p}\right)$ is the quadratic residue symbol, so that

$$\left(\frac{-1}{p}\right) = \begin{cases} 0 & \text{if } p = 2, \\ 1 & \text{if } p \equiv 1[4], \\ -1 & \text{if } p \equiv 3[4]. \end{cases}$$

$$\left(\frac{-3}{p}\right) = \begin{cases} 0 & \text{if } p = 3, \\ 1 & \text{if } p \equiv 1[3], \\ -1 & \text{if } p \equiv 2[3]. \end{cases}$$

We start by the following special case due to Shioda. Let p be a prime such that $p \equiv 3 \pmod{4}$ and define

$$\Gamma'_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) \mid \left(\frac{a}{p}\right) = 1 \right\}.$$

Then, $\Gamma'_0(p)$ is a subgroup of $\Gamma_0(p)$, and we have

Proposition 3.3.6. *Let p be a prime such that $p \equiv 3 \pmod{4}$, we have*

1. $-I \notin \Gamma'_0(p)$ and $\Gamma_0(p) = \{\pm I\} \cdot \Gamma'_0(p)$.
2. All $\Gamma'_0(p)$ -cusps are of the first kind.

Proof. For 1, we have $-I \notin \Gamma'_0(p)$ since -1 is not a square \pmod{p} when $p \equiv 3 \pmod{4}$. In order to prove that $\Gamma_0(p) = \{\pm I\}\Gamma'_0(p)$, it suffices to prove that for all $\sigma \in \Gamma_0(p)$, either $\sigma \in \Gamma'_0(p)$ or $-\sigma \in \Gamma'_0(p)$. Indeed, let $\sigma \in \Gamma_0(p)$, and suppose that $\sigma \notin \Gamma'_0(p)$. Write $\sigma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$, then $\left(\frac{a}{p}\right) = -1$ since $p \equiv 3 \pmod{4}$, so that $\left(\frac{-a}{p}\right) = 1$, and hence $-\sigma = \begin{pmatrix} -a & -b \\ -pc & -d \end{pmatrix} \in \Gamma'_0(p)$.

As for 2, in order to prove that $C_0(p)$ has no cusps of the second kind, it suffices to show that $\text{tr}\sigma = 2$ for all $\sigma \in \Gamma'_0(p)$. Let $\sigma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$, so that $\left(\frac{a}{p}\right) = 1$ and $ad - pbc = 1$. Suppose that $\text{tr}\sigma = -2$, i.e. $a + d = -2$, we have $a^2 + ad = -2a$, and

since $ad \equiv 1 \pmod{p}$, we have $a^2 + 2a + 1 = (a + 1)^2 \equiv 0 \pmod{p}$, so that $a \equiv -1 \pmod{p}$. Hence $\left(\frac{-1}{p}\right) = 1$ which is a contradiction since $p \equiv 3 \pmod{4}$. \square

According to the previous proposition we can construct a modular elliptic surface $X_0(P)$ with base curve $C_0(P)$. That is the modular elliptic surface associated with $\Gamma'_0(p)$.

In the following proposition we characterize the integers n for which there exists a subgroup Γ of $\Gamma_0(n)$ such that $\Gamma_0(n) = \{\pm I\} \cdot \Gamma$ and $-I \notin \Gamma$.

Proposition 3.3.7. *Let n be an integer, $n \geq 3$. Then, the following are equivalent:*

1. *There exists a subgroup Γ of $\Gamma_0(n)$ such that $\Gamma_0(n) = \{\pm I\} \cdot \Gamma$ and $-I \notin \Gamma$.*
2. *$P\Gamma_0(n)$ has no elliptic elements of order 2.*
3. *$4|n$ or n has a prime divisor p with $p \equiv 3 \pmod{4}$.*

Proof. (1 \Rightarrow 2): Suppose that $P\Gamma_0(n)$ has an elliptic element of order 2, say $\bar{\sigma}$, with a lift $\sigma \in \Gamma_0(n)$. Then, $\bar{\sigma}^2 = \bar{I}$, and so $\sigma^2 = \pm I$. However, $\text{SL}_2(\mathbb{Z})$ has no elliptic element of order 2. Therefore, $\sigma^2 = -I$. Now, $\Gamma_0(n) = \{\pm I\} \cdot \Gamma$ implies that either $\sigma \in \Gamma$ or $-\sigma \in \Gamma$. In both cases, we get $\sigma^2 = -I \in \Gamma$ which contradicts 1.

(2 \Rightarrow 3): Since $P\Gamma_0(n)$ has no elliptic element of order 2, $s_2 = 0$. By Theorem 3.5.5,

$$s_2 = \begin{cases} 0, & \text{if } 4|n; \\ \prod_{p|n} \left(1 + \left(\frac{-1}{p}\right)\right), & \text{otherwise.} \end{cases}$$

Therefore, either $4|n$ or n has a prime divisor p with $p \equiv 3 \pmod{4}$.

(3 \Rightarrow 1): Suppose that $4|n$ or $p|n$, for some prime $p \equiv 3 \pmod{4}$. If $4|n$, then we take $\Gamma = \Gamma_0(n) \cap \Gamma_1(4)$. Note that $-I \notin \Gamma$ since $-I \notin \Gamma_1(4)$. Now, to show that $\Gamma_0(n) = \{\pm I\}\Gamma$, it suffices to show that for any $\sigma \in \Gamma_0(n)$, either $\sigma \in \Gamma$ or $-\sigma \in \Gamma$. Indeed, let $\sigma = \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \in \Gamma_0(n)$, then $ad - nbc = 1$, and since $4|n$, $ad \equiv 1 \pmod{4}$. Hence, either $a \equiv d \equiv 1 \pmod{4}$ or $a \equiv d \equiv -1 \pmod{4}$. If $a \equiv d \equiv 1 \pmod{4}$, then $\sigma \in \Gamma_1(4)$. If $a \equiv d \equiv -1 \pmod{4}$, then $-a \equiv -d \equiv 1 \pmod{4}$,

so that $-\sigma \in \Gamma_1(4)$. Thus, we have either $\sigma \in \Gamma$ or $-\sigma \in \Gamma$. If there is a prime $p \equiv 3 \pmod{4}$ dividing n , we take $\Gamma = \Gamma_0(n) \cap \Gamma'_0(p)$, where $\Gamma'_0(p)$ is the subgroup of $\Gamma_0(p)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\left(\frac{a}{p}\right) = 1$. Let $\sigma = \begin{pmatrix} a & b \\ nc & d \end{pmatrix}$. Since $p|n$ then $\sigma \in \Gamma_0(p)$. If $\left(\frac{a}{p}\right) = 1$, then $\sigma \in \Gamma'_0(p)$ and we are done. If $\left(\frac{a}{p}\right) = -1$, then $\left(\frac{-a}{p}\right) = 1$, so that $-\sigma \in \Gamma'_0(p)$. Hence, we have either $\sigma \in \Gamma$ or $-\sigma \in \Gamma$, which proves that $\Gamma_0(n) = \{\pm I\}\Gamma$. \square

Corollary 3.3.8. *Let n be an integer, $n \geq 3$, and let $C_0(n)$ be the modular curve $P\Gamma_0(n) \setminus \overline{\mathbb{H}}$. Then the following are equivalent:*

1. $4|n$ or n has a prime divisor p , $p \equiv 3 \pmod{4}$.
2. There exists a modular elliptic surface X with base curve $C_0(n)$.

Proof. (1 \Rightarrow 2): This follows from the previous discussion.

(2 \Rightarrow 1): Suppose that there exists a modular elliptic surface X with base $C_0(n)$. We need to prove that $4|n$ or n has a prime divisor p , $p \equiv 3 \pmod{4}$. According to Proposition 3.3.7 it suffices to prove that $P\Gamma_0(n)$ has no elliptic elements of order 2. Let $J_n : C_0(n) \rightarrow \mathbb{P}^1$ be the natural map. If $P\Gamma_0(n)$ has an elliptic point v of order 2, then $J_n(v) = 1$ and $e_{J_n}(v) = 1$ and then by Table 2.4.3 the fiber C_v over v is of type *III* or *III** this is a contradiction, since a modular elliptic surface has only singular fibers of types I_b , I_b^* and IV^* . \square

Next, let n be an integer having a prime divisor p , $p \equiv 3 \pmod{4}$. We are going to study the modular elliptic surface associated with $\Gamma_0(n) \cap \Gamma'_0(p)$, which we denote by $X_0(n, p)$.

By Proposition 3.5.6, the cusps of $\Gamma'_0(p)$ are all of the first kind, therefore the cusps of $\Gamma_0(n) \cap \Gamma'_0(p)$ are also of the first kind. We can also prove that if $v \in C_0(n)$ is a cusp and if $r = p/q \in \mathbb{Q} \cup \{\infty\}$ is a representative of v , then the generator of the stabilizer of r in Γ is the subgroup $\langle \sigma_r \rangle$ generated by σ_r , where

$$\sigma_r = \begin{pmatrix} 1 - pqn/d\delta & p^2n/d\delta \\ -q^2n/d\delta & 1 + pqn/d\delta \end{pmatrix},$$

where $d = \gcd(q, n)$ and $\delta = \gcd(d, n/d)$. Also, we notice that σ_r is conjugate to $\begin{pmatrix} 1 & n/d\delta \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$, and therefore $X_0(n, p)$ has singular fibers of the type $I_{\frac{n}{d\delta}}$, with $d|n$ and $\delta = (d, n/d)$, and their number is exactly the number of cusps in $C_0(n)$ which is given by

$$t'(n) = \sum_{d|n} \varphi\left(d, \frac{n}{d}\right).$$

Moreover, the number of elliptic points in $C_0(n)$ is

$$s_3 = \begin{cases} 0, & \text{if } 9|n; \\ \prod_{q|n} \left(1 + \left(\frac{-3}{q}\right)\right), & \text{otherwise.} \end{cases}$$

so that $X_0(n, p)$ has s_3 singular fiber of the type IV^* . To compute the numerical invariants of $X_0(n, p)$ it suffices to compute the genus g_n of $C_0(n)$ and the Euler characteristic $e(X_0(n, p))$.

The genus g_n and the Euler characteristic $e(X_0(n, p))$ are given by

$$g_n = 1 + \frac{\mu}{12} - \frac{t'}{2} - \frac{s_3}{3}, \quad (40)$$

$$e(X_0(n, p)) = \mu(n) + 6t_2 + 8s_3 = \mu(n) + 8s_3, \quad (41)$$

since $t_2 = 0$ (no cusps of the second kind). Therefore, g_n and $e(X_0(n, p))$ are determined and we can compute all the other numerical invariants.

Chapter 4

Modularity of elliptic surfaces

In previous chapters, we have defined the notion of modular elliptic surfaces. It turns out that these surfaces satisfy many properties, in particular, they have a finite Mordell-Weil group and only few types of singular fibers occur. It is natural to ask when a general elliptic surface is modular. In this chapter, we introduce a theorem of Nori which provides sufficient conditions for an elliptic surface to be modular. We will provide a direct proof which is essentially the same as Nori's original proof, but without some heavy technical tools which were introduced to achieve the proof. Namely, we are going to bypass the notion of generalized modular elliptic surfaces which were a central theme in Nori's proof.

4.1 Nori's theorem

4.1.1 The statement of the theorem

If $\pi : X \rightarrow C$ is an elliptic surface, as previously we denote the rank of the Mordell-Weil theorem by r , and the Picard number by ρ .

Theorem 4.1.1. *(Nori, [3], Theorem 3.6) Let $\pi : X \rightarrow C$ be an elliptic surface such that:*

1. $r = 0$ and $\rho = h^{1,1}$,
2. X has no singular fibers of type II^* and III^* .

Then $\pi : X \longrightarrow C$ is a modular elliptic surface.

Nori proved this theorem as a corollary to a series of results, and by introducing the notion of generalized modular surfaces. Here we provide a direct proof inspired from Nori's proof but without the notion of generalized modular elliptic surfaces.

We start with an intermediate result.

Theorem 4.1.2. *Let $\pi : X \longrightarrow C$ be an elliptic surface. Then conditions 1 and 2 in the statement of Nori's theorem are equivalent to the following two statements:*

- a) *The only singular fibers of X are of types I_b, I_b^* or IV^* .*
- b) *The genus g of the base curve C satisfies the following equation:*

$$g = 1 + \frac{\mu}{12} - \frac{\sum_{b \geq 1} (\nu(I_b) + \nu(I_b^*))}{2} - \frac{\nu(IV^*)}{3},$$

where μ is the degree of the homological invariant J , and $\nu(T)$ denotes the number of singular fibers of type T .

Proof. We suppose 1) and 2) and we will prove a) and b).

By Theorem 2.6.4 and using the notations therein, we have $l = b_2 - \rho + r$. Since $r = 0$ and $\rho = h^{1,1}$, we have $l = b_2 - h^{1,1}$. Therefore, $l = 2P_g$ since $b_2 - h^{1,1} = 2P_g$. Again using Theorem 2.6.4, we have

$$\nu(I_0^*) + \nu(II) + \nu(III) + \nu(IV) \leq l - 2P_g,$$

and so

$$\nu(I_0^*) = \nu(II) = \nu(III) = \nu(IV) = 0.$$

But condition 2) says $\nu(II^*) = \nu(III^*) = 0$. This proves a).

Using Theorem 2.6.2, we have

$$e(X) = \mu + 6 \cdot \sum_{b \geq 0} \nu(I_b^*) + 2\nu(II) + 10\nu(II^*) + 3\nu(III) + 9\nu(III^*) + 4\nu(IV) + 8\nu(IV^*),$$

which, according to a), reduces to

$$e(X) = \mu + 6 \sum_{b \geq 0} \nu(I_b^*) + 8\nu(IV^*). \quad (42)$$

On the other hand, we have $e(X) = 12P_g - 12g + 12$ (By Corollary 2.6.3). Since $l = 2P_g$, we have $e(X) = 6l - 12g + 12$. Recalling the expression for $l = 4g + 2t - t_1 - 4$ where

$$t = \sum_{b \geq 1} (\nu(I_b) + \nu(I_{b^*})) + \nu(IV^*)$$

and

$$t_1 = \sum_{b \geq 1} \nu(I_b),$$

we get

$$e(X) = 12g - 12 + 6 \sum_{b \geq 1} \nu(I_b) + 12 \sum_{b \geq 1} \nu(I_{b^*}) + 12\nu(IV^*). \quad (43)$$

Finally, by comparing (42) and (43) we obtain

$$2g - 2 = \frac{\mu}{6} - \sum_{b \geq 1} (\nu(I_b) + \nu(I_{b^*})) - \frac{2}{3}\nu(IV^*).$$

This proves b).

Conversely, let us suppose that a) and b) hold. First we remark that a) implies 2) immediately. By Theorem 2.6.2 and condition a) we have

$$e(X) = \mu + 6 \sum_{b \geq 0} \nu(I_b^*) + 8\nu(IV^*) \quad (44)$$

but according to Corollary 2.6.3 we have $e(X) = 12P_g - 12g + 12$. It follows that

$$2P_g - 2g + 2 = \frac{\mu}{6} + \sum_{b \geq 1} \nu(I_b^*) + \frac{4}{3}\nu(IV^*).$$

Hence,

$$\frac{\mu}{6} = 2P_g - 2g + 2 - \sum_{b \geq 1} \nu(I_b^*) - \frac{4}{3}\nu(IV^*). \quad (45)$$

Using b) we have

$$\frac{\mu}{6} = 2g - 2 + \sum_{b \geq 1} \nu(I_b^*) + \sum_{b \geq 1} \nu(I_b) + \frac{2}{3}\nu(IV^*). \quad (46)$$

Now comparing (45) and (46), we have

$$2P_g = 4g - 4 + 2 \sum_{b \geq 1} \nu(I_b^*) + \sum_{b \geq 1} \nu(I_b) + 2\nu(IV^*) = 4g - 4 + 2t - t_1 = l.$$

By Theorem 2.6.4, we have $r \leq l - 2P_g$ so that $r = 0$. Again by Theorem 2.6.4 we have $l = b_2 - \rho + r$. Replacing l by $2P_g$ and r by 0 we obtain $2P_g = b_2 - \rho$. In the meantime, we have $2P_g = b_2 - h^{1,1}$ and therefore $\rho = h^{1,1}$. This yields 1. \square

4.1.2 The proof of Nori's theorem

We now provide a direct proof to Nori's theorem by using Theorem 4.1.2. We assume that $\pi : X \rightarrow C$ is an elliptic surface and that the conditions of Theorem 4.1.2 hold.

Lemma 4.1.3. *The curve C is modular.*

Proof. Let $J : C \rightarrow \mathbb{P}^1$ be the functional invariant of the elliptic surface X , and $\bar{\rho} : \pi_1(C \setminus \Sigma, O) \rightarrow \mathrm{PSL}_2(\mathbb{Z})$ the representation of Proposition 2.5.2. If we set $\Lambda = \bar{\rho}(\pi_1(C \setminus \Sigma))$ then, by proposition 2.5.3, there exists a map $\bar{\omega} : C \rightarrow \bar{H}/\Lambda$ such that the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\bar{\omega}} & \bar{H}/\Lambda \\ & \searrow J & \downarrow J_\Lambda \\ & & \mathbb{P}^1 \end{array}$$

The Hurwitz formula for the map J gives

$$2g - 2 = -2\mu + \sum_{v \in C} (e(v) - 1),$$

where $\mu = \deg(J)$ and $e(v)$ is the ramification index of J at v . By Theorem 4.1.2 we have:

$$2g - 2 = \frac{\mu}{6} - t' - \frac{2}{3}s_3,$$

where $t' = \sum_{b \geq 1} (\nu(I_b) + \nu(I_{b^*}))$, and $s_3 = \nu(IV^*)$.

Therefore, we have

$$\frac{13}{6}\mu - t' - \frac{2}{3}s_3 = \sum_{v \in C} (e(v) - 1).$$

In other terms

$$\begin{aligned} & \frac{13}{6}\mu - t' - \frac{2}{3}s_3 = \\ & \sum_{v \in J^{-1}(0)} (e(v) - 1) + \sum_{v \in J^{-1}(1)} (e(v) - 1) + \sum_{v \in J^{-1}(\infty)} (e(v) - 1) + \sum_{J(v) \notin \{0,1,\infty\}} (e(v) - 1). \end{aligned} \quad (47)$$

Since $\mu = \sum_{v \in J^{-1}(\infty)} e(v)$ and $t' = |J^{-1}(\infty)|$ we have

$$\sum_{v \in J^{-1}(\infty)} (e(v) - 1) = \mu - t',$$

and equation (47) becomes

$$\frac{7}{6}\mu - \frac{2}{3}s_3 = \sum_{v \in J^{-1}(0)} (e(v) - 1) + \sum_{v \in J^{-1}(1)} (e(v) - 1) + \sum_{J(v) \notin \{0,1,\infty\}} (e(v) - 1). \quad (48)$$

Writing the right hand side as

$$\frac{2}{3}(\mu - s_3) + \frac{\mu}{2},$$

equation (48) becomes

$$\sum_{v \in J^{-1}(0)} (e(v) - 1) - \frac{2}{3}(\mu - s_3) + \left(\sum_{v \in J^{-1}(1)} (e(v) - 1) - \frac{\mu}{2} \right) = - \sum_{J(v) \notin \{0,1,\infty\}} (e(v) - 1) \leq 0. \quad (49)$$

We claim that

$$\sum_{v \in J^{-1}(0)} (e(v) - 1) - \frac{2}{3}(\mu - s_3) \geq 0 \quad \text{and} \quad \sum_{v \in J^{-1}(1)} (e(v) - 1) - \frac{\mu}{2} \geq 0.$$

We will prove the first inequality. The second inequality has a similar proof. If $v \in C$ is such that $J(v) = 0$, then, by the table of Theorem 2.4.3, we have

- If $e(v) \equiv 0 \pmod{3}$ then $C_v \in \{I_0, I_0^*\}$,
- If $e(v) \equiv 1 \pmod{3}$ then $C_v \in \{II, IV^*\}$.

- If $e(v) \equiv 2 \pmod{3}$ then $C_v \in \{IV, II^*\}$.

However the only possible fibers are of types I_0, IV^*, I_b and I_b^* , hence, when $J(v) = 0$, we have $e(v) \equiv 0 \pmod{3}$ when C_v is of type I_0 , and $e(v) \equiv 1 \pmod{3}$ when C_v is of type IV^* . Therefore, we have

$$\mu = \sum_{v \in J^{-1}(0)} e(v) = \sum_{k_i \geq 1} 3k_i + \sum_{l_j \geq 0} (3l_j + 1)$$

with

$$s_3 = \nu(IV^*) = \sum_{l_j \geq 0} (3l_j + 1).$$

It follows that $\mu - s_3 = \sum_{k_i \geq 1} 3k_i > 0$ and then $\frac{2}{3}(\mu - s_3) = \sum_{k_i \geq 1} 2k_i$. Thus

$$\begin{aligned} \sum_{v \in J^{-1}(0)} (e(v) - 1) - \frac{2}{3}(\mu - s_3) &= \sum_{k_i \geq 1} (3k_i - 1) + \sum_{l_j \geq 0} 3l_j - \sum_{k_i \geq 1} 2k_i \\ &= \sum_{k_i \geq 1} k_i - 1 + \sum_{l_j \geq 0} 3l_j \geq 0. \end{aligned}$$

Now going back to (49) and using the inequalities of the claim, we obtain

$$\sum_{J(v) \notin \{0,1,\infty\}} (e(v) - 1) = 0, \quad \sum_{v \in J^{-1}(0)} (e(v) - 1) = \frac{2}{3}(\mu - s_3), \quad \sum_{v \in J^{-1}(1)} (e(v) - 1) = \frac{\mu}{2}.$$

The first equality implies that J is only ramified over $0, 1$ and ∞ . The second equality implies that $\sum_{k_i \geq 1} k_i - 1 + \sum_{l_j \geq 0} 3l_j = 0$, so that $k_i = 1$ for all i and $l_j = 0$ for all j . Hence $e(v) = 1$ or 3 when $J(v) = 0$. Similarly, we prove that $e(v) = 2$ when $J(v) = 1$. We now apply Proposition 2.5.3 to this situation, namely the fact that the map $\bar{\omega} : C \rightarrow \Lambda \backslash \overline{\mathbb{H}}$ is an isomorphism. We conclude that the curve C can be viewed as a modular curve, and the elliptic surface X can be considered as an elliptic surface with base curve $\Lambda \backslash \overline{\mathbb{H}}$, and its functional invariant is the natural map J_Λ . \square

Lemma 4.1.4. *The elliptic surface $\pi : X \rightarrow C$ is modular.*

Proof. Let $\rho : \pi_1(C \setminus \Sigma) \rightarrow \mathrm{SL}_2(\mathbb{Z})$ be the homological invariant of $\pi : X \rightarrow C$,

and let $\Gamma = \rho(\pi_1(C \setminus \Sigma))$, then we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{SL}_2(\mathbb{Z}) \\ & \nearrow \rho & \downarrow s \\ \pi_1(C \setminus \Sigma, O) & \xrightarrow{\bar{\rho}} & \mathrm{PSL}_2(\mathbb{Z}) \end{array}$$

We have $-I \notin \Gamma$, otherwise there exists $\beta \in \pi_1(C \setminus \Sigma)$ such that $\rho(\beta) = -I$. In this case we have $\bar{\rho}(\beta) = I$ and hence $\beta \in \mathrm{Ker} \bar{\rho}$. Therefore, it suffices to prove that $\rho(\mathrm{Ker} \bar{\rho})$ does not contain $-I$. By inspection of Table 2.4.3 and keeping in mind that the only singular fibers are of types I_b , I_b^* and IV^* , we observe that $\mathrm{Ker} \bar{\rho}$ is generated by the following elements:

- α_v^3 when $J(v) = 0$ and $e_v = 1$, in which case $[\rho(\alpha_v)] = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$,
- β_v when $J(v) = 0$ and $e_v = 3$, in which case $[\rho(\beta_v)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
- γ_v when $J(v) = 1$ and $e_v = 2$, in which case $[\rho(\gamma_v)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It follows that $\rho(\beta_v) = \rho(\gamma_v) = I$, and $\rho(\alpha_v^3) = I$. Therefore $\rho(\mathrm{Ker} \bar{\rho}) = \{I\}$. Since $\Lambda = P\Gamma$ and $-I \notin \Gamma$, Λ and Γ are isomorphic by the canonical map $s : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{PSL}_2(\mathbb{Z})$, and hence $\rho = s^{-1} \circ \bar{\rho}$.

Let X_Γ be the modular elliptic surface associated with Γ . It has $\Lambda \setminus \overline{\mathbb{H}}$ as its base curve, J_Λ as its functional invariant, and ρ as its homological invariant. By the previous Lemma, the surface X can be viewed as an elliptic surface over $\Lambda \setminus \overline{\mathbb{H}}$ with functional invariant J_Λ . It follows that X and X_Γ are elliptic surfaces over $\Lambda \setminus \overline{\mathbb{H}}$ with the same homological and functional invariants. Therefore, they are the same since by Kodaira's Theorem (Theorem 2.5.4), there is a unique elliptic surface with given homological and functional invariants. Thus X is modular. \square

This concludes the proof of Nori's theorem.

The modularity in question is more topological than arithmetic. It would be interesting to investigate more arithmetic aspects of these surfaces such as zeta functions and L -functions.

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