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Ordered Groups and Crossed Products of AF-related C^*-Algebras

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ORDERED GROUPS AND CROSSED PRODUCTS OF AF-RELATED C*-ALGEBRAS

By
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ABSTRACT

Let C be the canonical diagonal MASA in an AF algebra A. We show that an automorphism ϕ of C extends to an approximately inner automorphism α of A iff p is Murray-von Neumann equivalent in A to $\phi(p)$ for every projection $p \in C$. We determine some classes of AF algebras for which the ordered K_0-groups of the two crossed products, arising from these dynamical systems, are isomorphic and we show that the entropies of ϕ and α are equal. Along the way, we obtain results about when $K_0(A \ltimes_\alpha \mathbb{Z})$ (equipped with natural order) is pre-order isomorphic with $K_0(A)/\mathrm{Im}(id - K_0(\alpha))$ (equipped with the quotient order) for a large class of automorphisms α of the AF algebra A.

I. Putnam has proved that $C(X) \ltimes_\phi \mathbb{Z}$ is a (simple) AT algebra when X is the Cantor set and ϕ is a minimal self-homeomorphism of X. We extend his results to non-commutative AF algebras that have the set of extremal tracial states homeomorphic with the Cantor set, and satisfy a certain lifting property. The proof is based (as in the commutative case) on constructions of intermediate AF subalgebras (associated to closed subsets of $\delta_e T(A)$) of the crossed product and on the lifting.

We study the structure of the crossed products by $\mathbb{Z}$ of algebras of type $C(X) \otimes B$, where X is the Cantor set, B is a unital, simple, topological tracial rank zero or one C*-algebra and the automorphism is $\alpha_T \otimes \beta$, where T is a minimal homeomorphism of X and $\beta \in \mathrm{Aut}_0$. Using the work of Matui and Lin we show that the crossed product algebra has still topological tracial rank zero or one.

The last topic addressed in this thesis concerns the natural map $\widehat{\ }: A_{sa} \to \mathrm{Aff}T(A)$, where $T(A)$ is the simplex of tracial states of a unital C*-algebra A. Considering the natural orders on A_{sa} and on $\mathrm{Aff}T(A)$, we found examples of C*-algebras for which the natural maps lifts positive functions to positive elements in A.

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Chapter 1

INTRODUCTION

C*-algebras theory has experienced considerable expansion due to the introduction of the methods from ordered groups. For example, ordered K-theory is crucial to the Elliott theorem for AF algebras ([11]) and his subsequent classification programme, Effros-Handelman-Shen Theorem on the range of Elliott invariant ([9]), and the topological characterization, given by Giordano, Putnam and Skau, of the isomorphism of the crossed product C*-algebras arising from minimal dynamical systems on the Cantor set ([16]).

In this work we study relations between pre-ordered groups and crossed product C*-algebras of AF algebras by $\mathbb{Z}$. Some definitions are in order before stating the problems addressed in this work.

Let us recall ([23]) the construction of the crossed product of a C*-algebra by an action of a locally compact group. Let A be a C*-algebra, let G be a locally compact group with a left Haar measure ds and let $\gamma : G \longrightarrow \mathrm{Aut}(A)$ be a group homomorphism such that the following function is norm continuous on G for all $a \in A$: $t \mapsto \gamma_t(a)$. The triple (A, G, γ) is called a *C*-dynamical system.* Let $K(G, A)$ be the linear space of continuous functions from G to A with compact support, and endow this space with the following involution and multiplication:

$f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*)$

$f * g(s) := \int_G f(t)\alpha_t(g(t^{-1}s))dt$

for all $f, g \in K(G, A)$. In this way, $K(G, A)$ becomes a *-algebra. With the norm

$||f|| := \int_G ||f(s)||ds$, $K(G,A)$ is a normed algebra and denote by $L^1(G,A)$ its completion. The universal representation (π, H) of $L^1(G,A)$ is the direct sum of all non-degenerate representations of $L^1(G,A)$ and the full crossed product $A \ltimes_\alpha G$ is the norm closure of $\pi(L^1(G,A))$ in $B(H)$.

If $\alpha \in \mathrm{Aut}(A)$, one can define an action of $\mathbb{Z}$ on A. Define $\gamma_\alpha : \mathbb{Z} \longrightarrow \mathrm{Aut}(A)$ by $\gamma_\alpha(m) := \alpha^m$ for every $m \in \mathbb{Z}$. As above, one can consider the associated crossed product $A \ltimes_{\gamma_\alpha} \mathbb{Z}$, which from now on will be denoted $A \ltimes_\alpha \mathbb{Z}$.

Following [6], an approximately finite dimensional C*-algebra (abbreviated AF) is a separable C*-algebra such that there is an ascending sequence $\{A_n\}_{n\geq 0}$ of finite dimensional C*-subalgebras in A satisfying $A = \overline{\bigcup_n A_n}$. Strătilă and Voiculescu (see [32]) associated to each unital AF algebra a MASA (i.e., maximal abelian sub-algebra) C in A, a conditional expectation $P : A \longrightarrow C$ and a subgroup U of $\mathcal{U}(A)$ for which:

(i) $u^*Cu = C$, for all $u \in U$,

(ii) $P(u^*xu) = u^*P(x)u$ for all $u \in U$ and $x \in A$,

(iii) A is the closed linear manifold spanned of UC.

Let us recall briefly the construction (see [32]). If $A = \overline{\bigcup_n A_n}$ (where we assume $A_0 = \mathbb{C}1$) define inductively an ascending sequence $\{C_n\}_{n\geq 0}$ of commutative C*-subalgebras of A:

$C_0 := A_0$, $C_{n+1} := \langle C_n, D_{n+1}\rangle$, the C*-algebra generated by C_n and D_{n+1}, for $n \geq 0$, where D_{n+1} is an arbitrary MASA in $A_n' \bigcap A_{n+1}$. It turns out that C_n is a MASA in A_n. If $\{q_i\}_{i\in I_n}$ is the set of minimal projections of C_n, then for each $x \in A$, $P_n(x) := \sum_{i\in I_n} q_i x q_i$ defines a conditional expectation from A_n onto C_n such that $P_{n+1}|_{A_n} = P_n|_{A_n}$. Defining C to be $\overline{\bigcup_{n\geq 0} C_n}$, then C is a MASA in A and the above P_n's give rise to a conditional expectation $P : A \to C$. Construct for each $n \geq 0$ the group of unitary elements $\mathtt{U}_n := \{u \in U(A) | u^*C_n u = C_n\}$ and let $\mathtt{U} := \bigcup_{n\geq 0} \mathtt{U}_n$. Then they constructed a subgroup U of $\mathtt{U}$ such that $\mathtt{U}$ is the semi-direct product of its normal subgroup $\mathtt{U} \bigcap C$ by U.

To each C*-algebra A (unital or not), one can associate an abelian group $K_0(A)$ ([2]). This is done as follows. On the collection of all projections in $M_\infty(A) := \bigcup_{n=1}^{\infty} M_n(A)$, consider the Murray-von Neumann equivalence relation ($p \sim q$ iff there is partial isometry $v \in M_\infty(A)$ such that $p = v^*v$ and $q = vv^*$). Denoting by $[\cdot]$ the equivalence classes, let $V(A) := \{[p] | p = p^* = p^2 \in M_\infty(A)\}$. If the addition on $V(A)$ is defined by $[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$, then $V(A)$ is an abelian semigroup. Let $K_{00}(A)$ be the Grothendieck group of $V(A)$. The universal property of the Grothendieck groups implies that morphism $\pi_* : V(\tilde{A}) \mapsto V(\mathbb{C})$ induced by the canonical map $\pi : \tilde{A} \mapsto \mathbb{C}$ extends to a group morphism (denoted again π_*) between the K_{00}-groups:

$$\begin{array}{ccc} V(\tilde{A}) & \xrightarrow{\pi_*} & V(\mathbb{C}) \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_{\mathbb{C}}} \\ K_{00}(A) & \xrightarrow[\pi_*]{} & K_{00}(\mathbb{C}) \end{array}$$

where $\tilde{A}$ is the unitization of A, $\pi : \tilde{A} \mapsto \mathbb{C}$ is the canonical map and $i_A : V(A) \mapsto K_{00}(A)$ is given by $i_A([p]) := [p] - [0]$. Then one defines $K_0(A) := \mathrm{Ker}\{\pi_* : K_{00}(\tilde{A}) \mapsto K_{00}(\mathbb{C})\}$. Denote $K_0(A)^+ := i_A(V(A))$ and call the elements of $K_0(A)^+$ positive elements.

The main references for the K-theory used in this work are [2] and [13]. Because we use pre-ordered groups extensively let us recall the definition.

Definition 1.0.1. *A pre-ordered group (G, G^+) is an abelian group $(G, +)$ that contains a distinguished semigroup G^+ (the positive cone) containing the neutral element 0, satisfying $G = G^+ - G^+$. If $G^+ \bigcap -G^+ = \{0\}$, then (G, G^+) is called an ordered group.*

From the above definitions, it follows that $(K_0(A), K_0(A)^+)$ is a pre-ordered group. The positive cone G^+ induces a pre-order on G by $y \geq x$ if $y - x \in G^+$. An element $u \in G^+$ is an *order unit* if for any $g \in G$ there exists a $n \in \mathbb{N}^*$ such that $x \leq nu$. A *state* on a pre-ordered group with order unit (G, G^+, u) is a pre-order-preserving morphism $f : G \longmapsto \mathbb{R}$ with $f(u) = 1$.

The aim of the first section in Chapter 2 is to determine when an automorphism of C can be extended to an approximately inner automorphism of A, and to determine the connection between the ordered K_0-theory of the two crossed products arising in this manner. If α is an approximately inner extension of $\varphi \in \mathrm{Aut}(C)$ then the natural K_0-map from $K_0(C \ltimes_\varphi \mathbb{Z})$ to $K_0(A \ltimes_\alpha \mathbb{Z})$ lifts positives to positives exactly when the

K_0-map of the inclusion of A into $A \ltimes_\alpha \mathbb{Z}$ lifts positives to positives. Thus the next section is the natural continuation of section 1. This problem is related to a theorem proved in full generality in the commutative case by M. Boyle and D. Handelman in [5].

If X is a totally disconnected compact space and σ is a homeomorphism of X, then one can construct the crossed product, $C(X) \ltimes_{\alpha_\sigma} \mathbb{Z}$, associated to $C(X)$ and $\mathbb{Z}$ and the automorphism $\alpha_\sigma : C(X) \longrightarrow C(X)$ given by $\alpha_\sigma(f)(x) := f(\sigma^{-1}(x))$. Since $C(X)$ is an AF algebra, by the Pimsner-Voiculescu six term exact sequence, $\frac{K_0(C(X))}{H_{\alpha_\sigma}} \simeq K_0(C(X) \ltimes_{\alpha_\sigma} \mathbb{Z})$ as abelian groups, where $H_{\alpha_\sigma} := \{x - K_0(\alpha_\sigma)(x); x \in K_0(C(X))\}$. Of course $K_0(C(X)) \simeq C(X, \mathbb{Z})$ as ordered groups (see [29]). Now $C(X, \mathbb{Z})/H_{\alpha_\sigma}$ can be made into a pre-ordered abelian group, with cone

$$\{[f] \in C(X, \mathbb{Z})/H_{\alpha_\sigma}; \text{there is nonnegative function } f_0 \in C(X, \mathbb{Z}) \text{ such that } [f] = [f_0]\}.$$

After I. Putnam proved that for X the Cantor set and a minimal homeomorphisms σ, one has that $K_0(C(X))/H_{\alpha_\sigma} \simeq K_0(C(X) \ltimes_{\alpha_\sigma} \mathbb{Z})$ as ordered abelian groups (the order on $K_0(C(X) \ltimes_{\alpha_\sigma} \mathbb{Z})$ is the natural order, see [2]), see [29], M. Boyle and D. Handelman proved in [5] that the pre-ordered isomorphism above still holds for every zero-dimensional compact space and every homeomorphism. We address the same question in section 2 of Chapter 2, replacing $C(X)$ by noncommutative unital AF algebras. If the algebra is simple the answer is affirmative. Non simple examples are also given. Since AF algebras are exact, for $\alpha \in \mathrm{Aut}(A)$ and $\varphi \in \mathrm{Aut}(C)$ considered in section 1, one may consider their entropies $ht(\alpha)$ and $ht(\varphi)$ as defined by N. Brown in [3]. In section 3 we show the equalities of the these entropies. Section 4 of Chapter 2 contains two results on AF-embeddability of certain crossed products of commutative AF algebras by $\mathbb{Z}$.

Chapter 3 deals with the generalization to the noncommutative case of constructions of I. Putnam and Y.T. Poon regarding the intermediate C*-subalgebras of the crossed product algebra $C(X) \ltimes_T \mathbb{Z}$, where X is a compact totally disconnected space and T a self-homeomorphism of X. Let us recall briefly one of the constructions. For any nonempty closed subset Y of X, denote by A_Y the C*-subalgebra of $C(X) \ltimes_T \mathbb{Z}$ generated by $C(X)$ and $uC_0(X \backslash Y)$, where u is the canonical unitary implementing the action of $\mathbf{Z}$ on $C(X)$ and $C_0(X\backslash Y) := \{f \in C(X) | f|_Y = 0\}$. Using the first return map, it was proved ([26]) that A_Y is an AF algebra if and only if $X = \bigcup_{n\in\mathbb{Z}} T^n(W)$ for any clopen subset W of X containing Y. Putnam used the appropriate intermediate C*-algebras to show that, when T is minimal and X is the Cantor set, then $C(X) \ltimes_T \mathbb{Z}$ is a simple AT algebra.

We consider in Chapter 3 the analogous constructions for AF algebras that are not necessary commutative, but have a totally disconnected boundary (i.e., the set of all extremal tracial states). For some automorphisms α of A, we associate to each closed subset I of $\delta_e T(A)$ an AF intermediate subalgebra of the crossed product $A \ltimes_\alpha \mathbb{Z}$. In the non-commutative case, some natural conditions (such as the natural map$\widehat{\ }$: $A_{sa} \longrightarrow \mathrm{Aff}(T(A))$ lifts positives to positives and torsion free K-groups) are needed in order to prove the structure of the crossed product in this case (the crossed product $A \ltimes_\alpha \mathbb{Z}$ is an AT algebra). The theorem and the main lemma of section 4, suggested by Professor Hiroki Matui, deal with the structure of crossed products by $\mathbb{Z}$ of algebras of the type $C(X) \otimes B$, where X is totally disconnected, B is a unital, simple, topological tracial rank zero or one C*-algebra and the automorphism is $\alpha_T \otimes \beta$, where T is a minimal homeomorphism of X and β an approximately inner automorphism of B. Roughly speaking, the crossed product algebra has still topological tracial rank zero or one. The old idea of finding a nice "large" subalgebra of the crossed product C*-algebra is exploited. Both results mentioned above are based on a suitable approximation of finite subsets of $C(X) \otimes B$.

Given a unital C*-algebra A, let $T(A)$ be the set of all tracial states of A and let $AffT(A)$ be the real Banach space of all continuous affine functions on $T(A)$. There is a natural map$\widehat{\ }$: $A_{sa} \longmapsto AffT(A)$, given by $\widehat{a}(\tau) := \tau(a)$ for all $\tau \in T(A)$. It was proved in [7] that the natural map is onto. Equip $AffT(A)$ with the positive cone $AffT(A)^+ := \{f \in AffT(A) | 0 \leq f\}$. If $f > 0$ then it is easy to check (see 4.1.10) that f can be lifted back in A (through the natural map) to a positive element. Chapter 4 contains examples of C*-algebras for which the natural map lifts positives to positives and their properties (such as hereditary behavior under taking quotients, taking crossed products or taking tensor products) and furnishes examples for Chapter 3. Some counterexamples are also given.

Chapter 2

ORDERED GROUPS AND CROSSED PRODUCTS

2.1 Masas and crossed products

Let C be a Strătilă-Voiculescu MASA in a unital AF algebra A. An automorphism α on A is called inner if there is an unitary $u \in A$ such that $\alpha(x) = uxu^*$ for all $x \in A$. An automorphism α on A is called approximately inner if, for every finite subset F of A and every $\epsilon > 0$, there is an inner automorphism β on A such that $||\alpha(a) - \beta(a)|| < \epsilon$ for all $a \in F$. Denote by $Aut(A)_0$ the group of all approximately inner automorphisms of A. If $\varphi \in \mathrm{Aut}(C)$ has an extension to $\alpha \in \mathrm{Aut}_0(A)$ then one has for every two projections p, q in C that $\varphi(p) = q$ implies $p \sim_A q$. Denote this condition by $\bigcirc$. We shall prove the converse:

Theorem 2.1.1. *Let C be a Stratila-Voiculescu MASA in A. Then for every $\varphi \in Aut(C)$ which satisfies $\bigcirc$, there exists $\alpha \in Aut_0(A)$ such that $\alpha|_C = \varphi$.*

Proof. Suppose that $A = \overline{\bigcup_n B_n}$ and that $C = \overline{\bigcup_n C_n}$, as in the beginning of the Introduction, where the MASA C was defined.

Let $p_1, p_2, ..., p_r$ be the minimal (equivalent in A, orthogonal) projections of C_1 (here $r = 1$; we shall use r to see how the general construction works). These projections are equivalent in B_1. From condition $\bigcirc$, it follows that $p_1 \sim_A \varphi(p_1), \ldots, p_r \sim_A$

$\varphi(p_r)$. Thus $\varphi(p_1) \sim_A \cdots \sim_A \varphi(p_r)$. These projections are still orthogonal. Since $C = \overline{\bigcup_{n=1} C_n}$ is an AF algebra, there exists $n_1 > 1$ and $P_1, \ldots, P_r \in C_{n_1}$ such that $\varphi(p_1)$ is close in C to P_1, $\ldots$, $\varphi(p_r)$ is close in C to P_r. Since C is abelian, we get $\varphi(p_i) = P_i$ for every i. So we can suppose that $C_{n_1} \ni \varphi(p_1) \sim_A \cdots \sim_A \varphi(p_r)$. By enlarging n_1 if necessary (we are dealing with AF algebras), we may suppose that the equivalence is realized in B_{n_1}. Let $B_{n_1} \ni E_{ij}^{n_1}$ be the implementing partial isometries. Define $\phi_1 : B_1 \longrightarrow B_{n_1}$ by $\phi_1(e_{ij}^1) = E_{ij}^{n_1}$. Of course $\phi_1|_{C_1} = \varphi|_{C_1}$. It is a *-morphism. Do the same thing to $B_{n_1} \supseteq C_{n_1}$ and φ^{-1} and obtain $\psi_1 : B_{n_1} \longrightarrow B_{n_2}$ a *-morphism for some $n_2 > n_1$ such that $\psi_1|_{C_{n_1}} = \varphi^{-1}|_{C_{n_1}}$. Of course $\psi_1 \circ \phi_1(c) = c$ for every c in C_1.

Now there exists a $w \in U(B_{n_2})$ such that $\mathrm{Ad}(w) \circ \psi_1 \circ \phi_1$ is the inclusion of B_1 in B_{n_2}, denoted i_{1n_2}. Thus $\psi_1 \circ \phi_1 = \mathrm{Ad}(w^*) \circ i_{1n_2}$, i.e., the following diagram commutes:

$$\begin{tikzcd} B_1 \arrow[r, "L"] \arrow[dr, "\phi_1"'] & B_{n_2} \\ & B_{n_1} \arrow[u, "\psi_1"'] \end{tikzcd}$$

where $L := \mathrm{Ad}(w^*) \circ i_{1n_2}$.

Continuing this process, we obtain the following diagram:

$$\begin{tikzcd} B_1 \arrow[r, "\mathrm{Ad}(w^*)\circ i_{1n_2}"] \arrow[d, "id"'] & B_{n_2} \arrow[r, "\mathrm{Ad}(u^*)\circ i_{n_2n_4}"] \arrow[d, "\mathrm{Ad}(w)"] & B_{n_4} \arrow[r] \arrow[d, "\mathrm{Ad}(w)\circ\mathrm{Ad}(u)"] & \cdots \arrow[d] \\ B_1 \arrow[r, "i_{1n_2}"] & B_{n_2} \arrow[r, "i_{n_2n_4}"] & B_{n_4} \arrow[r] & \cdots \end{tikzcd}$$

Let X and Y be the inductive limits of the upper and lower sequences, respectively. The commutativity of the diagram above gives rise to an isomorphism $\alpha : X \longmapsto Y$ such that $\alpha|_C = \varphi$ (because $\phi|_{C_{n_k}} = \varphi|_{C_{n_k}}$, for every $k \geq 1$). To show that $X = Y = A$ we have to use Lemma 2.1 from [13].

Since $\mathrm{id}, \mathrm{Ad}(w), \mathrm{Ad}(w) \circ \mathrm{Ad}(u), \ldots$ are inner, we have only to show the commutativity of the squares, which is evident. The same arguments applies for the bottom sequence. Since for all is, we have that $p_i \sim_A \varphi(p_i)$ we see that α is a limit of inner automorphisms (relative to A). $\square$

Examples of automorphisms which have the property ◯ include those of the form $\varphi(x) = uxu^*$ where $u \in U$, see Introduction or [32].

Suppose φ is given on C satisfying condition ◯ and consider the associated approximately inner automorphism $\alpha \in \mathrm{Aut}(A)$ constructed as above. The following diagram comes from the Pimsner-Voiculescu exact sequence [24] :

$$\begin{tikzcd}
K_1(C \ltimes_\varphi \mathbb{Z}) \arrow[d] & 0 \arrow[l] & 0 \arrow[l] \\
K_0(C) \arrow[r, "id-K_0(\varphi)"] \arrow[d] & K_0(C) \arrow[r, "i_1"] \arrow[d, "j_1"'] & K_0(C \ltimes_\varphi \mathbb{Z}) \arrow[u] \arrow[d, "j"'] \\
K_0(A) \arrow[r, "id-K_0(\alpha)"'] & K_0(A) \arrow[r, "i_2"'] & K_0(A \ltimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \ltimes_\alpha \mathbb{Z}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where j_1 is the induced map from the inclusion of C in A, j is the induced map from the canonical embedding of $C \ltimes_\varphi \mathbf{Z}$ in $A \ltimes_\alpha \mathbf{Z}$, $i_1 = K_0(i_C)$ and $i_2 = K_0(i_A)$ (here i_C is the embedding of C in $C \ltimes_\varphi \mathbf{Z}$ and i_A is the embedding of A in $A \ltimes_\alpha \mathbf{Z}$).

Proposition 2.1.2. *The map j is always onto. In fact, one has $j_1(K_0(C)^+) = K_0(A)^+$.*

Proof. As the right square is commutative, it is sufficient to show that j_1 is onto (recall that i_1 and i_2 are already onto by the Pimsner-Voiculescu sequence). We now show that $j_1(K_0(C)^+) = K_0(A)^+$.

The inclusion "$\subseteq$"is always true.

Now suppose $K_0(A)^+ \ni x = [q]_0^A$ with q a projection in $M_n(A)$. Since $\Gamma(A)$ (the scale) generates $K_0(A)^+$, we can find projections $p_1, \ldots, p_n \in A$ such that $[q]_0^A = [p_1]_0^A + \ldots + [p_n]_0^A$. Thus it is sufficient to lift an element x of the form $K_0(A)^+ \ni x = [p]_0^A$, where p is a projection in A. Suppose that $A = \overline{\bigcup_{n=1} A_n}$. Then there exist a $n_0 \in \mathbf{N}$ and q in A_{n_0}, a projection such that $[p]_0^A = [q]_0^A$. But then $[q]_0^A = [r]_0^A$ for some projection r in C_{n_0}. So $j_1([r]_0^C) = [r]_0^A = [q]_0^A = [p]_0^A = x$. $\square$

Proposition 2.1.3. *The map j is injective if and only if $Ker(j_1) \subseteq Im(id - K_0(\varphi))$. In other words ($\alpha$ being approximately inner) j is injective if and only if the following sequence is exact:*

$$\begin{array}{ccc} K_0(C) & \xrightarrow{id-K_0(\varphi)} & K_0(C) \\ & & \Big\downarrow{\scriptstyle j_1} \\ & & K_0(A) \end{array}$$

Proof. $\Longleftarrow$ Suppose $j(x) = 0$ for some $x \in K_0(C \ltimes_\varphi \mathbf{Z})$. As i_1 is onto, there exists y in $K_0(C)$ such that $i_1(y) = x$. Since the right square is commutative, it follows that $i_2 \circ j_1(y) = 0$. Denote $K_0(A) \ni z := j_1(y)$. So $i_2(z) = 0$. Since α is approximately inner, i_2 is injective also, so $z = 0$. Thus $j_1(y) = 0$, i.e., $y \in \mathrm{Ker}(j_1) \subseteq H_\alpha$. The exactness of the second row in the diagram (the Pimsner-Voiculescu sequence) implies that $x = i_1(y) = 0$, so j is injective.

$\Longrightarrow$ Suppose that j is injective. Taking $x \in \mathrm{Ker}(j_1)$ we see $j \circ i_1(x) = i_2 \circ j_1(x) = 0$. Since j is injective we get $i_1(x) = 0$, so $x \in \mathrm{Ker}(i_1) = H_\varphi$ by the Pimsner-Voiculescu sequence. $\square$

Remark 2.1.4. *If the map $j_1 : K_0(C) \longrightarrow K_0(A)$ is injective and $\varphi \in Aut(C)$ satisfies $\bigcirc$ then $\varphi = id$.*

Proof. Suppose that $j_1 : K_0(C) \longrightarrow K_0(A)$ is injective. On taking an arbitrary $\varphi \in \mathrm{Aut}(A)$ satisfying $\bigcirc$, it follows that j is injective (indeed, using the same notation as in Proposition 3, we have j_1 is injective $\Rightarrow y = 0 \Rightarrow x = i_1(y) = 0 \Rightarrow j$ is injective). Using Proposition 3, we see that $\{0\} = \mathrm{Ker}(j_1) = H_\alpha$, thus $K_0(\varphi) = \mathrm{id}$, so by [14] (Theorem 3.8) we have that φ is approximately inner. Since C is commutative, the only possibility is $\varphi = \mathrm{id}$. $\square$

Since $\alpha(C) = C$, one knows that $C \ltimes_\varphi \mathbf{Z}$ is embedded in $A \ltimes_\alpha \mathbf{Z}$. Then $j(K_0(C \ltimes_\varphi \mathbf{Z})^+) \subseteq K_0(A \ltimes_\alpha \mathbf{Z})^+$. To determine when is j an order isomorphism, the next goal is to investigate when $j(K_0(C \ltimes_\varphi \mathbf{Z})^+) \supseteq K_0(A \ltimes_\alpha \mathbf{Z})^+$. We are working with order and not pre-order isomorphisms (since α is approximately inner, $A \ltimes_\alpha \mathbb{Z}$ is stably finite, by Theorem 0.2 of [4]), and thus the cone $(K_0(A \ltimes_\alpha \mathbb{Z}))^+)$ is strict. Because of Proposition 2.1.2 and Theorem 5.2 of [5] ($i_1 : K_0(C) \longrightarrow K_0(C \ltimes_\varphi \mathbb{Z})$ lifts positives to positives), we obtain that $j(K_0(C \rtimes_\varphi \mathbf{Z})^+) \supseteq K_0(A \rtimes_\alpha \mathbf{Z})^+$ is equivalent to $K_0(i_2) : K_0(A) \longrightarrow K_0(A \rtimes_\alpha \mathbb{Z})$ lifting positives to positives. From now on, we shall use i instead of i_2 for the canonical inclusion of A in $A \rtimes_\alpha \mathbf{Z}$. Recall that for finite dimensional C*-algebras, the following is true.

Proposition 2.1.5. *If A is a finite-dimensional C^*algebra and $\alpha \in Aut(A)$, then we can lift positives to positives in the following situation, $K_0(i) : K_0(A) \longrightarrow K_0(A \rtimes_\alpha \mathbb{Z})$.*

Proof. If $K_0(A \rtimes_\alpha \mathbb{Z})^+ \ni x = [b]_0^{A \rtimes_\alpha \mathbb{Z}}$ then there exist $0 \leq y_1 = [p]_0^A$ and $0 \leq y_2 = [q]_0^A$ such that $K_0(i)(y_1 - y_2) = x$, where p and $q \in P_n(A)$ for some $n \in \mathbb{N}$ (because $K_0(i)$ is onto). We can even suppose that $b \in P_n(A \rtimes_\alpha \mathbb{Z})$. Finite-dimensionality of A implies that α is inner ($\alpha = \mathrm{Ad}(u)$, for some unitary u in A). Let $\bar{\alpha} \in \mathrm{Aut}(M_n(A))$ be the induced automorphism on $M_n(A)$; we see that $\bar{\alpha}$ is also inner. By the universal property of crossed products, we can find an isomorphism $\psi : M_n(A) \rtimes_{\bar{\alpha}} \mathbb{Z} \longrightarrow M_n(A \rtimes_\alpha \mathbb{Z})$ such that the following diagram is commutative:

$$\begin{array}{ccc} M_n(A) & \xrightarrow{i_n} & M_n(A) \rtimes_{\bar{\alpha}} \mathbb{Z} \\ \| & & \downarrow \psi \\ M_n(A) & \xrightarrow{i^n} & M_n(A \rtimes_\alpha \mathbb{Z}) \end{array}$$

where i_n and i^n are the canonical inclusions. Indeed, if $\varphi : M_n(A) \longrightarrow M_n(A) \rtimes_\alpha \mathbb{Z}$ is given by $\varphi(x) = x$ and $v := \begin{pmatrix} u & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix} \in U(M_n(A \rtimes_\alpha \mathbb{Z}))$, then $\varphi(\bar{\alpha}(a)) = v\varphi(a)v^*$ for all $a \in M_n(A)$. It follows that there exists a *-morphism $\psi : M_n(A) \rtimes_{\bar{\alpha}} \mathbb{Z} \longrightarrow M_n(A \rtimes_\alpha \mathbb{Z})$ such that $\psi(xw^n) = \varphi(x)v^n$, where w is the canonical unitary implementing $\bar{\alpha}$. It turns out that ψ is an automorphism.

Now we notice that $\psi \circ i_n = i^n \circ id$: for every $x \in M_n(A)$, we have that $\psi \circ i_n(x) = \psi(x) = x$ and $i^n \circ id(x) = i^n(x) = x$. Let τ be an arbitrary trace on $M_n(A)$ and let $\tilde{\tau}$ be its canonical extension on $M_n(A) \rtimes_{\bar{\alpha}} \mathbb{Z}$ (since $\bar{\alpha}$ is inner, it will follow that τ is $\bar{\alpha}$-invariant). Defining $\bar{\tau} : M_n(A \rtimes_\alpha \mathbb{Z}) \longrightarrow \mathbb{C}$ by $\bar{\tau} := \tilde{\tau} \circ \psi^{-1}$, we see that $\bar{\tau}$ is a trace since $\tilde{\tau}$ is a trace. Then $\tau(p) - \tau(q) = K_0(\tau)([p]_0^A - [q]_0^A) = K_0(\tilde{\tau}) \circ K_0(i_n)([p]_0^A - [q]_0^A) =$

$K_0(\tilde{\tau} \circ i_n)([p]_0^A - [q]_0^A) = K_0(\bar{\tau} \circ i^n)([p]_0^A - [q]_0^A) = K_0(\bar{\tau})([b]_0^{A \rtimes_\alpha \mathbb{Z}}) \geq 0$ since $[b]_0^{A \rtimes_\alpha \mathbb{Z}} \geq 0$ and $\bar{\tau}$ is trace on $M_n(A \rtimes_\alpha \mathbb{Z})$. Since τ was chosen arbitrarily and $M_n(A)$ is finite-dimensional, it follows that $[p]_0^A - [q]_0^A \geq 0$, i.e. $x = K_0(i)(y_1 - y_2)$ and $y_1 - y_2 \geq 0$. □

The next step is

Proposition 2.1.6. *Let A be a unital AF algebra and $u \in U(A)$, a unitary. Then one can lift positives to positives in the following situation, $K_0(i) : K_0(A) \longrightarrow K_0(A \rtimes_{Adu} \mathbb{Z})$.*

Proof. The following chain of isomorphisms $A \rtimes_{\mathrm{Ad}u} \mathbb{Z} \simeq A \otimes C(T) = (\lim_n A_n) \otimes C(T) = \lim_n (A_n \otimes C(T))$ implies the following chain of order isomorphisms $K_0(A \rtimes_{\mathrm{Ad}u} \mathbb{Z}) \simeq K_0(\lim_n A_n \otimes C(T)) \simeq \lim_n K_0(A_n \otimes C(T)) \simeq \lim_n K_0(A_n) \simeq K_0(\lim_n A_n) \simeq K_0(A)$, where A is the inductive limit of an ascending sequence of finite dimensional C*-algebras $(A_n)_n$. We have used the continuity of the K_0 functor and the previous proposition. □

Proposition 2.1.7. *Let A be a unital AF algebra and let $\alpha \in Aut(A)$ be an approximately inner automorphism such that $A \rtimes_\alpha \mathbb{Z}$ is a simple C*-algebra. Then one can lift positives to positives in the following situation, $K_0(i) : K_0(A) \longrightarrow K_0(A \rtimes_\alpha \mathbb{Z})$.*

Proof. Let $0 \neq y = [P]_0^{A \rtimes_\alpha \mathbb{Z}}$ be a positive element in $K_0(A \rtimes_\alpha \mathbb{Z})$, where P is a non-zero projection in $M_k(A \rtimes_\alpha \mathbb{Z})$. Since $K_0(i)$ is onto, there is an $x \in K_0(A)$ such that $K_0(i)(x) = y$. Suppose that x is not positive. Then there exists a tracial state $\tau \in T(A)$ such that $K_0(\tau)(x) \leq 0$. Since α is approximately inner, we have that τ is α invariant, thus it will extend to a tracial state on the crossed product, denoted $\bar{\tau}$. We have $0 \geq K_0(\tau)(x) = K_0(\bar{\tau})(y) \geq 0$, and follows that $K_0(\bar{\tau})(y) = 0$, i.e., $K_0(\bar{\tau})([P]_0^{A \rtimes_\alpha \mathbb{Z}}) = 0$. Every tracial state on the crossed product can be extended it to a tracial state on matrices over the crossed product, and since the crossed product is simple, it follows that $M_k(A \rtimes_\alpha \mathbb{Z})$ is simple, contradicting $\bar{\tau}(P) = 0, P \neq 0$. □

If A is simple and α not inner (in contrast to 2.1.6), then by Theorem 3.1 in [17], the crossed product algebra is simple.

If $\alpha \in \mathrm{Aut}(A)$, then α is homotopic to identity if and only if α is approximately inner [8] (Theorem IV.5.7). Thus we obtain:

Theorem 2.1.8. *Let A be an unital AF algebra and C a Stratila-Voiculescu MASA of A. Suppose $\varphi \in Aut(C)$ satisfies $\bigcirc$ and α denotes its extension to an element of $Aut_0(A)$. If $A \ltimes_\alpha \mathbb{Z}$ is simple and $Ker(j_1) \subseteq H_\varphi$ then $K_0(C \ltimes_\varphi \mathbb{Z}) \simeq K_0(A \ltimes_\alpha \mathbb{Z})$ as ordered abelian groups. [Remark: this is the isomorphism induced by the inclusion $C \ltimes_\varphi \mathbb{Z} \hookrightarrow A \ltimes_\alpha \mathbb{Z}$].*

Proof. Use Proposition 2.1.3 for injectivity and Proposition 2.1.7 for $j(K_0(C \ltimes_\varphi \mathbf{Z})^+) \supseteq K_0(A \ltimes_\alpha \mathbf{Z})^+$, taking in account that the map $j_1 : K_0(C) \longrightarrow K_0(A)$ lifts positives to positives, see Proposition 2.1.2. See also the paragraph after Remark 2.1.4. $\square$

Remark 2.1.9. *One can define a Stratila-Voiculescu MASA in an arbitrary AF algebra (unital or not). Using the exact sequence $0 \longmapsto A \longmapsto \tilde{A} \longmapsto C(\mathrm{T}) \longmapsto 0$, where $\tilde{A}$ is the unitization of A, the conclusions of Theorems 2.1.6 and 2.1.7 and 2.1.8 are still valid.*

Proposition 2.1.10. *Let $\varphi \in Aut(C)$ satisfy $\bigcirc$. Then $C \ltimes_\varphi \mathbb{Z}$ is AF-embeddable .*

Proof. Since the associated automorphism α (see Theorem 2.1.1) is approximately inner, by Theorem 0.2 of [4], $A \ltimes_\alpha \mathbb{Z}$ is AF-embeddable and then $C \ltimes_\varphi \mathbb{Z}$ is AF-embeddable. $\square$

We shall prove next that another class of automorphisms satisfies the lifting property. Let us recall (see [10]) the following

Definition 2.1.11. *Let A be a unital C^*-algebra and $\alpha \in Aut(A)$. Then α satisfies the Rokhlin property if for every $k \in \mathbb{N}$ there are positive integers $k_1, \ldots, k_m \geq k$ such that for every finite subset, $F \subset A$, and every $\epsilon > 0$, there exist projections $e_{i,j}$, $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$ in A for which*

$$\sum_{i=1}^{m} \sum_{j=0}^{k_i-1} e_{i,j} = 1_A,$$

$$||\alpha(e_{i,j}) - e_{i,j+1}|| < \epsilon,$$

$$||[x, e_{i,j}]|| < \epsilon$$

for $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$ and all $x \in F$ (where $e_{i,k_i} = e_{i,0}$). For all i above, the set of projections $\{e_{i,j}; 0 \leq j \leq k_i - 1\}$ is called a Rohlin tower.

It was proved ([48]) that the following two conditions for a single automorphisms α of a unital AF algebra are equivalent:

i) For any $\epsilon > 0$ there is a $u \in U(A)$ and an increasing sequence $(A_n)_n$ of finite dimensional C*-subalgebras of A such that $||u - 1|| < \epsilon$, $A = \overline{\bigcup_n A_n}$ and $A_n = \mathrm{Ad}u \circ \alpha(A_n)$ for every n.

ii) There exist an increasing sequence $(A_n)_n$ of finite dimensional C*-subalgebras of A such that $A = \overline{\bigcup_n A_n}$ and $\lim_{n\to\infty}\mathrm{dist}(\alpha(A_n), A_n) = 0$.

The distance is defined as follows ([48]). Let B and C be C*-subalgebras of a C*-algebra A. By $B \subset^{\delta} C$, one understands that for any $x \in B$ there exists a $y \in C$ such that $||x - y|| \leq \delta||x||$. By definition, $\mathrm{dist}(B, C) := \inf\{0 < \delta; B \subset^{\delta} C, C \subset^{\delta} B\}$. It is known ([48]) that if α is approximately inner and α has the Rohlin property, then α satisfies condition i). The following proposition will be improved in the next section.

Proposition 2.1.12. *If A is a unital AF algebra and $\alpha \in Aut(A)$ is approximately inner and satisfies the Rokhlin property, then one can lift positives to positives in the following situation, $K_0(i) : K_0(A) \longrightarrow K_0(A \rtimes_{\alpha} \mathbb{Z})$.*

Proof. Since A is AF, one can write A as a inductive limit of an ascending sequence of finite dimensional C*-subalgebras $(A_n)_n$, hence $A = \overline{\bigcup_n A_n}$. From the results above, for $\epsilon = 1/2$, there is a $u \in A$ such that if $\alpha_n := \mathrm{Ad}u \circ \alpha|_{A_n}$, we have that $\alpha_n(A_n) = A_n$ for every n. Since $\alpha_{n+1}|_{A_n} = \mathrm{Ad}u \circ \alpha|_{A_n} = \alpha_n|_{A_n}$ we get the following commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow[\hookrightarrow]{} & A_{n+1} \\ {\scriptstyle \alpha_n}\downarrow & & {\scriptstyle \alpha_{n+1}}\downarrow \\ A_n & \xrightarrow[\hookrightarrow]{} & A_{n+1} \end{array}$$

for all n. It follows that the following diagram is also commutative for all n

$$\begin{array}{ccc} A_n \rtimes_{\alpha_n} \mathbb{Z} & \xrightarrow[\hookrightarrow]{} & A_{n+1} \rtimes_{\alpha_{n+1}} \mathbb{Z} \\ \downarrow & & \downarrow \\ A_n \rtimes_{\alpha_n} \mathbb{Z} & \xrightarrow[\hookrightarrow]{} & A_{n+1} \rtimes_{\alpha_{n+1}} \mathbb{Z} \end{array}$$

It is now easy to see that $A \rtimes_{\mathrm{Ad}u\circ\alpha} \mathbb{Z} = \lim_n(A_n \rtimes_{\alpha_n} \mathbb{Z})$. By continuity of the K_0 functor, $K_0(A \rtimes_{\mathrm{Ad}u\circ\alpha}\mathbb{Z})^+ = \lim_n K_0(A_n \rtimes_{\alpha_n} \mathbb{Z})^+$. From lemma 2.1.5, and $A \rtimes_\alpha \mathbb{Z} \simeq A \rtimes_{\mathrm{Ad}u\circ\alpha} \mathbb{Z}$, the conclusion follows. □

Some examples of automorphisms satisfying Rokhlin property are listed below.

Example 2.1.13. *1) It follows from [19] (see Theorem 1.3) that any UHF algebra admits an automorphism satisfying the Rokhlin property.*

2) If A and B are two unital AF algebras, $\alpha \in Aut(A)$ satisfies the Rohlin property, and $\beta \in Aut(B)$, then $\alpha \bigotimes \beta \in Aut(A \bigotimes B)$ satisfies the Rohlin property. Indeed, use the density of the algebraic tensor product and just tensor the appropriate Rokhlin towers for α with 1_B.

There exists a weaker property for automorphisms, defined by Osaka and Phillips in [20], namely, the tracial Rokhlin property. The following definition is from [20].

Definition 2.1.14. *Let A be a unital, simple and stably finite C*-algebra A. The automorphism α of A is said to have the tracial Rokhlin property if for every finite set $F \subseteq A$, every $\epsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ satisfying:*

i) $||\alpha(e_j) - e_{j+1}|| < \epsilon$ *for* $0 \leq j \leq n-1$.

ii) $||e_j a - a e_j|| < \epsilon$ *for* $0 \leq j \leq n$ *and all* $a \in F$.

iii) If $e := \sum_{j=0}^{n} e_j$ *then* $1 - e$ *is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of* A *generated by* x.

See [20] for a discussion about the connections between the Rokhlin and tracial Rokhlin properties. The facts from [20] (Corollary 1.14. and Proposition 2.4) which we shall need are:

1. If A is a simple unital stably finite C*-algebra and $\alpha \in \mathrm{Aut}(A)$ has the tracial Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is simple.

2. Suppose that A is a simple unital infinite dimensional C*-algebra with real rank 0 and that the order on projections is determined by traces (i.e. $\forall p, q \in P_\infty(A)$ such that $\tau(p) < \tau(q), \forall \tau \in T(A)$ one has $p \lesssim q$). If $\gamma : \Gamma \longrightarrow \mathrm{Aut}(A)$ is an action of a countable amenable group, then the following holds: for any two projections $p, q \in M_\infty(A)$ such that $\tau(p) < \tau(q)$ $\forall$ Γ-invariant tracial states τ on A there exists $r \in M_\infty(A \rtimes_\gamma \mathbb{Z})$ such that $p = r^*r$, $rr^* \leq q$, and $rr^* \in M_\infty(A)$, (in particular, $p \lesssim q$ in $M_\infty(A \rtimes_\gamma \mathbb{Z})$). Here we extend τ in the obvious way to $M_\infty(A)$.

Proposition 2.1.15. *Let A be a unital simple AF algebra and $\alpha \in Aut(A)$ satisfying the tracial Rokhlin property. Then the map $K_0(i) : K_0(A) \longrightarrow K_0(A \rtimes_\alpha \mathbb{Z})$ lifts positives to positives.*

Proof. Letting $0 \neq x \in K_0(A \rtimes_\alpha \mathbb{Z})^+$, one can suppose that $x := [P]_0^{A\rtimes_\alpha\mathbb{Z}}$ for a non-zero projection $P \in M_\infty(A \rtimes_\alpha \mathbb{Z})$. Since $K_0(i)$ is onto one can find two projections $p, q \in M_n(A)$, for some $n \in \mathbb{N}$, such that $K_0(i)([p]_0^A - [q]_0^A) = x$, hence $[P]_0^{A\rtimes_\alpha\mathbb{Z}} = [p]_0^{A\rtimes_\alpha\mathbb{Z}} - [q]_0^{A\rtimes_\alpha\mathbb{Z}}$.

Since A is simple and AF, it follows that $M_n(A) \otimes A$ is simple and AF, and because α has the tracial Rokhlin property, $id_{M_n(A)} \otimes \alpha : M_n(A) \otimes A \longrightarrow M_n(A) \otimes A$ also has the Rokhlin property (see the second part of Example 2.1.13). From Fact 1 above, $M_n(A) \otimes A \rtimes_{id_{M_n(A)} \otimes \alpha} \mathbb{Z}$ is a simple C*-algebra. If τ is any α-invariant tracial state of A, denote its extension to $M_n(A)$ again by τ, and the extension of the latter trace to the corresponding crossed product algebra by $\bar{\tau}$. From the simplicity of the crossed product algebra, $0 < \bar{\tau}(P) = K_0(\bar{\tau})([P]_0^{A\rtimes_\alpha\mathbb{Z}}) = K_0(\tau)([p]_0^A - [q]_0^A) = \tau(p) - \tau(q)$. Since for AF algebras the strict order on projections is determined by traces, see [21], we can apply Fact 2 to get a projection $r \in M_\infty(M_n(A) \otimes A \rtimes_{id_{M_n(A)} \otimes \alpha} \mathbb{Z})$ such that $q = r^*r$, $rr^* \leq p$ and $rr^* \in M_\infty(M_n(A))$. Thus $[P]_0^{A\rtimes_\alpha\mathbb{Z}} = [p]_0^{A\rtimes_\alpha\mathbb{Z}} - [q]_0^{A\rtimes_\alpha\mathbb{Z}} = [p]_0^{A\rtimes_\alpha\mathbb{Z}} - [r^*r]_0^{A\rtimes_\alpha\mathbb{Z}} = [p]_0^{A\rtimes_\alpha\mathbb{Z}} - [rr^*]_0^{A\rtimes_\alpha\mathbb{Z}} = [p - rr^*]_0^{A\rtimes_\alpha\mathbb{Z}}$ and because $p - rr^* \in M_\infty(A)$ (so $K_0(i)([p - rr^*]_0^A) = x$) one gets the lifting. $\square$

Fact 2 above yields the most general result.

Theorem 2.1.16. *Let A be a unital simple AF algebra and $\alpha \in Aut(A)$. Then $K_0(i) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$ lifts positives to positives.*

Proof. We can assume that A is infinite dimensional, otherwise we can use Proposition 2.1.5. If α is inner, we can use Proposition 2.1.6. If α is not inner, we get that $A \ltimes_\alpha \mathbb{Z}$ is a simple C*-algebra (see [17]) and the proof is word for word that of the previous proposition (the tracial Rokhlin property was needed in the proposition above only for the simplicity of the crossed product C*-algebra $A \ltimes_\alpha \mathbb{Z}$). □

2.2 (Pre)-Order Isomorphism

We shall now investigate another related problem. Given an AF algebra A and $\alpha \in \mathrm{Aut}(A)$, we have the Pimsner-Voiculescu exact sequence (see [2] or [24]):

$$\begin{tikzcd} K_0(A) \arrow[r, "id-K_0(\alpha)"'] & K_0(A) \arrow[r, "i"'] & K_0(A \ltimes_\alpha \mathbb{Z}) \arrow[d] \\ K_1(A \ltimes_\alpha \mathbb{Z}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Thus there exists a group isomorphism $T : K_0(A \ltimes_\alpha \mathbf{Z}) \longrightarrow K_0(A)/H_\alpha$, where $H_\alpha := \mathrm{Im}(K_0(\alpha) - id)$ and $T(x) = [y]$ (because for every $x \in K_0(A \ltimes_\alpha \mathbf{Z})$ there exists an $y \in K_0(A)$ such that $K_0(i)(y) = x$). We shall find some conditions when this group isomorphism is a (pre) order isomorphism. We consider $K_0(A \ltimes_\alpha \mathbf{Z})$ equipped with the natural order and $K_0(A)/H_\alpha$ with the quotient order.

M. Boyle and D. Handelman proved in [5] that this group isomorphism is a pre-order isomorphism in the commutative case: $A = C(X)$, X a zero dimensional space, S a homeomorphism of X, $\alpha(f)(x) = f(S^{-1}(x))$ for $f \in C(X)$ and $x \in X$. An example of a noncommutative C*-algebra for which the group isomorphism is a pre-order isomorphism (see [10], Remark 4.4 for all details) is the following: choose $\mathbb{R} \ni \lambda > 1$, and set $G_\lambda := \bigcup_{n\in\mathbb{Z}} \lambda^n\mathbb{Z}$. Then G_λ becomes a dimension group. Suppose that $\{1, \lambda, ..., \lambda^n, \dots\}$ is a linearly independent set over $\mathbb{Q}$, that is λ is transcendental over $\mathbb{Q}$, and let $\sigma : G_\lambda \longrightarrow G_\lambda$ be the order automorphism given by multiplication with λ.

Then $G_\lambda/(1-\lambda)G_\lambda \simeq \mathbb{Z}$. Now let A be a nonuniital AF algebra such that $K_0(A) \simeq G_\lambda$ ([9]), let $\alpha \in \mathrm{Aut}(A)$ such that $K_0(\alpha) = \sigma$ (see [11]). Since $A \ltimes_\alpha \mathbb{Z} \simeq O(\infty) \bigotimes \mathbb{K}$, (see [10], Remark 4.4) we have that $K_0(A \ltimes_\alpha \mathbb{Z}) \simeq K_0(O(\infty)) \simeq \mathbb{Z}$, with positive cone $\mathbb{Z}$. Also $K_0(A)/(id - K_0(\alpha))K_0(A) \simeq \mathbb{Z}$, with positive cone $\mathbb{Z}$ (since $K_0(A)$ is totally ordered). It is only a pre-order isomorphism because $(1-\lambda)G_\lambda$ is not an order ideal.

From the Pimsner-Voiculescu exact sequence, we obtain that T^{-1} is well defined and T^{-1} is always positive. Recalling the definition $T^{-1}([y]) = x$, where $y \in K_0(A)$ and $x = K_0(i)(y)$. Suppose that $[y] \geq 0$, then there exists $K_0(A) \ni y_0 \geq 0$ such that $[y] = [y_0]$. Thus $y - y_0 \in H_\alpha$, i.e., $y - y_0 = K_0(\alpha)(x) - x$ for some $x \in K_0(A)$. Then $x = K_0(i)(y) = K_0(i)(y_0) + K_0(i)(K_0(\alpha)(x) - x) = K_0(i)(y_0) \geq 0$, since $y_0 \geq 0$.

Next, we can show that T is positive if and only if we can lift positives to positives for the following map $K_0(i) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$.

$\Longrightarrow$ Letting $0 \leq x \in K_0(A \ltimes_\alpha \mathbb{Z})$ we have $[y] = T(x) \geq 0$ and $K_0(i)(y) = x$. Therefore there exists $y' \geq 0$ in $K_0(A)$ such that $y - y' \in H_\alpha = \mathrm{Ker}(K_0(i))$. It follows that $x = K_0(i)(y) = K_0(i)(y')$ and $y' \geq 0$.

$\Longleftarrow$ Take $0 \leq x \in K_0(A \ltimes_\alpha \mathbb{Z})$. Suppose there exists $y \in K_0(A)^+$ such that $K_0(i)(y) = x$. Then $T(x) = [y] \geq 0$, since $y - y = 0 \in H_\alpha$.

We discuss the simplest cases when T is a pre-order isomorphism (see also [27], [5]).

Remark 2.2.1. *a) If A is a unital AF algebra and $K_0(A)$ is totally ordered, then we can lift positives to positives, and consequently T is a pre-order isomorphism. (If A is unital, one can drop the condition that $A \ltimes_\alpha \mathbb{Z}$ is stably finite).*

b) If $H_\alpha \bigcap K_0(A)^+ = \{0\}$, then T is an order isomorphism if T is a pre-order isomorphism.

Proof. Indeed, if $0 < x \in K_0(A \ltimes_\alpha \mathbb{Z})$ and $K_0(A)$ is totally ordered then since $K_0(i)$ is onto(because A is AF), there is $y \in K_0(A)$ such that $K_0(i)(y){=}x$. If $y < 0$, then $0 < x \leq 0$, so $K_0(A \ltimes_\alpha \mathbb{Z})^+ \bigcap (-(K_0(A \ltimes_\alpha \mathbb{Z})^+)) \neq \{0\}$, contradicting the fact that $A \ltimes_\alpha \mathbb{Z}$ is stably finite. For b) we again notice (using [4], Theorem 0.2) that $H_\alpha \bigcap K_0(A)^+ = 0$ implies that $A \ltimes_\alpha \mathbb{Z}$ is stably finite, thus the cone $(K_0(A \ltimes_\alpha \mathbb{Z}))^+$ is strict. Also $H_\alpha \bigcap K_0(A)^+ = 0$ implies that H_α is an order ideal. $\square$

Example 2.2.2. *Matroid algebras, all UHF algebras, and $\mathbb{K}$ satisfy the conditions in 2.2.1.*

In addition to the examples of the previous section wherein T is a pre-order isomorphism, we have the following:

Proposition 2.2.3. *Let A be a simple, unital AF algebra and $\alpha \in Aut(A)$ such that $A \ltimes_\alpha \mathbb{Z}$ is simple. If all traces of A are α-invariant, then T is an order isomorphism.*

Proof. If $0 \neq x \in K_0(A \ltimes_\alpha \mathbb{Z})^+$, select $y \in K_0(A)$ such that $K_0(i_A)(y) = x$ (this is always possible since i_2 is onto). For every $\tau \in S(K_0(A))$, we have that $\tau(y) = K_0(\bar{\tau})(x)$, where $\bar{\tau}$ is obtained in the following way: τ corresponds to a trace on A, again denoted τ; since it is α-invariant, it admits a canonical extension $\bar{\tau}$ on the crossed product. If $K_0(\bar{\tau})(x) > 0$ for all $\tau \in S(K_0(A))$, then $\tau(y) > 0$ for all $\tau \in S(K_0(A))$. Since A is simple and unital, we see that $y \geq 0$. We remark that $K_0(\bar{\tau})(x) = 0$ does not hold for some τ, because the crossed product is simple, it remains that $y \geq 0$. □

Example 2.2.4. *All approximately inner automorphisms satisfy the condition on invariance in the above proposition. An example other than UHF (see [2]) can be constructed as follows. Let A be the simple AF algebra corresponding to the dimension group D^2 with the strict order from the first coordinate, where D is the group of dyadic rationals. To the order isomorphism f of D^2 given by $f(x,y) := (x, -2y)$ corresponds an automorphism α of A such that $K_0(\alpha) = f$. The fact that the crossed product $A \ltimes_\sigma \mathbb{Z}$ is simple follows from a result of Kishimoto, Theorem 3.1. of [17]. Since A has a unique tracial state the conditions in the above proposition hold.*

Let us point out the link between lifting positives and intertwining.

Proposition 2.2.5. *Suppose that A is an AF algebra and $\alpha, \beta \in Aut(A)$ such there exists an isomorphism $\varphi : A \ltimes_\alpha \mathbb{Z} \longrightarrow A \ltimes_\beta \mathbb{Z}$ implemented by some isomorphism $\psi \in Aut(A)$ (i.e. such that $\varphi \circ i_\alpha = i_\beta \circ \psi$). Then one can lift positives in the situation $K_0(i_\alpha) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$ iff one can lift positives in the situation $K_0(i_\beta) : K_0(A) \longrightarrow K_0(A \ltimes_\beta \mathbb{Z})$.*

Proof. From the statement, the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{i_\alpha} & A \ltimes_\alpha \mathbb{Z} \\ \Big\downarrow{\scriptstyle \psi} & & {\scriptstyle \varphi}\Big\downarrow \\ A & \xrightarrow{i_\beta} & A \ltimes_\beta \mathbb{Z} \end{array}$$

and one has the corresponding commutative diagram at level of K_0-groups

$$\begin{array}{ccc} K_0(A) & \xrightarrow{K_0(i_\alpha)} & K_0(A \ltimes_\alpha \mathbb{Z}) \\ \Big\downarrow{\scriptstyle K_0(\psi)} & & {\scriptstyle K_0(\varphi)}\Big\downarrow \\ K_0(A) & \xrightarrow{K_0(i_\beta)} & K_0(A \ltimes_\beta \mathbb{Z}). \end{array}$$

If $0 \leq x \in K_0(A \ltimes_\beta \mathbb{Z})$ then there exists $0 \leq y \in K_0(A \ltimes_\alpha \mathbb{Z})$ such that $K_0(\varphi)(y) = x$, because $K_0(\varphi)$ is an pre-order isomorphism. Let us suppose that one can lift in the situation $K_0(i_\alpha) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$. Then, there exists $0 \leq z \in K_0(A)$ such that $K_0(i_\alpha)(z) = y$. It follows that $K_0(\psi)(z) \in K_0(A)^+$ (since $K_0(\psi)$ is an order isomorphism) and $x = K_0(i_\beta)(K_0(\psi)(z))$ because the second diagram above is commutative. Therefore one can lift in the second situation. Now replacing ψ and φ by ψ^{-1} and φ^{-1}, and arguing in a parallel fashion, the converse follows. □

One link between behavior under extensions and lifting positives is given in the next proposition. It resembles Lemma 1.5 of [33].

Proposition 2.2.6. *Let I, A, B be three unital AF algebras, let γ, α, β be automorphisms on I, A, B respectively and suppose that the following diagram with exact rows is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\psi} & A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \gamma} & & {\scriptstyle \alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & I & \xrightarrow{\psi} & A & \xrightarrow{\pi} & B & \longrightarrow & 0. \end{array}$$

Suppose that I is simple, that $K_0(i_B) : K_0(B) \longrightarrow K_0(B \ltimes_\beta \mathbb{Z})$ lifts positives to positives and that $\delta : K_1(B \ltimes_\beta \mathbb{Z}) \longrightarrow K_0(I \ltimes_\gamma \mathbb{Z})$ satisfies $Im(\delta) \cap K_0(I \ltimes_\gamma \mathbb{Z})^+ \neq \{0\}$ (where δ is the index map from K-theory), then $K_0(i_A) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$ lifts positives to positives.

Proof. From the universal property of the crossed product, there exist natural morphisms $\bar{\psi} : I \rtimes_\gamma \mathbb{Z} \longrightarrow A \rtimes_\alpha \mathbb{Z}$ and $\bar{\pi} : A \rtimes_\alpha \mathbb{Z} \longrightarrow B \rtimes_\beta \mathbb{Z}$ given by $\bar{\psi}(cu_I^n) = \psi(c)u_A^n$, where $c \in I$, $n \in \mathbb{Z}$, $\bar{\pi}(au_A^n) = \pi(a)u_B^n$, where $a \in A$, $n \in \mathbb{Z}$ and u_I, u_A, u_B are the canonical unitaries implementing γ, α, β respectively. These morphisms make the following short sequence exact

$$0 \longrightarrow I \rtimes_\gamma \mathbb{Z} \xrightarrow{\bar{\psi}} A \rtimes_\alpha \mathbb{Z} \xrightarrow{\bar{\pi}} B \rtimes_\beta \mathbb{Z} \longrightarrow 0.$$

Using the six-term exact sequence in K-theory, the following commutative diagram has exact rows

$$\begin{tikzcd}
K_0(I) \arrow[r, "K_0(\psi)"] \arrow[d, "K_0(i_I)"'] & K_0(A) \arrow[r, "K_0(\pi)"] \arrow[d, "K_0(i_A)"] & K_0(B) \arrow[d, "K_0(i_B)"'] \\
K_0(I \rtimes_\gamma \mathbb{Z}) \arrow[r, "K_0(\bar{\psi})"'] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "K_0(\bar{\pi})"'] & K_0(B \rtimes_\beta \mathbb{Z}).
\end{tikzcd}$$

If $x \in K_0(A \rtimes_\alpha \mathbb{Z})^+$ then $y := K_0(\bar{\pi})(x) \in K_0(B \rtimes_\beta \mathbb{Z})^+$. From the hypotheses, there is a positive element $z \in K_0(B)$ such that $K_0(i_B)(z) = y$. Now let $0 \leq t \in K_0(A)$ be such that $K_0(\pi)(t) = z$. Denoting $x' := K_0(i_A)(t)$, we get from the commutativity of the right square that $x' \in K_0(A \rtimes_\alpha \mathbb{Z})^+$ and $K_0(\bar{\pi})(x - x') = 0$.

First suppose that γ is not inner. It follows that $I \rtimes_\gamma \mathbb{Z}$ is simple by [17]. From the hypotheses we know there is an $m \in K_1(B \rtimes_\beta \mathbb{Z})$ such that $0 \neq \delta(m) \in K_0(I \rtimes_\gamma \mathbb{Z})^+$, so $\delta(m)$ is an order unit. There is an $r \in K_0(I \rtimes_\gamma \mathbb{Z})$ such that $K_0(\bar{\psi})(r) = x - x'$. Since $\delta(m)$ is an order unit there is a $N \in \mathbb{N}$ such that $r' := r + N\delta(m) \geq 0$ and notice that $K_0(\bar{\psi})(r') = x - x'$. Using Prop. 2.1.16 we get that there is $r'' \in K_0(I)^+$ such that $K_0(i_I)(r'') = r'$. Hence $x - x' = K_0(i_A) \circ K_0(\psi)(r'')$ and it follows that $x = x' + K_0(i_A)(K_0(\psi)(r'')) = K_0(i_A)(t + K_0(\psi)(r''))$ where $t + K_0(\psi)(r'') \geq 0$, as a sum of two positive elements.

Now suppose that γ is inner. In this situation we know that by Proposition 2.1.6 that $K_0(I \rtimes_\gamma \mathbb{Z})$ is order isomorphic to $K_0(I)$. As the latter is simple, it follows that $\delta(m)$ is an order unit, and the proof continues as in the first case.

□

Remark 2.2.7. *Since for all automorphisms $\alpha \in Aut(A)$ one has that $A \rtimes_\alpha \mathbb{Z} \simeq$*

$A \rtimes_{\alpha^{-1}} \mathbb{Z}$ one obtains that $K_0(i_\alpha) : K_0(A) \longrightarrow K_0(A \rtimes_\alpha \mathbb{Z})$ lifts positives to positives if and only if $K_0(i_{\alpha^{-1}}) : K_0(A) \longrightarrow K_0(A \rtimes_{\alpha^{-1}} \mathbb{Z})$ lifts positives to positives, where i_α and $i_{\alpha^{-1}}$ are the usual inclusion maps of A in the crossed product algebras.

We close this section with another approach for lifting positives. We will recall the original proof (see [24]) of the P-V sequence which involves the (exact sequence of) Toeplitz algebra associated to the original algebra and the automorphism, and we shall make a connection with the lifting.

Denote by $C^*(S)$ the C*-algebra generated by a non-unitary isometry S, i.e. $I = S^*S$ and $SS^* \neq I$, where I is the unit element. Pimsner and Voiculescu defined the Toeplitz algebra for A and α to be the C*-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes C^*(S)$ generated by $A \otimes I$ and $u \otimes S$, and denoted it $\mathcal{T}(A, \alpha)$. They were able to show that the following sequence is exact:

$$0 \longrightarrow A \otimes \mathbb{K} \xrightarrow{\psi} \mathcal{T}(A, \alpha) \xrightarrow{\pi} A \rtimes_\alpha \mathbb{Z} \longrightarrow 0$$

and that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "i"'] & \mathcal{T}(A, \alpha) \arrow[d, "\pi"] \\ & A \rtimes_\alpha \mathbb{Z} \end{tikzcd}$$

here i is the inclusion of A in $A \rtimes_\alpha \mathbb{Z}$, $d : A \longrightarrow \mathcal{T}(A, \alpha)$ is given by $d(a) := a \otimes I$, and $\mathbb{K}$ is the C*-algebra of compact operators on the Hilbert space with basis indexed by $\{0, 1, 2, \ldots, \}$. Using the morphism $\varphi : \mathbb{K} \longrightarrow C^*(S)$, $\varphi(e_{ij}) := S^i P S^{*j}$, $P := I - SS^*$ they constructed the morphism $\psi : A \otimes \mathbb{K} \longrightarrow \mathcal{T}(A, \alpha)$, $\psi(a \otimes e_{ij}) := u^i a u^{*j} \otimes \varphi(e_{ij})$. Of course, $(e_{ij})_{i,j \geq 0}$ is the system of matrix-units for $\mathbb{K}$. They noticed that $\mathcal{T}(A, \alpha)/\psi(A \otimes \mathbb{K}) \simeq A \rtimes_\alpha \mathbb{Z}$ and π is the canonical quotient map. One of the main parts of their work was to prove that the abelian groups $K_0(A)$ and $K_0(\mathcal{T}(A, \alpha))$ are isomorphic (the isomorphism being $K_0(d)$).

Let us specialize by assuming that A is a unital AF algebra. Before we state the next theorem, we remark that $K_0(d)(K_0(A)^+) \subseteq K_0(\mathcal{T}(A, \alpha))^+$, because d is a

*-morphism, and that the following diagram is commutative, because the diagram of the level of algebras is commutative:

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(d)"] \arrow[dr, "K_0(i)"'] & K_0(\mathcal{T}(A,\alpha)) \arrow[d, "K_0(\pi)"] \\ & K_0(A \ltimes_\alpha \mathbb{Z}) \end{tikzcd}$$

Theorem 2.2.8. *Let A be a unital AF algebra, and let $\alpha \in Aut(A)$. Keeping the constructions and notation above we get:*

(i) If $K_0(d)$ is an order isomorphism, then the map $K_0(i) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$ lifts positives to positives.

(ii) If α is approximately inner and the map $K_0(i) : K_0(A) \longrightarrow K_0(A \ltimes_\alpha \mathbb{Z})$ lifts positives to positives, then $K_0(d)$ is an order isomorphism.

Proof. (i) Let $0 \neq x \in K_0(A \ltimes_\alpha \mathbb{Z})^+$; we have that $x := [P]_0^{A \ltimes_\alpha \mathbb{Z}}$ for some $n \in N$ and some projection $P \in M_n(A \ltimes_\alpha \mathbb{Z})$. Without loss of generality, we can assume that $n = 1$, because $M_n(A)$ is still an AF algebra, $M_n(A \ltimes_\alpha \mathbf{Z})$ and $M_n(A) \ltimes_{\overline{\alpha}} \mathbf{Z}$ are isomorphic, where $\overline{\alpha}$ is the induced automorphism and $K_0(A) \simeq K_0(M_n(A))$. By Lemma 9.7 in [13] (because $\psi(A \otimes K)$ is an AF ideal of $\mathcal{T}(A, \alpha)$), there exists a projection $e \in \mathcal{T}(A, \alpha)$ such that $\pi(e) = P$. Since $K_0(d)$ is an order isomorphism, there is $0 \leq y = [q]_0^A \in K_0(A)^+$ such that $K_0(d)(y) = [e]_0^{\mathcal{T}(A,\alpha)}$ for some projection $q \in M_\infty(A)$. It follows that $K_0(i)(y) = K_0(\pi) \circ K_0(d)(y) = K_0(\pi)([e]_0^{\mathcal{T}(A,\alpha)}) = [\pi(e)]_0^{A \ltimes_\alpha \mathbb{Z}} = [P]_0^{A \ltimes_\alpha \mathbb{Z}} = x$.

(ii) Let $0 \neq x \in K_0(\mathcal{T}(A, \alpha))^+$. Then $y := K_0(\pi)(x) \in K_0(A \ltimes_\alpha \mathbb{Z})^+$. By assumption there is $0 \leq z$ in $K_0(A)$ such that $K_0(i)(z) = y$. Since $K_0(d)$ and $K_0(i)$ are group isomorphisms (α is approximately inner), we get from the commutative diagram above that $K_0(\pi)$ is injective. From $K_0(\pi)(x) = y = K_0(i)(z) = K_0(\pi) \circ K_0(d)(z)$, we have that $x = K_0(d)(z)$. □

2.3 Entropy

We study in this section how the entropy of φ is related to the entropy of α. Since AF algebras are exact, we use [3], whence the following definitions are taken.

Definition 2.3.1. *Let A be an exact C^*-algebra and let $\pi : A \longrightarrow B(H)$ be a faithful $*$-representation of A. Define the* finite parts *of A, the* completely positive approximations *of (π, A) and the* completely positive δ-rank of ω with respect to π, *respectively:*

$\mathrm{Pf(A)} := \{\omega; \omega \subset \mathrm{A}$ *is a finite set*$\}$.

$\mathrm{CPA}(\pi, A) := \{(\varphi, \psi, B); \varphi : A \longrightarrow B, \psi : B \longrightarrow B(H)$ *are contractive completely positive maps and* $dim(B) < \infty\}$.

$\mathrm{rcp}(\pi, \omega, \delta) := \inf\{\mathrm{rank}(B); (\varphi, \psi, B) \in CPA(\pi, A)$ *and* $||\psi \circ \varphi(x) - \pi(x)|| < \delta$ *for all* $x \in \omega\}$, *where* $\mathrm{rank}(B)$ *is the the dimension of a maximal abelian subalgebra of* B, *and* $\omega \in \mathrm{Pf(A)}$.

The $\delta-$rank *of* ω *is defined to be* ∞ *if there is no such an approximation.*

If $\alpha \in \mathrm{Aut}(A)$, the topological entropy of α is defined as follows (see [3]):

Definition 2.3.2.

$$\mathrm{ht}(\pi, \alpha, \omega, \delta) := \limsup_{n \to \infty} n^{-1} \log(\mathrm{rcp}(\pi, \omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{n-1}(\omega), \delta)),$$

$$\mathrm{ht}(\pi, \alpha, \omega) := \sup_{\delta > 0} ht(\pi, \alpha, \omega, \delta),$$

$$\mathrm{ht}(\pi, \alpha) := \sup_{\omega \in \mathrm{Pf}(A)} \mathrm{ht}(\pi, \alpha, \omega) \text{ -\textit{the topological entropy of} } \alpha.$$

It can be proved (see [3]) that $\mathrm{ht}(\pi_1, \alpha) = \mathrm{ht}(\pi_2, \alpha)$ for any two faithful $*$-representations π_1, π_2 of A; the entropy will be denoted $\mathrm{ht}(\alpha)$.

Let us recall here the main result in [3]:

Theorem 2.3.3. *If A is a unital C^*-algebra and $\alpha \in Aut(A)$ then*

$$ht_A(\alpha) = ht_{A \ltimes_\alpha \mathbf{Z}}(Ad(u)),$$

where u is the canonical unitary in $A \ltimes_\alpha \mathbf{Z}$ implementing α.

Theorem 2.3.4. *If A is an unital AF algebra and C is a Stratila-Voiculescu MASA thereof, $\varphi \in Aut(C)$ satisfies $\bigcirc$, and $\alpha \in Aut_0(A)$ is the corresponding isomorphism from Theorem 2.1.1, then $ht_C(\varphi) = ht_A(\alpha)$.*

Proof. Since $\alpha|_C = \varphi$ and the entropy is monotonic (see [3]), $\mathrm{ht}_C(\varphi) \leq \mathrm{ht}_A(\alpha)$. Let us prove the reverse inequality. Take $\pi : A \longrightarrow B(H)$ to be a faithful *-representation of A. Then $\pi|_C : C \longrightarrow B(H)$ is also a faithful *-representation of C. We shall prove that $(\triangle)$ $\mathrm{rcp}(\pi, \omega, \delta) \leq \mathrm{rcp}(\pi|_C, \omega, \delta)$ for every $\omega \in Pf(C)$ and $\delta > 0$. By definition of entropy, this clearly implies the theorem. Let $(\varphi, \psi, B) \in CPA(\pi|_C, C)$ i.e., $\varphi : C \longrightarrow B, \psi : B \longrightarrow B(H)$ are contractive completely positive maps and $\dim(B) < \infty$. One knows (see [3]) that

$$(\perp) \quad \mathrm{rcp}(\pi|_C, \omega, \delta) = \inf\{\mathrm{rank}(B) : (\varphi, \psi, B) \in CPA(\pi|_C, C) \text{ and } ||\psi \circ \varphi(x) - \pi|_C(x)|| < \delta \text{ for all } x \in \omega\}$$

where $\mathrm{rank}(B)$ is the dimension of a maximal abelian subalgebra of B. Now we have that $(\varphi \circ P, \psi, B) \in CPA(\pi, A)$. Indeed, since $P : A \longrightarrow C$ is linear, positive and C is abelian one knows that P is completely positive (see [34], Corollary 3.5). Thus $\varphi \circ P$ is a completely positive map (as a composition of such maps). P is also a completely contractive map (see Lemma 2.3.5 below). It follows that $\varphi \circ P$ is completely contractive (as a composition of such maps).

As $P(y) = y$ for all $y \in C$ and $\omega \subseteq C$, it follows that for all $x \in \omega$, $||\psi \circ (\varphi \circ P)(x) - \pi(x)|| = ||\psi \circ \varphi(x) - \pi(x)|| = ||\psi \circ \varphi(x) - \pi|_C(x)|| < \delta$, where the last inequality is valid because of $(\perp)$. □

Lemma 2.3.5. *$P : A \longrightarrow C$ is completely contractive.*

Proof. We already know that $||P|| \leq 1$. Let $P_n : M_n(A) \longrightarrow M_n(C)$ be the induced map by P. This map is unital (since $P(1) = 1$) and it is also a 2-positive map (since P is completely positive). By Proposition 3.2 of [22], we see that P_n is contractive. Since n was chosen arbitrarily we get the conclusion. □

Remark 2.3.6. *A easier proof can be given as follows. Since P is completely positive, it follows that P_n is positive, and from $P_n(1) = 1$ it follows that P_n is contractive.*

Remark 2.3.7. *Theorem 2.3.1. is valid for any extension $\alpha \in Aut(A)$ of $\varphi \in Aut(C)$.*

2.4 The chain recurrence set and AF-embeddability

In this section, we show that the chain recurrent set (associated to a homeomorphism T of a metrizable totaly disconnected compact space X) is the maximal (with respect with to inclusion) closed, T-invariant subset Y of X such that $C(Y) \ltimes_{T|_Y} \mathbb{Z}$ is AF embeddable.

Let us recall (see [5] for example) that for $\epsilon > 0$, an ϵ-chain from $x \in X$ to $y \in X$ is a finite sequence of elements $x_0, x_1, \ldots, x_n \in X$ satisfying $x = x_0, x_n = y$ and for $0 \leq k < n$ one has $\epsilon > d(T(x_k), x_{k+1})$. The chain recurrent set, denoted $ch(T)$, is the set of all $x \in X$ such that for every $\epsilon > 0$ there is an ϵ-chain from x to x. It was noted in [5] that $ch(T)$ is a closed, T-invariant subset of X and moreover $ch(T|_{ch(T)}) = ch(T)$. Of course one has that $ch(T) = ch(T^{-1})$.

M. Pimsner proved (see [4], Theorem 4.7) that $C(X) \ltimes_T \mathbb{Z}$ is AF embeddable if and only if $X = ch(T)$.

Proposition 2.4.1. *Let X be a metrizable totaly disconnected compact space and T a homeomorphism of it. Then $ch(T)$ is the maximal (with respect to inclusion) closed, T-invariant subset $Y \subseteq X$ such that $C(Y) \ltimes_{T|_Y} \mathbb{Z}$ is AF embeddable.*

Proof. First, $C(ch(T)) \ltimes_{T|_{ch(T)}} \mathbb{Z}$ is AF embeddable by Pimsner's Theorem and the fact that $ch(T|_{ch(T)}) = ch(T)$. Next suppose that there is a closed, T-invariant subset Y of X such that $C(Y) \ltimes_{T|_Y} \mathbb{Z}$ is AF embeddable and such that $ch(T) \subsetneqq Y$. From Pimsner's Theorem we see that $ch(T|_Y) = Y$. Hence $ch(T) \subsetneqq ch(T|_Y) \subseteq ch(T)$, a contradiction. The proof is complete. $\square$

If T is minimal then $ch(T) = X$ and the crossed product $C(X) \ltimes_T \mathbb{Z}$ is AF embeddable by Pimsner's Theorem. We close this section by exhibiting some other examples of homeomorphisms for which the crossed product is AF embeddable.

Proposition 2.4.2. *Let X be the Cantor set, $T, S : X \longrightarrow X$ be two homeomorphisms such that T is minimal, $S^m \circ T = T \circ S^m$, for some integer $m \geq 1$ and there is a $y_0 \in X$ such that $S^m(y_0) = y_0$. Then $C(X) \ltimes_S \mathbb{Z}$ is AF embeddable.*

Proof. Denote by A_{y_0} the C*-subalgebra (of $C(X) \ltimes_T \mathbb{Z}$) generated by $C(X)$ and $uC_0(X \backslash \{y_0\})$ where u is the canonical unitary implementing the action of $\mathbb{Z}$ on $C(X)$ generated by T. Then A_{y_0} is unital simple AF algebra, c.f. [29]. Let $\alpha_{S^m} : C(X) \ltimes_T \mathbb{Z} \longrightarrow C(X) \ltimes_T \mathbb{Z}$ be the natural automorphism induced by S^m. Since $S^m(y_0) = y_0$ it follows that $\alpha_{S^m}(A_{y_0}) = A_{y_0}$. From the embedding $C(X) \hookrightarrow A_{y_0}$, we get the embedding $C(X) \ltimes_{S^m} \mathbb{Z} \hookrightarrow A_{y_0} \ltimes_{\alpha_S|_{A_{y_0}}} \mathbb{Z}$. The last algebra is AF embeddable from the Cor. 4.10 of [4]. So $C(X) \ltimes_{S^m} \mathbb{Z}$ is AF embeddable, and since for an AF algebra A and $\alpha \in \mathrm{Aut}(A)$ one has (see Thm. 4.1.2 of [4]) that $A \ltimes_\alpha \mathbb{Z}$ is AF embeddable iff for every $k \in \mathbb{Z}$, $A \ltimes_{\alpha^k} \mathbb{Z}$ is AF embeddable iff there is $k \in \mathbb{Z} \backslash \{0\}$ such that $A \ltimes_{\alpha^k} \mathbb{Z}$ is AF embeddable, we see that $C(X) \ltimes_S \mathbb{Z}$ is AF embeddable. $\square$

Chapter 3

A_Y AND RELATED CROSSED PRODUCTS

3.1 Extension

Let X be a zero-dimensional space, let T be a homeomorphism on X and consider the crossed product C*-algebra $C(X) \rtimes_{\alpha_T} \mathbb{Z}$ (see [23]) (i.e., the C*-algebra generated by $C(X)$ and an unitary u such that $ufu^* = f \circ T^{-1}$ for all $f \in C(X)$). Let Y be a non-empty closed subset X. Following Putnam [29], Poon (see [26]) denoted by A_Y the C*-subalgebra of $C(X) \rtimes_{\alpha_T} \mathbb{Z}$ generated by $C(X)$ and $uC_0(X \backslash Y)$. In the same paper, he found necessary and sufficient conditions for A_Y to be an AF-algebra. Namely, A_Y is AF if and only if for every clopen subset W of X that contains Y one has that $\bigcup_{n \in \mathbf{Z}} T^n(W) = X$.

Denote by $D(X,T)$ the collection of all non empty closed subsets Y of X such that $\bigcup_{n \in \mathbf{Z}} T^n(W) = X$ for all clopen subsets W of X that contain Y. Denote by $E(X,T)$ the set of minimal elements of $D(X,T)$ with respect to inclusion. For a closed T-invariant subset $Y \subseteq X$, we first observe that one can extend the canonical automorphism α_T of $C(X)$ to A_Y. This section contains some results on the ordered K-theory of crossed products of $C(X)$ and A_Y by $\mathbb{Z}$.

Proposition 3.1.1. *One has that $Ad(u)(A_Y) \subseteq A_Y$ and $Ad(u^*)(A_Y) \subseteq A_Y$. Therefore $Ad(u)|_{A_Y} \in Aut(A_Y)$ (with inverse $Ad(u^*)|_{A_Y}$) and it extends α_T; for all $f \in$*

$C(X)$, one has $ufu^* = \alpha_T(f)$.

Proof. Since A_Y is generated by $C(X)$ and $uC_0(X\backslash Y)$, it is sufficient to show that

$$\mathrm{Ad}(u)(C(X)) \subseteq A_Y,\ \mathrm{Ad}(u)(uC_0(X\backslash Y)) \subseteq A_Y,\ \mathrm{Ad}(u^*)(C(X)) \subseteq A_Y$$

and

$$\mathrm{Ad}(u^*)(uC_0(X\backslash Y)) \subseteq A_Y.$$

Since for all $f \in C(X)$, we have $ufu^* = \alpha_T(f) \in C(X) \subseteq A_Y$ and $u^*fu = (\alpha_T)^{-1}(f) \in C(X) \subseteq A_Y$; half of the inclusions are checked. Letting $g \in C_0(X\backslash Y)$, one has $u(ug)u^* = u(ugu^*) = u\alpha_T(g) \in uC_0(X\backslash Y) \subseteq A_Y$, because for all $t \in Y$, $\alpha_T(g)(t) = g(T^{-1}(t)) = 0$ (since Y is T-invariant, $T^{-1}(t) \in Y$). The same type of arguments show that the last inclusion is also true. □

Examples of nonempty T-invariant subsets of X can be constructed as follows. For $\emptyset \neq M \subseteq X$, consider $X_M := \bigcap_{n=0}^{\infty} \overline{\bigcup_{k>n} T^k(M)}$. Since X is compact and $\emptyset \neq M$, it follows that X_M is a non-empty T-invariant closed subset of X.

Let Y be a non-empty T-invariant subsets of X and denote $\beta := \mathrm{Ad}(u)|_{A_Y} \in \mathrm{Aut}(A_Y)$. We shall investigate when $K_0(C \rtimes_{\alpha_T} \mathbb{Z}) \simeq K_0(A_Y \rtimes_\beta \mathbb{Z})$. From the Pimsner-Voiculescu exact sequence ([2] or [24]) the following commutative diagram with exact rows arises:

$$\begin{array}{ccccc}
K_0(C(X)) & \xrightarrow{\mathrm{id}-K_0(\alpha_T)} & K_0(C(X)) & \xrightarrow{K_0(i_{C(X)})} & K_0(C(X) \rtimes_{\alpha_T} \mathbb{Z}) \\
{\scriptstyle K_0(i_1)}\downarrow & & {\scriptstyle K_0(i_1)}\downarrow & & \downarrow{\scriptstyle K_0(j_1)} \\
K_0(A_Y) & \xrightarrow{\mathrm{id}-K_0(\beta)} & K_0(A_Y) & \xrightarrow{K_0(i_{A_Y})} & K_0(A_Y \rtimes_\beta \mathbb{Z}) \\
{\scriptstyle K_0(i)}\downarrow & & {\scriptstyle K_0(i)}\downarrow & & \downarrow{\scriptstyle K_0(j)} \\
K_0(C(X) \rtimes_{\alpha_T} \mathbb{Z}) & \xrightarrow{0} & K_0(C(X) \rtimes_{\alpha_T} \mathbb{Z}) & \xrightarrow{K_0(i_{C(X) \rtimes_{\alpha_T} \mathbb{Z}})} & K_0((C(X) \rtimes_{\alpha_T} \mathbb{Z}) \rtimes_{\mathrm{Ad}(u)} \mathbb{Z})
\end{array}$$

where $i : A_Y \hookrightarrow C(X) \rtimes_{\alpha_T} \mathbb{Z}$ and $i_1 : C(X) \hookrightarrow A_Y$ are the canonical inclusions, all the other K_0-maps are obtained in the same way from canonical inclusions, u is the canonical unitary implementing α_T, and $S := \mathrm{id} - K_0(\beta)$. For the next proposition, recall the definition of topological stable rank [30]. For a unital C*-algebra A the *topological stable rank* of A, $\mathrm{tsr}(A)$, is defined to be the smallest integer n such that $\overline{\mathrm{Lg}_n(A)} = A^n$, where Lg_n is the set of all n-tuples in A^n which generate A as a left ideal. If there is no such n, then the topological stable rank of A is defined to be ∞. If A in not unital, then $\mathrm{tsr}(A)$ is defined to be $\mathrm{tsr}(\tilde{A})$ where $\tilde{A}$ is the C*-algebra obtained from A by adjoining a unit. Thus, the invertible elements of A are dense in A iff $\mathrm{tsr}(A) = 1$.

Theorem 3.1.2. *Let $\emptyset \neq Y$ be a T-invariant subset of X such that $Y \in E(X,T)$. If $tsr(C(X) \ltimes_{\alpha_T} \mathbb{Z}) = 1$, then*

a) $K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}) \simeq K_0(A_Y \ltimes_\beta \mathbb{Z})$,

b) $A_Y \ltimes_\beta \mathbb{Z}$ is AF embeddable, and therefore $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ is AF embeddable.

Proof. From the Pimsner-Voiculescu exact sequence, the rows of the diagram above are exact. From [25], Prop.3.4., $K_0(i)$ is a bijective map. Thus $K_0(i_{A_Y})$ is one to one (indeed, if $K_0(i_{A_Y})(y) = 0$ for some $y \in K_0(A_Y)$, then $K_0(i_{C(X)\ltimes_{\alpha_T}\mathbb{Z}}) \circ K_0(i)(y) = K_0(j) \circ K_0(i_{A_Y})(y) = 0$; as $K_0(i_{C(X)\ltimes_{\alpha_T}\mathbb{Z}})$ is one to one, $y = 0$). Since $K_0(i_{A_Y})$ is onto (from the P-V exact sequence and A_Y being AF), we obtain:

a) $K_0(i) \circ K_0(i_{A_Y})^{-1} : K_0(A_Y \ltimes_\beta \mathbb{Z}) \longrightarrow K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$ is an isomorphism;

b) β is approximately inner since $K_0(i_{A_Y})$ is injective.

Therefore by [4], (Theorem 0.2), $A_Y \ltimes_\beta \mathbb{Z}$ is AF embeddable, and therefore $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ is AF embeddable. One also obtains that the map $K_0(j)$ is one to one (the other three K_0-maps in the right-down commutative square are one to one). ☐

Theorem 3.1.3. *Suppose $\emptyset \neq Y$ is a T-invariant subset of X such that $Y \in D(X,T)$. If $C(X)$ is an S-V MASA in A_Y and β is approximately inner, then $K_0(j_1) : K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}) \longrightarrow K_0(A_Y \ltimes_\beta \mathbb{Z})$ is an isomorphism.*

Proof. Since $C(X)$ is a MASA in A_Y, $K_0(i_1)$ is onto, and therefore $K_0(j_1)$ is onto (because $K_0(i_{A_Y})$ is onto). As β is approximately inner, the inverse $K_0(i_{A_Y})^{-1}$: $K_0(A_Y \ltimes_\beta \mathbb{Z}) \longrightarrow K_0(A_Y)$ exists. We shall prove that $K_0(i) \circ K_0(i_{A_Y})^{-1} \circ K_0(j_1) = \mathrm{id}_{K_0(C(X)\ltimes_{\alpha_T}\mathbb{Z})}$, and from this relation it follows that $K_0(j_1)$ is an injective map.

Indeed, since $C(X)$ is an AF algebra and $K_0(i_{C(X)})$ is onto, an arbitrary element $g \in K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$ has the form $g = [p]_0^{\alpha_T,C} - [q]_0^{\alpha_T,C}$, with $p, q \in P_\infty(C(X))$.

Then because $C(X) \ltimes_{\alpha_T} \mathbb{Z} \hookrightarrow A_Y \ltimes_\beta \mathbb{Z}$ and $C(X) \hookrightarrow A_Y$, one has $K_0(i) \circ K_0(i_{A_Y})^{-1} \circ K_0(j_1)([p]_0^{\alpha_T,C} - [q]_0^{\alpha_T,C}) = K_0(i) \circ K_0(i_{A_Y})^{-1}([p]_0^{\beta,A_Y} - [q]_0^{\beta,A_Y}) = K_0(i)([p]_0^{A_Y} - [q]_0^{A_Y}) = [p]_0^{\alpha_T,C} - [q]_0^{\alpha_T,C} = g$. Since $K_0(j_1)$ is onto and injective, it is an isomorphism. $\square$

The next proposition deals with conditions under which $C(X)$ is a MASA in $C(X) \ltimes_{\alpha_T} \mathbb{Z}$.

Proposition 3.1.4. *If α_T has infinite order and for every $i \in \mathbb{Z}^*$ and every proper ideal I of $C(X)$ one has $Im(id - \alpha_T^i) \nsubseteq I$, then $C(X)$ is a MASA in $C(X) \ltimes_{\alpha_T} \mathbb{Z}$. The converse also holds; if $C(X)$ is a MASA in $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ and α_T has infinite order, then $Im(id - \alpha_T^i) \nsubseteq I$ for every $i \in \mathbb{Z}^*$.*

Proof. Given an element of the following form $A := \sum_{i=-n}^n f_i u^i$, $f_i \in C(X)$ such that $gA = Ag$ for all $g \in C(X)$, one needs to show that $f_i = 0$ for all $i \neq 0$. Using $ufu^* = \alpha_T(f)$, one gets that $f_i(g - \alpha_T^i(g)) = 0$ for every $g \in C(X)$ and every i. We shall prove $f_1 = 0$, the arguments for the the rest being the same. Suppose that $\mathrm{supp}(f_1) \neq \emptyset$. Then for every $g \in C(X)$, one has $g - \alpha_T(g)|_{\mathrm{supp}(f_1)} = 0$. Thus $\mathrm{Im}(\mathrm{id} - \alpha_T) \subseteq C_0(X \backslash Y) := \{f : X \longrightarrow \mathbb{C}; f\text{continuous}, f|_{\mathrm{supp}(f_1)=0}\}$; the latter is a proper ideal in $C(X)$ (if $X = \mathrm{supp}(f_1)$ then $C_0(X\backslash Y) = 0$ and one has $\mathrm{id} = \alpha_T$, and then the order of $\alpha_T < \infty$, a contradiction). Thus $\mathrm{supp}(f_1) = \emptyset$, and it follows that $f_1 = 0$

Conversely, assume that $C(X)$ is a MASA in $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ and α_T has infinite order. Suppose that for some $i \in \mathbb{Z}^*$ and for some proper ideal I of $C(X)$, one has $\mathrm{Im}(\mathrm{id} - \alpha_T^i) \subseteq I$. Setting $I = C_0(X\backslash Y)$, $Y \neq X$(otherwise the order of α_T is less than i) and there exists a $g \in C(X)$ and $t \in X\backslash Y$ such that $g(t) - \alpha_T^i(g)(t) \neq 0$. Choosing a function $f \in C(X)$ such that $f(t) \neq 0$, one sees that fu^i does not commute with g, yielding a contradiction. $\square$

Proposition 3.1.5. *If $tsr(C(X) \ltimes_{\alpha_T} \mathbb{Z}) = 1$, then for any T-invariant subsets in $E(X,T)$, Y_1 and Y_2, one has $A_{Y_1} \simeq A_{Y_2}$.*

Proof. By [25], Theorem 3.4, $K_0(i_{Y_1}) : K_0(A_{Y_1}) \longrightarrow K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$ and $K_0(i_{Y_2}) : K_0(A_{Y_2}) \longrightarrow K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$ are group isomorphisms. Moreover $K_0(i_{Y_1})$ and $K_0(i_{Y_2})$ are order isomorphisms. Indeed, from [5] one knows that $C(X)/H_{\alpha_T} \simeq$

$K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$ (pre-order isomorphism), where H_{α_T} is the range of the map $id - K_0(\alpha) : K_0(A) \longmapsto K_0(A)$. But then (see the discussion at the beginning of section 2 of Chapter 2) one can lift in the following situation: $K_0(i_{C(X)}) : K_0(C(X)) \longrightarrow K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z})$, where $i_{C(X)}$ is the canonical inclusion. Since $C(X) \subseteq A_{Y_1} \cap A_{Y_2}$, one obtains that $(K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}))^+ \subseteq K_0(i_{Y_1})(K_0(A_{Y_1})^+)$ and $(K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}))^+ \subseteq K_0(i_{Y_2})(K_0(A_{Y_2})^+)$. However, it is always true that

$$K_0(i_{Y_1})(K_0(A_{Y_1})^+) \subseteq (K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}))^+$$

and

$$K_0(i_{Y_2})(K_0(A_{Y_2})^+) \subseteq (K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}))^+.$$

Therefore $K_0(i_{Y_1})$ and $K_0(i_{Y_2})$ are pre-order isomorphisms. Since $\mathrm{tsr}(C(X) \ltimes_{\alpha_T} \mathbb{Z}) = 1$ it follows that $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ is stably finite, thus the cone $(K_0(C(X) \ltimes_{\alpha_T} \mathbb{Z}))^+$ is proper and one gets actually order isomorphisms. Now $f : K_0(A_{Y_1}) \longrightarrow K_0(A_{Y_2})$ is an order isomorphism, where $f := K_0(i_{Y_2})^{-1} \circ K_0(i_{Y_1})$ (f sends $[1_{A_{Y_1}}]_0$ to $[1_{A_{Y_2}}]_0$). By Elliott's classification of AF algebras (see [11]), $A_{Y_1} \simeq A_{Y_2}$.

$\square$

We continue the analysis of the sets $E(X,T)$ and $D(X,T)$ to obtain some information about the topological stable rank of certain crossed products. We need the following lemma.

Lemma 3.1.6. *If X is a zero dimensional space and $T : X \longrightarrow X$ is a homeomorphism, then:*

a) $D(X,T) = D(X,T^{-1})$,

b) $E(X,T) = E(X,T^{-1})$,

c) $T^k(D(X,T)) \subseteq D(X,T)$ for every $k \in \mathbb{Z}$,

d) $T^k(E(X,T)) \subseteq E(X,T)$ for every $k \in \mathbb{Z}$.

e) If $D(X,T) \ni Y_1 \subseteq Y_2$ and Y_2 is closed then $Y_2 \in D(X,T)$.

Proof. a) If $\emptyset \neq Y \in D(X,T)$ let $W \supseteq Y$ be a clopen subset of X. It follows from [26] that $\bigcup_{n\in\mathbb{Z}} T^n(W) = X$. But $\bigcup_{n\in\mathbb{Z}}(T^{-1})^n(W) = \bigcup_{n\in\mathbb{Z}} T^{-n}(W) = \bigcup_{n\in\mathbb{Z}} T^n(W) = X$. Therefore by [26], one obtains $Y \in D(X,T^{-1})$, and so $D(X,T) \subseteq D(X,T^{-1})$. Similarly, $D(X,T^{-1}) \subseteq D(X,T)$.

b) Let $Y \in E(X,T) \subseteq D(X,T)$. By a) one knows that $Y \in D(X,T^{-1})$. If $\emptyset \neq Y_1 \subsetneqq Y$ with $Y_1 \in D(X,T^{-1}) = D(X,T)$, then a contradiction results because $Y \in E(X,T)$. Thus $Y \in E(X,T^{-1})$ and therefore $E(X,T) \subseteq E(X,T^{-1})$. The other inclusion is obtained in the same way.

c) Choose $Y \in D(X,T)$; suppose that W is a clopen subset of X such that $W \supseteq T^k(Y)$. Then $T^{-k}(W)$ is again a clopen subset of X with $T^{-k}(W) \supseteq Y \in D(X,T)$. Thus $\bigcup_{n\in\mathbb{Z}} T^n(T^{-k}(W)) = X$, that is, $\bigcup_{n\in\mathbb{Z}} T^{n-k}(W) = X$. It follows that $\bigcup_{i\in\mathbb{Z}} T^i(W) = X$, and therefore $T^k(Y) \in D(X,T)$.

d) By c), $T^k(E(X,T)) \subseteq T^k(D(X,T)) \subseteq D(X,T)$. Let $Y \in E(X,T)$ and suppose that $\emptyset \neq Y_1 \subsetneqq T^k(Y)$ with $Y_1 \in D(X,T)$. Then (T^{-k} being injective) $\emptyset \neq T^{-k}(Y_1) \subsetneqq Y$. From c), $T^{-k} \in D(X,T)$, and a contradiction arises. Thus $T^k(Y) \in E(X,T)$.

e) If W is an arbitrary clopen subset of X such that $W \supseteq Y_2$, then $W \supseteq Y_1$. Since $Y_1 \in D(X,T)$, we obtain that $X = \bigcup_{n\in\mathbb{Z}} T^n(W)$. It now follows that $Y_2 \in D(X,T)$. □

A consequence of Lemma 3.1.6 is the following well-known result that can also be proven using the structure of the crossed product C*-algebras.

Corollary 3.1.7. *Let $T : X \longrightarrow X$ be a homeomorphism of a zero-dimensional space X. Then $tsr(C(X) \ltimes_{\alpha_T} \mathbb{Z}) = 1$ if and only if $tsr(C(X) \ltimes_{\alpha_{T^{-1}}} \mathbb{Z}) = 1$.*

Proof. If $\operatorname{tsr}(C(X) \ltimes_{\alpha_T} \mathbb{Z}) = 1$ let Y be a T-invariant subset in $E(X,T)$. By [25] one obtains that for each clopen subset Q of Y there exists a T-invariant subset A of X such that $Q = A \bigcap Y$. By Lemma 3.1.6(b), Y will be also a T^{-1}-invariant subset of $E(X,T^{-1})$ and of course the clopen subset Q is also T^{-1}-invariant(because it is T-invariant) satisfying $Q = A \bigcap Y$. Thus by the same result [25], Theorem 3.4, one gets $\operatorname{tsr}(C(X) \ltimes_{\alpha_{T^{-1}}} \mathbb{Z}) = 1$. The converse follows from interchanging T with T^{-1}. □

For the next consequence we keep the notations introduced after Proposition 3.1.1.

Proposition 3.1.8. *If $Y \in D(X,T)$ then $X_Y \in D(X,T)$.*

Proof. Since $Y \neq \emptyset$ it follows that X_Y (see the example following Proposition 3.1.1) is nonempty, and we already know that X_Y is closed. Setting $C_n := \overline{\bigcup_{k>n} T^k(Y)}$ for all $n > 0$, we notice that C_n is a nonempty closed subset of X such that $C_n \supseteq C_{n+1}$ for every $n > 0$. From the definition of X_Y we see that $X_Y = \bigcap_{n=0}^{\infty} C_n$. As in [26], it is sufficient to show that $C_n \in D(X,T)$ for every $n > 0$ (because then it will follow that $A_{X_Y} = \lim_n A_{C_n}$, and inductive limits of AF algebras are again AF algebras).

Since for every $n > 0$ one has that $T^{n+1}(Y) \subseteq C_n$ it will follow from lemma 3.1.6 c) and e) that $C_n \in D(X,T)$ as desired. $\square$

It would be interesting to find the connection between the dimension groups $K_0(A_Y)$ and $K_0(A_{X_Y})$.

Suppose that $Y \in D(X,T^2)$. Since $D(X,T^2) \subseteq D(X,T)$ one can construct from Y (at least) two AF algebras: $A_Y^T \subseteq A \ltimes_{\alpha_T} \mathbb{Z}$ and $A_Y^{T^2} \subseteq A \ltimes_{\alpha_{T^2}} \mathbb{Z} = A \ltimes_{\alpha_T^2} \mathbb{Z}$, namely $A_Y^T := < C(X), uC_0(X \backslash Y) >$, $A_Y^{T^2} := < C(X), vC_0(X \backslash Y) >$ where u, and v are the canonical unitaries implementing α_T, respectively α_{T^2}. It will be apparent from the proposition below that A_Y^T is not generally isomorphic to $A_Y^{T^2}$. Instead, we shall obtain some embeddings. Recall that there is an embedding $f : C(X) \ltimes_{\alpha_T^2} \mathbb{Z} \longrightarrow C(X) \ltimes_{\alpha_T} \mathbb{Z}$ (that is, an injective *-morphism) given by $C(X) \ni f \mapsto f \in C(X)$ and $v \mapsto u^2$. Also, we shall use the following well known lemma:

Lemma 3.1.9. *$C_0(X \backslash P) C_0(X \backslash P') = C_0(X \backslash (P \bigcup P'))$, where (as usual) $C_0(X \backslash Z)$ is the ideal of $C(X)$ consisting of all functions vanishing on the closed subset $Z \subseteq X$.*

Proof. If $f \in C_0(X \backslash P) C_0(X \backslash P')$, then $f = f_1 f_2$ where $f_1|_P = 0 = f_2|_{P'}$. Thus one has $f|_P = 0 = f|_{P'}$, and it follows that $f|_{P \bigcup P'} = 0$, i.e., $f \in C_0(X \backslash (P \bigcup P'))$. If $0 \leq g \in C_0(X \backslash (P \bigcup P'))$, then $g|_{P \bigcup P'} = 0$. From $g|_P = 0 = g|_{P'}$, one gets that g (and also $\sqrt{g}$) $\in C_0(X \backslash P)$ and g (and also $\sqrt{g}$) $\in C_0(X \backslash P')$. Since $g = \sqrt{g}\sqrt{g} \in C_0(X \backslash P) C_0(X \backslash P')$ one obtains the other inclusion (taking into account that positive elements span C*-algebras). $\square$

Proposition 3.1.10. *If $Y \in D(X,T^2)$ and $S \in D(X,T)$ are such that $Y \supseteq S \bigcup T^{-1}(S)$, then there exists an embedding (i.e., an injective *-morphism) $g : A_Y^{T^2} \hookrightarrow A_S^T$. In particular, if $Y \in D(X,T^2)$ is T-invariant, then $A_Y^{T^2}$ is embedded in A_Y^T.*

Proof. First, we notice that $A_Y^{T^2}$ and A_S^T are AF algebras. Define $g := f|_{A_Y^{T^2}}$, and let us prove that indeed $\mathrm{Im} g \subseteq A_S^T$. From the definition of f one sees that $\mathrm{Im} g \subseteq <$

$C(X), u^2C_0(X\backslash Y) >$. Now

$$\begin{aligned}
A_S^T &=< C(X), uC_0(X\backslash S) > \\
&\supseteq uC_0(X\backslash S)uC_0(X\backslash S) \\
&= u^2(\alpha_T)^{-1}(C_0(X\backslash S))C_0(X\backslash S) \\
&= u^2C_0(X\backslash T^{-1}(S))C_0(X\backslash S) \\
&= u^2C_0(X\backslash(S \cup T^{-1}(S))) \\
&\supseteq u^2C_0(X\backslash Y).
\end{aligned}$$

We used lemma 3.1.8 above and $Y \supseteq S \cup T^{-1}(S)$ at the last equality. Therefore $\mathrm{Img} \subseteq A_S^T$. □

We now prove a result about the entropies of α_T and β (for notation and definition see 2.3.2).

Proposition 3.1.11. *If Y is a T-invariant closed subset of X, then $ht_{C(X)}(\alpha_T) = ht_{A_Y}(\beta)$.*

Remark 3.1.12. *In Proposition 3.1.10, we do not require that A_Y be an AF algebra.*

Proof. Since $C(X)$ is an AF algebra (and is thus exact) and $\mathbb{Z}$ is an amenable locally compact group, it follows that $C(X) \ltimes_{\alpha_T} \mathbb{Z}$ is an exact C*-algebra. Then A_Y is exact (as a C*-subalgebra of an exact C*-algebra). From monotonicity of entropy in exact C*-algebras and $\beta|_{C(X)} = \alpha_T$, $\mathrm{Ad}(u)|_{A_Y} = \beta$, it follows that $ht_{C(X)}(\alpha_T) \leq ht_{A_Y}(\beta) \leq ht_{C(X)\ltimes_{\alpha_T}\mathbb{Z}}(\mathrm{Ad}(u))$. By the main result in [3] that $ht_{C(X)}(\alpha_T) = ht_{C(X)\ltimes_{\alpha_T}\mathbb{Z}}(\mathrm{Ad}(u))$, the equality in the statement follows. □

3.2 Noncommutative case

Let A be a unital AF algebra, $\alpha \in \mathrm{Aut}(A)$ and let $T(A)$ denote the set of the tracial states of A. Since A is unital, $T(A)$ is a non-empty convex compact set. Denote by $\delta_e T(A)$, the set of extremal tracial states of A, equipped with the weak topology, which is again a non-empty set. Suppose that $\delta_e T(A)$ is a compact, totally disconnected, infinite set, that every quotient of A is finite in the sense of [7], and that $\{0\} = A_+ \bigcap_{\tau\in\delta_e T(A)} \mathrm{Ker}(\tau)$. Examples of such C*-algebras are $C(X) \otimes B$ where X is a compact, totally disconnected space and B is a unital AF algebra with a unique tracial state.

Let $\emptyset \neq I \subseteq \delta_e T(A)$ be a clopen subset (i.e., open and closed subset) such that

$$(**) \qquad \delta_e T(A) = \bigcup_{i\in\mathbb{Z}} I \circ \alpha^i$$

where for all $k \in \mathbb{Z}$, one defines $I \circ \alpha^k := \{\tau \circ \alpha^k; \tau \in I\}$. Since $I \circ \alpha^k$ is also clopen, there exists a positive integer n such that

$$(*) \qquad \delta_e T(A) = \bigcup_{i=1}^{n} I \circ \alpha^i.$$

Here we notice for later use that $\delta_e T(A) \circ \alpha = \delta_e T(A)$ (if τ is a pure trace, then so is $\tau \circ \alpha$). Since $I \subseteq \delta_e T(A)$, one can define a map $\lambda : I \longrightarrow \mathbb{Z}_+^*$ by $\lambda(\tau) := \min\{k \in \{1, \ldots, n\}; \tau \in I \circ \alpha^k\}$. From (*), one sees that $\mathrm{Im}(\lambda)$ is finite, $\mathrm{Im}(\lambda) = \{J_1, \ldots, J_m\}$. For $1 \leq k \leq m$ and $1 \leq j \leq J_k$, define the following sets, $Y(k, j) := \lambda^{-1}(J_k) \circ \alpha^{-j}$. We can prove a result similar to that in [29] or [26]: the function λ is upper semicontinuous and lower semicontinuous, and therefore continuous.

Indeed, suppose that $\lambda(\tau) = n$, where $\tau \in I$. Then $\tau \in I \circ \alpha^n$, thus $\tau \circ \alpha^{-n} \in I$, which is open. Thus there is a neighborhood V of $\tau \circ \alpha^{-n}$ such that $V \subseteq I$. Define $C := (V \circ \alpha^n) \bigcap I$ and observe that C is a neighborhood of τ. For every $\gamma \in C$, one has that $\gamma \in V \circ \alpha^n \subseteq I \circ \alpha^n$, and we obtain $\lambda(\gamma) \leq n$ (since $\gamma \in I$). Thus λ is upper semi-continuous. Since I is also closed, it follows similarly that λ is lower semi-continuous.

The next propositions are the natural extensions of results from [29] (Lemma 3.1) and [26](Lemma 2.2) to the noncommutative case.

Proposition 3.2.1. $\bigcup_{k=1}^{m} Y(k, 1) = I \circ \alpha^{-1}$.

Proof. If $\chi \in Y(k, 1)$ for some $1 \leq k \leq m$ then $\chi = \tau \circ \alpha^{-1}$, $\tau \in \lambda^{-1}(J_k) \subseteq I$, and $\chi \in I \circ \alpha^{-1}$. If $\chi \in I \circ \alpha^{-1}$ then $\chi = \tau \circ \alpha^{-1}$, $\tau \in I$. Suppose $\lambda(\tau) = J_{k_0}$ for some $1 \leq k_0 \leq m$. Then $\chi = \tau \circ \alpha^{-1} \in Y(k_0, 1) \subseteq \bigcup_{k=1}^{m} Y(k, 1)$ as $\tau \in \lambda^{-1}(J_{k_0})$. ☐

Proposition 3.2.2. $Y(k,j) \circ \alpha^{-1} = Y(k,j+1)$ *for* $1 \leq j < J_k$.

Proof. If $\chi \in Y(k,j) \circ \alpha^{-1}$ then $\chi = \gamma_0 \circ \alpha^{-1}$, $\gamma_0 \in Y(k,j)$, i.e. $\gamma_0 = \tau_0 \circ \alpha^{-j}$, $\lambda(\tau_0) = J_k$. Thus $\chi = \tau_0 \circ \alpha^{-1-j}$, $\tau_0 \in \lambda^{-1}(J_k)$, so $\chi \in Y(k,j+1)$. If $Y(k,j+1) \ni \chi$, then $\chi = \gamma_0 \circ \alpha^{-1-j}$ and $\lambda(\gamma_0) = J_k$. Thus $\chi = (\gamma_0 \circ \alpha^{-j}) \circ \alpha^{-1} \in Y(k,j) \circ \alpha^{-1}$ because $\gamma_0 \in \lambda^{-1}(J_k)$. □

Proposition 3.2.3. $\bigcup_{k=1}^{m} Y(k, J_k) = I$.

Proof. If $Y(k, J_k) = \lambda^{-1}(J_k) \circ \alpha^{-J_k} \ni \chi$ for some $k \in \{1, \ldots, m\}$ then $\chi = \tau \circ \alpha^{-J_k}$, $\tau \in \lambda^{-1}(J_k)$. Thus $\tau \in I \circ \alpha^{J_k}$, $\tau = \tau_0 \circ \alpha^{J_k}$, $\tau_0 \in I$ and it follows that $\chi = \tau_0 \in I$. Let $I \ni \eta$. Now since $\delta_e T(A) = \bigcup_{i=1}^{n} I \circ \alpha^i$, on multiplying with α^{-1-n}, we obtain that $\delta_e T(A) = \bigcup_{k=-n}^{-1} I \circ \alpha^k$ (because $\delta_e T(A) \circ \alpha = \delta_e T(A)$ and $(I \circ \alpha^p) \circ \alpha^s = I \circ \alpha^{p+s}$ for every $p, s \in \mathbb{Z}$) . Let s be the maximum of $s \in \{-n, -(n-1), \ldots, -1\}$ such that $\eta \in I \circ \alpha^s$. Define $J := -s \geq 1$. We need to show:

a) $J = J_p$ for some p,

b) $\eta \in Y(p, J_p)$.

Let us first check a). Since $\eta \in I \circ \alpha^s$, it follows that $\eta = \eta_0 \circ \alpha^s$, $y_0 \in I$, thus $\eta \circ \alpha^{-s} = y_0 \in I$. We shall prove that $\lambda(\eta \circ \alpha^{-s}) = J$ and then $J = J_p$ for some $p \in \{1, \ldots, m\}$. Since $s = -J$, $\eta \circ \alpha^{-s} = \eta \circ \alpha^J \in I \circ \alpha^J$. If there exists $1 \leq s_1 < J$ such that $\eta \circ \alpha^{-s} \in I \circ \alpha^{s_1}$ then $\eta \circ \alpha^{-s} = \gamma_0 \circ \alpha^{s_1}$, $\gamma_0 \in I$, $\eta = \gamma_0 \circ \alpha^{s_1+s}$, $\gamma_0 \in I$, that is, $\eta \in I \circ \alpha^{s_1+s}$. A contradiction results from $0 = J + s > s_1 + s > s$ and the maximality of s.

For b) one needs to show that $\eta \in Y(p, J_p)$. One has that $\eta = (\eta \circ \alpha^{J_p}) \circ \alpha^{-J_p}$ and it is sufficient to prove that $\eta \circ \alpha^{J_p} \in \lambda^{-1}(J_p)$, i.e. $\lambda(\eta \circ \alpha^{J_p}) = J_p$, but this is exactly a). □

Proposition 3.2.4. *For* $1 \leq k \leq m$ *and* $1 \leq j \leq J_k$ *the sets* $Y(k,j)$ *are pairwise disjoint.*

Proof. a) If $Y(k,j) \bigcap Y(k,i) \ni \chi$ for some $1 \leq j < i \leq J_k$ then $\chi = \gamma \circ \alpha^{-j} = \tau \circ \alpha^{-i}$ where $\lambda(\gamma) = J_k = \lambda(\tau)$. It follows that $\tau = \gamma \circ \alpha^{i-j} \in I \circ \alpha^{i-j}$ and $1 \leq i - j < J_k$. Therefore $J_k = \lambda(\tau) < J_k$, a contradiction.

b) Suppose that $Y(k,j) \bigcap Y(s,i) \neq \emptyset$ for some $k < s$, $1 \leq j \leq J_k$, $1 \leq i \leq J_s$ and let χ be in the intersection. Of course $j \neq i$ because $k < s \Rightarrow J_k \neq J_s$. One has that $\chi = \tau \circ \alpha^{-j} = \gamma \circ \alpha^{-i}$ where $\lambda(\tau) = J_k$ and $\lambda(\gamma) = J_s$. If $i < j$ then $\tau = \gamma \circ \alpha^{j-i} \in I \circ \alpha^{j-i}$ and $0 < j - i < J_k$. Thus $J_k = \lambda(\tau) < J_k$, a contradiction. If $j < i$ then $\gamma = \tau \circ \alpha^{i-j} \in I \circ \alpha^{i-j}$ and $0 < i - j < J_s$. Therefore $J_s = \lambda(\gamma) < J_s$, a contradiction. $\square$

Proposition 3.2.5. $\delta_e T(A) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{J_k} Y(k,j)$.

Proof. The inclusion $\supseteq$ is always true since $Y(k,j) = \lambda^{-1}(J_k) \circ \alpha^{-j}$, $\lambda^{-1}(J_k) \subseteq I$, and $\delta_e T(A) \circ \alpha = \delta_e T(A)$. If $\chi \in \delta_e T(A)$, let j be the largest $j \in \{-n, -(n-1), \ldots, -1\}$ such that $\chi \in I \circ \alpha^j$. (Recall that $\delta_e T(A) = \bigcup_{k=-n}^{-1} I \circ \alpha^k$.) Thus $\chi = i \circ \alpha^j$, $i \in I$, i.e., $\chi \circ \alpha^{-j} = i \in I$. Let $\lambda(i) =: J_k$ for some $1 \leq k \leq m$. Then $\chi = i \circ \alpha^j \in \lambda^{-1}(J_k) \circ \alpha^j = Y(k, -j)$, if $1 \leq -j \leq J_k$. Suppose $-j > J_k$, since $\lambda(i) = J_k$ one has $i = i' \circ \alpha^{J_k}$, for some $i' \in I$. Thus $\chi = i' \circ \alpha^{J_k + j} \in I \circ \alpha^{J_k + j}$ and $j < J_k + j < 0$, contradicting the maximality of j. $\square$

Definition 3.2.6. *Let* $J := \{a \in A | \tau(a^*a) = 0, \forall \tau \in I\}$, *and for* $1 \leq k \leq m$, *define* $A(k, J_k) := \{a \in A | \tau(a^*a) = 0, \forall \tau \in Y(k, J_k)^c\}$, *where the complement of the subset* $Y(k, J_k)$ *is taken with respect to* $\delta_e T(A)$.

Remark 3.2.7. *For* $1 \leq k \leq m$, $A(k, J_k)$ *and* J *are ideals of* A, *and are thus AF.*

We shall use the following notation: $A(k, J_k) = I_{Y(k,J_k)^c}$ and $J = I_I$.

Definition 3.2.8. *For every* $1 \leq k \leq m$, $1 \leq i, j \leq J_k$ *and* $a \in A(k, J_k)$, *define the element* $e_{ij}^k \otimes a := u^{i-j} \alpha^{j-J_k}(a) \in A \ltimes_\alpha \mathbb{Z}$.

Our next goal is to prove that $e_{ij}^k \otimes a \in \langle A, uJ \rangle$ for every $1 \leq k \leq m$, $1 \leq i, j \leq J_k$, and $a \in A(k, J_k)$. If $i = j$ there is nothing to show. Let us assume for the moment that $J_k \geq i > j \geq 1$. Also, we can take $a \in A(k, J_k)$ to be positive. We require the following elementary fact. If n is a positive integer, then $u^n x^n = [u\alpha^{n-1}(x)] \ldots [u\alpha(x)][ux]$. For $n = 1$ this is trivial. Suppose that for some n we have $u^n x^n = [u\alpha^{n-1}(x)] \ldots [u\alpha(x)][ux]$. Then $u^{n+1}x^{n+1} = uu^n x^n x =$

$u\{[u\alpha^{n-1}(x)]\dots[u\alpha(x)][ux]\}x = [u\alpha^{n}(x)]\dots[u\alpha(x)][ux]$; the last equality is valid since for every $y \in A$, we have $uy = \alpha(y)u$ whithin $A \ltimes_\alpha \mathbb{Z}$. For $0 \le a \in A(k, J_k)$,

$$e^k_{ij} \otimes a = u^{i-j}\alpha^{j-J_k}(a) = u^{i-j}x = u^{i-j}y^{i-j} = [u\alpha^{i-j-1}(y)]\dots[u\alpha(y)][uy]$$

where $x := \alpha^{j-J_k}(a) \ge 0$ and $0 \le y := \sqrt[i-j]{x}$. We would like to prove that $y, \alpha(y), \dots, \alpha^{i-j-1}(y) \in I_I$. If this holds, it will follow that $e^k_{ij} \bigotimes a \in \langle A, uJ\rangle$ for $i > j$. This case implies the other one —if $t = u^{i-j}\alpha^{j-J_k}(a)$ for $i < j$ and $a \in A(k, J_k)$ then $t^* = \alpha^{j-J_k}(a^*)u^{j-i} = u^{j-i}\alpha^{j-J_k-j+i}(a^*) = u^{j-i}\alpha^{i-J_k}(a^*)$, and one is reduced to the first case because $a^* \in A(k, J_k)$.

Let us show that $y \in I_I$. We have that $y \in \alpha^{j-J_k}(A(k, J_k)) = \alpha^{j-J_k}(I_{Y(k,J_k)^c}) = I_{Y(k,J_k)^c \circ \alpha^{J_k-j}}$. It is sufficient to prove that $I_{Y(k,J_k)^c \circ \alpha^{J_k-j}} \subseteq I_I$, or in other words, that $Y(k, J_k)^c \circ \alpha^{J_k-j} \supseteq I = \bigcup_{z=1}^m Y(z, J_z)$. Thus, again, it is sufficient to show that $Y(k, J_k)^c \supseteq I \circ \alpha^{j-J_k}$. As a general idea why this should be true we observe that $j - J_k$ is negative and the exponent of α in 3.2.1 is again negative, so we can use that equation.

If $j - J_k = -1$ then $\bigcup_{z=1}^m Y(z, 1) = I \circ \alpha^{-1}$ (by Proposition 3.2.1 in this section). Of course, $Y(k, J_k)^c \supseteq \bigcup_{z=1}^m Y(z, 1)$ from Proposition 3.2.4 and $J_k \ne 1$(indeed, $J_k > j \ge 1$).

If $j - J_k = -2$ then $I \circ \alpha^{-2} = \{\bigcup_{z=1}^m Y(z, 1)\} \circ \alpha^{-1}$. If for some z, one has $J_z = 1$ (thus $z = 1$) then $\{\bigcup_{z=1}^m Y(z, 1)\} \circ \alpha^{-1} = Y(1, 1) \circ \alpha^{-1} \bigcup(\cup_{z=2}^m Y(z, 2))$ (if $J_z \ne 1$ for all $z \in \{1, \dots, m\}$ there will be no problems, as will follow). Since $J_k \ne 2$ (as $J_k = 2 + j > 2$) it follows that $Y(k, J_k) \bigcap Y(z, 2) = \emptyset$ for all $2 \le z \le m$. It remains to show that $Y(1, 1) \circ \alpha^{-1} \bigcap Y(k, J_k) = \emptyset$. Suppose to the contrary that $x \in Y(1, 1) \circ \alpha^{-1} \bigcap Y(k, J_k)$. Then $x = (\gamma \circ \alpha^{-1}) \circ \alpha^{-1} = \tau \circ \alpha^{-J_k}$, where $\lambda(\gamma) = 1$ and $\lambda(\tau) = J_k$. Then $\tau = \gamma \circ \alpha^{J_k-2} \in I \circ \alpha^{J_k-2}$. Since $1 \le j = J_k - 2 < J_k$, it follows that $J_k = \lambda(\tau) < J_k - 2$, a contradiction. Thus also in this case, $Y(k, J_k)^c \supseteq I \circ \alpha^{j-J_k}$.

If $j - J_k = -3$ we need to show that $Y(k, J_k)^c \supseteq I \circ \alpha^{-3} = \{\bigcup_{z=1}^m Y(z, 1)\} \circ \alpha^{-2}$. If for some z, $J_z = 1$, and for some t, $J_t = 2$, then $I \circ \alpha^{-3} = Y(1, 1) \circ \alpha^{-2} \bigcup Y(2, 1) \circ \alpha^{-2} \bigcup_{z=3}^m Y(z, 3)$(if $J_z \ne 1, 2$ for all $z \in \{1, \dots, m\}$ there will be no problem as can be seen). As $J_k \ne 3$ (since $J_k = j + 3 > 3$), one has that $Y(k, J_k) \bigcap Y(z, 3) = \emptyset$ for all $3 \le z \le m$. It remains to show that $Y(1, 1) \circ \alpha^{-2}(\bigcap Y(k, J_k)) = \emptyset$ and

$Y(2,1) \circ \alpha^{-2}(\bigcap Y(k, J_k)) = \emptyset$.

a) Suppose to the contrary that there is an $x \in Y(1,1) \circ \alpha^{-2} \bigcap Y(k, J_k)$. Then $x = (\gamma \circ \alpha^{-1}) \circ \alpha^{-2} = \tau \circ \alpha^{-J_k}$, where $\lambda(\gamma) = 1$ and $\lambda(\tau) = J_k$. Then $\tau = \gamma \circ \alpha^{J_k - 3} \in I \circ \alpha^{J_k-3}$. Since $1 \leq j = J_k - 3 < J_k$ we conclude that $J_k = \lambda(\tau) < J_k$, a contradiction.

b) If $x \in Y(2,1) \circ \alpha^{-2} \bigcap Y(k, J_k)$ then $x = (\gamma \circ \alpha^{-1}) \circ \alpha^{-2} = \tau \circ \alpha^{-J_k}$, where $\lambda(\gamma) = 2$ and $\lambda(\tau) = J_k$. It follows that $\tau = \gamma \circ \alpha^{J_k-3}$ and as above a contradiction results.

We shall not deal with the rest of the cases, but remark that the same arguments apply, since we are dealing only with negative exponents of α, i.e., $1 - J_k \leq j - J_k \leq -1$. Similarly, we also see that $\alpha(y), \ldots, \alpha^{i-j-1}(y) \in I_I$ again because of the negative exponents of α. Indeed, the last statement is equivalent to $Y(k, J_k)^c \supseteq I \circ \alpha^{j+1-J_k}, \ldots, Y(k, J_k)^c \supseteq I \circ \alpha^{i-1-J_k}$, and one can apply the same arguments as above since $j - J_k \leq 1 + j - J_k \leq \cdots \leq i - 1 - J_k < 0$. Thus we have shown the following

Proposition 3.2.9. *For every $1 \leq k \leq m$, $1 \leq i, j \leq J_k$ and $a \in A(k, J_k)$, one has that $e^k_{ij} \otimes a \in < A, uJ >$.*

One can check that $\{e^k_{ij} \otimes a^k_{ij} : 1 \leq k \leq m, 1 \leq i, j \leq J_k$ and $a^k_{ij} \in A(k, J_k)\}$ generates a C*-subalgebra B of $< A, uJ >$ isomorphic to $\bigoplus_{k=1}^m M_{J_k}(A(k, J_k))$. Define the map $T : B \longrightarrow \bigoplus_{k=1}^m M_{J_k}(A(k, J_k)) \simeq \bigoplus_{k=1}^m M_{J_k}(\mathbb{C}) \overline{\bigotimes} A(k, J_k)$ by $T(e^k_{ij} \bigotimes a^k_{ij}) = E^k_{ij} \overline{\bigotimes} a^k_{ij}$, where $\{E^k_{ij}\}_{ij}$ is the matrix unit system for $M_{J_k}(\mathbb{C})$, and $\overline{\bigotimes}$ is the usual tensor product. Obviously T is a linear, onto and injective map (if $E^k_{ij} \overline{\bigotimes} a^k_{ij} = 0$ then $a^k_{ij} = 0$ and thus $e^k_{ij} \bigotimes a^k_{ij} = u^{i-j} \alpha^{j-J_k}(a^k_{ij}) = 0$). The following chains of equalities (using $uau^* = \alpha(a)$) show that T is a *-isomorphism :

$$(T(u^{i-j}\alpha^{j-J_k}(a^k_{ij})))^* = (E^k_{ij} \overline{\bigotimes} a^k_{ij})^* = (E^k_{ij})^* \overline{\bigotimes} (a^k_{ij})^* = E^k_{ji} \overline{\bigotimes} (a^k_{ij})^*$$

and

$$T((u^{i-j}\alpha^{j-J_k}(a^k_{ij}))^*) = T(\alpha^{j-J_k}((a^k_{ij})^*)(u^{i-j})^*) = T(\alpha^{j-J_k}((a^k_{ij})^*)u^{j-i})$$

$$= T(u^{j-i}\alpha^{i-j}(\alpha^{j-J_k}((a^k_{ij})^*))) = T(u^{j-i}\alpha^{i-J_k}((a^k_{ij})^*)) = E^k_{ji} \overline{\bigotimes} (a^k_{ij})^*.$$

What we would like to show now is that: $B =< A, uJ >$. Thus we need to show that $A \subseteq B$ and $uJ \subseteq B$.

Here we impose a new condition on A. It is well known [7] that the natural map $\widehat{\ }: A_{sa} \longrightarrow \mathrm{Aff}(T(A))$ given by $\widehat{a}(\tau) = \tau(a)$ for all $\tau \in T(A)$ is surjective. Suppose that the natural map lifts positive functions in $\mathrm{Aff}(T(A))$ to positive elements in A. Examples of such AF algebras include $C(X)$ and $C(X, D)$, where X is a zero dimensional space and D is a unital AF algebra with unique tracial state (4.1.12). See Chapter 4 for other examples.

If A is a simple AF-algebra, $J = 0$ ($J = A$ is impossible), thus $B =< A, uJ >=< A >$, which is an AF-algebra. Therefore we shall also suppose that A is a non-simple AF-algebra.

The notation from the following lemma will be used in the sequel.

Lemma 3.2.10. $A = \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$, *where* $\Omega := \{(k, j); 1 \leq k \leq m, 1 \leq j \leq J_k\}$.

Proof. We have already obtained the following partition of $\delta_e(T(A))$

$$\delta_e(T(A)) = \bigcup_{(k,j)\in\Omega} Y(k, j).$$

For arbitrary $(k, j) \in \Omega$, let $g_{(k,j)} : \delta_e(T(A)) \longrightarrow \{0, 1\}$ be the characteristic function of $Y_{(k,j)}$. From the corollary 11.15 of [15] (since is $\delta_e(T(A))$ compact and $g_{(k,j)}$ is continuous), one can find $f_{(k,j)} \in \mathrm{Aff}(T(A))$ such that $f_{(k,j)}|_{\delta_e(T(A))} = g_{(k,j)}$ and $||f_{(k,j)}|| = ||g_{(k,j)}|| = 1$. Since $g_{(k,j)} \geq 0$, one can choose $f_{(k,j)} \geq 0$. Let $0 \leq a_{(k,j)} \in A$ be such that $f_{(k,j)} = \widehat{a_{(k,j)}}$. Since $f_{(k,j)} \neq 0$, we have that $a_{(k,j)} \neq 0$. Also we obtain that $a_{(k,j)} \in I_{Y(k,j)^c}$ because i) $0 \leq a_{(k,j)}$ and ii) for every $\gamma \in Y(k, j)^c$, we have $0 = g_{(k,j)}(\gamma) = f_{(k,j)}(\gamma) = \widehat{a_{(k,j)}}(\gamma) = \gamma(a_{(k,j)})$. It follows from $1_{C(\delta_e(T(A)))} = \sum_{(k,j)\in\Omega} g_{(k,j)}$ that $\hat{1} = \widehat{\sum_{(k,j)\in\Omega} a_{(k,j)}}$ in $\mathrm{Aff}(T(A))$.

Suppose to the contrary that $A \neq S := \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$. Then A/S is a unital AF algebra and it has at least one tracial state, let us say that $R : A/S \longrightarrow \mathbb{C}$ is an extremal tracial state of A/S. Then $R \circ \pi : A \longrightarrow \mathbb{C}$ is an extremal tracial state of A, where $\pi : A \longrightarrow A/S$ is the canonical quotient map. It follows that for all $(k, j) \in \Omega$ one has $R \circ \pi(a_{(k,j)}) = 0$, i.e. $\widehat{a_{(k,j)}}(R \circ \pi) = 0$. Plugging $R \circ \pi$ in $\hat{1} = \widehat{\sum_{(k,j)\in\Omega} a_{(k,j)}}$ we get that $1 = 0$, a contradiction. Hence $A = S$.

□

Lemma 3.2.11. $J = \sum_{(k,j)\in\Omega'} I_{Y(k,j)^c}$, *where* $\Omega' := \{(k,j); 1 \le k \le m, 1 \le j \le J_k - 1\}$

Proof. From $I = \bigcup_{k=1}^m Y(k,J_k) \subseteq Y(k,j)^c$ for all $(k,j) \in \Omega'$, it follows that $J = I_I = I_{\bigcup_{k=1}^m Y(k,J_k)} \supseteq I_{Y(k,j)^c}$ for all $(k,j) \in \Omega'$. Thus $J \supseteq \sum_{(k,j)\in\Omega'} I_{Y(k,j)^c}$. To prove the reverse inclusion, consider $x \in (I_{\bigcup_{k=1}^m Y(k,J_k)})^+$. From the lemma above, $x = \sum_{(k,j)\in\Omega} x_{(k,j)}$, where $0 \le x_{(k,j)} \in I_{Y(k,j)^c}$. It follows that $x = \sum_{(k,j)\in\Omega'} x_{(k,j)} + x_{(1,J_1)} + \cdots + x_{(m,J_m)}$. We shall prove that $x_{(1,J_1)} = \cdots = x_{(m,J_m)} = 0$, and the assertion will follow. Since $x_{(1,J_1)} \in I_{Y(1,J_1)^c}$ we have that $\gamma(x_{(1,J_1)}) = 0$ for all $\gamma \in Y(1,J_1)^c$. For all $\delta \in Y(1,J_1)$ we have $0 = \delta(\sum_{(k,j)\in\Omega'} x_{(k,j)}) + \delta(x_{(1,J_1)}) + \cdots + \delta(x_{(m,J_m)}) \ge 0$. We get that $\delta(x_{(1,J_1)}) = 0$ for all $\delta \in Y(1,J_1)$. We obtain that $x_{(1,J_1)} \in A_+ \bigcap \{\bigcap_{\tau\in\delta_e T(A)} \mathrm{Ker}(\tau)\} = 0$. The same argument applies to get $x_{(2,J_2)} = \cdots = x_{(m,J_m)} = 0$. □

Lemma 3.2.12. *For every* $1 \le k \le m$ *and* $1 \le i \le J_k$, $\alpha^{i-J_k}(A(k,J_k)) = A(k,i)$ *and* $a_{(k,i)} = \alpha^{i-J_k}(a_{(k,J_k)})$.

Proof. Let us start by showing that $\alpha^{i-J_k}(A(k,J_k)) = A(k,i)$. Choose $x \in A(k,J_k)$; we need to show that $\alpha^{i-J_k}(x) \in A(k,i) := \{a \in A; \tau(a^*a) = 0, \text{ for every } \tau \in Y(k,i)^c\}$. Thus if $\gamma \in Y(k,i)^c\}$ is arbitrary, we need to show that $\gamma(\alpha^{i-J_k}(x^*x)) = \gamma \circ \alpha^{i-J_k}(x^*x)) = 0$, and it is sufficient now to prove that $Y(k,i)^c \circ \alpha^{i-J_k} \subseteq Y(k,J_k)^c$, since $\gamma \in Y(k,i)^c$. Let us prove the contrapositive $\{Y(k,i)^c \circ \alpha^{i-J_k}\}^c \supseteq Y(k,J_k)$. Taking $\tau \in Y(k,J_k)$, we can write $\tau = \tau_0 \circ \alpha^{-J_k}$, with $\lambda(\tau_0) = J_k$. Suppose to the contrary that $\tau \in Y(k,i)^c \circ \alpha^{i-J_k}$. Then $\tau = \mu \circ \alpha^{i-J_k}$ for some $\mu \in Y(k,i)^c$. We obtain that $\tau_0 \circ \alpha^{-J_k} = \mu \circ \alpha^{i-J_k}$, i.e. $\tau_0 = \mu \circ \alpha^i$. Suppose that $\mu \in Y(s,t)$ for some $1 \le s \le m$ and $1 \le t \le J_s$ such that $t \ne i$ or $s \ne k$.

i) If $\mu \in Y(s,t)$ with $t \ne i$ then $\mu = \mu_0 \circ \alpha^{-t}$, where $\lambda(\mu_0) = s$. We obtain $\tau_0 = \mu_0 \circ \alpha^{i-t}$. If $i > t$, then $Y(k,i) \ni \tau_0 \circ \alpha^{-i} = \mu_0 \circ \alpha^{-t} \in Y(s,t)$, a contradiction from Proposition 3.2.4, since $t \ne i$. If $i < t$ then $Y(k,1) \ni \tau_0 \circ \alpha^{-1} = \mu_0 \circ \alpha^{i-t-1} \in Y(s,t-i+1)$, a contradiction by Proposition 3.2.4, since $1 \ne t-i+1$ and $1 < t-i+1 \le J_s - i + 1 \le J_s$.

ii) If $\mu \in Y(s,t)$, $t = i$, but $s \neq k$, then $\tau_0 = \mu \circ \alpha^i = \mu_0 \circ \alpha^{-i} \circ \alpha^i = \mu_0$. It follows that $\tau_0 = \mu_0$, thus $J_k = \lambda(\tau_0) = \lambda(\mu_0) = J_s$, a contradiction (again from Proposition 3.2.4) since $s \neq k$. Therefore we have shown $\alpha^{i-J_k}(A(k,J_k)) \subseteq A(k,i)$.

Let us prove the reverse inclusion $A(k,i) \subseteq \alpha^{i-J_k}(A(k,J_k))$. For $x \in A(k,i) := \{a \in A; \tau(a^*a) = 0 \text{ for every } \tau \in Y(k,i)^c\}$, we need to show that $x \in \alpha^{i-J_k}(A(k,J_k))$. Since $x = \alpha^{i-J_k}(\alpha^{J_k-i}(x))$, it is sufficient to show that $\alpha^{J_k-i}(x) \in A(k,J_k)$. Let $\gamma \in Y(k,J_k)^c$ be arbitrary; we need to show that $\gamma(\alpha^{J_k-i}(x^*)\alpha^{J_k-i}(x)) = 0$, i.e., $\gamma \circ \alpha^{J_k-i}(x^*x) = 0$. Since $x \in A(k,i)$, it is sufficient to prove that $\gamma \circ \alpha^{J_k-i} \in Y(k,i)^c$. Let us prove that $Y(k,J_k)^c \circ \alpha^{J_k-i} \subseteq Y(k,i)^c$.

For $\tau \in Y(k,i)$ one has that $\tau = \tau_0 \circ \alpha^{-i}$, where $\lambda(\tau_0) = J_k$. Supposing to the contrary that $\tau \in Y(k,J_k)^c \circ \alpha^{J_k-i}$, one can write $\tau = \mu \circ \alpha^{J_k-i}$, where $\mu \in Y(k,J_k)^c$. It follows that $\tau_0 \circ \alpha^{-i} = \mu \circ \alpha^{J_k-i}$, i.e., $\tau_0 = \mu \circ \alpha^{J_k}$. Let us say that $\mu \in Y(s,t)$ for some $1 \leq s \leq m$ and $1 \leq t \leq J_s$ with $s \neq k$ or $t \neq J_k$.

i) The case $t \neq J_k$ splits into two subcases:

a) If $t < J_k$, since $\mu \in Y(s,t)$, one gets that $\mu = \mu_0 \circ \alpha^{-t}$, with $\lambda(\mu_0) = J_s$. Thus $\tau_0 = \mu_0 \circ \alpha^{-t+J_k}$ implies that $Y(k, J_k+1-t) \ni \tau_0 \circ \alpha^{-1+t-J_k} = \mu \circ \alpha^{-1} \in Y(s,1)$, a contradiction (see Proposition 3.2.4) since $J_k + 1 - t \neq 1$ and $1 < J_k + 1 - t \leq J_k$.

b) If $t > J_k$ then $Y(k,J_k) \ni \tau_0 \circ \alpha^{-J_k} = \mu_0 \circ \alpha^{-t} \in Y(s,t)$, a contradiction (see Proposition 3.2.4) since $1 \leq t \leq J_s$.

ii) If $t = J_k$ but $s \neq k$, one has that $\tau_0 = \mu_0$. It follows that $J_k = \lambda(\tau_0) = \lambda(\mu_0) = J_s$, a contradiction since $s \neq k$.

Before starting to show the last equalities in the statement we remark that the condition $A^+ \cap_{\tau \in \delta_e T(A)} \mathrm{Ker}(\tau) = \{0\}$ implies that the collection of elements $a_{(k,j)} \in I_{Y(k,j)^c}$, $(k,j) \in \Omega$, is a set of orthogonal projections summing to 1 (these projections come from $A = \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$).

To prove the second set of equalities, we start by eliminating the trivial case—when $J_k = i$—thus we shall suppose that $1 \leq i < J_k$. Set $x := a_{(k,i)} - \alpha^{i-J_k}(a_{(k,J_k)})$. Then $x = 1x = (\sum_{k=1}^m \sum_{i=1}^{J_k} a_{(k,i)})x = a_{(k,i)} - (\sum_{k=1}^m \sum_{i=1}^{J_k} a_{(k,i)})\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)} - a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ because $a_{(k,j)}$ (for $(k,j) \in \Omega$) are mutually orthogonal. Thus $\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$. Observe that $a_{(k,i)} = a_{(k,i)}1 = a_{(k,i)}\alpha^{i-J_k}(1) = a_{(k,i)}\alpha^{i-J_k}(\sum_{k'=1}^m \sum_{i'=1}^{J_{k'}} a_{(k',i')}) = a_{(k,i)} \sum_{k'=1}^m \sum_{i'=1}^{J_{k'}} \alpha^{i-J_k}(a_{(k',i')}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$

because $a_{(k,i)} \in A(k,i) = \alpha^{i-J_k}(A(k,J_k))$ and the projections $a_{(k,j)}$ $((k,j) \in \Omega)$ are mutually orthogonal. It follows from $\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ and $a_{(k,i)} = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ that $a_{(k,i)} = \alpha^{i-J_k}(a_{(k,J_k)})$.

□

From the lemmas above, we can write $1 = \sum_{k=1}^{m}\sum_{i=1}^{J_k} a_{(k,i)}$ for some $a_{(k,i)} \in I_{Y_{(k,i)^c}}$. Thus for every $a \in A$, one has $a = \sum_{k=1}^{m}\sum_{i=1}^{J_k} aa_{(k,i)} = \sum_{k=1}^{m}\sum_{i=1}^{J_k} a\alpha^{i-J_k}(a_{(k,J_k)}) = \sum_{k=1}^{m}\sum_{i=1}^{J_k} \alpha^{i-J_k}(a_i^k)$, where $a_i^k := \alpha^{J_k-i}(a)a_{(k,J_k)} \in A$. It follows that for every $a \in A$, one has $a = \sum_{k=1}^{m}\sum_{i=1}^{J_k} e_{ii}^k \bigotimes a_i^k \in B$.

Now let $x \in J = \sum_{(k,j)\in\Omega'} I_{Y(k,j)^c}$; we show that $ux \in B$. Write $x = \sum_{(k,j)\in\Omega'} x_{(k,j)}$, where $x_{(k,j)} \in I_{Y(k,j)^c} = \alpha^{j-J_k}(I_{Y(k,J_k)^c})$. Thus

$$ux = u\sum_{(k,j)\in\Omega'} \alpha^{j-J_k}(b_{(k,J_k)})$$

for some $b_{(k,J_k)} \in I_{Y(k,J_k)^c}$. We obtain

$$ux = \sum_{(k,j)\in\Omega'} u^{j+1-j}\alpha^{j-J_k}(b_{(k,J_k)}) = \sum_{(k,j)\in\Omega'} e_{j+1,j}^k \bigotimes b_{(k,J_k)} \in B$$

since $b_{(k,J_k)} \in I_{Y(k,J_k)^c}$. Therefore one has

Proposition 3.2.13. *$\langle A, uJ\rangle$ is an AF algebra.*

The next step is to compute $K_0(\langle A, uJ\rangle)$.

The K-theory of $A(k,J_k)$ determines the K-theory of $\langle A, uJ\rangle$. We do the general case here. If $\emptyset \neq F \subseteq T(A)$ define $I_F := \{a \in A; \tau(a^*a) = 0 \text{ for all } \tau \in F\}$. To compute the K_0-theory of I_F, we recall the following short exact sequence

$$0 \longrightarrow K_0(I_F) \xrightarrow{K_0(i)} K_0(A) \xrightarrow{K_0(\pi)} K_0(A/I_F) \longrightarrow 0$$

where $i : I_F \hookrightarrow A$ is the canonical inclusion and $\pi : A \longrightarrow A/I_F$ is the canonical quotient map. We shall identify $K_0(I_F)$ with its image in $K_0(A)$, namely $K_0(i)(K_0(I_F)) \subseteq K_0(A)$. Our first task is to show that $(K_0(i)(K_0(I_F)))^+ \subseteq (\bigcap_{\tau\in F}(\mathrm{Ker}K_0(\tau))^+)$, where $(\mathrm{Ker}K_0(\tau))^+ := K_0(A)^+ \bigcap \mathrm{Ker}K_0(\tau)$.

Since the positive cone of an AF algebra is generated by its scale, it is sufficient to show that $K_0(\tau)([p]_0^A) = 0$ for every $p \in P(I_F)$. One has that $K_0(\tau)([p]_0^A) =$

$\tau(p) = 0$ since $\tau \in F$ and $p \in I_F$. Thus one obtains that $K_0(i)(K_0(A)) \subseteq < \bigcap_{\tau \in F}(\mathrm{Ker}K_0(\tau))^+ >$, the order ideal generated by $\bigcap_{\tau \in F}(\mathrm{Ker}K_0(\tau))^+$. Let us prove that $(K_0(i)(K_0(I_F)))^+ \supseteq < \bigcap_{\tau \in F}(\mathrm{Ker}K_0(\tau))^+ >$. If $x \in \bigcap_{\tau \in F}(\mathrm{Ker}K_0(\tau))^+$, then $x = [p]_0^A$ for some projection $p \in P_n(A)$ and some $n \in \mathbb{N}$. Since $K_0(A)^+$ is generated by its scale, one can assume that $[p]_0^A = [p_1]_0^A + \cdots + [p_k]_0^A$ for some projections $p_1, \ldots, p_k \in A$. Therefore $0 = \tau([p]_0^A) = \tau(p_1) + \cdots + \tau(p_k)$, for every $\tau \in F$, and thus $\tau(p_1) = \cdots = \tau(p_k) = 0$ for every $\tau \in F$. It follows that $p_1, \ldots, p_k \in I_F$ and from

$$[p]_0^A = \left[\begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_k \end{pmatrix}\right]_0^A$$

it follows that $[p]_0^A \in K_0(i)(K_0(I_F))$.

Therefore we have shown

Proposition 3.2.14. *If $\emptyset \neq F \subseteq T(A)$, then $K_0(I_F) \simeq < \bigcap_{\tau \in F}(KerK_0(\tau))^+ >$.*

Remark 3.2.15. *Proposition 3.2.14 extends the commutative case, i.e., when $F \subseteq \delta_e T(C(X))$, the isomorphism is valid if F is either $\emptyset$ or $T(A)$.*

Using Proposition 3.2.14 (and the fact that $B =< A, uJ >$)), one has

Proposition 3.2.16. *If I is a non-empty clopen subset of $\delta_e(T(A))$ then $K_0(\langle A, uJ\rangle)$ is order isomorphic to $\oplus_{k=1}^m G_k$, where for all $k \in \{1, \ldots, m\}$,*

$$G_k := < \cap_{\tau \in Y(k,J_k)^c} KerK_0(\tau)^+ > .$$

Proof. We have that $K_0(< A, uJ >) = K_0(B) \simeq K_0(\oplus_{k=1}^m M_{J_k}(A(k, J_k)))$

$$\simeq \oplus_{k=1}^m K_0(M_{J_k}(A(k, J_k))) \simeq \oplus_{k=1}^m K_0(A(k, J_k))) \simeq \oplus_{k=1}^m G_k.$$

□

The next remark shall be used in section 3 of this chapter; it is parallel to [29].

Remark 3.2.17. *If $\emptyset \neq I$ is a clopen subset of $\delta_e(T(A))$ and $\mathcal{P}$ is a partition of $\delta_e(T(A))$ we shall define a C^*-subalgebra of $A \ltimes_\alpha \mathbb{Z}$ and prove that it is finite dimensional. By $C(\mathcal{P})$, we mean the following C^*-subalgebra: for any $X \in \mathcal{P}$ consider the characteristic function $\chi_X : \delta_e(T(A)) \longrightarrow \{0,1\}$; take its positive affine extension to $T(A)$ and lift it to a positive element $a_X \in A$. From the condition $A^+ \cap \{\cap_{\tau \in \delta_e(T(A))} Ker(\tau)\} = \{0\}$, it is easy to see that a_X is a projection. Define $C(\mathcal{P}) := \langle a_X, X \in \mathcal{P} \rangle$. Of course a_{I^c} is defined to be the projection $\sum_{k=1}^m \sum_{i=2}^{J_k} a_{(k,i)}$. Indeed, using the properties (Lemma 3.2.12) of the orthogonal projections $a_{(k,i)}$, $1 \leq k \leq m$, $1 \leq i \leq J_k$, one can prove as in [29] that $\{e_{ij}^k \otimes a_{k,J_k}\}$ is a system of matrix units and that $\langle C(\mathcal{P}), ua_{I^c} \rangle$ is included in the finite dimensional C^*-algebra generated by all $e_{ij}^k \otimes a_{k,J_k}$. So, the C^*-algebra $\langle C(\mathcal{P}), ua_{I^c} \rangle$ is itself finite dimensional. Also note that $\langle C(\mathcal{P}), ua_{I^c} \rangle \subseteq \langle A, uJ \rangle$, where J is defined in 3.2.6.*

We begin now to carry out the construction of AF algebras for closed subsets $I \subseteq \delta_e(T(A))$. We require some preparation similar to that in [26].

If $0 \neq P$ is a nonzero α-invariant ideal of A, there is a canonical *-morphism $\pi : A \ltimes_\alpha \mathbb{Z} \longrightarrow (A/P) \ltimes_{\bar{\alpha}} \mathbb{Z}$, where $\bar{\alpha} : A/P \longrightarrow A/P$ is given by $\bar{\alpha}([x]) := [\alpha(x)]$ for all $[x] \in A/P$. Denote by u and $\bar{u}$ the corresponding canonical unitaries implementing α, respectively $\bar{\alpha}$ (i.e. $uau^* = \alpha(a)$ for all $a \in A$ and $\bar{u}y\bar{u}^* = \bar{\alpha}(y)$ for all $y \in A/P$). What we mean by 'x is the positive element of A obtained from the characteristic function χ_V' is explained now, and it will be used in the sequel. For any clopen subset $V \subseteq \delta_e(T(A))$, let $\chi_V : \delta_e(T(A)) \mapsto \{0,1\}$ be the characteristic function of V. Then χ_V extends to an affine, real, positive function on $T(A)$, call it f, and the latter has the form $\hat{x}$ for some $0 \leq x \in A$, where $\hat{}$ is the natural map. We observe that $\widehat{\alpha(x)}|_{\delta_e(T(A))} = \chi_{V \circ \alpha^{-1}}$. If $\tau \in V \circ \alpha^{-1}$, then $\tau = v \circ \alpha^{-1}$ for some $v \in V$, and thus $\widehat{\alpha(x)}(\tau) = \tau(\alpha(x)) = \tau \circ \alpha(x) = v \circ \alpha^{-1} \circ \alpha(x)$. We obtain $\widehat{\alpha(x)}(\tau) = v(x) = \hat{x}(v) = \chi_V(v) = 1$. Similarly, if τ does not belong to $V \circ \alpha^{-1}$, then $\widehat{\alpha(x)}(\tau) = 0$.

Lemma 3.2.18. *Let D be a C^*-subalgebra of $A \ltimes_\alpha \mathbb{Z}$ containing A. Suppose $ux \in D$ for the element x constructed from the characteristic function χ_{V^c}, V a clopen subset of $\delta_e(T(A))$, such that $\delta_e(T(A)) \neq \bigcup_{n \in \mathbb{Z}} V \circ \alpha^n$. Then D is not AF.*

Proof. Denote $W := \delta_e(T(A))\backslash(\bigcup_{n\in\mathbb{Z}} V\circ\alpha^n)$. Then $W \neq \emptyset$ and consider the following ideal $I_W := \{a \in A; \tau(a^*a) = 0 \text{ for all } \tau \in W\} \subseteq A$. We prove first that $I_W \neq 0$ and I_W is α -invariant. To see that $I_W \neq 0$, notice that $I_W \supseteq I_O$ where $O := \delta_e(T(A))\backslash V \supseteq \delta_e(T(A))\backslash \bigcup_{n\in\mathbb{Z}} V \circ \alpha^n = W$ and O is clopen; also $I_O \neq (0)$ (just consider the positive element $0 \neq b \in A$ obtained from the characteristic function χ_{O^c}; then $b \in I_O$), and $\alpha(I_W) = I_{W\circ\alpha^{-1}} = I_W$.

From $\delta_e(T(A))\backslash V \supseteq W$ it will follow that $\pi(uc) = \bar{u}$, where $\bar{u}$ is the canonical unitary in $A/I_W \ltimes_{\bar{\alpha}} \mathbb{Z}$, c is the element of A obtained from the characteristic function $\chi_{\delta_e(T(A))\backslash V}$ and $\pi : A \ltimes_\alpha \mathbb{Z} \mapsto A/I_W \ltimes_{\bar{\alpha}} \mathbb{Z}$ is the canonical map. Indeed, $\pi(uc) = \bar{u}\pi(c) = \bar{u}1 = \bar{u}$ (we have $1 \geq c \geq 0$, thus $1 - c \geq 0$; from $\widehat{1-c}|_W = 0$ we obtain that $1 - c \in I_W$, so $\pi(1-c) = 0$).

Thus the map $\pi : D \longrightarrow A/I_W \ltimes_{\bar{\alpha}} \mathbb{Z}$ is onto. Hence, the following quotient

$$D/\mathrm{Ker}(\pi) \simeq A/I_W \ltimes_{\bar{\alpha}} \mathbb{Z}$$

is not AF because A/I_W is a unital AF algebra. Hence D is not AF. □

In order to obtain that $\langle A, uI_I\rangle$ is an AF algebra for closed I, we need to impose another condition on A: any nonzero quotient of A is finite in the sense of Cuntz-Pedersen [7], i.e., $\{0\} = A^+ \bigcap \{\sum_{n=1}^{\infty} x_n x_n^* - x_n^* x_n; x_n \in A\}$.

Theorem 3.2.19. *Suppose that any nonzero quotient of A is finite in the sense of Cuntz-Pedersen and let I be a nonempty closed subset of $\delta_e(T(A))$. Then $\langle A, uI_I\rangle$ is an AF algebra if and only $\delta_e(T(A)) = \bigcup_{n\in Z} W \circ \alpha^n$ for every clopen subset $W \subseteq \delta_e(T(A))$ containing I.*

Proof. To prove necessity, suppose to the contrary that there is a clopen subset $\delta_e(T(A)) \supseteq W \supseteq I$ such that $\delta_e(T(A)) \neq \bigcup_{n\in Z} W \circ \alpha^n$. If $x \in A$ is the element obtained from the characteristic function $\chi_{\delta_e(T(A))\backslash W}$, then $ux \in uI_W \subseteq uI_I \subseteq < A, uI_I >$. Using the lemma above we get that $\langle A, uI_I\rangle$ is not AF.

To prove the other implication, suppose that I is a closed subset of $\delta_e(T(A))$ such that for every clopen subset $W \subseteq \delta_e(T(A))$ containing I, one has $\delta_e(T(A)) = \bigcup_{n\in Z} W \circ \alpha^n$. There exists a decreasing sequence of clopen subsets $I_1 \supseteq I_2 \supseteq \ldots$ of $\delta_e(T(A))$ such that $\bigcap_{n=1}^{\infty} I_n = I$. This yields an increasing sequence of C*-algebras

$\langle A, uI_{I_1}\rangle \subseteq \langle A, uI_{I_2}\rangle \subseteq \ldots$ such that the closure of $\bigcup_{n=1}^{\infty}\langle A, uI_{I_n}\rangle$ equals the C*-algebra $\langle A, uI_I\rangle$. For this see Remark 3.2.20.

Since $\langle A, uI_{I_n}\rangle$ is an AF algebra for all $n \geq 1$, it follows that $\langle A, uI_I\rangle$ is an AF algebra. $\square$

Remark 3.2.20. *Here we show that $I_I = \overline{\bigcup I_{I_n}}$. Since $I \subseteq I_n$ for every $n \geq 1$ we get that $I_I \supseteq I_{I_n}$ for every $n \geq 1$, thus $I_I \supseteq \overline{\bigcup I_{I_n}}$.*

Let $0 \neq x \in (I_I)^+ \setminus \overline{\bigcup I_{I_n}}$. Let $\epsilon > 0$ be given. Since $\widehat{x}|_I = 0$ and $\delta_e(T(A))$ is totaly disconnected, there exist (after using the lifting hypotheses) elements $a_{n_1} \in (I_{I_{n_1}})_{sa}, \ldots, a_{n_k} \in (I_{I_{n_k}})_{sa}$ such that $||\widehat{x} - \sum_{i=1}^{k}\widehat{a_{n_i}}|| < \epsilon/4$. Hence in A/A_0, the following inequality holds: $||[x - \sum_{i=1}^{k} a_{n_i}]|| < \epsilon/4$. Set $a := \sum_{i=1}^{k} a_{n_i} \in J := \overline{\bigcup I_{I_n}}$; then there exists $c \in A_0$ such that $||x - a - c|| < \epsilon/4$. It follows that $||[x] - [c]|| < \epsilon/4$ in A/J. Since $[c] \in (A/J)_0$ and the latter is a closed subspace in A/J ([7]), $0 \neq [x] \in (A/J)_0 \cap (A/J)^+$, a contradiction, since we have supposed that any quotient of A is finite in the sense of Cuntz and Pedersen. Therefore $I_I \subseteq J$, and we have proved the equality of the two ideals of A.

Observe that in general A/I_I is a finite C*-algebra in the sense of [7] for any nonempty $I \subseteq T(A)$. Suppose to the contrary that A/I_I is not finite. Then by Theorem 3.8 in [7], there is $x \in A^+$ such that $\tau(x) = 0$ for all finite traces on A that vanish on I_I but x is not in I_I^+. So $x \in I_{F_{I_I}}$ where by F_K we understand the closed face of all traces that vanish on an ideal K of A. Since $I \subseteq F_{I_I}$ we get that $I_I \supseteq I_{F_{I_I}} \ni x$, hence $x \in A^+ \cap I_I = I_I^+$ a contradiction.

Denote by $D(A, \alpha)$ the set of all closed subsets $I \subseteq \delta_e(T(A))$ such that for every clopen subset $W \subseteq \delta_e(T(A))$ containing I, one has $\delta_e(T(A)) = \bigcup_{n \in Z} W \circ \alpha^n$.

Remark 3.2.21. *In analogy with the commutative case, an automorphism $\alpha \in Aut(A)$ is minimal if the only α-invariant ideals of A are (0) and A. If $\alpha \in Aut(A)$ is minimal then we shall prove that condition (*) holds (see the beginning of section 3.2) if $\delta_e(T(A))$ is totally disconnected and the natural map, $\widehat{\ }: A \longmapsto AffT(A)$, lifts positives to positives. Set $P := I_{\bigcup_{i\in\mathbb{Z}} I \circ \alpha^i} = \{x \in A; \tau(x^*x) = 0$ for all $\tau \in \bigcup_{i\in\mathbb{Z}} I \circ \alpha^i\}$. Then P is an α-invariant ideal of A and $P \neq A$. We obtain that $P = (0)$; suppose to* *the contrary that condition (*) does not hold. Just use the second part of the following paragraph to obtain a nonzero element in P, a contradiction.*

The analogy above is natural in the following sense. If A is a unital AF C*-algebra and $\delta_e(T(A))$ is compact, then the automorphism α of A gives rise to a homeomorphism $\delta\alpha : \delta_e(T(A)) \longrightarrow \delta_e(T(A))$ defined by the formula $\delta\alpha(\tau) := \tau \circ \alpha$, for every $\tau \in \delta_e(T(A))$. Let us verify that minimality of $\delta\alpha$ implies minimality of α. Suppose to the contrary that α is not minimal, thus there is a proper ideal I of A such that $\alpha(I) = I$. The closed face $F := \{\tau \in T(A); \tau|_I = 0\}$ of $T(A)$ is nonempty because A/I is AF, so it admits at least one tracial state. Hence $\delta_e(F) = F \cap \delta_e(T(A))$ is a nonempty closed subset of $\delta_e(T(A))$ (and different from $\delta_e(T(A))$ since a tracial state evaluated in 1 is never 0). The minimality of $\delta\alpha$ is contradicted because $\delta\alpha(\delta_e(F)) = \delta_e(F)$. Indeed, if $\tau \in \delta_e(F)$ then $\delta\alpha(\tau) = \tau \circ \alpha \in \delta_e(T(A))$ and moreover for each $i \in I$ one has $\tau \circ \alpha(i) = \tau(i')$ for some $i' \in I$, therefore $(\tau \circ \alpha)|_I = 0$ since $\tau|_I = 0$.

In the case that the boundary $\delta_e(T(A))$ is totally disconnected and the natural map lifts positives to positives, one can prove the converse as follows. Suppose by absurdum that there is a nonempty closed subset $V \subsetneq \delta_e(T(A))$ such that $\delta\alpha(V) = V$. The ideal $I_V := \{x \in A; \tau(x^*x) = 0 \text{ for all } \tau \in V\}$ is different from A. There exist clopen subsets $(I_n)_{n\in\mathbb{N}}$ in $\delta_e(T(A))$ such that $V = \cap_{n\in\mathbb{N}} I_n$ and $I_n \supseteq I_{n+1}$ for every $n \in \mathbb{N}$. Hence $I_V \supseteq I_1 \neq 0$ because the natural map lifts positives to positives. Now it easy to see (using $\delta\alpha(V) = V$) that the proper ideal I_V is α-invariant, a contradiction.

Let us discuss here the simplicity of the crossed product algebras involved here. As in [1], one says that the action is *topologically free* if for any $z_1, \dots, z_m \in \mathbb{Z}\backslash\{0\}$ one has that $\cap_{k=1}^m \{x \in \hat{A}; z_k x \neq x\}$ is dense in the spectrum of A, $\hat{A}$.

Proposition 3.2.22. *Let A be a unital AF algebra with totally disconnected boundary space $\delta_e(T(A))$, $\{0\} = A^+ \cap \{\cap_{\tau\in\delta_e(T(A))} Ker(\tau)\}$ and such that the natural map lifts positives to positives. If $\alpha \in Aut(A)$ is minimal and topologically free, then $A \ltimes_\alpha \mathbb{Z}$ is simple.*

Proof. We shall follow the proof of the commutative case (see [8]). If I is a nonzero ideal of the crossed product, let $a \in I$ be a nonzero positive element. Letting E :

$A \rtimes_\alpha \mathbb{Z} \longrightarrow A$ be the canonical faithful expectation, it follows that $0 \neq x := E(a) \geq 0$ and that $x \in I$ (see Lemma 3.2.23 below). Since $x \geq 0$, $\widehat{x}|_{\delta_e(T(A))} \geq 0$. Then $\widehat{x} > 0$ on a nonempty open subset O of $\delta_e(T(A))$ since $\{0\} = A^+ \cap \{\cap_{\tau \in \delta_e(T(A))} \mathrm{Ker}(\tau)\}$. Since $\delta\alpha : \delta_e(T(A)) \longrightarrow \delta_e(T(A))$ is minimal (see the Remark above 3.2.21), $\delta_e(T(A))$ is covered by $(\delta\alpha)^j(O)$, $\{1 \leq j \leq n\}$ for some $n \in \mathbb{N}$.

Define $0 \leq x_n := \sum_{j=0}^{n} u^j x u^{-j} = \sum_{j=0}^{n} \alpha^j(x) \in I$. Then $\widehat{x_n} > 0$ and $x_n \in I \cap A$. The ideal generated in A by x_n is A (if not, let J be the ideal generated by x_n, and notice that A/J is unital and AF, thus has at least one tracial state— just compose it with the canonical map $\pi : A \longrightarrow A/J$ and obtain a tracial state on A that kills x_n, a contradiction). Thus $A \subseteq A \cap I \subseteq A$, and it follows that $A = A \cap I$, so $A \subseteq I$. Since $1 \in A$ is invertible, it follows that $I = A \rtimes_\alpha \mathbb{Z}$. $\square$

Lemma 3.2.23. *If α is topologically free, then for each nonzero ideal I of the crossed product algebra $A \rtimes_\alpha \mathbb{Z}$, one has that $E(I) \subseteq I \cap A$, where E is the canonical faithful expectation.*

Proof. If $I \cap A = 0$ then by Theorem 1 in [1], we get that $I \subseteq 0 = I_\lambda$ ($\mathbb{Z}$ is amenable), a contradiction with $I \neq 0$.

So $I \cap A \neq 0$. Suppose by absurdum that there is a nonzero positive element $a \in I$ such that $E(a)$ is not in $A \cap I$. Thus $E(a)$ is different than 0. Then the rest of the proof of Theorem 1 in [1] works and we get $||E(a)|| < ||E(a)||$, a contradiction. $\square$

We generalize the work of Poon [25] concerning stable rank for some crossed products. If P is a nonzero α-invariant ideal of A, one can construct the crossed product $P \rtimes_{\alpha|_P} \mathbb{Z}$, where $\alpha|_P : P \longrightarrow P$ is the restriction map. There is an exact sequence associated to P,

$$\dagger \qquad 0 \longrightarrow P \rtimes_{\alpha|_P} \mathbb{Z} \xrightarrow{i} A \rtimes_\alpha \mathbb{Z} \xrightarrow{\pi} A/P \rtimes_{\bar{\alpha}} \mathbb{Z} \longrightarrow 0$$

where $\bar{\alpha} : A/P \longrightarrow A/P$ is the canonical automorphism given by $\bar{\alpha}([x]) := [\alpha(x)]$ for all $[x] \in A/P$, and i and π are the canonical morphisms. The following lemma was proved in [28]

Lemma 3.2.24. *Let*

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Then tsr(A) = 1 if and only if tsr(B) = tsr(I) = 1 and $K_1(\pi) : K_1(A) \longrightarrow K_1(B)$ is onto.

We get the following by applying this lemma to the exact sequence (†).

Proposition 3.2.25. *Let P be a nonzero α-invariant ideal of A. Then tsr($A \ltimes_\alpha \mathbb{Z}$) = 1 if and only if tsr($P \ltimes_{\alpha|_P} \mathbb{Z}$) = tsr($(A/P) \ltimes_{\bar{\alpha}} \mathbb{Z}$) = 1 and $K_1(\pi) : K_1(A \ltimes_\alpha \mathbb{Z}) \longrightarrow K_1((A/P) \ltimes_{\bar{\alpha}} \mathbb{Z})$ is onto.*

Let I be an α-invariant subset of $D(A,\alpha)$. Let $P := I_I = \{x \in A; \tau(x^*x) = 0 \text{ for all } \tau \in I\}$. Then P is an α-invariant ideal of A.

Proposition 3.2.26. *Let I be an α-invariant subset of $D(A,\alpha)$ and let $\pi : A \ltimes_\alpha \mathbb{Z} \longrightarrow A/P \ltimes_{\bar{\alpha}} \mathbb{Z}$ be defined as in (†). The following are equivalent:*

1) tsr($A \ltimes_\alpha \mathbb{Z}$) = 1.

2) tsr($A/P \ltimes_{\bar{\alpha}} \mathbb{Z}$) = 1 and $K_1(\pi) : K_1(A \ltimes_\alpha \mathbb{Z}) \longrightarrow K_1(A/P \ltimes_{\bar{\alpha}} \mathbb{Z})$ is onto.

Proof. Since P is α-invariant we get that $P \ltimes_{\alpha|_P} \mathbb{Z}$ is an ideal of the AF algebra $\langle A, uP\rangle$, thus $\mathrm{tsr}(P \ltimes_{\alpha|_P} \mathbb{Z}) = 1$. □

3.3 AT algebras

In this section we prove Theorem 3.3.17 about the structure of the crossed product algebra obtained from certain action on AF algebras, generalizing the approximation in [28]. The conditions on α in this theorem assure us that the crossed product algebra is simple.

In [1] it was proved that if the action is $\mathbb{Z}$-simple (i.e., no nontrivial α^n invariant ideals, for any $n \neq 0$) and is topologically free (i.e., for any $z_1, \ldots, z_m \in \mathbb{Z}\backslash\{0\}$, $\bigcap_{k=1}^m \{x \in \hat{A}; z_k x \neq x\}$ is dense in the spectrum of A, $\hat{A}$), then $A \ltimes_\alpha \mathbb{Z}$ is simple. Hence we can substitute the condition on the Connes spectrum by the condition that the action is topologically free.

Let $\emptyset \neq I \subseteq \delta_e(T(A))$ be a clopen subset and $\mathcal{P}$ be a partition of $\delta_e(T(A))$. We shall define a C*-subalgebra of the crossed product as follows. Set $A_I :=< C(\mathcal{P}), ua_{I^c} >$. For $X \in \mathcal{P}$ let $\chi_X : \delta_e(T(A)) \longrightarrow \{0,1\}$ be the characteristic function (it is continuous and positive). Take its positive affine continuous extension f_X to $T(A)$ and lift it back in A to a positive element $0 \neq a_X$, i.e. $\widehat{a_X} = f_X$. Define $C(\mathcal{P}) :=< a_X; X \in \mathcal{P} >\subseteq A$.

In constructing the rest of A_I, we shall use the minimality of α. Set $J := \bigcup_{i\in\mathbb{Z}} I \circ \alpha^i \subseteq \delta_e(T(A))$, where $I \circ \alpha^i := \{\tau \circ \alpha^i; \tau \in I\}$. Put $P := I_J := \{x \in A; \tau(x^*x) = 0, \text{ for all } \tau \in J\}$ and note that P is an ideal of A different from A. Observe that P is α-invariant. Indeed, if $x \in P$, we must show that $\alpha(x) \in P$. Take $\gamma \in J$, so $\gamma = \tau \circ \alpha^i$ for some $i \in \mathbb{Z}$ and $\tau \in I$. Then $\gamma(\alpha(x)^*\alpha(x)) = \gamma \circ \alpha(x^*x) = \tau \circ \alpha^{i+1}(x^*x) = 0$ since $\tau \circ \alpha^{i+1} \in J$.

Since P is α-invariant, from minimality, $P = 0$. Suppose that $J \neq \delta_e(T(A))$. We shall find a contradiction. Since I is open, we get that $I \circ \alpha^i$ is open for all $i \in \mathbb{Z}$, thus J is open, i.e., J^c is closed and different from $\delta_e(T(A))$. If J^c were not the empty set, it would follow that J^c is a proper $\delta\alpha$ invariant closed subset of $\delta_e(T(A))$, a contradiction by the discussion after Remark 3.2.21.

There exists a positive integer n such that (*)

$$\delta_e T(A) = \bigcup_{i=1}^{n} I \circ \alpha^i$$

Here we notice that $\delta_e T(A) \circ \alpha = \delta_e T(A)$ for later use (if τ is a pure trace so is $\tau \circ \alpha$). Since $I \subseteq \delta_e T(A)$, one can define a map $\lambda : I \longrightarrow Z_+^*$ by $\lambda(\tau) := \min\{k \in \{1, \ldots, n\}; \tau \in I \circ \alpha^k\}$. From (*), one sees that $\mathrm{Im}(\lambda)$ is finite, $\mathrm{Im}(\lambda) = \{J_1, \ldots, J_m\}$. For $1 \leq k \leq m$ and $1 \leq j \leq J_k$, define the following sets, $Y(k,j) := \lambda^{-1}(J_k) \circ \alpha^{-j}$. We can prove a result similar to that in [29] or [26]: the function λ is upper semi-continuous and lower semi-continuous, and therefore continuous. Suppose that $\lambda(\tau) = n$, where $\tau \in I$. Then we obtain that $\tau \in I \circ \alpha^n$, thus $\tau \circ \alpha^{-n} \in I$, which is open. Thus there is a neighborhood V of $\tau \circ \alpha^{-n}$ such that $V \subseteq I$. Define $C := (V \circ \alpha^n) \bigcap I$ and notice that C is a neighborhood of τ. For every $\gamma \in C$ one have that $\gamma \in V \circ \alpha^n \subseteq I \circ \alpha^n$ and we get $\lambda(\gamma) \leq n$ (since $\gamma \in I$). Thus λ is upper semi-continuous. Since I is also closed, it follows similarly that λ is lower semi-continuous.

The next propositions are the natural extensions of results from [29](Lemma 3.1) and [26](Lemma 2.2) to the non-commutative case.

Proposition 3.3.1. $\bigcup_{k=1}^{m} Y(k,1) = I \circ \alpha^{-1}$.

Proof. If $\chi \in Y(k,1)$ for some $1 \leq k \leq m$ then $\chi = \tau \circ \alpha^{-1}$, $\tau \in \lambda^{-1}(J_k) \subseteq I$, thus $\chi \in I \circ \alpha^{-1}$. If $\chi \in I \circ \alpha^{-1}$ then $\chi = \tau \circ \alpha^{-1}$, $\tau \in I$. Let us say that $\lambda(\tau) = J_{k_0}$ for some $1 \leq k_0 \leq m$. Then $\chi = \tau \circ \alpha^{-1} \in Y(k_0, 1) \subseteq \bigcup_{k=1}^{m} Y(k,1)$ as $\tau \in \lambda^{-1}(J_{k_0})$. □

Proposition 3.3.2. $Y(k,j) \circ \alpha^{-1} = Y(k, j+1)$ *for* $1 \leq j < J_k$.

Proof. If $\chi \in Y(k,j) \circ \alpha^{-1}$ then $\chi = \gamma_0 \circ \alpha^{-1}$, $\gamma_0 \in Y(k,j)$, i.e. $\gamma_0 = \tau_0 \circ \alpha^{-j}$, $\lambda(\tau_0) = J_k$. Thus $\chi = \tau_0 \circ \alpha^{-1-j}$, $\tau_0 \in \lambda^{-1}(J_k)$, so $\chi \in Y(k, j+1)$. If $Y(k, j+1) \ni \chi$, then $\chi = \gamma_0 \circ \alpha^{-1-j}$ and $\lambda(\gamma_0) = J_k$. Thus $\chi = (\gamma_0 \circ \alpha^{-j}) \circ \alpha^{-1} \in Y(k,j) \circ \alpha^{-1}$ because $\gamma_0 \in \lambda^{-1}(J_k)$. □

Proposition 3.3.3. $\bigcup_{k=1}^{m} Y(k, J_k) = I$.

Proof. If $Y(k, J_k) = \lambda^{-1}(J_k) \circ \alpha^{-J_k} \ni \chi$ for some $k \in \{1, \ldots, m\}$ then $\chi = \tau \circ \alpha^{-J_k}$, $\tau \in \lambda^{-1}(J_k)$. Thus $\tau \in I \circ \alpha^{J_k}$, $\tau = \tau_0 \circ \alpha^{J_k}$, $\tau_0 \in I$ and it follows that $\chi = \tau_0 \in I$. Let $I \ni \eta$. Now since $\delta_e T(A) = \bigcup_{i=1}^{n} I \circ \alpha^i$, on multiplying with α^{-1-n}, we obtain that $\delta_e T(A) = \bigcup_{k=-n}^{-1} I \circ \alpha^k$ (because $\delta_e T(A) \circ \alpha = \delta_e T(A)$ and $(I \circ \alpha^p) \circ \alpha^s = I \circ \alpha^{p+s}$ for every $p, s \in \mathbb{Z}$). Let s be the maximum of $s \in \{-n, -(n-1), \ldots, -1\}$ such that $\eta \in I \circ \alpha^s$. Define $J := -s \geq 1$. We need to show:

a) $J = J_p$ for some p,

b) $\eta \in Y(p, J_p)$.

a) Since $\eta \in I \circ \alpha^s$, it follows that $\eta = \eta_0 \circ \alpha^s$, $\eta_0 \in I$, thus $\eta \circ \alpha^{-s} = \eta_0 \in I$. We shall prove that $\lambda(\eta \circ \alpha^{-s}) = J$ and then $J = J_p$ for some $p \in \{1, \ldots, m\}$. Since $s = -J$, $\eta \circ \alpha^{-s} = \eta \circ \alpha^J \in I \circ \alpha^J$. If there exists $1 \leq s_1 < J$ such that $\eta \circ \alpha^{-s} \in I \circ \alpha^{s_1}$ then $\eta \circ \alpha^{-s} = \gamma_0 \circ \alpha^{s_1}$, $\gamma_0 \in I$, $\eta = \gamma_0 \circ \alpha^{s_1+s}$, $\gamma_0 \in I$, i.e. $\eta \in I \circ \alpha^{s_1+s}$. A contradiction results from $0 = J + s > s_1 + s > s$ and the maximality of s.

b) We must show that $\eta \in Y(p, J_p)$. One has that $\eta = (\eta \circ \alpha^{J_p}) \circ \alpha^{-J_p}$ and it is sufficient to prove that $\eta \circ \alpha^{J_p} \in \lambda^{-1}(J_p)$, i.e. $\lambda(\eta \circ \alpha^{J_p}) = J_p$, but this is exactly a). □

Proposition 3.3.4. *For $1 \leq k \leq m$ and $1 \leq j \leq J_k$ the sets $Y(k,j)$ are pairwise disjoint.*

Proof. a) If $Y(k,j) \bigcap Y(k,i) \ni \chi$ for some $1 \leq j < i \leq J_k$ then $\chi = \gamma \circ \alpha^{-j} = \tau \circ \alpha^{-i}$ where $\lambda(\gamma) = J_k = \lambda(\tau)$. It follows that $\tau = \gamma \circ \alpha^{i-j} \in I \circ \alpha^{i-j}$ and $1 \leq i - j < J_k$. Therefore $J_k = \lambda(\tau) < J_k$, a contradiction.

b) Suppose that $Y(k,j) \bigcap Y(s,i) \neq \emptyset$ for some $k < s$, $1 \leq j \leq J_k$, $1 \leq i \leq J_s$ and let χ be in the intersection. Of course $j \neq i$ because $k < s \Rightarrow J_k \neq J_s$. One has that $\chi = \tau \circ \alpha^{-j} = \gamma \circ \alpha^{-i}$ where $\lambda(\tau) = J_k$ and $\lambda(\gamma) = J_s$. If $i < j$ then $\tau = \gamma \circ \alpha^{j-i} \in I \circ \alpha^{j-i}$ and $0 < j - i < J_k$. Thus $J_k = \lambda(\tau) < J_k$, a contradiction. If $j < i$ then $\gamma = \tau \circ \alpha^{i-j} \in I \circ \alpha^{i-j}$ and $0 < i - j < J_s$. Therefore $J_s = \lambda(\gamma) < J_s$, a contradiction. $\square$

Proposition 3.3.5. $\delta_e T(A) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{J_k} Y(k,j)$.

Proof. The inclusion $\supseteq$ is always true since $Y(k,j) = \lambda^{-1}(J_k) \circ \alpha^{-j}$, $\lambda^{-1}(J_k) \subseteq I$, and $\delta_e T(A) \circ \alpha = \delta_e T(A)$. If $\chi \in \delta_e T(A)$, let j be the largest $j \in \{-n, -(n-1), \ldots, -1\}$ such that $\chi \in I \circ \alpha^j$.(Recall that $\delta_e T(A) = \bigcup_{k=-n}^{-1} I \circ \alpha^k$). Thus $\chi = i \circ \alpha^j$, $i \in I$, i.e. $\chi \circ \alpha^{-j} = i \in I$. Let $\lambda(i) =: J_k$ for some $1 \leq k \leq m$. Then $\chi = i \circ \alpha^j \in \lambda^{-1}(J_k) \circ \alpha^j = Y(k, -j)$, if $1 \leq -j \leq J_k$. Suppose $-j > J_k$, since $\lambda(i) = J_k$ one has $i = i' \circ \alpha^{J_k}$, for some $i' \in I$. Thus $\chi = i' \circ \alpha^{J_k + j} \in I \circ \alpha^{J_k + j}$ and $j < J_k + j < 0$, contradicting the maximality of j. $\square$

Definition 3.3.6. *Let $J := \{a \in A | \tau(a^*a) = 0, \forall \tau \in I\}$ and for $1 \leq k \leq m$ define $A(k, J_k) := \{a \in A | \tau(a^*a) = 0, \forall \tau \in Y(k, J_k)^c\}$, where the complement of the subset $Y(k, J_k)$ is taken with respect to $\delta_e T(A)$.*

Remark 3.3.7. *For $1 \leq k \leq m$, $A(k, J_k)$ and J are ideals of A, and are thus AF.*

We shall use the following notation: $A(k, J_k) = I_{Y(k,J_k)^c}$ and $J = I_I$.

By splitting up the towers $\{Y_{(k,j)}; 1 \leq j \leq J_k\}$ we can make the partition $\{Y_{(k,j)}; 1 \leq j \leq J_k, 1 \leq k \leq m\}$ finer than $\mathcal{P}$.

Set $\Omega := \{(k,j); 1 \leq k \leq m, 1 \leq j \leq J_k\}$ and for any $(k,j) \in \Omega$ let $g_{(k,j)} : \delta_e T(A) \longrightarrow \{0,1\}$ be the characteristic function of $Y_{(k,j)}$. From Corollary 11.15 of [15], there are $f_{(k,j)} \in Aff(T(A))$ such that $f_{(k,j)}|_{\delta_e T(A)} = g_{(k,j)}$ and $||f_{(k,j)}|| = ||g_{(k,j)}|| = 1$. Since $g_{(k,j)} \geq 0$, we can choose $f_{(k,j)} \geq 0$. Let $0 \leq a_{(k,j)} \in A$ be such that $f_{(k,j)} = \widehat{a_{(k,j)}}$. Since $f_{(k,j)} \neq 0$, we get that $a_{(k,j)} \neq 0$. We also obtain that $a_{(k,j)} \in I_{Y(k,j)^c}$ because i) $0 \leq a_{(k,j)}$ and ii) for every $\gamma \in Y(k,j)^c$ we have $0 = g_{(k,j)}(\gamma) = f_{(k,j)}(\gamma) = \widehat{a_{(k,j)}}(\gamma) = \gamma(a_{(k,j)})$.

Definition 3.3.8. *For every $1 \leq k \leq m$, $1 \leq i,j \leq J_k$ define the element $e^k_{ij} \otimes a_{(k,J_k)} := u^{i-j}\alpha^{j-J_k}(a_{(k,J_k)}) \in A \ltimes_\alpha \mathbb{Z}$.*

Define $a_{I^c} := \sum_{\Omega_0} a_{(k,i)}$ where $\Omega_0 := \Omega\backslash\{(1,1),(2,1),\ldots,(m,1)\}$.

So, now the algebra $A_I :=< C(\mathcal{P}), ua_{I^c} >$ is defined. We shall show that A_I is included in the finite dimensional C*-algebra generated by the matrix unit system $e^k_{ij} \otimes a_{(k,J_k)}$, where $1 \leq k \leq m$, $1 \leq i,j \leq J_k$.

We require some lemmas about the projections arising in this fashion.

Lemma 3.3.9. $A = \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$, *where $\Omega := \{(k,j); 1 \leq k \leq m, 1 \leq j \leq J_k\}$.*

Proof. The proof resembles the proofs given in the previous section. We have already obtained the following partition of $\delta_e(T(A))$,

$$\delta_e(T(A)) = \bigcup_{(k,j)\in\Omega} Y(k,j).$$

For arbitrary $(k,j) \in \Omega$ let $g_{(k,j)} : \delta_e(T(A)) \longrightarrow \{0,1\}$ be the characteristic function of $Y_{(k,j)}$. From corollary 11.15 of [15] (since is $\delta_e(T(A))$ compact and $g_{(k,j)}$ is continuous), one can find $f_{(k,j)} \in \mathrm{Aff}(T(A))$ such that $f_{(k,j)}|_{\delta_e(T(A))} = g_{(k,j)}$ and $||f_{(k,j)}|| = ||g_{(k,j)}|| = 1$. Since $g_{(k,j)} \geq 0$ one can choose $f_{(k,j)} \geq 0$. Let $0 \leq a_{(k,j)} \in A$ be such that $f_{(k,j)} = \widehat{a_{(k,j)}}$. Since $f_{(k,j)} \neq 0$, we have that $a_{(k,j)} \neq 0$. Also we obtain that $a_{(k,j)} \in I_{Y(k,j)^c}$ because i) $0 \leq a_{(k,j)}$ and ii) for every $\gamma \in Y(k,j)^c$ we have $0 = g_{(k,j)}(\gamma) = f_{(k,j)}(\gamma) = \widehat{a_{(k,j)}}(\gamma) = \gamma(a_{(k,j)})$. It follows from $1_{C(\delta_e(T(A)))} = \sum_{(k,j)\in\Omega} g_{(k,j)}$ that $\hat{1} = \widehat{\sum_{(k,j)\in\Omega} a_{(k,j)}}$ in $\mathrm{Aff}(T(A))$. Suppose to the contrary that $A \neq S := \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$. Then A/S is a unital AF algebra and it has at least one tracial state, let us say that $R : A/S \longrightarrow \mathbb{C}$ is an extremal tracial state of A/S. Then $R \circ \pi : A \longrightarrow \mathbb{C}$ is an extremal tracial state of A, where $\pi : A \longrightarrow A/S$ is the canonical quotient map. It follows that for all $(k,j) \in \Omega$ one has $R \circ \pi(a_{(k,j)}) = 0$, i.e. $\widehat{a_{(k,j)}}(R \circ \pi) = 0$. Plugging $R \circ \pi$ in $\hat{1} = (\widehat{\sum_{(k,j)\in\Omega} a_{(k,j)}})$ we get that $1 = 0$, a contradiction. Hence $A = S$.

$\square$

Lemma 3.3.10. *For every $1 \le k \le m$ and $1 \le i \le J_k$, $\alpha^{i-J_k}(A(k, J_k)) = A(k, i)$ and $a_{(k,i)} = \alpha^{i-J_k}(a_{(k,J_k)})$.*

Proof. Let us begin by showing that $\alpha^{i-J_k}(A(k, J_k)) = A(k, i)$. Letting $x \in A(k, J_k)$ we need to show that $\alpha^{i-J_k}(x) \in A(k,i) := \{a \in A; \tau(a^*a) = 0$, for every $\tau \in Y(k,i)^c\}$. Thus if $\gamma \in Y(k,i)^c\}$ is arbitrary, we need to show that $\gamma(\alpha^{i-J_k}(x^*x)) = \gamma \circ \alpha^{i-J_k}(x^*x)) = 0$, and it is sufficient now to prove that $Y(k,i)^c \circ \alpha^{i-J_k} \subseteq Y(k, J_k)^c$, since $\gamma \in Y(k,i)^c$. Let us prove the contrapositive $\{Y(k,i)^c \circ \alpha^{i-J_k}\}^c \supseteq Y(k, J_k)$. Taking $\tau \in Y(k, J_k)$ we can write $\tau = \tau_0 \circ \alpha^{-J_k}$, with $\lambda(\tau_0) = J_k$. Suppose to the contrary that $\tau \in Y(k,i)^c \circ \alpha^{i-J_k}$. Then $\tau = \mu \circ \alpha^{i-J_k}$, for some $\mu \in Y(k,i)^c$. We obtain that $\tau_0 \circ \alpha^{-J_k} = \mu \circ \alpha^{i-J_k}$, i.e. $\tau_0 = \mu \circ \alpha^i$. Let us say that $\mu \in Y(s,t)$ for some $1 \le s \le m$ and $1 \le t \le J_s$ such that $t \ne i$ or $s \ne k$.

i) If $\mu \in Y(s,t)$ with $t \ne i$ then $\mu = \mu_0 \circ \alpha^{-t}$, where $\lambda(\mu_0) = s$. We get that $\tau_0 = \mu_0 \circ \alpha^{i-t}$. If $i > t$ then $Y(k,i) \ni \tau_0 \circ \alpha^{-i} = \mu_0 \circ \alpha^{-t} \in Y(s,t)$, a contradiction from Proposition 3.2.4 since $t \ne i$. If $i < t$ then $Y(k, 1) \ni \tau_0 \circ \alpha^{-1} = \mu_0 \circ \alpha^{i-t-1} \in Y(s, t-i+1)$ a contradiction by Proposition 3.2.4., since $1 \ne t-i+1$ and $1 < t-i+1 \le J_s - i + 1 \le J_s$.

ii) If $\mu \in Y(s,t)$, $t = i$ but $s \ne k$ then $\tau_0 = \mu \circ \alpha^i = \mu_0 \circ \alpha^{-i} \circ \alpha^i = \mu_0$. It follows that $\tau_0 = \mu_0$, thus $J_k = \lambda(\tau_0) = \lambda(\mu_0) = J_s$, a contradiction (again from Proposition 3.2.4) since $s \ne k$. Therefore we have shown $\alpha^{i-J_k}(A(k, J_k)) \subseteq A(k,i)$.

Let us prove the reverse inclusion $A(k,i) \subseteq \alpha^{i-J_k}(A(k, J_k))$. Letting $x \in A(k,i) := \{a \in A; \tau(a^*a) = 0$ for every $\tau \in Y(k,i)^c\}$, we need to show that $x \in \alpha^{i-J_k}(A(k, J_k))$. Since $x = \alpha^{i-J_k}(\alpha^{J_k-i}(x))$, it is sufficient to show that $\alpha^{J_k-i}(x) \in A(k, J_k)$. Let $\gamma \in Y(k, J_k)^c$ be arbitrary; we need to show that $\gamma(\alpha^{J_k-i}(x^*)\alpha^{J_k-i}(x)) = 0$, i.e. $\gamma \circ \alpha^{J_k-i}(x^*x) = 0$. Since $x \in A(k,i)$ it is sufficient to prove that $\gamma \circ \alpha^{J_k-i} \in Y(k,i)^c$. Let us prove that $Y(k, J_k)^c \circ \alpha^{J_k-i} \subseteq Y(k,i)^c$.

For $\tau \in Y(k,i)$ one has that $\tau = \tau_0 \circ \alpha^{-i}$, where $\lambda(\tau_0) = J_k$. Supposing to the contrary that $\tau \in Y(k, J_k)^c \circ \alpha^{J_k-i}$, one can write $\tau = \mu \circ \alpha^{J_k-i}$, where $\mu \in Y(k, J_k)^c$.

It follows that $\tau_0 \circ \alpha^{-i} = \mu \circ \alpha^{J_k - i}$, i.e. $\tau_0 = \mu \circ \alpha^{J_k}$. Let us say that $\mu \in Y(s,t)$ for some $1 \leq s \leq m$ and $1 \leq t \leq J_s$ with $s \neq k$ or $t \neq J_k$.

i) The case $t \neq J_k$ splits in two subcases:

a) If $t < J_k$ since $\mu \in Y(s,t)$, one gets that $\mu = \mu_0 \circ \alpha^{-t}$, with $\lambda(\mu_0) = J_s$. Thus $\tau_0 = \mu_0 \circ \alpha^{-t+J_k}$ implies that $Y(k, J_k + 1 - t) \ni \tau_0 \circ \alpha^{-1+t-J_k} = \mu \circ \alpha^{-1} \in Y(s,1)$, a contradiction (see Proposition 3.2.4.) since $J_k + 1 - t \neq 1$ and $1 < J_k + 1 - t \leq J_k$.

b) If $t > J_k$ then $Y(k, J_k) \ni \tau_0 \circ \alpha^{-J_k} = \mu_0 \circ \alpha^{-t} \in Y(s,t)$, a contradiction(see Proposition 3.2.4) since $1 \leq t \leq J_s$.

ii) If $t = J_k$ but $s \neq k$ one has that $\tau_0 = \mu_0$. It follows that $J_k = \lambda(\tau_0) = \lambda(\mu_0) = J_s$, a contradiction since $s \neq k$.

Before starting to show the last equalities in the statement, let us remark that the condition $A^+ \cap_{\tau \in \delta_e T(A)} \mathrm{Ker}(\tau) = \{0\}$ implies that $a_{(k,j)} \in I_{Y(k,j)^c}$, $(k,j) \in \Omega$, are orthogonal projections summing up to 1 (these projections are coming from $A = \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$).

To prove the second set of equalities, we start by eliminating the trivial case when $J_k = i$ thus we shall suppose that $1 \leq i < J_k$. Denote $x := a_{(k,i)} - \alpha^{i-J_k}(a_{(k,J_k)})$. Then $x = 1x = (\sum_{k=1}^{m}\sum_{i=1}^{J_k} a_{(k,i)})x = a_{(k,i)} - (\sum_{k=1}^{m}\sum_{i=1}^{J_k} a_{(k,i)})\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)} - a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ because $a_{(k,j)}$, $(k,j) \in \Omega$ are mutually orthogonal. Thus $\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$. Remark that $a_{(k,i)} = a_{(k,i)}1 = a_{(k,i)}\alpha^{i-J_k}(1) = a_{(k,i)}\alpha^{i-J_k}(\sum_{k'=1}^{m}\sum_{i'=1}^{J_{k'}} a_{(k',i')}) = a_{(k,i)}\sum_{k'=1}^{m}\sum_{i'=1}^{J_{k'}} \alpha^{i-J_k}(a_{(k',i')}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ because $a_{(k,i)} \in A(k,i) = \alpha^{i-J_k}(A(k,J_k))$ and the projections $a_{(k,j)}$, $(k,j) \in \Omega$ are mutually orthogonal. It follows from $\alpha^{i-J_k}(a_{(k,J_k)}) = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ and $a_{(k,i)} = a_{(k,i)}\alpha^{i-J_k}(a_{(k,J_k)})$ that $a_{(k,i)} = \alpha^{i-J_k}(a_{(k,J_k)})$.

□

Of course $C(\mathcal{P}) \subseteq$ span $\{e^k_{i,i} \otimes a_{(k,J_k)}; 1 \leq k \leq m, 1 \leq i \leq J_k\}$ and from the lemmas above one gets that $1 = \sum_{(k,i)\in\Omega} a_{(k,i)} = \sum_{(k,i)\in\Omega} e^k_{i,i} \otimes a_{(k,J_k)}$ and

$$ua_{I^c} = u\sum_{(k,j)\in\Omega_o} a_{(k,j)} = \sum_{(k,j)\in\Omega_o} u\alpha^{j-J_k}(a_{(k,J_k)}) = \sum_{(k,j)\in\Omega_o} u^{j+1-j}\alpha^{j-J_k}(a_{(k,J_k)}) = \sum_{(k,j)\in\Omega_o} e^k_{j+1,j} \otimes a_{(k,J_k)}.$$

It turns out that $A_I =$ span $\{e^k_{i,j} \otimes a_{(k,J_k)}; 1 \leq j \leq J_k, 1 \leq k \leq m\}$.

Let us note here that $\mathcal{P}_1 \leq \mathcal{P}_2$ imply $C(\mathcal{P}_1) \subseteq C(\mathcal{P}_2)$. Indeed, if $X \in \mathcal{P}_1$, let $a_X \in C(\mathcal{P}_1)$ be its corresponding projection and let $\overline{X}_1, \ldots, \overline{X}_n$ be the other elements of $\mathcal{P}_1$. Then $1 = a_X + a_{\overline{X}_1} + \cdots + a_{\overline{X}_n}$. Let us say that X is partitioned by some $Y_1, \ldots, Y_m \in \mathcal{P}_2$. Denote the rest of elements of $\mathcal{P}_2$ by $Y_{m+1}, \ldots, Y_k$ and we have that $1 = a_{Y_1} + \cdots + a_{Y_k}$. Multiplying both sides of the last equation with a_X we get that $a_X = a_{Y_1} + \cdots + a_{Y_m}$. If we multiply both sides of $1 = a_X + a_{\overline{X}_1} + \cdots + a_{\overline{X}_n}$ with $a_{Y_1} + \cdots + a_{Y_m}$ we get that $a_{Y_1} + \cdots + a_{Y_m} = (a_{Y_1} + \cdots + a_{Y_m})a_X$. Hence it follows that $a_X = a_{Y_1} + \cdots + a_{Y_m} \in C(\mathcal{P}_2)$.

Lemma 3.3.11. *Suppose that $I_1 \supseteq I_2$ are nonempty clopen subsets of $\delta_e(T(A))$ and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions such that $\mathcal{P}_1 \leq \mathcal{P}_2$. If $a_{I_1} \in C(\mathcal{P}_2)$ then $A_{I_1} \subseteq A_{I_2}$.*

Proof. Since $1 = a_{I_2^c} + a_{I_2}$ by multiplying both sides with $a_{I_1^c}$ we get that $a_{I_1^c} = a_{I_1^c}a_{I_2^c} + a_{I_1^c}a_{I_2} = a_{I_1^c}a_{I_2^c} + 0 \in A_{I_2}$. ☐

Remark 3.3.12. *Let I be a nonempty closed subset of $\delta_e(T(A))$. Consider the C^*-subalgebra of $A \rtimes_\alpha \mathbb{Z}$ generated by $Z(A)$ and uI_I. Start by choosing an increasing sequence of partitions of $\delta_e(T(A))$, let us say, $\mathcal{P}_1 \leq \mathcal{P}_2 \leq \ldots$, whose union generates the topology of $\delta_e(T(A))$. Let $(I_n)_{n \geq 1}$ be a sequence of decreasing clopen subsets of $\delta_e(T(A))$ such that their intersection is I. We shall define new partitions $\mathcal{P}_n^{'}$ and finite dimensional C^*-algebras A_n for each $n \geq 1$. Set $\mathcal{P}_1^{'} := \mathcal{P}_1$ and $A_1 := A_{I_1}$. Assuming that we defined $\mathcal{P}_n^{'}$ and $A_n = A_{I_n}$ where this last algebra is computed using the partition $\mathcal{P}_n^{'}$, let us define $\mathcal{P}_{n+1}^{'} := \mathcal{P}_n^{'} \bigvee \mathcal{P}_{n+1} \bigvee \{I_n, \delta_e(T(A)) \backslash I_n\}$. Then $\mathcal{P}_{n+1}^{'} \geq \mathcal{P}_{n+1}, \mathcal{P}_n^{'}$ and $a_{\delta_e(T(A))\backslash I_n} \in C(\mathcal{P}_{n+1}^{'})$. Define now $A_{n+1} := A_{I_{n+1}}$ where the last algebra uses the partition $\mathcal{P}_{n+1}^{'}$.*

The last lemma tells us that $A_n \subseteq \langle Z(A), uI_I \rangle$ and $A_n \subseteq A_{n+1}$ for all $n \geq 1$. If any nonzero quotient of A is finite in the sense of Cuntz-Pedersen ([7]), then as in Remark 3.2.20, one gets that $I_I = \overline{\bigcup_n I_{I_n}}$. Let us suppose now that $Z(A)$ is generated by its projections. If e is a central projection in A, p is a projection in A and $\tau \in \delta_e(T(A))$ then $\tau(ep) = \tau(e)\tau(p)$. (Indeed, if $\tau(e) = 0$, since $ep \leq e$ it will follow that $\tau(ep) = 0$, so one may assume that $\tau(e) \neq 0$; setting $\tau_e(p) := \frac{\tau(ep)}{\tau(e)}$ one remarks that $\tau_e(p) \leq \frac{\tau(p)}{\tau(e)}$ and since τ is extremal one gets the desired equality). So for every extremal tracial state τ one has that $\tau|_{Z(A)}$ is multiplicative on projections, and thus $\tau|_{Z(A)} : Z(A) \longrightarrow \mathbb{C}$ is multiplicative, thus $\tau|_{Z(A)} \in \delta_e(T(Z(A)))$.

Let x be a projection in $Z(A)$, and let $\epsilon > 0$ be given. Since $\widehat{x}|_{\delta_e(T(A))} \geq 0$ and $\delta_e(T(A))$ is totally disconnected, it follows that there are some real numbers $\lambda_1, \ldots, \lambda_n$ and some clopen subsets $E_1, \ldots, E_n$ of $\delta_e(T(A))$ such that $||\widehat{x}|_{\delta_e(T(A))} - \sum_{i=1}^n \lambda_i \chi_{E_i}|| < \epsilon$. Using the lifting hypotheses we get that there are some central projections $P_1, \ldots, P_n$ such that $||\widehat{x}|_{\delta_e(T(A))} - \sum_{i=1}^n \lambda_i \widehat{P_i}|_{\delta_e(T(A))}|| < \epsilon$. Setting $y := x - \sum_{i=1}^n \lambda_i P_i$, we have that $||\widehat{y}|_{\delta_e(T(A))}|| < \epsilon$. Recalling that $||y|| = \sup_{\gamma \in \delta_e(T(Z(A)))} |\gamma(y)| = \sup_{\gamma \in (T(Z(A)))} |\gamma(y)|$ and $||\widehat{y}|| = \sup_{\tau \in T(A)} |\tau(y)| = \sup_{\tau \in \delta_e T(A)} |\tau(y)|$, it follows from the discussion above and from $y \in Z(A)$ that $||y|| < \epsilon$, and therefore $\langle Z(A), uI_I \rangle = \lim_n A_n$.

Recalling that A is unital, let us fix an α-invariant tracial state τ of A. Additionally suppose that τ is faithful. Consider the GNS construction associated to τ. There is a Hilbert space H_τ, a representation $\pi_\tau : A \longrightarrow B(H_\tau)$, and a vector $\xi_\tau \in (H_\tau)_1$ such that $\tau(a) = (\xi_\tau, \pi_\tau(a)(\xi_\tau))$ for all $a \in A$ and $H_\tau = \overline{\pi_\tau(A)\xi_\tau}$. Because we shall use H_τ extensively, let us recall its construction. Let $V_\tau := \{a \in A; \tau(a^*a) = 0\}$ be the canonical ideal associated to τ. Then A/V_τ becomes a prehilbert space via $([b], [a]) := \tau(b^*a)$ for all $a, b \in A$. Then H_τ is the completion of A/V_τ and $\xi_\tau := [1] \in A/V_\tau$. The representation is obtained by extending $L_a : A/V_\tau \longrightarrow A/V_\tau$, $L_a([b]) := [ab]$ to $\pi_\tau^a : H_\tau \longrightarrow H_\tau$, so $\pi_\tau : A \longrightarrow B(H_\tau)$, $\pi_\tau(a) := \pi_\tau^a$ is the representation.

Now define $\overline{\pi_\tau} : A \longrightarrow B(L^2(\mathbb{Z}, H_\tau)$ by $(\overline{\pi_\tau}(a)\xi)(t) := \pi_\tau(\alpha^{-t}(a))(\xi(t))$ and $\lambda : \mathbb{Z} \longrightarrow \mathcal{U}(B(L^2(\mathbb{Z}, H_\tau))$ by $(\lambda(\xi)(m) := \underline{\xi}(m)$ for all $\xi \in L^2(\mathbb{Z}, H_\tau), m \in \mathbb{Z}$, where $\underline{\xi}(m) := \lim_n[\alpha(b_n)]$ if $\xi(m) := \lim_n[b_n]$ and it follows that $\lambda^{-1}(\xi)(m) := \underline{\underline{\xi}}(m)$, where $\underline{\underline{\xi}}(m) = \lim_n[\alpha^{-1}(b_n)]$ if $\xi(m) := \lim_n[b_n]$. From these definitions, it easy to see that the following relation holds for all $a \in A$: $\lambda \circ \overline{\pi_\tau}(a) \circ \lambda^{-1} = \overline{\pi_\tau}(\alpha(a))$.

We shall use either notation that has already appeared in previous sections, λ or u, depending on which is more convenient. We are associating certain C*-subalgebras to partitions of $\delta_e(T(A))$ and at the end, the type of these algebras will give the approximation result.

Let $\mathcal{P}$ be a partition of $\delta_e(T(A))$ and $\epsilon > 0$ be given and choose $0 < N \in \mathbb{N}$ such that $\pi/N < \epsilon$. (Here π means the real number 3.1415..., not be confused with the canonical quotient map or with a representation.) For arbitrary $\tau_0 \in \delta_e(T(A))$, the elements $\tau_0, \tau_0 \circ \alpha^{-1}, \ldots, \tau_0 \circ \alpha^{-N}$ are mutually disjoint, thus we can find a clopen neighbourhood I of τ_0 such that

(i) $I, I \circ \alpha^{-1}, \ldots, I \circ \alpha^{-N}$ are mutually disjoint,

(ii) for $n \in \{0, 1, \ldots, N\}$, the clopen subset $I \circ \alpha^{-n}$ lies in an element of $\mathcal{P}$. Since α is minimal (remark 3.2.19) and I is clopen, we can apply the results in the previous section 3.2; we obtain the ideals $I_{Y_{(k,j)^c}}$, $1 \leq k \leq m$ and $1 \leq j \leq J_k$, $A = \sum_{(k,j)\in\Omega} I_{Y(k,j)^c}$, where $\Omega := \{(k,j); 1 \leq k \leq m, 1 \leq j \leq J_k\}$ with all their properties. By splitting the towers $\{Y_{(k,j)}; 1 \leq j \leq J_k\}$, we can make the partition $\mathcal{P}_0 := \{Y_{(k,j)}; 1 \leq j \leq J_k, 1 \leq k \leq m\}$ finer than $\mathcal{P}$.

From the previous section, $A_0 :=< C(\mathcal{P}_0), ua_{I^c} >$ is a finite dimensional C*-algebra.

Define two new automorphisms of A as follows

$$\psi_0(x) := \begin{cases} \alpha(x), & \text{if } x \notin \sum_{k=1}^{m} I_{Y_{(k,J_k)^c}}, \\ \alpha^{1-J_k}(x) & \text{if } x \in I_{Y_{(k,J_k)^c}}. \end{cases} \quad \text{and } \phi_0 := \psi_0^{-1} \circ \alpha.$$

To see (for example) why ψ_0 is injective, we must use the properties from the previous section. Suppose by absurdum that there are nonzero elements $x \notin \sum_{k=1}^{m} I_{Y_{(k,J_k)^c}}$ and $y \in I_{Y_{(k',J_{k'})^c}}$ such that $\psi_0(x) = \psi_0(y)$. From $Y_{(k,j)^c} \circ \alpha^{-1} = Y_{(k,j+1)^c}$, we obtain by iteration that $Y_{(k,1)^c} \circ \alpha^{-(J_k-1)} = Y_{(k,J_k)^c}$, hence

$$I_{Y_{(k',1)^c} \circ \alpha^{-(J_{k'}-1)}} = I_{Y_{(k',J_{k'})^c}} \qquad (\nabla).$$

It follows that $y \in I_{Y_{(k',1)^c} \circ \alpha^{-(J_{k'}-1)}} = \alpha^{J_{k'}-1}(I_{Y_{(k,1)^c}})$. Now $\psi_0(x) = \psi_0(y)$ implies that $\alpha(x) = \alpha^{1-J_k}(y) \in I_{Y_{(k,1)^c}}$. If $x \in I_{Y_{(s,t)^c}}$, then $\alpha(x) \in I_{Y_{(s,t+1)^c}}$ which is disjoint from $\sum_{k=1}^{m} I_{Y_{(k,1)^c}}$ because x, y are nonzero. This yields a contradiction.

Associated to these two automorphisms of A are two unitary operators $u_0 : L^2(Z, H_\tau) \longrightarrow L^2(\mathbb{Z}, H_\tau)$ and $v_0 : L^2(\mathbb{Z}, H_\tau) \longrightarrow L^2(\mathbb{Z}, H_\tau)$ given as follows. For $\xi \in L^2(\mathbb{Z}, H_\tau)$ and $m \in \mathbb{Z}$, we know that $\xi(m) \in H_\tau$, and thus $\xi(m) = \lim_n [a_n]$ for some $(a_n)_{n\geq 1}$ in A. Define $\bar{\xi}(m) := \lim_n [\phi_0(a_n)]$ and observe that $\bar{\xi} \in L^2(\mathbb{Z}, H_\tau)$. Then let $u_0(\xi)(m) := \bar{\xi}(m)$. The fact that $\bar{\xi}(m)$ is well-defined uses the invariance of the trace. Indeed, since τ is α-invariant we have that τ is ϕ_0-invariant (see the definition above of ϕ_0). Thus one obtains a well-defined map $\overline{\phi_0} : A/V_\tau \longrightarrow A/V_\tau$, $\overline{\phi_0}([a]) := [\phi_0(a)]$ which will assure that $\bar{\xi}(m)$ is well-defined. To construct v_0, use the other automorphism, ψ_0, instead of ϕ_0.

We have to show that these operators are unitary. Given $\xi_1, \xi_2 \in L^2(\mathbb{Z}, H_\tau)$, we need to show that $< u_0(\xi_1), u_0(\xi_2) >=< \xi_1, \xi_2 >$. From the definition of u_0, we need to prove that $< \overline{\xi_1}, \overline{\xi_2} >=< \xi_1, \xi_2 >$, where $\overline{\xi_i}(m) := \lim_n[\phi_0(a_n)]$ if $\xi_i(m) = \lim_n[a_n]$, where $(a_n)_{n\geq 1} \in A$, $m \in \mathbb{Z}$ and $i \in \{1, 2\}$. We get that $< u_0(\xi_1), u_0(\xi_2) >=< \overline{\xi_1}, \overline{\xi_2} >= \int_{\mathbb{Z}} < \overline{\xi_1}(m), \overline{\xi_2}(m) > dm = \int_{\mathbb{Z}} < \lim_n[\phi_0(a_n^1)], \lim_k[\phi_0(a_k^2)] > dm = \int_{\mathbb{Z}} \lim_n \lim_k < [\phi_0(a_n^1)], [\phi_0(a_k^2)] > dm = \int_{\mathbb{Z}} \lim_n \lim_k \tau((\phi_0(a_n^1))^* \phi_0(a_k^2))dm = \int_{\mathbb{Z}} < \xi_1(m), \xi_2(m) > dm =< \xi_1, \xi_2 >$ because τ is ϕ_0-invariant. Here dm denotes Haar measure on $\mathbb{Z}$ and $<,>$ is the scalar product. The same argument applies to yield that v_0 is unitary.

We now remark that $(v_0 \circ u_0)(\xi)(m) = v_0(\lim_n[\phi_0(b_n)]) = \lim_n[\psi_0 \circ \phi_0(b_n)] = \lim_n[(\psi_0 \circ \phi_0)(b_n)] = \lim_n[\alpha(b_n)] = \lambda(\xi)(m)$, if $\xi(m) := \lim_n[b_n]$, and thus $v_0 \circ u_0 = \lambda$. We shall show bellow that $v_0 \in< C(\mathcal{P}_0), ua_{I^c} >$, hence $u_0 \in A \ltimes_\alpha \mathbb{Z}$.

Notice the following equalities:

$$v_0 \circ \overline{\pi_\tau}(a_{(k,i)}) = \begin{cases} \lambda \circ \overline{\pi_\tau}(a_{(k,i)}) & \text{if } 1 \leq i < J_k \\ \lambda^{1-J_k} \circ \overline{\pi_\tau}(a_{(k,J_k)}) & \text{if } i = J_k. \end{cases} \qquad (\aleph)$$

Let us check one case; the others can be proved similarly, as will be apparent from this case. Suppose $m = 0$. Then for $1 \leq i < J_k$, we have

$$\begin{aligned} v_0 \circ \overline{\pi_\tau}(a_{(k,i)})(\xi)(m) &= v_0(\lim_n[\alpha^{-m}(a_{(k,i)})x_n]) = \lim_n[\psi_0(\alpha^{-m}(a_{(k,i)})x_n)] \\ &= \lim_n[\psi_0(a_{(k,i)}x_n)] = \lim_n[\alpha(a_{(k,i)}x_n)] \end{aligned}$$

and $\lambda \circ \overline{\pi_\tau}(a_{(k,i)})(\xi)(m) = \lambda(\lim_n[a_{(k,i)}x_n]) = \lim_n[\alpha(a_{(k,i)}x_n)]$, where $\xi(m) := \lim_n[x_n]$. Also,

$$\begin{aligned} v_0 \circ \overline{\pi_\tau}(a_{(k,J_k)})(\xi)(m) &= v_0(\lim_n[a_{(k,J_k)}x_n]) \\ &= \lim_n[\psi_0(a_{(k,J_k)}x_n)] \\ &= \lim_n[\alpha^{1-J_k}(a_{(k,J_k)}x_n)] = \lambda^{1-J_k}(\xi)(m). \end{aligned}$$

Taking into account definition 3.3.8, we see, for example, $(\hbar)$ $\lambda\circ\overline{\pi_\tau}(a_{(k,i)}) = e^k_{i+1,i}\otimes a_{(k,J_k)} \in \langle C(\mathcal{P}_0), ua_{I^c}\rangle$, thus from $1 = \sum_{k=1}^m \sum_{i=1}^{J_k} a_{(k,i)}$, we have $v_0 \in \langle C(\mathcal{P}_0), ua_{I^c}\rangle$.

We shall now split the Hilbert space $L^2(\mathbb{Z}, H_\tau)$ as follows. Recall that $A/V_\tau = \sum_{k=1}^m \sum_{i=1}^{J_k} I_{Y(k,i)^c}/V_\tau = (\sum_{k=1}^m I_{Y(k,J_k)^c} \oplus \sum_{k=1}^m \sum_{i=1,i\neq J_k}^{J_k} I_{Y(k,i)^c})/V_\tau \simeq R_1/V_\tau \oplus R_2/V_\tau$ where we denoted for simplicity $R_1 := \sum_{k=1}^m I_{Y(k,J_k)^c}$ and $R_2 := \sum_{k=1}^m \sum_{i=1,i\neq J_k}^{J_k} I_{Y(k,i)^c}$. The completion of R_i/V_τ is denoted H_i^I, with $i \in \{1,2\}$ and it is easy to check that $H_\tau := H_1^I \oplus H_2^I$, hence $L^2(\mathbb{Z}, H_\tau) = L^2(\mathbb{Z}, H_1^I) \oplus L^2(\mathbb{Z}, H_2^I)$. From the definition of ψ_0, we see that $\psi_0 = \alpha$ off $\sum_{k=1}^m I_{Y_{(k,J_k)^c}}$, thus it follows from the definitions of ϕ_0 and ψ_0, that $v_0 = u$ on $L^2(\mathbb{Z}, H_2^I)$ and $u_0 = 1$ on $L^2(\mathbb{Z}, H_2^I)$.

For $Z := Y_{(1,J_1)}$, repeating the previous construction (defining a new map $\lambda' : Z \longrightarrow \mathbb{Z}$ etc) yields clopen subsets $Z_{(k,j)}$, m', $J_1', \ldots, J'_{m'}$ that satisfy the analogous relations. We ask that the partition $\mathcal{P}_1 := \{Z_{(k,j)}; 1 \le k \le m', 1 \le j \le J_k'\}$ be finer than $\mathcal{P}_0$.

We obtain as above a new finite dimensional C*-subalgebra $A_1 :=< C(\mathcal{P}_1), ua_{Z^c} >$ of $A \ltimes_\alpha \mathbb{Z}$. Since $I \supseteq Z = Y_{(1,J_1)}$, we see that $A_1 \supseteq A_0$. Analogously, we define automorphisms ψ_1 and ϕ_1, and unitary operators u_1 and v_1, which satisfy $v_1 \circ u_1 = \lambda$, $v_1 \in A_1$, $u_1 \in A \ltimes_\alpha \mathbb{Z}$, $v_1 = u$ on $L^2(\mathbb{Z}, H_2^Z)$, $u_1 = 1$ on $L^2(\mathbb{Z}, H_2^Z)$ $(\sharp)$. The last relation tells us that

$$u_1 \circ (e^1_{J_1,J_1} \otimes a_{(1,J_1)}) = (e^1_{J_1,J_1} \otimes a_{(1,J_1)}) \circ u_1 \qquad (\S).$$

We also have that $a_{(1,J_1)} = e^1_{J_1,J_1} \otimes a_{(1,J_1)}$. Set $B := \langle A_0, u_1\rangle$; we shall prove that $B \simeq M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_m}(\mathbb{C})$.

Define $q_1 := \sum_{j=1}^{J_1} e^1_{j,j} \otimes a_{(1,J_1)}$; so $q_1 = \sum_{j=1}^{J_1} u^{j-j}\alpha^{j-J_1}(a_{(1,J_1)}) = \sum_{j=1}^{J_1} \alpha^{j-J_1}(a_{1,J_1}) = \sum_{j=1}^{J_1} a_{(1,J_1)}$ from Lemma 3.3.10. Put $\hat{u} := \sum_{j=1}^{J_1} (e^1_{j,J_1} \otimes a_{(1,j)}) \circ u_1 \circ (e^1_{(J_1,j)} \otimes a_{(1,J_1)})$; here again we recall that $e^k_{(i,j)} \otimes a_{(k,i)} := \lambda^{i-j} \circ \overline{\pi_\tau}(\alpha^{j-J_k}(a_{(k,i)}))$. Now notice that $\hat{u} = \sum_{j=1}^{J_1} e^1_{j,J_1} \otimes a_{1,J_1} u_1 e^1_{J_1,j} \otimes a_{(1,J_1)} = \sum_{j=1}^{J_1} u^{j-J_1}\alpha^{J_1-J_1}(a_{1,J_1}) u_1 u^{J_1-j}\alpha^{j-J_1}(a_{1,J_1}) = \sum_{j=1}^{J_1} u^{j-J_1} a_{1,J_1} u_1 u^{J_1-j} \alpha^{j-J_1}(a_{1,J_1})$. From these two definitions and the properties of the projections $a_{(k,i)}$, $1 \le k \le m$ and $1 \le i \le J_k$, it follows easily that q_1 is the unit of $\langle \{e^1_{(i,j)} \otimes a_{(1,J_1)}; 1 \le i,j \le J_1\}, \hat{u}\rangle$ and that $\hat{u}$ is a unitary element in

$\langle\{e^1_{(i,j)} \otimes a_{(1,J_1)}; 1 \leq i,j \leq J_1\}, \hat{u}\rangle$. Indeed,

$$\begin{aligned}
q_1 e^1_{i_1,i_2} \otimes a_{(1,J_1)} &= \sum_{j=1}^{J_1} a_{(1,J_1)} u^{i_1-i_2} \alpha^{i_2-J_1}(a_{1,J_1}) \\
&= \sum_{j=1}^{J_1} a_{(1,J_1)} \alpha^{i_1-i_2}(\alpha^{i_2-J_1}(a_{(1,J_1)})) u^{i_1-i_2} \\
&= \sum_{j=1}^{J_1} a_{(1,J_1)} \alpha^{i_1-J_1}(a_{(1,J_1)}) u^{i_1-i_2} = a_{(1,J_1)} u^{J_1-i_2} = u^{J_1-i_2} \alpha^{-J_1+i_2}(a_{(1,J_1)}) \\
&= e^1_{i_1,i_2} \otimes a_{(1,J_1)},
\end{aligned}$$

where the last sum sign vanishes because we must have $J_1 = i_1$. The other relation to be checked is

$$\begin{aligned}
q_1 \hat{u} &= \sum_{i=1}^{J_1} a_{(1,i)} \sum_{j=1}^{J_1} u^{j-J_1} a_{1,J_1} u_1 u^{J_1-j} \alpha^{j-J_1}(a_{1,J_1}) \\
&= \sum_{i=1}^{J_1} a_{(1,i)} \sum_{j=1}^{J_1} a_{(1,j)} u^{j-J_1} u_1 u^{J_1-j} \alpha^{j-J_1}(a_{1,J_1}) \\
&= \sum_{j=1}^{J_1} a_{(1,j)} u^{j-J_1} u_1 u^{J_1-j} \alpha^{j-J_1}(a_{1,J_1}) = \sum_{j=1}^{J_1} u^{j-J_1} a_{1,J_1} u_1 u^{J_1-j} \alpha^{j-J_1}(a_{1,J_1}) = \hat{u},
\end{aligned}$$

where we have used the properties of the orthogonal projections $a_{(k,i)}$ to cancel out one of the sums. The verification of $\hat{u}\hat{u}^* = q_1$ is the following:

$$\begin{aligned}
\hat{u}\hat{u}^* &= \sum_{j=1}^{J_1}(e^1_{j,J_1} \otimes a_{(1,j)}) \circ u_1 \circ (e^1_{(J_1,j)} \otimes a_{(1,J_1)}) \sum_{i=1}^{J_1} \alpha^{i-J_1}(a_{(1,J_1)})u^{i-J_1}u_1^* a_{(1,J_1)}u^{J_1-i} \\
&= \sum_{j=1}^{J_1} u^{j-J_1}a_{(1,J_1)}u_1 u^{J_1-j}\alpha^{j-J_1}(a_{(1,J_1)})u^{j-J_1}u_1^* a_{(1,J_1)}u^{J_1-j} \\
&= \sum_{j=1}^{J_1} u^{j-J_1}a_{(1,J_1)}u_1 \alpha^{J_1-j}(\alpha^{j-J_1}(a_{(1,J_1)}))u^{J_1-j}u^{j-J_1}u_1^* a_{(1,J_1)}u^{J_1-j} \\
&= \sum_{j=1}^{J_1} u^{j-J_1}a_{(1,J_1)}u_1 a_{(1,J_1)}u_1^* a_{(1,J_1)}u^{J_1-j} = \sum_{j=1}^{J_1} u^{j-J_1}a_{(1,J_1)}a_{(1,J_1)}a_{(1,J_1)}u^{J_1-j} \\
&= \sum_{j=1}^{J_1} u^{j-J_1}a_{(1,J_1)}u^{J_1-j} = \sum_{j=1}^{J_1} a_{(1,j)}u^{j-J_1}u^{J_1-j} \\
&= \sum_{j=1}^{J_1} a_{(1,j)} = q_1
\end{aligned}$$

where we have used (§).

From the definitions of A_0 and of B, we see that $\hat{u} \in B$. In fact, $B = \langle A_0, \hat{u}\rangle$. This can be seen from the following chain of equalities:

$$\begin{aligned}
u_1 &= u_1 1 = u_1(a_{(1,J_1)} + 1 - a_{(1,J_1)}) = u_1 a_{(1,J_1)} + u_1(1 - a_{(1,J_1)}) \\
&= u_1 e^1_{J_1,J_1} \otimes a_{1,J_1} + u_1(1 - a_{(1,J_1)}) = (e^1_{J_1,J_1} \otimes a_{1,J_1})u_1(e^1_{J_1,J_1} \otimes a_{1,J_1}) + u_1(1 - a_{(1,J_1)}) \\
&= (e^1_{J_1,J_1} \otimes a_{1,J_1})u_1(e^1_{J_1,J_1} \otimes a_{1,J_1}) + 1 - a_{(1,J_1)} \qquad (\diamond)
\end{aligned}$$

by consecutive use of (§) and ($\sharp$). So, $u_1 = (e^1_{J_1,J_1} \otimes a_{1,J_1})u_1(e^1_{J_1,J_1} \otimes a_{1,J_1}) + 1 - e^1_{J_1,J_1} \otimes a_{1,J_1} = (e^1_{J_1,J_1} \otimes a_{1,J_1})\hat{u}(e^1_{J_1,J_1} \otimes a_{1,J_1}) + 1 - e^1_{J_1,J_1} \otimes a_{1,J_1}$, using the properties of the orthogonal projections $a_{(k,i)}$ and the definition of $\hat{u}$.

To verify that $B \simeq M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_m}(\mathbb{C})$, we shall use the following two lemmas.

Lemma 3.3.13. *If $1 \in D$ is a unital C^*-algebra, $1 \in E$ is a C^*-subalgebra of D (the same unit), and $(e_{i,j})$, $1 \leq i,j \leq n$ is a system of matrix units in D such that $ee_{i,j} = e_{i,j}e$ for all $e \in E$, for all i,j then $M_n(E) =< E, (e_{i,j}), 1 \leq i,j \leq n >$.*

Lemma 3.3.14. *If D is a C^*-algebra, E and F are $*$-invariant subsets of D such that $EF = FE = 0$ then $< E, F > = < E > + < F > \simeq < E > \oplus < F >$.*

Indeed, the two facts we need to prove in order to apply the lemmas above are:

1. $\hat{u}$ commutes with $e^1_{i,j} \otimes a_{(1,J_1)}$ for all $1 \leq i, j \leq J_1$.

2. $\sigma(\hat{u}) = \mathbb{T}$ in $< \{e^1_{(i,j)} \otimes a_{(1,J_1)}; 1 \leq i, j \leq J_1\}, \hat{u} >$.

The definition of $\hat{u}$ and the properties of orthogonal projections $a_{k,i}$, $1 \leq i \leq J_k$ implies 1.

In proving 2, we first need to show that $[u]_1 \neq 0$ in $K_1(A \ltimes_\alpha \mathbb{Z})$. This can be seen from [24], Lemma 2.3, where it was proved that $\delta([(1-F) + Fxu^*F)]_1) = [F \otimes e_{00}]_0$, where F is a projection in some $M_k(A)$, $x \in M_k(A)$ and $\{e_{ij}, 0 \leq i, j\}$ is a system of matrix units for the algebra of compact operators (see the end of section 2.2). If $[u]_1 = 0$, it follows that $[1 \otimes e_{00}]_0 = 0$ in $K_0(A \otimes \mathbb{K})$, hence $1 \otimes e_{00} = 0$, a contradiction. Now, from $u = u_1 v_1$, we have that $[u]_1 = [u_1]_1 + [v_1]_1$ in $K_1(A \ltimes_\alpha \mathbb{Z})$. Since $v_1 \in A_1$ and A_1 is AF, it follows that $[v_1]_1 = 0$, so we conclude that $[u_1]_1 \neq 0$; by [31], $\sigma(u_1) = \mathbb{T}$. Since $\hat{u}$ is a unitary in the unital C*-algebra $< q_1, \hat{u} >$, by spectral theory, $\sigma(\hat{u}) \subseteq \mathbb{T} = \sigma(u_1)$, and by compactness of the spectrum, it is sufficient to prove that $\sigma(u_1) \subseteq \sigma(\hat{u}) \cup \{1\}$. Select λ outside $\sigma(\hat{u}) \cup \{1\}$ and let $c \in < q_1, \hat{u} >$ be the inverse of $\hat{u} - \lambda q_1$, i.e. $q_1 = c(\hat{u} - \lambda q_1) = (\hat{u} - \lambda q_1)c$. Define $b := (e^1_{J_1,J_1} \otimes a_{(1,J_1)})c(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + (1/1-\lambda)[1 - (e^1_{J_1,J_1} \otimes a_{(1,J_1)})]$. From ($\diamond$) (see the discussion before Lemma 3.3.13)

$$
\begin{aligned}
u_1 - \lambda 1 &= (e^1_{J_1,J_1} \otimes a_{(1,J_1)})\hat{u}(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + 1 - e^1_{J_1,J_1} \otimes a_{(1,J_1)} - \lambda 1 \\
&= (e^1_{J_1,J_1} \otimes a_{(1,J_1)})(\hat{u} - \lambda q_1)(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + \\
&\lambda(e^1_{J_1,J_1} \otimes a_{(1,J_1)})q_1(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + 1 - (e^1_{J_1,J_1} \otimes a_{(1,J_1)}) - \lambda 1 \\
&= (e^1_{J_1,J_1} \otimes a_{(1,J_1)})(\hat{u} - \lambda q_1)(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + \lambda(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) \\
&+ 1 - (e^1_{J_1,J_1} \otimes a_{(1,J_1)}) - \lambda 1 \\
&= (e^1_{J_1,J_1} \otimes a_{(1,J_1)})(\hat{u} - \lambda q_1)(e^1_{J_1,J_1} \otimes a_{(1,J_1)}) + (1-\lambda)[1 - (e^1_{J_1,J_1} \otimes a_{(1,J_1)})].
\end{aligned}
$$

Since $e^1_{J_1,J_1} \otimes a_{(1,J_1)}$ commutes with $\hat{u}$ and q_1, one can see easily that b is an inverse of $u_1 - \lambda 1$, thus λ is not in $\sigma(u_1)$.

For any partition $\mathcal{R}$ of $\delta_e(T(A))$, we denote by $\mathcal{C}(\mathcal{R})$ the following C*-algebra. For any $X \in \mathcal{R}$ consider the characteristic function $\chi_X : \delta_e(T(A)) \longrightarrow \{0,1\}$, take its positive affine extension on $T(A)$ and let $a_X \in A$ be its positive lift. We just put $\mathcal{C}(\mathcal{R}) :=< a_X; X \in \mathcal{R} >$. Recall that we have that $A_0 \subseteq B$. The idea for the end of the proof is to twist B by a unitary z (to obtain an approximating unitary as in the statement). This unitary z will be chosen to commute with $\mathcal{C}(\mathcal{P})$ and u_1, and so that zv_0z^* approximates v_1. It will follow that $\mathcal{C}(\mathcal{P}) = z\mathcal{C}(\mathcal{P})z^* \subseteq zBz^*$ and that $zv_0u_1z^* = zv_0z^*u_1$ approximates $v_1u_1 = u$, zBz^* is the sought-for C*-subalgebra.

We know that $v_0 \in< C(\mathcal{P}), ua_{I^c} >$, which is a finite dimensional C*-algebra. The same is true for v_1, it lies in a finite dimensional C*-algebra. Hence we can assume that the spectrum of $v_1v_0^*$ is a finite subset of $\mathbb{T}$. Thus we can find a strictly positive θ such that $(\pi+\theta)/N < \epsilon$ and $-e^{i\theta}$ is not in $\sigma(v_1v_0^*)$. If $g(z) := e^{(1/N)Log_\theta(z)} = e^{(1/N)iArg_\theta(z)}$ for $z \in \sigma(v_1v_0^*)$ then by spectral theory we can define $w := g(v_1v_0^*)$ and observe that $w^N = v_1v_0^*$ and $||w-1|| = ||g-1|| < (\pi+\theta)/N < \epsilon$.

If $I' := I \circ \alpha^{-1}$, notice that on $L^2(\mathbb{Z}, H_2^{I'})$ one has that $v_1v_0^* = v_1v_0^{-1} = uu^{-1} = 1$ because $v_1 = u$ on $L^2(\mathbb{Z}, H_2^{Z})$ and $v_0 = u$ on $L^2(Z, H_2^{I})$. From this, $v_1v_0^*$ is a unitary when restricted to $L^2(\mathbb{Z}, H_1^{I'})$. Since every polynomial in $v_1v_0^*$ and $v_0v_1^*$ leaves $L^2(\mathbb{Z}, H_2^{I'})$ and $L^2(\mathbb{Z}, H_1^{I'})$ invariant, as above, by spectral theory, w will satisfy the same properties. Now the fact that w is an unitary will imply that $w(L^2(\mathbb{Z}, H_1^{I'})) = L^2(\mathbb{Z}, H_1^{I'})$. The next step is to define z by the following formula

$$z := \begin{cases} u^{j-1}w^{N-j+1}u^{-j+1} & \text{on } L_j^2 \\ 1 & \text{on } L_c^2. \end{cases}$$

where $L_j^2 := L^2(\mathbb{Z}, H_1^{I\circ\alpha^{-j}})$, $1 \le j \le N$ and $L_c^2 := L^2(\mathbb{Z}, H_2^{I"})$ where $I" := \{I \circ \alpha^{-1}\} \cup \cdots \cup \{I \circ \alpha^{-N}\}$. From the definition of λ, $\lambda(L_j^2) = L_{j+1}^2$, hence z is surjective. Since z is clearly an isometry, we have that z is a unitary, and because $v_1v_0^* \in A_1$, by spectral theory, w remains inside A_1. As in the commutative case, the following formula is easy to prove

$$z = 1 - \sum_{s=1}^{N} \alpha^s(\sum a_{(k,J_k)}) +$$

$$\sum_{j=1}^{N}[\alpha^j(\sum_{k=1}^{m} a_{(k,J_k)})](u\sum_{k=1}^{m}\sum_{j=1}^{J_k-1} a_{(k,j)})^{j-1}w^{N-j+1}[(u\sum_{k=1}^{m}\sum_{j=1}^{J_k-1} a_{(k,j)})^*]^{j-1}[\alpha^j(\sum_{k=1}^{m} a_{(k,J_k)})].$$

Thus we obtain that $z \in A_1$.

We need the following two lemmas.

Lemma 3.3.15. *z commutes with $C(\mathcal{P})$.*

Proof. If $\mathcal{P} = \{X_1, \dots, X_r\}$, it is sufficient to show that $za_{X_i} = a_{X_i}z$ for all $i \in \{1, \dots, r\}$. First notice that $z = 1$ on L_c^2 and $z(L_j^2) = L_j^2$ for all $1 \le j \le N$. Indeed, the first fact is just from the definition of z and the second fact is proved as follows. Iterating $\lambda(L_j^2) = L_{j+1}^2$ we have that $\lambda^{j-1}w^{N-j+1}\lambda^{-j+1}(L_j^2) = \lambda^{j-1}w^{N-j+1}(L_1^2)$. Since $L_1^2 = L^2(H_1^{I\circ\alpha^{-1}})$ and $I \circ \alpha^{-1} = I'$ we get that $\lambda^{j-1}w^{N-j+1}\lambda^{-j+1}(L_j^2) = \lambda^{j-1}w^{N-j+1}(L^2(Z, H_1^{I'})) = \lambda^{j-1}(L^2(Z, H_1^{I'})$ because $w(L^2(\mathbb{Z}, H_1^{I'})) = L^2(\mathbb{Z}, H_1^{I'})$. Hence $\lambda^{j-1}w^{N-j+1}\lambda^{-j+1}(L_j^2) = \lambda^{j-1}(L^2(Z, H_1^{I'}) = L_j^2$ by iterating again $\lambda(L_j^2) = L_{j+1}^2$. Since each of the sets $I \circ \alpha^{-j}$ lies in an element of $\mathcal{P}$ we get from the above relations that $z(L^2(\mathbb{Z}, H_1^{X_i})) = L^2(\mathbb{Z}, H_1^{X_i})$ and $z(L^2(\mathbb{Z}, H_2^{X_i})) = L^2(\mathbb{Z}, H_2^{X_i})$, from where it follows that, in this order, $a_{X_i}za_{X_i} = za_{X_i}$ and $a_{X_i}z = a_{X_i}za_{X_i}$, which completes the proof of this lemma. □

Lemma 3.3.16. *zBz^* contains a unitary that approximates u.*

Proof. (a) Notice two facts: v_0 takes $L^2(\mathbb{Z}, H_1^I)$ onto $L^2(\mathbb{Z}, H_1^{I\circ\alpha^{-1}})$, and $z = 1$ on $L^2(\mathbb{Z}, H_1^I)$. Indeed, the verifications for the first are as follows. If $m := \lim_s[a_s]$ then $(v_0\xi)(m) = \overline{\xi}(m) = \lim_s[\psi_0(a_s)] = \lim_s[\alpha^{1-J_k}(b_s^k)]$, where $a_s = \sum_{k=1}^{J_k} b_s^k$ with $b_s^k \in I_{Y(k,J_k)^c}$. Using 3.3.1 and 3.3.10, we get that $v_0(L^2(\mathbb{Z}, H_1^I)) \subseteq L^2(\mathbb{Z}, H_1^{I\circ\alpha^{-1}})$. The other inclusion is proved in a similar way. The second fact follows from the definition of z.

From the definition of z (when $j = 1$) we see that on $L^2(\mathbb{Z}, H_1^I)$, $z = w^N = v_1v_0^*$; thus on $L^2(\mathbb{Z}, H_1^I)$ we get that (from the two facts above) $zv_0 - v_1z = v_1v_0^*v_0 - v_1 = 0$.

(b) Using the definition of v_0 and of ψ_0 (first branch) we see that if $1 \le j \le N$, then v_0 and u agree on $L_j^2 = L^2(\mathbb{Z}, H_1^{I\circ\alpha^{-j}})$. The same arguments yields that v_1 and u agree on L_j^2. It follows from the definition of z and $\lambda(L_j^2) = L_{j+1}^2$ that on L_j^2

we have $zv_0 - v_1z = u^j w^{N-j} u^{-j} u - uu^{j-1} w^{N-j+1} u^{-j+1}$. Hence $||(zv_0 - v_1z)|_{L_j^2}|| = ||(1-w)_{L_j^2}|| \leq ||1-w|| < \epsilon$.

(c) As in (b) we notice that v_0, v_1 and u agree on $L^2(\mathbb{Z}, H_2^{\hat{I}})$ where $\hat{I} := I \cup \{I \circ \alpha^{-1}\} \cup \cdots \cup \{I \circ \alpha^{-N}\}$. Since we have chosen the subsets $I, I \circ \alpha^{-1}, \ldots, I \circ \alpha^{-N}$ mutually disjoint we have from the definition of z that $z = 1$ on $L^2(\mathbb{Z}, H_2^{\hat{I}})$. It follows that $zv_0 - v_1z = 0$ on $L^2(\mathbb{Z}, H_2^{\hat{I}})$ because also $z = 1$ on L_c^2 (the very definition of z).

Since $I, I \circ \alpha^{-1}, \ldots, I \circ \alpha^{-N}$ are mutually disjoint we get from (a), (b) and (c) that $||zv_0 - v_1z|| < \epsilon$.

The approximating unitary is $u' := zv_0z^*u_1$. From the definition of z we have that $\begin{cases} z = 1 \text{ on } L^2(\mathbb{Z}, H_1^I) \\ z(L^2(\mathbb{Z}, H_2^I)) = L^2(\mathbb{Z}, H_2^I). \end{cases}$

From ($\sharp$) and $I \supseteq Z$ we get that $u_1 = 1$ on $L^2(\mathbb{Z}, H_2^I)$ and $u_1(L^2(\mathbb{Z}, H_1^I)) = L^2(\mathbb{Z}, H_1^I)$. Hence z commutes with u_1. This relation shows that $u' \in zBz^*$. Indeed, $u' = zv_0z^*u_1 = zv_0u_1z^* \in zBz^*$. Moreover, $||u - u'|| = ||v_1u_1 - zv_0z^*u_1|| = ||v_1zu_1z^* - zv_0z^*u_1|| = ||v_1z - zv_0|| < \epsilon$ since the unitaries z^* and u_1 commutes. $\square$

Even if $K_0(A)$ is torsion-free, the groups $K_i(A \ltimes_\alpha \mathbb{Z})$, $i \in \{0, 1\}$ can have torsion, thus the torsion condition in the result(s) below is necessary:

Theorem 3.3.17. *Let A be a unital AF algebra with compact, zero dimensional boundary space that is an infinite set, such that $\{0\} = A^+ \cap \{\cap_{\tau \in \delta_e T(A)} Ker(\tau)\}$ and the natural map$\hat{}$: $A_{sa} \longrightarrow Aff(T(A))$ lifts positives to positives. Let $\alpha \in Aut(A)$ be an automorphism such that $K_*(A \ltimes_\alpha \mathbb{Z})$ is torsion free, $\Gamma(\alpha) = \mathbb{T}$ and the action giving the crossed product is $\mathbb{Z}$-simple. Let u be the canonical unitary implementing α. Then for any partition $\mathcal{P}$ of $\delta_e T(A)$ and any $\epsilon > 0$ there exists a C^*-subalgebra D of $A \ltimes_\alpha \mathbb{Z}$ such that*

(i) $D \simeq M_{J_1}(C(\mathrm{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_m}(\mathbb{C})$ for a suitable choice of integers $J_1, J_2, \ldots, J_m$,

(ii) $C(\mathcal{P}) \subseteq D$,

(iii) there exists $u' \in \mathcal{U}(D)$ such that $||u - u'|| < \epsilon$.

Proof. If one considers $D := zBz^*$ (where z and B were defined in the paragraphs preceding this theorem) then (iii) is just Lemma 3.3.16. To see that (ii) holds, note that Lemma 3.3.15 and $C(\mathcal{P}) \subseteq B$ implies that $C(\mathcal{P}) = zC(\mathcal{P})z^* \subseteq zBz^* = D$. Lemmas 3.3.13 and 3.3.14 and the fact that $B \simeq M_{J_1}(C(\mathrm{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_m}(\mathbb{C})$ give us (i). $\square$

This result was known ([28]) when A was $C(X)$, X the Cantor set and α arose from a minimal homeomorphism of X. The proof above mimics the original proof of Putnam. Examples of algebras satisfying the above requirements are $C(X) \otimes B$, where X is the Cantor set, B is a simple unital AF algebra with unique tracial state, $\alpha_\varphi \otimes \beta$, where φ is a minimal homeomorphism of X and β is an approximately inner automorphism of B.

As in [28] one gets:

Corollary 3.3.18. *$A \rtimes_\alpha \mathbb{Z}$ is an AT algebra if A and α are as in the above theorem.*

3.4 Minimal dynamical systems tensored with tracial rank 0 or 1 C*-algebras

In this section, we consider crossed products by $\mathbb{Z}$ of C*-algebras of the form $C(X) \otimes B$ (and thus nonsimple) and automorphisms of the form $\alpha_\varphi \otimes \beta$, where B is a simple tracial rank zero or one C*-algebra, β an approximately inner automorphism of B, X the Cantor set and φ a minimal self homeomorphism. We prove that the crossed product is tracial rank zero or one. First we shall deal with the case of tracial rank 0 C*-algebras and at the end of the section we shall indicate how the proofs can be changed in order to get the corresponding result for tracial rank one C*-algebras.

Results on the tracial rank of crossed products of certain simple C*-algebras can be found in [42] ($A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero if A is unital, simple, has tracial rank zero and α has a certain Rokhlin property), [43] ($C(X) \rtimes_T \mathbb{Z}$ has tracial rank zero when the image of $K_0(A)$ in $\mathrm{Aff}(T(A))$ is dense), where X is an infinite compact metric space with finite covering dimension and T is a minimal homeomorphism of X). The class of non-simple C*-algebras considered in this work has its roots in [39] and [38], where H. Lin and H. Matui have studied crossed products of $C(X \times \mathrm{T})$, (X is the Cantor set and T is the circle).

Let us recall some definitions needed in this section. An ordered group (G, G^+) is called weakly unperforated if $nx > 0$ for some $n > 0$ implies that $x > 0$.

A C*-algebra A is said to satisfy Blackadar's second fundamental comparability question if the order on projections of $M_\infty(A)$ is determined by traces, i.e., if $p, q \in M_\infty(A)$ are projections and $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then p is Murray-von Neumann equivalent to a subprojection of q.

To define the tracial topological rank ([37]) we need some preliminary constructions. For $0 < \sigma_1 < \sigma_2 \leq 1$, define

$$f_{\sigma_1}^{\sigma_2}(t) := \begin{cases} 1 & \text{if } t \geq \sigma_2, \\ \text{linear} & \text{if } \sigma_1 \leq t \leq \sigma_2, \\ 0 & \text{if } 0 < t \leq \sigma_1 \end{cases}$$

and remark that continuity of $f_{\sigma_1}^{\sigma_2}$ is needed for continuous functional calculus in the definitions below. Let us denote by I^0 the class of all finite dimensional C*-algebras and by I^k the class of all unital C*-algebras which are unital hereditary C*-subalgebras of C*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and $F \in I^0$. A C*-subalgebra B of a C*-algebra A is said to be *hereditary* if for any $a \in A^+$ and $b \in B^+$ the inequality $a \leq b$ implies that $a \in B$. By $\mathrm{Her}(b)$, we mean the smallest hereditary C*-subalgebra of A containing b, that is, the closure of bAb.

For two positive elements $a, b \in A$ we write $[a] \leq [b]$ (see [37] or [41], page 201) if there is a partial isometry $w \in A^{**}$ such that, for each $x \in \mathrm{Her}(a)$, $w^*x, xw \in A$, $ww^* = P_{[a]}$, where $P_{[a]}$ is the open projection in A^{**} corresponding to $\mathrm{Her}(a)$, and $w^*xw \in \mathrm{Her}(b)$. For a positive integer n we write $n[a] \leq [b]$ if there are mutually orthogonal positive elements $b_1, \ldots, b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_j]$ for all $1 \leq j \leq n$.

A unital C*-algebra A is said to have *tracial topological rank* (see [37]) no more than k if for every $\epsilon > 0$, and every finite subset F containing a nonzero positive element b, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, and every integer $n > 0$, there exist a nonzero projection $p \in A$ and a C*-subalgebra $B \in I^k$ with $1_B = p$ such that

(1) $||[x, p]|| < \epsilon$ for all $x \in F$,

(2) $pxp \in_\epsilon B$ for all $x \in F$,

(3) $n[1-p] \leq [p]$ and $n[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

Condition (2) means that for $x \in F$ there is a $y \in B$ such that $||pxp - y|| < \epsilon$.

If A has tracial topological rank no more than k we write $TR(A) \leq k$, and if $TR(A) \not\leq k-1$ we say $TR(A) = k$. If A is a non unital C*-algebra, we say $TR(A) = k$ if $TR(\tilde{A}) = k$.

For simple C*-algebras, this definition simplifies as follows (see [37]). A unital simple C*-algebra A has tracial topological rank no more than k if for every $\epsilon > 0$, every finite subset F and every nonzero element $a \in A^+$, there is a nonzero projection $p \in A$ and a C*-subalgebra $B \in I^k$ with $1_B = p$ such that

(1) $||px - xp|| < \epsilon$ for all $x \in F$,

(2) $pxp \in_\epsilon B$ for all $x \in F$,

(3) $[1-p] \leq [a]$, or in other words, $1-p$ is equivalent to a projection in $\mathrm{Her}(a)$.

Examples of C*-algebras having topological tracial rank zero include $(\otimes_{\mathbb{Z}} A \ltimes_\sigma \mathbb{Z})$ where A is a unital simple AF algebra with a unique tracial state and σ is the shift (see [18]); all simple AH algebras with slow dimension growth and with real rank zero (see [12]).

Examples of C*-algebras having topological tracial rank one are include $(A \otimes B) \ltimes_\gamma \mathbb{Z}$, where A is the unital AF-algebra with $K_0 = \mathbb{Q}$, B is a unital simple AT-algebra with $K_0(B) = \mathbb{Q}$, $K_1(B) = \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{Aff}((T(B)) = C_{\mathbb{R}}([0,1])$; $\alpha \in \mathrm{Aut}(A)$ such that α^m is uniformly outer for each $m \neq 0$ and $\beta \in \mathrm{Aut}(B)$ is such that $K_0(\beta) = \mathrm{id}_{K_0(B)}$, $K_1(\beta)(a,b) = (-a,b)$ for all $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ and $\tau \circ \beta(b) = \tau(b)$ for all $\tau \in T(B)$ and all $b \in B$; and $\gamma := \alpha \otimes \beta$ (see [45]).

Lemma 3.4.1. *Any inductive limit of tracial rank 0 C*-algebras is a tracial rank 0 C*-algebra.*

Proof. Without loss of generality, we may assume that $A := \overline{\cup_n A_n}$ and $A_n \subseteq A_{n+1}$. Let $\epsilon > 0$, $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, $F \in A$, a finite subset containing a nonzero positive element b. We can assume $F \subseteq A_m$ for some $m \in \mathbb{N}$. Since $\mathrm{TR}(A_m) = 0$ we get that there is a non zero projection $p \in A_m \subseteq A$ and a C*-subalgebra $B \in I^0$, of $A_m \subseteq A$, with $1_B = p$ such that

1) $||xp - px|| < \epsilon$ for all $x \in F$,

2) $pxp \in_\epsilon B$ for all $x \in F$ and

3) $[f_{\sigma_2}^{\sigma_1}((1-p)b(1-p))] \leqslant [f_{\sigma_3}^{\sigma_4}(pbp)]$.

It follows from the tracial rank 0 definition ([37]) that A has tracial rank 0. $\square$

Let X be the Cantor set and $\varphi : X \longrightarrow X$ be a minimal homeomorphism. One associates to these data an automorphism of $C(X)$ given by $\alpha_\varphi : C(X) \longrightarrow C(X)$, $\alpha_\varphi(f) := f \circ \varphi^{-1}$, for all $f \in C(X)$. Let B be a unital, simple, separable, tracial rank 0 C*-algebra and $\beta \in \mathrm{Aut}_0 B$ be an approximately inner automorphism of B. Then one can form $C(X) \otimes B$ and consider the natural automorphism $\alpha_\varphi \otimes \beta \in \mathrm{Aut}(C(X) \otimes B)$. Denote by C the crossed product C*-algebra $C(X) \otimes B \rtimes_{\alpha_\varphi \otimes \beta} \mathbb{Z}$. Denote by u the canonical implementing unitary. We regard $C(X) \otimes B$ as a C*-subalgebra of C. For later use, we shall denote M_φ the set of φ -invariant measures on X, and recall that if φ is free, then there is a canonical bijection between M_φ and the tracial state space of $C(X) \rtimes_{\alpha_\varphi} \mathbb{Z}$.

Definition 3.4.2. *Fix an element $x_0 \in X$ and denote by A_{x_0} the C*-subalgebra of C which is generated by $C(X) \otimes B$ and $u(C(X \backslash \{x_0\}) \otimes B)$ where $C(X \backslash \{x_0\})$ is the C*-subalgebra of $C(X)$ consisting of all functions vanishing at x_0. It is known ([47]) that $\widetilde{A_{x_0}} := A_{x_0} \cap (C(X) \rtimes_{\alpha_\varphi} \mathbb{Z})$ is a unital AF algebra. Let us say that $A_{x_0} \cap (C(X) \rtimes_{\alpha_\varphi} \mathbb{Z}) = \lim_n D_n$ where $(D_n)_{n \geqslant 1}$ is an ascending sequence of finite dimensional C*-subalgebras. Then, regarding $C(X)$ and the implementing unitary of α_φ in the crossed product, $A_{x_0} = \lim_n (D_n \otimes B)$.*

Lemma 3.4.3. *A_{x_0} has tracial rank 0.*

Proof. Since D_n finite dimensional, we know that $\mathrm{TR}(D_n) = 0$. Since $\mathrm{TR}(D_n \otimes B) \leqslant \mathrm{TR}(D_n) + \mathrm{TR}(B)$ (see [36]), we get that $\mathrm{TR}(D_n \otimes B) = 0$. The result follows from Lemma 3.4.1. $\square$

Lemma 3.4.4. *A_{x_0} is simple.*

Proof. Let I be a nonzero ideal in A_{x_0}. Then $I \cap \widetilde{A_{x_0}}$ is an ideal in $\widetilde{A_{x_0}}$, which is simple (Lemma 2.5. of [40]). So, either $I \cap \widetilde{A_{x_0}} = 0$ or $I \cap \widetilde{A_{x_0}} = \widetilde{A_{x_0}}$. As in Lemma 2.5. of [40] we notice first that $I \cap (C(X) \otimes B) \neq 0$. Letting $\{x_0\} = \cap_n Y_n$, with

$Y_n \supseteq Y_{n+1}$, Y_n clopen for each n, we get that $\widetilde{A_{x0}} = \lim_n A_{Y_n}$, where A_{Y_n} is the C*-subalgebra of $C(X) \ltimes_{\alpha_\varphi} \mathbb{Z}$ generated by $C(X)$ and $\overline{u}C_0(X \backslash \{Y_n\})$, where $\overline{u}$ is the canonical unitary implementing α_φ. Since $I \neq 0$ there is an $m \in \mathbb{N}$ and nonzero element $a \in I \cap < C(X) \otimes B, u(C_0(X \backslash Y_m) \otimes B) >$.

Let us say that $a = \sum_{n=-N}^{n} (h_n \otimes a_n) u^n$, with $h_n \in C(X)$ and $a_n \in B$ for $-N \leqslant n \leqslant N$. Then $a^* a = \sum_{n=-N}^{n} (f_n \otimes b_n) u^n$ and $f_0 \otimes b_0 \neq 0$. One can choose an $y \in X$ such that $f_0(y) \neq 0$. Since φ is minimal there is a neighborhood V of y such that the subsets $\varphi^n(V)$ are disjoint $(-N \leqslant n \leqslant N)$ and choose a $g \in C(X)$ satisfying $\mathrm{supp}(g) \subseteq V$ and $g(y) \neq 0$. Then $g \otimes 1 \in C(X) \otimes B$ and the following equalities hold $(g \otimes 1) a^* a (g \otimes 1) = \sum_{n=-N}^{N} (g \otimes 1)(f_n \otimes b_n) u^n (g \otimes 1) = \sum_{n=-N}^{N} (g \otimes 1)(f_n \otimes b_n)(\alpha_\varphi \otimes \beta)^n (g \otimes 1) u^n = \sum_{n=-N}^{N} (g \otimes 1)(f_n \otimes b_n)((g \circ \varphi^{-n}) \otimes 1) u^n = \sum_{n=-N}^{N} ((g f_n (g \circ \varphi^{-n})) \otimes b_n) u^n = (g^2 \otimes 1)(f_0 \otimes b_0) \neq 0$. Since $(g^2 \otimes 1)(f_0 \otimes b_0) \in I \cap (C(X) \otimes B)$ we get that $I \cap (C(X) \otimes B) \neq 0$.

Case 1. If $I \cap \widetilde{A_{x0}} = \widetilde{A_{x0}}$ it follows that $\widetilde{A_{x0}} \subseteq I$, which implies that $I \cap (C(X) \otimes B) = C(X) \otimes J$, where J is an ideal of B. Since β is approximately inner, we get that $C(X) \otimes J$ is an $\alpha_\varphi \otimes \beta$ -invariant ideal. From the minimality of $\alpha_\varphi \otimes \beta$ (see Lemma 3.4.7), we get that $C(X) \otimes J$ is either 0 or $C(X) \otimes B$. We saw above in the proof that the first case is not possible, the second one will imply that $1 \in C(X) \otimes B \subseteq I$, which implies $I = A_{x0}$.

A shorter way to prove this case is via the observation that B unital implies that $\widetilde{A_{x0}}$ is unital and contained in I, so $1 \in I$.

Case 2. If $I \cap \widetilde{A_{x0}} = 0$ we have that $I \cap (C(X) \otimes B) = 0$, which is impossible again by the discussion above. □

Lemma 3.4.5. *There is a canonical bijection between $T(A_{x0})$ and the set of $\alpha_\varphi \otimes \beta$-invariant tracial states of $C(X) \otimes B$.*

Proof. Since $T((C(X) \otimes B) \ltimes_{\alpha_\varphi \otimes \beta} \mathbb{Z})$ is in bijective correspondence with $\alpha_\varphi \otimes \beta$-invariant tracial states of $C(X) \otimes B$, it is sufficient to show that every $\tau \in T(A_{x0})$ extends to a tracial state on $(C(X) \otimes B) \ltimes_{\alpha_\varphi \otimes \beta} \mathbb{Z}$. Let $f \otimes b \in C(X) \otimes B$ with $0 \leqslant f \leqslant 1$, $0 \leqslant b \in B$, and let $n \in \mathbb{N}$ be chosen arbitrarily. Since φ is minimal there is a clopen neighborhood U of x such that the subsets $U, \varphi^{-1}(U), \ldots, \varphi^{-n}(U)$ are mutually disjoint. Set $p := \chi_U \otimes 1 \in C(X) \otimes B$. Then the projections $p, u^*pu, \dots, (u^n)^*pu^n$ are mutually equivalent and disjoint (by the choice of U) in A_{x_0}. So, $1 \geqslant \tau(\sum_{k=0}^{n}(u^k)^*pu^k)$ and it follows that $\tau(p) \leqslant 1/(n+1) < 1/n$. Since $p(1-(f\otimes b)) \geqslant 0$ we see that $\tau(f\otimes b) \leqslant \tau(p)+\tau((1-p)(f\otimes b)) = \tau(p)+\tau(u(1-p)(f\otimes b)u^*) < 1/n+\tau(u(f\otimes b)u^*)$ because $(1-p)(f\otimes b) \leqslant f\otimes b$. In a similar fashion one can obtain $\tau(u(f\otimes b)u^*) < 1/n+\tau(f\otimes b)$. Because n was chosen arbitrarily, we get that $\tau(f\otimes b) = \tau(u(f\otimes b)u^*)$, and the last says that τ extends to a tracial state on the crossed product algebra. □

Remark 3.4.6. *If $\mu \in M_\varphi$ and $\tau \in T(B)$ then $\tau_\mu \otimes \tau$ is $\alpha_\varphi \otimes \beta$-invariant, as β is approximately inner. Conversely, if $\gamma\otimes\tau$ is $\alpha_\varphi\otimes\beta$-invariant, then γ is α_φ invariant. Indeed, letting $f \in C(X)$, we obtain from $\gamma\otimes\tau = (\gamma\otimes\tau)\circ(\alpha_\varphi\otimes\beta)$ evaluated at $f\otimes 1$ that $\gamma(f) = \gamma(f\circ\varphi^{-1})$.*

Lemma 3.4.7. *The crossed product algebra $(C(X)\otimes B) \ltimes_{\alpha_\varphi\otimes\beta} \mathbb{Z}$ is simple if φ is minimal, β is approximately inner and B is simple.*

Proof. We first show that $C(X)\otimes B$ is $\mathbb{Z}$ -simple. Suppose there is a proper ideal I of $C(X)\otimes B$ such that $\alpha_\varphi^m \otimes \beta^m(I) = I$ for all $m \in \mathbb{Z}\backslash\{0\}$. Since I can be written as $I = C_0(U)\otimes B$, for some nonvoid closed subset U of X, we get that $\varphi^{-m}(U) = U$ for all $m \in \mathbb{Z}\backslash\{0\}$. Minimality of φ implies that $U = X$, so I would not be proper as we have supposed. Thus the action is $\mathbb{Z}$ -simple.

To finish the proof it is sufficient (see [44], Theorem 7.2) to show that for all nonzero integers m, the automorphism $\alpha_\varphi^m \otimes \beta^m$ is properly outer. If this were not the case, it would follow (see [44]) that there is a nonzero $\alpha_\varphi^m \otimes \beta^m$-invariant ideal I of $C(X)\otimes B$ such that $\alpha_\varphi^m \otimes \beta^m|_I = \mathrm{Ad}u \circ \exp\delta$ for some unitary $u \in M(I)$ and some *-derivation δ of I. From Corollary 8.7.8. of [46], one sees that $\alpha_\varphi^m \otimes \beta^m|_I$ is approximately inner.

Since B is simple, the ideal I has the form $C_0(U)\otimes B$ for some non-void closed subset U of X. Let $x_0 \in U$. Since $\alpha_\varphi^m \otimes \beta^m|_I$ is approximately inner it will leave invariant any ideal, in particular (for $C_0(\{x_0\})\otimes B$) we get that $\varphi^{-m}(x_0) = x_0$, a contradiction because φ is assumed to be minimal.

□

Proposition 3.4.8. *If φ is minimal, β is approximately inner, and B is simple, then $(C(X) \otimes B) \rtimes_{\alpha_\varphi \otimes \beta} \mathbb{Z}$ is almost tracially AF.*

Proof. Pick an $x_0 \in X$. Then A_{x_0} is a unital, simple C*-algebra with tracial rank 0 (lemmas 3.4.3, 3.4.4). Take a finite subset $F \subset C$ and take $\epsilon > 0$. Because the set of all elements of the form $\sum_{k=-n}^{n} c_k u^k$, $n \in \mathbb{N}$, $c_k \in C(X) \otimes B$ is dense in C, we may assume there is a $n \in \mathbb{N}$ such that $F \subseteq E_n$, where $E_n := \{\sum_{k=-n}^{n} c_k u^k; c_k \in C(X) \otimes B\}$. Since φ is minimal, there is a clopen neighborhood U of x_0 such that $\varphi^{1-n}(U), \varphi^{2-n}(U), \ldots, U, \varphi(U), \ldots, \varphi^n(U)$ are mutually disjoint and $\mu(U) < \epsilon/2n$ for all $\mu \in M_\varphi$. Setting $V := \cup_{k=1-n}^{n} \varphi^k(U)$, we have that $p := \chi_{V^c} \otimes 1$ is a projection in $C(X) \otimes B$. Because of the choice of U, we get that $u^k p, u^{1-k} p \in A_{x_0}$ for $0 \leqslant k \leqslant n$. From here and the definition of E_n, we obtain that $ap \in A_{x_0}$ for all $a \in E_n$. Using Lemmas 3.4.5 and 3.4.6, we get that $\tau(1-p) = \tau(\chi_V \otimes 1) < \epsilon$ for all $\tau \in T(A_{x_0})$. $\square$

Remark 3.4.9. *Using Theorem 3.10. of [38] (which says that simple, almost tracially AF and unital C*-algebras have real rank 0) one obtains that $(C(X) \otimes B) \rtimes_{\alpha_\varphi \otimes \beta} \mathbb{Z}$ has real rank 0. As usual, we regard $C(X)$ and B as C*-subalgebras of $C(X) \otimes B$.*

The approximations in the following lemma will be used in the proof of the main result of this section.

Lemma 3.4.10. *Given any $N \in \mathbb{N}, \epsilon > 0$ and a finite subset F of $C(X) \otimes B$, there exists $\mathbb{N} \ni M > N$, a clopen neighborhood U of x_0, and a partial isometry $w \in A_{x_0}$ satisfying the following relations:*

(1) $\varphi^{-N+1}(U), \varphi^{-N+2}(U), \ldots, U, \varphi(U), \ldots, \varphi^M(U)$ are mutually disjoint and $\mu(U) < \epsilon$ for all φ -invariant measure μ.

(2) $w^*w = \chi_U \otimes 1$ and $ww^* = \chi_{\varphi^M(U)} \otimes 1$.

(3) $(u^*)^i w u^i \in A_{x_0}$ for all $0 \leqslant i \leqslant N-1$.

(4) $||wa - aw|| < \epsilon$ for all $a \in F$.

Proof. We may assume that $F = F_1 \bigcup F_2$ where $F_1 \subset C(X)$ and $F_2 \subset B$. By continuity, there is a clopen neighborhood O of x_0 such that $|f(x_0) - f(y)| < \epsilon/2$ for all $y \in O$ and all $f \in F_1$. From the minimality of φ we can find an $M > N$ so that $\varphi^M(x_0) \in O$. There is always a clopen neighborhood U of x_0 such that condition (1) is satisfied. Shrinking (since $\varphi^M(x_0) \in O$) U we may suppose that $U \bigcup \varphi^M(U) \subset O$. Using the minimality of φ one can define a continuous function $n : U \longrightarrow \mathbb{N}$ by $n(x) := \min\{k \in \mathbb{N}; \varphi^{-k}(x) \in \varphi^M(U)\}$. Since n is continuous and U is clopen, we see that $\{n^{-1}(k); k \in \mathbb{N}\}$ forms a finite partition of U. Define $w_1 := \sum_{k\in\mathbb{N}} u^{-k}(\chi_{n^{-1}(k)} \otimes 1)$. From (1) we see that $w_1^* w_1 = \chi_U \otimes 1$ and $w_1 w_1^* = \chi_{\varphi^M(U)} \otimes 1$, and from its definition we can see that w_1 is a partial isometry in A_{x_0}.

For each element $b \in B$ one has $w_1^* b w_1 = \sum_{k\in\mathbb{N}}(\chi_{n^{-1}(k)} \otimes \beta^k)$. Indeed, supposing that $w_1 := \sum_{m=1}^n u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1)$ we see that $w_1^* b w_1 = [\sum_{m=1}^n u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1)]^*(1 \otimes b) \sum_{l=1}^n u^{-k_l}(\chi_{n^{-1}(k_l)} \otimes 1) = [\sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes 1) u^{k_m}](1 \otimes b) \sum_{l=1}^n u^{-k_l}(\chi_{n^{-1}(k_l)} \otimes 1) = \sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes 1)(1 \otimes \beta^{k_m}(b)) u^{k_m} \sum_{l=1}^n u^{-k_l}(\chi_{n^{-1}(k_l)} \otimes 1)$. From (1) we need to impose $m = l$ (the other terms will be zero), so the last sum is equal to $\sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes 1)(1 \otimes \beta^{k_m}(b)) u^{k_m} u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1) = \sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes \beta^{k_m}) = \sum_{k\in\mathbb{N}}(\chi_{n^{-1}(k)} \otimes \beta^k)$.

Using that $n(U)$ and F_2 are finite subsets and that β is approximately inner, there exist some unitaries $v_k \in B$ such that $||\beta^k(b) - v_k b v_k^*|| < \epsilon/2t$ for all $k \in n(U)$ and all $b \in F_2$, where t us the cardinality of $n(U)$. Define $w_2 := \sum_{k\in N}(\chi_{n^{-k}} \otimes v_k) \in C(X) \otimes B \subseteq A_{x_0}$. One can check that $w_2^* w_2 = w_2 w_2^* = \chi_U \otimes 1$. Indeed, writing w_2 as $\sum_{m=1}^n \chi_{n^{-1}(k_m)} \otimes v_{k_m}$ we see that $w_2^* w_2 = \sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes v_{k_m}^*) \sum_{l=1}^n \chi_{n^{-1}(k_l)} \otimes v_{k_l}$. Since $\{n^{-1}(k); k \in \mathbb{N}\}$ is a partition of U we can impose $m = l$ and we have that $w_2^* w_2 = \sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes v_{k_m}^*)(\chi_{n^{-1}(k_m)} \otimes v_{k_m}) = \sum_{m=1}^n (\chi_{n^{-1}(k_m)} \otimes 1) = \chi_U \otimes 1$. The other equality may be checked similarly to this one. So w_2 is a partial isometry.

Setting $w := w_1 w_2 \in A_{x_0}$, we have that w is a partial isometry. Let us check condition (2).

$$
\begin{aligned}
w^*w &= (w_1w_2)^*w_1w_2 = w_2^*w_1^*w_1w_2 = w_2^*(\chi_U \otimes 1)w_2 \\
&= \left[\sum_{m=1}^{n} \chi_{n^{-1}(k_m)} \otimes v_{k_m}\right]^* (\chi_U \otimes 1) \sum_{l=1}^{n} \chi_{n^{-1}(k_l)} \otimes v_{k_l} \\
&= \sum_{m=1}^{n} \chi_{n^{-1}(k_m)} \otimes v_{k_m}^* \sum_{l=1}^{n} \chi_{n^{-1}(k_l)} \otimes v_{k_l} \\
&= \sum_{m=1}^{n} \chi_{n^{-1}(k_m)} \otimes 1 = \chi_U \otimes 1,
\end{aligned}
$$

again because $\{n^{-1}(k); k \in \mathbb{N}\}$ is a partition of U. Notice, as in [47], $\varphi^M(U) = \bigcup_{m=1}^{n} \varphi^{-k_m}(n^{-1}(k_m))$. (If $y \in \phi^{-k_m}(n^{-1}(k_m))$ then $y = \phi^{-k_m}(z)$ where $n(z) = k_m$, i.e. $\phi^{-k_m}(z) \in \phi^M(U)$, hence $y \in \phi^M(U)$. The other inclusion is proved in a similar fashion.) We have the following chain of equalities

$$
\begin{aligned}
ww^* &= w_1w_2w_2^*w_1^* = w_1(\chi_U \otimes 1)w_1^* \\
&= \sum_{m=1}^{n} u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1)(\chi_U \otimes 1)[\sum_{l=1}^{n} u^{-k_l}(\chi_{n^{-1}(k_l)} \otimes 1)]^* \\
&= \sum_{m=1}^{n} u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1) \sum_{l=1}^{n}(\chi_{n^{-1}(k_l)} \otimes 1)u^{k_l} =^{m=l} \sum_{m=1}^{n} u^{-k_m}(\chi_{n^{-1}(k_m)} \otimes 1)u^{k_m} \\
&= \sum_{m=1}^{n}((\alpha_\varphi)^{-k_m} \otimes \beta^{-k_m})(\chi_{n^{-1}(k_m)} \otimes 1) \\
&= \sum_{m=1}^{n}(\alpha_\varphi)^{-k_m}(\chi_{n^{-1}(k_m)}) \otimes 1 \\
&= \sum_{m=1}^{n} \chi_{\varphi^{-k_m}(n^{-1}(k_m))} \otimes 1 \\
&= \chi_{\bigcup_{m=1}^{n} \varphi^{-k_m}(n^{-1}(k_m))} \otimes 1 \\
&= \chi_{\varphi^M(U)} \otimes 1.
\end{aligned}
$$

Let us check condition (4). Letting $1 \otimes b \in F_2$ we have

$$\begin{aligned}
||w(1 \otimes b) - (1 \otimes b)w|| &= ||w_1 w_2(1 \otimes b) - (1 \otimes b)w_1 w_2|| \\
&\leqslant ||w_2(1 \otimes b) - w_1^*(1 \otimes b)w_1 w_2|| \\
&= ||w_2(1 \otimes b) - \sum_{k \in \mathbb{N}} (\chi_{n^{-1}(k)} \otimes \beta^k(b))w_2|| \\
&= ||\sum_{l \in \mathbb{N}} (\chi_{n^{-1}(l)} \otimes v_l)(1 \otimes b) - \sum_{k \in \mathbb{N}} (\chi_{n^{-1}(k)} \otimes \beta^k(b)) \sum_{l \in \mathbb{N}} (\chi_{n^{-1}(l)} \otimes v_l)|| \\
&= ||\sum_{l \in \mathbb{N}} (\chi_{n^{-1}(l)} \otimes v_l)(1 \otimes b) - \sum_{k=l \in \mathbb{N}} \chi_{n^{-1}(l)} \otimes \beta^l(b)v_l|| \\
&= ||\sum_{l \in \mathbb{N}} (\chi_{n^{-1}(l)} \otimes v_l b) - \sum_{l \in \mathbb{N}} \chi_{n^{-1}(l)} \otimes \beta^l(b)v_l|| \\
&= ||\sum_{l \in \mathbb{N}} \chi_{n^{-1}(l)} \otimes (v_l b - \beta^l(b)v_l)|| \\
&\leqslant \sum_{l \in \mathbb{N}} ||\chi_{n^{-1}(l)}|| \, ||v_l b v_l^* - \beta^l(b)|| \, ||v_l|| \\
&< 1 \frac{\epsilon}{2t} t 1 = \epsilon/2.
\end{aligned}$$

Let $f \otimes 1 \in F_1$. Then

$$\begin{aligned}
||w(f \otimes 1) - (f \otimes 1)w|| &= ||w_1 w_2(f \otimes 1) - (f \otimes 1)w_1 w_2|| \\
&\leqslant ||w_1 w_2(f \otimes 1)w_2^* - (f \otimes 1)w_1|| \\
&= ||w_1 \sum_{k \in \mathbb{N}} (\chi_{n^{-1}(k)} \otimes v_k)(f \otimes 1) \sum_{l \in \mathbb{N}} (\chi_{n^{-1}(l)} \otimes v_l^*) - (f \otimes 1)w_1|| \\
&=^{k=l} ||w_1 \sum_{k \in \mathbb{N}} (\chi_{n^{-1}(k)} f) \otimes 1 - (f \otimes 1)w_1|| \\
&= ||w_1(\chi_U f \otimes 1) - (f \otimes 1)w_1||
\end{aligned}$$

$$= ||\sum_{k \in \mathbb{N}} u^{-k}(\chi_{n^{-1}(k)} \otimes 1)(\chi_U f \otimes 1) - (f \otimes 1) \sum_{l \in \mathbb{N}} u^{-l}(\chi_{n^{-1}(l)} \otimes 1)||$$

$$= ||\sum_{k \in \mathbb{N}} u^{-k}(\chi_{n^{-1}(k)} f \otimes 1) - \sum_{l \in \mathbb{N}} u^{-l}(f \circ \varphi^{-l} \otimes 1)(\chi_{n^{-1}(l)} \otimes 1)||$$

$$= || \sum_{k \in \mathbb{N}} u^{-k}(\chi_{n^{-1}(k)} f \otimes 1) - \sum_{l \in \mathbb{N}} u^{-l}((f \circ \varphi^{-l})\chi_{n^{-1}(l)}) \otimes 1)||$$

$$= || \sum_{k \in \mathbb{N}} u^{-k}[(\chi_{n^{-1}(k)}(f - f \circ \varphi^{-k})) \otimes 1]||.$$

The last norm is less or equal than ϵ because of the choice (regarding F_1) of O and the choice of U (recall that $U \bigcup \varphi^M(U) \subset O$).

We shall prove that that condition (3) also holds. Since $w_2 \in C(X) \otimes B$ it is sufficient to show that $(u^*)^i w_1 u^i \in A_{x_0}$ for all $0 \leqslant i \leqslant N-1$. Since $(u^*)^i A_{\varphi^i(x_0)} u^i =< C(X) \otimes B, (u^*)^i u(C(X \backslash \{\varphi^i(x_0)\}) \otimes B) u^i >=< C(X) \otimes B, u(u^*)^i (C(X \backslash \{\varphi^i(x_0)\}) \otimes B) u^i >=< C(X) \otimes B, u(C(X \backslash \{x_0\}) \otimes B) >= A_{x_0}$ it is sufficient to prove that $w_1 \in A_{\varphi^i(x_0)}$ for all $i = 0, 1, \ldots, M-1$. But this is valid from the definition of w_1 and because $U, \varphi(U), \ldots, \varphi^M(U)$ were chosen mutually disjoint. $\square$

Remark 3.4.11. *The following discussion is a consequence of that preceding Theorem 3.12 of [38]. Since C has real rank 0 (see Remark 3.4.9.), and A_{x_0} has property SP, and because unital, simple C^*-algebras with tracial rank 0 have property SP, one can conclude that C $(= (C(X) \otimes B) \ltimes_{\alpha_\varphi \otimes \beta} \mathbb{Z})$ has stable rank one.*

Remark 3.4.12. *$K_0(C)$ is weakly unperforated.*

Proof. Let us remark first that $K_0(i) : K_0(A_{x_0}) \longrightarrow K_0(C)$ is an abelian group isomorphism. Indeed, using the notation introduced before Lemma 3.4.3, we get (because of [47], and since D_n is finite dimensional) that $K_0(A_{x_0}) \simeq \lim_n K_0(D_n \otimes B) \simeq \lim_n K_0(D_n) \otimes K_0(B) \simeq (\lim_n K_0(D_n)) \otimes K_0(B) \simeq \frac{K_0(C(X))}{H_{\alpha_\varphi}} \otimes K_0(B)$. Using the six-term exact sequence of Pimsner and Voiculescu, $K_1(C(X)) = 0$ and that β is approximately inner, we obtain that the last group is isomorphic to $\frac{K_0(C(X) \otimes B)}{H_{\alpha_\varphi \otimes \beta}} \simeq K_0(C)$.

More is true: $K_0(i)$ is an order isomorphism. Of course, it is order preserving. For the last part, suppose that $0 < \eta \in K_0(C)$ and let $[p]_0 - [q]_0 \in K_0(A_{x_0})$ be the unique lifting for some projections p and q in matrices over A_{x_0}. For any $\sigma \in T(A_{x_0})$, there is (by Lemma 3.4.5) an $\alpha_\varphi \otimes \beta$ invariant tracial state τ of C such that $\tau \circ i = \sigma$. Since C is simple and $\eta > 0$ one has that $0 < K_0(\tau)(\eta) = K_0([p]_0 - [q]_0)$. Since

$\sigma(p) > \sigma(q)$ for all $\sigma \in T(A_{x_0})$ and A_{x_0} has tracial rank 0 (Lemma 3.4.3) we conclude (see Theorem 3.7.2 in [37]) that $p \succeq q$, whence it folllows that $[p]_0 - [q]_0 > 0$.

Now it is routine to prove (using the above) that $K_0(C)$ satisfies Blackadar's Second Fundamental Question and that $K_0(C)$ is weakly unperforated. $\square$

Theorem 3.4.13. *If φ is minimal, β is approximately inner and the simple unital C^*-algebra B has tracial rank 0 then C has tracial rank 0.*

Proof. As always, the proof runs in parallel with its counterpart in [38], which in this case is Theorem 5.6. The following will be shown: given $\epsilon > 0$, a finite subset $F \subset C(X) \otimes B$, and a nonzero positive element $c \in C$, there is a projection $e \in A_{x_0}$ so that the following hold:

(i) $||ae - ea|| < \epsilon$ for all $a \in F \cup \{u\}$,

(ii) for any $a \in F \cup \{u\}$, there is an element $b \in eA_{x_0}e$ such that $||eae - b|| < \epsilon$,

(iii) $1 - e$ is equivalent to a projection in $\overline{cAc}$.

Since A_{x_0} is a simple unital C*-algebra with tracial rank 0, we obtain that $eA_{x_0}e$ is a simple unital C*-algebra with tracial rank 0 (see [37] for hereditary subalgebras of tracial rank 0 C*-algebras). Hence, if the above is shown, by [38] or [35], it follows that C has tracial rank 0. Since C has real rank 0 (Remark 3.4.9.), has stable rank one (Remark 3.4.11), and is weakly unperforated $K_0(C)$, it is sufficient to prove the following: given any $\epsilon > 0$ and given any finite subset $F \subset C(X) \otimes B$, there is a projection $e \in A_{x_0}$ such that the following conditions hold:

(a) $||ae - ea|| < \epsilon$ for all $a \in F \cup \{u\}$,

(b) given any $a \in F \cup \{u\}$ there is an element $b \in eA_{x_0}e$ such that $||eae - b|| < \epsilon$,

(c) $\tau(1 - e) < \epsilon$ for all $\tau \in T(C)$.

Toward this, we may assume that $F = F^*$. Pick $N \in \mathbb{N}$ such that $\frac{2\pi}{N} < \epsilon$. From Lemma 3.4.10 (with N, $\frac{\epsilon}{2}$, and a finite subset $G := \cup_{i=0}^{N-1} u^i F(u^i)^*$), we can find $M > N$, a clopen neighborhood U of x_0, and a partial isometry $w \in A_{x_0}$. Set $p := \chi_U \otimes 1 \in C(X) \otimes B$ and $q := \chi_{\varphi^M(U)} \otimes 1 \in C(X) \otimes B$ and remark that $p \perp q$. Again, as in [38], one can define (for $t \in [0, \pi]$), the following continuous path of projections $P(t) := p\cos^2 t + w \sin t \cos t + w^* \sin t \cos t + q \sin^2 t$ satisfying $P(0) = p = P(\pi)$. Evidently $P(t) = P(t)^*$ and from $pq = 0$ we see $w^*p = wq = qw^* = 0$ which in turns implies that

$$
\begin{aligned}
P(t)^2 =& p\cos^4 t + w^2\sin^2 t\cos^2 t + (w^*)^2\sin^2 t\cos^2 t + q\sin^4 t + pw\sin t\cos^3 t \\
& + wp\sin t\cos^3 t + pw^*\sin t\cos^3 t + w^*p\sin t\cos^3 t + 0 + 0 + ww^*\sin^2 t\cos^2 t \\
& + w^*w\sin^2 t\cos^2 t + wq\sin^3 t\cos t + qw\sin^3 t\cos t + w^*q\sin^3 t\cos t \\
& + qw^*\sin^3 t\cos t \\
& = q\sin^2 t + p\cos^2 t + w^2\sin^2 t\cos^2 t + (w^*)^2\sin^2 t\cos^2 t + pv\sin t\cos^3 t \\
& + w\sin t\cos^3 t + w^*\sin t\cos^3 t + w^*p\sin t\cos^3 + wq\sin^3 t\cos t + w\sin^3 t\cos t \\
& + (w^*)\sin^3 t\cos t + qw^*\sin^3 t\cos t \\
& = q\sin^2 t + p\cos^2 t + w\sin t\cos t + w^*\sin t\cos t + 0 = P(t).
\end{aligned}
$$

From Lemma 3.4.10 (4), and $p, q \in C(X) \otimes B$, we see that $||(u^i)^*P(t)u^if - f(u^i)^*P(t)u^i|| < \epsilon$ for all $t \in [0,\pi], 0 \leqslant i \leqslant N-1$ and $f \in F$. Define the following element $e := 1 - \left(\sum_{i=0}^{M-N} u^ip(u^i)^* + \sum_{i=1}^{N-1}(u^i)^*P(\frac{i\pi}{N})u^i\right)$. Since the projections appearing in the definition of e are orthogonal (by the choice of U), we see that e is a projection. From Lemma 3.4.10 (3) and the definition of $P(t)$, we see that $e \in A_{x_0}$.

From Lemma 3.4.10 (3) and the definitions of e and of p we get that $efe \in A_{x_0}$ for all $f \in C(X) \otimes B$. Since $e \perp p$ it follows that $eue = eue - eupe = eu(1-p)e$. Thus $eue = eu(1-p)e \in eA_{x_0}e \subseteq A_{x_0}$ because $e \in A_{x_0}$.

For all $\tau \in T(C)$ one can check from the definition of e and the choice of U that $\tau(1-e) < M\tau(p) < \frac{\epsilon}{2}$.

It remains to show $||fe - ef|| < \epsilon$ for all $f \in F$ and $||ue - eu|| < \epsilon$. The following computations use the definition of e:

$$
\begin{aligned}
||ue - eu|| \leqslant& ||e - u^*eu|| \\
=& ||1 - \sum_{i=0}^{M-N} u^i p(u^i)^* - \sum_{i=1}^{N-1} (u^i)^* P(\frac{i\pi}{N}) u^i - 1 + \sum_{i=0}^{M-N} u^* u^i p(u^i)^* u \\
&+ \sum_{i=1}^{N-1} (u^{i+1})^* P(\frac{i\pi}{N}) u^{i+1}|| \\
=& ||u^{-1} p(u^{-1})^* - u^{M-N} p(u^{M-N})^* \\
&- u^* P(\frac{\pi}{N}) u - \cdots - (u^{N-1})^* P(\frac{(N-1)\pi}{N}) u^{N-1} \\
&+ (u^2)^* P(\frac{\pi}{N}) u^2 + \cdots + (u^N)^* P(\frac{(N-1)\pi}{N}) u^N||
\end{aligned}
$$

and the result follows from the estimate in Theorem 5.6 of [38]:
$||P(\frac{i\pi}{N}) - P(\frac{(i-1)\pi}{N})|| < \frac{2\pi}{N}$.

For the other estimate, let $g \otimes b \in F$. Then

$$
\begin{aligned}
||fe - ef|| =& \\
& ||\sum_{i=1}^{N-1} (g \otimes b)(u^i)^* [w \sin i\pi/N \cos i\pi/N + w^* \sin i\pi/N \cos i\pi/N] u^i \\
& - \sum_{i=1}^{N-1} (u^i)^* [w \sin i\pi/N \cos i\pi/N + w^* \sin i\pi/N \cos i\pi/N] u^i (g \otimes b)||.
\end{aligned}
$$

The general estimate will be simplified by taking the case of w and $i = 1$. Indeed, we get that $||(g\otimes b)u^*w \sin i\pi/N \cos i\pi/N u - u^*w \sin i\pi/N \cos i\pi/N u(g\otimes b)|| \leqslant ||(g \otimes b)u^*wu - u^*wu(g \otimes b)|| = ||u^* [u(g \otimes b)u^*w - wu(g \otimes b)u^*] u||$. From the form of G and the previous lemma, we obtain the desired estimate.

$\square$

Let us indicate here how the proof can be modified to obtain the following tracial rank one version of Proposition 3.4.13.

Proposition 3.4.14. *If φ is minimal, β is approximately inner and the simple unital C^*-algebra B has tracial rank one, then C has tracial rank one.*

Using [36] one can get that A_{x_0} has tracial rank one. One can prove (exactly as in Lemma 3.4.8) that there is a unital simple sub-algebra D of C with tracial rank one such that the following holds: for every finite subset $F \subset C$ and every $\epsilon > 0$, there is a projection $p \in D$ such that

(1) for each $a \in F$, there is an element $b \in D$ such that $||ap - b|| < \epsilon$,

(2) $\tau(1-p) < \epsilon$ for each tracial state $\tau \in T(D)$.

From this it follows that C has property (SP). Since A_{x_0} has also property (SP) because this algebra has tracial rank one, it follows as in Remark 3.4.11 that C has stable rank one. Remark 3.4.12 is still valid since C*-algebras with tracial rank less or equal than one satisfy Blackadar's Second Comparability Question (see [37], Theorem 3.7.2). Hence the proof of Proposition 3.4.13 applies, yielding 3.4.14.

Chapter 4

LIFTINGS

4.1 Examples

In this chapter we shall investigate the following problem. Given a unital (AF) algebra A, let $\widehat{\ }: A \longrightarrow Aff(T(A))$ be the natural map defined by $\widehat{a}(\tau) := \tau(a)$ for every $a \in A$. Here $Aff(T(A))$ is the real Banach space of all affine continuous real-valued functions on $T(A)$ equipped with the supremum norm. Also $Aff(T(A))$ is a partially ordered real vector space equipped with the pointwise ordering, the constant function 1 being an order unit. We shall recall another map which will be useful. For every tracial state $\tau \in T(A)$ and every $n \geq 1$ denote again by τ the extension to $M_n(A)$ which is again a tracial state. Define $\widehat{\widehat{\ }}: K_0(A) \longrightarrow Aff(T(A))$ by $\widehat{\widehat{[p]_0}}(\tau) := \tau(p)$. It was proved in [7] that $\widehat{\ }$ is an onto map. Actually, it was proved that for every $f \in Aff(T(A))$ there is a self-adjoint element $a \in A$ such that $\widehat{a} = f$ regardless whether A is AF. We shall investigate when $\widehat{\ }$ lifts positives to positives.

Remark 4.1.1. *If A is a simple unital C^*- algebra with at least two traces then $\widehat{\ }$ can not lift positives to positives; since $T(A)$ is a Choquet simplex, for any $\tau_1 \neq \tau_2$, extremal tracial states, there is a continuous, affine function $g : T(A) \longrightarrow [0,1]$ such that $g(\tau_1) = 0$ and $g(\tau_2) = 1$. If there were a positive lift for g, then $I_{\tau_1} := \{a \in A; \tau_1(a^*a) = 0\}$ would be a proper ideal of A, contradicting the simplicity of A. This lifting is possible for all commutative C^*-algebras (independent of being AF) [7].*

In what follows we shall investigate the quotients of unital (not necessarily AF) C*-algebras for which the lifting is possible. Let A be a non-simple unital C*-algebra and let I be an ideal of A. Define the following canonical maps $\pi : A \longrightarrow A/I$, $\overline{\pi} : T(A/I) \longrightarrow T(A)$, $\overline{\pi}(\overline{\tau}) := \overline{\tau} \circ \pi \in T(A)$ and $\overline{\overline{\pi}} : Aff(T(A)) \longrightarrow Aff(T(A/I))$ given by $\overline{\overline{\pi}}(f)(s) := f \circ \overline{\pi}(s) = f(s \circ \pi)$ for all $s \in T(A/I)$. The following diagram is commutative

$$(\star) \qquad \begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\widehat{\ }"'] & A/I \arrow[d, "\widehat{\ }"] \\ Aff(T(A)) \arrow[r, "\overline{\overline{\pi}}"] & Aff(T(A/I)) \end{tikzcd}$$

Indeed, for every $s \in T(A/I)$ one has that $(\overline{\overline{\pi}} \circ \widehat{\ })(a)(s) = \overline{\overline{\pi}}(\widehat{a})(s) = \widehat{a}(s \circ \pi) = s \circ \pi(a)$ and $\widehat{\ } \circ \pi(a)(s) = \widehat{[a]}(s) = s([a]) = s(\pi(a))$.

Lemma 4.1.2. *The canonical map $\overline{\overline{\pi}}$ is onto.*

Proof. Since the diagram is commutative, π and both $\widehat{\ }$ maps are onto it follows that $\overline{\overline{\pi}}$ is onto. □

The next observation is that $\overline{\overline{\pi}}$ lifts positives to positives. For this we need the following extension result proved in [15] (11.23).

Lemma 4.1.3. *Let K be a Choquet simplex, let F be a closed face of K and let $g \in Aff(F)$. Then there exist $f \in Aff(K)$ such that $f|_F = g$ and $||g|| = ||f||$. If g is positive, then f can be chosen to be positive .*

Let $F := \overline{\pi}(T(A/I))$. Since $\overline{\pi}$ is injective, it follows that $T(A/I)$ and F are homeomorphic by $\overline{\pi}_1$, where $\overline{\pi}_1(s) := \overline{\pi}(s)$ for every $s \in T(A/I)$. From this we see that $Aff(T(A/I)) \simeq Aff(F)$. It is very easy to check that F is a closed face of $T(A)$. Letting $R : Aff(T(A)) \longrightarrow Aff(F)$ be given by restriction, $R(f) := f|_F$ for all $f \in Aff(T(A))$ and $T : Aff(F) \longrightarrow Aff(T(A/I))$ be given by $T(h) := h \circ (\overline{\pi}_1)$ for all $h \in Aff(F)$, we have that $T \circ R = \overline{\overline{\pi}}$. Indeed, $T \circ R(f)(s) = (T(R(f)))(s) = T(f|_F)(s) = (f|_F \circ \overline{\pi}_1)(s) = f|_F(\overline{\pi}_1(s)) = f|_F(s \circ \pi) = f(s \circ \pi) = \overline{\overline{\pi}}(s)$ for all $f \in Aff(T(A))$ and all $s \in T(A/I)$. Using Lemma 4.1.2 (since A is unital, $T(A)$ is a Choquet simplex) for our F we get the following

Proposition 4.1.4. *Let A be a unital C^*-algebra and let I be an ideal of A. If $\widehat{\ }: A \longrightarrow Aff(T(A))$ lifts positives to positives, then $\widehat{\ }: A/I \longrightarrow Aff(T(A/I))$ also lifts positives to positives.*

Proof. By the discussion above (and lemma 4.1.3), any positive of $Aff(T(A/I))$ lifts to a positive of $Aff(T(A))$. By the lifting hypothesis we can lift the latter positive to a positive of A. Since the diagram $\star$ is commutative, the proof is complete. □

Remark 4.1.5. *It follows from the proposition above and Remark 4.1.1 that if the natural map lifts positives to positives for a unital C^*-algebra A, then each simple quotient of A has unique trace.*

For crossed products the proof of lifting is much simpler:

Proposition 4.1.6. *Suppose that A is a unital C^*-algebra for which the natural map $\widehat{\ }: A \longrightarrow Aff(T(A))$ lifts positives to positives. If α is an approximately inner automorphism of A, then the corresponding natural map associated to the crossed product lifts positives to positives.*

Proof. Since α is approximately inner, we have that $T(A)$ is homeomorphic to $T(A \ltimes_\alpha \mathbb{Z})$. From here we get that the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\widehat{\ }"] \arrow[d, "i"'] & Aff(T(A)) \arrow[d, "\simeq"] \\ A \ltimes_\alpha \mathbb{Z} \arrow[r, "\widehat{\ }"] & Aff(T(A \ltimes_\alpha \mathbb{Z})) \end{tikzcd}$$

Since $Aff(T(A))$ is order isomorphic to $Aff(T(A \ltimes_\alpha \mathbb{Z}))$, the conclusion follows.

□

Definition 4.1.7. *Let A and B be two unital C^*-algebras and let $\varphi, \psi : A \longrightarrow B$ be two *-morphisms. We say that φ and ψ are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if for every $\epsilon > 0$ and every finite subset F of A there is a unitary $w \in B$ such that $||w\varphi(a)w^* - \psi(a)|| < \epsilon$ for all $a \in F$.*

Proposition 4.1.8. *Let A and B be two unital C^*-algebras and let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ be two *-morphisms such that $\varphi \circ \psi \approx_u id_B$. If the natural map of A lifts positives to positives then the natural map of B lifts positives to positives.*

Proof. Let $0 \leq g \in \mathrm{Aff}(T(A))$ and set $f := \overline{\overline{\psi}}(g) \in \mathrm{Aff}(T(A))^+$. From the lifting hypothesis, there is a positive element $a \in A$ such that $\widehat{a} = f$. If $b := \varphi(a)$, then $b \in B^+$. It is clear that $\overline{\overline{\varphi}}(f) = g$ will imply that $\widehat{b} = g$ and the proof will be complete.

To prove that $\overline{\overline{\varphi}}(f) = g$ it is sufficient to show $\overline{\overline{\varphi}}(\overline{\overline{\psi}}(g)) = g$, or in other words we need to prove that for any $s \in T(B)$ the following equality holds $\overline{\overline{\varphi}}(\overline{\overline{\psi}}(g))(s) = g(s)$. We have that $\overline{\overline{\varphi}}(\overline{\overline{\psi}}(g))(s) = (\overline{\overline{\psi}}(g) \circ \overline{\varphi})(s) = \overline{\overline{\psi}}(g)(\overline{\varphi}(s)) = \overline{\overline{\psi}}(g)(s \circ \varphi) = (g \circ \overline{\psi})(s \circ \varphi) = g(\overline{\psi}(s \circ \varphi)) = g((s \circ \varphi) \circ \psi) = g(s \circ \varphi \circ \psi)$, where as usual, $\overline{\psi}(r) := r \circ \psi$, for $r \in T(A)$ and $\overline{\overline{\psi}}(k) := k \circ \overline{\psi}$ for every $k \in \mathrm{Aff}(T(B))$.

We know that $g = \widehat{x}$ for some selfadjoint $x \in B$, thus it is sufficient to show that $s \circ \varphi \circ \psi(x) = s(x)$ for all $s \in T(B)$. Leting $\epsilon > 0$ be arbitrary we get that there is a unitary $u \in B$ such that $||u\varphi \circ \psi(x)u^* - x|| < \epsilon$. Thus $|s \circ \varphi \circ \psi(x) - s(x)| = |s(u\varphi \circ \psi(x)u^*) - s(x)| \leq ||u\varphi \circ \psi(x)u^* - x|| < \epsilon$. Thus we obtain that $s \circ \varphi \circ \psi(x) = s(x)$ for all $s \in T(B)$ and the proof is done. □

Let us recall some facts from the paper [7] which will lead to the fact that strictly positive affine positive functions on $T(A)$, A unital, can be lifted (by the natural map) to a positive element.

For a unital C*-algebra A, denote $A_0 := \{x - y; x, y \in A^+, x \equiv y\}$, where for x, y positives elements in A we say $x \equiv y$ if there is a sequence of elements $\{x_n\}_n$ in A such that $x = \sum_{n=1}^{\infty} x_n^* x_n$ and $y = \sum_{n=1}^{\infty} x_n x_n^*$. Then $\equiv$ is an equivalence relation on A^+ which gives rise to a new equivalence relation on A_{sa} as follows: for $x, y \in A_{sa}$ define $x \sim y$ if there is a positive element $c \in A^+$ such that $x + c \equiv y + c$. Moreover, $x \sim y$ in A_{sa} if and only if $x - y \in A_0$, thus $\mathcal{A} := A_{sa}/A_0 = A_{sa}/\sim$. Of course, $\mathcal{A}$ is a Banach space with the quotient norm, $|||[a]||| := \inf\{||a - z||; z \in A_0\}$ and one can define a closed convex cone $\mathcal{A}_+$ in $\mathcal{A}$ by $\mathcal{A}_+ := \overline{\{[a]; a \in A_+\}}$. With this cone $\mathcal{A}$ is an order unit space with order unit $u := [1]$. The crucial result we need is the fact that $\mathcal{A}$ is a copy of $\mathrm{Aff}T(A)$. The map $\phi : \mathcal{A} \longrightarrow \mathrm{Aff}T(A)$, given by $\phi([a])(\tau) := \tau(a)$ for all $\tau \in T(A)$ is a linear isometry such that $\phi(\mathcal{A}_+) = \mathrm{Aff}T(A)^+$ and $\phi(u) = 1$.

Lemma 4.1.9. *Let $f \in \mathit{Aff}T(A)^+$. Then for every $\epsilon > 0$ there is a lifting $a_\epsilon \in A_{sa}$ such that $a_\epsilon \geq -1\epsilon$.*

Proof. Let $\epsilon > 0$ be fixed. By the discussion above there is a $0 \leq [a]$ such that

$\phi([a]) = f$. Since $[a]$ is positive, there is a sequence of positive elements $\{a_n\}_n \in A^+$ such that $\lim_{n\to\infty}[a_n] = [a]$. So there is an index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $||[a - a_n]|| < \epsilon/2$. Thus, for all $n \geq n_0$, one has that $\epsilon/2 > \inf_{z\in A_0} ||a - a_n - z||$. It follows that there is a $z_0 \in A_0$ such that $\epsilon/2 \geq ||a - a_{n_0} - z_0||$ and the proof is complete because $-\epsilon/2 < -||a - a_{n_0} - z_0|| \leq a - a_{n_0} - z_0$, thus $-\epsilon/2 \leq -\epsilon/2 + a_{n_0} \leq a - z_0$ and $\widehat{a} = \widehat{a - z_0} = f$. $\square$

Proposition 4.1.10. *Let A be a unital C*-algebra and $0 < f \in AffT(A)$. Then there is a positive element $a \in A^+$ such that $\widehat{a} = f$.*

Proof. Since $f > 0$ let $0 < \delta := \inf_{\tau\in T(A)} f(\tau)$. Then $f - \delta/2 > 0$ and by the above lemma there is a self adjoint lifting $a \in A_{sa}$ for $f - \delta/2$ such that $a \geq -1\delta/4$. It follows that $\widehat{a + 1\delta/2} = f$ and $a + 1\delta/2 \geq -1\delta/4 + 1\delta/2 = 1\delta/4 \geq 0$. $\square$

For finite dimensional C*-algebras, the lifting holds even if the positive $f \in$ Aff$T(A)$ assumes the value 0 somewhere.

Remark 4.1.11. *When A is a finite dimensional C*-algebra, we have that $A = M_{n_1} \oplus \cdots \oplus M_{n_m}$ and denote the tracial state of M_{n_i} by τ_i for every $1 \leq i \leq m$. For every tracial state τ of A, there is a vector $(t_1, \ldots, t_m) \in \mathbb{R}^m$ such that $\tau = \tau_{(t_1,\ldots,t_m)}$, where $\tau_{(t_1,\ldots,t_m)}(a_1, \ldots, a_m) := \sum_{i=1}^m t_i\tau_i(a_i)$. Let $0 \leq f \in AffT(A)$ and suppose that $0 = f(\tau_{(1,0,\ldots,0)}) = f(\tau_{(0,1,0,\ldots,0)}) = \cdots = f(\tau_{\underbrace{(0,0,\ldots,1,0,\cdots,0)}_{k}})$, $f(\tau_{\underbrace{(0,0,\ldots,1,0,\cdots,0)}_{k+1}}) = x_{k+1} > 0, \ldots, f(\tau_{(0,0,\ldots,0,1)}) = x_m > 0$, for some $1 < k < m$. Then $0 \leq b := 0 \oplus 0 \cdots \oplus 0 \oplus x_{k+1}1 \cdots \oplus x_m 1 \in A^+$ and $\widehat{b} = f$.*

Another class of C*-algebras for which the natural map lifts positives to positives is exhibited in the next

Proposition 4.1.12. *For any compact space X and any unital C*-algebra B with unique tracial state, the natural map $\widehat{\ }: C(X) \otimes B \longrightarrow AffT(C(X) \otimes B)$ lifts positives to positives.*

Proof. Denote by $T : B \longrightarrow \mathbb{C}$ the unique tracial state of B and by $\epsilon_x, x \in X$ the extremal tracial states of $C(X)$. Of course, the following diagram is commutative

$$
\begin{tikzcd}
C(X) \otimes B \arrow[r, "\widehat{}"] & Aff(T(C(X) \otimes B)) \\
C(X) \arrow[u, "M"] \arrow[r, "\widehat{}"] & Aff(T(C(X))) \arrow[u, "\overline{\overline{M}}"']
\end{tikzcd}
$$

where $M : C(X) \longrightarrow C(X) \otimes B$ is given by $M(h) := h \otimes 1$ for every $h \in C(X)$, $\overline{M} : T(C(X) \otimes B) \longrightarrow T(C(X))$ is given by $\overline{M}(S) := S \circ M$ for every $S \in T(C(X) \otimes B)$ and $\overline{\overline{M}} : \mathrm{Aff}T(C(X)) \longrightarrow \mathrm{Aff}T(C(X) \otimes B)$ is given by $\overline{\overline{M}}(k) := k \circ \overline{M}$ for every $k \in \mathrm{Aff}T(C(X))$.

Suppose that $0 \leq f \in \mathrm{Aff}T(C(X) \otimes B)$. Then, the function $g : T(C(X)) \longrightarrow \mathbb{R}$ defined by $g(\epsilon_x) := f(\epsilon_x \otimes T)$ for all $x \in X$ is affine, positive and satisfies $\overline{\overline{M}}(g) = f$. Indeed, for all $x \in X$ we have that $\overline{\overline{M}}(g)(\epsilon_x \otimes T) = (g \circ \overline{M})(\epsilon_x \otimes T) = g((\epsilon_x \otimes T) \circ M)$. Since the natural map is onto let $a \in C(X)$ a self-adjoint element such that $\widehat{a} = g$. From $(\epsilon_x \otimes T) \circ M(a) = \epsilon_x \otimes T(a \otimes 1) = \epsilon_x(a) = \widehat{a}(\epsilon_x) = g(\epsilon_x) = f(\epsilon_x \otimes T)$ we get that $\overline{\overline{M}}(g) = f$.

Since the natural map for commutative C*-algebras lifts positives to positives, M is a *-morphism (thus preserves positivity), and the diagram commutes, we obtained the desired result. □

Remark 4.1.13. *In the above proposition, $C(X)$ can be replaced by any unital C*-algebra for which the natural map lifts the positives to positives.*

Let us give a non-trivial

Example 4.1.14. *Let A be a unital AF algebra with unique tracial state, $\tau : A \longrightarrow \mathbb{C}$, for instance, the CAR algebra, and write $A = \overline{\cup_{n=1}^{\infty} A_n}$, where $1 \in (A_n)_{n \in \mathbb{N}}$ is an increasing sequence of finite dimensional C*-algebras. Denote $A' := \{f : \mathbb{N} \cup \{\infty\} \longrightarrow A; f(n) \in A_n \text{ for all } n \in \mathbb{N}\}$, where $\mathbb{N} \cup \{\infty\}$ is the one point compactification of $\mathbb{N}$. Then A' is an AF algebra; it is not simple because it has A as a quotient. Let $0 \leq g \in \mathrm{Aff}T(A')$. Since the natural map is surjective, let $f \in A'$ be a self-adjoint lift. For all $i \in \mathbb{N} \cup \{\infty\}$ denote $r_i := \tau(f(i)) \geq 0$. From the fact that τ is continuous, one can define a positive element $f' : \mathbb{N} \cup \{\infty\} \longrightarrow A$ in A' by $f'(i) := r_i 1$ for all $i \in \mathbb{N} \cup \{\infty\}$. It follows from the form of extremal tracial states of A' that $g = \widehat{f} = \widehat{f'}$, hence we have found a positive lift for g.*

Let us indicate here that indeed the extremal tracial states appearing above have the desired form. Setting $I := \{f \in A'; f(\infty) = 0\} \equiv \{f \in A'; \lim_n ||f(n)|| = 0\}$ *we see that the map* $K : A'/I \longrightarrow A$ *given by* $K([f]) := f(\infty)$ *is an isomorphism. Our goal is to show that all extremal tracial states of* A' *have the form* $\tau_n : A' \longrightarrow \mathbb{C}$, *where* $\tau_n(f) := \tau(f(n))$, *for all* $f \in A'$, *here* $n \in \mathbb{N} \cup \{\infty\}$. *Let* δ *be an extremal tracial state of* A'. *If* $\delta|_I = 0$, *there is an extremal tracial state* $\overline{\delta} : A'/I \longrightarrow \mathbb{C}$ *such that* $\overline{\delta} \circ \pi = \delta$, *where* $\pi : A' \longrightarrow A'/I$ *is the canonical projection. The uniqueness of the tracial state (let us say* τ*) of* A *tells us that* $\overline{\delta} = \tau \circ \tau$. *Hence,* $\delta(f) = \overline{\delta} \circ \pi(f) = \tau \circ K \circ \pi(f) = \tau \circ K([f]) = \tau(f(\infty)) = \tau_\infty(f)$. *Suppose now that* $\delta|_I \neq 0$. *For each* $n \in \mathbb{N}$ *let* $T_n : A_n \longrightarrow I$ *be given by* $T_n(x)(k) = \begin{cases} x & \textit{if}, k = n \\ 0 & \textit{otherwise.} \end{cases}$ *It is a one to one morphism and denote* $\widetilde{A_n} := T_n(A_n)$. *It follows that for each* n, $\delta|_{\widetilde{A_n}} = \begin{cases} 0 & \textit{or,} \\ \tau_{A_n} & . \end{cases}$ *Since* δ *is extremal and* $\delta|_I \neq 0$ *we see that there is only one* $n_0 \in \mathbb{N}$ *such that* $\delta|_{\widetilde{A_{n_0}}} = \tau|_{A_{n_0}}$ *from where the desired form follows.*

Thus, the general example is constructed as follows. Let X *be a compact space and* B *a* C^**-algebra with unique tracial state such that* $B = \overline{\cup_{n=1}^{\infty} B_n}$, *where* B_n *are* C^**-subalgebras containing the unit of* B. *If* $x \in X$ *and* φ *is a homeomorphism of* X *define* $A := \{f : X \longrightarrow B; f \textit{ is continuous }, f(\varphi^n(x)) \in B_n \textit{ for all } n \in \mathbb{N}\}$. *Then* A *is a* C^**-algebra for which the natural map lifts positives to positives.*

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