An Invariant for
Locally Finite Dimensional Semisimple Algebras

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To my extended family and to Nelly.

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Like an element in a set, defined by relations, my existence is the people around me. Thanks.

ABSTRACT

Complete invariants were found for the category of unital direct limits of finite dimensional semisimple complex algebras and the category of unital direct limits of finite dimensional semisimple real algebras by G. A. Elliott ([E]) and by K. R. Goodearl and D. E. Handelman ([GH]) respectively. We are naturally led to consider similar complete invariants for other algebras of this type. For other fields, the situation is much more complicated, since the set of division rings containing a field F that is neither real closed nor algebraically closed is infinite (even ignoring the noncommutative ones). So let $\Omega = \{D_i\}$ be a finite set of finite dimensional division algebras, we shall only study the categories of unital direct limits of finite direct products of matrix algebras involving just this set of division rings.

The conjecture of [GH] concerning a proposed complete invariant for direct limit algebras is simplified, and we show that this invariant (essentially a diagram of ordered K_0-groups) is complete, establishing the conjecture.

INTRODUCTION

Let F be a field and let Ω be a finite set of finite dimensional division F-algebras. Let $\mathcal{R}$ be the category of all unital direct limits, $R = \lim R_i \to R_{i+1}$ where the R_i are finite direct products of matrix algebras over division algebras in Ω. For $F = \mathbf{C}$, the complete invariant found by G. A. Elliott is the dimension group with order unit $(K_0(R), [R])$ and for $F = \mathbf{R}$, the complete invariant found by K. R. Goodearl and D. E. Handelman is the triple of dimension groups with order unit

$$(K_0(R), [R]) \longrightarrow (K_0(R \otimes_{\mathbf{R}} \mathbf{C}), [R \otimes_{\mathbf{R}} \mathbf{C}]) \longrightarrow (K_0(R \otimes_{\mathbf{R}} \mathbf{H}), [R \otimes_{\mathbf{R}} \mathbf{H}]),$$

induced by the inclusions $\mathbf{R} \to \mathbf{C} \to \mathbf{H}$, where $\mathbf{H}$ is the quaternions. These results suggest that a complete invariant for $\mathcal{R}$ should be a diagram of dimension groups with order unit $(K_0(R \otimes_F D_i^{op}), [R \otimes_F D_i^{op}])$, where D_i's are in Ω and D_i^{op} is the reverse of D_i, as conjectured in [GH, XVI.6]. In fact, let K/F be a Galois extension that contains all the centres of the division algebras in Ω. Then the diagram

$$(K_0(R), [R]) \xrightarrow{K_0(\mu)} (K_0(R^K), [R^K]) \left\langle \begin{array}{l} \xrightarrow{K_0(\nu_1)} (K_0(R^{M_z(D_1^{op})}), [R^{M_z(D_1^{op})}]) \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_\omega)} (K_0(R^{M_z(D_\omega^{op})}), [R^{M_z(D_\omega^{op})}]) \end{array} \right. \qquad (*)$$

is a complete invariant, where:

(i) R^X is the notation for $R \otimes_F X$ and $D_1, \ldots, D_\omega$ are all the division algebras in Ω.

(ii) $K_0(R^K)$ is also an ordered $\mathrm{Gal}(K/F)$-module induced by the group homomorphism $\mathrm{Gal}(K/F) \to \mathrm{End}(K_0(R^K))$ defined as $\tau \mapsto K_0(\mathrm{id}_R \otimes \tau)$.

(iii) The diagram above is induced by the inclusions

$$F \longrightarrow K \left\langle \begin{array}{l} \xrightarrow{\lambda_1} M_z(D_1^{op}) \\ \quad\cdots\cdots\cdots \\ \xrightarrow{\lambda_\omega} M_z(D_\omega^{op}) \end{array} \right.$$

with $z = [K : F]$ and $\lambda_i : K \to M_z(D_i^{op})$ is any unital $\mathbf{Z}(D_i)$-algebra map, where $\mathbf{Z}(D_i)$ denotes the (separable) centre of D_i.

Let $\kappa(R)$ denote the diagram $(*)$. Our classification theorem is as following:

Theorem 3.4.1. *Let $R, S \in \mathcal{R}$. Then $R \cong S$ if and only if $\kappa(R) \cong \kappa(S)$. Moreover, if f is any isomorphism of $\kappa(R)$ onto $\kappa(S)$, then there is an isomorphism ρ of R onto S such that $\kappa(\rho) = f$.*

The usual (in fact the only) method that has been used to prove the classification theorems up to now is to prove the invariant is faithful and full in the finite dimensional simple case and then apply the interweaving argument. Hence we have to prove the following two conditions:

(I) If $R, S \in \mathcal{R}$ are finite dimensional simple and $\phi, \psi : R \to S$ are unital algebra homomorphisms, then $\kappa(\phi) = \kappa(\psi)$ if and only if there exists an inner automorphism θ of S such that $\phi = \theta\psi$;

(II) If $R, S \in \mathcal{R}$ are finite dimensional simple and $f : \kappa(R) \to \kappa(S)$ is a morphism, then there exists a unital algebra homomorphism ϕ such that $f = \kappa(\phi)$.

That condition (I) is true is based on two observations:

(I.a) If $K_0(\phi \otimes \mathrm{id}_K) = K_0(\psi \otimes \mathrm{id}_K)$ then $K_0(\phi \otimes \mathrm{id}_{\mathbf{Z}(S)}) = K_0(\psi \otimes \mathrm{id}_{\mathbf{Z}(S)})$ because the map $K_0(S \otimes \mathbf{Z}(S)) \to K_0(S \otimes K)$ is injective.

$$\begin{array}{ccc} R \otimes \mathbf{Z}(S) & \longrightarrow & R \otimes K \\ \phi^{\mathbf{Z}(S)} \big\downarrow \psi^{\mathbf{Z}(S)} & & \phi^K \big\downarrow \psi^K \\ S \otimes \mathbf{Z}(S) & \longrightarrow & S \otimes K \end{array}$$

(I.b) If $K_0(\phi \otimes \mathrm{id}_{\mathbf{Z}(S)}) = K_0(\psi \otimes \mathrm{id}_{\mathbf{Z}(S)})$ then there exists an inner automorphism θ of S such that $\phi = \theta\psi$ because there is a projection $\pi : S \otimes \mathbf{Z}(S) \to S$ such that the composite map of the inclusion $S \to S \otimes \mathbf{Z}(S)$ and π is the identity map id_S and the composite map of $\phi \otimes \mathrm{id}_{\mathbf{Z}(S)}$ or $\psi \otimes \mathrm{id}_{\mathbf{Z}(S)}$ with π is an $\mathbf{Z}(S)$-algebra homomorphism into a central simple $\mathbf{Z}(S)$-algebra.

$$\begin{array}{ccc} R & \longrightarrow & R \otimes \mathbf{Z}(S) \\ \phi \big\downarrow \psi & & \phi^{\mathbf{Z}(S)} \big\downarrow \psi^{\mathbf{Z}(S)} \\ S & \longrightarrow & S \otimes \mathbf{Z}(S) \\ & & \pi \big\downarrow \\ & & S \end{array}$$

Condition (II) is true because:

(II.b) The matrix sizes of simple components of $R \otimes R^{op}$ are sufficiently large such that if there is a homomorphism $K_0(R \otimes R^{op}) \to K_0(S \otimes R^{op})$, the matrix size of S also has to be large (so that we can recover embeddings $R \to S$).

(II.a) The entries of the matrix representing $K_0(R \otimes R^{op}) \to K_0(S \otimes R^{op})$ are not arbitrary but rather being permuted by $\mathrm{Gal}(K/F)$ through the maps $K_0(R \otimes K) \to K_0(R \otimes M_x(R^{op}))$ and $K_0(R \otimes K) \to K_0(R \otimes M_y(S^{op}))$.

It becomes clear that in order to prove (II), we first need to classify explicitly all unital F-algebra homomorphisms between finite dimensional simple F-algebras. By Noether-Skolem Theorem, results of B. Fein, D. J. Saltman, M. Schacher ([FSS]), and from the $\mathrm{Gal}(K/F)$-module structure of $K_0(- \otimes K)$, we obtain the following result:

Proposition 2.3. *Let D and D' be division F-algebras with separable centres E and E' respectively. Then there is a unital F-algebra map $\phi : M_n(D) \to M_t(D')$ if and only if there are nonnegative integers $a_1, a_2, \ldots, a_k$ such that*

$$t = n.\deg(D)^2(a_1[EE' : E']/s_1 + \cdots + a_k[\sigma_k(E)E' : E']/s_k)$$

where $EE', \sigma_2(E)E', \ldots, \sigma_k(E)E'$ are all pairwise inequivalent composita of E and E' over F and s_i's are the matrix sizes of $D' \otimes_{E'} E'\sigma_i(E) \otimes_E D^{op}$.

In fact, there exists an inner automorphism θ of $M_t(D')$ such that

$$\theta\phi(x) = \begin{pmatrix} \left.\begin{matrix}\psi_1(x\otimes 1) & & \\ & \ddots & \\ & & \psi_1(x\otimes 1)\end{matrix}\right\} a_1 \text{ times} & & \\ & \ddots & \\ & & a_k \text{ times} \left\{\begin{matrix}\psi_k(x\otimes 1) & & \\ & \ddots & \\ & & \psi_k(x\otimes 1)\end{matrix}\right. \end{pmatrix}$$

where ψ_i's are the unique (up to an inner automorphism) E'-algebra maps $M_n(D \otimes_E \sigma_i(E)E') \to M_{n.r_i}(D')$ and

$$r_i = \frac{\deg(D)^2.[\sigma_i(E)E' : E']}{s_i}.$$

The proof of this Proposition was greatly improved by Dr. Edhard Neher.

This thesis is organized into five chapters. Chapter 1 contains background materials. The aim of Chapter 2 is to prove (I) and (II). Our main result, the classification theorem, is proved in Chapter 3. In this chapter we also show that our invariant is minimal and that by omitting the order units, we obtain an invariant for Morita-equivalence in this category. In Chapter 4, we study the case where objects in the category involve only a Galois extension K/F and a finite set of central division F-algebras with K as a common subfield. Then as an application, we use these results to determine whether an object in the category is of certain types. An interesting result is the following

Corollary 4.2.6. *Suppose that for all $D' \in \Omega$, we have $[K : F] = deg(D') = p$, where p is a prime integer. Then given $R \in \mathcal{R}$, the following conditions are equivalent:*

(a) R is a direct limit of finite direct product of matrix algebras over D only.

(b) $K_0(\mu_R)(K_0(R)) = pK_0(R^K)$ and $K_0(\nu_{R,D})(K_0(R^K)) = pK_0(R^{D^{op}})$.

(c) $K_0(\nu_{R,D})$ is a group isomorphism of $K_0(R^K)$ onto $pK_0(R^{D^{op}})$.

This enables us to determine whether an algebra R is as described in (a) using only $K_0(\nu_{R,D})$.

Let $G = \mathrm{Gal}(K/F)$ be a finite abelian group. In Chapter 5, we show that given a locally representable action α of G on a AF C^*-algebra A, there exists a direct limit of finite direct product of matrix algebras over K, namely R, such that $K_0(A \times_\alpha G) \cong K_0(R^K)$ as an ordered $\mathbb{Z}G$-module and vice versa. Then by a result by D. E. Handelman and W. Rossmann ([HR]), the following is a straightforward consequence

Proposition 5.1. *If H is a $|G|$-divisible dimension group and is also a countable ordered $\mathbb{Z}G$-module, then there exists an F-algebra R as above such that $H = K_0(R^K)$ as ordered $\mathbb{Z}G$-module.*

Motivated by this observation on $\mathbb{Z}G$-modules, we attempt to find a simple description for the invariant of algebras of this type. We obtain some computational results.

We end the thesis with an appendix containing remarks and questions about an attempt of finding a noncomputational description for the invariants discussed in Chapter 5. We also discuss a possible connection between an invariant for general actions of finite abelian groups on locally semisimple F-algebras and a derivation of our invariant.

The material in Chapters 4,5 is anologous to parts of ([GH]). However, these, $[\mathbf{C} : \mathbf{R}] = \deg(\mathbf{H}) = 2$ and the magic number "2" sometimes disappears or does not tell us why it appears. As we later find out, it can be buried in $\deg(\mathbf{H})/[\mathbf{C} : \mathbf{R}]$, $\deg(\mathbf{H})^2/[\mathbf{C} : \mathbf{R}]$, $[\mathbf{C} : \mathbf{R}]^2$, etc. Its magic is revealed when certain results are no longer valid for $[K : F] > 2$, and in difficulties we encounter when we attempt to find a simple description of the invariant in Chapter 5.

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Chapter 1

Background

The first two sections 1.1 and 1.2 contain general results on finite dimensional semisimple algebras and their Grothendieck groups K_0. The last section 1.3 contains computational results about embedding into finite dimensional central simple algebras which are to be used in Chapter 2.

1.1. Finite dimensional algebras and tensor products.

Theorem 1.1.1. Wedderburn's Structure Theorem

Let A be a right or left semisimple R-algebra.

(a) *There exist natural numbers $n(1), \ldots, n(r)$ and R-division algebras $D_1, \ldots, D_r$ such that*

$$A \cong M_{n(1)}(D_1) \times \cdots \times M_{n(r)}(D_r). \tag{0}$$

(b) *The pairs $(n(1), D_1), \ldots, (n(r), D_r)$ appearing in (0) are uniquely determined (up to isomorphism) by A.*

(c) *Conversely, if $n(1), \ldots, n(r)$ are natural numbers and $D_1, \ldots, D_r$ are division algebras over R, then $M_{n(1)}(D_1) \times \cdots \times M_{n(r)}(D_r)$ is a right and left semisimple R-algebra.*

Proof. [P, Theorem 3.5].

An F-algebra A is **central simple** if A is simple and $\mathbf{Z}(A) = F$, where $\mathbf{Z}(A)$ denotes the centre of A. We shall denote by $\Delta(F)$ the class of all finite dimensional central simple F-algebras. If B is a subalgebra, then $\mathrm{C}_A(B)$ denotes the centralizer of B in A. Let A^{op} be the algebra whose additive group is that of A but where the product is defined by $a * b = ba$—that is, * is the product in A^{op}, ba is that in A. Clearly A^{op} is anti-isomorphic to A. We call A^{op} the **reverse** of A.

Unless otherwise stated, all tensor products are taken over F.

Proposition 1.1.2. *Let B and C be subalgebras of the finite dimensional F-algebra A such that $C \subseteq \mathrm{C}_A(B)$ and B is central simple. The following conditions are equivalent.*

(a) $A = BC$.

(b) $\dim_F A = (\dim_F B)(\dim_F C)$.

(c) The inclusion mappings of B and C into A induce an isomorphism $B \otimes C \cong A$.

Proof. [P, Proposition 12.4 a].

Proposition 1.1.3. *Let B and C be F-algebras.*

(a) If $B \otimes C$ is simple, then B and C are simple.

(b) If B is central simple and C is simple, then $B \otimes C$ is simple.

Proof. [P, Lemma 12.4 b].

Proposition 1.1.4. *Let B and C be F-algebras. Denote $B \otimes C$ by A.*

(a) $\mathrm{C}_A(B \otimes F) = \mathbf{Z}(B) \otimes C$.

(b) $\mathbf{Z}(A) = \mathbf{Z}(B) \otimes \mathbf{Z}(C)$.

Proof. [P, Lemma 12.4 c].

Proposition 1.1.5. *Let B and C be central simple F-algebras, and suppose that E/F is a field extension. Then*

(a) $B \otimes C$ is central simple.

(b) $B \otimes E$ is a central simple E-algebra.

(c) B^{op} is central simple.

(d) If $\dim_F B = n < \infty$, then $B^{op} \otimes B \cong M_n(F)$.

Proof. [P, Proposition 12.4 b].

Proposition 1.1.6. *Every finite dimensional central simple F-algebra is separable.*

Proof. [P, Corollary 12.4].

Let E and L be fields that contain F as a subfield. A **compositum** of E and L over F is a triple (K, ϕ, ψ) in which K is a field that contains F, and $\phi : E \to K$, $\psi : L \to K$ are F-algebra homomorphisms such that $K = \phi(E)\psi(L)$. Two such composita (K, ϕ, ψ) and (K', ϕ', ψ') are equivalent if there is an F-algebra homomorphism $\gamma : K \to K'$ such that $\phi' = \gamma\phi$ and $\psi' = \gamma\psi$. In this case, $\gamma(K) = \gamma(\phi(E)\psi(L)) = \phi'(E)\psi'(L) = K'$, so that γ is a field isomorphism. Clearly, equivalence of composita is an equivalence relation.

Lemma 1.1.7. *If E/F is a finite, separable field extension and L/F is a field extension, then $E \otimes_F L = K_1 \times \cdots \times K_r$, where K_i/L is a field extension such that $[E : F] = \sum_{i=1}^{r} [K_i : L]$. Write $1_{E\otimes_F L} = e_1 + \cdots + e_r$, where $e_i = 1_{K_i}$, and define $\phi_i : E \to K_i$, $\psi_i : L \to K_i$ by $\phi_i(x) = e_i(x \otimes 1)$, $\psi_i(y) = e_i(1 \otimes y)$. The triples (K_i, ϕ_i, ψ_i) are pairwise inequivalent composita of E and L over F, and every compositum of K and L over F is equivalent to one of the (K_i, ϕ_i, ψ_i).*

Proof. [P, Lemma 18.1].

Lemma 1.1.8. *Let D be a division algebra over F. If $x \in D$, then there is a subfield E of D such that $x \in E$. If $\dim_F D < \infty$, then the subalgebra $F[x] = \{\Phi(x) : \Phi \in F[\mathrm{x}]\}$ is* *a subfield of D.*

Proof. [P, Lemma 13.1b].

It follows from Lemma 1.1.8 that if $D \in \Delta(F)$ and D is a division algebra, then every subalgebra B of D is also a division algebra. Indeed, if $0 \neq x \in B$, then $x^{-1} \in F[x] \subseteq B$. So the centre of D is a field.

Definition. *Let A and B be elements of $\Delta(F)$. We say $A \sim B$ if for some integers n and m, we have $M_m(A) \cong M_n(B)$.*

By Wedderburn's Theorem, $A \cong M_n(D_1)$ and $B \cong M_m(D_2)$ for some division algebras $D_1, D_2 \in \Delta(F)$. Then $A \sim B$ if and only if $D_1 \cong D_2$. Hence "$\sim$" is an equivalence relation. We say the **matrix size** of A is n.

Let $B(F)$ denote the set of equivalence classes of $\Delta(F)$. If $[A]$ denotes the class of A we define a product in $B(F)$ by $[A][B] = [A \otimes B]$. Proposition 1.1.5 assures us that $B(F)$ is closed under this product. From the properties of tensor product, this product in $B(F)$ is associative and commutative. Finally, $[F]$ acts as a unit element for $B(F)$ and $[A^{op}]$ is the inverse of $[A]$ by Proposition 1.1.5.

Proposition 1.1.9. *$B(F)$ is a torsion abelian group.*

Proof. [H, Theorems 4.1.4 and 4.4.4].

We call $B(F)$ the **Brauer group** of F. Since for any $A \in \Delta(F)$, we have $[A] = [D]$ for a unique (up to isomorphism) division algebra D, this invariant $B(F)$ of the field F measures the division algebras lying above F and having F as its centre.

Proposition 1.1.10. *If $\phi : F \to E$ is a homomorphism of fields, then ϕ induces a group homomorphism $\phi_* : B(F) \to B(E)$ by $\phi_*([A]) = [A \otimes_\phi E]$. The correspondences $F \mapsto B(F)$ and $\phi \mapsto \phi_*$ define a functor from the category of fields to the category of abelian groups.*

Proof. [P, Proposition 12.5c].

The kernel of ϕ_* is called the **relative Brauer group** of E/F and is denoted by $B(E/F)$. Then $A \in B(E/F)$ if and only if $A \otimes E \cong M_n(E)$ as E-algebras.

Let A be an F-algebra. We say that the field extension E/F is a **maximal subfield** of A if there is no subfield L of A such that E is a proper subfield of L.

Proposition 1.1.11. *Let $D \in \Delta(F)$ be a division algebra and E/F be a subfield of D. Then $D \otimes E = M_n(C_D(E))$, where $n = [E : F]$. Moreover, if E/F is a maximal subfield of D, then $D \otimes E = M_n(E)$, where $n = [E : F] = [D : E] = [D : F]^{1/2}$.*

Proof. [R, p.94-96], [H, Corollary 4.2.1], [H, Theorem 4.2.2].

Corollary 1.1.12. *If $A \in \Delta(F)$, then $dim_F A = m^2$ for some $m \in \mathbb{N}$. For any subfield E/F of A, $[E : F]$ divides m.*

The natural number m is called the **degree** of A. It shall be denoted by $\deg(A)$. Explicitly $\deg(A) = (\dim_F A)^{1/2}$.

1.2. Some results about K_0.

The following is a set of basic definitions and concepts relevant to classifying algebras by means of ordered abelian groups and modules.

A **pre-ordered abelian group** is an abelian group G equipped with a particular pre-order relation $\leq$ (i.e., a reflexive, translative relation) which is **translation-invariant** (that is, $x \leq y$ implies $x + z \leq y + z$). The **positive cone** of G is the set $G^+ = \{x \in G \mid x \geq 0\}$. A **partially ordered abelian group** is an abelian group equipped with a particular translation-invariant **partial** order relation (i.e., as above and also antisymmetric).

Given pre-ordered abelian groups G and H, a **positive homomorphism** from G to H is any group homomorphism $f : G \to H$ such that $f(G^+) \subseteq H^+$. Equivalently, a group homomorphism $f : G \to H$ is positive if and only if f preserves the pre-ordering.

An **order unit** in a pre-ordered abelian group G is an element $u \geq 0$ such that for any $x \in G$ there is a positive integer n for which $x \leq nu$. Given pre-ordered abelian groups G and H and order units $u \in G$ and $v \in H$, a **normalized positive homomorphism** from (G, u) to (H, v) is any positive homomorphism $f : G \to H$ such that $f(u) = v$.

By **the category of pre-ordered abelian groups with order unit**, we mean the category whose objects are all pairs (G, u), where G is a pre-ordered abelian group and u is an order unit in G, and whose morphisms are all normalized positive homomorphisms between these objects. We shall denote this category by $\mathcal{P}$.

Given any objects (G, u) and (H, v) in $\mathcal{P}$, we shall use the phrase "map from (G, u) to (H, v)" only to refer to morphisms in $\mathcal{P}$.

Given a ring R with unit, the **Grothendieck group** $K_0(R)$ is an abelian group consisting of expression $[A] - [B]$ where A and B are finitely generated projective right R-modules. Two such expressions $[A] - [B]$ and $[C] - [D]$ are equal in $K_0(R)$ if and only if $A \oplus D$ and $B \oplus C$ are **stably isomorphic**, that is, $A \oplus D \oplus F \cong B \oplus C \oplus F$ for some finitely generated projective right R-module F. The sum of the expressions $[A] - [B]$ and $[C] - [D]$ is the expression $[A \oplus C] - [B \oplus D]$. The set of all expressions $[A]$ (that is, $[A] - [0]$), for finitely generated projective right R-modules A, is denoted by $K_0(R)^+$. Given any $x, y \in K_0(R)$, we define $x \leq y$ if and only if $y - x \in K_0(R)^+$. Then $(K_0(R), \leq)$ is a pre-ordered abelian group, and $[R]$ is an order unit in $K_0(R)$. (See [G1, Chapter 7]). Since the finitely generated projective right R-modules are exactly the direct summands of the free right R-modules of finite rank, they are just the right R-modules of the form eR^n, where n is a positive integer and e is an idempotent $n \times n$ matrix over R. Consequently, $K_0(R)$ may be constructed using idempotent matrices in place of finitely generated projective modules, as in [G2, Chapter 18] .

Given a unital ring homomorphism $\phi : R \to S$, we make S into a left R-module via

ϕ, i.e., $r * s := \phi(r)s$ for $r \in R,\ s \in S$, and then use the functor $(-) \otimes_R S$ from right R-modules to right S-modules. This in turn induces a normalized positive homomorphism $K_0(\phi) : (K_0(R), [R]) \to (K_0(S), [S])$, where

$$K_0(\phi)([A] - [B]) = [A \otimes_R S] - [B \otimes_R S]$$

for all finitely generated projective right R-modules A and B. The assignments $R \mapsto (K_0(R), [R])$ and $\phi \mapsto K_0(\phi)$ define a functor K_0 from the category of rings with unit to the category $\mathcal{P}$.

Proposition 1.2.1. *Arbitrary products and direct limits exist in $\mathcal{P}$. The functor K_0 preserves all finite products and all direct limits.*

Proof. [G1, Propositions 15.11, 15.13]; [G2, Propositions 18.6, 18.7].

Recall that over a (von Neumann) regular ring R, every finitely generated projective right module is a finite direct sum of submodules isomorphic to principle right ideals eR, for idempotents $e \in R$ [G1, Proposition 2.6]. Consequently, $K_0(R)$ is generated (as a group) by the set $\{[eR] \mid e \text{ is an idempotent in } R\}$, and $K_0(R)^+$ is generated (as a semigroup with zero) by the same set.

A **unit regular ring** is a unital ring R in which for each $x \in R$ there exists a unit (invertible element) $u \in R$ such that $xux = x$. Over a unit regular ring R, finitely generated projective right modules cancel from direct sums [G1, Theorem 4.5]. Consequently, given finitely generated projective right R-modules A, B, C, D, we have $[A] - [B] = [C] - [D]$ in $K_0(R)$ if and only if $A \oplus D \cong B \oplus [C]$, and we have $[A] - [B] \leq [C] - [D]$ in $K_0(R)$ if and only if $A \oplus D$ embeds in $B \oplus C$ [G1, Proposition 15.2]. Moreover, $K_0(R)$ is partially ordered [G1, Proposition 15.2].

A **matricial algebra** over a field F is any F-algebra that is isomorphic to a finite direct product of full matrix algebras over F. (In case F is algebraically closed, the matricial F-algebras are exactly the finite dimensional semisimple F-algebras). An **ultramatricial F-algebra** is any F-algebra that is isomorphic to a direct limit of a (countable) sequence of matricial F-algebras (in the category of F-algebras). Any ultramatricial F-algebra is a directed union of matricial F-subalgebras, i.e., the union of an ascending sequence $R_1 \subseteq R_2 \subseteq \ldots$ of matricial subalgebras.

Observe that the field F is unit regular, and hence so is every full matrix algebra over F, by [G1, Corollary 4.7]. As unit regularity is preserved in direct products and unital direct limits, all unital ultramatricial F-algebras are unit regular.

Proposition 1.2.2. *Let R be a matricial algebra over a field F, and let S be a unit regular F-algebra.*

(a) Given any map $f : (K_0(R), [R]) \to (K_0(S), [S])$, there exists a unital F-algebra map $\phi : R \to S$ such that $K_0(\phi) = f$.

(b) Let $\phi, \psi : R \to S$ be unital F-algebra maps. Then $K_0(\phi) = K_0(\psi)$ if and only if there exists an inner automorphism θ of S such that $\theta\phi = \psi$.

Proof. [G1, Lemma 15.23].

Theorem 1.2.3. [Elliott] *Let R and S be unital ultramatricial algebras over a field F. Then $R \cong S$ (as F-algebra) if and only if $(K_0(R), [R]) \cong (K_0(S), [S])$ in $\mathcal{P}$. Moreover, if $f : (K_0(R), [R]) \to (K_0(S), [S])$ is any isomorphism in $\mathcal{P}$, there exists an F-algebra isomorphism $\phi : R \to S$ such that $K_0(\phi) = f$.*

Proof. [E, Theorem 4.3].

Proposition 1.2.4. *Let $R = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where D_i's are division rings, then $(K_0(R), [R]) \cong (\mathbb{Z}, n_1) \times \cdots \times (\mathbb{Z}, n_r)$ in the category $\mathcal{P}$.*

Proof. [G1, Lemma 15.22].

A partially ordered abelian group G is **directed** if G, as a partially ordered set, is directed (upward or downward, which are equivalent). Equivalently, G is directed if and only if G is generated (as a group) by G^+ [G1, Proposition 15.16]. The group G is

unperforated provided that whenever $x \in G$ satisfies $nx \geq 0$ for some positive integer n, then $x \geq 0$. Note that any unperforated partially ordered abelian group is torsion-free. The group G satisfies the **Riesz interpolation property** provided that whenever $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for all $i, j = 1, 2$, there exists $z \in G$ satisfying $x_i \leq z \leq y_j$ for all $i, j = 1, 2$. (In other words, the intersection of two overlapped closed intervals is nonempty).

An **interpolation group** is a partially ordered abelian group satisfying the Riesz interpolation property. A **dimension group** is a directed, unperforated, interpolation group.

Proposition 1.2.5. [Elliott; Effros, Handelman, Shen] *Let F be a field, and let (G, u) be an object in $\mathcal{P}$. Then $(G, u) \cong (K_0(R), [R])$ for some unital ultramatricial F-algebra R if and only if G is a countable dimension group.*

Proof. [E, Theorems 5.1, 5.5]; [EHS, Theorem 2.2].

1.3. Embedding into central simple algebras.

In the next two Propositions, we let L/F be a field extension of finite index and A/L be central simple. If A and B are unital F-algebras, then an embedding of A into B is a unital F-algebra map. We also say "A is embeddable in B".

Proposition 1.3.1. *Suppose that $A \cong M_n(\Delta)$, where Δ/L is central simple. Let A/L be embeddable in $M_m(D)$, where D is a F-division ring. Then $n \mid m$ and Δ is embeddable in $M_k(D)$, where $m = nk$.*

Proof. [FSS, Proposition 2].

Proposition 1.3.2. *If $[A] = [D \otimes_F L]$ in $B(L)$, where D is a F-division algebra, then there is an integer w such that A/L is (unitally) embeddable in $B = M_w(D)$ so that A*

$= \mathbf{C}_B(L)$ *and* $deg(B) = deg(A) \cdot [L : F]$.

Proof. [FSS, Proposition 5].

Here is a slight extension of the Noether-Skolem Theorem.

Theorem 1.3.3. *Let R and S be finite dimensional simple unital algebras over a field F, and let $\phi_1, \phi_2 : R \to S$ be unital F-algebra maps. If either R or S is central simple, then there exists an inner automorphism θ of S such that $\phi_2 = \theta\phi_1$.*

Proof. [GH, Theorem 4.3].

Proposition 1.3.4. *Let θ be an inner automorphism of a ring R. Then $K_0(\theta)$ is the identity map on $K_0(R)$.*

Proof. Folklore.

Proposition 1.3.5. *Let $\phi : M_{k_1}(D_1) \times \cdots \times M_{k_n}(D_n) \to M_t(D)$ be a unital F-algebra map, where D_i's and D are division algebras. By Proposition 1.2.4, $K_0(\phi)$ is (isomorphic to) the map*

$$(\mathbf{Z}, k_1) \times \cdots \times (\mathbf{Z}, k_n) \xrightarrow{(a_1,\ldots,a_n)} (\mathbf{Z}, t)$$

where a_i's are non-negative integers. Then there are unital F-algebra maps $\psi_i : M_{k_i}(D_i) \to M_{a_ik_i}(D)$ and an inner automorphism θ of $M_t(D)$ such that

$$\theta\phi(x_1, \ldots, x_n) = diag(\psi_1(x_1), \ldots, \psi_n(x_n)).$$

Proof. Let $R = M_{k_1}(D_1) \times \cdots \times M_{k_n}(D_n)$. There are central orthogonal idempotents $e_1, \ldots, e_n$ in R such that $e_1 + \cdots + e_n = 1$ and $e_iR = M_{k_i}(D_i)$. Let $f_i = \phi(e_i)$. Since $\{f_1, \ldots, f_n\}$ is a set of orthogonal idempotents summing to 1, there exists an inner automorphism θ of $M_t(D)$ such that $\theta(f_i)$'s are the matrix diag$(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, where 1 appears a_ik_i times. Let $\psi_i : e_iR \to \theta(f_i)M_t(D)\theta(f_i)$ be the restriction of $\theta\phi$ to e_iR. ■

Corollary 1.3.6. *Let D in Proposition 1.3.5 be central and ϕ' be another F-algebra map. Then $K_0(\phi) = K_0(\phi')$ if and only if there exists an inner automorphism θ of $M_t(D)$ such that $\phi = \theta\phi'$.*

Proof. By the proof of [GH, Proposition 4.8], we can assume $\phi(e_i) = \phi'(e_i) = f_i$ because $K_0(\phi) = K_0(\phi')$. Let $\psi_i' : e_iR \to \theta(f_i)M_t(D)\theta(f_i)$ be the restriction of $\theta\phi'$ to e_iR. Then by Theorem 1.3.3, there exist inner automorphisms θ_i of $\theta(f_i)M_t(D)\theta(f_i)$ such that $\psi_i = \theta_i\psi_i'$. So there are units $u_i \in \theta(f_i)M_t(D)\theta(f_i)$ such that $\theta_i(s) = u_isu_i^{-1}$ for all $s \in \theta(f_i)M_t(D)\theta(f_i)$. Then $u = u_1 + \cdots + u_n$ is a unit of $M_t(D)$, the rule $\theta(s) = usu^{-1}$ defines an inner automorphism θ of $M_t(D)$, and $\psi_i = \theta\psi_i'$ for all $i = 1, \ldots, n$. ∎

SUBLEMMA 1.3.7. *If A is central simple over F and $e \neq 0$ is an idempotent in A then in B(F), [A] = [eAe] (here we are identifying F and Fe).*

Proof. [H, Sublemma 4.4.3].

Chapter 2

Homomorphisms Between Finite Dimensional Simple Algebras

The aim of this chapter is to classify all (unital) F-algebra maps $\phi : M_n(D) \to M_t(D')$ up to an inner automorphism of $M_t(D')$, where D and D' are division F-algebras with centres E and E' respectively. The main results are in Propositions 2.2, 2.3, and 2.4.4.

In Proposition 2.2, we assume that D' is a central division F-algebra and we find all possible positive integers t dividing n so that ϕ exists (uniquely by Theorem 1.3.3). In Proposition 2.3, we no longer assume that D' is central but rather that the centres of D and D' are separable and we prove that ϕ is completely and explicitly determined by $K_0(\phi \otimes_F \mathrm{id}_K)$. In Proposition 2.4.4, we prove that given a pair of homomorphisms (f, g) such that the diagram below commutes, there exists an F-algebra map $\phi : D \to M_\bullet(D')$ such that $K_0(\phi \otimes_F \mathrm{id}_K) = g$, and $K_0(\phi \otimes_F \mathrm{id}_{D^{op}}) = f$.

$$\begin{array}{ccc}
(K_0(D \otimes_F K), [D \otimes_F K]) & \xrightarrow{K_0(-)} & (K_0(D \otimes_F D^{op}), [D \otimes_F D^{op}]) \\
g \downarrow & & f \downarrow \\
(K_0(D' \otimes_F K), [D' \otimes_F K]) & \xrightarrow{K_0(-')} & (K_0(D' \otimes_F D^{op}), [D' \otimes_F D^{op}]),
\end{array}$$

Propositions 2.3 and 2.4.4 will play crucial roles in achieving our objective, which is to find a complete invariant for the category $\mathcal{R}$ of direct limits of finite direct products of matrix algebras over division F-algebras.

We shall use matrix notation, when convenient, for (positive) homomorphisms into or out of direct products of (pre-ordered) abelian groups, and we shall freely use the notation $(a, b, c, \ldots)^{tr}$ (for column vectors) and $\operatorname{diag}(a, b, c, \ldots)$ (for diagonal matrices). Since our homomorphisms act on the left of their arguments, elements of direct products must be thought of as column vectors, and matrices must be multiplied on the left of column vectors. For instance, given an abelian group G, the 1×2 matrix $(1, 1)$ represents the sum map $G \times G \to G$, while the 2×1 matrix $(1, 1)^{tr}$ represents the diagonal map $G \to G \times G$.

2.1. Preliminary results.

In this section, we shall investigate the group action of $\operatorname{Gal}(K/F)$ on $K_0(K \otimes_F D)$, and the map $\lambda \otimes_F \operatorname{id}_D$, where $\lambda : K \to M_{[K:Z(D)]}(D)$ is the composition of the left regular map $K \to \operatorname{End}_{Z(D)}(K) \cong M_{[K:Z(D)]}(Z(D))$ and the inclusion map $M_\bullet(Z(D)) \to M_\bullet(D)$. These results are in Lemmas 2.1.1 and 2.1.2 respectively.

We introduce more notation that will be used throughout this work.

Notation.

(i) Let $\phi, \psi : R \to S$ be F-algebra maps. We write $\phi \sim \psi$ to mean "there exists an inner automorphism θ of S such that $\theta\phi = \psi$".

(ii) If R and X are F-algebras, then we use R^X to denote $R \otimes_F X$. If $\phi : R \to S$ is an F-map, then we use ϕ^X to denote $\phi \otimes_F \operatorname{id}_X : R^X \to S^X$ which is defined by $x \otimes y \mapsto \phi(x) \otimes y$.

Definition. Let E/F and E'/F be *separable* field extensions and K/F be their Galois closure. Let $\sigma, \sigma' \in \operatorname{Gal}(K/F)$. Then the compositum of $\sigma(E)$ and $\sigma'(E')$ over F, denoted by $\sigma(E)\sigma'(E')$, is both a (left) E-algebra with the scalar operation

$$e.r := \sigma(e)r$$

and a (right) E'-algebra with the scalar operation

$$r.e' := r\sigma(e')$$

where $e \in E$, $e' \in E'$, and $r \in \sigma(E)\sigma'(E')$. As a result, if D is an F-algebra with centre E, then $D \otimes_E \sigma(E)\sigma'(E')$ is also both a E-algebra and a E'-algebra with the property

$$e \otimes r = 1 \otimes \sigma(e)r$$

where $e \in E$ and $r \in \sigma(E)\sigma'(E')$.

Lemma 2.1.1. *Let D be a division F-algebra with separable centre E and let K/F be a Galois extension containing E/F with $[E : F] = k$. Let $\tau \in Gal(K/F)$ and let $\tau^D : K^D \to K^D$ be the map defined by $x \otimes y \mapsto \tau(x) \otimes y$. Then with the isomorphism*

$$\begin{aligned} K^D &\cong (K \otimes_F E) \otimes_E D \\ &\cong (KE \times \cdots \times \sigma_m(K)E) \otimes_E D \\ &\cong KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D \end{aligned}$$

τ^D induces an (right) E-algebra isomorphism with the same name

$$KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D \xrightarrow{\tau^D} KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D\ ,$$

and $K_0(\tau^D)([e_i]) = [e_j]$ if and only if $\sigma_j\tau\sigma_i^{-1} \in Gal(K/E)$, where $\sigma_* \in Gal(K/F)$ and e_*'s are central idempotents such that $e_*(D \otimes_F K) \cong \sigma_*(K)E \otimes_E D$ for $* = i, j$.

In other words, $Gal(K/F)$ permutes $K_0(D^K)$ the same way it acts on the set of left cosets of $Gal(K/E)$ by left multiplication.

Proof. By Lemma 1.1.7, we have the isomorphism

$$K \otimes_F E \cong KE \times \cdots \times \sigma_m(K)E$$

defined by $x \otimes 1 \mapsto (x, \ldots, \sigma_m(x))$ and $1 \otimes y \mapsto (y, \ldots, y)$, where $\{[1], \ldots, [\sigma_m]\}$ is the set of all right cosets of $\mathrm{Gal}(K/E)$ in $\mathrm{Gal}(K/F)$. This induces the E-algebra isomorphism

$$K \otimes_F D \cong (K \otimes_F E) \otimes_E D \cong KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D$$

defined by $x \otimes 1 \mapsto (x \otimes 1, \ldots, \sigma_m(x) \otimes 1)$ and $1 \otimes y \mapsto (1 \otimes y, \ldots, 1 \otimes y)$. Hence τ^D maps $(x \otimes 1, \ldots, \sigma_m(x) \otimes 1) \mapsto (\tau(x) \otimes 1, \ldots, \sigma_m\tau(x) \otimes 1)$. As $\tau^D(e_i)$ is also a central idempotent of K^D, there is an j such that the composite map

$$\begin{aligned}\sigma_i(K)E \otimes_E D &\xrightarrow{q} KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D\\ &\xrightarrow{\tau^D} KE \otimes_E D \times \cdots \times \sigma_m(K)E \otimes_E D\\ &\xrightarrow{p} \sigma_j(K)E \otimes_E D\end{aligned}$$

is a unital (right) E-algebra map, where q is the inclusion map and p is the projection map. Then under this map,

$$\sigma_i(x) \otimes 1 \mapsto \sigma_j\tau(x) \otimes 1 \qquad \forall x \in K.$$

On the other hand, as an (right) E-algebra map,

$$x \otimes 1 = 1 \otimes x = (1 \otimes 1).x \mapsto (1 \otimes 1).x = 1 \otimes x = x \otimes 1 \qquad \forall x \in E.$$

Therefore,

$$x \otimes 1 = \sigma_i(\sigma^{-1}(x)) \otimes 1 \mapsto \sigma_j\tau(\sigma^{-1}(x)) \otimes 1 = x \otimes 1 \qquad \forall x \in E.$$

Hence $\sigma_j\tau\sigma_i^{-1} \in \mathrm{Gal}(K/E)$. Since this implies $\sigma_j\tau(K)E$ and $\sigma_i(K)E$ are equivalent composita of K and E over F, $\tau^D(e_i) = e_j$ where e_j is a central idempotent of K^D such that $e_jK^D \cong \sigma_j(K)E \otimes_E D$. ■

Let D and D' be division F-algebras with separable centres E and E' respectively and let K/F be any Galois extension containing them. Let $z = [K : E']$ and let $\lambda : K \to M_z(D')$ be the composition of the (left) regular algebra map $K \to M_z(E')$ and the inclusion map $M_z(E') \hookrightarrow M_z(D')$. Let $\lambda^D : K^D \to M_z(D')^D$ be defined by $x \otimes y \mapsto \lambda(x) \otimes y$ and let $\{1, \sigma_2, \ldots, \sigma_k\} \subseteq \{1, \sigma_2, \ldots, \sigma_m\}$ such that $EE', E\sigma_2(E'), \ldots, E\sigma_k(E')$ are all nonequivalent composita of E and E' over F. Then λ^D induces the (left) E', (right) E-algebra map with the same name

$$KE\otimes_E D\times\cdots\times K\sigma_m(E)\otimes_E D \xrightarrow{\lambda^D} M_z(D'\otimes_{E'}E'E)\otimes_E D\times\cdots\times M_z(D'\otimes_{E'}E'\sigma_k(E))\otimes_E D.$$

Lemma 2.1.2. *Let d and s_i be the matrix sizes of $D \otimes_E K$ and $D' \otimes_{E'} E'\sigma_i(E) \otimes_E D$ respectively. Then the matrix representing $K_0(\lambda^D)$ is determined by*

$$K_0(\lambda^D)_{i,j} = \begin{cases} \frac{s_i[K:E'\sigma_i(E)]}{d} & \text{if } E'\sigma_i(E) \sim E'\sigma_j(E) \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq m$. In particular, if $E = E'$, the first row is $(s_1[K:E]/d, 0, \ldots, 0)$.

Proof. We have the (left) K-algebra isomorphism

$$\begin{aligned} K \otimes_F E &\cong K \otimes_{E'} (E' \otimes_F E) \\ &\cong K \otimes_{E'} (E'E \times E'\sigma_2(E) \times \cdots \times E'\sigma_k(E)) \\ &\cong K \otimes_{E'} E'E \times K \otimes_{E'} E'\sigma_2(E) \times \cdots \times K \otimes_{E'} E'\sigma_k(E). \end{aligned}$$

For each $i = 1, \ldots, k$,

$$K \otimes_{E'} E'\sigma_i(E) \cong K\sigma_i(E) \times K\tau_{2,i}\sigma_i(E) \times \cdots \times K\tau_{n(i),i}\sigma_i(E)$$

as (left) K-algebras , where $n(i) = [E'\sigma_i(E) : E']$ and $\tau_{j(i),i} \in \mathrm{Gal}(K/E')$ for all $j(i) = 1, \ldots, n(i)$. Hence $E'\tau_{j(i),i}\sigma_i(E)$ and $E'\tau_{k(i'),i'}\sigma_{i'}(E)$ are equivalent composita of E and E' over F if and only if $i = i'$. (*)

With the (left) E', (right) E-algebra isomorphisms

$$K^D \cong K^E \otimes_E D \quad \text{and} \quad D'^D \cong D' \otimes_{E'} (E' \otimes_F E) \otimes_E D,$$

λ^D induces the (left) E', (right) E-algebra map

$$\begin{aligned} &KE \otimes_E D \times \cdots \times K\sigma_m(E) \otimes_E D \\ &\cong (K \otimes_{E'} E'E) \otimes_E D \times \cdots \times (K \otimes_{E'} E'\sigma_k(E)) \otimes_E D \\ &\xrightarrow{\lambda^D} M_z(D' \otimes_{E'} E'E) \otimes_E D \times \cdots \times M_z(D' \otimes_{E'} E'\sigma_k(E)) \otimes_E D. \end{aligned}$$

Let e_i be the central idempotent of K^D satisfying

$$e_i K^D \cong (K \otimes_{E'} E'\sigma_i(E)) \otimes_E D.$$

Then $\lambda^D(e_i) = \sum x_j.\mathrm{I}_z \otimes y_j$, where I_z denotes the $z \times z$ identity matrix. Hence $\lambda^D(e_i)$ is the central idempotent of $M_z(D')^D$ such that

$$\lambda^D(e_i) M_z(D')^D \cong M_z(D' \otimes_{E'} E'\sigma_i(E)) \otimes_E D.$$

Then since λ^D is one-to-one and by (*), $K_0(\lambda^D)$ is a matrix such that the (i,j)-entry equals a strictly positive integer if $E'\sigma_i(E)$, $E'\sigma_j(E)$ are equivalent composita, and equals 0 otherwise, for $1 \leq i \leq k$ and $1 \leq j \leq m$.

Consider the restriction of λ^D to the map

$$K \otimes_{E'} (E'\sigma_i(E)) \otimes_E D) \to M_z(D') \otimes_{E'} (E'\sigma_i(E) \otimes_E D).$$

This is the tensor product of the E'-algebra map $\lambda' : K \to M_z(D')$ and the identity map of $E'\sigma_i(E)) \otimes_E D$ denoted by id. Then $\forall \tau \in \mathrm{Gal(K/E')}$, we have $\lambda'\tau \sim \lambda'$ by Theorem 1.3.3. Hence

$$(\lambda' \otimes_{E'} \mathrm{id})(\tau \otimes_{E'} \mathrm{id}) = (\lambda'\tau) \otimes_{E'} \mathrm{id} \sim \lambda' \otimes_{E'} id.$$

Then by Lemma 2.1.1, K_0 of this restriction map is a row of equal entries. Since $K \otimes_{E'} (E'\sigma_i(E)) \otimes_E D)$ splits into $[E'\sigma_i(E) : E']$ matrix algebras $D \otimes_E K$ and $z = [K : E']$, the (i,j)-entry of λ^D is $s_i[K : E'\sigma_i(E)]/d$.

Finally, if $E = E'$ then EE and $E\sigma_j(E)$ are equivalent composita if and only if $\sigma_j \in \mathrm{Gal}(K/E)$, i.e., $j = 1$. ■

Let D and D' be division F-algebras with separable centres E and E' respectively and let K/F be any Galois extension containing them. Let $r = \deg(D')^2$ and let $\mu : D' \to M_r(K)$ be the composition of the E'-algebra map $D' \hookrightarrow D' \otimes_{E'} D'^{op} \cong M_r(E')$ and the inclusion map $M_r(E') \hookrightarrow M_r(K)$. Then $\mu^D : D'^D \to M_r(K)^D$, defined by $x \otimes y \mapsto \mu(x) \otimes y$ induces the E, E'-algebra map with the same name

$$D'\otimes_{E'}E'E\otimes_E D\times\cdots\times D'\otimes_{E'}E'\sigma_k(E)\otimes_E D \xrightarrow{\mu^D} M_r(K)E\otimes_E D\times\cdots\times M_r(K)\sigma_m(E)\otimes_E D.$$

Similar to Lemma 2.1.2, we have

Lemma 2.1.3. *Let d and s_i be the matrix sizes of $D \otimes_E K$ and $D' \otimes_{E'} E'\sigma_i(E) \otimes_E D$ respectively. Then the matrix representing $K_0(\mu^D)$ is determined by*

$$K_0(\mu^D)_{i,j} = \begin{cases} \frac{dr}{s_i} & \text{if } E'\sigma_i(E) \sim E'\sigma_j(E) \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq m$. In particular, if $E = E'$, the first column is $(dr/s_1, 0, \ldots, 0)$.

2.2. Embedding into a central simple algebra.

Proposition 2.2. *Let D be a finite-dimensional division F-algebra with centre E (which need not be separable) and let D' be a finite-dimensional central division F-algebra. Define D'', a central division E-algebra, by $D^{op} \otimes_F D' \cong M_s(D'')$, and let $d = deg(D)$. Then s divides $d^2[E:F]$ and there exists an F-embedding $f : D \to M_t(D')$ if and only if $d^2[E:F]/s$ divides t. In this case, $f \sim diag(\phi, \ldots, \phi)$, where ϕ is the minimal embedding*

$$D \hookrightarrow D \otimes_E D'' \hookrightarrow M_{d^2[E:F]/s}(D').$$

Proof. By taking the tensor product $D \otimes_E$ on both sides of

$$D^{op} \otimes_F D' \cong M_s(D'')$$

we obtain

$$M_{d^2}(D' \otimes_F E) \cong M_s(D'' \otimes_E D).$$

Then since $E \hookrightarrow M_{[E:F]}(F)$ as F-algebras, we have

$$M_s(D'' \otimes_E D) \hookrightarrow M_{d^2[E:F]}(D').$$

Hence s divides $d^2[E:F]$ and there exists an F-embedding

$$D \hookrightarrow D'' \otimes_E D \hookrightarrow M_{d^2[E:F]/s}(D').$$

Conversely, let $M = (D')^t$ and $A = D^{op} \otimes_F D'$. If we have an embedding $D \hookrightarrow M_t(D')$, then

$$D^{op} \hookrightarrow M_t(D')^{op} = \mathrm{End}_{D'}(M)$$

(see [Jacobson, Basic Algebra II, p.224]). Hence M is also a left A-module via $(d \otimes d')m = dd'm$. Let $S \cong (D'')^s$, the minimal left ideal of A. Then $M \cong (S)^r$. From

$$\dim_F S = s\dim_F D'' \quad \text{and} \quad \dim_F D \dim_F D' = s^2 \dim_F D''$$

and $M \cong (S)^r \cong (D')^t$, we obtain

$$t\dim_F D' = r\dim_F S = rs\dim_F D'' = r\frac{\dim_F D \dim_F D'}{s}.$$

Hence

$$t = r\frac{\dim_F D}{s} = r\frac{d^2[E:F]}{s}. \blacksquare$$

Example 1. The (left) regular embedding $D \to M_{[D:F]}(F)$ is the minimal embedding, for any finite-dimensional division F-algebra D.

Example 2. Let D be a finite-dimensional central division F-algebra, let E/F be a field extension, and let $D \otimes_F E \cong M_s(D')$. Then s divides $[E:F]$ and E is embeddable in D if and only if $s = [E:F]$ — in this case, $D \otimes_F E \cong M_{[E:F]}(\mathrm{C}_D(E))$. As a result, E is a maximal subfield of D if and only if $D \otimes_F E \cong M_{[E:F]}(E)$.

Proof. By applying Proposition 2.2 with $d = 1$ and $s = [E:F]$. $\blacksquare$

Example 3. Let D and D' be finite-dimensional central division F-algebras and let $D^{op} \otimes_F D' \cong M_s(D'')$. Then s divides both $\deg(D)^2$ and $\deg(D')^2$, and D is embeddable in D' if and only if $s = \deg(D)^2$. In this case, $D'' \cong \mathrm{C}_{D'}(D)$ and $D' \cong D \otimes_F D''$.

Proof. The first statement is obtained by applying Proposition 2.2 with $E = F$ while the second statement is obtained by tensoring both sides of $D^{op} \otimes_F D' \cong M_{deg(D)^2}(D'')$ with D, we obtain $D' \cong D \otimes_F D''$. $\blacksquare$

2.3. Embedding into a simple algebra.

From now on we consider the case

$$\begin{array}{ccccc} D & & K & & D' \\ & \backslash & / \quad \backslash & / & \\ \mathrm{Z}(D) = E & & & & E' = \mathrm{Z}(D') \\ & & \backslash \; / & & \\ & & F & & \end{array}$$

where K/F is a Galois extension containing separable E/F and E'/F. Observe that the composite map

$$D' \xrightarrow{\mu'} D'^{E'} \cong D' \times \cdots \xrightarrow{\pi} D'$$

is the identity map, where $\mu'(x) = x \otimes 1$, and π is the projection map. Hence

$$\phi = \pi\mu'\phi = \pi\phi^{E'}\mu$$

because $\mu'\phi = \phi^{E'}\mu$, where $\mu : D \to D^{E'}$ is defined by $\mu(x) = x \otimes 1$. Therefore ϕ is completely determined by $\pi\phi^{E'}$. Since this map is an E'-algebra map into a simple central E'-algebra, by Lemma 2.2 it can be recovered from its induced K_0 map, which is embedded in $K_0(\phi^{E'})$. Therefore,

Proposition 2.3. *Let D and D' be division F-algebras with separable centres E and E' respectively. Then there is a unital F-algebra map $\phi : M_n(D) \to M_t(D')$ if and only if there are nonnegative integers $a_1, a_2, \ldots, a_k$ such that*

$$t = n.\deg(D)^2(a_1[EE' : E']/s_1 + \cdots + a_k[\sigma_k(E)E' : E']/s_k)$$

where $EE', \sigma_2(E)E', \ldots, \sigma_k(E)E'$ are all pairwise inequivalent composita of E and E' over F and s_i's are the matrix sizes of $D' \otimes_{E'} E'\sigma_i(E) \otimes_E D^{op}$.

In fact, there exists an inner automorphism θ of $M_t(D')$ such that

$$\theta\phi(x) = \begin{pmatrix} \psi_1(x\otimes 1) & & & & & \\ & \ddots & & & & \\ & & \psi_1(x\otimes 1) & & & \\ & & & \ddots & & \\ & & & & \psi_k(x\otimes 1) & \\ & & & & & \ddots \\ & & & & & & \psi_k(x\otimes 1) \end{pmatrix} \begin{matrix} \} a_1 \text{ times} \\ \\ a_k \text{ times} \{ \end{matrix}$$

where ψ_i's are the unique (up to an inner automorphism) E'-algebra maps $M_n(D \otimes_E \sigma_i(E)E') \to M_{n.r_i}(D')$ and

$$r_i = \frac{\deg(D)^2.[\sigma_i(E)E' : E']}{s_i}.$$

Consequently, if $\phi, \varphi : M_n(D) \to M_t(D')$ are unital F-algebra maps, then the following conditions are equivalent

(a) $\phi \sim \varphi$,

(b) $K_0(\phi^{E'}) = K_0(\varphi^{E'})$,

(c) $K_0(\phi^K) = K_0(\varphi^K)$ *for any Galois extension* K/F *containing* E *and* E'.

Proof. From the commutative diagram

$$\begin{array}{ccc} M_n(D) & \xrightarrow{\mu} & M_n(D)^{E'} \\ \phi\big\downarrow & & \phi^{E'}\big\downarrow \\ M_t(D') & \xrightarrow{\mu'} & M_t(D')^{E'} \end{array}$$

where μ and μ' are the maps $x \mapsto x \otimes_F 1$, we have the commutative diagram

$$\begin{array}{ccc} M_n(D) & \xrightarrow{\nu} & M_n(D \otimes_E EE') \times \cdots \times M_n(D \otimes_E \sigma_k(E)E') \\ \phi\big\downarrow & & \psi\big\downarrow \\ M_t(D') & \xrightarrow{\nu'} & M_t(D') \times \cdots \times M_t(D' \otimes_{E'} E'\tau_p(E')) \end{array}$$

where $\nu(x) = (x \otimes_E 1, \ldots, x \otimes_E 1)$, $\nu'(x) = (x, x \otimes_{E'} 1, \ldots, x \otimes_{E'} 1)$, and ψ is the E'-algebra map derived from $\phi^{E'}$.

Let $\pi : M_t(D') \times \cdots \times M_t(D' \otimes_{E'} E'\tau_p(E')) \to M_t(D')$ be the projection map. By Proposition 1.3.5 and Proposition 2.2, there exist nonnegative integers $a_1, \ldots, a_k$ such that

$$\pi\psi \sim \operatorname{diag}(a_1.\psi_1, \ldots, a_k.\psi_k)$$

where $a_i.\psi_i := \operatorname{diag}(\overbrace{\psi_i, \cdots, \psi_i}^{a_i \text{ times}})$ and ψ_i's are E'-algebra maps $M_n(D \otimes_E \sigma_i(E)E') \to M_{n.r_i}(D')$, with

$$r_i = \frac{\deg(D)^2.[\sigma_i(E)E' : E']}{s_i}. \tag{1}$$

Here s_i is defined by $D' \otimes_{E'} E'\sigma_i(E) \otimes_E D^{op} \cong M_{s_i}(T_i)$, where T_i is a division $\sigma_i(E)E'$-algebra. Then we have

$$\begin{aligned} \phi(x) &= \pi\nu'\phi(x) \\ &= \pi\psi\nu(x) \\ &= \pi\psi(x \otimes 1, \ldots, x \otimes 1) \\ &\sim \operatorname{diag}(a_1.\psi_1(x \otimes 1), \ldots, a_k.\psi_k(x \otimes 1)) \ . \end{aligned} \tag{2}$$

Let t_i be the matrix size of $D \otimes_E \sigma_i(E)E'$. Then $K_0(\pi\psi)$, which is the first row of $K_0(\psi)$, is

$$(a_1 \deg(D)^2[EE' : E']/s_1t_1, \ldots, a_k \deg(D)^2[\sigma_k(E)E' : E']/s_kt_k) \tag{3}$$

and

$$t = n.\deg(D)^2(a_1[EE' : E']/s_1 + \cdots + a_k[\sigma_k(E)E' : E']/s_k). \tag{4}$$

As a result, the equivalence of (a) and (b) is clear. Now (a) certainly implies (c). So we only need to prove that (c) implies (b). Let consider the commutative diagram

$$\begin{array}{ccc} M_n(D)^{E'} & \longrightarrow & M_n(D)^K \\ \phi^{E'}, \varphi^{E'} \Big\downarrow & & \phi^K = \varphi^K \Big\downarrow \\ M_t(D')^{E'} & \xrightarrow{\iota} & M_t(D')^K \end{array}$$

where $\iota(x \otimes y) = x \otimes y$. By Lemma 2.1.3, $K_0(\iota)$ is one-to-one. Therefore if $K_0(\phi^K) = K_0(\varphi^K)$, then $K_0(\phi^{E'}) = K_0(\varphi^{E'})$. ■

2.4. $K_0(D^K)$ as an ordered $\mathbb{Z}Gal(K/F)$-module.

Let D, D', K, and ϕ be as in Proposition 2.3. Then it is easily seen that with the action of $\mathrm{Gal}(K/F)$ on $K_0(D^K)$ as in Lemma 2.1.1, $K_0(D^K)$ is a cyclic ordered $\mathrm{Gal}(K/F)$-module and $K_0(\phi^K)$ is a normalized ordered $\mathrm{Gal}(K/F)$-module homomorphism.

Begin with a map $g : (K_0(D^K), [D^K]) \to (K_0(D'^K), [D'^K])$. To recover from g an F-algebra map $\phi : D \to M_t(D')$, for some positive integer t, it is necessary for g to be a normalized ordered $\mathrm{Gal}(K/F)$-module homomorphism and its entries (when g is in matrix form) have to be sufficiently large. In order to achieve the latter, the proof of Proposition 2.2 suggests that we should also consider the map $(K_0(D^{D^{op}}), [D^{D^{op}}]) \to (K_0(D'^{D^{op}}), [D'^{D^{op}}])$. However the entries of the matrix $K_0(\phi^{D^{op}})$ are not arbitrary but rather are permuted by $\mathrm{Gal}(K/F)$ through the map $(-)^K \xrightarrow{(-)^\lambda} (-)^{M_s(D^{op})}$. Therefore, in order to recover an F-algebra map ϕ, it is natural to consider commutative diagrams of the form

$$\begin{array}{ccc}(K_0(D^K),[D^K]) & \xrightarrow{K_0(-)} & (K_0(D^{D^{op}}),[D^{D^{op}}]) \\ g\downarrow & & f\downarrow \\ (K_0(D'^K),[D'^K]) & \xrightarrow{K_0(-)} & (K_0(D'^{D^{op}}),[D'^{D^{op}}])\end{array}$$

where g is a normalized ordered Gal(K/F)-module homomorphism, and f is a morphism in $\mathcal{P}$. This is studied in Proposition 2.4.4, the main result of this section. Let us begin this section by verifying the Gal(K/F)-module structure of $K_0((-)^K)$.

Proposition 2.4.1. *The map Gal$(K/F) \to$ Aut$(K_0(K^D))$, defined by $\tau \mapsto K_0(\tau^D)$, is a group homomorphism. With this action, $K_0(D^K)$ is an ordered cyclic (left) $\mathbb{Z}$Gal(K/F)-module with the usual ordering of $\mathbb{Z}$Gal(K/F). If $\phi : D \to M_t(D')$ is a unital F-algebra map, then $K_0(\phi^K) : (K_0(D^K),[D^K]) \to (K_0(D'^K),[M_t(D')^K])$ is a normalized order-preserving module homomorphism.*

Proof. The first statement is easily verified. The second statement follows from Lemma 2.1.1. The last statement follows from (2) and (3) in the proof of Proposition 2.3. ■

Definition. Let e_i and e'_i be the central idempotents of D^K and D'^K such that $e_i(D^K) \cong D \otimes_E E\sigma_i(K)$ and $e'_1(D'^K) \cong D' \otimes_{E'} E'\sigma'_i(K)$ Let u and v be primitive idempotents of D^K and D'^K such that $d[u] = [e_1]$ and $d'[v] = [e'_1]$, where d and d' denote the matrix size of $D \otimes_E K$ and $D' \otimes_{E'} K$ respectively. Then the positive elements $[u]$ and $[v]$ are generators for the ordered $\mathbb{Z}$Gal(K/F)-modules $K_0(D^K)$ and $K_0(D'^K)$ by Lemma 2.1.1.

Let $g : K_0(D^K) \to K_0(D'^K)$ be an order-preserving $\mathbb{Z}$Gal(K/F)-module homomorphism. Then g is determined by $g([u]) = \chi[v]$, for some $\chi \in \mathbb{Z}$Gal(K/F). In the Lemma 2.4.2, we shall investigate which possible elements $\chi \in \mathbb{Z}$Gal(K/F) give rise to an F-algebra map ϕ.

Lemma 2.4.2. *Let D and D' be division F-algebras with separable centres E and E'*

respectively and let K/F be a Galois extension containing them. Let $g : K_0(D^K) \to K_0(D'^K)$ be an ZGal(K/F)-order-preserving module homomorphism and let $[\mathbf{1}], \ldots, [\tau_m]$ be all the left cosets of Gal(K/E') in Gal(K/F). Then $g([u]) = (c_1\mathbf{1}+c_2\tau_2+\cdots+c_m\tau_m).[v]$ for some nonnegative integers $c_1, \ldots, c_m$ such that $c_i = c_j$ if $E\tau_i(E')$ and $E\tau_j(E')$ are equivalent composita of E and E' over F. As a result, if $\phi : D \to M_t(D')$ is the F-algebra map defined in (2), that is

$$\phi(x) = diag(a_1.\psi_1(x \otimes 1), \ldots, a_k.\psi_k(x \otimes 1)),$$

then

$$K_0(\phi^K)([u]) = (c_1\mathbf{1} + \cdots + c_m\tau_m).[v] \quad \text{where} \quad c_i = \frac{a_j d' \deg(D)^2}{ds_j} \tag{5}$$

for $i = 1, \ldots, m,\ j = 1, \ldots, k$, and $\sigma_j^{-1}\tau_i \in$ Gal(K/E').

Recall that ψ_i is the E'-algebra map from $D \otimes_E E\sigma_i(E')$ into D', and d, d', s_i are the matrix sizes of $D \otimes_E K$, $D' \otimes_{E'} K$, and $D' \otimes_{E'} \sigma_i(E')E \otimes_E D^{op}$ respectively.

Proof. Let $g([u]) = \chi[v]$, where $\chi \in$ ZGal(K/F). By Lemma 2.1.1, for all $\xi \in$ Gal(K/E'), $\xi[v] = [v]$, hence we can write $\chi = c_1\mathbf{1}+c_2\tau_2+\cdots+c_m\tau_m$, where $\{[\mathbf{1}], \ldots, [\tau_m]\}$ is the set of all left cosets of Gal(K/E') and c_i's are nonnegative integers. Then again by Lemma 2.1.1, for all $\xi \in$ Gal(K/E), $\xi.g([u]) = g(\xi.[u]) = g([u])$. Therefore $c_i = c_j$ if there exists $\xi \in$ Gal(K/E) such that $[\xi\tau_i] = [\tau_j]$. That is if $E\tau_i(E')$ and $E\tau_j(E')$ are equivalent composita of E and E' over F.

Now let $\phi : D \to M_t(D')$ be the F-algebra map

$$\phi(x) = \mathrm{diag}(a_1.\psi_1(x \otimes 1), \ldots, a_k.\psi_k(x \otimes 1)).$$

Then we have the commutative diagram

$$\begin{array}{ccc} D^{E'} & \xrightarrow{\iota} & D^K \\ \phi^{E'}\big\downarrow & & \phi^K\big\downarrow \\ M_t(D'^{\,E'}) & \xrightarrow{\iota'} & M_t(D'^{\,K}) \end{array}$$

which induces

$$\begin{array}{ccc} D\otimes_E EE'\times\cdots\times D\otimes_E E\sigma_k(E') & \xrightarrow{\nu} & D\otimes_E EK\times\cdots\times D\otimes_E E\sigma_l(K) \\ \phi^{E'}\Big\downarrow & & \phi^{K}\Big\downarrow \\ M_t(D')\otimes_{E'} E'E'\times\cdots\times M_t(D')\otimes_{E'} E'\tau_n(E') & \xrightarrow{\nu'} & M_t(D')\otimes_{E'} E'K\times\cdots\times M_t(D')\otimes_{E'} E'\tau_m(K) \end{array}$$

with the corresponding K_0 diagram

$$\begin{array}{ccc} (\mathbf{Z},t_1)\times\cdots\times(\mathbf{Z},t_k) & \xrightarrow{M} & (\mathbf{Z},d)\times\cdots\times(\mathbf{Z},d) \\ B\Big\downarrow & & C\Big\downarrow \\ (\mathbf{Z},t)\times\cdots\times(\mathbf{Z},t.*) & \xrightarrow{N} & (\mathbf{Z},d't)\times\cdots\times(\mathbf{Z},d't), \end{array}$$

where * is a positive integer. By (3), the first row of B is $(b_1,\ldots,b_k)$, where

$$b_i=\frac{a_i\deg(D)^2[E\sigma_i(E'):\sigma_i(E')]}{s_it_i},$$

with s_i and t_i are matrix sizes of $D'\otimes_{E'}\sigma_i(E')E\otimes_E D^{op}$ and $D\otimes_E E\sigma_i(E')$. By Lemma 2.1.3, the first row of N is $(d',0,\ldots,0)$. Hence the first row of $CM=NB$ is $(d'b_1,\ldots,d'b_k)$. Let $K_0(\phi^K)([u])=(c_1\mathbf{1}+\cdots+c_m\tau_m).[v]$. We shall find the first row of C:

Let the first row of C be $(x_1,\ldots,x_l)$ and let $\sigma_j^{-1}\tau_i\in \mathrm{Gal}(K/E')$. By Lemma 2.1.1, $\sigma_j.[e_j]=[e_1]=d[u]$, that is $\sigma_j^{-1}.[u]=(1/d)[e_j]$. Then

$$\begin{aligned} K_0(g)((1/d)[e_j])=K_0(g)(\sigma_j^{-1}.[u])&=\sigma_j^{-1}.K_0(g)([u])\\ &=\sigma_j^{-1}(c_1\mathbf{1}+\cdots+c_m\tau_m).[v]\\ &=(c_i\mathbf{1}+\cdots).[v]\ . \end{aligned}$$

Consequently, $x_j=c_i$. Moreover, if $E\sigma_j(E')$ and $E\sigma_p(E')$ are equivalent composita of E, E' over F, then there exist $\xi\in\mathrm{Gal}(K/E)$ such that $\sigma_j^{-1}\xi\sigma_p\in\mathrm{Gal}(K/E')$. With $\sigma_j^{-1}\tau_i\in \mathrm{Gal}(K/E')$, we have $\tau_i^{-1}\xi\sigma_p\in\mathrm{Gal}(K/E')$. Therefore $x_j=x_p=c_i$ because $(\sigma_p^{-1}\xi^{-1})\tau_i\in \mathrm{Gal}(K/E')$ and $[\sigma_p^{-1}\xi^{-1}]=[\sigma_p^{-1}]$.

Now by Lemma 2.1.3, the (i,j)-entry of M is d/t_j if $E\sigma_i(E')$ and $E\sigma_j(E')$ are equivalent composita of E and E' over F, and is 0 otherwise. Since there are exactly $[E\sigma_j(E'):\sigma_j(E')]$ positive entries on the j^{th}-column, the first row of CM is

$$([EE':E']x_1d/t_1,\ldots,[E\sigma_k(E'):\sigma_k(E')]x_kd/t_k).$$

Hence

$$c_i = x_j = \frac{d'b_jt_j}{d[E\sigma_j(E'):\sigma_j(E')]} = \frac{a_jd'\deg(D)^2}{ds_j}. \blacksquare$$

Corollary 2.4.3. *Let E and E' be subfields of K/F and let $f : K_0(E^K) \to K_0(E'^K)$ be an order-preserving module homomorphism. Then there exists an F-algebra map $\phi : E \to M_t(E')$, for some positive integer t, such that $K_0(\phi^K) = f$.*

Proof. In this case, (5) is reduced to $c_i = a_j$. $\blacksquare$

Since $d'\deg(D)^2/ds_j \neq 1$ in general, not all ordered $\mathbb{Z}\mathrm{Gal}(K/F)$-module homomorphisms $g : K_0(D^K) \to K_0(D'^K)$ can be converted into F-algebra homomorphisms $D \to M_t(D')$. For instance, if $d'\deg(D)^2/ds_1 = 2$, then the module homomorphism defined by $g([u]) = [v]$ can not be of the form $K_0(\phi^K)$ for any algebra map ϕ because $a_1 = 1/2$ from (5). Hence as discussed at the begining of this section, g needs to be embedded into the diagram below.

Proposition 2.4.4. *Let D and D' be division F-algebras with separable centres E and E' respectively and let K/F be any Galois extension containing them. Let $\lambda : K \to M_z(D^{op})$ be any E-algebra map. For any commutative diagram*

$$\begin{array}{ccc} (K_0(D^K), [D^K]) & \xrightarrow{K_0(\mathrm{id}_D^\lambda)} & (K_0(D^{M_z(D^{op})}), [D^{M_z(D^{op})}]) \\ g\Big\downarrow & & f\Big\downarrow \\ (K_0(M_t(D')^K), [M_t(D')^K]) & \xrightarrow{K_0(\mathrm{id}_{D'}^\lambda)} & (K_0(M_t(D')^{M_z(D^{op})}), [M_t(D')^{M_z(D^{op})}]). \end{array}$$

where g, f are morphisms in the category $\mathcal{P}$ such that g is an order-preserving $\mathbb{Z}G$-module homomorphism, there exists an F-algebra map $\phi : D \to M_t(D')$ satisfying $K_0(\phi^K) = g$ and $K_0(\phi^{M_z(D^{op})}) = f$.

Proof. With the isomorphisms

$$D \otimes_F D^{op} \cong D \otimes_E EE \otimes_E D^{op} \times \cdots \times D \otimes_E E\sigma_n(E) \otimes_E D^{op}$$

$$D \otimes_F M_y(K) \cong M_y(D \otimes_E EK) \times \cdots \times M_y(D \otimes_E E\sigma_n(K)) \times \cdots$$

$$D' \otimes_F D^{op} \cong D' \otimes_{E'} E'E \otimes_E D^{op} \times \cdots \times D' \otimes_{E'} E'\tau_k(E) \otimes_E D^{op}$$

$$D' \otimes_F M_y(K) \cong M_y(D' \otimes_{E'} EK) \times \cdots \times M_y(D' \otimes_{E'} E'\tau_k(K)) \times \cdots \quad ,$$

the diagram above becomes

$$\begin{array}{ccc} (\mathbf{Z}, d) \times \cdots \times (\mathbf{Z}, d) & \xrightarrow{M} & (\mathbf{Z}, z\tau_1) \times \cdots \times (\mathbf{Z}, z\tau_n) \\ C\big\downarrow & & A\big\downarrow \\ (\mathbf{Z}, td') \times \cdots \times (\mathbf{Z}, td') & \xrightarrow{N} & (\mathbf{Z}, tzs_1) \times \cdots \times (\mathbf{Z}, tzs_k). \end{array}$$

Let the first column of A be $(a_1, \ldots, a_k)^{tr}$. As the first column of M is $(z\tau_1/d, 0, \ldots, 0)^{tr}$ by Lemma 2.1.2, the first column of AM is $z\tau_1/d.(a_1, \ldots, a_k)^{tr}$. Let Σ_i be the sum of entries on the i^{th}-row of N and let the first column of C be $(c_1, \ldots, c_k, \ldots)^{tr}$. Then by Lemma 2.1.2 and the proof of Lemma 2.4.2, the first column of NC is $(c_1\Sigma_1, \ldots, c_k\Sigma_k, \ldots)^{tr}$. Note that $\Sigma_i td' = tzs_i$ and $\tau_1 = \deg(D)^2$. Hence

$$c_i = \frac{a_i z\tau_1}{d\Sigma_i} = \frac{a_i d' \deg(D)^2}{ds_i}.$$

Now let $\phi : D \to M_t(D')$ be defined by

$$\phi(x) = diag(a_1.\psi_1(x \otimes 1), \ldots, a_k.\psi_k(x \otimes 1)).$$

By Lemma 2.4.2, $K_0(\phi^K)([u]) = g([u])$. Hence $K_0(\phi^K) = g$ because they are both module homomorphisms and $[u]$ is a generator of $K_0(D^K)$. Then the pair $(g, K_0(\phi^{M_z(D^{op})}))$ makes the diagram above commute. By Lemma 2.1.2, there exists a positive integer n such that $n.K_0(D^{M_z(D^{op})}) \subseteq K_0(\mathrm{id}_D^\lambda)(K_0(D^K))$. Then $f(n.x) = K_0(\phi^{M_z(D^{op})})(n.x)$ for all $x \in K_0(D^{M_z(D^{op})})$. Therefore $K_0(\phi^{M_z(D^{op})}) = f$ because $K_0(D'^{D^{op}})$ is a free abelian group. ■

Chapter 3

Classification of Locally Finite Dimensional Semisimple Algebras

Our main objective is to find a complete invariant $\kappa(-)$ for the category $\mathcal{R}$ of all direct limits of finite direct products of matrix algebras over a finite number of division F-algebras. That is if given an isomorphism $f : \kappa(R) \to \kappa(S)$, with $R, S \in \mathcal{R}$, we can recover an F-algebra isomorphism $\phi : R \to S$ such that $\kappa(\phi) = f$.

Based on the work done in [GH] for the case $F = \mathbf{R}$, the invariant $\kappa(-)$ should be a diagram of $(K_0(- \otimes_F X), [- \otimes_F X])$'s with induced inclusion maps, where X's are the division algebras involved. Let $\mathcal{R}_0$ denote the subcategory of all finite direct products of matrix algebras. The standard technique is to prove that for all $R \in \mathcal{R}_0$ and $S \in \mathcal{R}$

(i) if ϕ and ϕ' are F-algebra maps from R into S such that $\kappa(\phi) = \kappa(\phi')$, then ϕ differs from ϕ' only by an inner automorphism of R, and

(ii) given a morphism $f : \kappa(R) \to \kappa(S)$, there exists an F-algebra map $\phi : R \to S$ such that $\kappa(\phi) = f$,

then apply the usual interweaving argument: Let $R = R_m \xrightarrow{\alpha_{nm}} R_n \xrightarrow{\alpha_{\cdot n}} \cdots$ and $S = S_m \xrightarrow{\beta_{nm}} S_n \xrightarrow{\beta_{\cdot n}} \cdots$. Let $f : \kappa(R) \to \kappa(S)$ be an isomorphism. Then f^{-1} induces a morphism from $\kappa(S_1)$ into $\kappa(R)$. From (ii), there exists an F-algebra map $\sigma : S_1 \to R_{n(1)}$ such that $\kappa(\sigma)$ is a restriction of f^{-1}. With a similar argument, f induces an F-algebra map $\rho : R_{n(1)} \to S_{m(2)}$. Since $\kappa(\rho)$ is a restriction of f, we have $\kappa(\rho\sigma) = \kappa(\beta_{m(2)1})$, hence we can assume $\rho\sigma = \beta_{m(2)1}$ by (ii). Continuing in this manner, we construct two sequences of F-algebra maps $\sigma_i : S_{m(i)} \to R_{n(i)}$ and $\rho_i : R_{n(i)} \to S_{m(i+1)}$ such that $\rho_i\sigma_i = \beta_{m(i+1)m(i)}$ and $\sigma_{i+1}\rho_i = \alpha_{m(i+1)m(i)}$. Finally, we construct an F-algebra map ϕ from the sequence $\{\rho_i\}$. We can see that ϕ^{-1} can be constructed from the sequence $\{\sigma_i\}$.

As in the cases $F = \mathbf{C}$ and $F = \mathbf{R}$, by removing the order units $[- \otimes_F X]$'s from $\kappa(-)$ we obtain an invariant for Morita-equivalence in $\mathcal{R}$.

In section 3.1, we introduce $\kappa(-)$ and some of its functorial properties. We prove (ii) in Theorem 3.2.1, and (i) in Theorem 3.3.1. Finally, the interweaving argument is used in Theorem 3.4.1 to prove that $\kappa(-)$ is a complete invariant of $\mathcal{R}$. Theorem 3.4.4 is about the invariant for Morita-equivalence in $\mathcal{R}$ obtained by removing the order units. Section 3.5 is to show that $\kappa(-)$ is minimal. We also show that our invariant and the invariants found by G. A. Elliott, K. R. Goodearl and D. E. Handelman for $F = \mathbf{C}$ and $F = \mathbf{R}$ respectively are the same.

3.1. Description of the invariant.

Definition. Let Ω be a *finite set of finite dimensional division F-algebras* with *separable* centres. We can assume that Ω includes a Galois extension K/F which contains all the centres of the division algebras in Ω. Then within the category of unital F-algebras, let $\mathcal{R}_0$ denote the full subcategory whose objects are all finite dimensional semisimple F-algebras of matrices over division algebras taken from Ω, and let $\mathcal{R}$ denote the full subcategory whose objects are all unital direct limits of (countable) sequences of objects from $\mathcal{R}_0$. Note that a unital F-algebra is an object of $\mathcal{R}$ if and only if it is a countable ascending union of finite dimensional semisimple F-subalgebras.

Observe that for any object $R \in \mathcal{R}$ and any idempotent $g \in R$, the algebra gRg is an object of $\mathcal{R}$.

By [G1, Lemma 15.22], K_0 of any $R \in \mathcal{R}_0$ is isomorphic (as an ordered group) to a finite product of copies of $\mathbf{Z}$, and so is a countable dimension group. Since the class of countable dimension groups is closed under countable direct limits, we see by [G1, Proposition 15.11,15.13] that $K_0(R)$, for all $R \in \mathcal{R}$, is a countable dimension group.

Proposition 3.1.1. *Let $R, S \in \mathcal{R}$. Then*

(a) *The map $Gal(K/F) \to Aut(K_0(R^K))$, defined by $\tau \mapsto K_0(\mathrm{id}_R \otimes_F \tau)$, is a group homomorphism. With this action, $K_0(R^K)$ is an ordered (left) $\mathbf{Z}Gal(K/F)$-module with the usual ordering on $\mathbf{Z}Gal(K/F)$.*

(b) *If $\phi : R \to S$ is a unital F-algebra map, then $K_0(\phi^K) : (K_0(R^K), [R^K]) \to (K_0(S^K), [S^K])$ is a normalized order-preserving module homomorphism.*

(c) *If $R = R_1 \xrightarrow{\alpha_1} R_2 \xrightarrow{\alpha_2} \cdots$, then as ordered $\mathbf{Z}Gal(K/F)$-modules*

$$K_0(R^K) = K_0(R_1^K) \xrightarrow{K_0(\alpha_1^K)} K_0(R_2^K) \xrightarrow{K_0(\alpha_2^K)} \cdots \quad .$$

Proof. (a) and (b) are easily verified and (c) is true because the funtors K_0 and $(-)^K$ preserve finite products and arbitrary direct limits. ■

Definition. Let $|\Omega \setminus \{F, K\}| = \omega$. Let $\mathcal{K}$ denote the category whose objects are all diagrams in $\mathcal{P}$ of the form

$$(G_1, u_1) \xrightarrow{g_1} (G_2, u_2) \begin{cases} \xrightarrow{g_{21}} (G_{31}, u_{31}) \\ \cdots\cdots \quad \cdots\cdots \\ \xrightarrow{g_{2\omega}} (G_{3\omega}, u_{3\omega}) \end{cases}$$

where G_2 is an ordered $\mathbf{Z}\mathrm{Gal}(K/F)$-module (note that there are no maps connecting any of the (G_{3*}, u_{3*})). A morphism in $\mathcal{K}$ is an ordered set $(f_1, , f_2, f_{31}, \ldots, f_{3\omega})$ of morphisms in $\mathcal{P}$, where f_2 is an ordered $\mathbf{Z}\mathrm{Gal}(K/F)$-module homomorphism, such that the diagrams below commute

$$\begin{array}{ccccc} (G_1, u_1) & \xrightarrow{g_1} & (G_2, u_2) & \xrightarrow{g_{2i}} & (G_{3i}, u_{3i}) \\ f_1 \downarrow & & f_2 \downarrow & & f_{3i} \downarrow \\ (H_1, v_1) & \xrightarrow{h_1} & (H_2, v_2) & \xrightarrow{h_{2i}} & (H_{3i}, v_{3i}) \end{array}$$

for all $i = 1, \ldots, \omega$.

Definition. Let $R, S \in \mathcal{R}$

(i) Let $\mu : R \to R^K$ and $\mu' : S \to S^K$ denote the maps defined by $x \mapsto x \otimes 1$.

(ii) For all $D_i \in \Omega \setminus \{F, K\}$, we let $\nu_i : R^K \to R^{M_z(D_i^{op})}$ be the map defined by $x \otimes y \mapsto x \otimes \lambda_{D_i^{op}}(y)$, where $z = [K : F]$ and $\lambda_{D_i^{op}} : K \to M_z(D_i^{op})$ is any $Z(D_i^{op})$-algebra map. Similarly, we define ν_i' for S.

Definition. For any algebra $R \in \mathcal{R}$, we define $\kappa(R)$ to be the object

$$(K_0(R), [R]) \xrightarrow{K_0(\mu)} (K_0(R^K), [R^K]) \left\langle \begin{array}{l} \xrightarrow{K_0(\nu_1)} (K_0(R^{M_z(D_1^{op})}), [R^{M_z(D_1^{op})}]) \\ \cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_\omega)} (K_0(R^{M_z(D_\omega^{op})}), [R^{M_z(D_\omega^{op})}]) \end{array} \right.$$

in $\mathcal{K}$. For any F-algebra map $\phi : R \to S$ in $\mathcal{R}$, there are commutative diagrams

$$\begin{array}{ccccc} R & \xrightarrow{\mu} & R^K & \xrightarrow{\nu_i} & R^{M_z(D_i^{op})} \\ \phi \big\downarrow & & \phi^K \big\downarrow & & \phi^{M_z(D_i^{op})} \big\downarrow \\ S & \xrightarrow{\mu'} & S^K & \xrightarrow{\nu_i'} & S^{M_z(D_i^{op})} \end{array}$$

which induce the commutative diagrams

$$\begin{array}{ccccc} (K_0(R), [R]) & \xrightarrow{K_0(\mu)} & (K_0(R^K), [R^K]) & \xrightarrow{K_0(\nu_i)} & (K_0(R^{M_z(D_i^{op})}), [R^{M_z(D_i^{op})}]) \\ K_0(\phi) \big\downarrow & & K_0(\phi^K) \big\downarrow & & K_0(\phi^{M_z(D_i^{op})}) \big\downarrow \\ (K_0(S), [S]) & \xrightarrow{K_0(\mu')} & (K_0(S^K), [S^K]) & \xrightarrow{K_0(\nu_i')} & (K_0(S^{M_z(D_i^{op})}), [S^{M_z(D_i^{op})}]) \end{array}$$

for all $i = 1, \ldots, \omega$. Thus $(K_0(\phi), K_0(\phi^K), K_0(\phi^{M_z(D_1^{op})}), \ldots, K_0(\phi^{M_z(D_\omega^{op})}))$ is a morphism from $\kappa(R)$ to $\kappa(S)$ in $\mathcal{K}$. We denote this morphism by $\kappa(\phi)$.

Observe that κ is a functor from $\mathcal{R}$ to $\mathcal{K}$.

Proposition 3.1.2. *The categories $\mathcal{R}$ and $\mathcal{K}$ both possess finite products and countable direct limits, and the functor κ preserves them.*

Proof. This is immediate from Proposition 1.2.1 and the observation that the functors $(-)^K$ and $(-)^{M_z(D_i^{op})}$ preserve finite products and arbitrary direct limits. ■

3.2. Fullness of the invariant.

The objective of this section is to prove the following theorem, which requires a number of steps.

Theorem 3.2.1. *Let $R \in \mathcal{R}_0$ and $S \in \mathcal{R}$, and let $f : \kappa(R) \to \kappa(S)$ be a map in $\mathcal{K}$. Then there exists an F-algebra map $\phi : R \to S$ such that $\kappa(\phi) = f$.*

Lemma 3.2.2. *Let $R \in \mathcal{R}_0$ and $f : \kappa(R) \to L$ be a map in $\mathcal{K}$. Suppose that L together with the maps $t_n : L_n \to L$ is a direct limit for a sequence $L_1 \to L_2 \to \cdots$ in $\mathcal{K}$. Then $f = t_n f'$ for some n and some map $f' : \kappa(R) \to L_n$.*

Proof. Write out each L_n in the form

$$(G_{n1}, u_{n1}) \xrightarrow{g_{n1}} (G_{n2}, u_{n2}) \left\langle \begin{array}{l} \xrightarrow{g_{n21}} (G_{n31}, u_{n31}) \\ \cdots\cdots \quad \cdots\cdots\cdots \\ \xrightarrow{g_{n2\omega}} (G_{n3\omega}, u_{n3\omega}) \end{array} \right.$$

and write out L in the form

$$(H_1, v_1) \xrightarrow{h_1} (H_2, v_2) \left\langle \begin{array}{l} \xrightarrow{h_{21}} (H_{31}, v_{31}) \\ \cdots\cdots \quad \cdots\cdots \\ \xrightarrow{h_{2\omega}} (H_{3\omega}, v_{3\omega}) \end{array} \right.$$

then each t_n is a $(\omega{+}2)$-tuple $(t_{n1}, t_{n2}, t_{n31}, \ldots, t_{n3\omega})$, and f is a tuple $(f_1, f_2, f_{31}, \ldots, f_{3\omega})$.

By Proposition 1.2.4, $K_0(R)$ is a free abelian group with a basis $\{x_1, \ldots, x_m\}$ such that $K_0(R)^+ = \mathbb{Z}^+ x_1 + \cdots + \mathbb{Z}^+ x_m$. Since each $f_1(x_p)$ is in H_1^+, there exist $n \in \mathbb{N}$ and elements $y_p \in G_{n1}^+$ such that each $f_1(x_p) = t_{n1}(y_p)$. Define a homomorphism $f_1' : K_0(R) \to G_{n1}$ such that $f_1'(x_p) = y_p$ for $p = 1, \ldots, m$. Then f_1' is a positive homomorphism, and $t_{n1} f_1' = f_1$. As

$$t_{n1} f_1'([R]) = f_1([R]) = v_1 = t_{n1}(u_{n1}),$$

we may assume, after possibly increasing n, that $f_1'([R]) = u_{n1}$. Thus f_1' is now a map from $(K_0(R), [R])$ to (G_{n1}, u_{n1}).

Similarly, after possibly increasing n some more, there exist maps

$$f_2' : (K_0(R^K), [R^K]) \to (G_{n2}, u_{n2}) \text{ and } f_{3i}' : (K_0(R^{M_i(D_i^{op})}), [R^{M_i(D_i^{op})}]) \to (G_{n3i}, u_{n3i})$$

such that $t_{n2}f_2' = f_2$ and $t_{n3i}f_{3i}' = f_{3i}$, for all $i = 1, \ldots, \omega$. Since $K_0(R)$ is finitely generated and

$$t_{n2}f_2'K_0(\mu) = f_2K_0(\mu) = h_1f_1 = h_1t_{n1}f_1' = t_{n2}g_{n1}f_1'$$

we may assume, after increasing n once again, that $f_2'K_0(\mu) = g_{n1}f_1'$. Similarly, we may assume that $f_{3i}'K_0(\nu_i) = g_{n2i}f_2'$ for all $i = 1, \ldots, \omega$.

Since f_2 and t_{n2} are Gal(K/F)-module homomorphisms and $t_{n2}f_2' = f_2$, for all $\sigma \in$ Gal(K/F),

$$t_{n2}f_2'\sigma = f_2\sigma = \sigma f_2 = \sigma t_{n2}f_2' = t_{n2}\sigma f_2',$$

and by increasing n again we may assume $f_2'\sigma = \sigma f_2'$ for all $\sigma \in$ Gal(K/F). So f_2' is also a Gal(K/F)-module homomorphism. Therefore $f' = (f_1', f_2', f_{31}', \ldots, f_{3\omega}')$ is a map from $\kappa(R)$ to L_n such that $t_nf' = f$. ■

Lemma 3.2.3. *Let $R_1, \ldots, R_m, S \in \mathcal{R}$, let*

$$r : \kappa(R_1 \times \cdots \times R_m) \to \kappa(R_1) \times \cdots \times \kappa(R_m)$$

be the natural isomorphism, and let

$$f : \kappa(R_1 \times \cdots \times R_m) \to \kappa(S)$$

be a map in $\mathcal{K}$. Then there exists $S_1, \ldots, S_m \in \mathcal{R}$, maps $g_n : \kappa(R_n) \to \kappa(S_n)$ for n = 1,…,m, and a map

$$\phi : S_1 \times \cdots \times S_m \to S$$

such that $f = \kappa(\phi)s^{-1}(g_1 \times \cdots \times g_m)r$, where

$$s : \kappa(S_1 \times \cdots \times S_m) \to \kappa(S_1) \times \cdots \times \kappa(S_m)$$

is the natural isomorphism.

Proof. By induction, we may assume that $m = 2$.

Set $x_1 = [(1,0)(R_1 \times R_2)]$ and $x_2 = [(0,1)(R_1 \times R_2)]$ in $K_0(R_1 \times R_2)$. Then $f_1(x_1)$ and $f_1(x_2)$ are elements of $K_0(S)^+$ whose sum is $[S]$. For $n = 1, 2$, there is a finitely generated projective right S-module A_n such that $f_1(x_n) = [A_n]$, and $[A_1 \oplus A_2] = [S]$. By unit-regularity, $A_1 \oplus A_2 \cong S$ and so there exist orthogonal idempotents $e_1, e_2 \in S$ such that $e_1 + e_2 = 1$ and each $e_n S \cong A_n$. Thus each $f_1(x_n) = [e_n S]$.

Set $S_n = e_n S e_n$ for $n = 1, 2$, and let $\phi : S_1 \times S_2 \to S$ to be the sum map. For $n = 1, 2$, there is a positive homomorphism $t_n : K_0(S_n) \to K_0(S)$ such that

$$t_n([B]) = [B \otimes_{S_n} (e_n S)]$$

for all finitely generated projective right S_n-module B. Set

$$W_n = \{y \in K_0(S) \mid -m[e_n S] \leq y \leq m[e_n S] \quad \text{for some } m \in \mathbb{N}\},$$

and observe that t_n gives an embedding of $K_0(S_n)$ into W_n (as ordered groups). Using the Riesz decomposition properties [G2,Proposition 21.2] in $K_0(S)$, we infer that W_n is generated (as a subgroup of $K_0(S)$) by the elements $[aS] = t_n([aS_n])$ for idempotents $a \in S_n$. Hence, t_n gives an isomorphism of $(K_0(S_n), [S_n])$ onto $(W_n, [e_n S])$.

As $f_1([(1,0)(R_1 \times R_2)]) = [e_1 S]$, we see that $f_1 r_1^{-1}$ maps $K_0(R_1) \times \{0\}$ into W_1, and similarly $f_1 r_1^{-1}$ maps $\{0\} \times K_0(R_2)$ into W_2. Consequently, there exist unique maps

$$g_{n1} : (K_0(R_n), [R_n]) \to (K_0(S_n), [S_n])$$

for $n = 1, 2$ such that $f_1 r_1^{-1} = (t_1, t_2)(g_{11} \times g_{21})$. Given idempotents $a_n \in S_n$ for $n = 1, 2$, we compute that

$$K_0(\phi)([(a_1, a_2)(S_1 \times S_2)]) = [(a_1 + a_2)S] = [a_1 S] + [a_2 S] = (t_1, t_2)s_1([(a_1, a_2)(S_1 \times S_2)]).$$

Thus $K_0(\phi) = (t_1, t_2)s_1$, whence $f_1 r_1^{-1} = K_0(\phi)s_1^{-1}(g_{11} \times g_{21})$.

In the same manner (for each $D \in \Omega$, we define t_n^D and W_n^D similar to t_n and W_n), there exist unique maps

$$g_{n2} : (K_0(R_n^K), [R_n^K]) \to (K_0(S_n^K), [S_n^K])$$
$$g_{n3i} : (K_0(R_n^{M_s(D_i^{op})}), [R_n^{M_s(D_i^{op})}]) \to (K_0(S_n^{M_s(D_i^{op})}), [S_n^{M_s(D_i^{op})}])$$

for $n = 1, 2$ and for all $i = 1, \ldots, \omega$, such that

$$f_2 r_2^{-1} = K_0(\phi^K) s_2^{-1}(g_{12} \times g_{22}) \quad \text{and} \quad f_{3i} r_{3i}^{-1} = K_0(\phi^{M_2(D_i^{op})}) s_{3i}^{-1}(g_{13i} \times g_{23i}).$$

Thus $f = \kappa(\phi)s^{-1}(g_{11} \times g_{21}, g_{12} \times g_{22}, g_{131} \times g_{231}, \ldots, g_{13\omega} \times g_{23\omega})r$. Now we shall show that $(g_{n1}, g_{n2}, g_{n31}, \ldots, g_{n3\omega})$ are morphisms in $\mathcal{K}$.

Observe that

$$\begin{aligned} K_0(\phi^K)s_2^{-1}(g_{12} \times g_{22})(K_0(\mu^{R_1}) \times K_0(\mu^{R_2})) &= f_2 r_2^{-1}(K_0(\mu^{R_1}) \times K_0(\mu^{R_2})) \\ &= K_0(\mu_S) f_1 r_1^{-1} \\ &= K_0(\mu_S) K_0(\phi) s_1^{-1}(g_{11} \times g_{21}) \\ &= K_0(\phi^K) K_0(\mu_{(S_1 \times S_2)}) s_1^{-1}(g_{11} \times g_{21}) \\ &= K_0(\phi^K) s_2^{-1}(K_0(\mu^{S_1}) \times K_0(\mu^{S_2}))(g_{11} \times g_{21}), \end{aligned}$$

Observe also that since ϕ is the sum map and s is the natural isomorphism, $K_0(\phi^K)s_2^{-1}$ restricts to injections of $K_0(S_1^K) \times \{0\}$ and $\{0\} \times K_0(S_2^K)$ into $K_0(S^K)$, whence

$$g_{n2} K_0(\mu^{R_n}) = K_0(\mu^{S_n}) g_{n1}$$

for $n = 1, 2$. Similarly,

$$g_{n3i} K_0(\nu_i^{R_n}) = K_0(\nu_i^{S_n}) g_{n2}$$

for $n = 1, 2$ and $i = 1, \ldots, \omega$. Therefore each $g_n = (g_{n1}, g_{n2}, g_{n31}, \ldots, g_{n3\omega})$ is a map from $\kappa(R_n)$ to $\kappa(S_n)$, and

$$\begin{aligned} f &= \kappa(\phi)s^{-1}(g_{11} \times g_{21}, g_{12} \times g_{22}, g_{131} \times g_{231}, \ldots, g_{13\omega} \times g_{23\omega})r \\ &= \kappa(\phi)s^{-1}(g_1 \times g_2)r. \end{aligned}$$

Now we need to prove that g_{n2} is a ZG-module homomorphism. Let $\xi \in$ ZG and $x \in K_0(R_1^K)$. Since $K_0(\phi^K)s^{-1}(g_{12} \times g_{22}) = f_2 r_2^{-1}$ and $K_0(\phi^K)s^{-1}$ are module homomorphisms, we have

$$\begin{aligned} K_0(\phi^K)s^{-1}(g_{12} \times g_{22})(\xi.x, 0) &= \xi.K_0(\phi^K)s^{-1}(g_{12} \times g_{22})(x, 0) \\ &= K_0(\phi^K)s^{-1}(\xi.(g_{12} \times g_{22})(x, 0)) \end{aligned}$$

Since $K_0(\phi^K)s_2^{-1}$ restricts to injections of $K_0(S_1^K) \times \{0\}$ and $\{0\} \times K_0(S_2^K)$ into $K_0(S^K)$, whence $(g_{12} \times g_{22})(\xi.x, 0) = \xi.(g_{12} \times g_{22})(x, 0)$. Then $g_{12}(\xi.x) = \xi.g_{12}(x)$. Similarly, $g_{22}(\xi.x) = \xi.g_{22}(x)$. ■

Proof of Theorem 3.2.1. By Lemma 3.2.3, we can assume that $R = M_n(D_i)$ for some $D_i \in \Omega$. By Proposition 3.1.2 and Lemma 3.2.2, we can assume that $S = M_t(D_j)$ for some $D_j \in \Omega$. By the hypothesis, we have the commutative diagram

$$\begin{array}{ccc}
(K_0(R^K), [R^K]) & \xrightarrow{K_0(\nu_i)} & (K_0(R^{M_i(D_i^{op})}), [R^{M_i(D_i^{op})}]) \\
f_2 \downarrow & & \downarrow f_{3i} \\
(K_0(S^K), [S^K]) & \xrightarrow{K_0(\nu_i')} & (K_0(S^{M_i(D_i^{op})}), [S^{M_i(D_i^{op})}])
\end{array}$$

Then by Proposition 2.4.4, there exists a map $\phi : R \to S$ in $\mathcal{R}$ such that $\kappa(\phi^K) = f_2$. Then for $k = 1, \ldots, \omega$, we have the commutative diagrams

$$\begin{array}{ccc}
(K_0(R^K), [R^K]) & \xrightarrow{K_0(\nu_k)} & (K_0(R^{M_{z(k)}(D_k^{op})}), [R^{M_{z(k)}(D_k^{op})}]) \\
K_0(\phi^K) \downarrow & & \downarrow K_0(\phi^{M_{z(k)}(D_k^{op})}) \\
(K_0(S^K), [S^K]) & \xrightarrow{K_0(\nu_k')} & (K_0(S^{M_{z(k)}(D_k^{op})}), [S^{M_{z(k)}(D_k^{op})}])
\end{array}$$

By Lemma 2.1.2, there exists a positive integer m such that $m.K_0(R^{M_{z(k)}(D_k^{op})}) \subseteq K_0(\nu_k)(K_0(R^K))$. Then $f_{3k}(m.x) = K_0(\phi^{M_{z(k)}(D_k^{op})})(m.x)$ for all $x \in K_0(R^{M_{z(k)}(D_k^{op})})$. Therefore $K_0(\phi^{M_{z(k)}(D_k^{op})}) = f_{3k}$ because $K_0(S^{D_k^{op}})$ is a free group. Hence $\kappa(\phi) = f$. ∎

3.3. Faithfulness of the invariant (up to inner automorphism).

In this section we prove that the maps whose existence is guaranteed by Theorem 3.2.1 are unique up to inner automorphisms. The precise result is the following theorem

Theorem 3.3.1. *Let $R \in \mathcal{R}_0$ and $S \in \mathcal{R}$, and let $\phi_1, \phi_2 : R \to S$ be maps in $\mathcal{R}$. Then the following conditions are equivalent:*

(a) There exists an inner automorphism θ of S such that $\phi_2 = \theta\phi_1$.

(b) $K_0(\phi_1^K) = K_0(\phi_2^K)$.

(c) $\kappa(\phi_1) = \kappa(\phi_2)$.

Lemma 3.3.2. *Let R and S be unit-regular rings, and let $\phi_1, \phi_2 : R \to S$ be unital ring maps. Let $e \in R$ and $f \in S$ be idempotents such that each $\phi_q(e) = f$, and let $\psi_q : eRe \to fSf$ be the restriction of ϕ_q. If $K_0(\phi_1) = K_0(\phi_2)$, then $K_0(\psi_1) = K_0(\psi_2)$.*

Proof. [GH, Lemma 4.7].

Proof of Theorem 3.3.1. (a) implies (c) is an immediate consequence of Proposition 1.3.5. Obviously (c) implies (b).

Now let $K_0(\phi_1^K) = K_0(\phi_2^K)$. First, we suppose that R is simple. Since R is finite dimensional, there is a finite dimensional semisimple subalgebra $T \subseteq S$ such that $\phi_1(R) \subseteq T$ and $\phi_2(R) \subseteq T$. Since any inner automorphism of T extends to an inner automorphism of S, we may replace S by T. Thus, without loss of generality, $S = S_1 \times \cdots \times S_t$ where each S_m is a finite dimensional simple algebra.

The maps ϕ_1 and ϕ_2 may be expressed as $\phi_1 = (\phi_{11}, \ldots, \phi_{1t})$ and $\phi_2 = (\phi_{21}, \ldots, \phi_{2t})$, where ϕ_{1m} and ϕ_{2m} are maps from R to S_m. Then $K_0(\phi_1^K) = K_0(\phi_2^K)$ implies $K_0(\phi_{1i}^K) = K_0(\phi_{2i}^K)$ for $i = 1, \ldots, t$ because each ϕ_{1i} (resp. ϕ_{2i}) is a composite map of ϕ_1 (resp. ϕ_2) and the projection map $S \to S_i$. By Proposition 2.3, each S_i has an inner automorphism θ_i such that $\phi_{2i} = \theta_i \phi_{1i}$. Then $\theta = \theta_1 \times \cdots \times \theta_t$ is an inner automorphism of S such that $\phi_2 = \theta\phi_1$.

Now let $R \in \mathcal{R}_0$. There are central orthogonal idempotents $e_1, \ldots, e_t \in R$ such that $e_1 + \cdots + e_t = 1$ and each e_iR is a simple algebra. Set $f_i = \phi_1(e_i)$ and $g_i = \phi_2(e_i)$. Then because $K_0(\phi_1^K) = K_0(\phi_2^K)$ implies $K_0(\phi_1) = K_0(\phi_2)$ [ref. Lemma 4.2(a)], we have

$$[f_iS] = K_0(\phi_1)([e_iR]) = K_0(\phi_2)([e_iR]) = [g_iS],$$

whence $f_iS \cong g_iS$. There exist elements $x_i \in f_iSg_i$ and $y_i \in g_iSf_i$ such that $x_iy_i = f_i$ and $y_ix_i = g_i$. Set $x = x_1 + \cdots + x_t$ and $y = y_1 + \cdots + y_t$. Since $\{f_1, \ldots, f_t\}$ and $\{g_1, \ldots, g_t\}$ are sets of orthogonal idempotents summing to 1, we compute that $xy = yx = 1$. Let ρ be the inner automorphism of S given by the rule $\rho(s) = ysx$, and observe that

$$\rho\phi_1(e_i) = \rho(f_i) = yf_ix = y_if_ix_i = g_i = \phi_2(e_i)$$

for all i.

Now $K_0(\rho\phi_1) = K_0(\phi_1) = K_0(\phi_1)$ and similarly $K_0(\rho^K\phi_1^K) = K_0(\phi_1^K)$, by Proposition 1.3.5, so it suffices to find an inner automorphism θ' of S such that $\phi_2 = \theta'\rho\phi_1$. Thus we may replace ϕ_1 by $\rho\phi_1$. In other words, there is no loss of generality in assuming that $\phi(e_i) = \phi'(e_i) = f_i$ for all $i = 1, \ldots, t$.

Let $\psi_{qi} : e_iR \to f_iSf_i$ denote the restriction of ϕ_q to e_iR, for $q = 1, 2$ and $i = 1, \ldots, t$. Note that for all q, i, the map

$$\psi_{qi}^K : (e_iR)^K = \mu_R(e_i)R^K \to \mu_S(f_i)S^K\mu_S(f_i) = (f_iSf_i)^K$$

equals the restriction of ϕ_q^K to $(e_iR)^K$. By Lemma 3.3.2, $K_0(\psi_{1i}^K) = K_0(\psi_{2i}^K)$ for each $i = 1, \ldots, t$.

For each $i = 1, \ldots, t$, the first part of this proof shows that $\psi_{2i} = \theta_i\psi_{1i}$ for some inner automorphism θ_i of f_iSf_i. There is a unit $u_i \in f_iSf_i$ such that $\theta_i(s) = u_isu_i^{-1}$ for all $s \in f_iSf_i$. Then $u = u_1 + \cdots + u_t$ is a unit of S, the rule $\theta(s) = usu^{-1}$ defines an inner automorphism of S, and $\psi_{2i} = \theta\psi_{1i}$ for all $i = 1, \ldots, t$. Therefore

$$\phi_2(r) = \psi_{21}(e_1r) + \cdots + \psi_{2t}(e_tr) = \theta\psi_{11}(e_1r) + \cdots + \theta\psi_{1t}(e_tr) = \theta\phi_1(r)$$

for all $r \in R$, so that $\phi_2 = \theta\phi_1$. ■

3.4. Completeness of the invariant.

We now prove that the invariant $\kappa(-)$ classifies the algebras in $\mathcal{R}$ up to isomorphism. Then by removing the order units from $\kappa(-)$, we obtain an invariant that classifies the algebras in $\mathcal{R}$ up to Morita equivalence (Theorem 3.4.4).

Theorem 3.4.1. *Let $R, S \in \mathcal{R}$. Then $R \cong S$ if and only if $\kappa(R) \cong \kappa(S)$. Moreover, if f is any isomorphism of $\kappa(R)$ onto $\kappa(S)$, then there is an isomorphism ρ of R onto S such that $\kappa(\rho) = f$.*

Proof. $R \cong S$ implies $\kappa(R) \cong \kappa(S)$ follows from the functoriality of κ.

Conversely, let $f : \kappa(R) \to \kappa(S)$ be an isomorphism in $\mathcal{K}$.

Write R as the union of a sequence $R_1 \subseteq R_2 \subseteq \ldots$ of finite dimensional semisimple unital subalgebras, and write S as the union of a similar sequence $S_1 \subseteq S_2 \subseteq \ldots$. For $n \in \mathbb{N}$, let $\phi_n : R_n \to R$ and $\psi_n : S_n \to S$ be the inclusion maps. For $m \leq n$, let $\alpha_{nm} : R_m \to R_n$ and $\beta_{nm} : S_m \to S_n$ be the inclusion maps.

The following technical property is basic to our induction procedure:

(I) Given any map $\sigma : S_m \to R_n$ such that $\kappa(\phi_n\sigma) = f^{-1}\kappa(\psi_m)$, there exists an integer $s > m$ and a map $\rho : R_n \to S_s$ such that $\rho\sigma = \beta_{sm}$ and $\kappa(\psi_s\rho) = f\kappa(\phi_n)$:

By Theorem 3.2.1, there exists a map $\rho' : R_n \to S$ such that $\kappa(\rho') = f\kappa(\phi_n)$. Since

$$\kappa(\psi_m) = f\kappa(\phi_n\sigma) = f\kappa(\phi_n)\kappa(\sigma) = \kappa(\rho')\kappa(\sigma) = \kappa(\rho'\sigma),$$

Theorem 3.3.1 says that $\psi_m = \theta\rho'\sigma$ for some inner automorphism θ of S. As R_n is finite dimensional, there exists an integer $s > m$ such that $\theta\rho'(R_n) \subseteq S_s$. Then $\theta\rho'$ defines a map $\rho : R_n \to S_s$ such that $\psi_s\rho = \theta\rho'$, and $\psi_s\rho\sigma = \theta\rho'\sigma = \psi_m = \psi_s\beta_{sm}$, whence $\rho\sigma = \beta_{sm}$. In addition,

$$\kappa(\psi_s\rho) = \kappa(\theta\rho') = \kappa(\rho') = f\kappa(\phi_n),$$

by Theorem 3.3.1.

By symmetry, the following property holds as well:

(II) Given any map $\rho : R_n \to S_m$ such that $\kappa(\psi_m\rho) = f\kappa(\phi_n)$, there exists an integer $t > n$ and a map $\sigma : S_m \to R_t$ such that $\sigma\rho = \alpha_{tn}$ and $\kappa(\phi_t\sigma) = f^{-1}\kappa(\psi_m)$.

We now proceed by induction to choose positive integers $n(1) < n(2) < \ldots$ and maps $\rho_q : R_{n(q)} \to S$ for all q such that

(a) $\rho_q(R_{n(q)}) \supseteq S_q$ for all q;

(b) $\kappa(\rho_q) = f\kappa(\phi_{n(q)})$ for all q;

(c) For all $q > 1$, the map ρ_{q-1} is injective, and $\rho_q\alpha_{n(q),n(q-1)} = \rho_{q-1}$.

To start, we use Theorem 3.2.1 to obtain a map $\sigma' : S_1 \to R$ such that $\kappa(\sigma') = f^{-1}\kappa(\psi_1)$. Since S_1 is finite dimensional, there exists a positive integer $n(1)$ such that $\sigma'(S_1) \subseteq R_{n(1)}$. Then σ' defines a map $\sigma : S_1 \to R_{n(1)}$ such that $\phi_{n(1)}\sigma = \sigma'$, and

$$\kappa(\phi_{n(1)}\sigma) = f^{-1}\kappa(\psi_1).$$

By (I), there exists an integer $s > 1$ and a map $\rho : R_{n(1)} \to S_s$ such that $\rho\sigma = \beta_{s1}$ and $\kappa(\psi_s\rho) = f\kappa(\phi_{n(1)})$. Setting $\rho_1 = \psi_s\rho$, we obtain a map $\rho_1 : R_{n(1)} \to S$ such that $\rho_1\sigma = \psi_s\beta_{s1} = \psi_1$ and $\kappa(\rho_1) = f\kappa(\phi_{n(1)})$. Note that

$$\rho_1(R_{n(1)}) \supseteq \rho_1\sigma(S_1) = \psi_1(S_1) = S_1.$$

Thus properties (a) and (b) hold for the case $q = 1$.

Now suppose that integers $n(1) < \ldots < n(q)$ and maps $\rho_1, \ldots, \rho_q$ have been chosen, for some q, satisfying the appropriate cases of (a),(b),(c). Since $R_{n(q)}$ is finite dimensional, there exists an integer $m > q$ such that $\rho_q(R_{n(q)}) \subseteq S_m$. Then ρ_q defines a map $\rho' : R_{n(q)} \to S_m$ such that $\psi_m\rho' = \rho_q$, and $\kappa(\psi_m\rho') = f\kappa(\phi_{n(q)})$. By (II), there exists an integer $n(q+1) > n(q)$ and a map $\sigma : S_m \to R_{n(q+1)}$ such that

$$\sigma\rho' = \alpha_{n(q+1),n(q)} \quad \text{and} \quad \kappa(\phi_{n(q+1)}\sigma) = f^{-1}\kappa(\psi_m).$$

As $\sigma\rho' = \alpha_{n(q+1),n(q)}$, we see that ρ' is injective, whence the map $\rho_q = \psi_m\rho'$ is injective.

By (I), there exists an integer $s > m$ and a map $\rho : R_{n(q+1)} \to S_s$ such that $\rho\sigma = \beta_{sm}$ and $\kappa(\psi_s\rho) = f\kappa(\phi_{n(q+1)})$. Setting $\rho_{q+1} = \psi_s\rho$, we obtain a map $\rho_{q+1} : R_{n(q+1)} \to S$ such that

$$\rho_{q+1}\sigma = \psi_s\rho\sigma = \psi_s\beta_{sm} = \psi_m \quad \text{and} \quad \kappa(\rho_{q+1}) = f\kappa(\phi_{n(q+1)}).$$

Since $m > q$, we see that

$$\rho_{q+1}(R_{n(q+1)}) \supseteq \rho_{q+1}\sigma(S_m) = \psi_m(S_m) = S_m \supseteq S_{q+1}$$

$$\rho_{q+1}\alpha_{n(q+1),n(q)} = \rho_{q+1}\sigma\rho' = \psi_m\rho' = \rho_q.$$

This completes the induction step.

As $n(1) < n(2) < \ldots$, the union of the subalgebras $R_{n(q)}$ equals R. Hence, the compatible maps ρ_q induce a map $\rho : R \to S$ such that $\rho\phi_{n(q)} = \rho_q$ for all q. Since each ρ_q is injective, ρ must be injective. On the other hand, $S_q \subseteq \rho_q(R_{n(q)}) \subseteq \rho(R)$ for all q, whence $\rho(R) = S$. Therefore ρ is an isomorphism of R onto S.

Finally, we have

$$\kappa(\rho)\kappa(\phi_{n(q)}) = \kappa(\rho\phi_{n(q)}) = \kappa(\rho_q) = f\kappa(\phi_{n(q)})$$

for all q. As $\{R, \phi_{n(q)}\}$ is a direct limit for the sequence of inclusion maps

$$R_{n(1)} \to R_{n(2)} \to \cdots ,$$

Proposition 3.1.2 says that $\{\kappa(R), \kappa(\phi_{n(q)})\}$ is a direct limit for the corresponding sequence

$$\kappa(R_{n(1)}) \to \kappa(R_{n(2)}) \to \cdots .$$

Thus $\kappa(\rho) = f$, by the uniqueness property of direct limits. ■

Corollary 3.4.2. *If $R \in \mathcal{R}$, then κ induces a surjective group homomorphism of $Aut(R)$ onto $Aut(\kappa(R))$.*

Remark. By Corollary 2.4.3, if Ω contains only fields, then $\kappa(R)$ is reduced to $(K_0(R), [R]) \xrightarrow{K_0(\mu)} (K_0(R^K), [R^K])$. If Ω contains only central division F-algebras, then $\kappa(R)$ is reduced to

$$\begin{array}{ccc}
(K_0(R), [R]) & \xrightarrow{K_0(\nu_1)} & (K_0(R^{D_1^{op}}), [R^{D_1^{op}}]) \\
1\downarrow & & \\
(K_0(R), [R]) & \xrightarrow{K_0(\nu_2)} & (K_0(R^{D_2^{op}}), [R^{D_2^{op}}]) \\
1\downarrow & & \\
\vdots & & \vdots \\
1\downarrow & & \\
(K_0(R), [R]) & \xrightarrow{K_0(\nu_\omega)} & (K_0(R^{D_\omega^{op}}), [R^{D_\omega^{op}}])
\end{array}$$

We now throw away the order units and obtain an invariant for Morita-equivalence in $\mathcal{R}$.

Definition. Let $\mathcal{L}$ denote the category whose objects are all diagrams of the form

$$G_1 \xrightarrow{g_1} G_2 \left\langle \begin{array}{l} \xrightarrow{g_{21}} G_{31} \\ \cdots\cdots \quad \cdots \\ \xrightarrow{g_{2\omega}} G_{3\omega} \end{array} \right.$$

in the category of pre-ordered abelian groups and positive homomorphisms, and G_2 is an ordered ZGal(K/F)-module. Morphisms in $\mathcal{L}$ are all ordered sets $(f_1, , f_2, f_{31}, \ldots, f_{3\omega})$ of positive homomorphisms, with f_2 being an ordered $\mathbb{Z}\mathrm{Gal}(K/F)$-module homomorphism, giving rise to commutative diagrams

$$\begin{array}{ccccc} G_1 & \xrightarrow{g_1} & G_2 & \xrightarrow{g_{2i}} & G_{3i} \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_{3i}}\downarrow \\ H_1 & \xrightarrow{h_1} & H_2 & \xrightarrow{h_{2i}} & H_{3i} \end{array}$$

for all $i = 1, \ldots, \omega$.

Definition. For any algebra $R \in \mathcal{R}$, we define $\mathcal{X}_R$ to be the object in $\mathcal{L}$ as below.

$$K_0(R) \xrightarrow{K_0(\mu_R)} K_0(R^K) \left\langle \begin{array}{l} \xrightarrow{K_0(\nu_{R,1})} K_0(R^{M_i(D_1^{op})}) \\ \cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_{R,\omega})} K_0(R^{M_i(D_\omega^{op})}) \end{array} \right.$$

By using the same argument as in [GH, Proposition 5.3 and Theorem 5.4], we obtain the followings:

Proposition 3.4.3. *Let $R \in \mathcal{R}$, let A be a finitely generated projective generator in Mod-R, and set $S = End_R(A)$. Set $A^X = A \otimes_R R^X$. Then $\kappa(S)$ is isomorphic to the diagram*

$$(K_0(R), [A]) \xrightarrow{K_0(\mu_R)} (K_0(R^K), [A^K]) \left\langle \begin{array}{l} \xrightarrow{K_0(\nu_{R,1})} (K_0(R^{M_i(D_1^{op})}), [A^{M_i(D_1^{op})}]) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_{R,\omega})} (K_0(R^{M_i(D_\omega^{op})}), [A^{M_i(D_\omega^{op})}]) \end{array} \right.$$

Proof. The functor $(-) \otimes_S A$ gives a category equivalence from Mod-S to Mod-R [S, Proposition IV.10.7], and so restricts to a category equivalence from the subcategory of finitely generated projective right S-modules to the subcategory of finitely generated projective right R-modules. Hence $(-) \otimes_S A$ induces an isomorphism

$$f_F : (K_0(S), [S]) \to (K_0(R), [A])$$

in $\mathcal{P}$, such that $f_F([B] - [C]) = [B \otimes_S A] - [C \otimes_S A]$.for all finitely generated projective right S-modules B, C.

As A is a finitely generated projective generator in Mod-R, it follows immediately that A^K is a finitely generated projective generator in Mod-R^K, and that A^{D_i} is a finitely generated projective generator in Mod-R^{D_i}). If S_K and S_i denote the endomorphism rings of A^K and A^{D_i}, then the functors $(-) \otimes_{S_K} A^K$ and $(-) \otimes_{S_i} A^{D_i}$ induce isomorphisms

$$g_K : (K_0(S_K), [S_K]) \to (K_0(R^K), [A^K]) \quad \text{and} \quad g_i : (K_0(S_i), [S_i]) \to (K_0(R^{D_i}), [A^{D_i}])$$

in $\mathcal{P}$.

There is a natural F-algebra map $\phi_K : S^K \to S_K$ such that

$$\phi_K(s \otimes \alpha)(a \otimes r) = sa \otimes \alpha r$$

for all $s \in S$, $\alpha \in K$, $a \in A$, $r \in R^K$, and it is easily checked that ϕ_K is an isomorphism. A natural F-algebra isomorphism $\phi_i : S^{D_i} \to S_i$ is constructed in the same manner. Consequently, we obtain isomorphisms

$$f_K = g_K K_0(\phi_K) : (K_0(S^K), [S^K]) \to (K_0(R^K), [A^K])$$
$$f_i = g_i K_0(\phi_i) : (K_0(S^{D_i}), [S^{D_i}]) \to (K_0(R^{D_i}), [A^{D_i}])$$

in $\mathcal{P}$. Moreover, f_K is a Gal(K/F)-map because $K_0(\phi_K)$ is.

Given any idempotent $e \in S$, we compute that

$$\begin{aligned} f_K K_0(\mu_S)([eS]) &= g_K([\phi_K \mu_S(e) S_K]) = [\phi_K \mu_S(e) S_K \otimes_{S_K} A^K] = [(\phi_K \mu_S(e))(A^K)] \\ &= [(\phi_K(e \otimes 1))(A \otimes_R R^K)] = [eA \otimes_R R^K] = K_0(\mu_R)([eA]) \\ &= K_0(\mu_R)([eS \otimes_S A]) = K_0(\mu_R) f_F([eS]). \end{aligned}$$

Since $K_0(S)$ is generated (as a group) by the elements $[eS]$ for idempotents $e \in S$, we conclude that $f_K K_0(\mu_S) = K_0(\mu_R) f_F$. A similar argument (incorporating the action of the map $\phi_i \nu_{S,i} (\phi_K)^{-1} : S_K \to S_i$) shows that $f_i K_0(\nu_{S,i}) = K_0(\nu_{R,i}) f_K$.

Therefore, $(f_F, f_K, f_1, \ldots, f_\omega)$ is an isomorphism of $\kappa(S)$ onto the given diagram. ■

Theorem 3.4.4. *Let $R, S \in \mathcal{R}$. Then R and S are Morita-equivalent (as F-algebras) if and only if $\mathcal{X}_R \cong \mathcal{X}_S$.*

Proof. If R and S are Morita-equivalent F-algebras, there exists a finitely generated projective generator A in Mod-R such that $S \cong \mathrm{End}_R(A)$ as F-algebras. In view of Proposition 3.4.3, $\kappa(S)$ is isomorphic to the diagram

$$Y: \quad (K_0(R),[A]) \xrightarrow{K_0(\mu_R)} (K_0(R^K),[A^K]) \left\langle \begin{array}{l} \xrightarrow{K_0(\nu_{R,1})} (K_0(R^{M_2(D_1^{op})}),[A^{M_2(D_1^{op})}]) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_{R,\omega})} (K_0(R^{M_2(D_\omega^{op})}),[A^{M_2(D_\omega^{op})}]) \end{array} \right.$$

from which we see that $\mathcal{X}_R \cong \mathcal{X}_S$.

Conversely, let $(f_F, f_K, f_1, \ldots, f_\omega) : \mathcal{X}_S \to \mathcal{X}_R$ be an isomorphism in $\mathcal{L}$. Then $f_F([S])$ is an order-unit in $K_0(R)$, whence $f_F([S]) = [A]$ for some finitely generated projective generator A in Mod-R. Observe that

$$f_K([S^K]) = f_K K_0(\mu_S)([S]) = K_0(\mu_R) f_F([S]) = K_0(\mu_R)([A]) = [A^K],$$

and similarly $f_i([S^{D_i}]) = [A^{D_i}]$. Consequently, $(f_F, f_K, f_1, \ldots, f_\omega)$ provides an isomorphism of $\kappa(S)$ onto Y in $\mathcal{K}$. On the other hand, if $T = \mathrm{End}_R(A)$, then $\kappa(T) \cong Y$ by Proposition 3.4.3, and so $\kappa(S) \cong \kappa(T)$. Then $S \cong T$ (as F-algebras) by Theorem 3.4.1, and therefore R and S are Morita-equivalent F-algebras. ■

3.5. Minimality of the invariant.

In this section we shall examine the necessity for each of $K_0(R)$, $K_0(R^K)$, and $K_0(R^{D_i^{op}})$ in $\kappa(R)$ of a given set Ω. We shall show that the appearance of $K_0(R)$ and $K_0(R^{D_i^{op}})$'s in $\kappa(R)$ is necessary only if $F, D_i \in \Omega$. On the other hand, the appearance of $K_0(R^K)$ and its module structure is crucial. Finally we show that our invariant is essentially the same as those invariants found by G. A. Elliott, K. R. Goodearl and D. E. Handelman in the cases $F = \mathbf{C}$ and $F = \mathbf{R}$.

The necessity of $K_0(R)$: So the first question is

(a) if $F \notin \Omega$, then can we discard $K_0(R)$ from $\kappa(R)$?

The answer is "Yes". Denote this new invariant by κ'. Then κ' satisfies Theorem 3.4.1, if it satisfies Theorems 3.2.1 and 3.3.1. Certainly, κ' satisfies Theorem 3.3.1. Now we note that Proposition 3.1.2 and Lemma 3.2.2 still hold for κ'. Therefore κ' satisfies Theorem 3.2.1 (hence Theorem 3.4.1) if the following is true:

Proposition 3.5. *Let $R = M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m)$ and $S = M_t(D)$, where D_i's and D are division F-algebras different from F. If $f : \kappa'(R) \to \kappa'(S)$ is a map in the category corresponding to κ', then there exists an F-algebra map $\phi : R \to S$ such that $\kappa'(\phi) = f$.*

Proof. We prove this for $m = 2$. Let $R_1 = M_{n_1}(D_1)$ and $R_2 = M_{n_2}(D_2)$. We write out $\kappa'(R)$ as

$$(K_0(R^K), [R^K]) \begin{array}{l} \xrightarrow{K_0(\nu_1)} (K_0(R^{M_t(D_1^{op})}), [R^{M_t(D_1^{op})}]) \\ \cdots\cdots\cdots\cdots\cdots \\ \xrightarrow{K_0(\nu_2)} (K_0(R^{M_t(D_2^{op})}), [R^{M_t(D_2^{op})}]) \end{array}$$

and similarly for $\kappa'(S)$. Let $f = (g, h, k, l)$ where

$g : (K_0(R^K), [R^K]) \to (K_0(S^K), [S^K])$,

$h : (K_0(R^{M_{t_1}(D_1^{op})}), [R^{M_{t_1}(D_1^{op})}]) \to (K_0(S^{M_{t_1}(D_1^{op})}), [S^{M_{t_1}(D_1^{op})}])$,

$k : (K_0(R^{M_{t_2}(D_2^{op})}), [R^{M_{t_2}(D_2^{op})}]) \to (K_0(S^{M_{t_2}(D_2^{op})}), [S^{M_{t_2}(D_2^{op})}])$.

and l represents all other similar maps. Let write in the matrix forms $\nu^R_{D'}$=diag$(\nu^{R_1}_{D'}, \nu^{R_2}_{D'})$ for all $D' \in \Omega \setminus \{K\}$, $g = (g_1, g_2)$, $h = (h_1, h_2)$, $k = (k_1, k_2)$ and $l = (l_1, l_2)$ corresponding to R_1 and R_2. Then

$$\nu^S_{D_1^{op}} g = h \nu^R_{D_1^{op}} \quad \text{implies} \quad \nu^S_{D_1^{op}} g_i = h_i \nu^{R_i}_{D_1^{op}}$$
$$\nu^S_{D_2^{op}} g = k \nu^R_{D_2^{op}} \quad \text{implies} \quad \nu^S_{D_2^{op}} g_i = k_i \nu^{R_i}_{D_2^{op}}$$

for $i = 1, 2$. Then proceed as in the proof of Proposition 2.4.4, there exist F-algebra maps $\phi_i : R_i \to S$ such that $K_0(\phi_i^K) = (g_i)$ for i= $1, 2$. We can check that the map $\phi : R \to S$ defined by $\phi(x, y) = diag(\phi_1(x), \phi_2(y))$ satisfies $\kappa'(\phi) = f$ (as in the proof of Theorem 3.2.1 or Proposition 2.4.4, $K_0(\phi^K) = g$ forces other components of $\kappa'(\phi)$ and f to be identical). ■

Note that if $F \in \Omega$, then we can not discard $K_0(R)$ from $\kappa(R)$ [GH, Section 11].

The necessity of $K_0(R^K)$: The second and third questions are

(b) if $K \notin \Omega$, then can we eliminate $K_0(R^K)$ from $\kappa(R)$? and

(c) do we need the ordered $\mathbf{Z}\mathrm{Gal}(K/F)$-module structure of $K_0(R^K)$?

The answer is "No" for (b) and is "Yes" for (c) as given in the next example

Example 3.5.1. Let K/F be a field extension with $\mathrm{Gal}(K/F) = S_4$. Let E/F be the subfield fixed by the subgroup S_3. Then $[E : F] = 4$ and $E \otimes_F E \cong E \times E'$, where E'/F is a field extension of index 12. Let $\Omega = \{E, F\}$. Let $p \in \mathbf{N}$ and R_p be the direct limit of the sequence

$$E \times M_p(F) \to E \times M_{p+4}(F) \to M_m(E) \times M_{p+8}(F) \to \cdots,$$

where each of the maps $E \times M_{p+4n}(F) \to E \times M_{p+4n+4}(F)$ is given by the rule $(x, y) \mapsto (x, diag(\lambda(x), y))$, with $\lambda : E \to M_4(F)$ being the regular map.

Each morphism $\kappa(E \times M_q(F)) \to \kappa(E \times M_{q+4}(F))$ is isomorphic to the diagram shown in Figure 3-1. (Note that we use the regular embedding $K \to M_6(E)$.)

$$\begin{array}{ccccc}
(\mathbf{Z}^2, (1,q)^{tr}) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} & (\mathbf{Z}^5, (1,1,1,1,q)^{tr}) & \xrightarrow{\begin{pmatrix} 6&0&0&0&0 \\ 0&2&2&2&0 \\ 0&0&0&0&6 \end{pmatrix}} & (\mathbf{Z}^3, (6,6,6q)^{tr}) \\
\Big\downarrow \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&1&1&1&1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 1&3&1 \end{pmatrix} \\
(\mathbf{Z}^2, (1,q+4)^{tr}) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} & (\mathbf{Z}^5, (1,1,1,1,q+4)^{tr}) & \xrightarrow{\begin{pmatrix} 6&0&0&0&0 \\ 0&2&2&2&0 \\ 0&0&0&0&6 \end{pmatrix}} & (\mathbf{Z}^3, (6,6,6q+24)^{tr})
\end{array}$$

Figure 3-1

Let $G_1 = \mathbf{Z}^2$ as abelian group with

$$G_1^+ = \{(a, b)^{tr} \in G_1 \mid a > 0\} \cup \{(0, b)^{tr} \in G_1 \mid b \geq 0\};$$

$G_2 = \mathbf{Z}^5$ as abelian group with

$$G_2^+ = \{(a_1, \ldots, a_5)^{tr} \in G_2 \mid a_1, \ldots, a_4 > 0\} \cup \{(0, \ldots, 0, a_5)^{tr} \in G_2 \mid a_5 \geq 0\};$$

$G_3 = \mathbf{Z}^3$ as abelian group with

$$G_2^+ = \{(a_1, a_2, a_3)^{tr} \in G_2 \mid a_1, a_2 > 0\} \cup \{(0, 0, a_3)^{tr} \in G_2 \mid a_3 \geq 0\}.$$

A direct limit in $\mathcal{K}$ for the sequence of morphisms described above is the diagram

$$\mathrm{K}_p : (G_1, (1,p)^{tr}) \xrightarrow{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} (G_2, (1,1,1,1,p)^{tr}) \xrightarrow{\begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}} (G_3, (6m, 6m, 6p)^{tr})$$

together with the morphism shown in Figure 3-2.

$$\begin{array}{ccccc}
(\mathbf{Z}^2, (1, p+4n)^{tr}) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} & (\mathbf{Z}^5, (1,1,1,1,p+4n)^{tr}) & \xrightarrow{\begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}} & (\mathbf{Z}^3, (6, 6, 6p+24n)^{tr}) \\
\Big\downarrow \begin{pmatrix} 1 & 0 \\ -4n & 1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ -n & \cdots & -n & -n \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -n & -3n & 1 \end{pmatrix} \\
(G_1, (1,p)^{tr}) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} & (G_2, (1,1,1,1,p)^{tr}) & \xrightarrow{\begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}} & (G_3, (6,6,6p)^{tr})
\end{array}$$

Figure 3-2

Note that the vertical maps are the n^{th} powers of the counterparts in Figure 3-1. With this morphism, the module structure of G_2 is determined. Thus $\kappa(R_p) \cong \mathrm{K}_p$. For positive integers p, q, if $(G_2, (1,1,1,1,p)^{tr}) \cong (G_2, (1,1,1,1,q)^{tr})$ as modules, then there exists an 5×5 invertible matrix A such that

$$A(1, 0, \ldots, 0)^{tr}, \ldots, A(0, \ldots, 0, 1)^{tr} > 0$$

which implies that the first four rows of A have only nonnegative entries. Then

$$A(1, 0, \ldots, 0, x)^{tr} > 0, \quad \forall x \in \mathbf{Z}$$

implies that the last column of A is of the form $(0,\ldots,0,a)^{tr}$. Then because $det(A) = 1$, we have $a = 1$. Since ZGal(K/F) permutes $K_0(E^K)$ but acts trivially on $K_0(F^K)$, the last row of A is $(b,\ldots,b,1)$. Then

$$A(1,1,1,1,p)^{tr} = (1,1,1,1,q)^{tr}$$

implies that $b = (q-p)/4$. Hence $p \equiv q (\text{mod } 4)$. Conversely, if $p \equiv q(\text{mod } 4)$, then $(G_2,(1,1,1,1,p)^{tr}) \cong (G_2,(1,1,1,1,q)^{tr})$ by the matrix

$$\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{q-p}{4} & \cdots & \frac{q-p}{4} & 1 \end{pmatrix}.$$

In particular, in case $R_p \cong R_q$, we have $(G,(1,1,1,1,p)^{tr}) \cong (G,(1,1,1,1,q)^{tr})$ and so $p \equiv q(\text{mod } 4)$. Conversely, if $p \equiv q(\text{mod } 4)$, there exists an isomorphism

$$\left(\begin{pmatrix} 1 & 0 \\ q-p & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{q-p}{4} & \cdots & \frac{q-p}{4} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{q-p}{4} & \frac{3(q-p)}{4} & 1 \end{pmatrix} \right)$$

from K_p onto K_q, whence $R_p \cong R_q$.

On the other hand, if $p \equiv q(\text{mod } 2)$, then we have the triple of ordered group isomorphisms

$$\left(\begin{pmatrix} 1 & 0 \\ q-p & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ q-p & 0\ldots & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q-p & 0 & 1 \end{pmatrix} \right).$$

Note that the middle map is not a module homomorphism. This answers both (a) and (b).

The necessity of each $K_0(R^{D_i^{op}})$: Finally, suppose the set Ω contains division F-algebras D's with the same centre E such that these $[D]$'s form a subgroup of the Brauer group $B(E)$. Then instead of including all of $K_0(R^{D^{op}})$'s into $\kappa_\Omega(R)$, can we simplify $\kappa_\Omega(R)$ by just including only $K_0(R^{D_o^{op}})$'s into $\kappa_\Omega(R)$, where $[D_o]$'s are the generators of this subgroup? The answer is "No" by means of the following example.

Example 3.5.2. Let D be a central division F-algebra of degree 5. Then there are central division F-algebras D_1 and D_2 such that $[D_1]$ and $[D_2]$ are distinct elements in the cyclic group generated by $[D]$ and $D \otimes D_i \cong M_5(D_i')$ for $i = 1, 2$. Consider $\Omega = \{F, D, D_1, D_2\}$ and $\kappa'(R)$ defined by

$$\kappa'(R) := K_0(R) \to K_0(R^D).$$

Then we have

$$\kappa'(D_i) \cong (K_0(D_i), [D_i]) \to (K_0(D_i^{D^{op}}), [D_i^{D^{op}}]) \cong (\mathbf{Z}, 1) \xrightarrow{5} (\mathbf{Z}, 5).$$

Hence $\kappa'(D_1) \cong \kappa'(D_2)$, but $D_1 \ncong D_2$.

The cases $F = \mathbf{C}$ and $F = \mathbf{R}$: It is easily seen that if $F = \mathbf{C}$ then our invariant becomes $K_0(-)$ as found by G. A. Elliott. Now let take a look at the invariant found by K. R. Goodearl and D. E. Handelman for the case $F = \mathbf{R}$. By the proof of [Lemma 3.7, GH] and an analogy of it, we can see that if the triple (f, g, h) of morphisms in $\mathcal{P}$ makes the diagram

$$\begin{array}{ccccc} (K_0(R), [R]) & \xrightarrow{K_0(\sigma_R)} & (K_0(R^{\mathbf{C}}), [R^{\mathbf{C}}]) & \xrightarrow{K_0(\tau_R)} & (K_0(R^{\mathbf{H}}), [R^{\mathbf{H}}]) \\ f\downarrow & & g\downarrow & & h\downarrow \\ (K_0(S), [S]) & \xrightarrow{K_0(\sigma_S)} & (K_0(S^{\mathbf{C}}), [S^{\mathbf{C}}]) & \xrightarrow{K_0(\tau_S)} & (K_0(S^{\mathbf{H}}), [S^{\mathbf{H}}]) \end{array}$$

commutative then g is also an module homomorphism even though the module structure of $K_0((-)^{\mathbf{C}})$ is not declared. In a sense, the inclusions $\mathbf{R} \to \mathbf{C} \to \mathbf{H}$ define the module structure on $K_0((-)^{\mathbf{C}})$. Thus our invariant and the invariant in [GH] are essentially the same. This suggests a new question:

Can the module structure on $K_0((-)^K)$ be omitted if Ω contains a large enough number of subfields and division algebras (central or not)? The answer is "No"; this results from examining the case $[K : F] = 3$.

Chapter 4

Some Properties of the Invariant and Applications

Motivated by interesting results in [GH, VI and VII], we shall study the D-"branch" of the diagram $\kappa(R)$, which is the triple

$$\kappa_D(R) \equiv (K_0(R), [R]) \xrightarrow{K_0(\mu_R)} (K_0(R^K), [R^K]) \xrightarrow{K_0(\nu_{R,D})} (K_0(R^{M_r(D^{op})}), [R^{M_r(D^{op})}]),$$

here r denotes a positive integer. After that, we use these properties to determine whether a given algebra $R \in \mathcal{R}$ could be constructed as a direct limit of finite dimensional semisimple F-algebra in which only F-matrix algebras appear, or only K-matrix algebras appear, or both K-matrix algebras and D-matrix algebras appear, etc. We refer to them as types f,k,fd, etc.

In this chapter, we only consider the case $\Omega = \{F, K, D_1, \ldots, D_\omega\}$, where K/F is a Galois extension and D_i's are central division F-algebras containing K as a subfield. Corollary 4.2.6 is quite interesting. It states that if degrees of all D_i are $[K : F] = p$, a prime integer, then an algebra is of type d depends solely on $K_0(\nu_{R,D})$. One point worth being mentioned here is that in general, the invariant $\kappa(-)$ is expected to be a diagram of K_0-groups associated to the lattice of inclusions between the division algebras occuring in Ω. Proposition 4.1.5 shows that these inclusions are already incorporated into $\kappa(-)$ in this particular case.

4.1. Some properties of the invariant.

The main results of this section are Propositions 4.1.7 and 4.1.9, where relationships between $K_0(\mu_R)$ and $K_0(\nu_{R,D})$ exist in special cases.

Proposition 4.1.1. *Let $n \in \mathbf{N}$.*

(a) $\kappa_D(M_n(F)) \cong (\mathbf{Z}, n) \xrightarrow{1} (\mathbf{Z}, n) \xrightarrow{1} (\mathbf{Z}, n)$

(b) $\kappa_D(M_n(K)) \cong (\mathbf{Z}, n) \xrightarrow{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}} (\mathbf{Z}, n) \times \cdots \times (\mathbf{Z}, n) \xrightarrow{(1,\ldots,1)} (\mathbf{Z}, n[K:F])$

(c) $\kappa_D(M_n(D')) \cong (\mathbf{Z}, n) \xrightarrow{[K:F]} (\mathbf{Z}, n[K:F]) \xrightarrow{s/[K:F]} (\mathbf{Z}, n.s)$

where s is the matrix size of $D' \otimes D$.

Proof. Follows from Proposition 1.1.11 and Lemmas 2.1.2, 2.1.3. ■

Lemma 4.1.2. *If $R \in \mathcal{R}$, then*

(a) $K_0(\mu_R)$ and $K_0(\nu_{R,D}\mu_R)$ are one to one.

(b) $[K:F]K_0(R^K)^+ \subseteq K_0(\mu_R)(K_0(R)^+) + ker(K_0(\nu_{R,D}))$.

(c) $K_0(\mu_R)(K_0(R)) + [K:F]K_0(R^K) = K_0(\mu_R)(K_0(R)) + ker(K_0(\nu_{R,D}))$ for all $R \in \mathcal{R}$ if and only if $[K:F] \leq 2$.

Proof. As these properties are preserved by finite direct products and countable direct limits, it suffices to consider the cases that R is one of $M_n(F), M_n(K)$ and $M_n(D')$, for some $n \in \mathbf{N}$. In these cases, (a) and (b) are clear from Proposition 4.1.1. Moreover, (b) implies

$$\begin{array}{lrcl} & K_0(\mu_R)(K_0(R)) + [K:F]K_0(R^K) & \subseteq & K_0(\mu_R)(K_0(R)) + \ker(K_0(\nu_{R,D})). \\ \text{Hence} & K_0(\mu_R)(K_0(R)) + [K:F]K_0(R^K) & = & K_0(\mu_R)(K_0(R)) + \ker(K_0(\nu_{R,D})) \\ \text{if and only if} & \ker(K_0(\nu_{R,D})) & \subseteq & K_0(\mu_R)(K_0(R)) + [K:F]K_0(R^K). \end{array}$$

We can easily check from Proposition 4.1.1 that this is true for $R = M_n(F)$ or $R = M_n(D')$, but true for $R = M_n(K)$ if and only if $[K:F] \leq 2$. ■

For any algebra $R \in \mathcal{R}$, we may tensor the identity map on R with an embedding $K \to M_{[K:F]}(F)$ to obtain a map $R^K \to M_{[K:F]}(R)$. As

$$(K_0(M_{[K:F]}(R)), [M_{[K:F]}(R)]) \cong (K_0(R), [K:F].[R]),$$

we obtain a morphism $(K_0(R^K), [R^K]) \to (K_0(R), [K:F].[R])$, which we may view as an auxiliary map for the diagram $\kappa_D(R)$. This map is uniquely determined by the diagram $\kappa_D(R)$, and is to obtained directly from the diagram, as follows

Proposition 4.1.3. *Given any $R \in \mathcal{R}$, there is a unique morphism*

$$s_{R,D} : (K_0(R^K), [R^K]) \to (K_0(R), [K:F].[R])$$

in $\mathcal{P}$ such that $K_0(\nu_{R,D}\mu_R)s_{R,D} = [K:F]K_0(\nu_{R,D})$. Moreover, $s_{R,D}K_0(\mu_R)$ is multiplication by $[K:F]$ on $K_0(R)$ and $ker(s_{R,D}) = ker(K_0(\nu_{R,D}))$.

Proof. We relabel $\kappa_D(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

By Lemma 4.1.2, f_1 and f_2f_1 are one to one, while also

$$[K:F]G_2^+ \subseteq f_1(G_1^+) + \ker(f_2) \quad \text{and hence} \quad [K:F]G_2 \subseteq f_1(G_1) + \ker(f_2).$$

Consequently, $[K:F]f_2(G_2^+) \subseteq f_2f_1(G_1^+)$ and $[K:F]f_2(G_2) \subseteq f_2f_1(G_1)$. Let $s_{R,D} : G_2 \to G_1$ be the composition of the homomorphisms

$$G_2 \xrightarrow{f_2} f_2(G_2) \xrightarrow{[K:F]} [K:F]f_2(G_2) \xrightarrow{\subseteq} f_2f_1(G_1) \xrightarrow{(f_2f_1)^{-1}} G_1.$$

Since $[K:F]f_2(G_2^+) \subseteq f_2f_1(G_1^+)$, we have $s_{R,D}(G_2^+) \subseteq G_1^+$. As $[K:F]f_2(u_2) = f_2f_1([K:F]u_1)$, we also have $s_{R,D}(u_2) = [K:F]u_1$. Thus $s_{R,D}$ is a morphism from (G_2, u_2) to (G_1, u_1).

From the abbreviated expression $s_{R,D} = (f_2f_1)^{-1}([K:F]f_2)$ it is clear that $f_2f_1s_{R,D} = [K:F]f_2$. As f_2f_1 is one to one, the uniqueness of $s_{R,D}$ is immediate from this relation.

Note that $f_2f_1s_{R,D}f_1 = [K:F]f_2f_1$, whence $s_{R,D}f_1 = [K:F]$. Finally, using $s_{R,D} = (f_2f_1)^{-1}([K:F]f_2)$ again, we conclude that $\ker(s_{R,D}) = \ker(f_2)$. ■

Lemma 4.1.4. *If $R \in \mathcal{R}$, then*

(a) $deg(D)^2 K_0(R^{D^{op}})^+ \subseteq K_0(\nu_{R,D})[[K:F]K_0(R^K)^+ \cap K_0(\mu_R)(K_0(R)^+)]$.

(b) $deg(D)^2 K_0(R^{D^{op}}) \subseteq K_0(\nu_{R,D}\mu_R)(K_0(R))$.

(c) $(deg(D)^2/[K:F])K_0(R^{D^{op}}) \subseteq K_0(\nu_{R,D})(K_0(R^K))$.

Proof. Reduce to the finite dimensional simple case and use Proposition 4.1.1. ■

In parallel with Proposition 4.1.3, we may tensor the identity map on R with embeddings $D^{op} \to M_{p_D}(K)$, where $p_D = \deg(D)^2/[K:F]$ (Proposition 2.2). So there are auxiliary maps associated with any diagram $\kappa(R)$, which could be constructed as K_0 of the map $R^{D^{op}} \to M_{p_D}(R^K)$ composed with the isomorphism

$$(K_0(M_{p_D}(R^K)), [M_{p_D}(R^K)]) \cong (K_0(R^K), p_D[R^K]).$$

Once again, this map may be obtained directly from the diagram $\kappa(R)$, as follows

Proposition 4.1.5. *Given any $R \in \mathcal{R}$, there are unique morphisms*

$$t_{R,D} : (K_0(R^{D^{op}}), [R^{D^{op}}]) \to (K_0(R^K), p_D[R^K])$$

in $\mathcal{P}$ such that $t_{R,D}K_0(\nu_{R,D}) = (p_D/[K:F])K_0(\mu_R)s_{R,D}$. Moreover, $K_0(\nu_{R,D})t_{R,D}$ is multiplication by p_D on $K_0(R^{D^{op}})$ and $t_{R,D}K_0(\nu_{R,D}\mu_R) = p_D K_0(\mu_R)$. Also, $s_{R,D}t_{R,D}K_0(\nu_{R,D}\mu_R)$ and $K_0(\nu_{R,D}\mu_R)s_{R,D}t_{R,D}$ each equal multiplication by $\deg(D)^2$.

Proof. We relabel $\kappa_D(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

Then by Lemma 4.1.4, $p_D[K:F]G_3^+ \subseteq f_2([K:F]G_2^+ \cap f_1(G_1^+))$ and so $p_D[K:F]G_3 \subseteq f_2([K:F]G_2 \cap f_1(G_1))$ as well. Since $f_2 f_1$ is one to one by Lemma 4.1.2, the restriction of f_2 to $f_1(G_1)$ is one to one.

Let $t_{R,D} : G_3 \to G_2$ be the composition of the homomorphisms

$$G_3 \xrightarrow{p_D[K:F]} p_D[K:F]G_3 \xrightarrow{\subseteq} f_2([K:F]G_2 \cap f_1(G_1)) \xrightarrow{f_2^{-1}} [K:F]G_2 \cap f_1(G_1) \xrightarrow{1/[K:F]} G_2.$$

From $p_D[K:F]G_3^+ \subseteq f_2([K:F]G_2^+ \cap f_1(G_1^+))$ we obtain $t_{R,D}(G_3^+) \subseteq G_2^+$. As $p_D[K:F]u_3 = f_2(p_D[K:F]u_2)$ and $p_D[K:F]u_2 \in [K:F]G_2 \cap f_1(G_1)$, we also have $t_{R,D}(u_3) = p_D u_2$. Thus $t_{R,D}$ is a morphism from (G_3, u_3) to $(G_2, p_D u_2)$.

Given any $x \in G_2$, by Proposition 4.1.1 we have $[K:F]x = f_1(y) + z$ for some $y \in G_1$ and some $z \in \ker(f_2)$. Then by Proposition 4.1.3, $[K:F]f_2(x) = f_2 f_1(y)$ and so $s_{R,D}(x) = y$. Since $p_D[K:F]f_2(x) = f_2(p_D f_1(y))$ with $p_D f_1(y) \in [K:F]G_2 \cap f_1(G_1)$, we see that $t_{R,D} f_2(x) = (p_D/[K:F])f_1(y) = (p_D/[K:F])f_1 s_{R,D}(x)$. Thus $t_{R,D} f_2 = (p_D/[K:F])f_1 s_{R,D}$. If $t : G_3 \to G_2$ is any homomorphism satisfying $t f_2 = (p_D/[K:F])f_1 s_{R,D}$, then t and $t_{R,D}$ agree on $f_2(G_2)$. As $p_D G_3 \subseteq f_2(G_2)$ by Lemma 4.1.4, we find that $p_D t = p_D t_{R,D}$, and hence $t = t_{R,D}$ because G_2 is torsion-free. Therefore $t_{R,D}$ is unique.

Note that $f_2 t_{R,D} f_2 = (p_D/[K:F])f_2 f_1 s_{R,D} = p_D f_2$ by Proposition 4.1.3. Since $p_D G_3 \subseteq f_2(G_2)$, it follows that $p_D f_2 t_{R,D} = {p_D}^2$, and so $f_2 t_{R,D} = p_D$. Also $t_{R,D} f_2 f_1 = (p_D/[K:F])f_1 s_{R,D} f_1 = p_D f_1$ by Proposition 4.1.3. Finally, $s_{R,D} t_{R,D} f_2 f_1 = p_D s_{R,D} f_1 = p_D[K:F]$ and $f_2 f_1 s_{R,D} t_{R,D} = [K:F]f_2 t_{R,D} = p_D[K:F]$. ■

Remark. If the division algebras in Ω are not central then Lemma 4.1.2(b) and Propositions 4.1.3, 4.1.5 are not true.

Lemma 4.1.6. *Let m be a divisor of $[K:F]$. For any $R \in \mathcal{R}$,*

(a) $mK_0(R^{D^{op}})^+ \cap K_0(\nu_{R,D}\mu_R)(K_0(R)) \subseteq mK_0(\nu_{R,D})(K_0(R^K)^+)$.

(b) $mK_0(R^{D^{op}}) \cap K_0(\nu_{R,D}\mu_R)(K_0(R)) \subseteq mK_0(\nu_{R,D})(K_0(R^K))$.

Let $[K:F]^2$ divide the matrix size of $D^{op} \otimes D'$, for all central division algebras $D' \in \Omega$. For any $R \in \mathcal{R}$,

(c) $[K:F]K_0(R^K)^+ \cap K_0(\mu_R)(K_0(R)) \subseteq [K:F]K_0(\mu_R)(K_0(R)^+) + K_0(\nu_{R,D})^{-1}([K:F]^2 K_0(R^{D^{op}})^+)$.

(d) $[K:F]K_0(R^K) \cap K_0(\mu_R)(K_0(R)) \subseteq [K:F]K_0(\mu_R)(K_0(R)) + K_0(\nu_{R,D})^{-1}([K:F]^2 K_0(R^{D^{op}}))$.

Let s denote the matrix size of $D^{op} \otimes D'$. Let $gcd([K:F], \frac{deg(D)^2}{s}) = 1$, for all central division algebras $D' \in \Omega$. For any $R \in \mathcal{R}$,

(e) $[K:F]K_0(R^K)^+ \cap K_0(\mu_R)(K_0(R)) \subseteq [K:F]K_0(\mu_R)(K_0(R)^+) + K_0(\nu_{R,D})^{-1}(deg(D)^2 K_0(R^{D^{op}})^+)$.

(f) $[K:F]K_0(R^K)\cap K_0(\mu_R)(K_0(R)) \subseteq [K:F]K_0(\mu_R)(K_0(R))+K_0(\nu_{R,D})^{-1}(deg(D)^2K_0(R^{D^{op}}))$.

Proof. Reduce to the finite dimensional simple case and use Proposition 4.1.1. Note that $\gcd([K:F], \frac{\deg(D)^2}{s}) = 1$ implies that $[K:F]^2$ divides s because $[K:F]^2$ divides $\deg(D)^2$. ■

Remark. In fact, the conditions on $[K:F]^2$ and s are forced upon by (c),(d) and (e),(f) respectively. In turn, conditions in (c),(d),(e), and (f) are chosen so that we are able to obtain further results.

Proposition 4.1.7. *Let $R \in \mathcal{R}$, consider the following conditions:*

(a) $K_0(\mu_R)(K_0(R)) = K_0(R^K)$.

(b) $K_0(\mu_R)$ *is an ordered group isomorphism.*

(c) $K_0(\nu_{R,D})$ *is a group isomorphism.*

(d) $K_0(\nu_{R,D})$ *is an ordered group isomorphism.*

We have (d) $\Rightarrow$ (c), and (a) $\Leftrightarrow$ (b). If $\deg(D)$ and $[K:F]$ have the same prime factors, then (b) $\Rightarrow$ (d). If $[K:F]^2$ divides the matrix size of $D^{op} \otimes D'$, for all central division algebras $D' \in \Omega$, then (c) $\Rightarrow$ (a).

Proof. We relabel $\kappa_D(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

(a) $\Rightarrow$ (b): As f_1 is one to one by Lemma 4.1.2, it must be a group isomorphism, and as f_1 is positive, $f_1(G_1^+) \subseteq G_2^+$. On the other hand, $G_2^+ = f_1 f_1^{-1}(G_2^+)$ beccause f_1 is surjective. Using Proposition 4.1.3, we compute that

$$[K:F]f_1^{-1}(G_2^+) = s_{R,D} f_1 f_1^{-1}(G_2^+) = s_{R,D}(G_2^+) \subseteq G_1^+.$$

Since G_1 is unperforated, $f_1^{-1}(G_2^+) \subseteq G_1^+$, and thus $G_2^+ = f_1(G_1^+)$. Therefore f_1 is an ordered group isomorphism.

(b) $\Rightarrow$ (a), and (d) $\Rightarrow$ (c): priories.

(c) $\Rightarrow$ (a): Since f_2 is one to one, Lemma 4.1.2(b) shows that $[K:F]G_2 \subseteq f_1(G_1)$ (because G_2 is directed). In view of Lemma 4.1.6(d),

$$[K:F]G_2 = [K:F]G_2 \cap f_1(G_1) \subseteq [K:F]f_1(G_1) + f_2^{-1}([K:F]^2G_3).$$

As f_2 is a group isomorphism, $f_2^{-1}([K:F]^2G_3) = [K:F]^2G_2 \subseteq [K:F]f_1(G_1)$. Hence $[K:F]G_2 \subseteq [K:F]f_1(G_1)$ and therefore $f_1(G_1) = G_2$.

(b) $\Rightarrow$ (d): Since f_2f_1 is one to one by Lemma 4.1.2, we first see that f_2 is one to one. By Lemma 4.1.4(c), $p_DG_3 \subseteq f_2(G_2)$. Let m be a divisor of $[K:F]$. Using Lemma 4.1.6(a), we obtain

$$p_DG_3^+ = p_DG_3^+ \cap f_2(G_2) \subseteq mG_3^+ \cap f_2f_1(G_1) \subseteq mf_2(G_2^+),$$

and so $(p_D/m)G_3^+ \subseteq f_2(G_2^+)$. Thus by repeating the procedure on the common prime factors of $[K:F]$ and $\deg(D)$, we finally have $G_3^+ \subseteq f_2(G_2^+)$. Then $f_2(G_2^+) = G_3^+$ (because f_2 is positive) and consequently, $f_2(G_2) = G_3$ (because G_3 is directed). Thus f_2 is an ordered group isomorphism. ■

Example. Let $[K:F] = \deg(D') = \deg(D) = 2$ and let $D \neq D'^{op}$. Then $\kappa_D(D')$ is isomorphic to

$$(\mathbf{Z},1) \xrightarrow{2} (\mathbf{Z},2) \xrightarrow{1} (\mathbf{Z},2).$$

Hence $\kappa_D(R)$ satisfies (c) but not (a). Now suppose that $\deg(D)$ has prime factors which do not divide $[K:F]$. Consider

$$R = D \to M_{[K:F]}(D) \to M_{[K:F]^2}(D) \to M_{[K:F]^3}(D) \to \cdots .$$

Then $\kappa_D(R)$ is isomorphic to

$$(\mathbf{Z}[1/[K:F]],1) \xrightarrow{[K:F]} (\mathbf{Z}[1/[K:F]],1) \xrightarrow{p_D} (\mathbf{Z}[1/[K:F]],1).$$

Since $\mathbf{Z}[1/[K:F]] = [K:F]\mathbf{Z}[1/[K:F]]$ but $\mathbf{Z}[1/[K:F]] \neq p_D\mathbf{Z}[1/[K:F]]$ (p_D and $[K:F]$ do not have the same prime factors), $\kappa_D(R)$ satisfies (b) but not (d).

Proposition 4.1.8. *Let $R \in \mathcal{R}$ and let $q_D = (deg(D)/[K:F])^2$. Suppose that for all central division algebras $D' \in \Omega$, we have $gcd([K:F], \frac{deg(D)^2}{s}) = 1$, where s is the matrix size of $D' \otimes D^{op}$. Then $t_{R,D}$ induces a group isomorphism of $K_0(R^{D^{op}})/(1/q_D)K_0(\nu_{R,D})(K_0(R^K))$ onto the torsion subgroup of $K_0(R^K)/K_0(\mu_R)(K_0(R))$. Here $(1/q_D)K_0(\nu_{R,D})(K_0(R^K))$ denotes the subgroup $\{z \in K_0(R^{D'^{op}}) \mid q_D z \in K_0(\nu_{R,D})(K_0(R^K))\}$.*

Proof. We relabel $\kappa_D(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

As $t_{R,D} f_2 = q_D f_1 s_{R,D}$, we see that $t_{R,D}$ induces a homomorphism

$$t^o_{R,D} : G_3/(1/q_D)f_2(G_2) \to G_2/f_1(G_1).$$

Since $p_D G_3 = f_2 t_{R,D}(G_3)$, the group $G_3/f_2(G_2)$ is torsion. Hence if A denotes the torsion subgroup of $G_2/f_1(G_1)$, then $t^o_{R,D}$ maps $G_3/(1/q_D)f_2(G_2)$ into A.

Given a coset $z + (1/q_D)f_2(G_2)$ in $\ker(t^o_{R,D})$, we have $t_{R,D}(z) = f_1(x)$ for some $x \in G_1$. Then $p_D z = f_2 t_{R,D}(z) = f_2 f_1(x)$, and so $p_D z = [K:F]f_2(y)$ for some $y \in G_2$, by Lemma 4.1.6(b). Hence, $q_D z = f_2(y)$ and so $z + (1/q_D)f_2(G_2) = 0$. Thus $t^o_{R,D}$ is one to one.

Given a coset $y + f_1(G_1)$ in A, we have $my = f_1(x)$ for some $m \in \mathbb{N}$ and some $x \in G_1$. Then $s_{R,D}(my) = s_{R,D} f_1(x) = [K:F]x$, whence $[K:F]my = f_1([K:F]x) = f_1 s_{R,D}(my)$. Consequently, $[K:F]q_D y = q_D f_1 s_{R,D}(y) = t_{R,D} f_2(y)$. Since $[K:F]y \in f_1(G_1)$, Lemma 4.1.6(f) shows that $[K:F]y = [K:F]f_1(v) + w$ for some $v \in G_1$ and some $w \in f_2^{-1}(\deg(D)^2 G_3)$. Then

$$\begin{aligned}[K:F]^2 q_D y &= t_{R,D} f_2([K:F]y) = [K:F]t_{R,D} f_2 f_1(v) + t_{R,D} f_2(w) \\ &= [K:F]p_D f_1(v) + t_{R,D}(\deg(D)^2 z)\end{aligned}$$

for some $z \in G_3$, whence $y = f_1(v) + t_{R,D}(z)$, and so

$$y + f_1(G_1) = t^o_{R,D}(z + (1/q_D)f_2(G_2)).$$

Therefore $t^o_{R,D}$ is an isomorphism of $G_3/(1/q_D)f_2(G_2)$ onto A. ∎

Proposition 4.1.9. *Let $R \in \mathcal{R}$, consider the following conditions:*

(a) $K_0(\mu_R)(K_0(R)) = [K:F]K_0(R^K)$.

(b) $K_0(\mu_R)$ *is an ordered group isomorphism of* $K_0(R)$ *onto* $[K:F]K_0(R^K)$.

(c) $K_0(\nu_{R,D})$ *is a group isomorphism of* $K_0(R^K)$ *onto* $p_D K_0(R^{D^{op}})$.

(d) $K_0(\nu_{R,D})$ *is an ordered group isomorphism of* $K_0(R^K)$ *onto* $p_D K_0(R^{D^{op}})$.

We have (c) ⇒ (d), (d) ⇒ (a), and (a) ⇒ (b). If $\deg(D)$ *and* $[K:F]$ *have the same prime factors and if* $D \subseteq D'$ *for all* $D' \in \Omega$, *then (b) ⇒ (c).*

Note that the condition above is equivalent to: $\deg(D)$ and $[K:F]$ have the same prime factors and $\gcd([K:F], \frac{\deg(D)^2}{s}) = 1$, for all central division algebras $D' \in \Omega$.

Proof. Relabel $\kappa_D(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

(a) ⇒ (b): As f_1 is one to one by Lemma 4.1.2, it must be a group isomorphism of G_1 onto $[K:F]G_2$. Then $f_1(G_1^+) \subseteq G_2^+ \cap [K:F]G_2 = [K:F]G_2^+$, because f_1 is positive and G_2 is unperforated. Now from $f_1(G_1) = [K:F]G_2$ we have $[K:F]G_2^+ = f_1 f_1^{-1}([K:F]G_2^+)$. Using Proposition 4.1.3, we compute that

$$[K:F]f_1^{-1}([K:F]G_2^+) = s_{R,D} f_1 f_1^{-1}([K:F]G_2^+) = s_{R,D}([K:F]G_2^+) \subseteq [K:F]G_2^+,$$

whence $f_1^{-1}([K:F]G_2^+) \subseteq G_1^+$ and so $[K:F]G_2^+ \subseteq f_1(G_1^+)$. Thus

$$f_1(G_1^+) = [K:F]G_2^+ = G_2^+ \cap [K:F]G_2,$$

and therefore f_1 is an ordered group isomorphism of G_1 onto $[K:F]G_2$.

(b) ⇒ (c): By Lemma 4.1.2, $f_2 f_1$ is one to one. Since $[K:F]\ker(f_2) \subseteq [K:F]G_2 = f_1(G_1)$, it follows that $[K:F]\ker(f_2) = \{0\}$, whence f_2 is one to one. According to Proposition 4.1.8, $t_{R,D}$ induces a group isomorphism of $G_3/(1/q_D)f_2(G_2)$ onto the torsion subgroup of $G_2/f_1(G_1)$. As $G_2/f_1(G_1) = G_2/[K:F]G_2$ is a torsion group, it follows that $G_2 = t_{R,D}(G_3) + [K:F]G_2$. Using Proposition 4.1.5, we find that

$$f_2(G_2) = f_2 t_{R,D}(G_3) + [K:F]f_2(G_2) = p_D G_3 + [K:F]f_2(G_2).$$

Hence

$$f_2(G_2) = p_D G_3 + [K:F](p_D G_3 + [K:F] f_2(G_2)) = p_D G_3 + [K:F]^2 f_2(G_2) = \dots$$
$$= p_D G_3 + [K:F]^n f_2(G_2),$$

for all $n \in \mathbb{N}$. Since there exists an $n \in \mathbb{N}$ such that p_D divides $[K:F]^n$, we have $f_2(G_2) = p_D G_3$. Thus f_2 gives a group isomorphism of G_2 onto $p_D G_3$.

(c) $\Rightarrow$ (d): We have $f_2(G_2^+) \subseteq G_3^+ \cap p_D G_3 = p_D G_3^+$ because f_2 is positive and G_3 is unperforated. Conversely, using Proposition 4.1.5 we see that $p_D G_3^+ = f_2 t_{R,D}(G_3^+) \subseteq f_2(G_2^+)$. Thus $f_2(G_2^+) = p_D G_3^+ = G_3^+ \cap p_D G_3$.

(d) $\Rightarrow$ (a): Since f_2 is one to one and G_2 is directed, Lemma 4.1.2(b) shows that $[K:F]G_2 \subseteq f_1(G_1)$. As $f_2 f_1(G_1) \subseteq f_2(G_2) = p_D G_3$, we have

$$f_2 f_1(G_1) = p_D G_3 \cap f_2 f_1(G_1) \subseteq [K:F] f_2(G_2) = f_2([K:F]G_2)$$

by Lemma 4.1.6(b). Then $f_1(G_1) \subseteq [K:F]G_2$, and therefore $f_1(G_1) = [K:F]G_2$. ■

Example. Suppose that $\deg(D)$ has prime factors which do not divide $[K:F]$. Let

$$R = F \to M_{[K:F]}(F) \to M_{[K:F]^2}(F) \to M_{[K:F]^3}(F) \to \cdots.$$

Then $\kappa_D(R)$ is isomorphic to

$$(\mathbb{Z}[1/[K:F]], 1) \xrightarrow{1} (\mathbb{Z}[1/[K:F]], 1) \xrightarrow{1} (\mathbb{Z}[1/[K:F]], 1).$$

Since $\mathbb{Z}[1/[K:F]] = [K:F]\mathbb{Z}[1/[K:F]]$ but $\mathbb{Z}[1/[K:F]] \neq p_D \mathbb{Z}[1/[K:F]]$ (p_D and $[K:F]$ do not have the same prime factors), $\kappa_D(R)$ satisfies (a) but not (d). Hence (b) $\not\Rightarrow$ (c). On the other hand, by letting $[K:F] = \deg(D') = \deg(D) = 2$ and $D \neq D'^{op}$. Then $\kappa_D(D')$ is isomorphic to

$$(\mathbb{Z}, 1) \xrightarrow{2} (\mathbb{Z}, 2) \xrightarrow{1} (\mathbb{Z}, 2).$$

Hence (b) $\not\Rightarrow$ (c) even though $[K:F] = \deg(D)$.

4.2. Applications.

In this section, we shall try to use conditions on $\kappa(R)$ to determine the type of R. Corollary 4.2.6 is an interesting result, which says that if $[K:F]$ is a prime number and K is a maximal subfield of all $D_i \in \Omega$ then R could be constructed with only D if and only if $K_0(\nu_{R,D})$ is a group isomorphism of $K_0(R^K)$ onto $[K:F]K_0(R^{D^{op}})$.

Definition. Let w be one of the letter combinations f,k,d_i and let W be the corresponding set $\{F\},\{K\},\{D_i\}$. We shall say that an algebra $R \in \mathcal{R}$ is of **type** w provided R is isomorphic to a direct limit of a sequence $R_1 \to R_2 \to \cdots$ of maps from $\mathcal{R}$, such that each R_n is isomorphic to a direct product of matrix algebras $M_\bullet(D_\bullet)$ where $D_\bullet \in$ W.

In particular, an algebra $R \in \mathcal{R}$ is of type f if and only if R is an ultramatricial F-algebra. However, if R is of type k, then R need not be an ultramatricial K-algebra.

Lemma 4.2.1. *If $K_0(\nu_{R,D})$ is one to one for some $D \in \Omega$, then Gal(K/F) acts trivially on $K_0(R^K)$.*

Proof. Let $\sum x \otimes y$ be an idempotent in R^K and let $\sigma \in \text{Gal}(K/F)$. There exists an inner automorphism θ of D^{op} such that $\theta_{|K} = \sigma$ by Noether-Skolem Theorem. Then $id_R \otimes \theta$ is an inner automorphism of $R^{D^{op}}$ such that $(\text{id}_R \otimes \theta)(\sum x \otimes y) = \sum x \otimes \sigma(y)$. By Proposition 1.3.5, $K_0(\text{id}_R \otimes \theta)$ is the identity map of $K_0(R^{D^{op}})$, whence $[\sum x \otimes y] = [\sum x \otimes \sigma(y)]$ in $K_0(R^{D^{op}})$. Hence for all $\sigma \in \text{Gal}(K/F)$,

$$\begin{aligned} K_0(\nu_{R,D})(\sigma.[\textstyle\sum x \otimes y]) &= K_0(\nu_{R,D})([\textstyle\sum x \otimes \sigma(y)]) \\ &= [\textstyle\sum x \otimes \sigma(y)] \quad \in K_0(R^{D^{op}}) \\ &= [\textstyle\sum x \otimes y] \\ &= K_0(\nu_{R,D})([\textstyle\sum x \otimes y]). \end{aligned}$$

Because $K_0(\nu_{R,D})$ is one to one, we have $\sigma.[\sum x \otimes y] = [\sum x \otimes y]$ in $K_0(R^K)$, whence Gal(K/F) acts trivially on $K_0(R^K)$. ∎

Theorem 4.2.2. *Given $R \in \mathcal{R}$, consider the following conditions:*

(a) R is an ultramatricial F-algebra.

(b) $K_0(\mu_R)$ *is surjective.*

(c) $K_0(\nu_{R,D})$ *is a group isomorphism for all* $D \in \Omega$.

(d) There exists a countable dimension group (G, u) *with order unit such that* $\kappa(R)$ *is isomorphic to*

$$(G,u) \xrightarrow{1} (G,u) \begin{cases} \xrightarrow{1} (G,u) \\ \cdots\cdots \quad \cdots\cdots \\ \xrightarrow{1} (G,u) \end{cases}$$

where Gal(K/F) acts trivially on (G, u)*. Any such diagram above is isomorphic to* $\kappa(S)$ *for some unital ultramatricial F-algebra S.*

Then (a)⇔(d), and (a)⇔(b)∧(c). If $\deg(D)$ *and* $[K : F]$ *have the same prime factors for all* $D \in \Omega$*, then (b)⇒(a). If there exists* $D \in \Omega$ *such that* $[K : F]^2$ *divides the matrix size of* $D^{op} \otimes D'$ *for all* $D' \in \Omega$*, then (c)⇒(a).*

Proof. (a)⇒(b),(c),(d): Reduce to the case that $R = M_n(F)$ for some $n \in \mathbb{N}$, and use Proposition 4.1.1.

Given an arbitrary countable dimension group (G, u) with order unit. By Theorem 1.2.5, there exists a unital ultramatricial F-algebra S such that $(K_0(S), [S]) \cong (G, u)$. Applying the implication (a)⇒(d) to the algebra S, we see that $\kappa(S)$ is isomorphic to the given diagram.

(d)⇒(a): By previous paragraph, there exists a unital ultramatricial F-algebra S such that $\kappa(S) \cong \kappa(R)$. Then $R \cong S$ by Theorem 3.4.1, and therefore R is a unital ultramatricial F-algebra.

(b)∧(c)⇒(d): Set $(G, u) = (K_0(R), [R])$ and observe that G is a countable dimension group. As $K_0(\mu_R)$ and $K_0(\nu_{R,D})$ are isomorphisms, $\kappa(R)$ is isomorphic to the given diagram.

(b)⇒(a): By Proposition 4.1.7 and by (b)∧(c)⇒(d).

(c)⇒(a): By Proposition 4.1.7, $K_0(\mu_R)$ is an isomorphism in $\mathcal{P}$, whence $\kappa(R)$ is isomorphic to the given diagram. ■

Theorem 4.2.3. *Given $R \in \mathcal{R}$, the following conditions are equivalent:*

(a) There is an (unital) F-algebra embedding of K into the centre of R.

(b) R is isomorphic (as an F-algebra) to some unital ultramatricial K-algebra.

(c) $R \cong S^K$ for some unital ultramatricial F-algebra S.

(d) There exists a countable dimension group (G, u) with order unit such that $\kappa(R)$ is isomorphic to

$$(G,u) \xrightarrow{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}} (G,u)\times\cdots\times(G,u) \begin{array}{l} \xrightarrow{(1\dots1)} (G,[K:F]u) \\ \quad\cdots\,(1\dots1)\,\cdots \qquad \cdots\cdots \\ \xrightarrow{} (G,[K:F]u) \end{array}$$

where Gal(K/F) permutes elements of $(G,u)\times\cdots\times(G,u)$ the same way it permutes elements of $K_0(K^K)$. Conversely, any such diagram is isomorphic to $\kappa(S^K)$ for some unital ultramatricial F-algebra S.

Proof. (a)⇒(d): Let R be the union of a sequence $R_1 \subseteq R_2 \subseteq \dots$ of finite dimensional semisimple unital subalgebras. Then since $[K : F]$ is finite, we can assume that there is an (unital) F-algebra embedding of K into the centre of R_n for all n. Hence each R_n is a finite direct product of $M_\bullet(K)$. Then by applying Proposition 4.1.1, we see that $\kappa(R)$ is isomorphic to the given diagram.

Given an arbitrary countable dimension group (G, u) with order unit, there exists a unital ultramatricial F-algebra S such that $(K_0(S), [S]) \cong (G, u)$ by Theorem 1.2.5. Then Theorem 4.2.2 shows that $(K_0(S^K), [S^K]) \cong (G, u)$. Applying the implication (a)⇒(d) to the algebra S^K, we see that $\kappa(S^K)$ is isomorphic to the given diagram.

(d)⇒(c): By the previous paragraph, there exists a unital ultramatricial F-algebra S such that $\kappa(S^K) \cong \kappa(R)$. Then $R \cong S^K$ by Theorem 3.4.1.

(c)⇒(b)⇒(a): Clear. ■

There are algebras in $\mathcal{R}$ of type k which are not isomorphic to S^K for some ultramatricial F-algebra S. For instance, let $\lambda : M_n(K) \to M_{n[K:F]}(F)$ be the regular maps and

$\iota : M_n(F) \to M_n(K)$ be the inclusion maps. Then the direct limit

$$F \xrightarrow{\iota} K \xrightarrow{\lambda} M_{[K:F]}(F) \xrightarrow{\iota} M_{[K:F]}(K) \xrightarrow{\lambda} M_{[K:F]^2}(F) \xrightarrow{\iota} \cdots$$

is isomorphic to both

$$K \xrightarrow{\iota\lambda} M_{[K:F]}(K) \xrightarrow{\iota\lambda} M_{[K:F]^2}(K) \xrightarrow{\iota\lambda} \cdots \quad \text{and}$$

$$F \xrightarrow{\lambda\iota} M_{[K:F]}(F) \xrightarrow{\lambda\iota} M_{[K:F]^2}(F) \xrightarrow{\lambda\iota} \cdots$$

whence this direct limit, let denote by R, is an ultramatricial F-algebra of type k. Since $(K_0(R), [R])$ is isomorphic to

$$(\mathbf{Z}, 1) \xrightarrow{[K:F]} (\mathbf{Z}, [K:F]) \xrightarrow{[K:F]} (\mathbf{Z}, [K:F]^2) \xrightarrow{[K:F]} \cdots$$

we have $(K_0(R), [R]) \cong (\mathbf{Z}[1/[K:F]], 1)$, where $\mathbf{Z}[1/[K:F]]$ is the unital subring of $\mathbf{Q}$ generated by $1/[K:F]$. By Theorem 4.2.2, $(K_0(R^K), [R^K]) \cong (\mathbf{Z}[1/[K:F]], 1)$. Since $\mathbf{Z}[1/[K:F]] \ncong \mathbf{Z}[1/[K:F]] \times \mathbf{Z}[1/[K:F]] \times \cdots$, we conclude from Theorem 4.2.3 that R can not be isomorphic to a K-algebra. This example is very similar to [GH, Example 7.3].

Theorem 4.2.4. *Given $R \in \mathcal{R}$. Let $D \neq F$ be a central division in Ω and let s' denote the matrix size of $D \otimes D'^{op}$. Then the following conditions are equivalent:*

(a) R is of type d.

(b) $R \cong S^D$ for some unital ultramatricial F-algebra S.

(c) $K_0(\mu_R)(K_0(R)) = [K:F]K_0(R^K)$ and $K_0(\nu_{R,D'})(K_0(R^K)) = (s'/[K:F])K_0(R^{D'^{op}})$ for all $D' \in \Omega$.

(d) $K_0(\nu_{R,D'})$ is a group isomorphism of $K_0(R^K)$ onto $(s'/[K:F])K_0(R^{D'^{op}})$ for all $D' \in \Omega$.

(e) There exists a countable dimension group (G, u) with order unit such that $\kappa(R)$ is isomorphic to

$$(G,u) \xrightarrow{[K:F]} (G,u) \begin{cases} \xrightarrow{s_1/[K:F]} (G,u) \\ \quad \cdots \qquad\qquad \cdots \\ \xrightarrow{s_\omega/[K:F]} (G,u) \end{cases}$$

where Gal(K/F) acts trivially on $(G, [K:F]u)$. Conversely, any such diagram is isomorphic to $\kappa(S^D)$ for some unital ultramatricial F-algebra S.

Proof. (a)⇒(c): Reduce to the case $R = M_n(D)$ for some $n \in \mathbb{N}$ and apply Proposition 4.1.1.

(c)⇒(d): By Proposition 4.1.9, $K_0(\mu_R)$ is an ordered group isomorphism of $K_0(R)$ onto $[K:F]K_0(R^K)$. By Lemma 4.1.2, $K_0(\nu_{R,D'})K_0(\mu_R)$ is one to one for all $D' \in \Omega$. Since $[K:F]\ker(K_0(\nu_{R,D'})) \subseteq [K:F]K_0(R^K) = K_0(\mu_R)(K_0(R))$, it follows that $[K:F]\ker(K_0(\nu_{R,D'})) = \{0\}$, whence $K_0(\nu_{R,D'})$ is one to one for all $D' \in \Omega$.

(d)⇒(e): Set $(G,u) = (K_0(R),[R])$, and relabel $\kappa_{D'}(R)$ as

$$(G,u) \xrightarrow{f_1} (G_2,u_2) \xrightarrow{f_{2,D'}} (G_{3,D'},u_{3,D'})$$

where $\mathrm{Gal}(K/F)$ acts trivially on (G_2,u_2) by Lemma 4.2.1. Then by Proposition 4.1.9, f_1 gives an ordered group isomorphism of G onto $[K:F]G_2$, and $f_{2,D'}$ gives an ordered group isomorphism of G_2 onto $s'/[K:F]G_{3,D'}$. Hence, $f_{2,D'}f_1$ gives an ordered group isomorphism of G onto $s'G_{3,D'}$. Consequently, there is an isomorphism $(1, (1/[K:F])f_1, (1/s')f_{2,D'}f_1)$ from the diagram

$$(G,u) \xrightarrow{[K:F]} (G,[K:F]u) \xrightarrow{s'/[K:F]} (G,s'u),$$

where $\mathrm{Gal}(K/F)$ acts trivially on $(G,[K:F]u)$, onto $\kappa_{D'}(R)$.

Given an arbitrary countable dimension group (G,u) with order unit. By Theorem 1.2.5, there exists a unital ultramatricial F-algebra S such that $(K_0(S),[S]) \cong (G,u)$. Then Theorem 4.2.2 shows that $(K_0(S^D),[S^D]) \cong (G,u)$. Applying the implication (a)⇒(e) to the algebra S^D, we see that $\kappa(S^D)$ is isomorphic to the given diagram.

(e)⇒(b): By the previous paragraph there exists a unital ultramatricial F-algebra S such that $\kappa(S^D) \cong \kappa(R)$. Then $R \cong S^D$ by Theorem 3.4.1.

(b)⇒(a): Clear. ■

Corollary 4.2.5. *Let R be an ultramatricial F-algebra. Then $R \cong R^D$ if and only if $K_0(R)$ is deg(D)-divisible.*

Proof. By Theorem 4.2.2, $\kappa_{D'}(R)$ is isomorphic to the diagram

$$X_{D'}: \quad (G,u) \xrightarrow{1} (G,u) \xrightarrow{1} (G,u)$$

where $(G,u) = (K_0(R),[R])$. Then $(K_0(R^D),[R^D]) \cong (G,u)$, whence Theorem 4.2.4 shows that $\kappa_{D'}(R^D)$ is isomorphic to the diagram

$$Y_{D'}: \quad (G,u) \xrightarrow{[K:F]} (G,[K:F]u) \xrightarrow{s'/[K:F]} (G,s'u).$$

If $R \cong R^D$, then $X_D \cong Y_D$, whence $G = \deg(D)^2/[K:F]G$. Hence G is $\deg(D)$-divisible. Conversely, if $G = \deg(D)G$, then for all $D' \in \Omega$, the map $(1,[K:F],s') : X_{D'} \to Y_{D'}$ is an isomorphism because both $[K:F]$ and s' divide $\deg(D)$, and so $\kappa(R) \cong \kappa(R^{D^{op}})$. In this case $R \cong R^D$ by Theorem 3.4.1. (Throughout the proof, $\mathrm{Gal}(K/F)$ acts trivially).
■

Corollary 4.2.6. *Suppose that for all $D' \in \Omega$, we have $[K:F] = deg(D') = p$, where p is a prime integer. Then given $R \in \mathcal{R}$, the following conditions are equivalent:*

(a) R is of type d.

(b) $K_0(\mu_R)(K_0(R)) = pK_0(R^K)$ and $K_0(\nu_{R,D})(K_0(R^K)) = pK_0(R^{D^{op}})$.

(c) $K_0(\nu_{R,D})$ is a group isomorphism of $K_0(R^K)$ onto $pK_0(R^{D^{op}})$.

Proof. Let $D' \neq D$ and consider the triple

$$(K_0(R^{D'^{op}}),[R^{D'^{op}}]) \xrightarrow{t_{R,D'}} (K_0(R^K),p[R^K]) \xrightarrow{K_0(\nu_{R,D})} (K_0(R^{D^{op}}),p[R^{D^{op}}]).$$

By reducing to the finite dimensional case, we have

(i) $t_{R,D'}$ and $K_0(\nu_{R,D})t_{R,D'}$ are one to one.

(ii) $pK_0(R^K)^+ \subseteq t_{R,D'}(K_0(R^{D'^{op}})^+) + \ker(K_0(\nu_{R,D}))$.

(iii) $pK_0(R^{D^{op}}) \cap K_0(\nu_{R,D})t_{R,D'}(K_0(R^{D'^{op}})) \subseteq pK_0(\nu_{R,D})(K_0(R^K))$.

(b)⇒(c): It suffices to prove that $K_0(\nu_{R,D})$ is one to one. Since $p\ker(K_0(\nu_{R,D})) \subseteq pK_0(R^K) = K_0(\mu_R)(K_0(R))$ and $K_0(\nu_{R,D})t_{R,D'}$ is one to one, it follows that $p\ker(K_0(\nu_{R,D})) = \{0\}$, whence $K_0(\nu_{R,D})$ is one to one.

(c)⇒(a): Since $K_0(\nu_{R,D})$ is one to one and $K_0(R^K)$ is directed, from (ii) we have $pK_0(R^K) \subseteq t_{R,D'}(K_0(R^{D'^{op}}))$. As $K_0(\nu_{R,D})t_{R,D'}(K_0(R^{D'^{op}})) \subseteq K_0(\nu_{R,D})(K_0(R^K)) = pK_0(R^{D^{op}})$, from (iii) we obtain

$$K_0(\nu_{R,D})t_{R,D'}(K_0(R^{D'^{op}})) = pK_0(R^{D^{op}} \cap K_0(\nu_{R,D})t_{R,D'}(K_0(R^{D'^{op}})) \subseteq K_0(\nu_{R,D})(pK_0(R^K)).$$

Hence $t_{R,D'}(K_0(R^{D'^{op}})) \subseteq pK_0(R^K)$, and therefore $t_{R,D'}(K_0(R^{D'^{op}})) = pK_0(R^K)$. Then by Proposition 4.1.5, we have

$$pK_0(R^{D'^{op}}) = K_0(\nu_{R,D'})t_{R,D'}(K_0(R^{D'^{op}})) = K_0(\nu_{R,D'})(pK_0(R^K)),$$

whence $K_0(R^{D'^{op}}) = K_0(\nu_{R,D'})(K_0(R^K))$. On the other hand, $K_0(\mu_R)(K_0(R)) = pK_0(R^K)$ by Proposition 4.1.9. Therefore R is of type d by Theorem 4.2.4. ■

The following parallels the work done in [GH, VII]. Now we assume that $\Omega = \{F, K, D\}$, where D is a central division F-algebra containing K. We shall determine whether a given algebra could be constructed as a direct limit of finite dimensional semisimple F-algebra in which only F and D-matrix algebras appear, or only F and K-matrix algebras, etc. We refer to them as types fk,fd,kd, each of which requires a preparatory lemma. The first lemma will also be used in our characterization of type k.

Lemma 4.2.7. *Let $\phi : R \to T$ be a map in $\mathcal{R}_0$. If*

$$K_0(\phi^{D^{op}})(K_0(R^{D^{op}})) \subseteq K_0(\nu_T)(K_0(T^K)),$$

then ϕ factors through an algebra in $\mathcal{R}_0$ of type fk. If in addition R and T are both of type kd, then ϕ factors through an algebra in $\mathcal{R}_0$ of type k.

Proof. Write $T = T_1 \times \ldots \times T_n$ where each T_m is a simple algebra, and write $\phi = (\phi_1, \ldots, \phi_n)$ where each ϕ_m is a map from R to T_m. If each ϕ_m factors through an algebra $S_m \in \mathcal{R}_0$ of type fk (respectively, type k), then ϕ factors through the algebra $S_1 \times \ldots \times S_n$ of type fk (respectively, type k). Thus we may assume that T is one of $M_t(F), M_t(K), M_t(D)$ (respectively, $M_t(K)$ or $M_t(D)$), for some $t \in \mathbb{N}$. In the first two cases T is of type fk (respectively, in the first case T is of type k) and there is nothing to prove. Therefore we may assume that $T = M_t(D)$).

We now proceed by induction on the number of $M_\bullet(D)$) that appears in R. It suffices to show that when $R = R' \times M_n(D)$ for some $R' \in \mathcal{R}_0$ and some $n \in \mathbb{N}$, and $S = S' \times M_{pn}(K)$, there exist maps $\psi : R \to S$ and $\eta : S \to T$ such that $\eta\psi = \phi$ and

$$K_0(\eta^{D^{op}})(K_0(S^{D^{op}})) \subseteq K_0(\nu_T)(K_0(T^K)).$$

In view of Theorems 3.2.1 and 3.3.1, it is enough to find morphisms $h : \kappa(R) \to \kappa(S)$ and $k : \kappa(S) \to \kappa(T)$ such that $kh = \kappa(\phi)$ and

$$k_3(K_0(S^{D^{op}})) \subseteq K_0(\nu_T)(K_0(T^K)).$$

The diagrams $\kappa(R)$, $\kappa(S)$, and $\kappa(T)$ may of course be replaced by any isomorphic diagrams, and for $\kappa(M_\bullet(D))$ and $\kappa(M_\bullet(K))$ we shall use Proposition 4.1.1. Then $\kappa(\phi)$ takes the form shown in Figure 4-1, where $f_1 = K_0(\mu_{R'})$ and $f_2 = K_0(\nu_{R'})$, while g_1, g_2, g_3 are maps in $\mathcal{P}$ and a, b, c are positive integers.

$$\begin{array}{ccccc}
(G_1,u_1)\times(\mathbb{Z},n) & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & [K:F] \end{pmatrix}} & (G_2,u_2)\times(\mathbb{Z},[K:F]n) & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ 0 & p_D \end{pmatrix}} & (G_3,u_3)\times(\mathbb{Z},\deg(D)^2 n) \\
\Big\downarrow{\scriptstyle (g_1,a)} & & \Big\downarrow{\scriptstyle (g_2,b)} & & \Big\downarrow{\scriptstyle (g_3,c)} \\
(\mathbb{Z},t) & \xrightarrow{[K:F]} & (\mathbb{Z},[K:F]t) & \xrightarrow{p_D} & (\mathbb{Z},\deg(D)^2 t)
\end{array}$$

Figure 4-1

The hypothesis $K_0(\phi^{D^{op}})(K_0(R^{D^{op}})) \subseteq K_0(\nu_T)(K_0(T^K))$ says that $g_3(G_3)+c\mathbb{Z} \subseteq p_D\mathbb{Z}$. In particular, $c = p_D d$ for some $d \in \mathbb{Z}^+$. From the commutativity of the diagram, $a = b = c$.

Now let h and k be the top and bottom morphisms in Figure 4-2. It is clear that the middle vertical maps are module homomorphisms and that $kh = \kappa(\phi)$. Since $g_3(G_3) + c\mathbb{Z} \subseteq p_D\mathbb{Z}$, we also have

$$k_3(K_0(S^{D^{op}})) \subseteq K_0(\nu_T)(K_0(T^K)). \blacksquare$$

$$
\begin{array}{ccccc}
(G_1,u_1)\times(\mathbf{Z},n) & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & [K:F] \end{pmatrix}} & (G_2,u_2)\times(\mathbf{Z},[K:F]n) & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ 0 & p_D \end{pmatrix}} & (G_3,u_3)\times(\mathbf{Z},\deg(D)^2 n) \\
\begin{pmatrix} 1 & 0 \\ 0 & p_D \end{pmatrix}\Big\downarrow & & \begin{pmatrix} 1 & 0 \\ 0 & q_D \\ \vdots & \vdots \\ 0 & q_D \end{pmatrix}\Big\downarrow & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big\downarrow \\
(G_1,u_1)\times(\mathbf{Z},p_D n) & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}} & (G_2,u_2)\times(\mathbf{Z},p_D n)^{[K:F]} & \xrightarrow{\begin{pmatrix} f_2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{pmatrix}} & (G_3,u_3)\times(\mathbf{Z},\deg(D)^2 n) \\
(g_1,d)\Big\downarrow & & (g_2,d,\ldots,d)\Big\downarrow & & (g_3,c)\Big\downarrow \\
(\mathbf{Z},t) & \xrightarrow{[K:F]} & (\mathbf{Z},[K:F]t) & \xrightarrow{p_D} & (\mathbf{Z},\deg(D)^2 t)
\end{array}
$$

Figure 4-2

Theorem 4.2.8. *Given $R \in \mathcal{R}$, consider the following conditions:*

(a) R is of type fk.

(b) coker$(K_0(\mu_R))$ is torsion-free.

(c) $K_0(\nu_R)$ is surjective.

Then (a)⇔(c)⇒(b). Moreover, if K is a maximal subfield of D then (b)⇒(c).

Proof. (a)⇒(c): Reduce to the cases that $R = M_n(F)$ or $R = M_n(K)$, for some $n \in \mathbf{N}$, and apply Proposition 4.1.1.

(b)⇔(c): By Proposition 4.1.8. Note that $q_D = 1$ in this case.

(c)⇒(a): Write R as the direct limt of a sequence

$$R(1) \xrightarrow{\phi_1} R(2) \xrightarrow{\phi_2} R(3) \xrightarrow{\phi_3} \cdots$$

of maps from $\mathcal{R}_0$. Since $K_0(R(1)^{D^{op}})$ is finitely generated, it follows from (c) that

$$K_0(\phi_{m-1}^{D^{op}}\phi_{m-2}^{D^{op}}\cdots\phi_1^{D^{op}})(K_0(R(1)^{D^{op}})) \subseteq K_0(\nu_{R(m)})(K_0(R(m)^K))$$

for some integer $m \geq 1$. Hence, after telescoping the sequence, we may assume that

$$K_0(\phi_1^{D^{op}})(K_0(R(1)^{D^{op}})) \subseteq K_0(\nu_{R(2)})(K_0(R(2)^K)).$$

Similarly, we may assume that $K_0(\phi_n^{D^{op}})(K_0(R(n)^{D^{op}})) \subseteq K_0(\nu_{R(n+1)})(K_0(R(n+1)^K))$ for all $n \in \mathbb{N}$.

By Lemma 4.2.5, each ϕ_n factors into a composition

$$R(n) \xrightarrow{\psi_n} S(n) \xrightarrow{\eta_n} R(n+1)$$

where S_n is an algebra in $\mathcal{R}_0$ of type fk. Now R is isomorphic to the direct limit of the sequence

$$S_1 \xrightarrow{\psi_2\eta_1} S_2 \xrightarrow{\psi_3\eta_2} S_3 \xrightarrow{\psi_4\eta_3} \cdots$$

and therefore R is of type fk. ■

Remark. Again let $R = D \to M_{[K:F]}(D) \to M_{[K:F]^2}(D) \to \cdots$ as in the proof of Proposition 4.1.7. Then $coker(K_0(\mu_R)) = 0$ (hence torsion-free) but $K_0(\nu_R)$ is not surjective (hence R is not of type fk) if $\deg(D)$ and $[K:F]$ do not have the same prime factors. One then can take $S = R \times K$ so that $coker(K_0(\mu_S)) \neq 0$ but still torsion-free and $K_0(\nu_S)$ is still not surjective. From this counter-example, one may ask whether (b)⇒(c) if $\deg(D)$ and $[K:F]$ have the same prime factors?

Lemma 4.2.9. *Let $\phi : R \to T$ be a map in $\mathcal{R}_0$. If*

$$ker(K_0(\nu_R)) \subseteq ker(K_0(\phi^K)),$$

then ϕ factors through an algebra in $\mathcal{R}_0$ of type fd.

Proof. Proceeding as in Lemma 4.2.7, it suffices to show that if $R = R' \times M_n(K)$ and $S = R' \times M_{[K:F]n}(F)$ while $T = M_t(K)$, for some $R' \in \mathcal{R}_0$ and some $n, t \in \mathbb{N}$, then there exist maps $h : \kappa(R) \to \kappa(S)$ and $k : \kappa(S) \to \kappa(T)$ such that $kh = \kappa(\phi)$ and $\ker(K_0(\nu_S)) \subseteq \ker(q_2)$.

The diagram $\kappa(\phi)$ takes the form shown in Figure 4-3.

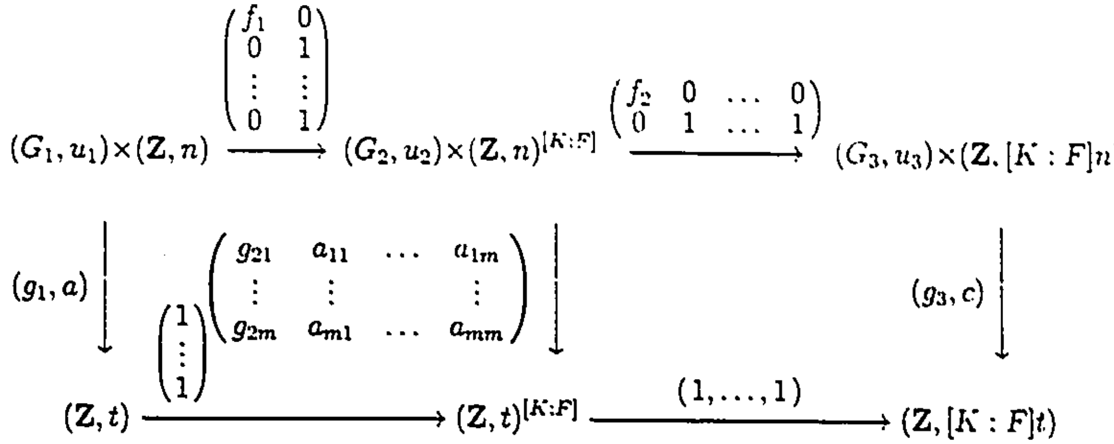

Figure 4-3

where $m = [K : F]$. We have,

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix} \in \ker \begin{bmatrix} f_2 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 1 \end{bmatrix} \subseteq \ker \begin{bmatrix} g_{21} & a_{11} & \ldots & a_{1[K:F]} \\ \vdots & \vdots & & \vdots \\ g_{2[K:F]} & a_{[K:F]1} & \ldots & a_{[K:F][K:F]} \end{bmatrix}$$

and hence $a_i := a_{i1} = \ldots = a_{i[K:F]}$ for each $i = 1, \ldots, [K : F]$. From the commutativity of the diagram (the left square) , $[K : F]a_i = a$ for all $i = 1, \ldots, [K : F]$. Hence $b := a_1 = \ldots = a_{[K:F]}$. Thus $c = a = [K : F]b$ (from the right square).

Now let h and k be the top and bottom morphisms of Figure 4-4. It is clear that the middle vertical maps are module homomorphisms and $kh = \kappa(\phi)$. Since

$$\ker \begin{bmatrix} f_2 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 1 \end{bmatrix} \subseteq \ker \begin{bmatrix} g_{21} & a_{11} & \ldots & a_{1[K:F]} \\ \vdots & \vdots & & \vdots \\ g_{2[K:F]} & a_{[K:F]1} & \ldots & a_{[K:F][K:F]} \end{bmatrix}$$

we obtain $\ker(f_2) \subseteq \ker(g_{21}, \ldots, g_{2[K:F]})^{tr}$, and therefore

$$\ker \begin{bmatrix} f_2 & 0 \\ 0 & 1 \end{bmatrix} \subseteq \ker \begin{bmatrix} g_{21} & b \\ \vdots & \vdots \\ g_{2[K:F]} & b \end{bmatrix}. \blacksquare$$

$$\begin{array}{ccccc}
(G_1,u_1)\times(\mathbf{Z},n) & \xrightarrow{\begin{pmatrix} f_1 & 0\\ 0 & 1\\ \vdots & \vdots \\ 0 & 1\end{pmatrix}} & (G_2,u_2)\times(\mathbf{Z},n)^{[K:F]} & \xrightarrow{\begin{pmatrix} f_2 & 0 & \dots & 0\\ 0 & 1 & \dots & 1\end{pmatrix}} & (G_3,u_3)\times(\mathbf{Z},[K:F]n)\\
{\scriptstyle \begin{pmatrix}1&0\\0&[K:F]\end{pmatrix}}\downarrow & & {\scriptstyle\begin{pmatrix}1&0&\dots&0\\0&1&\dots&1\end{pmatrix}}\downarrow & & {\scriptstyle\begin{pmatrix}1&0\\0&1\end{pmatrix}}\downarrow\\
(G_1,u_1)\times(\mathbf{Z},[K:F]n) & \xrightarrow{\begin{pmatrix} f_1 & 0\\ 0 & 1\end{pmatrix}} & (G_2,u_2)\times(\mathbf{Z},[K:F]n) & \xrightarrow{\begin{pmatrix} f_2 & 0\\ 0 & 1\end{pmatrix}} & (G_3,u_3)\times(\mathbf{Z},[K:F]n)\\
{\scriptstyle (g_1,b)}\downarrow & & {\scriptstyle\begin{pmatrix}g_{21} & b\\ \vdots & \vdots\\ g_{2[K:F]} & b\end{pmatrix}}\downarrow & & {\scriptstyle (g_3,c)}\downarrow\\
(\mathbf{Z},t) & \xrightarrow{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}} & (\mathbf{Z},t)^{[K:F]} & \xrightarrow{(1,\dots,1)} & (\mathbf{Z},[K:F]t)
\end{array}$$

Figure 4-4

Theorem 4.2.10. *Let $R \in \mathcal{R}$. Then R is of type fd if and only if $K_0(\nu_R)$ is injective.*

Proof. As Theorem 4.2.8, using Lemma 4.2.9 in place of Lemma 4.2.7. ∎

Corollary 4.2.11. *If $R \in \mathcal{R}$ and R^K has comparability of projections (equivalently, $K_0(R^K)$ is totally ordered), then R is of type fd.*

Proof. If not, then $K_0(\nu_R)$ is not injective, and since $K_0(R^K)$ is totally ordered, $\ker(K_0(\nu_R))$ must contain a nonzero positive element. Observe, however, that this is impossible: the condition $\ker(K_0(\nu_R))^+ = \{0\}$ clearly holds for matrix algebras over F, K, D by Proposition 4.1.1, and it is preserved by finite direct products and countable direct limits, whence it holds for all algebras in $\mathcal{R}$. Therefore R must be of type fd. ∎

Lemma 4.2.12. *Let $\phi : R \to T$ be a map in $\mathcal{R}_0$. If*

$$K_0(\phi^{D^{op}}\nu_R\mu_R)(K_0(R)) \subseteq [K:F]K_0(\nu_T)(K_0(T^K)),$$

then ϕ factors through an algebra in $\mathcal{R}_0$ of type kd.

Proof. Proceeding as in Lemma 4.2.7, it suffices to show that if $R = R' \times M_n(F)$ and $S = R' \times M_n(K)$ while $T = M_t(F)$, for some $R' \in \mathcal{R}_0$ and some $n, t \in \mathbf{N}$, then there exist maps $h : \kappa(R) \to \kappa(S)$ and $k : \kappa(S) \to \kappa(T)$ such that $kh = \kappa(\phi)$ and

$$k_3 K_0(\nu_S \mu_S)(K_0(S)) \subseteq [K : F] K_0(\nu_T)(K_0(T^K)).$$

The diagram $\kappa(\phi)$ takes the form shown in Figure 4-5.

$$\begin{array}{ccccc}
(G_1, u_1) \times (\mathbf{Z}, n) & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}} & (G_2, u_2) \times (\mathbf{Z}, n) & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ 0 & 1 \end{pmatrix}} & (G_3, u_3) \times (\mathbf{Z}, n) \\
\downarrow (g_1, a) & & \downarrow (g_2, b) & & \downarrow (g_3, c) \\
(\mathbf{Z}, t) & \xrightarrow{1} & (\mathbf{Z}, t) & \xrightarrow{1} & (\mathbf{Z}, t)
\end{array}$$

Figure 4-5

The hypothesis $K_0(\phi^{D^{op}} \nu_R \mu_R)(K_0(R)) \subseteq [K : F] K_0(\nu_T)(K_0(T^K))$ says that

$$g_3 f_2 f_1(G_1) + c\mathbf{Z} \subseteq [K : F]\mathbf{Z}.$$

In particular, $c = [K : F]d$ for some $d \in \mathbf{Z}^+$. From the commutativity of the diagram, $a = b = c$.

Now let h and k be the top and bottom morphisms of Figure 4-6. It is clear that the middle vertical maps are module homomorphisms and $kh = \kappa(\phi)$. Since $g_3 f_2 f_1(G_1) + [K : F]d\mathbf{Z} \subseteq [K : F]\mathbf{Z}$ we also have

$$k_3 K_0(\nu_S \mu_S)(K_0(S)) \subseteq [K : F] K_0(\nu_T)(K_0(T^K)). \blacksquare$$

Theorem 4.2.13. *Let $R \in \mathcal{R}$. Then R is of type kd if and only if*

$$K_0(\nu_R \mu_R)(K_0(R)) = [K : F] K_0(\nu_R)(K_0(R^K)).$$

Proof. As Theorem 4.2.8, using Lemma 4.2.12 in place of Lemma 4.2.7. ■

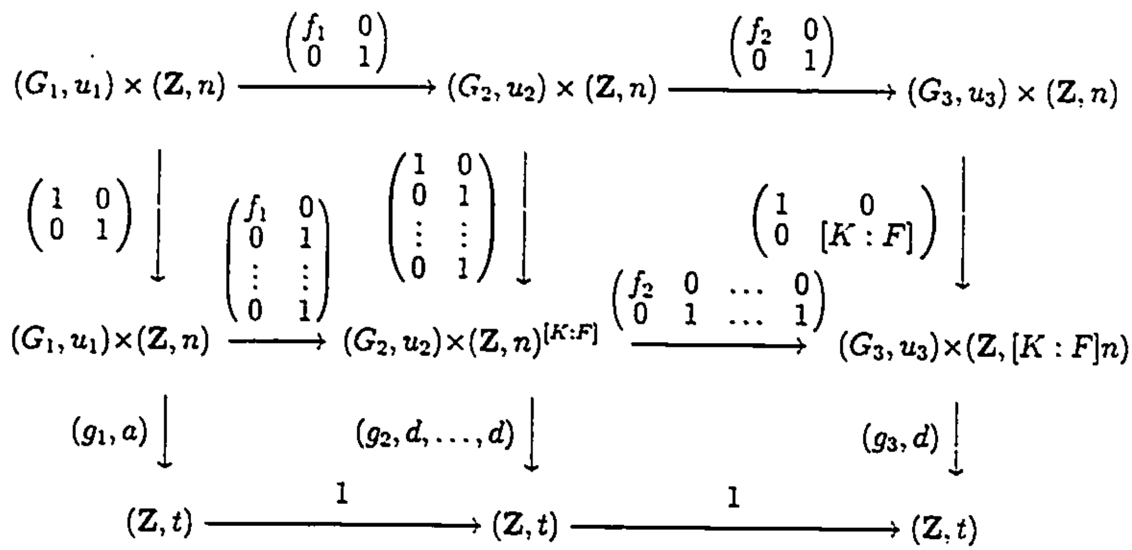

Figure 4-6

Theorem 4.2.14. *Given $R \in \mathcal{R}$, the following conditions are equivalent:*

(a) R is of type k.

(b) R is of type fk and of type kd.

(c) $K_0(\nu_R\mu_R)(K_0(R)) = [K : F]K_0(R^{D^{op}})$.

Proof. (a)⇒(b): A priori.

(b)⇒(c): By Theorems 4.2.8 and 4.2.13, $K_0(\nu_R)$ is surjective and $K_0(\nu_R\mu_R)(K_0(R)) = [K : F]K_0(\nu_R)(K_0(R^K))$, from which the desired relation is clear.

(c)⇒(b): By Lemma 4.1.6(b),

$$[K : F]K_0(R^{D^{op}}) \cap K_0(\nu_R\mu_R)(K_0(R)) \subseteq [K : F]K_0(\nu_R)(K_0(R^K)),$$

whence $[K : F]K_0(R^{D^{op}}) \subseteq [K : F]K_0(\nu_R)(K_0(R^K))$ and so $K_0(\nu_R)$ is surjective. Then also $K_0(\nu_R\mu_R)(K_0(R)) = [K : F]K_0(\nu_R)(K_0(R^K))$. Now Theorems 4.2.8 and 4.2.13 show that R is of type fk and of type kd.

(b)⇒(a): Since R is of type kd, we may write it as the direct of a sequence

$$R(1) \xrightarrow{\phi_1} R(2) \xrightarrow{\phi_2} R(3) \xrightarrow{\phi_3} \cdots$$

of maps from $\mathcal{R}_0$, where each $R(n)$ is of type kd. As R is also of type fk, $K_0(\nu_R)$ is surjective by Theorem 4.2.8. Hence, after telescoping the sequence, we may assume that

$$K_0(\phi_n^{D^{op}})(K_0(R(n)^{D^{op}})) \subseteq K_0(\nu_{R(n+1)})(K_0(R(n+1)^K))$$

for all $n \in \mathbb{N}$. By Lemma 4.2.7, each ϕ_n factors as $R(n) \to S_n \to R(n+1)$ where S_n is an algebra in $\mathcal{R}_0$ of type k. Now R is isomorphic to a direct limit of the algebras S_n, and therefore R is of type k. ■

Theorem 4.2.15. *Let $R \in \mathcal{R}$. Then*

(a) $K_0(R)$ is deg(D)*-divisible if and only if $K_0(R^{D^{op}})$ is* deg(D)*-divisible.*

(b) If $K_0(R^K)$ is deg(D)*-divisible, then $K_0(R)$ and $K_0(R^{D^{op}})$ are* deg(D)*-divisible. In this case R is of type k.*

Proof. Relabel $\kappa(R)$ as

$$(G_1, u_1) \xrightarrow{f_1} (G_2, u_2) \xrightarrow{f_2} (G_3, u_3).$$

From Proposition 4.1.3 and 4.1.5, we have positive homomorphisms $s : G_2 \to G_1$ and $t : G_3 \to G_2$ such that $sf_1 = [K : F]$ and $f_2t = \deg(D)^2/[K : F]$ while $stf_2f_1 = \deg(D)^2$ and $f_2f_1st = \deg(D)^2$. So t and st are injective.

(a): If G_1 is deg(D)-divisible, then $stf_2f_1 = \deg(D)^2$ implies that stf_2f_1 is surjective, hence st is surjective. So st is an isomorphism, hence G_3 is deg(D)-divisible. Similarly, if G_3 is deg(D)-divisible , then $f_2f_1st = \deg(D)^2$ implies that f_2f_1 is an isomorphism [ref. Lemma 4.1.2(a)]. Hence G_1 is deg(D)-divisible.

(b): If G_2 is deg(D)-divisible, then for all $x \in G_1$, there exists $y \in G_2$ such that $\deg(D)^2y = f_1(x)$. Thus $\deg(D)^2s(y) = sf_1(x) = [K : F]x$, hence G_1 is deg(D)-divisible. Therefore G_3 is deg(D)-divisible by (a). In addition, as $f_2f_1st = \deg(D)^2$, we have $[K : F]G_3 = \deg(D)^2G_3 \subseteq f_2f_1(G_1)$. By Theorem 4.2.14, R is of type k. ■

Remark. Assume that G_1 and G_3 are deg(D)-divisible and $\ker(f_2) \subseteq \deg(D)G_2$ (for example, R is of type fd or $\deg(D) = 2$ [GH, Theorem 7.14]). Then $f_2 t = \deg(D)^2/[K:F]$ implies that f_2 is surjective. Hence $G_2/\ker(f_2)$ is deg(D)-divisible, and so G_2 is deg(D)-divisible. This argument is from the proof of [GH, Theorem 7.14], based on [GH, Lemma 6.1(c)] which is true only if $[K:F] = 2$. Now we shall give an example wherein G_1 (hence G_3) is deg(D)-divisible but G_2 is not.

Example. Let $\deg(D) = [K:F] = 3$. First we construct by induction a countable sequence $\chi_1, \chi_2, \ldots$ of elements of $\mathbb{Z}\mathrm{Gal}(K/F)$ satisfying:

(a) Let x denote a generator of $\mathrm{Gal}(K/F)$. Each $\chi_n = a_n + b_n x + c_n x^2$ where a_n, b_n, c_n are integers such that $a_n + b_n + c_n \equiv 0 \ mod\ 3$.

(b) Let $\chi_n \chi_{n-1} \cdots \chi_1 = A_n + B_n x + C_n x^2$. Then for all $n \in \mathbb{N}$, $\{A_n, B_n, C_n\}$ are not mutually congruent modulo 3.

Suppose we already have $\chi_1, \ldots, \chi_n$ satisfying (a) and (b). Let $\chi_{n+1} = a + bx + cx^2$ and consider $\chi_{n+1}\chi_n\chi_{n-1}\cdots\chi_1$:

$$(a+bx+cx^2)(A_n+B_nx+C_nx^2) = aA_n+bC_n+cB_n+(aB_n+bA_n+cC_n)x+(aC_n+bB_n+cA_n)x^2.$$

Then

$$aA_n + bC_n + cB_n \equiv aB_n + bA_n + cC_n \equiv aC_n + bB_n + cA_n \pmod 3$$

implies

$$a(A_n-B_n)+b(C_n-A_n)+c(B_n-C_n) \equiv a(A_n-C_n)+b(C_n-B_n)+c(B_n-A_n) \equiv 0 \ mod\ 3.$$

We can easily check that this is true for all $a+b+c \equiv 0 \ mod\ 3$ if and only if $A_n \equiv B_n \equiv C_n \ mod\ 3$. So by induction hypothesis (b), there exists $\chi_{n+1} = a + bx + cx^2$ such that $\chi_1, \ldots, \chi_{n+1}$ satisfies (a) and (b).

Now let $R = K \xrightarrow{\chi_1} M_{3\cdot}(K) \xrightarrow{\chi_2} M_{3\cdot\cdot}(K) \xrightarrow{\chi_3} \cdots$. Here we denote $M_{3\cdot}(K) \xrightarrow{\chi_n} M_{3\cdot\cdot}(K)$ to be the map

$$t \mapsto diag(\overbrace{t\ldots t}^{a_n \text{ times}}\ \overbrace{x(t)\ldots x(t)}^{b_n \text{ times}}\ \overbrace{x^2(t)\ldots x^2(t)}^{c_n \text{ times}}).$$

Then G_1 is isomorphic to

$$\mathbf{Z} \xrightarrow{a_1+b_1+c_1} \mathbf{Z} \xrightarrow{a_2+b_2+c_2} \cdots$$

hence 3-divisible from (a). On the other hand, G_2 is isomorphic to

$$\mathbf{Z}^3_{(1)} \xrightarrow{\begin{pmatrix} a_1 & b_1 & c_1 \\ c_1 & a_1 & b_1 \\ b_1 & c_1 & a_1 \end{pmatrix}} \mathbf{Z}^3_{(2)} \xrightarrow{\begin{pmatrix} a_2 & b_2 & c_2 \\ c_2 & a_2 & b_2 \\ b_2 & c_2 & a_2 \end{pmatrix}} \cdots$$

hence not 3-divisible because from (b), there is no $g \in G_2$ such that $3g = (1, 0, \ldots, 0)^{tr} \in \mathbf{Z}^3_{(1)}$.

Chapter 5

Locally Representable Actions on AF C^*-algebras and Algebras of type k

In section 5.1, we shall describe the connection between the classification of locally representable actions of finite abelian groups G on AF C^*-algebras and the classification of F-algebras of type k where $\mathrm{Gal}(K/F) = G$. With this connection as motivation, in section 5.2 we study these algebras. K.R.Goodearl and D.E.Handelman derive a simple description of algebras of this type for $K = \mathbf{C}$ in [GH, X]. We obtain an equational description for algebras of type k for any abelian Galois extension K/F. However, an attempt to derive a simple description similar to that of [GH, X] is unsuccessful, even when the underlying dimension group is simple.

5.1. Locally representable actions on AF C^*-algebras.

Let G be a compact group, and $\overline{A}$ an AF (approximately finite dimensional) C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(\overline{A})$ be a point norm continuous group homomorphism, that is, for all $a \in \overline{A}$ the map $G \to \overline{A}$ defined by $g \mapsto \alpha(g)(b)$ be continuous. This condition allows us to form the crossed product $\overline{A} \times_\alpha G$: If G is finite, this is the algebra of formal sums

$$\left\{\sum a_g g \mid g \in G, a_g \in \overline{A}\right\},$$

with multiplication defined via $ga = a^{\alpha(g)}g$; if G is infinite, $\overline{A} \times_\alpha G$ is the C^* norm-closure of the set of continuous functions $f : G \to \overline{A}$, with multiplication defined via a twisted convolution:

$$(f_1 * f_2)(g) = \int_G f_1(h)\alpha_h(f_2(h^{-1}g))dh\ ,$$

where α_h is $\alpha(h)$, and the measure on G is normalized Haar measure.

Now $K_0(\overline{A} \times_\alpha G)$ admits a natural structure as a module over $K_0(G)$ by taking tensor products over $\mathbf{C}$ (e.g. see [W]). Furthermore, if $\overline{A} \times_\alpha G$ is stably finite (as will be the case when A is AF), then $K_0(\overline{A} \times_\alpha G)$ admits a natural partial ordering [GH2] and becomes a partially ordered module over the partially ordered ring $K_0(G)$.

If A is a *-algebra, $U(A)$ will denote the group of unitary elements of A.

Let A be a finite dimensional C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be a (continuous) group homomorphism of a special form: There exists a representation $\gamma : G \to U(A)$ such that $Ad\gamma(g) = \alpha(g)$ for all $g \in G$. We call such an action α **representable.** If A is now an algebra written as a unital union of finite dimensional *-subalgebras, and $\alpha : G \to \mathrm{Aut}(A)$ is a point-norm continuous homomorphism such that for some increasing nest $A^1 \subset A^2 \subset A^3 \subset \ldots$ of finite dimensional C^*-subalgebras of $\overline{A}$, with dense union, and we have that both $\alpha(G)A^i \subset A^i$ and α/A^i is representable for all i, then we say α is **locally representable.** Then by [HR, Theorem III.1], $(K_0(A \times_\alpha G), [A_\alpha])$ is a complete invariant for the locally representable action on the AF algebra A up to automorphism. Here A_α is A viewed as a (projective) $(A \times_\alpha G)$-module via the action

$$\left(\sum a_g g\right)a = \sum a_g a^{\alpha(g)}.$$

In the representable case, if we write $A = \prod_{i=1}^m A_i$ in terms of simple components, then $K_0(A \times_\alpha G) \cong K_0(G)^m$ as $K_0(G)$-modules. If α is locally representable, we can write the AF C^*-algebra A as direct limit of finite dimensional unital C^* -subalgebras $A^1 \subset A^2 \subset \ldots$ with $\alpha A^n = A^n$ and the restrictions of α to A^n are representable. Then as $K_0(G)$-modules

$$K_0(A \times_\alpha G) \cong \varinjlim K_0(A^n \times_\alpha G) \cong \varinjlim K_0(G)^{m(n)},$$

where $m(n)$ is the number of simple components in A^n, and the maps in the last direct limit are given by matrices of characters (see [HR, II]).

If G is a *finite abelian group*, then $\hat{G}$, the dual group, is isomorphic to G and $K_0(G) \cong \mathbf{Z}\hat{G}$ (the integral group ring), with

$$K_0(G)^+ = \left\{\sum a_g \hat{g} \mid a_g \geq 0\right\},$$

as characters are sums of the irreducible characters, and the multiplication of the irreducibles is simply the usual group multiplication in $\hat{G}$.

Therefore, if $\alpha : G \to \mathrm{Aut}(A)$ is a locally representable where G is a finite abelian group and A is a AF C^*-algebra, then as ordered $\mathbf{Z}\hat{G}$-modules

$$K_0(A \times_\alpha G) \cong \varinjlim (\mathbf{Z}\hat{G})^{m(n)}, \tag{6}$$

where the maps in the direct limit are given by matrices of elements of $\mathbf{Z}\hat{G}$.

Now let K/F be a Galois extension such that $\mathrm{Gal}(K/F) = G$. Let B be a unital AF F-algebra of type k. Write B as a direct limit of a sequence $B_1 \to B_2 \to \cdots$, where each B_n is a finite product of matrix algebras over K and the connecting maps are unital F-algebra maps. Then as ordered $\mathbf{Z}G$-modules

$$K_0(B^K) \cong \varinjlim (\mathbf{Z}G)^{m(n)}, \tag{7}$$

where the maps in the direct limit are given by matrices of elements of $\mathbf{Z}G$.

By identifying G and $\hat{G}$, we shall show that given a locally representable action α on a finite abelian group G into an AF C^*-algebra A, we can construct a corresponding ultramatricial F-algebra of type k, where $\mathrm{Gal}(K/F) = G$, such that

$$K_0(B^K) \cong K_0(A \times_\alpha G)$$

as ordered ZG-modules and then we go in reverse.

By comparing (6) and (7), it would be sufficient to reconstruct A and α from $K_0(A\times_\alpha G)$ and B from $K_0(B^K)$ (up to automorphisms). The former is shown in [HR, Example II.4] where we replace -1 by proper roots of unity corresponding to $\hat{g}$. The latter has been done in Chapter 2, but we shall repeat here. If $\chi = a_1 + a_2g_2 + \cdots$ is an element of ZG^+ (that is, $a_1, a_2, \ldots \in Z^+$), then an entry χ corresponds to a block diagonal map $M_\bullet(K) \to M_\bullet(K)$ consisting of a_1 identity blocks followed by a_2 blocks of the automorphism g_2^{-1} (as an element of $\mathrm{Gal}(K/F)$), and so on.

Let E/F be the subfield of K/F fixed by G_E, a subgroup of G. Then the map $\sum a_g g \mapsto \sum a_g g_{|E}$, where $g_{|E}$ denotes the restriction of g on E, is an ordered ring homomorphism $K_0(G) \to K_0(G/G_E)$ induced by the group homomorphism $G \to G/G_E \cong \mathrm{Gal}(E/F)$. This corresponds to factoring out an ideal $I_E \triangleleft K_0(G)$. It is straightforward to check that there exists a commutative diagram as given in Figure 5-1.

$$\begin{array}{ccccc}
K_0(A\times_\alpha G)/I_F K_0(A\times_\alpha G) \cong K_0(A) & \longrightarrow & K_0(A\times_\alpha G) & \longrightarrow & K_0(A\times_\alpha G)/I_E K_0(A\times_\alpha G) \\
\downarrow & & \downarrow & & \downarrow \\
K_0(B^F) \cong K_0(B) & \longrightarrow & K_0(B^K) & \longrightarrow & K_0(B^{M_\bullet(E)})
\end{array}$$

Figure 5-1

in which the vertical maps are ordered group isomorphisms (the first and the last maps) and module isomorphisms (the middle map). The isomorphism $K_0(A\times_\alpha G)/I_F K_0(A\times_\alpha G) \cong K_0(A)$ is induced by the map $K_0(G) \to Z$ given by $\chi \mapsto \chi(1)$ (see [HR, Appendix]).

In the group action, the obvious question arises, which partially ordered ZG-modules arise as $K_0(A\times_\alpha G)$'s. Similarly, the problem for type k unital AF F-algebra is to determine which partially ordered ZG-modules arise as $K_0(B^K)$. Hence, the problems for group actions and for type k algebras are equivalent. As a result, we have

Proposition 5.1. *If $|G|= n$ and H is a countable ordered ZG-module such that*

(a) H is a dimension group,

(b) H is n-divisible as an abelian group.

Then $H = K_0(R^K)$ as ordered $\mathbf{Z}G$-modules for some $R \in \mathcal{R}$ of type k.

Proof. A straightforward result of [HR, Proposition IV.4] and the observation above. ■

In both cases, the ordered $\mathbf{Z}G$-module is flat (being a direct limit of free modules) and a dimension group.

5.2. Algebras of type k.

In this section, we let $G = \mathrm{Gal}(K/F) =< g >$, a cyclic group of order 3, and we choose $\Omega = \{K\}$. That is all $R \in \mathcal{R}$ are of type k, and $\mathcal{K}$ is the category whose each object (H, v) is both a pre-ordered abelian groups with unit and an ordered $\mathbf{Z}G$-modules and whose morphisms are ordered $\mathbf{Z}G$-module homomorphisms. We denote $\mathcal{L}$ to be the category whose objects are pre-ordered abelian groups which are also ordered $\mathbf{Z}G$-modules and morphisms are ordered $\mathbf{Z}G$-module homomorphisms. Let $\mathcal{L}_0$ denote the subcategory whose objects are $K_0(R^K)$, where $R \in \mathcal{R}_0$.

As in [GH, VIII], we use row(Z) to denote the collection of all finite rows of integers of arbitrary length, including length 0 (the empty row), and we use mat($\mathbf{Z}^+$) to denote the set of all nonnegative integer matrices. Given a partially ordered abelian group H, we use col(H) (and col(H^+)) to denote the collection of all finite columns with entries from H (and H^+ respectively), of arbitrary length including length 0. We shall not specify the sizes of rows, columns or matrices unless necessary, assuming only that rows, columns and matrices have the correct sizes for the indicated sums and products to be performed. We denote identity matrices by I. In a formula including ax where $a \in$ row(Z) and $x \in$ col(H^+), we allow a, x to have length 0 in order to include the case where no ax term appears.

Theorem 5.2.1. *An object $X \in \mathcal{K}$ is isomorphic to $\kappa(R)$ for some $R \in \mathcal{R}$ if and only if*

X is isomorphic (in the category $\mathcal{K}$) to some (H, v) where

(a) (H, v) is both a countable dimension group with order unit and an ordered ZG-module.

(b) $v \in (1+g+g^2)(H)$.

(c) Given any $Y \in \mathcal{L}_0$, any morphism $p : Y \to H$ in $\mathcal{L}$, and any $y \in ker(p)$, there exist $Y' \in \mathcal{L}_0$ and morphisms $q : Y \to Y'$ and $p' : Y' \to H$, in $\mathcal{L}$ such that $p'q = p$ and $y \in ker(q)$.

Proof. If $(H, v) \cong \kappa(R^K)$ for some $R \in \mathcal{R}$, then by reducing to the case of finite dimensional semisimple algebras, (H, v) satisfies (a) and (b). Moreover, H must be a direct limit in $\mathcal{L}$ of a sequence

$$Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots$$

where each $Y_n \in \mathcal{L}_0$. Let $h_n : Y_n \to H$ (for $n = 1, 2, \ldots$) be the associated natural maps.

Consider any $Y \in \mathcal{L}_0$, any morphism $p : Y \to H$ in $\mathcal{L}$, and any $y \in \ker(p)$. As in the proof of Lemma 3.2.2, there exists Y_n and a morphism $q : Y \to Y_n$ in $\mathcal{L}_0$ such that $h_n q = p$. Now $q(y) \in \ker(h_n)$, and so by increasing n, we may assume that $q(y) = 0$. This proves (c).

Conversely, assume that $(H, v) \in \mathcal{K}$ satisfying (a),(b),(c). We next prove a stronger version of (c):

(c') Given any $Y \in \mathcal{L}_0$ and any morphism $p : Y \to H$ in $\mathcal{L}$, there exist $Y' \in \mathcal{L}_0$ and morphisms $q : Y \to Y'$ and $p' : Y' \to H$ in $\mathcal{L}$ such that $p'q = p$ and $\ker(p) \subseteq \ker(q)$.

Since $\ker(p)$ is a subgroup of finitely generated abelian group, it is finitely generated. Applying (c) by induction on the number of generators of $\ker(p)$, we obtain (c').

Next list H^+ (with repetitions allowed) as $\{x_1, x_2, x_3, \ldots\}$, with $(1+g+g^2)(x_1) = v$, possible by (a),(b). For $n = 1, 2, \ldots$, let W_n be $K_0(K^K) \in \mathcal{L}_0$, that is, $W_n = Z^3$ such that $g(1,0,0)^{tr} = (0,1,0)^{tr}$ and $g^2(1,0,0)^{tr} = (0,0,1)^{tr}$. Let w_n be the map (x_n, gx_n, g^2x_n) from W_n to H. Since $gx_n, g^2x_n \in H^+$ by (a), w_n is a positive group homomorphism. As W_n is a cyclic module generated by $(1,0,0)^{tr}$, we can easily see that w_n is a module homomorphism. So w_n is a morphism in $\mathcal{L}_0$. Then $x_n \in w_n(W_n^+)$.

We now construct objects $Y_1, Y_2, \ldots$ in $\mathcal{L}_0$ and morphisms $p_n : Y_n \to H$ and $q_n : Y_n \to Y_{n+1}$ such that for all $n \in \mathbb{N}$:

(i) $x_n \in p_n(Y_n^+)$;

(ii) $p_{n+1}q_n = p_n$;

(iii) $\ker(p_n) \subseteq \ker(q_n)$.

To start, set $Y_1 = W_1$ and $p_1 = w_1$.

Now consider the morphism $(p_1, w_2) : Y_1 \times W_2 \to H$. By (c'), there exist $Y_2 \in \mathcal{L}_0$ and morphisms $q' : Y_1 \times W_2 \to Y_2$ and $p_2 : Y_2 \to H$ such that $p_2 q' = (p_1, w_2)$ and $\ker(p_1, w_2) \subseteq \ker(q')$. Set

$$q_1 = q'(1,0)^{tr} : Y_1 \to Y_2,$$

and note that $p_2 q_1 = p_2 q'(1,0)^{tr} = (p_1, w_2)(1,0)^{tr} = p_1$. We also have

$$x_2 \in w_2(W_2^+) = (p_1, w_2)(\{0\} \times W_2^+) \subseteq p_2 q'(Y_1^+ \times W_2^+) \subseteq p_2(Y_2^+).$$

In addition, since $\ker(p_1, w_2) \subseteq \ker(q')$, we obtain

$$q_1(\ker(p_1)) = q'(\ker(p_1) \times \{0\}) \subseteq q'(\ker(p_1, w_2)) = \{0\},$$

whence $\ker(p_1) \subseteq \ker(q_1)$.

Continuing in the same manner, we obtain Y_n, p_n, q_n as described. From (i) it follows that

$$H^+ = \bigcup_{n=1}^{\infty} p_n(Y_n^+).$$

Consequently, using (ii) and (iii) we see that H together with the morphisms p_n is a direct limit in $\mathcal{L}$ for the sequence

$$Y_1 \xrightarrow{q_1} Y_2 \xrightarrow{q_2} \cdots.$$

Since $(1 + g + g^2)(x_1) = v$, we have $v = p_1(v_1)$ where $v_1 = (1,1,1)^{tr} \in Y_1^+$. For all $n > 1$, set $v_n = q_{n-1}q_{n-2}\cdots q_1(v_1)$. Then $p_n(v_n) = v$. Let X_n be the ideal of (dimension group) Y_n generated by v_n. Then using $^\circ$ to denote restrictions of maps, the morphisms p_n and q_n restrict to morphisms $p_n^o : (X_n, v_n) \to (H, v)$ and $q_n^o : (X_n, v_n) \to (X_{n+1}, v_{n+1})$ in $\mathcal{K}$, and (H, v) together with the morphisms p_n^o is a direct limit in $\mathcal{K}$ for the sequence

$$(X_1, v_1) \xrightarrow{q_1^o} (X_2, v_2) \xrightarrow{q_2^o} \cdots.$$

This is because if $u \in H$, then there exist $m \in \mathbb{N}$ such that $mv - u \geq 0$. By (i), there exists $x \in Y_n^+$, for some n, such that $p_n(x) = mv - u = mp_n(v_n) - u$. Then $u = p_n(mv_n - x)$. Since $mv_n - x \leq mv_n$, we have $mv_n - x \in X_n$, and hence $u \in p_n(X_n)$.

So X is a direct limit for this sequence. Since $v_1 \in (1 + g + g^2)(Y_1^+)$ and each q_n is a $\mathbb{Z}G$-module map, $v_n \in (1+g+g^2)(Y_n^+)$ for all n. Then each (X_n, v_n) is isomorphic to an object in $\kappa(\mathcal{R}_0)$. Hence, there exist $R_n \in \mathcal{R}_0$ and isomorphisms $h_n : (X_n, v_n) \to \kappa(R_n)$. By Theorem 3.2.1, there exist unital F-algebra maps $f_n : R_n \to R_{n+1}$ for all n such that $\kappa(f_n) = h_{n+1} q_n^o h_n^{-1}$, whence $(h_1, h_2, \ldots)$ provides an isomorphism between the sequences

$$(X_1, v_1) \xrightarrow{q_1^o} (X_2, v_2) \xrightarrow{q_2^o} \cdots \quad \text{and} \quad \kappa(R_1) \xrightarrow{\kappa(f_1)} \kappa(R_2) \xrightarrow{\kappa(f_2)} \cdots.$$

Therefore, $X \cong \kappa(R)$ if R is the direct limit of the sequence

$$R_1 \xrightarrow{f_1} R_2 \xrightarrow{f_2} \cdots \quad \blacksquare$$

Proposition 5.2.2. *Let H be both a dimension group and an ordered $\mathbb{Z}G$-module. Then the condition (c) in Theorem 5.2.1 is equivalent to the following condition:*

(c#) Given any $x \in \mathrm{col}(H^+)$ and any $a_1, a_2, a_3 \in \mathrm{row}(\mathbb{Z})$ such that $a_1x + a_2gx + a_3g^2x = 0$, there exist $y \in \mathrm{col}(H^+)$ and $b_1, b_2, b_3 \in \mathrm{mat}(\mathbb{Z}^+)$ satisfying the relations in Figure 5-1.

$$\begin{bmatrix} x \\ gx \\ g^2x \end{bmatrix} = \begin{bmatrix} b_1 & b_3 & b_2 \\ b_2 & b_1 & b_3 \\ b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} y \\ gy \\ g^2y \end{bmatrix}$$

$$(a_1, a_2, a_3) \begin{bmatrix} b_1 & b_3 & b_2 \\ b_2 & b_1 & b_3 \\ b_3 & b_2 & b_1 \end{bmatrix} = 0$$

Figure 5-1

Proof. (c) $\Rightarrow$ (c#): Let m be the length of x, and let $Y = \mathbb{Z}^m \times \mathbb{Z}^m \times \mathbb{Z}^m$ such that

$$g(a, 0, 0)^{tr} = (0, a, 0)^{tr} \text{ and } g^2(a, 0, 0)^{tr} = (0, 0, a)^{tr} \text{for all } a \in \mathbb{Z}^m. \qquad (*)$$

Then $p = (x^{tr}, (gx)^{tr}, (g^2x)^{tr})$ is a morphism from Y to H in $\mathcal{L}$, and the column $y = (a_1, a_2, a_3)^{tr}$ is an element of $\ker(p)$.

By (c), there exist $Y' \in \mathcal{L}_0$ and morphisms $q : Y \to Y'$ and $p' : Y' \to H$ in $\mathcal{L}$ such that $p'q = p$ and $q(y) = 0$. We may assume that $Y' = \mathbf{Z}^s \times \mathbf{Z}^s \times \mathbf{Z}^s$, for some nonnegative integer s, and G acts on Y' as in (*). Then the $\mathbf{Z}G$-module map q is the matrix

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{bmatrix}$$

where each b_i is a nonnegative integer matrix of appropriate size. Now p' has the form $(y^{tr}, (gy)^{tr}, (g^2y)^{tr})$ for some $y \in \mathrm{col}(H^+)$.

As $p'q = p$ and $q(y) = 0$, we obtain the relations in Figure 5-2. Transposing these equations concludes the proof of (c$^\#$).

$$(y^{tr}, (gy)^{tr}, (g^2y)^{tr}) \begin{bmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{bmatrix} = (x^{tr}, (gx)^{tr}, (g^2x)^{tr})$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{bmatrix} \begin{bmatrix} a_1^{tr} \\ a_2^{tr} \\ a_3^{tr} \end{bmatrix} = 0.$$

Figure 5-2

(c$^\#$) $\Rightarrow$ (c): we may assume that $Y = \mathbf{Z}^m \times \mathbf{Z}^m \times \mathbf{Z}^m$ as in the previous part. Also p must have the form $(x^{tr}, (gx)^{tr}, (g^2x)^{tr})$. Then $y = (a_1, a_2, a_3)^{tr}$ for some $a_1, a_2, a_3 \in \mathrm{row}(\mathbf{Z})$ of length m. From $p(y) = 0$, we obtain

$$a_1x + a_2gx + a_3g^2x = 0.$$

Now there exist y and b_1, b_2, b_3 as given by (c$^\#$). Let s be the length of y and let $Y' = \mathbf{Z}^s \times \mathbf{Z}^s \times \mathbf{Z}^s$ as in (c) $\Rightarrow$ (c$^\#$). Then

$$q = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{bmatrix}$$

is a morphism from Y to Y', and $p' = (y^{tr}, (gy)^{tr}, (g^2y)^{tr})$ is a morphism from Y' to H. From the relations given by (c$^\#$), we see that $p'q = p$ and $q(y) = 0$. ■

Corollary 5.2.3. *Let H be both a dimension group and an ordered $\mathbf{Z}G$-module. Then the condition (c) in Theorem 5.2.1 is equivalent to the following condition:*

(d1) Given $p \in \text{row}(\mathbf{Z})$ *and* $x \in \text{col}(H^+)$ *with* $p(x+gx+g^2x)=0$, *there exist* $y \in \text{col}(H^+)$ *and* $q_1, q_2, q_3 \in \text{mat}(\mathbf{Z}^+)$ *such that* $x = q_1y + q_2gy + q_3g^2$ *and* $p(q_1+q_2+q_3)=0$, *and*

(d2) given $p_1, p_2, p_3 \in \text{row}(\mathbf{Z})$ *and* $x \in \text{col}(H^+)$ *with* $p_1+p_2+p_3=0$ *and* $p_1x + p_2gx + p_3g^2x = 0$, *there exist* $y \in \text{col}(H^+)$ *and* $q_1, q_2, q_3 \in \text{mat}(\mathbf{Z}^+)$ *such that*

$$x = q_1y + q_2gy + q_3g^2y \qquad \text{and} \qquad (p_1\ p_2\ p_3)\begin{bmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{bmatrix} = 0.$$

Proof. (c) is equivalent to ($c^\#$) which implies both (d1) and (d2).

Conversely, suppose that H satisfies both (d1) and (d2). Let $x \in \text{col}(H^+)$ and $a_1, a_2, a_3 \in \text{row}(\mathbf{Z})$ such that $a_1x + a_2gx + a_3g^2x = 0$ Then we also have $a_1gx + a_2g^2x + a_3x = 0$ and $a_1g^2x + a_2x + a_3gx = 0$. Hence $(a_1+a_2+a_3)(x+gx+g^2x)=0$. Then by (d1) there exist $z \in \text{col}(H^+)$ and $q_1, q_2, q_3 \in \text{mat}(\mathbf{Z}^+)$ such that $x = q_1z + q_2gz + q_3g^2z$ and $(a_1+a_2+a_3)(q_1+q_2+q_3)=0$. Then

$$(a_1\ a_2\ a_3)\begin{bmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{bmatrix}\begin{bmatrix} z \\ gz \\ g^2z \end{bmatrix} = (a_1\ a_2\ a_3)\begin{bmatrix} x \\ gx \\ g^2x \end{bmatrix} = 0.$$

Since $(a_1+a_2+a_3)(q_1+q_2+q_3)=0$, (d2) shows that there exist $y \in \text{col}(H^+)$ and $r_1, r_2, r_3 \in \text{mat}(\mathbf{Z}^+)$ such that

$$\begin{bmatrix} z \\ gz \\ g^2z \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_3 & r_1 & r_2 \\ r_2 & r_3 & r_1 \end{bmatrix}\begin{bmatrix} y \\ gy \\ g^2y \end{bmatrix} \quad \text{and} \quad (a_1a_2a_3)\begin{bmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{bmatrix}\begin{bmatrix} r_1 & r_2 & r_3 \\ r_3 & r_1 & r_2 \\ r_2 & r_3 & r_1 \end{bmatrix} = 0.$$

Now by letting

$$b_1 = q_1r_1 + q_2r_3 + q_3r_2$$

$$b_2 = q_1r_2 + q_2r_1 + q_3r_3$$

$$b_3 = q_1r_3 + q_2r_2 + q_3r_1$$

we have

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{bmatrix}\begin{bmatrix} r_1 & r_2 & r_3 \\ r_3 & r_1 & r_2 \\ r_2 & r_3 & r_1 \end{bmatrix} = 0.$$

Hence

$$\begin{bmatrix} x \\ gx \\ g^2x \end{bmatrix} = \begin{bmatrix} b_1 & b_3 & b_2 \\ b_2 & b_1 & b_3 \\ b_3 & b_2 & b_1 \end{bmatrix}\begin{bmatrix} y \\ gy \\ g^2y \end{bmatrix}$$

$$(a_1, a_2, a_3)\begin{bmatrix} b_1 & b_3 & b_2 \\ b_2 & b_1 & b_3 \\ b_3 & b_2 & b_1 \end{bmatrix} = 0. \blacksquare$$

Remarks. We can generalize the results in this section to any finite abelian group G by

(i) Replacing the condition (b) in Theorem 5.2.1 with $v \in (\sum_{g \in G} g)(H)$;

(ii) Replacing the condition $a_1 x + a_2 g x + a_3 g^2 x = 0$ in Proposition 5.2.2 ($c^{\#}$) with $a_1 x + a_2 g_2 x + \cdots + a_n g_n x = 0$, where $|G| = n$, and replacing the matrix

$$\begin{bmatrix} b_1 & b_3 & b_2 \\ b_2 & b_1 & b_3 \\ b_3 & b_2 & b_1 \end{bmatrix}$$

with the matrix formed by filling in the multiplication table of G with $b_1, b_2, \ldots, b_3 \in$ mat($\mathbf{Z}^+$).

(iii) Replacing the conditions ($d1$) and ($d2$) in Corollary 5.2.3 with conditions similar to those in Remark (ii) above.

Appendix

Remarks and Problems

App.1. The results in section 5.2 are too computational to be practical. In the case $|G|=2$, K. E. Goodearl and D. E. Handelman derive much better descriptions of the $\mathbf{Z}G$-module $K_0(R^K)$, especially when $K_0(R^K)$ is also a countable simple dimension group [GH, Theorem 10.6]. By imitating their work, we derive the following result for $|G|=3$:

Lemma A.1. *Let H be both a dimension group and an ordered $\mathbf{Z}G$-module. Assume that whenever $a, b \in H^+$ with $a + ga + g^2a \leq b + gb + g^2b$, there exist $c, d, e \in H^+$ such that $a = c + d + e$ and $c + gd + g^2e \leq b$.*

Given $p \in row(\mathbf{Z})$ and $x \in col(H^+)$ with $p(x+gx+g^2x) = 0$, there exist $y \in col(H^+)$ and $q_1, q_2, q_3 \in mat(\mathbf{Z}^+)$ such that $x = q_1y + q_2gy + q_3g^2y$ and $p(q_1 + q_2 + q_3) = 0$.

Proof. See App.4. ■

The condition imposed on H in Lemma A.1 is satisfied by all $\kappa(R)$, for $R \in \mathcal{R}$, as following:

By reducing R to the finite dimensional simple case, we let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ such that $a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$. Then by reducing b_1, we can assume that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Then by applying g^k for some $k = 1, 2$ or 3, we may assume that $a_1 \geq b_1$. Then

if $a_2 \geq b_2$, we let

$$c = (b_1, b_2, a_3);\ d = (0, a_2 - b_2, 0);\ e = (a_1 - b_1, 0, 0);$$

if $a_3 \geq b_3$, we let

$$c = (b_1, a_2, b_3);\ d = (a_1 - b_1, 0, 0);\ e = (0, 0, a_3 - b_3);$$

if $a_2 < b_2$ and $a_3 < b_3$, we let

$$c = (b_1, a_2, a_3);\ d = (b_2 - a_2, 0, 0);\ e = (b_3 - a_3, 0, 0).\ ■$$

A similar result can be derived when H is simple as a dimension group (i.e., H is a dense subgroup of $\mathbf{R}^n$) with g acting on H by "shifting". For example, H is a dense subgroup of $\mathbf{R}^3$ such that if $(a,b,c) \in H$, then $(c,a,b) \in H$ and $g(a,b,c) = (c,a,b)$. Hence condition (d1) of Corollary 5.2.3 is also satisfied in this case.

To simplify condition (d2) of Corollary 5.2.3, an attempt to imitate [GH, Lemma 10.2] was made. That was to prove the following statement:

A.2. *Let H be both a dimension group and an ordered $\mathbf{Z}G$-module. Assume that*

(i) whenever $p_1, p_2, p_3 \in \mathbf{Z}$, and $x \in H^+$ with $p_1+p_2+p_3 = 0$ and $p_1x+p_2gx+p_3g^2x = 0$, there exist $y \in H^+$ such that $x = y + gy + g^2y$;

(ii) whenever $a, b \in H^+$ with $a, ga, g^2a \leq b$, there exist $c \in H^+$ such that $c = gc$ and $a \leq c \leq b$.

Given $p_1, p_2, p_3 \in row(\mathbf{Z})$ and $x \in col(H^+)$ with $p_1 + p_2 + p_3 = 0$ and $p_1x + p_2gx + p_3g^2x = 0$, there exist $y \in col(H^+)$ and $q_1, q_2, q_3 \in mat(\mathbf{Z}^+)$ such that

$$x = q_1y + q_2gy + q_3g^2y \qquad \text{and} \qquad (p_1p_2p_3)\begin{bmatrix} q_1 & q_2 & q_3 \\ q_3 & q_1 & q_2 \\ q_2 & q_3 & q_1 \end{bmatrix} = 0.$$

Again we note that the conditions (i) and (ii) above are satisfied if H is a simple dimension group where g acts on H by "shifting". The attempt was stalled when in the proof we have to prove the statement:

If $a \leq x$ and $a \leq gx + g^2x$ then there exists $z = gz$ such that $a \leq z \leq x$ (see **App.5**) which is not true if H is simplicial group or a countable simple dimension group with g acting on H by "shifting".

App.2. An analogy of [GH, Theorem 10.6] for the countable simple dimension group case would be:

A.3. *Let H be both a countable simple dimension group and an ordered $\mathbf{Z}G$-module. Let v be an order-unit. Then (H, v) is isomorphic (as ordered $\mathbf{Z}G$-modules with order-unit) to $(K_0(R^K), [R^K])$ for some $R \in \mathcal{R}$ of type k if and only if* *(iii)* $ker(1+g+g^2) = \{(k+mg+ng^2)h \mid h \in H,\ \ k,m,n \in \mathbf{Z},\ \text{ and }\ k+m+n=0\}$,

(iv) $ker(1-g) = (1+g+g^2)(H)$,

(v) $(1+g+g^2)(H^+) = (1+g+g^2)(H)^+$,

(vi) $v \in (1+g+g^2)(H)^+$.

Let H be a simplicial group. Then conditions (ii) and (iv) implies that g acts on H by "shifting". This in turn implies condition (d2) of Corollary 5.2.3 (by taking $col(H^+) \ni y = (e_1, e_2, \ldots, e_n)^T$, where e_i's are atoms of H). Now if H is a countable simple dimension group with finitely many states, then condition (ii) is satisfied. Condition (iv) will ensure that $H = H_1 \times H_2$ where g acts on H_1 by "shifting" and H_2 is a 3-divisible group (with g acting trivially). Then we possibly derive a similar condition to ($c^{\#}$) on H_2 by using $b_i \in mat(\mathbf{R}^+)$ instead of $b_i \in mat(\mathbf{Z}^+)$. Meanwhile H_2 is isomorphic to some $K_0(R_2^K)$ for some $R_2 \in \mathcal{R}$ of type k by Proposition 5.1. Therefore a question arises:

A.4. *Let H be a countable simple dimension group with finitely many states. Then $(H, v) \cong (K_0(R^K), [R^K])$ for some $R \in \mathcal{R}$ of type k if and only if the conditions (iv),(v), and (vi) are satisfied ?*

Since the column x in the hypothesis of Proposition 5.2.2 is a finite column, the assumption that H having finitely many states might not be necessary.

App.3. In chapter 5, we assume G to be abelian and by identifying $K_0(G)$ with $\mathbf{Z}G$, we obtain a connection between locally representable actions and algebras of type k. When G is not abelian, we might hope to maintain the connection through abelian subgroups of G. However, $K_0(B^E)$ is a $\mathbf{Z}$Gal(E/F)-module, but $K_0(A \times_\alpha G)/I_E K_0(A \times_\alpha G)$ is not (see Figure 5-1). From section 3.5, we know that the set of all $K_0(B^E)$'s as dimension groups alone (without any module structures) is not an invariant for algebras of this type. Hence, this attempt fails. However, this gives rise to the following question:

A.5. *Can we, by transferring the ordered $\mathbf{Z}$Gal(K/F)-module structure from $K_0(R^K)$ to $K_0(R^{D^{op}})$ through the maps $K_0(R^K) \to K_0(R^{M_2(D^{op})})$, obtain an invariant which is* *the set of all $K_0(R^{D^{op}})$'s as ordered $\mathbb{Z}Gal(K/F)$-modules without any connecting maps between them?* Note that this transferring of module structure is possible because $\mathrm{Gal}(K/F)$ preserves the kernels of the maps above by Lemma 2.1.2.

The answer is "Yes". We can reduce R to the finite dimensional simple case. Then the faithfulness of this invariant follows Proposition 2.3(b) and the inclusion $K_0(R^E) \to K_0(R^{D^{op}})$. Fullness of this invariant holds by a very similar argument to the proof of Proposition 2.4.4. Finally, we apply the routine interweaving argument.

An application for this new invariant would be found in its connection to the classification of actions of finite abelian groups on locally semisimple F-algebras (not necessarily locally representable):

Let G be a cyclic group of prime order p acting on a locally semisimple F-algebra by action α. As G is compact, this algebra is the union of an ascending nest of its finite dimensional subalgebras A_n stable with respect to α (see [Ha2]). Both $K_0(A_n)$ and $K_0(A_n \times_\alpha G)$ are ordered G-modules. The natural map $K_0(A_n) \to K_0(A_n \times_\alpha G)$ carries 1-orbits of $K_0(A_n)$ to p-orbits of $K_0(A_n \times_\alpha G)$ and vice versa. Now let E/F and L/F be Galois extensions with the same Galois group G such that $E \otimes_F L$ is a field (E and F are linearly disjoint over F). Let R be a finite direct product of matrix algebras over E and L. Both $K_0(R^E)$ and $K_0(R^L)$ are ordered G-modules and the natural map $K_0(R^E) \to K_0(R^{M_p(L)})$ carries 1-orbits of $K_0(R^E)$ to p-orbits of $K_0(R^L)$ and vice versa. For a finite abelian group which is not of prime order, the action of G on $K_0(-)$ can be more complicated but we can still expect to have a similar connection.

App.4. Proof of Lemma A.1. Write $p = (p_1, \dots, p_n)$ and $x = (x_1, \dots, x_n)^{tr}$.

First suppose that $p_i \geq 0$ for all i. Renumber the p_i and the x_i so that for some $t \leq n$ we have $p_i = 0$ for $i \leq t$ while $p_i > 0$ for $i > t$. Then

$$p_{t+1}(x_{t+1} + gx_{t+1} + g^2x_{t+1}) + \cdots + p_n(x_n + gx_n + g^2x_n) = 0,$$

whence $x_i + gx_i + g^2x_i = 0$ and so $x_i = 0$ for $i = t+1, \dots, n$. In particular, if all $p_i > 0$ then all $x_i = 0$, in which case we just take $y = 0$ and $q_1, q_2, q_3 = 0$. Otherwise, $p_1 = 0$

and $t \geq 1$. Set $y_j = x_j$ for $j = 1, \dots, t$ and $q_{1ii} = 1$ for $i = 1, \dots, t$, while $q_{1ij} = 0$ for all other i, j and $q_2, q_3 = 0$. Then

$$q_{1i1}y_1 + q_{2i1}gy_1 + q_{3i1}g^2y_1 + \cdots + q_{1it}y_t + q_{2it}gy_t + q_{3it}g^2y_t = \begin{cases} x_i & \text{for } i = 1, \dots, t \\ 0 (= x_i) & \text{for } i = t+1, \dots, n, \end{cases}$$

while

$$p_1(q_{11j} + q_{21j} + q_{31j}) + \cdots + p_n(q_{1nj} + q_{2nj} + q_{3nj}) = \begin{cases} p_j (= 0) & \text{for } j = 1, \dots, t \\ 0 & \text{for } j = t+1, \dots, n. \end{cases}$$

If $p_i \leq 0$ for all i, multiply through by -1 and use the result of the preceding paragraph.

In the general case, we define **degree** of p to be the ordered pair (π, ν) where π is the maximum of $| p_1 |, \dots, | p_n |$ and ν is the number of times $\pm\pi$ appears in p. These degrees lie in $\mathbf{Z}^+ \times \mathbf{N}$, which we order lexicographically, and we proceed by induction on degree. In case p has degree $(0, \nu)$ for some ν, all $p_i = 0$, and this case has been covered above.

Now assume that p has degree (π, ν) with $\pi > 0$, and that the lemma holds for all relations in which the coefficient row has smaller degree. Since we already covered the cases in which all the $p_i \geq 0$ or all the $p_i \leq 0$, we may assume that at least one $p_i > 0$ and at least one $p_i < 0$.

After renumbering the p_i and the x_i, and multiplying through by -1 if necessary, we may assume that $p_1 = \pi$ and $p_2, \dots, p_m \geq 0$, while $p_{m+1}, \dots, p_n < 0$, where $1 \leq m < n$. Note that $\pi \geq -p_i > 0$ for $i = m+1, \dots, n$. Now

$$\begin{aligned} \pi(x_1 + gx_1 + g^2x_1) = p_1(x_1 + gx_1 + g^2x_1) &\leq \sum_{i=1}^{m} p_i(x_i + gx_i + g^2x_i) \\ &= -\sum_{i=m+1}^{n} p_i(x_i + gx_i + g^2x_i) \\ &\leq \sum_{i=m+1}^{n} \pi(x_i + gx_i + g^2x_i), \end{aligned}$$

and hence, since H is unperforated,

$$x_1 + gx_1 + g^2x_1 \leq \sum_{i=m+1}^{n} (x_i + gx_i + g^2x_i).$$

By hypothesis, there exist $c, d, e \in H^+$ such that $x_1 = c + d + e$ and

$$c + gd + g^2e \leq x_{m+1} + \cdots + x_n.$$

By Riesz decomposition, there exist $c_{m+1}, \ldots, c_n, gd_{m+1}, \ldots, gd_n, g^2e_{m+1}, \ldots, g^2e_n$ such that

$$c = c_{m+1} + \cdots + c_n,$$

$$gd = gd_{m+1} + \cdots + gd_n$$

$$\text{and} \quad g^2e = g^2e_{m+1} + \cdots + g^2e_n$$

while $c_i + gd_i + g^2e_i \leq x_i$ for $i = m+1, \ldots, n$. Set $z_i = x_i - c_i - gd_i - g^2e_i$ for $i = m+1, \ldots, n$, and note that each $z_i \in H^+$. Now

$$\sum_{i=m+1}^{n} \pi(c_i + gc_i + g^2c_i + d_i + gd_i + g^2d_i + e_i + ge_i + g^2e_i) + \sum_{i=2}^{n} p_i(x_i + gx_i + g^2x_i)$$

$$= \pi(c + gc + g^2c + d + gd + g^2d + e + ge + g^2e) + \sum_{i=2}^{n} p_i(x_i + gx_i + g^2x_i)$$

$$= \sum_{i=1}^{n} p_i(x_i + gx_i + g^2x_i) = 0$$

and consequently

$$\sum_{i=m+1}^{n} (\pi + p_i)(c_i + gc_i + g^2c_i) + \sum_{i=m+1}^{n} (\pi + p_i)(d_i + gd_i + g^2d_i)$$

$$+ \sum_{i=m+1}^{n} (\pi + p_i)(e_i + ge_i + g^2e_i) + \sum_{i=2}^{m} p_i(x_i + gx_i + g^2x_i) + \sum_{i=m+1}^{n} p_i(z_i + gz_i + g^2z_i) = 0. \quad (\dagger)$$

Since $\pi > \pi + p_i \geq 0$ for $i = m+1, \ldots, n$, we see that $\pm\pi$ appears in the coefficient list of the relation (†) exactly $\nu - 1$ times, whence the coefficient list of (†) has degree less than (π, ν). By induction, there exist $y \in col(H^+)$ and $r_1, r_2, r_3, s_1, s_2, s_3, t_1, t_2, t_3, u_1, u_2, u_3 \in mat(\mathbf{Z}^+)$ such that the relations of Figure A-1 hold.

$$(c_{m+1}, \ldots, c_n, d_{m+1}, \ldots, d_n, e_{m+1}, \ldots, e_n, x_2, \ldots, x_m, z_{m+1}, \ldots, z_n)^{tr}$$

$$= \begin{bmatrix} r_1 \\ s_1 \\ t_1 \\ u_1 \end{bmatrix} y + \begin{bmatrix} r_2 \\ s_2 \\ t_2 \\ u_2 \end{bmatrix} gy + \begin{bmatrix} r_3 \\ s_3 \\ t_3 \\ u_3 \end{bmatrix} g^2y$$

$$(\pi + p_{m+1}, \ldots, \pi + p_n, \pi + p_{m+1}, \ldots, \pi + p_n, \pi + p_{m+1}, \ldots, \pi + p_n, p_2, \ldots, p_n) \begin{bmatrix} r_1 + r_2 + r_3 \\ s_1 + s_2 + s_3 \\ t_1 + t_2 + t_3 \\ u_1 + u_2 + u_3 \end{bmatrix} = 0.$$

Figure A-1

Write $y = (y_1, y_2, \ldots, y_t)^{tr}$, write $r_k = (r_{kij})$, $s_k = (s_{kij})$, and $t_k = (t_{kij})$ (for $k = 1, 2, 3$; $i = m+1, \ldots, n$ and $j = 1, \ldots, t$), and write $u_k = (u_{kij})$ (for $k = 1, 2, 3$; $i = 2, \ldots, n$ and $j = 1, \ldots, t$).

Now for $k = 1, 2, 3$ and $j = 1, \ldots, t$ we define

$$q_{k1j} = \sum_{i=m+1}^{n} (r_{kij} + s_{kij} + t_{kij})$$
$$q_{kij} = u_{kij} \qquad \text{for } i = 2, \ldots, m$$

and define

$$q_{1ij} = u_{1ij} + r_{1ij} + s_{3ij} + t_{2ij}$$

$$q_{2ij} = u_{2ij} + r_{2ij} + s_{1ij} + t_{3ij}$$

$$q_{3ij} = u_{3ij} + r_{3ij} + s_{2ij} + t_{1ij}$$

for $i = m + 1, \ldots, n$ and $j = 1, \ldots, t$. By Figure A-1, we compute that

$$\begin{aligned}
&\sum_{j=1}^{t}(q_{11j}y_j + q_{21j}gy_j + q_{31j}g^2y_j) \\
=&\sum_{j=1}^{t}\sum_{i=m+1}^{n}(r_{1ij}y_j + s_{1ij}y_j + t_{1ij}y_j + r_{2ij}gy_j + s_{2ij}gy_j + t_{2ij}gy_j + r_{3ij}g^2y_j + s_{3ij}g^2y_j + t_{3ij}g^2y_j) \\
=&\sum_{i=m+1}^{n}(c_i + d_i + e_i) = c + d + e = x_1,
\end{aligned}$$

$$\sum_{j=1}^{t}(q_{1ij}y_j + q_{2ij}gy_j + q_{3ij}g^2y_j) = \sum_{j=1}^{t}(u_{1ij}y_j + u_{2ij}gy_j + u_{3ij}g^2y_j) = x_i$$

for $i = 2, \ldots, m$,

$$\begin{aligned}
\sum_{j=1}^{t}(q_{1ij}y_j + q_{2ij}gy_j + q_{3ij}g^2y_j) =&\sum_{j=1}^{t}(u_{1ij}y_j + u_{2ij}gy_j + u_{3ij}g^2y_j) \\
&+\sum_{j=1}^{t}(r_{1ij}y_j + r_{2ij}gy_j + r_{3ij}g^2y_j) \\
&+\sum_{j=1}^{t}g(s_{3ij}g^2y_j + s_{1ij}y_j + s_{2ij}gy_j) \\
&+\sum_{j=1}^{t}g^2(t_{2ij}gy_j + t_{3ij}g^2y_j + t_{1ij}y_j) \\
=&z_i + c_i + gd_i + g^2e_i = x_i
\end{aligned}$$

for $i = 2, \ldots, m$, and

$$\sum_{i=1}^{n} p_i(q_{1ij} + q_{2ij} + q_{3ij})$$

$$= p_1 \sum_{i=m+1}^{n} (r_{1ij} + s_{1ij} + t_{1ij} + r_{2ij} + s_{2ij} + t_{2ij} + r_{3ij} + s_{3ij} + t_{3ij}) + \sum_{i=2}^{m} p_i(u_{1ij} + u_{2ij} + u_{3ij})$$

$$+ \sum_{i=m+1}^{n} p_i(u_{1ij} + r_{1ij} + s_{3ij} + t_{2ij} + u_{2ij} + r_{2ij} + s_{1ij} + t_{3ij} + u_{3ij} + r_{3ij} + s_{2ij} + t_{1ij})$$

$$= \sum_{i=m+1}^{n} (\pi + p_i)(r_{1ij} + r_{2ij} + r_{3ij}) + \sum_{i=m+1}^{n} (\pi + p_i)(s_{1ij} + s_{2ij} + s_{3ij})$$

$$+ \sum_{i=m+1}^{n} (\pi + p_i)(t_{1ij} + t_{2ij} + t_{3ij}) + \sum_{i=2}^{n} p_i(u_{1ij} + u_{2ij} + u_{3ij}) = 0$$

for $j = 1, \ldots, t$.

This completes the induction step. ■

App.5. The stalled proof of A.2. Write $p_k = (p_{k1}, \ldots, p_{kn})$ for $k = 1, 2, 3$ and $x = (x_1, \ldots, x_n)^{tr}$. If all $p_{ki} = 0$, just take $y = x$, $q_1 = \mathbf{I}$, $q_2 = q_3 = 0$.

If $n = 1$ and $p_1 \neq 0$, by (i) we have $x = y + gy + g^2y$ for some $y \in H^+$. In this case, take $q_1 = q_2 = q_3 = 1$.

In the general case, we define the degree of the triple (p_1, p_2, p_3) to be the ordered pair (π, ν) where π is the maximum of $| p_{ki} |$ for $k = 1, 2, 3$ and $i = 1, \ldots, n$; and ν is the number of times π appears in $| p_{ki} |$. As in Lemma A.1, we proceed by induction on this degree. If (p_1, p_2, p_3) has degree $(0, \nu)$ for some ν, then all $p_{ki} = 0$, and this case is covered above. Now assume that (p_1, p_2, p_3) has degree (π, ν) with $\pi > 0$, and that the Conjecture holds in cases of lower degree. Since the case $n = 1$ is covered above, we may assume that $n > 1$.

We may renumber the p_{ki}, x_i so that $| p_{1i} | + | p_{2i} | + | p_{3i} | > 0$ for $i = 1, \ldots, m$ and equal 0 for $i = m+1, \ldots, n$. Let $p'_k = (p_{k1}, \ldots, p_{km})$ for $k = 1, 2, 3$ and $x' = (x_1, \ldots, x_m)^{tr}$. Then

$$p'_1 x' + p'_2 g x' + p_3 g^2 x' = 0.$$

Suppose there exist $y \in col(H^+)$ and $q_1, q_2, q_3 \in mat(\mathbf{Z}^+)$ such that

$$x' = q_1 y + q_2 g y + q_3 g^2 y \quad \text{and} \quad p_1' q_1 + p_2' q_2 + p_3' q_3 = 0.$$

Then the relations of Figure A-2 hold.

$$x = \begin{bmatrix} q_1 & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} y \\ x_{m+1} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} gy \\ gx_{m+1} \\ \vdots \\ gx_n \end{bmatrix} + \begin{bmatrix} q_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g^2 y \\ g^2 x_{m+1} \\ \vdots \\ g^2 x_n \end{bmatrix}$$

$$p_1 \begin{bmatrix} q_1 & 0 \\ 0 & \mathbf{I} \end{bmatrix} + p_2 \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix} + p_3 \begin{bmatrix} q_3 & 0 \\ 0 & 0 \end{bmatrix} = \left[(p_1' q_1 + p_2' q_2 + p_3' q_3) p_{1(m+1)} \cdots p_{1n} \right] = 0.$$

Figure A-2

Thus there is no loss of generality in assuming that all $| p_{1i} | + | p_{2i} | + | p_{3i} | > 0$.

For any $i = 1, \ldots, n$, the relations in (ii) still hold if we replace x_i by gx_i; the triple (p_{1i}, p_{2i}, p_{3i}) by (p_{2i}, p_{3i}, p_{1i}); and the i^{th}-row of q_1 by the i^{th}-row of q_3, the i^{th}-row ofq_2 by the i^{th}-row of q_1, and the i^{th}-row of q_3 by the i^{th}-row of q_2.

So after renumbering the p_{ki}, x_i and multiplying through by -1 if necessary, we may now assume that

$$p_{11} = \pi;$$

$$p_{11}, \ldots, p_{1m} > 0, \quad \text{and} \quad p_{k1}, \ldots, p_{km} \leq 0;$$

$$\text{while} \quad p_{1(m+1)}, \ldots, p_{1n} < 0, \quad \text{and} \quad p_{k(m+1)}, \ldots, p_{kn} \geq 0$$

where $1 \leq m \leq n$ and $k = 2, 3$. Since

$$\begin{aligned} \pi x_1 = p_{11} x_1 &\leq \sum_{i=1}^{m} p_{1i} x_i + \sum_{i=m+1}^{n} p_{2i} g x_i + \sum_{i=m+1}^{n} p_{3i} g^2 x_i \\ &= \sum_{i=m+1}^{n} -p_{1i} x_i + \sum_{i+1}^{m} -p_{2i} g x_i + \sum_{i=1}^{m} -p_{3i} g^2 x_i \\ &\leq \sum_{i=m+1}^{n} \pi x_i + \sum_{i=1}^{m} \pi g x_i + \sum_{i=1}^{m} \pi g^2 x_i \end{aligned}$$

we obtain

$$x_1 \leq \sum_{i=m+1}^{n} x_i + \sum_{i=1}^{m} g x_i + \sum_{i=1}^{m} g^2 x_i$$

and consequently $x_1 = a+b$ for some $a, b \in H^+$ with $a \le gx_1 + g^2x_1$, and $b \le \sum_{i=m+1}^{n} x_i + \sum_{i=2}^{m} gx_i + \sum_{i=2}^{m} g^2x_i$.

Now we need to introduce $z = gz$ such that $a \le z \le x_1$ so that

$$p_{11}x_1 + p_{21}gx_1 + p_{31}g^2x_1 = p_{11}(x_1 - z) + p_{21}g(x_1 - z) + p_{31}g^2(x_1 - z).$$

However, a only satisfies $a \le x_1$ and $a \le gx_1 + g^2x_1$.

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