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ON THE CONTROLLABILITY OF LINEAR AND NONLINEAR SYSTEMS

by

Williams R. Colmenares M.

A thesis
presented to the School of Graduate Studies and Research
at the University of Ottawa
in partial fulfillment of the
requirements for the degree of
Master of Applied Science
in
The Department of Electrical Engineering, Faculty of
Science and Engineering.

OTTAWA, Ontario, 1983
Williams R. Colmenares M., 1983



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to
my wife Nene

and

my parents Alba and Guillermo.

...and God said let there be light,
and there was light and God saw the
light. that it was good....

Genesis 1, 3-4

ABSTRACT

A short survey of the controllability theory of linear and nonlinear systems is presented.

Regarding the controllability theory of linear systems, a global finite time null controllability counter example is exhibited.

An algorithm for synthesizing a stabilizing feedback control based on Lie algebraic methods is illustrated and applied to the satellite attitude control problem.

Various aspects of satellite attitude control are investigated and the efficiency of the above mentioned algorithm is evaluated.

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Chapter I INTRODUCTION

Controllability theory attempts to define and isolate the theoretical limits to which a system can be controlled. Researchers are aware of its fundamental interconnections with other aspects of control including the existence of optimal controls, their feedback synthesis, stabilization and observability.

Extensive research has been done in this area and certainly our knowledge of the controllability of linear systems is substantial. On the other hand, our knowledge of the controllability of nonlinear systems is very limited and most of the available results in this area of nonlinear theory are in some sense local. So that, while the controllability theory for linear systems is fairly complete, a comprehensive theory capable of handling nonlinear systems with the detail available in the linear case is yet to be done.

In this research we briefly survey the controllability theory of linear and nonlinear systems. Our aim is to show

those results which, in the author's opinion, are best known and of major importance.

In the course of this research on controllability the author discovered a counterexample to a very well established result [1, p. 98] from which numerous other results have been derived, see for instance [15, thm. 3] and [25, thm. 1].

controllability and asymptotic stability do not necessarily imply global null-controllability. This counter example is one of the major contributions of this thesis.

Also, the controllability properties of the attitude dynamics of the satellite problem are studied and a 'modified' feedback control is synthesized. Here, the primary objective is to evaluate the effectiveness of the synthesizing algorithm and not the satellite attitude control problem itself.

Great emphasis is placed on the geometric approach (Lie Algebraic Methods) when studying nonlinear systems. The reason is that the approach provides an analytic characterization of the controllability space for some special cases of nonlinear systems. This cannot be said for most other approaches commonly used when studying controllability for nonlinear systems. It is believed that the controllability space for an even broader class of systems may be characterized by using the same approach.

Also, the approach naturally accommodates controllability concepts and therefore, allows the use of such geometric theory as a tool in the controllability research.

In the following an outline of the thesis will be presented.

In chapter II a review of the controllability theory of linear systems is presented. The Reachable and Attainable sets are defined and some of their properties exhibited. Controllability criteria for linear and "mildly" nonlinear systems are introduced. The term "mildly" nonlinear systems, refers to systems with additive nonlinear control and uncertain dynamic systems.

In chapter III the global finite time null controllability counter example is presented along with some comments on the reasons why the original theory is erroneous.

In chapter IV a review of the controllability theory of nonlinear systems is presented, controllability criteria are generated and an algorithm for synthesizing a 'modified' feedback control is illustrated.

In chapter V the effectiveness of the algorithm mentioned above is evaluated by using it to generate a control for the attitude dynamics of the satellite problem. The satellite dynamics are presented, the controllability properties of the model are analyzed and the control policies are computed. Afterwards, the algorithm is slightly modified to reduce the regulation time. Several figures are exhibited to illustrate the numerical results.

In the concluding part of this chapter critical comments are also presented on the efficiency and effectiveness of the Lie Algebraic Methods and its associated stabilizing algorithm in the study of nonlinear systems. Proposals for further research are also included in this chapter.

In Appendix A the Fortran codes used to solve the variety of problems considered in the satellite attitude control are exhibited.

Chapter II
ON THE CONTROLLABILITY OF LINEAR SYSTEMS

2.1 INTRODUCTION

Frequently, control systems of the form

$$\dot{x} = f(t, x, u) \quad (S)$$

are used to model the behavior of physical, biological or social systems. In analysing such systems, the first question that comes to mind is where the system can be driven to, by the available inputs (controls). The aim of controllability theory is to answer this question.

In this chapter a brief review of the controllability theory of linear systems is presented and some of the well-known results are given.

Throughout the chapter, unless otherwise stated, we shall consider control systems of the form :

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad ; \quad t \in [t_0, \infty) \quad (2.1) \\ x(t_0) &= x_0 \end{aligned}$$

where $x(t) \in R^n$; $u(t) \in R^m$; $t \geq t_0$

$A(\bullet)$ and $B(\bullet)$ are $n \times n$ and $n \times m$ matrix valued functions, respectively, with components summable over finite intervals.

$u(\bullet) \in M(\Omega)$ the space of measurable functions on $[t_0, \infty)$ with values in $\Omega \subseteq R^m$, a compact and convex set.

2.1.1 The Reachable and Attainable Sets

By definition the reachable set for the system (2.1) is given by

$$R(t; t_0) = \{ y(t; u, t_0) \in R^n : y(t; u, t_0) = \int_{t_0}^t X^{-1}(s) B(s) u(s) ds ; \\ u(\bullet) \in M(\Omega) \} \quad (2.2)$$

where $X(t)$ is the state transition matrix of (2.1); and the attainable set

$$A(t; t_0, x_0) = X(t) \{ X^{-1}(t_0) x_0 + R(t; t_0) \} \quad (2.3)$$

is defined as the set of states that may be attained from $x_0 \in R^n$ starting at time $t=t_0$ by using some admissible control $u(\bullet) \in M(\Omega)$.

Liapunov's theorem on the range of a vector measure allows us to state some of the properties of the attainable and reachable sets.

Theorem 2.1 [2, Thm. 1, pp. 69] For system (2.1) the attainable set (2.3) (and the reachable set (2.2)) is compact, convex and varies continuously with respect to t .

Most of the results we present in the next sections rely heavily on such properties of the reachable and attainable sets.

2.1.2 Basic Concepts

An initial condition $x(t_0) = x_0 \in R^n$ is said to be controllable to a point $x_1 \in R^n$ if there exists a time $T \in [t_0, \infty)$ and a control $u(\cdot) \in M(\cdot)$ such that the solution $x(\cdot; x_0, t_0, u)$ of (2.1) satisfies :

$$x(T; x_0, t_0, u) = x_1. \quad (2.4)$$

The domain

$$C(x_1, t_0) = \{x_0 \in R^n : x_0 \text{ is controllable to } x_1\} \quad (2.5)$$

of controllability to x_1 is defined as the set of initial states from which the system (2.1) can be driven to x_1 in finite time by using some admissible control. If $C(x_1, t_0)$ contains an open neighborhood of x_1 then the system is said to be locally controllable to x_1 . If x_1 is set to be 0 and again $C(0, t_0)$ contains an open neighborhood of the origin,

the system is said to be locally null controllable. If $C(0, t_0) \cong \mathbb{R}^n$ then the system is globally null controllable.

The system is said to be controllable at t_0 if every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ is controllable to every $x_1 \in \mathbb{R}^n$ in finite time. If the system is controllable at t_0 for every $t_0 \in [0, \infty)$, then the system is simply said to be controllable.

In this chapter as well as in the rest of this thesis $|\cdot|$ will denote the euclidean norm, i.e., for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$|x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$

Next we investigate the properties that are required in order for the system (2.1) to be controllable in any of the senses defined above.

2.2 LINEAR SYSTEMS. BASIC RESULTS ON CONTROLLABILITY

In this and succeeding sections we present some results on the controllability of linear systems and mildly nonlinear in the control and/or disturbance systems.

2.2.1 Linear Autonomous Systems

For linear systems the majority of results are for autonomous systems, i.e., the matrices A and B are constant; and the basic results are:

Theorem 2.2 [2, Thm. 5, pp. 81] System (2.1) with A and B constant and $\Omega \subseteq \mathbb{R}^n$ is controllable if and only if the $n \times nm$ matrix Q

$$Q = [B, AB, A^2B, \dots, A^{n-1}B] \quad (2.6)$$

is such that

$$\text{rank } Q = n. \quad (2.7)$$

Corollary 2.2.1 [2, Corollary 2, pp. 84] Let A and B in system (2.1) be constant. Assume Ω to be a subset of \mathbb{R}^n with 0 in its interior. The system (2.1) is locally null controllable if and only if the rank condition (2.7) is satisfied.

Many variations on the above theme may be obtained by changing or imposing further assumptions on matrix A; however, the algebraic results of theorem 2.2 and its corollary 2.2.1 form the cornerstone for the study of almost all questions relating to controllability of linear autonomous systems. It is in general well known that the

null controllability space $C(0, t_0)$ is the subspace of R^n generated by the columns of Q [14]. Nevertheless, it is important to mention an additional result on null controllability of autonomous systems since it implies that the combined action of the system itself (without control) and that of the control will determine whether or not the system is globally null controllable.

Corollary 2.2.2 [2, Corollary 3, pp. 85] System (2.1) is globally null controllable if it is locally null controllable (Corol. 2.2.1) and asymptotically stable.

In Corollary (2.2.2), the asymptotic stability together with the local null controllability property implies that the autonomous system without control ($u=0$) will arrive in a neighborhood of the origin in finite time, and then with some admissible control can be steered to the origin in finite time; hence, the global null controllability property. It should be mentioned that a system is said to be asymptotically stable if for every $x(t_0)=x_0 \in R^n$ the solution of (2.1) without control ($u=0$) satisfies :

$$\lim_{t \rightarrow \infty} x(t; x_0, t_0, 0) = 0 \quad (2.8)$$

A linear autonomous system is asymptotically stable if and only if the eigenvalues λ of A satisfy $\text{Re } \lambda < 0$.

2.2.2 Time Varying Systems

For linear time varying systems the most familiar results are 1) that of Kalman, and 2) that of Lasalle and Hermes which are stated below in theorems 2.3 and 2.4.

Theorem 2.3 [22, Thm. 2.4, pp. 52] Let Ω be the whole of \mathbb{R}^m then the system (2.1) is controllable on $[t_0, t_1]$ if and only if the controllability gramian

$$W(t_1, t_0) = \int_{t_0}^{t_1} X(t_1, s) B(s) B'(s) X'(t_1, s) ds \quad (2.9)$$

is positive definite.

Theorem 2.4 [1, Thm. 19.3, pp.96] Let $\Omega = \mathbb{R}^n$ and suppose $A(t)$ has $k-2$ continuous derivatives and $B(t)$ has $k-1$ continuous derivatives. Let Γ denote the operator given by:

$$(\Gamma B)(t^*) \triangleq -A(t^*)B(t^*) + (d/dt)B(t) \Big|_{t=t^*} \quad (2.10)$$

If there exists a positive integer k and a time $t_1 \geq 0$ such that

$$\text{rank} [B(t_1), (\Gamma B)(t_1), \dots, (\Gamma^{k-1} B)(t_1)] = n \quad (2.11)$$

then the linear system (2.1) is controllable at t_1 . If in theorem 2.4 Ω is constrained to be the unit cube C^m in \mathbb{R}^m , condition (2.10) is a sufficient condition only for local null controllability [1].

Unfortunately, as can be seen, the controllability criteria expressed in theorems (2.3) and (2.4) are not as simple as those of linear time invariant systems, and condition (2.10) is not even necessary. Besides being cumbersome, conditions (2.9) and (2.10) require Ω to be the whole of R^m , which is seldom satisfied in real life problems. Many other criteria have been introduced which do not need this condition, but require either Ω containing 0 or the existence of an element which lies in the Kernel of $B(t)$: The criterion for controllability to the origin given by Barmish and Schmitendorf [3] does not require Ω to satisfy either of the above conditions. In order to present the results of [3] let us introduce the scalar function $J: R^n \times R \times R^n \rightarrow R$ given by:

$$J(x_0, T, \lambda) = x_0' X'(T, t_0) \lambda + \int_{t_0}^T \sup\{w' B'(s) X'(T, s) \lambda : w \in \Omega\} ds. \quad (2.12)$$

Using the function J one can characterize the null controllability subspace as follows:

Theorem 2.5 [3, Thm. 2.3, pp. 330] Let ρ be a compact set. Pick any subset Λ of R^n which contains 0 as an interior point. Then for system (2.1) the initial condition $x(t_0) = x_0 \in R^n$ is controllable to the origin if and only if

$$\min\{J(x_0, T, \lambda) : \lambda \in \Lambda\} = 0 \quad (2.13)$$

for some $T \in [t_0, \infty)$.

From theorem 2.5, easily follows:

Corollary 2.5.1 [3, Lemma 6.1, pp. 336] There exists a solution to the time optimal control problem if and only if there is some finite time $t_f \in [t_0, \infty)$ such that

$$\min \{J(x_0, t_f, \lambda) : \lambda \in \Lambda\} = 0. \quad (2.14)$$

Furthermore, the time optimal cost is given by :

$$\min \{ t_f : \min \{ J(x_0, t_f, \lambda) = 0 : \lambda \in \Lambda \} \}. \quad (2.15)$$

Also from theorem 2.5 necessary and sufficient conditions for global null controllability may be derived.

Theorem 2.6 [3, Thm. 2.1, pp. 329] Suppose Ω is a compact set which contains zero. The system (2.1) is globally null controllable at t_0 if and only if

$$\int_{t_0}^{\infty} \sup \{ w' B'(s) X'(t_0, s) \lambda : w \in \Omega \} ds = \infty \quad (2.16)$$

for all $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$.

For linear time varying systems an analogue version of Corollary 2.2.2 in global null controllability was introduced by Lasalle and Hermes [1] who expressed the results in terms of 'proper systems' which we now define. System (2.1) is said to be proper at time t_0 if

$$\eta' X^{-1}(t) B(t) = 0 \quad (2.17)$$

on $[t_0, t_0 + \delta]$ for each $\delta > 0$ implies $\eta = 0$.

(*) The time optimal control problem being the problem of:

- (1) determining whether there exists a control u^* that steers the initial condition to x_1 in minimum time, and

The following theorem by Lasalle and Hermes [1] shows the relationship between properness at some $t_0 \in [0, \infty)$ and local null controllability.

Theorem 2.7 [1, Corollary 17.4, pp. 78] Let Ω be the unit cube in \mathbb{R}^m . System (2.1) is proper at t_0 if and only if the origin is an interior point of $R(t, t_0)$ for each $t > t_0$.

Clearly, system (2.1) is locally null controllable if and only if it is proper at t for some $t \in [0, \infty)$ (recall the properties of the reachable set established in theorem 2.1).

The general version of Corollary 2.2.2 on global null controllability arises as a combination of asymptotic stability and the result of theorem 2.7.

Corollary 2.7.1 [1, Remark pp. 78] If the uncontrolled system :

$$\dot{x}(t) = A(t)x(t) \quad (2.18)$$

is asymptotically stable and system (2.1) is proper at time t_0 , then there is for each initial state $x_0 \in \mathbb{R}^n$ an admissible control u that brings the system to the origin in finite time, i.e., the system (2.1) is globally null controllable.

The idea behind Corollary 2.7.1 is similar to that of linear time invariant systems (see Corollary 2.2.2); but unfortunately, Corollary 2.7.1 is incorrect. A detailed explanation of this statement as well as a counter example supporting it will be presented in Chapter III.

2.3 CONTROLLABILITY RESULTS FOR MILDLY NONLINEAR SYSTEMS

In previous sections we have been studying a variety of questions of controllability for linear systems. In this section similar questions on the controllability of mildly nonlinear systems of the form:

$$i) \dot{x} = Ax + f(t, u) \quad \text{and} \quad (2.19a)$$

$$ii) \dot{x} = Ax + f(t, u) + g(t, q) \quad (2.19b)$$

will be analyzed.

2.3.1 Controllability Results for Systems of the Form

$$\dot{x} = Ax + f(t, u).$$

Consider the following mildly nonlinear system of the form:

$$\dot{x}(t) = A(t)x(t) + f(t, u(t)); \quad t \in [t_0, \infty) \quad (2.20)$$

$$x(t_0) = x_0$$

where $x(t) \in R^n$; $u(t) \in R^m$; $t \geq t_0$.

$A(\cdot)$ is a matrix valued function with components summable over finite intervals.

$u(\cdot) \in M(\bar{U})$ the space of measurable functions on $[t_0, \infty)$ with values in a compact subset \bar{U} of R^m .

$f: [t_0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a function continuous in its second argument and integrable over finite subinterval of $[t_0, \infty)$ for each selection $u(\cdot) \in M(\Omega)$.

Our main objective will be to characterize the controllability space to some given target set Z .

Systems of the form (2.20) were studied by Barmish and Schmitendorf [4] who characterized the controllability space to Z . Their result is in terms of a scalar function $\bar{J}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by:

$$\bar{J}(x_0, T, \lambda) = \lambda' X(T, t_0) x_0 + \int_{t_0}^T \max\{\lambda' X(T, s) f(s, w) : w \in \Omega\} ds - \inf\{\lambda' z : z \in Z\}. \quad (2.21)$$

Using this function \bar{J} one can characterize the null controllability space as follows:

Theorem 2.8 [4, Thm. 3.2.1, pp. 541] Consider the system (2.20) with admissible controls $u(\cdot) \in M(\Omega)$. Suppose the target set Z is a closed and convex subset of \mathbb{R}^n . Then the system (2.20) starting from the initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ is controllable to the target Z if and only if

$$\min\{\bar{J}(x_0, T, \lambda) : \lambda \in \Lambda\} = 0 \quad (2.22)$$

for some $T \in [t_0, \infty)$ and any $\Lambda \subseteq \mathbb{R}^n$ containing 0 in its interior.

For system (2.20) global controllability criteria may be obtained from theorem 2.8. The criterion for global controllability to Z will be described in terms of two time functions $V : R^n \times R \rightarrow R$; $W : R \rightarrow R$, which we define by :

$$V(Z(0), t) \triangleq \int_{t_0}^t \max_{w \in \Omega} \{ f'(s, w) \cdot Z(s) \} ds - \inf_{z \in Z} \{ z' \cdot Z(t) \} \quad (2.23)$$

and

$$W(t) = \min \{ V(Z(0), t) : \|Z(0)\|=1 \}$$

where $Z(t)$ are the solutions ($Z(t) = X'(T, t)\lambda$) of the adjoint system $Z' = -A'(t)Z(t)$ $t \in [0, \infty)$

Theorem 2.9 [4, Thm. 4.1, pp. 542] Consider the system (2.20) under the same hypothesis as in theorem 2.8. Then a necessary condition for global controllability to Z is that

$$\sup_{t \geq 0} V(Z(0), t) = \infty ; \forall Z(0) \in R^n, Z(0) \neq 0. \quad (2.24)$$

and a sufficient condition is that

$$\sup_{t \geq 0} W(t) = \infty. \quad (2.25)$$

Note that theorem 2.9 only requires Ω to be compact, so in a sense results (2.24) and (2.25) are more general than the one obtained for linear systems (2.1) in (2.16). On the other hand (2.16) is better in the sense that it is preferred to have in just one criterion necessary and sufficient conditions which theorem 2.9 fails to provide.

It is important to point out that if we assume 0 \in Ω conditions (2.24) and (2.25) merge in just one necessary and sufficient condition for global controllability to Z

$$\int_{t_0}^{\infty} \max_{w \in \Omega} \{ f'(s, w) Z(s) \} ds = \infty. \quad (2.26)$$

2.3.2 Controllability Results for Uncertain Dynamic Systems

In the following, we will let some additive disturbance appear in system (2.20) and we shall present some results that characterize the controllability space of such type of systems. For the disturbance, (or uncertainty) we will assume no knowledge except for those of Lebesgue measurability and bounded to be within a certain prespecified set. Such problems are often referred to in the literature as set theoretic problems. Conceptually, for these type of systems where the uncertainty is restricted to lie in a prespecified set, what we wish is to guarantee the transfer of the state to the target in the presence of any possible admissible perturbation.

We shall consider systems of the form

$$\dot{x}(t) = A(t)x(t) + f(t, u(t)) + g(t, q(t)) \quad (2.27)$$

$$x(t_0) = x_0$$

where similar hypothesis to those of system (2.20) hold for the state $x(t)$, the control $u(t)$, the matrix $\lambda(t)$, the function $f(\cdot, \cdot)$ and the target Z . Additionally, we assume $g: [t_0, \infty) \times R^p \rightarrow R^n$ to be continuous in its second argument and measurable and integrable over finite subintervals of $[t_0, \infty)$ for each selection $q(\cdot) \in M(Q)$, the space of measurable functions with values in Q a compact subset of R^p . A perturbation $q(\cdot)$ is said to be admissible whenever $q(\cdot) \in M(Q)$.

Here we will pursue two objectives. The first, to characterize the controllability space for the case in which given the initial condition, $x_0 \in R^n$, and the perturbation $q(\cdot) \in M(Q)$ there exists a control $u(\cdot) \in M(\Omega)$ that transfers x_0 to the target Z (such controllability concept will be referred to as weak controllability). The controllability criterion for this case is described in terms of a functional $\tilde{J}: R^n \times R \times R^n \rightarrow R$ given by:

$$\begin{aligned} \tilde{J}(x_0, T, \lambda) = & \lambda' X(T, t_0) x_0 + \int_{t_0}^T \max\{\lambda' X(T, s) f(s, w) : w \in \Omega\} ds + \\ & \int_{t_0}^T \min\{\lambda' X(T, s) g(s, q) : q \in Q\} ds - \\ & \inf\{\lambda' z : z \in Z\}. \end{aligned} \quad (2.28)$$

Using the functional \tilde{J} we can characterize the weakly controllable space to Z as follows:

Theorem 2.10 [7, Thm. 2.1, pp. 732] Consider the system (2.27) with admissible controls $u(\cdot) \in M(\Omega)$. Suppose the

target set Z is a closed and convex subset of R^n . Then for any given admissible perturbation $q(\bullet) \in M(Q)$ there exists a control $u(\bullet) \in M(\Omega)$ that steers the initial condition $x_0 \in R^n$ to the target Z if and only if

$$\min\{ \tilde{J}(x_0, T, \lambda) : \lambda \in \Lambda \} = 0 \quad (2.29)$$

for some $T \in [t_0, \infty)$ and any $\Lambda \subseteq R^n$ containing 0 in its interior. Similar results for slightly more general systems of the form

$$\dot{x}(t) = \lambda(t)x(t) + h(t, u(t), q(t)) \quad (2.30)$$

were derived by Schmitendorf and Elenbogen in [7]. Global controllability criteria, similar to those of theorem 2.9 may also be found in [7].

The second objective will be to characterize the controllability space for the case when given the initial condition $x_0 \in R^n$ there exists a control $u(\bullet) \in M(\Omega)$ that steers x_0 to the target Z in the presence of any admissible perturbation. Note that opposite to the previous case, we look for just one control that guarantees the controllability property (such controllability concept will be referred to, as strong controllability). Barmish [5] characterized the controllability space for the case in question. In order to introduce the result, we require the definition of the envelope of a function and of the modified functional $J^*: R^n \times R \times R^n \rightarrow R$.

Definition: Let Λ be a compact convex set in \mathbb{R}^n containing 0 in its interior. Suppose $F: \Lambda \rightarrow \mathbb{R}$ is a real valued function; then one can define $\text{env}_\Lambda F: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the envelope of F with respect to Λ , by

$$(\text{env}_\Lambda F)(\lambda) = \sup\{G(\lambda) : G \text{ is an affine linear function on } \mathbb{R}^n \text{ and } G(\lambda) \leq F(\lambda) \text{ for all } \lambda \in \Lambda\}$$

Definition: Let $J^*: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be given by:

$$\begin{aligned} J^*(x_0, T, \lambda) = & \lambda' X(T, t_0) x_0 + \int_{t_0}^T \max\{\lambda' X(T, s) f(s, w) : w \in W\} ds \\ & + \text{env}_\Lambda \left\{ \int_{t_0}^T \min\{\lambda' X(T, s) g(s, q) : q \in Q\} ds - \right. \\ & \left. - \inf\{\lambda' z : z \in Z\} \right\}. \end{aligned} \quad (2.31)$$

Using the functional J^* we can characterize the strongly controllable space to Z as follows:

Theorem 2.11 [5, Thm. 5.1, pp. 403] Let $T \in [t_0, \infty)$ and x_0 be given. Then for system (2.27) there exists a control u that steers the initial condition x_0 to the target Z in the presence of any admissible perturbation if and only if

$$\min\{J^*(x_0, T, \lambda) : \lambda \in \Lambda\} \geq 0 \quad (2.32)$$

where Λ is any arbitrary compact and convex subset of \mathbb{R}^n containing 0 in its interior.

Global controllability results similar to those of theorem 2.9 may be derived for system (2.27) when 'strong' controllability is required; such results may be found in [5].

As has been shown, the criteria for controllability become more complex as the complexity of the system or the requirement for a stronger control increases; and from the simple test for linear time invariant systems (2.7) we arrived at the more complex and difficult to compute criterion (2.30) for some semilinear uncertain dynamic systems.

Some numerical methods on how to compute the proposed controllability criteria may be found in [3] and [5].

2.4 SUMMARY

In this chapter, a review of the controllability theory of linear systems has been presented. The Reachable and Attainable sets have been defined and some of their properties are exhibited. The now classic results on controllability of linear systems of Kalman and Lee and Markus as well as the new results of Barmish and Schmitendorf have been introduced. Additionally, some criteria for mildly nonlinear systems, i.e., systems with additive nonlinear control, and uncertain dynamic systems, i.e., systems with additive disturbance, have also been introduced.

Chapter III

A GLOBAL FINITE TIME NULL CONTROLLABILITY COUNTEREXAMPLE

3.1 INTRODUCTION

It has been generally accepted that a control system

$$\dot{x}(t) = f(x, t, u) \quad (3.1)$$

is globally finite-time null controllable if it is asymptotically stable and locally null controllable.

This notion can be found in Lee and Markus [2] for nonlinear autonomous systems, and in Lasalle and Hermes [1] for linear systems, and has been widely used and extended; see for instance, Chukwu [15] and Khambadkone [25].

Lasalle and Hermes [1] studied systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (3.2)$$

$$x(t_0) = x_0$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $t \geq t_0$

$A(\bullet)$ and $B(\bullet)$ are $n \times n$ and $n \times m$ matrix valued functions respectively, with components summable over finite intervals.

$u(\bullet) \in M(C^m)$ the space of measurable functions on $[t_0, \infty)$ with values in C^m the unit cube in \mathbb{R}^m .

As mentioned in chapter II (Corollary 2.7.1), Lasalle and Hermes concluded that if system (3.2) was asymptotically stable and proper at t_0 (which implies local null controllability), then system (3.2) was globally finite time null controllable.

In this chapter we present a counter example to Corollary 2.7.1 [23], i.e., we present an example of a one dimensional system which is asymptotically stable, locally null controllable (in fact proper) but fails to be globally finite time null controllable.

3.2 THE COUNTER EXAMPLE

Consider the scalar system

$$\dot{x}(t) = -x(t) + e^{-pt}u(t) \quad (3.3)$$

$$x(0) = x_0 \quad ; \quad p > 1, \quad u(\bullet) \in M[-1, +1]$$

where $x(t) \in R$, $t \geq 0$ and $M[-1,+1]$ is the space of measurable functions with values in the closed interval $[-1,+1]$.

Clearly the system is asymptotically stable. Further, the transition function being $X(t) = e^{-t}$, and $B(t) = e^{-pt}$, $t \geq 0$, the system is proper since

$$B'(t) (X'(t))^{-1} n = n e^{-(p-1)t} = 0 \quad (3.4)$$

if and only if $n=0$ for any interval $[t, t+\delta]$, $0 \leq t < \infty$, $\delta > 0$. Consequently the system (3.3) is locally null controllable (in finite time) [1, Corollary 17.4]. Combining these facts (asymptotic stability and local null controllability) one would be tempted to conclude [1, p. 78] that system (3.3) is globally finite time null controllable. Let us check now whether, in fact, system (3.3) is globally null controllable.

By definition the reachable set for the system (3.3) is given by

$$R(t) = \left\{ \xi \in R : \xi = \int_{t_0}^t e^{-(p-1)s} u(s) ds; u \in M[-1,+1] \right\} \quad (3.5)$$

for each $t \geq 0$. Clearly $R(t)$ is symmetric, convex and compact for all $t \geq 0$ (Theorem 2.1). A point $\xi \in R(t)$ is a boundary point of $R(t)$ if and only if

$$u(t) = \text{sign}(B'(t) (X'(t))^{-1} n) = \text{sign}(n e^{-(p-1)t}), \quad (3.6)$$

for $n \neq 0$ [1, Lemma 13.1].

Since $X^{-1}(t)B(t) = e^{-(p-1)t} > 0$ for all $t \in [0, \infty)$, the boundary points are attained by controls $u(t)=1$ or $u(t)=-1$. Hence it follows from (3.5) that, for any $p > 1$,

$$R(t) = \left\{ \xi \in \mathbb{R} : \frac{\xi}{(p-1)}(1-e^{-(p-1)t}) \leq \xi \leq \frac{1}{(p-1)}(1-e^{-(p-1)t}) \right\}. \quad (3.7)$$

Consequently, an initial state x_0 can be steered to the origin if and only if $x_0 \in R(t)$ for some $t \in [0, \infty)$ or equivalently

$$x_0 \in \left\{ \xi : \frac{-1}{(p-1)} < \xi < \frac{1}{(p-1)} \right\}. \quad (3.8)$$

In other words, initial states $\{x_0\}$, for which $|x_0| \geq \left(\frac{1}{p-1}\right)$, cannot be steered to the zero state in finite time and hence the system cannot be globally finite time null controllable even though the system is asymptotically stable, locally null controllable and proper. In fact, for the given example, one can verify the above conclusion directly. Take an initial state $x_0 = \xi^*$ with $|\xi^*| \geq \left(\frac{1}{p-1}\right)$, $p > 1$, and let the uncontrolled system run til time $t_0 > 0$ so that $|e^{-t_0}\xi^*| < \left(\frac{1}{p-1}\right)$ and then apply control from t_0 onwards. The result is, for $t > t_0$,

$$\begin{aligned} x(t) &= e^{-(t-t_0)}\{e^{-t_0}\xi^*\} + \int_{t_0}^t e^{-(t-s)} e^{-ps} u(s) ds \\ &= e^{-t} \left\{ \xi^* + \int_{t_0}^t e^{-(p-1)s} u(s) ds \right\}. \end{aligned} \quad (3.9)$$

Since $\left| \int_{t_0}^t e^{-(p-1)s} u(s) ds \right| < \left(\frac{1}{p-1}\right)$ independently of $t \in [t_0, \infty)$ and $u \in M$, the conclusion follows.

By use of the necessary and sufficient conditions for global finite time null controllability given by Barmish and Schmitendorf [3] (integral (2.16)), the same conclusion can be reached from the following inequality

$$\int_0^{\infty} \sup_{u \in C^m} \{ B'(s)(X'(s))^{-1} \lambda, u \} ds = \int_0^{\infty} |\lambda| e^{-(p-1)s} ds = \frac{|\lambda|}{(p-1)} < \infty \quad (3.10)$$

for $p > 1$. The integral on the left of (3.10) should diverge for all $\lambda \in R$, $\lambda \neq 0$ for global finite time null controllability.

If in (3.3), instead of $B(t) = e^{-pt}$; $p > 1$ we use

$$B(t) = \begin{cases} p & 0 \leq t \leq t^* ; p \neq 0 \\ 0 & t > t^* ; 0 < t^* < \infty \end{cases} \quad (3.11)$$

we will arrive at the same conclusions as for system (3.3); i.e., the following system:

$$\begin{aligned} \dot{x}(t) &= -x(t) + B(t)u(t) ; t \geq 0 & (3.12) \\ x(0) &= x_0 ; B(t) \text{ as in (3.11)} ; u(\cdot) \in M[-1, +1] \end{aligned}$$

is asymptotically stable, null controllable, in fact proper at T for any $T < t^*$, but it is not globally finite time null controllable since the controllability integral (2.16)

$$\int_0^{\infty} \sup \{ w' B'(s) X'(0, s) \lambda : w \in R \} ds = \int_0^{t^*} |\lambda| e^s ds \quad (3.13)$$

fails to diverge for any $\lambda \in R$.

Systems (3.3) and (3.12) bear one property in common, in both cases the control influence over the system is made ineffective by the killing action of $B(t)$, either because it is rapidly diminished (system 3.3) or nullified completely after some time t^* (system 3.12). It should be said, nonetheless, that if $p \leq 1$ for system (3.3) or the control admissible set $\Omega \in \mathbb{R}^m$ then we would have obtained the global finite time null controllability property. Consequently this killing action is not entirely reflected in the properties of asymptotic stability and properness (let alone asymptotic stability and local null controllability). Unlike the properties mentioned above, the integral condition (2.16) bears this property.

3.3 COMMENTS AND CONCLUSIONS

The conditions that a linear time varying system is asymptotically stable and locally null controllable (or proper) do not guarantee the global finite time null controllability. This conclusion has been supported by an example, and it is believed that similar conclusions may hold for nonlinear time varying systems.

Although the system may be asymptotically stable and proper, the effectiveness of the control may be stifled by the action of $B(t)$, therefore, preventing the system from

being globally finite time null controllable. Note for instance, that in system (3.3) the control parameter $B(t)$, through which control is exerted in the system decays at a rate ($p > 1$) faster than the asymptotic decay rate of the uncontrolled system itself, and in system (3.12) the control parameter $B(t)$, and therefore the control action itself vanishes after $t^* > t$. So that, the properties of asymptotic stability and properness alone (or more generally asymptotic stability and local null controllability) cannot guarantee global finite time null controllability because they fail to recognize this killing action. On the other hand, the controllability integral (2.16) does reflect this action. It should be said that when unlimited control is available only properness at some time $t \in [t_0, \infty)$ is necessary in order to have global finite time null controllability.

Finally, for time invariant systems this problem does not arise and the general belief :

Asymptotic Stability + Local Null Controllability \implies

\implies Global Null Controllability.

remains true, since the domain of null controllability remains invariant with respect to the initial time t_0 .

Chapter IV

ON THE CONTROLLABILITY OF NONLINEAR SYSTEMS

4.1 INTRODUCTION

In Chapter II the basic results on the controllability for linear systems of the form (2.1) were presented. Those results were developed by studying the geometry of the sets of reachability and attainability. In this chapter a similar attempt will be made to introduce the basic results on controllability for nonlinear systems, i.e., systems of the form

$$\dot{x} = f(t, x, u) \quad (4.1)$$

where $x \in \mathbb{R}^n$, the state space; $u \in U$, the admissible control set; and f is a mapping from $[t_0, \infty) \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$. A precise description of the system will be given later in the chapter.

Unfortunately, studying the geometry of the attainable set for nonlinear systems is more difficult. This is due to the inherent difficulties associated with some phenomena that arise only in nonlinear systems. Among them, we may cite: 1) nonuniqueness of the equilibrium points ($\dot{x}=0$); 2)

singularities developed by the solutions of some nonlinear systems (e.g. the attainable set may not be pathwise connected); and 3) the critical dependence of solutions on the parameters, i.e., variations in the parameters may determine the existence of none, one or more than one smooth solutions. Also one can usually obtain closed-form expressions for solutions of linear systems whereas this is not always possible in the case of nonlinear systems.

In succeeding sections, we present results on the controllability of nonlinear systems (usually local in some sense). Most of those results concentrate upon rather special classes of nonlinear problems.

A comprehensive controllability theory for nonlinear systems giving similar information as available in the linear case is yet to be developed.

4.2 CONTROLLABILITY OF NONLINEAR SYSTEMS VIA RELATED LINEAR SYSTEMS

In this section we will determine some controllability properties of nonlinear systems by studying similar properties of related linear systems.

4.2.1 Time Invariant Systems

Consider the nonlinear autonomous system in R^n

$$\dot{x} = f(x, u) \text{ in } C^1(R^n \times \Omega) \quad (4.2)$$

where Ω is the restraint set in R^m .

The domain $C(0)$ of null controllability is defined as the set of all initial states $x_0 \in R^n$, such that if u^* is an admissible control and $x(t)$ is the solution of (4.2) with $u = u^*$ and $x(t_0) = x_0$, then at some positive time $t \geq t_0$ we have $x(t) = 0$. If $C(0)$ contains an open neighborhood of zero, then system (4.2) is said to be locally null controllable (near the origin).

The system (4.2) is globally asymptotically stable if for each $\epsilon > 0$ there exists $\delta = \delta(t_0, \epsilon)$ such that $|\dot{x}_0| < \delta$ implies that every solution $x(t)$ of (4.2), with some admissible control $u^*(\cdot) \in M(\Omega)$ initiating at $x(t_0) = x_0$, satisfies $|x(t)| \leq \epsilon$ on $t_0 \leq t < \infty$, and every solution of (4.2) with this admissible u^* can be extended over $t_0 \leq t < \infty$ and tends toward the origin as $t \rightarrow \infty$.

The system is globally finite time null controllable if every initial state $x_0 \in R^n$ may be steered to the origin in finite time. It has been believed that global finite time null controllability comes as a result of global asymptotic stability and local null controllability, even though this is not always the case (c.f. Chapter III).

Given the nonlinear system model (4.2) the first temptation in analyzing questions of local and global null controllability is to linearize the process (4.2) around the origin ($x=0, u=0$). Such a procedure yields the linearized dynamics:

$$\begin{aligned}\dot{x}(t) &= Fx(t) + Gu(t) \\ x(t_0) &= x_0\end{aligned}\tag{4.3}$$

where $F = (df/dx)(0,0)$ and $G = (df/du)(0,0)$.

Since controllability criteria for systems of the form (4.3) are readily available (see for instance theorem 2.1) we would clearly like to be able to conclude something about the controllability properties of (4.2) in a neighborhood of the origin by studying the corresponding properties of the linearized system (4.3). The following theorem is a typical result in this direction.

Theorem 4.1 [2, Thm. 1, pp. 366] Let, in system (4.2) $0 \in \text{int}(\Omega)$ and $f(0,0)=0$. Then system (4.2) is locally null controllable if the linearized system (4.3) is controllable, i.e.,

$$\text{rank } [G, FG, F^2G, \dots, F^{n-1}G] = n.\tag{4.4}$$

The problem with the above type of linearized result is that it provides only sufficient conditions, and that one may have the nonlinear system actually locally controllable and yet the rank condition (4.4) fails to be satisfied.

Global finite time null controllability criteria may also be derived for systems of the form (4.2) by combining global asymptotic stability criteria with that of local controllability (e.g. Theorem 4.1). By doing so we obtain

Theorem 4.2 [2, Corollary to thm. 7, pp. 397].

Assume

- (i) System (4.2) satisfies $f(0,0)=0$ and $0 \in \text{int}\{\Omega\}$;
 - (ii) System (4.3) satisfies the rank condition (4.4);
- and
- (iii) There exist a scalar function $V(x)$ and an m -vector $U(x)$ in $C^1(\mathbb{R}^n)$, such that

- (a) $V(x) \geq 0$, and $V(x)=0$ if and only if $x=0$;
- (b) $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ as $|x| \rightarrow \infty$;
- (c) $U(x) \subseteq \Omega$ for all $x \in \mathbb{R}^n$;
- (d) $(\partial V / \partial x^i) f^i(x, U(x)) < 0$ for $x \neq 0$, $i=1, \dots, n$.

Then the system is globally finite time null controllable.

4.2.2 Time Varying Systems

In this section we will study the questions of null controllability for systems of the form:

$$\dot{x} = f(t, x, u) \quad (4.5)$$

where $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is such that $f(t,x,u)$ and $(\partial f/\partial x)(t,x,u)$ are continuous. The controls $u(\cdot)$ are assumed to belong to the set $M(\Omega)$ of bounded measurable functions with values in $\Omega \subseteq \mathbb{R}^m$. No attempt will be made to redefine the controllability concepts previously defined to account for time varying systems since no major change is involved.

The following theorem by Yorke [16] not only expands the result of theorem 4.1 to account for time varying systems but relaxes some of the requirements. In the theorem he relates some controllability properties of systems like (4.5) to those of systems of the form

$$\dot{x}(t) = L(t)x(t) + v(t) ; v(t) \text{ in } Q(t) \subseteq \mathbb{R}^m \quad (4.6)$$

where v is a measurable function, $L(t)$ is $(\partial f/\partial x)(t,0,0)$ an $n \times n$ matrix which is continuous in t ; $Q(t)$ is the unbounded closed convex cone of the set $f(t,0,\Omega) \equiv \{f(t,0,u) : u \in \Omega\}$. The precise statement of the theorem is given by:

Theorem 4.3.

[16, Thm. 1, pp. 334] Suppose $0 \in \Omega$ and $f(t,0,0) = 0$. If system (4.6) is locally null controllable (near the origin) then system (4.5) is locally null controllable (near the origin).

In a sense the result of theorem 4.3 is quite different from that of theorem 4.1, since theorem 4.1 involves a representation of f which is useful only for considering small values of x and u (because f was linearized around

$x=0, u=0$). It is certainly reasonable to consider x small since we are discussing local null controllability criteria, but in many situations, particularly if bang-bang controls are desirable, it is helpful to be able to use control values u for which $f(x,u)$ is either 0 or large and not rely on having u almost 0. Let us remark that theorem 4.3 fails to provide a necessary condition.

Similarly to the autonomous case (thm 4.2) one could be tempted to combine a result on global asymptotic stability with the local null controllability result of theorem 4.3 to obtain a criterion on global finite time null controllability. Chukwu in [15] did so and established

Theorem 4.4 [15, Thm. 3, pp. 809] Suppose 1) the hypothesis of theorem 4.2 are satisfied, and the system (4.6) is locally null controllable. Further suppose 2) that for some admissible control $u^*(\cdot) \in M(\Omega)$ there exists a symmetric positive definite $n \times n$ constant matrix A such that the eigenvalues $\lambda_k(x, u^*, T), k=1, \dots, n$ of the matrix

$$\{AJ + J'A\}/2 ; (J = af/ax) \quad (4.7)$$

satisfy

$$\lambda_k \leq -\delta < 0 ; k=1, \dots, n \quad (4.8)$$

for all (x,t) in R^{n+1} where $\delta > 0$ is a constant; and finally suppose 3) that there are constant $r > 0$ and $p, 1 \leq p \leq 2$ such that

$$\int_t^{t+r} \|f(s; 0, u^*)\|^p ds \rightarrow 0 \text{ as } t \rightarrow \infty \quad (4.9)$$

Then the system (4.5) is globally finite time null controllable.

Unfortunately, the counter example of chapter II shows that such a result is incorrect. In fact, by taking the matrix A of (4.7) to be the identity and $u^*=0$, system (2.3) satisfies all of the requirements of theorem 4.4 but, nevertheless, fails to be globally finite time null controllable as shown in chapter II.

In the next section we will present a rather different approach to solving the questions of controllability of nonlinear systems.

4.3 CONTROLLABILITY OF NONLINEAR SYSTEMS VIA LIE ALGEBRAIC METHODS

In this section we will present some results on the controllability theory of nonlinear systems obtained through geometric methods (specifically Lie algebraic). The motivation for utilizing them is that they provide a good frame for the study of nonlinear systems, since the notation and methodology of differential geometry, involving manifolds, vector fields, trajectory curves, etc., are highly suited to the requirements of the problems of control systems. Specially, Lie theory being merely and interplay between the geometric interpretation of a system

$$\left(\frac{dx_i}{dt}\right) = A_i(x(t)) \quad (4.10)$$

of ordinary differential equations as the integral curves of a vector field $X = A_i(x) \{ \partial / \partial x_i \}$ and the interpretation of X as a generator of a one parameter group of transformations, say, solutions of the differential equation (4.10), enable us to use all of the properties of vector fields and integral curves to obtain an effective characterization of the controllability space and therefore of the controllability properties of nonlinear systems.

Here, we will be dealing with systems of the form

$$\dot{x} = f(x, u) \quad (4.11)$$

where it will be assumed that the state x evolves on a C^∞ (or analytic when specified) manifold \mathcal{M} of dimension n , i.e., a Hausdorff topological space with countable cover (U^α, x_α) of coordinate charts such that :

- (i) U^α is an open subset of \mathcal{M}
- (ii) $x_\alpha = \text{col}(x_\alpha^1, \dots, x_\alpha^n) : U^\alpha \rightarrow \mathbb{R}^n$ is a homeomorphism onto its range.
- (iii) If (U^α, x_α) and (U^β, x_β) are two such coordinate charts, then the change of coordinates $x_\beta \circ x_\alpha^{-1} : x_\alpha(U^\alpha \cap U^\beta) \rightarrow x_\beta(U^\alpha \cap U^\beta)$ is C^∞ (analytic).

For a fixed u in (4.11) $f(\cdot, u)$ is considered to be the local coordinate representation of a C^∞ (analytic) vector field on \mathcal{M} , i.e., a mapping which assigns to each point x in \mathcal{M} a

tangent vector in $T_x \mathcal{V}$, the tangent space to \mathcal{V} at x . System (4.11) is called smooth (analytic) if the above conditions are satisfied. It is assumed that u belongs to $M(\Omega)$, the set of bounded measurable functions with values in Ω a subset of \mathbb{R}^m such that $\text{int}\{\Omega\} \neq \emptyset$. We also assume that system (4.11) is complete, that is for every bounded measurable control u and every $x_0 \in \mathcal{V}$ there exists a solution $x(t) = x(t; x_0, u)$ satisfying $x(t) \in \mathcal{V}$ for all t in $[0, \infty)$. To simplify notation we assume that the manifold \mathcal{V} admits globally defined coordinates $x = (x^1, \dots, x^n)$, allowing us to identify points in \mathcal{V} with their coordinates representation and to describe control systems in the familiar fashion.

Given a point x^* in \mathcal{V} we say that x^* is reachable from x_0 at T if there exists an admissible control $u(\cdot) \in M(\Omega)$ such that the solution $x(t)$ of (4.11) satisfies $x(t_0) = x_0$, $x(T) = x^*$. The set of states reachable from x_0 at time T is denoted by:

$$R(T, x_0) = \{x : x \text{ is reachable from } x_0 \text{ at time } T\} \quad (4.12)$$

and the reachable set from x_0

$$R(x_0) = \bigcup_{t \geq 0} R(t, x_0) \quad (4.13)$$

Given two states x^* and x we say that x^* is weakly reachable from x if and only if there exist states x^1, x^2, \dots, x^k , $k \geq 1$ such that $x^1 = x^*$, $x^k = x$ and either x^i is reachable from x^{i-1} or x^{i-1} is reachable from x^i , $i = 1, \dots, k$.

It should be said that weak reachability is a symmetric relation, i.e., x^* is weakly reachable from x if and only if x is weakly reachable from x^* . The system (4.11) is said to be weakly reachable if every $x \in \mathbb{R}$ is weakly reachable from any $y \in \mathbb{R}$. Since weak reachability is a global concept we can define a local version of it (local weak reachability) as: given $x_0 \in \mathbb{R}$, we denote by $WR(x_0)$ the set of states weakly reachable from x_0 , then the system (4.11) is locally weakly reachable at x_0 if for every neighborhood U of x_0 , $U \cap WR(x_0)$ is also a neighborhood of x_0 . Intuitively, local weak controllability of x_0 implies that the set of points reachable from x_0 going either forward or backward in time is a neighborhood of x_0 .

The local weak controllability concept lends itself to a simple algebraic test; however, to introduce it we require a few additional notions which we present in the next section.

Let $x^*(\bullet) = x(\bullet; x_0, u^*)$ denote the solution of (4.11) corresponding to the control u^* and with initial condition $x_0 \in \mathbb{R}$, the system is said to be locally controllable along $x^*(\bullet)$ at time $t^1 \geq 0$ if all points in some neighborhood of $x^*(t^1)$ may be attained by a solution of (4.11) starting at x_0 by the use of an admissible control. The system is simply called locally controllable along $x^*(\bullet)$ if it is locally controllable at t for every $t \in [0, \infty)$, i.e., $x^*(t)$ is an interior point of $R(t, x_0)$ for every t in $(0, \infty)$.

4.3.1 Notation and Mathematical Preliminaries

Let $X(x), Y(x)$ be two C^∞ vector fields on \mathcal{V} . Then the Jacobi bracket of X and Y , denoted $[X, Y]$, is given by [21]:

$$[X, Y](\bullet) = \left(\frac{\partial Y}{\partial X} Y - \frac{\partial Y}{\partial X} X \right)(\bullet) . \quad (4.14)$$

Usually the Jacobi bracket is defined as the negative of the right hand side of (4.14). Defining it this way does not alter the results and avoids taking into consideration some negative signs that would appear.

It is easy to prove that the Jacobi bracket satisfies:

$$[c_1 X_1 + c_2 X_2, c_3 X_3 + c_4 X_4] = c_1 c_3 [X_1, X_3] \\ + c_2 c_3 [X_2, X_3] + c_1 c_4 [X_1, X_4] + c_2 c_4 [X_2, X_4] \quad (4.15a)$$

$$[X, Y] = -[Y, X] \quad (4.15b)$$

$$[X, [Y, W]] = [[X, Y], W] + [Y, [X, W]] \quad (4.15c)$$

where X, X_1, \dots, X_4, Y, W denote vector fields and c_1, \dots, c_4 denote real constant.

The set of all C^∞ vector fields on \mathcal{V} is an infinite-dimensional vector space denoted by $V(\mathcal{V})$ and becomes a Lie algebra under the multiplication defined by the Jacobi bracket. By Lie Algebra we mean a vector space with a multiplication $(X, Y) \rightarrow [X, Y]$ defined for any two elements X and Y satisfying (4.15a) to (4.15c).

Each constant control $u \in \Omega$ defines a vector field $f(\cdot, u)$ in $V(\mathbb{M})$. Let F_0 denote the subset of all such vector fields, that is, F_0 is the set of all vector fields generated from $f(\cdot, \cdot)$ through the use of constant controls. F will denote the smallest subalgebra of $V(\mathbb{M})$ containing F_0 , i.e., the elements of F are linear combinations of elements of the form :

$$[f^1, [f^2, \dots [f^i, f^{i-1}] \dots]] \quad (4.16)$$

where $f^i(\cdot) = f(\cdot, u^i)$ for some constant $u^i \in \Omega$. We let $F(x)$ be the space of tangent vectors spanned by the vector fields of F at x .

Given a vector field $X \in V(\mathbb{M})$, $(\exp. tX)(x_0)$ denotes the solution to :

$$\begin{aligned} \dot{x} &= X(x) \\ x(0) &= x_0 \end{aligned} \quad (4.17)$$

at time t . Finally, for two vector fields X and Y in $V(\mathbb{M})$ we denote the Lie product $[X, Y]$ by $(\text{ad}X, Y)$ and inductively $(\text{ad}^{k+1}X, Y) = [X, (\text{ad}^k X, Y)]$ and $(\text{ad}^0 X, Y) = Y$.

With the above background we are now ready to introduce some of the basic results on the controllability for nonlinear systems obtained by means of the theory of differential geometry.

4.3.2 Basic Results

In this section we present some of the basic results on the controllability of nonlinear systems obtained using the geometric approach. Although the assumptions on the system are rather strong (C^∞ or analytic), none of the results, except for analytic systems, provide but sufficient conditions on a certain aspect of the controllability theory.

Let us start our discussion by stating the relationship between the Lie Algebra F (4.16) and the reachable set.

Theorem 4.5 [10, Thm. 2.2, pp. 730] If the space of tangent vectors $F(x)$ spanned by the vector fields F at x is of dimension n , then system (4.11) is locally weakly controllable at x .

The condition of system (4.11) being locally weakly controllable guarantees that the interior of the reachable set from x , for every $x \in \mathcal{X}$, is not empty, i.e.,

$$\text{int}\{R(x)\} \neq \emptyset \quad \forall x \in \mathcal{X} \quad (4.18)$$

For C^∞ systems the converse of theorem 4.5 is not quite true, but we do have :

Theorem 4.6 [10, Thm. 2.5, pp. 731] If the system (4.11) is locally weakly controllable then the rank condition, $\dim F(x) = n$, is satisfied for all $x \in \mathcal{X}_0$, an open dense subset of \mathcal{X} .

Nonetheless, if we strengthen our assumption and consider the system to be analytic we may state :

Corollary 4.6.1 .

[10, Thm. 2.6, pp. 731] The system is locally weakly controllable if and only if the rank condition, $\dim F(x) = n$, is satisfied for all $x \in \mathcal{X}$.

The criteria for the system (4.11) to be locally controllable along some reference trajectory $x^*(t)$ can be derived when the system considered is C^∞ . The following result provides such criterion.

Theorem 4.7 [11, Proposition 4, pp. 256] Consider the system (4.11) and let u^* be an admissible control which generates the reference trajectory $x^*(\cdot) = x(\cdot; x_0, u^*)$. Suppose there exists a time $t_1 \geq 0$ and an integer k such that $u^*(t_1) \in \text{int}(\mathcal{U})$, u^* is constant in a neighborhood of t_1 and if $x = x^*(t_1)$

$$\text{rank}\{Y^1(x_1), \dots, Y^m(x_1), (\text{ad}^k X, Y^1)(x_1), \dots, (\text{ad}^k X, Y^m)(x_1)\} = n \quad (4.19)$$

where $X(x) = f(x, u^*(t_1))$ and $Y^i(x) = (\partial/\partial u_i) f(x, u^*(t_1))$. Then the system (4.11) is locally controllable along $x^*(\cdot)$ at time t_1 .

Remark If $t_2 > t_1$, the map $x(t_2; \cdot, u^*)$ considered as a map of initial data, carries a neighborhood of $x(t_1; x_0, u^*)$ homeomorphically onto a neighborhood of $x(t_2; x_0, u^*)$. Thus if

a system is locally controllable along $x^*(\cdot)$ at time t_1 , this is also true for any $t_2 \geq t_1$.

A simpler criterion for local controllability along some reference trajectory may be obtained if we specialize the system under consideration to:

$$\begin{aligned} \dot{x} &= X(x) + Y(x)u, & u \in [-1, 1] & \quad (4.20) \\ x(0) &= x_0 \end{aligned}$$

where $x \in \mathbb{R}^n$; $X(\cdot)$ and $Y(\cdot)$ are C^∞ vector fields on \mathbb{R}^n . The basic result is contained in the following theorem:

Theorem 4.8 [11, Proposition 2, pp. 255] Let $x(\cdot) = x(\cdot, x_0, 0)$ be the solution of (4.20) corresponding to the control $u=0$; a sufficient condition that the system (4.20) is locally controllable along $x(\cdot)$ at time $t_1 \geq 0$ is that there exists an integer k such that:

$$\dim \text{span}\{S(x(t_1)) = [\text{ad}^i X, Y)(x(t_1)); i=0, \dots, k]\} = n \quad (4.21)$$

The results of theorem 4.8 can be extended to systems of the form:

$$\dot{x} = X(x) + \sum_{j=1}^m u_j Y_j(x); \quad -1 \leq u_j \leq 1 \quad (4.22)$$

by simply substituting in the rank condition (4.21) the vector $(\text{ad}^i X, Y)$ by the matrix $[(\text{ad}^i X, Y_j)]$, $j=1, \dots, m$.

It follows immediately from theorem 4.8 that if the initial condition $x(0) = p$, is such that $X(p) = 0$, then \dim

span $S(p) = n$ is a sufficient condition that all points in a neighborhood of p be attainable by a solution of (4.20) initiating from p . Since reversing the time merely introduces a minus sign in certain elements of $S(p)$ but does not change the dimension of its span, it follows that dimension span $S(p) = n$ is a sufficient condition that all points in a neighborhood of p be steerable to p in finite time. This is therefore, a sufficient condition for local controllability around the rest solution p .

In practice, one is not only interested in knowing that it is possible to steer all points in some neighborhood of p to p , but what is preferred is a method to construct a feedback control to do the task. In the next section we will describe an algorithm which computes a 'modified' feedback control (modified in the sense of discrete measurement of the state), obtained from the rank condition (4.21).

Before starting the next section let us include another result which, under strong assumptions, relates the Lie algebra F to the reachable set.

Theorem 4.9 [11, Theorem, pp. 256] Let F_0 be symmetric, i.e., if $X \in F_0 \rightarrow -X \in F_0$; then every pair of points of \mathcal{R} can be joined by a solution of (4.11) in finite time if the rank condition $\{\dim F(x) = n\}$ is satisfied. Recall that F is the smallest subalgebra containing F_0 .

As it has been shown, results on controllability for nonlinear systems are not as complete as those of linear processes. Moreover, even though we consider a limited class of systems (C^∞ and analytic), controllability criteria turned out to be only sufficient in most of the cases and not necessary and sufficient which is what is desirable. Other approaches have been given to the problem, like for instance Vinter's [17] characterization of the reachable set by embedding solutions of (4.1) in a space of linear functionals; but with no greater success than the approaches presented in this chapter.

4.4 SYNTHESIS OF A STABILIZING FEEDBACK CONTROL VIA LIE ALGEBRAIC METHODS

Consider the control system on R^n

$$\dot{x} = X(x) + Y(x)u \quad (4.23)$$

where X, Y are C^∞ vector fields, $p \in R^n$ is a rest solution, i.e., $X(p) = 0$ and the control u assumes values in $[-1, 1]$. If

$$\dim \text{span } S(p) = \dim \text{span } \{(\text{ad}^j X, Y)(p); j=0, \dots, n-1\} = n$$

then system (4.23) is locally controllable around p .

In most of the cases one is interested not only in knowing that the system is locally controllable, but also in synthesizing a stabilizing control. Here we give a

constructive algorithm to generate a modified (in the sense of discrete measurement of the state variable) feedback control.

For a vector field W on R^n , we denote the solution at time t of

$$\dot{x} = W(x(t))$$

$$x(0) = p$$

by $(\exp tW)(p)$. The algorithm is based on the fact that by composing maps of the form $(\exp \epsilon(X \pm u(\epsilon)Y))(p)$, one can form maps, denoted $q_k^\pm(\epsilon)p$ such that $q_k^\pm(0) = p$ and $(d/d\epsilon)q_k^\pm(\epsilon)p|_{\epsilon=0} = \pm(\text{ad}^k X, Y)(p)$. Furthermore, the maps $q_k^\pm(\epsilon)p$ correspond to admissible trajectories of system (4.23) resulting from controls with at most $2^k - 1$ switches.

Now let $(\alpha_1, \dots, \alpha_n) \in R^n$ and consider the composition

$$q(\epsilon)p = q_{n-1}^{c_{n-1}}(|\alpha_n|\epsilon) \circ \dots \circ q_0^{c_0}(|\alpha_1|\epsilon)p \quad (4.24)$$

where

$$c_i = \begin{cases} + & \text{if } \alpha_{i+1} \geq 0 \\ - & \text{if } \alpha_{i+1} < 0 \end{cases}$$

then for $\epsilon \geq 0$, $q(\epsilon)p$ correspond to a trajectory of (4.23) resulting from a control with at most $\sum_1^{n-1} (2^{\nu}-1) = (2^n - n)$ switches, and $(d/d\epsilon)q(\epsilon)p|_{\epsilon=0} = \sum_1^n \alpha_\nu (\text{ad}^{\nu-1} X, Y)(p)$. In particular if $\dim \text{span } S(p) = n$ and we are at a point q^1 near p with $q^1 - p = \sum_1^n \alpha_\nu (\text{ad}^{\nu-1} X, Y)(p)$, one can prescribe a control over the interval $[0, \epsilon]$, which drives the solution in the

direction $-\sum_{p=0}^n (\text{ad}^{p-1} X, Y)(p)$. This is the essence of the algorithm. It will be used in the next chapter to synthesize a control for the satellite (attitude controlled) problem.

4.4.1 Preliminaries

In this section we show how to obtain the directions $(\text{ad}^j X, Y)(p)$; $j=0, \dots, n-1$ via composition of solutions of (4.23). The ensuing formulae are based on the Baker-Campbell-Hausdorff formula [19].

Lemma 4.10 [12, Lemma 2, pp. 355] let

$$q_0^\pm(\epsilon, k)p = (\exp \epsilon(X \pm \epsilon^k Y))p \quad (4.25a)$$

$$q_1^\pm(\epsilon, k)p = q_0^\mp(\epsilon, k-1) \circ q_0^\pm(\epsilon, k-1)p \quad (4.25b)$$

when $k=1$

$$q_1^\pm(\epsilon, 1)p = \exp(2\epsilon X \pm \epsilon^2[X, Y] + O(\epsilon^3))p$$

and inductively for $m=2, \dots$

$$q_m^\pm(\epsilon, k) = q_{m-1}^\mp(\epsilon, k) \circ q_{m-1}^\pm(\epsilon, k)p \quad (4.25c)$$

then

$$q_m^\pm(\epsilon, m)p = \exp(2^m \epsilon X + a_m \epsilon^{2m} (\text{ad}^m X, Y) + O(\epsilon^{2m}))p \quad (4.25d)$$

where $\ln(a_m) = (m(m-1)/2)\ln(2)$, and

$$\left. \begin{aligned} (d^j/d\epsilon^j) q_m^\pm(\epsilon, m)p \Big|_{\epsilon=0} = \begin{cases} 0 & \text{if } 1 \leq j \leq 2m-1 \\ +a_m(2m)! (\text{ad}^m X, Y)(p) & \text{if } j=2m \end{cases} \end{aligned} \right\}$$

Note that

$$(d/d\epsilon) q_0^\pm(\epsilon, 0) = \pm Y(p) \text{ and } (d^2/d\epsilon^2) q_1^\pm(\epsilon, 1) = \pm [X, Y](p).$$

Since we are interested in obtaining the directions $\pm(\text{ad}^j X, Y)(p)$ $j=0, \dots, (n-1)$ as first derivatives we reparametrize the forms (4.25a) to (4.25d) as follows:

Corollary 4.10.1 [12, Corollary 1, pp. 356].

Let

$$q_0^\pm(\xi)p = q_0^\pm(\xi, 0)$$

and for $m=1, 2, \dots$

$$q_m^\pm(\xi)p = q_m^\pm((\xi/a_m)^{1/m}, m)p \quad (4.26)$$

then

$$(d/d\xi)q_m^\pm(\xi)p|_{\xi=0} = \pm(\text{ad}^m X, Y)(p) \quad (4.27)$$

For computing purposes, it is useful to have an explicit list of the first several functions $q_m^\pm(\xi)p$. These are :

$$q_0^\pm(\xi)p = (\exp \xi(X \pm Y))p \quad (4.28a)$$

$$q_1^\pm(\xi)p = (\exp \xi^{1/2}(X \mp Y)) \bullet (\exp \xi^{1/2}(X \pm Y))p \quad (4.28b)$$

$$\begin{aligned} q_2^\pm(\xi)p = & \exp((\xi/2)^{1/2}(X \pm (\xi/2)^{1/2}Y)) \bullet \\ & \exp(2(\xi/2)^{1/2}(X \mp (\xi/2)^{1/2}Y)) \bullet \\ & \exp((\xi/2)^{1/2}(X \pm (\xi/2)^{1/2}Y))p \end{aligned} \quad (4.28c)$$

$$q_3^\pm(\xi)p = q_2^\mp((\xi/8)^{1/6}, 3) \bullet q_2^\pm((\xi/8)^{1/6}, 3)p \quad (4.28d)$$

The maps $q_m^\pm(\xi)p$ have been defined for $\xi \geq 0$. Since $(d/d\xi)q_m^\pm(\xi)p|_{\xi=0} = \pm(\text{ad}^m X, Y)(p)$, if we define

$$q_m(\delta)p = \begin{cases} q_m^+(\delta)p & \text{if } \delta \geq 0 \\ q_m^-(\delta)p & \text{if } \delta < 0 \end{cases} \quad (4.29)$$

$m = 0, \dots, (n-1)$, then $q_m(\cdot)p$ is defined, and is continuously differentiable, for δ in a neighborhood of 0. Moreover, since $q_m^+(\delta)p$, $\delta \geq 0$ is the end point of an admissible trajectory of (4.23), the composition

$$q_{n-1}(\alpha_n) \circ \dots \circ q_0(\alpha_1)p \quad (4.30)$$

is the end point of an admissible trajectory of (4.23) corresponding to a control with at most $\sum_1^{n-1} (2^i - 1) = 2^n - n$ switches.

The following theorem establishes the relation between such compositions and local controllability to the rest state p .

Theorem 4.11 [12, Thm. 1, pp. 356] Consider the system (4.23). Assume:

- (i) $x(0) = p$ and $X(p) = 0$
- (ii) $\dim \text{span } S(p) = n$

and let $q_m^+(\delta)p$ and $q_m(\delta)p$ be defined as in (4.26) and (4.29) above. Then every point in some neighborhood of p can be attained from p by a trajectory of (4.23) of the form (4.30). Similarly, p can be attained from any point q^1 in some neighborhood of p by a trajectory of (4.23) of the form :

$$q_{n-1}(\alpha_n) \circ \dots \circ q_0(\alpha_1)q^1 \quad (4.31)$$

4.4.2 The Feedback Control Algorithm

In this section we will assume for system (4.23) that the initial condition $x(0) = p$ is such that $X(p) = 0$ and that $\dim \text{span } S(p) = n$. Further, for any $\epsilon > 0$, let $U(p, \epsilon)$ denote a neighborhood p such that $x \in U(p, \epsilon)$ implies :

$$\|X(x)\| \leq \epsilon; \quad |(\text{ad}^j X, Y)(x) - (\text{ad}^j X, Y)(p)| \leq \epsilon \quad (4.32)$$

for $j = 0, \dots, (n-1)$. Then the algorithm is as follows :

Given $q^i \in U(p, \epsilon)$, express $q^i - p$ as a linear combination of $(\text{ad}^j X, Y)(p)$ $j = 0, \dots, (n-1)$, i.e.,

$$q^i - p = \sum_1^n \alpha_j(q^i) (\text{ad}^{j-1} X, Y)(p) \quad (4.33)$$

and define

$$q^{i+1} = q_{n-1}(-\alpha_n(q^i)\epsilon) \dots \circ q_0^i(-\alpha_1(q^i)\epsilon)q^i. \quad (4.34)$$

It can be proved [12] that for $\epsilon > 0$ sufficiently small, $q^i \rightarrow p$ as $i \rightarrow \infty$.

To gain insight into why the algorithm is expected to work, we note that :

$$\begin{aligned} q_{n-1}(-\alpha_n(q^i)\epsilon) \dots \circ q_0^i(-\alpha_1(q^i)\epsilon)q^i - q^i &\approx \\ &\approx -\epsilon \sum_1^n \alpha_j(q^i) (\text{ad}^{j-1} X, Y)(q^i) \quad (4.35) \end{aligned}$$

a direction which is 'nearly' the negative of $q^i - p$.

In the next chapter a satellite attitude control will be synthesized using the above algorithm.

4.5 SUMMARY

In this chapter, a review of the controllability theory of nonlinear system has been presented. Several approaches as to how to focus the controllability questions have been introduced, and so controllability criteria generated via related linear systems and via Lie Algebraic Methods. An algorithm to synthesize a feedback control for nonlinear systems has been exhibited.

Chapter V

SATELLITE ATTITUDE CONTROL VIA LIE ALGEBRAIC METHODS

In this chapter, a variety of satellite attitude control problems are investigated in an effort to: 1) evaluate and improve the stabilizing algorithm presented in chapter IV and 2) study certain controllability aspects. The system model to be used is that of a three-axis, zero-momentum attitude control for a satellite and is presented in the next section. In it the control is performed by momentum-exchange devices. The fundamental principle involved is that, for any vehicle the total external torque is equal to the rate of change of the total angular momentum of the vehicle. Thus, in the absence of any external torque, if the angular momentum of any part of the vehicle (the control element) is changed by some means, the angular momentum of the remainder of the vehicle will change by an equal and opposite amount.

5.1 SYSTEM MODEL

The satellite angular momentum dynamics, using both reaction jets and flywheels, can be described as follows [20]:

$$\begin{cases} I_x \dot{\beta} + (I_z - I_y)(\gamma - w_0)\zeta + C_x \dot{\Omega}_x - C_y \Omega_y \zeta + C_z \Omega_z \gamma = T_x \\ I_y \dot{\gamma} + (I_x - I_z)\beta \zeta + C_x \Omega_x \zeta + C_y \dot{\Omega}_y - C_z \Omega_z \beta = T_y \\ I_z \dot{\zeta} + (I_y - I_x)(\gamma - w_0)\beta - C_x \Omega_x \gamma + C_y \Omega_y \beta + C_z \dot{\Omega}_z = T_z \end{cases} \quad (5.1)$$

where β, γ, ζ represent the angular velocities of the satellite body (spacecraft), $\Omega_x, \Omega_y, \Omega_z$ are the angular velocities of the flywheels, and T_x, T_y, T_z are the (applied) torques due to reaction jets. The rest of the symbols are the parameters of the system (satellite and flywheels moments of inertia).

For the purpose of regulation we consider the torques (T_x, T_y, T_z) and the flywheel acceleration ($\dot{\Omega}_x, \dot{\Omega}_y, \dot{\Omega}_z$) as the control variables, and we define them by the vector

$$u = (u_1, u_2, u_3, u_4, u_5, u_6)' = (T_x, T_y, T_z, \dot{\Omega}_x, \dot{\Omega}_y, \dot{\Omega}_z)' \quad (5.2)$$

where $\Omega_x, \Omega_y, \Omega_z$ are the flywheel angular velocities in the x, y and z directions, respectively.

Using (5.1) and (5.2), the overall system model is given by

$$\left\{ \begin{array}{l}
 I_x \dot{\beta} + (I_z - I_y)(\gamma - w_0)\zeta + C_x u_4 - C_y \Omega_y \zeta + C_z \Omega_z \gamma = u_1 \\
 I_y \dot{\gamma} + (I_x - I_z)\beta \zeta + C_x \Omega_x \zeta + C_y u_5 - C_z \Omega_z \beta = u_2 \\
 I_z \dot{\zeta} + (I_y - I_x)(\gamma - w_0)\beta - C_x \Omega_x \gamma + C_y \Omega_y \beta + C_z u_6 = u_3 \\
 \dot{\Omega}_x = u_4 \\
 \dot{\Omega}_y = u_5 \\
 \dot{\Omega}_z = u_6
 \end{array} \right. \quad (5.3)$$

The objective is to design a discrete feedback control law that brings the satellite's body rates to a rest (desired) state ($\beta, \gamma, \zeta = 0$) following a sudden perturbation. Further, the control law shall be obtained by using the feedback control algorithm presented in 4.4.2. Once the control law is synthesized, a slight modification is introduced in the algorithm so that it performs the regulation more efficiently (less time). Comments on the controllability of the system, with jets and flywheels, as well as comparisons between the depicted algorithms are presented.

Regarding the controllability of (5.3) we next study two cases: the first when we consider only jets as controls, i.e., $u_4, u_5, u_6 = 0$; the second, when we consider only flywheels i.e., $u_1, u_2, u_3 = 0$. The control possibilities, as well as limitations, of both type of devices are analyzed. At this point we should stress that our main objective is to determine the efficiency of the algorithm presented in 4.4.2., since the controllability properties of system (5.1) are well known (see for instance [18]).

5.2 SATELLITE ATTITUDE CONTROL USING JETS ONLY

When there is only jets actuating the system, the model reduces to:

$$\begin{cases} I_x \dot{\theta} = -(I_z - I_x)(\gamma - \omega_0)\theta + u_1 \\ I_y \dot{\gamma} = -(I_x - I_z)\theta + u_2 \\ I_z \dot{\xi} = -(I_y - I_x)(\gamma - \omega_0)\xi + u_3 \end{cases} \quad (5.4)$$

or in matrix form

$$\dot{x} = X(x) + u_1 Y_1 + u_2 Y_2 + u_3 Y_3 \quad (5.5)$$

where

$$X(x) = \begin{pmatrix} -((I_z - I_x)(\gamma - \omega_0)\theta)/I_x \\ -(I_x - I_z)\theta/I_x \\ -((I_y - I_x)(\gamma - \omega_0)\xi)/I_z \end{pmatrix}$$

and

$$Y_1 = \begin{bmatrix} 1/I_x \\ 0 \\ 0 \end{bmatrix} ; \quad Y_2 = \begin{bmatrix} 0 \\ 1/I_x \\ 0 \end{bmatrix} ; \quad Y_3 = \begin{bmatrix} 0 \\ 0 \\ 1/I_z \end{bmatrix}$$

Now, since the vector field $X(\bullet)$ at the desired state $x(\theta, \gamma, \xi=0)$ satisfies:

$$X(x) = 0 \quad (5.6)$$

and

$$\dim \text{Span } S(x) = \text{rank } [Y_1, Y_2, Y_3] = 3 \quad (5.7)$$

we can conclude (thm 4.8) that system (5.4) is locally controllable around the origin (desired state).

5.2.1 The Feedback Control Algorithm

Given an initial perturbation $q^1 \in \mathbb{R}^3$ we now write

$$q^1 = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 \quad (5.8a)$$

and find

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \cdot \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \end{bmatrix} \quad (5.8b)$$

our algorithm to proceed from q^1 to q^2 (and in general from q^i to q^{i+1}) then becomes

$$q^2 = q_0^{(1)}(|\alpha_1| \delta) \cdot q_0^{(2)}(|\alpha_2| \delta) \cdot q_0^{(3)}(|\alpha_3| \delta) q^1 \quad (5.8c)$$

where again

$$e_i = \begin{cases} - & \text{if } \alpha_i \geq 0 \\ + & \text{if } \alpha_i < 0 \end{cases} \quad i=1,2,3$$

Once q' is obtained we compute the new a coefficients corresponding to q' from (5.78b) and repeat the iterative procedure until the desired convergence is achieved.

The system block diagram is shown in figure BG.1.

We next give the numerical results of a typical computer run using algorithm (5.8). The initial perturbation was set to be

$$q' = (0.5, 0.6, 0.5)$$

the control bounds were chosen as $|u_i| \leq 20$ $i=1,2,3$ (actually only bang-bang control policies were used), while the value of δ was 0.005.

For numerical values, the physical parameters of the whecon system [24] have been used in this research. Throughout the sequel these quantities remain unchanged. They are the moments of inertia:

$$I_x = 645 \text{ slug-ft}^2$$

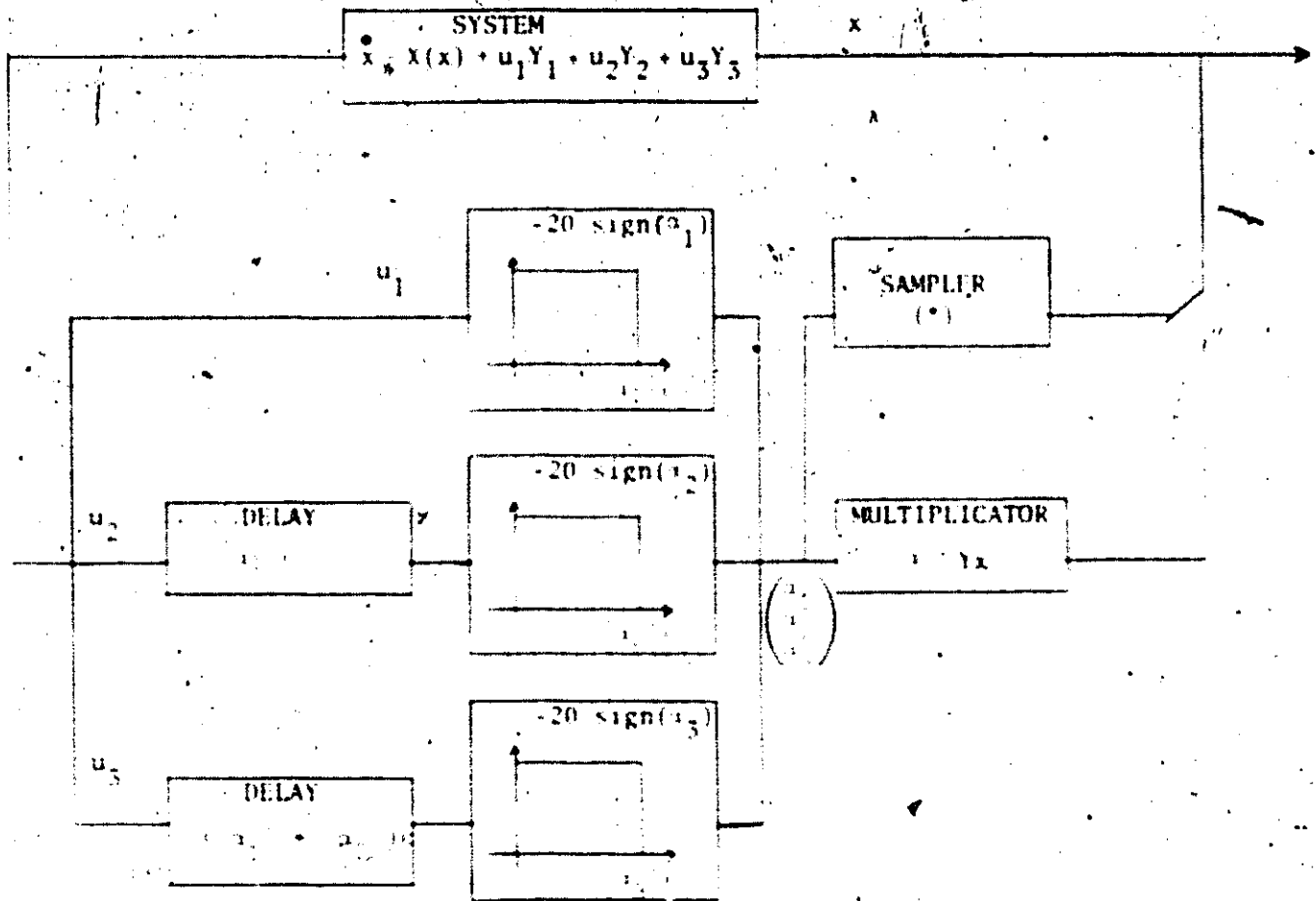
$$I_y = 100 \text{ slug-ft}^2$$

$$I_z = 669 \text{ slug-ft}^2$$

and the orbital rate

$$\omega_0 = 7.29 \times 10^{-3} \text{ rad/sec.}$$

The Fortran code used to solve this program is listed in Appendix A. Figures 5.1 and 5.3 illustrate the state trajectories and the control sequence for the case above indicated.



$\tau = 0.005$

$X(x), Y_1, Y_2, Y_3$ as in (5.5)

$Y = (Y_1, Y_2, Y_3)^{-1}$ as in (5.8b)

(*) Sampling Period $(\tau_{x_1} + \tau_{x_2} + \tau_{x_3})$

FIG. BG.1. Block diagram of the satellite attitude control system using jets only. The feedback control laws (u_1, u_2, u_3) are generated using the algorithm of section 5.2.1.

In fig. 5.3, one drawback of this algorithm is apparent. The controls acting upon the system are applied sequentially, i.e., firstly the control corresponding to the x direction (u_1), secondly the one corresponding to the y direction (u_2), thirdly the one corresponding to the z direction (u_3) and then back to u_1 . Such sequential use of the control is an implicit characteristic of the algorithm. In the next section a slight modification to this algorithm will be introduced in order to avoid this problem.

5.2.2 The Modified Algorithm

One drawback of algorithm 4.4.2 is the sequential use of the controls (u_1, u_2, u_3) which represents a regulation time for every iteration (the time the system takes to go from q^j to q^{j+1} determined by (5.8c)) of:

$$\text{Regulation time from the } j^{\text{th}} \text{ iteration} = \sum |a_i^j| t_i$$

Considerable time savings could be achieved if instead of using the controls sequentially, one would use them all at the same time. The proposed modified algorithm accomplishes such time reduction and at the same time preserves the regulation property of the original algorithm.

When we consider only jets as controls the modified algorithm is as follows:

Given the initial perturbation q^1 we express it as a linear combination of the vectors Y_1, Y_2, Y_3 from (5.5) and obtain the α coefficients as in (5.8b). Then the coefficients are ordered according to their absolute value. Let $P: (\alpha_1^1, \alpha_2^1, \alpha_3^1) \rightarrow (\alpha_{m1}^1, \alpha_{m2}^1, \alpha_{m3}^1)$ be such ordering map where α_{m1}^1 is the α coefficient with the maximum absolute value corresponding to the first iteration. Similarly α_{m2}^1 is the α coefficient with the second largest absolute value and α_{m3}^1 the α coefficient with the third largest (or minimum) absolute value. Further, let $PY: (Y_1, Y_2, Y_3) \rightarrow (Y_{m1}, Y_{m2}, Y_{m3})$ be the map which assigns to every α_{mi}^1 its corresponding Y_i vector ($i=1, 2, 3$).

The algorithm to proceed from q^1 to q^2 becomes:

$$\begin{aligned}
 q^2 = & \exp[(|\alpha_{m1}^1| - |\alpha_{m2}^1|) \& (X - Y_{m1} \text{sign}(\alpha_{m1}^1))] \cdot \\
 & \exp[(|\alpha_{m2}^1| - |\alpha_{m3}^1|) \& (X - Y_{m2} \text{sign}(\alpha_{m2}^1) - Y_{m1} \text{sign}(\alpha_{m1}^1))] \cdot \\
 & \exp[|\alpha_{m3}^1| \& (X - Y_{m3} \text{sign}(\alpha_{m3}^1) - Y_{m2} \text{sign}(\alpha_{m2}^1) - Y_{m1} \text{sign}(\alpha_{m1}^1))] q^1
 \end{aligned}
 \tag{5.9}$$

The Campbell-Baker-Hausdorff formula [19] enables us to rewrite (5.9) in the form:

$$\begin{aligned}
 q^2 = & \exp[|\alpha_{m1}^1| \& X - \alpha_{m1}^1 \& Y_{m1} - \alpha_{m2}^1 \& Y_{m2} - \alpha_{m3}^1 \& Y_{m3} + O(\&)] q^1 \\
 = & \exp[|\alpha_{m1}^1| \& X - \& \sum_1^3 \alpha_i^1 Y_i + O(\&)] q^1
 \end{aligned}
 \tag{5.10}$$

hence, like before, if q^j is sufficiently near zero, so that $|X(q^j)|$ is small compared to ϵ , the trajectory moves essentially in the direction

$$-\sum_1^m a_1^j Y_1$$

which is the direction from q^j to the desired state.

Once q^j is obtained, we compute the new a coefficients corresponding to q^j and then repeat the iterative process until the desired convergence is achieved.

The system block diagram is shown in figure BG.2.

The verification that an iterative application of this algorithm yields a sequence of points q^0, q^1, \dots which converges to zero if the original disturbance q^0 is sufficiently small is exactly the same as that of algorithm 4.4.2 and may be found in [12].

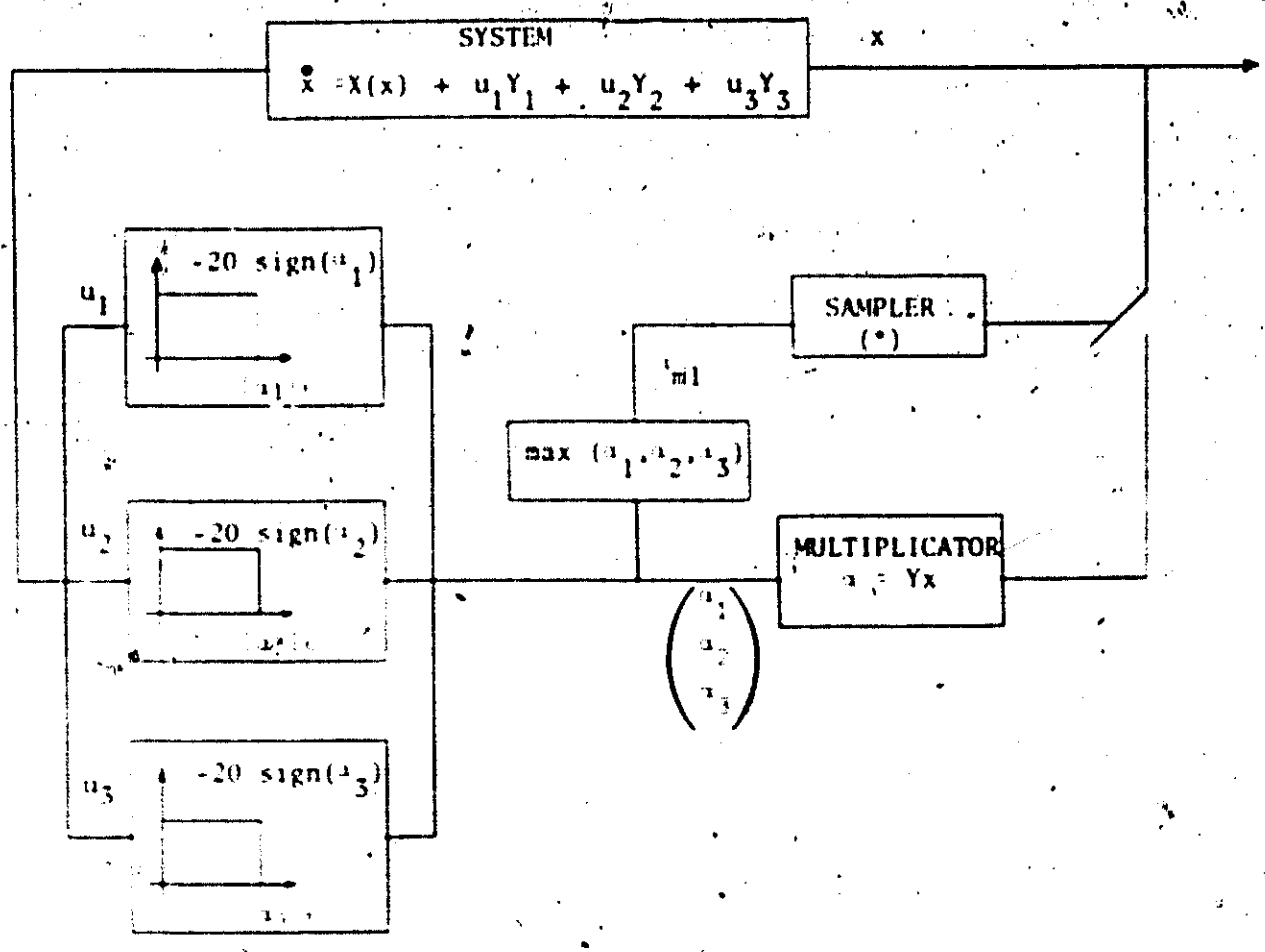
By using the modified algorithm the regulation time for every iteration (the time the system takes to go from q^j to q^{j+1} determined by (5.9)) reduces to:

$$\text{Regulation time from the } j^{\text{th}} \text{ iteration} = |a_{n1}^j| \epsilon < \sum_1^3 |a_i^j| \epsilon;$$

but the computational burden increases since for every iteration an ordering of the a coefficients is required.

We next give the numerical results of a computer run using algorithm (5.9) when the original data was chosen to be the same as in the previous case (5.2.1), i.e.,

$$q^0 = (0.5, 0.6, 0.5)$$



0.005
 $X(x), Y_1, Y_2, Y_3$ as in (5.5)
 $Y = (Y_1, Y_2, Y_3)^{-1}$ as in (5.8b)
 (*) Sampling Period T_m

FIG. 8G.2. Block diagram of the satellite attitude control system using jets only. The feedback control laws (u_1, u_2, u_3) are generated using the algorithm of section 5.2.2.

and $\epsilon = 0.005$, $|u_i| \leq 20$, $i = 1, 2, 3$.

The Fortran code used to solve this program is listed in Appendix A. Figures 5.2 and 5.4 illustrate the state trajectories and the control sequence for the above case.

From the graphical results, it is apparent that by applying all the controls at the same time, the regulation time reduces considerably.

Remark. In the algorithms of sections 5.2.1 and 5.2.2 while the magnitude of the controls is decided a priori by the system analyst, the α coefficients (computed from (5.18b)) determine the sign and period of time the controls are to be used on the system, i.e., α_1 determines the sign and acting time of the thruster u_1 ; similarly α_2 of u_2 and α_3 of u_3 . Eventually, once the controls have been applied, the system should attain a state which is closer (than the original (initial) perturbation) to the final desired state (closer in the sense of euclidean distance and the original perturbation being the state the system is, before the use of the controls). Repetitive use of such a procedure, computing the α coefficients and applying the respective control law, should drive the system to the desired state.

The conceptual background for doing so, is exactly the same in both, algorithm 5.2.1 and 5.2.2, but algorithm 5.2.1 calls for a sequential use of the controls (see figures BG.1 and 5.3) whereas algorithm 5.2.2 does not. In algorithm 5.2.2 the control policies on every axis (u_1, u_2, u_3)

determined by the a coefficients are input on the system simultaneously (see figures BG.2 and 5.4).

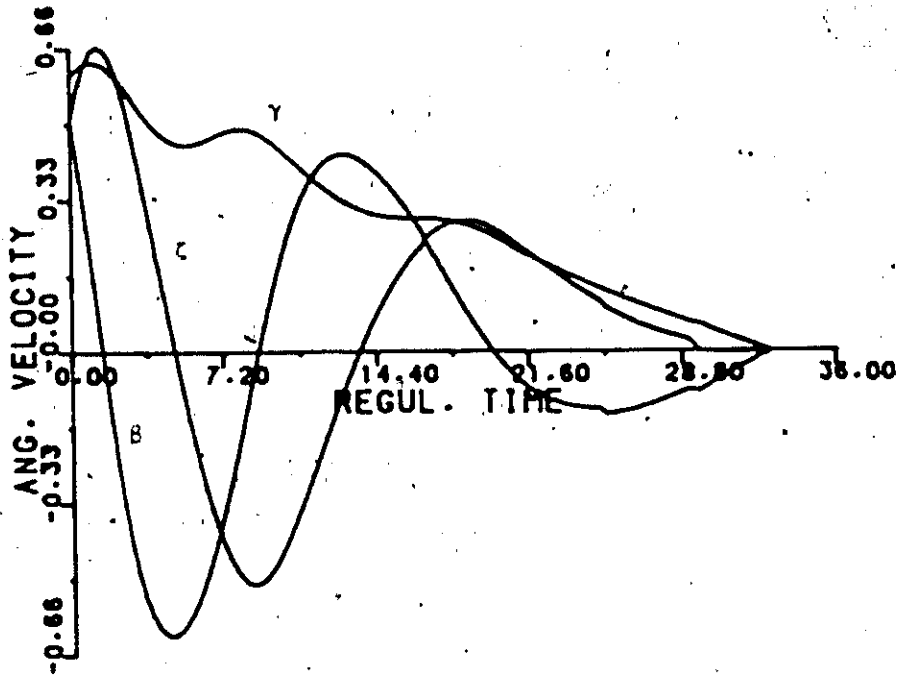


FIG. 5.1. Profiles of the satellite angular velocities (B, Y, Z) when only jets are used. The feedback control laws are synthesized via the algorithm of section 5.2.1.

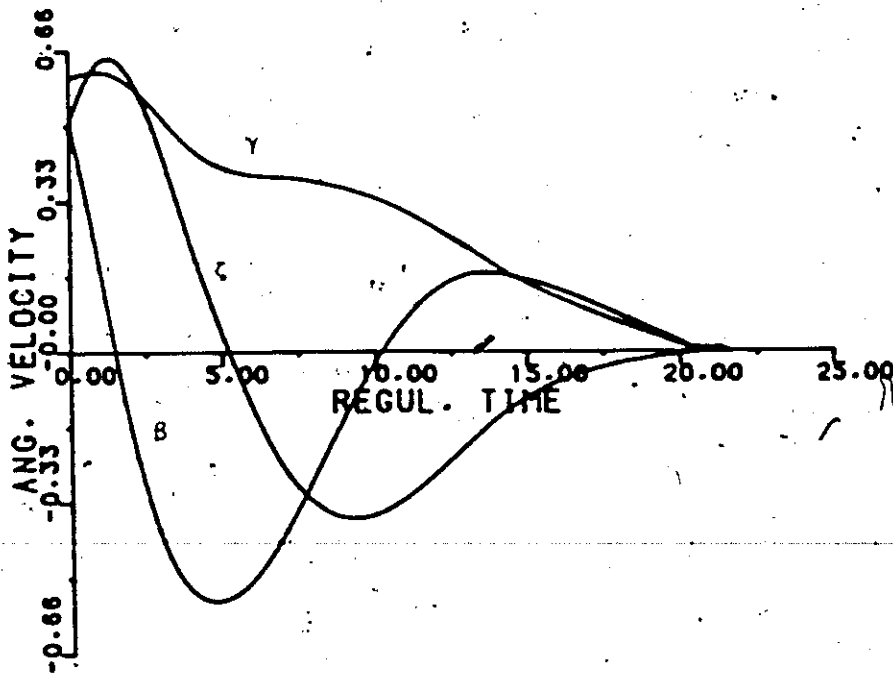


FIG. 5.2. Profiles of the satellite angular velocities (B, Y, Z) when only jets are used. The feedback control laws are synthesized via the algorithm of section 5.2.2.

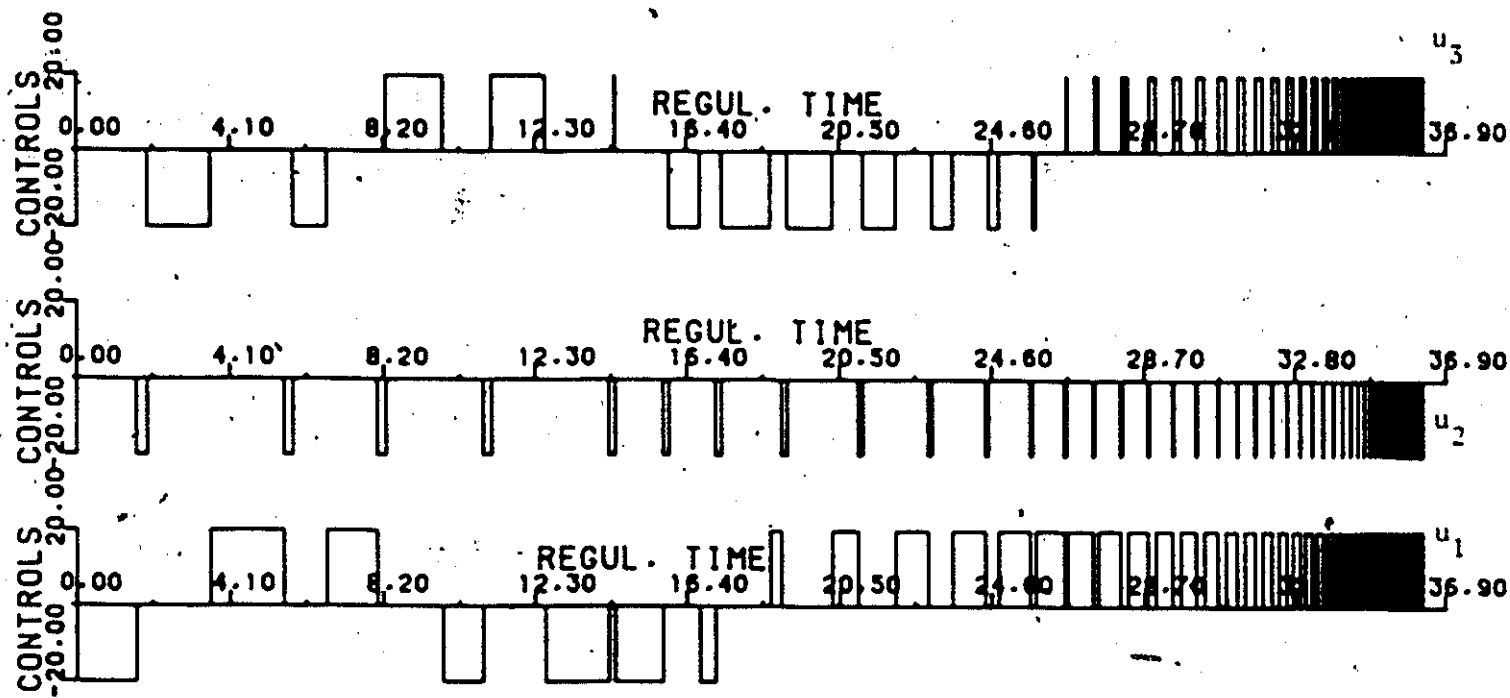


FIG. 5.3. Satellite attitude control using jets only. Controls' profile; the control policies are generated using the algorithm of section 5.2.1.

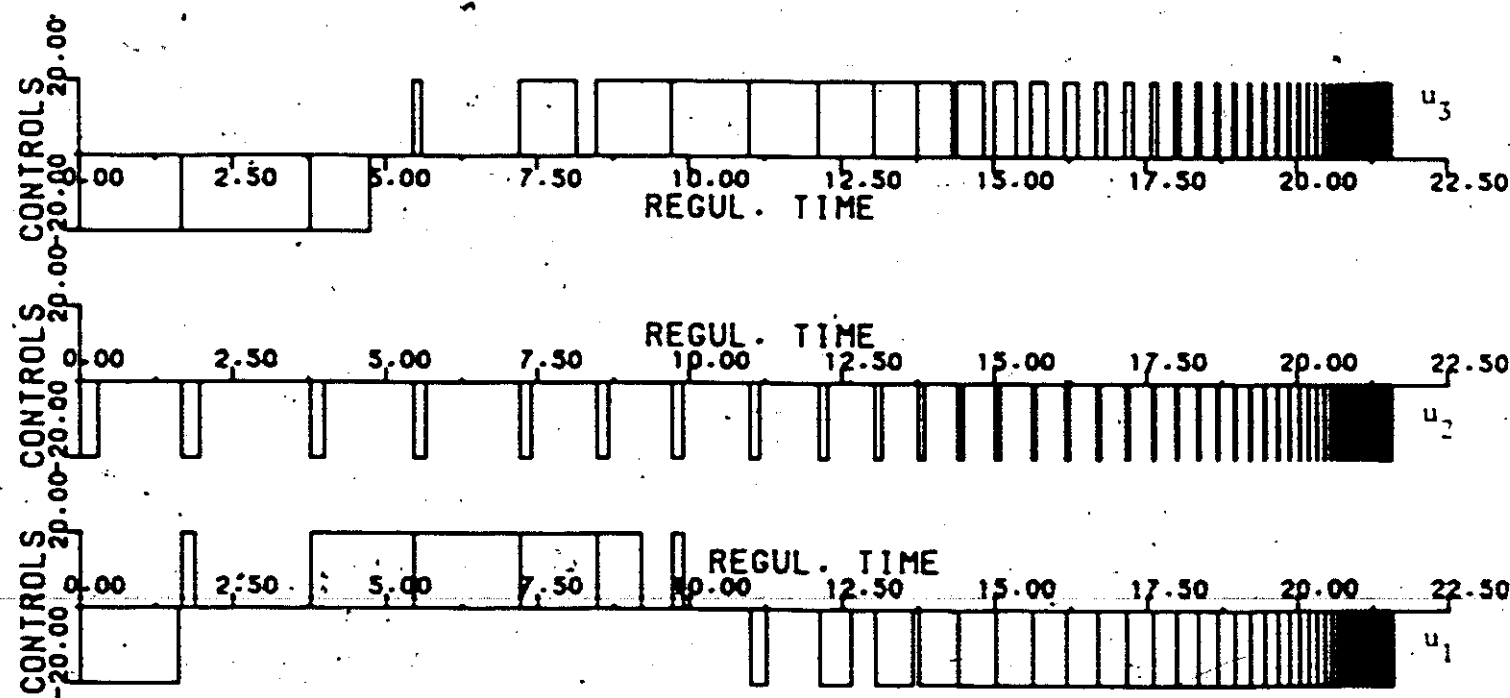


FIG. 5.4. Satellite attitude control using jets only. Controls' profile; the control policies are generated using the algorithm of section 5.2.2.

5.3 SATELLITE ATTITUDE CONTROL USING FLYWHEELS ONLY

When only flywheels are used, the system model reduces to:

$$\left\{ \begin{array}{l} I_x \dot{\beta} = - (I_z - I_y) (\gamma - w_0) \zeta + C_y \Omega_y \zeta - C_z \Omega_z \gamma - C_x u_4 \\ I_y \dot{\gamma} = - (I_x - I_z) \beta \zeta - C_x \Omega_x \zeta + C_z \Omega_z \beta - C_y u_5 \\ I_z \dot{\zeta} = - (I_y - I_x) (\gamma - w_0) \beta + C_x \Omega_x \gamma - C_y \Omega_y \beta - C_z u_6 \\ \dot{\hat{n}}_x = u_4 \\ \dot{\hat{n}}_y = u_5 \\ \dot{\hat{n}}_z = u_6 \end{array} \right. \quad (5.11)$$

or in matrix form:

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}) + u_4 \mathbf{Y}_4 + u_5 \mathbf{Y}_5 + u_6 \mathbf{Y}_6 \quad (5.12)$$

where

$$\mathbf{X}(\mathbf{x}) = \begin{pmatrix} \{ -(I_z - I_y) (\gamma - w_0) \zeta + C_y \Omega_y \zeta - C_z \Omega_z \gamma \} / I_x \\ \{ -(I_x - I_z) \beta \zeta - C_x \Omega_x \zeta + C_z \Omega_z \beta \} / I_y \\ \{ -(I_y - I_x) (\gamma - w_0) \beta + C_x \Omega_x \gamma - C_y \Omega_y \beta \} / I_z \end{pmatrix}$$

and

$$Y_4 = \begin{bmatrix} -C_x/I_x \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ; \quad Y_5 = \begin{bmatrix} 0 \\ -C_y/I_y \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \quad Y_6 = \begin{bmatrix} 0 \\ 0 \\ -C_z/I_z \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Again our aim is to drive the satellite's angular velocities to the rest position ($\beta, \gamma, \zeta = 0$). For the flywheels, let us assume for the moment that their final angular velocities at the desired state are $\bar{\Omega}_x, \bar{\Omega}_y, \bar{\Omega}_z = 0$. The necessity of such an argument will become evident later in this section (see (5.13b)). Accordingly, we define the desired state \bar{x} as:

$$\bar{x} = (0, 0, 0, \bar{\Omega}_x, \bar{\Omega}_y, \bar{\Omega}_z)'$$

Since $X(\bar{x}) = 0$ it follows:

$$(\text{ad } X, Y_i)(\bar{x}) = \left(\frac{\partial X}{\partial x} \right)_{\bar{x}} Y_i = X_x(\bar{x}) Y_i ; \quad i=4,5,6.$$

Computing the Lie Brackets we have:

$$[X, Y_i](\bar{x}) = \begin{pmatrix} 0 \\ -C_x C_z \bar{\Omega}_z / I_x I_y \\ [-\omega_0 C_x (I_y - I_x) + C_x C_y \bar{\Omega}_y] / I_x I_z \\ 0 \\ 0 \\ 0 \end{pmatrix} ;$$

$$[X, Y_5](x) = \begin{pmatrix} C_2 C_3 \ddot{\eta}_2 / I_x I_y \\ 0 \\ -C_1 C_3 \ddot{\eta}_2 / I_x I_z \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

and

$$[X, Y_6](x) = \begin{pmatrix} [-C_2 \omega_0 (I_z - I_y) - C_1 C_3 \ddot{\eta}_2] / I_x I_z \\ C_1 C_2 \ddot{\eta}_x / I_x I_z \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By theorem 4.8, the system is locally controllable if

$$\text{rank}\{A(x), [Y_4, Y_5, Y_6], [X, Y_4], [X, Y_5], [X, Y_6]\}(x) = 6 \quad (5.13a)$$

or equivalently by computing the determinant of A if

$$\det A = \frac{C_2^2 C_3^2 C_1^2 \omega_0 \ddot{\eta}_x \ddot{\eta}_2 (I_z - I_x)}{I_x^2 I_y I_z} \neq 0 \quad (5.13b)$$

which is true if $I_x \neq I_z$ and the final flywheel angular velocities (at x) are $\ddot{\eta}_x \neq 0$ and $\ddot{\eta}_2 \neq 0$. The first condition is immediately satisfied under our numerical assumptions and for the second it is well known [20] that when flywheels are used, they attain a certain saturating velocity, so it would be logical to assume a final flywheel angular velocity different from 0.

From (5.13) and according to the algorithm 4.4.2, we could try to control all the six states of (5.11), i.e., drive the angular velocities $\dot{\alpha}, \dot{\gamma}, \dot{\zeta}$ to 0 and $\alpha_x, \alpha_y, \alpha_z$ to some nominal value $\alpha_x^0, \alpha_y^0, \alpha_z^0 \neq 0$, even though our aim is to drive only the body rates $\dot{\alpha}, \dot{\gamma}, \dot{\zeta}$ to 0.

The procedure to follow would be very similar to the one introduced for the jets and the algorithm to proceed from one iteration to the next is:

$$q^i = q_1^{e_1} (|a_1^i| \delta) \cdot q_2^{e_2} (|a_2^i| \delta) \cdot q_3^{e_3} (|a_3^i| \delta) \cdot q_4^{e_4} (|a_4^i| \delta) \cdot q_5^{e_5} (|a_5^i| \delta) \cdot q_6^{e_6} (|a_6^i| \delta) \cdot q^1 \quad (5.14)$$

where

$$e_i = \begin{cases} - & \text{if } a_i \geq 0 \\ + & \text{if } a_i < 0 \end{cases} \quad i=1, \dots, 6$$

Nevertheless, the algorithm leads to numerical difficulties if we attempt to use it to drive all six states to $x=(0,0,0,\alpha_x^0,\alpha_y^0,\alpha_z^0)$. The problem, it is believed, arises because the order of magnitude of the remainder terms $O(\delta)$ is not negligible. The convergence of the algorithm is based partly on the possibility of neglecting such remainder terms.

Ultimately, our aim is to control the angular velocities $\dot{\alpha}, \dot{\gamma}, \dot{\zeta}$ to the rest position. This task may be achieved if

instead we consider only the first three components of the vectors Y_4, Y_5, Y_6 , i.e.,

$$\text{Proj}[Y_4, Y_5, Y_6] = \begin{bmatrix} -C_x/I_x & 0 & 0 \\ 0 & -C_y/I_y & 0 \\ 0 & 0 & -C_z/I_z \end{bmatrix} \quad (5.15)$$

and then, proceed as in 5.2, i.e., given the initial perturbation to the angular velocity q^1 , compute the coefficients from:

$$\begin{bmatrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{bmatrix} = \begin{bmatrix} -I_x/C_x & 0 & 0 \\ 0 & -I_y/C_y & 0 \\ 0 & 0 & -I_z/C_z \end{bmatrix} \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \end{bmatrix} \quad (5.16)$$

By doing so we reduce the number of functions required from 6 in (5.14) to simply 3 in (5.17) below. It allows us to obtain a best approximation from the Baker-Campbell-Hausdorff formula.

5.3.1 The Feedback Control Algorithm

In this section, as in 5.2.1, we will compute a stabilizing control using algorithm 4.4.2.

The algorithm to proceed from q^1 to q^2 is:

$$q^2 = \exp [|\alpha_3^1| \&(X - Y_6 \text{sgn}(\alpha_3^1))] \cdot \exp [|\alpha_2^1| \&(X - Y_5 \text{sgn}(\alpha_2^1))] \cdot \exp [|\alpha_1^1| \&(X - Y_4 \text{sgn}(\alpha_1^1))] q^1 \quad (5.17)$$

The system block diagram is shown in figure BG.3. Clearly, it is very similar to the one of section 5.2.1 depicted in figure BG.1.

We next give the results of a typical computer run using algorithm (5.17). The initial perturbation was set to

$$q' = (0.004, 0.003, 0.002).$$

The control bounds were chosen as $|u_4| \leq 10$; $|u_5| \leq 15$; $|u_6| \leq 20$ (again bang-bang controls only) while the value of ϵ was 0.005.

For the system (5.12) all the constants are as shown for (5.4), and the values chosen for the parameters C_x , C_y , C_z were those of [24] and are as follows:

$$C_x = 0.10758 \text{ slug-ft}^2$$

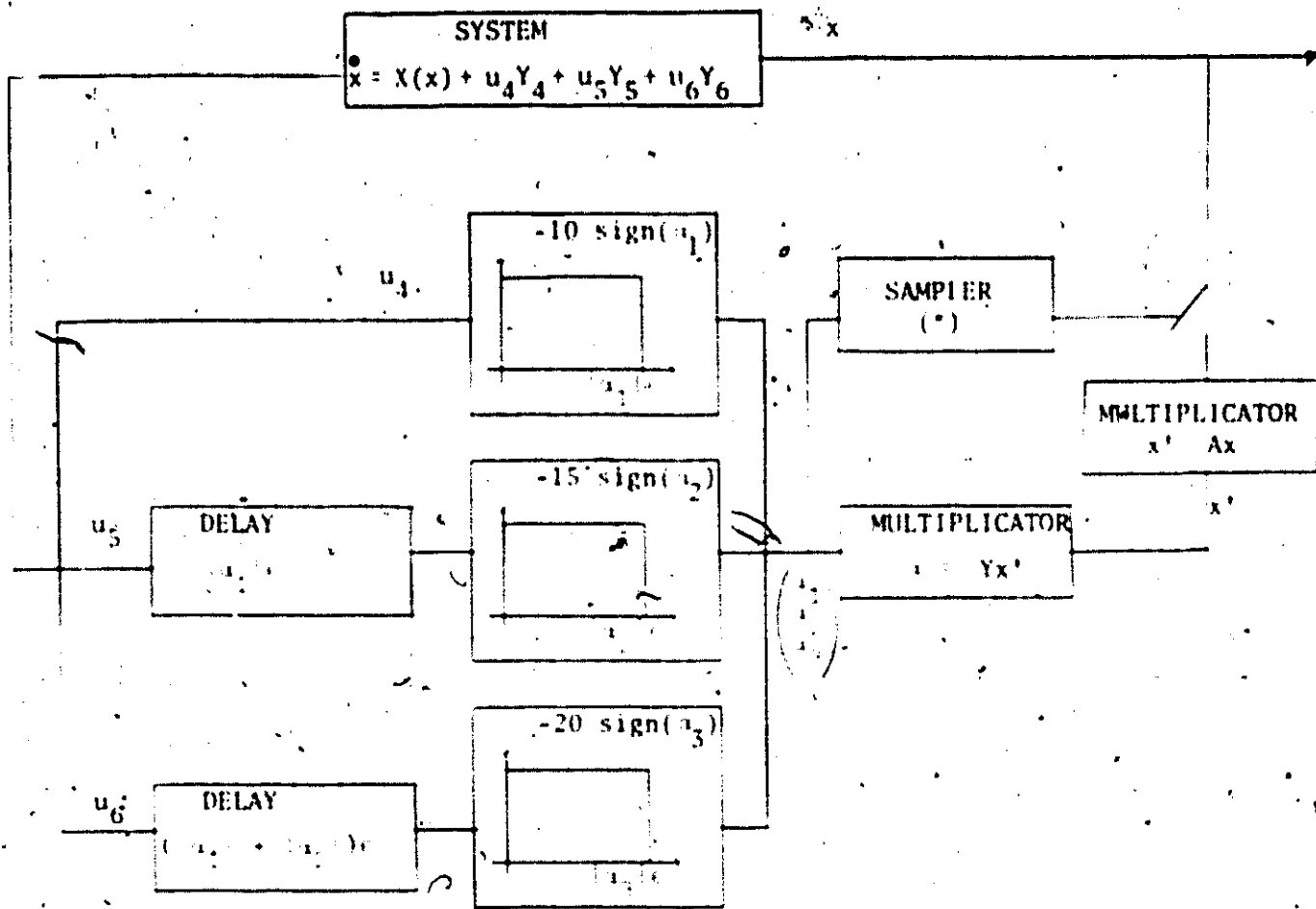
$$C_y = 0.01668 \text{ slug-ft}^2$$

$$C_z = 0.11156 \text{ slug-ft}^2 .$$

Those correspond to a maximum allowable flywheel velocity of 157 rpm and to a maximum slewrate of 0.00262 rad/sec.

The Fortran code used to solve this program is listed in Appendix A. Figures 5.5 and 5.7 illustrate the state trajectories and the control sequence, and figure 5.9 illustrates the flywheels angular velocities.

As in 5.2.1, the synthesized controls are applied on the system sequentially (fig. 5.7) which means, consequently, a long regulation time. In the next section the same type of



$\epsilon = 0.005$

$X(x), Y_4, Y_5, Y_6$ as in (5.12)

$Y = (\text{Proj} \{Y_4, Y_5, Y_6\})^{-1}$ as in (5.16)

(*) Sampling Period $(|a_1| + |a_2| + |a_3|)\epsilon$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

FIG. BG.3. Block diagram of the satellite attitude control system using flywheels only. The feedback control laws (u_4, u_5, u_6) are generated using the algorithm of section 5.3.1.

modification as in 5.2.2 is introduced to regulate in a shorter time.

5.3.2 The Modified Algorithm

The procedure to follow is exactly the same as in 5.2.2 and therefore will not be repeated here. The algorithm to proceed from q^i to q^{i+1} (and in general from q^v to q^{v+1}) is that of (5.9) with the ordering map defined by

$$P: (\alpha_1^i, \alpha_2^i, \alpha_3^i) \rightarrow (\alpha_{m1}^i, \alpha_{m2}^i, \alpha_{m3}^i)$$

where α_1^i , α_2^i , and α_3^i are computed from (5.16). Accordingly, the ordering map for the Y vectors is defined by:

$$PY: (Y_4, Y_5, Y_6) \rightarrow (Y_{m1}, Y_{m2}, Y_{m3})$$

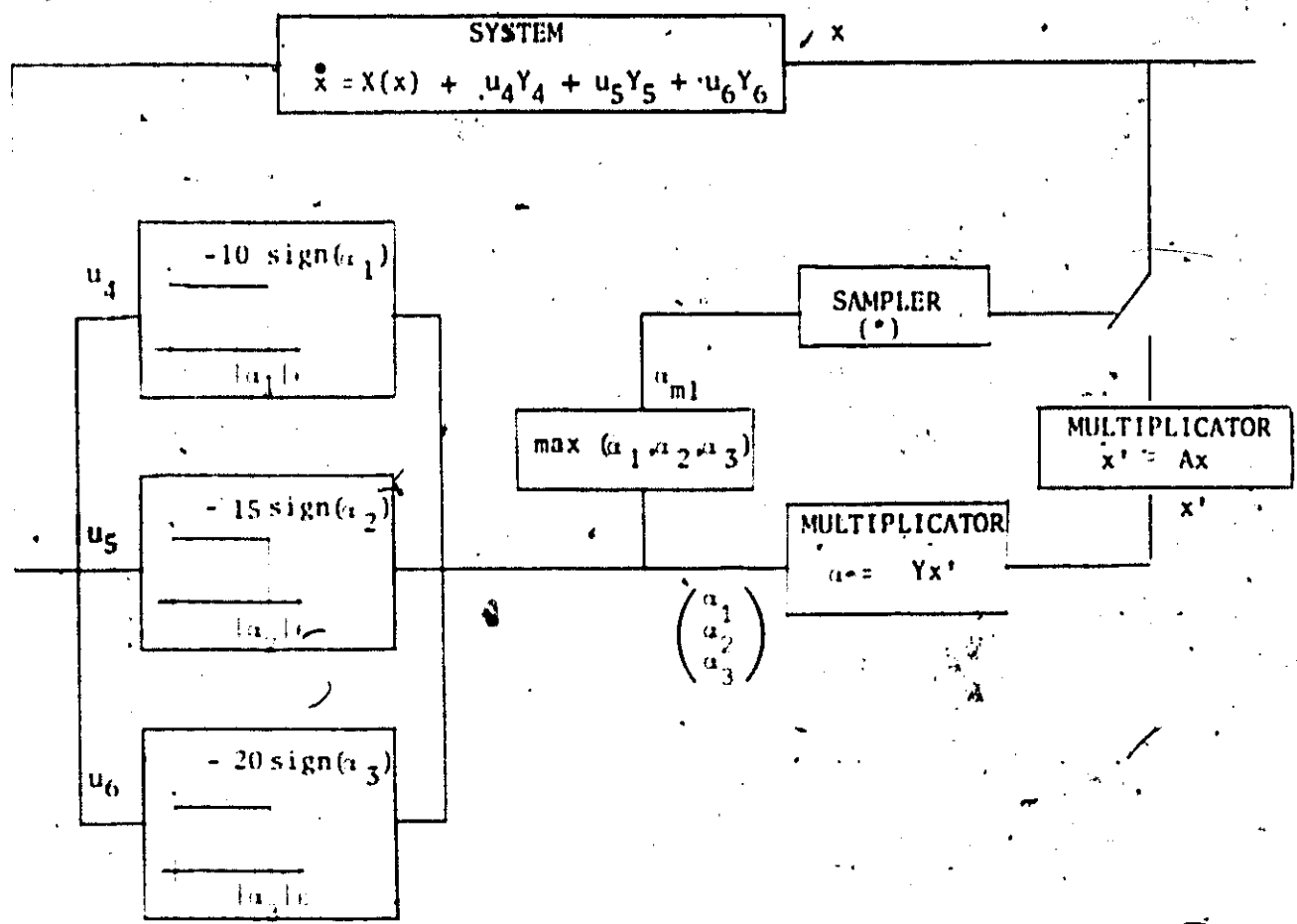
Again the regulation time between iterations is reduced from $\sum_1^3 |\alpha_i^j| \epsilon$ to $|\alpha_{m1}^j| \epsilon$ and similarly the computation burden increases due to the additional requirement of ordering of the α coefficients.

The system block diagram is shown in figure BG.4.

We next give the results of the computer run with the initial data set to that of 5.3.1, i.e.,

$$q^i = (0.004, 0.003, 0.002);$$

$$\epsilon = 0.005; (|u_4| \leq 10; |u_5| \leq 15; |u_6| \leq 20)$$



$c = 0.005$

$X(x), Y_4, Y_5, Y_6$ as in (5.12)

$Y = (\text{Proj}[Y_4, Y_5, Y_6])^{-1}$ as in (5.16)

(*) Sampling period $|\alpha_{m1}|c$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

FIG. B6.4. Block diagram of the satellite attitude control system using flywheels only. The feedback control laws (u_4, u_5, u_6) are generated using the algorithm of section 5.3.2.

The Fortran code used to solve this program is listed in Appendix A. Figures 5.6 and 5.8 illustrate the state trajectories and the control sequence and figure 5.10 illustrates the flywheel angular velocities.

As in 5.2.2, from the graphical results it is apparent that by applying the controls in all three directions simultaneously the regulation time reduces considerably.

The basic difference between algorithms 5.3.1 and 5.3.2 is the manner in which controls are applied on the sistem (in one case sequentially whereas in the other simultaneously). For a detailed explanation of the difference the reader is referred to the remark on page 65.

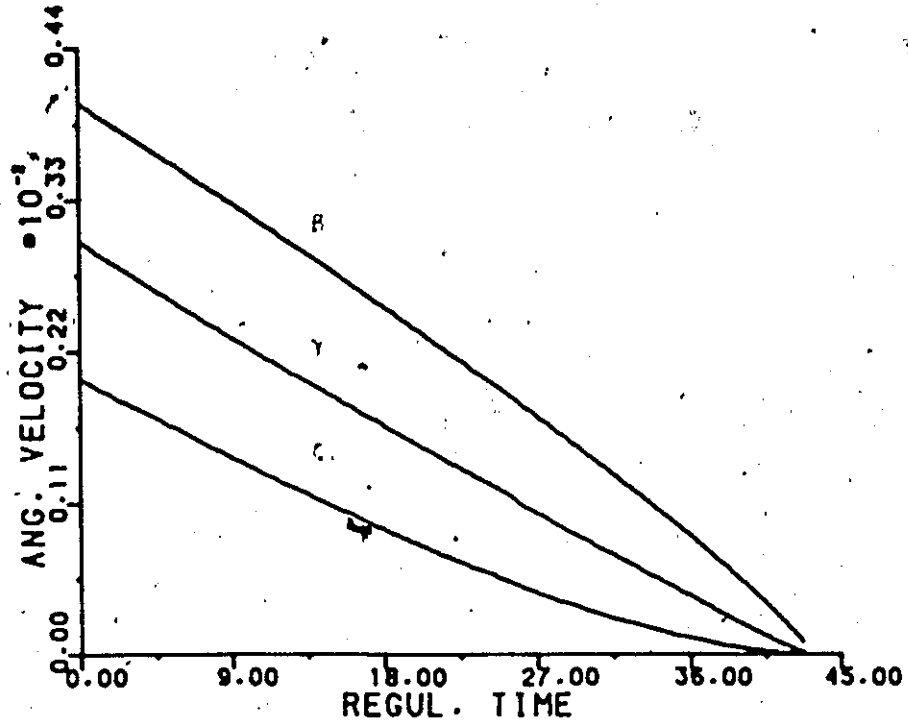


FIG. 5.5. Profiles of the satellite angular velocities (B, γ , ζ) when only flywheels are used. The feedback control laws are synthesized via the algorithm of section 5.3.1.

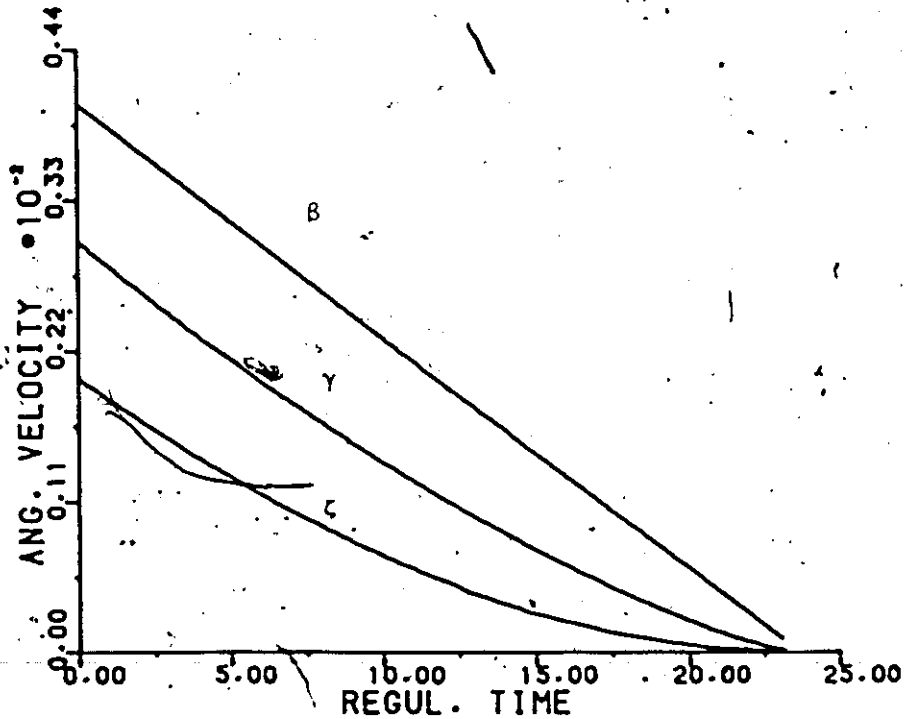


FIG. 5.6. Profiles of the satellite angular velocities (B, γ , ζ) when only flywheels are used. The feedback control laws are synthesized via the algorithm of section 5.3.2.

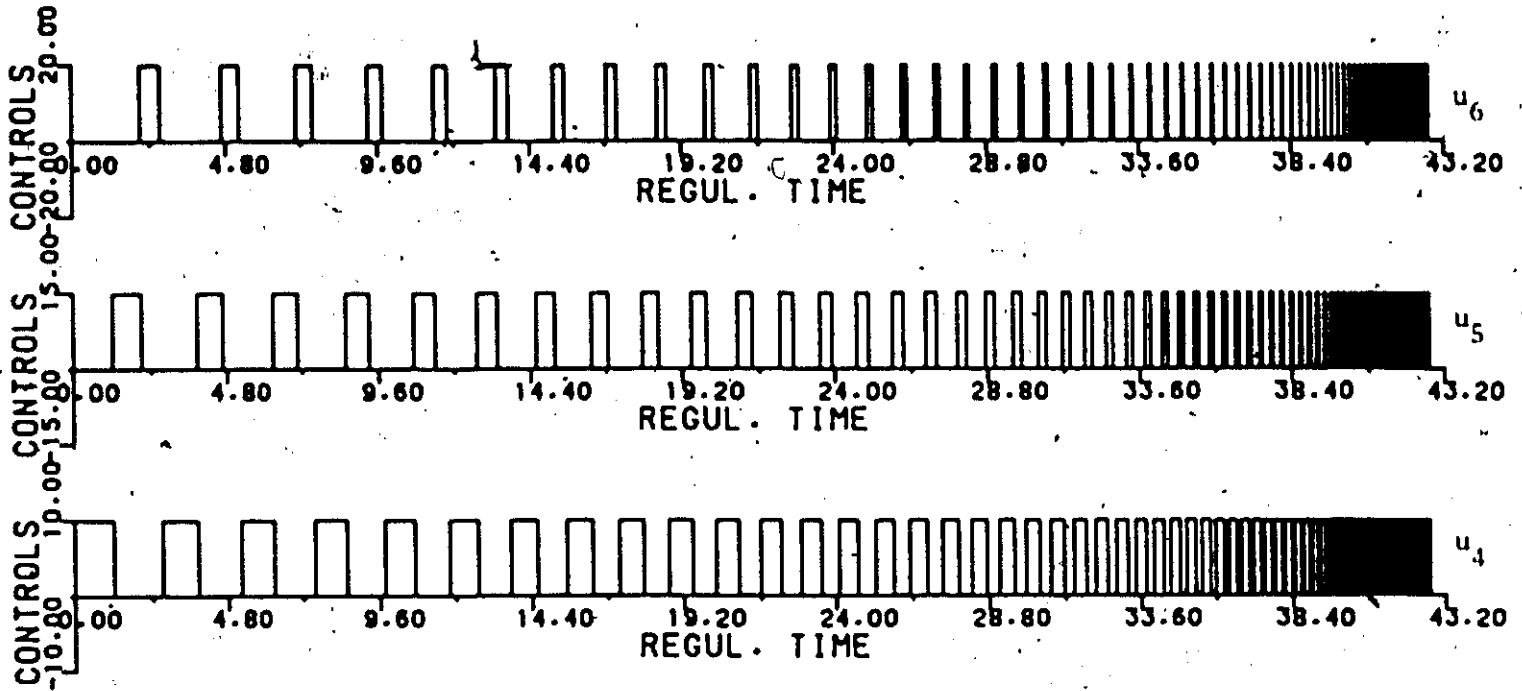


FIG. 5.7. Satellite attitude control using flywheels only. Controls' profile; the control policies are generated using the algorithm of section 5.3.1.

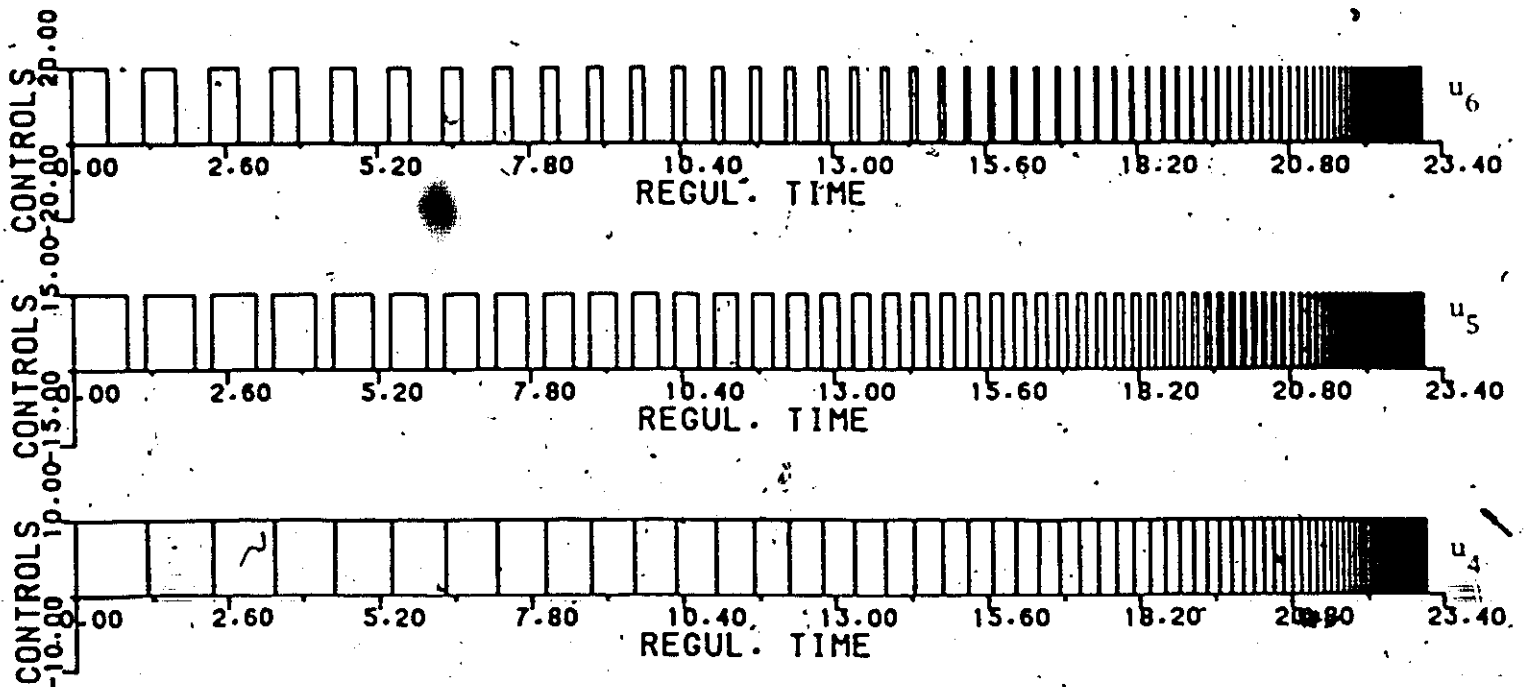


FIG. 5.8. Satellite attitude control using flywheels only. Controls' profile; the control policies are generated using the algorithm of section 5.3.2.

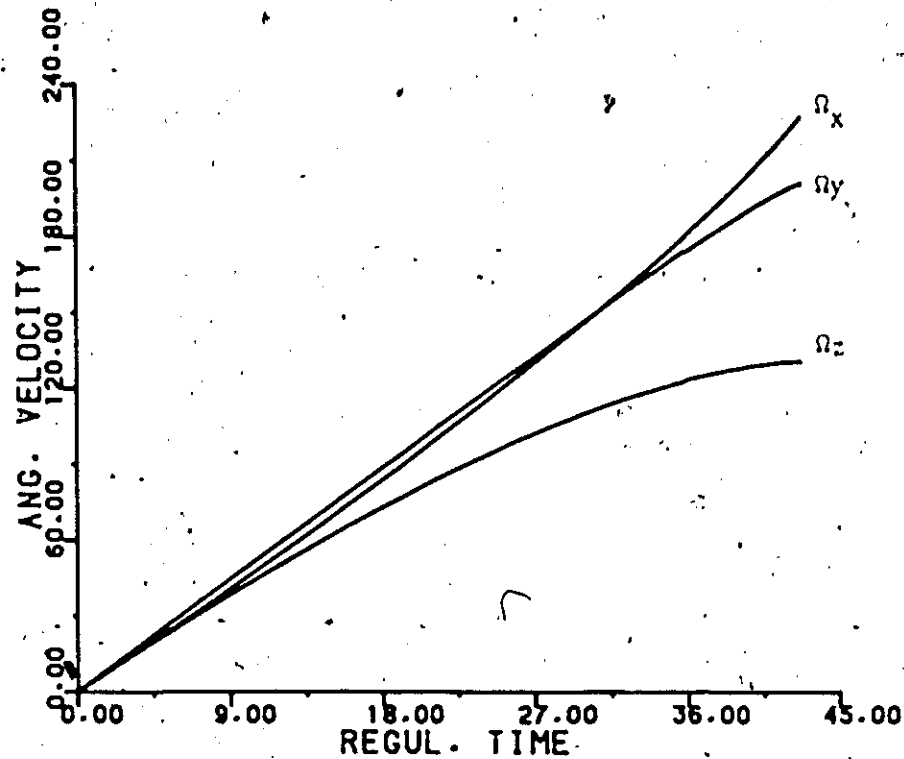


FIG. 5.9. Satellite attitude control using flywheels only. Profile of the flywheels angular velocities, when algorithm of section 5.3.1. is utilized to generate the control policies.

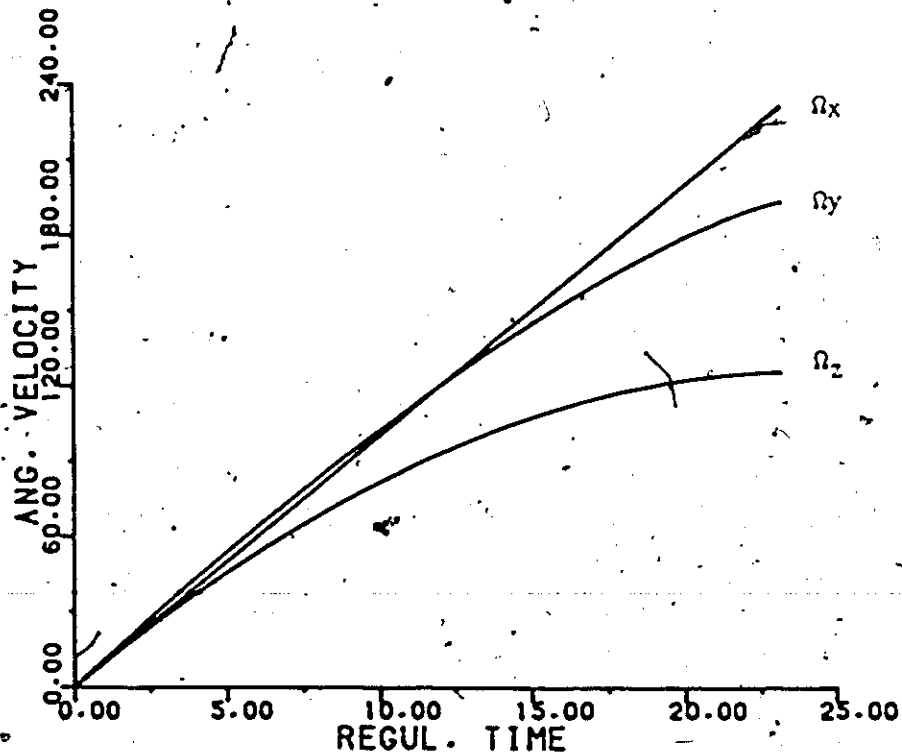


FIG. 5.10. Satellite attitude control using flywheels only. Profile of the flywheels angular velocities, when algorithm of section 5.3.2 is utilized to

5.4 COMMENTS AND CONCLUSIONS

In this chapter the algorithm introduced in 4.4.2 was used to synthesize a discrete feedback attitude control for the satellite model introduced in (5.1), in an effort to evaluate the validity and efficiency of such algorithm. Also, a slight modification is introduced in the algorithm in order to reduce the regulation time.

The algorithm provides a general method to synthesize a control for systems of the form :

$$\dot{x} = X(x) + \sum_1^m u_j Y_j$$

whenever the conditions of 4.4.2 are satisfied. It is the author's belief that the geometric idea of moving along the reference frame of the Lie bracket vectors toward the final desired position can be further exploited. One problem with the algorithm, though, is its lacking of any sort of optimality criterion. In some cases what is desirable is not just to have a stabilizing control but to have a stabilizing control optimal in some sense. The modified algorithm introduced by the author reduces the regulation time but yet lacks of optimality criteria.

On the other hand both algorithms, Hermes' [12] and the author's 5.2.2 and 5.3.2, are simple and its implementation in the computer is simple as well.

Another drawback of algorithm 4.4.2, as reported in [12], arises when higher order Lie brackets $((ad^m X, Y); m \geq 2)$ are required. In this case the remainder terms $O(\epsilon)$ in the Baker-Campbell-Hausdorff expansion may not be negligible and as a result difficulties in the convergence of the algorithm may be encountered as experienced with flywheel controls [see section 5.3, pp. 72].

It should be pointed out that for the case of attitude control via jets only (section 5.2) a smooth proportional feedback control $(u_i = -K_i x_i, i=1,2,3)$ is easily obtained from Liapunov's stability theory by simply defining the Liapunov function as:

$$V(x) = \frac{1}{2} I_x \beta^2 + \frac{1}{2} I_y \gamma^2 + \frac{1}{2} I_z \zeta^2 .$$

The numerical results for the computer run with the same initial data as in 5.2.1 and $k_1 = k_2 = k_3 = 1$ are shown in figure 5.11. It should be mentioned that the time is being reduced considerably because the order of magnitude of the control is originally much higher than in case 5.2.1.

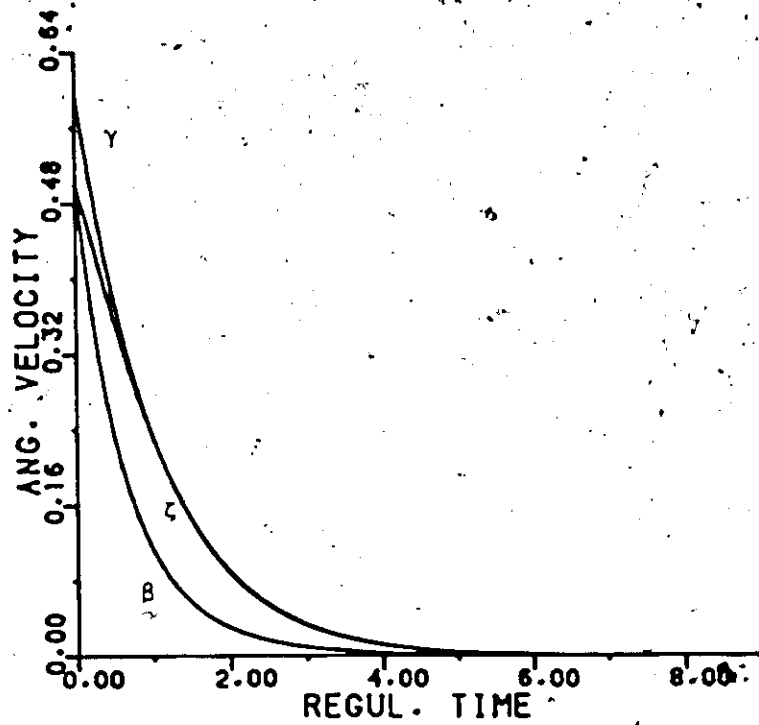


FIG. 5.11. Satellite attitude control using jets only. Profile of the satellite angular velocities when a proportional feedback control is utilized.

AD

About the satellite's controllability properties, our results are consistent with those of previous studies [18,24]. We sum up our results in the following.

The system controlled only by jets is locally controllable near the origin; the jets are able to control larger perturbations than the flywheels; however, the use of jets reduces the useful life of the satellite and also the regulation by jets is poor near the origin.

The system controlled only by flywheels is also locally controllable. The flywheels are more accurate and smoother than the jets; however for large perturbations, flywheels are not so effective. Even for small perturbations they attain very high velocities, so that having the limitation of a maximum allowable velocity prevents their use for large perturbations.

Regarding the Lie Algebraic approach to study the controllability properties of systems, if we specialize to linear time invariant systems the Lie algebraic controllability rank condition (4.21) reduces to the well-known linear controllability condition (2.6).

If we consider analytic systems, or systems with F_0 symmetric, the geometric approach also provides analytic characterization of the controllability space. Even for C^∞ systems, it is possible to relate the reachable set to the tangent space generated by the Lie algebra associated to the system itself [see theorems 4.5, 4.8, 4.9]. Hence, even

though no major result has been obtained yet and those available were obtained under rather strong conditions, there is still great interest in such methods because they could provide an analytic characterization of the controllability space for a fairly wide class of nonlinear systems.

For further research we would like to propose the following:

- 1) To determine when the Lie algebra F [see section 4.3.1] generated by vector fields of the form $f(\cdot, u)$ ($\dot{x}=f(x, u); u=\text{constant}$) has a finite dimensional basis $m < n$. We can reduce, in such cases, the degree of differentiability required from m to n .
- 2) For C^∞ systems the controllability condition

$$\text{rank } F(x) = n$$

is satisfied in an open dense subset of the state space \mathcal{X} if the system is locally weakly controllable [thm. 4.6]. Since $\text{rank } F(x) = n$ is a sufficient condition for local weak controllability, it would be interesting to determine whether there exist C^∞ -locally weakly controllable systems where the controllability condition is satisfied not only on an open dense subset of \mathcal{X} but in the whole of \mathcal{X} . By studying such systems, a generalization of theorem 4.6 might be obtained and condition $\text{rank } F(x) = n$ would provide the necessary and sufficient condition for local weak controllability.

3) It would be desirable to modify the algorithm 4.4.2 so that it could generate not only a stabilizing control but also a time optimal one.

Chapter VI
CONCLUSIONS

In this thesis a brief review of the Controllability theory of linear and nonlinear systems is presented.

Regarding the controllability theory of linear systems, a counterexample to a very well established result is shown on chapter III. The result in question was obtained by Lasalle and Hermes [1, page 78] and is quoted below:

"if the uncontrolled system $\dot{x} = A(t)x$ is asymptotically stable and the system (17.1)* is proper at time t_0 , then there is for each initial state x_0 in R^n an admissible control u that brings the system to the origin in finite time."

Also, a certain number of other results have been derived from this result; we cite, for instance, Chukwu [15, thm. 3] and Khambadkone [25, thm. 1]. Nevertheless; such a conclusion is not always true, i.e., for time varying systems the asymptotic stability and the local null controllability properties combined do not guarantee the global null controllability of the system.

(*) System (17.1) being, as in Lasalle and Hermes [1, page 71] the system:

$$\dot{x} = A(t)x + B(t)u \quad u(\cdot) \in M(C^m)$$

To support our conclusion we constructed an example of a one dimensional system (see Chapter III) which is asymptotically stable, and locally null controllable but it is not globally null controllable.

In the second part of our research we applied an algorithm based on Lie Algebraic Methods to generate a stabilizing attitude control for the satellite problem. Two cases were studied; in the first; only jets were used as controls and in the second, only flywheels.

The numerical results obtained for both cases are illustrated in Chapter V.

One drawback of the algorithm is that it calls for sequential application of the available controls. Since a simultaneous application of the controls do not change the regulation properties of the original algorithm, we introduced such a modification in the algorithm in order to obtain some savings in the regulation times. From our numerical results, also included in chapter V, it is obvious that, considerable savings in the regulation time may be achieved by doing so.

Finally, further comments and conclusions as well as proposals for further research are included in the later part of chapters III and V.

Appendix A

FORTRAN CODES

In the following the Fortran Codes used in Chapter V of this thesis are exhibited.

```

*****
*
* FORTRAN CODE FOR THE SATELLITE ATTITUDE CONTROLLED PROBLEM
* USING ONLY THE JETS AS CONTROLS.
* THE SYNTHESIZING ALGORITHM IS DERIVED USING HERMES' APPROACH.
* [12] WHICH UTILIZES LIE ALGEBRAIC METHODS AND IS DEPICTED IN
* 5.2.1.
*
*****

```

```

COMMON/ML/X(3),U(3)
COMMON/MPTA/Yx,Yy,Yz,Wo-SG(3)

```

Y(1), X(2), X(3) ARE THE SATELLITE'S BODY RATES ON THE x, y, z DIRECTIONS RESPECTIVELY.
 Yx, Yy, Yz ARE THE MOMENTS OF INERTIA OF THE SATELLITE AND Wo IS THE ANGULAR VELOCITY OF THE REFERENCE FRAME.
 ALFA (1,2,3) ARE THE COEFFICIENTS OF THE PERTURBATION RELATIVE TO THE LIE BRACKET REFERENCE SYSTEM.
 TI(1,2,3) ARE THE INTEGRATION TIMES AND STP(1,2,3) ARE THE STEPS OF INTEGRATION:

```

DIMENSION ALFA(3),TI(3),STP(3)
READ(5,100)(X(I)-I=1,3)
READ(5,101) Yx,Yy,Yz,Wo
100 FORMAT(3(F3.1,1X))
101 FORMAT(3(F5.1,1X),F9.7)
TOT=0
PP=.005
M=50

```

CHECKING WHETHER THE ORIGINAL PERTURBATION IS WITHIN THE ADMISSIBLE PERTURBATION SET.

```

F1=(-(Yz-Yy)*X(2)-Wo)*X(3))/Yx
F2=(-(Yx-Yz)*X(1)*X(3))/Yy
F3=(-(Yx-Yy)*X(2)-Wo)*X(1))/Yz

```

```

DIS=SQRT(F1**2+F2**2+F3**2)
IF(DIS.GT.EP) GO TO 102
WRITE(6,230) DIS
230 FORMAT(1X,F12.6,/)

```

COMPUTING THE ALPHA (α) COEFFICIENT AND THE INTEGRATION TIMES.

```

1 ALFA(1)=X(1)*Yz
ALFA(2)=X(2)*Yy
ALFA(3)=X(3)*Yz
DO 400, J=1,3
TI(J)=ABS(ALFA(J))*EP
STP(J)=TI(J)/M
SG(J)=-1
IF(ALFA(J).GE.0) SG(J)=1
400 CONTINUE
TOT=TOT+TI(1)+TI(2)+TI(3)
WRITE(6,201)(ALFA(J),J=1,3)

```

DETERMINING THE CONTROL SEQUENCE AND PERFORMING THE NUMERICAL INTEGRATION.

```

U(1)=20
U(2)=0
U(3)=0
CALL RKM(STP(1),M)
U(1)=0
U(2)=20
U(3)=0
CALL RKM(STP(2),M)
U(1)=0
U(2)=0
U(3)=20
CALL RKM(STP(3),M)
D=SQRT(X(1)**2+X(2)**2+X(3)**2)
WRITE(6,202)(X(J),J=1,3),D
WRITE(6,250)(TI(J),J=1,3),TOT
DO 401 J=1,3
401 SG(J)=-SG(J)
WRITE(6,203)(SG(J),J=1,3)
201 FORMAT(1X,3(1X,F12.6),/)
202 FORMAT(1X,4(1X,F11.6),/)
203 FORMAT(1X,3(1X,F4.1),/)
250 FORMAT(1X,4(1X,F7.3),/)

```

CONVERGENCE CRITERION.

```

      IF(D.GE.0.001) GO TO 1.
      WRITE(6,204) TOT
204  FORMAT(1X,F7.3)
102  STOP
      END

```

SUBROUTINE OF INTEGRATION. ROUNGE, KUNTA METHOD.

```

SUBROUTINE RKM(STP,N)
COMMON /ML/X(3),U(3)
COMMON/NENE/ F(3)
DIMENSION F1(3),F2(3),F3(3),XX(3)
DO 300 K=1,N
CALL FEU(X,U)
DO 410 I=1,3
F1(I)=F(I)
410  XX(I)=X(I)+STP*F(I)/2
CALL FEU(XX,U)
DO 411 I=1,3
F2(I)=F(I)
411  XX(I)=X(I)+STP*F(I)/2
CALL FEU(XX,U)
DO 412 I=1,3
F3(I)=F(I)
412  YY(I)=X(I)+STP*F(I)
CALL FEU(YY,U)
DO 413 I=1,3
413  X(I)=X(I)+STP*(F1(I)+2*F2(I)+2*F3(I)+F(I))/6
300  CONTINUE
RETURN
END

```

```

SUBROUTINE FEU(Z,U)
COMMON/HARIA/Yx,Yy,Yz,Wo,S6(3)
COMMON/NENE/F(3)
DIMENSION Z(3),U(3)
F(1)=-((Yz-Yy)*Z(2)-Wo)*Z(3)-S6(1)*U(1)/Yx
F(2)=-((Yx-Yz)*Z(1)*Z(3)-S6(2)*U(2))/Yy
F(3)=-((Yy-Yx)*Z(2)-Wo)*Z(1)-S6(3)*U(3)/Yz
RETURN
END

```

```

XENTRY
$STOP
$END

```

* FORTRAN CODE FOR THE SATELLITE ATTITUDE CONTROLLED PROBLEM
* USING ONLY THE JETS AS CONTROLS.
* THE SYNTHESIZING ALGORITHM IS DERIVED VIA LIE ALGEBRAIC
* METHODS AND DEPICTED IN SECTION 5.2.2.
* HERMES ALGORITHM [12] IS MODIFIED TO REDUCE THE REGULATION
* TIME

COMMON/ML/X(3),U(3).
COMMON/MARIA/Yx,Yy,Yz,Wo,SR(3)

X(1), X(2), X(3) ARE THE SATELLITE'S BODY RATES ON THE X, Y, Z
DIRECTIONS RESPECTIVELY.
Yx, Yy, Yz ARE THE MOMENTS OF INERTIA OF THE SATELLITE AND Wo
IS THE ANGULAR VELOCITY OF THE REFERENCE FRAME.
ALFA(1,2,3) ARE THE COEFFICIENTS OF THE PERTURBATION RELATIVE
TO THE LIE BRACKET REFERENCE SYSTEM.
TI(1,2,3) ARE THE INTEGRATION TIMES AND STP(1,2,3) ARE THE STEPS
OF INTEGRATION.

DIMENSION ALFA(3),TI(3),STP(3),W(3),WR(3),U(3),M(3)
READ(5,100)(X(I),I=1,3)
READ(5,101) Yx,Yy,Yz,Wo
100 FORMAT(3(F3.1,1X))
101 FORMAT(7(F3.1,1X),F9.7)
101=0
SR=0.05
M=50

CHECKING WHETHER THE ORIGINAL PERTURBATION IS WITHIN THE
ADMISSIBLE PERTURBATION SET.

F1=(-(Yz-Yy)*(X(2)-Wo)*X(3))/Yx
F2=(-(Yx-Yz)*(X(1)-Wo)*X(3))/Yy
F3=(-(Yx-Yy)*(X(2)-Wo)*X(1))/Yz
DIS=SQRT(F1**2+F2**2+F3**2)
IF(DIS,ST,EP) GO TO 102
WRITE(6,950) DIS
220 FORMAT(1X,F10.4)

COMPUTING THE ALPHA (α) COEFFICIENTS AND THE INTEGRATION TIMES.

```

1  ALFA(1)=X(1)*YX
  ALFA(2)=X(2)*Yy
  ALFA(3)=X(3)*Yz
  DO 400 J=1,3
  TI(J)=ABS(ALFA(J))*EP
  SG(J)=-1
  IF (ALFA(J).GT.0) SG(J)=1
400 CONTINUE
  WRITE (4,201) (ALFA(J),J=1,3)

```

ORDERING OF THE INTEGRATION TIMES.

```

206 IF (TI(1)-TI(2)) 206-206-207
206 IF (TI(1)-TI(3)) 208,208-208
208 TMAX=TI(2)
  TMIN=TI(1)
  TMIN=TI(3)
  GO TO 216
207 IF (TI(1)-TI(3)) 210,210-211
210 TMAX=TI(3)
  TMIN=TI(1)
  TMIN=TI(2)
  GO TO 216
209 IF (TI(2)-TI(3)) 212-212-213
212 TMIN=TI(1)
  TMIN=TI(2)
  TMAX=TI(3)
  GO TO 214
213 TMIN=TI(1)
  TMIN=TI(3)
  TMAX=TI(2)
  GO TO 216
211 IF (TI(2)-TI(3)) 214-214-215
214 TMAX=TI(1)
  TMIN=TI(3)
  TMIN=TI(2)
  GO TO 216
215 TMAX=TI(1)
  TMIN=TI(2)
  TMIN=TI(3)
216 TOT=TOT+TMAX
  STP(1)=TMIN/M
  STP(2)=(TMIN-TMIN)/M

```

STP(3)=(TMAX-TMED)/M

DETERMINING THE CONTROL SEQUENCE AND PERFORMING THE NUMERICAL INTEGRATION.

```

DO 111 J=1,3
  U(J)=TI(J)-TMIN
111  W(J)=TI(J)-TMED
  DO 205 I=1,3
    U(I)=0
    IF(UV(I).GT.0) U(I)=1
    W(I)=1
    IF(W(I).LE.0) W(I)=0
205  CONTINUE
    U(1)=20
    U(2)=20
    U(3)=20
    CALL RKM(STP(1),M)
    U(1)=20*U(1)
    U(2)=20*U(2)
    U(3)=20*U(3)
    CALL RKM(STP(2),M)
    U(1)=20*U(1)
    U(2)=20*U(2)
    U(3)=20*U(3)
    CALL RKM(STP(3),M)
    D=SQRT(Y(1)**2+Y(2)**2+X(3)**2)
    WRITE(6,202) (X(J),J=1,3),D
    WRITE(6,217) (TI(J),J=1,3),TOT
  DO 401 J=1,3
401  SG(J)=-SG(J)
    WRITE(6,203) (SG(J),J=1,3)
201  FORMAT(1X,3(1X,F12.6),/)
202  FORMAT(1X,4(1X,F11.6),/)
203  FORMAT(1X,3(1X,F4.1),/)
217  FORMAT(1X,4(1X,F7.3),/)

```

CONVERGENCE CRITERION.

```

IF(D.GE.0.001) GO TO 1
WRITE(6,204) TOT
204  FORMAT(1X,F7.3)
111  STOP
END

```

SUBROUTINE OF INTEGRATION: RUNGE KUTTA METHOD.

```

SUBROUTINE RKM(STP,N)
COMMON /M/ X(3),H(3)
COMMON /NENE/ F(3)
DIMENSION F1(3),F2(3),F3(3),XX(3)
DO 300 I=1,N
CALL FEV(X,H)
DO 410 I=1,3
F1(I)=F(I)
410 XX(I)=X(I)+STP*F1(I)/2
CALL FEV(XX,H)
DO 411 I=1,3
F2(I)=F(I)
411 XX(I)=X(I)+STP*F2(I)/2
CALL FEV(XX,H)
DO 412 I=1,3
F3(I)=F(I)
412 XX(I)=X(I)+STP*F3(I)
CALL FEV(XX,H)
DO 413 I=1,3
F4(I)=F(I)
413 XX(I)=X(I)+STP*F4(I)
300 CONTINUE
RETURN
END

```

```

SUBROUTINE FEV(Z,UU)
COMMON /M/IA/ Yx,Yy,Yz,Uo,SG(Z)
COMMON /NENE/ F(3)
DIMENSION Z(3),III(3)
F(1)=-(-Yy-Yy)*Z(2)-Uo)*Z(3)-SG(1)*III(1)/Y
F(2)=-(-Yx-Yx)*Z(1)+Uo)*Z(3)-SG(2)*III(2)/Y
F(3)=-(-Yx-Yx)*Z(1)-Uo)*Z(2)-SG(3)*III(3)/Y
RETURN
END

```

* FORTRAN-CODE FOR THE SATELLITE ATTITUDE CONTROLLED PROBLEM *
* USING ONLY THE FLYWHEELS AS CONTROLS. *
* THE SYNTHESIZING ALGORITHM IS DERIVED USING KERMES' APPROACH *
* WHICH UTILIZED LIE ALGEBRAIC METHODS AND IS DEPICTED IN *
* FIG. 1. *
*

* THE PROGRAM COMPUTES THE SATELLITE'S BODY RATES IN THE N-BODY *
* STRUCTURE. *
* (A) (B) (C) ARE THE FLYWHEELS ANGULAR VELOCITIES IN THE N- *
* COORDINATE SYSTEMS, RESPECTIVELY. *
* (D) (E) (F) ARE THE MOMENTS OF INERTIA OF THE SATELLITE AND *
* (G) (H) (I) ARE PARAMETERS OF THE SYSTEM. *
* (J) (K) (L) (M) ARE THE COEFFICIENTS OF THE PERTURBATION RELATIVE *
* TO THE LIE THEORY REFERENCE SYSTEM. *
* (N) (O) (P) ARE THE INTEGRATION TIMES AND (Q) (R) (S) ARE THE STEPS *
* OF INTEGRATION. *
*
*-----

```
100 DIMENSION ALFA(3),TI(3),STP(3),SG(3)
101 READ(5,100) (X(I),I=1,6)
102 READ(5,101) Cx,Cy,Cz,Yx,Yy,Yz,Mo
103 FORMAT(3(F6.3,1X),3(F7.2,1X))
104 FORMAT(3(F8.6,1X),3(F5.1,1X),F9.7)
105 TOT=0
106 SF=1005
107 N=50
```

* CHECKING WHETHER THE ORIGINAL PERTURBATION IS WITHIN THE *
* ADMISSIBLE PERTURBATION SET. *
*-----

```
108 F1=(-(Yc-Yy)*(X(2)-Uo)*X(3)+Cx*YX(5)*X(2)-Cy*YX(6)*X(2))/Yx
109 F2=(-(Yx-Yz)*YX(1)*X(3)-Cx*X(4)*X(3)+Cz*YX(6)*X(1))/Yy
110 F3=(-(Yy-Yx)*(X(2)-Uo)*X(1)-Cx*X(5)*X(1)+Cz*YX(4)*X(2))/Yz
111 IF (ABS(F1)+ABS(F2)+ABS(F3)) GO TO 102
112 WRITE(6,230) BIS
113 FORMAT(1Y,5I2,6F7)
```

COMPUTING THE ALPHA AND COEFFICIENTS AND THE INTEGRATION TIMES.

```

1  ALPHA(1)=X(1)*Y(1)
2  ALPHA(2)=X(2)*Y(2)
3  ALPHA(3)=X(3)*Y(3)
4  DO 401 I=1,3
5  T1(1)=X(I)*Y(I)*Z(I)
6  T1(2)=X(I)*Y(I)*Z(I)
7  T1(3)=X(I)*Y(I)*Z(I)
8  T1(4)=X(I)*Y(I)*Z(I)
9  T1(5)=X(I)*Y(I)*Z(I)
10 IF (ALFA(I).NE.0) SG(I)=1
400 CONTINUE
TOT=TOT+T1(1)+T1(2)+T1(3)
WRITE(A-201) (ALFA(I),I=1,3)

```

DETERMINING THE CONTROL SEQUENCE AND PERFORMING THE NUMERICAL INTEGRATION.

```

11 I=1
12 DO 100 I=1,3
13 DO 101 J=1,3
14 CALL SUBROUTINE STR(I,J)
15 I=I+1
16 DO 101 J=1,3
17 CALL SUBROUTINE STR(I,J)
18 I=I+1
19 DO 101 J=1,3
20 CALL SUBROUTINE STR(I,J)
21 I=I+1
22 DO 101 J=1,3
23 CALL SUBROUTINE STR(I,J)
24 I=I+1
25 DO 101 J=1,3
26 CALL SUBROUTINE STR(I,J)
27 I=I+1
28 DO 101 J=1,3
29 CALL SUBROUTINE STR(I,J)
30 I=I+1
31 DO 101 J=1,3
32 CALL SUBROUTINE STR(I,J)
33 I=I+1
34 DO 101 J=1,3
35 CALL SUBROUTINE STR(I,J)
36 I=I+1
37 DO 101 J=1,3
38 CALL SUBROUTINE STR(I,J)
39 I=I+1
40 DO 101 J=1,3
41 CALL SUBROUTINE STR(I,J)
42 I=I+1
43 DO 101 J=1,3
44 CALL SUBROUTINE STR(I,J)
45 I=I+1
46 DO 101 J=1,3
47 CALL SUBROUTINE STR(I,J)
48 I=I+1
49 DO 101 J=1,3
50 CALL SUBROUTINE STR(I,J)
51 I=I+1
52 DO 101 J=1,3
53 CALL SUBROUTINE STR(I,J)
54 I=I+1
55 DO 101 J=1,3
56 CALL SUBROUTINE STR(I,J)
57 I=I+1
58 DO 101 J=1,3
59 CALL SUBROUTINE STR(I,J)
60 I=I+1
61 DO 101 J=1,3
62 CALL SUBROUTINE STR(I,J)
63 I=I+1
64 DO 101 J=1,3
65 CALL SUBROUTINE STR(I,J)
66 I=I+1
67 DO 101 J=1,3
68 CALL SUBROUTINE STR(I,J)
69 I=I+1
70 DO 101 J=1,3
71 CALL SUBROUTINE STR(I,J)
72 I=I+1
73 DO 101 J=1,3
74 CALL SUBROUTINE STR(I,J)
75 I=I+1
76 DO 101 J=1,3
77 CALL SUBROUTINE STR(I,J)
78 I=I+1
79 DO 101 J=1,3
80 CALL SUBROUTINE STR(I,J)
81 I=I+1
82 DO 101 J=1,3
83 CALL SUBROUTINE STR(I,J)
84 I=I+1
85 DO 101 J=1,3
86 CALL SUBROUTINE STR(I,J)
87 I=I+1
88 DO 101 J=1,3
89 CALL SUBROUTINE STR(I,J)
90 I=I+1
91 DO 101 J=1,3
92 CALL SUBROUTINE STR(I,J)
93 I=I+1
94 DO 101 J=1,3
95 CALL SUBROUTINE STR(I,J)
96 I=I+1
97 DO 101 J=1,3
98 CALL SUBROUTINE STR(I,J)
99 I=I+1
100 DO 101 J=1,3
101 CALL SUBROUTINE STR(I,J)

```

CONVERGENCE CRITERION.

IF (ABS(TOT-TOTPREV).GT.1E-6) GO TO 100


```

*****
*
* FORTRAN CODE FOR THE SATELLITE ATTITUDE CONTROLLED PROBLEM
* USING ONLY THE FLYWHEELS AS CONTROLS,
* THE SYNTHESIZING ALGORITHM IS DERIVED VIA LIE-ALGEBRAIC
* METHODS AND DEPICTED IN SECTION 5.3.2.
* HERMES' ALGORITHM (12) IS MODIFIED TO REDUCE THE REGULATION
* TIME
*
*****

```

```

COMMON/ML /X(6),U(3)
COMMON/MARIA/CX,CY,CZ,YX,YZ,MO,S6(3)

```

```

-----
X(1), X(2), X(3) ARE THE SATELLITE'S BODY RATES IN THE X, Y, Z
DIRECTIONS RESPECTIVELY.
X(4), X(5), X(6) ARE THE FLYWHEEL ANGULAR VELOCITIES IN THE
X, Y, Z DIRECTION RESPECTIVELY.
YX, YZ ARE THE MOMENTS OF INERTIA AND CX, CY, CZ, MO ARE
PARAMETERS OF THE SYSTEM.
ALFA(1-3) ARE THE COEFFICIENTS OF THE PERTURBATION RELATIVE
TO THE LIE BRACKET REFERENCE SYSTEM.
TI(1-3) ARE THE INTEGRATION TIMES AND STP(1-3) ARE THE STEPS
OF INTEGRATION.

```

```

-----
DIMENSION ALFA(3),TI(3),STP(3),UU(3),UU(3),UU(3),UU(3)
DIMENSION P(3,00,3)
REAL(5,100) (X(1),Y(1-6))
REAL(5,100) (CX,CY,CZ,YX,YZ,MO)
100 FORMAT(3E12,3,1Y1,3(E7,2,1Y))
101 FORMAT(3E12,3,1Y1,3(E7,2,1Y))
TOT=0
EP=0.005
M=50

```

```

-----
CHECKING WHETHER THE ORIGINAL PERTURBATION IS WITHIN THE
ADMISSIBLE PERTURBATION SET.

```

```

-----
F1=(-(Yz-Yy)*X(2)-Uo)*X(3)+Cx*X(5)*X(2)-Cz*X(6)*X(2))/Yx
F2=(-(Yy-Yz)*X(1)+Y(3)-Cy*X(4)*X(3)+Cz*X(6)*X(1))/Yy
F3=(-(Yx-Yz)*X(2)-Uo)*X(1)-Cx*X(5)*X(1)+Cy*X(4)*X(2))/Yz
S=S+(SQRT(F1**2+F2**2+F3**2))
IF (S.GT.EP) GO TO 102

```

```

WRITE(6-230) NIS
FORMAT(1X,F12.6-/)

```

COMPUTING THE ALPHA (α) COEFFICIENTS AND THE INTEGRATION TIMES.

```

1  ALFA(1) = X(1)*Y(1)*Z(1)
ALFA(2) = X(2)*Y(2)*Z(2)
ALFA(3) = X(3)*Y(3)*Z(3)
DO 400 I=1,3
TIME=ALFA(I)*T(1)
SG(I)=1
IF(ALFA(I).GE.0) SG(I)=1
400 CONTINUE
WRITE(6-201)(ALFA(J), J=1,3)

```

ORDERING OF THE INTEGRATION TIMES.

```

187 IF(TI(1)-TI(2)) 204,206,207
188 IF(TI(1)-TI(3)) 208,208,209
189 THAY=TI(2)
THED=TI(1)
THIN=TI(3)
GO TO 214
207 IF(TI(1)-TI(3)) 210,210,211
210 THAY=TI(3)
THED=TI(1)
THIN=TI(2)
GO TO 214
208 IF(TI(2)-TI(3)) 212,212,213
212 THIN=TI(1)
THED=TI(2)
THAY=TI(3)
GO TO 214
217 THIN=TI(1)
THED=TI(2)
THAY=TI(3)
GO TO 214
214 IF(TI(2)-TI(3)) 214,214,215
215 THAY=TI(1)
THED=TI(3)
THIN=TI(2)
GO TO 214
215 THAY=TI(1)
THED=TI(3)
THIN=TI(2)
216 TOT=TOT+THAY

```

```

WRITE(6-218) (TI(J), J=1,3),TOT
STP(1)=TMIN/M
STP(2)=(TMED-TMIN)/M
STP(3)=(TMAX-TMED)/M

```

DETERMINING THE CONTROL SEQUENCE AND PERFORMING THE NUMERICAL INTEGRATION.

```

DO 111 J=1,3
VV(J)=TI(J)-TMIN
111 MU(J)=TI(J)-TMED
DO 205 I=1,3
U(I)=0.
IF(U(I).GT.0) U(I)=1.
M(J)=1.
IF(MU(I).IF.0) M(J)=0
205 CONTINUE
I(1)=10
I(2)=15
I(3)=20
CALL RKN(STP(1),M)
U(1)=10*U(1)
U(2)=15*U(2)
U(3)=20*U(3)
CALL RKN(STP(2),M)
U(1)=10*U(1)
U(2)=15*U(2)
U(3)=20*U(3)
CALL RKN(STP(3),M)
D=SQRT(X(1)**2+X(2)**2+X(3)**2)
WRITE(6-202) (X(J), J=1,6),D
DO 401 J=1,3
401 SG(J)=-SG(J)
WRITE(6-203) (SG(J), J=1,3)
201 FORMAT(1X,3(1X,F12.6),/)
202 FORMAT(1X,7(1X,F11.6),/)
203 FORMAT(1X,3(1X,F4.1),/)
218 FORMAT(1X,4(1X,F9.4),/)

```

CONVERGENCE CRITERION.

```

IF(D.GE.0.0001) GO TO 1
DO 219 I=1,3
SG(I)=1
219 U(I)=X(I+3)/DELTA
ST=DELTA/M

```

```

CALL RKM(STP-N)
D=SQRT(X(1)**2+X(2)**2+X(3)**2)
WRITE(6,202) (X(I),I=1,6),D
WRITE(6,204) TOT
204 FORMAT(/,1X,F7.3)
102 STOP
END

```

SUBROUTINE OF INTEGRATION, RUNGE-KUTTA METHOD.

```

SUBROUTINE RKM(STP-N)
COMMON /ML/X(6),U(3)
COMMON /NENE/ F(6)
DIMENSION F1(6),F2(6),F3(6),XY(6)
DO 300 K=1,N
CALL FEU(X,U)
DO 410 I=1,6
F1(I)=F(I)
410 XY(I)=X(I)+STP*F1(I)/2
CALL FEU(XY,U)
DO 411 I=1,6
F2(I)=F(I)
411 XY(I)=X(I)+STP*F2(I)/2
CALL FEU(XY,U)
DO 412 I=1,6
F3(I)=F(I)
412 XX(I)=X(I)+STP*F3(I)
CALL FEU(XX,U)
DO 413 I=1,6
413 X(I)=X(I)+STP*(F1(I)+2*F2(I)+2*F3(I)+F(I))/6
300 CONTINUE
RETURN
END

```

SUBROUTINE FEU(Z,U)

```

COMMON /MATA/Cx,Cy,Cz,Yx,Yy,Yz,Uo,S6(3)
COMMON /NENE/ F(6)

```

DIMENSION Z(5),U(3)

```

F(1)=-((Yz-Yx)*(Z(2)-Uo)*Z(3)+Cx*Z(5)*Z(2)-Cz*Z(4)*Z(2)+S6(1)*Cy*U(1)
*U(1))/Yx
F(2)=-((Yx-Yz)*Z(1)*Z(3)-Cx*Z(4)*Z(3)+Cz*Z(1)*Z(1)+S6(2)*Cy*U(1)*U(1)
*/Yy
F(3)=-((Yy-Yx)*(Z(2)-Uo)*Z(1)-Cy*Z(5)*Z(1)+Cz*Z(4)*Z(2)+S6(3)*Cz*U(1)
*U(1))/Yz
F(4)=-S6(1)*U(1)

```

```

F(6)=-S6(3)*U(3)
RETURN
END

```

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