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LA THÈSE A ÉTÉ
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Application of the Galerkin method to Finite Deflection
Buckling and Vibration of Isotropic and Skewed Sandwich
Plates

by

Badal Das

A thesis
presented to the University of Ottawa
in partial fulfillment of the
requirements for the degree of
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in
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ABSTRACT

Variational methods are widely used for the solution of complex differential equations in mechanics, for which exact solutions are not possible. The Galerkin method, although known to be the most rapidly converging was applied in the past only for the solution of linear differential equations in applied mechanics. In the present study, the suitability of the method for the solution of large deflection, stability and free vibration analysis of plates is studied. The method is first applied to investigate these behaviours for clamped rectangular, isotropic homogeneous plates. After the validity of the method for the solution of these problems is established, it is then extended for the solution of similar problems for clamped skew sandwich plates. The governing differential equations of sandwich plates in terms of displacements in rectangular coordinates are first established and then transformed into skew coordinates. Whenever possible results of the present method are compared with the existing solutions in the technical literature obtained by much more laborious methods and close agreements are found. The effects of skew and shear rigidity of the core on the behaviour of sandwich plates are also discussed. It is shown that the stiffness of sandwich plates

tends to increase with increase in the angle of skew and increase in the shear rigidity of the core. The parameters considered for the analysis of the problems of sandwich plates are the aspect ratio of the plates, poisson's ratio, skew angle and various shear rigidities of the core. To determine the deflection characteristics of sandwich plate on elastic foundation various foundation moduli were also taken into consideration.

The definite integrals involved in the formation of the Galerkin algebraic equations from the governing differential equations were evaluated by using the trapezoidal rule. The nonlinear algebraic equation obtained in the large deflection analysis are solved by the Newton-Raphson iterative procedure and the algebraic equations obtained in the eigen value problems are solved by using IMSL subroutine EIGZF.

Simplicity in formulation and quick convergence are the obvious advantages of the method found in comparison with other numerical methods requiring extensive computer facilities.

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List of Symbols

x, y, z	rectangular cartesian coordinates
α, β	oblique coordinates
u, v, w	effective displacements in the x, y and z direction
θ_x, θ_y	effective change of slope of the normal to the undeformed middle surface in x and y direction respectively
$\sigma_x, \sigma_y, \sigma_z$	normal stresses
$\tau_x, \tau_y, \tau_{xy}$	shear stresses
$\epsilon, \epsilon_x, \epsilon_y, \epsilon_z$	normal strains
$\gamma, \gamma_x, \gamma_y, \gamma_z$	shear strains
E	modulus of elasticity of the isotropic material
G	shear modulus of elasticity of the isotropic material
ν	Poisson's ratio of the isotropic material
h	plate thickness
D, D'	flexural rigidity of the plate for isotropic plate
	$D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{for sandwich plate}$
	$D = \frac{th^2 E_f}{2(1-\nu^2)}, \quad D' = \left(\frac{t^3}{6} + \frac{th^2}{2} \right) \frac{E_f}{(1-\nu^2)}$
q, p	lateral load per unit area
$2a, 2b$	dimensions of the plate in x and y direction respectively

λ	aspect ratio of the plate, (a/b)
U, V, W	dimensionless displacements in the x, y and z direction respectively
K	dimensionless foundation modulus
Q	dimensionless load parameter
ξ, η	dimensionless parameters in directional coordinates for rectangular plate $\xi = x/a, \eta = y/b$ for skew plate $\xi = \alpha/a, \eta = \beta/b$
N_x, N_y	normal forces in x and y directions
N_{xy}	shear force parallel to x y plane
M_x, M_y	bending moments in the x and y direction
M_{xy}	torsional moment
Q_x, Q_y	transverse shear force per unit run in the x z plane and y z planes respectively
∇	Laplacian operator
H	twisting moment in the sandwich plate, $(S_+ - S_-)h/2$
S	shearing force parallel to the plane of the plate
V_x, V_y	shear stress resultants
t	face thickness of the sandwich plate
E_f	modulus of elasticity of the face layer of the sandwich plate
E_c	modulus of elasticity of the core layer of the sandwich plate
G_f	shear modulus of elasticity of the face layer of the sandwich plate
G_c	shear modulus of elasticity of the core layer of the sandwich plate

$(h-t)$	thickness of the core of the sandwich plate
ν_f	Poisson's ratio of the face layer of the sandwich plate
ν_c	Poisson's ratio of the core layer of the sandwich plate
F	Airy's stress function
$\bar{\sigma}_z$	the effective transverse normal stress
θ	dimensionless ratio, h/a
μ	dimensionless ratio, t/a
e	effective transverse normal strain for the sandwich plate, $(W_+ - W_-)/h$
M_x	$(N_{x+} - N_{x-})h/2$
M_y	$(N_{y+} - N_{y-})h/2$
N_x	$N_{x+} + N_{x-}$
N_y	$N_{y+} + N_{y-}$
P	$P_+ + P_-$
q	$(p_+ - p_-)/2$
S	$S_+ + S_-$
u	$(u_+ + u_-)/2$
v	$(v_+ + v_-)/2$
w	$(w_+ + w_-)/2$
θ_x	$(u_+ - u_-)/h$
θ_y	$(v_+ - v_-)/h$
$\bar{\sigma}_z$	$(\sigma_{z+} + \sigma_{z-})/2$
λ^n	eigenvalue corresponding to the n^{th} mode of vibration

ρ	mass per unit surface area of the plate
μ	dimensionless parameter, $\frac{thE_f}{2a^2(1-\nu^2)G_c}$
U_1	G_c , transverse shear stiffness
α	phase angle

Note:

- 1) Partial differentiation is denoted by a comma in subscript.
- 2) Subscripts (+) and (-) are quantities referring to the upper and lower membranes of sandwich plates where $z = \pm h/2$.
- 3) All symbols not given here are defined immediately following the formula in which they first appear.

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Chapter I

INTRODUCTION

Skewed plates are often used as components of large scale structures such as triangular dams and building floor systems. In the field of reinforced concrete such plates are also of considerable interest and practical importance, especially in the case of skewed slabs for highway bridges that cross rivers, railways or other highways at an oblique angle. In general, the deflection of skewed plates in the aforementioned structures are generally small with respect to their thickness and hence for the design of such plates a linear analysis is sufficient. In contrast, the requirements of skewed plates or panels for the aircraft industry are different. In the case of design of swept wings and skin panels for aircrafts, for example, the weight is of primary importance; for those cases when skewed plates are used for their component parts, they must be thin and as a result their deflections are usually quite large in comparison with the plate thickness. Due to the high strength to weight ratio sandwich plates gained increasing popularity in the aerospace industry. Hence in order to obtain design charts

for skew sandwich plates used for various components in the aerospace industry, the nonlinear analysis (large deflection) must be used.

Very often specification merely ensuring that plates withstand applied lateral loads is not a sufficient criteria for design. Rather, the designer must also keep in mind that thin plates may, under certain circumstances, buckle under the action of inplane forces or vibrate excessively due to possible resonance caused due to the matching of the driving periodic force and the plate's natural frequencies. To avoid buckling and the resonance of a plate, a buckling and a free vibration analysis would also be required.

In contrast to the square and rectangular plates, skewed plates have not received nearly as much attention. This may perhaps due to its relatively difficult mathematical model and its absence of orthogonal relationships.

The fundamental problems of mechanics are governed by both differential equations and minimum principles (i.e. in the equilibrium condition the potential energy of the mechanical system is minimum). The governing differential equations and the equations led by minimum principles are usually rather complex and do not lend themselves for easy exact solutions. Confronted with this problem researchers frequently have to resort to numerical methods to effect a solution. Variational methods were exploited by engineers and scientists as a very effective tool for solving their

problems. These methods gained popularity in recent years due to the development of high speed digital computers.

Galerkin method although known to be the most rapidly converging variational method was not used extensively for solving highly complex differential equations. This was mainly due to the fact that it involves a process of definite integrations. This method can be considered as a means for the approximate solutions of differential equations. According to the method the substitution of the assumed displacement functions satisfying the boundary conditions of the problem, into the governing differential equation results in an error function ϵ . Then this method consists in choosing n coefficients in the displacement function in such a manner that n distinct weighted means of the error, taken throughout a certain range of representation be zero. In the mechanical application this can be interpreted as a generalised force and the multipliers used to weight the errors are the virtual displacements corresponding to increments of each of the generalised coordinates in turn. Thus the vanishing of the weighted means can be interpreted as the vanishing of the virtual work in the appropriate displacements.

In the present study while assuming the approximate displacement functions, care was taken to ensure that these functions satisfy the boundary conditions of the problem. For the case of skew sandwich plates care was taken to

ensure that the assumed functions for the displacements also satisfy the condition of polar symmetry.

OBJECT AND SCOPE

The main objective of the thesis is to study the applicability of the Galerkin method to large deflection and eigen-value problems. To study the applicability of the method, the problem of large deflection, stability and vibration of the clamped homogeneous isotropic plates is first studied. After the validity of the method has been established the same method is then extended to analyse the corresponding problems of skew sandwich plates. The large deflection analysis of the skew sandwich plate and the homogeneous plate on elastic foundations are also studied.

OUTLINE OF THE THESIS

Since the main objective of the thesis is to study the applicability of the Galerkin method for large deflection, stability and vibration of homogeneous plates and sandwich plates, existing literature related to these types of plates is briefly reviewed in chapter 2. Chapter 3 is devoted to the formulation of the Galerkin algebraic equations from the

governing differential equations. Since the process of the Galerkin method entails the evaluation of the definite integrals, the trapezoidal rule used in carrying out the definite integrals is also discussed in the same chapter.

In chapter 4 the Galerkin method is used in the formulation of the large deflection problem of clamped isotropic rectangular homogeneous plates, due to lateral loads with and without elastic foundations. The same formulation was then extended for the skew sandwich plates. The nonlinear algebraic equations obtained for both the isotropic and skew sandwich plates were solved by using the Newton-Raphson iterative procedure. For the linear analysis of plates the load was kept small. This also gave an opportunity to check the governing differential equations used for the analysis.

In chapter 5, the Galerkin method is used to solve the buckling problem of clamped rectangular isotropic homogeneous plates and clamped skew sandwich plates. In chapter 6 this method is further applied for the free vibration analysis of both the isotropic and skew sandwich plates. The Galerkin method used in chapter 5 and 6 leads to eigen value equations which were then solved by using the IMSL subroutine EIGZF.

Finally in the last chapter conclusions drawn in the present study are summarized.

For all cases considered in this study, whenever possible, comparisons are made with other investigators. All computations involved in this thesis were programmed in fortran IV for the Amdahl 470. The programs are included in the appendix.

The page contains several hand-drawn scribbles. A large, irregular loop is drawn in the upper center. To the right of this, there is a horizontal line with a small loop at its left end. In the lower left, there is a small, wavy horizontal line. Near the bottom center, there is a small, curved mark. At the very bottom, there is a small, dark, irregular shape.

Chapter II

LITERATURE REVIEW

The governing differential equations for the large deflection analysis of isotropic plates were first obtained by Von Karman[56] in 1910. In 1940, Rostovsev[49] modified the Von Karman equations and obtained the governing differential equations for the case of large deflection of orthotropic plates. According to A.E.H Love, James Bernoulli appears to have attempted to discover a theoretical basis for certain nodal figures for vibrating plates which was reported by E.F.F Chaldni in a paper published at Leipzig in 1787. Mdlle Sophie Germain in 1821 gave the equation for flexural vibration of isotropic plates in the form which is now accepted.

Sandwich plate is a three layered plate consisting of two thin sheets of high strength material between which a low average strength and density material is sandwiched. Sandwich construction gained considerable attention especially in the aerospace industry after World War II because of its light weight, high stiffness to weight ratio and good thermal and accoustical properties.

The theories of sandwich plates were analysed by several authors. The early theory of William, et al[61] accounts for the transverse shear effects in the core of the sandwich plate by assuming that a linear element initially straight and normal to the middle plane of the core will remain straight after deformation but will deviate from the normal to the deformed middle plane by an amount represented by a parameter, additional parameters being necessary for orthotropic structures of dissimilar facings. This tilting method was further used by March and Ericksen in their analysis of sandwich plates [8,9].

Subsequent theory of Libove and Batdroff[31] and Reissner[46] led to the deflection equations of sandwich plates on the basis of a simplified model. This consists in assuming the core in a state of antiplane stress. Reissner[47] noted that with such simplifying assumptions the equilibrium equations for the sandwich plates have the same form as the equations for homogeneous plates with transverse shear deformations taken into account. Various effects previously omitted were incorporated by Eringen[11] who generalized the variational approach by Hoff[21]. The theory then accounts for the flexural rigidity as well as transverse deformation of the core, including the flexural rigidities of the two facings about their own middle planes.

In 1948 Reissner[48] realizing the inadequacy of the linear small deflection theory in the treatment of sandwich

panels with a relatively soft core derived a system of differential equations for the finite deflection of isotropic sandwich plates. This derivation can be interpreted as an extension of the classical Von Karman theory[56] by the inclusion of transverse shear deformation and curvature effects. In this derivation, Reissner used a simplified model consisting of two facings acting as membranes and a core resisting transverse shear and normal stresses. His differential equations were later verified by Wang[58] using the method of complementary energy. In 1950 Hoff[21] utilized the energy principle in conjunction with the calculus of variations to derive a set of differential equations for bending and buckling of sandwich plates..

As noted in the survey by Lui M Habib[19] the buckling of sandwich plates was investigated by several authors for various types of loading conditions. A system of equations that accounts for both membrane and plate behaviour of the facings in general buckling was given by Eringen[12] and Gerard[14]. The local compressive buckling of sandwich construction with wrinkling of the facings was noted by Gough et al[17] and further investigated among others by Cox[5] and Goodier[15]. Goodier and Neou[16] in the process of general evaluation of the problem of local buckling observed that the critical wrinkling stresses is not affected by the compressive stress in the core or an increase in the core thickness.

Yu[64,65] investigated the simple thickness shear mode of vibration of sandwich plates on the basis of a compressive sandwich theory. A shear coefficient was introduced and determined by matching the lower frequency for the antisymmetrical simple thickness shear mode calculated from the sandwich plate theory with the corresponding fundamental frequency from the elastic theory. This coefficient appears to vary and is close to unity in magnitude. This theory assumes the transverse normal displacement as constant throughout the thickness of the plate, while the displacements in the plane of the plate are assumed to vary linearly through the thickness, with the slopes in the facings equal but different from that in the core. The theory includes the effects of transverse shear deformation and both the rotatory and translatory inertias of the core and facings, the flexural rigidities of the core and the flexural and extensional rigidities of the facings. On the basis of this theory the free flexural vibrations of sandwich plates were investigated by Yu[66] and the relative importance of these special effects was discussed in detail. In comparison to homogeneous plates, the transverse shear effects were found to be relatively important, and the frequency of the simple thickness shear mode much lower and closer to the first branch of the frequency curve. It was also noted that except for plates with very thin facings over low frequency ranges the flexural rigidities of the facings about their own middle planes should be included.

In 1961, Falgout[13] derived the differential equations for free vibration of sandwich plates with isotropic facings and core by superposing bending deflections and deflections due to transverse shear. The flexural vibration of sandwich plates was also considered by Mindlin[36] neglecting the flexure and transverse shear deformations of the facings and their rotatory inertia about their own middle planes. The resulting equations then have the same form, except for constants, as those for flexural motions of homogeneous plates with rotatory inertia and shear deformation taken into account.

For the sandwich plates, the shear effects in the facings, the rotatory inertia of the facings about their own middle planes, and the flexural rigidity of the core were judged to be negligible by Yu[67]. The most important effects in this simplified model were the shear effect in the core, the rotatory and translatory inertias of the core, the translatory inertias of the facings and its rotatory effects about the middle plane of the sandwich plate and the joint effect of the flexural and extensional rigidities of the facings. In a further study Yu[68] showed that, in the case of an isotropic sandwich plate with similar facings, the basic system of equations can be separated into two independent set of equations one of which yields a frequency equation which is a generalization of plane strain vibration, while the other leads to the modes independent of the transverse deflection.

Yu[69] also studied the nonlinear flexural vibration of sandwich plates. The nonlinear equations of motion and boundary conditions were derived from a simplified version of the equations of motion of the nonlinear theory of elasticity by means of a variational procedure. The flexural rigidities of the facings were neglected but the transverse shear effect of the core was considered. The free vibration of a sandwich plate in plane strain with hinged edges was considered and a solution was found in terms of an elliptic integral.

An attempt is made here to review some of the research works which employ numerical schemes as a method of solution:

Ritz Method : Way[59] first used the Ritz method for the solution of large deflection of uniformly loaded clamped homogeneous plates. Weil and Newmark[60] used the same method for the large deflection of clamped elliptical plates. Ku[26] used the same method for the analysis of small deflection problems of clamped skew plates on elastic foundation subjected to uniformly distributed and concentrated loads. March[33] and Ericksen[10] used the same method to solve the small deflection problem of clamped rectangular sandwich plates.

Lord Rayleigh[32] in his classic work gives a method for the approximation of frequencies of the dynamical systems. Later W. Ritz[57] produced what is known as the Raleigh Ritz method for approximating frequencies in the vibrating system. Ritz applied his technique to the square homogeneous plates with all edges free. D Young[6] used the Raleigh Ritz method in his work on square homogeneous plates, with all edges clamped, cantilevered plate with two adjacent edges free and two adjacent edges clamped. The Raleigh Ritz method was found to give estimates of the frequencies which are higher than the actual values.

Energy Method: Hoff[21] utilized the principle of minimum potential energy in conjunction with the calculus of variations to solve the bending and buckling problem subjected to edgewise compression. Alwan[1] used the variational principle of complementary energy to analyze the problem of nonlinear deflection of rectangular sandwich plates with isotropic faces and orthotropic cores. The effect of transverse normal stress deformation of the core was neglected. The energy principle was used by Ueng[54] in conjunction with the Langrange's principle to obtain the upper and lower bounds of the natural frequencies of an all clamped rectangular sandwich panel. Raville and Ueng[55] presented the natural frequencies for free vibrations of a

simply supported sandwich plate. Experimental results were found in close agreement with the theoretical results.

Galerkin Method: The Galerkin method was applied to the small deflection of clamped homogeneous plates of various planform on elastic foundations by Ng[40]. This method was used by Bolton[2] on the large deflection problem of circular homogeneous plates with various boundary conditions. Munakata[38], using this method studied the free vibration and buckling problem of a clamped rectangular plate.

Series Solution: Fourier series method proves to be extremely powerful for the case of rectangular plates with simply supported edges. In 1820 Navier presented a paper to the French Academy of Sciences on the solution of the small deflection problem of simply supported rectangular plates by double Fourier series. Levy [29], using a similar method also solved the corresponding large deflection problem. Using the Fourier series solution Yen[62] solved the small deflection problem of a simply supported rectangular sandwich plate. Pohlie [45] also used the same method and solved the static problem of an orthotropic sandwich plate. Alwan[1] was the first to investigate the large deflection problem of sandwich plates by means of the Fourier series.

Kennedy[24] traced Taylor's[51] work and investigated the small deflection problem of clamped skew sandwich plate subjected to uniformly distributed load by introducing the idea of treating the partial deflections due to bending and shear deformation separately. A series representation was used for the assumed lateral displacement.

Finite Element Method: This method gained popularity due to improvement in computing facilities in last two decades. Among others, Haskell[20] and Melliere[35] presented solutions for the large deflection of rectangular homogeneous plate with various boundary conditions.

Kwok[27] and Monforton[37] used the finite element method for the static analysis of skew sandwich plates.

In using this method quite a large number of finite elements are often required to obtain sufficiently accurate answers. In the case of large deflection problems the solution often involves cycles of iterations. Consequently, the use of the finite element method has been found to be rather inefficient for the analysis of large deflection problems of homogeneous and orthotropic plates.

Perturbation Method: This method was applied by many investigators to the large deflection problem of a variety

of uniformly loaded clamped plates. The method was first used by Chien[4] in analysing a clamped circular plate subjected to uniform pressure. Subsequent works utilizing this method are those by Chan[3], Kennedy and Ng[25], Nash[39] and Ng[41]

Kan[22] used this method for the solution of large deflection problem of uniformly loaded clamped rectangular sandwich plates. Ng[42] also utilized the same method to obtain solutions for large deflection problem of circular and elliptical sandwich plates on elastic foundations.

Chapter III

THE GALERKIN METHOD

3.1 INTRODUCTORY COMMENTS

The fundamental problems of mechanics are known to be governed by both the differential equation and by so called minimum principles. In view of this the problem of solving the boundary value problem generally turns out to be equivalent to the problem of finding the function giving the minimum of the 'integral' by which the potential energy is expressed. The method of reducing the problem of solving a differential equation to the equivalent problem of seeking an approximate function to give a minimum value of an 'integral' is called the variational method.

In the theory of elasticity, the variational principle introduced by La-grange is based on the virtual work theorem. The objective of the variational methods is to find from a group of admissible functions those which represent the deflections of the elastic body pertinent to its stable equilibrium conditions.

In 1915 B.G. Galerkin, a Russian investigator described his method for solving differential equations in his treatise 'Rods and Plates' (Vesterik, Ingeneroff, 1915, p.897)[7]. The detailed discussion of this method is given in the following section. In his method Galerkin generalized and simplified the virtual work principle, which states: the virtual work of internal and external forces must vanish. It was pointed out by E.P.Grossman[18] that the Galerkin's method in application to mechanics leads to the same result as the La-grange's principle of virtual work but employs a special coordinate system[7].

After the publication of Galerkin's method large amount of work has been published in which the method was used extensively for the practical solutions of extremely diverse types of problems.

The variational methods were exploited by the engineers and scientists as a very effective tool for the solution of their problems. These methods have experienced tumultuous progress from the time of their origin. The appearance of new methods such as the finite element method and recent modifications of some classical methods were mainly due to rapid development of computers whose exploitation enabled relatively fast solutions of equations containing large number of unknowns.

3.2 GALERKIN METHOD

The Galerkin method belongs to the same general class as the Ritz method for it seeks to obtain an approximate solution of the differential equations with given boundary conditions by choosing functions which satisfy the boundary conditions exactly and then proceeds to specialize the chosen functions in such a way as to secure approximate satisfaction of the differential equations. The basic idea of this method can be expounded quite briefly [23]. Let it be required to determine the solution of the equation;

$$L(U) = 0 \quad (3.2.1)$$

where L is the differential operator whose solution U satisfies the homogeneous boundary condition i.e. the solution U is identically equal to zero both on the boundary and at all points of the domain 'R' sufficiently close to the boundary.

We shall seek an approximate solution of the problem in the form;

$$\bar{U}(x,y) = \sum_{i=1}^n c_i \phi_i(x,y) \quad (3.2.2)$$

where, $\phi_i(x,y), i=1,2,3 - - - - n$, is a system of functions chosen before hand satisfying the boundary condition of the problem and C_i are the undetermined coefficients. We can always consider a function $\phi_i(x,y)$ to be linearly independent and complete in the given region. By

linear independent we mean none of the functions can be expressed as a linear combination of the rest $(n-1)$ functions. By completeness we mean, it is possible to find a real number n for which the norm of the function $(U-\bar{U})$ is less than a small number ϵ ; as

$$\left[\iint_R \{U - (c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n)\}^2 \right]^{1/2} < \epsilon \quad (3.2.3)$$

In order that $\bar{U}$ be the exact solution of the given equation (3.2.1) it is necessary that $L(\bar{U})$ be identically equal to zero.

This requirement and if $L(\bar{U})$ is considered to be continuous at every point of its domain, i.e

$$||L\bar{U} - LU|| = [\iint (L\bar{U} - LU)(L\bar{U} - LU)]^{0.5} \stackrel{n \rightarrow \infty}{=} 0 \quad (3.2.4)$$

is equivalent to the requirement of orthogonality of the expression $L(\bar{U})$ to all the functions $\phi_i(x,y)$, $i=1,2,\dots,n$ of the system.

Since we choose only n constants C_i , $i=1,2,\dots,n$ we can only in general satisfy n conditions of orthogonality.

Stating this condition we arrive at the system of equations

$$\iint_R L\left(\sum_{j=1}^n c_j \phi_j(x,y)\right) \phi_i(x,y) dx dy = 0 \quad (3.2.5)$$

where $i=1,2,\dots,n$

These n equations yield the n unknown coefficients C_i . Whether or not these equations would be linear depends on the operator L . If L is linear these equations are linear and if L is nonlinear these equations would also be nonlinear.

In mechanical applications $L(U)$ is obtained from the equilibrium condition of an infinitesimal element. The equilibrium of the structural system is obtained by integrating the differential equation over the entire domain of the structure. So $L(\bar{U})$ can be interpreted as a generalized force and $\phi_i(x,y)$, the multipliers, are the virtual displacements and $\bar{U}$ as the displacement. Although $\phi_i(x,y)$, components of $\bar{U}$ are interrelated its arbitrary variations are not interrelated. Thus the condition " $L(\bar{U})$ should be orthogonal to each of the functions $\phi_i(x,y)$ " can be interpreted as the virtual work requiring the appropriate displacement be zero. Hence the virtual work of the external and internal forces are obtained directly from the differential equations of the equilibrium without determining the actual potential energy of the system. So this method can be considered as a perfectly universal method. It can be applied to differential equations of diverse types, elliptical, hyperbolic, parabolic etc, even though they are utterly unconnected with the variational problem[23]. There is probably scarcely a mechanical problem concerning elastic or other continuously deformable bodies to which this method cannot be applied with success.

But the only drawback of this method is that it requires the governing differential equations of the mechanical problem to be solved. Moreover, this process involves the evaluation of the definite integrals to form the equations for the evaluation of the constants C_i in the assumed displacement functions.

Fortunately the governing differential equations of most of the structural problems are readily available. Also, with the rapid development of the computers the numerical integration is not tedious any more. In the present study the numerical integrations are carried out by the use of trapezoidal rule described below.

In the trapezoidal rule the original function is approximated by a set of straight lines. The region to be integrated is divided into uniformly spaced sections or panels. The panel width $dx=(b-a)/n$, where a and b are the limit of the integration and n is the number of panels.

In the case of a one dimensional function $f(x)$, if the integral is fitted by a single straight line i.e if there is only one panel as illustrated in figure 1, then the calculated area is $(b-a)(f(a)+f(b))/2$.

In the above equation $f(a)$ is the value of the function at the extreme left $f(b)$ is the value at the extreme right. For the more general case the area is divided into n panels. The panels can be numbered 1 to n . Thus there are $n+1$ edges which can be numbered 0 through n .

The area of the first panel = $(f(a)+f(1))dx/2$

The area of the last panel = $(f(n-1)+f(b))dx/2$

The area of the i th panel = $(f(i-1)+f(i))dx/2$

Hence the total area = sum of the area of each panel

$$= (f(a) + f(b) + \sum_{i=1}^{n-1} f(i))dx/2$$

Similarly if f is a two dimensional function then the value of the integral is the volume under the surface defined by the function $f(x,y)$. The axes are divided into n number of panels. Let us take a simple case where the plane is divided into four panels as in figure 2.

The volume over area $A_1 = (f(1)+f(2)+f(4)+f(5))dxdy/4$

The volume over area $A_2 = (f(4)+f(5)+f(7)+f(8))dxdy/4$

The volume over area $A_3 = (f(2)+f(3)+f(5)+f(6))dxdy/4$

The volume over area $A_4 = (f(5)+f(6)+f(8)+f(9))dxdy/4$

Hence the total volume = $1/4 \sum f(\text{corner points})$
 $+ 1/2 \sum f(\text{boundary points except the corners})$
 $+ \sum f(\text{interior points})$

Throughout the study this technique is used for evaluating definite integrals and it has been found that the technique does not require extensive computer time or memory even for a very fine mesh grid.

3.3 GALERKIN METHOD IN EIGEN VALUE PROBLEMS

For the eigen value problem Galerkin method assumes a solution in the form of a series of n functions for the deflected shape of the vibrating or buckled elastic body satisfying all the boundary conditions. In general, the series solution will not satisfy the differential equation defining the eigen value problem unless by some coincidence the series is composed of eigen functions. Substituting the series function into the differential equation an error function is obtained. For the similar argument given in the previous section Galerkin method insists that this error function be orthogonal to each of the independent weighting functions. The weighting functions are exactly the n functions chosen for the deflected shape of the problem.

Let us consider the eigen value problem,

$$L[W] = \lambda M[W] \quad (3.3.1)$$

where L and M are the differential operators. The function W is subject to the boundary condition which in general do not depend on the eigen values λ . We assume a solution of the eigenvalue problem in the form:

$$W_n = \sum_{j=1}^n a_j U_j \quad (3.3.2)$$

where a_j are the coefficients to be determined and U_j are the functions satisfying the boundary conditions.

Substituting (3.3.2) in (3.3.1) we get the error function so that

$$\epsilon = L[W_n] - \lambda^n M[W_n]$$

where λ^n is the corresponding estimate of the eigen value.

The condition that each of the weighted function U_j be orthogonal to ϵ leads to;

$$\begin{aligned} \int \epsilon U_r dD &= 0 \\ \text{or, } \sum_{j=1}^n \left[\int U_r L(U_j) dD - \lambda^n \int U_r M(U_j) dD \right] &= 0 \end{aligned} \quad (3.3.3)$$

where $r=1,2,\dots,n$ and D is the domain of the problem.

These are called Galerkin equations and represent an eigen value problem for an n degrees of freedom system.

Chapter IV

LARGE DEFLECTION ANALYSIS OF CLAMPED, ISOTROPIC HOMOGENEOUS AND SKEW SANDWICH PLATES

4.1 FORMULATION OF THE LARGE DEFLECTION PROBLEM OF HOMOGENEOUS PLATE

DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATIONS

The classical theory of the bending of a thin elastic plate expresses the relation between the transverse deflection of the middle surface of the plate w and the lateral loading of intensity q by the equation,

$$D \nabla^4 w = q \quad (4.1.1)$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate and $\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

It is well known that this theory is restricted in application for, on the one hand, its basic assumptions can be questioned unless the plate is thin, and on the other hand it neglects an effect which must be appreciable when w

has values comparable with the thickness. This is the membrane effect of the curvature, whereby tension or compression in the middle surface tends to oppose or to reinforce q . The effect is negligible when w is very small, provided no stresses act initially in the plane of the middle surface; but even so, it operates when w is small because stretching the middle surface is a necessary consequence of the transverse deflection. When the deflection gets larger and larger the membrane effect becomes more and more prominent until for very large values of w the membrane effect is predominant whereas the bending stiffness is comparatively negligible.

Small transverse deflections of thin plates are governed by a single linear equation but for the deflection range in the neighbourhood of $1/2$ the plate thickness significant amount of stretching of the middle surface of the plate will take place resulting in a much stiffer plate and hence the design of plates employing the linear theory can be over conservative. Large deflection entails necessarily the stretching of the plate middle surface, resulting in tensile forces interacting with the curvature.

BASIC ASSUMPTIONS

- (1) Points which lie on a normal to the mid-plane of the undeflected plate lie on a normal to the midplane of the deflected plate.

- (2) Stresses normal to the mid-plane of the plate arising from the applied loading are negligible in comparison with the stresses in the plane of the plate.
- (3) The slope of the deflected plate in any direction is small so that its square may be neglected in comparison with unity.
- (4) The mid-plane of the plate is a neutral plane i.e. any mid plane stresses arising from the deflection of the plate are ignored. (This assumption is valid only for small deflection theory)

Let us consider the equilibrium of a small element cut out from the middle plane of the plate with sides dx and dy . It is assumed that u, v, w are the displacements in x, y and z directions respectively.

If N_x and N_y are the inplane forces per unit length of the plate, neglecting body forces the equilibrium of the plate element in the x and y directions yields respectively (fig. 3),

$$N_{x,x} + N_{xy,y} = 0 \quad (4.1.2)$$

$$N_{xy,x} + N_{y,y} = 0 \quad (4.1.3)$$

Secondly equilibrium of the plate element in the z direction;

(a) Forces in the z direction due to inplane forces:

From figure 3 it can also be seen that the net contribution of the inplane forces N_x, N_y, N_{xy} in the plate element (taking downward positive) is;

$$(N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) dx dy \quad (4.1.4)$$

(b) Forces in the z directions due to lateral loads;

Let Q_x, Q_y be the shear force per unit length. Also shown on figure 4 are the directions of the bending and twisting moments acting per unit length, q is the intensity of the distributed load. Writing the equilibrium equation of all the elements in the downward direction we get;

$$Q_{x,x} dx dy + Q_{y,y} dx dy + q dx dy = 0$$

or, $Q_{x,x} + Q_{y,y} + q = 0 \quad (4.1.5)$

Also taking moments about the x axis we obtain;

$$Q_y - M_{y,y} + M_{xy,x} = 0 \quad (4.1.6)$$

The moments of the load q and the moment due to change of the force Q_y are neglected in this equation, since they are small quantities of higher order than those retained.

Similarly by taking moments about the Y axis we obtain

$$M_{x,x} - M_{xy,y} - Q_x = 0 \quad (4.1.7)$$

Further by substitution the well known moment and curvature relationships;

$$M_x = -D(w_{,xx} + \nu w_{,yy})$$

$$M_y = -D(w_{,yy} + \nu w_{,xx})$$

$$M_{xy} = D(1-\nu)w_{,xy}$$

into equations (4.1.6) and (4.1.7) we get,

$$Q_y = -D(w_{,yyy} + \nu w_{,xxy}) \quad (4.1.8)$$

similarly,

$$Q_x = -D(w_{,xxx} + \nu w_{,xyy}) \quad (4.1.9)$$

Substituting equations (4.1.8) and (4.1.9) into equation (4.1.5) we get;

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = q/D \quad (4.1.10)$$

which is the well known expression for the analysis of the small deflection of plates.

If we now add to this equation of equilibrium in the vertical direction due to lateral loads, the effects due to the inplane forces i.e equation (4.1.4) we get;

$$D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = q + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} \quad (4.1.11)$$

In terms of stresses equations (4.1.2), (4.1.3) and (4.1.11) can be written as;

$$\sigma_{x,x} + \tau_{xy,y} = 0 \quad (4.1.12)$$

$$\sigma_{y,y} + \tau_{xy,x} = 0 \quad (4.1.13)$$

$$D\nabla^4 w = q + h(\sigma_x w_{,xx} + \sigma_y w_{,yy} + 2\tau_{xy} w_{,xy}) \quad (4.1.14)$$

From the theory of elasticity the equations of plane strain are;

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy} \end{aligned} \quad (4.1.15)$$

and the equations of compatibility are;

$$\begin{aligned} \epsilon_x &= u_{,x} + \frac{1}{2}(w_{,x})^2 \\ \epsilon_y &= v_{,y} + \frac{1}{2}(w_{,y})^2 \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x} w_{,y} \end{aligned} \quad (4.1.16)$$

Substituting equations (4.1.15) and (4.1.16) into equations (4.1.12), (4.1.13) and (4.1.14), the three general nonlinear

and coupled differential equations in rectangular coordinates governing the large deflection behaviour of thin elastic plates are obtained.

$$u_{,xx} + w_{,x} w_{,xx} + v(v_{,xy} + w_{,y} + w_{,xy}) + \frac{1}{2} (1-v)(u_{,yy} + v_{,xy} + w_{,x}(w_{,y})^2 + w_{,y} w_{,xy}) = 0 \quad (4.1.17)$$

$$v_{,yy} + w_{,y} v_{,yy} + v(u_{,xy} + w_{,x} w_{,xy}) + \frac{1}{2} (1-v)(v_{,xx} + u_{,xy} + w_{,y} w_{,xx} + w_{,x} w_{,xy}) = 0 \quad (4.1.18)$$

$$\begin{aligned} D \nabla^2 \nabla^2 w = q + h \left[\frac{E}{(1-v)^2} \left\{ u_{,x} + \frac{1}{2} (w_{,x})^2 + v \left[v_{,y} + \frac{1}{2} (w_{,y})^2 \right] \right\} w_{,xx} \right. \\ \left. + \frac{E}{(1-v)^2} \left\{ v_{,y} + \frac{1}{2} (w_{,y})^2 + v \left[u_{,x} + \frac{1}{2} (w_{,x})^2 \right] \right\} w_{,yy} \right. \\ \left. + \frac{E}{1+v} \left\{ (u_{,y} + v_{,x} w_{,x} w_{,y}) w_{,xy} \right\} \right] \end{aligned} \quad (4.1.19)$$

where, ∇ is the Laplacian operator, ν being the Poisson's ratio; D , flexural rigidity; E , the modulus of elasticity of the plate material; q , the intensity of the uniformly distributed load and h is the thickness of the plate. The comma notation signifies differentiation.

For the ease of computation these equations are again transformed into dimensionless form by using the following dimensionless ratios.

$$\begin{aligned} \lambda = a/b \quad \xi = x/a \quad \eta = y/b \quad U = au/h^2 \\ V = av/h^2 \quad W = w/h \quad Q = qa^4/Dh \end{aligned} \quad (4.1.20)$$

where $2a$, $2b$ are the lengths of the plate in x and y directions respectively.

In the dimensionless form the equations are:

$$2U_{,\xi\xi} + (1+\nu)\lambda V_{,\xi\eta} + (1-\nu)\lambda^2 U_{,\eta\eta} + 2W_{,\xi}W_{,\xi\xi} + (1-\nu)\lambda^2 W_{,\xi}W_{,\eta\eta} + (1+\nu)\lambda^2 W_{,\eta}W_{,\xi\eta} = 0 \quad (4.1.21)$$

$$(1-\nu)V_{,\xi\xi} + (1+\nu)\lambda U_{,\xi\eta} + 2\lambda^2 V_{,\eta\eta} + 2\lambda^3 W_{,\eta}W_{,\eta\eta} + (1-\nu)\lambda W_{,\eta}W_{,\xi\xi} + (1+\nu)\lambda W_{,\xi}W_{,\xi\eta} = 0 \quad (4.1.22)$$

$$\begin{aligned} W_{,\xi\xi\xi\xi} + 2\lambda^2 W_{,\xi\xi\eta\eta} + \lambda^4 W_{,\eta\eta\eta\eta} - 12W_{,\xi\xi}U_{,\xi} - 6W_{,\xi\xi}(W_{,\xi})^2 - \\ 12\nu\lambda W_{,\xi\xi}V_{,\eta} - 6\nu\lambda^2 W_{,\xi\xi}(W_{,\eta})^2 - 12\lambda^3 W_{,\eta\eta}V_{,\eta} - \\ 6\lambda^4 W_{,\eta\eta}(W_{,\eta})^2 - 12\lambda^2\nu W_{,\eta\eta}U_{,\xi} - 6\lambda^2\nu W_{,\eta\eta}(W_{,\xi})^2 - \\ 12(1-\nu)\lambda^2 W_{,\xi\eta}U_{,\eta} - 12(1-\nu)\lambda W_{,\xi\eta}V_{,\xi} - \\ 12(1-\nu)\lambda^2 W_{,\xi\eta}W_{,\xi}W_{,\eta} - Q = 0 \end{aligned} \quad (4.1.23)$$

Hence the problem of large deflection of thin homogeneous isotropic plates is reduced to finding a solution to equations (4.1.21) to (4.1.23). The governing differential equations for the thin isotropic plate resting on a Winkler type foundation, in dimensionless form can be obtained by keeping the first two equations (4.1.21 and 4.1.22) unchanged and adding KW to the left hand side of the third equation (4.1.23), where $K=ka^4/D$, k = foundation modulus.

BOUNDARY CONDITIONS AND METHOD OF SOLUTION

The boundary conditions of a clamped plate are;

$$W = W_{,\xi} = U = V = 0 \quad @ \quad \xi = \pm 1$$

$$W = W_{,\eta} = U = V = 0 \quad @ \quad \eta = \pm 1$$

Galerkin method, as described in chapter 3, is used to solve equations (4.1.21), (4.1.22) and (4.1.23). The three displacement functions U, V and W are approximated by linearly independent functions as follows;

$$U = (1-\xi)^2(1-\eta)^2\xi(C_{11}+C_{12}\xi^2+C_{13}\eta^2+C_{14}\xi^2\eta^2)$$

$$= C_{11}F_{11}+C_{12}F_{12}+C_{13}F_{13}+C_{14}F_{14}$$

$$V = (1-\xi)^2(1-\eta)^2\eta(C_{21}+C_{22}\xi^2+C_{23}\eta^2+C_{24}\xi^2\eta^2)$$

$$= C_{21}F_{21}+C_{22}F_{22}+C_{23}F_{23}+C_{24}F_{24}$$

$$W = (1-\xi)^2(1-\eta)^2(C_{31}+C_{32}\xi^2+C_{33}\eta^2)$$

$$= C_{31}F_{31}+C_{32}F_{32}+C_{33}F_{33}$$

The chosen functions are selected such that they satisfy the boundary conditions of the problem. These functions are then substituted into the three governing differential equations. After substitution of these equations into the governing differential equations we get the error or residual functions, $\epsilon_1(U, V, W)$, $\epsilon_2(U, V, W)$ and $\epsilon_3(U, V, W)$, from equations (4.1.21), (4.1.22) and (4.1.23) respectively. These error functions are in general different from zero since the

approximating functions are not the exact solution of the governing differential equations of the plate. In order that the chosen functions yield the approximate solutions of the problem, Galerkin method requires that:

(a) Since ϵ_1 is obtained from the differential equation (4.1.21), which is formed as a result of the equilibrium of the forces in the ξ direction and U represents the displacements in that particular direction; this error function should be orthogonal to each of the independent functions approximating the displacement U .

(b) Similarly each of the functions chosen for V and W must also be orthogonal to the error function ϵ_2 and ϵ_3 respectively.

These requirements give rise to following 11 nonlinear equations in C 's which are as follows.

$$\begin{array}{llll} \iint_A \epsilon_1 F_{11} dA = 0 & \iint_A \epsilon_1 F_{12} dA = 0 & \iint_A \epsilon_1 F_{13} dA = 0 & \iint_A \epsilon_1 F_{14} dA = 0 \\ \iint_A \epsilon_2 F_{21} dA = 0 & \iint_A \epsilon_2 F_{22} dA = 0 & \iint_A \epsilon_2 F_{23} dA = 0 & \iint_A \epsilon_2 F_{24} dA = 0 \\ \iint_A \epsilon_3 F_{31} dA = 0 & \iint_A \epsilon_3 F_{32} dA = 0 & \iint_A \epsilon_3 F_{33} dA = 0 & \end{array}$$

The definite integrals of the above equations were solved by using the Trapezoidal rule mentioned in chapter 3. These nonlinear equations in C 's are then solved by the Newton-Raphson iterative procedure.

4.2 FORMULATION OF THE LARGE DEFLECTION PROBLEM OF SKEW SANDWICH PLATES

DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATIONS

Sandwich plates are a type of three layer construction consisting of two very thin layers of high strength material between which a thicker layer of comparatively low average strength material is sandwiched. The two thin sheets are termed as face sheets or skins and the middle layer is called the core. Sandwich construction is an efficient way of obtaining light weight structures with comparatively high strength. Due to their high strength to weight ratio sandwich structures have become increasingly popular in various areas of structural design. This is particularly true for the aero-space industry.

In sandwich construction, since the elastic modulus of the core layer in the plane of the plate is of negligible magnitude in comparison to that of the facings, the normal stresses in the core are of little importance in resisting bending moments, even though the usual ratio of the thickness of the facings to that of the core is between one-tenth to one-hundredth. On the other hand, the core performs the task of transmitting shearing forces and undergoes considerable shearing deformations because of its low modulus of shear. As a consequence of such shearing

deformations, the differential equations governing the deflection problem are different from those of the general homogeneous structure.

As in the case of homogeneous plates, sandwich plates can be analyzed by the general linear theory when the plate deflection to thickness ratio is small. But for large deflection the nonlinear theory must be used in order to obtain a more realistic solution.

Reissner[48] first obtained two coupled nonlinear differential equations of the fourth order by considering the equilibrium and compatibility of an infinitesimal element of the sandwich plate. Those equations are;

$$\nabla^2 \nabla^2 F = 2tE_f[(w_{,xy})^2 - w_{,xx}(w_{,yy})] \quad (4.2.1)$$

$$D\nabla^2 \nabla^2 w = [1 - (tE_f/2(1-\nu_f^2)G_c)\nabla^2][p + F_{,yy}w_{,xx} - 2F_{,xy}w_{,xy} + F_{,xx}w_{,yy}] \quad (4.2.2)$$

where,

F = membrane stress function

t = thickness of the facings

E = modulus of elasticity of the facing material

W = out of plane displacement

D = flexural rigidity of sandwich plate

$$= \frac{1}{2} t h^2 E_f / (1 - \nu_f^2)$$

ν_f = Poisson's ratio of the facing material

G_c = shear modulus of the core material

p = lateral load

It is obvious that Reissner's equations are of a very complex nature and no exact solution can be probable. For the present study, a set of differential equations which are more suitable for an approximate solution of the problem is derived. In deriving the equations Reissner's steps are followed.

Let us consider a sandwich plate of two face layers of thickness t and a core layer of thickness $(h-t)$ shown in Fig.5. Assuming t is small compared to h , the elastic constants E_f , G_f of face layers are large compared to the values of the constants E_c , G_c for the core layer, so that tE_f , tG_f are large compared to the values hE_c and hG_c .

On the basis of the assumption $t \ll h$, it can be assumed that the stresses in the faces parallel to their plane are distributed uniformly over the thickness of the face layers. From the assumption $hE_c \ll tE_f$, the face parallel stresses in the core layer and their effect on the deformation of the composite plate can be neglected. Thus the sandwich plate treated here is a combination of two plates without bending stiffness (the face layers) and of a third plate (the core layer) offering resistance to the transverse shear stress and transverse normal stresses.

In the following derivation we shall restrict ourselves to isotropic face and core material of the sandwich plate.

For a sandwich plate subjected to some arbitrary distributed load p the strain displacement relationships for the face membrane with the notation of figure 5, are of following form, where subscripts '+' and '-' are used to denote upper and lower facings respectively.

$$\epsilon_x^\pm = u^\pm_{,x} + \frac{1}{2} (w^\pm_{,x})^2 \quad (4.2.3)$$

$$\epsilon_y^\pm = v^\pm_{,y} + \frac{1}{2} (w^\pm_{,y})^2 \quad (4.2.4)$$

$$\gamma^\pm = u^\pm_{,y} + v^\pm_{,x} + w^\pm_{,x} w^\pm_{,y} \quad (4.2.5)$$

and for the core layers are;

$$\epsilon_z = w_{,z} \quad (4.2.6)$$

$$\gamma_x = u_{,z} + w_{,x} \quad (4.2.7)$$

$$\gamma_y = v_{,z} + w_{,y} \quad (4.2.8)$$

Stress strain relationships for the facings are (by Hooke's law);

$$\epsilon_x^\pm = \frac{1}{E_f t} (N_x^\pm - \nu_f N_y^\pm) \quad (4.2.9)$$

$$\epsilon_y^\pm = \frac{1}{E_f t} (N_y^\pm - \nu_f N_x^\pm) \quad (4.2.10)$$

$$\gamma^\pm = \frac{1}{G_f t} S^\pm \quad (4.2.11)$$

and those for the core are;

$$\epsilon_z = \sigma_z / E_c \quad (4.2.12)$$

$$\gamma_x = \tau_x / G_c \quad (4.2.13)$$

$$\gamma_y = \tau_y / G_c \quad (4.2.14)$$

With the notation of fig.5 the equilibrium equation for the face layers are:

$$N_{x\pm,x} + S_{\pm,y} \mp \tau_{x\pm} = 0 \quad (4.2.15)$$

$$S_{\pm,x} + N_{y\pm,y} \mp \tau_{y\pm} = 0 \quad (4.2.16)$$

$$(N_{x\pm} w_{\pm,x})_{,x} + (S_{\pm} w_{\pm,x})_{,y} + (S_{\pm} w_{\pm,y})_{,x} + (N_{y\pm} w_{\pm,y})_{,y} + p_{\pm} \mp \sigma_{z\pm} \mp \tau_{x\pm} w_{\pm,x} \mp \tau_{y\pm} w_{\pm,y} = 0 \quad (4.2.17)$$

and for the core layer the equilibrium equations are:

$$\tau_{x,z} = 0 \quad (4.2.18)$$

$$\tau_{y,z} = 0 \quad (4.2.19)$$

$$\tau_{x,x} + \tau_{y,y} + \sigma_{z,z} = 0 \quad (4.2.20)$$

The differential equations (4.2.18 to 4.2.20) and (4.2.6 to 4.2.8) for the core layer must be integrated and the results of the integration must be combined with the remaining equations for the face layers in such a way that a set of differential equations for the composite plate is obtained.

From equations (4.2.18) and (4.2.19) it follows that τ_x and τ_y do not vary across the thickness of the core. Thus the transverse shear resultants can be defined by the following equations:

$$V_x = h\tau_x \quad (4.2.21)$$

$$V_y = h\tau_y \quad (4.2.22)$$

Integration of equation (4.2.20) gives

$$V_{x,x} + V_{y,y} + \sigma_{z+} - \sigma_{z-} = 0 \quad (4.2.23)$$

From equations (4.2.6) to (4.2.8) and (4.2.12) to (4.2.14) and also from the fact that σ_z varies linearly over the thickness, we finally get,

$$w_+ - w_- = h(\sigma_{z+} + \sigma_{z-})/2E_c \quad (4.2.24)$$

$$\frac{v_x}{G_c} = \left[\frac{h}{2} (w_+ + w_-) + \frac{h^2}{12E_c} (v_{x,x} + v_{y,y}) \right]_{,x} + u_+ - u_- \quad (4.2.25)$$

$$\frac{v_y}{G_c} = \left[\frac{h}{2} (w_+ + w_-) + \frac{h^2}{12E_c} (v_{x,x} + v_{y,y}) \right]_{,y} + v_+ - v_- \quad (4.2.26)$$

Equations (4.2.24), (4.2.25) and (4.2.26) are the stress strain relations for the core layer in a form suitable for use in the derivation of the equations for the composite plate.

Let us define some appropriate variables as follows:

$$\begin{aligned} \theta_x &= (u_+ - u_-)/h \\ \theta_y &= (v_+ - v_-)/h \end{aligned} \quad (4.2.27)$$

representing effective changes of slope of the normal to the undeformed middle surface; let

$$w = \frac{1}{2} (w_+ + w_-) \quad (4.2.28)$$

represent the effective transverse deflection of the middle surface.

$$u = \frac{1}{2} (u_+ + u_-) \quad v = \frac{1}{2} (v_+ + v_-) \quad (4.2.29)$$

represent the essential tangential component of the displacements of the middle surface, and let

$$e = (w_+ - w_-)/h \quad (4.2.30)$$

represents the effective transverse normal strain of the composite plate.

In addition to the transverse shear stress resultants V_x and V_y defined by equations (4.2.21), (4.2.22) we define the stress resultants and the couples as follows:

$$\begin{aligned} N_x &= N_{x+} + N_{x-} & N_y &= N_{y+} + N_{y-} & (4.2.31) \\ S &= S_+ + S_- \end{aligned}$$

$$\begin{aligned} M_x &= (N_{x+} - N_{x-})/h & M_y &= (N_{y+} - N_{y-})/h & (4.2.32) \\ H &= (S_+ - S_-)/h \end{aligned}$$

Finally, we write the effective transverse normal stress

$$\bar{\sigma}_z = \frac{1}{2} (\sigma_{z+} + \sigma_{z-}) \quad (4.2.33)$$

and the external load terms p and q by means of the following relations

$$p = p_+ + p_- \quad q = (p_+ - p_-)/2 \quad (4.2.34)$$

We now obtain the differential equations for the composite plate by combining the six equations of equilibrium, equations (4.2.15) to (4.2.17), by means of suitable additions and subtractions.

From equation (4.2.15),

$$N_{x,x} + S_{,y} = 0 \quad (4.2.35)$$

$$M_{x,x} + H_{,y} - V_x = 0 \quad (4.2.36)$$

From equation (4.2.16),

$$S_{,x} + N_{y,y} = 0 \quad (4.2.37)$$

$$H_{,x} + M_{y,y} - V_y = 0 \quad (4.2.38)$$

From equation (4.2.17) after some transformations;

$$\begin{aligned} p + v_{x,x} + v_{y,y} + N_x w_{,xx} + 2S w_{,xy} + N_y w_{,yy} \\ + M_{x,e,xx} + 2H_{e,xx} + 2H_{e,xy} + M_{y,e,yy} - V_x e_{,x} \\ - V_y e_{,y} = 0 \end{aligned} \quad (4.2.39)$$

$$\begin{aligned} q - \sigma_z + \frac{h}{4} (N_x e_{,xx} + 2S e_{,xy} + N_y e_{,yy}) + \frac{1}{h} (M_x w_{,xx} \\ + 2H w_{,xy} + M_y w_{,yy}) = 0 \end{aligned} \quad (4.2.40)$$

Equations (4.2.35) and (4.2.37) are the usual equations of horizontal equilibrium for the elements of the plate. Equations (4.2.36) and (4.2.38) are the usual equations of moment equilibrium.

Equation (4.2.39) is the condition of transverse force equilibrium and contains terms that do not occur when homogeneous isotropic plates are considered. The significance of equation (4.2.40) is that it gives the local change of thickness of the plate caused by the external loads.

We next obtain the stress-strain relations combining equations (4.2.3) to (4.2.5) according to equations (4.2.31) and (4.2.32). The resulting equations after some transformations,

$$u_{,x} + \frac{1}{2} [(w_{,x})^2 + \frac{h^2}{4} (e_{,x})^2] = (N_x - \nu_f N_y) / 2tE_f \quad (4.2.41)$$

$$v_{,y} + \frac{1}{2} [(w_{,y})^2 + \frac{h^2}{4} (e_{,x})^2] = (N_y - \nu_f N_x) / 2tE_f \quad (4.2.42)$$

$$u_{,y} + v_{,x} + w_{,x} w_{,y} + \frac{h^2}{4} e_{,x} e_{,y} = S / 2tG_f \quad (4.2.43)$$

$$\theta_{x,x} + w_{,x} e_{,x} = (M_x - \nu_f M_y) / \frac{1}{2} t h^2 E_f \quad (4.2.44)$$

$$\theta_{y,y} + w_{,y} e_{,y} = (M_y - \nu_f M_x) / \frac{1}{2} t h^2 E_f \quad (4.2.45)$$

$$\theta_{x,y} + \theta_{y,x} + w_{,x} e_{,y} + w_{,y} e_{,x} = H / \frac{1}{2} t h^2 G_f \quad (4.2.46)$$

In addition to the above six stress strain relations we have equations (4.2.24), (4.2.25) and (4.2.26) which may be written in the following form;

$$w_{,x} + \theta_x = \frac{v_x}{hG_c} - \frac{h}{12E_c} (v_{x,x} + v_{y,y})_{,x} \quad (4.2.47)$$

$$w_{,y} + \theta_y = \frac{v_y}{hG_c} - \frac{h}{12E_c} (v_{x,x} + v_{y,y})_{,y} \quad (4.2.48)$$

$$e = \frac{\bar{\sigma}_z}{E_c} \quad (4.2.49)$$

The following discussion will be restricted to cases corresponding to the relation,

$$e = \frac{q}{E_c} \quad (4.2.50)$$

This should be true in most cases of practical interest. Also, by assuming that $q \approx p$ it can be seen that terms involving e in the above equation are negligible provided;

$$\frac{p}{E_c} \ll 1 \quad (4.2.51)$$

which is true for all practical cases.

Since the effect of local change of thickness of the plate is negligibly small so long as the above assumptions are valid, it can be concluded that as far as the present investigation is concerned equation (4.2.40) will be of negligible influence in the solution of the problem and hence can be discarded. Similarly for all other equations, terms involving e can be neglected without introducing appreciable error, thus reducing the problem to a set of five equations with five unknowns.

From the assumption e is small compared to the total deformation, the quantity W , may be set equal to zero and

$$v_x = hG_c (w_x + \theta_x) \quad (4.2.52)$$

$$v_y = hG_c (w_y + \theta_y) \quad (4.2.53)$$

from which,

□

$$v_{x,x} = hG_c (w_{,xx} + \theta_{x,x}) \quad (4.2.54)$$

$$v_{y,y} = hG_c (w_{,yy} + \theta_{y,y}) \quad (4.2.55)$$

With equations (4.2.52) through (4.2.55) and the stress strain relations, the equations of equilibrium may be written in the following form[42];

$$\begin{aligned} 2u_{,xx} + (1-\nu) u_{,yy} + (1+\nu)(v_{,xy}) + (w_{,x}^2 + vw_{,y}^2)_{,x} \\ + (1-\nu)(w_{,xy}w_{,y} + w_{,x}w_{,yy}) = 0 \end{aligned} \quad (4.2.56)$$

$$\begin{aligned} 2v_{,yy} + (1-\nu)v_{,xx} + (1+\nu) v_{,xy} + (w_{,y}^2 + vw_{,x}^2)_{,y} \\ + (1-\nu)(w_{,xy}w_{,y} + w_{,xx}w_{,y}) = 0 \end{aligned} \quad (4.2.57)$$

$$\frac{thE_f}{(1-\nu)^2} [2\theta_{x,xx} + (1-\nu)\theta_{x,yy} + (1+\nu)\theta_{y,xy}] - 4G_c(w_{,x} + \theta_{x,x}) = 0 \quad (4.2.58)$$

$$\frac{thE_f}{(1-\nu)^2} [2\theta_{y,yy} + (1-\nu)\theta_{y,xx} + (1+\nu)\theta_{x,xy}] - G_c(w_{,y} + \theta_{y,y}) = 0 \quad (4.2.59)$$

$$\begin{aligned} p + hG_c(w_{,xx} + w_{,yy} + \theta_{x,x} + \theta_{y,y} + \frac{thE_f}{(1-\nu)^2} \{w_{,xx}[2(u_{,x} + vw_{,y}) \\ + w_{,x}^2 + vw_{,y}^2] + w_{,yy}[2(v_{,y} + vu_{,x}) + w_{,y}^2 + vw_{,x}^2] \\ + 2(1-\nu)w_{,xy}(u_{,y} + v_{,x} + w_{,x}w_{,y})\} + \frac{thE_f}{2(1-\nu)^2} \\ [w_{,xx}(\theta_{x,x} + \nu\theta_{y,y}) + w_{,yy}(\theta_{y,y} + \nu\theta_{x,x}) + (1-\nu)w_{,xy}(\theta_{x,y} + \theta_{y,x})] = 0 \end{aligned} \quad (4.2.60)$$

These equations (4.2.56) to (4.2.60) are the differential equations governing the large deflection behaviour of sandwich plates in terms of the displacements, where E_f is the Young's modulus of the face material; G_c being the shear modulus of the core; t the thickness of the face layer; $(h-t)$ the thickness of the core layer; p the intensity of the uniformly distributed load; ν the Poisson's ratio of the face material; Θ_x and Θ_y are the change of slope of the normal to the middle surface of the plate in the x and y directions respectively and the comma notation signifies differentiation.

From these equations, it can be seen that the first fourth order differential equation of Reissner is replaced by two differential equations of the second order such as (4.2.56) and (4.2.57). Reissner derived his first equation following Von Karman's procedure. The equations (4.2.35) and (4.2.37) were satisfied by setting;

$$N_x = F_{,yy} \quad N_y = F_{,xx} \quad S = -F_{,xy} \quad (4.2.61)$$

Reissner then obtained his first equation by means of equation (4.2.46) to (4.2.48), in the same form as of homogeneous plate. From the equations (4.2.56) to (4.2.60) it can be seen that the second fourth order equation of Reissner is replaced by three second order equations such as equations (4.2.58) to (4.2.60). In Reissner's derivation of his second equation, the equation of moment equilibrium, equations (4.2.36) and (4.2.38) were introduced into

equation (4.2.39) to replace the derivatives of the transverse shear stress resultants V_x and V_y then the stress strain relationships in equations (4.2.44) to (4.2.46), (4.2.47) and (4.2.48) were used. The unknown functions θ_x and θ_y were eliminated by means of suitable addition and subtractions. Thus the equations (4.2.56) to (4.2.60) derived here in terms of $\theta_x, \theta_y, u, v, w$ are actually Reissner's equations (4.2.1) and (4.2.2) written in a modified form. In the derivation presented here the unknowns θ_x, θ_y, u and v etc. are not eliminated as was done by Reissner.

THE OBLIQUE COORDINATE SYSTEM

In investigating the large deflection of skew sandwich plates, it is often advantageous to adopt a coordinate system parallel to the edges of the plate, namely the oblique coordinate system α and β as shown in figure 6.

By the transformation, $x = \alpha \cos\phi$ and $y = \beta + \alpha \sin\phi$

in which ϕ is the skew angle, the following relationships between the rectangular and oblique coordinate system holds;

$$\begin{aligned}
u_{,x} &= u_{,\alpha} \sec \phi - u_{,\beta} \tan \phi \\
u_{,xx} &= u_{,\alpha\alpha} \sec^2 \phi - 2u_{,\alpha\beta} \sec \phi \tan \phi + u_{,\beta\beta} \tan^2 \phi \\
u_{,xy} &= u_{,\alpha\beta} \sec \phi - u_{,\beta\beta} \tan \phi \\
u_{,y} &= u_{,\beta} \\
u_{,yy} &= u_{,\beta\beta}
\end{aligned}$$

Similar relationships hold for all other displacements. Using these transformation relationships and by defining further the dimensionless quantities,

$$\begin{aligned}
\lambda &= a/b & \xi &= \alpha/a & \eta &= \beta/b & \mu &= t/a & \theta &= h/a \\
W &= w/h & U &= 12au/h^2 & V &= 12av/h^2 & Q &= 12a^3(1-\nu^2)p/th^2E_f
\end{aligned}$$

the equations (4.2.56) to (4.2.60) can be rewritten into oblique dimensionless form as follows;

$$A1K1 \theta_{x,\xi\xi} + A1K2 \theta_{x,\eta\eta} + A1K3 \theta_{x,\xi\eta} + A1K4 \theta_{y,\xi\eta} + A1K5 W_{,\xi} + A1K6 \theta_{x,\eta} + A1K7 \theta_{y,\eta\eta} + A1K8 W_{,\eta} = 0 \quad (4.2.62)$$

$$A2K1 \theta_{y,\eta\eta} + A2K2 \theta_{y,\xi\xi} + A2K3 \theta_{y,\xi\eta} + A2K4 \theta_{x,\xi\eta} + A2K5 W_{,\eta} + A2K6 \theta_{y,\eta} + A2K7 \theta_{x,\eta\eta} = 0 \quad (4.2.63)$$

$$A3K1 U_{,\xi\xi} + A3K2 U_{,\eta\eta} + A3K3 U_{,\xi\eta} + A3K4 V_{,\xi\eta} + A3K5 W_{,\xi} W_{,\xi\xi} + A3K6 W_{,\xi\eta} W_{,\xi} + A3K7 W_{,\xi} W_{,\eta\eta} + A3K8 W_{,\xi\eta} W_{,\eta} + A3K9 W_{,\xi\xi} W_{,\eta} + A3K10 V_{,\eta\eta} + A3K11 W_{,\eta} W_{,\eta\eta} = 0 \quad (4.2.64)$$

$$A4K1 V_{,\eta\eta} + A4K2 V_{,\xi\xi} + A4K3 V_{,\xi\eta} + A4K4 U_{,\xi\eta} + A4K5 W_{,\eta\eta} W_{,\eta} + A4K6 W_{,\xi\eta} W_{,\eta} + A4K7 W_{,\xi\xi} W_{,\eta} + A4K8 W_{,\xi} W_{,\eta\eta} + A4K9 W_{,\xi\eta} W_{,\xi} + A4K10 U_{,\eta\eta} = 0 \quad (4.2.65)$$

$$\begin{aligned} & A5K1 W_{,\xi\xi} U_{,\xi} + A5K2 W_{,\xi\xi} V_{,\eta} + A5K3 W_{,\xi\xi} (W_{,\xi})^2 + A5K4 W_{,\xi\xi} (W_{,\eta})^2 \\ & + A5K5 W_{,\xi\xi} U_{,\eta} + A5K6 W_{,\xi\xi} W_{,\xi} W_{,\eta} + A5K7 W_{,\eta\eta} V_{,\eta} \\ & + A5K8 W_{,\eta\eta} V_{,\xi} + A5K9 W_{,\eta\eta} U_{,\xi} + A5K10 W_{,\eta\eta} (W_{,\eta})^2 \\ & + A5K11 W_{,\eta\eta} W_{,\xi} W_{,\eta} + A5K12 W_{,\eta\eta} (W_{,\xi})^2 + A5K13 W_{,\xi\eta} V_{,\xi} \\ & + A5K14 W_{,\xi\eta} V_{,\eta} + A5K15 W_{,\xi\eta} U_{,\xi} + A5K16 W_{,\xi\eta} (W_{,\eta})^2 \\ & + A5K17 W_{,\xi\eta} W_{,\xi} W_{,\eta} + A5K18 W_{,\xi\eta} (W_{,\xi})^2 + A5K19 W_{,\xi\eta} U_{,\eta} \\ & + A5K20 W_{,\xi\xi} \theta_{x,\xi} + A5K21 W_{,\xi\xi} \theta_{y,\eta} + A5K22 W_{,\xi\xi} \theta_{x,\eta} \\ & + A5K23 W_{,\xi\xi} + A5K24 W_{,\eta\eta} \theta_{y,\eta} + A5K25 W_{,\eta\eta} \theta_{y,\xi} \\ & + A5K26 W_{,\eta\eta} \theta_{x,\xi} + A5K27 W_{,\eta\eta} + A5K28 W_{,\xi\eta} \theta_{y,\xi} \\ & + A5K29 W_{,\xi\eta} \theta_{y,\eta} + A5K30 W_{,\xi\eta} \theta_{x,\xi} + A5K31 W_{,\xi\eta} \theta_{x,\eta} \\ & + A5K32 W_{,\xi\eta} + A5K33 \theta_{x,\xi} + A5K34 \theta_{y,\eta} + A5K35 W_{,\eta\eta} U_{,\eta} \\ & + A5K36 W_{,\eta\eta} \theta_{x,\eta} + A5K37 \theta_{x,\eta} + Q = 0 \quad (4.2.66) \end{aligned}$$

The coefficients A1K1 to A5K37 in the above equations are given in table 1, where for brevity $s=\sin\phi$, $c=\cos\phi$, $2a$ and $2b$ are the oblique dimensions of the plate. Hence the problem of large deflection of skew sandwich plate is reduced to finding a solution to equations (4.2.62) to (4.2.66).

The governing differential equations for the large deflection of skew sandwich plate on a Winkler type foundation can be obtained very easily by adding $'-KW'$ to the left hand side of the last differential equation 4.2.66 keeping the remaining equations 4.2.62 to 4.2.65 unchanged. Where

$K = 12 ak \frac{1-\nu^2}{\mu\theta E_f}$ and $k =$ modulus of elastic support reaction per unit area per unit deflection.

METHOD OF SOLUTION AND BOUNDARY CONDITIONS

The boundary conditions for the clamped skew sandwich plate are;

$$W = W_{,\xi} = U = V = \theta_x = \theta_y = 0 \quad @ \xi = \pm 1$$

$$W = W_{,\eta} = U = V = \theta_x = \theta_y = 0 \quad @ \eta = \pm 1$$

Galerkin method is then applied to solve the differential equations for the large deflection of skewed sandwich

plates. The five unknown displacements are approximated by the following functions;

$$U = (1-\xi^2)(1-\eta^2)\xi(S_{11} + S_{12}\xi^2 + S_{13}\eta^2)(1-\xi\eta)$$

$$V = (1-\xi^2)(1-\eta^2)\eta(S_{21} + S_{22}\xi^2 + S_{23}\eta^2)(1-\xi\eta)$$

$$\theta_x = (1-\xi^2)(1-\eta^2)\xi(S_{31} + S_{32}\xi^2 + S_{33}\eta^2)(1-\xi\eta)$$

$$\theta_y = (1-\xi^2)(1-\eta^2)\eta(S_{41} + S_{42}\xi^2 + S_{43}\eta^2)(1-\xi\eta)$$

$$W = (1-\xi^2)^2(1-\eta^2)^2(S_{51} + S_{52}\xi^2 + S_{53}\eta^2 + S_{54}\xi^2\eta^2)(1-\xi\eta)$$

These functions are chosen such that they satisfy all the boundary conditions of the problem, including the requirement of polar symmetry and have the same order of derivatives as is called for by the differential equations. These approximate displacement expressions are then substituted into the governing differential equations. As a result of this substitution we get five error functions from the five differential equations. Applying the Galerkin's orthogonality condition, "each of these error functions must be orthogonal to each of the displacement functions in a particular direction", As was done in the case of homogeneous plates, we get 16 Galerkin's algebraic equations in terms of the constants in the displacement functions. The integrations involved in this process were evaluated by the trapezoidal rule described in the previous chapter. The Galerkin's algebraic equations obtained in this analysis are nonlinear and are solved by using Newton-Raphson iterative procedure.

4.3 DISCUSSIONS AND CONCLUSIONS

During this investigation numerical experimentation was carried out for the case of homogeneous plates by varying the number of terms in U and V . The solutions were first obtained with 5 terms (one for U and one for V) and subsequently with 11 terms. It was found that there was no significant difference between the two set of results. The results reported herein for the homogeneous plates are the results obtained with the use of 11 terms.

In the case of skew sandwich plates the experiment was carried out with 7, 8 and 16 terms. The 7 term solutions were obtained by taking only the first terms in U, V, θ_x, θ_y and 3 terms in W . The 8 term solutions were obtained by adding the fourth term for W . The 16 term solutions were obtained by further adding 2 terms for each of the displacements U, V, θ_x and θ_y . It was found that although the effort to get the 16 term solution is double to that of the 7 term solution, there was no significant difference in the results. The results reported herein for the case of sandwich plates, unless otherwise stated, are the results obtained by using 16 terms.

For the purpose of demonstrating the accuracy and convergence characteristics of the present method the solutions obtained for the linear analysis of clamped isotropic homogeneous plates and that of clamped skew sandwich plates ~~are~~ presented in Tables 2 and 3 respectively. It is found that in both the cases the results are in excellent agreement with other investigators. It is also found that the centre deflection varies very little (less than 5%) with even 100% increase in the number of terms. Unlike other numerical methods where the convergence deteriorates rapidly with the increase in the skew angle the present method shows excellent convergence even for high skew angles.

* In Table 4 the Galerkin solution for the clamped isotropic homogeneous plates are found to be in close agreement with that of Ng[43] whose work was based on the method of collocation least squares.

Table 5 shows the numerical comparison of the linear centre deflection for the clamped skew sandwich plates between the Galerkin method and those obtained by Kwok[27] who used a high precision finite element for the analysis. Kwok used in his analysis a precise element with five degrees of freedom per node, with fully clamped edge conditions. It is found that the results are in excellent agreement.

Table 6 shows the dimensionless centre deflection of the nonlinear analysis of the clamped skew sandwich plate for various skew angle ϕ , aspect ratio λ and core rigidity G_c . No comparison for these results can be made as no data were readily available.

The nonlinear load deflection characteristics of the plates are shown graphically in figures 8 to 17. Figure 8 and figure 9 respectively show nonlinear load-deflection characteristics for the clamped homogeneous plates and the clamped sandwich plates for various aspect ratios. It is observed that the deflection differs very little when the aspect ratio, $a/b = 2/3$ and $a/b = 1/3$. Therefore it can be concluded that as the aspect ratio becomes less than $1/3$, the plates can be considered as infinitely long plates.

Figure 10 shows the nonlinear load deflection characteristics of the clamped skew sandwich plates for various skew angles. Figure 11 shows the variation of centre deflection with the angle of skew for various core rigidities. It is interesting to observe that the deflection of the skew sandwich plate decreases with the increase in the skew angle. This is due to the increasing rigidity of the plate at the obtuse corners. As is expected, it is found that the more rigid is the core the less is the deflection. It can also be observed that as the rigidity of the core of the sandwich plate is increased, the rate of the decrease of central deflection due to increase of the skew angle decreases.

Figures 12,13 and 14 show the nonlinear load-deflection characteristics for the clamped isotropic homogeneous plates resting on elastic foundations for various aspect ratios. Figures 15,16 and 17 show the corresponding nonlinear load-deflection curves for the clamped skew sandwich plates resting on elastic foundations, for different skew angles. As is expected, it is found that for both the isotropic and sandwich plates the deflection decreases due to increase in the foundation modulus. It can also be observed that the curves tend to become linear with the increase of the elastic foundation. This is due to increase in the apparent stiffness of the plate with an increase in the intensity of the foundation.

From the present study it can be concluded that Galerkin's method is found to be powerful and at the same time very simple to apply for the large deflection analysis of sandwich and layered plates.

Chapter V

BUCKLING OF CLAMPED, HOMOGENEOUS AND SKEW SANDWICH PLATES

To demonstrate the versatility of the Galerkin method the problem of buckling of clamped homogeneous rectangular plates and clamped skew sandwich plates under inplane loads will be investigated in this chapter. The application of the Galerkin method to eigen value problems was briefly discussed in chapter 3.

5.1 GOVERNING DIFFERENTIAL EQUATIONS

When a flat plate is compressed in its middle plane just as in the case of columns, the flat form of the plate becomes unstable and begins to buckle at a certain critical value of the inplane forces. To investigate such stability problems it is assumed that the plate buckles slightly under the action of the forces applied in its middle plane, then the magnitude of the forces necessary to keep the plate in such a buckled shape is calculated. The differential equation in such case for the homogeneous plate is obtained

from equation (4.1.11) by putting the lateral load equal to zero, since there is no laterally applied load. Similarly the differential equation governing the buckling of the sandwich plate can be obtained from equation (4.2.2) by using equation (4.2.61), [48].

Thus the equations for the buckling of plates in the cartesian coordinates are;

for homogeneous plate ;

$$w_{,xxxx} + 2w_{,xxyy} = w_{,yyyy} = \frac{1}{D} (N_x w_{,xx} + N_y w_{,yy}) \quad (5.1.1)$$

$$\text{where, } D = \frac{Eh^3}{12(1-\nu^2)}$$

and for sandwich plate ;

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = \frac{1}{D} \left(1 - \frac{\text{th} E_f}{2(1-\nu^2)G_c} \nu^2\right) (N_x w_{,xx} + N_y w_{,yy}) \quad (5.1.2)$$

$$\text{where, } D = \frac{\text{th}^2 E_f}{2(1-\nu^2)}$$

For investigating the buckling load for a skew sandwich plate a coordinate system parallel to the edges of the skew plate is adopted. This coordinate system namely the oblique coordinate system α and β are shown in the figure 6.

By the transformation

$$x = a \cos \phi \quad \text{and} \quad y = \beta + a \sin \phi$$

in which ϕ is the skew angle, the following relationships between the rectangular and oblique coordinate system hold;

$$w_{,x} = w_{,\alpha} \text{Sec} \phi - w_{,\beta} \tan \phi$$

$$w_{,xx} = w_{,\alpha\alpha} \text{Sec}^2 \phi - 2w_{,\alpha\beta} \text{Sec} \phi \tan \phi + w_{,\beta\beta} \tan^2 \phi$$

$$w_{,xy} = w_{,\alpha\beta} \text{Sec}\phi - w_{,\beta\beta} \tan\phi$$

$$w_{,y} = w_{,\beta}$$

$$w_{,yy} = w_{,\beta\beta}$$

$$w_{,xxxx} = w_{,\alpha\alpha\alpha\alpha} \text{Sec}^4\phi - 4w_{,\alpha\alpha\alpha\beta} \text{Sec}^3\phi \tan\phi + 6w_{,\alpha\alpha\beta\beta} \text{Sec}^2\phi \tan^2\phi - w_{,\alpha\beta\beta\beta} \text{Sec}\phi \tan^3\phi + w_{,\beta\beta\beta\beta} \tan^4\phi$$

$$w_{,xxyy} = w_{,\alpha\alpha\beta\beta} \text{Sec}^2\phi - 2w_{,\alpha\beta\beta\beta} \text{Sec}\phi \tan\phi + w_{,\beta\beta\beta\beta} \tan^2\phi$$

$$w_{,yyyy} = w_{,\beta\beta\beta\beta}$$

Using these transformation relationships the equation (5.1.2) can be transformed into skew coordinate system which then takes the following form;

$$\begin{aligned} & w_{,\alpha\alpha\alpha\alpha} (\text{Sec}^4\phi) + w_{,\alpha\alpha\alpha\beta} (-4\text{Sec}^3\phi \tan\phi) + w_{,\alpha\alpha\beta\beta} (6\text{Sec}^2\phi \tan^2\phi + 2\text{Sec}^2\phi) \\ & + w_{,\alpha\beta\beta\beta} (-4\text{Sec}^3\phi \tan\phi) + w_{,\beta\beta\beta\beta} (\text{Sec}^4\phi) \\ & = \frac{1}{D} \{ N_{\alpha} [w_{,\alpha\alpha} \text{Sec}^2\phi - 2w_{,\alpha\beta} \text{Sec}\phi \tan\phi + w_{,\beta\beta} \tan^2\phi] + N_{\beta} [w_{,\beta\beta}] \} \\ & - \frac{\frac{thE_f}{2(1-\nu^2)} G_c}{D} \{ w_{,\alpha\alpha\alpha\alpha} (N_{\alpha} \text{Sec}^4\phi) + w_{,\alpha\alpha\alpha\beta} N_{\alpha} (-4\text{Sec}^3\phi \tan\phi) \\ & + w_{,\alpha\alpha\beta\beta} [(6\text{Sec}^2\phi \tan^2\phi + \text{Sec}^2\phi) N_{\alpha} + N_{\beta} \text{Sec}^2\phi] \\ & + w_{,\alpha\beta\beta\beta} [(-4\text{Sec}\phi \tan^3\phi - 2\text{Sec}\phi \tan\phi) N_{\alpha} - 2N_{\beta} \text{Sec}\phi \tan\phi] \\ & + w_{,\beta\beta\beta\beta} [(\tan^4\phi + \tan^2\phi) N_{\alpha} + N_{\beta} (\text{Sec}^2\phi)] \} \end{aligned} \quad (5.1.3)$$

For the ease of computation it is convenient to render the equations (5.1.1) and (5.1.3) dimensionless. To get further computational advantage we can assume that there exists a relationship between the two inplane loads N_x and N_y for the homogeneous plates and N_α and N_β for the skew sandwich plates, so that ;

$$\begin{aligned} N_y &= r N_x && \text{for the homogeneous plate} \\ N_\beta &= r N_\alpha && \text{for the skew sandwich plate} \end{aligned} \quad (5.1.4)$$

Once the relationship is established now the only interest is to determine the critical buckling load N_x or N_α for the homogeneous and skew sandwich plates respectively.

Equation (5.1.1) and (5.1.3) are transformed into dimensionless form by using the following dimensionless quantities.

for homogeneous plate;

$$\xi = x/a \quad \eta = y/b \quad \lambda = a/b \quad N_\xi = N_x a^2/D \quad \text{and} \quad N_\eta = N_y a^2/D \quad W = \frac{w}{h}$$

$$\text{for sandwich plate; } \xi = \frac{\alpha}{a} \quad \eta = \frac{\beta}{b} \quad \lambda = \frac{a}{b}$$

$$\mu = \frac{thE_f}{2a^2(1-\nu)^2 G_c} \quad N_\xi = \frac{N_\alpha a^2}{D} \quad N_\eta = \frac{N_\beta a^2}{D} \quad W = \frac{w}{h}$$

Using the relationship in equation (5.1.4) and substituting the above dimensionless quantities we get the dimensionless form of the differential equations for the buckling of the plates;

for homogeneous plate;

$$W_{,\xi\xi\xi\xi} + 2\lambda^2 W_{,\xi\xi\eta\eta} + \lambda^4 W_{,\eta\eta\eta\eta} = N_\xi (W_{,\xi\xi} + \lambda^2 r W_{,\eta\eta}) \quad (5.1.5)$$

and for skew sandwich plate;

$$\begin{aligned}
 & W_{,\xi\xi\xi\xi} \text{Sec}^4 \phi + \lambda W_{,\xi\xi\xi\eta} (-4\text{Sec}^3 \phi \tan \phi) + \lambda^2 W_{,\xi\xi\eta\eta} (6\text{Sec}^2 \phi \tan^2 \phi + 2\text{Sec}^2 \phi) \\
 & + \lambda^3 W_{,\xi\eta\eta\eta} (-4\text{Sec}^3 \phi \tan \phi) + \lambda^4 W_{,\eta\eta\eta\eta} \text{Sec}^4 \phi \\
 & = N_{\xi} [W_{,\xi\xi} + W_{,\xi\eta} (-2\lambda \text{Sec} \phi \tan \phi) + W_{,\eta\eta} \lambda^2 (r + \tan^2 \phi)] \\
 & - \mu [W_{,\xi\xi\xi\xi} (\text{Sec}^4 \phi) + W_{,\xi\xi\xi\eta} (-4\lambda \text{Sec}^3 \phi \tan \phi) \\
 & + W_{,\xi\xi\eta\eta} \lambda^2 (6\text{Sec}^2 \phi \tan^2 \phi + \text{Sec}^2 \phi + r \text{Sec}^2 \phi) \\
 & + W_{,\xi\eta\eta\eta} \lambda^3 (-4\text{Sec} \phi \tan^3 \phi - 2\text{Sec} \phi \tan \phi - 2r \text{Sec} \phi \tan \phi) \\
 & + W_{,\eta\eta\eta\eta} \lambda^4 (\text{Sec}^2 \phi \tan^2 \phi + r \text{Sec}^2 \phi)] \quad (5.1.6)
 \end{aligned}$$

5.2 BOUNDARY CONDITIONS AND APPROXIMATE FUNCTIONS

In the Galerkin method to find an approximate solution of the problem under consideration, a linear combination of linearly independent functions must be chosen for the displacements, satisfying all the boundary conditions.

The boundary conditions for the clamped plates are;

$$W_{,\xi} = W = 0 \quad @ \xi = \pm 1$$

$$W_{,\eta} = W = 0 \quad @ \eta = \pm 1$$

Satisfying the boundary conditions the following functions are chosen to approximate the displacement W ;

for rectangular plate,

$$W = (1-\xi^2)^2 (1-\eta^2)^2 (C_1 + C_2 \xi^2 + C_3 \eta^2 + C_4 \xi^2 \eta^2 + C_5 \xi^4 + C_6 \eta^4 + C_7 \xi^4 \eta^4) \quad (5.2.1)$$

for skew plate,

$$W = (1-\xi^2)^2 (1-\eta^2)^2 (1-\xi\eta) (C_1 + C_2 \xi^2 + C_3 \eta^2 + C_4 \xi^2 \eta^2 + C_5 \xi^4 + C_6 \eta^4 + C_7 \xi^4 \eta^4) \quad (5.2.2)$$

The term $(1-\xi\eta)$ is used to ensure polar symmetry. These expressions for W are symmetric about the ξ and η axis. So when these expressions are used it can be expected that the plate buckles in one half wave in both ξ and η directions; which is only possible for aspect ratio equal to 1. For aspect ratios greater than 1 the plate can be expected to buckle in more than one half waves in the longer direction although it will buckle into a single half wave in the shorter direction. This argument also holds for the skew sandwich plate. To approximate the buckled shape of the plates for aspect ratios greater than 1 the following expressions are chosen for the displacement W .

for rectangular plate,

$$W = (1-\xi^2)^2 (1-\eta^2)^2 \xi (C_1 + C_2 \xi^2 + C_3 \eta^2 + C_4 \xi^2 \eta^2 + C_5 \xi^4 + C_6 \eta^4 + C_7 \xi^4 \eta^4) \quad (5.2.3)$$

for skew plate,

$$W = (1-\xi^2)^2 (1-\eta^2)^2 (1-\xi\eta) \xi (C_1 + C_2 \xi^2 + C_3 \eta^2 + C_4 \xi^2 \eta^2 + C_5 \xi^4 + C_6 \eta^4 + C_7 \xi^4 \eta^4) \quad (5.2.4)$$



Since this is not known for what aspect ratio the plate will buckle into more than a single half wave in the longer direction so for aspect ratios greater than 1 both the expressions for W will be used to seek the buckling load of a particular plate. It is obvious that the lowest buckling load obtained from the two expressions is the critical load for the plate under consideration.

5.3 RESULTS AND CONCLUSIONS

Following the procedure of the Galerkin method, the displacement expressions for W are substituted into the differential equation. This substitution leads to an error function ϵ in terms of the constants C_i , $i=1,2,3, \dots, 7$. In the Galerkin method it is required that this error function be orthogonal to each of the independent functions constituting the expressions for W . This requirement leads to 7 equations which when written in matrix form become;

$$[A][C] = N_\epsilon [B][C]$$

defining an eigen value problem. The eigen values are determined from the Galerkin equations by using IMSL subroutine EIGZF. To get the Galerkin equations the integrations necessary were evaluated by using the trapezoidal rule.

For the isotropic homogeneous plates the buckling coefficients C_r depend on the aspect ratio a/b , axial load ratio r , and also on the elastic constant D . But in the case of sandwich plates the buckling coefficient also depends on another extra variable μ . For both types of plates results were obtained for various practical combinations of these variables.

Numerical computations for both the homogeneous and skew sandwich plates, for the case of uniaxial buckling were carried out by varying the number of terms in the assumed approximate displacement function W . Results were first computed for 3 terms in W , then the results were obtained for 6 and 7 terms in W . To show the convergence characteristic of the present method all the results obtained are tabulated in table 7. It can be seen from table 7 that the 3 term solution gives satisfactory results. The difference in the solutions obtained for 3 and 7 term solutions is less than 0.5%, although the effort involve to obtain a 7 term solution is almost double to that of the 3 term solution. As is expected the results improve with the increase in the number of terms in the assumed displacement function. The results obtained for both the plates are compared with those obtained by Levy[30] and Thurston[52] respectively. Levy obtained his solutions by using a Fourier series method. Thurston used the La-grangian multiplier method to Hoff's energy expression for the

sandwich plate. It appears the error in the present analysis for the sandwich plate is only 7%. This discrepancy may be due to the different differential equation used in the present study. For the homogeneous plate the present solution is in excellent agreement with those obtained by Levy.

Table 8 shows the critical load coefficients C_r for clamped, isotropic, homogeneous plates under various combination of loads and aspect ratios. The results obtained are found to be in very good agreement with those obtained by Sa[50]. The present results are also compared with those obtained by Timoshenko and Gere[53] for the case of biaxial buckling and those obtained by Levy[30] and Maubetsch[34] for uniaxial buckling and a very close agreement is found. Sa obtained his solutions by using the method of collocation least square. The Ritz solutions obtained by Levy and Maubetsch appears to be upperbound.

Table 9 shows the critical buckling load for the skew sandwich plate for different aspect ratios and skew angles. To ensure polar symmetry equations (5.2.2) and (5.1.4) for deflection W are used for analyzing the skew sandwich plate. For the rectangular sandwich plate (when $\phi = 0$ deg.), since it is not required to ensure polar symmetry equations (5.2.1) and (5.2.3) for deflection W are used on the differential equation (5.1.2) after transforming the equation into dimensionless form. The results show an

increase in the buckling load with the increase in the skew angle. This is because of the increasing rigidity of the plate at the obtuse corners. No comparison of these results were made possible as no results by other investigators were readily available during this investigation.

From the study presented here it can be concluded that the present method is very simple to apply for obtaining the approximate buckling loads for sandwich and layered plates.

Chapter VI

FREE VIBRATION OF CLAMPED HOMOGENEOUS AND SKEW SANDWICH PLATES

In many design problems, specifications merely ensuring that plates will withstand applied static loads prove to be inadequate. Rather, the designer must be concerned with the possibility that large cyclic displacements and stresses will be induced by periodic or random time varying forces acting on the plate's lateral surfaces. Random forces are to be expected for example; on the surface of plate exposed to tangential gas flow, as in aircraft components or in stationary structures exposed to high wind velocities. Such forces can also be expected when plates are subjected to tangential liquid flow, for example when plates are used in the hulls of surface ships or submarines or even in fixed submerged structures such as offshore oil drilling equipment. Regular or periodic excitation forces are likely to be experienced when plates form part of a structure housing rotating or reciprocating machinery. The source of such periodic forces could be reciprocating engines of compressors.

It is well known that associated with each plate natural frequency there exists a distinct or characteristic mode shape which the plate acquires as it vibrates. There is usually little that can be done to change the nature of the driving forces. The most commonly used techniques for resolving design problems in order to avoid resonance is to alter the plate geometry or boundary conditions so that its natural frequencies are removed from the frequency region where driving energy is available. So the ability to conduct an accurate free vibration analysis of plates is absolutely essential if the designer is concerned with the possible resonance between the plate and a driving force system. In fact free vibration analysis is the essential first step toward obtaining solutions for the forced vibration of rectangular plates.

In this chapter we in general shall restrict ourselves to the free vibration analysis of clamped isotropic homogeneous and clamped skew sandwich plates.

6.1 THE DIFFERENTIAL EQUATIONS

The differential equations governing the pure bending (linear) of homogeneous plates is well known and also derived in chapter 4;

$$w(x,y),_{xxxx} + 2w(x,y),_{xxyy} + w(x,y),_{yyyy} = \frac{q}{D} \quad (6.1.1)$$

where q is the applied static loading.

In effect this equation states that the sum of the transverse forces acting on a differential plate element including the lateral loading is equal to zero.

In the free vibration analysis of plate there will be no surface loading q but there will exist an inertial body force which must be taken into consideration. This inertial force is due to the oscillatory nature of the plate motion. Let us focus our attention on the differential plate element in figure 7. It will be appreciated that an inertial force of magnitude $\rho \Delta A \frac{\partial^2 w}{\partial t^2}$ continues to act on the plate element and opposes acceleration. The infinitesimal area ΔA is the product of the lengths dx and dy ; and ρ is the mass per unit surface area. The lateral load q of equation (6.1.1) must be replaced by this inertial force in order to develop the governing differential equation for the free vibration analysis of rectangular plates. We thus obtain the governing differential equation as follows;

$$-w(x,y,t),_{xxxx} + 2w(x,y,t),_{xxyy} + w(x,y,t),_{yyyy} + \rho w(x,y,t),_{tt} = 0 \quad (6.1.2)$$

Displacement W is now expressed as a function of the coordinates x, y and time t .

The governing differential equation for the free vibration analysis of rectangular sandwich plate was derived by Falgout[13] by superposing the bending deflection and the deflection due to transverse shear:

$$w(x,y,t),_{xxxx} + 2w(x,y,t),_{xxyy} + w(x,y,t),_{yyyy} = \rho \left[\frac{1}{U_1} w(x,y,t),_{xxtt} + \frac{1}{U_1} w(x,y,t),_{yytt} - \frac{1}{D'} w(x,y,t),_{tt} \right] \quad (6.1.3)$$

$$\text{where, } U_1 = G_c C \quad D' = \frac{E_f}{(1-\nu^2)} (t^3/6 + th^2/2)$$

It is reasonable to assume that the displacement function $W(x,y,t)$, be expressed as a product of two functions one involving only the space coordinates and the other involving the variable time. The equality is therefore,

$$w(x,y,t) = w(x,y)T(t) \quad (6.1.4)$$

substituting (6.1.4) into equations (6.1.2) and (6.1.3) we get;

from equation (6.1.2),

$$\frac{D}{\rho} [w(x,y),_{xxxx} + 2w(x,y),_{xxyy} + w(x,y),_{yyyy}] / w = -T(t),_{tt} / T(t) \quad (6.1.5)$$

and from equation (6.1.3)

$$\frac{w(x,y),_{xxxx} + 2w(x,y),_{xxyy} + w(x,y),_{yyyy} + w(x,y),_{yyyy}}{-\frac{\rho}{U_1} w(x,y),_{xx} - \frac{\rho}{U_1} w(x,y),_{yy} + \frac{\rho}{D} w(x,y)} = \frac{-T(t),_{tt}}{T(t)} \quad (6.1.6) \quad 71$$

In equations (6.1.5) and (6.1.6) the left hand side is a function of the variables x and y only while the right hand side is a function of t only. These equations can only be valid if both sides of each equation are equal to a constant. Let us denote this constant by a positive real quantity ω^2 . From the right hand side of the equations we then obtain ;

$$T(t),_{tt} + \omega^2 T(t) = 0 \quad (6.1.7)$$

The solution of this ordinary differential equation is well known;

$$T(t) = A \sin(\omega t + \alpha)$$

where A and α are the constants to be determined.

It is clear that $T(t)$ varies sinusoidally with time. The quantity ω which dictates the frequency of these sinusoidal oscillations is known as the circular frequency. The quantity α , represents a phase angle and can be neglected when conducting free vibration studies. In fact the reference can be always selected such that α is always zero.

From the left hand side of the equation (6.1.5) and (6.1.6) we obtain respectively;

$$w(x,y),_{xxxx} + 2w(x,y),_{xxyy} + w(x,y),_{yyyy} - \frac{\omega^2 \rho}{D} w(x,y) = 0 \quad (6.1.8)$$

$$w(x,y),_{xxxx} + 2w(x,y),_{xxyy} + w(x,y),_{yyyy} = \frac{\rho\omega^2}{D'} \left[-\frac{D'}{U_1} (w(x,y),_{xx} + w(x,y),_{yy}) + w \right] \quad (6.1.9)$$

Equations (6.1.8) and (6.1.9) are the homogeneous partial differential equations involving the mode shape expression $w(x,y)$, the plate properties and the circular frequency of oscillation ω . It is only due to the presence of w in these equations they are coupled with equation (6.1.7). The time variable therefore enters equations (6.1.8) and (6.1.9) implicitly.

In investigating the free vibration analysis of skew sandwich plate a oblique coordinate system namely α and β parallel to the sides of the plate are adopted as in figure 6. Using the relationship between the oblique and rectangular coordinate system shown in previous chapter the equation (6.1.9) can be transformed into oblique coordinate system. Which takes the following form;

$$\begin{aligned} & w,_{\alpha\alpha\alpha\alpha} \sec^4 \phi + w,_{\alpha\alpha\alpha\beta} (-4\sec^3 \phi \tan \phi) + w,_{\alpha\alpha\beta\beta} (6\sec^2 \phi \tan^2 \phi + 2\sec^2 \phi) \\ & + w,_{\alpha\beta\beta\beta} (-4\sec^3 \phi \tan \phi) + w,_{\beta\beta\beta\beta} \sec^4 \phi \\ & = \frac{\rho\omega^2}{D'} \left[-\frac{D'}{U_1} \{ w,_{\alpha\alpha} (\sec^2 \phi) + w,_{\alpha\beta} (-2\sec \phi \tan \phi) + w,_{\beta\beta} (\sec^2 \phi) \} + w \right] \end{aligned} \quad (6.1.10)$$

For further computational advantage these equations (6.1.8) and (6.1.10) are again transformed into dimensionless forms using the following dimensionless ratios.;

for the homogeneous plate ;

$$\xi = x/a \quad \eta = y/b \quad \lambda = a/b$$

and for the skew sandwich plate;

$$\lambda = a/b, \quad \xi = \frac{x}{a}, \quad \eta = \frac{y}{b}$$

Thus in the dimensionless form the governing differential equations for the plates can be written as;

for rectangular homogeneous plate;

$$W_{,\xi\xi\xi\xi} + 2\lambda^2 W_{,\xi\xi\eta\eta} + \lambda^4 W_{,\eta\eta\eta\eta} = \frac{\omega^2 a^4 \rho}{D} W \quad (6.1.11)$$

and for skew sandwich plate;

$$\begin{aligned} & \sec^4 \phi W_{,\xi\xi\xi\xi} + 2\lambda^2 W_{,\xi\xi\xi\eta} \lambda (-4\sec^3 \phi \tan \phi) + W_{,\xi\xi\eta\eta} \lambda^2 (6\sec^2 \phi \tan^2 \phi + 2\sec^2 \phi) \\ & + W_{,\xi\eta\eta\eta} \lambda^3 (-4\sec^3 \phi \tan \phi) + W_{,\eta\eta\eta\eta} \lambda^4 (\sec^4 \phi) \\ & = \frac{\rho \omega^2 a^4}{D'} \left[-\frac{D'}{U_1} \left\{ \frac{W_{,\xi\xi}}{a^2} \sec^2 \phi + \frac{W_{,\xi\eta}}{ab} (-2\sec \phi \tan \phi) \right. \right. \\ & \quad \left. \left. + \frac{W_{,\eta\eta}}{b^2} (\sec^2 \phi) \right\} + W \right] \end{aligned} \quad (6.1.12)$$

6.2 BOUNDARY CONDITIONS AND APPROXIMATE FUNCTIONS

Since the plates are fixed the boundary conditions used in the previous chapter will be valid for the present investigation. Before choosing the displacement functions we can place all the vibration modes of the clamped plate into 4 categories;

- (1) Modes symmetric about both axis
- (2) Modes antisymmetric with the ξ axis
- (3) Modes antisymmetric with the η axis

(4) Modes antisymmetric with both the axis

It is evident that the mode shape associated with any eigen value will be composed of an integral number of half sine waves running in the direction of each axis. The higher eigenvalues, with associated higher frequencies, will have more complicated shapes involving higher number of these half sine waves. The simplest mode of all free vibration modes with the lowest of all frequencies will involve a half sine wave running in each of the two coordinate directions. This is known as the fundamental mode.

The functions chosen for the approximation of the deflected shape satisfying all the boundary condition of the problem, for modes of above 4 categories are as follows;

for the homogeneous plate;

Mode category 1

$$W = (1-\xi^2)^2(1-\eta^2)(c_1+c_2\xi^2 + c_3\eta^2 + c_4\xi^2\eta^2 + c_5\xi^4 + c_6\eta^4 + c_7\xi^4\eta^4)$$

Mode category 2

$$W = (1-\xi^2)^2(1-\eta^2)^2\xi(c_1 + c_2\xi^2 + c_3\eta^2 + c_4\xi^2\eta^2 + c_5\xi^4 + c_6\eta^4 + c_7\xi^4\eta^4)$$

Mode category 3

$$W = (1-\xi^2)^2(1-\eta^2)^2\eta(c_1+c_2\xi^2 + c_3\eta^2+c_4\xi^2\eta^2 + c_5\xi^4+c_6\eta^4 + c_7\xi^4\eta^4)$$

for mode category 4

$$W = (1-\xi^2)^2(1-\eta^2)^2\xi\eta(c_1 + c_2\xi^2 + c_3\eta^2 + c_4\xi^2\eta^2 + c_5\xi^4 + c_6\eta^4 + c_7\xi^4\eta^4)$$

and for the skew sandwich plate;

Mode category 1

$$W=(1-\xi^2)^2(1-\eta^2)^2(1-\xi\eta)(c_1+c_2\xi^2+c_3\eta^2+c_4\xi^2\eta^2+c_5\xi^4+c_6\eta^4+c_7\xi^4\eta^4)$$

Mode category 2

$$W=(1-\xi^2)^2(1-\eta^2)^2(1-\xi\eta)\xi(c_1+c_2\xi^2+c_3\eta^2+c_4\xi^2\eta^2+c_5\xi^4+c_6\eta^4+c_7\xi^4\eta^4)$$

Mode category 3

$$W=(1-\xi^2)^2(1-\eta^2)^2(1-\xi\eta)\eta(c_1+c_2\xi^2+c_3\eta^2+c_4\xi^2\eta^2+c_5\xi^4+c_6\eta^4+c_7\xi^4\eta^4)$$

for mode category 4

$$W=(1-\xi^2)^2(1-\eta^2)^2(1-\xi\eta)\xi\eta(c_1+c_2\xi^2+c_3\eta^2+c_4\xi^2\eta^2+c_5\xi^4+c_6\eta^4+c_7\xi^4\eta^4)$$

The $(1-\xi\eta)$ term is used to ensure polar symmetry of the skew sandwich plate.

6.3 RESULTS AND CONCLUSIONS

Galerkin method is used to obtain the solution for the free vibration analysis of the plates. For the homogeneous and sandwich plates results were obtained for different aspect ratios λ ranging from 1 to 2. All the solutions presented for a particular mode category are obtained by using all the 7 terms in the assumed displacement function for that mode category. For the sandwich plate results were obtained by varying the shear modulus of the core and the skew angle.

To determine the convergence characteristic of the method for free vibration analysis numerical experimentation was carried out by varying the number of terms in the displacement function W . Results thus obtained are shown in Table 10. For the fundamental frequency it can be seen that the result varies very little less than 1% even with 200% increase in the number of terms. For the other frequencies the convergence is good. It can be seen that the higher the number of terms in the displacement function the more accurate are the lower frequencies.

Table 11 shows the frequency for 28 modes of vibration of a rectangular homogeneous plate for different aspect ratios. The results obtained in the present study are compared with those obtained by Odman[44] and Young[6] and a very close agreement is found.

Table 12 shows the frequencies for the rectangular sandwich plates for different aspect ratios. Table 13 shows the frequencies for the skew sandwich plate for various aspect ratio and skew angle $\phi=30$ degree. In these tables The fundamental frequencies in each of the four mode category are found in close agreement with the finite element solution obtained by Lam[28]. Because of the lack of data in hand no comparison can be made possible for the higher frequencies in each mode category.

Table 14 shows the variation of natural frequencies of a clamped rectangular sandwich plate for different aspect

ratios and core rigidities. Table 15 shows the corresponding variation for the clamped skew sandwich plate for different aspect ratios and skew angles. It can be seen that the frequencies tend to increase with the increase in the core rigidity, because of the increase in the stiffness of the plate. It is also observed that the frequency increases with the increase in the skew angle of the plate. This is due to the increasing rigidity of the plate at the obtuse corners with the increase in the skew angle, which contributes to increase the apparent stiffness of the plate. Due to lack of data in hand no comparison of these results can be made possible.

From the tables presented for the free vibration analysis it can be found that the frequencies in the 2nd and 3rd mode category are same. These are called degenerate modes. This degeneracy disappears with the deviation of the aspect ratio from 1.

From this study it can be observed that the Galerkin method is sufficiently accurate and easy to apply for the approximate solution of free vibration problems. The computer time and memory requirement for free vibration analysis are also found to be small. Thus it can be concluded that the Galerkin method is suitable for the free vibration analysis of sandwich and layered plates.



Chapter VII

CONCLUSION

As a result of the present investigations the following conclusions can be drawn.

1. For small and large deflection analysis the Galerkin method is found to yield results which are in close agreement with those obtained by much more laborious methods. The same comment is also true for eigen value problems.
2. With the increase of the skew angle the plate becomes much more stiff due to the increasing rigidity at the obtuse corners. As a result of this, the plate tends to deflect less at high skew angles under the same lateral loadings.
3. For a given aspect ratio a/b , the maximum centre deflection of the plate decreases with the increasing value of the foundation modulus. This is to be expected since the object of the foundation modulus is to reduce the center deflection.
4. With the increasing value of the foundation modulus the load-deflection curves tend to become linear due to increasing rigidity of the plate contributed by the foundation modulus.

5. When the aspect ratio of the plate^{is} less than $1/3$, the plates can be considered as infinitely long plates.
6. The maximum center deflection decreases with the increasing core rigidity of the sandwich plate, due to overall stiffness increase of the plate structure.
7. To determine the critical load for the buckling of plates, the Galerkin method has been found to yield excellent results even with only small number of terms adopted in the displacement function.
8. Due to the increasing rigidity of the plate with increase in the angle of skew it has been found that the critical buckling load is very sensitive to increase in the skew angle.
9. For the vibration analysis this method is found to yield results which are in excellent agreement with those obtained by much more laborious methods. The higher the number of terms the better is the result obtained by this method for the lower frequencies.
10. The increasing rigidity of the plate due to increase in the skew angle or in the shear rigidity tend to increase the frequency of the plate.
11. For aspect ratio 1 the frequencies in the 2nd and 3rd mode category are found to be of same magnitude. These are called degenerate modes. This degeneracy disappears with the deviation of aspect ratio from 1.

Finally from the present study it can be concluded that, the present method is extremely versatile and sufficiently accurate for all types of analysis considered. The integrals in the formulation of the Galerkin equation can be evaluated in closed form. Simplicity in the formulation and computation is the advantage found in the present method compared to other much more laborious numerical methods. Computing time and memory requirements are small, thus making it ideally suited for the computers with relatively smaller capacity and speed.

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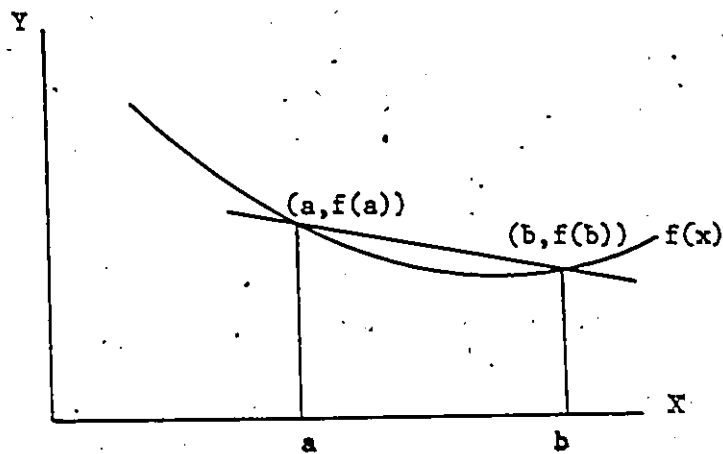


Fig. 1 Trapezoidal rule applied to one dimensional case

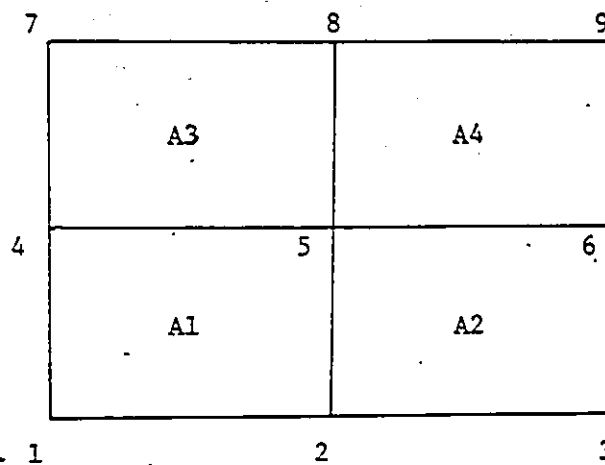


Fig. 2 Trapezoidal rule applied to two dimensional case

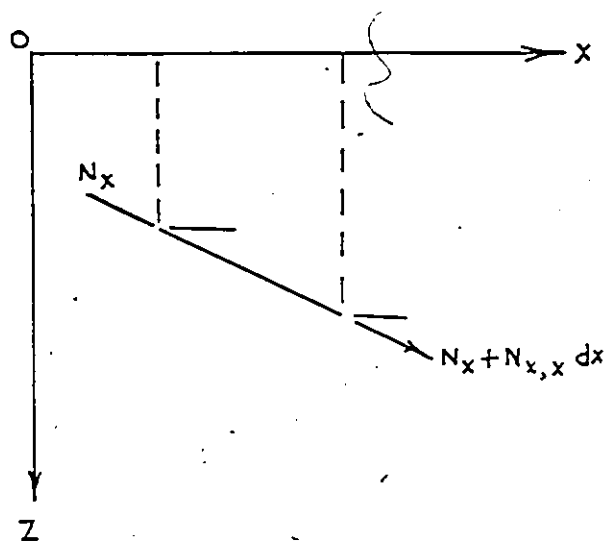
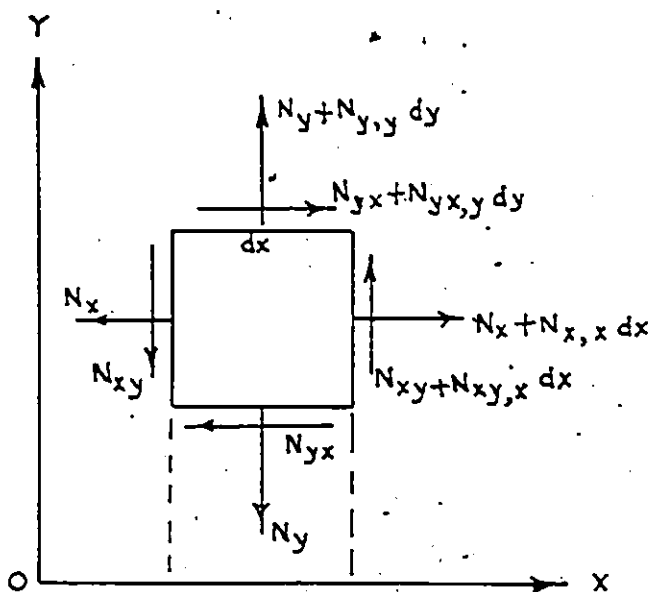


Fig. 3 Inplane forces on plate element

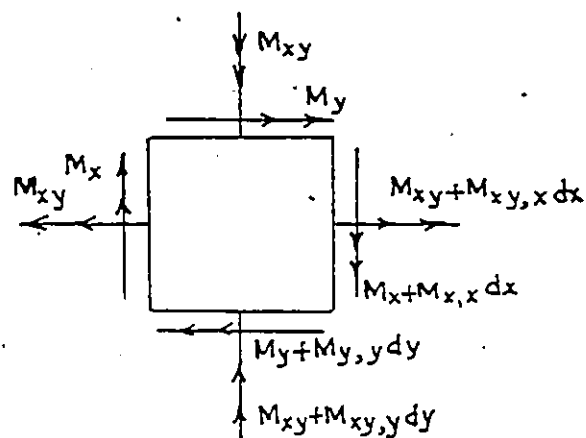
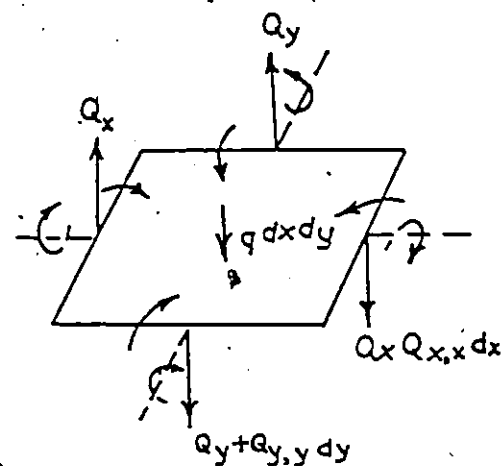


Fig. 4 Moment and Shear forces on plate element

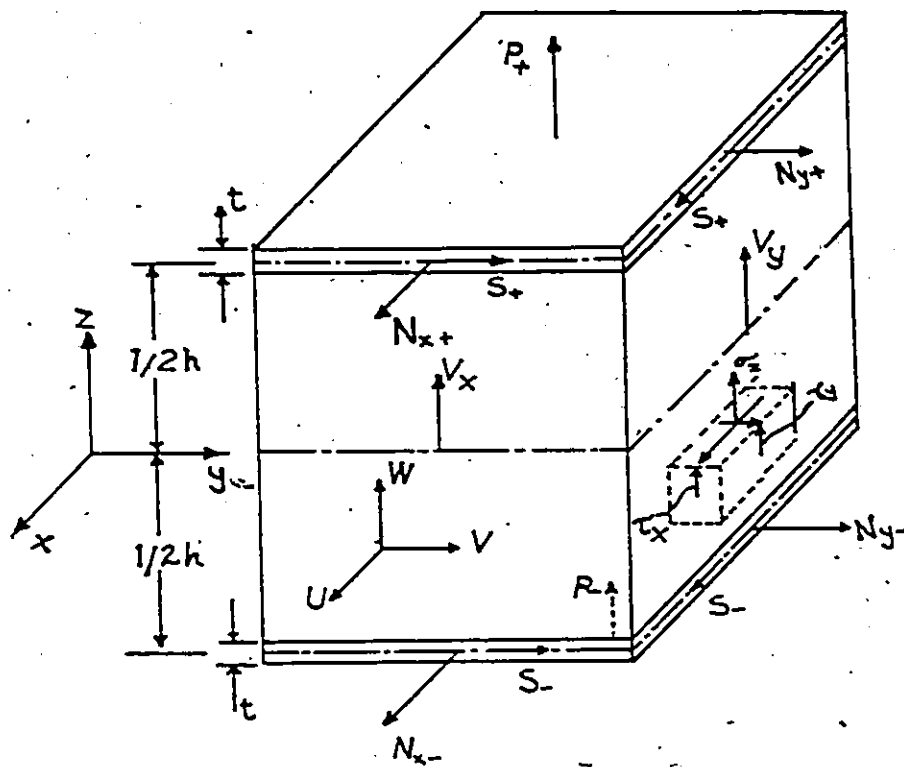


Fig. 5 Element of Sandwich plate

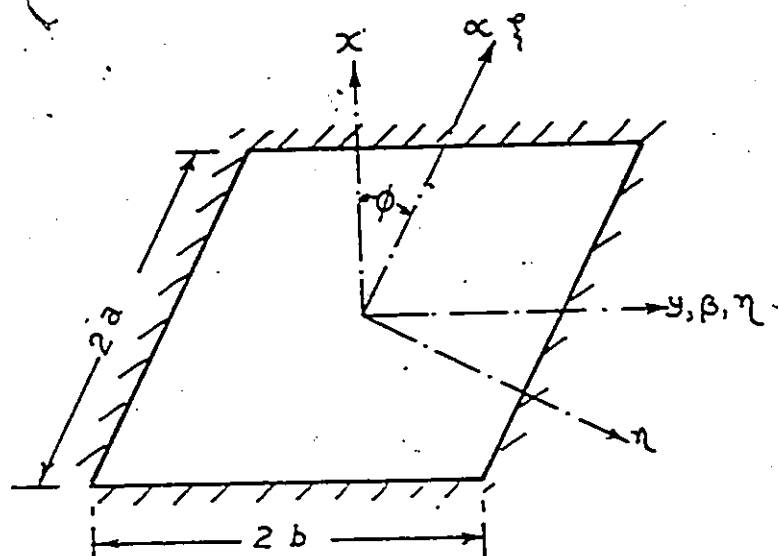


Fig. 6 Rectangular and oblique coordinate system

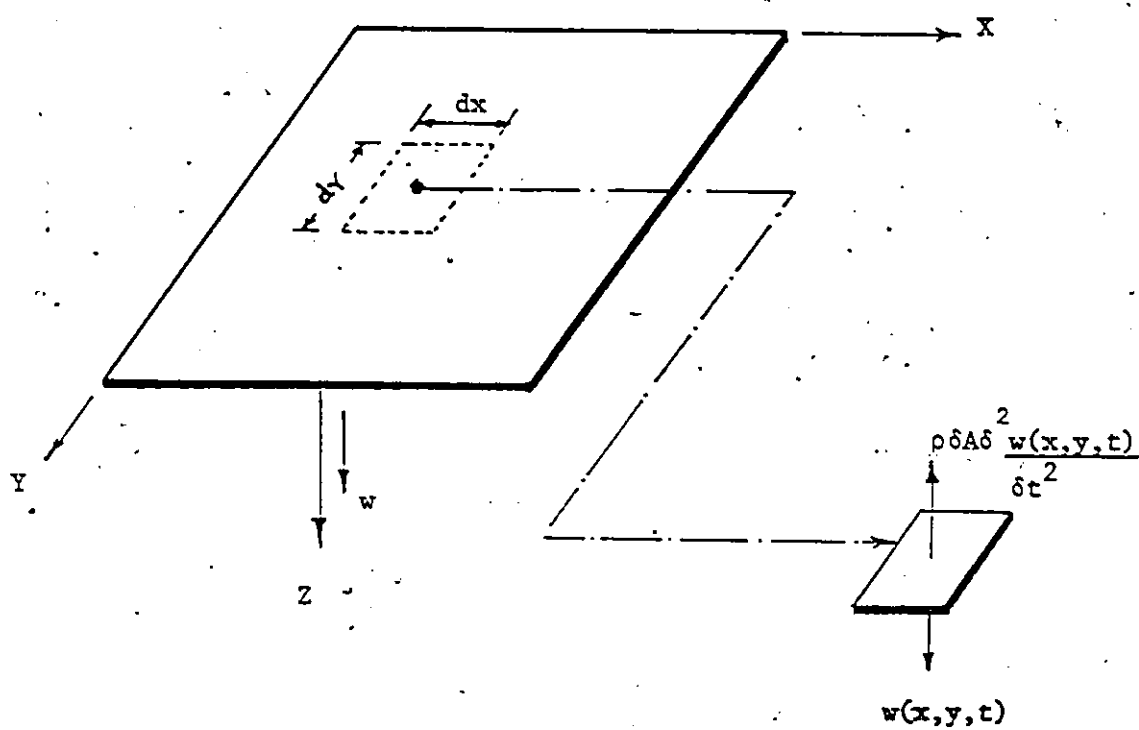


Fig. 7 Rectangular plate element subjected to an inertial force

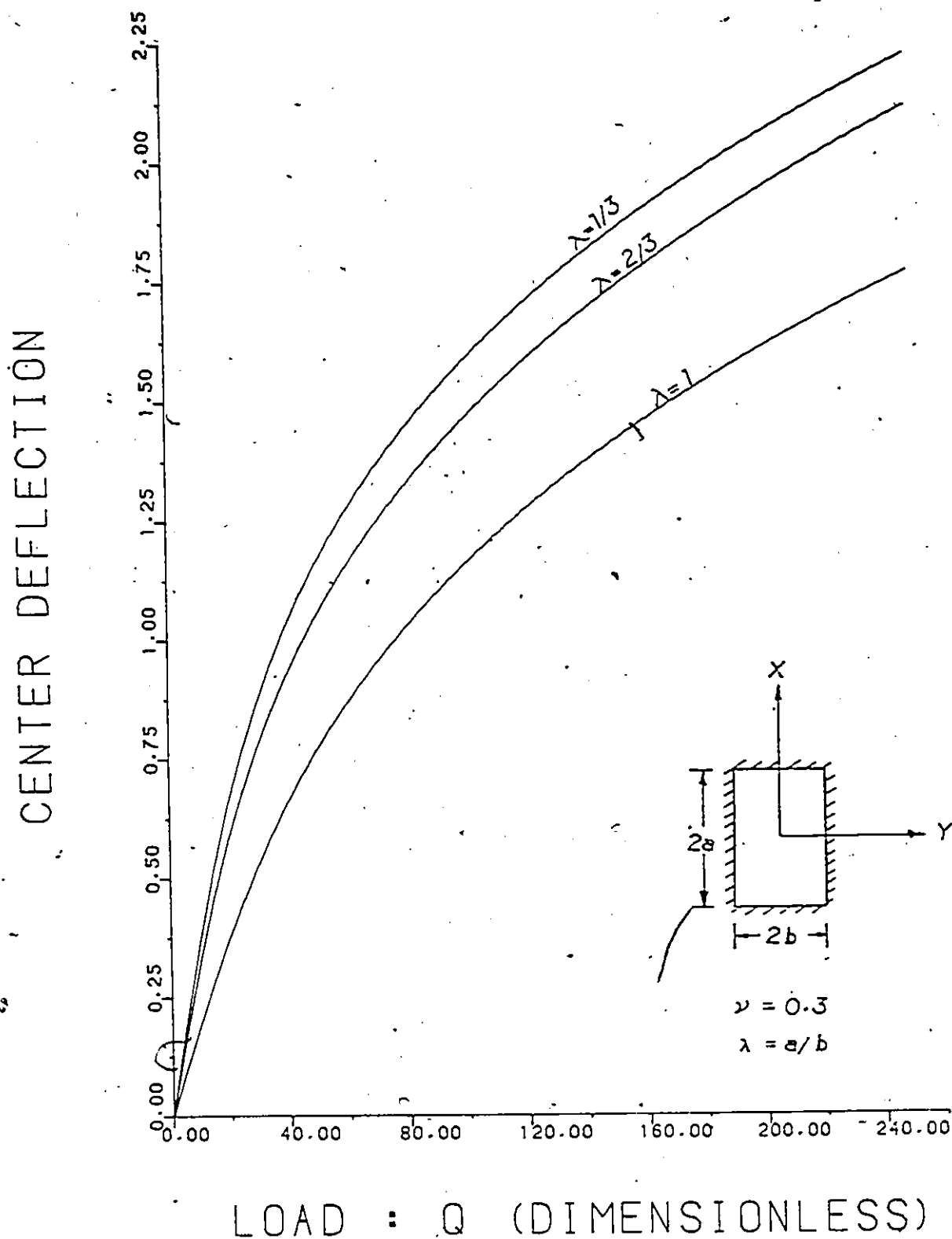


FIG. 8 Variation of central deflection with lateral pressure of clamped rectangular isotropic, homogeneous plate for various aspect ratio.

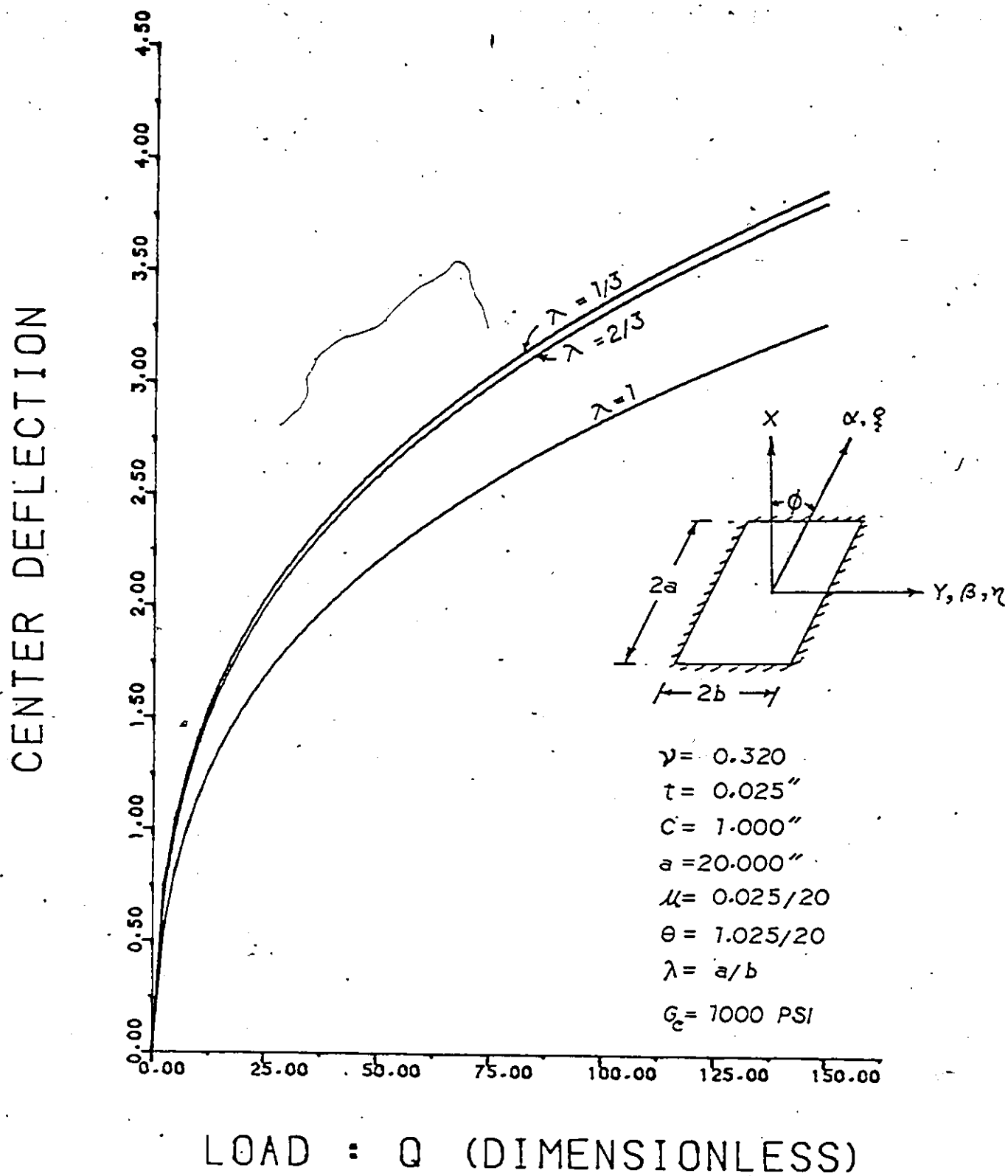


FIG. 9 Variation of central deflection with lateral pressure of clamped rectangular sandwich plate for various aspect ratio.

CENTER DEFLECTION

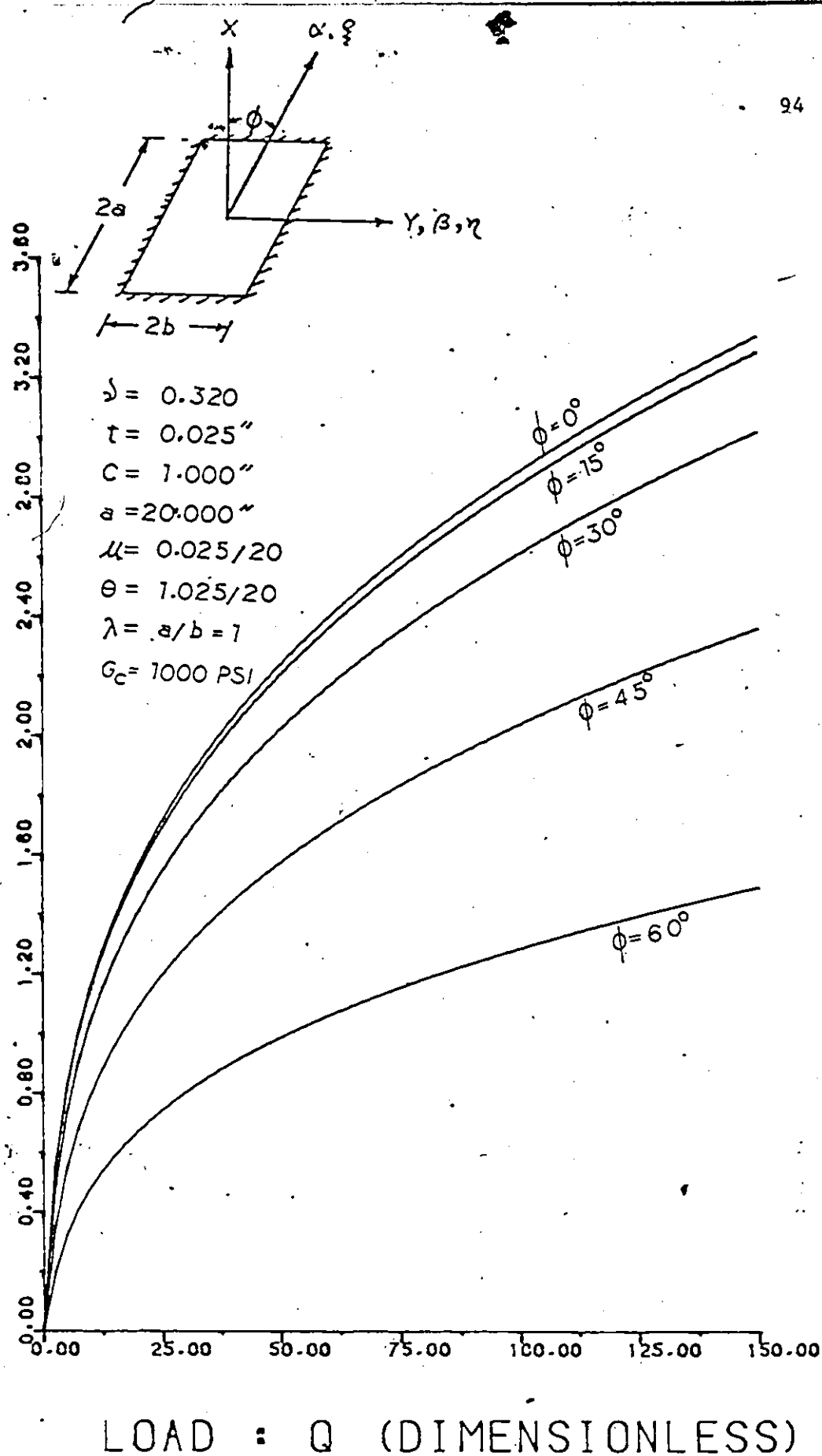


FIG. 10 Variation of central deflection with lateral pressure of clamped rectangular sandwich plate for various skew angle.

CENTER DEFLECTION

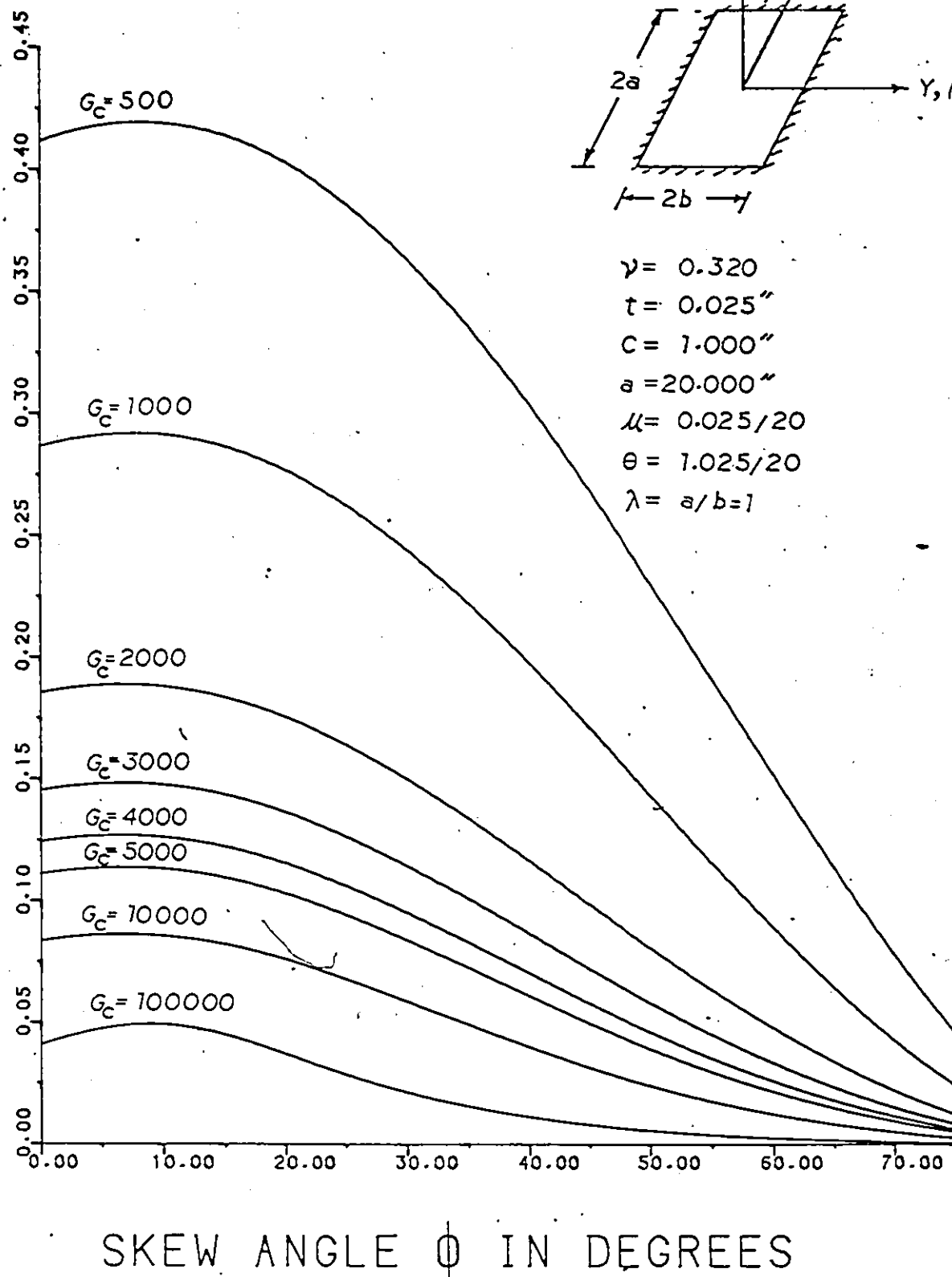


FIG. 11 Variation of central deflection with shear modulus of core G_c (psi) and skew angles of clamped skew sandwich plate.

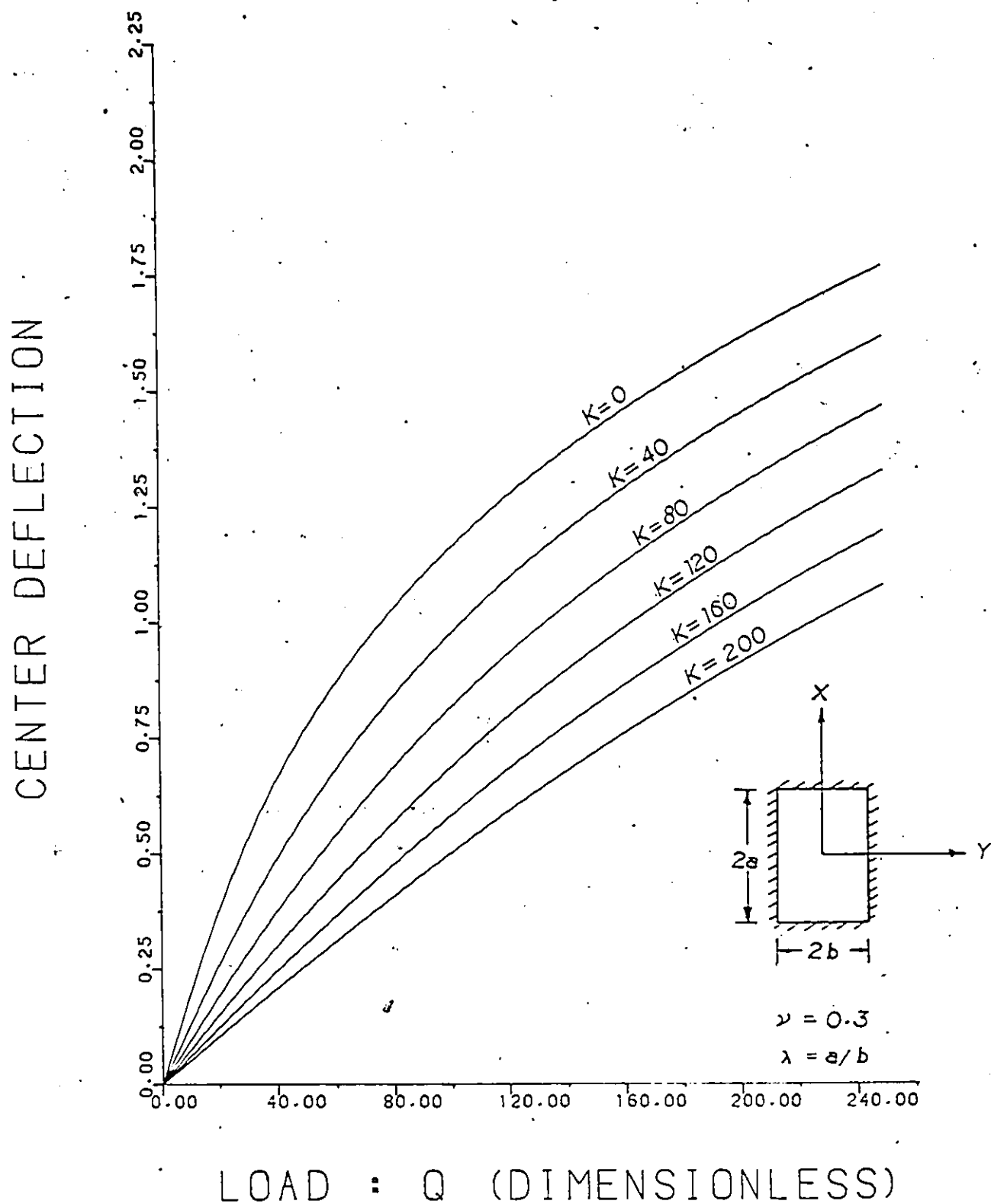


FIG.12 Variation of central deflection with lateral pressure and foundation modulus for clamped rectangular isotropic homogeneous plate of aspect ratio 1.

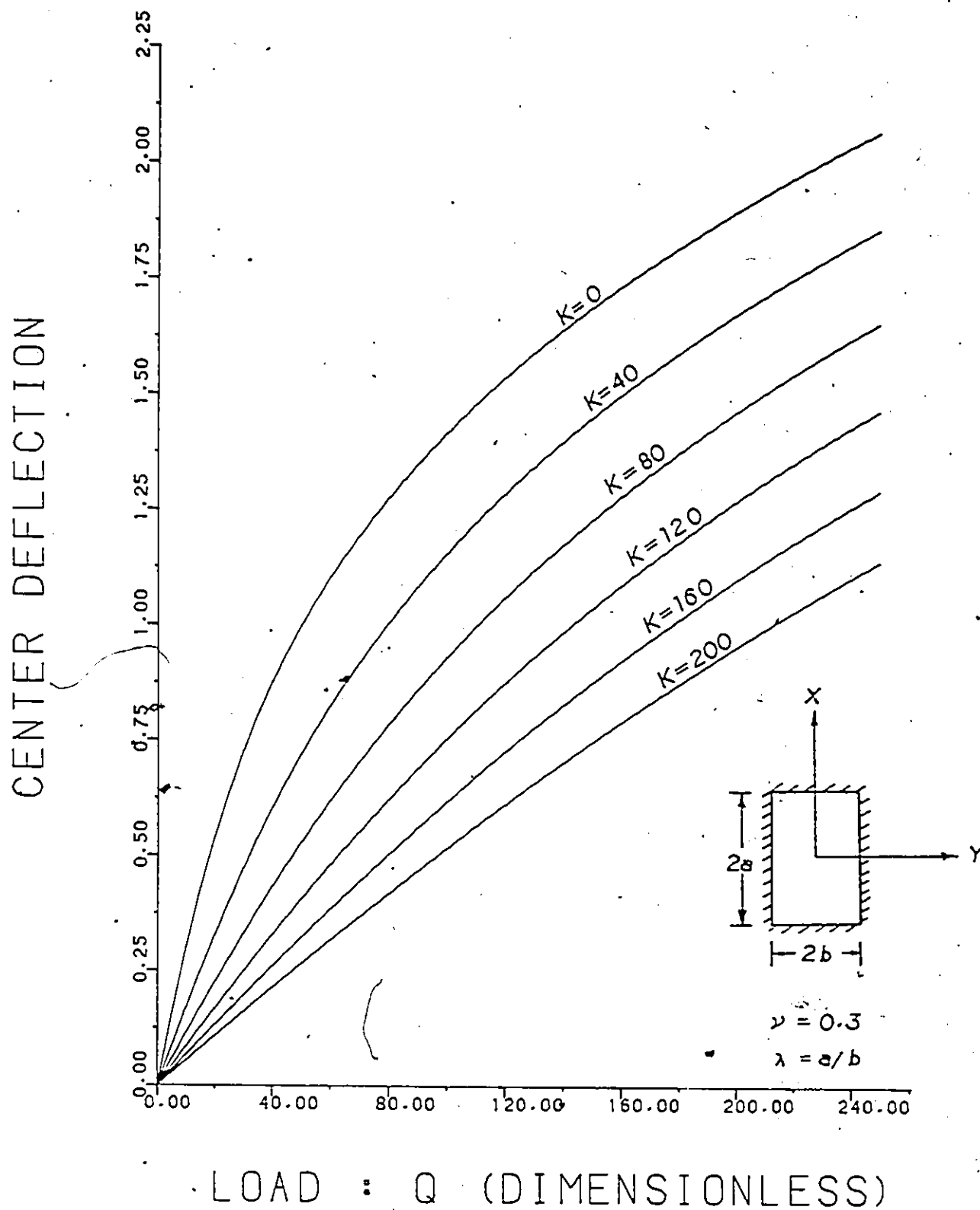


FIG. 13 Variation of central deflection with lateral pressure and foundation modulus for clamped rectangular isotropic homogeneous plate of aspect ratio $3/4$.

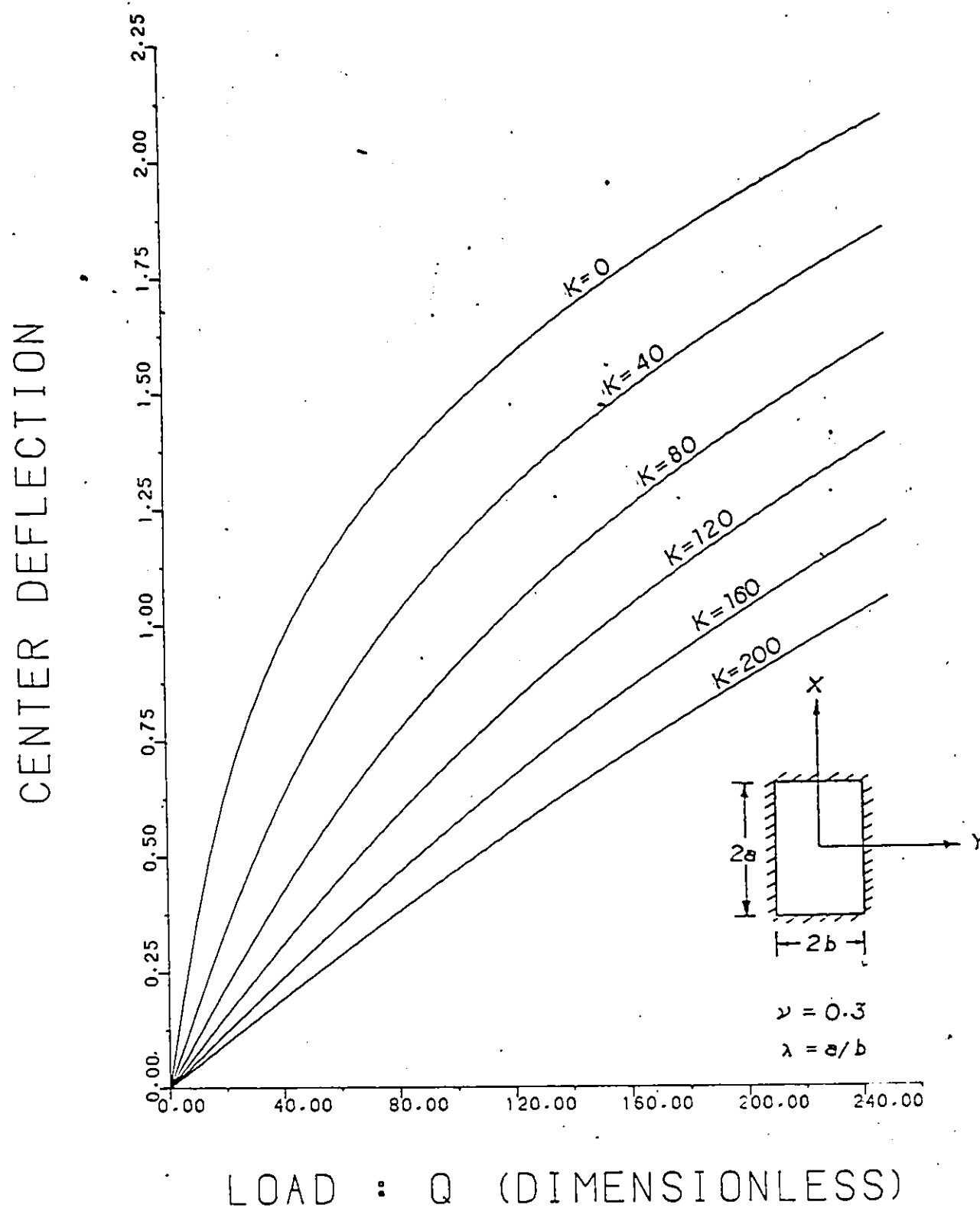


FIG. 14 Variation of central deflection with lateral pressure and foundation modulus for clamped rectangular isotropic homogeneous plate of aspect ratio $1/2$.

CENTER DEFLECTION

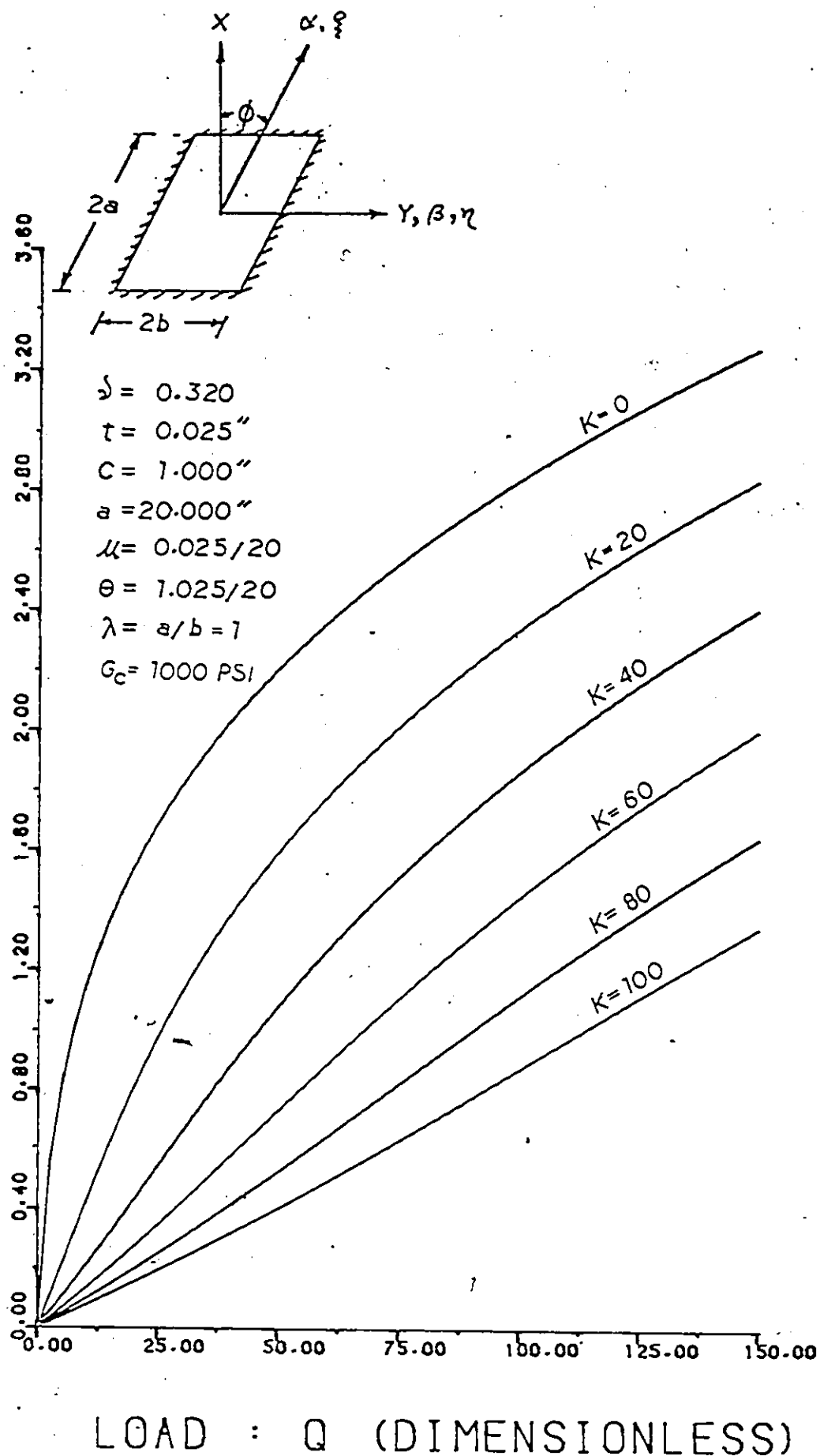


FIG.15 Variation of central deflection with lateral pressure and foundation modulus for clamped square sandwich plate.

CENTER DEFLECTION

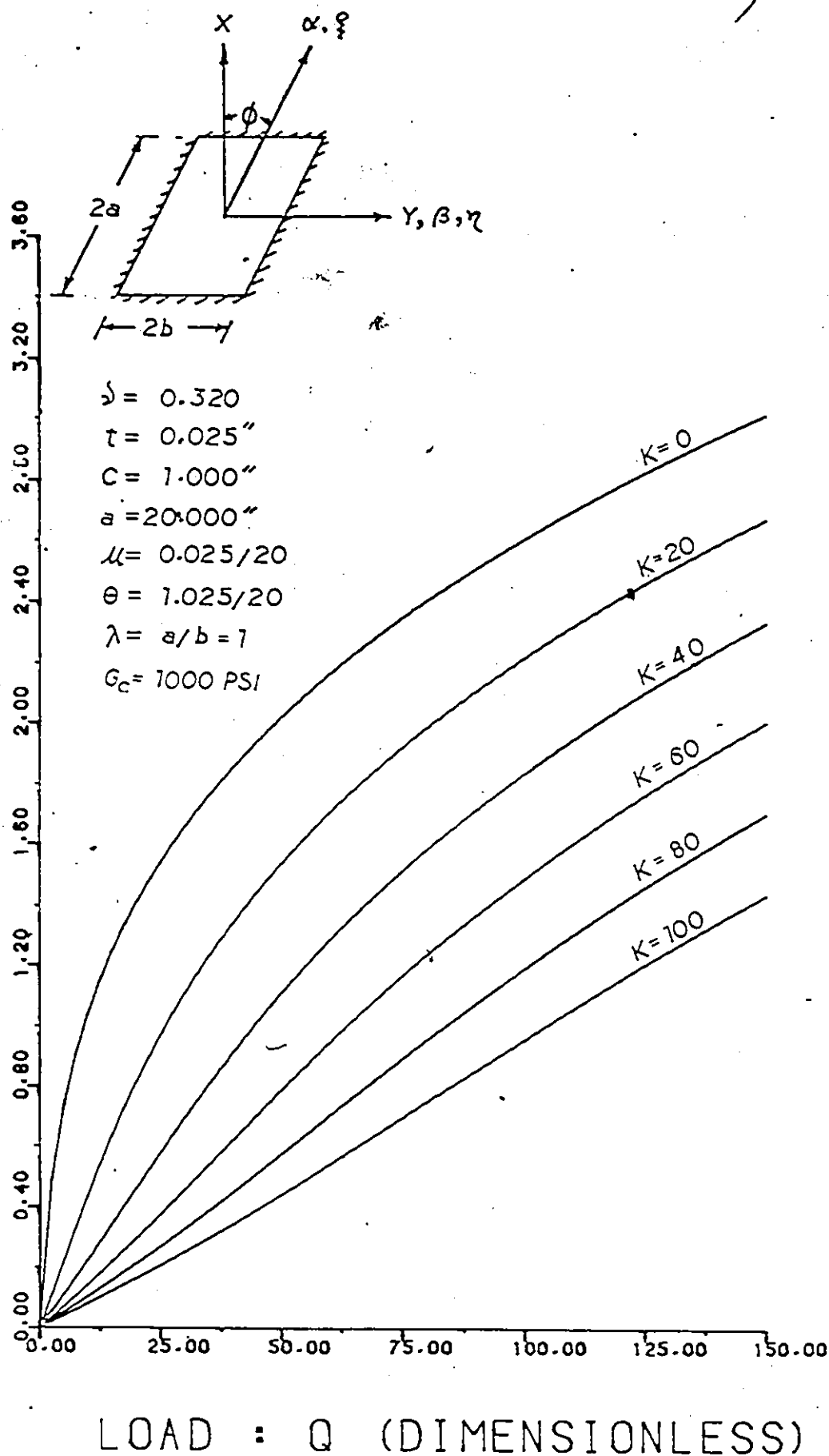


FIG.16 Variation of central deflection with lateral pressure and foundation modulus for clamped skew sandwich plate with skew angle 30° .

CENTER DEFLECTION

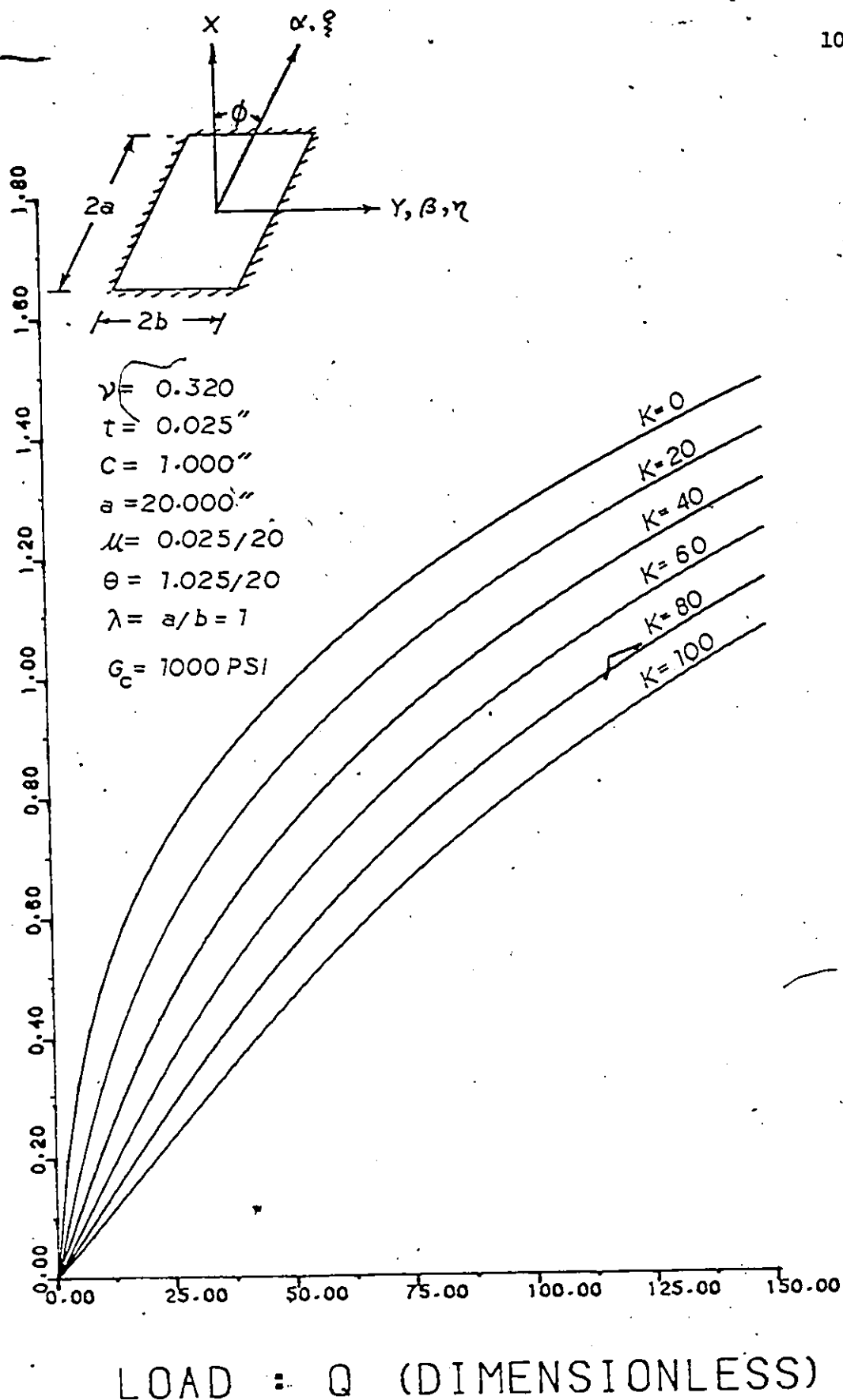


FIG17 Variation of central deflection with lateral pressure and foundation modulus for clamped skew sandwich plate with skew angle 60.

Table 1 Coefficients for the differential equations
for the skew sandwich plate

$$s = \sin \phi \quad c = \cos \phi \quad F = E_f \mu \theta / (1-\nu)^2 \quad J = \frac{(1-\nu^2) G_c}{E_f \mu}$$

$$A1K1 = 2F/c^2$$

$$A1K2 = (2\lambda^2 s^2/c^2 + \lambda^2 (1-\nu))F$$

$$A1K3 = -4\lambda s/c^2 F$$

$$A1K4 = (1+\nu) \lambda F/c$$

$$A1K5 = -4G_c \theta/c$$

$$A1K6 = -4G_c$$

$$A1K7 = -(1+\nu) \lambda^2 F s/c$$

$$A1K8 = 4G_c \theta \lambda s/c$$

$$A2K1 = F(2\lambda^2 + (1-\nu) \lambda^2 s^2/c^2)$$

$$A2K2 = F(1-\nu)/c^2$$

$$A2K3 = -2(1-\nu) \lambda F s/c^2$$

$$A2K4 = (1+\nu) \lambda F/c$$

$$A2K5 = -4G_c \lambda \theta$$

$$A2K6 = -4G_c$$

$$A2K7 = -\lambda^2 (1+\nu) (s/c) F$$

$$A3K1 = 2/c^2$$

$$A3K2 = \lambda^2 (1-\nu) + 2\lambda^2 s^2/c^2$$

$$A3K3 = -4\lambda s/c^2$$

$$A3K4 = \lambda(1+\nu)/c$$

$$A3K5 = 24/c^3$$

$$A3K6 = -48\lambda s/c^3$$

$$A3K7 = 24\lambda^2 s^2/c^3 + 12\lambda^2 (1-\nu)/c$$

$$A3K8 = 48\lambda^2 s^2/c^3 + 12\lambda^2 (1+\nu)/c$$

$$A3K9 = -24\lambda s/c^3$$

$$A3K10 = -\lambda^2 (1+\nu) s/c$$

$$A3K11 = -24\lambda^3 (s^3/c^3 + s/c)$$

$$A4K1 = 2\lambda^2 + (1-\nu) \lambda^2 s^2/c^2$$

$$A4K2 = (1-\nu)/c^2$$

$$A4K3 = -2\lambda(1-\nu) s/c^2$$

$$A4K4 = \lambda(1+\nu)/c$$

$$A4K5 = 24\lambda^3 (1 + s^2/c^2)$$

$$A4K6 = 12\lambda^2 (-3 s/c^2 + \nu s/c^2)$$

$$A4K7 = 12\lambda(1-\nu)/c^2$$

$$A4K8 = 12\lambda^2 (-s/c^2 - \nu s/c^2)$$

$$A4K9 = 12\lambda(1+\nu)/c^2$$

$$A4K10 = -\lambda^2 (1+\nu) s/c$$

Table 1 Continued

A5K1 = $2\theta/c^3$	A5K20 = $6/c^3$
A5K2 = $2\theta\lambda v/c^2$	A5K21 = $6v\lambda/c^2$
A5K3 = $12\theta/c^4$	A5K22 = $-6\lambda s/c^3$
A5K4 = $12\lambda^2\theta(s^2/c^4 + v/c^2)$	A5K23 = $12J/c^2$
A5K5 = $-2\lambda\theta s/c^3$	A5K24 = $6\lambda^3(1+s^2/c^2)$
A5K6 = $24\lambda\theta s/c^4$	A5K25 = $6\lambda^2(-(1-v) s/c^2)$
A5K7 = $2\theta\lambda^3(1+s^2/c^2)$	A5K26 = $6\lambda^2(s^2/c^3 + v/c)$
A5K8 = $-2\theta\lambda^2(1-v) s/c^2$	A5K27 = $12\lambda^2J(1+s^2/c^2)$
A5K9 = $2\lambda^2\theta(s^2/c^3 + v/c)$	A5K28 = $6\lambda(1-v)/c^2$
A5K10 = $12\lambda^4\theta(1+s^2/c^2)^2$	A5K29 = $6\lambda^2(-vs/c^2 - s/c^2)$
A5K11 = $12\lambda^3\theta(-2 s/c^4)$	A5K30 = $-12\lambda s/c^3$
A5K12 = $12\theta\lambda^2(s^2/c^4 + v/c^2)$	A5K31 = $6\lambda^2(2 s^2/c^3 + (1-v)/c)$
A5K13 = $2\lambda\theta(1-v)/c^2$	A5K32 = $-24\lambda J s/c^2$
A5K14 = $2\theta\lambda^2(-s/c^2 - vs/c^2)$	A5K33 = $12J/c\theta$
A5K15 = $-4\lambda\theta s/c^3$	A5K34 = $12J\lambda/\theta$
A5K16 = $-24\lambda^3\theta(s^3/c^4 + s^2/c^3)$	A5K35 = $-2\lambda^3\theta(s/c + s^3/c^3)$
A5K17 = $12\theta\lambda^2(4 s^2/c^4 + 2(1-v)/c^2)$	A5K36 = $-6\lambda^3(s^3/c^3 + s/c)$
A5K18 = $-24\lambda\theta s/c^4$	A5K37 = $-12J\lambda s/c\theta$
A5K19 = $\theta\lambda^2(4 s^2/c^3 + 2(1-v)/c)$	

Table 2

Comparison of the max. small deflection coefficient α of the
clamped isotropic homogeneous plate

$$W_{\max} = \alpha q a^4 / D \quad Q = 1$$

$\lambda = a/b$	5 term sol.	11 term sol.	Ref[43]
1	.020200	.02198	.020201
3/4	.031355	.03134	.031412
1/2	.039889	.03987	.035090

Table 3

Comparison of max. centre deflection of clamped skew sandwich plate

$$E_f = 10^7 \text{ psi} \quad \nu = 0.32 \quad C = 1'' \quad t = .025'' \quad 2a = 40'' \quad G_c = 10000 \text{ psi}$$

$$q = 1 \text{ psi} \quad \lambda = a/b = 1$$

	Angle of Skew			
	15°	30°	45°	60°
	(in)	(in)	(in)	(in)
7 term sol'n	.02757	.01956	.01029	.00393
8 term sol'n	.02772	.01993	.01065	.00417
16 term sol'n	.02811	.01996	.01071	.00419
Kwok[27]	.02950	.02170	.01260	.00520
Kennedy[24]	.03210	.02330	.01380	.00580

Table 4

Variation of max. small deflection coefficients α of a clamped rectangular plate with the dimensionless foundation modulus K

$$w_{\max} = \alpha q a^4 / D \quad K = k a^4 / D$$

K	$\lambda = a/b = 1$		$\lambda = 3/4$		$\lambda = 1/2$	
	Present	Ref [43]	Present	Ref [43]	Present	Ref [43]
0	.020198	.020201	.031340	.031412	.03987	.040545
20	.016007	.016033	.022095	.022201	.02490	.025525
40	.013210	.013245	.016910	.017036	.01780	.018411
60	.011203	.011251	.013602	.013741	.01370	.014307
80	.009730	.009756	.011315	.011463	.01100	.011656
100	.008534	.008594	.009643	.009797	.00920	.009812
120	.007602	.007665	.008371	.008530	.00782	.008459
140	.006841	.006903	.007371	.007534	.00679	.007427
160	.006208	.006377	.006567	.006734	.00599	.006616
180	.005674	.005745	.005911	.006076	.00534	.005963
200	.005217	.005291	.005360	.005528	.00481	.005425

Table 5

Comparison of centre deflection of clamped skew sandwich plates
with various skew angles, aspect ratio and core rigidities

$E = 10^7 \text{ psi}$ $\nu = 0.32$ $c = 1''$ $t = .025$ $2a = 40''$ $\lambda = a/b$
 G_c = shear modulus of core q = load intensity = 1 psi

ϕ	G_c (psi)	$\lambda = 1.0$		$\lambda = 1.5$		$\lambda = 2.0$	
		Present	Kwok[27]	Present	Kwok[27]	Present	Kwok[27]
		$w_c(\text{in})$	$w_c(\text{in})$	$w_c(\text{in})$	$w_c(\text{in})$	$w_c(\text{in})$	$w_c(\text{in})$
15°	1,000	.108037		.06151		.0373	
	10,000	.02811	.0295	.0122	.0130	.005999	.0065
30°	1,000	.09034		.05166		.03126	
	10,000	.01996	.0217	.00894	.0097	.0045	.0049
45°	1,000	.06141		.0355		.02154	
	10,000	.01065	.0126	.0051	.0058	.00272	.0030
60°	1,000	.03098		.01814		.01105	
	10,000	.00417	.0052	.00216	.0025	.00123	.0013

Table 6

Dimensionless deflection, W at the centre of the skew sandwich
plate
for various skew angles ϕ , aspect ratio λ and core rigidities G_c

$Q = 50$, $E = 10^7 \text{ psi}$, $\nu = 0.32$, $t = .025''$, $c = 1''$

ϕ	$\lambda = 1.0$		$\lambda = 1.5$		$\lambda = 2.0$	
	$G_c = 1000$	$G_c = 10,000$	$G_c = 1000$	$G_c = 10,000$	$G_c = 1000$	$G_c = 10,000$
15°	2.23757	1.77806	1.48499	1.05630	1.00609	0.63180
30°	2.02070	1.50564	1.32781	0.87941	0.88942	0.52018
45°	1.57303	1.03199	1.02836	0.59150	0.68504	0.34770
60°	0.98665	0.51047	0.64721	0.28973	0.43173	0.17135

Table 7 Convergence table for the buckling load coefficient C_r of rectangular plates with aspect ratio $\lambda = 1$, $N_y/N_x = 0$ and $N_\alpha = N_x = C_r \pi^2 D/4a^2$

	No. of terms				
	3 terms sol.	6 terms sol.	7 terms sol.	Levy[30]	Thurston[52]
Homogeneous plate	10.087	10.081	10.074	10.07	-----
Sandwich plate with $G_c = 3500 \text{ psi}$ $E_f = 9.9 \times 10^6 \text{ psi}$ $t = .012 \text{ in.}$ $c = .244 \text{ in.}$ $a = 37.27/2 \text{ in.}$	8.481	8.475	8.465	----	7.94

Table 8 Comparison of buckling load coefficient C_r for clamped rectangular homogeneous plate with various aspect ratio. $N_x = C_r \pi^2 D / 4a^2$

N_y / N_x	$\lambda = 1$		$\lambda = 1.25$		$\lambda = 1.50$		$\lambda = 1.75$		$\lambda = 2.0$	
	Present Solution	Sa Ref.[50]	Present Solution	Sa Ref.[50]	Present Solution	Sa Ref.[50]	Present Solution	Sa Ref.[50]	Present Solution	Sa Ref.[50]
0	10.074	9.957	14.470	14.996	18.790	18.802	24.840	24.712	31.670	31.462
Levy Ref.[30]	—	10.07	—	14.45	—	18.74	—	24.84	—	31.52
Maulbetich Ref.[34]	—	10.48	—	14.65	—	19.01	—	25.02	—	32.24
1/8	9.138	9.109	13.336	13.281	17.643	17.679	22.952	22.943	29.785	29.533
1/4	8.329	8.339	12.380	12.261	16.606	16.652	21.254	21.302	27.283	27.236
1/2	7.031	7.063	9.928	9.965	13.953	13.932	18.406	18.481	23.022	23.118
1.0	5.304	5.333	6.864	6.847	9.273	9.326	12.194	12.261	15.694	15.769
Timoshenko Ref.[53]	—	5.33	—	—	—	—	—	—	—	—
2.0	3.515	3.532	4.307	4.325	5.459	5.478	6.933	6.952	8.701	8.721
4.0	2.082	2.085	2.430	2.435	2.983	2.987	3.707	3.709	4.584	4.584
8.0	1.142	1.139	1.296	1.296	1.562	1.562	1.919	1.917	2.353	2.351

Table 9 Buckling load coefficient C_r for clamped skew sandwich plates for various aspect ratio λ and skew angle ϕ

$$t = 0.012 \text{ in.} \quad c = 0.244 \text{ in.} \quad 2a = 37.27 \text{ in.} \quad E_f = 9.9 \times 10^6 \text{ psi}$$

$$G_c = 3500.0 \text{ psi} \quad \mu = 0.013748 \quad N_a = C_r \pi^2 D / 4a^2 \quad \phi = 0.3$$

O	λ	N_y/N_x						
		8	4	2	1	1/2	1/4	1/8
0	1.0	0.9647	1.7623	2.9788	4.4949	5.9576	7.0492	7.7179
	1.25	1.0352	1.9457	3.4624	5.6331	8.0761	9.8357	10.2905
	1.50	1.1642	2.2304	4.1080	7.0539	10.6094	11.9644	12.7459
	1.75	1.3230	2.5662	4.8356	8.6259	12.5960	14.6377	15.5287
	2.00	1.4933	2.9202	5.5882	10.2415	14.8711	16.9843	18.0233
15	1.00	1.0447	1.8961	3.1762	4.7421	6.2218	7.3066	7.9641
	1.25	1.1199	2.0937	3.6932	5.9331	8.3824	10.2691	10.8579
	1.50	1.2606	2.4026	4.3845	7.4171	11.0198	12.6178	13.3964
	1.75	1.4338	2.7662	5.1611	9.0484	13.3148	15.3125	16.2345
	2.00	1.6187	3.1477	5.9608	10.7110	15.6094	17.7652	18.8217
30	1.00	1.4798	2.5884	4.1119	5.7807	7.2010	8.1706	8.7401
	1.25	1.5686	2.8386	4.7567	7.1446	9.4497	11.1295	12.0905
	1.50	1.7437	3.2233	5.5899	8.7916	12.1418	14.0887	14.7523
	1.75	1.9581	3.6676	6.5023	10.5531	14.9251	16.6876	17.5985
	2.00	2.1838	4.1251	7.4197	12.3039	17.1781	19.1917	20.0527
45	1.00	2.8396	4.5720	6.5450	8.3091	9.5749	10.3487	10.7779
	1.25	2.9533	4.9522	7.4683	9.9782	11.9487	13.2172	13.9337
	1.50	3.1914	5.4875	8.5583	11.8492	14.5976	16.3963	17.3623
	1.75	3.4778	6.0738	9.6805	13.7331	17.2507	19.3099	19.9683
	2.00	3.7703	6.6509	10.7528	15.5106	19.7073	21.5495	22.2741
60	1.00	7.1416	9.8184	12.0507	13.5775	14.4862	14.9839	15.2445
	1.25	7.2536	10.4320	13.3392	15.4788	16.8135	17.5631	17.9602
	1.50	7.5205	11.1369	14.6464	17.3642	19.1185	20.1199	20.6525
	1.75	7.8353	11.8268	15.8537	19.0835	21.2180	22.4422	23.0820
	2.00	8.1450	12.4520	16.9124	20.5778	23.0373	24.4181	25.0453

Table 10 Convergence table for the free vibration analysis of
clamped square plates in the first mode category
f = frequency parameter ω = circular frequency = $(f/a^2)(D/\rho)^{0.5}$

	Mode No.	No. of terms				
		3 term sol.	6 term sol.	7 term sol.	Young[63]	Lam[28]
Homogeneous plate frequency parameter, f	1	9.00	8.999	8.996	8.998	----
	2	34.324	33.097	32.974	32.912	-----
	3	34.663	33.106	33.104	33.063	-----
	4	-----	56.044	55.280	-----	-----
	5	-----	90.584	90.557	-----	-----
	6	-----	90.729	90.584	-----	-----
	7	-----	-----	114.275	-----	-----
Sandwich plate frequency, ω . t=0.016in. c=0.25in. a=20in. $G_c=1000$ psi $E_f=10^7$ psi	1	68.389	68.362	68.356	-----	70.29
	2	187.234	176.810	176.810	-----	-----
	3	190.500	179.006	178.967	-----	-----
	4	-----	250.450	244.118	-----	-----
	5	-----	357.696	357.696	-----	-----
	6	-----	360.462	360.441	-----	-----
	7	-----	-----	403.108	-----	-----

Table 11 Comparison of frequency parameter f of clamped rectangular homogeneous plates for various aspect ratios $\lambda = a/b$

$$\omega = f/a^2 \left(\frac{D}{\rho}\right)^{1/2}$$

Mode no.	$\lambda=1.0$			$\lambda=1.25$	$\lambda=1.50$		$\lambda=1.75$	$\lambda=2.0$	
	Present solution	Young ref.[63]	Odman ref.[44]	Present solution	Present solution	Odman ref.[44]	Present solution	Present solution	Odman ref.[44]
First 7 eigen values for doubly symmetric modes (category 1)									
1	8.996	8.998	8.9997	11.675	15.191	15.1931	19.503	24.579	24.580
2	32.9743	32.912	---	34.9263	37.4748	---	40.7556	44.818	44.78
3	33.104	33.063	32.976	49.915	70.641	---	95.150	99.104	---
4	55.280	---	55.015	71.236	91.351	90.484	96.236	123.542	---
5	90.557	---	---	92.009	93.925	---	115.700	143.986	142.38
6	90.584	---	---	140.012	192.298	---	247.021	311.242	---
7	114.275	---	---	148.406	201.318	---	272.902	355.685	---
First 7 eigen values for modes antisymmetric about y axis (category 2)									
1	18.349	18.353	18.351	20.512	23.459	23.468	27.229	31.828	31.830
2	41.293	---	41.256	51.695	57.002	56.706	59.944	63.585	63.34
3	52.962	---	52.632	57.706	78.097	77.895	102.396	130.548	130.35
4	71.345	---	74.088	86.475	105.413	---	128.202	142.656	159.49
5	93.721	---	---	114.856	137.373	---	140.068	154.952	---
6	97.285	---	---	136.528	144.388	---	179.808	222.674	---
7	135.113	---	---	146.456	206.964	---	278.595	361.312	---
First 7 eigen values for modes antisymmetric about x axis (category 3)									
1	18.349	18.353	18.351	26.761	37.196	37.423	49.611	63.985	---
2	41.293	---	41.256	48.604	58.046	---	69.626	83.318	---
3	52.962	---	52.632	81.140	110.299	---	119.864	131.566	---
4	71.345	---	74.088	96.811	115.640	---	156.429	203.476	---
5	93.721	---	---	102.763	128.396	---	166.022	209.687	---
6	97.285	---	---	128.820	173.381	---	227.502	290.508	---
7	135.113	---	---	209.809	301.140	---	409.090	533.650	---
First 7 eigen values for doubly antisymmetric modes (category 4)									
1	27.055	27.068	27.059	34.904	44.893	44.893	56.967	71.081	71.08
2	60.828	---	---	67.830	76.763	76.556	87.805	100.977	100.80
3	61.054	---	---	88.772	123.058	---	162.062	172.776	---
4	95.229	---	92.845	122.664	153.345	152.533	163.768	210.752	238.35
5	141.105	---	---	146.525	157.326	---	198.852	247.264	---
6	141.432	---	---	215.803	307.072	---	415.020	539.621	---
7	257.680	---	---	333.853	432.983	---	554.163	696.415	---

Table 12 Natural frequencies of vibration of clamped rectangular sandwich plates for various aspect ratios $\lambda = a/b$

$$E_f = 10^7 \text{ psi} \quad \nu = 0.34 \quad \rho = 500.02 \times 10^{-6} \text{ lb-sec}^2/\text{in}^4$$

$$c = 0.25 \text{ in.} \quad t = 0.016 \text{ in.} \quad b = 20 \text{ in.} \quad G_c = 1000 \text{ psi.}$$

Mode no.	$\lambda=1.00$		$\lambda=1.50$		$\lambda=2.00$	
	Present solution	Lam ref. [28]	Present solution	Lam ref. [28]	Present solution	Lam ref. [28]
First 7 frequencies for doubly symmetric modes (category 1)						
1	68.356	70.29	53.410	55.67	49.327	51.18
2	176.810	-----	108.434	-----	79.672	-----
3	178.967	-----	172.327	-----	149.568	-----
4	244.118	-----	201.751	-----	170.694	-----
5	357.696	-----	217.1454	-----	187.858	-----
6	360.441	-----	345.778	-----	333.371	-----
7	403.108	-----	358.135	-----	357.917	-----
First 7 frequencies for modes antisymmetric about y axis (category 2)						
1	118.381	123.87	75.807	81.84	60.760	65.21
2	203.692	-----	146.340	-----	103.984	-----
3	239.930	-----	182.748	-----	176.092	-----
4	292.664	-----	222.862	-----	195.563	-----
5	364.829	-----	283.047	-----	197.938	-----
6	424.503	-----	332.611	-----	307.861	-----
7	461.141	-----	360.056	-----	358.638	-----
First 7 frequencies for modes antisymmetric about x axis (category 3)						
1	118.381	123.87	109.642	114.65	107.122	89.62
2	203.692	---	148.395	-----	128.055	-----
3	239.930	-----	234.374	-----	178.285	-----
4	292.664	-----	236.226	-----	234.848	-----
5	364.829	---	260.653	-----	250.069	-----
6	424.503	-----	379.364	-----	370.179	-----
7	461.141	-----	461.028	-----	461.091	-----
First 7 frequencies for doubly antisymmetric modes (category 4)						
1	154.717	165.54	124.761	121.49	115.026	111.67
2	257.356	-----	178.292	-----	145.641	-----
3	260.408	-----	243.629	-----	216.209	-----
4	333.680	-----	279.871	-----	238.915	-----
5	459.776	-----	293.852	-----	262.225	-----
6	465.737	-----	459.015	-----	446.425	-----
7	527.776	-----	461.885	-----	460.950	-----

Table 13. Natural frequencies of vibration of clamped skew sandwich plates for various aspect ratios $\lambda=a/b$

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$$E_f = 10^7 \text{ psi} \quad \lambda = 0.34 \quad \rho = 500.02 \times 10^{-6} \text{ lb-sec}^2/\text{in}^4 \quad G_c = 1000 \text{ psi}$$

$$C = 0.25 \text{ in.} \quad t = 0.016 \text{ in.} \quad b = 20 \text{ in.} \quad \phi = 30 \text{ degree}$$

Mode no.	$\lambda=1.00$		$\lambda=1.50$		$\lambda=2.00$	
	Present solution	Lam ref. [28]	Present solution	Lam ref. [28]	Present solution	Lam ref. [28]
First 7 frequencies for doubly symmetric modes (category 1)						
1	86.143	84.35	67.903	67.83	63.095	62.98
2	212.721	---	138.398	---	105.382	---
3	222.036	---	209.357	---	189.325	---
4	319.468	---	258.183	---	207.534	---
5	429.674	---	274.513	---	244.224	---
6	434.875	---	429.164	---	421.034	---
7	520.030	---	443.617	---	430.251	---
First 7 frequencies for modes antisymmetric about y axis (category 2)						
1	148.399	131.65	98.850	94.41	80.88	77.12
2	258.991	---	189.939	---	134.304	---
3	289.143	---	229.513	---	218.628	---
4	382.988	---	296.223	---	242.804	---
5	453.796	---	346.162	---	265.413	---
6	550.109	---	440.740	---	437.102	---
7	614.937	---	507.350	---	476.370	---
First 7 frequencies for modes antisymmetric about x axis (category 3)						
1	148.399	158.46	136.364	130.81	133.120	101.26
2	258.991	---	194.645	---	168.538	---
3	289.143	---	282.031	---	233.201	---
4	382.988	---	298.111	---	281.246	---
5	453.796	---	344.367	---	325.675	---
6	550.109	---	547.117	---	540.083	---
7	614.937	---	557.617	---	550.870	---
First 7 frequencies for doubly antisymmetric modes (category 4)						
1	200.484	175.33	161.922	138.64	148.388	130.40
2	315.675	---	231.412	---	192.631	---
3	330.117	---	298.379	---	274.785	---
4	430.851	---	349.723	---	292.506	---
5	562.197	---	381.217	---	336.530	---
6	570.929	---	539.070	---	508.893	---
7	644.179	---	560.620	---	557.781	---

Table 14 Natural frequencies of vibration of clamped rectangular sandwich plates for various aspect ratio and shear rigidities

$$E_f = 10^7 \text{ psi} \quad \nu = 0.34 \quad \rho = 500.02 \times 10^{-6} \text{ lb-sec}^2/\text{in}^4$$

$$c = 0.25 \text{ in.} \quad t = 0.016 \text{ in.} \quad b = 20 \text{ in.}$$

Mode no.	$\lambda=1.00$		$\lambda=1.50$		$\lambda=2.00$	
	$G_c = 5000 \text{ psi}$	$G_c = 10000 \text{ psi}$	$G_c = 5000 \text{ psi}$	$G_c = 10000 \text{ psi}$	$G_c = 5000 \text{ psi}$	$G_c = 10000 \text{ psi}$
First 7 frequencies for doubly symmetric modes (category 1)						
1	77.581	79.009	58.808	59.605	53.709	54.341
2	253.499	271.975	137.374	142.865	94.905	97.482
3	254.997	273.320	243.431	260.203	199.032	209.455
4	388.689	432.292	301.983	328.475	240.101	256.354
5	597.365	679.654	316.038	341.153	273.403	294.806
6	598.490	680.280	571.134	647.458	536.757	600.934
7	705.423	820.643	593.309	673.605	591.479	670.789
First 7 frequencies for modes antisymmetric about y axis (category 2)						
1	150.942	157.169	88.868	91.010	68.703	69.925
2	306.224	333.505	200.401	212.356	131.299	136.456
3	376.676	417.031	264.677	284.932	251.359	269.454
4	483.424	545.494	341.704	374.685	276.355	295.433
5	621.963	713.420	439.017	484.179	291.353	315.451
6	664.748	737.558	483.818	522.541	432.295	461.625
7	812.855	953.573	613.018	687.458	596.697	678.405
First 7 frequencies for modes antisymmetric about x axis (category 3)						
1	150.942	157.169	137.184	142.270	133.137	137.883
2	306.224	333.505	203.866	216.137	168.594	176.846
3	376.676	417.031	357.899	392.152	253.721	271.783
4	483.424	545.494	367.922	406.299	365.089	402.788
5	621.963	713.420	408.886	451.760	383.342	420.071
6	664.748	737.558	566.321	617.949	541.453	587.242
7	813.855	953.573	810.504	948.003	809.421	946.172
First 7 frequencies for doubly antisymmetric modes (category 4)						
1	213.367	226.414	162.442	169.944	146.330	152.296
2	418.465	469.280	259.427	279.616	199.693	211.615
3	421.161	471.731	385.846	428.531	322.043	349.898
4	588.370	685.071	466.445	526.681	374.886	414.981
5	828.697	979.099	472.896	530.789	425.798	477.296
6	833.195	982.870	817.383	959.461	813.206	952.577
7	1063.560	1362.163	898.706	1124.123	857.206	1059.584

Table 15 Natural frequencies of vibration of clamped skew sandwich plate for various aspect ratio and skew angle

$$E_f = 10^7 \text{ psi} \quad \nu = 0.34 \quad \rho = 500.02 \times 10^{-6} \text{ lb-sec}^2/\text{in}^4$$

$$G_c = 1000 \text{ psi} \quad c = 0.25 \text{ in.} \quad t = 0.016 \text{ in.} \quad b = 20 \text{ in.}$$

Mode No.	$\lambda=1.0$		$\lambda=1.5$		$\lambda=2.0$	
	45 deg.	60 deg.	45 deg.	60 deg.	45 deg.	60 deg.
First 7 frequencies for doubly symmetric modes (category 1)						
1	122.266	212.617	96.916	171.289	89.797	158.586
2	279.147	428.622	190.909	310.279	150.100	254.297
3	292.090	447.538	272.645	414.901	255.058	400.335
4	433.102	675.645	343.686	530.308	270.278	413.885
5	546.105	804.290	367.310	569.685	329.008	512.241
6	554.862	821.722	544.798	802.567	542.270	799.341
7	681.643	1027.338	580.670	876.702	550.870	827.670
First 7 frequencies for modes antisymmetric about y axis (category 2)						
1	202.532	326.299	140.769	240.608	116.796	204.648
2	349.235	542.363	243.470	381.178	186.282	304.211
3	372.314	560.120	305.399	472.493	287.769	441.617
4	512.522	789.933	402.029	628.610	322.324	499.058
5	587.506	877.512	447.709	671.960	359.408	560.928
6	693.560	1012.319	563.574	834.290	556.301	820.694
7	792.595	1174.320	653.338	967.750	608.126	895.196
First 7 frequencies for modes antisymmetric about x axis (category 3)						
1	202.532	326.299	184.000	293.409	178.729	283.445
2	349.235	542.363	270.320	435.940	234.555	380.706
3	372.314	560.120	360.241	534.554	318.950	505.722
4	512.422	789.933	398.194	617.534	358.679	531.652
5	587.506	877.512	456.700	696.156	427.205	646.542
6	693.560	1012.319	689.385	1004.376	683.012	997.776
7	792.595	1174.320	710.010	1043.930	691.929	1006.797
First 7 frequencies for doubly antisymmetric modes (category 4)						
1	276.700	444.737	224.089	363.381	203.653	328.560
2	418.122	643.013	317.331	503.842	268.156	433.179
3	434.234	659.122	386.910	582.224	365.818	554.867
4	579.621	896.603	466.584	720.725	381.231	587.838
5	715.548	1052.970	504.514	767.425	447.727	684.970
6	729.288	1075.090	704.293	1032.260	675.824	1021.787
7	859.665	1317.780	724.842	1114.393	702.807	1042.199

Appendix

COMPUTER PROGRAMS

FILE: TPROG FORTRAN A

```

CC
CC
CC PROGRAM FOR THE DEFINITE INTEGRATION
CC N1 = NUMBER OF INTEGRATIONS
CC N2 = NUMBER OF NODES IN THE X AND Y DIRECTIONS
CC F = THE FUNCTION TO BE INTEGRATED AND SHOULD BE
CC      SUPPLIED
CC
CC
CC
CC (U1,U2,U3), (V1,V2,V3) AND (W1,W2,W3,W4) ARE THE
CC      DISPLACEMENT FUNCTIONS FOR U,V AND W
CC
CC
CC
CC U1X= THE FUNCTION U1 DIFFERENTIATED W.R.TO X ONCE
CC U1XX= THE FUNCTION U1 DIFFERENTIATED W.R.TO X TWICE
CC SIMILARLY U1Y, U1YY, U1XY ETC.
CC
CC

```

```

10      IMPLICIT REAL *8 (A-H,O-Z)
1      DIMENSION AREA(N1)
      COMMON/KOUNT/K
      EXTERNAL F
      DO 10 I=1,N1
      K=I
      DX=1.0/(N2-1)
      DY=1.0/(N2-1)
      CALL INTRGL(I,N,DX,DY,F,AREA)
      WRITE (6,1) I, AREA(I)
      CONTINUE
1      FORMAT(5X,I4,F12.6)
      STOP
      END

```

```

CC
CC
CC

```

FUNCTION F

```

FUNCTION F(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/KOUNT/K
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
EXTERNAL U1X,U2X,U3X,U1XX,U2XX,U3XX,U1Y,U2Y,U3Y,U1YY,U2YY,U3YY
1,U1XY,U2XY,U3XY,V1X,V2X,V3X,V1XX,V2XX,V3XX,V1Y,V2Y,V3Y,V1YY
2,V2YY,V3YY,V1XY,V2XY,V3XY,W1X,W2X,W3X,W4X,W1Y,W2Y,W3Y,W4Y,W1XX
3,W2XX,W3XX,W4XX,W1YY,W2YY,W3YY,W4YY,W1XY,W2XY,W3XY,W4XY
4,U1,U2,U3
X2=X*X
X3=X2*X
X4=X3*X
X4=X3*X
X5=X4*X
X6=X5*X
X7=X6*X
Y2=Y*Y
Y3=Y2*Y
Y4=Y3*Y
Y5=Y4*Y

```

FILE: TPROG: FORTRAN A

```
Y6=Y5*Y
Y7=Y6*Y
AAX=(1.-X2)*(1.-Y2)*(1.-X*Y)*X
GO TO (1,2,3,4,5,6,7,8.....N1),K
1  F=U1XX(X,Y)*AAX
   RETURN
2  F=U2XX(X,Y)*AAX
   RETURN
3  F=U3XX(X,Y)*AAX
   RETURN
4  F=U1YY(X,Y)*AAX
   RETURN
5  F=U2YY(X,Y)*AAX
   RETURN
   END
```

FILE: TPROG FORTRAN A

```

FUNCTION U1X(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U1X=1.-3.*X2-Y2+3.*X2*Y2-2.*X*Y+4.*X3*Y+2.*X*Y3-4.*X3*Y3
  RETURN
END
FUNCTION U2X(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U2X=3.*X2-5.*X4-3.*X2*Y2+5.*X4*Y2-4.*X3*Y+6.*X5*Y+4.*X3*Y3
  1-6.*X5*Y3
  RETURN
END
FUNCTION U3X(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U3X=Y2-3.*X2*Y2-Y4+3.*X2*Y4-2.*X*Y3+4.*X3*Y3+2.*X*Y5-4.*X3*Y5
  RETURN
END
FUNCTION U1XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U1XX=-6.*X+6.*X*Y2-2.*Y+12.*X2*Y+2.*Y3-12.*X2*Y3
  RETURN
END
FUNCTION U2XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U2XX=6.*X-20.*X3-6.*X*Y2+20.*X3*Y2-12.*X2*Y+30.*X4*Y+12.*X2*Y3
  1-30.*X4*Y3
  RETURN
END
FUNCTION U3XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U3XX=-6.*X*Y2+6.*X*Y4-2.*Y3+12.*X2*Y3+2.*Y5-12.*X2*Y5
  RETURN
END
FUNCTION U1Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U1Y=-2.*X*Y+2.*X3*Y-X2+X4+3.*X2*Y2-3.*X4*Y2
  RETURN
END
FUNCTION U2Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U2Y=-2.*X3*Y+2.*X5*Y-X4+X6+3.*X4*Y2-3.*X6*Y2
  RETURN
END
FUNCTION U3Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  U3Y=2.*X*Y-2.*X3*Y-4.*X*Y3+4.*X3*Y3-3.*X2*Y2+3.*X4*Y2+5.*X2*Y4
  1-5.*X4*Y4

```

```
RETURN
END
FUNCTION U1YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U1YY=-2.*X+2.*X3+6.*X2*Y-6.*X4*Y
RETURN
END
FUNCTION U2YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U2YY=-2.*X3+2.*X5+6.*X4*Y-6.*X6*Y
RETURN
END
FUNCTION U3YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U3YY=2.*X-2.*X3-12.*X*Y2+12.*X3*Y2-6.*X2*Y+6.*X4*Y+20.*X2*Y3
1-20.*X4*Y3
RETURN
END
FUNCTION U1XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U1XY=-2.*Y+6.*X2*Y-2.*X+4.*X3+6.*X*Y2-12.*X3*Y2
RETURN
END
FUNCTION U2XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U2XY=-6.*X2*Y+10.*X4*Y-4.*X3+6.*X5+12.*X3*Y2-18.*X5*Y2
RETURN
END
FUNCTION U3XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
U3XY=2.*Y-6.*X2*Y-4.*Y3+12.*X2*Y3-6.*X*Y2+12.*X3*Y2+10.*X*Y4
1-20.*X3*Y4
RETURN
END
FUNCTION V1X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V1X=-2.*X*Y+2.*X*Y3-Y2+3.*X2*Y2+Y4-3.*X2*Y4
RETURN
END
FUNCTION V2X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V2X=2.*X*Y-4.*X3*Y-2.*X*Y3+4.*X3*Y3-3.*X2*Y2+5.*X4*Y2+3.*X2*Y4
1-5.*X4*Y4
RETURN
END
FUNCTION V3X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
```



```
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V3X=-2.*X*Y3+2.*X*Y5-Y4+3.*X2*Y4+Y6-3.*X2*Y6
RETURN
END
FUNCTION V1XX(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V1XX=-2.*Y+2.*Y3+6.*X*Y2-6.*X*Y4
RETURN
END
FUNCTION V2XX(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V2XX=2.*Y-12.*X2*Y-2.*Y3+12.*X2*Y3-6.*X*Y2+20.*X3*Y2+6.*X*Y4
1-20.*X3*Y4
RETURN
END
FUNCTION V3XX(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V3XX=-2.*Y3+2.*Y5+6.*X*Y4-6.*X*Y6
RETURN
END
FUNCTION V1Y(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V1Y=1.-X2-3.*Y2+3.*X2*Y2-2.*X*Y+2.*X3*Y+4.*X*Y3-4.*X3*Y3
RETURN
END
FUNCTION V2Y(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V2Y=X2-X4-3.*X2*Y2+3.*X4*Y2-2.*X3*Y+2.*X5*Y+4.*X3*Y3-4.*X5*Y3
RETURN
END
FUNCTION V3Y(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V3Y=3.*Y2-3.*X2*Y2-5.*Y4+5.*X2*Y4-4.*X*Y3+4.*X3*Y3+6.*X*Y5
1-6.*X3*Y5
RETURN
END
FUNCTION V1YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V1YY=-6.*Y+6.*X2*Y-2.*X+2.*X3+12.*X*Y2-12.*X3*Y2
RETURN
END
FUNCTION V2YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V2YY=-6.*X2*Y+6.*X4*Y-2.*X3+2.*X5+12.*X3*Y2-12.*X5*Y2
RETURN
END
FUNCTION V3YY(X,Y)
```

```
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V3YY=6.*Y-6.*X2*Y-20.*Y3+20.*X2*Y3-12.*X*Y2+12.*X3*Y2
1+30.*X*Y4-30.*X3*Y4
RETURN
END
FUNCTION V1XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V1XY=-2.*X+6.*X*Y2-2.*Y+6.*X2*Y+4.*Y3-12.*X2*Y3
RETURN
END
FUNCTION V2XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V2XY=2.*X-4.*X3-6.*X*Y2+12.*X3*Y2-6.*X2*Y+10.*X4*Y+12.*X2*Y3
1-20.*X4*Y3
RETURN
END
FUNCTION V3XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
V3XY=-6.*X*Y2+10.*X*Y4-4.*Y3+12.*X2*Y3+6.*Y5-18.*X2*Y5
RETURN
END
FUNCTION W1X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W1X=-4.*X+8.*X*Y2-4.*X*Y4+4.*X3-8.*X3*Y2+4.*X3*Y4-Y+2.*Y3-Y5
1+6.*X2*Y-12.*X2*Y3+6.*X2*Y5-5.*X4*Y+10.*X4*Y3-5.*X4*Y5
RETURN
END
FUNCTION W2X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W2X=2.*X-4.*X*Y2+2.*X*Y4-8.*X3+16.*X3*Y2-8.*X3*Y4+6.*X5
1-12.*X5*Y2+6.*X5*Y4-3.*X2*Y+6.*X2*Y3-3.*X2*Y5+10.*X4*Y-20.*X4*Y3
2+10.*X4*Y5-7.*X6*Y+14.*X6*Y3-7.*X6*Y5
RETURN
END
FUNCTION W3X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W3X=-4.*X*Y2+8.*X*Y4-4.*X*Y6+4.*X3*Y2-8.*X3*Y4+4.*X3*Y6-Y3
1+2.*Y5-Y7+6.*X2*Y3-12.*X2*Y5+6.*X2*Y7-5.*X4*Y3+10.*X4*Y5-5.*X4*Y7
RETURN
END
FUNCTION W4X(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W4X=2.*X*Y2-4.*X*Y4+2.*X*Y6-8.*X3*Y2+16.*X3*Y4-8.*X3*Y6+6.*X5*Y2
1-12.*X5*Y4+6.*X5*Y6-3.*X2*Y3+6.*X2*Y5-3.*X2*Y7+10.*X4*Y3-20.*
2X4*Y5+10.*X4*Y7-7.*X6*Y3+14.*X6*Y5-7.*X6*Y7
RETURN
END
```

```
FUNCTION W1XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W1XX=-4.*Y2-4.*Y4+12.*X2-24.*X2*Y2+12.*X2*Y4+12.*X*Y-24.*X*Y3
  1+12.*X*Y5-20.*X3*Y+40.*X3*Y3-20.*X3*Y5
  RETURN
END
FUNCTION W2XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W2XX=2.-4.*Y2+2.*Y4-24.*X2+48.*X2*Y2-24.*X2*Y4+30.*X4-60.*X4*Y2
  1+30.*X4*Y4-6.*X*Y+12.*X*Y3-6.*X*Y5+40.*X3*Y-80.*X3*Y3+40.*X3*Y5
  2-42.*X5*Y+84.*X5*Y3-42.*X5*Y5
  RETURN
END
FUNCTION W3XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W3XX=-4.*Y2+8.*Y4-4.*Y6+12.*X2*Y2-24.*X2*Y4+12.*X2*Y6+12.*X*Y3
  1-24.*X*Y5+12.*X*Y7-20.*X3*Y3+40.*X3*Y5-20.*X3*Y7
  RETURN
END
FUNCTION W4XX(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W4XX=2.*Y2-4.*Y4+2.*Y6-24.*X2*Y2+48.*X2*Y4-24.*X2*Y6+30.*X4*Y2
  1-60.*X4*Y4+30.*X4*Y6-6.*X*Y3+12.*X*Y5-6.*X*Y7+40.*X3*Y3
  2-80.*X3*Y5+40.*X3*Y7-42.*X5*Y3+84.*X5*Y5-42.*X5*Y7
  RETURN
END
FUNCTION W1Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W1Y=-4.*Y+4.*Y3+8.*X2*Y-8.*X2*Y3-4.*X4*Y+4.*X4*Y3-X+6.*X*Y2
  1-5.*X*Y4+2.*X3-12.*X3*Y2+10.*X3*Y4-X5+6.*X5*Y2-5.*X5*Y4
  RETURN
END
FUNCTION W2Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W2Y=-4.*X2*Y+4.*X2*Y3+8.*X4*Y-8.*X4*Y3-4.*X6*Y+4.*X6*Y3-X3
  1+6.*X3*Y2-5.*X3*Y4+2.*X5-12.*X5*Y2+10.*X5*Y4-X7+6.*X7*Y2
  2-5.*X7*Y4
  RETURN
END
FUNCTION W3Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
  COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
  W3Y=2.*Y-8.*Y3+6.*Y5-4.*X2*Y+16.*X2*Y3-12.*X2*Y5+2.*X4*Y
  1-8.*X4*Y3+6.*X4*Y5-3.*X*Y2+10.*X*Y4-7.*X*Y6+6.*X3*Y2
  2-20.*X3*Y4+14.*X3*Y6-3.*X5*Y2+10.*X5*Y4-7.*X5*Y6
  RETURN
END
FUNCTION W4Y(X,Y)
  IMPLICIT REAL *8 (A-H,O-Z)
```

```
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W4Y=2.*X2*Y-8.*X2*Y3+6.*X2*Y5-4.*X4*Y+16.*X4*Y3-12.*X4*Y5
1+2.*X6*Y-8.*X6*Y3+6.*X6*Y5-3.*X3*Y2+10.*X3*Y4-7.*X3*Y6
2+6.*X5*Y2-20.*X5*Y4+14.*X5*Y6-3.*X7*Y2+10.*X7*Y4-7.*X7*Y6
RETURN
END
FUNCTION W1YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W1YY=-4.+12.*Y2+8.*X2-24.*X2*Y2-4.*X4+12.*X4*Y2+12.*X*Y
1-20.*X*Y3-24.*X3*Y+40.*X3*Y3+12.*X5*Y-20.*X5*Y3
RETURN
END
FUNCTION W2YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W2YY=-4.*X2+12.*X2*Y2+8.*X4-24.*X4*Y2-4.*X6+12.*X6*Y2
1+12.*X3*Y-20.*X3*Y3-24.*X5*Y+40.*X5*Y3+12.*X7*Y-20.*X7*Y3
RETURN
END
FUNCTION W3YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W3YY=2.-24.*Y2+30.*Y4-4.*X2+48.*X2*Y2-60.*X2*Y4+2.*X4
1-24.*X4*Y2+30.*X4*Y4-6.*X*Y+40.*X*Y3-42.*X*Y5+12.*X3*Y
2-80.*X3*Y3+84.*X3*Y5-6.*X5*Y+40.*X5*Y3-42.*X5*Y5
RETURN
END
FUNCTION W4YY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W4YY=2.*X2-24.*X2*Y2+30.*X2*Y4-4.*X4+48.*X4*Y2-60.*X4*Y4
1+2.*X6-24.*X6*Y2+30.*X6*Y4-6.*X3*Y+40.*X3*Y3-42.*X3*Y5
2+12.*X5*Y-80.*X5*Y3+84.*X5*Y5-6.*X7*Y+40.*X7*Y3-42.*X7*Y5
RETURN
END
FUNCTION W1XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W1XY=16.*X*Y-16.*X*Y3-16.*X3*Y+16.*X3*Y3-1.*Y2-5.*Y4+6.*X2
1-36.*X2*Y2+30.*X2*Y4-5.*X4+30.*X4*Y2-25.*X4*Y4
RETURN
END
FUNCTION W2XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W2XY=-8.*X*Y+8.*X*Y3+32.*X3*Y-32.*X3*Y3-24.*X5*Y+24.*X5*Y3
1-3.*X2+18.*X2*Y2-15.*X2*Y4+10.*X4-60.*X4*Y2+50.*X4*Y4
2-7.*X6+42.*X6*Y2-35.*X6*Y4
RETURN
END
FUNCTION W3XY(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W3XY=-8.*X*Y+32.*X*Y3-24.*X*Y5+8.*X3*Y-32.*X3*Y3+24.*X3*Y5-3.*Y2
```

1+10.*Y4-7.*Y6+18.*X2*Y2-60.*X2*Y4+42.*X2*Y6-15.*X4*Y2+50.*X4*Y4
2-35.*X4*Y6

RETURN

END

FUNCTION W4XY(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

W4XY=4.*X*Y-16.*X*Y3+12.*X*Y5-16.*X3*Y+64.*X3*Y3-48.*X3*Y5

1+12.*X5*Y-48.*X5*Y3+36.*X5*Y5-9.*X2*Y2+30.*X2*Y4-21.*X2*Y6

2+30.*X4*Y2-100.*X4*Y4+70.*X4*Y6-21.*X6*Y2+70.*X6*Y4-49.*X6*Y6

RETURN

END

FUNCTION U1(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

U1=(1.-X2)*(1.-Y2)*(1.-X*Y)*X

RETURN

END

FUNCTION U2(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

U2=(1.-X2)*(1.-Y2)*(1.-X*Y)*X*X2

RETURN

END

FUNCTION U3(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

U3=(1.-X2)*(1.-Y2)*(1.-X*Y)*X*Y2

RETURN

END

FUNCTION V1(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

V1=(1.-X2)*(1.-Y2)*(1.-X*Y)*Y

RETURN

END

FUNCTION V2(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

V2=(1.-X2)*(1.-Y2)*(1.-X*Y)*X2*Y

RETURN

END

FUNCTION V3(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

V3=(1.-X2)*(1.-Y2)*(1.-X*Y)*Y3

RETURN

END

FUNCTION W1(X,Y)

IMPLICIT REAL *8 (A-H,O-Z)

COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7

W1=(1.-X2)*(1.-X2)*(1.-Y2)*(1.-Y2)*(1.-X*Y)

RETURN

END

FUNCTION W2(X,Y)

```
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W2=(1.-X2)*(1.-X2)*(1.-Y2)*(1.-Y2)*(1.-X*Y)*X2
RETURN
END
FUNCTION W3(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W3=(1.-X2)*(1.-X2)*(1.-Y2)*(1.-Y2)*(1.-X*Y)*Y2
RETURN
END
FUNCTION W4(X,Y)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON/INTER/X2,X3,X4,X5,X6,X7,Y2,Y3,Y4,Y5,Y6,Y7
W4=(1.-X2)*(1.-X2)*(1.-Y2)*(1.-Y2)*(1.-X*Y)*X2*Y2
RETURN
END
```

FORTRAN IV G LEVEL 21

INTRGL

```

0001      SUBROUTINE INTRGL(L,N,DX,DY,F,AREA)
0002      IMPLICIT REAL *8 (A-H,O-Z)
0003      EXTERNAL F
0004      M=N-1
0005      SUM =0.0
0006      X=-1.0
0007      Y=-1.0
0008      CALL F (L,X,Y,A1)
0009      X=-1.0
0010      Y=1.0
0011      CALL F (L,X,Y,A2)
0012      X=1.0
0013      Y=1.0
0014      CALL F (L,X,Y,A3)
0015      X=1.0
0016      Y=-1.0
0017      CALL F (L,X,Y,A4)
0018      SUM=SUM +A1+A2+A3+A4
0019      SUM=SUM*0.25
0020      X=-1.0+DX
0021      DO 1 I=2,M
0022          Y=-1.0
0023          CALL F (L,X,Y,A1)
0024          Y=1.0
0025          CALL F (L,X,Y,A2)
0026          SUM=SUM+A1+A2
0027          X=X+DX
0028      1 CONTINUE
0029          Y=-1.0+DY
0030          DO 2 I=2,M
0031              X=-1.0
0032              CALL F (L,X,Y,A1)
0033              X=1.0
0034              CALL F (L,X,Y,A2)
0035              SUM=SUM+A1+A2
0036              Y=Y+DY
0037      2 CONTINUE
0038          SUM=SUM*0.50
0039          X=-1.0
0040          DO 3 I=2,M
0041              Y=-1.0
0042              X=X+DX
0043              DO 3 J=2,M
0044                  Y=Y+DY
0045                  CALL F (L,X,Y,A)
0046                  SUM=SUM+A
0047      3 CONTINUE
0048          AREA=SUM*DX*DY
0049          RETURN
0050          END

```

```

CC
CC PROGRAM FOR THE SOLUTION OF NONLINEAR ALGEBRAIC
CC EQUATION BY THE NEWTON RAPHSON ITERATIVE PROCEDURE
CC APPLIED TO LARGE DEFLECTION OF HOMOGENEOUS PLATE
C
C Y ARE THE UNKNOWNNS IN THE EQUATIONS TO BE DETERMINED
C DATA IS THE FIRST SUPPLIED VALUES FOR THE UNKNOWNNS
C ERR = ARE THE ALGEBRAIC EQUATIONS
C AA IS THE MATRIX WHERE THE DATA OBTAINED BY USING
C THE GALERKIN METHOD ARE STORED
C H = ASPECT RATIO OF THE PLATE
C G = POISSONS RATIO OF THE PLATE
C A(I,J) IS THE JACOBIAN MATRIX OBTAINED FROM THE
C SYSTEM OF EQUATIONS
      REAL *8 A(11,11),C(11),DD(11,11),ERR(11),Y(11),AA(11,100)
      1,CC(11,24),B(11,100),BA(11,127),AB(11,127),AB1(11,13),WW(51,12)
      DATA Y/11*0.0D00/
      DO 111 K=1,8
      DO 111 L=1,39
      READ(5,4) AA(K,L)
111      CONTINUE
4      FORMAT(4X,E25.10)
      DO 112 K=9,11
      DO 112 L=1,72
      READ(5,4) AA(K,L)
CCC      WRITE (6,52) K,L,AA(K,L)
52      FORMAT(9X,I4,I4,E25.10)
112      CONTINUE
      DO 222 K=9,11
      DO 222 L=1,127
      READ(5,4) AB(K,L)
222      CONTINUE
      DO 1001 KQ=1,51
      AKQ=KQ
      Q=5.*(AKQ-1.0)
      H=0.0
      G=0.3
      DO 113 L=1,4
      DO 114 M=1,13
114      B(L,M)=AA(L,M)
      DO 115 M=14,17
115      B(L,M)=AA(L,M)*(G*H+((1.0-G)*H/2.0))
      DO 116 M=18,26
116      B(L,M)=AA(L,M)*(G*H*H+(1.-G)*H*H/2.)
      DO 117 M=27,39
117      B(L,M)=AA(L,M)*H*H*(1.0-G)/2.0
C      WRITE (6,4) (L,Q(L),L=1,31)
      CC(L,1)=B(L,1)+B(L,27)
      CC(L,2)=B(L,2)+B(L,28)
      CC(L,3)=B(L,3)+B(L,29)
      CC(L,4)=B(L,4)+B(L,30)
      CC(L,5)=B(L,14)
      CC(L,6)=B(L,15)
      CC(L,7)=B(L,16)
      CC(L,8)=B(L,17)

```



```

CC(L,9)=B(L,5)+B(L,18)+B(L,31)
CC(L,10)=B(L,6)+B(L,8)+B(L,19)+B(L,21)+B(L,32)+B(L,34)
CC(L,11)=B(L,7)+B(L,11)+B(L,20)+B(L,24)+B(L,33)+B(L,37)
CC(L,12)=B(L,9)+B(L,22)+B(L,35)
CC(L,13)=B(L,10)+B(L,12)+B(L,23)+B(L,25)+B(L,36)+B(L,38)
CC(L,14)=B(L,13)+B(L,26)+B(L,39)
CCC  WRITE(6,5) (CC(L,J),J=1,14)
5    FORMAT(/10X,E25.10)
113  CONTINUE
DO 121 L=5,8
DO 122 M=1,4
122  B(L,M)=AA(L,M)*H*H
DO 123 M=5,13
123  B(L,M)=AA(L,M)*H*H*H
DO 124 M=14,26
124  B(L,M)=AA(L,M)*(G*H+(1.-G)*H/2.)
DO 125 M=27,30
125  B(L,M)=AA(L,M)*(1.0-G)/2.0
DO 126 M=31,39
126  B(L,M)=AA(L,M)*H*(1.-G)/2.
CC(L,1)=B(L,14)
CC(L,2)=B(L,15)
CC(L,3)=B(L,16)
CC(L,4)=B(L,17)
CC(L,5)=B(L,1)+B(L,27)
CC(L,6)=B(L,2)+B(L,28)
CC(L,7)=B(L,3)+B(L,29)
CC(L,8)=B(L,4)+B(L,30)
CC(L,9)=B(L,5)+B(L,18)+B(L,31)
CC(L,10)=B(L,6)+B(L,8)+B(L,19)+B(L,21)+B(L,32)+B(L,34)
CC(L,11)=B(L,7)+B(L,11)+B(L,20)+B(L,24)+B(L,33)+B(L,37)
CC(L,12)=B(L,9)+B(L,22)+B(L,35)
CC(L,13)=B(L,10)+B(L,12)+B(L,23)+B(L,25)+B(L,36)+B(L,38)
CC(L,14)=B(L,13)+B(L,26)+B(L,39)
121  CONTINUE
DO 131 L=9,11
DO 132 M=1,12
132  B(L,M)=AA(L,M)*(-12.)
DO 133 M=13,24
133  B(L,M)=(-12.*G*H)*AA(L,M)
DO 134 M=25,36
134  B(L,M)=(-12.*H*H*H)*AA(L,M)
DO 135 M=37,48
135  B(L,M)=(-12.*H*H*G)*AA(L,M)
DO 136 M=49,60
136  B(L,M)=(-12.*H*H*(1.-G))*AA(L,M)
DO 137 M=61,72
137  B(L,M)=(-12.*H*(1.-G))*AA(L,M)
C    WRITE (6,5) (B(L,M),M=1,72)
CC(L,1)=B(L,1)+B(L,37)+B(L,49)
CC(L,2)=B(L,2)+B(L,38)+B(L,50)
CC(L,3)=B(L,3)+B(L,39)+B(L,51)
CC(L,4)=B(L,4)+B(L,40)+B(L,52)
CC(L,5)=B(L,5)+B(L,41)+B(L,53)
CC(L,6)=B(L,6)+B(L,42)+B(L,54)

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FILE: TPROG5 FORTRAN A

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CC(L,7)=B(L,7)+B(L,43)+B(L,55)
CC(L,8)=B(L,8)+B(L,44)+B(L,56)
CC(L,9)=B(L,9)+B(L,45)+B(L,57)
CC(L,10)=B(L,10)+B(L,46)+B(L,58)
CC(L,11)=B(L,11)+B(L,47)+B(L,59)
CC(L,12)=B(L,12)+B(L,48)+B(L,60)
CC(L,13)=B(L,13)+B(L,25)+B(L,61)
CC(L,14)=B(L,14)+B(L,26)+B(L,62)
CC(L,15)=B(L,15)+B(L,27)+B(L,63)
CC(L,16)=B(L,16)+B(L,28)+B(L,64)
CC(L,17)=B(L,17)+B(L,29)+B(L,65)
CC(L,18)=B(L,18)+B(L,30)+B(L,66)
CC(L,19)=B(L,19)+B(L,31)+B(L,67)
CC(L,20)=B(L,20)+B(L,32)+B(L,68)
CC(L,21)=B(L,21)+B(L,33)+B(L,69)
CC(L,22)=B(L,22)+B(L,34)+B(L,70)
CC(L,23)=B(L,23)+B(L,35)+B(L,71)
CC(L,24)=B(L,24)+B(L,36)+B(L,72)
131  CONTINUE
      DO 224 L=9,11
      DO 99 M=1,3
99      BA(L,M)=AB(L,M)
      DO 225 M=4,6
225      BA(L,M)=AB(L,M)*2.*H*H
      DO 226 M=7,9
226      BA(L,M)=AB(L,M)*H*H*H*H
      DO 227 M=14,31
227      BA(L,M)=AB(L,M)*(-6.0)
      DO 228 M=35,52
228      BA(L,M)=AB(L,M)*(-6.*H*H*G)
      DO 229 M=59,76
229      BA(L,M)=AB(L,M)*(-6.*H*H*G)
      DO 230 M=83,109
230      BA(L,M)=AB(L,M)*(-12.*H*H*(1.-G))
      DO 231 M=110,127
231      BA(L,M)=AB(L,M)*(-6.*H*H*H*H)
      AB1(L,1)=BA(L,1)+BA(L,4)+BA(L,7)
      AB1(L,2)=BA(L,2)+BA(L,5)+BA(L,8)
      AB1(L,3)=BA(L,3)+BA(L,6)+BA(L,9)
      AB1(L,4)=BA(L,14)+BA(L,35)+BA(L,59)+BA(L,83)+BA(L,110)
      AB1(L,5)=BA(L,15)+BA(L,23)+BA(L,36)+BA(L,44)+BA(L,60)+BA(L,68)
      1+BA(L,111)+BA(L,119)+BA(L,87)+BA(L,93)+BA(L,95)
      AB1(L,6)=BA(L,16)+BA(L,30)+BA(L,37)+BA(L,51)+BA(L,61)+BA(L,75)
      1+BA(L,112)+BA(L,126)+BA(L,91)+BA(L,103)+BA(L,107)
      AB1(L,7)=BA(L,17)+BA(L,20)+BA(L,38)+BA(L,41)+BA(L,62)+BA(L,65)
      1+BA(L,113)+BA(L,116)+BA(L,84)+BA(L,86)+BA(L,92)
      AB1(L,8)=BA(L,18)+BA(L,26)+BA(L,39)+BA(L,47)+BA(L,63)+BA(L,71)
      1+BA(L,114)+BA(L,122)+BA(L,85)+BA(L,89)+BA(L,101)
      AB1(L,9)=BA(L,19)+BA(L,24)+BA(L,29)+BA(L,40)+BA(L,45)+BA(L,50)
      1+BA(L,64)+BA(L,69)+BA(L,74)+BA(L,115)+BA(L,120)+BA(L,125)
      2+BA(L,88)+BA(L,90)+BA(L,94)+BA(L,98)+BA(L,102)+BA(L,104)
      AB1(L,10)=BA(L,21)+BA(L,42)+BA(L,66)+BA(L,117)+BA(L,96)
      AB1(L,11)=BA(L,22)+BA(L,31)+BA(L,43)+BA(L,52)+BA(L,67)+BA(L,
176)+BA(L,118)+BA(L,127)+BA(L,100)+BA(L,106)+BA(L,108)
      AB1(L,12)=BA(L,25)+BA(L,27)+BA(L,46)+BA(L,48)+BA(L,70)+BA(L,72)

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FILE: TPROG5 FORTRAN A

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1+BA(L,121)+BA(L,123)+BA(L,97)+BA(L,99)+BA(L,105)
AB1(L,13)=BA(L,28)+BA(L,49)+BA(L,73)+BA(L,124)+BA(L,109)
CCC      WRITE(6,5) (AB1(L,M),M=1,13)
224      CONTINUE
        DO 1 I=1,11
        DO 1 J=1,11
1      A(I,J)=0.0
        ITER=0
12      A(1,1)=CC(1,1)
        A(1,2)=CC(1,2)
        A(1,3)=CC(1,3)
        A(1,4)=CC(1,4)
        A(1,5)=CC(1,5)
        A(1,6)=CC(1,6)
        A(1,7)=CC(1,7)
        A(1,8)=CC(1,8)
        A(1,9)=2.*CC(1,9)*Y(9)+CC(1,11)*Y(11)+CC(1,10)*Y(10)
        A(1,10)=CC(1,10)*Y(9)+2.*CC(1,12)*Y(10)+CC(1,13)*Y(11)
        A(1,11)=CC(1,11)*Y(9)+CC(1,13)*Y(10)+2.*CC(1,14)*Y(11)
        A(2,1)=CC(2,1)
        A(2,2)=CC(2,2)
        A(2,3)=CC(2,3)
        A(2,4)=CC(2,4)
        A(2,5)=CC(2,5)
        A(2,6)=CC(2,6)
        A(2,7)=CC(2,7)
        A(2,8)=CC(2,8)
        A(2,9)=2.*CC(2,9)*Y(9)+CC(2,11)*Y(11)+CC(2,10)*Y(10)
        A(2,10)=CC(2,10)*Y(9)+2.*CC(2,12)*Y(10)+CC(2,13)*Y(11)
        A(2,11)=CC(2,11)*Y(9)+CC(2,13)*Y(10)+2.*CC(2,14)*Y(11)
        A(3,1)=CC(3,1)
        A(3,2)=CC(3,2)
        A(3,3)=CC(3,3)
        A(3,4)=CC(3,4)
        A(3,5)=CC(3,5)
        A(3,6)=CC(3,6)
        A(3,7)=CC(3,7)
        A(3,8)=CC(3,8)
        A(3,9)=2.*CC(3,9)*Y(9)+CC(3,11)*Y(11)+CC(3,10)*Y(10)
        A(3,10)=CC(3,10)*Y(9)+2.*CC(3,12)*Y(10)+CC(3,13)*Y(11)
        A(3,11)=CC(3,11)*Y(9)+CC(3,13)*Y(10)+2.*CC(3,14)*Y(11)
        A(4,1)=CC(4,1)
        A(4,2)=CC(4,2)
        A(4,3)=CC(4,3)
        A(4,4)=CC(4,4)
        A(4,5)=CC(4,5)
        A(4,6)=CC(4,6)
        A(4,7)=CC(4,7)
        A(4,8)=CC(4,8)
        A(4,9)=2.*CC(4,9)*Y(9)+CC(4,11)*Y(11)+CC(4,10)*Y(10)
        A(4,10)=CC(4,10)*Y(9)+2.*CC(4,12)*Y(10)+CC(4,13)*Y(11)
        A(4,11)=CC(4,11)*Y(9)+CC(4,13)*Y(10)+2.*CC(4,14)*Y(11)
        A(5,1)=CC(5,1)
        A(5,2)=CC(5,2)
        A(5,3)=CC(5,3)

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FILE: TPROG5 FORTRAN A

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A(5,4)=CC(5,4)
A(5,5)=CC(5,5)
A(5,6)=CC(5,6)
A(5,7)=CC(5,7)
A(5,8)=CC(5,8)
A(5,9)=2.*CC(5,9)*Y(9)+CC(5,11)*Y(11)+CC(5,10)*Y(10)
A(5,10)=CC(5,10)*Y(9)+2.*CC(5,12)*Y(10)+CC(5,13)*Y(11)
A(5,11)=CC(5,11)*Y(9)+CC(5,13)*Y(10)+2.*CC(5,14)*Y(11)
A(6,1)=CC(6,1)
A(6,2)=CC(6,2)
A(6,3)=CC(6,3)
A(6,4)=CC(6,4)
A(6,5)=CC(6,5)
A(6,6)=CC(6,6)
A(6,7)=CC(6,7)
A(6,8)=CC(6,8)
A(6,9)=2.*CC(6,9)*Y(9)+CC(6,11)*Y(11)+CC(6,10)*Y(10)
A(6,10)=CC(6,10)*Y(9)+2.*CC(6,12)*Y(10)+CC(6,13)*Y(11)
A(6,11)=CC(6,11)*Y(9)+CC(6,13)*Y(10)+2.*CC(6,14)*Y(11)
A(7,1)=CC(7,1)
A(7,2)=CC(7,2)
A(7,3)=CC(7,3)
A(7,4)=CC(7,4)
A(7,5)=CC(7,5)
A(7,6)=CC(7,6)
A(7,7)=CC(7,7)
A(7,8)=CC(7,8)
A(7,9)=2.*CC(7,9)*Y(9)+CC(7,11)*Y(11)+CC(7,10)*Y(10)
A(7,10)=CC(7,10)*Y(9)+2.*CC(7,12)*Y(10)+CC(7,13)*Y(11)
A(7,11)=CC(7,11)*Y(9)+CC(7,13)*Y(10)+2.*CC(7,14)*Y(11)
A(8,1)=CC(8,1)
A(8,2)=CC(8,2)
A(8,3)=CC(8,3)
A(8,4)=CC(8,4)
A(8,5)=CC(8,5)
A(8,6)=CC(8,6)
A(8,7)=CC(8,7)
A(8,8)=CC(8,8)
A(8,9)=2.*CC(8,9)*Y(9)+CC(8,11)*Y(11)+CC(8,10)*Y(10)
A(8,10)=CC(8,10)*Y(9)+2.*CC(8,12)*Y(10)+CC(8,13)*Y(11)
A(8,11)=CC(8,11)*Y(9)+CC(8,13)*Y(10)+2.*CC(8,14)*Y(11)
A(9,1)=CC(9,1)*Y(9)+CC(9,5)*Y(10)+CC(9,9)*Y(11)
A(9,2)=CC(9,2)*Y(9)+CC(9,6)*Y(10)+CC(9,10)*Y(11)
A(9,3)=CC(9,3)*Y(9)+CC(9,7)*Y(10)+CC(9,11)*Y(11)
A(9,4)=CC(9,4)*Y(9)+CC(9,8)*Y(10)+CC(9,12)*Y(11)
A(9,5)=CC(9,13)*Y(9)+CC(9,17)*Y(10)+CC(9,21)*Y(11)
A(9,6)=CC(9,14)*Y(9)+CC(9,18)*Y(10)+CC(9,22)*Y(11)
A(9,7)=CC(9,15)*Y(9)+CC(9,19)*Y(10)+CC(9,23)*Y(11)
A(9,8)=CC(9,16)*Y(9)+CC(9,20)*Y(10)+CC(9,24)*Y(11)
A(9,9)=CC(9,1)*Y(1)+CC(9,2)*Y(2)+CC(9,3)*Y(3)+CC(9,4)*Y(4)
1+CC(9,13)*Y(5)+CC(9,14)*Y(6)+CC(9,15)*Y(7)+CC(9,16)*Y(8)
2+AB1(9,1)+AB1(9,4)*3.*Y(9)*Y(9)+AB1(9,5)*Y(10)*Y(10)
3+AB1(9,6)*Y(11)*Y(11)+2.*AB1(9,7)*Y(9)*Y(10)+AB1(9,9)*Y(10)*Y(11)
4+AB1(9,8)*2.*Y(9)*Y(11)
A(9,10)=CC(9,5)*Y(1)+CC(9,6)*Y(2)+CC(9,7)*Y(3)+CC(9,8)*Y(4)

```

```
1+CC(9,17)*Y(5)+CC(9,18)*Y(6)+CC(9,19)*Y(7)+CC(9,20)*Y(8)
2+AB1(9,2)+AB1(9,5)*2.*Y(10)*Y(9)+AB1(9,7)*Y(9)*Y(9)
3+AB1(9,9)*Y(9)*Y(11)+AB1(9,10)*3.*Y(10)*Y(10)+AB1(9,11)*Y(11)*
4Y(11)+AB1(9,12)*2.*Y(11)*Y(10)
A(9,11)=CC(9,9)*Y(1)+CC(9,10)*Y(2)+CC(9,11)*Y(3)+CC(9,12)*Y(4)
1+CC(9,21)*Y(5)+CC(9,22)*Y(6)+CC(9,23)*Y(7)+CC(9,24)*Y(8)
2+AB1(9,3)+AB1(9,6)*2.*Y(11)*Y(9)+AB1(9,9)*Y(9)*Y(10)
3+AB1(9,11)*2.*Y(10)*Y(11)+AB1(9,8)*Y(9)*Y(9)+AB1(9,12)*Y(10)*Y(10)
4+AB1(9,13)*3.*Y(11)*Y(11)
A(10,1)=CC(10,1)*Y(9)+CC(10,5)*Y(10)+CC(10,9)*Y(11)
A(10,2)=CC(10,2)*Y(9)+CC(10,6)*Y(10)+CC(10,10)*Y(11)
A(10,3)=CC(10,3)*Y(9)+CC(10,7)*Y(10)+CC(10,11)*Y(11)
A(10,4)=CC(10,4)*Y(9)+CC(10,8)*Y(10)+CC(10,12)*Y(11)
A(10,5)=CC(10,13)*Y(9)+CC(10,17)*Y(10)+CC(10,21)*Y(11)
A(10,6)=CC(10,14)*Y(9)+CC(10,18)*Y(10)+CC(10,22)*Y(11)
A(10,7)=CC(10,15)*Y(9)+CC(10,19)*Y(10)+CC(10,23)*Y(11)
A(10,8)=CC(10,16)*Y(9)+CC(10,20)*Y(10)+CC(10,24)*Y(11)
A(10,9)=CC(10,1)*Y(1)+CC(10,2)*Y(2)+CC(10,3)*Y(3)+CC(10,4)*Y(4)
1+CC(10,13)*Y(5)+CC(10,14)*Y(6)+CC(10,15)*Y(7)+CC(10,16)*Y(8)
2+AB1(10,1)+AB1(10,4)*3.*Y(9)*Y(9)+AB1(10,5)*Y(10)*Y(10)
3+AB1(10,6)*Y(11)*Y(11)+AB1(10,7)*2.*Y(9)*Y(10)+AB1(10,9)*
4Y(10)*Y(11)+AB1(10,8)*2.*Y(9)*Y(11)
A(10,10)=CC(10,5)*Y(1)+CC(10,6)*Y(2)+CC(10,7)*Y(3)+CC(10,8)*Y(4)
1+CC(10,17)*Y(5)+CC(10,18)*Y(6)+CC(10,19)*Y(7)+CC(10,20)*Y(8)
2+AB1(10,2)+AB1(10,5)*2.*Y(9)*Y(10)+AB1(10,7)*Y(9)*Y(9)
3+AB1(10,9)*Y(9)*Y(11)+AB1(10,10)*3.*Y(10)*Y(10)+AB1(10,11)*
4Y(11)*Y(11)+AB1(10,12)*2.*Y(10)*Y(11)
A(10,11)=CC(10,9)*Y(1)+CC(10,10)*Y(2)+CC(10,11)*Y(3)+CC(10,12)
1*Y(4)
1+CC(10,21)*Y(5)+CC(10,22)*Y(6)+CC(10,23)*Y(7)+CC(10,24)*Y(8)
2+AB1(10,3)+2.*AB1(10,6)*Y(9)*Y(11)+AB1(10,9)*Y(9)*Y(10)
3+2.*AB1(10,11)*Y(10)*Y(11)+AB1(10,8)*Y(9)*Y(9)+AB1(10,12)*Y(10)
4*Y(10)+AB1(10,13)*3.*Y(11)*Y(11)
A(11,1)=CC(11,1)*Y(9)+CC(11,5)*Y(10)+CC(11,9)*Y(11)
A(11,2)=CC(11,2)*Y(9)+CC(11,6)*Y(10)+CC(11,10)*Y(11)
A(11,3)=CC(11,3)*Y(9)+CC(11,7)*Y(10)+CC(11,11)*Y(11)
A(11,4)=CC(11,4)*Y(9)+CC(11,8)*Y(10)+CC(11,12)*Y(11)
A(11,5)=CC(11,13)*Y(9)+CC(11,17)*Y(10)+CC(11,21)*Y(11)
A(11,6)=CC(11,14)*Y(9)+CC(11,18)*Y(10)+CC(11,22)*Y(11)
A(11,7)=CC(11,15)*Y(9)+CC(11,19)*Y(10)+CC(11,23)*Y(11)
A(11,8)=CC(11,16)*Y(9)+CC(11,20)*Y(10)+CC(11,24)*Y(11)
A(11,9)=CC(11,1)*Y(1)+CC(11,2)*Y(2)+CC(11,3)*Y(3)+CC(11,4)*Y(4)
1+CC(11,13)*Y(5)+CC(11,14)*Y(6)+CC(11,15)*Y(7)+CC(11,16)*Y(8)
2+AB1(11,1)+AB1(11,4)*3.*Y(9)*Y(9)+AB1(11,5)*Y(10)*Y(10)
3+AB1(11,6)*Y(11)*Y(11)+AB1(11,7)*2.*Y(9)*Y(10)+AB1(11,9)*Y(10)*
4*Y(11)+AB1(11,8)*2.*Y(9)*Y(11)
A(11,10)=CC(11,5)*Y(1)+CC(11,6)*Y(2)+CC(11,7)*Y(3)+CC(11,8)*Y(4)
1+CC(11,17)*Y(5)+CC(11,18)*Y(6)+CC(11,19)*Y(7)+CC(11,20)*Y(8)
2+AB1(11,2)+AB1(11,4)*2.*Y(9)*Y(10)+AB1(11,7)*Y(9)*Y(9)
3+AB1(11,9)*Y(9)*Y(11)+AB1(11,10)*3.*Y(10)*Y(10)+AB1(11,11)*Y
4(11)*Y(11)+AB1(11,12)*2.*Y(10)*Y(11)
A(11,11)=CC(11,9)*Y(1)+CC(11,10)*Y(2)+CC(11,11)*Y(3)+CC(11,12)
1*Y(4)
1+CC(11,21)*Y(5)+CC(11,22)*Y(6)+CC(11,23)*Y(7)+CC(11,24)*Y(8)
2+AB1(11,3)+AB1(11,6)*2.*Y(9)*Y(11)+AB1(11,9)*Y(9)*Y(10)
```

```

      3+AB1(11,11)*2.*Y(10)*Y(11)+AB1(11,8)*Y(9)*Y(9)+AB1(11,12)*
      4Y(10)*Y(10)+AB1(11,13)*3.*Y(11)*Y(11)
      CALL EQTN(Y,CC,ERR,AB1,Q)
      DO 10 I=1,11
10      C(I)=-ERR(I)
      BIG=DABS(C(1))
      DO 31 I=2,11
      IF(BIG .LT. DABS(C(I))) BIG=DABS(C(I))
31      CONTINUE
      DO 11 I=1,11
      DO 11 J=1,11
11      DD(I,J)=A(I,J)
      C
      CALL SIST(DD,C,11)
      DO 44 I=1,11
44      Y(I)=C(I)+Y(I)
      ITER=ITER+1
      IF(ITER .GT. 15) GO TO 100
50      FORMAT(11F10.7)
      IF(BIG .GT. 1.0E-10) GO TO 12
100     WRITE(6,50) (Y(I),I=1,11)
      DO 1002 KKK=1,11
      WW(KQ,1)=Q
1002     WW(KQ,KKK+1)=Y(KKK)
1001     CONTINUE
      DO 1004 KL=1,51
      WRITE (6,1005) WW(KL,1),WW(KL,10)
1004     CONTINUE
1005     FORMAT(1X,2F15.7)
      WRITE(6,1003) ((WW(I,J),J=1,12),I=1,51)
1003     FORMAT(1X,12F10.6)
      STOP
      END

```

```

C
C   THE NONLINEAR ALGEBRAIC EQUATIONS
C

```

```

      SUBROUTINE EQTN(Y,CC,ERR,AB1,Q)
      REAL *8 Y(11),CC(11,24),ERR(11),AB1(11,13)
      ERR(1)=CC(1,1)*Y(1)+CC(1,2)*Y(2)+CC(1,3)*Y(3)
      1+CC(1,4)*Y(4)+CC(1,5)*Y(5)+CC(1,6)*Y(6)+CC(1,7)*Y(7)
      2+CC(1,8)*Y(8)+CC(1,9)*Y(9)*Y(9)+CC(1,10)*Y(9)*Y(10)
      3+CC(1,11)*Y(9)*Y(11)+CC(1,12)*Y(10)*Y(10)+CC(1,13)*Y(10)*Y(11)
      4+CC(1,14)*Y(11)*Y(11)
      ERR(2)=CC(2,1)*Y(1)+CC(2,2)*Y(2)+CC(2,3)*Y(3)
      1+CC(2,4)*Y(4)+CC(2,5)*Y(5)+CC(2,6)*Y(6)+CC(2,7)*Y(7)
      2+CC(2,8)*Y(8)+CC(2,9)*Y(9)*Y(9)+CC(2,10)*Y(9)*Y(10)
      3+CC(2,11)*Y(9)*Y(11)+CC(2,12)*Y(10)*Y(10)+CC(2,13)*Y(10)*Y(11)
      4+CC(2,14)*Y(11)*Y(11)
      ERR(3)=CC(3,1)*Y(1)+CC(3,2)*Y(2)+CC(3,3)*Y(3)
      1+CC(3,4)*Y(4)+CC(3,5)*Y(5)+CC(3,6)*Y(6)+CC(3,7)*Y(7)
      2+CC(3,8)*Y(8)+CC(3,9)*Y(9)*Y(9)+CC(3,10)*Y(9)*Y(10)
      3+CC(3,11)*Y(9)*Y(11)+CC(3,12)*Y(10)*Y(10)+CC(3,13)*Y(10)*Y(11)
      4+CC(3,14)*Y(11)*Y(11)
      ERR(4)=CC(4,1)*Y(1)+CC(4,2)*Y(2)+CC(4,3)*Y(3)
      1+CC(4,4)*Y(4)+CC(4,5)*Y(5)+CC(4,6)*Y(6)+CC(4,7)*Y(7)

```

```
2+CC(4,8)*Y(8)+CC(4,9)*Y(9)*Y(9)+CC(4,10)*Y(9)*Y(10)
3+CC(4,11)*Y(9)*Y(11)+CC(4,12)*Y(10)*Y(10)+CC(4,13)*Y(10)*Y(11)
4+CC(4,14)*Y(11)*Y(11)
ERR(5)=CC(5,1)*Y(1)+CC(5,2)*Y(2)+CC(5,3)*Y(3)
1+CC(5,4)*Y(4)+CC(5,5)*Y(5)+CC(5,6)*Y(6)+CC(5,7)*Y(7)
2+CC(5,8)*Y(8)+CC(5,9)*Y(9)*Y(9)+CC(5,10)*Y(9)*Y(10)
3+CC(5,11)*Y(9)*Y(11)+CC(5,12)*Y(10)*Y(10)+CC(5,13)*Y(10)*Y(11)
4+CC(5,14)*Y(11)*Y(11)
ERR(6)=CC(6,1)*Y(1)+CC(6,2)*Y(2)+CC(6,3)*Y(3)
1+CC(6,4)*Y(4)+CC(6,5)*Y(5)+CC(6,6)*Y(6)+CC(6,7)*Y(7)
2+CC(6,8)*Y(8)+CC(6,9)*Y(9)*Y(9)+CC(6,10)*Y(9)*Y(10)
3+CC(6,11)*Y(9)*Y(11)+CC(6,12)*Y(10)*Y(10)+CC(6,13)*Y(10)*Y(11)
4+CC(6,14)*Y(11)*Y(11)
ERR(7)=CC(7,1)*Y(1)+CC(7,2)*Y(2)+CC(7,3)*Y(3)
1+CC(7,4)*Y(4)+CC(7,5)*Y(5)+CC(7,6)*Y(6)+CC(7,7)*Y(7)
2+CC(7,8)*Y(8)+CC(7,9)*Y(9)*Y(9)+CC(7,10)*Y(9)*Y(10)
3+CC(7,11)*Y(9)*Y(11)+CC(7,12)*Y(10)*Y(10)+CC(7,13)*Y(10)*Y(11)
4+CC(7,14)*Y(11)*Y(11)
ERR(8)=CC(8,1)*Y(1)+CC(8,2)*Y(2)+CC(8,3)*Y(3)
1+CC(8,4)*Y(4)+CC(8,5)*Y(5)+CC(8,6)*Y(6)+CC(8,7)*Y(7)
2+CC(8,8)*Y(8)+CC(8,9)*Y(9)*Y(9)+CC(8,10)*Y(9)*Y(10)
3+CC(8,11)*Y(9)*Y(11)+CC(8,12)*Y(10)*Y(10)+CC(8,13)*Y(10)*Y(11)
4+CC(8,14)*Y(11)*Y(11)
ERR(9)=CC(9,1)*Y(1)*Y(9)+CC(9,2)*Y(2)*Y(9)+CC(9,3)*Y(3)*Y(9)
1+CC(9,4)*Y(4)*Y(9)+CC(9,5)*Y(1)*Y(10)+CC(9,6)*Y(2)*Y(10)
2+CC(9,7)*Y(3)*Y(10)+CC(9,8)*Y(4)*Y(10)+CC(9,9)*Y(1)*Y(11)
3+CC(9,10)*Y(2)*Y(11)+CC(9,11)*Y(3)*Y(11)+CC(9,12)*Y(4)*Y(11)
4+CC(9,13)*Y(5)*Y(9)+CC(9,14)*Y(6)*Y(9)+CC(9,15)*Y(7)*Y(9)
5+CC(9,16)*Y(8)*Y(9)+CC(9,17)*Y(5)*Y(10)+CC(9,18)*Y(6)*Y(10)
6+CC(9,19)*Y(7)*Y(10)+CC(9,20)*Y(8)*Y(10)+CC(9,21)*Y(5)*Y(11)
7+CC(9,22)*Y(6)*Y(11)+CC(9,23)*Y(7)*Y(11)+CC(9,24)*Y(8)*Y(11)
8+AB1(9,1)*Y(9)+AB1(9,2)*Y(10)+AB1(9,3)*Y(11)+AB1(9,4)*
9(Y(9)**3)+AB1(9,5)*Y(9)*Y(10)*Y(10)+AB1(9,6)*Y(9)*Y(11)*Y(11)+
1AB1(9,7)*Y(9)*Y(9)*Y(10)+AB1(9,9)*Y(9)*Y(10)*Y(11)+AB1(9,10)*
2(Y(10)**3)+AB1(9,11)*Y(10)*Y(11)*Y(11)+AB1(9,8)*Y(11)*Y(9)*Y(9)
3+AB1(9,12)*Y(11)*Y(10)*Y(10)+AB1(9,13)*(Y(11)**3)-1.137777*Q
ERR(10)=CC(10,1)*Y(1)*Y(9)+CC(10,2)*Y(2)*Y(9)+CC(10,3)*Y(3)*Y(9)
1+CC(10,4)*Y(4)*Y(9)+CC(10,5)*Y(1)*Y(10)+CC(10,6)*Y(2)*Y(10)
2+CC(10,7)*Y(3)*Y(10)+CC(10,8)*Y(4)*Y(10)+CC(10,9)*Y(1)*Y(11)
3+CC(10,10)*Y(2)*Y(11)+CC(10,11)*Y(3)*Y(11)+CC(10,12)*Y(4)*Y(11)
4+CC(10,13)*Y(5)*Y(9)+CC(10,14)*Y(6)*Y(9)+CC(10,15)*Y(7)*Y(9)
5+CC(10,16)*Y(8)*Y(9)+CC(10,17)*Y(5)*Y(10)+CC(10,18)*Y(6)*Y(10)
6+CC(10,19)*Y(7)*Y(10)+CC(10,20)*Y(8)*Y(10)+CC(10,21)*Y(5)*Y(11)
7+CC(10,22)*Y(6)*Y(11)+CC(10,23)*Y(7)*Y(11)+CC(10,24)*Y(8)*Y(11)
8+AB1(10,1)*Y(9)+AB1(10,2)*Y(10)+AB1(10,3)*Y(11)+AB1(10,4)*
9(Y(9)**3)+AB1(10,5)*Y(9)*Y(10)*Y(10)+AB1(10,6)*Y(9)*Y(11)*Y(11)+
1AB1(10,7)*Y(9)*Y(9)*Y(10)+AB1(10,9)*Y(9)*Y(10)*Y(11)+AB1(10,10)*
2(Y(10)**3)+AB1(10,11)*Y(10)*Y(11)*Y(11)+AB1(10,8)*Y(11)*Y(9)*Y(9)
3+AB1(10,12)*Y(11)*Y(10)*Y(10)+AB1(10,13)*(Y(11)**3)-0.16253964*Q
ERR(11)=CC(11,1)*Y(1)*Y(9)+CC(11,2)*Y(2)*Y(9)+CC(11,3)*Y(3)*Y(9)
1+CC(11,4)*Y(4)*Y(9)+CC(11,5)*Y(1)*Y(10)+CC(11,6)*Y(2)*Y(10)
2+CC(11,7)*Y(3)*Y(10)+CC(11,8)*Y(4)*Y(10)+CC(11,9)*Y(1)*Y(11)
3+CC(11,10)*Y(2)*Y(11)+CC(11,11)*Y(3)*Y(11)+CC(11,12)*Y(4)*Y(11)
4+CC(11,13)*Y(5)*Y(9)+CC(11,14)*Y(6)*Y(9)+CC(11,15)*Y(7)*Y(9)
5+CC(11,16)*Y(8)*Y(9)+CC(11,17)*Y(5)*Y(10)+CC(11,18)*Y(6)*Y(10)
```

FILE: TPROG5 FORTRAN A

```

6+CC(11,19)*Y(7)*Y(10)+CC(11,20)*Y(8)*Y(10)+CC(11,21)*Y(5)*Y(11)
7+CC(11,22)*Y(6)*Y(11)+CC(11,23)*Y(7)*Y(11)+CC(11,24)*Y(8)*Y(11)
8+AB1(11,1)*Y(9)+AB1(11,2)*Y(10)+AB1(11,3)*Y(11)+AB1(11,4)*
9(Y(9)**3)+AB1(11,5)*Y(9)*Y(10)*Y(10)+AB1(11,6)*Y(9)*Y(11)*Y(11)+
1AB1(11,7)*Y(9)*Y(9)*Y(10)+AB1(11,9)*Y(9)*Y(10)*Y(11)+AB1(11,10)*
2(Y(10)**3)+AB1(11,11)*Y(10)*Y(11)*Y(11)+AB1(11,8)*Y(11)*Y(9)*Y(9)
3+AB1(11,12)*Y(11)*Y(10)*Y(10)+AB1(11,13)*(Y(11)**3)-0.16253964*Q
RETURN
END

```

C
C
C
CSUBROUTINE FOR THE SOLUTION OF A SYSTEM OF LINEAR
EQUATION

```

SUBROUTINE SIST(A,B,N)
REAL *8 A(N,N),B(N)
REAL *8 D,BA,H
NM1=N-1
D=1.0D00
DO 7 K=1,NM1
BA=A(K,K)
J=K
KP1=K+1
DO 1 I=KP1,N
IF(DABS(BA) .GE. DABS(A(I,K))) GO TO 1
BA=A(I,K)
J=I
1 CONTINUE
D=D*BA
IF(J .LE. K) GO TO 3
IF(DABS(D) .EQ. 0.0D00) GO TO 9
DO 2 I=K,N
H=A(K,I)
A(K,I)=A(J,I)/BA
2 A(J,I)=H
H=B(K)
B(K)=B(J)/BA
B(J)=H
GO TO 5
3 DO 4 I=KP1,N
4 A(K,I)=A(K,I)/BA
B(K)=B(K)/BA
5 DO 7 I=KP1,N
DO 6 J=KP1,N
6 A(I,J)=A(I,J)-A(I,K)*A(K,J)
7 B(I)=B(I)-A(I,K)*B(K)
D=D*A(N,N)
IF(DABS(D) .EQ. 0.0D00) GO TO 9
B(N)=B(N)/A(N,N)
DO 8 I=1,NM1
K=N-I
KP1=K+1
DO 8 J=KP1,N
8 B(K)=B(K)-A(K,J)*B(J)
RETURN
9 WRITE (6,10)

```


FILE: TPROG5 FORTRAN A

137

10 FORMAT(' SINGULAR SOLUTION')
 CALL EXIT
 END

```

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C      PROGRAM FOR BUCKLING PROBLEM                                C
C      USED AFTER THE NECESSARY DEFINITE INTEGRATIONS              C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C      AM REPRESENTS MEW IN THE DIFFERENTIAL EQUATION
C      H = ASPECT RATIO
C      R = AXIAL LOAD RATIO NY/NX
C      QQ = EIGEN VALUE
C      A AND B ARE THE MATRICES IN THE TWO SIDE OF THE EIGEN
C      VALUE PROBLEM FORMED FROM THE RESULTS OF THE DEFINITE
C      INTEGRATIONS WHICH ARE STORED IN X AND Y MATRICES
C
      INTEGER N,IA,IB,IJOB,IZ,IER
      REAL*8 A(7,7),B(7,7),WK(7),BETA(7),X(7,27),Y(7,18),P(7),Q
1,X1(7,27)
      COMPLEX*16 Z(7,7),ALFA(7),E(7),ZN
      IA=7
      IB=7
      IZ=7
      N= 7
      IJOB=1
      READ (5,1) ((X(I,J),J=1,27),I=1,7)
      READ (5,1) ((Y(I,J),J=1,18),I=1,7)
      DO 101 IK=1,7
      DO 101 JK=1,5
      XJK=JK
      XIK=IK
      H=1.0+0.25*(XJK-1.0)
      R=XIK-1.0
      DO 25 I=1,7
      DO 25 J=1,7
      JJ=J+9
      KK=J+18
      X1(I,J)=X(I,J)
      X1(I,JJ)=X(I,JJ)*2.*H*H
      X1(I,KK)=X(I,KK)*H*H*H*H
      A(I,J)=X1(I,J)+X1(I,JJ)+X1(I,KK)
      AM=0.013748
      B(I,J)=Y(I,J)+R*H*H*Y(I,JJ)-AM*(X(I,J)+X(I,JJ))*(R*H*H+H*H)
1+X(I,KK)*R*H*H*H*H)
25      CONTINUE
      CALL EIGZF (A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
      DO 5 J=1,7
      E(J)=ALFA(J)/BETA(J)
      P(J)=E(J)
5      CONTINUE
      Q=P(1)
      DO 55 J=2,7
      IF (DABS(P(J))-DABS(Q)) 56,57,55
56      Q=P(J)
      GO TO 55,
57      Q=P(J)
55      CONTINUE
      QQ=Q*0.40528
      WRITE(6,103) R,H,QQ

```

```
103        FORMAT(3F12.6)
101        CONTINUE
2        FORMAT(2F20.7)
1        FORMAT(4X,F12.7)
         STOP
         END
```

```

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C      PROGRAM FOR VIBRATION PROBLEM
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CC
CC      DEG = SKEW ANGLE
CC      T1= FACE THICKNESS
CC      C0= CORE THICKNESS
CC      GM = SHEAR RIGIDITY OF THE CORE
CC      ANEW = POISSONS RATIO OF THE FACE MATERIAL
CC      EF = MODULUS OF ELASTICITY OF THE FACE MATERIAL
CC      XL,YL = DIMENSIONS OF THE PLATE IN X AND Y DIRECTIONS
CC      H = ASPECT RATIO
CC      U1 AND DP ARE THE ELASTIC CONSTANTS
CC      A AND B ARE THE MATRICES DEFINING TWO SIDES OF THE
CC      EIGEN VALUE PROBLEM WHICH ARE FORMED FROM THE RESULTS
CC      OF THE DEFINITE INTEGRATIONS STORED IN MATRICES X
CC              AND Y
CC      QQ = THE REQUIRED EIGEN VALUE
CC
      INTEGER N,IA,IB,IJOB,IZ,IER
      REAL*8  QQ(7),A(7,7),B(7,7),WK(7),BETA(7),X(7,63),X1(7,63)
      COMPLEX*16  Z(7,7),ALFA(7),E(7),ZN
      IA=7
      IB=7
      IZ=7
      N=7
      IJOB=1
      DEG=0.0
      PHI=DEG*3.1416/180.0
      C1=COS(PHI)
      S=SIN(PHI)
      T1=0.016
      C0=0.25
      GM=1000.00
      ANEW=0.34
      EF=10.0**7
      XL=20.00
      YL=20.00
      H=1.00
      DP=(EF/(1.-ANEW*ANEW))*(((T1**3)/6.0)+(T1/2.0)*((C0+T1)**2))
      U1=GM*C0
      DEN=500.02/(10.0**6)
      READ (5,1) ((X(I,J),J=1,63),I=1,7)
      DO 151 I=1,7
      DO 151 J=1,7
      J1=J+7
      J2=J+14
      J3=J+21
      J4=J+28
      J5=J+35
      J6=J+42
      J7=J+49
      J8=J+56
      X1(I,J)=X(I,J)/(C1*C1*C1*C1)
      X1(I,J1)=X(I,J1)*(-4.*H*S/(C1*C1*C1*C1))

```

```
X1(I,J3)=X(I,J3)*H*H*((6.*S*S/(C1*C1*C1*C1))+2.0/(C1*C1))
X1(I,J2)=X(I,J2)*H*H*H*(-4.*S/(C1*C1*C1*C1))
X1(I,J4)=X(I,J4)*H*H*H*H/(C1*C1*C1*C1)
A(I,J)=X1(I,J)+X1(I,J1)+X1(I,J2)+X1(I,J3)+X1(I,J4)
151       CONTINUE
DO 152 I=1,7
DO 152 J=1,7
J1=J+7
J2=J+14
J3=J+21
J4=J+28
J5=J+35
J6=J+42
J7=J+49
J8=J+56
X1(I,J5)=X(I,J5)/(C1*C1*XL*XL)
X1(I,J6)=X(I,J6)/(C1*C1*YL*YL)
X1(I,J7)=X(I,J7)*(-2.*S/(C1*C1*XL*YL))
X1(I,J8)=X(I,J8)
B(I,J)=X1(I,J8)-((X1(I,J5)+X1(I,J6)+X1(I,J7))*(DP/U1))
B(I,J)=B(I,J)*(XL*XL*XL*XL*DEN/DP)
152       CONTINUE
CALL EIGZF (A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
DO 5 J=1,7
E(J)=ALFA(J)/BETA(J)
5       CONTINUE
DO 6 J=1,7
ZN=Z(7,J)
DO 6 I=1,7
Z(I,J)=Z(I,J)/ZN
6       CONTINUE
2       FORMAT(2F20.7)
1       FORMAT(4X,F12.7)
DO 101 II=1,7
QQ(II)=E(II)
QQ(II)=QQ(II)**0.5
WRITE(6,103) QQ(II)
103       FORMAT(2X,F12.6)
101       CONTINUE
STOP
END
```

PURPOSE - EIGENVALUES AND (OPTIONALLY) EIGENVECTORS OF THE SYSTEM $A \cdot X = \text{LAMBDA} \cdot B \cdot X$ WHERE A AND B ARE REAL MATRICES

USAGE - CALL EIGZF (A, IA, B, IB, N, IJOB, ALFA, BETA, Z, IZ, WK, IER)

ARGUMENTS

A - THE INPUT REAL GENERAL MATRIX OF ORDER N. INPUT A IS DESTROYED IF IJOB IS EQUAL TO 0 OR 1.

IA - THE INPUT ROW DIMENSION OF MATRIX A EXACTLY AS SPECIFIED IN THE DIMENSION STATEMENT IN THE CALLING PROGRAM.

B - THE INPUT REAL GENERAL MATRIX OF ORDER N. INPUT B IS DESTROYED IF IJOB IS EQUAL TO 0 OR 1.

IB - THE INPUT ROW DIMENSION OF MATRIX B EXACTLY AS SPECIFIED IN THE DIMENSION STATEMENT IN THE CALLING PROGRAM.

N - THE INPUT ORDER OF THE MATRICES A AND B.

IJOB - INPUT OPTION PARAMETER. WHEN
IJOB = 0, COMPUTE EIGENVALUES ONLY
IJOB = 1, COMPUTE EIGENVALUES AND EIGENVECTORS.
IJOB = 2, COMPUTE EIGENVALUES, EIGENVECTORS AND PERFORMANCE INDEX.
IJOB = 3, COMPUTE PERFORMANCE INDEX ONLY.
IF THE PERFORMANCE INDEX IS COMPUTED, IT IS RETURNED IN WK(1). THE ROUTINES HAVE PERFORMED (WELL, SATISFACTORILY, POORLY) IF WK(1) IS (LESS THAN 1, BETWEEN 1 AND 100, GREATER THAN 100).

ALFA - OUTPUT VECTORS OF LENGTH N.
BETA - ALFA IS TYPE COMPLEX AND BETA IS TYPE REAL. IF A AND B WERE SIMULTANEOUSLY REDUCED TO TRIANGULAR FORM BY UNITARY EQUIVALENCES, ALFA AND BETA WOULD CONTAIN THE DIAGONAL ELEMENTS OF THE RESULTING MATRICES. (SEE MOLER-STEWART REFERENCE).
THE J-TH EIGENVALUE IS THE COMPLEX NUMBER GIVEN BY $\text{ALFA}(J)/\text{BETA}(J)$.
NOTE - THE ROUTINE TREATS ALFA AS A REAL VECTOR OF LENGTH $2 \cdot N$. AN APPROPRIATE EQUIVALENCE STATEMENT MAY BE REQUIRED. SEE DOCUMENT EXAMPLE.

Z - THE OUTPUT N BY N COMPLEX MATRIX CONTAINING THE EIGENVECTORS.
THE EIGENVECTOR IN COLUMN J OF Z CORRESPONDS TO THE EIGENVALUE $\text{ALFA}(J)/\text{BETA}(J)$.
IF IJOB = 0, Z IS NOT USED.
NOTE - THE ROUTINE TREATS Z AS A REAL VECTOR OF LENGTH $2 \cdot N \cdot N$. AN APPROPRIATE EQUIVALENCE STATEMENT MAY BE REQUIRED. SEE DOCUMENT EXAMPLE.

- IZ - THE INPUT ROW DIMENSION OF MATRIX Z EXACTLY AS SPECIFIED IN THE DIMENSION STATEMENT IN THE CALLING PROGRAM. IZ MUST BE GREATER THAN OR EQUAL TO N IF IJOB IS NOT EQUAL TO ZERO.
- WK - WORK AREA, THE LENGTH OF WK DEPENDS ON THE VALUE OF IJOB AS FOLLOWS,
 IJOB = 0, THE LENGTH OF WK IS AT LEAST N.
 IJOB = 1, THE LENGTH OF WK IS AT LEAST N.
 IJOB = 2, THE LENGTH OF WK IS AT LEAST $2*N*N$.
 IJOB = 3, THE LENGTH OF WK IS AT LEAST 1.
- IER - ERROR PARAMETER.. (OUTPUT)
 TERMINAL ERROR
 IER = 128+J, INDICATES THAT EQZTF FAILED TO CONVERGE ON EIGENVALUE J. EIGENVALUES J+1,J+2,...,N HAVE BEEN COMPUTED CORRECTLY. EIGENVALUES 1,...,J MAY BE INACCURATE. IF IJOB = 1 OR 2 EIGENVECTORS MAY BE INACCURATE. THE PERFORMANCE INDEX IS SET TO 1000.
 WARNING ERROR (WITH FIX)
 IER = 66, INDICATES IJOB IS LESS THAN 0 OR IJOB IS GREATER THAN 3. IJOB IS RESET TO 1.
 IER = 67, INDICATES IJOB IS NOT EQUAL TO 0, AND IZ IS LESS THAN THE ORDER OF MATRIX A. IJOB IS RESET TO 0.
- PRECISION/HARDWARE - SINGLE AND DOUBLE/H32
 - SINGLE/H36,H48,H60
- REQD. IMSL ROUTINES - EQZQF,EQZTF,EQZVF,UERTST,UGETIO,VHSH2C,VHSH2R,VHSH3R
- NOTATION - INFORMATION ON SPECIAL NOTATION AND CONVENTIONS IS AVAILABLE IN THE MANUAL INTRODUCTION OR THROUGH IMSL ROUTINE UHELP

Algorithm

EIGZF computes eigenvalues and (optionally) eigenvectors for the generalized eigenproblem $Ax = \lambda Bx$ where A and B are real N by N matrices. It can also compute a performance index.

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ can be calculated from the output by setting $\lambda_I = ALFA(I)/BETA(I)$ when $BETA(I) \neq 0$. If $BETA(I) = 0$ then λ_I is regarded as being infinite. The eigenvectors are returned in the complex matrix Z so that column M of Z contains the eigenvector corresponding to λ_M .

EIGZF calls IMSL routine EQZQF to reduce A to upper Hessenberg form and B to upper triangular form. Then, EQZTF is called to further transform A to quasi-upper triangular form (upper Hessenberg with no

two consecutive subdiagonal elements being nonzero) while retaining B in upper triangular form. EQZVF is called to compute ALFA(I) and BETA(I), $I=1,2,\dots,N$ and, optionally, the associated eigenvectors.

The performance index is defined as follows:

$$P = \max_{1 \leq j \leq N} \frac{\|S_j A z^j - \alpha_j B z^j\|_\infty}{(|S_j| \|A\|_1 \|z^j\|_1 + |\alpha_j| \|B\|_1 \|z^j\|_1) \text{EPS}}$$

where the max is taken over the j eigenvalues $\lambda_j = \alpha_j / S_j$ and associated eigenvectors z^j . Here, $\alpha_j = \text{ALFA}(j)$, $S_j = \text{BETA}(j)$, and z^j denotes column j of Z . EPS specifies the relative precision of floating point arithmetic. When P is less than 1, the performance of the routines is considered to be excellent in the sense that the residuals $SAz - \alpha Bz$ are as small as can be expected. When P is between 1 and 100 the performance is good. When P is greater than 100 the performance is considered poor.

See reference:

Moler, C. B., and Stewart, G. W., "An algorithm for generalized matrix eigenvalue problems", SIAM J. Numer. Anal., 10, 1973, 241-256.

Programming Notes

1. A and B are preserved when IJOB=2 or 3. In all other cases A and B are destroyed.
2. The eigenvalues are unordered except that complex conjugate pairs of eigenvalues appear consecutively. That is, if λ_M and λ_{M+1} are such a pair then $\text{ALFA}(M+1)/\text{BETA}(M+1)$ is the complex conjugate of $\text{ALFA}(M)/\text{BETA}(M)$. ALFA is type COMPLEX and BETA is type REAL. It is not necessarily true that $\text{ALFA}(M+1)$ is the conjugate of $\text{ALFA}(M)$ for such a pair.
3. The eigenvectors are normalized so that the largest component has absolute value 1.
4. When IJOB=3 (i.e., to compute a performance index only) the eigenvalues, ALFA and BETA, and eigenvectors, Z , are assumed to be input.
5. If parameter IJOB is not in the range 0 to 3, computation continues with IJOB reset to 1 and IER=66 is returned. If IJOB is greater than zero (indicating that eigenvectors are desired) and IZ is less than N , computation continues with IJOB reset to 0 and IER=67 is returned.

Example

In example 1, EIGZF is called to compute eigenvalues, eigenvectors, and a performance index by setting input IJOB=2. For machines which require equivalencing, see example 2.

Example 1

```

INTEGER  IA,IB,N,IJOB,IZ,IER
REAL     A(3,3),B(3,3),BETA(3),WK(18)
COMPLEX  ALFA(3),Z(3,3)

```

Input:

```

IA  = 3
IB  = 3
IZ  = 3
N   = 3
IJOB = 2

```

$$A = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -10.0 & 2.0 & 0.0 \\ 5.0 & 1.0 & 0.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.5 & 0.0 & 0.0 \\ 3.0 & 3.0 & 0.0 \\ 4.0 & 0.5 & 1.0 \end{bmatrix}$$

```
CALL EIGZF (A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
```

Output:

```
IER = 0
```

$$Z = \begin{bmatrix} -0.25205+0.19169i & -0.25205-0.19169i & 0.0+0.0i \\ -0.08799-0.72598i & -0.08799+0.72598i & 0.0+0.0i \\ 1.00000+0.00000i & 1.00000+0.00000i & 1.0+0.0i \end{bmatrix} \quad (\text{eigenvectors})$$

```
WK(1) < 10 \quad (\text{performance index})
```

The eigenvalues are as follows (where $\lambda_j = \widetilde{ALFA}_j / BETA_j$):

$$\begin{aligned} \lambda_1 &= 0.83333 + 1.9930i \\ \lambda_2 &= 0.83333 - 1.9930i \\ \lambda_3 &= 0.50000 + 0.0000i \end{aligned}$$

Example 2

For machines which require an equivalence statement in situations where an array is of one type in the calling program but of another type in the subroutine, EIGZF should be called as follows.

```

INTEGER  IA,IB,N,IJOB,IZ,IER
REAL     A(3,3),B(3,3),BETA(3),WK(18),RALFA(6),RZ(18)
COMPLEX  ALFA(3),Z(3,3)
EQUIVALENCE (ALFA(1),RALFA(1)),(Z(1,1),RZ(1))

```

Input:

```

IA  = 3
IB  = 3
IZ  = 3
N   = 3
IJOB = 2

```

$$A = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -10.0 & 2.0 & 0.0 \\ 5.0 & 1.0 & 0.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.5 & 0.0 & 0.0 \\ 3.0 & 3.0 & 0.0 \\ 4.0 & 0.5 & 1.0 \end{bmatrix}$$

CALL EIGZF (A,IA,B,IB,N,IJOB,RALFA,BETA,RZ,IZ,WK,IER)

• Output:

IER = 0

$$Z = \begin{bmatrix} -0.25205+0.19169i & -0.25205-0.19169i & 0.0+0.0i \\ -0.08799-0.72598i & -0.08799+0.72598i & 0.0+0.0i \\ 1.00000+0.00000i & 1.00000+0.00000i & 1.0+0.0i \end{bmatrix} \quad (\text{eigenvectors})$$

WK(1) < 10 (performance index)

The eigenvalues are as follows (where $\lambda_j = \text{ALFA}_j / \text{BETA}_j$):

$$\begin{aligned} \lambda_1 &= 0.83333 + 1.9930i \\ \lambda_2 &= 0.83333 - 1.9930i \\ \lambda_3 &= 0.50000 + 0.0000i \end{aligned}$$