Koszul Algebras and Koszul Duality

by

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Abstract

In this thesis, we present a detailed exposition of Koszul algebras and Koszul duality. We begin with an overview of the required concepts of graded algebras and homological algebra. We then give a precise treatment of Koszul and quadratic algebras, together with their dualities. We fill in some arguments that are omitted in the literature and work out a number of examples in full detail to illustrate the abstract concepts.

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Chapter 1

Introduction

Homological methods are indispensable in the study of algebraic topology and algebraic geometry. The major technique in homological algebra is to construct a *complex* of modules for a specific algebra, and then compute its homology and cohomology groups, which describe invariants of the topological structure associated to this algebra. In the early 1950s, Jean-Louis Koszul in [10] introduced a complex (today, we call it the *Koszul complex*) for the theory of Lie algebra cohomology. Since then, Koszul complexes have played a significant role in homological algebra and have been applied to various mathematical and physical theories (e.g., supersymmetry, a topic of active research in quantum field theory). However, computing the cohomology groups for arbitrary algebras can be extremely difficult. In 1970, Stewart Priddy in [16] constructed a resolution, called the *Koszul resolution*, for a class of positively graded algebras which came to be called *Koszul algebras*. For a Koszul algebra A, the Koszul resolution is an exact sequence of projective graded modules which are generated by their lowest homogeneous pieces. Consequently, Koszul algebras, compared to other algebras, provide an "easiest" way to compute the cohomology groups Ext_A^n derived from the functor Hom_A.

The class of Koszul algebras is not small. Many well-known algebras in topology and geometry are contained in this class. In [16], Stewart Priddy gave several examples of Koszul algebras, including 1) for a graded Lie algebra L over a field $\Bbbk$, the universal enveloping algebra $U(L)$ is Koszul; 2) for a graded restricted Lie algebra L of characteristic 2, the universal enveloping algebra $U(L)$ is Koszul; 3) a quadratic Poincaré-Birkhoff-Witt (PBW) algebra is Koszul; 4) for a prime number p, the mod p Steenrod algebra A_p is Koszul. In [7], David Eisenbud stated that the homogeneous coordinate rings of many homogeneous spaces over an algebraically closed field $\Bbbk$ such as the

Grasmannian, flag manifolds, etc. are Koszul algebras, and furthermore, the homogeneous coordinate ring of an algebraic variety (over an algebraically closed field $\Bbbk$) embedded in projective space by a "sufficiently" ample line bundle is a Koszul algebra. In quantum group theory, the quantum affine n-space $A_q^{n|0}$, a quadratic algebra over a field $\Bbbk$, is a Koszul algebra (see [14] for the special case: the quantum plane $A_q^{2|0}$ and [15, Example 1 in Section 4.2, p 83] for the general case). In [15], Alexander Polishchuk and Leonid Positselski explained a connection between Koszul algebras and one-dependent discrete-time stochastic processes. In summary, the notions and theories of Koszul algebras and Koszul duality appear frequently in many mathematical areas, such as algebraic geometry, topology, representation theory, noncommutative algebra, and combinatorial algebra. In recent decades, we can also see many applications of the theory of Koszul algebras and duality to theoretical physics (see, for example, [2], [4] and [5]).

There are several references on the theory of Koszul algebras and duality. However, almost all of them assume the readers have a strong background in homological algebra and category theory. This thesis is intended to introduce the basic notions of Koszul algebras and Koszul duality, assuming only a knowledge of algebra typically taught in first year graduate algebra courses.

In Chapter 2, we introduce the concepts of associative algebras and coalgebras. Then, we review the definitions and properties of graded algebras, tensor algebras, symmetric algebras, and exterior algebras.

In Chapter 3, we review the concepts and properties of complexes, resolutions, cohomology groups, and the functor Ext. Then, we deduce the natural isomorphism of two functors: Ext_R^n and E_R^n, and finally describe the Yoneda product and Ext-algebra.

In Chapter 4, we define Koszul algebras and quadratic algebras, describe some properties of these algebras, and then give some typical examples of Koszul algebras.

In Chapter 5, we define the dual of a quadratic algebra, then construct a Koszul complex of a quadratic algebra, and finally show that the dual of a Koszul algebra is also a Koszul algebra.

Throughout this thesis, $\Bbbk$ denotes a field, all rings have a multiplicative unit $1 \neq 0$, and an R-module will always mean a left R-module unless indicated otherwise.

Chapter 2

Algebras and Coalgebras

Before introducing the notions of Koszul algebras and Koszul duality, we recall some concepts that we will use in their definitions.

2.1 Algebras

First of all, we summarize briefly the definitions of algebras and coalgebras. For further details, we refer the reader to textbooks such as [12] and [3].

Definition 2.1.1 (Algebra)**.** An *associative algebra* over a field $\Bbbk$ is a vector space A over $\Bbbk$ with a binary map $p\colon A \otimes A \to A$, called a product, such that p is associative:

$$p \circ (p \otimes \mathrm{id}_A) = p \circ (\mathrm{id}_A \otimes p).$$

Denote $xy = p(x \otimes y)$. Then associativity is read as

$$(xy)z = x(yz)$$

for all $x, y, z \in A$. Furthermore, in this thesis, an associative algebra A is always assumed to be *unital*, i.e., there is a map $e\colon \Bbbk \to A$, called a unit, such that

$$p \circ (e \otimes \mathrm{id}_A) = p \circ (\mathrm{id}_A \otimes e) = \mathrm{id}_A .$$

Denote $1 = 1_A = e(1_{\Bbbk})$.Then unitality is read as

$$1x = x1 = x$$

for all $x \in A$. An associative algebra A is *commutative* if the multiplication p is commutative:

$$p(x \otimes y) = p(y \otimes x)$$

or

$$xy = yx.$$

for all $x, y \in A$.

Note that one can also consider algebras that are not associative. For instance, in general, Lie algebras are not associative.

Definition 2.1.2 (Morphism)**.** A map $\varphi\colon A_1 \to A_2$ between two associative algebras A_1 and A_2 is called *algebra morphism* (or simply *morphism*) if φ is linear over $\Bbbk$, and

$$\varphi(xy) = \varphi(x)\varphi(y)$$

for all $x, y \in A_1$ and furthermore, $\varphi(1_{A_1}) = 1_{A_2}$.

2.2 Coalgebras

Coalgebras are dual to algebras. In categorical language, the definitions of algebras and coalgebras are related by the reversing of arrows.

Definition 2.2.1 (Coalgebra)**.** A *coassociative coalgebra* over a field $\Bbbk$ is a vector space C over $\Bbbk$ with a $\Bbbk$-linear map $\Delta\colon C \to C \otimes C$, called a coproduct, such that Δ is coassociative:

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta.$$

A coassociative coalgebra C is *counital* if there is a map $\epsilon\colon C \to \Bbbk$, called a counit, such that

$$(\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C = (\epsilon \otimes \mathrm{id}_C) \circ \Delta.$$

A coassociative coalgebra (C, Δ) is *cocommutative* if

$$\Delta = \tau \circ \Delta$$

where $\tau\colon C \otimes C \to C \otimes C$ is a $\Bbbk$-linear map, called the *switching map*, defined by

$$\tau(x \otimes y) = y \otimes x. \tag{2.1}$$

Notation 2.2.2 (Sweedler's Notation)**.** Suppose that (C, Δ) is a coassociative coalgebra. Then for $c \in C$, we have $\Delta(c) = \sum_{i=1}^{m} c_i \otimes \hat{c}_i$ for some $c_1, \cdots, c_m, \hat{c}_1, \cdots, \hat{c}_m \in C$. *Sweedler's* $\sum$*-notation* denotes the sum $\sum_{i=1}^{m} c_i \otimes \hat{c}_i$ by

$$\sum c_{\underline{1}} \otimes c_{\underline{2}} := \sum_{i=1}^{m} c_i \otimes \hat{c}_i.$$

where $c_{\underline{1}}$ and $c_{\underline{2}}$ are formal symbols.

Definition 2.2.3 (Coalgebra Morphism)**.** Let (C, Δ, ϵ) and $(\hat{C}, \hat{\Delta}, \hat{\epsilon})$ be coassociative and counital coalgebras. A $\Bbbk$-linear map $f \colon C \to \hat{C}$ is a *coalgebra morphism* if

$$\begin{aligned} \hat{\Delta} \circ f &= (f \otimes f) \circ \Delta, \quad \text{and} \\ \hat{\epsilon} \circ f &= \epsilon, \end{aligned}$$

that reads, in Sweedler's notation, that for all $c \in C$,

$$\begin{gathered} \sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}}, \\ \hat{\epsilon}(f(c)) = \epsilon(c). \end{gathered}$$

2.3 Graded Algebras

Definition 2.3.1 (Graded Ring)**.** Let R be a ring and let $(G, \cdot)$ be a monoid. A *G-grading* on R is a family $\{R_d\}_{d \in G}$ such that

- each R_d is a subgroup of $(R, +)$,
- $R = \bigoplus_{d \in G} R_d$,
- $R_i R_j \subseteq R_{i \cdot j}$ for all $i, j \in G$.

A *G-graded ring* is a ring R with a G-grading of R. An ideal I of the graded ring R is called a *graded ideal* if $I = \bigoplus_{d \in G} (I \cap R_d)$.

Example 2.3.2 ($\mathbb{N}$-graded ring)**.** Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the monoid of nonnegative integers under addition. An *$\mathbb{N}$-graded ring* is a ring R with a $\mathbb{N}$-grading of R, i.e.,

$$R = \bigoplus_{i \in \mathbb{N}} R_i$$

where R_i are abelian additive groups, satisfying

$$R_i R_j \subseteq R_{i+j}$$

for all $i, j \in \mathbb{N}$. An $\mathbb{N}$-graded ring is also called a *positively graded* ring.

Example 2.3.3. Let $R = \Bbbk[X_1, \cdots, X_n]$ be the polynomial ring in n indeterminates. Define R_d, for $d \in \mathbb{N}$, by

$$R_d = \mathrm{Span}_{\Bbbk}\{X_1^{i_1} \cdots X_n^{i_n} \mid i_1 + \cdots + i_n = d\}.$$

In other words, R_d is the set of homogeneous polynomials of degree d and $R_0 = \Bbbk$. Then

$$R = \bigoplus_{d=0}^{\infty} R_d$$

is an $\mathbb{N}$-grading on R.

Proposition 2.3.4. *Let $R = \bigoplus_{d \in G} R_d$ be a G-graded ring. Let I be a graded ideal in R and let $I_d = I \cap R_d$ for all $d \in G$. Then R/I is a graded ring whose homogeneous component of degree d is isomorphic to R_d/I_d.*

Proof. The map

$$\begin{aligned} R = \bigoplus_{d \in G} R_d &\to \bigoplus_{d \in G} (R_d/I_d) \\ (r_d)_{d \in G} &\mapsto (r_d + I_d)_{d \in G} \end{aligned}$$

is an isomorphism of graded rings with kernel $I = \bigoplus_{d \in G} I_d$. $\square$

Definition 2.3.5 (Graded Algebra)**.** An associative algebra A is a *graded algebra* if A is graded as a ring. If $A = \bigoplus_{d=0}^{\infty} A_d$ is an $\mathbb{N}$-graded algebra, we let

$$A_+ = \bigoplus_{d>0} A_d.$$

Definition 2.3.6. An $\mathbb{N}$-graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$ is said to be *generated in degrees 0 and 1* if A_0 and A_1 generate A as an algebra.

Remark 2.3.7. Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded $\Bbbk$-algebra with $A_0 = \Bbbk$. Recall that $A_1 A_{i-1} \subseteq A_i$ since A is a graded ring. If A is generated by A_0 and A_1, then any element in A_i is a sum of monomials, in which we have a factor from A_1 on the left, and hence, the factor on the right is in A_{i-1}. This implies that $A_i \subseteq A_1 A_{i-1}$. We thus obtain an equivalent definition of Definition 2.3.6: The graded algebra A is generated in degrees 0 and 1 if $A_i = A_1 A_{i-1}$ for all $i \in \mathbb{N}_+$.

Lemma 2.3.8. *Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded $\Bbbk$-algebra with $A_0 = \Bbbk$. The following are equivalent.*

1. *A is generated in degrees 0 and 1.*
2. *$A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{N}$.*
3. *$A_m = A_1 A_1 \cdots A_1 = A_1^m$ for each $m \in \mathbb{N}$.*
4. *$A_+ = AA_1$.*

Proof.
($1 \implies 2$) Suppose that A is generated in degrees 0 and 1. Then, $A_d = A_1 A_{d-1} = A_{d-1} A_1$ for all $d \in \mathbb{N}_+$. Hence,

$$A_{i+j} = A_1 A_{i+j-1} = A_1 A_1 A_{i+j-2} = A_2 A_{i+j-2} = \cdots = A_i A_{i+j-i} = A_i A_j.$$

($2 \implies 1$) Suppose that $A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{N}$. Then,

$$A_1 A_{i-1} = A_{1+i-1} = A_i \text{ for all } i \in \mathbb{N}.$$

($1 \iff 3$) By Definition 2.3.6. the statement 1 implies that $A_d = A_1 A_{d-1} = A_{d-1} A_1$ for all $d \in \mathbb{N}_+$. Then the equivalence follows from induction on m.
($2 \implies 4$) Suppose that $A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{N}$. Then,

$$AA_1 = \bigoplus_{d=0}^{\infty} A_d A_1 = \bigoplus_{d=0}^{\infty} A_{d+1} = \bigoplus_{d=1}^{\infty} A_d = A_+.$$

($4 \implies 1$) Suppose that $A_+ = AA_1$. Then,

$$\bigoplus_{d=1}^{\infty} A_d = AA_1 = \bigoplus_{d=0}^{\infty} A_d A_1 \subseteq \bigoplus_{d=0}^{\infty} A_{d+1} = \bigoplus_{d=1}^{\infty} A_d.$$

Thus,

$$\bigoplus_{d=0}^{\infty} A_d A_1 = \bigoplus_{d=0}^{\infty} A_{d+1}.$$

This implies that $A_d A_1 = A_{d+1}$ for all $d \in \mathbb{N}$. $\square$

Definition 2.3.9 (Graded module)**.** Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded ring. An R-module M is called a *graded R-module* if there is a family of additive subgroups $\{M_j \mid j \in \mathbb{Z}\}$ of M such that

- $M = \bigoplus_{j\in\mathbb{Z}} M_j$ as abelian groups;
- $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$.

Let N be a submodule of M and define $N_j = N \cap M_j$ for all $j \in \mathbb{Z}$. The submodule N is a *graded submodule of M* if $N = \bigoplus_{j\in\mathbb{Z}} N_j$.

Definition 2.3.10 (Graded Homomorphism)**.** Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded ring. Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ and $N = \bigoplus_{j\in\mathbb{Z}} N_j$ be two graded R-modules. Let $f\colon M \to N$ be an R-module homomorphism. Then, f is said to be *homogeneous of degree d* if $f(M_j) \subseteq N_{j+d}$ for all j.

Notation 2.3.11 (Shifted Grading)**.** Let R be a $\mathbb{Z}$-graded ring. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded R-module. Denote by $M(n) = \bigoplus_{j\in\mathbb{Z}} M(n)_j$ the same R-module M with *shifted grading* $M(n)_j = M_{j+n}$.

Remark 2.3.12. Let R be a $\mathbb{Z}$-graded ring. We can easily verify the following properties.

1. Let M be a $\mathbb{Z}$-graded R-module. Then, for $n \in \mathbb{Z}$, $M(n)$ is a $\mathbb{Z}$-graded R-module.
2. If $f\colon M \to N$ is a homogeneous homomorphism of degree d, then $f\colon M(-d) \to N$ is a homogeneous homomorphism of degree 0.
3. Let $\{M^{(t)}\}$ be a family of graded R-modules. Then, $\bigoplus_t M^{(t)}$ is a graded R-module.

Notation 2.3.13 (Categories of Graded modules)**.** For a ring R, we introduce the following notation.

- Denote by R-Mod the category of left R-modules.
- Denote by R-Modf the full subcategory of R-Mod with $\mathrm{obj}_{R\text{-Modf}}$ equal to the class of all finitely generated left R-modules.

- If R is a $\mathbb{Z}$-graded ring, denote by R-grMod the category of graded R-modules, with $\mathrm{obj}_{R\text{-grMod}}$ equal to the class of all graded R-modules and $\mathrm{Mor}_{R\text{-grMod}}(M, N) = \hom_R(M, N)$ for $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and $N = \bigoplus_{i\in\mathbb{Z}} N_i$, where

$$\hom_R(M, N) = \{f \in \mathrm{Hom}_R(M, N) \mid f(M_i) \subseteq N_i \text{ for all } i \in \mathbb{Z}\}.$$

- If R is a $\mathbb{Z}$-graded ring, denote by R-grModf the full subcategory of R-grMod with $\mathrm{obj}_{R\text{-grModf}}$ equal to the class of all finitely generated graded R-modules.

2.4 Tensor Algebras, Symmetric Algebras and Exterior Algebras

In this section, we recall the concepts of tensor algebras, symmetric algebras and exterior algebras of a finite dimensional vector space V over a field $\Bbbk$. For further details, we refer the reader to Section 11.5 of [6].

Definition 2.4.1 (Tensor Algebras)**.** For each $i \in \mathbb{N}_+$, denote

$$T_i(V) = V^{\otimes i}, \quad T_0(V) = \Bbbk.$$

The elements of $T_i(V)$ are called *i-tensors*. Denote

$$T(V) = \bigoplus_{i=0}^{\infty} T_i(V).$$

Then, $T(V)$ is an $\Bbbk$-algebra containing V, with product defined by

$$(w_1 \otimes \cdots \otimes w_r)(w_1' \otimes \cdots \otimes w_t') = w_1 \otimes \cdots \otimes w_r \otimes w_1' \otimes \cdots \otimes w_t',$$

and extended linearly over $\Bbbk$. With the above product, the algebra $T(V)$ is called the *tensor algebra* of the vector space V.

Remark 2.4.2. In Definition 2.4.1, we have that

1. with respect to the product defined in Definition 2.4.1, $T_i(V)T_j(V) \subseteq T_{i+j}(V)$ so that the tensor algebra $T(V)$ has a grading structure with the homogeneous component $T_i(V)$ of degree i;

2. (*Universal Property*): if A is an $\Bbbk$-algebra and $\varphi\colon V \to A$ is a $\Bbbk$-linear map, then there is a unique $\Bbbk$-algebra homomorphism $\Phi\colon T(V) \to A$ such that $\Phi|_V = \varphi$ (see [6, p. 442–443]).

3. (*Induced Map*): if $\varphi\colon V \to U$ is any $\Bbbk$-linear map for two vector spaces V and U, then there is an induced map on the i^{th} tensor powers:

$$T_i(\varphi)\colon w_1 \otimes \cdots \otimes w_i \mapsto \varphi(w_1) \otimes \cdots \otimes \varphi(w_i)$$

(see [6, p. 450]).

Definition 2.4.3 (Symmetric Algebras)**.** Denote by $S(V)$ the quotient $\Bbbk$-algebra $T(V)/C(V)$, where $C(V)$ is the ideal generated by all elements of the form $w_1 \otimes w_2 - w_2 \otimes w_1$ for all $w_1, w_2 \in V$. The algebra $S(V)$ is called the *symmetric algebra* of the vector space V.

Remark 2.4.4. The quotient ring $S(V) = T(V)/C(V)$ defined in Definition 2.4.3 has following properties.

1. By the definition of $C(V)$, it is clear that $S(V)$ is a commutative ring.

2. Note that the ideal $C(V)$ in Definition 2.4.3 is generated by homogeneous tensors of degree 2. Hence, $C(V)$ is a graded ideal. By Proposition 2.3.4, the quotient ring $S(V)$ is graded with the i^{th} homogeneous component $S_i(V)$, for all $i \in \mathbb{N}$, defined by $S_i(V) = T_i(V)/C_i(V)$, where $C_i(V) = C(V) \cap T_i(V)$.

Theorem 2.4.5 (cf. [6, p. 445])**.** *Let $S(V)$ be the symmetric algebra of V.*

1. *The i^{th} homogeneous component $S_i(V)$ of V is equal to $V^{\otimes i}$ modulo the submodule generated by all elements of the form*

$$(w_1 \otimes \cdots \otimes w_i) - (w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(i)})$$

for all $w_j \in V$ and all permutations σ in the symmetric group S_i.

2. *(*Universal Property for Symmetric Multilinear Maps*): If $\varphi\colon V^{\times i} = V \times \cdots \times V \to U$ is a symmetric i-multilinear map to a $\Bbbk$-vector space U, then there is a unique $\Bbbk$-linear map $\Phi\colon S_i(V) \to U$ such that $\varphi = \Phi \circ \iota$, where $\iota\colon V^{\times i} \to S_i(V)$ is the canonical map defined by $\iota\big((w_1, \cdots, w_k)\big) = w_1 \otimes \cdots \otimes w_k \mod C(V)$.*

3. *(*Universal Property for Maps to Commutative $\Bbbk$-algebras*): If A is a commutative $\Bbbk$-algebra and $\varphi\colon V \to A$ is a $\Bbbk$-linear map, then there is a unique $\Bbbk$-algebra homomorphism $\Phi\colon S(V) \to A$ such that $\Phi|_V = \varphi$.*

Definition 2.4.6 (Exterior Algebras)**.** Denote by $\bigwedge(V)$ the quotient $\Bbbk$-algebra $T(V)/A(V)$, where $A(V)$ is the ideal of $T(V)$ generated by all elements of the form $w \otimes w$ for all $w \in V$. The image of $w_1 \otimes \cdots \otimes w_i$ in $\bigwedge(V)$ is denoted by $w_1 \wedge \cdots \wedge w_i$. With the *exterior product*:

$$(w_1 \wedge \cdots \wedge w_r) \wedge (w'_1 \wedge \cdots \wedge w'_t) = w_1 \wedge \cdots \wedge w_r \wedge w'_1 \wedge \cdots \wedge w'_t,$$

the algebra $\bigwedge(V)$ is called the *exterior algebra* of V.

Remark 2.4.7. The exterior algebra has following properties.

1. Note that the ideal $A(V)$ in Definition 2.4.6 is generated by homogeneous elements, hence, $A(V)$ is a graded ideal. By Proposition 2.3.4, the exterior algebra $\bigwedge(V) = T(V)/A(V)$ is graded with the i^{th} homogeneous component $\bigwedge^i(V)$ for all $i \in \mathbb{N}$, defined by $\bigwedge^i(V) = T_i(V)/A_i(V)$, where $A_i(V) = A(V) \cap T_i(V)$.

2. The exterior product is anticommutative on simple tensors:

$$w \wedge w' = -w' \wedge w \quad \text{for all } w, w' \in V,$$

   since in $\bigwedge(V)$,

$$\begin{aligned} 0 &= (w + w') \otimes (w + w') = w \otimes w + w \otimes w' + w' \otimes w + w' \otimes w' \\ &= w \otimes w' + w' \otimes w. \end{aligned}$$

3. Suppose that the characteristic of the ground field $\Bbbk$ is not 2. If $w \otimes w' = -w' \otimes w$ for all $w, w' \in V$, then $w \otimes w = 0$.

4. If $\varphi\colon V \to U$ is any $\Bbbk$-linear map of vector spaces, then there is an induced map on i^{th} exterior powers:

$$\bigwedge^i(\varphi)\colon w_1 \wedge \cdots \wedge w_i \mapsto \varphi(w_1) \wedge \cdots \wedge \varphi(w_i).$$

   (see [6, p. 450]).

Theorem 2.4.8 (cf. [6, p. 447–448]). *Let $\bigwedge(V)$ be the exterior algebra of V.*

1. *The i^{th} homogeneous component $\bigwedge^i(V)$ of V is equal to $V^{\otimes i}$ modulo the submodule generated by all elements of the form*

$$w_1 \otimes \cdots \otimes w_i \quad \textit{where } w_r = w_t \textit{ for some } r \neq t.$$

In particular,

$$w_1 \wedge \cdots \wedge w_i = 0 \quad \textit{if } w_r = w_t \textit{ for some } r \neq t.$$

2. *(*Universal Property for Alternating Multilinear Maps*): A multilinear map $f\colon V^{\times i} = V \times \cdots \times V \to U$ for some $\Bbbk$-spaces V and U is said to be* alternating *if f vanishes on $(v_1, \cdots, v_i)$ in which there are two components $v_m = v_k$ for some $m \neq k$. If $\varphi\colon V^{\times i} \to U$ is an alternating i-multilinear map to an $\Bbbk$-vector space U, then there is a unique $\Bbbk$-linear map $\Phi\colon \bigwedge^i(V) \to U$ such that $\varphi = \Phi \circ \iota$, where $\iota\colon V^{\times i} \to \bigwedge^i(V)$ is the canonical map defined by $\iota\big((w_1, \cdots, w_i)\big) = w_1 \wedge \cdots \wedge w_i$.*

Theorem 2.4.9 (cf. [7, Corollary A2.3, p. 578–579]). *Let V be a $\Bbbk$-vector space with a basis $\{x_1, \cdots, x_r\}$.*

1. *The homogeneous component $T_d(V)$ is the $\Bbbk$-vector space of dimension r^d with basis the set of all words of length d in $\{x_1, \cdots, x_r\}$.*

2. *The algebra $S(V)$ is isomorphic to the polynomial ring on "variables" x_i. The homogeneous component $S_d(V)$ is the $\Bbbk$-vector space of dimension $\binom{d+r-1}{r-1}$ with basis the set of monomials of degree d in the x_i.*

3. *The homogeneous component $\bigwedge^d(V)$ is the $\Bbbk$-vector space of dimension $\binom{r}{d}$ with basis*

$$\{x_{i_1} \wedge \cdots \wedge x_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq r \textit{ for all } i_j \in \{1, \cdots, r\}\}.$$

Chapter 3

A Review of Homological Algebra

Throughout this chapter, R denotes a ring. In order to define Koszul algebras, we need some concepts such as projective resolution, the functor Ext etc. In this chapter, we review some definitions and concepts in homological algebra that we will use to introduce the notions of Koszul algebras and Koszul duality.

3.1 Projective and Injective Resolutions

First of all, we recall the concepts of projective and injective resolutions. For further details, we refer the reader to standard algebra textbooks (e.g. [6]).

Definition 3.1.1 (Free module)**.** An R-module M is said to be *free* on the subset S of M if, for every nonzero $x \in M$, there are unique nonzero elements $r_1, \cdots, r_n$ in R and unique pairwise distinct elements $a_1, \cdots, a_n$ in S such that $x = r_1a_1 + \cdots + r_na_n$ for some $n \in \mathbb{N}_+$. Here, S is called a *basis* or *set of free generators* for M.

Definition 3.1.2 (Direct Summand)**.** Let M be an R-module. An R-submodule L of M is called a *direct summand* of M if there is an R-submodule N of M such that $M = L \oplus N$.

Definition 3.1.3 (Projective module)**.** An R-module P is called *projective* if P is a direct summand of a free R-module.

Example 3.1.4. Let R be the ring of $n \times n$-matrices with entries in a field $\Bbbk$, where n is an integer ≥ 2. Then, R acts naturally on $\Bbbk^n$ by matrix multiplication, so that $\Bbbk^n$ become an R-module. Consider the ring R as an R-module. We have $R \cong \Bbbk^n \oplus \cdots \oplus \Bbbk^n$ (n summands). This implies that the R-module $\Bbbk^n$ is projective. But, as $\Bbbk$-spaces, $\dim(R) = n^2 \neq n = \dim(\Bbbk^n)$ since $n > 1$. This implies that the R-module $\Bbbk^n$ is not free.

Notation 3.1.5 (Induced Homomorphisms)**.** Let D, L, and M be R-modules. Given an R-module homomorphism $\psi\colon L \to M$, denote by ψ' the induced homomorphism of abelian groups defined by

$$\begin{aligned} \psi'\colon \operatorname{Hom}_R(D, L) &\to \operatorname{Hom}_R(D, M) \\ f &\mapsto f' = \psi \circ f \end{aligned}$$

where $f' = \psi \circ f$ can be illustrated by the following commutative diagram.

$$\begin{tikzcd} D \arrow[d, "f"'] \arrow[dr, dashed, "f'"] & \\ L \arrow[r, "\psi"'] & M \end{tikzcd}$$

Definition 3.1.6 (Exact Sequence)**.**

1. Let X, Y and Z be R-modules. Let $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ be R-module homomorphisms. We call

   $$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

   a *pair of homomorphisms.* This pair is said to be *exact at* Y if $\operatorname{im}\alpha = \ker\beta$.

2. A sequence

   $$\cdots \to X_{j-1} \to X_j \to X_{j+1} \to \cdots$$

   with R-modules X_i for $i \in \mathbb{Z}$ is said to be an *exact sequence* if every pair of homomorphisms in this sequence is exact.

The following theorem shows that there are several equivalent definitions of projective module.

Theorem 3.1.7 (cf. [6, Proposition 10.30, p. 389–390])**.** *Let P be an R-module. Then the following are equivalent:*

1. *For any R-modules L, M, N, if*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(P, L) \xrightarrow{\psi'} \mathrm{Hom}_R(P, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(P, N) \to 0$$

is also a short exact sequence, in other words, the functor $\mathrm{Hom}_R(P, -)$ *is exact.*

2. *For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then for every R-module homomorphism $f \in \mathrm{Hom}_R(P, N)$, there is a* lift *$F \in \mathrm{Hom}_R(P, M)$ making the following diagram*

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

commute.

3. *If P is a quotient of the R-module M (i.e., $P = M/\ker\varphi$ for some $\varphi \in \mathrm{Hom}_R(M, P)$), then P is isomorphic to a direct summand of M.*

4. *For every short exact sequence $0 \to L \to M \to P \to 0$, $M \cong L \oplus P$.*

5. *The module P is a direct summand of a free R-module (i.e., P is a projective module by Definition 3.1.3).*

Definition 3.1.8 (Projective Resolution)**.** Let A be an R-module. A *projective resolution* of A is an exact sequence

$$\cdots \to P^{(n)} \xrightarrow{d_n} P^{(n-1)} \to \cdots \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} A \to 0 \tag{3.1}$$

such that each $P^{(i)}$ is a projective R-module.

Remark 3.1.9. Every R-module A has a free, hence projective resolution $P^\bullet$. Furthermore, if A is finitely generated, then A admits a finitely generated resolution $P^\bullet$, i.e., every $P^{(i)}$ in (3.1) is a finitely generated R-module. For a proof of these facts, see [6, p. 779].

Definition 3.1.10 (Injective module)**.** An R-module Q is called *injective* if for any R-module M such that $Q \subseteq M$, Q is a direct summand of M.

Notation 3.1.11 (Induced Homomorphisms)**.** Let D, N, and M be R-modules. Given an R-module homomorphism $\varphi\colon M \to N$, denote by φ'' the induced homomorphism of abelian groups defined by

$$\begin{aligned} \varphi''\colon \operatorname{Hom}_R(N, D) &\to \operatorname{Hom}_R(M, D) \\ f &\mapsto f'' = f \circ \varphi \end{aligned}$$

where $f'' = f \circ \varphi$ can be illustrated by the following commutative diagram.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, dashed, "f''"'] & N \arrow[d, "f"] \\ & D \end{tikzcd}$$

The following theorem shows that there are several equivalent definitions of injective module.

Theorem 3.1.12 (cf. [6, Proposition 10.34, p. 394–395])**.** *Let Q be an R-module. Then the following are equivalent:*

1. *For any R-modules L, M, N, if*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is a short exact sequence, then

$$0 \to \operatorname{Hom}_R(N, Q) \xrightarrow{\varphi''} \operatorname{Hom}_R(M, Q) \xrightarrow{\psi''} \operatorname{Hom}_R(L, Q) \to 0$$

is also a short exact sequence, in other words, the functor $\operatorname{Hom}_R(-, Q)$ *is exact.*

2. *For any R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is exact, then given $f \in \mathrm{Hom}_R(L, Q)$, there is a* lift *$F \in \mathrm{Hom}_R(M, Q)$ making the diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\psi"] \arrow[d, "f"] & M \arrow[dl, dashed, "F"] \\ & Q & \end{tikzcd}$$

commute.

3. *For every short exact sequence $0 \to Q \to M \to N \to 0$, $M \cong Q \oplus N$.*

4. *The R-module Q is injective.*

Definition 3.1.13 (Injective Resolution)**.** Let D be an R-module. An *injective resolution* of D is an exact sequence

$$0 \to D \xrightarrow{\varepsilon} Q^{(0)} \xrightarrow{d_0} Q^{(-1)} \xrightarrow{d_{-1}} \cdots \to Q^{(-n)} \xrightarrow{d_{-n}} \cdots \tag{3.2}$$

such that each $Q^{(i)}$ is an injective R-module.

3.2 Complexes, Cohomology Groups, and the Functor Ext

Definition 3.2.1 (Complex and Cohomology Group)**.** Let $C^\bullet$ be a sequence of abelian group homomorphisms:

$$0 \to C^{(0)} \xrightarrow{d_1} C^{(1)} \to \cdots \to C^{(n-1)} \xrightarrow{d_n} C^{(n)} \xrightarrow{d_{n+1}} \cdots .$$

- The sequence $C^\bullet$ is called a *cochain complex* if $d_{n+1} \circ d_n = 0$ for all n.
- If $C^\bullet$ is a cochain complex, the n^{th} *cohomology group $H^n(C^\bullet)$ of $C^\bullet$* is the quotient group $\ker d_{n+1} / \operatorname{im} d_n$:

$$H^n(C^\bullet) = \ker d_{n+1} / \operatorname{im} d_n.$$

Let $P^\bullet$ be the projective resolution (3.1). Then, for any R-module D, $P^\bullet$ yields a sequence $\mathrm{Hom}_R(P^\bullet, D)$:

$$\begin{aligned} 0 \to \mathrm{Hom}_R(A, D) \xrightarrow{\epsilon''} \mathrm{Hom}_R(P^{(0)}, D) \xrightarrow{d_1''} \mathrm{Hom}_R(P^{(1)}, D) \xrightarrow{d_2''} \cdots \\ \cdots \to \mathrm{Hom}_R(P^{(n-1)}, D) \xrightarrow{d_n''} \mathrm{Hom}_R(P^{(n)}, D) \xrightarrow{d_{n+1}''} \cdots \end{aligned} \tag{3.3}$$

where the maps d_i'' and ϵ'' in (3.3) are induced by the corresponding ones d_i and ϵ in (3.1) as in Notation 3.1.11. This sequence is a cochain complex although it is not necessarily exact (see [6, p. 779]).

Definition 3.2.2 (Ext)**.** Let A and D be R-modules. For any projective resolution of A as in (3.1), let $d_n''\colon \mathrm{Hom}_R(P^{(n-1)}, D) \to \mathrm{Hom}_R(P^{(n)}, D)$ for all $n \in \mathbb{N}$ as in (3.3). Define

$$\mathrm{Ext}_R^n(A, D) = \ker d_{n+1}'' / \operatorname{im} d_n''$$

for $n \in \mathbb{N}_+$ and

$$\mathrm{Ext}_R^0(A, D) = \ker d_1''.$$

The group $\mathrm{Ext}_R^n(A, D)$ is called the n^{th} *cohomology group derived from the functor* $\mathrm{Hom}_R(-, D)$. When $R = \mathbb{Z}$, denote $\mathrm{Ext}^n(A, D) = \mathrm{Ext}_{\mathbb{Z}}^n(A, D)$.

Remark 3.2.3.

1. Up to isomorphism, the cohomology groups $\mathrm{Ext}_R^n(A, D)$ defined in Definition 3.2.2 do not depend on the choice of projective resolution of A (see [6, p. 780]).

2. If all the terms of the resolution (3.1) are in the categories R-Mod, R-Modf, R-grMod, R-grModf, we use the notations $\mathrm{Ext}^n_{R\text{-Mod}}$, $\mathrm{Ext}^n_{R\text{-Modf}}$, $\mathrm{Ext}^n_{R\text{-grMod}}$, $\mathrm{Ext}^n_{R\text{-grModf}}$ respectively to denote Ext_R^n.

Let $Q^\bullet$ be the injective resolution (3.2). Then, for any R-module A, $Q^\bullet$ yields a sequence $\mathrm{Hom}_R(A, Q^\bullet)$:

$$\begin{aligned} 0 \to \mathrm{Hom}_R(A, D) \xrightarrow{\varepsilon'} \mathrm{Hom}_R(A, Q^{(0)}) \xrightarrow{d_0'} \mathrm{Hom}_R(A, Q^{(-1)}) \xrightarrow{d_{-1}'} \cdots \\ \cdots \mathrm{Hom}_R(A, Q^{(-n)}) \xrightarrow{d_{-n}'} \cdots \end{aligned} \tag{3.4}$$

where the d_i' and ε' in (3.4) are induced by the corresponding ones d_i and ε in (3.2) as in Notation 3.1.5. This sequence is a cochain complex although it is not necessarily exact.

Definition 3.2.4 (EXT)**.** Let A and D be R-modules. For any injective resolution of D as in (3.2), let $d'_{-n}\colon \operatorname{Hom}_R(A, Q^{(-n)}) \to \operatorname{Hom}_R(A, Q^{(-(n+1))})$ for all $n \in \mathbb{N}$ as in (3.4). Define

$$\mathrm{EXT}^n_R(A, D) = \ker d'_{-n} / \operatorname{im} d'_{-(n-1)}$$

for $n \in \mathbb{N}_+$ and

$$\mathrm{EXT}^0_R(A, D) = \ker d'_0.$$

The group $\mathrm{EXT}^n_R(A, D)$ is called the n^{th} *cohomology group derived from the functor* $\operatorname{Hom}_R(A, -)$. When $R = \mathbb{Z}$, denote $\mathrm{EXT}^n(A, D) = \mathrm{EXT}^n_{\mathbb{Z}}(A, D)$

Remark 3.2.5.

1. Up to isomorphism, the cohomology groups $\mathrm{EXT}^n_R(A, D)$ defined in Definition 3.2.4 do not depend on the choice of injective resolution of D (see [6, p. 786, p. 780]).

2. There is a natural isomorphism between the two cohomology groups $\operatorname{Ext}^n_R(A, D)$ and $\mathrm{EXT}^n_R(A, D)$ (see [6, p. 786]). From now on, we will identify $\operatorname{Ext}^n_R(A, D)$ and $\mathrm{EXT}^n_R(A, D)$.

3. For a fixed R-module D and fixed integer $n \geq 0$, $\operatorname{Ext}^n_R(-, D)$ defines a contravariant functor from the category of R-modules to the category of abelian groups (see [9, p. 91, p. 140]).

4. For a fixed R-module A and fixed integer $n \geq 0$, $\operatorname{Ext}^n_R(A, -)$ defines a covariant functor from the category of R-modules to the category of abelian groups (see [9, p. 91, p. 140]).

Definition 3.2.6 (Natural Transformation)**.** Let F, G be two functors from the category $\mathfrak{C}$ to the category $\mathfrak{D}$. A *natural transformation* t from F to G is a rule assigning to each $X \in \operatorname{obj}(\mathfrak{C})$ a morphism $t_X\colon F(X) \to G(X)$ such that for any morphism $f\colon X \to Y$ in $\mathfrak{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{t_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{t_Y} & G(Y) \end{array}$$

commutes. If t_X is an isomorphism for each $X \in \operatorname{obj}(\mathfrak{C})$, then t is called a *natural isomorphism* and then we write $F \cong G$.

Definition 3.2.7 (Adjoint Functor)**.** Let $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{C}$ be functors such that there is a natural isomorphism

$$\eta\colon \operatorname{Mor}_{\mathfrak{D}}(F-,-) \to \operatorname{Mor}_{\mathfrak{C}}(-,G-)$$

of functors $\mathfrak{C}^{\mathrm{op}} \times \mathfrak{D} \to \mathfrak{Set}$, where $\mathfrak{Set}$ denotes the category of sets. Then, we say that F is *left adjoint* to G and G is *right adjoint* to F.

Theorem 3.2.8 (cf. [9, Theorem 4.12.1 in p. 162 and the application in p. 164])**.** *Let R and S be two rings. Let $U\colon S$-*Mod $\to R$-Mod *be a functor.*

1. *If U has a left adjoint $F\colon R$-*Mod $\to S$-Mod *and if U preserves surjections, then F sends projectives to projectives.*
2. *If U has a right adjoint $\bar{F}\colon R$-*Mod $\to S$-Mod *and if U preserves injections, then $\bar{F}$ sends injectives to injectives.*
3. *Let $U^\gamma\colon S$-*Mod $\to R$-Mod *be a functor induced by a ring isomorphism $\gamma\colon R \to S$. Then, we have a natural isomorphism*

$$\theta_\gamma\colon \operatorname{Ext}^n_S(-,-) \to \operatorname{Ext}^n_R(U^\gamma(-), U^\gamma(-)).$$

Remark 3.2.9. From the proof of Theorem 3.2.8 described in [9, p. 162–164], we see that Theorem 3.2.8 holds also with categories of modules replaced everywhere by categories of graded modules.

Now, we recall some properties for the functor Ext that can be found in some standard algebra textbooks (e.g., Chapter 17 of [6], Chapter IV of [9]).

Proposition 3.2.10 (Properties of Ext)**.** *Let A, A_i and D, D_i, $i \in \mathcal{I}$, be R-modules, where $\mathcal{I}$ is an index set.*

1. $\operatorname{Ext}^0_R(A,D) \cong \operatorname{Hom}_R(A,D)$.
2. $\operatorname{Ext}^1_R(M,Q) = 0$ *for all R-modules M* $\iff$ $\operatorname{Ext}^n_R(M,Q) = 0$ *for all R-modules M and for all $n \in \mathbb{N}$* $\iff$ *Q is an injective R-module.*
3. $\operatorname{Ext}^1_R(P,N) = 0$ *for all R-modules N* $\iff$ $\operatorname{Ext}^n_R(P,N) = 0$ *for all R-modules N and for all $n \in \mathbb{N}$* $\iff$ *P is a projective R-module.*
4. $\operatorname{Ext}^n_R(\bigoplus_{i\in\mathcal{I}} A_i, D) \cong \prod_{i\in\mathcal{I}} \operatorname{Ext}^n_R(A_i, D)$ *for every $n \in \mathbb{N}$.*
5. $\operatorname{Ext}^n_R(A, \prod_{i\in\mathcal{I}} D_i) \cong \prod_{i\in\mathcal{I}} \operatorname{Ext}^n_R(A, D_i)$ *for every $n \in \mathbb{N}$.*

3.3 Yoneda Product

The modules in this section are all R-modules.

Definition 3.3.1 (Yoneda Primitive Equivalence)**.** Two exact sequences from an R-module B to an R-module A of length ℓ

$$\alpha\colon 0 \to B \to X_1 \to \cdots \to X_\ell \to A \to 0$$
$$\alpha'\colon 0 \to B \to X'_1 \to \cdots \to X'_\ell \to A \to 0$$

are *primitively equivalent* if there is a commutative diagram

$$\begin{array}{ccccccc} \alpha\colon 0 \to & B \to & X_1 \to \cdots \to & X_\ell \to & A \to 0 \\ & \| & \downarrow & \downarrow & \| \\ \alpha'\colon 0 \to & B \to & X'_1 \to \cdots \to & X'_\ell \to & A \to 0. \end{array}$$

Remark 3.3.2. If $\ell = 1$, by the Short Five Lemma (see [6, Proposition 10.24, p. 383]), the primitive equivalence in Definition 3.3.1 gives an isomorphism between the two exact sequences α and α'. This implies that for $\ell = 1$, the primitive equivalence is an equivalence relation. But, if $\ell > 1$, in general, the primitive equivalence is not symmetric, so it is not an equivalence relation.

Definition 3.3.3 (Yoneda Equivalence)**.** The *Yoneda equivalence* is the equivalence relation that is generated by the primitive equivalence in Definition 3.3.1. The set of Yoneda equivalence classes of exact sequences of length ℓ from B to A is denoted by $E^\ell_R(A, B)$.

3.3.1 Natural isomorphism from Ext^1_R to E^1_R

Now, we give the description and interpretation for the elements of Ext by $E^\ell_R(A, B)$. Consider the case $\ell = 1$ first.

Functoriality of $E^1_R(-, B)$

Let

$$\alpha\colon 0 \to B \xrightarrow{q} X \xrightarrow{p} A \to 0$$

be a short exact sequence and let $v\colon A' \to A$ be a R-module homomorphism. Define the *pull-back* (or *fibered product*) of X and A' over A by

$$X' = \ker(-p, v) = \{x \oplus a' \in X \oplus A' \mid -p(x) + v(a') = 0\},$$

where $(-p, v)\colon X \oplus A' \to A$ is defined by $(-p, v)(x \oplus a') = -p(x) + v(a')$ for $x \in X$ and $a' \in A'$. We will construct a short exact sequence

$$\alpha'\colon 0 \to B \xrightarrow{\hat{q}} X' \xrightarrow{\hat{p}} A' \to 0$$

and $f\colon X' \to X$ such that the diagram

$$\begin{array}{lccccccccc} \alpha'\colon & 0 & \longrightarrow & B & \xrightarrow{\hat{q}} & X' & \xrightarrow{\hat{p}} & A' & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle v} & & \\ \alpha\colon & 0 & \longrightarrow & B & \xrightarrow{q} & X & \xrightarrow{p} & A & \longrightarrow & 0 \end{array} \tag{3.5}$$

commutes.

Let $(x, a') \in X'$. Then $p(x) = v(a') \in A$. Define $\hat{p}\colon X' \to A'$ by $\hat{p}(x, a') = a'$ and define $f\colon X' \to X$ by $f(x, a') = x$. Then $p \circ f(x, a') = p(x) = v(a') = v \circ \hat{p}(x, a')$.

Let $b \in B$. Then $q(b) \in X$. Since α is exact, $p \circ q(b) = 0$. This implies that $(q(b), 0) \in \ker(-p, v) = X'$. Define $\hat{q}\colon B \to X'$ by $\hat{q}(b) = (q(b), 0)$. Then, $f \circ \hat{q}(b) = f(q(b), 0) = q(b)$. We conclude that the diagram (3.5) commutes.

Since q is injective, $\hat{q}$ is an injection. For all $a' \in A'$, $v(a') \in A$. Since p is surjective, there is $x' \in X$ such that $p(x') = v(a')$. Thus, $(x, a') \in \ker(-p, v) = X'$. This implies that $\hat{p}$ is a surjection.

For $b \in B$, $\hat{q}(b) = (q(b), 0) \in \ker \hat{p}$. Hence, $\operatorname{im} \hat{q} \subseteq \ker \hat{p}$. Conversely, for $(x, a') \in \ker \hat{p}$, we have $0 = \hat{p}(x, a') = a'$ and $x \in X$. So, $p(x) = p \circ f(x, 0) = v \circ \hat{p}(x, 0) = v(0) = 0$, i.e., $x \in \ker p = \operatorname{im} q$. Hence, there is $b \in B$ such that $q(b) = x$. Thus, $(x, a') = (q(b), 0) = \hat{q}(b) \in \operatorname{im} \hat{q}$. It follows that $\ker \hat{p} \subseteq \operatorname{im} \hat{q}$. We conclude that α' is exact.

Let B be fixed. For each $\alpha \in E^1_R(A, B)$ and each $v\colon A' \to A$, define $v_*([\alpha]) = [\alpha'] \in E^1_R(A', B)$. This gives the contravariant functor E_B defined by $E_B(v) = v_*$. In this sense, with B fixed, $E^1_R(-, B)$ can be viewed as a contravariant functor.

Functoriality of $E_R^1(A,-)$

Let
$$\alpha\colon 0 \to B \xrightarrow{q} X \xrightarrow{p} A \to 0$$
be a short exact sequence and let $u\colon B \to B'$ be an R-module homomorphism. Define the *push-out* (or *fibered coproduct*) of X and B' under B to be

$$X'' = \operatorname{coker}(-q,u) = (X \oplus B')/N,$$

where $(-q,u)\colon B \to X \oplus B'$ is defined by $(-q,u)(b) = (-q(b),u(b))$ for all $b \in B$, and N is the subgroup of $X \oplus B'$ formed by all elements $(-q(b),u(b))$ for $b \in B$. We will construct a short exact sequence

$$\alpha''\colon 0 \to B' \xrightarrow{\tilde{q}} X'' \xrightarrow{\tilde{p}} A \to 0$$

and $g\colon X \to X''$ such that the diagram

$$\begin{tikzcd}
\alpha\colon & 0 \arrow[r] & B \arrow[r, "q"] \arrow[d, "u"] & X \arrow[r, "p"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\alpha''\colon & 0 \arrow[r] & B' \arrow[r, "\tilde{q}"] & X'' \arrow[r, "\tilde{p}"] & A \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

commutes. Define

$$\begin{aligned}
&\tilde{q}\colon B' \to X'' && \tilde{q}(b') = (0,b') + N,\\
&g\colon X \to X'' && g(x) = (x,0) + N,\\
&\tilde{p}\colon X'' \to A && \tilde{p}\big((x,b') + N\big) = p(x)
\end{aligned}$$

for any $b' \in B'$. We verify that $\tilde{p}$ is well-defined as follows. If $(x_1,b_1') + N = (x_2,b_2') + N$ for $x_1,x_2 \in X$ and $b_1',b_2' \in B'$, then $(x_1 - x_2, b_1' - b_2') \in N$, so that $x_1 - x_2 = -q(b)$, $b_1' - b_2' = u(b)$ for some $b \in B$. This implies that $x_2 - x_1 \in \operatorname{im} q = \ker p$ since α is exact. Hence, $p(x_2 - x_1) = 0$, i.e., $p(x_2) = p(x_1)$.

Then, it is straightforward to verify that α'' is exact and the diagram (3.6) commutes.

Let A be fixed. For each $\alpha \in E_R^1(A,B)$ and each $u\colon B \to B'$, define $u^*([\alpha]) = [\alpha''] \in E_R^1(A,B')$. This gives the covariant functor E_A defined by $E_A(u) = u^*$. In this sense, with A fixed, $E_R^1(A,-)$ can be viewed as a covariant functor.

Proposition 3.3.4. *For any $\alpha \in E^1_R(A,B)$ and any R-module homomorphisms $v\colon A' \to A$, $u\colon B \to B'$, we have that $u^* \circ v_*([\alpha]) = v_* \circ u^*([\alpha])$.*

Proof. In fact, $v_* \circ u^*([\alpha])$ is illustrated by the following commutative diagram.

$$\begin{tikzcd}
\alpha\colon 0 \arrow[r] & B \arrow[r, "q"] \arrow[d, "u"] & X \arrow[r, "p"] \arrow[d, "g'"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
u^*([\alpha])\colon 0 \arrow[r] & B' \arrow[r, "\tilde{q}"] & Y' \arrow[r, "\tilde{p}"] & A \arrow[r] & 0 \\
v_*[u^*[\alpha]]\colon 0 \arrow[r] & B' \arrow[r] \arrow[u, "\mathrm{id}"] & Z' \arrow[r] \arrow[u, "f'"] & A' \arrow[r] \arrow[u, "v"] & 0
\end{tikzcd}$$

where

$$\begin{aligned} Y' &= \operatorname{coker}(-q,u) = (X \oplus B')/N_2 \text{ with} \\ N_2 &= \{(-q(b), u(b)) \mid b \in B\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{p}\colon Y' &\to A \\ (x,b') + N_2 &\mapsto p(x) \text{ for } x \in X, b' \in B, \end{aligned}$$

and finally,

$$\begin{aligned} Z' &= \ker(-\tilde{p}, v) \\ &= \{(y,a) \in Y' \oplus A' \mid -\tilde{p}(y) + v(a) = 0\} \\ &= \{\big((x,b) + N_2, a\big) \in \big((X \oplus B')/N_2\big) \oplus A' \mid -p(x) + v(a) = 0\}. \end{aligned}$$

On the other hand, $u^* \circ v_*([\alpha])$ is illustrated by the following commutative diagram.

$$\begin{tikzcd}
\alpha\colon 0 \arrow[r] & B \arrow[r, "q"] & X \arrow[r, "p"] & A \arrow[r] & 0 \\
v_*([\alpha])\colon 0 \arrow[r] & B \arrow[r, "\hat{q}"] \arrow[u, "\mathrm{id}"] \arrow[d, "u"] & Y \arrow[r, "\hat{p}"] \arrow[u, "g"] \arrow[d, "f"] & A' \arrow[r] \arrow[u, "v"] \arrow[d, "\mathrm{id}"] & 0 \\
u^*[v_*[\alpha]]\colon 0 \arrow[r] & B' \arrow[r] & Z \arrow[r] & A' \arrow[r] & 0,
\end{tikzcd}$$

where

$$Y = \ker(-p, v) = \{(x,a) \in X \oplus A' \mid -p(x) + v(a) = 0\},$$

and

$$\begin{aligned} \hat{q}\colon B &\to Y \\ b &\mapsto (q(b), 0), \end{aligned}$$

and finally,

$$\begin{aligned} Z &= \operatorname{coker}(-\hat{q}, u) = (Y \oplus B')/N_1 \\ &= \{(x, a, b) + N_1 \in (X \oplus A' \oplus B')/N_1 \mid -p(x) + v(a) = 0\}, \end{aligned}$$

with

$$N_1 = \{(-\hat{q}(b), u(b)) \mid b \in B\} = \{(-q(b), 0, u(b)) \mid b \in B\}.$$

Define $\tau\colon Z' \to Z$ by

$$\tau\big((x, b) + N_2, a\big) = (x, a, b) + N_1,$$

for all $x \in X$, $b \in B$, $a \in A'$ such that $-p(x) + v(a) = 0$. Then, we see that τ is an R-module isomorphism. It follows that $u^* \circ v_*([\alpha]) = v_* \circ u^*([\alpha])$. $\square$

Recall that $E^1_R(-,-)$ is a contravariant functor in the first argument and a covariant functor in the second. Thus, we conclude that $E^1_R(-,-)$ can be viewed as a bifunctor from the category of R-modules to the category of sets.

Abelian group structure on $E^1_R(A, B)$

Let $\alpha\colon 0 \to B \xrightarrow{q} X \xrightarrow{p} A \to 0$ and $\alpha'\colon 0 \to B \xrightarrow{q'} X' \xrightarrow{p'} A \to 0$ be two short exact sequences. Then,

$$\alpha \oplus \alpha'\colon 0 \to B \oplus B \xrightarrow{q \oplus q'} X \oplus X' \xrightarrow{p \oplus p'} A \oplus A \to 0$$

is a short exact sequence. Let $d\colon A \to A \oplus A$ be the diagonal map $d(a) = (a, a)$. Let $s\colon B \oplus B \to B$ be the sum map $s(b, b') = b + b'$. By the functoriality of $E^1_R(-, B \oplus B)$, we have the following commutative diagram.

$$\begin{array}{ccccccccc} d_*([\alpha \oplus \alpha'])\colon 0 & \longrightarrow & B \oplus B & \xrightarrow{\hat{q}} & Y & \xrightarrow{\hat{p}} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle d} & & \\ \alpha \oplus \alpha'\colon 0 & \longrightarrow & B \oplus B & \xrightarrow{q \oplus q'} & X \oplus X' & \xrightarrow{p \oplus p'} & A \oplus A & \longrightarrow & 0 \end{array} \tag{3.7}$$

where

$$\begin{aligned} Y &= \ker\big(-(p \oplus p'), d\big) \\ &= \{(x, x', a) \in X \oplus X' \oplus A \mid -(p \oplus p')(x, x') + d(a) = 0\}, \end{aligned}$$

and

$$\begin{aligned} \hat{q}\colon B \oplus B &\to Y \\ (b, b') &\mapsto \big((q \oplus q')(b, b'), 0\big). \end{aligned}$$

Then, by the functoriality of $E^1_R(A \oplus A, -)$, we have the following commutative diagram.

$$\begin{tikzcd} d_*([\alpha \oplus \alpha'])\colon 0 \arrow[r] & B \oplus B \arrow[r, "\hat{q}"] \arrow[d, "s"] & Y \arrow[r, "\hat{p}"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ s^* d_*([\alpha \oplus \alpha'])\colon 0 \arrow[r] & B \arrow[r] & Z \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{3.8}$$

where

$$Z = \operatorname{coker}(-\hat{q}, s) = (Y \oplus B)/N_1$$

with

$$N_1 = \{(-\hat{q}(b, b'), s(b, b')) \mid (b, b') \in B \oplus B\}.$$

By Proposition 3.3.4, $s^* d_*([\alpha \oplus \alpha']) = d_* s^*([\alpha \oplus \alpha'])$. We define an operation $+$ on $E^1_R(A, B)$ by setting $[\alpha] + [\alpha'] = s^* d_*([\alpha \oplus \alpha']) = d_* s^*([\alpha \oplus \alpha'])$.

Remark 3.3.5. Consider the case where $r = 0 \in R$. We have the following commutative diagram:

$$\begin{tikzcd} 0_*[\alpha]\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & B \oplus A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "0"] & 0 \\ \alpha\colon & 0 \arrow[r] & B \arrow[r, "q"] & X \arrow[r, "p"] & A \arrow[r] & 0. \end{tikzcd}$$

This gives the zero element $0\colon 0 \to B \to B \oplus A \to A \to 0$ in $E^1_R(A, B)$.

Since $X \oplus X' \cong X' \oplus X$, we see that $[\alpha] + [\alpha'] = [\alpha'] + [\alpha]$. Since $(X \oplus X') \oplus X'' \cong X \oplus (X' \oplus X'')$, we see that $([\alpha] + [\alpha']) + [\alpha''] = [\alpha] + ([\alpha'] + [\alpha''])$. We conclude that $E^1_R(A, B)$ is an abelian additive group.

For any R-module homomorphisms $f, g\colon M \to N$, define the sum $+$ of f and g by $(f + g)(x) = f(x) + g(x)$. Then, the sum can be written as

$$f + g = s \circ (f \oplus g) \circ d.$$

Lemma 3.3.6. *For any R-module homomorphisms $u, u_1, u_2 \colon B \to B'$ and $v, v_1, v_2 \colon A' \to A$, and any $\alpha, \alpha' \in E^1_R(A, B)$, we have that*

$$\begin{aligned}(u_1 \oplus u_2)^* &= u_1^* \oplus u_2^*,\\ (v_1 \oplus v_2)_* &= v_{1*} \oplus v_{2*},\end{aligned}$$

and

$$\begin{aligned}u^* \circ s^* &= s^* \circ (u \oplus u)^*,\\ v_* \circ d_* &= d_* \circ (v \oplus v)_*,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(u_1 + u_2)^*([\alpha]) &= d_* \circ s^* \circ (u_1 \oplus u_2)^*([\alpha] \oplus [\alpha]),\\ (v_1 + v_2)_*([\alpha]) &= d_* \circ s^* \circ (v_1 \oplus v_2)_*([\alpha] \oplus [\alpha]).\end{aligned}$$

Remark 3.3.7. By a slight abuse of notation, we use s above to denote two different, but similar maps (and similarly for d). For example, in (3.9), on the left $s \colon B \oplus B \to B$ and $d \colon A' \to A' \oplus A'$, on the right $s \colon B' \oplus B' \to B'$ and $d \colon A \to A \oplus A$.

Proof. The verifications of these equalities are straightforward computations, involving the diagrams used to prove functoriality of $E^1_R(-, B)$ and $E^1_R(A, -)$. We refer the reader to [13, p. 70] for details. □

Proposition 3.3.8 (cf. [13, Theorem III 2.1, p. 69])**.** *For any R-module homomorphisms $u \colon B \to B'$ and $v \colon A' \to A$, and any $\alpha, \alpha' \in E^1_R(A, B)$,*

$$u^*([\alpha] + [\alpha']) = u^*([\alpha]) + u^*([\alpha']), \quad v_*([\alpha] + [\alpha']) = v_*([\alpha]) + v_*([\alpha']).$$

For any R-module homomorphisms $u_1, u_2 \colon B \to B'$ and $v_1, v_2 \colon A' \to A$, and any $\alpha \in E^1_R(A, B)$,

$$(u_1 + u_2)^*([\alpha]) = u_1^*([\alpha]) + u_2^*([\alpha]), \quad (v_1 + v_2)_*([\alpha]) = v_{1*}([\alpha]) + v_{2*}([\alpha]).$$

Proof. By Lemma 3.3.6 and Proposition 3.3.4, we obtain that

$$\begin{aligned}u^*([\alpha] + [\alpha']) &= u^* \circ d_* \circ s^*([\alpha] \oplus [\alpha']) = d_* \circ u^* \circ s^*([\alpha] \oplus [\alpha'])\\ &= d_* \circ s^* \circ (u \oplus u)^*([\alpha] \oplus [\alpha']) = d_* \circ s^*\big(u^*([\alpha]) \oplus u^*([\alpha'])\big)\\ &= u^*([\alpha]) + u^*([\alpha']),\end{aligned}$$

$$\begin{aligned}v_*([\alpha] + [\alpha']) &= v_* \circ s^* \circ d_*([\alpha] \oplus [\alpha']) = s^* \circ v_* \circ d_*([\alpha] \oplus [\alpha'])\\ &= s^* \circ d_* \circ (v \oplus v)_*([\alpha] \oplus [\alpha']) = s^* \circ d_*\big(v_*([\alpha]) \oplus v_*([\alpha'])\big)\\ &= v_*([\alpha]) + v_*([\alpha']),\end{aligned}$$

and

$$\begin{aligned}(u_1+u_2)^*([\alpha]) &= d_* \circ s^* \circ (u_1^* \oplus u_2^*)([\alpha]\oplus[\alpha])\\ &= d_* \circ s^*\big(u_1^*([\alpha]) \oplus u_2^*([\alpha])\big) = u_1^*([\alpha]) + u_2^*([\alpha]),\\ (v_1+v_2)_*([\alpha]) &= d_* \circ s^* \circ (v_{1*} \oplus v_{2*})([\alpha]\oplus[\alpha])\\ &= d_* \circ s^*\big(v_{1*}([\alpha]) \oplus v_{2*}([\alpha])\big) = v_{1*}([\alpha]) + v_{2*}([\alpha]). \quad \square\end{aligned}$$

Canonical isomorphism from $\mathrm{Ext}^1_R(A,B)$ to $E^1_R(A,B)$ as abelian groups

Let

$$P^\bullet\colon \cdots \xrightarrow{\varphi_1} P^{(1)} \xrightarrow{\varphi_0} P^{(0)} \xrightarrow{h} A \to 0$$

be a projective resolution of A. Then, from Section 3.1, we see that this resolution induces a cochain complex $\mathrm{Hom}_R(P^\bullet, B)$:

$$\mathrm{Hom}_R(P^\bullet,B)\colon 0 \to \mathrm{Hom}_R(A,B) \xrightarrow{\hat{h}} \mathrm{Hom}_R(P^{(0)},B) \xrightarrow{\hat{\varphi}_0} \mathrm{Hom}_R(P^{(1)},B) \xrightarrow{\hat{\varphi}_1} \cdots .$$

Then

$$\mathrm{Ext}^1_R(A,B) = H_1(\mathrm{Hom}_R(P^\bullet,B)) = \ker\hat{\varphi}_1/\operatorname{im}\hat{\varphi}_0.$$

Suppose that $\mu \in \mathrm{Ext}^1_R(A,B)$. Then, $\mu = u + \operatorname{im}\hat{\varphi}_0$ for some $u \in \mathrm{Hom}_R(P^{(1)},B)$ with $\hat{\varphi}_1(u) = 0$ in $\mathrm{Hom}_R(P^{(2)},B)$. Then, $u \circ \varphi_1 = \hat{\varphi}_1(u) = 0$. Hence,

$$0 = u \circ \varphi_1(P^{(2)}) = u(\operatorname{im}\varphi_1) = u(\ker\varphi_0),$$

so that $\ker\varphi_0 \subseteq \ker u$. By the First Isomorphism Theorem for modules, $P_1/\ker\varphi_0 \overset{\tau}{\cong} \operatorname{im}\varphi_0 = \ker h$. Then, we obtain an exact sequence:

$$\beta\colon 0 \to P^{(1)}/\ker\varphi_0 \to P^{(0)} \xrightarrow{h} A \to 0$$

i.e., $\beta \in E^1_R(A, P^{(1)}/\ker\varphi_0)$. For any $p \in P^{(1)}/\ker\varphi_0$, $p = p_1 + \ker\varphi_0$ for some $p_1 \in P^{(1)}$. Since $\ker\varphi_0 \subseteq \ker u$, $u(p) = u(p_1) + u(\ker\varphi_0) = u(p_1) \in B$. Hence, $u \in \mathrm{Hom}_R(P^{(1)}/\ker\varphi_0, B)$. By the functoriality of $E^1_R(A,-)$, we have $\mu' = u^*([\beta]) \in E^1_R(A,B)$, where $u^*([\beta])$ is illustrated by the following commutative diagram:

$$\begin{tikzcd}
\beta\colon 0 \arrow[r] & P^{(1)}/\ker\varphi_0 \arrow[r, "\iota"] \arrow[d, "u"] & P^{(0)} \arrow[r, "h"] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mu'\colon 0 \arrow[r] & B \arrow[r] & P'_0 \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

where $P_0' = \operatorname{coker}(-\iota, u)$ and $(-\iota, u)\colon P^{(1)}/\ker\varphi_0 \to P^{(0)} \oplus B$.

Define $\epsilon\colon \operatorname{Ext}^1_R(A,B) \to E^1_R(A,B)$ by

$$\epsilon(\mu) = \mu'. \tag{3.10}$$

- ϵ is surjective.

Let $\mu'\colon 0 \to B \xrightarrow{\zeta} X \xrightarrow{\xi} A \to 0 \in E^1_R(A,B)$. Note that every R-module has a projective resolution (see [6, p. 779]). So, there is a projective resolution $P^\bullet$ of A:

$$\cdots \xrightarrow{\varphi_1} P^{(1)} \xrightarrow{\varphi_0} P^{(0)} \xrightarrow{h} A \to 0$$

By Theorem 3.1.7, there is $F \in \operatorname{Hom}_R(P^{(0)}, X)$ making the diagram

$$\begin{tikzcd} & P^{(0)} \arrow[dl, dashed, "F"'] \arrow[d, "h"] & \\ X \arrow[r, "\xi"] & A \arrow[r] & 0 \end{tikzcd}$$

commute. Thus, the follow diagram commutes.

$$\begin{tikzcd} \beta\colon 0 \arrow[r] & P^{(1)}/\ker\varphi_0 \arrow[r, "\iota"] \arrow[d, dashed, "u"] & P^{(0)} \arrow[r, "h"] \arrow[d, "F"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \mu'\colon 0 \arrow[r] & B \arrow[r, "\zeta"] & X \arrow[r, "\xi"] & A \arrow[r] & 0, \end{tikzcd}$$

where $\iota = \iota_0 \circ \tau\colon P^{(1)}/\ker\varphi_0 \to P^{(0)}$, $\tau\colon P^{(1)}/\ker\varphi_0 \cong \ker h$ is the isomorphism given by the First Isomorphism Theorem: $P^{(1)}/\ker\varphi_0 \cong \operatorname{im}\varphi_0$ and the exactness of $P^\bullet$: $\operatorname{im}\varphi_0 = \ker h$, $\iota_0\colon \ker h \hookrightarrow P^{(0)}$ is the inclusion and $u \in \operatorname{Hom}_R(P^{(1)}/\ker\varphi_0, B)$ is defined by $\zeta \circ u = F \circ \iota$. Note that $\mathrm{id} \circ h \circ \iota = \xi \circ F \circ \iota$. Since β is exact, $h \circ \iota = 0$. So, $\xi \circ F \circ \iota = \mathrm{id} \circ h \circ \iota = 0$. This implies that $F \circ \iota \in \ker\xi = \operatorname{im}\zeta$ since μ' is exact. Note that ζ is injective. So, u is well-defined by $u = \zeta^{-1} \circ F \circ \iota$. Then, we can verify that $X \cong \operatorname{coker}(-\iota, u)$ for $(-\iota, u)\colon P^{(1)}/\ker\varphi_0 \to P^{(0)} \oplus B$. Hence, $\mu' = u^*([\beta])$. This implies that there is $\mu = u + \operatorname{im}\hat{\varphi}_0 \in \operatorname{Ext}^1_R(A,B)$, such that $\epsilon(\mu) = \mu'$.

- ϵ is injective.

Let $\mu_1'\colon (0 \to B \to X \to A \to 0) \in E^1_R(A,B)$ and let $\mu_2'\colon (0 \to B \to Y \to A \to 0) \in E^1_R(A,B)$. Suppose that $\mu_1' = \mu_2'$. By the Short Five Lemma (see [6, Proposition 10.24, p. 383]), $X \cong Y$. This implies that $u_1 = u_2$, consequently, $\mu_1 = \mu_2$.

- ϵ is a homomorphism of abelian groups.

Given a projective resolution $P^\bullet\colon \cdots \xrightarrow{\varphi_1} P^{(1)} \xrightarrow{\varphi_0} P^{(0)} \xrightarrow{h} A \to 0$, we can compute $\mathrm{Ext}^1_R(A,B)$ from $P^\bullet$ and obtain an exact sequence $\beta\colon 0 \to P^{(1)}/\ker\varphi_0 \to P^{(0)} \xrightarrow{h} A \to 0$. Let $\mu_1 = u_1 + \mathrm{im}\,\hat{\varphi}_0 \in \mathrm{Ext}^1_R(A,B)$ and $\mu_2 = u_2 + \mathrm{im}\,\hat{\varphi}_0 \in \mathrm{Ext}^1_R(A,B)$. Then, $\mu_1+\mu_2 = u_1+u_2+\mathrm{im}\,\hat{\varphi}_0$. By Proposition 3.3.8, $(u_1+u_2)^*[\beta] = u_1^*[\beta]+u_2^*[\beta]$. This gives the following commutative diagram:

$$\begin{tikzcd}
\beta\colon 0 \arrow[r] & P^{(1)}/\ker\varphi_0 \arrow[r, "\iota"] \arrow[d, "u_1+u_2"] & P^{(0)} \arrow[r, "h"] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mu_1'+\mu_2'\colon 0 \arrow[r] & B \arrow[r] & P_0'' \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

It follows that $\epsilon(\mu_1+\mu_2) = \mu_1'+\mu_2' = \epsilon(\mu_1)+\epsilon(\mu_2)$. We conclude that ϵ is an isomorphism of abelian groups.

Natural isomorphism between the functors Ext^1_R and E^1_R

The canonical isomorphism ϵ gives an isomorphism naturally in both A and B. Thus, we conclude that there is a natural isomorphism of set-valued bifunctors $\epsilon\colon \mathrm{Ext}^1_R(-,-) \to E^1_R(-,-)$.

Module structure on $E^1_R(A,B)$ provided that R is commutative

Suppose that R is commutative. We will define an action of R on $E^1_R(A,B)$.

Let $r \in R$ with $r \neq 0$. Then, the action $r\colon M \to M$, defined by $r(m) = rm$, is an endomorphism of any R-module M.

By the functoriality in B, the action $r\colon B \to B$, induces a map r^* on $E^1_R(A,B)$ with $r^*([\alpha]) = [\alpha'']$; by the functoriality in A, the action $r\colon A \to A$, induces a map r_* on $E^1_R(A,B)$ with $r_*([\alpha]) = [\alpha']$; both actions can be illustrated in the following commutative diagram.

$$
\begin{array}{lccccccccc}
\alpha' \colon & 0 & \longrightarrow & B & \xrightarrow{\hat{q}} & X' & \xrightarrow{\hat{p}} & A & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle r} & & \\
\alpha \colon & 0 & \longrightarrow & B & \xrightarrow{q} & X & \xrightarrow{p} & A & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
\alpha'' \colon & 0 & \longrightarrow & B & \xrightarrow{\tilde{q}} & X'' & \xrightarrow{\tilde{p}} & A & \longrightarrow & 0
\end{array}
$$

Set $\alpha r = \alpha'$ and $r\alpha = \alpha''$. From the diagrams (3.11) and (3.12), we see that for $r_1, r_2 \in R$, the two sequences $r_1^* \circ r_2^*([\alpha])$ and $(r_1 r_2)^*([\alpha])$ are same.

$$
\begin{array}{lccccccccc}
\alpha \colon & 0 & \longrightarrow & B & \xrightarrow{q} & X & \xrightarrow{p} & A & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle r_2} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
r_2^*[\alpha] \colon & 0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle r_1} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
r_1^*[r_2^*[\alpha]] \colon & 0 & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0
\end{array}
\tag{3.11}
$$

$$
\begin{array}{lccccccccc}
\alpha \colon & 0 & \longrightarrow & B & \xrightarrow{q} & X & \xrightarrow{p} & A & \longrightarrow & 0 \\
& & & \Big\downarrow{\scriptstyle r_1 r_2} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
(r_1 r_2)^*[\alpha] \colon & 0 & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0
\end{array}
\tag{3.12}
$$

The identification of $r_1^* \circ r_2^*([\alpha])$ and $(r_1 r_2)^*([\alpha])$ gives $(r_1 r_2)\alpha = (r_1 r_2)^*([\alpha]) = r_1^* \circ r_2^*([\alpha]) = r_1(r_2 \alpha)$ for all $r_1, r_2 \in R$ and all $\alpha \in E_R^1(A, B)$. Similarly, $\alpha(r_1 r_2) = (\alpha r_1) r_2$ for all $r_1, r_2 \in R$ and all $\alpha \in E_R^1(A, B)$.

Remark 3.3.9. We see from above diagrams that the left action r on α is defined using $r \colon B \to B$ while the right action r is defined using $r \colon A \to A$. Then $r\alpha = \alpha''$ and $\alpha r = \alpha'$ are two different sequences. So, we cannot conclude that $r\alpha = \alpha r$. Now, we will focus on the left actions.

We have already defined an addition $+$ on $E_R^1(A, B)$ and we also have the zero element $0 \colon 0 \to B \to B \oplus A \to A \to 0$ in $E_R^1(A, B)$. Now, it remains to verify the two laws of distribution. By Proposition 3.3.8,

$$
\begin{aligned}
r(\alpha + \alpha') &= r\alpha + r\alpha', \\
(r_1 + r_2)\alpha &= r_1\alpha + r_2\alpha,
\end{aligned}
$$

for any $r, r_1, r_2 \in R$ and any $\alpha, \alpha' \in E^1_R(A,B)$. We conclude that $E^1_R(A,B)$ is an R-module.

R-module isomorphism: $E^1_R(A,B) \cong \operatorname{Ext}^1_R(A,B)$ provided that R is commutative

We have already proved that $\epsilon\colon \operatorname{Ext}^1_R(A,B) \to E^1_R(A,B)$ defined by (3.10) is an isomorphism of abelian groups. It remains to prove that ϵ is an R-module homomorphism. In fact, given a projective resolution

$$P^\bullet\colon \cdots \xrightarrow{\varphi_1} P^{(1)} \xrightarrow{\varphi_0} P^{(0)} \xrightarrow{h} A \to 0,$$

we can compute $\operatorname{Ext}^1_R(A,B)$ from $P^\bullet$ and obtain an exact sequence

$$\beta\colon 0 \to P^{(1)}/\ker\varphi_0 \to P^{(0)} \xrightarrow{h} A \to 0.$$

Let $\mu = u + \operatorname{im}\hat\varphi_0 \in \operatorname{Ext}^1_R(A,B)$ and $r \in R$. Then, $r\mu = ru + \operatorname{im}\hat\varphi_0$. Using the module structure on $E^1_R(A,B)$, we have $(ru)^*[\beta] = ru^*[\beta]$. This gives the following commutative diagram:

$$\begin{tikzcd}
\beta\colon 0 \arrow[r] & P^{(1)}/\ker\varphi_0 \arrow[r, "\iota"] \arrow[d, "ru"] & P^{(0)} \arrow[r, "h"] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
r\mu'\colon 0 \arrow[r] & B \arrow[r] & P_0''' \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

It follows that $\epsilon(r\mu) = r\mu' = r\epsilon(\mu)$. We conclude that ϵ is an R-module isomorphism.

3.3.2 Natural isomorphism from Ext^n_R to E^n_R

Now, we consider the general case $\ell \in \mathbb{N}_+$. The general case is analogous to the case $\ell = 1$, but it has numerous verifications. Here we just give a brief outline of the arguments involved. More details can be found, for example, in Chapter III of [13] and Chapter IV of [9].

$E^n_R(-,B)$ is a contravariant functor

Let $\alpha\colon (0 \to B \to X_1 \to \cdots \to X_{n-1} \xrightarrow{\varphi} X_n \xrightarrow{p} A \to 0) \in E^n_R(A,B)$. Let $v\colon A' \to A$ be an R-module homomorphism. Define $X'_n = \ker(-p,v)\colon X_n \oplus A' \to A$, as the *pull-back* of (p,v), illustrated by the following commutative diagram:

$$\begin{array}{ccccccc}
X_{n-1} & \xrightarrow{\varphi'} & X'_n & \xrightarrow{p'} & A' & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle v} & & \\
X_{n-1} & \xrightarrow{\varphi} & X_n & \xrightarrow{p} & A & \longrightarrow & 0,
\end{array}$$

where $\varphi' \colon X_{n-1} \to X'_n$ is defined by $\varphi'(x) = \big(\varphi(x), 0\big)$. Then, we obtain an exact sequence:

$$\alpha_v \colon 0 \to B \to X_1 \to \cdots \to X_{n-1} \xrightarrow{\varphi'} X'_n \xrightarrow{p'} A' \to 0.$$

One can prove that if $\tilde{\alpha} \in [\alpha]$, then $\tilde{\alpha}_v \in [\alpha_v]$. Thus, v induces a map $v_* \colon E^n_R(A, B) \to E^n_R(A', B)$ by $v_*([\alpha]) = [\alpha_v]$. One can also verify that $1_* = 1$ and $(v \circ v')_* = v'_* \circ v_*$. This implies that $E^n_R(-, B)$ is a contravariant functor from the category of R-modules to the category of sets.

$E^n_R(A, -)$ is a covariant functor

Let $\alpha \colon (0 \to B \xrightarrow{q} X_1 \xrightarrow{\psi} X_2 \to \cdots \to X_n \to A \to 0) \in E^n_R(A, B)$. Let $u \colon B \to B'$ be an R-module homomorphism. Define $X'_1 = \operatorname{coker}(-q, u) = (X_1 \oplus B')/N \colon B \to X_1 \oplus B'$, as the *push-out* of (q, u), illustrated by the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & B & \xrightarrow{q} & X_1 & \xrightarrow{\psi} & X_2 \\
 & & \Big\downarrow{\scriptstyle u} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & B' & \xrightarrow{q'} & X'_1 & \xrightarrow{\psi'} & X_2,
\end{array}$$

where $\psi' \colon X'_1 \to X_2$ is defined by $\psi'\big((x, b') + N\big) = \psi(x)$. Then, we obtain an exact sequence:

$$\alpha_u \colon 0 \to B' \xrightarrow{q'} X'_1 \xrightarrow{\psi'} X_2 \to \cdots \to X_n \to A \to 0.$$

One can prove that if $\tilde{\alpha} \in [\alpha]$, then $\tilde{\alpha}_u \in [\alpha_u]$. Thus, u induces a map $u^* \colon E^n_R(A, B) \to E^n_R(A, B')$ by $u^*([\alpha]) = [\alpha_u]$. One can also verify that $1^* = 1$ and $(u \circ u')^* = u^* \circ u'^*$. This implies that $E^n_R(A, -)$ is a covariant functor from the category of R-modules to the category of sets.

Remark 3.3.10. It can be shown that $u^* \circ v_* = v_* \circ u^* \colon E^n_R(A, B) \to E^n_R(A', B')$.

This implies that $E^n_R(-, -)$ is a bifunctor from the category of R-modules to the category of sets.

$E^n_R(A,B)$ **is naturally isomorphic to** $\operatorname{Ext}^n_R(A,B)$

Proposition 3.3.11 (cf. [13, Theorem III 6.4, p. 89], [9, Theorem IV 9.1, p. 150])**.** *There is a natural isomorphism of set-valued bifunctors*

$$\epsilon_n\colon E^n_R(-,-) \to \operatorname{Ext}^n_R(-,-),$$

for $n \in \mathbb{N}$.

3.3.3 Yoneda product

In this subsection, let $\otimes$ denote the tensor product of abelian groups. As they are naturally isomorphic, we identify $\operatorname{Ext}^i_R(M,N)$ and $E^i_R(M,N)$ for any $i \in \mathbb{N}$. Recall that, for all $i \in \mathbb{N}$, $\operatorname{Ext}^i_R(M,N)$ is an abelian group.

Definition 3.3.12 (Yoneda Product)**.** The *Yoneda product* is the $\mathbb{Z}$-bilinear map

$$\mu\colon \operatorname{Ext}^t_R(N,L) \times \operatorname{Ext}^s_R(M,N) \to \operatorname{Ext}^{s+t}_R(M,L)$$

defined for all $s,t \in \mathbb{N}$ as follows. For $s = t = 0$,

$$\begin{aligned} \mu\colon \operatorname{Hom}_R(N,L) \times \operatorname{Hom}_R(M,N) &\to \operatorname{Hom}_R(M,L) \\ (u,v) &\mapsto u \circ v. \end{aligned}$$

For $s,t \in \mathbb{N}_+$, let

$$\alpha\colon 0 \to L \to X_1 \to \cdots \to X_t \xrightarrow{p} N \to 0$$

and

$$\beta\colon 0 \to N \xrightarrow{q} Y_1 \to \cdots \to Y_s \to M \to 0$$

be exact sequences. Then

$$\mu\colon \operatorname{Hom}_R(N,L) \times \operatorname{Ext}^s_R(M,N) \to \operatorname{Ext}^s_R(M,L)$$

is defined by the functoriality of $E^s_R(M,-)$:

$$\begin{tikzcd} \beta\colon & 0 \arrow[r] & N \arrow[r, "q"] \arrow[d, "u"] & Y_1 \arrow[r] \arrow[d, "g"] & Y_2 \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \arrow[r] & M \arrow[r] & 0 \\ \mu(u,\beta)\colon & 0 \arrow[r] & L \arrow[r, "\tilde{q}"] & Y_1'' \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

and

$$\mu\colon \operatorname{Ext}^t_R(N, L) \times \operatorname{Hom}_R(M, N) \to \operatorname{Ext}^t_R(M, L)$$

is defined by the functoriality of $E^t_R(-, L)$:

$$\begin{array}{llllllllll}
\alpha\colon & 0 \to L \cdots & \longrightarrow & X_{t-1} & \longrightarrow & X_t & \xrightarrow{p} & N & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle v} & & \\
\mu\big(\alpha, v\big)\colon & 0 \to L \cdots & \longrightarrow & X_{t-1} & \longrightarrow & X'_t & \xrightarrow{\tilde{p}} & M & \longrightarrow & 0,
\end{array}$$

and $\mu\big([\alpha], [\beta]\big)$ is defined to be the Yoneda equivalence class of the exact sequence

$$\alpha\beta\colon 0 \to L \to X_1 \to \cdots \to X_t \xrightarrow{qp} Y_1 \to \cdots \to Y_s \to M \to 0.$$

Let α_1 and α_2 be two Yoneda equivalent sequences from L to N and let β_1 and β_2 be two Yoneda equivalent sequences from N to M. Then, by a straightforward computation, we see that their Yoneda products $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are Yoneda equivalent. This implies that the Yoneda product is well-defined. We also see from this definition that the Yoneda product satisfies the associative law: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Remark 3.3.13. Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded $\Bbbk$-algebra with $A_0 = \Bbbk$. Then, $A_+ = \bigoplus_{d>0}^{\infty} A_d$ is an ideal, called the *augmentation ideal*, of the ring A. Then, via the canonical projection $\rho\colon A \to A/A_+$, $a \mapsto a + A_+$, called the *canonical augmentation map*, the ring A acts naturally on the quotient A/A_+, hence on A_0 via the isomorphism $A_0 \cong A/A_+$, by

$$\begin{aligned} A \times A_0 &\to A_0 \\ (a, x) &\mapsto \rho(a)x. \end{aligned}$$

for any $a \in A$ and $x \in A_0$. Writing $a \cdot x = \rho(a)x$, A_0 can be viewed as an A-module.

Definition 3.3.14. Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded ring such that $A_0 = \Bbbk$. Viewing A_0 as an A-module, we can consider $\operatorname{Ext}^i_A(A_0, A_0)$ for each $i \in \mathbb{N}$. Define

$$E(A) = \bigoplus_{i=0}^{\infty} \operatorname{Ext}^i_A(A_0, A_0).$$

Note that $\operatorname{Ext}^0_A(\Bbbk, \Bbbk) = \operatorname{Hom}_A(\Bbbk, \Bbbk) \cong \Bbbk$. The direct sum $E(A)$, with the Yoneda product, is an $\mathbb{N}$-graded $\Bbbk$-algebra with $E(A)_i = \operatorname{Ext}^i_A(\Bbbk, \Bbbk)$ (see [7, p. 654–655]), called the Ext-*algebra*.

Chapter 4

Koszul Algebras and Quadratic Algebras

In this chapter, we introduce our main object of study: Koszul algebras. We define these algebras and give some examples and properties in Section 4.1. Then, in Section 4.2, we introduce quadratic algebras and show the relation between Koszul algebras and quadratic algebras. We refer the reader interested in further references to [1], [11], and [15].

4.1 Koszul Algebras

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded $\Bbbk$-algebra. Recall, from Section 3.3.3, that for an $\mathbb{N}$-graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$, A_0 can be viewed as an A-module.

Definition 4.1.1 (Koszul Algebras)**.** A *Koszul algebra* A is an $\mathbb{N}$-graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$ that satisfies following conditions:

1. $A_0 = \Bbbk$;

2. $A_0 \cong A/A_+$, considered as a graded A-module, admits a graded projective resolution

$$\cdots \to P^{(2)} \to P^{(1)} \to P^{(0)} \to A_0 \to 0,$$

such that $P^{(i)}$ is generated as a $\mathbb{Z}$-graded A-module by its degree i component, i.e., for the decomposition of A-modules:

$$P^{(i)} = \bigoplus_{j\in\mathbb{Z}} P_j^{(i)},$$

one has that $P^{(i)} = AP_i^{(i)}$.

Remark 4.1.2. A more general definition of Koszul algebra allows for A_0 to be an arbitrary semisimple commutative ring. Especially, when we discuss the direct sum of Koszul algebras, the condition 1 in Definition 4.1.1 should be that $A_0 = \Bbbk^n$, or more generally, $A_0 = \Bbbk_1 \oplus \cdots \oplus \Bbbk_n$ for some $n \in \mathbb{N}_+$, where $\Bbbk$ and $\Bbbk_j$, $j \in \{1, \cdots, n\}$, are fields. In this thesis, we assume that $A_0 = \Bbbk$. However, if $A = \bigoplus_{d=0}^{\infty} A_d$ is *locally finitely generated*, i.e., each component A_d is in the category A_0-grModf, many propositions and theorems still hold when A_0 is a semisimple ring.

Remark 4.1.3. Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded algebra with $A_0 = \Bbbk$. Let M be a vector space over $\Bbbk$.

1. The tensor product $A \otimes_{\Bbbk} M$ is naturally an A-module via left multiplication on the left factor.
2. Let $\mathcal{B}$ be a basis of the vector space M. Then, $A \otimes_{\Bbbk} M$ is a free left A-module with basis $1 \otimes_{\Bbbk} \mathcal{B}$ (see [12, p. 5]). In fact, $A \otimes_{\Bbbk} \Bbbk^n \cong A^n$ as left A-modules (see [6, Corollary 10.18, p. 373]). If $\dim(M) = n$, then $M \cong \Bbbk^n$ as vector spaces over $\Bbbk$.

Example 4.1.4. Consider the polynomial ring $\Bbbk[x] = \bigoplus_{j=0}^{\infty} \Bbbk[x]_j$, that is graded by degree of polynomials (Example 2.3.3), i.e., $\Bbbk[x]_j$ is the set of polynomials of degree j. Recalling the notation for grading shifts from Notation 2.3.11, we have $\Bbbk[x](-1)_j = \Bbbk[x]_{j-1}$. Then, we have a graded free $\Bbbk[x]$-resolution of $\Bbbk$:

$$0 \to \Bbbk[x](-1) \xrightarrow{x} \Bbbk[x] \xrightarrow{x \mapsto 0} \Bbbk \to 0,$$

where $\xrightarrow{x}$ denotes multiplication by x. Note that as $\Bbbk[x]$-modules, $\Bbbk[x]$ is generated in degree 0 and $\Bbbk[x](-1)$ is generated in degree 1. This resolution shows that $\Bbbk[x]$ is Koszul.

Example 4.1.5. Consider the polynomial ring $A = \Bbbk[x, y]/(xy) = \bigoplus_{i \in \mathbb{N}} A_i$ with $A_0 = \Bbbk$ and $A_i = \Bbbk \bar{x}^i + \Bbbk \bar{y}^i$ for $i \geq 1$, where we denote $\bar{z} = z + (xy) \in A$ for every $z \in \Bbbk[x, y]$. Then, $\bar{x}A + \bar{y}A = A_+ = \bigoplus_{i=1}^{\infty} A_i$. Let $\rho \in \hom_A(A, \Bbbk)$ be the canonical augmentation map. Then $\ker \rho = A_+ = \bar{x}A + \bar{y}A$. Define $d_1 \colon A^2 \to A$ by $d_1(a_1, a_2) = \bar{x}a_1 + \bar{y}a_2$. We can verify that d_1 is an A-module homomorphism and $\ker d_1 = \bar{y}A \times \bar{x}A$. For $i > 1$, define $d_i \colon A^2 \to A^2$ by

$$d_i(a_1, a_2) = \begin{cases} (\bar{y}a_1, \bar{x}a_2) & \text{if } i \text{ is even,} \\ (\bar{x}a_1, \bar{y}a_2) & \text{if } i \text{ is odd.} \end{cases}$$

Then, it is easily verified that d_i is an A-module homomorphism and

$$\ker d_i = \begin{cases} \bar{x}A \times \bar{y}A & \text{if } i \text{ is even,} \\ \bar{y}A \times \bar{x}A & \text{if } i \text{ is odd.} \end{cases}$$

Consequently, we have a graded free A-resolution of $\Bbbk$:

$$\cdots \to A(-i)^2 \xrightarrow{d_i} \cdots \to A(-3)^2 \xrightarrow{d_3} A(-2)^2 \xrightarrow{d_2} A(-1)^2 \xrightarrow{d_1} A \xrightarrow{\rho} \Bbbk \to 0.$$

This proves that $A = \Bbbk[x, y]/(xy)$ is Koszul.

Let V be the vector space over a field $\Bbbk$ with basis $\{x_1, \cdots, x_n\}$. Denote by $\Bbbk\langle x_1, \cdots, x_n\rangle$ the tensor algebra $T(V)$ and denote by $\Bbbk[x_1, \cdots, x_n]$ the symmetric algebra $S(V)$.

Remark 4.1.6. The symmetric algebra $\Bbbk[x_1, \cdots, x_n]$ is the commutative polynomial ring in n indeterminates. By Definition 2.4.3, we have

$$\Bbbk[x_1, \cdots, x_n] = \Bbbk\langle x_1, \cdots, x_n\rangle/I,$$

where I is the ideal in $\Bbbk\langle x_1, \cdots, x_n\rangle$ that is generated by all commutators $x_ix_j - x_jx_i$. For $A = \Bbbk\langle x_1, \cdots, x_n\rangle$ or $A = \Bbbk[x_1, \cdots, x_n]$, $A = \bigoplus_{i=0}^{\infty} A_i$ is a graded algebra where A_i is the additive group of all elements of degree i. In addition, for $A = \Bbbk\langle x_1, \cdots, x_n\rangle$, $\dim A_i = n^i$; for $A = \Bbbk[x_1, \cdots, x_n]$, $\dim A_i = \binom{i+n-1}{n-1}$.

Example 4.1.7 (cf. [8, p. 338–339])**.** Let $A = \Bbbk\langle x_1, \cdots, x_n\rangle$. Define $d\colon A^n \to A$ by $d(a_1, \cdots, a_n) = a_1x_1 + \cdots + a_nx_n$. Then, d is an A-module homomorphism. Furthermore, d is injective and $\operatorname{im} d = Ax_1 + \cdots + Ax_n = A_+$. Hence, we have a graded free A-module resolution:

$$0 \to A(-1)^n \xrightarrow{d} A \to \Bbbk \to 0.$$

This proves that $\Bbbk\langle x_1, \cdots, x_n\rangle$ is Koszul.

Recall that for a vector space V over a field $\Bbbk$ with basis $\{x_1, \cdots, x_n\}$, the exterior algebra $\bigwedge(V) = \bigoplus_{i=0}^{n} \bigwedge^i(V)$ is an $\mathbb{N}$-graded $\Bbbk$-algebra, where $\bigwedge^i(V)$ is the abelian additive group with the basis

$$\{x_{m_1} \wedge \cdots \wedge x_{m_i} \mid 1 \leq m_1 < \cdots < m_i \leq n\}.$$

The following proposition is well known, but its proof is often omitted in the literature.

Proposition 4.1.8. *Let $\Bbbk$ be a field of characteristic 0. Let V be a finite-dimensional vector space over $\Bbbk$.*

1. *The symmetric algebra $S(V)$ is Koszul.*

2. *The exterior algebra $\bigwedge(V)$ is Koszul.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a basis of V. By Theorem 2.4.9, we identify the symmetric algebra $S(V)$ with the polynomial ring $\Bbbk[x_1, \cdots, x_n]$. Define $d\colon S(V) \otimes_{\Bbbk} \bigwedge(V) \to S(V) \otimes_{\Bbbk} \bigwedge(V)$ by

$$d = \sum_{r=1}^{n} x_r \otimes \phi_r,$$

where $x_r\colon S(V) \to S(V)$ is defined by $x_r(f) = fx_r$, $\phi_r\colon \bigwedge(V) \to \bigwedge(V)$ is right contraction by x_r, explicitly,

$$\phi_r(x_{m_1} \wedge \cdots \wedge x_{m_i}) = \begin{cases} (-1)^{i-k} x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i}, \text{ if } r = m_k; \\ 0, \text{ if } r \notin \{m_1, \cdots m_i\}. \end{cases}$$

With the action of $S(V)$ on $S(V) \otimes_{\Bbbk} \bigwedge(V)$ defined by $g \cdot (f \otimes y) = (gf) \otimes y$, $S(V) \otimes_{\Bbbk} \bigwedge(V)$ is an $S(V)$-module and then,

$$d \in \mathrm{Hom}_{S(V)}\big(S(V) \otimes_{\Bbbk} \textstyle\bigwedge(V), S(V) \otimes_{\Bbbk} \bigwedge(V)\big).$$

For $j \in \mathbb{N}$ and $i \in \{0, \cdots, n\}$, let $d_{j,i}$ be the restriction of d to $S_j(V) \otimes \bigwedge^i(V)$. Denote $d_{\bullet,i} = \sum_{j=0}^{\infty} d_{j,i}$. Then,

$$d_{\bullet,i} \in \mathrm{Hom}_{S(V)}\left(S(V) \otimes_{\Bbbk} \bigwedge^{i}(V), S(V) \otimes_{\Bbbk} \bigwedge^{i-1}(V)\right).$$

Moreover,

$$
\begin{aligned}
&d^2\Big(\big(1 \otimes (x_{m_1} \wedge \cdots \wedge x_{m_i})\big)\Big)\\
&\quad = d\left(\sum_{k=1}^{i}(-1)^{i-k} x_{m_k} \otimes (x_{m_1} \wedge \cdots \wedge x_{m_{k-1}} \wedge x_{m_{k+1}} \wedge \cdots \wedge x_{m_i})\right)\\
&\quad = \sum_{k=1}^{i}(-1)^{i-k} x_{m_k} \sum_{t=1}^{k-1}(-1)^{i-t+1} x_{m_t} \otimes (x_{m_1} \wedge \cdots \wedge \hat{x}_{m_t} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i})\\
&\qquad + \sum_{k=1}^{i}(-1)^{i-k} x_{m_k} \sum_{t=k+1}^{i}(-1)^{i-t} x_{m_t} \otimes (x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge \hat{x}_{m_t} \wedge \cdots \wedge x_{m_i})\\
&\quad = \sum_{t<k}(-1)^{i-k}(-1)^{i-t+1} x_{m_k} x_{m_t} \otimes (x_{m_1} \wedge \cdots \wedge \hat{x}_{m_t} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i})\\
&\qquad + \sum_{t>k}(-1)^{i-k}(-1)^{i-t} x_{m_k} x_{m_t} \otimes (x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge \hat{x}_{m_t} \wedge \cdots \wedge x_{m_i})\\
&\quad = 0.
\end{aligned}
$$

This shows that $d^2 = 0$, consequently, $\operatorname{im} d_{\bullet,i} \subseteq \ker d_{\bullet,i-1}$.

Define $\alpha\colon S(V) \otimes_{\Bbbk} \bigwedge(V) \to S(V) \otimes_{\Bbbk} \bigwedge(V)$ to be the linear map over $\Bbbk$:

$$\alpha = \sum_{s=1}^{n} \frac{\partial}{\partial x_s} \otimes \psi_s,$$

where $\psi_s\colon \bigwedge(V) \to \bigwedge(V)$ is defined by

$$\psi_s(g) = g \wedge x_s.$$

With the action of $\bigwedge(V)$ on $S(V) \otimes_{\Bbbk} \bigwedge(V)$ defined by $z \cdot (f \otimes y) = f \otimes (z \wedge y)$, $S(V) \otimes_{\Bbbk} \bigwedge(V)$ is a $\bigwedge(V)$-module and then

$$\alpha \in \operatorname{Hom}_{\bigwedge(V)}\Big(S(V) \otimes_{\Bbbk} \bigwedge(V), S(V) \otimes_{\Bbbk} \bigwedge(V)\Big).$$

For $j \in \mathbb{N}$ and $i \in \{0, \cdots, n\}$, let $\alpha_{j,i}$ be the restriction of α to $S_j(V) \otimes \bigwedge_i(V)$. Denote $\alpha_{j,\bullet} = \sum_{i=0}^{\infty} \alpha_{j,i}$. Then,

$$\alpha_{j,\bullet} \in \operatorname{Hom}_{\bigwedge(V)}\Big(S_j(V) \otimes_{\Bbbk} \bigwedge(V), S_{j-1}(V) \otimes_{\Bbbk} \bigwedge(V)\Big).$$

Moreover, for $f \in S(V)$ and $g \in \bigwedge(V)$,

$$\begin{aligned}\alpha^2(f \otimes g) &= \alpha\left(\sum_{s=1}^{n} \frac{\partial f}{\partial x_s} \otimes (g \wedge x_s)\right)\\ &= \sum_{s=1}^{n}\sum_{r=1}^{n} \frac{\partial^2 f}{\partial x_r \partial x_s} \otimes (g \wedge x_s \wedge x_r) = 0,\end{aligned}$$

since $x_r \wedge x_s = -x_s \wedge x_r$. This shows that $\alpha^2 = 0$, consequently, $\operatorname{im}\alpha_{j,\bullet} \subseteq \ker\alpha_{j-1,\bullet}$.

Then, for $f \in S_j(V)$ and a homogeneous element $x_{m_1} \wedge \cdots \wedge x_{m_i}$ of degree i in $\bigwedge(V)$,

$$\begin{aligned}&d \circ \alpha\big(f \otimes (x_{m_1} \wedge \cdots \wedge x_{m_i})\big)\\ &\quad= d\left(\sum_{s \notin \{m_1,\cdots,m_i\}} \frac{\partial f}{\partial x_s} \otimes \big(x_{m_1} \wedge \cdots \wedge x_{m_i} \wedge x_s\big)\right)\\ &\quad= \sum_{s \notin \{m_1,\cdots,m_i\}} \frac{\partial f}{\partial x_s} x_s \otimes (x_{m_1} \wedge \cdots \wedge x_{m_i})\\ &\qquad+ \sum_{s \notin \{m_1,\cdots,m_i\}} \sum_{k=1}^{i} (-1)^{i-k+1} \frac{\partial f}{\partial x_s} x_{m_k} \otimes \big(x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i} \wedge x_s\big),\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\alpha \circ d\big(f \otimes (x_{m_1} \wedge \cdots \wedge x_{m_i})\big)\\ &\quad= \alpha\left(\sum_{k=1}^{i} (-1)^{i-k} f x_{m_k} \otimes (x_{m_1} \wedge \cdots \wedge x_{m_{k-1}} \wedge x_{m_{k+1}} \wedge \cdots \wedge x_{m_i})\right)\\ &\quad= \sum_{k=1}^{i} (-1)^{i-k} \left(\sum_{s=1}^{n} \frac{\partial (f x_{m_k})}{\partial x_s} \otimes \big(x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i} \wedge x_s\big)\right)\\ &\quad= \sum_{k=1}^{i} (-1)^{i-k} \left(\sum_{s=1}^{n} \frac{\partial f}{\partial x_s} x_{m_k} \otimes \big(x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i} \wedge x_s\big)\right)\\ &\qquad+ \sum_{k=1}^{i} (-1)^{i-k} f \otimes \big((x_{m_1} \wedge \cdots \wedge \hat{x}_{m_k} \wedge \cdots \wedge x_{m_i}) \wedge x_k\big).\end{aligned} \tag{4.2}$$

We see that

$$\sum_{k=1}^{i}(-1)^{i-k}\left(\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}x_{m_k}\otimes\big(x_{m_1}\wedge\cdots\wedge\hat{x}_{m_k}\wedge\cdots\wedge x_{m_i}\wedge x_s\big)\right)$$
$$=\sum_{k=1}^{i}\sum_{\substack{s\notin\\ \{m_1,\cdots,m_{k-1},\\ m_{k+1},\cdots,m_i\}}}(-1)^{i-k}\frac{\partial f}{\partial x_s}x_{m_k}\otimes\big(x_{m_1}\wedge\cdots\wedge\hat{x}_{m_k}\wedge\cdots\wedge x_{m_i}\wedge x_s\big)$$
$$=\sum_{s\notin\{m_1,\cdots,m_i\}}\sum_{k=1}^{i}(-1)^{i-k}\frac{\partial f}{\partial x_s}x_{m_k}\otimes\big(x_{m_1}\wedge\cdots\wedge\hat{x}_{m_k}\wedge\cdots\wedge x_{m_i}\wedge x_s\big)$$
$$+\sum_{k=1}^{i}\frac{\partial f}{\partial x_{m_k}}x_{m_k}\otimes\big(x_{m_1}\wedge\cdots\wedge x_{m_i}\big).$$

The sum of (4.1) and (4.2) gives that

$$d\circ\alpha\big(f\otimes(x_{m_1}\wedge\cdots\wedge x_{m_i})\big)+\alpha\circ d\big(f\otimes(x_{m_1}\wedge\cdots\wedge x_{m_i})\big)$$
$$=\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}x_s\otimes\big(x_{m_1}\wedge\cdots\wedge x_{m_i}\big)+if\otimes\big(x_{m_1}\wedge\cdots\wedge x_{m_i}\big).$$

where $i=1+\cdots+1$ (i times) $\in\Bbbk$. Since f is a homogeneous polynomial of degree j,

$$\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}x_s=jf,$$

where $j=1+\cdots+1$ (j times) $\in\Bbbk$. We conclude that

$$d_{j,i+1}\circ\alpha_{j,i}+\alpha_{j+1,i}\circ d_{j,i}=(j+i)\,\mathrm{id}\,.$$

where $j+i\in\Bbbk$ is a constant which depends on the degree of the homogeneous elements. Recall that the characteristic of $\Bbbk$ is 0. Then, $j+i$ is a nonzero element in $\Bbbk$, hence invertible. Then, if $d_{j,i}(f_j\otimes y_i)=0$ for $f_j\in S_j(V)$ and $y_i\in\bigwedge^i(V)$, we have $d_{j,i+1}\circ\alpha_{j,i}((j+i)^{-1}f_j\otimes y_i)=f_j\otimes y_i$. Note that if $f\otimes y\in\ker d_{\bullet,i}$, then for every homogeneous term f_j of f, we have $d_{j,i}(f_j\otimes y)=0$. This implies that $\operatorname{im} d_{\bullet,i+1}\supseteq\ker d_{\bullet,i}$. Similarly, $\operatorname{im}\alpha_{j+1,\bullet}\supseteq\ker\alpha_{j,\bullet}$.

We conclude that $\operatorname{im} d_{\bullet,i+1}=\ker d_{\bullet,i}$ and $\operatorname{im}\alpha_{j+1,\bullet}=\ker\alpha_{j,\bullet}$. Therefore, we obtain a graded free $S(V)$-module resolution of $\Bbbk$:

$$0\to S(V)\otimes_{\Bbbk}\bigwedge^{n}(V)\xrightarrow{d_{\bullet,n}}\cdots\xrightarrow{d_{\bullet,2}}S(V)\otimes_{\Bbbk}\bigwedge^{1}(V)\xrightarrow{d_{\bullet,1}}S(V)\xrightarrow{\rho}\Bbbk\to 0.$$

This proves that $S(V)$ is Koszul. We also obtain a graded free $\bigwedge(V)$-module resolution of $\Bbbk$:

$$\cdots \to S_\ell(V) \otimes_\Bbbk \bigwedge(V) \xrightarrow{\alpha_{\ell,\bullet}} \cdots \xrightarrow{\alpha_{2,\bullet}} S_1(V) \otimes_\Bbbk \bigwedge(V) \xrightarrow{\alpha_{1,\bullet}} \bigwedge(V) \xrightarrow{\rho'} \Bbbk \to 0.$$

This proves that $\bigwedge(V)$ is Koszul. $\square$

Definition 4.1.9 (Pure module)**.** A $\mathbb{Z}$-graded module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ over a $\mathbb{Z}$-graded ring is called *pure of degree* n if $M = M_{-n}$.

Now, let A be an $\mathbb{N}$-graded ring with $A_0 = \Bbbk$.

Lemma 4.1.10. *Let ℓ be a fixed integer. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded A-module such that $M_i = 0$ for all $i < \ell$. Then M admits a graded exact sequence of A-modules*

$$P^{(1)} \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} M \to 0$$

such that $P^{(n)}$, $n \in \{0, 1\}$, is a free graded A-module living only in degrees $\geq n + \ell$, i.e., $P^{(n)} = \bigoplus_{j\geq n+\ell} P^{(n)}_j$.

Proof. Let $P^{(0)} = A\otimes_\Bbbk M$. Recall that $A\otimes_\Bbbk M$ is a free A-module (see Remark 4.1.3). Then, $P^{(0)}$ is a graded free A-module living only in degree $\geq \ell$. In addition, we have an exact sequence: $P^{(0)} \xrightarrow{\epsilon} M \to 0$, where $\epsilon\colon A\otimes_\Bbbk M \to M$ is the unique $\Bbbk$-linear homomorphism satisfying

$$\epsilon(a \otimes m) = am$$

for all $a \in A$ and $m \in M$ and extended by linearity. Let $K^{(0)} = \ker \epsilon$. Then, $K^{(0)}$ is also a graded A-module living only in degree $\geq \ell$ and we have an exact sequence:

$$0 \to K^{(0)} \xrightarrow{\iota_1} P^{(0)} \xrightarrow{\epsilon} M \to 0,$$

where $\iota_1\colon \ker \epsilon \to P^{(0)}$ is the inclusion. For any element f of degree ℓ, $f \in A_0 \otimes_\Bbbk M_\ell$. Note that $A_0 \otimes_\Bbbk M_\ell = \Bbbk \otimes_\Bbbk M \cong M$. So, the restriction of ϵ to the space $A_0 \otimes_\Bbbk M_\ell$ is just scalar multiplication. Consequently, $\epsilon(f) = 0 \iff f = 0$. We conclude that $K^{(0)} = \ker \epsilon$ is a graded A-module living only in degree $\geq \ell + 1$.

Now, let $P^{(1)} = A \otimes_\Bbbk K^{(0)}$. Then, $P^{(1)}$ is a graded free A-module living only in degree $\geq \ell+1$. In addition, we have an exact sequence: $P^{(1)} \xrightarrow{\epsilon_1} K^{(0)} \to 0$, where $\epsilon_1 \colon A \otimes_{\Bbbk} K^{(0)} \to K^{(0)}$ is an A-module homomorphism satisfying

$$\epsilon_1(a \otimes x) = ax$$

for all $a \in A$ and $x \in K^{(0)}$. Then, we have an exact sequence:

$$P^{(1)} \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} M \to 0,$$

where $d_1 = \iota_1 \circ \epsilon_1$. $\square$

Lemma 4.1.11. *Let ℓ be a fixed integer. Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded A-module such that $M_i = 0$ for all $i < \ell$. Then M admits a graded free resolution of A-modules*

$$\cdots \to P^{(n)} \xrightarrow{d_n} P^{(n-1)} \to \cdots \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} M \to 0 \tag{4.3}$$

such that every A-module $P^{(n)}$ lives only in degrees $\geq n+\ell$, i.e., $P^{(n)} = \bigoplus_{j\geq n+\ell} P^{(n)}_j$.

Proof. By Lemma 4.1.10, we have exact sequences:

$$0 \to K^{(2)} \xrightarrow{\iota_2} P^{(1)} \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} M \to 0,$$

and

$$P^{(3)} \xrightarrow{d_3} P^{(2)} \xrightarrow{\epsilon_2} K^{(2)} \to 0,$$

where $K^{(2)} = \ker d_1$ is a graded A-module living only in degree $\geq \ell+2$, $\iota_2 \colon K^{(2)} \to P^{(1)}$ is the inclusion, and $\epsilon_2 \colon P^{(2)} = A \otimes_{\Bbbk} K^{(2)} \to K^{(2)}$ is the unique $\Bbbk$-linear homomorphism defined by

$$\epsilon_2(a \otimes x) = ax$$

for all $a \in A$ and $x \in K^{(2)}$. Then, we have an exact sequence

$$P^{(3)} \xrightarrow{d_3} P^{(2)} \xrightarrow{d_2} P^{(1)} \xrightarrow{d_1} P^{(0)} \xrightarrow{\epsilon} M \to 0,$$

where $d_2 = \iota_2 \circ \epsilon_2$. Repeating this process and applying induction on n, we have a graded free resolution (4.3). $\square$

Remark 4.1.12. From the proof of Lemma 4.1.11, we see that the differential d_j is injective on $P^{(j)}_{j+\ell}$ for all j. In particular, for the case where $\ell = 0$, d_j is injective on $P^{(j)}_j$ for all j.

Remark 4.1.13. Lemma 4.1.11 allows us to construct a free resolution (4.3) for the computation of $\mathrm{Ext}^n_{A\text{-grMod}}(M,N)$ if M lives only in degree $\geq \ell$. Since $\hom_A(M,N) = \hom_A\big(M(u), N(u)\big)$ for any $\mathbb{Z}$-graded A-module N and $u \in \mathbb{Z}$, by Definition 3.2.2, $\mathrm{Ext}^n_{A\text{-grMod}}(M,N) = \mathrm{Ext}^n_{A\text{-grMod}}(M(\ell), N(\ell))$, where $M(\ell)$ lives only in degree ≥ 0. This allows us to compute $\mathrm{Ext}^n_{A\text{-grMod}}(M,N)$ by assuming $\ell = 0$.

Proposition 4.1.14 (cf. [1, Lemma 2.1.2])**.** *Let M, N be two pure A-modules of degrees m, n respectively. Then, $\mathrm{Ext}^i_A(M,N) = 0$ for all $i > m-n$.*

Proof. By Remark 4.1.13, it is enough to prove the proposition for the case where $m = 0$. Fix $i \in \mathbb{Z}$ such that $i > m-n = -n$, Then $\mathrm{Ext}^i_A(M,N) = \mathrm{Ext}^i_A(M_0, N_{-n})$. By Lemma 4.1.11, $M = M_0$ admits a graded projective resolution $P^\bullet$ given by (4.3) where $\ell = 0$ and $P^{(j)} = \bigoplus_{t\geq j} P^{(j)}_t$. From (3.3), the resolution $P^\bullet$ induces a complex $\hom_A(P^\bullet, N)$. Since $\hom_A(\bigoplus_{t\geq i} P^{(i)}_t, N_{-n}) = 0$ if $i > -n$, we have that $\mathrm{Ext}^i_A(M,N) = 0$ if $i > -n$ by Definition 3.2.2. □

Lemma 4.1.15. *If , for some $i \in \mathbb{N}$, there is an exact sequence:*

$$0 \to K \xrightarrow{\iota} P^{(i)} \xrightarrow{d_i} \cdots \to P^{(1)} \to P^{(0)} \to A_0 \to 0,$$

where K is a graded A-module living only in degrees $\geq \ell$ for some $\ell \in \mathbb{N}_+$ with the inclusion $\iota \in \hom(K, P^{(i)})$ and $P^{(t)}$ are projective graded A-modules with the differentials $d_t \in \hom(P^{(t)}, P^{(t-1)})$ for all $t \in \{1, \cdots, i\}$, then,

$$\mathrm{Ext}^{i+1}_A(A_0, N) = \hom_A(K, N)$$

for any N in A-grMod.

Proof. By Lemma 4.1.11, K admits a free graded A-module resolution $\hat{P}^\bullet$:

$$\cdots \to P^{(i+2)} \xrightarrow{d_{i+2}} P^{(i+1)} \xrightarrow{d'_{i+1}} K \to 0.$$

This gives a projective graded A-module resolution of A_0:

$$\cdots \to P^{(i+2)} \xrightarrow{d_{i+2}} P^{(i+1)} \xrightarrow{d_{i+1}} P^{(i)} \xrightarrow{d_i} \cdots \to P^{(1)} \to P^{(0)} \to A_0 \to 0,$$

where $d_{i+1} = \iota \circ d'_{i+1}$. Then, this yields a complex

$$\begin{aligned} 0 \to \hom_A(A_0, N) \to \hom_A(P^{(0)}, N) \to \cdots \xrightarrow{d_i^*} \hom_A(P^{(i)}, N) \\ \xrightarrow{d_{i+1}^*} \hom_A(P^{(i+1)}, N) \xrightarrow{d_{i+2}^*} \hom_A(P^{(i+2)}, N) \to \cdots \end{aligned}$$

for any N in A-grMod. By Definition 3.2.2,

$$\operatorname{Ext}_A^{i+1}(A_0, N) = \ker d_{i+2}^* / \operatorname{im} d_{i+1}^* = \operatorname{Ext}_A^0(K, N) = \hom_A(K, N). \qquad \square$$

Lemma 4.1.16. *Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A-module. Suppose that M is living only in degree $\geq j$ and $\hom_A\big(M, A_0(-n)\big) = 0$ for all $n \in \mathbb{Z}$ unless $n = j$. Then $M = AM_j$.*

Proof. Since M is an A-module, certainly $AM_j \subseteq M$ for all $j \in \mathbb{Z}$. It remains to prove $M \subseteq AM_j$. We proceed by contradiction.

Suppose that M is not contained in AM_j. Then, there is an integer $n > j$ such that $M_n \nsubseteq AM_j$ and $M_i \subseteq AM_j$ for all $i < n$. Now, choose a nonzero $x \in M_n$ such that $x \notin AM_j$. Then, $x \notin AM_i$ for all $i < n$. Let V be a vector space complement to $\Bbbk x$ in M_n such that V contains $A_{n-i}M_i$ for all $i < n$, i.e.,

$$M_n = V \oplus \Bbbk x,$$

with $A_{n-i}M_i \subseteq V$ for all $i < n$. Let

$$N = V + \sum_{i \neq n} M_i.$$

We see that N is a submodule of M. Then, we have a $\Bbbk$-space isomorphism:

$$M/N \cong \Bbbk x \cong A_0(-n).$$

Thus, the quotient map $M \to M/N$ is a nonzero A-module homomorphism. This contradiction shows that $x \in AM_j$. We conclude that $M \subseteq AM_j$. $\square$

Proposition 4.1.17 (cf. [1, Proposition 2.1.3])**.** *The following conditions are equivalent.*

1. *A is a Koszul algebra.*

2. *For $i \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have $\operatorname{Ext}_A^i(A_0, A_0(-n)) = 0$ unless $i = n$.*

Proof. $(1 \Longrightarrow 2)$

Suppose that A is Koszul. By Definition 4.1.1, A_0 admits a graded projective resolution $P^\bullet$:

$$\cdots \to P^{(2)} \to P^{(1)} \to P^{(0)} \to A_0 \to 0,$$

such that $P^{(i)} = AP_i^{(i)}$. Then, $\hom_A\big(P^{(i)}, A_0(-n)\big) = \hom_A\big(AP_i^{(i)}, A_0(-n)\big) = 0$ unless $i = n$. By Definition 3.2.2,

$\operatorname{Ext}_A^i(A_0, A_0(-n)) = \ker d_{i+1}^* / \operatorname{im} d_i^* = \operatorname{Ext}_A^0(P^{(i)}, A_0(-n)) = \hom_A(P^{(i)}, A_0(-n))$.

Hence, $\operatorname{Ext}_A^i\big(A_0, A_0(-n)\big) = 0$ unless $i = n$.

$(2 \Longrightarrow 1)$

We construct a graded projective resolution of A_0 satisfying the conditions of Koszul algebra by induction. From the proof of of Lemma 4.1.11, there is an exact sequence: $P^{(0)} \xrightarrow{\epsilon} A_0 \to 0$, where $P^{(0)} = \bigoplus_{j=0}^{\infty} P_j^{(0)}$ is a graded free (hence projective) A-module with $P^{(0)} = AP_0^{(0)}$. Now, suppose that for $i \in \mathbb{N}_+$, there is an exact sequence:

$$P^{(i)} \xrightarrow{d_i} \cdots \to P^{(1)} \to P^{(0)} \to A_0 \to 0,$$

with the free graded A-module

$$P^{(t)} = AP_t^{(t)}$$

and the differential $d_t \in \hom(P^{(t)}, P^{(t-1)})$ for all $t \in \{0, 1, \cdots, i\}$. Put

$$K = \ker(d_i).$$

Then, by a similar argument in the proof of Lemma 4.1.10, we see that $K = \bigoplus_{j\in\mathbb{Z}} K_j$ is in A-grMod living only in degree $\geq i+1$ and we have an exact sequence $K^\bullet$:

$$0 \to K \xrightarrow{\iota} P^{(i)} \xrightarrow{d_i} \cdots \to P^{(1)} \to P^{(0)} \to A_0 \to 0.$$

By Lemma 4.1.15,

$$\operatorname{Ext}_A^{i+1}(A_0, N) = \hom_A(K, N)$$

for any N in A-grMod. Let $N = A_0(-n)$. From $\operatorname{Ext}_A^{i+1}(A_0, A_0(-n)) = 0$ unless $i+1 = n$, we have that $\hom_A(K, A_0(-n)) = 0$ unless $n = i+1$. By Lemma 4.1.16, we have $K = AK_{i+1}$. Put $P^{(i+1)} = A \otimes_{A_0} K_{i+1}$. This completes the induction step. $\square$

4.2 Quadratic Algebras

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be an $\mathbb{N}$-graded $\Bbbk$-algebra. Suppose that A_1 is a vector space over $\Bbbk$. Consider the tensor algebra of A_1:

$$T(A_1) = T_{\Bbbk}(A_1) = \bigoplus_{i=0}^{\infty} (A_1)^{\otimes i}.$$

Definition 4.2.1 (Relation of A of degree i). Let $\pi\colon T(A_1) \to A$ be the canonical map defined by linearly extending the multiplication $x_1 \otimes \cdots \otimes x_j \mapsto x_1 \cdots x_j$ over $\Bbbk$ for all $j \in \mathbb{N}_+$. Define $R_i = \ker \pi \cap A_1^{\otimes i}$ for $i \in \mathbb{N}_+$. The elements of R_i are called *degree i relations of A*. In particular, R_2 is the set of *quadratic relations of A*.

Definition 4.2.2 (Quadratic Algebras). A *quadratic algebra* A is an $\mathbb{N}$-graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$ that satisfies following conditions:

1. $A_0 = \Bbbk$,
2. $A \cong T(A_1)/I$ for an ideal I generated by the quadratic relation set $R = (\ker \pi) \cap (A_1 \otimes_{\Bbbk} A_1)$, i.e., $I = (R) = T(A_1)RT(A_1)$.

Remark 4.2.3. A more general definition of quadratic algebra allows A_0 to be an arbitrary semisimple commutative ring. For simplicity, we restrict attention to the case $A_0 = \Bbbk$.

From now on, $A = \bigoplus_{d=0}^{\infty} A_d$ is an $\mathbb{N}$-graded $\Bbbk$-algebra with $A_0 = \Bbbk$.

Example 4.2.4.

1. The tensor algebra $T(V)$ is quadratic since $T(V) = T(V)/(0)$.
2. The symmetric algebra is quadratic since $S(V) = T(V)/I$, where I is generated by $\{v \otimes w - w \otimes v \mid v, w \in V\}$.
3. The exterior algebra is quadratic since $\bigwedge(V) = T(V)/I$, where I is generated by $\{v \otimes v \mid v \in V\}$.

Notation 4.2.5. From now on, we specify the notation π_k and m as follows.

- For $k \in \mathbb{N}_+$, denote by π_k the canonical map $\pi\colon T_{\Bbbk}(A_1) \to A$ restricted to $A_1^{\otimes k}$. By Definition 4.2.1, $\ker \pi_k$ is the set of relations of A of degree k.
- Denote by m the $\Bbbk$-linear map $A \otimes_{\Bbbk} A_1 \to A_+$ induced by the multiplication $A \times A_1 \to A_+$.

Lemma 4.2.6. *Suppose that A is generated in degrees 0 and 1. Then for any $k \in \mathbb{N}_+$, $\pi_k \colon A_1^{\otimes k} \to A_k$ is surjective.*

Proof. Note that by Lemma 2.3.8, $A_k = \prod_{i=1}^{k} A_1$. The lemma follows from the definition of π. $\square$

Proposition 4.2.7 (cf. [1, Proposition 2.3.2, p. 482])**.** *Suppose that A is generated in degrees 0 and 1. For an element $x \in \ker \pi_n$ with $n > 2$, if $x \notin T(A_1)R_kT(A_1)$ for all $k < n$, then there exists a nonzero element $\bar{p}_x \in \ker m/(A_+ \ker m)$ with $\deg \bar{p}_x > 2$.*

Proof. Let $x \in \ker \pi_n$ with $n > 2$. Suppose that $x \notin T(A_1)R_kT(A_1)$ for all $k < n$. Consider the natural surjections:

$$A_1^{\otimes n} = A_1^{\otimes n-1} \otimes_{\Bbbk} A_1 \xrightarrow{p} A_{n-1} \otimes_{\Bbbk} A_1 \xrightarrow{m} A_n,$$

where $p = \pi_{n-1} \otimes \mathrm{id}$. Since A is generated in degrees 0 and 1, by Lemma 2.3.8, $A_j = \prod_{i=1}^{j} A_1$ for all $j \in \mathbb{N}_+$. Then, $m(p(x)) = \pi_n(x) = 0$. Hence $p(x) \in \ker m$. We claim that $p(x) \notin A_+ \ker m$. This claim implies that $\bar{p}(x) = p(x) + A_+ \ker m$ is a nonzero element in $\ker m/(A_+ \ker m)$ with $\deg \bar{p}(x) > 2$.

Assume that $p(x) \in A_+ \ker m$. Since $A_+ \ker m = A_1 A \ker m \subseteq A_1 \ker m$, we have $p(x) \in A_1 \ker m$. This implies that there is some $y \in A_1 \otimes_{\Bbbk} \ker m$ such that $p(y) = p(x)$, consequently, $p(x - y) = 0$. Hence, $x - y \in R_{n-1} \otimes_{\Bbbk} A_1$. This implies that $x \in A_1 \otimes_{\Bbbk} R_{n-1} + R_{n-1} \otimes_{\Bbbk} A_1$, i.e., $x \in T(A_1)R_{n-1}T(A_1)$. This contradiction proves the claim. $\square$

Lemma 4.2.8. *We have that*

$$A_+ = AA_1 \quad \Longleftrightarrow \quad \hom_A \big(A_+, A_0(-n)\big) = 0 \text{ for } n \neq 1.$$

Proof. Suppose that $A_+ = AA_1$. Then, any element $a \in A_+$ can be written as $a = \alpha_1 u_1 + \cdots + \alpha_t u_t \in A_+$ for some $t \in \mathbb{N}_+$ with $\alpha_j \in A$ and $u_j \in A_1$ for all $j \in \{1, \cdots, t\}$. Let $f \in \hom_A \big(A_+, A_0(-n)\big)$. Then, if $n \neq 1$,

$$f(a) = \alpha_1 f(u_1) + \cdots + \alpha_t f(u_t) = 0.$$

It follows that $\hom_A \big(A_+, A_0(-n)\big) = 0$ for $n \neq 1$.

Conversely, note that A_+ is a graded A-module that is living only in degree ≥ 1. By Lemma 4.1.16, we obtain $A_+ = AA_1$. $\square$

Proposition 4.2.9 (cf. [1, Proposition 2.3.1])**.** *The following conditions are equivalent.*

1. $\mathrm{Ext}^1_A\big(A_0, A_0(-n)\big) = 0$ *unless* $n = 1$.
2. *The graded ring* A *is generated in degrees* 0 *and* 1.

Proof. By Lemma 2.3.8,

$$2. \quad \iff \quad A_+ = AA_1.$$

By Lemma 4.2.8,

$$A_+ = AA_1 \quad \iff \quad \hom_A\big(A_+, A_0(-n)\big) = 0 \text{ for } n \neq 1.$$

Recall that we have an exact sequence of A-modules:

$$0 \to A_+ \xrightarrow{d_1} A \xrightarrow{\rho} A_0 \to 0, \tag{4.4}$$

where d_1 is the inclusion. By Lemma 4.1.15,

$$\mathrm{Ext}^1_A\big(A_0, A_0(-n)\big) = \hom_A\big(A_+, A_0(-n)\big).$$

This completes the proof. □

Lemma 4.2.10. *Recall, from Remark 4.1.3, that* $A \otimes_{\Bbbk} A_1$ *is naturally an* A*-module via the action on the left factor. For the induced multiplication* $m\colon A \otimes_{\Bbbk} A_1 \to A_+$*, give* $\ker m$ *the* A*-module structure as a submodule of* $A \otimes_{\Bbbk} A_1$*. Then, for* $n \in \mathbb{N}$*, we have an isomorphism of* $\Bbbk$*-vector spaces:*

$$\hom_A\big(\ker m, A_0(-n)\big) \cong \hom_{A_0}\big(\ker m/(A_+ \ker m), A_0(-n)\big).$$

Proof. Suppose that $a \in A_+$ and $x \in \ker m$. Let

$$f \in \hom_A\big(\ker m, A_0(-n)\big).$$

If $f(ax) \neq 0$, then ax is of degree n, hence x must be of degree $< n$, consequently, $f(x) = 0$, thus $f(ax) = af(x) = 0$. This contradiction shows that $f(A_+ \ker m) = 0$. Define $\bar{f}\colon \ker m/(A_+ \ker m) \to A_0(-n)$ by

$$\bar{f}(x + A_+ \ker m) = f(x).$$

Then, $\bar{f}$ is well-defined and $\bar{f} \in \hom_{A_0}\big(\ker m/(A_+ \ker m), A_0(-n)\big)$. Now, define $\phi\colon \hom_A\big(\ker m, A_0(-n)\big) \to \hom_{A_0}\big(\ker m/(A_+ \ker m), A_0(-n)\big)$, by

$\phi(f) = \bar{f}$.

Conversely, for each $\bar{g} \in \hom_{A_0}\big(\ker m/(A_+ \ker m), A_0(-n)\big)$, define $g\colon \ker m \to A_0(-n)$ by

$$g(x) = \bar{g}(x + A_+ \ker m).$$

Then, g is an A_0-module homomorphism. Consider $ay \in \ker m$ with $a \in A$ and $y \in \ker m$. If $\bar{g}(ay + A_+ \ker m) \neq 0$, then $ay \in \ker m$ is of degree n and $ay \notin A_+ \ker m$. So, $a \in A_0$. Hence, $g(ay) = ag(y)$. Consequently, $g \in \hom_A\big(\ker m, A_0(-n)\big)$. Therefore, ϕ gives an isomorphism of $\Bbbk$-spaces. □

Proposition 4.2.11 (cf. [1, Proposition 2.3.2]). *Suppose that A is generated in degrees* 0 *and* 1. *If* $\operatorname{Ext}^2_A\big(A_0, A_0(-n)\big) = 0$ *for all* $n \neq 2$, *then A is quadratic.*

Proof. Since A is generated in degrees 0 and 1, we have that $AA_1 = A_+$ and an exact sequence (4.4) (see the proof of Proposition 4.2.9). Since $AA_1 = A_+$, we see that $\operatorname{im} m = A_+ = \ker \rho$, where ρ is given in (4.4) and $m\colon A \otimes_{\Bbbk} A_1 \to A_+$ is the induced multiplication. Hence, we obtain an exact sequence

$$0 \to \ker m \to A \otimes_{\Bbbk} A_1 \xrightarrow{m} A \xrightarrow{\rho} A_0 \to 0.$$

Note that $\ker m$ is in A-grMod. By Lemma 4.1.15,

$$\operatorname{Ext}^2_A\big(A_0, A_0(-n)\big) = \hom_A\big(\ker m, A_0(-n)\big).$$

Since $A_0 = \Bbbk$, the restriction of m to $A_0 \otimes_{\Bbbk} A_1$ is injective. This implies that $\ker m \subseteq A_+ \otimes_{\Bbbk} A_1$.

Suppose that

$$\operatorname{Ext}^2_A\big(A_0, A_0(-n)\big) = 0 \text{ for all } n \neq 2.$$

Then, by Lemma 4.2.10,

$$\hom_{\Bbbk}\big(\ker m/(A_+ \ker m), A_0(-n)\big) = 0 \text{ for all } n \neq 2.$$

This implies that any nonzero element in $\ker m/(A_+ \ker m)$ is of degree 2. By Lemma 4.2.6, the canonical map $\pi\colon T(A_1) \to A$ is surjective. Hence, $T(A_1)/\ker \pi \cong \pi\big(T(A_1)\big) = A$. From Proposition 4.2.7, the relations in $\ker \pi$ must be quadratic. Therefore, the graded algebra A is quadratic. □

Proposition 4.2.12 (cf. [1, Corollary 2.3.3])**.** *If A is Koszul, then A is quadratic.*

Proof. Suppose that $A = \bigoplus_{i=0}^{\infty} A_i$ is a Koszul algebra. By Proposition 4.1.17, $\mathrm{Ext}^1_A\big(A_0, A_0(-n)\big) = 0$ unless $n = 1$, and $\mathrm{Ext}^2_A\big(A_0, A_0(-n)\big) = 0$ unless $n = 2$. By Proposition 4.2.9, A is generated in degrees 0 and 1. By Proposition 4.2.11, A is quadratic. $\square$

Example 4.2.13. Consider the polynomial ring $\mathbb{Q}[x]/(x^3+2)$. Note that x^3+2 is irreducible in $\mathbb{Q}[x]$. So, $x^3+2 \notin \mathbb{Q}[x]P\mathbb{Q}[x]$ for any polynomial P with $\deg P = 2$. This implies that $\mathbb{Q}[x]/(x^3+2)$ is not quadratic, consequently, is not Koszul.

Chapter 5

Koszul Duality

Let V be a finite-dimensional $\Bbbk$-vector space and let $\dim V = n$. Denote $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. Then V^* is also a $\Bbbk$-vector space via $(af)(v) = a\big(f(v)\big)$ for any $f \in V^*$, $v \in V$ and $a \in \Bbbk$. Moreover, we can identify $V^* \otimes_{\Bbbk} V^*$ with $(V \otimes_{\Bbbk} V)^*$ by the rule

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v)\psi(w)$$

for all $\varphi, \psi \in V^*$ and $v, w \in V$. Applying induction, we can identify $(V^*)^{\otimes j}$ with $(V^{\otimes j})^*$ for $j \in \mathbb{N}_+$.

Here we state some well-known and useful facts for this chapter.

1. Suppose that $\mathcal{B}_V = \{v_1, \cdots, v_n\}$ is a basis of V. For each $i \in \{1, \cdots, n\}$, define $\varphi_i \in V^*$ by

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad j \in \{1, \cdots, n\}. \tag{5.1}$$

   Then, $\mathcal{B}_{V^*} = \{\varphi_1, \cdots, \varphi_n\}$ is a basis of V^*. We call it the *dual basis* to $\mathcal{B}$.

2. There is a canonical isomorphism of $\Bbbk$-vector spaces from V to $V^{**} = (V^*)^*$. For any $v \in V$, define $\mathrm{ev}_v \in V^{**}$ by $\mathrm{ev}_v(f) = f(v)$ for $f \in V^*$. Then, the map $v \mapsto \mathrm{ev}_v$ is the canonical isomorphism of V with V^{**}. In this sense, we identify V^{**} with V.

Throughout this chapter, $A = \bigoplus_{d=0}^{\infty} A_d$ is a locally finite-dimensional graded $\Bbbk$-algebra with $A_0 = \Bbbk$ and $A_1 = V$. This means that every graded component A_d is a finite-dimensional vector space over $\Bbbk$. In addition, if A is generated in degree 0 and 1, by Lemma 2.3.8, $A_d = V^d$, so that $A = \Bbbk \oplus (\bigoplus_{d=1}^{\infty} V^d)$.

5.1 Duality for Quadratic Algebras

Let $T(V)$ be the tensor algebra of V over $\Bbbk$.

Definition 5.1.1 (Perpendicular Subspace)**.** For any subspace $Q \subseteq V^{\otimes j}$, define the *perpendicular subspace*

$$Q^{\perp} = \{\alpha \in (V^*)^{\otimes j} \mid \alpha(q) = 0 \text{ for all } q \in Q\}.$$

Denote by $\langle Q \rangle$ the ideal of $T(V)$ generated by Q, i.e.,

$$\langle Q \rangle = T(V)QT(V).$$

Denote by $\langle Q^{\perp} \rangle$ the ideal of $T(V^*)$ generated by $Q^{\perp}$, i.e.,

$$\langle Q^{\perp} \rangle = T(V^*)Q^{\perp}T(V^*).$$

Remark 5.1.2. In general, $\langle Q^{\perp} \rangle \neq \langle Q \rangle^{\perp}$.

Recall that the homogeneous component $T_j(V) = V^{\otimes j}$ is also a vector space over $\Bbbk$ with $\dim(V^{\otimes j}) = n^j$ (see [6, Proposition 11.32]).

Lemma 5.1.3. *We have* $\dim(Q^{\perp})^{\perp} = \dim Q$*.*

Proof. Let $\dim Q = \ell$ and let $\mathcal{B}_Q = \{q_1, \cdots, q_\ell\}$ be a basis of Q. Then, we can extend $\mathcal{B}_Q$ to a basis $\mathcal{B}_{V^{\otimes j}} = \{q_1, \cdots, q_\ell, q_{\ell+1}, \cdots, q_{n^j}\}$ of $V^{\otimes j}$. Then, we have the dual basis $\mathcal{B}_{(V^{\otimes j})^*} = \{\alpha_1, \cdots, \alpha_\ell, \alpha_{\ell+1}, \cdots, \alpha_{n^j}\}$ of $(V^*)^{\otimes j}$. The $Q^{\perp}$ has a basis given by $\{\alpha_{\ell+1}, \cdots, \alpha_{n^j}\}$, and so $\dim Q^{\perp} = n^j - \ell$. It follows that $\dim(Q^{\perp})^{\perp} = n^j - (n^j - \ell) = \ell$. □

Corollary 5.1.4. *Under the identification of a vector space with its double dual, we have* $Q = (Q^{\perp})^{\perp}$*.*

Proof. For every $v \in Q$ and any $\alpha \in Q^{\perp}$, we have $\mathrm{ev}_v(\alpha) = \alpha(v) = 0$. This implies that for every $v \in Q$, $\mathrm{ev}_v \in (Q^{\perp})^{\perp}$. Under the identification of v with ev_v, we obtain $Q \subseteq (Q^{\perp})^{\perp}$. By Lemma 5.1.3, we have $Q = (Q^{\perp})^{\perp}$. □

Definition 5.1.5 (Quadratic Dual Algebra)**.** Let $A = T(V)/\langle Q \rangle$ be a quadratic algebra, where $\langle Q \rangle$ is generated by the quadratic relation set $Q \subseteq V \otimes_{\Bbbk} V$. Define $A^! = T(V^*)/\langle Q^\perp \rangle$. We call $A^!$ the *quadratic dual algebra* to the quadratic algebra A.

Remark 5.1.6.

1. We see, by the definition, that $A^!$ is a quadratic algebra. Thus, we can write
$$A^! = \bigoplus_{d=0}^{\infty} A^!_d$$
as a graded $\Bbbk$-algebra generated in degree 0 and 1 with $A^!_0 = \Bbbk$ and $A^!_1 = V^*$.

2. Some references call $A^!$ the *Koszul dual* of the quadratic algebra A.

Proposition 5.1.7. *Let A be a quadratic algebra. Under the identification of a vector space with its double dual, we have $(A^!)^! = A$.*

Proof. This follows immediately from Corollary 5.1.4. $\square$

Example 5.1.8. Consider the symmetric algebra $S(V) = T(V)/\langle Q_S \rangle$, where $\langle Q_S \rangle$ is generated by $Q_S = \{v \otimes w - w \otimes v \mid v, w \in V\}$. Let $Q_E = \{\varphi \otimes \varphi \mid \varphi \in V^*\}$. Then, for any $\varphi \in V^*$ and any $v, w \in V$,

$$(\varphi \otimes \varphi)(v \otimes w - w \otimes v) = \varphi(v)\varphi(w) - \varphi(w)\varphi(v) = 0.$$

This implies that $Q_E \subseteq Q_S^\perp$. Regarding v, w as elements in V^{**}, we have that

$$(v \otimes w - w \otimes v)(\varphi \otimes \varphi) = 0.$$

This implies that $Q_S \subseteq Q_E^\perp$. Thus,

$$Q_E \subseteq Q_S^\perp \subseteq (Q_E^\perp)^\perp = Q_E.$$

It follows that

$$Q_S^\perp = Q_E = \{\varphi \otimes \varphi \mid \varphi \in V^*\}.$$

Hence,

$$\big(S(V)\big)^! = T(V^*)/\langle Q_E \rangle = \bigwedge(V^*),$$
$$\big(\bigwedge(V^*)\big)^! = \big(S(V)\big)^{!!} = S(V).$$

Example 5.1.9.

1. Recall that the tensor algebra $T(V) = T(V)/\langle Q\rangle$ is a quadratic algebra, where $\langle Q\rangle$ is the ideal generated by the quadratic relation set $Q = \{0\} \subseteq V \otimes_{\Bbbk} V$. Then, $Q^{\perp} = (V \otimes_{\Bbbk} V)^* = V^* \otimes_{\Bbbk} V^*$. Thus,

$$\begin{aligned}\big(T(V)\big)^! &= T(V^*)/\langle Q^{\perp}\rangle = T(V^*)/\big(T(V^*)(V^* \otimes_{\Bbbk} V^*)T(V^*)\big)\\ &\cong \Bbbk \oplus V^*,\end{aligned}$$

where $\Bbbk \oplus V^* = \bigoplus_{i\in\mathbb{N}} V_i^*$ is a graded algebra with $V_0^* = \Bbbk$, $V_1^* = V^*$ and $V_i^* = 0$ for all $i \geq 2$. This gives another isomorphism of algebras:

$$(\Bbbk \oplus V^*)^! \cong \big(T(V)\big)^{!!} = T(V).$$

2. Let $V = \Bbbk x$. Then, $T(V) = T_{\Bbbk}(\Bbbk x) = \Bbbk[x]$. Thus,

$$\begin{aligned}(\Bbbk[x])^! &\cong \Bbbk[x]/\langle x^2\rangle,\\ \big(\Bbbk[x]/\langle x^2\rangle\big)^! &\cong \Bbbk[x].\end{aligned}$$

3. Let V be the vector space over $\Bbbk$ with basis $\{x_1, \cdots, x_n\}$. Consider the quadratic algebra $S(V) = \Bbbk[x_1, \cdots, x_n] = T(V)/\langle Q_S\rangle$, where $Q_S = \{v \otimes w - w \otimes v \mid v, w \in V\}$. Note that if $x_i \otimes x_j - x_j \otimes x_i = 0$ for all $1 \leq i < j \leq n$, then $v \otimes w - w \otimes v = 0$, where $v = a_1x_1 + \cdots + a_nx_n$ and $w = b_1x_1 + \cdots + b_nx_n$ for some $a_i, b_i \in \Bbbk$. Recall that we denote by $\Bbbk\langle x_1, \cdots, x_n\rangle$ the tensor algebra $T(V)$. Then,

$$S(V) = \Bbbk\langle x_1, \cdots, x_n\rangle/\langle x_ix_j - x_jx_i \mid 1 \leq i < j \leq n\rangle.$$

Let V^* be the dual space of V with the dual basis $\{\xi_1, \cdots, \xi_n\}$. Then, $\xi_i \otimes \xi_i, (\xi_i + \xi_j) \otimes (\xi_i + \xi_j) \in Q_E = \{\varphi \otimes \varphi \mid \varphi \in V^*\}$ for all $i, j \in \{1, \cdots, n\}$. This implies that $\xi_i \otimes \xi_j + \xi_j \otimes \xi_i \in (Q_E)$ for all $i, j \in \{1, \cdots, n\}$, where (Q_E) is the vector space over $\Bbbk$ generated by Q_E. Conversely, if $\xi_i \otimes \xi_i = 0$ and $\xi_i \otimes \xi_j + \xi_j \otimes \xi_i = 0$ for all $1 \leq i < j \leq n$, then $\varphi \otimes \varphi = 0$, where $\varphi = a_1\xi_1 + \cdots + a_n\xi_n$ for some $a_i \in \Bbbk$. We thus obtain

$$\begin{aligned}\bigwedge(V^*) &= T(V^*)/\langle Q_E\rangle\\ &= T(V^*)/\langle \xi_i \otimes \xi_i, \xi_i \otimes \xi_j + \xi_j \otimes \xi_i \mid 1 \leq i < j \leq n\rangle.\end{aligned}$$

By Example 5.1.8, we have

$$\big(S(V)\big)^! = \Bbbk\langle \xi_1, \cdots, \xi_n\rangle/\langle \xi_i^2, \xi_i\xi_j + \xi_j\xi_i \mid 1 \leq i < j \leq n\rangle.$$

4. Let V be the vector space over the field $\Bbbk$ with basis $\{x, y\}$. Consider the quadratic algebra $A = \Bbbk[x, y]/\langle x^2 \rangle$. Then, $A = T(V)/\langle Q \rangle$, where $Q = \{x \otimes x, x \otimes y - y \otimes x\}$ is the quadratic relation set. Let V^* be the dual space of V with the dual basis $\{\xi, \eta\}$ Then, $T(V^*) = \Bbbk\langle \xi, \eta \rangle$ and $Q^\perp$ is generated by $\{\xi \otimes \eta + \eta \otimes \xi, \eta \otimes \eta\}$. Thus,

$$A^! = T(V^*)/\langle Q^\perp \rangle = \Bbbk\langle \xi, \eta \rangle / \langle \xi\eta + \eta\xi, \eta^2 \rangle.$$

5.2 The Koszul Complex of a Quadratic Algebra

Let $A = \bigoplus_{i=0}^{\infty} A_i = T(V)/\langle Q \rangle$ be a quadratic algebra, where $\langle Q \rangle = T(V)QT(V)$ is the ideal generated by the quadratic relation set $Q \subseteq V \otimes_\Bbbk V$. Let $A^! = \bigoplus_{i=0}^{\infty} A_i^! = T(V^*)/\langle Q^\perp \rangle$ be the quadratic dual algebra of A. Then, by Proposition 2.3.4,

$$A_i = V^{\otimes i}/\langle Q \rangle_i \qquad A_i^! = (V^*)^{\otimes i}/\langle Q^\perp \rangle_i$$

where

$$\begin{aligned} \langle Q \rangle_i &= \sum_{\nu=0}^{i-2} V^{\otimes \nu} \otimes_\Bbbk Q \otimes_\Bbbk V^{\otimes(i-\nu-2)}, \\ \langle Q^\perp \rangle_i &= \sum_{\nu=0}^{i-2} (V^*)^{\otimes \nu} \otimes_\Bbbk Q^\perp \otimes_\Bbbk (V^*)^{\otimes(i-\nu-2)}. \end{aligned} \tag{5.2}$$

Lemma 5.2.1. *Let W be a finite-dimensional $\Bbbk$-vector space. Let W_1 and W_2 be subspaces of W. Then,*

$$\begin{aligned} (W_1 + W_2)^\perp &= W_1^\perp \cap W_2^\perp, \\ (W_1 \cap W_2)^\perp &= W_1^\perp + W_2^\perp. \end{aligned}$$

Proof. Since $W_i \subseteq W_1 + W_2$ for $i \in \{1, 2\}$, we have $(W_1 + W_2)^\perp \subseteq W_i^\perp$. Hence, $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$.

Let $f \in W_1^\perp \cap W_2^\perp$. Then, for all $w_i \in W_i$, $i \in \{1, 2\}$, we have $f(w_i) = 0$. So, for any $w = w_1 + w_2 \in W_1 + W_2$ with $w_i \in W_i$, we have $f(w) = f(w_1) + f(w_2) = 0$. Thus, $W_1^\perp \cap W_2^\perp \subseteq (W_1 + W_2)^\perp$. We conclude that

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp.$$

This gives that

$$
\begin{aligned}
(W_1 \cap W_2)^\perp &= \big((W_1^\perp)^\perp \cap (W_2^\perp)^\perp\big)^\perp \\
&= \big((W_1^\perp + W_2^\perp)^\perp\big)^\perp = W_1^\perp + W_2^\perp. \quad \square
\end{aligned}
$$

Corollary 5.2.2. *Let $\langle Q\rangle_i$ and $\langle Q^\perp\rangle_i$ be the subspaces in $V^{\otimes i}$ and $(V^*)^{\otimes i}$ respectively, given by (5.2). Then, by Lemma 5.2.1, the perpendicular subspaces $\langle Q\rangle_i^\perp$ in $(V^*)^{\otimes i}$ of $\langle Q\rangle_i$ and $\langle Q^\perp\rangle_i^\perp$ in $(V^{**})^{\otimes i}$ of $\langle Q^\perp\rangle_i$ are given by*

$$
\begin{aligned}
\langle Q\rangle_i^\perp &= \bigcap_{\nu=0}^{i-2} (V^*)^{\otimes \nu} \otimes_{\Bbbk} Q^\perp \otimes_{\Bbbk} (V^*)^{\otimes(i-\nu-2)}, \\
\langle Q^\perp\rangle_i^\perp &= \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes(i-\nu-2)}).
\end{aligned}
$$

Remark 5.2.3. We notice that $\langle Q\rangle_i^\perp = \big(\langle Q\rangle_i\big)^\perp$, and in general,

$$
\begin{aligned}
\langle Q\rangle_i^\perp &\neq \langle Q^\perp\rangle_i, \\
\langle Q\rangle_i &\neq \langle Q^\perp\rangle_i^\perp.
\end{aligned}
$$

For $i \in \mathbb{N}$, define a $\Bbbk$-linear map $\hat{d}_i \colon A \otimes_{\Bbbk} V^{\otimes i} \to A \otimes_{\Bbbk} V^{\otimes i-1}$ by extending linearly

$$\hat{d}_i(a \otimes w_1 \otimes \cdots \otimes w_i) = aw_1 \otimes w_2 \otimes \cdots \otimes w_i,$$

for all $a \in A$ and $w_1, \cdots, w_i \in V$, i.e.,

$$\hat{d}_i = m \otimes \mathrm{id}^{\otimes(i-1)},$$

where $m \colon A \otimes_{\Bbbk} V \to A$ is the canonical map induced by multiplication. Then, define

$$d_i = \hat{d}_i|_{A \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes(i-\nu-2)})}. \tag{5.3}$$

Note that $\bigcap_{\nu=0}^{i-2}(V^{\otimes \nu} \otimes Q \otimes V^{\otimes(i-\nu-2)})$ is a subspace of $Q \otimes V^{\otimes(i-2)}$, which is a subspace of $V^{\otimes i}$. For a nonzero $w \in Q \otimes V^{\otimes(i-2)}$, w is a finite sum of simple tensors in $Q \otimes V^{\otimes(i-2)}$: $w = \sum_{j=1}^k q^{(j)} \otimes w_3^{(j)} \otimes \cdots \otimes w_i^{(j)}$ with $w_3^{(j)}, \cdots, w_i^{(j)} \in V$ and $q^{(j)} \in Q$. Since Q is a subspace of $V \otimes V$, $q^{(j)}$ is a finite sum of simple tensors $w_1^{(j_k)} \otimes w_2^{(j_k)}$ such that $w_1^{(j_k)} \otimes w_2^{(j_k)} \in Q$ with $w_1^{(j_k)}, w_2^{(j_k)} \in V$. We thus conclude that w is a finite sum of simple tensors

$w_1 \otimes w_2 \otimes w_3 \otimes \cdots \otimes w_i$ in $V^{\otimes i}$ such that $w_1 \otimes w_2 \in Q$. Since the quadratic relation set $Q = \ker \pi$, where $\pi \colon V \otimes V \to V^2$ is the canonical map induced by multiplication, we have that $w_1 w_2 = 0$. Then,

$$\begin{aligned}\hat{d}_{i-1}\hat{d}_i(a \otimes w_1 \otimes w_2 \otimes w_3 \otimes \cdots \otimes w_i) &= \hat{d}_{i-1}(aw_1 \otimes w_2 \otimes w_3 \otimes \cdots \otimes w_i) \\ &= aw_1w_2 \otimes w_3 \otimes \cdots \otimes w_i = 0.\end{aligned}$$

This shows that $d_{i-1}d_i = 0$. We thus obtain a complex:

$$\cdots \to A \otimes_{\Bbbk} \langle Q^{\perp} \rangle_i^{\perp} \xrightarrow{d_i} A \otimes_{\Bbbk} \langle Q^{\perp} \rangle_{i-1}^{\perp} \to \cdots \to A \otimes_{\Bbbk} \langle Q^{\perp} \rangle_2^{\perp} \to A \otimes_{\Bbbk} V \to A. \tag{5.4}$$

Remark 5.2.4. The concept of Koszul algebra was introduced by Stewart Priddy in [16]. We call the complex (5.4) the *Koszul complex* of the quadratic algebra A. In some references (e.g. [7]), the complex (5.4) is also called the *Priddy complex* of A.

Denote

$$K^{(i)} = A \otimes_{\Bbbk} \langle Q^{\perp} \rangle_i^{\perp} \text{ for } i \geq 2,$$
$$K^{(1)} = A \otimes_{\Bbbk} V, \quad K^{(0)} = A.$$

Then, for each $i \in \mathbb{N}$, $K^{(i)}$ is a graded free A-module given by

$$K^{(i)} = \big(\bigoplus_{j \in \mathbb{Z}} A_{j-i}\big) \otimes_{\Bbbk} \langle Q^{\perp} \rangle_i^{\perp} = \bigoplus_{j \in \mathbb{Z}} \big(A_{j-i} \otimes \langle Q^{\perp} \rangle_i^{\perp}\big) = \bigoplus_{j \in \mathbb{Z}} K_j^{(i)}$$

with the j^{th} graded piece

$$K_j^{(i)} = A_{j-i} \otimes \langle Q^{\perp} \rangle_i^{\perp}.$$

Then, we have that $K^{(i)} \cong A \otimes_{\Bbbk} K_i^{(i)}$ since $K_i^{(i)} = A_0 \otimes_{\Bbbk} \langle Q^{\perp} \rangle_i^{\perp} = \Bbbk \otimes_{\Bbbk} \langle Q^{\perp} \rangle_i^{\perp} \cong \langle Q^{\perp} \rangle_i^{\perp}$.

Note that for the differential $d_i \colon K^{(i)} \to K^{(i-1)}$ in the Koszul complex, $\ker d_i = \bigoplus_{j \in \mathbb{Z}} Z_j^{(i)}$ is in A-grMod, where $Z_j^{(i)}$ is the homogeneous component of degree j for $\ker d_i$, i.e., $Z_j^{(i)} = \ker d_i \cap K_j^{(i)}$. We see then that $Z_j^{(i)} = \ker d_j^{(i)}$, where $d_j^{(i)} \colon K_j^{(i)} \to K_j^{(i-1)}$ is the map induced by d_i from the homogeneous component $K_j^{(i)}$ to the homogeneous component $K_j^{(i-1)}$, defined by

$$d_j^{(i)} = d_i|_{K_j^{(i)}}.$$

Lemma 5.2.5. *The induced map* $d_i^{(i)}\colon K_i^{(i)} \to K_i^{(i-1)}$ *is injective for* $i \geq 2$.

Proof. Note that

$$K_i^{(i)} = A_0 \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-2)}) = \mathbb{k} \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-2)}).$$

So, $d_i^{(i)} = m \otimes \mathrm{id}^{\otimes (i-1)}$, where $m\colon \mathbb{k} \otimes_{\mathbb{k}} V \to V$ is the canonical isomorphism $\mathbb{k} \otimes_{\mathbb{k}} V \cong V$. This shows that $d_i^{(i)}$ is injective. $\square$

Corollary 5.2.6. *We have that* $Z_j^{(i)} = 0$ *for all* $j \leq i$.

Proof. By Lemma 5.2.5, $d_i^{(i)}$ is injective. So, $Z_i^{(i)} = \ker d_i^{(i)} = 0$. Note that A is $\mathbb{N}$-graded. Hence, $A_{j-i} = \{0\}$ for all $j < i$, consequently, $Z_j^{(i)} = 0$ for all $j < i$. $\square$

Lemma 5.2.7. *We have that* $Z_{i+1}^{(i)} = \ker d_{i+1}^{(i)} \subseteq Q \otimes V^{\otimes (i-1)}$.

Proof. Note that

$$K_{i+1}^{(i)} = A_1 \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-2)}) = V \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-2)}).$$

So, $d_{i+1}^{(i)} = \pi \otimes \mathrm{id}^{\otimes (i-1)}$ with $Q = \ker \pi$. Thus, $Z_{i+1}^{(i)} = \ker d_{i+1}^{(i)} = \ker(\pi \otimes \mathrm{id}^{\otimes (i-1)}) \subseteq \ker \pi \otimes V^{\otimes (i-1)} = Q \otimes V^{\otimes (i-1)}$. $\square$

Lemma 5.2.8. *We have that* $Z_{i+1}^{(i)} = \langle Q^{\perp} \rangle_{i+1}^{\perp}$.

Proof. Obviously,

$$\langle Q^{\perp} \rangle_{i+1}^{\perp} = \bigcap_{\nu=0}^{i-1} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-1)}) \subseteq \ker d_{i+1}^{(i)} = Z_{i+1}^{(i)}.$$

By Lemma 5.2.7,

$$\begin{aligned} Z_{i+1}^{(i)} &= \ker d_{i+1}^{(i)} \\ &\subseteq \left(V \otimes \bigcap_{\nu=0}^{i-2} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-2)}) \right) \cap \left(Q \otimes V^{\otimes (i-1)} \right) \\ &= \bigcap_{\nu=0}^{i-1} (V^{\otimes \nu} \otimes Q \otimes V^{\otimes (i-\nu-1)}) = \langle Q^{\perp} \rangle_{i+1}^{\perp}. \quad \square \end{aligned}$$

Theorem 5.2.9 (cf. [1, Theorem 2.6.1]). *Let $A = \bigoplus_{i=0}^{\infty} A_i = T(V)/\langle Q\rangle$ be a quadratic algebra, where $\langle Q\rangle = T(V)QT(V)$ is the ideal generated by the quadratic relation set $Q \subseteq V \otimes_{\Bbbk} V$. Then, A is Koszul if and only if the Koszul complex (5.4) of A is a resolution of $\Bbbk$.*

Proof. If the Koszul complex (5.4) of A is a resolution of $\Bbbk$, by Definition 4.1.1, A is Koszul.

Conversely, suppose that A is a Koszul algebra. Consider first the sequence:

$$A \otimes_{\Bbbk} V \xrightarrow{d_1} A \xrightarrow{\rho} \Bbbk \to 0, \tag{5.5}$$

where $d_1(a \otimes v) = av$ for $a \in A$ and $v \in V$, and $\rho\colon A \to A/A_+ \cong \Bbbk$ is the canonical augmentation map. We see that the sequence (5.5) is exact. Now, we apply induction on i to prove that the Koszul complex

$$\cdots \to K^{(i+1)} \xrightarrow{d_{i+1}} K^{(i)} \xrightarrow{d_i} K^{(i-1)} \xrightarrow{d_{i-1}} \cdots$$

is a resolution of $\Bbbk$. Suppose that $\ker d_{i-1} = \operatorname{im} d_i$ for all $i \in \{1, \cdots, p\}$. Then, we have an exact sequence:

$$0 \to \ker d_p \xrightarrow{\iota} K^{(p)} \xrightarrow{d_p} \cdots \to A \otimes V \xrightarrow{d_1} A \xrightarrow{\rho} \Bbbk \to 0,$$

where ι is the inclusion map. By Corollary 5.2.6, $\ker d_i = \bigoplus_{j\in\mathbb{Z}} Z_j^{(i)}$ is in A-grMod living only in degree $\geq i+1$. By Lemma 4.1.15,

$$\operatorname{Ext}_A^{p+1}\big(\Bbbk, \Bbbk(-\ell)\big) = \hom_A\big(\ker d_p, \Bbbk(-\ell)\big)$$

for all $\ell \in \mathbb{N}$. Since A is Koszul, by Proposition 4.1.17, $\operatorname{Ext}_A^{p+1}(\Bbbk, \Bbbk(-\ell)) = 0$ unless $\ell = p+1$. Hence, $\hom_A\big(\ker d_p, \Bbbk(-\ell)\big) = 0$ unless $\ell = p+1$. By Lemma 4.1.16, we see that $\ker d_p$ is generated by $Z_{p+1}^{(p)}$, i.e., $\ker d_p = AZ_{p+1}^{(p)}$. Then, by Lemma 5.2.8, for any $w \in Z_{p+1}^{(p)}$, we have $w \in \langle Q^\perp\rangle_{p+1}^\perp$. Recall that $d_{p+1}\colon A \otimes \langle Q^\perp\rangle_{p+1}^\perp \to A \otimes \langle Q^\perp\rangle_p^\perp$ is defined by $d_{p+1}(a \otimes w) = aw$ for $a \in A$ and $w \in \langle Q^\perp\rangle_{p+1}^\perp$. Hence, we have that

$$d_{p+1}\big(1_{\Bbbk} \otimes w\big) = w.$$

This implies that $\ker d_p \subseteq \operatorname{im} d_{p+1}$. From $d_p \circ d_{p+1} = 0$, we conclude that $\ker d_p = \operatorname{im} d_{p+1}$. This completes the induction step. $\square$

Example 5.2.10 (cf. [17, Lemma 3.7, p. 16])**.** Here we give an example of that a quadratic algebra is not Koszul. Let V be the vector space over $\Bbbk$ with basis $\{x, y\}$. Denote $\Bbbk\langle x, y\rangle = T(V)$. Consider the quadratic algebra $A = \Bbbk\langle x, y\rangle/\langle Q\rangle$, where Q is the set of quadratic relations given by $Q = \{y^2 + axy, yx\}$, with $a \in \Bbbk^\times = \Bbbk \setminus \{0\}$. We will show that the Koszul complex

$$K^\bullet\colon \cdots \to K^{(i+1)} \xrightarrow{d_{i+1}} K^{(i)} \xrightarrow{d_i} K^{(i-1)} \xrightarrow{d_{i-1}} \cdots$$

is not exact. Recall that $K^{(3)} = A \otimes_\Bbbk \langle Q^\perp\rangle_3^\perp$, where

$$\langle Q^\perp\rangle_3^\perp = (Q \otimes_\Bbbk V) \cap (V \otimes_\Bbbk Q).$$

If $v \in \langle Q^\perp\rangle_3^\perp$, then

$$\begin{aligned} v &= (a_1x + b_1y) \otimes (y^2 + axy) + (c_1x + e_1y) \otimes (yx) \\ &= (y^2 + axy) \otimes (a_2x + b_2y) + (yx) \otimes (c_2x + e_2y) \end{aligned}$$

for some $a_i, b_i, c_i, e_i \in \Bbbk$, $i = 1, 2$. This gives

$$\begin{aligned} v &= a_1xy^2 + a_1ax^2y + b_1y^3 + b_1ayxy + c_1xyx + e_1y^2x \\ &= a_2y^2x + b_2y^3 + a_2axyx + b_2axy^2 + c_2yx^2 + e_2yxy. \end{aligned}$$

Comparing the coefficients of every term, we obtain that

$$\begin{aligned} a_1 &= b_2a, \quad a_1a = 0, \quad b_1 = b_2, \\ b_1a &= e_2, \quad c_1 = a_2a, \quad e_1 = a_2. \end{aligned}$$

This gives $a_1 = c_2 = b_1 = b_2 = e_2 = 0$ and $c_1 = e_1a$. Hence, $v = e_1(y^2x + axyx)$. This implies that $\langle Q^\perp\rangle_3^\perp$ is a vector space over $\Bbbk$ generated by $y^2x + axyx$. Now, we show that $\operatorname{im} d_3 \subsetneq \ker d_2$ in the complex $K^\bullet$. Recall that $K^{(2)} = A \otimes_\Bbbk \langle Q^\perp\rangle_2^\perp$, where

$$\langle Q^\perp\rangle_2^\perp = (Q),$$

is a vector space over $\Bbbk$ generated by $Q = \{y^2 + axy, yx\}$. Since $A = T(V)/\langle Q\rangle$, we have that in A, $y^2 = -axy$ and hence, $-axy^2 = y^3 = -ayxy = -a(yx)y = 0$. Recall the definition (5.3) for the differential $d_2\colon A \otimes_\Bbbk (Q) \to A \otimes V$. We obtain that

$$\begin{aligned} d_2\big((-axy) \otimes (y^2 + axy)\big) &= d_2\big((-axy) \otimes \big((y + ax)y\big)\big) \\ &= -axy(y + ax) \otimes y = \big(-axy^2 - a^2x(yx)\big) \otimes y = 0. \end{aligned}$$

Hence, $(-axy) \otimes (y^2 + axy) \in \ker d_2$. But, since $\langle Q^\perp\rangle_3^\perp = \langle y^2x + axyx\rangle$, for $w \in K^{(3)} = A \otimes_\Bbbk \langle Q^\perp\rangle_3^\perp$, $w = \alpha \otimes b(y^2x + axyx)$ for some $b \in \Bbbk$ and $\alpha \in A$.

Recall the definition (5.3) for the differential $d_3\colon A\otimes_{\Bbbk} Q^{\perp}\rangle_3^{\perp} \to A\otimes_{\Bbbk}(Q)$. We obtain that

$$\begin{aligned} d_3(w) &= d_3\left(b\alpha\otimes(y^2x+axyx)\right) = (b\alpha y+b\alpha ax)\otimes yx \\ &= b(\alpha y+\alpha ax)\otimes yx. \end{aligned}$$

This implies that all the elements in $\operatorname{im} d_3$ are of the form $(\alpha y+\alpha ax)\otimes yx$ for some $\alpha\in A$. Note that y^2x+axy and yx are linearly independent over $\Bbbk$ in the space (Q) and by Remark 4.1.3, $A\otimes_{\Bbbk}(Q)$ is defined to be a free left A-module with basis $1\otimes_{\Bbbk}\mathcal{B}$, where $\mathcal{B}$ is a basis of the vector space $(Q)\cong\Bbbk^2$. Note that in A, $-axy\neq 0$ and $y+ax\neq 0$. Thus, $(-axy)\otimes(y^2+axy)\notin \operatorname{im} d_3$. It follows that $\operatorname{im} d_3 \subsetneqq \ker d_2$. Consequently, the Koszul complex $K^{\bullet}$ is not exact. By Theorem 5.2.9, this quadratic algebra A is not Koszul.

Lemma 5.2.11. $\dim A_i^! = \dim\left(\bigcap_{\nu=0}^{i-2}(V^{\otimes\nu}\otimes Q\otimes V^{\otimes(i-\nu-2)})\right)$.

Proof. Recall that

$$\dim\langle Q^{\perp}\rangle_i + \dim\langle Q^{\perp}\rangle_i^{\perp} = \dim(V^*)^{\otimes i}$$

and

$$\dim A_i^! = \dim(V^*)^{\otimes i} - \dim\langle Q^{\perp}\rangle_i.$$

The lemma follows. □

Proposition 5.2.12. *There is a canonical isomorphism of* $\Bbbk$*-vector spaces from* $(A_i^!)^*$ *to* $\langle Q^{\perp}\rangle_i^{\perp} = \bigcap_{\nu=0}^{i-2}(V^{\otimes\nu}\otimes Q\otimes V^{\otimes(i-\nu-2)})$.

Proof. Define $\mathrm{ev}_v\in(A_i^!)^*$ for every $v\in\langle Q^{\perp}\rangle_i^{\perp}\subseteq\left((V^*)^{\otimes i}\right)^*$ by

$$\mathrm{ev}_v(\bar{\varphi}) = v(\varphi).$$

for $\bar{\varphi}=\varphi+\langle Q^{\perp}\rangle_i\in A_i^! = (V^*)^{\otimes i}/\langle Q^{\perp}\rangle_i$ with $\varphi\in(V^*)^{\otimes i}$. Note that for every $\varphi\in\langle Q^{\perp}\rangle_i$, we have $v(\varphi)=0$ since $v\in\langle Q^{\perp}\rangle_i^{\perp}$. Hence, ev_v is well-defined. Then, define a $\Bbbk$-linear map $\Psi\colon\langle Q^{\perp}\rangle_i^{\perp}\to(A_i^!)^*$ by

$$\Psi(v) = \mathrm{ev}_v.$$

Suppose that $v\in\ker\Psi$. Then, $\mathrm{ev}_v=\Psi(v)=0$. This implies that $v(\varphi)=\mathrm{ev}_v(\bar{\varphi})=0$ for all $\bar{\varphi}=\varphi+\langle Q^{\perp}\rangle_i\in A_i^!$. If $\langle Q^{\perp}\rangle_i=(V^*)^{\otimes i}$, then $\langle Q^{\perp}\rangle_i^{\perp}=\left((V^*)^{\otimes i}\right)^{\perp}=0$, consequently, $v=0$. Now, assume that $\langle Q^{\perp}\rangle_i\neq(V^*)^{\otimes i}$. Then, for each nonzero $v\in\langle Q^{\perp}\rangle_i^{\perp}$, there is $\varphi\notin\langle Q^{\perp}\rangle_i$ such that $v(\varphi)=1_{\Bbbk}$, hence, $v \notin \ker \Psi$. This contradiction shows that $\ker \Psi = 0$, i.e., Ψ is injective. Then, we have $\Psi\left(\langle Q^{\perp}\rangle_i^{\perp}\right) \subseteq (A_i^!)^*$. By Lemma 5.2.11, we have

$$\dim(A_i^!)^* = \dim A_i^! = \dim\langle Q^{\perp}\rangle_i^{\perp}.$$

Hence, Ψ is surjective. Thus,

$$\Phi = \Psi^{-1} \tag{5.6}$$

is a canonical isomorphism from $(A_i^!)^*$ to $\bigcap_{\nu=0}^{i-2}(V^{\otimes\nu} \otimes Q \otimes V^{\otimes(i-\nu-2)})$. $\square$

We obtain a complex that is isomorphic to the complex (5.4):

$$\cdots \to A\otimes_{\Bbbk}(A_i^!)^* \xrightarrow{D_i} A\otimes_{\Bbbk}(A_{i-1}^!)^* \to \cdots \to A\otimes_{\Bbbk}(A_2^!)^* \to A\otimes_{\Bbbk}V \to A, \tag{5.7}$$

with the differential $D_i = (\mathrm{id}_A \otimes \Phi_{i-1})^{-1} \circ d_i \circ (\mathrm{id}_A \otimes \Phi_i)$, where d_i is defined by (5.3) and Φ_i is the canonical isomorphism from $(A_i^!)^*$ to $\bigcap_{\nu=0}^{i-2}(V^{\otimes\nu} \otimes Q \otimes V^{\otimes(i-\nu-2)})$, defined in Proposition 5.2.12.

5.3 Koszul Duality

Let $A = \bigoplus_{i=0}^{\infty} A_i = T(V)/\langle Q\rangle$ be a quadratic algebra, where $\langle Q\rangle = T(V)QT(V)$ is the ideal generated by the quadratic relation $Q \subseteq V \otimes_{\Bbbk} V$. Let $A^! = \bigoplus_{i=0}^{\infty} A_i^! = T(V^*)/\langle Q^{\perp}\rangle$ be the quadratic dual algebra of A. Consider the $\mathbb{N}\times\mathbb{N}$-graded space

$$A \otimes_{\Bbbk} (A^!)^* = \bigoplus_{i,j\geq 0} A_i \otimes_{\Bbbk} (A_j^!)^*.$$

Applying the functor $\mathrm{Hom}_{\Bbbk}(-,\Bbbk)$ to every component of above space, we obtain another $\mathbb{N}\times\mathbb{N}$-graded space

$$A^! \otimes_{\Bbbk} A^* = \bigoplus_{i,j\geq 0} A_j^! \otimes_{\Bbbk} A_i^*.$$

We recall the following fact of linear algebra.

Lemma 5.3.1. *Let*

$$\cdots \to V_{i+1} \xrightarrow{h_{i+1}} V_i \xrightarrow{h_i} V_{i-1} \to \cdots$$

be an exact sequence of finite-dimensional vector spaces over a field $\Bbbk$. Then, the sequence of dual spaces

$$\cdots \leftarrow V_{i+1}^* \xleftarrow{h_{i+1}^*} V_i^* \xleftarrow{h_i^*} V_{i-1}^* \leftarrow \cdots$$

is also exact, where $h_i^* \colon V_{i-1}^* \to V_i^*$ is given by

$$h_i^*(\alpha) = \alpha \circ h_i$$

for $\alpha \in V_{i-1}^* = \mathrm{Hom}_{\Bbbk}(V_{i-1}, \Bbbk)$.

Proposition 5.3.2 (cf. [1, Proposition 2.9.1])**.** *If A is Koszul, then $A^!$ is Koszul.*

Proof. Since A is Koszul, noting that the complex (5.7) is isomorphic to the complex (5.4), by Theorem 5.2.9, we have the exact sequence:

$$\left(\bigoplus_{i,j\geq 0} A_i \otimes_{\Bbbk} (A_j^!)^*\right)_{\bullet} : \cdots \to A \otimes_{\Bbbk} (A_j^!)^* \xrightarrow{D_i} A \otimes_{\Bbbk} (A_{j-1}^!)^* \to \cdots .$$

By the definition (5.3) of d_j, the differential d_j maps $A_{i-j} \otimes_{\Bbbk} \langle Q^{\perp}\rangle_j^{\perp}$ to $A_{i-j+1} \otimes_{\Bbbk} \langle Q^{\perp}\rangle_{j-1}^{\perp}$. So, the differential D_j has degree $(1, -1)$. Note that

$$D_{i-\ell+1} \colon A_{\ell-1} \otimes_{\Bbbk} (A_{i-\ell+1}^!)^* \to A_\ell \otimes_{\Bbbk} (A_{i-\ell}^!)^*.$$

By Lemma 5.3.1, we obtain an exact sequence $\left(\left(\bigoplus_{i,j\geq 0} A_i \otimes_{\Bbbk} (A_j^!)^*\right)_{\bullet}\right)^* = \left(\bigoplus_{i,j\geq 0} A_j^! \otimes_{\Bbbk} A_i^*\right)_{\bullet}$:

$$\cdots \to \mathrm{Hom}_{\Bbbk}\left(A_\ell \otimes_{\Bbbk} (A_{i-\ell}^!)^*, \Bbbk\right) \xrightarrow{D_\ell^*} \mathrm{Hom}_{\Bbbk}\left(A_{\ell-1} \otimes_{\Bbbk} (A_{i-\ell+1}^!)^*, \Bbbk\right) \to \cdots$$

where the differential is given by

$$\begin{aligned} D_\ell^* \colon A_{i-\ell}^! \otimes_{\Bbbk} A_\ell^* &\to A_{i-\ell+1}^! \otimes_{\Bbbk} A_{\ell-1}^*, \\ f &\mapsto f \circ D_{i-\ell+1}. \end{aligned}$$

Thus, the differential D_ℓ^* has degree $(1, -1)$. This sequence is precisely the free $A^!$-module resolution:

$$\cdots \to A^! \otimes_{\Bbbk} A_\ell^* \xrightarrow{D_\ell^*} A^! \otimes_{\Bbbk} A_{\ell-1}^* \to \cdots \to A^! \otimes V^* \xrightarrow{D_1^*} A^! \to \Bbbk \to 0.$$

Hence, $A^!$ is Koszul. $\square$

Corollary 5.3.3. *The quadratic algebra A is Koszul if and only if its dual $A^!$ is Koszul.*

Proof. This follows from the fact that $(A^!)^! = A$ and Proposition 5.3.2. $\square$

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