A CLASS OF CONTRACTIVITY PRESERVING HERMITE–BIRKHOFF–TAYLOR HIGH ORDER TIME DISCRETIZATION METHODS

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Abstract

In this thesis, we study the contractivity preserving, high order, time discretization methods for solving non-stiff ordinary differential equations. We construct a class of one-step, explicit, contractivity preserving, multi-stage, multi-derivative, Hermite-Birkhoff-Taylor methods of order $p = 5, 6, \ldots, 15$, that we denote by CPHBT, with nonnegative coefficients by casting s-stage Runge-Kutta methods of order 4 and 5 with Taylor methods of order $p - 3$ and $p - 4$, respectively.

The constructed CPHBT methods are implemented using an efficient variable step algorithm and are compared to other well-known methods on a variety of initial value problems. The results show that CPHBT methods have larger regions of absolute stability, require less function evaluations and hence they require less CPU time to achieve the same accuracy requirements as other methods in the literature. Also, we show that the contractivity preserving property of CPHBT is very efficient in suppressing the effect of the propagation of discretization errors when a long-term integration of a standard N-body problem is considered.

The formulae of 49 CPHBT methods of various orders are provided in Butcher form.

Résumé

Dans cette thèse, nous étudions des solveurs numériques d'ordre élevé, qui préserve la propriété de contractivité pour résoudre des équations différentielles ordinaires non-raides. Nous construisons une classe de méthodes explicites, multi-étage, multi-dérivées, Hermite-Birkhoff-Taylor d'ordre $p = 5, 6, \ldots, 15$, à un pas. Ces méthodes sont à coefficients positifs ou nuls et combinent des méthodes Runge-Kutta à s-étages d'ordre 4 et 5 et des méthodes de Taylor d'ordre $p-3$ et $p-4$, respectivement.

Ces méthodes sont implémentées en utilisant un algorithme efficace à pas variable. Nous les comparons aux autres méthodes bien connues pour des problèmes standards. Nos résultats montrent que les méthodes CPHBT ont des régions de stabilité absolue plus grandes, qu'elles nécessitent moins d'évaluations de fonction et, donc, sont plus rapides que les autres méthodes connues. Nous montrons, de plus, que la propriété de préservation de la contractivité de CPHBT supprime très efficacement l'effet de la propagation des erreurs de discrétisation dans les intégrations à long terme des problèmes standards à N-corps.

Les formules des 49 méthodes CPHBT de divers ordres sont fournies sous la forme Butcher.

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Dedication

I want to dedicate this work to Dr. Rémi Vaillancourt. Aside from being my supervisor, he was like a grandfather to me. I have never seen him without a smile on his face. I wish him a complete and quick recovery.

To my family and friends. Without you, this work wouldn't be possible...

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Chapter 1

Introduction

Differential equations can be traced back to Isaac Newton [39] when he investigated the solution of the differential equation [22]

$$\frac{dy}{dx} = 1 - 3x + y + x^2 + xy \tag{1.0.1}$$

and obtained a solution recursively as an infinite series. Since then, enormous efforts were directed towards solving differential equations arising in different fields such as chemistry, physics, biology, astronomy and weather/climate prediction. However, only a small percentage of the differential equations in the literature can be solved analytically. Hence, numerical methods play a vital role in obtaining accurate approximations to the exact solution of differential equations.

Huge efforts were directed towards developing, implementing and analyzing various numerical methods satisfying different accuracy and stability properties. In this thesis, we establish a new class of high order, explicit, one-step, multi-stage, multi-derivative, Hermite-Birkhoff-Taylor time discretizations satisfying certain stability properties, in particular, the contractivity preserving property. These methods are denoted by CPHBT.

1.1 The advantages of higher order methods

In general, high order methods generate numerical solutions with higher accuracy compared to lower order methods. However, in applications, the required accuracy of the numerical solution is usually known and it is in general not extremely high. Then, why do we search and derive higher order methods? There are at least two good reasons:

- The number of steps required: Let M_1 be a lower order method, with fixed step-size for simplicity, such that the global error $\mathrm{GE}(M_1)=\max_x \|y_{\mathrm{exact}}(x)-y_{\mathrm{num}}(x)\|$ is of order $\mathcal{O}(h^{m_1})$. Similarly, let M_2 be a higher order method with $\mathrm{GE}(M_2)$ of order $\mathcal{O}(h^{m_2})$ where $m_2 > m_1$. If we require an accuracy $\mathrm{TOL} = 10^{-n}$, then M_1 and M_2 require step sizes h_{m_1} and h_{m_2}, respectively, such that

$$h_{m_1} \approx 10^{\frac{-n}{m_1}} < h_{m_2} \approx 10^{\frac{-n}{m_2}}.$$

  Hence, the higher order method M_2 will require fewer integration steps than the lower order method M_1 to achieve the same accuracy 10^{-n} for the same problem configuration. Therefore, if the higher order method M_2 is established efficiently with minimal function evaluations, then it will require less CPU time to achieve the required accuracy in comparison with the lower order method M_1. This can be a very important property when dealing with real time applications or when the computing resources are limited and hence, minimizing the computing demand is essential.

- Long-term integration problems: Many applications require very large integration intervals to study the long-term behaviour/results of the model under investigation. For instance, astronomical simulations require extremely large integration interval up to 1 billion years [50] where the time unit is days, i.e., $t \in [0, 3.65242 \times 10^{11}]$. Indeed, solving problems over long integration periods will result in losing accuracy due to the propagation of discretization errors. As an example, consider the modified oscillatory initial value problem [27]

$$y' = 2y\cos(x) \qquad y(0) = 1, \tag{1.1.1}$$

with exact solution $y(x) = e^{2\sin(x)}$. We solved problem (1.1.1) using a 4-stage Runge-Kutta method of order 4, RK(4,4), and a 13-stage Runge-Kutta method of order 8, RK(13,8), with coefficients given in Table (1.1). We choose fixed step sizes such that both methods use the same number of function evaluations. In Figure (1.1), we plot the numerical solution obtained by RK(4,4) and RK(13,8) with the exact solution over different integration intervals. For instance, if we

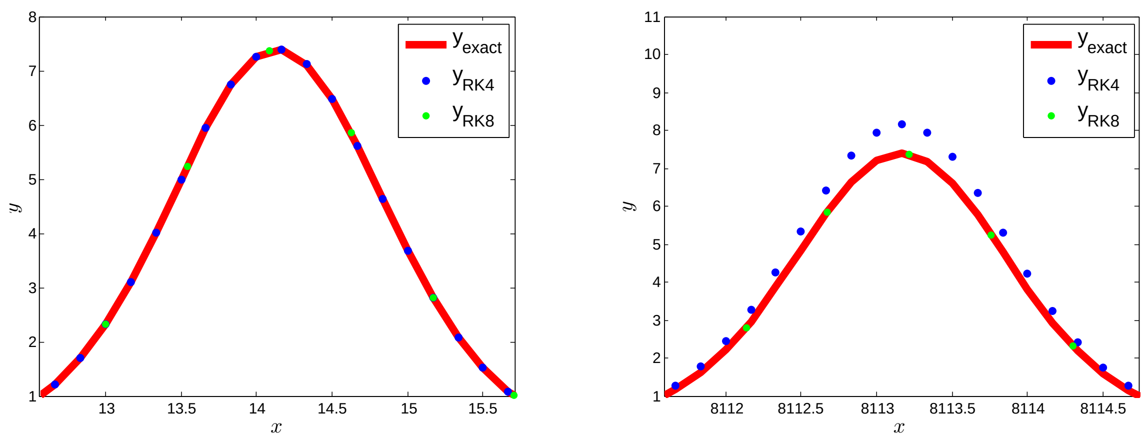


Figure 1.1: The numerical solutions obtained by RK(4,4) and RK(13,8) over different integration intervals using the same number of function evaluations.

require accuracy TOL $= 10^{-4}$, then RK(4,4) method is sufficient over the interval $t \in [0, 5\pi]$. However, when we consider a relatively longer integration interval $t \in [0, 2580\pi]$, RK(4,4) method loses accuracy due to the propagation of discretization errors while RK(13,8) method still satisfies the required tolerance as shown in Figure(1.1). This illustrates the need of higher order methods when a long-term integration problem is considered.

RK(4,4)

$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

RK(13,8)

$\frac{1}{18}$	$\frac{1}{18}$												
$\frac{1}{12}$	$\frac{1}{48}$	$\frac{1}{16}$											
$\frac{1}{8}$	$\frac{1}{32}$	0	$\frac{3}{32}$										
$\frac{5}{16}$	$\frac{5}{16}$	0	$-\frac{75}{64}$	$\frac{75}{64}$									
$\frac{3}{8}$	$\frac{3}{80}$	0	0	$\frac{3}{16}$	$\frac{3}{20}$								
$\frac{59}{400}$	$\frac{29443841}{614563806}$	0	0	$\frac{77736538}{692538347}$	$-\frac{28693883}{1125000000}$	$\frac{23124283}{1800000000}$							
$\frac{93}{200}$	$\frac{16016141}{946692911}$	0	0	$\frac{61564180}{158732637}$	$\frac{22789713}{633445777}$	$\frac{545815736}{2771057229}$	$-\frac{180193667}{1043307555}$						
$\frac{5490023248}{9719169821}$	$\frac{39632708}{573591083}$	0	0	$-\frac{433636366}{683701615}$	$-\frac{421739975}{2616292301}$	$\frac{100302831}{723423059}$	$\frac{790204164}{839813087}$	$\frac{800635310}{3783071287}$					
$\frac{13}{20}$	$\frac{246121993}{1340847787}$	0	0	$-\frac{37695042795}{15268766246}$	$-\frac{309121744}{1061227803}$	$-\frac{12992083}{490766935}$	$\frac{6005943493}{2108947869}$	$\frac{393006217}{1396673457}$	$\frac{123872331}{1001029789}$				
$\frac{1201146811}{1299019798}$	$-\frac{1028468189}{846180014}$	0	0	$\frac{8478235783}{508512852}$	$\frac{1311729495}{1432422823}$	$-\frac{10304129995}{1701304382}$	$-\frac{48777925059}{3047939560}$	$\frac{15336726248}{1032824649}$	$-\frac{45442868181}{3398467696}$	$\frac{3065993473}{597172653}$			
1	$\frac{185892177}{718116043}$	0	0	$-\frac{3185094517}{667107341}$	$-\frac{477755414}{1098053517}$	$-\frac{703635378}{230739211}$	$\frac{5731566787}{1027545527}$	$\frac{5232866602}{850066563}$	$-\frac{4093664535}{808688257}$	$\frac{3962137247}{1805957418}$	$\frac{65686358}{487910083}$		
1	$\frac{403863854}{491063109}$	0	0	$-\frac{5068492393}{434740067}$	$-\frac{411421997}{543043805}$	$\frac{652783627}{914296604}$	$\frac{11173962825}{925320556}$	$-\frac{13158990841}{6184727034}$	$\frac{3936647629}{1978049680}$	$-\frac{160528059}{685178525}$	$\frac{248638103}{1413531060}$	0	
	$\frac{14005451}{335480064}$	0	0	0	0	$-\frac{59238493}{1068277825}$	$\frac{181606767}{758867731}$	$\frac{561292985}{797845732}$	$-\frac{1041891430}{1371343529}$	$\frac{760417239}{1151165299}$	$\frac{118820643}{751138087}$	$\frac{1}{4}$	

Table 1.1: The coefficients of RK(4,4) and RK(13,8) in Butcher tables.

1.2 Contractivity preserving methods

In 1988, Shu in [52] and Shu and Osher in [53] established strong stability preserving Runge-Kutta methods, denoted by SSPRK. The derivation of such methods was motivated by the observation that the solution of some partial differential equations of the form

$$u_x = \Phi(x, u, u_y, u_{yy}, \ldots) \tag{1.2.1}$$

such as hyperbolic conservation laws, satisfies the monotonicity property

$$\|u(x + \Delta x)\| \leq \|u(x)\|, \tag{1.2.2}$$

where $\|\cdot\|$ is some norm such as the total variation norm [21, 52, 53]. Strong stability preserving high order time discretization methods are designed for the time evolution of hyperbolic partial differential equations with discontinuous solutions satisfying a discrete form of inequality (1.2.2) as follows:

$$\|u_{n+1}\| \leq \|u_n\|, \tag{1.2.3}$$

where u_n, u_{n+1} are numerical approximations of $u(x_n)$ and $u(x_n + \Delta x)$, respectively. Assuming that the SSP property (1.2.3) holds for the forward Euler method

$$u_{n+1} = u_n + \Delta x_{\mathrm{FE}} f(x_n, u_n), \tag{1.2.4}$$

Shu and Osher show that the SSP property (1.2.3) holds for the designed SSPRK methods by expressing them as convex combination of the forward Euler method with a modified step size, $\Delta x \leq c\Delta x_{\mathrm{FE}}$, where c is called the SSP coefficient. Since then, significant efforts were directed towards deriving and optimizing SSP methods [52, 53, 20, 17, 28, 45, 47, 46, 55]. The designed SSP methods were shown to be more stable and suppressed the spurious oscillations and overshoot that may occur at discontinuities in comparison with other non-SSP methods as shown by Gottlieb and Shu in [20]. Also, most SSP methods have the same form and the same computational cost as traditional ODE solvers.

In this thesis, we are interested in the solution of non-stiff initial value problems of the form:

$$\frac{dy}{dx} = f(x, y(x)), \qquad y(x_0) = y_0, \tag{1.2.5}$$

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$. The exact solution of some IVP of the form (1.2.5) naturally preserve the contractivity property

$$\|y(x + \Delta x) - \tilde{y}(x + \Delta x)\| \leq \|y(x) - \tilde{y}(x)\|, \tag{1.2.6}$$

where $y(x)$ and $\tilde{y}(x)$ are two solutions with different neighbouring initial conditions $y(x_0)$ and $\tilde{y}(x_0)$, respectively. Throughout this work, we assume that f is a sufficiently smooth function such that for each $x_0 \in \mathbb{R}$ and $y_0 = y(x_0) \in \mathbb{R}^N$, problem (1.2.5) has a unique solution $y : [x_0, \infty) \to \mathbb{R}^N$. Also, we suppose that there exists a norm $\| \cdot \|$ on $\mathbb{R}^N$ such that inequality (1.2.6) holds. Under these assumptions, the initial value problem (1.2.5) is said to be *dissipative* [31].

Hence, when we design a numerical method, we require that the numerical solution of the IVP (1.2.5) satisfies a discrete form of the contractivity preserving property (1.2.6) as follows:

$$\|y_{n+1} - \tilde{y}_{n+1}\| \leq \|y_n - \tilde{y}_n\|, \tag{1.2.7}$$

where y_n and $\tilde{y}_n$ are two numerical solutions with different neighbouring initial conditions $y_0 = y(x_0)$ and $\tilde{y}_0 = \tilde{y}(x_0)$, respectively. Moreover, the contractivity preserving property (1.2.7) is a desired property to reduce the effect of the propagation of discretization errors. Since all numerical methods introduce discretization errors, one can consider the introduction of such errors to the numerical solution as jumping from one solution $\{y_n\}_{n=0}^{N_{\text{end}}}$ to another neighbouring perturbed solution $\{\tilde{y}_n\}_{n=0}^{N_{\text{end}}}$. Then, the contractivity preserving property (1.2.7) will guarantee that the perturbed numerical solution $\{\tilde{y}_n\}_{n=0}^{N_{\text{end}}}$ by the discretization errors will not "wander away" from the exact solution, i.e., it will minimize the effect of the propagation of the discretization errors [33]. Kraaijevanger investigated in [31] the contractivity preserving Runge-Kutta methods and established the necessary and sufficient conditions for Runge-Kutta methods to be contractive. However, the designed contractive Runge-Kutta methods achieved limited order of accuracy with order barrier equals to 6 ($p \leq 6$) for implicit contractive Runge-Kutta methods. This order barrier is even more severe for explicit contractive Runge-Kutta methods ($p \leq 4$) [31, page 516]. One of the goals of this work is to overcome this order barrier by casting Runge-Kutta methods with the Taylor series method.

The Taylor series method is a successful classical method that has been investigated extensively. It has been shown that it is a competitive candidate in astronomical simulations [3], solving general problems [8, 29, 34], sensitivity analysis of ODEs/DAEs [2] and validating solutions of ODEs [38, 25]. The main computational cost in solving ODEs by the Taylor method in terms of function evaluations and CPU time lies in the repeated evaluation of the Taylor coefficients of the functions involved. In [11], Deprit and Zahar showed that recursive computation of Taylor coefficients is very effective in achieving high accuracy with little computing time and large step sizes. Hence, the addition of higher order derivatives to Runge-Kutta methods can be implemented efficiently using the notion of automatic differentiation [34]. In 2013, Nguyen-Ba *et al.* [40], extended the forward Euler method used by Shu in [52] and Shu and Osher in [53], to the expanded 9-derivative series:

$$y_{n+1} = y_n + \Delta x f(x_n, y_n) + \sum_{m=2}^{9} \eta_m \Delta x^m f^{(m-1)}(x_n, y_n), \tag{1.2.8}$$

and designed a two-step, contractivity preserving, explicit, 9-derivative, Hermite-Obrechkoff method of order 13, denoted by HO(13), by expressing the method as a convex combination of the expanded 9-derivative series (1.2.8). The designed 2-steps, contractivity preserving (CP) HO(13) method has a larger region of absolute stability compared to the Taylor method of order 13, T(13). Also, it reduces the number of high order derivatives required by 4 compared to T(13) and hence it requires less CPU time to achieve the same accuracy. Moreover, the CP HO(13) method is successful in suppressing the error growth in long-term integration of Kepler's two-body problem with eccentricity $\varepsilon = 0.5$ [40]. As shown in Figure (1.2), Nguyen-Ba *et al.* plot the relative error in position and energy of HO(13) compared to Taylor method of order 13 over an integration interval of 10000 periods. It was shown that the contractivity preserving property increases the efficiency of the designed 2-steps, explicit, 9-derivatives, HO(13) method.

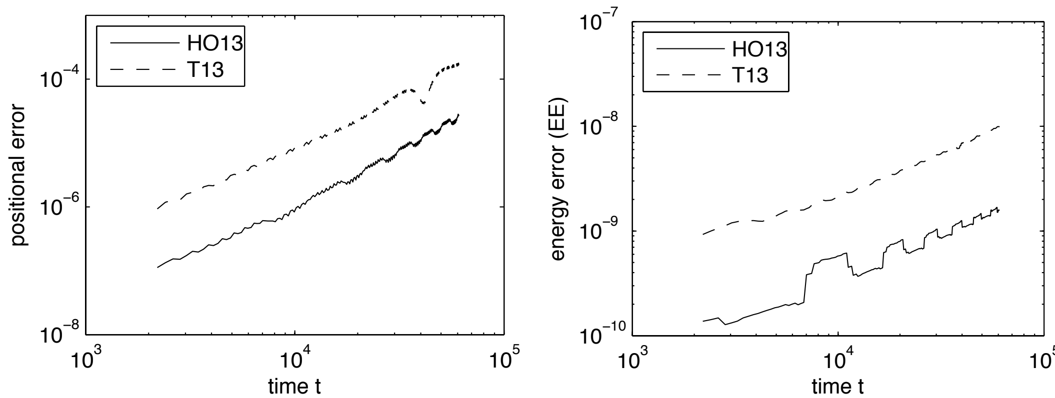


Figure 1.2: The propagation of error in position (left) and energy error (right) for Kepler's two-body problem.

In this work, we design one-step methods by the addition of off-step points, where only the derivative $y'(x)$ is considered, and the order of high order derivatives $y^{(j)}(x_n)$ necessary is reduced. The CPHBT methods take advantage of this fact. The first set of newly designed CP methods are obtained by casting s-stage Runge-Kutta methods of order 4, RK(s,4) for $s = 5, 6$ with Taylor series methods of order d, T(d) for $d = 2, 3, \ldots, 13$.

Also, to break the Kraaijevanger order barrier [31, page 516], we construct the second set of newly designed CP methods. The construction casts s-stage Runge-Kutta methods of order 5, RK(s,5), $s = 7, 8, 9, 10$ with Taylor series methods of order $p - 4$, T($p - 4$).

These two sets of newly designed CP methods incorporate a function evaluation at each off-step point similar to Runge-Kutta methods and high order derivatives information as in Taylor series methods to achieve better performance than widely used methods existing in the literature including Runge-Kutta and Taylor series methods. Moreover, the designed methods overcome the order barriers of CP RK methods and require significantly less high order derivatives compared to the Taylor series methods of the same order. In addition, a variable step algorithm with error estimation formula is used to minimize the number of function evaluations required to achieve the user defined tolerance, TOL, and to optimize the performance of CPHBT(d, s, p). To test and analyze the performance of the designed methods, we consider more than 30 test problems consisting of single equations, small systems, orbit equations, higher order equations and Hamiltonian problems using C++, Fortran and MATLAB codes. We show that the designed CPHBT(d, s, p) methods require significantly less step points, less function evaluations and less CPU time to achieve the user defined tolerance compared to well known, widely used ODEs solvers such as the Dormand-Prince Runge Kutta pair DP(8,7)13M [43], Taylor series methods of order p, T(p), and the Taylor method introduced by Martin Lara, T(p)L [34]. The designed methods have significantly larger regions of absolute stability compared to other methods of the same order. Also, the contractivity preserving property minimizes the growth of discretization errors when long-term integration problems are considered. These results make our methods competitive candidates for astronomical computations [3].

1.3 Thesis outline

This chapter included a brief introduction and the motivation together with the contributions of this work. The rest of the thesis can be outlined as follows:

In Chapter 2, we include a brief background of the theory and the definitions to make this work self contained. We also discuss and describe the Runge-Kutta methods, Taylor methods, automatic differentiation and we list the DETEST problems [27] used in this work.

In Chapter 3, we present an introduction to the newly designed methods and a detailed derivation of the order conditions of $\text{CPHBT}_{\text{RK4}}$ methods by means of rooted trees, B-series, elementary differentials and elementary weights. Moreover, we formulate and prove the contractivity preserving property and represent the CPHBT methods in different forms to facilitate and simplify the optimization problem. Finally, we formulate the optimization problem and we construct 24 new $\text{CPHBT}_{\text{RK4}}$ methods.

In Chapter 4, we test, investigate and compare the newly designed methods to other well known methods to show the efficiency, accuracy and stability of the designed CPHBT methods. We also study the propagation of discretization errors of the CPHBT methods when applied to a long-term integration problem.

In Chapter 5, we present the second set of methods, $\text{CPHBT}_{\text{RK5}}$, and we derive the order conditions and formulate the optimization process. Also, we study the gain in the contractivity preserving coefficients, c_{cp}, by analyzing the efficient CP coefficient, $c_{\text{cp}}^{\text{eff}}$, compared to $\text{CPHBT}_{\text{RK4}}$ presented in Chapter 4.

In Chapter 6, we present some numerical simulations and results by comparing $\text{CPHBT}_{\text{RK5}}$ to other methods including $\text{CPHBT}_{\text{RK4}}$. We also study the propagation of discretization errors compared to other methods including $\text{CPHBT}_{\text{RK4}}$ and a well-known, well tested Runge-Kutta-Nyström pair designed by Philip Sharp in 2013 [49].

Finally, in Appendices A and B, we list the coefficients of 24 $\text{CPHBT}_{\text{RK4}}$ methods constructed in Chapter 3 and of 27 $\text{CPHBT}_{\text{RK5}}$ methods constructed in Chapter 5, respectively.

1.4 Contributions

In this section, we list the contributions of this thesis. We investigate and test our results rigorously and we are hoping that these results will be a great addition to the field of ODE solvers in general and to the contractivity preserving time discretization field in particular. Our main contribution is the derivation of two sets of new one-step, explicit, multi-stage, multi-derivatives, contractivity preserving, HBT methods. More precisely, our contributions can be summarized as follows:

1. We present the formulae of the new variable step, explicit, s-stage, d-derivative, one-step, contractivity preserving Hermite-Birkhoff-Taylor methods in different forms such as the Butcher form, the Shu-Osher form and the Canonical Shu-Osher form in a compact vector notation.

2. We formulate the contractivity preserving property of CPHBT(d, s, p) presented in two theorems utilizing the properties of a modified version of the Canonical Shu-Osher form.

3. We derive the order condition of CPHBT$_{\mathrm{RK4}}(d, s, p)$ and CPHBT$_{\mathrm{RK5}}$ (d, s, p) methods of orders $p = 5, 6, \ldots, 15$. Also, establishing the elementary weights of the whole class of CPHBT that can be used recursively to obtain the order conditions of any CPHBT method by means of rooted trees, B-series, elementary weights and elementary differentials.

4. We formulate the nonlinear optimization problem and obtaining the nonnegative coefficients $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{\gamma}_0)$ of 23 new optimal CPHBT$_{\mathrm{RK4}}(d, s, p)$ methods for $d = 2, 3, \ldots, 13$, $p = 5, 6, \ldots, 15$ and $s = 5, 6$.

5. We formulate the nonlinear optimization problem and obtaining the nonnegative coefficients $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{\gamma}_0)$ of 26 new optimal CPHBT$_{\mathrm{RK5}}(d, s, p)$ methods for $d = 2, 3, \ldots, 10$, $p = 6, 7, \ldots, 14$ and $s = 7, 8, 9, 10$.

6. We study the performance of the two new sets of methods, CPHBT$_{\mathrm{RK4}}$ and CPHBT$_{\mathrm{RK5}}$, compared to well known and widely used methods applied to more than 30 test problems by:

(a) Showing that, in general, the designed methods have large regions of absolute stability and a fairly good percentage efficiency gain (PEG) in terms of the number of steps, number of function evaluations and CPU time compared to other well known methods of the same order.

(b) Analyzing the maximum global error (MGE) and the maximum global energy error (MGEE) for various test problems.

(c) Testing the performance of the variable step algorithm by considering the mean, median and standard deviation of the difference $|\mathrm{MGE} - \mathrm{TOL}|$ of 21 different test problems over a range of user defined tolerances.

(d) Analyzing the contractivity preserving coefficients percentage efficiency gain (c_{cp} PEG) and the effective contractivity preserving coefficient, $c_{\mathrm{cp}}^{\mathrm{eff}}$.

(e) Investigating and analyzing the propagation of discretization errors of the numerical solution in long-term integration of a standard N-body problem (interval lengths of up to 800,000 periods).

Chapter 2

Preliminary background and notations

In this chapter we will introduce briefly the background material used in this thesis. We will follow the notations and work presented in [18, 22, 30, 33].

2.1 Runge-Kutta methods

2.1.1 Formulation and classification

An s-stage Runge-Kutta method is written in the Butcher form as follows:

$$\begin{aligned} Y_i &= y_n + \Delta x \sum_{j=1}^{s} a_{ij} F_j, \quad i = 1, 2, \ldots, s, \\ y_{n+1} &= y_n + \Delta x \sum_{j=1}^{s} b_j F_j, \end{aligned} \tag{2.1.1}$$

where a_{ij}, b_j and c_j are the Runge-Kutta coefficients, y_{n+1} is an approximation of $y(x_{n+1})$, Y_j is the stage value and $F_j = f\left(x_n + \Delta x c_j, Y_j\right)$ is the stage derivative for $j = 1, 2, \ldots, s$. In the literature, Runge-Kutta methods are represented by their coefficients summarized in a table called the Butcher tableau as follows:

$$\begin{array}{c|cccc}
c_1 & a_{1,1} & a_{1,2} & \dots & a_{1,m} \\
c_2 & a_{2,1} & a_{2,2} & \dots & a_{2,m} \\
\vdots & \vdots & \vdots & & \vdots \\
c_m & a_{m,1} & a_{m,2} & \dots & a_{m,m} \\
\hline
 & b_1 & b_2 & \dots & b_m
\end{array}$$

The Runge-Kutta method (2.1.1) is classified as:

- Explicit: if $a_{ij} = 0$ for all $j \geq i$, $i = 1, 2, \dots, m$, i.e, the matrix $A = [a_{ij}] \in \mathcal{M}_{m,m}(\mathbb{R})$ is strictly lower triangular.
- Semi-implicit: if $a_{ij} = 0$ for all $j > i$, $i = 1, 2, \dots, m$, i.e, A is a lower triangular matrix.
- Implicit: if $a_{ij} \neq 0$ for some $j > i$.

Explicit Runge-Kutta methods are widely used numerical methods for solving nonstiff problems since they are efficient and easy to implement if the function f is not computationally costly. The main advantage of explicit Runge-Kutta methods is that each stage depends only on previously computed stage values. However, these methods have bounded stability regions and hence they are not suitable for stiff problems. On the other hand, implicit and semi-implicit Runge-Kutta methods require solving a nonlinear system to evaluate the current stage. Indeed, implicit Runge-Kutta methods are expensive to implement, but they can have unbounded regions of stability and hence they are suitable for solving stiff problems.

2.1.2 Order of accuracy

Let us consider the autonomous initial value problem:

$$\frac{dy}{dx} = f(y(x)), \qquad y(x_0) = y_0. \tag{2.1.2}$$

where the function $f : \mathbb{R}^N \to \mathbb{R}^N$. The Taylor expansion of the exact solution is given by:

$$
\begin{aligned}
y(x_1) = y(x_0) + hf(y(x_0)) + \frac{1}{2!}h^2 f'(y(x_0))f(y(x_0)) \\
+ \frac{1}{3!}h^3\Big(f''(y(x_0))\big(f(y(x_0)), f(y(x_0))\big) + \big(f'(y(x_0))\big)^2 f(y(x_0))\Big) + \mathcal{O}(h^4),
\end{aligned}
\tag{2.1.3}
$$

where $x_1 = x_0 + h$. Similarly, the Taylor expansion of the numerical solutions generated by the Runge-Kutta method (2.1.1) is given by:

$$
\begin{aligned}
y_1 =& y_0 + h\big(\sum_i b_i\big) f_0 + h^2\big(\sum_i b_i c_i\big) f_0' f_0 + \frac{1}{2}\big(\sum_i b_i c_i^2\big) h^3 f_0''(f_0, f_0) \\
&+ \big(\sum_{ij} b_i a_{ij} c_j\big) h^3 \big(f_0'\big)^2 f_0 + \mathcal{O}(h^4),
\end{aligned}
\tag{2.1.4}
$$

where f_0, f_0' and f_0'' are numerical approximations to $f(y(x_0))$, $f'(y(x_0))$ and $f''(y(x_0))$, respectively. Then, by forcing the Taylor expansion of the numerical solution to be equal to the Taylor expansion of the exact solution, we obtain the order condition of Runge-Kutta method up to order $p = 3$

$$
\begin{aligned}
\sum_i b_i = 1, \qquad \sum_i b_i c_i = \frac{1}{2}, \\
\sum_i b_i c_i^2 = \frac{1}{3}, \qquad \sum_{ij} b_i a_{ij} c_j = \frac{1}{6}.
\end{aligned}
\tag{2.1.5}
$$

Indeed, this approach involves tedious computations for higher order methods as A. Cayley (1857) said: "But without a more convenient notation, it would be difficult to find the corresponding expressions...This, however, can be at once effected by means of the analytical forms called trees" [22]. For higher order methods, we utilize the framework of the B-series, rooted trees, elementary differentials and elementary weights.

2.1.3 Rooted trees

The theory of rooted trees is a large branch of graph theory and it is out of the scope of this thesis. We will just summarize the relevant definitions, topics and results, in particular, the one-to-one correspondence between rooted trees and Taylor expansion derivatives. For more details, we refer the reader to [7, 22, 36].

Definition 2.1.1 *[33] A rooted tree is a pair of nodes (P, S) where S is a finite set of "Sons" or "Edges" and P is a finite set of "Parents", also known as "Vertices", such that:*

- *A tree t is a connected graph with one node in P considered as the root of the tree which never appears as a son.*
- *Other than the root, every parent node appears only once as a son in the tree.*
- *The order of a tree t is the number of its nodes and denoted by $r(t)$.*

Notation 2.1.1

- *T denotes the set of all rooted trees and T_k is the set of all rooted trees of order k.*
- *τ denotes the unique tree of order 1.*
- *$t = [t_1 t_2 \dots t_n]$ denotes the rooted tree generated by connecting the roots of $t_1, t_2, \dots, t_n$ to one more parent which represents the root of the tree t. This process is called grafting.*
- *If $t = [t_1 t_2 \dots t_n]$, then $t_1, t_2, \dots, t_n$ are the rooted trees generated by removing the root of the tree t.*
- *$t = [{t_1}^{n_1} {t_2}^{n_2} \dots {t_k}^{n_k}]$ is the rooted tree generated by grafting $n_1, n_2, \dots, n_k$ duplicates of the trees $t_1, t_2, \dots, t_k$, respectively.*

Definition 2.1.2 *[33, page 164] Let $t = [t_1{}^{n_1} t_2{}^{n_2} \dots t_k{}^{n_k}]$, then the order $r(t)$, the symmetry $\sigma(t)$ and the density $\gamma(t)$ of a rooted tree t are defined by*

$$\begin{aligned} r(\tau) &= \sigma(\tau) = \gamma(\tau) = 1 \\ r(t) &= 1 + \sum_{i=1}^{k} n_i r(t_i), \\ \sigma(t) &= \prod_{i=1}^{k} n_i! \sigma(t_i)^{n_i}, \\ \gamma(t) &= r(t) \prod_{i=1}^{k} \gamma(t_i)^{n_i}. \end{aligned}$$

Moreover, we define the number of essentially different ways of labelling a tree monotonically by

$$\alpha(t) = \frac{r(t)!}{\sigma(t)\gamma(t)}. \tag{2.1.6}$$

Table 2.1 presents the values of $r(t)$, $\sigma(t)$, $\gamma(t)$ and $\alpha(t)$ of trees up to order 4 [33]. For more details, we refer to [22, page 147] and [33, page 165].

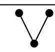
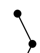
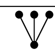
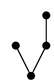
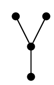
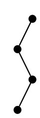
Notation 2.1.2 *We denote n copies of a tree t_1 by t_1^n,*

$$\underbrace{[[[\dots[}_{k\text{-times}} \text{ by } [_k \text{ and } \underbrace{]]]\dots]}_{k\text{-times}} \text{ by }]_k.$$

2.1.4 Elementary differentials and elementary weights

In this section, following Butcher's book [6], we will introduce the elementary differentials and elementary weights of Runge-Kutta methods. Recall that since $f : \mathbb{R}^N \to \mathbb{R}^N$, then its first derivative evaluated at y is a linear operator given by the matrix of partial derivatives of f. Similarly, the m-th derivative of f, $f^{(m)}(y)$, is a multilinear

Table 2.1: Values of $r(t)$, $\sigma(t)$, $\gamma(t)$ and $\alpha(t)$ of trees up to order 4

$r(t)$	t	Notation	$\sigma(t)$	$\gamma(t)$	$\alpha(t)$
1		τ	1	1	1
2		$[\tau]$	1	2	1
3		$[\tau^2]$	2	3	1
3		$[_2\tau]_2$	1	6	1
4		$[\tau^3]$	6	4	1
4		$[\tau[\tau]_2$	1	8	3
4		$[_2\tau^2]_2$	2	12	1
4		$[_3\tau]_3$	1	24	1

operator. For instance, if $w_1, w_2, \ldots, w_m \in \mathbb{R}^N$, then

$$f^{(m)}(y)\big(w_1, w_2, \ldots, w_m\big) = \begin{bmatrix} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_m=1}^{N} f^1_{y_{j_1} y_{j_2} \cdots y_{j_m}} w_1^{j_1} w_2^{j_2} \ldots w_m^{j_m} \\ \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_m=1}^{N} f^2_{y_{j_1} y_{j_2} \cdots y_{j_m}} w_1^{j_1} w_2^{j_2} \ldots w_m^{j_m} \\ \vdots \\ \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_m=1}^{N} f^N_{y_{j_1} y_{j_2} \cdots y_{j_m}} w_1^{j_1} w_2^{j_2} \ldots w_m^{j_m} \end{bmatrix},$$

where the superscripts correspond to the component number. Using this notation, we define the elementary differentials as follows [6]:

Definition 2.1.3 (Elementary differentials) *If $f : \mathbb{R}^N \to \mathbb{R}^N$ and $t \in T$ is a rooted tree, then the elementary differential $F : T \times \mathbb{R}^N \to \mathbb{R}^N$ is defined recursively* *as follows:*

$$F(\tau)(y) = f(y), \tag{2.1.7}$$

$$F\left([t_1 t_2 \dots t_m]\right)(y) = f^{(m)}(y)\left(F(t_1)(y), F(t_2)(y), \dots, F(t_m)(y)\right). \tag{2.1.8}$$

Rooted trees are particularly useful in expressing the derivatives of $y(x)$ in terms of f and its partial derivatives. This is neatly summed up by Butcher [7] in the following theorem stating that the m-th derivative of y is a linear combination of all elementary differentials of order m.

Theorem 2.1.1 *If $y'(x) = f(y(x))$ where $f : \mathbb{R}^N \to \mathbb{R}^N$. Then, the k-th derivative of the exact solution y is given by*

$$y^{(k)} = \sum_{t \in T_k} \alpha(t) F(t)(y). \tag{2.1.9}$$

Moreover, the quantities in equations (2.1.5) are called the elementary weights of the Runge-Kutta method (2.1.1) and can be defined recursively as follows [33]:

Definition 2.1.4 (Elementary weights) *The elementary weights of an s-stage Runge-Kutta method corresponding to the rooted tree $t = [t_1 t_2 \dots t_m] \in T$, denoted by $\Phi_{RK}(t)$, is defined recursively by*

$$\Phi_i(\tau) = \sum_{j=1}^{s} a_{ij}, \tag{2.1.10}$$

$$\Phi_i(t) = \sum_{j=1}^{s} a_{ij} \Phi_i(t_1) \Phi_i(t_2) \dots \Phi_i(t_m), \tag{2.1.11}$$

$$\Phi_{RK}(t) = \sum_{i=1}^{s} b_i \Phi_i(t_1) \Phi_i(t_2) \dots \Phi_i(t_m). \tag{2.1.12}$$

It is worth noting that different methods have different elementary weights. The above definition is strictly for an s-stage Runge-Kutta method of the form (2.1.1).

2.1.5 The B-series and the order conditions

The B-series is named after John Charles Butcher [7] in his elegant interpretation of the Taylor expansions of the exact and the numerical solutions.

Definition 2.1.5 (B-series) *Let $a : T \to \mathbb{R}$ be a real valued function on T. Then, a B-series is defined as follows:*

$$B(a, y) = \sum_{t \in T} \frac{h^{r(t)}}{r(t)!} \alpha(t) a(t) F(t)(y). \tag{2.1.13}$$

We can rewrite the Taylor expansion of the exact solution in (2.1.3) as a B-series as follows [6, page 155]:

Theorem 2.1.2 *The Taylor expansion of the exact solution $y_1 = y(x_0 + h)$ is equal to*

$$y(x_1) = y(x_0) + \sum_{i=1}^{p} h^i \sum_{t \in T_i} \frac{1}{\sigma(t)\gamma(t)} F(t)(y(x_0)) + \mathcal{O}(h^{p+1}). \tag{2.1.14}$$

Similarly, by using the elementary weights (2.1.10)-(2.1.12), we can rewrite the Taylor expansion of the numerical solution in (2.1.4) as a B-series as follows [6, page 160]:

Theorem 2.1.3 *The Taylor expansion of the numerical solution of the Runge-Kutta method is given by*

$$y_1 = y_0 + \sum_{i=1}^{p} h^i \sum_{t \in T_i} \frac{1}{\sigma(t)} \Phi_{RK}(t) F(t)(y_0) + \mathcal{O}(h^{p+1}). \tag{2.1.15}$$

Hence, by comparing equations (2.1.14) and (2.1.15), we get the following theorem [33, page169]:

Theorem 2.1.4 *The Runge-Kutta method has order p if and only if $\Phi_{RK}(t) = \frac{1}{\gamma(t)}$ holds for all rooted trees of order $r(t) \leq p$ and does not hold for some tree of order $p + 1$.*

Finally, we state a very important theorem for the derivation of the order condition of higher order methods [22]:

Theorem 2.1.5 *If $a : T \to \mathbb{R}$, $a(\phi) = 1$ and $t = [t_1, t_2, \dots, t_m]$, then*

$$hf(B(a, y)) = B(a', y),$$

where $a'(\phi) = 0, a'(\tau) = 1$ *and* $a'(t) = r(t)a(t_1)a(t_2)\dots a(t_m)$.

2.1.6 Linear stability of Runge-Kutta methods

In this section, we consider the vector notation of the Runge-Kutta method (2.1.1) operating on scalar problems as follows:

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{e}y_n + h\boldsymbol{A}\boldsymbol{F}, \\ y_{n+1} &= y_n + h\boldsymbol{b}^T\boldsymbol{F}, \end{aligned} \tag{2.1.16}$$

where $\boldsymbol{Y} = [Y_1, Y_2, \dots, Y_s]^T \in \mathbb{R}^s$, $\boldsymbol{F} = [F_1, F_2, \dots, F_s]^T \in \mathbb{R}^s$, $\boldsymbol{e} = [1, 1, \dots, 1]^T \in \mathbb{R}^s$, $\boldsymbol{A} = (a_{ij}) \in \mathcal{M}_{s,s}(\mathbb{R})$, and $\boldsymbol{b} = [b_1, b_2, \dots, b_s]^T \in \mathbb{R}^s$. Applying the Runge-Kutta method (2.1.16) to the Dahlquist test equation [33, page 198]

$$y' = \lambda y, \qquad y(0) = 1, \lambda \in \mathbb{C},$$

we get the following:

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{e}y_n + z\boldsymbol{A}\boldsymbol{Y}, \\ y_{n+1} &= y_n + z\boldsymbol{b}^T\boldsymbol{Y}, \end{aligned} \tag{2.1.17}$$

where $z = \lambda h$. Solving the two equations simultaneously for y_{n+1} in terms of y_n, we obtain:

$$y_{n+1} = \left[1 + z\boldsymbol{b}^T(\boldsymbol{I} - z\boldsymbol{A})^{-1}\boldsymbol{e}\right]y_n, \tag{2.1.18}$$

where $R(z) = 1 + z\boldsymbol{b}^T(\boldsymbol{I} - z\boldsymbol{A})^{-1}\boldsymbol{e}$ is the stability function of the Runge-Kutta method provided that $(\boldsymbol{I} - z\boldsymbol{A})^{-1}$ is invertible [33]. Moreover, Dekker and Verwer derived an alternative form of the stability function as a ratio of two determinants as follows [18]:

$$R(z) = \frac{|\boldsymbol{I} - z\boldsymbol{A} + z\boldsymbol{e}\boldsymbol{b}^T|}{|\boldsymbol{I} - z\boldsymbol{A}|}. \tag{2.1.19}$$

Definition 2.1.1 *The stability region is the set of all values of z in the complex plain such that $|R(z)| \leq 1$.*

We note that for explicit Runge-Kutta methods, $(\boldsymbol{I} - z\boldsymbol{A})$ is a lower triangular matrix and $|\boldsymbol{I} - z\boldsymbol{A}| = 1$. Then, by (2.1.19), the stability functions of explicit Runge-Kutta methods are always polynomials and hence their regions of absolute stability are finite. Indeed, for implicit Runge-Kutta methods, $(\boldsymbol{I} - z\boldsymbol{A})$ is a polynomial in z, so the stability function (2.1.19) is a rational function in z and it is possible to have an infinite region of absolute stability.

2.2 Taylor series method

Consider the non-autonomous initial value problem

$$y'(x) = f(x, y) \quad y(x_0) = y_0, \tag{2.2.1}$$

where $f : [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ and $y : [a, b] \to \mathbb{R}^N$. The Taylor expansion of degree p of y about x_n evaluated at x_{n+1} is given by

$$y(x_{n+1}) = y(x_n) + hy^{(1)}(x_n) + \frac{h^2}{2!}y^{(2)}(x_n) + \ldots + \frac{h^p}{p!}y^{(p)}(x_n) + \mathcal{O}(h^{p+1}) \tag{2.2.2}$$

where $h = \Delta x$ and

$$\begin{aligned} y^{(2)} &= f_x + f_y y' = f_x + f_y f, \\ y^{(3)} &= f_{xx} + 2f_{xy}f + f_{yy}(f, f) + f_y(f_x + f_y f), \\ &\vdots \end{aligned} \tag{2.2.3}$$

The Taylor series method of order p, $\mathrm{T}(p)$, is equivalent to the truncated Taylor series expansion of degree p of the exact solution in (2.2.2) and is given as follows:

$$y_{n+1} = y_n + hy_n^{(1)} + \frac{h^2}{2!}y_n^{(2)} + \ldots + \frac{h^p}{p!}y_n^{(p)}, \tag{2.2.4}$$

where y_n and y_{n+1} are approximations of $y(x_n)$ and $y(x_{n+1})$, respectively. This method is commonly used in many fields such as the astronomical computations field since it is very successful in reaching high order accuracy. However, as we can see in (2.2.3), the repeated computation of the Taylor coefficients $Y^{[i]} = \frac{1}{i!}y^{(i)}$ and $F^{[i]} = \frac{1}{i!}(f(x, y(x))^{(i)}$ of $y(x)$ and $f(x, y(x))$, respectively, can be very costly for higher derivatives. So, the Taylor coefficients are computed efficiently by an extension of Newton's approach which has been rediscovered several times (Steffensen 1956 [57]). Throughout this work, we use the notation $Y^{[i]} = \frac{1}{i!}y^{(i)}$ and $F^{[i]} = \frac{1}{i!}(f(x, y(x))^{(i)}$ to refer to the normalized i-th derivative of $y(x)$ and $f(x, y(x))$, respectively.

2.2.1 Automatic differentiation

The exact solution can be written in terms of its Taylor coefficients as follows:

$$y(t_0 + h) = \sum_{i=0}^{p} \frac{h^i}{i!}y^{(i)} + \mathcal{O}(h^{p+1}) = \sum_{i=0}^{p} h^i Y^{[i]} + \mathcal{O}(h^{p+1}).$$

Then, from (2.2.1),

$$Y^{[i+1]} = \frac{1}{i+1}F^{[i]}. \tag{2.2.5}$$

Now suppose that $f(x, y)$ is the composition of a finite sequence of algebraic operations and elementary functions (multiplication, division, ln, sin, cos, ...). This leads to a finite sequence of series that forms f [22, page 46]. For each given series $p = \sum_{i=0}^{\infty} P^{[i]}h^i$, $q = \sum_{i=0}^{\infty} Q^{[i]}h^i$, $r = \sum_{i=0}^{\infty} R^{[i]}h^i$, we can find formulae to generate the i-th Taylor coefficient from the preceding ones as follows:

a) If $r = p \pm q$, then

$$R^{[i]} = P^{[i]} \pm Q^{[i]}, \quad i = 0, 1, \ldots \tag{2.2.6}$$

b) If $r = pq$, then the Cauchy product yields

$$R^{[i]} = \sum_{j=0}^{i} P^{[j]} Q^{[i-j]}, \quad i = 0, 1, \ldots \tag{2.2.7}$$

c) If $r = \frac{p}{q}$, $q \neq 0$, then by writing $p = rq$, using formula b) and solving for $R^{[i]}$, we get:

$$R^{[i]} = \frac{1}{Q_0} \left[P^{[i]} - \sum_{j=1}^{i} R^{[i-j]} Q^{[j]} \right], \quad i = 0, 1, \ldots \tag{2.2.8}$$

d) If $r = p^{\alpha}$, then

$$R^{[i]} = \frac{1}{iP_0} \sum_{j=0}^{i-1} (i\alpha - j(\alpha + 1)) P^{[i-j]} R^{[j]}. \tag{2.2.9}$$

e) If $r = e^{p}$, then

$$R^{[i]} = \frac{1}{i} \sum_{j=0}^{i-1} (i - j) R^{[j]} P^{[i-j]}. \tag{2.2.10}$$

f) If $r = \ln(p)$, then

$$R^{[i]} = \frac{1}{P_0} \left[P^{[i]} - \frac{1}{i} \sum_{j=1}^{i-1} (i - j) P^{[j]} R^{[i-j]} \right]. \tag{2.2.11}$$

g) If $r = \cos(p)$ and $q = \sin(p)$, then

$$R^{[i]} = \frac{-1}{i} \sum_{j=1}^{i} j Q^{[i-j]} P^{[j]}, \tag{2.2.12}$$

$$Q^{[i]} = \frac{1}{i} \sum_{j=1}^{i} j R^{[i-j]} P^{[j]}. \tag{2.2.13}$$

There are formulae for several other elementary functions including inverse trigonometric functions. For more details and proves of the above formulas, refer to [37, 29, 34]

2.3 DETEST problems

In this thesis, we consider a bank of test problems called DETEST for the purpose of testing the designed contractivity preserving methods and comparing them to well known, well tested ordinary differential equations solvers in the literature. This bank of problems is accepted in the numerical ODE community as a standard comparison tool between higher order ODE solvers and has been used by several authors [23, 41, 40, 51]. In our work, we utilize 20 of the DETEST problems of classes A, B, D and E. We describe these classes briefly below [27]:

2.3.1 Problem class A: Single equations

- **A1**: The negative exponential.

$$y' = -y, \qquad y(0) = 1. \qquad \text{(solution: } y = e^{-x}\text{)}$$

- **A2**: A special case of the Riccati equation [10, page 73]:

$$y' = \frac{-y^3}{2}, \qquad y(0) = 1. \qquad \text{(solution: } y = \frac{1}{\sqrt{x+1}}\text{)}$$

- **A3**: An oscillatory problem:

$$y' = y\cos x, \qquad y(0) = 1. \qquad \text{(solution: } y = e^{\sin x}\text{)}$$

- **A4**: A logistic curve [10, page 97]:

$$y' = \frac{y}{4}\big(1 - \frac{y}{20}\big), \qquad y(0) = 1. \qquad \text{(solution: } y = \frac{20}{1 + 19e^{-\frac{x}{4}}}\text{)}$$

- **A5**: A spiral curve [10, page 38]:

$$y' = \frac{y - x}{y + x}, \qquad y(0) = 4. \qquad \text{(solution in polar coordinates: } r = 4e^{\frac{\pi}{2}-\theta}\text{)}$$

2.3.2 Problem class B: Small systems

- **B1**: The growth of two conflicting populations [10, page 102].

$$\begin{aligned} y_1' &= 2(y_1 - y_1 y_2), & y_1(0) &= 1, \\ y_2' &= -(y_2 - y_1 y_2), & y_2(0) &= 3. \end{aligned}$$

- **B2**: A linear chemical reaction [16, page 175]:

$$\begin{aligned} y_1' &= -y_1 + y_2, & y_1(0) &= 2, \\ y_2' &= y_1 - 2y_2 + y_3, & y_2(0) &= 0, \\ y_3' &= y_2 - y_3, & y_3(0) &= 1. \end{aligned}$$

- **B3**: A nonlinear chemical reaction [16, page 177]:

$$\begin{aligned} y_1' &= -y_1, & y_1(0) &= 1, \\ y_2' &= y_1 - y_2^2, & y_2(0) &= 0, \\ y_3' &= y_2^2, & y_3(0) &= 0. \end{aligned}$$

- **B4**: The integral surface of a torus [9, page 9]:

$$\begin{aligned} y_1' &= -y_2 - \frac{y_1 y_3}{\sqrt{y_1^2 + y_2^2}}, & y_1(0) &= 3, \\ y_2' &= y_1 - \frac{y_2 y_3}{\sqrt{y_1^2 + y_2^2}}, & y_2(0) &= 0, \\ y_3' &= \frac{y_1}{\sqrt{y_1^2 + y_2^2}}, & y_3(0) &= 0. \end{aligned}$$

- **B5**: The Euler equations of motion for a rigid body without external forces [32]:

$$\begin{aligned} y_1' &= y_2 y_3, & y_1(0) &= 0, \\ y_2' &= -y_1 y_3, & y_2(0) &= 1, \\ y_3' &= -0.51 y_1 y_2, & y_3(0) &= 1. \end{aligned}$$

2.3.3 Problem class D: Orbit equations

- **D1**:

$$\begin{aligned} y_1' &= y_3, & y_1(0) &= 1-\varepsilon, \\ y_2' &= y_4, & y_2(0) &= 0, \\ y_3' &= \frac{-y_1}{(y_1^2+y_2^2)^{\frac{3}{2}}}, & y_3(0) &= 0, \\ y_4' &= \frac{-y_2}{(y_1^2+y_2^2)^{\frac{3}{2}}}, & y_4(0) &= \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \end{aligned}$$

  where $\varepsilon = 0.1$ is the eccentricity of the orbit.

- **D2**: As above with $\varepsilon = 0.3$.

- **D3**: As above with $\varepsilon = 0.5$.

- **D4**: As above with $\varepsilon = 0.7$.

- **D5**: As above with $\varepsilon = 0.9$.

All the D class problems are derived from the orbit equations

$$\begin{aligned} x'' &= \frac{-x}{r^3}, \qquad x(0) = 1-\varepsilon, \qquad x'(0) = 0, \\ y'' &= \frac{-y}{r^3}, \qquad y(0) = 0, \qquad y'(0) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \\ r^2 &= x^2 + y^2, \end{aligned}$$

with solution

$$\begin{aligned} x &= \cos u - \varepsilon, & x' &= \frac{-\sin u}{1-\varepsilon\cos u}, \\ y &= \sqrt{1-\varepsilon^2}\sin u, & y' &= \frac{\sqrt{1-\varepsilon^2}\cos u}{1-\varepsilon\cos u}, \end{aligned}$$

where $u - \varepsilon \sin u - t = 0$.

2.3.4 Problem class E: Higher order equations

- **E1**:

$$y_1' = y_2, \qquad y_1(0) = J_{\frac{1}{2}}(1) = 0.673967071418030,$$
$$y_2' = -\left(\frac{y_2}{x+1} + \left(1 - \frac{0.25}{(x+1)^2}\right)y_1\right), \qquad y_2(0) = J'_{\frac{1}{2}}(1) = 0.09540051444747446.$$

This is an ODE system derived from Bessel's equation of order 1/2 with the origin shifted one unit to the left [10, page 4,69]:

$$(x+1)^2 y'' + (x+1)y' + ((x+1)^2 - 0.25)y = 0.$$

- **E2**:

$$y_1' = y_2, \qquad y_1(0) = 2,$$
$$y_2' = (1 - y_1^2)y_2 - y_1, \qquad y_2(0) = 0.$$

This is an ODE system derived from the Van der Pol's equation [10, page 358, 531]:

$$y'' - (1 - y^2)y' + y = 0.$$

- **E3**:

$$y_1' = y_2, \qquad y_1(0) = 0,$$
$$y_2' = \frac{y_1^3}{6} - y_1 + 2\sin(2.78535x), \qquad y_2(0) = 0.$$

This is an ODE system derived from the Duffing's equation [10, page 390]:

$$y'' + y - \frac{y^3}{6} = 2\sin(2.78535x).$$

- **E4**:

$$y_1' = y_2, \qquad y_1(0) = 30,$$
$$y_2' = 0.32 - 0.4y_2^2, \qquad y_2(0) = 0.$$

This is an ODE system derived from the falling body equation [10, page 60]:

$$y'' = 0.32 - 0.4y'^2.$$

- **E5**:

$$\begin{aligned} y_1' &= y_2, & y_1(0) &= 0, \\ y_2' &= \frac{\sqrt{1 + y_2^2}}{25 - x}, & y_2(0) &= 0. \end{aligned}$$

This is an ODE system derived from a linear pursuit equation [10, page 117]:

$$1 + (y')^2 = (25 - x)^2 (y'')^2.$$

Chapter 3

CP s-Stage HBT methods based on combining T(d) and RK(s,4) methods

3.1 Introduction

In this chapter, we consider the solution of non-stiff autonomous initial value problem of the form

$$\frac{dy}{dx} = f(x, y(x)), \qquad y(x_0) = y_0, \tag{3.1.1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a sufficiently smooth function. We present the formulae of the new explicit contractivity preserving, one-step, s-stages, explicit, Hermite-Birkhoff-Taylor ODE solver of orders $p = 5, 6, \ldots, 15$, denoted by $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$, with nonnegative coefficients by combining an s-stages Runge-Kutta method of order 4, RK$(s, 4)$, for $s = 5, 6$, with a Taylor series method, T(d), of order d. The newly designed methods incorporate the function evaluations at off-step points similar to Runge-Kutta methods and the high order derivatives information as in Taylor series methods to achieve better performance than widely used methods existing in the literature including Runge-Kutta and Taylor series methods.

For the analysis of the new $\mathrm{CPHBT_{RK4}}(d,s,p)$ methods, we define the forward Euler expanded series, FES(d),

$$y_{n+1} = y_n + \Delta x f(x_n, y_n) + \sum_{m=2}^{d} \eta_m \Delta x^m f^{(m-1)}(x_n, y_n). \tag{3.1.2}$$

We are interested in the contractivity preserving property,

$$\|y_{n+1} - \tilde{y}_{n+1}\| \leq \|y_n - \tilde{y}_n\|, \tag{3.1.3}$$

where y_n, $\tilde{y}_n$ are two numerical solutions with different neighbouring initial conditions $y(x_0) = y_0$, $\tilde{y}(x_0) = \tilde{y}_0$. Throughout this work, we assume that $\|\cdot\|$ is a norm such that f is dissipative with respect to $\|\cdot\|$ [31]. To achieve inequality (3.1.3), we suppose that there exists a step-size ΔFES(d) such that

$$\begin{aligned}
\left\| y_{n+1} - \tilde{y}_{n+1} \right\| &= \Bigg\| y_n + \sum_{m=1}^{d} \eta_m (\Delta x)^m f^{(m-1)}(x_n, y_n) \\
&\qquad - \tilde{y}_n - \sum_{m=1}^{d} \eta_m (\Delta x)^m f^{(m-1)}(x_n, \tilde{y}_n) \Bigg\| \\
&\leq \|y_n - \tilde{y}_n\|
\end{aligned} \tag{3.1.4}$$

holds for all $\Delta x \leq \Delta$FES(d).

3.2 Formulation of $\mathrm{CPHBT_{RK4}}(d,s,p)$ in Butcher form

We call our new method $\mathrm{CPHBT_{RK4}}(d,s,p)$ because we use Hermite-Birkhoff interpolation polynomials to define the predictors P_i and obtain the stages Y_i to order $p-3$ as follows:

$$Y_i = y_n + \Delta x \sum_{j=1}^{i-1} a_{i,j} F_j + \sum_{m=2}^{d} (\Delta x)^m \gamma_{i,m}\, y_n^{(m)}, \qquad i = 2, 3, \ldots, s. \tag{3.2.1}$$

Also, we use Hermite-Birkhoff interpolation polynomials to define the integration formula (IF) and obtain y_{n+1} to order p as follows:

$$y_{n+1} = y_n + \Delta x \sum_{j=1}^{s} b_j F_j + \sum_{m=2}^{d} (\Delta x)^m \gamma_{s+1,m}\, y_n^{(m)}, \tag{3.2.2}$$

where in the formula of CPHBT$_{\text{RK4}}(d, s, p)$ we have the Runge-Kutta coefficients $a_{i,j}$, b_j, c_j for $i = 2, 3, \ldots, s$, $j = 1, 2, \ldots, i-1$, and the Taylor coefficients $\gamma_{i,m}$ for $i = 2, 3, \ldots, s{+}1$ and $m = 2, 3, \ldots, d$. The high order derivatives $y_n^{(m)}$, $m = 2, 3, \ldots, d$, are evaluated only once per time step integration at the step point x_n. Note that the CPHBT$_{\text{RK4}}(d, s, p)$ formula (3.2.1)-(3.2.2) is given in the Butcher form. Following the literature of Runge-Kutta methods, we represent CPHBT$_{\text{RK4}}(d, s, p)$ by the coefficient triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{\gamma}_0)$ where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{2,1} & 0 & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ a_{s,1} & a_{s,2} & \ldots & a_{s,s-1} & 0 \end{bmatrix}, \tag{3.2.3}$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \ldots & b_{s-1} & b_s \end{bmatrix}^T, \tag{3.2.4}$$

and $\boldsymbol{\gamma}_0$ is a $(s+1) \times (d-1)$ matrix ,

$$\boldsymbol{\gamma}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,d} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{s+1,2} & \gamma_{s+1,3} & \cdots & \gamma_{s+1,d} \end{bmatrix}, \tag{3.2.5}$$

and the Butcher tableau

$$
\begin{array}{c|cccccc}
c_1 & & & & & \\
c_2 & a_{2,1} & & & & \\
c_3 & a_{3,1} & a_{3,2} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \dots & b_{s-1} & b_s
\end{array}
$$

3.3 Derivation of the order conditions

In this section, we will derive the order conditions of $\mathrm{CPHBT}_{\mathrm{RK4}}(d,s,p)$ with $p = 5, 6, \dots, 15$ by forcing the Taylor expansion series of the numerical solution to be equal to that of the exact solution. In this following theoretical framework, we will restrict ourself to the following autonomous initial value problem:

$$\frac{dy}{dx} = f(y), \qquad y(x_0) = y_0, \tag{3.3.1}$$

where function $f : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by using the substitution $z = [x \quad y]^T$,

$$z' = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 \\ f(x,y) \end{bmatrix} = f(z), \tag{3.3.2}$$

where $z_0 = [x_0 \quad y(x_0)]^T$ to transform the equation (3.1.1) to the autonomous form (3.3.1). We will derive the order conditions by studying the accuracy of the new $\mathrm{CPHBT}_{\mathrm{RK4}}(d,s,p)$ after performing one integration step from y_0 to y_1.

By Theorem 2.1.2, the Taylor expansion of the exact solution about y_0 is given by:

$$
\begin{aligned}
y &= y_0 + hy_0^{(1)} + \frac{h^2}{2!}y_0^{(2)} + \frac{h^3}{3!}y_0^{(3)} + \frac{h^4}{4!}y_0^{(4)} + \mathcal{O}(h^5) \\
&= y_0 + hF(\tau)(y_0) + \frac{h^2}{2!}F([\tau])(y_0) + \frac{h^3}{3!}\Big(F([\tau^2])(y_0) + F([_2\tau]_2)(y_0)\Big) \\
&\quad + \frac{h^4}{4!}\Big(F([\tau^3])(y_0) + 3F([\tau[\tau]_2)(y_0) + F([_2\tau^2]_2)(y_0) + F([_3\tau]_3)(y_0)\Big) + \mathcal{O}(h^5),
\end{aligned}
$$

where $h = \Delta x$ and $F(\tau)(y_0), F([\tau])(y_0), \ldots, F([_3\tau]_3)(y_0)$ are elementary differentials corresponding to the rooted trees $\tau, [\tau], \ldots, [_3\tau]_3$, respectively, as discussed briefly in Section 2.1.4 and the subscript in $[_n$ and $]_n$. For more details, we refer to [22]. We utilize the framework of B-series with the elementary differentials, elementary weights and rooted trees to express the Taylor series expansion of the exact solution and the numerical solution as B-series and compare the coefficient sequences, also called elementary weights, to derive the order conditions.

Theorem 3.3.1 *The elementary weights $\Psi : T \to \mathbb{R}$ of an s-stage, d-derivative CPHBT method* (3.2.1)-(3.2.2) *corresponding to the rooted tree $t = [t_1\ t_2\ \ldots\ t_n] \in T$ is defined by*

$$\Psi(t) \quad = \quad \sum_{j=1}^{s} b_j \tilde{S}'_j(t) + \sum_{m=2}^{d} \gamma_{s+1,m} \Phi^{(m)}(t), \tag{3.3.3}$$

where $\Phi^{(m)} : T \to \mathbb{R}$ for $m = 2, 3, \ldots, d$, is given by

$$\Phi^{(m)}(t) = \begin{cases} m! & \text{if } r(t) = m, \\ 0 & \text{otherwise}, \end{cases} \tag{3.3.4}$$

$\tilde{S}'_i : T \to \mathbb{R}$ for $i = 2, 3, \ldots, s$, are computed recursively as follows

$$\tilde{S}'_i(\phi) \quad = \quad 0, \quad \tilde{S}'_i(\tau) = 1, \tag{3.3.5}$$

$$\tilde{S}'_i(t) \quad = \quad r(t)\tilde{S}_i(t_1)\tilde{S}_i(t_2)\ldots\tilde{S}_i(t_n), \tag{3.3.6}$$

and $\tilde{S}_i : T \to \mathbb{R}$ for $i = 2, 3, \ldots, s$, is given by

$$\tilde{S}_i(t) \quad = \quad \sum_{j=1}^{i-1} a_{ij} \tilde{S}'_j(t) + \sum_{m=2}^{d} \gamma_{im} \Phi^{(m)}(t). \tag{3.3.7}$$

The CPHBT method is of order p if and only if

$$\Psi(t) = 1 \qquad \forall t \in T_i \quad \text{with} \quad i \leq p, \tag{3.3.8}$$

where T_i is the set of all rooted trees of order i.

Proof: Let $y = y(h)$ be the Taylor expansion of the exact solution of the IVP (3.3.1). We can express $y(h)$ as a B-series with elementary weights $\Phi : T \to \mathbb{R}$ as follows

$$y(h) = B(\Phi, y_0) = y_0 + \sum_{i=1}^{p} \frac{h^i}{i!} \sum_{t\in T_i} \alpha(t)\Phi(t)F(t)(y_0) + \mathcal{O}(h^{p+1}), \tag{3.3.9}$$

where $F(t)(y_0)$ is elementary differentials given in Definition 2.1.3 and $\alpha : T \to \mathbb{R}$ is defined in equation (2.1.6). Then,

$$hf(y(h)) = hy'(h) = \sum_{i=1}^{p} \frac{h^i}{(i-1)!} \sum_{t\in T_i} \alpha(t)\Phi(t)F(t)(y_0) + \mathcal{O}(h^{p+1}). \tag{3.3.10}$$

Moreover, by theorem 2.1.5, we have

$$hf(y(h)) = hf(B(\Phi, y_0)) = B(\Phi', y_0) = \sum_{i=1}^{p} \frac{h^i}{i!} \sum_{t\in T_i} \alpha(t)\Phi'(t)F(t)(y_0) + \mathcal{O}(h^{p+1}), \tag{3.3.11}$$

where

$$\begin{cases} \Phi'(\phi) = 0, \quad \Phi'(\tau) = 1, \\ \Phi'([t_1\ t_2\ \ldots\ t_n]) = r([t_1\ t_2\ \ldots\ t_n])\Phi(t_1)\Phi(t_2)\ldots\Phi(t_n). \end{cases} \tag{3.3.12}$$

By comparing equations (3.3.10) and (3.3.11), we obtain

$$\Phi(t) = \frac{\Phi'(t)}{i} \qquad \forall t \in T_i, i \geq 1. \tag{3.3.13}$$

Then, by (3.3.12), we get the following

$$\Phi(t) = 1 \qquad \forall t \in T_i, i \geq 1. \tag{3.3.14}$$

Similarly, let

$$y_1(h) = B(\Psi, y_0) = y_0 + \sum_{i=1}^{p} \frac{h^i}{i!} \sum_{t\in T_i} \alpha(t)\Psi(t)F(t)(y_0) + \mathcal{O}(h^{p+1}) \tag{3.3.15}$$

be the Taylor expansion of the numerical solution generated by $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$ expressed as a B-series with elementary weights $\Psi : T \to \mathbb{R}$. Then, $\mathrm{CPHBT}_{\mathrm{RK4}}$ (d, s, p) is of order p if and only if

$$\Psi(t) = 1 \qquad\qquad \forall t \in T_i \ \text{ with } \ i \leq p. \tag{3.3.16}$$

To derive the order conditions (3.3.16), we need to find an expression for the elementary weights $\Psi(t)$ of $\mathrm{CPHBT}_{\mathrm{RK4}}(d,s,p)$. To do so, we write for $i=1,\ldots,s$ the stages Y_i as B-series $Y_i=B(\tilde{S}_i,y_0)$; where $\tilde{S}_i : T\to\mathbb{R}$ are the elementary weights. Then,

$$hF_i = hf(Y_i) = hf(B(\tilde{S}_i,y_0)) = B(\tilde{S}_i',y_0),$$

where

$$\begin{cases} \tilde{S}_i'(\phi)=0, \quad \tilde{S}_i'(\tau)=1, \\ \tilde{S}_i'([t_1\ t_2\ \ldots\ t_n]) = r([t_1\ t_2\ \ldots\ t_n])\tilde{S}_i(t_1)\tilde{S}_i(t_2)\ldots\tilde{S}_i(t_n). \end{cases} \tag{3.3.17}$$

Moreover, by an extension of theorem 2.1.5, we get

$$h^m(y^{(m)})\big(B(\Phi,y_0)\big) = \sum_{t\in T_m} h^m\alpha(t)F(t)(y_0) \tag{3.3.18}$$

which is a B-series, $B(\Phi^{(m)},y_0)$, with elementary weights

$$\Phi^{(m)}(t) = \begin{cases} m! & \text{if } r(t)=m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.19}$$

Hence, if the numerical solution y_1 is generated by $\mathrm{CPHBT}_{\mathrm{RK4}}(d,s,p)$ method defined in (3.2.1) and (3.2.2), then

$$\begin{aligned} \tilde{S}_i(t) &= \sum_{j=1}^{i-1} a_{ij}\tilde{S}_j'(t) + \sum_{m=2}^{d}\gamma_{im}\Phi^{(m)}(t), \qquad i=2,3,\ldots,s, && (3.3.20)\\ \Psi(t) &= \sum_{j=1}^{s} b_j\tilde{S}_j'(t) + \sum_{m=2}^{d}\gamma_{s+1,m}\Phi^{(m)}(t), && (3.3.21) \end{aligned}$$

where $\tilde{S}_j'(t)$ and $\Phi^{(m)}(t)$ are defined recursively in (3.3.17) and (3.3.19), respectively. ∎

By using equations (3.3.17)-(3.3.21), we can compute $\Psi(t)$ recursively for all $t\in T$. As an example, we will derive one of the order conditions of $\mathrm{CPHBT}_{\mathrm{RK4}}(2,5,5)$

of order 5. Let us consider a tree of order 5, say $t_{14} = [[\tau^2]\tau]$. Then, by equations (3.3.21) and (3.3.19), the elementary weight $\Psi(t_{14})$ is computed recursively as follows:

$$\begin{aligned}\Psi(t_{14}) &= \sum_{j=1}^{5} b_j \tilde{S}'_j(t_{14}) + \sum_{m=2}^{2} \gamma_{im} \Phi^{(m)}(t_{14}) \\ &= \sum_{j=1}^{5} b_j \tilde{S}'_j(t_{14}). \end{aligned} \tag{3.3.22}$$

Also, by (3.3.17),

$$\tilde{S}'_j(t_{14}) = 5\tilde{S}_j([\tau^2])\tilde{S}_j(\tau). \tag{3.3.23}$$

But,

$$\tilde{S}_j(\tau) = \sum_{k=1}^{j-1} a_{jk} = c_j. \tag{3.3.24}$$

$$\begin{aligned}\tilde{S}'_j([\tau^2]) &= \sum_{k=1}^{j-1} a_{jk}\tilde{S}'_k([\tau^2]) = \sum_{k=1}^{j-1} a_{jk} 3.\big(\tilde{S}_k(\tau)\big)^2 = 3\sum_{k=1}^{j-1} a_{jk}\Big(\sum_{l=1}^{k-1} a_{kl}\Big)^2 \\ &= 3\sum_{k=1}^{j-1} a_{jk} c_k^2. \end{aligned} \tag{3.3.25}$$

Substituting equations (3.3.24) and (3.3.25) into (3.3.23) we get

$$\tilde{S}'_j(t_{14}) = 15 c_j \sum_{k=1}^{j-1} a_{jk} c_k^2. \tag{3.3.26}$$

Finally, substituting equation (3.3.26) into (3.3.22) we get the elementary weight corresponding to the rooted tree $t_{14} = [[\tau^2]\tau]$:

$$\Psi(t_{14}) = \tilde{S}'_j(t_{14}) = 15\sum_{j=1}^{5} b_j c_j \left[\sum_{k=1}^{j-1} a_{jk} c_k^2\right]. \tag{3.3.27}$$

Hence, by equation (3.3.16), the order condition of $\mathrm{CPHBT}_{\mathrm{RK4}}(2,5,5)$ corresponding to the rooted tree t_{14} is given by

$$\Psi(t_{14}) = 1 \Longleftrightarrow \sum_{j=1}^{5} b_j c_j \left[\sum_{k=1}^{j-1} a_{jk} c_k^2\right] = \frac{1}{15}, \tag{3.3.28}$$

which is the order condition (3.3.34) given below for $p = 5, s = 5$ and $d = 2$. To derive the order conditions of a class of s-stage $\mathrm{CPHBT_{RK4}}(d, s, p)$ of order $p = 5, 6, \ldots, 15$ with $s = 5, 6$, we must consider all rooted trees of order $r(t) \le p$. However, as shown in Table (3.1) [35, page 9], the number of order conditions grows rapidly and hence reducing the number of independent order conditions is crucial.

Table 3.1: Cardinality of trees of order p and the number of order conditions.

order p	1	2	3	4	5	6	7	8	9	10	11	12	13
Card(T_p)	1	1	2	4	9	20	48	115	286	719	1842	4766	12486
No. of conditions	1	2	4	8	17	37	85	200	486	1205	3047	7813	20299

To do so, we reduce the number of the rooted trees to be considered by imposing the following simplifying conditions [22, 41]:

$$\sum_{j=1}^{i-1} a_{i,j} \frac{c_j^k}{k!} + \gamma_{i,k+1} = \frac{1}{(k+1)!} c_i^{k+1}, \tag{3.3.29}$$

for $i = 2, 3, \ldots, s$ and $k = 0, 1, \ldots, p-4$. Here, we have $c_1 = 0$, the convention that $c_1^0 = 1$ and $\gamma_{i,1} = 0$ for $i = 2, 3, \ldots, s$. If condition (3.3.29) is satisfied for some k, then both trees given in Figure (3.1) generate equivalent order conditions. Hence, by enforcing these simplifying conditions for $i = 2, 3, \ldots, s$ and $k = 0, 1, \ldots, p-4$, all order conditions generated from trees of order $r = 1, 2, \ldots, p-2$ are equivalent to those generated by the bushy trees $t = [\tau^{r-1}]$ of order $r = 1, 2, \ldots, p-2$ and the number of independent order conditions generated from trees of orders $r = p-1, p$ is reduced considerably. For instance, when we derive the order conditions of $\mathrm{CPHBT_{RK4}}(d, s, 12)$ of order 12, the simplifying conditions (3.3.29) reduces the number of order conditions from 7813 independent conditions to only 52 independent conditions including the simplifying conditions. The remaining trees will lead to the following sets of order conditions for $\mathrm{CPHBT_{RK4}}(d, s, p)$:

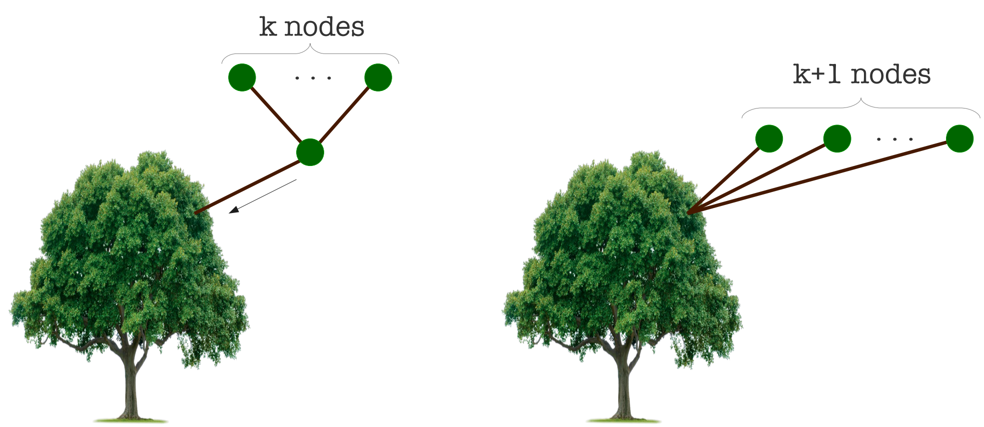


Figure 3.1: An example of the use of the simplifying conditions. These trees generate equivalent order conditions provided that conditions (3.3.29) are satisfied.

$$\sum_{i=1}^{s} b_i = 1, \tag{3.3.30}$$

$$\sum_{i=2}^{s} b_i \frac{c_i^k}{k!} + \gamma_{s+1,k+1} = \frac{1}{(k+1)!}, \quad k = 1, \ldots, p-4, \tag{3.3.31}$$

$$\sum_{i=2}^{s} b_i c_i^k = \frac{1}{k+1}, \qquad k = p-3,\ p-2,\ p-1, \tag{3.3.32}$$

$$\sum_{i=3}^{s} b_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-3} \right] = \frac{1}{(p-1)(p-2)}, \tag{3.3.33}$$

$$\sum_{i=3}^{s} b_i c_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-3} \right] = \frac{1}{p(p-2)}, \tag{3.3.34}$$

$$\sum_{i=3}^{s} b_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-2} \right] = \frac{1}{p(p-1)}, \tag{3.3.35}$$

$$\sum_{i=4}^{s} b_i \left[\sum_{j=3}^{i-1} a_{i,j} \left[\sum_{k=2}^{j-1} a_{j,k}\, c_k^{p-3} \right] \right] = \frac{1}{p(p-1)(p-2)}, \tag{3.3.36}$$

where an induction process similar to the one shown in [5, 36] is applied to derive the order conditions of $\mathrm{CPHBT_{RK4}}(d,s,p)$ of order p. As an example, let us derive the order conditions for $p = 4, 5, 6$. For $p = 4$, the predictors are of order at least 1 and hence the simplifying conditions (3.3.29) are enforced for $i = 2, 3, \ldots, s$ and $k = 0$ only. Hence, we have to consider all trees in Table 3.2 of order $r \leq 4$ to derive the order conditions of $\mathrm{CPHBT_{RK4}}(d,s,4)$ from equations (3.3.16)-(3.3.21). It is worth mentioning that these order conditions are similar to the order conditions of $\mathrm{RK}(s,4)$ with added Taylor coefficients. In Table 3.2, $\phi_{\mathrm{RK}}(t_i)$ is the elementary weights of Runge-Kutta method given in Definition 2.1.4 corresponding to a tree t_i. Also, $r(t_i)$ and $\alpha(t_i)$ are given in Definition 2.1.2 and formula (2.1.6), respectively.

For $p = 5$, the predictors are of order at least 2 and hence, applying the simplifying conditions for $i = 2, 3, \ldots, s$ and $k = 0, 1$, the following pairs of trees will generate the same order conditions: $(t_3,t_4), (t_5,t_6), (t_7,t_8), (t_9,t_{10}), (t_{11},t_{12}), (t_{13},t_{14}), (t_{15},t_{17}),$ $(t_{16},t_{17}), (t_{18},t_{19}), (t_{20},t_{21}), (t_{22},t_{23}), (t_{24},t_{26}), (t_{25},t_{26}), (t_{27},t_{28}), (t_{29},t_{30}), (t_{31},t_{34}),$ $(t_{32},t_{34}), (t_{33},t_{34}), (t_{35},t_{37})$ and (t_{26},t_{27}). Then, we can ignore the trees on the left component of each pair and reduce Table (3.2) to get Table (3.3). Considering all trees of order $r \leq 5$, we can derive the order conditions of $\mathrm{CPHBT_{RK4}}(d,s,5)$.

Finally, for $p = 6$, the predictors are of order at least 3. Then, applying the simplifying conditions (3.3.29) for $i = 2, 3, \ldots, s$ and $k = 0, 1, 2$, the following pairs of trees generate the same order conditions: $(t_6,t_8), (t_{10},t_{12}), (t_{14},t_{17}), (t_{19},t_{21}), (t_{23},t_{26}),$ (t_{28},t_{30}) and (t_{34},t_{37}). So, Table (3.3) is reduced to Table (3.4) from which we can derive the order conditions of $\mathrm{CPHBT_{RK4}}(d,s,6)$.

Table 3.2: Notations and some functions on trees of order 1 to 6.

i	notation	$r(t_i)$	$\phi_{\mathrm{RK}}(t_i)$	$\alpha(t_i)$
1	τ	1	$\sum_i b_i$	1
2	$[\tau]$	2	$\sum_i b_i c_i$	1
3	$[_2\tau]_2$	3	$\sum_{ij} b_i a_{ij} c_j$	1
4	$[\tau^2]$	3	$\sum_i b_i c_i^2$	1
5	$[_3\tau]_3$	4	$\sum_{ijk} b_i a_{ij} a_{jk} c_k$	1
6	$[_2\tau^2]_2$	4	$\sum_{ij} b_i a_{ij} c_j^2$	1
7	$[[\tau]\tau]$	4	$\sum_{ij} b_i c_i a_{ij} c_j$	3
8	$[\tau^3]$	4	$\sum_i b_i c_i^3$	1
9	$[_4\tau]_4$	5	$\sum_{ijkl} b_i a_{ij} a_{jk} a_{kl} c_l$	1
10	$[_3\tau^2]_3$	5	$\sum_{ijk} b_i a_{ij} a_{jk} c_k^2$	1
11	$[_2[\tau]\tau]_2$	5	$\sum_{ijk} b_i a_{ij} c_j a_{jk} c_k$	3
12	$[_2\tau^3]_2$	5	$\sum_{ij} b_i a_{ij} c_j^3$	1
13	$[[_2\tau]_2\tau]$	5	$\sum_{ijk} b_i c_i a_{ij} a_{jk} c_k$	4
14	$[[\tau^2]\tau]$	5	$\sum_{ij} b_i c_i a_{ij} c_j^2$	4
15	$[[\tau]^2]$	5	$\sum_i b_i (\sum_j a_{ij} c_j)^2$	3
16	$[[\tau]\tau^2]$	5	$\sum_{ij} b_i c_i^2 a_{ij} c_j$	6
17	$[\tau^4]$	5	$\sum_i b_i c_i^4$	1
18	$[_5\tau]_5$	6	$\sum_{ijklm} b_i a_{ij} a_{jk} a_{kl} a_{lm} c_m$	1
19	$[_4\tau^2]_4$	6	$\sum_{ijkl} b_i a_{ij} a_{jk} a_{kl} c_l^2$	1
20	$[_3[\tau]\tau]_3$	6	$\sum_{ijkl} b_i a_{ij} a_{jk} c_k a_{kl} c_l$	3
21	$[_3\tau^3]_3$	6	$\sum_{ijk} b_i a_{ij} a_{jk} c_k^3$	1
22	$[_2[_2\tau]_2\tau]_2$	6	$\sum_{ijkl} b_i a_{ij} c_j a_{jk} a_{kl} c_l$	4
23	$[_2[\tau^2]\tau]_2$	6	$\sum_{ijk} b_i a_{ij} c_j a_{jk} c_k^2$	4
24	$[_2[\tau]^2]_2$	6	$\sum_i b_i a_{ij} (\sum_k a_{jk} c_k)^2$	3
25	$[_2[\tau]\tau^2]_2$	6	$\sum_{ijk} b_i a_{ij} c_j^2 a_{jk} c_k$	6
26	$[_2\tau^4]_2$	6	$\sum_{ij} b_i a_{ij} c_j^4$	1
27	$[[_3\tau]_3\tau]$	6	$\sum_{ijkl} b_i c_i a_{ij} a_{jk} a_{kl} c_l$	5
28	$[[_2\tau^2]_2\tau]$	6	$\sum_{ijk} b_i c_i a_{ij} a_{jk} c_k^2$	5
29	$[[[\tau]\tau]\tau]$	6	$\sum_{ijk} b_i c_i a_{ij} c_j a_{jk} c_k$	15
30	$[[\tau^3]\tau]$	6	$\sum_{ij} b_i c_i a_{ij} c_j^3$	5
31	$[[_2\tau]_2[\tau]]$	6	$\sum_i b_i (\sum_{jk} a_{ij} a_{jk} c_k)(\sum_j a_{ij} c_j)$	10
32	$[[\tau^2][\tau]]$	6	$\sum_i b_i (\sum_j a_{ij} c_j^2)(\sum_j a_{ij} c_j)$	10
33	$[[_2\tau]_2\tau^2]$	6	$\sum_{ijk} b_i c_i^2 a_{ij} a_{jk} c_k$	10
34	$[[\tau^2]\tau^2]$	6	$\sum_{ij} b_i c_i^2 a_{ij} c_j^2$	10
35	$[[\tau]^2\tau]$	6	$\sum_i b_i c_i (\sum_j a_{ij} c_j)^2$	15
36	$[[\tau]\tau^3]$	6	$\sum_{ijk} b_i c_i^3 a_{ij} c_j$	10
37	$[\tau^5]$	6	$\sum_i b_i c_i^5$	1

Table 3.3: Notations and some functions on the remaining trees after applying the simplifying conditions (3.3.29) for $k = 0, 1$.

i	notation	$r(t_i)$	$\phi_{\mathrm{RK}}(t_i)$	$\alpha(t_i)$
1	τ	1	$\sum_i b_i$	1
2	$[\tau]$	2	$\sum_i b_i c_i$	1
4	$[\tau^2]$	3	$\sum_i b_i c_i^2$	1
6	$[_2\tau^2]_2$	4	$\sum_{ij} b_i a_{ij} c_j^2$	1
8	$[\tau^3]$	4	$\sum_i b_i c_i^3$	1
10	$[_3\tau^2]_3$	5	$\sum_{ijk} b_i a_{ij} a_{jk} c_k^2$	1
12	$[_2\tau^3]_2$	5	$\sum_{ij} b_i a_{ij} c_j^3$	1
14	$[[\tau^2]\tau]$	5	$\sum_{ij} b_i c_i a_{ij} c_j^2$	4
17	$[\tau^4]$	5	$\sum_i b_i c_i^4$	1
19	$[_4\tau^2]_4$	6	$\sum_{ijkl} b_i a_{ij} a_{jk} a_{kl} c_l^2$	1
21	$[_3\tau^3]_3$	6	$\sum_{ijk} b_i a_{ij} a_{jk} c_k^3$	1
23	$[_2[\tau^2]\tau]_2$	6	$\sum_{ijk} b_i a_{ij} c_j a_{jk} c_k^2$	4
26	$[_2\tau^4]_2$	6	$\sum_{ij} b_i a_{ij} c_j^4$	1
28	$[[_2\tau^2]_2\tau]$	6	$\sum_{ijk} b_i c_i a_{ij} a_{jk} c_k^2$	5
30	$[[\tau^3]\tau]$	6	$\sum_{ij} b_i c_i a_{ij} c_j^3$	5
34	$[[\tau^2]\tau^2]$	6	$\sum_{ij} b_i c_i^2 a_{ij} c_j^2$	10
37	$[\tau^5]$	6	$\sum_i b_i c_i^5$	1

Table 3.4: Notations and some functions on the remaining trees after applying the simplifying conditions (3.3.29) for $k = 0, 1, 2$.

i	notation	$r(t_i)$	$\phi_{\text{RK}}(t_i)$	$\alpha(t_i)$
1	τ	1	$\sum_i b_i$	1
2	$[\tau]$	2	$\sum_i b_i c_i$	1
4	$[\tau^2]$	3	$\sum_i b_i c_i^2$	1
8	$[\tau^3]$	4	$\sum_i b_i c_i^3$	1
12	$[_2\tau^3]_2$	5	$\sum_{ij} b_i a_{ij} c_j^3$	1
17	$[\tau^4]$	5	$\sum_i b_i c_i^4$	1
21	$[_3\tau^3]_3$	6	$\sum_{ijk} b_i a_{ij} a_{jk} c_k^3$	1
26	$[_2, \tau^4]_2$	6	$\sum_{ij} b_i a_{ij} c_j^4$	1
30	$[[\tau^3]\tau]$	6	$\sum_{ij} b_i c_i a_{ij} c_j^3$	5
37	$[\tau^5]$	6	$\sum_i b_i c_i^5$	1

3.4 Formulation of CPHBT$_{\text{RK4}}(d,s,p)$ methods in Shu-Osher form

Let $\sigma = [0, c_2, c_3, \ldots, c_s]^T$ define the $s-1$ off-step points $x_n + c_j\Delta x$, $j = 2, 3, \ldots, s$. Then, $F_1 = f_n$ and $F_j := f(x_n + c_j\Delta x, Y_j)$, $j = 2, 3, \ldots, s$, where we always choose $c_1 = 0$.

To be able to write CPHBT$_{\text{RK4}}(d,s,p)$ methods as a convex combination of the forward Euler expanded series FES(d) method (3.1.2), we restrict ourself to nonnegative coefficient triplets $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{\gamma}_0)$ and rewrite the formulas (3.2.1) and (3.2.2) in the modified Shu-Osher form as in [53, 18] for $2 \le i \le s+1$,

$$
\begin{aligned}
Y_i &= v_i y_n + \left[\sum_{j=1}^{i-1} \alpha_{i,j} Y_j + \Delta x\, \beta_{i,j} F_j\right] + \sum_{m=2}^{d} (\Delta x)^m \delta_{i,m}\, y_n^{(m)}, \\
y_{n+1} &= Y_{s+1}, \qquad (3.4.1)
\end{aligned}
$$

where

$$v_i = 1 - \sum_{j=1}^{i-1} \alpha_{ij}, \tag{3.4.2}$$

$$\beta_{ij} = a_{ij} - \sum_{l=j+1}^{i-1} \alpha_{il} a_{lj}, \tag{3.4.3}$$

$$\delta_{im} = \gamma_{im} - \sum_{j=1}^{i-1} \alpha_{ij} \gamma_{jm}. \tag{3.4.4}$$

Also, we require that v_i, α_{ij}, β_{ij} and δ_{im} to be nonnegative for $i = 2, 3, \ldots, s+1$, $j = 1, 2, \ldots, i-1$ and $m = 2, 3, \ldots, d$. We note that equation (3.4.2) is the consistency condition of CPHBT$_{\text{RK4}}(d, s, p)$.

Definition 3.4.1 (Contractivity preserving coefficient $c_{\mathbf{fcp}}$) *Let the forward Euler expanded series FES(d)* (3.1.2) *be contractive for all $\Delta x \leq \Delta FED(d)$. Then, a d-derivative CPHBT is a contractivity preserving method with contractivity preserving coefficient c_{fcp} if the method is contractive for all $\Delta x \leq c_{fcp} \Delta FED(d)$.*

The following theorem proves that under certain conditions on δ_{im} and the step-size Δx, the contractivity preserving property (3.1.3) of CPHBT$_{\text{RK4}}(d, s, p)$ method holds.

Theorem 3.4.1 *The new one-step, d-derivative, s-stages, explicit CPHBT$_{RK4}(d, s, p)$ method of order p (3.4.1) satisfies the contractivity preserving property (3.1.3) provided that*

- *All the coefficients of CPHBT$_{RK4}(d, s, p)$ are nonnegative.*
- *f satisfies the FES(d) condition* (3.1.4).
- *$\Delta x \leq c_{fcp} \Delta FES(d)$ where c_{fcp} is defined by:*

$$c_{fcp} = \min_{\substack{i = 2, 3, \ldots, s+1 \\ j = 1, 2 \ldots, i-1}} \left\{ \frac{\alpha_{ij}}{\beta_{ij}} \right\}. \tag{3.4.5}$$

- $\delta_{i,m}$ *satisfies the following conditions:*

$$\frac{\delta_{i,m}}{\alpha_{i,1}} \leq \left[\frac{1}{c_{fcp}}\right]^m \frac{1}{m!}, \tag{3.4.6}$$

for $i = 2,3,\ldots,s+1$ *and* $m = 2,3,\ldots,d$.

Proof: We rewrite the stages of the new CPHBT$_{\text{RK4}}(d,s,p)$ in (3.4.1) as follows:

$$\begin{aligned} Y_i = v_i y_n &+ \sum_{j=2}^{i-1} \alpha_{i,j}\left[Y_j + \Delta x\,\frac{\beta_{i,j}}{\alpha_{i,j}} F_j\right] + \alpha_{i,1}\Bigg[y_n + \Delta x\,\frac{\beta_{i,1}}{\alpha_{i,1}} f_n \\ &+ \sum_{m=2}^{d} (\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, y_n^{(m)}\Bigg], \quad i = 2,3,\ldots,s+1 \end{aligned}$$

Then, by applying the same norm or convex functional $\|\cdot\|$ as in (3.1.4) we get

$$\begin{aligned} \|Y_i - \tilde{Y}_i\| &= \Bigg\| v_i(y_n - \tilde{y}_n) + \sum_{j=2}^{i-1} \alpha_{i,j}\left[Y_j + \Delta x\,\frac{\beta_{i,j}}{\alpha_{i,j}} F_j - (\tilde{Y}_j + \Delta x\,\frac{\beta_{i,j}}{\alpha_{i,j}} \tilde{F}_j)\right] \\ &+ \alpha_{i,1}\left[y_n + \Delta x\,\frac{\beta_{i,1}}{\alpha_{i,1}} f_n + \sum_{m=2}^{d}(\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, y_n^{(m)} - (\tilde{y}_n + \Delta x\,\frac{\beta_{i,1}}{\alpha_{i,1}} \tilde{f}_n + \sum_{m=2}^{d}(\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, \tilde{y}_n^{(m)})\right]\Bigg\| \\ &\leq v_i \Big\| y_n - \tilde{y}_n \Big\| + \sum_{j=2}^{i-1} \alpha_{i,j} \Big\| Y_j + \Delta x\,\frac{\beta_{i,j}}{\alpha_{i,j}} F_j - (\tilde{Y}_j + \Delta x\,\frac{\beta_{i,j}}{\alpha_{i,j}} \tilde{F}_j)\Big\| \\ &+ \alpha_{i,1}\Big\| y_n + \Delta x\,\frac{\beta_{i,1}}{\alpha_{i,1}} f_n + \sum_{m=2}^{d}(\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, y_n^{(m)} - (\tilde{y}_n + \Delta x\,\frac{\beta_{i,1}}{\alpha_{i,1}} \tilde{f}_n + \sum_{m=2}^{d}(\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, \tilde{y}_n^{(m)})\Big\|. \end{aligned}$$

However, by the assumptions of the theorem, $\Delta x \frac{\beta_{i,j}}{\alpha_{i,j}} \leq \Delta\text{FES}(d)$ and $(\Delta x)^m \frac{\delta_{i,m}}{\alpha_{i,1}} \leq \left[\frac{\Delta x}{c_{\text{fcp}}}\right]^m \frac{1}{m!}$. Then, inequality (3.1.4) holds and we get

$$\begin{aligned} \|Y_i - \tilde{Y}_i\| &\leq v_i\,\|y_n - \tilde{y}_n\| + \sum_{j=2}^{i-1} \alpha_{i,j}\,\|y_n - \tilde{y}_n\| + \alpha_{i,1}\,\|y_n - \tilde{y}_n\| \\ &= \|y_n - \tilde{y}_n\|\,. \end{aligned}$$

This is valid for $i = 2,3,\ldots,s+1$. Hence, the contractivity preserving property (3.1.3) follows by taking $i = s+1$. ∎

Remark 3.4.1 *In the above theorem:*

- *We use the notations* $\tilde{f}_n = f(\tilde{y}_n)$ *and* $\tilde{F}_j = f(\tilde{Y}_j)$.

- *We use the convention that* $\frac{\alpha_{ij}}{0} = +\infty$ *in the evaluation of* c_{fcp} *in (3.4.5).*

- *Enforcing the conditions of the theorem with the order conditions, which are derived in Section 3.3, will generate a feasible* $CPHBT_{RK4}(d,s,p)$ *method with a feasible contractivity preserving coefficient* c_{fcp} *defined in* (3.4.5).

- *We are interested in an optimal* $CPHBT_{RK4}(d,s,p)$ *with an optimal contractivity preserving coefficient* c_{cp}*, i.e., with the largest contractivity preserving coefficient that will allow a maximum stepsize while maintaining the contractivity preserving property* (3.1.3).

- *To improve the optimization process of the coefficient of* $CPHBT_{RK4}(d,s,p)$*, we need to present the method in different forms.*

3.5 CPHBT$_{\mathbf{RK4}}(d,s,p)$ in vector notation

We are interested in a more compact form of the modified Shu-Osher form of CPHBT$_{\text{RK4}}$ (d,s,p) (3.4.1). Let $\boldsymbol{v} \in \mathbb{R}^{s+1}$, $\boldsymbol{v} = [0, v_2, v_3, \ldots, v_{s+1}]^T$, and define the matrices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{(s+1)\times(s+1)}$, and $\boldsymbol{\delta} \in \mathbb{R}^{(s+1)\times(d-1)}$ as follows:

$$(\boldsymbol{\alpha})_{ij} = \begin{cases} \alpha_{ij} & i = 2,3,\ldots,s+1, \quad j = 1,2,\ldots,i-1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.1}$$

$$(\boldsymbol{\beta})_{ij} = \begin{cases} \beta_{ij} & i = 2,3,\ldots,s+1, \quad j = 1,2,\ldots,i-1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.2}$$

$$(\boldsymbol{\delta})_{ij} = \begin{cases} \delta_{im} & i = 2,3,\ldots,s+1, \quad m = 2,3,\ldots,d, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.3}$$

Also, let $\boldsymbol{Y}, \boldsymbol{F} \in \mathbb{R}^{(s+1)\times N}$ and $\boldsymbol{f}_{\mathrm{B}} \in \mathbb{R}^{(d-1)\times N}$:

$$\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_{s+1}]^T, \qquad \boldsymbol{F} = [F_1, F_2, \ldots, F_{s+1}]^T,$$
$$\boldsymbol{f}_{\mathrm{B}} = [(\Delta x)^2 y_n^{(2)}, (\Delta x)^3 y_n^{(3)}, \ldots, (\Delta x)^d y_n^{(d)}]^T,$$

with the following N-vectors: Y_j, F_j for $j = 1, 2, \ldots, s+1$, $y_n^{(j)}$ for $j = 2, 3, \ldots, d$, $Y_1 = y_n$, $F_1 = f_n$, $Y_{s+1} = y_{n+1}$ and $F_{s+1} = f_{n+1}$, where N is the dimension of ODE system. Hence, we can rewrite $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$ (3.4.1) compactly as follows:

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{v} y_n^T + \boldsymbol{\alpha} \boldsymbol{Y} + \Delta x \boldsymbol{\beta} \boldsymbol{F} + \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}, \\ y_{n+1} &= Y_{s+1}. \end{aligned} \tag{3.5.4}$$

Note that the consistency condition (3.4.2) becomes

$$\boldsymbol{v} + \boldsymbol{\alpha} \boldsymbol{e}_{s+1} = \boldsymbol{e}_{s+1}, \tag{3.5.5}$$

where the $(s+1)$-vector $\boldsymbol{e}_{s+1}$ is

$$\boldsymbol{e}_{s+1} = [1, 1, 1, \ldots, 1]^T \in \mathbb{R}^{s+1}.$$

3.6 The Butcher form in vector notation

This subsection describes a generalized result for the new $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$, using the result for RK methods, following closely section 3.2.1 of [18, pp. 31–32] and our published work [42].

Recall that the coefficients matrix $\boldsymbol{\alpha} \in \mathbb{R}^{(s+1)\times(s+1)}$ is defined in (3.5.1). If we let $\boldsymbol{\alpha} = \boldsymbol{0}$, then the Shu-Osher form (3.5.4) becomes,

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{v} y_n^T + \Delta x \boldsymbol{\beta} \boldsymbol{F} + \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}, \\ y_{n+1} &= Y_{s+1}. \end{aligned} \tag{3.6.1}$$

which is the Butcher form (3.2.1) and (3.2.2). The elements $\boldsymbol{v}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ of (3.6.1) are then denoted by $\boldsymbol{v}_0$, $\boldsymbol{\beta}_{\boldsymbol{0}}$, $\boldsymbol{\gamma}_0$, respectively. Here, the consistency condition (3.5.5) becomes

$$\boldsymbol{v}_0 = \boldsymbol{e}_{s+1}. \tag{3.6.2}$$

Hence the Butcher form (3.6.1) can be rewritten as

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{e}_{s+1} y_n^T + \Delta x \boldsymbol{\beta}_0 \boldsymbol{F} + \boldsymbol{\gamma}_0 \boldsymbol{f}_{\mathrm{B}}, \\ y_{n+1} &= Y_{s+1}. \end{aligned} \tag{3.6.3}$$

To find the relation between the Shu-Osher coefficients and the Butcher coefficients, we can solve (3.5.4) for $\boldsymbol{Y}$,

$$\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{v} y_n^T + \Delta x (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\beta} \boldsymbol{F} + (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}, \tag{3.6.4}$$

where $(\boldsymbol{I} - \boldsymbol{\alpha})$ is invertible since $\boldsymbol{\alpha}$ is a strictly lower triangular matrix. Comparing (3.6.4) with (3.6.3), we have the following relations between the modified Shu-Osher coefficients and the Butcher coefficients,

$$\begin{aligned} \boldsymbol{v}_0 &= \boldsymbol{e}_{s+1} = (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{v}, \\ \boldsymbol{\beta}_0 &= (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\beta}, \\ \boldsymbol{\gamma}_0 &= (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\delta}. \end{aligned} \tag{3.6.5}$$

Remark 3.6.1

- *These relations will enable us to transform a Shu-Osher form of $CPHBT_{RK4}(d, s, p)$ into its Butcher form and vice versa.*
- *The relations* (3.6.5) *confirm the previously derived equations* (3.4.2)-(3.4.4).
- *The form* (3.6.3) *is the Butcher form* (3.2.1) *and* (3.2.2) *with* $\boldsymbol{\gamma}_0$ *defined in* (3.2.5) *and the following matrix,*

$$\boldsymbol{\beta}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{2,1} & 0 & 0 & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ a_{s,1} & a_{s,2} & \dots & a_{s,s-1} & 0 & 0 \\ b_1 & b_2 & \dots & b_{s-1} & b_s & 0 \end{bmatrix}. \tag{3.6.6}$$

3.7 $\mathbf{CPHBT_{RK4}}(d,s,p)$ in the canonical Shu-Osher form

A more useful way to represent the results of $\mathrm{CPHBT_{RK4}}(d,s,p)$ is to consider the canonical form of (3.5.4). To derive its canonical Shu-Osher form, we require the ratio $r = \frac{\alpha_{i,j}}{\beta_{i,j}}$ to be constant for $i = 2,3,4,\ldots,s+1$ and $j = 1,2,3,\ldots,i-1$, such that $\beta_{i,j} \neq 0$ [18, 42]. Generally, if the particular Shu-Osher form of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is sparse, the Shu-Osher form of $\mathrm{CPHBT_{RK4}}(d,s,p)$ will allow a reduced-storage implementation, similar to the work of Ketcheson on optimal explicit SSP Runge-Kutta methods [30].

We shall denote the coefficient matrices of this special form by $\boldsymbol{\alpha}_r$, $\boldsymbol{\beta}_r$, and require that $\boldsymbol{\alpha}_r = r\boldsymbol{\beta}_r$. Substituting this relation into (3.6.5), we can solve for $\boldsymbol{\beta}_r$ in terms of $\boldsymbol{\beta}_0$ and r. Thus we find

$$\begin{aligned}(\boldsymbol{I} - r\boldsymbol{\beta}_r)^{-1}\boldsymbol{\beta}_r = \boldsymbol{\beta}_0 \quad &\Leftrightarrow \quad \boldsymbol{\beta}_r = \boldsymbol{\beta}_0 - r\boldsymbol{\beta}_r\boldsymbol{\beta}_0 \\ &\Leftrightarrow \quad \boldsymbol{\beta}_r(\boldsymbol{I} + r\boldsymbol{\beta}_0) = \boldsymbol{\beta}_0.\end{aligned}$$

Hence, the coefficients for this form are given by

$$\boldsymbol{v}_r = (\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\boldsymbol{v}_0 \qquad = (\boldsymbol{I} - \boldsymbol{\alpha}_r)\boldsymbol{v}_0, \tag{3.7.1}$$

$$\boldsymbol{\beta}_r = \boldsymbol{\beta}_0(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1} \qquad = \boldsymbol{\beta}_0(\boldsymbol{I} - \boldsymbol{\alpha}_r), \tag{3.7.2}$$

$$\boldsymbol{\alpha}_r = r\boldsymbol{\beta}_r = r\boldsymbol{\beta}_0(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1} \quad = r\boldsymbol{\beta}_0(\boldsymbol{I} - \boldsymbol{\alpha}_r), \tag{3.7.3}$$

$$\boldsymbol{\delta}_r = (\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\boldsymbol{\gamma}_0 \qquad = (\boldsymbol{I} - \boldsymbol{\alpha}_r)\boldsymbol{\gamma}_0, \tag{3.7.4}$$

where the identity $(\boldsymbol{I} - \boldsymbol{\alpha}_r) = (\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}$ follows from

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{\alpha}_r)(\boldsymbol{I} + r\boldsymbol{\beta}_0) &= (\boldsymbol{I} - r\boldsymbol{\beta}_r)(\boldsymbol{I} + r\boldsymbol{\beta}_0) \\ &= \boldsymbol{I} + r\boldsymbol{\beta}_0 - r\boldsymbol{\beta}_r - r^2\boldsymbol{\beta}_r\boldsymbol{\beta}_0 = \boldsymbol{I}\end{aligned}$$

since $r\boldsymbol{\beta}_r = r\boldsymbol{\beta}_0 - r^2\boldsymbol{\beta}_r\boldsymbol{\beta}_0$.

Then, using (3.6.5) and (3.7.2), we have

$$\begin{aligned} \boldsymbol{\beta}_r &= \boldsymbol{\beta}_0 \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \\ &= \boldsymbol{\beta}_0 \left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right) \\ &= \left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right) \boldsymbol{\beta}_0 \\ &= \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\beta}_0. \end{aligned} \tag{3.7.5}$$

As in [18], we will refer to the form obtained by the relations (3.7.1)–(3.7.4) as the *canonical Shu-Osher form* of CPHBT$_{\text{RK4}}(d, s, p)$:

$$\boldsymbol{Y} = \boldsymbol{v}_r y_n^T + \boldsymbol{\alpha}_r \boldsymbol{Y} + \Delta x \boldsymbol{\beta}_r \boldsymbol{F} + \boldsymbol{\delta}_r \boldsymbol{f}_{\text{B}}. \tag{3.7.6}$$

Then, (3.7.6) can be written in terms of the Butcher array:

$$\begin{aligned} \boldsymbol{Y} = \left[\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{v}_0 y_n^T\right] + \Big[r\boldsymbol{\beta}_0 \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{Y} \\ + \Delta x \boldsymbol{\beta}_0 \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{F}\Big] + \left[\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\gamma}_0 \boldsymbol{f}_{\text{B}}\right]. \end{aligned} \tag{3.7.7}$$

Using (3.7.5) and (3.7.7), we have

$$\boldsymbol{Y} = \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \left[\boldsymbol{v}_0 y_n^T + \boldsymbol{\beta}_0 \left(r\boldsymbol{Y} + \Delta x \boldsymbol{F}\right) + \boldsymbol{\gamma}_0 \boldsymbol{f}_{\text{B}}\right]. \tag{3.7.8}$$

Here the consistency condition is

$$\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{v}_0 + r \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\beta}_0 \boldsymbol{e}_{s+1} = \boldsymbol{e}_{s+1}. \tag{3.7.9}$$

Remark 3.7.1

- *Since* $\boldsymbol{\beta}_0$ *is strictly lower triangular, then* $\boldsymbol{I} + r\boldsymbol{\beta}_0$ *is invertible and the transformations* (3.7.1)-(3.7.4) *are always defined.*
- *For* $r = 0$*, the consistency condition* (3.7.9) *is equivalent to* (3.6.2) *and the canonical Shu-Osher form* (3.7.6) *or* (3.7.8) *corresponds to the Butcher form* (3.6.3)*, with coefficient vector* $\boldsymbol{v}_0$ *and coefficient matrices* $\boldsymbol{\beta}_0$ *and* $\boldsymbol{\gamma}_0$.

- *Generally, the sparse canonical Shu-Osher forms* (3.7.6) *or* (3.7.8) *will allow for reduced-storage implementation which enhance the performance of CPHBT$_{RK4}$* (d, s, p).

- *The relations* (3.7.1)–(3.7.4) *will enable us to transform easily a Butcher form of CPHBT$_{RK4}(d, s, p)$ method into its canonical Shu-Osher form and vice versa.*

Recall that in Theorem 3.4.1, the contractivity preserving coefficient c_{fcp} is given in (3.4.5) as a minimum function $\min(\cdot)$. Indeed, we are interested in optimal CPHBT methods with maximal c_{fcp} to maximize the step-size in (3.4.5). However, the $\min(\cdot)$ function is not smooth or even sensitive to its parameters which will generate various problems especially if the optimization process uses gradient information. To avoid these issues, we state an improved version of Theorem 3.4.1 where we use a modified form of the *canonical Shu-Osher form* in which we consider a fixed ratio

$$\tilde{r} = \frac{\alpha_{ij}}{\beta_{ij}} \qquad i = 3, 4, \ldots, s+1, \quad j = 2, 3, \ldots, i-1,$$

such that the feasible contractivity preserving coefficient c_{fcp} defined in 3.4.1 is equal to $\tilde{r}$ and should satisfy

$$c_{\text{fcp}} = \tilde{r} \leq r_{i1} = \frac{\alpha_{i1}}{\beta_{i1}} \qquad i = 2, 3, \ldots, s+1.$$

Moreover, conditions (3.4.6) with c_{fcp} replaced by $r_{i1} = \frac{\alpha_{i1}}{\beta_{i1}}$ are imposed on δ_{im} for $i = 2, 3, \ldots, s+1$ and $m = 2, 3, \ldots, d$. We then get the following theorem:

Theorem 3.7.1 *The new one-step, d-derivative, s-stages, explicit CPHBT$_{RK4}(d, s, p)$ method (3.4.1) satisfies the contractivity preserving property (3.1.3) provided that*

- *All the coefficients of CPHBT$_{RK4}(d, s, p)$ are nonnegative.*

- *f satisfies the FES(d) condition* (3.1.4).

- $\Delta x \leq c_{fcp} \Delta FES(d)$ *where* $\Delta FES(d)$ *is given in* (3.1.4) *and* c_{fcp} *is defined by:*

$$c_{fcp} = \tilde{r} = \left\{ \frac{\alpha_{ij}}{\beta_{ij}} \right\} \qquad i = 3, 4, \ldots, s+1, \;\; j = 2, 3, \ldots, i-1. \tag{3.7.10}$$

- $\tilde{r}$ *satisfies the conditions:*

$$\tilde{r} \leq r_{i1} = \frac{\alpha_{i1}}{\beta_{i1}} \qquad i = 2, 3, \ldots, s+1. \tag{3.7.11}$$

- $\delta_{i,m}$ *satisfies the following conditions:*

$$\delta_{i,m} r_{i1}^{m} m! - \alpha_{i,1} \leq 0 \qquad i = 2, 3, \ldots, s+1, \;\; m = 2, 3, \ldots, d. \tag{3.7.12}$$

The conditions on $\delta_{i,m}$ in the form (3.7.12) are more numerically stable and will enhance the performance of the optimization software. The proof of Theorem 3.7.1 is a straightforward extension of the proof of Theorem 3.4.1.

3.8 Formulation of the optimization problem of CPHBT$_{\text{RK4}}(d, s, p)$

As we mentioned before, any set of nonnegative coefficients $(\boldsymbol{v}_{\tilde{r}}, \boldsymbol{\alpha}_{\tilde{r}}, \boldsymbol{\beta}_{\tilde{r}}, \boldsymbol{\delta}_{\tilde{r}})$ satisfying the order conditions (3.3.30)-(3.3.36), the simplifying conditions (3.3.29) and the conditions of Theorem 3.7.1 will generate a feasible CPHBT$_{\text{RK4}}(d, s, p)$ of order p. In this section, we formulate an optimization process to find an optimal CPHBT$_{\text{RK4}}$ (d, s, p) with optimal contractivity preserving coefficient c_{cp} as follows:

$$\underset{\boldsymbol{v}_{\tilde{r}}, \boldsymbol{\alpha}_{\tilde{r}}, \boldsymbol{\beta}_{\tilde{r}}, \boldsymbol{\delta}_{\tilde{r}}}{\text{maximize}} \; \tilde{r} = c_{cp} \tag{3.8.1}$$

subject to

$$\left(\boldsymbol{I} + \tilde{r}\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{v}_0 \geq 0, \tag{3.8.2}$$

$$\left(\boldsymbol{I} + \tilde{r}\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\beta}_0 \geq 0, \tag{3.8.3}$$

$$\left(\boldsymbol{I}+\tilde{r}\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{\gamma}_0 \geq 0, \tag{3.8.4}$$

$$c_i \leq 1 \qquad i=2,3,\dots,s+1, \tag{3.8.5}$$

$$\tilde{r} \leq r_{i1} \qquad i=2,3,\dots,s+1, \tag{3.8.6}$$

$$\delta_{i,m} r_{i1}^m m! - \alpha_{i,1} \leq 0, \qquad \begin{cases} i=2,3,\dots,s+1, \\ m=2,3,\dots,d, \end{cases} \tag{3.8.7}$$

together with

- The simplifying conditions (3.3.29),
- The set of order conditions (3.3.30)-(3.3.36),

where inequalities (3.8.2)-(3.8.4) are taken component-wise and the coefficient matrices $\boldsymbol{v}_0, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0$ and $\boldsymbol{\delta}$ are defined in equations (3.6.5), (3.4.4) and (3.5.3). However, the objective function in this form is not smooth and hence using an optimization method depending on the gradient will not give reliable results. Moreover, Spiteri [56] showed that even optimization methods that do not use gradient information fail to converge to the same optimum. To avoid this problem, we follow a standard reformulation of the optimization process by introducing a dummy parameter z and the objective function

$$F : \mathbb{R} \times \mathbb{R}^{s+1} \times \mathbb{R}^{(s+1)\times(s+1)} \times \mathbb{R}^{(s+1)\times(s+1)} \times \mathbb{R}^{(s+1)\times(d-1)} \to \mathbb{R}$$
$$\text{such that} \quad F(z,\boldsymbol{v},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}) = z. \tag{3.8.8}$$

Then the optimization problem is as follows

$$\underset{z,\boldsymbol{v},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}}{\text{maximize}} \; F(z,\boldsymbol{v},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}) = z \tag{3.8.9}$$

subject to

- The constraint $\alpha_{ij} - z\beta_{ij} = 0 \quad \text{for} \quad i=3,4,\dots,s+1,\ j=2,3,\dots,i-1,$

- The constraints (3.8.2)-(3.8.7) with $\tilde{r}$ replaced by z,
- The simplifying conditions (3.3.29),
- The set of order conditions (3.3.30)-(3.3.36).

3.9 Construction of optimal $\mathbf{CPHBT_{RK4}}(d,s,p)$

The objective function (3.8.8) of the optimization process (3.8.9) is a projection function and hence it is continuous. Let $\mathrm{FS} \subset \mathbb{R} \times \mathbb{R}^{s+1} \times \mathbb{R}^{(s+1)\times(s+1)} \times \mathbb{R}^{(s+1)\times(s+1)} \times \mathbb{R}^{(s+1)\times(d-1)}$ be the feasible set of the optimization problem (3.8.9). We managed to find bounds for all the variables in the feasible set FS, component-wise, except the first component corresponding to z, in particular, $\mathrm{FS} \subset \mathbb{R}\times[0,1]^{s+1}\times[0,1]^{(s+1)\times(s+1)}\times[0,1]^{(s+1)\times(s+1)} \times [0,1]^{(s+1)\times(d-1)}$. However, following [55, 54, 18] and our published work in [42], our numerical search for optimal solutions is computationally stable and the variable z is bounded. Since $\mathrm{CPHBT_{RK4}}(d,s,p)$ methods contain many free parameters, the MATLAB Optimization Toolbox was used to search for the methods with largest contractivity preserving coefficient c_{cp}. Several authors, [55, 54, 18], have successfully used this technique to find optimal Runge-Kutta methods. In particular, the MATLAB function `fmincon` from the Optimization Toolbox is used to solve the optimization problem above. Although no analytic proof of global optimality is provided, the obtained methods are a result of performing an extensive numerical search with a large number of initial points similar to Spiteri and Ruuth's approach in [55, 54]. In this work, depending on the size of the feasible set, the MATLAB Optimization Toolbox was used to tolerance in the range $(10^{-15}, 10^{-12})$ on the objective function provided that all the constraints were satisfied to tolerance in a the range $(10^{-17}, 10^{-14})$.

We managed to obtain s-stage, d-derivative $\mathrm{CPHBT_{RK4}}(d,s,p)$ methods of order p for $p = 5, 6, \dots, 15$, $s = 5, 6$ and $d = 2, 3, \dots, 13$. In Tables 3.5, we summarize these methods with their optimal contractivity preserving coefficients c_{cp} and interval of absolute stability compared to Taylor series methods of the same order. Moreover, we managed to obtain a contractivity preserving $\text{CPHBT}_{\text{RK4}}(2, 5, 5)$ of order 5 with $c_{cp} = 1.0625$ and $|\alpha| = 5.68$ which is given in Section A of the Appendix. This shows that by adding only one more derivative to $\text{RK}(s, 4)$, we can break the Kraaijevanger order barrier [31, page 516] and obtain a contractivity preserving method of order 5 with nonnegative coefficients. The formulae of the new $\text{CPHBT}_{\text{RK4}}(d, s, p)$ with their contractivity preserving coefficient, $c_{cp}(\text{CPHBT}_{\text{RK4}}(d, s, p))$, and abscissa vector σ are given in Section A of the Appendix.

Table 3.5: The contractivity preserving coefficients c_{cp} and the interval of absolute stability $(\alpha, 0)$ of $\text{CPHBT}_{\text{RK4}}(d, s, p)$ compared to $\text{T}(p)$.

$s \backslash p$	5	6	7	8	9	10	11	12	13	14	15
$\text{T}(p)$	T(5)	T(6)	T(7)	T(8)	T(9)	T(10)	T(11)	T(12)	T(13)	T(14)	T(15)
$\lvert\alpha\rvert$	3.2	3.55	3.95	4.3	4.7	5.05	5.45	5.8	6.17	6.55	6.925
$\boldsymbol{s = 5}$											
(d, p)	(3,5)	(4,6)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,12)	(11,13)	(12,14)	(13,15)
c_{cp}	1.6544	1.6289	1.5486	1.4715	1.4116	1.3639	1.3253	1.2936	1.2674	1.2452	1.2261
$\lvert\alpha\rvert$	6	6.16	5.24	6.52	5.68	6.8	6.8	6.88	6.8	7.83	7.83
$\boldsymbol{s = 6}$											
(d, p)	(2,5)	(3,6)	(4,7)	(5,8)	(6,9)	(7,10)	(8,11)	(9,12)	(10,13)	*	*
c_{cp}	1.837	1.0078	1.2976	1.1226	1.0678	0.6112	0.654	0.993	0.979	*	*
$\lvert\alpha\rvert$	5.56	5.36	5.48	5.84	5.24	5.56	5.52	6.28	6.68	*	*

Chapter 4

Numerical results for the designed $\mathbf{CPHBT_{RK4}}(d,s,p)$ methods obtained from $\mathbf{T}(d)$ and $\mathbf{RK}(s,4)$ methods

4.1 Introduction

In this chapter, we will test the methods derived in chapter 3 and compare them to different well known and widely used numerical methods. The s-stage, one step, d-derivatives $\mathrm{CPHBT_{RK4}}(d,s,p)$ method of order p requires the evaluation of the d derivatives $y^{(1)}, y^{(2)}, \ldots, y^{(d)}$ at each integration step. These derivatives were computed recursively using the automatic differentiation technique that is described briefly in Section 2.2.1 and in more details in [22, page 46-49] and [34]. All numerical tests and simulations were obtained using two systems: a Macbook pro 6,2 with a 10.6.8 Mac OSX, a 2.53 GHz Intel Core i5 processor and a 4 GB 1067 MHz DDR3 memory. The second machine is a Sun Microsystems with AMD Opteron(tm) processor 152 and a 2GB memory. All the machine dependent numerical results including the CPU time analysis were performed in C++ codes using the Macbook pro machine while the rest of the machine independent results were obtained using the Sun Microsystems machine using MATLAB and FORTRAN codes.

In the following sections, we test the 6-stage, 9-derivatives $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$ of order 12, which satisfies the objective function and constraints to tolerances 10^{-15} and 10^{-17}, respectively. This method, presented below, is chosen to be a representative of the class of $\mathrm{CPHBT}_{\mathrm{RK4}}(d,s,p)$ methods that are constructed in the previous chapter. The rest of the designed methods are expected to give results similar to the ones presented in this chapter.

$\mathbf{CPHBT_{RK4}}(9,6,12)$ with $c_{\mathrm{cp}} = 9.9163130315966042\times10^{-1}$, and abscissa vector $\sigma = [0\ 8.5568256645013996\times10^{-1}\ 7.1829693393055971\times10^{-1}\ 8.0835544845753426\times10^{-1}\ 8.8980234528386681\times10^{-1}\ 9.7567281938112871\times10^{-1}]^T$.

$$
\begin{aligned}
Y_2 =& y_n + 8.5568256645013996 \times 10^{-1} h_n f_n + 3.6609632726334906 \times 10^{-1} h_n^2 y_n^{(2)} + 1.0442074829355763 \times 10^{-1} h_n^3 y_n^{(3)} \\
&+ 2.2337753472618866 \times 10^{-2} h_n^4 y_n^{(4)} + 3.8228052440362082 \times 10^{-3} h_n^5 y_n^{(5)} + 5.4518463370932600 \times 10^{-4} h_n^6 y_n^{(6)} \\
&+ 6.6643569508796518 \times 10^{-5} h_n^7 y_n^{(7)} + 7.1282175743356624 \times 10^{-6} h_n^8 y_n^{(8)} + 6.7772127869139232 \times 10^{-7} h_n^9 y_n^{(9)}. \\
Y_3 =& y_n + 7.0032387510619154 \times 10^{-1} h_n f_n + 1.7973058824368105 \times 10^{-2} h_n F_2 + 2.4259600954522675 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 5.5187737782493251 \times 10^{-2} h_n^3 y_n^{(3)} + 9.2151107182336153 \times 10^{-3} h_n^4 y_n^{(4)} + 1.1919736246642830 \times 10^{-3} h_n^5 y_n^{(5)} \\
&+ 1.2205437013104496 \times 10^{-4} h_n^6 y_n^{(6)} + 9.7761743593585443 \times 10^{-6} h_n^7 y_n^{(7)} + 5.5977694224758458 \times 10^{-7} h_n^8 y_n^{(8)} \\
&+ 1.2156801810285854 \times 10^{-8} h_n^9 y_n^{(9)}. \\
Y_4 =& y_n + 5.9253804964276735 \times 10^{-1} h_n f_n + 3.7790846964182783 \times 10^{-3} h_n F_2 + 2.1203831411834850 \times 10^{-1} h_n F_3 \\
&+ 1.7117909772661233 \times 10^{-1} h_n^2 y_n^{(2)} + 3.1950954904935901 \times 10^{-2} h_n^3 y_n^{(3)} + 4.2991986294050764 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+ 4.3995842070519627 \times 10^{-4} h_n^5 y_n^{(5)} + 3.5189487487470358 \times 10^{-5} h_n^6 y_n^{(6)} + 2.2401550747435463 \times 10^{-6} h_n^7 y_n^{(7)} \\
&+ 1.1920417110245276 \times 10^{-7} h_n^8 y_n^{(8)} + 6.5142505217891534 \times 10^{-9} h_n^9 y_n^{(9)}. \\
Y_5 =& y_n + 6.5225547192463695 \times 10^{-1} h_n f_n + 7.3327692956782933 \times 10^{-4} h_n F_2 + 4.1142711343481492 \times 10^{-2} h_n F_3 \\
&+ 1.9567088508618058 \times 10^{-1} h_n F_4 + 2.0752234507578865 \times 10^{-1} h_n^2 y_n^{(2)} + 4.2604870773856586 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+ 6.2756226953662323 \times 10^{-3} h_n^4 y_n^{(4)} + 6.9432484726877225 \times 10^{-4} h_n^5 y_n^{(5)} + 5.8166925177690642 \times 10^{-5} h_n^6 y_n^{(6)} \\
&+ 3.5517957695381351 \times 10^{-6} h_n^7 y_n^{(7)} + 1.3567209577595762 \times 10^{-7} h_n^8 y_n^{(8)} + 1.2639876550905203 \times 10^{-9} h_n^9 y_n^{(9)}. \\
Y_6 =& y_n + 7.2075952122256659 \times 10^{-1} h_n f_n + 1.5003061651449836 \times 10^{-4} h_n F_2 + 8.4172968211253581 \times 10^{-3} h_n F_3 \\
&+ 4.0031846060581737 \times 10^{-2} h_n F_4 + 2.0631412466034044 \times 10^{-1} h_n F_5 + 2.5385547529513375 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 5.7816607679521070 \times 10^{-2} h_n^3 y_n^{(3)} + 9.4732176422683261 \times 10^{-3} h_n^4 y_n^{(4)} + 1.1701172028993881 \times 10^{-3} h_n^5 y_n^{(5)} \\
&+ 1.0997733968369118 \times 10^{-4} h_n^6 y_n^{(6)} + 7.5740152977001563 \times 10^{-6} h_n^7 y_n^{(7)} + 3.2206534412851753 \times 10^{-7} h_n^8 y_n^{(8)} \\
&+ 2.5860379801980058 \times 10^{-10} h_n^9 y_n^{(9)}. \\
y_{n+1} =& y_n + 6.3769782436600186 \times 10^{-1} h_n f_n + 8.4507585951567590 \times 10^{-4} h_n F_2 + 4.7415786695281098 \times 10^{-2} h_n F_3 \\
&+ 2.2550450654635593 \times 10^{-1} h_n F_4 + 1.5037770460171211 \times 10^{-2} h_n F_5 + 1.9783881742796200 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 3.9512215213133563 \times 10^{-2} h_n^3 y_n^{(3)} + 5.6542687574121630 \times 10^{-3} h_n^4 y_n^{(4)} + 6.0866317220988314 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+ 5.0062835142051295 \times 10^{-5} h_n^6 y_n^{(6)} + 3.0966865898048209 \times 10^{-6} h_n^7 y_n^{(7)} + 1.3443315671020195 \times 10^{-7} h_n^8 y_n^{(8)} \\
&+ 3.2502867248596942 \times 10^{-9} h_n^9 y_n^{(9)}.
\end{aligned}
$$

Our method is compared with the widely used Dormand-Prince Runge-Kutta pair , DP(8,7)13M, of order 8 [43], Taylor method of order 12, T(12), and the Taylor method of order 12 introduced by Martin Lara, T(12)L [34]. Also, we will use the standard bank of differential equations test problems, DETEST, which is used in the literature as a comparison tool of higher order numerical methods [27]. In particular, we concentrate on the 20 nonstiff DESTEST problems in classes A, B, D, E which are described briefly in Section 2.3.

4.2 Stability region of CPHBT$_{\mathbf{RK4}}(d, s, p)$

To obtain the region of absolute stability of CPHBT$_{\mathrm{RK4}}(d, s, p)$, we write the method in Butcher form in compact vector notation

$$\boldsymbol{Y} = \boldsymbol{e}_{s+1} y_n^T + h \boldsymbol{\beta}_0 \boldsymbol{F} + \boldsymbol{\gamma}_0 \boldsymbol{f}_{\mathrm{B}}, \tag{4.2.1}$$

$$y_{n+1} = Y_{s+1}. \tag{4.2.2}$$

where

$$\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_{s+1}]^T, \qquad \boldsymbol{F} = [F_1, F_2, \ldots, F_{s+1}]^T,$$
$$\boldsymbol{f}_{\mathrm{B}} = [h^2 y_n^{(2)}, h^3 y_n^{(3)}, \ldots, h^d y_n^{(d)}]^T,$$

Definition 4.2.1 *[23] Given the Dahlquist test equation*

$$y' = \lambda y, \qquad y_0 = 1,$$

the stability region of a numerical method is defined as

$$S = \{z \in \mathbb{C} : |R(z)| \leq 1\}, \quad z = \lambda h.$$

where $R(z)$ is the stability function of the numerical method $y_{n+1} = R(z) y_n$.

Applying method (4.2.2) to the Dahlquist test equation we get

$$\boldsymbol{Y} = \boldsymbol{e}_{s+1} y_n + z\boldsymbol{\beta}_0 \boldsymbol{Y} + \boldsymbol{\gamma}_0 \hat{\boldsymbol{z}} y_n, \tag{4.2.3}$$

$$y_{n+1} = Y_{s+1}. \tag{4.2.4}$$

where $\hat{\boldsymbol{z}} = [z^2, z^3, \ldots, z^d]^T$. Solving (4.2.3) for $\boldsymbol{Y}$, we get the following

$$(I - z\boldsymbol{\beta}_0)\boldsymbol{Y} = (\boldsymbol{e}_{s+1} + \boldsymbol{\gamma}_0 \hat{\boldsymbol{z}}) y_n \tag{4.2.5}$$

$$\boldsymbol{Y} = (I - z\boldsymbol{\beta}_0)^{-1}(\boldsymbol{e}_{s+1} + \boldsymbol{\gamma}_0 \hat{\boldsymbol{z}}) y_n, \tag{4.2.6}$$

where $(I - z\boldsymbol{\beta}_0)$ is invertible since $\boldsymbol{\beta}_0$ is a strictly lower triangular matrix by definition. Hence, the stability function of CPHBT$_{\text{RK4}}(d, s, p)$ is

$$R(z) = \left[(I - z\boldsymbol{\beta}_0)^{-1}\right]_{s+1} (\boldsymbol{e}_{s+1} + \boldsymbol{\gamma}_0 \hat{\boldsymbol{z}}), \tag{4.2.7}$$

where $[X]_{s+1}$ denotes the $(s+1)$-th row of a matrix X. Also, another form of the stability function can be obtained from the Butcher form (3.2.1)-(3.2.2) as $R(z) = 1 + \sum_{j=1}^{d+2} s_j z^j$ where

$$\begin{aligned}
s_1 &= \sum_j b_j, \\
s_2 &= \sum_j b_j \sum_k a_{jk} + \gamma_{s+1,2}, \\
s_3 &= \sum_j b_j \Big(\sum_k a_{jk} \sum_l a_{kl} + \gamma_{j,2}\Big) + \gamma_{s+1,3}, \\
s_q &= \sum_j b_j \Big(\sum_k a_{jk}\gamma_{k,q-2} + \gamma_{j,q-1}\Big) + \gamma_{s+1,q} \quad \text{for } q = 4, 5, \ldots, d+2.
\end{aligned}$$

We used the scanning technique [33] to plot the region of absolute stability of CPHBT$_{\text{RK4}}$ $(9, 6, 12)$ which is easier to implement than the boundary locus technique. In Figure (4.1), we have the region of absolute stability of CPHBT$_{\text{RK4}}(9, 6, 12)$ with an interval of absolute stability $(-6.28, 0)$ compared to Taylor method of order 12, T(12), with interval of absolute stability $(-5.72, 0)$.

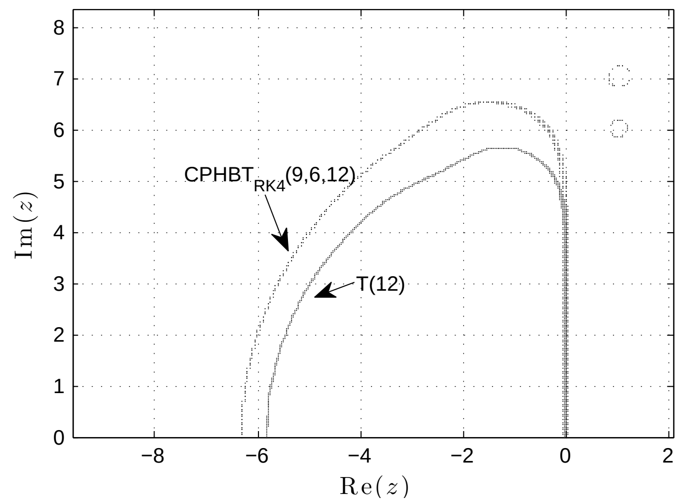


Figure 4.1: The region of absolute stability of CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ and T(12) methods.

Definition 4.2.1 (Scaled interval of absolute stability) *The scaled interval of absolute stability of a method M is given by $\frac{I_{abs}}{l}$ where I_{abs} and l are the interval of absolute stability and the number of function evaluations per integration step of the method M, respectively.*

The scaled interval of absolute stability of the new CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ is equal to $-6.28/14 = -0.4486$ which is larger than the scaled interval of absolute stability of DP(8,7)13M ($-5.12/13 = -0.3938$) [41].

4.3 Variable step algorithm of the CPHBT$_{\mathbf{RK4}}(d, s, p)$ methods

The use of a variable step formulation will improve the performance of CPHBT$_{\mathrm{RK4}}$ (d, s, p) method of order p since it will allow an automatic optimization of the step-size based on certain error control formula. To design such error control formula, we assume that the root criterion for convergence is applicable to the Taylor series of order p as follows:

$$\sqrt[p]{\|Y^{[p]}\|_\infty h^p} < k < 1 \qquad \forall p \geq p_0 \tag{4.3.1}$$

for some number p_0 where h is the step-size and $y(x+h) \approx \sum_{i=0}^{p} Y^{[i]} h^i = \sum_{i=0}^{p} \frac{y^{(i)}}{i!} h^i$ is the Taylor series method of order p, T(p). If E_p denotes the local truncation error at order p, then

$$|E_p| < k^{p+1} + k^{p+2} + \ldots = k^{p+1}(1 + k + k^2 + \ldots) = \frac{k^{p+1}}{1-k}. \tag{4.3.2}$$

Our goal is to design a step control algorithm that takes a user defined tolerance, TOL, and optimize the step-size to obtain a numerical solution with a maximum global error smaller than TOL. To force the local truncation error E_p to be less than the user's tolerance, it is sufficient that

$$\frac{k^{p+1}}{1-k} = \text{TOL}. \tag{4.3.3}$$

By solving this implicit equation for k, we get the values of k as a function of TOL and p, $k = k(\text{TOL}, p)$. Since we are interested in high order methods, TOL is usually very small and hence equation (4.3.3) can be simplified to $k = \text{TOL}^{\frac{1}{p+1}}$. However, to get a better estimate of the values k, we use Newton-Raphson method to solve equation (4.3.3) implicitly with initial points $k_0(\text{TOL}, p) = \text{TOL}^{\frac{1}{p+1}}$. In Figure (4.2), we plot the value of k as a function of p for different value of TOL.

Then, from (4.3.1), we have the following step control for the Taylor method of order p:

$$h_{T(p)} = \min\left\{ k(\text{TOL}, p-1)\|Y^{[p-1]}\|^{\frac{-1}{p-1}}, k(\text{TOL}, p)\|Y^{[p]}\|^{\frac{-1}{p}} \right\}. \tag{4.3.4}$$

However, for all the designed CPHBT$_{\text{RK4}}(d, s, p)$ methods, we have $d < p - 1$ and hence we do not have access to the higher order derivatives values $Y^{[p]}$ and $Y^{[p-1]}$. To overcome this issue, we assume that the function $f(t, y)$ is analytic on a ball of radius ρ such that [29, 41]: $\|Y^{[p]}\|_\infty \approx \frac{M}{\rho^p}$ for some M. Then, we get

$$\rho^{p-q} \approx \frac{\|Y^{[q]}\|_\infty}{\|Y^{[p]}\|_\infty} \iff \rho \approx \left(\frac{\|Y^{[q]}\|_\infty}{\|Y^{[p]}\|_\infty} \right)^{\frac{1}{p-q}}. \tag{4.3.5}$$

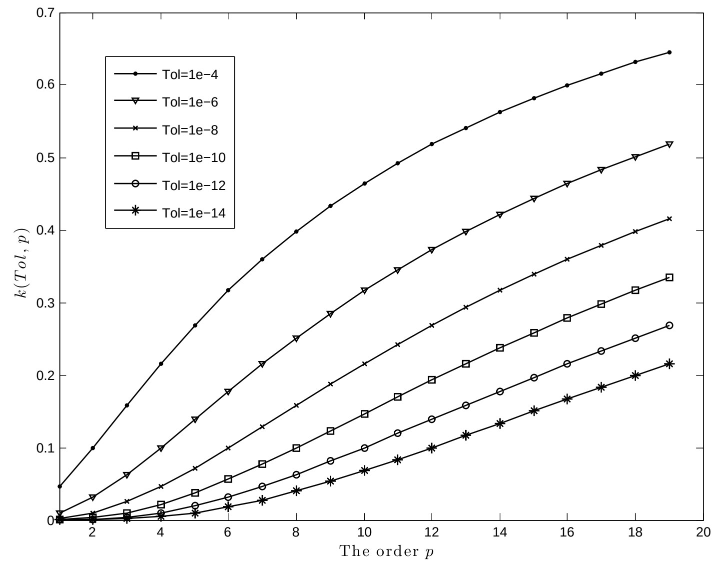


Figure 4.2: The graph of k as a function of p for different values of TOL.

To get a better estimate of ρ for $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$ and to avoid problems, we take

$$\rho_{est} = \min\left\{ \frac{\|Y^{[d-1]}\|_\infty}{\|Y^{[d]}\|_\infty}, \left(\frac{\|Y^{[d-2]}\|_\infty}{\|Y^{[d]}\|_\infty}\right)^{\frac{1}{2}}, \left(\frac{\|Y^{[d-3]}\|_\infty}{\|Y^{[d-1]}\|_\infty}\right)^{\frac{1}{2}} \right\} \tag{4.3.6}$$

In our research, we find that it is sufficient to estimate the error and control the step size of a $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$ method using the information from the estimated $(p-1)$-th derivative of the form:

$$\|Y^{[p-1]}\|_\infty \approx \frac{\|Y^{[d]}\|_\infty}{\rho_{est}^{p-d-1}}. \tag{4.3.7}$$

Using this estimation, the step control is of the form:

$$h_{n+1} = \eta k(\mathrm{TOL}, p-1) \left(\frac{\|Y^{[d]}\|_\infty}{\rho_{est}^{p-d-1}}\right)^{\frac{-1}{p-1}}, \tag{4.3.8}$$

where η is a control factor.

4.4 Testing the step control algorithm

For the purpose of testing the step control, we will consider the following problems:

- HH: The Hénon Heiles problem is a well known problem [3, 24] described by the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) + \frac{1}{2}(x^2 + y^2) + \varepsilon y(x^2 - \frac{1}{3}y^2)$$

  with $\varepsilon = 1$ and the initial values:
  $x(0) = 0$, $y(0) = 0.52$, $X(0) = 0.371956090598519$, $Y(0) = 0$.

- EqMP: The equatorial main problem in the artificial satellite theory [3, 58]. This problem accepts, due to the axial symmetry, the polar component Λ of the angular momentum as an integral. Other parameters of the problem are the gravitational constant μ of the planet, the oblateness coefficient J_2 and the scaling factor α that is the equatorial radius of the planet. The Hamiltonian function in cylindrical coordinates is

$$\mathcal{H} = \frac{1}{2}\big(P^2 + \frac{\Lambda^2}{\rho^2} + Z^2\big) - \frac{\mu}{r} + \frac{\alpha^2 J_2 \mu P_2(u)}{r^3},$$

  where $u = z/r$, $r = \sqrt{\rho^2 + z^2}$, and $P_2(x) = (3x^2 - 1)/2$ with initial values $\rho(0) = 0.3$, $z(0) = 2$, $P(0) = 0$ and $Z(0) = -1$.

The maximum global error (MGE) of a method is taken in the uniform norm

$$\text{MGE} = \max_n \{\|y_n - z_n\|_\infty\}, \tag{4.4.1}$$

where y_n is the numerical solution and z_n is the exact solution at $x = x_n$ or a reference solution obtained by DP(8,7)13M with stringent tolerance 5×10^{-14}. This choice of MGE is ideal to monitor the error uniformly over the whole integration interval as opposed to the end point error defined by the difference between the numerical solution and the exact or reference solution at the endpoint of the integration interval only.

In our numerical simulations, $\eta = 1.4$ gave optimal results for $\text{CPHBT}_{\text{RK4}}(9, 6, 12)$ in terms of the number of steps required and the difference $|\text{TOL} - \text{MGE}|$. In Table 4.1, we test the step control algorithm (4.3.8) of $\text{CPHBT}_{\text{RK4}}(9, 6, 12)$ compared to T(12) with step control algorithm (4.3.4). We evaluate how responsive the algorithm is to the user defined tolerance TOL compared to MGE by computing the mean, median and standard deviation of the difference $|\text{MGE} - \text{TOL}|$ for the Hénon Heiles problem, Equatorial main problem, the 20 DETEST problems A,B,D,E and the Kepler problem with eccentricity $\varepsilon = 0.99$. It can be seen that the step control algorithm with high order derivative estimation of $\text{CPHBT}_{\text{RK4}}(9,6,12)$ performs favourably over T(12). To study the Hamiltonian problems in this work, we define

Table 4.1: The mean, median and standard deviation of the absolute value of the difference $|\text{MGE} - \text{TOL}|$ of $\text{CPHBT}_{\text{RK4}}(9,6,12)$ and T(12) for the Hénon Heiles, Equatorial main problem, the 20 DETEST problems A,B,D,E and Kepler's two body problem with eccentricity $\varepsilon = 0.99$.

Method	**Mean**	**Median**	**Standard deviation**
$\mathbf{CPHBT_{RK4}}(9,6,12)$	5.98×10^{-5}	2.64×10^{-10}	5.98×10^{-4}
T(12)	7.0×10^{-5}	6.57×10^{-10}	6.46×10^{-4}

the maximum global energy error (MGEE) as follows:

$$\text{MGEE} = \max_n \left\{ \left| \frac{\mathcal{H}_n - \mathcal{H}_0}{\mathcal{H}_0} \right| \right\}, \tag{4.4.2}$$

where $\mathcal{H}_0$ and $\mathcal{H}_n$ are the value of the Hamiltonian at the initial point and the numerical value of the Hamiltonian at t_n, respectively. In Table 4.2, we list the number of steps required to achieve certain user defined tolerances and the associated MGEE. We notice that $\text{CPHBT}_{\text{RK4}}(9,6,12)$ requires considerably less step points to achieve the required user defined tolerance TOL and achieves better maximum global energy error (MGEE). We note that in general, T(12)L requires less step points than T(12), but T(12) achieves better MGEE than T(12)L.

Table 4.2: The maximum global energy error (MGEE) and the number of steps (NS) for $\text{CPHBT}_{\text{RK4}}(9,6,12)$ compared to T(12) and T(12)L for the listed problems.

Problem	**TOL**	**T(12)L** NS	**T(12)L** MGEE	**T(12)** NS	**T(12)** MGEE	$\mathbf{CPHBT_{RK4}}(9,6,12)$ NS	$\mathbf{CPHBT_{RK4}}(9,6,12)$ MGEE
D1	10^{-4}	44	7.52×10^{-4}	48	6.46×10^{-4}	37	2.2×10^{-4}
$\varepsilon=0.1$,	10^{-7}	73	1.15×10^{-6}	82	4.45×10^{-7}	61	7.68×10^{-8}
$t_{\mathbf{end}}=16\pi$	10^{-10}	122	1.43×10^{-9}	142	2.93×10^{-10}	105	1.91×10^{-11}
D2	10^{-4}	67	7.12×10^{-4}	68	6.65×10^{-4}	51	1.02×10^{-4}
$\varepsilon=0.3$,	10^{-7}	111	7.17×10^{-7}	118	4.67×10^{-7}	81	2.73×10^{-8}
$t_{\mathbf{end}}=16\pi$	10^{-10}	188	7.8×10^{-10}	206	2.67×10^{-10}	152	9.99×10^{-12}
D3	10^{-4}	83	1.26×10^{-3}	89	8.58×10^{-4}	65	1.39×10^{-3}
$\varepsilon=0.5$,	10^{-7}	139	1.55×10^{-6}	154	5.47×10^{-7}	113	2.68×10^{-7}
$t_{\mathbf{end}}=16\pi$	10^{-10}	235	1.87×10^{-9}	271	3.39×10^{-10}	197	1.08×10^{-10}
D4	10^{-4}	115	8.03×10^{-4}	116	6.81×10^{-4}	84	7.87×10^{-4}
$\varepsilon=0.7$,	10^{-7}	194	1.24×10^{-6}	202	1.11×10^{-6}	148	3.33×10^{-7}
$t_{\mathbf{end}}=16\pi$	10^{-10}	327	1.5×10^{-9}	356	6.39×10^{-10}	259	1.4×10^{-10}
D5	10^{-4}	167	2.73×10^{-3}	169	1.33×10^{-3}	120	2.92×10^{-3}
$\varepsilon=0.9$,	10^{-7}	273	7.81×10^{-7}	300	3.11×10^{-7}	217	1.68×10^{-7}
$t_{\mathbf{end}}=16\pi$	10^{-10}	461	1.32×10^{-9}	528	2.56×10^{-10}	379	5.48×10^{-11}
Kepler,	10^{-4}	306	9.36×10^{-3}	306	9.46×10^{-3}	198	1.82×10^{-2}
$\varepsilon=0.99$,	10^{-7}	488	6.48×10^{-6}	509	4.43×10^{-6}	364	3.08×10^{-6}
$t_{\mathbf{end}}=16\pi$	10^{-10}	824	7.36×10^{-9}	897	2.95×10^{-9}	644	1.24×10^{-9}
HH	10^{-4}	56	1.47×10^{-3}	61	1.39×10^{-3}	47	1.14×10^{-4}
$t_{\mathbf{end}}=70$	10^{-7}	92	8.93×10^{-7}	104	2.29×10^{-7}	79	8.33×10^{-7}
	10^{-10}	155	3.29×10^{-9}	180	2.71×10^{-10}	138	6.8×10^{-10}
EqMP	10^{-4}	146	3.08×10^{-3}	157	2.49×10^{-3}	121	1.62×10^{-4}
$t_{\mathbf{end}}=70$	10^{-7}	245	1.09×10^{-5}	269	8.72×10^{-7}	213	3.89×10^{-7}
	10^{-10}	411	4.93×10^{-9}	474	1.08×10^{-9}	369	3.25×10^{-10}

4.5 Number of steps and number of function evaluations analysis of CPHBT$_{\mathbf{RK4}}(9,6,12)$

The main advantage of the step control formula (4.3.8) is to satisfy the user defined tolerance TOL while minimizing the number of steps and hence maximizing the average step-size required. To show such property, we plot the maximum global energy error (MGEE) as a function of the number of steps for the DETEST problems of class D and for the Hénon Heiles problem, Equatorial main problem and Kepler equation with eccentricity $\varepsilon = 0.99$. As we can see in Figure 4.3, CPHBT$_{\text{RK4}}(9,6,12)$ requires considerably less steps than T(12) to achieve the required user defined tolerance. To analyze the percentage efficiency gain of the number of steps, we consider the following formula:

Definition 4.5.1 (NSPEG) *[41] Let $NS(A,i,j)$ be the number of steps required by a method A to solve problem i and obtain maximum global error MGE=10^{-j}. Then, the number of steps percentage efficiency gain of a method A over a method B for a problem i is defined by*

$$NS\ PEG_i(A,B) = 100\left(\frac{\sum_j NS(B,i,j) - NS(A,i,j)}{\sum_j NS(A,i,j)}\right). \tag{4.5.1}$$

To compute the NS PEG of CPHBT$_{\text{RK4}}(9,6,12)$ over T(12), we use the MATLAB function `Polyfit` to fit the data $(\log_{10}(\text{MGEE}), \log_{10}(\text{NS}))$ and obtain a model in a least squares sense, i.e., by minimizing the sum of squares of the deviations of the data from the model. In Table 4.3, we list the number of steps percentage efficiency gain of CPHBT$_{\text{RK4}}(9,6,12)$ over T(12) for various problems. We show that CPHBT$_{\text{RK4}}(9,6,12)$ has good NS PEG over T(12) for the listed problems. However, due to the 6 off-step function evaluations, CPHBT$_{\text{RK4}}(9,6,12)$ requires 14 function evaluations per integration step while T(12) method requires only 12 function evaluations, so CPHBT$_{\text{RK4}}(9,6,12)$ might still require more function evaluations to solve

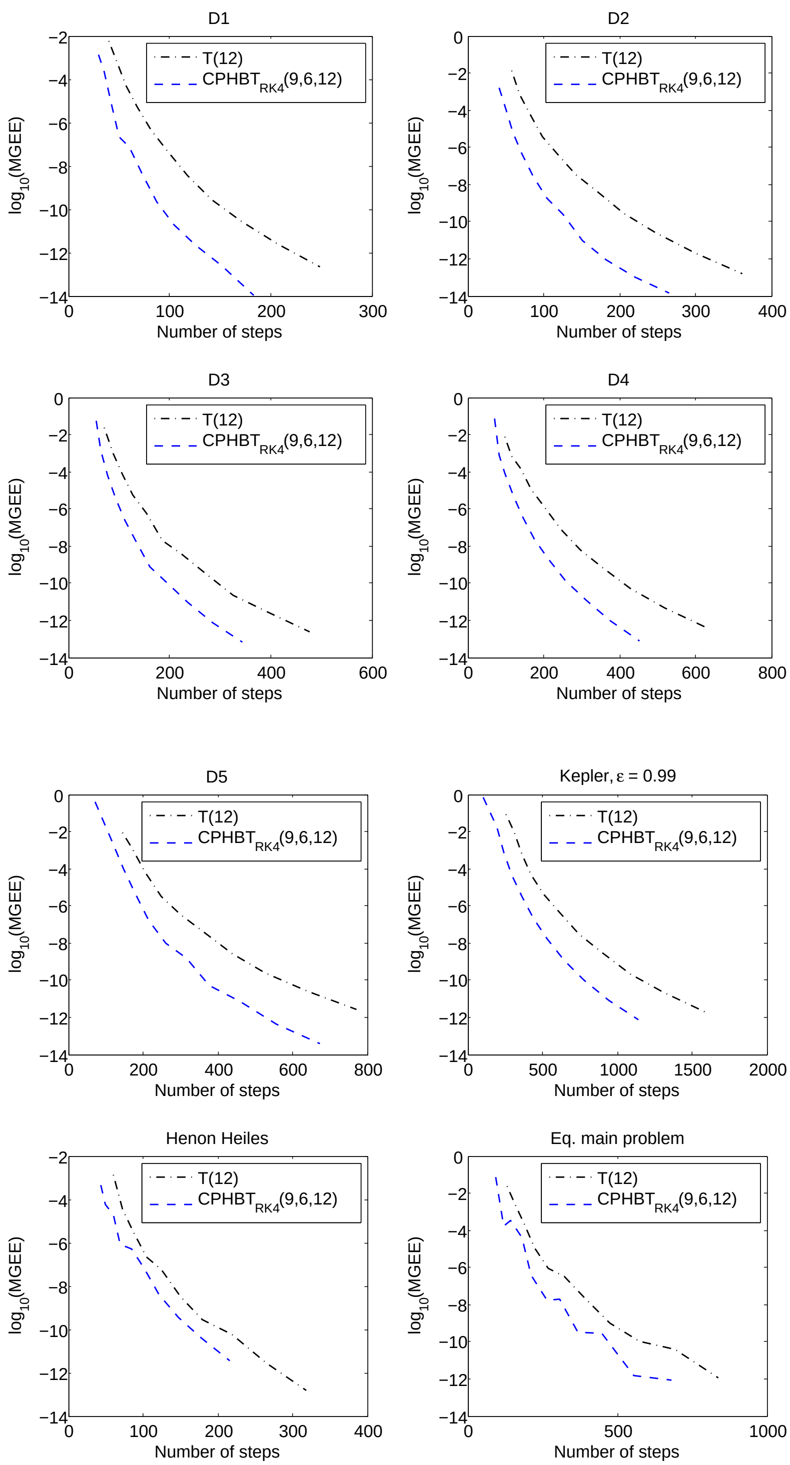


Figure 4.3: The number of steps versus $\log_{10}(\text{MGEE})$ for CPHBT$_{\text{RK4}}(9, 6, 12)$ and T(12) for the listed problems.

Table 4.3: The NS PEG of CPHBT$_{\mathrm{RK4}}(9,6,12)$ over T(12) for the listed problems.

Problem	**NS PEG**	**Problem**	**NS PEG**
D1 ($\varepsilon = 0.1$)	% 66	B2	% 72
D2 ($\varepsilon = 0.3$)	% 72	B3	% 12
D3 ($\varepsilon = 0.5$)	% 49	B4	% 32
D4 ($\varepsilon = 0.7$)	% 56	B5	% 52
D5 ($\varepsilon = 0.9$)	% 56	E1	% 128
Kepler problem ($\varepsilon = 0.99$)	% 48	E2	% 54
Hénon Heiles	% 21	E3	% 24
Equatorial main problem	% 37	E4	% 18
B1	% 38	E5	% 71

the ODE problems. Hence, we compute the NFE PEG according to the following definition:

Definition 4.5.2 (NFEPEG) *[41] Let $NFE(A,i,j)$ be the number of function evaluations required by a method A to solve problem i and obtain maximum global error $MGE{=}10^{-j}$. Then, the number of function evaluations percentage efficiency gain of a method A over a method B for a problem i is defined by*

$$NFE\ PEG_i(A,B) = 100\left(\frac{\sum_j NFE(B,i,j) - NFE(A,i,j)}{\sum_j NFE(A,i,j)}\right). \tag{4.5.2}$$

In Table 4.4, we list the number of function evaluations percentage efficiency gain (NFEPEG) for the same problems. As noticed, NS PEG of CPHBT$_{\mathrm{RK4}}(9,6,12)$ over T(12) listed in Table 4.3 compensates the difference in function evaluations per integration step and hence we still get high NFEPEG.

Moreover, in Table 4.2, we noticed that the Taylor method introduced by Lara, T(12)L [34], requires less steps than T(12). In Figure 4.4, we plot the maximum global

Table 4.4: The NFE PEG of $\text{CPHBT}_{\text{RK4}}(9,6,12)$ over T(12) for the listed problems.

Problem	**NFE PEG**	**Problem**	**NFE PEG**
D1 ($\varepsilon = 0.1$)	% 42	B2	% 47
D2 ($\varepsilon = 0.3$)	% 47	B3	% -4
D3 ($\varepsilon = 0.5$)	% 28	B4	% 13
D4 ($\varepsilon = 0.7$)	% 34	B5	% 30
D5 ($\varepsilon = 0.9$)	% 34	E1	% 95
Kepler problem ($\varepsilon = 0.99$)	% 27	E2	% 32
Hénon Heiles	% 4	E3	% 6
Equatorial main problem	% 17	E4	% 1
B1	% 18	E5	% 47

energy error (MGEE) as a function of the number of steps for $\text{CPHBT}_{\text{RK4}}(9,6,12)$ and T(12)L. Moreover, in Table 4.5, we list the NS PEG and NFE PEG for $\text{CPHBT}_{\text{RK4}}(9,6,12)$ method over T(12)L method. We notice that $\text{CPHBT}_{\text{RK4}}(9,6,12)$ still performs favourably over T(12)L.

Table 4.5: The NS PEG and NFE PEG of $\text{CPHBT}_{\text{RK4}}(9,6,12)$ over T(12)L for the listed problems.

Problem	**NS PEG**	**NFEPEG**
D1 ($\varepsilon = 0.1$)	% 59	% 36
D2 ($\varepsilon = 0.3$)	% 67	% 43
D3 ($\varepsilon = 0.5$)	% 51	% 29
D4 ($\varepsilon = 0.7$)	% 53	% 31
D5 ($\varepsilon = 0.9$)	% 50	% 29
Kepler problem ($\varepsilon = 0.99$)	% 41	% 21
Hénon Heiles	% 16	% -1
Equatorial main problem	% 28	% 10

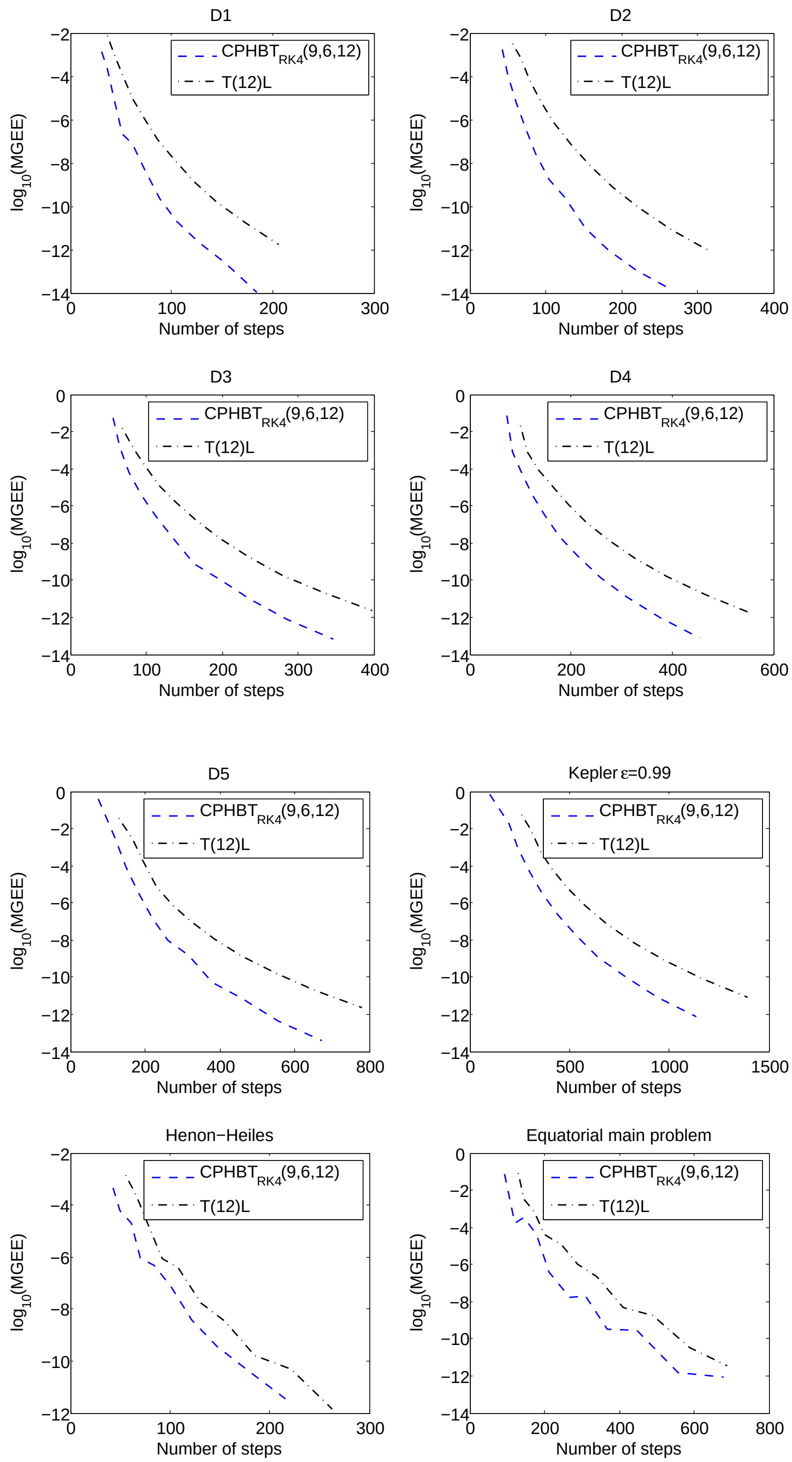


Figure 4.4: The number of steps versus $\log_{10}(\text{MGEE})$ for CPHBT$_{\text{RK4}}(9, 6, 12)$ and T(12)L for the listed problems.

4.6 CPU time analysis of $\mathrm{CPHBT_{RK4}}(9,6,12)$

Since our method $\mathrm{CPHBT_{RK4}}(9,6,12)$ consists of 6 stages and 9 derivatives that are computed at each integration step using recurrence formulae, we test the CPU time of our method against DP(8,7)13M and T(12) using C++ codes. For the CPU time analysis, we consider the DETEST problems E2, B1, B5, the five problems in class D, Kepler problem with eccentricity $\varepsilon = 0.99$ together with HH, Aren, EqMP and the following two problems:

- GAD: A Galactic Dynamics model described in [3, 4]. This is a Hamiltonian problem given by:

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \Omega(p_1 q_2 - p_2 q_1) + A \ln\bigl(C + \frac{q_1^2}{a^2} + \frac{q_2^2}{b^2} + \frac{q_3^2}{c^2}\bigr),$$

  where q_1, q_2, q_3 are the coordinates, p_1, p_2, p_3 are moments with initial values

$$q_1(0) = 2.5, \quad q_2(0) = q_3(0) = 0,$$

$$p_1(0) = 0, \quad p_2(0) = \frac{1}{40}\bigl(25 + \sqrt{6961 - 3200 \ln 5}\bigr), \quad p_3(0) = 0.2,$$

  and parameters configuration $a = 1.25$, $b = 1$, $c = 0.75$, $A = 1$, $C = 1$, $\Omega = 0.25$.

- Aren: Arenstorf orbits are a particular case of the restricted three-body problem described in [3, 1]. Two bodies of masses $1-\mu$ and μ are in circular rotation in a plane and a third body of negligible mass is moving around the same plane. As in [58, 23], the equations are the following:

$$x'' = x + 2y' - \mu' \frac{x+\mu}{D_1} - \mu \frac{x-\mu'}{D_2}, \qquad y'' = y - 2x' - \mu' \frac{y}{D_1} - \mu \frac{y}{D_2},$$

$$D_1 = \bigl((x+\mu)^2 + y^2\bigr)^{\frac{3}{2}}, \qquad D_2 = \bigl((x-\mu')^2 + y^2\bigr)^{\frac{3}{2}},$$

  $x(0) = 0.994$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = -2.00158510637908252240537862224$, $\mu = 0.012277471$, $\mu' = 1 - \mu$.

In Figures 4.5 and 4.6, we plot the CPU time in seconds versus the maximum global error (MGE) $\|y_{n+1} - y(t_{n+1})\|_\infty$ where $y(t)$ is the exact solution if it is available or a reference solution obtained by DP(8,7)M13 with stringent tolerance of 5×10^{-14}. For all the considered test problems, it is shown that $\mathrm{CPHBT_{RK4}}(9,6,12)$ requires less CPU time than T(12) and DP(8,7)13M to achieve the same accuracy. We can analyze the CPU time percentage efficiency gain as follows:

Definition 4.6.1 (CPUPEG) *[41, 51] Let $CPU(A,i,j)$ be the CPU time required by a method A to solve problem i and obtain maximum global error MGE=10^{-j}. Then, the CPU percentage efficiency gain of a method A over a method B for a problem i is defined by*

$$CPUPEG_i(A,B) = 100\left(\frac{\sum_j CPU(B,i,j) - CPU(A,i,j)}{\sum_j CPU(A,i,j)}\right), \tag{4.6.1}$$

To compute the CPU PEG for $\mathrm{CPHBT_{RK4}}(9,6,12)$, DP(8,7)13M and T(12), we use the MATLAB function `Polyfit` to fit the data $(\log_{10}(\mathrm{MGE}), \log_{10}(\mathrm{CPU}))$ in a least squares sense. To determine the sufficient degree of the model, let $\mathrm{CPU_{A,i}}$ be the CPU time required by a method A of order p to solve a problem i with an average step-size h. Then, $\mathrm{CPU_{A,i}}$ is related inversely to h, i.e., $\mathrm{CPU_{A,i}} = \frac{c_1}{h}$ for some constant c_1. The maximum global error can be written as

$$\log_{10}(\mathrm{MGE}) = c_2 + p\log_{10}(h) = c_2 + p\log_{10}\left(\frac{c_1}{\mathrm{CPU_{A,i}}}\right) = c_3 - p\log_{10}\big(\mathrm{CPU_{A,i}}\big),$$

where $c_3 = c_2 + p\log_{10}(c_1)$ for some constant c_2. Hence, a linear least squares model would be sufficient to fit the data $(\log_{10}(\mathrm{MGE}), \log_{10}(\mathrm{CPU}))$. The CPU PEG for the 13 problems mentioned above are listed in Table 4.6. Since DP(8,7)13M is of order 8, we consider the tolerance range $10^{-4} \le \mathrm{TOL} \le 10^{-9}$ to compute the CPU PEG of $\mathrm{CPHBT_{RK4}}(9,6,12)$ over DP(8,7)13M while we consider the more stringent tolerance range $10^{-4} \le \mathrm{TOL} \le 10^{-13}$ to compute the CPU PEG of $\mathrm{CPHBT_{RK4}}(9,6,12)$ over T(12). The results confirm that $\mathrm{CPHBT_{RK4}}(9,6,12)$ performs well compared to

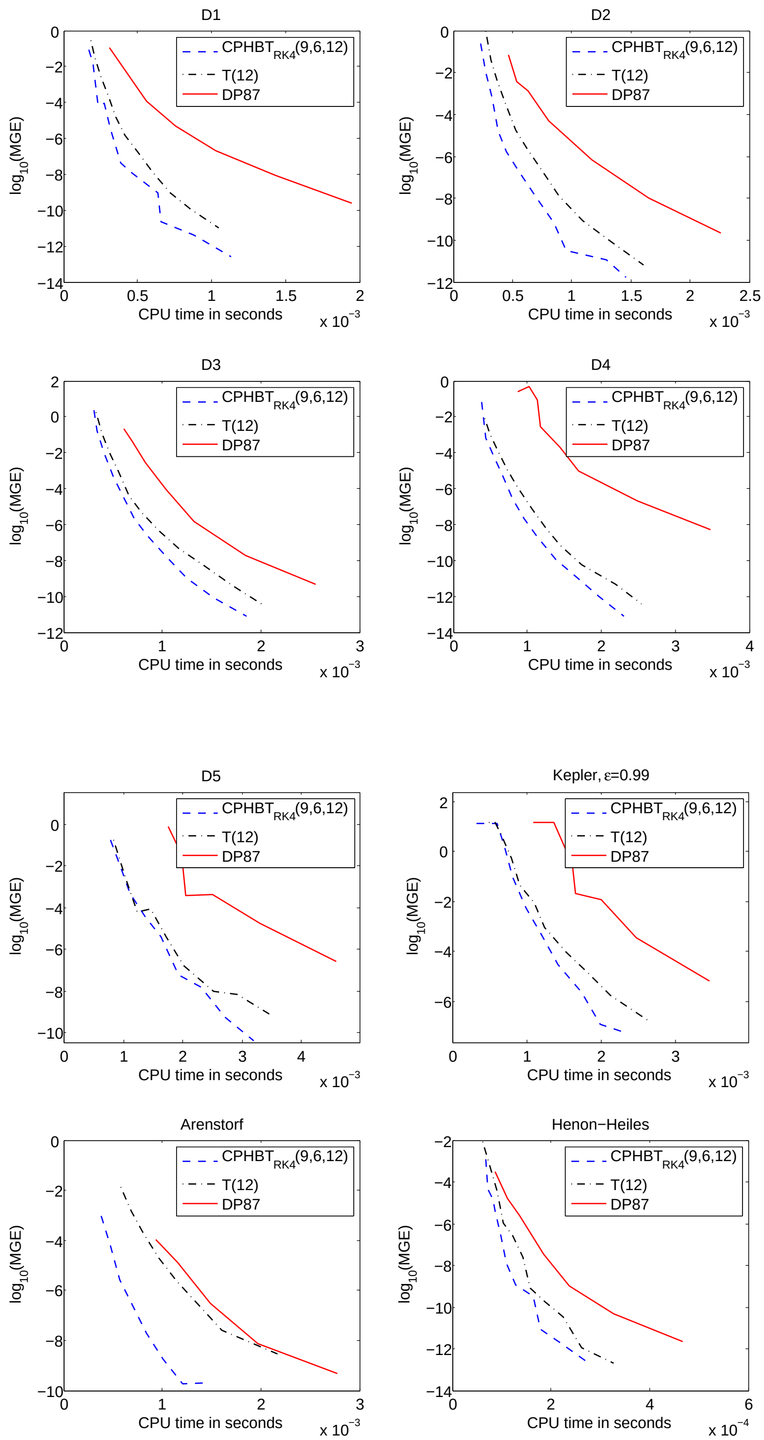


Figure 4.5: The CPU time in seconds versus $\log_{10}(\text{MGE})$ for CPHBT$_{\text{RK4}}(9, 6, 12)$, T(12) and DP(8,7)13M for the listed problems.

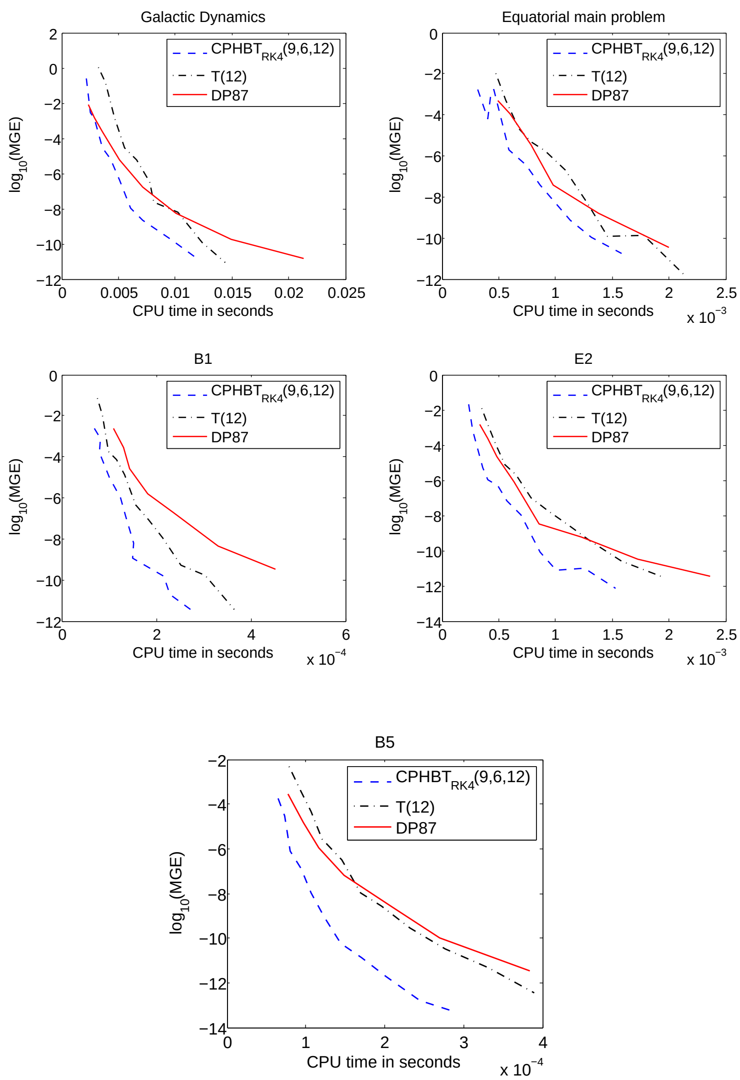


Figure 4.6: The CPU time in seconds versus $\log_{10}(\text{MGE})$ for CPHBT$_{\text{RK4}}(9, 6, 12)$, T(12) and DP(8,7)13M for the listed problems.

DP(8,7)13M and T(12). This is due to the fact that, for CPHBT$_{\text{RK4}}(9,6,12)$, the step size for the next integration step is chosen using the variable step algorithm (4.3.8) specifically to satisfy the given tolerance TOL, so there is no step rejection criteria as opposed to DP(8,7)13M. Also, it is superior to T(12) since CPHBT$_{\text{RK4}}(9,6,12)$ requires only 9 derivatives to reach order 12 compared to 12 derivatives in T(12) which enhances the performance of our method and minimizes the CPU time required.

Table 4.6: The CPU PEG of CPHBT$_{\text{RK4}}(9,6,12)$ over T(12) and DP(8,7)13M for the listed problems.

Problem	**Over T(12)** $10^{-4} \leq \text{TOL} \leq 10^{-13}$	**Over DP(8,7)13M** $10^{-4} \leq \text{TOL} \leq 10^{-9}$
D1 ($\varepsilon = 0.1$)	%27	%190
D2 ($\varepsilon = 0.3$)	%31	%233
D3 ($\varepsilon = 0.5$)	%24	%166
D4 ($\varepsilon = 0.7$)	%21	%292
D5 ($\varepsilon = 0.9$)	%22	%246
Kepler problem ($\varepsilon = 0.99$)	%28	%195
Arenstorf problem	%120	%187
Hénon Heiles	%21	%139
Galactic dynamics problem	%31	%126
Equatorial main problem	%24	%67
B1	%41	%196
E2	%52	%129
B5	%65	%125

Finally, we summarize some numerical results in Tables 4.7 and 4.8 by listing the CPU time in seconds, the maximum global error (MGE) and the maximum global energy error (MGEE) for 11 test problems.

Table 4.7: The CPU time in seconds, MGE, MGEE and NS of $\text{CPHBT}_{\text{RK4}}(9,6,12)$ and T(12) for the DETEST class D problems and the Kepler problem with $\varepsilon = 0.99$.

Problem	T	CPU time		MGE		MGEE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
D1	-3	1.78×10^{-4}	2.00×10^{-4}	8.72×10^{-2}	2.81×10^{-1}	1.71×10^{-3}	7.05×10^{-3}	28	38
	-4	2.04×10^{-4}	2.32×10^{-4}	1.47×10^{-2}	2.89×10^{-2}	3.16×10^{-4}	6.24×10^{-4}	33	45
	-5	2.39×10^{-4}	2.70×10^{-4}	7.39×10^{-5}	2.58×10^{-3}	1.33×10^{-6}	5.85×10^{-5}	40	54
$\varepsilon = 0.1$	-6	2.82×10^{-4}	3.21×10^{-4}	9.00×10^{-5}	2.25×10^{-4}	1.72×10^{-6}	5.08×10^{-6}	48	66
	-7	3.34×10^{-4}	3.84×10^{-4}	3.11×10^{-6}	1.62×10^{-5}	8.89×10^{-8}	3.77×10^{-7}	57	79
	-8	4.05×10^{-4}	4.57×10^{-4}	4.38×10^{-8}	1.44×10^{-6}	2.90×10^{-9}	3.20×10^{-8}	70	96
	-9	4.86×10^{-4}	5.48×10^{-4}	1.28×10^{-8}	1.60×10^{-7}	2.82×10^{-10}	3.60×10^{-9}	85	115
	-10	5.74×10^{-4}	6.55×10^{-4}	9.21×10^{-10}	1.23×10^{-8}	2.24×10^{-11}	2.78×10^{-10}	102	139
	-11	6.88×10^{-4}	7.82×10^{-4}	2.30×10^{-11}	9.02×10^{-10}	2.02×10^{-12}	2.14×10^{-11}	123	168
	-12	8.23×10^{-4}	9.44×10^{-4}	3.92×10^{-12}	1.05×10^{-10}	1.44×10^{-13}	2.41×10^{-12}	149	204
	-13	9.70×10^{-4}	1.14×10^{-3}	2.42×10^{-13}	9.83×10^{-12}	1.11×10^{-14}	2.16×10^{-13}	181	247
D2	-3	2.32×10^{-4}	2.85×10^{-4}	2.49×10^{-1}	9.62×10^{-1}	3.67×10^{-3}	1.04×10^{-2}	39	56
	-4	2.80×10^{-4}	3.27×10^{-4}	1×10^{-2}	4.08×10^{-2}	1.53×10^{-4}	7.52×10^{-4}	48	66
	-5	3.33×10^{-4}	3.84×10^{-4}	6.75×10^{-4}	4.00×10^{-3}	7.94×10^{-6}	5.41×10^{-5}	58	79
$\varepsilon = 0.3$	-6	3.95×10^{-4}	4.60×10^{-4}	1.54×10^{-5}	3.60×10^{-4}	2.97×10^{-7}	5.58×10^{-6}	70	96
	-7	4.70×10^{-4}	5.47×10^{-4}	1.62×10^{-6}	1.76×10^{-5}	3.53×10^{-8}	3.61×10^{-7}	84	115
	-8	5.71×10^{-4}	6.52×10^{-4}	2.02×10^{-7}	2.47×10^{-6}	2.33×10^{-9}	4.01×10^{-8}	102	139
	-9	6.73×10^{-4}	7.83×10^{-4}	1.32×10^{-8}	1.95×10^{-7}	1.74×10^{-10}	3.62×10^{-9}	123	168
	-10	8.15×10^{-4}	9.45×10^{-4}	6.53×10^{-10}	1.31×10^{-8}	1.17×10^{-11}	2.64×10^{-10}	149	203
	-11	9.75×10^{-4}	1.17×10^{-3}	3.28×10^{-11}	8.40×10^{-10}	7.59×10^{-13}	2.13×10^{-11}	180	246
	-12	1.13×10^{-3}	1.40×10^{-3}	1.12×10^{-11}	8.93×10^{-11}	1.40×10^{-13}	2.03×10^{-12}	217	297
	-13	1.34×10^{-3}	1.64×10^{-3}	1.58×10^{-12}	6.41×10^{-12}	1.51×10^{-14}	1.72×10^{-13}	263	360
D3	-3	3.07×10^{-4}	3.36×10^{-4}	2.29	1.79	4.36×10^{-2}	1.67×10^{-2}	53	74
	-4	3.43×10^{-4}	3.91×10^{-4}	1.62×10^{-1}	1.14×10^{-1}	1.50×10^{-3}	9.35×10^{-4}	61	87
	-5	4.20×10^{-4}	4.68×10^{-4}	8.59×10^{-3}	1.08×10^{-2}	6.31×10^{-5}	7.95×10^{-5}	74	104
$\varepsilon = 0.5$	-6	4.99×10^{-4}	5.55×10^{-4}	4.44×10^{-4}	8.41×10^{-4}	3.73×10^{-6}	5.63×10^{-6}	90	126
	-7	5.96×10^{-4}	6.58×10^{-4}	3.27×10^{-5}	4.20×10^{-5}	2.51×10^{-7}	3.31×10^{-7}	109	152
	-8	7.35×10^{-4}	7.98×10^{-4}	2.21×10^{-6}	4.78×10^{-6}	1.76×10^{-8}	4.29×10^{-8}	132	183
	-9	8.73×10^{-4}	9.56×10^{-4}	2.20×10^{-7}	4.75×10^{-7}	1.76×10^{-9}	3.58×10^{-9}	160	222
	-10	1×10^{-3}	1.16×10^{-3}	1.45×10^{-8}	4.74×10^{-8}	1.18×10^{-10}	3.46×10^{-10}	193	268
	-11	1.22×10^{-3}	1.45×10^{-3}	1.07×10^{-9}	4.81×10^{-9}	9.47×10^{-12}	3.22×10^{-11}	234	324
	-12	1.42×10^{-3}	1.69×10^{-3}	8.58×10^{-11}	4.18×10^{-10}	7.76×10^{-13}	3.04×10^{-12}	283	392
	-13	1.77×10^{-3}	2.02×10^{-3}	7.94×10^{-12}	3.30×10^{-11}	6.84×10^{-14}	2.53×10^{-13}	343	474
D4	-3	3.97×10^{-4}	4.61×10^{-4}	2.81	1.97	6.83×10^{-2}	7.75×10^{-3}	69	97
D4	-4	4.54×10^{-4}	5.24×10^{-4}	2.34×10^{-1}	3.38×10^{-1}	6.95×10^{-4}	1.06×10^{-3}	80	113
	-5	5.36×10^{-4}	6.26×10^{-4}	2.79×10^{-2}	3.76×10^{-2}	7.88×10^{-5}	1.11×10^{-4}	96	137
$\varepsilon = 0.7$	-6	6.58×10^{-4}	7.53×10^{-4}	1.58×10^{-3}	3.54×10^{-3}	4.35×10^{-6}	1.02×10^{-5}	118	166
	-7	7.80×10^{-4}	9.09×10^{-4}	8.69×10^{-5}	3.78×10^{-4}	2.57×10^{-7}	1.15×10^{-6}	144	200
	-8	9.35×10^{-4}	1.10×10^{-3}	8.05×10^{-6}	2.11×10^{-5}	2.26×10^{-8}	7.64×10^{-8}	174	242
	-9	1.09×10^{-3}	1.33×10^{-3}	6.68×10^{-7}	1.73×10^{-6}	1.96×10^{-9}	6.79×10^{-9}	210	292
	-10	1.29×10^{-3}	1.60×10^{-3}	3.65×10^{-8}	2.00×10^{-7}	1.16×10^{-10}	7.14×10^{-10}	256	353
	-11	1.53×10^{-3}	1.93×10^{-3}	3.69×10^{-9}	1.34×10^{-8}	1.21×10^{-11}	5.43×10^{-11}	308	427
	-12	1.82×10^{-3}	2.30×10^{-3}	3.03×10^{-10}	9.49×10^{-10}	9.95×10^{-13}	4.85×10^{-12}	373	517
	-13	2.24×10^{-3}	2.74×10^{-3}	2.29×10^{-11}	5.88×10^{-11}	8.08×10^{-14}	3.97×10^{-13}	451	626
D5	-3	4.53×10^{-4}	6.58×10^{-4}	4.52	4.55	3.21×10^{-1}	8.95×10^{-3}	82	144
	-4	6.36×10^{-4}	7.77×10^{-4}	3.31	1.95	2.48×10^{-3}	1.05×10^{-3}	117	168
	-5	7.65×10^{-4}	9.16×10^{-4}	1.75×10^{-1}	1.63×10^{-1}	8.28×10^{-5}	6.09×10^{-5}	145	204

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Table 4.7 – *Continued from previous page*

Problem	T	CPU time		MGE		MGEE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
$\varepsilon = 0.9$	−6	8.96×10^{-4}	1.12×10^{-3}	1.09×10^{-2}	2.26×10^{-3}	4.02×10^{-6}	2.81×10^{-6}	176	246
	−7	1.08×10^{-3}	1.39×10^{-3}	4.47×10^{-4}	6.22×10^{-5}	1.73×10^{-7}	2.97×10^{-7}	213	298
	−8	1.28×10^{-3}	1.66×10^{-3}	3.40×10^{-5}	8.26×10^{-5}	1.35×10^{-8}	2.74×10^{-8}	258	360
	−9	1.52×10^{-3}	2.16×10^{-3}	3.94×10^{-6}	3.86×10^{-6}	1.49×10^{-9}	2.56×10^{-9}	312	435
	−10	1.83×10^{-3}	2.59×10^{-3}	5.69×10^{-8}	1.65×10^{-7}	4.99×10^{-11}	3.12×10^{-10}	376	526
	−11	2.17×10^{-3}	2.95×10^{-3}	1.57×10^{-8}	9.55×10^{-9}	7.40×10^{-12}	3.47×10^{-11}	456	636
	−12	2.95×10^{-3}	3.51×10^{-3}	5.24×10^{-10}	6.43×10^{-9}	4.92×10^{-13}	2.29×10^{-12}	553	769
	−13	3.44×10^{-3}	4.32×10^{-3}	4.10×10^{-11}	7.50×10^{-10}	3.15×10^{-14}	2.41×10^{-13}	670	931
Kepler	−3	5.64×10^{-4}	1.14×10^{-3}	1.41×10	1.43×10	4.81×10^{-1}	6.76×10^{-2}	100	256
	−4	1.01×10^{-3}	1.29×10^{-3}	1.41×10	1.40×10	1.89×10^{-2}	6.58×10^{-3}	194	304
	−5	1.24×10^{-3}	1.48×10^{-3}	1.38×10	1.42×10	7.93×10^{-4}	5.58×10^{-4}	239	358
$\varepsilon = 0.99$	−6	1.50×10^{-3}	1.75×10^{-3}	8.93	8.65	4.73×10^{-5}	4.87×10^{-5}	295	423
	−7	1.77×10^{-3}	2.03×10^{-3}	9.73×10^{-1}	9.49×10^{-1}	3.15×10^{-6}	4.57×10^{-6}	360	507
	−8	2.08×10^{-3}	2.44×10^{-3}	6.30×10^{-2}	6.70×10^{-2}	2.11×10^{-7}	3.98×10^{-7}	438	612
	−9	2.46×10^{-3}	2.94×10^{-3}	4.85×10^{-3}	3.79×10^{-3}	1.63×10^{-8}	3.24×10^{-8}	530	739
	−10	2.97×10^{-3}	3.49×10^{-3}	3.65×10^{-4}	3.13×10^{-4}	1.27×10^{-9}	3.14×10^{-9}	641	894
	−11	3.48×10^{-3}	4.41×10^{-3}	3.03×10^{-5}	1.38×10^{-5}	1.08×10^{-10}	2.68×10^{-10}	776	1081
	−12	4.22×10^{-3}	4.99×10^{-3}	2.05×10^{-6}	2.00×10^{-6}	8.03×10^{-12}	2.27×10^{-11}	939	1308
	−13	5.08×10^{-3}	6.01×10^{-3}	2.25×10^{-7}	1.87×10^{-7}	7.11×10^{-13}	2.28×10^{-12}	1136	1583

Table 4.8: The CPU time in seconds, MGE and NS of CPHBT$_{\mathrm{RK4}}(9,6,12)$ and T(12) for the Arenstorf problem, B1, B5, E2 and the Galactic dynamics problems.

Problem	T	CPU time		MGE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
Aren	−3	2.53×10^{-4}	3.12×10^{-4}	1.31	1.59	29	40
$t\in[0,20]$	−4	3.14×10^{-4}	4.04×10^{-4}	2.87×10^{-1}	8.46×10^{-1}	37	50
	−5	3.60×10^{-4}	4.80×10^{-4}	2.32×10^{-2}	1.46×10^{-1}	45	63
	−6	4.33×10^{-4}	5.71×10^{-4}	9.13×10^{-4}	1.35×10^{-2}	55	76
	−7	5.13×10^{-4}	7.00×10^{-4}	6.14×10^{-5}	1.55×10^{-3}	67	92
	−8	6.33×10^{-4}	8.27×10^{-4}	2.77×10^{-6}	1.88×10^{-4}	81	111
	−9	7.33×10^{-4}	1.01×10^{-3}	2.71×10^{-7}	2.03×10^{-5}	98	135
	−10	8.75×10^{-4}	1.21×10^{-3}	1.97×10^{-8}	2.34×10^{-6}	119	163
	−11	1.04×10^{-3}	1.45×10^{-3}	1.80×10^{-9}	2.45×10^{-7}	144	197
	−12	1.24×10^{-3}	1.75×10^{-3}	2.71×10^{-10}	2.52×10^{-8}	174	238
	−13	1.47×10^{-3}	2.13×10^{-3}	1.41×10^{-10}	2.79×10^{-9}	210	288
B5	−3	7.03×10^{-5}	8.59×10^{-5}	1.73×10^{-4}	4.79×10^{-3}	13	18
$t\in[0,20]$	−4	8.01×10^{-5}	9.73×10^{-5}	2.90×10^{-5}	4.41×10^{-4}	16	22
	−5	8.77×10^{-5}	1.12×10^{-4}	8.10×10^{-7}	4.75×10^{-5}	19	26
	−6	1×10^{-4}	1.30×10^{-4}	1.27×10^{-7}	2.80×10^{-6}	23	31
	−7	1.18×10^{-4}	1.48×10^{-4}	1.11×10^{-8}	2.99×10^{-7}	28	38
	−8	1.35×10^{-4}	1.76×10^{-4}	8.42×10^{-10}	1.08×10^{-8}	34	46

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Table 4.8 – *Continued from previous page*

Problem	T	CPU time		MGE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
	-9	1.56×10^{-4}	2.07×10^{-4}	5.48×10^{-11}	2.84×10^{-9}	41	55
	-10	1.83×10^{-4}	2.44×10^{-4}	1.34×10^{-11}	2.69×10^{-10}	49	67
	-11	2.16×10^{-4}	2.85×10^{-4}	1.84×10^{-12}	3.17×10^{-11}	60	80
	-12	2.55×10^{-4}	3.41×10^{-4}	1.52×10^{-13}	4.83×10^{-12}	73	97
	-13	3.06×10^{-4}	4.06×10^{-4}	4.90×10^{-14}	3.69×10^{-13}	88	118
B1	-3	6.69×10^{-5}	7.99×10^{-5}	2.37×10^{-3}	7.40×10^{-2}	26	33
$t\in[0,20]$	-4	7.71×10^{-5}	8.63×10^{-5}	8.20×10^{-4}	1.09×10^{-2}	28	39
	-5	8.87×10^{-5}	1.02×10^{-4}	1.68×10^{-4}	1.86×10^{-4}	35	46
	-6	1.02×10^{-4}	1.16×10^{-4}	9.50×10^{-6}	7.12×10^{-5}	43	56
	-7	1.27×10^{-4}	1.36×10^{-4}	1.02×10^{-6}	1.32×10^{-5}	52	67
	-8	1.40×10^{-4}	1.59×10^{-4}	1.28×10^{-7}	4.55×10^{-7}	62	80
	-9	1.64×10^{-4}	1.86×10^{-4}	5.91×10^{-9}	8.37×10^{-8}	76	98
	-10	1.95×10^{-4}	2.19×10^{-4}	1.17×10^{-9}	1.09×10^{-8}	90	118
	-11	2.28×10^{-4}	2.58×10^{-4}	1.47×10^{-10}	5.25×10^{-10}	111	142
	-12	2.72×10^{-4}	3.07×10^{-4}	2.07×10^{-11}	1.73×10^{-10}	136	172
	-13	3.21×10^{-4}	3.73×10^{-4}	2.46×10^{-12}	3.58×10^{-12}	164	208
E2	-3	8.64×10^{-5}	1.11×10^{-4}	5.17×10^{-3}	2.23×10^{-3}	30	40
E2	-4	1.00×10^{-4}	1.33×10^{-4}	1.99×10^{-4}	4.88×10^{-4}	36	48
$t\in[0,20]$	-5	1.20×10^{-4}	1.55×10^{-4}	5.24×10^{-6}	5.71×10^{-5}	47	58
	-6	1.34×10^{-4}	1.79×10^{-4}	2.50×10^{-7}	2.01×10^{-6}	54	69
	-7	1.60×10^{-4}	2.08×10^{-4}	1.04×10^{-7}	4.20×10^{-7}	66	83
	-8	1.83×10^{-4}	2.45×10^{-4}	1.83×10^{-8}	3.88×10^{-8}	78	101
	-9	2.11×10^{-4}	2.96×10^{-4}	3.49×10^{-9}	7.02×10^{-9}	94	121
	-10	2.68×10^{-4}	3.44×10^{-4}	4.67×10^{-11}	1.04×10^{-9}	120	146
	-11	3.05×10^{-4}	4.15×10^{-4}	7.58×10^{-12}	1.14×10^{-10}	138	177
	-12	3.71×10^{-4}	4.97×10^{-4}	2.55×10^{-12}	8.70×10^{-12}	168	214
	-13	4.46×10^{-4}	6.05×10^{-4}	5.86×10^{-13}	1.21×10^{-12}	207	258
GAD	-3	1.72×10^{-3}	2.36×10^{-3}	5.39	4.50	259	367
$t\in[0,500]$	-4	2.06×10^{-3}	2.77×10^{-3}	2.50×10^{-1}	1.12	318	445
	-5	2.38×10^{-3}	3.25×10^{-3}	2.94×10^{-3}	1.48×10^{-1}	384	536
	-6	2.81×10^{-3}	3.87×10^{-3}	1.22×10^{-3}	1.41×10^{-3}	466	646
	-7	3.37×10^{-3}	4.70×10^{-3}	2.76×10^{-5}	2.77×10^{-5}	564	781
	-8	3.99×10^{-3}	5.59×10^{-3}	6.42×10^{-6}	5.55×10^{-6}	681	941
	-9	4.77×10^{-3}	6.64×10^{-3}	4.30×10^{-7}	3.26×10^{-7}	826	1138
	-10	5.63×10^{-3}	8.19×10^{-3}	1.09×10^{-8}	2.26×10^{-8}	999	1375
	-11	6.86×10^{-3}	9.75×10^{-3}	2.32×10^{-9}	6.41×10^{-9}	1209	1662
	-12	8.34×10^{-3}	1.16×10^{-2}	1.83×10^{-10}	1.16×10^{-10}	1461	2011
	-13	1.03×10^{-2}	1.41×10^{-2}	1.41×10^{-11}	9.10×10^{-12}	1772	2434

4.7 The propagation of error in a long-term integration problem for $\mathbf{CPHBT_{RK4}}(9,6,12)$

4.7.1 Fixed step-size configuration

As shown in the previous sections, the designed $\mathrm{CPHBT_{RK4}}(9,6,12)$ method perform favourably over T(12), T(12)L and DP(8,7)13M in terms of accuracy and efficiency. In particular, the CPU time, the number of steps and of function evaluations required to achieve the user defined tolerance TOL. In addition, $\mathrm{CPHBT_{RK4}}$ satisfies the contractivity preserving property which is a natural property occurring in dissipative systems of ODEs [33]. T. Nguyen-Ba *et al.* showed in [40] that the contractivity preserving property can reduce the propagation of discretization errors in the numerical method. To investigate this property, we test the propagation of error of $\mathrm{CPHBT_{RK4}}(9,6,12)$ method compared to Adams-Bashforth-Moulton, ABM(12,11), method of order 12 with predictor of order 11 and corrector of order 12 in PECE mode [48, page 135-140]. We will consider Kepler's two-body problem with eccentricities $\varepsilon = 0.3, 0.5, 0.7$ over the time period $t \in [0, 800000\pi]$. In this simulation, we choose fixed step sizes h_{CPHBT} and h_{ABM} such that the methods $\mathrm{CPHBT_{RK4}}(9,6,12)$ and ABM(12,11) use approximately an equal CPU time. In Figure (4.7), we plot 300 equidistant points of the energy error over the integration period in a log-log scale using the MATLAB function `loglog`. We note that this MATLAB function plots the data in log-log scale, but it leaves the axis linear with logarithmic ticks, i.e., it plots logarithmic data on linear axes. Also, we used the MATLAB `filter` command to remove some chattering in the collected data.

From Figure (4.7), we notice that the energy error (EE) grows asymptotically as a power law module αt^{β}. Then, we fit the collected data to the module αt^{β}. We are interested in the asymptotic behaviour of $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$ over very long integration intervals. In particular, we are interested in the change in the error ratio $\frac{EE_{\mathrm{ABM}}}{EE_{\mathrm{CPHBT}}} = \frac{\alpha_{\mathrm{ABM}}}{\alpha_{\mathrm{CPHBT}}} t^{(\beta_{\mathrm{ABM}} - \beta_{\mathrm{CPHBT}})}$. For very long integration intervals, it is sufficient to study the value of the exponents β_{ABM} and β_{CPHBT}. However, if $\beta_{\mathrm{ABM}} = \beta_{\mathrm{CPHBT}}$, then the energy error ratio is given by $\frac{EE_{\mathrm{ABM}}}{EE_{\mathrm{CPHBT}}} = \frac{\alpha_{\mathrm{ABM}}}{\alpha_{\mathrm{CPHBT}}}$. Solving Kepler's two body problem for $t \in [0, 800000\pi]$, we obtained $\beta_{\mathrm{ABM}} = \beta_{\mathrm{CPHBT}} = 1$ for eccentricities $\varepsilon = 0.3, 0.5, 0.7$. Then, the error ratio $\frac{EE_{\mathrm{ABM}}}{EE_{\mathrm{CPHBT}}} = \frac{\alpha_{\mathrm{ABM}}}{\alpha_{\mathrm{CPHBT}}} = 18, 21$ and 13 for $\varepsilon = 0.3, 0.5$ and 0.7, respectively. For instance, in Figure (4.8), we plot the energy error versus time for $\varepsilon = 0.3$ to show the error growth of the discretization errors of $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$ and ABM(12,11). We note that this error growth is an expected result since ABM(12,11) and $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$ are non-symplectic methods [44].

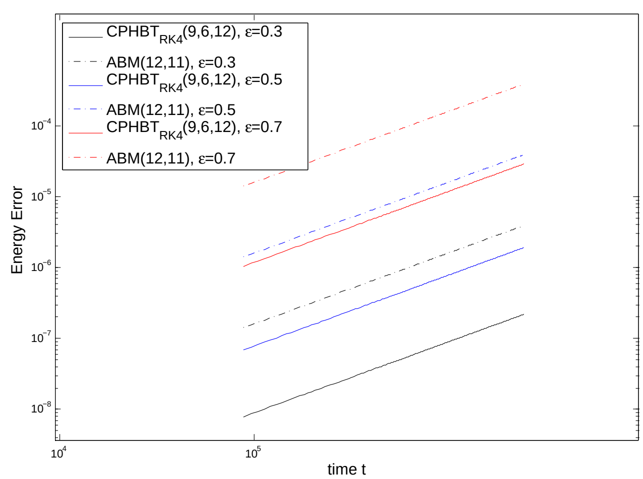


Figure 4.7: The energy error (EE) versus time in log-log scale for Kepler's two-body problem with eccentricity $\varepsilon = 0.3, 0.5, 0.7$, $t \in [0, 800000\pi]$ and fixed step-size configuration.

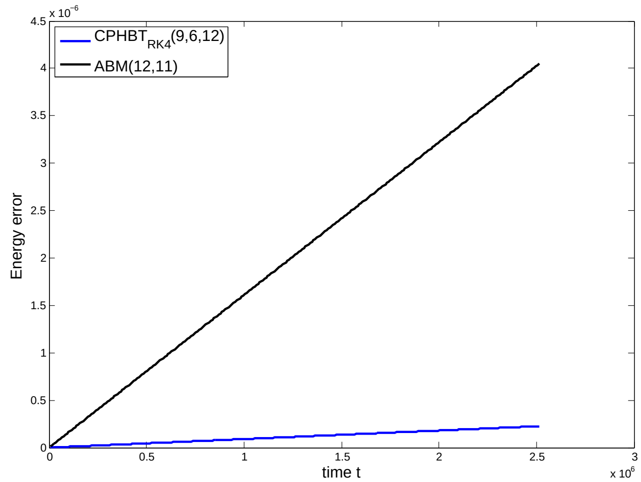


Figure 4.8: The energy error (EE) versus time for Kepler's two-body problem with eccentricity $\varepsilon = 0.3$, $t \in [0, 800000\pi]$ and fixed step-size configuration.

4.7.2 Variable step-size configuration

We can elaborate more in investigating the propagation of error in long-term integration of Kepler's two-body problem by considering a variable-step CPHBT$_{\text{RK4}}(9,6,12)$ and a variable-step T(12). To make this comparison fair, we choose the control factor η in the step control formula (4.3.8) of CPHBT$_{\text{RK4}}(9,6,12)$ such that both methods use approximately the same CPU time to solve Kepler's problem over the time period $t \in [0, 20000\pi]$. Figure (4.9) presents the energy error as a function of time for Kepler's two-body problem with eccentricities $\varepsilon = 0.3, 0.5, 0.7, 0.99$ using the MATLAB `loglog` function. As expected, the variable-step configuration managed to obtain noticeably better energy error for CPHBT$_{\text{RK4}}(9,6,12)$ compared to the fixed step-size configuration. Also, it is clear that CPHBT$_{\text{RK4}}(9,6,12)$ is a couple of orders more accurate than T(12). More precisely, CPHBT$_{\text{RK4}}(9,6,12)$ is 26, 46, 29 and 43 times more accurate than T(12) at the end of the integration interval with eccentricities $\varepsilon = 0.3, 0.5, 0.7$ and 0.99, respectively. In fact, CPHBT$_{\text{RK4}}(9,6,12)$ applied to Kepler's problem with the ultra high eccentricity ($\varepsilon = 0.99$) is about 5 times more accurate than T(12) applied to Kepler's problem with the lowest eccentricity ($\varepsilon = 0.3$). Analysing Figure (4.9), we note that the amplitude of the oscillations of the energy error of the CPHBT$_{\mathrm{RK4}}(9,6,12)$ is larger for the ultra high eccentricity $\varepsilon = 0.99$. This occurred since CPHBT$_{\mathrm{RK4}}(9,6,12)$ has no access to the 11th-derivative and the variable step-control (4.3.8) depends on an estimation of the (11)-th derivative. This estimation becomes less accurate for ultra high eccentricities and hence we have these oscillations. Also, we note the symplectic-method like behaviour of CPHBT$_{\mathrm{RK4}}(9,6,12)$ for eccentricity $\varepsilon = 0.5$ at the end of the integration period which is a desired property.

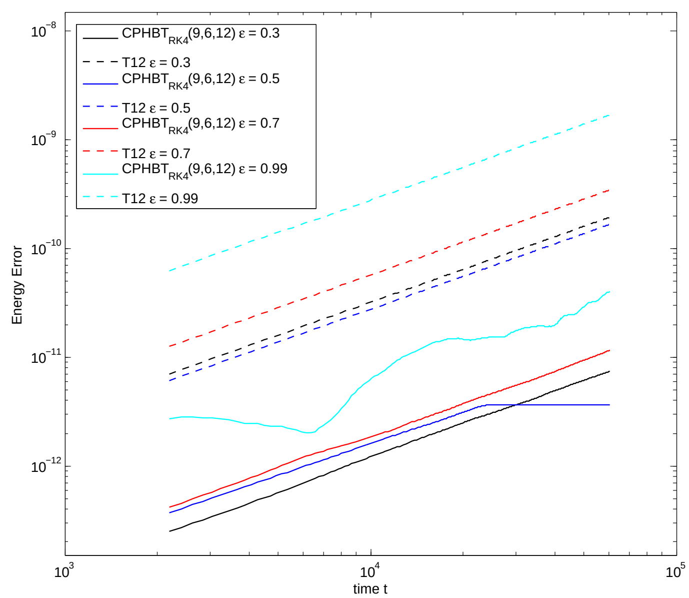


Figure 4.9: The energy error (EE) of T(12) and CPHBT$_{\mathrm{RK4}}(9,6,12)$ for Kepler's two-body problem with eccentricity $\varepsilon = 0.3, 0.5, 0.7, 0.99$ and $t \in [0, 20000\pi]$.

Chapter 5

CP s-Stage HBT methods based on combining T(d) and RK(s,5) methods

5.1 Introduction

In the previous chapters, we show that $\text{CPHBT}_{\text{RK4}}(d, s, p)$ performs favourably in comparison with many well known methods such as T(12), T(12)L, DP(8,7)M13 and Adams-Bashforth-Moulton methods in terms of the NS, NFE and CPU time PEG required to satisfy a given user defined tolerance TOL. Also, we show the effect of the contractivity preserving property of $\text{CPHBT}_{\text{RK4}}$ by considering the long term integration of Kepler's two-body problem where $\text{CPHBT}_{\text{RK4}}(d, s, p)$ was superior to the other considered well known methods. These results motivated us to investigate and design the second set of CPHBT methods by casting Runge-Kutta methods of order 5, RK(s,5), with Taylor method of order $p-4$, T($p-4$), and obtain $\text{CPHBT}_{\text{RK5}}(p-4, s, p)$. By considering RK($s$,5) instead of RK($s$,4), we managed to reduce the number of computationally expensive high order derivatives (HOD) required to reach order p from $p-2$ and $p-3$ HOD for CPHBT$_{\text{RK4}}$ to $p-4$ for the CPHBT$_{\text{RK5}}$. Moreover, CPHBT$_{\text{RK5}}(p-4,s,p)$ tends to have larger optimal contractivity preserving coefficients and larger stability regions.

5.2 Formulation of CPHBT$_{\text{RK5}}(p-4,s,p)$ in Butcher form

To obtain the explicit, one-step, s-stage, $(p-4)$-derivative CPHBT method of order p, Hermite-Birkhoff interpolation polynomials are used to define the following s formulae which performs the integration from x_n to x_{n+1}. HB polynomials are used as predictors P_i to obtain the stages Y_i to order $p-4$ as follows:

$$Y_i \;=\; y_n + \Delta x \sum_{j=1}^{i-1} a_{i,j} F_j + \sum_{m=2}^{p-4} (\Delta x)^m \gamma_{i,m}\, y_n^{(m)}, \qquad i \;=\; 2,3,\ldots,s. \tag{5.2.1}$$

Also, a Hermite-Birkhoff interpolation polynomial is used as an integration formula (IF) to obtain y_{n+1} to order p as follows:

$$y_{n+1} = y_n + \Delta x \sum_{j=1}^{s} b_j F_j + \sum_{m=2}^{p-4} (\Delta x)^m \gamma_{s+1,m}\, y_n^{(m)}, \tag{5.2.2}$$

Since CPHBT$_{\text{RK5}}(p-4,s,p)$ has a similar form as CPHBT$_{\text{RK4}}(d,s,p)$ with $d=p-4$, then all contractivity preserving formulations, theorems and the different forms of CPHBT$_{\text{RK4}}(d,s,p)$ can be easily extended to the CPHBT$_{\text{RK5}}(p-4,s,p)$ and hence these derivations will be omitted in this chapter.

5.3 The order conditions of CPHBT$_{\text{RK5}}(p-4,s,p)$

To derive the order conditions of CPHBT$_{\text{RK5}}(p-4,s,p)$ obtained by combining RK(s,5) of order 5 and Taylor method of order $p-4$, T$(p-4)$, we follow a similar approach as in Section 3.3. We note that the new set of methods, CPHBT$_{\text{RK5}}(p-4,s,p)$, have similar formula as the first set of methods, $\text{CPHBT}_{\text{RK4}}(d,s,p)$, with $d=p-4$. Then, the elementary weights of $\text{CPHBT}_{\text{RK5}}(p-4,s,p)$ are given by the recursive formulas (3.3.17)-(3.3.21) with $d=p-4$ and hence we can derive the order conditions of $\text{CPHBT}_{\text{RK5}}(p-4,s,p)$ by enforcing equation (3.3.16). As an example, we derive the order condition of $\text{CPHBT}_{\text{RK5}}(2,s,6)$ of order 6 associated with the rooted tree $t_{34}=[\tau^2[\tau^2]_2$. By equations (3.3.17)-(3.3.21), the elementary weight $\Psi(t_{34})$ is given by

$$\begin{aligned}\Psi(t_{34}) &= \sum_{j=1}^{s} b_j \tilde{S}_j'(t_{34}) + \sum_{m=2}^{2} \gamma_{im}\Phi^{(m)}(t_{34}) \\ &= \sum_{j=1}^{s} b_j \tilde{S}_j'(t_{34}) = 6\sum_{j=1}^{s} b_j \big(\tilde{S}_j(\tau)\big)^2 \tilde{S}_j([\tau^2]),\end{aligned} \tag{5.3.1}$$

with,

$$\tilde{S}_j(\tau) = \sum_{k=1}^{j-1} a_{jk}\tilde{S}_k'(\tau) = \sum_{k=1}^{j-1} a_{jk} = c_j, \tag{5.3.2}$$

$$\tilde{S}_j'([\tau^2]) = \sum_{l=1}^{j-1} a_{jl}\tilde{S}_l'([\tau^2]) = 3\sum_{l=1}^{j-1} a_{jl}\big(\tilde{S}_l(\tau)\big)^2 = 3\sum_{l=1}^{j-1} a_{jl}c_l^2. \tag{5.3.3}$$

Substituting equations (5.3.2) and (5.3.3) into (5.3.1), the elementary weight of $\text{CPHBT}_{\text{RK5}}(2,s,6)$ corresponding to the rooted tree $t_{34}=[\tau^2[\tau^2]_2$ is

$$\Psi(t_{34}) = 18\sum_{j=1}^{s} b_j c_j^2 \left[\sum_{l=1}^{j-1} a_{jl}c_l^2\right]. \tag{5.3.4}$$

Hence, by equation (3.3.16), the order condition of $\text{CPHBT}_{\text{RK5}}(2,s,6)$ corresponding to the rooted tree t_{34} is given by

$$\Psi(t_{34}) = 1 \Longleftrightarrow \sum_{j=1}^{s} b_j c_j^2 \left[\sum_{l=1}^{j-1} a_{jl}c_l^2\right] = \frac{1}{18}, \tag{5.3.5}$$

which is the order condition (5.3.20) given below for $p=6$.

To reduce the number of independent order conditions, we impose the following simplifying conditions [22, 41]:

$$\sum_{j=1}^{i-1} a_{i,j}\frac{c_j^k}{k!} + \gamma_{i,k+1} = \frac{1}{(k+1)!}\,c_i^{k+1}, \qquad \begin{cases} i = 2,3,\ldots,s, \\ k = 0,1,\ldots,p-5. \end{cases} \tag{5.3.6}$$

Again, we take $c_1 = 0$, the convention that $c_1^0 = 1$ and $\gamma_{i,1} = 0$ for $i = 2,3,\ldots,s$. By enforcing these simplifying conditions, all order conditions of CPHBT$_{\text{RK5}}(p-4,s,p)$ generated from trees of order $r = 1,2,\ldots,p-3$ are equivalent to those generated by the bushy trees of order $r = 1,2,\ldots,p-3$ and the number of independent order conditions generated from trees of orders $r = p-2, p-1, p$ is reduced considerably. For instance, when we derive the order conditions of CPHBT$_{\text{RK5}}(8,s,12)$ of order 12, the simplifying conditions (5.3.6) reduces the number of order conditions from 7813 independent conditions to only 95 independent conditions including the simplifying conditions. We note that, when deriving CPHBT$_{\text{RK4}}(d,s,12)$ by casting T(d) with RK(s,4) of order 4 in the previous chapter, the simplifying conditions (3.3.29) reduced the number of order conditions from 7813 independent conditions to 52 only. The remaining trees will lead to the following sets of order conditions for CPHBT$_{\text{RK5}}(p-4,s,p)$:

$$\sum_{i=1}^{s} b_i = 1, \tag{5.3.7}$$

$$\sum_{i=2}^{s} b_i c_i^k + k!\gamma_{s+1,k+1} = \frac{1}{k+1}, \quad k = 1,\ldots,p-5, \tag{5.3.8}$$

$$\sum_{i=2}^{s} b_i c_i^k = \frac{1}{k+1}, \qquad k = p-4, p-3,\ p-2,\ p-1, \tag{5.3.9}$$

$$\sum_{i=3}^{s} b_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-4}\right] = \frac{1}{(p-2)(p-3)}, \tag{5.3.10}$$

$$\sum_{i=4}^{s} b_i \left[\sum_{j=3}^{i-1} a_{i,j}\left[\sum_{k=2}^{j-1} a_{j,k}\, c_k^{p-4}\right]\right] = \frac{1}{(p-1)(p-2)(p-3)}, \tag{5.3.11}$$

$$\sum_{i=3}^{s} b_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-3} \right] = \frac{1}{(p-1)(p-2)}, \tag{5.3.12}$$

$$\sum_{i=3}^{s} b_i c_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-4} \right] = \frac{1}{(p-1)(p-3)}, \tag{5.3.13}$$

$$\sum_{i=5}^{s} b_i \left[\sum_{j=4}^{i-1} a_{i,j} \left[\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{p-4} \right) \right] \right] = \frac{1}{p(p-1)(p-2)(p-3)}, \tag{5.3.14}$$

$$\sum_{i=4}^{s} b_i \left[\sum_{j=3}^{i-1} a_{i,j} \left[\sum_{k=2}^{j-1} a_{j,k}\, c_k^{p-3} \right] \right] = \frac{1}{p(p-1)(p-2)}, \tag{5.3.15}$$

$$\sum_{i=4}^{s} b_i \left[\sum_{j=3}^{i-1} a_{i,j} c_j \left[\sum_{k=2}^{j-1} a_{j,k}\, c_k^{p-4} \right] \right] = \frac{1}{p(p-1)(p-3)}, \tag{5.3.16}$$

$$\sum_{i=3}^{s} b_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-2} \right] = \frac{1}{p(p-1)}, \tag{5.3.17}$$

$$\sum_{i=4}^{s} b_i c_i \left[\sum_{j=3}^{i-1} a_{i,j} \left[\sum_{k=2}^{j-1} a_{j,k}\, c_k^{p-4} \right] \right] = \frac{1}{p(p-2)(p-3)}, \tag{5.3.18}$$

$$\sum_{i=3}^{s} b_i c_i \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-3} \right] = \frac{1}{p(p-2)}, \tag{5.3.19}$$

$$\sum_{i=3}^{s} b_i c_i^2 \left[\sum_{j=2}^{i-1} a_{i,j}\, c_j^{p-4} \right] = \frac{1}{p(p-3)}. \tag{5.3.20}$$

5.4 Formulation of the optimization problem of CPHBT$_{\text{RK5}}(p-4,s,p)$

Considering the conditions of the contractivity preserving property in Theorem (3.7.1) together with the simplifying conditions (5.3.6) and the order conditions (5.3.7)-(5.3.20), we can obtain a feasible CPHBT$_{\text{RK5}}(p-4,s,p)$ with a feasible contractivity preserving coefficient c_{fcp} as mentioned in Remark 3.4.1. However, following our work in Section 3.8, we obtain an optimal CPHBT$_{\text{RK5}}(p-4,s,p)$ with largest contractivity preserving coefficient by solving the following nonlinear optimization problem:

$$\underset{z,\boldsymbol{v},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}}{\text{maximize}} \; F(z,\boldsymbol{v},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}) = z \tag{5.4.1}$$

subject to

$$\alpha_{ij} - z\beta_{ij} = 0, \qquad \begin{cases} i = 3,4,\ldots,s+1, \\ j = 2,3,\ldots,i-1, \end{cases} \tag{5.4.2}$$

$$\left(\boldsymbol{I} + z\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{v}_0 \geq 0, \tag{5.4.3}$$

$$\left(\boldsymbol{I} + z\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\beta}_0 \geq 0, \tag{5.4.4}$$

$$\left(\boldsymbol{I} + z\boldsymbol{\beta}_0\right)^{-1} \boldsymbol{\gamma}_0 \geq 0, \tag{5.4.5}$$

$$0 \leq c_i \leq 1 \qquad i = 2,3,\ldots,s+1, \tag{5.4.6}$$

$$z \leq z_{i1} = \frac{\alpha_{i1}}{\beta_{i1}} \qquad i = 2,3,\ldots,s+1, \tag{5.4.7}$$

$$\delta_{i,m} z_{i1}^m m! - \alpha_{i,1} \leq 0, \qquad \begin{cases} i = 2,3,\ldots,s+1, \\ m = 2,3,\ldots,p-4 \end{cases} \tag{5.4.8}$$

together with

- The simplifying conditions (5.3.6),
- The set of order conditions (5.3.7)-(5.3.20),

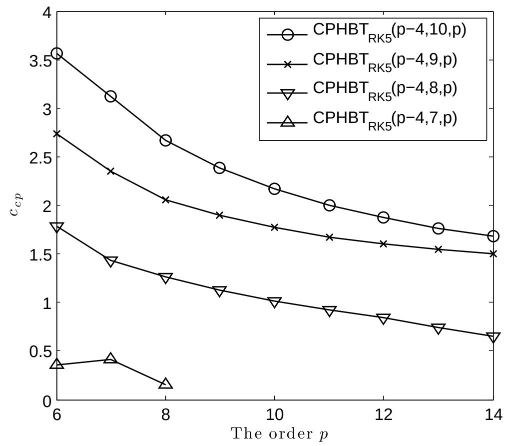
where the objective function is defined in (3.8.8), the inequalities (5.4.3)-(5.4.5) are taken component-wise and the coefficient matrices $\boldsymbol{v}_0, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0$ and $\boldsymbol{\delta}$ are defined in equations (3.6.5), (3.4.4) and (3.5.3).

5.5 Construction of optimal CPHBT$_{\mathbf{RK5}}(p-4, s, p)$

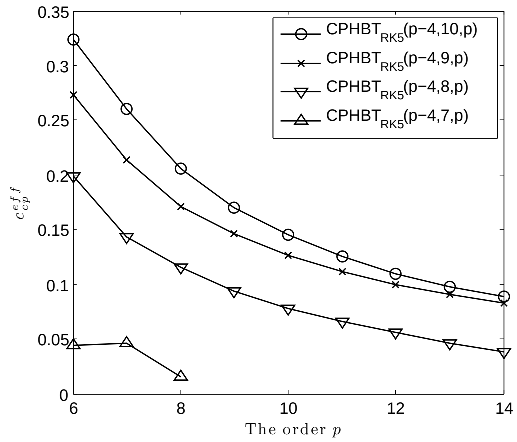
Indeed, the number of constraints in the above mentioned optimization problem is considerably larger than the optimization problem of CPHBT$_{\text{RK4}}$. Our numerical search for 6-stages methods failed to satisfy some of the constraints, especially the more restrictive constraints (5.4.8) on $\delta_{i,m}$ for $i = 2,3,\ldots,s+1$ and $m = 2,3,\ldots,p-4$. Hence, we consider $s = 7,8,9,10$ and obtain CPHBT$_{\text{RK5}}(p-4,s,p)$ for $p = 6,7,\ldots,14$. In Table 5.1, we summarize these results by listing the optimal contractivity preserving coefficients and the interval of absolute stability $(\alpha, 0)$ for the designed CPHBT$_{\text{RK5}}(p-4,s,p)$. For $s = 7$, the optimization process has a solution for $p = 6,7,8$. However, the resulting CPHBT$_{\text{RK5}}(p-4,7,p)$, for $p = 6,7,8$, have relatively small optimal contractivity preserving coefficients. Also, the optimization process fails to obtain an optimal CPHBT$_{\text{RK5}}(p-4,7,p)$ for $p = 9,10,\ldots,14$.

Table 5.1: The contractivity preserving coefficient c_{cp} and the interval of absolute stability $(\alpha, 0)$ of CPHBT$_{\text{RK5}}(d,s,p)$ compared to T(p).

$s\backslash p$	6	7	8	9	10	11	12	13	14
Taylor	T(6)	T(7)	T(8)	T(9)	T(10)	T(11)	T(12)	T(13)	T(14)
$\|\alpha\|$	3.55	3.95	4.30	4.70	5.05	5.45	5.80	6.17	6.55
$\boldsymbol{s=10}$									
(p,d)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(12,8)	(13,9)	(14,10)
c_{cp}	3.5646	3.12	2.6736	2.38564	2.1721	2.0022	1.8702	1.7654	1.685
$\|\alpha\|$	10.00	9.40	7.85	8.05	7.75	8.15	8.15	8.40	8.65
$\boldsymbol{s=9}$									
(p,d)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(12,8)	(13,9)	(14,10)
c_{cp}	2.7322	2.3553	2.053	1.902	1.7751	1.6745	1.5996	1.5444	1.4967
$\|\alpha\|$	8.05	8.00	7.65	8.65	8.80	8.95	8.67	8.75	9.05
$\boldsymbol{s=8}$									
(p,d)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(12,8)	(13,9)	(14,10)
c_{cp}	1.789	1.4302	1.266	1.1276	1.0103	0.9213	0.839	0.73982	0.6555
α	8.25	6.80	7.00	7.05	7.25	7.55	7.85	8.05	8.35
$\boldsymbol{s=7}$									
(p,d)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(12,8)	(13,9)	(14,10)
c_{cp}	0.3552	0.4169	0.1517	*	*	*	*	*	*
$\|\alpha\|$	4.55	4.45	4.70	*	*	*	*	*	*



(a) c_{cp} as a function of the order



(b) $c_{\text{cp}}^{\text{eff}}$ as a function of the order

Figure 5.1: The CP coefficient c_{cp} and the effective CP coefficient $c_{\text{cp}}^{\text{eff}}$ as a function of the order of CPHBT$_{\text{RK5}}(p-4, s, p)$ for $s = 7, 8, 9, 10$.

To analyze the optimal CP coefficients in Table 5.1, we plot, in Figure (5.1a), the optimal CP coefficient c_{cp} as a function of the order p for different number of stages s. We notice that the optimal CP coefficients decrease linearly as a function of p. However, to compare the CP coefficients of methods with different number of stages, we consider the effective contractivity preserving coefficient.

Definition 5.5.1 (Effective contractivity preserving coefficient) *The effective contractivity preserving coefficient of a CP method M is given by*

$$c_{cp}^{eff}(M) = \frac{c_{cp}(M)}{N}, \tag{5.5.1}$$

where N is the number of function evaluations of the method M per integration time step.

In Figure (5.1b), we plot the efficient CP coefficients $c_{\text{cp}}^{\text{eff}}$ as a function of p for different values of s. It is seen that, for lower order methods, $p = 6, 7, 8$, the difference in $c_{\text{cp}}^{\text{eff}}$ between CPHBT$_{\text{RK5}}(p-4, s, p)$ with $s = 10$ and $s = 8$ is considerably large. This is more

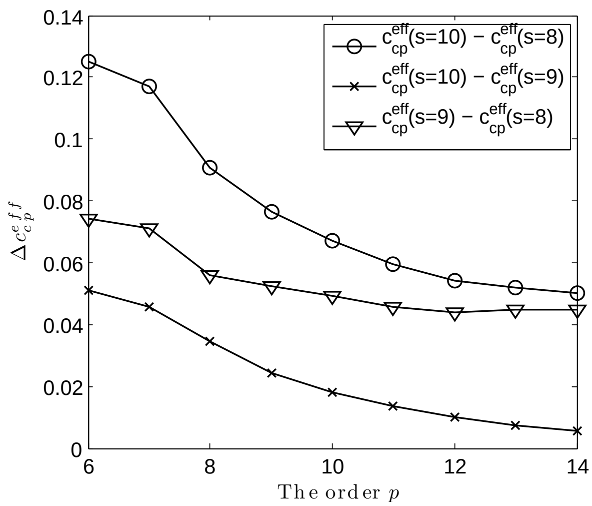


Figure 5.2: The difference of the efficient contractivity preserving coefficients as a function of the order of CPHBT$_{\text{RK5}}(p-4,s,p)$ for $s=8,9,10$.

obvious in Figure (5.2) where we plot the difference of $c_{\text{cp}}^{\text{eff}}$ as a function of p. However, this difference drops rapidly and for higher order methods, $p=12,13,14$, this difference is negligible. Hence, we recommend using the 10-stages CPHBT$_{\text{RK5}}(d,10,p)$ for $p=6,7,8$ which will allow considerably higher CP coefficient and the 8-stages CPHBT$_{\text{RK5}}(d,8,p)$ for $p=12,13,14$ which will be more computationally economical.

The formulae of the new CPHBT$_{\text{RK5}}(p-4,s,p)$ with their contractivity preserving coefficient, $c_{cp}(\text{CPHBT}_{\text{RK5}}(p-4,s,p))$, and abscissa vector σ are given in Section B of the Appendix .

Chapter 6

Numerical results for the designed $\mathbf{CPHBT_{RK5}}(p-4,s,p)$ methods obtained from $\mathbf{T}(p-4)$ and $\mathbf{RK}(s,5)$ methods

In this chapter, we present some numerical results obtained by testing the second set of $\mathrm{CPHBT_{RK5}}$ in terms of maximum global error (MGE), maximum global energy error (MGEE), number of steps (NS), number of function evaluations (NFE), CPU time and the propagation of errors in long-term integration problems. For consistency and comparison reasons, we will consider an 8-stage, 8-derivatives $\mathrm{CPHBT_{RK5}}(8,8,12)$ of order 12 as a representative of the designed $\mathrm{CPHBT_{RK5}}$ methods.

CPHBT$_{\mathbf{RK5}}(8,8,12)$ with $c_{\rm cp}=8.3899828718045832\times10^{-1}$, and abscissa vector $\sigma=[0\ 6.3442778287933732\times10^{-1}\ 6.4806091964118251\times10^{-1}\ 6.5778452073891203\times10^{-1}\ 7.4053651347951488\times10^{-1}\ 8.3110284752952135\times10^{-1}\ 9.2735138793776239\times10^{-1}\ 9.4800556307728689\times10^{-1}]^T$

$$
\begin{aligned}
Y_2 =& y_n + 6.3442778287933732\times10^{-1}h_nf_n + 2.0124930584459580\times10^{-1}h_n^2y_n^{(2)} + 4.2559383637664192\times10^{-2}h_n^3y_n^{(3)}\\
&+ 6.7502138504886089\times10^{-3}h_n^4y_n^{(4)} + 8.5650464142537658\times10^{-4}h_n^5y_n^{(5)} + 9.0565056780893895\times10^{-5}h_n^6y_n^{(6)}\\
&+ 8.2081411685491133\times10^{-6}h_n^7y_n^{(7)} + 6.5093410039040332\times10^{-7}h_n^8y_n^{(8)}\\
Y_3 =& y_n + 5.5991828761338081\times10^{-1}h_nf_n + 8.8142632027801662\times10^{-2}h_nF_2 + 1.5407134316854010\times10^{-1}h_n^2y_n^{(2)}\\
&+ 2.7623779892062172\times10^{-2}h_n^3y_n^{(3)} + 3.5981073656161508\times10^{-3}h_n^4y_n^{(4)} + 3.5759061708865212\times10^{-4}h_n^5y_n^{(5)}\\
&+ 2.7392899410268045\times10^{-5}h_n^6y_n^{(6)} + 1.5426932790737223\times10^{-6}h_n^7y_n^{(7)} + 4.8137564385466838\times10^{-8}h_n^8y_n^{(8)}\\
Y_4 =& y_n + 5.6601553493298773\times10^{-1}h_nf_n + 6.3191453543435779\times10^{-3}h_nF_2 + 8.5449840451580661\times10^{-2}h_nF_3\\
&+ 1.5695449429876798\times10^{-1}h_n^2y_n^{(2)} + 2.8219624670652730\times10^{-2}h_n^3y_n^{(3)} + 3.6553656460089476\times10^{-3}h_n^4y_n^{(4)}\\
&+ 3.5555085562573676\times10^{-4}h_n^5y_n^{(5)} + 2.5694849223444351\times10^{-5}h_n^6y_n^{(6)} + 1.2079358620528968\times10^{-6}h_n^7y_n^{(7)}\\
&+ 3.4510912524133941\times10^{-9}h_n^8y_n^{(8)}\\
Y_5 =& y_n + 5.2677084285373510\times10^{-1}h_nf_n + 1.0523109471714857\times10^{-3}h_nF_2 + 1.4229741127800948\times10^{-2}h_nF_3\\
&+ 1.9848361855080737\times10^{-1}h_nF_4 + 1.3374835757257769\times10^{-1}h_n^2y_n^{(2)} + 2.1544442788534502\times10^{-2}h_n^3y_n^{(3)}\\
&+ 2.4253119077743551\times10^{-3}h_n^4y_n^{(4)} + 1.9592717340050647\times10^{-4}h_n^5y_n^{(5)} + 1.0916080320101515\times10^{-5}h_n^6y_n^{(6)}\\
&+ 3.4266972684998814\times10^{-7}h_n^7y_n^{(7)} + 5.7470130863604662\times10^{-10}h_n^8y_n^{(8)}\\
Y_6 =& y_n + 5.7613702206133321\times10^{-1}h_nf_n + 1.9087312382352843\times10^{-4}h_nF_2 + 2.5810575738702186\times10^{-3}h_nF_3\\
&+ 3.6001895069534565\times10^{-2}h_nF_4 + 2.1619199970095976\times10^{-1}h_nF_5 + 1.5979263483339215\times10^{-1}h_n^2y_n^{(2)}\\
&+ 2.8029909245438807\times10^{-2}h_n^3y_n^{(3)} + 3.4138373578088666\times10^{-3}h_n^4y_n^{(4)} + 2.9427596137500677\times10^{-4}h_n^5y_n^{(5)}\\
&+ 1.6920775322864040\times10^{-5}h_n^6y_n^{(6)} + 4.9033154364877436\times10^{-7}h_n^7y_n^{(7)} + 1.0424203448578282\times10^{-10}h_n^8y_n^{(8)}\\
Y_7 =& y_n + 6.4862655064719066\times10^{-1}h_nf_n + 3.6769940695116786\times10^{-5}h_nF_2 + 4.9721685285373531\times10^{-4}h_nF_3\\
&+ 6.9354318727585934\times10^{-3}h_nF_4 + 4.1647387796267914\times10^{-2}h_nF_5 + 2.2960803082799633\times10^{-1}h_nF_6\\
&+ 2.0341342435066243\times10^{-1}h_n^2y_n^{(2)} + 4.0586746183670148\times10^{-2}h_n^3y_n^{(3)} + 5.6748112511148018\times10^{-3}h_n^4y_n^{(4)}\\
&+ 5.7092740164794271\times10^{-4}h_n^5y_n^{(5)} + 3.9719428341979857\times10^{-5}h_n^6y_n^{(6)} + 1.5555777745779016\times10^{-6}h_n^7y_n^{(7)}\\
&+ 2.0081263140683207\times10^{-11}h_n^8y_n^{(8)}\\
Y_8 =& y_n + 5.5209241164721989\times10^{-1}h_nf_n + 1.5031111898254198\times10^{-2}h_nF_2 + 2.9496090303136268\times10^{-3}h_nF_3\\
&+ 4.1142636987470324\times10^{-2}h_nF_4 + 2.4706224342114499\times10^{-1}h_nF_5 + 1.4493201023321448\times10^{-2}h_nF_6\\
&+ 7.5234349069562528\times10^{-2}h_nF_7 + 1.4607397169734107\times10^{-1}h_n^2y_n^{(2)} + 2.4353265219505451\times10^{-2}h_n^3y_n^{(3)}\\
&+ 2.8196597451252271\times10^{-3}h_n^4y_n^{(4)} + 2.3435194720078191\times10^{-4}h_n^5y_n^{(5)} + 1.3864058168038072\times10^{-5}h_n^6y_n^{(6)}\\
&+ 5.5819444876399311\times10^{-7}h_n^7y_n^{(7)} + 1.4291909676730649\times10^{-8}h_n^8y_n^{(8)}\\
y_{n+1} =& y_n + 5.4145653542767558\times10^{-1}h_nf_n + 5.5914233410541501\times10^{-3}h_nF_2 + 4.8662147845363705\times10^{-2}h_nF_3\\
&+ 3.8278997260764475\times10^{-2}h_nF_4 + 2.2986603756187235\times10^{-1}h_nF_5 + 1.5395891137679231\times10^{-3}h_nF_6\\
&+ 7.9920171403241869\times10^{-3}h_nF_7 + 1.2661325230917767\times10^{-1}h_nF_8 + 1.4079205087886684\times10^{-1}h_n^2y_n^{(2)}\\
&+ 2.3150059902996928\times10^{-2}h_n^3y_n^{(3)} + 2.6587905709737395\times10^{-3}h_n^4y_n^{(4)} + 2.2109640724393226\times10^{-4}h_n^5y_n^{(5)}\\
&+ 1.3203681536550432\times10^{-5}h_n^6y_n^{(6)} + 5.2841027574064201\times10^{-7}h_n^7y_n^{(7)} + 1.1157623293823810\times10^{-8}h_n^8y_n^{(8)}
\end{aligned}
$$

6.1 Stability region of CPHBT$_{\mathbf{RK5}}(8, 8, 12)$

Since the formula of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ is similar to CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ with different values of s and d, we can use formula (4.2.7) as the stability function of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ with $s = 8$ and $d = p - 4 = 8$. In Figure (6.1), we plot the stability regions of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$, CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ and T(12) with interval of absolute stability (-7.85, 0), (-6.28, 0) and (-5.72, 0), respectively. Also, by computing the area of the stability regions of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$, CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ and T(12), we get 76, 65 and 52 units square, respectively, excluding the moons. Hence, we notice the significant gain in stability by casting RK(s,5) with T($p-4$) compared to RK(s,4) with T(d), $d = p - 2, p - 3$.

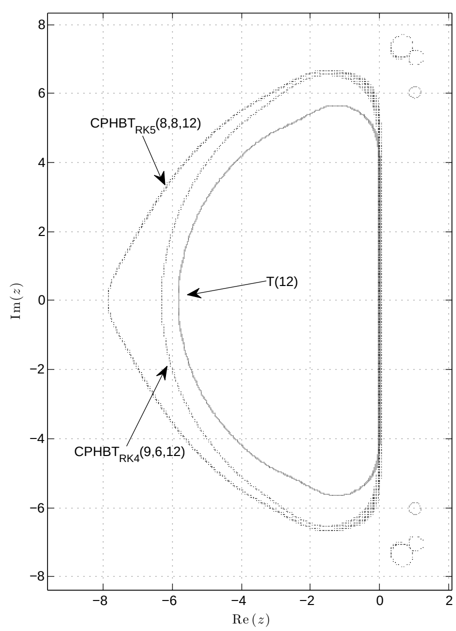


Figure 6.1: The region of absolute stability of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$, CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ and T(12) methods.

6.2 NS and NFE analysis of $\mathbf{CPHBT_{RK5}}(8,8,12)$

For consistency and comparison, we will consider the same test problems as in the numerical results of $\mathrm{CPHBT_{RK4}}$ to test the second set of methods $\mathrm{CPHBT_{RK5}}$. The problems are composed of two sets. The first set consists of Hamiltonian problems such as the 5 D-class DETEST problems, Kepler's two-body problem with ultra high eccentricity $\varepsilon = 0.99$, the HH and EqMP test problems. The second set of problems consists of the B-class, E-class DETEST problems together with Arenstorf and the Galactic dynamics problems. The user defined tolerance, TOL, is the desired accuracy for a certain problem. Some numerical integration problems may run for hundreds of hours or even a few years in some astronomical simulations. Then, we are interested in reaching a desired or acceptable accuracy, the user defined tolerance, using minimal number of integration steps (NS), function evaluations (NFE) and CPU time.

In Table 6.1, we test $\mathrm{CPHBT_{RK5}}(8,8,12)$ method on the set of Hamiltonian problems compared to T(12) and T(12)L. We show that although $\mathrm{CPHBT_{RK5}}(8,8,12)$ requires less integration steps compared to T(12) and T(12)L in all the considered test problems, it achieves remarkably higher accuracy. Also, comparing similar results for $\mathrm{CPHBT_{RK4}}(9,6,12)$ in Table 4.2, we notice that $\mathrm{CPHBT_{RK5}}(8,8,12)$ performs favourably considering the NS and MGEE simultaneously. These results confirms that the error estimation and step control formula (4.3.8) with $d = 8$ and $s = 8$ is performing well in estimating the error and optimizing the step size for $\mathrm{CPHBT_{RK5}}(8,8,12)$ to achieve the user defined tolerance using minimal NFE and hence minimal CPU time.

Moreover, since $\mathrm{CPHBT_{RK5}}(8,8,12)$ is of order 12, we extend the data given in Table 6.1 to $10^{-13} \leq$TOL$\leq 10^{-4}$ and we plot the $\log_{10}$(MGEE) as a function of the number of steps (NS) in Figure (6.2) for $\mathrm{CPHBT_{RK5}}(8,8,12)$, T(12) and T(12)L. Indeed, $\mathrm{CPHBT_{RK5}}(8,8,12)$ clearly performs better than T(12) and T(12)L even for the Kepler two-body problem with ultra high eccentricity $\varepsilon = 0.99$. To measure

Table 6.1: The maximum global energy error (MGEE) and the number of steps (NS) for $\text{CPHBT}_{\text{RK5}}(8, 8, 12)$ compared to T(12) and T(12)L for the listed problems.

Problem	**TOL**	**T(12)L** NS	**T(12)L** MGEE	**T(12)** NS	**T(12)** MGEE	$\mathbf{CPHBT_{RK5}}(8, 8, 12)$ NS	$\mathbf{CPHBT_{RK5}}(8, 8, 12)$ MGEE
D1	10^{-4}	44	7.52×10^{-4}	48	6.46×10^{-4}	33	1.96×10^{-5}
$\varepsilon = 0.1$,	10^{-7}	73	1.15×10^{-6}	82	4.45×10^{-7}	58	1.02×10^{-8}
$t_{\text{end}} = 16\pi$	10^{-10}	122	1.43×10^{-9}	142	2.93×10^{-10}	102	5.18×10^{-12}
D2	10^{-4}	67	7.12×10^{-4}	68	6.65×10^{-4}	48	1.21×10^{-4}
$\varepsilon = 0.3$,	10^{-7}	111	7.17×10^{-7}	118	4.67×10^{-7}	84	2.93×10^{-9}
$t_{\text{end}} = 16\pi$	10^{-10}	188	7.8×10^{-10}	206	2.67×10^{-10}	148	1.14×10^{-12}
D3	10^{-4}	83	1.26×10^{-3}	89	8.58×10^{-4}	64	1.12×10^{-4}
$\varepsilon = 0.5$,	10^{-7}	139	1.55×10^{-6}	154	5.47×10^{-7}	111	2.49×10^{-8}
$t_{\text{end}} = 16\pi$	10^{-10}	235	1.87×10^{-9}	271	3.39×10^{-10}	197	1.59×10^{-11}
D4	10^{-4}	115	8.03×10^{-4}	116	6.81×10^{-4}	82	3.97×10^{-4}
$\varepsilon = 0.7$,	10^{-7}	194	1.24×10^{-6}	202	1.11×10^{-6}	144	1.55×10^{-7}
$t_{\text{end}} = 16\pi$	10^{-10}	327	1.5×10^{-9}	356	6.39×10^{-10}	258	4.16×10^{-11}
D5	10^{-4}	167	2.73×10^{-3}	169	1.33×10^{-3}	122	4.34×10^{-3}
$\varepsilon = 0.9$,	10^{-7}	273	7.81×10^{-7}	300	3.11×10^{-7}	212	1.14×10^{-7}
$t_{\text{end}} = 16\pi$	10^{-10}	461	1.32×10^{-9}	528	2.56×10^{-10}	376	2.38×10^{-11}
Kepler,	10^{-4}	306	9.36×10^{-3}	306	9.46×10^{-3}	213	1.86×10^{-2}
$\varepsilon = 0.99$,	10^{-7}	488	6.48×10^{-6}	509	4.43×10^{-6}	359	1.42×10^{-6}
$t_{\text{end}} = 16\pi$	10^{-10}	824	7.36×10^{-9}	897	2.95×10^{-9}	639	4.85×10^{-10}
HH	10^{-4}	56	1.47×10^{-3}	61	1.39×10^{-3}	48	2.19×10^{-5}
$t_{\text{end}} = 70$	10^{-7}	92	8.93×10^{-7}	104	2.29×10^{-7}	78	2.80×10^{-7}
	10^{-10}	155	3.29×10^{-9}	180	2.71×10^{-10}	143	2.04×10^{-10}
EqMP	10^{-4}	146	3.08×10^{-3}	157	2.49×10^{-3}	107	2.58×10^{-2}
$t_{\text{end}} = 70$	10^{-7}	245	1.09×10^{-5}	269	8.72×10^{-7}	188	7.18×10^{-6}
	10^{-10}	411	4.93×10^{-9}	474	1.08×10^{-9}	342	1.24×10^{-9}

the percentage of the efficiency gain of our method, we consider the NS PEG and NFE PEG defined in (4.5.1) and (4.5.2), respectively, with respect to T(12) and T(12)L in Tables 6.2 and 6.3, respectively.

Theorem 6.2.1 *[29, page 102] Let the normalized d-th derivative of a function f be given by* $f^{[d]}(x) = \frac{1}{d!} f^{(d)}(x)$*, then the the required number of arithmetic operations to compute the normalized derivative* $f^{[d]}$ *up to order d is* $\mathcal{O}(d^2)$*.*

We note that although the NFE PEG_{B2} and NFE PEG_{E4} of $\text{CPHBT}_{\text{RK5}}(8,8,12)$ over T(12) are negative, $\text{CPHBT}_{\text{RK5}}(8,8,12)$ is expected to perform considerably faster and use less CPU time since it uses only 7 recursive higher order derivatives computations while T(12) and T(12)L require 11 recursive higher order derivatives computations.

Table 6.2: The NS PEG and NFE PEG of $\text{CPHBT}_{\text{RK5}}(8,8,12)$ over T(12) for the listed problems.

Problem	**NS PEG**	**NFE PEG**	**Problem**	**NS PEG**	**NFE PEG**
D1 ($\varepsilon = 0.1$)	% 80	% 44	B2	% 22	% -2
D2 ($\varepsilon = 0.3$)	% 187	% 130	B3	% 17	% -6
D3 ($\varepsilon = 0.5$)	% 67	% 34	B4	% 42	% 14
D4 ($\varepsilon = 0.7$)	% 143	% 94	B5	% 58	% 26
D5 ($\varepsilon = 0.9$)	% 142	% 94	E1	% 153	% 102
Kepler ($\varepsilon = 0.99$)	% 62	% 30	E2	% 32	% 6
HH	% 25	% 0	E3	% 36	% 9
EqMP	% 40	% 12	E4	% 11	% -11
B1	% 39	% 11	E5	% 59	% 27

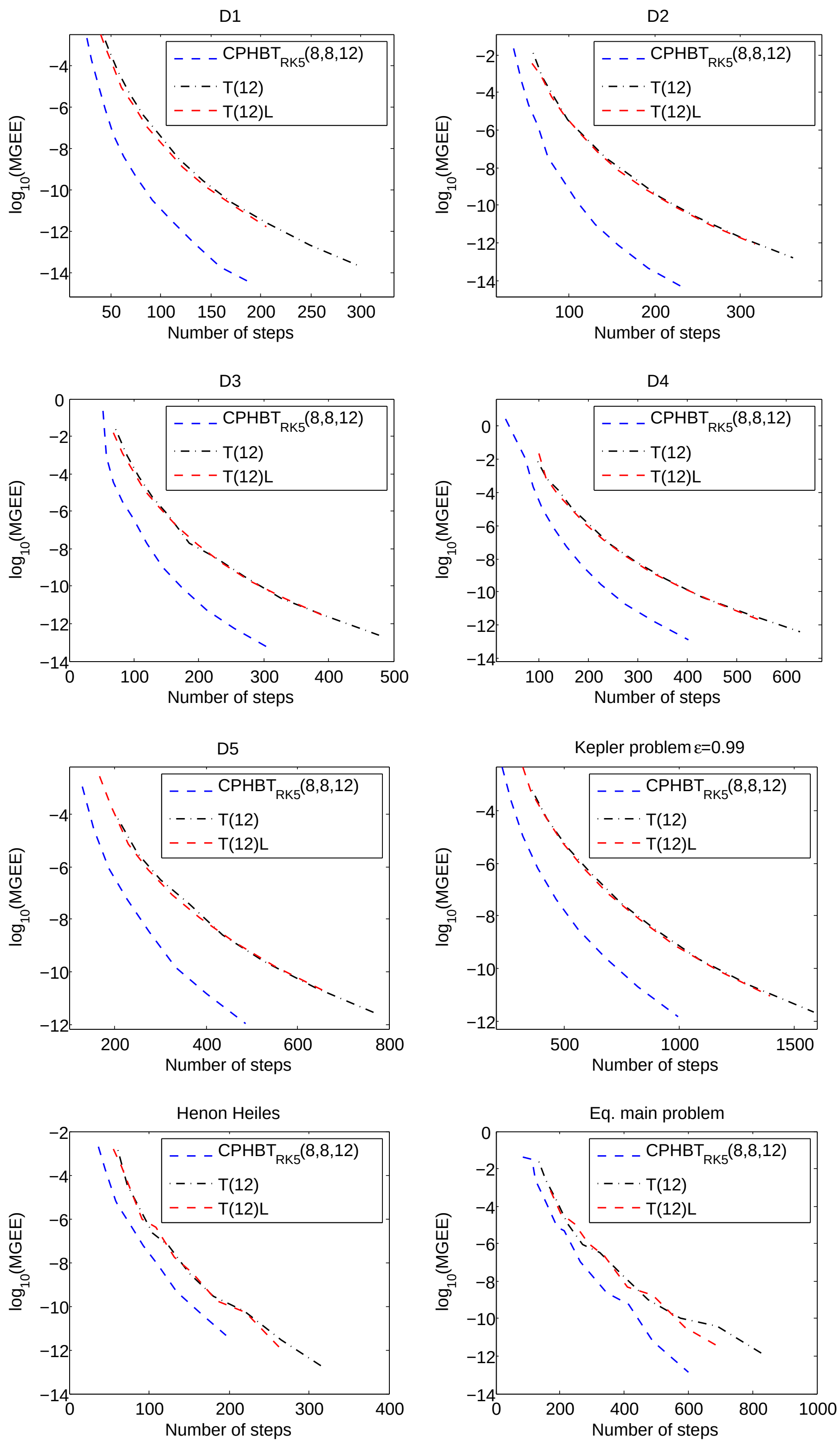


Figure 6.2: The number of steps versus $\log_{10}(\mathrm{MGEE})$ for $\mathrm{CPHBT_{RK5}}(8,8,12)$, T(12) and T(12)L for the listed problems.

Table 6.3: The NS PEG and NFE PEG of $\mathrm{CPHBT_{RK5}}(8, 8, 12)$ method over T(12)L method for the listed problems.

Problem	**NS PEG**	**NFEPEG**
D1 ($\varepsilon = 0.1$)	% 73	% 38
D2 ($\varepsilon = 0.3$)	% 88	% 50
D3 ($\varepsilon = 0.5$)	% 65	% 32
D4 ($\varepsilon = 0.7$)	% 62	% 30
D5 ($\varepsilon = 0.9$)	% 58	% 26
Kepler problem ($\varepsilon = 0.99$)	% 57	% 26
Hénon Heiles	% 27	% 2
Equatorial main problem	% 31	% 5

6.3 CPU time analysis of $\mathbf{CPHBT_{RK5}}(8, 8, 12)$

Although computer resources are improving rapidly, the mathematical models and the physical problems are getting more complicated and computationally expensive to solve. These numerical integrations are so computationally demanding that computer resources are under huge demand and pressure. Hence, one of our main goals is to reach a desired or acceptable accuracy using minimal computational resources. We measure the CPU time required to achieve the user defined tolerance for Kepler's two-body problem with eccentricities $\varepsilon = 0.1, 0.3, 0.5, 0.7, 0.9$ and 0.99, Aren, HH, Gady, EqMP, described in sections 4.4 and 4.6, together with B1, B5 and E2 of the DETEST problems compared to T(12) and DP(8,7)13M. These results are depicted in Figure (6.3) and (6.4) as we plot the $\log_{10}(\mathrm{MGE})$ as a function of the CPU time in seconds. From the figures, it is clear that $\mathrm{CPHBT_{RK5}}(8, 8, 12)$ requires less CPU time to achieve the same or even better maximum global error in all of the cases in hand. To measure the percentage of the efficiency gain of $\mathrm{CPHBT_{RK5}}(8, 8, 12)$

over T(12), DP(8,7)13M and CPHBT$_{\mathrm{RK4}}(9, 6, 12)$, we compute the CPU PEG for all the consider test problems summarized in Table 6.4. As expected from analyzing the Figures (6.3) and (6.4), CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ has large positive CPU PEG over T(12) and DP(8,7)13M. However, the CPU PEG of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ over CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ is positive for 7 problems and negative for 6 problems. This is a consequence of the direct function evaluations versus recursive computation of higher order derivatives. In CPHBT$_{\mathrm{RK5}}(8, 8, 12)$, we have 15 functions evaluations composed of 8 direct function evaluations and 7 recursive computations of higher order derivatives. While in CPHBT$_{\mathrm{RK4}}(9, 6, 12)$, we have 14 functions evaluations composed of 6 direct function evaluations and 8 recursive computations of higher order derivatives. Since the computational complexity of the higher order derivatives is quadratic [29, page 102], we expect that CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ will require more CPU time compared to CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ when applied to problems with computationally expensive functions. And for less expensive functions, CPHBT$_{\mathrm{RK4}}(9, 6, 12)$ will require less CPU time compared to CPHBT$_{\mathrm{RK5}}(8, 8, 12)$.

Finally, we summarize some of the numerical results. In Table 6.5, we list the CPU time, MGE, MGEE and NS of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ and T(12) for the D-class DETEST problems together with Kepler's two-body problem with eccentricity $\varepsilon = 0.99$. Also, in Table 6.6, we list the CPU time, MGE and NS of both methods for Arenstorf problem, Galactic dynamics problem together with B1, B5 and E2 of the DETEST problems. We note the remarkable performance of CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ compared to T(12). In all the considered problems, CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ requires less integration steps to achieve better maximum global error and maximum global energy error and still requires less CPU time although CPHBT$_{\mathrm{RK5}}(8, 8, 12)$ has 15 function evaluations per integration step while T(12) has only 12 function evaluations per integration step.

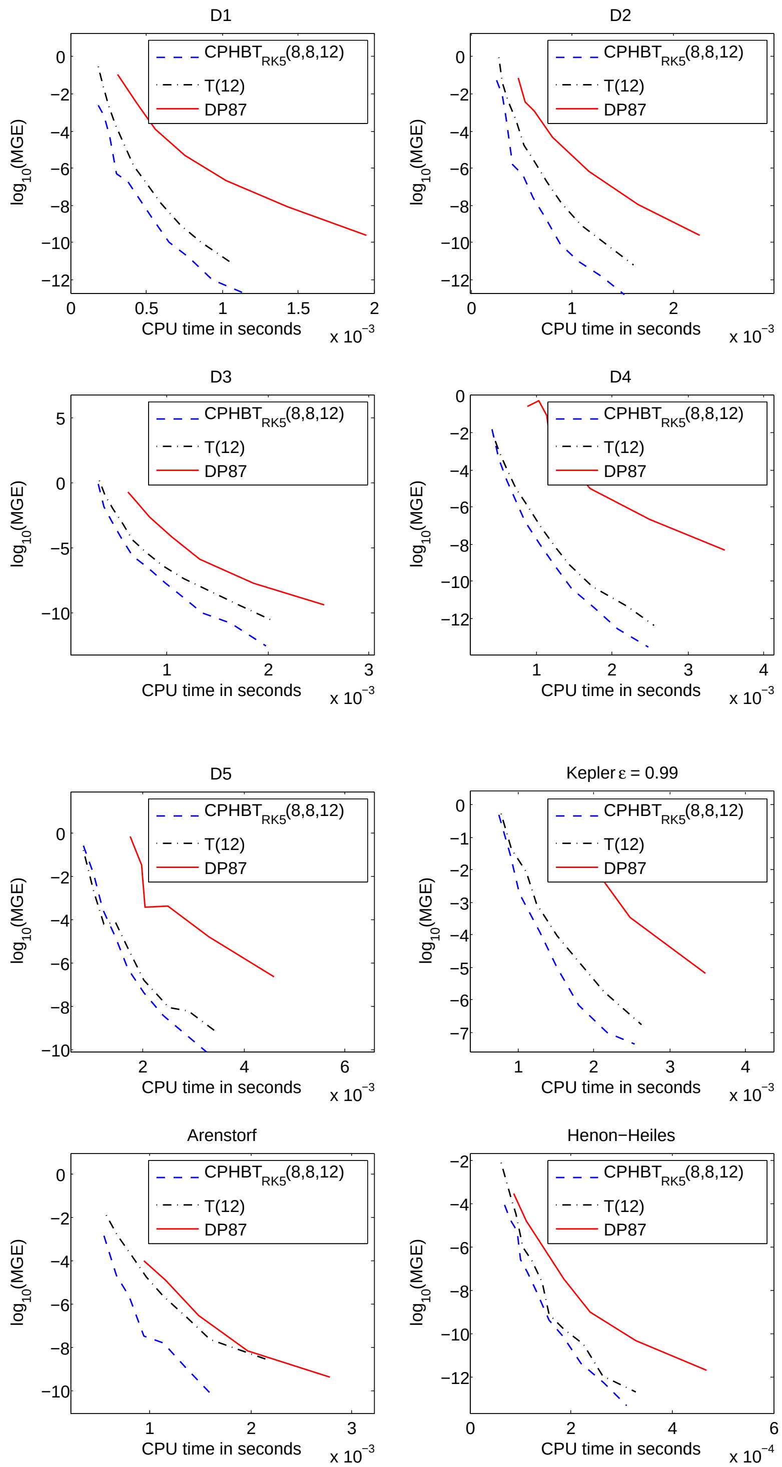


Figure 6.3: The CPU time in seconds versus $\log_{10}(\mathrm{MGE})$ for CPHBT$_{\mathrm{RK5}}(8, 8, 12)$, T(12) and DP(8,7)13M for the listed problems.

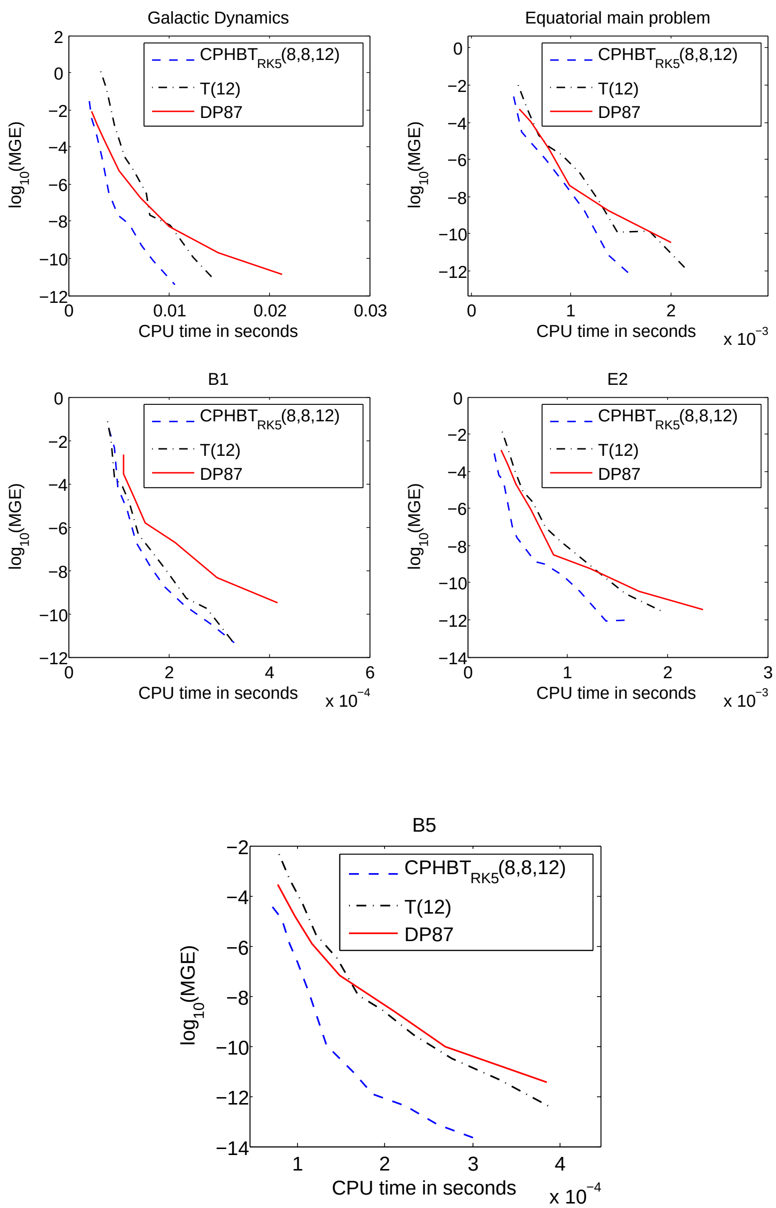


Figure 6.4: The CPU time in seconds versus $\log_{10}$(MGE) for CPHBT$_{RK5}(8, 8, 12)$, T(12) and DP(8,7)13M for the listed problems.

Table 6.4: The CPU PEG of CPHBT$_{\mathrm{RK5}}(8,8,12)$ method over T(12), DP(8,7)13M and CPHBT$_{\mathrm{RK4}}(9,6,12)$ methods for the listed problems.

Problem	**T(12)** $10^{-4} \leq \text{TOL} \leq 10^{-13}$	**DP(8,7)13M** $10^{-4} \leq \text{TOL} \leq 10^{-9}$	**CPHBT$_{\mathbf{RK4}}(9,6,12)$** $10^{-4} \leq \text{TOL} \leq 10^{-13}$
D1 ($\varepsilon = 0.1$)	%28	%206	% 0.33
D2 ($\varepsilon = 0.3$)	%40	%167	% 6.93
D3 ($\varepsilon = 0.5$)	%34	%138	% 7.48
D4 ($\varepsilon = 0.7$)	%21	%246	% 1.13
D5 ($\varepsilon = 0.9$)	%14	%175	% -5.86
Kepler ($\varepsilon = 0.99$)	%50	%189	% 17.22
Arenstorf	%69	%124	% -14.89
Hénon Heiles	%11	%61	% -8.22
Gady	%54	%83	% 17.57
EqMP	%33	%36	% 7.14
B1	%17	%70	% -10.88
E2	%49	%71	% -2.21
B5	%62	%54	% -2.18

Table 6.5: The CPU time in seconds, MGE, MGEE and NS of $\mathrm{CPHBT_{RK5}}(8,8,12)$ and T(12) for the DETEST class D problems and the Kepler problem with $\varepsilon = 0.99$.

Problem	TOL	CPU time		MGE		MGEE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
D1	-5	2.64×10^{-4}	2.70×10^{-4}	5.34×10^{-5}	2.58×10^{-3}	1.35×10^{-6}	5.85×10^{-5}	40	54
$\varepsilon = 0.1$	-6	3.17×10^{-4}	3.21×10^{-4}	4.70×10^{-7}	2.25×10^{-4}	2.37×10^{-8}	5.08×10^{-6}	48	66
	-7	3.92×10^{-4}	3.84×10^{-4}	1.84×10^{-7}	1.62×10^{-5}	1.02×10^{-8}	3.77×10^{-7}	58	79
	-8	4.76×10^{-4}	4.57×10^{-4}	2.40×10^{-8}	1.44×10^{-6}	9.31×10^{-10}	3.20×10^{-8}	70	96
	-9	5.65×10^{-4}	5.48×10^{-4}	1.65×10^{-9}	1.60×10^{-7}	6.78×10^{-11}	3.60×10^{-9}	84	115
	-10	6.63×10^{-4}	6.55×10^{-4}	1.06×10^{-10}	1.23×10^{-8}	5.18×10^{-12}	2.78×10^{-10}	102	139
	-11	7.82×10^{-4}	7.82×10^{-4}	1.70×10^{-11}	9.02×10^{-10}	7.19×10^{-13}	2.14×10^{-11}	123	168
	-12	9.24×10^{-4}	9.44×10^{-4}	1.01×10^{-12}	1.05×10^{-10}	4.64×10^{-14}	2.41×10^{-12}	149	204
	-13	1.07×10^{-3}	1.14×10^{-3}	1.97×10^{-13}	9.83×10^{-12}	2.44×10^{-15}	2.16×10^{-13}	180	247
D2	-5	3.59×10^{-4}	3.84×10^{-4}	6.38×10^{-4}	4.00×10^{-3}	8.88×10^{-6}	5.41×10^{-5}	56	79
$\varepsilon = 0.3$	-6	4.34×10^{-4}	4.60×10^{-4}	1.50×10^{-6}	3.60×10^{-4}	5.26×10^{-8}	5.58×10^{-6}	69	96
	-7	5.20×10^{-4}	5.47×10^{-4}	4.12×10^{-7}	1.76×10^{-5}	2.93×10^{-9}	3.61×10^{-7}	84	115
	-8	6.20×10^{-4}	6.52×10^{-4}	2.64×10^{-8}	2.47×10^{-6}	2.36×10^{-10}	4.01×10^{-8}	101	139
	-9	7.42×10^{-4}	7.83×10^{-4}	2.13×10^{-9}	1.95×10^{-7}	1.44×10^{-11}	3.62×10^{-9}	122	168
	-10	8.98×10^{-4}	9.45×10^{-4}	7.36×10^{-11}	1.31×10^{-8}	1.14×10^{-12}	2.64×10^{-10}	148	203
	-11	9.97×10^{-4}	1.17×10^{-3}	9.43×10^{-12}	8.40×10^{-10}	1.15×10^{-13}	2.13×10^{-11}	178	246
	-12	1.20×10^{-3}	1.40×10^{-3}	1.59×10^{-12}	8.93×10^{-11}	3.73×10^{-14}	2.03×10^{-12}	216	297
	-13	1.54×10^{-3}	1.64×10^{-3}	1.22×10^{-13}	6.41×10^{-12}	4.44×10^{-15}	1.72×10^{-13}	262	360
D3	-5	4.67×10^{-4}	4.68×10^{-4}	1.08×10^{-3}	1.08×10^{-2}	9.10×10^{-6}	7.95×10^{-5}	76	104
$\varepsilon = 0.5$	-6	5.50×10^{-4}	5.55×10^{-4}	6.65×10^{-5}	8.41×10^{-4}	5.82×10^{-7}	5.63×10^{-6}	92	126
	-7	6.82×10^{-4}	6.58×10^{-4}	2.62×10^{-6}	4.20×10^{-5}	2.49×10^{-8}	3.31×10^{-7}	111	152
	-8	8.17×10^{-4}	7.98×10^{-4}	3.30×10^{-7}	4.78×10^{-6}	2.85×10^{-9}	4.29×10^{-8}	135	183
	-9	9.55×10^{-4}	9.56×10^{-4}	2.10×10^{-8}	4.75×10^{-7}	1.96×10^{-10}	3.58×10^{-9}	162	222
	-10	1.10×10^{-3}	1.16×10^{-3}	1.67×10^{-9}	4.74×10^{-8}	1.59×10^{-11}	3.46×10^{-10}	197	268
	-11	1.32×10^{-3}	1.45×10^{-3}	1.07×10^{-10}	4.81×10^{-9}	1.12×10^{-12}	3.22×10^{-11}	237	324
	-12	1.53×10^{-3}	1.69×10^{-3}	1.74×10^{-11}	4.18×10^{-10}	1.50×10^{-13}	3.04×10^{-12}	288	392
	-13	1.77×10^{-3}	2.02×10^{-3}	3.42×10^{-13}	3.30×10^{-11}	5.77×10^{-15}	2.53×10^{-13}	348	474
D4	-5	6.15×10^{-4}	6.26×10^{-4}	9.86×10^{-3}	3.76×10^{-2}	2.65×10^{-5}	1.11×10^{-4}	100	137
$\varepsilon = 0.7$	-6	7.38×10^{-4}	7.53×10^{-4}	7.52×10^{-4}	3.54×10^{-3}	2.08×10^{-6}	1.02×10^{-5}	120	166
	-7	8.48×10^{-4}	9.09×10^{-4}	5.58×10^{-5}	3.78×10^{-4}	1.55×10^{-7}	1.15×10^{-6}	144	200
	-8	1.01×10^{-3}	1.10×10^{-3}	3.25×10^{-6}	2.11×10^{-5}	9.27×10^{-9}	7.64×10^{-8}	176	242
	-9	1.20×10^{-3}	1.33×10^{-3}	1.85×10^{-7}	1.73×10^{-6}	5.43×10^{-10}	6.79×10^{-9}	213	292
	-10	1.38×10^{-3}	1.60×10^{-3}	1.44×10^{-8}	2.00×10^{-7}	4.16×10^{-11}	7.14×10^{-10}	258	353
	-11	1.66×10^{-3}	1.93×10^{-3}	1.26×10^{-9}	1.34×10^{-8}	3.90×10^{-12}	5.43×10^{-11}	312	427
	-12	1.95×10^{-3}	2.30×10^{-3}	8.33×10^{-11}	9.49×10^{-10}	2.76×10^{-13}	4.85×10^{-12}	378	517
	-13	2.30×10^{-3}	2.74×10^{-3}	1.07×10^{-11}	5.88×10^{-11}	2.75×10^{-14}	3.97×10^{-13}	457	626
D5	-5	8.37×10^{-4}	9.16×10^{-4}	2.62×10^{-1}	1.63×10^{-1}	8.03×10^{-5}	6.09×10^{-5}	144	204
$\varepsilon = 0.9$	-6	9.71×10^{-4}	1.12×10^{-3}	1.26×10^{-2}	2.26×10^{-3}	3.95×10^{-6}	2.81×10^{-6}	176	246
	-7	1.09×10^{-3}	1.39×10^{-3}	3.15×10^{-4}	6.22×10^{-5}	1.14×10^{-7}	2.97×10^{-7}	212	298
	-8	1.37×10^{-3}	1.66×10^{-3}	1.70×10^{-5}	8.26×10^{-5}	6.53×10^{-9}	2.74×10^{-8}	257	360
	-9	1.63×10^{-3}	2.16×10^{-3}	5.39×10^{-7}	3.86×10^{-6}	2.55×10^{-10}	2.56×10^{-9}	312	435
	-10	1.89×10^{-3}	2.59×10^{-3}	4.63×10^{-8}	1.65×10^{-7}	2.38×10^{-11}	3.12×10^{-10}	376	526
	-11	2.30×10^{-3}	2.95×10^{-3}	4.05×10^{-9}	9.55×10^{-9}	1.79×10^{-12}	3.47×10^{-11}	457	636
	-12	2.73×10^{-3}	3.51×10^{-3}	7.90×10^{-11}	6.43×10^{-9}	1.23×10^{-13}	2.29×10^{-12}	553	769
	-13	3.19×10^{-3}	4.32×10^{-3}	3.50×10^{-10}	7.50×10^{-10}	8.33×10^{-14}	2.41×10^{-13}	670	931
Kepler	-5	6.29×10^{-4}	1.48×10^{-3}	1.19×10	1.42×10	2.08×10^{-4}	5.58×10^{-4}	109	358
$\varepsilon = 0.99$	-6	7.50×10^{-4}	1.75×10^{-3}	4.90×10^{-1}	8.65	8.76×10^{-6}	4.87×10^{-5}	131	423

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Table 6.5 – *Continued from previous page*

Problem	T	CPU time		MGE		MGEE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
	-7	9.09×10^{-4}	2.03×10^{-3}	2.95×10^{-2}	9.49×10^{-1}	5.09×10^{-7}	4.57×10^{-6}	158	507
	-8	1.04×10^{-3}	2.44×10^{-3}	1.97×10^{-3}	6.70×10^{-2}	3.48×10^{-8}	3.98×10^{-7}	191	612
	-9	1.21×10^{-3}	2.94×10^{-3}	1.24×10^{-4}	3.79×10^{-3}	2.44×10^{-9}	3.24×10^{-8}	231	739
	-10	1.47×10^{-3}	3.49×10^{-3}	9.57×10^{-6}	3.13×10^{-4}	1.92×10^{-10}	3.14×10^{-9}	279	894
	-11	1.72×10^{-3}	4.41×10^{-3}	6.69×10^{-7}	1.38×10^{-5}	1.51×10^{-11}	2.68×10^{-10}	337	1081
	-12	1.96×10^{-3}	4.99×10^{-3}	9.93×10^{-8}	2.00×10^{-6}	1.53×10^{-12}	2.27×10^{-11}	407	1308
	-13	2.34×10^{-3}	6.01×10^{-3}	4.46×10^{-8}	1.87×10^{-7}	2.56×10^{-13}	2.28×10^{-12}	492	1583

Table 6.6: The CPU time in seconds, MGE and NS of $\mathrm{CPHBT_{RK5}}(8,8,12)$ and T(12) for the Arenstorf problem, B1, B5, E2 and the Galactic dynamics problems.

Problem	TOL	CPU time		MGE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
Aren	1×10^{-3}	2.40×10^{-4}	3.12×10^{-4}	1.93	1.59	26	40
$t\in[0,20]$	1×10^{-4}	3.12×10^{-4}	4.04×10^{-4}	2.65×10^{-1}	8.46×10^{-1}	36	50
	1×10^{-5}	3.91×10^{-4}	4.80×10^{-4}	2.01×10^{-1}	1.46×10^{-1}	45	63
	1×10^{-6}	4.53×10^{-4}	5.71×10^{-4}	8.57×10^{-4}	1.35×10^{-2}	55	76
	1×10^{-7}	5.27×10^{-4}	7.00×10^{-4}	1.44×10^{-3}	1.55×10^{-3}	65	92
	1×10^{-8}	6.41×10^{-4}	8.27×10^{-4}	2.18×10^{-5}	1.88×10^{-4}	80	111
	1×10^{-9}	7.72×10^{-4}	1.01×10^{-3}	2.53×10^{-6}	2.03×10^{-5}	97	135
	1×10^{-10}	9.26×10^{-4}	1.21×10^{-3}	3.49×10^{-8}	2.34×10^{-6}	117	163
	1×10^{-11}	1.16×10^{-3}	1.45×10^{-3}	1.75×10^{-8}	2.45×10^{-7}	141	197
	1×10^{-12}	1.39×10^{-3}	1.75×10^{-3}	1.12×10^{-9}	2.52×10^{-8}	172	238
	1×10^{-13}	1.59×10^{-3}	2.13×10^{-3}	7.56×10^{-11}	2.79×10^{-9}	208	288
B5	1×10^{-3}	7.25×10^{-5}	8.59×10^{-5}	3.53×10^{-5}	4.79×10^{-3}	14	18
$t\in[0,20]$	1×10^{-4}	8.83×10^{-5}	9.73×10^{-5}	1.25×10^{-5}	4.41×10^{-4}	17	22
	1×10^{-5}	9.68×10^{-5}	1.12×10^{-4}	1.15×10^{-6}	4.75×10^{-5}	21	26
	1×10^{-6}	1.12×10^{-4}	1.30×10^{-4}	2.25×10^{-7}	2.80×10^{-6}	24	31
	1×10^{-7}	1.22×10^{-4}	1.48×10^{-4}	1.19×10^{-8}	2.99×10^{-7}	28	38
	1×10^{-8}	1.41×10^{-4}	1.76×10^{-4}	1.01×10^{-10}	1.08×10^{-8}	35	46
	1×10^{-9}	1.64×10^{-4}	2.07×10^{-4}	9.11×10^{-12}	2.84×10^{-9}	43	55
	1×10^{-10}	1.86×10^{-4}	2.44×10^{-4}	1.23×10^{-12}	2.69×10^{-10}	51	67
	1×10^{-11}	2.23×10^{-4}	2.85×10^{-4}	4.19×10^{-13}	3.17×10^{-11}	61	80
	1×10^{-12}	2.63×10^{-4}	3.41×10^{-4}	7.48×10^{-14}	4.83×10^{-12}	75	97
	1×10^{-13}	3.10×10^{-4}	4.06×10^{-4}	2.29×10^{-14}	3.69×10^{-13}	91	118
B1	1×10^{-3}	7.87×10^{-5}	7.99×10^{-5}	6.98×10^{-4}	7.40×10^{-2}	26	33
$t\in[0,20]$	1×10^{-4}	8.52×10^{-5}	8.63×10^{-5}	3.48×10^{-2}	1.09×10^{-2}	28	39

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Table 6.6 – *Continued from previous page*

Problem	TOL	CPU time		MGE		NS	
		CPHBT	T(12)	CPHBT	T(12)	CPHBT	T(12)
	1×10^{-5}	1.01×10^{-4}	1.02×10^{-4}	4.40×10^{-3}	1.86×10^{-4}	35	46
	1×10^{-6}	1.05×10^{-4}	1.16×10^{-4}	5.57×10^{-4}	7.12×10^{-5}	40	56
	1×10^{-7}	1.27×10^{-4}	1.36×10^{-4}	1.88×10^{-7}	1.32×10^{-5}	53	67
	1×10^{-8}	1.43×10^{-4}	1.59×10^{-4}	9.24×10^{-8}	4.55×10^{-7}	62	80
	1×10^{-9}	1.90×10^{-4}	1.86×10^{-4}	2.32×10^{-7}	8.37×10^{-8}	79	98
	1×10^{-10}	2.00×10^{-4}	2.19×10^{-4}	1.14×10^{-10}	1.09×10^{-8}	93	118
	1×10^{-11}	2.41×10^{-4}	2.58×10^{-4}	8.59×10^{-10}	5.25×10^{-10}	114	142
	1×10^{-12}	2.90×10^{-4}	3.07×10^{-4}	6.76×10^{-12}	1.73×10^{-10}	142	172
	1×10^{-13}	3.48×10^{-4}	3.73×10^{-4}	5.10×10^{-12}	3.58×10^{-12}	171	208
E2	1×10^{-3}	1.05×10^{-4}	1.11×10^{-4}	2.08×10^{-4}	2.23×10^{-3}	35	40
$t \in [0, 20]$	1×10^{-4}	1.16×10^{-4}	1.33×10^{-4}	6.07×10^{-5}	4.88×10^{-4}	40	48
	1×10^{-5}	1.30×10^{-4}	1.55×10^{-4}	7.26×10^{-6}	5.71×10^{-5}	47	58
	1×10^{-6}	1.47×10^{-4}	1.79×10^{-4}	6.55×10^{-8}	2.01×10^{-6}	59	69
	1×10^{-7}	1.70×10^{-4}	2.08×10^{-4}	2.57×10^{-8}	4.20×10^{-7}	66	83
	1×10^{-8}	2.11×10^{-4}	2.45×10^{-4}	3.78×10^{-10}	3.88×10^{-8}	88	101
	1×10^{-9}	2.41×10^{-4}	2.96×10^{-4}	4.58×10^{-10}	7.02×10^{-9}	102	121
	1×10^{-10}	2.99×10^{-4}	3.44×10^{-4}	7.28×10^{-11}	1.04×10^{-9}	126	146
	1×10^{-11}	3.53×10^{-4}	4.15×10^{-4}	8.20×10^{-12}	1.14×10^{-10}	150	177
	1×10^{-12}	4.28×10^{-4}	4.97×10^{-4}	2.39×10^{-13}	8.70×10^{-12}	185	214
	1×10^{-13}	5.00×10^{-4}	6.05×10^{-4}	9.90×10^{-14}	1.21×10^{-12}	226	258
GAD	1×10^{-3}	1.84×10^{-3}	2.36×10^{-3}	6.83×10^{-1}	4.50	268	367
$t \in [0, 500]$	1×10^{-4}	2.01×10^{-3}	2.77×10^{-3}	3.11×10^{-2}	1.12	325	445
	1×10^{-5}	2.37×10^{-3}	3.25×10^{-3}	3.53×10^{-3}	1.48×10^{-1}	397	536
	1×10^{-6}	2.79×10^{-3}	3.87×10^{-3}	5.71×10^{-4}	1.41×10^{-3}	476	646
	1×10^{-7}	3.38×10^{-3}	4.70×10^{-3}	3.06×10^{-5}	2.77×10^{-5}	573	781
	1×10^{-8}	4.08×10^{-3}	5.59×10^{-3}	3.65×10^{-7}	5.55×10^{-6}	697	941
	1×10^{-9}	4.61×10^{-3}	6.64×10^{-3}	2.19×10^{-8}	3.26×10^{-7}	841	1138
	1×10^{-10}	5.59×10^{-3}	8.19×10^{-3}	6.56×10^{-9}	2.26×10^{-8}	1023	1375
	1×10^{-11}	6.51×10^{-3}	9.75×10^{-3}	4.99×10^{-10}	6.41×10^{-9}	1233	1662
	1×10^{-12}	7.82×10^{-3}	1.16×10^{-2}	7.87×10^{-11}	1.16×10^{-10}	1490	2011
	1×10^{-13}	9.65×10^{-3}	1.41×10^{-2}	3.60×10^{-12}	9.10×10^{-12}	1807	2434

6.4 The propagation of error in a long-term integration problem for $\mathbf{CPHBT_{RK5}}(8,8,12)$

In the previous sections, we showed that $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ performs very well in terms of NS, NFE, MGE, MGEE and CPU time required to solve several problems often used to test higher-order ODE solvers. However, one of the main interests of this work is the contractivity preserving property and the propagation of discretization errors in long-term integration problems. Since in some astronomical simulation problems, experts need to run the simulations for very large intervals, it is important to use a numerical method that prevents or minimizes the propagation of discretization errors to achieve satisfactory results. In the following subsections, we show that $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ succeeds in achieving these properties.

6.4.1 Fixed step-size configuration

To be consistent with the previous chapters, we compare $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ in a fixed step-size configurations to the Adams-Bashforth-Moulton method, ABM(12,11), of order 12 with predictor of order 11 and corrector of order 12 in PECE mode and $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$ in fixed step-size configurations. The fixed step sizes $h_{\mathrm{CPHBT}_{\mathrm{RK5}}}$, $h_{\mathrm{CPHBT}_{\mathrm{RK4}}}$ and h_{ABM} are chosen so that all three methods use approximately equal CPU time when applied to Kepler's two-body problem with eccentricities $\varepsilon = 0.3, 0.5$ and 0.7 over the integrations interval $t \in [0, 20000\pi]$. We plot the energy error ($\Delta E(t) = \frac{E(t)-E(0)}{E(0)}$) as a function of time in log-log scale using the MATLAB function `loglog`. We plot 300 equidistant points spanning the integration period and we use the MATLAB filter command to remove some chattering. We plot the results in Figure (6.5). We show that the fixed step size $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ performs remarkably better than the fixed step size ABM(12,11) and $\mathrm{CPHBT}_{\mathrm{RK4}}(9,6,12)$. On average, $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ is 40, 40 and 128 times more accurate that ABM(12,11) for eccentricities $\varepsilon = 0.3, 0.5$ and 0.7, respectively. Also, CPHBT$_{\mathrm{RK5}}(8,8,12)$ is 10, 10 and 26 times more accurate than CPHBT$_{\mathrm{RK4}}(9,6,12)$ for eccentricities $\varepsilon = 0.3, 0.5$ and 0.7, respectively. For instance, in Figure (6.6), we plot the EE versus time of CPHBT$_{\mathrm{RK5}}(8,8,12)$, CPHBT$_{\mathrm{RK4}}(9,6,12)$ and ABM(12,11) for Kepler's problem with $\varepsilon = 0.7$.

Experts in astronomical computations tend to have problems with integration intervals up to 1 billion years long [50]. Hence, it is important to estimate the maximum long-integration interval possible for a method to maintain a desired accuracy. We fit the collected data to the module αt^{β} and we analyze the values of the exponent β. We notice that the values of β for CPHBT$_{\mathrm{RK4}}(9,6,12)$ and CPHBT$_{\mathrm{RK5}}(8,8,12)$

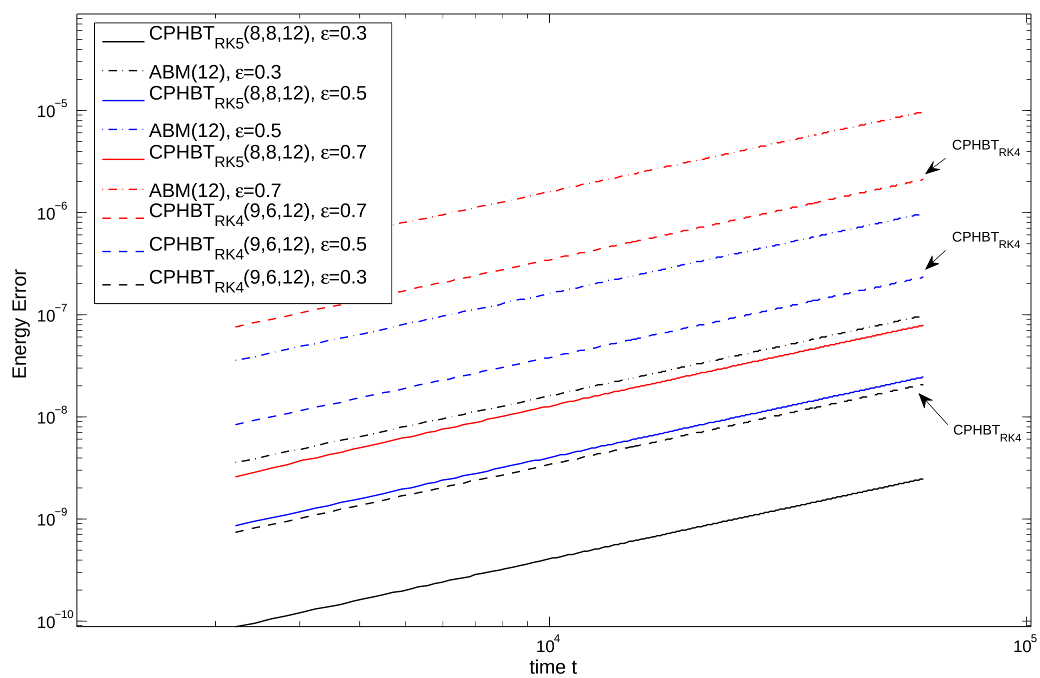


Figure 6.5: The energy error (EE) of ABM(12), CPHBT$_{\mathrm{RK5}}(8,8,12)$ and CPHBT$_{\mathrm{RK4}}(9,6,12)$ for Kepler's two-body problem with eccentricity $\varepsilon = 0.3, 0.5, 0.7$, $t \in [0, 20000\pi]$ and fixed step size configuration.

methods are slightly higher than ABM(12,11) method. The considered interval of

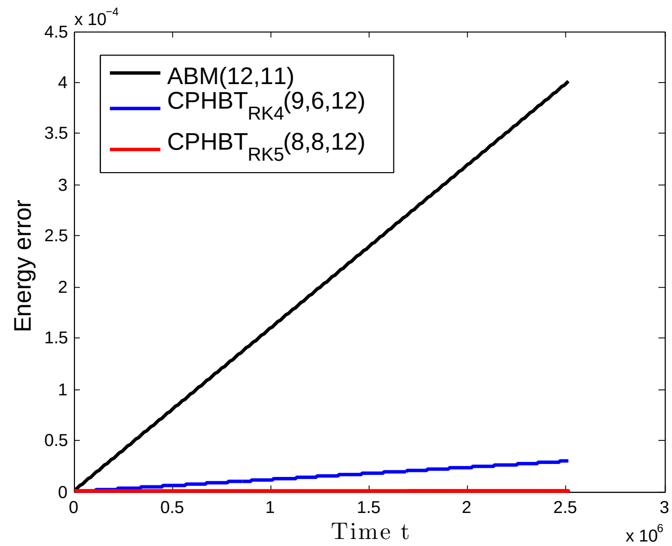


Figure 6.6: The energy error (EE) as a function of time for Kepler's two-body problem with eccentricity $\varepsilon = 0.7$, $t \in [0, 800000\pi]$ and fixed step size configuration.

integration in this simulation is relatively small to capture the asymptotic behaviour by the power law αt^{β}. We extend the integration interval by solving Kepler's two-body problem over the interval $t \in [0, t_{\text{end}}]$ for $t_{\text{end}} = 20000\pi, 40000\pi, \ldots, 800000\pi$. We fit the collected data (t, EE) to the power law αt^{β} and we plot the values of β in Figure (6.7) as a function of t_{end} for CPHBT$_{\text{RK4}}(9,6,12)$, CPHBT$_{\text{RK5}}(8,8,12)$ and ABM(12,11) methods. We notice that the value of β decreases for CPHBT$_{\text{RK4}}(9,6,12)$ and CPHBT$_{\text{RK5}}(8,8,12)$ methods with t_{end} while it increases for ABM(12,11). Indeed, a better estimation of the value of β is achieved as t_{end} increases. This result agrees with the expected behaviour of the designed CPHBT methods in terms of the propagation of discretization errors when long-term integration problems are considered.

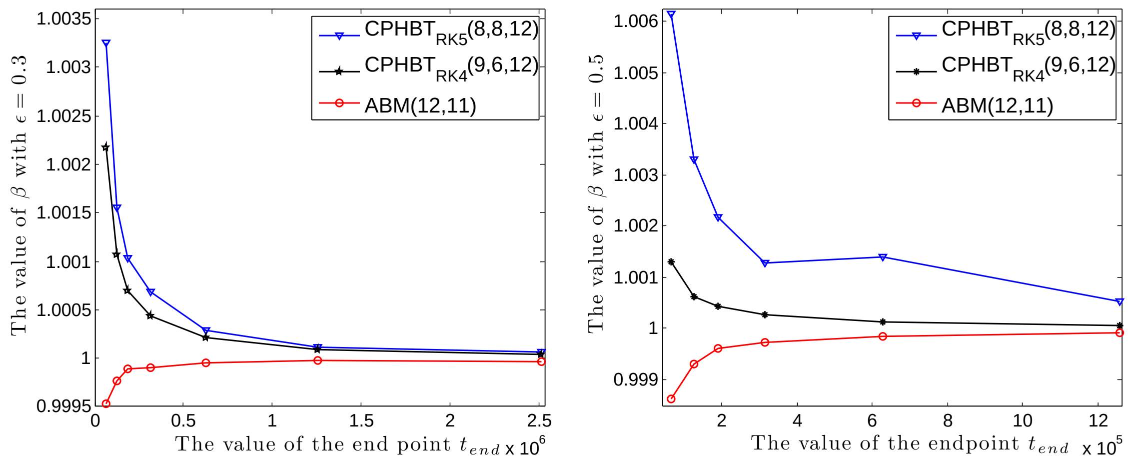
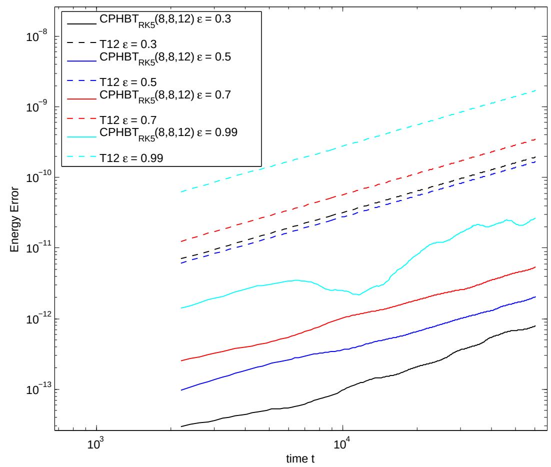


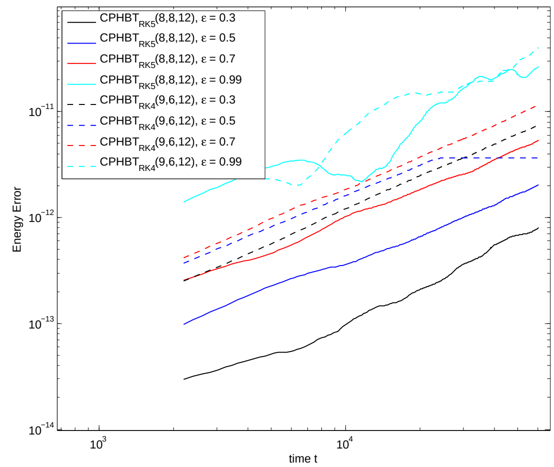
Figure 6.7: The values of β as a function of the endpoint t_{end}.

6.4.2 Variable step-size configuration

Since the error estimation and step control formula are a crucial component of the designed $\text{CPHBT}_{\text{RK5}}(8,8,12)$ and its performance, we will consider solving a long-term integration problem with a VS configuration compared to VS T(12) and VS $\text{CPHBT}_{\text{RK4}}(9,6,12)$. As mentioned previously, we managed to choose the control factor η in the step control formula (4.3.8) so that the three considered methods use approximately equal CPU time. For the variable step size case, we consider Kepler's two-body problem with higher eccentricities $\varepsilon = 0.3, 0.5, 0.7$ and 0.99 over the integration interval $t \in [0, 20000\pi]$. In Figure (6.8a), we plot the energy error as a function of time using the MATLAB function `loglog`. We notice the remarkable effect of the contractivity preserving property of $\text{CPHBT}_{\text{RK5}}(8,8,12)$ in suppressing the propagation of discretization errors in the long-term integration problem even for ultra high eccentricity $\varepsilon = 0.99$ which is consider a challenging test problem. By computing the ratio of the energy error at the endpoint, $\text{CPHBT}_{\text{RK5}}(8,8,12)$ is 273 times more accurate than T(12) for $\varepsilon = 0.3$. Also, it is 83, 65 and 70 times more accurate than T(12) for eccentricities $\varepsilon = 0.5, 0.7$ and 0.99, respectively. Moreover, in Figure (6.8b), we plot similar results comparing $\text{CPHBT}_{\text{RK5}}(8,8,12)$ to $\text{CPHBT}_{\text{RK4}}(6,6,12)$ designed



(a) VS T(12)



(b) VS CPHBT$_{\mathrm{RK4}}(9,6,12)$

Figure 6.8: The energy error (EE) for Kepler's two-body problem with eccentricity $\varepsilon = 0.3, 0.5, 0.7, 0.99$ and $t \in [0, 20000\pi]$.

in chapter 3. We show that for all the considered eccentricities, CPHBT$_{\mathrm{RK5}}(8,8,12)$ clearly performs better than CPHBT$_{\mathrm{RK4}}(6,6,12)$. However, for the ultra high eccentricity $\varepsilon = 0.99$, the two methods have closer accuracy, although CPHBT$_{\mathrm{RK5}}(8,8,12)$ is still more accurate. This is the case since the estimated high order derivative in the step control formula (4.3.8) is less accurate for ultra high eccentricities. This explains the oscillations of the curves in Figure (6.8). Note that the T(12) curves are smoother since, for T(12) method, we compute the required high order derivative recursively and hence the step control formula does not rely on an estimated high order derivative. These results are very encouraging to investigate casting Runge-Kutta methods of order 6, RK$(s,6)$, with Taylor methods of order $d-5$, T$(d-5)$, as a future project.

6.4.3 CPHBT$_{\mathbf{RK5}}(8, 8, 12)$ compared to Runge-Kutta-Nyström methods of order 12

In this section, we will consider the special second order system of non-stiff ordinary differential equations of the form

$$y''(x) = f(x, y(x)), \qquad \text{where } y(x_0), y'(x_0) \text{ are given.} \tag{6.4.1}$$

This second order ODE system can be transformed to a first order ODE without loss of generality. However, many authors [13, 15, 12] have shown that it is more efficient to use explicit s-stage Runge-Kutta-Nyström pair, denoted by RKN$(p, q)s$M, given by the two Runge-Kutta methods of orders q and p where $p > q$ and often $p = q + 1$ or $p = q + 2$. The RKN$(p, q)s$M methods are described by the following formula

$$\hat{y}_{n+1} = \hat{y}_n + h_n \hat{y}'_n + h_n^2 \sum_{i=1}^{s} \hat{b}_i \mathbf{g}_i, \qquad \hat{y}'_{n+1} = \hat{y}'_n + h_n \sum_{i=1}^{s} \hat{b}'_i \mathbf{g}_i, \tag{6.4.2}$$

$$y_{n+1} = \hat{y}_n + h_n \hat{y}'_n + h_n^2 \sum_{i=1}^{s} b_i \mathbf{g}_i, \qquad y'_{n+1} = \hat{y}'_n + h_n \sum_{i=1}^{s} b'_i \mathbf{g}_i, \tag{6.4.3}$$

$$\mathbf{g}_i = f\big(x_n + c_i h_n, \hat{y}_n + c_i h_n \hat{y}'_n + h_n^2 \sum_{j=1}^{i-1} a_{ij} \mathbf{g}_j\big), \quad i = 1, 2, \ldots, s, \tag{6.4.4}$$

where $\hat{y}_0 = y(x_0)$ and $\hat{y}'_0 = y'(x_0)$. These methods are integrated with a step control formula given by

$$h_{n+1} = \text{stfac } h_n \left(\frac{\text{TOL}}{\|\sigma_{n+1}\|_\infty} \right)^{\frac{1}{1+p}}, \tag{6.4.5}$$

where $\sigma_{n+1} = \max\{|y_{n+1} - \hat{y}_{n+1}|, |y'_{n+1} - \hat{y}'_{n+1}|\}$, TOL is the user defined tolerance and stfac $= 0.9$ is a step control factor. These methods are widely used in astronomical computations when the derivatives are computationally "cheap" to evaluate [49]. Several authors developed different optimal Runge-Kutta-Nyström methods. In 2013, Sharp *et al.* [49] presented a new optimal explicit 17 stages Runge-Kutta-Nyström pair of order 12, RKN(12,10)17M, with applications in the astronomical simulations field. In our numerical comparison of different Runge-Kutta-Nyström methods, Sharp's RKN(12,10)17M pair performed favourably. Hence, we will compare $\text{CPHBT}_{\text{RK5}}(8,8,12)$ with RKN(12,10)17M in terms of the long integration of Kepler's two-body problem over the integration interval $t \in [0, t_{\text{end}}]$ with t_{end} up to 800,000 π. In Figures (6.9a)-(6.9d), we plot the energy error of $\text{CPHBT}_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M as a function of time for different eccentricities using the MATLAB function `loglog`.

For eccentricity $\varepsilon = 0.1$, given in Figure (6.9a), the control factor η in the step control formula (4.3.8) was chosen so that RKN(12,10)17M and $\text{CPHBT}_{\text{RK5}}(8,8,12)$ use approximately equal CPU time. With an integration interval of 200,000 periods, we show that $\text{CPHBT}_{\text{RK5}}(8,8,12)$ is more accurate than RKN(12,10)17M and it is visible that the gap between the energy error of the two methods increases with time. To study the error growth, we fit the collected data (EE, t) to the power model αt^{β} to get $\beta = 0.711$ and 0.981 for $\text{CPHBT}_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M, respectively, with error ratio $\frac{EE_{\text{RKN}}}{EE_{\text{CPHBT}}} = 4.41$ at $t = t_{\text{end}}$. However, for eccentricities $\varepsilon = 0.3, 0.5, 0.7$, it becomes more challenging to control the CPU time required by RKN(12,10)17M due to the high number of rejected steps that can not be controlled. Hence, for that range of eccentricities, our study will focus on the error growth of $\text{CPHBT}_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M only and we will ignore the accuracy difference between the two methods since the CPU time required by the two methods is not normalized.

The problem with eccentricity $\varepsilon = 0.3$ and $t_{\text{end}} = 35000\pi$ is depicted in Figure (6.9b). $\text{CPHBT}_{\text{RK5}}(8,8,12)$ is less accurate than RKN(12,10)17M for the first half of the considered integration interval. However, we are interested only in the propagation of the discretization errors in this configuration and since the $\text{CPHBT}_{\text{RK5}}(8,8,12)$ is contractivity preserving, it minimizes the error growth over the integration interval and after approximately 4859 periods, $\text{CPHBT}_{\text{RK5}}(8,8,12)$ surpasses RKN(12,10)17M and becomes the more accurate method. For this problem, $\beta = 0.689$ and 1.418 for

CPHBT$_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M, respectively. Also, the error ratio $\frac{EE_{\text{RKN}}}{EE_{\text{CPHBT}}}$ is equal to 1.4 and in general, this ratio will increase with t_{end} due to the contractivity preserving property of CPHBT$_{\text{RK5}}(8,8,12)$.

The problem with eccentricity $\varepsilon = 0.5$ and $t_{\text{end}} = 700{,}000\,\pi$ is depicted in Figure (6.9c). CPHBT$_{\text{RK5}}(8,8,12)$ method prevents the growth of the error considerably and the method is more accurate than RKN(12,10)17M after 232,538 periods with error ratio at the end of the integration interval equals 1.54. For this problem, we have $\beta = 0.0436$ and 1.418 for CPHBT$_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M, respectively.

Finally, Kepler's problem with eccentricity $\varepsilon = 0.7$ and t_{end}=500,000 π is depicted in Figure (6.9d). It is shown that CPHBT$_{\text{RK5}}(8,8,12)$ surpasses RKN(12,10)17M method after 148,044 periods with error ratio equals 1.523 at $t = t_{\text{end}}$. The growth of error of CPHBT$_{\text{RK5}}(8,8,12)$ is very low in this configuration where $\beta = 0.004$ and 0.859 for CPHBT$_{\text{RK5}}(8,8,12)$ and RKN(12,10)17M, respectively. We managed to obtain similar results for CPHBT$_{\text{RK4}}(9,6,12)$ compared to RKN(12,10)17M which we do not include here. Note that this number of periods is considered to be small in applications where experts use integration intervals far longer than the ones considered in this section. For instance, Sharp considers in [50] the Jovian problem which describes the Newtonian gravitational forces interaction of the Sun, Jupiter, Saturn, Uranus and Neptune over an integration period of 10 million years, where the time unit is taken to be one day. This shows that the contractivity preserving property and the suppression of error growth achieved by the designed CPHBT methods in long term integration problems can be very useful in solving realistic physical and astronomical problems.

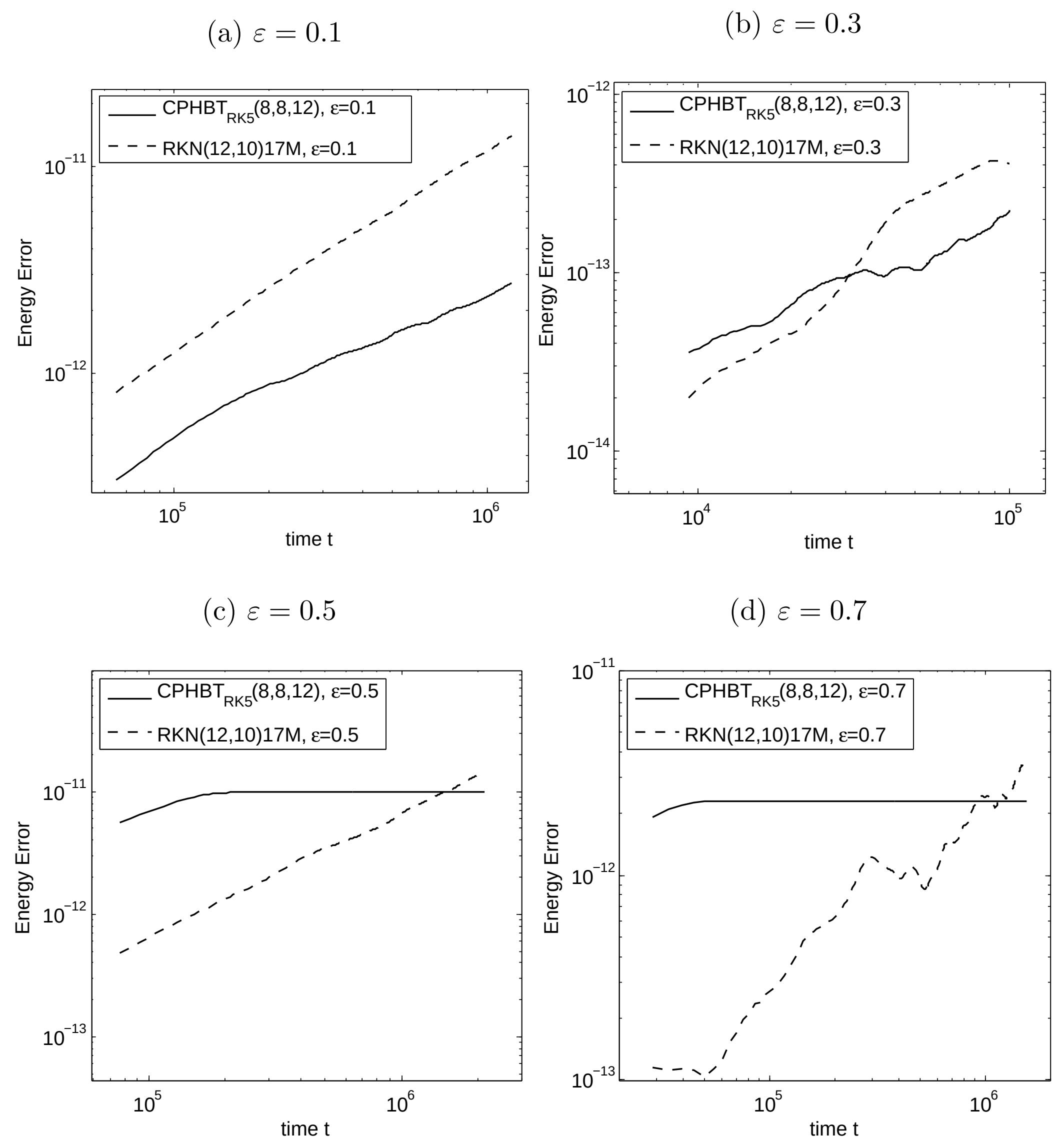


Figure 6.9: The energy error of Kepler's two-body problem as a function of time of $\mathrm{CPHBT}_{\mathrm{RK5}}(8,8,12)$ compared to Runge-Kutta-Nyström for different eccentricities.

Chapter 7

Conclusion and future work

In this thesis, we considered a new family of one-step, explicit, multi-derivative, multi-stage, contractivity preserving, Hermite-Birkhoff-Taylor methods. The first set of methods, $\mathrm{CPHBT}_{\mathrm{RK4}}(d, s, p)$, derived by casting s-stage Runge-Kutta methods of order 4, RK(s,4), with Taylor method of order d, T(d), proved to be very efficient compared to T(p), T(p)L by Martin Lara, DP(8,7)13M and ABM methods. In general, the $\mathrm{CPHBT}_{\mathrm{RK4}}$ is favourable in terms of the following comparison aspects: stability, accuracy, efficiency and the propagation of discretization errors in long-term integration problems. We obtained the second set of methods, $\mathrm{CPHBT}_{\mathrm{RK5}}(p-4, s, p)$, by casting s-stage Runge-Kutta methods of order 5, RK(s,5), with the Taylor method of order $p-4$, T($p-4$). These methods are more challenging to obtain due to the higher number of order conditions and contractivity preserving restrictions in the optimization problem. $\mathrm{CPHBT}_{\mathrm{RK5}}$ methods have larger regions of absolute stability and perform better than $\mathrm{CPHBT}_{\mathrm{RK4}}$ in terms of the NS, NFE percentage efficiency gain. Also, on average, $\mathrm{CPHBT}_{\mathrm{RK4}}$ and $\mathrm{CPHBT}_{\mathrm{RK5}}$ require the same CPU time.

Since one of the main interests in this thesis is the contractivity preserving property and the propagation of errors, we compare $\mathrm{CPHBT}_{\mathrm{RK5}}$ to a well-known Runge-Kutta-Nyström pair, RKN, presented by Philip Sharp in 2013 [49]. $\mathrm{CPHBT}_{\mathrm{RK5}}$ utilizes the contractivity preserving property and performs better than the considered RKN pair in terms of the propagation of discretization errors in the long-term integration of Kepler's two-body problem where we consider integration intervals up to t =800,000 π. This result can be very interesting to experts in the field of astronomical computations since they consider extremely large integration intervals, up to $t = 3.65242 \times 10^{11}$ [50], and so the propagation of discretization errors is of great importance. Since the main problem in the construction of the CPHBT methods is the contractivity preserving property and the main problem in the construction of the RKN methods is achieving maximum accuracy, the considered RKN method had higher accuracy than CPHBT. However, in the considered long-term integration problems, CPHBT surpasses RKN method in terms of accuracy after a relatively small period of time.

The promising results in this work will be followed up by the following ideas that will be addressed in our future work:

- The advantages of $\mathrm{CPHBT_{RK5}}$ over $\mathrm{CPHBT_{RK4}}$ in terms of stability, accuracy and even the value of the contractivity preserving coefficients motivated us to investigate a third set of CPHBT methods by casting s-stage Runge-Kutta methods of order 6, RK(s,6), with Taylor method of order $p-5$, T($p-5$). Although such methods will be more complex to derive due to the higher number of constraints in the optimization problem, we believe that such methods exist and will have better stability and accuracy properties. Also, by construction, these methods will require less higher order derivatives compared to $\mathrm{CPHBT_{RK5}}$ and $\mathrm{CPHBT_{RK4}}$ which will increase their computational efficiency.

- Since the main drawback of methods using higher order derivative information is that we need to formulate and code the recurrence formulas of the higher order derivatives described in Section 2.2.1 for each problem. This issue can be solved by integrating the designed methods with some modern general purpose algebraic manipulator similar to the work done by Lara, Elipe and Palacios's in [34], and Jorba and Zou's in [29]. This will result in an application that takes the differential equation as an input and provides a FORTRAN or C++ code as an output with the recurrence formulas of the automatic differentiation process.

- We believe that the performance and stability of the RKN pair can be improved by adding higher order derivative information. Such methods can be thought of as combining two Hermite-Birkhoff-Taylor methods to create a pair of HBT methods. Such methods can be divided into two categories:

  1. CPHBTpq pair where we consider two contractivity preserving HBT methods of orders p and q with $p = q + 1$ or $p = q + 2$ and use a step controller similar to RKN methods in (6.4.5). Such methods estimate the error more accurately and allow a better automatic step-size control. Moreover, these methods add minimal computational cost since the same function and higher order derivatives are used for both methods.

  2. Non-CP high accuracy HBT pair composed of two HBT methods designed by choosing the free parameters specifically to minimize the principal error and achieve better accuracy. This special consideration of the free parameters will be possible by dropping the large number of contractivity preserving conditions in the optimization problem.

Appendices

Appendix A

$\mathbf{CPHBT_{RK4}}(d, s, p)$ formulae

A.1 Five stages $\mathbf{CPHBT_{RK4}}(d, 5, p)$ methods formulae

$\mathbf{CPHBT_{RK4}(2,5,5)}$ with $c_{cp} = 1.0625338959788060$, and abscissa vector
$\sigma = [0\ 4.2850362365796774 \times 10^{-1}\ 6.5248338991294597 \times 10^{-1}\ 6.0260786608926109 \times 10^{-1}\ 9.4283575658959828 \times 10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 4.2850362365796774 \times 10^{-1} h_n f_n + 9.1807677744004626 \times 10^{-2} h_n^2 y_n^{(2)},\\
Y_3 &= y_n + 2.4783303603362317 \times 10^{-1} h_n f_n + 4.0465035387932280 \times 10^{-1} h_n F_2 + 3.9473144104375933 \times 10^{-2} h_n^2 y_n^{(2)},\\
Y_4 &= y_n + 3.0711977059612233 \times 10^{-1} h_n f_n + 8.8846520080429953 \times 10^{-2} h_n F_2 + 2.0664157541270883 \times 10^{-1} h_n F_3\\
&\quad + 8.6668687102258411 \times 10^{-3} h_n^2 y_n^{(2)},\\
Y_5 &= y_n + 2.0871142715810950 \times 10^{-1} h_n f_n + 5.2743466069405748 \times 10^{-2} h_n F_2 + 1.2267214191931065 \times 10^{-1} h_n F_3\\
&\quad + 5.5870872144277239 \times 10^{-1} h_n F_4 + 5.1450602154364091 \times 10^{-3} h_n^2 y_n^{(2)},\\
y_{n+1} &= y_n + 2.5075377432281282 \times 10^{-1} h_n f_n + 2.0839397966111808 \times 10^{-1} h_n F_2 + 6.4981764618288446 \times 10^{-2} h_n F_3\\
&\quad + 2.9595862644754489 \times 10^{-1} h_n F_4 + 1.7991185495023562 \times 10^{-1} h_n F_5 + 2.0328576290332894 \times 10^{-2} h_n^2 y_n^{(2)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(3,5,5)}$ with $c_{cp} = 1.6544514297191719$, and abscissa vector
$\sigma = [0\ 5.4808274228565224{\times}10^{-1}\ 6.3139226452853348{\times}10^{-1}\ 5.9512887026466110{\times}10^{-1}\ 8.7603815620983694{\times}10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 5.4808274228565224 \times 10^{-1} h f_n + 1.5019734619568034 \times 10^{-1} h^2 y_n^{(2)} + 9.7913207802001970 \times 10^{-3} h^3 y_n^{(3)},\\
Y_3 &= y_n + 3.8116145601471918 \times 10^{-1} h f_n + 2.5023080851381441 \times 10^{-1} h F_2 + 6.2180908118627441 \times 10^{-2} h^2 y_n^{(2)}\\
&\quad + 1.3438194283833214 \times 10^{-2} h^3 y_n^{(3)},
\end{aligned}
$$

$$Y_4 = y_n + 3.3460998717488410\times10^{-1}hf_n + 7.6275703390041152\times10^{-2}hF_2 + 1.8424317969973586\times10^{-1}hF_3$$
$$+1.8954070972911352\times10^{-2}h^2y_n^{(2)} + 1.3566189439262331\times10^{-2}h^3y_n^{(3)},$$
$$Y_5 = y_n + 2.5774902310335085\times10^{-1}hf_n + 5.4523963216387562\times10^{-2}hF_2 + 1.3170207426930505\times10^{-1}hF_3$$
$$+4.3206309562079354\times10^{-1}hF_4 + 1.3548889391988866\times10^{-2}h^2y_n^{(2)} + 1.0825240235151289\times10^{-2}h^3y_n^{(3)},$$
$$y_{n+1} = y_n + 4.0368154940504869\times10^{-1}hf_n + 7.1984917418738248\times10^{-2}hF_2 + 5.9113253061746168\times10^{-2}hF_3$$
$$+1.9392750836898545\times10^{-1}hF_4 + 2.7129277174548150\times10^{-1}hF_5 + 7.0147979821784903\times10^{-2}h^2y_n^{(2)}$$
$$+5.6284771897167256\times10^{-3}h^3y_n^{(3)}.$$

CPHBT$_{\mathbf{RK4}}$(3,5,6) with c_{cp}= 6.8211668346068355 $\times10^{-1}$, and abscissa vector
$\sigma = [0\ 4.7866009264327564\times10^{-1}\ 6.8440248420931726\times10^{-1}\ 6.8225905894506400\times10^{-1}\ 9.6045235789866867\times10^{-1}]^T$.

$$Y_2 = y_n + 4.7866009264327564\times10^{-1}h_nf_n + 1.1455774214463461\times10^{-1}h_n^2y_n^{(2)} + 1.8278073155985095\times10^{-2}h_n^3y_n^{(3)},$$
$$Y_3 = y_n + 2.6377998718488588\times10^{-1}h_nf_n + 4.2062249702443144\times10^{-1}h_nF_2 + 3.2868176802382097\times10^{-2}h_n^2y_n^{(2)}$$
$$+5.2442281844169098\times10^{-3}h_n^3y_n^{(3)},$$
$$Y_4 = y_n + 4.3058405574487157\times10^{-1}h_nf_n + 5.6110204779506276\times10^{-2}h_nF_2 + 1.9556479842068614\times10^{-1}h_nF_3$$
$$+7.2035962075298554\times10^{-2}h_n^2y_n^{(2)} + 6.9956961270428622\times10^{-4}h_n^3y_n^{(3)},$$
$$Y_5 = y_n + 3.1787654927202391\times10^{-1}h_nf_n + 2.0990314695794404\times10^{-2}h_nF_2 + 7.3159003400556089\times10^{-2}h_nF_3$$
$$+5.4842649053029424\times10^{-1}h_nF_4 + 2.6947994920294642\times10^{-2}h_n^2y_n^{(2)} + 4.8578091073209944\times10^{-4}h_n^3y_n^{(3)},$$
$$y_{n+1} = y_n + 3.3737569831798614\times10^{-1}h_nf_n + 1.8139912869065516\times10^{-1}h_nF_2 + 4.0662218957669478\times10^{-2}h_nF_3$$
$$+3.0481877832632642\times10^{-1}h_nF_4 + 1.3574417570736286\times10^{-1}h_nF_5 + 4.7000966108577895\times10^{-2}h_n^2y_n^{(2)}$$
$$+2.8097543373721955\times10^{-3}h_n^3y_n^{(3)}$$

CPHBT$_{\mathbf{RK4}}$(4,5,6) with c_{cp} = 1.6289806645490685, and abscissa vector
$\sigma = [0\ 6.0466839315262999\times10^{-1}\ 6.8897630604729898\times10^{-1}\ 6.6703533001073756\times10^{-1}\ 8.8273882977431872\times10^{-1}]^T$.

$$Y_2 = y_n + 6.0466839315262999\times10^{-1}h_nf_n + 1.8281193283889177\times10^{-1}h_n^2y_n^{(2)} + 3.6846865892939729\times10^{-2}h_n^3y_n^{(3)}$$
$$+1.0708970275330335\times10^{-5}h_n^4y_n^{(4)},$$
$$Y_3 = y_n + 4.6451021267513215\times10^{-1}h_nf_n + 2.2446609337216683\times10^{-1}h_nF_2 + 1.0161662315069432\times10^{-1}h_n^2y_n^{(2)}$$
$$+1.3473090632113511\times10^{-2}h_n^3y_n^{(3)} + 2.3322068777870491\times10^{-3}h_n^4y_n^{(4)},$$
$$Y_4 = y_n + 4.5990915568820728\times10^{-1}h_nf_n + 5.5457713234330748\times10^{-2}h_nF_2 + 1.5166846108819954\times10^{-1}h_nF_3$$
$$+8.4438563327518451\times10^{-2}h_n^2y_n^{(2)} + 3.3287290094947950\times10^{-3}h_n^3y_n^{(3)} + 3.8749278184591696\times10^{-3}h_n^4y_n^{(4)},$$
$$Y_5 = y_n + 3.7679653094714194\times10^{-1}h_nf_n + 3.4175595121912289\times10^{-2}h_nF_2 + 9.3465085677299234\times10^{-2}h_nF_3$$
$$+3.7830161802796525\times10^{-1}h_nF_4 + 5.2213244509045596\times10^{-2}h_n^2y_n^{(2)} + 2.0513160075384253\times10^{-3}h_n^3y_n^{(3)}$$
$$+2.5907000696281622\times10^{-3}h_n^4y_n^{(4)},$$
$$y_{n+1} = y_n + 5.0933497961429763\times10^{-1}h_nf_n + 3.9695675606679123\times10^{-2}h_nF_2 + 3.8824652749983236\times10^{-2}h_nF_3$$
$$+1.5714348142153173\times10^{-1}h_nF_4 + 2.5500121060750836\times10^{-1}h_nF_5 + 1.1932828954879460\times10^{-1}h_n^2y_n^{(2)}$$

$+1.5883590473209471 \times 10^{-2} h_n^3 y_n^{(3)} + 1.0807291773231296 \times 10^{-3} h_n^4 y_n^{(4)}.$

CPHBT$_{\mathbf{RK4}}$(5,5,7) with $c_{cp} = 1.5486363112234323$, and abscissa vector
$\sigma = [0\ 5.9448263083486175 \times 10^{-1}\ 7.4611168636626923 \times 10^{-1}\ 7.1527878030425218 \times 10^{-1}\ 8.9279153752926677 \times 10^{-1}]^T.$

$$
\begin{aligned}
Y_2 &= y_n + 5.9448263083486175 \times 10^{-1} h_n f_n + 1.7670479918216928 \times 10^{-1} h_n^2 y_n^{(2)} + 3.5015977966320645 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+5.2040976756684626 \times 10^{-3} h_n^4 y_n^{(4)} + 5.9507643942744467 \times 10^{-4} h_n^5 y_n^{(5)}, \\
Y_3 &= y_n + 4.4634974224850704 \times 10^{-1} h_n f_n + 2.9976194411776214 \times 10^{-1} h_n F_2 + 1.0013805510285899 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+1.6255197473429017 \times 10^{-2} h_n^3 y_n^{(3)} + 2.4158578353375188 \times 10^{-3} h_n^4 y_n^{(4)} + 4.2386402042934049 \times 10^{-4} h_n^5 y_n^{(5)}, \\
Y_4 &= y_n + 5.3628836349796449 \times 10^{-1} h_n f_n + 5.6747759257548616 \times 10^{-2} h_n F_2 + 1.2224265754873902 \times 10^{-1} h_n F_3 \\
&+1.3086963418977443 \times 10^{-1} h_n^2 y_n^{(2)} + 1.6939482097573323 \times 10^{-2} h_n^3 y_n^{(3)} + 4.5734464140764362 \times 10^{-4} h_n^4 y_n^{(4)} \\
&+2.1457557270600997 \times 10^{-4} h_n^5 y_n^{(5)}, \\
Y_5 &= y_n + 4.6032156704749760 \times 10^{-1} h_n f_n + 2.9757623371333759 \times 10^{-2} h_n F_2 + 6.4102107481229298 \times 10^{-2} h_n F_3 \\
&+3.3861023962920611 \times 10^{-1} h_n F_4 + 9.0819923799791646 \times 10^{-2} h_n^2 y_n^{(2)} + 8.8827952849606998 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+3.4009877667845440 \times 10^{-4} h_n^4 y_n^{(4)} + 1.6672900302383718 \times 10^{-4} h_n^5 y_n^{(5)}. \\
y_{n+1} &= y_n + 5.7497384573678145 \times 10^{-1} h_n f_n + 3.2441120753385122 \times 10^{-2} h_n F_2 + 2.4932874462414000 \times 10^{-2} h_n F_3 \\
&+1.3170435307193448 \times 10^{-1} h_n F_4 + 2.3594780597548484 \times 10^{-1} h_n F_5 + 1.5725407467662181 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+2.6268526463189373 \times 10^{-2} h_n^3 y_n^{(3)} + 2.7874661505894647 \times 10^{-3} h_n^4 y_n^{(4)} + 1.6007340660752472 \times 10^{-4} h_n^5 y_n^{(5)}.
\end{aligned}
$$

CPHBT$_{\mathbf{RK4}}$(6,5,8) with $c_{cp} = 1.4715349175532293$, and abscissa vector
$\sigma = [0\ 6.3812458308410325 \times 10^{-1}\ 7.6732516263606776 \times 10^{-1}\ 7.5421376503889004 \times 10^{-1}\ 9.0470574170528428 \times 10^{-1}]^T.$

$$
\begin{aligned}
Y_2 &= y_n + 6.3812458308410325 \times 10^{-1} h_n f_n + 2.0360149176813030 \times 10^{-1} h_n^2 y_n^{(2)} + 4.3307705683279879 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+6.9089279083680054 \times 10^{-3} h_n^4 y_n^{(4)} + 8.8175134821709183 \times 10^{-4} h_n^5 y_n^{(5)} + 1.2838317254671800 \times 10^{-4} h_n^6 y_n^{(6)}, \\
Y_3 &= y_n + 4.9720293695307349 \times 10^{-1} h_n f_n + 2.7012222568299432 \times 10^{-1} h_n F_2 + 1.2202231996152318 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+2.0301341079021675 \times 10^{-2} h_n^3 y_n^{(3)} + 2.7462593735600987 \times 10^{-3} h_n^4 y_n^{(4)} + 3.5049112355876980 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+9.9487210341961491 \times 10^{-5} h_n^6 y_n^{(6)}, \\
Y_4 &= y_n + 5.9370283825951065 \times 10^{-1} h_n f_n + 4.5654695694174262 \times 10^{-2} h_n F_2 + 1.1485623108520505 \times 10^{-1} h_n F_3 \\
&+1.6715374183417167 \times 10^{-1} h_n^2 y_n^{(2)} + 2.8395948317622623 \times 10^{-2} h_n^3 y_n^{(3)} + 2.8566635028080207 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+5.9238241315113858 \times 10^{-5} h_n^5 y_n^{(5)} + 1.6833518626430792 \times 10^{-5} h_n^6 y_n^{(6)}, \\
Y_5 &= y_n + 5.2693740981356185 \times 10^{-1} h_n f_n + 2.0530226230123742 \times 10^{-2} h_n F_2 + 5.1649110179471089 \times 10^{-2} h_n F_3 \\
&+3.0558899548212765 \times 10^{-1} h_n F_4 + 1.2603430877803559 \times 10^{-1} h_n^2 y_n^{(2)} + 1.7115259039665234 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+1.2845983766675317 \times 10^{-3} h_n^4 y_n^{(4)} + 4.2784131142800703 \times 10^{-5} h_n^5 y_n^{(5)} + 4.5316048150476212 \times 10^{-5} h_n^6 y_n^{(6)}, \\
y_{n+1} &= y_n + 6.2838771624152567 \times 10^{-1} h_n f_n + 2.2913478782924837 \times 10^{-2} h_n F_2 + 1.9961417389470458 \times 10^{-2} h_n F_3 \\
&+1.1810444492173085 \times 10^{-1} h_n F_4 + 2.1063294266434826 \times 10^{-1} h_n F_5 + 1.9042461736685512 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+3.6333015991553086 \times 10^{-2} h_n^3 y_n^{(3)} + 4.7308596338484195 \times 10^{-3} h_n^4 y_n^{(4)} + 4.1480659453566184 \times 10^{-4} h_n^5 y_n^{(5)}
\end{aligned}
$$

$+2.0390815678371275 \times 10^{-5} h_n^6 y_n^{(6)}.$

CPHBT$_{\mathbf{RK4}}$(7,5,9) with $c_{cp} = 1.4115607214001771$, and abscissa vector
$\sigma = [0\ 6.7264209027272404 \times 10^{-1}\ 7.8570115394476325 \times 10^{-1}\ 7.8339576219887797 \times 10^{-1}\ 9.1412683756610613 \times 10^{-1}]^T.$

$$
\begin{aligned}
Y_2 &= y_n + 6.7264209027272404 \times 10^{-1} h_n f_n + 2.2622369080322974 \times 10^{-1} h_n^2 y_n^{(2)} + 5.0722525417031622 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+8.5295263801058816 \times 10^{-3} h_n^4 y_n^{(4)} + 1.1474636906701524 \times 10^{-3} h_n^5 y_n^{(5)} + 1.2863872923407095 \times 10^{-4} h_n^6 y_n^{(6)} \\
&+1.6871283691865543 \times 10^{-5} h_n^7 y_n^{(7)}, \\
Y_3 &= y_n + 5.3985006213700604 \times 10^{-1} h_n f_n + 2.4585109180775722 \times 10^{-1} h_n F_2 + 1.4329335936566504 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+2.5221656768449830 \times 10^{-2} h_n^3 y_n^{(3)} + 3.4086352785824706 \times 10^{-3} h_n^4 y_n^{(4)} + 3.9820862126158530 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+4.4641979895002042 \times 10^{-5} h_n^6 y_n^{(6)} + 5.8549048724342979 \times 10^{-6} h_n^7 y_n^{(7)}, \\
Y_4 &= y_n + 6.3712202767851545 \times 10^{-1} h_n f_n + 3.7684224334290320 \times 10^{-2} h_n F_2 + 1.0858951018607231 \times 10^{-1} h_n F_3 \\
&+1.9618756122956357 \times 10^{-1} h_n^2 y_n^{(2)} + 3.8086849790783764 \times 10^{-2} h_n^3 y_n^{(3)} + 5.0035703763330226 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+4.1310702391352867 \times 10^{-4} h_n^5 y_n^{(5)} + 6.8427533622897276 \times 10^{-6} h_n^6 y_n^{(6)} + 8.9746839854607379 \times 10^{-7} h_n^7 y_n^{(7)}, \\
Y_5 &= y_n + 5.7925228138991269 \times 10^{-1} h_n f_n + 1.4764652723196561 \times 10^{-2} h_n F_2 + 4.2545294101237273 \times 10^{-2} h_n F_3 \\
&+2.7756460935175958 \times 10^{-1} h_n F_4 + 1.5701178533660620 \times 10^{-1} h_n^2 y_n^{(2)} + 2.5667427335795582 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+2.6654164614515119 \times 10^{-3} h_n^4 y_n^{(4)} + 1.6185504288187001 \times 10^{-4} h_n^5 y_n^{(5)} + 4.8332984267274632 \times 10^{-6} h_n^6 y_n^{(6)} \\
&+3.5162749957115332 \times 10^{-7} h_n^7 y_n^{(7)}. \\
y_{n+1} &= y_n + 6.6992609764748168 \times 10^{-1} h_n f_n + 1.6928908398811920 \times 10^{-2} h_n F_2 + 1.6276124483944193 \times 10^{-2} h_n F_3 \\
&+1.0618509589795518 \times 10^{-1} h_n F_4 + 1.9068377357180699 \times 10^{-1} h_n F_5 + 2.1833062483429713 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+4.5559398104867747 \times 10^{-2} h_n^3 y_n^{(3)} + 6.7074212297902388 \times 10^{-3} h_n^4 y_n^{(4)} + 7.1620368169263062 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+5.3466285655376258 \times 10^{-5} h_n^6 y_n^{(6)} + 2.2949248887023122 \times 10^{-6} h_n^7 y_n^{(7)}.
\end{aligned}
$$

CPHBT$_{\mathbf{RK4}}$(8,5,10) with $c_{cp} = 1.3638646257964513$, and abscissa vector
$\sigma = [0\ 7.0080215042708205 \times 10^{-1}\ 8.0155292312263227 \times 10^{-1}\ 8.0617794551281863 \times 10^{-1}\ 9.2177638433749054 \times 10^{-1}]^T.$

$$
\begin{aligned}
Y_2 &= y_n + 7.0080215042708205 \times 10^{-1} h_n f_n + 2.4556182702161131 \times 10^{-1} h_n^2 y_n^{(2)} + 5.7363418813182784 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+1.0050101815031959 \times 10^{-2} h_n^4 y_n^{(4)} + 1.4086265927971036 \times 10^{-3} h_n^5 y_n^{(5)} + 1.6452809089683067 \times 10^{-4} h_n^6 y_n^{(6)} \\
&+1.6471662843737338 \times 10^{-5} h_n^7 y_n^{(7)} + 1.3780368036111647 \times 10^{-6} h_n^8 y_n^{(8)}, \\
Y_3 &= y_n + 5.7598930912692958 \times 10^{-1} h_n f_n + 2.2556361399570268 \times 10^{-1} h_n F_2 + 1.6316807853692550 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+3.0441420822447194 \times 10^{-2} h_n^3 y_n^{(3)} + 4.2604690652900757 \times 10^{-3} h_n^4 y_n^{(4)} + 4.9033569512550257 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+5.0615131326159361 \times 10^{-5} h_n^6 y_n^{(6)} + 5.0673132682173783 \times 10^{-6} h_n^7 y_n^{(7)} + 4.2393680864481340 \times 10^{-7} h_n^8 y_n^{(8)}, \\
Y_4 &= y_n + 6.7134687083218469 \times 10^{-1} h_n f_n + 3.1720695085314318 \times 10^{-2} h_n F_2 + 1.0311037959531959 \times 10^{-1} h_n F_3 \\
&+2.2008308241789187 \times 10^{-1} h_n^2 y_n^{(2)} + 4.6412646367557459 \times 10^{-2} h_n^3 y_n^{(3)} + 6.9302909374724855 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+7.4549437034508953 \times 10^{-4} h_n^5 y_n^{(5)} + 5.2301785173365859 \times 10^{-5} h_n^6 y_n^{(6)} + 7.1260916703504024 \times 10^{-7} h_n^7 y_n^{(7)} \\
&+3.7931187385561651 \times 10^{-7} h_n^8 y_n^{(8)},
\end{aligned}
$$

$$
\begin{aligned}
Y_5 = y_n &+ 6.2138885317264780\times10^{-1}h_nf_n + 1.0977009014961151\times10^{-2}h_nF_2 + 3.5681549956889096\times10^{-2}h_nF_3\\
&+2.5372897219299267\times10^{-1}h_nF_4 + 1.8399178764908977\times10^{-1}h_n^2y_n^{(2)} + 3.3924417553441152\times10^{-2}h_n^3y_n^{(3)}\\
&+4.2316163710359516\times10^{-3}h_n^4y_n^{(4)} + 3.5592113113033997\times10^{-4}h_n^5y_n^{(5)} + 1.8099135778786784\times10^{-5}h_n^6y_n^{(6)}\\
&+4.9543809927693444\times10^{-7}h_n^7y_n^{(7)} + 6.7811743002780728\times10^{-7}h_n^8y_n^{(8)},\\
y_{n+1} = y_n &+ 7.0321426874850679\times10^{-1}h_nf_n + 1.2920781087526969\times10^{-2}h_nF_2 + 1.3481818905558848\times10^{-2}h_nF_3\\
&+9.5868258479031063\times10^{-2}h_nF_4 + 1.7451487277937641\times10^{-1}h_nF_5 + 2.4198813337169503\times10^{-1}h_n^2y_n^{(2)}\\
&+5.3869206897701916\times10^{-2}h_n^3y_n^{(3)} + 8.6163531919612318\times10^{-3}h_n^4y_n^{(4)} + 1.0347489795011107\times10^{-3}h_n^5y_n^{(5)}\\
&+9.3680192323108188\times10^{-5}h_n^6y_n^{(6)} + 6.0871403686808235\times10^{-6}h_n^7y_n^{(7)} + 2.3175101190842453\times10^{-7}h_n^8y_n^{(8)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(9,5,11)}$ with $c_{cp} = 1.3253079734478708$, and abscissa vector
$\sigma = [0\ 7.2429567207324930\times10^{-1}\ 8.1528856428413632\times10^{-1}\ 8.2450162786848580\times10^{-1}\ 9.2812027230881411\times10^{-1}]^T$.

$$
\begin{aligned}
Y_2 = y_n &+ 7.2429567207324930\times10^{-1}h_nf_n + 2.6230211029201994\times10^{-1}h_n^2y_n^{(2)} + 6.3328094420063374\times10^{-2}h_n^3y_n^{(3)}\\
&+1.1467066177274499\times10^{-2}h_n^4y_n^{(4)} + 1.6611092807154917\times10^{-3}h_n^5y_n^{(5)} + 2.0052237714382313\times10^{-4}h_n^6y_n^{(6)}\\
&+2.0748212845587281\times10^{-5}h_n^7y_n^{(7)} + 1.8784800959141834\times10^{-6}h_n^8y_n^{(8)} + 6.1629758220391547\times10^{-32}h_n^9y_n^{(9)},\\
Y_3 = y_n &+ 6.0694710911169403\times10^{-1}h_nf_n + 2.0834145517244243\times10^{-1}h_nF_2 + 1.8144690723140125\times10^{-1}h_n^2y_n^{(2)}\\
&+3.5671362222369919\times10^{-2}h_n^3y_n^{(3)} + 5.2153006558404869\times10^{-3}h_n^4y_n^{(4)} + 6.1269157584519765\times10^{-4}h_n^5y_n^{(5)}\\
&+6.1805077935152860\times10^{-5}h_n^6y_n^{(6)} + 5.7289258156145483\times10^{-6}h_n^7y_n^{(7)} + 5.1867952173481764\times10^{-7}h_n^8y_n^{(8)}\\
&+4.8361850237535367\times10^{-8}h_n^9y_n^{(9)},\\
Y_4 = y_n &+ 6.9914916425821572\times10^{-1}h_nf_n + 2.7122831285393719\times10^{-2}h_nF_2 + 9.8229632324876429\times10^{-2}h_nF_3\\
&+2.4017102195620088\times10^{-1}h_n^2y_n^{(2)} + 5.3655667295174354\times10^{-2}h_n^3y_n^{(3)} + 8.6657866470045818\times10^{-3}h_n^4y_n^{(4)}\\
&+1.0558933168246529\times10^{-3}h_n^5y_n^{(5)} + 9.6416084843446254\times10^{-5}h_n^6y_n^{(6)} + 5.8887917418002241\times10^{-6}h_n^7y_n^{(7)}\\
&+6.7524041561148269\times10^{-8}h_n^8y_n^{(8)} + 6.3108523635658873\times10^{-9}h_n^9y_n^{(9)},\\
Y_5 = y_n &+ 6.5601718623534822\times10^{-1}h_nf_n + 8.3876075469791270\times10^{-3}h_nF_2 + 3.0377050120939483\times10^{-2}h_nF_3\\
&+2.3333842840554728\times10^{-1}h_nF_4 + 2.0747453644503186\times10^{-1}h_n^2y_n^{(2)} + 4.1640348954393273\times10^{-2}h_n^3y_n^{(3)}\\
&+5.8451373697182120\times10^{-3}h_n^4y_n^{(4)} + 5.9060465527598365\times10^{-4}h_n^5y_n^{(5)} + 4.1732461687485503\times10^{-5}h_n^6y_n^{(6)}\\
&+1.8210810464844811\times10^{-6}h_n^7y_n^{(7)} + 4.6434535379714043\times10^{-8}h_n^8y_n^{(8)} + 8.3766562521193928\times10^{-8}h_n^9y_n^{(9)},\\
y_{n+1} = y_n &+ 7.3049821038076390\times10^{-1}h_nf_n + 1.0112276063254677\times10^{-2}h_nF_2 + 1.1320389876402904\times10^{-2}h_nF_3\\
&+8.6956500782500348\times10^{-2}h_nF_4 + 1.6111262289707823\times10^{-1}h_nF_5 + 2.6221866991856319\times10^{-1}h_n^2y_n^{(2)}\\
&+6.1303457437452448\times10^{-2}h_n^3y_n^{(3)} + 1.0412678352601583\times10^{-2}h_n^4y_n^{(4)} + 1.3533693769707648\times10^{-3}h_n^5y_n^{(5)}\\
&+1.3736913097097669\times10^{-4}h_n^6y_n^{(6)} + 1.0797188914801632\times10^{-5}h_n^7y_n^{(7)} + 6.2101845671912387\times10^{-7}h_n^8y_n^{(8)}\\
&+2.1237133463449921\times10^{-8}h_n^9y_n^{(9)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(10,5,12)}$ with $c_{cp} = 1.2936747608596961$, and abscissa vector
$\sigma = [0\ 7.4423533270870701\times10^{-1}\ 8.2727137725735045\times10^{-1}\ 8.3958210167620639\times10^{-1}\ 9.3347415242277998\times10^{-1}]^T$.

$$
Y_2 = y_n + 7.4423533270870701\times10^{-1}h_nf_n + 2.7694311522601994\times10^{-1}h_n^2y_n^{(2)} + 6.8703617167207562\times10^{-2}h_n^3y_n^{(3)}
$$

$$+1.2782914845182090 \times 10^{-2} h_n^4 y_n^{(4)} + 1.9026993765582326 \times 10^{-3} h_n^5 y_n^{(5)} + 2.3600935059291095 \times 10^{-4} h_n^6 y_n^{(6)}$$
$$+2.5092356794411568 \times 10^{-5} h_n^7 y_n^{(7)} + 2.3343273134168098 \times 10^{-6} h_n^8 y_n^{(8)} + 1.9303209608353133 \times 10^{-7} h_n^9 y_n^{(9)},$$
$$Y_3 = y_n + 6.3373243388363965 \times 10^{-1} h_n f_n + 1.9353894337371072 \times 10^{-1} h_n F_2 + 1.9815044590081149 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+4.0761767781779923 \times 10^{-2} h_n^3 y_n^{(3)} + 6.2187225817218143 \times 10^{-3} h_n^4 y_n^{(4)} + 7.5493903097584335 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+7.6953920168282153 \times 10^{-5} h_n^6 y_n^{(6)} + 6.9375002702269951 \times 10^{-6} h_n^7 y_n^{(7)} + 5.8446057720170153 \times 10^{-7} h_n^8 y_n^{(8)}$$
$$+4.8330690236536808 \times 10^{-8} h_n^9 y_n^{(9)},$$
$$Y_4 = y_n + 7.2225823203532746 \times 10^{-1} h_n f_n + 2.3493031744528232 \times 10^{-2} h_n F_2 + 9.3830837896350716 \times 10^{-2} h_n F_3$$
$$+2.5734114193506746 \times 10^{-1} h_n^2 y_n^{(2)} + 6.0022528028879356 \times 10^{-2} h_n^3 y_n^{(3)} + 1.0235356888342359 \times 10^{-2} h_n^4 y_n^{(4)}$$
$$+1.3449693410170329 \times 10^{-3} h_n^5 y_n^{(5)} + 1.3878586049175356 \times 10^{-4} h_n^6 y_n^{(6)} + 1.1027980823930676 \times 10^{-5} h_n^7 y_n^{(7)}$$
$$+5.9693209307913861 \times 10^{-7} h_n^8 y_n^{(8)} + 5.8666975243812925 \times 10^{-9} h_n^9 y_n^{(9)},$$
$$Y_5 = y_n + 6.8495172659353409 \times 10^{-1} h_n f_n + 6.5578371213013062 \times 10^{-3} h_n F_2 + 2.6191909097590612 \times 10^{-2} h_n F_3$$
$$+2.1577267961035410 \times 10^{-1} h_n F_4 + 2.2797972598516167 \times 10^{-1} h_n^2 y_n^{(2)} + 4.8739909969234767 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+7.4320582456993464 \times 10^{-3} h_n^4 y_n^{(4)} + 8.4429693271739087 \times 10^{-4} h_n^5 y_n^{(5)} + 7.1757565381194631 \times 10^{-5} h_n^6 y_n^{(6)}$$
$$+4.3691730384821213 \times 10^{-6} h_n^7 y_n^{(7)} + 1.6662742729244420 \times 10^{-7} h_n^8 y_n^{(8)} + 4.0042415191055253 \times 10^{-9} h_n^9 y_n^{(9)},$$
$$y_{n+1} = y_n + 7.5326323653346827 \times 10^{-1} h_n f_n + 8.0766991972185277 \times 10^{-3} h_n F_2 + 9.6193379315639461 \times 10^{-3} h_n F_3$$
$$+7.9245476678979715 \times 10^{-2} h_n F_4 + 1.4979524965876967 \times 10^{-1} h_n F_5 + 2.7966815457617183 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+6.7944413511295737 \times 10^{-2} h_n^3 y_n^{(3)} + 1.2080199971464114 \times 10^{-2} h_n^4 y_n^{(4)} + 1.6626117337741018 \times 10^{-3} h_n^5 y_n^{(5)}$$
$$+1.8220341179994698 \times 10^{-4} h_n^6 y_n^{(6)} + 1.6023302134113757 \times 10^{-5} h_n^7 y_n^{(7)} + 1.1129105640963680 \times 10^{-6} h_n^8 y_n^{(8)}$$
$$+5.7414486926078978 \times 10^{-8} h_n^9 y_n^{(9)} + 1.7817546307671103 \times 10^{-9} h_n^{10} y_n^{(10)}.$$

CPHBT$_{\mathbf{RK4}}$(11,5,13) with $c_{cp} = 1.2673611773316924$, and abscissa vector
$\sigma = [0\ 7.6139325620392972 \times 10^{-1}\ 8.3780031020389356 \times 10^{-1}\ 8.5222369871008397 \times 10^{-1}\ 9.3805874408685674 \times 10^{-1}]^T$.

$$Y_2 = y_n + 7.6139325620392972 \times 10^{-1} h_n f_n + 2.8985984529641151 \times 10^{-1} h_n^2 y_n^{(2)} + 7.3565777151000683 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+1.4003121652543265 \times 10^{-2} h_n^4 y_n^{(4)} + 2.1323764784099337 \times 10^{-3} h_n^5 y_n^{(5)} + 2.7059617839153468 \times 10^{-4} h_n^6 y_n^{(6)}$$
$$+2.9432872197410008 \times 10^{-5} h_n^7 y_n^{(7)} + 2.8012488002275147 \times 10^{-6} h_n^8 y_n^{(8)} + 2.3698354949361984 \times 10^{-7} h_n^9 y_n^{(9)}$$
$$+1.8043767641571234 \times 10^{-8} h_n^{10} y_n^{(10)} + 8.5594650392506139 \times 10^{-11} h_n^{11} y_n^{(11)},$$
$$Y_3 = y_n + 6.5711748459198460 \times 10^{-1} h_n f_n + 1.8068282561190885 \times 10^{-1} h_n F_2 + 2.1338399495609203 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+4.5637284013214777 \times 10^{-2} h_n^3 y_n^{(3)} + 7.2361254053377420 \times 10^{-3} h_n^4 y_n^{(4)} + 9.0958252435044374 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+9.5014000541886179 \times 10^{-5} h_n^6 y_n^{(6)} + 8.5927253846436024 \times 10^{-6} h_n^7 y_n^{(7)} + 7.0208417953977293 \times 10^{-7} h_n^8 y_n^{(8)}$$
$$+5.4266957458083771 \times 10^{-8} h_n^9 y_n^{(9)} + 4.1318495443290518 \times 10^{-9} h_n^{10} y_n^{(10)} + 2.4752718955087072 \times 10^{-10} h_n^{11} y_n^{(11)},$$
$$Y_4 = y_n + 7.4181729111079597 \times 10^{-1} h_n f_n + 2.0571362731684226 \times 10^{-2} h_n F_2 + 8.9835044867603700 \times 10^{-2} h_n F_3$$
$$+2.7221589100945970 \times 10^{-1} h_n^2 y_n^{(2)} + 6.5668739780347679 \times 10^{-2} h_n^3 y_n^{(3)} + 1.1660680737255626 \times 10^{-2} h_n^4 y_n^{(4)}$$
$$+1.6139491487261023 \times 10^{-3} h_n^5 y_n^{(5)} + 1.7922291689476943 \times 10^{-4} h_n^6 y_n^{(6)} + 1.6066454837184573 \times 10^{-5} h_n^7 y_n^{(7)}$$
$$+1.1313167290317045 \times 10^{-6} h_n^8 y_n^{(8)} + 5.5019292109915785 \times 10^{-8} h_n^9 y_n^{(9)} + 4.7042531818549840 \times 10^{-10} h_n^{10} y_n^{(10)}$$
$$+1.4938442428573119 \times 10^{-9} h_n^{11} y_n^{(11)},$$

$$
\begin{aligned}
Y_5 &= y_n + 7.0947049768247195\times10^{-1}h_nf_n + 5.2280671526493284\times10^{-3}h_nF_2 + 2.2830944811726872\times10^{-2}h_nF_3\\
&+2.0052923444000856\times10^{-1}h_nF_4 + 2.4597295008641859\times10^{-1}h_n^2y_n^{(2)} + 5.5226045260474227\times10^{-2}h_n^3y_n^{(3)}\\
&+8.9545295160175769\times10^{-3}h_n^4y_n^{(4)} + 1.1037044310105602\times10^{-3}h_n^5y_n^{(5)} + 1.0544578772260942\times10^{-4}h_n^6y_n^{(6)}\\
&+7.7367116529844821\times10^{-6}h_n^7y_n^{(7)} + 4.1359516900426941\times10^{-7}h_n^8y_n^{(8)} + 1.3982766119759095\times10^{-8}h_n^9y_n^{(9)}\\
&+3.1944360120070275\times10^{-10}h_n^{10}y_n^{(10)} + 3.7965000759175295\times10^{-10}h_n^{11}y_n^{(11)},\\
y_{n+1} &= y_n + 7.7253885611207529\times10^{-1}h_nf_n + 6.5613596859926222\times10^{-3}h_nF_2 + 8.2600743784096535\times10^{-3}h_nF_3\\
&+7.2550058930072492\times10^{-2}h_nF_4 + 1.4008965089344996\times10^{-1}h_nF_5 + 2.9484273056732124\times10^{-1}h_n^2y_n^{(2)}\\
&+7.3883623134179580\times10^{-2}h_n^3y_n^{(3)} + 1.3617367451870756\times10^{-2}h_n^4y_n^{(4)} + 1.9575734011503957\times10^{-3}h_n^5y_n^{(5)}\\
&+2.2674169067858434\times10^{-4}h_n^6y_n^{(6)} + 2.1493803462258279\times10^{-5}h_n^7y_n^{(7)} + 1.6679643346447017\times10^{-6}h_n^8y_n^{(8)}\\
&+1.0378645904175825\times10^{-7}h_n^9y_n^{(9)} + 4.8539110734279988\times10^{-9}h_n^{10}y_n^{(10)} + 1.3786127413521402\times10^{-10}h_n^{11}y_n^{(11)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(12,5,14)}$ with $c_{cp} = 1.2451925611501480$, and abscissa vector
$\sigma = [0\ 7.7632668708461672\times10^{-1}\ 8.4711629296060575\times10^{-1}\ 8.6298190985272327\times10^{-1}\ 9.4203279976041476\times10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 7.7632668708461672\times10^{-1}h_nf_n + 3.0134156253988820\times10^{-1}h_n^2y_n^{(2)} + 7.7979832309164415\times10^{-2}h_n^3y_n^{(3)}\\
&+1.5134456218996893\times10^{-2}h_n^4y_n^{(4)} + 2.3498564514642067\times10^{-3}h_n^5y_n^{(5)} + 3.0404271234827019\times10^{-4}h_n^6y_n^{(6)}\\
&+3.3719495944221956\times10^{-5}h_n^7y_n^{(7)} + 3.2721680720676250\times10^{-6}h_n^8y_n^{(8)} + 2.8225237766359073\times10^{-7}h_n^9y_n^{(9)}\\
&+2.1912005327333162\times10^{-8}h_n^{10}y_n^{(10)} + 1.5464431366499340\times10^{-9}h_n^{11}y_n^{(11)} + 1.7390528459355455\times10^{-25}h_n^{12}y_n^{(12)},\\
Y_3 &= y_n + 6.7770019112604785\times10^{-1}h_nf_n + 1.6941610183455783\times10^{-1}h_nF_2 + 2.2728076582364704\times10^{-1}h_n^2y_n^{(2)}\\
&+5.0263844856410216\times10^{-2}h_n^3y_n^{(3)} + 8.2455604151750288\times10^{-3}h_n^4y_n^{(4)} + 1.0712264510548155\times10^{-3}h_n^5y_n^{(5)}\\
&+1.1514264438243281\times10^{-4}h_n^6y_n^{(6)} + 1.0601581460525035\times10^{-5}h_n^7y_n^{(7)} + 8.6431254902898253\times10^{-7}h_n^8y_n^{(8)}\\
&+6.4689977221419615\times10^{-8}h_n^9y_n^{(9)} + 4.6224617592496038\times10^{-9}h_n^{10}y_n^{(10)} + 3.2623094761214780\times10^{-10}h_n^{11}y_n^{(11)}\\
&+-2.9661580898378634\times10^{-26}h_n^{12}y_n^{(12)},\\
Y_4 &= y_n + 7.5861671462281699\times10^{-1}h_nf_n + 1.8181036851906369\times10^{-2}h_nF_2 + 8.6184158377999837\times10^{-2}h_nF_3\\
&+2.8524645950242217\times10^{-1}h_n^2y_n^{(2)} + 7.0714034257402067\times10^{-2}h_n^3y_n^{(3)} + 1.2960180087437946\times10^{-2}h_n^4y_n^{(4)}\\
&+1.8642827123618478\times10^{-3}h_n^5y_n^{(5)} + 2.1766696392552757\times10^{-4}h_n^6y_n^{(6)} + 2.0964859544199930\times10^{-5}h_n^7y_n^{(7)}\\
&+1.6633795560796135\times10^{-6}h_n^8y_n^{(8)} + 1.0524449548138731\times10^{-7}h_n^9y_n^{(9)} + 4.6488644582067275\times10^{-9}h_n^{10}y_n^{(10)}\\
&+3.5009577140865102\times10^{-11}h_n^{11}y_n^{(11)} + 6.6546935735709924\times10^{-12}h_n^{12}y_n^{(12)},\\
Y_5 &= y_n + 7.3049944276265633\times10^{-1}h_nf_n + 4.2381172587350486\times10^{-3}h_nF_2 + 2.0090084631935245\times10^{-2}h_nF_3\\
&+1.8720515510708807\times10^{-1}h_nF_4 + 2.6184943407401290\times10^{-1}h_n^2y_n^{(2)} + 6.1135822014528315\times10^{-2}h_n^3y_n^{(3)}\\
&+1.0394945455378521\times10^{-2}h_n^4y_n^{(4)} + 1.3608093050715548\times10^{-3}h_n^5y_n^{(5)} + 1.4096277604309622\times10^{-4}h_n^6y_n^{(6)}\\
&+1.1629057440832313\times10^{-5}h_n^7y_n^{(7)} + 7.5075020483724947\times10^{-7}h_n^8y_n^{(8)} + 3.5748960089103080\times10^{-8}h_n^9y_n^{(9)}\\
&+1.0836803673146441\times10^{-9}h_n^{10}y_n^{(10)} + 2.3690611768201968\times10^{-11}h_n^{11}y_n^{(11)} + 1.5512519933084427\times10^{-12}h_n^{12}y_n^{(12)},\\
y_{n+1} &= y_n + 7.8906382044293033\times10^{-1}h_nf_n + 5.4081693776149497\times10^{-3}h_nF_2 + 7.1591103602263477\times10^{-3}h_nF_3\\
&+6.6710638106725520\times10^{-2}h_nF_4 + 1.3165826171250275\times10^{-1}h_nF_5 + 3.0814041998117864\times10^{-1}h_n^2y_n^{(2)}\\
&+7.9208815139396579\times10^{-2}h_n^3y_n^{(3)} + 1.5029800345764550\times10^{-2}h_n^4y_n^{(4)} + 2.2360349572021613\times10^{-3}h_n^5y_n^{(5)}
\end{aligned}
$$

$$+2.7012053626807768 \times 10^{-4} h_n^6 y_n^{(6)} + 2.7028355639019234 \times 10^{-5} h_n^7 y_n^{(7)} + 2.2582926828093552 \times 10^{-6} h_n^8 y_n^{(8)}$$
$$+1.5684180247762641 \times 10^{-7} h_n^9 y_n^{(9)} + 8.8394113567827505 \times 10^{-9} h_n^{10} y_n^{(10)} + 3.7806401270802183 \times 10^{-10} h_n^{11} y_n^{(11)}$$
$$+9.8978948106354690 \times 10^{-12} h_n^{12} y_n^{(12)}.$$

$\mathbf{CPHBT_{RK4}(13,5,15)}$ with $c_{cp} = 1.2262986831284113$, and abscissa vector
$\sigma = [0\ 7.8945008400500616 \times 10^{-1}\ 8.5541293327597789 \times 10^{-1}\ 8.7225362024039466 \times 10^{-1}\ 9.4551343353416728 \times 10^{-1}]^T$.

$$Y_2 = y_n + 7.8945008400500616 \times 10^{-1} h_n f_n + 3.1161571756775558 \times 10^{-1} h_n^2 y_n^{(2)} + 8.2001684803714975 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+1.6184059239211204 \times 10^{-2} h_n^4 y_n^{(4)} + 2.5553013851874564 \times 10^{-3} h_n^5 y_n^{(5)} + 3.3621381553239094 \times 10^{-4} h_n^6 y_n^{(6)}$$
$$+3.7917717845098518 \times 10^{-5} h_n^7 y_n^{(7)} + 3.7417681922613932 \times 10^{-6} h_n^8 y_n^{(8)} + 3.2821546818977962 \times 10^{-7} h_n^9 y_n^{(9)}$$
$$+2.5910972893416393 \times 10^{-8} h_n^{10} y_n^{(10)} + 1.8595836115780913 \times 10^{-9} h_n^{11} y_n^{(11)} + 1.2233736986455471 \times 10^{-10} h_n^{12} y_n^{(12)}$$
$$+9.3890971495703128 \times 10^{-13} h_n^{13} y_n^{(13)},$$
$$Y_3 = y_n + 6.9594920485873601 \times 10^{-1} h_n f_n + 1.5946372841724191 \times 10^{-1} h_n F_2 + 2.3997698941316317 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+5.4630663523690601 \times 10^{-2} h_n^3 y_n^{(3)} + 9.2333170846932849 \times 10^{-3} h_n^4 y_n^{(4)} + 1.2360156120588611 \times 10^{-3} h_n^5 y_n^{(5)}$$
$$+1.3667680414458034 \times 10^{-4} h_n^6 y_n^{(6)} + 1.2882799965251786 \times 10^{-5} h_n^7 y_n^{(7)} + 1.0637674020342335 \times 10^{-6} h_n^8 y_n^{(8)}$$
$$+7.9125388731318910 \times 10^{-8} h_n^9 y_n^{(9)} + 5.4704887879243722 \times 10^{-9} h_n^{10} y_n^{(10)} + 3.6364187405410597 \times 10^{-10} h_n^{11} y_n^{(11)}$$
$$+2.3923092271155322 \times 10^{-11} h_n^{12} y_n^{(12)} + 5.3568094170901934 \times 10^{-12} h_n^{13} y_n^{(13)},$$
$$Y_4 = y_n + 7.7322282225153660 \times 10^{-1} h_n f_n + 1.6197972325394164 \times 10^{-2} h_n F_2 + 8.2832825663463810 \times 10^{-2} h_n F_3$$
$$+2.9676942802592282 \times 10^{-1} h_n^2 y_n^{(2)} + 7.5252365958003165 \times 10^{-2} h_n^3 y_n^{(3)} + 1.4149479729502739 \times 10^{-2} h_n^4 y_n^{(4)}$$
$$+2.0974655627435429 \times 10^{-3} h_n^5 y_n^{(5)} + 2.5413399492630354 \times 10^{-4} h_n^6 y_n^{(6)} + 2.5700144448996192 \times 10^{-5} h_n^7 y_n^{(7)}$$
$$+2.1880956432618300 \times 10^{-6} h_n^8 y_n^{(8)} + 1.5584657448848323 \times 10^{-7} h_n^9 y_n^{(9)} + 8.9579945532981467 \times 10^{-9} h_n^{10} y_n^{(10)}$$
$$+3.6258140555709643 \times 10^{-10} h_n^{11} y_n^{(11)} + 2.4300547239940600 \times 10^{-12} h_n^{12} y_n^{(12)} + 8.8685076268800971 \times 10^{-12} h_n^{13} y_n^{(13)},$$
$$Y_5 = y_n + 7.4872523429822313 \times 10^{-1} h_n f_n + 3.4856149180045094 \times 10^{-3} h_n F_2 + 1.7824658977864404 \times 10^{-2} h_n F_3$$
$$+1.7547792534007522 \times 10^{-1} h_n F_4 + 2.7593740803590405 \times 10^{-1} h_n^2 y_n^{(2)} + 6.6519096675664482 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+1.1747004447371975 \times 10^{-2} h_n^4 y_n^{(4)} + 1.6109114993668023 \times 10^{-3} h_n^5 y_n^{(5)} + 1.7709264100931299 \times 10^{-4} h_n^6 y_n^{(6)}$$
$$+1.5835170250404947 \times 10^{-5} h_n^7 y_n^{(7)} + 1.1500300757481820 \times 10^{-6} h_n^8 y_n^{(8)} + 6.6284754947005383 \times 10^{-8} h_n^9 y_n^{(9)}$$
$$+2.8438595799926996 \times 10^{-9} h_n^{10} y_n^{(10)} + 7.8023293953868674 \times 10^{-11} h_n^{11} y_n^{(11)} + 1.6404768410107448 \times 10^{-12} h_n^{12} y_n^{(12)}$$
$$+1.9826744357737219 \times 10^{-12} h_n^{13} y_n^{(13)},$$
$$y_{n+1} = y_n + 8.0338280277017116 \times 10^{-1} h_n f_n + 4.5140069134017424 \times 10^{-3} h_n F_2 + 6.2564522381039706 \times 10^{-3} h_n F_3$$
$$+6.1592721638890638 \times 10^{-2} h_n F_4 + 1.2425401643943255 \times 10^{-1} h_n F_5 + 3.1987625055807828 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+8.3999051300493746 \times 10^{-2} h_n^3 y_n^{(3)} + 1.6326317613882917 \times 10^{-2} h_n^4 y_n^{(4)} + 2.4973376221763557 \times 10^{-3} h_n^5 y_n^{(5)}$$
$$+3.1184821185802373 \times 10^{-4} h_n^6 y_n^{(6)} + 3.2509535101654037 \times 10^{-5} h_n^7 y_n^{(7)} + 2.8644435849314006 \times 10^{-6} h_n^8 y_n^{(8)}$$
$$+2.1401417063684133 \times 10^{-7} h_n^9 y_n^{(9)} + 1.3452589547262228 \times 10^{-8} h_n^{10} y_n^{(10)} + 6.9289769665790717 \times 10^{-10} h_n^{11} y_n^{(11)}$$
$$+2.7301204968760894 \times 10^{-11} h_n^{12} y_n^{(12)} + 6.6287359467519055 \times 10^{-13} h_n^{13} y_n^{(13)}.$$

A.2 Six stages CPHBT$_{\mathbf{RK4}}(d,6,p)$ methods formulae

CPHBT$_{RK4}$**(2,6,5)** with $c_{cp} = 1.2976127867974319$, and abscissa vector $\sigma = [0 \ 2.4215115018574479 \times 10^{-1}$ $4.3091247352923828 \times 10^{-1}$ $7.1755217613698619 \times 10^{-1}$ $6.3789666822945823 \times 10^{-1}$ $9.1162795439326216 \times 10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 2.4215115018574479 \times 10^{-1} h_n f_n + 2.9318589768139568 \times 10^{-2} h_n^2 y_n^{(2)}, \\
Y_3 &= y_n + 9.5008189423197248 \times 10^{-2} h_n f_n + 3.3590428410604101 \times 10^{-1} h_n F_2 + 1.1503171172946225 \times 10^{-2} h_n^2 y_n^{(2)}, \\
Y_4 &= y_n + 5.2968877458690576 \times 10^{-2} h_n f_n + 1.8727304425722113 \times 10^{-1} h_n F_2 + 4.7731025442107444 \times 10^{-1} h_n F_3 \\
&\quad + 6.4132373003349891 \times 10^{-3} h_n^2 y_n^{(2)}, \\
Y_5 &= y_n + 2.1843373430033702 \times 10^{-1} h_n f_n + 1.6095619405664588 \times 10^{-1} h_n F_2 + 9.2533337192536869 \times 10^{-2} h_n F_3 \\
&\quad + 1.6597340267993835 \times 10^{-1} h_n F_4 + 5.5120066613837957 \times 10^{-3} h_n^2 y_n^{(2)}, \\
Y_6 &= y_n + 1.4414214514717352 \times 10^{-1} h_n f_n + 1.9641646471846821 \times 10^{-1} h_n F_2 + 4.7408116750919459 \times 10^{-2} h_n F_3 \\
&\quad + 8.5034071941227127 \times 10^{-2} h_n F_4 + 4.3862715583547374 \times 10^{-1} h_n F_5 + 6.7263572444967524 \times 10^{-3} h_n^2 y_n^{(2)}, \\
y_{n+1} &= y_n + 1.3801086263538129 \times 10^{-1} h_n f_n + 1.8655696308174072 \times 10^{-1} h_n F_2 + 2.1052813888604691 \times 10^{-1} h_n F_3 \\
&\quad + 4.1158361077699136 \times 10^{-2} h_n F_4 + 2.1230519068675860 \times 10^{-1} h_n F_5 + 6.3887148255630834 \times 10^{-3} h_n^2 y_n^{(2)}.
\end{aligned}
$$

CPHBT$_{\mathbf{RK4}}$**(3,6,6)** with $c_{cp} = 1.0078102279098364$, and abscissa vector $\sigma = [0 \ 3.2291182006524632 \times 10^{-1}$ $4.6763210226643609 \times 10^{-1}$ $6.5685844113763181 \times 10^{-1}$ $6.9432613831537349 \times 10^{-1}$ $9.2341217279277621 \times 10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 3.2291182006524632 \times 10^{-1} h_n f_n + 5.2136021768920369 \times 10^{-2} h_n^2 y_n^{(2)} + 5.6117792267867941 \times 10^{-3} h_n^3 y_n^{(3)}, \\
Y_3 &= y_n + 1.7641243269853876 \times 10^{-1} h_n f_n + 2.9121966956789730 \times 10^{-1} h_n F_2 + 1.5301617996088940 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 1.8605794145771419 \times 10^{-3} h_n^3 y_n^{(3)}, \\
Y_4 &= y_n + 1.7393198747646021 \times 10^{-1} h_n f_n + 1.0957613789392612 \times 10^{-1} h_n F_2 + 3.7335031576724548 \times 10^{-1} h_n F_3 \\
&\quad + 5.7574826797553089 \times 10^{-3} h_n^2 y_n^{(2)} + 7.0007327045361735 \times 10^{-4} h_n^3 y_n^{(3)}, \\
Y_5 &= y_n + 3.7936802472901365 \times 10^{-1} h_n f_n + 2.3395090535713092 \times 10^{-2} h_n F_2 + 7.9712285966478183 \times 10^{-2} h_n F_3 \\
&\quad + 2.1185073708416852 \times 10^{-1} h_n F_4 + 5.7057873130557581 \times 10^{-2} h_n^2 y_n^{(2)} + 1.4946938136978940 \times 10^{-4} h_n^3 y_n^{(3)}, \\
Y_6 &= y_n + 2.9895332904772254 \times 10^{-1} h_n f_n + 6.7783447326436388 \times 10^{-2} h_n F_2 + 3.4564112731177286 \times 10^{-2} h_n F_3 \\
&\quad + 9.1860779928356881 \times 10^{-2} h_n F_4 + 4.3025050375908325 \times 10^{-1} h_n F_5 + 2.9219955896308131 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 3.9079562914262118 \times 10^{-4} h_n^3 y_n^{(3)}, \\
y_{n+1} &= y_n + 2.2541962441999902 \times 10^{-1} h_n f_n + 1.3923603156968034 \times 10^{-1} h_n F_2 + 2.1224145797680849 \times 10^{-1} h_n F_3 \\
&\quad + 4.1742717395043566 \times 10^{-2} h_n F_4 + 1.9551135099759831 \times 10^{-1} h_n F_5 + 2.1005362324719831 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 8.3314370352840692 \times 10^{-4} h_n^3 y_n^{(3)}.
\end{aligned}
$$

CPHBT$_{\mathbf{RK4}}$**(4,6,7)** with $c_{cp} = 1.2976127867974319$, and abscissa vector $\sigma = [0 \ 5.8175020127045940 \times 10^{-1}$ $6.7572512798376727 \times 10^{-1}$ $6.7629581434464170 \times 10^{-1}$ $8.0159352160890140 \times 10^{-1}$ $1 \ ]^T$.

$$
Y_2 = y_n + 5.8175020127045940 \times 10^{-1} h_n f_n + 1.6921664833910996 \times 10^{-1} h_n^2 y_n^{(2)} + 3.2813939743196588 \times 10^{-2} h_n^3 y_n^{(3)}
$$

$$+4.7723790125203357\times10^{-3}h_n^4y_n^{(4)},$$

$$Y_3=y_n+4.5301985728846905\times10^{-1}h_nf_n+2.2270527069529814\times10^{-1}h_nF_2+9.8743388243357461\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.3737743768911162\times10^{-2}h_n^3y_n^{(3)}+1.3791469365360038\times10^{-3}h_n^4y_n^{(4)},$$

$$Y_4=y_n+4.9709212539320363\times10^{-1}h_nf_n+4.0176733493457517\times10^{-2}h_nF_2+1.3902695545798058\times10^{-1}h_nF_3$$
$$+1.1137118418379514\times10^{-1}h_n^2y_n^{(2)}+1.3014846925126651\times10^{-2}h_n^3y_n^{(3)}+2.4880245871400700\times10^{-4}h_n^4y_n^{(4)},$$

$$Y_5=y_n+4.6430377340366097\times10^{-1}h_nf_n+1.4266679898329707\times10^{-2}h_nF_2+4.9368201401443879\times10^{-2}h_nF_3$$
$$+2.7365486690546686\times10^{-1}h_nF_4+9.4545467766839364\times10^{-2}h_n^2y_n^{(2)}+9.5776585998835016\times10^{-3}h_n^3y_n^{(3)}$$
$$+8.8349269035725003\times10^{-5}h_n^4y_n^{(4)},$$

$$Y_6=y_n+4.4620004854413337\times10^{-1}h_nf_n+7.1311781552299914\times10^{-3}h_nF_2+2.4676620061979952\times10^{-2}h_nF_3$$
$$+1.3678596722254405\times10^{-1}h_nF_4+3.8520618601611267\times10^{-1}h_nF_5+7.7890263135432028\times10^{-2}h_n^2y_n^{(2)}$$
$$+4.7873780215476032\times10^{-3}h_n^3y_n^{(3)}+4.4161247176497030\times10^{-5}h_n^4y_n^{(4)},$$

$$y_{n+1}=y_n+4.4397514115499548\times10^{-1}h_nf_n+7.5590360796050110\times10^{-2}h_nF_2+3.8064580818115215\times10^{-2}h_nF_3$$
$$+2.1099731207317030\times10^{-1}h_nF_4+1.5687520953351156\times10^{-1}h_nF_5+8.7359952367507501\times10^{-2}h_n^2y_n^{(2)}$$
$$+9.2837831189798437\times10^{-3}h_n^3y_n^{(3)}+4.6810842957626948\times10^{-4}h_n^4y_n^{(4)}.$$

CPHBT$_{\mathbf{RK4}}$(5,6,8) with $c_{cp}=1.1225568710407938$, and abscissa vector $\sigma=[0\ 5.9208218739488050\times10^{-1}\ 7.1901190437089968\times10^{-1}\ 7.0621996522392172\times10^{-1}\ 8.2709525423172647\times10^{-1}\ 9.9491542708620018\times10^{-1}]^T$.

$$Y_2=y_n+5.9208218739488050\times10^{-1}h_nf_n+1.7528065831515316\times10^{-1}h_n^2y_n^{(2)}+3.4593518527750174\times10^{-2}h_n^3y_n^{(3)}$$
$$+5.1205515298989122\times10^{-3}h_n^4y_n^{(4)}+6.0635747009814994\times10^{-4}h_n^5y_n^{(5)},$$

$$Y_3=y_n+4.4296705227179839\times10^{-1}h_nf_n+2.7604485209910123\times10^{-1}h_nF_2+9.5047819463601760\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.3566913531582599\times10^{-2}h_n^3y_n^{(3)}+1.5867362585522914\times10^{-3}h_n^4y_n^{(4)}+1.8789565495598578\times10^{-4}h_n^5y_n^{(5)},$$

$$Y_4=y_n+5.5795991165715542\times10^{-1}h_nf_n+3.5073730263139015\times10^{-2}h_nF_2+1.1318632330362730\times10^{-1}h_nF_3$$
$$+1.4722447483885984\times10^{-1}h_n^2y_n^{(2)}+2.3298966273411572\times10^{-2}h_n^3y_n^{(3)}+2.1390391016160716\times10^{-3}h_n^4y_n^{(4)}$$
$$+2.3873662085468692\times10^{-5}h_n^5y_n^{(5)},$$

$$Y_5=y_n+5.1377386222119970\times10^{-1}h_nf_n+1.0576002609036548\times10^{-2}h_nF_2+3.4129784359564028\times10^{-2}h_nF_3$$
$$+2.6861560504192616\times10^{-1}h_nF_4+1.2153999252821983\times10^{-1}h_n^2y_n^{(2)}+1.6639281461178327\times10^{-2}h_n^3y_n^{(3)}$$
$$+1.2498087514679335\times10^{-3}h_n^4y_n^{(4)}+7.1987755676071587\times10^{-6}h_n^5y_n^{(5)},$$

$$Y_6=y_n+5.1995897059784735\times10^{-1}h_nf_n+4.1715419032875940\times10^{-3}h_nF_2+1.3461969599408076\times10^{-2}h_nF_3$$
$$+1.0595130256039248\times10^{-1}h_nF_4+3.5137164242526475\times10^{-1}h_nF_5+1.1733639834272332\times10^{-1}h_n^2y_n^{(2)}$$
$$+1.3320585234133830\times10^{-2}h_n^3y_n^{(3)}+4.9296787931861323\times10^{-4}h_n^4y_n^{(4)}+2.8394465321876344\times10^{-6}h_n^5y_n^{(5)},$$

$$y_{n+1}=y_n+4.9097666002944390\times10^{-1}h_nf_n+7.4072713392728778\times10^{-2}h_nF_2+2.5025183291748233\times10^{-2}h_nF_3$$
$$+1.9695860601929985\times10^{-1}h_nF_4+1.4587115075975474\times10^{-1}h_nF_5+1.1164949111933453\times10^{-1}h_n^2y_n^{(2)}$$
$$+1.4996390713219837\times10^{-2}h_n^3y_n^{(3)}+1.2229106818657672\times10^{-3}h_n^4y_n^{(4)}+5.0419128961382324\times10^{-5}h_n^5y_n^{(5)}.$$

$\mathbf{CPHBT_{RK4}(6,6,9)}$ with $c_{cp} = 1.0677538948928231$, and abscissa vector $\sigma = [0\ 8.0726307132940789\times10^{-1}$ $6.2830676593582724\times10^{-1}$ $7.5250397719891582\times10^{-1}$ $8.5338851860941456\times10^{-1}$ $9.6977571275092012\times10^{-1}\ ]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 8.0726307132940789\times10^{-1}h_nf_n + 3.2583683316609435\times10^{-1}h_n^2y_n^{(2)} + 8.7678680897969749\times10^{-2}h_n^3y_n^{(3)} \\
&\quad +1.7694940307951539\times10^{-2}h_n^4y_n^{(4)} + 2.8568943719974992\times10^{-3}h_n^5y_n^{(5)} + 3.8437755420040018\times10^{-4}h_n^6y_n^{(6)}, \\
Y_3 &= y_n + 6.0215464503282234\times10^{-1}h_nf_n + 2.6152120903004880\times10^{-2}h_nF_2 + 1.7627305461843149\times10^{-1}h_n^2y_n^{(2)} \\
&\quad +3.2818055753360324\times10^{-2}h_n^3y_n^{(3)} + 4.2004695763476979\times10^{-3}h_n^4y_n^{(4)} + 3.5321587753450615\times10^{-4}h_n^5y_n^{(5)} \\
&\quad +1.0733369952717966\times10^{-5}h_n^6y_n^{(6)}, \\
Y_4 &= y_n + 4.7185216983821010\times10^{-1}h_nf_n + 7.6240354091905898\times10^{-3}h_nF_2 + 2.7302777195151523\times10^{-1}h_nF_3 \\
&\quad +1.0543131920422481\times10^{-1}h_n^2y_n^{(2)} + 1.4643402080623579\times10^{-2}h_n^3y_n^{(3)} + 1.4052741303366180\times10^{-3}h_n^4y_n^{(4)} \\
&\quad +1.0297177683596529\times10^{-4}h_n^5y_n^{(5)} + 7.6204106310126564\times10^{-6}h_n^6y_n^{(6)}, \\
Y_5 &= y_n + 5.4758871848532087\times10^{-1}h_nf_n + 1.9154073700206448\times10^{-3}h_nF_2 + 6.8593517546604393\times10^{-2}h_nF_3 \\
&\quad +2.3529087520746864\times10^{-1}h_nF_4 + 1.4243465364471233\times10^{-1}h_n^2y_n^{(2)} + 2.2801565971915409\times10^{-2}h_n^3y_n^{(3)} \\
&\quad +2.3854694120173945\times10^{-3}h_n^4y_n^{(4)} + 1.4892084080236166\times10^{-4}h_n^5y_n^{(5)} + 1.9144967070365330\times10^{-6}h_n^6y_n^{(6)}, \\
Y_6 &= y_n + 6.1703688111968735\times10^{-1}h_nf_n + 5.4384124236589372\times10^{-4}h_nF_2 + 1.9475744107839574\times10^{-2}h_nF_3 \\
&\quad +6.6806092475674789\times10^{-2}h_nF_4 + 2.6591315380535258\times10^{-1}h_nF_5 + 1.8035761908078810\times10^{-1}h_n^2y_n^{(2)} \\
&\quad +3.2241826798008771\times10^{-2}h_n^3y_n^{(3)} + 3.7116650428930804\times10^{-3}h_n^4y_n^{(4)} + 2.4273425769940520\times10^{-4}h_n^5y_n^{(5)} \\
&\quad +5.4358267800177604\times10^{-7}h_n^6y_n^{(6)}, \\
y_{n+1} &= y_n + 5.3005588964943273\times10^{-1}h_nf_n + 2.1860901690953570\times10^{-3}h_nF_2 + 7.8287061394509277\times10^{-2}h_nF_3 \\
&\quad +2.6854186593390089\times10^{-1}h_nF_4 + 2.6742407320733554\times10^{-2}h_nF_5 + 1.3280651415556025\times10^{-1}h_n^2y_n^{(2)} \\
&\quad +2.0441621260746994\times10^{-2}h_n^3y_n^{(3)} + 2.0799858047412328\times10^{-3}h_n^4y_n^{(4)} + 1.3637746665177018\times10^{-4}h_n^5y_n^{(5)} \\
&\quad +4.6867084951139577\times10^{-6}h_n^6y_n^{(6)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(7,6,10)}$ with $c_{cp} = 1.0372974566873341$, and abscissa vector $\sigma = [0\ 8.2577571013225815\times10^{-1}$ $6.6492858252664744\times10^{-1}$ $7.7518460288634661\times10^{-1}$ $8.6833765675039798\times10^{-1}$ $9.7245357209238958\times10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 8.2577571013225815\times10^{-1}h_nf_n + 3.4095276172221639\times10^{-1}h_n^2y_n^{(2)} + 9.3850169644238646\times10^{-2}h_n^3y_n^{(3)} \\
&\quad +1.9374797621000809\times10^{-2}h_n^4y_n^{(4)} + 3.1998474528301011\times10^{-3}h_n^5y_n^{(5)} + 4.4039271711260439\times10^{-4}h_n^6y_n^{(6)} \\
&\quad +5.1952229815818279\times10^{-5}h_n^7y_n^{(7)}, \\
Y_3 &= y_n + 6.4186009478336448\times10^{-1}h_nf_n + 2.3068487743282944\times10^{-2}h_nF_2 + 2.0201561308255761\times10^{-1}h_n^2y_n^{(2)} \\
&\quad +4.1132216628267090\times10^{-2}h_n^3y_n^{(3)} + 5.9799749477806637\times10^{-3}h_n^4y_n^{(4)} + 6.3621558608725737\times10^{-4}h_n^5y_n^{(5)} \\
&\quad +4.6222016613351427\times10^{-5}h_n^6y_n^{(6)} + 1.2431588634386864\times10^{-6}h_n^7y_n^{(7)}, \\
Y_4 &= y_n + 5.1966918829608832\times10^{-1}h_nf_n + 5.9713118050173956\times10^{-3}h_nF_2 + 2.4954410278524083\times10^{-1}h_nF_3 \\
&\quad +1.2959561348694473\times10^{-1}h_n^2y_n^{(2)} + 2.0434776116002699\times10^{-2}h_n^3y_n^{(3)} + 2.2581519014919457\times10^{-3}h_n^4y_n^{(4)} \\
&\quad +1.8440360636623072\times10^{-4}h_n^5y_n^{(5)} + 1.1964636630130731\times10^{-5}h_n^6y_n^{(6)} + 7.8937035355585419\times10^{-7}h_n^7y_n^{(7)}, \\
Y_5 &= y_n + 5.9025551353408701\times10^{-1}h_nf_n + 1.3615701504434488\times10^{-3}h_nF_2 + 5.6900696641896030\times10^{-2}h_nF_3 \\
&\quad +2.1981987642397149\times10^{-1}h_nF_4 + 1.6764490833238621\times10^{-1}h_n^2y_n^{(2)} + 3.0033493926366672\times10^{-2}h_n^3y_n^{(3)}
\end{aligned}
$$

$+3.7070628800189459\times10^{-3}h_n^4y_n^{(4)}+3.1682343145137337\times10^{-4}h_n^5y_n^{(5)}+1.6640646719525480\times10^{-5}h_n^6y_n^{(6)}$
$+1.7999112190777956\times10^{-7}h_n^7y_n^{(7)},$

$Y_6=y_n+6.5959195228523460\times10^{-1}h_nf_n+3.4294678829149643\times10^{-4}h_nF_2+1.4331917572174004\times10^{-2}h_nF_3$
$+5.5367342327302151\times10^{-2}h_nF_4+2.4281941311938737\times10^{-1}h_nF_5+2.0925092469737130\times10^{-1}h_n^2y_n^{(2)}$
$+4.1804562901158367\times10^{-2}h_n^3y_n^{(3)}+5.7318318926584504\times10^{-3}h_n^4y_n^{(4)}+5.3856577059708036\times10^{-4}h_n^5y_n^{(5)}$
$+2.9849091421330360\times10^{-5}h_n^6y_n^{(6)}+4.5335436561352374\times10^{-8}h_n^7y_n^{(7)},$

$y_{n+1}=y_n+5.7301509266165696\times10^{-1}h_nf_n+1.5669421778113602\times10^{-3}h_nF_2+6.5483296241495959\times10^{-2}h_nF_3$
$+2.5297634189316814\times10^{-1}h_nF_4+2.1519888656418292\times10^{-2}h_nF_5+1.5728953245544339\times10^{-1}h_n^2y_n^{(2)}$
$+2.7136973568336459\times10^{-2}h_n^3y_n^{(3)}+3.2275735374709693\times10^{-3}h_n^4y_n^{(4)}+2.7006268212665887\times10^{-4}h_n^5y_n^{(5)}$
$+1.5135902842873110\times10^{-5}h_n^6y_n^{(6)}+4.5646672801272238\times10^{-7}h_n^7y_n^{(7)}.$

CPHBT$_{\mathbf{RK4}}$(8,6,11) with $c_{\rm cp}$= 6.5356080481107670 $\times10^{-1}$, and abscissa vector $\sigma=[0\ 6.5087863532715440\times10^{-1}$ $7.0654875715362375\times10^{-1}$ $9.1376178971858601\times10^{-1}$ $8.3344995221094431\times10^{-1}$ $9.4535874251759922\times10^{-1}]^T$.

$Y_2=y_n+6.5087863532715440\times10^{-1}h_nf_n+2.1182149896266930\times10^{-1}h_n^2y_n^{(2)}+4.5956696059258097\times10^{-2}h_n^3y_n^{(3)}$
$+7.4780579037986702\times10^{-3}h_n^4y_n^{(4)}+9.7346162466438175\times10^{-4}h_n^5y_n^{(5)}+1.0560089563415098\times10^{-4}h_n^6y_n^{(6)}$
$+9.8190524056687467\times10^{-6}h_n^7y_n^{(7)}+-3.8116902287253406\times10^{-15}h_n^8y_n^{(8)},$

$Y_3=y_n+5.5085339965904634\times10^{-1}h_nf_n+1.5569535749457744\times10^{-1}h_nF_2+1.4826679130482112\times10^{-1}h_n^2y_n^{(2)}$
$+2.5806545148937987\times10^{-2}h_n^3y_n^{(3)}+3.2285794663505047\times10^{-3}h_n^4y_n^{(4)}+3.0303664563469384\times10^{-4}h_n^5y_n^{(5)}$
$+2.1227228536452358\times10^{-5}h_n^6y_n^{(6)}+9.9915125481966478\times10^{-7}h_n^7y_n^{(7)}+1.1559045373283776\times10^{-8}h_n^8y_n^{(8)},$

$Y_4=y_n+4.2735262500270949\times10^{-1}h_nf_n+4.4923939710489399\times10^{-2}h_nF_2+4.4148522500538717\times10^{-1}h_nF_3$
$+7.6309434573329699\times10^{-2}h_n^2y_n^{(2)}+7.4461544457900507\times10^{-3}h_n^3y_n^{(3)}+1.0305197693432373\times10^{-3}h_n^4y_n^{(4)}$
$+3.8839689249606062\times10^{-4}h_n^5y_n^{(5)}+1.1693420065543834\times10^{-4}h_n^6y_n^{(6)}+2.4507406128613122\times10^{-5}h_n^7y_n^{(7)}$
$+3.9134087348729651\times10^{-6}h_n^8y_n^{(8)},$

$Y_5=y_n+7.6634878604078260\times10^{-1}h_nf_n+1.4948999079956653\times10^{-3}h_nF_2+1.4690969369559067\times10^{-2}h_nF_3$
$+5.0915296892607043\times10^{-2}h_nF_4+2.8944207404606009\times10^{-1}h_n^2y_n^{(2)}+7.1251382214370801\times10^{-2}h_n^3y_n^{(3)}$
$+1.2698454883163554\times10^{-2}h_n^4y_n^{(4)}+1.7085935086651174\times10^{-3}h_n^5y_n^{(5)}+1.7222370110735698\times10^{-4}h_n^6y_n^{(6)}$
$+1.1567646237306843\times10^{-5}h_n^7y_n^{(7)}+1.3022353203800895\times10^{-7}h_n^8y_n^{(8)},$

$Y_6=y_n+6.6681279687701434\times10^{-1}h_nf_n+2.6070833006391805\times10^{-4}h_nF_2+2.5620833013539905\times10^{-3}h_nF_3$
$+8.8795523746943023\times10^{-3}h_nF_4+2.6684360163447274\times10^{-1}h_nF_5+2.1435706707299398\times10^{-1}h_n^2y_n^{(2)}$
$+4.3729946547701411\times10^{-2}h_n^3y_n^{(3)}+6.2396386988725150\times10^{-3}h_n^4y_n^{(4)}+6.4077743813611347\times10^{-4}h_n^5y_n^{(5)}$
$+4.5965559989036774\times10^{-5}h_n^6y_n^{(6)}+2.0173803685982059\times10^{-6}h_n^7y_n^{(7)}+4.6822093160349545\times10^{-8}h_n^8y_n^{(8)},$

$y_{n+1}=y_n+5.3151195913554916\times10^{-1}h_nf_n+2.8854850608260395\times10^{-2}h_nF_2+2.7877293318834395\times10^{-1}h_nF_3$
$+7.9098735495526445\times10^{-4}h_nF_4+2.3770332752933908\times10^{-2}h_nF_5+1.3486677637220160\times10^{-1}h_n^2y_n^{(2)}$
$+2.1479796786362360\times10^{-2}h_n^3y_n^{(3)}+2.3659105517097076\times10^{-3}h_n^4y_n^{(4)}+1.8599838398004252\times10^{-4}h_n^5y_n^{(5)}$
$+1.0266109091019705\times10^{-5}h_n^6y_n^{(6)}+3.6473009496940092\times10^{-7}h_n^7y_n^{(7)}+6.2751677741251466\times10^{-9}h_n^8y_n^{(8)}.$

$\mathbf{CPHBT_{RK4}}(9,6,12)$ with $c_{\rm cp}$= 9.9163130315966042 $\times 10^{-1}$, and abscissa vector $\sigma = [0\ 8.5568256645013996 \times 10^{-1}$ $7.1829693393055971 \times 10^{-1}$ $8.0835544845753426 \times 10^{-1}$ $8.8980234528386681 \times 10^{-1}$ $9.7567281938112871 \times 10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 8.5568256645013996 \times 10^{-1} h_n f_n + 3.6609632726334906 \times 10^{-1} h_n^2 y_n^{(2)} + 1.0442074829355763 \times 10^{-1} h_n^3 y_n^{(3)} \\
&\quad + 2.2337753472618866 \times 10^{-2} h_n^4 y_n^{(4)} + 3.8228052440362082 \times 10^{-3} h_n^5 y_n^{(5)} + 5.4518463370932600 \times 10^{-4} h_n^6 y_n^{(6)} \\
&\quad + 6.6643569508796518 \times 10^{-5} h_n^7 y_n^{(7)} + 7.1282175743356624 \times 10^{-6} h_n^8 y_n^{(8)} + 6.7772127869139232 \times 10^{-7} h_n^9 y_n^{(9)}, \\
Y_3 &= y_n + 7.0032387510619154 \times 10^{-1} h_n f_n + 1.7973058824368105 \times 10^{-2} h_n F_2 + 2.4259600954522675 \times 10^{-1} h_n^2 y_n^{(2)} \\
&\quad + 5.5187737782493251 \times 10^{-2} h_n^3 y_n^{(3)} + 9.2151107182336153 \times 10^{-3} h_n^4 y_n^{(4)} + 1.1919736246642830 \times 10^{-3} h_n^5 y_n^{(5)} \\
&\quad + 1.2205437013104496 \times 10^{-4} h_n^6 y_n^{(6)} + 9.7761743593585443 \times 10^{-6} h_n^7 y_n^{(7)} + 5.5977694224758458 \times 10^{-7} h_n^8 y_n^{(8)} \\
&\quad + 1.2156801810285854 \times 10^{-8} h_n^9 y_n^{(9)}, \\
Y_4 &= y_n + 5.9253804964276735 \times 10^{-1} h_n f_n + 3.7790846964182783 \times 10^{-3} h_n F_2 + 2.1203831411834850 \times 10^{-1} h_n F_3 \\
&\quad + 1.7117909772661233 \times 10^{-1} h_n^2 y_n^{(2)} + 3.1950954904935901 \times 10^{-2} h_n^3 y_n^{(3)} + 4.2991986294050764 \times 10^{-3} h_n^4 y_n^{(4)} \\
&\quad + 4.3995842070519627 \times 10^{-4} h_n^5 y_n^{(5)} + 3.5189487487470358 \times 10^{-5} h_n^6 y_n^{(6)} + 2.2401550747435463 \times 10^{-6} h_n^7 y_n^{(7)} \\
&\quad + 1.1920417110245276 \times 10^{-7} h_n^8 y_n^{(8)} + 6.5142505217891534 \times 10^{-9} h_n^9 y_n^{(9)}, \\
Y_5 &= y_n + 6.5225547192463695 \times 10^{-1} h_n f_n + 7.3327692956782933 \times 10^{-4} h_n F_2 + 4.1142711343481492 \times 10^{-2} h_n F_3 \\
&\quad + 1.9567088508618058 \times 10^{-1} h_n F_4 + 2.0752234507578865 \times 10^{-1} h_n^2 y_n^{(2)} + 4.2604870773856586 \times 10^{-2} h_n^3 y_n^{(3)} \\
&\quad + 6.2756226953662323 \times 10^{-3} h_n^4 y_n^{(4)} + 6.9432484726877225 \times 10^{-4} h_n^5 y_n^{(5)} + 5.8166925177690642 \times 10^{-5} h_n^6 y_n^{(6)} \\
&\quad + 3.5517957695381351 \times 10^{-6} h_n^7 y_n^{(7)} + 1.3567209577595762 \times 10^{-7} h_n^8 y_n^{(8)} + 1.2639876550905203 \times 10^{-9} h_n^9 y_n^{(9)}, \\
Y_6 &= y_n + 7.2075952122256659 \times 10^{-1} h_n f_n + 1.5003061651449836 \times 10^{-4} h_n F_2 + 8.4172968211253581 \times 10^{-3} h_n F_3 \\
&\quad + 4.0031846060581737 \times 10^{-2} h_n F_4 + 2.0631412466034044 \times 10^{-1} h_n F_5 + 2.5385547529513375 \times 10^{-1} h_n^2 y_n^{(2)} \\
&\quad + 5.7816607679521070 \times 10^{-2} h_n^3 y_n^{(3)} + 9.4732176422683261 \times 10^{-3} h_n^4 y_n^{(4)} + 1.1701172028993881 \times 10^{-3} h_n^5 y_n^{(5)} \\
&\quad + 1.0997733968369118 \times 10^{-4} h_n^6 y_n^{(6)} + 7.5740152977001563 \times 10^{-6} h_n^7 y_n^{(7)} + 3.2206534412851753 \times 10^{-7} h_n^8 y_n^{(8)} \\
&\quad + 2.5860379801980058 \times 10^{-10} h_n^9 y_n^{(9)}, \\
y_{n+1} &= y_n + 6.3769782436600186 \times 10^{-1} h_n f_n + 8.4507585951567590 \times 10^{-4} h_n F_2 + 4.7415786695281098 \times 10^{-2} h_n F_3 \\
&\quad + 2.2550450654635593 \times 10^{-1} h_n F_4 + 1.5037770460171211 \times 10^{-2} h_n F_5 + 1.9783881742796200 \times 10^{-1} h_n^2 y_n^{(2)} \\
&\quad + 3.9512215213133563 \times 10^{-2} h_n^3 y_n^{(3)} + 5.6542687574121630 \times 10^{-3} h_n^4 y_n^{(4)} + 6.0866317220988314 \times 10^{-4} h_n^5 y_n^{(5)} \\
&\quad + 5.0062835142051295 \times 10^{-5} h_n^6 y_n^{(6)} + 3.0966865898048209 \times 10^{-6} h_n^7 y_n^{(7)} + 1.3443315671020195 \times 10^{-7} h_n^8 y_n^{(8)} \\
&\quad + 3.2502867248596942 \times 10^{-9} h_n^9 y_n^{(9)}.
\end{aligned}
$$

$\mathbf{CPHBT_{RK4}(10,6,13)}$ with $c_{\rm cp}$= 9.7891787765377547 $\times 10^{-1}$, and abscissa vector $\sigma = [0\ 8.7216793659067693 \times 10^{-1}$ $7.3455764850691607 \times 10^{-1}$ $8.1757120453712473 \times 10^{-1}$ $8.9651910639916765 \times 10^{-1}$ $9.7496976750604369 \times 10^{-1}]^T$.

$$
\begin{aligned}
Y_2 &= y_n + 8.7216793659067693 \times 10^{-1} h_n f_n + 3.8033845480841949 \times 10^{-1} h_n^2 y_n^{(2)} + 1.1057300177878189 \times 10^{-1} h_n^3 y_n^{(3)} \\
&\quad + 2.4109556701009359 \times 10^{-2} h_n^4 y_n^{(4)} + 4.2055164640070510 \times 10^{-3} h_n^5 y_n^{(5)} + 6.1131943611852479 \times 10^{-4} h_n^6 y_n^{(6)} \\
&\quad + 7.6167601599609963 \times 10^{-5} h_n^7 y_n^{(7)} + 8.3038674902740692 \times 10^{-6} h_n^8 y_n^{(8)} + 8.0470744163497083 \times 10^{-7} h_n^9 y_n^{(9)} \\
&\quad + 7.0184002892993493 \times 10^{-8} h_n^{10} y_n^{(10)}, \\
Y_3 &= y_n + 7.2012730214911580 \times 10^{-1} h_n f_n + 1.4430346357800395 \times 10^{-2} h_n F_2 + 2.5720178408283345 \times 10^{-1} h_n^2 y_n^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&+6.0569734092327014\times10^{-2}h_n^3y_n^{(3)}+1.0535273068813885\times10^{-2}h_n^4y_n^{(4)}+1.4342568517131792\times10^{-3}h_n^5y_n^{(5)}\\
&+1.5749689808795212\times10^{-4}h_n^6y_n^{(6)}+1.4073976601458831\times10^{-5}h_n^7y_n^{(7)}+1.0031357604274832\times10^{-6}h_n^8y_n^{(8)}\\
&+5.1753607888684203\times10^{-8}h_n^9y_n^{(9)}+9.9142792981533817\times10^{-10}h_n^{10}y_n^{(10)},\\
Y_4&=y_n+6.1666636939405672\times10^{-1}h_nf_n+2.7984750143857798\times10^{-3}h_nF_2+1.9810636012868230\times10^{-1}h_nF_3\\
&+1.8625005501485570\times10^{-1}h_n^2y_n^{(2)}+3.6569540601767869\times10^{-2}h_n^3y_n^{(3)}+5.2202276081466138\times10^{-3}h_n^4y_n^{(4)}\\
&+5.7333986969055597\times10^{-4}h_n^5y_n^{(5)}+4.9955600014877659\times10^{-5}h_n^6y_n^{(6)}+3.5105648832820824\times10^{-6}h_n^7y_n^{(7)}\\
&+2.0199739121807016\times10^{-7}h_n^8y_n^{(8)}+1.0036569808495538\times10^{-8}h_n^9y_n^{(9)}+5.2663391534795842\times10^{-10}h_n^{10}y_n^{(10)},\\
Y_5&=y_n+6.6879859693051713\times10^{-1}h_nf_n+5.2130954680464753\times10^{-4}h_nF_2+3.6903933852158038\times10^{-2}h_nF_3\\
&+1.9029526606968783\times10^{-1}h_nF_4+2.1873058782821597\times10^{-1}h_n^2y_n^{(2)}+4.6342355201246521\times10^{-2}h_n^3y_n^{(3)}\\
&+7.0893782334120270\times10^{-3}h_n^4y_n^{(4)}+8.2350328604301074\times10^{-4}h_n^5y_n^{(5)}+7.3925833067888403\times10^{-5}h_n^6y_n^{(6)}\\
&+5.0587192636392605\times10^{-6}h_n^7y_n^{(7)}+2.4688209624789656\times10^{-7}h_n^8y_n^{(8)}+6.9903818955404693\times10^{-9}h_n^9y_n^{(9)}\\
&+9.8103176312352046\times10^{-11}h_n^{10}y_n^{(10)},\\
Y_6&=y_n+7.4176380457262048\times10^{-1}h_nf_n+9.7315866934121088\times10^{-5}h_nF_2+6.8890706838519791\times10^{-3}h_nF_3\\
&+3.5523517465872519\times10^{-2}h_nF_4+1.9069605891676469\times10^{-1}h_nF_5+2.7013206313673471\times10^{-1}h_n^2y_n^{(2)}\\
&+6.4058587138924733\times10^{-2}h_n^3y_n^{(3)}+1.1045878291482814\times10^{-2}h_n^4y_n^{(4)}+1.4611266868308343\times10^{-3}h_n^5y_n^{(5)}\\
&+1.5174660155378573\times10^{-4}h_n^6y_n^{(6)}+1.2335402746447030\times10^{-5}h_n^7y_n^{(7)}+7.5037597758261154\times10^{-7}h_n^8y_n^{(8)}\\
&+2.8660479124037351\times10^{-8}h_n^9y_n^{(9)}+1.8313487083344824\times10^{-11}h_n^{10}y_n^{(10)},\\
y_{n+1}&=y_n+6.5942861217613780\times10^{-1}h_nf_n+5.8392170607958266\times10^{-4}h_nF_2+4.1336300376774014\times10^{-2}h_nF_3\\
&+2.1315077980703817\times10^{-1}h_nF_4+1.3450052318548556\times10^{-2}h_nF_5+2.1255576091346373\times10^{-1}h_n^2y_n^{(2)}\\
&+4.4405639039154392\times10^{-2}h_n^3y_n^{(3)}+6.7132711449553002\times10^{-3}h_n^4y_n^{(4)}+7.7509361264270515\times10^{-4}h_n^5y_n^{(5)}\\
&+7.0071732288657962\times10^{-5}h_n^6y_n^{(6)}+4.9751023352591012\times10^{-6}h_n^7y_n^{(7)}+2.7100069746079071\times10^{-7}h_n^8y_n^{(8)}\\
&+1.0517882730774634\times10^{-8}h_n^9y_n^{(9)}+2.3004926750783143\times10^{-10}h_n^{10}y_n^{(10)}.
\end{aligned}
$$

Appendix B

$\mathbf{CPHBT_{RK5}}(p-4, s, p)$ formulae

B.1 Eight stages $\mathbf{CPHBT_{RK5}}(p-4, 8, p)$ methods formulae

$\mathbf{CPHBT_{RK5}(2,8,6)}$ with $c_{cp} = 1.7887086633648770$, and abscissa vector $\sigma = [0\ 2.7374840363797043{\times}10^{-1}$ $4.5658640304921172{\times}10^{-1}$ $4.1283532228170239{\times}10^{-1}$ $4.8070171447096005{\times}10^{-1}$ $7.3916285851534802{\times}10^{-1}$ $8.6532176997823984{\times}10^{-1}$ $9.1643089653967069{\times}10^{-1}]^T$

$$Y_2 = y_n + 2.7374840363797043 \times 10^{-1} h_n f_n + 3.7469094247168593 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$Y_3 = y_n + 1.5070365407323477 \times 10^{-1} h_n f_n + 3.0588274897597695 \times 10^{-1} h_n F_2 + 2.0500657492140875 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$Y_4 = y_n + 2.0780487165097125 \times 10^{-1} h_n f_n + 7.2507797907122584 \times 10^{-2} h_n F_2 + 1.3252265272360861 \times 10^{-1} h_n F_3 + 4.8595664037269087 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_5 = y_n + 2.0134651993002337 \times 10^{-1} h_n f_n + 2.6509116516975090 \times 10^{-2} h_n F_2 + 4.8450767277317522 \times 10^{-2} h_n F_3 + 2.0439531074664413 \times 10^{-1} h_n F_4 + 1.7766752782009357 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_6 = y_n + 1.4313900660529966 \times 10^{-1} h_n f_n + 1.8845099736681054 \times 10^{-2} h_n F_2 + 3.4443227901431346 \times 10^{-2} h_n F_3 + 1.4530284380710670 \times 10^{-1} h_n F_4 + 3.9743268046482932 \times 10^{-1} h_n F_5 + 1.2630229602692438 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_7 = y_n + 1.9083046103368298 \times 10^{-1} h_n f_n + 1.3521350617302660 \times 10^{-1} h_n F_2 + 1.6347321865646934 \times 10^{-2} h_n F_3 + 6.8963108460088407 \times 10^{-2} h_n F_4 + 1.8862805482612396 \times 10^{-1} h_n F_5 + 2.6533931761967083 \times 10^{-1} h_n F_6 + 1.4639231496678199 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$Y_8 = y_n + 1.7812861593143697 \times 10^{-1} h_n f_n + 1.5786156799529541 \times 10^{-1} h_n F_2 + 1.7935856764543159 \times 10^{-1} h_n F_3 + 3.5742513219759790 \times 10^{-2} h_n F_4 + 9.7762985479441469 \times 10^{-2} h_n F_5 + 8.6121364308031828 \times 10^{-2} h_n F_6 + 1.8145528196027358 \times 10^{-1} h_n F_7 + 1.2390232299531368 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$y_{n+1} = y_n + 1.8075533857622461 \times 10^{-1} h_n f_n + 9.5820862600417209 \times 10^{-2} h_n F_2 + 1.1893847441760788 \times 10^{-1} h_n F_3$$

$$+1.0826361778870985 \times 10^{-1} h_n F_4 + 1.4395480408615213 \times 10^{-1} h_n F_5 + 1.4141603745982614 \times 10^{-1} h_n F_6$$
$$+5.1666554932071694 \times 10^{-2} h_n F_7 + 1.5918431013899040 \times 10^{-1} h_n F_8 + 1.0450037630755806 \times 10^{-2} h_n^2 y_n^{(2)}$$

$\mathbf{CPHBT_{RK5}(3,8,7)}$ with $c_{cp} = 1.4301700462580107$, and abscissa vector $\sigma = [0\ 4.1901165948273023{\times}10^{-1}\ 4.4757325832024575{\times}10^{-1}\ 4.5277129402054028{\times}10^{-1}\ 6.0594401398694331{\times}10^{-1}\ 7.5615644157974560{\times}10^{-1}\ 8.9787029959899645{\times}10^{-1}\ 9.0803080383130441{\times}10^{-1}]^T$

$$Y_2 = y_n + 4.1901165948273023 \times 10^{-1} h_n f_n + 8.7785385391235732 \times 10^{-2} h_n^2 y_n^{(2)} + 1.2261033337037569 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$Y_3 = y_n + 3.0569128968756548 \times 10^{-1} h_n f_n + 1.4188196863268027 \times 10^{-1} h_n F_2 + 4.0710711654244683 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$+2.4879517684760167 \times 10^{-3} h_n^3 y_n^{(3)}$$
$$Y_4 = y_n + 2.9965216785588289 \times 10^{-1} h_n f_n + 2.5829099290801412 \times 10^{-2} h_n F_2 + 1.2729002687385593 \times 10^{-1} h_n F_3$$
$$+3.4706616508130984 \times 10^{-2} h_n^2 y_n^{(2)} + 4.5292262207792885 \times 10^{-4} h_n^3 y_n^{(3)}$$
$$Y_5 = y_n + 2.4325155235107554 \times 10^{-1} h_n f_n + 1.0990984940026231 \times 10^{-2} h_n F_2 + 5.4165371879008183 \times 10^{-2} h_n F_3$$
$$+2.9753610481683335 \times 10^{-1} h_n F_4 + 2.0019944028470162 \times 10^{-2} h_n^2 y_n^{(2)} + 1.9273090641787306 \times 10^{-4} h_n^3 y_n^{(3)}$$
$$Y_6 = y_n + 2.9365213309221716 \times 10^{-1} h_n f_n + 4.7870137955370318 \times 10^{-3} h_n F_2 + 2.3591187126543709 \times 10^{-2} h_n F_3$$
$$+1.2958888090560947 \times 10^{-1} h_n F_4 + 3.0453722665983812 \times 10^{-1} h_n F_5 + 3.0115048157946720 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$+8.3942022747099970 \times 10^{-5} h_n^3 y_n^{(3)}$$
$$Y_7 = y_n + 3.7473634586941112 \times 10^{-1} h_n f_n + 4.6547631248905569 \times 10^{-2} h_n F_2 + 9.7181246971988992 \times 10^{-3} h_n F_3$$
$$+5.3382684697295089 \times 10^{-2} h_n F_4 + 1.2545069172417800 \times 10^{-1} h_n F_5 + 2.8803482136200770 \times 10^{-1} h_n F_6$$
$$+6.1246335999288588 \times 10^{-2} h_n^2 y_n^{(2)} + 4.7322051061649837 \times 10^{-3} h_n^3 y_n^{(3)}$$
$$Y_8 = y_n + 2.8695124981194142 \times 10^{-1} h_n f_n + 6.1487913773827806 \times 10^{-2} h_n F_2 + 2.1504415616235822 \times 10^{-2} h_n F_3$$
$$+1.1812602474343520 \times 10^{-1} h_n F_4 + 2.7759921777488938 \times 10^{-1} h_n F_5 + 4.1534676597778750 \times 10^{-2} h_n F_6$$
$$+1.0082730551319606 \times 10^{-1} h_n F_7 + 3.3240802553153995 \times 10^{-2} h_n^2 y_n^{(2)} + 1.6428972860892126 \times 10^{-3} h_n^3 y_n^{(3)}$$
$$y_{n+1} = y_n + 2.7682104631993659 \times 10^{-1} h_n f_n + 3.5675932299269984 \times 10^{-2} h_n F_2 + 9.5468361801232846 \times 10^{-2} h_n F_3$$
$$+1.1437308074564843 \times 10^{-1} h_n F_4 + 2.3746250685919018 \times 10^{-1} h_n F_5 + 1.1857151878401898 \times 10^{-2} h_n F_6$$
$$+2.8777262045268905 \times 10^{-2} h_n F_7 + 1.9956465805105117 \times 10^{-1} h_n F_8 + 3.0633482765549919 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$+1.3929472061346105 \times 10^{-3} h_n^3 y_n^{(3)}$$

$\mathbf{CPHBT_{RK5}(4,8,8)}$ with $c_{cp} = 1.2659158342655910$, and abscissa vector $\sigma = [0\ 4.6549546831062921{\times}10^{-1}\ 5.1034026450176229{\times}10^{-1}\ 5.1168465455700396{\times}10^{-1}\ 6.4457785369866316{\times}10^{-1}\ 7.7055098081628315{\times}10^{-1}\ 9.0748644687260571{\times}10^{-1}\ 9.2028905345425827{\times}10^{-1}]^T$

$$Y_2 = y_n + 4.6549546831062921 \times 10^{-1} h_n f_n + 1.0834301550886601 \times 10^{-1} h_n^2 y_n^{(2)} + 1.6811060914161780 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+1.9563681682590629 \times 10^{-3} h_n^4 y_n^{(4)}$$
$$Y_3 = y_n + 3.6380282758188648 \times 10^{-1} h_n f_n + 1.4653743691987589 \times 10^{-1} h_n F_2 + 6.2011079961807464 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$+6.2764731280630180 \times 10^{-3} h_n^3 y_n^{(3)} + 3.6291424141126434 \times 10^{-4} h_n^4 y_n^{(4)}$$
$$Y_4 = y_n + 3.8008660937586147 \times 10^{-1} h_n f_n + 2.0592060949110085 \times 10^{-2} h_n F_2 + 1.1100598423203252 \times 10^{-1} h_n F_3$$
$$+6.4674258445319321 \times 10^{-2} h_n^2 y_n^{(2)} + 5.6417097613508565 \times 10^{-3} h_n^3 y_n^{(3)} + 5.0998245469019597 \times 10^{-5} h_n^4 y_n^{(4)}$$

$$Y_5 = y_n + 3.2115506150902295\times10^{-1}h_nf_n + 7.2269700862404749\times10^{-3}h_nF_2 + 3.8958554436157630\times10^{-2}h_nF_3$$
$$+2.7723726766724216\times10^{-1}h_nF_4 + 4.2636008402430772\times10^{-2}h_n^2y_n^{(2)} + 2.4853235196206987\times10^{-3}h_n^3y_n^{(3)}$$
$$+1.7898295627819029\times10^{-5}h_n^4y_n^{(4)}$$
$$Y_6 = y_n + 3.8530610883393435\times10^{-1}h_nf_n + 2.5006658695534340\times10^{-3}h_nF_2 + 1.3480383372227078\times10^{-2}h_nF_3$$
$$+9.5929243455099109\times10^{-2}h_nF_4 + 2.7333457928546917\times10^{-1}h_nF_5 + 6.3559837715972345\times10^{-2}h_n^2y_n^{(2)}$$
$$+4.8851319371224404\times10^{-3}h_n^3y_n^{(3)} + 6.1931428791837387\times10^{-6}h_n^4y_n^{(4)}$$
$$Y_7 = y_n + 4.7060010974195571\times10^{-1}h_nf_n + 7.0904412156906488\times10^{-3}h_nF_2 + 4.9406394610072499\times10^{-3}h_nF_3$$
$$+3.5158629546919035\times10^{-2}h_nF_4 + 1.0017861057183432\times10^{-1}h_nF_5 + 2.8951801633519869\times10^{-1}h_nF_6$$
$$+1.0029241367077595\times10^{-1}h_n^2y_n^{(2)} + 1.1781452863415523\times10^{-2}h_n^3y_n^{(3)} + 6.9695929781549127\times10^{-4}h_n^4y_n^{(4)}$$
$$Y_8 = y_n + 3.6025096680073893\times10^{-1}h_nf_n + 4.5922132089482856\times10^{-2}h_nF_2 + 1.3662872530663393\times10^{-2}h_nF_3$$
$$+9.7227874686875840\times10^{-2}h_nF_4 + 2.7703480179345696\times10^{-1}h_nF_5 + 3.3845068498470743\times10^{-2}h_nF_6$$
$$+9.2345337054569654\times10^{-2}h_nF_7 + 5.6914710641376057\times10^{-2}h_n^2y_n^{(2)} + 4.7972871137271361\times10^{-3}h_n^3y_n^{(3)}$$
$$+1.9315336079997545\times10^{-4}h_n^4y_n^{(4)}$$
$$y_{n+1} = y_n + 3.4837195819006461\times10^{-1}h_nf_n + 2.4099023203914976\times10^{-2}h_nF_2 + 7.6677095876881346\times10^{-2}h_nF_3$$
$$+1.0917946698940810\times10^{-1}h_nF_4 + 2.3315884398441686\times10^{-1}h_nF_5 + 7.7032074763077095\times10^{-3}h_nF_6$$
$$+2.1017989395798298\times10^{-2}h_nF_7 + 1.7979241488320813\times10^{-1}h_nF_8 + 5.3025873534678025\times10^{-2}h_n^2y_n^{(2)}$$
$$+4.2639564123841071\times10^{-3}h_n^3y_n^{(3)} + 1.5707173962518938\times10^{-4}h_n^4y_n^{(4)}$$

$\mathbf{CPHBT_{RK5}(5,8,9)}$ with $c_{cp} = 1.1275980963815175$, and abscissa vector $\sigma = [0\ 5.2916638242436365{\times}10^{-1}\ 5.4717079927184487{\times}10^{-1}\ 5.5332133691141594{\times}10^{-1}\ 6.6326676999053613{\times}10^{-1}\ 7.8739643637687862{\times}10^{-1}\ 9.0889684717619790{\times}10^{-1}\ 9.3096254350452368{\times}10^{-1}]^T$

$$Y_2 = y_n + 5.2916638242436365\times10^{-1}h_nf_n + 1.4000853014404391\times10^{-1}h_n^2y_n^{(2)} + 2.4695935801625393\times10^{-2}h_n^3y_n^{(3)}$$
$$+3.2670647521826084\times10^{-3}h_n^4y_n^{(4)} + 3.4576416721172421\times10^{-4}h_n^5y_n^{(5)}$$
$$Y_3 = y_n + 4.3540355099259298\times10^{-1}h_nf_n + 1.1176724827925191\times10^{-1}h_nF_2 + 9.0554471342437354\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.1655079335988349\times10^{-2}h_n^3y_n^{(3)} + 9.7471550769325714\times10^{-4}h_n^4y_n^{(4)} + 4.3576151932388834\times10^{-5}h_n^5y_n^{(5)}$$
$$Y_4 = y_n + 4.3732579965100798\times10^{-1}h_nf_n + 1.2982573102755517\times10^{-2}h_nF_2 + 1.0301296415765253\times10^{-1}h_nF_3$$
$$+8.9846623763868719\times10^{-2}h_n^2y_n^{(2)} + 1.0996058893167219\times10^{-2}h_n^3y_n^{(3)} + 7.7247007639110593\times10^{-4}h_n^4y_n^{(4)}$$
$$+5.0616847664132701\times10^{-6}h_n^5y_n^{(5)}$$
$$Y_5 = y_n + 3.8792222679216631\times10^{-1}h_nf_n + 3.5645703301504829\times10^{-3}h_nF_2 + 2.8283835011049294\times10^{-2}h_nF_3$$
$$+2.4349613785716995\times10^{-1}h_nF_4 + 6.7867456158971939\times10^{-2}h_n^2y_n^{(2)} + 6.6229909856647569\times10^{-3}h_n^3y_n^{(3)}$$
$$+3.2855397504491643\times10^{-4}h_n^4y_n^{(4)} + 1.3897654337183664\times10^{-6}h_n^5y_n^{(5)}$$
$$Y_6 = y_n + 4.3083455174960550\times10^{-1}h_nf_n + 1.0936206691822964\times10^{-3}h_nF_2 + 8.6775638315206328\times10^{-3}h_nF_3$$
$$+7.4705331796727642\times10^{-2}h_nF_4 + 2.7208536832984259\times10^{-1}h_nF_5 + 8.2868519700826371\times10^{-2}h_n^2y_n^{(2)}$$
$$+8.6269295556025069\times10^{-3}h_n^3y_n^{(3)} + 4.1131347750743502\times10^{-4}h_n^4y_n^{(4)} + 4.2638412567534908\times10^{-7}h_n^5y_n^{(5)}$$
$$Y_7 = y_n + 5.2935554175905541\times10^{-1}h_nf_n + 3.3382123178179248\times10^{-4}h_nF_2 + 2.6487749625921583\times10^{-3}h_nF_3$$
$$+2.2803360053260068\times10^{-2}h_nF_4 + 8.3052447128292697\times10^{-2}h_nF_5 + 2.7070290204121572\times10^{-1}h_nF_6$$

$+1.3056674571497795\times10^{-1}h_n^2y_n^{(2)}+1.9019610970213398\times10^{-2}h_n^3y_n^{(3)}+1.6459596301799176\times10^{-3}h_n^4y_n^{(4)}$

$+6.3393707741640448\times10^{-5}h_n^5y_n^{(5)}$

$Y_8=y_n+4.1615991632964217\times10^{-1}h_nf_n+3.2250212169529668\times10^{-2}h_nF_2+8.7569597838020544\times10^{-3}h_nF_3$

$+7.5388853240494588\times10^{-2}h_nF_4+2.7457483165616464\times10^{-1}h_nF_5+2.8959269078241059\times10^{-2}h_nF_6$

$+9.4872501246649477\times10^{-2}h_nF_7+7.8625982598329147\times10^{-2}h_n^2y_n^{(2)}+8.5493668994264475\times10^{-3}h_n^3y_n^{(3)}$

$+5.5262940489358221\times10^{-4}h_n^4y_n^{(4)}+1.9341622037383198\times10^{-5}h_n^5y_n^{(5)}$

$y_{n+1}=y_n+4.0891568072911599\times10^{-1}h_nf_n+1.4798464472168647\times10^{-2}h_nF_2+7.3040596973247995\times10^{-2}h_nF_3$

$+6.5967009037703833\times10^{-2}h_nF_4+2.4025939675171498\times10^{-1}h_nF_5+2.0724001835159978\times10^{-2}h_nF_6$

$+1.7037085411893846\times10^{-2}h_nF_7+1.5925776478899478\times10^{-1}h_nF_8+7.6280468399285331\times10^{-2}h_n^2y_n^{(2)}$

$+8.2394175322924122\times10^{-3}h_n^3y_n^{(3)}+5.2577863418829592\times10^{-4}h_n^4y_n^{(4)}+1.6300118228193040\times10^{-5}h_n^5y_n^{(5)}$

$\mathbf{CPHBT_{RK5}(6,8,10)}$ with $c_{cp}=1.0102662158967612$, and abscissa vector $\sigma=[0\ 5.6834582459174177\times10^{-1}$ $5.8872834307651145\times10^{-1}$ $5.9266716271678299\times10^{-1}$ $6.9119669892206426\times10^{-1}$ $8.0301319377855152\times10^{-1}$ $9.1514325426020959\times10^{-1}$ $9.3812312749673699\times10^{-1}]^T$

$Y_2=y_n+5.6834582459174177\times10^{-1}h_nf_n+1.6150848816543345\times10^{-1}h_n^2y_n^{(2)}+3.0597558294982948\times10^{-2}h_n^3y_n^{(3)}$

$+4.3474986249139929\times10^{-3}h_n^4y_n^{(4)}+4.9417653817764127\times10^{-4}h_n^5y_n^{(5)}+4.6810528680743980\times10^{-5}h_n^6y_n^{(6)}$

$Y_3=y_n+4.8192504380628043\times10^{-1}h_nf_n+1.0680329927023102\times10^{-1}h_nF_2+1.1259932177794925\times10^{-1}h_n^2y_n^{(2)}$

$+1.6759338754692318\times10^{-2}h_n^3y_n^{(3)}+1.7375921636098861\times10^{-3}h_n^4y_n^{(4)}+1.2505020041908140\times10^{-4}h_n^5y_n^{(5)}$

$+5.0508450441324238\times10^{-6}h_n^6y_n^{(6)}$

$Y_4=y_n+4.8972551120061442\times10^{-1}h_nf_n+1.0025618163404805\times10^{-2}h_nF_2+9.2916033352763772\times10^{-2}h_nF_3$

$+1.1522686229824411\times10^{-1}h_n^2y_n^{(2)}+1.6974534376519225\times10^{-2}h_n^3y_n^{(3)}+1.6740791101138286\times10^{-3}h_n^4y_n^{(4)}$

$+1.0068007848309577\times10^{-4}h_n^5y_n^{(5)}+4.7412246776079993\times10^{-7}h_n^6y_n^{(6)}$

$Y_5=y_n+4.4170144047095189\times10^{-1}h_nf_n+2.2889737592531457\times10^{-3}h_nF_2+2.1213890125468052\times10^{-2}h_nF_3$

$+2.2599239456639111\times10^{-1}h_nF_4+9.1148019954699053\times10^{-2}h_n^2y_n^{(2)}+1.1300393810174062\times10^{-2}h_n^3y_n^{(3)}$

$+8.7775864021303685\times10^{-4}h_n^4y_n^{(4)}+3.6777668015721458\times10^{-5}h_n^5y_n^{(5)}+1.0824807704508198\times10^{-7}h_n^6y_n^{(6)}$

$Y_6=y_n+4.8533481192208222\times10^{-1}h_nf_n+5.8673275445721412\times10^{-4}h_nF_2+5.4377574822569302\times10^{-3}h_nF_3$

$+5.7928641433436297\times10^{-2}h_nF_4+2.5372525018631881\times10^{-1}h_nF_5+1.0917380670713579\times10^{-1}h_n^2y_n^{(2)}$

$+1.4481234883226475\times10^{-2}h_n^3y_n^{(3)}+1.1482193711736190\times10^{-3}h_n^4y_n^{(4)}+4.1900801523674731\times10^{-5}h_n^5y_n^{(5)}$

$+2.7747234791951545\times10^{-8}h_n^6y_n^{(6)}$

$Y_7=y_n+5.7596733708092829\times10^{-1}h_nf_n+1.5220123159727727\times10^{-4}h_nF_2+1.4105798246979706\times10^{-3}h_nF_3$

$+1.5026961600401012\times10^{-2}h_nF_4+6.5817521303050255\times10^{-2}h_nF_5+2.5676865321953485\times10^{-1}h_nF_6$

$+1.5723918021563499\times10^{-1}h_n^2y_n^{(2)}+2.6320266248509510\times10^{-2}h_n^3y_n^{(3)}+2.8685272382114706\times10^{-3}h_n^4y_n^{(4)}$

$+1.8939605862363163\times10^{-4}h_n^5y_n^{(5)}+4.7867983502074729\times10^{-6}h_n^6y_n^{(6)}$

$Y_8=y_n+4.6717806055159911\times10^{-1}h_nf_n+2.4846798799337950\times10^{-2}h_nF_2+5.7262405846307237\times10^{-3}h_nF_3$

$+6.1001863115636198\times10^{-2}h_nF_4+2.6718584447718841\times10^{-1}h_nF_5+2.3107059611988029\times10^{-2}h_nF_6$

$+8.9077260356356547\times10^{-2}h_nF_7+1.0163922419313526\times10^{-1}h_n^2y_n^{(2)}+1.3309204816804832\times10^{-2}h_n^3y_n^{(3)}$

$+1.1229597574844965 \times 10^{-3} h_n^4 y_n^{(4)} + 6.0183295758714790 \times 10^{-5} h_n^5 y_n^{(5)} + 1.8757825874629922 \times 10^{-6} h_n^6 y_n^{(6)}$

$y_{n+1} = y_n + 4.6009981224021063 \times 10^{-1} h_n f_n + 1.0450633390164462 \times 10^{-2} h_n F_2 + 6.3775666473099110 \times 10^{-2} h_n F_3$

$+5.3711089532930169 \times 10^{-2} h_n F_4 + 2.3525253298364060 \times 10^{-1} h_n F_5 + 1.7771279587692007 \times 10^{-2} h_n F_6$

$+1.3122298402439624 \times 10^{-2} h_n F_7 + 1.4581668738982345 \times 10^{-1} h_n F_8 + 9.9001948787816429 \times 10^{-2} h_n^2 y_n^{(2)}$

$+1.2907611426296940 \times 10^{-2} h_n^3 y_n^{(3)} + 1.0921126938947714 \times 10^{-3} h_n^4 y_n^{(4)} + 5.8019066124036735 \times 10^{-5} h_n^5 y_n^{(5)}$

$+1.5545407629659841 \times 10^{-6} h_n^6 y_n^{(6)}$

$\mathbf{CPHBT_{RK5}(7,8,11)}$ with c_{cp}= $9.2129728884463769\times10^{-1}$, and abscissa vector $\sigma = [0\ 6.0297956070795589\times10^{-1}$ $6.2246735280161303\times10^{-1}$ $6.2721585402548885\times10^{-1}$ $7.1725083296018455\times10^{-1}$ $8.1750732724067177\times10^{-1}$ $9.2145408226271541\times10^{-1}$ $9.4359073748436306\times10^{-1}]^T$

$Y_2 = y_n + 6.0297956070795589 \times 10^{-1} h_n f_n + 1.8179217531577974 \times 10^{-1} h_n^2 y_n^{(2)} + 3.6538988670684183 \times 10^{-2} h_n^3 y_n^{(3)}$

$+5.5080658343405310 \times 10^{-3} h_n^4 y_n^{(4)} + 6.6425022342823090 \times 10^{-4} h_n^5 y_n^{(5)} + 6.6754884653819362 \times 10^{-5} h_n^6 y_n^{(6)}$

$+5.7502615748100375 \times 10^{-6} h_n^7 y_n^{(7)}$

$Y_3 = y_n + 5.2275798880583790 \times 10^{-1} h_n f_n + 9.9709363995775083 \times 10^{-2} h_n F_2 + 1.3361009415128178 \times 10^{-1} h_n^2 y_n^{(2)}$

$+2.2071066092381960 \times 10^{-2} h_n^3 y_n^{(3)} + 2.6121204824921308 \times 10^{-3} h_n^4 y_n^{(4)} + 2.2955069014028275 \times 10^{-4} h_n^5 y_n^{(5)}$

$+1.4559775069042984 \times 10^{-5} h_n^6 y_n^{(6)} + 5.2823033730031047 \times 10^{-7} h_n^7 y_n^{(7)}$

$Y_4 = y_n + 5.3257206458381301 \times 10^{-1} h_n f_n + 7.9626957446034940 \times 10^{-3} h_n F_2 + 8.6681093697072331 \times 10^{-2} h_n F_3$

$+1.3794237005676432 \times 10^{-1} h_n^2 y_n^{(2)} + 2.2883897347505455 \times 10^{-2} h_n^3 y_n^{(3)} + 2.6731651048587427 \times 10^{-3} h_n^4 y_n^{(4)}$

$+2.2283291777839195 \times 10^{-4} h_n^5 y_n^{(5)} + 1.1768232275583431 \times 10^{-5} h_n^6 y_n^{(6)} + 4.2183976372388931 \times 10^{-8} h_n^7 y_n^{(7)}$

$Y_5 = y_n + 4.8712964712512519 \times 10^{-1} h_n f_n + 1.5527768419036632 \times 10^{-3} h_n F_2 + 1.6903370308221921 \times 10^{-2} h_n F_3$

$+2.1166503868493372 \times 10^{-1} h_n F_4 + 1.1300662181773756 \times 10^{-1} h_n^2 y_n^{(2)} + 1.6306629033030547 \times 10^{-2} h_n^3 y_n^{(3)}$

$+1.5865847127968832 \times 10^{-3} h_n^4 y_n^{(4)} + 1.0267554545023126 \times 10^{-4} h_n^5 y_n^{(5)} + 3.6865593355713411 \times 10^{-6} h_n^6 y_n^{(6)}$

$+8.2261464814109908 \times 10^{-9} h_n^7 y_n^{(7)}$

$Y_6 = y_n + 5.3394585248586235 \times 10^{-1} h_n f_n + 3.3469548433758759 \times 10^{-4} h_n F_2 + 3.6434609015132064 \times 10^{-3} h_n F_3$

$+4.5623640647023067 \times 10^{-2} h_n F_4 + 2.3395967772193568 \times 10^{-1} h_n F_5 + 1.3526572059024625 \times 10^{-1} h_n^2 y_n^{(2)}$

$+2.1138175452081728 \times 10^{-2} h_n^3 y_n^{(3)} + 2.1873690255423948 \times 10^{-3} h_n^4 y_n^{(4)} + 1.4402149254392764 \times 10^{-4} h_n^5 y_n^{(5)}$

$+4.5266625519821800 \times 10^{-6} h_n^6 y_n^{(6)} + 1.7731162085502080 \times 10^{-9} h_n^7 y_n^{(7)}$

$Y_7 = y_n + 6.1466569211176680 \times 10^{-1} h_n f_n + 7.5004843543933383 \times 10^{-5} h_n F_2 + 8.1649507586870312 \times 10^{-4} h_n F_3$

$+1.0224201367451179 \times 10^{-2} h_n F_4 + 5.2430074035513821 \times 10^{-2} h_n F_5 + 2.4324261482857093 \times 10^{-1} h_n F_6$

$+1.8111442955716084 \times 10^{-1} h_n^2 y_n^{(2)} + 3.3446727742397285 \times 10^{-2} h_n^3 y_n^{(3)} + 4.2090177415037011 \times 10^{-3} h_n^4 y_n^{(4)}$

$+3.5943014443224030 \times 10^{-4} h_n^5 y_n^{(5)} + 1.8138491920708396 \times 10^{-5} h_n^6 y_n^{(6)} + 2.1848779610322565 \times 10^{-7} h_n^7 y_n^{(7)}$

$Y_8 = y_n + 5.1224193922056838 \times 10^{-1} h_n f_n + 1.9290958904697862 \times 10^{-2} h_n F_2 + 4.0074074265021246 \times 10^{-3} h_n F_3$

$+5.0181001332280040 \times 10^{-2} h_n F_4 + 2.5732998798355028 \times 10^{-1} h_n F_5 + 1.8406002266277335 \times 10^{-2} h_n F_6$

$+8.2133440350487003 \times 10^{-2} h_n F_7 + 1.2428150226233631 \times 10^{-1} h_n^2 y_n^{(2)} + 1.8658301349196486 \times 10^{-2} h_n^3 y_n^{(3)}$

$+1.8901469202658502 \times 10^{-3} h_n^4 y_n^{(4)} + 1.3124087778235076 \times 10^{-4} h_n^5 y_n^{(5)} + 6.0432806583916487 \times 10^{-6} h_n^6 y_n^{(6)}$

$+1.7115986230225067 \times 10^{-7} h_n^7 y_n^{(7)}$

$$y_{n+1} = y_n + 5.0391147935251557 \times 10^{-1} h_n f_n + 7.4829467774940222 \times 10^{-3} h_n F_2 + 5.5784735913450331 \times 10^{-2} h_n F_3$$
$$+4.5189152394032664 \times 10^{-2} h_n F_4 + 2.3173160626157041 \times 10^{-1} h_n F_5 + 1.0381051394452100 \times 10^{-2} h_n F_6$$
$$+1.0236719716589524 \times 10^{-2} h_n F_7 + 1.3528230818989537 \times 10^{-1} h_n F_8 + 1.2064033301795153 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+1.7963252571564069 \times 10^{-2} h_n^3 y_n^{(3)} + 1.8186186599538341 \times 10^{-3} h_n^4 y_n^{(4)} + 1.2714116670449867 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+5.8000201243385346 \times 10^{-6} h_n^6 y_n^{(6)} + 1.3695545589541550 \times 10^{-7} h_n^7 y_n^{(7)}$$

$\mathbf{CPHBT_{RK5}}(8,8,12)$ with c_{cp}= 8.3899828718045832×10^{-1}, and abscissa vector $\sigma = [0\ 6.3442778287933732{\times}10^{-1}\ 6.4806091964118251{\times}10^{-1}\ 6.5778452073891203{\times}10^{-1}\ 7.4053651347951488{\times}10^{-1}\ 8.3110284752952135{\times}10^{-1}\ 9.2735138793776239{\times}10^{-1}\ 9.4800556307728689{\times}10^{-1}]^T$

$$Y_2 = y_n + 6.3442778287933732 \times 10^{-1} h_n f_n + 2.0124930584459580 \times 10^{-1} h_n^2 y_n^{(2)} + 4.2559383637664192 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+6.7502138504886089 \times 10^{-3} h_n^4 y_n^{(4)} + 8.5650464142537658 \times 10^{-4} h_n^5 y_n^{(5)} + 9.0565056780893895 \times 10^{-5} h_n^6 y_n^{(6)}$$
$$+8.2081411685491133 \times 10^{-6} h_n^7 y_n^{(7)} + 6.5093410039040332 \times 10^{-7} h_n^8 y_n^{(8)}$$
$$Y_3 = y_n + 5.5991828761338081 \times 10^{-1} h_n f_n + 8.8142632027801662 \times 10^{-2} h_n F_2 + 1.5407134316854010 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+2.7623779892062172 \times 10^{-2} h_n^3 y_n^{(3)} + 3.5981073656161508 \times 10^{-3} h_n^4 y_n^{(4)} + 3.5759061708865212 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+2.7392899410268045 \times 10^{-5} h_n^6 y_n^{(6)} + 1.5426932790737223 \times 10^{-6} h_n^7 y_n^{(7)} + 4.8137564385466838 \times 10^{-8} h_n^8 y_n^{(8)}$$
$$Y_4 = y_n + 5.6601553493298773 \times 10^{-1} h_n f_n + 6.3191453543435779 \times 10^{-3} h_n F_2 + 8.5449840451580661 \times 10^{-2} h_n F_3$$
$$+1.5695449429876798 \times 10^{-1} h_n^2 y_n^{(2)} + 2.8219624670652730 \times 10^{-2} h_n^3 y_n^{(3)} + 3.6553656460089476 \times 10^{-3} h_n^4 y_n^{(4)}$$
$$+3.5555085562573676 \times 10^{-4} h_n^5 y_n^{(5)} + 2.5694849223444351 \times 10^{-5} h_n^6 y_n^{(6)} + 1.2079358620528968 \times 10^{-6} h_n^7 y_n^{(7)}$$
$$+3.4510912524133941 \times 10^{-9} h_n^8 y_n^{(8)}$$
$$Y_5 = y_n + 5.2677084285373510 \times 10^{-1} h_n f_n + 1.0523109471714857 \times 10^{-3} h_n F_2 + 1.4229741127800948 \times 10^{-2} h_n F_3$$
$$+1.9848361855080737 \times 10^{-1} h_n F_4 + 1.3374835757257769 \times 10^{-1} h_n^2 y_n^{(2)} + 2.1544442788534502 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+2.4253119077743551 \times 10^{-3} h_n^4 y_n^{(4)} + 1.9592717340050647 \times 10^{-4} h_n^5 y_n^{(5)} + 1.0916080320101515 \times 10^{-5} h_n^6 y_n^{(6)}$$
$$+3.4266972684998814 \times 10^{-7} h_n^7 y_n^{(7)} + 5.7470130863604662 \times 10^{-10} h_n^8 y_n^{(8)}$$
$$Y_6 = y_n + 5.7613702206133321 \times 10^{-1} h_n f_n + 1.9087312382352843 \times 10^{-4} h_n F_2 + 2.5810575738702186 \times 10^{-3} h_n F_3$$
$$+3.6001895069534565 \times 10^{-2} h_n F_4 + 2.1619199970095976 \times 10^{-1} h_n F_5 + 1.5979263483339215 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+2.8029909245438807 \times 10^{-2} h_n^3 y_n^{(3)} + 3.4138373578088666 \times 10^{-3} h_n^4 y_n^{(4)} + 2.9427596137500677 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+1.6920775322864040 \times 10^{-5} h_n^6 y_n^{(6)} + 4.9033154364877436 \times 10^{-7} h_n^7 y_n^{(7)} + 1.0424203448578282 \times 10^{-10} h_n^8 y_n^{(8)}$$
$$Y_7 = y_n + 6.4862655064719066 \times 10^{-1} h_n f_n + 3.6769940695116786 \times 10^{-5} h_n F_2 + 4.9721685285373531 \times 10^{-4} h_n F_3$$
$$+6.9354318727585934 \times 10^{-3} h_n F_4 + 4.1647387796267914 \times 10^{-2} h_n F_5 + 2.2960803082799633 \times 10^{-1} h_n F_6$$
$$+2.0341342435066243 \times 10^{-1} h_n^2 y_n^{(2)} + 4.0586746183670148 \times 10^{-2} h_n^3 y_n^{(3)} + 5.6748112511148018 \times 10^{-3} h_n^4 y_n^{(4)}$$
$$+5.7092740164794271 \times 10^{-4} h_n^5 y_n^{(5)} + 3.9719428341979857 \times 10^{-5} h_n^6 y_n^{(6)} + 1.5555777745779016 \times 10^{-6} h_n^7 y_n^{(7)}$$
$$+2.0081263140683207 \times 10^{-11} h_n^8 y_n^{(8)}$$
$$Y_8 = y_n + 5.5209241164721989 \times 10^{-1} h_n f_n + 1.5031111898254198 \times 10^{-2} h_n F_2 + 2.9496090303136268 \times 10^{-3} h_n F_3$$
$$+4.1142636987470324 \times 10^{-2} h_n F_4 + 2.4706224342114499 \times 10^{-1} h_n F_5 + 1.4493201023321448 \times 10^{-2} h_n F_6$$
$$+7.5234349069562528 \times 10^{-2} h_n F_7 + 1.4607397169734107 \times 10^{-1} h_n^2 y_n^{(2)} + 2.4353265219505451 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+2.8196597451252271 \times 10^{-3} h_n^4 y_n^{(4)} + 2.3435194720078191 \times 10^{-4} h_n^5 y_n^{(5)} + 1.3864058168038072 \times 10^{-5} h_n^6 y_n^{(6)}$$

$$+5.5819444876399311\times10^{-7}h_n^7y_n^{(7)}+1.4291909676730649\times10^{-8}h_n^8y_n^{(8)}$$

$$y_{n+1}=y_n+5.4145653542767558\times10^{-1}h_nf_n+5.5914233410541501\times10^{-3}h_nF_2+4.8662147845363705\times10^{-2}h_nF_3$$
$$+3.8278997260764475\times10^{-2}h_nF_4+2.2986603756187235\times10^{-1}h_nF_5+1.5395891137679231\times10^{-3}h_nF_6$$
$$+7.9920171403241869\times10^{-3}h_nF_7+1.2661325230917767\times10^{-1}h_nF_8+1.4079205087886684\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.3150059902996928\times10^{-2}h_n^3y_n^{(3)}+2.6587905709737395\times10^{-3}h_n^4y_n^{(4)}+2.2109640724393226\times10^{-4}h_n^5y_n^{(5)}$$
$$+1.3203681536550432\times10^{-5}h_n^6y_n^{(6)}+5.2841027574064201\times10^{-7}h_n^7y_n^{(7)}+1.1157623293823810\times10^{-8}h_n^8y_n^{(8)}$$

CPHBT$_{\mathbf{RK5}}$(9,8,13) with c_{cp}= 7.3981614589567168$\times10^{-1}$, and abscissa vector $\sigma=[0$ 6.6316884336031645$\times10^{-1}$ 6.6552534456124690$\times10^{-1}$ 6.8255131739962571$\times10^{-1}$ 7.5805741411028338$\times10^{-1}$ 8.4399678029601544$\times10^{-1}$ 9.3183807109044126$\times10^{-1}$ 9.5217739155443293$\times10^{-1}]^T$

$$Y_2=y_n+6.6316884336031645\times10^{-1}h_nf_n+2.1989645740192998\times10^{-1}h_n^2y_n^{(2)}+4.8609493104756339\times10^{-2}h_n^3y_n^{(3)}$$
$$+8.0590753296531353\times10^{-3}h_n^4y_n^{(4)}+1.0689055329839458\times10^{-3}h_n^5y_n^{(5)}+1.1814414099506766\times10^{-4}h_n^6y_n^{(6)}$$
$$+1.1192787619071024\times10^{-5}h_n^7y_n^{(7)}+9.2783850241462522\times10^{-7}h_n^8y_n^{(8)}+6.8368176274608356\times10^{-8}h_n^9y_n^{(9)}$$

$$Y_3=y_n+5.9338241815166537\times10^{-1}h_nf_n+7.2142926409581462\times10^{-2}h_nF_2+1.7361905106301259\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.3265548928368548\times10^{-2}h_n^3y_n^{(3)}+4.6674045755896853\times10^{-3}h_n^4y_n^{(4)}+5.0662692232004232\times10^{-4}h_n^5y_n^{(5)}$$
$$+4.3571527678842963\times10^{-5}h_n^6y_n^{(6)}+2.9509158674938910\times10^{-6}h_n^7y_n^{(7)}+1.4706424099751308\times10^{-7}h_n^8y_n^{(8)}$$
$$+3.6489803912536636\times10^{-9}h_n^9y_n^{(9)}$$

$$Y_4=y_n+5.8984518218526627\times10^{-1}h_nf_n+4.6972541598545676\times10^{-3}h_nF_2+8.8008881054504912\times10^{-2}h_nF_3$$
$$+1.7125093694557295\times10^{-1}h_n^2y_n^{(2)}+3.2473882146496276\times10^{-2}h_n^3y_n^{(3)}+4.4911981767817204\times10^{-3}h_n^4y_n^{(4)}$$
$$+4.7725109633802245\times10^{-4}h_n^5y_n^{(5)}+3.9658876073627216\times10^{-5}h_n^6y_n^{(6)}+2.5172192199980150\times10^{-6}h_n^7y_n^{(7)}$$
$$+1.0591526881452161\times10^{-7}h_n^8y_n^{(8)}+2.3758632260515969\times10^{-10}h_n^9y_n^{(9)}$$

$$Y_5=y_n+5.6077185743100488\times10^{-1}h_nf_n+6.4158452687660924\times10^{-4}h_nF_2+1.2020881644968782\times10^{-2}h_nF_3$$
$$+1.8462309050743317\times10^{-1}h_nF_4+1.5288510762824156\times10^{-1}h_n^2y_n^{(2)}+2.6794068830641254\times10^{-2}h_n^3y_n^{(3)}$$
$$+3.3530122818893069\times10^{-3}h_n^4y_n^{(4)}+3.1302595652995976\times10^{-4}h_n^5y_n^{(5)}+2.1876053777132641\times10^{-5}h_n^6y_n^{(6)}$$
$$+1.0876560797805447\times10^{-6}h_n^7y_n^{(7)}+3.1298087505204534\times10^{-8}h_n^8y_n^{(8)}+3.2451017460208804\times10^{-11}h_n^9y_n^{(9)}$$

$$Y_6=y_n+6.0514823130406725\times10^{-1}h_nf_n+9.8931071971960921\times10^{-5}h_nF_2+1.8535963031667672\times10^{-3}h_nF_3$$
$$+2.8468517380923878\times10^{-2}h_nF_4+2.0842750423588552\times10^{-1}h_nF_5+1.7743482031882651\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.3250583050177897\times10^{-2}h_n^3y_n^{(3)}+4.4051727701609929\times10^{-3}h_n^4y_n^{(4)}+4.2758191550126715\times10^{-4}h_n^5y_n^{(5)}$$
$$+2.9947870321764276\times10^{-5}h_n^6y_n^{(6)}+1.3611818064025165\times10^{-6}h_n^7y_n^{(7)}+2.4514137781210259\times10^{-8}h_n^8y_n^{(8)}$$
$$+5.0036629033489305\times10^{-12}h_n^9y_n^{(9)}$$

$$Y_7=y_n+6.8028409245184629\times10^{-1}h_nf_n+1.5646617872971343\times10^{-5}h_nF_2+2.9315878690437289\times10^{-4}h_nF_3$$
$$+4.5024884901171793\times10^{-3}h_nF_4+3.2964218905007205\times10^{-2}h_nF_5+2.1377846583869312\times10^{-1}h_nF_6$$
$$+2.2546532755835413\times10^{-1}h_n^2y_n^{(2)}+4.8126851293900992\times10^{-2}h_n^3y_n^{(3)}+7.3481188463329350\times10^{-3}h_n^4y_n^{(4)}$$
$$+8.3834980190962340\times10^{-4}h_n^5y_n^{(5)}+7.1713057626288326\times10^{-5}h_n^6y_n^{(6)}+4.3701054524888691\times10^{-6}h_n^7y_n^{(7)}$$
$$+1.5387781817848601\times10^{-7}h_n^8y_n^{(8)}+7.9114271444266149\times10^{-13}h_n^9y_n^{(9)}$$

$$Y_8=y_n+5.8398742230001055\times10^{-1}h_nf_n+1.1649277211928055\times10^{-2}h_nF_2+2.1293072296099053\times10^{-3}h_nF_3$$

$$+3.2703032354900081\times10^{-2}h_nF_4+2.3942979971427841\times10^{-1}h_nF_5+1.1235879128497262\times10^{-2}h_nF_6$$
$$+7.1042673615208732\times10^{-2}h_nF_7+1.6467200046732158\times10^{-1}h_n^2y_n^{(2)}+2.9589568566844537\times10^{-2}h_n^3y_n^{(3)}$$
$$+3.7562065838250957\times10^{-3}h_n^4y_n^{(4)}+3.5155677490805048\times10^{-4}h_n^5y_n^{(5)}+2.4425228281457076\times10^{-5}h_n^6y_n^{(6)}$$
$$+1.2271212794368014\times10^{-6}h_n^7y_n^{(7)}+4.2007882514736903\times10^{-8}h_n^8y_n^{(8)}+9.4143172045521872\times10^{-10}h_n^9y_n^{(9)}$$
$$y_{n+1}=y_n+5.7297654428736045\times10^{-1}h_nf_n+4.5726354926864943\times10^{-3}h_nF_2+4.1606203089191648\times10^{-2}h_nF_3$$
$$+3.0761927239199700\times10^{-2}h_nF_4+2.2521832219645466\times10^{-1}h_nF_5+9.7837829091380890\times10^{-4}h_nF_6$$
$$+6.1861300569995819\times10^{-3}h_nF_7+1.1769985934719381\times10^{-1}h_nF_8+1.5889121037821258\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.8180322710702352\times10^{-2}h_n^3y_n^{(3)}+3.5514524826100167\times10^{-3}h_n^4y_n^{(4)}+3.3309427938371308\times10^{-4}h_n^5y_n^{(5)}$$
$$+2.3537072281314291\times10^{-5}h_n^6y_n^{(6)}+1.2272569288667992\times10^{-6}h_n^7y_n^{(7)}+4.3732385698700383\times10^{-8}h_n^8y_n^{(8)}$$
$$+8.3449144497480801\times10^{-10}h_n^9y_n^{(9)}$$

$\mathbf{CPHBT_{RK5}(10,8,14)}$ with $c_{\mathrm{cp}}=6.5550155934415277\times10^{-1}$, and abscissa vector $\sigma=[0\ 6.8743747920332909\times10^{-1}\ 6.8165288240522792\times10^{-1}\ 7.0377738407340129\times10^{-1}\ 7.7328324041797447\times10^{-1}\ 8.5462249268765023\times10^{-1}\ 9.3546470856506980\times10^{-1}\ 9.5572657772867442\times10^{-1}]^T$

$$Y_2=y_n+6.8743747920332909\times10^{-1}h_nf_n+2.3628514390671371\times10^{-1}h_n^2y_n^{(2)}+5.4143754566809039\times10^{-2}h_n^3y_n^{(3)}$$
$$+9.3051115385027350\times10^{-3}h_n^4y_n^{(4)}+1.2793364839468260\times10^{-3}h_n^5y_n^{(5)}+1.4657730792954273\times10^{-4}h_n^6y_n^{(6)}$$
$$+1.4394676438784999\times10^{-5}h_n^7y_n^{(7)}+1.2369300106282390\times10^{-6}h_n^8y_n^{(8)}+9.4479116495247067\times10^{-8}h_n^9y_n^{(9)}$$
$$+6.4948485680850304\times10^{-9}h_n^{10}y_n^{(10)}$$
$$Y_3=y_n+6.2120348133900594\times10^{-1}h_nf_n+6.0449401066221861\times10^{-2}h_nF_2+1.9077014215736315\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.8505113954916646\times10^{-2}h_n^3y_n^{(3)}+5.7228853186649551\times10^{-3}h_n^4y_n^{(4)}+6.6392002282162667\times10^{-4}h_n^5y_n^{(5)}$$
$$+6.1995684052762329\times10^{-5}h_n^6y_n^{(6)}+4.7073819635834642\times10^{-6}h_n^7y_n^{(7)}+2.8592455428420127\times10^{-7}h_n^8y_n^{(8)}$$
$$+1.2788461540299406\times10^{-8}h_n^9y_n^{(9)}+2.5735616541680215\times10^{-10}h_n^{10}y_n^{(10)}$$
$$Y_4=y_n+6.1041415462319337\times10^{-1}h_nf_n+3.5584840009883177\times10^{-3}h_nF_2+8.9804745449219556\times10^{-2}h_nF_3$$
$$+1.8398940430604682\times10^{-1}h_n^2y_n^{(2)}+3.6392395097117218\times10^{-2}h_n^3y_n^{(3)}+5.2885419748241831\times10^{-3}h_n^4y_n^{(4)}$$
$$+5.9780149685586951\times10^{-4}h_n^5y_n^{(5)}+5.4074022179514692\times10^{-5}h_n^6y_n^{(6)}+3.9332904749005875\times10^{-6}h_n^7y_n^{(7)}$$
$$+2.2297920441770781\times10^{-7}h_n^8y_n^{(8)}+8.5000012238922041\times10^{-9}h_n^9y_n^{(9)}+1.5149839859968509\times10^{-11}h_n^{10}y_n^{(10)}$$
$$Y_5=y_n+5.9018736186801246\times10^{-1}h_nf_n+4.0245764293754988\times10^{-4}h_nF_2+1.0156742637272670\times10^{-2}h_nF_3$$
$$+1.7253667826975175\times10^{-1}h_nF_4+1.7035603550416945\times10^{-1}h_n^2y_n^{(2)}+3.1882609501606200\times10^{-2}h_n^3y_n^{(3)}$$
$$+4.3166862418130862\times10^{-3}h_n^4y_n^{(4)}+4.4539590186311990\times10^{-4}h_n^5y_n^{(5)}+3.5746749036108047\times10^{-5}h_n^6y_n^{(6)}$$
$$+2.2128832033088962\times10^{-6}h_n^7y_n^{(7)}+9.9835189556224902\times10^{-8}h_n^8y_n^{(8)}+2.6694819158442599\times10^{-9}h_n^9y_n^{(9)}$$
$$+1.7134121650513836\times10^{-12}h_n^{10}y_n^{(10)}$$
$$Y_6=y_n+6.3051966982793684\times10^{-1}h_nf_n+5.2785616678847070\times10^{-5}h_nF_2+1.3321399977516684\times10^{-3}h_nF_3$$
$$+2.2629598721828601\times10^{-2}h_nF_4+2.0008829852345425\times10^{-1}h_nF_5+1.9359433098030873\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.8283831051971813\times10^{-2}h_n^3y_n^{(3)}+5.4193049886180093\times10^{-3}h_n^4y_n^{(4)}+5.7437299870234161\times10^{-4}h_n^5y_n^{(5)}$$
$$+4.5849840630121496\times10^{-5}h_n^6y_n^{(6)}+2.6369978523694175\times10^{-6}h_n^7y_n^{(7)}+9.1184599173144341\times10^{-8}h_n^8y_n^{(8)}$$
$$+3.5012709037604951\times10^{-10}h_n^9y_n^{(9)}+2.2461076587311806\times10^{-13}h_n^{10}y_n^{(10)}$$

$$Y_7 = y_n + 7.0629444574613620 \times 10^{-1} h_n f_n + 6.9138847600715215 \times 10^{-6} h_n F_2 + 1.7448431994134819 \times 10^{-4} h_n F_3$$
$$+2.9640354243238849 \times 10^{-3} h_n F_4 + 2.6207658920789721 \times 10^{-2} h_n F_5 + 1.9981717026911863 \times 10^{-1} h_n F_6$$
$$+2.4430320723377635 \times 10^{-1} h_n^2 y_n^{(2)} + 5.4853558576865524 \times 10^{-2} h_n^3 y_n^{(3)} + 8.9187902082405075 \times 10^{-3} h_n^4 y_n^{(4)}$$
$$+1.1059686459078408 \times 10^{-3} h_n^5 y_n^{(5)} + 1.0673147070963020 \times 10^{-4} h_n^6 y_n^{(6)} + 7.9448859946511097 \times 10^{-6} h_n^7 y_n^{(7)}$$
$$+4.3052134849878947 \times 10^{-7} h_n^8 y_n^{(8)} + 1.3740389362875697 \times 10^{-8} h_n^9 y_n^{(9)} + 2.9419511358507244 \times 10^{-14} h_n^{10} y_n^{(10)}$$
$$Y_8 = y_n + 6.1011827827339371 \times 10^{-1} h_n f_n + 9.5434083764857746 \times 10^{-3} h_n F_2 + 1.7323608697954003 \times 10^{-3} h_n F_3$$
$$+2.9428311882405054 \times 10^{-2} h_n F_4 + 2.2818205362290719 \times 10^{-1} h_n F_5 + 8.8853037118449656 \times 10^{-3} h_n F_6$$
$$+6.7836860991842371 \times 10^{-2} h_n F_7 + 1.8075237232162897 \times 10^{-1} h_n^2 y_n^{(2)} + 3.4400852707815531 \times 10^{-2} h_n^3 y_n^{(3)}$$
$$+4.6806819099197870 \times 10^{-3} h_n^4 y_n^{(4)} + 4.7807659344261766 \times 10^{-4} h_n^5 y_n^{(5)} + 3.7280568657588136 \times 10^{-5} h_n^6 y_n^{(6)}$$
$$+2.1973911650392532 \times 10^{-6} h_n^7 y_n^{(7)} + 9.3794560823245422 \times 10^{-8} h_n^8 y_n^{(8)} + 2.6767209502936749 \times 10^{-9} h_n^9 y_n^{(9)}$$
$$+5.1613016981502102 \times 10^{-11} h_n^{10} y_n^{(10)}$$
$$y_{n+1} = y_n + 6.0065699935454753 \times 10^{-1} h_n f_n + 3.7680619448774463 \times 10^{-3} h_n F_2 + 3.5353133566138645 \times 10^{-2} h_n F_3$$
$$+2.5108426567251499 \times 10^{-2} h_n F_4 + 2.1969938294266037 \times 10^{-1} h_n F_5 + 6.4002170259951161 \times 10^{-4} h_n F_6$$
$$+4.8863904576247896 \times 10^{-3} h_n F_7 + 1.0988758346430022 \times 10^{-1} h_n F_8 + 1.7561002722285013 \times 10^{-1} h_n^2 y_n^{(2)}$$
$$+3.3100134539233494 \times 10^{-2} h_n^3 y_n^{(3)} + 4.4848450133697969 \times 10^{-3} h_n^4 y_n^{(4)} + 4.6017247128083777 \times 10^{-4} h_n^5 y_n^{(5)}$$
$$+3.6572067812901306 \times 10^{-5} h_n^6 y_n^{(6)} + 2.2504897198419219 \times 10^{-6} h_n^7 y_n^{(7)} + 1.0423472480310328 \times 10^{-7} h_n^8 y_n^{(8)}$$
$$+3.3492133034856635 \times 10^{-9} h_n^9 y_n^{(9)} + 5.8305044112316857 \times 10^{-11} h_n^{10} y_n^{(10)}$$

B.2 Nine stages CPHBT$_{\mathbf{RK5}}(p-4,9,p)$ methods formulae

CPHBT$_{\mathbf{RK5}}$(2,9,6) with $c_{cp} = 2.7322083370534886$, and abscissa vector
$\sigma = [0 \ 2.7850633886554327\times10^{-1} \ 4.3666434505105711\times10^{-1} \ 4.1325187469127583\times10^{-1} \ 4.1975726870492769\times10^{-1} \ 6.0472995999901091\times10^{-1} \ 8.3273650601125326\times10^{-1} \ 8.7020090463847888\times10^{-1} \ 9.2342683963440797\times10^{-1}]^T$

$$Y_2 = y_n + 2.7850633886554327 \times 10^{-1} h_n f_n + 3.8782890394144412 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$Y_3 = y_n + 1.8869166885671015 \times 10^{-1} h_n f_n + 2.4797267619434699 \times 10^{-1} h_n F_2 + 2.6275912933855904 \times 10^{-2} h_n^2 y_n^{(2)}$$
$$Y_4 = y_n + 2.0778537219087789 \times 10^{-1} h_n f_n + 8.2983695585448505 \times 10^{-2} h_n F_2 + 1.2248280691494942 \times 10^{-1} h_n F_3$$
$$+8.7931960633594822 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_5 = y_n + 2.0700637552097195 \times 10^{-1} h_n f_n + 3.0893713688697588 \times 10^{-2} h_n F_2 + 4.5598701551223829 \times 10^{-2} h_n F_3$$
$$+1.3625847794403434 \times 10^{-1} h_n F_4 + 3.2735886209151453 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_6 = y_n + 1.6695043903817541 \times 10^{-1} h_n f_n + 2.3368490403888331 \times 10^{-2} h_n F_2 + 3.4491574252511446 \times 10^{-2} h_n F_3$$
$$+1.0306805346773554 \times 10^{-1} h_n F_4 + 2.7685140283670023 \times 10^{-1} h_n F_5 + 2.4761938640649917 \times 10^{-3} h_n^2 y_n^{(2)}$$
$$Y_7 = y_n + 1.4321707907410533 \times 10^{-1} h_n f_n + 2.0046469822405589 \times 10^{-2} h_n F_2 + 2.9588317021333278 \times 10^{-2} h_n F_3$$

$$+8.8416093114480995\times10^{-2}h_nF_4+2.3749472885647357\times10^{-1}h_nF_5+3.1397381812245456\times10^{-1}h_nF_6$$
$$+2.1241828082375928\times10^{-3}h_n^2y_n^{(2)}$$
$$Y_8=y_n+1.9095575804764064\times10^{-1}h_nf_n+1.2167650304983745\times10^{-1}h_nF_2+9.3726057675183175\times10^{-2}h_nF_3$$
$$+4.0771050983459310\times10^{-2}h_nF_4+1.0951524046615264\times10^{-1}h_nF_5+1.4478181624206363\times10^{-1}h_nF_6$$
$$+1.6877447817414196\times10^{-1}h_nF_7+1.2893199562551104\times10^{-2}h_n^2y_n^{(2)}$$
$$Y_9=y_n+1.9460307859365930\times10^{-1}h_nf_n+1.5314754717909948\times10^{-1}h_nF_2+1.8569069873836250\times10^{-1}h_nF_3$$
$$+1.9160304858302873\times10^{-2}h_nF_4+5.1466551470873881\times10^{-2}h_nF_5+6.8040035030300150\times10^{-2}h_nF_6$$
$$+7.9315356757162697\times10^{-2}h_nF_7+1.7200326700664717\times10^{-1}h_nF_8+1.6227963812261926\times10^{-2}h_n^2y_n^{(2)}$$
$$y_{n+1}=y_n+1.8427119881003809\times10^{-1}h_nf_n+7.7241703325365438\times10^{-2}h_nF_2+1.0107399017006455\times10^{-1}h_nF_3$$
$$+9.6593019052518214\times10^{-2}h_nF_4+1.1115540987927246\times10^{-1}h_nF_5+1.4695016016904025\times10^{-1}h_nF_6$$
$$+1.1027538931873983\times10^{-1}h_nF_7+5.5129510087042373\times10^{-2}h_nF_8+1.1730961918791873\times10^{-1}h_nF_9$$
$$+1.0780643484936181\times10^{-2}h_n^2y_n^{(2)}$$

$\mathbf{CPHBT_{RK5}(3,9,7)}$ with $c_{cp}=2.3552862054682677$, and abscissa vector
$\sigma=[0\ \ 3.5479303261404727\times10^{-1}\ \ 4.7990751966446193\times10^{-1}\ \ 4.6673251054265519\times10^{-1}\ \ 4.8961115951347806\times10^{-1}$
$6.4488679474433286\times10^{-1}\ \ 8.2815090647536549\times10^{-1}\ \ 9.0004478502198249\times10^{-1}\ \ 9.2426227753295565\times10^{-1}]^T$

$$Y_2=y_n+3.5479303261404727\times10^{-1}h_nf_n+6.2939047995736200\times10^{-2}h_n^2y_n^{(2)}+7.4434452360827730\times10^{-3}h_n^3y_n^{(3)}$$
$$Y_3=y_n+2.5098698137594377\times10^{-1}h_nf_n+2.2892053828851822\times10^{-1}h_nF_2+3.3936201708224471\times10^{-2}h_n^2y_n^{(2)}$$
$$+4.0133075712903681\times10^{-3}h_n^3y_n^{(3)}$$
$$Y_4=y_n+3.0222226889725828\times10^{-1}h_nf_n+5.7628039065779596\times10^{-2}h_nF_2+1.0688220257961729\times10^{-1}h_nF_3$$
$$+3.7180018718717442\times10^{-2}h_n^2y_n^{(2)}+1.0103027331248872\times10^{-3}h_n^3y_n^{(3)}$$
$$Y_5=y_n+3.0724992902467363\times10^{-1}h_nf_n+1.7839671329300824\times10^{-2}h_nF_2+3.3087076983405017\times10^{-2}h_nF_3$$
$$+1.3143448217609854\times10^{-1}h_nF_4+3.6306669782328362\times10^{-2}h_n^2y_n^{(2)}+3.1275519684904429\times10^{-4}h_n^3y_n^{(3)}$$
$$Y_6=y_n+2.5980255675048991\times10^{-1}h_nf_n+1.1318743132080025\times10^{-2}h_nF_2+2.0992770463848908\times10^{-2}h_nF_3$$
$$+8.3391286475428289\times10^{-2}h_nF_4+2.6938143792248576\times10^{-1}h_nF_5+2.3035506745713814\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.9843391007676127\times10^{-4}h_n^3y_n^{(3)}$$
$$Y_7=y_n+2.5181407048648125\times10^{-1}h_nf_n+8.0569619732129144\times10^{-3}h_nF_2+1.4943174464330806\times10^{-2}h_nF_3$$
$$+5.9359985131702521\times10^{-2}h_nF_4+1.9175238595877861\times10^{-1}h_nF_5+3.0222432846085934\times10^{-1}h_nF_6$$
$$+1.6397244792832322\times10^{-2}h_n^2y_n^{(2)}+1.4125017672263519\times10^{-4}h_n^3y_n^{(3)}$$
$$Y_8=y_n+2.7490954020159269\times10^{-1}h_nf_n+1.2928410620206351\times10^{-1}h_nF_2+1.0293912577114263\times10^{-2}h_nF_3$$
$$+4.0891344672653700\times10^{-2}h_nF_4+1.3209256856529214\times10^{-1}h_nF_5+1.2997675601540520\times10^{-1}h_nF_6$$
$$+1.8259655678786096\times10^{-1}h_nF_7+3.5433968143793480\times10^{-2}h_n^2y_n^{(2)}+2.2665370531949050\times10^{-3}h_n^3y_n^{(3)}$$
$$Y_9=y_n+2.6149955125942981\times10^{-1}h_nf_n+8.9096133009681486\times10^{-2}h_nF_2+1.0455385300155579\times10^{-1}h_nF_3$$
$$+3.6013890068621844\times10^{-2}h_nF_4+1.1633677692124676\times10^{-1}h_nF_5+1.5618501809313759\times10^{-1}h_nF_6$$
$$+4.8290681056484125\times10^{-2}h_nF_7+1.1228637412279825\times10^{-1}h_nF_8+2.9798481321905837\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.5619838562924732\times10^{-3}h_n^3y_n^{(3)}$$

$$
\begin{aligned}
y_{n+1} = y_n &+ 2.6721981299333869 \times 10^{-1} h_n f_n + 5.2300895821946906 \times 10^{-2} h_n F_2 + 7.6945005129481270 \times 10^{-2} h_n F_3 \\
&+ 9.8036880047318628 \times 10^{-2} h_n F_4 + 1.0581829338095995 \times 10^{-1} h_n F_5 + 1.5780136138191742 \times 10^{-1} h_n F_6 \\
&+ 6.4459428045119296 \times 10^{-2} h_n F_7 + 3.7107515076252998 \times 10^{-2} h_n F_8 + 1.4031080812366478 \times 10^{-1} h_n F_9 \\
&+ 2.8722143107697995 \times 10^{-2} h_n^2 y_n^{(2)} + 1.2743632434136043 \times 10^{-3} h_n^3 y_n^{(3)}
\end{aligned}
$$

$\mathbf{CPHBT_{RK5}(4,9,8)}$ with $c_{cp} = 2.0529679460210106$, and abscissa vector
$\sigma = [0\ 4.1460243717406275\times10^{-1}\ 5.2225879099263739\times10^{-1}\ 5.1467340708555709\times10^{-1}\ 5.4344753041944771\times10^{-1}\ 6.7549071231038582\times10^{-1}\ 8.2947465311655355\times10^{-1}\ 9.2216229381097015\times10^{-1}\ 9.3055413832164524\times10^{-1}]^T$

$$
\begin{aligned}
Y_2 = y_n &+ 4.1460243717406275 \times 10^{-1} h_n f_n + 8.5947590455336342 \times 10^{-2} h_n^2 y_n^{(2)} + 1.1878026824006885 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+ 1.2311647175130367 \times 10^{-3} h_n^4 y_n^{(4)} \\
Y_3 = y_n &+ 3.0707904438767880 \times 10^{-1} h_n f_n + 2.1517974660495853 \times 10^{-1} h_n F_2 + 4.7163075011632576 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+ 5.2472029497165583 \times 10^{-3} h_n^3 y_n^{(3)} + 5.4387578282485386 \times 10^{-4} h_n^4 y_n^{(4)} \\
Y_4 = y_n &+ 3.7466083312144899 \times 10^{-1} h_n f_n + 4.2900118760984041 \times 10^{-2} h_n F_2 + 9.7112455203124018 \times 10^{-2} h_n F_3 \\
&+ 6.3940030742456883 \times 10^{-2} h_n^2 y_n^{(2)} + 5.7907839643113199 \times 10^{-3} h_n^3 y_n^{(3)} + 1.0843183916023694 \times 10^{-4} h_n^4 y_n^{(4)} \\
Y_5 = y_n &+ 3.8055414526896320 \times 10^{-1} h_n f_n + 1.1143372952122831 \times 10^{-2} h_n F_2 + 2.5225112141389089 \times 10^{-2} h_n F_3 \\
&+ 1.2652490005697251 \times 10^{-1} h_n F_4 + 6.4754501612111368 \times 10^{-2} h_n^2 y_n^{(2)} + 5.5944824231781033 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+ 2.8165339829921753 \times 10^{-5} h_n^4 y_n^{(4)} \\
Y_6 = y_n &+ 3.3712047906097031 \times 10^{-1} h_n f_n + 5.8009631035678619 \times 10^{-3} h_n F_2 + 1.3131566666956406 \times 10^{-2} h_n F_3 \\
&+ 6.5865719478884269 \times 10^{-2} h_n F_4 + 2.5357198400000686 \times 10^{-1} h_n F_5 + 4.7178278894073944 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+ 2.9123485554912415 \times 10^{-3} h_n^3 y_n^{(3)} + 1.4662176152122870 \times 10^{-5} h_n^4 y_n^{(4)} \\
Y_7 = y_n &+ 3.4629342637616772 \times 10^{-1} h_n f_n + 3.3955404946384498 \times 10^{-3} h_n F_2 + 7.6864419889640006 \times 10^{-3} h_n F_3 \\
&+ 3.8553894190689331 \times 10^{-2} h_n F_4 + 1.4842603281656333 \times 10^{-1} h_n F_5 + 2.8511931724953071 \times 10^{-1} h_n F_6 \\
&+ 4.5492113050630001 \times 10^{-2} h_n^2 y_n^{(2)} + 1.7047164889895806 \times 10^{-3} h_n^3 y_n^{(3)} + 8.5823701987407336 \times 10^{-6} h_n^4 y_n^{(4)} \\
Y_8 = y_n &+ 3.6681574297060787 \times 10^{-1} h_n f_n + 1.4769444554327010 \times 10^{-1} h_n F_2 + 3.2407092041002571 \times 10^{-3} h_n F_3 \\
&+ 1.6254849764957939 \times 10^{-2} h_n F_4 + 6.2578448047528268 \times 10^{-2} h_n F_5 + 1.2021020870306121 \times 10^{-1} h_n F_6 \\
&+ 2.0536788957744442 \times 10^{-1} h_n F_7 + 6.8342301681705117 \times 10^{-2} h_n^2 y_n^{(2)} + 8.0942855459555183 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+ 5.4762347495797596 \times 10^{-4} h_n^4 y_n^{(4)} \\
Y_9 = y_n &+ 3.3824896499966395 \times 10^{-1} h_n f_n + 5.1958362534998893 \times 10^{-2} h_n F_2 + 5.8709978876240243 \times 10^{-2} h_n F_3 \\
&+ 3.3582767124999899 \times 10^{-2} h_n F_4 + 1.2928802654051971 \times 10^{-1} h_n F_5 + 1.9556350396190098 \times 10^{-1} h_n F_6 \\
&+ 3.6538671403563990 \times 10^{-2} h_n F_7 + 8.6663862879757428 \times 10^{-2} h_n F_8 + 5.0888840971576421 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+ 4.2522042950117224 \times 10^{-3} h_n^3 y_n^{(3)} + 1.6234154659505514 \times 10^{-4} h_n^4 y_n^{(4)} \\
y_{n+1} = y_n &+ 3.3700647201442324 \times 10^{-1} h_n f_n + 3.6816076090899930 \times 10^{-2} h_n F_2 + 6.6534363197990234 \times 10^{-2} h_n F_3 \\
&+ 8.5188754288484617 \times 10^{-2} h_n F_4 + 1.0365029367932606 \times 10^{-1} h_n F_5 + 1.5529050705249040 \times 10^{-1} h_n F_6 \\
&+ 5.1824548696949933 \times 10^{-2} h_n F_7 + 2.4724331619166373 \times 10^{-2} h_n F_8 + 1.3896465336026942 \times 10^{-1} h_n F_9 \\
&+ 4.9816502099013497 \times 10^{-2} h_n^2 y_n^{(2)} + 3.9036576666184285 \times 10^{-3} h_n^3 y_n^{(3)} + 1.4056070271231856 \times 10^{-4} h_n^4 y_n^{(4)}
\end{aligned}
$$

CPHBT$_{\mathbf{RK5}}$(5,9,9) with $c_{cp} = 1.9020010277318113$, and abscissa vector
$\sigma = [0\ 4.7455276167291471{\times}10^{-1}\ 5.6674886390240065{\times}10^{-1}\ 5.1147389582735725{\times}10^{-1}\ 6.1432770859660712{\times}10^{-1}\ 7.1768991263553961{\times}10^{-1}\ 8.1316420747753548{\times}10^{-1}\ 9.1620732886606948{\times}10^{-1}\ 9.3470878058632800{\times}10^{-1}]^T$

$$
\begin{aligned}
Y_2 &= y_n + 4.7455276167291471\times10^{-1}h_nf_n + 1.1260016180569510\times10^{-1}h_n^2y_n^{(2)} + 1.7811572583236554\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +2.1131327397781192\times10^{-3}h_n^4y_n^{(4)} + 2.0055859548863181\times10^{-4}h_n^5y_n^{(5)}\\
Y_3 &= y_n + 3.7141752416078644\times10^{-1}h_nf_n + 1.9533133974161412\times10^{-1}h_nF_2 + 6.7907110651677566\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +8.3460191704484103\times10^{-3}h_n^3y_n^{(3)} + 8.1968275223723948\times10^{-4}h_n^4y_n^{(4)} + 7.4511611411723054\times10^{-5}h_n^5y_n^{(5)}\\
Y_4 &= y_n + 4.3491434243959659\times10^{-1}h_nf_n + 2.0738617083130333\times10^{-2}h_nF_2 + 5.5820936304630413\times10^{-2}h_nF_3\\
&\quad +8.9324752813713007\times10^{-2}h_n^2y_n^{(2)} + 1.1000601320504635\times10^{-2}h_n^3y_n^{(3)} + 7.8854624066291689\times10^{-4}h_n^4y_n^{(4)}\\
&\quad +7.9110079281636096\times10^{-6}h_n^5y_n^{(5)}\\
Y_5 &= y_n + 3.6910193052084223\times10^{-1}h_nf_n + 8.4434018334683062\times10^{-3}h_nF_2 + 2.2726616439811907\times10^{-2}h_nF_3\\
&\quad +2.1405575980248470\times10^{-1}h_nF_4 + 6.2328209678721547\times10^{-2}h_n^2y_n^{(2)} + 6.0413041650083966\times10^{-3}h_n^3y_n^{(3)}\\
&\quad +3.2104420210370344\times10^{-4}h_n^4y_n^{(4)} + 3.2208424784299360\times10^{-6}h_n^5y_n^{(5)}\\
Y_6 &= y_n + 3.9988113339489240\times10^{-1}h_nf_n + 3.4804533319469188\times10^{-3}h_nF_2 + 9.3681349617031583\times10^{-3}h_nF_3\\
&\quad +8.8235890831806857\times10^{-2}h_nF_4 + 2.1672430011519028\times10^{-1}h_nF_5 + 7.2308269239915568\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +7.2774866075617332\times10^{-3}h_n^3y_n^{(3)} + 3.6601682529166220\times10^{-4}h_n^4y_n^{(4)} + 1.3276629677025455\times10^{-6}h_n^5y_n^{(5)}\\
Y_7 &= y_n + 4.7535248634568861\times10^{-1}h_nf_n + 1.3937670699328329\times10^{-3}h_nF_2 + 3.7515222216769201\times10^{-3}h_nF_3\\
&\quad +3.5334557684980436\times10^{-2}h_nF_4 + 8.6788181988257884\times10^{-2}h_nF_5 + 2.1054369216699886\times10^{-1}h_nF_6\\
&\quad +1.0533625430419623\times10^{-1}h_n^2y_n^{(2)} + 1.3634115886401907\times10^{-2}h_n^3y_n^{(3)} + 9.6598549009061874\times10^{-4}h_n^4y_n^{(4)}\\
&\quad +5.3167008658552745\times10^{-7}h_n^5y_n^{(5)}\\
Y_8 &= y_n + 4.2081773745920698\times10^{-1}h_nf_n + 1.5138981758373782\times10^{-1}h_nF_2 + 1.4968136093992199\times10^{-3}h_nF_3\\
&\quad +1.4098076380669596\times10^{-2}h_nF_4 + 3.4627472332277205\times10^{-2}h_nF_5 + 8.4004477432591373\times10^{-2}h_nF_6\\
&\quad +2.0977293406818728\times10^{-1}h_nF_7 + 8.7674739763968404\times10^{-2}h_n^2y_n^{(2)} + 1.1528553023546974\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +9.9164491431011923\times10^{-4}h_n^4y_n^{(4)} + 5.7749561716070669\times10^{-5}h_n^5y_n^{(5)}\\
Y_9 &= y_n + 3.8902781485146753\times10^{-1}h_nf_n + 3.3164629422596521\times10^{-2}h_nF_2 + 7.9496426303215101\times10^{-3}h_nF_3\\
&\quad +7.4796753896972726\times10^{-2}h_nF_4 + 1.8371517967900325\times10^{-1}h_nF_5 + 1.2672826068266901\times10^{-1}h_nF_6\\
&\quad +3.4031650040606709\times10^{-2}h_nF_7 + 8.5294849382690843\times10^{-2}h_nF_8 + 6.8705842172362322\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +6.9557495896525072\times10^{-3}h_n^3y_n^{(3)} + 4.1504059005451451\times10^{-4}h_n^4y_n^{(4)} + 1.2651067583006226\times10^{-5}h_n^5y_n^{(5)}\\
y_{n+1} &= y_n + 3.9382271106335764\times10^{-1}h_nf_n + 2.1488617027391115\times10^{-2}h_nF_2 + 3.5603525100765504\times10^{-2}h_nF_3\\
&\quad +8.7359508298123487\times10^{-2}h_nF_4 + 1.3858150189818005\times10^{-1}h_nF_5 + 1.4674605412553202\times10^{-1}h_nF_6\\
&\quad +9.3058815227011257\times10^{-3}h_nF_7 + 2.3323693147551176\times10^{-2}h_nF_8 + 1.4376850781639791\times10^{-1}h_nF_9\\
&\quad +7.1171297717510337\times10^{-2}h_n^2y_n^{(2)} + 7.4891218237655623\times10^{-3}h_n^3y_n^{(3)} + 4.6795998683294159\times10^{-4}h_n^4y_n^{(4)}\\
&\quad +1.4271591384369294\times10^{-5}h_n^5y_n^{(5)}
\end{aligned}
$$

CPHBT$_{\mathbf{RK5}}$(6,9,10) with $c_{cp} = 1.7750965780291403$, and abscissa vector
$\sigma = [0\ 5.1570811068367106{\times}10^{-1}\ 6.0284732063650370{\times}10^{-1}\ 5.5422164284318964{\times}10^{-1}\ 6.4684203893028136{\times}10^{-1}$

$7.4032023261514557\times10^{-1}$ $8.2588059877586173\times10^{-1}$ $9.2225087223511748\times10^{-1}$ $9.4121605629782168\times10^{-1}]^T$

$$Y_2 = y_n + 5.1570811068367106\times10^{-1}h_nf_n + 1.3297742771246074\times10^{-1}h_n^2y_n^{(2)} + 2.2859179336389174\times10^{-2}h_n^3y_n^{(3)} + 2.9471660468371164\times10^{-3}h_n^4y_n^{(4)} + 3.0397548677708631\times10^{-4}h_n^5y_n^{(5)} + 2.6127103996660038\times10^{-5}h_n^6y_n^{(6)}$$

$$Y_3 = y_n + 4.1256216712366539\times10^{-1}h_nf_n + 1.9028515351283828\times10^{-1}h_nF_2 + 8.3580848990047599\times10^{-2}h_n^2y_n^{(2)} + 1.1211323486321000\times10^{-2}h_n^3y_n^{(3)} + 1.1534730559727862\times10^{-3}h_n^4y_n^{(4)} + 1.0272021217507149\times10^{-4}h_n^5y_n^{(5)} + 8.8250701381885558\times10^{-6}h_n^6y_n^{(6)}$$

$$Y_4 = y_n + 4.8535311054212743\times10^{-1}h_nf_n + 1.7388607230158226\times10^{-2}h_nF_2 + 5.1479925070904028\times10^{-2}h_nF_3 + 1.1357883402025423\times10^{-1}h_n^2y_n^{(2)} + 1.6705768444891900\times10^{-2}h_n^3y_n^{(3)} + 1.6539014006865126\times10^{-3}h_n^4y_n^{(4)} + 1.0119549654505958\times10^{-4}h_n^5y_n^{(5)} + 8.0645113703631931\times10^{-7}h_n^6y_n^{(6)}$$

$$Y_5 = y_n + 4.2174521421585059\times10^{-1}h_nf_n + 6.1910911591465259\times10^{-3}h_nF_2 + 1.8329064815911622\times10^{-2}h_nF_3 + 2.0057666873937258\times10^{-1}h_nF_4 + 8.3795957260193579\times10^{-2}h_n^2y_n^{(2)} + 1.0148327169794486\times10^{-2}h_n^3y_n^{(3)} + 7.9257729502767572\times10^{-4}h_n^4y_n^{(4)} + 3.6029943957728955\times10^{-5}h_n^5y_n^{(5)} + 2.8713124856428131\times10^{-7}h_n^6y_n^{(6)}$$

$$Y_6 = y_n + 4.5566210386039169\times10^{-1}h_nf_n + 2.2352112998135056\times10^{-3}h_nF_2 + 6.6174656031373381\times10^{-3}h_nF_3 + 7.2415544355673905\times10^{-2}h_nF_4 + 2.0338990749612912\times10^{-1}h_nF_5 + 9.7199580982187506\times10^{-2}h_n^2y_n^{(2)} + 1.2454065345630058\times10^{-2}h_n^3y_n^{(3)} + 9.9440089444084289\times10^{-4}h_n^4y_n^{(4)} + 4.1912911823385969\times10^{-5}h_n^5y_n^{(5)} + 1.0366492671849418\times10^{-7}h_n^6y_n^{(6)}$$

$$Y_7 = y_n + 5.2520323243930289\times10^{-1}h_nf_n + 1.0417833105992604\times10^{-3}h_nF_2 + 3.0842566089337433\times10^{-3}h_nF_3 + 3.3751308229336759\times10^{-2}h_nF_4 + 6.9711901881672506\times10^{-2}h_nF_5 + 1.9308811630601663\times10^{-1}h_nF_6 + 1.3189745635677677\times10^{-1}h_n^2y_n^{(2)} + 2.0506217694819508\times10^{-2}h_n^3y_n^{(3)} + 2.0885100425046964\times10^{-3}h_n^4y_n^{(4)} + 1.2395658518860790\times10^{-4}h_n^5y_n^{(5)} + 4.8315964830185729\times10^{-8}h_n^6y_n^{(6)}$$

$$Y_8 = y_n + 4.8160736979522900\times10^{-1}h_nf_n + 1.2944787689101189\times10^{-1}h_nF_2 + 1.1121942190469233\times10^{-3}h_nF_3 + 1.2170845249778512\times10^{-2}h_nF_4 + 2.5138366907274501\times10^{-2}h_nF_5 + 6.9628281285082125\times10^{-2}h_nF_6 + 2.0314593788769464\times10^{-1}h_nF_7 + 1.1551811976682201\times10^{-1}h_n^2y_n^{(2)} + 1.7830784894433446\times10^{-2}h_n^3y_n^{(3)} + 1.8828214439303677\times10^{-3}h_n^4y_n^{(4)} + 1.3164419626471749\times10^{-4}h_n^5y_n^{(5)} + 6.0035508378518922\times10^{-6}h_n^6y_n^{(6)}$$

$$Y_9 = y_n + 4.4120999522091908\times10^{-1}h_nf_n + 2.4739063686188282\times10^{-2}h_nF_2 + 5.8019299208501443\times10^{-3}h_nF_3 + 6.3491061190048700\times10^{-2}h_nF_4 + 1.7704473963752998\times10^{-1}h_nF_5 + 1.2051140577870244\times10^{-1}h_nF_6 + 2.8734214467594502\times10^{-2}h_nF_7 + 7.9683646395988600\times10^{-2}h_nF_8 + 9.0543543407499344\times10^{-2}h_n^2y_n^{(2)} + 1.1124002273538746\times10^{-2}h_n^3y_n^{(3)} + 8.7027776793679285\times10^{-4}h_n^4y_n^{(4)} + 4.2457800434923912\times10^{-5}h_n^5y_n^{(5)} + 1.1473515834171237\times10^{-6}h_n^6y_n^{(6)}$$

$$y_{n+1} = y_n + 4.4589835580469794\times10^{-1}h_nf_n + 1.5754140812292369\times10^{-2}h_nF_2 + 3.0747857902913268\times10^{-2}h_nF_3 + 7.5574838469094283\times10^{-2}h_nF_4 + 1.3320083218132878\times10^{-1}h_nF_5 + 1.4051161271529586\times10^{-1}h_nF_6 + 6.7716604769356599\times10^{-3}h_nF_7 + 1.8778679318565160\times10^{-2}h_nF_8 + 1.3276202231887677\times10^{-1}h_nF_9 + 9.3401515770851526\times10^{-2}h_n^2y_n^{(2)} + 1.1904708255722525\times10^{-2}h_n^3y_n^{(3)} + 9.8856816632667249\times10^{-4}h_n^4y_n^{(4)} + 5.1728355684992869\times10^{-5}h_n^5y_n^{(5)} + 1.3695249809596008\times10^{-6}h_n^6y_n^{(6)}$$

$\mathbf{CPHBT_{RK5}(7,9,11)}$ with $c_{cp} = 1.6749618903443189$, and abscissa vector
$\sigma = [0\ 5.5349307290500882{\times}10^{-1}\ 6.3339587301574651{\times}10^{-1}\ 5.8923919315732054{\times}10^{-1}\ 6.7357340535177346{\times}10^{-1}$
$7.5859117482848781{\times}10^{-1}\ 8.3891817899559773{\times}10^{-1}\ 9.2796718875020578{\times}10^{-1}\ 9.4646481360042800{\times}10^{-1}]^T$

$$
\begin{aligned}
Y_2 &= y_n + 5.5349307290500882\times10^{-1}h_nf_n + 1.5317729087691467\times10^{-1}h_n^2y_n^{(2)} + 2.8260856475575958\times10^{-2}h_n^3y_n^{(3)}\\
&+3.9105470733984881\times10^{-3}h_n^4y_n^{(4)} + 4.3289214327900367\times10^{-4}h_n^5y_n^{(5)} + 3.9933800436655182\times10^{-5}h_n^6y_n^{(6)}\\
&+3.1575831309228082\times10^{-6}h_n^7y_n^{(7)}\\
Y_3 &= y_n + 4.5394720656420928\times10^{-1}h_nf_n + 1.7944866645153731\times10^{-1}h_nF_2 + 1.0127157215372252\times10^{-1}h_n^2y_n^{(2)}\\
&+1.4864589513659779\times10^{-2}h_n^3y_n^{(3)} + 1.6350304282155611\times10^{-3}h_n^4y_n^{(4)} + 1.4781919435209951\times10^{-4}h_n^5y_n^{(5)}\\
&+1.2002889521766912\times10^{-5}h_n^6y_n^{(6)} + 9.4907374359172831\times10^{-7}h_n^7y_n^{(7)}\\
Y_4 &= y_n + 5.2761668848221699\times10^{-1}h_nf_n + 1.4241341068557845\times10^{-2}h_nF_2 + 4.7381163606545658\times10^{-2}h_nF_3\\
&+1.3570789625895074\times10^{-1}h_n^2y_n^{(2)} + 2.2411703161789778\times10^{-2}h_n^3y_n^{(3)} + 2.6137465439289376\times10^{-3}h_n^4y_n^{(4)}\\
&+2.1849027196454133\times10^{-4}h_n^5y_n^{(5)} + 1.1714082854658347\times10^{-5}h_n^6y_n^{(6)} + 7.5320051962342271\times10^{-8}h_n^7y_n^{(7)}\\
Y_5 &= y_n + 4.6597504466909534\times10^{-1}h_nf_n + 4.4886882407174505\times10^{-3}h_nF_2 + 1.4933935707906488\times10^{-2}h_nF_3\\
&+1.8817573673405413\times10^{-1}h_nF_4 + 1.0402649582069610\times10^{-1}h_n^2y_n^{(2)} + 1.4582688516970886\times10^{-2}h_n^3y_n^{(3)}\\
&+1.4011879755371677\times10^{-3}h_n^4y_n^{(4)} + 9.2533671366378496\times10^{-5}h_n^5y_n^{(5)} + 3.6921288316437648\times10^{-6}h_n^6y_n^{(6)}\\
&+2.3739915356709439\times10^{-8}h_n^7y_n^{(7)}\\
Y_6 &= y_n + 5.0193632831540291\times10^{-1}h_nf_n + 1.4317737014441882\times10^{-3}h_nF_2 + 4.7635334108704344\times10^{-3}h_nF_3\\
&+6.0023119596882536\times10^{-2}h_nF_4 + 1.9043641980388765\times10^{-1}h_nF_5 + 1.2027972368234083\times10^{-1}h_n^2y_n^{(2)}\\
&+1.7960986664129559\times10^{-2}h_n^3y_n^{(3)} + 1.8096735754856695\times10^{-3}h_n^4y_n^{(4)} + 1.2104338086585174\times10^{-4}h_n^5y_n^{(5)}\\
&+4.4432149877852939\times10^{-6}h_n^6y_n^{(6)} + 7.5724097240522035\times10^{-9}h_n^7y_n^{(7)}\\
Y_7 &= y_n + 5.6677048059386492\times10^{-1}h_nf_n + 6.4450294980898323\times10^{-4}h_nF_2 + 2.1442713549794296\times10^{-3}h_nF_3\\
&+2.7018988823378779\times10^{-2}h_nF_4 + 5.8606228622709908\times10^{-2}h_nF_5 + 1.8373370665085578\times10^{-1}h_nF_6\\
&+1.5540194243328581\times10^{-1}h_n^2y_n^{(2)} + 2.7022828575077748\times10^{-2}h_n^3y_n^{(3)} + 3.2548174489206694\times10^{-3}h_n^4y_n^{(4)}\\
&+2.7226339860377137\times10^{-4}h_n^5y_n^{(5)} + 1.3713175664631010\times10^{-5}h_n^6y_n^{(6)} + 3.4086674447163797\times10^{-9}h_n^7y_n^{(7)}\\
Y_8 &= y_n + 5.3694236638718695\times10^{-1}h_nf_n + 1.0914372126551128\times10^{-1}h_nF_2 + 6.9592113621790301\times10^{-4}h_nF_3\\
&+8.7689859577520513\times10^{-3}h_nF_4 + 1.9020593227555255\times10^{-2}h_nF_5 + 5.9630591807825123\times10^{-2}h_nF_6\\
&+1.9376500896815707\times10^{-1}h_nF_7 + 1.4394343931682246\times10^{-1}h_n^2y_n^{(2)} + 2.5145396516901154\times10^{-2}h_n^3y_n^{(3)}\\
&+3.1099124022150025\times10^{-3}h_n^4y_n^{(4)} + 2.7394913035445759\times10^{-4}h_n^5y_n^{(5)} + 1.6085013922218056\times10^{-5}h_n^6y_n^{(6)}\\
&+5.7724274122129328\times10^{-7}h_n^7y_n^{(7)}\\
Y_9 &= y_n + 4.8714385554005740\times10^{-1}h_nf_n + 1.7727651324468435\times10^{-2}h_nF_2 + 4.1759949988426515\times10^{-3}h_nF_3\\
&+5.2619815089260329\times10^{-2}h_nF_4 + 1.6585742253032318\times10^{-1}h_nF_5 + 1.2096374522945873\times10^{-1}h_nF_6\\
&+2.4006752476647762\times10^{-2}h_nF_7 + 7.3969576411369503\times10^{-2}h_nF_8 + 1.1217475465089716\times10^{-1}h_n^2y_n^{(2)}\\
&+1.5892457752385913\times10^{-2}h_n^3y_n^{(3)} + 1.5009632308611337\times10^{-3}h_n^4y_n^{(4)} + 9.4939619072855879\times10^{-5}h_n^5y_n^{(5)}\\
&+3.8479036235082055\times10^{-6}h_n^6y_n^{(6)} + 9.3758559148423162\times10^{-8}h_n^7y_n^{(7)}\\
y_{n+1} &= y_n + 4.9051037546958509\times10^{-1}h_nf_n + 1.1539108800372367\times10^{-2}h_nF_2 + 2.7075828031432568\times10^{-2}h_nF_3
\end{aligned}
$$

$+6.4309480876208094\times10^{-2}h_nF_4+1.2575447572169510\times10^{-1}h_nF_5+1.3731881095084394\times10^{-1}h_nF_6$
$+4.9564763030656572\times10^{-3}h_nF_7+1.5271888731613748\times10^{-2}h_nF_8+1.2326355511518330\times10^{-1}h_nF_9$
$+1.1470158283529716\times10^{-1}h_n^2y_n^{(2)}+1.6736276000627756\times10^{-2}h_n^3y_n^{(3)}+1.6654678829641925\times10^{-3}h_n^4y_n^{(4)}$
$+1.1476887995161095\times10^{-4}h_n^5y_n^{(5)}+5.1739876869203847\times10^{-6}h_n^6y_n^{(6)}+1.2101439828293541\times10^{-7}h_n^7y_n^{(7)}$

CPHBT$_{\text{RK5}}$(8,9,12) with $c_{cp}=1.5995737614658707$, and abscissa vector
$\sigma=[0\ 5.8367999774694079\times10^{-1}\ 6.5725027014461568\times10^{-1}\ 6.2183948708365344\times10^{-1}\ 6.9962116599250190\times10^{-1}$
$7.7829229916174081\times10^{-1}\ 8.6160885547579769\times10^{-1}\ 9.3520342976171444\times10^{-1}\ 9.5096630143764826\times10^{-1}]^T$

$Y_2=y_n+5.8367999774694079\times10^{-1}h_nf_n+1.7034116988493439\times10^{-1}h_n^2y_n^{(2)}+3.3141577884883255\times10^{-2}h_n^3y_n^{(3)}$
$+4.8360190262946795\times10^{-3}h_n^4y_n^{(4)}+5.6453751487436812\times10^{-4}h_n^5y_n^{(5)}+5.4918209234989108\times10^{-5}h_n^6y_n^{(6)}$
$+4.5792371775063518\times10^{-6}h_n^7y_n^{(7)}+3.3410114318120176\times10^{-7}h_n^8y_n^{(8)}$
$Y_3=y_n+4.8836409404132047\times10^{-1}h_nf_n+1.6888617610329523\times10^{-1}h_nF_2+1.1741347591512433\times10^{-1}h_n^2y_n^{(2)}$
$+1.8551331692256076\times10^{-2}h_n^3y_n^{(3)}+2.1780506950966664\times10^{-3}h_n^4y_n^{(4)}+2.0531436353425514\times10^{-4}h_n^5y_n^{(5)}$
$+1.6614647453514115\times10^{-5}h_n^6y_n^{(6)}+1.2370621301726277\times10^{-6}h_n^7y_n^{(7)}+9.0256052668998137\times10^{-8}h_n^8y_n^{(8)}$
$Y_4=y_n+5.6262163714230973\times10^{-1}h_nf_n+1.2594977616775168\times10^{-2}h_nF_2+4.6622872324568479\times10^{-2}h_nF_3$
$+1.5534784191100795\times10^{-1}h_n^2y_n^{(2)}+2.7860463867455307\times10^{-2}h_n^3y_n^{(3)}+3.6066062401904730\times10^{-3}h_n^4y_n^{(4)}$
$+3.5142485155277841\times10^{-4}h_n^5y_n^{(5)}+2.5542722797616065\times10^{-5}h_n^6y_n^{(6)}+1.2222830676209110\times10^{-6}h_n^7y_n^{(7)}$
$+6.7310006619442525\times10^{-9}h_n^8y_n^{(8)}$
$Y_5=y_n+5.0532054593618048\times10^{-1}h_nf_n+3.5757857469196491\times10^{-3}h_nF_2+1.3236498500529456\times10^{-2}h_nF_3$
$+1.7748833580887233\times10^{-1}h_nF_4+1.2357882541768619\times10^{-1}h_n^2y_n^{(2)}+1.9289880821388456\times10^{-2}h_n^3y_n^{(3)}$
$+2.1246639526612851\times10^{-3}h_n^4y_n^{(4)}+1.7080073975800049\times10^{-4}h_n^5y_n^{(5)}+9.7999809207078964\times10^{-6}h_n^6y_n^{(6)}$
$+3.4701311148650835\times10^{-7}h_n^7y_n^{(7)}+1.9109693531674208\times10^{-9}h_n^8y_n^{(8)}$
$Y_6=y_n+5.4241543624197430\times10^{-1}h_nf_n+1.0292604288048770\times10^{-3}h_nF_2+3.8100168988777503\times10^{-3}h_nF_3$
$+5.1088553272486889\times10^{-2}h_nF_4+1.7994903231959705\times10^{-1}h_nF_5+1.4209952653295749\times10^{-1}h_n^2y_n^{(2)}$
$+2.3658028690527105\times10^{-2}h_n^3y_n^{(3)}+2.7561026860724652\times10^{-3}h_n^4y_n^{(4)}+2.3051671296732626\times10^{-4}h_n^5y_n^{(5)}$
$+1.3278214232465064\times10^{-5}h_n^6y_n^{(6)}+4.2741243031389195\times10^{-7}h_n^7y_n^{(7)}+5.5005676376680081\times10^{-10}h_n^8y_n^{(8)}$
$Y_7=y_n+6.0002706446909093\times10^{-1}h_nf_n+3.1268549794314038\times10^{-4}h_nF_2+1.1574689921584644\times10^{-3}h_nF_3$
$+1.5520512857713888\times10^{-2}h_nF_4+5.4667848098050587\times10^{-2}h_nF_5+1.8992327556084079\times10^{-1}h_nF_6$
$+1.7452778065469110\times10^{-1}h_n^2y_n^{(2)}+3.2400280401219887\times10^{-2}h_n^3y_n^{(3)}+4.2328335093514335\times10^{-3}h_n^4y_n^{(4)}$
$+4.0049341906073151\times10^{-4}h_n^5y_n^{(5)}+2.6519246389432526\times10^{-5}h_n^6y_n^{(6)}+1.0178503760659854\times10^{-6}h_n^7y_n^{(7)}$
$+1.6710520317498831\times10^{-10}h_n^8y_n^{(8)}$
$Y_8=y_n+5.4727474760898787\times10^{-1}h_nf_n+3.9205580775729061\times10^{-2}h_nF_2+8.3544758888557911\times10^{-3}h_nF_3$
$+1.1202524761431940\times10^{-1}h_nF_4+1.4352320839379899\times10^{-2}h_nF_5+4.9861845317674888\times10^{-2}h_nF_6$
$+1.6412921171676759\times10^{-1}h_nF_7+1.4900305023790236\times10^{-1}h_n^2y_n^{(2)}+2.6643902890896059\times10^{-2}h_n^3y_n^{(3)}$
$+3.4540667897824491\times10^{-3}h_n^4y_n^{(4)}+3.3443491039311933\times10^{-4}h_n^5y_n^{(5)}+2.3545176224436274\times10^{-5}h_n^6y_n^{(6)}$
$+1.0620371539432916\times10^{-6}h_n^7y_n^{(7)}+2.0952223829430315\times10^{-8}h_n^8y_n^{(8)}$

$$
\begin{aligned}
Y_9 = y_n &+ 5.2867882886172857 \times 10^{-1} h_n f_n + 1.9601590854701780 \times 10^{-2} h_n F_2 + 3.6320894911546879 \times 10^{-3} h_n F_3 \\
&+ 4.8702722949589189 \times 10^{-2} h_n F_4 + 1.3193638196589913 \times 10^{-1} h_n F_5 + 1.3619355772732789 \times 10^{-1} h_n F_6 \\
&+ 1.7097379646841691 \times 10^{-2} h_n F_7 + 6.5123749940405318 \times 10^{-2} h_n F_8 + 1.3411583809649510 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 2.1429194481804102 \times 10^{-2} h_n^3 y_n^{(3)} + 2.3709034783270739 \times 10^{-3} h_n^4 y_n^{(4)} + 1.8709472466339362 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+ 1.0412749803897490 \times 10^{-5} h_n^6 y_n^{(6)} + 3.9640341854826657 \times 10^{-7} h_n^7 y_n^{(7)} + 1.0475470860895419 \times 10^{-8} h_n^8 y_n^{(8)} \\
y_{n+1} = y_n &+ 5.2788010881971537 \times 10^{-1} h_n f_n + 9.5039305060613394 \times 10^{-3} h_n F_2 + 2.2553078705528255 \times 10^{-2} h_n F_3 \\
&+ 6.0141544718733363 \times 10^{-2} h_n F_4 + 1.2150793075161179 \times 10^{-1} h_n F_5 + 1.2881800313019395 \times 10^{-1} h_n F_6 \\
&+ 3.1322878461796108 \times 10^{-3} h_n F_7 + 1.1930853420199575 \times 10^{-2} h_n F_8 + 1.1453226210177671 \times 10^{-1} h_n F_9 \\
&+ 1.3419085778117998 \times 10^{-1} h_n^2 y_n^{(2)} + 2.1628446754516500 \times 10^{-2} h_n^3 y_n^{(3)} + 2.4410917167951196 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+ 1.9995910334281173 \times 10^{-4} h_n^5 y_n^{(5)} + 1.1788949870597435 \times 10^{-5} h_n^6 y_n^{(6)} + 4.6672112029656844 \times 10^{-7} h_n^7 y_n^{(7)} \\
&+ 9.7672516174141138 \times 10^{-9} h_n^8 y_n^{(8)}
\end{aligned}
$$

$\mathbf{CPHBT_{RK5}(9,9,13)}$ with $c_{cp} = 1.5443902726738195$, and abscissa vector
$\sigma = [0\ 6.1066993624474653\times10^{-1}\ 6.7895376046654976\times10^{-1}\ 6.4934433993014595\times10^{-1}\ 7.2147094241035892\times10^{-1}\ 7.9458966163650724\times10^{-1}\ 8.7535102291325484\times10^{-1}\ 9.4105152870534625\times10^{-1}\ 9.5475871341761676\times10^{-1}]^T$

$$
\begin{aligned}
Y_2 = y_n &+ 6.1066993624474641 \times 10^{-1} h_n f_n + 1.8645888551658135 \times 10^{-1} h_n^2 y_n^{(2)} + 3.7954945243559066 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+ 5.7944859980142652 \times 10^{-3} h_n^4 y_n^{(4)} + 7.0770367899569123 \times 10^{-4} h_n^5 y_n^{(5)} + 7.2028893422078366 \times 10^{-5} h_n^6 y_n^{(6)} \\
&+ 6.2836971076914714 \times 10^{-6} h_n^7 y_n^{(7)} + 4.7965811401690332 \times 10^{-7} h_n^8 y_n^{(8)} + 3.2545865545108939 \times 10^{-8} h_n^9 y_n^{(9)} \\
Y_3 = y_n &+ 5.1951749939939029 \times 10^{-1} h_n f_n + 1.5943626106715947 \times 10^{-1} h_n F_2 + 1.3312617304485150 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 2.2435507182649295 \times 10^{-2} h_n^3 y_n^{(3)} + 2.8028099845287175 \times 10^{-3} h_n^4 y_n^{(4)} + 2.7846791177572877 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+ 2.3219549848022926 \times 10^{-5} h_n^6 y_n^{(6)} + 1.7122421265687045 \times 10^{-6} h_n^7 y_n^{(7)} + 1.1810708593305828 \times 10^{-7} h_n^8 y_n^{(8)} \\
&+ 8.0138274040883032 \times 10^{-9} h_n^9 y_n^{(9)} \\
Y_4 = y_n &+ 5.9297427463681229 \times 10^{-1} h_n f_n + 1.1137657639168600 \times 10^{-2} h_n F_2 + 4.5232407654164981 \times 10^{-2} h_n F_3 \\
&+ 1.7331188994748031 \times 10^{-1} h_n^2 y_n^{(2)} + 3.3130172448993675 \times 10^{-2} h_n^3 y_n^{(3)} + 4.6255715669057500 \times 10^{-3} h_n^4 y_n^{(4)} \\
&+ 4.9700804935723397 \times 10^{-4} h_n^5 y_n^{(5)} + 4.1850143554276770 \times 10^{-5} h_n^6 y_n^{(6)} + 2.7019247595202553 \times 10^{-6} h_n^7 y_n^{(7)} \\
&+ 1.1705055307300311 \times 10^{-7} h_n^8 y_n^{(8)} + 5.5981785704673662 \times 10^{-10} h_n^9 y_n^{(9)} \\
Y_5 = y_n &+ 5.3937585704423252 \times 10^{-1} h_n f_n + 2.8813543890392110 \times 10^{-3} h_n F_2 + 1.1701795884148899 \times 10^{-2} h_n F_3 \\
&+ 1.6751193509293830 \times 10^{-1} h_n F_4 + 1.4178269862709225 \times 10^{-1} h_n^2 y_n^{(2)} + 2.4040114925646401 \times 10^{-2} h_n^3 y_n^{(3)} \\
&+ 2.9254707352807173 \times 10^{-3} h_n^4 y_n^{(4)} + 2.6776933385940752 \times 10^{-4} h_n^5 y_n^{(5)} + 1.8613742229795332 \times 10^{-5} h_n^6 y_n^{(6)} \\
&+ 9.4808013172561088 \times 10^{-7} h_n^7 y_n^{(7)} + 3.0281423236619320 \times 10^{-8} h_n^8 y_n^{(8)} + 1.4482700866935133 \times 10^{-10} h_n^9 y_n^{(9)} \\
Y_6 = y_n &+ 5.7670422799400756 \times 10^{-1} h_n f_n + 7.5675662180887178 \times 10^{-4} h_n F_2 + 3.0733503508182844 \times 10^{-3} h_n F_3 \\
&+ 4.3995201213626364 \times 10^{-2} h_n F_4 + 1.7006012545624624 \times 10^{-1} h_n F_5 + 1.6187610002239944 \times 10^{-1} h_n^2 y_n^{(2)} \\
&+ 2.9229108196148750 \times 10^{-2} h_n^3 y_n^{(3)} + 3.7689256870251328 \times 10^{-3} h_n^4 y_n^{(4)} + 3.6221913552116286 \times 10^{-4} h_n^5 y_n^{(5)} \\
&+ 2.5983890465007017 \times 10^{-5} h_n^6 y_n^{(6)} + 1.3159303174327268 \times 10^{-6} h_n^7 y_n^{(7)} + 3.7685331377958376 \times 10^{-8} h_n^8 y_n^{(8)} \\
&+ 3.8037250205744040 \times 10^{-11} h_n^9 y_n^{(9)} \\
Y_7 = y_n &+ 6.2720288693577209 \times 10^{-1} h_n f_n + 2.1699778192136842 \times 10^{-4} h_n F_2 + 8.8127436215981400 \times 10^{-4} h_n F_3
\end{aligned}
$$

$$+1.2615497246289542\times10^{-2}h_nF_4+4.8764251218642647\times10^{-2}h_nF_5+1.8567011536846936\times10^{-1}h_nF_6$$
$$+1.9148350193222305\times10^{-1}h_n^2y_n^{(2)}+3.7579924671996903\times10^{-2}h_n^3y_n^{(3)}+5.2568450950897464\times10^{-3}h_n^4y_n^{(4)}$$
$$+5.4588241547962740\times10^{-4}h_n^5y_n^{(5)}+4.1954026524626122\times10^{-5}h_n^6y_n^{(6)}+2.2308520205329562\times10^{-6}h_n^7y_n^{(7)}$$
$$+6.2769335607886703\times10^{-8}h_n^8y_n^{(8)}+1.0907071953073876\times10^{-11}h_n^9y_n^{(9)}$$
$$Y_8=y_n+5.7492750348042798\times10^{-1}h_nf_n+2.5782638153924475\times10^{-3}h_nF_2+1.0470880297813708\times10^{-2}h_nF_3$$
$$+1.4989130199999628\times10^{-1}h_nF_4+1.1234443717370101\times10^{-2}h_nF_5+4.2775197177770889\times10^{-2}h_nF_6$$
$$+1.4917393821657490\times10^{-1}h_nF_7+1.6410059596948864\times10^{-1}h_n^2y_n^{(2)}+3.0821998582136266\times10^{-2}h_n^3y_n^{(3)}$$
$$+4.2374217479771896\times10^{-3}h_n^4y_n^{(4)}+4.4551729645637235\times10^{-4}h_n^5y_n^{(5)}+3.5891702862382763\times10^{-5}h_n^6y_n^{(6)}$$
$$+2.0993782424190732\times10^{-6}h_n^7y_n^{(7)}+7.2239584718039931\times10^{-8}h_n^8y_n^{(8)}+1.2959261011563081\times10^{-10}h_n^9y_n^{(9)}$$
$$Y_9=y_n+5.6457324474389903\times10^{-1}h_nf_n+1.7873376320914888\times10^{-2}h_nF_2+3.0015459445024849\times10^{-3}h_nF_3$$
$$+4.2967316676138428\times10^{-2}h_nF_4+1.1454456123833853\times10^{-1}h_nF_5+1.3952543192472175\times10^{-1}h_nF_6$$
$$+1.3532787827691763\times10^{-2}h_nF_7+5.8740448741409990\times10^{-2}h_nF_8+1.5429910501305300\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.6918996689930572\times10^{-2}h_n^3y_n^{(3)}+3.3200362089750067\times10^{-3}h_n^4y_n^{(4)}+3.0175549229113295\times10^{-4}h_n^5y_n^{(5)}$$
$$+2.0340585279463778\times10^{-5}h_n^6y_n^{(6)}+9.9554411542525170\times10^{-7}h_n^7y_n^{(7)}+3.4524193462865353\times10^{-8}h_n^8y_n^{(8)}$$
$$+8.9837877535148504\times10^{-10}h_n^9y_n^{(9)}$$
$$y_{n+1}=y_n+5.6067082694593062\times10^{-1}h_nf_n+7.6369912480236694\times10^{-3}h_nF_2+1.9534074784199648\times10^{-2}h_nF_3$$
$$+5.4727913745692226\times10^{-2}h_nF_4+1.1816054010190716\times10^{-1}h_nF_5+1.2050960357368530\times10^{-1}h_nF_6$$
$$+2.2328476804335490\times10^{-3}h_nF_7+9.6919035744758263\times10^{-3}h_nF_8+1.0683529834565186\times10^{-1}h_nF_9$$
$$+1.5245420495996173\times10^{-1}h_n^2y_n^{(2)}+2.6566078074664216\times10^{-2}h_n^3y_n^{(3)}+3.2958801359931960\times10^{-3}h_n^4y_n^{(4)}$$
$$+3.0487397408904472\times10^{-4}h_n^5y_n^{(5)}+2.1284033306127121\times10^{-5}h_n^6y_n^{(6)}+1.0982435063017648\times10^{-6}h_n^7y_n^{(7)}$$
$$+3.8787483566763066\times10^{-8}h_n^8y_n^{(8)}+7.3459532317902453\times10^{-10}h_n^9y_n^{(9)}$$

CPHBT$_{\mathbf{RK5}}$(10,9,14) with $c_{cp}=1.4966663165833098$, and abscissa vector
$\sigma=[0\ 6.3574862054060821\times10^{-1}\ 6.9962186576632690\times10^{-1}\ 6.7117475903227730\times10^{-1}\ 7.3824913421514082\times10^{-1}$
$8.0611771463251769\times10^{-1}\ 8.8231297820273324\times10^{-1}\ 9.4532082024719333\times10^{-1}\ 9.5793362338703303\times10^{-1}]^T$

$$Y_2=y_n+6.3574862054060821\times10^{-1}h_nf_n+2.0208815425964313\times10^{-1}h_n^2y_n^{(2)}+4.2825755099388581\times10^{-2}h_n^3y_n^{(3)}$$
$$+6.8066036820115519\times10^{-3}h_n^4y_n^{(4)}+8.6545778028109378\times10^{-4}h_n^5y_n^{(5)}+9.1702264991640363\times10^{-5}h_n^6y_n^{(6)}$$
$$+8.3285126384120962\times10^{-6}h_n^7y_n^{(7)}+6.6185505262818893\times10^{-7}h_n^8y_n^{(8)}+4.6752604078466976\times10^{-8}h_n^9y_n^{(9)}$$
$$+2.9722903549566594\times10^{-9}h_n^{10}y_n^{(10)}$$
$$Y_3=y_n+5.4840816005316295\times10^{-1}h_nf_n+1.5121370571316395\times10^{-1}h_nF_2+1.4860147271520072\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.6515575128995845\times10^{-2}h_n^3y_n^{(3)}+3.5067263738485861\times10^{-3}h_n^4y_n^{(4)}+3.6755273415446410\times10^{-4}h_n^5y_n^{(5)}$$
$$+3.2003416999293519\times10^{-5}h_n^6y_n^{(6)}+2.4118119596952440\times10^{-6}h_n^7y_n^{(7)}+1.6420979714970234\times10^{-7}h_n^8y_n^{(8)}$$
$$+1.0582692555151785\times10^{-8}h_n^9y_n^{(9)}+6.7267823106731442\times10^{-10}h_n^{10}y_n^{(10)}$$
$$Y_4=y_n+6.1989570087659529\times10^{-1}h_nf_n+9.4635400324778653\times10^{-3}h_nF_2+4.1815518123204144\times10^{-2}h_nF_3$$
$$+1.8996629525259737\times10^{-1}h_n^2y_n^{(2)}+3.8245097968973740\times10^{-2}h_n^3y_n^{(3)}+5.6634776008910809\times10^{-3}h_n^4y_n^{(4)}$$
$$+6.5316173676251480\times10^{-4}h_n^5y_n^{(5)}+6.0365777300597296\times10^{-5}h_n^6y_n^{(6)}+4.4951670205235307\times10^{-6}h_n^7y_n^{(7)}$$

$$+2.6181694376066425\times10^{-7}h_n^8y_n^{(8)}+1.0373504730511371\times10^{-8}h_n^9y_n^{(9)}+4.2098812000265442\times10^{-11}h_n^{10}y_n^{(10)}$$

$$Y_5=y_n+5.6801307249977362\times10^{-1}h_nf_n+2.2393204375027007\times10^{-3}h_nF_2+9.8946423871721090\times10^{-3}h_nF_3$$
$$+1.5810209889069235\times10^{-1}h_nF_4+1.5804560091215011\times10^{-1}h_n^2y_n^{(2)}+2.8574404916064541\times10^{-2}h_n^3y_n^{(3)}$$
$$+3.7489778157134934\times10^{-3}h_n^4y_n^{(4)}+3.7657588335121236\times10^{-4}h_n^5y_n^{(5)}+2.9640806644274963\times10^{-5}h_n^6y_n^{(6)}$$
$$+1.8229773390128648\times10^{-6}h_n^7y_n^{(7)}+8.3890543419711525\times10^{-8}h_n^8y_n^{(8)}+2.4577408891125720\times10^{-9}h_n^9y_n^{(9)}$$
$$+9.9616771084859449\times10^{-12}h_n^{10}y_n^{(10)}$$

$$Y_6=y_n+6.0510994128216322\times10^{-1}h_nf_n+5.3688864160976782\times10^{-4}h_nF_2+2.3722916209292424\times10^{-3}h_nF_3$$
$$+3.7905794850753509\times10^{-2}h_nF_4+1.6019279823706184\times10^{-1}h_nF_5+1.7920824428801982\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.4425629812624300\times10^{-2}h_n^3y_n^{(3)}+4.7838374138026258\times10^{-3}h_n^4y_n^{(4)}+5.0620386128954217\times10^{-4}h_n^5y_n^{(5)}$$
$$+4.1579291843853056\times10^{-5}h_n^6y_n^{(6)}+2.6223023325908214\times10^{-6}h_n^7y_n^{(7)}+1.1927180537869958\times10^{-7}h_n^8y_n^{(8)}$$
$$+3.1231771531451345\times10^{-9}h_n^9y_n^{(9)}+2.3883635416174762\times10^{-12}h_n^{10}y_n^{(10)}$$

$$Y_7=y_n+6.5140350724245877\times10^{-1}h_nf_n+1.4303606034849948\times10^{-4}h_nF_2+6.3201792915207203\times10^{-4}h_nF_3$$
$$+1.0098733963850757\times10^{-2}h_nF_4+4.2552507966378078\times10^{-2}h_nF_5+1.7748317504054514\times10^{-1}h_nF_6$$
$$+2.0744028826265570\times10^{-1}h_n^2y_n^{(2)}+4.2756028815299413\times10^{-2}h_n^3y_n^{(3)}+6.3510833516394975\times10^{-3}h_n^4y_n^{(4)}$$
$$+7.1377166221590221\times10^{-4}h_n^5y_n^{(5)}+6.1551750570844538\times10^{-5}h_n^6y_n^{(6)}+3.9822823742306943\times10^{-6}h_n^7y_n^{(7)}$$
$$+1.7570580466179177\times10^{-7}h_n^8y_n^{(8)}+3.6349234965342158\times10^{-9}h_n^9y_n^{(9)}+6.3629975603262839\times10^{-13}h_n^{10}y_n^{(10)}$$

$$Y_8=y_n+6.1527305742706173\times10^{-1}h_nf_n+1.8095860295016977\times10^{-3}h_nF_2+7.9830880003632712\times10^{-3}h_nF_3$$
$$+1.2755817808199893\times10^{-1}h_nF_4+9.2320151896684889\times10^{-3}h_nF_5+3.8505904646060150\times10^{-2}h_nF_6$$
$$+1.4495899087253908\times10^{-1}h_nF_7+1.8871129438824891\times10^{-1}h_n^2y_n^{(2)}+3.8293971091644889\times10^{-2}h_n^3y_n^{(3)}$$
$$+5.7378008445576134\times10^{-3}h_n^4y_n^{(4)}+6.6824490015696873\times10^{-4}h_n^5y_n^{(5)}+6.1644614560532883\times10^{-5}h_n^6y_n^{(6)}$$
$$+4.4556095273682269\times10^{-6}h_n^7y_n^{(7)}+2.3764039752040628\times10^{-7}h_n^8y_n^{(8)}+7.5707086785176945\times10^{-9}h_n^9y_n^{(9)}$$
$$+8.0499938880985079\times10^{-12}h_n^{10}y_n^{(10)}$$

$$Y_9=y_n+5.9611486618214826\times10^{-1}h_nf_n+1.3442041437963377\times10^{-2}h_nF_2+2.1828621764766291\times10^{-3}h_nF_3$$
$$+3.4878982428829738\times10^{-2}h_nF_4+1.0411076376083531\times10^{-1}h_nF_5+1.4076886452819071\times10^{-1}h_nF_6$$
$$+1.1843881787456869\times10^{-2}h_nF_7+5.4591361085132066\times10^{-2}h_nF_8+1.7294326594411871\times10^{-1}h_n^2y_n^{(2)}$$
$$+3.2288313182393676\times10^{-2}h_n^3y_n^{(3)}+4.3143112302548876\times10^{-3}h_n^4y_n^{(4)}+4.3288857164897499\times10^{-4}h_n^5y_n^{(5)}$$
$$+3.3154054406687788\times10^{-5}h_n^6y_n^{(6)}+1.9205195160648545\times10^{-6}h_n^7y_n^{(7)}+8.1356149730429533\times10^{-8}h_n^8y_n^{(8)}$$
$$+2.4505496855661689\times10^{-9}h_n^9y_n^{(9)}+5.9797282504588469\times10^{-11}h_n^{10}y_n^{(10)}$$

$$y_{n+1}=y_n+5.8969700264687930\times10^{-1}h_nf_n+5.9263765358245945\times10^{-3}h_nF_2+1.7599863961789489\times10^{-2}h_nF_3$$
$$+4.7066104995185667\times10^{-2}h_nF_4+1.1114691305701783\times10^{-1}h_nF_5+1.1831704599391640\times10^{-1}h_nF_6$$
$$+1.7775281476166377\times10^{-3}h_nF_7+8.1930639537699974\times10^{-3}h_nF_8+1.0027610070800021\times10^{-1}h_nF_9$$
$$+1.6952664531670161\times10^{-1}h_n^2y_n^{(2)}+3.1468527515514912\times10^{-2}h_n^3y_n^{(3)}+4.2053855785304135\times10^{-3}h_n^4y_n^{(4)}$$
$$+4.2620777807727682\times10^{-4}h_n^5y_n^{(5)}+3.3503944431217883\times10^{-5}h_n^6y_n^{(6)}+2.0419345291708860\times10^{-6}h_n^7y_n^{(7)}$$
$$+9.3785363603542721\times10^{-8}h_n^8y_n^{(8)}+2.9917683775701943\times10^{-9}h_n^9y_n^{(9)}+5.1763384829666555\times10^{-11}h_n^{10}y_n^{(10)}$$

B.3 Ten stages $\text{CPHBT}_{\text{RK5}}(p-4,10,p)$ methods formulae

$\mathbf{CPHBT_{RK5}(2,10,6)}$ with $c_{cp} = 3.5646029568785740$, and abscissa vector $\sigma = [0\ 2.7842964332218451\times10^{-1}\ 4.1816904481179606\times10^{-1}\ 3.9067928644511601\times10^{-1}\ 3.8637915334868184\times10^{-1}\ 5.2647383807775272\times10^{-1}\ 7.0948880078541965\times10^{-1}\ 8.6092081271814713\times10^{-1}\ 8.6331609198841319\times10^{-1}\ 9.2359094107564599\times10^{-1}]^T$

$$Y_2 = y_n + 2.7842964332218451\times10^{-1}h_nf_n + 3.8761533140259471\times10^{-2}h_n^2y_n^{(2)}$$
$$Y_3 = y_n + 2.0829660140341305\times10^{-1}h_nf_n + 2.0987244340838301\times10^{-1}h_nF_2 + 2.8997965458053533\times10^{-2}h_n^2y_n^{(2)}$$
$$Y_4 = y_n + 2.0788727118384684\times10^{-1}h_nf_n + 7.8226619945594530\times10^{-2}h_nF_2 + 1.0456539531567460\times10^{-1}h_nF_3 + 1.0808531059356920\times10^{-2}h_n^2y_n^{(2)}$$
$$Y_5 = y_n + 2.0035382524896286\times10^{-1}h_nf_n + 3.1407830653775524\times10^{-2}h_nF_2 + 4.1982796017569143\times10^{-2}h_nF_3 + 1.1263470142837431\times10^{-1}h_nF_4 + 4.3396034925764770\times10^{-3}h_n^2y_n^{(2)}$$
$$Y_6 = y_n + 1.7321599331601423\times10^{-1}h_nf_n + 2.3780494143120792\times10^{-2}h_nF_2 + 3.1787347741816220\times10^{-2}h_nF_3 + 8.5281562014141582\times10^{-2}h_nF_4 + 2.1240844086265992\times10^{-1}h_nF_5 + 3.2857384063319909\times10^{-3}h_n^2y_n^{(2)}$$
$$Y_7 = y_n + 1.5228416193073463\times10^{-1}h_nf_n + 2.0906795741995821\times10^{-2}h_nF_2 + 2.7946079775225816\times10^{-2}h_nF_3 + 7.4975910376689242\times10^{-2}h_nF_4 + 1.8674043778337787\times10^{-1}h_nF_5 + 2.4663541517739626\times10^{-1}h_nF_6 + 2.8886810050869206\times10^{-3}h_n^2y_n^{(2)}$$
$$Y_8 = y_n + 1.7211789900772700\times10^{-1}h_nf_n + 1.7189877825122258\times10^{-2}h_nF_2 + 2.2977681656987981\times10^{-2}h_nF_3 + 6.1646306545852791\times10^{-2}h_nF_4 + 1.5354076014900933\times10^{-1}h_nF_5 + 2.0278729892720190\times10^{-1}h_nF_6 + 2.3066098860624595\times10^{-1}h_nF_7 + 2.3751164054973081\times10^{-3}h_n^2y_n^{(2)}$$
$$Y_9 = y_n + 1.8487661073348063\times10^{-1}h_nf_n + 1.1892874300183151\times10^{-1}h_nF_2 + 1.5897185683123877\times10^{-1}h_nF_3 + 2.6573936974398974\times10^{-2}h_nF_4 + 6.6186967022478962\times10^{-2}h_nF_5 + 8.7415721100029306\times10^{-2}h_nF_6 + 9.9431260021363793\times10^{-2}h_nF_7 + 1.2093099630359121\times10^{-1}h_nF_8 + 1.6432322059672003\times10^{-2}h_n^2y_n^{(2)}$$
$$Y_{10} = y_n + 1.8604197008738452\times10^{-1}h_nf_n + 9.5175947307226436\times10^{-2}h_nF_2 + 1.2722153356038512\times10^{-1}h_nF_3 + 4.4559371989602158\times10^{-2}h_nF_4 + 8.1796442454806009\times10^{-2}h_nF_5 + 1.0803176727439978\times10^{-1}h_nF_6 + 9.4146663729170951\times10^{-2}h_nF_7 + 5.6213339927445892\times10^{-2}h_nF_8 + 1.3040390474522515\times10^{-1}h_nF_9 + 1.3150410733447590\times10^{-2}h_n^2y_n^{(2)}$$
$$y_{n+1} = y_n + 1.8166312024945286\times10^{-1}h_nf_n + 6.3485441638502421\times10^{-2}h_nF_2 + 8.4860886311297734\times10^{-2}h_nF_3 + 9.0655938213552140\times10^{-2}h_nF_4 + 9.4477412549241357\times10^{-2}h_nF_5 + 1.2474549960713693\times10^{-1}h_nF_6 + 1.2862535040010073\times10^{-1}h_nF_7 + 4.1753746470125870\times10^{-2}h_nF_8 + 6.0207984191906955\times10^{-2}h_nF_9 + 1.2952462036868309\times10^{-1}h_nF_{10} + 1.0429630883138145\times10^{-2}h_n^2y_n^{(2)}$$

$\mathbf{CPHBT_{RK5}(3,10,7)}$ with $c_{cp} = 3.1200969859848842$, and abscissa vector $sigma = [0\ 3.2050285760086444{\times}10^{-1}\ 4.4361640980594369{\times}10^{-1}\ 4.9699659275932151{\times}10^{-1}\ 4.3080263699822630{\times}10^{-1}\ 5.5962554634995565{\times}10^{-1}\ 7.0303366569480485{\times}10^{-1}\ 8.3252919142375004{\times}10^{-1}\ 9.0103587809232788{\times}10^{-1}\ 9.2299674761687311{\times}10^{-1}]^T$

$$
\begin{aligned}
Y_2 &= y_n + 3.2050285760086444 \times 10^{-1} h_n f_n + 5.1361040865160010 \times 10^{-2} h_n^2 y_n^{(2)} + 5.4871201222128525 \times 10^{-3} h_n^3 y_n^{(3)} \\
Y_3 &= y_n + 2.3894270785911881 \times 10^{-1} h_n f_n + 2.0467370194682488 \times 10^{-1} h_n F_2 + 3.2799253174852494 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 4.0380325680298120 \times 10^{-3} h_n^3 y_n^{(3)} \\
Y_4 &= y_n + 2.6384659966337665 \times 10^{-1} h_n f_n + 9.0864055769972821 \times 10^{-2} h_n F_2 + 1.4228593732597206 \times 10^{-1} h_n F_3 \\
&\quad + 3.1260240397285581 \times 10^{-2} h_n^2 y_n^{(2)} + 1.7926681003588467 \times 10^{-3} h_n^3 y_n^{(3)} \\
Y_5 &= y_n + 3.0792904785269803 \times 10^{-1} h_n f_n + 2.0166165944127908 \times 10^{-2} h_n F_2 + 3.1577735115338887 \times 10^{-2} h_n F_3 \\
&\quad + 7.1129688086061446 \times 10^{-2} h_n F_4 + 3.6972528105890981 \times 10^{-2} h_n^2 y_n^{(2)} + 3.9785928087344042 \times 10^{-4} h_n^3 y_n^{(3)} \\
Y_6 &= y_n + 2.5889870899479361 \times 10^{-1} h_n f_n + 1.3678009624687621 \times 10^{-2} h_n F_2 + 2.1418080463590086 \times 10^{-2} h_n F_3 \\
&\quad + 4.8244795809858680 \times 10^{-2} h_n F_4 + 2.1738595145702558 \times 10^{-1} h_n F_5 + 2.5077182662621995 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 2.6985412537690751 \times 10^{-4} h_n^3 y_n^{(3)} \\
Y_7 &= y_n + 2.4990868717298748 \times 10^{-1} h_n f_n + 9.9767410793949567 \times 10^{-3} h_n F_2 + 1.5622349235462385 \times 10^{-2} h_n F_3 \\
&\quad + 3.5189757094079489 \times 10^{-2} h_n F_4 + 1.5856132664727501 \times 10^{-1} h_n F_5 + 2.3377480446560542 \times 10^{-1} h_n F_6 \\
&\quad + 2.0376083350789522 \times 10^{-2} h_n^2 y_n^{(2)} + 1.9683161600008567 \times 10^{-4} h_n^3 y_n^{(3)} \\
Y_8 &= y_n + 2.7201857717838834 \times 10^{-1} h_n f_n + 3.6032833524090833 \times 10^{-2} h_n F_2 + 1.0729474639132179 \times 10^{-2} h_n F_3 \\
&\quad + 2.4168426950863291 \times 10^{-2} h_n F_4 + 1.0890037774518724 \times 10^{-1} h_n F_5 + 1.6055721182919455 \times 10^{-1} h_n F_6 \\
&\quad + 2.2012228955689364 \times 10^{-1} h_n F_7 + 2.6712536887582496 \times 10^{-2} h_n^2 y_n^{(2)} + 6.3476958226611768 \times 10^{-4} h_n^3 y_n^{(3)} \\
Y_9 &= y_n + 2.4715921561947546 \times 10^{-1} h_n f_n + 1.0217860189175355 \times 10^{-1} h_n F_2 + 1.3730578806730506 \times 10^{-1} h_n F_3 \\
&\quad + 1.2005028944805181 \times 10^{-2} h_n F_4 + 5.4093391745733480 \times 10^{-2} h_n F_5 + 7.9752562267518520 \times 10^{-2} h_n F_6 \\
&\quad + 1.0933994433729180 \times 10^{-1} h_n F_7 + 1.5920134521844481 \times 10^{-1} h_n F_8 + 2.8962157560692869 \times 10^{-2} h_n^2 y_n^{(2)} \\
&\quad + 1.9780802820534545 \times 10^{-3} h_n^3 y_n^{(3)} \\
Y_{10} &= y_n + 2.4695196732085400 \times 10^{-1} h_n f_n + 6.0181404002769251 \times 10^{-2} h_n F_2 + 8.7664481989114793 \times 10^{-2} h_n F_3 \\
&\quad + 7.7400070405320612 \times 10^{-2} h_n F_4 + 7.6895109292245703 \times 10^{-2} h_n F_5 + 1.1337026195981609 \times 10^{-1} h_n F_6 \\
&\quad + 1.2158190033865408 \times 10^{-1} h_n F_7 + 4.6114405330680071 \times 10^{-2} h_n F_8 + 9.2837146977418517 \times 10^{-2} h_n F_9 \\
&\quad + 2.5227343509045069 \times 10^{-2} h_n^2 y_n^{(2)} + 1.1763728132955210 \times 10^{-3} h_n^3 y_n^{(3)} \\
y_{n+1} &= y_n + 2.5830759520583763 \times 10^{-1} h_n f_n + 4.3428428180664114 \times 10^{-2} h_n F_2 + 6.4984111796381180 \times 10^{-2} h_n F_3 \\
&\quad + 9.1206283256506898 \times 10^{-2} h_n F_4 + 8.9121350143994416 \times 10^{-2} h_n F_5 + 1.0855393058951872 \times 10^{-1} h_n F_6 \\
&\quad + 1.3327288860399417 \times 10^{-1} h_n F_7 + 2.1190182583713087 \times 10^{-2} h_n F_8 + 4.2659903795695203 \times 10^{-2} h_n F_9 \\
&\quad + 1.4727532584369460 \times 10^{-1} h_n F_{10} + 2.7071045291873909 \times 10^{-2} h_n^2 y_n^{(2)} + 1.1793895663086526 \times 10^{-3} h_n^3 y_n^{(3)}
\end{aligned}
$$

$\mathbf{CPHBT_{RK5}(4,10,8)}$ with $c_{cp} = 2.6736188344238125$, and abscissa vector $\sigma = [0\ 3.7402489357294960{\times}10^{-1}\ 4.9449816435270666{\times}10^{-1}\ 4.9878873711571220{\times}10^{-1}\ 5.0283264680252127{\times}10^{-1}\ 6.2541121459979643{\times}10^{-1}\ 7.3668628792253987{\times}10^{-1}\ 8.4269412254028120{\times}10^{-1}\ 9.1992200220240106{\times}10^{-1}\ 9.3148685278833943{\times}10^{-1}]^T$

$$
Y_2 = y_n + 3.7402489357294960 \times 10^{-1} h_n f_n + 6.9947310506128138 \times 10^{-2} h_n^2 y_n^{(2)} + 8.7206784559228766 \times 10^{-3} h_n^3 y_n^{(3)}
$$

$$+8.1543770784011716\times10^{-4}h_n^4y_n^{(4)}$$

$$Y_3 = y_n + 2.7840897317942642\times10^{-1}h_nf_n + 2.1608919117328024\times10^{-1}h_nF_2 + 4.1441480543247346\times10^{-2}h_n^2y_n^{(2)}$$
$$+5.0382859173385046\times10^{-3}h_n^3y_n^{(3)} + 6.0697878358535848\times10^{-4}h_n^4y_n^{(4)}$$

$$Y_4 = y_n + 3.4704003821033153\times10^{-1}h_nf_n + 5.5567651163863953\times10^{-2}h_nF_2 + 9.6181047741516729\times10^{-2}h_nF_3$$
$$+5.6050065770380464\times10^{-2}h_n^2y_n^{(2)} + 5.0359836972812223\times10^{-3}h_n^3y_n^{(3)} + 1.5608548084707833\times10^{-4}h_n^4y_n^{(4)}$$

$$Y_5 = y_n + 3.6705629478183960\times10^{-1}h_nf_n + 1.4349851482314973\times10^{-2}h_nF_2 + 2.4837899777951287\times10^{-2}h_nF_3$$
$$+9.6588600760415450\times10^{-2}h_nF_4 + 6.0593531632119314\times10^{-2}h_n^2y_n^{(2)} + 5.1337551944944755\times10^{-3}h_n^3y_n^{(3)}$$
$$+4.0307686608820573\times10^{-5}h_n^4y_n^{(4)}$$

$$Y_6 = y_n + 3.2186672374877234\times10^{-1}h_nf_n + 8.5441500969952264\times10^{-3}h_nF_2 + 1.4788915694248303\times10^{-2}h_nF_3$$
$$+5.7510525706333054\times10^{-2}h_nF_4 + 2.2270089935344747\times10^{-1}h_nF_5 + 4.4393892024205163\times10^{-2}h_n^2y_n^{(2)}$$
$$+3.0567267540746335\times10^{-3}h_n^3y_n^{(3)} + 9.3670480396297045\times10^{-5}h_n^4y_n^{(4)}$$

$$Y_7 = y_n + 3.4472015925654997\times10^{-1}h_nf_n + 1.6078577223738736\times10^{-2}h_nF_2 + 8.3090577508470979\times10^{-3}h_nF_3$$
$$+3.2311921256089532\times10^{-2}h_nF_4 + 1.2512307677925649\times10^{-1}h_nF_5 + 2.1014349565605805\times10^{-1}h_nF_6$$
$$+5.0771852337739710\times10^{-2}h_n^2y_n^{(2)} + 3.5583179514058717\times10^{-3}h_n^3y_n^{(3)} + 7.7216333997365640\times10^{-5}h_n^4y_n^{(4)}$$

$$Y_8 = y_n + 3.5804520915881827\times10^{-1}h_nf_n + 7.5726447810212802\times10^{-2}h_nF_2 + 4.5308761104523735\times10^{-3}h_nF_3$$
$$+1.7619484241412197\times10^{-2}h_nF_4 + 6.8228814438993993\times10^{-2}h_nF_5 + 1.1458990571318592\times10^{-1}h_nF_6$$
$$+2.0395338506720576\times10^{-1}h_nF_7 + 5.9491055666249036\times10^{-2}h_n^2y_n^{(2)} + 5.3156914161634631\times10^{-3}h_n^3y_n^{(3)}$$
$$+1.8808733796739743\times10^{-4}h_n^4y_n^{(4)}$$

$$Y_9 = y_n + 3.0622827155293281\times10^{-1}h_nf_n + 1.1786919340158487\times10^{-1}h_nF_2 + 1.0726270981899948\times10^{-1}h_nF_3$$
$$+1.4380399058469868\times10^{-2}h_nF_4 + 5.5685942078426302\times10^{-2}h_nF_5 + 6.7279099082940677\times10^{-2}h_nF_6$$
$$+8.8647678380208539\times10^{-2}h_nF_7 + 1.6256870882883856\times10^{-1}h_nF_8 + 4.6449201078373012\times10^{-2}h_n^2y_n^{(2)}$$
$$+4.6252927899254970\times10^{-3}h_n^3y_n^{(3)} + 3.0847116933835497\times10^{-4}h_n^4y_n^{(4)}$$

$$Y_{10} = y_n + 3.2864696617127187\times10^{-1}h_nf_n + 3.1151576604196412\times10^{-2}h_nF_2 + 3.0012731014500662\times10^{-2}h_nF_3$$
$$+3.0863431686783370\times10^{-2}h_nF_4 + 1.1951401781436777\times10^{-1}h_nF_5 + 1.9512862937953399\times10^{-1}h_nF_6$$
$$+8.1787714230854527\times10^{-2}h_nF_7 + 3.4653598481100327\times10^{-2}h_nF_8 + 7.9728187405730497\times10^{-2}h_nF_9$$
$$+4.7017672666169102\times10^{-2}h_n^2y_n^{(2)} + 3.5126209425953032\times10^{-3}h_n^3y_n^{(3)} + 1.1516521344988865\times10^{-4}h_n^4y_n^{(4)}$$

$$y_{n+1} = y_n + 3.2427827894719286\times10^{-1}h_nf_n + 3.3842469714701388\times10^{-2}h_nF_2 + 4.5121298517174925\times10^{-2}h_nF_3$$
$$+6.1887573202160040\times10^{-2}h_nF_4 + 1.1369218819461296\times10^{-1}h_nF_5 + 1.2177771423612925\times10^{-1}h_nF_6$$
$$+1.1828871171632893\times10^{-1}h_nF_7 + 1.2850304192812181\times10^{-2}h_nF_8 + 2.9564937143943128\times10^{-2}h_nF_9$$
$$+1.3869652413494440\times10^{-1}h_nF_{10} + 4.6469585328660794\times10^{-2}h_n^2y_n^{(2)} + 3.5534596678701018\times10^{-3}h_n^3y_n^{(3)}$$
$$+1.2558765850119953\times10^{-4}h_n^4y_n^{(4)}$$

CPHBT$_{\mathbf{RK5}}$(5,10,9) with $c_{cp} = 2.3856368028474391$, and abscissa vector $\sigma = [0\ 4.1917529055823749\times10^{-1}\ 5.2405698244208032\times10^{-1}\ 5.2786402398851617\times10^{-1}\ 5.5210608401389483\times10^{-1}\ 6.6238678709395604\times10^{-1}\ 7.5564179985879198\times10^{-1}\ 8.7471065097018452\times10^{-1}\ 9.1293992993193451\times10^{-1}\ 9.4085301374204600\times10^{-1}]^T$

$$Y_2 = y_n + 4.1917529055823749\times10^{-1}h_nf_n + 8.7853962107291420\times10^{-2}h_n^2y_n^{(2)} + 1.2275403364338754\times10^{-2}h_n^3y_n^{(3)}$$

$$
\begin{aligned}
&+1.2863864429915658 \times 10^{-3} h_n^4 y_n^{(4)} + 1.0784428220223343 \times 10^{-4} h_n^5 y_n^{(5)} \\
&Y_3 = y_n + 3.1927855855928888 \times 10^{-1} h_n f_n + 2.0477842388279147 \times 10^{-1} h_n F_2 + 5.1479805092022442 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+5.9968652973965393 \times 10^{-3} h_n^3 y_n^{(3)} + 6.2896137839045377 \times 10^{-4} h_n^4 y_n^{(4)} + 6.5966496453727342 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_4 = y_n + 3.9845683886584027 \times 10^{-1} h_n f_n + 4.2470777551617149 \times 10^{-2} h_n F_2 + 8.6936407571058749 \times 10^{-2} h_n F_3 \\
&+7.5957881974197103 \times 10^{-2} h_n^2 y_n^{(2)} + 8.8448953503901511 \times 10^{-3} h_n^3 y_n^{(3)} + 6.2829070580332420 \times 10^{-4} h_n^4 y_n^{(4)} \\
&+1.3681365173263413 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_5 = y_n + 4.1395781011349519 \times 10^{-1} h_n f_n + 1.0695319063338957 \times 10^{-2} h_n F_2 + 2.1892997274723675 \times 10^{-2} h_n F_3 \\
&+1.0555995756233706 \times 10^{-1} h_n F_4 + 8.0732868467274163 \times 10^{-2} h_n^2 y_n^{(2)} + 9.3963716502202723 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+6.2734856233036425 \times 10^{-4} h_n^4 y_n^{(4)} + 3.4453469935243566 \times 10^{-6} h_n^5 y_n^{(5)} \\
&Y_6 = y_n + 3.7635523717243458 \times 10^{-1} h_n f_n + 5.4890890744295475 \times 10^{-3} h_n F_2 + 1.1236000668659346 \times 10^{-2} h_n F_3 \\
&+5.4175850792410128 \times 10^{-2} h_n F_4 + 2.1513060938602241 \times 10^{-1} h_n F_5 + 6.3816531843595342 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+6.0766139965727026 \times 10^{-3} h_n^3 y_n^{(3)} + 3.2197002435864144 \times 10^{-4} h_n^4 y_n^{(4)} + 1.2107854095285966 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_7 = y_n + 3.9987371489811863 \times 10^{-1} h_n f_n + 4.0363174140173659 \times 10^{-2} h_n F_2 + 5.0647418645017161 \times 10^{-3} h_n F_3 \\
&+2.4420317125707088 \times 10^{-2} h_n F_4 + 9.6972315668539366 \times 10^{-2} h_n F_5 + 1.8894753616175156 \times 10^{-1} h_n F_6 \\
&+7.4337842526428721 \times 10^{-2} h_n^2 y_n^{(2)} + 8.0368721906856969 \times 10^{-3} h_n^3 y_n^{(3)} + 4.9701743471249754 \times 10^{-4} h_n^4 y_n^{(4)} \\
&+1.5205695046221745 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_8 = y_n + 4.0872238091506313 \times 10^{-1} h_n f_n + 6.8170982705768782 \times 10^{-2} h_n F_2 + 2.7428480455627701 \times 10^{-3} h_n F_3 \\
&+1.3225001568141824 \times 10^{-2} h_n F_4 + 5.2516067673533284 \times 10^{-2} h_n F_5 + 1.0232592186139944 \times 10^{-1} h_n F_6 \\
&+2.2700744820071542 \times 10^{-1} h_n F_7 + 7.7255262932959409 \times 10^{-2} h_n^2 y_n^{(2)} + 8.0725948047622696 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+4.1128808277233618 \times 10^{-4} h_n^4 y_n^{(4)} + 2.0149783909574135 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_9 = y_n + 3.6632244181168760 \times 10^{-1} h_n f_n + 6.3506989130738406 \times 10^{-2} h_n F_2 + 1.5193797672158917 \times 10^{-2} h_n F_3 \\
&+3.9514679317807250 \times 10^{-2} h_n F_4 + 1.5691155592380779 \times 10^{-1} h_n F_5 + 9.4680692606829001 \times 10^{-2} h_n F_6 \\
&+6.2114217982995774 \times 10^{-2} h_n F_7 + 1.1469555548590983 \times 10^{-1} h_n F_8 + 6.4679715890237882 \times 10^{-2} h_n^2 y_n^{(2)} \\
&+7.3483179241706914 \times 10^{-3} h_n^3 y_n^{(3)} + 5.8373042939625420 \times 10^{-4} h_n^4 y_n^{(4)} + 1.9155773629254956 \times 10^{-5} h_n^5 y_n^{(5)} \\
&Y_{10} = y_n + 3.9009898424943285 \times 10^{-1} h_n f_n + 1.9829294933082171 \times 10^{-2} h_n F_2 + 7.9048999082521332 \times 10^{-3} h_n F_3 \\
&+3.1105251357254757 \times 10^{-2} h_n F_4 + 1.2351798046019984 \times 10^{-1} h_n F_5 + 1.9683080587195864 \times 10^{-1} h_n F_6 \\
&+6.0671517984667363 \times 10^{-2} h_n F_7 + 2.3824265778407835 \times 10^{-2} h_n F_8 + 8.7070013198790513 \times 10^{-2} h_n F_9 \\
&+6.8980165700845669 \times 10^{-2} h_n^2 y_n^{(2)} + 6.9204899744523193 \times 10^{-3} h_n^3 y_n^{(3)} + 3.9302518860676939 \times 10^{-4} h_n^4 y_n^{(4)} \\
&+1.0207238601932514 \times 10^{-5} h_n^5 y_n^{(5)} \\
&y_{n+1} = y_n + 3.7969414299658161 \times 10^{-1} h_n f_n + 3.0786519618046827 \times 10^{-2} h_n F_2 + 2.9936545631869378 \times 10^{-2} h_n F_3 \\
&+3.5350898317343823 \times 10^{-2} h_n F_4 + 1.4037731177484225 \times 10^{-1} h_n F_5 + 1.0479447431208680 \times 10^{-1} h_n F_6 \\
&+1.1882430037345164 \times 10^{-1} h_n F_7 + 7.2018856995588751 \times 10^{-3} h_n F_8 + 2.6320571166776104 \times 10^{-2} h_n F_9 \\
&+1.2671335010944271 \times 10^{-1} h_n F_{10} + 6.6492573396444224 \times 10^{-2} h_n^2 y_n^{(2)} + 6.8100997046303853 \times 10^{-3} h_n^3 y_n^{(3)} \\
&+4.1580497337864356 \times 10^{-4} h_n^4 y_n^{(4)} + 1.2435129293901815 \times 10^{-5} h_n^5 y_n^{(5)}
\end{aligned}
$$

$\mathbf{CPHBT_{RK5}(6,10,10)}$ with $c_{cp} = 2.1720633064988792$, and abscissa vector $\sigma = [0\ 4.6039173766619429{\times}10^{-1}\ 5.5302830860306673{\times}10^{-1}\ 5.6209265279347487{\times}10^{-1}\ 5.9225669436654071{\times}10^{-1}\ 6.8850812576069620{\times}10^{-1}\ 7.7342683799476519{\times}10^{-1}\ 8.7769081324673848{\times}10^{-1}\ 9.0825588600218488{\times}10^{-1}\ 9.4767740940606027{\times}10^{-1}]^T$

$$
\begin{aligned}
Y_2 &= y_n + 4.6039173766619429\times10^{-1}h_nf_n + 1.0598027605564893\times10^{-1}h_n^2y_n^{(2)} + 1.6264147817201060\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +1.8719698188052591\times10^{-3}h_n^4y_n^{(4)} + 1.7236788754768480\times10^{-4}h_n^5y_n^{(5)} + 1.3226125210988296\times10^{-5}h_n^6y_n^{(6)}\\
Y_3 &= y_n + 3.6132178875054827\times10^{-1}h_nf_n + 1.9170651985251846\times10^{-1}h_nF_2 + 6.4660057261344620\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +7.8726150054111484\times10^{-3}h_n^3y_n^{(3)} + 7.7948579410064224\times10^{-4}h_n^4y_n^{(4)} + 7.2208891059488535\times10^{-5}h_n^5y_n^{(5)}\\
&\quad +6.6889816437436232\times10^{-6}h_n^6y_n^{(6)}\\
Y_4 &= y_n + 4.4217295214897484\times10^{-1}h_nf_n + 3.5254487688933786\times10^{-2}h_nF_2 + 8.4665212955566269\times10^{-2}h_nF_3\\
&\quad +9.4920940796228037\times10^{-2}h_n^2y_n^{(2)} + 1.2915391162833080\times10^{-2}h_n^3y_n^{(3)} + 1.1992281429935860\times10^{-3}h_n^4y_n^{(4)}\\
&\quad +7.1610561278309339\times10^{-5}h_n^5y_n^{(5)} + 1.2300917949620015\times10^{-6}h_n^6y_n^{(6)}\\
Y_5 &= y_n + 4.5622205129411858\times10^{-1}h_nf_n + 1.1288866968807590\times10^{-2}h_nF_2 + 1.9377108551817013\times10^{-2}h_nF_3\\
&\quad +1.0536866755179752\times10^{-1}h_nF_4 + 1.0024365149740622\times10^{-1}h_n^2y_n^{(2)} + 1.3819049751153016\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +1.2779774284152013\times10^{-3}h_n^4y_n^{(4)} + 7.2338199563864983\times10^{-5}h_n^5y_n^{(5)} + 3.7403959923412801\times10^{-7}h_n^6y_n^{(6)}\\
Y_6 &= y_n + 4.2936992760642373\times10^{-1}h_nf_n + 4.9048411337496416\times10^{-3}h_nF_2 + 8.4190591793399383\times10^{-3}h_nF_3\\
&\quad +4.5781084695607766\times10^{-2}h_nF_4 + 2.0003321314557515\times10^{-1}h_nF_5 + 8.5903372303297323\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +1.0275017622930144\times10^{-2}h_n^3y_n^{(3)} + 7.6507879351271992\times10^{-4}h_n^4y_n^{(4)} + 3.1429848340212647\times10^{-5}h_n^5y_n^{(5)}\\
&\quad +6.0074457683547491\times10^{-7}h_n^6y_n^{(6)}\\
Y_7 &= y_n + 4.3300381569011170\times10^{-1}h_nf_n + 6.6435740383584810\times10^{-2}h_nF_2 + 3.2278675322640863\times10^{-3}h_nF_3\\
&\quad +1.7552469193160541\times10^{-2}h_nF_4 + 7.6692739691321274\times10^{-2}h_nF_5 + 1.7651420550432273\times10^{-1}h_nF_6\\
&\quad +8.9903601519914320\times10^{-2}h_n^2y_n^{(2)} + 1.1513548428163798\times10^{-2}h_n^3y_n^{(3)} + 9.6126300509783737\times10^{-4}h_n^4y_n^{(4)}\\
&\quad +5.0430435551193395\times10^{-5}h_n^5y_n^{(5)} + 2.0848667576175740\times10^{-6}h_n^6y_n^{(6)}\\
Y_8 &= y_n + 4.4785330301293264\times10^{-1}h_nf_n + 9.3632253191585371\times10^{-2}h_nF_2 + 1.4777465102325000\times10^{-3}h_nF_3\\
&\quad +8.0356767546666610\times10^{-3}h_nF_4 + 3.5110619412418026\times10^{-2}h_nF_5 + 8.0809775674879417\times10^{-2}h_nF_6\\
&\quad +2.1077143869002393\times10^{-1}h_nF_7 + 9.7280061614219523\times10^{-2}h_n^2y_n^{(2)} + 1.2916217297737793\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +1.0597905552383016\times10^{-3}h_n^4y_n^{(4)} + 4.6755707895646095\times10^{-5}h_n^5y_n^{(5)} + 2.7705782652093099\times10^{-6}h_n^6y_n^{(6)}\\
Y_9 &= y_n + 4.1549387261441162\times10^{-1}h_nf_n + 2.3647591471200748\times10^{-2}h_nF_2 + 7.6569321725213878\times10^{-3}h_nF_3\\
&\quad +4.1636797275277970\times10^{-2}h_nF_4 + 1.8192540433328505\times10^{-1}h_nF_5 + 9.9086850141829316\times10^{-2}h_nF_6\\
&\quad +4.3591268426073013\times10^{-2}h_nF_7 + 9.5217169567585858\times10^{-2}h_nF_8 + 8.0684451450822189\times10^{-2}h_n^2y_n^{(2)}\\
&\quad +9.5144845480209968\times10^{-3}h_n^3y_n^{(3)} + 7.4156466535279196\times10^{-4}h_n^4y_n^{(4)} + 3.8631835160384212\times10^{-5}h_n^5y_n^{(5)}\\
&\quad +7.9331697678630540\times10^{-7}h_n^6y_n^{(6)}\\
Y_{10} &= y_n + 4.3763390277249853\times10^{-1}h_nf_n + 9.2120730972329069\times10^{-3}h_nF_2 + 5.2018171784455136\times10^{-3}h_nF_3\\
&\quad +2.8101541238680864\times10^{-2}h_nF_4 + 1.2278524254092209\times10^{-1}h_nF_5 + 2.1155264271500743\times10^{-1}h_nF_6\\
&\quad +9.6894653137520586\times10^{-3}h_nF_7 + 2.1164868449822866\times10^{-2}h_nF_8 + 1.0233585609969797\times10^{-1}h_nF_9\\
&\quad +8.8739109617458012\times10^{-2}h_n^2y_n^{(2)} + 1.0702065220994247\times10^{-2}h_n^3y_n^{(3)} + 8.0838842151547666\times10^{-4}h_n^4y_n^{(4)}
\end{aligned}
$$

$+3.5581913117153501 \times 10^{-5} h_n^5 y_n^{(5)} + 4.7936445753874328 \times 10^{-7} h_n^6 y_n^{(6)}$

$y_{n+1} = y_n + 4.2874765408285315 \times 10^{-1} h_n f_n + 2.9934536284237673 \times 10^{-2} h_n F_2 + 1.4084147752983455 \times 10^{-2} h_n F_3$
$+3.2263306724999316 \times 10^{-2} h_n F_4 + 1.4096941900277368 \times 10^{-1} h_n F_5 + 9.5186704878037223 \times 10^{-2} h_n F_6$
$+1.1152927180530137 \times 10^{-1} h_n F_7 + 5.3387686834779008 \times 10^{-3} h_n F_8 + 2.5813884222197575 \times 10^{-2} h_n F_9$
$+1.1613230656313875 \times 10^{-1} h_n F_{10} + 8.6820488889440550 \times 10^{-2} h_n^2 y_n^{(2)} + 1.0748368278050662 \times 10^{-2} h_n^3 y_n^{(3)}$
$+8.7055632998371719 \times 10^{-4} h_n^4 y_n^{(4)} + 4.4595570001044773 \times 10^{-5} h_n^5 y_n^{(5)} + 1.1596790903710386 \times 10^{-6} h_n^6 y_n^{(6)}$

$\mathbf{CPHBT_{RK5}(7,10,11)}$ with $c_{cp} = 2.0021964644692973$, and abscissa vector $\sigma = [0\ 4.9945147980645271{\times}10^{-1}$ $5.9080930475477056{\times}10^{-1}\ 5.9113608342369661{\times}10^{-1}\ 6.2321561215669419{\times}10^{-1}\ 7.1033462044206053{\times}10^{-1}$ $7.8879640954182384{\times}10^{-1}\ 8.8221230399786266{\times}10^{-1}\ 9.1653145201639363{\times}10^{-1}\ 9.5208401644445351{\times}10^{-1}]^T$

$Y_2 = y_n + 4.9945147980645271 \times 10^{-1} h_n f_n + 1.2472589034042773 \times 10^{-1} h_n^2 y_n^{(2)} + 2.0764843500234657 \times 10^{-2} h_n^3 y_n^{(3)}$
$+2.5927579535354006 \times 10^{-3} h_n^4 y_n^{(4)} + 2.5899135933464112 \times 10^{-4} h_n^5 y_n^{(5)} + 2.1558936279461875 \times 10^{-5} h_n^6 y_n^{(6)}$
$+1.5382346611186077 \times 10^{-6} h_n^7 y_n^{(7)}$

$Y_3 = y_n + 3.9800702447330860 \times 10^{-1} h_n f_n + 1.9280228028146187 \times 10^{-1} h_n F_2 + 7.8232433095773074 \times 10^{-2} h_n^2 y_n^{(2)}$
$+1.0323450063861638 \times 10^{-2} h_n^3 y_n^{(3)} + 1.0731506582757368 \times 10^{-3} h_n^4 y_n^{(4)} + 9.9977927836434171 \times 10^{-5} h_n^5 y_n^{(5)}$
$+9.1337660207652033 \times 10^{-6} h_n^6 y_n^{(6)} + 8.2879641376949653 \times 10^{-7} h_n^7 y_n^{(7)}$

$Y_4 = y_n + 4.8547562783646392 \times 10^{-1} h_n f_n + 3.6851739173122657 \times 10^{-2} h_n F_2 + 6.8808716414109947 \times 10^{-2} h_n F_3$
$+1.1566244899360778 \times 10^{-1} h_n^2 y_n^{(2)} + 1.7822548584443598 \times 10^{-2} h_n^3 y_n^{(3)} + 1.9576636757058807 \times 10^{-3} h_n^4 y_n^{(4)}$
$+1.5666226704316900 \times 10^{-4} h_n^5 y_n^{(5)} + 8.4437860816390366 \times 10^{-6} h_n^6 y_n^{(6)} + 1.4587263817773807 \times 10^{-7} h_n^7 y_n^{(7)}$

$Y_5 = y_n + 4.9652540892988356 \times 10^{-1} h_n f_n + 7.7155217110938387 \times 10^{-3} h_n F_2 + 1.4406243974307890 \times 10^{-2} h_n F_3$
$+1.0456843754140892 \times 10^{-1} h_n F_4 + 1.2001980127728457 \times 10^{-1} h_n^2 y_n^{(2)} + 1.8595674217509409 \times 10^{-2} h_n^3 y_n^{(3)}$
$+2.0300883241024569 \times 10^{-3} h_n^4 y_n^{(4)} + 1.5827443890422077 \times 10^{-4} h_n^5 y_n^{(5)} + 7.8352359851429787 \times 10^{-6} h_n^6 y_n^{(6)}$
$+3.0540851861225888 \times 10^{-8} h_n^7 y_n^{(7)}$

$Y_6 = y_n + 4.7398256984694004 \times 10^{-1} h_n f_n + 2.9124069014954540 \times 10^{-3} h_n F_2 + 5.4379789139953469 \times 10^{-3} h_n F_3$
$+3.9471840087793600 \times 10^{-2} h_n F_4 + 1.8852982469183607 \times 10^{-1} h_n F_5 + 1.0679226296094230 \times 10^{-1} h_n^2 y_n^{(2)}$
$+1.4915051174609376 \times 10^{-2} h_n^3 y_n^{(3)} + 1.3960764301845275 \times 10^{-3} h_n^4 y_n^{(4)} + 8.6073801938428627 \times 10^{-5} h_n^5 y_n^{(5)}$
$+2.9575958972631555 \times 10^{-6} h_n^6 y_n^{(6)} + 4.0385194995008435 \times 10^{-8} h_n^7 y_n^{(7)}$

$Y_7 = y_n + 4.8108625957486334 \times 10^{-1} h_n f_n + 5.1060347160797478 \times 10^{-2} h_n F_2 + 2.1527636899271135 \times 10^{-3} h_n F_3$
$+1.5625942185417640 \times 10^{-2} h_n F_4 + 7.4634375641532941 \times 10^{-2} h_n F_5 + 1.6423672128928535 \times 10^{-1} h_n F_6$
$+1.1191245363886120 \times 10^{-1} h_n^2 y_n^{(2)} + 1.6394910419712711 \times 10^{-2} h_n^3 y_n^{(3)} + 1.6364781064023928 \times 10^{-3} h_n^4 y_n^{(4)}$
$+1.1055187535626911 \times 10^{-4} h_n^5 y_n^{(5)} + 4.6433741682703354 \times 10^{-6} h_n^6 y_n^{(6)} + 1.6776010587140489 \times 10^{-7} h_n^7 y_n^{(7)}$

$Y_8 = y_n + 4.9220922931102734 \times 10^{-1} h_n f_n + 9.1837459053609752 \times 10^{-2} h_n F_2 + 8.4893402535703958 \times 10^{-4} h_n F_3$
$+6.1620297952498931 \times 10^{-3} h_n F_4 + 2.9431777040758767 \times 10^{-2} h_n F_5 + 6.4766115095649807 \times 10^{-2} h_n F_6$
$+1.9695675967621015 \times 10^{-1} h_n F_7 + 1.1943002195933335 \times 10^{-1} h_n^2 y_n^{(2)} + 1.8429585818819622 \times 10^{-2} h_n^3 y_n^{(3)}$
$+1.9242745255058241 \times 10^{-3} h_n^4 y_n^{(4)} + 1.3048998994229195 \times 10^{-4} h_n^5 y_n^{(5)} + 4.9261300404997578 \times 10^{-6} h_n^6 y_n^{(6)}$
$+2.8698704937944090 \times 10^{-7} h_n^7 y_n^{(7)}$

$$Y_9 = y_n + 4.5876508149936096\times10^{-1}h_nf_n + 2.0090962397717457\times10^{-2}h_nF_2 + 5.0422841282365136\times10^{-3}h_nF_3$$
$$+3.6599669828572955\times10^{-2}h_nF_4 + 1.7481144332709331\times10^{-1}h_nF_5 + 8.8693002183960354\times10^{-2}h_nF_6$$
$$+3.7481583911058346\times10^{-2}h_nF_7 + 9.5047424740393696\times10^{-2}h_nF_8 + 1.0000179942055155\times10^{-1}h_n^2y_n^{(2)}$$
$$+1.3565878481506373\times10^{-2}h_n^3y_n^{(3)} + 1.2581113077140122\times10^{-3}h_n^4y_n^{(4)} + 8.2480771719662128\times10^{-5}h_n^5y_n^{(5)}$$
$$+3.7638459745053936\times10^{-6}h_n^6y_n^{(6)} + 6.9373992033581475\times10^{-8}h_n^7y_n^{(7)}$$
$$Y_{10} = y_n + 4.8173462554269009\times10^{-1}h_nf_n + 4.9615839060525285\times10^{-3}h_nF_2 + 3.0897323432745858\times10^{-3}h_nF_3$$
$$+2.2426975701201220\times10^{-2}h_nF_4 + 1.0711823385746128\times10^{-1}h_nF_5 + 2.1258110744484707\times10^{-1}h_nF_6$$
$$+7.1272269161922657\times10^{-3}h_nF_7 + 1.8073357794255035\times10^{-2}h_nF_8 + 9.4971172938479370\times10^{-2}h_nF_9$$
$$+1.0929906695204024\times10^{-1}h_n^2y_n^{(2)} + 1.5188116786421256\times10^{-2}h_n^3y_n^{(3)} + 1.3971181047360247\times10^{-3}h_n^4y_n^{(4)}$$
$$+8.4683462785529871\times10^{-5}h_n^5y_n^{(5)} + 3.0686721337326254\times10^{-6}h_n^6y_n^{(6)} + 2.9016973908007801\times10^{-8}h_n^7y_n^{(7)}$$
$$y_{n+1} = y_n + 4.7118468535254887\times10^{-1}h_nf_n + 2.6616399649699123\times10^{-2}h_nF_2 + 5.3718462024021163\times10^{-3}h_nF_3$$
$$+3.8991812515442772\times10^{-2}h_nF_4 + 1.2824577017254413\times10^{-1}h_nF_5 + 8.3005062698383730\times10^{-2}h_nF_6$$
$$+1.1339235746421306\times10^{-1}h_nF_7 + 3.9301934972959763\times10^{-3}h_nF_8 + 2.0652226916684495\times10^{-2}h_nF_9$$
$$+1.0860964553078571\times10^{-1}h_nF_{10} + 1.0635238731802167\times10^{-1}h_n^2y_n^{(2)} + 1.5044966693082962\times10^{-2}h_n^3y_n^{(3)}$$
$$+1.4574163190206374\times10^{-3}h_n^4y_n^{(4)} + 9.8129406438187720\times10^{-5}h_n^5y_n^{(5)} + 4.3371319996718675\times10^{-6}h_n^6y_n^{(6)}$$
$$+9.9760925164565553\times10^{-8}h_n^7y_n^{(7)}$$

$\mathbf{CPHBT_{RK5}(8,10,12)}$ with $c_{cp} = 1.8701607518699632$, and abscissa vector $\sigma = [0\ 5.3471339241832860\times10^{-1}\ 6.1419048198419735\times10^{-1}\ 6.2247490469133238\times10^{-1}\ 6.5190649075772933\times10^{-1}\ 7.3122243548862309\times10^{-1}\ 8.0109358144540077\times10^{-1}\ 8.8597616135642421\times10^{-1}\ 9.2329867900410201\times10^{-1}\ 9.5546018664238141\times10^{-1}]^T$

$$Y_2 = y_n + 5.3471339241832860\times10^{-1}h_nf_n + 1.4295920601575873\times10^{-1}h_n^2y_n^{(2)} + 2.5480734008705690\times10^{-2}h_n^3y_n^{(3)}$$
$$+3.4062224307760238\times10^{-3}h_n^4y_n^{(4)} + 3.6427055025833058\times10^{-4}h_n^5y_n^{(5)} + 3.2463390281120529\times10^{-5}h_n^6y_n^{(6)}$$
$$+2.4798013638025933\times10^{-6}h_n^7y_n^{(7)} + 1.6574787497031028\times10^{-7}h_n^8y_n^{(8)}$$
$$Y_3 = y_n + 4.3875596113411597\times10^{-1}h_nf_n + 1.7543452085008132\times10^{-1}h_nF_2 + 9.4807786288959292\times10^{-2}h_n^2y_n^{(2)}$$
$$+1.3535194138059323\times10^{-2}h_n^3y_n^{(3)} + 1.4590677128069084\times10^{-3}h_n^4y_n^{(4)} + 1.3077100325014241\times10^{-4}h_n^5y_n^{(5)}$$
$$+1.0650953501719131\times10^{-5}h_n^6y_n^{(6)} + 8.4650691154109263\times10^{-7}h_n^7y_n^{(7)} + 6.7188948790357645\times10^{-8}h_n^8y_n^{(8)}$$
$$Y_4 = y_n + 5.2152779527722093\times10^{-1}h_nf_n + 2.7471660102639177\times10^{-2}h_nF_2 + 7.3475449311472235\times10^{-2}h_nF_3$$
$$+1.3392011728977682\times10^{-1}h_n^2y_n^{(2)} + 2.2413014655429917\times10^{-2}h_n^3y_n^{(3)} + 2.7184380562997631\times10^{-3}h_n^4y_n^{(4)}$$
$$+2.4957345184822578\times10^{-4}h_n^5y_n^{(5)} + 1.7275397905943342\times10^{-5}h_n^6y_n^{(6)} + 8.1502592760525529\times10^{-7}h_n^7y_n^{(7)}$$
$$+1.0275574247934990\times10^{-8}h_n^8y_n^{(8)}$$
$$Y_5 = y_n + 5.3232762977294423\times10^{-1}h_nf_n + 9.5349847453887089\times10^{-3}h_nF_2 + 1.3294428795617059\times10^{-2}h_nF_3$$
$$+9.6749447443779307\times10^{-2}h_nF_4 + 1.3900313759997429\times10^{-1}h_n^2y_n^{(2)} + 2.3560123338002857\times10^{-2}h_n^3y_n^{(3)}$$
$$+2.8798592041378076\times10^{-3}h_n^4y_n^{(4)} + 2.6463194562377713\times10^{-4}h_n^5y_n^{(5)} + 1.8100428572917558\times10^{-5}h_n^6y_n^{(6)}$$
$$+8.1025988346648693\times10^{-7}h_n^7y_n^{(7)} + 3.2740652255152965\times10^{-9}h_n^8y_n^{(8)}$$
$$Y_6 = y_n + 5.1442333173771493\times10^{-1}h_nf_n + 3.1594079500815012\times10^{-3}h_nF_2 + 4.4050960908963840\times10^{-3}h_nF_3$$
$$+3.2057835600391937\times10^{-2}h_nF_4 + 1.7717676410953836\times10^{-1}h_nF_5 + 1.2749029855225688\times10^{-1}h_n^2y_n^{(2)}$$

$$
\begin{aligned}
&+2.0020617548819012\times10^{-2}h_n^3y_n^{(3)}+2.1916651740869499\times10^{-3}h_n^4y_n^{(4)}+1.7132060698762117\times10^{-4}h_n^5y_n^{(5)}\\
&+9.1401358816081493\times10^{-6}h_n^6y_n^{(6)}+2.6847882674322473\times10^{-7}h_n^7y_n^{(7)}+1.0864561870145116\times10^{-9}h_n^8y_n^{(8)}\\
Y_6&=y_n+5.1964822075710193\times10^{-1}h_nf_n+3.9652528247946586\times10^{-2}h_nF_2+2.0509692496128418\times10^{-3}h_nF_3\\
&+1.4925811757302131\times10^{-2}h_nF_4+8.2491752151681808\times10^{-2}h_nF_5+1.4232429928175541\times10^{-1}h_nF_6\\
&+1.2749029855225688\times10^{-1}h_n^2y_n^{(2)}+2.0020617548819012\times10^{-2}h_n^3y_n^{(3)}+2.1916651740869499\times10^{-3}h_n^4y_n^{(4)}\\
&+1.7132060698762117\times10^{-4}h_n^5y_n^{(5)}+9.1401358816081493\times10^{-6}h_n^6y_n^{(6)}+2.6847882674322473\times10^{-7}h_n^7y_n^{(7)}\\
&+1.0864561870145116\times10^{-9}h_n^8y_n^{(8)}\\
Y_7&=y_n+5.1964822075710193\times10^{-1}h_nf_n+3.9652528247946586\times10^{-2}h_nF_2+2.0509692496128418\times10^{-3}h_nF_3\\
&+1.4925811757302131\times10^{-2}h_nF_4+8.2491752151681808\times10^{-2}h_nF_5+1.4232429928175541\times10^{-1}h_nF_6\\
&+1.3127446676438828\times10^{-1}h_n^2y_n^{(2)}+2.1158350158284184\times10^{-2}h_n^3y_n^{(3)}+2.3873674395731206\times10^{-3}h_n^4y_n^{(4)}\\
&+1.9262476197671143\times10^{-4}h_n^5y_n^{(5)}+1.0645081283694613\times10^{-5}h_n^6y_n^{(6)}+3.5328680609032152\times10^{-7}h_n^7y_n^{(7)}\\
&+1.2340853721043418\times10^{-8}h_n^8y_n^{(8)}\\
Y_8&=y_n+5.2946979632995927\times10^{-1}h_nf_n+8.3906932745341514\times10^{-2}h_nF_2+1.0660779532144420\times10^{-3}h_nF_3\\
&+7.7583215112884451\times10^{-3}h_nF_4+3.1189277217761519\times10^{-2}h_nF_5+4.8893296627847894\times10^{-2}h_nF_6\\
&+1.8369245897101127\times10^{-1}h_nF_7+1.3888736567384863\times10^{-1}h_n^2y_n^{(2)}+2.3567830309430295\times10^{-2}h_n^3y_n^{(3)}\\
&+2.8163400074592516\times10^{-3}h_n^4y_n^{(4)}+2.3914877120945224\times10^{-4}h_n^5y_n^{(5)}+1.3535805705222980\times10^{-5}h_n^6y_n^{(6)}\\
&+4.5412610436241338\times10^{-7}h_n^7y_n^{(7)}+3.4076583769479442\times10^{-8}h_n^8y_n^{(8)}\\
Y_9&=y_n+4.9849196150774777\times10^{-1}h_nf_n+1.7916870786036133\times10^{-2}h_nF_2+4.2919119431825831\times10^{-3}h_nF_3\\
&+3.1234144419735491\times10^{-2}h_nF_4+1.7053817882509290\times10^{-1}h_nF_5+7.2609733375995786\times10^{-2}h_nF_6\\
&+3.2784100923279984\times10^{-2}h_nF_7+9.5431777223031408\times10^{-2}h_nF_8+1.1949908777399414\times10^{-1}h_n^2y_n^{(2)}\\
&+1.8136292956138802\times10^{-2}h_n^3y_n^{(3)}+1.9258802957906722\times10^{-3}h_n^4y_n^{(4)}+1.4874787245809315\times10^{-4}h_n^5y_n^{(5)}\\
&+8.3756352989960089\times10^{-6}h_n^6y_n^{(6)}+3.3110374516147796\times10^{-7}h_n^7y_n^{(7)}+7.0920596446912391\times10^{-9}h_n^8y_n^{(8)}\\
Y_{10}&=y_n+5.2130758528105836\times10^{-1}h_nf_n+4.1196717623323978\times10^{-3}h_nF_2+2.3295646947607135\times10^{-3}h_nF_3\\
&+1.6953274222414810\times10^{-2}h_nF_4+9.3354354818154212\times10^{-2}h_nF_5+2.0838066061489829\times10^{-1}h_nF_6\\
&+5.5606064323499727\times10^{-3}h_nF_7+1.5667570421008806\times10^{-2}h_nF_8+8.7786898395403809\times10^{-2}h_nF_9\\
&+1.2964534489352778\times10^{-1}h_n^2y_n^{(2)}+2.0163281073706636\times10^{-2}h_n^3y_n^{(3)}+2.1505657748201617\times10^{-3}h_n^4y_n^{(4)}\\
&+1.6109889318801374\times10^{-4}h_n^5y_n^{(5)}+8.2430743981103078\times10^{-6}h_n^6y_n^{(6)}+2.5957903951167403\times10^{-7}h_n^7y_n^{(7)}\\
&+1.5690541839096717\times10^{-9}h_n^8y_n^{(8)}\\
y_{n+1}&=y_n+5.1121146549413532\times10^{-1}h_nf_n+2.0386385220343931\times10^{-2}h_nF_2+9.2834499181217583\times10^{-3}h_nF_3\\
&+2.8885741716967287\times10^{-2}h_nF_4+1.2608846259566284\times10^{-1}h_nF_5+6.9551015529840027\times10^{-2}h_nF_6\\
&+1.1210368019316443\times10^{-1}h_nF_7+3.0072300306260027\times10^{-3}h_nF_8+1.6849798025876470\times10^{-2}h_nF_9\\
&+1.0263277127526207\times10^{-1}h_nF_{10}+1.2627272492050823\times10^{-1}h_n^2y_n^{(2)}+1.9837749524817122\times10^{-2}h_n^3y_n^{(3)}\\
&+2.1887780169247391\times10^{-3}h_n^4y_n^{(4)}+1.7575037140596972\times10^{-4}h_n^5y_n^{(5)}+1.0182985071431401\times10^{-5}h_n^6y_n^{(6)}\\
&+3.9712985318020886\times10^{-7}h_n^7y_n^{(7)}+8.2049614302710573\times10^{-9}h_n^8y_n^{(8)}
\end{aligned}
$$

$\mathbf{CPHBT_{RK5}(9,10,13)}$ with $c_{cp}=1.7654302957846233$, and abscissa vector $\sigma=[0\ 5.6643414477234810\times10^{-1}\ 6.3837393023407563\times10^{-1}\ 6.4995581616873621\times10^{-1}\ 6.7103706226296034\times10^{-1}\ 7.4311512104624933\times10^{-1}\ 8.1062633633402681\times10^{-1}\ 8.8770470308259519\times10^{-1}\ 9.3474586237612645\times10^{-1}\ 9.5795425726853622\times10^{-1}]^T$

$$\begin{aligned}
Y_2 &= y_n + 5.6643414477234810\times10^{-1}h_nf_n + 1.6042382018199070\times10^{-1}h_n^2y_n^{(2)} + 3.0289843128632952\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +4.2893003469639474\times10^{-3}h_n^4y_n^{(4)} + 4.8592123474085186\times10^{-4}h_n^5y_n^{(5)} + 4.5873729837859642\times10^{-5}h_n^6y_n^{(6)}\\
&\quad +3.7120638468893955\times10^{-6}h_n^7y_n^{(7)} + 2.6282996380664340\times10^{-7}h_n^8y_n^{(8)} + 1.6541762863262582\times10^{-8}h_n^9y_n^{(9)}\\
Y_3 &= y_n + 4.7486575732562625\times10^{-1}h_nf_n + 1.6350817290844946\times10^{-1}h_nF_2 + 1.1114402531656344\times10^{-1}h_n^2y_n^{(2)}\\
&\quad +1.7127887245994563\times10^{-2}h_n^3y_n^{(3)} + 1.9670959847143022\times10^{-3}h_n^4y_n^{(4)} + 1.8213975374559906\times10^{-4}h_n^5y_n^{(5)}\\
&\quad +1.4545852383025418\times10^{-5}h_n^6y_n^{(6)} + 1.0715328211991059\times10^{-6}h_n^7y_n^{(7)} + 7.7085841270141556\times10^{-8}h_n^8y_n^{(8)}\\
&\quad +5.5443107639465361\times10^{-9}h_n^9y_n^{(9)}\\
Y_4 &= y_n + 5.5445384549799015\times10^{-1}h_nf_n + 2.1392872161899845\times10^{-2}h_nF_2 + 7.4109098508846227\times10^{-2}h_nF_3\\
&\quad +1.5179431175733762\times10^{-1}h_n^2y_n^{(2)} + 2.7229056707161546\times10^{-2}h_n^3y_n^{(3)} + 3.5744927231526019\times10^{-3}h_n^4y_n^{(4)}\\
&\quad +3.6200464952811031\times10^{-4}h_n^5y_n^{(5)} + 2.8836927193539255\times10^{-5}h_n^6y_n^{(6)} + 1.7745415435483015\times10^{-6}h_n^7y_n^{(7)}\\
&\quad +7.5165697717878540\times10^{-8}h_n^8y_n^{(8)} + 7.2539766133039854\times10^{-10}h_n^9y_n^{(9)}\\
Y_5 &= y_n + 5.6470252219513994\times10^{-1}h_nf_n + 1.3750941519969839\times10^{-2}h_nF_2 + 1.0711668721084006\times10^{-2}h_nF_3\\
&\quad +8.1871929826766610\times10^{-2}h_nF_4 + 1.5730517963286400\times10^{-1}h_n^2y_n^{(2)} + 2.8678606817661555\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +3.8208682297448562\times10^{-3}h_n^4y_n^{(4)} + 3.9195669146348266\times10^{-4}h_n^5y_n^{(5)} + 3.1526806986788384\times10^{-5}h_n^6y_n^{(6)}\\
&\quad +1.9459778312623941\times10^{-6}h_n^7y_n^{(7)} + 8.0823197452525613\times10^{-8}h_n^8y_n^{(8)} + 4.1612180311619090\times10^{-10}h_n^9y_n^{(9)}\\
Y_6 &= y_n + 5.4709428183968178\times10^{-1}h_nf_n + 4.0065347225447835\times10^{-3}h_nF_2 + 3.1209988497946919\times10^{-3}h_nF_3\\
&\quad +2.3854565098420249\times10^{-2}h_nF_4 + 1.6503874053580778\times10^{-1}h_nF_5 + 1.4559671425610465\times10^{-1}h_n^2y_n^{(2)}\\
&\quad +2.4918868642019951\times10^{-2}h_n^3y_n^{(3)} + 3.0464262564009596\times10^{-3}h_n^4y_n^{(4)} + 2.7795051225905305\times10^{-4}h_n^5y_n^{(5)}\\
&\quad +1.8997073346867843\times10^{-5}h_n^6y_n^{(6)} + 9.2606744982055426\times10^{-7}h_n^7y_n^{(7)} + 2.6594308031932635\times10^{-8}h_n^8y_n^{(8)}\\
&\quad +1.2124307637929809\times10^{-10}h_n^9y_n^{(9)}\\
Y_7 &= y_n + 5.6150300314603108\times10^{-1}h_nf_n + 2.2047279862419181\times10^{-2}h_nF_2 + 1.3987198346471108\times10^{-3}h_nF_3\\
&\quad +1.0690761181227611\times10^{-2}h_nF_4 + 7.3964448877575847\times10^{-2}h_nF_5 + 1.4102212343212597\times10^{-1}h_nF_6\\
&\quad +1.5379920896741664\times10^{-1}h_n^2y_n^{(2)} + 2.7108721841640988\times10^{-2}h_n^3y_n^{(3)} + 3.4040772935866096\times10^{-3}h_n^4y_n^{(4)}\\
&\quad +3.1643867398743926\times10^{-4}h_n^5y_n^{(5)} + 2.1631070931198815\times10^{-5}h_n^6y_n^{(6)} + 1.0120968360769677\times10^{-6}h_n^7y_n^{(7)}\\
&\quad +2.5941430191409516\times10^{-8}h_n^8y_n^{(8)} + 6.4575370769244129\times10^{-10}h_n^9y_n^{(9)}\\
Y_8 &= y_n + 5.6469366296970547\times10^{-1}h_nf_n + 7.0628685878739064\times10^{-2}h_nF_2 + 2.0155510450644096\times10^{-3}h_nF_3\\
&\quad +1.5405354480294267\times10^{-2}h_nF_4 + 2.2251777836773803\times10^{-2}h_nF_5 + 4.2400881585230879\times10^{-2}h_nF_6\\
&\quad +1.7030878928678733\times10^{-1}h_nF_7 + 1.5820655188210189\times10^{-1}h_n^2y_n^{(2)} + 2.8919542300045245\times10^{-2}h_n^3y_n^{(3)}\\
&\quad +3.8018296617436803\times10^{-3}h_n^4y_n^{(4)} + 3.7135827457491192\times10^{-4}h_n^5y_n^{(5)} + 2.6573047702678404\times10^{-5}h_n^6y_n^{(6)}\\
&\quad +1.2908713722463772\times10^{-6}h_n^7y_n^{(7)} + 3.8901765460266653\times10^{-8}h_n^8y_n^{(8)} + 3.1597721519677465\times10^{-9}h_n^9y_n^{(9)}\\
Y_9 &= y_n + 5.2274625398639141\times10^{-1}h_nf_n + 1.8717419697859103\times10^{-2}h_nF_2 + 4.0442871272310830\times10^{-3}h_nF_3\\
&\quad +3.0911485455876811\times10^{-2}h_nF_4 + 1.9709856924238928\times10^{-1}h_nF_5 + 1.4773891053246037\times10^{-2}h_nF_6
\end{aligned}$$

$$+3.3854950950856257\times10^{-2}h_nF_7+1.1259900486227652\times10^{-1}h_nF_8+1.3296233301514793\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.1822926715599468\times10^{-2}h_n^3y_n^{(3)}+2.5833852769707099\times10^{-3}h_n^4y_n^{(4)}+2.3334707362756188\times10^{-4}h_n^5y_n^{(5)}$$
$$+1.6544350421996948\times10^{-5}h_n^6y_n^{(6)}+9.2256266420882364\times10^{-7}h_n^7y_n^{(7)}+3.8114058091853727\times10^{-8}h_n^8y_n^{(8)}$$
$$+7.6966328682629719\times10^{-10}h_n^9y_n^{(9)}$$
$$Y_{10}=y_n+5.5531192978880639\times10^{-1}h_nf_n+5.3764637578485371\times10^{-3}h_nF_2+1.5631810589904864\times10^{-3}h_nF_3$$
$$+1.1947778940949956\times10^{-2}h_nF_4+8.0570920327995371\times10^{-2}h_nF_5+1.6731364385117536\times10^{-1}h_nF_6$$
$$+5.1202385489403604\times10^{-2}h_nF_7+1.4039855599479078\times10^{-2}h_nF_8+7.0628098453887372\times10^{-2}h_nF_9$$
$$+1.4864140122137892\times10^{-1}h_n^2y_n^{(2)}+2.5263114945332763\times10^{-2}h_n^3y_n^{(3)}+3.0139230915772709\times10^{-3}h_n^4y_n^{(4)}$$
$$+2.6220053893687010\times10^{-4}h_n^5y_n^{(5)}+1.6620242481144206\times10^{-5}h_n^6y_n^{(6)}+7.3826610214186177\times10^{-7}h_n^7y_n^{(7)}$$
$$+2.1078596933910890\times10^{-8}h_n^8y_n^{(8)}+1.8626100464508011\times10^{-10}h_n^9y_n^{(9)}$$
$$y_{n+1}=y_n+5.4549943335143936\times10^{-1}h_nf_n+1.6044307468663078\times10^{-2}h_nF_2+1.1112257019290070\times10^{-2}h_nF_3$$
$$+1.7048730209331123\times10^{-2}h_nF_4+1.1759027601172016\times10^{-1}h_nF_5+6.3764920206474354\times10^{-2}h_nF_6$$
$$+1.1616987855667295\times10^{-1}h_nF_7+2.4316881895940735\times10^{-3}h_nF_8+1.2232712198989968\times10^{-2}h_nF_9$$
$$+9.8105796787824953\times10^{-2}h_nF_{10}+1.4470082321380714\times10^{-1}h_n^2y_n^{(2)}+2.4660993945625851\times10^{-2}h_n^3y_n^{(3)}$$
$$+2.9995764659038575\times10^{-3}h_n^4y_n^{(4)}+2.7265581508943048\times10^{-4}h_n^5y_n^{(5)}+1.8745611941477079\times10^{-5}h_n^6y_n^{(6)}$$
$$+9.5451738579298906\times10^{-7}h_n^7y_n^{(7)}+3.3330769972075675\times10^{-8}h_n^8y_n^{(8)}+6.2522836798074345\times10^{-10}h_n^9y_n^{(9)}$$

$\mathbf{CPHBT_{RK5}(10,10,14)}$ with $c_{cp}=1.6845263293032087$, and abscissa vector $\sigma=[0\ 5.9343005761842926\times10^{-1}$ $6.6032312085122502\times10^{-1}$ $6.7035766911908345\times10^{-1}$ $6.9757269007435319\times10^{-1}$ $7.6612963307130821\times10^{-1}$ $8.2552005004722695\times10^{-1}$ $8.9691172353251647\times10^{-1}$ $9.3998752728239798\times10^{-1}$ $9.6102884854375048\times10^{-1}]^T$

$$Y_2=y_n+5.9343005761842926\times10^{-1}h_nf_n+1.7607961664250613\times10^{-1}h_n^2y_n^{(2)}+3.4830312349864452\times10^{-2}h_n^3y_n^{(3)}$$
$$+5.1673385661619876\times10^{-3}h_n^4y_n^{(4)}+6.1329080461028796\times10^{-4}h_n^5y_n^{(5)}+6.0657532919456007\times10^{-5}h_n^6y_n^{(6)}$$
$$+5.1422861793406494\times10^{-6}h_n^7y_n^{(7)}+3.8144839796207175\times10^{-7}h_n^8y_n^{(8)}+2.5151438309009976\times10^{-8}h_n^9y_n^{(9)}$$
$$+1.4925619484902157\times10^{-9}h_n^{10}y_n^{(10)}$$
$$Y_3=y_n+5.0535595019487956\times10^{-1}h_nf_n+1.5496717065634544\times10^{-1}h_nF_2+1.2605113495379075\times10^{-1}h_n^2y_n^{(2)}$$
$$+2.0699850180014245\times10^{-2}h_n^3y_n^{(3)}+2.5240790744149640\times10^{-3}h_n^4y_n^{(4)}+2.4539978387501469\times10^{-4}h_n^5y_n^{(5)}$$
$$+2.0094837324664023\times10^{-5}h_n^6y_n^{(6)}+1.4609531725622789\times10^{-6}h_n^7y_n^{(7)}+9.9575684975054986\times10^{-8}h_n^8y_n^{(8)}$$
$$+6.6606958767845225\times10^{-9}h_n^9y_n^{(9)}+4.4547456036785224\times10^{-10}h_n^{10}y_n^{(10)}$$
$$Y_4=y_n+5.8322457938957528\times10^{-1}h_nf_n+1.8037219764057973\times10^{-2}h_nF_2+6.9095869965450099\times10^{-2}h_nF_3$$
$$+1.6836027341600762\times10^{-1}h_n^2y_n^{(2)}+3.1967682168035506\times10^{-2}h_n^3y_n^{(3)}+4.4703389616453110\times10^{-3}h_n^4y_n^{(4)}$$
$$+4.8755394490671526\times10^{-4}h_n^5y_n^{(5)}+4.2691671444262734\times10^{-5}h_n^6y_n^{(6)}+3.0207989079127973\times10^{-6}h_n^7y_n^{(7)}$$
$$+1.6822693206402487\times10^{-7}h_n^8y_n^{(8)}+6.5128616296313862\times10^{-9}h_n^9y_n^{(9)}+5.1850482333369511\times10^{-11}h_n^{10}y_n^{(10)}$$
$$Y_5=y_n+5.9208869215473492\times10^{-1}h_nf_n+6.9403773956583162\times10^{-3}h_nF_2+1.0274033710763762\times10^{-2}h_nF_3$$
$$+8.8269586813196299\times10^{-2}h_nF_4+1.7322882393716457\times10^{-1}h_n^2y_n^{(2)}+3.3278833204487601\times10^{-2}h_n^3y_n^{(3)}$$
$$+4.6995817706428156\times10^{-3}h_n^4y_n^{(4)}+5.1649571436544371\times10^{-4}h_n^5y_n^{(5)}+4.5448390433200733\times10^{-5}h_n^6y_n^{(6)}$$
$$+3.2182676339911780\times10^{-6}h_n^7y_n^{(7)}+1.7786896769317757\times10^{-7}h_n^8y_n^{(8)}+6.6455794310577777\times10^{-9}h_n^9y_n^{(9)}$$

$$
\begin{aligned}
&+1.8416446761273598\times10^{-11}h_n^{10}y_n^{(10)}\\
Y_6 &= y_n+5.7841507532521941\times10^{-1}h_nf_n+1.8634924880678789\times10^{-3}h_nF_2+2.7585797645732213\times10^{-3}h_nF_3\\
&+2.3700398778623545\times10^{-2}h_nF_4+1.5939208671482422\times10^{-1}h_nF_5+1.6347459009250773\times10^{-1}h_n^2y_n^{(2)}\\
&+2.9911749890355704\times10^{-2}h_n^3y_n^{(3)}+3.9501502209028268\times10^{-3}h_n^4y_n^{(4)}+3.9604582544246685\times10^{-4}h_n^5y_n^{(5)}\\
&+3.0690725796041343\times10^{-5}h_n^6y_n^{(6)}+1.8131663928410011\times10^{-6}h_n^7y_n^{(7)}+7.6191053016103847\times10^{-8}h_n^8y_n^{(8)}\\
&+1.7843501303036769\times10^{-9}h_n^9y_n^{(9)}+4.9448190264126259\times10^{-12}h_n^{10}y_n^{(10)}\\
Y_7 &= y_n+5.9043043232493331\times10^{-1}h_nf_n+1.7958730528799361\times10^{-2}h_nF_2+1.4586072801606044\times10^{-3}h_nF_3\\
&+1.2531656559352036\times10^{-2}h_nF_4+8.3581028538824861\times10^{-2}h_nF_5+1.1955959481515684\times10^{-1}h_nF_6\\
&+1.7081859040696901\times10^{-1}h_n^2y_n^{(2)}+3.2043520019335005\times10^{-2}h_n^3y_n^{(3)}+4.3369550953010101\times10^{-3}h_n^4y_n^{(4)}\\
&+4.4422177001322740\times10^{-4}h_n^5y_n^{(5)}+3.4876686890540442\times10^{-5}h_n^6y_n^{(6)}+2.0486024982127579\times10^{-6}h_n^7y_n^{(7)}\\
&+8.1869942644563346\times10^{-8}h_n^8y_n^{(8)}+1.6561472495679913\times10^{-9}h_n^9y_n^{(9)}+4.5290135142888129\times10^{-11}h_n^{10}y_n^{(10)}\\
Y_8 &= y_n+5.8900894046938557\times10^{-1}h_nf_n+8.6657695471225551\times10^{-2}h_nF_2+3.9802886706887414\times10^{-4}h_nF_3\\
&+3.4196737741949992\times10^{-3}h_nF_4+2.2807826719538012\times10^{-2}h_nF_5+3.2625759324504865\times10^{-2}h_nF_6\\
&+1.6199379890659854\times10^{-1}h_nF_7+1.7360999944418690\times10^{-1}h_n^2y_n^{(2)}+3.3817549911252742\times10^{-2}h_n^3y_n^{(3)}\\
&+4.8305235381998508\times10^{-3}h_n^4y_n^{(4)}+5.2911112323407179\times10^{-4}h_n^5y_n^{(5)}+4.4915956888063516\times10^{-5}h_n^6y_n^{(6)}\\
&+2.8892171954995373\times10^{-6}h_n^7y_n^{(7)}+1.3114966443703605\times10^{-7}h_n^8y_n^{(8)}+3.9158387856764826\times10^{-9}h_n^9y_n^{(9)}\\
&+2.5778593577831624\times10^{-10}h_n^{10}y_n^{(10)}\\
Y_9 &= y_n+5.5631602197664731\times10^{-1}h_nf_n+1.7211673796407412\times10^{-2}h_nF_2+3.2004516719069138\times10^{-3}h_nF_3\\
&+2.7496750998476687\times10^{-2}h_nF_4+1.8488586449448124\times10^{-1}h_nF_5+1.9238705072271198\times10^{-2}h_nF_6\\
&+2.8220794727845982\times10^{-2}h_nF_7+1.0341726454436123\times10^{-1}h_nF_8+1.5126470023518393\times10^{-1}h_n^2y_n^{(2)}\\
&+2.6675948167045008\times10^{-2}h_n^3y_n^{(3)}+3.4118853719250276\times10^{-3}h_n^4y_n^{(4)}+3.3487074568373207\times10^{-4}h_n^5y_n^{(5)}\\
&+2.5965582862179711\times10^{-5}h_n^6y_n^{(6)}+1.6047369990862359\times10^{-6}h_n^7y_n^{(7)}+7.7702807418301099\times10^{-8}h_n^8y_n^{(8)}\\
&+2.7290451952662450\times10^{-9}h_n^9y_n^{(9)}+5.0521164706530579\times10^{-11}h_n^{10}y_n^{(10)}\\
Y_{10} &= y_n+5.8532833450920740\times10^{-1}h_nf_n+4.0529692379392655\times10^{-3}h_nF_2+2.7535633275909705\times10^{-3}h_nF_3\\
&+2.3657299949851572\times10^{-2}h_nF_4+6.6773909399830647\times10^{-2}h_nF_5+1.5682791565300769\times10^{-1}h_nF_6\\
&+4.5278658792413586\times10^{-2}h_nF_7+1.1328429325961174\times10^{-2}h_nF_8+6.5027768347948267\times10^{-2}h_nF_9\\
&+1.6631147411677050\times10^{-1}h_n^2y_n^{(2)}+3.0315896169288176\times10^{-2}h_n^3y_n^{(3)}+3.9396077751724448\times10^{-3}h_n^4y_n^{(4)}\\
&+3.8244988042746535\times10^{-4}h_n^5y_n^{(5)}+2.8128828547723106\times10^{-5}h_n^6y_n^{(6)}+1.5466195375570637\times10^{-6}h_n^7y_n^{(7)}\\
&+6.0872653315290231\times10^{-8}h_n^8y_n^{(8)}+1.5906008354109543\times10^{-9}h_n^9y_n^{(9)}+1.1210073202001079\times10^{-11}h_n^{10}y_n^{(10)}\\
y_{n+1} &= y_n+5.7477438181988205\times10^{-1}h_nf_n+1.2723618159852982\times10^{-2}h_nF_2+1.3564521219555059\times10^{-2}h_nF_3\\
&+1.9218057567157325\times10^{-2}h_nF_4+1.1485428369921558\times10^{-1}h_nF_5+5.4975225293228937\times10^{-2}h_nF_6\\
&+1.0629387862938201\times10^{-1}h_nF_7+1.7516258263226465\times10^{-3}h_nF_8+1.0054731789283433\times10^{-2}h_nF_9\\
&+9.1789675996120271\times10^{-2}h_nF_{10}+1.6138949160591556\times10^{-1}h_n^2y_n^{(2)}+2.9319726209364978\times10^{-2}h_n^3y_n^{(3)}\\
&+3.8422803320804179\times10^{-3}h_n^4y_n^{(4)}+3.8258855495751858\times10^{-4}h_n^5y_n^{(5)}+2.9602804594150724\times10^{-5}h_n^6y_n^{(6)}\\
&+1.7789840626988628\times10^{-6}h_n^7y_n^{(7)}+8.0703132790948057\times10^{-8}h_n^8y_n^{(8)}+2.5468406137515121\times10^{-9}h_n^9y_n^{(9)}
\end{aligned}
$$

$$+4.3658347039740745 \times 10^{-11} h_n^{10} y_n^{(10)}$$

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