A Categorical Approach to Superdifferential Operators

Edward Poon, Professor Hadi Salmasian

Introduction

The 1970's was a turning point in theoretical physics since it was in that era that physicists and mathematicians began applying the tools of abstract mathematics to study the fundamental building blocks of matter.

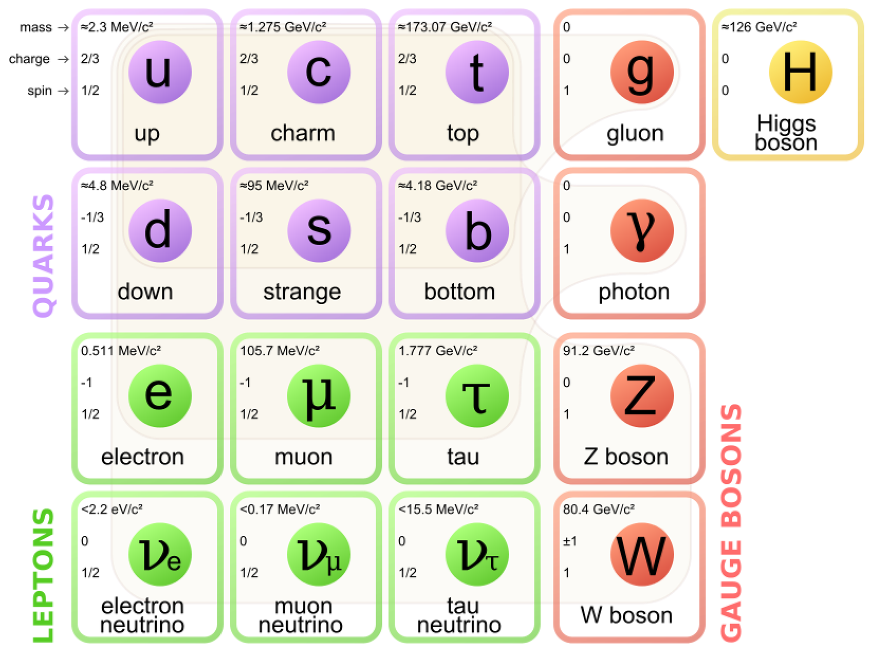

Figure: The Standard Model of elementary physics

The idea behind supergeometry is to replace geometric spaces with their graded analogues. Intuitively, this involves studying spaces that can be broken-down into smaller and nicer parts. Moreover, supergeometry is about using categories of graded objects which usually turn out to carry extra symmetry, making it intimately tied to particle physics through supersymmetry, string theory, the Higgs boson, etc.

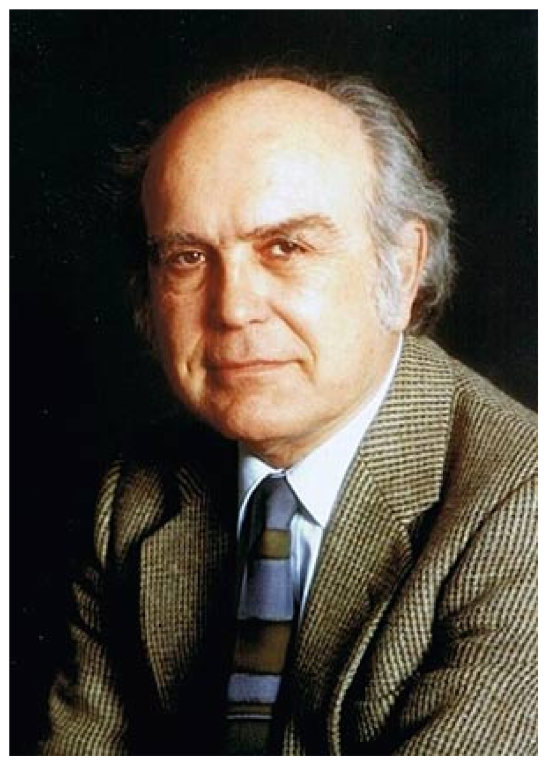

Figure: Bruno Zumino, architect of supersymmetry

The study of these space is often conducted using category theory, and the emerging idea to use categories to obtain invariants for geometric and algebraic objects. Many results in math can be stated and proved in much simpler ways using the language of categories, making it an excellent framework to carry out explorations in supergeometry.

Goal

Over the years, the applications of category theory have grown immensely in several fields of mathematics. In particular, braided monoidal categories have found applications in quantum field theory and string theory. The goal of this project is to develop a suitable adaption of the existing theory of differential operators to the categorical supergeometric settings.

Observations

In supergeometry, we are interested in $\mathbb{Z}_2$-graded or super vector spaces, i.e, spaces that can be written in the form $V = V_0 \oplus V_1$ and linear maps that preserve this grading.

The category of super vector spaces over a fixed field $\mathbb{F}$, $\mathbb{F}$-$\mathbf{SVect}$, is a symmetric monoidal category where the monoidal product is the tensor product of vector spaces and the associator is the usual associative isomorphism

$$(U \otimes V) \otimes W \to U \otimes (V \otimes W).$$

The identity object is the underlying field $\mathbb{F}$ and the left and right unitors are the isomorphisms

$$\mathbb{F} \otimes V \to V, \qquad V \otimes \mathbb{F} \to V.$$

However, the braiding in $\mathbb{F}$-$\mathbf{SVect}$ is slightly different from $\mathbf{Vect}$ since we have to respect the parity of the elements:

$$\gamma_{V\otimes W}: V \otimes W \to W \otimes V, \quad (u \otimes v) \mapsto -1^{|u||v|} v \otimes u.$$

One example of such spaces are Grassmann algebras, which appears in many fields of mathematics through the definition of the cross product and determinant in linear algebra and differential forms in differential geometry. We define the Grassmann algebra of a vector space V by

$$\Lambda(V) = \Big(\bigoplus_{n=0}^{\infty} V^{\otimes n}\Big)/\{(v \otimes w) + (w \otimes v): v, w \in V\}.$$

Many algebraic structures can be generalized as objects within categories. For our purposes, we are interested in ring objects. Similar to how a ring must satisfy the axiom

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

a ring object is equipped with a morphism $m: R \times R \to R$ such that the diagram below commutes

$$\begin{array}{ccc}
(R \times R) \times R & \xrightarrow{\cong} & R \times (R \times R) \\
\downarrow{\scriptstyle m \times \mathrm{id}_R} & & \downarrow{\scriptstyle \mathrm{id}_R \times m} \\
R \times R & & R \times R \\
& \searrow^{m} \quad R \quad {}^{m}\swarrow &
\end{array}$$

A general correspondence of morphisms

Let R be a superring. The functor $F_R: \mathbf{Grass} \to \mathbf{Vect}$ defined by

$$\Lambda(V) \mapsto (\Lambda(V)_0 \otimes R_0),$$

$$\begin{array}{c}
(\Lambda(V) \xrightarrow{\phi} \Lambda(W)) \\
\downarrow \\
(\Lambda(V)_0 \otimes R_0) \xrightarrow{\phi \otimes \mathrm{id}_R} (\Lambda(W)_0 \otimes R_0).
\end{array}$$

It turns out that F_R is a ring object in $\mathbf{Grass}^{\mathbf{Vect}}$, the category of functors from $\mathbf{Grass}$ to $\mathbf{Vect}$.

Moreover, given another superring S, we discover a fact that is akin to the Yoneda Lemma. There is a canonical bijective correspondence

$$\mathrm{Nat}(F_R, F_S) \cong \mathrm{Mor}_{\mathbf{Ring}}(R, S).$$

This means that we can construct a natural transformation $\eta: F_R \Rightarrow F_S$ from a ring homomorphism $f: R \to S$ by defining its behavior on $\mathbf{ObGrass}$ like so:

$$\begin{array}{ccc}
(\Lambda(V) \otimes R)_0 & \xrightarrow{\mathrm{id}_{\Lambda(V)} \otimes f} & (\Lambda(V) \otimes S)_0 \\
{\scriptstyle \phi \otimes \mathrm{id}_R}\downarrow & & \downarrow{\scriptstyle \phi \otimes \mathrm{id}_S} \\
(\Lambda(W) \otimes R)_0 & \xrightarrow[\mathrm{id}_{\Lambda(W)} \otimes f]{} & (\Lambda(W) \otimes S)_0
\end{array}$$

Conclusion and next steps

The next step would be to investigate how this functor behaves in the setting of supermanifolds, where one can use different techniques deepen the theory of supergeometry.

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