A REFINED FINITE ELEMENT ANALYSIS OF FOLDED PLATE STRUCTURES

by

GH. ATTAR-HASSAN

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ABSTRACT

A triangular finite element has been developed for solving folded plate structures. The element is formed by the combination of an in-plane and a plate bending high precision triangular element. It has 36 degrees of freedom corresponding to the generalized nodal displacements, u, $\partial u/\partial x$, $\partial u/\partial y$, v, $\partial v/\partial x$, $\partial v/\partial y$, w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x^2$, $\partial^2 w/\partial x \partial y$ and $\partial^2 w/\partial y^2$. The in-plane displacements u and v are represented by a cubic polynomial whereas the lateral displacement w is represented by a quintic polynomial.

A computer program has been written to analyse prismatic folded plate structures with any number of plates, elements and nodal points. The structure can have various boundary conditions as well as different material properties. A uniform line load distributed along the ridge is considered although with only minor modifications, the program can accomodate various types of loadings.

Some difficulties arise when assembling inclined elements meeting along the ridge where the in-plane and the lateral displacements are coupled. This is due to the difference in order between the polynomials representing the in-plane and the lateral displacements respectively. To overcome these difficulties, a new method of linking elements is presented. In this method, a new global rotation matrix is developed in which the corresponding terms to the second derivative of the in-plane displacements - which are not considered as degrees of freedom - are set equal to zero. As a result the second derivative of the lateral displacement, which is no longer controlled by the second derivative of the in-plane displacement, takes very large values. To compensate for the removed control, a new load vector, called the "reduced consistent load vector" is developed. In

this vector, the terms that correspond to the second derivatives of the lateral displacement are omitted.

The element developed in this thesis is used to solve a simply supported folded plate structure subjected to a uniform vertical line load distributed along its ridge. The variation of the effect of the in-plane action and the slab action with the slope angle α is studied by varying α between zero and 30 degrees. The first value which represents the limiting case of the flat plate shows the accuracy of the plate bending finite element. The 30° angle is a commonly occuring value and it is felt that it represents a critical value with regard to the error of the method presented here.

Three different load vectors are used to obtain the finite element results: a lumped load, a fully consistent load and the reduced consistent load vector. The reliability of these results is examined by comparing them with those calculated by the direct stiffness method based on the elasticity theory. This method is presented in some detail in this dissertation.

The mid-span vertical deflection of the ridge and the strain energy of the structure are determined for different values of the slope angle α, namely α equal to 0°, 10°, 20° and 30°. For α equal to zero and 30°, the longitudinal moment, the transverse moment and the longitudinal stress are also calculated. It is found that the fully consistent load vector yields excellent values for the flat plate case but does not give good moment values for the folded plate case. The longitudinal stresses which govern in the folded plate case are determined with an error of 1%, using the reduced consistent load vector. The moments predicted by this same vector are in error by 10%. However this latter error is not considered significant, since the moments have little importance in the folded plate structure design.

Whereas full compatibility is maintained for the flat plate case, horizontal discontinuity is created along the ridge for the folded plate case. This does not prevent the results from converging rapidly toward the exact values.

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NOTATIONS

α	Slope angle
$\beta = \frac{3-\nu}{1+\nu}$	
δ	The deflection
$\mu = \frac{m\pi B}{2L}$	
ν	Poisson's ratio
ξ, η	Local coordinates
σ_x	Longitudinal stress
τ_{xy}	Shearing stress
B	Plate width
D	Flexural rigidity of Plate = $\frac{Eh^3}{12(1-\nu^2)}$
E	Young's modulus
h	Plate thickness
L	Plate span
m	Number of terms in Fourier series
M	Transverse moment per unit length
N	Normal force per unit length
P	External load
P_n	Perpendicular component of the external load
P_t	Tangential component of the external load
Q	Shearing force per unit length
t	Element thickness
T	Tangential force per unit length
u, v	In-plane displacements
U_e	Element strain energy
V_e	Element virtual work

$\bar{u}_1$, $\bar{v}_1$, $\bar{u}_2$ and $\bar{v}_2$	Displacement amplitudes at the plate edges 1 and 2 respectively, for the mth term of Fourier series
u_x, u_y, v_x, v_y	Derivatives of u, v and w with respect to x, y
w_x, w_y, w_{xx}, w_{xy}, w_{yy}	Coordinate system
w	Lateral displacement
u_ξ, u_η, v_ξ, v_η	Derivatives of u, v and w with respect to ξ, η
w_ξ, w_η, $w_{\xi\xi}$, $w_{\xi\eta}$, $w_{\eta\eta}$	Coordinate system
w, y, z	Plate coordinates
X, Y, Z	Global coordinates (elasticity method)
x′, y′, z′	Global coordinates (finite element method)

Elasticity Method

{d}	Displacement vector with respect to local system
{D}	Displacement vector with respect to global system
{f}	Force vector with respect to local system
{F}	Force vector with respect to global system
$[k]_{EL}$	Plate stiffness matrix with respect to local system
$[T]_{EL}$	Transformation matrix

Finite Element Method

{A}	Column vector a_i (in-plane displacements)
{F}	Force vector (folded plate element)
$\{\bar{F}\}$	Force vector of the complete structure
$\{P\}_p$	Load vector (in-plane element)
$\{P\}_b$	Load vector (bending element)
{P}	Load vector (folded plate element)
$\{P_2\}_p$	Force vector (in-plane element)

$[K_1]_p$	Element stiffness matrix with respect to local system (in-plane element)
$[K_2]_p$	Element stiffness matrix with respect to the plate system (in-plane element)
$[K]_p$	Reduced element stiffness matrix (in-plane element)
$[K_1]$	Stiffness matrix with respect to the plate system (folded plate element)
$[K]$	Stiffness matrix with respect to global system (folded plate element)
$\{\bar{K}\}$	Stiffness matrix of the complete structure
$[R]_p$	Rotation matrix (in-plane element)
$[R]_b$	Rotation matrix (bending element)
$[R]$	Global rotation matrix (folded plate element)
$[T]_p$	Transformation matrix (in-plane element)
$[T]_b$	Transformation matrix (bending element)
$[T_2]$	Inverse of transformation matrix (20 x 18)
$\{V_1\}$	Element in-plane displacement vector with respect to local system
$\{V_2\}$	Element in-plane displacement vector with respect to the plate system
$\{W_1\}$	Displacement vector with respect to the plate system (folded plate element)
$\{W\}$	Displacement vector with respect to global system (folded plate element)
$\{\bar{W}\}$	Displacement vector of the complete structure

CHAPTER I
INTRODUCTION

1.1 Definition

The folded plate structure is a prismoidal shell composed of a series of plates rigidly connected along their common edges, called "Ridges", and supported on two end diaphragms. These structures are extensively used to provide economical means of covering large areas, without intermediate supports. As compared to cylindrical shells, prismatic folded plate structures require simpler formwork and are less likely to buckle; but due to their greater bending moments, they are heavier and less suitable to cover very large spans. Folded plates are mainly used for roof structures, but they are also frequently used as components for bins, floors, bridges, airplanes, missiles and foundations [20]*. Typical folded plate structures are shown in Fig. 1.1.

Folded plates may be made of reinforced concrete, prestressed concrete, or timber combined with plywood sheeting and steel.

1.2. Object and Scope

The object of this thesis is to investigate the use of two high precision triangular in-plane and plate bending finite elements in the analysis of folded plate structures. The main characteristics of these elements, as well as the different methods used for the solution of folded plates, are discussed briefly in this dissertation. A new rotation matrix and a special consistent load vector have been developed to overcome the difficulties created by the difference in order between the polynomials used to describe the in-plane and the normal displacements.

*Numerals in parentheses refer to corresponding items in the list of References (Appendix I).

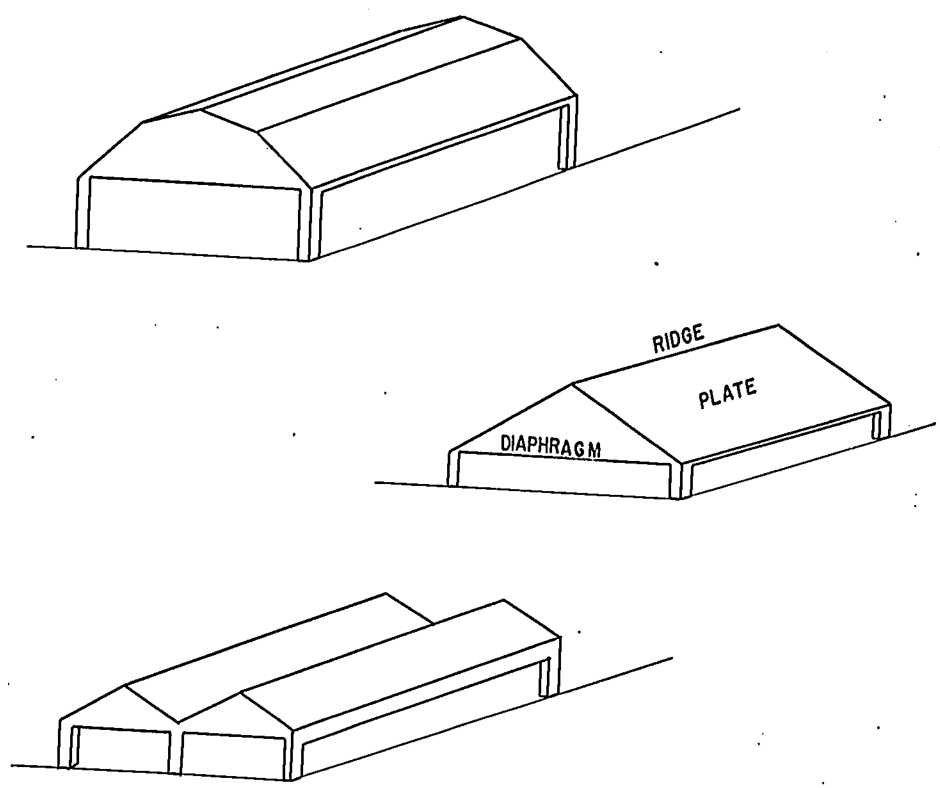


FIG. 1.1 SOME TYPES OF FOLDED PLATE STRUCTURES

The results are compared with an exact solution to judge the reliability of the finite element solution, and to evaluate the additional error due to the incompatibility along the ridges. The exact solution is also discussed in some detail in this thesis.

A computer program was developed to handle prismatic folded plate structures with arbitrary boundary conditions and number of plates. This program is tested by analysing a simply supported prismatic folded plate with different slopes, and all the results are compared with the results obtained by the exact solution.

1. 3 Historical Review

The first papers on the analysis of folded plate structures were published in Germany in 1930 by G. Ehlers [12]. In his method, the plate elements were assumed to be connected by hinged joints. He also assumed these elements to act, in the longitudinal direction, as beams supported at the end diaphragms; that is, the component of the external load, parallel to the plane of the plate, causes this plate to bend in its plane as a beam. The longitudinal stresses are therefore linearly distributed along the transverse section of the plate. Unlike ordinary beams the longitudinal edges of the plates are not free as they are joined monolithically to the adjacent plates. In order to satisfy this monolithic action, Ehlers assumed shear forces to act between the plates. These forces of such magnitude as to prevent relative shift along the edges of the plates. However, he neglected the relative displacements of the joints, or rather the internal forces and stresses resulting from these displacements. Craemer [9], [10], and Gruber [16], [17] developed the analysis of folded plates by using a strip theory approach. They analysed a unit width strip in the transverse direction [16], and considered the effect of the rigidity (the monolithic action) and the relative displacements of the joints. They also considered the actual stress distributions in the planes of the plates.

In 1947 Winter and Pei [26] introduced the folded plate analysis in the United States. Their technique was similar to previous methods in neglecting the effect of the relative displacements of the joints. However, they realized the similarity between the simultaneous equations obtained by equating the stresses along the common joint between two adjacent plates and the three moment equation for continuous beams. This allowed them to use an iteration procedure similar to the moment distribution method by establishing-for any folded plate structure-a conjugate, continuous beam. The spans and moments of inertia of this conjugate beam are made respectively equal to the width and first moment of area of the section of the different plates. If the fixed end moments applied at each end of the nth span are equal to the actual fixed end moments divided by the width of the nth plate, the resulting support moments, from the moment distribution, are equivalent to the normal force at the respective joints. On the other hand, if these fixed end moments are equal to the shearing forces divided by the width of the nth plate, the resulting support moments are equal to the edge shear forces at the corresponding joints.

I. Gaafar [14] appears to be one of the first researchers who presented an experimental investigation of folded plate structures. He substantiated his proposed method of analysis by studying the deformations and joint displacements in small-scale models. In his method, the analysis was first performed without considering the relative joint displacements, using the procedure presented by Winter and Pei [26]. Then, the correction due to the effect of the relative joint displacements was added. This last step was easily accomplished by choosing the relative transverse displacements between each pair of consecutive joints as unknowns and then calculating the corresponding (slab) end moments in terms of these displacements. Consequently the shear forces, the normal forces and the longitudinal stresses were determined in terms of these unknowns also. The plate deflections due to this previous step were added to those calculated in the original step. Therefore the total plate deflections were expressed in terms of the applied load and the relative transverse deflections. These deflections were calculated from the geometrical relations between the relative joint deflections and the total deflections. It was found that there is only one set of the relative displacement values which satisfies the algebraic relations between the relative displacements and the total displacements imposed by the geometrical requirements of the cross sections of the plates and the equilibrium conditions.

After calculating the values of the relative displacements, the slab moments, shears and longitudinal stresses were easily obtained since they were expressed in terms of these relative displacements. These moment and force values were then added to the corresponding values from the initial analysis.

In 1958, D. Yitzhaki [27] presented the "Method of Particular Loading". This method is based on the resolution of the folded plate structure into several compatible elementary structures assembled together and satisfying the equilibrium and the geometrical conditions. A matrix formulation of this method was presented by A. C. Scordelis [23]. This made possible the use of the digital computer in the analysis of folded plate structures.

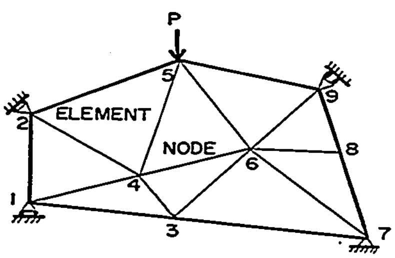
The ASCE Task committee on folded plate construction [21] summarized, in 1963, the work done on folded plates up to that time. They recommended a modified version of Gaafar's method for the analysis of simply supported prismatic folded plates. In this method the analysis is divided into two parts. The first is the elementary analysis which consists of the transverse slab and the longitudinal plate analysis. In the transverse slab analysis, the surface loads are considered as carried transversely by the plates acting as continuous one-way slabs spanning between the supports at the joints. In the longitudinal plate analysis, however, all loads carried transversely to the joints are considered as transferred longitudinally to the end supporting members by the plates acting as inclined simple beams. The second is the correction analysis to determine the corrections due to the relative joint displacements. The final results are obtained by the superposition of those obtained from the two steps.

The previous methods are generally referred to as the "Ordinary method". The basic steps and assumptions of this method will be discussed in the following sections.

Goldberg and Leve [15] presented a general method of analysis using the theory of elasticity to define the plane stresses due to in-plane loads and the classical plate theory for stresses due to loads normal to the plane of the plates. They expressed the displacements, the forces and the applied loading in terms of Fourier series. By chosing the displacements functions which satisfy the boundary conditions along the diaphragms, the two-dimensional problem was reduced to a one dimensional problem. This method is called "The Exact Method" or "The Elasticity Method". It was programmed and applied by De-Fries, Skene and Scordelis, A. C. [11]. This method is considered as an exact solution for comparison with the results obtained in this thesis. The method is presented in detail in Chapter 4.

The finite-element method, which has proven to be one of the most powerful tools in the solution of plane stress and plate bending problems, has been used in the last few years to analyse folded plate structures. The method is based on representing a continuous medium by a gridwork of elements. The elements are commonly triangular or rectangular in shape, and are considered to be joined together at their vertices, or nodes as shown in Fig. 1-2. At these nodes equilibrium and compatibility are satisfied. In 1966 Abu-Chazaleh [1] developed a rectangular element and used it to analyse a prismatic folded plate, while the rectangular finite element developed by Zienkiewicz and Cheung [28], was used by Rockey and Evans [22]. Another element was developed by J. H. Argyris and D. W. Scharpf [2] and was used in the analysis of folded plate structures. Their element used complete fifth order polynomials for all three displacements. However, they did not present any numerical example to show the accuracy of the results that can be obtained by using this element.

Recently, Kam, S., Lo, and S. C. Scordelis [19] presented their "Finite segment method", in which the basic structural element was a finite segment formed by dividing each plate longitudinally. This method was based on the ordinary method but included the effect of shearing stresses and transverse normal stresses in the calculation of deflections. Y. K. Cheung [5] has extended his "Finite strip method" used in the analysis of elastic slabs to analyse folded plate structures. This method is similar to the finite element method, however the structure is divided into longitudinal strip instead of arbitrary elements. A set of displacement functions satisfying the boundary conditions is assumed to represent the different displacements within each strip.



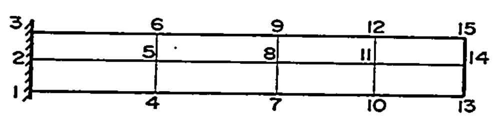


FIG. 1.2 FINITE ELEMENT SUBDIVISION

CHAPTER II
DIFFERENT METHODS OF ANALYSIS

2.1 General

The rigidity of the folded plate structure is due to the resistance of the plates to forces acting in their planes. This can be explained by the small displacements of the joints. Consider a section through a typical joint as shown in Fig. 2-1. The total deflection δ due to the external load is resolved, in the same manner as the load, into two components, $\delta \cos \alpha$ and $\delta \sin \alpha$, respectively perpendicular and parallel to slab C. While this slab is flexible perpendicular to its plane, it is very stiff and acts as a relatively deep beam parallel to that plane. Hence the deflection $\delta \sin \alpha$ would be very small. Since the perpendicular deflection $\delta \cos \alpha$ cannot occur without the parallel component $\delta \sin \alpha$, it would be very small too. Consequently the total deflection is very small.

From above, we can see that the structural action of the folded plate can be divided into two parts: the transverse slab action, and the in-plane plate action. Many different methods have been used to analyse folded plate structures, using different concepts to deal with the slab and plate actions*. These methods may be grouped into three main categories:

---

* The elementary beam theory has been used to analyse folded plate structures as if they were beams spanning between the two end diaphragms. It has been shown [11], [24], however, that the stresses obtained by this method were considerably in error from the actual stresses.

1. The ordinary method
2. The elasticity method
3. The finite element method.

2.2. The Ordinary Method

This method has been fully described in the literature [14], [21], [23], [24], [27]. As in most methods of analysis, the following simplifying assumptions are made:

1. The material is considered homogeneous, isotropic and obeying Hooke's law.
2. Longitudinal edge joints are completely monolithic and continuous (rigidly connected).
3. The principle of superposition holds; that is, the structure may be analysed separately for the effect of the relative displacements of the joints.
4. The supporting end diaphragms are rigid in their own plane and completely flexible normal to that plane.

In the analysis, the external loads p applied between joints, are resolved into two components, one p_n perpendicular and the other p_t parallel to the plates on which they act, as shown in Fig. 2-2. The perpendicular components p_n are resisted by the transverse slab-action, in which the structure in the transverse direction is considered as a continuous one way slab supported on rigid supports at the joints. The reactions of these slabs at each joint are resolved into their components parallel to the adjoining plates. These components together with the tangential components of the external loads are resisted by the plate-action. Under these loads, the plates are considered to act as simple beams between the two end diaphragms (as explained in the previous chapter). Shear forces are assumed, acting between the plates, to satisfy

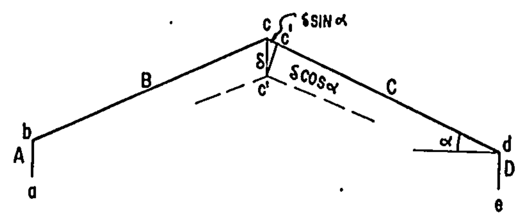


FIG.2.1 SECTION THROUGH A TYPICAL RIDGE

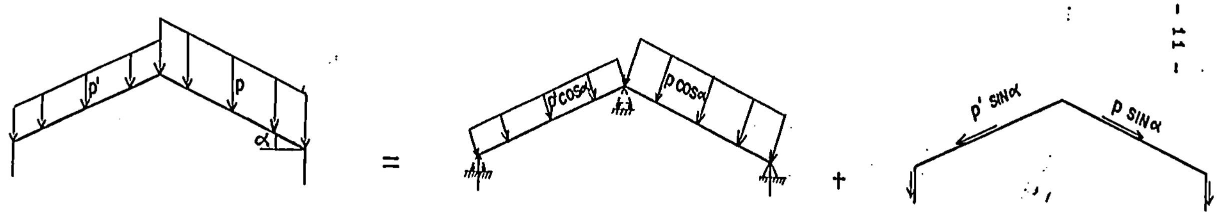


FIG.2.2 SLAB AND PLATE ACTION OF THE FOLDED PLATE

compatibility. The longitudinal stresses and the plate deflections are then computed. Lastly, the internal forces and stresses are corrected for forces and stresses due to the effect of the relative displacements of the joints.

The last step can be performed separately [14]. Alternatively, the external loads can be divided initially between the slab system and the plate system such as to produce the same displacements in both systems at each joint [23], [27].

2. 3. The Elasticity Method

The basic assumptions underlying this method are the same assumptions previously mentioned. The classical thin plate theory and the elasticity theory are used to determine the stresses and displacements due to normal and in-plane loadings respectively. Four displacements are assumed at each point of the joint: two components of translation and a rotation, all lying in the plane normal to the joint, and a translation in the direction of the joint. These joint displacements are expanded into Fourier half-range series. The assumed functions satisfy the boundary conditions in the longitudinal direction. The problem, then can be considered as a one-dimensional problem. Consequently, four distributed generalized forces are assumed at each longitudinal edge, namely, a force in the plane of the plate and normal to its edge, a shear in the direction of the thickness, a longitudinal shear, and an edge moment. Each of these forces is linearly dependent upon the four components of displacement at both edges of the plate, and may also be expanded into Fourier series. The force-displacement relationships allow the forces to be written in terms of the displacements. By writing the joint equilibrium equations, four simultaneous equations are obtained at each joint. Therefore, 4n simultaneous equations are obtained for each harmonic of Fourier expansion, where n is the number of joints with unknown forces and displacements. The solution of these equations gives the required displacements. This method with the derivation of the necessary formulas is described in more detail in Chapter IV.

2.4. The Finite Element Method

The finite element method is a numerical procedure which employs a physical approach to reduce the continuous problem into a model with a finite number of degrees of freedom. In this method the behaviour of the continuous structure is approximated by that of a model composed of a finite number of discrete elements interconnected at a finite number of nodal points or nodes. At these nodes, the basic conditions of continuity and equilibrium are established.

The finite-element analysis is normally carried out by the stiffness method of analysis. The deformed shape of the element is limited to certain patterns by assuming displacement functions, which will represent approximately its possible deformed shape under the loading being investigated.* . The forces at the nodes of the element are calculated in terms of the displacements at these nodes. This relationship between element forces and displacements is expressed in terms of the element stiffness matrix. The basic step in the finite element analysis is the evaluation of this element stiffness matrix. Then, the stiffness matrix of the complete structure is obtained by summing the stiffness matrices of individual elements. This matrix is singular. Therefore it cannot be inverted to solve the force-displacement relationships for the displacements due to

*This is the displacement method approach of the finite-element method.

given external loads. This singularity is avoided by applying boundary conditions to prevent the rigid body displacements. The resulting matrix equations can then be solved.

The use of the "ordinary method" and the "elasticity method" in the analysis of folded plate structures is limited to isotropic, homogeneous materials and to simply supported structures only. The length-width ratios of the plates should be more than two [14] or even three [24], [27], when using the ordinary method to meet the assumption of one-way slab action.

The solution by the elasticity method requires considerable computational effort, especially when the structure is subjected to loads other than the most simple ones such as uniform loads or concentrated load at midspan, where a large number of separate harmonics must be used to represent the applied loads.

The finite-element method can be applied to folded plates with orthotropic properties, different loading and boundary conditions; it is therefore considered the most convenient method for the analysis of folded plate structures.

The basic step in applying the finite-element method is the selection of an appropriate finite element with satisfactory performance. Several in-plane, plate bending and shell elements of different shapes, with different displacement functions and consequently with different number of degrees of freedom have been developed by many researchers [1],[3], [4], [6], [7], [8], [18], [25], [28], [29]. Only some of these elements are conforming elements, that is, the transverse displacements and slopes are continuous between elements. This property leads to monotonic convergence of the finite-element solution to the exact one, as the number of elements is increased [13]. The fully conforming triangular elements developed in the National Research Council of Canada [7], [8] have been tested on plane stress, plate bending and shell roof problems and have been proven to yield excellent results. Such elements are used by the author here for the analysis of folded plate structures.

The element used in this study is treated as a combination of two elements, an in-plane element and a plate bending element. The in-plane element possesses twenty degrees of freedom, namely u, v and their derivatives at each node, plus u and v at the centroid. The displacements are described by cubic polynomials with ten terms each. This gives a total of twenty parameters consistent with the number of degrees of freedom of the element. In the plate bending element, the displacement function is taken as a quintic polynomial. Eighteen degrees of freedom are used, w and its first and second derivatives at each node. Three conditions ensuring cubic variations of normal slope along the edges are used to reduce the number of independent parameters to eighteen. In Fig. 2-3 a typical subdivision of finite elements is shown. In this figure two types of nodal points can be recognized; nodes between coplanar elements and nodes connecting inclined elements meeting along the ridge. In assembling the coplanar elements such as 6 and 7, the in-plane displacements and the normal displacement are uncoupled, and hence no interaction exists between the in-plane element and the plate bending element. These two elements are compatible elements when used to solve in-plane and plate bending problems, respectively. Complete compatibility is, therefore, maintained along the edges of the elements. A lack of compatibility occurs when assembling non-coplanar elements such as 8 and 9 which meet along the ridges. Here the in-plane and normal displacements are coupled, as shown in

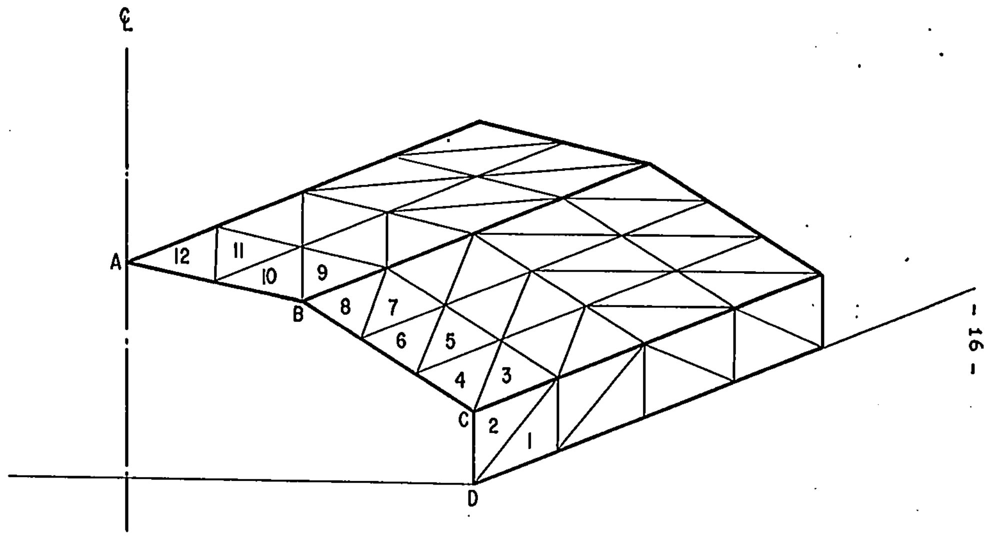


FIG. 2.3 TYPICAL FINITE-ELEMENT SUBDIVISION

Fig. 2-1. The lateral displacement of one plate has a component parallel to the other. This displacement is represented by a quintic polynomial and its component is hence also a fifth order displacement. This parallel component should be added to the in-plane displacement of the second plate. The in-plane displacement, however, is represented by a cubic polynomial and does not match with this fifth order component of the lateral displacement. This causes the incompatibility along the ridge.

It should be noted that these difficulties arise from the use of high precision elements with higher order derivatives as degrees of freedom. They have not occurred in previous works [1], [22], since simpler non-conforming elements have been used.

In this thesis, these difficulties are investigated and a method of assembly for these high precision elements is developed. A new rotation matrix is formed in which the second derivatives of the in-plane displacements u and v are assumed equal to zero. A special consistent load vector has been developed with the corresponding terms of the second derivatives of the normal displacement omitted.

CHAPTER III

FINITE ELEMENT ANALYSIS OF FOLDED PLATE STRUCTURES

3.1. General

This chapter describes briefly the triangular finite element used in this analysis. As mentioned before, this element is treated as composed of two separate in-plane and plate bending elements. A full description of these elements can be found in references [7] and [8]. The required rotation matrix for assembling non-coplanar elements, and the consistent load vectors for linear loadings along the ridges are derived here. Boundary and continuity conditions are also discussed.

The procedure described in this chapter can be used for the analysis of folded plate structures of variable thickness, material properties and with any type of boundary conditions, provided that the assumptions of linear elasticity and small-deflection theory hold.

3.2. The In-Plane Finite-Element

The in-plane displacements within this triangular element are represented by two complete cubic polynomials in the form:

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 \quad (3.1.a)$$

and

$$v = a_{11} + a_{12}\xi + a_{13}\eta + a_{14}\xi^2 + a_{15}\xi\eta + a_{16}\eta^2 + a_{17}\xi^3 + a_{18}\xi^2\eta + a_{19}\xi\eta^2 + a_{20}\eta^3 \quad (3.1.b)$$

or

$$u = \sum_{i=1}^{10} a_i\, \xi^{mi}\, \eta^{ni} \quad (3.2.a)$$

$$v = \sum_{i=11}^{20} a_i\, \xi^{pi}\, \eta^{qi} \quad (3.2.b)$$

To define the twenty parameters in Equations (3. 1) or (3. 2), twenty generalized coordinates are chosen, namely u, v and their first derivatives u_ξ, u_η, v_ξ and v_η at each node plus u and v at the centroid of the element, as shown in Fig (3-1). It can be easily shown that the assumed functions ensure continuous displacements between adjacent elements, and hence, the element is conforming. The generalized coordinates can be written in the form:

$$\{V_1\} = [T]_p^* \quad \{A\} \tag{3. 3}$$

where:

$$\{V_1\}^T = [u_1, u_{\xi 1}, u_{\eta 1}, v_1\ v_{\xi 1}, \ldots\ v_{\eta 3}, u_c, v_c] \tag{3. 4}$$

$$\{A\}^T = [a_1, a_2, \ldots\ a_{20}] \tag{3. 5}$$

and

$[T]_p$ is: is the in-plane transformation matrix defined in Appendix II.

The transformation matrix $[T]_p$ has a non-zero determinant and can be inverted to give:

$$\{A\} = [T^{-1}]_p \{v_1\} \tag{3. 6}$$

Using the following strain energy expression:

$$U = \frac{Et}{2(1-\nu^2)} \int\int [u^2{}_\xi + v^2{}_\eta + 2_\nu\, u_\xi v_\eta + \frac{1-\nu}{2} (u_\eta + v_\xi)^2]\, d\xi d\eta \tag{3. 7}$$

The element strain energy is found to be:

$$U_e = \frac{Et}{2(1-\nu^2)} \{A\}^T [k]_p \{A\} \tag{3. 8}$$

where:

E : is Young's modulus, t is the plate thickness, ν is Poisson's

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*Subscript p denotes in-plane element.

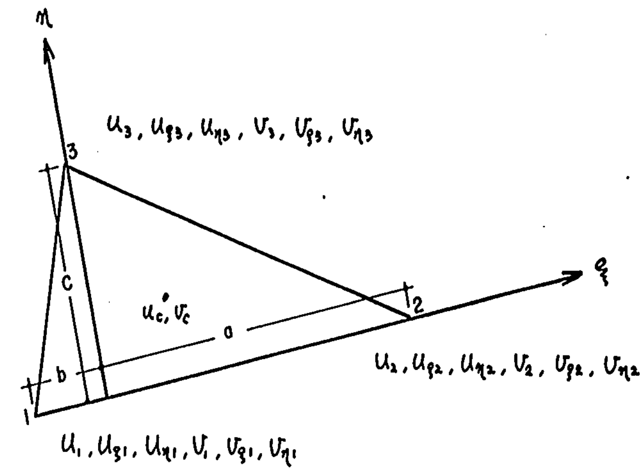

FIG. 3.1 LOCAL COORDINATE SYSTEM AND GENERALIZED COORDINATES OF THE IN-PLANE ELEMENT

ratio and $[k]_p$ is the stiffness matrix given by:

$$k_{ij} = m_i m_j\, G(m_i+m_j-2,\ n_i+n_j) + q_i q_j\, G(p_i+p_j,\ q_i+q_j-2)$$
$$+ \frac{1-\nu}{2}\,[n_i n_j\, G(m_i+m_j,\ n_i+n_j-2) + p_i p_j\, G(p_i+p_j-2,\ q_i+q_j)\,]$$
$$+ [\frac{1-\nu}{2}\, n_j p_i + \nu m_j q_i\,]\ G(m_j+p_i-1,\ n_j+q_i-1)$$
$$+ [\frac{1-\nu}{2}\, n_i p_j + \nu m_i q_j\,]\ G(m_i+p_j-1,\ n_i+q_j-1) \qquad (3.9)$$

in which:

$$G\,(m,n) = C^{(n+1)}\,[\,a^{m+1} - (-b)^{m+1}\,]\ \frac{m!\ n!}{(m+n+2)!} \qquad (3.10)$$

m_i and n_i are zero for $i > 10$,

p_i and q_i are zero for $i < 11$

and a, b, c are the geometrical properties of the triangular element shown in Fig. (3-1).

By substituting Equation (3.6) in Equation (3.8), the following expression is obtained:

$$U_e = \frac{Et}{2(1-\nu^2)}\ \{V_1\}^T\, [T^{-1}]_p^T\, [k]_p\, [T^{-1}]\{V_1\} \qquad (3.11)$$

Defining $[K_1]_p = [T^{-1}]_p^T\, [k]_p\, [T^{-1}]$, Equation (3.11) becomes:

$$U_e = \frac{Et}{2(1-\nu^2)}\ \{V_1\}^T\, [K_1]_p\, \{V_1\} \qquad (3.12)$$

where: $[K_1]_p$ is the stiffness matrix expressed in terms of the generalized displacements. This matrix must be transformed to the global coordinate system, shown in Fig. (3.2) before assembling the elements in the plane of each plate. This transformation can be done by using the relation:

$$\{V_1\} = [R]_p\, \{V_2\} \qquad (3.13)$$

where:

$$\{v_2\}^T = [u_1, u_{x1}, u_{y1}, v_1 \ldots u_c, v_c]$$

and $[R]_p$: is the 20 x 20 rotation matrix, defined in Appendix II.

Substituting Equation (3. 13) into Equation (3. 12), the following expression can be obtained:

$$U_e = \frac{Et}{2(1-\nu^2)} \{V_2\}^T [K_2]_p \{V_2\} \tag{3. 14}$$

where:

$$[K_2]_p = [R]_p^T [T^{-1}]_p^T [k]_p [T^{-1}]_p [R]_p \tag{3. 15}$$

is the 20 x 20 stiffness matrix in the plate global coordinate system.

The general equilibrium equation for an element can be written in the form:

$$[K_2]_p \{V_2\} = \{P_2\}_p \tag{3. 16}$$

or in the partitioned form:

$$\left[\begin{array}{c|c} K_o & K_{oc} \\ \hline K_{oc}{}^T & K_c \end{array}\right] \left\{\begin{array}{c} V \\ \hline V_c \end{array}\right\} = \left\{\begin{array}{c} P_o \\ \hline P_c \end{array}\right\} \tag{3. 17}$$

where:

$$\{V_c\} = \begin{Bmatrix} u_c \\ v_c \end{Bmatrix}$$

and $\{V\}^T = [u_1, u_{x1}, u_{y1}, v_1, \ldots v_{x3}, v_{y3}]$

From equation (3. 17), the following two equations can be obtained:

$$[K_o]\{V\} + [K_{oc}]\{V_c\} = \{P_o\} \tag{3. 18a}$$

$$[K_{oc}]^T \{V\} + [K_c]\{V_c\} = \{P_c\} \tag{3. 18b}$$

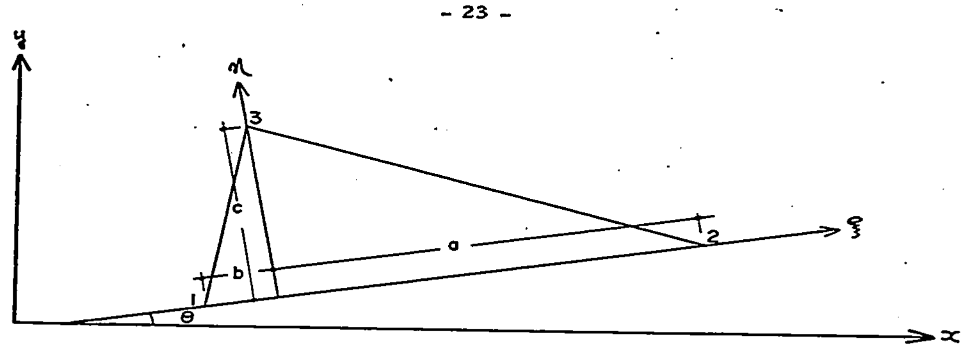

FIG. 3.2 PLATE GLOBAL COORDINATE SYSTEM

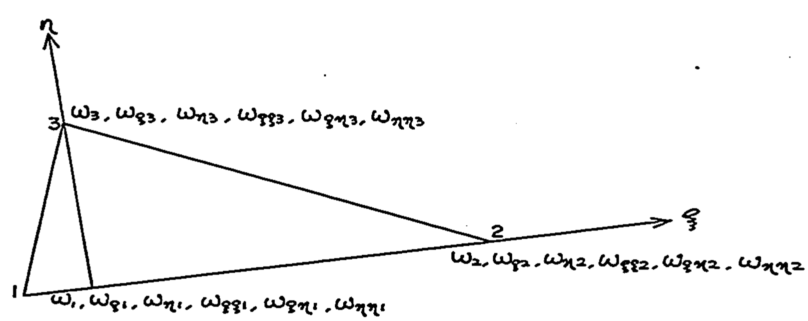

FIG. 3.3 GENERALIZED COORDINATES OF THE PLATE BENDING ELEMENT

Since the displacements at the centroid of each element are independent of the adjacent elements, they can be eliminated from Equations(3. 18), thus:

$$\{V_c\} = [K_c^{-1}]\{P_c\} - [K_c^{-1}][K_{oc}]^T\{V\} \qquad (3.19)$$

Substituting into Equation (3. 18a):

$$[K_o]\{V\} + [K_{oc}][K_c^{-1}]\{P_c\} - [K_{oc}][K_c^{-1}][K_{oc}]^T\{V\} = \{P_o\} \qquad (3.20)$$

or

$$\left([K_o] - [K_{oc}][K_c^{-1}][K_{oc}]^T\right)\{V\} = \{P_o\} - [K_c^{-1}]\{P_c\} \qquad (3.21)$$

By defining:

$$[K]_p = [K_o] - [K_{oc}][K_c^{-1}][K_{oc}]^T \qquad (3.22a)$$

and

$$\{P\}_p = \{P_o\} - [K_{oc}][K_c^{-1}]\{P_c\} \qquad (3.22.b)$$

Equation (3. 21) can be written in the form:

$$[K]_p \; \{V\} = \{P\}_p \qquad (3.23)$$

where $[K]_p$ now, is the 18 x 18 element stiffness matrix in the plate global coordinate system.

3. 3 Plate-Bending Element

The triangular plate bending element* has 18 degrees of freedom, the normal displacement w, and its first and second derivatives at each node, as shown in Fig. (3-3). The following complete quintic polynomial in ξ and η is chosen as a displacement function, in which the term $\xi^4\eta$ is omitted to ensure a cubical variation of the normal slope along the element edge 1-2.

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* This element is described in detail in reference [7].

$$w(\xi,\eta) = b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2 + b_7\xi^3$$
$$+ b_8\xi^2\eta + b_9\xi\eta^2 + b_{10}\eta^3 + b_{11}\xi^4 + b_{12}\xi^3\eta + b_{13}\xi^2\eta^2 + b_{14}\xi\eta^3$$
$$+ b_{15}\eta^4 + b_{16}\xi^5 + b_{17}\xi^3\eta^2 + b_{18}\xi^2\eta^3 + b_{19}\xi\eta^4 + b_{20}\eta^5 \ldots \quad (3.24)$$

This equation can be written in the form:

$$w = \sum_{i=1}^{20} b_i\, \xi^{ri}\, \eta^{si} \quad (3.25)$$

Another two conditions are set to provide cubic variation of normal slopes along the remaining two edges 2-3 and 3-1. These conditions reduce the number of independent parameters to 18, and yield the following relations:

$$5b^4c\ b_{16} + (3b^2c^3 - 2b^4c)\ b_{17} + (2bc^4 - 3b^3c^2)b_{18} + (c^5 - 4b^2c^3)b_{19} - 5bc^4\ b_{20} = 0 \quad (3.26a)$$

$$5a^4cb_{16} + (3a^2c^3 - 2a^4c)b_{17} + (-2ac^4 - 3a^3c^2)b_{18} + (c^5 - 4a^2c^3)b_{19} - 5ac^4b_{20} = 0 \quad (3.26b)$$

By using the expressions (3.26) with the bending strain energy expression η the following expression can be obtained:

$$U_e = \frac{1}{2} D \iint w^2_{\xi\xi} + w^2_{\eta\eta} + 2\nu w_{\xi\xi} w_{\eta\eta} + 2(1-\nu)w^2_{\xi\eta}\ d\xi d\eta \ldots \quad (3.27)$$

Following the same procedure used in section (3.2), a similar equation to Equation (3.15) is obtained:

$$[K]_b = [R]_b^T\ [T_2]^T\ [k]_b\ [T_2]\ [R]_b^* \quad (3.28)$$

where:

$$k_{ij} = r_ir_j(r_{i-1})\ G(r_i+r_{j-4}, s_i+s_j) + s_is_j(s_{i-1})(s_{j-1})$$
$$G(r_i+r_j,\ s_i+s_{j-4}) + \{2(1-\nu)r_ir_j\,s_is_j + \nu r_is_j(r_{i-1})(s_{j-1})$$
$$+ \nu r_js_i(r_{j-1})(s_{i-1})\}\ G(r_i+r_{j-2},\ s_i+s_{j-2}) \quad (3.20)$$

* The subscript b denotes plate bending.

$$G(r,s) = C^{s+1}\left\{a^{r+1} - (-b)^{r+1}\right\}\frac{r!\ s!}{(r+s+2)!} \qquad (3.30)$$

$[R]_b$ is the bending rotation matrix defined in Appendix II, and $[T_2]$ is a 20 x 18 matrix consisting of the first eighteen columns of $[T^{-1}]_b$, where $[T]_b$ is the bending transformation matrix defined in Appendix II.

3.4. Folded Plate Element

In a folded plate, the element is, generally, subjected to both in-plane and bending forces. These forces are uncoupled, provided the deformatinns are small, and hence the folded plate element can be formed by a simple superposition of the in-plane and the bending finite elements. Its stiffness matrix will be the combination of their corresponding stiffness matrices. Since these matrices are derived in the previous sections in terms of the plate global coordinate system, the folded plate element equilibrium equation, relative to this system, can therefore be written as:

$$\{F_1\}_{36x1} = [K_1]\{W_1\}_{36x1} \qquad (3.31)$$

where:

$$\{w_1\}^T = [u_1, u_{x1}, u_{y1}, v_1, v_{x1}, v_{y1}, w_1, w_{x1}, w_{y1}, w_{xx1}, w_{xy1}, w_{yy1}, \ldots w_{xy3}, w_{yy3}]$$

$\{F_1\}^T$ is the corresponding force matrix and $[K_1]$ is the 36 x 36 stiffness matrix relative to the plate global coordinate system. This stiffness matrix $[K_1]$ is simply given by:

$$[K_1] = \begin{bmatrix} K_{11}p & 0 & K_{12}p & 0 & K_{13}p & 0 \\ 0 & K_{11}b & 0 & K_{12}b & 0 & K_{13}b \\ K_{21}p & 0 & K_{22}p & 0 & K_{23}p & 0 \\ 0 & K_{21}b & 0 & K_{22}b & 0 & K_{23}b \\ K_{31}p & 0 & K_{32}p & 0 & K_{33}p & 0 \\ 0 & K_{31}b & 0 & K_{32}b & 0 & K_{33}b \end{bmatrix}$$

$K_{11}p$, $K_{12}p$, $K_{13}p$ ... $K_{32}p$ and $K_{33}p$ are 6 x 6 submatrices formed by subdividing the in-plane stiffness matrix, defined in Equation (3.22a), into nine 6 x 6 submatrices. Similarly, $K_{11}b$, $K_{12}b$ ... $K_{33}b$ are formed by subdividing the bending element stiffness matrix defined in Equation (3.28) into 6 x 6 submatrices.

3.5. The Element Assemblage

The stiffness matrix of the whole structure is obtained merely by adding together the appropriate components of the element stiffness matrices. The matrix $[K_1]$ defined in Equation (3.32) can be used in assembling the coplanar elements. However, the elements meeting along the ridges are not coplanar. Before the requirements of compatibility and equilibrium are considered and the process of assembly performed, the forces and the displacements of the nodes lying on the ridges must be expressed in a common coordinate system. This can be easily done by a rotation matrix relating the displacements in the plate global system to that of the common system. These two systems are shown in Fig (3-4) and will be referred to simply as the plate and the global system respectively. The global system is denoted by x', y', z' and the corresponding displacements by u', v' and w'. As it can be seen

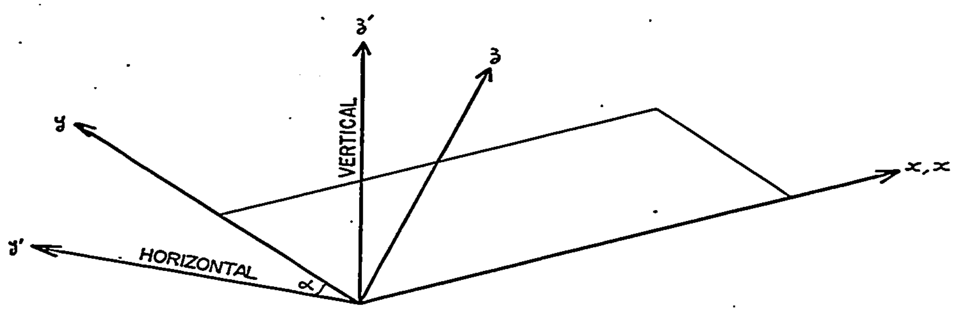


FIG. 3.4 PLATE AND GLOBAL COORDINATE SYSTEM

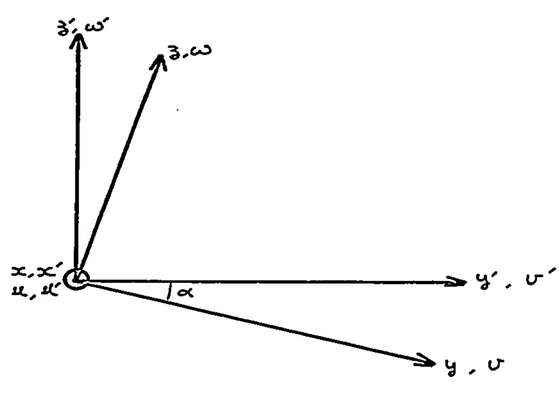


FIG. 3.5 THE PRIMED AND UNPRIMED DISPLACEMENTS

from Fig (3-4), the rotation is performed about the x axis. Hence, x is equal to x′ and consequently u and its derivatives with respect to x are equal to u′ and its derivatives with respect to x′ respectively.

3.5.1. Continuity Conditions:

For conformity, it is necessary to ensure the continuity of the in-plane displacements u and v, the lateral displacement w and the normal slope between adjoining elements.

Consider the element shown in Fig. 3-1. The in-plane displacements u and v vary cubically along the edge 1-2 and are uniquely defined by u_1, $u_{\xi 1}$, u_2, $u_{\xi 2}$ and v_1, $v_{\xi 1}$, v_2, $v_{\xi 2}$. The normal displacement also varies quintically along this edge and is uniquely defined by w_1, $w_{\xi 1}$, $w_{\xi\xi 1}$, w_2, $w_{\xi 2}$, $w_{\xi\xi 2}$. Similarly, the normal slope which varies cubically can be uniquely defined by $w_{\eta 1}$, $w_{\xi\eta 1}$, $w_{\eta 2}$ and $w_{\xi\eta 2}$. The displacements and normal slope of the adjacent element along this edge are uniquely defined by similar parameters. Therefore, to ensure continuity between these two elements, it is sufficient that u, u_ξ, v, v_ξ, w, w_ξ, w_η, $w_{\xi\xi}$ and $w_{\xi\eta}$ be made continuous at the nodes 1 and 2.

Before assembling elements and applying the continuity conditions, it is necessary to express all the degrees of freedom and consequently the constraint equations in terms of a common coordinate system, that is, the plate coordinate system for coplanar elements and the global system for inclined elements. Since the x-axis coincides with the x′-axis and the plates are assumed rigidly connected along the ridges, the continuity equations of the u displacement and the normal slope can be expressed in terms of the plate coordinate system. Hence, for the elements meeting along the ridge, the constraint equations become:

$$u_I = u^*_{II} \qquad (3.33a)$$

* The subscript I and II denote the consecutive plates I and II joining along the ridge.

$$u_{xI} = u_{xII} \quad (3.33b)$$

$$w_{yI} = w_{yII} \quad (3.33c)$$

$$w_{xyI} = w_{xyII} \quad (3.33d)$$

in the plate coordinate system and:

$$v'_{I} = v'_{II} \quad (3.34a)$$

$$v'_{xI} = v'_{xII} \quad (3.34b)$$

$$w'_{I} = w'_{II} \quad (3.34c)$$

$$w'_{xI} = w'_{xII} \quad (3.34d)$$

$$w'_{xxI} = w'_{xxII} \quad (3.34e)$$

in the global coordinate system.

3.5.2 Global Rotation Matrix

As mentioned before, this rotation matrix relates the node displacements in the plate system to those of the global system. However, only the displacements involved in Equations (3.34) have to be rotated since equations (3.33) are expressed in terms of the plate coordinates. The remaining degrees of freedom which do not appear in Equations (3.33) or Equations (3.34) are not equal in consecutive plates. Therefore no condition need be imposed on the values of these degrees of freedom and the assembly of the elements can be performed without rotating them. Consider Fig. (3-5), the relations between the primed (global) and the unprimed (plate) displacements can be written in the form:

$$v = v'\cos\alpha - w'\sin\alpha \quad (3.35a)$$

$$w = v'\sin\alpha + w'\cos\alpha \quad (3.35b)$$

$$v_x = v'_x\cos\alpha - w'_x\sin\alpha \quad (3.35c)$$

$$w_x = v'_x\sin\alpha + w'_x\cos\alpha \quad (3.35d)$$

$$w_{xx} = v'_{xx}\sin\alpha + w'_{xx}\cos\alpha \quad (3.35e)$$

It should be noted here, however, that in Equations(3. 35a) and (3. 35b), the displacement v′ represented by a cubic polynomial is added to w′ which, in turn, is represented by a quintic one. These two displacements do not match together, hence causing some discontinuity along the ridges and the element is not a conforming element any more. Further, in Equation (3. 35e), the second derivative of the v displacement is included but it is not considered as a degree of freedom. Different methods can be used to overcome this difficulty: A new in-plane element can be developed in which the second derivatives of the in-plane displacements are considered as degrees of freedom or the second derivatives can be calculated in terms of the existing degrees of freedom. Alternatively, the second derivatives of the in-plane displacements can be assumed equal to zero. This is the easiest way, and is adopted in this dissertation. Thus, the second derivative of v, which appears in Equation (3. 35e) was neglected. This is felt to be a reasonable assumption since folded plate structures are usually subjected to a uniformly distributed load (e. g. own weight and snow loading). Thus, the first term in equation (3. 35e) is negligible. Accepting this assumption of neglecting the second derivative of v, and considering equation (3. 35), the 12 x 12 rotation matrix becomes:

$$[R_1] = \begin{bmatrix} 1 &&&&&&&&&&& \\ & 1 &&&&&&&&&& \\ && 1 &&&&&&&&& \\ &&& \cos\alpha &&& -\sin\alpha &&&&& \\ &&&& \cos\alpha &&& -\sin\alpha &&&& \\ &&&&& 1 &&&&&& \\ &&& \sin\alpha &&& \cos\alpha &&&&& \\ &&&& \sin\alpha &&& \cos\alpha &&&& \\ &&&&&&&& 1 &&& \\ &&&&&&&&& \cos\alpha && \\ &&&&&&&&&& 1 & \\ &&&&&&&&&&& 1 \end{bmatrix} \qquad (3.36)$$

and the global element stiffness matrix can be written in the form:

$$[K]_{36x36} = [R]^T [K_1] [R] \qquad (3.37)$$

where [R] is the 36 x 36 rotation matrix given by:

$$[R] = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix} \qquad (3.38)$$

in which, R_{11}, R_{22} and R_{33} are 12 by 12 submatrices. If any of the nodes 1, 2 and 3 is lying or the ridge, its corresponding matrix (R_{11}, R_{22} or R_{33}) is equal to $[R_1]$ Otherwise it is equal to a 12 by 12 unity matrix.

The element equilibrium equation will be:

$$\{F\}_{36x1} = [K] \{W\} \qquad (3.39)$$

where {F} and {W} are the global force and displacement matrices.

By using the element stiffness matrices $[K_1]$ and [K] for coplanar and non-coplanar elements respectively, the stiffness matrix of the total structure can be formulated by considering the compatibility conditions at each nodal point, thus giving rise to the following equilibrium equation for the complete structure:

$$\{\bar{F}\} = [\bar{K}] \{\bar{W}\} \qquad (3.40)$$

where $[\bar{K}]$ is the stiffness matrix for the complete structure obtained by appropriate addition of the component stiffness matrices.

At this stage, the stiffness matrix $[\bar{K}]$ is singular and Equation (3.40) cannot be solved for the displacements $\{\bar{W}\}$ since rigid body displacements have not yet been eliminated. This can be done by considering the boundary conditions and prescribed displacements, and thus the number of equations can be reduced. This is accomplished by eliminating from the total stiffness matrix the row and column which correspond to the appropriate degree of freedom constrained. The total stiffness matrix may then be compressed and hence its dimensions reduced. However, this technique leads to considerable programming difficulties. A simpler scheme to apply the boundary conditions and prescribed displacement is to delete all terms of the corresponding row and column, except the diagonal term. This is set to one and the value of the prescribed displacement is placed in the corresponding position in the load vector. The different boundary conditions will be discussed later in this dissertation.

3.6.1 Load Vector

In the finite element analysis, two load vectors can be used to represent the force matrices in the equilibrium equations. The lumped load vector, obtained by replacing the actual forces acting on the structure by statically equivalent concentrated forces acting at the nodes, and the consistent load vector which produces the same virtual work done by the applied load.

3.6.2. Consistent Load Vector

Similar to the stiffness matrix, the consistent load vector is composed of an in-plane load and a bending load sub-matrix:

$$\{P\} = \left\{\frac{P_p}{P_b}\right\} \qquad (3.41)$$

If $Q_u(\xi,\eta)$, $Q_v(\xi,\eta)$ and $Q_w(\xi,\eta)$ are the applied loads in the u, v and w directions, respectively, the load matrices will be given by:

$$\{P\}_p = [R]_p^T [T^{-1}]_p^T \{Q\}_p \qquad (3.42a)$$

$$\{P\}_b = [R]_b^T [T_2]^T \{Q\}_b \qquad (3.42b)$$

where the entries in the vectors $\{Q\}_p$ and $\{Q\}_b$ are

$$Q_{pi} = \begin{cases} Q_u \quad {}^{mi}\ \eta^{ni}\ d\ d_\eta & i \leq 11 \\ Q_v \quad {}^{pi}\ \eta^{qi}\ d\ d_\eta & 10 < i \leq 20 \end{cases} \tag{3.43a}$$

and

$$Q_{bi} = \quad Q_w \quad {}^{ri}\ \eta^{si}\ d\ d_\eta \quad 1 \leq i \leq 20 \tag{3.43b}$$

More details about the consistent load vector can be found in references [7] and [8].

In the special case of a uniformly distributed load of intensity Q_o along the ridges where the load and consequently the virtual work are independent of y and have no components in the u direction, the displacement functions can be written as:

$$v(x) = c_1 + c_2x + c_3x + c_4x^3 \tag{3.44a}$$

$$w(x) = c_5 + c_6x + c_7x^2 + c_8x^3 + c_9x^4 + c_{10}x^5 \tag{3.44b}$$

and hence, the generalized displacements of the element nodes i and j (Fig. 3-6), lying on the ridges, can be expressed in the matrix form:

$$\{W\}_{ij} = [X]\ \{c\} \tag{3.45}$$

where

$$\{W\}_{ij}^T = [v_i,\ v_{xi},\ w_i,\ w_{xi},\ w_{xyi},\ \ldots\ w_{xxj}]$$

$$\{C\}^T = [c_1,\ c_2,\ \ldots\ c_{10}]$$

and:

$$[x] = \begin{bmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & & & \\ 1 & l & l^2 & l^3 & 0 & 0 & 0 & & & \\ 0 & 1 & 2l & 3l^2 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & l & l^2 & l^3 & l^4 & l^5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2l & 3l^2 & 4l^3 & 5l^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6l & 12l^2 & 20l^3 \end{bmatrix}$$

l, being the length of the edge ij.

By solving equation (3.45) for $\{c\}$ and substituting into equation (3.44), the displacement $v_{(x)}$ and $w_{(x)}$, can be expressed in terms of the generalized displacements at the nodes i and j:

$$v_{(x)} = 1 - \frac{3}{l^2}x^2 + \frac{2}{l^3}x^3 \quad v_i + x - \frac{2}{l}x^2 + \frac{1}{l^2}x^3 \quad v_{xi}$$
$$+ \frac{3}{l^2}x^2 - \frac{2}{l^3}x^3 \; v_j + - \frac{1}{l}x^2 + \frac{1}{l^2}x^3 \quad v_{xj} \qquad (3.46a)$$

and

$$w_{(x)} = 1 - \frac{10}{l^3}x^3 + \frac{15}{l^4}x^4 - \frac{6}{l^5}x^5 \quad w_i$$
$$+ x - \frac{6}{l^2}x^3 + \frac{8}{l^3}x^4 - \frac{3}{l^4}x^5 \quad w_{xi}$$
$$+ \frac{x^2}{2} - \frac{3}{2l}x^3 + \frac{3}{2l^2}x^4 - \frac{1}{2l^3}x^5 \quad w_{xxi}$$
$$+ \frac{10}{l^3}x^3 - \frac{15}{l^4}x^4 + \frac{6}{l^5}x^5 \quad w_j$$
$$+ \frac{-4}{l^2}x^4 + \frac{7}{l^3}x^4 - \frac{3}{l^4}x^5 \quad w_{xj}$$
$$+ \frac{x^3}{2l} - \frac{1}{l^2}x^4 + \frac{1}{2l^3}x^5 \quad w_{xxj} \qquad (3.46b)$$

Since there is no load component in the u direction, the element virtual work can be calculated using the formula:

$$V_e = \int_0^l v_{(x)} q_v \, dx + \int_0^l q_w w_{(x)} \, dx \qquad (3.47a)$$

or

$$V_e = q_v \int_0^l v_{(x)} + q_w \int_0^l w_{(x)} \, dx \qquad (3.47b)$$

where q_v and q_w are the in-plane and normal components of Q_o. These components are shown in Fig. 3-6 and are given by:

$$q_v = Q_o \sin \alpha \qquad (3.48a)$$

$$q_w = Q_o \cos \alpha \qquad (3.48b)$$

By substituting Equation (3.46) into Equation (3.47) and performing the integration, the following relation is obtained:

$$V_e = \{W\}_{ij}^T \{P_o\} \qquad (3.49)$$

where $\{P_o\}$ is the 10 x 1 consistent load vector given by:

$$\{P_o\}^T = \Big[q_v \frac{l}{2},\ q_v \frac{l^2}{12},\ q_w \frac{l}{2},\ q_w \frac{l^2}{10},\ q_w \frac{l^3}{120},\ q_v \frac{l}{2}$$
$$- q_v \frac{l^2}{12},\ q_w \frac{l}{2},\ -q_w \frac{l^2}{10},\ q_w \frac{l^3}{120} \Big] \qquad (3.50)$$

This matrix is inserted into the element force matrix in the rows corresponding to the generalized displacements appearing in $\{W\}_{ij}$. The remaining terms are set equal to zero. This element force matrix is expressed in terms of the plate coordinate system and must be rotated to the global system by premultiplying it by the inverse of the global rotation matrix derived in section 3. 5. 2. This force vector will be referred to as "The fully consistent load vector".

As pointed out in section 2. 1, the plates are flexible perpendicular to their planes and the lateral displacement can be very large if not controlled by the in-plane displacement. The latter displacement, however, is very small due to the high stiffness of the plate in its own plane.

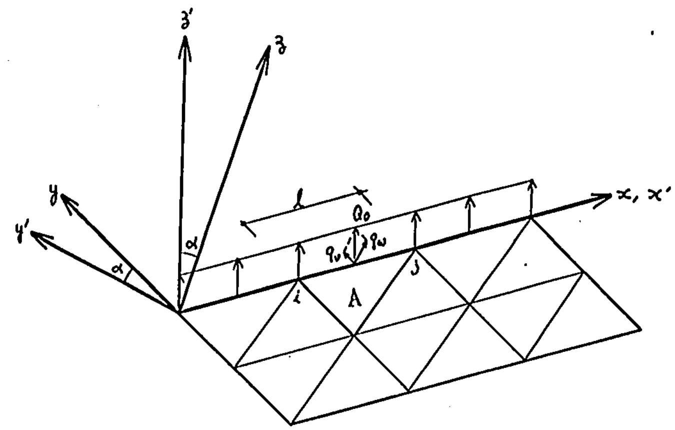


FIG. 3.6 FORCES ACTING ON THE ELEMENT EDGE ALONG THE RIDGES

Consequently it follows that the lateral displacement is also kept small. Similarly, the derivatives of the lateral displacement are controlled by the derivatives of the in-plane displacement. Since the control of the second derivative is removed by neglecting the second term in equation (3.35e), the second derivative of the lateral displacement will take large values. As a result, the strain energy and the bending moments which are functions of this second derivative will be very large. This will be shown by the numerical examples. To compensate for the removed control of the in-plane displacement, the effect of the second derivative of the lateral displacement should be neglected. This is due to the fact that the in-plane and the lateral displacements are of the same order (as mentioned in section 2.1) and consequently the term $v'_{xx} \sin \alpha$ in equation (3.35e) is of the same order as $w'_{xx} \cos \alpha$ for the common values of the slope angle α.

A new load vector has been developed in which the generalized load terms corresponding to w_{xx} are set to zero. This load vector will be referred to as "the reduced consistent load vector". It will be shown by the illustrative examples in Chapter V, that this load vector yields better results for strain energy and moments than the fully consistent load vector.

CHAPTER IV
THE EXACT SOLUTION

4.1. General

The convergence of the finite-element solution to the exact results can be proved theoretically by the theorem of minimum potential energy when conforming elements are used. In this dissertation, the element used in the analysis is fully conforming when employed to solve in-plane or plate bending problems. Nevertheless, some discontinuities are created when the element is used to analyse folded plate structures, as described in the previous chapter, and hence the conformity of the element is lost. The reliability of the analysis therefore, must be established by comparison with an exact solution. The "direct stiffness analysis" [11], based on the "elasticity method" [15] provides such a solution, and is discussed in some detail in the following sections.

As mentioned in the previous chapter, each plate in the structure is considered separately and four displacements and four forces are assumed at each edge of the plate. The force-displacement relationships are established by using the plate theory and the two-dimensional elasticity theory. These relations define the stiffness of the individual plate. The stiffness of the total structure which defines the stiffness or the equilibrium relation for the structure, is formed by the summation of the stiffness of the plate elements.

To define the configuration of the structure, two right-hand coordinate systems are used, as shown in Fig 4-1 and Fig 4-2. The local system has y and z axes parallel and perpendicular to the plate, respectively, whereas in the global system the Y and Z axes are horizontal and vertical. The n-axis is common in both systems and

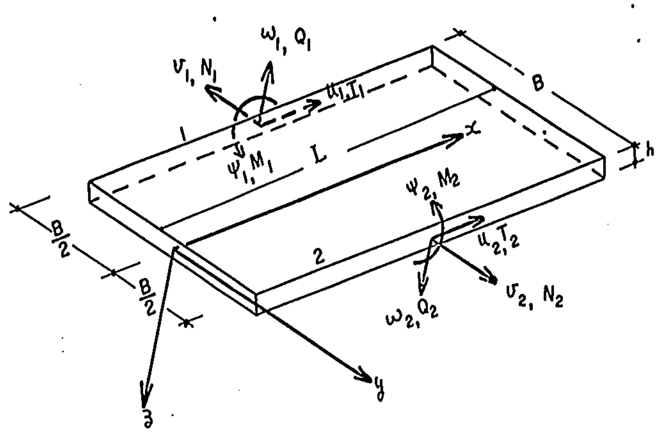


FIG. 4.1 POSITIVE PLATE EDGE DISPLACEMENTS AND FORCES IN THE LOCAL COORDINATES

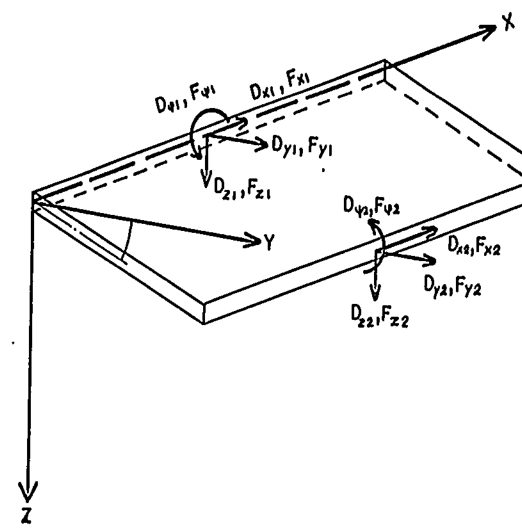


FIG. 4.2 POSITIVE PLATE EDGE DISPLACEMENTS AND FORCES IN THE GLOBAL COORDINATES

is parallel to the longitudinal direction of the plate. The forces and displacements of each individual plate can be expressed in either the local or global system. . Since the global system is independent of the position of the individual plates, it is used with its corresponding forces and displacements to define the structural behaviour of the whole structure.

4.2 Analysis of the Plate Element

Three displacements are needed, in general, to represent the behavior of the individual plates in the folded plate structures: two in-plane displacements u and v, respectively parallel and perpendicular to the longitudinal direction of the plate, and w, normal to the plane of the plate. The elasticity method deals with simply supported folded plate structures only. For such structures the boundary conditions along the end diaphragms can be easily satisfied by cosine and sine type displacement expressions. Moreover, by assuming the solution in the longitudinal direction in a Fourier series, form the problem can be reduced to one dimensional problem. The displacements can be expressed in terms of the Fourier series in the following form:

$$u_{(x,y)} = \sum_{m=o}^{\infty} u_{om}(y) \cos \frac{m\pi x}{L} \tag{4.1a}$$

$$v_{(x,y)} = \sum_{m=o}^{\infty} v_{om}(y) \sin \frac{m\pi x}{L} \tag{4.1b}$$

$$w_{(x,y)} = \sum_{m=o}^{\infty} w_{om}(y) \sin \frac{m\pi x}{L} \tag{4.1c}$$

Consequently the slope can be written in the form:

$$\psi_{(x,y)} = \frac{\partial w}{\partial y} = \sum_{m=o}^{\infty} \psi_{om}(y) \sin \frac{m\pi x}{L} \tag{4.1d}$$

It can be seen that the assumed functions satisfy the boundary conditions at the diaphragms, and the problem is reduced to one dimensional problem in which the terms $u_{om}(y)$, $v_{om}(y)$ and $w_{om}(y)$ have to be found.

In the "elasticity method", where it is assumed that the loads are applied at the joints, the analysis aims to define the relations between the forces and displacements at the edges of the plates and to evaluate these edge forces. Each plate is assumed to be, at its longitudinal edges, subjected to four element displacements and forces· two in-plane displacements in the u and v directions with their corresponding forces T and N; and the lateral displacement parallel to w and the normal slope with the corresponding force and moment Q and M, as shown in Fig. 4-1.

The mth terms of the displacements defined in Equation (4. 1) must satisfy the following plate homogeneous differential equation and the in-plane equilibrium equations:*

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0 \qquad (4.2a)$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad (4.2b)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \qquad (4.2c)$$

where:

$$\sigma_x = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) \qquad (4.3a)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \qquad (4.3b)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (4.3c)$$

and

E : Young's modulus

ν: Poisson's ratio

---

* The subscript m will be omitted for simplicity

Substituting Equations 4.3 into Equations 4.2 gives the following governing equations:

$$\frac{E}{2(1+\nu)}\left[\frac{\partial^2 u_o}{\partial y^2} - \frac{2}{1-\nu}\left(\frac{m\pi}{L}\right)^2 u_o + \frac{1+\nu}{1-\nu}\frac{m\pi}{L}\frac{\partial v_o}{\partial y}\right]\cos\frac{m\pi x}{L} = 0 \qquad (4.4a)$$

$$\frac{E}{2(1+\nu)}\left[-\frac{1+\nu}{1-\nu}\frac{m\pi}{L}\frac{\partial u_o}{\partial y} + \frac{2}{1-\nu}\frac{\partial^2 v_o}{\partial y^2} - \left(\frac{m\pi}{L}\right)^2 v_o\right]\sin\frac{m\pi x}{L} = 0 \qquad (4.4b)$$

$$\left[w_o\left(\frac{m\pi}{L}\right)^4 - 2\frac{\partial^2 w_o}{\partial y^2}\left(\frac{m\pi}{L}\right)^2 + \frac{\partial^4 w_o}{\partial y^4}\right]\sin\frac{m\pi x}{L} = 0 \qquad (4.4c)$$

To satisfy equations (4.4), u_o, v_o and w_o must be of the form:

$$u_o = c_1\cosh\frac{m\pi y}{L} + c_2\sinh\frac{m\pi y}{L} + c_3\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} + c_4\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} \qquad (4.5a)$$

$$v_o = c_5\cosh\frac{m\pi y}{L} + c_6\sinh\frac{m\pi y}{L} + c_7\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} + c_8\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} \qquad (4.5b)$$

$$w_o = c_9\sinh\frac{m\pi y}{L} + c_{10}\cosh\frac{m\pi y}{L} + c_{11}\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} + c_{12}\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} \qquad (4.5c)$$

The twelve constants $c_1 \ldots c_{12}$ can be found by considering the following four cases:

1. In-plane translations normal to the plate edges.
2. In-plane translations tangential to the plate edges.
3. Normal translations of the plate edges.
4. Rotations of the plate edges.

The solution of each of the previous cases is composed of the sum of a symmetrical case and an anti-symmetrical case. This gives eight boundary conditions that should be satisfied by the displacements defined in Equations 4. 5. The equilibrium equations 4. 2b and 4. 2c provide four relations connecting $c_1 \ldots c_4$ to $c_5 \ldots c_8$. These relations plus the eight boundary conditions are sufficient to define the twelve constants. The following expression is, therefore, obtained for the u displacement:

$$
\begin{aligned}
u = \frac{1}{2}\Bigg\{ & \bar{u}_1\Bigg[-\frac{\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} + (\beta-\mu\coth\mu)\cosh\frac{m\pi y}{L}}{\mu\cosh\mu - \beta\cosh\mu} \\
& + \frac{\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} + (\beta-\mu\tanh\mu)\sinh\frac{m\pi y}{L}}{\mu\,\mathrm{sech}\,\mu + \beta\sinh\mu}\Bigg] \\
& - \bar{u}_2\Bigg[\frac{\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} + (\beta-\mu\coth\mu)\cosh\frac{m\pi y}{L}}{\mu\,\mathrm{csch}\,\mu - \beta\cosh\mu} \\
& + \frac{\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} + (\beta-\mu\tanh\mu)\sinh\frac{m\pi y}{L}}{\mu\,\mathrm{sech}\,\mu + \beta\sinh\mu}\Bigg] \\
& + \bar{v}_1\Bigg[\frac{\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} - \mu\tanh\mu\cosh\frac{m\pi y}{L}}{\mu\,\mathrm{sech}\,\mu\;\beta\sinh\mu} \\
& - \frac{\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} - \mu\coth\mu\sinh\frac{m\pi y}{L}}{\mu\,\mathrm{csch}\,\mu + \beta\cosh\mu}\Bigg] \\
& - \bar{v}_2\Bigg[\frac{\frac{m\pi y}{L}\sinh\frac{m\pi y}{L} - \mu\tanh\mu\cosh\frac{m\pi y}{L}}{\mu\,\mathrm{sech}\,\mu - \beta\sinh\mu} \\
& + \frac{\frac{m\pi y}{L}\cosh\frac{m\pi y}{L} - \mu\coth\mu\sinh\frac{m\pi y}{L}}{\mu\,\mathrm{csch}\,\mu + \beta\cosh\mu}\Bigg]\Bigg\}\cos\frac{m\pi y}{L}
\end{aligned}
\qquad (4.6)
$$

where:

$\beta = \frac{3-\nu}{1+\nu}$; $\mu = \frac{m\pi B}{2L}$,

B and L are the plate width and span respectively,

$\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$ and $\bar{v}_2$: are the displacements amplitude at edges

1 and 2, respectively, for the mth term of Fourier series.

Similar expressions can be obtained for v and w displacements. The forces can be expressed in terms of the displacements by using the following relationships:

$$N = h \times \sigma_y \qquad (4.7a)$$

$$T = h \times \tau_{xy} \qquad (4.7b)$$

$$M = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \qquad (4.7c)$$

$$Q = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^2 w}{\partial x^2 \partial y}\right] \qquad (4.7d)$$

where:

$D = \dfrac{Eh^3}{12(1-\nu)^2}$ ... the flexural rigidity of the plate.

h : the plate thickness.

Finally, by using Equations 4.3 and Equations 4.7 the relations between edge forces and displacements are obtained and can be written in matrix form as follows:

$$\{f\} = [k]_{EL} \{d\} \qquad (4.8)$$

where:

$$\{f\}^{t*} = \left[M_1 \sin\frac{m\pi x}{L};\ M_2 \sin\frac{m\pi x}{L};\ Q_1 \sin\frac{m\pi x}{L}\right.$$
$$Q_2 \sin\frac{m\pi x}{L};\ T_1 \cos\frac{m\pi x}{L};\ T_2 \cos\frac{m\pi x}{L};$$
$$\left.N_1 \sin\frac{m\pi x}{L};\ N_2 \sin\frac{m\pi x}{L}\right]$$

$$\{d\}^t = \left[\psi_1 \sin\frac{m\pi x}{L};\ \psi_2 \sin\frac{m\pi x}{L};\ \bar{w}_1 \sin\frac{m\pi x}{L};\ \bar{w}_2 \sin\frac{m\pi x}{L}\right.$$
$$\left.\bar{u}_1 \cos\frac{m\pi x}{L};\ \bar{u}_2 \cos\frac{m\pi x}{L};\ \bar{v}_1 \sin\frac{m\pi x}{L};\ \bar{v}_2 \cos\frac{m\pi x}{L}\right]$$

*Superscript t denotes transpose.

$[k]_{EL}^{*}$: the plate stiffness matrix corresponding to the mth term and is defined in Appendix II.

4.3. Analysis of the Structure

As pointed out in the definition of the folded plate, the structure is composed of a number of plates joined along their edges. Compatibility and equilibrium must be satisfied between the plate elements. Before considering these requirements, however, the plate edge forces and displacements have to be expressed in terms of the common global system of coordinates. This can be performed by using the following relations:

$$\{d\} = [T]_{EL} \{D\} \qquad (4.9a)$$

$$\{f\} = [T]_{EL} \{f\} \qquad (4.9b)$$

where, $\{d\}$ and $\{f\}$ are respectively, the displacement and force matrices in the local coordinate system,

$\{D\}^t = [Dy1, Dz1, Dx1, D\psi 1, Dy2, DZ2, Dx2, D\psi 2]$; the global displacement matrix.

$\{f\}^t = [Fy1, Fz1, Fx1, F\psi 1, Fy2, Fz2, Fx2, F\psi 2]$, the global force matrix.

and $[T]_{EL}$ is the transformation matrix and defined in Appendix II. Noting that $[T]_{EL}^{-1} = [T]_{EL}^{t}$ and substituting Equations (4.9) into Equations (4.8), the following relation will be obtained:

$$\{F\} = [T]_{EL} [k]_{EL} [T]_{EL} \{D\} \qquad (4.10)$$

or

$$\{F\} = [K]_{EL} \{D\} \qquad (4.11)$$

where: $[K]_{EL} = [T]_{EL}^{t} [k]_{EL} [T]_{EL}$ is the plate stiffness matrix in

*Subscript EL denotes elasticity method.

the global coordinate system.

The overall stiffness matrix is easily formed by assembling the individual global plate stiffness matrices. Since the boundary conditions have already been taken into consideration in the displacement functions, this matrix is not singular and can be inverted to solve the force-displacement relations for the displacements. The external loading is also expressed in terms of a Fourier series, and the previous procedure has to be repeated for each harmonic of the series.

The internal forces and moments will be obtained by substituting the corresponding displacements for each plate into Equation (4. 8), after transforming them to the local coordinate system by making use of Equation (4. 9a). The longitudinal stresses and moments are calculated by using the following expressions:

$$\sigma_x = E \frac{\partial u}{\partial x} + \nu \frac{N}{h} \tag{4. 13}$$

$$M_x = \nu M_y + D\,(1-\nu^2)\,\frac{\partial^2 w}{\partial x^2} \tag{4. 14}$$

where:

M_y : the transverse moment

D : the flexural rigidity of the plate

ν : Poisson's ratio

E : Young's modulus

N : Normal force per unit length

h : plate thickness

and $\frac{\partial^2 w}{\partial x^2}$: the second derivative of the normal displacement with respect to x.

CHAPTER V
RESULTS AND CONCLUSIONS

5.1. Illustrative Examples

To demonstrate the applicability of the finite element described in Chapter III, a two-slab simply supported folded plate structure has been analysed. This two-slab structure is a simple form of the folded plates, nevertheless it has been chosen because an "exact" solution by the elasticity method can be obtained. The results obtained by the "exact" method and by the finite element technique are presented together below.

The geometry and loading of the structure are shown in Fig (5-1). The loading is a uniform, vertical line load along the ridge of the structure. This line load has been chosen for two reasons. First, in general the in-plane stresses, as opposed to the bending stresses or moments, are critical from the design point of view for folded plate structures. The line load accentuates the in-plane action and consequently the in-plane stresses. The second reason relates to the inherent error which results from the neglect of the second derivative of the in-plane displacement v, as discussed in Chapter III. This error becomes more significant as the in-plane loads become larger since large effect of the second derivatives of the in-plane displacement are associated with large in-plane loads caused by such a line load along the ridge.

Thus the line load on the ridge of the folded plate structure represents a severe loading case both from the point of view of the structure itself as well as from the method of analysis used in this thesis.

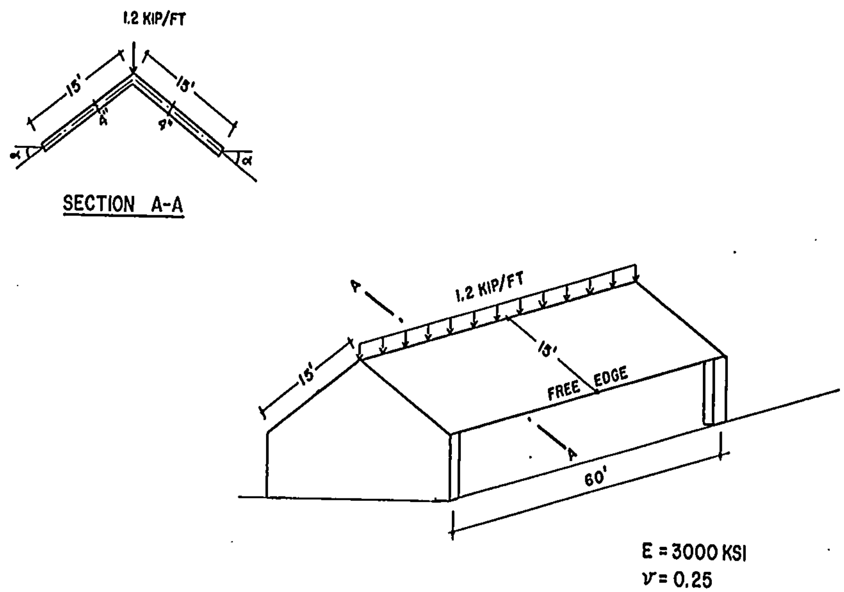


FIG. 5.1 SIMPLY SUPPORTED FOLDED PLATE STRUCTURE

It is obvious that the folded plate behavior changes from a pure bending action to a pure in-plane action as the slope angle α changes from zero to 90°. Full details about these two extreme cases can be found in references [7] and [8]. To show the behavior of the structure and the variation of the error between these two cases, four different values of the slope angle α have been considered; namely, zero degrees, 10°, 20° and 30°. The first value is the limiting case of a flat plate whereas the last is a common value for a folded plate structure.

The structure has been analysed by both the finite element method and the elasticity method. The results from the elasticity method were used for comparison with those obtained from the finite element method. In order to calculate the errors with at least two significant figures, sixty five terms of Fourier series have been used to approximate very accurately the exact solutions. At this stage, seven figures became constant for the strain energy and four for the deflection and the convergence was very slow. A series of different mesh sizes were used in the analysis to show the convergence of the finite element solution. These gridworks are shown in Fig. 5-2. Due to symmetry, only one-quarter of the structure need to be considered.

5.2. Discussion of Results

A series of comparative analyses has been carried out by using the three load vectors defined in sections (3.6.1) and (3.6.2). The mid-span vertical deflection of the ridge and the strain energy of the structure have been calculated for various slope angles, α, between zero and 30 degrees with particular attention has been paid to the two limiting values.

The folded plate structure with α equal to zero corresponds to the flat plate. In analysing the flat plate case, errors due to the

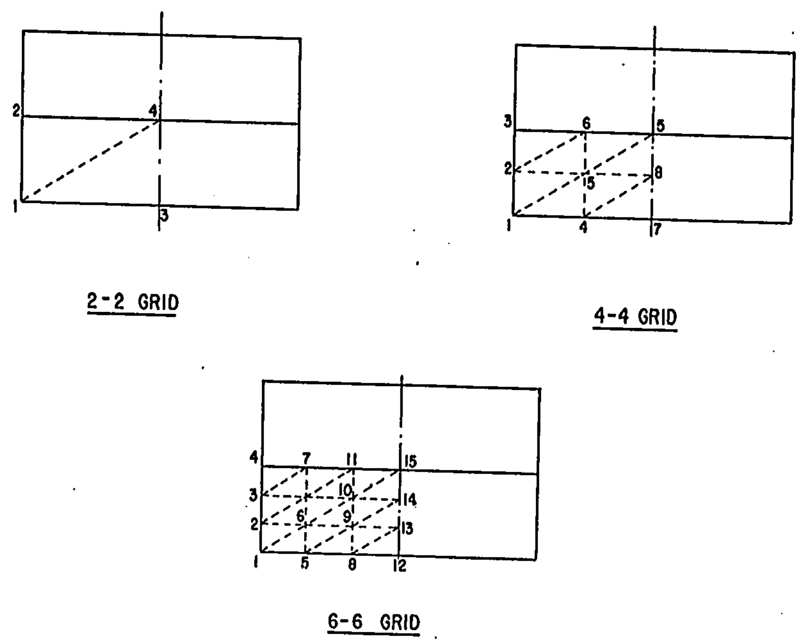


FIG.5.2 THE FINITE ELEMENT IDEALIZATIONS

neglect of the second derivative of the in-plane displacement - as discussed in Chapter III - do not arise, since there is no in-plane action. So the general accuracy of the finite element method can be studied.

The case of α equal to 30° is of interest. This angle represents a limiting design value since very few folded plate structures have larger slopes. Moreover, the relative errors* for the case α equal to 90° are expected to be small since this case represents the pure in-plane action and the element is a conforming one. The studies of the conforming element for the flat plate case in this thesis and for the pure in-plane action in reference (8) have shown small relative errors. Thus, the error curves shown in figures (5.12) and (5.13) after attaining a maximum, drop to low error values when α equal to 90°. It is then reasonable to assume that these maximum values are not much larger than the errors corresponding to α equal to 30°.

Therefore the folded plate with α equals 30° is a critical case for the finite element method, as well as commonly occurring in practical folded plate design.

The results and convergence studies are presented in figures (5.3) through (5.26) and tables 1 to 6. These figures and tables show how the central deflection of the ridge and the strain energy of the structure vary with mesh size, angle of slope and the load vector. For the two slope angles, α equal to zero and 30°, the variation of the longitudinal moment, longitudinal stress and transverse moment is also shown.

---

*The relative error is defined as the error relative to the best elastic theory solution.

5.2.1. Vertical Deflection

The values of the central deflection of the ridge are presented in Table 1 for the flat plate case, and in Tables 2 and 3 and Figure (5.3) for the folded plate cases. It can be seen that these values are in good agreement with the exact solution, even for the coarsest grid. Excellent results are obtained from the fully consistent load vector, especially for the flat plate case where the in-plane and bending actions are uncoupled and the element is a conforming one.

Table 3 shows a rapid decrease in the vertical deflection as the slope angle α increases from 0° (the limiting flat plate case) to 30°. This decrease in the vertical deflection is relatively large even for small increases of the slope angle α, as it can be seen from the table. This is mainly due to the quick interference of the in-plane action associated with the increase of the stiffness of the structure due to the large stiffness of the plates in their planes.

The percentage error of the central deflection of the ridge is plotted against the angle of slopes in Figure (5-12). This graph shows an approximately constant error for the values obtained by the lumped load vector, and slightly decreasing error for the values corresponding to the reduced consistent load vector. The increasing error of the values calculated using the fully consistent load vector can be explained by the increasing relative importance of the omitted term that corresponds to the second derivative of the in-plane displacement, v, with the increase of the angle of the slope α.

Figures (5-5) through (5-8) show the relative error plotted against the number of elements per side (both scales logarithmic). The error curve of the fully consistent load vector is quite flat for α equal to zero. This is due to the excellence of the results which are very close to the exact values for all gridworks considered (This can be seen from Fig. 5-3). As α increases, the relative error increases and the curve becomes less flat and parallel to the error curves corresponding to the lumped load and the reduced consistent load vectors. These two curves are parallel to the N^{-2} curve as shown in the previous figures. This value of the error curve was found in reference [6] for the simply supported square plate under a central point load.

In Figure (5-9), the central deflection of the ridge is plotted against the total number of degrees of freedom (calculated before considering the boundary conditions) required by the different load vectors. From this figure, it is obvious that the fully consistent load vector provides the best results for any number of degrees of freedom.

5.2.2. Strain Energy

The strain energy appears to converge from below to the exact solution when using a conforming element, as shown in Table 1 and 4 for the flat plate case with a fully consistent load vector. As can be seen from Figures 5-4 and 5-13, this load vector yields poor results for the folded plate cases where the conformity of the element is lost along the ridges. These large values of the strain energy were expected, as mentioned before, since the second derivative of the lateral deflection, included in the strain energy formula, is not controlled by the second derivative of the in-plane displacement, and hence it takes very large values. Figures (5-5), (5-6), (5-7), (5-8) and (5-13) show that the error of the strain energy values increases with the slope angle. Again, the percentage error is constant for the lumped load vector, and decreases slightly with α for the reduced consistent load vector.

For comparison, the strain energy is plotted against the total number of degrees of freedom in Figure (5-10). It is clear that the best results are predicted by the reduced consistent load vector. The convergence of the strain energy calculated by the fully consistent load vector is very significant between the values determined using 2 x 2 grid and 4 x 4 grid. While the relative error of the values obtained by the reduced consistent load vector is proportional to N^{-2} for all values of α, the proportionality is varying with α for the lumped and fully consistent load vectors.

5.2.3. Horizontal Discontinuity

Horizontal and vertical discontinuities should be expected in general, along the ridges where the in-plane and the lateral displacements are coupled. As mentioned before, these two displacements do not match together since they are represented by two functions of different order. This causes the discontinuities between the nodes. In the problem analysed in this thesis, the symmetry of the structure and the loading system provide symmetric displacements, that is, continuous vertical displacement, and symmetrical horizontal displacements.

To show the horizontal discontinuity between the elements along the ridge, the horizontal deflection which should be zero due to symmetry is plotted against the distance along the ridge in Figures (5-14) through (5-16). Figure (5-15) shows that the fully consistent load vector yields large horizontal deflections, almost five times the maximum vertical deflection, associated with the closing of the plates along the ridge. The reduced consistent load and the lumped load vectors yield very small horizontal deflections of the order of 4/1000 of the maximum vertical deflection. While the former causes the separation of the plates along the ridge, the latter causes closure of the plates near the diaphragms and separation at the central part of the ridge.

5. 2. 4. Longitudinal Moments

For the flat plate case, the examination of Table 1 and Figures (5-17), (5-18) and (5-19) shows the excellent agreement between the exact values and the values predicted by the fully consistent load vector. The maximum moment values calculated with the lumped and the reduced consistent load vectors converge monotonically towards the exact value. Since the stress boundary conditions have not been satisfied, the moments at the supports are not zero, but they converge to zero as the number of elements is increased. This is clearly shown in Figure (5-19).

The calculated moments for the folded plate case are presented in Tables 2 and 5 and Figures (5-20) and (5-21). The best results are given by the reduced consistent load vector. Again the moments at the supports are not zero.

The relative errors in the finite element solutions for the maximum longitudinal moment along the ridge are plotted against N, the number of elements per side, in Figure (5-26). The errors of the fully consistent load vector are very large and are outside the range of the graph. However, the values of the moments obtained by this load vector are given numerically in Tables 2 and 5. Again the large values of the moments are caused by the large values of the second derivative of the lateral displacement which is no longer controlled by the second derivative of the in-plane displacement.

The relative error of the longitudinal moment for the flat plate case (α equal to zero) has been found by Cowper et al [7] to be of the order of 10^{-4}. The same error has reached the value of 10^{-1} for the folded plate case (α equal to 30°) as shown in Figure (5-26). This increase, however, is not considered to be significant since the longitudinal moments have most importance in the flat plate case, where the relative error is small. In the folded plate case, where the relative error is relatively large, the in-plane stresses become much more important than the longitudinal moments. This can be seen in Tables 1 and 2.

It can be seen in Figures (5-17) through (5-21) that the curves of the longitudinal moment along the ridge of the folded plate structure have cusps corresponding to the nodal points. This is expected with the use of refined finite element [7].

5.2.5. Transverse Moments

The values of the transverse moments are shown in Tables 1 and 2. As for the longitudinal moment, good results are obtained by the fully consistent load vector for the flat plate case but poor ones for the folded plate structures. Reasonable results are calculated with the lumped and the reduced consistent load vectors. In general, the same comments made in the previous section for the longitudinal moment apply also to the transverse moment.

5.2.6. Longitudinal Stresses *

The in-plane longitudinal stresses are presented in Tables 2 and 6, and Figures (5-22) to (5-24). It can be seen from these tables and graphs that, in general, the best longitudinal stress values along the ridge are obtained using the reduced consistent load vector. However, the fully consistent load vector generally yields quite good values, and gives better values than the reduced consistent load vector for the stresses at the supports and mid-span. Again, the stresses at the supports are not equal to zero but do converge rapidly to zero as the finite element grid is refined.

As mentioned in section (5.2.4), the longitudinal stress does not exist in the flat plate case, but its importance increases with the

*The longitudinal stresses at mid-plane.

slope angle and it becomes the governing factor in the folded plate cases, as shown in Tables 1, 2 and 6.

In Figure (5-11), the in-plane longitudinal stresses at the center of the ridge are plotted against the total number of degrees of freedom, and Figure (5-25) shows the relative error of these stresses. From these two figures it appears that for a given number of elements or degrees of freedom, the fully consistent load vector yields the best accuracy for the stresses at mid-span of the ridge.

5.3. Conclusions

An arbitrary triangular element, formed by the combination of an in-plane and a plate bending element has been presented. This element has 36 degrees of freedom corresponding to the generalized nodal displacements,

$$u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y} \text{ and } \frac{\partial^2 w}{\partial y^2}.$$

A method for linking the elements at the nodes along the ridges has also been presented. One lumped load and two consistent load vectors were used with this element to predict strain energy, vertical displacement, longitudinal moment and stress, and transverse moment along the ridge of a simply supported folded plate structure with different slope angles. The finite element results were compared with those obtained by the elasticity, or the so-called exact method.

It was found that for the flat plate case, where the in-plane and the slab actions are uncoupled, the fully consistent load vector yields excellent results.

Because of the difference in order between the in-plane and the lateral displacement functions, the conformity of the element was lost. This causes some discontinuities along the ridge of the structure.

In general, the longitudinal moments and stresses, transverse moments and vertical ridge deflections computed by using the reduced consistent load vector are in good agreement with the exact values.

The fully consistent load vector yields vertical deflections and longitudinal stresses with good accuracy but, longitudinal and transverse moments with large error.

The errors of the central vertical deflection of the ridge and the strain energy obtained by the lumped load vector, are almost independent of the slope angle. However, these same errors determined with the fully consistent load vector increase with increasing slope angle, and when determined with the reduced consistent load vector decrease with increasing slope angle.

The strain energy computed by the fully consistent load vector converges from above to the exact solution for the folded plate cases because of the loss of conformity of the element.

Generally, the convergence of the results toward the exact values is quite rapid.

Finally, the reduced consistent load vector has been proven to be the best load vector to be used for the analysis of folded plate structures and yields, in general, good results even for the coarsest gridwork. From the engineering point of view, the moments and stresses are of most interest. The moments calculated with this load vector were in error by 10%. However, this error is not so significant since the moments have little importance compared to the longitudinal stresses for the folded plate structures. These stresses were obtained with an error of about 0.5%.

Thus, the element has been proved to yield good results when using the reduced consistent load vector, and is a powerful and reliable element to be used in the analysis of folded plate structures.

	LOAD VECTOR	MESH SIZE				EXACT SOLUTION
		2 - 2	4 - 4	6 - 6	8 - 8	
VERTICAL DISPLACE-MENTS	Lumped Fully Consistent Reduced Consist.	47.046704 58.506260 61.033542	55.630754 58.502042 59.124233	57.222247 58.501997 58.774799	58.502002*	58.503760 (IN)
TRANVER. MOMENTS	Lumped Fully Consistent Reduced Consist.	7.99163 5.73033 9.13462	6.48768 5.54728 6.62501	6.17657 5.55224 6.22870		5.55534 (KIP-FT)/FT
LONGIT. MOMENTS	Lumped Fully Consistent Reduced Consist.	20.8629 18.5121 24.2087	20.0846 18.6186 20.9762	19.7447 18.6315 20.1362		18.6399 (KIP-FT)/FT
LONGIT. STRESSES	Lumped Fully Consistent Reduced Consist.	0.0 0.0 0.0	0.0 0.0 0.0	0.0 0.0 0.0		0.0 (KIP/SQ.IN)
STRAIN ENERGY	Lumped Fully Consistent Reduced Consist.	211.710170 337.373721 365.519898	303.341063 337.416991 344.718834	322.021893 337.422971 340.661798		337.43514 (KIP-IN)/4.0

* This value is given by G.R. Cowper and G.M. Lindberg at the N.R.C. of Canada (private communication)

TABLE 1. RESULTS AT THE CENTER OF THE RIDGE FOR $\alpha = 0^\circ$

	LOAD VECTOR	MESH SIZE 2 - 2	MESH SIZE 4 - 4	MESH SIZE 6 - 6	EXACT SOLUTION
VERTICAL DISPLACE-MENTS	Lumped	0.110207	0.128781	0.132350	0.135264 (IN)
	Fully Consistent	0.134044	0.134930	0.135107	
	Reduced Consist.	0.139212	0.136309	0.135718	
TRANVER. MOMENTS	Lumped	0.017514	0.013038	0.011446	0.010982 (KIP-FT)/FT
	Fully Consistent	-3.84611	-1.20801	-0.762333	
	Reduced Consist.	0.017103	0.012327	0.011620	
LONGIT. MOMENTS	Lumped	0.045341	0.042599	0.039357	0.03709 (KIP-FT)/FT
	Fully Consistent	-6.23001	-2.51632	-1.62620	
	Reduced Consist.	0.045639	0.040602	0.039316	
LONGIT. STRESSES	Lumped	-0.285465	-0.296793	-0.300450	-0.30457 (KIP/SQ.IN)
	Fully Consistent	-0.327795	-0.308490	-0.305769	
	Reduced Consist.	-0.332287	-0.310764	-0.306796	
STRAIN ENERGY	Lumped	0.495931	0.705834	0.749169	0.785652 (KIP-IN)/4.0
	Fully Consistent	8.040997	1.563061	1.009182	
	Reduced Consist.	0.832870	0.798415	0.791415	

TABLE 2. RESULTS AT THE CENTER OF THE RIDGE FOR $\alpha = 30^{\circ}$

ANGLE OF SLOPE (degrees)	LOAD VECTOR	MESH SIZE 2 - 2	4 - 4	6 - 6	EXACT DEFLECT. (inches)
0.0	Lumped Fully Consistent Reduced Consist.	47.046704 58.506260 61.033542	55.630754 58.502042 59.124233	57.222247 58.501997 58.774799	58.503760
10.0	Lumped Fully Consistent. Reduced Consist.	0.898195 1.101999 1.144331	1.049781 1.102443 1.113798	1.079107 1.102642 1.107651	1.102910
20.0	Lumped Fully Consistent Reduced Consist.	0.234896 0.287154 0.298179	0.274492 0.287979 0.290924	0.282109 0.288153 0.289458	0.288325
30.0	Lumped Fully Consistent Reduced Consist.	0.110207 0.134044 0.139212	0.128781 0.134930 0.136309	0.132350 0.135107 0.135718	0.135264

TABLE 3. COMPARISON OF THE VERTICAL DISPLACEMENT AT THE CENTER OF THE RIDGE

ANGLE OF SLOPE (degrees)	LOAD VECTOR	MESH SIZE			EXACT STRAIN ENERGY (KIP-IN)/4.0
		2 - 2	4 - 4	6 - 6	
0.0	Lumped Fully Consistent Reduced Consist.	211.710170 337.373721 365.519898	303.341063 337.416991 344.718834	322.021893 337.422971 340.661798	337.435146
10.0	Lumped Fully Consistent Reduced Consist.	4.041877 11.894261 6.919002	5.752755 6.986355 6.539936	6.106206 6.569844 6.465088	6.404008
20.0	Lumped Fully Consistent Reduced Consist.	1.057034 7.813961 1.795698	1.504421 2.331023 1.706813	1.596790 1.862927 1.689106	1.664573
30.0	Lumped Fully Consist. Reduced Consist.	0.495931 8.040996 0.832870	0.705834 1.563061 0.798415	0.749169 1.009182 0.791415	0.785652

TABLE 4. COMPARISON OF THE STRAIN ENERGY OF THE STRUCTURE

DISTANCE ALONG THE RIDGE (FT)	LOAD VECTOR	MESH SIZE			EXACT LONGITUD. MOMENT (KIP-FT/FT)
		2 - 2	4 - 4	6 - 6	
0.0	Lumped Fully Consistent Reduced Consist.	-0.008330 -5.40543 0.020397	-0.020528 -2.585000 0.023078	-0.030985 -1.71106 0.030837	0.0
10.0	Lumped Fully Consistent Reduced Consist.	0.009535 1.854606 0.021518	0.025325 1.049820 0.021533	0.032281 -1.60312 0.024358	0.02393
20.0	Lumped Fully Consistent Reduced Consist.	0.023318 2.44192 0.030142	0.025664 0.808443 0.033158	0.036878 -1.59208 0.035406	0.03204
30.0	Lumped Fully Consistent Reduced Consist.	-0.045341 -6.23001 0.045639	0.042598 -2.51632 0.040602	0.039357 -1.62620 0.039316	0.03709

TABLE 5. LONGITUDINAL MOMENT ALONG THE RIDGE FOR $\alpha = 30^{\circ}$

DISTANCE ALONG THE RIDGE (FT)	LOAD VECTOR	MESH SIZE			EXACT LONGITUD. STRESS (KIP/SQ.IN)
		2 - 2	4 - 4	6 - 6	
0.0	Lumped Fully Consistent Reduced Consist.	0.026831 -0.013885 -0.033943	0.022440 -0.002421 -0.017552	0.021243 -0.002014 -0.174123	0.0
10.0	Lumped Fully Consistent Reduced Consist.	-0.106603 -0.156841 -0.174298	-0.159169 -0.172479 -0.176389	-0.175103 -0.175681 -0.174123	-0.173007
20.0	Lumped Fully Consistent Reduced Consist.	-0.210863 -0.261675 -0.273950	-0.258812 -0.269238 -0.270923	-0.267071 -0.272435 -0.273533	-0.271073
30.0	Lumped Fully Consistent Reduced Consist.	-0.285465 -0.327795 -0.332287	-0.296793 -0.308490 -0.310764	-0.300450 -0.305769 -0.306796	-0.304575

TABLE 6. LONGITUDINAL STRESS ALONG THE RIDGE FOR $\alpha = 30^{\circ}$

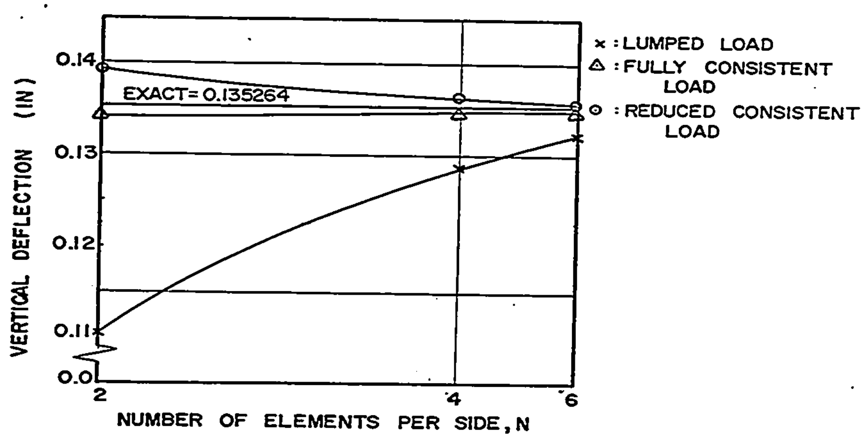


FIG. 5.3 VERTICAL DEFLECTION AT THE CENTER OF THE RIDGE $\alpha = 30°$

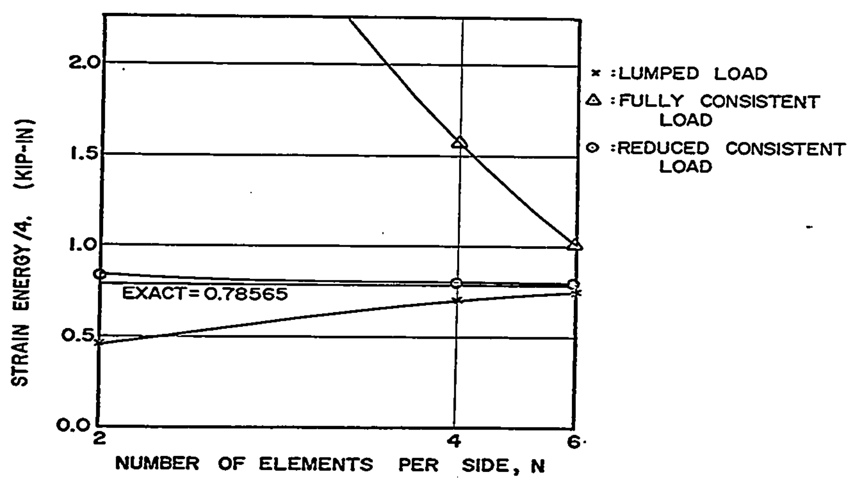


FIG. 5.4 STRAIN ENERGY $\alpha = 30°$

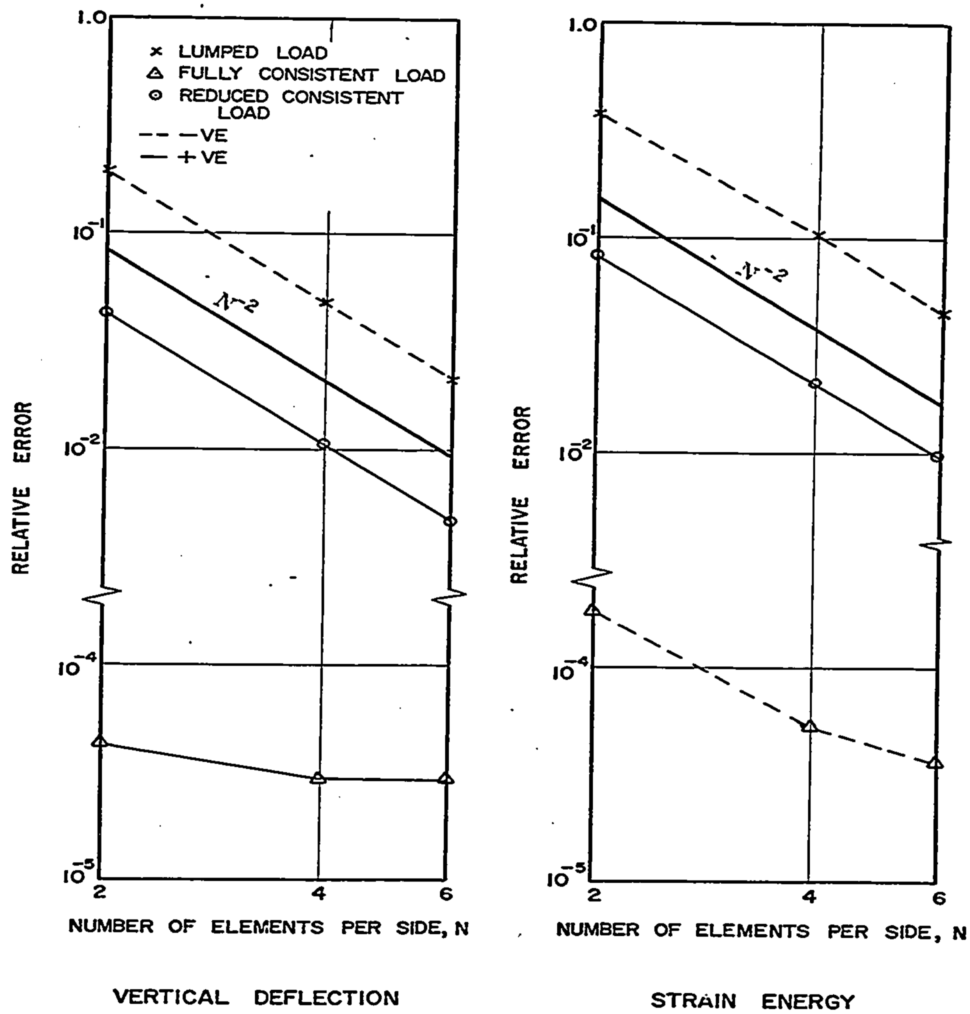


FIG. 5.5 RELATIVE ERROR OF THE FINITE ELEMENT SOLUTION, FLAT PLATE $\alpha = 0.0°$

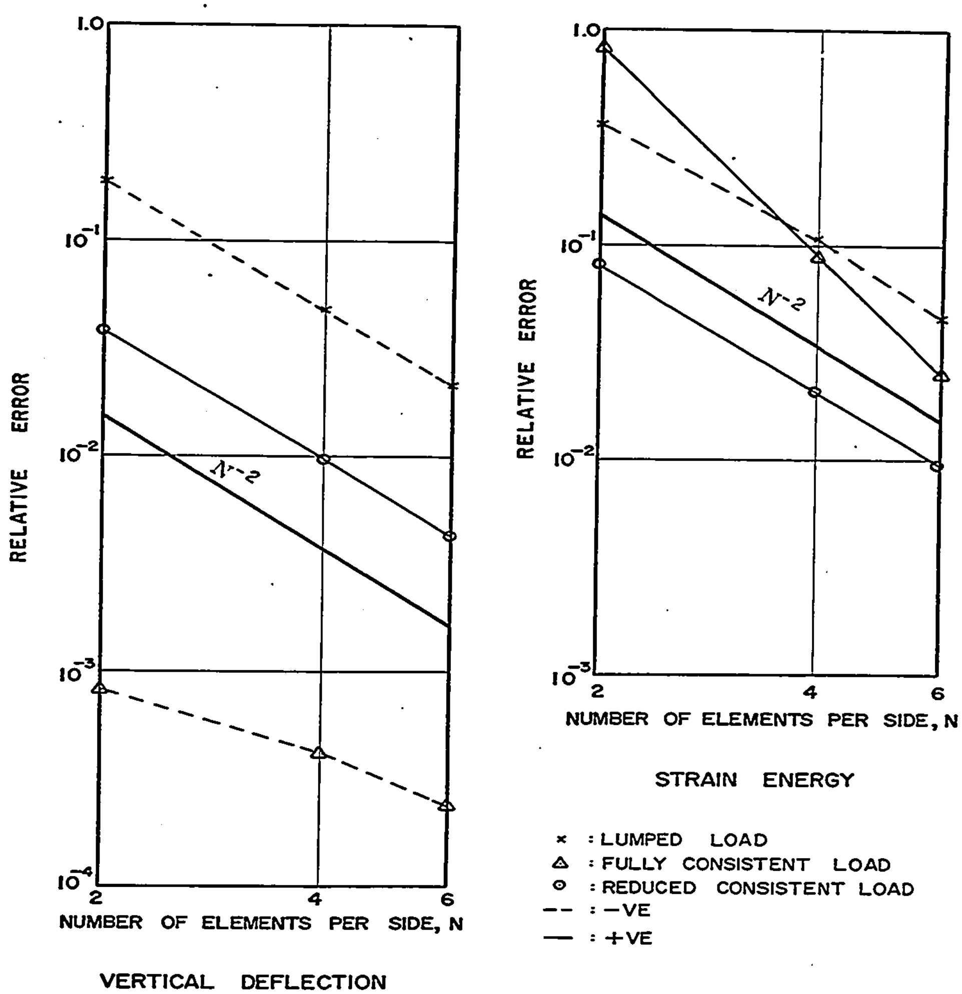


FIG. 5.6 RELATIVE ERROR OF THE FINITE ELEMENT SOLUTION $\alpha = 10°$

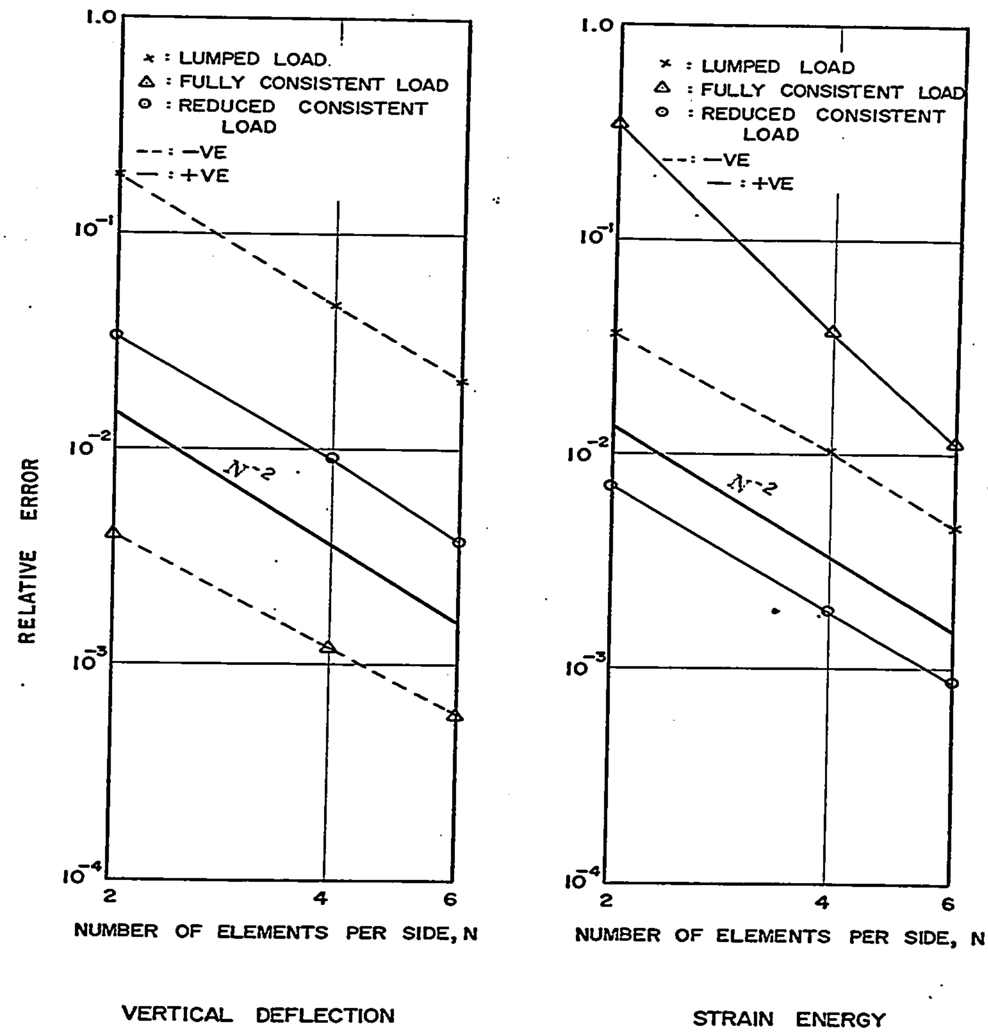
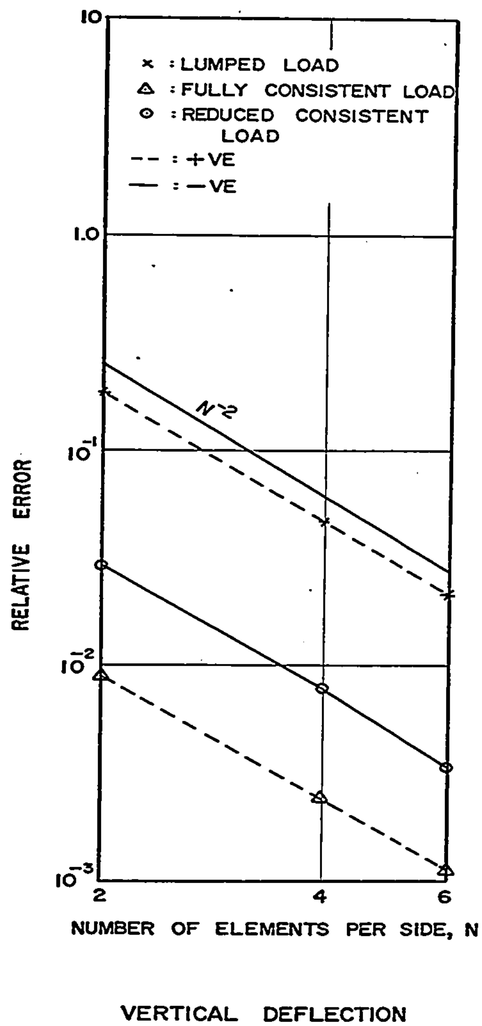


FIG.5.7 RELATIVE ERROR OF FINITE ELEMENT SOLUTION $\alpha = 20°$



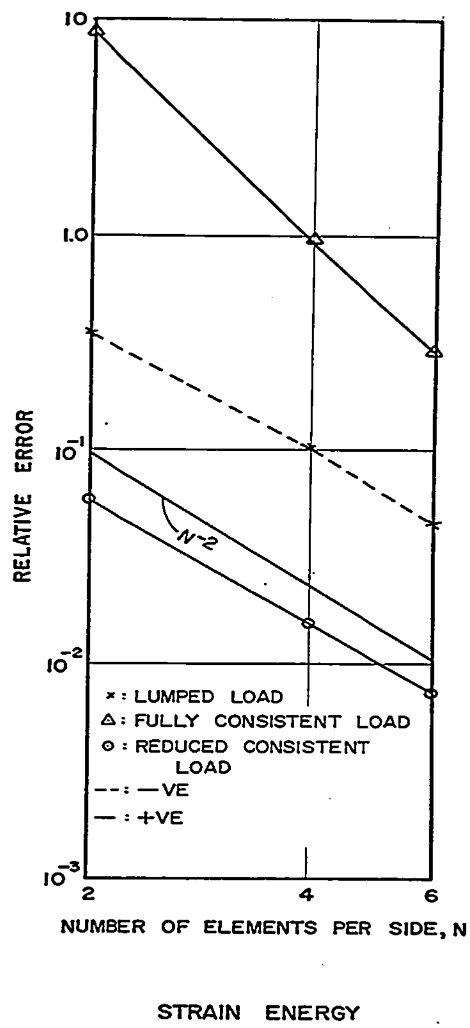


FIG. 5.8 RELATIVE ERROR OF THE FINITE ELEMENT SOLUTION $\alpha = 30°$

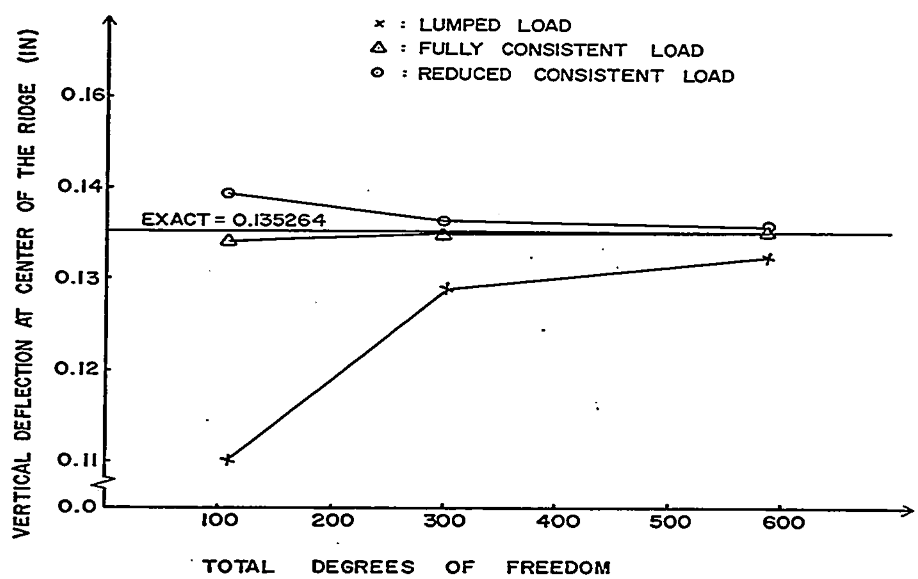


FIG. 5.9 VERTICAL DEFLECTION COMPARISIONS, $\alpha = 30°$

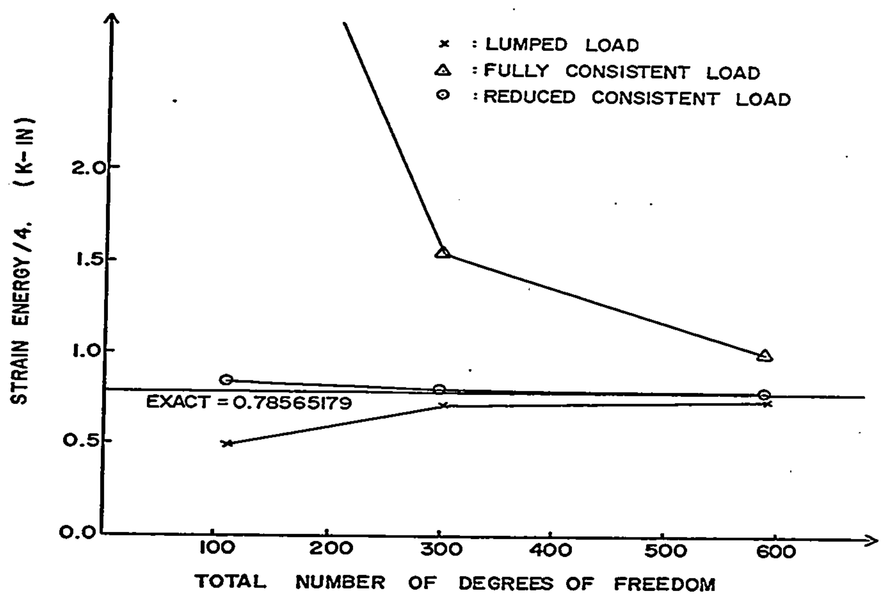


FIG. 5.10 STRAIN ENERGY COMPARISONS. $\alpha = 30°$

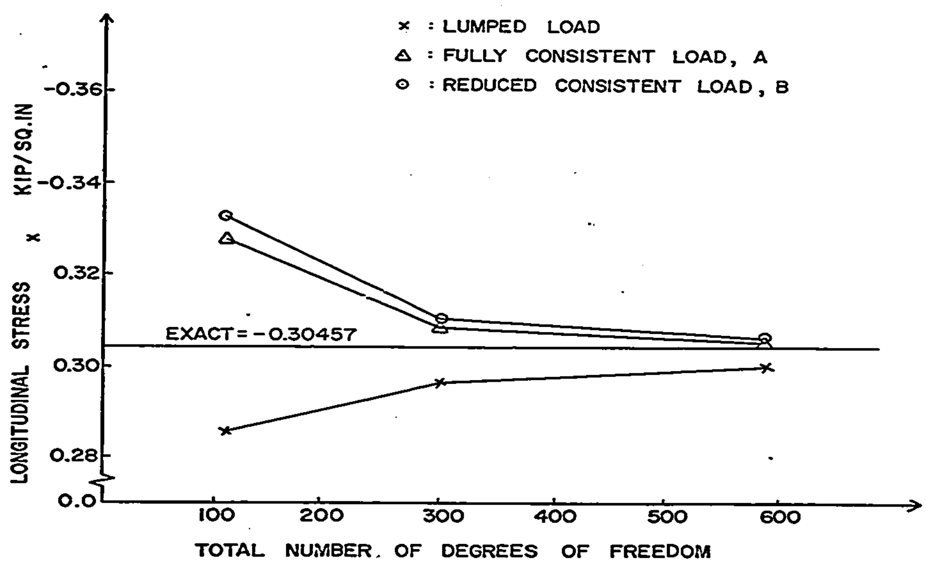


FIG. 5.11 LONGITUDINAL STRESS COMPARISONS. α = 30°

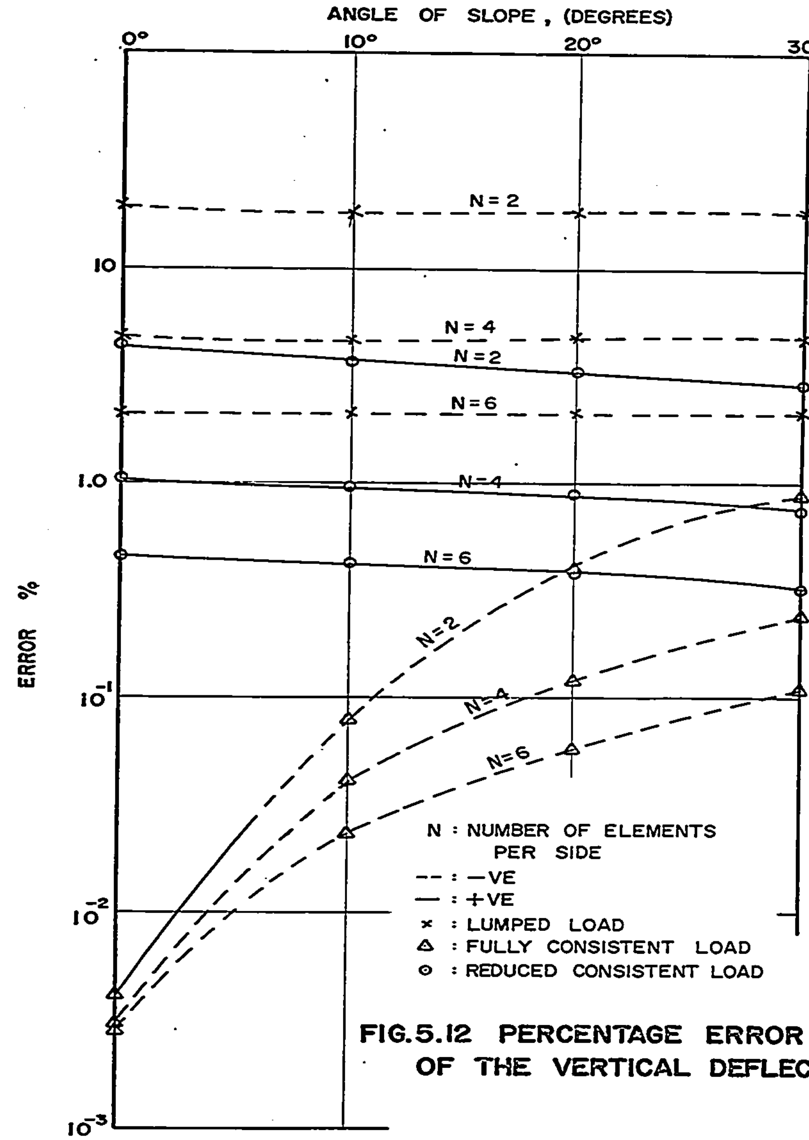


FIG. 5.12 PERCENTAGE ERROR OF THE VERTICAL DEFLECTION

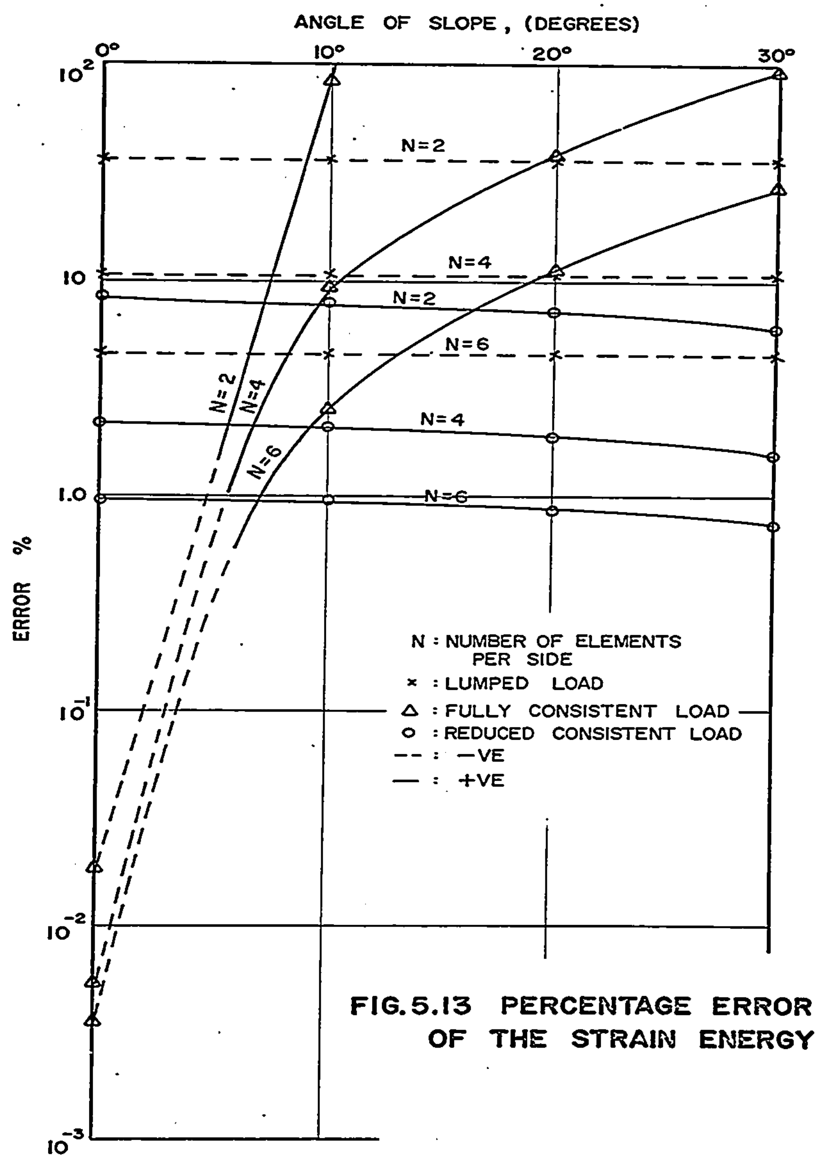


FIG. 5.13 PERCENTAGE ERROR OF THE STRAIN ENERGY

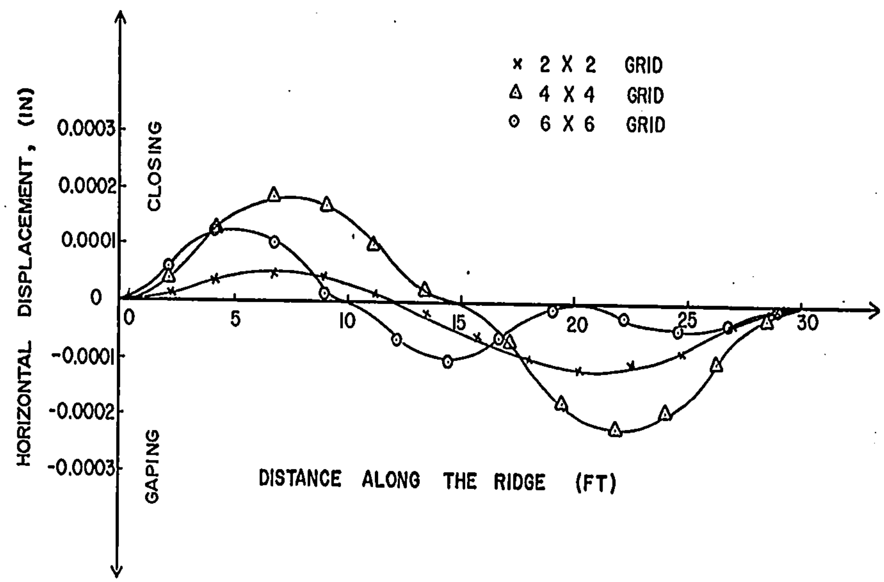


FIG. 5.14 HORIZONTAL DISCONTINUITY ALONG THE RIDGE.
LUMPED LOAD. $\alpha = 30°$

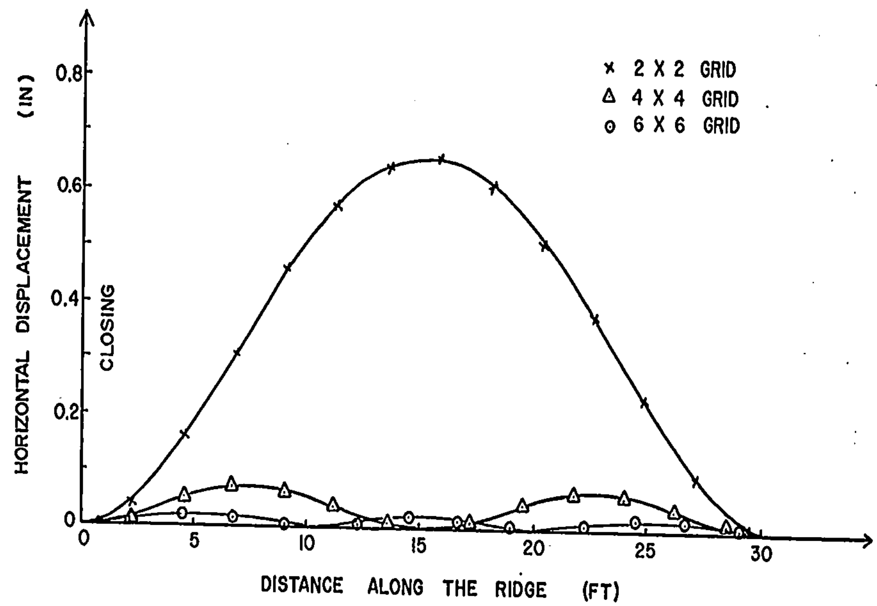


FIG. 5.15 HORIZONTAL DISCONTINUTY ALONG THE RIDGE
FULLY CONSISTENT LOAD $\alpha = 30°$

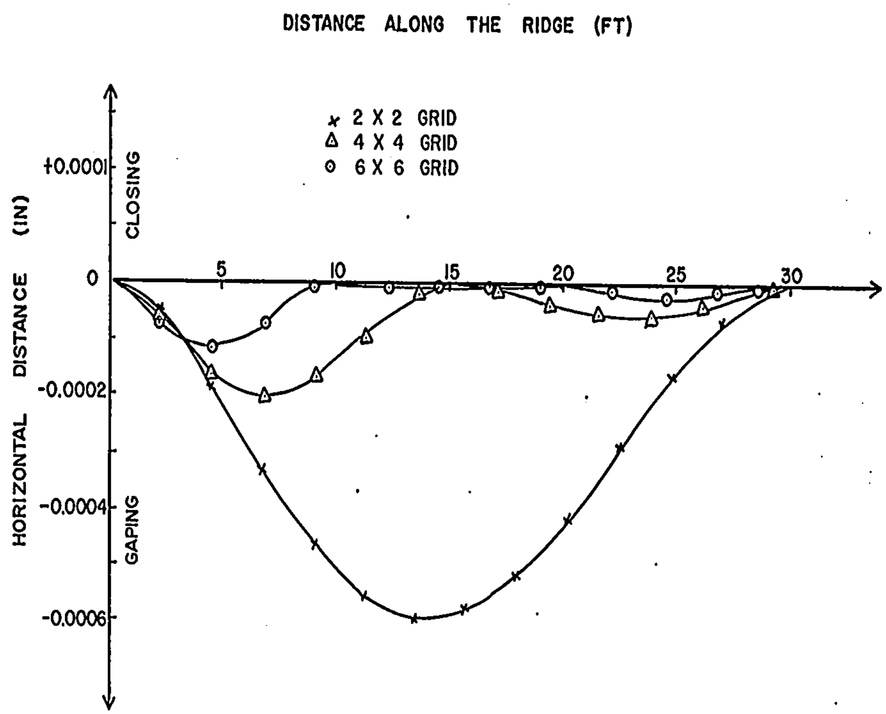


FIG. 5.16 HORIZONTAL DISCONTINUITY ALONG THE RIDGE
REDUCED CONSISTENT LOAD. $\alpha = 30°$

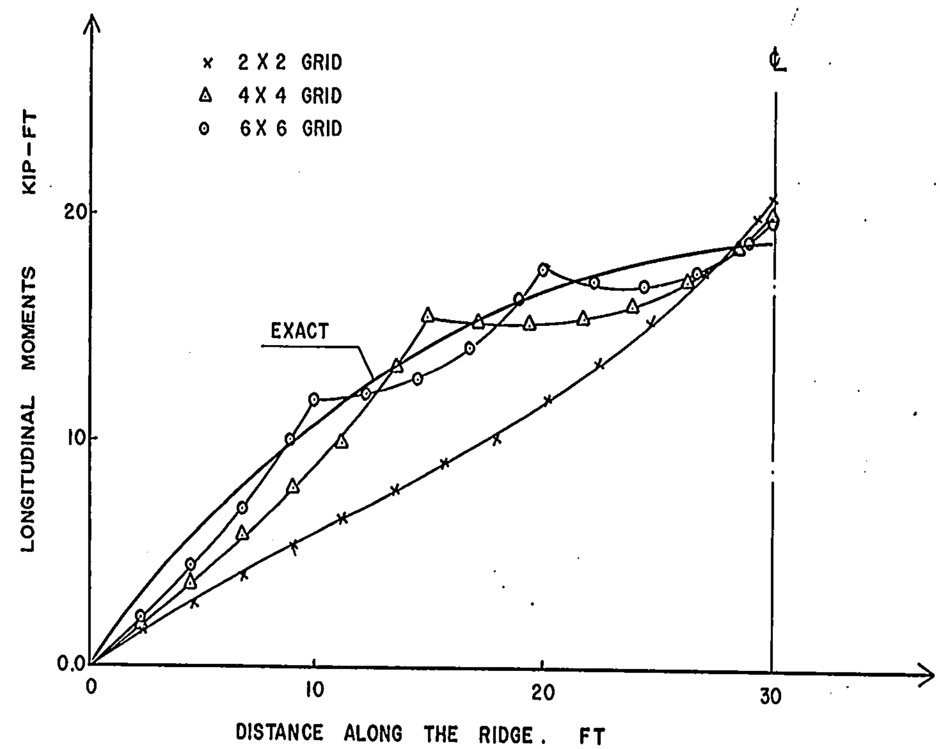


FIG. 5.17 LONGITUDINAL MOMENTS ALONG THE RIDGE $\alpha = 0.0$
LUMPED LOAD

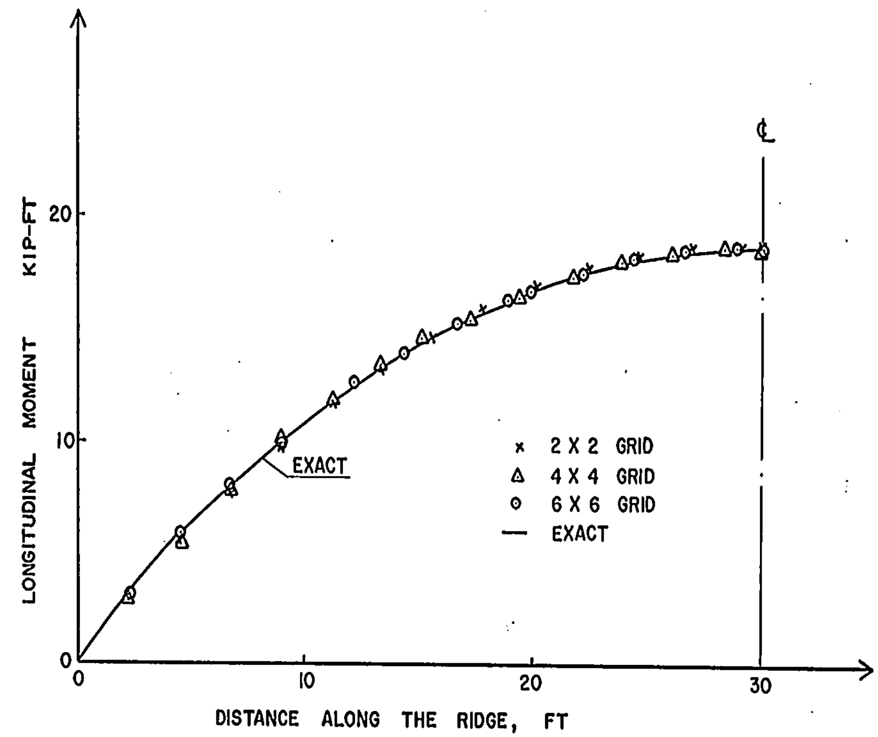


FIG. 5.18 LONGITUDINAL MOMENTS ALONG THE RIDGE, $\alpha = 0.0$
FULLY CONSISTENT LOAD

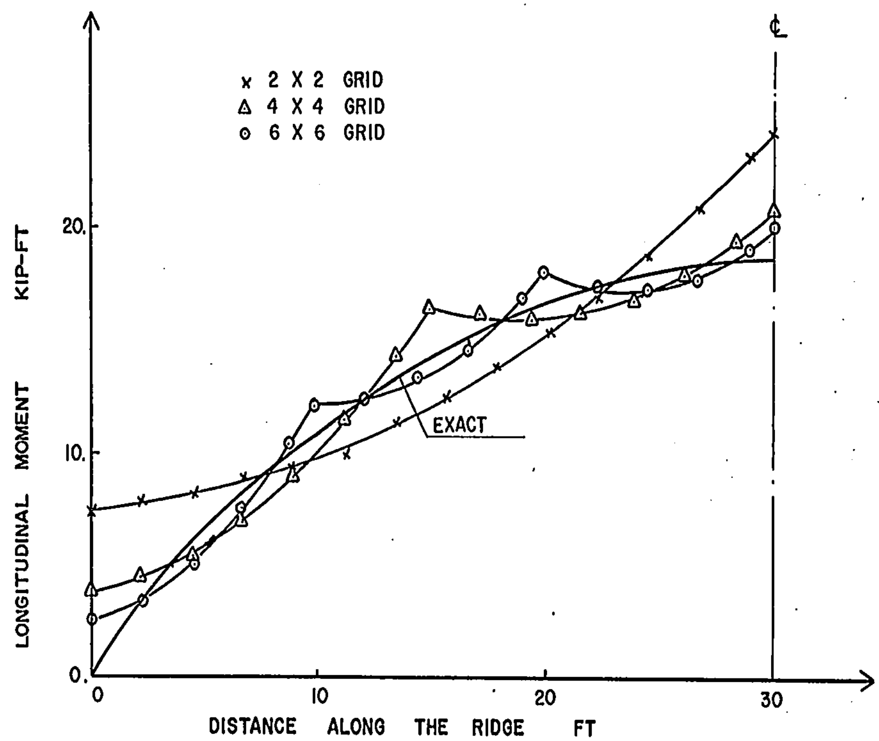


FIG. 5.19 LONGITUDINAL MOMENTS ALONG THE RIDGE $\alpha = 0.0$
REDUCED CONSISTENT LOAD

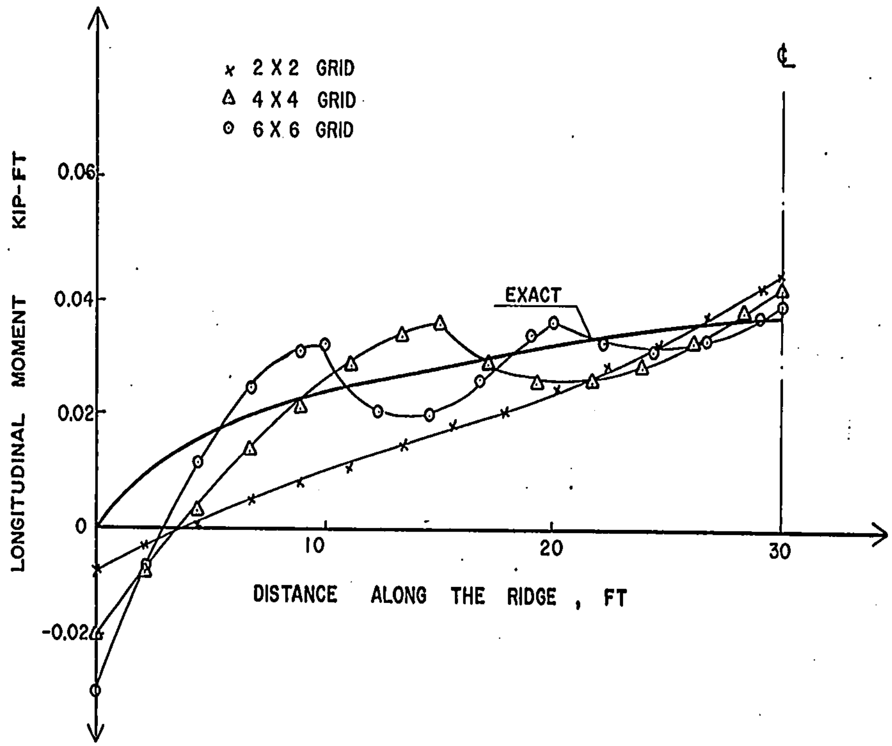


FIG. 5.20 LONGITUDINAL MOMENTS ALONG THE RIDGE $\alpha = 30°$
LUMPED LOAD

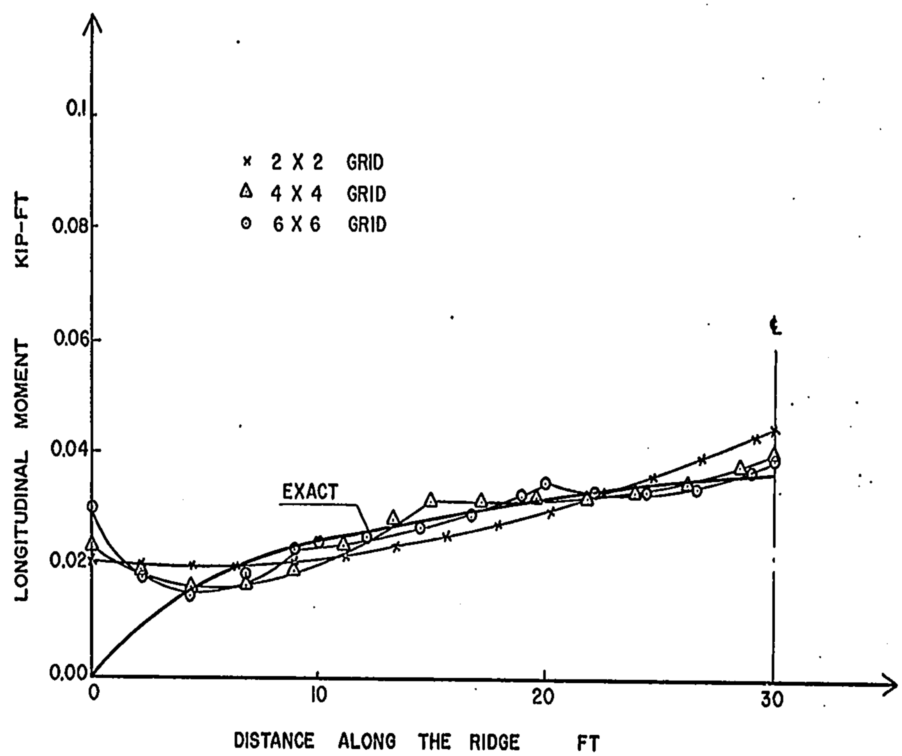


FIG. 5.21 LONGITUDINAL MOMENTS ALONG THE RIDGE, $\alpha = 30°$
REDUCED CONSISTENT LOAD

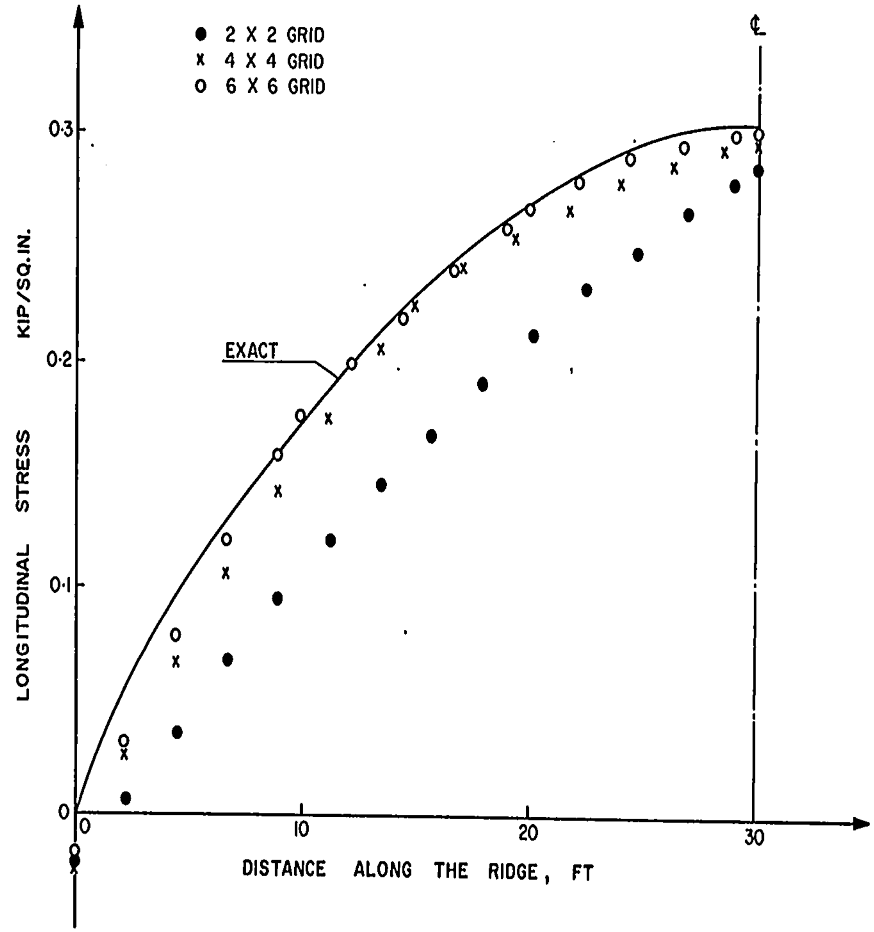


FIG. 5.22 LONGITUDINAL STRESSES ALONG THE RIDGE (LUMPED LOAD) $\alpha = 30°$

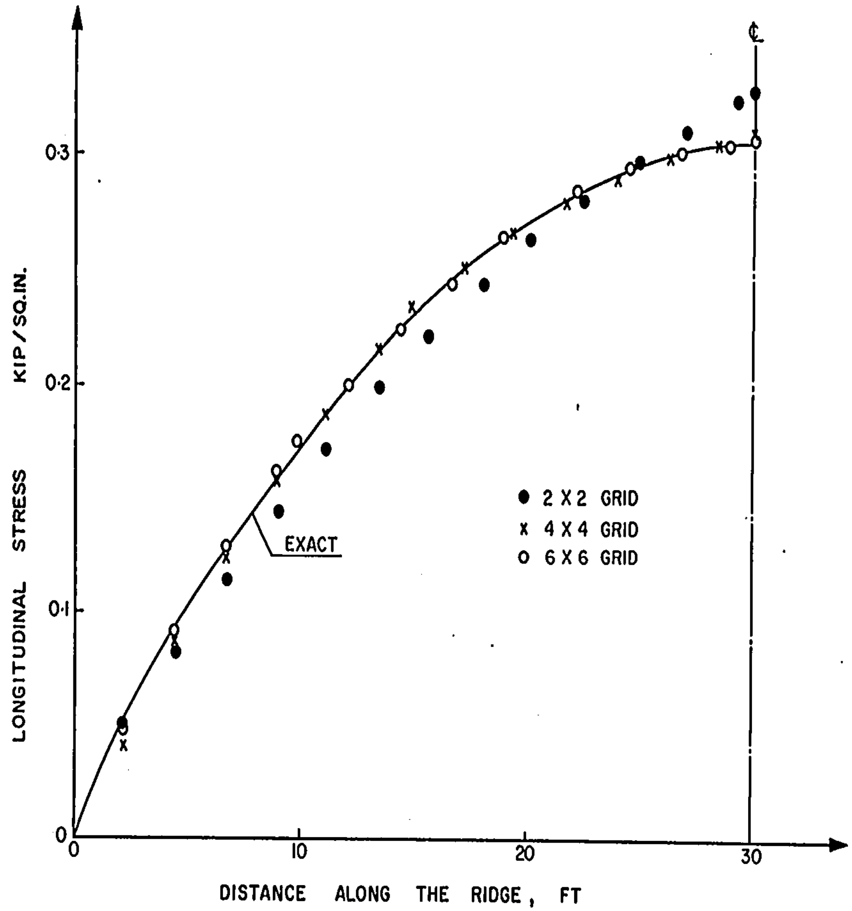


FIG. 5.23 LONGITUDINAL STRESSES ALONG THE RIDGE $\alpha = 30°$
(FULLY CONSISTENT LOAD)

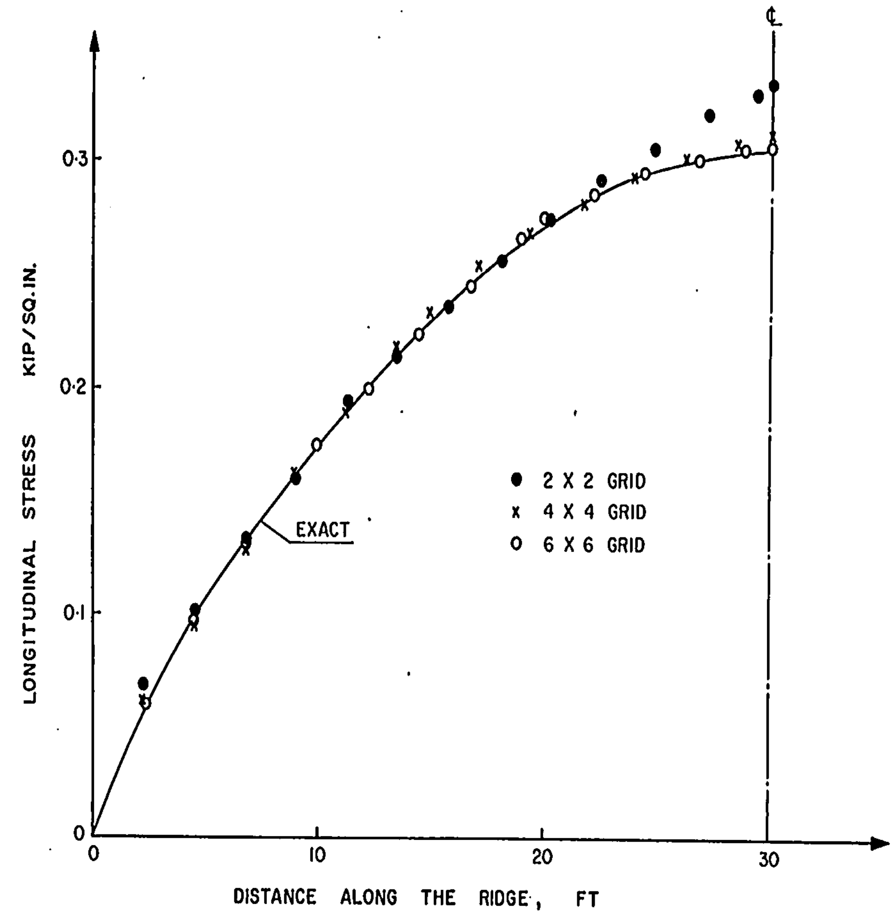
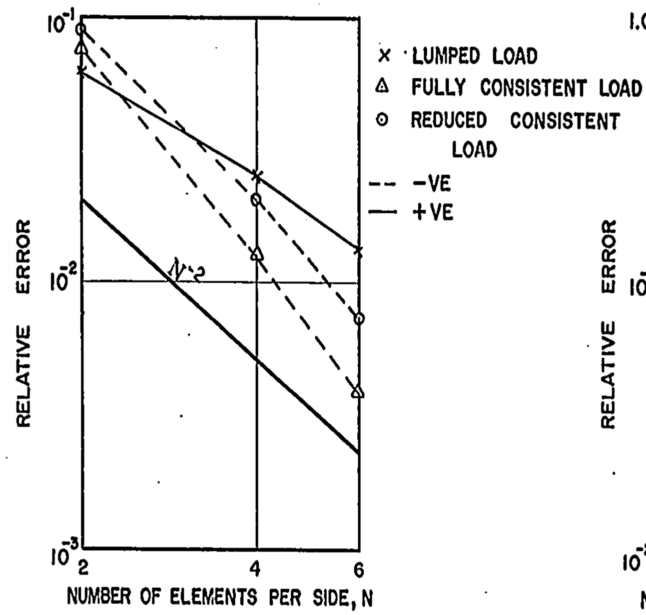
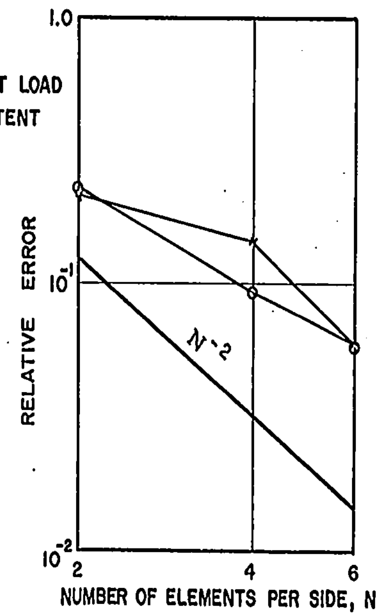


FIG. 5.24 LONGITUDINAL STRESSES ALONG THE RIDGE $\alpha = 30°$
(REDUCED CONSISTENT LOAD)





RELATIVE ERROR OF FINITE ELEMENT SOLUTION $\alpha = 30°$

FIG.5.25 LONGITUDINAL STRESSES **FIG.5.26 LONGITUDINAL MOMENTS**

APPENDIX I

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APPENDIX II

TABLE 7

IN-PLANE TRANSFORMATION MATRIX

$$[T]_p = \begin{bmatrix} S_1 & 0 \\ 0 & S_1 \\ S_2 & 0 \\ 0 & S_2 \\ S_3 & 0 \\ 0 & S_3 \\ S_4 & 0 \\ 0 & S_4 \end{bmatrix}$$

where

$$[S_1] = \begin{bmatrix} 1 & -b & 0 & b^2 & 0 & 0 & -b^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2b & 0 & 0 & 3b^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -b & 0 & 0 & b^2 & 0 & 0 \end{bmatrix}$$

$$[S_2] = \begin{bmatrix} 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 \end{bmatrix}$$

$$[S_3] = \begin{bmatrix} 1 & 0 & c & 0 & 0 & c^2 & 0 & 0 & 0 & c^3 \\ 0 & 1 & 0 & 0 & c & 0 & 0 & 0 & c^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2c & 0 & 0 & 0 & 3c^2 \end{bmatrix}$$

$$[S_4] = \begin{bmatrix} 1 & \frac{a-b}{3} & \frac{c}{3} & \frac{(a-b)^2}{9} & \frac{(a-b)c}{9} & \frac{c^2}{9} & \frac{(a-b)^3}{27} & \frac{(a-b)^2c}{27} & \frac{(a-b)c^2}{27} & \frac{c^3}{27} \end{bmatrix}$$

a, b and c are defined in Fig. (3-1)

TABLE 8

IN-PLANE ROTATION MATRIX $[R]_p$

$$[R]_p = \left[\begin{array}{c|c|c|c} R_1^p & 0 & 0 & 0 \\ \hline 0 & R_1^p & 0 & 0 \\ \hline 0 & 0 & R_1^p & 0 \\ \hline 0 & 0 & 0 & R_2^p \\ \hline \end{array}\right]$$

where

$$[R_1^p] = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta & 0 & 0 \\ 0 & \cos^2\theta & \sin\theta\cos\theta & 0 & \sin\theta\cos\theta & \sin^2\theta \\ 0 & -\sin\theta\cos\theta & \cos^2\theta & 0 & -\sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & -\sin\theta\cos\theta & -\sin^2\theta & 0 & \cos^2\theta & \sin\theta\cos\theta \\ 0 & \sin^2\theta & -\sin\theta\cos\theta & 0 & -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

$$[R_2^p] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The angle θ is shown in Figure (3-1)

TABLE 9

BENDING TRANSFORMATION MATRIX [T]

1	$-b$	0	b^2	0	0	$-b^3$	0	0	0	$-b^4$	0	0	0	0	$-b^5$	0	0	0	0
0	1	0	$-2b$	0	0	$3b^2$	0	0	0	$-4b^3$	0	0	0	0	$5b^4$	0	0	0	0
0	0	1	0	$-b$	0	0	b^2	0	0	0	$-b^3$	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$-6b$	0	0	0	$12b^2$	0	0	0	0	$-20b^3$	0	0	0	0
0	0	0	0	1	0	0	$-2b$	0	0	0	$3b^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$-2b$	0	0	0	$2b^2$	0	0	0	$-2b^3$	0	0	0
1	a	0	a^2	0	0	a^3	0	0	0	a^4	0	0	0	0	a^5	0	0	0	0
0	1	0	$2a$	0	0	$3a^2$	0	0	0	$4a^3$	0	0	0	0	$5a^4$	0	0	0	0
0	0	1	0	a	0	0	a^2	0	0	0	a^3	0	0	0	0	0	0	0	0
0	0	0	2	0	0	$6a$	0	0	0	$12a^2$	0	0	0	0	$20a^3$	0	0	0	0
0	0	0	0	1	0	0	$2a$	0	0	0	$3a^2$	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	$2a$	0	0	0	$2a^2$	0	0	0	$2a^3$	0	0	0
1	0	c	0	0	c^2	0	0	0	c^3	0	0	0	0	c^4	0	0	0	0	c^5
0	1	0	0	c	0	0	0	c^2	0	0	0	0	c^3	0	0	0	0	c^4	0
0	0	1	0	0	$2c$	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0	0	0	0	$5c^4$
0	0	0	2	0	0	0	$2c$	0	0	0	0	$2c^2$	0	0	0	0	$2c^3$	0	0
0	0	0	0	1	0	0	0	$2c$	0	0	0	0	$3c^2$	0	0	0	0	$4c^3$	0
0	0	0	0	0	2	0	0	0	$6c$	0	0	0	0	$12c^2$	0	0	0	0	$20c^3$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5a^4c$	$3a^2c^3-2a^4c$	$-2ac^4+3a^3c^2$	$c^5-4a^2c^3$	$5ac^4$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$5b^4c$	$3b^2c^3-2b^4c$	$2bc^4-3b^3c^2$	$c^5-4b^2c^3$	$-5bc^4$

TABLE 10

BENDING ROTATION MATRIX

$$[R]_b = \begin{bmatrix} R_1^b & 0 & 0 \\ 0 & R_1^b & 0 \\ 0 & 0 & R_1^b \end{bmatrix}$$

$$[R_1^b] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \\ 0 & 0 & 0 & -\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & \sin\theta\cos\theta \\ 0 & 0 & 0 & \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

The angle θ is shown in Figure 3-1.

TABLE 11

PLATE TRANSFORMATION MATRIX (Elasticity Method)

$$[T]_{EL} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{V}{B} & -\frac{H}{B} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{V}{B} & \frac{H}{B} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{H}{B} & \frac{V}{B} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{H}{B} & -\frac{V}{B} & 0 & 0 \end{bmatrix}$$

where:

B: plate width

H: Horizontal projection of the plate width

V: Vertical projection of the plate width

TABLE 12

PLATE STIFFNESS MATRIX

$$[k]_{EL} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & & & & \\ k_{21} & k_{22} & k_{23} & k_{24} & & \text{zeros} & & \\ k_{31} & k_{32} & k_{33} & k_{34} & & & & \\ k_{41} & k_{42} & k_{43} & k_{44} & & & & \\ & & & & k_{55} & k_{56} & k_{57} & k_{58} \\ & & & & k_{65} & k_{66} & k_{67} & k_{68} \\ & \text{zeros} & & & k_{75} & k_{76} & k_{77} & k_{78} \\ & & & & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix}$$

where:

$$k_{11} = k_{22} = \frac{Eh^3}{12(1-\nu^2)} \frac{m\pi}{L} \left[\frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu + \sinh\mu} - \frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu - \cosh\mu} \right]$$

$$k_{12} = k_{21} = \frac{Eh^3}{12(1-\nu^2)} \frac{m\pi}{L} \left[\frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu + \sinh\mu} + \frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu - \cosh\mu} \right]$$

$$k_{13} = k_{31} = \frac{Eh^3}{12(1-\nu^2)} \frac{m^2\pi^2}{L^2} \left[\frac{\cosh\mu}{\mu\,\mathrm{csch}\,\mu + \cosh\mu} - \frac{\sinh\mu}{\mu\,\mathrm{sech}\,\mu - \sinh\mu} - (1-\nu) \right]$$

$$k_{14} = k_{41} = -\frac{Eh^3}{12(1-\nu^2)} \frac{m^2\pi^2}{L^2} \left[\frac{\cosh\mu}{\mu\,\mathrm{csch}\,\mu + \cosh\mu} + \frac{\sinh\mu}{\mu\,\mathrm{sech}\,\mu - \sinh\mu} \right]$$

$$k_{23} = k_{32} = k_{14}$$

$$k_{24} = k_{42} = k_{13}$$

$$k_{33} = k_{44} = \frac{Eh^3}{12(1-\nu^2)} \frac{m^3\pi^3}{L^3} \left[\frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu + \cosh\mu} - \frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu - \sinh\mu} \right]$$

$$k_{34} = k_{43} = \frac{Eh^3}{12(1-\nu^2)} \frac{m^3\pi^3}{L^3} \left[\frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu + \cosh\mu} + \frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu - \sinh\mu} \right]$$

$$k_{55} = k_{66} = \frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[- \frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu - \frac{3-\nu}{1+\nu}\cosh\mu} + \frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu + \frac{3-\nu}{1+\nu}\sinh\mu} \right]$$

$$k_{56} = k_{65} = \frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[\frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu - \frac{3-\nu}{1+\nu}\cosh\mu} + \frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu + \frac{3-\nu}{1+\nu}\sinh\mu} \right]$$

$$k_{57} = k_{75} = -\frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[\frac{\cosh\mu}{\mu\,\mathrm{csch}\,\mu - \frac{3-\nu}{1+\nu}\cosh\mu} - \frac{\sinh\mu}{\mu\,\mathrm{sech}\,\mu + \frac{3-\nu}{1+\nu}\sinh\mu} + (1+\nu) \right]$$

$$k_{58} = k_{85} = -\frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[\frac{\cosh\mu}{\mu\,\mathrm{csch}\,\mu - \frac{3-\nu}{1+\nu}\cosh\mu} + \frac{\sinh\mu}{\mu\,\mathrm{sech}\,\mu + \frac{3-\nu}{1+\nu}\sinh\mu} \right]$$

$$k_{67} = k_{76} = k_{58}$$

$$k_{68} = k_{86} = k_{57}$$

$$k_{77} = k_{88} = \frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[- \frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu - \frac{3-\nu}{1+\nu}\sinh\mu} + \frac{\sinh\mu}{\mu\,\mathrm{csch}\,\mu + \frac{3-\nu}{1+\nu}\cosh\mu} \right]$$

$$k_{78} = k_{87} = -\frac{Eh}{(1+\nu)^2} \frac{m\pi}{L} \left[\frac{\cosh\mu}{\mu\,\mathrm{sech}\,\mu - \frac{3-\nu}{1+\nu}\sinh\mu} + \frac{\sinh\mu}{\mu\,\mathrm{scsh}\,\mu + \frac{3-\nu}{1+\nu}\cosh\mu} \right]$$

in which:

E : Young's modulus

h : Plate thickness

ν : Poisson's ratio

L : Plate span

$\mu = \frac{m\pi B}{2L}$

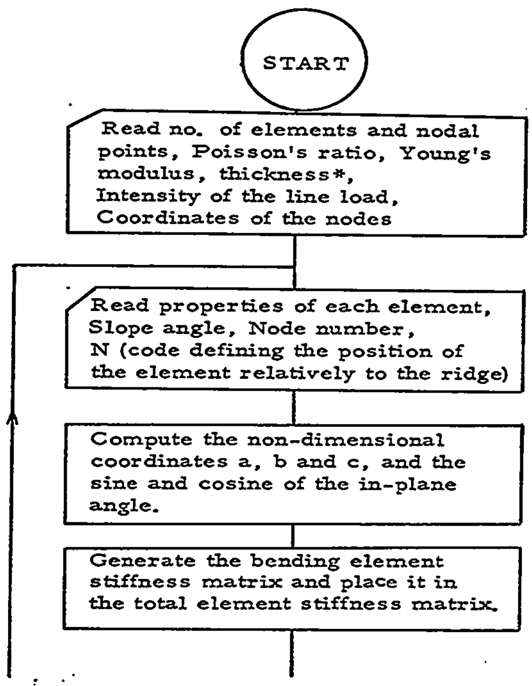
and B : Plate width

APPENDIX III
THE COMPUTER PROGRAM

The computer program developed by the author and used in this thesis to analyse the simply supported folded plate structure described in Chapter 5, is written in Fortran IV for an IBM-360 model 65 computer system. The flow chart describing the general steps of the program, the listing and a typical output are presented in this appendix. This program can handle prismatic folded plate structures with different properties and boundary conditions, under the effect of a line load along the ridges. By changing the dimensions of the arrays corresponding to the complete structure, the program can analyse structures with any number of plates, nodes and elements. Slight modifications are needed to use the same program to solve folded plate structures subjected to any type of loading system.

The flow chart and the listing with the comment statements make this program self explanatory. However, the restraints corresponding to the different boundary conditions and the equations used to calculate the stresses are described later in this appendix.

FLOW CHART



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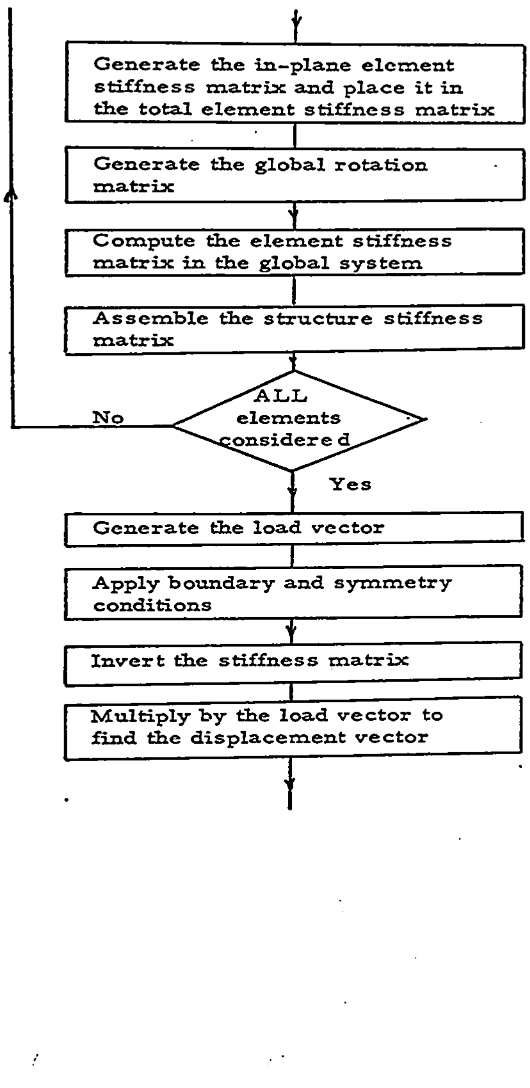
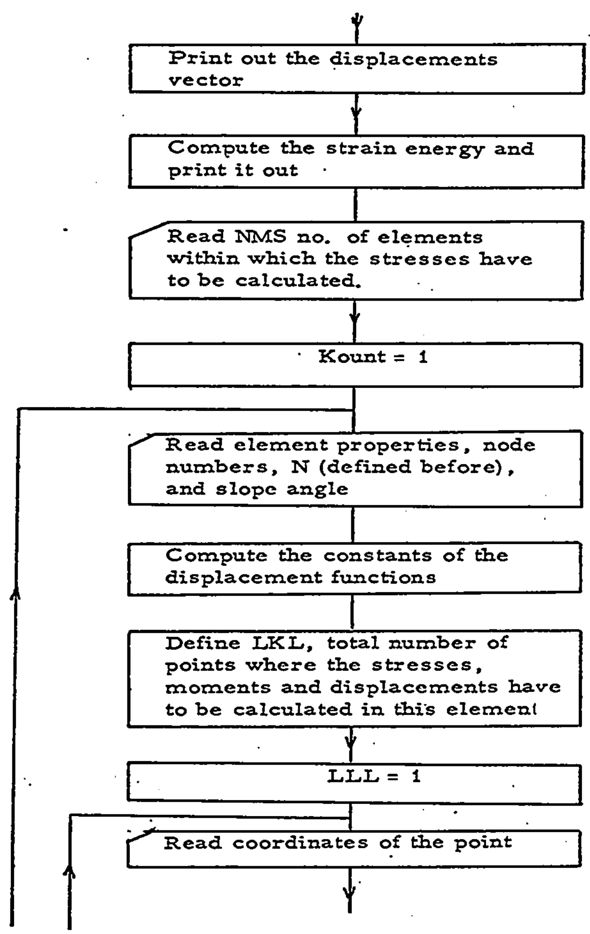
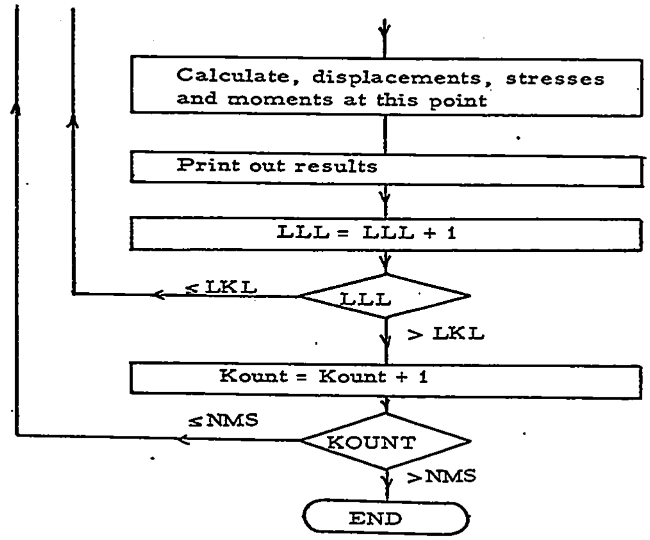
* These values must be read for each element if they are variable.


Generate the in-plane element stiffness matrix and place it in the total element stiffness matrix
Generate the global rotation matrix
Compute the element stiffness matrix in the global system
Assemble the structure stiffness matrix
ALL elements considered
No
Yes
Generate the load vector
Apply boundary and symmetry conditions
Invert the stiffness matrix
Multiply by the load vector to find the displacement vector


Print out the displacements vector
Compute the strain energy and print it out
Read NMS no. of elements within which the stresses have to be calculated.
Kount = 1
Read element properties, node numbers, N (defined before), and slope angle
Compute the constants of the displacement functions
Define LKL, total number of points where the stresses, moments and displacements have to be calculated in this element
LLL = 1
Read coordinates of the point


Calculate, displacements, stresses and moments at this point
Print out results
LLL = LLL + 1
LLL
≤ LKL
> LKL
Kount = Kount + 1
KOUNT
≤NMS
>NMS
END

I. Boundary and symmetry conditions:

To satisfy the boundary and symmetry conditions for the problem described in Chapter 5 and solved by using the previous computer program, the following restraints on the displacements have to be applied:

a) Along the free edge:

No restraint.

b) Along the diaphragms:

$$v = 0$$
$$v_{\eta} = 0$$
$$w = 0$$
$$w_{\eta} = 0$$
$$w_{\eta\eta} = 0$$

c) Along the longitudinal center line:

$$v' = 0$$
$$v'_{x} = 0$$
$$w_{\eta} = 0$$
$$w_{\xi\eta} = 0$$

d) Along the transversal center line:

$$u = 0$$
$$u_{\eta} = 0$$
$$v_{x} = 0$$
$$w_{x} = 0$$
$$w_{\xi\eta} = 0$$

II. Moments and stresses:

The moments and stresses are calculated by using the following expressions:

$$M_{xx} = \frac{Eh^3}{12(1-\nu^2)} \left[w_{xx} + \nu w_{yy} \right]$$

$$M_{yy} = \frac{Eh^3}{12(1-\nu^2)} \left[w_{yy} + \nu w_{xx} \right]$$

$$\sigma_x = \frac{E}{1-\nu^2} (u_x + \nu v_y)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)} (u_y + v_x)$$

```
G LEVEL  19                    MAIN                    DATE = 71205          18/46/12
C*****************************************************************************
C
C
C                 GHASSAN ATTAR-HASSAN
C
C*****************************************************************************
C
C
C          DEPARTMENT OF CIVIL ENGINEERING
C
C          UNIVERSITY OF OTTAWA
C
C               JUNE 1971
C
C               M.A.SC.THESIS
C
C*****************************************************************************
C
C
C          PROGRAM ANALYSING PRISMATIC FOLDED PLATE STRUCTURES,USING
C          THE FINITE-ELEMENT METHOD.
C
C          THE REFINED IN-PLANE AND BENDING ELEMENTS, DEVELOPED
C          IN THE NATIONAL RESEARCH COUNCIL OF CANADA, ARE USED
C          TO FORM THE TRIANGULAR ELEMENT USED IN THIS ANALYSIS.
C
C          THIS ELEMENT HAS 36 DEGREES OF FREEDOM,NAMELY U,UX,UY,
C          V,VX,VY,W,WX,WY,WXX,WXY AND WYY, AT EACH NODE.
C
C          THIS PROGRAM CAN HANDLE ANY PRISMATIC FOLDED PLATE
C          STRUCTURE WITH DIFFERENT ELEMENT PROPERTIES AND
C          DIFFERENT BOUNDARY CONDITIONS,UNDER THE EFFECT OF A
C          LINE LOAD UNIFORMLY DISTRIBUTED ON THE RIDGES.
C
C          THE PROGRAM CAN BE MODIFIED EASILY TO HANDLE ANY TYPE
C          OF LOADING.
C
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
C*****************************************************************************
C
      COMMON SJ(20,20), SA(18,18),SB(2,2),SC(18,2),SD(2,18),SE(18,18),
     @SF(18,2),SG(20,20),SH(20,20),SI(20,20),SK(20,20),SM(20),SN(20),SO(
     @20),SP(20),SP1(18,18),DISP(48)
```

```
      DIMENSION WS(36,36),R(12,12),KODE(48),FORCE(48),A1(48,48)
      DIMENSION RO(36,36),RWS(36,36),X(4),Y(4)
C
C              THE COMMON AND DIMENSIONS STATMENTS HAVE TO BE CHANGED
C              EVERY TIME WE CHANGE THE NUMBER OF ELEMENTS AND NODES.
C
C**********************************************************************
C
      DATA KODE/0,0,0,1,0,1,1,0,1,0,0,1,0,0,0,1,1,1,1,0,1,0,1,1,1,0,1,0,
     11,0,0,1,0,0,1,0,1,0,1,1,1,0,0,1,1,0,1,0/
C
C              THIS DATA STATMENT IS USED TO DEFINE THE BOUNDARY
C              CONDITIONS. IF THE PRESCRIBED DISPLACEMENTS IS EQUAL
C              TO ZERO, THE CORRESPONDING TERM IN KODE IS SET EQUAL
C              TO ONE.
C
C**********************************************************************
C
C         READ THE STRUCTURE PROPERTIES.
C
      READ(1,75)NM,ANU,YM,TH,PZERO,NOD
C
C         NM   -THE NUMBER OF ELEMENTS.
C
C         YM   -YOUNG'S MODULUS.
C
C         ANU  -POISSON'S RATIO.
C
C         TH   -THICKNESS OF THE STRUCTURE.
C
C         PZERO-THE LINE LOAD INTENSITY.
C
C         NOD  -THE NUMBER OF NODES.
C
C**********************************************************************
      WRITE(3,325) NM,NOD,YM,ANU,TH,PZERO
      M1=12*NOD
      DO 1 I=1,12
      DO 1 J=1,12
    1 R(I,J)=0.D0
      DO 2 I=1,M1
      FORCE(I)=0.D0
      DISP(I)=0.D0
```

```
G LEVEL  19                    MAIN                 DATE = 71205          18/46/12
      DO 2 J=1,M1
      A1(I,J)=0.D0
      DO 2 K=1,NOD
      X(K)=0.D0
      Y(K)=0.D0
    2 CONTINUE
C*********************************************************************
C
C        READ THE COORDINATES OF THE NODES.
C
      READ(1,50)(X(I),Y(I),I=1,NCD)
C
C*********************************************************************
      WRITE(3,125)
      WRITE(3,350) (I,X(I),Y(I),I=1,NOD)
      KCUNT=1
    3 CONTINUE
      IF(KOUNT.GT.NM) GO TO 9
C*********************************************************************
C
C        READ THE SLOPE ANGLE, ALPHA.
C
      READ(1,100) ALPHA
C
C*********************************************************************
C
C        READ THE NODE NUMBERS CORRESPONDING TO EACH ELEMENT,
C        N1,N2 AND N3.N IS A CODE NUMBER USED TO INDICATE THE
C        NODES LYING ON THE RIDGES.IF NODE NUMBER 1 IS ON THE RIDGE,
C        N IS EQUAL TO 1. IF NODE NUMBER 2 THEN N EQUAL TO 2.IF NODE
C        NUMBER 3 ,N=3. IF NODE NUMER 1 AND 2 , N=4. IF NODES 1
C        AND 3 , N=5. IF NODES 2 AND 3 , N=6.WHEN NONE OF THE NODES
C        LIES CN THE RIDGE, N IS EQUAL TO ZERO.
C
C
      READ(1,150) N1,N2,N3,N
C
C*********************************************************************
C
C        CALL ABC TO CALCULATE THE ELEMENT DIMENSIONS, A,B, AND C,
C        AND TO CALCULATE THE SINE AND COSIN OF THE IN-PLANE ANGLE.
C
      CALL        ABC(N1,N2,N3,X,Y,NOD,A,B,C,COSIN,SINUS)
```

```
C
C***********************************************************************
      A=A*12.D0
      B=B*12.D0
      C=C*12.D0
      WRITE(3,175) A,B,C,ALPHA
      WRITE(3,150) N1,N2,N3,N
      DO 4 I=1,36
      DO 4 J=1,36
    4 WS(I,J)=0.D0
C***********************************************************************
C
C          CALL BENDEG TO FORM THE BENDING ELEMENT STIFFNESS MATRIX.
C
      CALL BENDEG(ANU,A,B,C,YM,TH,THETA,COSIN,SINUS)
C
C***********************************************************************
C
C          PLACE THE BENDING ELEMENT STIFFNESS MATRIX IN THE APPROPRIATE
C          ROWS AND COLUMNS IN THE TOTAL ELEMENT STIFFNESS MATRIX.
C
      DO 5 I=1,18
      DO 5 J=1,18
      IF(I.LT.6.OR.I.EQ.6) II=I+6
      IF(J.LT.6.OR.J.EQ.6) JJ=J+6
      IF(I.GT.6) II=I+12
      IF(J.GT.6) JJ=J+12
      IF(I.GT.12) II=I+18
      IF(J.GT.12) JJ=J+18
    5 WS(II,JJ)=SE(I,J)
C
C***********************************************************************
C
C          CALL PSESGH TO FORM THE IN-PLANE ELEMENT STIFFNESS MATRIX.
C
      CALL PSESGH(ANU,A,B,C,TH,THETA,YM,COSIN,SINUS)
C
C***********************************************************************
C
C          PLACE THE IN-PLANE ELEMENT STIFFNESS MATRIX IN THE APPROPRIATE
C          ROWS AND COLUMNS IN THE TOTAL ELEMENT STIFFNESS MATRIX.
C
      DO 6 I=1,18
```

```
      DO 6 J=1,18
      IF(I.LT.6.OR.I.EQ.6) II=I
      IF(J.LT.6.OR.J.EQ.6) JJ=J
      IF(I.GT.6) II=I+6
      IF(J.GT.6) JJ=J+6
      IF(I.GT.12) II=I+12
      IF(J.GT.12) JJ=J+12
    6 WS(II,JJ)=SP1(I,J)
C
C***********************************************************************
C
C        CALL GLORCT TO FORM THE 12 BY 12 GLOBAL ROTATION MATRIX.
C
      CALL GLCROT(ALPHA,R)
C
C***********************************************************************
C
C        CALL ROTGHA TO FORM THE APPROPRIATE ELEMENT GLOBAL ROTATION
C        MATRIX.
C
      CALL ROTGHA(RWS,R,N)
C
C***********************************************************************
C
C        ROTATE THE ELEMENT STIFFNESS MATRIX TO THE GLOBAL COORDINATE
C        SYSTEM.
C
      DO 7 I=1,36
      DO 7 J=1,36
      RO(I,J)=0.D0
      DO 7 L=1,36
    7 RO(I,J)=RO(I,J)+WS(I,L)*RWS(L,J)
      DO 8 I=1,36
      DO 8 J=1,36
      WS(I,J)=0.D0
      DO 8 L=1,36
    8 WS(I,J)=WS(I,J)+RWS(L,I)*RO(L,J)
C
C***********************************************************************
C
C        CALL SETUP TO ASSEMBLE THE ELEMENT STIFFNESS MATRICES,
C        TO FORM THE STIFFNESS MATRIX FOR THE WHOLE STRUCTURE.
C
```

```
      CALL SETUP(N1,N2,N3,WS,M1,M1,36,36,A1)
C
C*************************************************************************
      KOUNT=KOUNT+1
      GO TO 3
    9 CONTINUE
C*************************************************************************
C
C        CALL LOADVE TO FORM THE CONSISTENT LOAD VECTOR.
C
C        WHEN USING THE LUMPED LOAD VECTOR, THE FORCE MATRIX SHOULD
C         BE READ FROM CARDS.
C
      CALL LOADVE(PZERO,M1,NM,FORCE,ALPHA)
C
C*************************************************************************
C
C        APPLY THE BOUNDARY CONDITIONS.
C
      DO 11 I=1,M1
      IF(KODE(I).EQ.0) GO TO 11
      FORCE(I)=0.D0
      DO 10 J=1,M1
      A1(I,J)=0.D0
      A1(J,I)=0.D0
      A1(I,I)=1.D0
   10 CONTINUE
   11 CONTINUE
C
C*************************************************************************
C
C        CALL MATIN TO INVERT THE STIFFNESS MATRIX OF THE STRUCTURE.
C
C
      CALL MATIN(A1,M1)
C
C*************************************************************************
C
C        CALCULATE THE DISPLACEMENTS BY MULTIPLYING THE FORCE MATRIX
C        BY THE INVERSION OF THE STIFFNESS MATRIX.
C
      DO 12 I=1,M1
      DO 12 J=1,M1
```

```
  12   DISP(I)=DISP(I)+A1(I,J)*FORCE(J)
C
C**********************************************************************
C
C         WRITE THE DISPLACEMENTS.
C
       WRITE(3,375)
       KW=0
  13   CONTINUE
       IF(KW.GT.2) GO TO 15
       K1=KW+1
       K2=KW+2
       WRITE(3,225) K1,K2
       DO 14 J=1,12
       I=J+12*KW
  14   WRITE(3,275) DISP(I),DISP(I+12)
       KW=KW+2
       GO TO 13
  15   CONTINUE
C
C**********************************************************************
C
C         CALCULATE THE STRAIN ENERGY.
C
       STEN=0.D0
       DO 16 I=1,M1
  16   STEN=STEN+0.5D0*DISP(I)*FORCE(I)
C
C**********************************************************************
       WRITE(3,300) STEN
C**********************************************************************
C
C         CALL STRESS TO CALCULATE THE INTERNAL FORCES AND MOMENTS.
C
       CALL STRESS(NM,ANU,YM,TH,PZERO,NOD,M ,WS,R,X,Y)
C
C**********************************************************************
  50   FORMAT(2F20.5)
  75   FORMAT(I5,4F10.5,I5)
  100  FORMAT(F10.5)
  125  FORMAT(///20X,'COORDINATES OF THE NODES',/20X,'*****************
      1*****',///15X,'NODE',16X,'X',14X,'Y',/33X,'(FT)',11X,'(FT)')
  150  FORMAT(4I10)
```

```
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  175 FORMAT(///25X,'ELEMENT PROPERTIES',/25X,'******************',/ 5X,
     1'A = ',F6.2,' IN',5X,'B = ',F6.2,' IN',5X,'C= ',F6.2,' IN',/5X,'AL
     2PHA.THE ANGLE WITH THE HORIZONTAL = ',F6.2,' DEGREES',//20X,'NODES
     3 NUMBER',//9X,'I',9X,'J',9X,'K',9X,'N',/)
  200 FORMAT(//5X,' ROW  ',I5)
  225 FORMAT(///,2(18X,' NODE',I4))
  250 FORMAT(/9D13.4)
  275 FORMAT(/4F30.12)
  300 FORMAT(///15X,' STRAIN ENERGY =',F20.15,' KIP-IN',///)
  325 FORMAT(////   20X,'PROPERTIES OF THE STRUCTURE',/20X,'***********
     1***************',///15X,'NUMBER OF ELEMENTS =',I5,/15X,'NUMBER OF 
     2NODES =',I5,/15X,'YOUNG S MODULUS =',F6.1,'KIP/SQ.IN.',/15X,'POISS
     3ON S RATIO =',F5.2,/15X,'THICKNESS OF THE STRUCTURE =',F5.2,'IN',
     4/15X,'LOAD ALONG THE RIDGE =',F7.3,'KIP/FT.')
  350 FORMAT(/I18,F20.2,F15.2)
  375 FORMAT(1H1,//30X,'RESULTS',/30X,'*******',//10X,'NOTE.THE GENERALI
     1ZED DISPLACEMENTS ARE AS FOLLOWS .',//10X,'U, UX, UY, V, VX, VY, W
     2, WX, WY, WXX, WXY AND WYY')
      RETURN
      END
```

```
      SUBROUTINE ABC(N1,N2,N3,X,Y,NOD,A,B,C,COSIN,SINUS)
C**********************************************************************
C
C
C             THIS SUBROUTINE CALCULATES THE ELEMENT DIMENSIONS
C             AND THE SINE AND COSIN OF THE IN-PLANE ANGLE.
C
C
C**********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(NOD),Y(NOD)
      Y1=Y(N1)
      X2=X(N2)
      X3=X(N3)
      X1=X(N1)
      Y2=Y(N2)
      Y3=Y(N3)
      RR=(X2-X1)**2+(Y2-Y1)**2
      RR=DSQRT(RR)
      A=((X2-X3)*(X2-X1)+(Y2-Y3)*(Y2-Y1))/RR
      B=((X3-X1)*(X2-X1)+(Y3-Y1)*(Y2-Y1))/RR
      C=((X2-X1)*(Y3-Y1)-(X3-X1)*(Y2-Y1))/RR
      COSIN=(X2-X1)/RR
      SINUS=(Y2-Y1)/RR
      RETURN
      END
```

```
      SUBROUTINE BENDEG(ANU,A,B,C,YM,T,TH,COSIN,SINUS)
C*************************************************************************
C
CC
C         THIS SUBROUTINE GIVES THE BENDING ELEMENT STIFFNESS MATRIX.
C
C
C*************************************************************************
C
C         THE DEGREES OF FREEDOM USED IN THIS SUBROUTINE ARE-
C         W-WX-WY-WXX-WXY-WYY-******************
C
C*************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON BTT(20,20),BR(18,18),SB(2,2),SC(18,2),SD(2,18),S(18,18),SF(
     118,2),BT(20,20),SH(20,20),SK(20,20),TKR(20,20),SM(20),SN(20),SO(2
     22),SP(20)
      DO 1 I=1,20
      SM(I)=0.D0
      SN(I)=0.D0
      DO 1 J=1,20
      BT(I,J)=0.D0
      SK(I,J)=0.D0
      BTT(I,J)=0.D0
      TKR(I,J)=0.D0
    1 CONTINUE
      DO 5 I=1,18
      DO 5 J=1,18
      S(I,J)=0.D0
      BR(I,J)=0.D0
    5 CONTINUE
      SM(2)=1.D0
      SM(4)=2.D0
      SM(5)=1.D0
      SM(7)=3.D0
      SM(8)=2.D0
      SM(9)=1.D0
      SM(11)=4.D0
      SM(12)=3.D0
      SM(13)=2.D0
      SM(14)=1.D0
      SM(16)=5.D0
      SM(17)=3.D0
```

```
      SM(18)=2.DO
      SM(19)=1.DO
      SN(3)=1.DO
      SN(5)=1.DO
      SN(6)=2.DO
      SN(8)=1.DO
      SN(9)=2.DO
      SN(10)=3.DO
      SN(12) =1.DO
      SN(13)=2.DO
      SN(14)=3.DO
      SN(15)=4.DO
      SN(17)=2.DO
      SN(18)=3.DO
      SN(19)=4.DO
      SN(20)=5.DO
      CALL PBTRAG(A,B,C,BT)
      CALL PBROTG(COSIN,SINUS,BR)
      DO 10 I=1,20
      DO 10 J=1,20
      A1=SM(I)+SM(J)-4.DO
      B1=SN(I)+SN(J)
      F1=0.DO
      CALL FACTRF(A,B,C,A1,B1,F1)
      A2=SM(I)+SM(J)
      B2=SN(I)+SN(J)-4.DO
      F2=0.DO
      CALL FACTRF(A,B,C,A2,B2,F2)
      F3=0.DO
      A3=SM(I)+SM(J)-2.DO
      B3=SN(I)+SN(J)-2.DO
      CALL FACTRF(A,B,C,A3,B3,F3)
   10 SK(I,J)=SM(I)*SM(J)*(SM(I)-1.DO)*(SM   (J)-1.DO)*F1+SN(I)*SN(J)*
     &(SN(I)-1.DO)*(SN(J)-1.DO)*F2+(2.DO*(1.DO-ANU)*SM(I)*SM(J)*SN(I)*SN
     &(J)+(ANU*SM(I)*SN(J)*(SM(I)-1.DO)*(SN(J)-1.DO))+ ANU*SM(J)*SN(I)*
     &(SM(J)-1.DO)*(SN(I)-1.DO))*F3
      CALL MATIN(BT,20)
      DO 15 I=1,20
      DO 15 J=1,18
      BTT(I,J)=0.DO
      DO 15 L=1,18
   15 BTT(I,J)=BTT(I,J)+BT(I,L)*BR(L,J)
      DO 20 I=1,20
```

```
      DO 20 J=1,18
      DO 20 L=1,20
   20 TKR(I,J)=TKR(I,J)+SK(I,L)*BTT(L,J)
      DO 25 I=1,18
      DO 25 J=1,20
      SK(I,J)=0.D0
   25 SK(I,J)=BT(J,I)
      DO 30 I=1,18
      DO 30 J=1,18
      BTT(I,J)=0.D0
      DO 30 L=1,20
   30 BTT(I,J)=BTT(I,J)+SK(I,L)*TKR(L,J)
      DO 35 I=1,18
      DO 35 J=1,18
      SK(I,J)=0.D0
   35 SK(I,J)=BR(J,I)
      DO 40 I=1,18
      DO 40 J=1,18
      TKR(I,J)=0.D0
      DO 40 L=1,18
   40 TKR(I,J)=TKR(I,J)+SK(I,L)*BTT(L,J)
      ZZZ=           T*T*  YM*T/(12.D0*(1.D0-ANU**2))
      DO 45 I=1,18
      DO 45 J=1,18
   45 S(I,J)=TKR(I,J)*ZZZ
      RETURN
      END
```

```
LEVEL  19                    PBROTG               DATE = 71205          18/46/12
      SUBROUTINE PBROTG(C,S,R)
C***************************************************************************
C
C
C         THIS SUBROUTINE GIVES THE ROTATION MATRIX FOR THE OUT-PLANE
C         DISPLACEMENTS.............
C
C
C***************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  R(18,18)
      DO 1 I=1,18
      DO 1 J=1,18
    1 R(I,J)=0.D0
      R(1,1)=1.D0
      R(2,2)=C
      R(2,3)=S
      R(3,2)=-S
      R(3,3)=C
      R(4,4)=C*C
      R(4,5)=2.D0*C*S
      R(4,6)=S*S
      R(5,4)=-C*S
      R(5,5)=C*C-S*S
      R(5,6)=S*C
      R(6,4)=S*S
      R(6,5)=-2.D0*S*C
      R(6,6)=C*C
      DO 2 I=7,12
      DO 2 J=7,12
      II=I-6
      JJ=J-6
    2 R(I,J)=R(II,JJ)
      DO 3 I=13,18
      DO 3 J=13,18
      II=I-6
      JJ=J-6
    3 R(I,J)=R(II,JJ)
      RETURN
      END
```

```
      SUBROUTINE PBTRAG(A,B,C,T)
C**********************************************************************
C
C
C         THIS SUBROUTINE GIVES THE TRANSFORMATION MATRIX FOR THE
C         OUTPLANE DISPLACEMENTS.......
C
C
C**********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  T(20,20)
      DO 1 I=1,20
      DO 1 J=1,20
    1 T(I,J)=0.D0
      T(1,1)=1.D0
      T(1,2)=-B
      T(1,4)=B*B
      T(1,7)=-B**3
      T(1,11)=B**4
      T(1,16)=-(B**5)
      T(2,2)=1.D0
      T(2,4)=-2.D0*B
      T(2,7)=3.D0*B*B
      T(2,11)=-4.D0*B**3
      T(2,16)=5.D0*B**4
      T(3,3)=1.D0
      T(3,5)=-B
      T(3,8)=B*B
      T(3,12)=-(B**3)
      T(4,4)=2.D0
      T(4,7)=-6.D0*B
      T(4,11)=12.D0*B**2
      T(4,16)=-20.D0*B**3
      T(5,5)=1.D0
      T(5,8)=-2.D0*B
      T(5,12)=3.D0*B*B
      T(6,6)=2.D0
      T(6,9)=-2.D0*B
      T(6,13)=2.D0*B*B
      T(6,17)=-2.D0*B**3
      T(7,1)=1.D0
      T(7,2)=A
      T(7,4)=A*A
```

```
      T(7,7)=A**3
      T(7,11)=A**4
      T(7,16)=A**5
      T(8,2)=1.D0
      T(8,4)=2.D0*A
      T(8,7)=3.D0*A*A
      T(8,11)=4.D0*A**3
      T(8,16)=5.D0*A**4
      T(9,3)=1.D0
      T(9,5)=A
      T(9,8)=A*A
      T(9,12)=A**3
      T(10,4)=2.D0
      T(10,7)=6.D0*A
      T(10,11)=12.D0*A*A
      T(10,16)=20.D0*A**3
      T(11,5)=1.D0
      T(11,8)=2.D0*A
      T(11,12)=3.D0*A*A
      T(12,6)=2.D0
      T(12,9)=2.D0*A
      T(12,13)=2.D0*A*A
      T(12,17)=2.D0*A**3
      T(13,1)=1.D0
      T(13,3)=C
      T(13,6)=C*C
      T(13,10)=C**3
      T(13,15)=C**4
      T(13,20)=C**5
      T(14,2)=1.D0
      T(14,5)=C
      T(14,9)=C*C
      T(14,14)=C**3
      T(14,19)=C**4
      T(15,3)=1.D0
      T(15,6)=2.D0*C
      T(15,10)=3.D0*C*C
      T(15,15)=4.D0*C**3
      T(15,20)=5.D0*C**4
      T(16,4)=2.D0
      T(16,8)=2.D0*C
      T(16,13)=2.D0*C*C
      T(16,18)=2.D0*C**3
```

```
      T(17,5)=1.D0
      T(17,9)=2.D0*C
      T(17,14)=3.D0*C*C
      T(17,19)=4.D0*C**3
      T(18,6)=2.D0
      T(18,10)=6.D0*C
      T(18,15)=12.D0*C*C
      T(18,20)=20.D0*C**3
      T(19,16)=5.D0*A**4*C
      T(19,17)=(3.D0*A*A*C**3)-(2.D0*A**4*C)
      T(19,18)=(-2.D0*A*C**4)+(3.D0*A**3*C*C)
      T(19,19)=C**5-4.D0*A*A*C**3
      T(19,20)=5.D0*A*C**4
      T(20,16)=5.D0*B**4*C
      T(20,17)=(3.D0*B*B*C**3)-(2.D0*B**4*C)
      T(20,18)=(2.D0*B*C**4)-(3.D0*B**3*C*C)
      T(20,19)=C**5-4.D0*B*B*C**3
      T(20,20)=-5.D0*B*C**4
      RETURN
      END
```

```
LEVEL  19                    FACTRF              DATE = 71205          18/46/12

      SUBROUTINE FACTRF(A,B,C,SM,SN,F)                                  00006060
C***********************************************************************00006070
C                                                                       00006080
C                                                                       00006090
C          THIS SUBROUTINE CALCULATES THE FACTOR -F- IN THE STIFFNESS   00006100
C          MATRIX FORMULA....                                           00006110
C                                                                       00006120
C                                                                       00006130
C***********************************************************************00006140
      IMPLICIT REAL *8(A-H,O-Z)                                         00006150
      SUMM=1.D0                                                         00006160
      SUMN=1.D0                                                         00006170
      IF(SM.LT.0.0.OR.SN.LT.0.0) GO TO 9                                00006180
      M=SM                                                              00006190
      N=SN                                                              00006200
      IF(M.EQ.0) GO TO 2                                                00006210
      DO 1 I=1,M                                                        00006220
    1 SUMM=SUMM*I                                                       00006230
      IF(M.GT.0) GO TO 3                                                00006240
    2 SUMM=1.D0                                                         00006250
    3 IF(N.EQ.0) GO TO 5                                                00006260
      DO 4 I=1,N                                                        00006270
    4 SUMN=SUMN*I                                                       00006280
      IF(N.GT.0) GO TO 6                                                00006290
    5 SUMN=1.D0                                                         00006300
    6 CONTINUE                                                          00006310
      K=M+N+2                                                           00006320
      SUMK=1.D0                                                         00006330
      DO 7 I=1,K                                                        00006340
    7 SUMK=SUMK*I                                                       00006350
      N1=N+1                                                            00006360
      M1=M+1                                                            00006370
      F=C**N1*(A**M1-(-B)**M1)*SUMN*SUMM/SUMK                           00006380
      IF(SM)9,8,8                                                       00006390
    8 IF(SN)9,10,10                                                     00006400
    9 F=0.D0                                                            00006410
   10 CONTINUE                                                          00006420
      RETURN                                                            00006430
      END                                                               00006440
```

```
LEVEL  19                    SINPLG              DATE = 71205          18/46/12
      SUBROUTINE SINPLG(A,B,C,ANU)
C***********************************************************************
C
C
C              THIS SUBROUTINE CALCULATE THE IN-PLANE ELEMENT
C              STIFFNESS MATRIX IN THE LOCAL SYSTEM.
C
C
C***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(20,20),S1(18,18),SC(2,2),SOC(18,2),SCT(2,18),SE(18,18),
     1SOCM(18,2),TR(20,20),AU(20,20),R(20,20),S(20,20),SM(20),SN(20),P(20
     20),Q(20),SO(18,18)
      DO 1 I=1,20
      P(I)=0.D0
      Q(I)=0.D0
      SM(I)=0.D0
      SN(I)=0.D0
      DO 1 J=1,20
      R(I,J)=0.D0
      TR(I,J)=0.D0
      AU(I,J)=0.D0
      T(I,J)=0.D0
      S(I,J)=0.D0
    1 CONTINUE
      DO 5 I=1,2
      DO 5 J=1,2
      SC(I,J)=0.D0
      DO 5 L=1,18
      SOC(L,J)=0.D0
      SOCM(L,J)=0.D0
      SCT(J,L)=0.D0
      DO 5 K=1,18
      S1(L,K)=0.D0
      SO(L,K)=0.D0
    5 CONTINUE
       SM(2)=1.D0
      SM(4)=2.D0
      SM(5)=1.D0
      SM(7)=3.D0
      SM(8)=2.D0
      SM(9)=1.D0
      SN(3)=1.D0
```

```
      SN(5)=1.DO
      SN(6)=2.DO
      SN(8)=1.DO
      SN(9)=2.DO
      SN(10)=3.DO
      F1=0.DO
      F2=0.DO
      F3=0.DO
      F4=0.DO
      F5=0.DO
      F6=0.DO
      DO 10 I=11,20
      J=I-10
      P(I)=SM(J)
      Q(I)=SN(J)
   10 CONTINUE
      DO 15 I=1,20
      DO 15 J=1,20
      SM1=SM(I)+SM(J)-2.DO
      SN1=SN(I)+SN(J)
      CALL FACTRF(A,B,C,SM1,SN1,F1)
      P1=P(I)+P(J)
      Q1=Q(I)+Q(J)-2.DO
      CALL FACTRF(A,B,C,P1,Q1,F2)
      SM2=SM(I)+SM(J)
      SN2=SN(I)+SN(J)-2.DO
      CALL FACTRF(A,B,C,SM2,SN2,F3)
      P2=P(I)+P(J)-2.DO
      Q2=Q(I)+Q(J)
      CALL FACTRF(A,B,C,P2,Q2,F4)
      SMP=SM(J)+P(I)-1.DO
      SNQ=SN(J)+Q(I)-1.DO
      CALL FACTRF(A,B,C,SMP,SNQ,F5)
      SMP1=SM(I)+P(J)-1.DO
      SNQ1=SN(I)+Q(J)-1.DO
      CALL FACTRF(A,B,C,SMP1,SNQ1,F6)
      S2=(1.DO-ANU)*0.5DO*((SN(I)*SN(J)*F3)+(P(I)*P(J)*F4))
      SF=(SM(I)*SM(J)*F1)+(Q(I)*Q(J)*F2)
      S3=((0.5DO*(1.DO-ANU)*SN(J)*P(I))+(ANU*SM(J)*Q(I)))*F5
      S4=((0.5DO*(1.DO-ANU)*SN(I)*P(J))+(ANU*SM(I)*Q(J)))*F6
   15 S(I,J)=SF+S2+S3+S4
      RETURN
      END
```

```
      SUBROUTINE  PSESGH(ANU,A,B,C,TH,THETA,YM,COSIN,SINUS)
C***********************************************************************
C
C
C        THIS SUBROUTINE GIVES THE IN-PLANE ELEMENT STIFFNESS MATRIX
C
C
C***********************************************************************
C
C        THE DEGREES OF FREEDOM USED IN THIS SUBROUTINE ARE-
C        U-V-UX-VX-UY-VY...............
C
C***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(20,20),S1(18,18),SC(2,2),SOC(18,2),SCT(2,18),SE(18,18),
     1SOCM(18,2),TR(20,20),AU(20,20),R(20,20),S(20,20),SM(20),SN(20),P(2
     20),Q(20),SO(18,18)
      CALL SINPLG(A,B,C,ANU)
      CALL TRANGH(A,B,C,T)
      CALL COORGH(COSIN,SINUS,R)
      CALL MATIN(T,20)
      CALL MATMGH(T,R,TR,20,20,20)
      CALL MATMGH(S,TR,AU,20,20,20)
      DO 20 I=1,20
      DO 20 J=1,20
      TR(I,J)=0.D0
   20 S(I,J)=0.D0
      CALL TRANSP(T,20,20,S)
      CALL MATMGH(S,AU,TR,20,20,20)
      DO 25 I=1,20
      DO 25 J=1,20
   25 T(I,J)=0.D0
      CALL TRANSP(R,20,20,T)
      DO 30 I=1,20
      DO 30 J=1,20
   30 S(I,J)=0.D0
      CALL MATMGH (T,TR,S,20,20,20)
      DO 35 I=1,18
      DO 35 J=1,18
      SO(I,J)=S(I,J)
      DO 35 L=1,2
      L1=L+18
      SOC(I,L)=S(I,L1)
```

```
      DO 35 K=1,2
      K1=K+18
   35 SC(L,K)=S(L1,K1)
      CALL MATIN(SC,2)
      CALL MATMGH(SOC,SC,SOCM,18,2,2)
      CALL TRANSP(SOC,18,2,SCT)
      CALL MATMGH(SOCM,SCT,S1,18,2,18)
      DO 40 I=1,18
      DO 40 J=1,18
   40 SO(I,J)=SC(I,J)-S1(I,J)
      DO 45 I=1,18
      DO 45 J=1,18
   45 SO(I,J)=SO(I,J)*YM*TH/(1.DO-ANU**2)
      RETURN
      END
```

LEVEL 19 TRANGH DATE = 71205 18/46/12

```
      SUBROUTINE TRANGH(A,B,C,T)
C***********************************************************************
C
C
C         THIS SUBROUTINE GIVES THE TRANSFORMATION MATRIX FOR THE
C         IN-PLANE DISPLACEMENTS........
C
C
C***********************************************************************
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION T(20,20)
      DO 1 I=1,20
      DO 1 J=1,20
    1 T(I,J)=0.D0
      T(1,1)=1.D0
      T(1,2)=-B
      T(1,4)=B*B
      T(1,7)=(-B)**3
      T(2,2)=1.D0
      T(2,4)=-2.D0*B
      T(2,7)=3.D0*B**2
      T(3,3)=1.D0
      T(3,5)=-B
      T(3,8)=B**2
      DO 2 I=4,6
      DO 2 J=11,20
      II=I-3
      JJ=J-10
    2 T(I,J)=T(II,JJ)
      T(7,1)=1.D0
      T(7,2)=A
      T(7,4)=A**2
      T(7,7)=A**3
      T(8,2)=1.D0
      T(8,4)=2.D0*A
      T(8,7)=3.D0*A**2
      T(9,3)=1.D0
      T(9,5)=A
      T(9,8)=A**2
      DO 3 I=10,12
      DO 3 J=11,20
      II=I-3
      JJ=J-10
```

```
   3    T(I,J)=T(II,JJ)
        T(13,1)=1.D0
        T(13,3)=C
        T(13,6)=C**2
        T(13,10)=C**3
        T(14,2)=1.D0
        T(14,5)=C
        T(14,9)=C**2
        T(15,3)=1.D0
        T(15,6)=2.D0*C
        T(15,10)=3.D0*C**2
        DO 4 I=16,18
        DO 4 J=11,20
        II=I-3
        JJ=J-10
   4    T(I,J)=T(II,JJ)
        T(19,1)=1.D0
        T(19,2)=(A-B)/3.D0
        T(19,3)=C/3.D0
        T(19,4)=(A-B)**2/9.D0
        T(19,5)=(A-B)*C/9.D0
        T(19,6)=C*C/9.D0
        T(19,7)=(A-B)**3/27.D0
        T(19,8)=(A-B)**2*C/27.D0
        T(19,9)=(A-B)*C*C/27.D0
        T(19,10)=C**3/27.D0
        DO 5 J=11,20
        JJ=J-10
        T(20,J)=T(19,JJ)
   5    CONTINUE
        RETURN
        END
```

```
      SUBROUTINE COORGH(C,S,R)
C***********************************************************************
C
C
C           THIS SUBROUTINE FORMS THE IN-PLANE ROTATION MATRIX....
C
C
C***********************************************************************
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION  R(20,20)
      DO 1 I=1,20
      DO 1 J=1,20
    1 R(I,J)=0.D0
      R(1,1)=C
      R(1,4)=S
      R(2,2)=C*C
      R(2,3)=C*S
      R(2,5)=C*S
      R(2,6)=S*S
      R(3,2)=-S*C
      R(3,3)=C*C
      R(3,5)=-S*S
      R(3,6)=C*S
      R(4,1)=-S
      R(4,4)=C
      R(5,2)=-C*S
      R(5,3)=-S*S
      R(5,5)=C*C
      R(5,6)=C*S
      R(6,2)=S*S
      R(6,3)=-S*C
      R(6,5)=-S*C
      R(6,6)=C*C
      DO 2 I=7,12
      DO 2 J=7,12
      II=I-6
      JJ=J-6
    2 R(I,J)=R(II,JJ)
      DO 3 I=13,18
      DO 3 J=13,18
      II=I-6
      JJ=J-6
    3 R(I,J)=R(II,JJ)


      R(19,19)=C
      R(19,20)=S
      R(20,19)=-S
      R(20,20)=C
      RETURN
      END
```

```
LEVEL  19                    MATIN                  DATE = 71205          18/46/12
      SUBROUTINE MATIN(A,N)
C*********************************************************************
C
C
C         THIS IS THE INVERSION SUBROUTINE
C
C
C*********************************************************************
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION  A(N,N),INDEX(48,2)
      DO 108 I=1,N
 108  INDEX(I,1)=0
      II=0
 109  AMAX=-1.0
      DO 110 I=1,N
      IF(INDEX(I,1))110,111,110
 111  DO 112 J=1,N
      IF(INDEX(J,1))112,113,112
 113  TEMP=DABS(A(I,J))
      IF(TEMP-AMAX)112,112,114
 114  IROW=I
       ICOL=J
      AMAX=TEMP
 112  CONTINUE
 110  CONTINUE
      IF(AMAX)225,115,116
 116  INDEX(ICOL,1)=IROW
       IF(IROW-ICOL)119,118,119
 119  DO 120 J=1,N
      TEMP=A(IROW,J)
      A(IROW,J)=A(ICOL,J)
 120  A(ICOL,J)=TEMP
      II=II+1
      INDEX(II,2)=ICOL
 118  PIVOT=A(ICOL,ICOL)
      A(ICOL,ICOL)=1.
      PIVOT=1./PIVOT
      DO 121 J=1,N
 121  A(ICOL,J)=A(ICOL,J)*PIVOT
      DO 122 I=1,N
      IF(I-ICOL)123,122,123
 123  TEMP=A(I,ICOL)
      A(I,ICOL)=0.
```

```
LEVEL  19                 MATIN               DATE = 71205          18/46/12
        DO 124 J=1,N
  124   A(I,J)=A(I,J)-A(ICOL,J)*TEMP
  122   CONTINUE
        GO TO 109
  125   ICOL=INDEX(II,2)
        IROW=INDEX(ICOL,1)
        DO 126 I=1,N
        TEMP=A(I,IROW)
        A(I,IROW)=A(I,ICOL)
  126   A(I,ICOL)=TEMP
        II=II-1
  225   IF(II)125,127,125
  115   WRITE(6,150)
  150   FORMAT(1H0,2X,11H ZERO PIVOT)
  127   CONTINUE
        RETURN
        END
```

```
LEVEL  19                    MATMGH               DATE = 71205          18/46/12
      SUBROUTINE MATMGH(A,B,C,M,N,K)
C*************************************************************************
C
C
C              THIS IS THE MATRIX MULTIPLICATION SUBROUTINE.
C
C
C*************************************************************************
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION A(M,N) ,B(N,K) ,C(M,K)
      DO 2 I=1,M
      DO 2 J=1,K
      C(I,J)=0.D0
      DO 1 L=1,N
    1 C(I,J) =C(I,J)+A(I,L)*B(L,J)
    2 CONTINUE
      RETURN
      END
```

LEVEL 19 TRANSP DATE = 71205 18/46/12

```
      SUBROUTINE TRANSP(O,M,N,T)
C**********************************************************************
C
C
C                THIS SUBROUTINE GIVES THE TRANSPOSE OF THE MATRICES.
C
C
C**********************************************************************
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION O(M,N), T(N,M)
      DO 1 I=1,N
      DO 1 J=1,M
    1 T(I,J)=0.D0
      DO 2 I=1,M
      DO 2 J=1,N
    2 T(J,I)=O(I,J)
      RETURN
      END
```

LEVEL 19 ROTGHA DATE = 71205 18/46/12

```
      SUBROUTINE ROTGHA(C,B,N)
C***********************************************************************
C
C
C               THIS SUBROUTINE FORMS THE APPROPRIATE 36 BY 36
C               ELEMENT ROTATION MATRIX BY PLACING THE 12 BY 12
C               GLOBAL ROTATION MATRIX IN THE CORRESPONDING ROWS
C               AND COLUMNS OF THE NODES LYING ON THE RIDGES.
C
C
C***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(12,12),C(36,36)
      DO 1 I=1,36
      DO 1 J=1,36
      C(I,J)=0.D0
    1 CONTINUE
      DO 10 I=1,36
   10 C(I,I)=1.D0
      IF(N.EQ.0) GO TO 19
      K=1
      GO TO (11,12,13,11,11,12),N
   11 M=0
      GO TO (15,12,13,15,15,12) ,N
   12 M=12
      GO TO (15,15,13,15,15,15),N
   13 M=24
   15 CONTINUE
      IF(K.GT.2) GO TO 19
      DO 16 I=1,12
      DO 16 J=1,12
   16 C(I+M,J+M)=B(I,J)
      GO TO (19,19,19,17,18,17),N
   17 M=M+12
      K=K+1
      GO TO 15
   18 M=M+24
      K=K+1
      GO TO 15
   19 CONTINUE
      RETURN
      END
```

```
LEVEL  19                    GLOROT              DATE = 71205          18/46/12
      SUBROUTINE  GLOROT(B,R)
C*************************************************************************
C
C
C           THIS SUBROUTINE FORMS THE 12 BY 12 GLOBAL ROTATION MATRIX.
C
C
C*************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION R(12,12)
      A=B*3.141592653589793/180.D0
      C=DCOS(A)
      S=DSIN(A)
      R(1,1)=1.D0
      R(2,2)=1.D0
      R(3,3)=1.D0
      R(4,4)=C
      R(4,7)=-S
      R(5,5)=C
      R(5,8)=-S
      R(6,6)=1.D0
      R(7,4)=S
      R(7,7)=C
      R(8,5)=S
      R(8,8)=C
      R(9,9)=1.D0
      R(10,10)=C
      R(11,11)=1.D0
      R(12,12)=1.D0
      RETURN
      END
```

```
LEVEL  19                    MAIN                  DATE = 71205          18/46/12
C***********************************************************************
C
C
C             THIS SUBROUTINE ASSEMBLES THE ELEMENT STIFFNESS MATRICES
C             TO FORM THE TOTAL STIFFNESS MATRIX.
C
C
C***********************************************************************
      SUBROUTINE SETUP(I,J,K,B,M1,N1,M2,N2,A)
      REAL*8 A(M1,N1),  B(M2,N2)
      DO 10 M=1,12
      I1=12*I-12+M
      J1=12*J-12+M
      K1=12*K-12+M
      DO 10 N=1,12
      I2=12*I-12+N
      J2=12*J-12+N
      K2=12*K-12+N
      A(I1,I2)=A(I1,I2)+B(M,N)
      A(I1,J2)=A(I1,J2)+B(M,N+12)
      A(I1,K2)=A(I1,K2)+B(M,N+24)
      A(J1,I2)=A(J1,I2)+B(M+12,N)
      A(J1,J2)=A(J1,J2)+B(M+12,N+12)
      A(J1,K2)=A(J1,K2)+B(M+12,N+24)
      A(K1,I2)=A(K1,I2)+B(M+24,N)
      A(K1,J2)=A(K1,J2)+B(M+24,N+12)
   10 A(K1,K2)=A(K1,K2)+B(M+24,N+24)
      RETURN
      END
```

```
LEVEL  19                    COMFOC              DATE = 71205          18/46/12
      SUBROUTINE  COMFOC(A,N,N1,N2,F,M1)
C*****************************************************************************
C
C
C                 THIS SUBROUTINE IS USED TO APPLY THE CONTINUITY
C                 CONDITIONS BETWEEN THE ELEMENTS ON OPPOSITE
C                 SIDES OF THE RIDGE.IT IS USED ONLY WHEN SOLVING
C                 THE WHOLE STRUCTURE.
C
C
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(N,N) ,F(N)
      DO 3 I=1,12
      IF(I.EQ.10.OR.I.EQ.7) GO TO 3
      J1=12*N2-I+1
      J2=12*N1-I+1
      F(J2)=F(J2)+F(J1)
      F(J1)=0.D0
      DO 1 L=1,M1
      A(J2,L)=A(J2,L)+A(J1,L)
    1 A(J1,L)=0.D0
      DO 2 L=1,M1
      A(L,J2)=A(L,J2)+A(L,J1)
    2 A(L,J1)=0.D0
      A(J1,J1)=1.D0
    3 CONTINUE
      RETURN
      END
```

```
LEVEL  19                    LOADVE              DATE = 71205          18/46/12
      SUBROUTINE LOADVE(P,N,NN,F,A)
C*************************************************************************
C
C
C               THIS SUBROUTINE FORMS THE CONSISTENT LOAD VECTOR.
C
C
C*************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  F(N),L(3)
      DO 1 I=1,N
    1 F(I)=0.D0
      K=1
C*************************************************************************
C
C          NMR  -THE NUMBER OF ELEMENTS LYING ON THE RIDGES.
C
      READ(1,50) NMR
C
C*************************************************************************
    2 CONTINUE
      IF(K.GT.NMR) GO TO 5
      DO 3 I=1,NN
    3 L(NN)=0
C*************************************************************************
C
C          SL    -LENGTH OF THE ELEMENT EDGE LYING ON THE RIDGE.
C
C          L     -ARRAY CONTAINING THE NODES NUMBER WHICH ARE LYING
C                 ON THE RIDGES.
C
      READ(1,100) SL,(L(I),I=1,NN)
C
C*************************************************************************
      SL=SL*12.D0
      B= A*3.141592653589793/180.D0
      C=DCOS(B)
      S=DSIN(B)
      NN1=NN-1
      DO 4 I=1,NN1
      N1=L(I)
      N2=L(I+1)
      I1=N1*12-5
```

```
      I2=I1+1
      I3=I1+3
      I4=I1-3
      I5=I4+1
      J1= N2*12-5
      J2=J1+1
      J3=J1+3
      J4=J1-3
      J5=J4+1
C**********************************************************************
C
      S1=P*SL*C.5D0
      S2=P*SL*SL*(1.D0/12.D0+C*C/60.D0)
      S3=0.D0
      S4=0.D0
      S5= -P*SL*SL*S*C/60.D0
C
C              THE S1,S2,...S5 SHOULD BE CHANGE FOR THE FULLY
C              CONSISTENT LOAD VECTOR.AS INDICATED IN
C              THE TEXT.
C
C**********************************************************************
      F(I1)= F(I1)+S1
      F(I2)=F(I2)+S2
      F(I3)=F(I3)+S3
      F(I4)=F(I4)+S4
      F(I5)= F(I5)+S5
      F(J1)=F(J1)+S1
      F(J2)=F(J2)-S2
      F(J3)=F(J3)+S3
      F(J4)=F(J4)+S4
      F(J5)=F(J5)-S5
    4 CONTINUE
      K=K+1
      GO TO 2
    5 CONTINUE
   50 FORMAT(I5)
  100 FORMAT(F10.5,10I5)
      RETURN
      END
```

```
LEVEL  19                    DISPGH              DATE = 71205          18/46/12
      SUBROUTINE DISPGH(X,Y,BD,PD,D)
C***********************************************************************
C
C
C               THIS SUBROUTINE CALCULATES THE DISPLACEMENTS AT ANY POINT
C               WITHIN THE ELEMENT.
C
C
C***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION BD(20),PD(36),D(12)
      DO 1 I=1,12
    1 D(I)=0.D0
      X2=X*X
      X3=X2*X
      X4=X3*X
      X5=X4*X
      Y2=Y*Y
      Y3=Y2*Y
      Y4=Y3*Y
      Y5=Y4*Y
      D(1)=PD(1)+PD(2)*X+PD(3)*Y+PD(4)*X2+PD(5)*X*Y+PD(6)*Y2+PD(7)*X3+PD
     1(8)*X2*Y+PD(9)*X*Y2+PD(10)*Y3
      D(2)=PD(2)+2.D0*PD(4)*X+PD(5)*Y+3.D0*PD(7)*X2+2.D0*PD(8)*X*Y+PD(9)
     1*Y2
      D(3)=PD(3)+PD(5)*X+2.D0*PD(6)*Y+PD(8)*X2+2.D0*PD(9)*X*Y+3.D0*PD(10
     1)*Y2
      D(4)=PD(11)+PD(12)*X+PD(13)*Y+PD(14)*X2+PD(15)*X*Y+PD(16)*Y2+PD(17
     1)*X3+PD(18)*X2*Y+PD(19)*X*Y2+PD(20)*Y3
      D(5)=PD(12)+2.D0*PD(14)*X+PD(15)*Y+3.D0*PD(17)*X2+2.D0*PD(18)*X*Y+
     1PD(19)*Y2
      D(6)=PD(13)+PD(15)*X+2.D0*PD(16)*Y+PD(18)*X2+2.D0*PD(19)*X*Y+3.D0*
     1PD(20)*Y2
      D(7)=BD(1)+BD(2)*X+BD(3)*Y+BD(4)*X2+BD(5)*X*Y+BD(6)*Y2+BD(7)*X3+BD
     1(8)*X2*Y+ BD(9)*X*Y2+BD(10)*Y3+BD(11)*X4+BD(12)*X3*Y+BD(13)*X2*Y2+
     2BD(14)*X*Y3+BD(15)*Y4+BD(16)*X5+BD(17)*X3*Y2+BD(18)*X2*Y3+BD(19)*X
     3*Y4+BD(20)*Y5
      D(8)=BD(2)+2.D0*BD(4)*X+BD(5)*Y+3.D0*BD(7)*X2+2.D0*BD(8)*X*Y+BD(9)
     1*Y2+4.D0*BD(11)*X3+3.D0*BD(12)*X2*Y+2.D0*BD(13)*X*Y2+BD(14)*Y3+5.D
     20*BD(16)*X4+3.D0*BD(17)*X2*Y2+2.D0*BD(18)*X*Y3+BD(19)*Y4
      D(9)=BD(3)+BD(5)*X+2.D0*BD(6)*Y+BD(8)*X2+2.D0*BD(9)*X*Y+3.D0*BD(10
     1)*Y2+BD(12)*X3+2.D0*BD(13)*X2*Y+3.D0*BD(14)*X*Y2+4.D0*BD(15)*Y3+2.
     2D0*BD(17)*X3*Y+3.D0*BD(18)*X2*Y2+4.D0*BD(19)*X*Y3+5.D0*BD(20)*Y4
```

```
      D(10)=2.D0*BD(4)+6.D0*BD(7)*X+2.D0*BD(8)*Y+12.D0*BD(11)*X2+6.D0*BD
     1(12)*X*Y+2.D0*BD(13)*Y2+20.D0*BD(16)*X3+6.D0*BD(17)*X*Y2+2.D0*BD(1
     28)*Y3
      D(11)=BD(5)+2.D0*BD(8)*X+2.D0*BD(9)*Y+3.D0*BD(12)*X2+4.D0*BD(13)*X
     1*Y+3.D0*BD(14)*Y2+6.D0*BD(17)*X2*Y+6.D0*BD(18)*X*Y2+4.D0*BD(19)*Y3
      D(12)=2.D0*BD(6)+2.D0*BD(9)*X+6.D0*BD(10)*Y+2.D0*BD(13)*X2+6.D0*BD
     1(14)*X*Y+12.D0*BD(15)*Y2+2.D0*BD(17)*X3+6.D0*BD(18)*X2*Y+12.D0*BD(
     219)*X*Y2+20.D0*BD(20)*Y3
      WRITE(3,150) (D(I),I=1,12)
      VXX=2.D0*PD(14)+6.D0*PD(17)*X+2.D0*PD(18)*Y
      VXY= PD(15)+2.D0*PD(18)*X+2.D0*PD(19)*Y
      VYY=2.D0*PD(16)+2.D0*PD(19)*X+6.D0*PD(20)*Y
      WRITE(3,200) VXX,VXY,VYY
  100 FORMAT(4F25.15)
  150 FORMAT(3F25.15)
  200 FORMAT(///2X,'VXX=',F20.15,'    VXY=',F20.15,'      VYY=',F20.15)
      RETURN
      END
```

```
      SUBROUTINE STRESS(NM,ANU,YM,TH,PZERO,NOD,M1,GR,SR,X,Y)
C*************************************************************************
C
C
C               THIS SUBROUTINE CALCULATES THE INTERNAL FORCES
C               AND STRESSES FOR GIVEN ELEMENTS.
C
C
C*************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(20,20),S1(18,18), SC(2,2),SOC(18,2),SCT(2,18),SE(18,18),
     &SOCM(18,2),TR(20,20),AU(20,20),R(20,20),S(20,20),SM(20),SN(20),P(2
     &0),Q(20),SO(18,18),DIS(M1)
      DIMENSION X(NOD),Y(NOD)
      DIMENSION SXS(6,6),D1(12),DISP(36),SR(12,12),GR(36,36),D(36)
C*************************************************************************
C
C          NMS  -THE NUMBER OF ELEMENTS WITHIN WHICH WE WANT
C                TO CALCULATE THE STRESSES.
C
      READ(1,400) NMS
C
C*************************************************************************
      KOUNT=1
    1 CONTINUE
      IF(KOUNT.GT.NMS) GO TO 28
      DO 2 I=1,20
      DO 2 J=1,20
    2 R(I,J)=0.D0
      DO 3 I=1,12
      D1(I)=0.D0
      DO 3 J=1,12
    3 SR(I,J)=0.D0
      DO 4 I=1,36
      DISP(I)=0.D0
      D(I)=0.D0
      DO 4 J=1,36
      GR(I,J)=0.D0
    4 CONTINUE
      DO 5 I=1,18
      DO 5 J=1,18
      S1(I,J)=0.D0
    5 CONTINUE
```

```
      READ(1,150) ALPHA
      READ(1,175) N1,N2,N3,N
C*********************************************************************
C
C              CHOOSE THE CORRESPONDING DISPLACEMENTS FOR THE ELEMENT.
C              AND PUT THEM INTO DISP.
C
      DO 6 I=1,12
      I1=12*N1-12+I
      I2=12*N2-12+I
      I3=12*N3-12+I
      DISP(I)=DIS(I1)
      DISP(I+12)=DIS(I2)
    6 DISP(I+24)=DIS(I3)
C
C*********************************************************************
      WRITE(3,100) N1,N2,N3
      CALL      ABC(N1,N2,N3,X,Y,NOD,A,B,C,COSIN,SINUS)
      A=A*12.D0
      B=B*12.D0
      C=C*12.D0
      CALL GLCROT (ALPHA,SR)
      CALL RCTGHA(GR,SR,N)
      DO 7 I=1,36
      DO 7 J=1,36
    7 D(I)=D(I)+GR(I,J)*DISP(J)
      WRITE(3,200)
      DO 8 I=1,6
      DISP(I)=D(I)
      DISP(I+6)=D(I+12)
      DISP(I+12)=D(I+24)
      DISP(I+18)=D(I+6)
      DISP(I+24)=D(I+18)
    8 DISP(I+30)=D(I+30)
      WRITE(3,500)
      DO 9 I=1,18
    9 WRITE(3,125) DISP(I),DISP(I+18)
      CALL PBRCTG(COSIN,SINUS,S1)
      DO 10 I=1,6
      DO 10 J=1,6
      SXS(I,J)=0.D0
   10 SXS(I,J)=S1(I,J)
      DO 11 I=1,18
```

```
      D(I)=0.D0
      DO 11 J=1,18
   11 D(I)=D(I)+S1(I,J)*DISP(J+18)
      WRITE(3,225)
      WRITE(3,350) (D(I),I=1,18)
      CALL PBTRAG(A,B,C,R)
      CALL MATIN(R,20)
      DO 12 I=19,36
   12 D(I)=0.D0
      DO 13 I=1,20
      GR(I,1)=0.D0
      DO 13 J=1,20
   13 GR(I,1)=GR(I,1)+R(I,J)*D(J)
      WRITE(3,450)
      WRITE(3,350) (GR(I,1),I=1,20)
      CALL SINPLG(A,B,C,ANU)
      CALL TRANGH(A,B,C,T)
      CALL COORGH(COSIN,SINUS,R)
      CALL MATIN(T,20)
      CALL MATMGH(T,R,TR,20,20,20)
      CALL MATMGH(S,TR,AU,20,20,20)
      CALL TRANSP(T,20,20,S)
      CALL MATMGH(S,AU,TR,20,20,20)
      CALL TRANSP(R,20,20,AU)
      CALL MATMGH(AU,TR,S,20,20,20)
      DO 14 I=1,18
      DO 14 J=1,18
      SC(I,J)=S(I,J)
      DO 14 L=1,2
      L1=L+18
      SOC(I,L)=S(I,L1)
      DO 14 K=1,2
      K1=K+18
   14 SC(L,K)=S(L1,K1)
      CALL MATIN(SC,2)
      CALL TRANSP(SOC,18,2,SCT)
      DO 15 I=1,2
      D(I)=0.D0
      DO 15 J=1,18
   15 D(I)=D(I)+SCT(I,J)*DISP(J)
      DO 16 I=1,2
      G(I)=0.D0
      DO 16 J=1,2
```

```
 16   Q(I)=Q(I)+SC(I,J)*D(J)
      D(19)=-Q(1)
      D(20)=-Q(2)
      DO 17 I=1,18
 17   D(I)=DISP(I)
      DO 18 I=1,20
      SN(I)=0.D0
      DO 18 J=1,20
 18   SN(I)=SN(I)+R(I,J)*D(J)
      WRITE(3,475)
      WRITE(3,350) (SN(I),I=1,20)
      DO 19 I=1,20
      D(I)=0.D0
      DO 19 J=1,20
 19   D(I)=D(I)+T(I,J)*SN(J)
      WRITE(3,425)
      WRITE(3,350) (D(I),I=1,20)
      LLL=1
      LKL=17
      CALL MATIN(SXS,6)
 20   CONTINUE
      IF(LLL.GT.LKL) GO TO 27
      READ(1,550) XI,YI
      IF(LLL.LT.4) GO TO 21
      YI=((A/12.D0)-XI)*0.5D0
 21   CONTINUE
      XI=XI*12.D0
      YI=YI*12.D0
      WRITE(3,250) XI,YI
      WRITE(3,525)
      DO 22 I=1,20
 22   P(I)=GR(I,1)
      CALL DISPGH(XI,YI,P,D,D1)
      DO 23 I=1,6
 23   SM(I)=D1(I+6)
      DO 24 I=1,6
      SN(I)=0.D0
      DO 24 J=1,6
      Q(J)=D1(J)
 24   SN(I)=SN(I)+R(J,I)*D1(J)
      WRITE(3,275)
      WRITE(3,350) (SM(I),I=1,6)
      WRITE(3,300)
```

```
      WRITE(3,350) (Q(I),I=1,6)
      DO 25 I=1,6
      D1(I)=0.D0
      DO 25 J=1,6
   25 D1(I)=D1(I)+SXS(I,J)*SM(J)
      RIG=YM*TH**3/(12.D0*(1.D0-ANU**2))
      SMXX=RIG*(D1(4)+ANU*D1(6))
      SMYY=RIG*(D1(6)+ANU*D1(4))
      SIGYN=YM*(SN(6)+ANU*SN(2))/(1.D0-ANU**2)-6.D0*SMYY/(TH*TH)
      SIGYP=YM*(SN(6)+ANU*SN(2))/(1.D0-ANU**2)+6.D0*SMYY/(TH*TH)
      SIGXP=YM*(SN(2)+ANU*SN(6))/(1.D0-ANU**2)+6.D0*SMXX/(TH*TH)
      SIGXN=YM*(SN(2)+ANU*SN(6))/(1.D0-ANU**2)-6.D0*SMXX/(TH*TH)
      GS=YM/(2.D0*(1.D0+ANU))
      SHEAR=GS*(SN(3)+SN(5))
      WRITE(3,325)
      WRITE(3,375) SMXX,SMYY,SIGXP,SIGXN,SIGYP,SIGYN,SHEAR
      DO 26 I=1,6
   26 SN(I+6)=D1(I)
      WRITE(3,575)
      WRITE(3,350) (SN(I),I=1,12)
      LLL=LLL+1
      GO TO 20
   27 CONTINUE
      KOUNT=KOUNT+1
      GO TO 1
   28 CONTINUE
  100 FORMAT(2X,'STRESSES IN THE ELEMENT DEFINED BY THE NODES ',I2,', ',
     1I2,' AND',I2,//)
  125 FORMAT(2F25.15)
  150 FORMAT(F10.5)
  175 FORMAT(4I5)
  200 FORMAT(/12X,'GLOBAL DISPLACEMENTS OF THE ELEMENT',//)
  225 FORMAT(////40X,'LOCAL BENDING DISPLACEMENTS',///)
  250 FORMAT(////12X,'DISPLACEMENTS AT X=',F15.10,'   AND     Y=',F15.10,/
     1//)
  275 FORMAT(///40X,' LOCAL BENDING DISPLACEMENTS',///)
  300 FORMAT(///40X,' LOCAL IN-PLANE DISPLACEMENTS',///)
  325 FORMAT(///9X,'SMXX',12X,'SMYY',12X,'SIGXP',10X,'SIGXN',10X,'SIGYP'
     1,10X,'SIGYN',11X,'SHEAR')
  350 FORMAT(4F25.15)
  375 FORMAT(//7D16.6)
  400 FORMAT(I5,4F10.5,I5)
  425 FORMAT(////40X,' IN-PLANE COEFFICIENTS',///)
```

```
  450 FORMAT(////40X,' BENDING COEFFICIENTS',///)                    00014980
  475 FORMAT(////40X,' LOCAL IN-PLANE DISPLACEMENTS',///)            00014990
  500 FORMAT(//5X,'IN-PLANE DISPLACEMENTS',5X,'BENDING DISPLACEMENTS')  00015000
  525 FORMAT(//15X,'NOTE. DISPLACEMENTS AND DISTANCES ARE GIVEN IN INCHE00015010
     1S',/15X,'MOMENTS IN (KIP-IN/IN) AND STRESSES IN (KIP/SQ.IN)',//)  00015020
  550 FORMAT(2F20.15)                                                00015030
  575 FORMAT(/40X,'GLOBAL DISPLACEMENTS AT THIS POINT',//)           00015040
      RETURN                                                         00015050
      END                                                            00015060
```

PRCPERTIES OF THE STRUCTURE
****

NUMBER CF ELEMENTS = 2
NUMBER CF NODES = 4
YOUNG S MCDULUS =3000.0KIP/SQ.IN.
POISSON S RATIC = 0.25
THICKNESS OF THE STRUCTURE = 4.00IN
LOAD ALONG THE RIDGE = -0.050KIP/FT.

CCORDINATES OF THE NODES
****

NODE	X (FT)	Y (FT)
1	0.0	0.0
2	C.0	15.00
3	3C.00	0.0
4	30.00	15.00

ELEMENT PROPERTIES
****

A = 0.0 IN B = 360.00 IN C= 180.00 IN
ALPHA.THE ANGLE WITH THE HORIZONTAL = -30.00 DEGREES

NCDES NUMBER

I	J	K	N
1	3	4	3

ELEMENT PROPERTIES
****

A = 321.99 IN   B = 80.50 IN   C= 161.00 IN
ALPHA.THE ANGLE WITH THE HORIZONTAL = -30.00 DEGREES

NODES NUMBER

I	J	K	N
1	4	2	6

RESULTS
****

NOTE.THE GENERALIZED CISPLACEMENTS ARE AS FOLLOWS .

U, UX, UY, V, VX, VY, W, WX, WY, WXX, WXY AND WYY

NODE 1	NODE 2
-0.025292174878	0.024806358402
0.000009882310	-0.000010607209
0.000312217077	0.000318534079
0.0	0.0
-0.000323907940	0.0
0.0	0.0
0.0	0.0
-0.000539288698	-0.000664778681
0.0	0.0
-0.000000134051	0.000001380034
0.000000113059	0.0
0.0	0.0

NODE 3	NODE 4
0.0	0.0
0.000111038252	-0.000109277501
0.0	0.0
-0.068764469397	0.0

-0.000C26960837	0.000021751527
-0.123144142643	-0.139212476924
0.0	0.0
0.000C56879669	0.0
0.000C02510910	0.000002985168
0.0	0.0
-0.000000704145	0.000000355806

STRAIN ENERGY = 0.832869926702295 KIP-IN

TRESSES IN THE ELEMENT DEFINED BY THE NODES 1, 4 AND 2

GLOBAL DISPLACEMENTS OF THE ELEMENT

IN-PLANE DISPLACEMENTS	BENDING DISPLACEMENTS
-0.025292174878015	0.0
0.000009882310334	-0.000539288697786
0.000312217076561	0.0
0.0	-0.C00000134051326
-0.000323907939967	0.000000113059080
0.0	0.0
0.0	-0.120561541539977
-0.000109277501088	0.0
0.0	0.0
-0.069606238462022	0.000002585231226
0.0	0.0
0.000021751526551	0.C00000355806007
0.024806358402255	0.0
-0.000010607208572	-0.000575715225901
0.000318534079136	0.0
0.0	0.000001195144453
-0.000332389340651	0.0
0.0	0.0

LOCAL BENDING DISPLACEMENTS

0.0	-0.000482354475098	0.000241177237549	-0.000000016793797
0.000000121455978	-0.000000117257529	-0.120561541539977	0.0
0.0	0.000002139346182	-0.000000891770088	0.000000801691051
0.0	-0.000514935352319	0.000257467676159	0.000000956115562
-0.000000478057781	0.000000239028891		

BENDING COEFFICIENTS

-0.037932095419880	-0.000449888688491	0.000233837873308	0.000000372926939
-0.000000286679740	-0.000000190006721	0.000000001131736	-0.000000002215963
0.000000000315135	0.000000001319105	-0.000000000002350	0.000000000002643
0.000000000020295	-0.000000000014640	0.000000000002063	0.000000000000003
-0.000000000000048	-0.000000000000015	0.000000000000051	-0.000000000000016

LOCAL IN-PLANE DISPLACEMENTS

-0.022622008930422	0.000003229502905	0.000310602325109	0.011311004465211
-0.000325522691420	0.000006652807429	-0.031128856171828	-0.000083071695560
0.000052411611056	-0.062257712343656	0.000052411611056	-0.000004454278976
0.022187481464666	-0.000014027871464	0.000325548014868	-0.011093740732333
-0.000325375404919	0.000003420662892	-0.010330741277599	-0.034906191957140

IN-PLANE COEFFICIENTS

-0.022270397463655	0.000004003501067	0.000261030970844	-0.000000023196480
-0.000000622223821	-0.000000119171831	-0.000000000231922	-0.000000000079737
0.000000003169164	0.000000001323169	-0.013785796169687	-0.000294314140939
0.000008872855194	0.000000262757416	0.000000013785129	0.000000180109797
0.000000000057079	-0.000000000171353	-0.000000001283975	-0.000000000815926

DISPLACEMENTS AT X= -80.4984471900 AND Y= 0.0

NOTE. DISPLACEMENTS AND DISTANCES ARE GIVEN IN INCHES
MOMENTS IN (KIP-IN/IN) AND STRESSES IN (KIP/SQ.IN)

-0.022622008930422	0.000003229502905	0.000310602325109
0.011311004465211	-C.000325522691420	0.000006652807429
-0.000000000000000	-0.000482354475098	0.000241177237549
-C.000000016793797	0.000000121455978	-0.000000117257529

VXX= 0.000000249867838 VXY= 0.000000041372401 VYY= 0.000000566935551

LOCAL BENDING DISPLACEMENTS

-0.000000000000000	-0.000482354475098	0.000241177237549	-0.000000016793797
0.000000121455978	-0.000000117257529		

LOCAL IN-PLANE DISPLACEMENTS

-0.022622008930422	0.000003229502905	0.000310602325109	0.011311004465211
-0.000325522691420	0.000006652807429		

SMXX	SMYY	SIGXP	SIGXN	SIGYP	SIGYN	SHEAR
-C.228781D-02	-0.571952D-03	0.307655D-01	0.324813D-01	0.769137D-02	0.812033D-02	-0.140290D-(

GLOBAL DISPLACEMENTS AT THIS POINT

-0.000323907939967	-0.000000000000000	-0.000000000000000	-0.000539288697786
0.000000000000000	-0.000000134051326	0.000000113059080	-0.000000000000000

DISPLACEMENTS AT X= 321.9937887600 AND Y= 0.0

NOTE. DISPLACEMENTS AND DISTANCES ARE GIVEN IN INCHES
MOMENTS IN (KIP-IN/IN) AND STRESSES IN (KIP/SQ.IN)

-0.031128856171828	-0.000083071695560	0.000052411611056
-0.062257712343656	0.000052411611056	-0.000004454278976
-0.120561541539978	-0.000000000000000	-0.000000000000000
0.000002139346182	-0.000000891770088	0.000000801691051

VXX= 0.000001628102809 VXY= -0.000000096563957 VYY= -0.000000466644230

LOCAL BENDING DISPLACEMENTS

-0.120561541539978	-0.000000000000000	-0.000000000000000	0.000002139346182
-0.000000891770088	0.000000801691051		

LOCAL IN-PLANE DISPLACEMENTS

-0.031128856171828	-0.000083071695560	0.000052411611056	-0.062257712343656
0.000052411611056	-0.000004454278976		

SMXX	SMYY	SIGXP	SIGXN	SIGYP	SIGYN	SHEAR
0.456394D-01	0.171027D-01	-0.315172D 00	-0.349402D 00	-0.114036D-01	-0.242306D-01	0.325769D-

END OF
REEL