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THE COORDINATE ALGEBRA OF EXTENDED AFFINE LIE ALGEBRAS OF TYPE A_1

By

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submitted to the School of Graduate Studies and Research

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Abstract

In this thesis we classify the coordinate algebra of extended affine Lie algebras, EALA's for short, of type A_1 . The coordinate algebras of EALA's of the other reduced types were already described in [6], [7] and [2]. In Chapter 1 we show that the coordinate algebra of EALA's of type A_1 is a certain \mathbf{Z}^n -graded Jordan algebra called a *Jordan torus*. The main result in this thesis is the classification of Jordan tori. For this purpose we study more general objects, *division graded alternative or Jordan algebras* in Chapter 2. In Chapter 3 we classify Jordan tori. They fall into five classes namely three types of *Hermitian tori*, *Clifford tori* and the *Albert torus*. Our second goal is the classification of division \mathbf{Z}^n -graded alternative algebras in Chapter 4, which generalizes the classification of alternative tori in [7]. In the associative case, we get a generalization of quantum tori, and in the nonassociative case, we obtain the *Cayley torus* or 3 closely related *octonion rings*. In the final chapter we classify *graded forms* and *derivations* of Jordan tori. These results serve as a preparation of classifying tame EALA's of type A_1 .

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Dedication

I dedicate this work to my father.

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Introduction

In this thesis we study the coordinate algebra of extended affine Lie algebras of type A_1 and related topics. Extended affine Lie algebras, EALA's for short, form a new class of infinite dimensional Lie algebras, which were first introduced by Høegh-Krohn and Torresani in 1990 [11] (under the name of irreducible quasi-simple Lie algebras) as a generalization of the finite dimensional simple Lie algebras and the affine Kac-Moody Lie algebras, and systematically studied in the recent memoir [1].

Let us first explain the construction of affine Kac-Moody Lie algebras of type $A_l^{(1)}$. Let \mathbb{C} be the field of complex numbers, $\mathbb{C}[t^{\pm 1}]$ the algebra of Laurent polynomials over \mathbb{C} in one variable t and $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$ the set of $(l+1) \times (l+1)$ matrices over \mathbb{C} of trace 0, which is well-known to be a finite dimensional simple Lie algebra of type A_l . Note that the bilinear form $\varepsilon(\cdot, \cdot)$ on $\mathbb{C}[t^{\pm 1}]$ defined by $(r, s \in \mathbb{Z})$

$$\varepsilon(t^r, t^s) = \begin{cases} 1 & \text{if } r + s = 0 \\ 0 & \text{otherwise} \end{cases}$$

is symmetric, invariant and nondegenerate. The *loop algebra*

$$L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \cong \mathfrak{sl}_{l+1}(\mathbb{C}[t^{\pm 1}])$$

also has a symmetric invariant nondegenerate bilinear form (\cdot, \cdot) defined by

$$(x \otimes f, y \otimes g) = \text{tr}(xy)\varepsilon(f, g)$$

for $x \otimes f, y \otimes g \in L\mathfrak{g}$, where tr is the usual trace form of matrices. Let

$$d = t \frac{d}{dt}$$

be the so-called degree derivation on $\mathbb{C}[t^{\pm 1}]$. Using this d and the bilinear form (\cdot, \cdot) on $L\mathfrak{g}$, we can define a 1-dimensional central extension

$$\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$$

of $L\mathfrak{g}$, where the Lie bracket is defined as

$$[x \otimes f + ac, y \otimes g + bc] := [x, y] \otimes fg + (x \otimes df, y \otimes g)c$$

for $x \otimes f, y \otimes g \in L\mathfrak{g}$ and $a, b \in \mathbb{C}$. The affine Kac-Moody Lie algebra $\widehat{L\mathfrak{g}}$ of type $A_l^{(1)}$ is defined as follows:

$$\widehat{L\mathfrak{g}} = \widetilde{L\mathfrak{g}} \oplus \mathbb{C}d$$

contains $\widetilde{L\mathfrak{g}}$ as a subalgebra, and for $x \otimes f + ac \in \widetilde{L\mathfrak{g}}$ and $b \in \mathbb{C}$, we have

$$[x \otimes f + ac, bd] := bx \otimes df.$$

Note that $\widetilde{L\mathfrak{g}}$ is an ideal of $\widehat{L\mathfrak{g}}$ and $\mathbb{C}c$ is the centre of $\widehat{L\mathfrak{g}}$. We point out some further properties of $\widehat{L\mathfrak{g}}$. The bilinear form (\cdot, \cdot) on $L\mathfrak{g}$ can be extended to a symmetric form on $\widehat{L\mathfrak{g}}$ defined by

$$(L\mathfrak{g}, \mathbb{C}c \oplus \mathbb{C}d) = \{0\}. \quad (c, c) = (d, d) = 0 \text{ and } (c, d) = 1.$$

Then, again the new form

(EA1) (\cdot, \cdot) on $\widehat{L\mathfrak{g}}$ is a nondegenerate symmetric invariant bilinear form.

Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , e.g. the subalgebra of all diagonal matrices of $\mathfrak{sl}_{l+1}(\mathbb{C})$, and put

$$\mathcal{H} := (\mathfrak{h} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Then \mathcal{H} is a finite dimensional abelian subalgebra of $\widehat{L\mathfrak{g}}$. As in the case of semisimple Lie algebras, \mathcal{H} decomposes $\widehat{L\mathfrak{g}}$. With the notation $\mathcal{L}_\alpha = \{x \in \widehat{L\mathfrak{g}} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$, we have

$$\widehat{L\mathfrak{g}} = \bigoplus_{\alpha \in R} \mathcal{L}_\alpha \quad \text{and} \quad \mathcal{L}_0 = \mathcal{H}$$

where $R = \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_\alpha \neq 0\}$ is the root system of $(\widehat{L\mathfrak{g}}, \mathcal{H})$. Note that we consider 0 as a root. Thus we have

(EA2) $\widehat{L\mathfrak{g}}$ contains a nontrivial self-centralizing finite dimensional ad-diagonalizable abelian subalgebra.

The root system R can be described in terms of the root system Δ of \mathfrak{g} relative to \mathfrak{h} , which has type A_l . We extend $\alpha \in \Delta \subset \mathfrak{h}^*$ to a linear form on \mathcal{H} , also denoted α , by $\alpha(\mathbb{C}c \oplus \mathbb{C}d) = 0$. We define $\theta \in \mathcal{H}^*$ by $\theta|_{(\mathfrak{h} \otimes 1) \oplus \mathbb{C}c} = 0$ and $\theta(d) = 1$. Then

one can check that $R = \Delta + \mathbb{Z}\theta$. The restriction of (\cdot, \cdot) to \mathcal{H} is still a nondegenerate symmetric bilinear form, and so the dual space \mathcal{H}^* is equipped with the induced bilinear form, again denoted (\cdot, \cdot) . This allows us to speak of a non-isotropic root, i.e., $(\alpha, \alpha) \neq 0$. One can show that $R^\times := (\Delta \setminus \{0\}) + \mathbb{Z}\theta$ is the set of non-isotropic roots and that

(EA3) for any non-isotropic root α , $\text{ad } \mathcal{L}_\alpha$ is (locally) nilpotent.

Also, it is clear that

(EA4) R is a discrete subset of \mathcal{H}^* .

Moreover, R^\times is indecomposable relative to the form (\cdot, \cdot) , and for any isotropic root σ , there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$, namely,

(EA5) R is irreducible.

Finally, let \mathbb{R} be the field of real numbers and \mathcal{V} the \mathbb{R} -span of the root system R . Then $\dim_{\mathbb{R}} \mathcal{V} = l + 1$ and the restriction of (\cdot, \cdot) to \mathcal{V} is positive semidefinite with nullity 1, i.e., the radical is 1-dimensional. (For further details on affine Kac-Moody Lie algebras (see e.g. [22]).)

An EALA is defined as a complex Lie algebra which satisfies the properties (EA1)-(EA5) above. In particular, \mathfrak{g} and $\widehat{\mathfrak{Lg}}$ are EALA's of type A_l with nullity 0 and 1, respectively. In general an EALA of type A_l has *nullity* n , i.e., the \mathbb{R} -vector space \mathcal{V} spanned by the root system, defined by the same way as above, is $(l + n)$ -dimensional, and after a suitable adjustment by a constant factor,

the form on \mathcal{V} is positive semidefinite with an n -dimensional radical.

The *core* of an EALA is defined as the subalgebra generated by the non-isotropic root spaces. For example, the core of $\widehat{\mathfrak{Lg}}$ is $\widetilde{\mathfrak{Lg}}$. One has a description of an EALA \mathcal{L} of type A_l ($l \geq 2$), D_l and E_l due to Berman, Gao and Krylyuk [6]. Their description is a 2-step process:

- A) describe the core \mathcal{L}_c , and then,
- B) describe how \mathcal{L}_c sits in \mathcal{L} .

This program is currently being worked out for the other types of EALA's. In particular, Allison and Gao [2] describe the cores of all non-simply laced reduced types, i.e., B_l , C_l , G_2 and F_4 . The main topic of this thesis is a description of the cores of EALA's of type A_1 .

For motivation, let us first describe the core of an EALA of type A_l , $l \geq 3$ ([6]). An $n \times n$ matrix $q = (q_{ij})$ over \mathbb{C} such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix*. The *quantum torus* $\mathbb{C}_q = \mathbb{C}_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by a quantum matrix q is defined as the associative algebra over \mathbb{C} with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$. Note that \mathbb{C}_q is commutative if and only if $q = 1$ where 1 is the quantum matrix whose entries are all 1. In this case, the quantum torus \mathbb{C}_1 becomes the algebra of Laurent polynomials $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ in n variables. We note that \mathbb{C}_q is a unital \mathbb{Z}^n -graded associative algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ over \mathbb{C} satisfying

- (1) $A_\alpha A_\beta = A_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$, i.e., A is strongly graded,
- (2) $\dim_{\mathbb{C}} A_\alpha = 1$ for all $\alpha \in \mathbb{Z}^n$.

Conversely, any such algebra is isomorphic to \mathbb{C}_q for some q . We observe that all nonzero homogeneous elements are invertible. Taking \mathbb{C}_q as a coordinate algebra we define

$$\mathfrak{sl}_{l+1}(\mathbb{C}_q) = \{X \in M_{l+1}(\mathbb{C}_q) \mid \text{tr}(X) \in [\mathbb{C}_q, \mathbb{C}_q]\}$$

where $[\mathbb{C}_q, \mathbb{C}_q]$ is the span of all commutators $[a, b] = ab - ba$. It is shown in [6] that the core of any EALA of type A_l , $l \geq 3$ with nullity n , is a central extension of $\mathfrak{sl}_{l+1}(\mathbb{C}_q)$.

The Lie algebras $\mathfrak{sl}_3(\mathbb{C}_q)$ are examples of cores of an EALA of type A_2 , but do not give all possibilities. Rather, there exists a construction which associates to every alternative algebra A a Lie algebra $\mathfrak{psl}_3(A)$, and it is shown in [7] that a Lie algebra is the core of an EALA of type A_2 if and only if it is a central extension of $\mathfrak{psl}_3(A)$ where A is a unital \mathbb{Z}^n -graded alternative algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ over \mathbb{C} satisfying (1) and (2) above. These alternative algebras have been classified in [7]. Besides \mathbb{C}_q , there exists up to isomorphisms one more type, the *Cayley torus* \mathbb{O}_t . It is defined as $\mathbb{O}_t = (\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}], t_1, t_2, t_3)$, i.e., the octonion algebra over $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ obtained by the Cayley-Dickson process with the structure constants t_1, t_2 and t_3 .

Now we consider our main topic, EALA's of type A_1 . The Tits-Kantor-Koecher construction which associates to every Jordan algebra J a Lie algebra $\text{TKK}(J)$, called

the *Tits-Kantor-Koecher algebra* of J . comes into play. We show:

Theorem 1. *The core of any EALA of type A_1 is a central extension of $\mathrm{TKK}(J)$ where J is a unital \mathbb{Z}^n -graded Jordan algebra $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ over \mathbb{C} satisfying*

- (T1) $\{\alpha \in \mathbb{Z}^n \mid J_\alpha \neq (0)\}$ generates \mathbb{Z}^n .
- (T2) all nonzero homogeneous elements are invertible,
- (T3) $\dim_{\mathbb{C}} J_\alpha \leq 1$ for all $\alpha \in \mathbb{Z}^n$.

Such a graded Jordan algebra over a field F of characteristic $\neq 2$ is called a *Jordan n -torus* or simply a *Jordan torus*. We classify Jordan tori not only over \mathbb{C} but over any F . This will lead to analogs of extended affine Lie algebras over general fields. The simplest example of Jordan tori is the plus algebra F_q^+ of a quantum torus F_q , which is defined on the space F_q with the new product $x \cdot y := \frac{1}{2}(xy + yx)$ for $x, y \in F_q$. We note that $\mathfrak{sl}_2(F_q) \cong \mathrm{TKK}(F_q^+)$. To state our main result, we briefly describe other examples of Jordan n -tori.

(a) Let $\varepsilon = (\varepsilon_{ij})$ be a quantum matrix such that $\varepsilon_{ij} = 1$ or -1 for all i, j . We call such an ε *elementary*. Define an involution $*$ on $F_\varepsilon = F_\varepsilon[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $t_i^* = t_i$ for all i . Then the symmetric elements $H(F_\varepsilon, *)$ form a Jordan torus. Also, let E be a quadratic field extension of F with the nontrivial Galois automorphism σ_E . Let $\xi = (\xi_{ij})$ be a quantum matrix such that $\xi_{ij}\sigma_E(\xi_{ij}) = 1$ for all i, j . Define a σ_E -semilinear involution σ on $E_\xi = E_\xi[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over F such that $\sigma(t_i) = t_i$ for all i . Then the symmetric elements $H(E_\xi, \sigma)$ form a Jordan torus over F .

(b) Let $2 \leq m \leq n$ and let $S^{(m)}$ be any *semilattice* in \mathbb{Z}^m (see §2.4 for the precise definition). One can construct a Jordan algebra $J_{S^{(m)}}(\{a_\varepsilon\}_{\varepsilon \in I})$ over $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by a certain symmetric bilinear form which depends on $S^{(m)}$ and a family $(a_\varepsilon)_{\varepsilon \in I} \in F$ of nonzero elements in F (the details are in 3.3.8). Then $J_{S^{(m)}}(\{a_\varepsilon\}_{\varepsilon \in I})$ is a Jordan n -torus called a *Clifford torus*. Clifford tori are a slight generalization of a construction which already appeared in [1].

(c) Suppose that F contains a primitive 3rd root of unity ω . Let $\omega = (\omega_{ij})$ be a quantum matrix such that $\omega_{12} = \omega$, $\omega_{21} = \omega^2$ and $\omega_{ij} = 1$ for the other i, j . Let $\mathbb{A}_t = (F_\omega, t_3)$ be the first Tits construction, using the quantum torus $F_\omega = F_\omega[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and the structure constant t_3 (details are in §3.4). The central closure of \mathbb{A}_t is a 27-dimensional exceptional Jordan division algebra over a field. Then \mathbb{A}_t is a Jordan torus, which is called the *Albert torus*. This torus was independently found in [1] and

[32] (and it is a coordinate algebra of EALA's of type G_2 (see [1] and [2])).

We can now state our main result:

Theorem 2. *Let J be a Jordan torus over F . Then J is isomorphic to one of the five tori*

$$F_q^+, H(F_\epsilon, *), H(E_\xi, \sigma), J_{S(m)}(\{a_\epsilon\}_{\epsilon \in I}) \text{ or } \mathbb{A}_t.$$

Since the Jordan tori turn out to be strongly prime, we can use Zelmanov's Prime Structure Theorem [21] as the first step of our proof. Thus, a Jordan torus is either of Hermitian, Clifford or Albert type. For each type we then determine the possible Jordan tori.

To explain our second result, let G be a group. We define a *division G -graded algebra* as a G -graded alternative or Jordan algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ over a field of F (ch. $F \neq 2$ if \mathcal{T} is Jordan) satisfying the following conditions (G1) and (G2)

- (G1) $\{g \in G \mid \mathcal{T}_g \neq (0)\}$ generates G ,
- (G2) all nonzero homogeneous elements are invertible.

We note that if \mathcal{T} is a division G -graded alternative algebra, then $\{g \in G \mid \mathcal{T}_g \neq (0)\} = G$ and \mathcal{T} is strongly graded. Also, we define a special class of division G -graded algebras called *G -tori*, by adding one more condition

- (G3) $\dim_F \mathcal{T}_g \leq 1$ for all $g \in G$.

Thus Jordan \mathbb{Z}^n -tori are our Jordan n -tori defined above. For a G -graded alternative algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$, we note that \mathcal{T} is a G -torus if and only if \mathcal{T} is strongly graded and $\dim_F \mathcal{T}_g = 1$ for all $g \in G$. This is not true for G -graded Jordan algebras.

The second goal of this thesis is the classification of division \mathbb{Z}^n -graded alternative algebras, which will lead to generalizations of EALA's of type A_2 . Since they are strongly prime, we can apply Slater's Theorem classifying prime alternative algebras [33]. Namely, a strongly prime alternative algebra is either associative or an *octonion ring*, i.e., its central closure is an octonion algebra. We first classify division \mathbb{Z}^n -graded associative algebras, which are crossed product algebras over some associative division algebra D over F . As a natural generalization of quantum tori F_q , we construct a division \mathbb{Z}^n -graded associative algebra $D_{\varphi, q}$ over F determined by D , an n -tuple φ of automorphisms of D and a quantum matrix q over D , and obtain that any division \mathbb{Z}^n -graded associative algebra is graded isomorphic to $D_{\varphi, q}$ for some D , φ and q (the details are in §4.2). Then we classify division \mathbb{Z}^n -graded octonion rings. Let us

present four such octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 . Let F be a field of characteristic $\neq 2$ and K any field extension of F . We define $\mathbb{O}_i = (K[t_1^{\pm 1}, \dots, t_n^{\pm 1}], \mu_1, \mu_2, \mu_3)$, i.e. the octonion algebra over $K[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ obtained by the Cayley-Dickson process with the structure constants μ_1, μ_2 and μ_3 , for

- $[i = 1]: 0 \neq \mu_1, \mu_2, \mu_3 \in K$ such that (K, μ_1, μ_2, μ_3) is an octonion division algebra.
- $[i = 2]: 0 \neq \mu_1, \mu_2 \in K$ and $\mu_3 = t_1$ such that (K, μ_1, μ_2) is a quaternion division algebra,
- $[i = 3]: 0 \neq \mu_1 \in K, \mu_2 = t_1$ and $\mu_3 = t_2$ such that (K, μ_1) is a field.
- $[i = 4]: \mu_1 = t_1, \mu_2 = t_2$ and $\mu_3 = t_3$.

Note that \mathbb{O}_4 is the Cayley torus over K . Our second result is the following:

Theorem 3. *A division \mathbb{Z}^n -graded alternative algebra is graded isomorphic to $D_{\varphi, q}$ or to one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 .*

As a corollary, we obtain the classification of alternative tori over any field, which generalizes a result in [7]. Namely, an alternative torus is isomorphic to F_q or the Cayley torus $\mathbb{O}_t = (F[t_1^{\pm 1}, \dots, t_n^{\pm 1}], t_1, t_2, t_3)$.

This thesis consists of five chapters with Chapter 1 serving as review and motivation. In Chapter 2 we study division graded algebras and tori, and in Chapter 3 we classify Jordan tori. In Chapter 4 we classify division \mathbb{Z}^n -graded alternative algebras. In Chapter 5 we classify graded forms and derivations of tori. Let us explain the contents in more details:

In §1.1 we review the basics of alternative and Jordan algebras, especially invertibility, centres, primeness and central closures. In §1.2 we give the definition and the basic structure of an EALA and prove Theorem 1 above.

In §2.1 we summarize some basic properties of graded algebras. In §2.2 we show that any division G -graded algebra for a totally ordered abelian group G is a domain and that any unit is homogeneous. In §2.3 we prove that if G is abelian, the centre \mathcal{Z} of a division G -graded algebra \mathcal{T} is graded by a subgroup H of G , which we call the *central grading group of \mathcal{T}* . Moreover, \mathcal{T} becomes a G/H -graded algebra over \mathcal{Z} . If G is a totally ordered abelian group, the central closure $\overline{\mathcal{T}} = \overline{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{T}$ is also a G/H -graded algebra over $\overline{\mathcal{Z}}$ where $\overline{\mathcal{Z}}$ is the field of fractions of \mathcal{Z} . Some properties of the graded algebra $\overline{\mathcal{T}}$ are derived in Proposition 2.3.5. In §2.4 we review semilattices, and also study the relation between isomorphisms and graded isomorphisms. We show

that any isomorphism between two division \mathbb{Z}^n -graded algebras is ‘almost’ a graded isomorphism (see Definition 2.4.2 and Theorem 2.4.3). In §2.5 we classify G -tori for a cyclic group G (2.5.3). We review quantum tori which are the associative tori, and prove some properties about their *toral gradings* (see 2.5.10-12). In §2.6 we define the *central degree* of a prime alternative or Jordan algebra (see Definition 2.6.3). We determine the central grading groups of division \mathbb{Z}^n -graded algebras of central degree 2 or 3 (see Proposition 2.6.8), which will be crucial for the classification of division \mathbb{Z}^n -graded octonion rings and division \mathbb{Z}^n -graded Jordan algebras of Clifford or Albert type. Moreover, for this very purpose we determine the quantum tori of central degree 2 or 3.

In §3.0 we state Zelmanov’s Prime Structure Theorem, and see that a Jordan torus is either of Hermitian, Clifford or Albert type. In §3.1 we show that a Hermitian torus is graded isomorphic to either F_q^+ , $H(F_\epsilon, *)$ or $H(E_\xi, \sigma)$. In §3.2 we classify F_ϵ and $H(F_\epsilon, *)$ for all elementary quantum matrices ϵ . It turns out that $H(F_\epsilon, *)$ are divided into three types (see 3.2.18). In §3.3 we first classify division \mathbb{Z}^n -graded Jordan algebras of Clifford type, and as a corollary we get in 3.3.10 that any Clifford torus is graded isomorphic to $J_{S(m)}(\{a_\epsilon\}_{\epsilon \in I})$. In §3.4 we classify special Jordan tori of central degree 3 and then show that any Jordan torus of Albert type is isomorphic (not graded in general) to the Albert torus A_t . As a corollary, we obtain the classification of Jordan tori of central degree 3, which was used in the classification of cores of EALA’s of type G_2 in [2].

In §4.0 we review crossed product algebras and observe that these algebras are precisely the *predivision graded* (a generalization of division grading) associative algebras. In §4.1 we classify predivision \mathbb{Z}^n -graded associative algebras as a natural generalization of quantum tori. In §4.2 we review the Cayley-Dickson process over a ring, and prove some lemmas for the preparation to classify division \mathbb{Z}^n -graded octonion rings in the next section. In §4.3 we show that the four octonion rings \mathbb{O}_1 , \mathbb{O}_2 , \mathbb{O}_3 and \mathbb{O}_4 described above are in fact non-isomorphic division \mathbb{Z}^n -graded alternative algebras, and prove that these four exhaust the division \mathbb{Z}^n -graded alternative but not associative algebras.

In §5.1 we show that any Jordan torus admits a decomposition $J = Z(J) \oplus (J, J, J)$ where $Z(J)$ is the centre of J and (J, J, J) is the space spanned by associators. As a corollary, we get the classification of the graded and nondegenerate graded forms of

tori. In §5.2 we classify the derivations of tori and get a decomposition into the inner derivations and the degree derivations. Simultaneously, we obtain the homogeneous dimensions of the inner derivations for F_q^+ and $H(E_\xi, \sigma)$ using the result for F_q in [6]. Then in §5.3 we determine the dimensions for the rest of Jordan tori, i.e., $H(F_\epsilon, *)$, Clifford tori and the Albert torus. These results will be useful in the classification of EALA's of type A_1 .

Chapter 1 Review

In this chapter, the basic concepts and known results needed in this thesis are overviewed. For the convenience of the reader, we give proofs of those results for which there is no convenient reference. The motivation of the main topic of the thesis, the classification of Jordan tori (see Theorem 1.2.9), is presented.

§ 1.1 ALTERNATIVE AND JORDAN ALGEBRAS

In this section, an algebra over a field F is a “linear” nonassociative algebra A defined as a vector space over F with an F -bilinear map $A \times A \longrightarrow A$, called *multiplication*. Throughout we assume that A is unital in the sense that there exists an element 1 in A , called an *identity element*, such that $1x = x = x1$ for all $x \in A$.

For an algebra A and $x, y, z \in A$ let us define the *commutator* $[x, y] = xy - yx$ and the *associator* $(x, y, z) = (xy)z - x(yz)$.

An algebra A is *alternative* if for all $x, y \in A$,

$$(x, x, y) = 0 = (x, y, y).$$

It is easy to check the following identity, for all $x, y \in A$,

$$(x, y, x) = 0.$$

called the *flexible law*. Thus we denote

$$xyx := (xy)x = x(yx).$$

We note that the associator on A is *alternative*, i.e.,

$$(x_1, x_2, x_3) = \text{sgn}(\pi)(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$$

for all $\pi \in S_3$ (the symmetric group of 3 letters). Moreover, if the characteristic of F is not 2, the converse is also true, i.e., the associator is alternative $\implies A$ is alternative.

Any associative algebra is clearly alternative. But there are alternative, nonassociative algebras. For example, the octonions are alternative but not associative. The reader will find some more material on octonions over a ring in §4.2.

An algebra J over a field F of characteristic $\neq 2$ satisfying the following two identities is called a (*linear*) *Jordan algebra* over F : for all $x, y \in J$ we have

$$[x, y] = 0 \text{ (commutativity)} \quad \text{and} \quad (x, y, x^2) = 0 \text{ (Jordan identity)}.$$

Let us define the so-called *U-operator* for $x \in J$, i.e., $U_x : J \longrightarrow J$ by

$$U_x y = U_x(y) = 2x(xy) - x^2 y \quad \text{for all } y \in J.$$

The *plus algebra* A^+ of an alternative algebra A over F of characteristic $\neq 2$ is an example of a Jordan algebra: for $x, y \in A^+ = (A, \cdot)$ with the new multiplication \cdot on A defined as $x \cdot y := \frac{1}{2}(xy + yx)$. For this example, the *U-operator* is given by $U_x y = xyx$. A Jordan algebra is called *special* if it is isomorphic to a subalgebra of the plus algebra of some associative algebra. It is well-known that the plus algebra of an alternative algebra is special (see [13, page 15]). A Jordan algebra is called *exceptional* if it is not special.

Remark 1.1.1. It is well known that both alternative and Jordan algebras are *power associative*, i.e., the subalgebra generated by any element is associative (so it is commutative) (see e.g. [33, page 37, 68]).

An element x in an alternative algebra A is called *invertible* if there exists $y \in A$ such that $xy = yx = 1$. In this case y is unique and is denoted by x^{-1} .

An element x in a Jordan algebra J is called *invertible* if there exists $y \in J$ such that $xy = 1$ and $x^2 y = x$. In this case y is unique and is denoted by x^{-1} .

We denote the subset of invertible elements of an algebra A by A^\times . We will use the following properties of invertible elements on several occasions.

Lemma 1.1.2. ([33, page 204, 303-4]) (1) *Let A be an alternative algebra over F and $x \in A^\times$. Then for all $y \in A$, we have*

$$x(x^{-1}y) = x^{-1}(xy) = y.$$

(2) Let A be an alternative algebra over F of characteristic $\neq 2$. Then

$$A^\times = (A^+)^\times.$$

and for $x \in A^\times = (A^+)^\times$, x^{-1} in the alternative algebra A and the Jordan algebra A^+ coincide.

(3) Let J be a Jordan algebra and $x \in J$. Then $x \in J^\times \iff U_x$ is invertible \iff there exists $y \in J$ such that $U_x y = 1$. In these cases, we have $U_x^{-1} = U_{x^{-1}}$ and $y = x^{-2}$.

(4) Let $x \in A^\times$ where A is either an alternative or a Jordan algebra. Then the subalgebra of A generated by x and x^{-1} is commutative and associative. \square

For the statement (2) in Lemma 1.1.2, the reference [33] only says that it holds for associative algebras. However, this even holds for alternative algebras using the theorem by Artin:

Theorem 1.1.3. ([33, page 36]) *The subalgebra of an alternative algebra generated by two elements is associative.* \square

Next we recall some basic notions for alternative and Jordan algebras.

Definition 1.1.4. (0) Let A be an arbitrary algebra and $0 \neq x \in A$. Then x is called a *zero-divisor* if $xy = 0$ for some $0 \neq y \in A$.

(1) Let A be an alternative algebra. Then A is called

- (i) an *alternative domain* if there is no zero-divisor in A ,
- (ii) *nondegenerate* if $xAx = 0$ implies $x = 0$ for all $x \in A$ where $xAx = \{xax \mid a \in A\}$,
- (iii) *prime* if $IK = (0)$ implies $I = (0)$ or $K = (0)$ for all ideals I, K of A where $IK = \{\sum_{x,y} xy \mid x \in I, y \in K\}$,
- (iv) *strongly prime* if A is nondegenerate and prime. (The reader is advised that "strongly prime" means something different in the theory of associative rings.)

(2) Let J be a Jordan algebra. Then J is called

- (i) a *Jordan domain* if $U_x y = 0$ implies $x = 0$ or $y = 0$ for all $x, y \in J$,
- (ii) *nondegenerate* if $U_x = 0$ implies $x = 0$ for all $x \in J$,
- (iii) *prime* if $U_I K = (0)$ implies $I = (0)$ or $K = (0)$ for all ideals I, K of J where $U_I K = \{\sum_{x,y} U_x y \mid x \in I, y \in K\}$,

(iv) *strongly prime* if J is nondegenerate and prime.

We note here that for an associative algebra A , A is prime $\iff A^+$ is prime. The direction \Leftarrow is easy and the direction \Rightarrow follows from [33, Exercise 5, page 57].

The nondegeneracy and primeness generalize the notion of a domain. Namely,

Lemma 1.1.5. (1) *An alternative domain is strongly prime.*

(2) *A Jordan domain is strongly prime.*

Therefore, we have for any alternative or Jordan algebra:

$$\begin{array}{ccccc} \text{domain} & \implies & \text{strongly prime} & \implies & \text{prime} \\ & & \downarrow & & \\ & & \text{nondegenerate} & & \end{array}$$

Proof. For (1), let A be an alternative domain. Then, $xAx = 0$ for $x \in A$ implies $x1x = x^2 = 0$. So $x = 0$ since A is an alternative domain. Hence A is nondegenerate. Suppose that $IK = (0)$ for ideals $I \neq (0)$ and $K \neq (0)$ of A . Then for $0 \neq y \in I$ and $0 \neq z \in K$, we have $yz \in IK = (0)$, which is absurd since A is an alternative domain. Hence A is prime.

For (2), let J be a Jordan domain. Then $U_x = 0$ for $x \in J$ implies $U_x 1 = x^2 = 0$. So $x = 0$ since J is a Jordan domain. Hence J is nondegenerate. Suppose that $U_I K = (0)$ for ideals $I \neq (0)$ and $K \neq (0)$ of J . Then for $0 \neq y \in I$ and $0 \neq z \in K$, we have $U_y z = 0$, which is absurd since J is a Jordan domain. Hence J is prime. \square

Remark 1.1.6. An alternative algebra A is called *semiprime* if $I^2 = (0)$ implies $I = (0)$ for all ideals I of A . One can easily prove that for any associative algebra

$$\text{nondegenerate} \iff \text{semiprime}.$$

Thus we have for any associative algebra:

$$\text{prime} \iff \text{strongly prime} \implies \text{nondegenerate}.$$

We mention that there exist prime degenerate alternative algebras (see [25]) and prime degenerate Jordan algebras (see e.g. [21, page 201]).

For an arbitrary algebra A , the *centre* $Z(A)$ of A is defined as

$$Z(A) = \{z \in A \mid [z, x] = (z, x, y) = (x, z, y) = 0 \text{ for all } x, y \in A\}.$$

One can check that $(x, y, z) = 0$ for all $z \in Z(A)$ and $x, y \in A$. If A is alternative, then $(x, z, y) = 0$ is not needed since the associator is alternative. If A is Jordan, then $[z, x] = 0$ is not needed since a Jordan algebra is commutative. Moreover, for an arbitrary algebra A , the following identity holds: for all $x, y, z \in A$,

$$(1.1.7) \quad [xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y)$$

(which can be easily verified by expanding both sides). Also, for a commutative algebra A (e.g. a Jordan algebra), the following identity holds: for all $x, y, z \in A$.

$$(1.1.8) \quad (x, y, z) + (z, y, x) = 0$$

(which can be easily verified by expanding both sides). Hence, for a commutative algebra A and $z \in A$, we have

$$\begin{aligned} (z, x, y) = 0 \quad \text{for all } x, y \in A &\iff (x, y, z) = 0 \quad \text{for all } x, y \in A \quad \text{by (1.1.8)} \\ &\implies (x, z, y) = 0 \quad \text{for all } x, y \in A \quad \text{by (1.1.7)}. \end{aligned}$$

Consequently, the centre of an alternative algebra A is

$$Z(A) = \{z \in A \mid [z, x] = (z, x, y) = 0 \text{ for all } x, y \in A\},$$

and the centre of a Jordan algebra J is

$$Z(J) = \{z \in J \mid (z, x, y) = 0 \text{ for all } x, y \in J\}.$$

The centre is always a commutative associative subalgebra of A ([33, page 136]). Any algebra A can be considered as an algebra over $Z(A)$.

Lemma 1.1.9. *Let A be an alternative or Jordan algebra and $Z = Z(A)$ be the centre of A . Suppose that $z \in Z$ is invertible in A . Then $z^{-1} \in Z$.*

Proof. For all $x, y \in A$, we have $0 = [1, x] = [zz^{-1}, x] = z[z^{-1}, x]$ and $0 = (1, x, y) = (zz^{-1}, x, y) = z(z^{-1}, x, y)$ since $1, z \in Z$. So we get $[z^{-1}, x] = (z^{-1}, x, y) = 0$ since $z \in Z$ and it is invertible. Hence $z^{-1} \in Z$ in both cases. \square

The following lemma is true for semiprime associative algebras by [20, Corollary 3.4, page 12]. However, for the convenience of the reader we prove it here in the following special case.

Lemma 1.1.10. *Assume that $\text{ch.}F \neq 2$. Let A be an associative domain over F . Then we have $Z(A) = Z(A^+)$.*

Proof. We use a well-known identity for an arbitrary associative ring A , which can be easily verified by expanding both sides:

$$(1.1.10a) \quad [x, [y, z]] = (y, x, z)^\circ \quad \text{for all } x, y, z \in A,$$

where $(y, x, z)^\circ = (y \circ x) \circ z - y \circ (x \circ z)$ is the associator of the multiplication \circ . $x \circ y = xy + yx$. From this identity, it is clear that $Z(A) \subset Z(A^+)$.

Let $x \in Z(A^+)$. One can easily verify that

$$(1.1.11) \quad [x, y]^2 = x \circ U_y x - U_x y^2 - U_y x^2.$$

The R.H.S of (1.1.11) $\times 2$ is equal to

$$\begin{aligned} & x \circ (y \circ (y \circ x) - y^2 \circ x) - (x \circ (x \circ y^2) - x^2 \circ y^2) - (y \circ (y \circ x^2) - y^2 \circ x^2) \\ & = 4x^2 \circ y^2 - 2x^2 \circ y^2 - 2x^2 \circ y^2 + x^2 \circ y^2 - 2x^2 \circ y^2 + x^2 \circ y^2 = 0 \end{aligned}$$

since $x \in Z(A^+)$ and $x \circ x = 2x^2$ for all $x \in A$. Hence we get $[x, y]^2 = 0$. Therefore, we obtain $[x, y] = 0$ for all $y \in A$, i.e., $x \in Z(A)$, since A is a domain. \square

The following lemma is well-known, for example, for Jordan algebras (see [14, Proposition 7.6.5, page 7.24]).

Lemma 1.1.12. *Let A be an alternative or a Jordan algebra and Z the centre of A . If A is prime, then a nonzero element of Z is not a zero-divisor in A (Definition 1.1.4(0)). In particular, Z is an integral domain.*

Proof. Let $0 \neq z \in Z$. Then for $za \in zA$ and $a' \in A$, we have

$$(za)a' = z(aa') \in zA \quad \text{and} \quad a'(za) = a'(az) = (a'a)z = z(a'a) \in zA$$

since z is in the centre. Hence zA is a nonzero ideal of A .

Let $l_z : A \rightarrow A$ be the left multiplication defined by $l_z(x) = zx$ for all $x \in A$. We claim that $\ker l_z$ is an ideal of A . For $u \in \ker l_z$ and $x \in A$, we have

$$l_z(xu) = z(xu) = (zx)u = (xz)u = x(zu) = 0 \quad \text{and} \quad l_z(ux) = z(ux) = (zu)x = 0$$

since z is in the centre. Hence, $xu, ux \in \ker l_z$ and our claim is proven.

Since $z \in Z$, we have

$$(zA) \ker l_z = A(z \ker l_z) = (0).$$

So, if A is alternative, we get $\ker l_z = (0)$, i.e., z is not a zero-divisor in A .

If A is a Jordan algebra, then we have, for $u \in \ker l_z$ and any $x \in A$,

$$U_{zx}u = U_{xz}u = U_x U_z u = U_x(2z(zu) - z^2u) = U_x(z(zu)) = 0$$

since $z \in Z$ and $\ker l_z$ is an ideal of A . This implies $U_{zA} \ker l_z = (0)$. Hence, by primeness of the Jordan algebra A , we get $\ker l_z = (0)$. Thus z is not a zero-divisor in A . \square

Definition 1.1.13. For an algebra A with the property that the centre $Z = Z(A)$ is an integral domain (e.g. a prime alternative or Jordan algebra), we define the tensor algebra \bar{A} over \bar{Z} as $\bar{A} = \bar{Z} \otimes_Z A$ where \bar{Z} is the field of fractions of Z , and call it the *central closure of A* .

The central closure $\bar{A} = \bar{Z} \otimes_Z A$ can be identified with the *ring of quotients of A modulo $Z^* := Z \setminus \{0\}$* (see e.g. [33, page 186]):

We define a relation \sim on $Z^* \times A$ by $(z, a) \sim (z', a') \iff$

$$(1.1.14) \quad \text{there exists some } z_1 \in Z^* \text{ such that } z_1(z'a - za') = 0$$

for $(z, a), (z', a') \in Z^* \times A$. Note that if Z does not contain zero-divisors in A , then

$$(1.1.15) \quad (z, a) \sim (z', a') \iff z'a = za'.$$

One can check that \sim is an equivalence relation. e.g., for the transitivity, let $(z, a) \sim (z', a')$ and $(z', a') \sim (z'', a'')$. So there exist some $z_1, z_2 \in Z^*$ such that $z_1(z'a - za') = 0$ and $z_2(z''a' - z'a'') = 0$. Then we have $z_1 z_2 z'(z''a - za'') = z_2 z''(z_1 z'a) - z_1 z(z_2 z'a'') = z_2 z'' z_1 z a' - z_1 z z_2 z'' a' = 0$, and hence $(z, a) \sim (z'', a'')$.

We denote $(Z^* \times A) / \sim$ by $(Z^*)^{-1}A$ and the equivalence class of (z, a) by $z^{-1}a$, and define for $z, z_1 \in Z^*$ and $z' \in Z$,

$$\begin{aligned} z^{-1}a + z_1^{-1}a_1 &:= (zz_1)^{-1}(z_1a + za_1), \\ (z^{-1}a)(z_1^{-1}a_1) &:= (zz_1)^{-1}(aa_1) \\ \frac{z'}{z} \cdot (z_1^{-1}a_1) &:= (zz_1)^{-1}(z'a_1). \end{aligned}$$

Then $(Z^*)^{-1}A$ becomes a \overline{Z} -algebra with

$$0 = z^{-1}0 \quad \text{and} \quad 1 = z^{-1}z \quad \text{for any } z \in Z^*.$$

In fact, it is well-known that $(Z^*)^{-1}A$ is a \overline{Z} -module with $0 = z^{-1}0$ for any $z \in Z^*$, which is called a Z -localization of A (see e.g. [17, page 397]). To show that the multiplication is well-defined, suppose that $z^{-1}a = u^{-1}b$ and $z_1^{-1}a_1 = u_1^{-1}b_1$ for $u^{-1}b, u_1^{-1}b_1 \in (Z^*)^{-1}A$. So there exist some $z', z'_1 \in Z$ such that $z'(ua - zb) = 0$ and $z'_1(u_1a_1 - z_1b_1) = 0$. Then we have $z'z'_1(uu_1aa_1 - zz_1bb_1) = (z'ua)(z'_1u_1a_1) - z'z'_1zz_1bb_1 = (z'zb)(z'_1z_1b_1) - z'z'_1zz_1bb_1 = 0$, and hence $(zz_1)^{-1}(aa_1) = (uu_1)^{-1}(bb_1)$. It is straightforward to check that $1 := 1^{-1}1 = z^{-1}z$ for all $z \in Z^*$ and that 1 is an identity element for $(Z^*)^{-1}A$. Also the distributive law holds. Thus $(Z^*)^{-1}A$ is a (nonassociative) ring.

Since $(Z^*)^{-1}A$ is a \overline{Z} -module, we can identify \overline{Z} with $\overline{Z}.1$. Indeed clearly the subfield $(Z^*)^{-1}Z = \{z^{-1}z' \mid z \in Z^*, z' \in Z\}$ of $(Z^*)^{-1}A$ is contained in the centre of $(Z^*)^{-1}A$. We note here that \overline{Z} embeds into $(Z^*)^{-1}A$ via $z'/z \mapsto z^{-1}z'$, and so the image is $(Z^*)^{-1}Z$. Indeed, this map is clearly a ring homomorphism. If $z^{-1}z' = 0$, there exists some $z'' \in Z^*$ such that $z''z' = 0$ by $0 = 1^{-1}0$ and (1.1.14). Since Z is an integral domain, we get $z' = 0$ and $z'/z = 0$. So this map is injective and hence an embedding. Identifying $\overline{Z} = \overline{Z}.1$ with $(Z^*)^{-1}Z$, $(Z^*)^{-1}A$ becomes a \overline{Z} -algebra.

Note that if Z does not contain zero-divisors in A , then A embeds into $(Z^*)^{-1}A$ via $a \mapsto 1^{-1}a$ for $a \in A$. In fact, this map is a Z -algebra homomorphism. If $1^{-1}a = 0$, then $a = 0$ by $0 = 1^{-1}0$ and (1.1.15). So this map is injective and hence an embedding. Moreover, we claim that the centre $Z((Z^*)^{-1}A)$ of $(Z^*)^{-1}A$ is equal to $\overline{Z}.1$. For, let $z^{-1}a \in Z((Z^*)^{-1}A)$. Then for all $b, c \in A$, identifying A with the Z -subalgebra of $(Z^*)^{-1}A$ by the above, we have $[z^{-1}a, b] = (z^{-1}a, b, c) = (b, z^{-1}a, c) = 0$. Since $0 = [z^{-1}a, b] = z^{-1}ab - z^{-1}ba = z^{-1}[a, b]$, we get $[a, b] = 0$. Also, since $0 = (z^{-1}a, b, c) = (z^{-1}ab)c - z^{-1}a(bc) = z^{-1}(a, b, c)$, we get $(a, b, c) = 0$, and similarly get $(b, a, c) = 0$. Hence $a \in Z$ and $z^{-1}a \in \overline{Z}.1$, which proves our claim.

The ring of quotients $(Z^*)^{-1}A$ of A modulo Z^* has the following universal property: Let B be an algebra over F and $Z(B)$ the centre. Let $f : A \rightarrow B$ be an F -algebra homomorphism (resp. anti-homomorphism) such that all elements of $f(Z^*)$ are invertible in B and $f(Z^*) \subset Z(B)$. Then there exists a unique F -algebra homomorphism (resp. anti-homomorphism) $\bar{f} : (Z^*)^{-1}A \rightarrow B$ such that

$\bar{f} \circ \iota = f$ where $\iota : A \longrightarrow (Z^*)^{-1}A$ is the natural map defined by $\iota(a) = 1^{-1}a$. In fact, we define $\bar{f}(z^{-1}a) = f(z)^{-1}f(a)$. Then \bar{f} is well-defined: Suppose that $z^{-1}a = z'^{-1}a'$. Then there exists $z_1 \in Z^*$ such that $z_1(z'a - za') = 0$, and so $f(z_1)f(z')f(a) = f(z_1)f(z)f(a')$. Hence

$$\begin{aligned} f(z)^{-1}f(a) &= (f(z_1)f(z')f(z))^{-1}f(z_1)f(z')f(a) \\ &= (f(z_1)f(z')f(z))^{-1}f(z_1)f(z)f(a') = f(z)^{-1}f(a'). \end{aligned}$$

One can readily check the other properties.

Finally, it is well-known that the map $\varphi : \bar{A} \longrightarrow (Z^*)^{-1}A$ defined by $\varphi((z'/z) \otimes a) = z^{-1}(z'a)$ is a \bar{Z} -algebra isomorphism (see [33, Exercise 2, page 192]). In fact, it is clear that φ is a surjective \bar{Z} -algebra homomorphism by the universal property of tensor algebras. For the injectivity, let $x := \sum_{i=1}^m (z'_i/z_i) \otimes a_i \in \bar{A}$ for $z_i \in Z^*$, $z'_i \in Z$ and $a_i \in A$, and suppose that $\varphi(x) = 0$. Put $z := z_1 \cdots z_m$ and $z_i^\wedge := z_1 \cdots \hat{z}_i \cdots z_m$ where \hat{z}_i means that z_i is omitted. Then we have $z'_i/z_i = (z'_i z_i^\wedge)/z$, and so

$$x = \sum_{i=1}^m \frac{z'_i}{z_i} \otimes a_i = \sum_{i=1}^m \frac{z'_i z_i^\wedge}{z} \otimes a_i = \sum_{i=1}^m \frac{1}{z} \otimes z'_i z_i^\wedge a_i = \frac{1}{z} \otimes \sum_{i=1}^m z'_i z_i^\wedge a_i.$$

Hence we get $\varphi(x) = z^{-1} \sum_{i=1}^m z'_i z_i^\wedge a_i = 0$, and so there exists some $z'' \in Z^*$ such that $z'' \sum_{i=1}^m z'_i z_i^\wedge a_i = 0$ by (1.1.14). Thus we obtain $z'' \cdot x = (1/z) \otimes (z'' \sum_{i=1}^m z'_i z_i^\wedge a_i) = 0$, and hence $x = z''^{-1}(z'' \cdot x) = 0$, i.e., φ is injective.

Via this identification, the following lemma becomes easy:

Lemma 1.1.16. *Let A be an algebra. Suppose that the centre $Z = Z(A)$ does not contain zero-divisors of A (e.g. a prime alternative or Jordan algebra). Then we have:*

- (i) A embeds into \bar{A} via $a \mapsto 1 \otimes a$ for all $a \in A$,
- (ii) \bar{A} is a central over \bar{Z} , i.e., $Z(\bar{A}) = \bar{Z} \cdot 1$,
- (iii) A does not have zero-divisors $\iff \bar{A}$ does not have zero-divisors,
- (iv) A is a Jordan domain $\iff \bar{A}$ is a Jordan domain.

(v) *For an algebra homomorphism (resp. anti-homomorphism) $f : A \longrightarrow B$ such that all elements of $f(Z^*)$ are invertible in B and $f(Z^*) \subset Z(B)$, the map $\bar{f} : \bar{A} \longrightarrow \bar{B}$ defined by $\bar{f}(z \otimes x) = f(z)^{-1}f(x)$ is an algebra homomorphism (resp. anti-homomorphism).*

Proof. For (i) and (ii) are clear from the identification $1 \otimes a$ with $1^{-1}a = 0$ and the identification of \bar{A} with $(Z^*)^{-1}A$.

For (iii). let $z_1^{-1}a_1, z_2^{-1}a_2 \in \bar{A}$. Then $(z_1^{-1}a_1)(z_2^{-1}a_2) = (z_1z_2)^{-1}(a_1a_2) = 0 \iff 1^{-1}(a_1a_2) = 0 \iff a_1a_2 = 0$.

For (iv). let $z_1^{-1}a_1, z_2^{-1}a_2 \in \bar{A}$. Then $U_{z_1^{-1}a_1}(z_2^{-1}a_2) = (z_1^2z_2)^{-1}(U_{a_1}a_2) = 0 \iff U_{a_1}a_2 = 0$.

(v) is clear by the universal property of $(Z^*)^{-1}A$ above. \square

Remark 1.1.17. It is also true that A is semiprime (nondegenerate, prime) $\iff \bar{A} = (Z^*)^{-1}A$ is semiprime (nondegenerate, prime, respectively) for any algebra A (see [33, Proposition 2, page 186]).

§1.2 EXTENDED AFFINE LIE ALGEBRAS OF TYPE A_1

We start by reviewing extended affine Lie algebras following [1] and [2]. Let \mathcal{L} be a Lie algebra over \mathbb{C} (the field of complex numbers). Assume that

(EA1) \mathcal{L} has a nondegenerate invariant symmetric bilinear form (\cdot, \cdot) .

Here 'invariant' means that (\cdot, \cdot) satisfies $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathcal{L}$.

(EA2) \mathcal{L} has a nontrivial finite dimensional self-centralizing ad-diagonalizable abelian subalgebra \mathcal{H} .

We will be assuming three further axioms about the triple $(\mathcal{L}, (\cdot, \cdot), \mathcal{H})$. To describe them we need some further notation. Because of (EA2), we have

$$\mathcal{L} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{L}_\alpha \quad \text{and} \quad \mathcal{L}_0 = \mathcal{H}$$

where $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$ and \mathcal{H}^* is the complex dual space of \mathcal{H} . Let $R = \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_\alpha \neq (0)\}$. R is called the *root system* of \mathcal{L} . Note that since $\mathcal{H} \neq (0)$, we have $0 \in R$. Also,

$$\alpha, \beta \in R, \alpha + \beta \neq 0 \implies (\mathcal{L}_\alpha, \mathcal{L}_\beta) = \{0\}.$$

Thus, $-R = R$ by nondegeneracy. Moreover, (\cdot, \cdot) is nondegenerate on \mathcal{H} . As in the classical theory of finite-dimensional complex semisimple Lie algebras, we can transfer (\cdot, \cdot) to a form on \mathcal{H}^* . Let

$$R^\times = \{\alpha \in R \mid (\alpha, \alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R \mid (\alpha, \alpha) = 0\}.$$

The elements of R^\times (resp. R^0) are called *non-isotropic* (resp. *isotropic*) roots. We have $R = R^\times \cup R^0$. We further require that

(EA3) $\alpha \in R^\times$, $x_\alpha \in \mathcal{L}_\alpha \implies \text{ad } x_\alpha$ acts locally nilpotently on \mathcal{L} .

(EA4) R is a discrete subset of \mathcal{H}^* .

(EA5) R is irreducible. That is,

(a) $R^\times = R_1 \cup R_2$, $(R_1, R_2) = (0) \implies R_1 = \emptyset$ or $R_2 = \emptyset$

(b) $\sigma \in R^0 \implies$ there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R^\times$.

If \mathcal{L} satisfies (EA1)-(EA5), the triple $(\mathcal{L}, (\cdot, \cdot), \mathcal{H})$, or simply the algebra \mathcal{L} itself, is called an *extended affine Lie algebra* or EALA for short.

Example 1.2.1. Any finite-dimensional complex simple Lie algebra is an EALA. Indeed, let H be a Cartan subalgebra and let (\cdot, \cdot) be the Killing form of L . Then (EA1)-(EA5) express well-known properties of L (see e.g. [12]). Note that $R^\times = R \setminus \{0\}$ in this case.

Let t_α be the unique element of \mathcal{H} so that $(t_\alpha, h) = \alpha(h)$ for $h \in \mathcal{H}$, and put $h_\alpha = \frac{2}{(\alpha, \alpha)} t_\alpha$. Then there exist nonzero $e_\alpha \in \mathcal{L}_\alpha$ and $f_\alpha \in \mathcal{L}_{-\alpha}$ so that $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$ and $[e_\alpha, f_\alpha] = h_\alpha$. In other words, $\{e_\alpha, h_\alpha, f_\alpha\}$ is an sl_2 -triplet. Thus we can use sl_2 -theory. Assuming only (EA1), (EA2) and (EA3), one can show that some well-known properties of finite dimensional semisimple Lie algebras over \mathbb{C} are also true for EALA's.

Lemma 1.2.2. ([1, Lemma I.1.21 and Theorem I.1.29]) *Let $\alpha \in R^\times$, $\beta \in R$ and $r = 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)}$. Then*

(i) $2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

(ii) $\dim_{\mathbb{C}} \mathcal{L}_\alpha = 1$.

(iii) *Assume that $\text{ade}_\alpha(e_\beta) = 0$. Then $r \geq 0$,*

$$(\text{ad } f_\alpha)^i(e_\beta) \neq 0 \quad \text{for all } i = 0, 1, \dots, r.$$

$$\text{and } (\text{ad } f_\alpha)^{r+1}(e_\beta) = 0. \quad \square$$

In the following, \mathcal{L} is an EALA with root system R . We recall some of the properties of R that we will need. First, note that if a nondegenerate invariant symmetric bilinear form on \mathcal{L} is multiplied by a nonzero complex number, then we still have such a form. Since the axioms are invariant under such a change, we may as well assume that there is some non-isotropic root $\alpha \in R^\times$ with $(\alpha, \alpha) \in \mathbb{R}_{>0}$. Then if $\beta \in R^\times$ we have $2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ so that $(\beta, \alpha) \in \mathbb{R}$, and hence, since $2 \frac{(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ we get $(\beta, \beta) \in \mathbb{R}$ if $(\alpha, \beta) \neq 0$. It now follows, using (EA5)(a), that $(\alpha, \beta) \in \mathbb{R}$ for any $\alpha, \beta \in R$. That is,

our form is real valued on the real linear span of the roots. From now on *we assume that our form is scaled so that there is at least one $\alpha \in R^\times$ with $(\alpha, \alpha) > 0$* . Let \mathcal{V} be the real span of R in \mathcal{H} . Then it was proven in [1, Theorem I.2.14] that the real valued symmetric bilinear form $(\cdot, \cdot)|_{\mathcal{V}}$ is positive semidefinite on \mathcal{V} . This was a conjecture of Kac. Thus one can define the following concepts. Let

$$\mathcal{V}^0 = \{\alpha \in \mathcal{V} \mid (\alpha, \beta) = 0 \text{ for all } \beta \in \mathcal{V}\}.$$

The *nullity* of R or of \mathcal{L} is defined to be the real dimension n of \mathcal{V}^0 . Let $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and let $\bar{\cdot} : \mathcal{V} \rightarrow \bar{\mathcal{V}}$ be the canonical projection. Then (\cdot, \cdot) induces a positive definite symmetric bilinear form on $\bar{\mathcal{V}}$ so that, relative to this form, the image \bar{R} of R in $\bar{\mathcal{V}}$ is a finite irreducible (possibly non-reduced) root system [1, Proposition I.2.19]. (Note however that $0 \in \bar{R}$. Contrary to the usual convention, we are requiring that a finite root system contains 0.) The *type* of R or of \mathcal{L} is defined to be the type of the finite root system \bar{R} . We next want to lift \bar{R} to a finite root system \dot{R} inside \mathcal{V} . To do this, we fix a choice of a fundamental system $\bar{\Pi} = \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\}$ for \bar{R} . Also, for $i = 1, \dots, l$, we choose a fixed preimage $\dot{\alpha}_i \in R$ of $\bar{\alpha}_i$ under $\bar{\cdot}$. Let $\dot{\Pi} = \{\dot{\alpha}_1, \dots, \dot{\alpha}_l\}$ and $\dot{\mathcal{V}} = \text{Span}_{\mathbb{R}} \dot{\Pi}$. Then we have $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and $\bar{\cdot}$ restricts to an isometry of $\dot{\mathcal{V}}$ onto $\bar{\mathcal{V}}$. Let

$$\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \dot{\alpha} + \sigma \in R \text{ for some } \sigma \in \mathcal{V}^0\}.$$

In other words, \dot{R} is the image of R under the projection of \mathcal{V} onto $\dot{\mathcal{V}}$ with kernel \mathcal{V}^0 . Put $\dot{R}^\times = \dot{R} \setminus \{0\}$. For each $\dot{\alpha} \in \dot{R}^\times$, we define

$$S_{\dot{\alpha}} = \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R\}.$$

Then, by definition, we have $R = R^0 \cup \left(\bigcup_{\dot{\alpha} \in \dot{R}^\times} (\dot{\alpha} + S_{\dot{\alpha}})\right)$. From [1, Chapter II], we have the following results:

$$(1.2.3) \quad S_{\dot{\alpha}} \subset R^0, \text{ and } S_{\dot{\alpha}} \text{ depends only on the length of } \dot{\alpha}.$$

Moreover, if $\dot{\alpha} \in \dot{R}^\times$ has minimal length, then

$$R^0 = S_{\dot{\alpha}} + S_{\dot{\alpha}}.$$

Also, if $\dot{\alpha} \in \dot{R}^\times$ is reduced in \dot{R} (i.e., $\frac{1}{2}\dot{\alpha} \notin \dot{R}$), then $S := S_{\dot{\alpha}}$ is a discrete spanning set in \mathcal{V}^0 . We put

$$(1.2.4) \quad \Lambda = \langle R^0 \rangle := \text{the subgroup of } \mathcal{V}^0 \text{ generated by } R^0.$$

Then Λ is a lattice in \mathcal{V}^0 [1, Corollary II.2.31], that is, Λ is a discrete subgroup of \mathcal{V}^0 which spans \mathcal{V}^0 , and hence Λ is the \mathbb{Z} -span of a basis of \mathcal{V}^0 . It follows from (1.2.3) and (1.2.4) that

$$R \subset \Lambda \cup \left(\bigcup_{\dot{\alpha} \in R^\times} (\dot{\alpha} + \Lambda) \right) = \dot{R} + \Lambda.$$

Thus,

$$(1.2.5) \quad \mathcal{L} = \bigoplus_{\dot{\alpha} \in \dot{R}, \sigma \in \Lambda} \mathcal{L}_{\dot{\alpha} + \sigma},$$

and the description of R is reduced to the description of $S_{\dot{\alpha}}$ for each $\dot{\alpha} \in \dot{R}$. In particular, if R^\times has only one root length, it is shown in [1] that all $S_{\dot{\alpha}}$ are equal, say $S := S_{\dot{\alpha}}$, and so

$$R = (S + S) \cup (\dot{R}^\times + S).$$

Clearly $0 \in S$, and it is shown in [1] that $2\sigma - \tau \in S$ whenever $\sigma, \tau \in S$. In general, a subset S of a lattice Λ which has the following three properties.

- (i) $0 \in S$,
- (ii) $2\sigma - \tau \in S$ for all $\sigma, \tau \in S$,
- (iii) S generates Λ ,

is called a *semilattice in* Λ . So our S defined above is a semilattice in Λ . The following definition is important.

Definition 1.2.6. The *core* of \mathcal{L} is defined to be the subalgebra \mathcal{L}_c of \mathcal{L} generated by the spaces \mathcal{L}_α , $\alpha \in R^\times$.

One can check that \mathcal{L}_c is an ideal of \mathcal{L} . Moreover,

$$\mathcal{L}_c = \bigoplus_{\dot{\alpha} \in \dot{R}^\times, \sigma \in \Lambda} \mathcal{L}_{\dot{\alpha} + \sigma} \oplus \sum_{\dot{\alpha} \in \dot{R}^\times, \sigma, \tau \in \Lambda} [\mathcal{L}_{\dot{\alpha} + \sigma}, \mathcal{L}_{-\dot{\alpha} + \tau}],$$

and hence \mathcal{L}_c is perfect, i.e., $\mathcal{L}_c = [\mathcal{L}_c, \mathcal{L}_c]$.

For the remainder of this section, let

\mathcal{L} be an extended affine Lie algebra of type A_1 .

In this case, \overline{R} is of type A_1 and so is \dot{R} . Thus, we have $\dot{\Pi} = \{\dot{\alpha}\}$, $\dot{R} = \{0, \pm\dot{\alpha}\}$ and

$$R = (S + S) \cup (\dot{\alpha} + S) \cup (-\dot{\alpha} + S)$$

where S is a semilattice in Λ . We put

$$\mathcal{L}^\sigma := \mathcal{L}_{-\dot{\alpha}+\sigma} \oplus \mathcal{L}_\sigma \oplus \mathcal{L}_{\dot{\alpha}+\sigma} \quad \text{for } \sigma \in \Lambda.$$

Then, by (1.2.5), we have

$$\mathcal{L} = \bigoplus_{\sigma \in \Lambda} \mathcal{L}^\sigma \quad \text{and} \quad [\mathcal{L}^\sigma, \mathcal{L}^\tau] \subset \mathcal{L}^{\sigma+\tau} \quad \text{for } \sigma, \tau \in \Lambda.$$

In other words, \mathcal{L} is a Λ -graded Lie algebra. Moreover, \mathcal{L}_c is generated by homogeneous elements and so \mathcal{L}_c is a Λ -graded subalgebra of \mathcal{L} . Thus,

$$\mathcal{L}_c = \bigoplus_{\sigma \in \Lambda} (\mathcal{L}_c)^\sigma,$$

where for $\sigma \in \Lambda$, and

$$(\mathcal{L}_c)^\sigma := \mathcal{L}_c \cap \mathcal{L}^\sigma = \mathcal{L}_{-\dot{\alpha}+\sigma} \oplus \sum_{\tau, \nu \in \Lambda, \tau+\nu=\sigma} [\mathcal{L}_{\dot{\alpha}+\tau}, \mathcal{L}_{-\dot{\alpha}+\nu}] \oplus \mathcal{L}_{\dot{\alpha}+\sigma}.$$

Also, let

$$\begin{aligned} (\mathcal{L}_c)_{-\dot{\alpha}} &:= \bigoplus_{\sigma \in \Lambda} \mathcal{L}_{-\dot{\alpha}+\sigma}, & (\mathcal{L}_c)_{\dot{\alpha}} &:= \bigoplus_{\sigma \in \Lambda} \mathcal{L}_{\dot{\alpha}+\sigma} \\ \text{and } (\mathcal{L}_c)_0 &:= \sum_{\tau, \nu \in \Lambda} [\mathcal{L}_{\dot{\alpha}+\tau}, \mathcal{L}_{-\dot{\alpha}+\nu}]. \end{aligned}$$

Then we have

$$\mathcal{L}_c = (\mathcal{L}_c)_{-\dot{\alpha}} \oplus (\mathcal{L}_c)_0 \oplus (\mathcal{L}_c)_{\dot{\alpha}}.$$

Let $\dot{\mathcal{G}} := \langle e_{\dot{\alpha}}, h_{\dot{\alpha}}, f_{\dot{\alpha}} \rangle \cong sl_2(\mathbb{C})$ and $\dot{\mathcal{H}} := \mathbb{C}h_{\dot{\alpha}}$, which are both subalgebras of \mathcal{L}_c . Obviously, for each $\varepsilon = 0, -1, 1$

$$(\mathcal{L}_c)_{\varepsilon\dot{\alpha}} = \{x \in \mathcal{L}_c \mid [h, x] = \varepsilon\dot{\alpha}(h)x \text{ for all } h \in \dot{\mathcal{H}}\},$$

and \mathcal{L}_c is generated as an algebra by $(\mathcal{L}_c)_{-\dot{\alpha}}$ and $(\mathcal{L}_c)_{\dot{\alpha}}$. Therefore, \mathcal{L}_c is graded by the root system A_1 as defined in [5]. By the description of such Lie algebras (see [4] or [23]), \mathcal{L}_c is a central extension of the TKK-algebra of a unital Jordan algebra, called the *coordinate algebra of \mathcal{L}* . The Jordan algebra J is defined as follows. Let $J := (\mathcal{L}_c)_{\dot{\alpha}}$ as a \mathbb{C} -vector space and define a multiplication on J by

$$(1.2.7) \quad xy := \frac{1}{2} [[x, f_{\dot{\alpha}}], y] \quad \text{for } x, y \in J.$$

Then one can check that this multiplication is commutative and satisfies the Jordan identity in §1.1, and so J is a Jordan algebra over \mathbb{C} . Note that $e_{\dot{\alpha}}$ is the identity element of J . Our goal is to describe the structure of J . We put $J_{\sigma} := \mathcal{L}_{\dot{\alpha}+\sigma}$. Then

$$1 := e_{\dot{\alpha}} \in J_0,$$

and one can easily see that

$$J = \bigoplus_{\sigma \in \Lambda} J_{\sigma} \text{ is a } \Lambda\text{-graded Jordan algebra over } \mathbb{C}.$$

i.e., $J_{\sigma}J_{\tau} \subset J_{\sigma+\tau}$. Also, by 1.2.2(ii), we have

$$\dim_{\mathbb{C}} J_{\sigma} = 1 \quad \text{if } \sigma \in S \quad \text{and} \quad J_{\sigma} = (0) \quad \text{otherwise.}$$

These conditions are not enough to classify J . To obtain a crucial property of J , the ‘invertibility of nonzero homogeneous elements’, we first prove the following easy consequence of sl_2 -theory:

Lemma 1.2.8. *For $\sigma, \tau \in S$ and $\varepsilon = \pm 1$, we have*

$$[e_{\varepsilon\dot{\alpha}+\sigma}, [e_{\varepsilon\dot{\alpha}+\sigma}, e_{-\varepsilon\dot{\alpha}+\tau}]] \neq 0,$$

and hence, $[\mathcal{L}_{\varepsilon\dot{\alpha}+\sigma}, [\mathcal{L}_{\varepsilon\dot{\alpha}+\sigma}, \mathcal{L}_{-\varepsilon\dot{\alpha}+\tau}]] = \mathcal{L}_{\varepsilon\dot{\alpha}+2\sigma+\tau}$.

Proof. We only show the case $\varepsilon = 1$ since the case $\varepsilon = -1$ is done by the same manner. Take $\alpha := -\dot{\alpha} - \sigma$ and $\beta := -\dot{\alpha} + \tau$ in 1.2.2(iii). Then $[e_{\alpha}, e_{\beta}] \in \mathcal{L}_{-2\dot{\alpha}-\sigma+\tau} = (0)$ since A_1 is reduced. Also, we have $2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = 2$, and hence by 1.2.2(iii), $[f_{-\dot{\alpha}-\sigma}, [f_{-\dot{\alpha}-\sigma}, e_{-\dot{\alpha}+\tau}]] \neq 0$. Since $\dim_{\mathbb{C}} \mathcal{L}_{\dot{\alpha}+\sigma} = 1$ (see 1.2.2(ii)), there exists some $0 \neq c \in \mathbb{C}$ such that $f_{-\dot{\alpha}-\sigma} = ce_{\dot{\alpha}+\sigma}$, and so $0 \neq [e_{\dot{\alpha}+\sigma}, [e_{\dot{\alpha}+\sigma}, e_{-\dot{\alpha}+\tau}]] \in \mathcal{L}_{\varepsilon\dot{\alpha}+2\sigma+\tau}$. \square

The following fact is well-known in the theory of Jordan pairs [23]:

$$(*) \quad U_x y = \frac{1}{2} [x, [x, \bar{y}]] \quad \text{where} \quad \bar{y} := \frac{1}{2} [f_{\dot{\alpha}}, [f_{\dot{\alpha}}, y]] \quad \text{for all } x, y \in J.$$

For the convenience of the reader, we show this by direct calculation from (1.2.7), using the skew symmetry and the Jacobi identity of the bracket $[\cdot, \cdot]$. Put $f := f_{\dot{\alpha}}$.

$$\begin{aligned} U_x y &= 2x(xy) - x^2 y = [[x, f], xy] - \frac{1}{2} [[x^2, f], y] \\ &= \frac{1}{2} [[x, f], [[x, f], y]] - \frac{1}{4} [[[[x, f], x], f], y], \end{aligned}$$

and

$$[[[[x, f], x], f], y] = [[[[x, f], x], y], f] + [[[[x, f], x], [f], y]] = [[[[x, f], x], [f], y]]$$

since $[[[x, f], x], y] \in ((\mathcal{L}_c)_{\dot{\alpha}}, (\mathcal{L}_c)_{\dot{\alpha}}) = (0)$. Moreover, we have

$$\begin{aligned} [[[[x, f], x], [f], y]] &= [[[[x, f], [f], y], x] + [[x, f], [x, [f], y]] \\ &= -[x, [[x, f], [f], y]] + [[x, f], [[x, f], y]] \end{aligned}$$

since $[x, y] \in ((\mathcal{L}_c)_{\dot{\alpha}}, (\mathcal{L}_c)_{\dot{\alpha}}) = (0)$. Thus we get

$$U_x y = \frac{1}{4} [[x, f], [[x, f], y]] + \frac{1}{4} [x, [[x, f], [f], y]].$$

On the other hand,

$$\begin{aligned} \frac{1}{2} [x, [x, \bar{y}]] &= \frac{1}{4} [x, [x, [f, [f, y]]]] \\ &= \frac{1}{4} [x, [[x, f], [f], y]] + \frac{1}{4} [x, [f, [x, [f], y]]], \end{aligned}$$

and

$$\begin{aligned} [x, [f, [x, [f], y]]] &= [[x, f], [x, [f], y]] + [f, [x, [x, [f], y]]] \\ &= [[x, f], [x, [f], y]] \quad \text{since } [x, [x, [f], y]] = 0 \\ &= [[x, f], [[x, f], y]] \quad \text{since } [x, y] = 0. \end{aligned}$$

Thus we obtain

$$\frac{1}{2} [x, [x, \bar{y}]] = \frac{1}{4} [x, [[x, f], [f], y]] + \frac{1}{4} [[x, f], [[x, f], y]] = U_x y.$$

Now, for $\sigma \in S$, let $0 \neq x \in J_{\sigma}$. Since S is a semilattice, there exists $0 \neq y \in J_{-2\sigma}$. Then, by (*) and 1.2.8, we have $0 \neq \frac{1}{2} [x, [x, \bar{y}]] = U_x y \in J_{2\sigma-2\sigma} = J_0$. Since $\dim_{\mathbb{C}} J_{-2\sigma} = \dim_{\mathbb{C}} J_0 = 1$ and $1 \in J_0$, there exists $c \in \mathbb{C}$ such that $U_x(cy) = 1$. Hence by 1.1.2(3), x is invertible. Thus any nonzero element in J_{σ} for all $\sigma \in S$ is invertible. Consequently, we have obtained some necessary conditions of the coordinate algebra J . For convenience, we put $\Lambda = \mathbb{Z}^n$ as a free abelian group of rank n .

Theorem 1.2.9. *The core of an extended affine Lie algebra of type A_1 is isomorphic to a central extension of the TKK algebra constructed from a unital \mathbb{Z}^n -graded Jordan algebra $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ over \mathbb{C} satisfying*

- (T1) $\text{supp}(J) := \{\alpha \in \mathbb{Z}^n \mid J_\alpha \neq (0)\}$ generates \mathbb{Z}^n ,
- (T2) any $0 \neq x \in J_\alpha$ for any $\alpha \in \text{supp}(J)$ is invertible,
- (T3) $\dim_{\mathbb{C}} J_\alpha = 1$ for all $\alpha \in \text{supp}(J)$. \square

We call such an algebra a *Jordan torus over \mathbb{C}* . In Chapter 3, we will classify Jordan tori in a more general setting.

Remark 1.2.10. By the argument above, we know that $1 \in J_0$ and that $S = \text{supp}(J)$ is a semilattice in $\Lambda = \mathbb{Z}^n$. However, we do not need to assume these since they are easy consequences of the definition of Jordan tori (see 2.1.2 and 2.1.3(2)).

Chapter 2 Graded algebras

Unless mentioned otherwise all algebras in this chapter are either unital alternative algebras over a field F or unital linear Jordan algebras over F of characteristic $\neq 2$.

§ 2.1 DIVISION G -GRADED ALGEBRAS

For a group G , an algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ is called G -graded if $\mathcal{T}_g \mathcal{T}_h \subset \mathcal{T}_{gh}$ for all $g, h \in G$. We will refer to \mathcal{T}_g as a *homogeneous space* and an element of \mathcal{T}_g as a *homogeneous element of degree g* . A graded algebra is called *of strong type* or *strongly graded* if $\mathcal{T}_h \mathcal{T}_g = \mathcal{T}_{hg}$ for all $h, g \in G$. *predivision graded* if every nonzero homogeneous space contains an invertible element and *division graded* if any nonzero element of every homogeneous space is invertible. We begin by introducing a simple method of constructing a new G -graded algebra from a G -graded algebra.

Lemma 2.1.1. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a (predivision) G -graded algebra over F and Z a commutative associative algebra over F . Then $\mathcal{T}_Z := Z \otimes_F \mathcal{T} = \bigoplus_{g \in G} (Z \otimes_F \mathcal{T}_g)$ is a (predivision) G -graded algebra over F .*

Proof. Since Z is commutative and associative, \mathcal{T}_Z is an algebra over F , alternative or Jordan depending on \mathcal{T} . Clearly we have $(Z \otimes_F \mathcal{T}_g)(Z \otimes_F \mathcal{T}_h) \subset Z \otimes_F \mathcal{T}_{gh}$ for $g, h \in G$, and so \mathcal{T}_Z is a G -graded algebra over F . If $x \in \mathcal{T}_g$ is invertible in \mathcal{T} , then $1 \otimes x$ is invertible in \mathcal{T}_Z . Hence if \mathcal{T} is predivision, then so is \mathcal{T}_Z . \square

For a G -graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$, let

$$S := \text{supp}(\mathcal{T}) = \{g \in G \mid \mathcal{T}_g \neq (0)\}.$$

Lemma 2.1.2. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a G -graded algebra. Then $1 \in \mathcal{T}_e$ where e is the identity element of G . In particular, $e \in S$.*

Proof. Let $1 = \sum_{g \in G} x_g \in \mathcal{T}$. For any $u \in \mathcal{T}_h$, $h \in G$, we have $u = 1u = \sum_{g \in G} x_g u \in \mathcal{T}_h$ since 1 is a left identity element. Since G is a group, we have $x_e u = u$ (and $x_g u = 0$ if $g \neq e$). Thus x_e is a left identity element. Hence we have $1 = x_e 1 = x_e \in \mathcal{T}_e$ since 1 is a right identity element. \square

If \mathcal{T} is predivision graded, then S has the following properties:

Lemma 2.1.3. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a G -graded algebra.*

(0) *For $h \in S = \text{supp}(\mathcal{T})$, suppose that $x \in \mathcal{T}_h$ is invertible. Then $x^{-1} \in \mathcal{T}_{h^{-1}}$.*

(1) *If \mathcal{T} is a predivision G -graded alternative algebra, then S is a subgroup of G .*

(2) *If \mathcal{T} is a predivision G -graded Jordan algebra, then S satisfies the following properties: for $h, k \in S$.*

$$(2.1.4) \quad e \in S, \quad h^2k \in S \quad \text{and} \quad h^{-1} \in S.$$

Proof. For (0), let $x^{-1} = \sum_{g \in G} x_g$. Then

$$\sum_{g \in G} x x_g = \sum_{g \in G} x_g x = 1 \in \mathcal{T}_e \quad \text{and} \quad \sum_{g \in G} x^2 x_g = x \in \mathcal{T}_h.$$

Since $g = h^{-1}$ is the unique solution of $hg = gh = e$ and $h^2g = h$, we have $x x_{h^{-1}} = x_{h^{-1}} x = 1$ and $x^2 x_{h^{-1}} = x$. Thus, by the uniqueness of the inverse, we obtain $x^{-1} = x_{h^{-1}} \in \mathcal{T}_{h^{-1}}$ in both alternative and Jordan cases.

For (1), let $h, k \in S$. Since \mathcal{T} is predivision graded, there exist invertible elements $x \in \mathcal{T}_h$ and $y \in \mathcal{T}_k$. Hence $0 \neq xy \in \mathcal{T}_{hk}$ by 1.1.2(1), and so we have $hk \in S$. Thus, by (0), S is a subgroup.

For (2), let $h, k \in S$. Since \mathcal{T} is predivision graded, there exist invertible elements $x \in \mathcal{T}_h$ and $y \in \mathcal{T}_k$. Hence $0 \neq U_x y \in \mathcal{T}_{h^2k}$ by 1.1.2(3), and so we have $h^2k \in S$. Thus, by 2.1.1 and (0), S satisfies (2.1.4). \square

For a G -graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$, we note that $\mathcal{T} = \bigoplus_{g \in \langle S \rangle} \mathcal{T}_g$, where $\langle S \rangle$ is the subgroup of G generated by S . From now on,

we always assume that $\langle S \rangle = G$.

Thus, if \mathcal{T} is a predivision G -graded alternative algebra, then by 2.1.3(1), this assumption is equivalent to saying that $S = G$, or every homogeneous space is nonzero.

Lemma 2.1.5. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a G -graded algebra over F .*

(0) *For $g \in G$, $h \in S = \text{supp}(\mathcal{T})$ and an invertible element $y \in \mathcal{T}_h$, if $(x, y^{-1}, y) = (y, y^{-1}, x) = 0$ for all $x \in \mathcal{T}$, then we have $\mathcal{T}_{gh} = \mathcal{T}_g y$ and $\mathcal{T}_{hg} = y \mathcal{T}_g$.*

(1) *If \mathcal{T} is a predivision G -graded alternative algebra, then it is of strong type. Also, if $(x_g)_{g \in G}$ is a family of invertible elements, then we have $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_e x_g$ and $\dim_F \mathcal{T}_g = \dim_F \mathcal{T}_h$ for all $g, h \in G$.*

(2) If \mathcal{T} is a predivision G -graded Jordan algebra, then it satisfies

$$U_{\mathcal{T}_g}\mathcal{T}_h = \mathcal{T}_{g^2h} \quad \text{for all } g, h \in S.$$

Proof. For (0), we have by 2.1.2 and our assumption $(x, y^{-1}, y) = 0$,

$$\mathcal{T}_{gh} = \mathcal{T}_{gh}(y^{-1}y) = (\mathcal{T}_{gh}y^{-1})y \subset (\mathcal{T}_{gh}\mathcal{T}_{h^{-1}})y \subset \mathcal{T}_gy \subset \mathcal{T}_{gh}.$$

Hence we get $\mathcal{T}_{gh} = \mathcal{T}_gy$, and similarly $\mathcal{T}_{hg} = y\mathcal{T}_g$ by $(y, y^{-1}, x) = 0$.

For (1), $\mathcal{T}_{hg} = \mathcal{T}_hy$ holds for any $g, h \in G = S$ and any invertible $y \in \mathcal{T}_g$ by 1.1.2(1). Since $\mathcal{T}_{hg} = \mathcal{T}_hy \subset \mathcal{T}_h\mathcal{T}_g \subset \mathcal{T}_{hg}$, we obtain $\mathcal{T}_h\mathcal{T}_g = \mathcal{T}_{hg}$.

Define a map $\varphi : \mathcal{T}_g = \mathcal{T}_e x_g \longrightarrow \mathcal{T}_h = \mathcal{T}_e x_h$ by $\varphi(xx_g) = xx_h$ for all $x \in \mathcal{T}_e$, which is well-defined by the invertibility of x_g and 1.1.2(1). Then φ is clearly an isomorphism of F -vector spaces by the invertibility of x_h and 1.1.2(1). Hence we get $\dim_F \mathcal{T}_g = \dim_F \mathcal{T}_h$ for all $g, h \in G$.

For (2), it is enough to show that $\mathcal{T}_{g^2h} \subset U_{\mathcal{T}_g}\mathcal{T}_h$ for all $h, g \in S$. Take $x \in \mathcal{T}_{g^2h}$. Since $g^{-1} \in S$, there exists an invertible element $y \in \mathcal{T}_{g^{-1}}$ so that $U_y x \in \mathcal{T}_h$. Note that $y^{-1} \in \mathcal{T}_g$ by 2.1.3. Since $U_y^{-1} = U_{y^{-1}}$ by 1.1.2(3), we get $x = U_y^{-1}U_y x = U_{y^{-1}}U_y x \in U_{\mathcal{T}_g}\mathcal{T}_h$. \square

Remark 2.1.6. For a G -graded alternative algebra, we have shown in 2.1.5

$$\text{predivision graded} \implies \text{strong type.}$$

However, this is not true in general, even for a division G -graded Jordan algebra (e.g. Example 2.5.13). Also, the converse, i.e., strong type \implies predivision graded, is not true even for associative algebras (see §5.0 and [24, Exercise 3, page 18]).

We introduce a simple method of constructing a division G -graded Jordan algebra from a division G -graded alternative algebra for an abelian group G .

Lemma 2.1.7. *Let \mathcal{T} be a predivision (resp. division) G -graded alternative algebra over F for an abelian group G with $\text{ch.}F \neq 2$. Then \mathcal{T}^+ is a predivision (resp. division) G -graded Jordan algebra over F .*

Proof. Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$. First of all, \mathcal{T}^+ is a Jordan algebra over F as mentioned in the beginning of §1.1. Since $\mathcal{T}^+ = \mathcal{T}$ as a vector space, \mathcal{T}^+ has the same G -grading

as \mathcal{T} as vector spaces, and so $\text{supp}(\mathcal{T}^+) = \text{supp}(\mathcal{T})$. For $x \in \mathcal{T}_g$ and $y \in \mathcal{T}_h$, we have $x \cdot y = 1/2(xy + yx) \in \mathcal{T}_{gh} + \mathcal{T}_{hg} = \mathcal{T}_{gh}$ since G is abelian, and so \mathcal{T}^+ is a G -graded algebra. Since the invertible elements of \mathcal{T} and \mathcal{T}^+ coincide by 1.1.2, the proof is finished. \square

We give an example of a division G -graded Jordan algebra where G is not abelian.

Example 2.1.8. First we prove the following:

Claim. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a predivision G -graded Jordan algebra. Then for all $g, h \in S$, we have the relation $g^2h = hg^2$.*

Proof. If $gh = hg$, then the relation is clear. Suppose that $gh \neq hg$. Let $x \in \mathcal{T}_g$ and $y \in \mathcal{T}_h$ be invertible. The commutativity of \mathcal{T} implies that $xy = yx \in \mathcal{T}_{gh} \cap \mathcal{T}_{hg}$, which forces $xy = 0$ by our assumption, $gh \neq hg$. Thus we have $0 \neq U_{xy} = 2x(xy) - x^2y = -x^2y = -yx^2 \in \mathcal{T}_{g^2h} \cap \mathcal{T}_{hg^2}$. Hence we obtain $g^2h = hg^2$. \square

By this claim, we need to find a nonabelian group G satisfying $g^2h = hg^2$ for all $g, h \in G$. Here is such an example; let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8, or the group of 8 units of the real quaternion $\mathbb{H} = \mathbb{R}.1 \oplus \mathbb{R}.i \oplus \mathbb{R}.j \oplus \mathbb{R}.k$.

We construct a division G -graded Jordan algebra over \mathbb{Q} (the field of rational numbers). Let

$$\mathcal{D} = (\mathbb{Q}(\sqrt{2}), -\sqrt{2}, -\sqrt{2})$$

be the quaternion algebra over a field $\mathbb{Q}(\sqrt{2})$ with structure constants $\mu_1 = \mu_2 = -\sqrt{2}$ (see §4.2). Since $(\mathbb{Q}(\sqrt{2}), -\sqrt{2}) = \mathbb{Q}(\sqrt{2})(-\sqrt{2})$ is a quadratic field extension of $\mathbb{Q}(\sqrt{2})$ adding the negative number $-\sqrt{2}$, all the norms of nonzero elements of $\mathbb{Q}(\sqrt{2})(-\sqrt{2})$ are a positive real numbers. Hence $-\sqrt{2}$ is not a norm of an element of $\mathbb{Q}(\sqrt{2})(-\sqrt{2})$. This implies that \mathcal{D} is a division algebra over $\mathbb{Q}(\sqrt{2})$ (see e.g. [18, page 232]). Let $i := ((0, 1), (0, 0))$ and $j := ((0, 0), (1, 0))$ in $\mathcal{D} = \mathbb{Q}(\sqrt{2})^4$, and $k := ij$ so that $i^2 = j^2 = k^2 = -\sqrt{2}$ and

$$(*) \quad ij + ji = jk + kj = ki + ik = 0.$$

Let $\mathcal{D}_1 = \mathbb{Q}.1$, $\mathcal{D}_{-1} = \mathbb{Q}.\sqrt{2}$, $\mathcal{D}_i = \mathbb{Q}.i$, $\mathcal{D}_{-i} = \mathbb{Q}.\sqrt{2}i$, $\mathcal{D}_j = \mathbb{Q}.j$, $\mathcal{D}_{-j} = \mathbb{Q}.\sqrt{2}j$, $\mathcal{D}_k = \mathbb{Q}.k$ and $\mathcal{D}_{-k} = \mathbb{Q}.\sqrt{2}k$. One can verify that $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is a division G -graded associative algebra over \mathbb{Q} . Moreover, one can check that $\mathcal{D}^+ = \bigoplus_{g \in G} \mathcal{D}_g$ is a division G -graded Jordan algebra over \mathbb{Q} by (*) even though G is not abelian (\mathcal{D}^+ is clearly not of strong type).

§ 2.2 PRIMENESS AND UNITS

Recall that a group G is an *ordered group* ([18, page 100]) if G is totally ordered by $<$ and satisfies, for all $g, h, k \in G$,

$$g < h \implies gk < hk \quad \text{and} \quad kg < kh.$$

Lemma 2.2.1. *Let $G = (G, <)$ be an ordered group. Then for any $g, h, k, l \in G$,*

$$g < h \quad \text{and} \quad k < l \implies gk < hl \quad \text{and} \quad kg < lh.$$

Proof. We have $gk < hk$ from $g < h$ and $hk < hl$ from $k < l$, and so we get $gk < hl$. Similarly or by symmetry, we get the second inequality. \square

Example 2.2.2. \mathbb{Z}^n is an ordered abelian group with respect to the lexicographic order: for $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$,

$$\alpha < \beta \Leftrightarrow \exists i \geq 1 \quad \text{s.t.} \quad (\alpha_1, \dots, \alpha_{i-1}) = (\beta_1, \dots, \beta_{i-1}) \quad \text{and} \quad \alpha_i < \beta_i.$$

An ordered group is torsion-free. However, there are examples of torsion-free groups which are not ordered groups. Also, it is known that an abelian group is an ordered group if and only if it is torsion-free ([18, page 101]).

To get some properties of division G -graded algebras for an ordered group G , we prepare some terminology. Let (x_1, \dots, x_r) be an ordered set of r elements in some (non-associative) algebra. Denote by B_r the set of all distributions of brackets on an ordered set of r elements. Define $(x_1, \dots, x_r)^b$ for $b \in B_r$ as the product with the corresponding distribution of brackets. For example, $\sum_{b \in B_4} (x_1, x_2, x_3, x_4)^b = x_1((x_2x_3)x_4) + x_1(x_2(x_3x_4)) + (x_1x_2)(x_3x_4) + ((x_1x_2)x_3)x_4 + (x_1(x_2x_3))x_4$. Also, for the group of permutations S_r ,

$$\begin{aligned} \sum_{\sigma \in S_3, b \in B_3} (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})^b &= (x_1x_2)x_3 + x_1(x_2x_3) + (x_1x_3)x_2 + x_1(x_3x_2) \\ &\quad + (x_2x_1)x_3 + x_2(x_1x_3) + (x_2x_3)x_1 + x_2(x_3x_1) \\ &\quad + (x_3x_1)x_2 + x_3(x_1x_2) + (x_3x_2)x_1 + x_3(x_2x_1). \end{aligned}$$

Lemma 2.2.3. Let $G = (G, <)$ be an ordered group and $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ a division G -graded algebra over F . Let f be a (non-associative) polynomial in r -variables X_1, \dots, X_r over F satisfying:

- (i) $f = \sum_{b \in B_r} a_b (X_1 \cdots X_r)^b$ if G is an arbitrary group, or
- (ii) $f = \sum_{\sigma \in S_r, b \in B_r} a_b (X_{\sigma(1)} \cdots X_{\sigma(r)})^b$ if G is an abelian group,

where $a_b \in F$.

Suppose that $f(x^{(1)}, \dots, x^{(r)}) = 0$ for some nonzero elements $x^{(1)}, \dots, x^{(r)}$ in \mathcal{T} . Then there exist nonzero homogeneous elements $x_{h_1}^{(1)}, \dots, x_{h_r}^{(r)}$ in \mathcal{T} , i.e., $0 \neq x_{h_i}^{(i)} \in \mathcal{T}_{h_i}$ for some $h_i \in G$, $i = 1, \dots, r$, such that $f(x_{h_1}^{(1)}, \dots, x_{h_r}^{(r)}) = 0$.

Also, if some of those $x^{(1)}, \dots, x^{(r)}$ happen to be equal, we can choose the corresponding $x_{h_i}^{(i)}$ to be equal: if $x^{(i)} = x^{(j)}$ for some $1 \leq i, j \leq r$, then we can take $x_{h_i}^{(i)} = x_{h_j}^{(j)}$.

Proof. Let $x^{(i)} = \sum_{g_i \in G} x_{g_i}^{(i)}$ be the decomposition into homogeneous elements and let $h_i = \min\{g \in G \mid x_g^{(i)} \neq 0\}$ for $i = 1, \dots, r$. Then for the case (i), we have

$$f(x^{(1)}, \dots, x^{(r)}) = f(x_{h_1}^{(1)}, \dots, x_{h_r}^{(r)}) + \sum_{(g_1, \dots, g_r) \neq (h_1, \dots, h_r)} f(x_{g_1}^{(1)}, \dots, x_{g_r}^{(r)}).$$

The first term of the R.H.S. is an element of degree $h_1 \cdots h_r$ and each term in the sum is of degree $g_1 \cdots g_r$ since the order of $(1 \cdots r)$ is fixed in the first (i). For the case (ii), i.e., G is abelian, this holds even if the order of $(1 \cdots r)$ is not fixed.

Now, we claim that $h_1 \cdots h_r < g_1 \cdots g_r$. We use induction on r . Clearly, this is true for $r = 1$. Assume that $r > 1$ and that $h_1 \cdots h_{r-1} < g_1 \cdots g_{r-1}$. Note that we chose h_r to be $h_r \leq g_r$, and so if $h_r = g_r$, then we get $h_1 \cdots h_r < g_1 \cdots g_r$. If $h_r < g_r$, then we also get $h_1 \cdots h_r < g_1 \cdots g_r$ by 2.2.1. Thus we proved our claim. So $f(x_{h_1}^{(1)}, \dots, x_{h_r}^{(r)})$ is the homogeneous component of $0 = f(x^{(1)}, \dots, x^{(r)})$, which forces $f(x_{h_1}^{(1)}, \dots, x_{h_r}^{(r)}) = 0$. The last statement is clear since our h_i and $x_{h_i}^{(i)}$ above are uniquely determined by $x^{(i)}$. \square

Using this lemma, we get a basic property of division G -graded algebras for an ordered group G .

Theorem 2.2.4. Let G be an ordered group. Then:

- (1) A division G -graded alternative algebra is an alternative domain.
- (2) If G is abelian, a division G -graded Jordan algebra is a Jordan domain.

Proof. Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a division G -graded algebra over F and x, y nonzero elements in \mathcal{T} . Moreover, let $x = x_{g_1} + \cdots + x_{g_m}$ and $y = y_{h_1} + \cdots + y_{h_l}$ be the decompositions into nonzero homogeneous elements in \mathcal{T} .

For (1), suppose that $xy = 0$. Applying 2.2.3(1) for $f = X_1 X_2$, we get $x_{g_i} y_{h_j} = 0$ for some i and j . Since x_{g_i} and y_{h_j} are invertible, this is a contradiction by 1.1.2(1). Hence $xy \neq 0$.

For (2), we need to consider the linearization of the U -operator, i.e.,

$$\{xyz\} := \frac{1}{2}(U_{x+z} - U_x - U_z)(y) = (xy)z + (zy)x - (xz)y.$$

Suppose that $U_x y = 0$. Applying 2.2.3(2) and the last statement of 2.2.3 for $f = \{X_1 X_2 X_3\}$ and $f(x, y, x) = \{xyx\} = U_x y = 0$, we get $\{x_{g_i} y_{h_j} x_{g_i}\} = U_{x_{g_i}} y_{h_j} = 0$ for some i and j . Since x_{g_i} and y_{h_j} are invertible, this is a contradiction by 1.1.2(3). Hence $U_x y \neq 0$. \square

Thus, by 1.1.5, 2.2.4 and 2.2.2, we get the following:

Corollary 2.2.5. *Let $G = (G, <)$ be an ordered group. Then a division G -graded alternative algebra is strongly prime, and if G is abelian, and so is a division G -graded Jordan algebra. In particular, a division \mathbb{Z}^n -graded algebra is strongly prime. \square*

As another property of division G -graded algebras for an ordered group G , we first define the following:

Definition 2.2.6. An element in a graded algebra is called a *trivial unit* if it is invertible and homogeneous.

It is an interesting question whether or not units of a specified group ring are all trivial (see e.g. [18, page 95]). For our case, we have the following:

Theorem 2.2.7. *Let $G = (G, <)$ be an ordered group and $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ a division G -graded algebra. If*

(1) \mathcal{T} is alternative, or (2) \mathcal{T} is Jordan and G is abelian,
then \mathcal{T} has only trivial units.

Proof. Suppose that there exists $x \in \mathcal{T}$ such that x is invertible but not homogeneous. Let y be the inverse of x . We decompose x and y into nonzero homogeneous elements, i.e.,

$$x = x_{g_1} + x_{g_2} + \cdots + x_{g_m} \quad \text{and} \quad y = y_{h_1} + \cdots + y_{h_l}$$

so that

$$g_1 < g_2 < \cdots < g_m \quad \text{with} \quad m > 1 \quad \text{and} \quad h_1 < \cdots < h_l.$$

Note that if $l = 1$, then m has to be 1 by 2.1.3(0), and so we have $l > 1$.

Now, we have

$$1 = xy = x_{g_1}y_{h_1} + \left(\sum_{(i,j) \neq (1,1), (m,l)} x_{g_i}y_{h_j} \right) + x_{g_m}y_{h_l}$$

where $1 \leq i \leq m$ and $1 \leq j \leq l$, and by 2.2.1,

$$g_1h_1 < g_ih_j < g_mh_l \quad \text{for all} \quad i, j.$$

In case (1), we get $x_{g_1}y_{h_1} \neq 0$ and $x_{g_m}y_{h_l} \neq 0$ by 1.1.2(1). So 1 is not homogeneous, which contradicts 2.1.2. Hence there are no non-trivial units in \mathcal{T} .

For the case (2), we use the identity $U_x y^2 = 1$ instead of $xy = 1$. Thus we have

$$\begin{aligned} 1 &= \{xy^2x\} \\ &= \{x_{g_1}y_{h_1}^2x_{g_1}\} + \left(\sum_{(i,j,s,t) \neq (1,1,1,1), (m,l,l,m)} \{x_{g_i}(y_{h_j}y_{h_s})x_{g_t}\} \right) + \{x_{g_m}y_{h_l}^2x_{g_m}\}, \end{aligned}$$

where $1 \leq i, t \leq m$ and $1 \leq j, s \leq l$. Also, since G is abelian, we have

$$\{x_{g_i}(y_{h_j}y_{h_s})x_{g_t}\} \in \mathcal{T}_{g_i h_j h_s g_t} \quad \text{for all} \quad i, j, s, t.$$

and by 2.2.1,

$$g_1h_1^2g_1 < g_ih_jh_sg_t < g_mh_l^2g_m.$$

Since $y_{h_j}^2 = U_{y_{h_j}} 1 \neq 0$, we get

$$0 \neq U_{x_{g_1}} y_{h_1}^2 = \{x_{g_1}y_{h_1}^2x_{g_1}\} \in \mathcal{T}_{g_1^2h_1^2}, \quad \text{and} \quad 0 \neq U_{x_{g_m}} y_{h_l}^2 = \{x_{g_m}y_{h_l}^2x_{g_m}\} \in \mathcal{T}_{g_m^2h_l^2},$$

by 1.1.2(3). So 1 is not homogeneous, which contradicts 2.1.2. Hence there are no non-trivial units in \mathcal{T} . \square

For later use, we prove a corollary of Theorem 2.2.4 and 2.2.7 for division \mathbb{Z}^n -graded algebras although it is true more generally under the assumption of 2.2.4 and 2.2.7.

Corollary 2.2.8. *Let $\mathcal{T} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{T}_\alpha$ be a division \mathbb{Z}^n -graded algebra. Suppose that $0 \neq x \in \mathcal{T}$ and $x^m \in \mathcal{T}_\beta$ for some $\beta \in \mathbb{Z}^n$. Then we have $\beta \in m\mathbb{Z}^n$ and $x \in \mathcal{T}_{\frac{1}{m}\beta}$.*

Proof. We have $x^m \neq 0$ for the case where \mathcal{T} is alternative because \mathcal{T} is a domain (see 2.2.4). If \mathcal{T} is Jordan, then we have $x^m = U_x \cdots U_x x$ if m is odd and $U_x \cdots U_x 1$ if m is even. Since \mathcal{T} is a domain (see 2.2.4), we have $x^m \neq 0$. Since $x^m \in \mathcal{T}_\beta$, x^m is invertible. Then x is invertible by [33, Lemma 9, page 209] for the alternative case and by [13, Theorem 13(6), page 52] for the Jordan case. Hence by 2.2.7, we have $x \in \mathcal{T}_\gamma$ for some $\gamma \in \mathbb{Z}^n$, and so $m\gamma = \beta$. \square

§ 2.3 CENTRE AND CENTRAL CLOSURE

We consider the centre of a division G -graded algebra. In general, a subalgebra \mathcal{S} of a graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ is called *homogeneous* if $\mathcal{S} = \bigoplus_{g \in G} (\mathcal{S} \cap \mathcal{T}_g)$.

Lemma 2.3.1. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a G -graded algebra for a group G and $\mathcal{Z} = Z(\mathcal{T})$ the centre of \mathcal{T} . Put $\mathcal{Z}_g := \mathcal{Z} \cap \mathcal{T}_g$ and $H := \{g \in G \mid \mathcal{Z}_g \neq (0)\}$. Assume that G is abelian if \mathcal{T} is alternative. Then*

(i) \mathcal{Z} is a homogeneous subalgebra.

(ii) If \mathcal{T} is a division G -graded algebra, then H is a subgroup of G contained in the centre of G and \mathcal{Z} is a division H -graded commutative associative algebra.

Proof. For (i), let $z = \sum_{g \in G} z_g \in \mathcal{Z}$ be the decomposition of z into homogeneous elements. We need to show $[z_g, x] = (z_g, x, y) = 0$ for all $g \in G$ and $x, y \in \mathcal{T}$. Let $x = \sum_{g \in G} x_g$ and $y = \sum_{g \in G} y_g$ be the decompositions into homogeneous elements. Since $[\cdot, \cdot]$ and (\cdot, \cdot, \cdot) are multilinear, it is enough to show $[z_g, x_h] = (z_g, x_h, y_k) = 0$ for all $h, k \in G$. Fix $h, k \in G$. By $(\sum_{g \in G} z_g, x_h, y_k) = 0$, we have $\sum_{g \in G} (z_g, x_h, y_k) = 0$. Since $(z_g, x_h, y_k) \in \mathcal{T}_{ghk}$, each term of $\sum_{g \in G} (z_g, x_h, y_k) = 0$ is in a different homogeneous space. Therefore, $\sum_{g \in G} (z_g, x_h, y_k) = 0$ forces $(z_g, x_h, y_k) = 0$ for all $g \in G$. So if \mathcal{T} is a Jordan algebra, then \mathcal{Z} is a homogeneous subalgebra.

If \mathcal{T} is alternative, we are assuming that G is abelian. Hence we have $[z_g, x_h] \in \mathcal{T}_{gh} + \mathcal{T}_{hg} = \mathcal{T}_{gh}$. Thus we get $[z_g, x_h] = 0$ for all $g \in G$ by the same argument as the above. So if \mathcal{T} is an alternative algebra and G is abelian, then \mathcal{Z} is a homogeneous subalgebra.

For (ii), take $0 \neq z \in \mathcal{Z}_h$ for $h \in H$. Then we know $0 \neq z^{-1} \in \mathcal{Z} \cap \mathcal{T}_{h^{-1}} = \mathcal{Z}_{h^{-1}}$ by 1.1.9 and 2.1.3(0), and get $h^{-1} \in H$. For $0 \neq z' \in \mathcal{Z}_{h'}$ for $h' \in H$, it is clear

that $0 \neq zz' \in \mathcal{Z}_{hh'}$ by the invertibility of z and the fact that $z \in Z$. Hence we get $hh' \in H$. Therefore, H is a subgroup of G , and \mathcal{Z} is a division H -graded commutative associative algebra.

Finally, we show that if \mathcal{T} is a Jordan algebra, then H is contained in the centre of G . For $h \in H$, we take any nonzero elements $z \in \mathcal{Z}_h$. Then for any $g \in S = \text{supp}(\mathcal{T})$ and $0 \neq x \in \mathcal{T}_g$, we have $(x^{-1}z)x = z \in \mathcal{T}_{g^{-1}hg} \cap \mathcal{T}_h$ since $z \in \mathcal{Z}$. So $\mathcal{T}_{g^{-1}hg} \cap \mathcal{T}_h \neq (0)$, which implies $g^{-1}hg = h$. Hence h belongs to the centralizer of S , which is the centre of G since S generates G . \square

In 2.3.1, whenever $H = \{g \in G \mid \mathcal{Z}_g \neq (0)\}$ is a subgroup of G , we call it the *central grading group of G in \mathcal{T}* .

Lemma 2.3.2. *Let \mathcal{T} be an algebra over F with centre Z , A a commutative associative F -algebra, $\mathcal{T}_A = A \otimes \mathcal{T}$ with centre $Z(\mathcal{T}_A)$ and $Z_A := A \otimes_F Z$. Then we have $Z(\mathcal{T}_A) = Z_A$. In particular, assuming that $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ is division G -graded algebra and that G is abelian if \mathcal{T} is alternative, the central grading groups of \mathcal{T} and \mathcal{T}_A coincide.*

Proof. It is clear that $Z_A \subset Z(\mathcal{T}_A)$. Conversely, let $\bar{z} \in Z(\mathcal{T}_A)$. Then $\bar{z} = \sum_i a_i \otimes u_i$ where $\{a_i\}$ is a basis of A over F and $u_i \in \mathcal{T}$. Hence for $v, w \in \mathcal{T}$:

$$0 = (\bar{z}, v, w) = \sum_i a_i \otimes (u_i, v, w),$$

whence $(u_i, v, w) = 0$. Similarly, we get $[u_i, v] = 0$ and all $u_i \in Z$. Thus we obtain $Z(\mathcal{T}_A) = Z_A$. Let H and H' be the central grading groups of \mathcal{T} and \mathcal{T}_A , respectively. Then we have

$$\bigoplus_{h \in H} (A \otimes_F \mathcal{T}_h) = A \otimes_F \left(\bigoplus_{h \in H} \mathcal{T}_h \right) = Z_A = Z(\mathcal{T}_A) = \bigoplus_{h' \in H'} (A \otimes_F \mathcal{T}_{h'}),$$

which forces $H = H'$. \square

There is a method of constructing a division G -graded algebra of strong type from a division G -graded commutative associative algebra.

Lemma 2.3.3. *Let $\mathcal{Z} = \bigoplus_{g \in G} \mathcal{Z}_g$ be a division G -graded commutative associative algebra over F . Then:*

- (i) G is abelian, $K := \mathcal{Z}_e$ is a field and $\dim_K \mathcal{Z}_g = 1$ for all $g \in G$,

(ii) for a division algebra D over K , $\mathcal{Z} \otimes_K D = \bigoplus_{g \in G} (\mathcal{Z}_g \otimes_K D)$ is a division G -graded algebra over K of strong type.

Proof. For (i), since \mathcal{Z} is associative, we have $\mathcal{Z}_g \mathcal{Z}_h = \mathcal{Z}_{gh}$ and $\mathcal{Z}_h \mathcal{Z}_g = \mathcal{Z}_{hg}$ for all $g, h \in G$, by 2.1.5(1). Since \mathcal{Z} is commutative, we have $\mathcal{Z}_{gh} = \mathcal{Z}_g \mathcal{Z}_h = \mathcal{Z}_h \mathcal{Z}_g = \mathcal{Z}_{hg}$. Hence $gh = hg$, and so G is abelian. It follows from 2.1.2 that $K = \mathcal{Z}_e$ is a field. Since $\mathcal{Z} = \bigoplus_{g \in G} Kx_g$ by 2.1.5(1), each $Kx_g = \mathcal{Z}_g$ is a 1-dimensional vector space over K .

For (ii), since \mathcal{Z} is commutative and associative, $\mathcal{Z} \otimes_K D$ is an algebra over K , alternative or Jordan depending on D . For $g, h \in G$, we have

$$(\mathcal{Z}_g \otimes_K D)(\mathcal{Z}_h \otimes_K D) = (\mathcal{Z}_g \mathcal{Z}_h) \otimes_K D = \mathcal{Z}_{gh} \otimes_K D.$$

Hence $\mathcal{Z} \otimes_K D = \bigoplus_{g \in G} (\mathcal{Z}_g \otimes_K D)$ is a strongly G -graded algebra over K . Also, any nonzero element in $\mathcal{Z}_g \otimes_K D$ can be written as $x \otimes d$ for some $0 \neq x \in \mathcal{Z}_g$ and $0 \neq d \in D$ since $\dim_K \mathcal{Z}_g = 1$. Then $x^{-1} \otimes d^{-1}$ is the inverse of $x \otimes d$, and so $\mathcal{Z} \otimes_K D$ is a division G -graded algebra over K . Thus we have shown (ii). \square

Let $A = \bigoplus_{g \in G} A_g$ be an associative division G -graded algebra over F . Suppose that the group G acts on a set S . Then a left A -module M is called a graded A -module of type S if

- (1) $M = \bigoplus_{s \in S} M_s$,
- (2) for each $g \in G$ and $s \in S$, $A_g M_s \subset M_{g.s}$.

Let $\text{supp}(M) = \{s \in S \mid M_s \neq (0)\}$. The following lemma seems to be well-known, for example it is proven on page 29, Theorem 3 and page 30, Corollary 2 in [8], but for the convenience of the reader we include the proof.

Lemma 2.3.4a. *Let $A = \bigoplus_{g \in G} A_g$ be an associative division G -graded algebra over F and $M = \bigoplus_{s \in S} M_s$ a graded A -module of type S . Then:*

- (i) $M_{g.s} = a_g M_s = A_g M_s$ for all $g \in G$, $s \in S$ and any $0 \neq a_g \in A_g$.
- (ii) For the G -orbit $\bar{s} := G.s \in S/G$ of s , we have $AM_s = \bigoplus_{s' \in G.s} M_{s'}$. In particular, $\bar{s} = \bar{s}'$ iff $AM_s = AM_{s'}$. Hence letting $M_{\bar{s}} := AM_s$,

$$M = \bigoplus_{\bar{s} \in S/G} M_{\bar{s}}.$$

(iii) Suppose that A is commutative and that G acts freely on S , i.e., $g.s = s$ for some $g \in G$ and $s \in S \implies g = e$. Then for any $s \in \text{supp}(M)$, $M_{\bar{s}}$ is a free

A -module, and any A_e -basis of the A_e -vector space M_s is an A -basis of $M_{\bar{s}}$, and so $\text{rank}_A M_{\bar{s}} = \dim_{A_e} M_{s'}$ for all $\bar{s} \in S/G$ and all $s' \in \bar{s}$. In particular, M is a free A -module and there exists a basis of M consisting of homogeneous elements.

Proof. (i): Since $(a_g, a_g^{-1}, M_s) = 0$, we have, as in 2.1.5(0), $M_{g.s} = a_g M_s$. Thus we get $M_{g.s} = a_g M_s \subset A_g M_s \subset M_{g.s}$, and hence $M_{g.s} = a_g M_s = A_g M_s$.

(ii): By (i), we have $M_{\bar{s}} = AM_s = (\bigoplus_{g \in G} A_g)M_s = \bigoplus_{g \in G} A_g M_s = \bigoplus_{g \in G} M_{g.s} = \bigoplus_{s' \in \bar{s}} M_{s'}$, and hence in particular

$$M = \bigoplus_{s \in S} M_s = \bigoplus_{\bar{s} \in S/G} \left(\bigoplus_{s' \in \bar{s}} M_{s'} \right) = \bigoplus_{\bar{s} \in S/G} M_{\bar{s}}$$

(iii): Let $(x_g)_{g \in G}$ be a family of nonzero homogeneous elements of A . Then, by 2.1.5(1), $\{x_g\}_{g \in G}$ is an A_e -basis of A . Let $\{m_i\}_{i \in I}$ be an A_e -basis of M_s . Clearly $\{m_i\}_{i \in I}$ generates $M_{\bar{s}} = AM_s$ as a A -module. Suppose $\sum_{i \in I} x_i m_i = 0$ for $x_i \in A$. Let $x_i = \sum_{g \in G} a_{gi} x_g$ for $a_{gi} \in A_e$. Then $0 = \sum_{i \in I} x_i m_i = \sum_{g \in G} \sum_{i \in I} a_{gi} x_g m_i$, and so $\sum_{i \in I} a_{gi} x_g m_i = 0$ for all $g \in G$ since G acts freely on S . Since A is commutative and x_g is invertible, we get $\sum_{i \in I} a_{gi} m_i = 0$ and hence $a_{gi} = 0$ for all $g \in G$ and $i \in I$. Hence $x_i = 0$ for all $i \in I$. Therefore, $M_{\bar{s}}$ is a free A -module with basis $\{m_i\}_{i \in I}$. Since $M_{\bar{s}} = M_{\bar{s}'}$ for any $s' \in \bar{s}$, we obtain $\dim_{A_e} M_s = \text{rank}_A M_{\bar{s}} = \text{rank}_A M_{\bar{s}'} = \dim_{A_e} M_{s'}$. Since $M = \bigoplus_{\bar{s} \in S/G} M_{\bar{s}}$, M is a free A -module with a basis consisting of homogeneous elements. \square

A division G -graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ over F is clearly a graded \mathcal{Z} -module of type G where $\mathcal{Z} = Z(\mathcal{T})$ is the centre of \mathcal{T} . Also, \mathcal{Z} is commutative and the central grading group acts freely on G , and so the graded module \mathcal{T} satisfies the assumptions of (iii) in Lemma 2.3.4a(iii). Thus we have the following corollary:

Corollary 2.3.4. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a division G -graded algebra over F . Assume that G is abelian if \mathcal{T} is alternative. Let $\mathcal{Z} = Z(\mathcal{T})$ be the centre of \mathcal{T} and H the central grading group of G in \mathcal{T} , and so $\mathcal{Z} = \bigoplus_{h \in H} \mathcal{Z}_h$ for $\mathcal{Z}_h = \mathcal{Z} \cap \mathcal{T}_h$. Let $K := Z_e$ which is a field. Then:*

(i) $\mathcal{T}_{hg} = z_h \mathcal{T}_g = \mathcal{Z}_h \mathcal{T}_g$ for all $h \in H, g \in G$ and any $0 \neq z_h \in \mathcal{Z}_h$.

(ii) For $\bar{g} \in G/H$, we have $\mathcal{Z} \mathcal{T}_{\bar{g}} = \bigoplus_{g' \in \bar{g}} \mathcal{T}_{g'}$ and in particular, if $\bar{g} = \bar{g}'$, then $\mathcal{Z} \mathcal{T}_{\bar{g}} = \mathcal{Z} \mathcal{T}_{\bar{g}'}$.

(iii) Let $\mathcal{T}_{\bar{g}} := \mathcal{Z} \mathcal{T}_{\bar{g}}$. Then $\mathcal{T} = \bigoplus_{\bar{g} \in G/H} \mathcal{T}_{\bar{g}}$, and \mathcal{T} is a predivision G/H -graded algebra over \mathcal{Z} .

(iv) For $g \in \text{supp}(\mathcal{T})$, $\mathcal{T}_{\bar{g}}$ is a free \mathcal{Z} -module. and any K -basis of the K -vector space \mathcal{T}_g becomes a \mathcal{Z} -basis of $\mathcal{T}_{\bar{g}}$, and so $\text{rank}_{\mathcal{Z}} \mathcal{T}_{\bar{g}} = \dim_K \mathcal{T}_{g'}$ for all $\bar{g} \in G/H$ and all $g' \in \bar{g}$. In particular, \mathcal{T} is a free \mathcal{Z} -module and there exists a basis of \mathcal{T} consisting of homogeneous elements.

Proof. We only need to show that the graded \mathcal{Z} -module $\mathcal{T} = \bigoplus_{\bar{g} \in G/H} \mathcal{T}_{\bar{g}}$ is a predivision G/H -graded algebra. For $k \in G$, we have $\mathcal{T}_{\bar{g}}\mathcal{T}_{\bar{k}} = (\mathcal{Z}\mathcal{T}_{\bar{g}})(\mathcal{Z}\mathcal{T}_{\bar{k}}) = \mathcal{Z}\mathcal{T}_{\bar{g}}\mathcal{T}_{\bar{k}} \subset \mathcal{Z}\mathcal{T}_{\bar{gk}} = \mathcal{T}_{\bar{gk}} = \mathcal{T}_{\bar{g}\bar{k}}$. Since $\mathcal{T}_{\bar{g}} \neq (0) \iff \mathcal{T}_{\bar{g}} \neq (0)$, we get $\langle \text{supp}(\mathcal{T}) \rangle = G/H$. Thus $\mathcal{T} = \bigoplus_{\bar{g} \in G/H} \mathcal{T}_{\bar{g}}$ is a G/H -graded algebra over \mathcal{Z} . Since $\mathcal{T}_g \subset \mathcal{T}_{\bar{g}}$, \mathcal{T} is a predivision G/H -graded algebra over \mathcal{Z} . \square

We prove some properties of the central closures of division G -graded algebras for an ordered abelian group G . Recall that such an algebra \mathcal{T} is prime by 2.2.5. and so the central closure $\bar{\mathcal{T}}$ makes sense (see 1.1.13). Also, we identify \mathcal{T} with a subalgebra of $\bar{\mathcal{T}}$ via $t \mapsto 1 \otimes t$ (see 1.1.16(i)).

Proposition 2.3.5. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a division G -graded algebra over F for an ordered abelian group G . The notations are the same as in 2.3.4 above. Let $\bar{\mathcal{T}} = \bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{T}$ be the central closure of \mathcal{T} where $\bar{\mathcal{Z}}$ is the field of fractions of \mathcal{Z} . Then:*

(i) *Let $\bar{\mathcal{T}}_{\bar{g}} := \bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{T}_{\bar{g}}$ for $\bar{g} \in G/H$. Then $\bar{\mathcal{T}} = \bigoplus_{\bar{g} \in G/H} \bar{\mathcal{T}}_{\bar{g}}$, and $\bar{\mathcal{T}}$ is a predivision G/H -graded algebra over $\bar{\mathcal{Z}}$. Moreover, if \mathcal{T} is Jordan of strong type, and so is $\bar{\mathcal{T}}$.*

(ii) *Any K -basis of \mathcal{T}_g becomes a $\bar{\mathcal{Z}}$ -basis of $\bar{\mathcal{T}}_{\bar{g}}$, and hence $\dim_{\bar{\mathcal{Z}}} \bar{\mathcal{T}}_{\bar{g}} = \dim_K \mathcal{T}_{g'}$ for all $\bar{g} \in G/H$ and all $g' \in \bar{g}$. Also, there exists a $\bar{\mathcal{Z}}$ -basis of $\bar{\mathcal{T}}$ which is a \mathcal{Z} -basis of \mathcal{T} .*

(iii) *$\bar{\mathcal{Z}} \cap \mathcal{T} = \mathcal{Z}$, and in particular, $\bar{\mathcal{Z}} \cap \mathcal{T}_h = \mathcal{Z}_h$ for all $h \in H$.*

Proof. For (i), we have

$$\bar{\mathcal{T}} = \bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{T} = \bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \left(\bigoplus_{\bar{g} \in G/H} \mathcal{T}_{\bar{g}} \right) = \bigoplus_{\bar{g} \in G/H} (\bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{T}_{\bar{g}}) = \bigoplus_{\bar{g} \in G/H} \bar{\mathcal{T}}_{\bar{g}}.$$

That $\bar{\mathcal{T}}$ is a G/H -graded algebra over $\bar{\mathcal{Z}}$ is shown as in 2.3.4(iii), and from this the second statement will be clear. Since $\mathcal{T}_g \subset \bar{\mathcal{T}}_{\bar{g}}$, $\bar{\mathcal{T}}$ is predivision G/H -graded.

For (ii), let $\{t_i\}_{i \in I}$ be a K -basis of \mathcal{T} . Then, by 2.3.4(iv), this is a \mathcal{Z} -basis of $\mathcal{T}_{\bar{g}}$. It is straightforward to check that $\{t_i\}_{i \in I} = \{1 \otimes t_i\}_{i \in I}$ is a $\bar{\mathcal{Z}}$ -basis of $\bar{\mathcal{T}}_{\bar{g}}$, and so $\dim_{\bar{\mathcal{Z}}} \bar{\mathcal{T}}_{\bar{g}} = \dim_K \mathcal{T}_{g'}$ for all $\bar{g} \in G/H$ and all $g' \in \bar{g}$. The second assertion follows from the \mathcal{Z} -space decomposition $\bar{\mathcal{T}} = \bigoplus_{\bar{g} \in G/H} \bar{\mathcal{T}}_{\bar{g}}$.

For (iii), it is clear that $\overline{\mathcal{Z}} \cap \mathcal{T} \supset \mathcal{Z}$. Let $t \in \overline{\mathcal{Z}} \cap \mathcal{T}$. We have $t = 1 \otimes t = z'/z \otimes 1$ for $z' \in \mathcal{Z}$, $z \neq 0$. Multiplying both sides by z , we get $z \otimes t = z' \otimes 1$, and so $1 \otimes zt = 1 \otimes z'$, and hence $zt = z' \in \mathcal{Z}$. Let $t = \sum_{g \in G} t_g$ be the homogeneous decomposition of t and $U = U(t) := \{g \in G \mid t_g \neq 0\}$. We show that $t \in \mathcal{Z}$ by induction on $\sharp U$ where \sharp means the order of a set. If $\sharp U = 0$, i.e., $U = \emptyset$, then $t = 0 \in \mathcal{Z}$. Thus we assume that $\sharp U > 0$. Let $z = \sum_{k \in H} z_k$ be the homogeneous decomposition of z and $V := \{k \in H \mid z_k \neq 0\}$. Then we have $zt = \sum_{k \in V, g \in U} z_k t_g \in \mathcal{Z}$. Since U and V are finite subsets of totally ordered sets, there exist unique minimal elements $g_0 \in U$ and $k_0 \in V$. Since \mathcal{Z} is a homogeneous subalgebra, $z_{k_0} \in \mathcal{Z}$, and so $z_{k_0} t_{g_0} \neq 0$. Moreover, $z_{k_0} t_{g_0}$ is the only element of degree $k_0 g_0$ among $z_k t_g$ for $k \in V, g \in U$, and so $z_{k_0} t_{g_0} \in \mathcal{Z}$. Since $z_{k_0} \in \mathcal{Z}$ is invertible, we get $t_{g_0} = z_{k_0}^{-1} (z_{k_0} t_{g_0}) \in z_{k_0}^{-1} \mathcal{Z} = \mathcal{Z}$ (see 1.1.9). Also, for $t' := t - t_{g_0}$, we have $zt' = zt - z t_{g_0} \in \mathcal{Z}$ and $\sharp U(t') = \sharp U(t) - 1$. Hence by induction, we get $t' \in \mathcal{Z}$. Therefore, we obtain $t = t' + t_{g_0} \in \mathcal{Z}$. \square

§ 2.4 ISOMORPHISMS

First we show some properties of a subset S of a group G satisfying 2.1.4:

Lemma 2.4.1. *Let G be a group and S a subset of G satisfying (2.1.4):*

$$(2.1.4) \quad e \in S, \quad h^2 k \in S \quad \text{and} \quad h^{-1} \in S \quad \text{for } h, k \in S.$$

Then:

- (i) For any $m \in \mathbb{Z}$ and $g \in S$, we have $g^m \in S$.
- (ii) Assume that G is abelian. Then we have

$$\langle S \rangle^2 S \subset S \quad \text{where} \quad \langle S \rangle^2 = \{g^2 \mid g \in \langle S \rangle\},$$

and in particular, $\langle S \rangle^2 \subset S$ (see [1. Lemma 1.3 page 23] for the case $G = \mathbb{Z}^n$).

- (iii) If $G = \mathbb{Z}^n$, then there exists a basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n such that each $\sigma_i \in S$.

Proof. For (i), we write g^m as $\prod_{j=1}^{|m|} g^{\varepsilon_j}$ where

$$\varepsilon_j = \begin{cases} 1 & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ -1 & \text{if } m < 0. \end{cases}$$

Since $e \in S$ and $g^{-1} \in S$, we have $g^\varepsilon \in S$. Thus, since $g^{2\varepsilon}h \in S$ for $h \in S$ (in particular $g^{2\varepsilon} = g^{2\varepsilon}e \in S$) and

$$g^m = \begin{cases} \prod_{j=1}^{|l|} g^{2\varepsilon} & \text{if } m = 2l \text{ for } l \in \mathbb{Z} \\ (\prod_{j=1}^{|l|} g^{2\varepsilon})g^\varepsilon & \text{if } m = 2l + 1 \text{ for } l \in \mathbb{Z}. \end{cases}$$

we obtain $g^m \in S$ by induction.

One can show (ii) in the same way as the case $G = \mathbb{Z}^n$ in [1, Lemma 1.3, page 23]: Let $g \in \langle S \rangle$, and so $g = s_1 \cdots s_m$ for $s_i \in S$. Then we have $g^2 = (s_1 \cdots s_m)^2 = s_1^2 \cdots s_m^2$ since G is abelian. Hence for $s \in S$, we get $g^2s = s_1^2 \cdots s_m^2s \in S$ inductively, and obtain $\langle S \rangle^2 S \subset S$.

(iii) was shown in [1, Proposition 1.11, page 24]. \square

For a predivision \mathbb{Z}^n -graded Jordan algebra $\mathcal{T} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{T}_\alpha$ and $S = \text{supp}(\mathcal{T})$, we recall $\langle S \rangle = \mathbb{Z}^n$, and from 2.1.3 that S satisfies (2.1.4), i.e., S has the following properties:

- (i) $\mathbf{0} \in S$;
- (ii) $2\alpha - \beta \in S$ if $\alpha, \beta \in S$;
- (iii) $\langle S \rangle = \mathbb{Z}^n$.

A subset S of \mathbb{Z}^n satisfying (i)-(iii) is called a *semilattice in \mathbb{Z}^n* . The definition of semilattices in [1] is slightly more general than this, that is, S is a discrete subset of a real vector space satisfying (i) and (ii), and $\langle S \rangle$ is a lattice in the real vector space.

We give a definition for isomorphisms of general G -graded algebras.

Definition 2.4.2. (1) Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ and $\mathcal{T}' = \bigoplus_{g \in G} \mathcal{T}'_g$ be two G -graded algebras. Then \mathcal{T} and \mathcal{T}' are called *G -graded isomorphic*, denoted $\mathcal{T} \cong_G \mathcal{T}'$, if there exists an isomorphism φ (as algebras) from \mathcal{T} onto \mathcal{T}' such that $\varphi(\mathcal{T}_g) = \mathcal{T}'_g$ for all $g \in G$.

(2) Given an automorphism p of a group G , we always obtain a new G -graded algebra by renaming the degrees. More precisely, for $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$, we put $\tilde{\mathcal{T}}_g := \mathcal{T}_{p(g)}$. Then $\tilde{\mathcal{T}} := \bigoplus_{g \in G} \tilde{\mathcal{T}}_g$ with the same algebra structure as \mathcal{T} is a G -graded algebra. Note that $\mathcal{T} = \tilde{\tilde{\mathcal{T}}}$ as algebras. We call $\tilde{\mathcal{T}}$ the *p -regradation of \mathcal{T}* .

Using Theorem 2.2.7, we prove the following two properties of isomorphisms in the class of certain division graded algebras.

Theorem 2.4.3. Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ and $\mathcal{T}' = \bigoplus_{g \in G} \mathcal{T}'_g$ be division G -graded algebras for an ordered group G and let $\varphi : \mathcal{T} \xrightarrow{\sim} \mathcal{T}'$ be an isomorphism.

(i) If \mathcal{T} is a Jordan algebra, assume that G is abelian. Then $\varphi(\mathcal{T}_e) = \mathcal{T}'_e$, in particular, \mathcal{T}_e and \mathcal{T}'_e are isomorphic division algebras.

(ii) If \mathcal{T} is a Jordan algebra, assume that $G = \mathbb{Z}^n$. Then there exists an automorphism p for G such that $\varphi(\mathcal{T}_g) \subset \mathcal{T}'_{p(g)}$ for all $g \in G$. In other words, φ is a graded isomorphism from $\tilde{\mathcal{T}}$ onto \mathcal{T}' where $\tilde{\mathcal{T}}$ is the p^{-1} -regradation of \mathcal{T} .

(iii) Moreover, under the hypotheses of (ii) let H and H' be the central grading groups of \mathcal{T} and \mathcal{T}' , respectively. Then $p|_H$ is an isomorphism from H onto H' , and hence $G/H \cong G/H'$.

Proof. Assume that G is abelian if \mathcal{T} is a Jordan algebra. Let $S := \text{supp}(\mathcal{T})$ and $0 \neq x \in \mathcal{T}_g$ for $g \in S$. Since $\varphi(x)$ is invertible in \mathcal{T}' , $\varphi(x) \in \mathcal{T}'_{g'}$ for some $g' \in S' := \text{supp}(\mathcal{T}')$ by Theorem 2.2.7. Let $0 \neq y \in \mathcal{T}_g$. By what we have just shown, one gets $0 \neq \varphi(y) \in \mathcal{T}'_{g''}$ for some $g'' \in S'$. We claim that $g' = g''$. Indeed, if $x + y = 0$, then $\varphi(x) + \varphi(y) = 0$. So $0 \neq \varphi(y) = -\varphi(x) \in \mathcal{T}'_{g'} \cap \mathcal{T}'_{g''}$. If $x + y \neq 0$, then $x + y$ is invertible, and so $\varphi(x) + \varphi(y) = \varphi(x + y)$ is homogeneous forcing $g' = g''$. Thus we have a well-defined map $p : S \rightarrow S'$ defined by $\varphi(\mathcal{T}_g) \subset \mathcal{T}'_{p(g)}$ for all $g \in S$. Applying the same argument for φ^{-1} , we obtain a map $q : S' \rightarrow S$ defined by $\varphi^{-1}(\mathcal{T}'_k) \subset \mathcal{T}_{q(k)}$ for all $k \in S'$. Since $\mathcal{T}_g = \varphi^{-1}\varphi(\mathcal{T}_g) \subset \mathcal{T}_{qp(g)}$ and $\mathcal{T}'_k = \varphi\varphi^{-1}(\mathcal{T}'_k) \subset \mathcal{T}'_{pq(k)}$ for all $g \in S$ and $k \in S'$, we get $qp = \text{id}_S$ and $pq = \text{id}_{S'}$. Hence p is bijective from S onto S' and $\varphi(\mathcal{T}_g) = \mathcal{T}'_{p(g)}$ for all $g \in S$. Also, since $\varphi(1) = 1$, we have $p(e) = e$, and so $\varphi(\mathcal{T}_e) = \mathcal{T}'_{p(e)} = \mathcal{T}'_e$. Thus $\varphi|_{\mathcal{T}_e}$ is an isomorphism between two division algebras \mathcal{T}_e and \mathcal{T}'_e , and we have shown (i).

(ii): If \mathcal{T} is alternative or $S = G$, then for all $g, h \in G$ and for $0 \neq x \in \mathcal{T}_g$ and $0 \neq y \in \mathcal{T}_h$, we have

$$0 \neq \varphi(xy) = \varphi(x)\varphi(y) \in \mathcal{T}'_{p(gh)} \cap \mathcal{T}'_{p(g)p(h)}.$$

Hence we get $p(gh) = p(g)p(h)$. Therefore, p is an automorphism of G such that $\varphi(\mathcal{T}_g) = \mathcal{T}'_{p(g)}$ for all $g \in G$. So we are done in this case.

Let $G = \mathbb{Z}^n$ and let $\mathcal{T} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{T}_\alpha$ with support S and $\mathcal{T}' = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{T}'_\alpha$ with support S' be isomorphic division \mathbb{Z}^n -graded Jordan algebras through φ . Then, by the above argument, there exists a bijective map p from S onto S' such that $\varphi(\mathcal{T}_\alpha) = \mathcal{T}'_{p(\alpha)}$ for all $\alpha \in S$. For $\alpha, \beta \in S$ and for $0 \neq x \in \mathcal{T}_\alpha$ and $0 \neq y \in \mathcal{T}_\beta$, we

have

$$0 \neq \varphi(U_x y) = U_{\varphi(x)} \varphi(y) \in \mathcal{T}'_{p(2\alpha+\beta)} \cap \mathcal{T}'_{2p(\alpha)+p(\beta)},$$

and so we get $p(2\alpha + \beta) = 2p(\alpha) + p(\beta)$ for all $\alpha, \beta \in S$. Then we need the following lemma:

Lemma 2.4.4. *Let S and S' be semilattices in \mathbb{Z}^n . Suppose that a map $p : S \rightarrow S'$ is bijective and satisfies $p(2\alpha + \beta) = 2p(\alpha) + p(\beta)$ for all $\alpha, \beta \in S$. Then there exists a unique group automorphism \bar{p} of \mathbb{Z}^n such that $\bar{p}|_S = p$.*

Using this lemma, we can conclude the proof of (ii). For, we have an automorphism \bar{p} of \mathbb{Z}^n such that $\varphi(\mathcal{T}_\alpha) = \mathcal{T}'_{\bar{p}(\alpha)}$ for all $\alpha \in S$. If $\alpha \in \mathbb{Z}^n \setminus S$, i.e., $\mathcal{T}_\alpha = (0)$, then $\bar{p}(\alpha) \in \mathbb{Z}^n \setminus S'$, i.e., $\mathcal{T}_{\bar{p}(\alpha)} = (0)$. Hence $\varphi(\mathcal{T}_\alpha) = (0) = \mathcal{T}'_{\bar{p}(\alpha)}$ for all $\alpha \in \mathbb{Z}^n \setminus S$. Thus we obtain $\varphi(\mathcal{T}_\alpha) = \mathcal{T}'_{\bar{p}(\alpha)}$ for all $\alpha \in G$.

Proof of 2.4.4. We choose a basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n such that each $\sigma_i \in S$ by 2.4.1(iii). We define a map $\bar{p} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ by $\bar{p}(\sum_{i=1}^n m_i \sigma_i) = \sum_{i=1}^n m_i p(\sigma_i)$ for $\sum_{i=1}^n m_i \sigma_i \in \mathbb{Z}^n$. Then \bar{p} is a well-defined endomorphism. We prove $\bar{p}|_S = p$.

Claim 1. $p(m\alpha) = mp(\alpha)$ for any $m \in \mathbb{Z}$ and $\alpha \in S$ ($m\alpha \in S$ by 2.4.1(i)).

Proof. Since $p(\mathbf{0}) = p(2 \cdot \mathbf{0} + \mathbf{0}) = 2p(\mathbf{0}) + p(\mathbf{0})$, we have $p(\mathbf{0}) = \mathbf{0}$, and so Claim 1 is done for $m = 0$. Also, for $m = -1$, we have $p(-\alpha) = p(2(-\alpha) + \alpha) = 2p(-\alpha) + p(\alpha)$, and hence $p(-\alpha) = -p(\alpha)$. Next for $m = 2$, we have $p(2\alpha) = p(2\alpha + \mathbf{0}) = 2p(\alpha) + p(\mathbf{0}) = 2p(\alpha)$. Therefore, we get $p(2\varepsilon\alpha) = 2\varepsilon p(\alpha)$ where $\varepsilon = 1, 0$ or -1 . We show that $p(2l\varepsilon\alpha) = 2l\varepsilon p(\alpha)$ by induction on l . We have already seen the case $l = 0$ and 1. For $l > 1$, we have

$$\begin{aligned} p(2l\varepsilon\alpha) &= 2p(\varepsilon\alpha) + p(2(l-1)\varepsilon\alpha) \quad \text{since } 2(l-1)\varepsilon\alpha \in S, \\ &= 2\varepsilon p(\alpha) + 2(l-1)\varepsilon p(\alpha) \quad \text{by induction,} \\ &= 2l\varepsilon p(\alpha). \end{aligned}$$

Thus we have $2p(l\varepsilon\alpha) = p(2l\varepsilon\alpha) = 2l\varepsilon p(\alpha) = 2(l\varepsilon p(\alpha))$. Hence we get $p(l\varepsilon\alpha) = l\varepsilon p(\alpha)$. Since $m\alpha = |m|\varepsilon\alpha$, we obtain $p(m\alpha) = p(|m|\varepsilon\alpha) = |m|\varepsilon p(\alpha) = mp(\alpha)$, which proves Claim 1.

Claim 2. Suppose that $\sum_{i=1}^n m_i \sigma_i \in S$. Then,

$$p\left(\sum_{i=1}^n 2m_i \sigma_i\right) = \sum_{i=1}^n 2m_i p(\sigma_i)$$

$$(\sum_{i=1}^n 2m_i \sigma_i = 2 \sum_{i=1}^n m_i \sigma_i \in S).$$

Proof. We prove this by induction on n . We have already shown the case $n = 1$ in Claim 1. Assume $n > 1$. By 2.4.1(i), we have $m_i \sigma_i \in S$. So we have, inductively,

$$2m_1 \sigma_1, \quad 2m_1 \sigma_1 + 2m_2 \sigma_2, \quad \dots, \quad \text{and} \quad 2m_1 \sigma_1 + \dots + 2m_{n-1} \sigma_{n-1} \in S.$$

Thus we have

$$\begin{aligned} p\left(\sum_{i=1}^n 2m_i \sigma_i\right) &= p\left(\sum_{i=1}^{n-1} 2m_i \sigma_i\right) + 2f(m_n \sigma_n) \\ &= \sum_{i=1}^{n-1} 2m_i p(\sigma_i) + 2m_n p(\sigma_n) \quad \text{by induction and by Claim 1} \\ &= \sum_{i=1}^n 2m_i p(\sigma_i). \quad \text{which proves Claim 2.} \end{aligned}$$

By Claim 1 and 2, we get $p(\sum_{i=1}^n m_i \sigma_i) = \sum_{i=1}^n m_i p(\sigma_i)$ since

$$2p\left(\sum_{i=1}^n m_i \sigma_i\right) = p\left(\sum_{i=1}^n 2m_i \sigma_i\right) = \sum_{i=1}^n 2m_i p(\sigma_i) = 2 \sum_{i=1}^n m_i p(\sigma_i).$$

Thus

$$\tilde{p}\left(\sum_{i=1}^n m_i \sigma_i\right) = \sum_{i=1}^n m_i p(\sigma_i) = p\left(\sum_{i=1}^n m_i \sigma_i\right),$$

and so we have shown $\tilde{p}|_S = p$. Then, since $\tilde{p}(S) = p(S) = S'$ and S' generates \mathbb{Z}^n , we get surjectivity of \tilde{p} . Finally, it is well-known that an epimorphism on a free abelian group of finite rank is an automorphism (e.g. [17, Exercise 4, page 175]). The uniqueness of \tilde{p} is clear since $\langle S \rangle = \mathbb{Z}^n$. So the lemma has been established.

(iii): Since φ maps the centre of T onto the centre of T' , $p|_H$ is an isomorphism from H onto H' . Also, $\pi \circ p : G \longrightarrow G/p(H) = G/H$ is an epimorphism with kernel H where π is the canonical epimorphism. Hence we get $G/H \cong G/H'$. \square

We prove a corollary of 2.4.3 for division \mathbb{Z}^n -graded Jordan algebras. For this purpose we introduce two concepts about semilattices. Let S and S' be semilattices in \mathbb{Z}^n . If there exists an automorphism f of \mathbb{Z}^n such that $f(S) = S'$, then we say that S is *isomorphic* to S' , denoted $S \cong S'$. If there exists $\rho \in S$ such that the semilattice $S + \rho$ (this is a semilattice by [1, Corollary 1.9, page 24]) is isomorphic to S' , then we say that S is *similar* to S' , denoted $S \sim S'$. The definition of isomorphisms and similarity in [1, Definition 4.1, page 43] are slightly more general than ours: for semilattices $S, S' \subset V$ where V is a real vector space, S and S' are *isomorphic* if there exists $\varphi \in \text{GL}(V)$ such that $\varphi(S) = S'$, and *similar* if there exists $\sigma' \in S'$ such that S and $S' + \sigma'$ are isomorphic.

Also, we recall the concept of isotopes of Jordan algebras. Let J be a unital Jordan algebra and suppose that u is an invertible element in J . Define a new product \cdot_u on J by $x \cdot_u y = \{xuy\}$ for $x, y \in J$ defined in 2.2.4. Then this product satisfies commutativity and the Jordan identity, and so $J^{(u)} := (J, \cdot_u)$ is a Jordan algebra with unit u^{-1} . We call $J^{(u)}$ an *isotope* of J . Let J' be another Jordan algebra. If an isotope of J is isomorphic to J' , then we say that J is *isotopic* to J' , denoted $J \sim J'$.

Corollary 2.4.5. *Let \mathcal{J} and \mathcal{J}' be division \mathbb{Z}^n -graded Jordan algebras. Put $S := \text{supp}(\mathcal{J})$ and $S' := \text{supp}(\mathcal{J}')$. If $\mathcal{J} \cong \mathcal{J}'$, then $S \cong S'$, and if $\mathcal{J} \sim \mathcal{J}'$, then $S \sim S'$.*

Moreover, let H and H' be the central grading groups of \mathcal{T} and \mathcal{T}' , respectively. Then $\mathcal{J} \sim \mathcal{J}'$ implies $H \cong H'$ and $\mathbb{Z}^n/H \cong \mathbb{Z}^n/H'$.

Proof. If $\mathcal{J} \cong \mathcal{J}'$, then there is an automorphism p of \mathbb{Z}^n such that $p(S) = S'$ by 2.4.3, and so $S \cong S'$. If $\mathcal{J} \sim \mathcal{J}'$, then $\mathcal{J}^{(u)} \cong \mathcal{J}'$ for some invertible element u in \mathcal{J} . Let $\mathcal{J} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{J}_\alpha$. By 2.2.7, u lies in \mathcal{J}_γ for some $\gamma \in S$. Let $\alpha' = \alpha - \gamma$ for all $\alpha \in \mathbb{Z}^n$. Then we have $\mathcal{J}_{\alpha'} \cdot_u \mathcal{J}_{\beta'} \subset \mathcal{J}_{(\alpha+\beta)'}$ and

$$\text{supp}(\mathcal{J}^{(u)}) = S - \gamma$$

which generates \mathbb{Z}^n , and so $\mathcal{J}^{(u)} = \bigoplus_{\alpha' \in \mathbb{Z}^n} \mathcal{J}_{\alpha'}$ is a \mathbb{Z}^n -graded Jordan algebra. Moreover, x is invertible in $\mathcal{J} \iff x$ is invertible in $\mathcal{J}^{(u)}$ since $U_x^{(u)} = U_x U_u$ (see [13, page 58]). Hence $\mathcal{J}^{(u)} = \bigoplus_{\alpha' \in \mathbb{Z}^n} \mathcal{J}_{\alpha'}$ is a division \mathbb{Z}^n -graded Jordan algebra. Therefore, since $\mathcal{J}^{(u)} \cong \mathcal{J}'$, we get $S - \gamma \cong S'$, i.e., $S \sim S'$. \square

§2.5 G-TORI AND TORI

In this subsection we will consider the following class of algebras.

Definition 2.5.1. A predivision G -graded algebra $T = \bigoplus_{g \in G} T_g$ is called an *alternative* or *Jordan G -torus over F* if $\dim_F T_g \leq 1$ for all $g \in G$.

We note that a G -torus is a division G -graded algebra by one dimensionality of the nonzero homogeneous spaces.

Lemma 2.5.2. *Let G be an arbitrary group and $T = \bigoplus_{g \in G} T_g$ a G -graded algebra over F . Suppose that T satisfies the following two properties: for all $g, h \in G$,*

$$(S1) \quad \dim_F T_g = 1, \quad (S2) \quad T_g T_h = T_{gh}.$$

Then T is a G -torus over F . Moreover, if $0 \neq x \in T$ is homogeneous, then for $y \in T$, $xy = 0$ implies $y = 0$.

Proof. Let $x \in T_g$. Then we have $F1 = T_e = T_g T_{g^{-1}} = FxT_{g^{-1}}$ by 2.1.2, 2.1.3(0) and (S1-2). So there exists a nonzero element $u \in T_{g^{-1}}$ such that $xu = 1$.

Suppose that T is an alternative algebra. Since $ux \in T_e = F1$, we have $ux = c1$ for some $c \in F$. Then, by the flexible law, we get $x = (xu)x = x(ux) = cx$. Hence $c = 1$, i.e., $xu = ux = 1$. Therefore x is invertible.

Suppose that T is a Jordan algebra. Since T is commutative, we have $xu = ux = 1$. We need to show another condition of invertibility in Jordan algebras, namely, $x^2u = x$. Since $x^2u \in T_g = Fx$, we can put $x^2u = dx$ for some $d \in F$. Then we have $dx^2 = x(dx) = x(x^2u) = x^2(xu) = x^2$ by the Jordan Identity. Hence $d = 1$, and x is invertible.

Thus we have shown that any nonzero homogeneous element is invertible. Consequently, T is a division G -graded algebra over F with (S1), and so it is a G -torus.

The last assertion is clear if T is alternative since x is invertible. In the Jordan case, let $y = \sum_{h \in G} y_h$. By $xy = \sum_{h \in G} xy_h = 0$, we get $xy_h = 0$ for all $h \in G$. If y_h is nonzero for some $h \in G$, then by (S1), $T_g T_h = FxT_h = (0)$, which contradicts (S2). Therefore, $y_h = 0$ for all $h \in G$, i.e., $y = 0$. \square

By 2.1.5(1) and 2.5.2, a G -graded alternative algebra $T = \bigoplus_{g \in G} T_g$ over F is an alternative G -torus if and only if T satisfies (S1-2). However, a Jordan G -torus satisfies neither (S1) nor (S2) in general, and we will give examples in Chapter 3. We call a Jordan G -torus satisfying (S1-2) a *Jordan G -torus of strong type*.

We can classify G -tori for a cyclic group G .

Proposition 2.5.3. *Let C_r be a cyclic group of order r . Then any C_r -torus T over F is commutative and associative. More precisely, if $r < \infty$, then T is isomorphic to an r -dimensional commutative associative algebra $F[x]$ generated by an element x whose minimal polynomial is $\lambda^r - a$ for some nonzero $a \in F$. Otherwise T is isomorphic to the algebra $F[t, t^{-1}]$ of Laurent polynomials.*

Proof. Let $C_r = \langle g \rangle$ and $S = \text{supp}(T)$. We first show $S = C_r$. Since $\langle S \rangle = C_r$, we have $C_r^2 \subset S$ where $C_r^2 = \{g^{2i} \mid i \in \mathbb{Z}^n\}$ by 2.4.1(ii). Hence if $C_r^2 = C_r$, we are done. Otherwise $C_r^2 \neq S$, and so $g^{2i+1} \in S$ for some $i \in \mathbb{Z}^n$. Then we have $g^{-2i}g^{2i+1} = g \in S$ by 2.4.1(ii), and hence $S = C_r$.

Thus $\dim_F T = |S| = |C_r| = r$. Let $0 \neq x \in T_g$ which exists since $S = C_r$. Since any power of an invertible element is nonzero (see e.g. [13, page 52]), T is generated by x and x^{-1} if $r = \infty$ and by x if $r < \infty$. In any case, T is commutative and associative by 1.1.2(4).

Suppose that $r < \infty$. Since $x^r \in T_e = F1$, $x^r = a1$ for some nonzero $a \in F$. So x is a root of a polynomial $\lambda^r - a$ (identifying F with $F1$). Since $T = \bigoplus_{i=0}^{r-1} Fx^i$, $\{x^i \mid i = 0, 1, \dots, r-1\}$ is a basis of T . So $\lambda^r - a$ is the minimal polynomial of x .

Suppose that $r = \infty$. Since $T = \bigoplus_{i \in \mathbb{Z}} Fx^i$, $\{x^i \mid i \in \mathbb{Z}\}$ is a basis of T . It is clear that T is isomorphic to the algebra of Laurent polynomials $F[t, t^{-1}]$ through $x^i \mapsto t^i$ for all $i \in \mathbb{Z}$. \square

Remark 2.5.4. (1) By 2.3.3, a division G -graded commutative associative algebra $Z = \bigoplus_{g \in G} Z_g$ over F is a G -torus over the field Z_e .

(2) Let $T = \bigoplus_{g \in G} T_g$ be a G -torus over F and let K be a field extension of F . Then $T_K = K \otimes_F T = \bigoplus_{g \in G} (K \otimes_F T_g)$ is a G -torus over K with $\text{supp}(T_K) = \text{supp}(T)$. Indeed, T_K is a predivision G -graded algebra over F by 2.1.0. Since $\dim_K(K \otimes_F T_g) = \dim_F T_g \leq 1$, T_K is a G -torus over K . It is clear that $\text{supp}(T_K) = \text{supp}(T)$.

We show that the central closure of a G -torus is a G/H -torus for the central grading group H if G is an ordered abelian group.

Lemma 2.5.5. *Let $T = \bigoplus_{g \in G} T_g$ be a G -torus over F where G is an ordered abelian group, $Z = Z(T)$ the centre of T and H the central grading group of G in T . Then for the central closure $\bar{T} = \bar{Z} \otimes_Z T$ (\bar{Z} is the field of fractions of Z), we have $\bar{T} = \bigoplus_{\bar{g} \in G/H} \bar{T}_{\bar{g}}$, which is a G/H -torus over \bar{Z} where $\bar{T}_{\bar{g}} = \bar{Z} \otimes_Z ZT_g$. Moreover, if*

T is a Jordan G -torus of strong type, then $\bar{T} = \bigoplus_{\bar{g} \in G/H} \bar{T}_{\bar{g}}$ is a Jordan G/H -torus over \bar{Z} of strong type.

Proof. We know that $\bar{T} = \bigoplus_{\bar{g} \in G/H} \bar{T}_{\bar{g}}$ is a predivision G/H -graded algebra over \bar{Z} by 2.3.5(i). Also, since $Z_e = F1$, we have $\dim_{\bar{Z}} \bar{T}_{\bar{g}} = \dim_F T_g \leq 1$ for all $g \in G$ by 2.3.5(ii). Hence \bar{T} is a G/H -torus over \bar{Z} . The second statement follows from 2.3.5(i). \square

Remark 2.5.6. In 2.5.5, G/H cannot be a nontrivial cyclic group. If it were cyclic, the G/H -torus \bar{T} is commutative and associative by 2.5.3. Since T embeds into \bar{T} by 1.1.16(i), T becomes commutative and associative, whence $Z = T$ and $H = G$. Hence G/H is the trivial group.

Now we specify G to be \mathbb{Z}^n .

Definition 2.5.7. A \mathbb{Z}^n -torus, i.e., a predivision \mathbb{Z}^n -graded algebra $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_{\alpha}$ over F satisfying $\dim_F T_{\alpha} \leq 1$ for all $\alpha \in \mathbb{Z}^n$, is called an n -torus over F or simply a torus when this abbreviation does not create confusion.

A torus clearly generalizes the algebra of Laurent polynomials in n -variables, and so it is a commutative associative torus. Also, from 2.5.3, we know that a 1-torus is isomorphic to $F[t, t^{-1}]$, the algebra of Laurent polynomials. As a noncommutative version of the algebra of Laurent polynomials, we recall quantum tori (see for example [6]).

Definition 2.5.8. An $n \times n$ matrix $\mathbf{q} = (q_{ij})$ over F such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix*. The *quantum torus* $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by a quantum matrix \mathbf{q} is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$.

We notice that $F_{\mathbf{q}}$ is commutative if and only if $\mathbf{q} = \mathbf{1}$ where $\mathbf{1}$ is the quantum matrix whose entries are all 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra of Laurent polynomials $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

It is easily seen that $F_{\mathbf{q}}$ is an associative torus. Namely, for any basis $\{\epsilon_1, \dots, \epsilon_n\}$ of \mathbb{Z}^n , declare the degree of t_i as ϵ_i for all $i = 1, \dots, n$. Then we have $F_{\mathbf{q}} = \bigoplus_{\alpha \in \mathbb{Z}^n} F t_{\alpha}$ where $t_{\alpha} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_n \epsilon_n$, $\alpha_i \in \mathbb{Z}$. So the \mathbb{Z}^n -graded structure of $F_{\mathbf{q}}$ depends on the choice of a basis of \mathbb{Z}^n . One can check

that the multiplication rule of F_q is the following: for $\alpha = \alpha_1 \varepsilon_1 + \cdots + \alpha_n \varepsilon_n$ and $\beta = \beta_1 \varepsilon_1 + \cdots + \beta_n \varepsilon_n$, $\alpha_i, \beta_i \in \mathbb{Z}$.

$$(2.5.9) \quad t_\alpha t_\beta = \prod_{i < j} q_{ij}^{\alpha_j \beta_i} t_{\alpha + \beta}.$$

Conversely, any associative torus is graded isomorphic to some quantum torus F_q . In fact, let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be an associative torus over F . Choose a basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathbb{Z}^n and choose a nonzero element $x_i \in T_{\varepsilon_i}$ for all $i = 1, \dots, n$. Since $x_i x_j, x_j x_i \in T_{\varepsilon_i + \varepsilon_j}$ and $\dim_F T_{\varepsilon_i + \varepsilon_j} = 1$, there exists a unique nonzero $q_{ij} \in F$ such that $x_j x_i = q_{ij} x_i x_j$. Then the $n \times n$ matrix $q = (q_{ij})$ is clearly a quantum matrix over F , and one can check that $T \cong_{\mathbb{Z}^n} F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, via $x_i \mapsto t_i$ specifying the degree of t_i as ε_i for $i = 1, \dots, n$.

Note that $q = (q_{ij})$ is not determined by T , but by the chosen basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathbb{Z}^n with its order. When we are considering the order of a basis of \mathbb{Z}^n , we denote it by $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$.

Definition 2.5.10. For a quantum torus $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the \mathbb{Z}^n -grading determined by an ordered basis $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ of \mathbb{Z}^n , i.e., declaring the degree of t_i to be ε_i , is called the $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ -grading of F_q . The \mathbb{Z}^n -gradings obtained in this way make F_q into a torus as defined in 2.5.7. They will be called *toral gradings*.

We summarize the above arguments as a theorem, which was also shown in [7, Lemma 1.8].

Theorem 2.5.11. *Any quantum torus is an associative torus. Moreover, for an associative n -torus T , any ordered basis $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ of \mathbb{Z}^n determines a quantum matrix q and T is graded isomorphic to F_q with the $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ -grading.*

A commutative associative n -torus is graded isomorphic to the algebra of Laurent polynomials $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with any toral grading.

Proof. We only need to show the last statement. For a commutative associative n -torus T , any basis of \mathbb{Z}^n determines the quantum matrix 1 . Thus T is graded isomorphic to $F_1 = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with any toral grading. \square

When $F_q \cong F_\eta$, we say that q is equivalent to η and denote this by $q \cong \eta$. Note that this is an equivalence relation. Also, we have that $q \cong 1$ implies $q = 1$. The following lemma is useful later to classify a certain type of quantum tori.

Lemma 2.5.12. *Let \mathfrak{q} and $\eta = (\eta_{ij})_{1 \leq i, j \leq n}$ be quantum matrices, and let $F_{\mathfrak{q}}$ respectively F_{η} be the corresponding quantum tori. Then the following are equivalent:*

- (i) $\mathfrak{q} \cong \eta$, i.e., $F_{\mathfrak{q}} \cong F_{\eta}$ as algebras,
- (ii) for any toral grading of $F_{\mathfrak{q}}$, there exists a basis $\langle \sigma_1, \dots, \sigma_n \rangle$ of \mathbb{Z}^n and nonzero homogeneous elements $x_i \in F_{\mathfrak{q}}$ of degree σ_i such that $x_j x_i = \eta_{ij} x_i x_j$ for all $1 \leq i < j \leq n$,
- (iii) there exist toral gradings of $F_{\mathfrak{q}}$ and F_{η} such that $F_{\mathfrak{q}} \cong_{\mathbb{Z}^n} F_{\eta}$.

Proof. We prove (i) \implies (ii) \implies (iii) \implies (i).

Suppose that (i) holds, i.e., there exists an isomorphism φ from $F_{\mathfrak{q}}$ onto F_{η} . Let $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ and $\langle \varepsilon'_1, \dots, \varepsilon'_n \rangle$ be any bases of \mathbb{Z}^n . Consider the associative tori $F_{\mathfrak{q}} = \bigoplus_{\alpha \in \mathbb{Z}^n} (F_{\mathfrak{q}})_{\alpha}$ with an $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ -grading and $F_{\eta} = F_{\eta}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} (F_{\eta})_{\alpha}$ with an $\langle \varepsilon'_1, \dots, \varepsilon'_n \rangle$ -grading. Then, by 2.4.3, there exists an automorphism p of \mathbb{Z}^n such that $\varphi((F_{\mathfrak{q}})_{\alpha}) = (F_{\eta})_{p(\alpha)}$ for all $\alpha \in \mathbb{Z}^n$. Let $\sigma_i := p^{-1}(\varepsilon'_i)$ and $x_i := \varphi^{-1}(t_i) \in (F_{\mathfrak{q}})_{\sigma_i}$ for $i = 1, \dots, n$. Then $\langle \sigma_1, \dots, \sigma_n \rangle$ is a basis of \mathbb{Z}^n , and we have

$$x_j x_i = \varphi^{-1}(t_j) \varphi^{-1}(t_i) = \varphi^{-1}(t_j t_i) = \varphi^{-1}(\eta_{ij} t_i t_j) = \eta_{ij} x_i x_j$$

for all $1 \leq i < j \leq n$. So (ii) holds.

Suppose that (ii) holds. Consider $F_{\mathfrak{q}}$ as an associative torus with a toral grading. Since $\langle \sigma_1, \dots, \sigma_n \rangle$ is a basis of \mathbb{Z}^n , the nonzero homogeneous elements x_i of degree σ_i determine a quantum matrix, which is η by the assumption (ii). Thus $F_{\mathfrak{q}}$ is graded isomorphic to the quantum torus $F_{\eta} = F_{\eta}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with the $\langle \sigma_1, \dots, \sigma_n \rangle$ -grading via $x_i \mapsto t_i$, by 2.5.11. Hence (iii) holds.

Finally, (iii) implies $F_{\mathfrak{q}} \cong F_{\eta}$, i.e., $\mathfrak{q} \cong \eta$. \square

We shall give an example of an alternative torus which is not associative in Chapter 4. We close this section by giving an example of a Jordan torus.

Example 2.5.13. The plus algebra A^+ of an associative torus $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_{\alpha}$ over F is a predivision \mathbb{Z}^n -graded Jordan algebra over F by 2.1.7. Since $\dim_F A_{\alpha} = 1$ for all $\alpha \in \mathbb{Z}^n$, we have $\dim_F (A^+)_{\alpha} = 1$, and hence A^+ is a Jordan torus over F . By 2.5.11, we have $A^+ \cong_{\mathbb{Z}^n} F_{\mathfrak{q}}^+$ for some quantum torus $F_{\mathfrak{q}} = F_{\mathfrak{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a toral grading. From (2.5.9) the multiplication rule of $F_{\mathfrak{q}}^+$, using the notation there.

is the following:

$$\begin{aligned}
t_\alpha \cdot t_\beta &= \frac{1}{2} \left(\prod_{i < j} q_{ij}^{\alpha_j \beta_i} + \prod_{i < j} q_{ij}^{\beta_j \alpha_i} \right) t_1^{\alpha_1 + \beta_1} \dots t_n^{\alpha_n + \beta_n} \\
(2.5.14) \quad &= \frac{1}{2} \prod_{i < j} q_{ij}^{\alpha_j \beta_i} \left(1 + \prod_{i, j} q_{ij}^{\alpha_i \beta_j} \right) t_{\alpha + \beta}.
\end{aligned}$$

Note that $\text{supp}(F_q^+) = \mathbb{Z}^n$, and by (2.5.14), F_q^+ is a Jordan torus of strong type if and only if

$$(2.5.15) \quad \prod_{i, j} q_{ij}^{\alpha_i \beta_j} \neq -1 \quad \text{for all } \alpha, \beta \in \mathbb{Z}^n.$$

For example, let $\mathbf{h} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $(\alpha_1, \alpha_2) = (1, 0)$ and $(\beta_1, \beta_2) = (0, 1)$. Then $\prod_{i, j} q_{ij}^{\alpha_i \beta_j} = (-1)^{\alpha_1 \beta_2} (-1)^{\alpha_2 \beta_1} = (-1)^1 (-1)^0 = -1$. Hence $F_{\mathbf{h}}^+$ is not of strong type. Also, let $\mathbf{i} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ where i is a square root of -1 , $(\alpha_1, \alpha_2) = (1, 0)$ and $(\beta_1, \beta_2) = (0, 2)$. Then $\prod_{i, j} q_{ij}^{\alpha_i \beta_j} = i^{\alpha_1 \beta_2} (-i)^{\alpha_2 \beta_1} = i^2 (-i)^0 = -1$. Hence $F_{\mathbf{i}}^+$ is not of strong type. In general for a 2×2 quantum matrix $\mathbf{q} = \begin{pmatrix} 1 & q \\ q^{-1} & 1 \end{pmatrix}$, we have

$$\begin{aligned}
\prod_{i, j} q_{ij}^{\alpha_i \beta_j} &= q^{\alpha_1 \beta_2} (q^{-1})^{\alpha_2 \beta_1} \neq -1 \quad \text{for all } (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{Z}^2 \\
\iff q^l &\neq -q^m \quad \text{for all } l, m \in \mathbb{Z} \\
\iff q^k &\neq -1 \quad \text{for all } k \in \mathbb{Z} \\
\iff &\text{either } q \text{ is not a root of unity or } q \text{ is a primitive } r\text{th root of unity, } r \text{ odd.}
\end{aligned}$$

For an $n \times n$ quantum matrix, $n \geq 3$, the situation is more complicated. We just give one example of F_q^+ which is not of strong type and where all q_{ij} , $i \neq j$ are not roots of unity, namely,

$$\mathbf{q} = \begin{pmatrix} 1 & 2 & 2 \\ \frac{1}{2} & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

Indeed, let $(\alpha_1, \alpha_2, \alpha_3) = (1, -1, 0)$ and $(\beta_1, \beta_2, \beta_3) = (0, 0, 1)$. Then $\prod_{i, j} q_{ij}^{\alpha_i \beta_j} = q_{12}^{\alpha_1 \beta_2} q_{21}^{\alpha_2 \beta_1} q_{13}^{\alpha_1 \beta_3} q_{31}^{\alpha_3 \beta_1} q_{23}^{\alpha_2 \beta_3} q_{32}^{\alpha_3 \beta_2} = 2^0 (\frac{1}{2})^0 2^1 (\frac{1}{2})^0 (-2)^{-1} (-\frac{1}{2})^0 = -1$. Note that this corresponds to the fact that $(t_1 t_2^{-1}) \cdot t_3 = 0$ in $F_q^+ = F_q[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]^+$. Hence F_q^+ is not of strong type.

§2.6 CENTRAL DEGREE

Recall that in this chapter our algebras are assumed to be alternative or Jordan, and so they are power associative by 1.1.1, even strictly power associative (see [13, page 222]). Hence one can define the generic degree for a finite dimensional algebra over an arbitrary base field F (ch. $F \neq 2$ for our Jordan case). We first review the definition of the generic minimal polynomial (see [13, page 222]):

Let A be a finite dimensional algebra (alternative or Jordan) over F , $\{u_1, \dots, u_n\}$ a basis of A over F , $L = F(\xi_1, \dots, \xi_n)$ where the ξ_i are algebraically independent over F , and $A_L := L \otimes A$. Let $x = \sum_{i=1}^n \xi_i \otimes u_i$ and $m(\lambda)_x$ the minimal polynomial of x (in A_L), which we write as

$$m(\lambda)_x = \lambda^m - \sigma_1(\xi)\lambda^{m-1} + \sigma_2(\xi)\lambda^{m-2} - \dots + (-1)^m \sigma_m(\xi)$$

where $\sigma_i(\xi) = \sigma_i(\xi_1, \dots, \xi_n) \in L$. For $a = \sum_{i=1}^n a_i u_i \in A$, the specialization $\xi_i = a_i$ of $m(\lambda)_x$ gives $m(\lambda)_a \in F[\lambda]$, and $m(\lambda)_a$ is called the *generic minimal polynomial of a* . It is known that $\deg m(\lambda)_x (= \deg m(\lambda)_a)$ and $m(\lambda)_a$ itself do not depend on the choice of a basis of A (see [13, page 223]). Thus the degree of the generic minimal polynomial of a is invariant for any $a \in A$. Also, the elements $\sigma_1(a)$ and $\sigma_m(a)$ of A are called the *generic trace* and the *generic norm* of a , respectively. It is well-known that the map $\sigma_1 : A \rightarrow F$ is F -linear with $\sigma_1(1) = m$ and that the map $\sigma_m : A \rightarrow F$ satisfies $\sigma_m(\alpha a) = \alpha^m \sigma_m(a)$ for $\alpha \in F$ and $a \in A$ with $\sigma_m(1) = 1$ (see [13, Theorem 1, page 224]). We denote $\sigma_1 = \text{tr}$ and $\sigma_m = n$ by convention.

Definition 2.6.1. For a finite dimensional algebra its *degree* is defined to be the degree of the generic minimal polynomial. Also, for a commutative associative unital ring Φ and a Φ -algebra A , we say A has degree ≤ 2 if there exist a Φ -linear form $\text{tr} : A \rightarrow \Phi$ and a Φ -quadratic map $n : A \rightarrow \Phi$ with $n(1) = 1$ such that

$$a^2 - \text{tr}(a)a + n(a)1 = 0$$

for all $a \in A$ (see [19]). In case Φ is a field and $\dim_{\Phi} A \neq 1$, we say that the degree ≤ 2 algebra A has degree 2. This definition of degree ≤ 2 is consistent for the case of finite dimensional algebras. For example, Φ has degree ≤ 2 by the second definition, and in case Φ is a field, Φ has degree 1.

Example 2.6.2. An r^2 -dimensional central simple associative algebra has degree r (see e.g. [13, page 233]).

Definition 2.6.3. We say that a prime algebra P over a field F has *central degree* r if the central closure $\overline{P} = \overline{Z} \otimes_Z P$ is a finite dimensional algebra over \overline{Z} of degree r for $r \geq 3$ or $r = 1$, and *central degree 2* if the central closure \overline{P} (not necessarily finite dimensional) has degree 2 over \overline{Z} . Note that \overline{P} is always central over \overline{Z} by 1.1.16(ii).

Example 2.6.4. Assume that F contains a primitive r -th root of unity $\zeta = \zeta_r$. Let ζ be an $n \times n$ quantum matrix

$$\zeta = \zeta_r^{(n)} = \begin{pmatrix} 1 & \zeta & 1 & \cdots & 1 \\ \zeta^{-1} & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

where the $(1, 2)$ -entry is ζ , the $(2, 1)$ -entry is ζ^{-1} and the other entries are all 1. Let $F_\zeta = F_\zeta[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the quantum torus determined by ζ and $Z = Z(F_\zeta)$ the centre of F_ζ . One finds that $Z = F[t_1^{\pm r}, t_2^{\pm r}, t_3^{\pm 1}, \dots, t_n^{\pm 1}]$, the algebra of Laurent polynomials in the variables $t_1^r, t_2^r, t_3, \dots, t_n$. So for an $(\epsilon_1, \dots, \epsilon_n)$ -grading of F_ζ , the central grading group of F_ζ is $r\mathbb{Z}\epsilon_1 + r\mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3 + \cdots + \mathbb{Z}\epsilon_n$, the central closure \overline{F}_ζ is a $(\mathbb{Z}_r \times \mathbb{Z}_r)$ -torus over \overline{Z} by 2.5.5, where $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$. So the dimension of \overline{F}_ζ over \overline{Z} is r^2 . Since \overline{F}_ζ has no zero divisors by 2.2.4(1) and 1.1.16(iii), the finite dimensional algebra \overline{F}_ζ has no radical, and hence it is semisimple. By Wedderburn's Structure Theorem, \overline{F}_ζ is a division algebra since it has no zero divisors. Hence \overline{F}_ζ has degree r over \overline{Z} by 2.6.2. Thus, in case Φ is a field, F_ζ has central degree r .

We show that this example has the following strong property:

Lemma 2.6.5. *Let tr be the generic trace of the central closure \overline{F}_ζ over \overline{Z} . Then $tr(F_\zeta) \subset Z$ (identifying F_ζ with a subalgebra of \overline{F}_ζ).*

Proof. For $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ with $(\bar{0}, \bar{0}) \neq (\bar{i}, \bar{j}) \in \mathbb{Z}_r \times \mathbb{Z}_r$, let $m = m(\bar{i}, \bar{j})$ be the order of (\bar{i}, \bar{j}) in the group $\mathbb{Z}_r \times \mathbb{Z}_r$. Note that $m > 1$ and that m divides r .

Claim. $\{1, t_1^i t_2^j, \dots, (t_1^i t_2^j)^{m-1}\}$ is linearly independent over \overline{Z} .

Proof. One can check that $(t_1^i t_2^j)^k = \zeta^{kij} t_1^{ki} t_2^{kj}$ for $k \in \mathbb{Z}$, and so $(t_1^i t_2^j)^k = u_k t_1^{p_k} t_2^{q_k}$ for some $0 \neq u_k \in Z$ and $0 \leq p_k, q_k \leq r-1$ ($p_k \equiv ki$, $q_k \equiv kj \pmod{r}$). Since m is the

order of (\bar{i}, \bar{j}) , all $(p_1, q_1), \dots, (p_k, q_k), \dots, (p_{m-1}, q_{m-1})$ are distinct and not $(0, 0)$. Hence

$$1. t_1^i t_2^j = u_1 t_1^{p_1} t_2^{q_1} \dots (t_1^i t_2^j)^k = u_k t_1^{p_k} t_2^{q_k} \dots (t_1^i t_2^j)^{m-1} = u_{m-1} t_1^{p_{m-1}} t_2^{q_{m-1}}$$

are linearly independent over \bar{Z} since $\{t_1^p t_2^q \mid 0 \leq p, q \leq r-1\}$ is a basis of \bar{F}_ζ over \bar{Z} and all $u_1, \dots, u_k, \dots, u_{m-1}$ are not zero.

Note that $(t_1^i t_2^j)^m \in \mathcal{Z} \subset \bar{Z}$. So the minimal polynomial of $t_1^i t_2^j$ over \bar{Z} is $f(\lambda) := \lambda^m - (t_1^i t_2^j)^m$ by the Claim. Since \bar{F}_ζ is a division algebra, $f(\lambda)$ is irreducible over \bar{Z} . Recall that the minimal polynomial and the generic minimal polynomial of an element have the same irreducible factors (see [13, page 224]). So the generic minimal polynomial of $t_1^i t_2^j$ is a power of $f(\lambda)$. Since the degree of the generic minimal polynomial is r , the generic minimal polynomial of $t_1^i t_2^j$ is $f(\lambda)^{\frac{r}{m}}$. Since $m > 1$ and $-\text{tr}(t_1^i t_2^j)$ is the coefficient of λ^{r-1} in $f(\lambda)^{\frac{r}{m}}$, we have $\text{tr}(t_1^i t_2^j) = 0$ for $(\bar{i}, \bar{j}) \neq (\bar{0}, \bar{0})$. Thus we get

$$\text{tr}(t_1^i t_2^j) = \begin{cases} r t_1^i t_2^j & \text{if } i \equiv j \equiv 0 \pmod{r} \\ 0 & \text{otherwise.} \end{cases}$$

and we obtain $\text{tr}(F_\zeta) \subset \mathcal{Z}$.

To describe the central grading group of a division \mathbb{Z}^n -graded algebra of central degree 2, we prove the following:

Lemma 2.6.6. *Let $T = \bigoplus_{g \in G} T_g$ be a G -graded algebra over F which has degree ≤ 2 . Let $e \neq g \in \text{supp}(G)$. If $0 \neq x \in T_g$ is not nilpotent, then $\text{tr}(x) = 0$, $x^2 = -n(x)1 \in F1$ and the order of g is 2.*

Proof. We have

$$x^2 - \text{tr}(x)x + n(x)1 = 0,$$

and so $x^2 + n(x)1 = \text{tr}(x)x \in T_g$. If $\text{tr}(x) \neq 0$, then $g^2 = g$ since $x^2 \neq 0$. Hence $g = e$, which contradicts our assumption. So we get $\text{tr}(x) = 0$. Then $x^2 = -n(x)1 \in F1$. Moreover, we have $0 \neq x^2 \in T_{g^2} \cap F1 \subset T_{g^2} \cap T_e$ by 2.1.2, and hence $g^2 = e$. \square

To describe the central grading group of a division \mathbb{Z}^n -graded algebra of central degree 3, we prove the following:

Lemma 2.6.7. *Let S be a semilattice in \mathbb{Z}^n . If $k\mathbb{Z}^n \subset S$ for some odd number k , then $S = \mathbb{Z}^n$.*

Proof. By 2.4.1(iii) there exists a basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n such that each $\sigma_i \in S$. Let $k = 2l + 1$. For any $m_1, \dots, m_n \in \{0, 1\}$, we have $k(m_1\sigma_1 + \dots + m_n\sigma_n) \in S$, and so

$$km_1\sigma_1 + \dots + km_n\sigma_n = m_1\sigma_1 + \dots + m_n\sigma_n + 2(lm_1\sigma_1 + \dots + lm_n\sigma_n) \in S.$$

Then, by 2.4.1(ii), we get $m_1\sigma_1 + \dots + m_n\sigma_n \in S$. Hence S contains all the cosets of $\mathbb{Z}^n/2\mathbb{Z}^n$, whence $S = \mathbb{Z}^n$. \square

Using the two lemmas above, we can show the following:

Proposition 2.6.8. *Let $\mathcal{T} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{T}_\alpha$ be a division \mathbb{Z}^n -graded algebra over F of central degree r for $r = 2, 3$. Let tr be the generic trace of the central closure $\overline{\mathcal{T}}$. Then:*

(i) *There exists a basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathbb{Z}^n such that the central grading group of \mathcal{T} is given as*

$$\Gamma := r\mathbb{Z}\varepsilon_1 + \dots + r\mathbb{Z}\varepsilon_m + \mathbb{Z}\varepsilon_{m+1} + \dots + \mathbb{Z}\varepsilon_n \quad \text{for some } 0 \leq m \leq n.$$

Moreover, for any $\alpha \in \mathbb{Z}^n \setminus \Gamma$, we have $\text{tr}(\mathcal{T}_\alpha) = \{0\}$.

(ii) *If $r = 3$, then $\text{supp}(\mathcal{T}) = \mathbb{Z}^n$.*

(iii) *If \mathcal{T} is a torus, then $m \geq 2$.*

Proof. Let Γ be the central grading group of \mathcal{T} , \mathcal{Z} the centre of \mathcal{T} and $\{\sigma_1, \dots, \sigma_n\}$ a basis of \mathbb{Z}^n in $S := \text{supp}(\mathcal{T})$. Recall that

$$\overline{\mathcal{T}} = \bigoplus_{\bar{\alpha} \in \mathbb{Z}^n/\Gamma} \overline{\mathcal{T}}_{\bar{\alpha}}$$

is a \mathbb{Z}^n/Γ -graded algebra over $\overline{\mathcal{Z}}$ where $\overline{\mathcal{T}}_{\bar{\alpha}} = \overline{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{Z}\mathcal{T}_\alpha$ (see 2.3.5(i)). Also, by 2.2.4, $\overline{\mathcal{T}}$ is an (alternative or Jordan) domain. We show (i) for the cases $r = 2$ and 3 separately.

[Case $r = 2$]: Since $\overline{\mathcal{T}}$ is a domain, there are no nonzero nilpotent elements in $\overline{\mathcal{T}}$. Hence for $\sigma_i \notin \Gamma$, i.e., $\bar{\sigma}_i \neq \bar{0}$, we have $2\sigma_i \in \Gamma$ by 2.6.6. If $\sigma_i \in \Gamma$, then $2\sigma_i \in \Gamma$. Thus we obtain $\mathbb{Z}(2\sigma_1) + \dots + \mathbb{Z}(2\sigma_n) = 2\mathbb{Z}^n \subset \Gamma$. Then the exponent of

\mathbb{Z}^n/Γ is 1 or 2, i.e., $\Gamma = \mathbb{Z}^n$ or $\mathbb{Z}^n/\Gamma \cong \mathbb{Z}_2^m$ for some $1 \leq m \leq n$, and there exists a basis $\{\epsilon_1, \dots, \epsilon_n\}$ of \mathbb{Z}^n such that $\Gamma = 2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n$, $0 \leq m \leq n$, by the Fundamental Theorem of finitely generated abelian groups. The second statement also follows from 2.6.6.

[Case $r = 3$]: For $\beta \in S$ such that $\overline{\beta} \neq \overline{0}$ in \mathbb{Z}^n/Γ , take $0 \neq x \in \overline{\mathcal{T}}_{\overline{\beta}}$. Since $\overline{\mathcal{T}}$ has degree 3 over $\overline{\mathcal{Z}}$, x satisfies

$$x^3 + z_1x^2 + z_2x + z_31 = 0 \quad \text{for some } z_1, z_2, z_3 \in \overline{\mathcal{Z}} \quad (z_1 = -\text{tr}(x)).$$

If $\overline{2\beta} = \overline{0}$, then $\overline{3\beta} = \overline{\beta}$ and therefore,

$$x^3 - z_2x = -z_1x^2 + z_31 \in \overline{\mathcal{T}}_{\overline{\beta}} \cap \overline{\mathcal{T}}_{\overline{0}} = (0).$$

Hence we get $x^3 - z_2x = x(x^2 - z_21) = 0$. Since $\overline{\mathcal{T}}$ is a Jordan domain, the subalgebra $\overline{\mathcal{Z}}[x]$ of $\overline{\mathcal{T}}$ generated by x is a commutative associative algebra domain over $\overline{\mathcal{Z}}$ by 1.1.1. So $x^2 - z_21 = 0$ since $x \neq 0$. Since $x \notin \overline{\mathcal{T}}_{\overline{0}}$, the polynomial $f(\lambda) = \lambda^2 - z_2$ is the minimal polynomial of x over $\overline{\mathcal{Z}}$. If $f(\lambda)$ is reducible over $\overline{\mathcal{Z}}$, say $f(\lambda) = (\lambda - a)(\lambda - b)$, $a, b \in \overline{\mathcal{Z}}$, then $(x - a1)(x - b1) = 0$ in $\overline{\mathcal{Z}}[x]$ which is a commutative associative domain. Hence, $x = a1$ or $x = b1$, and so $x \in \overline{\mathcal{Z}}1 \subset \overline{\mathcal{T}}_{\overline{0}}$, i.e., $\overline{\beta} = \overline{0}$, which is a contradiction. Therefore, $f(\lambda)$ is irreducible over $\overline{\mathcal{Z}}$. As already mentioned in 2.6.5, the minimal polynomial and the generic minimal polynomial of an element have the same irreducible factors (see [13, page 224]). Since $f(\lambda)$ is the irreducible minimal polynomial of x , the generic minimal polynomial of x has to be a power of $f(\lambda)$. However, this is impossible since the degree of the generic minimal polynomial of x is 3. Therefore, $\overline{2\beta} \neq \overline{0}$.

Now, $\overline{2\beta} \neq \overline{0}$ implies that $\overline{3\beta} \neq \overline{\beta}$. Since $\overline{\beta} \neq \overline{0}$, we have $\overline{3\beta} \neq \overline{2\beta}$. Hence $\{\overline{3\beta}, \overline{0}\} \cap \{\overline{2\beta}, \overline{\beta}\} = \emptyset$. So $(\overline{\mathcal{T}}_{\overline{3\beta}} + \overline{\mathcal{T}}_{\overline{0}}) \cap (\overline{\mathcal{T}}_{\overline{2\beta}} \oplus \overline{\mathcal{T}}_{\overline{\beta}}) = (0)$. Since

$$x^3 + z_31 = -z_1x^2 - z_2x \in (\overline{\mathcal{T}}_{\overline{3\beta}} + \overline{\mathcal{T}}_{\overline{0}}) \cap (\overline{\mathcal{T}}_{\overline{2\beta}} \oplus \overline{\mathcal{T}}_{\overline{\beta}}),$$

we get two equalities $x^3 + z_31 = 0$ and $-z_1x^2 - z_2x = 0$.

By the first identity, we have $0 \neq x^3 = -z_31 \in \overline{\mathcal{T}}_{\overline{3\beta}} \cap \overline{\mathcal{Z}}1 \subset \overline{\mathcal{T}}_{\overline{3\beta}} \cap \overline{\mathcal{T}}_{\overline{0}}$, and hence $\overline{3\beta} = \overline{0}$. Thus we obtain $\overline{3\beta} = \overline{0}$ for all $\beta \in S$. In particular, for $\sigma_i \notin \Gamma$, i.e., $\overline{\sigma_i} \neq \overline{0}$, we get $3\sigma_i \in \Gamma$. If $\sigma_i \in \Gamma$, then $3\sigma_i \in \Gamma$. Hence we have $\mathbb{Z}(3\sigma_1) + \dots + \mathbb{Z}(3\sigma_n) = 3\mathbb{Z}^n \subset \Gamma$, and so the exponent of \mathbb{Z}^n/Γ is 1 or 3, i.e., $\Gamma = \mathbb{Z}^n$ or

$\mathbb{Z}^n/\Gamma \cong \mathbb{Z}_3^m$ for some $1 \leq m \leq n$. and there exists a basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathbb{Z}^n such that $\Gamma = 3\mathbb{Z}\varepsilon_1 + \dots + 3\mathbb{Z}\varepsilon_m + \mathbb{Z}\varepsilon_{m+1} + \dots + \mathbb{Z}\varepsilon_n$, $0 \leq m \leq n$. by the Fundamental Theorem of finitely generated abelian groups.

By the second identity and by the same reason above. we have $-z_1x - z_21 = 0$. Then $-z_1x = z_21 \in \overline{\mathcal{T}}_{\beta} \cap \overline{\mathcal{Z}}1 \subset \overline{\mathcal{T}}_{\beta} \cap \overline{\mathcal{T}}_0 = (0)$. Hence $z_1 = 0$. i.e., $\text{tr}(x) = 0$ for any $\beta \in S \setminus \Gamma$ and $0 \neq x \in \mathcal{T}_{\beta}$. Therefore, for any $\alpha \in \mathbb{Z}^n \setminus \Gamma$. we have $\text{tr}(\mathcal{T}_{\alpha}) = \{0\}$.

For (ii), since we have $\Gamma \subset S$, we get $3\mathbb{Z}^n \subset S$. By 2.6.7. we obtain $S = \mathbb{Z}^n$.

For (iii). let \mathcal{T} be an n -torus over F . If $m = 0$, then $\Gamma = \mathbb{Z}^n$, and so $\mathcal{Z} = \mathcal{T}$ since \mathcal{T} is a torus. and hence $\overline{\mathcal{Z}} = \overline{\mathcal{T}}$. Therefore, the degree of $\overline{\mathcal{T}}$ over $\overline{\mathcal{Z}}$ is not 2 or 3. If $m = 1$. then $\mathbb{Z}/\Gamma = \mathbb{Z}_r$ is a nontrivial cyclic group, and $\overline{\mathcal{T}}$ is a \mathbb{Z}_r -torus over $\overline{\mathcal{Z}}$ by 2.5.5. which cannot happen by 2.5.6. Hence we get $m \geq 2$. (So n has to be ≥ 2 .) \square

We will classify associative tori of central degree $r = 2$ and 3. Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_{\alpha}$ be an associative torus over F of central degree r and $\overline{\mathcal{T}}$ the central closure over $\overline{\mathcal{Z}}$. Then, by 2.6.8(i), we have $\dim_{\overline{\mathcal{Z}}}\overline{\mathcal{T}} = r^m$. So $\overline{\mathcal{T}}$ is a finite dimensional central associative algebra over $\overline{\mathcal{Z}}$. Since $\overline{\mathcal{T}}$ does not have zero-divisors by 1.1.16(iii), it is a finite dimensional central associative division algebra over $\overline{\mathcal{Z}}$ of degree r by Wedderburn's Structure Theorem. Hence the dimension of $\overline{\mathcal{T}}$ over $\overline{\mathcal{Z}}$ has to be r^2 . Thus, by 2.6.8. there exists a basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n such that the central grading group of T is $\Gamma := r\mathbb{Z}\sigma_1 + r\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3 + \dots + \mathbb{Z}\sigma_n$. Also, it is clear that an associative torus whose central grading group is Γ has central degree r .

Now, we classify associative tori whose central grading group is Γ . Let $0 \neq t_i \in \mathcal{T}_{\sigma_i}$, for $1 \leq i \leq n$. Then since $t_i t_1 = t_1 t_i$, $t_i t_2 = t_2 t_i$ and $t_j t_i = t_i t_j$ for all $3 \leq i \leq n$ and $1 \leq j \leq n$. we can identify such a T with the quantum torus $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by

$$q = q(q) = \begin{pmatrix} 1 & q & 1 & \cdots & 1 \\ q^{-1} & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

where the (1,2)-entry is some $q \in F^{\times}$, the (2,1)-entry is q^{-1} and the other entries are all 1. Moreover, since $t_2^r \in Z$, we have $t_1 t_2^r = t_2^r t_1 = q^r t_1 t_2^r$, and so $q^r = 1$. If $q = 1$. then $q = q(1) = 1$, but the algebra of Laurent polynomials F_1 cannot have central degree r since $Z = Z(F_1) = F_1$. Hence $q \neq 1$ and q is the primitive 2nd or a

primitive 3rd root of unity. If $r = 2$, then $\text{ch.}F \neq 2$ and $q = -1$.

If $r = 3$, then $\text{ch.}F \neq 3$ and F has to contain a primitive 3rd root of unity, say ω . Since q can be either ω or ω^{-1} , let $q := q(\omega)$ and $q' := q(\omega^{-1})$. We claim that $F_q \cong F_{q'}$. Consider $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ as an n -torus with the $\langle \sigma_1, \dots, \sigma_n \rangle$ -grading. Since $\langle \sigma_2, \sigma_1, \sigma_3, \dots, \sigma_n \rangle$ is a basis of \mathbb{Z}^n , $t_1 t_2 = \omega^{-1} t_2 t_1$ (since $t_2 t_1 = \omega t_1 t_2$) and $t_j t_i = t_i t_j$ for $1 \leq i \leq n$ and $3 \leq j \leq n$, we get $F_q \cong F_{q'}$ by 2.5.12.

Remark 2.6.9. Note that $F_{q'}$ can be identified with the opposite algebra F_q^{op} of F_q . Then F_q and $F_{q'}$ are both algebras over their common centre

$$Z = F[t_1^{\pm 3}, t_2^{\pm 3}, t_3^{\pm 1}, \dots, t_n^{\pm 1}].$$

We showed that $F_q \cong F_{q'}$ over F , but we note that $F_q \not\cong F_{q'}$ over Z . In general, if A is an associative domain of central degree 3, then we always have $A \not\cong A^{op}$ over Z . For, if $A \cong A^{op}$ over Z , then $\bar{A} \cong \bar{A}^{op}$ over \bar{Z} , which cannot happen since \bar{A} is a central division associative algebra of degree 3. (See e.g. [29]; if $\bar{A} \cong \bar{A}^{op}$, then $[\bar{A}]^2 = 1$ in the Brauer group. But the order of \bar{A} in the Brauer group has to divide the degree.)

We summarize the above results as a proposition.

Proposition 2.6.10. (1) For an associative torus T over F and $r = 2, 3$, T has central degree $r \iff$ the central grading group of T is $r\mathbb{Z}\sigma_1 + r\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3 + \dots + \mathbb{Z}\sigma_n$ for some basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n .

(2) Let ω be a primitive 3rd root of unity,

$$h = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 1 & \omega & 1 & \cdots & 1 \\ \omega^2 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}.$$

(i) If $\text{ch.}F \neq 2$, then an associative torus over F of central degree 2 exists and any such torus is isomorphic to the quantum torus F_h . If $\text{ch.}F = 2$, no such torus exists.

(ii) If F contains a primitive 3rd root of unity, say $\omega \in F$, then an associative torus over F of central degree 3 exists, and any such torus is isomorphic to the quantum torus F_ω . If F does not contain a primitive 3rd root of unity, e.g. $\text{ch.}F = 3$ or $F = \mathbb{Q}, \mathbb{R}$, no such torus exists. \square

Chapter 3 Jordan tori

Our purpose in this chapter is to classify Jordan tori over a field F of characteristic $\neq 2$.

§ 3.0 ZELMANOV'S PRIME STRUCTURE THEOREM

By Zelmanov's Prime Structure Theorem ([21, page 200]), a strongly prime Jordan algebra \mathcal{J} over F is one of the following three types: (The new terminology used below will be explained in the following sections.)

Hermitian Type: \mathcal{J} is special and $q_{48}(\mathcal{J}) \neq \{0\}$,

Clifford Type: the central closure $\overline{\mathcal{J}}$ is a simple Jordan algebra over $\overline{\mathcal{Z}}$ determined by a symmetric bilinear form,

Albert Type: the central closure $\overline{\mathcal{J}}$ is an Albert algebra over $\overline{\mathcal{Z}}$.

Since Jordan tori or division \mathbb{Z}^n -graded division algebras are strongly prime by 2.2.5, the type of Jordan tori or division \mathbb{Z}^n -graded division algebras is defined as the above.

§ 3.1 HERMITIAN TYPE

We review the so-called Zelmanov polynomial q_{48} . Let $\mathcal{FSJ}(X)$ be a free special Jordan algebra on an infinite set X of variables over F . That is, $\mathcal{FSJ}(X)$ is the subalgebra of $\mathcal{FA}(X)^+$ generated by X where $\mathcal{FA}(X)$ is the free associative algebra on X . For $x, y, z, w \in X$, let

$$p_{16}(x, y, z, w) = [[D_{x,y}^2(z)^2, D_{x,y}(w)], D_{x,y}(w)] \in \mathcal{FA}(X)$$

where $D_{x,y}(z) = [[x, y], z]$. For 12 variables $x_i, y_i, z_i, w_i \in X$, $i = 1, 2, 3$, let

$$q_{48} = [[p_{16}(x_1, y_1, z_1, w_1), p_{16}(x_2, y_2, z_2, w_2)], p_{16}(x_3, y_3, z_3, w_3)] \in \mathcal{FA}(X).$$

By the identity $[x, [y, z]] = (y, x, z)^\circ$ (see 1.1.10a), we have $p_{16}, q_{48} \in \mathcal{FSJ}(X)$. Moreover, q_{48} is homogeneous in each variable, i.e., all monomials of q_{48} , the monomials not only of the associative product but also of the Jordan product, have the same partial degree in each variable. Namely, p_{16} has degree 6 in x , 6 in y , 2 in z and 2 in w . and so q_{48} has degree 6 in x_i , 6 in y_i , 4 in z_i and 2 in w_i . and hence the total degree in 12 variables is 48. For any Jordan algebra \mathcal{J} , we denote the evaluation of q_{48} on \mathcal{J} by $q_{48}(\mathcal{J})$.

Recall that a division \mathbb{Z}^n -graded Jordan algebra is strongly prime by 2.2.5. Thus, by §3.0, a division \mathbb{Z}^n -graded Jordan algebra \mathcal{J} of Hermitian Type is special and $q_{48}(\mathcal{J}) \neq \{0\}$. We prove a more precise version of the Hermitian Type classification in our case. For this purpose, we need to quote the existence of a nice ideal of $\mathcal{FSJ}(X)$, called Q_{48} .

An ideal $\mathcal{I} \triangleleft \mathcal{FSJ}(X)$ is called *formal* if for all permutations σ of X ,

$$p(x_1, \dots, x_n) \in \mathcal{I} \implies p(\sigma(x_1), \dots, \sigma(x_n)) \in \mathcal{I}.$$

For a formal ideal \mathcal{I} of $\mathcal{FSJ}(X)$ and any special Jordan algebra \mathcal{J} , it is well-known that the evaluation $\mathcal{I}(\mathcal{J})$ is an ideal of \mathcal{J} (see [21, page 144]). We define an *r-tad* $\{p_1 \cdots p_r\}$ for $p_1, \dots, p_n \in \mathcal{FSJ}(X)$ as

$$\{p_1 \cdots p_r\} = p_1 \cdots p_r + p_r \cdots p_1.$$

In particular, $\{p_1 p_2 p_3 p_4\}$ is called a *tetrad*. A formal ideal $\mathcal{H} \triangleleft \mathcal{FSJ}(X)$ is called *hermitian* if it is closed under tetrads, i.e.,

$$\{\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H}\} \subset \mathcal{H}.$$

An ideal $\mathcal{I} \triangleleft \mathcal{FSJ}(X)$ is called a *linearization-invariant T-ideal* if \mathcal{I} contains all the linearizations of any $p \in \mathcal{I}$ and $T(\mathcal{I}) \subset \mathcal{I}$ for any algebra endomorphism T of $\mathcal{FSJ}(X)$. Since for a permutation σ the map $(x_1, \dots, x_n) \mapsto (\sigma(x_1), \dots, \sigma(x_n))$ defines an endomorphism of $\mathcal{FSJ}(X)$, every linearization-invariant T-ideal is formal. Let Q_{48} be the linearization-invariant T-ideal of $\mathcal{FSJ}(X)$ generated by q_{48} . Then it is known that Q_{48} is an hermitian ideal (see [21, page 198]).

Now, let \mathcal{J} be any strongly prime special Jordan algebra with $q_{48}(\mathcal{J}) \neq \{0\}$. Then the evaluation $Q_{48}(\mathcal{J})$ is a nonzero ideal of \mathcal{J} . Since \mathcal{J} is special, there exists an

associative algebra \mathcal{A} with involution $*$ such that $\mathcal{J} \subset H(\mathcal{A}, *) = \{a \in \mathcal{A} \mid a^* = a\}$. Let \mathcal{P} be the associative subalgebra of \mathcal{A} generated by $Q_{48}(\mathcal{J})$. By the Special Hermitian Structure Theorem [21, page 146], we have $H(\mathcal{P}, *) = Q_{48}(\mathcal{J})$ and \mathcal{P} is $*$ -prime.

Proposition 3.1.1. *Let \mathcal{J} be a division \mathbb{Z}^n -graded Jordan algebra over F of Hermitian Type. Then $\mathcal{J} = H(\mathcal{P}, *)$ for some $*$ -prime associative algebra \mathcal{P} and \mathcal{P} is generated by \mathcal{J} .*

Proof. By the above observation, we already know

$$H(\mathcal{P}, *) = Q_{48}(\mathcal{J}) \triangleleft \mathcal{J}$$

for some $*$ -prime associative algebra \mathcal{P} and \mathcal{P} is generated by $Q_{48}(\mathcal{J})$. Thus we only need to show $Q_{48}(\mathcal{J}) = \mathcal{J}$.

Let B be a basis of \mathcal{J} over F such that B consists of homogeneous elements in \mathcal{J} . Recall that Q_{48} contains q_{48} and all the linearizations of q_{48} . If the evaluations of q_{48} and all the linearizations of q_{48} on B vanish, then we have $q_{48}(\mathcal{J}) = \{0\}$, which contradicts the fact that \mathcal{J} is of Hermitian Type. Hence there exist elements $b_1, \dots, b_m \in B$ and $q'_{48} \in Q_{48}$ where $q'_{48} = q_{48}$ or some linearization of q_{48} such that $q'_{48}(b_1, \dots, b_m) \neq 0$. Since q'_{48} is homogeneous in each variable, $q'_{48}(b_1, \dots, b_m)$ is a nonzero homogeneous element in \mathcal{J} , and hence it is invertible in \mathcal{J} . Thus the ideal $Q_{48}(\mathcal{J})$ contains an invertible element, and so we get $Q_{48}(\mathcal{J}) = \mathcal{J}$. \square

Let J be a Jordan torus over F of Hermitian Type. Then, by 3.1.1, there exists a $*$ -prime associative algebra P such that

$$(H) \quad J = H(P, *) \quad \text{and } J \text{ generates } P.$$

We call a Jordan torus J satisfying (H) an *Hermitian torus*. Thus

$$\text{a Jordan torus of Hermitian type} \implies \text{a Hermitian torus.}$$

However, the converse is not true in general (see Remark 3.3.11(4)). We will classify Hermitian tori in this section. We will see that F_q^+ (Example 2.5.13) is one class of Hermitian tori. We next construct two other types of Jordan tori which will be Hermitian tori.

Construction 3.1.2. Let E be a field extension of F with $[E : F] \leq 2$. say $E := F(\sqrt{a})$ for some $a \in F$ if $[E : F] = 2$ and $E = F$ if $[E : F] = 1$. Let σ_E be the non-trivial Galois automorphism of E over F if $[E : F] = 2$. Let $\xi = (\xi_{ij})$ be a quantum matrix over E (see 2.5.8) satisfying

$$(3.1.3) \quad \sigma_E(\xi_{ij})\xi_{ij} = 1 \quad (\iff \sigma_E(\xi_{ij}) = \xi_{ji}) \quad \text{for all } i, j.$$

Note that $\xi_{ij} = 1$ or -1 if $E = F$. Let $\theta_i \in E$ such that

$$(3.1.3a) \quad \sigma_E(\theta_i)\theta_i = 1 \quad \text{for } i = 1, \dots, n.$$

Then for the quantum torus $E_\xi = E_\xi[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over E , there exists a unique involution τ of F_ξ over F if $E = F$ or a unique σ_E -semilinear involution τ of E_ξ over F otherwise, such that

$$(3.1.4) \quad \tau(t_i) = \theta_i t_i \quad \text{for all } i.$$

In fact, let us consider E_ξ as an associative torus with an $(\epsilon_1, \dots, \epsilon_n)$ -grading. Then we have $E_\xi = \bigoplus_{\alpha \in \mathbb{Z}^n} E t_\alpha$ where $t_\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_n \epsilon_n \in \mathbb{Z}^n$. For convenience, let σ_F be the identity map on F . Thus we can define an F -linear map τ by

$$\tau(xt_\alpha) = \sigma_E(x)\theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n} t_n^{\alpha_n} \cdots t_1^{\alpha_1} = \sigma_E(x)\theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n} \prod_{i < j} \xi_{ij}^{\alpha_i \alpha_j} t_\alpha \quad \text{for } x \in E.$$

Then, by (3.1.3) and (3.1.3a), we have

$$\tau^2(xt_\alpha) = \sigma_E^2(x)\sigma_E(\theta_1)^{\alpha_1} \cdots \sigma_E(\theta_n)^{\alpha_n} \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n} \prod_{i < j} \sigma_E(\xi_{ij})^{\alpha_i \alpha_j} \xi_{ij}^{\alpha_i \alpha_j} t_\alpha = xt_\alpha.$$

By the multiplication table (2.5.9) of a quantum torus, for $y \in E$ and $\beta = \beta_1 \epsilon_1 + \cdots + \beta_n \epsilon_n \in \mathbb{Z}^n$, we have

$$\begin{aligned} \tau(xt_\alpha y t_\beta) &= \tau(xy \prod_{i < j} \xi_{ij}^{\alpha_j \beta_i} t_{\alpha+\beta}) \\ &= \sigma_E(xy)\theta_1^{\alpha_1+\beta_1} \cdots \theta_n^{\alpha_n+\beta_n} \prod_{i < j} \sigma_E(\xi_{ij})^{\alpha_j \beta_i} \xi_{ij}^{(\alpha_j+\beta_j)(\alpha_i+\beta_i)} t_{\alpha+\beta} \\ &= \sigma_E(x)\sigma_E(y)\theta_1^{\alpha_1+\beta_1} \cdots \theta_n^{\alpha_n+\beta_n} \prod_{i < j} \xi_{ij}^{(\alpha_i \alpha_j + \beta_i \beta_j + \beta_j \alpha_i)} t_{\alpha+\beta} \\ &= \sigma_E(x)\sigma_E(y)\theta_1^{\alpha_1+\beta_1} \cdots \theta_n^{\alpha_n+\beta_n} \prod_{i < j} \xi_{ij}^{(\alpha_i \alpha_j + \beta_i \beta_j)} t_\beta t_\alpha \\ &= \tau(y t_\beta) \tau(x t_\alpha). \end{aligned}$$

Hence τ is an involution of E_q over F if $E = F$ or a unique σ_E -semilinear involution τ of E_ξ over F otherwise, and it is the unique one satisfying (3.1.4). This is mentioned in [1, page 83] when $E = F$.

Thus the symmetric elements $J := H(E_\xi, \tau)$ form a Jordan algebra over F . (In [2, page 16], such a J is used to construct extended affine Lie algebras of type C .) Since τ is graded on $E_\xi = \bigoplus_{\alpha \in \mathbb{Z}^n} Et_\alpha$, i.e., $\tau(Et_\alpha) = Et_\alpha$ for all $\alpha \in \mathbb{Z}^n$, we have $J = \bigoplus_{\alpha \in \mathbb{Z}^n} (Et_\alpha \cap J)$ as F -vector spaces. Since $E_q^+ = \bigoplus_{\alpha \in \mathbb{Z}^n} Et_\alpha$ is a Jordan torus over E , J is a \mathbb{Z}^n -graded Jordan algebra over F . We call this \mathbb{Z}^n -grading of J a *toral grading induced from a toral grading of E_ξ* . We always consider a toral grading of J without mentioning. If $E = F$, then $\dim_F(Ft_\alpha \cap J) \leq 1$. Otherwise $Et_\alpha = (F + F\sqrt{a})t_\alpha$ for some $a \in E$, and so $\dim_F Et_\alpha = 2$ for all $\alpha \in \mathbb{Z}^n$. If $xt_\alpha \in J$ for some $x \in E$, then $\sqrt{a}xt_\alpha \notin J$, whence $\dim_F(Et_\alpha \cap J) \leq 1$. In general, the inverse of a symmetric element is also symmetric. Thus if $\text{supp}(J)$ generates \mathbb{Z}^n , then J is a Jordan n -torus over F . This is true if $E \neq F$. In fact, assume that $\dim_F(Et_\alpha \cap J) = 0$ for some $\alpha \in \mathbb{Z}^n$. Then for all $x \in E$, we have $xt_\alpha + \tau(xt_\alpha) = 0$, and so $\tau(xt_\alpha) = -xt_\alpha$ and therefore $\tau(\sqrt{a}xt_\alpha) = \sqrt{a}xt_\alpha$. Hence for $x \neq 0$, we have $0 \neq \sqrt{a}xt_\alpha \in Et_\alpha \cap J$ which is a contradiction. Therefore, we always have $\text{supp}(J) = \mathbb{Z}^n$. In addition, $H(E_\xi, \tau)$ is a Jordan torus over F of strong type if and only if the entries of ξ satisfies (2.5.15):

$$\prod_{i,j} \xi_{ij}^{\alpha, \beta_j} \neq -1 \quad \text{for all } \alpha, \beta \in \mathbb{Z}^n.$$

If $E = F$, then $H(F_\xi, \tau)$ is not a Jordan n -torus in general (see 3.2.14).

From now on, we denote $H(E_\xi, \tau)$ when $\theta_i = 1$ for all i and $E = F$ by

$$H(F_\epsilon, *),$$

i.e., ϵ is a quantum matrix whose entries are 1 or -1 , and $*$ is the unique involution of F_ϵ satisfying $t_i^* = t_i$ for all i . We call such a quantum matrix ϵ an *elementary quantum matrix*. Since $t_1, \dots, t_n \in J$, $\text{supp}(H(F_\epsilon, *))$ generates \mathbb{Z}^n . Thus $J := H(F_\epsilon, *) = \bigoplus_{\alpha \in \mathbb{Z}^n} (Ft_\alpha \cap J)$ is a Jordan n -torus over F . If $q \neq 1$, then $(t_i t_j)^* = -t_i t_j$ for some i, j , and so we have $Ft_i t_j \cap J = \{0\}$. Hence $\epsilon_i + \epsilon_j \notin \text{supp}(J)$. Therefore, $\text{supp}(J) = \mathbb{Z}^n$ if and only if $\epsilon = 1$, i.e., $J = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In particular, J is never of strong type unless $\epsilon = 1$. Also, J is generated by t_i and tetrads $\{t_{i_1} t_{i_2} t_{i_3} t_{i_4}\}$ as

a Jordan algebra (see e.g. [13, page 8, Cohn's Theorem]). In particular, J is finitely generated. We call the basis $\{t_\alpha \mid t_\alpha^* = t_\alpha\}_{\alpha \in \mathbb{Z}^n}$ above the *standard basis* of J . Clearly, the structure constants relative to the standard basis are 0 or ± 1 .

Remark 3.1.5. In general, let A be an associative algebra generated by $\{x_i\}_{i \in I}$ with $x_j x_i = \pm x_i x_j$. Then $\{x_i^2\}_{i \in I}$ is contained in the centre $Z = Z(A)$. Indeed, we check $[x_i^2, x_j] = 0$ for all x_j . If $x_j x_i = x_i x_j$, we have nothing to check. If $x_j x_i = -x_i x_j$, then we have $x_j x_i^2 = x_i x_i x_i = -x_i x_j x_i = x_i^2 x_j$. Hence $x_i^2 \in Z$. In particular, the subalgebra of A generated by $\{x_i^2\}_{i \in I}$ is commutative.

Therefore, the centre $Z = Z(J)$ of $J = H(F_\epsilon, *)$ contains $F[t_1^{\pm 2}, \dots, t_n^{\pm 2}]$. So for any homogeneous element in J , its square is in Z . If $\epsilon = 1$, then the central grading group Γ is \mathbb{Z}^n . Otherwise, since \mathbb{Z}^n/Γ has exponent 2, there exists a basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n such that

$$\Gamma = 2\mathbb{Z}\sigma_1 + \dots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \dots + \mathbb{Z}\sigma_n \quad \text{for some } 1 < m \leq n.$$

When $\theta_i = 1$ for all i and $E \neq F$, we denote $H(E_\xi, \tau)$ by

$$H(E_\xi, \sigma),$$

i.e., σ is a unique σ_E -semilinear involution τ of E_ξ over F satisfying $\sigma(t_i) = t_i$ for all i . We claim that

$$(3.1.5a) \quad H(E_\xi, \tau) \cong_{\mathbb{Z}^n} H(E_\xi, \sigma).$$

Indeed, let $\tau(t_i) = \theta_i t_i$ and $E = F(\sqrt{a})$. We put

$$t'_i := \begin{cases} \sqrt{a} t_i & \text{if } \theta_i = -1 \\ (1 + \theta_i) t_i & \text{otherwise.} \end{cases}$$

Then we have $\tau(t'_i) = t'_i$. Since $E_\xi[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \cong_{\mathbb{Z}^n} E_\xi[t_1'^{\pm 1}, \dots, t_n'^{\pm 1}]$ over E (or we can identify the two graded algebras), we get $H(E_\xi, \tau) \cong_{\mathbb{Z}^n} H(E_\xi, \sigma)$ over F (or $H(E_\xi, \tau) = H(E_\xi, \sigma)$).

We note that for $x \in E$ and $t \in J := H(E_\xi, \sigma)$,

$$\varphi : J_E = E \otimes_F J \ni x \otimes t \mapsto xt \in E_\xi^+$$

gives a \mathbb{Z}^n -graded isomorphism over E . In fact, if t has degree α , then xt has degree α , and so φ is \mathbb{Z}^n -graded. For $x, y \in E$ and $t, u \in J$, $(xt) \cdot (yu) = (xy)(t \cdot u)$ implies that φ is a homomorphism. For $\tilde{u} := \sum_{\alpha \in \mathbb{Z}^n} x_\alpha \otimes u_\alpha \in J_E$, $\varphi(\tilde{u}) = \sum_{\alpha \in \mathbb{Z}^n} x_\alpha u_\alpha = 0$ implies that $\tilde{u} = 0$. Hence φ is injective. The surjectivity follows from $\text{supp}(J) = \mathbb{Z}^n$. Thus we identify

$$J_E = E_\xi^+.$$

Finally, we claim that

$$Z(H(E_\xi, \sigma)) = H(Z(E_\xi), \sigma).$$

Indeed, the inclusion \supset follows from $Z(E_\xi^+) = Z(E_\xi)$ (see 1.1.10). For the other inclusion, let $t \in Z(H(E_\xi, \sigma))$, and let u_α and u_β be any nonzero homogeneous elements of degree α and β in E_ξ . Then there exist some $0 \neq x, y \in E$ such that $xu_\alpha, yu_\beta \in H(E_\xi, \sigma)$. Hence $(t, xu_\alpha, yu_\beta) = 0$, and so we get $(t, u_\alpha, u_\beta) = 0$. Therefore, $t \in Z(E_\xi^+) \cap H(E_\xi, \sigma) = Z(E_\xi) \cap H(E_\xi, \sigma) = H(Z(E_\xi), \sigma)$. By the claim, since $\text{supp}(H(E_\xi, \sigma)) = \mathbb{Z}^n$ and $Z(E_\xi^+) = Z(E_\xi)$, the central grading groups of $H(E_\xi, \sigma)$ and E_ξ coincide.

Example 3.1.6. Recall the quantum matrix ζ_r ($r > 1$) from 2.6.4:

$$\zeta = \zeta_r = \begin{pmatrix} 1 & \zeta_r & 1 & \cdots & 1 \\ \zeta_r^{-1} & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

where ζ_r is a primitive r -th root of unity. Let $\zeta_r \in \mathbb{C}$ and $E^{(r)} := \mathbb{R}(\zeta_r)$. Note that $E^{(r)} = \mathbb{C}$ if $r > 2$ and $E^{(2)} = \mathbb{R}$ otherwise. Consider the quantum torus $E_\zeta^{(r)} = E_\zeta^{(r)}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ determined by $\zeta = \zeta_r$. Take the complex conjugate $\bar{\cdot}$ over \mathbb{R} for $\sigma_{\mathbb{C}}$. Then ζ satisfies (3.1.3). Hence $H(E_\zeta^{(r)}, \sigma)$ is a Jordan torus. Thus $H(E_\zeta^{(r)}, \sigma)$ is an example of $H(E_\xi, \sigma)$ if $r > 2$, and of $H(F_\epsilon, *)$ if $r = 2$, i.e., $H(E_\zeta^{(2)}, \sigma) = H(\mathbb{R}_h, *)$ where $h = \zeta_2$. By 2.5.15, we note that $H(E_\zeta^{(r)}, \sigma)$ is of strong type if and only if r is odd.

The following series of lemmas serve as preparation for the classification of Hermitian tori. Also, if a \mathbb{Z}^n -graded algebra A is graded isomorphic to $B := F_q, F_q^+, H(F_\epsilon, *)$ or $H(E_\xi, \sigma)$ for some toral grading, we simply denote by $A \cong_{\mathbb{Z}^n} B$.

Lemma 3.1.7. *Let $\mathcal{J} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{J}_\alpha$ be a division \mathbb{Z}^n -graded Jordan algebra over F satisfying $\mathcal{J} = H(\mathcal{P}, *)$ for an associative algebra \mathcal{P} with involution $*$ such that \mathcal{P} is generated by \mathcal{J} . Suppose that there exists $v \in \mathcal{P}$ such that $vv^* = 0$ and $v + v^*$ is invertible in \mathcal{J} . Then there exists a division \mathbb{Z}^n -graded associative algebra \mathcal{B} such that $\mathcal{J} \cong_{\mathbb{Z}^n} \mathcal{B}^+$.*

In particular, if \mathcal{J} is a torus, then $\mathcal{J} \cong_{\mathbb{Z}^n} F_q^+$ for some quantum torus F_q .

Remark. This lemma is inspired by an analogous result in the classification of Jordan division algebras (see [14, page 8.24]).

Proof. For any $y \in \mathcal{J} = H(\mathcal{P}, *)$ we claim that $v^*yv = v^*v = yvy^* = 0$. Clearly we have $v^*yv \in H(\mathcal{P}, *) = \mathcal{J}$. Also,

$$U_{v^*yv}1 = (v^*yv)^2 = (v^*yv) \cdot (v^*yv) = v^*yvv^*yv = 0$$

since $vv^* = 0$. Hence $v^*yv = 0$ since \mathcal{J} is a Jordan domain by 2.2.4. In particular, $v^*v = 0$ for $y = 1$. By the same argument, we get $yvy^* = 0$, and so our claim is settled.

Now, since $v + v^*$ is invertible in \mathcal{J} , there exists $z \in \mathcal{J}$ such that

$$(v + v^*)z(v + v^*) = 1$$

(see 1.1.2(3)). By the claim, we have $e + e^* = 1$ where $e := vzv$ and $ee^* = e^*e = 0$. Also, we have $e = e(e + e^*) = e^2$ and $e^* = e^*(e + e^*) = e^{*2}$. Thus e and e^* are supplementary orthogonal idempotents in \mathcal{P} . By the claim, we have $e^*Je = eJe^* = \{0\}$. Since \mathcal{J} generates \mathcal{P} , we get $e^*\mathcal{P}e = e\mathcal{P}e^* = \{0\}$. Therefore, for $\mathcal{B} := e\mathcal{P}e$, we have $\mathcal{P} = \mathcal{B} \oplus \mathcal{B}^*$. Since $\mathcal{J} = H(\mathcal{P}, *)$, we obtain an isomorphism of Jordan algebras

$$f : \mathcal{J} = \{b + b^* \mid b \in \mathcal{B}\} \xrightarrow{\sim} \mathcal{B}^+.$$

Let $\mathcal{B}_\alpha := f(\mathcal{J}_\alpha)$ for all $\alpha \in \mathbb{Z}^n$. Then $\mathcal{B}^+ = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{B}_\alpha$ is a division \mathbb{Z}^n -graded Jordan algebra and $\mathcal{J} \cong_{\mathbb{Z}^n} \mathcal{B}^+$. We show that the associative algebra $\mathcal{B} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{B}_\alpha$ is a division \mathbb{Z}^n -graded algebra. In fact, we have $\mathbb{Z}^n = \langle \text{supp}(\mathcal{B}^+) \rangle = \langle \text{supp}(\mathcal{B}) \rangle$. Moreover, all nonzero elements of \mathcal{B}_α are invertible in \mathcal{B}^+ , and so are they in \mathcal{B} by 1.1.2(2). Hence we only need to show that $\mathcal{B}_\alpha \mathcal{B}_\beta \subset \mathcal{B}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$. Note that we have $\mathcal{B}_\alpha \cdot \mathcal{B}_\beta = \mathcal{B}_\alpha \circ \mathcal{B}_\beta \subset \mathcal{B}_{\alpha+\beta}$. If α or β is not contained in the support of \mathcal{B} , i.e., $\mathcal{B}_\alpha = (0)$ or $\mathcal{B}_\beta = (0)$, we have nothing to prove. Otherwise, for $0 \neq x \in \mathcal{B}_\alpha$ and

$0 \neq y \in \mathcal{B}_\beta$, xy and yx are invertible in \mathcal{B} and so are they in \mathcal{B}^+ by 1.1.2(2). Hence we have $xy \in \mathcal{B}_\gamma$ and $yx \in \mathcal{B}_\delta$ for some $\gamma, \delta \in \mathbb{Z}^n$ by 2.2.7. If $x \circ y = xy + yx \neq 0$, then $0 \neq x \circ y \in \mathcal{B}_{\alpha+\beta} \cap (\mathcal{B}_\gamma + \mathcal{B}_\delta)$, which forces $\alpha + \beta = \gamma = \delta$. So we get $xy \in \mathcal{B}_{\alpha+\beta}$. If $x \circ y = xy + yx = 0$, i.e., $yx = -xy$, then we have $[x^2, y^2] = 0$, and so $x^2 \cdot y^2 = x^2 y^2$. Thus we get

$$0 \neq (xy)^2 = (xy) \cdot (xy) = xyxy = -x^2 y^2 = -x^2 \cdot y^2 \in \mathcal{B}_{2\gamma} \cap \mathcal{B}_{2\alpha+2\beta}.$$

Hence $\gamma = \alpha + \beta$ and we obtain $xy \in \mathcal{B}_{\alpha+\beta}$. Therefore, $\mathcal{B} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{B}_\alpha$ is a division \mathbb{Z}^n -graded associative algebra.

In particular, if such a \mathcal{J} happens to be a torus, then $\mathcal{B} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{B}_\alpha$ is an associative torus since $\dim_F \mathcal{B}_\alpha = \dim_F \mathcal{J}_\alpha \leq 1$. Hence $\mathcal{B} \cong_{\mathbb{Z}^n} F_q$ for some quantum torus F_q by 2.5.11. Identifying them, we get $\mathcal{J} \cong_{\mathbb{Z}^n} F_q^+$. \square

Lemma 3.1.8. *Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be a Jordan torus over F satisfying $J = H(P, *)$ for an associative algebra P with involution $*$, which is generated by J (e.g. a Hermitian torus). Suppose that there exist an invertible element $u \in P$ so that $u^* = -u$ and $0 \neq y \in J_\gamma$ for some $\gamma \in \mathbb{Z}^n$ such that the following three conditions hold:*

$$(i) \quad u^2 \in J_{2\gamma}, \quad (ii) \quad uy^{-1}u \in J_\gamma, \quad (iii) \quad [u, y] \in J_{2\gamma}.$$

Then $J \cong_{\mathbb{Z}^n} F_q^+$ or $J_E := E \otimes_F J \cong_{\mathbb{Z}^n} E_q^+$ for some quantum torus F_q or E_q where E is a quadratic field extension of F .

Proof. By $\dim_F J_\alpha \leq 1$ for all $\alpha \in \mathbb{Z}^n$, there exist $a, b, c \in F$, $a, b \neq 0$, such that

$$\begin{aligned} (1) \quad & u^2 = ay^2, \\ (2) \quad & buy^{-1}u = y \quad (\iff \quad yu^{-1} = buy^{-1}), \\ (3) \quad & [u, y] = uy - yu = cu^2. \end{aligned}$$

By (3), we have $uyu^{-1} - y = cu$. By (2), we have $bu^2y^{-1} - y = cu$. By (1), we have $aby^2y^{-1} - y = cu$. Hence we get $cu = aby - y \in J$. Since $u \notin J$, we obtain $c = 0$ and hence

$$(4) \quad uy = yu.$$

Let $v := u + \sqrt{a} y$ if $\sqrt{a} \in F$. Otherwise, let $E := F(\sqrt{a})$, $J_E := E \otimes_F J$, $P_E := E \otimes_F P$, $*$:= $\text{id} \otimes *$ and $v := 1 \otimes u + \sqrt{a} \otimes y = u + \sqrt{a} y$. Then since $u^* = -u$, we have

$$vv^* = \begin{cases} (u + \sqrt{a} y)(-u + \sqrt{a} y) = -u^2 + ay^2 = 0 \\ (1 \otimes u + \sqrt{a} \otimes y)(-1 \otimes u + \sqrt{a} \otimes y) = 1 \otimes (-u^2 + ay^2) = 0. \end{cases}$$

by (4) and (1). Also, $v + v^* = 2\sqrt{a} y$ or $2\sqrt{a} \otimes y$ is invertible in J or in J_E . Thus, by 3.1.7, we get $J \cong_{\mathbb{Z}^n} F_q^+$ if $\sqrt{a} \in F$. If $\sqrt{a} \notin F$, then we can apply 3.1.7 for the Jordan torus $J_E = H(P_E, *)$ over E , and obtain $J_E \cong_{\mathbb{Z}^n} E_q^+$. \square

Lemma 3.1.9. *Let $E \supset F$ be fields with $[E : F] = 2$, and σ_E is the nontrivial Galois automorphism of E over F .*

(1) *Suppose that τ is a σ_E -semilinear involution of $E_q^+ = E_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^+$ over F such that $\sigma(t_i) = t_i$ for all i . Then:*

(i) $q \in M_n(F)$ and τ is an automorphism of E_q over F or

(ii) $q = \xi$ and τ is an involution of E_ξ over F , where ξ is defined by 3.1.3. In particular, $\tau = \sigma$ where σ is defined in 3.1.2.

(2) *Let J be a Jordan torus over F . Suppose that $J_E = E \otimes_F E \cong_{\mathbb{Z}^n} E_q^+$. Then:*

(a) $q \in M_n(F)$ and $J \cong_{\mathbb{Z}^n} F_q^+$ or

(b) $J \cong_{\mathbb{Z}^n} H(E_\xi, \sigma)$, which is a Jordan torus defined in 3.1.2.

Also, if $\xi \notin M_n(F)$, then on $E \otimes_F H(E_\xi, \sigma) = E_\xi^+$, we have $\sigma_E \otimes \text{id} = \sigma$.

Proof. We use a theorem in [14, Theorem 1.1.7, page 1.4], namely, if A and B are associative algebras. B has no zero-divisors and $f : A^+ \rightarrow B^+$ is a homomorphism of Jordan algebras, then $f : A \rightarrow B$ is either a homomorphism or anti-homomorphism of associative algebras.

(1): Since E_q has no zero-divisors, τ is an order 2 automorphism or an involution of the associative F -algebra E_q . If τ is an automorphism, then $H(E_q, \tau)$, the set of fixed points by τ , is an associative algebra and $J = H(E_q, \tau)^+$. Moreover,

$$t_j t_i - q_{ij} t_i t_j = 0 = \tau(t_j t_i - q_{ij} t_i t_j) = t_j t_i - \sigma_E(q_{ij}) t_i t_j,$$

which forces $\sigma_E(q_{ij}) = q_{ij}$, and so $q_{ij} \in F$ for all i, j , i.e., $q \in M_n(F)$.

If τ is an involution, then

$$0 = \tau(t_j t_i - q_{ij} t_i t_j) = t_i t_j - \sigma_E(q_{ij}) t_j t_i = t_i t_j - \sigma_E(q_{ij}) q_{ij} t_i t_j.$$

Hence we get $\sigma_E(q_{ij})q_{ij} = 1$ for all i, j and obtain $\mathfrak{q} = \xi$. In particular, $\tau = \sigma$ since σ is a unique σ_E -semilinear involution of E_ξ such that $\tau(t_i) = t_i$ for all i .

(2): Let $\tau := \sigma_E \otimes \text{id}$, which is a σ_E -semilinear involution of $J_E = E \otimes_F J$. Identifying J with $1 \otimes J$, we have $J = H(J_E, \tau)$. Also, we identify J_E with E_q^+ , and so $J = H(E_q^+, \tau)$. Since $\text{supp } J = \text{supp } J_E = \text{supp } E_q^+ = \Lambda$, one can choose $t_1, \dots, t_n \in E_q$ such that $E_q = E_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $\tau(t_i) = t_i$ for all i . Thus one can apply (1). For the case (i), one gets $H(E_q, \tau) = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, and so $J = F_q^+$. For the case (ii), one obtains $J = H(E_\xi, \sigma)$. The last statement follows from (1). \square

Finally we can prove the main result of this section. The reader is reminded that any Jordan division algebra of Hermitian Type is isomorphic to A^+ or $H(B, *)$ for some associative division algebra A or some associative division algebra B with involution $*$. We can prove an analogous result for Hermitian tori.

Theorem 3.1.10. *Let J be a Hermitian torus over F . Then, with the notation from 3.1.2,*

$$J \cong_{\mathbb{Z}^n} F_q^+, \quad H(F_\epsilon, *) \quad \text{or} \quad H(E_\xi, \sigma).$$

Proof. Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be a Hermitian torus over F , i.e., $J = H(P, *)$ for some $*$ -prime associative algebra P which is generated by J . Let $(\sigma_1, \dots, \sigma_n)$ be a basis of \mathbb{Z}^n such that each $\sigma_i \in S := \text{supp}(J)$ and let $0 \neq x_i \in J_{\sigma_i}$, $i = 1, \dots, n$.

We first consider the case where the following two conditions hold:

(A) for all $1 \leq i, j \leq n$, $[x_i, x_j] = 0$ or $x_i \circ x_j = 0$, i.e.,

(†) $x_j x_i = \pm x_i x_j$ for all $1 \leq i, j \leq n$,

(B) J is generated by r -tads $\{x_{i_1}^{\epsilon_1} \cdots x_{i_r}^{\epsilon_r}\}$ where $r > 0$, $\epsilon_k = \pm 1$ and $i_1, \dots, i_r \in \{1, \dots, n\}$.

We will show $J \cong_{\mathbb{Z}^n} H(F_\epsilon, *)$ in this case. Since J generates P and every r -tad is generated by 1-tads $2x_1^{\pm 1}, \dots, 2x_n^{\pm 1}$ as an associative algebra, P is generated by $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. Thus, by (†), there exist an elementary quantum matrix ϵ and an epimorphism φ from $F_\epsilon = F_\epsilon[t_1^\pm, \dots, t_n^\pm]$ onto P such that $\varphi(t_i) = x_i$ for $i = 1, \dots, n$. We give F_ϵ the $(\sigma_1, \dots, \sigma_n)$ -grading.

Now, we show injectivity of φ . Suppose that $\varphi(t) = 0$ for $t = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha t_\alpha$ where $a_\alpha \in F$, $t_\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ and $\alpha = \alpha_1 \sigma_1 + \cdots + \alpha_n \sigma_n \in \mathbb{Z}^n$. So we have

$$(1) \quad \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x_\alpha = 0.$$

where $x_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Note that all x_α are invertible in P since x_i is invertible in J and in P by 1.1.2(2). We need to show that all $a_\alpha = 0$. Put

$$M := \{\alpha \in \mathbb{Z}^n \mid x_\alpha \in J\} \quad \text{and} \quad N := \{\beta \in \mathbb{Z}^n \mid x_\beta \notin J\}.$$

Then we have $\sum_{\alpha \in M} a_\alpha x_\alpha = 0$ and $\sum_{\beta \in N} a_\beta x_\beta = 0$ since $x_\alpha \in J$ are symmetric and $x_\beta \notin J$ are skew relative to $*$ by (†) (note $\text{ch.}F \neq 2$).

Claim 1. *Assuming only (A), we have $x_\alpha \in J \implies x_\alpha \in J_\alpha$.*

Proof. By (†) and Remark 3.1.5, the subalgebra of A generated by $\{x_i^2\}_{i=1}^n$ is commutative, and so the Jordan product and the associative product coincide in the subalgebra. Therefore,

$$x_\alpha^2 = \pm x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} = \pm (\cdots (x_1^{2\alpha_1} \cdot x_2^{2\alpha_2}) \cdots x_n^{2\alpha_n}) \in J_{2\alpha_1 \sigma_1 + \cdots + 2\alpha_n \sigma_n} = J_{2\alpha}.$$

Hence we get $x_\alpha \in J_\alpha$ by 2.2.8.

By Claim 1, we obtain $a_\alpha = 0$ for all $\alpha \in M$. If $N = \emptyset$, we are done. Otherwise we pick any $\beta_0 \in N$. Multiply (1) by x_{β_0} . Then, by (†), we have

$$\sum_{\beta \in N} \pm a_\beta x_{\beta + \beta_0} = 0.$$

Applying the same argument for this equation instead of (1), we have

$$\sum_{\beta \in M_1} \pm a_\beta x_{\beta + \beta_0} = 0 \quad \text{and} \quad \sum_{\gamma \in N_1} \pm a_\gamma x_{\gamma + \beta_0} = 0,$$

where $M_1 := \{\beta \in N \mid x_{\beta + \beta_0} \in J\}$ and $N_1 := \{\gamma \in N \mid x_{\gamma + \beta_0} \notin J\}$. By Claim 1, we get $a_\beta x_{\beta + \beta_0} \in J_{\beta + \beta_0}$, and hence $a_\beta = 0$ for all $\beta \in M_1$. Since $N = M_1 \sqcup N_1$ (disjoint union) and $\beta_0 \in M_1$ (because $x_{2\beta_0} \in J$), we have $N_1 \subsetneq N$. If $N_1 = \emptyset$, we are done. Otherwise, repeating this method for the finite set N_1 , we get some $r > 1$ such that $N_r = \emptyset$ and $a_\beta = 0$ for all $\beta \in M_2 \sqcup \cdots \sqcup M_r = N_1$. Consequently, we obtain $a_\alpha = 0$ for all $\alpha \in \mathbb{Z}^n$. Thus $t = 0$ and φ is injective.

We have shown that φ is an isomorphism. Further, P is graded with $P_{\sigma_i} = J_{\sigma_i}$, and φ is a graded isomorphism. Also, through this isomorphism, we get an involution $*$ of F_ϵ such that $t_i^* = t_i$ for $i = 1, \dots, n$. Therefore, we obtain $J \cong_{\mathbb{Z}^n} H(F_\epsilon, *)$.

We consider the second case: the negation of (A), i.e.,

there exist some i, j such that $u := [x_i, x_j] \neq 0$ and $x_i \circ x_j \neq 0$.

We divide the case into two subcases:

$$(I) \quad u^2 = 0 \quad \text{and} \quad (II) \quad u^2 \neq 0.$$

(I) $u^2 = 0$: Note that $u^* = -u$, and so we have $uu^* = -u^2 = 0$. We need the following claim which can be proven in the same way as in the classification of Jordan division algebras (see [14, page 8.25]).

Claim 2. *There exists $y \in J$ such that for $v = yu$, $v + v^* \neq 0$.*

Proof. Otherwise, for all $y \in J$, we have $v + v^* = 0$ for $v = yu$, i.e., $yu = -u^*y^* = uy$. So for all $w \in P$, we have $(uy)(uw) = u^2yw = 0$, and hence we have $(uJ)(uP) = \{0\}$. Since P is generated by J , we get $(uP)^2 = \{0\}$. Then we have $(PuP)^2 = PuPPuP = PuPuP = P(uP)^2 = \{0\}$. Moreover, $(PuP)^* = Pu^*P = PuP$, and so PuP is a nonzero $*$ -ideal. This contradicts the fact that P is $*$ -prime ($*$ -semiprime is enough!).

Let $y \in J$ be such an element as in Claim 2 so that

$$v + v^* = yu - uy = [y, u] \neq 0.$$

Decompose y into nonzero distinct homogeneous elements, namely, $y = \sum_{\alpha \in \mathbb{Z}^n} y_\alpha$. Let $v_\alpha := y_\alpha u$ for all y_α . Suppose that $v_\alpha + v_\alpha^* = 0$ for all v_α . Then we have $[y_\alpha, u] = y_\alpha u - uy_\alpha = v_\alpha + v_\alpha^* = 0$ for all y_α . Therefore,

$$v + v^* = [y, u] = \left[\sum_{\alpha \in \mathbb{Z}^n} y_\alpha, u \right] = \sum_{\alpha \in \mathbb{Z}^n} [y_\alpha, u] = 0.$$

which contradicts our choice of y . Hence there exists some v_α such that $v_\alpha + v_\alpha^* \neq 0$. Now, recall the identity $[x, [y, z]] = (y, x, z)^\circ$ (see 1.1.10a), and so

$$0 \neq v_\alpha + v_\alpha^* = [y_\alpha, u] = [y_\alpha, [x_i, x_j]] = (x_i, y_\alpha, x_j)^\circ \in J_{\sigma_i + \alpha + \sigma_j}.$$

Hence $v_\alpha + v_\alpha^*$ is invertible in J . Also, we have $v_\alpha v_\alpha^* = y_\alpha u u^* y_\alpha^* = -y_\alpha u^2 y_\alpha = 0$. Therefore, by 3.1.7, we get $J \cong_{\mathbb{Z}^n} F_q^+$ for some q .

(II) $u^2 \neq 0$: Let $0 \neq y := x_i \circ x_j \in J_\gamma$ where $\gamma := \sigma_i + \sigma_j$. We show that these u and y satisfy the three conditions in 3.1.8. (Note that $u^* = -u$.) First we have by (1.1.11),

$$0 \neq u^2 = [x_i, x_j]^2 = x_i \circ U_{x_j} x_i - U_{x_i} x_j^2 - U_{x_j} x_i^2 \in J_{2\gamma}.$$

Hence u^2 is invertible in J and hence in P by 1.1.2(2). Thus, u is invertible in P and $u^2 \in J_{2\gamma}$. Secondly, by a well-known identity

$$U_{[x_i, x_j]} = U_{x_i \circ x_j} - 2U_{x_i} U_{x_j} - 2U_{x_j} U_{x_i},$$

(which can be easily verified by expanding both sides), we have

$$u y^{-1} u = U_u y^{-1} = U_{[x_i, x_j]} y^{-1} = (U_{x_i \circ x_j} - 2U_{x_i} U_{x_j} - 2U_{x_j} U_{x_i}) y^{-1} \in J_\gamma$$

since $y^{-1} \in J_{-\gamma}$. Thirdly, we have by $[x, [y, z]] = (y, x, z)^\circ$ (see 1.1.10a),

$$[u, y] = [[x_i, x_j], x_i \circ x_j] = -(x_i, x_i \circ x_j, x_j)^\circ \in J_{2\gamma}.$$

Thus u and y satisfy the conditions in 3.1.8, and we get $J \cong_{\mathbb{Z}^n} F_q^+$ or $J_E \cong_{\mathbb{Z}^n} E_q^+$. Therefore, by 3.1.9(2), $J \cong_{\mathbb{Z}^n} F_q^+$ or $H(E_\xi, \sigma)$.

We consider the final case: (A) with the negation of (B), i.e.,

assuming the relation (\dagger), J is not generated by r -tads $\{x_{i_1}^{\epsilon_1} \cdots x_{i_r}^{\epsilon_r}\}$.

By our assumption, there exist $\gamma \in \mathbb{Z}^n$ and $0 \neq y \in J_\gamma$ such that y is not generated by r -tads $\{x_{i_1}^{\epsilon_1} \cdots x_{i_r}^{\epsilon_r}\}$. Let $\gamma = \gamma_1 \sigma_1 + \cdots + \gamma_n \sigma_n$.

Claim 3. $u := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \notin J$.

Proof. Otherwise we have $2u = \{x_1^{\gamma_1} \cdots x_n^{\gamma_n}\}$, and by Claim 1, $u \in J_\gamma$. So we get $y = au$ for some $a \in F$ by $\dim_F J_\gamma \leq 1$, i.e., $y = \frac{1}{2}a\{x_1^{\gamma_1} \cdots x_n^{\gamma_n}\}$, which contradicts our setting of y .

Now, we show that these u and $y \in J_\gamma$ satisfy the conditions in 3.1.8. Observe first that $u^* = \pm u$ by (A) and hence $u^* = -u$ by Claim 3. Next, by definition, u is

clearly invertible, and by (†), we have $u^2 = \pm x_1^{2\gamma_1} \cdots x_n^{2\gamma_n} \in J$. Hence $u^2 \in J_{2\gamma}$ by Claim 1. Secondly we have

$$\begin{aligned} U_u y^{-1} &= u y^{-1} u = x_1^{\gamma_1} \cdots x_n^{\gamma_n} y^{-1} x_1^{\gamma_1} \cdots x_n^{\gamma_n} = \pm x_1^{\gamma_1} \cdots x_n^{\gamma_n} y^{-1} x_n^{\gamma_n} \cdots x_1^{\gamma_1} \\ &= \pm U_{x_1^{\gamma_1}} \cdots U_{x_n^{\gamma_n}} y^{-1} \in J_\gamma. \end{aligned}$$

In particular, we get a formula for u :

$$(2) \quad U_u = \pm U_{x_1^{\gamma_1}} \cdots U_{x_n^{\gamma_n}}.$$

Thirdly, since $u^* = -u$ and $[u, y]^* = [y^*, u^*] = -[y, u] = [u, y]$, we have $[u, y] \in J$. Also, we have by (1.1.11) and (2),

$$[u, y]^2 = y \circ U_u y - U_y u^2 - U_u y^2 \in J_{4\gamma}.$$

Hence we get $[u, y] \in J_{2\gamma}$ by 2.2.8. Thus, by 3.1.8, we get $J \cong_{\mathbf{Z}^n} F_q^+$ or $J_E \cong_{\mathbf{Z}^n} E_q^+$. Therefore, by 3.1.9(2), $J \cong_{\mathbf{Z}^n} F_q^+$ or $H(E_\xi, \sigma)$. \square

§ 3.2 ELEMENTARY QUANTUM TORI WITH GRADED INVOLUTION

For an elementary quantum matrix $\varepsilon = (\varepsilon_{ij})$ defined in 3.1.2, i.e., $\varepsilon_{ij} = \pm 1$, we call the quantum torus F_ε determined by ε an *elementary torus*. Note that ε is a symmetric matrix.

For convenience, we use the following notation:

Definition 3.2.1. For square matrices A_1, \dots, A_r of sizes $l_i, i = 1, \dots, r$, we define the square matrix $A_1 \times \cdots \times A_r$ of size $l_1 + \cdots + l_r$ to be

$$A_1 \times \cdots \times A_r = \begin{pmatrix} A_1 & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & A_2 & \mathbf{1} & & \vdots \\ \mathbf{1} & \mathbf{1} & A_3 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{1} \\ \mathbf{1} & \cdots & \cdots & \mathbf{1} & A_r \end{pmatrix}.$$

where $\mathbf{1}$'s are matrices of suitable sizes whose entries are all 1.

Also, we write $\mathbf{1} = \mathbf{1}_k$ if $\mathbf{1}$ is a square matrix of size k .

Recall from §2.5 that for quantum matrices q and η , we defined $q \cong \eta \iff F_q \cong F_\eta$.

Lemma 3.2.2. (1) Let $q = (q_{ij})$ be an $n \times n$ quantum matrix, σ a permutation on $\{1, \dots, n\}$, and put $\bar{q}_\sigma = (\bar{q}_{ij})$ where $\bar{q}_{ij} = q_{\sigma(i)\sigma(j)}$. Then $q \cong \bar{q}_\sigma$. In particular, for a transposition $(ij) \in S$, we have $q \cong \bar{q}_{(ij)}$.

(2) Let r , s and η be quantum matrices with $s \cong \eta$. Then:

$$(i) \quad r \times s \cong s \times r, \quad (ii) \quad r \times s \cong r \times \eta.$$

Proof. For (1), let $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, and so we have $t_j t_i = q_{ij} t_i t_j$. Hence the generators $\bar{t}_i := t_{\sigma(i)}$ satisfy $\bar{t}_j \bar{t}_i = t_{\sigma(j)} t_{\sigma(i)} = q_{\sigma(i)\sigma(j)} t_{\sigma(i)} t_{\sigma(j)} = q_{\sigma(i)\sigma(j)} \bar{t}_i \bar{t}_j$, and

$$F_q = F_{\bar{q}_\sigma}[\bar{t}_1^{\pm 1}, \dots, \bar{t}_n^{\pm 1}].$$

Thus we get $q \cong \bar{q}_\sigma$.

For (2), let r and s be the sizes of the matrices r and s , respectively, and let $n := r + s$ and $F_{r \times s} = F_{r \times s}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

(i) follows from (1): Take

$$\sigma = \begin{pmatrix} 1 & \cdots & s & s+1 & \cdots & n \\ r+1 & \cdots & n & 1 & \cdots & r \end{pmatrix}.$$

Then $s \times r = (\widetilde{r \times s})_\sigma$.

For (ii), we consider a toral grading $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ of $F_{r \times s}$ (see 2.5.10). Let $r \times \eta = (a_{ij})$. The subalgebra of $F_{r \times s}$ generated by $t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}$ can be identified with the $(n-r)$ -torus $F_s[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ with the $\langle \varepsilon_{r+1}, \dots, \varepsilon_n \rangle$ -grading. Our assumption $s \cong \eta$ and 2.5.12 imply that there exists a basis $\langle \sigma_{r+1}, \dots, \sigma_n \rangle$ of $\mathbb{Z}\varepsilon_{r+1} + \dots + \mathbb{Z}\varepsilon_n$ in \mathbb{Z}^n such that $x_j x_i = a_{ij} x_i x_j$ for all $r+1 \leq i, j \leq n$ where x_i is a nonzero element of degree σ_i . Note that all $x_1 := t_1, \dots, x_r := t_r$ commute with all t_{r+1}, \dots, t_n , and so all x_1, \dots, x_r commute with all x_{r+1}, \dots, x_n . Hence we get $x_j x_i = a_{ij} x_i x_j$ for all $1 \leq i, j \leq n$. Since $\langle \varepsilon_1, \dots, \varepsilon_r, \sigma_{r+1}, \dots, \sigma_n \rangle$ is a basis of \mathbb{Z}^n , we obtain $r \times s \cong r \times \eta$ by 2.5.12. \square

Our first result in this section is the following:

Theorem 3.2.3. Let ε be an $n \times n$ elementary quantum matrix. Then there exists $l \geq 0$ such that $\varepsilon \cong h_l^{(n)}$ where

$$h_l^{(n)} = \overbrace{h \times \cdots \times h}^{l\text{-times}} \times 1_{n-2l} \quad \text{and} \quad h = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The number l is an invariant of F_ϵ : if Γ is the central grading group of F_ϵ then $|\mathbb{Z}^n/\Gamma| = 4^l$.

Before we prove the theorem, we give some examples and a lemma.

Example 3.2.4. Let

$$\mathbf{m}^{(3)} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}^{(4)} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

and let

$$F_{\mathbf{m}^{(3)}} = F_{\mathbf{m}^{(3)}}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \quad \text{and} \quad F_{\mathbf{m}^{(4)}} = F_{\mathbf{m}^{(4)}}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]$$

be elementary quantum tori with an $\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ -grading and an $\langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle$ -grading, respectively.

In $F_{\mathbf{m}^{(3)}}$, t_1 commutes with $t_2 t_3$ which has degree $\epsilon_2 + \epsilon_3$, and in $F_{\mathbf{m}^{(4)}}$, t_1 commutes with $t_2 t_3$ and $t_2 t_4$ which has degree $\epsilon_2 + \epsilon_3$ and $\epsilon_2 + \epsilon_4$. Since $\langle \epsilon_1, \epsilon_2, \epsilon_2 + \epsilon_3 \rangle$ and $\langle \epsilon_1, \epsilon_2, \epsilon_2 + \epsilon_3, \epsilon_2 + \epsilon_4 \rangle$ are bases of \mathbb{Z}^3 and \mathbb{Z}^4 , respectively, we have by 2.5.12.

$$\mathbf{m}^{(3)} \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & * & * \\ 1 & * & * \end{pmatrix} \quad \text{and} \quad \mathbf{m}^{(4)} \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & * & * & * \\ 1 & * & * & * \\ 1 & * & * & * \end{pmatrix},$$

and the $*$ -parts of both matrices are some elementary matrices. Indeed in both algebras, we have $(t_2 t_3)t_2 = -t_2(t_2 t_3)$, and in $F_{\mathbf{m}^{(4)}}$, $(t_2 t_4)t_2 = -t_2(t_2 t_4)$ and $(t_2 t_3)(t_2 t_4) = -(t_2 t_4)(t_2 t_3)$. So we get

$$\mathbf{m}^{(3)} \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}^{(4)} \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

In both algebras, t_1 and t_2 commute with $t_1(t_2 t_3)$ which has degree $\epsilon_1 + \epsilon_2 + \epsilon_3$, and in $F_{\mathbf{m}^{(4)}}$, t_1 and t_2 commutes with $t_1(t_2 t_4)$ which has degree $\epsilon_1 + \epsilon_2 + \epsilon_4$. Since $\langle \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2 + \epsilon_3 \rangle$ and $\langle \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 + \epsilon_2 + \epsilon_4 \rangle$ are bases of \mathbb{Z}^3 and \mathbb{Z}^4 , respectively, we have by 2.5.12.

$$\mathbf{m}^{(3)} \cong \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{m}^{(4)} \cong \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & * & * \\ 1 & 1 & * & * \end{pmatrix}.$$

and the $*$ -part is \mathfrak{h} by $(t_1 t_2 t_4)(t_1 t_2 t_3) = -(t_1 t_2 t_3)(t_1 t_2 t_4)$. Thus we have shown

$$\mathfrak{m}^{(3)} \cong \mathfrak{h}_1^{(3)} = \mathfrak{h} \times (1) \quad \text{and} \quad \mathfrak{m}^{(4)} \cong \mathfrak{h}_2^{(4)} = \mathfrak{h} \times \mathfrak{h}.$$

Note that we also have shown

$$\begin{aligned} F_{\mathfrak{m}^{(3)}} &= F_{\mathfrak{h} \times (1)}[t_1^{\pm 1}, t_2^{\pm 1}, (t_1 t_2 t_3)^{\pm 1}] \quad \text{and} \\ F_{\mathfrak{m}^{(4)}} &= F_{\mathfrak{h} \times \mathfrak{h}}[t_1^{\pm 1}, t_2^{\pm 1}, (t_1 t_2 t_3)^{\pm 1}, (t_1 t_2 t_4)^{\pm 1}]. \end{aligned}$$

Lemma 3.2.5. *Let $\varepsilon = (\varepsilon_{ij})$ be an $n \times n$ elementary quantum matrix for $n \geq 3$. If $\varepsilon_{kp} = \varepsilon_{kq} = -1$ for some distinct $1 \leq k, p, q \leq n$, then there exists an elementary quantum matrix $\eta = (\eta_{ij})$ with*

$$\begin{aligned} \eta_{ij} &= \varepsilon_{ij} \quad \text{for all } i, j \neq q \quad (\eta_{qq} = \varepsilon_{qq} = 1), \\ \eta_{iq} &= \varepsilon_{ip} \varepsilon_{iq} \quad \text{for all } i \neq q \end{aligned}$$

such that $\varepsilon \cong \eta$. In particular,

(a) $\eta_{kq} = 1$ and $\eta_{ki} = \varepsilon_{ki}$ for all $i \neq q$.

(b) if $k = 2$ and $p = 1$, then $\eta_{i1} = \varepsilon_{i1}$ for all i , i.e., the first rows of ε and η are the same.

Proof. Let $F_\varepsilon = F_\varepsilon[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a $\langle \sigma_1, \dots, \sigma_n \rangle$ -grading. Since $\varepsilon_{kp} = \varepsilon_{kq} = -1$, we have $t_p t_k = -t_k t_p$ and $t_q t_k = -t_k t_q$. Hence t_k commutes with $t_p t_q$ which has degree $\sigma_p + \sigma_q$. Let

$$x_1 := t_1, \dots, x_{q-1} := t_{q-1}, x_q := t_p t_q, x_{q+1} := t_{q+1}, \dots, x_n := t_n.$$

Then the relations between x_i and x_j for $1 \leq i, j \leq n$ determine an elementary quantum matrix $\eta = (\eta_{ij})$, i.e., $x_j x_i = \eta_{ij} x_i x_j$. It is clear that $\eta_{ij} = \varepsilon_{ij}$ for all $i, j \neq q$. For $i \neq q$, we have $x_q x_i = (t_p t_q) t_i = \varepsilon_{ip} \varepsilon_{iq} t_i (t_p t_q) = \varepsilon_{ip} \varepsilon_{iq} x_i x_q$. Hence $\eta_{iq} = \varepsilon_{ip} \varepsilon_{iq}$. Since

$$\langle \sigma_1, \dots, \sigma_{q-1}, \sigma_p + \sigma_q, \sigma_{q+1}, \dots, \sigma_n \rangle$$

is a basis of \mathbb{Z}^n , we get $\varepsilon \cong \eta$ by 2.5.12. (a) and (b) are clear now. \square

Now we are ready to prove Theorem 3.2.3.

Proof of the 1st part of Theorem 3.2.3. We prove this by induction on n . When $n = 1$, ε has to be (1), and so the statement is clear. Let $n > 1$, $\varepsilon = (\varepsilon_{ij})$ and

$$N_k(\varepsilon) := \sharp\{i \mid \varepsilon_{ki} = -1, 1 \leq i \leq n\}$$

(We will use this notation only for $k = 1$ and 2). If $N_1(\varepsilon) = 0$, then we have $\varepsilon = (1) \times \varepsilon'$ for an elementary quantum matrix ε' of size $n - 1$. By induction, we have $\varepsilon' \cong \mathbf{h}_l^{(n-1)}$ for some $l \geq 0$. Then, by 3.2.2(2), we get

$$\varepsilon = (1) \times \varepsilon' \cong (1) \times \mathbf{h}_l^{(n-1)} \cong \mathbf{h}_l^{(n-1)} \times (1) = \mathbf{h}_l^{(n)}.$$

If $N_1(\varepsilon) > 1$, then by 3.2.5(a) for $k = 1$, there exists an elementary quantum matrix ε' such that $\varepsilon \cong \varepsilon'$ and $N_1(\varepsilon') = N_1(\varepsilon) - 1$. Repeating this, we obtain an elementary quantum matrix ν such that $\varepsilon \cong \nu$ and $N_1(\nu) = 1$, i.e., only one entry, say the $(1i_0)$ -entry, is -1 in the first column of ν . So if $N_1(\varepsilon) = 1$, we can also put $\nu = \varepsilon$. Then, by 3.2.2(1), we get

$$\varepsilon \cong \nu_{(2i_0)} =: \eta = (\eta_{ij}) = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & & & & \\ 1 & & * & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix},$$

$\eta_{12} = \eta_{21} = -1$, the other $\eta_{1i} = \eta_{i1} = 1$ and $*$ is some elementary quantum matrix of size $n - 1$.

If $n = 2$, we have $\eta = \mathbf{h}$ and we are done. We assume that $n > 2$. Note that $N_2(\eta) \geq 1$ since $\eta_{21} = -1$. If $N_2(\eta) > 1$, we can apply 3.2.5(b) for any $q > 2$ such that $\eta_{2q} = -1$, and get an elementary quantum matrix η' such that $\eta \cong \eta'$, $N_1(\eta') = N_1(\eta) = 1$ and $N_2(\eta') = N_2(\eta) - 1$. Repeating this, we obtain an elementary quantum matrix $\mu = (\mu_{ij})$ such that $\eta \cong \mu$, $N_1(\mu) = N_2(\mu) = 1$ and $\mu_{21} = \mu_{12} = -1$. Also, if $N_2(\eta) = 1$, we put $\eta = \mu$. Thus we have $\eta \cong \mu = \mathbf{h} \times \mu'$ for an elementary quantum matrix μ' of size $n - 2$. By induction, we have $\mu' \cong \mathbf{h}_{l'}^{(n-2)}$ for some $l' \geq 0$. Then, by 3.2.2(2)(ii), we get $\mu = \mathbf{h} \times \mu' \cong \mathbf{h} \times \mathbf{h}_{l'}^{(n-2)} = \mathbf{h}_l^{(n)}$ where $l = l' + 1$, and hence $\varepsilon \cong \eta \cong \mu \cong \mathbf{h}_l^{(n)}$. \square

Next we consider the centre $Z = Z(F_{\mathbf{h}_l^{(n)}})$ of $F_{\mathbf{h}_l^{(n)}}$. Note that $Z(F_{\mathbf{h}_0^{(n)}}) = F_{\mathbf{h}_0^{(n)}} = F_1 = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the algebra of Laurent polynomials.

Lemma 3.2.6. Let $l > 0$ and $F_{\mathbf{h}_l^{(n)}} = F_{\mathbf{h}_l^{(n)}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an elementary torus.

(i) Let $x := t_1^{\epsilon_1} \cdots t_{2l}^{\epsilon_{2l}}$, where $\epsilon_i = 0$ or 1 but not all ϵ_i are 0 . Then for $\epsilon_j \neq 0$, we have $xt_k = -t_k x$ where

$$k = \begin{cases} j+1 & \text{if } j \text{ is odd} \\ j-1 & \text{if } j \text{ is even.} \end{cases}$$

(ii) The centre of $F_{\mathbf{h}_l^{(n)}}$ is equal to

$$F[t_1^{\pm 2}, \dots, t_{2l}^{\pm 2}, t_{2l+1}^{\pm 1}, \dots, t_n^{\pm 1}],$$

the algebra of Laurent polynomials in the variables $t_1^2, \dots, t_{2l}^2, t_{2l+1}, \dots, t_n$. Hence for a toral grading $\langle \sigma_1, \dots, \sigma_n \rangle$ of $F_{\mathbf{h}_l^{(n)}}$, the central grading group of $F_{\mathbf{h}_l^{(n)}}$ is

$$2\mathbb{Z}\sigma_1 + \cdots + 2\mathbb{Z}\sigma_{2l} + \mathbb{Z}\sigma_{2l+1} + \cdots + \mathbb{Z}\sigma_n.$$

Proof. (i) is clear from the relation matrix $\mathbf{h}_l^{(n)}$.

For (ii), let $Z' = F[t_1^{\pm 2}, \dots, t_{2l}^{\pm 2}, t_{2l+1}^{\pm 1}, \dots, t_n^{\pm 1}]$. Then $Z \supset Z'$ by 3.1.5. For the other inclusion, it is enough to show that any monomial not in Z' is not contained in Z , but this is clear by (i). \square

We can now finish the proof of Theorem 3.2.3.

Proof of the 2nd part of Theorem 3.2.3. Suppose that $\epsilon \cong \mathbf{h}_l^{(n)}$. Then, by 2.5.12, $F_\epsilon \cong_{\mathbb{Z}^n} F_{\mathbf{h}_l^{(n)}}$ for some toral gradings. Let Γ be the central grading group of F_ϵ . By 3.2.6(ii), we have $\mathbb{Z}^n/\Gamma \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (the direct sum of $2l$ copies of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), and so $|\mathbb{Z}^n/\Gamma| = 4^l$. Thus, by 2.4.3(iii), we get $F_\epsilon \cong F_{\mathbf{h}_{l'}}^{(n)}$ if $l' \neq l$. \square

We always consider a quantum torus as an associative torus by giving it a toral grading. So we can consider the graded involutions of a quantum torus (see 2.5.10).

Let E be a field extension of F of $[E : F] \leq 2$, and if $[E : F] = 2$, let σ_E be the nontrivial Galois automorphism of E over F . Let $E_q = E_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the quantum torus over E determined by $\mathbf{q} = (q_{ij})$, and let τ be a graded involution of F_q if $E = F$ or a graded σ_E -semilinear involution of E_q over F otherwise. Then we have $\tau(t_i) = a_i t_i$ for some $a_i \in E$, $i = 1, \dots, n$. For convenience, we set $\sigma_E = \text{id}$ if $E = F$. Since $t_i = \tau^2(t_i) = a_i \sigma_E(a_i) t_i$, we get $a_i \sigma_E(a_i) = 1$ for all $1 \leq i \leq n$. Note that $a_i = \pm 1$ for all $1 \leq i \leq n$ if $E = F$. Moreover, we have

$$a_i a_j \sigma_E(q_{ij}) t_j t_i = \tau(q_{ij} t_i t_j) = \tau(t_j t_i) = a_i a_j t_i t_j = a_i a_j q_{ji} t_j t_i,$$

and hence $q_{ij}^{-1} = q_{ji} = \sigma_E(q_{ij})$. Note that $q_{ij} = \pm 1$ for all $1 \leq i, j \leq n$ if $E = F$. Thus \mathbf{q} has to be ξ as defined in 3.1.2. i.e., $\xi_{ij}\sigma_E(\xi_{ij}) = 1$. Note that if $E = F$, ξ is elementary.

Conversely, as we saw in 3.1.2, each $(\theta_1, \dots, \theta_n)$, $\theta_i \in E$, satisfying $\theta_i\sigma_E(\theta_i) = 1$, gives $E_\xi = E_\xi[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ a unique graded involution τ of F_ξ if $E = F$, or a graded σ_E -semilinear involution τ of E_ξ over F otherwise, such that $\tau(t_i) = \theta_i t_i$ for all $1 \leq i \leq n$. We call this τ of type $(\theta_1, \dots, \theta_n)$, denoted $\tau = (\theta_1, \dots, \theta_n)$. The graded involution of type $(1, \dots, 1)$ is called the *main involution*. We use the same notation as in 3.1.2, i.e., $\ast = (1, \dots, 1)$ if $E = F$ and $\sigma = (1, \dots, 1)$ if $E \neq F$. Thus we have shown the following:

Proposition 3.2.7. (i) Let $F_{\mathbf{q}} = F_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a quantum torus over F . Then there exists a graded involution τ of $F_{\mathbf{q}}$ if and only if \mathbf{q} is elementary. In this case, τ has type (a_1, \dots, a_n) , i.e., $\tau(t_i) = a_i t_i$ where $\theta_i = 1$ or -1 for all $1 \leq i \leq n$.

(ii) Let E be a field extension of F of $[E : F] = 2$ and σ_E the nontrivial Galois automorphism of E over F . Let $E_{\mathbf{q}} = E_{\mathbf{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a quantum torus over E . Then there exists a graded σ_E -semilinear involution τ of $E_{\mathbf{q}}$ if and only if $\mathbf{q} = \xi$ (see 3.1.3). In this case, τ has type $(\theta_1, \dots, \theta_n)$, i.e., $\tau(t_i) = \theta_i t_i$ where $\sigma_E(\theta_i)\theta_i = 1$ for all $1 \leq i \leq n$. \square

Recall the notion of isomorphism in the class of algebras with involution. Namely, for algebras with involution (A, τ) and (B, ρ) , an *isomorphism of algebras with involution* from (A, τ) onto (B, ρ) is an isomorphism f from A onto B satisfying $f\tau = \rho f$.

In the case of 3.2.7(ii), if $\xi = \epsilon$, the classification of Jordan tori $H(E_\epsilon, \tau)$ is easy. Indeed, by 3.2.3 we have $E_\epsilon \cong_{\mathbb{Z}^n} E_{h_i^{(n)}}$ over E . Through this graded isomorphism, we have $(E_\epsilon, \tau) \cong_{\mathbb{Z}^n} (E_{h_i^{(n)}}, \tau')$ over E for some graded σ_E -semilinear involution τ' of $E_{h_i^{(n)}}$ over F . Since $(E_{h_i^{(n)}}, \tau') \cong_{\mathbb{Z}^n} (E_{h_i^{(n)}}, \sigma)$ (see 3.1.5a), we get $(E_\epsilon, \tau) \cong_{\mathbb{Z}^n} (E_{h_i^{(n)}}, \sigma)$ over E and obtain

$$(3.2.7a) \quad H(E_\epsilon, \tau) \cong_{\mathbb{Z}^n} H(E_{h_i^{(n)}}, \sigma) \quad \text{over } F.$$

The classification of $H(F_\epsilon, \tau)$ is not so easy. First we prove a lemma similar to 2.5.12 for elementary tori with graded involutions.

Lemma 3.2.8. Let (F_ϵ, τ) and (F_η, ρ) be elementary quantum tori with graded involutions. Let $\eta = (\eta_{ij})_{1 \leq i, j \leq n}$ and $\rho = (\theta_1, \dots, \theta_n)$. Then the following are equivalent:

$$(i) (F_\epsilon, \tau) \cong (F_\eta, \rho),$$

(ii) for any toral grading of F_ϵ , there exists a basis $\langle \sigma_1, \dots, \sigma_n \rangle$ of \mathbb{Z}^n and nonzero homogeneous elements $x_i \in F_\epsilon$ of degree σ_i such that $x_j x_i = \eta_{ij} x_i x_j$ and $\tau(x_i) = a_i x_i$ for all $1 \leq i < j \leq n$,

$$(iii) \text{ there exist toral gradings of } F_\epsilon \text{ and } F_\eta \text{ such that } (F_\epsilon, \tau) \cong_{\mathbb{Z}^n} (F_\eta, \rho).$$

Proof. The proof is similar to 2.5.12. In fact, suppose that (i) holds, i.e., there exists an isomorphism φ from (F_ϵ, τ) onto (F_η, ρ) . Let $F_\eta = F_\eta[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Then we already showed in the proof of 2.5.12 that $x_i := \varphi^{-1}(t_i)$, $i = 1, \dots, n$, satisfy all statements of (ii) except $\tau(x_i) = a_i x_i$. Let $\tau(x_i) = b_i x_i$. Then since φ is graded, we have

$$b_i t_i = \varphi(b_i x_i) = \varphi(\tau(x_i)) = \rho(\varphi(x_i)) = \rho(t_i) = a_i t_i.$$

Hence $b_i = a_i$ for all i . So (ii) holds.

Suppose that (ii) holds. Then we showed in the proof of 2.5.12 that there exist toral gradings of F_ϵ and F_η and that there is a graded isomorphism ψ from F_ϵ onto $F_\eta = F_\eta[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ so that $\psi(x_i) = t_i$. We need to show that ψ is an isomorphism with involution. Since x_1, \dots, x_n generates F_ϵ , this follows from

$$\psi(\tau(x_i)) = \psi(a_i x_i) = a_i \psi(x_i) = a_i t_i = \rho(t_i) = \rho(\psi(x_i))$$

for all i . Hence (iii) holds.

Finally, it is clear that (iii) implies (i). \square

We have a lemma similar to 3.2.2. For this purpose, we introduce a notation. For graded involutions τ and ρ of type (a_1, \dots, a_n) and (b_1, \dots, b_n) , respectively, we denote the graded involution of type $(a_1, \dots, a_n, b_1, \dots, b_n)$ by $\tau \times \rho$.

Lemma 3.2.9. *Let (F_τ, τ) , (F_ρ, ρ) and (F_η, ρ_1) be elementary quantum tori with graded involutions. Assume that $(F_\rho, \rho) \cong (F_\eta, \rho_1)$. Then:*

$$(i) (F_{\tau \times \rho}, \tau \times \rho) \cong (F_{\rho \times \tau}, \rho \times \tau), \quad (ii) (F_{\tau \times \rho}, \tau \times \rho) \cong (F_{\tau \times \eta}, \tau \times \rho_1).$$

Proof. This can be proven similarly to 3.2.2. \square

We will classify elementary tori with graded involution. Let τ be a graded involution of an elementary quantum torus F_ϵ . Then, by 3.2.3 and 2.5.12, we have

$F_\varepsilon \cong_{\mathbb{Z}^n} F_{h_l^{(n)}}$ for some $l \geq 0$, and hence $(F_\varepsilon, \tau) \cong_{\mathbb{Z}^n} (F_{h_l^{(n)}}, \rho)$ for some graded involution ρ of $F_{h_l^{(n)}}$ and some toral grading of F_ε and $F_{h_l^{(n)}}$. Thus it is enough to classify $F_{h_l^{(n)}}$ with graded involutions. Besides the main involution $\ast = (1, \dots, 1)$, we define two specific graded involutions of the n -torus $F_{h_l^{(n)}}$, namely,

$$\tau_1 = (1, \dots, 1, -1, 1, \dots, 1),$$

where only the $2l + 1$ position is -1 , if $n - 2l \geq 1$

$$\tau_2 = (1, \dots, 1, -1, -1, 1, \dots, 1),$$

where only the $2l - 1$ and $2l$ positions are -1 , if $l \geq 1$.

Remark 3.2.10. Suppose $\tau = \ast, \tau_1, \tau_2$. By the universal property of the central closure $\overline{F}_{h_l^{(n)}}$ (see 1.1.16(v)), the natural extension $\overline{\tau}$ of τ defined by $\overline{\tau}(z \otimes x) = \tau(z) \otimes \tau(x)$ is an involution of $\overline{F}_{h_l^{(n)}}$. Since \ast and τ_2 fix \overline{Z} (see 3.2.6(ii)), they are involutions of first kind, while τ_1 does not, and so it is an involution of second kind.

Example 3.2.11. Recall the two elementary quantum matrices $\mathbf{m}^{(3)}$ and $\mathbf{m}^{(4)}$ defined in 3.2.4. We showed there

$$\begin{aligned} F_{\mathbf{m}^{(3)}} &:= F_{\mathbf{m}^{(3)}}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] = F_{\mathbf{h} \times (1)}[t_1^{\pm 1}, t_2^{\pm 1}, (t_1 t_2 t_3)^{\pm 1}] \\ &\cong F_{\mathbf{h} \times (1)} := F_{\mathbf{h} \times (1)}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}], \\ F_{\mathbf{m}^{(4)}} &:= F_{\mathbf{m}^{(4)}}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] = F_{\mathbf{h} \times \mathbf{h}}[t_1^{\pm 1}, t_2^{\pm 1}, (t_1 t_2 t_3)^{\pm 1}, (t_1 t_2 t_4)^{\pm 1}] \\ &\cong F_{\mathbf{h} \times \mathbf{h}} := F_{\mathbf{h} \times \mathbf{h}}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]. \end{aligned}$$

Since $(t_1 t_2 t_3)^\ast = t_3 t_2 t_1 = -t_1 t_2 t_3$, we have $(F_{\mathbf{m}^{(3)}}, \ast) = (F_{\mathbf{h} \times (1)}, \tau_1)$. Also, since $(t_1 t_2 t_4)^\ast = t_4 t_2 t_1 = -t_1 t_2 t_4$, we have $(F_{\mathbf{m}^{(4)}}, \ast) = (F_{\mathbf{h} \times \mathbf{h}}, \tau_2)$.

Lemma 3.2.12. *Let \ast be the main involution and τ_1 the graded involution of $F_{h_l^{(n)}}$ defined above. Then:*

- (i) $(F_{\mathbf{h}}, (1, -1)) \cong (F_{\mathbf{h}}, (-1, 1)) \cong (F_{\mathbf{h}}, \ast)$,
- (ii) $(F_{\mathbf{1}_2}, (-1, -1)) \cong (F_{\mathbf{1}_2}, \tau_1)$,
- (iii) $(F_{\mathbf{h} \times (1)}, (-1, -1, -1)) \cong (F_{\mathbf{h} \times (1)}, \tau_1)$,
- (iv) $(F_{\mathbf{h} \times \mathbf{h}}, (-1, -1, -1, -1)) \cong (F_{\mathbf{h} \times \mathbf{h}}, \ast)$.

Proof. Let $F_{h_l^{(n)}} = F_{h_l^{(n)}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with an $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ -grading. Then we note that $t_{i_1} \cdots t_{i_r}$ has degree $\varepsilon_{i_1} + \cdots + \varepsilon_{i_r}$.

For (i). we have $n = 2$ and $l = 1$. Let $\tau = (1, -1)$. Then we have $\tau(t_1) = t_1$ and $\tau(t_2) = -t_2$. Since $(t_1 t_2)t_1 = -t_1(t_1 t_2)$ and $\tau(t_1 t_2) = t_1 t_2$, and since $\langle \varepsilon_1, \varepsilon_1 + \varepsilon_2 \rangle$ is a basis of \mathbb{Z}^2 , we get $(F_{\mathbf{h}}, \tau) \cong (F_{\mathbf{h}}, *)$ by 3.2.8. The case $(-1, 1)$ can be proven in the same way.

For (ii). we have $n = 2$ and $l = 0$. Let $\tau = (-1, -1)$. Then we have $\tau(t_1) = -t_1$ and $\tau(t_2) = -t_2$. Since $(t_1 t_2) = t_1(t_1 t_2)$ and $\tau(t_1 t_2) = t_1 t_2$, and since $\langle \varepsilon_1, \varepsilon_1 + \varepsilon_2 \rangle$ is a basis of \mathbb{Z}^2 , we get $(F_{1_2}, \tau) \cong (F_{1_2}, \tau_1)$ by 3.2.8.

For (iii), we have $n = 3$ and $l = 1$. Let $\tau = (-1, -1, -1)$. Then we have $\tau(t_1) = -t_1$, $\tau(t_2) = -t_2$ and $\tau(t_3) = -t_3$. Since $(t_2 t_3)(t_1 t_2 t_3) = -(t_1 t_2 t_3)(t_2 t_3)$, $t_3(t_1 t_2 t_3) = (t_1 t_2 t_3)t_3$, $t_3(t_2 t_3) = (t_2 t_3)t_3$, $\tau(t_1 t_2 t_3) = t_1 t_2 t_3$ and $\tau(t_2 t_3) = t_2 t_3$, and since $\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_2 + \varepsilon_3, \varepsilon_3 \rangle$ is a basis of \mathbb{Z}^3 , we get $(F_{\mathbf{h} \times (1)}, \tau) \cong (F_{\mathbf{h} \times (1)}, \tau_1)$ by 3.2.8.

For (iv), we have $n = 4$ and $l = 2$. Let $\tau = (-1, -1, -1, -1)$. Then we have $\tau(t_1) = -t_1$, $\tau(t_2) = -t_2$, $\tau(t_3) = -t_3$ and $\tau(t_4) = -t_4$. Put $x_1 := t_1 t_2 t_4$, $x_2 := t_2 t_4$, $x_3 := t_1 t_3$ and $x_4 := t_1 t_3 t_4$. Then one can check that $x_j x_i = a_{ij} x_i x_j$ where $(a_{ij}) = \mathbf{h} \times \mathbf{h}$ and $\tau(x_i) = x_i$ for $1 \leq i, j \leq 4$. Also, one can check that

$$\langle \varepsilon_1 + \varepsilon_2 + \varepsilon_4, \varepsilon_2 + \varepsilon_4, \varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_3 + \varepsilon_4 \rangle$$

is a basis of \mathbb{Z}^4 . Hence by 3.2.8, we get $(F_{\mathbf{h} \times \mathbf{h}}, \tau) \cong (F_{\mathbf{h} \times \mathbf{h}}, *)$. \square

Now we classify elementary tori with graded involution.

Theorem 3.2.13. *Let τ be an arbitrary graded involution of an elementary quantum torus F_{ε} . Let $*$ be the main involution, and τ_1 and τ_2 the graded involutions of $F_{\mathbf{h}_l^{(n)}}$ defined above. Then:*

(i) (F_{ε}, τ) is graded isomorphic to one of $(F_{\mathbf{h}_l^{(n)}}, *)$, $(F_{\mathbf{h}_l^{(n)}}, \tau_1)$ or $(F_{\mathbf{h}_l^{(n)}}, \tau_2)$.

(ii) In particular, for $\tau = *$ we have

$$(F_{\varepsilon}, *) \cong (F_{\mathbf{h}_l^{(n)}}, \tau_1) \implies l \geq 1$$

$$(F_{\varepsilon}, *) \cong (F_{\mathbf{h}_l^{(n)}}, \tau_2) \implies l \geq 2,$$

and conversely, $(F_{\mathbf{h}_l^{(n)}}, \tau_1)$ for $l \geq 1$ and $(F_{\mathbf{h}_l^{(n)}}, \tau_2)$ for $l \geq 2$ are graded isomorphic to $(F_{\varepsilon}, *)$ for some elementary quantum matrix ε .

Proof. We have $(F_{\varepsilon}, \tau) \cong_{\mathbb{Z}^n} (F_{\mathbf{h}_l^{(n)}}, \rho)$ for some graded involution ρ of $F_{\mathbf{h}_l^{(n)}}$ as mentioned above. So we classify $(F_{\mathbf{h}_l^{(n)}}, \rho)$. Let $\rho = (a_1, \dots, a_n)$. Note that $\mathbf{h}_l^{(n)} =$

$\mathfrak{h}_l^{(2l)} \times \mathbf{1}_{n-2l}$. We consider $(F_{\mathfrak{h}_l^{(2l)}}, (a_1, \dots, a_{2l}))$ and $(F_{\mathbf{1}_{n-2l}}, (a_{2l+1}, \dots, a_n))$ separately. By 3.2.9, we can also decompose each of these into small pieces in order to apply Lemma 3.2.12. Thus, by 3.2.12(i) and (iv), we have

$$(F_{\mathfrak{h}_l^{(2l)}}, (a_1, \dots, a_{2l})) \cong (F_{\mathfrak{h}_l^{(2l)}}, *) \quad \text{or} \quad (F_{\mathfrak{h}_l^{(2l)}}, \tau_2),$$

and by 3.2.12(ii),

$$(F_{\mathbf{1}_{n-2l}}, (a_{2l+1}, \dots, a_n)) \cong (F_{\mathbf{1}_{n-2l}}, *) \quad \text{or} \quad (F_{\mathbf{1}_{n-2l}}, \tau_1).$$

Hence by 3.2.9, we get

$$(F_{\mathfrak{h}_l^{(n)}}, \rho) \cong (F_{\mathfrak{h}_l^{(n)}}, *), (F_{\mathfrak{h}_l^{(n)}}, \tau_1), (F_{\mathfrak{h}_l^{(n)}}, \tau_2) \text{ or } (F_{\mathfrak{h}_l^{(n)}}, (1, \dots, 1, -1, -1, -1, 1, \dots, 1)).$$

and the last one is isomorphic to $(F_{\mathfrak{h}_l^{(n)}}, \tau_1)$ by 3.2.12(iii). Hence by (i) \Rightarrow (iii) of 3.2.8, we have shown (i).

For (ii), let $F_{\mathfrak{h}_l^{(n)}} = F_{\mathfrak{h}_l^{(n)}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with an $(\epsilon_1, \dots, \epsilon_n)$ -grading.

For the first statement, we need to show that $(F_{\mathfrak{h}_0^{(n)}}, \tau_1) \not\cong (F_{\epsilon}, *)$ and that $(F_{\mathfrak{h}_1^{(n)}}, \tau_2) \not\cong (F_{\epsilon}, *)$.

Suppose that $(F_{\mathfrak{h}_0^{(n)}}, \tau_1) \cong (F_{\epsilon}, *)$. We have $\mathfrak{h}_0^{(n)} = \mathbf{1}$, which forces $\epsilon = \mathbf{1}$, and hence $*$ is the identity map. This is a contradiction since τ_1 is not the identity map. Therefore, we get $(F_{\mathfrak{h}_0^{(n)}}, \tau_1) \not\cong (F_{\epsilon}, *)$.

Suppose that $(F_{\mathfrak{h}_1^{(n)}}, \tau_2) \cong (F_{\epsilon}, *)$. By 3.2.8, there exists a basis $\langle \rho_1, \dots, \rho_n \rangle$ of \mathbb{Z}^n such that a nonzero element $x_i \in F_{\mathfrak{h}_1^{(n)}}$ of degree ρ_i are fixed by τ_2 for all $i = 1, \dots, n$. Let $\rho_i = \alpha_{i1}\epsilon_1 + \dots + \alpha_{in}\epsilon_n$ for $\alpha_{ij} \in \mathbb{Z}$. Then one can take $x_i = t_1^{\alpha_{i1}} \dots t_n^{\alpha_{in}}$. Since $\tau_2 = (-1, -1, 1, \dots, 1)$, we have, by the multiplication rule (2.5.9) of a quantum torus,

$$\tau_2(x_i) = (-1)^{\alpha_{i1} + \alpha_{i2}} t_1^{\alpha_{in}} \dots t_n^{\alpha_{i1}} = (-1)^{\alpha_{i1} + \alpha_{i2} + \alpha_{i1}\alpha_{i2}} x_i = x_i.$$

Hence α_{i1} and α_{i2} are both even for all $i = 1, \dots, n$. This implies that the determinant of the matrix (α_{ij}) is even. This is absurd since $\langle \rho_1, \dots, \rho_n \rangle$ is a basis of \mathbb{Z}^n . Therefore, we get $(F_{\mathfrak{h}_1^{(n)}}, \tau_2) \not\cong (F_{\epsilon}, *)$. Thus we have shown the first statement of (ii).

For the second statement, let U be the subalgebra of $(F_{\mathfrak{h}_l^{(n)}}, \tau_1)$ generated by t_{2l-1} , t_{2l} and t_{2l+1} , and let V be the subalgebra of $(F_{\mathfrak{h}_l^{(n)}}, \tau_2)$ generated by t_{2l-3} , t_{2l-2} , t_{2l-1} and t_{2l} . Then we have $(U, \tau_1 |_U) \cong (F_{\mathfrak{h} \times (1)}, \tau_1)$ and $(U, \tau_2 |_V) \cong (F_{\mathfrak{h} \times \mathfrak{h}}, \tau_2)$.

Recall that we showed $(F_{\mathbf{h} \times (1)}, \tau_1) \cong (F_{\mathbf{m}^{(3)}}, *)$ and $(F_{\mathbf{h} \times \mathbf{h}}, \tau_2) \cong (F_{\mathbf{m}^{(4)}}, *)$ in 3.2.11. Therefore, by 3.2.9 and (i) \Rightarrow (iii) of 3.2.8, we obtain

$$(3.2.13a) \quad (F_{\mathbf{h}_l^{(n)}}, \tau_1) \cong_{\mathbb{Z}^n} (F_{\mathbf{h}_{l-1}^{(n-3)} \times \mathbf{m}^{(3)}}, *).$$

$$(3.2.13b) \quad (F_{\mathbf{h}_l^{(n)}}, \tau_2) \cong_{\mathbb{Z}^n} (F_{\mathbf{h}_{l-2}^{(n-4)} \times \mathbf{m}^{(4)}}, *). \quad \square$$

Remark 3.2.14. We have shown that $H(F_{\epsilon}, *)$ for the main involution $*$ is a Jordan n -torus in 3.1.2. We claim that $H(F_{\epsilon}, \tau)$ for an arbitrary graded involution τ is also a Jordan n -torus. Indeed, $(F_{\mathbf{h}_0^{(n)}}, \tau_1) = (F_1, \tau_1)$ and $(F_{\mathbf{h}_1^{(n)}}, \tau_2)$ are the only algebras which are not isomorphic to $(F_{\epsilon}, *)$ by 3.2.13. One can easily see that $H(F_1, \tau_1)$ is a Jordan $2\mathbb{Z} \oplus \mathbb{Z}^{n-1}$ -torus, and that $H(F_{\mathbf{h}_1^{(n)}}, \tau_2)$ is a Jordan $2\mathbb{Z}^2 \oplus \mathbb{Z}^{n-2}$ -torus, which are both isomorphic to the algebra of Laurent polynomials in n -variables (so they are both Jordan n -tori, in particular). Because of this, to classify the Jordan tori $H(F_{\epsilon}, \tau)$, it is enough to consider the Jordan tori $H(F_{\epsilon}, *)$.

We note that $\text{supp}(H(F_1, \tau_1)) = 2\mathbb{Z} \oplus \mathbb{Z}^{n-1}$ and $\text{supp}(H(F_{\mathbf{h}_1^{(n)}}, \tau_2)) = 2\mathbb{Z}^2 \oplus \mathbb{Z}^{n-2}$ are not semilattices in \mathbb{Z}^n anymore. But they are lattices so semilattices in the original definition in [1].

We will classify the Jordan tori $H(F_{\epsilon}, *)$. By Theorem 3.2.13, we already know that $H(F_{\epsilon}, *)$ is graded isomorphic to one of the three Jordan tori $H(F_{\mathbf{h}_l^{(n)}}, *)$, $H(F_{\mathbf{h}_l^{(n)}}, \tau_1)$ for $l \geq 1$ or $H(F_{\mathbf{h}_l^{(n)}}, \tau_2)$ for $l \geq 2$. To show that any two of these three are not isomorphic, we consider the semilattices of the three tori.

We first define the *index* of a semilattice S in \mathbb{Z}^n as the number of cosets of $\mathbb{Z}^n/2\mathbb{Z}^n$ contained in S . This concept was introduced by Azam in [3, page 3] but the coset $2\mathbb{Z}^n$ is not counted there. That is, his index is just our index minus one. Recall the definition of similarity of semilattices in §2.4. The following lemma is proven in [3, Lemma 1.7, page 3]:

Lemma 3.2.15. *The index is a similarity invariant of semilattices.* \square

Thus, by 2.4.3(ii) and 3.2.15, the index of the support of a Jordan torus is an isomorphism invariant. Let $F_{\mathbf{h}_l^{(n)}} = F_{\mathbf{h}_l^{(n)}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a $\langle \sigma_1, \dots, \sigma_n \rangle$ -grading and let τ be an arbitrary graded involution of $F_{\mathbf{h}_l^{(n)}}$. We put

$$S_{n,l,\tau} := \text{supp}(H(F_{\mathbf{h}_l^{(n)}}, \tau)).$$

So we have $2\mathbb{Z}\sigma_1 + \cdots + 2\mathbb{Z}\sigma_n \subset S_{n,l,\tau}$. Let

$$\begin{aligned} I(S_{n,l,\tau}) &:= \{(\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \mid \epsilon_1\sigma_1 + \cdots + \epsilon_n\sigma_n \in S_{n,l,\tau}\} \\ &= \{(\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \mid \tau(t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}) = t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}\} \quad \text{and} \\ I(S_{n,l,\tau})^- &:= \{0,1\}^n \setminus I(S_{n,l,\tau}) \\ &= \{(\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \mid \tau(t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}) = -t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}\}, \quad \text{so} \\ (0) \quad 2^n &= \sharp\{0,1\}^n = \sharp I(S_{n,l,\tau}) + \sharp I(S_{n,l,\tau})^-. \end{aligned}$$

We note that $\sharp I(S_{n,l,\tau})$ is the index of the semilattice $S_{n,l,\tau}$ in \mathbb{Z}^n if $S_{n,l,\tau} = S_{n,l,*}$, S_{n,l,τ_1} for $l \geq 1$ or S_{n,l,τ_2} for $l \geq 2$. Thus, by 3.2.15 and 2.4.5, if $\sharp I(S_{n,l_0,*})$, $\sharp I(S_{n,l_1,\tau_1})$ and $\sharp I(S_{n,l_2,\tau_2})$ are all distinct for any l_0, l_1, l_2 , then the three tori $H(F_{h_l^{(n)}}, *)$, $H(F_{h_l^{(n)}}, \tau_1)$ for $l \geq 1$ or $H(F_{h_l^{(n)}}, \tau_2)$ for $l \geq 2$ are pairwise nonisomorphic. In fact, we can prove the following:

Lemma 3.2.16. *In the notation above, we have*

$$\begin{aligned} \sharp I(S_{n,l,*}) &= 2^{n-1} + 2^{n-l-1}, \\ \sharp I(S_{n,l,\tau_1}) &= 2^{n-1} \\ \sharp I(S_{n,l,\tau_2}) &= 2^{n-1} - 2^{n-l-1}. \end{aligned}$$

In particular, for arbitrary $l_0, l_1 \geq 0$ and $l_2 \geq 1$,

$$\sharp I(S_{n,l_0,*}) > \sharp I(S_{n,l_1,\tau_1}) > \sharp I(S_{n,l_2,\tau_2}).$$

Proof. For $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n$ and $t^\epsilon := t_1^{\epsilon_1} \cdots t_{2l}^{\epsilon_{2l}} t_{2l+1}^{\epsilon_{2l+1}} \cdots t_n^{\epsilon_n}$, we have

$$(t^\epsilon)^* = (t_2^{\epsilon_2} t_1^{\epsilon_1})(t_4^{\epsilon_4} t_3^{\epsilon_3}) \cdots (t_{2l}^{\epsilon_{2l}} t_{2l-1}^{\epsilon_{2l-1}}) t_{2l+1}^{\epsilon_{2l+1}} \cdots t_n^{\epsilon_n} = (-1)^{\sum_{i=1}^l \epsilon_{2i-1} \epsilon_{2i}} t^\epsilon.$$

Note that

$$t_{2i}^{\epsilon_{2i}} t_{2i-1}^{\epsilon_{2i-1}} = \begin{cases} t_{2i-1}^{\epsilon_{2i-1}} t_{2i}^{\epsilon_{2i}} & \text{if } (\epsilon_{2i-1}, \epsilon_{2i}) \neq (1, 1) \\ -t_{2i-1}^{\epsilon_{2i-1}} t_{2i}^{\epsilon_{2i}} & \text{if } (\epsilon_{2i-1}, \epsilon_{2i}) = (1, 1). \end{cases}$$

Hence for

$$\bar{l} = \begin{cases} l-1 & \text{if } l \text{ is even} \\ l & \text{if } l \text{ is odd,} \end{cases}$$

we obtain, by comparing the pairs $(\epsilon_{2i-1}, \epsilon_{2i}) = (1, 1)$,

$$\begin{aligned} \sharp I(S_{n,l,*}) &= 2^n - 2^{n-2l} \left(\binom{l}{1} 3^{l-1} + \binom{l}{3} 3^{l-3} + \dots + \binom{l}{l} 3^{l-l} \right) \\ &= 2^n - 2^{n-2l} (2^{2l-1} - 2^{l-1}) \\ (1) \quad &= 2^{n-1} + 2^{n-l-1}. \end{aligned}$$

by comparing the binomial expansions of $(3+1)^l$ and $(3-1)^l$.

Next we show $\sharp I(S_{n,l,\tau_1}) = 2^{n-1}$ for any $l \geq 0$. Let $A_0 := \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l+1} = 0\}$ and $A_1 := \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l+1} = 1\}$ so that

$$I(S_{n,l,\tau_1}) = (I(S_{n,l,\tau_1}) \cap A_0) \sqcup (I(S_{n,l,\tau_1}) \cap A_1).$$

Since $\tau_1(t_{2l+1}) = -t_{2l+1}$ and t_{2l+1} commutes with all t_i , we have $\sharp(I(S_{n,l,\tau_1}) \cap A_1) = \sharp I(S_{n-1,l,*})$ and $\sharp(I(S_{n,l,\tau_1}) \cap A_0) = \sharp I(S_{n-1,l,*})^-$. Thus, by (0), we get

$$\sharp I(S_{n,l,\tau_1}) = \sharp I(S_{n-1,l,*}) + \sharp I(S_{n-1,l,*})^- = 2^{n-1}.$$

Recall that τ_2 is defined only for $l \geq 1$, and so we can consider a partition of $\{0,1\}^n$ by the following four subsets B_k , $k = 1, 2, 3, 4$, namely,

$$\begin{aligned} B_1 &:= \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l-1} = \epsilon_{2l} = 0\}, & B_2 &:= \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l-1} = 1, \epsilon_{2l} = 0\}, \\ B_3 &:= \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l-1} = 0, \epsilon_{2l} = 1\}, & B_4 &:= \{\epsilon \in \{0,1\}^n \mid \epsilon_{2l-1} = \epsilon_{2l} = 1\}, \end{aligned}$$

so that

$$I(S_{n,l,\tau_2}) = \bigsqcup_{k=1}^4 (I(S_{n,l,\tau_2}) \cap B_k).$$

Since $\tau_2(t_{2l-1}) = -t_{2l-1}$, $\tau_2(t_{2l}) = -t_{2l}$ and $\tau_2(t_{2l-1}t_{2l}) = -t_{2l-1}t_{2l}$, and since t_{2l-1} , t_{2l} and $t_{2l-1}t_{2l}$ commute with all t_i for $i \neq 2l-1, 2l$, we have $\sharp(I(S_{n,l,\tau_2}) \cap B_1) = \sharp I(S_{n-2,l-1,*})$ and $\sharp(I(S_{n,l,\tau_2}) \cap B_k) = \sharp I(S_{n-2,l-1,*})^-$ for $k = 2, 3, 4$. Thus we get

$$\begin{aligned} \sharp I(S_{n,l,\tau_2}) &= \sharp I(S_{n-2,l-1,*}) + 3\sharp I(S_{n-2,l-1,*})^- \\ &= 2^{n-2} + 2\sharp I(S_{n-2,l-1,*})^- \quad \text{by (0)} \\ &= 2^{n-2} + 2(2^{n-2} - (2^{(n-2)-1} + 2^{(n-2)-(l-1)-1})) \quad \text{by (0) and (1)} \\ &= 2^{n-1} - 2^{n-l-1}. \quad \square \end{aligned}$$

Thus, by the inequalities of 3.2.15, any two of three tori $H(F_{\mathfrak{h}_l^{(n)}, *})$, $H(F_{\mathfrak{h}_l^{(n)}, \tau_1})$ for $l \geq 1$ or $H(F_{\mathfrak{h}_l^{(n)}, \tau_2})$ for $l \geq 2$ are not isomorphic as Jordan algebras. Moreover, by the index formulas of 3.2.15, l is an invariant for $H(F_{\mathfrak{h}_l^{(n)}, *})$ and $H(F_{\mathfrak{h}_l^{(n)}, \tau_2})$. To show that l is an invariant for $H(F_{\mathfrak{h}_l^{(n)}, \tau_1})$, we describe the centres of the three tori.

Proposition 3.2.17. *Let $F_{h_l^{(n)}} = F_{h_l^{(n)}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and let $A := H(F_{h_l^{(n)}}, *)$, $B := H(F_{h_l^{(n)}}, \tau_1)$ for $l \geq 1$, and $C := H(F_{h_l^{(n)}}, \tau_2)$ for $l \geq 2$. Then for the centres $Z(A)$, $Z(B)$ and $Z(C)$, we have*

$$\begin{aligned} Z(A) = Z(C) &= F[t_1^{\pm 2}, \dots, t_{2l}^{\pm 2}, t_{2l+1}^{\pm 1}, \dots, t_n^{\pm 1}] \quad \text{and} \\ Z(B) &= F[t_1^{\pm 2}, \dots, t_{2l+1}^{\pm 2}, t_{2l+2}^{\pm 1}, \dots, t_n^{\pm 1}]. \end{aligned}$$

Proof. The inclusions

$$\begin{aligned} Z(A), Z(C) &\subset F[t_1^{\pm 2}, \dots, t_{2l}^{\pm 2}, t_{2l+1}^{\pm 1}, \dots, t_n^{\pm 1}] \quad \text{and} \\ Z(B) &\subset F[t_1^{\pm 2}, \dots, t_{2l+1}^{\pm 2}, t_{2l+2}^{\pm 1}, \dots, t_n^{\pm 1}]. \end{aligned}$$

are clear. We show the other inclusion of each case. For $(\epsilon_1, \dots, \epsilon_{2l}, \epsilon) \in \{0, 1\}^{2l+1}$ let $x := t_1^{\epsilon_1} \cdots t_{2l}^{\epsilon_{2l}} \in F_{h_l^{(n)}}$. It is enough to show that $x \notin Z(A)$ or $Z(C)$ if $x \neq 1$ and $xt_{2l+1}^{\epsilon} \notin Z(B)$ if $xt_{2l+1}^{\epsilon} \neq 1$.

If $1 \neq x \in A$, then by 3.2.6, there exists k such that $xt_k = -t_k x$, and so $[x, t_k] = 2xt_k$. Hence by 1.1.10a, we have

$$(x, t_k, t_k)^\circ = [t_k, [x, t_k]] = 2[t_k, xt_k] = 2(t_k xt_k - xt_k^2) = 4t_k xt_k \neq 0.$$

Therefore, we get $x \notin Z(A)$ since $t_k \in A$.

For $x \in C$, whenever $\epsilon_j \neq 0$ for some $1 \leq j \leq 2l - 2$, we have, by 3.2.6 and the above, $(x, t_k, t_k)^\circ \neq 0$ for some $1 \leq k \leq 2l - 2$. Hence $x \notin Z(C)$ since $t_k \in C$. The remaining case is $x = t_{2l-1}^{\epsilon_{2l-1}} t_{2l}^{\epsilon_{2l}}$. However, we have $x \notin C$, and so of course $x \notin Z(C)$.

Similarly, whenever $\epsilon_j \neq 0$ for some $1 \leq j \leq 2l$, we have $xt_{2l+1}^{\epsilon} \notin Z(B)$. The remaining case is $t_{2l+1}^{\epsilon} \neq 1$, but $t_{2l+1}^{\epsilon} \notin B$, and so of course $t_{2l+1}^{\epsilon} \notin Z(B)$. \square

Theorem 3.2.18. *Let $J = H(F_\epsilon, *)$ be a Hermitian torus. Then:*

(i) *J is graded isomorphic to one of the following three tori*

$$H(F_{h_l^{(n)}}, *), \quad H(F_{h_l^{(n)}}, \tau_1) \text{ for } l \geq 1, \quad \text{or} \quad H(F_{h_l^{(n)}}, \tau_2) \text{ for } l \geq 2.$$

and any two of these three tori are not isomorphic as algebras and l is an isomorphism invariant,

(ii) *for the centres $Z = Z(J)$ of J and $Z(F_\epsilon)$ of F_ϵ , we have*

$$Z = H(Z(F_\epsilon), *).$$

and for the central grading groups $\Gamma = \Gamma(J)$ of J and $\Gamma(F_\epsilon)$ of F_ϵ . we have

$$\Gamma = \text{supp}(J) \cap \Gamma(F_\epsilon).$$

Proof. We already established (i) except the invariant of l for the $H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)$. But this is now clear from 3.2.17 and 2.4.3(iii).

(ii): We can identify Z with the centre of one of the three tori and identify $Z(F_\epsilon)$ with $Z(F_{\mathfrak{h}_l^{(n)}})$. Then $Z = H(Z(F_\epsilon), *)$ follows from 3.2.17 and 3.2.6. Also, for the case $J = H(F_{\mathfrak{h}_l^{(n)}}, *)$ or $H(F_{\mathfrak{h}_l^{(n)}}, \tau_2)$, we have $\Gamma = \Gamma(F_{\mathfrak{h}_l^{(n)}})$, and so we are done. For the case $J = H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)$, there exists a basis $\langle \epsilon_1, \dots, \epsilon_n \rangle$ of \mathbb{Z}^n such that

$$\begin{aligned} \Gamma &= 2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_{m+1} + \mathbb{Z}\epsilon_{m+2} + \dots + \mathbb{Z}\epsilon_n \subset \\ \Gamma(F_{\mathfrak{h}_l^{(n)}}) &= 2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n. \end{aligned}$$

Since $2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_{m-1} + k\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n \notin \text{supp}(J)$ for odd $k \in \mathbb{Z}$, we get $\Gamma = \text{supp}(J) \cap \Gamma(F_\epsilon)$. \square

Remark 3.2.19. By 3.2.16 and 3.2.17, we have

$$\begin{aligned} \dim_{\overline{\mathbb{Z}}} \overline{H(F_{\mathfrak{h}_l^{(n)}}, *)} &= (2^{n-1} + 2^{n-l-1})/2^{n-2l} = 2^{2l-1} + 2^{l-1} = 2^l(2^l + 1)/2, \\ \dim_{\overline{\mathbb{Z}}} \overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)} &= 2^{n-1}/2^{n-2l-1} = 2^{2l} = (2^l)^2, \\ \dim_{\overline{\mathbb{Z}}} \overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_2)} &= (2^{n-1} - 2^{n-l-1})/2^{n-2l} = 2^{2l-1} - 2^{l-1} = 2^l(2^l - 1)/2. \end{aligned}$$

Note that $\overline{F}_{\mathfrak{h}_l^{(n)}}$ is a central associative division algebra of dimension 4^l , and so its degree is 2^l . We mentioned that the natural extensions $\overline{*}$ and $\overline{\tau_2}$ of $*$ and τ_2 to $\overline{F}_{\mathfrak{h}_l^{(n)}}$ are involutions of first kind, while the $\overline{\tau_1}$ is of second kind in 3.2.10. Since $Z(H(F_{\mathfrak{h}_l^{(n)}}, *)) = Z(H(F_{\mathfrak{h}_l^{(n)}}, \tau_2)) = Z(F_{\mathfrak{h}_l^{(n)}})$ (see 3.2.17 and 3.2.6), it is clear that $\overline{H(F_{\mathfrak{h}_l^{(n)}}, *)} = H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{*})$ and $\overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_2)} = H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{\tau_2})$. We claim that $\overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)} = H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{\tau_1})$. In fact, $\overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)} \subset H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{\tau_1})$ is clear. Let $Z = Z(H(F_{\mathfrak{h}_l^{(n)}}, \tau_1))$. Note that by 3.2.17 and 3.2.6, we have $Z = H(Z(F_{\mathfrak{h}_l^{(n)}}), \tau_1)$. Let $\frac{1}{z} \otimes x \in H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{\tau_1})$. Then we have $\frac{1}{\tau_1(z)} \otimes \tau_1(x) = \frac{1}{z} \otimes x$ and $\tau_1(z)x = z\tau_1(x)$, and so $\tau_1(z)x \in H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)$. Since $\frac{1}{z} \otimes x = \frac{1}{z\tau_1(z)} \otimes \tau_1(z)x$ and $\tau_1(z\tau_1(z)) = z\tau_1(z)$, we get $\frac{1}{z} \otimes x \in \overline{H(Z(F_{\mathfrak{h}_l^{(n)}}), \tau_1)} \otimes_Z H(F_{\mathfrak{h}_l^{(n)}}, \tau_1) = \overline{Z} \otimes_Z H(F_{\mathfrak{h}_l^{(n)}}, \tau_1) = \overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)}$. Hence $\overline{H(F_{\mathfrak{h}_l^{(n)}}, \tau_1)} \supset H(\overline{F}_{\mathfrak{h}_l^{(n)}}, \overline{\tau_1})$.

Recall that there are two types of involutions of first kind for a finite dimensional central simple associative algebra of degree r . namely, *orthogonal type* and *symplectic type*. The dimension of the set of fixed points of an involution of orthogonal type or of symplectic type is $r(r+1)/2$ or $r(r-1)/2$, respectively (see [15]). Thus the extensions of $*$ and τ_2 to $\overline{F}_{\mathbf{h}_l^{(n)}}$ are orthogonal type and symplectic type, respectively.

In preparation for §3.3, we consider a Jordan torus $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ satisfying the following property: for all $\alpha, \beta \in \text{supp}(J)$,

$$(h) \quad J_\alpha J_\beta = \begin{cases} \{0\} & \text{if } \overline{\mathbf{0}} \neq \overline{\alpha}, \overline{\beta} \in \mathbb{Z}^n/\Gamma \text{ and } \overline{\alpha} \neq \overline{\beta} \\ J_{\alpha+\beta} & \text{otherwise} \end{cases}$$

where Γ is the central grading group of J . Note that the part 'otherwise' states $\overline{\alpha} = \overline{\mathbf{0}}$, $\overline{\beta} = \overline{\mathbf{0}}$ or $\overline{\alpha} = \overline{\beta}$, and any Jordan torus satisfies this statement. In fact, this is clear for $\overline{\alpha} = \overline{\mathbf{0}}$ or $\overline{\beta} = \overline{\mathbf{0}}$. If $\overline{\alpha} = \overline{\beta}$, then there exists $\gamma \in \Gamma$ such that $\beta = \alpha + \gamma$, and so $J_\beta = J_{\alpha+\gamma} = J_\alpha J_\gamma$. Hence, $J_\alpha J_\beta = J_\alpha (J_\alpha J_\gamma) = J_{2\alpha} J_\gamma = J_{2\alpha+\gamma} = J_{\alpha+\beta}$ since $\gamma \in \Gamma$.

We determine Hermitian tori satisfying (h). Recall that Hermitian tori are either F_q^+ , $H(F_\epsilon, *)$ or $H(E_\xi, \sigma)$ (see Theorem 3.1.10). For $F_q^+ = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^+$, if some $q_{ij} \neq \pm 1$, then $t_i, t_j \notin Z(F_q) = Z(F_q^+)$ and $t_i \circ t_j = t_i t_j + t_j t_i = (1 + q_{ij})t_i t_j \neq 0$. Hence to have the property (h), q has to be an elementary quantum matrix ϵ . Also, since $Z(H(E_\xi, \sigma)) = Z(E_\xi) \cap H(E_\xi, \sigma)$, $\xi = \epsilon$ for the same reason. Therefore, such tori are either F_ϵ^+ , $H(F_\epsilon, *)$ or $H(E_\epsilon, \sigma)$. Thus, by 3.2.3, 3.2.7a and 3.2.17, it is enough to consider $F_{\mathbf{h}_l^{(n)}}^+$, $H(F_{\mathbf{h}_l^{(n)}}, *)$, $H(E_{\mathbf{h}_l^{(n)}}, \sigma)$, $H(F_{\mathbf{h}_l^{(n)}}, \tau_1)$ for $l \geq 1$, and $H(F_{\mathbf{h}_l^{(n)}}, \tau_2)$ for $l \geq 2$. If $l \geq 2$, then $t_2 t_3 = t_3 t_2$ and $t_2 \cdot t_3 = t_2 t_3 \neq 0$. Hence $F_{\mathbf{h}_l^{(n)}}^+$ for $l \geq 2$ does not satisfy (h). By the same argument, $H(F_{\mathbf{h}_l^{(n)}}, *)$, $H(E_{\mathbf{h}_l^{(n)}}, \sigma)$, $H(F_{\mathbf{h}_l^{(n)}}, \tau_1)$ for $l \geq 2$, and $H(F_{\mathbf{h}_l^{(n)}}, \tau_2)$ for $l \geq 3$ do not satisfy (h) since t_2 and t_3 are fixed by the involutions $*$, σ , τ_1 and τ_2 .

We denoted $\mathbf{h}_1^{(n)}$ by \mathbf{h} in 2.6.10, and will use the same abbreviation unless it creates confusion. We claim that the rest of cases, i.e., $F_{\mathbf{h}}^+$, $H(F_{\mathbf{h}}, *)$, $H(E_{\mathbf{h}}, \sigma)$, $H(F_{\mathbf{h}}, \tau_1)$, $H(F_{\mathbf{h}_2^{(n)}}, \tau_2)$ and $F_{\mathbf{h}_0^{(n)}}^+ = H(F_{\mathbf{h}_0^{(n)}}, *) = H(E_{\mathbf{h}_0^{(n)}}, \sigma) = F_1^+ = F_1$, the algebra of Laurent polynomials, do satisfy (h).

The first three clearly satisfy (h). For $H(F_{\mathbf{h}}, \tau_1)$, it is enough to show that any Jordan product of two among $t_1, t_2, t_1 t_2 t_3$ is zero, and this is clear. For $H(F_{\mathbf{h}_2^{(n)}}, \tau_2)$, it

is enough to show that any Jordan product of two among $t_1, t_2, t_1t_2t_3, t_1t_2t_4, t_1t_2t_3t_4$ is zero, and this is easily checked.

For the last case, the algebra of Laurent polynomials F_1 , since $\Gamma = \mathbb{Z}^n$ in this case, the statement (†) is empty, and hence F_1 does satisfy (†).

Finally, by 3.2.13a and b, we have $(F_h, \tau_1) \cong_{\mathbb{Z}^n} (F_{\mathbf{m}^{(3)}}, *)$ and $(F_h, \tau_2) \cong_{\mathbb{Z}^n} (F_{\mathbf{m}^{(4)}}, *)$ where $\mathbf{m}^{(3)} := \mathbf{m}^{(3)} \times \mathbf{1}_{n-3}$ and $\mathbf{m}^{(4)} := \mathbf{m}^{(4)} \times \mathbf{1}_{n-4}$. Thus we obtain the following proposition which will be important in §3.3.

Proposition 3.2.20. *Hermitian tori satisfying (†) are graded isomorphic to one of the following six tori*

$$F_h^+, H(F_h, *), H(E_h, \sigma), H(F_{\mathbf{m}^{(3)}}, *), H(F_{\mathbf{m}^{(4)}}, *) \text{ or } F_1. \quad \square$$

Remark 3.2.21. The central grading groups of the first three tori in 3.2.20 are $2\mathbb{Z}^2 \oplus \mathbb{Z}^{n-2}$ and the rest are, in order, $2\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$, $2\mathbb{Z}^4 \oplus \mathbb{Z}^{n-4}$ and \mathbb{Z}^n .

§ 3.3 CLIFFORD TYPE

We review the notion of a Jordan algebra determined by a symmetric bilinear form. Let Φ be a commutative associative unital ring containing $\frac{1}{2}$, V a Φ -module, possibly with $V = (0)$, and f a symmetric bilinear form on V . One can define a multiplication on $\mathcal{J} := \Phi \oplus V$ as follows:

$$(3.3.1) \quad (a \oplus v)(b \oplus w) = (ab + f(v, w)) \oplus (av + bw)$$

for $a, b \in \Phi$ and $v, w \in V$. Then one can easily check that \mathcal{J} is a Jordan algebra over Φ with identity element $1 = 1 \oplus 0$, called a *Jordan algebra determined by f* . For $x = a + v \in \mathcal{J} = \Phi \oplus V$, we have $x^2 = (a^2 + f(v, v)) \oplus 2av$. Define two maps $\text{tr}, n : \mathcal{J} \rightarrow \Phi$ by

$$\text{tr}(a + v) = 2a$$

which is linear and

$$n(a + v) = a^2 - f(v, v)$$

which is quadratic. Then $a^2 + f(v, v) + 2av - 2a(a + v) + a^2 - f(v, v) = 0$, and so \mathcal{J} has the property

$$x^2 - \text{tr}(x)x + n(x)1 = 0$$

for all $x \in \mathcal{J}$, i.e., \mathcal{J} has degree ≤ 2 , as defined in 2.6.1. We call tr the trace and n the norm of \mathcal{J} . We define the bilinear form of n by

$$n(x, y) = n(x + y) - n(x) - n(y) \quad \text{for } x, y \in \mathcal{J}.$$

For $x = a + v \in \mathcal{J} = \Phi \oplus V$, $n(1, x) = n(1 + x) - n(1) - n(x) = (1 + a)^2 - f(v, v) - 1 - a^2 + f(v, v) = 2a = \text{tr}(x)$ and $n(v, w) = n(v + w) - n(v) - n(w) = -f(v + w, v + w) + f(v, v) + f(w, w) = -2f(v, w)$ for $v, w \in V$. Thus we get

$$\text{tr}(x) = n(1, x) \quad \text{for all } x \in \mathcal{J},$$

$$V = \{x \in \mathcal{J} \mid \text{tr}(x) = 0\},$$

$$\text{and } n(\cdot, \cdot) |_{V \times V} = -2f.$$

The invertibility of $v \in V$ can be characterized:

$$(3.3.2) \quad \begin{aligned} f(v, v) \text{ is invertible in } \Phi &\iff v^2 = f(v, v)1 \text{ is invertible in } \mathcal{J} \\ &\iff v \text{ is invertible in } \mathcal{J} \end{aligned}$$

(see e.g. [13, Theorem 13(6), page 52]).

Some properties of the centre $Z = Z(\mathcal{J})$ are available; for $u \in V$:

$$(3.3.3) \quad \begin{aligned} &\text{if there exists } 0 \neq v, w \in V \text{ such that} \\ &f(u, v)w \neq 0 \text{ and } f(u, w) = 0, \text{ then } u \notin Z. \end{aligned}$$

Indeed, this follows from $(v \cdot u) \cdot w = f(u, v)w \neq 0$ and $v \cdot (u \cdot w) = f(u, w)u = 0$.

Also, if $V = \Phi$, one can easily check that the Jordan algebra $\mathcal{J} = \Phi \oplus \Phi$ is associative, i.e., $Z(\mathcal{J}) = \mathcal{J}$.

Recall the notion of a Jordan torus of Clifford type in §3.0: the central closure is a Jordan algebra determined by a symmetric bilinear form. We call a Jordan torus of Clifford type a *Clifford torus*. We will classify the more general division \mathbb{Z}^n -graded Jordan algebras of Clifford type and use this to obtain the classification of Clifford tori.

Construction 3.3.4. Let $0 \leq m \leq n$. If $m = 0$, let $S^{(m)} := \{0\}$. If $m \geq 1$, let $S^{(m)}$ be a semilattice in \mathbb{Z}^m and $\{\sigma_1, \dots, \sigma_m\}$ a basis of \mathbb{Z}^m such that each $\sigma_i \in S^{(m)}$ (see 2.4.1(iii)). Note that every element of $S^{(m)}$ for $m \geq 1$ can be uniquely written as

$$2(\alpha_1\sigma_1 + \dots + \alpha_m\sigma_m) + \epsilon_1\sigma_1 + \dots + \epsilon_m\sigma_m$$

for $\alpha_i \in \mathbb{Z}$ and $\epsilon_i = 0$ or 1 , $i = 1, \dots, m$. We put

$$I := \begin{cases} \{\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0, 1\}^m \mid \epsilon_1 \sigma_1 + \dots + \epsilon_m \sigma_m \in S^{(m)}\} & \text{if } m \geq 1 \\ \{\mathbf{0}\} & \text{if } m = 0. \end{cases}$$

Note that $S^{(m)} = \sqcup_{\epsilon \in I} (\epsilon_1 \sigma_1 + \dots + \epsilon_m \sigma_m + 2\mathbb{Z}^m)$, and so $\sharp I$ is the index of the semilattice $S^{(m)}$ as defined in §3.2.

Let $\mathbb{Z}^n := \mathbb{Z}^m \oplus \mathbb{Z}^{n-m}$ and extend $\{\sigma_1, \dots, \sigma_m\}$ to a basis

$$\{\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_n\}$$

of \mathbb{Z}^n . Let F be a field of characteristic $\neq 2$, K a field extension of F ,

$$\begin{aligned} \Gamma &:= 2\mathbb{Z}\sigma_1 + \dots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \dots + \mathbb{Z}\sigma_n \quad \text{and} \\ \mathcal{Z} &:= K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma} Kz_\alpha \end{aligned}$$

the algebra of Laurent polynomials in n -variables over K , where $z_\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\sigma_1 + \dots + 2\alpha_m\sigma_m + \alpha_{m+1}\sigma_{m+1} + \dots + \alpha_n\sigma_n$. We consider \mathcal{Z} as an Γ -graded algebra with homogeneous space Kz_α .

For each $\epsilon \in I$,

- (1) let (V_ϵ, g_ϵ) be a pair consisting of a vector space V_ϵ over K and a symmetric *anisotropic bilinear form* g_ϵ of V_ϵ , i.e., $g_\epsilon(v, v) \neq 0$ for all $0 \neq v \in V_\epsilon$,
- (2) assume that $V_\epsilon \neq (0)$ if $\epsilon \neq \mathbf{0}$,
- (3) assume that $K \oplus V_0$ is a Jordan division algebra over K determined by g_0 .

We denote the family of pairs of K -vector spaces V_ϵ and symmetric anisotropic bilinear forms g_ϵ indexed by $\epsilon \in I$ by \mathcal{V}_I :

$$\mathcal{V}_I = \{(V_\epsilon, g_\epsilon)\}_{\epsilon \in I}.$$

For each $\epsilon \in I$, we extend g_ϵ to $\mathcal{Z} \otimes_K V_\epsilon$ as a \mathcal{Z} -bilinear map, denoted by f_ϵ , as follows:

$$f_\epsilon(z \otimes v, z' \otimes v') = zz' g_\epsilon(v, v') z^\epsilon$$

for $z, z' \in \mathcal{Z}$ and $v, v' \in V_\epsilon$ where

$$z^\epsilon = z_1^{\epsilon_1} \dots z_m^{\epsilon_m} \quad \text{for } \epsilon = (\epsilon_1, \dots, \epsilon_m).$$

Obviously f_ϵ is symmetric and the image is $\mathcal{Z}z^\epsilon$ for each $\epsilon \in I$.

Remark. It is easily seen that f_ϵ and f below are anisotropic. A Jordan theoretic reason will be given below in Remark 3.3.5.

We claim that the Jordan algebra

$$\mathcal{J} := \mathcal{Z} \oplus \left(\bigoplus_{\epsilon \in I} (\mathcal{Z} \otimes_K V_\epsilon) \right)$$

over \mathcal{Z} determined by the symmetric bilinear form

$$f := \bigoplus_{\epsilon \in I} f_\epsilon \quad \text{on} \quad V := \bigoplus_{\epsilon \in I} (\mathcal{Z} \otimes_K V_\epsilon)$$

is a division \mathbb{Z}^n -graded Jordan algebra over F . First we put

$$V_\epsilon := \{0\} \quad \text{and} \quad g_\epsilon = 0 \quad \text{for} \quad \epsilon \in \{0, 1\}^m \setminus I.$$

For $\alpha = \alpha_1 \sigma_1 + \cdots + \alpha_n \sigma_n \in \mathbb{Z}^n$, there exist unique $(\alpha'_1, \dots, \alpha'_m) \in \mathbb{Z}^m$ and $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0, 1\}^m$ such that

$$\alpha = 2(\alpha'_1 \sigma_1 + \cdots + \alpha'_m \sigma_m) + \epsilon_1 \sigma_1 + \cdots + \epsilon_m \sigma_m + \alpha_{m+1} \sigma_{m+1} + \cdots + \alpha_n \sigma_n.$$

We put

$$\alpha' := 2\alpha'_1 \sigma_1 + \cdots + 2\alpha'_m \sigma_m + \alpha_{m+1} \sigma_{m+1} + \cdots + \alpha_n \sigma_n \in \Gamma.$$

Since $\mathbb{Z}^n \ni \alpha \mapsto (\alpha', \epsilon) \in \Gamma \times \{0, 1\}^m$ is bijective, we write $\alpha = (\alpha', \epsilon)$.

Now we set

$$\mathcal{J}_\alpha = \mathcal{J}_{(\alpha', \epsilon)} := \begin{cases} Kz_{\alpha'} \otimes_K (K \oplus V_0) & \text{if } \epsilon = 0, \\ Kz_{\alpha'} \otimes_K V_\epsilon & \text{otherwise.} \end{cases}$$

Since $\mathcal{Z} \otimes_K V_\epsilon = \bigoplus_{\alpha' \in \Gamma} (Kz_{\alpha'} \otimes_K V_\epsilon)$ and

$$\mathcal{Z} \oplus (\mathcal{Z} \otimes_K V_0) = \mathcal{Z} \otimes_K (K \oplus V_0) = \bigoplus_{\alpha' \in \Gamma} (Kz_{\alpha'} \otimes_K (K \oplus V_0))$$

(identifying \mathcal{Z} with $\mathcal{Z} \otimes_K K$), we get

$$\begin{aligned} \mathcal{J} &= \mathcal{Z} \oplus \left(\bigoplus_{\epsilon \in I} (\mathcal{Z} \otimes_K V_\epsilon) \right) = (\mathcal{Z} \oplus (\mathcal{Z} \otimes_K V_0)) \oplus \left(\bigoplus_{\epsilon \in I \setminus \{0\}} \mathcal{Z} \otimes_K V_\epsilon \right) \\ &= \left(\bigoplus_{\alpha' \in \Gamma} (Kz_{\alpha'} \otimes_K (K \oplus V_0)) \right) \oplus \left(\bigoplus_{(\alpha', \epsilon) \in \Gamma \times (I \setminus \{0\})} (Kz_{\alpha'} \otimes_K V_\epsilon) \right) \\ &= \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{J}_\alpha \end{aligned}$$

as K -vector spaces.

For $\alpha, \beta \in \mathbb{Z}^n$, if $\alpha = (\alpha', \mathbf{0})$ and $\beta = (\beta', \mathbf{0})$, then for $a, b \in K$ and $x, y \in V_0$,

$$(z_{\alpha'} \otimes (a+x))(z_{\beta'} \otimes (b+y)) = z_{\alpha'+\beta'} \otimes (ab + g_0(x, y) + bx + ay) \in \mathcal{J}_{\alpha+\beta}.$$

If $\alpha = (\alpha', \mathbf{0})$ and $\beta = (\beta', \eta)$, $\eta \neq \mathbf{0}$, then for $a \in K$, $x \in V_0$ and $y \in V_\eta$,

$$(z_{\alpha'} \otimes (a+x))(z_{\beta'} \otimes y) = az_{\alpha'+\beta'} \otimes y \in \mathcal{J}_{\alpha+\beta}.$$

Finally, if $\alpha = (\alpha', \epsilon)$, $\epsilon \neq \mathbf{0}$ and $\beta = (\beta', \eta)$, $\eta \neq \mathbf{0}$, then for $x \in V_\epsilon$ and $y \in V_\eta$,

$$(z_{\alpha'} \otimes x)(z_{\beta'} \otimes y) = \begin{cases} g_\epsilon(x, y)z_{\alpha'+\beta'}z^\epsilon \otimes 1 & \text{if } \epsilon = \eta \\ 0 & \text{otherwise.} \end{cases}$$

which is in $\mathcal{J}_{\alpha+\beta}$. Therefore, we obtain $\mathcal{J}_\alpha \mathcal{J}_\beta \subset \mathcal{J}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$. Since $\text{supp}(\mathcal{J})$ contains the basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n , \mathcal{J} is a \mathbb{Z}^n -graded algebra over K .

Any $0 \neq az_{\alpha'} \otimes u \in \mathcal{J}_{(\alpha', \mathbf{0})}$ for $a \in K$ and $u \in K \oplus V_0$ has the inverse $a^{-1}z_{-\alpha'} \otimes u^{-1}$. Also, for any $0 \neq az_{\alpha'} \otimes x \in \mathcal{J}_{(\alpha', \epsilon)}$, $\epsilon \neq \mathbf{0}$, $a \in K$ and $x \in V_\epsilon$, we have $(az_{\alpha'} \otimes x)^2 = a^2g_\epsilon(x, x)z_{2\alpha'}z^\epsilon \otimes 1$, which is invertible since g_ϵ is anisotropic. Hence by 3.3.2, $az_{\alpha'} \otimes x$ is invertible. Thus we have shown that $\mathcal{J} = \mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ is a division \mathbb{Z}^n -graded Jordan algebra over K , and hence over F . Note that

$$\text{supp}(\mathcal{J}) = S^{(m)} + \mathbb{Z}^{n-m},$$

which is a semilattice in \mathbb{Z}^n . We call the \mathbb{Z}^n -grading of \mathcal{J} a *toral grading determined by* $\{\sigma_1, \dots, \sigma_n\}$.

Remark 3.3.5. We note that f is anisotropic. Indeed, the division \mathbb{Z}^n -graded Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus V$ is a Jordan domain (see 2.2.4). So for $0 \neq x \in V$, we have $0 \neq x^2 = f(x, x)$.

Let $Z(\mathcal{J})$ be the centre of \mathcal{J} and

$$s := \sum_{\epsilon \in \{0,1\}^m} \dim_K V_\epsilon,$$

with the usual convention that $s = \infty$ if one of the V_ϵ is ∞ -dimensional.

If $s > 1$, then there are three cases:

(1) for any $\epsilon \in I$, there exists $\epsilon \neq \eta \in I$ such that $V_\eta \neq (0)$, i.e., at least two of the V_ϵ 's are nonzero.

(2) $m = 1$, $V_0 = (0)$ and $\dim_K V_1 > 1$ where $\mathcal{V}_I = \{(V_0, g_0), (V_1, g_1)\}$,

(3) $m = 0$ and $\dim_K V_0 > 1$ where $\mathcal{V}_I = \{(V_0, g_0)\}$.

We show $Z(\mathcal{J}) = \mathcal{Z}$ for these three cases.

(1): For any $0 \neq x \in V_\epsilon$ and $0 \neq z \in \mathcal{Z}$, we have $f(z \otimes x, z \otimes x) = g_\epsilon(x, x)z^2z^\epsilon \neq 0$ and for $0 \neq y \in V_\eta$, $f(z \otimes x, 1 \otimes y) = 0$ since $\eta \neq \epsilon$. Hence by 3.3.3, $z \otimes x \notin Z(\mathcal{J})$. Therefore, $J_\alpha \cap Z(\mathcal{J}) = (0)$ for $\alpha \notin \Gamma$ and $(\mathcal{Z} \otimes_K V_0) \cap Z(\mathcal{J}) = (0)$. Since $Z(\mathcal{J})$ is homogeneous, we get $Z(\mathcal{J}) = \mathcal{Z}$.

(2): For any $0 \neq x \in V_1$, there exists $0 \neq y \in V_1$ such that $g_1(x, y) = 0$, and so $f(z \otimes x, 1 \otimes y) = 0$ for $z \in \mathcal{Z}$. Since $f(z \otimes x, z \otimes x) \neq 0$ for $z \neq 0$, we get by 3.3.3, $z \otimes x \notin Z(\mathcal{J})$ and hence $Z(\mathcal{J}) = \mathcal{Z}$.

(3): One can prove this in the same way as (2).

If $s \leq 1$, then there are also three cases:

(i) $m = 1$, $V_0 = (0)$ and $\dim_K V_1 = 1$,

(ii) $m = 0$ and $\dim_K V_0 = 1$

(iii) $m = 0$ and $V_0 = (0)$.

We have $Z(\mathcal{J}) = \mathcal{J}$ for these three cases: In fact, $\mathcal{J} = \mathcal{Z} \oplus (\mathcal{Z} \otimes_K V_1)$ for (i) and $\mathcal{J} = \mathcal{Z} \oplus (\mathcal{Z} \otimes_K V_0)$ for (ii) are both isomorphic to $\mathcal{Z} \oplus \mathcal{Z}$ as \mathcal{Z} -modules. Thus both are associative, and so $Z(\mathcal{J}) = \mathcal{J}$. For (iii), we have $\mathcal{J} = \mathcal{Z}$, and so $Z(\mathcal{J}) = \mathcal{J}$.

Consequently, we have shown

$$Z(\mathcal{J}) = \begin{cases} \mathcal{Z} & \text{if } s > 1 \\ \mathcal{J} & \text{otherwise.} \end{cases}$$

In particular, we have $Z(\mathcal{J}) \cap \mathcal{J}_0 = K$ if $s > 1$.

Thus if $s > 1$, the central closure $\overline{\mathcal{J}}$ of \mathcal{J} is

$$\overline{\mathcal{J}} = \overline{\mathcal{Z}} \otimes_{\mathcal{Z}} \left(\mathcal{Z} \oplus \left(\bigoplus_{\epsilon \in I} (\mathcal{Z} \otimes_K V_\epsilon) \right) \right) = \overline{\mathcal{Z}}.1 \oplus \left(\bigoplus_{\epsilon \in I} (\overline{\mathcal{Z}} \otimes_K V_\epsilon) \right)$$

and f can be naturally extended to $\bigoplus_{\epsilon \in I} (\overline{\mathcal{Z}} \otimes_K V_\epsilon)$ over $\overline{\mathcal{Z}}$, and so $\overline{\mathcal{J}}$ is a Jordan algebra over $\overline{\mathcal{Z}}$ determined by the extended bilinear form. If $s \leq 1$, then $\mathcal{J} = Z(\mathcal{J})$ is a division \mathbb{Z}^n -graded commutative associative algebra over F and $\overline{\mathcal{J}} = \overline{\mathcal{Z}}$. In any case, \mathcal{J} is of Clifford type.

Remark 3.3.6. (1) If $s = 0$, i.e., $m = 0$ and $V_0 = (0)$, then we have $\mathcal{J} = \mathcal{J}(S^{(m)}, K, \mathcal{V}_I) = \mathcal{Z} = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ as mentioned above.

(2) If $s = 1$ and $V_0 = (0)$, then

$$\mathcal{J} = K[t_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}] \supseteq \mathcal{Z} = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$$

where t_1 is any nonzero element of V_1 , and so $t_1^2 = g_1(t_1, t_1)z_1$.

(3) If $m = 0$, then $\mathcal{J}_\alpha = Kz_\alpha$ for all $\alpha \neq 0$. So we have

$$\mathcal{J} \cong K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \otimes_K \mathcal{J}_0.$$

Note that $\mathcal{J}_0 = Kz_0 \otimes_K (K \oplus V_0) \cong K \oplus V_0$ is a Jordan division algebra over K determined by g_0 .

(4) If $\dim_K V_0 = 1$, then $\mathcal{J}_0 \cong K \oplus K$ (not as algebras) is a quadratic field extension of K . So if $s > 1$, then

$$K = Z(\mathcal{J}) \cap \mathcal{J}_0 \subseteq Z(\mathcal{J}_0) = \mathcal{J}_0.$$

Moreover, if $m = 0$, we have

$$\mathcal{J} \cong \mathcal{J}_0[z_1^{\pm 1}, \dots, z_n^{\pm 1}],$$

the algebra of Laurent polynomials over the field \mathcal{J}_0 .

We now start the classification. Let $\mathcal{J} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{J}_\alpha$ be a division \mathbb{Z}^n -graded Jordan algebra over F of Clifford type, i.e., the central closure $\overline{\mathcal{J}}$ is a Jordan algebra over $\overline{\mathcal{Z}}$ determined by a symmetric bilinear form where $\mathcal{Z} = Z(\mathcal{J})$ is the centre of \mathcal{J} and $\overline{\mathcal{Z}}$ is the field of fractions of \mathcal{Z} . So we have $\dim_{\overline{\mathcal{Z}}} \overline{\mathcal{J}} = 1$ or $\overline{\mathcal{J}}$ has degree 2 over $\overline{\mathcal{Z}}$. We will show that $\mathcal{J} \cong_{\mathbb{Z}^n} \mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ as defined in 3.3.4 for both cases.

If $\dim_{\overline{\mathcal{Z}}} \overline{\mathcal{J}} = 1$, then $\mathcal{J} = \mathcal{Z}$ since \mathcal{J} embeds into $\overline{\mathcal{J}}$. Hence \mathcal{J} is isomorphic to the algebra of Laurent polynomials in n variables over the field \mathcal{J}_0 (see Remark 2.5.4(1)). We set $K := \mathcal{J}_0$. Then we have $\mathcal{J} \cong_{\mathbb{Z}^n} \mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ for $s = 0$ (see Remark 3.3.6(1)).

Suppose that $\overline{\mathcal{J}}$ has degree 2 over $\overline{\mathcal{Z}}$. Let tr and n be the generic trace and norm of $\overline{\mathcal{J}}$, and

$$W := \{x \in \overline{\mathcal{J}} \mid \text{tr}(x) = 0\}.$$

Then $\overline{\mathcal{J}} = \overline{\mathcal{Z}}1 \oplus W$ is a Jordan algebra over $\overline{\mathcal{Z}}$ determined by the symmetric bilinear form

$$h := -\frac{1}{2}n(\cdot, \cdot) |_{W \times W}.$$

Let Γ be the central grading group of \mathcal{J} . Then $\mathcal{Z} = \bigoplus_{\alpha \in \Gamma} \mathcal{Z}_\alpha$ where $\mathcal{Z}_\alpha = \mathcal{Z} \cap \mathcal{J}_\alpha$ and $\mathcal{J} = \bigoplus_{\overline{\alpha} \in \mathbb{Z}^n/\Gamma} \mathcal{J}_{\overline{\alpha}}$ is a \mathbb{Z}^n/Γ -graded algebra where $\mathcal{J}_{\overline{\alpha}} = \mathcal{Z}\mathcal{J}_\alpha$ (see 2.3.4). Let $K := \mathcal{Z}_0$ which is a field extension of F .

By 2.6.8, we have $\text{tr}(\mathcal{J}_{\overline{\alpha}}) = \{0\}$ for $\overline{\alpha} \neq \overline{0}$, and so $\mathcal{J}_{\overline{\alpha}} \subset W$. Let

$$V_0 := W \cap \mathcal{J}_0.$$

Then $\mathcal{Z}V_0 \subset W$. Since $(\overline{\mathcal{Z}} \setminus \{0\})1 = \{x \in \overline{\mathcal{J}} \mid \text{tr}(x) \neq 0\}$, we have, by 2.3.5(iii),

$$\{x \in \mathcal{J} \mid \text{tr}(x) \neq 0\} = \mathcal{J} \cap (\overline{\mathcal{Z}} \setminus \{0\})1 = (\mathcal{J} \cap \overline{\mathcal{Z}}1) \setminus \{0\} = \mathcal{Z} \setminus \{0\}.$$

Hence

$$\{x \in \mathcal{J}_0 \mid \text{tr}(x) \neq 0\} = \mathcal{Z}_0 \setminus \{0\} = K \setminus \{0\}.$$

In particular, $\text{tr}(\mathcal{J}_0) = K$ and $\text{tr}|_{\mathcal{J}_0}: \mathcal{J}_0 \rightarrow K$ is K -linear. Since $\text{tr}(1) = 2$ and $\text{ch}.K \neq 2$, we have $\mathcal{J}_0 = K \oplus V_0$ as K -vector spaces. Let

$$V := \mathcal{Z}V_0 \oplus \left(\bigoplus_{\overline{\alpha} \neq \overline{0}} \mathcal{J}_{\overline{\alpha}} \right) \subset W.$$

Since $\mathcal{J}_{\overline{0}} = \mathcal{Z}\mathcal{J}_0 = \mathcal{Z}(K \oplus V_0)$ and $\text{tr}(\mathcal{Z}V_0) = \{0\}$, we get $\mathcal{J}_{\overline{0}} = \mathcal{Z} \oplus \mathcal{Z}V_0$ and

$$\mathcal{J} = \mathcal{J}_{\overline{0}} \oplus \left(\bigoplus_{\overline{\alpha} \neq \overline{0}} \mathcal{J}_{\overline{\alpha}} \right) = \mathcal{Z} \oplus V$$

as direct sums of \mathcal{Z} -modules. Moreover, for all $x, y \in V$, we have, by 2.3.5(iii),

$$xy = h(x, y)1 \in \mathcal{J} \cap \overline{\mathcal{Z}}1 = \mathcal{Z}.$$

Therefore, $\mathcal{J} = \mathcal{Z} \oplus V$ is the Jordan algebra over \mathcal{Z} determined by

$$f := h |_{V \times V}.$$

Now, by 2.6.8, there exists a basis $\{\epsilon_1, \dots, \epsilon_n\}$ of \mathbb{Z}^n such that

$$\Gamma = 2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n.$$

Let $S := \text{supp}(\mathcal{J})$ and $\Lambda := \mathbb{Z}\epsilon_1 + \dots + \mathbb{Z}\epsilon_m$.

Claim. $S \cap \Lambda$ is a semilattice in Λ .

Proof. Since S is a semilattice in \mathbb{Z}^n (see 2.1.3) and Λ is a subgroup of \mathbb{Z}^n , we have $0, 2\alpha - \beta \in S \cap \Lambda$ for all $\alpha, \beta \in S \cap \Lambda$. We need to show that $S \cap \Lambda$ generates Λ . For any $\gamma = \gamma_1 \epsilon_1 + \cdots + \gamma_n \epsilon_n \in S$, we have $\gamma' := \gamma_{m+1} \epsilon_{m+1} + \cdots + \gamma_n \epsilon_n \in \Gamma \subset S$. So let $0 \neq x \in \mathcal{J}_\gamma$ and $0 \neq z \in \mathcal{Z}_{\gamma'}$. Then $0 \neq xz^{-1} \in \mathcal{J}_{\gamma - \gamma'}$, and hence $\gamma - \gamma' = \gamma_1 \epsilon_1 + \cdots + \gamma_m \epsilon_m \in S \cap \Lambda$. Since S generates \mathbb{Z}^n , we have for $\alpha \in \Lambda \subset \mathbb{Z}^n$,

$$\alpha = \sum_{\gamma \in U} l_\gamma \gamma = \sum_{\gamma \in U} l_\gamma (\gamma - \gamma')$$

where U is a finite subset of S and l_γ is a positive integer. Therefore, $S \cap \Lambda$ generates Λ . \square

Let

$$S^{(m)} := \begin{cases} S \cap \Lambda & \text{for } m > 0, \\ \{0\} & \text{for } m = 0. \end{cases}$$

Since $S^{(m)}$ for $m > 0$ is a semilattice in $\Lambda \subset \mathbb{Z}^n$, there exists a basis $\{\sigma_1, \dots, \sigma_m\}$ of Λ such that each $\sigma_i \in S^{(m)}$ (see 2.4.1(iii)). Let

$$I = \begin{cases} \{\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0, 1\}^m \mid \epsilon_1 \sigma_1 + \cdots + \epsilon_m \sigma_m \in S\} & \text{if } m > 0 \\ \{0\} & \text{if } m = 0. \end{cases}$$

Let $\sigma_j := \epsilon_j$ for $j = m + 1, \dots, n$. Then we have

$$\begin{aligned} \Gamma &= 2\mathbb{Z}\epsilon_1 + \cdots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\sigma_{m+1} + \cdots + \mathbb{Z}\sigma_n \\ &= 2\Lambda + \mathbb{Z}\sigma_{m+1} + \cdots + \mathbb{Z}\sigma_n \\ &= 2\mathbb{Z}\sigma_1 + \cdots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \cdots + \mathbb{Z}\sigma_n. \end{aligned}$$

For $0 \neq \epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0, 1\}^m$, let

$$V_\epsilon := \mathcal{J}_{\epsilon_1 \sigma_1 + \cdots + \epsilon_m \sigma_m}.$$

Then $V_\epsilon = (0)$ if $\epsilon \notin I$, and we get

$$V = \mathcal{Z}V_0 \oplus \left(\bigoplus_{\bar{\alpha} \neq \bar{0}} \mathcal{J}_{\bar{\alpha}} \right) = \mathcal{Z}V_0 \oplus \left(\bigoplus_{\epsilon \in I \setminus \{0\}} \mathcal{Z}V_\epsilon \right) = \bigoplus_{\epsilon \in I} \mathcal{Z}V_\epsilon$$

as direct sums of \mathcal{Z} -modules. Let $f_\epsilon := f|_{\mathcal{Z}V_\epsilon \times \mathcal{Z}V_\epsilon}$ for all $\epsilon \in I$. Note that if $\bar{\alpha} \neq \bar{\beta}$ and $\bar{\beta} \neq \bar{0}$, then $\bar{\alpha} + \bar{\beta} \neq \bar{0}$, and so

$$(\mathcal{J}_{\bar{\alpha}} \cap V)\mathcal{J}_{\bar{\beta}} \subset f(\mathcal{J}_{\bar{\alpha}} \cap V, \mathcal{J}_{\bar{\beta}})1 \subset \mathcal{J}_{\bar{\alpha}+\bar{\beta}} \cap \mathcal{Z} \subset \mathcal{J}_{\bar{\alpha}+\bar{\beta}} \cap \mathcal{J}_{\bar{0}} = (0).$$

Hence we have

$$f = \bigoplus_{\epsilon \in I} f_\epsilon.$$

Since \mathcal{Z} is the algebra of Laurent polynomials over K (see 2.5.4(1)), we identify

$$\mathcal{Z} = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma} Kz_\alpha$$

where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\sigma_1 + \cdots + 2\alpha_m\sigma_m + \alpha_{m+1}\sigma_{m+1} + \cdots + \alpha_n\sigma_n$. Then we have

$$f_\epsilon(V_\epsilon, V_\epsilon) \subset \mathcal{Z}_{2\epsilon_1\sigma_1 + \cdots + 2\epsilon_m\sigma_m}$$

for $\epsilon = (\epsilon_1, \dots, \epsilon_m)$, and so we can define a map

$$g_\epsilon : V_\epsilon \times V_\epsilon \longrightarrow K \quad \text{by} \quad g_\epsilon(v, w)z^\epsilon = f_\epsilon(v, w)$$

where

$$z^\epsilon := z_1^{\epsilon_1} \cdots z_m^{\epsilon_m}.$$

Then g_ϵ is obviously a symmetric K -bilinear form on V_ϵ . Since all nonzero elements in V_ϵ are invertible, $f_\epsilon|_{V_\epsilon}$ is anisotropic (see 3.3.2). Hence g_ϵ is anisotropic since z^ϵ is invertible.

Finally, by 2.3.4(iv), $\mathcal{J}_{\bar{\alpha}} = \mathcal{Z}\mathcal{J}_\alpha$ is a free \mathcal{Z} -module with $\text{rank}_{\mathcal{Z}} \mathcal{J}_{\bar{\alpha}} = \dim_K \mathcal{J}_\alpha$, and so one can easily show that there is a natural \mathcal{Z} -module isomorphism

$$\mathcal{Z}\mathcal{J}_\alpha \cong \mathcal{Z} \otimes_K \mathcal{J}_\alpha$$

via $zx \leftrightarrow z \otimes x$ for $z \in \mathcal{Z}$ and $x \in \mathcal{J}_\alpha$. Identifying them, we get

$$\mathcal{Z}V_\epsilon = \mathcal{Z} \otimes_K V_\epsilon \quad \text{and} \quad V = \bigoplus_{\epsilon \in I} \mathcal{Z}V_\epsilon = \bigoplus_{\epsilon \in I} (\mathcal{Z} \otimes_K V_\epsilon),$$

and $f = \bigoplus_{\epsilon \in I} f_\epsilon$ becomes exactly the same as f as defined in 3.3.4. Hence $\mathcal{J} = \mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ and $s > 1$ where s is defined in 3.3.4. Thus we have shown the following:

Theorem 3.3.7. *Any $\mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ as defined above is a division \mathbb{Z}^n -graded Jordan algebra over F of Clifford type and conversely any such algebra is graded isomorphic to some $\mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ for some toral grading. Moreover, m and K are invariants of \mathcal{J} .*

Proof. We only need to show the last statement, and it follows from 2.4.3(i) and (iii). \square

We now specialize to Clifford tori. Let J be a Clifford torus over F . If it has central degree 1, then $Z(J) = J$, and so J is a commutative associative torus, and hence $J \cong_{\mathbb{Z}^n} F[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. So we assume that J has central degree 2. By Theorem 3.3.7, we can identify J with $\mathcal{J}(S^{(m)}, K, \mathcal{V}_I) = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{J}_\alpha$. First of all, by 2.6.8(iii), we have $m \geq 2$. Secondly, $\dim_F \mathcal{J}_\alpha \leq 1$ implies $K = F$, $Z = Z(\mathcal{J}) = F[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, $V_0 = (0)$ and $\dim_F V_\epsilon = 1$ for all $\epsilon \in I \setminus \{0\}$. Let $0 \neq t_\epsilon \in V_\epsilon$ for $\epsilon \in I \setminus \{0\}$. Then $V = \bigoplus_{\epsilon \in I} (Z \otimes_F V_\epsilon)$ can be considered as the free Z -module with basis $\{t_\epsilon \mid \epsilon \in I\}$. For the convenience of the reader we repeat the construction 3.3.4 under the special conditions established for Clifford tori:

$$m \geq 2, \quad K = F, \quad V_0 = (0), \quad \dim_F V_\epsilon = 1 \text{ for all } \epsilon \in I \setminus \{0\}.$$

Construction 3.3.8. Let $2 \leq m \leq n$, $S^{(m)}$ a semilattice in \mathbb{Z}^m and $\{\sigma_1, \dots, \sigma_m\}$ a basis of \mathbb{Z}^m in $S^{(m)}$. We put

$$I = I(S^{(m)}) := \{\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{0, 1\}^m \mid \epsilon_1 \sigma_1 + \dots + \epsilon_m \sigma_m \in S^{(m)}\}.$$

Note that $S^{(m)} = \sqcup_{\epsilon \in I} (\epsilon_1 \sigma_1 + \dots + \epsilon_m \sigma_m + 2\mathbb{Z}^m)$ and that I always contains

$$0 = (0, \dots, 0), \quad \epsilon_1 := (1, 0, \dots, 0), \quad \epsilon_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad \epsilon_m := (0, \dots, 0, 1),$$

and so the index $\sharp I$ of the semilattice $S^{(m)}$ is $\geq m + 1$. Let $\mathbb{Z}^n := \mathbb{Z}^m \oplus \mathbb{Z}^{n-m}$ and extend $\{\sigma_1, \dots, \sigma_m\}$ to a basis

$$\{\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_n\}$$

of \mathbb{Z}^n . Let

$$\begin{aligned} \Gamma &:= 2\mathbb{Z}\sigma_1 + \dots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \dots + \mathbb{Z}\sigma_n \quad \text{and} \\ Z &:= F[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma} Fz_\alpha \end{aligned}$$

where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\sigma_1 + \cdots + 2\alpha_m\sigma_m + \alpha_{m+1}\sigma_{m+1} + \cdots + \alpha_n\sigma_n$.

Let V be a free Z -module with basis

$$\{t_\epsilon \mid t_\epsilon \neq 0, \epsilon \in I \setminus \{0\}\}.$$

Define a Z -bilinear form $f : V \times V \rightarrow Z$ by

$$(*) \quad f(t_\epsilon, t_\eta) = \begin{cases} a_\epsilon z^\epsilon & \text{if } \epsilon = \eta \\ 0 & \text{otherwise} \end{cases}$$

for all t_ϵ, t_η , where $0 \neq a_\epsilon \in F$ and $z^\epsilon = z_1^{\epsilon_1} \cdots z_m^{\epsilon_m}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Note that a_ϵ corresponds to $g_\epsilon(t_\epsilon, t_\epsilon) \neq 0$ in 3.3.4.

Let

$$J = J_{S(m)} := Z \oplus V$$

be the Jordan algebra over Z determined by f . We put

$$t_0 := 1$$

so that $\{t_\epsilon \mid \epsilon \in I\}$ is a Z -basis of J , i.e.,

$$J = \bigoplus_{\epsilon \in I} Z t_\epsilon$$

as a graded Z -module. Then it is clear that

$$\{z_\alpha t_\epsilon \mid \alpha \in \Gamma, \epsilon \in I\}$$

is an F -basis of J . For $\alpha \in \mathbb{Z}^n$, we write $\alpha = (\alpha', \epsilon) \in \Gamma \times \{0, 1\}^m$ as defined in 3.3.4, and put

$$t_\epsilon := 0 \quad \text{for } \epsilon \in \{0, 1\}^m \setminus I \quad \text{and} \quad t_\alpha := z_{\alpha'} t_\epsilon.$$

Then we get

$$J = \bigoplus_{(\alpha', \epsilon) \in \Gamma \times \{0, 1\}^m} F z_{\alpha'} t_\epsilon = \bigoplus_{\alpha \in \mathbb{Z}^n} F t_\alpha$$

as a graded F -vector space. By (3.3.1) and (*), we have

$$(3.3.9) \quad t_\alpha t_\beta = (z_{\alpha'} t_\epsilon)(z_{\beta'} t_\eta) = \begin{cases} a_\epsilon z_{\alpha'+\beta'} z^\epsilon = a_\epsilon t_{\alpha+\beta} & \text{if } \epsilon = \eta \neq 0 \\ z_{\alpha'+\beta'} t_\epsilon = t_{\alpha+\beta} & \text{if } \eta = 0 \\ z_{\alpha'+\beta'} t_\eta = t_{\alpha+\beta} & \text{if } \epsilon = 0 \\ 0 & \text{otherwise,} \end{cases}$$

so we obtain $t_\alpha t_\beta \subset Ft_{\alpha+\beta}$. For $\alpha = (\alpha', \epsilon) \in \Gamma \times I$, since $t_\alpha^2 = a_\epsilon z_{2\alpha'} z^\epsilon$ is invertible, t_α is invertible by 3.3.2. Since $\text{supp}(J)$ contains the basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n , $\text{supp}(J)$ generates \mathbb{Z}^n and hence $J = \bigoplus_{\alpha \in \mathbb{Z}^n} Ft_\alpha$ is a Jordan torus over F . Also, since J is generated by z_i and t_ϵ ($\epsilon \in I$), it is finitely generated as a Jordan algebra.

Let $Z(J)$ be the centre of J . We claim that $Z(J) = Z$. Since the centre $Z(J)$ is homogeneous and $\dim_F J_\alpha \leq 1$ for all $\alpha \in \mathbb{Z}^n$, it is enough to show that $t_\alpha \notin Z(J)$ for $\alpha \notin \Gamma$. Let $\alpha \notin \Gamma$, and so $\alpha = (\alpha', \epsilon)$ with $\epsilon \neq \mathbf{0}$. Since $m \geq 2$, there exists $\mathbf{0} \neq \eta \in I$ such that $\eta \neq \epsilon$. Then $t_\alpha^2 t_\eta = a_\epsilon z_{2\alpha'} z^\epsilon t_\eta \neq 0$, while $t_\alpha(t_\alpha t_\eta) = 0$ since $\eta \neq \epsilon$. Hence $(t_\alpha, t_\alpha, t_\eta) \neq 0$, and so we get $t_\alpha \notin Z(J)$. Thus we obtain $Z(J) = Z$.

Since the central closure $\bar{J} = \bar{Z} \otimes_Z (Z \oplus V) = \bar{Z}.1 \oplus (\bar{Z} \otimes_Z V)$ of J over \bar{Z} , f can be naturally extended to $\bar{Z} \otimes_Z V$ over \bar{Z} , and so \bar{J} is a Jordan algebra over \bar{Z} determined by the extended bilinear form. Hence J is a Clifford torus over F with $\text{supp}(J_{S^{(m)}}) = S^{(m)} + \mathbb{Z}^{n-m}$ (which is a semilattice in \mathbb{Z}^n).

We set $a_0 = 1$ for convenience, and call $J_{S^{(m)}} = J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$ the *Clifford torus determined by $S^{(m)}$ of type $\{a_\epsilon\}_{\epsilon \in I}$* and $\{t_\alpha\}_{\alpha \in \mathbb{Z}^n}$ defined above a *standard basis of $J_{S^{(m)}}$* . When $a_\epsilon = 1$ for all $\epsilon \in I$, we call the $J_{S^{(m)}}$ the *standard Clifford torus determined by $S^{(m)}$* . The structure of standard Clifford tori is very simple. Namely, by (3.3.9), the structure constants of a standard Clifford torus $J = J_{S^{(m)}}$, relative to the standard basis, are 0 or 1.

As we already mentioned, $J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$ is a different description of $\mathcal{J} = \mathcal{J}(S^{(m)}, K, \mathcal{V}_I)$ when \mathcal{J} is a Jordan torus over F of central degree 2. Thus, as a corollary of Theorem 3.3.7 we obtain

Corollary 3.3.10. *Any $J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$ as defined above is a Clifford torus and conversely any Clifford n -torus is graded isomorphic to some $J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$ for some toral grading or to the algebra of Laurent polynomials in n -variables. Also, m is an isomorphism invariant.*

Remark 3.3.11. (0) One can easily check that

$$J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I}) \cong_{\mathbb{Z}^n} J' \otimes_F F[z_{m+1}^{\pm 1}, \dots, z_n^{\pm 1}]$$

where J' is the Clifford torus as constructed in 3.3.8 for $n = m$. When $m = n$, the standard Clifford torus $J_{S^{(n)}}$ appeared in [1] as the first example of an extended affine Lie algebra of type A_1 graded by an arbitrary semilattice.

(1) For $J = J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$, let $t_i := t_{\sigma_i}$ for $i = 1, \dots, n$. Then we have $t_i^2 = a_{\epsilon_i} z_i$ for $i = 1, \dots, m$ and $t_i = z_i$ for $i = m+1, \dots, n$. Thus the centre $Z(J)$ of J is

$$Z(J) = Z = F[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = F[t_1^{\pm 2}, \dots, t_m^{\pm 2}, t_{m+1}^{\pm 1}, \dots, t_n^{\pm 1}].$$

(2) For $J = J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$, one can always make $a_{\epsilon_i} = 1$ for all $i = 1, \dots, m$ in the following sense: Simply change the variables z_i for $i = 1, \dots, m$ in the algebra of Laurent polynomials $Z(J) = F[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ into $z'_i := a_{\epsilon_i} z_i$. Then we get a new F -basis of J using z'_i instead of z_i in 3.3.8, and the new basis determines new constants $\{b_\epsilon\}_{\epsilon \in I}$. It is clear that $b_{\epsilon_i} = 1$ for all $i = 1, \dots, m$ and that $J \cong_{\mathbb{Z}^n} J_{S^{(m)}}(\{b_\epsilon\}_{\epsilon \in I})$.

In particular, for a minimal semilattice S_0 in \mathbb{Z}^m , i.e., $I(S_0) = \{\mathbf{0}, \epsilon_1, \dots, \epsilon_m\}$, any Clifford torus determined by S_0 is graded isomorphic to the standard one.

(3) If any element of F has a square root in F , e.g. F is an algebraically closed field, then one can make $a_\epsilon = 1$ for all $\epsilon \in I$ by switching t_ϵ to $1/\sqrt{a_\epsilon} t_\epsilon$. Thus for such a base field F , a Clifford torus $J = (J_{S^{(m)}}, \{a_\epsilon\}_{\epsilon \in I})$ is always graded isomorphic to the standard Clifford torus $J_{S^{(m)}}$. We state this as a corollary:

Corollary 3.3.12. *Suppose that any element of the base field F has a square root, e.g. F is an algebraically closed field. Then a Clifford n -torus is graded isomorphic to a standard Clifford torus $J_{S^{(m)}}$ as defined in 3.3.8, or to the algebra of Laurent polynomials in n -variables. \square*

(4) A Clifford torus $J = (J_{S^{(m)}}, \{a_\epsilon\}_{\epsilon \in I})$ is, by (3.3.9), never of strong type, even if we take $S^{(m)} = \mathbb{Z}^m$. Moreover, let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ and Γ the central grading group of J . Then J satisfies the following property: for all $\alpha, \beta \in \text{supp}(J)$,

$$(4) \quad J_\alpha J_\beta = \begin{cases} \{0\} & \text{if } \bar{\mathbf{0}} \neq \bar{\alpha}, \bar{\beta} \in \mathbb{Z}^n/\Gamma \text{ and } \bar{\alpha} \neq \bar{\beta} \\ J_{\alpha+\beta} & \text{otherwise.} \end{cases}$$

By 3.2.20, Hermitian tori satisfying (4) are isomorphic to one of the following six tori

$$F_h^+, \quad H(F_h, *), \quad H(E_h, \sigma), \quad H(F_{\mathbf{m}^{(3)}}, *), \quad H(F_{\mathbf{m}^{(4)}}, *) \quad \text{or} \quad F_1.$$

The last one $F_1 = F[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is a Clifford torus. We claim that first five tori are also Clifford tori. Let $(\sigma_1, \dots, \sigma_n)$ be a toral grading of F_h , $F_{\mathbf{m}^{(3)}}$ and $F_{\mathbf{m}^{(4)}}$.

Let

$$\begin{aligned}
S_1^{(2)} &:= (\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2) \cap \text{supp}(F_h^+) \\
S_2^{(2)} &:= (\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2) \cap \text{supp}(H(F_h, *)) \\
S_3^{(2)} &:= (\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2) \cap \text{supp}(H(E_h, \sigma)) \\
S_4^{(3)} &:= (\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3) \cap \text{supp}(H(F_{\mathbf{m}^{(3)}}, *)) \\
S_5^{(4)} &:= (\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3 + \mathbb{Z}\sigma_4) \cap \text{supp}(H(F_{\mathbf{m}^{(4)}}, *)).
\end{aligned}$$

Then these $S_i^{(m)}$ are semilattices in \mathbb{Z}^m and $\{\sigma_1, \dots, \sigma_m\}$ is a basis of \mathbb{Z}^m in $S_i^{(m)}$. Thus in the notations of 3.3.8, we have

$$\begin{aligned}
I(S_1^{(2)}) &= \{\mathbf{0}, \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2\}, \\
I(S_2^{(2)}) &= \{\mathbf{0}, \epsilon_1, \epsilon_2\}, \\
I(S_3^{(2)}) &= \{\mathbf{0}, \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2\}, \\
I(S_4^{(3)}) &= \{\mathbf{0}, \epsilon_1, \epsilon_2, \epsilon_3\}, \\
I(S_5^{(4)}) &= \{\mathbf{0}, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4\}.
\end{aligned}$$

Also, let $a \in F$ such that $E = F(\sqrt{a})$ and let

$$\begin{aligned}
J_1 &= J_{S_1^{(2)}}(\{a_\epsilon\}_{\epsilon \in I(S_1^{(2)})}) \quad \text{where } a_{\epsilon_1} = 1, a_{\epsilon_2} = 1, a_{\epsilon_1 + \epsilon_2} = -1, \\
J_2 &= J_{S_2^{(2)}}(\{a_\epsilon\}_{\epsilon \in I(S_2^{(2)})}) \quad \text{where } a_{\epsilon_1} = 1, a_{\epsilon_2} = 1, \\
J_3 &= J_{S_3^{(2)}}(\{a_\epsilon\}_{\epsilon \in I(S_3^{(2)})}) \quad \text{where } a_{\epsilon_1} = 1, a_{\epsilon_2} = 1, a_{\epsilon_1 + \epsilon_2} = -a, \\
J_4 &= J_{S_4^{(3)}}(\{a_\epsilon\}_{\epsilon \in I(S_4^{(3)})}) \quad \text{where } a_{\epsilon_1} = 1, a_{\epsilon_2} = 1, a_{\epsilon_3} = 1, \\
J_5 &= J_{S_5^{(4)}}(\{a_\epsilon\}_{\epsilon \in I(S_5^{(4)})}) \quad \text{where } a_{\epsilon_1} = 1, a_{\epsilon_2} = 1, a_{\epsilon_3} = 1, a_{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4} = 1
\end{aligned}$$

be Clifford tori over F . (J_2, J_4 and J_5 are standard Clifford tori.) Since the structure constants relative to the standard basis of J_1, J_2, J_4 or J_5 coincide with the structure constants relative to the standard basis of $F_h^+, H(F_h, *), H(F_{\mathbf{m}^{(3)}}, *)$ or $H(F_{\mathbf{m}^{(4)}}, *)$, respectively, we get $J_1 \cong_{\mathbb{Z}^n} F_h^+, J_2 \cong_{\mathbb{Z}^n} H(F_h, *), J_4 \cong_{\mathbb{Z}^n} H(F_{\mathbf{m}^{(3)}}, *)$ and $J_5 \cong_{\mathbb{Z}^n} H(F_{\mathbf{m}^{(4)}}, *)$. For $\alpha = \alpha_1\sigma + \dots + \alpha_n\sigma \in \mathbb{Z}^n$, let

$$s_\alpha = \begin{cases} 0 & \text{if } \alpha_1\alpha_2 \text{ is even} \\ 1 & \text{otherwise.} \end{cases}$$

Then $\{(\sqrt{a})^{s_\alpha} t_\alpha \mid \alpha \in \mathbb{Z}^n\}$ is a basis of $H(E_h, \sigma)$. The structure constants of $H(E_h, \sigma)$ relative to this basis and the structure constants of the standard basis of J_3 coincide, and so we obtain $J_3 \cong_{\mathbb{Z}^n} H(E_h, \sigma)$. Therefore, these six Hermitian tori F_h^+ , $H(F_h, *)$, $H(E_h, \sigma)$, $H(F_{m^{(3)}}, *)$, $H(F_{m^{(4)}}, *)$ and F_1 are also Clifford tori. Recall that Hermitian tori are not necessarily Jordan tori of Hermitian type (see §3.1). We now know that the Jordan tori of Hermitian type are Hermitian tori excluding the six tori above.

As a final remark on division \mathbb{Z}^n -graded Jordan algebras over F of Clifford type, we mention that their central closures are always (possibly infinite dimensional) division algebras:

First it is well-known that \mathcal{J} is a division algebra if and only if its norm n is *anisotropic*, i.e., $n(x) \neq 0$ for all $0 \neq x \in \mathcal{J}$ (see [14, page 8.28]). Moreover, we have the following:

Lemma 3.3.13. *A Jordan algebra \mathcal{J} determined by a symmetric bilinear form over F which is a Jordan domain is a division algebra.*

Proof. Let tr and n be the trace and norm of \mathcal{J} . If $n(x) = 0$ for $0 \neq x \in \mathcal{J}$, then we have $x^2 = \text{tr}(x)x$. If $\text{tr}(x) = 0$, then $x^2 = 0$, which contradicts the fact that \mathcal{J} is a Jordan domain. Hence we have $\text{tr}(x) \neq 0$. Let $e := \text{tr}(x)^{-1}x$. Then we have $e^2 = e$, and so e is an idempotent.

We claim that a Jordan domain has no nontrivial idempotents. Indeed, let e an idempotent. Then we have $e(e - 1) = 0$. Since the subalgebra generated by e is a commutative associative domain, we get $e = 0$ or $e = 1$. Hence our claim has been settled.

Since \mathcal{J} is a Jordan domain and $e \neq 0$, we get $e = 1$ by the claim. However, this implies $x = \text{tr}(x)1 \in F1$, and so $n(x) = \text{tr}(x)^2 \neq 0$, which contradicts our assumption. Hence we obtain $n(x) \neq 0$ for all $0 \neq x \in \mathcal{J}$. Thus \mathcal{J} is a division algebra. \square

Since the central closure of a division \mathbb{Z}^n -graded Jordan algebra is a Jordan domain, we have a corollary of 3.3.13:

Corollary 3.3.14. *The central closure of a division \mathbb{Z}^n -graded Jordan algebra of Clifford type is a division algebra. \square*

§ 3.4 ALBERT TYPE

We classify Jordan tori of Albert type i.e., Jordan tori whose central closure is an Albert algebra. An Albert algebra is defined as either a *first* or a *second Tits construction*, which are both 27-dimensional central simple exceptional Jordan algebras of degree 3. We recall the first Tits construction but not the second one since second Tits constructions do not occur in the class of Jordan tori.

Let A be a central simple associative algebra over a field F of characteristic $\neq 2$ of degree 3 with generic trace tr (see §2.6). For $a, b \in A$, let

$$\begin{aligned} a \cdot b &= \frac{1}{2}(ab + ba), \\ a \times b &= a \cdot b - \frac{1}{2}\text{tr}(a)b - \frac{1}{2}\text{tr}(b)a + \frac{1}{2}(\text{tr}(a)\text{tr}(b) - \text{tr}(a \cdot b))1, \\ \bar{a} &= a \times 1 = \frac{1}{2}(\text{tr}(a)1 - a). \end{aligned}$$

Note that

$$\bar{1} = 1 \quad \text{and} \quad a(a \times a) = (a \times a)a = n(a)$$

where n is the generic norm on A .

Let $0 \neq \mu \in F$. A first Tits construction (A, μ) over F obtained from A and the structure constant μ is the direct sum $A \oplus A \oplus A$ as F -spaces with the following F -bilinear multiplication:

For $(a_0, a_1, a_2), (b_0, b_1, b_2) \in (A, \mu) = A \oplus A \oplus A$,

$$(3.4.0) \quad \begin{aligned} (a_0, a_1, a_2)(b_0, b_1, b_2) &= (a_0 \cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}, \\ \bar{a}_0 b_1 + \bar{b}_0 a_1 + \mu^{-1} a_2 \times b_2, & \quad b_2 \bar{a}_0 + a_2 \bar{b}_0 + \mu a_1 \times b_1) \end{aligned}$$

We will always identify A with $(A, 0, 0)$ as F -spaces.

Remark 3.4.1. Let $x := (0, 1, 0)$ and $y := (0, 0, 1)$. Since $a = \text{tr}(a)1 + 2\bar{a} = \overline{\text{tr}(a)1 + 2a}$, we have $A \cdot x = (0, A, 0)$ and $A \cdot x^2 = (0, 0, \mu A) = (0, 0, A)$. Thus (A, μ) is generated by A and x . Also, since $x = \mu y^2$, (A, μ) is generated by A and y .

The following lemma is well-known, but for the convenience of the reader we prove it ([13, Exercise 1, page 422]).

Lemma 3.4.2. *Let (A, μ) be a first Tits construction. Let $a \in A$ be invertible and $x = (0, a, 0), y = (0, 0, a) \in (A, \mu)$. Then:*

(i) $0 \neq x^3 \in F1$, and there exists an isomorphism Φ from (A, μ) onto (A, x^3) over $F = F1$ (identify) such that $\Phi|_A = id$ and $\Phi(x) = (0, 1, 0)$,

(ii) $0 \neq y^{-3} \in F1$, and there exists an isomorphism Ψ from (A, μ) onto (A, y^{-3}) over $F = F1$ such that $\Psi|_A = id$ and $\Psi(y) = (0, 0, 1)$.

Proof. For (i). we have

$$x^3 = (0, a, 0)^3 = (0, 0, \mu a \times a)(0, a, 0) = (\overline{\mu a(a \times a)}, 0, 0) = \mu n(a)1 \in F1,$$

and $n(a) \neq 0$ since a is invertible (see [13, page 227]). So $x^3 \neq 0$ and (A, x^3) makes sense. Define a map Φ from (A, μ) into (A, x^3) by

$$\Phi : (a_0, a_1, a_2) \mapsto (a_0, a_1 a^{-1}, a a_2).$$

Then Φ is clearly a linear isomorphism. Using the formulas

$$a_1 a^{-1} \times b_1 a^{-1} = n(a)^{-1} a(a_1 \times b_1) \quad \text{and} \quad a a_2 \times a b_2 = n(a)(a_2 \times b_2) a^{-1}$$

(see [13, page 413]), we have

$$\begin{aligned} & \Phi((a_0, a_1, a_2)) \Phi((b_0, b_1, b_2)) = (a_0, a_1 a^{-1}, a a_2)(b_0, b_1 a^{-1}, a b_2) \\ & = (a_0 \cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}, \overline{a_0 b_1} a^{-1} + \overline{b_0 a_1} a^{-1} + x^{-3} a a_2 \times a b_2, \\ & \quad a b_2 \overline{a_0} + a a_2 \overline{b_0} + x^3 a_1 a^{-1} \times b_1 a^{-1}) \\ & = (a_0 \cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}, (\overline{a_0 b_1} + \overline{b_0 a_1} + \mu^{-1} a_2 \times b_2) a^{-1}, a(b_2 \overline{a_0} + a_2 \overline{b_0} + \mu a_1 \times b_1)) \\ & = \Phi((a_0, a_1, a_2) \cdot (b_0, b_1, b_2)). \end{aligned}$$

Hence Φ is an isomorphism satisfying the required conditions.

For (ii). we have

$$y^{-3} = (0, 0, a)^{-3} = (\overline{\mu^{-1} a(a \times a)}, 0, 0)^{-1} = \mu n(a)^{-1} 1 \neq 0,$$

and as in (i) we have an isomorphism Ψ from (A, μ) onto (A, y^{-3}) defined by

$$\Psi : (a_0, a_1, a_2) \mapsto (a_0, a_1 a, a^{-1} a_2). \quad \square$$

The theory of first Tits constructions over a ring does not seem to be much developed. As far as the author knows, the most general paper is [26]. For our purpose, we do not need this generality, but only a very special case. One might say that this is almost the classical case (i.e., Tits constructions over a field above).

Lemma 3.4.3. *Let \mathcal{A} be a prime associative algebra over F of central degree 3 (see 2.6.3), and $\mu \in \mathcal{Z}$ a unit where $\mathcal{Z} = Z(\mathcal{A})$ is the centre of \mathcal{A} . Assume that $\text{tr}(\mathcal{A}) \subset \mathcal{Z}$ where tr is the generic trace of the central closure $\overline{\mathcal{A}}$ over $\overline{\mathcal{Z}}$. Then, the subset $(\mathcal{A}, \mu) := \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$ of the first Tits construction $(\overline{\mathcal{A}}, \mu) = \overline{\mathcal{A}} \oplus \overline{\mathcal{A}} \oplus \overline{\mathcal{A}}$ is a \mathcal{Z} -subalgebra such that $\overline{(\mathcal{A}, \mu)} = (\overline{\mathcal{A}}, \mu)$.*

Proof. By the multiplication rule (3.4.0) of $(\overline{\mathcal{A}}, \mu)$ and our assumption $\text{tr}(\mathcal{A}) \subset \mathcal{Z}$, it is clear that (\mathcal{A}, μ) is a \mathcal{Z} -subalgebra of $(\overline{\mathcal{A}}, \mu)$. Let $Z(\mathcal{A}, \mu)$ be the centre of (\mathcal{A}, μ) . Then we have $\mathcal{Z} \subset Z(\mathcal{A}, \mu) \subset Z(\mathcal{A}) = \mathcal{Z}$, and so $Z(\mathcal{A}, \mu) = \mathcal{Z}$. Therefore, the central closure $\overline{(\mathcal{A}, \mu)}$ of (\mathcal{A}, μ) is given as $\overline{\mathcal{Z}} \otimes_{\mathcal{Z}} (\mathcal{A}, \mu)$, which is a $\overline{\mathcal{Z}}$ -subalgebra of $(\overline{\mathcal{A}}, \mu)$. Thus we only need to show that $(\overline{\mathcal{A}}, \mu) \subset \overline{(\mathcal{A}, \mu)}$. But this is clear because we have, for $z_i \in \mathcal{Z}$ and $a_i \in \mathcal{A}$, $i = 0, 1, 2$,

$$\left(\frac{1}{z_0} \otimes a_0, \frac{1}{z_1} \otimes a_1, \frac{1}{z_2} \otimes a_2 \right) = \frac{1}{z_0 z_1 z_2} (1 \otimes z_1 z_2 a_0, 1 \otimes z_0 z_2 a_1, 1 \otimes z_0 z_1 a_2)$$

Hence we get $(\overline{\mathcal{A}}, \mu) = \overline{(\mathcal{A}, \mu)}$. \square

We call this (\mathcal{A}, μ) a *first Tits construction over \mathcal{Z}* . This is a special type of the general first Tits construction studied in [26].

Remark 3.4.4. This (\mathcal{A}, μ) is generated as a \mathcal{Z} -algebra by \mathcal{A} and $x = (0, 1, 0)$ or by \mathcal{A} and $y = (0, 0, 1)$ as in the classical case (see Remark 3.4.1). Thus (\mathcal{A}, μ) is characterized as the \mathcal{Z} -subalgebra of $(\overline{\mathcal{A}}, \mu)$ generated by \mathcal{A} and x or by \mathcal{A} and y .

Construction 3.4.5. Let us first observe that for a field F of $\text{ch}.F \neq 2$ we have

$$(3.4.5a) \quad F \text{ has a primitive 3rd root of unity } \omega \iff \text{ch}.F \neq 3 \text{ and } \sqrt{-3} \in F,$$

which follows from the formula $2\omega + 1 = \sqrt{-3}$. Suppose this is the case, and hence $\text{ch}.F \neq 3$. Let $F_\omega = F_\omega[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ be the quantum torus over F defined in 2.6.10, which has central degree 3. Assume that $n \geq 3$. Let Z be the centre of F_ω and tr the generic trace of the central closure \overline{F}_ω of F_ω . Then, by 2.6.5, we have

$$(3.4.6) \quad \text{tr}(u_1^i u_2^j) = \begin{cases} 3u_1^i u_2^j & \text{if } i \equiv j \equiv 0 \pmod{3} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{tr}(F_\omega) \subset Z = F[u_1^{\pm 3}, u_2^{\pm 3}, u_3^{\pm 1}, \dots, u_n^{\pm 1}].$$

Thus, by 3.4.3, we have the first Tits construction $\mathbb{A}_t = (F_\omega, u_3)$ over Z . Namely,

$$\mathbb{A}_t = (F_\omega, u_3) = F_\omega \oplus F_\omega \oplus F_\omega.$$

Let $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ be a basis of \mathbb{Z}^n and

$$G := \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + 3\mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \dots + \mathbb{Z}\varepsilon_n.$$

Consider the G -grading of F_ω , i.e.,

$$F_\omega = \bigoplus_{\alpha \in G} F u_\alpha \quad \text{where} \quad u_\alpha = u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \dots u_n^{\alpha_n}$$

for $\alpha = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + 3\alpha_3 \varepsilon_3 + \alpha_4 \varepsilon_4 + \dots + \alpha_n \varepsilon_n$ and arbitrary $\alpha_i \in \mathbb{Z}$.

For $\alpha = \alpha_1 \varepsilon_1 + \dots + \alpha_n \varepsilon_n \in \mathbb{Z}^n$, we put

$$t_\alpha := \begin{cases} (u_\alpha, 0, 0) & \text{if } \alpha_3 \equiv 0 \pmod{3} \\ (0, u_{\alpha - \varepsilon_3}, 0) & \text{if } \alpha_3 \equiv 1 \pmod{3} \\ (0, 0, u_{\alpha + \varepsilon_3}) & \text{if } \alpha_3 \equiv 2 \pmod{3}. \end{cases}$$

Then we obtain $t_\alpha \neq 0$ for all $\alpha \in \mathbb{Z}^n$ and

$$\mathbb{A}_t = \bigoplus_{\alpha \in \mathbb{Z}^n} F t_\alpha.$$

Thus, \mathbb{A}_t is a \mathbb{Z}^n -graded vector space over F whose homogeneous spaces are all 1-dimensional over F . We note that $t_{\varepsilon_3} = (0, 1, 0)$, $t_{2\varepsilon_3} = t_{\varepsilon_3}^2 = (0, 0, u_3)$ and $t_{-\varepsilon_3} = t_{\varepsilon_3}^{-1} = (0, 0, 1)$.

For $\alpha = \alpha_1 \varepsilon_1 + \dots + \alpha_n \varepsilon_n$ and $\beta = \beta_1 \varepsilon_1 + \dots + \beta_n \varepsilon_n \in \mathbb{Z}^n$, one can calculate the multiplication table of \mathbb{A}_t using (3.4.0) and (3.4.6), as follows:

(I) $\alpha_3 \equiv \beta_3 \equiv 0$ (all \equiv below are mod 3):

$$t_\alpha t_\beta = \frac{1}{2}(\omega^{\alpha_1 \beta_2} + \omega^{\alpha_2 \beta_1}) t_{\alpha + \beta}$$

(II) $\alpha_3 \equiv 0, \beta_3 \equiv 1$:

$$t_\alpha t_\beta = \begin{cases} t_{\alpha + \beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2 \beta_1} t_{\alpha + \beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 0, \beta_3 \equiv -1$:

$$t_{\alpha}t_{\beta} = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_1\beta_2}t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(IV) $\alpha_3 \equiv 1, \beta_3 \equiv -1$:

$$t_{\alpha}t_{\beta} = \begin{cases} \omega^{\alpha_2\beta_1}t_{\alpha+\beta} & \text{if } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ -\frac{1}{2}\omega^{\alpha_2\beta_1}t_{\alpha+\beta} & \text{otherwise} \end{cases}$$

(V) $\alpha_3 \equiv \beta_3 \equiv 1$ or $\alpha_3 \equiv \beta_3 \equiv -1$:

$$t_{\alpha}t_{\beta} = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv \beta_1 \equiv \beta_2 \equiv 0 \\ -\frac{1}{2}t_{\alpha+\beta} & \text{if } [\alpha_1 \equiv \alpha_2 \equiv 0 \text{ and } (\beta_1 \neq 0 \text{ or } \beta_2 \neq 0)] \\ & \text{or } [(\alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0) \text{ and } \beta_1 \equiv \beta_2 \equiv 0] \\ -\frac{1}{2}\omega^{-\alpha_1\alpha_2}t_{\alpha+\beta} & \text{if } (\alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0) \text{ and } (\beta_1 \neq 0 \text{ or } \beta_2 \neq 0) \\ & \text{and } \alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv 0 \\ \frac{1}{2}(\omega^{\alpha_1\beta_2} + \omega^{\alpha_2\beta_1})t_{\alpha+\beta} & \text{otherwise.} \end{cases}$$

Therefore, \mathbb{A}_t is a \mathbb{Z}^n -graded algebra over F . Using $\omega^2 + \omega + 1 = 0$, one can check that the structure constants relative to the basis $\{t_{\alpha}\}_{\alpha \in \mathbb{Z}^n}$ are

$$\left\{ 1, \omega, \omega^2, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2} \right\}.$$

Hence by 2.5.2, \mathbb{A}_t is a Jordan torus over F of strong type, which is called the *Albert torus over F* . We call the grading of \mathbb{A}_t above a *toral grading determined by $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$* and $\{t_{\alpha}\}_{\alpha \in \mathbb{Z}^n}$ defined above a *standard basis of the Albert torus \mathbb{A}_t* . By 3.4.3, the central closure $\overline{\mathbb{A}}_t$ of \mathbb{A}_t is an Albert algebra $(\overline{F}_{\omega}, t_3)$ over $\overline{\mathbb{Z}}$, and so the Albert torus is in fact a Jordan torus of Albert type.

Unlike the other types, we will show that the Albert torus \mathbb{A}_t is the only Jordan torus of Albert type. We begin with the following general theorem:

Theorem 3.4.7. ([13, Theorem 2, page 156]) *A finite dimensional Jordan domain is a division algebra. \square*

By this theorem, a Jordan torus of Albert type is a division algebra since the central closure is a 27-dimensional Jordan domain (see 2.2.4 and 1.1.16). In particular, the central closure $\overline{\mathbb{A}}$ of the Albert torus \mathbb{A} is a division algebra.

Now, we first classify special Jordan tori of central degree 3. For this purpose we show a lemma which is known, and so we omit the explanation of some well-known concepts and facts in the proof. The reader can find these in [13] and [15].

Lemma 3.4.8. *Let J be a finite dimensional central special Jordan division algebra over F of degree r . Then:*

- (a) $r \neq 2^m$ for $m \geq 1 \implies \dim_F J = r^2$.
- (b) $r = 3 \iff \dim_F J = 9$.

Proof. (a): It is clear for $r = 1$, and so we assume that $r > 1$. If J be a finite dimensional central special Jordan division algebra over F of degree $r \neq 2$, then $J \cong D^+$ or $H(D, *)$ where D is a central associative division over the centre of degree r and $*$ is an involution of D (see Theorem 11 and Exercise 1 in [13, page 210]). Thus, if $J \cong D^+$, then we have $\dim_F J = r^2$. If $*$ is of the second kind, we know $\dim_F H(D, *) = r^2$ (see [15, page 190]). If $J \cong H(D, *)$ and $r \neq 2^m$ for $m \geq 1$, then there does not exist an involution of first kind on D . For, if one exists, then $D \cong D^{op}$ (the opposite algebra), and so the order of D in the Brauer group of F is 2. Since any prime factor of the degree of D divides the order (see [15, Theorem 2.7.5, page 61]), the degree of D has to be a power of 2. This is a contradiction. Hence (a) has been shown.

(b): By (a), we get $r = 3 \implies \dim_F J = 9$. Suppose that $\dim_F J = 9$. From the classification of the finite dimensional simple associative algebras with involution (see [15, page 190]), we have $\dim_F J = r^2$ ($J \cong D^+$ or $*$ is of the second kind), $r(r+1)/2$ ($*$ is orthogonal) or $r(r-1)/2$ ($*$ is symplectic). Since $r(r+1)/2$ or $r(r-1)/2$ is never 9, we get $9 = r^2$, i.e., $r = 3$. \square

Example. Let F be a field of characteristic $\neq 2$ and $\neq 3$.

If $\sqrt{-3} \in F$, i.e., if F contains a primitive 3rd root of unity ω (see 3.4.5a), then F_ω^+ , where the quantum matrix ω is defined in 2.6.10, is a special Jordan torus of central degree 3. In fact, we have, by 1.1.10, $Z(F_\omega^+) = Z(F_\omega)$. So by 3.4.7, the central closure $\overline{F_\omega^+}$ is a 9-dimensional central special Jordan division algebra over $\overline{Z(F_\omega^+)} = \overline{Z(F_\omega)}$. Hence by 3.4.8(b), it has degree 3.

If $\sqrt{-3} \notin F$, let $E = F(\sqrt{-3})$ and consider the Jordan n -torus $H(E_\omega, \sigma)$ defined in Construction 3.1.2. As shown there, $E \otimes_F H(E_\omega, \sigma) \cong_{\mathbb{Z}^n} E_\omega^+$, and so by 2.3.2, the central grading group of $H(E_\omega, \sigma)$ coincides with the one of E_ω , which is

$$3\mathbb{Z}\epsilon_1 + 3\mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3 + \cdots + \mathbb{Z}\epsilon_n.$$

Therefore, the central closure $\overline{H(E_\omega, \sigma)}$ is a 9-dimensional central special Jordan algebra, which by 3.4.7 is a division algebra. Another application of 3.4.8(b) then

proves that $H(E_\omega, \sigma)$ has central degree 3.

Let $E_\omega = E_\omega[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with an $(\epsilon_1, \dots, \epsilon_n)$ -grading. Since $\sigma(t_\alpha) = \omega^{\alpha_1 \alpha_2} t_\alpha$ for $\alpha = \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n \in \mathbb{Z}^n$ and $t_\alpha := t_1^{\alpha_1} \dots t_n^{\alpha_n}$, we have

$$H(E_\omega, \sigma) = \bigoplus_{\alpha \in \mathbb{Z}^n} F(1 + \omega^{\alpha_1 \alpha_2} t_\alpha).$$

We can show that the two tori in this Example are the only special Jordan tori of central degree 3.

Proposition 3.4.9. *Let J be a special Jordan torus over F of central degree 3. Then $\text{ch}.F \neq 3$, and*

$$J \cong_{\mathbb{Z}^n} \begin{cases} F_\omega^+ & \text{if } \omega \in F \\ H(E_\omega, \sigma) & \text{otherwise} \end{cases}$$

where $E = F(\sqrt{-3})$. Conversely, the algebras F_ω^+ and $H(E_\omega, \sigma)$ are special Jordan tori of central degree 3.

Proof. Only the 1st part remains to be proven. We know that a special Jordan torus J is either a Hermitian torus or a Clifford torus (see §3.0). Since a Clifford torus has central degree 1 or 2 (see §3.3), special Jordan tori of central degree 3 have to be Hermitian tori, i.e., F_q^+ , $H(F_\epsilon, *)$ or $H(E_\epsilon, \sigma)$ (see 3.1.10). Let $Z = Z(J)$ be the centre of J . By 2.6.8, there exists a basis $\{\epsilon_1, \dots, \epsilon_n\}$ of \mathbb{Z}^n such that the central grading group Γ of J is given by

$$\Gamma = 3\mathbb{Z}\epsilon_1 + \dots + 3\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n$$

for some $2 \leq m \leq n$ and $\text{supp}(J) = \mathbb{Z}^n$. In particular, there exists a homogeneous element $t \in J$ such that $t^2 \notin Z$ (for example, take t to be a nonzero element of degree ϵ_1). If $\varphi: J \rightarrow H(F_\epsilon, *)$, then by 2.2.7, $\varphi(t)$ is homogeneous in $H(F_\epsilon, *)$, and $\varphi(t)^2 \notin \varphi(Z)$ which is equal to the centre of $H(F_\epsilon, *)$. However, by 3.1.5, this cannot happen. Hence the only possible candidates of central degree 3 are F_q^+ and $H(E_\epsilon, \sigma)$.

Now, since the central closure \bar{J} is a finite dimensional central special Jordan division algebra over \bar{Z} of degree 3 (see 2.2.4, 1.1.16(iv) and 3.4.7), we have, by 3.4.8, $\dim_{\bar{Z}} \bar{J} = 9$. So m has to be 2. If $J \cong_{\mathbb{Z}^n} F_q^+$, then by 1.1.10, $Z = Z(F_q^+) = Z(F_q)$, and hence the central grading group of F_q is Γ . Thus, by Proposition 2.6.10(1)&(2)(ii), F has to contain a primitive 3rd root of unity ω , and so $\text{ch}.F \neq 3$, and $q = \omega$, i.e., $J \cong_{\mathbb{Z}^n} F_\omega^+$.

Suppose that $J \cong_{\mathbb{Z}^n} H(E_\xi, \sigma)$. We will identify them. So we have $J_E = E \otimes_F J = E_\xi^+$ (see 3.1.2). Hence by 2.3.2, the grading group of $Z(E_\xi^+) = Z(E_\xi)$ is still Γ . Then, by 2.6.10(1)&(2)(ii), we get $\text{ch}.E \neq 3$ (so $\text{ch}.F \neq 3$), $\omega \in E$ and $\xi = \omega$, i.e.. $J = H(E_\omega, \sigma)$. Since $\omega\sigma(\omega) = 1$, $\sigma(\sqrt{-3}) = -\sqrt{-3}$, and so $\sqrt{-3} \notin F$, and hence $E = F(\sqrt{-3})$. (See 3.1.2: $\xi_{ij}\sigma(\xi_{ij}) = 1$ for all i, j and σ is the nontrivial Galois automorphism of E over F .) \square

By Zelmanov's Prime Structure Theorem (see §3.0), if \mathcal{J} is a strongly prime exceptional Jordan algebra over F , then \mathcal{J} has central degree 3 and the central closure $\overline{\mathcal{J}}$ is an Albert algebra. Let Tr be the generic trace of $\overline{\mathcal{J}}$ over $\overline{\mathcal{Z}}$. For a subalgebra \mathcal{U} of \mathcal{J} , let

$$\mathcal{U}^\perp := \{x \in \mathcal{J} \mid \text{Tr}(\mathcal{U}x) = 0\} \subset \mathcal{J} \subset \overline{\mathcal{J}}.$$

Note that a Jordan domain which is exceptional is strongly prime (see 1.1.5), and we simply call such a domain an *exceptional Jordan domain*. Note also that by 3.4.7, the central closure of an exceptional Jordan domain is an Albert division algebra. The following lemma for an exceptional Jordan domain serves as preparation for the classification of Jordan tori of Albert type, and we will show that such a torus satisfies all the assumptions of the lemma.

Lemma 3.4.10. *Let \mathcal{J} be an exceptional Jordan domain over F , $\mathcal{Z} = Z(\mathcal{J})$ the centre of \mathcal{J} and Tr the generic trace of $\overline{\mathcal{J}}$. Let \mathcal{U} be a subdomain of \mathcal{J} and $Z(\mathcal{U})$ the centre of \mathcal{U} . We assume the following conditions:*

- (i) $\mathcal{Z} = Z(\mathcal{U})$ and \mathcal{U} has central degree 3,
- (ii) $\mathcal{U} = \mathcal{A}^+$ for an associative algebra \mathcal{A} over F ,
- (iii) $\text{Tr}(\mathcal{U}) \subset \mathcal{Z}$,
- (iv) there exists an element $x \in \mathcal{U}^\perp$ such that $x^2 \in \mathcal{U}^\perp$ and $z := x^3 \in \mathcal{Z}$ is invertible.

Then, \mathcal{J} contains a subalgebra \mathcal{J}' so that there exists a \mathcal{Z} -isomorphism $\varphi : (\mathcal{A}, z) \xrightarrow{\sim} \mathcal{J}'$ with $\varphi|_{\mathcal{A}} = \text{id}$ and $\varphi((0, 1, 0)) = x$, or a \mathcal{Z} -isomorphism $\psi : (\mathcal{A}, z^{-1}) \xrightarrow{\sim} \mathcal{J}'$ with $\psi|_{\mathcal{A}} = \text{id}$ and $\psi((0, 0, 1)) = x$ where (\mathcal{A}, z) and (\mathcal{A}, z^{-1}) are defined in 3.4.3.

Moreover, assume that

- (v) there exists an F -isomorphism f from \mathcal{A} onto \mathcal{A}^{op} (the opposite algebra) such that $f \circ \text{Tr} = \text{Tr} \circ f$ and $f(z) = z$.

Then there exists an F -isomorphism $\tilde{f} : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}, z^{-1})$ with $\tilde{f}|_{\mathcal{A}} = f$ and

$\bar{f}((0, 1, 0)) = (0, 0, 1)$. In particular, \mathcal{J} always contains a subalgebra F -isomorphic to (\mathcal{A}, z) .

Proof. As mentioned above, the central closure $\bar{\mathcal{J}}$ is an Albert division algebra over $\bar{\mathcal{Z}}$. By (i), $\bar{\mathcal{U}} = \overline{\mathcal{Z}(\mathcal{U})} \otimes_{\mathcal{Z}(\mathcal{U})} \mathcal{U} = \bar{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{U} \subset \bar{\mathcal{J}}$ is a central division subalgebra over $\bar{\mathcal{Z}}$ of degree 3, and $\text{tr} := \text{Tr} |_{\bar{\mathcal{U}}}$ is the generic trace of $\bar{\mathcal{U}}$. By (ii) and 1.1.10, we have

$$\bar{\mathcal{U}} = \overline{\mathcal{Z}(\mathcal{A}^+)} \otimes_{\mathcal{Z}(\mathcal{A}^+)} \mathcal{A}^+ = \overline{\mathcal{Z}(\mathcal{A})} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}^+ = (\overline{\mathcal{Z}(\mathcal{A})} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A})^+ = (\bar{\mathcal{A}})^+.$$

Hence $B := \bar{\mathcal{A}}$ is a central associative division algebra over $\bar{\mathcal{Z}}$ of degree 3, and so \mathcal{A} has central degree 3 with $\text{tr}(\mathcal{A}) = \text{tr}(\mathcal{U}) = \text{Tr}(\mathcal{U}) \subset \mathcal{Z}$ by (iii). Note that the generic trace of \mathcal{A} coincides with the generic trace tr of $\mathcal{U} = \mathcal{A}^+$ (see [13, page 230]).

Now, since $\bar{\mathcal{J}}$ contains $\bar{\mathcal{U}} = B^+$ for the central simple associative algebra B over $\bar{\mathcal{Z}}$ of degree 3, we have $\bar{\mathcal{J}} \cong (B, \mu)$ over $\bar{\mathcal{Z}}$ for some $0 \neq \mu \in \bar{\mathcal{Z}}$ (see [13, page 420]). Note that this isomorphism is the identity map on B . So we identify $\bar{\mathcal{J}}$ with (B, μ) . Since $\bar{\mathcal{U}}^\perp = B^\perp = (0, B, B)$ (see e.g. [28, page 349]), we have, by (iv), $x = (0, u, v)$ for some $u, v \in B$. By the multiplication rule (3.4.0) of (B, μ) , we have $x^2 = (2\bar{u}\bar{v}, u', v')$ for some $u', v' \in B$. Since $x^2 \in \mathcal{U}^\perp$, we have $2\bar{u}\bar{v} = 0$, and so $\text{tr}(uv)1 = uv$. Hence $0 = 2\bar{u}\bar{v} = \overline{2\text{tr}(uv)1} = 2\text{tr}(uv)1$, and so we get $uv = 0$. Since B is an associative division algebra, we have $u = 0$ or $v = 0$. If $u = v = 0$, then $x = 0$ which contradicts the invertibility of x^3 . Thus we obtain $x = (0, u, 0)$ or $x = (0, 0, v)$ for some nonzero $u, v \in B$. Then, by 3.4.2, there exists a $\bar{\mathcal{Z}}$ -isomorphism Φ from (B, μ) onto (B, z) or Ψ from (B, μ) onto (B, z^{-1}) such that $\Phi|_{\mathcal{A}} = \text{id}$ or $\Psi|_{\mathcal{A}} = \text{id}$ and $\Phi(x) = (0, 1, 0)$ or $\Psi(x) = (0, 0, 1)$. So, $\Phi(\mathcal{J})$ contains \mathcal{A} and $(0, 1, 0)$ in (B, z) , or $\Psi(\mathcal{J})$ contains \mathcal{A} and $(0, 0, 1)$ in (B, z^{-1}) . Then, by Remark 3.4.4(2), $\Phi(\mathcal{J})$ contains (\mathcal{A}, z) or $\Psi(\mathcal{J})$ contains (\mathcal{A}, z^{-1}) . Let $\mathcal{J}' = \Phi^{-1}((\mathcal{A}, z))$ or $\mathcal{J}' = \Psi^{-1}((\mathcal{A}, z^{-1}))$. Then $\varphi := \Phi^{-1}|_{(\mathcal{A}, z)}$ and $\psi := \Psi^{-1}|_{(\mathcal{A}, z^{-1})}$ are the required \mathcal{Z} -isomorphisms, and so we have shown the first statement.

For the second statement, first note that \mathcal{A}^{op} has central degree 3, the same centre as \mathcal{A} and the generic trace of \mathcal{A}^{op} coincides with the generic trace tr of \mathcal{A} . We use the well-known fact that there exists an isomorphism $g : (B, z) \xrightarrow{\sim} (B^{op}, z^{-1})$ over $\bar{\mathcal{Z}}$ defined by $g((a_0, a_1, a_2)) = (a_0, a_2, a_1)$ for $a_0, a_1, a_2 \in \bar{\mathcal{A}}$ (see [13, Exercise 2, page 422]). Using this,

$$h := g|_{(\mathcal{A}, z)} : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}^{op}, z^{-1})$$

is a \mathcal{Z} -isomorphism. Now, by our assumption (v), the F -isomorphism $f : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^{op}$ satisfies $f(z) = z$ and $f \circ \text{tr} = f \circ \text{Tr} = \text{Tr} \circ f = \text{tr} \circ f$. We define $\bar{f} : (\mathcal{A}, z^{-1}) \longrightarrow (\mathcal{A}^{op}, z^{-1})$ by $\bar{f}((a_0, a_1, a_2)) = (f(a_0), f(a_1), f(a_2))$. We claim that \bar{f} is an F -isomorphism. In fact, \bar{f} is an F -linear isomorphism since f is an F -linear isomorphism. So we only need to check that \bar{f} is a homomorphism. By 3.4.0. for $(a_0, a_1, a_2), (b_0, b_1, b_2) \in (\mathcal{A}, z^{-1})$,

$$\begin{aligned} \bar{f}((a_0, a_1, a_2)(b_0, b_1, b_2)) &= \bar{f}((a_0 \cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}, \\ &\overline{a_0 b_1} + \overline{b_0 a_1} + z a_2 \times b_2, \quad b_2 \overline{a_0} + a_2 \overline{b_0} + z^{-1} a_1 \times b_1)) \\ &= ((f(a_0 \cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}), \\ &f(\overline{a_0 b_1} + \overline{b_0 a_1} + z a_2 \times b_2), \quad f(b_2 \overline{a_0} + a_2 \overline{b_0} + z^{-1} a_1 \times b_1)). \end{aligned}$$

Since f is an isomorphism and $f \circ \text{tr} = \text{tr} \circ f$, we have for $a, b \in \mathcal{A}$.

$$\begin{aligned} f(a \cdot b) &= \frac{1}{2}(f(a)f(b) + f(b)f(a)) = f(a) \cdot f(b), \\ f(a \times b) &= f(a) \cdot f(b) - \frac{1}{2}f(\text{tr}(a))f(b) - \frac{1}{2}f(\text{tr}(b))f(a) \\ &\quad + \frac{1}{2}\left(f(\text{tr}(a))f(\text{tr}(b)) - f(\text{tr}(a \cdot b))\right)f(1) \\ &= f(a) \cdot f(b) - \frac{1}{2}\text{tr}(f(a))f(b) - \frac{1}{2}\text{tr}(f(b))f(a) \\ &\quad + \frac{1}{2}\left(\text{tr}(f(a))\text{tr}(f(b)) - \text{tr}(f(a) \cdot f(b))\right)1 = f(a) \times f(b), \\ f(\bar{a}) &= f(a \times 1) = f(a) \times 1 = \overline{f(a)}. \end{aligned}$$

Since $f(z) = z$ and $f(z^{-1}) = z^{-1}$, we get

$$\bar{f}((a_0, a_1, a_2)(b_0, b_1, b_2)) = \bar{f}((a_0, a_1, a_2))\bar{f}((b_0, b_1, b_2)),$$

and our claim is settled.

Consequently, we obtain the F -isomorphism $\bar{f} := \bar{f}^{-1} \circ h : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}, z^{-1})$ with $\bar{f}((0, 1, 0)) = (0, 0, 1)$.

For the last statement, the composition map $\psi \circ \bar{f}$ gives an F -isomorphism from (\mathcal{A}, z) onto \mathcal{J}' . \square

Remark 3.4.11. In 3.4.10, if $(\mathcal{A}, z) \cong (\mathcal{A}, z^{-1})$ over \mathcal{Z} , then $(\overline{\mathcal{A}}, z) \cong (\overline{\mathcal{A}}, z^{-1})$ over $\overline{\mathcal{Z}}$ by 3.4.3. Since $(\overline{\mathcal{A}}, z)$ is an Albert division algebra, this cannot happen by [27, page 204] (see 2.6.9). Hence we always have $(\mathcal{A}, z) \not\cong (\mathcal{A}, z^{-1})$ over \mathcal{Z} though it may happen that $(\mathcal{A}, z) \cong (\mathcal{A}, z^{-1})$ over F . For example, this is the case if (\mathcal{A}, z) is a Jordan torus of Albert type below.

We start to classify Jordan tori of Albert type. Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be a Jordan torus of Albert type, i.e., the central closure \overline{J} is an Albert algebra over $\overline{\mathcal{Z}}$. Recall that an Albert algebra is a 27-dimensional central simple exceptional Jordan algebra of degree 3. By 2.6.8, the fact that J has degree 3 implies that $\text{supp}(J) = \mathbb{Z}^n$ and that there exists a basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathbb{Z}^n such that the central grading group Γ of J is

$$\Gamma = 3\mathbb{Z}\varepsilon_1 + \dots + 3\mathbb{Z}\varepsilon_m + \mathbb{Z}\varepsilon_{m+1} + \dots + \mathbb{Z}\varepsilon_n \quad \text{for some } 2 \leq m \leq n.$$

Moreover, because \overline{J} is 27-dimensional, we have $m = 3$. Thus the n -torus J has to have $n \geq 3$ and

$$\Gamma = 3\mathbb{Z}\varepsilon_1 + 3\mathbb{Z}\varepsilon_2 + 3\mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \dots + \mathbb{Z}\varepsilon_n.$$

Let $U = \bigoplus_{\alpha \in G} J_\alpha$, where

$$G := \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + 3\mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \dots + \mathbb{Z}\varepsilon_n.$$

We claim that $Z(U) = Z(J)$. Since $U \subset J$, we have $Z(U) \supset Z(J)$ and so the central grading group G_1 of U lies in between G and Γ , i.e., $G > G_1 > \Gamma$. If $\Gamma \neq G_1 \neq G$, we get $|G/G_1| = 3$ since $|G/\Gamma| = 9$. Hence the grading group of the central closure \overline{U} is $G/G_1 = \mathbb{Z}_3$. But by 2.5.6, this cannot happen. Suppose then that $G_1 = G$. Then U is commutative and associative. We will show that this cannot happen, using the method of [13, Lemma 2, page 420]:

Suppose that U is commutative and associative. Since \overline{J} is an Albert division algebra by 3.4.7, the commutative associative subalgebra $\overline{\mathcal{Z}} \otimes_{\mathcal{Z}} U$ becomes a subfield of \overline{J} , which is 9-dimensional since $|G/\Gamma| = 9$. But this is impossible. Indeed, take an algebraic closure Ω containing $\overline{\mathcal{Z}}$. Then, in \overline{J}_Ω , we have $1 = e_1 + \dots + e_9$ where e_i for $i = 1, \dots, 9$ are orthogonal idempotents. Hence by Lemma 1 in [13, page 229], the degree of \overline{J}_Ω is ≥ 9 , which is a contradiction since the degree of \overline{J}_Ω is equal to the degree of \overline{J} which is 3 (see [13, page 223]).

Thus we get $G_1 = \Gamma$, i.e., $Z(U) = Z$. In particular, we have $\bar{U} = \bar{Z} \otimes_Z U \subset \bar{J}$. Note that \bar{U} is a central subalgebra of the division algebra \bar{J} which is 9-dimensional since $|G/\Gamma| = 9$. So by the classification of finite dimensional central simple Jordan algebras, \bar{U} is special (see [13, Corollary 2, page 204 and 207]). Hence by 3.4.8, \bar{U} has degree 3. So U is a special Jordan torus of central degree 3. Therefore by 3.4.9, $\text{ch.}F \neq 3$ and U can be identified with F_ω^+ if $\omega \in F$ and with $H(E_\omega, \sigma)$ otherwise, where $E = F(\sqrt{-3}) = F(\omega)$.

We first consider the case $U = F_\omega^+$. Let x be an arbitrary nonzero element in J_{ϵ_3} . Let u_i be an arbitrary nonzero element in J_{ϵ_i} , for $i \neq 3$ and $u_3 := x^3 \in J_{3\epsilon_3}$. Then $F_\omega = F_\omega[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ is a G -torus with the $(\epsilon_1, \epsilon_2, 3\epsilon_3, \epsilon_4, \dots, \epsilon_n)$ -grading.

Let Tr be the generic trace of \bar{J} and $U^\perp = \{y \in J \mid \text{Tr}(Uy) = 0\}$. We claim that $x, x^2 \in U^\perp$. Since Tr is Z -linear and U is a free Z -module with basis $\{u_1^i u_2^j \mid i, j = 0, 1, 2\}$, it is enough to show that $\text{Tr}((u_1^i u_2^j)x^k) = 0$ for all $i, j = 0, 1, 2$ and $k = 1, 2$. Since such $(u_1^i u_2^j)x^k$ are all homogeneous and their degrees are not contained in Γ , we get, by 2.6.8, $\text{Tr}((u_1^i u_2^j)x^k) = 0$. Hence our claim is settled.

Since $x^3 = u_3 \in Z$ is invertible and $\text{Tr}(F_\omega) \subset Z$ (see 2.6.5), we have shown the conditions (i)-(iv) for $\mathcal{J} = J$, $\mathcal{U} = F_\omega^+$ and $z = u_3$ in Lemma 3.4.10. Therefore, by 3.4.10, J contains a subalgebra J' so that

- Case (I) $\varphi : (F_\omega, u_3) \xrightarrow{\sim} J'$ is a Z -isomorphism
with $\varphi|_{F_\omega} = \text{id}$ and $\varphi((0, 1, 0)) = x$. or
- Case (II) $\psi : (F_\omega, u_3^{-1}) \xrightarrow{\sim} J'$ is a Z -isomorphism
with $\psi|_{F_\omega} = \text{id}$ and $\psi((0, 0, 1)) = x$.

Note that we do not give any grading of (F_ω, u_3) yet.

Case (I): We can give the same \mathbb{Z}^n -grading to (F_ω, u_3) as the one given to the Albert torus $\mathbb{A}_t = (F_\omega, u_3)$. In particular, recall that in $\mathbb{A}_t = \bigoplus_{\alpha \in \mathbb{Z}^n} Ft_\alpha$ (see 3.4.5), $t_{\epsilon_i} = u_i$ for $i \neq 3$ and $t_{\epsilon_3} = (0, 1, 0)$.

We have $\varphi(t_{\epsilon_i}) = u_i \in J_{\epsilon_i}$, for $i \neq 3$ and $\varphi(t_{\epsilon_3}) = x \in J_{\epsilon_3}$. Thus we have the injective Z -homomorphism $\varphi : \mathbb{A}_t \longrightarrow J$ with $\varphi(Ft_{\epsilon_i}) = J_{\epsilon_i}$, for all $i = 1, \dots, n$. Since \mathbb{A}_t is of strong type, we have, for any $\alpha = \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n \in \mathbb{Z}^n$,

$$0 \neq \varphi(u_1^{\alpha_1} \cdot (u_2^{\alpha_2} \cdot (t_{\epsilon_3}^{\alpha_3} \cdot (u_4^{\alpha_4} \cdots u_n^{\alpha_n}) \dots))) \in J_\alpha,$$

and hence φ is surjective. Therefore, we obtain $J \cong \mathbb{A}_t$ over Z . Note that φ is a \mathbb{Z}^n -graded isomorphism.

Case (II): We first show that $F_\omega = F_\omega[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ satisfies the condition (v) in 3.4.10. Clearly we have $F_\omega^{\text{op}} = F_\omega[u_2^{\pm 1}, u_1^{\pm 1}, u_3^{\pm 1}, \dots, u_n^{\pm 1}]$, and the F -linear map f from F_ω into F_ω^{op} defined by

$$f(u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \dots u_n^{\alpha_n}) = u_2^{\alpha_1} u_1^{\alpha_2} u_3^{\alpha_3} \dots u_n^{\alpha_n}$$

(exchange the first two variables and leave alone the remaining variables) for all $\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \alpha_3 \epsilon_3 + \alpha_4 \epsilon_4 + \dots + \alpha_n \epsilon_n \in G$ is an F -algebra isomorphism (see 2.6.9). By (3.4.6), it is clear that $f \circ \text{Tr} = \text{Tr} \circ f$ and $f(u_3) = u_3$. Hence we have shown the condition (v) in 3.4.10, and so there exists an F -isomorphism $\tilde{f}: \mathbb{A}_t = (F_\omega, u_3) \xrightarrow{\sim} (F_\omega, u_3^{-1})$ with $\tilde{f}|_{F_\omega} = f$ and $\tilde{f}((0, 1, 0)) = (0, 0, 1)$.

Now, we give a toral grading $\langle \epsilon_2, \epsilon_1, \epsilon_3, \dots, \epsilon_n \rangle$ to $\mathbb{A}_t = (F_\omega, u_3)$ as in 3.4.5. By 3.4.10, we get an injective F -homomorphism $\psi \circ \tilde{f}: \mathbb{A}_t \rightarrow J$ with $\psi \circ \tilde{f}(Ft_{\epsilon_i}) = J_{\epsilon_i}$ for all $i = 1, \dots, n$. Since \mathbb{A}_t is of strong type, we have, for any $\alpha = \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n \in \mathbb{Z}^n$,

$$0 \neq \psi \circ \tilde{f}(u_1^{\alpha_2} \cdot (u_2^{\alpha_1} \cdot (t_{\epsilon_3}^{\alpha_3} \cdot (u_4^{\alpha_4} \dots u_n^{\alpha_n}) \dots))) \in J_\alpha,$$

and hence $\psi \circ \tilde{f}$ is surjective. Therefore, we obtain $J \cong \mathbb{A}_t$ over F . Note that $\psi \circ \tilde{f}$ is graded.

We finally consider the case $U = H(E_\omega, \sigma)$ where $E = F(\omega)$. Let $J_E = E \otimes_F J$ be the Jordan torus over E (see 2.5.4(2)). Let $\tau := \bar{} \otimes \text{id}$ be the graded F -involution of J_E (automorphism of order 2) where $\bar{}$ is the nontrivial Galois automorphism of E over F . Then $U_E = E \otimes_F U = E_\omega^+$ is an E -subalgebra of J_E , and by 3.1.9(2), we have

$$\tau|_{U_E} = \tau|_{E_\omega} = \sigma,$$

($\tau|_{U_E} = \tau|_{E_\omega^+}$ is an involution of the Jordan algebra E_ω^+ , and $\tau|_{E_\omega} = \sigma$ is an involution of the associative algebra E_ω). Also, J is the set of fixed points of J_E by τ .

Since J is exceptional, and so is J_E . Hence J_E must be of Albert type since the other two types are special (see §3.0). Since J_E contains the subalgebra $U_E = E_\omega^+$, we can apply the previous argument. Namely, choose $0 \neq u_i \in J_{\epsilon_i} \subset E \otimes_F J_{\epsilon_i}$, $i \neq 3$ and $0 \neq x \in J_{\epsilon_3} \subset E \otimes_F J_{\epsilon_3}$. Let $u_3 := x^3$. $E_\omega = E_\omega[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ and $\mathbb{A}_t = (E_\omega, u_3)$ the Albert torus over E . Then we have two cases:

Case (I): There exists an isomorphism $J_E \xrightarrow{\sim} \mathbb{A}_t$ such that the restriction to E_ω is the identity map and $x \mapsto (0, 1, 0)$.

Case (II): There exists an isomorphism $l : J_E \xrightarrow{\sim} \mathbb{A}_t$ such that $l|_{E_\omega}$ is an automorphism of the associative algebra E_ω , $l(u_1) = u_2$, $l(u_2) = u_1$ and $l(x) = (0, 1, 0)$.

Case (I): Identifying $J_E = \mathbb{A}_t$ and $x = (0, 1, 0)$, we can consider the involution τ of \mathbb{A}_t . Since all u_i and x are contained in $J = 1 \otimes J$, we have $\tau(u_i) = u_i$ and $\tau(x) = x$. Also, one sees that

$$\tau(u_1 u_2) = \sigma(u_1 u_2) = u_2 u_1.$$

Let tr be the generic trace of E_ω (see 3.4.5). Since $\text{tr}(u_1) = \text{tr}(u_2) = \text{tr}(u_1 u_2) = 0$, we have $\overline{u_1} = -\frac{1}{2}u_1$, $\overline{u_2} = -\frac{1}{2}u_2$ and $\overline{u_1 u_2} = -\frac{1}{2}u_1 u_2$. Hence by (3.4.0),

$$\begin{aligned} (*) \quad (u_1 u_2) \cdot x &= (u_1 u_2) \cdot (0, 1, 0) = (0, -\frac{1}{2}u_1 u_2, 0) \\ &= u_1 \cdot (0, u_2, 0) = -2u_1 \cdot (u_2 \cdot x). \end{aligned}$$

On the other hand,

$$\begin{aligned} (**) \quad (\omega u_1 u_2) \cdot x &= (u_2 u_1) \cdot x = \tau(u_1 u_2) \cdot \tau(x) \\ &= \tau((u_1 u_2) \cdot x) = -2\tau(u_1 \cdot (u_2 \cdot x)) \quad \text{by } (*) \\ &= -2\tau(u_1) \cdot (\tau(u_2) \cdot \tau(x)) = -2u_1 \cdot (u_2 \cdot x), \end{aligned}$$

and so we get $(u_1 u_2) \cdot x = (\omega u_1 u_2) \cdot x = \omega(u_1 u_2) \cdot x$, which is absurd since $(u_1 u_2) \cdot x \neq 0$.

Case (II): Define the involution τ' of \mathbb{A}_t so that $l \circ \tau = \tau' \circ l$. Then we have

$$\tau'(u_1 u_2) = l \circ \tau \circ l^{-1}(u_1 u_2) = l \circ \tau(u_2 u_1) = l(u_1 u_2) = u_2 u_1,$$

and similarly, one gets $\tau'(u_1) = u_1$, $\tau'(u_2) = u_2$ and $\tau'(l(x)) = l(x)$. Therefore, if we change τ into τ' and x into $l(x)$ in (*) and (**), both equations hold. So we get a contradiction.

Therefore, the case $\omega \notin F$ cannot happen. Consequently, we have proven the following:

Theorem 3.4.12. *Let J be a Jordan n -torus of Albert type over F . Then $n \geq 3$. F contains a primitive 3rd root of unity and $J \cong_{\mathbb{Z}^n} \mathbb{A}_t$ for some toral grading. Conversely, \mathbb{A}_t is a Jordan n -torus of Albert type. \square*

Combined with Proposition 3.4.9. we get the following result which is used in [2. Proposition 2.17, page 15] to classify extended affine Lie algebras of type G_2 :

Corollary 3.4.13. *Let J be a Jordan torus over F of central degree 3. Then $\text{ch}.F \neq 3$ and*

$$\begin{cases} J \cong_{\mathbb{Z}^n} F_{\omega}^+ \text{ or } \mathbb{A}_t & \text{if } \omega \in F \\ J \cong_{\mathbb{Z}^n} H(E_{\omega}, \sigma) & \text{otherwise} \end{cases}$$

where $E = F(\omega)$. Conversely, the algebras F_{ω}^+ , $H(E_{\omega}, \sigma)$ and \mathbb{A}_t are Jordan tori of central degree 3. \square

By 3.1.10, 3.3.10 and 3.4.12, we complete the classification of Jordan tori:

Theorem 3.4.14. *Let J be a Jordan n -torus over F . Then J is graded isomorphic to one of the four special Jordan tori*

$$F_q^+, H(F_{\epsilon}, *), H(E_{\xi}, \sigma), \text{ or a Clifford torus } J_{S(m)}(\{a_{\epsilon}\}_{\epsilon \in I}),$$

or J is graded isomorphic to the Albert torus \mathbb{A}_t if $n \geq 3$ and F contains a primitive 3rd root of unity. \square

Also, by 3.3.12, we have the following:

Corollary 3.4.15. *Let J be a Jordan n -torus over an algebraically closed field F . Then J is graded isomorphic to one of the three special Jordan tori*

$$F_q^+, H(F_{\epsilon}, *) \text{ or a standard Clifford torus } J_{S(m)},$$

or J is graded isomorphic to the Albert torus \mathbb{A}_t if $n \geq 3$ and $\text{ch}.F \neq 3$. \square

Chapter 4 Graded alternative algebras

In this chapter we classify predivision \mathbb{Z}^n -graded associative algebras and division \mathbb{Z}^n -graded alternative algebras over any field.

§ 4.0 CROSSED PRODUCT ALGEBRAS

In this section we review crossed product algebras (see [24]).

Definition 4.0.1. Let R be a unital associative algebra over F and G a group. Let $R * G$ be the free left R -module with basis $\bar{G} = \{\bar{g} \mid g \in G\}$, a copy of G . Define a multiplication on $R * G$ by linear extension of

$$(\tau\bar{g})(s\bar{h}) = r\sigma_g(s)\tau(g, h)\overline{gh},$$

for $r, s \in R$ and $g, h \in G$, where

(action) $\sigma : G \longrightarrow \text{Aut}_F(R)$, the group of F -automorphisms of R ,

(twisting) $\tau : G \times G \longrightarrow U(R)$, the group of units of R .

are arbitrary maps and $\sigma_g := \sigma(g)$. This $R * G = (R, G, \sigma, \tau)$ is called a *crossed product algebra over F* if this multiplication is associative. It is easily seen that this is in fact an algebra over F . If there is no action or twisting, that is, if $\sigma_g = \text{id}$ and $\tau(g, h) = 1$ for all $g, h \in G$, then $R * G = R[G]$ is the ordinary *group algebra*. If the action is trivial, then $R * G =: R^t[G]$ is called a *twisted group algebra*. Finally, if the twisting is trivial, then $R * G =: RG$ is called a *skew group algebra*.

Remark 4.0.2. If a crossed product algebra $R * G$ is commutative, then the action is clearly trivial, and so $R * G = R^t[G]$.

The following lemma characterizes σ and τ (see [24, Lemma 1.1, page 2]). We denote by $I(d)$ the inner automorphism determined by $d \in U(R)$, i.e., $I(d)(r) = drd^{-1}$ for $r \in R$.

Lemma 4.0.3. *The associativity of $R * G$ is equivalent to the following two conditions: for all $g, h, k \in G$,*

- (i) $\sigma_g \sigma_h = I(\tau(g, h)) \sigma_{gh}$,
- (ii) $\sigma_g(\tau(h, k)) \tau(g, hk) = \tau(g, h) \tau(gh, k)$.

Proof. Straightforward. \square

Remark 4.0.4. If R is commutative, then the action $\sigma : G \longrightarrow \text{Aut}_F(R)$ becomes a group homomorphism by condition (i) in 4.0.3. So the action is really an “group action” in usual sense. Also, for a skew group algebra RG , the action becomes a group homomorphism for the same reason. Conversely, any group action $G \longrightarrow \text{Aut}_F(R)$ defines a skew group algebra RG .

If $d : G \longrightarrow U(R)$ assigns to each element $g \in G$ a unit d_g , then $\tilde{G} = \{d_g \bar{g} \mid g \in G\}$ yields another R -basis for $R * G$ so that $R * G$ is a crossed product algebra for the new basis. We call this a *diagonal change of basis* ([24, page 3]). Any crossed product algebra has an identity element. It is of the form $1 = u\bar{e}$ for some unit u in R ([24, Exercise 2, page 9]). We can and will assume that $1 = \bar{e}$, via a diagonal change of basis, and so $\tau(g, e) = \tau(e, g) = 1$ for all $g \in G$. The embedding of R into $R * G$ is then given by $r \mapsto r\bar{e}$. Also, we have ([24, page 3])

$$(4.0.5) \quad r\bar{g} \text{ is invertible if and only if } r \in U(R).$$

Now, it is clear that a crossed product algebra $R * G = \bigoplus_{g \in G} R\bar{g}$ is a predivision G -graded associative algebra (see §2.1). Conversely, suppose that $A = \bigoplus_{g \in G} A_g$ is a predivision G -graded associative algebra over F . Then, by Lemma 2.1.5(1), we have $A = \bigoplus_{g \in G} Rx_g$ where $R = A_e$ and an invertible element $x_g \in A_g$. Moreover, for $h \in H$, we have $x_g x_h = x_g x_h (x_{gh})^{-1} x_{gh}$. So we put $\tau(g, h) := x_g x_h (x_{gh})^{-1} \in U(R)$. Then we have $x_g x_h = \tau(g, h) x_{gh}$. Also, let $I(x_g)$ be the inner automorphism determined by x_g and let $\sigma_g := I(x_g) |_R$. Then, σ_g is clearly an F -automorphism of R and for $r, r' \in R$

$$(rx_g)(r'x_h) = r(x_g r' x_g^{-1}) x_g x_h = r \sigma_g(r') x_g x_h = r \sigma_g(r') \tau(g, h) x_{gh}.$$

Hence, A is a crossed product algebra $R * G$ determined by these σ and τ . So the two concepts, a crossed product algebra $R * G$ and a predivision G -graded associative

algebra. coincide (see [24, Exercise 2, page 18]). In particular, a division G -graded associative algebra is a crossed product algebra $R * G$ where R is a division algebra, and a quantum torus is a twisted group algebra $F^t[\mathbb{Z}^n]$.

Lemma 4.0.6. *Let $Z = \bigoplus_{h \in H} Z_h$ be a predivision H -graded commutative associative algebra over a field F for a group H (which is necessarily abelian). Then, Z is a twisted group algebra $R^t[H]$ where $R := Z_e$.*

If the abelian group H is free, then it is a group algebra $R[H]$. In particular, when $G = \mathbb{Z}^n$, it is nothing but the algebra of Laurent polynomials in n -variables over R , $Z = R[z_1^\pm, \dots, z_n^\pm]$ for any choice of nonzero element $z_i \in Z_{\epsilon_i}$, $i = 1, \dots, n$, where $\{\epsilon_1, \dots, \epsilon_n\}$ is basis of \mathbb{Z}^n .

Proof. See [24, Exercise 6, page 10]. \square

§ 4.1 PREDIVISION \mathbb{Z}^n -GRADED ASSOCIATIVE ALGEBRAS

Our purpose is to classify division \mathbb{Z}^n -graded associative algebras. However, since our method works for predivision \mathbb{Z}^n -graded associative algebras or a crossed product algebra $R * \mathbb{Z}^n$, we classify the associative algebras $R * \mathbb{Z}^n$.

For a G -graded algebra $S = \bigoplus_{g \in G} S_g$ over F in general, we denote by $\text{GrAut}_F(S)$ the group of graded automorphisms of S , i.e.,

$$\text{GrAut}_F(S) := \{\sigma \in \text{Aut}_F(S) \mid \sigma(S_g) = S_g \text{ for all } g \in G\}.$$

The following lemma is useful later.

Lemma 4.1.1. *Let $R * G = (R, G, \sigma, \tau)$ be a crossed product algebra over F and $(R * G) * M = (R * G, M, \eta, \xi)$ a crossed product algebra over F for a group M , an action η and a twisting ξ . Suppose that $\eta(M) \subset \text{GrAut}_F(R * G)$ and that $\xi(m, l) \in U(R)$ for all $m, l \in M$. Then, $(R * G) * M$ is a crossed product algebra $R * (G \times M) = (R, (G \times M), \sigma', \tau')$ over F for some action σ' and twisting τ' .*

Proof. We have

$$(R * G) * M = \bigoplus_{m \in M} (R * G)\bar{m} = \bigoplus_{m \in M} \left(\bigoplus_{g \in G} R\bar{g} \right) \bar{m} = \bigoplus_{(g, m) \in G \times M} R\bar{g}\bar{m}$$

as free R -modules. We define $\eta_m = \eta(m) |_{R1}$ an F -automorphism of R for every $m \in M$. Also for $h \in G$, \bar{h} is a unit in $R * G$ (see 4.0.5). Since η_m is a graded

automorphism of $R * G$ by our first assumption. $\eta_m(\bar{h}) = d_{m,h}\bar{h}$ for some $d_{m,h} \in U(R)$. Therefore, for $r\bar{g}\bar{m} \in R\bar{g}\bar{m}$ and $s\bar{h}\bar{l} \in R\bar{h}\bar{l}$, we have

$$\begin{aligned}
(r\bar{g}\bar{m})(s\bar{h}\bar{l}) &= r\bar{g}\eta_m(s\bar{h})\bar{m}\bar{l} \\
&= r\bar{g}\eta_m(s)\eta_m(\bar{h})\xi(m,l)\bar{m}\bar{l} \\
&= r\bar{g}\eta_m(s)d_{m,h}\bar{h}\xi(m,l)\bar{m}\bar{l} \\
&= r\bar{g}\eta_m(s)d_{m,h}\sigma_h(\xi(m,l))\bar{h}\bar{m}\bar{l} \quad (\text{by our second assumption}) \\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}(\xi(m,l))\bar{g}\bar{h}\bar{m}\bar{l} \\
&= r\sigma_g\eta_m(s)\sigma_g(d_{m,h})\sigma_{gh}(\xi(m,l))\tau(g,h)\bar{g}\bar{h}\bar{m}\bar{l}.
\end{aligned}$$

Thus we have the action

$$\sigma' : G \times M \longrightarrow \text{Aut}_F R \quad \text{by} \quad \sigma'_{(g,m)} = \sigma_g\eta_m,$$

and the twisting $\tau' : (G \times M) \times (G \times M) \longrightarrow U(R)$ by

$$\tau'((g,m),(h,l)) = \sigma_g(d_{m,h})\sigma_{gh}(\xi(m,l))\tau(g,h).$$

Since the crossed product algebra $(R * G) * M$ is associative, we get

$$(R * G) * M = R * (G \times M) = (R, G \times M, \sigma', \tau'). \quad \square$$

A triple (R, φ, \mathbf{q}) where R is a unital associative algebra over F ,

$$\varphi = (\varphi_1, \dots, \varphi_n)$$

is an n -tuple of F -automorphisms φ_i of R , and $\mathbf{q} = (q_{ij})$ is an $n \times n$ matrix over R satisfying, for all $1 \leq i < j < k \leq n$,

$$(G1) \quad q_{ii} = 1 \quad \text{and} \quad q_{ji}^{-1} = q_{ij},$$

$$(G2) \quad \varphi_j\varphi_i = I(q_{ij})\varphi_i\varphi_j,$$

$$(G3) \quad \varphi_k(q_{ij}) = q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki},$$

is called a \mathbb{Z}^n -grading triple, and a *division \mathbb{Z}^n -grading triple* if R is a division algebra.

For a \mathbb{Z}^n -grading triple, we introduce several notations and prove some identities.

Notations.

$$(N1) \quad \epsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n,$$

i.e., the i -th coordinate is 1 and the others are 0.

$$(N2) \quad q_{ij}^{(m)} := \begin{cases} q_{ij}\varphi_i(q_{ij})\varphi_i^2(q_{ij})\cdots\varphi_i^{m-1}(q_{ij}) = \prod_{l=0}^{m-1} \varphi_i^l(q_{ij}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^{-1}(q_{ji})\varphi_i^{-2}(q_{ji})\cdots\varphi_i^m(q_{ji}) = \prod_{l=-1}^m \varphi_i^l(q_{ji}), & \text{if } m < 0, \end{cases}$$

and $q_{ij}^{-{(m)}} := (q_{ij}^{(m)})^{-1}$.

For $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ and $k = 0, 1, 2, \dots, n$.

$$(N3) \quad \varphi^{(\alpha)_k} := \begin{cases} \text{id}, & \text{if } k = 0, 1 \\ \varphi_1^{\alpha_1} \cdots \varphi_{k-1}^{\alpha_{k-1}}, & \text{if } k > 1, \end{cases}$$

and $\varphi^\alpha := \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n}$.

$$(N4) \quad q_{\epsilon_j, \alpha} := \prod_{i=1}^{j-1} \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)}) \quad \text{with } \alpha_0 = q_{0j} = 1.$$

$$(N5) \quad q_{\epsilon_j, \alpha}^{(m)} := \begin{cases} \prod_{l=m-1}^0 \varphi_j^l(q_{\epsilon_j, \alpha}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \prod_{l=m}^{-1} \varphi_j^l(q_{\epsilon_j, \alpha}^{-1}), & \text{if } m < 0. \end{cases}$$

$$(N6) \quad q_{\alpha, \beta} := \prod_{j=n}^1 \varphi^{(\alpha)_j}(q_{\epsilon_j, \beta}^{(\alpha_j)}).$$

Lemma 4.1.2. For $m \in \mathbb{Z}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, we have

- (1) $\varphi_i^{-m}(q_{ij}^{-{(m)}}) = q_{ij}^{(-m)}$,
- (2) $\varphi_j \varphi_i^m = I(q_{ij}^{(m)}) \varphi_i^m \varphi_j$,
- (3) $\varphi_j \varphi^{(\alpha)_i} = \begin{cases} I(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lj}^{(\alpha_l)})) \varphi^{(\alpha)_i} \varphi_j & \text{for } j \geq i, \\ I(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l}(q_{lj}^{(\alpha_l)})) \varphi^{(\alpha+\epsilon_j)_i} & \text{for } j < i, \end{cases}$
- (4) $q_{ij}^{(m+1)} = q_{ij} \varphi_i(q_{ij}^{(m)})$ and $q_{ij}^{-{(m+1)}} = \varphi_i(q_{ij}^{-{(m)}}) q_{ji}$,
- (5) $\varphi_k(q_{ij}^{(m)}) = q_{jk} \varphi_j(q_{ik}^{(m)}) q_{ij}^{(m)} \varphi_i^m(q_{kj}) q_{ik}^{-{(m)}}$.

Proof. For (1), we have from (N2),

$$q_{ij}^{-(m)} = \begin{cases} \varphi_i^{m-1}(q_{ji}) \cdots \varphi_i(q_{ji})q_{ji} = \prod_{l=m-1}^1 \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ \varphi_i^m(q_{ij}) \cdots \varphi_i^{-2}(q_{ij})\varphi_i^{-1}(q_{ij}) = \prod_{l=m}^{-1} \varphi_i^l(q_{ij}), & \text{if } m < 0. \end{cases}$$

So we get

$$\varphi_i^{-m}(q_{ij}^{-(m)}) = \begin{cases} \varphi_i^{-1}(q_{ji}) \cdots \varphi_i^{-m}(q_{ji}) = \prod_{l=-1}^{-m} \varphi_i^l(q_{ji}), & \text{if } m > 0 \\ 1, & \text{if } m = 0 \\ q_{ij}\varphi_i(q_{ij}) \cdots \varphi_i^{-m-1}(q_{ij}) = \prod_{l=1}^{-m-1} \varphi_i^l(q_{ij}), & \text{if } m < 0, \end{cases}$$

which is exactly $q_{ij}^{(-m)}$.

For (2), the case $m = 0$ is clear. Assume that $m > 0$. Put $q := q_{ij}$ for simplicity. Then, we have

$$\begin{aligned} \varphi_j \varphi_i^m &= \varphi_j \varphi_i^{m-1} \varphi_i \\ &= I(q^{(m-1)}) \varphi_i^{m-1} \varphi_j \varphi_i \quad \text{by induction on } m \\ &= I(q^{(m-1)}) \varphi_i^{m-1} I(q) \varphi_i \varphi_j \quad \text{by (G2)} \\ &= I(q^{(m-1)}) I(\varphi_i^{m-1}(q)) \varphi_i^m \varphi_j \\ &= I(q^{(m)}) \varphi_i^m \varphi_j. \end{aligned}$$

Also, we have $(\varphi_j \varphi_i^m)^{-1} = (I(q_{ij}^{(m)}) \varphi_i^m \varphi_j)^{-1}$ for $m > 0$. So we have

$$\varphi_i^{-m} \varphi_j^{-1} = \varphi_j^{-1} \varphi_i^{-m} (I(q_{ij}^{-(m)})) = \varphi_j^{-1} I(\varphi_i^{-m}(q_{ij}^{-(m)})) \varphi_i^{-m} = \varphi_j^{-1} I(q_{ij}^{(-m)}) \varphi_i^{-m}.$$

by (1). Hence, we get $\varphi_j \varphi_i^{-m} = I(q_{ij}^{(-m)}) \varphi_i^{-m} \varphi_j$, and (2) holds for all $m \in \mathbb{Z}$.

For (3), when $j \geq i$, using (2), we have

$$\begin{aligned} \varphi_j \varphi_i^{(\alpha)_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\ &= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\ &= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} I(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} \varphi_j \varphi_3^{\alpha_3} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\ &= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} I(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} I(q_{3j}^{(\alpha_3)}) \varphi_3^{\alpha_3} \varphi_j \varphi_4^{\alpha_4} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\ &\dots\dots \\ &= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} I(q_{2j}^{(\alpha_2)}) \varphi_2^{\alpha_2} I(q_{3j}^{(\alpha_3)}) \varphi_3^{\alpha_3} \cdots I(q_{i-1,j}^{(\alpha_{i-1})}) \varphi_{i-1}^{\alpha_{i-1}} \varphi_j \\ &= I\left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lj}^{(\alpha_l)})\right) \varphi^{(\alpha)_i} \varphi_j. \end{aligned}$$

When $j < i$, we have

$$\begin{aligned}
\varphi_j \varphi_i^{(\alpha)_i} &= \varphi_j \varphi_1^{\alpha_1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \varphi_j \varphi_2^{\alpha_2} \cdots \varphi_j^{\alpha_j} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&\dots\dots \\
&= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots I(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} I(q_{jj}^{(\alpha_j)}) \varphi_j^{\alpha_j} \varphi_j \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= I(q_{1j}^{(\alpha_1)}) \varphi_1^{\alpha_1} \cdots I(q_{j-1,j}^{(\alpha_{j-1})}) \varphi_{j-1}^{\alpha_{j-1}} \varphi^{\alpha_j+1} \cdots \varphi_{i-1}^{\alpha_{i-1}} \\
&= I\left(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l}(q_{lj}^{(\alpha_l)})\right) \varphi^{(\alpha+\epsilon_j)_i}.
\end{aligned}$$

For the first formula of (4), the case $m = 0$ is clear. We put $q := q_{ij}$, $p := q^{-1}$ and $\varphi := \varphi_i$ for simplicity. For $m > 0$, we have

$$\begin{aligned}
q^{(m+1)} &= q\varphi(q)\varphi^2(q) \cdots \varphi^m(q) \\
&= q\varphi(q\varphi(q) \cdots \varphi^{m-1}(q)) = q\varphi(q^{(m)}).
\end{aligned}$$

For $m = -1$, we have $q^{(-1+1)} = 1$, while $q\varphi(q^{(-1)}) = q\varphi\varphi^{-1}(p) = 1$. For $m < -1$, we have

$$\begin{aligned}
q^{(m+1)} &= \varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= qp\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^{m+1}(p) \\
&= q\varphi(\varphi^{-1}(p)\varphi^{-2}(p) \cdots \varphi^m(p)) = q\varphi(q^{(m)}).
\end{aligned}$$

The second formula follows from the first since $q_{ij}^{-(m+1)} = (q_{ij}^{(m+1)})^{-1}$.

For (5), the case $m = 0$ is clear. Assume that $m > 0$. Then, we have

$$\begin{aligned}
&\varphi_k(q_{ij}^{(m)}) \\
&= \varphi_k(q_{ij})\varphi_k\varphi_i(q_{ij}^{(m-1)}) \quad \text{by (3)} \\
&= \varphi_k(q_{ij})q_{ik}\varphi_i\varphi_k(q_{ij}^{(m-1)})q_{ki} \quad \text{by (G2)} \\
&= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i(q_{kj})q_{ki}q_{ik}\varphi_i(q_{jk}\varphi_j(q_{ik}^{(m-1)}))q_{ij}^{(m-1)}\varphi_i^{m-1}(q_{kj})(q_{ik}^{-(m-1)})q_{ki}
\end{aligned}$$

by (G3) and induction on m

$$\begin{aligned}
&= q_{jk}\varphi_j(q_{ik})q_{ij}\varphi_i\varphi_j(q_{ik}^{(m-1)})\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m-1)})q_{ki} \\
&= q_{jk}\varphi_j(q_{ik})q_{ij}q_{ji}\varphi_j\varphi_i(q_{ik}^{(m-1)})q_{ij}\varphi_i(q_{ij}^{(m-1)})\varphi_i^m(q_{kj})\varphi_i(q_{ik}^{-(m)})
\end{aligned}$$

by (G2) and (3)

$$\begin{aligned} &= q_{jk} \varphi_j(q_{ik}) \varphi_j \varphi_i(q_{ik}^{(m-1)}) \varphi_i(q_{ij}^{(m)}) \varphi_i^m(q_{kj}) \varphi_i(q_{ik}^{-(m)}) \quad \text{by (3)} \\ &= q_{jk} \varphi_j(q_{ik}^{(m)}) q_{ij}^{(m)} \varphi_i^m(q_{kj}) q_{ik}^{-(m)} \quad \text{by (3)}. \end{aligned}$$

Also, we have $(\varphi_k(q_{ij}^{(m)}))^{-1} = (q_{jk} \varphi_j(q_{ik}^{(m)}) q_{ij}^{(m)} \varphi_i^m(q_{kj}) q_{ik}^{-(m)})^{-1}$ for $m > 0$, and so we have $\varphi_k(q_{ji}^{-(m)}) = q_{ik}^{(m)} \varphi_i^m(q_{jk}) q_{ij}^{-(m)} \varphi_j(q_{ik}^{-(m)}) q_{kj}$. Applying φ_i^{-m} in both hands, we get

$$\begin{aligned} \varphi_i^{-m} \varphi_k(q_{ij}^{-(m)}) &= \varphi_i^{-m} (q_{ik}^{(m)} \varphi_i^m(q_{jk}) q_{ij}^{-(m)} \varphi_j(q_{ik}^{-(m)}) q_{kj}) \\ &= \varphi_i^{-m} (q_{ik}^{(m)}) q_{jk} q_{ij}^{(-m)} \varphi_i^{-m} \varphi_j(q_{ik}^{-(m)}) \varphi_i^{-m}(q_{kj}) \quad \text{by (1)}. \end{aligned}$$

Then, by (1) and (2), we have

$$I(q_{ik}^{(-m)}) \varphi_k(q_{ij}^{(-m)}) = q_{ik}^{(-m)} q_{jk} q_{ij}^{(-m)} I(q_{ij}^{(-m)}) \varphi_j(q_{ik}^{(-m)}) \varphi_i^{-m}(q_{kj}),$$

and we obtain

$$\varphi_k(q_{ij}^{(-m)}) = q_{jk} \varphi_j(q_{ik}^{(-m)}) q_{ij}^{(-m)} \varphi_i^{-m}(q_{kj}) q_{ik}^{(-m)} \quad \text{for } m > 0.$$

Hence, (5) holds for all $m \in \mathbb{Z}$. \square

Now we are ready to state our theorem.

Theorem 4.1.3. *Let (R, φ, q) be a \mathbb{Z}^n -grading triple and let $R_{\varphi, q} := \bigoplus_{\alpha \in \mathbb{Z}^n} Rt_{\alpha}$ be a free left R -module with basis $\{t_{\alpha} \mid \alpha \in \mathbb{Z}^n\}$.*

Then, there exists a unique associative multiplication on $R_{\varphi, q}$ such that, for $t_i := t_{\epsilon_i}$, $i = 1, \dots, n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$(4.1.4) \quad t_{\alpha} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}, \quad t_i r = \varphi_i(r) t_i, \quad t_j t_i = q_{ij} t_i t_j.$$

For $rt_{\alpha}, r' t_{\beta} \in R_{\varphi, q}$, we have

$$rt_{\alpha} r' t_{\beta} = r \varphi^{\alpha}(r') q_{\alpha, \beta} t_{\alpha + \beta}.$$

*In particular, $R_{\varphi, q}$ is a crossed product algebra $R * \mathbb{Z}^n$ with*

$$\begin{aligned} (\text{action}) \quad \sigma : \mathbb{Z}^n &\longrightarrow \text{Aut}_F(R) \quad \text{by } \sigma(\alpha) = \varphi^{\alpha} \\ (\text{twisting}) \quad \tau : \mathbb{Z}^n \times \mathbb{Z}^n &\longrightarrow U(R) \quad \text{by } \tau(\alpha, \beta) = q_{\alpha, \beta}. \end{aligned}$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n$, there exists a \mathbb{Z}^n -grading triple (R, φ, \mathbf{q}) such that $R * \mathbb{Z}^n = R_{\varphi, \mathbf{q}}$.

Proof. We first consider a crossed product algebra $R * \mathbb{Z}$. Let $t := \bar{1} \in R * \mathbb{Z}$. Then, t^m is a unit in $R\bar{m}$ for all $m \in \mathbb{Z}$. Using the diagonal basis change, we can take an R -basis $\{t^m \mid m \in \mathbb{Z}\}$. So we have $t^m t^l = t^{m+l}$ for all $m, l \in \mathbb{Z}$. Hence, $R * \mathbb{Z} = R\mathbb{Z}$ is a skew group algebra. Let ψ be the action of 1. i.e., $t(r1) = \psi(r)t$ for $r \in R$. (Note that $1 = \bar{0}$.) Then, the action of m is ψ^m , i.e.,

$$t^m(r1) = \psi^m(r)t^m.$$

Conversely, it is clear that any F -automorphism ψ of R determines a skew group algebra $R\mathbb{Z}$ by the action $m \mapsto \psi^m$ (see Remark 4.0.4). We denote this $R\mathbb{Z}$ by $R[t, t^{-1}; \psi]$ and call it the *skew Laurent polynomial algebra of R over F* .

Let $R^{(1)} := R\mathbb{Z} = R[t_1, t_1^{-1}; \psi_1]$ be the skew Laurent polynomial algebra of R over F where $\psi_1 = \varphi_1$. Let ψ_2 be a graded F -automorphism ψ_2 of $R^{(1)}$ and $R^{(2)} := R^{(1)}\mathbb{Z} = R^{(1)}[t_2, t_2^{-1}; \psi_2]$. Then, by Lemma 4.1.1, we get $R^{(2)} = (R\mathbb{Z})\mathbb{Z} = R * \mathbb{Z}^2$. Repeating this process n times, we can construct $R * \mathbb{Z}^n$ inductively. Namely, for a crossed product algebra $R^{(k-1)} = R * \mathbb{Z}^{k-1}$, if we specify an F -graded automorphism ψ_k of $R^{(k-1)}$, then we can construct

$$R^{(k)} := R^{(k-1)}\mathbb{Z} = R^{(k-1)}[t_k, t_k^{-1}; \psi_k] = R * \mathbb{Z}^k,$$

and we obtain $R^{(n)} = R * \mathbb{Z}^n$. Thus, our task is to specify ψ_k on $R^{(k-1)}$ and to show that ψ_k is a graded F -automorphism where $k \geq 2$. We note that

$$\{t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}} \mid (\alpha_1, \dots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}\}$$

is a basis of the free R -module $R^{(k-1)}$. For convenience, we put

$$t^{(\alpha)_k} = t_1^{\alpha_1} \cdots t_{k-1}^{\alpha_{k-1}},$$

and define an F -linear transformation ψ_k on $R^{(k-1)}$ by

$$\psi_k(rt^{(\alpha)_k}) = \varphi_k(r) \left[\prod_{i=1}^{k-1} \varphi^{(\alpha)_i}(q_{ik}^{(\alpha_i)}) \right] t^{(\alpha)_k} \quad \text{for } r \in R,$$

which is clearly graded. If $\psi_k(rt^{(\alpha)_k}) = 0$, then $\varphi_k(r) = 0$, and hence $r = 0$. and so ψ_k is injective. Since

$$\psi_k\left(\varphi_k^{-1}\left(r\left[\prod_{i=1}^{k-1}\varphi^{(\alpha)_i}(q_{ik}^{(\alpha)_i})\right]^{-1}\right)t^{(\alpha)_k}\right) = rt^{(\alpha)_k}.$$

ψ_k is surjective. Therefore, ψ_k is an F -linear graded isomorphism on $R^{(k-1)}$. So it remains to prove that ψ_k is a homomorphism. For this purpose, we use a well-known lemma.

Lemma 4.1.5. *Let A and B be unital associative algebras over F and f a F -linear map from A into B . Let $\{t_i\}_{i \in I}$ be a generating set of the F -algebra A . Then, f is a homomorphism if and only if $f(t_i y) = f(t_i)f(y)$ for all $i \in I$ and $y \in A$. Moreover, if $\{t_i^{\pm 1}\}_{i \in I}$ is a generating set of A , then f is a homomorphism if and only if $f(t_i y) = f(t_i)f(y)$ and $f(t_i^{-1}) = f(t_i)^{-1}$ for all $i \in I$ and $y \in A$.*

Proof. Straightforward. \square

We have a generating set $R \cup \{t_1^{\pm 1}, \dots, t_{k-1}^{\pm 1}\}$ of $R^{(k-1)}$ over F , and

$$\begin{aligned}\psi_k(t_j^{-1}) &= q_{jk}^{(-1)}t_j^{-1} = \varphi_j^{-1}(q_{kj})t_j^{-1} \\ &= (t_j\varphi_j^{-1}(q_{jk}))^{-1} = (q_{jk}t_j)^{-1} = \psi_k(t_j)^{-1}.\end{aligned}$$

So by Lemma 4.1.5, we only need to show that, for all $r, r' \in R$ and $1 \leq j \leq k-1$,

- (i) $\psi_k(rr't^{(\alpha)_k}) = \psi_k(r)\psi_k(r't^{(\alpha)_k})$,
- (ii) $\psi_k(t_j r t^{(\alpha)_k}) = \psi_k(t_j)\psi_k(r t^{(\alpha)_k})$.

For (i), we have

$$\begin{aligned}\psi_k(rr't^{(\alpha)_k}) &= \varphi_k(rr') \prod_{i=1}^{k-1} \varphi^{(\alpha)_i}(q_{ik}^{(\alpha)_i})t^{(\alpha)_k} \\ &= \varphi_k(r)\varphi_k(r') \prod_{i=1}^{k-1} \varphi^{(\alpha)_i}(q_{ik}^{(\alpha)_i})t^{(\alpha)_k} \\ &= \psi_k(r)\psi_k(r't^{(\alpha)_k}).\end{aligned}$$

For (ii), we first note that there is the embedding of $R^{(j)}$ into $R^{(k-1)}$ for $1 \leq j \leq k-1$, and so we have

$$t_j t^{(\alpha)_j} = \psi_j(t^{(\alpha)_j})t_j = \varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\alpha)_i}(q_{ij}^{(\alpha)_i})t^{(\alpha)_j}t_j.$$

Thus, we have

$$\begin{aligned}
\psi_k(t_j r t^{(\alpha)_k}) &= \psi_k(\varphi_j(r) t_j t^{(\alpha)_k}) \\
&= \psi_k(\varphi_j(r) (\psi_j(t^{(\alpha)_j}) t_j^{\alpha_j+1} \dots t_{k-1}^{\alpha_{k-1}})) \\
&= \psi_k(\varphi_j(r) \prod_{i=1}^{j-1} \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)}) t^{(\alpha+\epsilon_j)_k}) \\
&= \varphi_k \varphi_j(r) \prod_{i=1}^{j-1} \varphi_k \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)}) \prod_{i=1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})}) t^{(\alpha+\epsilon_j)_k} \\
&:= ABC t^{(\alpha+\epsilon_j)_k},
\end{aligned}$$

where $A = \varphi_k \varphi_j(r)$, $B = \prod_{i=1}^{j-1} \varphi_k \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)})$ and $C = \prod_{i=1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i}(q_{ik}^{(\alpha_i+\delta_{ij})})$.
First of all, we have

$$A = \varphi_k \varphi_j(r) = q_{jk} \varphi_j \varphi_k(r) q_{kj} \quad \text{by (G2).}$$

Secondly, by Lemma 4.1.2(2) and (4), we have

$$\begin{aligned}
&\varphi_k \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)}) \\
&= \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right) \varphi^{(\alpha)_i} \varphi_k(q_{ij}^{(\alpha_i)}) \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right)^{-1} \\
&= \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right) \varphi^{(\alpha)_i} (q_{jk} \varphi_j(q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)} \varphi_i(q_{kj} q_{ik}^{-(\alpha_i)})) \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right)^{-1}.
\end{aligned}$$

Note that

$$\begin{aligned}
\varphi^{(\alpha)_i}(q_{ki}^{-(\alpha_i)}) \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right)^{-1} &= \left(\prod_{l=1}^i \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right)^{-1} \\
\text{and } \varphi^{(\alpha)_i} \varphi_i^{\alpha_i}(q_{kj}) &= \varphi^{(\alpha)_{i+1}}(q_{kj}).
\end{aligned}$$

So we have

$$\begin{aligned}
(\varphi_k \varphi^{(\alpha)_i}(q_{ij}^{(\alpha_i)})) (\varphi_k \varphi^{(\alpha)_{i+1}}(q_{i+1,j}^{(\alpha_{i+1})})) &= \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right) \varphi^{(\alpha)_i} (q_{jk} \varphi_j(q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}) \\
&\times \varphi^{(\alpha)_{i+1}} (\varphi_j(q_{i+1,k}^{(\alpha_{i+1})}) q_{i+1,j}^{(\alpha_{i+1})}) \left(\prod_{l=1}^i \varphi^{(\alpha)_l}(q_{lk}^{(\alpha_l)}) \right)^{-1}.
\end{aligned}$$

Thus, after cancellations, we get

$$\begin{aligned} B &= \prod_{i=1}^{j-1} \varphi_k \varphi^{(\alpha)_i} (q_{ij}^{(\alpha_i)}) \\ &= q_{jk} \left(\prod_{i=1}^{j-1} \varphi^{(\alpha)_i} (\varphi_j (q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}) \right) \varphi^{(\alpha)_j} (q_{kj}) \left(\prod_{i=1}^{j-1} \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) \right)^{-1}. \end{aligned}$$

Thirdly, we have

$$\begin{aligned} C &= \prod_{i=1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i+\delta_{ij})}) \\ &= \left(\prod_{i=1}^{j-1} \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) \right) \varphi^{(\alpha)_j} (q_{jk}^{(\alpha_j+1)}) \prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)}) \\ &= \left(\prod_{i=1}^{j-1} \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) \right) \varphi^{(\alpha)_j} (q_{jk} \varphi_j (q_{jk}^{(\alpha_j)})) \prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)}). \end{aligned}$$

by Lemma 4.1.2(4). Consequently, after cancellations and notifying $q_{ii} = 1$, we obtain

$$\begin{aligned} \psi_k(t_j r t^{(\alpha)_k}) &= ABC t^{(\alpha+\epsilon_j)_k} \\ &= q_{jk} \varphi_j \varphi_k(r) \prod_{i=1}^j \varphi^{(\alpha)_i} (\varphi_j (q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}) \prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)}) t^{(\alpha+\epsilon_j)_k}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \psi_k(t_j) \psi_k(r t^{(\alpha)_k}) &= q_{jk} t_j \varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) t^{(\alpha)_k} \\ &= q_{jk} \varphi_j (\varphi_k(r) \prod_{i=1}^{k-1} \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)})) t_j t^{(\alpha)_k} \\ &= q_{jk} \varphi_j \varphi_k(r) \prod_{i=1}^{k-1} \varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) \prod_{l=1}^{j-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) t^{(\alpha+\epsilon_j)_k}. \end{aligned}$$

We rewrite $D := \prod_{i=1}^{k-1} \varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)})$. To find an expression for D , we use a little lemma:

Lemma 4.1.6. *Let A be a unital associative algebra, $a_0 = 1, a_1, \dots, a_k \in A$ units and $b_1, \dots, b_k \in A$. Then we have*

$$(1) \quad \prod_{i=1}^k \left(I \left(\prod_{l=1}^{i-1} a_l \right) (b_i) \right) = \prod_{i=1}^k a_i b_i \left(\prod_{l=1}^{k-1} a_l \right)^{-1}.$$

$$(2) \quad \prod_{i=j+1}^k \left(I \left(\prod_{l=1}^{j-1} a_l \right) (b_i) \right) = I \left(\prod_{l=1}^{j-1} a_l \right) \left(\prod_{i=j+1}^k b_i \right).$$

Proof. (1) is straightforward and (2) is obvious. \square

By Lemma 4.1.2(3), we have, for $i < j$,

$$\varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) = I \left(\prod_{l=1}^{i-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) \right) (\varphi^{(\alpha)_i} \varphi_j (q_{ik}^{(\alpha_i)})).$$

So by Lemma 4.1.6(1), we get

$$\prod_{i=1}^j \varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) = \prod_{i=1}^j \varphi^{(\alpha)_i} (\varphi_j (q_{ik}^{(\alpha_i)}) (q_{ij}^{(\alpha_i)})) \left(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) \right)^{-1}.$$

By Lemma 4.1.2(3), we have, for $j < i$,

$$\varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) = I \left(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) \right) (\varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)})).$$

So by Lemma 4.1.6(2), we get

$$\prod_{i=j+1}^{k-1} \varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) = I \left(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) \right) \left(\prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)}) \right).$$

Hence we get

$$\begin{aligned} D &= \prod_{i=1}^{k-1} \varphi_j \varphi^{(\alpha)_i} (q_{ik}^{(\alpha_i)}) \\ &= \prod_{i=1}^j \varphi^{(\alpha)_i} (\varphi_j (q_{ik}^{(\alpha_i)}) q_{ij}^{(\alpha_i)}) \prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i} (q_{ik}^{(\alpha_i)}) t^{(\alpha+\epsilon_j)_k} \left(\prod_{l=1}^{j-1} \varphi^{(\alpha)_l} (q_{lj}^{(\alpha_l)}) \right)^{-1}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \psi_k(t_j)\psi_k(rt^{(\alpha)k}) \\ = q_{jk}\varphi_j\varphi_k(r) \prod_{i=1}^j \varphi^{(\alpha)_i}(\varphi_j(q_{ik}^{(\alpha_i)})q_{ij}^{(\alpha_i)}) \prod_{i=j+1}^{k-1} \varphi^{(\alpha+\epsilon_j)_i}(q_{ik}^{(\alpha_i)})t^{(\alpha+\epsilon_j)_k}. \end{aligned}$$

Hence we have shown (ii) and constructed a crossed product algebra $R * \mathbb{Z}^k = R^{(k)}$ for $k = 1, \dots, n$ from (R, φ, q) .

Let us put $R_{\varphi, q} := R^{(n)} = \bigoplus_{\alpha \in \mathbb{Z}^n} Rt_{\alpha}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $t_{\alpha} = t_1^{\alpha_1} \dots t_n^{\alpha_n}$. Since $\psi_k|_R = \varphi_k$ for $k = 1, \dots, n$, we have $t_i r = \varphi_i(r)t_i$. Also, we have $t_j t_i = \psi_j(t_i)t_j = q_{ij}t_i t_j$ for $1 \leq i < j \leq n$, and so $t_j t_i = q_{ij}t_i t_j$ for all $1 \leq i, j \leq n$. Hence, our $R_{\varphi, q}$ satisfies (4.1.4). The uniqueness of the multiplication on $R_{\varphi, q}$ is clear since $R \cup \{t_1^{\pm 1}, \dots, t_n^{\pm 1}\}$ is a generating set of $R_{\varphi, q}$.

Now, we can easily check that $\psi_j^{\alpha_j}(t^{(\beta)_j}) = q_{\epsilon_j, \beta}^{(\alpha_j)} t^{(\beta)_j}$. So for $rt_{\alpha}, r't_{\beta} \in R_{\varphi, q}$, we get

$$\begin{aligned} rt_{\alpha}r't_{\beta} &= r\varphi^{\alpha}(r')t_{\alpha}t_{\beta} \\ &= r\varphi^{\alpha}(r')t^{(\alpha)_n}t_n^{\alpha_n}t^{(\beta)_n}t_n^{\beta_n} \\ &= r\varphi^{\alpha}(r')t^{(\alpha)_n}\psi_n^{\alpha_n}(t^{(\beta)_n})t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\alpha}(r')t^{(\alpha)_n}q_{\epsilon_n, \beta}^{(\alpha_n)}t^{(\beta)_n}t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\alpha}(r')\varphi^{(\alpha)_n}(q_{\epsilon_n, \beta}^{(\alpha_n)})t^{(\alpha)_n}t^{(\beta)_n}t_n^{\alpha_n+\beta_n} \\ &\dots\dots \\ &= r\varphi^{\alpha}(r')\varphi^{(\alpha)_n}(q_{\epsilon_n, \beta}^{(\alpha_n)}) \dots \varphi^{(\alpha)_2}(q_{\epsilon_2, \beta}^{(\alpha_2)})t_1^{\alpha_1+\beta_1} \dots t_n^{\alpha_n+\beta_n} \\ &= r\varphi^{\alpha}(r')q_{\alpha, \beta}t_{\alpha+\beta}. \end{aligned}$$

Conversely, for any crossed product algebra $R * \mathbb{Z}^n = (R, \mathbb{Z}^n, \tau, \sigma) = \bigoplus_{\alpha \in \mathbb{Z}^n} R\bar{\alpha}$, we take a new R -basis $\{t_{\alpha} \mid \alpha \in \mathbb{Z}^n\}$ of $R * \mathbb{Z}^n$ where $t_{\alpha} = \bar{\epsilon}_1^{\alpha_1} \dots \bar{\epsilon}_n^{\alpha_n}$. We set $q_{ij} := \tau(\epsilon_j, \epsilon_i)$ for $1 \leq i \leq j \leq n$, $q_{ji} := q_{ij}^{-1}$ and $\varphi_i := \sigma(\epsilon_i)$. Then, one can easily check that the triple (R, φ, q) is a \mathbb{Z}^n -grading triple:

From our setting, (G1) is clear. The coefficient of $t_j t_i r$ ($i < j, r \in R$) in $t_i t_j$ has two expressions, namely, $\varphi_j \varphi_i(r)$ and $I(q_{ij})\varphi_i \varphi_j(r)$, by associativity. So we have (G2). Similarly, the coefficient of $t_k t_j t_i$ ($i < j < k$) in $t_i t_j t_k$ has two expressions, namely, $\varphi_k(q_{ij})q_{ik}\varphi_i(q_{jk})$ and $q_{jk}\varphi_j(q_{ik})q_{ij}$, by the associativity. So we have (G3).

Also, it is clear that $R * \mathbb{Z}^n = \bigoplus_{\alpha \in \mathbb{Z}^n} Rt_{\alpha}$ satisfies (4.1.4). Therefore, we obtain $R * \mathbb{Z}^n = R_{\varphi, \mathbf{q}}$. \square

Corollary 4.1.7. *Let (D, φ, \mathbf{q}) be a division \mathbb{Z}^n -grading triple. Then, $D_{\varphi, \mathbf{q}} = \bigoplus_{\alpha \in \mathbb{Z}^n} Dt_{\alpha}$ is a division \mathbb{Z}^n -graded algebra. Conversely, for any division \mathbb{Z}^n -graded algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_{\alpha}$, there exists a division \mathbb{Z}^n -grading triple (D, φ, \mathbf{q}) such that $A \cong_{\mathbb{Z}^n} D_{\varphi, \mathbf{q}}$. More precisely, we choose any nonzero element $t_i \in A_{\epsilon_i}$ for each $i = 1, \dots, n$, and set, for $1 \leq i, j \leq n$,*

$$D := A_0, \quad \varphi_i := I(t_i) \mid_D \quad \text{and} \quad q_{ij} := t_j t_i t_j^{-1} t_i^{-1}. \quad \square$$

For constructing a \mathbb{Z}^n -grading triple, (G3) seems annoying to check. In certain situation, we do not need (G3) for the \mathbb{Z}^n -grading triple. We use the notation $[a, b] = aba^{-1}b^{-1}$ for $a, b \in U(R)$.

Lemma 4.1.8. *Let R be a unital associative algebra over F , $\varphi = (I(d_1), \dots, I(d_n))$ an n -tuple of inner automorphisms φ_i of R for some $d_1, \dots, d_n \in U(R)$ and $\mathbf{q} = (q_{ij})$ an $n \times n$ matrix over F . Suppose that a triple (R, φ, \mathbf{q}) satisfies (G1) and (G2). Then, (R, φ, \mathbf{q}) is a \mathbb{Z}^n -grading triple.*

Proof. We only need to check (G3). By (G1) and (G2), we have, for all $1 \leq i, j \leq n$, $I(d_j)I(d_i) = I(q_{ij})I(d_i)I(d_j)$. So for all $r \in R$, $d_j d_i r d_i^{-1} d_j^{-1} = q_{ij} d_i d_j r d_j^{-1} d_i^{-1} q_{ji}$ and hence $r d_i^{-1} d_j^{-1} q_{ij} d_i d_j = d_i^{-1} d_j^{-1} q_{ij} d_i d_j r$, i.e., $d_i^{-1} d_j^{-1} q_{ij} d_i d_j =: c_{ij}$ is in the centre of R . Note that $c_{ji}^{-1} = c_{ij}$. Thus we have

$$q_{ij} = c_{ij}[d_j, d_i].$$

Using this identity, we get (G3): for all $1 \leq i < j < k \leq n$,

$$\begin{aligned} & q_{jk} \varphi_j(q_{ik}) q_{ij} \varphi_i(q_{kj}) q_{ki} \\ &= c_{jk}[d_k, d_j] d_j c_{ik}[d_k, d_i] d_j^{-1} c_{ij}[d_j, d_i] d_i c_{kj}[d_j, d_k] d_i^{-1} c_{ki}[d_i, d_k] \\ &= d_k c_{ij}[d_j, d_i] d_k^{-1} = \varphi_k(q_{ij}). \quad \square \end{aligned}$$

If R is a finite dimensional central simple associative algebra over F for example, the defining identities of a \mathbb{Z}^n -grading triple are just (G1) and (G2) by Lemma 4.1.8.

Finally, we give some examples.

Example 4.1.9. (1) Let F_q be an arbitrary quantum torus and D an arbitrary associative division algebra. Then it is easy to see that $D \otimes_F F_q$ is a division \mathbb{Z}^n -graded associative algebra and is isomorphic to $D_{1,q}$ where $\varphi = \mathbf{1} = \mathbf{1}_n := (\text{id}, \dots, \text{id})$. Note also if D is a field, then this example becomes a quantum torus over D . Conversely, for a division \mathbb{Z}^n -grading triple (D, φ, q) , if $\varphi = \mathbf{1}$, then we have $I(q_{ij}) = \text{id}$ for all q_{ij} , by (G2). Hence q_{ij} is in the centre of D , say K , and we can show that $D_{1,q} \cong D \otimes_K K_q$. Therefore, $D_{\varphi,q}$ is a tensor product with D and some quantum torus if and only if $\varphi = \mathbf{1}$.

Finally, for a \mathbb{Z}^n -grading triple (R, φ, q) , if $\varphi = \mathbf{1}$, then the crossed product algebra $R_{1,q}$ has the trivial action by Theorem 4.1.3. So, $R_{1,q} = R^t[\mathbb{Z}^n]$ is a twisted group algebra.

(2) Let

$$\mathbf{1} = \mathbf{1}_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

Then, by (G2), $(R, \varphi, \mathbf{1})$ is a \mathbb{Z}^n -grading triple if and only if

$$(*) \quad \varphi_j \varphi_i = \varphi_i \varphi_j \quad \text{for all } i, j.$$

Now, let $\mathbb{H} = \langle i, j \rangle$ be Hamilton's quaternion over \mathbb{R} (the field of real numbers), i.e., the unique quaternion division algebra over \mathbb{R} , where i and j are the standard generators. $\varphi = \varphi_3 = (I(i), I(j), I(ij))$ and $\mathbf{1} = \mathbf{1}_3$. Then one can easily check (*), and hence $\mathbb{H}_{\varphi, \mathbf{1}}$ is a division \mathbb{Z}^3 -graded associative algebra over \mathbb{R} .

Finally, for a \mathbb{Z}^n -grading triple (R, φ, q) , if $q = \mathbf{1}$, then a crossed product algebra $R_{\varphi, \mathbf{1}}$ has the trivial twisting by Theorem 4.1.3. So, $R_{\varphi, \mathbf{1}} = R\mathbb{Z}^n$ is a skew group algebra.

(3) Let $K = \mathbb{Q}(\zeta_5)$ be a cyclotomic extension of \mathbb{Q} (the field of rational numbers) where $\zeta := \zeta_5$ is a primitive 5th root of unity, and φ the automorphism of K defined by $\varphi(\zeta) = \zeta^2$. Let $\varphi = (\varphi, \varphi^2, \varphi^3)$ and

$$q = \begin{pmatrix} 1 & \zeta & \zeta^2 \\ \zeta^{-1} & 1 & \zeta^{-1} \\ \zeta^3 & \zeta & 1 \end{pmatrix}.$$

Then one can easily check that (K, φ, q) is a division \mathbb{Z}^3 -grading triple, and hence $K_{\varphi, q}$ is a division \mathbb{Z}^3 -graded associative algebra over \mathbb{Q} .

(4) Let $\mathbb{H} = \langle i, j \rangle$ again, and put $k := ij$. Let $\varphi = (I(d_1), I(d_2), I(d_3))$ where $d_1 = 1 + i$, $d_2 = 1 + j$ and $d_3 = 1 + k$. We put $q_{ij} = 2[d_j, d_i]$ for $1 \leq i < j \leq 3$, $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$. Then, (\mathbb{H}, φ, q) satisfies (G1) and (G2), and

$$q = \begin{pmatrix} 1 & 1 - i + j - k & 1 - i + j + k \\ (1 - i + j - k)^{-1} & 1 & 1 - i - j + k \\ (1 - i + j + k)^{-1} & (1 - i - j + k)^{-1} & 1 \end{pmatrix}.$$

By Lemma 4.1.8, this is a division \mathbb{Z}^3 -grading triple and hence $\mathbb{H}_{\varphi, q}$ is a division \mathbb{Z}^3 -graded associative algebra over \mathbb{R} .

§4.2. CAYLEY-DICKSON PROCESS OVER A RING

For the classification of division \mathbb{Z}^n -graded alternative algebras, we need to review the Cayley-Dickson process over a ring ([19, page 103]) and to prove several lemmas. In this section, our algebras are arbitrary non-associative unital algebras, not necessarily alternative, over a ring of scalars Φ (a commutative associative unital ring). Moreover, for an algebra B over Φ , we assume that B is *faithful*, i.e., for all $\alpha \in \Phi$, $\alpha B = 0 \implies \alpha = 0$. Since B has a unit, B is faithful if and only if, for all $\alpha \in \Phi$,

$$\alpha 1 = 0 \implies \alpha = 0.$$

Let $*$ be a *scalar involution* over Φ , i.e., an anti-automorphism of period 2 with $bb^* \in \Phi 1$ (hence by linearization, $b + b^* \in \Phi 1$) for all $b \in B$, and let $\mu \in \Phi$ be a *cancellable scalar*, i.e.,

$$\mu b = 0 \text{ for some } b \in B \implies b = 0.$$

Then we can construct a new algebra $B \oplus B$ with its product

$$(a, b)(c, d) = (ac + \mu db^*, a^*d + cb) \text{ for } a, b, c, d \in B.$$

Letting $v = (0, 1)$ we can write this algebra as $B \oplus vB$,

$$(4.2.1) \quad (a + vb)(c + vd) = (ac + \mu db^*) + v(a^*d + cb).$$

We say that this new algebra $B \oplus vB$ is obtained from $B = (B, *)$ by the *Cayley-Dickson process with structure constant μ* , and denote it by (B, μ) . We define a new map $*$ on $C := (B, \mu)$ by

$$(4.2.2) \quad (a + vb)^* = a^* - vb.$$

Then it is easily seen that this new $*$ is also a scalar involution of C . Hence, for a cancellable scalar ν , we can obtain another new algebra (C, ν) from $C = (C, *)$ by the Cayley-Dickson process with structure constant ν . We write this (C, ν) as (B, μ, ν) instead of $((B, \nu), \mu)$. Repeating this process, we can produce new algebras with scalar involutions from $B = (B, *)$ without limit.

Remark 4.2.3. (1) That μ is cancellable is used to obtain the injectivity of $B \oplus B \ni (a, b) \mapsto a + vb \in B \oplus vB$. For, we have

$$(4.2.4) \quad vb = 0 \implies v(vb) = \mu b = 0 \implies b = 0.$$

(2) If μ is invertible, then μ is cancellable, and $v = (0, 1)$ is invertible in the sense that there exists an element $u \in C$ such that $uv = vu = 1$. Namely, take $u = (0, \mu^{-1}1)$.

(3) We can observe that $C = (B, \mu) = B \oplus vB$ is a $\mathbb{Z}_2 (= \mathbb{Z}/2\mathbb{Z})$ -graded algebra by defining $B_0 := B$ and $B_1 := vB$. Also, we can see that $*$ is a *graded* scalar involution, i.e., $B_i^* = B_i$ for $i = 0, 1$.

We will use the following lemma from [19, Theorem 6.8, page 105]:

Lemma 4.2.5. *For the algebra (B, μ) , we have*

- (i) (B, μ) is commutative $\iff B$ is commutative with trivial involution.
- (ii) (B, μ) is associative $\iff B$ is commutative and associative.
- (iii) (B, μ) is alternative $\iff B$ is associative. \square

When Φ is a field of characteristic $\neq 2$ and we start the Cayley-Dickson process from $\Phi = (\Phi, \text{id})$ where id is the trivial scalar involution, i.e., the identity map, we call Φ , (Φ, μ_1) , (Φ, μ_1, μ_2) and $(\Phi, \mu_1, \mu_2, \mu_3)$ *composition algebras* over Φ where μ_1 , μ_2 and μ_3 are any nonzero elements in Φ .

When Φ is a field of characteristic $= 2$, we define a separable quadratic algebra Φ_μ for $\neq \mu \in \Phi$ as

$$\Phi_\mu := \Phi[X]/(X^2 - X - \mu).$$

Then, Φ_μ has a scalar involution $*$ over Φ . If $\text{ch.}\Phi = 2$, we call $\Phi_{\mu_1} = (\Phi_{\mu_1}, *)$, (Φ_{μ_1}, μ_2) and $(\Phi_{\mu_1}, \mu_2, \mu_3)$ *composition algebras* over Φ where μ_1 , μ_2 and μ_3 are any nonzero elements in Φ (see [30]).

We call (Φ, μ_1, μ_2) or (Φ_{μ_1}, μ_2) a *quaternion algebra* over Φ , and $(\Phi, \mu_1, \mu_2, \mu_3)$ or $(\Phi_{\mu_1}, \mu_2, \mu_3)$ an *octonion algebra* over Φ . Also, one can show that (Φ, μ_1) or Φ_{μ_1} is either $\Phi \boxplus \Phi$ (direct sum as algebras) or a separable quadratic field extension of Φ .

For an algebra $B = (B, *)$ with scalar involution $*$ and $x \in B$, we define the *norm* $n(a)$ of a as the unique scalar $aa^* = n(a)1$. Also, for $a, b \in B$, we define

$$n(a, b) := n(a + b) - n(a) - n(b)$$

so $n(a, b)1 = ab^* + ba^*$. Note that $n(\cdot, \cdot)$ is a symmetric bilinear form on B . In the same way, we have the norm n and the symmetric bilinear form $n(\cdot, \cdot)$ on the algebra (B, μ) obtained by the Cayley-Dickson process.

For $C = (B, \mu) = B \oplus vB$ and $a, b \in B$, we have $n(a, vb) = a(vb)^* + (vb)a^* = -a(vb) + (vb)a^* = -v(a^*b) + v(a^*b) = 0$, and so we obtain

$$(4.2.6) \quad n(B, vB) = 0 \quad \text{or equivalently} \quad vB \subset B^\perp,$$

where B^\perp is the orthogonal submodule relative to $n(\cdot, \cdot)$.

We recall that an algebra B , in general, has degree ≤ 2 if there exist a linear form tr called *trace* and a quadratic form n called *norm* such that for all $a \in B$

$$a^2 - \text{tr}(a)a + n(a)1 = 0, \quad \text{tr}(1) = 2, \quad n(1) = 1.$$

In particular, if Φ is a field, then the trace tr and norm n are unique ([19, page 90]).

If an algebra B over Φ has a scalar involution $*$, then one can easily check that B has degree ≤ 2 , that $a + a^* = \text{tr}(a)1$ for $a \in B$ and that the two norms from $*$ and the degree ≤ 2 -algebra coincide. In particular, if Φ is a field, then a scalar involution is unique. For, if $\bar{}$ is another scalar involution, then we have $a + \bar{a} = \text{tr}(a)1 = a + a^*$ for all $a \in B$, by the uniqueness of the trace above. Hence we get $\bar{} = *$. We state this as a lemma:

Lemma 4.2.7. *Let B be an algebra with scalar involution $*$ over a field. Then, B has degree ≤ 2 , and $*$ is the only scalar involution. We have $a^* = \text{tr}(a)1 - a$ for all $a \in B$, where tr is the trace of the degree ≤ 2 -algebra B . \square*

Remark 4.2.8. (1) All $(\Phi, \mu_1, \dots, \mu_r)$ for $r \in \mathbb{Z}_{>0}$ have a scalar involution and have degree ≤ 2 , and this holds in particular for all composition algebras. We use the fact that the norms of composition algebras are nondegenerate, which is originally a

part of the definition of composition algebras (see e.g. [33, Chapter 2]). Also, it is well-known that a quaternion algebra is a central simple non-commutative associative algebra of degree 2 and that an octonion algebra is a central simple alternative but not associative algebra of degree 2 (see e.g. [33]).

(2) Degree 2 does not in general imply the existence of a scalar involution (see [19, page 87]).

We have a method of constructing a scalar involution.

Lemma 4.2.9. *Let A be an arbitrary algebra over a field K with scalar involution $*$ and let R be a commutative associative algebra over K . Then, an R -algebra $A_R := A \otimes_K R$ is a free R -module and the transformation $* \otimes \text{id}$ of A_R defined by $(a \otimes r)^{* \otimes \text{id}} = a^* \otimes r$, for $a \in A$ and $r \in R$, is a scalar involution over R .*

Proof. Let $\{a_i\}_{i \in I}$ be a basis of A over K . For $s \in R$ and $a \otimes r \in A_R$, we define $s.(a \otimes r) := a \otimes sr$. Then, A_R is clearly an R -algebra and a free R -module with basis $\{a_i \otimes 1\}_{i \in I}$. Hence, $* := * \otimes \text{id} : A_R \rightarrow A_R$ defined by $(a_i \otimes 1)^* = a_i^* \otimes 1$ is a well-defined R -linear transformation on A_R satisfying $(a \otimes r)^* = a^* \otimes r$ for $a \in A$ and $r \in R$. It is straightforward to show that $*$ is an involution of A_R . We show that it is scalar over R . For $f_i \in R$, $i \in I$, we have

$$\begin{aligned} & \left(\sum_i a_i \otimes f_i \right) \left(\sum_i a_i \otimes f_i \right)^* = \left(\sum_i a_i \otimes f_i \right) \left(\sum_i a_i^* \otimes f_i \right) \\ &= \sum_{i,j} a_i a_j^* \otimes f_i f_j = \sum_i a_i a_i^* \otimes f_i^2 + \sum_{i < j} (a_i a_j^* \otimes f_i f_j + a_j a_i^* \otimes f_j f_i) \\ &= \sum_i n(a_i) 1 \otimes f_i^2 + \sum_{i < j} (a_i a_j^* + a_j a_i^*) \otimes f_j f_i \\ &= \sum_i 1 \otimes n(a_i) f_i^2 + \sum_{i < j} 1 \otimes n(a_i, a_j) f_j f_i \in 1 \otimes R = R1. \end{aligned}$$

Hence the involution $* = * \otimes \text{id}$ is scalar over R . \square

We show a property of division composition algebras.

Lemma 4.2.10. *Let $C = (C, *)$ be a division composition algebra with scalar involution $*$ over a field L . Let B be a subalgebra of C such that $\dim_L B = 1/2 \dim_L C$. Then, B is a division composition algebra with scalar involution $* = *|_B$ over L and $C = (B, \mu)$ for some $0 \neq \mu \in L$, except when $\text{ch.}L = 2$ and $\dim_L C = 2$.*

Proof. First, we note that if $\text{ch}.L = 2$ and $\dim_L C = 2$, then $C = L_\mu$ and $B = L$ which is not a composition algebra. So we assume that $\text{ch}.L \neq 2$ or $\dim_L C \neq 2$. Since $1 \in B$ and $b + b^* \in L1$ for $b \in B$, we have $b^* = l1 - b \in B$ for some $l \in L$. Hence $*$ $=|_B$ is a scalar involution on B . Since C is a division algebra, we have $n(b, b) \neq 0$ for all $0 \neq b \in B$. Hence $n|_{B \times B}$ is nondegenerate. Therefore, B is a division composition algebra over L and $C = (B, \mu)$ for some $0 \neq \mu \in L$ (see [33, page 32]). \square

We show a criterion in which a subring of an associative composition algebra is obtained by the Cayley-Dickson process.

Lemma 4.2.11. *Let $B = (B, *)$ be an associative composition algebra with scalar involution $*$ over a field L . Let $C = (B, \mu)$ for $0 \neq \mu \in L$ and R a subring of L . Suppose:*

- (i) *There exists $t \in C$ such that $n(t, B) = 0$, where n is the norm on C .*
- (ii) *$0 \neq t^2 \in R1 = R$ (we identify $L1$ with L).*
- (iii) *P is a subalgebra of B over R such that the restriction of $*$ to P is a scalar involution over R .*

Then the algebra generated by P and t over R is equal to the algebra obtained by the Cayley-Dickson process with structure constant t^2 over R :

$$\langle P, t \rangle = (P, t^2) \quad \text{and } t = (0, 1) \text{ in } (P, t^2).$$

Moreover, the restriction of the scalar involution $$ of C to (P, t^2) is a scalar involution over R .*

Proof. Since B is a composition algebra, the norm $n(\cdot, \cdot)$ on C is nondegenerate and the dimension of C over L is finite. From a well-known theorem in Linear Algebra, we have $\dim B + \dim B^\perp = \dim C$. By (4.2.6), we have $vB \subset B^\perp$. Since $\dim B + \dim vB = \dim C$, we get $vB = B^\perp$. Thus we have $t = ve$ for some $0 \neq e \in B$, by (i). Also, by (ii) we have $0 \neq t^2 = (ve)^2 = \mu n(e)1 \in R$, which is cancellable. Now, we have by (4.2.1),

$$tP = (ve)P = v(Pe) \subset vB$$

and hence, $\langle P, t \rangle = P \oplus tP$. By (iii), P has a scalar involution $*$ $= *|_P$ over R . We show that

$$(a + tb)(c + td) = (ac + t^2 db^*) + t(a^*d + cb) \quad \text{for all } a, b, c, d \in P.$$

In fact, we have, using $t = ve$. (4.2.1) and the associativity of P ,

$$\begin{aligned}
a(td) &= a((ve)d) = a(v(de)) = v(a^*(de)) = v((a^*d)e) \\
&= (ve)(a^*d) = t(a^*d), \\
(tb)c &= ((ve)b)c = (v(be))c = v(c(be)) \\
&= v((cb)e) = (ve)(cb) = t(cb), \\
(tb)(td) &= ((ve)b)((ve)d) = (v(be))(v(de)) = \mu(de)(be)^* \\
&= \mu(de)(e^*b^*) = \mu n(e)db^* = t^2db^*.
\end{aligned}$$

Hence $\langle P, t \rangle = P \oplus tP = (P, t^2)$ and $t = (0, 1)$ in (P, t^2) . Moreover, we have for $b \in P$,

$$(tb)^* = b^*(ve)^* = b^*(e^*v^*) = b^*(-e^*v) = -b^*(ve) = -v(be) = -(ve)b = -tb.$$

So, $*$ $|_{(P, t^2)}$ coincides with the scalar involution of (P, t^2) over R . \square

Corollary 4.2.12. *Let $B = (B, *)$ be an associative composition algebra with scalar involution $*$ over a field L and R a subring of L . Let $C = (B, \mu_1, \dots, \mu_m)$ for $0 \neq \mu_i \in L$, $1 \leq m \leq 3$, be an alternative algebra over L . Let $B_0 := B$ and $B_i := (B_{i-1}, \mu_i)$, $1 \leq i \leq m$. (All B_i have the restricted scalar involution $*$ of C .) Suppose:*

- (i) *There exist $t_1, \dots, t_m \in C$ such that $n(t_i, B_{i-1}) = 0$.*
- (ii) *$0 \neq t_i^2 \in R1 = R$ for all $1 \leq i \leq m$.*
- (iii) *P is a subalgebra of B over R such that the restriction of $*$ to P is a scalar involution over R .*

Then,

$$\begin{aligned}
\langle P, t_1, \dots, t_m \rangle &= (P, t_1^2, \dots, t_m^2) \\
&\text{and each } t_i = (0, 1) \text{ in } ((P, t_1^2, \dots, t_{i-1}^2), t_i^2).
\end{aligned}$$

Proof. The case $m = 1$ is done by Lemma 4.2.11. For $m > 1$, assume that the subalgebra P_{m-1} of B_{m-1} generated by P, t_1, \dots, t_{m-1} over R is equal to $(P, t_1^2, \dots, t_{m-1}^2)$ and that the restriction of the scalar involution $*$ of B_{m-1} to $(P, t_1^2, \dots, t_{m-1}^2)$ is a scalar involution over R . Then, since $C = (B_{m-1}, \mu_m)$ has the element t_m satisfying $n(t_m, B_{m-1}) = 0$ and $t_m^2 \in R$, we can apply Lemma 4.2.11 for $B = B_{m-1}$, $P = P_{m-1}$ and $t = t_m$. Namely,

$$\langle P, t_1, \dots, t_m \rangle = \langle P_{m-1}, t_m \rangle = (P_{m-1}, t_m^2) = (P, t_1^2, \dots, t_m^2),$$

and each t_i has the required form. \square

Finally, we show some properties for a division G -graded alternative algebra which has a scalar involution. Most of them were already shown in more general setting (see Lemma 2.6.6).

Lemma 4.2.13. *Let $A = \sum_{g \in G} A_g$ be a division G -graded alternative algebra over a field L which has a scalar involution $*$. Then we have the following:*

- (i) *For any $x \in A_g$, $g \neq e$, we have $\text{tr}(x) = 0$, $x^2 = -n(x)1 \in L1$, $x^* = -x$ and $*$ is graded, i.e., $A_g^* = A_g$ for all $g \in G$.*
- (ii) *The exponent of G is 2 or G is trivial, i.e., $G = \{e\}$.*
- (iii) *All homogeneous spaces are orthogonal to each other relative to the norm.*

Proof. Any $0 \neq x \in A_g$ is not nilpotent since x is invertible. Thus, by 2.6.6, we only need to show that $x^* = -x$, $*$ is graded and (iii).

In fact, by Lemma 4.2.7, we have $x^* = \text{tr}(x)1 - x = -x$. For $y \in A_e$, we have $y^* = \text{tr}(y)1 - y \in A_e$, and so $*$ is graded.

For (iii), let $y \in A_h$ where $g \neq h \in G$. Then, we have $xy^* \in A_{gh}$ since $*$ is graded. If $gh = e$, then $g = h$ since the exponent of G is 2. Thus we have $gh \neq e$ and by (i), we get $\text{tr}(xy^*) = 0$. Hence, $n(x, y)1 = xy^* + yx^* = \text{tr}(xy^*) = 0$. \square

§4.3. DIVISION \mathbb{Z}^n -GRADED ALTERNATIVE ALGEBRAS

In this section all algebras are unital. An alternative algebra A is called an *octonion ring* if the central closure \bar{A} is an octonion algebra over the field $\bar{\mathbb{Z}}$ ([33, page 193]), where we used the notation of §1.1. We first state Slater's theorem ([33, Theorem 9, page 194]):

Theorem 4.3.1. *A strongly prime algebra over a field is either an associative algebra or an octonion ring. \square*

Note that a division G -graded associative algebra over any field for any group G is a crossed product algebra (see §4.0). By Corollary 2.2.5, a division G -graded alternative algebra over a field for an ordered group G is strongly prime. Hence by Theorem 4.3.1, it is either a crossed product algebra or an octonion ring.

Let F be a field of an arbitrary characteristic in this section.

Theorem 4.3.2. *A division G -graded alternative algebra A over F for an ordered group G is either a semi-prime crossed product algebra or an octonion ring which embeds into an octonion division algebra over \bar{Z} .*

Proof. When A is associative, one can easily see that it is strongly prime if and only if it is semi-prime. So we only need to show that \bar{A} is a division algebra when A is not associative. So we can assume that \bar{A} is an octonion algebra. We know that by Proposition 2.2.4 and Lemma 1.1.16(iii), \bar{A} is a domain. Hence \bar{A} is a division algebra (see [33, Lemma 9, page 43]). \square

Now, we specify $G = \mathbb{Z}^n$, and classify division \mathbb{Z}^n -graded alternative algebras. Since \mathbb{Z}^n is an ordered abelian group, we can apply Theorem 4.3.2. We showed that any division \mathbb{Z}^n -graded associative algebra is isomorphic to $D_{\varphi, q}$ for some D , φ and q in Corollary 4.1.7. So we start to classify division \mathbb{Z}^n -graded alternative algebras over F which are not associative. By theorem 4.3.2, any such algebra is an octonion ring which embeds into an octonion division algebra. We first construct four examples of such octonion rings. For this purpose, we prove a lemma and its corollary.

Lemma 4.3.3. *Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a basis of a free abelian group Λ of rank n and*

$$\Gamma_i := \mathbb{Z}\varepsilon_1 + \dots + \mathbb{Z}\varepsilon_{i-1} + 2\mathbb{Z}\varepsilon_i + \mathbb{Z}\varepsilon_{i+1} + \dots + \mathbb{Z}\varepsilon_n.$$

Let $A = \bigoplus_{\alpha \in \Gamma_i} A_\alpha$ be a division Γ_i -graded associative algebra over a field K and R a subalgebra contained in the centre of A . Suppose

- (i) *A has a scalar involution $*$ over R which is graded: $A_\alpha^* = A_\alpha$ for all $\alpha \in \Gamma_i$,*
- (ii) *there exists $z \in A_{2\varepsilon_i} \cap R$ such that z is invertible in R .*

Then the algebra (A, z) obtained by the Cayley-Dickson process over R has a unique Λ -grading such that every homogeneous space of degree $\alpha \in \Gamma_i$ in (A, z) is equal to A_α and $v = (0, 1) \in (A, z)$ has degree ε_i . Moreover, by the Λ -grading, (A, z) becomes a division Λ -graded alternative algebra over K and the scalar involution $$ of (A, z) is again graded.*

Proof. We have

$$(A, z) = A \oplus vA = \left(\bigoplus_{\alpha \in \Gamma_i} A_\alpha \right) \oplus v \left(\bigoplus_{\alpha \in \Gamma_i} A_\alpha \right).$$

We first claim $v(\bigoplus_{\alpha \in \Gamma_i} A_\alpha) = \bigoplus_{\alpha \in \Gamma_i} vA_\alpha$. We have $v(\bigoplus_{\alpha \in \Gamma_i} A_\alpha) = \sum_{\alpha \in \Gamma_i} vA_\alpha$ and so we only need to show the sum in the right-hand side is direct. By (4.2.4),

$\sum_{\alpha \in \Gamma_i} vx_\alpha = v(\sum_{\alpha \in \Gamma_i} x_\alpha) = 0$ for $x_\alpha \in A_\alpha$ implies $\sum_{\alpha \in \Gamma_i} x_\alpha = 0$. So we get $x_\alpha = 0$ for all $\alpha \in \Gamma_i$, and our claim is settled. For $\alpha \in \Lambda$, define

$$A'_\alpha := \begin{cases} A_\alpha, & \text{if } \alpha \in \Gamma_i \\ vA_{\alpha-\varepsilon_i}, & \text{otherwise.} \end{cases}$$

Then we get

$$(A, z) = \left(\bigoplus_{\alpha \in \Gamma_i} A_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Gamma_i} vA_\alpha \right) = \bigoplus_{\alpha \in \Lambda} A'_\alpha.$$

the direct sum of F -vector spaces. By the multiplication of (A, z) , we have

$$A'_\alpha A'_\beta = \begin{cases} A_\alpha A_\beta = A_{\alpha+\beta} = A'_{\alpha+\beta}, & \text{if } \alpha, \beta \in \Gamma_i \\ A_\alpha (vA_{\beta-\varepsilon_i}) \subset v(A_\alpha^* A_{\beta-\varepsilon_i}) = vA_{\alpha+\beta-\varepsilon_i} = A'_{\alpha+\beta}, & \text{if } \alpha \in \Gamma_i, \beta \notin \Gamma_i \\ (vA_\beta) A_{\alpha-\varepsilon_i} \subset v(A_\beta A_{\alpha-\varepsilon_i}) = vA_{\beta+\alpha-\varepsilon_i} = A'_{\alpha+\beta}, & \text{if } \alpha \notin \Gamma_i, \beta \in \Gamma_i \\ (vA_{\alpha-\varepsilon_i})(vA_{\beta-\varepsilon_i}) \subset zA_{\beta-\varepsilon_i}^* A_{\alpha-\varepsilon_i} = A'_{\alpha+\beta}, & \text{if } \alpha \notin \Gamma_i, \beta \notin \Gamma_i \end{cases}$$

since $*$ is graded and $z \in A_{2\varepsilon_i}$. Hence $(A, z) = \bigoplus_{\alpha \in \Lambda} A'_\alpha$ is a Λ -graded algebra over F . Since $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis of Λ , the uniqueness of Λ -grading is clear. Also by Lemma 4.2.5, (A, z) is an alternative algebra. Moreover, we know that $0 \neq x \in A_\alpha$ is invertible. Since z is invertible, v is invertible in the alternative algebra (A, z) (see Remark 4.2.3(2)). Hence, vx is invertible (see [33, Lemma 9, page 205]). Consequently, (A, z) is a division Λ -graded alternative algebra over F . Finally, by (4.2.2), it is clear that $*$ on (A, z) is graded. \square

Corollary 4.3.4. *Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be a basis of \mathbb{Z}^n and*

$$\Gamma^{(m)} := 2\mathbb{Z}\varepsilon_1 + \dots + 2\mathbb{Z}\varepsilon_m + \mathbb{Z}\varepsilon_{m+1} + \dots + \mathbb{Z}\varepsilon_n \quad \text{for } m = 1, 2, 3.$$

Let $A = \bigoplus_{\alpha \in \Gamma^{(m)}} A_\alpha$ be a division $\Gamma^{(m)}$ -graded associative algebra over a field K and R a subalgebra contained in the centre of A . Suppose

- (i) A has a scalar involution $*$ over R which is graded,
- (ii) there exist $z_i \in A_{2\varepsilon_i} \cap R$ for all $1 \leq i \leq m$ such that each z_i is invertible in R ,
- (iii) A is commutative if $m = 2$, and A is commutative and $*$ is trivial if $m = 3$.

Then the algebra (A, z_1, \dots, z_m) obtained by the Cayley-Dickson process over R has a unique \mathbb{Z}^n -grading such that every homogeneous space of degree $\alpha \in \Gamma^{(m)}$ in (A, z_1, \dots, z_m) is equal to A_α and each $v_i = (0, 1) \in ((A, z_1, \dots, z_{i-1}), z_i)$ has degree ε_i . Also by the \mathbb{Z}^n -grading, (A, z_1, \dots, z_m) becomes a division \mathbb{Z}^n -graded alternative algebra over K

Moreover, assume that

(iv) A is not commutative if $m = 1$, $*$ is not trivial if $m = 2$ and $\text{ch.}K \neq 2$ if $m = 3$.

Then (A, z_1, \dots, z_m) is not associative.

Proof. For $m = 1$, we take $\Lambda = \mathbb{Z}^n$ in 4.3.3, and so $\Gamma^{(1)} = \Gamma_1$. Then we take $z = z_1$ in 4.3.3 and get the required division \mathbb{Z}^n -graded alternative algebra (A, z_1) over K . Moreover, (iv) and 4.2.5 implies that (A, z_1) is not associative.

To show the cases $m = 2$ and $m = 3$, we define the following: for $j = 1, 2$, let

$$\Gamma_j^{(m)} := \mathbb{Z}\epsilon_1 + \dots + \mathbb{Z}\epsilon_j + 2\mathbb{Z}\epsilon_{j+1} + \dots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n.$$

For $m = 2$, we take $\Lambda = \Gamma_1^{(2)}$, $\Gamma_i = \Gamma^{(2)}$ and $z = z_1$ in 4.3.3. Then, by (iii), we get the division $\Gamma_1^{(2)}$ -graded associative algebra (A, z_1) which has the graded scalar involution $*$ over R . We note that R is contained in the centre of (A, z_1) and that $z_2 \in A_{2\epsilon_2} \cap R = (A, z_1)_{2\epsilon_2} \cap R$ is invertible in R . Thus we can apply 4.3.3 again for $\Lambda = \mathbb{Z}^n$, $\Gamma_i = \Gamma_1^{(2)}$ and $z = z_2$, and get the required division \mathbb{Z}^n -graded alternative algebra (A, z_1, z_2) over K . Moreover, (iv) implies that (A, z_1) is not commutative and that (A, z_1, z_2) is not associative.

For $m = 3$, by the same way as the case $m = 2$, we can apply 4.3.3 three times. Namely,

1. take $\Lambda = \Gamma_1^{(3)}$, $\Gamma_i = \Gamma^{(3)}$ and $z = z_1$ and get the division $\Gamma_1^{(3)}$ -graded commutative associative algebra (A, z_1) ,
2. take $\Lambda = \Gamma_2^{(3)}$, $\Gamma_i = \Gamma_1^{(3)}$ and $z = z_2$ and get the division $\Gamma_2^{(3)}$ -graded associative algebra (A, z_1, z_2) ,
3. take $\Lambda = \mathbb{Z}^n$, $\Gamma_i = \Gamma_2^{(3)}$ and $z = z_3$ and get the required division \mathbb{Z}^n -graded alternative algebra (A, z_1, z_2, z_3) over K .

Moreover, (iv) implies that (A, z_1) has the non-trivial graded scalar involution, that (A, z_1, z_2) is not commutative and that (A, z_1, z_2, z_3) is not associative.

Now we are ready to construct four octonion rings.

Construction 4.3.5. Let K be a field extension of F and let $\{\epsilon_1, \dots, \epsilon_n\}$ be a basis of \mathbb{Z}^n .

- (1) Let \mathbb{O} be an octonion division algebra over K and

$$K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} Kz_\alpha$$

the algebra of Laurent polynomials over K where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_n \epsilon_n \in \mathbb{Z}^n$. Then, by 2.3.3(ii).

$$\mathbb{O}_1 := \mathbb{O} \otimes_K K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} (\mathbb{O} \otimes_K Kz_\alpha)$$

is a division \mathbb{Z}^n -graded alternative algebra over K , and hence over F . It is clearly alternative but not associative.

(2) Let \mathbb{H} be a quaternion division algebra over K with scalar involution $*$. Let

$$\begin{aligned} \Gamma^{(1)} &:= 2\mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \cdots + \mathbb{Z}\epsilon_n \\ R &:= K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma^{(1)}} Kz_\alpha \end{aligned}$$

where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \cdots + \alpha_n \epsilon_n \in \Gamma^{(1)}$. Then, by 2.3.3(ii),

$$\mathbb{H}_1 := \mathbb{H} \otimes_K R = \bigoplus_{\alpha \in \Gamma^{(1)}} (\mathbb{H} \otimes_K Kz_\alpha)$$

is a division $\Gamma^{(1)}$ -graded associative algebra over F and it is clearly associative but not commutative. By 4.2.9, \mathbb{H}_1 has the scalar involution $*$ over R defined by $(x \otimes z)^* = x^* \otimes z$ for $x \in \mathbb{H}$, $z \in R$, and so $*$ is clearly graded on \mathbb{H}_1 . Let $Z(\mathbb{H}_1)$ be the centre of \mathbb{H}_1 . Then, $R \subset Z(\mathbb{H}_1)$ (in fact they are equal), $z_1 \in R$ has degree $2\epsilon_1$ in \mathbb{H}_1 and z_1 is invertible in R . Thus we can apply Corollary 4.3.4. Namely, the algebra

$$\mathbb{O}_2 := (\mathbb{H}_1, z_1)$$

obtained by the Cayley-Dickson process over R is a division \mathbb{Z}^n -graded alternative algebra over K , and hence over F , which is not associative.

Table 1 All \equiv below are mod 2.

For $x \in \mathbb{H}$ and $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_n \epsilon_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_2 = (\mathbb{H}_1, z_1)$

$$xt_\alpha := \begin{cases} (x \otimes z_\alpha, 0) & \text{if } \alpha_1 \equiv 0 \\ (0, x \otimes z_{\alpha - \epsilon_1}) & \text{if } \alpha_1 \equiv 1. \end{cases}$$

Note that $t_{\epsilon_1} = (0, 1)$ and $t_{\epsilon_1}^2 = z_1$. Then we can write $\mathbb{O}_2 = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{H}t_\alpha$. For $x, y \in \mathbb{H} = (\mathbb{H}, *)$ and $\alpha, \beta \in \mathbb{Z}^n$, one can check that

$$(xt_\alpha)(yt_\beta) = \begin{cases} xyt_{\alpha+\beta}, & \text{if } \alpha_1 \equiv \beta_1 \equiv 0 \\ x^*yt_{\alpha+\beta}, & \text{if } \alpha_1 \equiv 0, \beta_1 \equiv 1 \\ yxt_{\alpha+\beta}, & \text{if } \alpha_1 \equiv 1, \beta_1 \equiv 0 \\ yx^*t_{\alpha+\beta}, & \text{if } \alpha_1 \equiv \beta_1 \equiv 1. \end{cases}$$

(3) Assume that $n \geq 2$. Let \mathbb{E} be a separable quadratic field extension of K . Hence the nontrivial Galois automorphism over K is a scalar involution denoted by $*$. Let

$$\begin{aligned}\Gamma^{(2)} &:= 2\mathbb{Z}\epsilon_1 + 2\mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3 + \cdots + \mathbb{Z}\epsilon_n \\ R &:= K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma^{(2)}} Kz_\alpha\end{aligned}$$

where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\epsilon_1 + 2\alpha_2\epsilon_2 + \alpha_3\epsilon_3 + \cdots + \alpha_n\epsilon_n \in \Gamma^{(2)}$. Then,

$$\mathbb{E}_1 := \mathbb{E} \otimes_K R = \bigoplus_{\alpha \in \Gamma^{(2)}} (\mathbb{E} \otimes_K Kz_\alpha)$$

is a division $\Gamma^{(2)}$ -graded commutative associative algebra over K , and it has the graded scalar involution $*$ over R by the same argument used in (2). We have $R \subset Z(\mathbb{E}_1) = \mathbb{E}_1$. Also, z_1 and z_2 satisfy the condition in Corollary 4.3.4. Thus we get a division \mathbb{Z}^n -graded alternative algebra

$$\mathbb{O}_3 := (\mathbb{E}_1, z_1, z_2)$$

over K , and hence over F , which is not associative.

Table 2 All \equiv below are mod 2.

For $x \in \mathbb{E}$ and $\alpha = \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \cdots + \alpha_n\epsilon_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_3 = (\mathbb{E}_1, z_1, z_2)$,

$$t_\alpha := \begin{cases} ((z_\alpha, 0), (0, 0)) & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ ((0, z_{\alpha-\epsilon_1}), (0, 0)) & \text{if } \alpha_1 \equiv 1 \text{ and } \alpha_2 \equiv 0 \\ ((0, 0), (z_{\alpha-\epsilon_2}, 0)) & \text{if } \alpha_1 \equiv 0 \text{ and } \alpha_2 \equiv 1 \\ ((0, 0), (0, z_{\alpha-\epsilon_1-\epsilon_2})) & \text{if } \alpha_1 \equiv 1 \text{ and } \alpha_2 \equiv 1. \end{cases}$$

Note that $t_{\epsilon_1} = (0, 1, 0, 0)$, $t_{\epsilon_2} = (0, 0, 1, 0)$, $t_{\epsilon_1}^2 = z_1$ and $t_{\epsilon_2}^2 = z_2$. Then we can write

$\mathbb{O}_3 = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathbb{E}t_\alpha$. For $x, y \in E = (E, *)$ and $\alpha, \beta \in \mathbb{Z}^n$, one can check that

(I) $\alpha_2 \equiv \beta_2 \equiv 0$:

$$(xt_\alpha)(yt_\beta) = \begin{cases} xyt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv 0, \beta_1 \equiv 1) \\ x^*yt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv 0, \beta_1 \equiv 1) \text{ or } (\alpha_1 \equiv \beta_1 \equiv 1) \end{cases}$$

(II) $\alpha_2 \equiv 0, \beta_2 \equiv 1$:

$$(xt_\alpha)(yt_\beta) = \begin{cases} x^*yt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv 0, \beta_1 \equiv 1) \\ xyt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv 1, \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv \beta_1 \equiv 1) \end{cases}$$

(III) $\alpha_2 \equiv 1, \beta_2 \equiv 0$:

$$(xt_\alpha)(yt_\beta) = \begin{cases} xyt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv 0, \beta_1 \equiv 1) \\ xy^*t_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv 1, \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv \beta_1 \equiv 1) \end{cases}$$

(IV) $\alpha_2 \equiv \beta_2 \equiv 1$:

$$(xt_\alpha)(yt_\beta) = \begin{cases} x^*yt_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv 0, \beta_1 \equiv 1) \\ x^*y^*t_{\alpha+\beta}, & \text{if } (\alpha_1 \equiv 1, \beta_1 \equiv 0) \text{ or } (\alpha_1 \equiv \beta_1 \equiv 1). \end{cases}$$

(4) Assume that $n \geq 3$ and that $\text{ch.}F \neq 2$. We can apply Corollary 4.3.4 for

$$\begin{aligned} \Gamma^{(3)} &:= 2\mathbb{Z}\epsilon_1 + 2\mathbb{Z}\epsilon_2 + 2\mathbb{Z}\epsilon_3 + \mathbb{Z}\epsilon_4 + \cdots + \mathbb{Z}\epsilon_n \\ A = R &:= K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma^{(3)}} Kz_\alpha \end{aligned}$$

where $z_\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\epsilon_1 + 2\alpha_2\epsilon_2 + 2\alpha_3\epsilon_3 + \alpha_4\epsilon_4 + \cdots + \alpha_n\epsilon_n \in \Gamma^{(3)}$. In fact, A has the trivial scalar involution, i.e., the identity map, and z_1, z_2, z_3 satisfy the condition in 4.3.4. Thus we get a division \mathbb{Z}^n -graded alternative algebra

$$\mathbb{O}_4 := (R, z_1, z_2, z_3)$$

over K , and hence over F , which is not associative. It is clear that every homogeneous space is 1-dimensional over K , and so \mathbb{O}_4 is an alternative torus over K . It is called the *octonion torus* or the *Cayley torus* over K (see [2]). This torus is first discovered in [7].

Remark. When $\text{ch.}F = 2$, we can still construct the division \mathbb{Z}^n -graded alternative algebra $(K[z_1^{\pm 1}, \dots, z_n^{\pm 1}], z_1, z_2, z_3)$ by Lemma 4.3.3. However, this is commutative associative, and isomorphic to the algebra of Laurent polynomials in n variables over K .

We present a slightly different description of the octonion torus. For this purpose, we call the algebra

$$(K[z_1^{\pm 1}, \dots, z_n^{\pm 1}], z_1, z_2),$$

which is associative (so a quantum torus over K) the *quaternion torus* over K . One can easily check that the quaternion torus over K is graded isomorphic to $K_{\mathbf{h}} = K_{\mathbf{h}}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ as defined in 2.6.10, where

$$\mathbf{h} = \mathbf{h}^{(n)} = \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix},$$

via

$$((0, 1), (0, 0)) \leftrightarrow u_1, \quad ((0, 0), (1, 0)) \leftrightarrow u_2, \quad z_3 \leftrightarrow u_3, \quad \dots, \quad z_n \leftrightarrow u_n.$$

We identify them. So the octonion torus over K is

$$\mathbb{O}_4 = (K_{\mathbf{h}}, u_3)$$

and the \mathbb{Z}^n -grading comes from the following G -grading of $K_{\mathbf{h}}$ (see 4.3.3):

$$G = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + 2\mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \cdots + \mathbb{Z}\varepsilon_n.$$

$$K_{\mathbf{h}} = \bigoplus_{\alpha \in G} K u_{\alpha}$$

where $u_{\alpha} = u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \cdots u_n^{\alpha_n}$ for $\alpha = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + 2\alpha_3 \varepsilon_3 + \alpha_4 \varepsilon_4 + \cdots + \alpha_n \varepsilon_n$.

Table 3 All \equiv below are mod 2.

For $\alpha = \alpha_1 \varepsilon_1 + \cdots + \alpha_n \varepsilon_n \in \mathbb{Z}^n$, we put in $\mathbb{O}_4 = (K_{\mathbf{h}}, u_3)$

$$t_{\alpha} := \begin{cases} (u_{\alpha}, 0), & \text{if } \alpha_3 \equiv 0 \\ (0, u_{\alpha - \varepsilon_3}), & \text{if } \alpha_3 \equiv 1. \end{cases}$$

Then we have $\mathbb{O}_4 = \bigoplus_{\alpha \in \mathbb{Z}^n} K t_{\alpha}$ as K -spaces. Note that $t_{\varepsilon_3} = (0, 1)$ and $t_{\varepsilon_3}^2 = u_3$.

For $\alpha, \beta \in \mathbb{Z}^n$, one can check that

(I) $\alpha_3 \equiv \beta_3 \equiv 0$:

$$t_{\alpha} t_{\beta} = (-1)^{\alpha_2 \beta_1} t_{\alpha + \beta}$$

(II) $\alpha_3 \equiv 0, \beta_3 \equiv 1$:

$$t_{\alpha} t_{\beta} = \begin{cases} t_{\alpha + \beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_2 \beta_1 + 1} t_{\alpha + \beta} & \text{otherwise} \end{cases}$$

(III) $\alpha_3 \equiv 1, \beta_3 \equiv 0$:

$$t_{\alpha}t_{\beta} = (-1)^{\alpha_1\beta_2}t_{\alpha+\beta}$$

(IV) $\alpha_3 \equiv 1, \beta_3 \equiv 1$:

$$t_{\alpha}t_{\beta} = \begin{cases} t_{\alpha+\beta} & \text{if } \alpha_1 \equiv \alpha_2 \equiv 0 \\ (-1)^{\alpha_1\beta_2+1}t_{\alpha+\beta} & \text{otherwise.} \end{cases}$$

Note that the centre of \mathbb{O}_4 is

$$K[u_1^{\pm 2}, u_2^{\pm 2}, u_3^{\pm 1}, \dots, u_n^{\pm 1}] = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

Also, the structure constants of \mathbb{O}_4 relative to $\{t_{\alpha}\}_{\alpha \in \mathbb{Z}^n}$ are 1 or -1 .

All the gradings of $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 determined by a basis $(\varepsilon_1, \dots, \varepsilon_n)$ of \mathbb{Z}^n described in 4.3.5 are called *toral gradings*.

We show that $\mathbb{O}_1, \mathbb{H}_1$ and \mathbb{E}_1 in 4.3.5 are also the algebras obtained by the Cayley-Dickson process over $K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. More generally, we have the following:

Lemma 4.3.6. *Let R be a commutative associative algebra over a field K . Let $0 \neq \mu_1, \dots, \mu_l \in K$. Then,*

$$(K, \mu_1, \dots, \mu_l) \otimes_K R \cong (R, \mu_1, \dots, \mu_l) \quad \text{as } R\text{-algebras.}$$

Proof. First, we show that $(K, \mu) \otimes_K R \cong (R, \mu)$ over K for $0 \neq \mu \in K$. A map $(K, \mu) \times R = (K \oplus vK) \times R \longrightarrow (R, \mu) = R \oplus uR$ defined by $(a + vb, f) \mapsto af + ubf$ for $a, b \in K$ and $f \in R$ is clearly an K -bilinear map. So we have the induced K -linear map $\theta : (K, \mu) \otimes_K R \longrightarrow (R, \mu)$ such that $\theta((a + vb) \otimes f) = (a + ub)f$, which is clearly onto. Observe that any element of $(K, \mu) \otimes_K R$ has the form $1 \otimes f + v \otimes g$ for some $f, g \in R$. Thus, $\theta(1 \otimes f + v \otimes g) = f + ug = 0$ implies $f = g = 0$. Hence θ is injective. Since

$$\theta((a + vb) \otimes 1)\theta(1 \otimes f) = (a + vb)f = f(a + vb) = \theta(1 \otimes f)\theta((a + vb) \otimes 1),$$

θ is a K -algebra isomorphism. Since $(K, \mu) \otimes_K R$ becomes R -algebra in natural way, θ is also an R -algebra isomorphism. Thus we obtained the case $l = 1$ of the statement. We note that (R, μ) has a scalar involution $*$ over R . Also by Lemma

4.2.9. $(K, \mu) \otimes_K R$ has a scalar involution $\bar{}$ over R . i.e.. $\overline{(a + vb) \otimes f} = (a - vb) \otimes f$. Thus. $\theta(\overline{(a + vb) \otimes f}) = af - ubf = \theta((a + vb) \otimes f)^*$, and so θ is an isomorphism with involution.

Suppose that $l > 1$. We put $B := (K, \mu_1, \dots, \mu_{l-1})$ and $T := (R, \mu_1, \dots, \mu_{l-1})$. We assume that $\theta : B \otimes_K R \xrightarrow{\sim} T$ is an R -algebra isomorphism with involution. Then.

$$\begin{aligned} (K, \mu_1, \dots, \mu_l) \otimes_K R &= (B, \mu_l) \otimes_K R = (B \oplus vB) \otimes_K R \\ &= (B \otimes_K R) \oplus (vB \otimes_K R) \quad \text{and} \quad (R, \mu_1, \dots, \mu_l) = (T, \mu_l) = T \oplus uT. \end{aligned}$$

Thus, $\eta : (B \otimes_K R) \oplus (vB \otimes_K R) \longrightarrow T \oplus uT$ defined by

$$\eta((a \otimes f) + (vb \otimes g)) = \theta(a \otimes f) + u\theta(b \otimes g)$$

is clearly an R -module isomorphism. Also, that θ is an R -algebra isomorphism with involution clearly insures that η is an R -algebra isomorphism with involution. So we get the statement for any l . \square

We prove our main result in this section:

Theorem 4.3.7. *A division \mathbb{Z}^n -graded alternative algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ over F which is not associative is graded isomorphic over K to one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 for some toral gradings where $K = Z_0$ is a field. In other words, when $\text{ch.}F \neq 2$, it is isomorphic over K to $(K[z_1^{\pm 1}, \dots, z_n^{\pm 1}], \mu_1, \mu_2, \mu_3)$ for*

- (1) $0 \neq \mu_1, \mu_2, \mu_3 \in K$ such that (K, μ_1, μ_2, μ_3) is an octonion division algebra over K .
- (2) $0 \neq \mu_1, \mu_2 \in K$ and $\mu_3 = z_1$ such that (K, μ_1, μ_2) is a quaternion division algebra over K ,
- (3) $0 \neq \mu_1 \in K$, $\mu_2 = z_1$ and $\mu_3 = z_2$ such that (K, μ_1) is a quadratic field extension of K or
- (4) $\mu_1 = z_1$, $\mu_2 = z_2$ and $\mu_3 = z_3$.

When $\text{ch.}F = 2$, after some modification we can get a similar description of (1), (2), (3) (type (4) is gone).

Also, these four algebras are all non-isomorphic.

Proof. We already know that A is an octonion ring whose central closure \bar{A} is an octonion division algebra over \bar{Z} . By Lemma 2.3.5(i), \bar{A} is a \mathbb{Z}^n/Γ -graded alternative

algebra where $\Gamma = \{\alpha \in \mathbb{Z}^n \mid Z_\alpha = Z \cap A_\alpha \neq 0\}$ is the central grading group of A .
i.e.,

$$\bar{A} = \bigoplus_{\bar{\alpha} \in \mathbb{Z}^n/\Gamma} \bar{A}_{\bar{\alpha}} \quad \text{where } \bar{A}_{\bar{\alpha}} = \bar{Z} \otimes_Z A_{\bar{\alpha}} \text{ and } A_{\bar{\alpha}} = ZA_{\alpha}.$$

Since \bar{A} is a division algebra, it is, in particular, a division \mathbb{Z}^n/Γ -graded algebra of dimension 8. Hence by 2.1.5(1), every homogeneous space has the same dimension. Also, since the octonion algebra \bar{A} has a scalar involution, \mathbb{Z}^n/Γ is either (0) or an elementary group of exponent 2, by Lemma 4.2.13. Therefore, we have four cases:

$$\mathbb{Z}^n/\Gamma \cong (0), \quad \mathbb{Z}_2, \quad \mathbb{Z}_2^2 \quad \text{or} \quad \mathbb{Z}_2^3.$$

By the fundamental theorem of finitely generated abelian groups, there exists a basis $\{\epsilon_1, \dots, \epsilon_n\}$ of \mathbb{Z}^n such that for $m = 0, 1, 2, 3$,

$$\Gamma = 2\mathbb{Z}\epsilon_1 + \dots + 2\mathbb{Z}\epsilon_m + \mathbb{Z}\epsilon_{m+1} + \dots + \mathbb{Z}\epsilon_n.$$

Let

$$0 \neq t_i \in A_{\epsilon_i} \quad \text{for } 1 \leq i \leq m.$$

By 2.3.4(iii), we have $A = \bigoplus_{\bar{\alpha} \in \mathbb{Z}^n/\Gamma} A_{\bar{\alpha}}$ and hence

$$A = \langle ZA_0, t_i \rangle_{1 \leq i \leq m}$$

as a Z -algebra. Also by Lemma 4.2.13(iii), we have

$$n(t_i, ZA_0) = 0.$$

Moreover, by Lemma 4.2.13(ii), $t_i^2 \in \bar{Z}1$, and so $t_i^2 \in \bar{Z}1 \cap A_{2\epsilon_i}$. Hence by Lemma 2.3.5(iii),

$$t_i^2 \in Z_{2\epsilon_i} \subset Z.$$

Choose $0 \neq z_j \in Z_{\epsilon_j}$, for $m < j \leq n$ and put $z_i := t_i^2$. Since Z is a commutative associative Γ -torus over K , we can write

$$Z = K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \bigoplus_{\alpha \in \Gamma} Kz_\alpha$$

where $z_\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ for $\alpha = 2\alpha_1\epsilon_1 + \dots + 2\alpha_m\epsilon_m + \alpha_{m+1}\epsilon_{m+1} + \dots + \alpha_n\epsilon_n \in \Gamma$ (see 2.5.4 and 2.5.11).

Case (I): $\mathbb{Z}^n/\Gamma = (0)$, i.e., $\Gamma = \mathbb{Z}^n$.

By Lemma 2.3.5(ii), we have $\dim_K A_0 = \dim_{\bar{Z}} \bar{A}_0 = \dim_{\bar{Z}} \bar{A} = 8$. Since A_0 is an alternative division algebra over K , it is strongly prime, in particular. Since A is not associative, neither is A_0 . Hence, A_0 is an octonion algebra over its centre, by Slater's theorem 4.3.1. However, since $\dim_K A_0 = 8$, A_0 is already central over K and it is an octonion division algebra over K . By 2.3.4(iv), we can identify (and always do below)

$$ZA_0 = A_0 \otimes_K Z.$$

Hence we get $A = ZA_0 = A_0 \otimes_K K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \mathbb{O}_1$ in Construction 4.3.5(1).

For the next two cases, we prove a lemma. Recall that A is a subalgebra of \bar{A} over Z , by identifying $x = 1 \otimes x \in \bar{A}$ for $x \in A$.

Lemma 4.3.8. *Let $*$ be the scalar involution of the octonion division \bar{Z} -algebra \bar{A} . Suppose that the K -algebra A_0 has a scalar involution $\bar{}$. Then, $\sigma := *|_{ZA_0}$ is a scalar involution of Z -algebra ZA_0 , and for $z_i \in Z$, $x_i \in A_0$, we have $\sigma(\sum_i z_i x_i) = \sum_i z_i \bar{x}_i$.*

Proof. By Lemma 2.3.5(ii), one can easily check that

$$\bar{A}_0 = \bar{Z} \otimes_Z A_0 \cong \bar{Z} \otimes_K A_0$$

over \bar{Z} via $y \otimes x \mapsto y \otimes x$ for $y \in \bar{Z}$ and $x \in A_0$. By Lemma 4.3.3, we can naturally extend $\bar{}$ to the subalgebra \bar{A}_0 of \bar{A} over \bar{Z} . Namely, we can define $\overline{y \otimes x} := y \otimes \bar{x}$, which is still a scalar involution over \bar{Z} . But \bar{A}_0 has another scalar involution, that is, $* := *|_{\bar{A}_0}$. So we get $\bar{} = *$ on \bar{A}_0 since a scalar involution over a field is unique

by Lemma 4.2.7. Thus for $z_i \in Z$, $x_i \in A_0$, we have

$$\begin{aligned} \sigma\left(\sum_i z_i x_i\right) &= (1 \otimes \sum_i z_i x_i)^* = \left(\sum_i z_i \otimes x_i\right)^* \\ &= \sum_i z_i \otimes \bar{x}_i = 1 \otimes \sum_i z_i \bar{x}_i = \sum_i z_i \bar{x}_i, \\ \text{and } \left(\sum_i z_i x_i\right)\sigma\left(\sum_i z_i x_i\right) &= \left(\sum_i z_i x_i\right)\left(\sum_i z_i \bar{x}_i\right) \\ &= \sum_i z_i^2 x_i \bar{x}_i + \sum_{i < j} z_i z_j (x_i \bar{x}_j + x_j \bar{x}_i) \\ &= \sum_i n(x_i) z_i^2 + \sum_{i < j} n(x_i, x_j) z_i z_j \in Z, \end{aligned}$$

since $n(x_i), n(x_i, x_j) \in K$. Hence, σ is a scalar involution of ZA_0 over Z . \square

We denote this σ also by $*$.

Case (II): $Z^n/\Gamma = \mathbb{Z}_2$, i.e., $\Gamma = 2\mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \cdots + \mathbb{Z}\epsilon_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\bar{Z}} \bar{A}_0 = 4$. Then, by Lemma 4.2.10, $\bar{A}_0 = (\bar{A}_0, *)$ is a quaternion division algebra over \bar{Z} with the restricted scalar involution $*$ and $\bar{A} = (\bar{A}_0, \mu)$ for some structure constant $0 \neq \mu \in \bar{Z}$. Also by Lemma 2.3.5(ii), A_0 is a 4-dimensional associative non-commutative division algebra over K . Hence it has to be central and has degree 2. It is well-known that a central simple associative algebra of degree 2 over a field is a cyclic algebra (see e.g. [29]). So it is a quaternion algebra. (See e.g. [29, page 236] in the case of characteristic $\neq 2$. In the case of characteristic $= 2$, it is easily seen that the cyclic algebra (L, σ, μ_2) is exactly (Φ_{μ_1}, μ_2) for some μ_1 , where the Galois field L corresponds to Φ_{μ_1} and the automorphism σ corresponds to the scalar involution $*$ of Φ_{μ_1} .) Hence, A_0 has a scalar involution over K , and by Lemma 4.3.8, ZA_0 has the scalar involution $* = *|_{ZA_0}$ over Z .

Thus we can apply Lemma 4.2.11 for $A = \langle ZA_0, t_1 \rangle$:

$$\begin{aligned} L &:= \bar{Z}, \quad R := Z, \quad B := (\bar{A}_0, *), \\ P &:= ZA_0 \subset B, \quad \text{having the scalar involution } * \text{ over } R, \\ C &:= \bar{A} = (B, \mu) \quad \text{and} \quad t := t_1. \end{aligned}$$

So we get

$$A = \langle ZA_0, t_1 \rangle = (ZA_0, z_1)$$

and $t_1 = (0, 1)$. Let $\mathbb{H} := A_0$ which is the quaternion division algebra over K described above, in which case the division Γ -graded associative algebra

$$ZA_0 = \mathbb{H} \otimes_K K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$$

is equal to \mathbb{H}_1 constructed in 4.3.5(2). Thus we get

$$A = (ZA_0, z_1) = (\mathbb{H}_1, z_1) = \mathbb{O}_2 \quad \text{in Construction 4.3.5(2),}$$

which have the same \mathbb{Z}^n -gradings (see the uniqueness part of \mathbb{Z}^n -gradings in 4.3.3).

Case (III): $\mathbb{Z}^n/\Gamma = \mathbb{Z}_2^2$, i.e., $\Gamma = 2\mathbb{Z}\epsilon_1 + 2\mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3 + \dots + \mathbb{Z}\epsilon_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\bar{Z}} \bar{A}_0 = 2$. Thus, $\bar{A}_0 = (\bar{A}_0, *)$ is a quadratic field extension of \bar{Z} with the restricted scalar involution $*$, and by Lemma 4.2.10,

$$\bar{A} = (\bar{A}_0, \mu_1, \mu_2)$$

for some structure constants $0 \neq \mu_1, \mu_2 \in \bar{Z}$. Also by Lemma 2.3.5(ii), A_0 is a quadratic field extension of K . Hence, A_0 has a scalar involution over K , and by Lemma 4.3.8, ZA_0 has the scalar involution $* = *|_{ZA_0}$ over Z . Also, if A_0 is not separable, then the scalar involution of A_0 is trivial. So, $*$ becomes trivial and \bar{A} becomes commutative and associative. This contradicts our setting. Therefore, A_0 is a separable quadratic field extension of K .

As in the Case (II), we can apply Corollary 4.2.12 and get

$$A = \langle ZA_0, t_1, t_2 \rangle = (ZA_0, z_1, z_2),$$

$t_1 = ((0, 1), (0, 0))$ and $t_2 = ((0, 0), (1, 0))$. Let $\mathbb{E} := A_0$ which is the separable quadratic field extension of K shown above, in which case the division Γ -graded commutative associative algebra

$$ZA_0 = \mathbb{E} \otimes_K K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$$

is equal to \mathbb{E}_1 constructed in 4.3.5(3). Thus we get

$$A = (ZA_0, z_1, z_2) = (\mathbb{E}_1, z_1, z_2) = \mathbb{O}_3 \quad \text{in Construction 4.3.5(3),}$$

which have the same \mathbb{Z}^n -gradings (see the uniqueness part of \mathbb{Z}^n -gradings in 4.3.4).

Case (IV): $\mathbb{Z}^n/\Gamma = \mathbb{Z}_2^3$, i.e., $\Gamma = 2\mathbb{Z}\epsilon_1 + 2\mathbb{Z}\epsilon_2 + 2\mathbb{Z}\epsilon_3 + \mathbb{Z}\epsilon_4 + \cdots + \mathbb{Z}\epsilon_n$.

Since every homogeneous space has the same dimension, we have $\dim_{\bar{Z}} \bar{A}_0 = 1$. Thus we have $\bar{Z}1 = \bar{A}_0 = (\bar{A}_0, *)$ with the restricted scalar involution $*$ which is the trivial involution, and by Lemma 4.2.10,

$$\bar{A} = (\bar{A}_0, \mu_1, \mu_2, \mu_3),$$

for some structure constants $0 \neq \mu_1, \mu_2, \mu_3 \in \bar{Z}$ if $\text{ch.}F \neq 2$.

Note that $A_0 = K$ and $ZA_0 = Z$ in this case (see 2.3.5(ii)). As in Case (II), we can apply Corollary 4.2.12 and get

$$A = \langle Z, t_1, t_2, t_3 \rangle = \langle Z, z_1, z_2, z_3 \rangle = \mathbb{O}_4 \quad \text{in Construction 4.3.5(4),}$$

which have the same \mathbb{Z}^n -gradings (see the uniqueness part of \mathbb{Z}^n -gradings in 4.3.4).

If $\text{ch.}F = 2$, then $\bar{A}_0 \oplus \bar{A}_{\epsilon_1} = \bar{A}_0(t)$ for any nonzero element $t \in \bar{A}_{\epsilon_1}$, which is a purely inseparable extension field of \bar{A}_0 by Lemma 4.2.13(i). Hence, $\bar{A}_0 \oplus \bar{A}_{\epsilon_1}$ is not a composition algebra, which contradicts Lemma 2.10. Therefore, Case (IV) cannot happen if $\text{ch.}F = 2$.

The different descriptions of the octonion rings follow from 4.3.6.

The last statement follows from Theorem 2.4.3(i). \square

Remark. One can easily see that the algebras described in (1), (2) and (3) of Theorem 4.3.7 excluding the 'such that' part are predivision \mathbb{Z}^n -graded alternative algebras over F which are not associative.

Example 4.3.9. We choose our base field F to be \mathbb{R} , the field of real numbers. Then there exist a unique quadratic field extension \mathbb{C} , the field of complex numbers, a unique quaternion division algebra \mathbb{H} , Hamilton's quaternion and a unique octonion division algebra \mathbb{O} , the algebras of Cayley numbers. Hence there exist four division \mathbb{Z}^n -graded alternative but not associative algebras over \mathbb{R} , namely, taking K to be \mathbb{R} in Construction 4.3.5. $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 , the Cayley torus over \mathbb{R} .

Moreover, if we assume that a homogeneous space is finite dimensional (and hence all homogeneous spaces are finite dimensional), then any division \mathbb{Z}^n -graded alternative but not associative algebra over \mathbb{R} is isomorphic to one of the four algebras above

or another \mathbb{O}_4 , the Cayley torus over \mathbb{C} , taking K to be \mathbb{C} in Construction 4.3.5. Indeed, we know that finite dimensional field extension K of \mathbb{R} is \mathbb{R} or \mathbb{C} . If $K = \mathbb{R}$, then we get the four algebras above. If $K = \mathbb{C}$, then the only finite dimensional division algebra over \mathbb{C} is \mathbb{C} . Hence we have $A_0 = K = \mathbb{C}$ and only Case (IV) appears in Theorem 4.3.7.

From Corollary 4.1.7 and Theorem 4.3.7 we obtain:

Corollary 4.3.10. *A division \mathbb{Z}^n -graded alternative algebra A over F is graded isomorphic to some $D_{\varphi, q}$ or to one of the four octonion rings $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ and \mathbb{O}_4 for some toral gradings.*

If a division \mathbb{Z}^n -graded alternative algebra $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ over F is a torus over F , then $A_0 = Z_0 = K = F$. So we have the following corollary which improves the result in [7], namely, the classification of alternative tori over any field:

Corollary 4.3.11. *Let A be an alternative torus over F . Then, A is graded isomorphic to either a quantum torus F_q or the Cayley torus*

$$\mathbb{O}_t = (F_{\mathbf{h}}[u_1^{\pm 1}, \dots, u_n^{\pm 1}], u_3)$$

for some toral grading. If $\text{ch}.F = 2$, then the Cayley torus does not exist, and so A is isomorphic to a quantum torus F_q .

Corollary 4.3.12. *Let A be an alternative torus over F of central degree 2. Then, A is graded isomorphic to either the quaternion torus $F_{\mathbf{h}}$ or the Cayley torus \mathbb{O}_t for some toral grading. If $\text{ch}.F = 2$, then such a torus does not exist.*

Proof. If A is not associative, then by 4.3.11, A is graded isomorphic to the Cayley torus unless $\text{ch}.F = 2$. If A is associative, we already classified such a torus in 2.6.10, i.e. $A \cong_{\mathbb{Z}^n} F_{\mathbf{h}}$ unless $\text{ch}.F = 2$. \square

Remark. One can check that the plus algebra \mathbb{O}_t^+ of the Cayley torus \mathbb{O}_t over F is graded isomorphic to a Clifford torus $J_{S^{(m)}}(\{a_\epsilon\}_{\epsilon \in I})$ (see §3.3) determined by the lattice $S^{(m)} = \mathbb{Z}^3$ and $a_\epsilon = 1$ or -1 .

Chapter 5 Graded forms and derivations of tori

In this chapter, we classify the graded forms (see Definition 5.1.4) and derivations of tori, where tori means alternative tori and Jordan tori. Recall that alternative tori are quantum tori (associative tori) F_q over any field F or the Cayley torus \mathbb{O}_t over F of $\text{ch.}F \neq 2$ (see 4.3.11). The Jordan tori over F of $\text{ch.}F \neq 2$ are the three classes of Hermitian tori F_q^+ , $H(F_\epsilon, *)$ and $H(E_\xi, \sigma)$ (see §3.1), a Clifford tori (see §3.3). or the Albert torus \mathbb{A}_t over F where F is a field containing a primitive 3rd root of unity (see §3.4).

§ 5.1 GRADED FORMS

We first show that certain division G -graded algebras admit a decomposition as the sum of into the centre and the commutator or associator subspace.

Lemma 5.1.1. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a division G -graded algebra over F , $\mathcal{Z} = Z(\mathcal{T})$ its centre and $H = H(\mathcal{T})$ its central grading group, in which case $\mathcal{Z} = \bigoplus_{h \in H} \mathcal{Z}_h$ where $\mathcal{Z}_h = \mathcal{Z} \cap \mathcal{T}_h$. Assume that $\mathcal{Z}_h = \mathcal{T}_h$ for some $h \in H$. Then:*

- (i) $\mathcal{Z} = \bigoplus_{h \in H} \mathcal{T}_h$,
- (ii) if \mathcal{T} is alternative, it is a G -torus over the field \mathcal{Z}_e ,
- (iii) if \mathcal{T} is associative, then for any $g \in G \setminus H$ and any element $t_g \in \mathcal{T}_g$,

$$(a) \quad t_g = [u, v] \quad \text{for some homogeneous elements } u, v \in \mathcal{T},$$

$$(b) \quad [\mathcal{T}, \mathcal{T}] = \bigoplus_{g \in G \setminus H} \mathcal{T}_g, \quad \text{and hence } \mathcal{T} = \mathcal{Z} \oplus [\mathcal{T}, \mathcal{T}],$$

- (iv) if \mathcal{T} is alternative but not associative, and $G = \mathbb{Z}^n$,

$$[\mathcal{T}, \mathcal{T}] = (\mathcal{T}, \mathcal{T}, \mathcal{T}) = \bigoplus_{g \in G \setminus H} \mathcal{T}_g$$

$$\text{and hence } \mathcal{T} = \mathcal{Z} \oplus [\mathcal{T}, \mathcal{T}] = \mathcal{Z} \oplus (\mathcal{T}, \mathcal{T}, \mathcal{T}).$$

Proof. (i): For any $k \in H$, we have $kh^{-1} \in H$. By 2.3.4(i) and our assumption, we have $\mathcal{T}_k = \mathcal{Z}_{kh^{-1}}\mathcal{T}_h = \mathcal{Z}_{kh^{-1}}\mathcal{Z}_h$. Since $\mathcal{Z} = \bigoplus_{h \in H} \mathcal{Z}_h$ is a division H -graded

associative algebra. it is of strong type (see 2.1.5(1)), and so $\mathcal{Z}_{kh^{-1}}\mathcal{Z}_h = \mathcal{Z}_k$. Hence we get $\mathcal{T}_k = \mathcal{Z}_k$ and $\mathcal{Z} = \bigoplus_{h \in H} \mathcal{T}_h$.

(ii): By (i), we have $\mathcal{T}_e = \mathcal{Z}_e$, and so $\mathcal{T} = \bigoplus_{g \in G} \mathcal{Z}_e x_g$ where $(x_g)_{g \in G}$ is any family of homogeneous nonzero elements (see 2.1.5(1)). Hence, \mathcal{T} is a G -torus over \mathcal{Z}_e .

(iii): Let $g \in G \setminus H$ and $0 \neq t_g \in \mathcal{T}_g$. Since $t_g \notin \mathcal{Z}$ and \mathcal{T} is associative, there exists $t_k \in \mathcal{T}_k$ such that $0 \neq [t_g, t_k]$. Hence $[t_g t_k^{-1}, t_k] = t_g - t_k t_g t_k^{-1} \neq 0$. So by (ii), $= z t_g$ for some nonzero $z \in \mathcal{Z}_e$, and therefore $t_g = [t_g t_k^{-1}, z^{-1} t_k] \in [\mathcal{T}, \mathcal{T}]$. This proves (a) and “ \supset ” of (b). For the other inclusion it is sufficient to prove that for $g, k \notin H$ but $gk \in H$ we have $[\mathcal{T}_g, \mathcal{T}_k] = 0$. Indeed, for $0 \neq t_g \in \mathcal{T}_g$, $0 \neq t_k \in \mathcal{T}_k$ we have by (i), $0 \neq t_g t_k = z \in \mathcal{T}_{gk} = \mathcal{Z}_{gk}$, whence $t_k = t_g^{-1} z$ and then $[t_g, t_k] = z[t_g, t_g^{-1}] = 0$.

(iv): By (ii) and 4.3.11, $\text{ch.} F \neq 2$ and \mathcal{T} is the Cayley torus over \mathcal{Z}_e . Thus the decompositions follow from [7, Remark, page 4321 and Lemma 1.24, page 4322]. \square

Corollary 5.1.2. *Let G be a totally ordered abelian group and let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a division G -graded associative algebra over F such that $\mathcal{Z}_h = \mathcal{T}_h$ for some $h \in H$, where $H = H(\mathcal{T})$ is the central grading group. Then $\mathcal{Z}(\mathcal{T}) = \mathcal{Z}(\mathcal{T}^+)$ and*

$$(\mathcal{T}^+, \mathcal{T}^+, \mathcal{T}^+) = \bigoplus_{g \in G \setminus H} (\mathcal{T})_g, \quad \text{and hence} \quad \mathcal{T}^+ = \mathcal{Z}(\mathcal{T}^+) \oplus (\mathcal{T}^+, \mathcal{T}^+, \mathcal{T}^+).$$

Proof. Our assumption that G is a totally ordered abelian group implies that \mathcal{T} is a domain (see 2.2.4), and so $\mathcal{Z}(\mathcal{T}^+) = \mathcal{Z}(\mathcal{T})$ (see 1.1.10) and $H(\mathcal{T}^+) = H$. By 1.1.10a and 5.1.1, we have

$$(\mathcal{T}^+, \mathcal{T}^+, \mathcal{T}^+) = (\mathcal{T}, \mathcal{T}, \mathcal{T})^\circ = [\mathcal{T}, [\mathcal{T}, \mathcal{T}]] = [\mathcal{T}, \mathcal{T}] = \bigoplus_{g \in G \setminus H} \mathcal{T}_g. \quad \square$$

Example 5.1.3. Let $F_q = \bigoplus_{\alpha \in \mathbb{Z}^n} (F_q)_\alpha$ be a quantum torus, $Z = Z(F_q)$ its centre and $\Gamma = \Gamma(F_q)$ its central grading group. Since F_q satisfies the conditions of 5.1.1 and 5.1.2, we have for any $\alpha \in \mathbb{Z}^n \setminus \Gamma$ and any element x of degree α ,

- (a) $x = [u, v]$ for some homogeneous elements $u, v \in F_q$,
- (b) $[F_q, F_q] = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} (F_q)_\alpha$ and hence $F_q = Z \oplus [F_q, F_q]$,

which was also shown in [6] Proposition 2.44(ii)&(iii). Moreover, by 5.1.2,

$$(F_q^+, F_q^+, F_q^+) = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} (F_q)_\alpha, \quad \text{and hence} \quad F_q^+ = Z \oplus (F_q^+, F_q^+, F_q^+).$$

Also, for the Cayley torus $\mathbb{O} = \mathbb{O}_t$, we have

$$[\mathbb{O}, \mathbb{O}] = (\mathbb{O}, \mathbb{O}, \mathbb{O}) = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma(\mathbb{O})} \mathbb{O}_\alpha,$$

$$\text{and hence } \mathbb{O} = Z(\mathbb{O}) \oplus [\mathbb{O}, \mathbb{O}] = Z(\mathbb{O}) \oplus (\mathbb{O}, \mathbb{O}, \mathbb{O}),$$

which was also shown in [7, Remark 1.22, page 4321, Lemma 1.24, page 4322 and Theorem 1.40, page 4328].

Definition 5.1.4. For a G -graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ over F , we call a symmetric invariant bilinear form (\cdot, \cdot) satisfying for all $g, h \in G$,

$$(\mathcal{T}_g, \mathcal{T}_h) = 0 \quad \text{if } gh \neq e$$

a *graded form*. The term ‘invariant’ (= associative) means that for all $x, y, z \in \mathcal{T}$,

$$(xy, z) = (x, yz).$$

We denote the linear space of all graded forms of \mathcal{T} by $\text{Gf}(\mathcal{T})$.

For a G -graded algebra $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ over F , let $f : \mathcal{T}_e \rightarrow F$ be a linear form. We define a linear form $\varepsilon_f : \mathcal{T} \rightarrow F$ by

$$\varepsilon_f|_{\mathcal{T}_g} = \begin{cases} f & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\varepsilon_f(x, y) := \varepsilon_f(xy) \quad \text{for } x, y \in \mathcal{T}$$

is clearly a bilinear form satisfying $\varepsilon_f(\mathcal{T}_g, \mathcal{T}_h) = 0$ if $gh \neq e$.

Lemma 5.1.5. \mathcal{T} is division graded and $f(1) \neq 0 \implies \varepsilon_f(\cdot, \cdot)$ is nondegenerate.

Proof. Suppose that $\varepsilon_f(x, \mathcal{T}) = \{0\}$. Let $x = \sum x_g$ be the homogeneous decomposition of x . If $x_g \neq 0$, then $0 = \varepsilon_f(x, x_g^{-1}) = \varepsilon_f(x_g x_g^{-1}) = \varepsilon_f(1) = f(1) \neq 0$, which is a contradiction. Hence $x_g = 0$ for all g , i.e., $x = 0$. Similarly, $\varepsilon_f(\mathcal{T}, x) = \{0\}$ implies that $x = 0$. \square

Observe that for all $x, y, z \in \mathcal{T}$,

$$\varepsilon_f(x, y) = \varepsilon_f(y, x) \iff \varepsilon_f([x, y]) = 0 \iff [T, T] \subset \ker \varepsilon_f$$

$$\varepsilon_f(xy, z) = \varepsilon_f(x, yz) \iff \varepsilon_f((x, y, z)) = 0 \iff (T, T, T) \subset \ker \varepsilon_f.$$

Hence,

$$(5.1.5a) \quad \varepsilon_f(\cdot, \cdot) \in \text{Gf}(\mathcal{T}) \iff [T, T] + (T, T, T) \subset \ker \varepsilon_f.$$

Proposition 5.1.6. *Let $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ be a unital G -graded algebra over F .*

- (a) *For any $(\cdot, \cdot) \in \text{Gf}(\mathcal{T})$ there exists some $f \in \mathcal{T}_e^*$ such that $(\cdot, \cdot) = \varepsilon_f(\cdot, \cdot)$.*
(b) *Suppose that*

$$(5.1.6a) \quad [\mathcal{T}, \mathcal{T}] + (\mathcal{T}, \mathcal{T}, \mathcal{T}) \subset \bigoplus_{g \neq e} \mathcal{T}_g.$$

Then \mathcal{T}_e is a commutative associative subalgebra, and the map $\varepsilon : \mathcal{T}_e^ \longrightarrow \text{Gf}(\mathcal{T})$ defined by $\varepsilon(f) = \varepsilon_f(\cdot, \cdot)$ is a linear isomorphism.*

Proof. (a): Define $\varphi(x) := (1, x)$ for $x \in \mathcal{T}$ and let $f = \varphi|_{\mathcal{T}_e}$. Then $\varphi \in \mathcal{T}^*$, $f \in \mathcal{T}_e^*$ and

$$\varphi|_{\mathcal{T}_g} = \begin{cases} f & \text{if } g = e \\ 0 & \text{otherwise,} \end{cases}$$

so $\varphi = \varepsilon_f$ as defined above. Therefore, for all $x, y \in \mathcal{T}$, $(x, y) = (1, xy) = \varphi(xy) = \varepsilon_f(xy) = \varepsilon_f(x, y)$, i.e., $(\cdot, \cdot) = \varepsilon_f(\cdot, \cdot)$.

(b): The first assertion follows from

$$[\mathcal{T}_e, \mathcal{T}_e] + (\mathcal{T}_e, \mathcal{T}_e, \mathcal{T}_e) \subset \left(\bigoplus_{g \neq e} \mathcal{T}_g \right) \cap \mathcal{T}_e = (0).$$

Since $\bigoplus_{g \neq e} \mathcal{T}_g \subset \ker \varepsilon_f$, we have $\varepsilon_f(\cdot, \cdot) \in \text{Gf}(\mathcal{T})$ for all $f \in \mathcal{T}_e^*$, i.e., ε is well-defined. Clearly ε is a linear injection and by (b) it is surjective, and hence an isomorphism. \square

The division graded algebras $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ in 5.1.1 or 5.1.2 satisfy (5.1.6a) above. Note that $\mathcal{T}_e = \mathcal{Z}_e$ for these algebras, and so $\mathcal{Z}_e^* = \mathcal{T}_e^* \cong \text{Gf}(\mathcal{T})$ since $\bigoplus_{g \in G \setminus H} \mathcal{T}_g \subset \bigoplus_{g \neq e} \mathcal{T}_g$. In particular, for an alternative torus $A = F_q$ or \mathcal{O}_t (see 5.1.3), $\varepsilon_f(\cdot, \cdot) \in \text{Gf}(A)$ and $\text{Gf}(A) \cong F$. (When $f(1) = 1$, the graded form $\varepsilon_f(\cdot, \cdot)$ on F_q , \mathcal{O}_t or a Clifford torus is defined in [6, Remark 2.45, page 363], [7, Definition 1.51, page 4333] or [1, page 81].)

We will show that any Jordan torus $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ satisfies (5.1.6a), and so $\text{Gf}(J) \cong F$. Since Jordan algebras are commutative, we only need to show that $(J, J, J) \subset \bigoplus_{\alpha \neq 0} J_\alpha$. In fact, we can prove a stronger property, namely, a decomposition similar to 5.1.3 for Jordan tori:

Theorem 5.1.7. *Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be a Jordan torus over F (ch. $F \neq 2$), $Z = Z(J)$ the centre and $\Gamma = \Gamma(J)$ the central grading group. Then we have*

$$(J, J, J) = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} J_\alpha, \quad \text{and hence } J = Z \oplus (J, J, J).$$

Proof. We already saw the case $J = F_q^+$ in 5.1.3. So we only need to prove the result in the cases where $J = H(F_\epsilon, *)$, $H(E_\xi, \sigma)$, a Clifford torus or the Albert torus. Let

$$J' := \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} J_\alpha.$$

If $J' = (0)$, then we are done, and so assume that $J' \neq (0)$.

We first consider the case $J = H(E_q, \sigma)$ for $E_q = E_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ where $q = \xi$ or $E = F$, $q = \epsilon$ and $\sigma = *$. Let $0 \neq t \in J'$ with t homogeneous. Then there exists i such that $t_i t = x t t_i$ for $1 \neq x \in E$. Otherwise $t \in Z(E_q)$ which contradicts the fact $Z = H(Z(E_q), \sigma)$ (see 3.1.2 and 3.2.18). Hence, $(t, t_i, t_i^{-1})^\circ = (t + t_i t t_i^{-1} + t_i^{-1} t t_i + t) - t = (x + x^{-1} - 2)t$. Since $x \neq 1$, $x + x^{-1} - 2 \neq 0$, and so $t = (x + x^{-1} - 2)^{-1} (t, t_i, t_i^{-1})^\circ$. Since t_i is fixed by σ , we get $t \in (J, J, J)$, and so $J' \subset (J, J, J)$. For the other inclusion, first note that $Z = H(Z(E_q), \sigma)$ and $Z(E_q^+) = Z(E_q)$ imply that $\Gamma(J) \subset \Gamma(E_q^+)$. Thus we have

$$(J, J, J) \subset (E_q^+, E_q^+, E_q^+) = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma(E_q^+)} (E_q^+)_\alpha \subset \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma(J)} (E_q^+)_\alpha.$$

Hence we get $(J, J, J) \subset J'$.

For the remaining cases, i.e., J is a Clifford torus or the Albert torus, we recall the generic trace tr for \mathcal{J} over \mathcal{Z} , described in §2.6. We have $\text{tr}((J, J, J)) = \{0\}$ (see e.g. [13, Corollary, page 227]). Note that for any $0 \neq x \in J_\gamma$, $\text{tr}(x) \neq 0$ for all $\gamma \in \Gamma$. So for $\alpha, \beta, \delta \in \mathbb{Z}^n$, if $\alpha + \beta + \delta \in \Gamma$, then $(J_\alpha, J_\beta, J_\delta) = \{0\}$. Hence we get $(J, J, J) \subset J'$.

We show the other inclusion in each case separately:

Suppose that J is a Clifford torus. Recall the standard basis $\{t_\alpha\}_{\alpha \in \mathbb{Z}^n}$ of J , constructed in 3.3.8. It is enough to show that $t := t_\alpha \in (J, J, J)$ for any $\alpha \in \mathbb{Z}^n \setminus \Gamma$. By 3.3.8, there exists some $u := t_\beta \in J'$ such that $tu = 0$. Hence we get $t = -(tu)u^{-1} + t(uu^{-1}) = -(t, u, u^{-1}) \in (J, J, J)$.

Suppose that $J = \mathbb{A}_t = F_\omega \oplus F_\omega \oplus F_\omega$ is the Albert torus (see 3.4.5). If $0 \neq t \in J' \cap (F_\omega \oplus 0 \oplus 0)$, then $t \notin Z = Z(F_\omega)$, and so by Example 5.1.3, $t \in (F_\omega^+, F_\omega^+, F_\omega^+) \subset (J, J, J)$. Suppose that $t \in 0 \oplus F_\omega \oplus 0$ or $0 \oplus 0 \oplus F_\omega$. Let $u \in F_\omega$ be a homogeneous element of trace 0. Then, by (3.4.0), one can easily check that $u^{-1} \cdot (u \cdot t) = 4^{-1}t$. Hence we get $t = 4 \cdot 3^{-1}(u^{-1}, u, t) \in (J, J, J)$ (Recall that $\text{ch}.F \neq 3$ is one of the assumptions needed to define F_ω). \square

Thus a Jordan torus satisfies (5.1.6a) as an alternative torus does, and so by 5.1.6. $\varepsilon_f(\cdot, \cdot) \in \text{Gf}(T)$ for any torus $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ over F . Moreover, since $T_0^* \cong F1$, we have

$$\text{Gf}(T) \cong F.$$

Let $f_1 : T_0 \rightarrow F$ be the linear form such that $f_1(1) = 1$. We call $\varepsilon(\cdot, \cdot) := \varepsilon_{f_1}(\cdot, \cdot)$ the *standard nondegenerate graded form* of a Jordan or an associative torus T . We can summarize the classification of graded forms for tori as follows:

Corollary 5.1.8. *A graded form of a (Jordan or an associative) torus is unique up to a scalar. That is, for any graded form (\cdot, \cdot) , there exists some $a \in F$ such that $(\cdot, \cdot) = a\varepsilon(\cdot, \cdot)$, where $\varepsilon(\cdot, \cdot) = \varepsilon_{f_1}(\cdot, \cdot)$ is the standard nondegenerate graded form, as defined above. Moreover, (\cdot, \cdot) is nondegenerate if and only if $a \neq 0$.*

Proof. The second assertion follows from 5.1.5. \square

§ 5.2 DERIVATIONS

We classify the derivations of tori in this section. First we review some basic concepts and prove some lemmas for certain division graded algebras.

Let Φ be a unital commutative associative ring. For a nonassociative algebra A over Φ , a linear map $d : A \rightarrow A$ is called a *derivation of A* if for all $x, y \in A$,

$$d(xy) = d(x)y + xd(y).$$

The set of all the derivations of A is denoted by $\text{Der}_\Phi A$ or $\text{Der}A$ unless it creates confusion. This is a Lie algebra over Φ with the usual commutator as the Lie algebra product. Let $Z = Z(A)$ be the centre of A . We can consider $\text{Der}_\Phi A$ as a Lie algebra over Z since zd is a derivation for any $z \in Z$ and $d \in \text{Der}_\Phi A$. Let $\text{Der}_Z A$ be the subalgebra of all Z -linear derivations in $\text{Der}_\Phi A$, which is a Z -subalgebra of $\text{Der}_\Phi A$. Note that for $d \in \text{Der}_\Phi A$,

$$(5.2.1) \quad d \in \text{Der}_Z A \iff d|_Z = 0.$$

Also, by easy calculations, we have, for any $d \in \text{Der}_\Phi A$ and $x, y, z \in A$, $d([x, y]) = [d(x), y] + [x, d(y)]$ and $d((x, y, z)) = (d(x), y, z) + (x, d(y), z) + (x, y, d(z))$. Hence any $d \in \text{Der}_\Phi A$ leaves Z invariant and clearly

$$(5.2.2) \quad d|_Z \in \text{Der}_\Phi Z.$$

Let $F := \Phi$ be a field and assume that Z does not contain any zero divisors of A (e.g. a prime alternative or Jordan algebra), as we assumed in 1.1.16. Let \bar{A} be the central closure of A over \bar{Z} where \bar{Z} is the field of fractions of the integral domain Z . Let

$$\overline{\text{Der}_Z A} := \bar{Z} \otimes_Z \text{Der}_Z A$$

be the Lie algebra over \bar{Z} . For $u \otimes d \in \overline{\text{Der}_Z A}$ and $v \otimes a \in \bar{A}$, we define

$$(u \otimes d)(v \otimes a) := uv \otimes d(a).$$

Then $u \otimes d$ is a derivation of \bar{A} . Since A embeds into \bar{A} via $a \mapsto 1 \otimes a$ (see 1.1.16), we can identify $\overline{\text{Der}_Z A}$ with a \bar{Z} -subalgebra of $\text{Der}_{\bar{Z}} \bar{A}$. Also, since A embeds into \bar{A} , $\text{Der}_Z A$ embeds into $\overline{\text{Der}_Z A}$ via $d \mapsto 1 \otimes d$. So we can identify $\text{Der}_Z A$ with a Z -subalgebra of $\overline{\text{Der}_Z A}$. Thus we have

$$(5.2.3) \quad \text{Der}_Z A \subset \overline{\text{Der}_Z A} \subset \text{Der}_{\bar{Z}} \bar{A}.$$

Lemma 5.2.4. *Let G be an ordered abelian group, $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ a division G -graded algebra over F (so \mathcal{T} is a prime alternative or Jordan algebra (see 2.2.5)) and Z its centre. Assume that the central closure $\bar{\mathcal{T}}$ is finite dimensional over \bar{Z} . Then $\overline{\text{Der}_Z \mathcal{T}} = \text{Der}_{\bar{Z}} \bar{\mathcal{T}}$.*

Proof. We only need to show $\overline{\text{Der}_Z \mathcal{T}} \supset \text{Der}_{\bar{Z}} \bar{\mathcal{T}}$. Choose a basis $\{x_1, \dots, x_r\}$ of $\bar{\mathcal{T}}$ over \bar{Z} which is a Z -basis of the free Z -module \mathcal{T} (see 2.3.5(ii)). For $d \in \text{Der}_{\bar{Z}} \bar{\mathcal{T}}$, we have $d(x_i) = \sum_{j=1}^r (z'_{ij}/z_{ij}) \otimes x_j$ for $z'_{ij} \in Z$ and $0 \neq z_{ij} \in Z$, $1 \leq i, j \leq r$. Let $z := \prod_{i,j} z_{ij}$. Then $zd(x_i) \in \mathcal{T}$ for all $i = 1, \dots, r$. Since $\{x_1, \dots, x_r\}$ is a Z -basis of \mathcal{T} , $zd \in \text{Der}_Z \mathcal{T}$, whence $d = z^{-1} \otimes zd \in \overline{\text{Der}_Z \mathcal{T}}$. \square

For an associative algebra A and any $x \in A$, the map $\text{ad } x : A \rightarrow A$ defined by $\text{ad } x(y) = [x, y]$ for all $y \in A$ is a derivation. For a Jordan algebra J and any $x \in A$, let $L_x : A \rightarrow A$ be the left multiplication map, i.e., $L_x(y) = xy$ for all $y \in A$. Then one can check that for any $u, v \in A$, $[L_u, L_v]$ is a derivation (see [13, page 35]). We call $\text{ad } x$ in the associative case and $\sum_i [L_{u_i}, L_{v_i}]$ in the Jordan case an *inner derivation*, and denote the set of all inner derivations by $\text{IDer } A$. There is also the concept of inner derivations for alternative algebras (see e.g. [13, page 300]). We note that $\text{IDer } A$ is an ideal of the Lie algebra $\text{Der } A$. Also, any inner derivation is Z -linear, i.e.,

$$(5.2.5) \quad \text{IDer } A \subset \text{Der}_Z A.$$

When A is an associative algebra, a derivation is clearly a derivation of A^+ . So we have $\text{Der}A \subset \text{Der}A^+$. Also, note that

$$(5.2.6) \quad [L_u, L_v] = 4 \text{ad}[u, v] \quad \text{so} \quad \text{IDer}A^+ \subset \text{IDer}A.$$

Lemma 5.2.7. *Let A be an associative algebra with centre $Z(A)$ and let A^+ be the corresponding Jordan algebra with centre $Z(A^+)$.*

(a) *If A is prime then $\text{Der}A = \text{Der}A^+$ and $\text{Der}_{Z(A)}A = \text{Der}_{Z(A^+)}A^+$.*

(b) *$A = Z(A) + [A, A] \iff \text{IDer}A = \text{IDer}A^+$.*

In particular, for a quantum torus F_q we have

$$\text{Der}F_q = \text{Der}F_q^+ \quad \text{and} \quad \text{IDer}F_q = \text{IDer}F_q^+.$$

Proof. (a): The first statement follows from [10, page 55]. The second statement follows from $Z(A^+) = Z(A)$ (see 1.1.10).

(b): Since $\text{IDer}A = \text{IDer}A^+ \iff \text{ad}A = \text{ad}[A, A]$, it is enough to show that $\text{ad}A = \text{ad}[A, A] \iff A = Z(A) + [A, A]$. The implication \Leftarrow is clear. For the other implication \Rightarrow , let $a \in A$. Then $\text{ad}a = \text{ad}x$ for some $x \in [A, A]$, and hence $0 = \text{ad}(a - x)$, and so $a - x \in Z(A)$. Therefore, $a = z + x$ for some $z \in Z(A)$.

The last statement follows from 2.2.5 and 5.1.3. \square

Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra for an abelian group G . Define for each $g \in G$,

$$(\text{Der}A)_g := \{d \in \text{Der}_F A \mid d(A_h) \subset A_{g+h} \text{ for all } h \in G\},$$

and so we have

$$\text{Der}A \supset \bigoplus_{g \in G} (\text{Der}A)_g.$$

It is known that

(5.2.8) the inclusion is an equality if A is finitely generated

(see [9, Proposition 1]). Also, if A is alternative or Jordan, then by the definition of inner derivations, we have

$$(5.2.9) \quad \text{IDer}A = \bigoplus_{g \in G} (\text{IDer}A)_g$$

where $(\text{IDer}A)_g = (\text{Der}A)_g \cap \text{IDer}A$.

Example 5.2.10. A quantum torus $F_q = \bigoplus_{\alpha \in \mathbb{Z}^n} (F_q)_\alpha$ is finitely generated over F . Hence by 5.2.8, we have

$$\text{Der}F_q = \bigoplus_{\alpha \in \mathbb{Z}^n} (\text{Der}F_q)_\alpha.$$

Since $(\text{Der}F_q)_\alpha \subset (\text{Der}F_q^+)_\alpha$ for all $\alpha \in \mathbb{Z}^n$,

$$\text{Der}F_q = \bigoplus (\text{Der}F_q)_\alpha \subset \bigoplus (\text{Der}F_q^+)_\alpha \subset \text{Der}F_q^+.$$

Hence by 5.2.7, we get

$$(5.2.11) \quad (\text{Der}F_q)_\alpha = (\text{Der}F_q^+)_\alpha$$

and $\text{Der}_F F_q^+ = \bigoplus_{\alpha \in \mathbb{Z}^n} (\text{Der}F_q^+)_\alpha$. More generally,

Lemma 5.2.12. *For an alternative or a Jordan torus T , $\text{Der}T = \bigoplus_{\alpha \in \mathbb{Z}^n} (\text{Der}T)_\alpha$.*

Proof. All tori except F_q^+ and $H(E_\xi, \sigma)$ are finitely generated: Tori of strong type, i.e., alternative tori and the Albert torus, are clearly finitely generated. See 3.1.2 for $H(F_\epsilon, *)$ and 3.3.8 for Clifford tori. For $H(E_\xi, \sigma)$, we prove the following general lemma:

Lemma 5.2.13. *Let A be an associative algebra over a field E of $\text{ch.}E \neq 2$ and let E/F be a quadratic extension with the nontrivial Galois automorphism σ_E . Assume that σ is a σ_E -involution of A , i.e., an F -involution of A such that $\sigma(xa) = \sigma_E(x)\sigma(a)$ for all $x \in E$ and $a \in A$. Let $H := \{a \in A \mid \sigma(a) = a\}$ and $K := \{a \in A \mid \sigma(a) = -a\}$. Then:*

(i) $A = H \oplus \sqrt{x}H = EH$ for a suitable $x \in F$, and $K = \sqrt{x}H$ and $\dim_F H = \dim_F K = \dim_E A$.

(ii) If $d \in \text{Der}_E A^+$, then $0 \neq d \iff d|_H \neq 0$.

(iii) The map $\varphi: \text{Der}_E A^+ \longrightarrow \text{Der}_E A^+$ defined by $\varphi(d) = \sigma d \sigma$ is an automorphism of the Lie algebra of order 2. Then $H_\varphi := \{d \in \text{Der}_E A^+ \mid \varphi(d) = d\}$ can be identified with $\text{Der}_F H$ by the restriction map, and

$$\text{Der}_E A^+ = \text{Der}_F H \oplus \sqrt{x} \text{Der}_F H = E \text{Der}_F H.$$

(iv) $(\text{IDer}A^+ \cap \text{Der}_F H) \subset \text{ad } L$ where $L = \{a \in A \mid a + \sigma(a) \in Z(A)\} = Z(A) \cap H$.

Suppose that $A = \bigoplus_{g \in G} A_g$ is G -graded for an abelian group G and assume that σ is graded. Let $H_g = A_g \cap H$ and $K_g = A_g \cap K$. Then:

(a) $H = \bigoplus_{g \in G} H_g$, $K = \bigoplus_{g \in G} K_g$, $A_g = H_g \oplus \sqrt{x}H_g = EH_g$, $K_g = \sqrt{x}H_g$ and $\dim_F H_g = \dim_F K_g = \dim_E A_g$.

(b) For all $g \in G$, $(\text{Der}_E A^+)_g = (\text{Der}_F H)_g \oplus \sqrt{x}(\text{Der}_F H)_g = E(\text{Der}_F H)_g$.

(c) If $\text{Der}_E A^+ = \bigoplus_{g \in G} (\text{Der}_E A^+)_g$, then $\text{Der}_F H = \bigoplus_{g \in G} (\text{Der}_F H)_g$.

(d) $((\text{IDer} A^+)_g \cap \text{Der}_F H) \subset \text{ad } L_g$ for all $g \in G$, where $L_g = A_g \cap L$.

(e) If $\dim_E A_g = 1$ for some $g \in G$, then $\dim_F (\text{IDer} H)_g \leq 1$.

Moreover, if $(\text{IDer} A^+)_g \neq (0)$, then $\dim_F (\text{IDer} H)_g = 1$ and

$$((\text{IDer} A^+)_g \cap \text{Der}_F H) = (\text{IDer} H)_g = F.[L_h, L_k]$$

for some homogeneous elements $h, k \in H$.

Proof. (i) (see [15, page 190]): Since $\text{ch.} E \neq 2$, $H \cap K = (0)$, and for any $a \in A$ we have

$$a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a))$$

and $a + \sigma(a) \in H$, $a - \sigma(a) \in K$. Thus $A = H \oplus K$. Since E is a quadratic extension, there exists $x \in F$ such that $E = F(\sqrt{x})$, and $\sigma(\sqrt{x}) = -\sqrt{x}$. Hence $K = \sqrt{x}H$ and the rest is clear. (ii) follows from (i).

(iii): Clearly, φ is an automorphism of the Lie algebra $\text{Der}_E A^+$ of order 2, and so by a similar argument as in (i), we have $\text{Der}_E A^+ = H_\varphi \oplus \sqrt{x}H_\varphi = EH_\varphi$. Thus we need to show that the restriction map $\Psi : H_\varphi \rightarrow \text{Der}_F H$ defined by $\Psi(d) = d|_H$ is an F -isomorphism. First of all, by the definition of φ , Ψ is a well-defined F -linear map. Since $A = H \oplus \sqrt{x}H$, Ψ is injective. For $d \in \text{Der}_F H$, define $\bar{d} : A^+ \rightarrow A^+$ by $\bar{d}(h + \sqrt{x}h') = d(h) + \sqrt{x}d(h')$ for $h, h' \in H$. Then one can check that $\bar{d} \in H_\varphi$, and so $\Psi(\bar{d}) = d$, whence Ψ is surjective.

(iv): For $d \in (\text{IDer} A^+ \cap \text{Der}_F H)$, there exists $a \in A$ such that $d = \text{ad } a$ since $\text{IDer} A^+ \subset \text{IDer} A$. Also, since $d \in \text{Der}_F H$, $d(h) = \text{ad } a(h) = [a, h] \in H$ for all $h \in H$, and so $[a, h] = \sigma([a, h]) = -[\sigma(a), h]$. Hence $\text{ad } (a + \sigma(a))|_H = 0$ and by (ii), $\text{ad } (a + \sigma(a)) = 0$, and so $a + \sigma(a) \in Z(A)$, i.e., $a \in L$, which proves (iv).

(a): Since σ is graded, all statements can be shown in the same way as (i).

(b),(c),(d): Since σ is graded, and so is φ . Hence (c) follows, and (b) and (d) can be shown in the same way as (i) and (iv).

(e): We have $(\text{IDer}H)_g \subset ((\text{IDer}A^+)_g \cap \text{Der}_F H) \subset \text{ad } L_g$. So for the first statement, it is enough to show that $\dim_F(\text{ad } L_g) \leq 1$. Since $\dim_E A_g = 1$, $A_g \subset Z(A)$ or $A_g \cap Z(A) = (0)$. Hence $\text{ad } L_g = (0)$ or $L_g = \{a \in A_g \mid a + \sigma(a) = 0\} = K_g$, and so $\dim_F(\text{ad } L_g) = 0$ or $\dim_F(\text{ad } L_g) \leq \dim_F K_g = 1$ (see (a)).

If $(\text{IDer}A^+)_g \neq (0)$, there exist some $u \in A_p$ and $v \in A_q$ such that $p + q = g$ and $[L_u, L_v] \neq 0$. By (a) we have $A_p = EH_p$ and $A_q = EH_q$, and so $[L_u, L_v] = \sum_{i,j} y_i z_j [L_{h_i}, L_{k_j}] \neq 0$ for some nonzero $y_i, z_j \in E$, $h_i \in H_p$ and $k_j \in H_q$. Thus there exist some i_0, j_0 such that, letting $h := h_{i_0}$ and $k := k_{j_0}$, $[L_h, L_k] \neq 0$. Then, by (ii), $0 \neq [L_h, L_k] |_{H} = [L_h, L_k] \in (\text{IDer}H)_g$, and by the above, the rest follows. \square

The Jordan torus $H(E_\xi, \sigma)$ is a special case of $H = H(A, \sigma)$ in this lemma, and so by (c), we have completed the proof of 5.2.12.

Lemma 5.2.14. *Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra over F for an abelian group G and let $Z = \bigoplus_{h \in H} Z_h$ be the centre of A , where $H = \{h \in G \mid A_h \cap Z \neq (0)\}$ is a subset of G and $Z_h = A_h \cap Z$. Assume that $\text{Der}A = \bigoplus_{g \in G} (\text{Der}A)_g$. Then $\text{Der}_Z A$ is G -graded, i.e.,*

$$\text{Der}_Z A = \bigoplus_{g \in G} (\text{Der}_Z A)_g$$

where $(\text{Der}_Z A)_g = (\text{Der}A)_g \cap \text{Der}_Z A$.

Proof. Let $d \in \text{Der}_Z A$ and let $d = \sum_{g \in G} d_g$ be the homogeneous decomposition. For any $z \in Z_h$, we have $0 = d(z) = \sum_{g \in G} d_g(z)$, which is a homogeneous decomposition, since $d_g(z) \in A_{g+h}$ and $g + h = g' + h \implies g = g'$. Hence $d_g(z) = 0$ and we get $d_g |_{Z_h} = 0$ for all $h \in H$, whence $d_g |_Z = 0$, and so $d_g \in \text{Der}_Z A$. \square

The following lemma is a corollary to 2.3.4a:

Lemma 5.2.15. *Let $T = \bigoplus_{g \in G} T_g$ be a division G -graded algebra over F for an abelian group G , and $Z = \bigoplus_{h \in H} Z_h$ its centre where H is the central grading group. Suppose that $\text{Der}T = \bigoplus_{g \in G} (\text{Der}T)_g$. Then:*

(i) *for $\bar{g} \in G/H$, we have $(\text{Der}T)_{\bar{g}} := Z(\text{Der}T)_g = \bigoplus_{g' \in \bar{g}} (\text{Der}T)_{g'}$, and in particular,*

$$(\text{Der}T)_{\bar{0}} = Z(\text{Der}T)_0 = \bigoplus_{h \in H} (\text{Der}T)_h,$$

(ii) for $\bar{g} \in G/H$, $(\text{Der}T)_{\bar{g}}$ is a free Z -module and for $\overline{(\text{Der}T)}_{\bar{g}} := \overline{Z} \otimes_Z (\text{Der}T)_{\bar{g}}$, we have

$$\dim_{\overline{Z}} \overline{(\text{Der}T)}_{\bar{g}} = \text{rank}_Z (\text{Der}T)_{\bar{g}} = \dim_{Z_0} (\text{Der}T)_{g'} \quad \text{for all } g' \in \bar{g}.$$

Proof. Like T in 2.3.4, $\text{Der}T = \bigoplus_{g \in G} (\text{Der}T)_g$ is a graded module of type G over the division H -graded commutative associative algebra $Z = \bigoplus_{h \in H} Z_h$. Thus all statements follow from 2.3.4a. \square

For a graded algebra, there are special type of derivations:

Lemma 5.2.16. *Let $A = \bigoplus_{g \in G} A_g$ be a G -graded algebra over F for an abelian group G . Let φ be any group homomorphism from G into the additive group of F . Then a linear transformation d on A defined by*

$$d|_{A_g} = \varphi(g) \text{id}|_{A_g} \quad \text{for all } g \in G$$

is a derivation. Such a $d = d_\varphi$ will be called the degree derivation determined by φ .

Proof. For $x_g \in A_g$ and $x_h \in A_h$, $g, h \in G$, with $x_{g+h} := x_g x_h \in A_{g+h}$, we have $d(x_g x_h) = d(x_{g+h}) = \varphi(g+h)x_{g+h} = \varphi(g)x_{g+h} + \varphi(h)x_{g+h} = \varphi(g)x_g x_h + \varphi(h)x_g x_h = d(x_g)x_h + x_g d(x_h)$. Hence d is a derivation. \square

Remark 5.2.17. If G is a torsion group, A is predivision G -graded and $\text{ch}.F = 0$, then there exists only the trivial degree derivation. In fact, let $d = d_\varphi$ be a degree derivation. For any $g \in \text{supp}(A)$, let $x_g \in A_g$ be an invertible element. Since there exists some $r \geq 1$ such that $rg = 0$ and $0 \neq x_0 := x_g^r \in A_0$, we have $0 = \varphi(0)x_0 = d(x_0) = d(x_g^r) = r\varphi(g)x_g^r = r\varphi(g)x_0$. Hence, $\varphi(g) = 0$ since $\text{ch}.F = 0$. Thus we get $d = 0$.

Proposition 5.2.18. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be an alternative or a Jordan torus over F . Then $(\text{Der}T)_0$ is equal to the set of all degree derivations. Moreover, $(\text{Der}T)_0$ is n -dimensional over F .*

Proof. Let $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ be a basis of \mathbb{Z}^n such that each $\varepsilon_i \in \text{supp}(T)$ (this is always the case if T is alternative, and follows from 2.4.1(iii) if T is Jordan). Also, let $\{t_\alpha \mid \alpha \in \mathbb{Z}^n\}$ be a set of homogeneous elements of T so that $T = \bigoplus_{\alpha \in \mathbb{Z}^n} Ft_\alpha$. Hence $t_{\varepsilon_i} \neq 0$ in particular, but t_α could be 0 in general.

Let

$$V := \{\varphi \mid \varphi : \mathbb{Z}^n \longrightarrow F \text{ is a group homomorphism}\},$$

which is a vector space over F . Let $\langle \varphi_1, \dots, \varphi_n \rangle$ be the basis of V dual to $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$, i.e., $\varphi_i(\alpha) = \alpha_i$ for $\alpha = \alpha_1 \varepsilon_1 + \dots + \alpha_n \varepsilon_n \in \mathbb{Z}^n$. Since any degree derivation is contained in $(\text{Der}T)_0$, we have a map

$$d : V \longrightarrow (\text{Der}T)_0 : \varphi \mapsto d_\varphi,$$

which is clearly linear. The image of d is the subspace of all degree derivations. We show that d is injective: Suppose that $d(\sum_{i=1}^n a_i \varphi_i) = 0$ for $a_i \in F$. Let $d_i := d_{\varphi_i}$ for all i . Then $\sum_{i=1}^n a_i d_i = 0$ and $0 = \sum_{i=1}^n a_i d_i(t_{\varepsilon_j}) = \sum_{i=1}^n a_i \varphi_i(\varepsilon_j)(t_{\varepsilon_j}) = a_j t_{\varepsilon_j}$. Hence we get $a_j = 0$ since $t_{\varepsilon_j} \neq 0$ for all j , and so $\sum_{i=1}^n a_i \varphi_i = 0$, whence d is injective.

Next we show that d is surjective: Let $0 \neq D \in (\text{Der}T)_0$. Since $D(t_\alpha) \in T_\alpha = Ft_\alpha$, we can define a map $\varphi : \text{supp}(T) \longrightarrow F$ by $D(t_\alpha) = \varphi(\alpha)t_\alpha$. We will show that φ extends to a homomorphism $\mathbb{Z}^n \longrightarrow F$. Then $D = d_\varphi$ is the degree derivation determined by φ and we are done. For $\alpha, \beta \in \text{supp}(T)$, whenever

$$(*) \quad t_\alpha t_\beta = c_{\alpha, \beta} t_{\alpha + \beta} \quad \text{for some } 0 \neq c_{\alpha, \beta} \in F,$$

we have

$$\begin{aligned} (\varphi(\alpha) + \varphi(\beta))c_{\alpha, \beta} t_{\alpha + \beta} &= \varphi(\alpha)t_\alpha t_\beta + \varphi(\beta)t_\alpha t_\beta = d(t_\alpha)t_\beta + t_\alpha d(t_\beta) \\ &= d(t_\alpha t_\beta) = d(c_{\alpha, \beta} t_{\alpha + \beta}) = \varphi(\alpha + \beta)c_{\alpha, \beta} t_{\alpha + \beta}. \end{aligned}$$

Hence we get $\varphi(\alpha) + \varphi(\beta) = \varphi(\alpha + \beta)$ for all $\alpha, \beta \in \text{supp}(T)$ with $t_\alpha t_\beta \neq 0$.

Now, for the case where T is of strong type, i.e., an alternative torus or the Albert torus, we have $\text{supp}(T) = \mathbb{Z}^n$ and therefore $(*)$ holds for all $\alpha, \beta \in \mathbb{Z}^n$. Hence we obtain that φ is a homomorphism.

For the case $J = H(F_\varepsilon, *)$ or a Clifford torus, the central grading group is

$$\Gamma = 2\mathbb{Z}\sigma_1 + \dots + 2\mathbb{Z}\sigma_m + \mathbb{Z}\sigma_{m+1} + \dots + \mathbb{Z}\sigma_n$$

for some basis $\{\sigma_1, \dots, \sigma_n\}$ of \mathbb{Z}^n and some $1 < m \leq n$ (see 3.1.5 and 3.3.8). Let $\alpha \in \mathbb{Z}^n \setminus \text{supp}(T)$. Since $2\alpha \in \Gamma \subset \text{supp}(T)$ and $\text{ch}.F \neq 2$, we can define

$$\varphi(\alpha) := \frac{1}{2}\varphi(2\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus \text{supp}(T).$$

Then we get a map $\varphi : \mathbb{Z}^n \longrightarrow F$. For $\alpha \in \text{supp}(T)$, α and $\beta = \alpha$ satisfy (*) since $t_\alpha^2 = ct_{2\alpha}$ for some $0 \neq c \in F$. Hence we obtain

$$(1) \quad \varphi(2\alpha) = 2\varphi(\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^n.$$

Since the centre $Z = \bigoplus_{\gamma \in \Gamma} T_\gamma$ is a torus of strong type, $2\alpha, 2\beta \in 2\mathbb{Z}^n \subset \Gamma$ satisfy (*). Hence

$$(2) \quad \varphi(2\alpha + 2\beta) = \varphi(2\alpha) + \varphi(2\beta) \quad \text{for all } \alpha, \beta \in \mathbb{Z}^n.$$

Then, by (1) and (2), we get

$$2\varphi(\alpha + \beta) = \varphi(2\alpha + 2\beta) = \varphi(2\alpha) + \varphi(2\beta) = 2\varphi(\alpha) + 2\varphi(\beta) = 2(\varphi(\alpha) + \varphi(\beta)).$$

Hence $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$ for all $\alpha \in \mathbb{Z}^n$ since $\text{ch.}F \neq 2$.

For the case $T = F_q^+$, we have by 5.2.11, $D \in (\text{Der}F_q^+)_0 = (\text{Der}F_q)_0$, and so by the case when $T = F_q$, φ is a homomorphism.

For the case $T = H(E_\xi, \sigma)$, we have by 5.2.13(b), $D \in (\text{Der}_F H(E_\xi, \sigma))_0 \subset (\text{Der}E_\xi^+)_0$, and so φ is a homomorphism as in the case $T = F_q^+$. \square

We call $\langle d_1, \dots, d_n \rangle$ defined in the proof the *standard basis* of $(\text{Der}T)_0$ relative to $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$.

Corollary 5.2.19. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be a torus over F and $Z = \bigoplus_{\gamma \in \Gamma} Fz_\gamma$ the centre. Then for all $\gamma \in \Gamma$, we have $(\text{Der}T)_\gamma = z_\gamma(\text{Der}T)_0 = z_\gamma \text{span}\langle d_1, \dots, d_n \rangle$, and in particular $\dim_F(\text{Der}T)_\gamma = n$. \square*

Corollary 5.2.20. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be a torus over F and $Z = \bigoplus_{\gamma \in \Gamma} Fz_\gamma$ the centre. Then for any $d \in \text{Der}Z$, there exists $\bar{d} \in Z(\text{Der}T)_0$ with $\bar{d}|_Z = d$.*

Proof. Let $d = \sum_{\gamma \in \Gamma} d_\gamma$ be the homogeneous decomposition. By 5.2.18, there exists some homomorphism $\varphi_\gamma : \Gamma \longrightarrow F$ such that $d_\gamma = z_\gamma d_{\varphi_\gamma}$ where $d_{\varphi_\gamma} \in (\text{Der}Z)_0$ is the degree derivation determined by φ_γ . If φ_γ can be extended to a homomorphism $\bar{\varphi}_\gamma : \mathbb{Z}^n \longrightarrow F$, then the degree derivation $d_{\bar{\varphi}_\gamma} \in (\text{Der}T)_0$ satisfies $d_{\bar{\varphi}_\gamma}|_Z = d_{\varphi_\gamma}$ and we get $\bar{d} := \sum_{\gamma \in \Gamma} z_\gamma d_{\bar{\varphi}_\gamma} \in Z(\text{Der}T)_0$ such that $\bar{d}|_Z = d$. Thus we are done if we show the following lemma:

Lemma 5.2.21. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be a torus over F and $\Gamma = \Gamma(T)$ the central grading group. Then any homomorphism $\varphi : \Gamma \rightarrow F$ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{Z}^n \rightarrow F$.*

Proof. By the Fundamental Theorem of Finitely Generated Abelian Groups, there exists a basis $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ of \mathbb{Z}^n such that

$$\Gamma = m_1 \mathbb{Z} \varepsilon_1 + \dots + m_n \mathbb{Z} \varepsilon_n$$

for some $m_1, \dots, m_n \geq 0$. If,

(#) for any nonzero m_i , $\text{ch}.F$ does not divide m_i ,

then we can define a homomorphism $\tilde{\varphi} : \mathbb{Z}^n \rightarrow F$ by

$$\tilde{\varphi}(\varepsilon_i) = \begin{cases} \frac{1}{m_i} \varphi(m_i \varepsilon_i) & \text{if } m_i \neq 0 \\ \text{any element in } F & \text{if } m_i = 0. \end{cases}$$

Then clearly $\tilde{\varphi}|_\Gamma = \varphi$. Thus our work is to show that each class of tori satisfies (#). For the Cayley torus, $H(F_\varepsilon, *)$ or a Clifford tori, we have all $m_i = 1$ or 2 , and $\text{ch}.F \neq 2$ in these cases (see 4.3.11, 3.1.2, 3.3.8). For the Albert torus, all $m_i = 1$ or 3 , and $\text{ch}.F \neq 3$ in this case (see 3.4.12). Thus we need to consider F_q, F_q^+ and $H(E_\xi, \sigma)$. Since $\Gamma(F_q^+) = \Gamma(F_q)$ and $\Gamma(H(E_\xi, \sigma)) = \Gamma(E_\xi)$ (see 3.1.2), we only need to show that F_q satisfies (#):

Suppose that $p := \text{ch}.F$ divides $m_k \neq 0$ for some k . Let $F_q = F_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with the $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ -grading and $q = (q_{ij})$. Then $t_k^{m_k} \in Z$, and so $t_k^{m_k} t_j = t_j t_k^{m_k} = q_{kj}^{m_k} t_k^{m_k} t_j$, whence $q_{kj}^{m_k} = 1$ for all j . Since $m_k = pl_k$ for some $l_k \geq 1$ and $\text{ch}.F = p$, we have $q_{kj}^{l_k} = 1$ for all j . Hence $t_k^{l_k}$ commutes with all t_j , and we get $t_k^{l_k} \in Z$. Thus

$$\Gamma \subseteq m_1 \mathbb{Z} \varepsilon_1 + \dots + l_k \mathbb{Z} \varepsilon_k + \dots + m_n \mathbb{Z} \varepsilon_n \subset \Gamma,$$

which is absurd. \square

Corollary 5.2.22. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be a torus over F with the notations as above. Then*

$$\text{Der}T = \text{Der}_Z T \oplus Z(\text{Der}T)_0.$$

In particular,

$$(\text{Der}_Z T)_\alpha = (\text{Der} T)_\alpha \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus \Gamma \quad \text{and} \quad \text{Der}_Z T = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} (\text{Der} T)_\alpha.$$

Proof. Suppose that $\text{Der}_Z T \cap Z(\text{Der} T)_0 \neq (0)$. Since both $\text{Der}_Z T$ and $Z(\text{Der} T)_0$ are homogeneous, there exist some $z_\gamma \in Z_\gamma$ and $d \in (\text{Der} T)_0$ such that $0 \neq z_\gamma d \in \text{Der}_Z T$. Then z_γ is invertible, and so $0 \neq d \in \text{Der}_Z T$. By 5.2.19, $d = a_1 d_1 + \cdots + a_n d_n$ for $a_1, \dots, a_n \in F$ and some $a_t \neq 0$, where $\{d_1, \dots, d_n\}$ is the standard basis of $(\text{Der} T)_0$ relative to $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$. Then, in the notation of the proof of 5.2.21, $m_t \varepsilon_t \in \Gamma$, and so for $0 \neq z \in Z_{m_t \varepsilon_t}$, we have $d(z) = a_t m_t z \neq 0$ since $\text{ch.} F \nmid m_t$ (see \sharp) in the proof of 5.2.21). This contradicts $d \in \text{Der}_Z T$. Hence $\text{Der}_Z T \cap Z(\text{Der} T)_0 = (0)$.

For any $d \in \text{Der} T$, we have $d' := d|_Z \in \text{Der} Z$ (see 5.2.2). So by 5.2.20, there exists $\tilde{d}' \in Z(\text{Der} T)_0$ such that $\tilde{d}'|_Z = d'$. Then $(d - \tilde{d}')|_Z = 0$, and hence $d - \tilde{d}' \in \text{Der}_Z T$. Therefore, $d = (d - \tilde{d}') + \tilde{d}' \in \text{Der}_Z T + Z(\text{Der} T)_0$. Thus we finish the proof. \square

We want to show that

$$\text{IDer} T = \text{Der}_Z T.$$

We first consider the case where the central closure \bar{T} is finite dimensional over \bar{Z} , namely, T is either $H(F_\varepsilon, *)$, a Clifford torus, the Albert torus or the Cayley torus. For the Cayley torus, this is shown for fields F of characteristic $\neq 2, 3$ in [7, Lemma 1.39(b), page 4327 and 4328]. (Note that if $\text{ch.} F = 2$, then the Cayley torus does not exist (see 4.3.11)). So we only consider the three Jordan tori above. The method we will use is the same as in the case of the Cayley torus in [7]. Note that we are assuming that $\text{ch.} F \neq 2$ for all Jordan algebras and that $\text{ch.} F \neq 3$ for the Albert torus (otherwise the Albert torus does not exist).

Lemma 5.2.23. *Let $T = \bigoplus_{\alpha \in \mathbb{Z}^n} T_\alpha$ be one of three Jordan tori $H(F_\varepsilon, *)$, a Clifford torus or the Albert torus. Then $\text{IDer} T = \text{Der}_Z T$.*

Proof. Since the central closures of the three Jordan tori are central division algebras (see 3.4.7) and their degrees are not divided by $\text{ch.} F$ (the central degree of $H(F_\varepsilon, *)$, a Clifford torus or the Albert torus is, respectively, 2^m , 2(or 1) or 3 (see §3.2 and §3.0), all derivations of the central closure are inner (see [13, Exercise 1, page 258] for degree 2, [13, Theorem 9, page 254] for special Jordan algebras of degree ≥ 3 and [13, Theorem 17, page 408] for exceptional Jordan algebras). So we have $\text{Der}_{\bar{Z}} \bar{T} = \text{IDer} \bar{T}$.

Since $\text{IDer}T \subset \text{Der}_Z T$ (see 5.2.5), we need to show that $\text{Der}_Z T \subset \text{IDer}T$. By 5.2.14, it is enough to show that $(\text{Der}_Z T)_\alpha \subset \text{IDer}T$ for all $\alpha \in \mathbb{Z}^n$. Let $d \in (\text{Der}_Z T)_\alpha$. Since $\text{Der}_Z T$ embeds into $\overline{\text{Der}_Z T} = \text{Der}_{\overline{Z}} \overline{T} = \text{IDer} \overline{T}$ (see 5.2.4), we have $d = \sum_i [L_{z_i \otimes u_i}, L_{z'_i \otimes u'_i}]$ for some $z_i \otimes u_i, z'_i \otimes u'_i \in \overline{T}$. Then, clearly there exists $0 \neq z \in Z$ such that $zd = \sum_i [L_{1 \otimes v_i}, L_{1 \otimes v'_i}] = \sum_i [L_{v_i}, L_{v'_i}]$ for some $v_i, v'_i \in T$. Hence, $zd \in \text{IDer}T$. Let $z = \sum_{\gamma \in \Gamma} z_\gamma$ be the homogeneous decomposition. Then we have $zd = \sum_\gamma z_\gamma d \in \text{IDer}T$. Since $\text{IDer}T$ is homogeneous, we get $z_\gamma d \in \text{IDer}T$ for all γ . Since $z \neq 0$, $z_{\gamma_0} \neq 0$ for some γ_0 . Since $z_{\gamma_0}^{-1} \in Z$, we obtain $d \in \text{IDer}T$. \square

We consider the remaining tori. i.e., $T = F_q, F_q^+$ or $H(E_\xi, \sigma)$. For the case $T = F_q = \bigoplus_{\alpha \in \mathbb{Z}^n} (F_q)_\alpha$, we have the following result from [6, Lemma 2.48, page 364]. (It is shown there over the field \mathbb{C} of complex numbers, but the proof works over any field.): for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$ and any $0 \neq x_\alpha \in (F_q)_\alpha$,

$$(5.2.24) \quad (\text{Der}F_q)_\alpha = F \cdot \text{ad } x_\alpha \neq (0).$$

In particular, $(\text{Der}_Z F_q)_\alpha = (\text{Der}F_q)_\alpha = (\text{IDer}F_q)_\alpha$ for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$ (see 5.2.22), and so $\text{IDer}F_q = \text{Der}_Z F_q$. Then, by 5.2.7, we have

$$\text{IDer}F_q^+ = \text{IDer}F_q = \text{Der}_Z F_q = \text{Der}_Z F_q^+.$$

Moreover, 5.2.24 and 5.1.3(a) imply that for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$, there exist $\beta, \delta \in \mathbb{Z}^n \setminus \Gamma$ and $0 \neq u_\beta, v_\delta \in F_q^+$ of degree β and δ such that $\beta + \delta = \alpha$ and

$$(5.2.25) \quad (\text{Der}F_q^+)_\alpha = F \cdot [L_{u_\beta}, L_{v_\delta}] \neq (0).$$

Finally, we consider the case $T = H(E_\xi, \sigma)$. We show that for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$, $(\text{Der}T)_\alpha = (\text{IDer}T)_\alpha$. First we note that $(\text{Der}T)_\alpha \neq (0)$. Otherwise by 5.2.13(b), we have $(\text{Der}E_\xi^+)_\alpha = (0)$, which is a contradiction (apply 5.2.25 for E_ξ^+ instead of F_q^+). Let $0 \neq d \in (\text{Der}T)_\alpha \subset (\text{Der}E_\xi^+)_\alpha$ (see 5.2.13(b)). Then, by 5.2.25 for E_ξ^+ , $d \in (\text{IDer}E_\xi^+)_\alpha$. Thus, by 5.2.13(e), we obtain $(\text{Der}T)_\alpha = (\text{IDer}T)_\alpha$, and so

$$\text{IDer}H(E_\xi, \sigma) = \text{Der}_Z H(E_\xi, \sigma)$$

(see 5.2.22). Moreover, by 5.2.13(e), for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$, there exist $\beta, \delta \in \mathbb{Z}^n \setminus \Gamma$ and $0 \neq u_\beta, v_\delta \in H(E_\xi, \sigma)$ of degree β and δ such that

$$(5.2.26) \quad (\text{Der}H(E_\xi, \sigma))_\alpha = F \cdot [L_{u_\beta}, L_{v_\delta}] \neq (0).$$

We state the above results together with 5.2.19 and 5.2.22 as a theorem.

Theorem 5.2.27. *Let T be an alternative or a Jordan torus over F excluding the Cayley torus over F of characteristic 3, Γ the central grading group and Z its centre. Then*

$$\text{IDer}T = \bigoplus_{\alpha \in \mathbb{Z}^n \setminus \Gamma} (\text{Der}T)_{\alpha}, \quad \text{Der}T = \text{IDer}T \oplus Z(\text{Der}T)_0.$$

Moreover, $(\text{Der}T)_0$ is equal to the set of degree derivations of T , and for all $\gamma \in \Gamma$, $\dim_F(\text{Der}T)_{\gamma} = n$. \square

We do not yet know $\dim_F(\text{Der}T)_{\alpha}$ for $\alpha \in \mathbb{Z}^n \setminus \Gamma$. We only know this for $T = F_q$, F_q^+ or $H(E_{\xi}, \sigma)$. Namely, for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$,

$$\dim_F(\text{Der}T)_{\alpha} = 1$$

(see 5.2.24, 5.2.25, 5.2.26). Also, in the case of the Cayley torus $\mathbb{O} = \mathbb{O}_t$ over F of $\text{ch.}F \neq 3$, we know that for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$,

$$\dim_F(\text{Der}\mathbb{O})_{\alpha} = 2$$

(see by [7, Theorem 1.40, page 4328]). In the next section we will find the homogeneous dimensions of $\text{IDer}T$ for $T = H(F_{\epsilon}, \star)$, a Clifford torus or the Albert torus.

§ 5.3 INNER DERIVATIONS

Let $A := F_{\epsilon}$ and $J := H(A, \star)$. By Theorem 5.2.27, for $\alpha \in \mathbb{Z}^n \setminus \Gamma(J)$, we know that $(\text{Der}J)_{\alpha} = (\text{IDer}J)_{\alpha}$, which is spanned by $[L_u, L_v]$ for $u \in J_{\beta}$ and $v \in J_{\gamma}$ with $\beta + \gamma = \alpha$. By 5.2.6, we have $[L_u, L_v] = 4 \text{ad}([u, v])|_J$, and so

$$(\text{Der}J)_{\alpha} \subset B_{\alpha} := \{\text{ad } x|_J \mid x \in A_{\alpha}, [x, J] \subset J\}.$$

If $\alpha \in \Gamma(A)$, then $\text{ad } A_{\alpha} = (0)$, and so $(\text{Der}J)_{\alpha} = (0)$. Suppose that $\alpha \in \text{supp}(J)$. Then for any $x \in A_{\alpha}$ and any $y \in J$, we have $(\text{ad } x(y))^* = ([x, y])^* = [y, x] = -\text{ad } x(y)$ since $x^* = x$ and $y^* = y$. Hence $\text{ad } x(y) = 0$ or $\text{ad } x(y) \notin J$, and so $B_{\alpha} = (0)$, whence $(\text{Der}J)_{\alpha} = (0)$.

Suppose that $\alpha \notin (\text{supp}(J) \cup \Gamma(A))$. By the similar argument using $x^* = -x$ for $\alpha \notin \text{supp}(J)$, we get $[A_{\alpha}, J] \subset J$, and so $B_{\alpha} = (\text{Der}J)_{\alpha}$. If $B_{\alpha} = (0)$, then $[A_{\alpha}, J] = (0)$. Since J generates A as an associative algebra, we get $[A_{\alpha}, A] = (0)$. But this

contradicts our assumption $\alpha \notin \Gamma(A)$. Hence $B_\alpha \neq (0)$, and so $\dim_F(\text{Der}J)_\alpha = 1$ since $\dim_F A_\alpha = 1$. Consequently, for $\alpha \in \mathbb{Z}^n \setminus \Gamma(J)$, we have

$$(\text{Der}J)_\alpha = (\text{IDer}J)_\alpha = \begin{cases} (0) & \text{if } \alpha \in \text{supp}(J) \cup \Gamma(A) \\ B_\alpha \neq (0) & \text{otherwise.} \end{cases}$$

Thus:

Theorem 5.3.1. *Let $J = H(F_\epsilon, *)$. Then for $\alpha \in \mathbb{Z}^n \setminus \Gamma(J)$,*

$$\dim_F(\text{Der}J)_\alpha = \dim_F(\text{IDer}J)_\alpha = \begin{cases} 0 & \text{if } \alpha \in \text{supp}(J) \cup \Gamma(F_\epsilon) \\ 1 & \text{otherwise. } \square \end{cases}$$

Note that always $\Gamma(J) \subset \text{supp}(J)$ but $\Gamma(F_\epsilon) \not\subset \text{supp}(J)$ happens when $*$ is of second kind (see 3.2.18). Also, one can directly (without Theorem 5.2.27) show that $\text{ad} A_\alpha \subset (\text{IDer}J)_\alpha$ for all $\alpha \notin \text{supp}(J)$. In fact, let $0 \neq y \in A_\alpha$. Since $y \notin Z(A)$, there exists t_i such that $[y, t_i] \neq 0$ where t_i is one of the standard generators, i.e., $A = F_\epsilon[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. So $y = c[y_\alpha t_i^{-1}, t_i]$ for some $0 \neq c \in F$, whence $\text{ad} y = c[L_{y t_i^{-1}}, L_{t_i}]$. Since $y t_i = -t_i y$, $y^* = -y$ and $t_i^* = t_i$, we have $y t_i^{-1}, t_i \in J$, and hence $\text{ad} y$ is inner on J .

Clifford Tori

Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be a Clifford torus over F , and so $\Gamma = 2\mathbb{Z}^m \oplus \mathbb{Z}^{n-m}$ for some $1 < m \leq n$. Let $\{t_\alpha\}_{\alpha \in \mathbb{Z}^n}$ be the standard basis of J described in 3.3.8. By 5.2.27, for $\alpha \in \mathbb{Z}^n \setminus \Gamma$, we know that $(\text{Der}J)_\alpha = (\text{IDer}J)_\alpha$, which is spanned by

$$D_{\beta, \gamma} := [L_{t_\beta}, L_{t_\gamma}]$$

for $\beta, \gamma \in \mathbb{Z}^n$ with $\beta + \gamma = \alpha$. If $\beta \equiv \beta_1 \pmod{\Gamma}$ and $\gamma \equiv \gamma_1 \pmod{\Gamma}$, then $D_{\beta_1, \gamma_1} = z D_{\beta, \gamma}$ for some $z \in Z$. Hence $(\text{Der}J)_{\bar{\alpha}} = Z(\text{Der}J)_\alpha$ is generated by $D_{\bar{\beta}, \bar{\gamma}}$ for $\bar{\beta}, \bar{\gamma} \in \mathbb{Z}^n / \Gamma = \mathbb{Z}_2^m$ with $\bar{\beta} + \bar{\gamma} = \bar{\alpha}$ over Z where $D_{\bar{\beta}, \bar{\gamma}}$ is defined as $D_{\beta, \gamma}$ for any choices of $\beta \in \bar{\beta}$ and $\gamma \in \bar{\gamma}$. Let

$$M_{\bar{\alpha}} = \{ \{ \bar{\beta}, \bar{\gamma} \} \subset \mathbb{Z}_2^m \mid \bar{\alpha} = \bar{\beta} + \bar{\gamma}, \bar{\beta}, \bar{\gamma} \neq \bar{0} \}.$$

Then $\text{rank}_Z(\text{Der}J)_{\bar{\alpha}} \leq \sharp M_{\bar{\alpha}}$. Since $\sharp \mathbb{Z}_2^m = 2^m$, we have

$$(1) \quad \sharp M_{\bar{\alpha}} = (2^m - 2)/2 = 2^{m-1} - 1.$$

Let $S := \text{supp}(J)$ be the semilattice in \mathbb{Z}^n , \bar{S} the image of S in \mathbb{Z}_2^m and $L := \mathbb{Z}_2^m \setminus \bar{S}$. For $\bar{\alpha} \in \mathbb{Z}_2^m \setminus \{\bar{\mathbf{0}}\}$, we define

$$P_{\bar{\alpha}} := \{\{\bar{\beta}, \bar{\gamma}\} \in M_{\bar{\alpha}} \mid \bar{\beta} \in L \text{ or } \bar{\gamma} \in L\}.$$

Since $t_{\beta} = 0$ for $\beta \in L$, we have

$$(2) \quad \text{rank}_{\mathbb{Z}}(\text{Der}J)_{\bar{\alpha}} \leq \#M_{\bar{\alpha}} - \#P_{\bar{\alpha}}.$$

To find $\#P_{\bar{\alpha}}$, we define

$$Q_{\bar{\alpha}} := \{\{\bar{\beta}, \bar{\gamma}\} \in M_{\bar{\alpha}} \mid \bar{\beta} \in L \text{ and } \bar{\gamma} \in L\}.$$

Then, by a simple counting argument, we obtain

$$(3) \quad \#P_{\bar{\alpha}} = \begin{cases} \#L - \#Q_{\bar{\alpha}} & \text{if } \bar{\alpha} \notin L \\ \#L - 1 - \#Q_{\bar{\alpha}} & \text{if } \bar{\alpha} \in L. \end{cases}$$

Theorem 5.3.2. *Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_{\alpha}$ be a Clifford torus over F with the notations above. Then for $\alpha \in \mathbb{Z}^n \setminus \Gamma$, we have*

$$\dim_F(\text{Der}J)_{\alpha} = \begin{cases} 2^{m-1} - 1 - l + q(\bar{\alpha}) & \text{if } \bar{\alpha} \notin L \\ 2^{m-1} - l + q(\bar{\alpha}) & \text{if } \bar{\alpha} \in L \end{cases}$$

where $l := \#L$ and $q(\bar{\alpha}) := \#Q_{\bar{\alpha}}$. In particular, if $S = \mathbb{Z}^n$ is a lattice, then for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$, $(\text{Der}J)_{\alpha}$ has constant dimension, namely, $\dim_F(\text{Der}J)_{\alpha} = 2^{m-1} - 1$.

Proof. We know that $\bar{J} = \bar{Z} \oplus V$ is a finite dimensional simple Jordan algebra of the nondegenerate symmetric bilinear form $n(\cdot, \cdot)$ over \bar{Z} where V is a vector space over \bar{Z} with basis indexed by $\bar{S} \setminus \{\bar{\mathbf{0}}\}$ (see 3.3.8). Let $s := \#\bar{S} \setminus \{\bar{\mathbf{0}}\} = \dim_{\bar{Z}} V$. Since $\text{Der}\bar{J}$ is the set of all skew symmetric linear transformations of V relative to $n(\cdot, \cdot)$ (see [16, page 5]), we have

$$(4) \quad \dim_{\bar{Z}} \text{Der}\bar{J} = \frac{s(s-1)}{2}.$$

Now, by (1)-(3), we have for $\bar{\alpha} \in \mathbb{Z}_2^m \setminus \{\bar{\mathbf{0}}\}$,

$$(5) \quad \text{rank}_{\mathbb{Z}}(\text{Der}J)_{\bar{\alpha}} \leq \begin{cases} 2^{m-1} - 1 - l + q(\bar{\alpha}) & \text{if } \bar{\alpha} \notin L \\ 2^{m-1} - l + q(\bar{\alpha}) & \text{if } \bar{\alpha} \in L. \end{cases}$$

By 5.2.22 and 5.2.15, we have $\text{Der}_Z J = \bigoplus_{\bar{\alpha} \neq \mathbf{0}} (\text{Der} J)_{\bar{\alpha}}$, and so

$$\dim_{\bar{Z}} \text{Der} \bar{J} = \dim_{\bar{Z}} \overline{\text{Der}_Z J} = \sum_{\bar{\alpha} \neq \mathbf{0}} \dim_{\bar{Z}} \overline{(\text{Der} J)_{\bar{\alpha}}} = \sum_{\bar{\alpha} \neq \mathbf{0}} \text{rank}_Z (\text{Der} J)_{\bar{\alpha}}.$$

Thus, by (5),

$$(6) \quad \dim_{\bar{Z}} \text{Der} \bar{J} \leq s(2^{m-1} - 1 - l) + l(2^{m-1} - l) + \sum_{\bar{\alpha} \in \mathbb{Z}_2^m \setminus \{\bar{\mathbf{0}}\}} q(\bar{\alpha}).$$

Since $\bar{\alpha} \neq \mathbf{0} \iff \bar{\beta} \neq \bar{\gamma}$ in \mathbb{Z}_2^m , we have

$$\sum_{\bar{\alpha} \in \mathbb{Z}_2^m \setminus \{\bar{\mathbf{0}}\}} q(\bar{\alpha}) = \binom{l}{2} = \frac{l(l-1)}{2}.$$

Note that $\bar{S} \cup L \cup \{\bar{\mathbf{0}}\} = \mathbb{Z}_2^m$, and so $l = 2^m - s - 1$. Hence the right-hand side of (6) is

$$s(s - 2^{m-1}) + (2^m - s - 1)(s + 1 - 2^{m-1}) + \frac{(2^m - s - 1)(2^m - s - 2)}{2} = \frac{s(s-1)}{2}.$$

Therefore, by (4) the inequalities of (5) have to be equal. Since $\dim_F(\text{Der} J)_{\bar{\alpha}} = \text{rank}_Z(\text{Der} J)_{\bar{\alpha}}$ for $\bar{\alpha} \in \mathbb{Z}_2^m \setminus \{\bar{\mathbf{0}}\}$ (see 5.2.15(ii)), we have finished the proof. \square

Example 5.3.3. (1) Let $m = 2$ and $\bar{S} = \{\bar{\epsilon}_1, \bar{\epsilon}_2\}$. So $L = \{\bar{\epsilon}_1 + \bar{\epsilon}_2\}$. Hence for any $\gamma \in 2\mathbb{Z}^2 \oplus \mathbb{Z}^{n-2}$, we have $\dim(\text{Der} J)_{\epsilon_1 + \gamma} = \dim(\text{Der} J)_{\epsilon_2 + \gamma} = 2 - 1 - 1 = 0$ and $\dim(\text{Der} J)_{\epsilon_1 + \epsilon_2 + \gamma} = 2 - 1 = 1$. These dimensions are known (see [31, page 13]).

(2) Let $m = 3$ and $\bar{S} = \{\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3\}$. So $L = \{\bar{\epsilon}_1 + \bar{\epsilon}_2, \bar{\epsilon}_1 + \bar{\epsilon}_3, \bar{\epsilon}_2 + \bar{\epsilon}_3, \bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3\}$, and $q(\bar{\epsilon}_1) = q(\bar{\epsilon}_2) = q(\bar{\epsilon}_3) = q(\bar{\epsilon}_1 + \bar{\epsilon}_2) = q(\bar{\epsilon}_1 + \bar{\epsilon}_3) = q(\bar{\epsilon}_2 + \bar{\epsilon}_3) = 1$ and $q(\bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3) = 0$. Hence for any $\gamma \in 2\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$, we have $\dim(\text{Der} J)_{\epsilon_1 + \gamma} = \dim(\text{Der} J)_{\epsilon_2 + \gamma} = \dim(\text{Der} J)_{\epsilon_3 + \gamma} = \dim(\text{Der} J)_{\epsilon_1 + \epsilon_2 + \epsilon_3 + \gamma} = 2^2 - 4 - 1 + 1 = 0$, $\dim(\text{Der} J)_{\epsilon_1 + \epsilon_2 + \gamma} = \dim(\text{Der} J)_{\epsilon_1 + \epsilon_3 + \gamma} = \dim(\text{Der} J)_{\epsilon_2 + \epsilon_3 + \gamma} = 2^2 - 4 + 1 = 1$.

(3) Let $m = 3$ and $\bar{S} = \{\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3, \bar{\epsilon}_1 + \bar{\epsilon}_2\}$. So $L = \{\bar{\epsilon}_1 + \bar{\epsilon}_3, \bar{\epsilon}_2 + \bar{\epsilon}_3, \bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3\}$, and $q(\bar{\epsilon}_1) = q(\bar{\epsilon}_2) = q(\bar{\epsilon}_1 + \bar{\epsilon}_2) = 1$ and $q(\bar{\epsilon}_3) = q(\bar{\epsilon}_1 + \bar{\epsilon}_3) = q(\bar{\epsilon}_2 + \bar{\epsilon}_3) = q(\bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3) = 0$. Hence for any $\gamma \in 2\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$, we have $\dim(\text{Der} J)_{\epsilon_1 + \gamma} = \dim(\text{Der} J)_{\epsilon_2 + \gamma} = \dim(\text{Der} J)_{\epsilon_1 + \epsilon_2 + \gamma} = 2^2 - 3 - 1 + 1 = 1$, $\dim(\text{Der} J)_{\epsilon_3 + \gamma} = 2^2 - 3 - 1 = 0$, and $\dim(\text{Der} J)_{\epsilon_1 + \epsilon_3 + \gamma} = \dim(\text{Der} J)_{\epsilon_2 + \epsilon_3 + \gamma} = \dim(\text{Der} J)_{\epsilon_1 + \epsilon_2 + \epsilon_3 + \gamma} = 2^2 - 3 = 1$.

(4) Let $m = 3$ and let $S = \mathbb{Z}^n$. Then for any $\alpha \in \mathbb{Z}^n$ with $\bar{\alpha} \neq \bar{\mathbf{0}}$ in \mathbb{Z}_2^3 , we have $\dim(\text{Der} J)_{\alpha} = 2^2 - 1 = 3$. This is the case where J is the plus algebra of the octonion

torus $\mathbb{O} = \mathbb{O}_t$. As we already mentioned in §5.2, $\dim(\text{Der}\mathbb{O})_\alpha = 2$ for all $\alpha \in \mathbb{Z}^n$ with $\bar{\alpha} \neq \bar{0}$ in \mathbb{Z}_2^3 (see [7]).

In all examples above, we obtained $q(\bar{\alpha}) \leq 1$. However, $q(\bar{\alpha})$ can be bigger than 1 in general.

Albert Torus

For a unital Jordan algebra J with the centre Z , we define a map

$$\{\cdot, \cdot\} : J \times J \longrightarrow \text{IDer}J \quad \text{by} \quad \{x, y\} := [L_x, L_y] \quad \text{for } x, y \in J.$$

Then this is a skew symmetric Z -bilinear map satisfying

$$(0) \quad \{xy, z\} + \{yz, x\} + \{zx, y\} = 0 \quad \text{for all } x, y, z \in J$$

(see e.g. [13]), and it is clear from the definition that

$$(1) \quad \text{for all } x \in Z \text{ and } y \in J, \quad \{x, y\} = \{y, x\} = 0.$$

Also, by the skew symmetry of $\{\cdot, \cdot\}$ and commutativity of J , (0) can be written as

$$(2) \quad \{xy, z\} = \{x, yz\} + \{y, xz\} \quad \text{for all } x, y, z \in J.$$

For $x, y \in J$, we have by (2), $\{x^2, y\} = \{x, xy\} + \{x, xy\} = 2\{x, xy\}$. Thus we have a formula $\{x^i, y\} = i\{x, x^{i-1}y\}$ for $i = 1, 2$. Since $\{1, y\} = 0$ (see (1)), this formula holds for $i = 0$. Thus $\{x^i, y\} = i\{x, x^{i-1}y\}$ for $i = 0, 1, 2$. However, in general it does not hold for $i = 3$ anymore. Note that if $x^3 \in Z$, then $\{x^3, y\} = 0$ (see (1)). Let

$$[n] = [n]_3 = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3} \\ 1 & \text{if } n \equiv 1 \pmod{3} \\ 2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Then since $\{\cdot, \cdot\}$ is Z -bilinear and $x^{3m} \in Z$ for all $m \in \mathbb{Z}$, we get the following formula:

$$(3) \quad \{x^n, y\} = [n]\{x, x^{n-1}y\} \quad \text{for all } n \in \mathbb{Z},$$

provided that $x^3 \in Z$ and x^{-1} is defined.

For convenience, we denote for $x_1, \dots, x_m \in J$,

$$x_1(x_2(x_3 \cdots (x_{m-1}x_m)\cdots)) \text{ by } x_1x_2x_3 \cdots x_m.$$

Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_\alpha$ be the Albert torus over F (ch. $F \neq 2, 3$), and so $\Gamma = 3\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$. Let $\{t_\alpha\}_{\alpha \in \mathbb{Z}^n}$ be the standard basis of J relative to the basis $\langle \varepsilon_1, \dots, \varepsilon_n \rangle$ of \mathbb{Z}^n , described in 3.4.5. Recall that the structure constants for this basis are

$$\left\{ 1, \omega, \omega^2, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2} \right\}$$

where ω is a primitive 3rd root of unity in 3.4.5. Let let $t_i = t_{\varepsilon_i}$. Then for $\alpha = \alpha_1\varepsilon_1 + \cdots + \alpha_n\varepsilon_n \in \mathbb{Z}^n$ and any ordered multiplication using t_1, \dots, t_n of degree α , say $(t)_\alpha \in J_\alpha$, there exists some

$$\rho \in \Omega := \left\{ \left(-\frac{1}{2} \right)^i \omega^j \mid i, j \in \mathbb{Z} \right\} \text{ such that } (t)_\alpha = \rho t_1^{\alpha_1} \cdots t_n^{\alpha_n}.$$

Now, as in the case of Clifford tori, for $\alpha = \alpha_1\varepsilon_1 + \cdots + \alpha_n\varepsilon_n \in \mathbb{Z}^n \setminus \Gamma$, we have $(\text{Der}J)_\alpha = (\text{IDer}J)_\alpha$, which is spanned by $\{t_1^{\beta_1} \cdots t_n^{\beta_n}, t_1^{\gamma_1} \cdots t_n^{\gamma_n}\}$ for $\beta = \beta_1\varepsilon_1 + \cdots + \beta_n\varepsilon_n$, $\gamma = \gamma_1\varepsilon_1 + \cdots + \gamma_n\varepsilon_n \in \mathbb{Z}^n$ with $\beta + \gamma = \alpha$. Also, since $\Gamma = 3\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$, we have $z := t_1^{\beta_1} \cdots t_n^{\beta_n} t_1^{\gamma_1} \cdots t_n^{\gamma_n} = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \in Z$ and

$$(4) \quad \{t_1^{\beta_1} \cdots t_n^{\beta_n}, t_1^{\gamma_1} \cdots t_n^{\gamma_n}\} = z \{t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3}, t_1^{\gamma_1} t_2^{\gamma_2} t_3^{\gamma_3}\}.$$

Lemma 5.3.4. For $0 \neq \bar{\alpha} \in \mathbb{Z}^n/\Gamma = \mathbb{Z}_3^3$, $\text{rank}_Z(\text{Der}J)_{\bar{\alpha}} \leq 2$.

Proof. By (4), it is enough to show that $d := \{t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3}, t_1^{\gamma_1} t_2^{\gamma_2} t_3^{\gamma_3}\}$ is a Z -linear combination of two elements in $(\text{Der}J)_{\bar{\alpha}}$ which are independent of β_i, γ_i , $i = 1, 2, 3$. By (2), there exist $\sigma_1, \sigma_2, \sigma_3 \in \Omega$ so that

$$d = \sigma_1 \{t_1^{\beta_1}, t_1^{\gamma_1} t_2^{\alpha_2} t_3^{\alpha_3}\} + \sigma_2 \{t_2^{\beta_2}, t_1^{\alpha_1} t_2^{\gamma_2} t_3^{\alpha_3}\} + \sigma_3 \{t_3^{\beta_3}, t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\gamma_3}\}.$$

Since $\Gamma = 3\mathbb{Z}^3 \oplus \mathbb{Z}^{n-3}$, we can apply formula (3): Let

$$\begin{aligned} A_1 &:= \{t_1, t_1^{\alpha_1-1} t_2^{\alpha_2} t_3^{\alpha_3}\}, \\ A_2 &:= \{t_2, t_1^{\alpha_1} t_2^{\alpha_2-1} t_3^{\alpha_3}\}, \\ A_3 &:= \{t_3, t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3-1}\}. \end{aligned}$$

Then there exist $\rho_1, \rho_2, \rho_3 \in \Omega$ so that

$$(5) \quad d = [\beta_1]\rho_1 A_1 + [\beta_2]\rho_2 A_2 + [\beta_3]\rho_3 A_3.$$

Hence d is an F -linear combination of the three elements $A_1, A_2, A_3 \in (\text{Der } J)_{\bar{\alpha}}$ which are independent of $\beta_i, \gamma_i, i = 1, 2, 3$. Thus if we show the existence of a nontrivial relation among A_1, A_2, A_3 , we are done.

By the skew symmetry of $\{\cdot, \cdot\}$ and by the same method as above, there exist some $\tau_1, \tau_2, \tau_3 \in \Omega$ so that

$$(6) \quad -d = \{t_1^{\gamma_1} t_2^{\gamma_2} t_3^{\gamma_3}, t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3}\} = [\gamma_1]\tau_1 A_1 + [\gamma_2]\tau_2 A_2 + [\gamma_3]\tau_3 A_3.$$

Add the two identities (5) and (6). Then letting $\delta_i := [\beta_i]\rho_i + [\gamma_i]\tau_i$ for $i = 1, 2, 3$, we get

$$(7) \quad \delta_1 A_1 + \delta_2 A_2 + \delta_3 A_3 = 0.$$

Since $\bar{\alpha} = \bar{\beta} + \bar{\gamma} \neq \bar{0}$ in $\mathbb{Z}^n / \Gamma = \mathbb{Z}_3^3$, there exists $1 \leq j \leq 3$ such that $[[\beta_j] + [\gamma_j]] \neq 0$ (this j depends on α not on β or γ). We claim that $\delta_j \neq 0$. Suppose that $\delta_j = [\beta_j]\rho_j + [\gamma_j]\tau_j = 0$. Then $[\beta_j] = -[\gamma_j]\tau_j/\rho_j$. Since $\tau_j/\rho_j \in \Omega$ and $[\beta_j], [\gamma_j] \in \mathbb{Z}$, $\tau_j/\rho_j = (-1/2)^k$ for some $k \in \mathbb{Z}$. Hence $(-2)^k[\beta_j] = -[\gamma_j]$ and $[[\beta_j] + [\gamma_j]] = [[\beta_j] - (-2)^k[\beta_j]] = 0$, which is a contradiction. Thus we have shown our claim. Therefore, (7) is a nontrivial relation among A_1, A_2, A_3 , and this finishes the proof. \square

Finally, we can prove the following:

Theorem 5.3.5. *Let $J = \bigoplus_{\alpha \in \mathbb{Z}^n} J_{\alpha}$ be the Albert torus over F with the notations above. Then for all $\alpha \in \mathbb{Z}^n \setminus \Gamma$, we have $\dim_F(\text{Der } J)_{\alpha} = 2$.*

Proof. We know that $\text{Der}_{\bar{\mathbb{Z}}}\bar{J} = \text{IDer}\bar{J}$ is a simple Lie algebra of type F_4 , and so $\dim_{\bar{\mathbb{Z}}}\text{Der}\bar{J} = 52$ (see [16, page 21]). Then, as in the case of Clifford tori,

$$52 = \dim_{\bar{\mathbb{Z}}}\text{Der}\bar{J} = \sum_{\bar{\alpha} \neq \bar{0}} \text{rank}_{\mathbb{Z}}(\text{Der } J)_{\bar{\alpha}}$$

(see 5.2.22 and 5.2.15(ii) and Lemma 5.3.4). This forces to $\text{rank}_{\mathbb{Z}}(\text{Der } J)_{\bar{\alpha}} = 2$. Therefore, by 5.2.15(ii), we get $\dim_F(\text{Der } J)_{\alpha} = \text{rank}_{\mathbb{Z}}(\text{Der } J)_{\bar{\alpha}} = 2$. \square

Bibliography

- [1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. vol. **126**, no. 603, Amer. Math. Soc., Providence, RI, 1997.
- [2] B. Allison and Y. Gao, *The root system and the core of an extended affine Lie algebra*, (to appear).
- [3] S. Azam, *Nonreduced extended affine root systems of Nullity 3*, Comm. Algebra **25** (1997), no. 11, 3617–3654.
- [4] G. Benkart and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), 1–45.
- [5] S. Berman and R.V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), 323–347.
- [6] S. Berman, Y. Gao and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras*, J. Funct. Anal. **135** (1996), 339–389.
- [7] S. Berman, Y. Gao, Y. Krylyuk and E. Neher, *The alternative tori and the structure of elliptic quasi-simple Lie algebras of type A_2* , Trans. Amer. Math. Soc. **347** (1995), 4315–4363.
- [8] M. Boulagouaz, *The graded and tame extensions*, Commutative ring theory, 27–40. Lecture Notes in Pure and Appl. Math., 153. Dekker, New York, 1994.
- [9] R. Farnsteiner, *Derivations and central extensions of finitely generated graded Lie algebras*, J. Algebras **118** (1988), 33–45.
- [10] I.N. Herstein, *Topics in ring theory*, Chicago Lectures in Mathematics, The university of Chicago press, 1969.
- [11] R. Høegh-Krohn and B. Torresani, *Classification and construction of quasi-simple Lie algebras*, J. Funct. Anal. **89** (1990), 106–136.
- [12] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, second edition, Springer-Verlag, New York, 1972.
- [13] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Coloq. Publ., vol. 39, Amer. Math. Soc., Providence, RI, 1968.
- [14] N. Jacobson, *Structure theory of Jordan algebras*, The University of Arkansas Lecture Notes in Mathematics, vol. 5, Fayetteville, AR, 1981.

- [15] N. Jacobson. *Finite-dimensional division algebras over fields*, Springer-Verlag, Berlin Heidelberg New York, 1996.
- [16] N. Jacobson. *Exceptional Lie algebras*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1971.
- [17] N. Jacobson, *Basic algebra II*, second edition, W.H. Freeman, 1989.
- [18] T.Y. Lam, *A first course in noncommutative rings*, Springer-Verlag, **131**, 1991.
- [19] K. McCrimmon, *Nonassociative algebras with scalar involution*, Pacific J. of Math. **116**(1) (1985), 85–109.
- [20] K. McCrimmon. *Jordan centroids*, Comm. Algebra **27** (1999), no. 2, 993–954.
- [21] K. McCrimmon and E. Zel'manov, *The structure of strongly prime quadratic Jordan algebras*, Adv. Math. **69** (1988), 132–222.
- [22] R.V. Moody and A. Pianzola, *Lie algebras with triangular decomposition*, John Wiley, New York, 1995.
- [23] E. Neher, *Lie algebras graded by 3-graded root systems*, Amer. J. Math. **118** (1996), 439–491.
- [24] D.S. Passman. *Infinite crossed products*, Academic press, San Diego, 1989.
- [25] S.V. Pchelintsev. *Nilpotent elements and nil-radicals of alternative algebras (Russian)*, Algebra i Logika **24** (1985), no. 6, 674–695, 749.
- [26] H.P. Petersson and M.L. Racine. *Jordan algebras of degree 3 and the Tits process*. J. Algebra **98**(1) (1986), 211–243.
- [27] H.P. Petersson and M.L. Racine. *Albert algebras*, Jordan algebras (Oberwolfach, 1992), 197–207, de Gruyter-Berlin, 1994.
- [28] H.P. Peterson and M.L. Racine, *An elementary approach to the Serre-Rost invariant of Albert algebras*, Indag. Mathem. (N.S.) **7** (1996), no 3, 343–365.
- [29] R.S. Pierce, *Associative Algebras*, Graduate Texts in Math., vol. 88, Springer-Verlag, Berlin Heidelberg New York, 1982.
- [30] R.D. Schafer, *An Introduction to nonassociative algebras*, Academic Press, New York, 1966.
- [31] S. Tan, *TKK algebras and vertex operator representations*, J. Algebra **211** (1999), no. 1, 298–342.
- [32] Y. Yoshii, *Jordan tori*, Math. Reports Acad. Sci. Canada, **18**(4) (1996), 153–158.
- [33] K.A. Zhevlakov, A.M. Slinko, J.P. Shestakov and A.I. Shirshov, *Rings that are nearly associative*. Academic Press, New York, 1982.