

An essay on Multiplayer Colonel Blotto Game
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Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 2 | Literature Review | 3 |
| 3 | The Model (Generalized Blotto Game) | 4 |
| 3.1 | Assumptions | 4 |
| 3.2 | Equilibrium analysis | 5 |
| 4 | Asymmetric Equilibria | 7 |
| 5 | Extension of the work | 7 |
| 5.1 | Contest Success Functions | 8 |
| 5.1.1 | Defining the Ratio Contest Success Function | 8 |
| 5.1.2 | Defining the Auction Contest Success Function | 8 |
| 6 | Conclusion | 9 |
| 7 | Bibliography | 10 |

Abstract: We consider a Colonel Blotto game with complete information and analyze it over multiple battlefields. Resources among players are symmetric and there exists full information on the budget constraints of the players. Given a general ratio contest success function we derive the pure strategy symmetric Nash equilibrium. The valuations of battlefields are ordered in a way that, at equilibrium the marginal utility of additional resources to players remains the same across all battlefields that have positive allocations.

Keywords: Colonel Blotto game, symmetric equilibrium, ratio contest success function, marginal utility, game theory.

1 Introduction

In the Colonel Blotto game, two Colonels have a pool of troops to fight against each other over a set of battlefields. All the battlefields have an associated weight and the respective payoffs of the colonels are calculated as the sum of the weights of the battlefields they win.

The game was initially proposed to study warlike situations and has now found prominence in analyzing different forms of competition. The most important is its use in the area of election campaigning. Previous researchers have dealt with the problem from the area of incomplete information or variations across battlefields. This has led to the derivation of mixed strategy Nash equilibrium solutions. In this paper, we analyze the game based on multiple players and multiple battlefields. The assumptions made are similar to Blackett (1958), but this paper extends the work of Blackett to multiple players, which is the novelty of this research. The fundamental goal here is to generalize the Colonel Blotto game to suit the various real-world applications. Over the years, many variants of this game have been analyzed. After Borel in 1921, Borel and Ville (1938) analysed the 3 battlefields game with 2 opponents. Likewise, numerous variants of the game have been narrated by previous researchers. It has been studied as symmetric, depending on the similarity of the budget among the opponents, and homogenous, depending on the battlefield valuations. In this game, we define resources in terms of budgetary allocation and resource allocations are normalized to 1. The players allocate resources across N , a set of n battlefields. Denote x as a profile of allocations $x_{i,j}$ for all $i=1,2,3\dots m$ and $j=1,2,3\dots n$. So, $\sum_{j=1}^n x_{i,j}=1$, where allocations are in the $(n-1)$ dimensional simplex, Δ^{n-1} .

Allocations have been studied previously by researchers as discrete, continuous, or binary. In this paper, we introduce the M player Blotto game, with n battlefields and with the allocation of resources by multiple players across multiple battlefields. Here, we generalize the number of players to $M \geq 3$ players and each player must distribute their resources without the knowledge of the opponent's action. The allocation of resources to a battlefield determines the probability with which the player wins the battlefield. So, the higher the amount a player allocates to a battlefield, the more likely he is to win.

We model conflicts where the payoff to each risk-neutral player is given by a Contest Success Function(CSF) (Roberson, 2010). The ratio CSF defines the probability that player i wins the j^{th} battlefield when the profile of allocations is x .

The remainder of the article is structured as follows. Section 2 studies the relevant literature. In Section 3, we generalize the Colonel Blotto game, Section 4 presents an example and then in Section 5, we show some possible extensions of the game and in Section 6 we conclude.

2 Literature Review

The Colonel Blotto game was first studied by Borel in 1921. In this game, players must allocate resources over a certain number of battlefields. The player who allocates the highest amount of resources to a battlefield wins it. However, this is done without the knowledge of the opponent's actions. The payoffs of the whole game are represented as a function of the sum of the wins across individual battlefields. Thus the game is useful in understanding the budgetary allocation of the players. This was analyzed by several previous papers (for example, Bellman 1969; Blackett 1958; Gross and Wagner 1950; Shubik and Weber 1981).

The literature on Colonel Blotto games traces its roots to electoral competition too, chiefly to the works of Hotelling (1929). Asymmetry of information among players and asymmetric valuations of battlefields have been analysed. Hotelling stated that in political campaigning it was found that the Republican and the Democratic parties in the USA were never going to be poles apart while coming up with their political manifestoes. Both parties fear the loss of a majority of votes except for the very loyal voters. Eventually, it was found that their respective manifestoes had something similar in them, speaking of their mutual views on protective tariffs. Also, credit here can be given to Downs (1957) who showed the difference when voting is costly and costless on the parts of citizens. He proved that when voting is cost less, citizens who are politically driven come forward to vote, and when it is costly, only some indifferent citizens who are rational come forward to vote. The modern-day election models are based on the Hotelling-Downs framework. Friedman (1958) however, analysed this issue for 2 contestants based on advertisement expenditure. As a continuation to the previous work, Karlin (1959) discussed the problem based on two candidates and made contributions to the existing framework as analysed by Friedman and made it continuous.

There has been a recent resurgence of interest in the Blotto games (for example, Golman and Page 2009; Hart 2008; Laslier 2006; Roberson 2006; Kovenock and Roberson, 2010). The primary appeal of the Colonel Blotto game is that it provides a theoretical background to analyse a broad set of environments. The game analysed in this paper is based on Borel's original theme and an extension of Blackett's work with a 2-player symmetric game. The paper by Adamos and Matros (2009) is very close to what we do here, but the main difference is the asymmetry of information as dealt with by the authors. They assumed that the battlefield valuations may vary across battlefields, but each battlefield's valuation is the same for all players. They also proved that the higher the valuation, the higher is the resource allocation of the player towards that battlefield. However, in our case, we are able to show that at any symmetric equilibrium the marginal utility of additional resources to all players remains the same across all battlefields with positive allocations, although the valuations of the battlefields are ordered. There would exist a cut-off such that the players invest in a battlefield if and only if the valuation is above that cut-off.

3 The Model (Generalized Blotto Game)

Let M represent the set of m risk neutral players and N represents the set of n battlefields. So, $x_{i,j}$ represents the amount of resources player i allocates to battlefield j .

3.1 Assumptions

- Each player has 1 unit of resources to allocate across the n battlefields.
- The valuations of the battlefields are ordered such that $v_1 \leq v_2 \leq \dots \leq v_n$.
- $\sum_{j=1}^n x_{i,j}=1$, where allocations are in the $(n - 1)$ dimensional simplex, Δ^{n-1} .
- x is a profile of allocations $x_{i,j}$ for all $i=1,2,3\dots m$ and $j=1,2,3\dots n$.
- $P_{i,j}(x)$ is the probability that player i wins battlefield j when the profile of allocations is x , which is called the Contest Success Function (Roberson, 2010).
- Let $u_i(x) = \sum_{j=1}^n P_{i,j}(x)v_j$ represent player i 's payoff when the profile of allocation is x , which is also the expected number of battlefields player i wins and let v_j denote the valuation of battlefield j such that $\sum_{j=1}^n v_j=1$.

Let us now define the Contest Success Function (CSF). The ratio CSF defines the probability that player i wins the j^{th} battlefield when he allocates $x_{i,j}$ resources and all the players, but i , say l allocate $x_{l,j}$ to the j^{th} battlefield (Roberson, 2010).

We define the General form CSF as:

$$P_{i,j}(x) = \frac{x_{i,j}^k}{\sum_{l=1}^m x_{l,j}^k}$$

where k is the noise parameter. Now, when $k=0$, any player who allocates a positive amount to a battlefield wins it with equal probability. When $k=1$ the probability of winning is proportional to the amount of resources a player allocates. When $k \rightarrow \infty$, the probability that the player who allocates the most resources wins the battlefield goes to one.

3.2 Equilibrium analysis

A strategy for player i is an allocation for i , x_i . x^* is a Nash Equilibrium if for each $i \in M$

$$u_i(x_i^*, x_{-i}^*) \geq \max_{x_i \in \Delta^{n-1}} u_i(x_i, x_{-i}^*) \quad (1)$$

So in equilibrium, i solves:

$$\begin{aligned} \max_{x_i \in \mathbb{R}^n} \quad & \sum_{j=1}^n v_j \left(\frac{x_{i,j}^k}{x_{i,j}^k + \sum_{l \neq i} x_{l,j}^k} \right) \\ \text{s.t.} \quad & \sum_{j=1}^n x_{i,j} = 1 \quad \text{for all } j, x_j \geq 0 \end{aligned} \quad (2)$$

Lemma. In a symmetric equilibrium, if a player allocates $x_{i,j} > 0$, then for each $j' > j$, he allocates $x_{i,j'} > 0$.

Proof. Suppose x^* is a symmetric Nash equilibrium with $x_{i,j}^* > 0$ and $x_{i,j'}^* = 0$ for some $j' > j$. Since x^* is a symmetric Nash equilibrium for each $l \in M$, $x_{l,j}^* = x_{i,j}^* = 0$ and $x_{l,j'}^* = x_{i,j'}^* = 0$. Let us consider a deviation, \hat{x}_i for player i such that:

$$\hat{x}_i = \begin{cases} x_{i,j}^* & \text{if } k = j' \\ 0 & \text{if } k = j \\ x_{i,k}^* & \text{otherwise} \end{cases}$$

By deviating to \hat{x}_i from x_i^* , i wins battlefield j' and also loses j with certainty. Hence, in this case, i has a net expected gain from deviation, which is $v_{j'} - v_j(\frac{1}{m}) > 0$. Also, if we assume that $1/m$ is the probability for m players to win a battlefield where no one invests, then the net expected loss from deviation is $v_{j'}(\frac{1}{m}) + v_j(\frac{1}{m}) < v_{j'}$ when $m \geq 2$. This contradicts the fact that x^* is a symmetric Nash equilibrium. Thus, if $x_{i,j}^* > 0$ then for each $j' > j$, $x_{i,j'}^* > 0$. ■

Theorem. At any symmetric equilibrium, there exists a cut-off j_0 such that $x_{i,j}$ is proportional to v_j for $j \geq j_0$ and 0 for $j < j_0$ for $k > 0$.

Proof. The Lagrangian for (2) can be written as:

$$\mathcal{L} = \sum_{j=1}^n v_j \frac{x_{i,j}^k}{x_{i,j}^k + \sum_{l \neq i} x_{l,j}^k} + \lambda \{1 - x_{i,j} - \sum_{t \neq j} x_{i,t}\}$$

where λ is the lagrange multiplier.

The first order condition with respect to $x_{i,j}$ is:

$$v_j \left(k x_{i,j}^{*k-1} \frac{\sum_{l \neq i} x_{l,j}^{*k}}{\{\sum_{l=1}^m x_{l,j}^{*k}\}^2} \right) = \lambda$$

According to Lemma, we have

$$v_j \left(kx_{i,j}^{*k-1} \frac{\sum_{l \neq i} x_{l,j}^{*k}}{\{\sum_{l=1}^m x_{l,j}^{*k}\}^2} \right) = \lambda$$

By symmetry, we have: $x_{l,j}^* = x_{i,j}^*$, for all $l \neq i$.

So, we have:

$$\begin{aligned} v_j \left(kx_{i,j}^{*k-1} \frac{(m-1)x_{i,j}^{*k}}{m^2 x_{i,j}^{*2k}} \right) &= \lambda \\ \implies v_j \left(kx_{i,j}^{*2k-1} \frac{(m-1)}{m x_{i,j}^{*2k}} \right) &= \lambda \\ \implies v_j \left(k \frac{(m-1)}{m^2 x_{i,j}^*} \right) &= \lambda \\ \implies x_{i,j}^* &= v_j \left(k \frac{(m-1)}{\lambda m^2} \right) \end{aligned}$$

$$\sum_{j=1}^n x_{i,j} = 1 \iff \sum_{j=j_0}^n x_{i,j} = 1$$

$$\begin{aligned} \implies \sum_{j=j_0}^n \left(k \frac{(m-1)}{\lambda m^2} v_j \right) &= 1 \\ \implies \left(k \frac{(m-1)}{\lambda m^2} \right) \sum_{j=j_0}^n v_j &= 1 \\ \implies \lambda &= \left(k \frac{(m-1)}{\lambda m^2} \right) \sum_{j=j_0}^n v_j \\ \implies x_{i,j}^* &= \left(k \frac{(m-1)}{\frac{k(m-1)}{m^2} m^2 + \sum_{j=j_0}^n v_j} \right) v_j \\ x_{i,j}^* &= \left(\frac{v_j}{\sum_{t=j_0}^n v_t} \right) \end{aligned}$$

for all i, j such that $j \geq j_0$.

So, for all i, j ,

$$x_{i,j}^* = \begin{cases} \left(\frac{v_j}{\sum_{t=j_0}^n v_t} \right) & \text{if } j \geq j_0 \\ 0 & \text{if } j < j_0 \end{cases}$$

■

4 Asymmetric Equilibria

Here, we try to show the relevance of our theorem in a 2 players, 2 battlefields game. There are cases where, even with symmetric players and homogenous battlefield valuations, we can derive symmetric as well as asymmetric pure strategy equilibria. Asymmetric equilibria however are beyond the scope of the paper but the difference can be explained with a simple example.

Suppose that:

- There are two colonels, Blotto and Lotto fighting over battlefields A and B .
- The valuations of battlefields A and B are such that $v_A = v_B$.
- Both players have 1 unit of resource.
- Players have two strategies, such that $x_{iA} \in [0, 1]$ where x_{iA} is the amount invested in battlefield A and $1 - x_{iB}$ is the amount invested in battlefield B and $i \in \{\text{Blotto, Lotto}\}$.

Our theorem suggests that it is optimal for both players to invest half of their respective resources in both battlefields and in that case, $x_{\text{Blotto},j} = x_{\text{Lotto},j} = \frac{1}{2}$ for all $j \in \{A, B\}$ is an equilibrium. That would be a case of pure strategy symmetric equilibrium, which has been studied in this paper. However, even with symmetric players, we can see that, $(x_{\text{Blotto},A} = 1, x_{\text{Lotto},A} = 0)$ and $(x_{\text{Blotto},B} = 0, x_{\text{Lotto},B} = 1)$ which would also be an asymmetric equilibrium. The asymmetric equilibrium thus suggests that, if Colonel Blotto invest in battlefield A then the optimal best response for Colonel Lotto is to invest in battlefield B and vice versa. This would be a case of asymmetric equilibrium even with symmetric players.

Hence, an open question here, for the m -player n -battlefield game is the characterization of all equilibria.

5 Extension of the work

The game we have described is based on a ratio CSF which has been defined already. This has multiple variations and can be extended in multiple directions. In our case, the CSF represents the probability that a player i wins battlefield j when the profile of allocations is x . The same can be done in the case of an auction CSF, where $x_{i,j}$ is the bid for player i and $x_{-i,j}$ represents all the bids of players except for player i . They contest for the prize j and player i wins if and only if:

$$x_{i,j} > \max \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{M,j}\} \quad (3)$$

where $x_{i,j}$ represents the bid of player i in the auction to win prize j over all other bids. This version of the plurality game has been used by Matros (2009). Going back to our CSF we see that we have k , a noise parameter. The higher the value of k , the better. Lower values of k imply the presence of more noise in the model.

A measure of the noise can be found in Friedman (1958) where the author studies the case of $k = 1$ and by Robson (2005) where cases have been analysed for $k \in (0, 1]$. The case of $k = \infty$ has been analysed by Baye, Kovenock and de Vries (2005). In the case where $k \rightarrow \infty$, the ratio CSF and the auction CSF can be used interchangeably. Also, no pure strategy Nash Equilibrium has been derived. But, the main point of contrast comes with the symmetry and asymmetry of information.

5.1 Contest Success Functions

We now see how the game can be extended when we use the ratio CSF. But to do that let us provide a background to the concept of CSF:

There are m risk neutral players who have to allocate their resources to all the n battlefields. Each player has 1 unit of resources to allocate across the n battlefields. $\sum_{j=1}^n x_{i,j}=1$, where allocations are in the $(n-1)$ dimensional simplex, Δ^{n-1} . x is a profile of allocations $x_{i,j}$ for all $i=1,2,3\dots m$ and $j=1,2,3\dots n$. $P_{i,j}(x)$ is the probability that i wins battlefield j when the profile of allocations is x . Let $u_i(x) = \sum_{j=1}^n P_{i,j}(x)v_j$ represent player i 's payoff when the profile of allocations is x , which is also the expected number of battlefields i wins, and let v_j denote the valuation of battlefield j such that $\sum_{j=1}^n v_j=1$.

Let us now define the Contest Success Function (CSF). The ratio CSF defines the probability that player i wins the j^{th} battlefield when he allocates $x_{i,j}$ resources and all the players, but i , say l allocate $x_{l,j}$ to the j^{th} battlefield (Roberson, 2010).

5.1.1 Defining the Ratio Contest Success Function

$$P_{i,j}(x) = \begin{cases} \frac{x_{i,j}^k}{\sum_{j=1}^m x_{l,j}^k} & \text{if } (x_{l,j}, x_{-l,j}) \neq 0 \\ (\frac{1}{m}) & \text{if } (x_{l,j}, x_{-l,j}) = 0 \end{cases}$$

The payoffs obtained from the auction CSF are continuous.

5.1.2 Defining the Auction Contest Success Function

The Auction Contest Success Function (CSF) can be defined as:

$$P_{i,j}(x) = \begin{cases} (1) & \text{if } x_{i,j} > \max \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{m,i}\} \\ (1/m) & \text{if } x_{i,j} = \max \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{m,i}\} \\ (0) & \text{if } x_{i,j} < \max \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{m,i}\} \end{cases}$$

The auction CSF is very important considering the fact that it can lead us to deriving a Pure Strategy or a Mixed strategy Nash Equilibrium. It gives us discontinuous and the ratio CSF gives us continuous payoffs. In case of an auction CSF too, the cases of majority and plurality can be analyzed. Let us analyze the case of plurality first. Let $u_{i,j}(x)$ be the payoff to player i in the contest j , when i allocates $x_{i,j}$ resources then, the component contest success function under plurality rule with auction CSF is represented as:

$$(u_{i,j}(x)) = \begin{cases} (v_j) & \text{if } x_{i,j} > \max \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{m,i}\} \\ (v_j/m) & \text{if } x_{i,j} = \{x_{1,j}, \dots, x_{i-1,j}, x_{i+1,j}, \dots, x_{m,i}\} \\ (0) & \text{otherwise} \end{cases}$$

and when we focus on the plurality objective over general form CSF, then the payoffs across component contests is the summation of component contests across all battlefields:¹ $u_i(x) = \sum_{j=1}^n (u_{i,j}(x))$ and in this case the game is a constant sum game, with a constant value of $\sum_{j=1}^n v_j$. Under the majoritarian rule, let player i 's payoff across the set of component contests be $\overline{u}_i(x_i, x_{-i})$:

$$\overline{u}_i(x_i, x_{-i}) = \begin{cases} (v_j) & \text{if } \sum_{j=1}^n (u_{i,j}(x)) > \{\sum_{j=1}^n v_j\}/m \\ (v_j/m) & \text{if } \sum_{j=1}^n (u_{i,j}(x)) = \{\sum_{j=1}^n v_j\}/m \\ (0) & \text{otherwise} \end{cases}$$

The Contest Success Function (CSF) is thus a primitive tool for analyzing Colonel Blotto games. The auction CSF leads to discontinuous payoffs whereas the ratio CSF helps in deriving continuous payoffs. Both the auction as well as the ratio CSF can be used to analyze the plurality as well as the majoritarian objectives. These objectives can further be used for the analysis of discrete or continuous Blotto games. So far such an attempt has been made by researchers like Roberson and Kovenock (2010) or Friedman (1958) but a complete characterization of the multiplayer game still remains an open question to researchers.

6 Conclusion

This paper analyzes the zero-sum Colonel Blotto game with heterogenous battlefield valuations with multiple players. We find that, at any symmetric pure strategy equilibrium there would exist a cut-off. For any battlefield with a valuation higher than the cut-off, the resource allocation by the players to a battlefield is proportional to the valuation. We also find that the marginal utility of additional resources for players remains the same across all battlefields that have positive allocations. The ratio Contest Success Function (Roberson, 2010) has been used as a tool for the analysis of this model and has been used to derive a pure strategy symmetric Nash equilibrium.

¹For more details, see Roberson (2010).

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