Reconstructing a graph from its edge-contractions

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Abstract

In this thesis, we investigate the contraction reconstruction conjecture. It states that all simple graphs with at least four edges are reconstructible, that is they are uniquely determined from their collection of single edge contraction minors, called the deck. Similar questions have been studied in the past, the vertex reconstruction conjecture being the most famous.

There are usually two steps to show that a class of graph is reconstructible. The first one is to show that the class is recognizable, meaning that it is possible to determine if a graph G belongs to that class by looking at its deck. In order to recognize some classes of graphs, we show that a wide range of graph properties are reconstructible. We investigate the connectivity of graphs, which is useful to recognize disconnected, separable, and 2-connected graphs. We also show that the number of cycles of various lengths, the degree sequence, the number of spanning trees, the planarity, the presence of cliques of various sizes, and the diameter are reconstructible. Knowing the lengths of cycles allows us to recognize the class of bipartite graphs, while knowing the degree sequence allows us to recognize regular graphs.

The second step in showing that a class of graph is reconstructible is called weak reconstruction. We say that a class of graph is weakly reconstructible if no two graphs in that class share the same deck. A class of graphs that is both weakly reconstructible and recognizable is reconstructible. In this thesis, we show that disconnected graphs, bipartite graphs, most separable graphs and most 2-edge connected graphs are reconstructible. We also show that distance regular graphs and some cubic graphs are reconstructible. We quickly delve into the theory of probabilities to give a proof that almost all graphs are reconstructible.

Finally, the relation between edge contraction and graph automorphisms is studied. We study the automorphism group of a graph in relation to those of its cards. We also study the concept of contraction pseudo-similarity. Two edges are contraction pseudo-similar if they are not similar, but their contractions yield isomorphic graphs. We completely characterize the graphs that contain contraction pseudo-similar edges.

Résumé

Dans cette thèse, la conjecture de la reconstruction de graphes par contraction est étudiée. Elle énonce que tous les graphes simples avec au moins quatre arêtes sont reconstructibles, c'est-à-dire qu'ils sont déterminés par l'ensemble de leurs mineures obtenus par une contraction. Cet ensemble se nomme le deck. Des questions similaires ont été étudiées dans le passé, la conjecture de la reconstruction de graphes par effacement de sommets étant la plus célèbre.

La reconstruction d'une famille de graphes comporte deux étapes. La première est la reconnaissance. Il faut montrer qu'il est possible de déterminer si un graphe appartient à cette famille en ne considérant que son deck. Dans cette thèse, nous montrons que plusieurs propriétés d'un graphe sont reconstructibles. Nous étudions la connectivité, ce qui est utile pour reconnaître des graphes déconnexes, séparables et 2-connexes. Nous montrons aussi qu'il est possible de reconstruire le nombre de cycles de différentes tailles, la séquence des degrés, le nombre d'arbres couvrants, la planarité, le nombre de cliques de différentes tailles ainsi que le diamètre. La reconstruction de certains cycles nous permet alors de reconnaître les graphes bipartis, alors que la reconstruction de la séquence des degrés nous permet de reconnaître les graphes réguliers.

La deuxième étape dans la reconstruction est la reconstruction faible. Une classe de graphes est faiblement reconstructible si aucune paire de graphes dans cette classe ne partagent le même deck. Une classe de graphes qui est à la fois reconnaissable et faiblement reconstructible est reconstructible. Dans cette thèse, nous montrons que les graphes déconnexes, bipartis, séparables et plusieurs graphes 2-connexes sur les arêtes sont reconstructibles. Nous montrons aussi que les graphes distance-réguliers et quelques graphes cubiques sont reconstructibles. Nous empruntons aussi quelques éléments de la théorie des probabilités pour montrer que presque tous les graphes sont reconstructibles.

Finalement, la relation entre la reconstruction par contraction et les groupes d'automorphismes de graphes est étudiée. Nous étudions le groupe d'automorphisme d'un graphe et sa relation avec ceux de ses cartes. Nous étudions aussi la pseudo-similarité. Deux arêtes sont pseudo-similaires si elles ne sont pas similaires, mais que leurs contractions donnent deux graphes isomorphes. Nous caractérisons les graphes qui possèdent de telles arêtes.

Remerciements

Je voudrais d'abord remercier mon superviseur, Mike Newman. Sans sa persévérance, son dévouement et ses bonnes idées, je n'aurais pas eu autant de succès ni de plaisir à compléter ce doctorat. J'aurai toujours des bons souvenirs de nos rencontres hebdomadaires.

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Chapter 1

Introduction

1.1 Reconstruction

Reconstruction problems in graph theory ask whether a graph G is determined, up to isomorphism, by a collection of graphs derived from G. There are many different reconstruction problems, and they differ by the collection of graphs given. The most famous reconstruction problem is undoubtedly the vertex reconstruction problem, which asks whether all graphs G, with two notable exceptions, are determined by the collection of graphs obtained by deleting a single vertex of G. Another famous one is the edge reconstruction problem, which asks whether all graphs G, with this time four exceptions, are determined by the collection of graphs obtained by deleting a single edge of G. The collection of graphs derived from G need not be subgraphs of G. For example, the vertex switching reconstruction problem asks whether a graph G is determined by the collection G_i of graphs, where G_i is obtained by removing all edges incident to the vertex v_i, and adding all non-edges incident to v_i. It should be mentioned that these problems are still open despite the considerable attention they were given.

The reconstruction problem studied in this thesis is the contraction reconstruction problem. In this problem, the collection of graphs derived from G is the set of all G/e, where e is an edge of G. Surprisingly, very little is known about contraction reconstruction, while a plethora of results can be found about the other reconstruction problems mentioned [3]. Results in contraction reconstruction can be found in [1, 15, 16, 17, 26].

We start with a more precise definition of the contraction reconstruction problem. The terms "cards", "decks", "reconstructible", "reconstruction" and "recognizable" refer to Definition 1.1.1, except in Chapter 2, where we survey other types of reconstruction problems.

Definition 1.1.1. *For a graph G, the multiset of unlabeled graphs $\{G/e : e \in E(G)\}$ is the* ***contraction deck*** *of G, denoted $\mathcal{C}(G)$. Any graph $H \in \mathcal{C}(G)$ is a* ***contrac-***

tion card. A ***contraction reconstruction*** *of G is a graph F such that there exists a bijection $\phi : E(G) \to E(F)$ where $G/e \cong F/\phi(e)$ for all $e \in E(G)$. A graph G is* ***contraction reconstructible*** *if every contraction reconstruction of G is isomorphic to G. A class $\mathcal{G}$ of graphs is* ***contraction recognizable*** *if every contraction reconstruction of any $G \in \mathcal{G}$ is also in $\mathcal{G}$. A class $\mathcal{G}$ is* ***contraction weakly reconstructible*** *if for every $G \in \mathcal{G}$, all contraction reconstructions of G that are in $\mathcal{G}$ are isomorphic to G.*

For brevity, we will often say that a property is reconstructible to mean that the class of graphs with that property is recognizable.

See Figure 1.1 for an example of a contraction deck. Since the main subject of this thesis is the study of contraction reconstruction, we omit the word "contraction" when discussing the concepts defined in Definition 1.1.1.

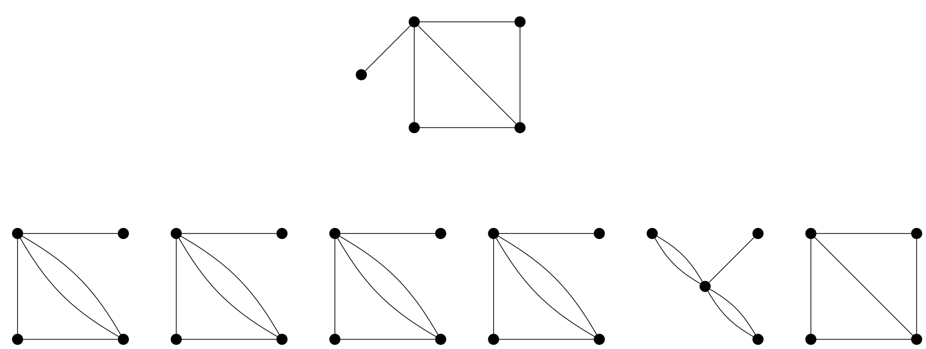

Figure 1.1: In the first row: a graph G. In the second row: the deck of G.

It follows from the definition that a class $\mathcal{G}$ of graphs is reconstructible if and only if it is recognizable and weakly reconstructible. A standard technique to show that a class is reconstructible is to show that it is both recognizable and weakly reconstructible. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are recognizable classes of graphs, then it follows from the definition that $\mathcal{G}_1 \cap \mathcal{G}_2$, $\mathcal{G}_1 \cup \mathcal{G}_2$ and the complements of $\mathcal{G}_1$ and $\mathcal{G}_2$ are also recognizable classes of graphs.

We can define a function on the set of all finite unlabeled graphs that maps a graph G to its deck. The reconstruction conjecture asks whether that function is injective, or if not, what graphs are mapped to the same deck. Some graphs are not reconstructible. For example, the ***empty graph*** E_n, which is the graph on n vertices with no edges, has $\mathcal{C}(E_n) = \emptyset$, for any $n \geq 0$.

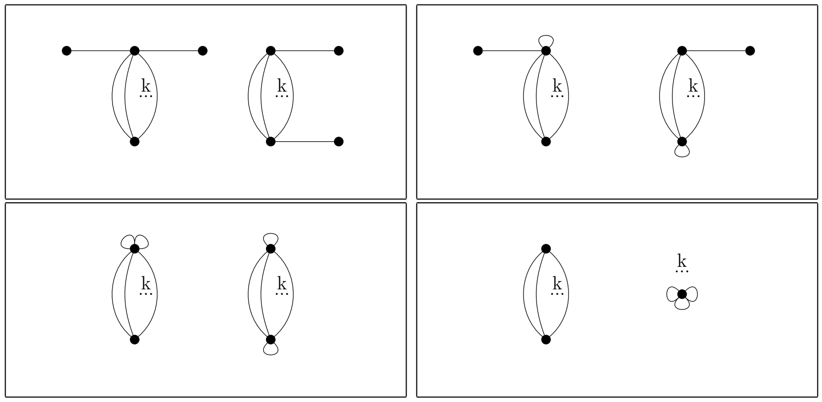

Figure 1.2: For any fixed $k \geq 0$, in each box are graphs that are contraction reconstructions of one another.

In Figure 1.2, each of the four pairs of graphs is composed of two graphs which are reconstructions of one another. Those graphs are therefore not reconstructible. Note that if two graphs G and H are reconstructions of one another, then adding an isolated vertex to each of G and H yields two graphs which are reconstructions of one another. Therefore, there are arbitrarily large, with respect to the number of vertices, non-reconstructible graphs. Since the number of parallel edges k in the graphs in Figure 1.2 is arbitrary, there are also arbitrarily large multigraphs, with respect to the number of edges, that are not reconstructible. On the other hand, one can choose $k = 0$ or 1. In both of those cases, the first pair of graphs in Figure 1.2 are simple graphs. They are shown in Figure 1.3.

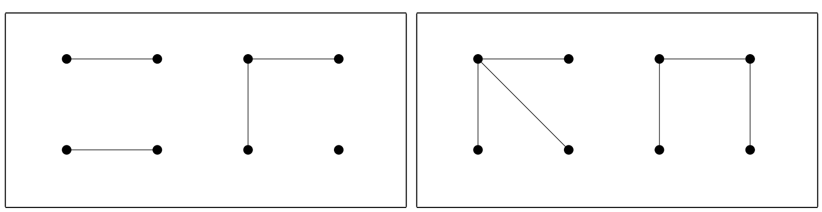

Figure 1.3: In each box are simple graphs that are contraction reconstructions of one another.

We mainly focus on simple graphs in this thesis, and attempt to shed some light on Conjecture 1.1.2.

Conjecture 1.1.2. *All simple graphs on at least four edges are contraction reconstructible.*

Some numerical evidence supporting this conjecture can be found in Appendix A.

Proofs that a certain class of graphs is reconstructible generally have two steps. The first step is to show that the class is recognizable, and the second step is to show that the class is weakly reconstructible. To show that a class is recognizable, it is often useful to determine if a graph has certain properties, many of which are shown to be reconstructible in this thesis.

In Chapter 2, we do a quick survey of other types of reconstruction problems on graphs. In Chapter 3, basic results about contraction reconstruction are presented. In Chapter 4, we study the reconstruction of connectivity properties, then show that most disconnected and most separable graphs are reconstructible. In Chapter 5, we study the cycles of graphs, and then show that bipartite graphs are reconstructible. In Chapter 6, the degree sequence of graphs is studied, which allows us to then show that a subclass of regular graphs, the distance regular graphs, are reconstructible. In Chapter 7, the relation between contraction reconstruction and graph automorphisms is studied. The main result of this chapter is a complete characterization of contraction pseudo-similar edges, which is analogous to a result in [8]. In Chapter 8, a probabilistic proof that almost all graphs are reconstructible is presented. We end this thesis with Chapter 9, where some open questions are discussed.

1.2 Graph definitions

In this section, we establish the basic graph definitions. Any reader familiar with graph theory may skip this section. A ***graph*** G is an ordered triple $(V(G), E(G), \psi)$, comprised of a set of ***vertices*** $V(G)$, a set of ***edges*** $E(G)$, along with an ***incidence relation*** between $E(G)$ and $V(G)$. The incidence relation ψ is a function from $E(G)$ to the subsets of $V(G)$ of size 1 or 2. We say an edge e is ***incident*** to the vertices in $\psi(e)$. Many edges may be incident to the same two vertices, but when it is not the case, edges that are incident to two vertices are often labeled by the two vertices. To facilitate the reading, for two vertices x and y, we write $xy \in E(G)$ to denote that there is an edge in $E(G)$ that is incident to both x and y. Similarly, if there are no edges incident to both x and y, then we write $xy \notin E(G)$. Graphs in this thesis may have parallel edges and loops, so for convenience, we may represent by xy an arbitrary edge linking x and y, even if there may be multiple edges linking the two vertices.

The vertices x and y are the ***ends*** of an edge xy, and the edge xy ***links*** the vertices x and y. Denote by $v(G)$ and $e(G)$ the quantities $|V(G)|$ and $|E(G)|$, respectively. A graph is ***finite*** if both $V(G)$ and $E(G)$ are finite. Henceforth all graphs are finite.

A vertex x and a vertex y are ***adjacent*** or ***neighbours*** if there is an edge of $E(G)$ that is incident to both x and y. These vertices are ***incident*** to the edge xy.

Two edges are ***incident*** if they share an end. Edges that are incident to only one vertex are ***loops***, while edges incident to two vertices are ***links***. Two edges which have the same ends are ***parallel***. The ***multiplicity*** of an edge e is the number of edges with the same two ends as e, including e itself. A graph is ***simple*** if its edge set does not contain any loops or parallel edges. In this thesis, the term graph refers to what is commonly known as ***multigraph***, which may or may not contain loops and parallel edges. For any graph G, an ***underlying simple graph*** is obtained from G by deleting all loops, and, for every pair of adjacent vertices, all but one link joining them. The ***degree*** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v, counting the loops twice.

Many operations can be made on a graph. For any vertex $v \in V(G)$, the graph $G-v$ has the vertex set $V(G)-\{v\}$, its edge set obtained by removing from $E(G)$ all edges incident to v, and the incidence relation of $G-v$ is the incidence relation on G restricted to the remaining edges. This operation is a ***vertex deletion***. For any edge $xy \in E(G)$, the graph $G-xy$ has vertex set $V(G)$, its edges set is $E(G)-\{xy\}$, and the incidence relation of $G-xy$ is the incidence relation on G restricted to $E(G)-xy$. This operation is an ***edge deletion***.

To ***identify*** two vertices x and y of a graph is to replace these vertices by a single vertex v, where any edge of G incident to x or y is now incident to v instead. To ***contract*** an edge xy is to identify x and y in $G-xy$: this operation is an ***edge contraction***. We denote the resulting graph by G/xy, whose vertex set is $V(G) \cup \{v^*\}-\{x,y\}$, where v^* is the ***contracted vertex***, and whose edge set is $E(G)-\{xy\}$. If ψ is the incidence relation on G, then the incidence relation ψ' on G/xy is defined as $\psi'(e)=\psi(e)$ if $x \notin \psi(e)$ and $y \notin \psi(e)$, and $\psi'(e)=\psi(e)\cup\{v^*\}-\{x,y\}$ otherwise. Note that G/xy may be a multigraph even if G is simple: if both x and y are adjacent to a third vertex v, then G/xy contains two parallel edges between v^* and v. More generally, we say that H is a ***contraction*** of G if H is obtained from G by a sequence of edge contractions.

To ***decontract*** a vertex v is to replace v by two adjacent vertices u and w, to replace each link incident to v by an edge incident to either u or w, the other end of the edge remaining unchanged, and to replace each loop on v by an edge linking u and w or by a loop on u or w. Note that while the contraction of an edge is unique, the decontraction of a vertex v is not, as there are many ways to split the edges incident to v into the edges incident to u and w. To ***subdivide*** an edge e is to delete that edge and add a vertex adjacent to the ends of e. It is a type of decontraction.

A graph H is a ***subgraph*** of G if H is obtained from G by a sequence of vertex and edge deletions. It is a ***proper*** subgraph if $V(H) \neq V(G)$ or $E(H) \neq E(G)$. It is a ***spanning subgraph*** if $V(H)=V(G)$. So $G-v$ and $G-e$ are proper subgraphs of G, for any $v \in V(G)$ and $e \in E(G)$, while G/e is a subgraph of G only in some cases, such as when e is a loop. For any two graphs $G_1=(V(G_1),E(G_1),\psi_1)$ and $G_2=(V(G_2),E(G_2),\psi_2)$, the ***intersection*** of G_1 and G_2, denoted by $G_1 \cap G_2$, is the graph G with $V(G) = V(G_1) \cap V(G_2)$, $E(G) = E(G_1) \cap E(G_2)$, and ψ is the restriction of ψ_1 to $E(G_1) \cap E(G_2)$, should it be equal to the restriction of ψ_2 to $E(G_1) \cap E(G_2)$. If those two restrictions are not equal, then $G_1 \cap G_2$ is undefined. However, if G_1 and G_2 are two subgraphs of the same graph G, then $G_1 \cap G_2$ is never undefined. The ***union*** of G_1 and G_2, denoted $G_1 \cup G_2$, is the graph G with $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, and $\psi(e)$ is equal to $\psi_1(e)$ if $e \in E(G_1)$, and equal to $\psi_2(e)$ if $e \in E(G_2)$. Note that $G_1 \cup G_2$ may be undefined, if for some $e \in E(G_1) \cap E(G_2)$, we have $\psi_1(e) \neq \psi_2(e)$. Again, when G_1 and G_2 are subgraphs of the same graph G, their union is never undefined.

An ***isomorphism*** between a graph G and a graph H is a pair of bijections (α, β), where $\alpha : V(G) \to V(H)$ and $\beta : E(G) \to E(H)$ such that for any two vertices $u, v \in V(G)$ and any edge $e \in E(G)$ linking u and v, $\alpha(u)$ and $\alpha(v)$ are the ends of $\beta(e)$. If $e \in E(G)$ is a loop attached to a vertex $v \in V(G)$ then $\beta(e) \in E(H)$ is a loop attached to $\alpha(v) \in V(H)$. The graphs G and H are ***isomorphic*** if such an isomorphism exists, and we denote this relation by $G \cong H$. Being isomorphic is an equivalence relation between graphs, and an ***unlabeled*** graph is a representative of an equivalence class of isomorphic graphs. For the sake of brevity, we may refer to an isomorphism (α, β) with a single function φ, and it is understood that if v is a vertex, then $\varphi(v)$ refers to $\alpha(v)$, and if e is an edge, then $\varphi(e)$ refers to $\beta(e)$. The relation between α and β when the pair is an isomorphism is presented in Chapter 7.

Graphs in this thesis are unlabeled. Labels may be assigned to vertices and edges of a graph, but only for the purpose of referring to them. An ***automorphism*** of G is an isomorphism between G and itself, and $Aut(G)$ denotes the group of automorphisms of G. For a vertex $v \in V(G)$ and a group Γ acting on $V(G)$, such as $Aut(G)$, $\{\varphi(v) : \varphi \in \Gamma\}$ is the ***orbit of*** v ***under*** Γ, and is denoted $\Gamma \cdot v$. Two vertices are ***similar*** if they belong in the same orbit under $Aut(G)$. Similarly, one can define the ***orbit of an edge*** xy ***under*** Γ by the set $\{\varphi(xy) : \varphi \in \Gamma\}$, and denote it $\Gamma \cdot xy$. Two edges are ***similar*** if they belong to the same orbit under $Aut(G)$. The relation of being in the same orbit is an equivalence relation. Therefore, the vertex orbits are a partition of $V(G)$, while the edge orbits are a partition of $E(G)$.

Chapter 2

A survey of reconstruction problems

In this chapter, we do a quick survey of other reconstruction problems, namely vertex reconstruction, edge reconstruction and switching reconstruction. We define the three reconstruction problems, and present the associated conjectures.

The main result shown here is Lemma 2.1.3, due to Kelly [12]. Kelly's Lemma states that every proper subgraph obtained by deleting vertices is vertex reconstructible, and every proper subgraph obtained by deleting edges is edge reconstructible. This very simple result is powerful and allows for the reconstruction of regular graphs, for instance. The principal interest in presenting this result here is that there are no such broad result that are known for the contraction reconstruction problem. Reconstructing subgraphs of a graph is straightforward for vertex reconstruction and edge reconstruction, but not in contraction reconstruction.

2.1 Vertex reconstruction

Vertex reconstruction was the first type of reconstruction to be studied. Even if our focus will be on contraction reconstruction, it is interesting to look at some simple results in vertex reconstruction, so as to appreciate the differences between the two subjects. The question was introduced by Kelly and Ulam in 1941, and was the subject of Kelly's doctoral thesis a year later [12].

The following definition is analogous to Definition 1.1.1.

Definition 2.1.1. *For a graph G, the multiset of unlabeled graphs $\{G-v : v \in V(G)\}$ is the **vertex deck** of G, denoted $\mathcal{V}(G)$. Any graph $H \in \mathcal{V}(G)$ is a **vertex card**. A **vertex reconstruction** of G is a graph F such that there exists a bijection $\phi : V(G) \to V(F)$ where $G - v \cong F - \phi(v)$ for all $v \in V(G)$. A graph G is **vertex reconstructible** if every vertex reconstruction of G is isomorphic to G. A class $\mathcal{G}$*

of graphs is ***vertex recognizable*** *if every vertex reconstruction of any $G \in \mathcal{G}$ is also in $\mathcal{G}$. The class $\mathcal{G}$ is* ***vertex weakly reconstructible*** *if for every $G \in \mathcal{G}$, all vertex reconstructions of G that are in $\mathcal{G}$ are isomorphic to G.*

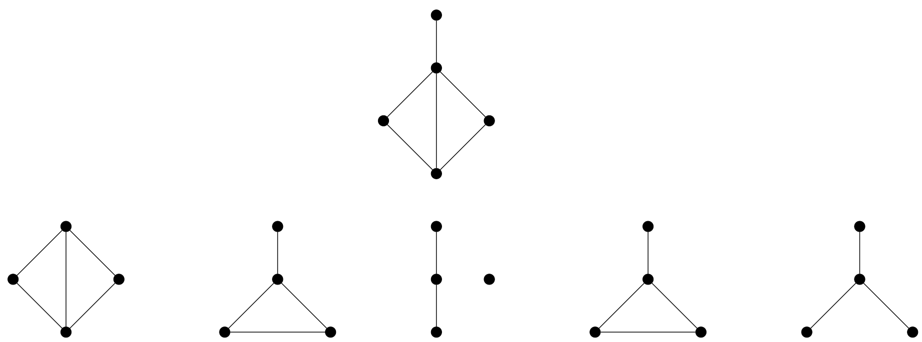

Figure 2.1: In the first row: a graph G. In the second row: the vertex deck of G.

In this section, the terms “cards”, “decks”, “reconstructible”, “reconstruction” and “recognizable” refer to Definition 2.1.1.

Two simple graphs are known not to be reconstructible. They are shown in Figure 2.2. From this example, we can see that one needs at least three cards to reconstruct a graph, as with only two cards, it is impossible to determine whether there is an edge between the two corresponding vertices or not. This brought on the conjecture of Kelly and Ulam. The way conjecture 2.1.2 is stated is due to Harary [9].

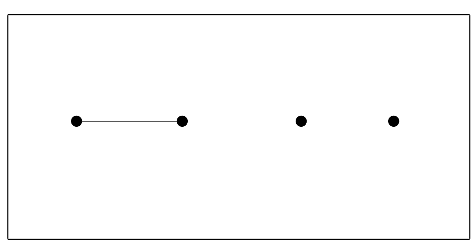

Figure 2.2: Two graphs that are not vertex reconstructible.

Conjecture 2.1.2. *All simple undirected graphs on at least three vertices are reconstructible.*

Many parameters of a graph G can be determined from $\mathcal{V}(G)$. Since a card is obtained by deleting a single vertex, the number of cards is equal to $v(G)$, so one can determine the number of vertices in G. Similarly, one can determine the number of edges in G, except when $v(G) = 2$, as every edge appears in all but two cards. And so,

$$e(G) = \frac{1}{v(G) - 2} \sum_{H \in \mathcal{V}(G)} e(H).$$

Since the right-hand side of the previous equation can be determined from the deck, so too can the left-hand side. It is possible to extend this formula to determine the number of subgraphs of G isomorphic to a certain graph F.

Lemma 2.1.3 (Kelly [13])**.** *Let F be a graph with $v(F) < v(G)$. Then, the number $s(F, G)$ of subgraphs of G isomorphic to F is*

$$s(F, G) = \frac{1}{v(G) - v(F)} \sum_{H \in \mathcal{V}(G)} s(F, H).$$

Kelly's lemma implies that all proper subgraphs of G can be determined from $\mathcal{V}(G)$. In particular, Lemma 2.1.3 can be used to determine the degree sequence of G.

Proposition 2.1.4 (Kelly [13])**.** *The degree sequence of a graph with at least three vertices is reconstructible.*

Proof: For any two graphs F and G, where $v(F) < v(G)$, the number of subgraphs of G that are isomorphic to F and include a given vertex v is $s(F, G) - s(F, G - v)$, which is reconstructible. Using this result with $F = K_2$, we conclude that the number of edges incident to any given vertex v is reconstructible. Therefore, the degree sequence is reconstructible. ∎

A direct consequence is that regular graphs are reconstructible. A graph is ***regular*** if all of its vertices have the same degree.

Corollary 2.1.5 (Kelly [13])**.** *Regular graphs are reconstructible.*

One can reconstruct a regular graph G by considering any card, adding a vertex and joining it to all the vertices with minimal degree. Proposition 2.1.4 assures that regularity is a reconstructible parameter. In other words, it is possible to determine if a graph is regular from its deck.

Some other classes of graphs were also shown to be vertex reconstructible, such as disconnected graphs, trees, maximal planar graphs, and some separable graphs [3].

Unfortunately, there does not seem to be any result analogous to Lemma 2.1.3 for contraction reconstruction. The existence of such a result would cause Conjecture 1.1.2 to be weaker or equivalent to Conjecture 2.1.2, because then $\mathcal{V}(G)$ could be recovered from $\mathcal{C}(G)$. If $\mathcal{V}(G)$ is reconstructible from $\mathcal{C}(G)$, then any vertex reconstructible graph is also contraction reconstructible. The existence of a relation between Conjecture 1.1.2 and Conjecture 2.1.2 is not yet known.

2.2 Edge reconstruction

In this section, a brief survey of edge reconstruction is presented. The edge reconstruction problem stands in the middle between the vertex reconstruction problem and the contraction reconstruction problem. The edge reconstruction problem is similar to the vertex reconstruction problem in the sense that both decks are collections of subgraphs. Indeed, both problems have their versions of Kelly's Lemma. Further connections between the two problems were established by considering the line graph.

On the other hand, a relation between the edge reconstruction problem and the contraction reconstruction problem can be seen in planar graphs, as the deletion of an edge in a planar graph corresponds to the contraction of an edge in the dual.

2.2.1 Definitions and elementary results

Definition 2.2.1 is similar to Definitions 1.1.1 and 2.1.1.

Definition 2.2.1. *For a graph G, the multiset of unlabeled graphs $\{G-e : e \in E(G)\}$ is the* ***edge deck*** *of G, denoted $\mathcal{E}(G)$. Any graph $H \in \mathcal{E}(G)$ is an* ***edge card****. An* ***edge reconstruction*** *of G is a graph F such that there exists a bijection $\phi : E(G) \to E(F)$ where $G - e \cong F - \phi(e)$ for all $e \in E(G)$. A graph G is* ***edge reconstructible*** *if every edge reconstruction of G is isomorphic to G. A class $\mathcal{G}$ of graphs is* ***edge recognizable*** *if every edge reconstruction of any $G \in \mathcal{G}$ is also in $\mathcal{G}$. The class $\mathcal{G}$ is* ***edge weakly reconstructible*** *if for every $G \in \mathcal{G}$, all edge reconstructions of G that are in $\mathcal{G}$ are isomorphic to G.*

Some graphs are not edge reconstructible. Figure 2.3 shows the only known simple counterexamples.

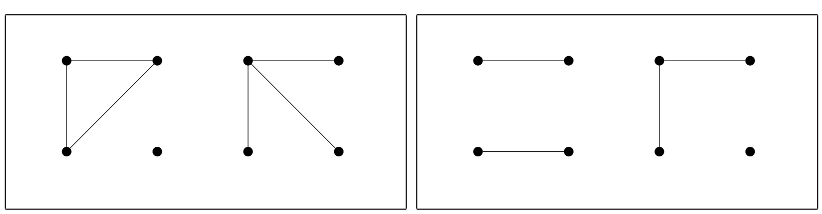

Figure 2.3: In each box are graphs that are edge reconstruction of one another.

All the counterexamples are on three or fewer edges, so Conjecture 2.2.2 is stated.

Conjecture 2.2.2. *All simple undirected graphs with at least four edges are edge reconstructible.*

The intuition that Conjecture 2.2.2 is weaker than Conjecture 2.1.2 was confirmed by Harary and Palmer [10]. The ***line graph*** of G has $E(G)$ as a vertex set, and two vertices are adjacent if their corresponding edges in G have an end in common.

Theorem 2.2.3 (Harary, Palmer [10])**.** *A graph is edge reconstructible if its line graph is vertex reconstructible.*

Lemma 2.2.4 is similar to Lemma 2.1.3, allowing us to determine most of the subgraphs of a graph by looking at its deck.

Lemma 2.2.4 (Kelly)**.** *Let F and G be graphs with $e(F) < e(G)$. Then, the number $s(F, G)$ of subgraphs of G isomorphic to F is*

$$s(F, G) = \frac{1}{e(G) - e(F)} \sum_{H \in \mathcal{E}(G)} s(F, H).$$

Another result similar to Lemma 2.2.4 is the counting theorem. This allows, among other things, to show that the edge reconstruction conjecture is weaker than the vertex reconstruction conjecture. A few definitions are needed: let $\mathcal{F}$ be a class of graphs and G a graph. Then, a subgraph of G is an ***$\mathcal{F}$-subgraph*** if it is isomorphic to a graph in $\mathcal{F}$. It is a ***maximal $\mathcal{F}$-subgraph*** if it is not contained in another $\mathcal{F}$-subgraph. An (F, G)***-chain of length*** n is a sequence $(X_0, ..., X_n)$ of $\mathcal{F}$-subgraphs of G such that $F \cong X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n \subsetneq G$. Two (F, G)-chains are ***isomorphic*** if they have the same length and their corresponding terms are isomorphic graphs. The ***rank*** of F in G is the length of the longest (F, G)-chain.

Theorem 2.2.5 (Kelly [3])**.** *Let $\mathcal{G}$ be an edge-recognizable class of graphs, and let $\mathcal{F}$ be any class of graphs such that, for every $G \in \mathcal{G}$, each $\mathcal{F}$-subgraph of G is*

1. *edge proper;*
2. *contained in a unique maximal $\mathcal{F}$-subgraph of G*

Then, for every $F \in \mathcal{F}$ and every $G \in \mathcal{G}$, the number $m(F, G)$ of maximal $\mathcal{F}$-subgraphs of G isomorphic to F is reconstructible.

Theorem 2.2.5 can be used to prove that vertex reconstruction implies edge reconstruction when a graph has no isolated vertices. Indeed, suppose G has no isolated vertices. Let $\mathcal{G}$ be the class of all edge reconstructions of G, which is a recognizable class of graphs, and let $\mathcal{F}$ be the class of all graphs with $v(G) - 1$ vertices. Since edge reconstructions of G have no isolated vertices, every $\mathcal{F}$-subgraph of G is edge proper and Theorem 2.2.5 applies. The maximal $\mathcal{F}$-subgraphs of G are exactly the cards in $\mathcal{V}(G)$. Since one can determine the cards of $\mathcal{V}(G)$ from $\mathcal{E}(G)$, this implies that $\mathcal{V}(G)$ is edge reconstructible, hence the class of graphs with no isolated vertices is weakly edge reconstructible if it is weakly vertex reconstructible.

Many classes of graphs were shown to be edge reconstructible; a list can be found in [20]. Regular graphs, cycle graphs and cacti, to name a few, are edge reconstructible. Furthermore, many parameters are edge reconstructible. The number of vertices, the degree sequence, the vertex and edge connectivity, the chromatic number and whether the graph is planar are all edge reconstructible.

2.2.2 Almost all graphs are edge reconstructible

Graphs with many edges have a large deck, and, broadly speaking, the deck contains more information. It is therefore reasonable to think that graphs with more edges are easily reconstructible. This intuition is confirmed by Theorem 2.2.6, found in [24]. It states that graphs with "enough" edges are edge reconstructible. The proof assumes a counterexample to conjecture 2.2.2 by taking two graphs G and H with isomorphic decks, and counts in a clever way the number of isomorphisms from G to H.

Theorem 2.2.6 (Müller [24])**.** *If G is a simple graph such that*

$$e(G) > 1 + v(G)\left(\log_2(v(G)) - 1\right),$$

then G is edge reconstructible.

Proof: For a graph G, let $\bar{\mathcal{E}}(G)$ be the multiset of graphs obtained from G by a sequence of edge deletions. Let $H \not\cong G$ such that $\bar{\mathcal{E}}(H) = \bar{\mathcal{E}}(G)$. For any $F \in \bar{\mathcal{E}}(G)$, define $P(F)$ to be the set of all bijections $g : V(G) \to V(H)$ such that $g(x)$ is adjacent to $g(y)$ in H if $xy \in E(F)$. In other words, $g \in P(F)$ if g maps the set of edges of F to edges of H. Similarly, for any $F \in \bar{\mathcal{E}}(H)$, define $Q(F)$ to be the set of all bijections $h : V(H) \to V(H)$ such that $h(e) \in E(H)$ for every $e \in E(F)$. For any bijection $g : V(G) \to V(H)$, let $r_g = \max\{e(F) \mid g \in P(F),\ F \in \bar{\mathcal{E}}(G)\}$. That is, r_g is equal to the number of edges of G that are mapped to edges of H by the bijection g. For any $r \geq 0$, let $[G, H]_r$ denote the number of bijections g from $V(G)$ to $V(H)$ with $r_g = r$. The number $[G, H]_r$ counts the number of bijections from $V(G)$ to $V(H)$ that maps exactly r edges of G to edges of H. By the inclusion-exclusion principle, that number is given by

$$[G, H]_r = \sum_{i=r}^{e(G)} (-1)^{i-r} \binom{i}{r} \sum_{\substack{F \in \bar{\mathcal{E}}(G) \\ e(F)=i}} |P(F)|.$$

Replacing G by H, one obtains

$$[H,H]_r = \sum_{i=r}^{e(G)} (-1)^{i-r} \binom{i}{r} \sum_{\substack{F\in\bar{\mathcal{E}}(H)\\ e(F)=i}} |Q(F)|.$$

Since $\bar{\mathcal{E}}(G) = \bar{\mathcal{E}}(H)$, the terms corresponding to $r \le i \le e(G)-1$ are equal in the two expressions. Note $|P(G)| = 0$, since it is equal to the number of isomorphisms from G to H. Note also that $Q(H) = Aut(H)$. We then get that

$$[H,H]_r - [G,H]_r = (-1)^{e(G)-r} \binom{e(G)}{r} |Aut(H)|.$$

Taking absolute values and summing over all possible values of r, one obtains

$$\sum_{r=0}^{e(G)} |[H,H]_r - [G,H]_r| = 2^{e(G)} |Aut(H)|.$$

And so,

$$2^{e(G)} \le 2^{e(G)} |Aut(H)| = \sum_{r=0}^{e(G)} |[H,H]_r - [G,H]_r| \le 2(v(G)!) \le 2\left(\frac{v(G)}{2}\right)^{v(G)}$$

Taking logarithms, one obtains $e(G) \le 1 + v(G)\left(\log_2(v(G)) - 1\right)$. Therefore, if a counterexample G to Conjecture 2.2.2 exists, it must satisfy

$$e(G) \le 1 + v(G)\left(\log_2(v(G)) - 1\right).$$

The previous inequality implies the theorem. ∎

A random graph will almost always have $e(G) > 1 + v(G)\left(\log_2(v(G)) - 1\right)$, so almost all graphs are edge reconstructible. Results and definitions about random graphs are presented in Chapter 8.

2.2.3 Pseudo-similarity

Recall that two edges a and b of a graph G are similar if there exists an automorphism of G that maps the ends of a to the ends of b. If two edges of a graph G are similar, then clearly their corresponding cards will be isomorphic. The converse, however, does not hold. There are graphs with edges $a, b \in E(G)$, for which $G - a \cong G - b$, but a is not similar to b in G. Such edges are called ***edge pseudo-similar***.

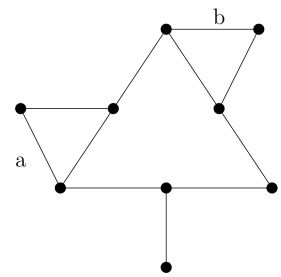

Figure 2.4: The edges a and b are pseudo-similar.

A method for creating graphs with pseudo-similar edges is explained in [8]. Let H be a graph and $\theta \in Aut(H)$. Let $a \in E(H)$ and $\theta^{\ell}(a) = b$. Then, in the graph $G = H - \{\theta(a), \theta^2(a), ..., \theta^{\ell-1}(a)\}$, the edges a and b are either similar or pseudo-similar. Exact conditions for a and b to be pseudo-similar, rather than similar in G, are not known. However, every graph G with pseudo-similar edges is obtained by the preceding algorithm.

Proposition 2.2.7 (Godsil and Kocay [8])**.** *Let G be a finite simple graph with pseudo-similar edges a and b. Then there is a graph H with the following properties:*

1. *G is a subgraph of H.*
2. *There is an automorphism θ of H which maps $G - a$ to $G - b$.*
3. *There is an integer $\ell \geq 1$ such that $\theta^{\ell}(a) = b$ and if $e \in E(H) - E(G)$, then $e = \theta^k(a)$ for some $k > \ell$.*

Proof: Let θ be an isomorphism from $G - b$ to $G - a$. The isomorphism θ can be seen as a permutation of $V(G)$ and therefore can be composed with itself. Denote by S the set of edges $\{\theta^k(a) : k \geq 0\}$. Notice that $b \in S$, since otherwise there is an integer $k \geq 1$ such that $\theta^k(a) = a$, implying that $\theta\left(\theta^{k-1}(a)\right) = a$, which contradicts the fact that θ maps onto $G - a$. We now claim that the edges in $E(G) - S$ are mapped onto themselves by θ. Indeed, suppose $e \in E(G) - S$. Since $G - S \subseteq G - b$, then $\theta(e) \in G - a$. Hence if $\theta(e) \in S$, then $\theta(e) = \theta^k(a)$ for some $k \geq 1$. But since θ is one-to-one, if follows that $e = \theta^{k-1}(a)$, hence $e \in S$, a contradiction. The claim follows. Suppose $\theta^{\ell}(a) = b$, and let $e_i = \theta^i(a)$, so that $e_0 = a$, and $e_{\ell} = b$. Let u_i and v_i be the ends of the edge e_i.

Since θ is an isomorphism between two graphs with the same vertex set, θ is a permutation of $V(G)$. Hence, there is an integer m such that $\{\theta^{m+1}(u_0), \theta^{m+1}(v_0)\} = \{u_0, v_0\}$ as sets. Clearly, $m > \ell$, since $\{\theta^k(u_0), \theta^k(v_0)\} = \{u_k, v_k\}$ for any $k \leq \ell$, and $\{u_k, v_k\} \neq \{u_0, v_0\}$, for otherwise $e_k = e_0$, a contradiction. Then the graph H is obtained by adding the edges $\theta^k(u_0)\theta^k(v_0)$ to G, for every $\ell < k \leq m$.

The graph G is a subgraph of H, and $V(G) = V(H)$. Therefore, the isomorphism θ can then be seen as a bijection on the graph H. One then needs to check that θ is an automorphism of H, by verifying that θ maps the edge set of H onto itself. Any edge in $G - b$ is mapped to an edge in H. Any edge in $E(H) - E(G)$ is of the form $\theta^k(u_0)\theta^k(v_0)$, which is mapped to $\theta^{k+1}(u_0)\theta^{k+1}(v_0)$, which is an edge by construction. This verifies all the stated properties of H. ∎

We will see in Chapter 7 that the analogous proposition for contraction pseudo-similar edges is not as straightforward as Proposition 2.2.7.

2.3 Switching reconstruction

The switching reconstruction problem asks whether a simple graph is determined by its switching deck. A card in the switching deck is obtained by swapping the set of neighbours and non-neighbours of a vertex. Unlike the cards in the vertex deck or the edge deck, a card in the switching deck of a graph G is not a subgraph of G. See [25] for an in-depth look at the results on switching reconstruction, and see [27] for the principal result on the switching conjecture.

Definition 2.3.1. *A graph is the* ***switch card*** *of a simple graph G with respect to the vertex v of G, denoted by $G \circledast v$, if $V(G \circledast v) = V(G)$, $E(G \circledast v) = E(G) - \{uv : uv \in E(G) \cup \{uv : uv \notin E(G)\}$.*

Using this definition, the switching deck is built, and the switching reconstruction conjecture, similar to the other reconstruction conjectures, asks if graphs are determined by their switching decks.

Definition 2.3.2. *For a simple graph G, the multiset of unlabeled graphs $\{G \circledast v : v \in V(G)\}$ is the* ***switching deck*** *of G, denoted $\mathcal{S}(G)$. Any graph $H \in \mathcal{S}(G)$ is a* ***switch card****. A* ***switching reconstruction*** *of G is a graph F such that there exists a bijection $\phi : V(G) \to V(F)$ where $G \circledast v \cong F \circledast \phi(v)$ for all $v \in V(G)$. A graph G is* ***switching reconstructible*** *if every switching reconstruction of G is isomorphic to G. A class $\mathcal{G}$ of graphs is* ***switching recognizable*** *if every switching reconstruction of any $G \in \mathcal{G}$ is also in $\mathcal{G}$. The class $\mathcal{G}$ is* ***switching weakly reconstructible*** *if for every $G \in \mathcal{G}$, all switching reconstructions of G that are in $\mathcal{G}$ are isomorphic to G.*

Some graphs are not switching reconstructible. See Figure 2.5 for the full list of known graphs that are not switching reconstructible.

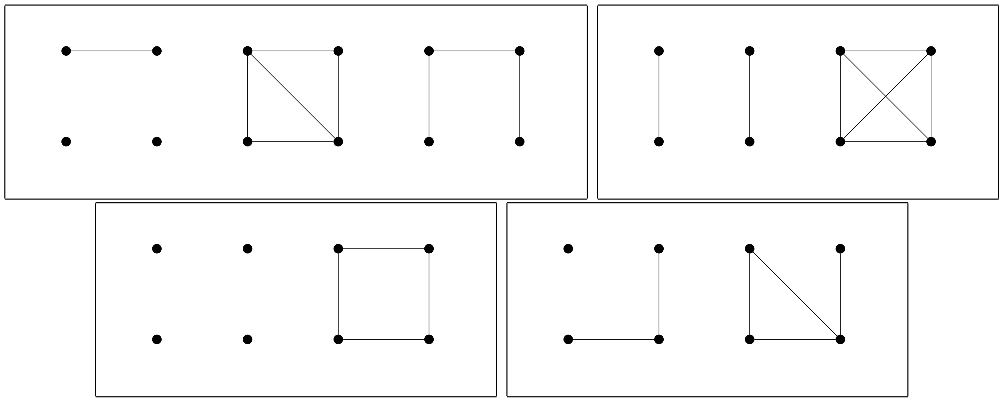

Figure 2.5: In each box are graphs that are switching reconstructions of one another.

Every graph that is known not to be switching reconstructible has four vertices. The following conjecture may then be stated.

Conjecture 2.3.3. *All simple undirected graphs with at least five vertices are switching reconstructible.*

Much work as been done on the switching conjecture. Perhaps the greatest advance on the conjecture is the following theorem.

Theorem 2.3.4 (Stanley [27])**.** *Let G be a simple graph such that $v(G) \not\equiv 0 \mod 4$. Then G is switching reconstructible.*

The problem remains open when $v(G) = 4m \geq 8$. Some progress has been made on the subject.

Theorem 2.3.5 (Niesink [25])**.** *Let G be a simple graph with $v(G) \equiv 0 \mod 4$, $v(G) \neq 4$. If G is not switching reconstructible, then*

$$\frac{1}{8}v(G)^2 + \frac{1}{4}v(G) - 2 \leq e(G) \leq \binom{v(G)}{2} - \frac{1}{8}v(G)^2 - \frac{1}{4}v(G) + 2.$$

In other words, graphs with very few edges and graphs with very few non-edges are switching reconstructible.

Chapter 3

Preliminary results on contraction reconstruction

In this chapter, we reconstruct some parameters of graphs, such as the number of spanning trees and the number of cliques of various sizes. From now on, the words "deck", "card", "reconstructible" and "recognizable" refer to the operation of contraction. See Definition 1.1.1 for details.

3.1 Elementary results

We start with a few simple propositions. Although these propositions are straightforward, they are used very extensively throughout this thesis.

Proposition 3.1.1. *The number of edges is reconstructible.*

Proof: The number of edges in a graph G is equal to the number of cards in its deck $\mathcal{C}(G)$. ∎

As seen in Figure 1.2, some small graphs are not reconstructible. In order to show that some property of a graph is reconstructible, it is best to avoid these small graphs by considering only graphs with four or more edges. Proposition 3.1.1 is therefore necessary.

Proposition 3.1.2. *Simple graphs with at least two edges are recognizable.*

Proof: If the graph G is not simple, then it has parallel edges or a loop. In both cases, one of the cards will contain a loop, unless G is a single loop. If the graph is simple, none of its cards will have a loop. Therefore, G is simple if and only if none of its cards have a loop. ∎

We now give conditions for the number of vertices to be reconstructible. The fourth pair of graphs in Figure 1.2 are reconstructions of another, and the two graphs do not have the same number of vertices, which shows that $v(G)$ is not always reconstructible. However, those two graphs are the only ones where the number of vertices is not reconstructible.

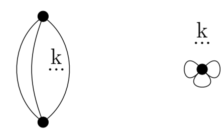

Figure 3.1: Two graphs whose number of vertices are not reconstructible. Adding isolated vertices to these two graphs yields two graphs whose number of vertices is still not reconstructible.

Proposition 3.1.3. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then $v(G)$ is reconstructible. In particular, if G is simple, then $v(G)$ is reconstructible.*

Proof: The contraction of a loop does not reduce the number of vertices, while the contraction of a link reduces the number of vertices by one. Therefore, if a deck contains two cards with different numbers of vertices, then the greater of those two numbers is equal to $v(G)$.

If every card of a deck has the same number of vertices, then either every edge of G is a loop, in which case $v(G) = v(H)$ for any $H \in \mathcal{C}(G)$, or every edge of G is a link, in which case $v(G) = v(H) + 1$ for any $H \in \mathcal{C}(G)$. Assume that every edge of G is a link or every edge of G is a loop. If a card contains a link, then every edge of G is a link, and $v(G) = v(H) + 1$ for any $H \in \mathcal{C}(G)$. We can assume that every edge of every card is a loop. If a card has more than one vertex with loops, then the decontraction of any vertex yields a graph with loops, hence every edge of G is a loop, which implies that $v(G) = v(H)$ for any $H \in \mathcal{C}(G)$. Therefore, every loop is attached to the same vertex in every card, which corresponds to the forbidden deck. ∎

The condition that G is not a graph composed of a graph in Figure 3.1 and isolated vertices can be verified in $\mathcal{C}(G)$. It is equivalent to the condition that the deck of G is not k copies of a graph with some number of isolated vertices and $k-1$ loops attached to one of those vertices for some $k \geq 1$.

Corollary 3.1.4. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then the number of loops is reconstructible.*

Proof: The number of loops in G is equal to the number of cards $H \in \mathcal{C}(G)$ such that $v(H) = v(G)$. By Proposition 3.1.3, $v(G)$ is reconstructible, and so one can reconstruct the number of cards of G with $v(G)$ vertices. ∎

A graph is ***disconnected*** if its vertices can be partitionned into two sets X and Y such that no edge has one end in X and one end in Y. Otherwise, the graph is ***connected***.

Proposition 3.1.5. *Connected graphs with at least one edge are recognizable.*

Proof: If a graph G is connected, then certainly G/e is connected for every $e \in E(G)$. Conversely, if G is disconnected, then there are two vertices $x, y \in V(G)$ such that there are no xy-paths in G. Therefore, there are certainly no xy-paths in G/e for any $e \in E(G)$. Assuming the deck is not empty, we conclude that G is connected if and only if every card of G is connected. ∎

For Proposition 3.1.5 to be true, the condition that G has at least one edge is necessary, since E_1 and E_r are reconstructions of one another for any $r \geq 2$, yet E_1 is connected, although trivially, while E_r is not.

For the following proposition, we define a ***cycle*** of length $k > 1$ of a graph G to be a subgraph whose vertex set are the distinct vertices $v_1, v_2, ..., v_k$ and whose edge set are the distinct edges $e_1, e_2, ..., e_k$, such that e_i is incident to v_i and v_{i+1} for $i = 1, 2, ..., k-1$ and e_k is incident to v_k and v_1. We often write that $v_1 v_2 ... v_k v_1$ is a cycle. A cycle of length 1 is a subgraph composed of one vertex and a loop. Note that while cycles are commonly defined to have a length of at least 3, in this thesis, we consider loops and pairs of parallel edges to be cycles. A ***forest*** is a graph with no cycles. A connected forest is a ***tree***.

Proposition 3.1.6. *Graphs with a cycle and at least two edges are recognizable.*

Proof: Suppose C is a cycle of a graph G. If $e \notin C$, then the edges of C either are a cycle in G/e if at most one end of e is incident to the vertices of C, or they contain two cycles if both ends of e are incident to the vertices of C. If $e \in C$, then C/e is a cycle of G/e unless C is a loop. Since G has at least two edges, there is a card of G with a cycle. Therefore, if G has a cycle, then so does one of its cards. Conversely, if G has no cycles, then G is a forest, and so are all of its cards. We conclude that G has a cycle if and only if one of its cards has a cycle. ∎

Propositions 3.1.5 and 3.1.6 show that forests are a recognizable class of graphs. If we restrict ourselves to loopless graphs, we can conclude that G has a cycle if and only if all of its cards have a cycle.

Proposition 3.1.7. *Graphs that are not in Figure 1.2 with a vertex of degree* 1 *are recognizable.*

Proof: Suppose first that G is loopless, which is recognizable by Corollary 3.1.4. If G has a vertex v of degree 1, then since $e(G) \geq 2$, there is an edge e that is not incident to v. Then, v is a vertex of degree 1 in G/e. Conversely, let H be a card of G with a vertex v of degree 1, and let v^* be the contracted vertex in H. If $v^* \neq v$, then v is a vertex of degree 1 in G. If $v^* = v$, then the decontraction of v^* yields two vertices v_1 and v_2 where $\deg(v_1) + \deg(v_2) - 2 = \deg(v) = 1$. This implies that one of v_1 or v_2 has a degree of 1. We therefore conclude that G has a vertex of degree 1 if and only if one of its cards has a vertex of degree 1.

Suppose then that G has at least one loop. Let $\mathcal{L}(G) \subset \mathcal{C}(G)$ be the set of cards corresponding to the contraction of loops. That set is reconstructible, since a card H is in $\mathcal{L}(G)$ if and only if $v(H) = v(G)$. To reconstruct G from such a card, one needs to add a loop to a vertex. If there exists a card $H \in \mathcal{L}(G)$ with at least two vertices of degree 1, then adding a loop to H will yield a graph with at least one vertex of degree 1, so G has such a vertex. If there exists a card $H \in \mathcal{L}(G)$ with no vertices of degree 1, then G does not have any vertices of degree 1. We may henceforth assume that every card of $\mathcal{L}(G)$ has exactly one vertex of degree 1.

Let $H \in \mathcal{L}(G)$ and $x \in V(H)$ be the vertex of degree 1. If there is a link e in G that is not adjacent to x, then G has a vertex of degree 1 if and only if G/e does, and the Proposition is proved for that case. If no such link exists, then G must be composed of some number of isolated vertices, and two vertices x and y linked by one edge, with $\ell(x)$ and $\ell(y)$ loops attached to x and y respectively. Without loss of generality, assume $\ell(x) \leq \ell(y)$. If there is a card in $\mathcal{L}(G)$ with a vertex with $k \geq 2$ loops, then we conclude that $\ell(x) = 0$ and $\ell(y) = k + 1$, or $\ell(x) = 1$ and $\ell(y) = k$. The latter is impossible since such a graph contains cards in $\mathcal{L}(G)$ with no vertices of degree 1. The proof is done in that case. Otherwise, we conclude that $\ell(x) = 1$ and $\ell(y) = 1$ or that $\ell(x) = 0$ and $\ell(y) = 2$ in G, which corresponds to graphs in Figure 1.2. ∎

Proposition 3.1.8. *Graphs that are not in Figure 1.2 with no vertices of degree* 1 *and a vertex of degree* 2 *are recognizable.*

Proof: By Proposition 3.1.7, one can recognize graphs that do not have a vertex of degree 1. Let G be such a graph.

Suppose first that G is loopless, which is recognizable by Corollary 3.1.4. If G has a vertex v of degree 2, then since $e(G) \geq 3$, there is an edge e that is not incident to v. Then v is a vertex of degree 2 in G/e. Conversely, let H be a card of G with a vertex v of degree 2, and let v^* be the contracted vertex in H. If $v^* \neq v$, then v is a vertex of degree 2 in G. If $v^* = v$, then the decontraction of v^* yields two vertices v_1 and v_2 where $\deg(v_1) + \deg(v_2) - 2 = \deg(v) = 2$. Since G has no vertices of degree 1, this implies that both v_1 and v_2 have degree 2. We therefore conclude that a graph G that has no vertices of degree 1, and at least three edges, has a vertex of degree 2 if and only if one of its cards has a vertex of degree 2.

If G has a loop, then let $\mathcal{L}(G) \subset \mathcal{C}(G)$ be the set of cards corresponding to the contraction of loops. Similarly to the proof of Proposition 3.1.7, if there exists a card $H \in \mathcal{L}(G)$ with at least two vertices of degree 2, then G must have a vertex of degree 2. If there exists a card of $\mathcal{L}(G)$ with no vertices of degree 2, then G does not have vertices of degree 2. We may henceforth assume that every card of $\mathcal{L}(G)$ has exactly one vertex of degree 2.

Let $H \in \mathcal{L}(G)$ and $x \in V(H)$ be the vertex of degree 2. If there is a link e in G that is not adjacent to x, then G has a vertex of degree 2 if and only if G/e does, and the Proposition is proved for that case. Otherwise, G is composed of two vertices, x and y, linked by two edges and with $\ell(x)$ and $\ell(y)$ loops attached to x and y, respectively. The graph G may also have isolated vertices. We may assume that $\ell(x) \leq \ell(y)$. If there is a card in $\mathcal{L}(G)$ with a vertex with $k \geq 2$ loops, then we conclude that $\ell(x) = 0$ and $\ell(y) = k + 1$, or $\ell(x) = 1$ and $\ell(y) = k$. The latter is impossible since such a graph contains cards in $\mathcal{L}(G)$ with no vertices of degree 2. The proof is done in that case. Otherwise, we conclude that $\ell(x) = 1$ and $\ell(y) = 1$ or that $\ell(x) = 0$ and $\ell(y) = 2$ in G, which corresponds to graphs in Figure 1.2. ∎

3.2 Contraction version of Kelly's Lemma

Kelly's Lemmas 2.1.3 and 2.2.4 are useful for the vertex and edge reconstruction problems. Whether one can reconstruct every proper subgraph using the contraction deck is not yet known, and such a result would constitute great progress towards the contraction reconstruction of many classes of graphs. While such a result is out of reach, the proof of Lemmas 2.1.3 and 2.2.4 can be altered in the following way.

Proposition 3.2.1. *Let $c(G, F)$ be the number of contractions of G isomorphic to F, then, for any F such that $e(F) < e(G)$:*

$$c(G, F) = \frac{1}{e(G) - e(F)} \sum_{H \in \mathcal{C}(G)} c(H, F).$$

In particular, $c(G, F)$ is reconstructible if $e(F) < e(G)$.

Proof: Let F_0 be a specific contraction of G isomorphic to F, and let $e_1, .., e_n$ correspond to the edges that need to be contracted in G to obtain F_0, with $n = e(G) - e(F_0)$. Then n cards will correspond to a contraction of one of those edges, in which case F_0 will be a contraction of any of those cards. For any other card, F_0 will not be a contraction of it. This yields the equation in the lemma, and since the right-hand side is reconstructible, so is the left-hand side. ∎

In other words, the number $c(F, G)$ counts the number of subsets of edges $E \subseteq E(G)$ such that $G/(E(G) - E)$ is isomorphic to F.

Corollary 3.2.2. *For any two graphs F and G such that $e(F) < e(G)$, the number of contractions of G which are isomorphic to F, and include a given edge e, is reconstructible.*

Proof: This number is $c(F, G) - c(F, G/e)$. ∎

By changing the concept of subgraph to contractions, we see that Proposition 3.2.1 is the exact analog of Lemma 2.1.3 and 2.2.4. However, very few uses for Proposition 3.2.1 and its corollary are known.

The number of connected components of a graph can be reconstructed using Proposition 3.2.1.

Proposition 3.2.3. *Let G be a graph with $e(G) \geq 1$. The number of connected components of G is reconstructible.*

Proof: Let E_i be the empty graph on i vertices, for any integer i. Since $e(G) \geq 1$ and $e(E_i) = 0$, Proposition 3.2.1 can be applied with $F = E_i$. Then, G has i connected components if and only if $c(G, E_i) \neq 0$. ∎

It should be noted that the number of connected components is the same for all cards, and is equal to the number of connected components of the graph.

A ***spanning tree*** of a graph G is a subgraph of G on $v(G)$ vertices, and is a tree. A ***spanning forest*** of a graph G is a subgraph of G on $v(G)$ vertices that is a forest. A spanning tree is considered a spanning forest, and furthermore deleting any number of edges from a spanning tree yields a spanning forest. We show that the number of spanning trees is reconstructible.

Proposition 3.2.4. *Let G be a graph not in Figure 3.1 with $v(G) > 1$. Then the number of spanning trees is reconstructible.*

Proof: We count the sets of edges A where $E(G)-A$ is a spanning tree, effectively counting the number of spanning trees. Such a set of edges has the property that $G/(E(G)-A)$ is a single vertex with $|A|$ loops, since $G/(E(G)-A)$ is obtained from G by contracting a spanning tree. Conversely, if a set of edges has been contracted from G to obtain a graph composed of a single vertex with $|A|$ loops, then those edges must have formed a spanning tree of G. Therefore, applying Proposition 3.2.1 with F being a graph composed of a single vertex and $e(G)-v(G)+1$ loops, we get that the number of spanning tree is reconstructible, provided that $v(G)$ is reconstructible. By Proposition 3.1.3, $v(G)$ is indeed reconstructible if G is not in Figure 3.1. ∎

Proposition 3.2.1 can also be used to show that the edge connectivity κ' of a graph is reconstructible by setting F to be equal to a graph with two vertices, joined by κ' parallel edges. Further results on edge connectivity and reconstruction can be found in Chapter 4.

3.3 Cliques and independent sets

A ***clique*** of a graph is a set of pairwise adjacent vertices. An ***independent set*** of a graph is a set of vertices, no two of which are adjacent. The ***clique number*** $\omega(G)$ of a graph G is the size of its largest clique and the ***independence number*** $\alpha(G)$ of a graph G is the size of its largest independent set. This section is dedicated to the reconstruction of those two parameters.

Denote by $\mathcal{C}^{(i)}(G) = \{G/A : A \subseteq E(G), |A| = i\}$ the ***i-deck of*** G. Note that for any $i \geq 1$, $\mathcal{C}^{(i)}(G)$ is reconstructible, as any card in $\mathcal{C}^{(i)}(G)$ appears i times in $\bigcup_{H\in\mathcal{C}(G)} \mathcal{C}^{(i-1)}(H)$, which is a reconstructible multiset. The notion of i-deck is helpful in reconstructing certain subgraphs. For instance, a clique of size k is the only graph whose $(k-1)$-deck contains a card composed of a single vertex with $\binom{k}{2}-(k-1)$ loops, and this observation helps reconstruct the number of cliques.

Proposition 3.3.1. *For a simple graph G, the number of cliques of size k in G is reconstructible for any positive integer k.*

Proof: If $k = 1$ or 2, then we are counting the number of vertices and edges respectively, which were already shown to be reconstructible by Propositions 3.1.1 and 3.1.3. A clique of size k is the only graph whose $(k-1)$-deck contains a card composed of a single vertex with $\binom{k}{2}-(k-1)$ loops. Each clique in G will appear as such in $\mathcal{C}^{(k-1)}(G)$ a number of times equal to the number of spanning trees in a clique of size k, which is k^{k-2}. Conversely, if a vertex v^* in $\mathcal{C}^{(k-1)}(G)$ has $\binom{k}{2}-(k-1)$ loops, then v^* must be decontracted at most $k-1$ times, since there $k-1$ contractions have occurred. The edges created by those decontractions form a tree and the $\binom{k}{2}-(k-1)$

loops are edges linking different vertices of that tree. In order for G to be simple, we conclude that there are k vertices in that tree, and they are all pairwise adjacent.

Therefore, the number of cliques of size k is equal to the number of subgraphs isomorphic to a single vertex with $\binom{k}{2} - (k-1)$ loops found throughout $\mathcal{C}^{(k-1)}(G)$, divided by k^{k-2}. ∎

Proposition 3.3.1 implies that $\omega(G)$ is reconstructible. For $\alpha(G)$, however, the problem is a little less straightforward, but nonetheless a partial result is obtained. It is shown, in the case of non-bipartite graphs, that not all contractions can decrease the independence number by one. In this case, the greatest independence number among the cards corresponds to the independence number of the graph. A graph G is ***bipartite*** if its vertices can be partitioned into two sets A and B, such that every edge has one end in A and one end in B.

Proposition 3.3.2. *For any simple graph G, $\alpha(G) = \alpha(H)$ or $\alpha(G) = \alpha(H) + 1$ for every $H \in \mathcal{C}(G)$. Furthermore, if G is not bipartite, then $\alpha(G)$ is reconstructible.*

Proof: Let $I = \{v_1, ..., v_k\}$ be an independent set in a card H. If the contracted vertex is not in I, then decontracting does not create adjacencies between the vertices of I. If the contracted vertex is in I, say v_1, decontracting that vertex yields two vertices, none of which are adjacent to the rest of I. This therefore implies $\alpha(H) \leq \alpha(G)$, for every card $H \in \mathcal{C}(G)$.

In the other direction, let I be an independent set in G. By contracting an edge not incident to any vertices of I, the set I remains independent. By contracting an edge incident to a vertex $v \in I$, there are two possible outcomes: the first is that I remains independent. The other outcome is that I is no longer an independent set: in that situation, $I - v$ is still independent, as the contracted edge was not incident to any vertices of $I - v$. This therefore shows that $\alpha(G)$ is either equal to $\alpha(H)$ or $\alpha(H) + 1$.

If two cards of $\mathcal{C}(G)$ have different independence numbers, then the greater of the two corresponds to $\alpha(G)$. However, if every card of $\mathcal{C}(G)$ have the same independence number, then that number is either $\alpha(G)$ or $\alpha(G) - 1$. If $\alpha(G/xy) = \alpha(G) - 1$ for every $xy \in E(G)$, then for every maximum independent set I of G, x or y is in I, and the other vertex, y or x, is adjacent to another vertex of I. Hence for every maximum independent set I of G, every edge of G is in a path of length two from one vertex of I to another vertex of I. One then sees that $G - I$ is also independent, thus G is bipartite. Therefore, it is impossible for a non-bipartite graph G that every contraction decreases the independence number, hence the maximum independent number found among the cards of a non-bipartite graph G is equal to the independence number of G. ∎

A consequence of this proof is that every graph H appears in the deck of another graph G which either is bipartite or it has the same independence number as H.

As shown in Figure 1.2, reconstructing $\alpha(G)$ is not always possible. While $\alpha(G)$ was not shown to be reconstructible for bipartite graphs, Theorem 5.2.5 will show that bipartite graphs with at least four edges are reconstructible, which implies that their independence number is reconstructible.

Chapter 4

Connectivity

In this chapter, we study the relation between the connectivity of a graph and contraction reconstruction. We find that graphs with a low connectivity can be decomposed into small parts, be it the connected components of a disconnected graph or the blocks of a separable graph. Such a decomposition is very useful for the reconstruction of graphs, as it allows us to narrow down the position of the contracted vertex in a card to a single part.

We first study the reconstructibility of the vertex connectivity. We then apply those results to recognize classes of graphs, such as disconnected and separable graphs. We then show that most disconnected graphs and most separable graphs are reconstructible. We also study the relation between the edge connectivity and contraction reconstruction. At the end of this chapter, we use results on the edge connectivity to reconstruct the Tutte polynomial.

4.1 Vertex connectivity

The goal of this section is to first define the vertex connectivity of a graph in terms of vertex cuts, and secondly to recognize some connectivity properties. In particular, it is shown that one can recognize graphs with a connectivity of 1 and 2, and also 3-connected graphs. Those results are applied later to reconstruct separable graphs.

4.1.1 Definitions and elementary results

The ***local connectivity*** between two distinct vertices x and y of a graph G is the maximum number of pairwise internally disjoint xy-paths, usually denoted by $p(x, y)$. The local connectivity is not defined when $x = y$. A non-trivial graph is k-***connected*** if $p(x, y) \geq k$ for every pair of vertices x and y. The ***connectivity*** of G, denoted $\kappa(G)$, is the maximum value of k such that G is k-connected. Note that connected graphs are those with $\kappa(G) \geq 1$ and disconnected graphs are those with $\kappa(G) = 0$.

An xy-***vertex-cut*** is a subset S of $V(G) - \{x, y\}$ such that x and y are in different connected components of $G - S$. The size of a minimum xy-vertex-cut is denoted by $c(x, y)$. This function is not defined if $x = y$ nor if x and y are adjacent. A well known theorem of Menger [4] relates the functions $p(x, y)$ and $c(x, y)$.

Theorem 4.1.1 (Menger, [4]). *Let G be a graph and $x, y \in V(G)$ be two distinct, nonadjacent vertices. The maximum number of pairwise internally disjoint xy-paths is equal to the minimum number of vertices in an xy-vertex-cut, that is,*

$$p(x, y) = c(x, y).$$

The connectivity $\kappa(G)$ of a graph G is equal to $\min\{p(x, y) \mid x, y \in V(G),\ x \neq y\}$. However, since $c(x, y)$ is only defined on pairs of nonadjacent and distinct vertices, one cannot conclude from Menger's theorem alone that $\kappa(G)$ is equal to the size of a minimum xy-vertex-cut of G. A theorem of Whitney [4] states that the minimum $p(x, y)$ taken over all pairs of distinct vertices is equal to the minimum $p(x, y)$ taken over all pairs of distinct nonadjacent vertices.

Theorem 4.1.2 (Whitney, [4]). *If G has at least one pair of nonadjacent vertices, then*

$$\kappa(G) = \min\{p(x, y) \mid x, y \in V(G),\ x \neq y,\ xy \notin E(G)\}.$$

One can conclude from Theorems 4.1.1 and 4.1.2 that if G has at least one pair of nonadjacent vertices, then $\kappa(G) = \min\{c(x, y) \mid x, y \in V(G),\ x \neq y,\ xy \notin E(G)\}$. If G has no pair of nonadjacent vertices, then G contains the complete graph $K_{v(G)}$ as a spanning subgraph. For any two vertices x, y in such a graph G, there are $v(G) - 2$ internally disjoint xy-paths of length 2, and as many xy-paths of length 1 as there are edges linking x and y. It is not possible to create a larger set of internally disjoint xy-paths than the one described here. Therefore, the connectivity of G is $v(G) - 2 + \mu$, where μ is the minimum number of parallel edges between two vertices of G. In particular, $\kappa(K_n) = n - 1$, and no simple graphs on n vertices has a connectivity greater than or equal to K_n, for $n \geq 2$.

In the context of contraction reconstruction, it is easier to think of connectivity in terms of vertex cuts than in terms of disjoint paths, hence the usefulness of Menger's theorem. We now exhibit a basic relation between the connectivity of a graph and the connectivity of its cards.

Lemma 4.1.3. *For any edge $e \in E(G)$,*

$$\kappa(G) \leq \kappa(G/e) + 1.$$

Furthermore, $\kappa(G) = \kappa(G/e) + 1$ if and only if e links two vertices in a minimum vertex cut of G.

Proof: Let $uv \in E(G)$. Let us first consider the case where the underlying simple graph of G/uv is complete, so $\kappa(G/uv) = v(G/uv) - 1$. Since

$$\kappa(G) \leq v(G) - 1 \leq v(G/uv) = \kappa(G/uv) + 1,$$

we get that the conclusion of the lemma holds.

We may now assume that the underlying simple graph G/uv is not complete, therefore G/uv contains two nonadjacent vertices x and y. Let $S \subseteq V(G/uv) - \{x, y\}$ be a minimum xy-vertex-cut of G/uv, so $\kappa(G/uv) = |S|$. If the contracted vertex v^* of G/uv is in S, then the set $(S - \{v^*\}) \cup \{u, v\}$ is an xy-vertex-cut of G, hence $\kappa(G) \leq \kappa(G/uv) + 1$. If the contracted vertex v^* of G/uv is not in S, then it is contained in one of the connected components of the disconnected graph $G/uv - S$. We conclude that $G - S$ is also disconnected, hence S is an xy-vertex-cut in G. Therefore, $\kappa(G) \leq \kappa(G/uv)$.

If e links two vertices in a minimum vertex cut of G, then surely $\kappa(G) = \kappa(G/e) + 1$. If $\kappa(G/e) + 1 = \kappa(G)$, then the decontraction of $v^* \in V(G/e)$ must increase the vertex connectivity, which implies that v^* is in a minimum vertex cut of G/e and so both ends of e are in a minimum vertex cut of G. ∎

The minimum possible vertex connectivity for a card of G is $\kappa(G) - 1$. Attach a vertex v to one of the vertices u of a complete graph on $n - 1$ vertices to create a graph G, whose vertex connectivity is 1. Then, G/uv is isomorphic to a complete graph on $n - 1$ vertices, whose vertex connectivity is $n - 2$. From this construction, one concludes that, in general, no non-trivial upper bound on $\kappa(G/e)$ can be stated in terms of $\kappa(G)$.

By Proposition 3.1.5, the class of of disconnected graphs is recognizable. Graphs with a vertex connectivity of 1 are studied next.

4.1.2 Separable graphs

A ***cut vertex*** of a connected graph is a vertex whose deletion results in a disconnected graph. Recall that a graph is ***empty*** if it contains no edges. A ***separation*** of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This vertex is called a ***separating vertex*** and is either a cut vertex or a vertex with a loop. A connected graph is ***separable*** if it has a separating vertex, and is ***nonseparable*** otherwise. A ***block*** of a graph is maximal nonseparable subgraph. A block is ***trivial*** if it contains exactly two vertices.

When the graph is loopless (such is the case of simple graphs), the concepts of cut vertices and separating vertices are equivalent. In particular, a connected loopless graph G has $\kappa(G) = 1$ if and only if G is separable.

The following lemma is standard, and states some basic properties of blocks which will be used throughout this chapter.

Lemma 4.1.4 (Bondy and Murty [4])**.** *Let G be a graph. Then,*

1. *Any two blocks of G have at most one vertex in common;*
2. *The blocks of G induce a partition of $E(G)$;*
3. *Each cycle of G is contained in a block of G;*
4. *Every pair of vertices in a non-trivial block is contained in a cycle.*

Proof:

1. By contradiction, suppose there are distinct blocks B_1 and B_2 with at least two vertices in common, so B_1 and B_2 are loopless. Let $B = B_1 \cup B_2$ and let $v \in B$. Then, $B - v = (B_1 - v) \cup (B_2 - v)$ is connected. Thus, B has no cut vertices, and, being loopless, is nonseparable. This contradicts the maximality of B_1 and B_2.
2. An edge by itself is a nonseparable subgraph, and is therefore contained in a block. Since any two blocks of G share at most one vertex, no edge is contained in more than one block. Thus the blocks of G form a partition of $E(G)$.
3. A cycle is a nonseparable graph, and is therefore contained in a block of G.
4. If a pair of vertices x and y is not part of any cycle, then one possibility is that the block contains only x and y. In that case, either there is one edge linking x and y, which contradicts the assumption that the block is non-trivial, or there are multiple edges linking x and y, in which case x and y is contained in a cycle of length 2. The other possibility is that the deletion of a vertex in an xy-path disconnects the block, which contradicts the assumption that a block is nonseparable.

∎

A direct consequence of Lemma 4.1.4 is that every vertex in a non-trivial block is contained in a cycle. Therefore, if B is a non-trivial block of G and $e \in E(B)$, then B/e contains a cycle, and so B/e is not a trivial block of G. The following proposition shows that separable graphs are recognizable.

Proposition 4.1.5. *A graph G with at least two edges is separable if and only if G has one of the following properties:*

1. *G has a vertex of degree* 1*,*
2. *G has a loop,*
3. *The underlying simple graph of every card of G is separable.*

In particular, separable graphs that are not in Figure 3.1 are recognizable.

Proof: We first note that a loopless graph G is separable if and only if its underlying simple graph is separable. In other words, adding parallel edges has no effect on whether a graph is separable or not.

By Propositions 3.1.1 and 3.1.5, connected graphs with at least two edges are recognizable. Furthermore, one can recognize a graph that is not in Figure 3.1. Let G be such a graph. If G has a vertex of degree 1 or if G has a loop, then G is separable. Having at least two edges and a vertex of degree 1 is a recognizable property by Proposition 3.1.7, and having loops is a recognizable property by Corollary 3.1.4. Therefore, the class of separable graphs with at least two edges and a vertex of degree 1 or a loop is recognizable.

Suppose then that G does not have any vertices of degree 1 nor loops. Suppose G is separable, and let v be a separating vertex of G. The vertex v induces a separation of G into at least two nonempty connected subgraphs, each pair of such subgraphs sharing only the vertex v. Each of those subgraphs must contain at least two edges, for otherwise G would have a vertex of degree 1 or a loop. Therefore, the contraction of any edge $e \in G$ does not reduce the number of nonempty connected subgraphs in the separation induced by v. We conclude that for any edge $e \in E(G)$, the vertex v induces a separation of G/e into at least two nonempty connected subgraphs, hence v is a cut vertex of G/e, and so G/e is separable. Therefore, if G is separable with no vertices of degree 1 nor loops, then all of its cards are separable too, which in turn implies that the underlying simple graph of every card of G is separable.

Suppose now that G has no vertices of degree 1 nor loops and that the underlying simple graph of G/e is separable for all $e \in E(G)$. We show that G is separable. Suppose G is not separable. Let G/xy be a card and let v_{xy} be the contracted vertex. Since the card has a separating vertex v but G does not, then $v = v^*$. In this case, $\{x, y\}$ is a vertex cut of G. This holds, by hypothesis, for every pair of adjacent vertices x and y. Furthermore, a card G/xy has only one cut vertex, namely the contracted vertex v_{xy}. Let $xy \in E(G)$ be such that the separation induced by v_{xy} in G/xy yields a nonempty connected subgraph F of minimum size with respect to the number of vertices. If $v(F) = 2$, then the vertex $v' \in V(F) - v_{xy}$ is only adjacent to x and y. Suppose, without loss of generality, that $v'x \in E(G)$. Then we see that the underlying simple graph of $G/v'x$ is not separable, which is a contradiction. Therefore, we may conclude that $v(F) \geq 3$. Let uw be an edge of F so that $u \neq v_{xy}$ and $w \neq v_{xy}$. Since any pair of adjacent vertices are a vertex cut of G, the vertex v_{uw} in $G/\{xy, uw\}$ induces a separation of the graph into at least two nonempty connected subgraphs, each pair sharing only v_{uw}. Since $uw \in E(F)$, there is a separation of F/uw into at least two nonempty connected subgraphs, only one of which contains v_{xy}. Let F' be one of the nonempty connected subgraphs of F/uw which does not contain v_{xy}. Then, F' is a nonempty subgraph in the separation of G/uw induced by v_{uw}, and is smaller than F. This contradicts the minimum choice of F. We conclude that G cannot be nonseparable, and is thus separable.

Therefore, if G is a simple connected graph with no vertices of degree 1, then G is separable if and only if all of its cards are, and so simple connected separable graphs with no vertices of degree 1 are recognizable. ∎

Corollary 4.1.6 follows directly from Proposition 4.1.5 and from the definition of simple separable graphs.

Corollary 4.1.6. *Simple graphs with at least two edges and with a connectivity of* 1 *are recognizable.*

Proof: By Proposition 4.1.5, simple separable graphs are recognizable. A simple graph G is separable if and only if $\kappa(G) = 1$. ∎

4.1.3 2-connected graphs

The next step is to show that graphs with $\kappa(G) = 2$ are recognizable. The strategy here is to show that 3-connected graphs are recognizable, hence, by complementarity, graphs with $\kappa(G) \leq 2$ are recognizable. Then, since graphs with $\kappa(G) \leq 1$ were shown to be recognizable, then so are the graphs with $\kappa(G) = 2$.

A theorem of Thomassen [4] is used to show that the class of 3-connected graphs is recognizable.

Theorem 4.1.7 (Thomassen, [4])**.** *Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that G/e is 3-connected*

Another theorem in [4] is of use here, which is somewhat the converse of Theorem 4.1.7.

Theorem 4.1.8 (Tutte, [4])**.** *Let H be a 3-connected graph, let v be a vertex of H of degree at least 4, and let G be a graph obtained from H by decontracting v into two vertices whose degrees are at least 3. Then G is 3-connected.*

From Theorems 4.1.7 and 4.1.8, we obtain the following result. Let $N(x)$ denote the set of neighbours of the vertex x.

Corollary 4.1.9. *A graph G is 3-connected if and only if there is a sequence $G_0, ..., G_n$ of graphs such that:*

1. *The underlying simple graph of G_0 is K_4 and $G_n = G$;*
2. *For any $i < n$, there is an edge $xy \in G_{i+1}$ such that $|N(x)| \geq 3$, $|N(y)| \geq 3$ in G_{i+1} and $G_i = G_{i+1}/xy$.*

Furthermore, every G_i is 3-connected.

Proof: Suppose G is 3-connected. If $v(G) = 4$, then since every 3-connected graph on four vertices must contain a subgraph isomorphic to K_4, we may take $G_n = G = G_0$. Suppose $v(G_{i+1}) \geq 5$ for some $i \leq n$, and suppose G_{i+1} is 3-connected. Then by Theorem 4.1.7, G_{i+1} contains an edge xy such that G_{i+1}/xy is 3-connected. Surely $|N(x)| \geq 3$ and $|N(y)| \geq 3$, since G_{i+1} is 3-connected. Then, let $G_i = G_{i+1}/xy$. Because $G_n = G$ is 3-connected, one has created, by induction, sequence $G_0, ..., G_n$ of 3-connected graphs with the desired properties.

Conversely, suppose there is a sequence $G_0, ..., G_n$ of graphs as described above. Let v^* be the contracted vertex in $G_i = G_{i+1}/xy$. Since $\deg(x) \geq 3$ and $\deg(y) \geq 3$ in G_{i+1}, then $\deg(v^*) \geq 4$ in G_i. By Theorem 4.1.8, if G_i is 3-connected, then so is G_{i+1}. Since G_0 is 3-connected, then, by induction, so are all the graphs in the sequence $G_0, ..., G_n$. In particular, G is 3-connected. ∎

In other words, G is 3-connected if there exists a sequence of edge contractions that transforms G into K_4, with possibly parallel edges and loops, where the edges contracted link vertices with at least three neighbours each. Note that every graph G_i in the sequence is also 3-connected. We can then show that 3-connected graphs are recognizable.

Proposition 4.1.10. *The class of simple 3-connected graphs is recognizable.*

Proof: We can deduce from Corollary 4.1.9 that a simple graph G is 3-connected if and only if the minimum degree of G is at least 3 and there exists a card $H \in \mathcal{C}(G)$ that is 3-connected. By Propositions 3.1.7 and 3.1.8, one can recognize a graph with no vertices of degree 1 and no vertices of degree 2, and by Proposition 3.1.5, one can recognize a connected graph. Since the graph is connected, it has no vertices of degree 0, hence the minimum degree of the graph is at least 3. Therefore, 3-connected graphs are recognizable. ∎

It is not known whether the proof of Proposition 4.1.10 generalizes to higher connectivities.

Proposition 4.1.5, along with Proposition 4.1.10, shows that the class of 2-connected graphs is recognizable, as is the class of graphs with vertex connectivity equal to 2. This will be useful later on when we show that 2-connected bipartite graphs are reconstructible in Theorem 5.2.5.

Proposition 4.1.11. *The class of* 2*-connected graphs is recognizable.*

Proof: By Proposition 3.1.5, graphs with a vertex connectivity of 0 are recognizable, and by Proposition 4.1.5, the class of graphs with a vertex connectivity of 1 is recognizable. By complementarity, the class of 2-connected graphs is also recognizable. ∎

Proposition 4.1.12. *The class of graphs G with $\kappa(G) = 2$ is recognizable.*

Proof: By Proposition 4.1.11, the class of 2-connected graphs is recognizable. By Proposition 4.1.10, the class of 3-connected graphs is recognizable. Since $\kappa(G) = 2$ if and only if G is 2-connected and G is not 3-connected, we conclude that the graphs with a vertex connectivity of 2 are recognizable. ∎

4.2 Reconstructing disconnected graphs

This section is dedicated to the reconstruction of disconnected graphs with at least two non-trivial components and at least three edges. While proving this theorem, we uncover a technique which we write in its most general form. Fragments of this technique can be found throughout this thesis.

4.2.1 Definition and main result

Recall that a graph is ***disconnected*** if its vertices can be partitioned into two sets X and Y such that no edge has one end in X and one end in Y. Equivalently, a graph G is disconnected if $\kappa(G) = 0$. Otherwise, the graph is ***connected***. A ***connected component*** of a graph G is a maximal connected subgraph of G. A connected graph has a single connected component, equal to the graph itself, while a disconnected graph has at least two connected components. A connected component is ***trivial*** if it has no edges, and is ***non-trivial*** otherwise.

Since every edge is contained within exactly one connected component of a graph, the difference between a graph G and G/e is in the connected component containing e. This observation is at the core of the proof of Theorem 4.2.1.

Theorem 4.2.1. *If G is a disconnected graph with at least three edges and at least two non-trivial components, then G is reconstructible.*

Proof: By Propositions 3.1.5 and 3.1.1, the class of disconnected graphs with at least three edges is recognizable. If G has a card with at least two non-trivial connected components, then G also has at least two non-trivial components. Conversely, if G has at least two non-trivial components, then since G has at least three edges, either G has at least three non-trivial components or G has two non-trivial components, one of which has at least two edges. In the first case, every card of G contains at least two non-trivial components. In the second case, a contraction within the largest non-trivial component yields a card with two non-trivial components. We conclude that a graph G with at least three edges has two non-trivial components if and only if a card of G does. Therefore, one can recognize a disconnected graph with at least three edges and at least two non-trivial components.

Over all cards in $\mathcal{C}(G)$, consider the connected component B with a maximum number of edges. Since G has at least two non-trivial components, and since there are no connected components with more edges than in B, B must be a component of G. Let $e \in E(B)$. Over all cards in $\mathcal{C}(G)$, consider a card H with the most connected components isomorphic to B/e and with the least connected components isomorphic to B. The graph G is reconstructed from H by replacing one connected component isomorphic to B/e by B. ∎

Note that one could reconstruct a disconnected graph given a subset $\mathcal{S}$ of $\mathcal{C}(G)$ with the property that all the non-trivial components have at least one edge corresponding to a card in $\mathcal{S}$. Indeed, according to the proof of Theorem 4.2.1, it only takes one card corresponding to the contraction of an edge in a connected component with at least two edges to recognize a disconnected graph with two non-trivial components, or any edge in any connected component to recognize a graph with more than two non-trivial connected components. Then, it takes one card corresponding to the contraction of any edge in a non-maximal connected component and one from a maximal connected component to reconstruct the graph. This observation will become useful when reconstructing separable graphs.

4.2.2 Reduction reconstruction

The underlying argument in the proof of Theorem 4.2.1 will be used often, and it is interesting to state it in as general as possible way. We start with a collection X of distinct ***connected objects***. An ***object*** is a finite collection of connected objects. The ***size*** of an object is a function from the set of objects to the nonnegative integers which satisfies the property that the size of an object is the sum of the sizes of its connected objects. For instance, in Theorem 4.2.1, the objects are disconnected graphs, the connected objects are the connected graphs, and the size of an object is its number of edges.

Let A be an object. A ***reduction*** is an operation that replaces one of the connected objects of A with a connected object of size one less than the original. An object whose size is 0 cannot be reduced. The ***reduction deck*** $\mathcal{R}(A)$ is the multiset of all objects obtained from A by a single reduction. The operation of edge contraction can be seen as a reduction on the set of graphs.

Theorem 4.2.2. *If A is an object of size at least 3 composed of at least two connected objects of size at least 1, then A is determined from $\mathcal{R}(A)$.*

Proof: We first prove that one can determine from $\mathcal{R}(A)$ if A has at least two connected objects of size at least 1. If $\mathcal{R}(A)$ contains an object composed of at least two connected objects of size at least 1, then A itself has two connected objects of size at least 1. Otherwise, either A contains exactly one connected object of size greater than 0, or A contains exactly two connected object of size 1 each. Therefore, one cannot determine if A has at least two connected objects of size at least 1 only if the size of A is less than 3.

The connected object X_0 with the maximum size throughout all of $\mathcal{R}(A)$ is part of A, for otherwise, X_0 was reduced from an even larger object, which will be part of one of the objects of $\mathcal{R}(A)$. Let X_1 be a connected object obtained by reducing X_0. Then, look through all $\mathcal{R}(A)$ for an object with the least number of X_0 connected objects and then the most X_1 connected objects. This object was obtained from A by reducing X_0 to X_1, thus A is determined. ∎

For instance, let $A = \{x_1, x_2, ..., x_k\}$ be a multiset of nonnegative integers, and let $\mathcal{R}(A)$ be the reduction deck of A, where a reduction consists of subtracting 1 from one of the non-zero integers of A. Then, Theorem 4.2.2 states that A can be determined from $\mathcal{R}(A)$ unless $A = \{1, 1\}$ or $A = \{2, 0\}$. Note that this counterexample is the analogue of the two graphs in Figure 4.1 that are not reconstructible.

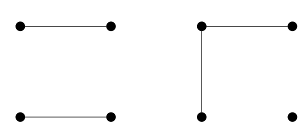

Figure 4.1: Two simple graphs that are contraction reconstructions of one another.

4.3 Reconstructing separable graphs

The next class of graphs we consider are the separable graphs. In a separable graph, every edge is contained in a single block, so an edge contraction affects only one block. This observation is similar to the one used in the proof of Theorem 4.2.1, which is that every edge is contained in exactly one connected component of a disconnected graph, hence every edge contraction affects only one connected component. The steps to reconstruct a separable graph are to first determine every block of the graph, and then to reattach them in the correct way.

It was shown, in Proposition 4.1.5, that separable graphs are recognizable. In this section, we show that some separable graphs are reconstructible. We first exhibit the proof that trees are reconstructible, found in [1], and then show, using different techniques, that most separable graphs with a cycle are reconstructible.

4.3.1 Trees

Recall that a graph is a ***tree*** if it is connected and does not contain any cycles. Equivalently, a tree is a connected graph on n vertices and $n-1$ edges. Every tree has at least two vertices of degree 1: such vertices are called ***leaves***. A tree is a separable graph, and thus has a connectivity of 1. A ***subtree*** is a subgraph of a tree that is also a tree.

Most trees were shown to be reconstructible in [1]. Instead of using the deck $\mathcal{C}(G)$, the authors have actually reconstructed trees with at least seven vertices from the set of isomorphically distinct cards, which is readily obtainable from the standard deck. Considering only the set of isomorphically distinct cards is not enough to reconstruct all trees with fewer than seven vertices. Figure 4.2 shows the only trees that are not reconstructible from their set of isomorphically distinct cards. The proof of this claim is the main focus of this section.

Notice that the trees in Figure 4.2not in the first box have distinct decks and are therefore reconstructible. The proof that trees with at least seven vertices are reconstructible from their set of isomorphically distinct cards relies on the observation that contracting an edge incident to a vertex of degree 1 is identical to the deletion of that vertex. One can then invoke the following result found in [21]:

Theorem 4.3.1 (Manvel [21]). *A tree is determined by its set of non-isomorphic maximal proper subtrees, except for the first four trees in Figure 4.2.*

The proof of Theorem 4.3.1 can be adapted to show that trees are reconstructible. It considers the centre of a tree G, which consists of all the vertices v such that $\max\{d(v,u) \mid u \in V(G)\}$ is minimum. Now it can be shown that the centre of a tree is exactly one vertex, in which case the tree is ***centred***, or the centre consists of two adjacent vertices, in which case the tree is said to be ***bicentred***. A maximal

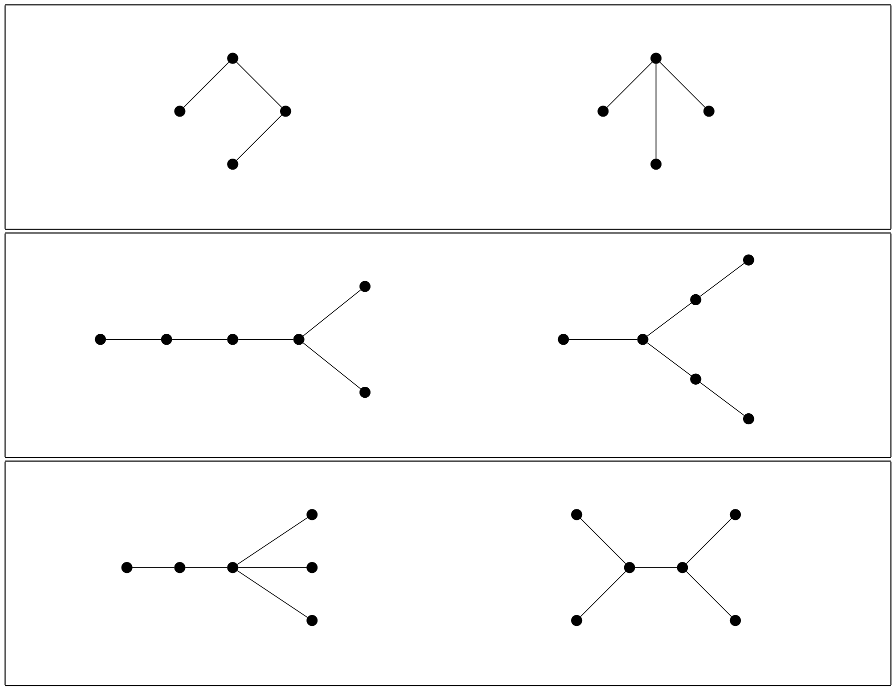

Figure 4.2: Each box contains two trees with identical sets of isomorphically distinct cards.

subgraph B of G containing a centre vertex v where v has exactly one neighbour in B is called a ***branch***. Theorem 4.3.1 essentially reconstructs these branches, which is enough to reconstruct the tree itself.

However, as mentioned before, Theorem 4.3.3, found in [1], does not aim to directly adapt Theorem 4.3.1 to the context of contraction reconstruction. It insteads reconstructs the deck used in Theorem 4.3.1 from a subset of $\mathcal{C}(G)$.

Before we show the proof of Theorem 4.3.3, a few definitions are required. The ***diameter*** of a graph G is the greatest distance between two vertices of G. We now define star graphs.

Definition 4.3.2. *For any integer $k \geq 1$, a k-**star** is the complete bipartite graph $K_{k,1}$. For any two integers $k_1 \geq 1$ and $k_2 \geq 1$, a (k_1, k_2)-**star** is a tree created from two linked vertices u and v by attaching k_1 leaves to u and k_2 leaves to v.*

Theorem 4.3.3 (Bhave, Kundu and Sampathkumar [1])**.** *If G is a tree with at least seven vertices, then G is reconstructible from its set of isomorphically distinct cards.*

Proof: According to the proof of Proposition 3.1.6, a graph G is a tree if and only if all of its cards are trees. Therefore, a graph G is a tree if and only if every isomorphically distinct card of G is a tree.

We now tackle some special cases. A card of G is a k-star if and only if G is either a $(k+1)$-star or a (k_1, k_2)-star, with $k_1 + k_2 = k$. We claim that one can easily check that those graphs have different isomorphically distinct decks. Indeed, since G has at least seven vertices, then $k \geq 5$, and so at least one of k_1 or k_2 is strictly greater than 1. This implies that a card of such a (k_1, k_2)-star graph contains two vertices at a distance of 3, while no cards of a $(k+1)$-star have two vertices at a distance of 3. Therefore k-stars and (k_1, k_2)-stars with at least seven vertices are reconstructible. In particular, we can henceforth assume that G is not a k-star, and so that G has a diameter of at least 3.

We say that a vertex of a tree is ***outer*** if it has a degree of 2 and lies on a path from a vertex of degree 1 to a vertex of degree at least 3. Let t be equal to the minimum number of outer vertices among the cards of G, and let k be equal to the maximum number of leaves among the cards. Since G has a diameter of at least 3, it must contain an edge e whose ends both have degrees greater than 1. Therefore, G/e has the same number of leaves as G. Furthermore, no cards of G can have more leaves than G. We conclude that G has k leaves.

Suppose there exists a card H with k leaves and at least one outer vertex. Then G also has at least one outer vertex, for otherwise the outer vertex of H is the result of the contraction of an edge between a leaf and a vertex of degree at least three. But in that case, G has $k+1$ leaves, which is a contradiction. If G has an outer vertex, but no card of G has k leaves and an outer vertex, then G is $(1, k-1)$-star. This graph has already been shown to be reconstructible. Therefore, G is not a $(1, k-1)$-star and has an outer vertex if and only if a card of G has k leaves and an outer vertex.

In this case, we show that a card is isomorphic to $G - v$, where v is a leaf, if and only if the card has $k-1$ leaves or the card has k leaves and t outer vertices. In one direction, contracting a leaf from a graph yields a card with either $k-1$ leaves if the leaf vertex is adjacent to a vertex of degree at least three, or a card with k leaves and t outer vertices otherwise. Conversely, if a card has $k-1$ leaves, then the card was obtained by contracting a leaf. If a card has k leaves and t outer vertices, then that card is obtained by the contraction of an edge incident to an outer vertex, which is equivalent to the contraction of a leaf. This, along with Theorem 4.3.1, shows that trees with at least one outer vertex are reconstructible.

Suppose then that G has no outer vertices. Every leaf of G is adjacent to vertices of degree at least 3. Then a card is isomorphic to $G - v$, where v is a leaf, if and only if the card has $k-1$ leaves. This, along with Theorem 4.3.1, completes the proof. ∎

Every part of this proof remains the same if one wishes to prove that trees are reconstructible from $\mathcal{C}(G)$. Of course, this affirmation is a corollary of Theorem 4.3.3. A simple verification on all trees with less than seven vertices allows us to state Corollary 4.3.4. Figures 4.3 and 4.4 lists all possible trees with four and five edges

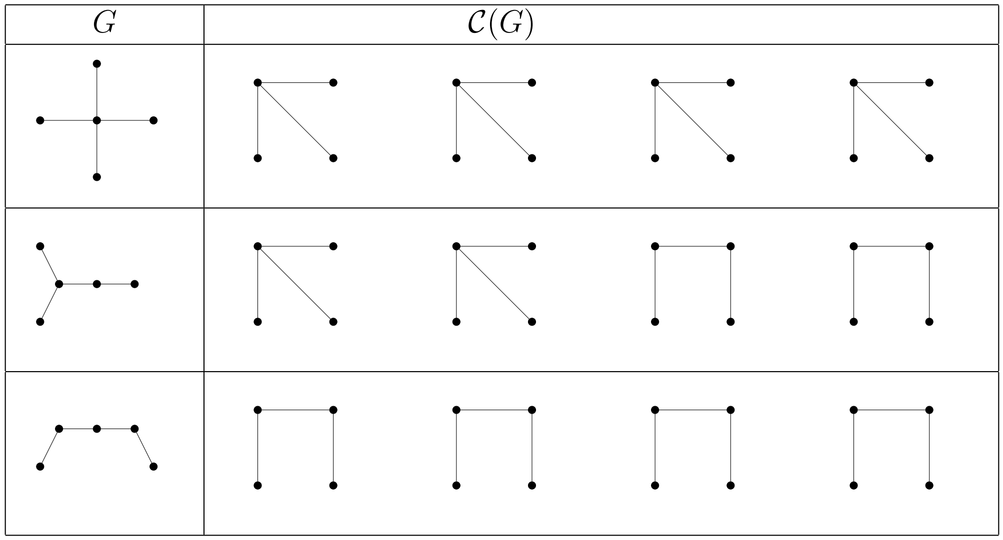

Figure 4.3: All trees with four edges and their decks. We see that they are all reconstructible.

and their decks. As one notices, those decks are all distinct.

Corollary 4.3.4. *Trees with more than three edges are reconstructible.*

Proof: The number of edges of a graph is reconstructible by Proposition 3.1.1. The class of trees with at least three edges is recognizable by Proposition 3.1.6. Trees with at least six edges are reconstructible by Theorem 4.3.3, and trees with four or five edges are reconstructible by Figures 4.3 and 4.4. ∎

We end this section with a discussion on how to further apply and adapt Theorem 4.3.1 to contraction reconstruction. A ***multi-tree*** is a graph whose underlying simple graph is a tree. Similar to trees, a multi-tree is either centred or bicentred. It is possible to consider a multi-tree G as a tree rooted at its centre, or its bicentre, and reconstruct each branch of G, since contractions can only affect one branch of a multi-tree.

While it is true that a contraction of a link of a centred multi-tree G may yield a card that is bi-centred, and vice versa, any multi-tree whose underlying simple graph is not a path contains links whose contraction does not change the centre. Manvel had considered a similar concept in [21]. We call those links ***non-essential***. Using these cards, a strategy for reconstructing the list of branches may be feasable.

For a centred multi-tree G, it certainly suffices to reconstruct the list of branches to reconstruct G. For a bicentred multi-tree, it would be necessary to also partition the list of branches into two parts, one part for each of the two centre vertices.

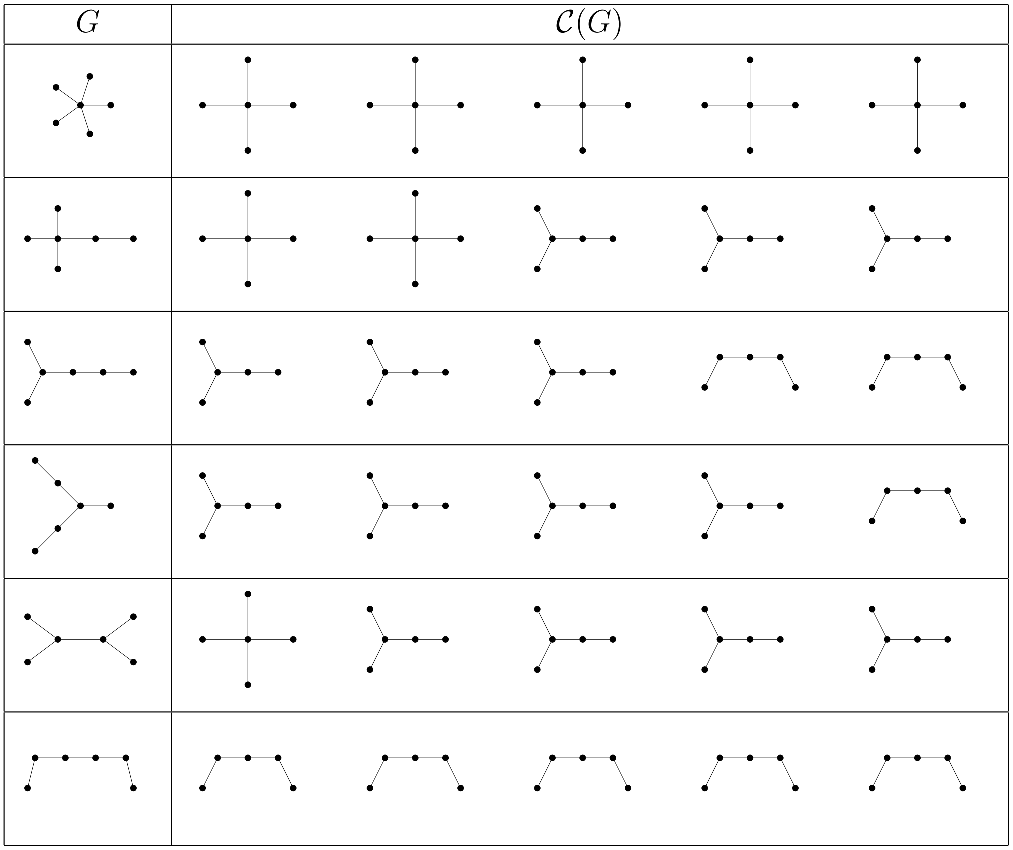

Figure 4.4: All trees with five edges and their decks. We see that they are all reconstructible.

4.3.2 Blocks of separable graphs

By Proposition 4.1.5, separable graphs are recognizable. It remains to show that they are weakly reconstructible to show that separable graphs are reconstructible. Since trees were shown to be reconstructible in Corollary 4.3.4, we assume here that separable graphs contain a cycle. It is important to understand the effect of an edge contraction on the blocks of a separable graph and its block structure.

From any graph G, one can create a bipartite graph $BT(G)$ with bipartition $(\mathcal{L}(G), S)$, where $\mathcal{L}(G)$ is the set of blocks of G and S is the set of separating vertices of G. A block B and a separating vertex v are adjacent if and only if $v \in V(B)$. Note that by the third statement of Lemma 4.1.4, $BT(G)$ does not have any cycles. Also, if G is connected, then so is $BT(G)$. Therefore, we call $BT(G)$ the ***block tree*** of G. We now discuss the relation between $BT(G)$ and $BT(G/e)$. In particular, Lemma 4.3.5 exhibits the relation between the number of vertices in $BT(G)$ and the number of vertices in $BT(G/e)$, for every edge $e \in E(G)$. Let $b(G)$ be the number of blocks in G.

Lemma 4.3.5. *Let G be a simple separable graph, and let $e \in E(G)$. Then*

1. $b(G/e) = b(G) - 1$ *if and only if e is the edge of a trivial block of G;*
2. $b(G/e) > b(G)$ *if and only if the ends of e are a vertex cut of a block of G;*
3. $b(G/e) = b(G)$ *if and only if e is not contained in a trivial block and the ends of e are not a vertex cut of G.*

Proof:

1. If B is a block of G that does not contain the edge e, then B is a block of G/e. Therefore, if $b(G/e) = b(G) - 1$, then the block B of G containing e must not contain any other edges, otherwise B/e would be a block of G/e. Hence e is the edge of a trivial block of G. Conversely, if e is the edge of a trivial block of G, then its contraction yields a card whose blocks are the blocks of G, minus the trivial block containing e and therefore $b(G/e) = b(G) - 1$.
2. Let B be the block of G containing e. Since every block of G that does not contain e is also a block of G/e, we have that $b(G/e) > b(G)$ if and only if B/e contains multiple blocks of G/e, which is the case if and only if B/e has a cut vertex. Since B does not have a cut vertex, B/e has a cut vertex if and only if the ends of e are a vertex cut of B.
3. If B is a block of G and $e \notin E(B)$, then B is a block of G/e. We conclude that $b(G/e) \geq b(G) - 1$. Therefore, by process of elimination, $b(G/e) = b(G)$ if and only if e is not contained in a trivial block and the ends of e are not a vertex cut of G.

∎

The contraction of an edge in a separable graph G can result in a graph with fewer blocks, the same number of blocks or more blocks than in G. Since we are trying to determine every block in the graph, the most desired outcome is when the number of blocks does not change. Fortunately, if a block is non-trivial, there is an edge within that block whose contraction preserves the number of blocks, as shown by Lemma 4.3.6.

Lemma 4.3.6. *Let G be a separable graph. Every edge e in a largest cycle of a loopless block B is such that B/e is a block of G/e.*

Proof: Let C be a largest cycle of a loopless block B, and let xy be an edge of C. Suppose B/xy is not a block of G/xy, so the contracted vertex v^* is a cut vertex of G/xy. This implies that $\{x, y\}$ is a 2-vertex cut. Neither x nor y is a cut vertex, since B is not separable. Let B_1' and B_2' be two connected components of $B - \{x, y\}$, and let B_1 be the subgraph of B induced by the vertices $V(B_1') \cup \{x, y\}$. Similarly, let B_2 be the subgraph induced by the vertices of $V(B_2') \cup \{x, y\}$. Suppose, without loss of generality, that $C \in B_1$. If there is no xy-path in B_2 except for the edge xy itself, then either x or y is a cut vertex of B, which is a contradiction. Therefore, there exists an xy-path p in B_2. The edges of $p \cup C - \{xy\}$ therefore contain a cycle of length greater than C in B, contradicting the choice of C. We therefore conclude that B/xy is a block of G/xy. ∎

A consequence of Lemma 4.3.6 is that for every non-trivial block B of G, there exists at least three edges of B such that their corresponding cards have $b(G)$ blocks.

We want to separate the deck $\mathcal{C}(G)$ into three parts, where each part corresponds to one of the three possibilities listed in Lemma 4.3.5. Knowing the number of blocks in G would certainly be sufficient to separate $\mathcal{C}(G)$.

Lemma 4.3.7. *If G is a simple separable graph with at least four edges, the number of blocks $b(G)$ is reconstructible.*

Proof: As seen in Lemma 4.3.5, a card of G can have $b(G) - 1$, $b(G)$, or more than $b(G)$ blocks. If G is a tree, then it has no non-trivial blocks and $b(G) = e(G)$. One can recognize if G is a tree by Proposition 3.1.6. If $\mathcal{C}(G)$ contains a card with a trivial block B, then B is a block of G. Therefore, the card corresponding to the contraction of the edge of B yields a card with $b(G) - 1$ blocks, which is the smallest number of blocks among all cards, and so $b(G)$ can be obtained. If not, then, by Lemma 4.3.6, the smallest number of blocks among all cards is $b(G)$. ∎

Now that the number of blocks is known, $\mathcal{C}(G)$ can be partitioned into $\mathcal{C}_{b-1}(G)$, $\mathcal{C}_b(G)$ and $\mathcal{C}_{>b}(G)$, where the subscripts "$b - 1$", "b"and "$> b$" correspond to the number of blocks in the cards. Note that if $\mathcal{C}(G) = \mathcal{C}_{b-1}(G)$, then G is a tree and has been shown to be reconstructible if it has enough edges by Corollary 4.3.4.

We may assume that $\mathcal{C}(G) \neq \mathcal{C}_{b-1}(G)$, or, in other words, that G has a non-trivial block, which certainly must contain a cycle. The next step is to determine $\mathcal{L}(G)$, the collection of all the blocks of G.

Lemma 4.3.8. *If G is a simple separable graph with at least four edges, then $\mathcal{L}(G)$ is reconstructible.*

Proof: If $\mathcal{C}(G) = \mathcal{C}_{b-1}(G)$, then G is a tree, hence is reconstructible by Corollary 4.3.4, and so is $\mathcal{L}(G)$. If $\mathcal{C}_{b-1}(G) \neq \emptyset$, then any card of $\mathcal{C}_{b-1}(G)$ allows us to reconstruct

$\mathcal{L}(G)$. We henceforth assume that $\mathcal{C}_{b-1}(G) = \emptyset$, so G has at least two non-trivial blocks.

According to Lemma 4.3.6, every non-trivial block of G contains an edge whose corresponding card is in $\mathcal{C}_b(G)$. Looking only at $\mathcal{C}_b(G)$, consider a block B with a maximum number of edges: B must be a block of G, for otherwise it was obtained by contracting a larger block, contradicting the maximality of B. Choose an edge $e \in E(B)$ such that B/e is not separable. By Lemma 4.3.6, such an edge exists. Then, consider a card in $\mathcal{C}_b(G)$ with the most blocks isomorphic to B/e and then the least blocks isomorphic to B. Therefore, $\mathcal{L}(G)$ is reconstructed by taking all the blocks from that card and replacing one B/e with B. ∎

Notice that the proof is roughly similar to the proof of Theorem 4.2.1. Unlike in the case of disconnected graphs, it does not suffice to know the collection of blocks to reconstruct a separable graph. It is necessary to know how they are attached to one another.

We now define the block degree of a vertex.

Definition 4.3.9. *For a separable graph G, the **block degree** of a vertex x is the number of blocks of G containing x.*

Non-trivial blocks play an important part in the proof of the reconstruction of separable graphs. We therefore introduce the following definition.

Definition 4.3.10. *Let G be a separable graph. Let $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_k = Z(G)$ be a sequence of graphs defined as follows. The graph G_k has no vertices of degree 1, and G_{i+1} is obtained from G_i by deleting a vertex of degree 1. Then G_k is the **trunk** of G and is denoted by $Z(G)$.*

Lemma 4.3.11. *The trunk of G is unique and is the minimal connected subgraph of G containing every vertex of every non-trivial block of G.*

Proof: We first show that $Z(G)$ is a minimal connected subgraph of G containing every vertex of every non-trivial block of G, then we show that such a minimal graph is unique.

If G is a tree, then $Z(G)$ is the empty graph, which is unique, and so the lemma is satisfied. We therefore assume that G has a non-trivial block.

The deletion of a vertex of degree 1 cannot disconnect a connected graph. Therefore, since $G_0 = G$ is connected, every G_i is connected and, in particular, $Z(G)$ is connected. Furthermore, if v is contained in a cycle, then $v \in Z(G)$. In particular, $Z(G)$ contains every vertex in every non-trivial block of G, and since $Z(G)$ is connected, $Z(G)$ contains a minimal connected subgraph F of G containing every vertex of every non-trivial block of G. Suppose $Z(G) \neq F$, so that there exists a vertex

$v \in Z(G) - F$. Because the blocks of G are arranged in a tree structure, this vertex v is not in any path linking two vertices in non-trivial blocks of $Z(G)$, for otherwise $v \in F$. Therefore, v is in a path linking a vertex of a non-trivial block to a vertex of degree 1 in $Z(G)$, which is a contradiction with the definition of $Z(G)$. We conclude that $Z(G) = F$.

We now show that $Z(G)$ is unique. Suppose F_1 and F_2 are two minimal connected subgraphs of G each containing every vertex of every non-trivial block of G. If v is a vertex in a non-trivial block of G or if v is in a path linking two vertices in non-trivial blocks of G, then $v \in F_1 \cap F_2$. Therefore, $F_1 \cap F_2$ is a connected subgraph of G containing every vertex of every non-trivial block of G. As to not contradict the minimal choices of F_1 and F_2, we conclude that $F_1 = F_1 \cap F_2 = F_2$, and so $Z(G)$ is unique. ∎

Equivalently, the trunk of G is the maximal subgraph of G with no vertices of degree 1.

Definition 4.3.12. *Let G be a separable graph. For any $v \in Z(G)$, the **limb** of v, denoted $T(v)$, is the maximal connected subgraph of G with $T(v) \cap Z(G) = \{v\}$, rooted at v. For any subset $F \subseteq Z(G)$, denote by $T(F)$ the subgraph induced by $\{T(v) \mid v \in F\}$.*

It follows from the definition that $T(v)$ is a tree. It also follows that $T(v) \cap T(u) \neq \emptyset$ if and only if $u = v$, so the limbs form a partition of $V(G)$. Every vertex of G is in the trunk or in a limb. We write $T(v) = \{v\}$ if v is the only vertex in its limb.

We end this section with a simple lemma that shows that one can identify the contracted block in some of the cards of $\mathcal{C}(G)$.

Lemma 4.3.13. *If G is a separable simple graph with at least four edges, and G is not a tree, then $\mathcal{C}(G)$ contains a card H for which it is possible to identify the block containing the contracted vertex.*

Proof: Since G is simple, the smallest block (in terms of number of vertices) of $\mathcal{L}(G)$ with a cycle has the desired property, as any contraction within that block yields a smaller block with a cycle, which is certainly not in $\mathcal{L}(G)$. ∎

4.3.3 Separable graphs with at least two non-trivial blocks

We present the proof that separable graphs with at least two non-trivial blocks are reconstructible. For any block B of a separable graph G, denote by $c_G(B)$ the set of cut vertices of G that are in B.

Theorem 4.3.14. *Let G be a simple separable graph with at least four edges and at least two non-trivial blocks. Then G is reconstructible.*

Proof: By Proposition 3.1.2, Lemma 4.1.5 and Proposition 3.1.1, simple separable graphs with at least four edges are recognizable. By Lemma 4.3.8, $\mathcal{L}(G)$ is reconstructible, and we can recognize a graph with at least two non-trivial blocks. We conclude that the class of graphs we described in this theorem is recognizable.

Recall that $Z(G) = G$ if and only if G has no vertices of degree 1, which is recognizable by Proposition 3.1.7. If G does have vertices of degree 1, consider a card G/e where $Z(G/e)$ has the maximum number of edges. Then $Z(G) = Z(G/e)$, as e must be an edge in $E(G) - E(Z(G))$. Therefore, one can determine if $Z(G) = G$ or not, and if $Z(G) \neq G$, one can reconstruct $Z(G)$. Recall that for any block B of $Z(G)$, $T(B)$ is the maximal connected subgraph of G such that $T(B) \cap Z(G) = B$. Let $T(B)'$ be the graph $T(B)$ with every vertex in $c_{Z(G)}(B)$ labeled by their block degrees in $Z(G)$.

Let $\mathcal{B} \subseteq \mathcal{L}(G)$ be the collection of blocks B of $Z(G)$ with $|c_{Z(G)}(B)| = 1$. Every block of $\mathcal{B}$ is non-trivial, they are leaves in the block tree graph of $Z(G)$, and since every nonempty tree contains at least two leaves, $|\mathcal{B}| \geq 2$. Define $\mathcal{B}' = \{T(B)' \mid B \in \mathcal{B}\}$. Consider the collection $\mathfrak{B}$ of cards G/e with the following three properties:

1. $e \in E(B)$ for some $B \in \mathcal{B}$;
2. $G/e \in \mathcal{C}_b(G)$;
3. The edge either links the cut vertex u of B in $Z(B)$ with a vertex v such that $T(v) = \{v\}$, or the edge is not incident to u.

A card G/e has the first property if and only if the total number of edges in the blocks of G/e with exactly one cut vertex is minimal, which can determined from the deck. By Lemma 4.3.6, one can determine if a card has the second property. Finally, a card G/e has the third property if and only if the total size of the limbs of the cut vertices of the blocks with only one cut vertex in $Z(G/e)$ is minimal, since any other edge contraction would increase the size of a limb attached to a cut vertex of such a block. This property can also be determined from the deck. Therefore, one can identify the collection $\mathfrak{B}$ in $\mathcal{C}(G)$.

Let F be a block of $\mathcal{B}$ with a minimum number of edges. A card of $\mathfrak{B}$ that corresponds to an edge e_0 of F has a non-trivial block F/e_0 containing only one cut vertex in $Z(G/e_0)$. Since F/e_0 contains too few edges to be a non-trivial block of G in $\mathcal{B}$, we can thus identify the block F/e_0 in G/e_0. Since $G/e_0 \in \mathfrak{B}$, the contraction of the edge e_0 did not change the limb of the vertex v in $c_{Z(G)}(F)$. Therefore, the subgraph $G - T(F) \cup \{T(v)\}$ is determined, and so $\mathcal{B}$ and $\mathcal{B}' - \{T(F)\}$ are determined. It remains to reconstruct $T(F)$, and replace $T(F/e_0)$ by $T(F)$ in G/e_0 to obtain G.

In every card $G/e \in \mathfrak{B}$, every graph of $\mathcal{B}'$ appears as a subgraph in the card, except for the subgraph $T(B)'$ that contains the edge e. Since $\mathcal{B}'$ contains at least two subgraphs, $T(F)$ is certainly a subgraph of one card of $\mathfrak{B}$. For every $B \in \mathcal{B}$, let $k(B) = |\{e \in E(B) \mid G/e \in \mathfrak{B}\}|$. Since $\mathcal{B}' - \{T(F)\}$ is known, one can easily check, for every edge e of every $T(B) \in \mathcal{B}' - \{T(F)\}$, if the three conditions stated above are satisfied, and hence determine $k(B)$ for every block of $\mathcal{B} - \{F\}$. Then $k(F) = |\mathfrak{B}| - \sum_{B\in\mathcal{B},B\neq F} k(B)$.

Let $\mathcal{A} = \{T(B)' \subset G/e \mid G/e \in \mathfrak{B},\ |c_{Z(G/e)}(B)| = 1\}$. Since there are at least two blocks in $\mathcal{B}$, $\mathcal{B}' \subseteq \mathcal{A}$. For every subgraph $T(B)' \in \mathcal{B}'$, there are $|\mathfrak{B}| - k(B)$ copies of $T(B)'$ in $\mathcal{A}$, and for each of the $k(B)$ edges $e \in E(B)$ such that $G/e \in \mathfrak{B}$, there is a graph $T(B)'/e$ in $\mathcal{A}$. Since every graph $T(B)' \in \mathcal{B}' - \{T(F)\}$ and their $k(B)$ edges $e \in E(B)$ such that $G/e \in \mathfrak{B}$ are known, one can easily determine, by elimination, the graphs of $\mathcal{A}$ that are $T(F)'$ or $T(F/e)'$ for some edge $e \in E(F)$, hence $T(F)'$ is determined.

Replace $T(F/e_0)'$ by $T(F)'$ in G/e_0 to reconstruct G. ∎

If G is a separable graph with an edge connectivity of at least 2, then G does not have any vertices of degree 1, hence $Z(G) = G$. In that case, G has at least two non-trivial blocks and G is reconstructible by Theorem 4.3.14. This remark will be useful when reconstructing graphs with an edge connectivity of 2, as one can then assume that the graphs are not separable.

4.3.4 Separable graphs with one non-trivial block

If G is separable and has only one non-trivial block B, Lemma 4.3.15 implies that one can reconstruct the multiset $\{T(v) : v \in Aut(B) \cdot u\}$, where $Aut(B) \cdot u$ is the orbit of the vertex $u \in V(B)$ under the automorphism group $Aut(B)$. For any subgraph F of B, recall that $T(F)$ is the subgraph of G induced by the vertex set $\{T(v) \mid v \in V(F)\}$. Denote by $t(v)$ the total number of edges in $T(v)$, and let $t(F) = \sum_{v\in V(F)} t(v)$.

Lemma 4.3.15. *Let G be a separable graph with exactly one non-trivial block B. Let F be a subgraph of B. If, for every $\sigma \in Aut(B)$, there is at least one vertex $v \in V(B) - V(\sigma(F))$ such that $T(v) \neq \{v\}$, then the multiset of subgraphs $\{T(\sigma(F)) \mid \sigma \in Aut(B)\}$ is reconstructible.*

Proof: One can separate the deck $\mathcal{C}(G)$ into two parts: $\mathcal{CB}$ are the cards corresponding to contractions within the block B, and $\mathcal{CT}$ are the cards corresponding to the contractions within limbs. Note that since G is separable, $G \neq B$, and so $\mathcal{CT} \neq \emptyset$. A card in $\mathcal{CB}$ has the same total number of edges in the limbs as G, and a card of $\mathcal{CT}$ has one fewer edge. Therefore, a card H belongs to $\mathcal{CB}$ if and only if the total number of edges in the limbs of H is maximum. Hence $t(G) = |\mathcal{CT}|$.

Let $\mathcal{F}$ denote the orbit of F under $Aut(B)$. Note that any subgraph in $\mathcal{F}$ can be seen as a subgraph of any card of $\mathcal{CT}$ since those cards contain the block B. Let $H_0 \in \mathcal{CT}$ be a card such that there exists an $F_0 \in \mathcal{F}$ with $t(F_0)$ maximum throughout the cards of $\mathcal{CT}$. Since we assume that there exists a vertex $v \in V(B) - V(F_0)$ such that $T(v) \neq \{v\}$, we conclude that the vertex to decontract in H_0 is not in $T(F_0)$, hence $T(F_0)$ is in $\{T(\sigma(F)) \mid \sigma \in Aut(B)\}$. Let $e \in E(T(F_0))$. Consider a card $H_1 \in \mathcal{CT}$ where the number of $F_1 \in \mathcal{F}$ such that $T(F_1) \cong T(F_0)$ is minimum and where the number of $F_2 \in \mathcal{F}$ such that $T(F_2) \cong T(F_0)/e$ is maximum. Then $\{T(\sigma(F)) \mid \sigma \in Aut(B)\}$ is reconstructed from H_1 by replacing one $T(F_0)/e$ with $T(F_0)$. ∎

Note that the argument used here is analogous to the one used to reconstruct disconnected graphs in Theorem 4.2.1.

It is therefore possible to reconstruct the number of subgraphs of G isomorphic to $T(F)$ for some induced subgraph $F \subseteq B$ provided, for instance, there are more vertices of B with non-trivial limbs than the number of vertices in F. In order to properly apply Lemma 4.3.15 to some subgraph F_1, it is necessary to recognize whether there is a vertex $v_i \in V(G) - V(F_i)$ such that $T(v_i) \neq \{v_i\}$ for every F_i in the orbit of F_1.

Corollary 4.3.16. *Let G be a simple separable graph composed of one non-trivial block B. Then, $\{T(v) \mid v \in V(B)\}$ is reconstructible.*

Proof: Suppose there is only one vertex $v \in V(B)$ such that $T(v) \neq \{v\}$. A graph G has that property if and only if every card of $\mathcal{CB}$ has the same property. In that case, the only non-trivial limb is found in every card of $\mathcal{CB}$ hence $\{T(v) \mid v \in V(B)\}$ is reconstructible. Suppose then that there are more than one vertex of $V(B)$ with a non-trivial limb. One can apply Lemma 4.3.15 to reconstruct $\{T(v) \mid v \in V(B)\}$. ∎

Recall that the ***distance*** between two vertices x and y, denoted by $d(x, y)$, is the length of a minimum xy-path.

Corollary 4.3.17. *Let G be a separable graph with exactly one non-trivial block B and let T_1 and T_2 be trees. If there are at least three vertices $x_1, x_2, x_3 \in V(B)$ such that $T(x_i) \neq \{x_i\}$, then the number of pairs of vertices $\{u, v\} \subseteq B$ such that $T(u) \cong T_1$, $T(v) \cong T_2$ and $d(u, v) = d$ is reconstructible for all integers d.*

Proof: For a fixed d, apply Lemma 4.3.15 to every subgraph F of B where $V(F) = \{u, v\}$ with $d(u, v) = d$. ∎

The following theorems, namely Theorems 4.3.18 and 4.3.21, uses Lemma 4.3.15 to reconstruct some separable graphs with one non-trivial block B, for some specific B.

Theorem 4.3.18. *Let G be a simple separable graph with exactly one non-trivial block B. If B is a cycle of length at least 3, then G is reconstructible.*

Proof: By Propositions 3.1.2 and 4.1.5, simple separable graphs are recognizable. By Lemma 4.3.8, the list of blocks of a simple separable graph is reconstructible, hence one can recognize simple separable graphs with exactly one non-trivial block that is a cycle.

By Lemma 4.3.15, the limbs $T(v)$ are reconstructible for any $v \in V(B)$, unless there exists exactly one vertex v where $T(v) \neq \{v\}$. Such is the case if and only if for every card of $\mathcal{CB}$, there is only one vertex of B with a non-trivial limb. Such a graph is reconstructible by taking any card in $\mathcal{CB}$, and subdividing any edge in the cycle. Assume then that at least two vertices of B have non-trivial limbs, and so $T(v)$ is reconstructible for any $v \in V(B)$.

Let $m = \max\{t(v) : v \in V(B)\}$. Suppose every vertex u of B adjacent to a vertex v of B with $t(v) = m$ has $t(u) = 0$. One can verify if G has that property using Corollary 4.3.17 using $d = 1$. Therefore, in G, the shortest paths on the cycle from a vertex with a tree of size m to a vertex with a tree of size at least 1 is of length at least 2 in G. Consider a card within $\mathcal{CB}$ where the shortest path between a vertex with a tree of size m to a vertex with a tree of size at least 1 is minimal : such a path does not exist in G, and one reconstructs G by subdividing that path.

Therefore, we may assume there exists a vertex v_1 with $t(v_1) = m$ that is adjacent to a vertex v_2 with $t(v_2) = m' \neq 0$. There exists a card in $\mathcal{CB}$ and a vertex v^* in that card with $t(v^*) = m + m'$. Since $m + m'$ exceeds the maximum size of a limb of any vertex of G, the vertex v^* is to be decontracted. Since $\{T(v) : v \in V(B)\}$ is known, it is also known that the decontraction of that vertex must yield two vertices v_1 and v_2, and $\{T(v_1), T(v_2)\}$ is known as a set. To complete de decontraction, one must choose which of the two limbs is attached to v_1. Let $\{S, R\} = \{T(v_1), T(v_2)\}$. There are then two possible reconstructions: one of these is G, and call the other H. We suppose H is a reconstruction of G, therefore $\mathcal{C}(G) = \mathcal{C}(H)$.

Since G and H are cycles of equal lengths with limbs attached to their vertices, G and H are entirely determined by a cyclic sequence of the limbs of the vertices of the cycle. Let $(S, R, T_1, T_2, ..., T_{k-1})$ be the cyclic sequence for G, with v_1 and v_2 the two vertices whose limbs are S and R in that cyclic sequence, respectively, and so $(R, S, T_1, T_2, ..., T_{k-1})$ is the cyclic sequence for H, with, again, v_1 and v_2 the two vertices whose limbs are S and R in that cyclic sequence. We show that these two cyclic sequences are equal, thus showing that $G \cong H$.

Let us first notice that $S \cong R$ implies that $G \cong H$. We henceforth assume $S \ncong R$.

By Corollary 4.3.17, the number of paths of length d on the cycle, starting at a vertex v with $T(v) = T_v$ and ending at a vertex u with $T(u) = T_u$, is reconstructible, for every d, and every tree T_v and T_u. Denote that quantity by $p(T_v, T_u, d)_G$. Since G and H are reconstructions, $p(T_v, T_u, d)_G = p(T_v, T_u, d)_H$ for any T_v, T_u and d.

In particular, $p(S, T, 1)_G$ is reconstructible for every tree T. Let us decompose that quantity into two parts for G,

$$p(S, T, 1)_G = p(S, T, 1)'_G + p(S, T, 1)''_G \tag{4.3.1}$$

where $p(S, T, d)'_G$ denotes the number of such paths of length d that intersect $\{v_1, v_2\}$ in zero or two places. Then, $p(S, T, d)''_G$ denotes the number of such paths of length d that intersect $\{v_1, v_2\}$ once. Similarly for H,

$$p(S, T, 1)_H = p(S, T, 1)'_H + p(S, T, 1)''_H$$

First, let us note that, $p(S, T, d)''_G \leq 2$. Furthermore, $p(S, T, 1)'_G$ may be obtained by looking at the two subsequences (S, R) and $(T_1, T_2, ..., T_{k-1})$ of the cyclic sequence of G. Since those two subsequences are equal for G and H, we conclude that $p(S, T, 1)'_G = p(S, T, 1)'_H$. And by (4.3.1), we conclude that $p(S, T, 1)''_G = p(S, T, 1)''_H$. Since $p(S, T, 1)''_G$ counts the number of neighbours u of v_1 in G with $u \neq v_2$ and $T(u) = T$, and it is equal to $p(S, T, 1)''_H$, the number of neighbours u of v_2 in H with $u \neq v_2$ and $T(u) = T$, we conclude that $T_1 \cong T_{k-1}$.

Suppose that $T_i \cong T_{k-i}$ for $i = 1, 2, ..., d$, for some integer $d \geq 1$. We show that it implies $T_{d+1} \cong T_{k-(d+1)}$, thus proving that $G \cong H$ by induction. Decompose $p(T, S, d+1)_G$ into

$$p(S, T, d+1)_G = p(S, T, d+1)'_G + p(S, T, d+1)''_G \tag{4.3.2}$$

and decompose $p(T, S, d+1)_H$ into

$$p(S, T, d+1)_H = p(S, T, d+1)'_H + p(S, T, d+1)''_H. \tag{4.3.3}$$

The quantity $p(S, T, d+1)'_G$ is obtained by looking at the two subsequences $(T_{k-d}, T_{k-(d-1)}, ..., T_{k-1}, S, R, T_1, T_2, ..., T_d)$ and $(T_1, T_2, ..., T_{k-1})$ of the cyclic sequence for G. Similarly, $p(S, T, d+1)'_H$ may be obtained by looking at the two subsequences $(T_{k-d}, T_{k-(d-1)}, ..., T_{k-1}, R, S, T_1, T_2, ..., T_d)$ and $(T_1, T_2, ..., T_{k-1})$ of the cyclic sequence for H. By the induction hypothesis, the two subsequences of G are equal to the two subsequences of H, thus $p(S, T, d+1)'_G = p(S, T, d+1)'_H$. By (4.3.2) and (4.3.3), we get $p(S, T, d+1)''_G = p(S, T, d+1)''_H$. We conclude that $T_d = T_{k-d}$. By induction, we have shown that the cyclic sequences of G and H are equal, and $G \cong H$. ∎

The key to the proof of Theorem 4.3.18 is that such graphs can be represented by a cyclic sequence of limbs.

Another structure for B allows the reconstruction of separable graphs with B as the only non-trivial block. This structure is presented in the following definition.

Definition 4.3.19. *A graph G is* ***fully subdivided*** *if its minimal degree is two, and if u is adjacent to v in G, then* $\deg(u) = 2$ *or* $\deg(v) = 2$.

Definition 4.3.20. *A* ***full path*** *is a path between two vertices of degrees not equal to 2, whose internal vertices are of degree 2. The* ***length*** *of a full path P, denoted $|P|$, is the number of edges in P.*

It follows from Definitions 4.3.19 and 4.3.20 that a fully subdivived graph either is a cycle, or it contains full paths. We now present Theorem 4.3.21, which shows that separable graphs with exactly one block, where that block is fully subdivided, are reconstructible. The proof is split into many cases, depending on the position of the limbs, and on the lengths of the full paths in that block.

Theorem 4.3.21. *Let G be a separable graph with exactly one non-trivial block B. If B is a fully subdivided graph, then G is reconstructible.*

Proof: By Lemma 4.3.8, the collection of blocks of G is reconstructible, and it is therefore possible to determine if B is a fully subdivided graph.

Let $P_1, P_2, ..., P_k$ be the full paths of B, and let $I(P_i)$ be the set of internal vertices of P_i. Let $t(P_i) = \sum_{v \in P_i} t(v)$ be the total number of edges on the limbs of the vertices of P_i. Denote by $|P_i|$ the number of vertices in P_i. Similarly, define $t(I(P_i)) = \sum_{v \in I(P_i)} t(v)$ and let $|I(P_i)|$ be the number of vertices in $I(P_i)$.

If B has no vertices of degree more than 2, then B is a cycle. We can then apply Theorem 4.3.18 to show that G is reconstructible. Otherwise, B has a vertex of degree at least 3, and since B is not separable, B must have at least two vertices of degree at least 3. This implies that B has at least three full paths.

First, let us observe that a limb of a vertex of degree at least 3 in B remains attached to a vertex of degree at least 3 in every card of $\mathcal{CB}$. Furthermore, there exists a full path P such that $t(I(P))$ is equal to $t(G)$ if and only if there exists a card $H \in \mathcal{CB}$ where every non-trivial limb is attached to internal vertices of the same full path. Therefore, one can recognize whether all non-trivial limbs are attached to internal vertices of the same full path. Suppose G contains a full path P such that every non-trivial limb of B is attached to a vertex of $I(P)$. Then, by considering a card of $\mathcal{CB}$ with a full path with non-trivial limbs, such that its length is maximum, one can reconstruct $T(P)$. In particular, $|I(P)|$ is reconstructible. Consider a card $H \in \mathcal{CB}$ containing a full path Q with $t(I(Q)) = t(I(P))$ and $|I(Q)| = |I(P)|$, and another full path Q' such that $|I(Q')|$ is minimal among all cards H. A card H satisfying these conditions corresponds to the contraction of an edge of $B - I(P)$ inside a full path of minimal length. One reconstructs G by subdividing an edge in the full path Q'.

Assume $t(I(P_i)) \neq t(G)$ for any i. Then by letting F be the internal vertices of a full path subgraph of B in Lemma 4.3.15, one can reconstruct the multiset $\{t(I(P_i)) \mid 1 \leq i \leq k\}$.

Suppose there exists a full path P of G such that $t(I(P)) = 0$. Consider, among every card of $\mathcal{CB}$ where $\sum_{v:\deg(v)=2} t(v)$ is maximum, a card with the smallest full path Q with $t(I(Q)) = 0$. These two conditions assures that the contraction occurred within a minimal full path P of G with $t(I(P)) = 0$. Subdivide any edge in the full path Q of the card in consideration to reconstruct G.

We may then assume that $t(I(P_i)) \geq 1$ for every i. Given this hypothesis, one can reconstruct the subgraphs $T(P_i)$, for any i, using Lemma 4.3.15. Similarly, one can reconstruct the subgraphs $T(I(P))$, for any i. We split the following argument into three cases.

In the first case, we assume that $\min\{|I(P_i)| : i = 1, 2, ..., k\} = 1$, hence there is a full path P with $|I(P)| = 1$ in G. Since $t(I(P)) \geq 1$, consider a card of $\mathcal{CT}$ with a full path Q such that $|I(Q)| = 1$ and $t(I(Q)) < \min\{t(I(P)) : |I(P)| = 1\}$. Then the full path Q is not in G: one reconstructs G by replacing the limb of the only vertex of $I(Q)$ by the limb in the collection $\{T(I(P)) : |I(P)| = 1\}$ that is missing from the card.

In the second case, we assume that $\min\{t(I(P_i)) : i = 1, 2, ..., k\} \geq 2$. Consider, among all the cards of $\mathcal{CT}$, a card H with a full path Q where $t(I(Q))$ is minimum. Let $\{Q_1, Q_2, ..., Q_k\}$ be the set of full paths of H. Since H is a card corresponding to the contraction in a limb, one can assume $P_i = Q_i$ for all i. Also, $Q = Q_i$ for some i, and since $t(I(Q_i))$ is minimum, H was obtained by contracting an edge on a limb of some vertex of $I(P_i)$. Therefore $T(I(Q_j)) = T(I(P_j))$ for all $j \neq i$. To reconstruct G from the card H, one needs to replace $T(I(Q_i))$ with $T(I(P_i))$. Let $\{q_1, q_2, ..., q_{r-1}\} = I(Q_i)$, such that q_j is adjacent to q_{j+1}, and let $\{p_1, p_2, ..., p_{r-1}\} = I(P_i)$, such that p_j is adjacent to p_{j+1}. To replace $T(I(Q_i))$ with $T(I(P_i))$, one can either replace $T(q_j)$ with $T(p_j)$ for all j, or replace $T(q_j)$ with $T(p_{r-j})$ for all j. Since $T(Q_i)$ is obtained from $T(P_i)$ by contracting exactly one edge, the first reconstruction option is to be considered only if there exists a t such that $T(q_t) = T(p_t)/e$ for some edge $e \in T(p_t)$ and $T(q_j) = T(p_j)$ if $j \neq t$. Similarly, the second reconstruction option is to be considered only if there exists a t' such that $T(q_{t'}) = T(p_{r-t'})/e$ for some edge $e \in T(p_{r-t'})$ and $T(q_j) = T(p_{r-j})$ if $j \neq t$. Therefore, for both reconstruction options to be considered, it is necessary that $T(q_j) = T(q_{r-j})$, for all j. In that case, $T(p_j) = T(p_{r-j})$ for all but one value of j, say j_0. If any of these limbs $T(p_j)$ is non-trivial, for $j \neq j_0$ and $j \neq r - j_0$, then a contraction within that limb yields a card with no such ambiguity, hence only one reconstruction option is valid, thus G is reconstructible. Otherwise, all of the limbs $T(p_j)$ are trivial except for $j = j_0$ and $r - j_0$. Then, for $T(q_j)$ to be equal to $T(q_{r-j})$ for all j and for such a property to hold for all cards corresponding to contractions within limbs of $I(P_i))$, it is necessary that $t(p_{j_0}) = 1$ and $t(p_{r-j_0}) = 0$, or vice versa, which contradicts the hypothesis that $t(I(P_i)) \geq 2$.

In the third case, we assume $\min\{t(I(P_i)) \mid i = 1, 2, ..., k\} = 1$ and $|I(P_i)| \geq 2$ for all i. This implies that there exists a full path P such that $t(I(P)) = 1$. There is only one leaf on one vertex of $I(P)$. One can certainly identify cards corresponding to the contraction of a leaf on $I(P)$, as they are the only cards of $\mathcal{CT}$ containing a full path with no non-trivial limbs on its internal vertices. We separate this case into two subcases, depending on $|I(P)|$.

Suppose that $\min\{|I(P_i)| : t(I(P_i)) = 1\} = 2$. Let P be a full path of G with $|I(P)| = 2$ and $t(I(P)) = 1$. The full path P has four vertices, say x_1, x_2, x_3 and x_4, where x_i is adjacent to x_{i+1}. Since $t(I(P)) = 1$, then $t(x_2) + t(x_3) = 1$. Suppose, without loss of generality, that $t(x_2) = 1$ and $t(x_3) = 0$. The cards G/x_1x_2, G/x_2x_3 and G/x_3x_4 have a unique full path Q such that $|I(Q)| = 1$. If $t(x_1) + 1 = t(x_4)$, then contract x_2x_3 to obtain a card with the full path Q with a single internal vertex v, and $t(v) = 1$. One then needs to subdivide one of the two edges of Q incident to v. Since $t(x_1) \neq t(x_4)$, only one subdivision yields a graph with the same multiset $\{T(P_i) : i = 1, 2, ..., k\}$ as G, so G is reconstructible. If $t(x_1) + 1 \neq t(x_4)$, then the contraction of x_1x_2 yields a card with a full path Q with no leaves on its internal vertex, and its two ends have $t(x_1) + 1$ and $t(x_4)$ leaves. Since those two numbers are distinct, only one decontraction within Q yields a graph with the same multiset $\{T(P_i) : i = 1, 2, ..., k\}$ as G.

Suppose every full path P with $t(I(P)) = 1$ has $|I(P)| \geq 3$. Choose such a path P where $|I(P)|$ is minimum. Then, $I(P)$ has a single vertex v with a leaf, and let d_1 and d_2 be the distances between v and each end of P, so $d_1 + d_2 = |I(P)| + 1$. Suppose $d_1 \geq 2$ and $d_2 \geq 2$. By contracting an edge in $I(P)$, one creates a full path Q with $t(I(Q)) = 1$, where the single vertex v with a leaf is at distances $d_1 - 1$ and d_2 from each end of Q, or at distances d_1 and $d_2 - 1$ from each end of Q. In at least one of those cases, the distances between v and each end of Q are not equal, hence only one subdivision yields a graph with the same multiset $\{T(P_i) : i = 1, 2, ..., k\}$ as G. If $d_1 = 1$, there are d_2 cards corresponding to the contraction of an edge in P with a full path Q with $t(I(Q)) = 1$, where the single vertex v with a leaf is at distances 1 and $d_2 - 1$ from each end of Q. Since $|I(P)| \geq 3$, we get $d_2 - 1 \geq 2$, and so the distances between v and each end of Q are not equal, hence only one subidivision yields a graph with the same multiset $\{T(P_i) : i = 1, 2, ..., k\}$ as G. ∎

Other separable graphs with exactly one non-trivial block B are shown to be reconstructible, based on the set $\{T(v) \mid v \in V(B)\}$ instead of on the structure of B itself. If a card in $\mathcal{CT}$ has a vertex $v_0 \in V(B)$ such that $T(v_0)$ is not isomorphic to any limb in G, then one needs to replace $T(v_0)$ by the missing limb. Since the limbs are rooted trees, there is only one way to replace a limb with another limb.

Definition 4.3.22. *A collection $\mathcal{A}$ of graphs is* ***closed under contraction*** *if for every $G \in \mathcal{A}$ and every $e \in E(G)$, $G/e \in \mathcal{A}$.*

Theorem 4.3.23. *Let G be a separable graph with exactly one non-trivial block B. If* *there exists a vertex $v \in V(B)$ such that $\{T(\varphi(v)) : \varphi \in Aut(B)\}$ is not closed under contraction, then G is reconstructible.*

Proof: First, assume there is exactly one vertex $v \in V(B)$ such that $T(v) \neq \{v\}$. This property is recognizable, as it holds if and only if every card of $\mathcal{CB}$ also has that property. One can reconstruct $T(v)$ from any card of $\mathcal{CB}$. Then, if $t(v) \geq 2$, from any card of $\mathcal{CT}$, replace the only limb of that card by $T(v)$ to reconstruct G. If there are more than one vertex with non-trivial limbs, then by Lemma 4.3.15, the collections $\mathcal{T}(v) = \{T(\varphi(v)) : \varphi \in Aut(B)\}$ are reconstructible for every $v \in V(B)$. Furthermore, the limbs $T(\varphi(v))$ are rooted at the vertex $\varphi(v)$. If $\mathcal{T}(v)$ is not closed under contraction, then there exists a limb $T \in \mathcal{T}(v)$ and an edge $e \in E(T)$ such that $T/e \notin \mathcal{T}(v)$. Then, in G/e, replace the unique limb isomorphic to T/e attached to a vertex in the orbit of v by T to reconstruct G. ∎

Note that similar strategies were already used in this thesis. For instance, the proof of Theorem 4.3.14 uses a card with a block that contains too few edges to be a block of G.

4.4 Edge connectivity

Results about the edge connectivity are discussed in this section. In particular, we show that the edge connectivity is reconstructible. We then introduce a new notion of "blocks", called edge blocks. The edge blocks of a graph are induced by the minimum edge cuts, and our hope is to use this partition of the vertices into edge blocks in a similar way to how the block structure of a separable graph was used. The main obstacle of this strategy is that some blocks of separable graphs have only one point of attachment to the rest of the graph, while if we consider 2-edge connected graphs, it is possible that every edge block has at least two points of attachment.

We start, as usual, with some definitions. Then, we show that the edge connectivity is reconstructible. Finally, we investigate the edge block structure, and then use that block structure to show some reconstruction results.

4.4.1 Definitions and elementary results

The ***local edge connectivity*** between two distinct vertices x and y in the same connected component is the maximum number of pairwise edge-disjoint xy-paths, which is denoted by $p'(x, y)$. The local edge connectivity is only defined when $x \neq y$. A graph is ***k-edge-connected*** if $p'(x, y) \geq k$ for any pair of distinct vertices x and y. The ***edge connectivity*** of a graph G is the maximum value of k for which G is

k-edge connected. The edge connectivity of G is denoted by $\kappa'(G)$. Similar to the vertex connectivity, one can redefine the edge connectivity in terms of edge cuts.

For two distinct vertices x and y in the same connected component, an xy-***edge-cut*** is a set of edges E such that x and y are in distinct connected components of $G-E$. Denote by $c'(x,y)$ the minimum size of such an edge cut. A version of Theorem 4.1.1 for the edge connectivity is now presented.

Theorem 4.4.1 (Menger, [4])**.** *For any graph G and any two distinct vertices x and y of G,*

$$p'(x,y) = c'(x,y).$$

Therefore, we have $\kappa'(G) = \min\{c'(x,y) \mid x, y \in G,\ x \neq y\}$. A k***-edge cut*** is an xy-edge cut of size k for some vertices x and y. A ***cut edge*** is an edge in a 1-edge cut. A k-edge cut C is ***minimal*** if no proper subset of C is an edge cut. If $\delta(G)$ denotes the minimum degree of G, then this basic inequality holds:

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

The goal of this section is to understand the relation between the edge connectivity of a graph and the deck. The following lemma is key.

Lemma 4.4.2. *Let $e \in E(G)$ and let k be a positive integer. Then $C \subset E(G)$ is a k-edge cut of G that does not contain e if and only if $C \subset E(G/e)$ is a k-edge cut of G/e.*

Proof: Let $C \subset E(G)$ be a k-edge cut, with $xy \notin C$. Now $G - C$ is disconnected, and so is $(G-C)/xy$. Since G/xy is connected, then C is a k-edge cut of G/xy. Let C' be a k-edge cut of G/xy, so that $G/xy - C'$ is disconnected. This implies that when decontracting the vertex v^* in $G/xy - C'$, the resulting graph is also disconnected. Adding C' back to that graph yields G, implying that C' is a k-edge cut of G. ∎

The parameter $\kappa'(G)$ is reconstructible in most cases. The two graphs in Figure 4.5 are reconstructions of one another, and their edge connectivities are not equal, since the edge connectivity of the graph on the left is equal to the number of edges, while the edge connectivity of a graph with only one vertex is undefined.

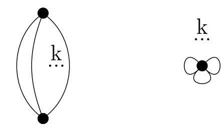

Figure 4.5: Two graphs whose edge connectivities are not reconstructible.

Proposition 4.4.3. *Let G be a graph and $k < e(G)$. The number of k-edge cuts of G is reconstructible.*

Proof: Let $C = \{e_1, ..., e_k\}$ be a k-edge cut of G. Then, by Lemma 4.4.2, C is a k-edge cut of G/e if and only if $e \notin C$. Therefore, every k-edge cut of G appears in $e(G) - k$ cards. Thus, the number of k-edge cuts of G, denoted by $c(G)$ is given by

$$c(G) = \frac{1}{e(G) - k} \sum_{H \in \mathcal{C}(G)} c(H) \tag{4.4.1}$$

where $c(H)$ is the number of k-edge cuts in the card H. Since $k < e(G)$, the right-hand side is well defined. Since the right-hand side of equation (4.4.1) is reconstructible, so too is the left-hand side. ∎

One cannot determine if there is a $e(G)$-edge cut in G or not, as Figure 4.5 shows. A consequence of Proposition 4.4.3 is that the edge connectivity of a graph is reconstructible for all but the graphs in Figure 4.5.

Corollary 4.4.4. *Let G be a graph not in Figure 4.5. The edge connectivity of G is reconstructible.*

Proof: The edge connectivity of a graph G is equal to $e(G)$ only if G has at least two vertices, and every edge is incident to every vertex. This is the case if and only if G is made of two vertices, and every edge links the two vertices. Therefore, one can assume that $\kappa'(G) \neq e(G)$. Proposition 4.4.3 counts the number of k-edge cuts of G for every $k < e(G)$. The minimum value of k for which there exists a k-edge cut is equal to $\kappa'(G)$. ∎

Note that $\kappa'(G)$ may be reconstructed using the analog of Kelly's Lemma for contraction, Proposition 3.2.1, by counting the number of contractions of G isomorphic to a graph with two vertices and k edges linking the two vertices. The minimum

k for which such a contraction exists is $\kappa'(G)$. However, the proof that we presented here makes use of Proposition 4.4.3, which is more general, as it shows that the number of k-edge cuts is reconstructible for any $k < e(G)$. For a connected graph G, the deletion of a k-edge cut may yield a graph with more than two connected components if $k > \kappa'(G)$. Therefore, it is not clear if the number of k-edge cuts can readily be obtained from Proposition 3.2.1 for any value of k.

4.4.2 Edge blocks

Studying graphs in terms of "blocks" was useful in the case of separable graphs. We will use a new definition for blocks in this section, which is now related to the edge connectivity rather than the vertex connectivity.

Definition 4.4.5. *Let $\mathcal{S}_G \subseteq E(G)$ be the union of all $\kappa'(G)$-edge cuts of G. A maximal connected subgraph of $G - \mathcal{S}_G$ is an* ***edge block*** *of G. An edge block is* ***trivial*** *if it has only one vertex. Denote by $\mathcal{EB}(G)$ the multiset of edge blocks of G. A vertex that is incident to an edge of a $\kappa'(G)$-edge cut is called an* ***attachment vertex****. Two edge blocks B_1 and B_2 are* ***adjacent*** *if a vertex of B_1 is adjacent to a vertex of B_2 in G. The* ***degree*** *of an edge block B is the number edges in a $\kappa'(G)$-edge cut with one end in B.*

Note that if B is an edge block of a graph G, then it is not necessarily true that B has a connectivity of at least $\kappa'(G)$. For instance, let G be a graph on four vertices x_1, x_2, x_3 and x_4. Link x_1 and x_2 with one edge, link x_3 to x_1 and x_2 with k edges each, and link x_4 to x_1 and x_2 with k edges each. Then $\kappa'(G) = 2k$, but $\{x_1, x_2\}$ is an edge block, with an edge connectivity of 1. See Figure 4.6 for this example.

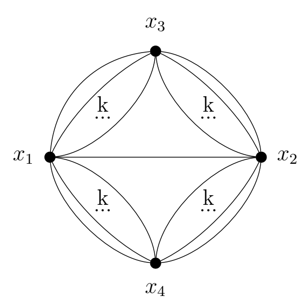

Figure 4.6: The graph has an edge connectivity of $2k$, yet one of its edge-blocks has an edge connectivity of 1.

From these definitions, it becomes clear that there are two disjoint sets of edges in a graph G: edges contained in an edge block and edges contained in a $\kappa'(G)$-edge cut. It is important to make the distinction between the two types, as we study the differences between a card corresponding to the contraction of an edge in an edge block and an edge in a $\kappa'(G)$-edge cut.

Definition 4.4.6. *An edge in a graph G is* ***external*** *if it is contained in a $\kappa'(G)$-edge cut. It is* ***internal*** *otherwise. A card is* ***external*** *if it corresponds to the contraction of an external edge, and is* ***internal*** *if it corresponds to the contraction of an internal edge. Denote by $\mathcal{C}_i(G)$ the set of internal cards and by $\mathcal{C}_e(G)$ the set of external cards of G.*

Even though the connectivity of a card of G may not be $\kappa'(G)$, an edge in a card is defined as external if it is contained in a $\kappa'(G)$-edge cut. Clearly, $\mathcal{C}(G) = \mathcal{C}_i(G) \cup \mathcal{C}_e(G)$ and $\mathcal{C}_i(G) \cap \mathcal{C}_e(G) = \emptyset$. Lemma 4.4.7 states that it is possible to distinguish the external cards from the internal cards.

Lemma 4.4.7. *Suppose G is not a graph in Figure 4.5. Let H be a card of G. Then, H is internal if and only if the number of $\kappa'(G)$-edge cuts in H is equal to the number of $\kappa'(G)$-edge cuts in G.*

Proof: Since G is not a graph in Figure 4.5, we may assume that $\kappa'(G) < e(G)$. Suppose $G/e \in \mathcal{C}(G)$ is internal. In other words, e does not belong to any $\kappa'(G)$-edge cuts of G. By Lemma 4.4.2, G/e contains as many $\kappa'(G)$-edge cuts as G. Conversely, if G/e is a card with the same number of $\kappa'(G)$-edge cuts, then by Lemma 4.4.2, e is not contained in any $\kappa'(G)$-edge cuts of G, hence G/e is internal. ∎

Corollary 4.4.8. *If G is a graph other than the ones in Figure 4.5, then $\mathcal{C}_i(G)$ and $\mathcal{C}_e(G)$ are reconstructible.*

Proof: By Proposition 4.4.3, the number of $\kappa'(G)$-edge cuts is reconstructible. Therefore, using Lemma 4.4.7, one can recognize whether any card is in $\mathcal{C}_i(G)$ or in $\mathcal{C}_e(G)$. ∎

Following a strategy similar to that used in the proof of Lemma 4.3.8, one can reconstruct $\mathcal{EB}(G)$ in certain cases, by considering only the cards in $\mathcal{C}_i(G)$. Lemma 4.4.9 shows how to find, among $\mathcal{C}_i(G)$, the deck of the disconnected graph whose connected components are the graphs of $\mathcal{EB}(G)$.

Lemma 4.4.9. *Let $\mathcal{S}_G \subseteq E(G)$ be the union of all $\kappa'(G)$-edge cuts of G, and $\mathcal{S}_H \subseteq E(H)$ be the union of all $\kappa'(G)$-edge cuts of H, for $H \in \mathcal{C}(G)$. Then the multiset $\{H - \mathcal{S}_H : H \in \mathcal{C}_i(G)\}$ is the deck of $G - \mathcal{S}_G$.*

Proof: By definition, an edge of G is in the subgraph $G - \mathcal{S}_G$ if and only if it is internal in G. Using Lemma 4.4.2, for any $e \in E(G - \mathcal{S}_G)$, the graph $(G - \mathcal{S}_G)/e$ is the graph $G/e - S_{G/e}$, with $G/e \in \mathcal{C}_i(G)$. Therefore, the deck of $G - \mathcal{S}_G$ is $\{H - \mathcal{S}_H : H \in \mathcal{C}_i(G)\}$. ∎

Reconstructing the graph $G - \mathcal{S}_G$ yields $\mathcal{EB}(G)$, as the list of the connected components of $G - \mathcal{S}_G$ is exactly $\mathcal{EB}(G)$.

Similar to the block degree of vertices in separable graph, we can think of the block degree of a vertex as the number of external edges that are incident to it.

Definition 4.4.10. *For any edge block B of a graph G, the **vertex-weighted edge block** B^+ of G is the block B with every vertex labeled by their number of incident external edges in G. For any graph G, the **set of vertex-weighted edge blocks** of G, denoted by $\mathcal{EB}(G)^+$, is the set $\mathcal{EB}(G)^+ = \{B^+ \mid B \in \mathcal{EB}(G)\}$.*

The vertex-weighted edge blocks $\mathcal{EB}(G)^+$ are much more useful than $\mathcal{EB}(G)$, as the vertex-weighted edge blocks show the points of attachment of every block to the rest of the graph. We now proceed to show that $\mathcal{EB}(G)^+$ is reconstructible, using a proof similar to that of Theorem 4.2.1.

Proposition 4.4.11. *If G is a graph with at least three internal edges and at least two non-trivial edge blocks, then $\mathcal{EB}(G)^+$ is reconstructible.*

Proof: By Corollary 4.4.8, it is possible to recognize a graph with at least three internal edges. Since G has at least three internal edges, the graph G has at least two non-trivial edge blocks if and only if there is an internal card of G with two non-trivial edge blocks. Therefore, one can recognize a graph with the conditions stated in the proposition.

By Corollary 4.4.8, one can partition the deck into $\mathcal{C}_i(G)$ and $\mathcal{C}_e(G)$. Consider a card $G/e\mathcal{C}_i(G)$ with an edge block B where $e(B)$ is maximal among all cards. Then B is an edge block of G, hence B^+ is one of the blocks of $\mathcal{EB}(G)^+$. Let $e \in E(B)$, and consider a card H of $\mathcal{C}_i(G)$ where the number of vertex-weighted edge blocks isomorphic to B^+ is minimal, and the number of vertex-weighted edge blocks isomorphic to B/e^+ is maximal. One can reconstruct $\mathcal{EB}(G)^+$ by taking all the vertex-weighted edge blocks in H and replacing one of the vertex-weighted edge blocks B/e^+ by B^+. ∎

When considering a vertex-weighted graph, we insist that automorphisms and isomorphisms also preserve the weights. In other words, any such function from one vertex-weighted graph to another must map a vertex with a weight of i to a vertex with a weight of i. Now that $\mathcal{EB}(G)^+$ is reconstructible, any $\kappa'(G)$-edge block in a card that is not isomorphic to any graph in $\mathcal{EB}(G)^+$ must have been contracted.

Identifying the contracted vertex is very useful in reconstruction, and if that is not possible, identifying the edge block of a card within which the contraction occurred is also useful.

Recall that a collection $\mathcal{A}$ of graphs is closed under contractions if for every $G \in \mathcal{A}$ and every $e \in E(G)$, $G/e \in \mathcal{A}$.

Proposition 4.4.12. *Let G be a graph such that $\mathcal{EB}(G)^+$ is not closed under contractions. Then there exists an internal card for which the contracted block can be identified.*

Proof: Since $\mathcal{EB}(G)^+$ is not closed under contractions, there exists a vertex-weighted edge block $B^+ \in \mathcal{EB}(G)^+$ and an edge $e \in B^+$ such that $B/e^+ \notin \mathcal{EB}(G)$. Therefore, in G/e, there is a unique vertex-weighted edge block that is not in $\mathcal{EB}(G)^+$, hence the contraction occurred within that edge block. ∎

The collection of all simple graphs is not closed under contractions. For instance, the graph K_3 contracts to a graph with parallel edges. In particular, if G is simple and $\mathcal{EB}(G)^+$ contains any graph with a cycle, then $\mathcal{EB}(G)^+$ is not closed under contractions, hence Proposition 4.4.12 can be applied.

From any card in $\mathcal{C}_i(G)$, the block structure of G can be determined. More precisely, the ***edge block graph*** of G is the graph obtained from G by contracting all of its internal edges. The edge block graph of G has the same edge connectivity as G, and every edge of the edge block graph is external. The vertices of the edge block graph of G correspond to edge blocks of G, and there are i edges between two vertices of the edge block graph if and only if there are i external edges in G linking two vertices of the corresponding edge blocks.

Proposition 4.4.13. *Let G be a simple graph with $v(G) > 2$. If $\mathcal{C}_i(G) \neq \emptyset$, then the edge block graph of G is reconstructible.*

Proof: By Corollary 4.4.8, $\mathcal{C}_i(G)$ is reconstructible. Take any card $H \in \mathcal{C}_i(G)$ and contract all of its internal edges to obtain the edge block graph of G. ∎

Note that if $\mathcal{C}_i(G) = \emptyset$, then G is equal to its edge block graph. We end this section with a lemma about the structure of the edge block graph, which will become useful in the next section. We will use the structure of the edge block graph to show that some graphs with $\kappa'(G) = 2$ are reconstructible.

Lemma 4.4.14. *If G' is the edge block graph of some graph G, then G' has a vertex of degree equal to $\kappa'(G)$.*

Proof: Let $k = \kappa'(G)$. Choose a k-edge cut of G', with edges $e_1, e_2, ..., e_k$. The removal of this k-edge cut yields a disconnected graph with exactly two connected components A and B. Suppose the chosen k-edge cut is such that $e(A)$ is minimum. If A has no edges, then A has a single vertex of degree k in G', and the proof is complete. Otherwise, A contains at least one edge e, which is external, and hence part of a k-edge cut. Since this edge cut is distinct from the previous one, it separates the graph G' into two parts, named C and D, such that when the two edge cuts are removed, the graph has at most 4 connected components: $A \cap C, A \cap D, B \cap C$ and $B \cap D$.

Suppose that all four parts are non-empty, so they all have at least one vertex. Let k_1, k_2, k_3, k_4, k_5 and k_6 be the number of edges between the different components, as described in Figure 4.7:

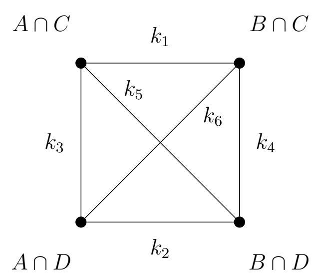

Figure 4.7: The structure of a graph and two of its k-edge cuts.

Since the graph G' has an edge connectivity equal to k, we have the following equations:

$$\begin{aligned}
k_1 + k_2 + k_5 + k_6 &= k \\
k_3 + k_4 + k_5 + k_6 &= k \\
k_1 + k_3 + k_5 &\geq k \\
k_1 + k_4 + k_6 &\geq k \\
k_2 + k_4 + k_5 &\geq k \\
k_2 + k_3 + k_6 &\geq k
\end{aligned} \tag{4.4.2}$$

The first two equations are obtained from the fact that the two edge cuts mentioned are of size k. The last four inequalities are obtained from the fact that since the graph has an edge connectivity of k, the degree of any connected subgraph must be at least k.

Equating the left-hand side of the first two equations yields $k_2 + k_1 = k_3 + k_4$. Summing the four inequalities yields

$$2k_1 + 2k_2 + 2k_3 + 2k_4 + 2k_5 + 2k_6 \geq 4k$$

$$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \geq 2k$$

Substituting $k_2 + k_1 = k_3 + k_4$, and then using the second equation, one obtains

$$\begin{aligned} 2k_3 + 2k_4 + k_5 + k_6 &\geq 2k \\ k_3 + k_4 &\geq k \end{aligned}$$

Using $k_3 + k_4 \geq k$ in the second equation, one obtains

$$k_5 + k_6 = k - (k_3 + k_4) \leq k - k = 0$$

Since k_5 and k_6 are non negative integers, one concludes that $k_5 = k_6 = 0$. The equations of (4.4.2) simplify to the following:

$$\begin{array}{rcl} k_1 + k_2 &=& k \\ k_3 + k_4 &=& k \\ k_1 + k_3 &\geq& k \\ k_1 + k_4 &\geq& k \\ k_2 + k_4 &\geq& k \\ k_2 + k_3 &\geq& k \end{array} \tag{4.4.3}$$

If $k_1 < k/2$, then

$$k + k > 2k_1 + (k_3 + k_4) = (k_1 + k_3) + (k_1 + k_4) \geq 2k$$

which is a contradiction. If $k_2 < k/2$, then

$$k + k > 2k_2 + (k_3 + k_4) = (k_2 + k_3) + (k_2 + k_4) \geq 2k$$

which is a contradiction. If $k_3 < k/2$, then

$$k + k > 2k_3 + (k_1 + k_2) = (k_1 + k_3) + (k_2 + k_3) \geq 2k$$

which is a contradiction. If $k_4 < k/2$, then

$$k + k > 2k_4 + (k_1 + k_2) = (k_1 + k_4) + (k_2 + k_4) \geq 2k$$

which is a contradiction. One must then assume that $k_1 = k_2 = k_3 = k_4 = k/2$. The connectivity k must then be even. Furthermore, the out degrees of $A \cap C$, $A \cap D$, $B \cap C$ and $B \cap D$ are equal to k. But then the edges linking $A \cap C$ to the rest of the graph forms a k-edge cut, and $e(A \cap C) < e(A)$, which contradicts the minimum choice of A.

Suppose now that at least one of the four parts is empty. Since the k-edge cut that induces the parts C and D is assumed to contain an edge of A, we get that

$A \cap C \neq \emptyset$ and $A \cap D \neq \emptyset$. So we can assume, without loss of generality, that $B \cap C = \emptyset$. Then $C \subset A$, but $C \neq A$, otherwise the edge cut defining the parts A and B would be the same as the one defining the parts C and D. Since $e(C) < e(A)$, we have obtained a contradiction to the minimum choice of A. ∎

An interesting observation that comes out of this proof is that if $\kappa'(G)$ is odd, and if E_1 and E_2 are two distinct $\kappa'(G)$-edge cuts of G', then $G' - E_1 - E_2$ has exactly three connected components. The maximum number of connected components that can be obtained by removing two $\kappa'(G)$-edge cuts is four, and a necessary condition for that maximum to be obtainable is that $\kappa'(G)$ be even.

A useful corollary to Lemma 4.4.14 is that in any graph G, there is an edge block of degree $\kappa'(G)$.

4.5 Reconstructing graphs with $\kappa'(G) = 2$

In this section, the results on the edge connectivity are applied to reconstruct graphs with $\kappa'(G) = 2$ and with non-trivial edge blocks of degree 2. While our hope is that the following results can be generalized to higher edge connectivities, we highlight some of the difficulties of extending to $\kappa'(G) > 2$.

To reconstruct graphs with non-trivial edge blocks of degree 2, the notion of sub-cuts and sub-blocks is introduced.

Definition 4.5.1. *Let G be a graph with $\kappa'(G) = 2$ that is not separable. Let B be an edge block of G. A **sub-cut** of B is a pair $(v, e) \in V(B) \times E(B)$ such that v is a separating vertex of $B - e$. If F is a connected component of $B - e - v$, then the subgraph of B induced by $V(F) \cup \{v\}$ is a **sub-block** of B with respect to (v, e). A sub-block A of an edge block B is **minimal** if it does not properly contain another sub-block of B.*

We will often write, for simplicity, that A is a sub-block of a graph G when we mean that A is a sub-block of a an edge block of G. See Figure 4.8 for an example of a sub-block and its corresponding sub-cut.

Lemma 4.5.2. *Let G be a graph with $\kappa'(G) = 2$ that is not separable. The edge sets of distinct minimal sub-blocks of G are disjoint.*

Proof: First, we observe that if A is a sub-block of an edge block B of the graph G with respect to (v, e), then e links a vertex of A to a vertex not in A. Furthermore, $B - e - v$ has exactly two connected components, one of which is $A - v$. If A is minimal, then A is a block of the separable graph $B - e$, and v is shared by the two blocks containing the ends of e.

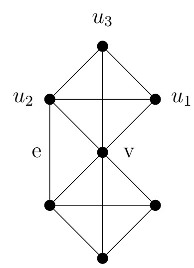

Figure 4.8: The subgraph induced by $\{u_1, u_2, u_3, v\}$ is a sub-block with respect to the sub-cut (v, e).

Let A_i be a minimal sub-block of the edge block B of G with respect to (v_i, e_i), for $i \in \{1, 2\}$. Let $e \in E(A_1) \cap E(A_2)$. If $e_1 \notin E(A_2)$ and $e_2 \notin E(A_1)$, then both A_1 and A_2 are single blocks of the separable graph $B - \{e_1, e_2\}$, and since they share an edge, we conclude that $A_1 = A_2$ using Proposition 4.1.4. If $e_1 \in E(A_2)$ and $e_2 \notin E(A_1)$, then A_1 is a single block of $B - \{e_1, e_2\}$, while A_2 is the union of at least one block of $B - \{e_1, e_2\}$. Since $e \in E(A_1) \cap E(A_2)$, A_2 contains the block of $B - \{e_1, e_2\}$ that contains e, which is A_1, hence $A_1 \subset A_2$. This contradicts the assumption that A_2 is minimal. If $e_1 \notin E(A_2)$ and $e_2 \in E(A_1)$, then a similar argument leads to the conclusion that A_1 is not minimal. Finally, suppose $e_1 \in E(A_2)$ and $e_2 \in E(A_1)$. If $A_1 - e_2$ is exactly one block of $B - \{e_1, e_2\}$, then since e_2 must link vertices in distinct blocks of $B - e_2$, it implies that $B - e_2$ is not a separable graph, hence A_2 is not a sub-block of B. We can therefore assume that both $A_1 - e_2$ and $A_2 - e_1$ are unions of multiple blocks of $G - \{e_1, e_2\}$. Then, since $e_2 \in E(A_1)$, A_1 contains both ends of e_2, and so A_1 contains the two blocks of $B - \{e_1, e_2\}$ that contain the ends of e_2. We conclude that $v_2 \in V(A_1)$. Then, since $B - \{e_1, e_2\}$ is separable, its block structure is a tree where v_1 and v_2 are separating vertices, and so one concludes that A_1 contains either A_2 or the other sub-block with respect to (v_2, e_2). This contradicts the minimality of A_1. ∎

We note here that it is not necessary for every vertex of G to be contained in a minimal sub-block of G. In other words, while the minimal sub-blocks are disjoint, they do not form a partition of $V(G)$. See Figure 4.9 for an example.

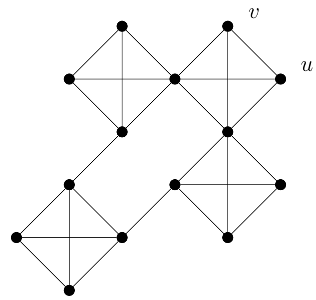

Figure 4.9: The sub-blocks of a graph do not necessarily form a partition of the vertices. Here, the vertices u and v are not part of any minimal sub-blocks of the graph.

We now present a series of theorems showing that various classes of graphs with an edge connectivity of 2 are reconstructible.

Theorem 4.5.3. *Let G be a graph with $\kappa'(G) = 2$, at least three internal edges and two non-trivial edge blocks. If there exists an edge block B of G such that B is a separable graph and at least one leaf block of B is non-trivial and contains at most one attachment vertex to the rest of G, then G is reconstructible.*

Proof: By Corollary 4.4.4, graphs with $\kappa'(G) = 2$ are recognizable. By Proposition 4.4.11, the list of vertex weighted edge blocks $\mathcal{EB}(G)^+$ is reconstructible. Therefore, one can determine from $\mathcal{EB}(G)^+$ if there exists an edge block B that satisfies the conditions of the theorem.

Consider a card $H \in \mathcal{C}_i(G)$ with the same number of separable edge blocks as G, and let B' be an edge block of H that is separable with a non-trivial leaf block that contains at most one attachment vertex. Since there is an edge e in a non-trivial leaf block of the separable edge block B, we conclude that H exists. Choose such an H where that non-trivial leaf block A is minimal with respect to the number of vertices. Then, a vertex of A is to be decontracted to obtain G, and it therefore remains to remove A, replace it with the correct block A', and attach A' to the rest of the card to reconstruct G. Note that A and A' are attached to the rest of H and G, respectively, by a cut-vertex and a vertex of attachment.

Since $\mathcal{EB}(G)^+$ is known, one can determine A', its cut-vertex, and its vertex of attachment. Therefore, one can replace A with A' in the card and reconstruct G. ∎

The key observation in the previous proof is that A' is a leaf block of a separable edge block B and hence is attached to the rest of the graph via two vertices: a cut-vertex of the edge block, and a vertex of attachment to another block of B. Since these two vertices play a different role in G, they cannot be confused for one another when reattaching A'. It therefore becomes straightforward to replace A' by the correct block A and to attach it to the rest of the graph. This idea of "asymetrical" points of attachment will feature in the proofs of the following theorems in this section, and we feel it is the main problem with generalizing this proof strategy to graphs with $\kappa'(G) > 2$.

Theorem 4.5.4. *Let G be a graph with $\kappa'(G) = 2$, at least three internal edges and two non-trivial edge blocks. If there exists an edge block B of G such that B has a minimal sub-block with no attachment vertices to other edge blocks of G, then G is reconstructible.*

Proof: By Corollary 4.4.4, graphs with $\kappa'(G) = 2$ are recognizable. By Proposition 4.4.11, the list of vertex weighted edge blocks $\mathcal{EB}(G)^+$ is reconstructible. Therefore, one can determine if G has an edge block B that has a minimal sub-block with no attachment vertices. Let B be such an edge block and let A be a minimal sub-block of B with respect to (v_0, e_0) with no attachment vertices. Since A is minimal, A is not separable.

Let C be a cycle of maximum length in A and let $e \in E(C)$ be such that e does not link an end of e_0 to v_0. Such an edge exists: if $e(A) = 1$, then B contains a vertex of degree 2 in G, which implies that B contains a 2-edge cut of the graph, which is a contradiction with the fact that B is an edge-block of G. Therefore, $e(A) \geq 2$ and at least one edge of A does not link e_0 to v_0. Suppose A/e is not a minimal sub-block of B/e. There exists a pair $(v', e') \in V(B/e) \times E(B/e)$ such that one of the sub-blocks of B/e with respect to (v', e') is contained in A/e. Since v' is a vertex of every sub-block with respect to (v', e'), we conclude that $v' \in V(A/e)$. Let $v^* \in V(A/e)$ denote the contracted vertex. If $v^* \neq v'$, then the decontraction of v^* in $B/e - e'$ yields a graph where v' is a separating vertex, which implies that there is a sub-block of B with respect to (v', e') that is contained in A, contradicting the minimality of A. Therefore, one concludes that $v^* = v'$.

Suppose $e' \notin E(A/e)$. Since e' links two vertices of distinct sub-blocks of B/e with respect to (v', e'), we conclude that one end of e' is in A/e. Since $e' \notin E(A/e)$, the other end of e' is not in A/e. We see that $e' = e_0$, for otherwise v_0 is not a separating vertex of $B - e_0$. Since $e' = e_0$ and $v' = v^*$, we get that v^* is a separating vertex of A/e, so the ends of e are a 2-vertex cut of A. If they are not a minimal 2-vertex cut of A, then one of the ends v'' of e is a separating vertex of A, hence there is a sub-block of B with respect to (v'', e_0) contained in A. This contradicts the minimality of A. Therefore, the ends of e are a minimal 2-vertex cut of A. However, since e is contained in C, a cycle of maximum length in A, we conclude from Proposition 4.3.6 that the ends of e cannot form a 2-vertex cut of A. This contradiction allows us to conclude that $e' \in E(A/e)$. It has now been shown that if A/e is not a minimal sub-block of B/e, and D is a minimal sub-block of B/e contained in A/e with respect to (v', e'), then both v' and e' are in A/e.

We now show that G is reconstructible. Consider a card $H \in \mathcal{C}_i(G)$ with an edge block B' isomorphic to B/e, where a sub-block of B' is isomorphic to A/e and has no attachment vertices to other edge blocks of H. Since e is an edge in a minimal sub-block, there is only one edge block B' in H isomorphic to B/e. One can then identify in B' the sub-cut corresponding to (v_0, e_0), as one sub-block of that sub-cut in B' is isomorphic to $B - A$, and the other sub-block, which we will call A', is isomorphic to A/e. It can be determined that A' needs to be removed and replaced by a sub-block A in order to reconstruct G. Since A' and A are attached to the rest of their graphs by only an edge and a vertex, one can replace A' by A in the card to reconstruct G. ∎

Theorem 4.5.5. *Let G be a graph with $\kappa'(G) = 2$, at least three internal edges and two non-trivial edge blocks. If there exists a non-trivial edge block B of G of degree 2, such that B is adjacent to two other edge blocks, then G is reconstructible.*

Proof: By Corollary 4.4.4, graphs with $\kappa'(G) = 2$ are recognizable. By Proposition 4.4.11, the list of vertex weighted edge blocks $\mathcal{EB}(G)^+$ is reconstructible. By Proposition 4.4.13, the edge block graph is reconstructible, hence one can determine if there is an edge block of G satisfiying the properties in the statement of the theorem. Let B be such an edge block. Furthermore, one can assume that no block of G is a separable graph with at least one non-trivial leaf block that contains at most one attachment vertex, for otherwise G is reconstructible by Theorem 4.5.3. In particular, any edge block of degree 2, such as B, cannot be separable. All the edge blocks that are linked to B by an external edge are either 2-connected graphs, or separable graphs where each of their leaf blocks have at least two attachment vertices.

Suppose there is a card $G/e \in \mathcal{C}_e(G)$ with a separable edge block B' that has a leaf block with exactly one attachment vertex. We first note that B' is unique with that property in G/e, since G itself has no such edge blocks. Then e must be an external edge incident to an edge block B of degree 2 such that B is adjacent to two distinct edge blocks. Call A the other edge block incident to e.

If B' has exactly one cut vertex v^*, then one reconstructs G by decontracting v^* in the following way: two neighbours of v^* are adjacent to the same vertex after the decontraction of v^* if and only if they both belong to the same block of B'.

If B' has more than one cut vertex, then A is separable. It implies, by hypothesis that all the leaf blocks of A have at least two attachment vertices. Therefore, only one leaf block F of B' has exactly one attachment vertex, and since B is not separable, one decontracts the separating vertex v^* of B' in F in the following way to reconstruct G: two neighbours of v^* are adjacent to the same vertex after the decontraction of v^* if and only if they both belong to F or if they both do not belong to F.

We may therefore suppose that there are no cards of $\mathcal{C}_e(G)$ with a separable edge block tghat has a leaf block with exactly one attachment vertex. We conclude that the two edge blocks adjacent to B are trivial, which means they only have one vertex each. In that case, consider a card $G/e \in \mathcal{C}_e(G)$ with an edge block B' that contains exactly two attachment vertices where the distance between B' and the nearest non-trivial block is minimum. To reconstruct G, it remains to decontract the attachment vertex v^* of B' which is nearest to a non-trivial block in the following way: two neighbours of v^* are adjacent to the same vertex after the decontraction if and only if they both belong to B' or if they both do not belong to B'. ∎

Theorem 4.5.6. *Let G be a graph with $\kappa'(G) = 2$, at least three internal edges and two non-trivial edge blocks. If there exists a non-trivial edge block B of G of degree 2 such that B is adjacent to exactly one other edge block, then G is reconstructible.*

Proof: By Corollary 4.4.4, graphs with $\kappa'(G) = 2$ are recognizable. By Proposition 4.4.11, the list of vertex weighted edge blocks $\mathcal{EB}(G)^+$ is reconstructible. By Proposition 4.4.13, the edge block graph is reconstructible, hence one can determine if there is an edge block of G satisfiying the properties in the statement of the theorem. One can assume that no edge blocks of G contain a minimal sub-block with no attachment vertices, for otherwise G is reconstructible by Theorem 4.5.4. Furthermore, one can assume that the edge blocks of degree 2 are not separable, for otherwise G is reconstructible by Theorem 4.5.3.

Consider a card $G/e \in \mathcal{C}_e(G)$ with an edge block B' that contains a minimal sub-block A with no attachment vertices. Such an edge block exists in G/e if e is an external edge incident to B. Furthermore, since G is assumed to have no edge blocks with a minimal sub-block with no attachment vertices, the edge block B' is unique in G/e with that property. Then, e must be an external edge linking an edge block B of degree 2 such that B is adjacent to only one other edge block. Let $v_0 \in V(B')$ and $e_0 \in E(B')$ be such that A_0 is a sub-block of B' with respect to (v_0, e_0), and $A_0 \cong B$. We show that the sub-cut (v_0, e_0), as described, is unique. Suppose (v_1, e_1) is another such sub-cut of B', and A_1 is a sub-block with respect to (v_1, e_1) with no vertices of attachment, such that $A_1 \cong B$. If $v_1 \in V(A_0)$ and $e_1 \in E(A_0)$, then both v_0 and e_0 must be in the same sub-block of B' with respect to (v_1, e_1), which implies that $v(A_1) < v(A_0)$, and so A_1 cannot be isomorphic to B. Similarly, if $v_1 \notin V(A_0)$ and $e_1 \notin E(A_0)$, one can conclude that $v(A_1) > v(A_0)$, which again leads to a contradiction. If $v_1 \in V(A_0)$ and $e_1 \notin E(A_0)$, then v_1 is a separating vertex of A_0, which contradicts the assumption that the edge blocks of degree 2 are not separable. Finally, a similar conclusion is obtained if $v_1 \notin V(A_0)$ and $e_1 \in E(A_0)$. Therefore, one can identify the sub-cut (v_0, e_0) uniquely.

Then, one reconstructs G by decontracting v_0 in the following way: two neighbours of v_0 are adjacent to the same vertex after the decontraction of v_0 if and only if they are in the same partition of $B' - e_0$ with respect to v_0. ∎

We gather Theorems 4.5.5 and 4.5.6 to conclude the following:

Corollary 4.5.7. *Let G be a graph with $\kappa'(G) = 2$, at least three internal edges and two non-trivial edge blocks. If there exists a non-trivial edge block of degree 2, then G is reconstructible.*

Proof: By Corollary 4.4.4, graphs with $\kappa'(G) = 2$ are recognizable. By Proposition 4.4.11, the list of vertex weighted edge blocks $\mathcal{EB}(G)^+$ is reconstructible, and it is therefore possible to determine if G has a non-trivial edge block of degree 2. If that edge block is adjacent to two distinct edge blocks, apply Theorem 4.5.5. Otherwise, that edge block is adjacent to exactly one other block, and apply Theorem 4.5.6. ∎

By Lemma 4.4.14, a graph G with $\kappa'(G) = 2$ is guaranteed to contain an edge block of degree 2. Graphs with an edge connectivity of 2 that have not yet shown to be reconstructible must have all of its edge blocks of degree 2 be trivial, or they must contain fewer than three internal edges.

4.6 Tutte polynomial

In this section, we study the Tutte polynomial and show that it is reconstructible. The two prerequisites for the proof is that the number of spanning trees and the edge connectivity is reconstructible. The number of spanning trees was shown to be reconstructible using the contraction version of Kelly's Lemma in Proposition 3.2.1 and the edge connectivity $\kappa'(G)$ is reconstructible by Proposition 4.4.4. The definition and properties of the Tutte polynomial that are presented here are taken from [4].

The ***Tutte polynomial*** of a graph G, denoted by $T(G, x, y)$, is defined as follows:

$$T(G, x, y) = \sum_{S \subseteq E(G)} (x-1)^{c(S)-c(G)} (y-1)^{|S|-v(G)+c(S)}$$

where $c(S)$ is the number of connected components of the spanning subgraph of G with edge set S. The Tutte polynomial satisfy the following properties:

Lemma 4.6.1. *([4]) Let G be a graph. The Tutte polynomial of G has the following properties.*

$T(G, x, y) = yT(G - e, x, y)$ *if e is a loop of G,*

$T(G, x, y) = xT(G/e, x, y)$ *if e is a cut-edge of G,*

$T(G, x, y) = T(G - e, x, y) + T(G/e, x, y)$ *otherwise.*

A and B are disjoint subgraphs of a graph G, then $T(A \cup B, x, y) = T(A, x, y)T(B, x, y)$

If G has no edges, then $T(G, x, y) = 1$.

The properties of the Tutte polynomial stated in Lemma 4.6.1 gives a way to compute the $T(G, x, y)$ of a graph G using a deletion-contraction algorithm. One remarkable property of the Tutte polynomial is that it is entirely defined by the recurrence relations and the base case mentioned above.

Many evaluations of the Tutte polynomial yield interesting graph parameters. We have the following [4]:

$T(G, 1, 1)$ is the number of spanning trees,

$T(G, 1, 2)$ is the number of connected spanning subgraphs,

$T(G, 2, 1)$ is the number of spanning forests,

$T(G, 0, 2)$ is the number of strong orientations,

$T(G, 2, 0)$ is the number of acyclic orientations,

$T(G, 1-x, 0) = (-1)^{v(G)-1}x^{c(G)-1}P(G, x)$, where $P(G, x)$ is the chromatic polynomial.

To understand these parameters, some definitions are required. Recall that a ***forest*** is a graph with no cycles. A connected forest is a ***tree***. A ***spanning subgraph*** of G is a subgraph H with $V(H) = V(G)$.

An ***orientation*** of a graph G is to assign a direction to every edge of G, so that edges can only be traversed in that assigned direction. An orientation is ***strong*** if every vertex can be reached from any other vertex. An ***oriented cycle*** is a cycle in an orientation of a graph, whose edges are oriented consistently. An orientation is ***acyclic*** if it contains no oriented cycles.

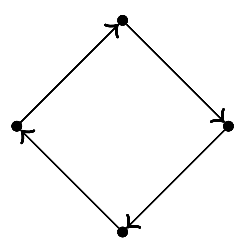

Figure 4.10: An oriented cycle.

A ***k-vertex colouring*** of a graph G is an assignment of k colours to its vertices. This colouring is a ***proper k-vertex colouring*** if no two adjacent vertices have the same colour. The ***chromatic number*** of G, denoted $\chi(G)$, is the least number k for which there exists a proper k-vertex colouring of G. Under a given proper colouring, a ***colour class*** is the set of all vertices of the same colour. It follows from the definition that graphs with loops cannot be properly coloured, and that every proper k-vertex colouring of a loopless graph is also a k-vertex proper colouring of its underlying simple graph.

This leads to the more general concept of chromatic polynomial. The ***chromatic polynomial*** of G, denoted by $P(G,x)$, is a function such that for any integer $k \geq 0$, $P(G,k)$ is equal to the number of proper k-vertex colourings of G. As the name suggests, $P(G,x)$ is a polynomial. While this is not obvious from the definition, it can be shown that $P(G,x)$ satisfy recurrence relations similar to those of $T(G,x,y)$. A deletion-contraction algorithm can be applied to $P(G,x)$ which allows us to conclude that it is indeed a polynomial.

We show that the Tutte polynomial is reconstructible. The proof is based on the edge connectivity $\kappa'(G)$ of the graph and the main idea is to apply the recurrence relations multiple times.

Proposition 4.6.2. *Let G be a graph that is not in Figure 3.1 and $e(G) \geq 4$. Then $T(G,x,y)$ is reconstructible.*

Proof: The proof is split according to $\kappa'(G)$. If $\kappa'(G) = 0$, then G is disconnected. By Theorem 4.2.1, if G has at least two non-trivial connected components, then G is reconstructible, and so is $T(G,x,y)$. Otherwise, G has only one non-trivial connected component B, and some number of isolated vertices. Since $T(K_1,x,y) = 1$ and that $T(A \cup B,x,y) = T(A,x,y)T(B,x,y)$ if A and B are disjoint subgraphs of G, we conclude that $T(G,x,y) = T(B,x,y)$. It therefore only remains to show that the Tutte polynomials of connected graphs are reconstructible.

Suppose $\kappa'(G) = 1$. By Proposition 4.4.4, this is a reconstructible property. By Corollary 4.4.8, one can reconstruct $\mathcal{C}_e(G)$, the set of external cards. If $G/e \in \mathcal{C}_e(G)$, then e is a cut-edge of G, which implies that $T(G,x,y) = xT(G/e,x,y)$. Since the right-hand side of the equation is reconstructible, so too is the left-hand side.

Suppose then that $\kappa'(G) \geq 2$. Since G is connected, there exists a set of $e(G) - v(G) + 1 = k$ edges of G whose complement in $E(G)$ is a tree. Let $\{e_1, e_2 ..., e_k\}$ be such a set. Since e_1 is not a cut-edge of G, we get that

$$T(G,x,y) = T(G - e_1,x,y) + T(G/e_1,x,y).$$

Since e_2 is not a cut-edge of $G - e_1$, we get that

$$T(G,x,y) = T(G - \{e_1,e_2\},x,y) + T(G/e_2 - e_1,x,y) + T(G/e_1,x,y).$$

More generally, e_j is not a cut-edge of $G - \{e_1, e_2, ..., e_j\}$, since the deletions of $\{e_1, ..., e_k\}$ yields a connected graph. Therefore, one can continue to apply the recurrence relation and obtain the following:

$$T(G,x,y) = T(G-\{e_1,...,e_k\},x,y) + \sum_{j=1}^{k} T(G/e_j - \{e_1,...,e_{j-1}\},x,y). \quad (4.6.1)$$

We now aim to show that the right-hand side is reconstructible. Since $G - \{e_1,...,e_k\}$ is a tree, we have that $T(G-\{e_1,...,e_k\},x,y) = x^{v(G)-1}$. Let $E(G) = \{e_1,...,e_m\}$ be an ordering of the edges of G such that $G-\{e_1,...,e_k\}$ is a tree. Let Γ be the set of permutations of $E(G)$ so that if $\varphi \in \Gamma$, then $G - \{\varphi(e_1),...,\varphi(e_k)\}$ is a tree. We get that $|\Gamma| = t(G)(v(G)-1)!(e(G)-v(G)+1)!$, where $t(G)$ is the number of spanning trees of G. By Proposition 3.2.4, the number of spanning tree is reconstructible and by Proposition 3.1.3, $v(G)$ is reconstructible, hence so is $|\Gamma|$. Therefore, equation (4.6.1) holds if one applies any permutation $\varphi \in \Gamma$ to the edge ordering $\{e_1,e_2,...,e_m\}$. Therefore, for any $\varphi \in \Gamma$:

$$T(G,x,y) = x^{v(G)-1} + \sum_{j=1}^{k} T(G/\varphi(e_j) - \{\varphi(e_1),...,\varphi(e_{j-1})\},x,y).$$

By summing over all permutations in Γ, one obtains:

$$|\Gamma|T(G,x,y) = |\Gamma|x^{v(G)-1} + \sum_{j=1}^{k}\sum_{\varphi\in\Gamma} T(G/\varphi(e_j) - \{\varphi(e_1),...,\varphi(e_{j-1})\},x,y). \quad (4.6.2)$$

We have shown that $|\Gamma|$ is reconstructible. It therefore remains to show that the summation on the right-hand side of equation (4.6.2) is reconstructible to complete the proof. Indeed, we claim that for any $j \le k$:

$$\sum_{\varphi\in\Gamma} T(G/\varphi(e_j) - \{\varphi(e_1),...,\varphi(e_{j-1})\},x,y)$$
$$= \sum_{H\in\mathcal{C}(G)} \sum_{\substack{A\subseteq E(H)\\ |A|=j-1\\ H-A \text{ is connected}}} T(H-A,x,y)(j-1)!(k-j)!(e(G)-k)!. \quad (4.6.3)$$

Since $\kappa'(G) \ge 2$, no edge of G is a cut-edge, therefore we sum over all cards of $\mathcal{C}(G)$. A factor of $(j-1)!$ is introduced to account for the permutations of the set A. A factor of $(k-j)!$ is introduced to account for the permutations of the set $\{e_{j+1},...,e_k\}$. Finally, a factor of $(e(G)-k)!$ is added to account for the permutations of the remaining edges $\{e_{k+1},...,e_{e(G)}\}$, which form a tree in G. Now, the right-hand side of equation (4.6.3) is reconstructible, which completes the proof. ∎

Note that the proof for the case where $\kappa'(G) \geq 2$ can be applied to graphs with $\kappa'(G) = 1$, with the adjustment that the sum on the right-hand side of equation (4.6.3) be taken over $\mathcal{C}_i(G)$ instead of $\mathcal{C}(G)$. However, using the recurrence relation $T(G, x, y) = xT(G/e, x, y)$ when e is a cut-edge is more direct.

A consequence of Proposition 4.6.2 is that all the parameters that can be deduced from the Tutte polynomial are reconstructible.

Corollary 4.6.3. *Let G be a graph with $e(G) \geq 4$. The following are reconstructible:*

The number of connected spanning subgraphs,

The number of spanning forests,

The number of strong orientations,

The number of acyclic orientations,

The chromatic polynomial $P(G, x)$,

The chromatic number $\chi(G)$.

Even if $T(G, 1, 1)$ gives the number of spanning trees of G, we cannot conclude that Proposition 4.6.2 implies that $t(G)$ is reconstructible since knowing $t(G)$ is necessary in the proof of Proposition 4.6.2.

In [6], many classes of graphs were shown to be T-unique, meaning that these graphs are entirely determined by their Tutte polynomials, and hence are reconstructible. We now define these classes of graphs.

We say that G is a ***complete multipartite*** graph if $V(G)$ can be partitioned into some number $V_1, V_2, ..., V_k$ of subsets such that for any $1 \leq i \leq k$, the set of neighbours of any vertex of V_i is exactly all the vertices not in V_i.

A graph is a ***wheel*** if it is obtained from a cycle by adding a vertex adjacent to all the vertices in the cycle.

The ***square of cycle*** is the graph obtained from a cycle by linking all the vertices at distance 2.

A ***ladder*** graph is obtained from two copies C and C' of a cycle by linking every vertex of C to its corresponding vertex in C'. A ***Möbius ladder*** is constructed from a cycle of length $2n$ by joining every pair of vertices at distance n.

The ***hypercube*** Q^n has as a vertex set all the binary strings of length n. Two vertices are adjacent if their binary strings differ in exactly one place. Hypercubes are part of a larger class of graphs called the distance regular graphs. They are studied in more detail in Chapter 6.

All of the graphs defined above have been shown to be T-unique in [6], which leads to the following theorem.

Theorem 4.6.4. *Let G be a graph such that $e(G) \geq 4$. If G belongs to any of the following classes, then G is reconstructible.*

Complete multipartite graphs, except $K_{1,p}$, for some $p \geq 3$,

Wheel graphs,

Square of cycle graphs,

Ladder graphs,

Möbius ladders,

Hypercubes.

By its definition, a graph G is bipartite if and only if $\chi(G) \leq 2$. Therefore, Proposition 4.6.2 allows us to recognize bipartite graphs. We explore bipartite graphs in more details in Chapter 5.

Corollary 4.6.5. *All bipartite graphs except the ones in Figure 3.1 are recognizable.*

It should also be mentioned that the proof of Proposition 4.6.2 can be altered to show that the Tutte polynomial is edge reconstructible. Indeed, by reversing the order $e_1, ..., e_m$ on the edges and by applying the recursive formula on the other polynomial on the right-hand side of equation (4.6.1), one obtains an equation similar to (4.6.1):

$$T(G, x, y) = T(G/\{e_m, ..., e_{k+1}\}, x, y) + \sum_{j=1}^{m-k} T((G - e_{m-j})/\{e_m, ..., e_{m-j+1}\}, x, y).$$

Since $e_m, ..., e_{k+1}$ are the edges of a spanning tree, $G/\{e_m, ..., e_{k+1}\}$ is made of a single vertex with k loops, hence its Tutte polynomial is y^k. By an argument similar to the one in the proof of Proposition 4.6.2, one can show that the Tutte polynomial is edge reconstructible. However, since the Tutte polynomial is vertex reconstructible [3], and that a graph is edge reconstructible if it is vertex reconstructible, it was already known that the Tutte polynomial is edge reconstructible.

Chapter 5

Cycles

In this chapter, we study the cycles of graphs in relation with the contraction reconstruction problem. We show that the number of cycles of certain lengths is reconstructible. We also show that bipartite graphs are reconstructible.

Recall that a ***cycle*** in a graph G is a sequence of distinct vertices $v_1, v_2, ..., v_k$, such that v_i is adjacent to v_{i+1}, and such that v_k is adjacent to v_1. The ***length*** of a cycle is the number of vertices in the sequence.

5.1 On the number of cycles

In this section, we reconstruct the number of cycles of various lengths. These results show, in particular, that simple graphs are recognizable. We also show that the girth of a graph is reconstructible.

5.1.1 Loops and parallel edges

The presence of loops or of parallel edges is what separates a simple graph from a multigraph. While most of the results in this thesis are about simple graphs, it is important to be able to determine when a deck corresponds to a graph that is not simple, and we prove that the number of loops and parallel edges are almost always reconstructible, which in turn shows that almost all simple graphs are recognizable.

Recall that the ***multiplicity*** of xy is the number of edges whose ends are $\{x, y\}$. If there are no edges linking x and y, we say that the multiplicity of xy is 0.

Proposition 5.1.1. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then the multiplicity of every link is reconstructible.*

Proof: A card of G corresponds to the contraction of a loop if and only if that card has $v(G)$ vertices. By Proposition 3.1.3, $v(G)$ is reconstructible and one can determine the cards of $\mathcal{C}(G)$ which correspond to the contraction of a loop. If such a card H exists, then the multiplicity of any edge of G is equal to the multiplicity in H. We may therefore assume that no such card exists, and so that G has no loops.

For a loopless graph G, the number of loops in a card G/e gives the number of edges parallel to e, and thus the multiplicity of e. ∎

The condition that G is not a graph composed of a graph in Figure 3.1 and isolated vertices is necessary: the graph with two vertices and k edges joining them is a reconstruction of the graph with one vertex and k loops. Those two graphs do not have the same edge multiplicities.

In particular, Proposition 5.1.1 reconstructs the number of edges in the underlying simple graph of any graph. The proof also shows that if $v(G)$ is reconstructible, then so is the number of loops and the number of parallel edges, which is the number of links $e \in E(G)$ for which there exists another links $e' \in E(G)$ with the same ends as e. In fact, it follows from Proposition 3.1.3, Corollary 3.1.4 and Proposition 5.1.1 that the number of vertices, the number of loops and the edge multiplicities are either all reconstructible, or none are reconstructible.

Corollary 5.1.2. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then the number of vertices, the number of loops, and the edge multiplicities, are reconstructible.*

Proof: Apply Proposition 3.1.3, Corollary 3.1.4 and Proposition 5.1.1. ∎

Note that Proposition 3.1.2 is not exactly a consequence of Corollary 5.1.2. If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then Proposition 3.1.2 shows that G is not simple for $k > 1$. However, Corollary 5.1.2 tells us nothing about G. In any other case, Proposition 3.1.2 is a consequence of Corollary 5.1.2.

We end this section with Proposition 5.1.3, which states the very minimal conditions under which the number of loops attached to each vertex of a graph is reconstructible. More specifically, Proposition 5.1.3 shows that unless G is a graph in Figure 5.1, then the number of loops attached to each vertex of G is reconstructible.

Figure 5.1: The only graphs where the number of loops attached to each vertex is not reconstructible.

Proposition 5.1.3. *Let G be a graph other than the ones in Figure 5.1. Then, the number of loops attached to each vertex is reconstructible.*

Proof: Denote by $\ell(v)$ the number of loops attached to the vertex v. We are looking to reconstruct the multiset $\{\ell(v) \mid v \in V(G)\}$. Since we assume that G is not, in particular, one of the first two graphs in Figure 5.1, we may assume, by Corollary 5.1.2, that the number of loops and the number of vertices are reconstructible. In particular, one may identify the subset $\mathcal{L}(G)$ of $\mathcal{C}(G)$ that corresponds to the contractions of loops.

Suppose there exists a card of $\mathcal{L}(G)$ with two vertices v_1 and v_2 such that $\ell(v_1) \neq 0$ and $\ell(v_2) \neq 0$. This implies that there are at least two vertices in G with loops attached. Then, let $k = \max\{\ell(v) \mid v \in V(H), H \in \mathcal{L}(G)\}$. Since there are at least two vertices of G with loops, we conclude that there is a vertex $v \in V(G)$ such that $\ell(v) = k$. Then, consider a card $H \in \mathcal{L}$ where the number of vertices with k loops is minimum. One reconstructs the multiset $\{\ell(v) \mid v \in V(G)\}$ by replacing one instance of $k-1$ in $\{\ell(v) \mid v \in V(H)\}$ with k.

We may then assume that no card of $\mathcal{L}(G)$ has two vertices with loops. If G has only one loop, then the number of loops attached to each vertex is certainly reconstructible. This leaves us with $\{l(v) \mid v \in V(G)\} = \{1, 1, 0, 0, ..., 0\}$ or $\{2, 0, 0, 0, ..., 0\}$. Let G_1 be a graph with two loops and $u_1, u_2 \in V(G_1)$ be such that $\ell(u_1) = \ell(u_2) = 1$. Let G_2 be a graph with two loops and $v \in V(G_2)$ be such that $\ell(v) = 2$.

Suppose G_1 and G_2 are reconstructions of one another. Then, $\mathcal{L}(G_1) \cong \mathcal{L}(G_2)$, which implies that removing all loops from G_1 and G_2 results in isomorphic graphs, which we will call G. This also implies that, in G, the degrees of u_1, u_2 and v are equal.

If there is a simple link whose ends are not $\{u_1, u_2\}$, then $\mathcal{C}(G_1)$ contains a card with exactly two vertices with one loop each. A card with such properties does not exist in $\mathcal{C}(G_2)$. Therefore, every edge that does not link u_1 to u_2 must have a multiplicity of at least 2 in G.

If there is a link that is not incident to either u_1 or u_2, then its contraction in G_1 yields a card with three vertices with loops, which does not exist in $\mathcal{C}(G_2)$. Therefore, every edge is incident to u_1 or to u_2 in G.

The number of cards in $\mathcal{C}(G_1) - \mathcal{L}(G_1)$ with only one vertex with loops is equal to the multiplicity of u_1u_2, while that number in $\mathcal{C}(G_2) - \mathcal{L}(G_2)$ is equal to the degree of v in G plus the number of simple edges not incident to v. Since the degrees of u_1, u_2 and v are equal, it implies that every link incident to u_1 in G is also incident to u_2. From this, we conclude that G_1 and G_2 are the third and fourth graphs of Figure 5.1. ∎

Note that one can recognize if G is a graph other than the ones in Figure 5.1, as no graph in Figure 5.1 is a reconstruction of a graph not found in Figure 5.1.

5.1.2 Girth

In this section, we show that the girth of G is reconstructible. The ***girth*** of G is the length of its smallest cycle. Knowing the girth of a graph is useful in identifying the contracted vertex in some cards. The presence of a cycle smaller than the girth of G in a card indicates that the contracted vertex is contained in that cycle.

Proposition 5.1.4. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then the girth of G is reconstructible. Furthermore, the number of cycles of minimal length is reconstructible.*

Proof: By Corollary 5.1.2, the number of loops is reconstructible. Should G have a loop, then the girth of G is 1. Otherwise, the girth of G is equal to 1 plus the length of the smallest cycle among all the cards in $\mathcal{C}(G)$.

If the girth of G is 1, then by Corollary 5.1.2, the number of loops is reconstructible, and so the number of cycles of minimal length is reconstructible. Otherwise, let $k \geq 2$ be the girth of G. Any cycle of length k in G appears as a cycle of length $k-1$ in exactly k cards. This implies that the total number of cycles of length $k-1$ throughout all the cards is equal to k times the number of cycles of length k in G. From this equality, we conclude that the number of cycles of minimal length is reconstructible. ∎

5.1.3 Reconstructing the number of cycles

In this section, we show that the number of cycles of various lengths is reconstructible.

Let $c_k(G)$ denote the number of cycles of length k in G. Let $c_k(G;e)$ be the number of cycles of length k in G passing through e. Finally, denote by $c_k(G;e)'$ denote the number of cycles of length k in G passing through both ends of e, but that does not contain e.

Lemma 5.1.5. *For any graph G and any $k \geq 1$,*

$$c_k(G/e) = c_k(G) - c_k(G;e) + c_{k+1}(G;e) - c_k(G;e)' \tag{5.1.1}$$

Proof: A cycle C of length k in G appears as a cycle of length k in G/e if and only if e is not an edge of C and the ends of e do not link two vertices in C, which accounts for $c_k(G) - c_k(G;e) - c_k(G;e)'$ on the right-hand side of equation (5.1.1). Furthermore, a cycle C of length $k+1$ appears as a cycle of length k in G/e if and only if $e \in E(C)$, which accounts for $c_{k+1}(G;e)$ on the right-hand side of equation (5.1.1). No cycles of length greater than $k+1$ or smaller than k in G can appear as a cycle of length k in G/e, and so equation (5.1.1) holds. ∎

While the left-hand side of equation (5.1.1) is determined from the deck, the individual terms on the right-hand side were not shown to be reconstructible. We use Lemma 5.1.5 to show the following two propositions.

Proposition 5.1.6. *Let G be a graph, and let g be the girth of G. If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, then $c_i(G)$ is reconstructible for any $g \le i \le 2g-2$.*

Proof: By summing equation (5.1.1) over all edges e, one obtains

$$\sum_{e\in E(G)} c_k(G/e) = e(G)c_k(G) - kc_k(G) + (k+1)c_{k+1}(G) - \sum_{e\in E(G)} c_k(G;e)'. \quad (5.1.2)$$

By Proposition 5.1.4, the girth g and $c_g(G)$ are reconstructible. Let $g \le i \le 2g-3$. Let C be a cycle of length i. If there exists an edge not in C linking two vertices of C, then that edge, along with edges of C, is contained in a cycle of length no more than $i/2+1 < g$. Since $c_j(G) = 0$ for any $j < g$, we conclude that no such edge exist. Therefore, equation (5.1.2) for $k = i$ becomes

$$\sum_{e\in E(G)} c_i(G/e) = e(G)c_i(G) - ic_i(G) + (i+1)c_{i+1}(G). \quad (5.1.3)$$

Since $c_g(G)$ is reconstructible, one can apply induction on equation (5.1.3) to show that $c_i(G)$ is reconstructible for $g \le i \le 2g-2$. ∎

Proposition 5.1.7. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, the length d of the smallest odd cycle is reconstructible. Furthermore, $c_i(G)$ is reconstructible for $i \le d$.*

Proof: By Corollary 5.1.2, the number of loops, $c_1(G)$, is reconstructible. If $c_1(G) \neq 0$, then $d = 1$, and the proposition is proven. Furthermore, by Proposition 5.1.1, $c_2(G)$ is reconstructible. The rest of the proof is done by induction.

Suppose $c_{2i}(G)$ is reconstructible for $i \le k$ and $c_{2i-1}(G) = 0$ for $i \le k$. We are looking to determine $c_{2k+1}(G)$. By equation (5.1.2), it only remains to reconstruct $\sum_{e\in E(G)} c_{2k}(G;e)'$. However, if C is a cycle of length $2k$ and $e \notin E(C)$ is an edge linking two vertices of C, then $E(C) \cup \{e\}$ contains a cycle of odd length, and less than $2k$. This is a contradiction, as there are no cycles of odd length less than $2k$. Therefore, $c_{2k+1}(G)$ is reconstructible.

Suppose $c_{2i}(G)$ is reconstructible for $i \le k$ and $c_{2i-1}(G) = 0$ for $i \le k+1$. We are looking to reconstruct $c_{2k+2}(G)$. Equation (5.1.2) then becomes

$$\sum_{e \in E(G)} c_{2k+1}(G/e) = (2k+2)c_{2k+2}(G)$$

hence $c_{2k+2}(G)$ is reconstructible. ∎

Proposition 5.1.7 is yet another proof that bipartite graphs are recognizable. We will study bipartite graphs in more detail later in this chapter.

5.1.4 Closed trails

A ***walk*** in a graph G is a sequence $v_0, e_0, v_1, ..., v_{k-1} e_k v_k$, whose terms are alternately vertices and edges of G, such that v_{i-1} and v_i are the ends of e_i, for $1 \leq i \leq k$. A walk is a ***trail*** if the edges $e_1, e_2, ..., e_k$ are distinct. A walk or a trail is ***closed*** if $v_0 = v_k$. We say that a set of edges E forms a walk or a trail if there exists a walk or a trail whose edges are E. The ***length*** of the walk or the trail $v_0, e_0, v_1, ..., v_{k-1} e_k v_k$ is $k+1$, which is the number of edges in the sequence. A graph G is ***eulerian*** if it has a closed trail of length $e(G)$.

We now reconstruct the number of closed trails of any length. The main idea is that if E is the set of edges of a closed trail in G and $e \in E$, then $E - e$ is the set of edges of a closed trail in G/e.

Proposition 5.1.8. *Let G be a graph other than the ones in Figure 5.1 and $k \geq 1$. The number of closed trails of length k is reconstructible.*

Proof: Let $t_k(G)$ denote the number of sets of edges of size k that form a closed trail in G. For any closed trail of length k, the contraction of an edge in it yields a closed trail of length $k-1$, while the contraction of an edge not in it yields a closed trail of length k. Therefore,

$$\sum_{e \in E(G)} t_k(G/e) = (k+1)t_{k+1}(G) + (e(G) - k)t_k(G).$$

By Corollary 3.1.4, the number of loops is reconstructible. That number is also $t_1(G)$. Therefore, using the previous equation and induction, one can reconstruct $t_k(G)$, for any $k \geq 2$.

∎

In particular, Proposition 5.1.8 allows the reconstruction of $t_{e(G)}(G)$, and should that number be greater than 0, G is eulerian. Therefore, eulerian graphs are recognizable. It is tempting to attempt to adapt the proof of Proposition 5.1.8 to show that the number of cycles of each length is reconstructible. However, one should be aware that while the contraction of an edge in a cycle of length k yields a cycle of length $k-1$, a contraction of an edge not in the cycle either preserves the cycle of length k, or destroys it if that edge links two vertices of that cycle. Therefore, an equation as simple as the one found in the proof of Proposition 5.1.8 does not seem to exist for cycles.

From Proposition 5.1.8, one can conclude that bipartite graphs are recognizable. The edge set of a closed trail can be partitioned into edge disjoint cycles [28]. Therefore, if a graph has a closed trail of odd length, then there must be a cycle of odd length in the partition of that closed trail. Conversely, any odd cycle is a closed trail of odd length. Therefore, a graph G is bipartite if and only if $t_{2k+1}(G) = 0$ for all $k \geq 0$.

Another consequence of Proposition 5.1.8 is that the length of the smallest odd cycle is reconstructible. If k is the smallest odd number for which $t_k(G) \neq 0$, then k is the length of the smallest odd cycle. Of course, this information was already shown to be reconstructible in Proposition 5.1.7.

5.2 Bipartite graphs

Recall that a graph G is ***bipartite*** if its vertices can be partitioned into two sets A and B, such that every edge has one end in A and one end in B. The parts A and B are a ***bipartition*** of G. Equivalently, a graph is bipartite if and only if it has no cycles of odd length. In this section, we show that bipartite graphs are reconstructible. The proof depends on the connectivity of the bipartite graph.

We start by reiterating that bipartite graphs are recognizable.

Proposition 5.2.1. *The class of bipartite graphs is recognizable. Furthermore, the number of cycles of each length in a bipartite graph is recognizable.*

Proof: Apply Proposition 5.1.7. ∎

We now show that bipartite graphs are weakly reconstructible. A few lemmas are first required. Lemma 5.2.2 is standard.

Lemma 5.2.2. *Let G be a connected bipartite graph. Then the bipartition of G is unique.*

Proof: Let (A_1, B_1) and (A_2, B_2) be two bipartitions of $V(G)$. Let $v \in V(G)$. Suppose, without loss of generality, that $v \in A_1 \cap A_2$. Since $v \in A_1$, the vertices of A_1 are all the vertices of G whose distance from v is an even number. Similarly, since $v \in A_2$, the vertices of A_2 are all the vertices of G whose distance from v is an even number. Since G is connected, the distance between v and any other vertex of G is an integer. Therefore, $A_1 = A_2$ and by complementarity, $B_1 = B_2$. ∎

Lemma 5.2.3. *Let G be a 2-connected graph, with a minimum degree of $\delta(G) \geq 3$, and let C be a cycle of G. There exists two edges $e_1, e_2 \in E(C)$ such that the intersection of all cycles containing e_i is precisely e_i. If G is a 2-connected graph with minimum degree $\delta(G) = 2$, but G is not a cycle and C is a cycle of G, then there exists two full paths $P_1, P_2 \subseteq C$ such that the intersection of all cycles containing P_i is precisely P_i*

Proof: Let G be a 2-connected graph, with a minimal degree of $\delta(G) \geq 3$, and C a cycle of G. Let us assume that at least one edge xy of C does not satisfy the property of the lemma. Let $v \in V(G) - \{x, y\}$ be a vertex contained in every cycle containing xy. In particular, $v \in C$. Now, removing the edge xy and the vertex v from G disconnects it, for otherwise there is a xy-path in $G - \{xy\}$ which does not go through v: this xy-path, along with the edge xy, forms a cycle not containing v, which is a contradiction. Therefore, the deletion of the edge xy and the vertex v yields a disconnected graph, with two connected components A' and B'. Suppose $x \in A'$ and $y \in B'$, then define $A = A' \cup \{v\}$ and $B = B' \cup \{v\}$. Note that there are no edges between A' and B' in G except for the edge xy.

Suppose x is adjacent to v in the cycle C. Since $\deg(x) \geq 3$, x must be adjacent to at least one vertex other than v and y, say u. Then, since G is 2-connected, there are two disjoint uv-paths, one being uxv. The other uv-path, named c, is entirely contained in $A - \{v, x\}$. But then $xucvx$ is a cycle, as is $xvc'yx$, where c' is a path in B between v and y. Both of these cycles go through the edge xv, and are otherwise disjoint. Therefore xv is an edge of C with the property that all cycles containing xv intersect only in xv.

Suppose that x is not adjacent to v in C, let c be the xv-path in $C - \{y\}$. Let $z \in c$ be the vertex adjacent to x. Suppose xz does not satisfy the conclusion of the lemma. There is a vertex v', other than x and z, such that v' is contained in every cycle containing xz. Since $\deg(x) \geq 3$, x is adjacent to at least three vertices, two being y and z. Let $u \in A$ be the third. Since G is 2-connected, there are two disjoint uv-paths, one being $uxcv$, and let c'' be the other. Then $xuc''vczx$ is a cycle containing xz, and so is $xyc'vcz$. These paths intersect only in c and v: this implies that $v' = v$ or $v' \in c$. But then the distance between z and v' in C is less than the distance between x and v in C, so one can repeat this argument until the distance between the end of the edge in consideration and the vertex contained in every cycle containing that edge is 1, and then one has found an edge of C satisfying the conditions of the lemma.

To find the second edge in C, repeat the previous argument with y instead of x. Since the edge found when using x is on the xv-path of C that does not contain

y, the second edge will be found on the yv-path of C that does not contain x. The second edge is therefore distinct from the first edge found.

If G has $\delta(G) = 2$ and since G is not a cycle, then G is the subdivision of a graph G' with $\delta(G') \geq 3$. One can then apply the previous argument to G' to conclude that there exists an edge $xy \in E(G')$ with the property that all cycles containing xy intersect only in xy. Going back to G, this implies the existence of a full path P with the property that all cycles containing P intersect only in P. ∎

If G is bipartite, the cards corresponding to edges whose existence is shown in Lemma 5.2.3 have a unique vertex, or a unique full path if $\delta(G) = 2$, contained in every odd cycle. In those cards, one can identify the full path containing the contracted vertex, or the contracted vertex itself if $\delta(G) \geq 3$. If, furthermore, the contracted vertex v^* is not a cut vertex of the card H, then $H - v^*$ is connected and bipartite. By Lemma 5.2.2, the bipartition (A, B) of $H - v^*$ is unique. It then becomes straightforward to decontract v^* into two vertices x and y reconstruct G. Indeed, it must be the case that every neighbour of v^* in A is linked to x and every neighbour of v^* in B is linked to y in G.

Lemma 5.2.4. *Let G be a 2-connected graph, with minimum degree $\delta(G) \geq 3$. There exists two cycles C_1 and C_2 of G that do not share any edges, such that if $xy \in E(C_1)$ or $xy \in E(C_2)$, then $\{x, y\}$ is not a 2-vertex cut of G.*

Proof: Let us assume that there exists an edge $x_0y_0 \in G$ such that $\{x_0, y_0\}$ is a 2-vertex cut of G. Therefore, $G - \{x_0, y_0\}$ is disconnected, and let A_0' and B_0' be two connected components. Define $A_0 = A_0' \cup \{x_0, y_0\}$ and $B_0 = B_0' \cup \{x_0, y_0\}$, where x_0 is adjacent to y_0 in both A_0 and B_0. If every cycle of A_0 contains x_0y_0, then A_0 itself is a cycle, which contradicts $\delta(G) \geq 3$. Therefore, there exists a cycle C_0 of A_0 that does not contain x_0y_0.

We recursively define x_ny_n in some subset A_{n-1} of G and in a cycle C_{n-1}. Suppose that $\{x_n, y_n\}$ is a 2-vertex cut of G. If no such edge exists, then C_{n-1} satisfies the lemma. Let A_n' be the connected component of $G - \{x_n, y_n\}$ such that $A_n' \subset A_{n-1}$ and let $A_n = A_n' \cup \{x_n, y_n\}$, where x_n is adjacent to y_n in A_n. Let C_n be a cycle of A_n that does not contain x_ny_n. Since G is finite and $A_n \subset A_{n-1}$, the sequence $A_0, A_1, ...$ must be finite, and so there exists a cycle $C \in A_0$ such that no edges xy of C is such that $\{x, y\}$ is a 2-vertex cut of G.

Repeat the previous argument with B_0 instead of A_0 to find another cycle, which does not share any edges with the first, that satisfies the lemma. ∎

It follows from Lemmas 5.2.3 and 5.2.4 that if G is 2-connected with $\delta(G) \geq 3$, then there exists an edge xy such that all cycles containing xy intersect only in xy, and that $G - \{x, y\}$ is connected. More precisely, those lemmas show that there are at least four such edges.

Theorem 5.2.5. *Simple 2-connected bipartite graphs are reconstructible.*

Proof: Using Proposition 3.1.2, Corollary 4.1.11 and Proposition 5.2.1, simple 2-connected bipartite graphs are recognizable. The proof is split into two cases, whether $\delta(G) = 2$ or $\delta(G) \geq 3$. According to Proposition 3.1.8, graphs with a minimal degree of 2 are recognizable. Since 2-connected graphs have no vertices of degree 1, it is therefore possible to determine if $\delta(G) \geq 3$.

Suppose $\delta(G) \geq 3$. Lemma 5.2.3 implies that there is a card $H \in \mathcal{C}(G)$ with at least one odd cycle, and a unique vertex $v^* \in V(H)$ contained in every odd cycle of H. By Lemma 5.2.4, one can chosse H such that $H - v^*$ is connected and bipartite, with a bipartition (H_1, H_2). By Lemma 5.2.2, this bipartition is unique. To obtain G from H, the vertex v^* must be decontracted, into two vertices, v_1 and v_2. All that remains is to separate the neighbours of v^* into the neighbours of v_1 and v_2. Let v_1 be added to part H_1, and v_2 be added to part H_2. Since G is known to be bipartite, every neighbour of v^* in H_1 becomes a neighbour of v_2 in G, and every neighbour of v^* in H_2 becomes a neighbour of v_1 in G. Hence, G is reconstructed.

If $\delta(G) = 2$, by Lemma 5.2.3, there exists a card with at least one odd cycle, and a unique full path P contained in every odd cycle of H, or a unique vertex $v^* \in V(H)$ contained in every odd cycle of H. In the latter, we refer to the previous argument. In the former, G is reconstructed from H by subdividing any edge in P. ∎

We now consider bipartite graphs that are not 2-connected. Some of these bipartite graphs were already shown to be reconstructible. In particular, Theorem 4.2.1 shows that disconnected bipartite graphs with at least two non-trivial connected components and at least three edges are reconstructible. It remains to show that separable bipartite graphs are reconstructible. Theorems 4.3.4 and 4.3.14 show that some separable bipartite graphs are reconstructible. We now generalize those results to all separable bipartite graphs with at least four edges. Some terminology and partial results in the following proof will be borrowed from the section on separable graphs.

Theorem 5.2.6. *Simple separable bipartite graphs with at least four edges are reconstructible.*

Proof: By Proposition 3.1.2, Lemma 4.1.5, Proposition 5.2.1 and Proposition 3.1.1, the class of simple separable bipartite graphs with at least four edges are recognizable. By Theorem 4.3.14, separable graphs with at least two non-trivial blocks are reconstructible. By Theorem 4.3.4, trees, which are separable graphs with no non-trivial blocks, are also reconstructible. So one can assume the bipartite graph we are reconstructing has only one non-trivial block B. Note that since G is bipartite, so is B. Furthermore, B is 2-connected.

By Lemma 4.3.8, the list of blocks of G is reconstructible, hence B is known. Furthermore, the cards of $\mathcal{C}(G)$ can be partitioned into $\mathcal{CB}$ and $\mathcal{CT}$, where $\mathcal{CB}$ are the cards corresponding to the contraction of edges in B, and $\mathcal{CT}$ are the cards corresponding to the contraction of edges in limbs of vertices of B. The proof is separated into two cases, whether $\delta(B) \geq 3$ or $\delta(B) = 2$. Note that since B is a block of G, it is not separable, and therefore B does not have any vertices of degree 1.

For any vertex $v \in V(B)$, recall that $T(v)$ denotes the limb of v in the graph G. For any subset of vertices $A \subseteq V(B)$, let $T(A)$ denote the collection of limbs of the vertices of A in the graph G. Recall that $T(v) = v$ if there are no trees attached to v.

Suppose $\delta(B) \geq 3$. According to Lemmas 5.2.3 and 5.2.4, there exists a card $G/xy \in \mathcal{CB}$ with a unique vertex v^* that is contained in every odd cycle of the non-trivial block, and $B/xy - v^*$ is connected. Let $\mathcal{CB}'$ be the collection of such cards. The vertex v^* is to be decontracted, into vertices x and y. Theorem 5.2.5 shows there is only one way to split the neighbours of v^* in B into the neighbours of x and y, since B is a 2-connected bipartite graph. Let (B_1, B_2) be the bipartition of B, and let $x \in B_1$ and $y \in B_2$. It remains to split $T(v^*)$ into $T(x)$ and $T(y)$. By Corollary 4.3.16, the multiset of all limbs of vertices of B is reconstructible. Therefore, the two trees T_1 and T_2 that are attached to x and y in G are known. This yields two possible reconstruction: $G = G_1$ is the first, with $T(x) = T_1$ and $T(y) = T_2$, and G_2 is the second, with $T(x) = T_2$ and $T(y) = T_1$. We now show that $G_1 \cong G_2$ or $\mathcal{C}(G_1) \neq \mathcal{C}(G_2)$.

Suppose only one vertex $v \in V(G)$ is such that $T(v) \neq \{v\}$. The graph G has that property if and only if every card of $\mathcal{CB}$ does too. If $T(v^*) = \{v^*\}$, then $G_1 \cong G_2$, and G is reconstructible. Therefore, one can assume that for every card $H \in \mathcal{CB}'$, we have $T(v^*) \neq \{v^*\}$. Therefore, in G_1, every edge that correspond to cards of $\mathcal{CB}'$ are incident to the only vertex $v \in V(B)$ with $T(v) \neq \{v\}$. By Lemmas 5.2.3 and 5.2.4, at least four such edges exist, and so these edges must form a star in G_1 with v at its center. By construction, the vertex v such that $T(v) \neq \{v\}$ in G_2 is one of the ends of that star, hence there exists a card of G_2 in $\mathcal{CB}'$ with the property that $T(v^*) = \{v^*\}$, hence G_2 is reconstructible. This implies that G_1 is also reconstructible.

We can henceforth assume that there are at least two vertices of B with non-trivial limbs. Suppose also that $|\mathcal{CT}| = 2$, so there are exactly two vertices v_1 and v_2 such that $t(v_1) = t(v_2) = 1$. If there exists a card of $\mathcal{CB}'$ such that $t(v^*) = 0$ or 2, then $G_1 \cong G_2$. Then one can assume that $t(v^*) = 1$ for every card of $\mathcal{CB}'$, in which case the edges corresponding to the cards of $\mathcal{CB}'$ in G_1 must form two stars, with v_1 and v_2 as their centers. Again, the edges corresponding to the cards of $\mathcal{CB}'$ in G_2 do not have that property, hence G_2 is reconstructible. This implies that G is reconstructible. We henceforth assume that $|\mathcal{CT}| \geq 3$ and that at least two vertices have non-trivial limbs.

By applying Lemma 4.3.15 with $F = B_1$ and $F = B_2$, we get that $T(B_1)$ and $T(B_2)$ are reconstructible, unless all non-trivial limbs are attached to vertices of the same part B_1 or B_2. Since $|\mathcal{CT}| \geq 3$, such is the case if and only if every card of $\mathcal{CT}$ satisfy that property as well. In that case, both G_1 and G_2 have that property if and only if $T(v^*) = \{v^*\}$ for every card in $\mathcal{CB}'$, which implies that $G_1 \cong G_2$. Otherwise, only one G_1 or G_2 has that property, and since that property is reconstructible, so are G_1 and G_2. Therefore, one can assume that $T(B_1)$ and $T(B_2)$ are reconstructible.

Suppose there are no automorphisms of B that swaps B_1 and B_2. This property is known since B is reconstructed. Then since $\mathcal{C}(G_1) \cong \mathcal{C}(G_2)$, we have that $T(B_1) = T(B_1) \cup T(y) - T(x)$ and $T(B_2) = T(B_2) \cup T(x) - T(y)$, which implies that $T(x) \cong T(y)$ and so $G_1 \cong G_2$.

Suppose now that there is an automorphism of B that swaps B_1 and B_2. The subgraph F induced by the vertices of B and $\{T(u) \mid u \notin \{x, y\}\}$ is the same in G_1 and G_2, and hence is reconstructible. If an automorphism of F that swaps B_1 and B_2 also swaps x and y, then $G_1 \cong G_2$. Otherwise, since the subgraph induced by the vertices of B and $T(B_1)$, the subgraph induced by the vertices of B and $T(B_2)$ and F are reconstructible by Proposition 4.3.15, one can then determine if $T(x) = T_1$ or T_2, and if $T(y) = T_1$ or T_2. We conclude that G is reconstructible.

If $\delta(B) = 2$, then according to the proof of 5.2.5, there exists a card $G/e \in \mathcal{CB}$ where B/e has a full path P which is contained in every odd cycle of B/e. Among all such cards, consider any card H where there is a vertex v in such a full path P with a limb $T(v)$ of maximum size. If v is unique in H, then $v = v^*$ is the vertex to be decontracted. Apply the proof for the case when $\delta(B) \geq 3$ to show that G is reconstructible. If v is not unique, it implies that in B, every full path P satisfying Lemma 5.2.3 has the following property: for every v in P where $T(v)$ is maximum, every neighbour u of v in P are such that $T(u) = \{u\}$. Indeed, if $T(v)$ is maximum and a neighbour u of v has $T(u) \neq \{u\}$, then in G/uv, the contracted vertex v^* is the only vertex on a full path P contained in every odd cycle with more than $|T(v)|$ vertices in its limb. In that case, consider a card H with a full path P contained in every odd cycle, and with the property that the distance between a vertex v of P with a maximal limb and any other vertex u of P with $T(u) \neq \{u\}$ is minimum. Subdivide the uv-path to reconstruct G. Finally, if $T(v) = \{v\}$ for every vertex v in such a full path P, then G is reconstructed by subdividing any edge of P. ∎

When we gather Theorems 4.2.1, 5.2.5 and 5.2.6, we conclude that almost all simple bipartite graphs are reconstructible.

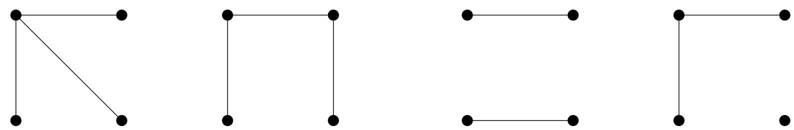

Figure 5.2: Simple bipartite graphs that are not reconstructible. One may add any number of isolated vertices to those graphs and they would still be not reconstructible.

Corollary 5.2.7. *Simple bipartite graphs that are not in Figure 5.2 are reconstructible.*

Proof: Bipartite graphs with a connectivity of 1 and at least four edges are reconstructible by Theorem 5.2.6. The 2-connected bipartite graphs are shown to be reconstructible in Theorem 5.2.5. This shows that connected bipartite graphs are reconstructible. Disconnected bipartite graphs with at least two non-trivial components and at least three edges are reconstructible by Theorem 4.2.1. Disconnected bipartite graphs with only one non-trivial components are reconstructible if and only if that non-trivial component is reconstructible. That non-trivial component is indeed reconstructible by Theorems 5.2.6 and 5.2.5 should it contain at least four edges.

It therefore remains to check that simple bipartite graphs with one, two and three edges are reconstructible, except for the ones in Figure 5.2. We can exclude graphs with one edge, since K_2 and the graph with one vertex and a loop are reconstructions of one another. Similarly, every simple graph with two edges are found in Figure 5.2 and are therefore excluded. If there are three edges, see Figure 5.3 for the list of graphs and their decks. ∎

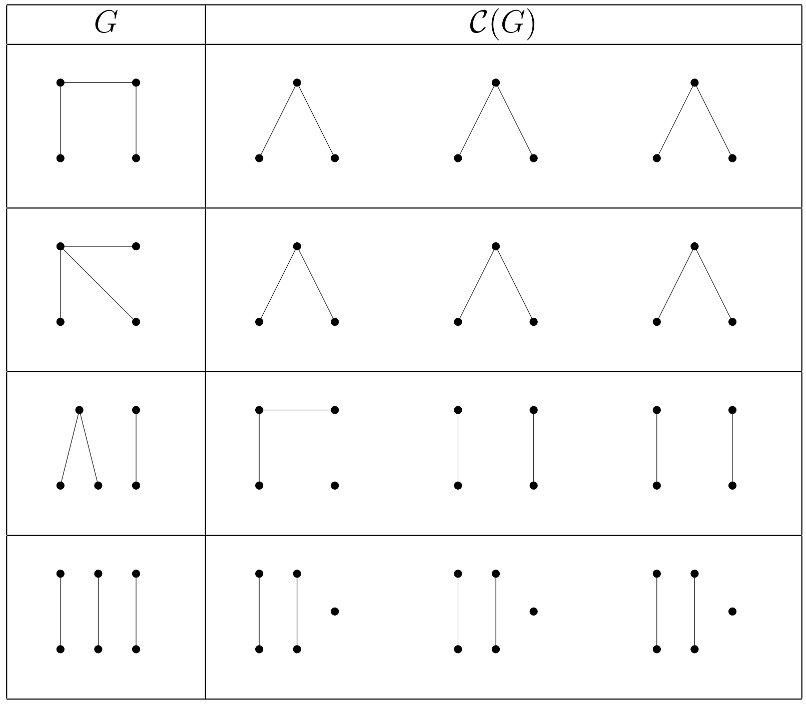

Figure 5.3: Every simple bipartite graph on three edges with no isolated vertices and their decks. The last two graphs are reconstructible.

5.3 Planar graphs

A graph is ***planar*** if it can be drawn on the plane in a way that its edges intersect only at their ends. Such a drawing is a ***planar embedding*** of the graph. A planar embedding $\widetilde{G}$ of a planar graph G may be seen as a graph isomorphic to G.

Two edges of a drawing of G are ***crossing edges*** if they meet at a point other than a vertex of $\widetilde{G}$. The ***crossing number*** of a graph G is the minimum number of crossing edges over all drawings of G. It follows that G is planar if and only if its crossing number is 0.

If G is planar, then every card of $\mathcal{C}(G)$ is planar. Indeed, in a planar embedding $\widetilde{G}$ of G, any edge can be continually shortened up to the point where its ends coincide, without breaking planarity. This operation corresponds to contracting the edge, which shows that every card of $\mathcal{C}(G)$ must be planar. The converse is false: in other words, if all cards of G are planar, it is not necessarily the case that G is planar as well. See Figure 5.4 for an example.

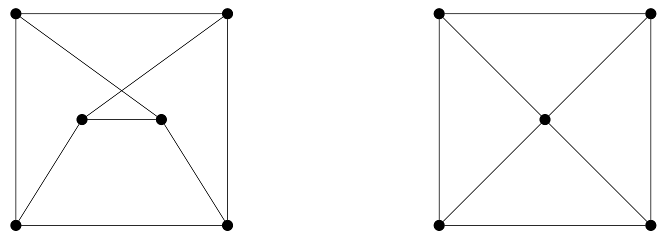

Figure 5.4: On the left: $K_{3,3}$, which is not planar. All of its cards are isomorphic to the wheel graph on the right, which is planar.

The graph $K_{3,3}$ has the property that all of its cards are planar graphs, while $K_{3,3}$ itself is not. Graphs with that property must have five or six vertices. Indeed, we claim that a graph G with at least seven vertices is planar if and only if G/e is planar for every $e \in E(G)$. Along with Proposition 5.3.3, which shows that planar graphs on five or six vertices are recognizable, it shows that planarity is a reconstructible property. We now prove this claim. To show that planarity is reconstructible, a few definitions are required. Recall that to ***subdivide an edge*** xy is to delete the edge xy, then add a new vertex, adjacent only to x and y. A graph G is an ***H-subdivision*** if it was obtained from H by a sequence of edge subdivisions. A famous theorem of Kuratowski [4] states that a graph is planar if and only if none of its subgraphs are K_5-subdivisions or $K_{3,3}$-subdivisions.

We now shoe that planar graphs are recognizable. This is done in Lemma 5.3.1, Proposition 5.3.2 and Proposition 5.3.3.

Lemma 5.3.1. *Graphs in Figure 3.1 are planar.*

This obvious result is necessary since Propositions 5.3.2 and 5.3.3 are based on the number of vertices in the graphs. Since $v(G)$ is not reconstructible only if G is in Figure 3.1, Lemma 5.3.1 is necessary.

Proposition 5.3.2. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, has at least seven vertices, then whether G is planar or not is recognizable.*

Proof: Let G be a graph. By Proposition 3.1.3, $v(G)$ is reconstructible, and so graphs on at least 7 vertices are recognizable. We show that planarity is a reconstructible property by showing that G is planar if and only if G/e is planar for every $e \in E(G)$. One of these implications was already discussed: if G is planar, then surely every card of G is planar. If G is not planar, then, by Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph, say F. And since $v(G) \geq 7$, either F is a proper subdivision of K_5 or $K_{3,3}$, in which case a contraction within F yields a card of G which is not planar, or G has an edge which is not contained in F, whose contraction yields a card containing a K_5-subdivision or a $K_{3,3}$-subdivision. ∎

To complete the proof that planarity is a reconstructible property, one only needs to consider graphs on five and six vertices. Graphs with fewer than five vertices are all planar, as their underlying simple graphs are subgraphs of the complete graph K_4, which is planar.

Proposition 5.3.3. *If G is not a graph composed of a graph in Figure 3.1 and isolated vertices, and has five or six vertices, then whether G is planar or not is recognizable.*

Proof: By Proposition 3.1.3, the number of vertices is reconstructible. We can therefore recognize graphs with five or six vertices. We first show that graphs with five or six vertices and a K_5-subdivision are recognizable. We then show that graphs with five or six vertices and a $K_{3,3}$-subdivision are recognizable. A graph is planar if and only if its underlying simple graph is also planar, which is why Propositions 5.1.1 and 5.1.3 are useful, as they show that the number of edges in the underlying simple graph is reconstructible.

We may assume that the graphs here are connected, because a graph is planar if and only if all of its connected components are planar.

Suppose $v(G) = 5$. Then G is not planar if and only if its underlying simple graph is K_5. Since K_5 is the only simple graph with five vertices and ten edges, it is possible to recognize, using Propositions 5.1.1 and 5.1.3, whether the underlying simple graph is K_5 or not. Therefore, one can recognize planar graphs with five vertices. We can then assume that $v(G) = 6$. If G has K_5 as a subgraph, then since G is connected, there is an edge e that is not contained in that subgraph. Then, G/e also has K_5 as a subgraph, hence we can recognize that G is not planar. Otherwise, G has a proper K_5-subdivision, and so there exists $e \in E(G)$ such that the underlying simple graph of G/e has a K_5-subdivision. In that case, we can also recognize that G is not planar.

We now show that graphs with six vertices and a $K_{3,3}$-subdivision are recognizable. We start by showing that if $v(G) = 6$ and the underlying simple graph of G contains at least 13 edges, then G is not planar. It suffices to show that the crossing number of K_6 is 3. Figure 5.5 shows that the crossing number of K_6 is at most 3. The crossing number of K_6 cannot be equal to 0, since K_5, which is not planar, is a subgraph of K_6. The crossing number cannot be equal to 1, since otherwise the removal of a vertex incident to the only crossing edge would yield a planar graph, but $K_6 - v \cong K_5$ for any vertex $v \in V(K_5)$, which is a contradiction. Suppose the crossing number of K_6 is 2. Every crossing involves two distinct edges and four distinct vertices. Since $v(K_6) = 6$, there must be a vertex involved in both crossings. The removal of that vertex yields a planar graph, but also yields K_5, which is a contradiction. Therefore, the crossing number of K_6 is 3. Since $e(K_6) = 15$, one must remove at least three edges from K_6 to obtain a planar graph, so we conclude that any graph with $v(G) = 6$ whose underlying simple graph contains at least 13 edges are not planar.

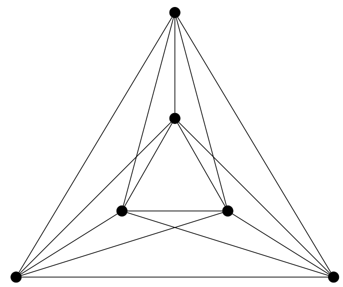

Figure 5.5: A drawing of K_6 with three crossings.

We are looking to determine if $K_{3,3}$ is a subgraph of G, so one can assume that the underlying simple graph of G contains at least 9 edges. From the previous argument, one can also assume it contains at most 12 edges. The only six non-isomorphic simple graphs with $v(G) = 6$ and $9 \leq e(G) \leq 12$ which contain a subgraph isomorphic to $K_{3,3}$ are shown in Figure 5.6. Some of these graphs have a card that is not planar, which shows that the graph itself is not planar. Those graphs in question are G_4 and G_6 of Figure 5.6, and the edges whose corresponding cards are not planar are indicated. We are then looking to show that the class of graphs whose underlying simple graph is isomorphic to any of the four remaining graphs in Figure 5.6 is recognizable.

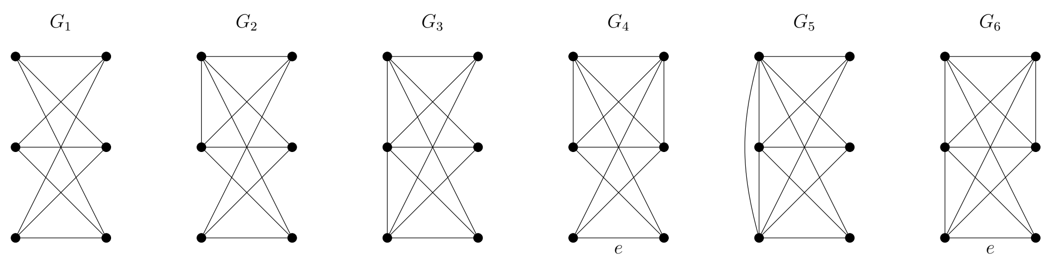

Figure 5.6: Graphs with subgraphs isomorphic to $K_{3,3}$.

If G is a graph and $e \in E(G)$ is a loop, then the underlying simple graph of G is isomorphic to the underlying simple graph of G/e. Therefore, if G has a loop, then the underlying simple graph of G is reconstructible and the planarity of G is determined. We henceforth assume that the graphs we are investigating are loopless.

The first graph G_1 in Figure 5.6 is $K_{3,3}$. If a graph's underlying simple graph is $K_{3,3}$ and that graph is loopless, then it is bipartite, hence reconstructible by Corollary 5.2.7. Therefore, the graphs whose underlying simple graph is $K_{3,3}$ are recognizable.

For the second graph G_2 in Figure 5.6, if we are to show that for any planar graph whose underlying simple graph has ten edges and six vertices, it contains a card whose underlying simple graph is not isomorphic to any of the underlying simple graphs of the cards of G_2, or vice versa, then we will have shown that the deck of a graph whose underlying simple graph is G_2 cannot belong to a planar graph. Figure 5.7 contains every simple planar graph with ten edges and six vertices.

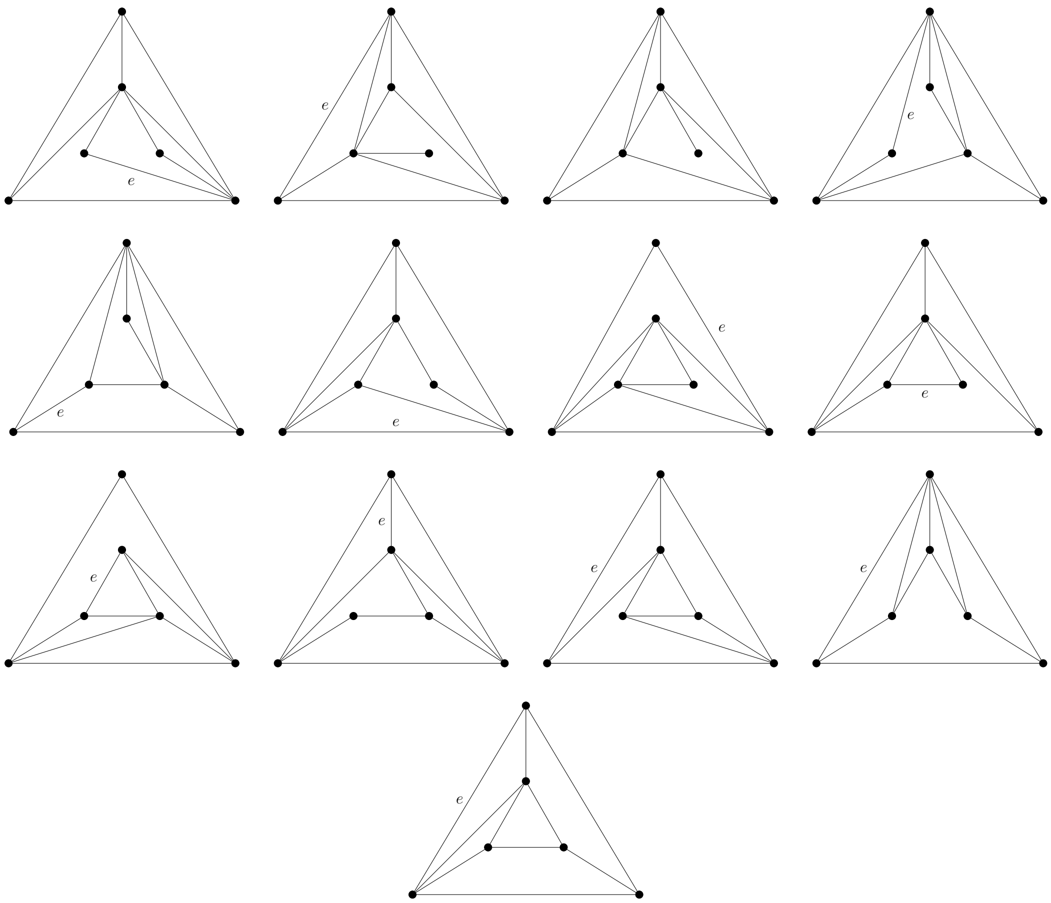

Figure 5.7: Planar graphs with 6 vertices and 10 edges.

In each graph of Figure 5.7, an edge e has been identified: the contraction of that edge yields a card whose underlying simple graph is not isomorphic to the underlying simple graph of any card of G_2 found in Figure 5.6.

For G_3, the same method is used. Figure 5.8 lists all simple planar graphs with six vertices and eleven edges. Again, an edge e has been identified in each graph: the contraction of that edge yields a card whose underlying simple graph is not isomorphic to the underlying simple graph of any card of G_3.

Finally, for G_5, the same method is used again. See Figure 5.9 for the list of

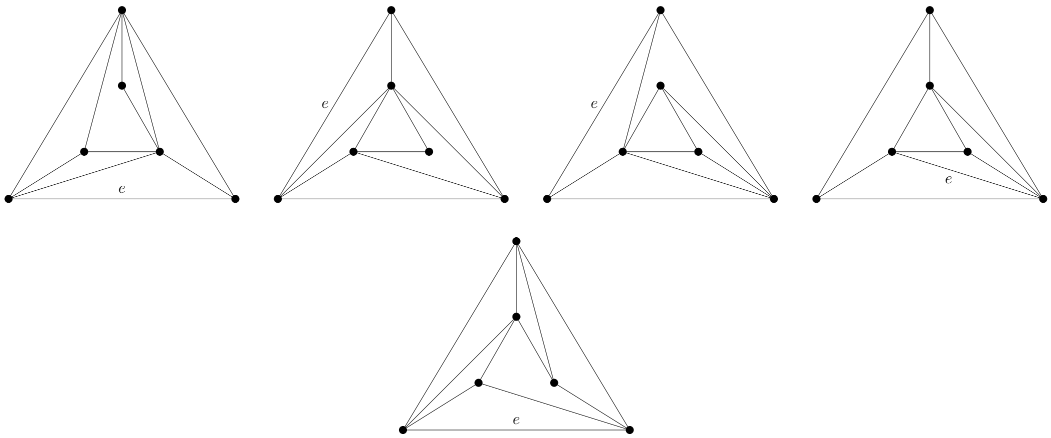

Figure 5.8: Planar graphs with 6 vertices and 11 edges.

all simple planar graphs with six vertices and twelve edges. For the second graph of Figure 5.9, there are no edges that have been identified. However, all the underlying simple cards of that graph are isomorphic, while the underlying simple cards of G_5 are not all isomorphic.

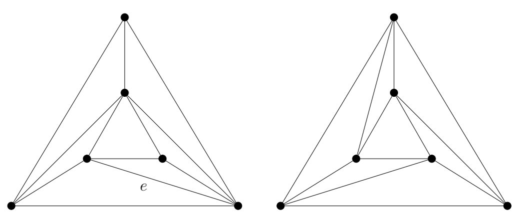

Figure 5.9: Planar graphs with 6 vertices and 12 edges.

∎

We have shown that the graphs in Figures 5.7, 5.8 and 5.9 are reconstructible by a computer program. See Appendix A for more details.

We gather the two previous propositions to conclude the following:

Corollary 5.3.4. *Planar graphs are recognizable.*

Proof: Graphs on fewer than five vertices are planar since their underlying simple graphs are subgraphs of K_4, which is planar. Planar graphs on five or six vertices are recognizable by Proposition 5.3.3, and planar graphs on more than six vertices are recognizable by Proposition 5.3.2. Finally, graphs whose number of vertices is not reconstructible are all planar, as stated in Lemma 5.3.1. ∎

A planar embedding of a graph partitions the plane into ***faces***, which are arcwise-connected open sets. Each planar graph has exactly one unbounded face, called the ***outer face***. Note that for any face on a planar embedding, there is a planar embedding where that face is the outer face. Let $f(\widetilde{G})$ denote the number of faces, including the outer face, in a planar embedding $\widetilde{G}$ of a graph G. Then, by Euler's formula:

$$v(\widetilde{G}) - e(\widetilde{G}) + f(\widetilde{G}) = 2.$$

Since both $v(\widetilde{G})$ and $e(\widetilde{G})$ are constant through all embeddings of the graph G, they are equal to $v(G)$ and $e(G)$ respectively. Therefore, $f(\widetilde{G})$ is also independent of the embedding, and we denote the number of faces simply by $f(G)$. Using Propositions 3.1.3 and 3.1.1, $v(G)$ and $e(G)$ are reconstructible if, in particular, G is simple, and so too is $f(G)$. The ***degree of a face*** f in an embedding $\widetilde{G}$ is the number of edges on its boundary, with cut edges counted twice. Let $f_k(\widetilde{G})$ denote the number of faces of degree k in $\widetilde{G}$. These parameters, unlike $f(G)$, are dependent on the embedding. Since every edge is counted in two faces exactly, or twice in the same face for cut edges, we get that

$$\sum_k k f_k(\widetilde{G}) = 2e(\widetilde{G}).$$

Since the right-hand side is independent of the embedding, so too is the left-hand side. We say that two faces are ***adjacent*** if they share an edge.

Having a unique planar embedding is very useful for reconstruction. Two embeddings are the same if the face boundaries, regarded as sets of edges, are identical. Lemma 5.3.5 links the connectivity of a graph to having a unique planar embedding.

Lemma 5.3.5 (Whitney [4])**.** *Let G be a simple 3-connected planar graph. Then G has a unique planar embedding.*

5.3.1 Maximal planar graphs

A simple graph G is ***maximal planar*** G is planar and $G \cup \{e\}$ is not planar for every $e \notin E(G)$. It follows that in a maximal planar graph, every face is a triangle. In this section, we show that maximal planar graphs are reconstructible, starting by showing that they are recognizable. These results are found in [17].

Proposition 5.3.6 (Lauri [17])**.** *Maximal planar graphs are recognizable.*

Proof: First, we show that a simple graph is maximal planar if and only if it is planar and $e = 3v - 6$. If G is a maximal planar graph, then certainly G is planar.

Furthermore, every face of G is a triangle, and since every edge is contained in exactly two faces, we have $2e = 3f$. Combining this equality with $v - e + f = 2$, we get that $e = 3v - 6$. Conversely, if G is a planar graph with $e = 3v - 6$, then using $v - e + f = 2$, we get that $f = 2v - 4$. Now $3f_3 + 4f_4 + 5f_5 + ... = 2e$, and since $f_3 + f_4 + f_5 + ... = f$, we get that $3f + f_4 + 2f_5 + ... = 2e$. Using $f = 2v - 4$ in the previous equation, along with $e = 3v - 6$, yields $f_4 + 2f_5 + ... = 0$. Therefore, all the faces of G are triangles, hence G is maximal planar.

By Corollary 5.3.4, planar graphs are recognizable. Since e and v are reconstructible, a graph with the property that $e = 3v - 6$ is also recognizable. Therefore, maximal planar graphs are recognizable. ∎

The next step is to show that maximal planar graphs are reconstructible. The idea of the proof is to find a card with a unique planar embedding. The concept of quasi-maximal planar graph is useful here.

Definition 5.3.7. *A graph G is* ***quasi-maximal planar*** *if its underlying simple graph is maximal planar, G has no loops, G has only two pairs of parallel edges and if those two pairs share a vertex.*

If G is a maximal planar graph and $uv \in E(G)$, then u and v have exactly two common neighbours if and only if G/uv is quasi-maximal planar. We now prove the existence of quasi-maximal planar cards in the deck of a maximal planar graph.

Lemma 5.3.8 (Lauri [17])**.** *Let G be a maximal planar graph with at least 4 vertices. There exists an edge $uv \in E(G)$ such that G/uv is a quasi-maximal planar graph.*

Proof: It suffices to show that in every maximal planar graph, there exists an edge uv such that u and v have exactly two common neighbours. Since an edge is contained in exactly two faces, it suffices to show that there exists a planar embedding of G and an edge uv such that the only triangles containing uv are faces. Suppose the contrary, so that for every edge $uv \in E(G)$, there exists a triangle containing uv that is not a face of G. Pick an edge uv, and let T be a triangle containing uv that is not a face. Choose an edge uv and a triangle T that is not a face such that the number of vertices inside T is minimal. Since T is not a face, choose an edge xy inside T. Let T' be a triangle containing xy that is not a face. If no such triangles exists, then we are done. If such a triangle T' exists, then it is strictly contained within T, and so the inside of T' contains fewer vertices than the inside of T, contradicting the minimal choice of uv and T. ∎

The other important fact needed to show that maximal planar graphs are reconstructible is that they have a unique planar embedding.

Lemma 5.3.9. *Maximal planar graphs have a unique planar embedding.*

Proof: If suffices to show that maximal planar graphs are 3-connected, since Lemma 5.3.5 states that 3-connected planar graphs have a unique embedding. Suppose G is a maximal planar graph, so that every face is a triangle. Let $\{u, v\} \subseteq V(G)$ be a 2-vertex cut of G. Let X and Y be connected components of $G - \{u, v\}$. The outer face of G must then contain u, v, and at least one vertex from both X and Y. The outer face of G therefore has at least four vertices, which is a contradiction. ∎

Therefore, quasi-maximal planar graphs also have a unique planar embedding, up to a permutation of the parallel edges. Furthermore, in a quasi-maximal planar card, the contracted vertex is the vertex shared by all parallel edges. It then remains to split the neighbours of the contracted vertex v^* into the neighbours of the two vertices that will come of the decontraction. The unique embedding, up to the position of parallel edges, is useful for that step, and since G is simple, parallel edges in a card are easily dealt with.

Theorem 5.3.10 (Lauri [17])**.** *Maximal planar graphs are reconstructible.*

Proof: By Proposition 5.3.6, maximal planar graphs are recognizable. Let G/xy be a card of $\mathcal{C}(G)$ such that G/xy is a quasi-maximal planar graph. Such a card exists by Lemma 5.3.8. By Lemma 5.3.9, G/xy has a unique planar embedding, up to the position of the parallel edges. Furthermore, G/xy contains exactly two pairs of parallel edges, and those pairs share the contracted vertex v^*. It remains to split the neighbours of v^* into the neighbours of x and y to reconstruct G. Since G is maximal planar and G/xy is quasi-maximal planar, they both have a unique planar embedding, and so the planar embedding of G/xy was obtained by contracting xy in the planar embedding of G. In particular, if a subset E of edges in G/xy forms a face, then either E is a face of G, or $E \cup \{xy\}$ is a face of G. It follows that two neighbours u and w of v^* in G/xy are attached to the same vertex x or y after the decontraction if and only if u, w and v^* are a triangle in G/xy. Therefore, it is possible to decide, for every pair of neighbours of v^*, whether they are attached or not to the same vertex x or y after the decontraction, hence G is reconstructible. ∎

Chapter 6

Degrees

In this chapter, we show that the degree sequence of a graph is reconstructible. Knowing the degree sequence allows us, in particular, to recognize regular graphs. We then show that some cubic graphs are reconstructible. We also show that distance regular graphs are reconstructible.

6.1 Reconstruction of the degree sequence

An important property of graphs is its degree sequence, which we now show to be reconstructible for every simple graph, with the exception of the known graphs that are not reconstructible. These graphs are shown again here in Figure 6.1.

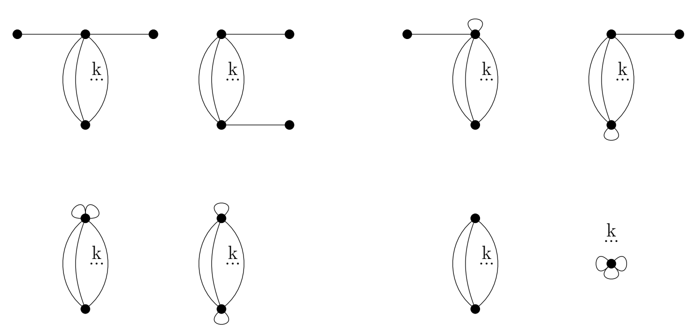

Figure 6.1: Graphs whose degree sequence are not reconstructible.

We can appreciate one of the differences between contraction reconstruction and edge and vertex deletion reconstruction. In the latter, obtaining the degree sequence is very straightforward using Proposition 2.1.4 and Lemma 2.2.4. For contraction reconstruction, it is harder to obtain that sequence. For any graph G, the degree sequence can be obtained by counting the subgraphs isomorphic to a star $K_{1,i}$, for all i. The fact that the degree sequence is hiding in G as a subset of subgraphs suggests that it is easy to reconstruct the degrees of a graph from decks composed of subgraphs of G. However, the contraction deck $\mathcal{C}(G)$ does not give any subgraphs of G directly. Despite this obstacle, one can still reconstruct the degree sequence from $\mathcal{C}(G)$, in almost all cases.

Proposition 6.1.1. *Let G be a simple connected graph which is not in Figure 6.1. The degree sequence of G is reconstructible.*

Proof: We may assume that $e(G) \geq 3$, since every simple graph with one or two edges appears in Figure 6.1. We may also assume that $\delta(G)$, the minimum degree, is greater than 0, for G is connected. We first show that $\delta(G)$ is reconstructible. Let $n_i(G)$ be the number of vertices in G with degree i. Let $n_{i,j}(G)$ be the number of edges linking a vertex of degree i to a vertex of degree j. If a vertex v in a card of G is of degree d, then it either corresponds to a vertex of degree d in G, or it was obtained by contracting an edge whose ends are vertices whose degrees sum to $d+2$. Therefore, for any integer $0 \leq i < e(G)$,

$$\sum_{H \in \mathcal{C}(G)} n_i(H) = (e(G) - i)n_i(G) + \sum_{\substack{\alpha,\beta \\ \alpha+\beta=i+2}} n_{\alpha,\beta}(G). \tag{6.1.1}$$

We claim that for any graph G with $e(G) \geq 3$, the minimum degree $\delta(G)$ is equal to the least positive integer i for which the left-hand side of (6.1.1) is not equal to 0. Since $\sum_{H \in \mathcal{C}(G)} n_i(H)$ is reconstructible for any $0 \leq i < e(G)$, then so would $\delta(G)$.

To prove the claim, we consider equation (6.1.1) for $i = 1$ and 2:

$$\begin{aligned} \sum_{H \in \mathcal{C}(G)} n_1(H) &= (e(G) - 1)n_1(G) + n_{1,2}(G) \\ \sum_{H \in \mathcal{C}(G)} n_2(H) &= (e(G) - 2)n_2(G) + n_{1,3}(G) + n_{2,2}(G) \end{aligned} \tag{6.1.2}$$

Because $e(G) \geq 3$, we get that $\sum_{H \in \mathcal{C}(G)} n_1(H) \neq 0$ if and only if $n_1(G) \neq 0$ or $n_{1,2}(G) \neq 0$. In both situations, G has a vertex of degree 1. Conversely, if $\delta(G) = 1$, then $n_1(G) \neq 0$ and so $\sum_{H \in \mathcal{C}(G)} n_1(H) \neq 0$. So $\delta(G) = 1$ if and only if $\sum_{H \in \mathcal{C}(G)} n_1(H) \neq 0$. Therefore, one can recognize if $\delta(G) = 1$ in simple graphs with at least three edges.

Suppose $\sum_{H \in \mathcal{C}(G)} n_1(H) = 0$, so $\delta(G) > 1$. Then $\sum_{H \in \mathcal{C}(G)} n_2(H) \neq 0$ if and only if at least one of the three terms on the right-hand side of (6.1.2) is nonzero. Since $\delta(G) > 1$, $n_{1,3}(G) = 0$, and so either $n_2(G) \neq 0$ or $n_{2,2}(G) \neq 0$. In both cases, $\delta(G) = 2$. Conversely, if $\delta(G) = 2$, then $n_2(G) \neq 0$ and $\sum_{H \in \mathcal{C}(G)} n_2(H) \neq 0$. Therefore, one concludes that $\delta(G) = 2$ if and only if $\delta(G) \neq 1$ and $\sum_{H \in \mathcal{C}(G)} n_2(H) \neq 0$. One can therefore recognize if $\delta(G) = 2$ in simple graphs with at least three edges.

Let $d \geq 3$ be such that for any $i < d$, $\delta(G) = i$ is equivalent to the statement that i is the smallest positive integer for which $\sum_{H \in \mathcal{C}(G)} n_i(H) \neq 0$. We show that $\delta(G) = d$ is equivalent to the statement that d is the smallest positive integer for which $\sum_{H \in \mathcal{C}(G)} n_d(H) \neq 0$, thereby completing the proof that $\delta(G)$ is recognizable. If $\delta(G) = d$, then the right-hand side of equation (6.1.1) with $i = d$ is not equal to 0, and so is $\sum_{H \in \mathcal{C}(G)} n_d(H)$. Furthermore, since every term in the right-hand side of equation (6.1.1) for $i < d$ involves a vertex of degree less than d, we conclude that $\sum_{H \in \mathcal{C}(G)} n_i(H) = 0$ for every $i < d$. Conversely, if d is the smallest number for which $\sum_{H \in \mathcal{C}(G)} n_d(H) \neq 0$, then we may assume that $\delta(G) \geq d$. But since $\sum_{H \in \mathcal{C}(G)} n_d(H) \neq 0$, and since every term in the right-hand side of equation (6.1.1) with $i = d$ involves a vertex of degree less than d except for $(e(G) - d)n_d(G)$, we conclude that $n_d(G) \neq 0$, and so that $\delta(G) = d$. We have shown that $\delta(G)$ is reconstructible for connected graphs.

We now show that $n_d(G)$ is reconstructible for every integer d. This is done by induction. Suppose $\delta(G) \geq 3$. Equation (6.1.1) with $i = \delta(G)$ simplifies to

$$\sum_{H \in \mathcal{C}(G)} n_{\delta(G)}(H) \quad = \quad (e(G) - \delta(G))n_{\delta(G)}(G)$$

and thus $n_{\delta(G)}(G)$ is reconstructible. Let $d < e(G)/2$ and suppose $n_i(G)$ is known for every $i < d$. Suppose also that $\sum_{H \in \mathcal{C}(G)} n_d(H)$ is not divisible by $e(G) - d$. In that case, the sum on the right-hand side of equation (6.1.1) must not be equal to 0. This implies that there is a pair of adjacent vertices u and v, in G, whose degrees sum to $d+2$. Since $\delta(G) \geq 3$, we conclude that $\deg(u) < d$ and $\deg(v) < d$. In that case, find a card $H \in \mathcal{C}(G)$ with $n_{\deg(u)}(H) = n_{\deg(u)}(G) - 1$ and $n_{\deg(v)}(H) = n_{\deg(v)}(G) - 1$. From this card, one can obtain the entire degree sequence of G, by taking the degree sequence of H, removing one of the degrees d, and adding the degrees $\deg(u)$ and $\deg(v)$.

We may therefore assume that $\sum_{H \in \mathcal{C}(G)} n_d(H)$ is divisible by $e(G) - d$, so

$$\sum_{H \in \mathcal{C}(G)} n_d(H) = k(e(G) - d)$$

for some positive integer k. Denote by Σ the sum on the right-hand side of equation (6.1.1). The possible solutions are $\Sigma = t(e(G) - d), n_d(G) = k - t$ for any constant $t \leq k$. If $t \geq 2$ then $\Sigma \geq 2(e(G) - d) > e(G)$, but since Σ counts a subset of the edges of G, this is a contradiction. Therefore, the possible solutions are $t = 0$ or 1, which yields, respectively,

$\Sigma = 0$ and $n_d(G) = k$;

$\Sigma = e(G) - d$ and $n_d(G) = k - 1$.

The first option occurs if and only there are no adjacent vertices u and v in G such that $\deg(u) + \deg(v) = d + 2$ or, equivalently, if every card H with $n_d(H) = k$ has no pairs of integers α and β such that $\alpha + \beta = d + 2$ and $n_\alpha(H) = n_\alpha(G) - 1$ and $n_\beta(H) = n_\alpha(G) - 1$. This allows for the reconstruction of $n_d(G)$, for every $d < e(G)/2$. Therefore, one can determine if the first option occurs, in which case $n_d(G)$ is reconstructible for every $d < e(G)/2$. By complementarity, one can determine if the second option occurs, in which case $n_d(G) = k - 1$ is reconstructible, and induction is applied.

It now remains to reconstruct $n_d(G)$ for $d > e(G)/2$. There cannot be more than one vertex with a degree of at least $e(G)/2$, otherwise G has a vertex of degree 1, or G is C_4, which contradicts $\delta(G) \geq 3$. Since G has only one vertex of degree at least $e(G)/2$, that last degree can be reconstructed using $\sum_{x \in V(G)} \deg(x) = 2e(G)$. This completes the proof that the degree sequence is reconstructible if $\delta(G) \geq 3$.

If $\delta(G) = 2$, then consider a card H where $n_2(H)$ is minimum. Then H was obtained from G by contracting an edge e incident to a vertex of degree 2. This contraction does not change the degree of the other vertex incident to e. Therefore, the degree sequence of H is equal to the degree sequence of G minus one vertex of degree 2.

If $\delta(G) = 1$, then G is separable. Therefore, by Theorem 4.3.14, G is reconstructible, save a few exceptions. In the case where G is reconstructible, its degree sequence certainly is reconstructible as well. In the other cases, G is a graph with a single non-trivial block B, and trees attached to its vertices. Furthermore, the list of blocks was shown to be reconstructible by Lemma 4.3.8. By Lemma 4.3.15, the list of limbs $\{T(\varphi(v)) \mid \varphi \in Aut(B)\}$ attached to every vertex v in some orbit of B is reconstructible, provided there are at least two vertices of G with non-trivial limbs. In that case, the degree of each vertex in $V(G) - V(B)$ is determined, as every tree $T(v)$ is determined, and the degree of each vertex v in $V(B)$ is also determined, as it is equal to the degree of v in B plus the degree of the root of a tree in $\{T(\varphi(v)) \mid \varphi \in Aut(B)\}$. If only one vertex of B has a non-trivial limb, then consider a card corresponding to the contraction of an edge in B, where the vertex v with the non-trivial limb has a minimum degree. Then, $\deg(v)$ equals the degree of the vertex in G with the non-trivial limbs, and since B and $T(v)$ are determined, so is the degree sequence. ∎

From Proposition 6.1.1, one can reconstruct various properties of a simple graph. For instance, a graph is ***regular*** if all of its vertices have the same degree. By Proposition 6.1.1, most regular graphs are recognizable. A graph is ***eulerian*** if it contains a closed trail of length $e(G)$. Equivalently, a graph is eulerian if and only if all of its vertices are of even degree, which is now a reconstructible property.

6.2 Regular graphs

6.2.1 1-regular and 2-regular graphs

Now that the degree sequence is reconstructible, we can conclude that the class of simple d-regular graphs is recognizable, for any $d \neq 1$. Indeed, simple 0-regular graphs are trivial, simple 2-regular graphs are cycles or a disjoint union of cycles, both of which are reconstructible. Then simple d-regular graphs for $d \geq 3$ are recognizable using Proposition 6.1.1. The class of simple 1-regular graphs is not recognizable: see Figure 6.2 for all the examples of 1-regular graphs that are not recognizable.

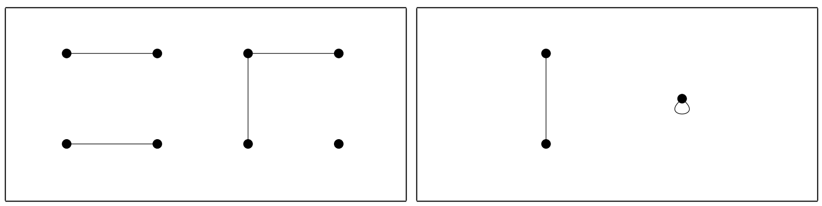

Figure 6.2: In each box, there is a 1-regular graph and a reconstruction of it that is not 1-regular.

However, the class of 1-regular graphs with at least three edges is reconstructible.

Theorem 6.2.1. 1-*regular graphs with at least three edges are reconstructible.*

Proof: If G is 1-regular, then G is the disjoint union of $e(G)$ copies of K_2. By Theorem 4.2.1, such graphs are reconstructible. ∎

Therefore, 1-regular graphs with two or fewer edges are not reconstructible while 1-regular graphs with at least three edges are reconstructible. The same is true of 2-regular graphs. Figure 6.3 shows that some small 2-regular graphs are not reconstructible. It furthermore shows that the class of 2-regular graphs is not recognizable.

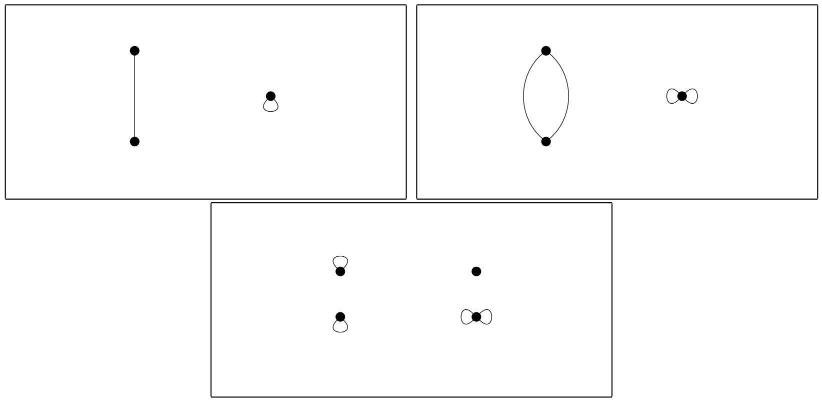

Figure 6.3: In each box, there is a 2-regular graph and a reconstruction of it that is not 2-regular.

Theorem 6.2.2 shows that the class of simple 2-regular graphs with at least three edges are reconstructible.

Theorem 6.2.2. *Simple* 2*-regular graphs are reconstructible except for the ones found in Figure 6.3.*

Proof: Let G be a 2-regular graph. Since G is not in Figure 6.3, we may assume that $e(G) \geq 3$. If G is disconnected, then it is reconstructible by Theorem 4.2.1. If G is connected, then since $e(G) \geq 3$, we may assume that G is simple. Therefore, by Proposition 6.1.1, one can recognize if G is 2-regular. Since $e(G)$ reconstructible by Proposition 3.1.1, and since there is only one isomorphically distinct connected 2-regular graph with $e(G)$ edges, G is reconstructible. ∎

6.2.2 Cubic graphs

It has been shown that 2-regular graphs, and most 1-regular graphs are reconstructble. Let us now consider 3-regular graphs, often referred to as ***cubic*** graphs. In this section, we show that some cubic graphs are reconstructible. First a standard lemma is required.

Lemma 6.2.3. *Any graph G with a minimal degree of at least* 3 *contains a cycle of even length.*

Proof: Assume, without loss of generality, that G is connected, since the cycle of even length we are seeking must entirely be contained in a connected component of G. Let $v \in V(G)$ be a vertex that is not a cut-vertex, that is $G - v$ is connected.

Such a vertex v exists. If G is 2-connected, then every vertex is not a cut vertex. If G is separable, then since $\delta(G) \geq 3$, the leaf blocks are non-trivial, and so they contain a vertex that is not a cut vertex. Let x, y and z be three distinct neighbours of v. Since $G - v$ is connected, let $x_1, x_2, ...x_m$ be a minimal xy-path in $G - v$, where $x_1 = x$ and $x_m = y$. Let $z_1, ..., z_k$ be a minimal path between z and the set $\{x_1, x_2, ..., x_m\}$ in $G - v$, with $z_1 = z$ and $z_k = x_j$. There are then three internally disjoint $x_j v$-paths in G. Two of these paths' lengths must have the same parity, yielding a cycle of even length. ∎

Theorem 6.2.4. *Simple cubic graphs with a triangle or with an even girth are reconstructible. Furthermore, every cubic graph has at most one nonisomorphic reconstruction.*

Proof: By Proposition 3.1.2, simple graphs are recognizable. By Proposition 6.1.1, the property of being cubic is reconstructible. By Proposition 5.1.4, the girth of a graph is also reconstructible. We can therefore recognize the class of cubic graphs with a triangle or with an even girth.

Furthermore, within each card of such a graph G, the contracted vertex v^* can be identified, as it is the only vertex of degree 4. Denote by e_1, e_2, e_3 and e_4 the four edges incident to the only vertex of degree 4 in each card. Suppose G has a triangle. Then, from a card containing parallel edges, say e_1 and e_2, one can reconstruct G uniquely by decontracting v^* and by splitting e_1 and e_2 evenly, and then splitting the remaining two edges evenly among the two new vertices.

Suppose now that G does not have a triangle. Let $2k$ be the girth of G. Consider a card with a cycle of length $2k - 1$. Now G does not have any cycles of length $2k - 1$, so this cycle certainly must contain the vertex v^* of degree 4, and the two edges incident to v^* in that cycle of length $2k-1$, say e_1 and e_2, must not be incident in G. All that remains is to decide whether e_3 is incident to e_1 or not. Therefore, G has at most one non-isomorphic reconstruction. Call that reconstruction H.

Applying the previous argument to every card corresponding to an edge e in a cycle of length $2k$, we conclude that every one of those cards generate two possible reconstructions, say $G^{(e)}$ and $H^{(e)}$. If G is the only common isomorphism class to every pair $\left(G^{(e)}, H^{(e)}\right)$ for every e in a cycle of length $2k$, then G is reconstructible. Otherwise, one may assume that $G^{(e)} \cong G^{(e')} \cong G \not\cong H \cong H^{(e)} \cong H^{(e')}$ for every pair of edges e, e', both contained within cycles of length $2k$.

Let $x_1, x_2, ..., x_{2k}$ be the vertices in a cycle C of length $2k$ in G, and let y_i be adjacent to x_i, and not equal to any of the x_j's. So, for example, x_2's neighbours are x_1, x_3 and y_2. In a card $G/x_i x_{i+1}$, the only choice to be made when decontracting the contracted vertex is to decide whether y_i is adjacent to x_i or x_{i+1}. If attaching y_i to x_i and y_{i+1} to x_{i+1} yields G, and that attaching y_i to x_{i+1} and y_{i+1} to x_{i+1} yields H

for any i, then the following procedure transforms G into H. Choose a transposition ϕ, of the type $(i, i+1)$, or $(2k, 1)$, then remove all of the $x_i y_i$ edges and add $x_i y_{\phi(i)}$ to transform G into H. Symmetrically, one can transform H into G in a similar fashion. One can then choose to repeat this operation as many times as wanted, each operation alternating the graph between G and H. Since the transpositions $(i, i+1)$ and $(2k, 1)$ generate the symmetric group S_{2k}, one can simplify this process to choosing any permutation $\phi \in S_{2k}$, removing all the $x_i y_i$, then adding $x_i y_{\phi(i)}$, to obtain either G or H from G. A permutation $\phi \in S_{2k}$ transforms G into G and H into H if and only if ϕ is the composition of an even number of transpositions of the type $(i, i+1)$, or $(2k, 1)$, or, equivalently, if and only if $\phi \in A_{2k}$, where A_{2k} is the alternating group on $2k$ elements.

Let $D_{2k} \subseteq S_{2k}$ be the automorphism group of the cycle C. Consider the rotation $\phi = (1, 2, ..., 2k)$, which is in D_{2k}. Removing all the $x_i y_i$ and adding $x_i y_{\phi(i)}$ in G yields a graph isomorphic to G, but since ϕ is not in A_{2k}, the resulting graph is also isomorphic to H, which leads to $G \cong H$. Thus, G is reconstructible. ∎

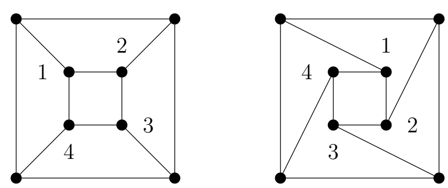

Figure 6.4: To the left is a cubic graph G. Using $\phi = (1234)$, we obtain the graph on the right, which is isomorphic to G.

Unfortunately, for an odd girth $2k+1$, we have that $D_{2k+1} \subseteq A_{2k+1}$, so the final part of the previous proof does not work. It also seems unlikely that this proof can be generalized to k-regular graphs, for $k > 3$.

6.3 Distance regular graphs

The class of distance regular graphs is proved to be reconstructible in this section. We first study a subclass of distance regular graphs, called the strongly regular graphs, before considering the distance regular graphs.

6.3.1 Strongly regular graphs

In this section, we study a class of regular graphs called the strongly regular graphs. We first define these graphs, then show that they are reconstructible. Definitions are based on [5].

A simple graph G is said to be ***strongly regular*** with parameters (v, k, a, c) if

1. $v(G) = v$;
2. G is k-regular;
3. any two adjacent vertices of G have exactly a common neighbours;
4. any two nonadjacent vertices of G have exactly c common neighbours.

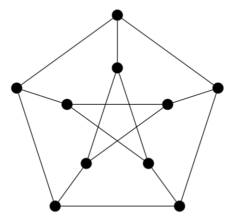

Figure 6.5: The Petersen graph is a strongly regular graph with the parameter set $(10, 3, 0, 1)$.

If $k = 1$, then the connected components of G are all isomorphic to K_2. From Theorem 6.2.1, we know that such graphs are reconstructible if $e(G) \geq 3$. If G is strongly regular and disconnected, then its connected components are all isomorphic to complete graphs on $k - 1$ vertices.

If $k = 2$ and G is strongly regular and connected, then $G = C_3, C_4$ or C_5, which are all reconstructible. Henceforth we focus on strongly regular graphs with $k \geq 3$. Furthermore, by counting the number of vertices at distance 2 from a fixed vertex, in two different ways, one obtains $k(k - a - 1) = (v - k - 1)c$. This implies that if $c = 0$, then either $k = 0$, in which case the graph is empty, or $k = a + 1$, in which case the connected components of the graph are complete. These special cases of strongly regular graphs are shown to be reconstructible.

Theorem 6.3.1. *Strongly regular graphs with at least three edges and $k = 2$ or $k = a + 1$ are reconstructible.*

Proof: If G is strongly regular with $k = 2$, then G is 2-regular and reconstructible by Theorem 6.2.2.

A graph G is strongly regular with $k = a + 1$ if and only if G is the disjoint union of complete graphs K_{k+1}. If $k = 1$, then since G has at least three non-trivial connected components, G is therefore reconstructible by Theorem 4.2.1. Otherwise, $k \geq 2$, so the complete graphs must be on at least three vertices. If G is connected, which can be determine by Proposition 3.1.5, then G is reconstructible: a complete graph K_n is the only simple graph on n vertices with $\binom{n}{2}$ edges. If G is not connected, then G is reconstructible by Theorem 4.2.1. ∎

The ***diameter*** of a graph is the maximum distance between two vertices, denoted by $\text{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}$. If G is disconnected, then $\text{diam}(G) = \infty$. It follows from the definition of a strongly regular graph that they are either disconnected if $c = 0$, or they have a diameter of at most 2 otherwise. Proposition 6.3.3 shows that the diameter of a graph is reconstructible, provided the graph is not bipartite. The following lemma is required for Proposition 6.3.3.

Lemma 6.3.2. *Let G be a connected graph. G contains an odd cycle if and only if for every vertex $x \in V(G)$, there exists an edge uv such that $d(x, u) = d(x, v)$.*

Proof: Orient the edges of the odd cycle, as to create an oriented cycle, and label each arc vu with the integer $d(u, x) - d(v, x)$. Since u and v are adjacent, we get $d(u, x) - 1 \leq d(v, x) \leq d(u, x) + 1$, so the possible labels for the arcs are $-1, 0$ or 1. The sum of the labels must be 0, since the edges form a cycle, and since the length of the cycle is odd, at least one edge uv has label 0. Hence $d(u, x) = d(v, x)$.

For the converse, suppose $d(u, x) = d(v, x)$. Let c_u and c_v be a minimum xu-path and xv-path, respectively. Now $v \notin c_u$ and $u \notin c_v$. Let $z \in c_u \cap c_v$ be the first vertex, starting at u or v, shared by both c_u and c_v. One can assume that the paths c_u and c_v are identical after they meet at z. There are therefore two disjoint paths, one zu-path and one zv-path. These paths are of equal lengths, and so they form an odd cycle with uv. ∎

Any edge whose end points are at the same distance of x cannot be contained in a minimum xy-path, for any $y \in V(G)$. Therefore, a consequence of Lemma 6.3.2 is that if G contains an odd cycle, then for every vertex x, there exists an edge uv which is not contained in any minimum xy-path for any vertex y.

Proposition 6.3.3. *If a graph G is not bipartite and has at least one edge, then the diameter of G is reconstructible.*

Proof: Proposition 3.1.5 shows that it is possible to recognize if a graph with at least one edge is disconnected, or, equivalently, if the diameter of a graph is ∞. By Proposition 5.2.1, bipartite graphs are recognizable. Assume that the graph is connected. Now, $\text{diam}(G/e) = \text{diam}(G) - 1$ if and only if e is contained in a minimal xy-path, for every pair of vertices x and y at distance $\text{diam}(G)$. Of course, $\text{diam}(G) - 1 \leq \text{diam}(G/e) \leq \text{diam}(G)$, since the distance between any two vertices can only decrease by at most 1 after a contraction, and the distance cannot increase. Therefore, for any card $H \in \mathcal{C}(G)$, either $\text{diam}(H) = \text{diam}(G)$ or $\text{diam}(H) = \text{diam}(G) - 1$.

If there exists two cards in $\mathcal{C}(G)$ with different diameters, then $\text{diam}(G)$ is equal to the greatest of these diameters. If every card has the same diameter, then one needs to determine whether $\text{diam}(G) = \text{diam}(H) + 1$ or $\text{diam}(G) = \text{diam}(H)$ for all $H \in \mathcal{C}(G)$. The first case occurs if and only if every edge of G is contained in a minimal xy-path, for every pair of vertices x and y at distance $\text{diam}(G)$. Since G is not bipartite, by Lemma 6.3.2, for every vertex $x \in V(G)$, there exists an edge uv such that u and v are at equal distance from x. Therefore, the edge uv is not contained in any minimal xy-path for any y. It is thus impossible for $\text{diam}(G) = \text{diam}(H) + 1$ for all $H \in \mathcal{C}(G)$ if G is not bipartite. Therefore, there always exists a card H such that $\text{diam}(H) = \text{diam}(G)$, so $\text{diam}(G)$ is reconstructible. ∎

Bipartite graphs were shown to be reconstructible in Corollary 5.2.7. We may therefore focus on strongly regular graphs that are not bipartite. Using Proposition 6.3.3, one can recognize the class of graphs with a diameter of 2, of which the strongly regular graphs are a subclass. To show that the class of strongly regular graphs is recognizable, we first show that the parameters v, k, a and c exist, and that their values can be reconstructed.

Proposition 6.3.4. *Non bipartite strongly regular graphs are recognizable. Furthermore, the parameters v, k, a and c of such graphs are reconstructible.*

Proof: We first show that the parameters v, k, a and c are reconstructible, then conclude that non bipartite strongly regular graphs are recognizable.

By Propositions 3.1.2 and 5.2.1, simple non bipartite graphs are recognizable. Furthermore, since G is simple, v is reconstructible by Proposition 3.1.3. By Proposition 6.1.1, we can determine if the graph is regular and find k.

Let $a(u, v)$ denote the number of parallel edges in G/uv. Then $a(u, v)$ counts the number of common neighbours of u and v in G. If the parameter $a(u, v)$ is constant throughout every card, then that constant is a.

We now show that any two vertices at distance 2 have c common neighbours. For any two adjacent vertices x and y, there are a neighbours of y at distance 1 from x, and therefore there are $k-a-1$ neighbours of y at distance 2 from x. By Theorem 6.3.1, we assume that $k \neq a+1$ and therefore we conclude that for any vertex x of G, there are vertices at distance 2 from x, and therefore there is an edge linking a neighbour y of x to a non neighbour z of x. In G/yz, the contracted vertex v^* can be identified, since G is k-regular with $k \geq 3$, and so v^* is the only vertex of G/yz of degree $2k-2$. The vertex x is adjacent to v^*, with only a single edge joining the two. Let A denote the set of common neighbours of v^* and x in G/yz. In A are the a common neighbours of y and x, and the $c(z, x) - 1$ common neighbours of z and x. Therefore, by the inclusion-exclusion principle, $|A| = a + c(z, x) - 1 - p$, where p denotes the number of common neighbours to x, y and z, which corresponds to the number of vertices of A linked by parallel edges to v^* in G/yz. Therefore, $c(z,x)$ can be determined, and if one goes through all the cards of $\mathcal{C}(G)$, one can determine $c(z,x)$ for every pair of vertices z and x at distance 2. Should $c(z,x)$ be constant throughout all pairs z and x, the parameter c is reconstructed.

The parameter c, as defined in the previous paragraph, represents the number of common neighbours between any two vertices at distance 2. For a graph to be strongly regular, every pair of nonadjacent vertices must have c common neighbours. It therefore remains to recognize if a graph has a diameter of 2, which is done using Proposition 6.3.3. ∎

Strongly regular graphs are part of a larger class of graphs called the amply regular graphs. A graph G is ***amply regular*** with parameters (v,k,a,c) if

1. $v(G) = v$,
2. G is k-regular,
3. any two adjacent vertices of G have exactly a common neighbours,
4. any two vertices at distance 2 of G have exactly c common neighbours.

It follows from the definition that an amply regular graph with a diameter of 2 is a strongly regular graph. The proof of Proposition 6.3.4 can be applied to amply regular graphs with $k \geq 3$.

Proposition 6.3.5. *Amply regular graphs are recognizable. Furthermore, the parameters v, k, a and c of such graphs are reconstructible.*

Proof: Apply the proof of Proposition 6.3.4 and omit the last paragraph of that proof. ∎

Since we do not need to determine the diameter of an amply regular graphs, Proposition 6.3.3 is not needed, hence Proposition 6.3.5 is valid for both bipartite and non bipartite amply regular graphs.

It remains to show that strongly regular graphs that are not bipartite are weakly reconstructible. Theorem 6.3.7 shows that, and combines the weak reconstruction and the recognizability of strongly regular graphs to conclude that they are reconstructible. This is done by determining, for every pair of vertices x and y adjacent to the contracted vertex v^* of a card, whether x and y are attached or not to the same vertex after the decontraction of v^*.

Definition 6.3.6. *Let $G/uv \in \mathcal{C}(G)$, let $v^* \in V(G/uv)$ be the contracted vertex and let $x, y \in V(G/uv)$ be two vertices linked to v^* by exactly one edge. We say that x and y are **separated** if $xu \in E(G)$ and $yv \in E(G)$ or if $xv \in E(G)$ and $yu \in E(G)$. We say that x and y are not separated if $xv \in E(G)$ and $yv \in E(G)$ or if $xu \in E(G)$ and $yu \in E(G)$.*

If follows from the definition being separated is a symmetric relation. It also follows that if x and y are separated and if y and z are separated, then x and z are not separated. Similarly, if x and y are separated and y and z are not, then x and z are separated. Furthermore, if one is able to determine, for every pair of vertices x and y linked to the contracted vertex v^* of a card, whether they are separated or not, then the graph is reconstructible.

Theorem 6.3.7. *Strongly regular graphs with at least one edge are reconstructible.*

Proof: First, by Corollary 5.2.7, simple bipartite graphs are reconstructible, and so are bipartite strongly regular graphs. We henceforth focus on non bipartite graphs. By Proposition 6.3.4, strongly regular graphs that are not bipartite are recognizable. Furthermore, the parameters v, k, a and c are reconstructible. By Theorem 6.3.1, we may assume that $k \geq 3$.

For any card $H \in \mathcal{C}(G)$, let v^* be the contracted vertex, which can be identified, since it is the only vertex of degree $2k-2$ in H, because $k \geq 3$. To reconstruct G from H, one needs to decontract v^* into two vertices v_1 and v_2. It is then sufficient to split the neighbours of v^* in H into the neighbours of the two vertices v_1 and v_2. First, let us observe that if two parallel edges link a vertex x to v^* in H, then x is adjacent to both v_1 and v_2 in G. We therefore focus on vertices that are linked to v^* by a single edge. Let x and y be such neighbours of v^* in H. If x and y are adjacent, then x and y are not separated if and only if x and y have exactly a common neighbours in H. If x and y are not adjacent, then x and y are not separated if and only if x and y have c common neighbours in H. It is possible to decide, for every pair x and y of neighbours of v^*, whether they are separated or not. Therefore, G is reconstructible. ∎

Notice that once the deck $\mathcal{C}(G)$ was recognized to be the deck of a strongly regular graph, any one card can be used to reconstruct G. This implies that two strongly regular graphs with the same parameters have no cards in common. Furthermore, two strongly regular graphs G_1 and G_2 with distinct parameters cannot share a card: if $v(G_1) \neq v(G_2)$, then $v(H_1) \neq v(H_2)$ for any two cards $H_1 \in \mathcal{C}(G_1)$ and $H_2 \in \mathcal{C}(G_2)$. If the vertices of G_1 and G_2 do not have the same degree, then the cards of G_1 do not have the same degree sequence as the cards of G_2. If the parameter a of G_1 is not the same as the parameter a of G_2, then the cards of G_1 do not have the same number of parallel edges as the cards of G_2. Finally, if G_1 and G_2 have the same parameters v, k and a, then since $k(k-a-1) = (v-k-1)c$, we conclude they have the same parameter c. This proves that two strongly regular graphs with distinct parameters cannot share a card, hence no two strongly regular graphs have a card in common.

6.3.2 Distance regular graphs

In this section, we introduce a class of graphs that generalizes the concept of strongly regular graphs. They are the distance regular graphs. We show that they are reconstructible. We first give a brief introduction to distance regular graphs, based on [5].

Recall that $\mathrm{diam}(G) = \max\{d(x,y) \mid x, y \in V(G)\}$ denotes the diameter of G. A simple regular graph is ***distance regular*** if there exists integers a_i, b_i, c_i, for $i = 1, 2, ..., \mathrm{diam}(G)$, such that for any two vertices x and y with $d(x,y) = i$, there are a_i neighbours of y at distance i from x, c_i neighbours of y at distance $i-1$ from x, and b_i neighbours of y at distance $i+1$ from x. Strongly regular graphs are distance regular, with $a_1 = a, c_1 = 1, b_1 = k-1-a, a_2 = k-c, c_2 = c$ and $b_2 = 0$. Since every vertex is of degree k, we see that $a_i + b_i + c_i = k$.

For two vertices x and y at distance d, denote by $a_d(x,y), b_d(x,y)$ and $c_d(x,y)$ the number of neighbours of y at distance $d, d+1$, and $d-1$ from x, respectively.

For any vertex x in a distance regular graph, the number k_i of vertices at distance i from x does not depend on the choice of x. The parameter k_i is only dependent on the parameters b_j and c_j for $j \leq i$. In [5], the following recursive formula for k_i is given:

$$k_0 = 1, \qquad k_1 = k, \qquad k_{i+1} = \frac{k_i b_i}{c_{i+1}}.$$

In particular, for any vertex x in a distance regular graph, there exists a vertex y such that $d(x,y) = \mathrm{diam}(G)$.

We start with a few lemmas that are useful in the proof of the weak reconstruction of distance regular graphs. Before the next lemma, we define a path between two vertices x and y to be ***geodesic*** if it is a minimum xy-path. In other words, a path of length t is geodesic if the ends of that path are at distance t in the graph.

Lemma 6.3.8. *Let G be a distance regular graph. Let t be the smallest integer such that $c_t \neq 1$ or $a_t \neq a_1$. If $c_t \neq 1$ and x and y are vertices on a cycle of length less than $2t$, then x is adjacent to y. If $c_t = 1$, $a_t \neq a_1$ and x and y are vertices on a cycle of length less than $2t+1$, then x is adjacent to y.*

Proof: This proof is done by induction on the length m of the cycle. If $m = 3$, then any two vertices on a cycle of length 3 are adjacent. Note that by definition, t is at least 2. If $c_2 = 1$ and $x_1x_2x_3x_4$ is a cycle of length $m = 4$, then x_2 and x_4 are two common neighbours of x_1 and x_3. However, since $c_2 = 1$, it implies that $x_1x_3 \in E(G)$. Similarly, x_1 and x_3 are two common neighbours of x_2 and x_4, which implies that $x_2x_4 \in E(G)$. Therefore, the vertices x_1, x_2, x_3 and x_4 are pairwise adjacent.

Suppose, by induction, that if C_0 is a cycle of length $i < m$, then every pair of vertices on C_0 are adjacent. Let C be a cycle of length m. Suppose $m < 2t$ and that m is even. Let x and y be vertices at distance $m/2$ on C. If $d(x,y) = m/2$ in G, then the two neighhours of x on C are at distance $m/2-1$ from y, which implies that $c_{m/2} \geq 2$, a contradiction with $c_{m/2} = 1$. Otherwise, $d(x,y) < m/2$ in G, which implies the existence of a geodesic path P of length less than $m/2$ between x and y. If that geodesic path P is disjoint from C except at its ends, then x and y are in a cycle of length less than m, which implies, by induction, that x and y are adjacent. The edge xy then splits the cycle C into two smaller cycles C_1 and C_2. By induction, every pair of vertices on C_1 are adjacent, and every pair of vertices on C_2 are adjacent. Furthermore, if $u \in C_1$ and $v \in C_2$, then they are adjacent, since $xuyv$ is a cycle of length 4. We conclude that the vertices of C are pairwise adjacent. Otherwise, the geodesic path P is not disjoint from C. Since the path P is strictly shorter than any xy-path on C, there exists two vertices z_1 and z_2 on $C \cap P$ that are not adjacent on C. Choose such z_1 and z_2 such that their distance is minimum. Then z_1 and z_2 are contained in a cycle of length less than m, which implies, by induction, that $z_1z_2 \in E(G)$. This further implies, by the argument in the case where P does not intersect C except at its ends, that the vertices of C are pairwise adjacent.

Suppose m is odd, and let x and y be vertices at distance $(m-1)/2$ on C. By a similar argument, if $d(x,y) < (m-1)/2$ in G, then the vertices of C are pairwise adjacent. Suppose $d(x,y) = (m-1)/2$ in G. Let $v \in C$ be the neighbour of x on C at distance $(m-3)/2$ from y, and let $u \in C$ be the other neighbour of x on C, which is at distance $(m-1)/2$ from y. We claim that the a_1 common neighbours of v and x must all be at distance $(m-1)/2$ from y. Indeed, suppose, by way of contradiction, that one of those neighbours z is at distance $(m-3)/2$ from y. In that case, let P_z be a geodesic zy-path, and let $P_v \subset C$ be a geodesic vy-path, both of length $(m-3)/2$. Let w be the vertex nearest to z in $P_v \cap P_z$. If $w = y$, then v, z and y are contained in a cycle of length $2(m-3)/2+1 = m-2 < m$, which implies, by induction, that they are pairwise adjacent. Therefore, $(m-1)/2 = d(x,y) = d(v,y)+1 = 2$, which implies that $m = 5$. In that case, $xvyz$ is a cycle of length 4, so, by induction, $xy \in E(G)$. This contradicts the assumption that $d(x,y) = (m-1)/2$. If $w \neq y$, then x and w are contained in a cycle of length less than m, which implies that x and w are adjacent. Since $w \neq v$, we conclude that $d(x,y) = d(y,w)+1 < (m-1)/2$, a contradiction. We have therefore shown that the a_1 common neighbours of v and x are all at distance $(m-1)/2$ from y. Since $a_{(m-1)/2} = a_1$, every neighbour of x at distance $(m-1)/2$ from y must be adjacent to v. In particular, u must be adjacent to v. The edge uv separates the cycle C into two smaller cycles C_1 and C_2. This implies, same as before, that the vertices of C are pairwise adjacent. ∎

A distance regular graph with the properties stated in Lemma 6.3.8 can only have triangles and cycles of length greater than $2t$ as induced cycles. The next Lemma shows that the minimum induced cycles of length greater than 3 are, in some sense, evenly spread across the graph.

Lemma 6.3.9. *Let G be a distance regular graph with $c_2 = 1$. Let t be the smallest integer such that $c_t \neq 1$ or $a_t \neq a_1$. If $c_t \neq 1$, then every geodesic path of length t is contained in a cycle of length $2t$. If $c_t = 1$ and $a_t \neq a_1$, then every geodesic path of length t is contained in an induced cycle of length $2t + 1$.*

Proof: Let x and y be the end vertices of a geodesic path P of length t, so $d(x, y) = t$ in G. Suppose $c_t \neq 1$, so $c_t \geq 2$. In that case, x has at least two neighbours at distance $t - 1$ from y, one of which is in P. Let $u \notin P$ be such a neighbour of x. Since $d(u, y) = t - 1$, there is a geodesic uy-path P' of length $t - 1$. We now show the claim that the two paths P' and P intersect only in their common end, the vertex y. Suppose otherwise the existence of another vertex $z \in P' \cap P$. It implies that the set of vertices in P and P' contain a cycle C of length less than $2t$, with $x, z \in C$. By Lemma 6.3.8, we conclude that $xz \in E(G)$, which contradicts the assumption that P is geodesic. The two paths P' and P are therefore disjoint except at their common end y, and they form a cycle C of length $2t$. If this cycle is not induced, then there is an edge which separates that cycle into two smaller cycles C_1 and C_2. The sum of the lengths of C_1 and C_2 is $2t + 2$. Since $c_2 = 1$, we have $2t + 2 \geq 8$, so one of C_1 or C_2 has a length of at least 4. By Lemma 6.3.8, any two vertices of C_1 must be adjacent and any two vertices of C_2 must be adjacent. This contradicts the assumption that P and P' are geodesic. Therefore, C is induced.

Suppose $c_t = 1$ and $a_t \neq a_1$. Let $u \in P$ be adjacent to x, and let M be the set of vertices adjacent to both u and x. Let $z \in M$. If $d(z, y) = t - 1$, then let P' be a minimum zy-path. If P' and P intersect elsewhere than in y, say at z', then z' and x are in a cycle of length at most $2(t - 1) + 1 = 2t - 1$. By Lemma 6.3.8, x and z' are therefore adjacent, hence $d(x, y) < t$. We may therefore assume that $d(z, y) = t$. In that case, $a_1 = |M| \leq a_t$. Since $a_t \neq a_1$, there therefore exists a neighbour v of x, such that $d(u, v) = 2$ and $d(y, v) = t$. Since $d(y, v) = t$, there exists a geodesic vy-path P'' of length t. We now prove claim that the paths P'' and P only intersect at y. Suppose otherwise, so P and P'' contain a cycle of length at most $2t$ containing x. By Lemma 6.3.8, any two vertices of that cycle must be adjacent, contradicting the assumption that P is geodesic. Therefore, the paths P'' and P intersect only at y, and those two paths, with the edge xv, form a cycle C of length $2t+1$. If this cycle C is not induced, then there is an edge which separates that cycle into two smaller cycles C_1 and C_2. The sum of the lengths of C_1 and C_2 is $2t+3$. Since $t \geq 2$, we have $2t + 3 \geq 7$, so one of C_1 or C_2 has a length of at least 4. By Lemma 6.3.8, any two vertices of C_1 must be adjacent and any two vertices of C_2 must be adjacent. This contradicts the assumption that P and P' are geodesic. Therefore, C is induced. ∎

For every vertex x in a distance regular graph G, there exists a vertex y such that $d(x, y) = \mathrm{diam}(G)$. This implies that any geodesic path in a distance regular graph G can be extended to a geodesic path of length $\mathrm{diam}(G)$. In particular, we conclude from Lemma 6.3.9 that if $t \geq 3$, then any geodesic path of length 2 or 3 is contained in an induced cycle of length $2t$ or $2t+1$, and in no smaller induced cycles.

We now show that distance regular graphs are recognizable, and that, furthermore, their parameters are reconstructible.

Proposition 6.3.10. *Let G be a simple k-regular graph, $k \geq 3$. The parameters (a_i, b_i, c_i) are reconstructible for any $i \leq \mathrm{diam}(G)$. In particular, distance regular graphs with $k \geq 3$ are recognizable.*

Proof: The proof is done by induction. We first show that (a_1, b_1, c_1) is reconstructible. We then show that if (a_i, b_i, c_i) is known for every $i < d$, then (a_d, b_d, c_d) is reconstructible. By Proposition 6.1.1, k is reconstructible.

Since G is simple, $c_1 = 1$. Since G is k-regular and $k \geq 3$, the contracted vertex in G/xy can be identified, since it is the only vertex of degree $2k - 2$. The parameter $a_1(x, y)$ is equal to the number of pairs of parallel edges in G/xy, which, equivalently, is equal to the number of triangles containing the edge xy in G. The value a_1 is well defined if and only if $a_1(x, y) = a_1(u, v)$ for any two edges xy and uv. Finally, since $b_1 = k - c_1 - a_1$, b_1 is reconstructible. Therefore, (a_1, b_1, c_1) is reconstructible.

Suppose (a_i, b_i, c_i) is known for every $i < d$. If $b_{d-1} = 0$, then $\mathrm{diam}(G) \leq d - 1$ and so there are no two vertices at distance d from each other, hence (a_d, b_d, c_d) are undefined. Otherwise, $b_{d-1} \neq 0$, and so for every vertex $x \in V(G)$, there is a vertex y at distance $d - 1$ from x and there is a neighbour z of y at distance d from x. In G/yz, let A be the neighbours of the contracted vertex v^* at distance $d - 1$ from x. Then, $|A| = a_{d-1} + c_d(x, z) - p - 1$, where p is the number of neighbours of v^* at distance $d - 1$ from x linked to v^* by parallel edges in G/yz, and so $c_d(x, z)$ is determined. Checking every card of G allows us to reconstruct $c_d(x, z)$ for every pair of vertices x and z at distance d, and should these values all be equal, then c_d exists and is reconstructible.

In G/yz, let B be the neighbours u of v^* at distance d from x with a minimum ux-path that is disjoint from v^*. Since B is obtained from a card, $|B|$ is determined. This set is decomposed into three pairwise disjoint sets, B_y, B_z and B_{yz}, where B_y are the vertices of B adjacent to y and not z, B_z are the vertices of B adjacent to z and not y, and B_{yz} are the vertices adjacent to both y and z. A vertex of B is in B_{yz} if and only if it is linked to v^* by parallel edges in G/yz, so $|B_{yz}|$ is determined from the card. When looking in G, a vertex u in B is at distance d from x, and thus has c_d neighbours at distance $d - 1$ from x. One such neighbour of u is y if and only if u is adjacent to y. Therefore, a vertex $u \in B$ is in B_y if and only if u is linked to v^* with exactly one edge and u has $c_d - 1$ neighbours at distance $d-1$ from x in $G/yz - v^*$. Therefore, $|B_y|$ is reconstructible, and so is $|B_z|$. For every neighbour u of z at distance d from x, there is an ux-path of length d that does not pass through y, in which case $u \in B_z$, or every ux-path of length d passes through y, in which case u is adjacent to y. The number of neighbours of z corresponding to the second case is given by the number of vertices in G/yz linked to v^* by parallel edges and at distance d from x. This number is reconstructible and denote it by p. Then, $p + |B_z| = a_d(z,x)$, so $a_d(z,x)$ is determined. Thus $a_d(z,x)$ is determined for every pair x and z at distance d, and so a_d is reconstructible. Since $b_d = k - c_d - a_d$, b_d is also reconstructible. ∎

For a given pair of vertices z and x at distance d, the parameters $a_d(z,x)$ and $c_d(z,x)$ may have been counted multiple times in different cards. However, our only interest is if these parameters are all equal, so it does not matter that we evaluate the same parameter multiple times. We require only that $a_d(z,x)$ and $c_d(z,x)$ are counted at least once throughout this process, for every ordered pair of vertices (z,x).

In this proof, the parameter c_i is used to determine a_i, which in turn is used to determine c_{i+1}. The fact that G is k-regular is used to show that the contracted vertex can be identified in every card. It is also used to reconstruct b_i, since $b_i = k - a_i - c_i$. The reconstruction of c_i and a_i is somewhat independent from the regularity of the graph, and from the existence of the parameters b_i. Therefore, the proof of Proposition 6.3.10 may be generalized to non-regular graphs, and state that c_i and a_i are reconstructible, while b_i may not be properly defined if the graph is not regular. We state it formally in Proposition 6.3.11.

Proposition 6.3.11. *Let $\mathcal{G}$ be the class of graphs where one can identify the contracted vertex in every card and for every $G \in \mathcal{G}$, there exist integers a_i, c_i, for $i = 1, 2..., \operatorname{diam}(G)$, such that for any two vertices x and y with $d(x,y) = i$ in G, there are a_i neighbours of y at distance i from x and c_i neighbours of y at distance $i-1$ from x. The class $\mathcal{G}$ is recognizable, and the parameters a_i and c_i are reconstructible for every graph in $\mathcal{G}$.*

We now show that distance regular graphs are reconstructible. This is done in Theorem 6.3.12. Before the next theorem, we introduce additional notation for distance regular graph, which can be found in [5]. For any $i, j, h \geq 0$, if x and y are two vertices at distance i, then $p^i_{j,h}$ denotes the number of vertices at distance j from x and at distance h from y. In distance regular graphs, these $p^i_{j,h}$ are independent from the choice of x and y. Furthermore, they are entirely defined by the parameters a_ℓ, b_ℓ, c_ℓ. In [5], one finds the following recursive relation:

$$p^i_{j+1,h} = \frac{1}{c_{j+1}} \left(p^i_{j,h-1} b_{h-1} + p^i_{j,h}(a_h - a_j) + p^i_{j,h+1} c_{h+1} - p^i_{j-1,h} b_{j-1} \right).$$

In particular, for $(i,j,h)=(2,1,2)$ and $(3,1,2)$, one obtains, respectively,

$$p^2_{2,2} = \frac{1}{c_2}\left(c_2b_1 + a_2(a_2-a_1) + b_2c_3 - k\right) \tag{6.3.1}$$

$$p^3_{2,2} = \frac{1}{c_2}c_3\left(a_2 - a_1 + a_3\right). \tag{6.3.2}$$

Theorem 6.3.12. *Let G be a distance regular graph, which is k-regular, with $k \geq 3$. Then G is reconstructible.*

Proof: By Proposition 6.3.10, the class of distance regular graphs with a degree k of at least 3 is recognizable. Let G be such a graph, let $G/uv \in \mathcal{C}(G)$, and let v^* be the contracted vertex of G/uv, which can be identified, since it is the only vertex of degree $2k-2$, which is greater than k. The strategy here is to decide, for every pair of neighbours x and y of v^*, whether they are separated or not. If one is able to do so, then G is reconstructible. One may first assume that x and y are both linked to v^* by a single edge, since a vertex linked to v^* by parallel edges is adjacent to both u and v in G. For two adjacent neighbours x and y of v^*, x and y are not separated if and only if they have a_1 common neighbours in G/uv.

Suppose $c_2 \neq 1$, so that between any two vertices at distance 2, there are at least two disjoint geodesic paths. If x and y are two nonadjacent neighbours of v^*, then x and y are not separated if and only if x and y have c_2 common neighbours in G/uv. We can henceforth assume that

$$c_2 = 1. \tag{6.3.3}$$

In particular, it implies that if $x_1x_2x_3x_4$ is a cycle in G, then $x_1x_3 \in E(G)$ and $x_2x_4 \in E(G)$. We may also assume that the only xy-path of length 2 in G/uv passes through v^*.

Let x and y again be nonadjacent neighbours of v^*, and let M_1 be the set of neighbours z of y in the card for which there exists a vertex w such that xv^*yzw is a cycle of length 5 in G/uv. Any such cycle must be induced: if $xy \in E(G/uv)$, then since $c_2 = 1$, we conclude that x and y are not separated. If $zx \in E(G/uv)$ or $wy \in E(G/uv)$, then x and y are at distance 2 in $G/uv - v^*$, and since $c_2 = 1$, we conclude that x and y are separated. If $zv^* \in E(G/uv)$, then y and z are not separated while x are separated z, which implies that x and y are separated. Similarly, if $wv^* \in E(G/uv)$, then x and w are not separated while y and w are separated, which implies that x and y are separated. Therefore, the cycle xv^*yzw is induced. If x and y are not separated, then M_1 contains the neighbours of y at distance 2 from x that are not adjacent to v^* in G/uv, so $|M_1| = a_2 - a_1$. If x and y are separated, then M_1 contains every neighbour of y at distance 2 from x, except for one end of uv, so $|M_1| = c_3 - 1$. Therefore, one cannot decide if x and y are separated or not only if

$$a_2 - a_1 = c_3 - 1 \tag{6.3.4}$$

and we henceforth assume the equality holds. By adding k on both sides of equation (6.3.4), and using the fact that $k = a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = ...$, we get that

$$a_3 + b_3 = a_1 + b_2. \tag{6.3.5}$$

Let M_2 represent the set of vertices z such that $d(x, z) = d(y, z) = 2$ in $G/uv - v^*$. Suppose first that x and y are not separated, so that u is adjacent to x and y. In G, let $N_u(i, j)$ be the set of neighbours w of u such that $d(x, w) = i$ and $d(y, w) = j$, for any integers i and j. Now $N_u(1, 1) = \emptyset$, for otherwise the existence of a vertex $w \in N_u(1, 1)$ would imply that x and y have w and u as common neighbours, which contradicts $c_2 = 1$. Furthermore, $|N_u(1, 2)| = |N_u(2, 1)| = a_1$. Since $N_u(1, 2) \cup N_u(2, 1) \cup N_u(2, 2) \cup \{x, y\}$ is the set of all the neighbours of u, one concludes that $|N_u(2, 2)| = k - 2a_1 - 2$. Since $|N_u(2, 2)| + |M_2| = p^2_{2,2}$, we conclude that $|M_2| = p^2_{2,2} - (k - 2a_1 - 2)$ if x and y are not separated.

Suppose now that x and y are separated, so that, without loss of generality, $xu \in E(G)$ and $yv \in E(G)$. Let A be the set of common neighbours of v and u in G. If $w \in A$, then $wx \notin E(G)$, for otherwise u and w would be common neighbours of x and v, which contradicts the assumption that $c_2 = 1$. Similarly, $wy \notin E(G)$. Therefore, $N_u(2, 2) \cup N_v(2, 2) \cup M_2$ is the set of all vertices at distance 2 from both x and y, where $N_u(2, 2) \cap N_v(2, 2) = A$. This implies that $|N_u(2, 2) - A| + |N_v(2, 2) - A| + |A| + |M_2| = p^3_{2,2}$. Now $|N_u(2, 2) - A| = |N_v(2, 2) - A| = a_2 - a_1$, and $|A| = a_1$, so one concludes that $|M_2| = p^3_{2,2} - 2a_2 + a_1$. Therefore, one cannot decide if x and y are separated or not only if $p^2_{2,2} - (k - 2a_1 - 2) = p^3_{2,2} - 2a_2 + a_1$, which, when combined with equations (6.3.1), (6.3.2), (6.3.3) and (6.3.4), becomes

$$b_2 = c_3 b_3. \tag{6.3.6}$$

Let us suppose $c_3 \neq 1$. Let M_3 denote the neighbours z of y in G/uv, such that $d(x, z) = 3$ and such that there is an xz-path of length 3 disjoint from y and v^*. If x and y are not separated, then any of the b_2 neighbours of y at distance 3 from x in G must have a minimum path to x that is disjoint from y, u and v, since $c_3 \neq 1$. Conversely, a vertex in M_3 corresponds to a neighbour of y at distance 3 from x in G. Thus, $|M_3| = b_2$ if x and y are not separated. Suppose now that x and y are separated, so that, without loss of generality, x is adjacent to u, while y is adjacent to v. In G, we may assume that every common neighbour z of y and v must be such that $d(z, x) = 3$, for otherwise, from G/uv, one can conclude that z and x are separated while z and y are not, since $c_2 = 1$, so x and y are separated. Therefore, all of the $a_3 - a_1$ neighbours of y at distance 3 from x that are not adjacent to v are in M_3. Since $c_3 \neq 1$, any vertex in M_3 must be at distance 3 from x in G. We conclude that $|M_3| = a_3 - a_1$ if x and y are separated. Therefore, one cannot decide if x and y are separated or not only if $b_2 = a_3 - a_1$. Along with equation (6.3.5), this yields $b_3 = 0$. By equation (6.3.6), we get that $b_2 = 0$ as well, hence G is a strongly regular graph.

Therefore, G is reconstructible by Theorem 6.3.7. We may henceforth assume that $c_3 = 1$.

Let t be the smallest integer for which $c_t \neq 1$ or $a_t \neq a_1$. If $c_3 = 1$, then by equation 6.3.4, $a_2 = a_1$, so $t \geq 3$. We may then apply Lemmas 6.3.8 and 6.3.9 to conclude that every geodesic path of length 2 or 3 is contained in an induced cycle of length $2t$ if $c_t \neq 1$, or in an induced cycle of length $2t + 1$ if $c_t = 1$ and $a_t \neq a_1$. Furthermore, no geodesic path of length 2 or 3 is contained in any smaller induced cycles. Suppose $c_t \neq 1$. In G/uv, we can therefore conclude that x and y are separated if and only if xv^*y is contained in an induced cycle of length $2t - 1$. If $c_t = 1$ and $a_t \neq a_1$, then we conclude that x and y are separated if and only if xv^*y is contained in an induced cycle of length $2t$. ∎

In the case where $c_2 = 1$ and $a_2 = a_1$, we use Lemma 6.3.8 to conclude that there is a gap in the list of the sizes of the induced cycles of a distance regular graph. Lemma 6.3.9 shows that that induced cycles are, in some sense, uniformly spread across a distance regular graph. Those are the two observations that allows the reconstruction.

Note that once the parameters of a distance regular graph G have been reconstructed, any one card is sufficient to reconstruct G. This implies that if G and H are two distance regular graphs, with the same parameters, then G and H have no cards in common. Furthermore, from the proof of Proposition 6.3.10, we see that each card of G is used to determine $a_i(x, y)$ and $c_i(x, y)$ for some pair of vertices x and y. It follows that if two distance regular graphs have a card in common, then they must have the same parameters. Together with the previous observation, we conclude that no two distance regular graphs have a card in common.

Chapter 7

Automorphisms

We discuss concepts related to the automorphism group of a graph in relation with edge contraction reconstruction. The main point of this chapter is the concept of contraction pseudo-similarity. It is an analogue of the work found in [8]. Two edges are contraction pseudo-similar if their corresponding cards are isomorphic, but the edges are not similar. We characterize all graphs with pairs of pseudo-similar edges.

7.1 Edge-automorphisms

An isomorphism between two graphs is commonly defined as a bijection $\theta : V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $\theta(x)\theta(y) \in E(H)$. While this definition is perfectly fine for simple graphs, a precision needs to be made for graphs that are not simple. In general, an isomorphism from G to H is defined as a pair of bijections (α, β), $\alpha : V(G) \to V(H)$ and $\beta : E(G) \to E(H)$, such that if x and y are the ends of an edge e, then $\alpha(x)$ and $\alpha(y)$ are the ends of $\beta(e)$. See [4] for more details.

For simple graphs, the bijection β is entirely defined by α, and is thus superfluous. We establish the conditions where the opposite occurs, that is when α is entirely defined by β in the case of simple graphs. If we consider graphs with parallel edges, then edge isomorphisms are not entirely defined by their corresponding vertex isomorphism, because the vertex isomorphism does not describe how a set of parallel edges is mapped to another set of parallel edges, which is required to define an edge isomorphism.

An ***edge-automorphism*** of a graph G is a bijection θ on $E(G)$ such that if two edges a, b share i ends in G, then $\theta(a)$ and $\theta(b)$ also share i ends, for $i = 0, 1, 2$. So edge automorphisms of G correspond to automorphisms of the line graph of G. The set of edge-automorphisms of G forms a group under composition of functions, and we denote the group by $Aut_1(G)$. Clearly, any automorphism $\varphi \in Aut(G)$ induces a bijection $\hat{\varphi}$ on the edge set of G, defined by $\hat{\varphi}(xy) = \varphi(x)\varphi(y)$ for any edge $xy \in E(G)$. Now $\hat{\varphi}$ has the property that any two edges a, b are incident if and only if $\hat{\varphi}(a)$ and $\hat{\varphi}(b)$

are incident. Thus, $\hat{\varphi}$ is in $Aut_1(G)$, and is called an ***induced edge-automorphism*** of G. The set of induced edge-automorphisms of G forms a group under composition of functions, and we denote the group by $Aut^*(G)$. Hence, $Aut^*(G)$ is a subgroup of $Aut_1(G)$.

Naturally, one can define edge-isomorphism in a similar way. Let G and H be non-trivial graphs. A bijection $\theta : E(G) \to E(H)$ is an ***edge-isomorphism*** if the following property holds: if $a, b \in E(G)$ share i ends in G, then $\theta(a)$ and $\theta(b)$ share i ends in H, for $i = 0, 1, 2$. The graphs G and H are said to be ***edge-isomorphic*** if such an edge-isomorphism exists.

Two questions naturally arise. First, is $Aut^*(G)$ isomorphic to $Aut(G)$? It can happen that two distinct automorphisms of G induce the same edge-automorphism. For instance, in $G = K_2$, there are 2 automorphisms, yet $|Aut^*(G)| = 1$. Consider also a graph with multiple isolated vertices. Then, any automorphism of G that permutes the isolated vertices and fixes every other vertex corresponds to the trivial edge-automorphism. A theorem found in [19] shows that the two previous examples are the only cases where $Aut(G)$ is not isomorphic to $Aut^*(G)$. We state this theorem as a proposition.

Proposition 7.1.1 (Lauri and Scapellato [19])**.** *Let G be a non-trivial simple graph. Then $Aut(G) \cong Aut^*(G)$ if and only if G has at most one isolated vertex and has no connected component isomorphic to K_2.*

Proof: Consider the mapping ψ that maps φ to $\hat{\varphi}$, from $Aut(G)$ to $Aut^*(G)$. This mapping is a homomorphism, since $\hat{\varphi}\hat{\pi}(uv) = \varphi\pi(u)\varphi\pi(v) = \widehat{\varphi\pi}(uv)$.

Suppose G has at least two isolated vertices u and v, or K_2 is a connected component of G, with vertices u and v. The automorphism that permutes u and v, and fixes the rest, is mapped to the trivial edge-automorphism of $Aut^*(G)$ by the homomorphism. Therefore the kernel is not trivial, and since the mapping is surjective by definition, we get $Aut^*(G) \cong Aut(G)/\psi \not\cong Aut(G)$.

Now suppose G has at most one isolated vertex and K_2 is not a connected component of G. If $Aut(G)$ is trivial, then every edge of G lies in its own orbit, so every automorphism of $Aut^*(G)$ must fix every edge, and thus $Aut^*(G)$ is trivial as well. Suppose then that $Aut(G)$ is not trivial, and let $\varphi \in Aut(G)$ be an automorphism such that $\varphi(u) = v \neq u$. The degrees of u and v are the same, and greater than 0, since G does not have two isolated vertices. There exists an edge $e \neq uv$ incident to u, for otherwise u and v would form a K_2 component of G, which would constitute a contradiction. Now e is not incident to v, since G is simple, but $\hat{\varphi}(e)$ is incident to v. Therefore, $\hat{\varphi}$ is not trivial, hence φ is not in the kernel of the homomorphism. ∎

The next natural question is whether $Aut_1(G)$ contains automorphisms that are not induced by an automorphism of G. In other words, is $Aut^*(G)$ a strict subgroup of $Aut_1(G)$? Figure 7.1 illustrates examples of graphs that are edge-isomorphic, but not isomorphic, as well as graphs whose induced edge-automorphism group is a strict subgroup of its edge-automorphism group.

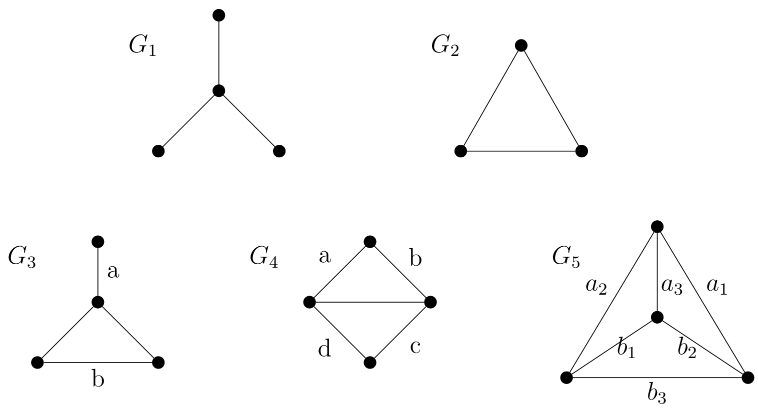

Figure 7.1: G_1 and G_2 are edge-isomorphic, but not isomorphic. G_3, G_4 and G_5 have edge-automorphisms that are not induced edge-automorphisms. For G_3, the edge-automorphism is the permutation (ab). For G_4, it is the permutation $(abcd)$, and for G_5, $(a_1b_1)(a_2b_2)(a_3b_3)$.

A theorem of Whitney [29] states that the graphs in Figure 7.1 is a complete list of connected graphs that give rise to edge-isomorphisms that are not induced by isomorphisms. We are, however, more interested in a corollary of this theorem of Whitney, found in [19].

Corollary 7.1.2 (Lauri and Scapellato [19])**.** *Let G be a non-trivial graph. Then $Aut_1(G) = Aut^*(G)$ if and only if both of these conditions hold:*

1. *not both G_1 and G_2 are components of G;*
2. *none of G_3, G_4 and G_5 are components of G.*

Moreover, $Aut_1(G) \cong Aut(G)$ if and only if both conditions hold and G has at most one isolated vertex and K_2 is not a component of G.

Therefore, in every connected graph but the five graphs shown in Figure 7.1, every edge-automorphism is determined by a unique automorphism, and vice versa, so one can consider the action of the edge-automorphism group on the graph instead of the widely used action of the automorphism group. Showing that a permutation θ of the vertices of a graph G is an automorphism can then be done by showing that the corresponding permutation on the edges of G maps incident edges to incident edges.

In particular, given an automorphism θ of G, we may choose to think of it as a permutation on vertices or on edges. In a slight abuse of notation, we will use the same symbol for both actions, writing $\theta(e)$ for the image of the edge e or $\theta(x)$ for the image of the vertex x. Of course, this assumes G has no connected component isomorphic to the ones found in Figure 7.1, and no connected component isomorphic to K_2, and at most one isolated vertex.

For any edge in $E(G)$, the edge ***orbit*** of e under a subgroup $\Gamma \subseteq Aut(G)$ is the set $\Gamma \cdot e = \{\theta(e) : \theta \in \Gamma\}$. The edge orbits are all disjoint, and they partition the set $E(G)$. An ***orbit transversal*** under Γ is a set of edges containing exactly one member from each orbit. Let θ be an automorphism of a graph G, and $\langle\theta\rangle$ the subgroup of $Aut(G)$ generated by θ. What form can the orbit of an edge under $\langle\theta\rangle$ take? Proposition 7.1.3 gives all the possible graphs.

Proposition 7.1.3. *For any edge $e \in E(G)$, and any automorphism $\theta \in Aut(G)$, the subgraph of G composed of all the edges in the orbit of e under $\langle\theta\rangle$ is either the disjoint union of isomorphic complete bipartite graphs, or the disjoint union of cycles of the same size. Furthermore, if e is incident to $\theta(e)$, the subgraph is either a star or a cycle.*

Proof: Denote by S the subgraph of G induced by the edges in the orbit of uv under the action of $\langle\theta\rangle$. Clearly S is an edge-transitive graph. Denote by m the number of edges in S and by n the number of vertices. Then, $\theta^m(uv) = uv$ and θ^m is the identity permutation on the edges of S. The proof is split according to whether S is vertex transitive or not.

Suppose that S is vertex transitive, thus regular of degree d. Since S is edge transitive and S has m edges, m is the least positive integer such that $\theta^m(uv) = uv$. Furthermore, $\theta^m(u) = u$ or $\theta^m(u) = v$. Suppose $\theta^m(u) = u$. Since S is vertex transitive, n is the least positive integer such that $\theta^n(u) = u$ and $\theta^n(v) = v$, hence n is the least positive integer such that $\theta^n(uv) = uv$. Therefore, $n = m$. Since $nd = 2m$, we conclude that $d = 2$, so S is the disjoint union of cycles, and because θ is an automorphism, the cycles must all have equal lengths. Suppose now that $\theta^m(u) = v$. Therefore, $\theta^{2m}(u) = u$ and $\theta^{2m}(v) = v$, hence $2m$ is a multiple of n. But $\theta^n(uv) = vu$, so n is a multiple of m. We conclude that either $n = 2m$, or $n = m$, a case already covered. If $n = 2m$, then $d = 1$, so S is the disjoint union of complete graphs K_2.

Suppose S is not vertex transitive. Then u and v are in different vertex orbits. This implies that S is bipartite, since every edge links a vertex in the orbit of u to a vertex in the orbit of v. Let $V_1 = \{x_0, x_1, x_2, ..., x_{\alpha-1}\}$ and $V_2 = \{y_0, y_1, y_2, ..., y_{\beta-1}\}$ be the two parts of the bipartite graph S. Then, an edge $x_i y_j$ can be seen as a pair of indices (i, j), with $0 \leq i < \alpha$ and $0 \leq j < \beta$. The group $\langle\theta\rangle$ acts cyclically on each part, mapping x_i and y_i to x_{i+1} and y_{i+1} respectively, and mapping $x_{\alpha-1}$ and $y_{\beta-1}$ to x_0 and y_0, respectively. Let $e = (0, 0) \in E(S)$. Then, for any $i \geq 0$, $\theta^i(e)$ is of the form (j, k), where

$$j \equiv i \mod \alpha$$

$$k \equiv i \mod \beta.$$

This implies $j \equiv k \mod \gcd(\alpha, \beta)$. In other words, the vertex $x_j \in V_1$ is adjacent to $y_k \in V_2$ if and only if $j \equiv k \mod \gcd(\alpha, \beta)$. Partition the vertices of V_1 into the parts $V_1(i) = \{x_j \in V_1 : j \equiv i \mod \gcd(\alpha, \beta)\}$, and partition the vertices of V_2, similarly, into the parts $V_2(i) = \{y_j \in V_2 : j \equiv i \mod \gcd(\alpha, \beta)\}$. The set of neighbours of any vertex in $V_1(i)$ is $V_2(i)$, and vice versa. Also, $|V_1(i)| = |V_1(j)|$ for any i and j. Similarly, $|V_2(i)| = |V_2(j)|$ for any i and j. Therefore, S is the disjoint union of isomorphic complete bipartite graphs.

Suppose that e is incident to $\theta(e)$. If S is disconnected, and e and $\theta(e)$ lie in a single connected component, then $\langle\theta\rangle$ only permutes edges of that component, and the orbit of e under $\langle\theta\rangle$ is connected. Therefore S is connected. Suppose that S is a connected bipartite graph. If θ does not permute the two parts of S, then $\langle\theta\rangle$ acts cyclically upon each part of S. Therefore, for e to be incident to $\theta(e)$, one of the parts must contain only one vertex, hence S is a star. If θ does permute the two parts, then S is vertex transitive, hence it is a cycle or the disjoint union of complete graphs K_2, the latter being impossible since it is disconnected. ∎

7.2 Pseudo-similarity

Recall that two edges a and b of a graph G are similar if an automorphism of G maps a to b. If two edges of a graph are similar, then their contractions are isomorphic. But is the converse true? In other words, is it possible for edges in different orbits to contract to isomorphic graphs? As we will see, the converse is not true, and one can construct examples of graphs with two edges in different orbits, yet their corresponding cards are isomorphic. We classify all types of what we call pseudo-similar edges.

Definition 7.2.1. *For a graph G, two edges a and b are **contraction pseudo-similar** if $G/a \cong G/b$, but a is not similar to b in G.*

The word "contraction" is used in the previous definition as to not confuse it with the definition of pseudo-similarity, which states that two edges are pseudo-similar if their deletion yield isomorphic graphs, yet no automorphism of G maps one edge to the other. Since this chapter is exclusively on edge contraction, we will henceforth omit the word "contraction".

Lemma 7.2.2. *Let G be a graph, $a, b \in E(G)$, and let $\varphi \in Aut(G)$. If a is pseudo-similar to b, then a is pseudo-similar to $\varphi(b)$.*

Proof: Since a is pseudo-similar to b, hence $G/a \cong G/b$, it follows that $\varphi(a)$ is pseudo-similar to $\varphi(b)$, hence $G/\varphi(a) \cong G/\varphi(b)$. Since $G/a \cong G/\varphi(a)$, we get that

$G/a \cong G/\varphi(b)$. Therefore, a is either pseudo-similar or similar to $\varphi(b)$. The edge a cannot be similar to $\varphi(b)$, since it would be similar to b as well. ∎

Lemma 7.2.2 states that if a is pseudo-similar to b, then every edge in the orbit of a is pseudo-similar to every edge in the orbit of b. While results in this section is written in terms of pseudo-similar edges, one should be aware that they can also be written in terms of pseudo-similar orbits.

Suppose a and b are pseudo-similar edges in a graph G, and denote $a = x_0y_0$ and $b = x_\ell y_\ell$. This choice of notation for the end vertices of a and b will become clear shortly. Let $\theta : G/b \to G/a$ be an isomorphism. Although this isomorphism may be seen as a bijection from $V(G/b)$ to $V(G/a)$, its action on $E(G/b)$ is entirely determined, with the exception of parallel edges. We henceforth allow an abuse of notation, and for any edge $uv \in E(G/b)$, we let $\theta(uv)$ denote an edge $\theta(u)\theta(v) \in E(G/a)$, by choosing an arbitrary but fixed bijection between the edges whose ends are $\{u, v\}$ and the edges whose ends are $\{\theta(u), \theta(v)\}$. Denote by v_b the contracted vertex of G/b and by v_a the contracted vertex of G/a. Therefore, $V(G/b) = V(G) \cup \{v_b\} - \{x_\ell, y_\ell\}$, and, similarly, $V(G/a) = V(G) \cup \{v_a\} - \{x_0, y_0\}$. The isomorphism θ may then be seen as a function on $V(G) \cup \{v_b\} - \{x_\ell, y_\ell\}$, and by the same abuse of notation, as a function on $E(G) - \{b\}$. It is therefore reasonable to consider, for any vertex $v \in V(G)$, the vertex $\theta^i(v)$, which can be defined recursively as the image of $\theta^{i-1}(v) \in V(G)$ by the function θ. Note that $\theta^i(v)$ may not exist: if $\theta^j(v) \notin V(G) \cup \{v_b\} - \{x_\ell, y_\ell\}$ for some $j < i$, then θ may not be applied to $\theta^j(v)$. Similarly, it is reasonable to consider, for any edge $e \in V(G)$, the edge $\theta^i(e)$, which exists provided $\theta^{i-1}(e) \neq b$.

This brings us to the notion of the pseudo-orbits $O(v)$ and $O(e)$, and of trajectories $\mathfrak{T}(v)$ and $\mathfrak{T}(e)$.

Definition 7.2.3. *Let $\theta : G/b \to G/a$ be an isomorphism. For any vertex $v \in V(G) - \{x_\ell, y_\ell\}$, if $\theta^i(v) = v$ for some $i > 0$, then the set $O(v) = \{\theta^j(v) : 0 \leq j < i\} \subseteq V(G) \cup \{v_b, v_a\}$ is the **pseudo-orbit of** v **under** θ. If i is the least positive integer for which $\theta^i(v)$ does not exist and k is the greatest negative integer for which $\theta^k(v)$ does not exist, then the set $\mathfrak{T}(v) = \{\theta^j(v) : k < j < i\} \subseteq V(G) \cup \{v_b, v_a\}$ is the **trajectory of** v **under** θ. A trajectory $\mathfrak{T}$ **starts at** v if $v \in \mathfrak{T}$ and $\theta^{-1}(v)$ does not exist. The trajectory $\mathfrak{T}$ **ends at** u if $u \in \mathfrak{T}$ and $\theta(u)$ does not exist.*

Definition 7.2.4. *Let $\theta : G/b \to G/a$ be an isomorphism. For any edge $e \in E(G) - \{b\}$, if $\theta^i(e) = e$ for some $i > 0$, then the set $O(e) = \{\theta^j(e) : 0 \leq j < i\} \subseteq E(G)$ is the **pseudo-orbit of** e **under** θ. If $\theta^i(e)$ does not exist for some $i > 0$, then the set $\mathfrak{T}(e) = \{\theta^j(e) : j < i\} \subseteq E(G)$ is the **trajectory of** e **under** θ. A trajectory $\mathfrak{T}$ **starts at** e if $e \in \mathfrak{T}$ and $\theta^{-1}(e)$ does not exist. The trajectory $\mathfrak{T}$ **ends at** f if $f \in T$ and $\theta(f)$ does not exist.*

Although the pseudo-orbit or trajectory of a vertex or edge depends on the isomorphism θ, we often omit the words "under θ" when the isomorphism is clear from the context. Note that the trajectory of a vertex v or an edge e includes $\theta^j(v)$ and $\theta^j(e)$, respectively, for negative values of j. Thus two trajectories are either equal or disjoint. Similarly, two pseudo-orbits are either equal or disjoint. Since G is finite, every vertex and every edge of G is in an pseudo-orbit or in a trajectory, hence together, the set of pseudo-orbits and trajectories induce a partition of the sets $V(G) \cup \{v_b, v_a\}$ and $E(G)$.

By definition, a trajectory must start at a vertex in $V(G/b) - V(G/a)$, which is equal to $\{v_b, x_0, y_0\}$, and must end at a vertex in $V(G/a) - V(G/b)$, which is equal to $\{v_a, x_\ell, y_\ell\}$. Similarly, a trajectory must start at an edge of $E(G/b) - E(G/a) = \{a\}$, and end at an edge of $E(G/a) - E(G/b) = \{b\}$. There are then exactly three distinct vertex trajectories and one edge trajectory. Let ℓ be the integer such that $\theta^\ell(a) = b$, and let $e_i = \theta^i(a)$ for $i = 0, 1, ..., \ell$, so that $e_0 = a$ and $e_\ell = b$. Let $e_i = x_i y_i$ so that $\theta(x_i) = x_{i+1}$ and $\theta(y_i) = y_{i+1}$. Since e_i and e_j are allowed to be incident, some vertices may have multiple labels.

7.2.1 Pseudo-similar edges in full paths

Graphs with pseudo-similar edges exist. First, recall that a ***full path*** is a path between two vertices of degrees not equal to two, whose internal vertices are of degree two. The ***length*** of a full path P, denoted $|P|$, is the number of edges in P.

Every edge of a graph is part of a full path. Any edge whose ends have degrees other than two are in a full path of length one. Equivalently, an edge is in a full path of length one or more if and only if one of its ends has a degree of at most two. If two edges a and b of a graph G are in the same full path, then $G/b \cong G/a$, but a may not be similar to b in G.

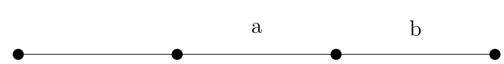

Figure 7.2: The edges a and b are pseudo-similar.

Proposition 7.2.5 settles the first case, where no power of θ maps v_b onto v_a.

Proposition 7.2.5. *Suppose there exists an isomorphism θ from G/b to G/a such that $\theta^i(v_b) \neq v_a$ for any integer i. Let $e_i = \theta^i(a)$. Then a and b are each in a full path of length at least one. Furthermore, there is a collection of full paths $\{P_0, P_1, ..., P_r\}$ of G and an integer $0 < p \leq r$ such that $a \in P_0$ and $b \in P_p$, with the following properties:*

1. *$\mathfrak{T}(a) \subseteq \bigcup_i P_i$ and $\mathfrak{T}(a) \cap P_i \neq \emptyset$ for $i = 0, 1, ..., r$;*

2. *Let $e \neq b$. If $e \in P_i$, then $\theta(e) \in P_{i+1}$. If $e \in P_r$, then $\theta(e) \in P_0$;*

3. *$|P_i| = |P_0|$ for $i \leq p$ and $|P_i| = |P_0| - 1$ for $i > p$;*

4. *For any i and j, P_i is similar to P_j in $G/\{a, e_1, e_2, ..., e_p\}$.*

Proof: Let $a = x_0y_0$ and $b = x_\ell y_\ell$. One needs to show that $\deg(x_\ell)$ or $\deg(y_\ell)$ is equal to 2, and that $\deg(x_0)$ or $\deg(y_0)$ is equal to 2. Note that $V(G/b) - V(G/a) = \{x_0, y_0, v_b\}$ and $V(G/a) - V(G/b) = \{x_\ell, y_\ell, v_a\}$, so v_b must eventually be mapped to an element of $V(G/a) - V(G/b)$ by a power of θ. This implies that, in G/a, $\theta^i(v_b) = x_0$ or y_0 for some $i \geq 1$. But then $\deg(x_0) = \deg(v_b) = \deg(x_0) + \deg(y_0) - 2$, or $\deg(y_0) = \deg(v_b) = \deg(x_0) + \deg(y_0) - 2$. This implies $\deg(y_0) = 2$ or $\deg(x_0) = 2$. When considering $\theta^{-1} : G/a \to G/b$, a similar argument shows that $\deg(x_\ell) = 2$ or $\deg(y_\ell) = 2$.

We define $\theta(P_i) = P_{i+1}$ cyclically, for any $i \neq p$. If $i = p$, then $\theta(P_p - \{b\}) = P_{p+1}$. One can then choose the full paths $P_0, P_1, ..., P_r$ with properties 1 and 2. From these observations, one concludes that property 3 holds.

By property 1, in $G/\{a, e_1, e_2, ..., e_p\}$, every full path in the collection have the same length. One can consider θ as a permutation of $V(G - \bigcup_i P_i)$. In this graph, the ends of P_i are mapped by θ to the ends of P_{i+1} cyclically. For $i = 0, 1, ..., r$, add a full path of length $|P_0| - 1$ linking the ends of P_i. The resulting graph is isomorphic to $G/\{a, e_1, e_2, ..., e_p\}$ and θ maps P_i to P_{i+1} cyclically, hence 4 is proved. ∎

7.2.2 If $\theta(v_b) \neq v_a$ and $\theta^k(v_b) = v_a$ for some $k \geq 2$.

Another type of pseudo-similar edges exist, and is shown in Figure 7.3. In this graph G, the automorphism group is trivial, yet G/b is isomorphic to G/a. The important thing to notice is that the isomorphism does not map v_b to v_a. This observation is central in constructing graphs with pseudo-similar edges. We constructed the graph in Figure 7.3 by adapting the method used in [8] to construct pairs of vertices that are pseudo-similar under vertex deletion. Take a graph H, and let $\theta \in Aut(H)$. Let a and b be edges of H such that $\theta^k(b) = a$ for some $k > 1$. Consider the set $C = \{b, \theta(b), \theta^2(b), ..., \theta^{k-1}(b)\}$, hence $\theta(C) = \{\theta(b), \theta^2(b), ..., \theta^{k-1}(b), a\}$. It is clear that $H/C \cong H/\theta(C)$. Let $G = H/(C - \{b\})$. We see that $G/b \cong H/\theta(C) \cong H/C \cong G/a$.

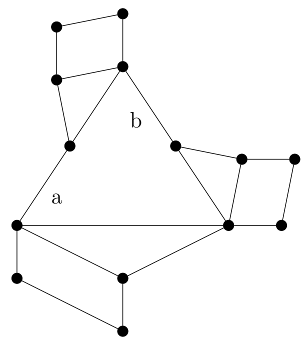

Figure 7.3: The edges a and b are pseudo-similar.

In no way does this construction prevent a and b from being similar in G, rather than pseudo-similar. A necessary condition for pseudo-similarity is the absence of an automorphism of H which maps the set $C - \{b\}$ to itself while also permuting a and b.

This case is the most similar to pseudo-similarity found in [8], in that the solution is to decontract some vertices to create a new graph H. However, for edge pseudo-similarity, the graph H has the property that a and b are similar, while in our case, such a precise conclusion cannot be drawn.

Proposition 7.2.6. *If there exists an isomorphism $\theta : G/b \to G/a$ such that $\theta(v_b) \neq v_a$ and $\theta^k(v_b) = v_a$ for some $k > 1$, then there exists a graph H, with $E(H) = E(G) \cup \{e_{\ell+1}, ..., e_{\ell+k-1}\}$ and $G = H/\{e_{\ell+1}, ..., e_{\ell+k-1}\}$. Also, there exists an isomorphism $\hat{\theta} : H/e_{\ell+k-1} \to H/a$ such that $\mathfrak{T}(a) = \{a, e_1, ..., e_{\ell+k-1}\}$, where $\hat{\theta}(e_i) = e_{i+1}$ for $0 \leq i \leq \ell + k$ and $\hat{\theta}(v_{e_{\ell+k-1}}) = v_a$.*

Proof: For any vertex $v \in V(G)$, let $I(v)$ be the set of edges incident to v. To construct H, decontract $\theta^i(v_b)$ in G to create the edge $e_{i+\ell}$ whose ends are $x_{i+\ell}$ and $y_{i+\ell}$, by partitioning the incident edges of $\theta^i(v_b)$ using the partition $I(x_{i+\ell}) = \theta^i(I(x_\ell))$ and $I(y_{i+\ell}) = \theta^i(I(y_\ell))$, for $1 \leq i \leq k-1$. Then $G = H/\{e_{\ell+1}, e_{\ell+2}, ..., e_{\ell+k-1}\}$. Let $\hat{\theta} : V(H/e_{\ell+k-1}) \to V(H/a)$ be the following bijection:

$\hat{\theta}(v) = \theta(v)$ for any $v \in V(G) - \{x_\ell, y_\ell\}$.

$\hat{\theta}(x_{\ell+i}) = x_{\ell+i+1}$ and $\hat{\theta}(y_{\ell+i}) = y_{\ell+i+1}$ for $0 \leq i < k-1$.

$\hat{\theta}(v_{e_{\ell+k-1}}) = v_a$.

We show that $\hat{\theta}$ is an isomorphism. Let $uv \in E(H/e_{\ell+k-1})$. If $u = x_{\ell+i}$ for some $i \geq 0$, then $uv \in I(x_{\ell+i})$ and $\hat{\theta}(u)\hat{\theta}(v) \in I(x_{\ell+i+1})$, hence $\hat{\theta}(u)\hat{\theta}(v)$ is an edge. A similar argument settles the cases where $u = y_{\ell+i}$, for any $i \geq 0$. By symmetry, it also settles the cases where $v = x_{\ell+i}$ and $v = u_{\ell+i}$ for any $i \geq 0$. The remaining case to cover is when u and v are in $V(G) - \{x_\ell, y_\ell\}$. In that case, $\hat{\theta}(u)\hat{\theta}(v) = \theta(u)\theta(v)$ is an edge.

From the definition of $\hat{\theta}$, one concludes that $\mathfrak{T}(a) = \{a, e_1, ..., e_{\ell+k-1}\}$, where $\hat{\theta}(e_i) = e_{i+1}$ for $0 \leq i \leq \ell + k$ and $\hat{\theta}(v_{e_{\ell+k-1}}) = v_a$. ∎

7.2.3 If $\theta(v_b) = v_a$.

Figure 7.4 shows yet a third type of graphs with pseudo-similar edges. The construction of such graphs is explained by Lemma 7.2.7.

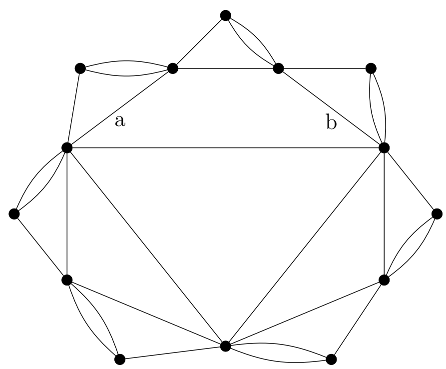

Figure 7.4: The edges a and b are pseudo-similar.

In this last case, $\theta(v_b) = v_a$. It was mentioned that the isomorphism θ may be seen as acting on the vertices $V(G) - \{x_\ell, y_\ell\}$. In the case where $\theta(v_b) = v_a$, we extend the action of θ to the whole graph G. Using the definition of isomorphism found in [4], the isomorphism θ is seen as a pair of bijection $\alpha : V(G/b) \rightarrow V(G/a)$ and $\beta : E(G/b) \rightarrow E(G/a)$ such that if the ends of an edge $e \in E(G/b)$ are x and y, then the ends of $\beta(e)$ are $\alpha(x)$ and $\alpha(y)$. Given an isomorphism $\theta : G/b \rightarrow G/a$, we denote by α and β the action of θ on vertices and edges respectively. We write $\theta = (\alpha, \beta)$ for brevity. We define $\hat{\theta}_1 = (\hat{\alpha}_1, \hat{\beta})$ or $\hat{\theta}_2 = (\hat{\alpha}_2, \hat{\beta})$, where $\hat{\alpha}_i : V(G) \rightarrow V(G)$ and $\hat{\beta} : E(G) \rightarrow E(G)$ are defined as follows:

$\hat{\alpha}_i(v) = \alpha(v)$ if $v \neq x_l$ and $v \neq y_l$;

$\hat{\alpha}_1(x_l) = x_0$ and $\hat{\alpha}_1(y_l) = y_0$;

$\hat{\alpha}_2(x_l) = y_0$ and $\hat{\alpha}_2(y_l) = x_0$;

$\hat{\beta}(e) = \beta(e)$ if $e \neq b$;

$\hat{\beta}(b) = a$.

Each $\hat{\alpha}_i$ is a permutation of $V(G)$ and $\hat{\beta}$ is a permutation of $E(G)$. However, $(\hat{\alpha}_1, \hat{\beta})$ and $(\hat{\alpha}_2, \hat{\beta})$ are not necessarily automorphisms of G. Lemma 7.2.7 states the lesser condition satisfied by those bijections.

Lemma 7.2.7. *Let $\theta = (\alpha, \beta)$ be an isomorphism from G/b to G/a such that $\alpha(v_b) = v_a$. Let $u, v \in V(G)$. If $|\{u,v\} \cap \{x_\ell, y_\ell\}| = 0$ or 2, then $\hat{\beta}(uv) = \hat{\alpha}_i(u)\hat{\alpha}_i(v)$ for all $i \in \{1,2\}$. If $|\{u,v\} \cap \{x_\ell, y_\ell\}| = 1$, then $\hat{\beta}(uv) = \hat{\alpha}_i(u)\hat{\alpha}_i(v)$ for some choice of $i \in \{1,2\}$.*

Proof: If $|\{u,v\} \cap \{x_\ell, y_\ell\}| = 2$, then $uv = b = x_\ell y_\ell$. By definition, $\hat{\beta}(x_\ell y_\ell) = a = \hat{\alpha}_i(x_\ell)\hat{\alpha}_i(y_\ell)$ for $i = 1$ and $i = 2$. If $|\{u,v\} \cap \{x_\ell, y_\ell\}| = 0$, then since $b \neq uv \in E(G)$, then $uv \in E(G/b)$. Then since (α, β) is an isomorphism from G/b to G/a, we get that $\beta(uv) = \alpha(u)\alpha(v)$. Finally, since none of $\alpha(u)$ and $\alpha(v)$ is equal to v_a, we conclude that $\hat{\beta}(uv) = \hat{\alpha}_i(u)\hat{\alpha}_i(v)$ for $i = 1$ and $i = 2$.

Suppose now that $|\{u,v\} \cap \{x_\ell, y_\ell\}| = 1$, and, without loss of generaliy, assume that $v = x_\ell$. Now, $uv \in E(G)$ becomes uv_b in G/b. This edge is mapped to $\alpha(u)v_a \in E(G/a)$, which then becomes $\alpha(u)x_0$ or $\alpha(u)y_0$ in G. This implies $\hat{\beta}(uv) = \alpha(u)\alpha_1(v)$ or $\alpha(u)\alpha_2(v)$, which is equal to $\hat{\alpha}(u)_i\hat{\alpha}_1(v)$ or $\hat{\alpha}(u)_i\hat{\alpha}_2(v)$ for any $i \in \{1,2\}$. ∎

Lemma 7.2.7 shows that while $(\hat{\alpha}_1, \hat{\beta})$ and $(\hat{\alpha}_2, \hat{\beta})$ are not necessarily automorphisms of G, they almost are, in the sense that only for edges uv incident to b can $\hat{\beta}(uv)$ not be equal to $\hat{\alpha}_i(u)\hat{\alpha}_i(v)$.

If such functions $(\hat{\alpha}_1, \hat{\beta})$ and $(\hat{\alpha}_2, \hat{\beta})$ exist on G as described above Lemma 7.2.7, then G/b and G/a are isomorphic, and v_b is mapped to v_a. Indeed, since $\hat{\beta}$ maps b to a, one can restrict $(\hat{\alpha}_1, \hat{\beta})$ to a function from $G/b \to G/a$. Since $\hat{\alpha}_1(\{x_\ell, y_\ell\}) = \{x_0, y_0\}$, its restriction maps v_b to v_a. We conclude from Lemma 7.2.7 that $\hat{\beta}(uv) = \hat{\alpha}_1(u)\hat{\alpha}_1(v)$ for every edge $uv \in G/b$. An identical conclusion can be drawn by replacing $(\hat{\alpha}_1, \hat{\beta})$ with $(\hat{\alpha}_2, \hat{\beta})$.

Lemma 7.2.7 is the key to creating graphs with pseudo-similar edges, such as the graph in Figure 7.4. In this example, some of the edges incident to b are mapped to edges by $(\hat{\alpha_1}, \hat{\beta})$, while the other edges incident to b are mapped to edges by $(\hat{\alpha}_2, \hat{\beta})$. One can apply the result of Lemma 7.2.7 to construct graphs G with edges a and b, with $G/b \cong G/a$ and v_b is mapped to v_a by an isomorphism.

We now exhibit a procedure to create any graph G with pseudo-similar edges a and b and an isomorphism $\theta : G/b \to G/a$ such that $\theta(v_b) = v_a$. Start with two permutations $\hat{\alpha}_1$ and $\hat{\alpha}_2$ of the vertices of an empty graph $E_{v(G)}$. Let $a = x_0y_0$ and let $b = x_\ell y_\ell$. Choose $\hat{\alpha}_1$ and $\hat{\alpha}_2$ such that $\hat{\alpha}_1(x_\ell) = x_0$, $\hat{\alpha}_1(y_\ell) = y_0$, $\hat{\alpha}_2(x_\ell) = y_0$, $\hat{\alpha}_2(y_\ell) = x_0$, and $\hat{\alpha}_1(v) = \hat{\alpha}_2(v)$ for any $v \notin \{x_\ell, y_\ell\}$. For any pair of vertices u and v in $E_{v(G)}$, define $\Omega(u,v) = \{uv, \hat{\alpha}_{i_1}(u)\hat{\alpha}_{i_1}(v), \hat{\alpha}_{i_2}(u)\hat{\alpha}_{i_2}(v), ..., \hat{\alpha}_{i_k}(u)\hat{\alpha}_{i_k}(v) = uv\}$, where each i_j is chosen, independently of the others, to be 1 or 2. Depending on the choice for i_j, many $\Omega(u,v)$ may exist. To create G, add the edges $\Omega(x_0,y_0)$, and for any number of pairs of vertices u and v, add $\Omega(u,v)$ to $E_{v(G)}$. For each $\Omega(u,v)$, the i_j can be chosen differently.

The functions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ have the properties described above Lemma 7.2.7, and, as mentioned before, their existence imply the existence of an isomorphism $(\alpha,\beta) : G/b \to G/a$ with $\alpha(v_b) = v_a$.

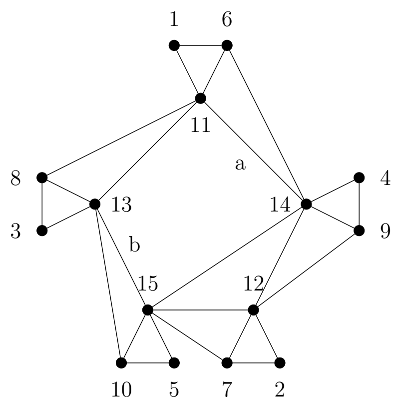

Figure 7.5: The edges a and b are pseudo-similar. The permutations on the vertex set are $\hat{\alpha}_1 = (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)$ and $\hat{\alpha}_2 = (1,2,3,4,5)(6,7,8,9,10)(11,12,13)(14,15)$.

Everything that has been established since we assumed that $\theta(v_b) = v_a$ does not rule out the possibility that a and b, as well as all the edges in $\mathfrak{T}(a)$, are similar. While we do not attempt to state necessary conditions for similarity, the following proposition states equivalent conditions for the edges of $\mathfrak{T}(a)$ to be pairwise similar in G via θ.

Proposition 7.2.8. *For any vertex $v \in V(G)$, let $I(v)$ be the set edges incident to v. Then, $\theta : G/b \to G/a$ is an isomorphism such that $\theta(v_b) = v_a$ and $\theta(I(x_\ell) - \{b\}) = I(x_0) - \{a\}$ if and only if the edges of $\mathfrak{T}(a)$ are pairwise similar in G via θ.*

Proof: Since $\theta(I(x_\ell) - \{b\}) = I(x_0) - \{a\}$, we have $\theta(I(y_\ell) - \{b\}) = I(y_0) - \{a\}$. Let $\theta = (\alpha, \beta)$ and let $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\beta}$ be defined as before. It would suffice to show that $(\hat{\alpha}_1, \hat{\beta})$ is an automorphism of G. By Lemma 7.2.7, it suffices to prove that $\hat{\beta}(x_\ell v) = \hat{\alpha}_1(x_\ell)\hat{\alpha}_1(v)$ and $\hat{\beta}(y_\ell v) = \hat{\alpha}_1(y_\ell)\hat{\alpha}_1(v)$ for every edge $x_\ell v$ and for every edge $y_\ell v$. Let $x_\ell v \in E(G)$, so $x_\ell v \in I(x_\ell)$. By hypothesis, $\hat{\beta}(x_\ell v) \in I(x_0)$, and so $\hat{\alpha}_1(x_\ell)\hat{\alpha}_1(v) = x_0\hat{\alpha}_1(v)$ is equal to $\hat{\beta}(x_\ell v)$. By symmetry, this argument settles the case of edges of the form $y_\ell v$.

Conversely, if the edges of $\mathfrak{T}(a)$ are pairwise similar in G via θ, then it follows that $\theta(b) = a$ in G and that $\theta(I(x_\ell) - \{b\}) = I(x_0) - \{a\}$. Since $\theta(b) = a$ in G, we get that $\theta(v_b) = v_a$. ∎

To summarize, edges a and b are pseudo-similar in G for one or more of the three following reasons: either a and b are in full paths of length at least one (Proposition 7.2.5), or some vertices of G need to be decontracted (Proposition 7.2.6), or some edges incident to the ends of b are not mapped correctly (Lemma 7.2.7).

7.2.4 Complementary edges

We now re-word the Conjecture 1.1.2 in a way that highlights possible connections with the concept of pseudo-similarity. This section is an adaptation of Lauri's work [18] on edge deletion pseudo-similarity. Suppose $G \not\cong H$ are contraction reconstructions of one another, thus $\mathcal{C}(G) \cong \mathcal{C}(H)$. For every $xy \in E(G)$, there is an edge $x'y' \in E(H)$ such that $G/xy \cong H/x'y'$. Suppose an isomorphism $\theta : G/xy \to H/x'y'$ is such that $\theta(v_{xy}) \neq v_{x'y'}$. Then obtain a graph K by decontracting $\theta(v_{xy})$ in H, which creates two new vertices x'' and y'', where the neighbours of x'' are $\theta(I(x))$ and the neighbours of y'' are $\theta(I(y))$. Therefore, $K/x''y'' \cong H$ and $K/x'y' \cong G$. Furthermore, since $\mathcal{C}(G) \cong \mathcal{C}(H)$, for every edge $e \in E(K) - \{x'y', x''y''\}$, there exists an edge $e' \in E(K) - \{x'y', x''y''\}$ such that $K/\{x'y', e\} \cong K/\{x''y'', e'\}$.

Definition 7.2.9. *Let K be a graph and let a, b be distinct edges of $E(K)$. We say a and b are **complementary edges** if for every edge $e \in E(K) - \{a, b\}$, there exists an edge $e' \in E(K) - \{a, b\}$ such that $K/\{a, e\} \cong K/\{b, e'\}$.*

Similar edges and pseudo-similar edges are complementary. Indeed, let θ be an isomorphism from K/a to K/b. For every edge $e \in E(K) - \{a, b\}$, the corresponding edge e' is equal to $\theta(e)$. Using this definition, one can state a weaker version of Conjecture 1.1.2 in the following way:

Conjecture 7.2.10. *If a and b are complementary edges in a simple graph K with at least five edges, then a and b are similar or pseudo-similar.*

For every edge $e \in E(K) - \{a, b\}$, there exists an edge $e' \in E(K) - \{a, b\}$ such that $(K/a)/e \cong (K/b)/e'$. Since $(K/a)/b \cong (K/b)/a$, we conclude that $\mathcal{C}(K/a) \cong \mathcal{C}(K/b)$. If Conjecture 7.2.10 is true, then a and b are pseudo-similar or similar in K, which implies that $K/a \cong K/b$. Therefore, if two graphs G and H have isomorphic decks, and a graph K as described above exists, then $G \cong H$ if Conjecture 7.2.10 holds.

The existence of the graph K relies on the existence of edges $xy \in E(G)$ and $x'y' \in E(H)$ such that $G/xy \cong H/x'y'$, and the existence of an isomorphism $\theta : G/xy \to H/x'y'$ such that $\theta(v_{xy}) \neq v_{x'y'}$. Such a pair of edges and such an isomorphism were not shown to always exist, hence Conjecture 7.2.10 is weaker. We define pairs of graphs G and H, whose graph K does not exist.

Definition 7.2.11. *Let G and H be reconstructions of one another. H is a **complementary reconstruction** of G if for every pair of cards $G/a \cong H/b$, the contracted vertex of G/a is mapped to the contracted vertex of H/b by every isomorphism.*

Graphs G and H in the third row of Figure 7.6 are complementary reconstructions, because the cards of G and H contain only one vertex. It is not clear whether simple complementary reconstructions exist or not. Of all the graphs in Figure 1.2, no pair of reconstructions are complementary. If simple complementary reconstructions do not exist, then Conjecture 7.2.10 is equivalent to Conjecture 1.1.2.

Conjecture 7.2.10 is restricted to graphs with at least five edges because Conjecture 1.1.2 is restricted to graphs with at least four edges. Every example in Figure 1.2 generates a graph K with complementary edges a and b, but K/a is not isomorphic to K/b. They are shown in Figure 7.6.

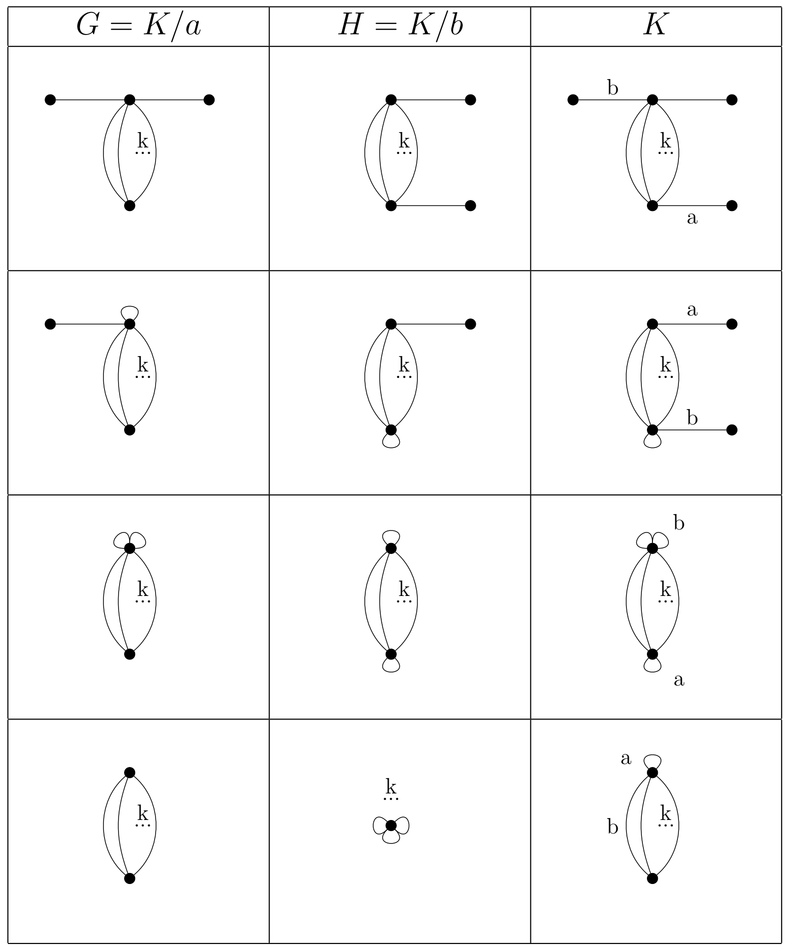

Figure 7.6: In each row, the graphs G and H are reconstructions of one another. In K, the edges a and b are complementary, yet $G = K/a$ is not isomorphic to $H = K/b$.

7.2.5 On the number of pseudo-similar edges

We now investigate the following question: can every edge have a pseudo-similar mate? By considering a path graph, it is easy to see that the answer is positive. If we exclude that possibility, and consider only simple graphs with a minimal degree of three, the question remains unanswered. However, there are simple graphs where "almost" every edge has a pseudo-similar mate, and a minimal degree of at least three. Denote by $ps(G)$ the number of edges with a pseudo-similar mate in G.

Proposition 7.2.12. *There exists an infinite sequence of simple graphs $G_1, G_2, \ldots$, where $\delta(G_n) \geq 3$ for all n and where $\frac{ps(G_n)}{e(G_n)} \to 1$ as $n \to \infty$.*

Proof: Start with the complete bipartite graph $K_{n,n}$, where the bipartitions are

$\{x_0, x_1, ..., x_{n-1}\}$ and $\{y_0, y_1 ..., y_{n-1}\}$. Link the vertices $x_0, x_1, ..., x_{n-1}$, in that order, with an oriented cycle, and do the same with $y_0, y_1, ..., y_{n-1}$. Then, replace each of the oriented edges with the graph found in Figure 7.7.

Figure 7.7: An oriented edge and its replacement.

Finally, contract x_0y_0 to obtain G_n. In G_n, the edge x_iy_j is pseudo-similar to the edge $x_{n-j}y_{n-i}$, so $ps(G_n) \geq n^2 - 1$. The total number of edges in G_n is $n^2 - 1 + 12n$, hence

$$\frac{ps(G_n)}{e(G_n)} \geq \frac{n^2 - 1}{n^2 - 1 + 12n}$$

which tends to 1 as n goes to infinity. ∎

7.3 Automorphism groups of the cards

What follows highlights the difficulty in finding a relation between the automorphism group of a graph and those of its cards. Such relations could, in principle, be useful in solving Conjecture 1.1.2. We show that for any finite set of finite groups $\Gamma_1, \Gamma_2, ..., \Gamma_k$, there exists a graph G whose automorphism group is Γ_1, and edges $e_2, e_3, ..., e_k$ such that $Aut(G/e_i) \cong \Gamma_i$. This is based on the work done in [11], which answered similar questions, but in the context of vertex reconstruction.

Definition 7.3.1. *Let $\Gamma_1, ..., \Gamma_k$ and Γ be finite groups. We say $\Gamma \to \{\Gamma_1, ..., \Gamma_k\}$ if there exists a graph G with a subset of its edges $e_1, ..., e_k$ such that $Aut(G) \cong \Gamma$ and $Aut(G/e_i) \cong \Gamma_i$ for $i = 1, ..., k$.*

Let I denote the trivial group. We first show that $I \to \{I\}$.

Lemma 7.3.2. $I \to \{I\}$.

Proof: It suffices to find a graph G with no non-trivial automorphisms, with an edge $e \in E(G)$, such that $Aut(G/e)$ is also the trivial group. Such a graph is shown in Figure 7.8. ∎

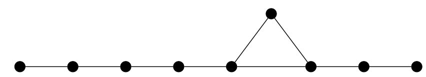

Figure 7.8: The automorphism group of this graph is trivial. Any contraction yields a graph with a trivial automorphism group.

We should mention that the graph in Figure 7.8 is not unique: there are infinitely many graphs G with $Aut(G) \cong I$ and $Aut(G/e) \cong I$ for some $e \in E(G)$. For example, if one subdivides the two leaves of the graph in Figure 7.8, one obtains a graph with similar properties.

Another way of constructing graphs with $I \to \{I\}$ is to take any graph H with no non-trivial automorphism and a minimum degree of at least 2, then to add a vertex incident to only one vertex of H. The resulting graph has no non-trivial automorphisms, and one of its cards is isomorphic to H, which also has no non-trivial automorphisms.

For the following proofs, one needs to define Cayley graphs for a group Γ, and construct, from these Cayley graphs, simple graphs with the same automorphism group Γ.

Definition 7.3.3. *Let $\Gamma = \{g_1, ..., g_n\}$ be a group of order n. The graph $C(\Gamma)$ is a complete directed graph on the vertex set Γ, where the edge $g_i g_j$ is coloured with colour $g_i^{-1} g_j$.*

Using this definition, $C(\Gamma)$ has antiparallel edges between every pair of vertices. Consider the automorphisms of $C(\Gamma)$ that preserve the colours of the edges, i.e automorphisms that map each edge of colour k to an edge of colour k, for every k. Under that automorphism group, $C(\Gamma)$ is vertex-transitive: every vertex has one in-edge and one out-edge of each colour. Furthermore, that automorphism group is Γ.

Lemma 7.3.4. *The automorphism group of $C(\Gamma)$ is isomorphic to Γ.*

Proof: Let $\varphi \in Aut(C(\Gamma))$. Denote by 1_Γ the identity element of the group Γ, which is also a vertex of $C(\Gamma)$. Suppose $\varphi(1_\Gamma) = g_i$. By definition, the vertex 1_Γ has an out-edge of colour g heading to the vertex g, for every $g \in \Gamma$. Since φ is an automorphism, it is colour-preserving, hence the edge $1_\Gamma g$, coloured by g, must be mapped to an out-edge of $\varphi(1_\Gamma)$ of colour g, hence $\varphi(g) = g_i g$ for every $g \in \Gamma$. Therefore, the automorphism φ is entirely determined by its action on the vertex 1_Γ. We henceforth denote a automorphism of $C(\Gamma)$ by φ_i, where $\varphi_i(1_\Gamma) = g_i$.

Let $\pi : Aut(C(\Gamma)) \to \Gamma$ be defined by $\pi(\varphi) = \varphi(1_\Gamma)$. We show that π is an isomorphism. This function is injective, since if $\varphi_i(1_\Gamma) = \varphi_j(1_\Gamma)$, then $g_i = g_j$. Because φ_i and φ_j are entirely determined by their action on 1_Γ, we conclude that

$\varphi_i = \varphi_j$. The function π is also surjective, since an automorphism φ_i exists for every $i = 1, 2, ..., n$. Finally, π is a homomorphism, since

$$\pi(\varphi_i \varphi_j) = \varphi_i \varphi_j(1_\Gamma) = \varphi_i(g_j) = g_i g_j = \varphi_i(1_\Gamma)\varphi_j(1_\Gamma) = \pi(\varphi_i)\pi(\varphi_j).$$

We conclude that π is an isomorphism. ∎

If $|\Gamma| > 3$, from $C(\Gamma)$, one can create a simple, undirected graph, which we name $Cay(\Gamma)$. First, take a 3-star graph, where w is the central vertex, and x_1, x_2 and x_3 are the other tree vertices, and let G_k be the graph obtained from a 3-star where wx_2 has been subdivided once, and wx_3 has been subdivided k times. To obtain $Cay(\Gamma)$, replace each directed edge uv coloured by g_k with G_k, by attaching u to x_1 in G_k, and v to x_2.

Lemma 7.3.5. *If $|\Gamma| > 3$, then $Aut(Cay(\Gamma)) \cong \Gamma$.*

Proof: Since $|\Gamma| = n > 3$, the only vertices of $Cay(\Gamma)$ with a degree greater than 3 are the vertices $g_1, g_2, ..., g_n$, hence any automorphism of $Cay(\Gamma)$ must permute those vertices. One can conclude that every automorphism of $Cay(\Gamma)$ is determined by its action on 1_Γ. The rest of the proof is identical to the one of Lemma 7.3.4. ∎

If $|\Gamma| \leq 3$, then we define $Cay(\Gamma)$ as shown in Figure 7.9. Note that there are many ways to define $Cay(\Gamma)$: the necessary conditions are that the automorphism group is Γ, and that the vertices corresponding to the group elements must have a degree of at least 2.

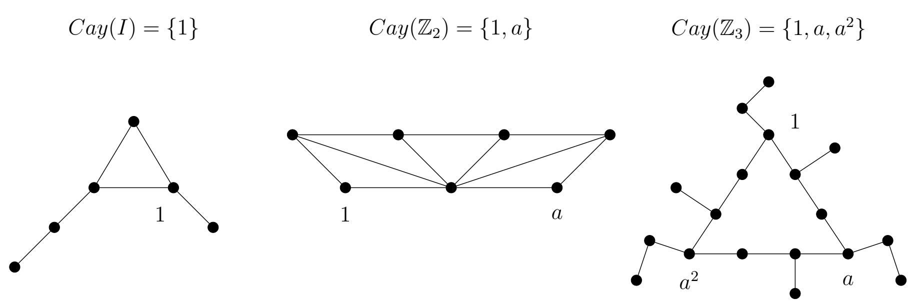

Figure 7.9: The graphs $Cay(I)$, $Cay(S_2)$ and $Cay(\mathbb{Z}_3)$.

Note that $Cay(\Gamma)$ is not necessarily the smallest graph whose automorphism group is Γ. For example, a graph on two vertices, K_2, has the same automorphism group as $Cay(S_2)$, which has seven vertices. For our purposes, the graphs $Cay(\Gamma)$ will suffice.

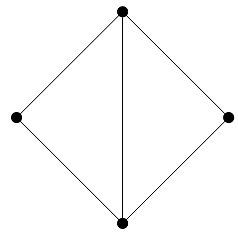

Figure 7.10: The graph $K_4 - e$: the smallest connected graph whose automorphism group is $S_2 \times S_2$.

Recall that we write $\Gamma \to \{\Gamma_1, ..., \Gamma_k\}$ if there exists a graph G with a subset of its edges $e_1, ..., e_k$ such that $Aut(G) \cong \Gamma$ and $Aut(G/e_i) \cong \Gamma_i$ for $i = 1, ..., k$.

Lemma 7.3.6. *For any group* Γ, $\Gamma \to \{\Gamma\}$.

Proof: Let $G = Cay(\Gamma)$. Create the graph G' from the disjoint union of G and a path of length three, and adding an edge between every vertex of G and one internal vertex v of the path. Note that v is the only vertex with degree $v(G)+2$. Therefore, we can observe that $Aut(G') = Aut(G) = \Gamma$, and that the contraction of the edge vw, where w is the end of the path of length 3, yields a graph whose automorphism group is still Γ. ∎

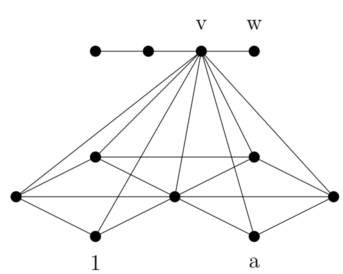

Figure 7.11: Example of the construction in the proof of Lemma 7.3.6. $K_2 = G = Cay(S_2)$ and $V(G) = \{1, a\}$.

Many other graphs could have been employed instead of a path of length three in the construction in Lemma 7.3.6. In particular, using any graph described in Lemma 7.3.2 would work.

Lemma 7.3.7. *For any group* Γ, $I \to \{\Gamma\}$.

Proof: Let $G \in Cay(\Gamma)$. Create the graph G' by adding a leaf to the vertex of G corresponding to 1_Γ, the identity element of Γ. Then, any automorphism of G' must fix 1_Γ and the new vertex, and, restricted to G, this automorphism must be trivial by Lemma 7.3.4. So $Aut(G') = I$ and by contracting the leaf, one obtains G, whose automorphism group is Γ. ∎

Theorem 7.3.8. *For any $k \geq 1$, for any groups $\Gamma_1, ..., \Gamma_k$ and Γ, $\Gamma \rightarrow \{\Gamma_1, ..., \Gamma_k\}$.*

Proof: The proof is by induction on k. For $k = 1$, consider $G = Cay(\Gamma)$ and H, a graph with the property that $I \rightarrow \{\Gamma_1\}$, which exists by Lemma 7.3.7. Let $\Gamma = \{g_1, ..., g_n\}$ be the subset of vertices of G corresponding to the elements of the group Γ. Build a graph G' as the disjoint union of G and $H_1, ..., H_n$, n copies of H. Attach g_i to every vertex of H_i, for $i = 1, ..., n$. Any automorphism of $Aut(G')$ must map $V(G)$ to itself, and since $Aut(H_i) = I$, we get that any automorphism of G' is determined by its action on $V(G)$, so $Aut(G') = Aut(G) = \Gamma$. Since H is chosen as to have $I \rightarrow \{\Gamma_1\}$, there is an edge $e \in H_1$ such that $Aut(H_1/e) \cong \Gamma_1$. Since H_1/e is not isomorphic to the other H_i's, any automorphism of G'/e must fix g_1, and so must fix G. This implies that $Aut(G'/e) = \Gamma_1$.

Suppose now that $\Gamma \rightarrow \{\Gamma_1, ..., \Gamma_k\}$ for any choice of groups $\Gamma, \Gamma_1, ..., \Gamma_k$. We will first show that $I \rightarrow \{\Gamma_1, ..., \Gamma_{k+1}\}$ for any choice of groups $\Gamma_1, ..., \Gamma_{k+1}$. Let $G = Cay(\Gamma_{k+1})$ and H be such that $I \rightarrow \{\Gamma_1, ..., \Gamma_k\}$. Such a graph H exists by the induction hypothesis. Let $\Gamma_{k+1} = \{g_1, ..., g_n\}$ be the subset of vertices of G corresponding to the elements of the group Γ_{k+1}. Link g_i to every vertex of a copy H_i of H, for $i = 1, ..., n$. Then, add a single vertex v, and link it to $1_{\Gamma_{k+1}}$, the vertex in G corresponding to the identity element of Γ_{k+1}. Clearly, any automorphism of the created graph G' must fix $1_{\Gamma_{k+1}}$, and so it must fix G entirely, as any vertex of G has a higher degree than any other vertex in the H_i's. Since $Aut(H_i) \cong I$, we conclude that $Aut(G') \cong I$. Similar to the construction for the case of $k = 1$, we can see that $Aut(G'/v1_{\Gamma_{k+1}}) \cong \Gamma_{k+1}$. Also, there are k edges in H_1 whose contraction yields graphs whose automorphism groups are $\Gamma_1, ..., \Gamma_k$, as an automorphism of the resulting graph will certainly fix G and the vertex v, and all the other H_i. Thus, we have shown $I \rightarrow \{\Gamma_1, ..., \Gamma_{k+1}\}$.

We now show that $\Gamma \rightarrow \{\Gamma_1, ..., \Gamma_{k+1}\}$. Let $G = Cay(\Gamma)$, and H such that $I \rightarrow \{\Gamma_1, ..., \Gamma_{k+1}\}$. Let $\Gamma = \{g_1, ..., g_n\}$ be the subset of vertices of G corresponding to the elements of the group Γ. Link g_i to every vertex of H_i, a copy of H, for $i = 1, ..., n$. The created graph G' will be such that $Aut(G') \cong \Gamma$, as all the H_i's have a trivial automorphism group. But H_1 has $k + 1$ edges whose contraction yield graphs with automorphism groups $\Gamma_1, ..., \Gamma_k$, respectively. When contracting one of those edges, any automorphism of the resulting graph must fix G, as one of its vertex is linked to a graph not isomorphic to the graphs linked to the other vertices of G. And so $\Gamma \rightarrow \{\Gamma_1, ..., \Gamma_{k+1}\}$. ∎

In this section, we have shown that given the automorphism group of some of the cards of a graph, no information about the automorphism group of a graph or the automorphism group of the other cards can be obtained. Looking at the proof of Theorem 7.3.8, we however see that the size of the graph G grows very quickly in terms of k, in the sense that in this construction, most edges "contract" to an unknown automorphism group.

We believe the real problem in this area is to find a reasonable bound on the number of edges in a graph with $\Gamma \to \{\Gamma_1, \Gamma_2, ..., \Gamma_k\}$. Determining the circumstances under which there exists a graph on m edges such that $\Gamma \to \{\Gamma_1, \Gamma_2, ..., \Gamma_m\}$ would be interesting.

Chapter 8

Reconstruction of random graphs

In this chapter, we show that almost all simple graphs are reconstructible. The proof is based upon properties that almost all graphs have, such as a sufficiently large minimal degree, and a diameter of two, but most importantly, it is based on the observation that almost no simple graphs have two large isomorphic vertex deleted subgraphs.

To show that almost all simple graphs are reconstructible, it is sufficient to show that a dominating class of simple graphs is weakly reconstructible. Indeed, let $\mathcal{G}_n$ be the collection of all simple graphs on n vertices, and suppose that $\mathcal{A}_n$ is a collection of graphs on n vertices, that are weakly reconstructible, and such that $\frac{|\mathcal{A}_n|}{|\mathcal{G}_n|} = 1 - \epsilon_n$, with $\epsilon_n \to 0$ as $n \to \infty$. Thus, almost all graphs are in some $\mathcal{A}_n$. Since $\mathcal{A}_n$ is weakly reconstructible, no two graphs in $\mathcal{A}_n$ have identical decks. Therefore, a graph G in $\mathcal{A}_n$ is reconstructible if no graph in $\mathcal{G}_n - \mathcal{A}_n$ is a reconstruction of G. The worst case scenario is when every graph in $\mathcal{G}_n - \mathcal{A}_n$ is the reconstruction of another graph in $\mathcal{A}_n$. Hence the proportion of simple graphs on n vertices that are not reconstructible is at most $2\frac{|\mathcal{G}_n| - |\mathcal{A}_n|}{|\mathcal{G}_n|} = 2\epsilon_n$, which tends to 0 as $n \to \infty$.

8.1 Random graph theory

We first give a brief introduction to random graph theory. Then, we introduce a class of graphs that is dense, meaning almost all simple graphs are in that class. Finally, in the next section, we show that simple graphs in this class are weakly reconstructible.

A ***finite probability space*** (Ω, P) consists of a finite set Ω, called the ***sample space***, and a probability function $P : \Omega \to [0, 1]$, satisfying $\sum_{\omega \in \Omega} P(\omega) = 1$. The probability space that is of interest here is where $\Omega = \mathcal{G}_n$, the set of all labeled simple graphs on the vertices $\{1, ..., n\}$, and where $P(G)$ is equal for every $G \in \mathcal{G}_n$. So, for every $G \in \mathcal{G}_n$,

$$P(G) = 2^{-\binom{n}{2}}.$$

The most intuitive way of viewing this probability space is to imagine the edges of the complete graph K_n as being considered for inclusion in G, one by one, each edge being chosen with probability $\frac{1}{2}$, and independently of each other. We say that ***almost all*** labeled graphs are in a class $\mathcal{P}$, or synonymously, we say that the class $\mathcal{P}$ is ***dense***, if

$$\sum_{G \in \mathcal{G}_n \cap \mathcal{P}} P(G) \to 1$$

as $n \to \infty$. In the context of reconstruction, we are more interested in dense classes of unlabeled graphs. It is well known that almost all graphs have no non-trivial automorphisms [7]. Using this observation, probability results about labeled graphs can be applied to unlabeled graphs. We start by showing that almost all graphs have certain properties.

Lemma 8.1.1. *Let k be a fixed positive integer. Almost all simple graphs have a minimum degree of at least k.*

Proof: The probability that a fixed vertex of a random graph on n vertices has degree at most $k-1$ is given by

$$\sum_{i=0}^{k-1} \binom{n-1}{i} \left(\frac{1}{2}\right)^{n-1}.$$

Therefore, the probability that every vertex of a random graph on n vertices has a degree of at most $k-1$, hence the minimum degree of the graph is less then k, is at most

$$n \sum_{i=0}^{k-1} \binom{n-1}{i} \left(\frac{1}{2}\right)^{n-1}$$

This probability tends to 0 as $n \to \infty$, hence almost all simple graphs have a minimum degree of at least k. ∎

Lemma 8.1.2. *For any fixed $k \geq 0$, almost all simple graphs have the property that any two vertices have at least k common neighbours.*

Proof: Let G be a random graph on n vertices, and let $x, y \in V(G)$. The probability that x and y have exactly i common neighbours is $\left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{n-i-2}$. So the probability that there exists a pair of vertices with less than k common neighbours is at most

$$\binom{n}{2} \sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{n-i-2}$$

which tends to 0 as $n \to \infty$. Therefore, for almost all simple graphs, every pair of vertices have at least two common neighbours. ∎

Lemma 8.1.3. *Let $k \geq 2$, $k_1 \geq 1$ and $k_2 \geq 1$ be fixed. Almost all simple graphs G have the following property: for any $U \subset V(G)$ of size $v(G) - k$ and any two disjoint subsets $X_1, X_2 \subseteq V(G) - U$ of size $|X_1| = k_1$ and $|X_2| = k_2$, the set of neighbours of X_1 in U is distinct from the set of neighbours of X_2 in U.*

Proof: Let G be a random graph on n vertices. Let $u \in U$. The probability that u is adjacent to a vertex of X_1 and a vertex of X_2 or that u is not adjacent to any vertices of $X_1 \cup X_2$ is

$$p = 1 - \left(\left(\frac{1}{2}\right)^{k_1}\left(1 - \left(\frac{1}{2}\right)^{k_2}\right) + \left(\frac{1}{2}\right)^{k_2}\left(1 - \left(\frac{1}{2}\right)^{k_1}\right)\right) < 1.$$

Therefore, the probability that for every $u \in U$, either u is adjacent to a vertex of X_1 and a vertex of X_2, or u is not adjacent to any vertices of $X_1 \cup X_2$, is p^{n-k}. We conclude that the probability that there exists a subset U, and two subsets $X_1, X_2 \subseteq V(G) - U$ of size $|X_1| = k_1$ and $|X_2| = k_2$ such that the neighbours of X_1 in U is equal to the neighbours of X_2 in U is at most

$$\binom{n}{k}\binom{k}{k_1}\binom{k-n_1}{k_2}p^{n-k}$$

which tends to 0 as $n \to \infty$. ∎

If we consider the situation where X_1 and X_2 intersect, Lemma 8.1.3 can instead be applied with $X_1 - X_2$ and $X_2 - X_1$, which are now disjoint. We then conclude that X_1 and X_2 have distinct neighbour sets in U. Lemma 8.1.3 states, in other words, that for a fixed subset U of the vertices of $V(G)$, the subsets of $V(G) - U$ are almost always distinguishable by their adjacency to U.

Almost all simple graphs have all the properties of Lemmas 8.1.1, 8.1.2 and 8.1.3. In particular, Lemma 8.1.2 implies that almost all simple graphs have a diameter of 2 and a triangle.

8.2 Almost all graphs are reconstructible

The main result of this section, and of this chapter, is Theorem 8.2.9, which states that almost all simple graphs are reconstructible. The proof hinges on a property that almost all graphs have, which is that all of its large vertex-deleted subgraphs are isomorphically distinct.

Definition 8.2.1. *For any positive integer k, a graph G has* ***property*** P_k *if for any two distinct subsets X and Y of $V(G)$ with $|X| = |Y| = k$, we have $G - X \ncong G - Y$.*

If G does not have property P_k for some k, then there exists two distinct subsets X and Y of $V(G)$ of size k such that $G - X \cong G - Y$. Let α be an isomorphism from $G - X$ to $G - Y$. If G has at least $4k + 2$ vertices, then there exists a vertex $v \in V(G)$ such that neither v nor $\alpha(v)$ is in $X \cup Y$. Then $G - (X \cup \{v\}) \cong G - (Y \cup \{\alpha(v)\})$, with $X \cup \{v\} \neq Y \cup \{\alpha(v)\}$, and so G does not have property P_{k+1}. Therefore, for any k and any graph G with a sufficiently large number of vertices with respect to k, if G does not have property P_k, then it does not have property P_{k+1}.

The class of graphs with property P_k was studied and shown to be dense in [2], [14] and [23], for any k.

Lemma 8.2.2 (Korshunov [14])**.** *For any fixed k, almost all graphs have property P_k.*

The property P_k is closely related to vertex deletion. We introduce an analogous property, for contractions.

Definition 8.2.3. *Let G be a graph and $A \subseteq E(G)$. The* ***support*** *of A, denoted* $\mathrm{supp}(A)$*, is the set of all vertices incident to at least one edge of A.*

Definition 8.2.4. *A graph G has property CP_k if $G/A \ncong G/B$ for any two distinct subsets A, B of $E(G)$, with $|A| = |B| = k$ and* $\mathrm{supp}(A) \neq \mathrm{supp}(B)$.

We now introduce some notation to represent sets of vertices in a graph and its contractions.

Definition 8.2.5. *For a subset of vertices $X \in G/A$, $X^{\uparrow G}$ denotes the vertices in G such that if X' is the subgraph of G induced by $X^{\uparrow G}$, then $V(X'/(A \cap E(X'))) = X$. For a subset of vertices $X \in G$, denote by $X^{\downarrow G/A}$ the vertices in G/A so that if X' is the subgraph of G induced by X, then $V(X'/(A \cap E(X'))) = X^{\downarrow G/A}$.*

In other words, if X is a subset of vertices of G, then $X^{\downarrow G/A}$ are the corresponding vertices in G/A. Conversely, if X is subset of vertices of G/A, then $X^{\uparrow G}$ are the vertices of G that are contracted to X. For instance, if $xy \in E(G)$ and $v^* \in V(G/xy)$ is the contracted vertex of G/xy, then $\{v^*\}^{\uparrow G} = \{x, y\}$, and $\{x, y\}^{\downarrow G/xy} = \{v^*\}$.

Note that $X \subseteq \left(X^{\downarrow G/A}\right)^{\uparrow G}$ with equality if and only if there are no edges of A with one end point in X and one end point not in X. Conversely, $Y = \left(Y^{\uparrow G}\right)^{\downarrow G/A}$. We now relate the properties P_k to the property CP_3.

Lemma 8.2.6. *Let $t \geq 1$. Let G be a graph with properties P_i for all $i \leq 6t$ and the property of Lemma 8.1.3. Then G has property CP_t.*

Proof: Suppose G is a graph that does not have property CP_t. This implies the existence of two distinct subsets of t edges $A, B \in E(G)$, with $\mathrm{supp}(A) \neq \mathrm{supp}(B)$, such that $G/A \cong G/B$. Let α be an isomorphism from G/A to G/B. Let $S \subseteq V(G/A)$ and $T \subseteq V(G/B)$ be defined as follows:

$$\begin{aligned} S &= \mathrm{supp}(A)^{\downarrow G/A} \cup \mathrm{supp}(B)^{\downarrow G/A} \cup \alpha^{-1}\left(\mathrm{supp}(A)^{\downarrow G/B}\right) \cup \alpha^{-1}\left(\mathrm{supp}(B)^{\downarrow G/B}\right) \\ T &= \alpha(S) = \alpha\left(\mathrm{supp}(A)^{\downarrow G/A}\right) \cup \alpha\left(\mathrm{supp}(B)^{\downarrow G/A}\right) \cup \mathrm{supp}(A)^{\downarrow G/B} \cup \mathrm{supp}(B)^{\downarrow G/B}. \end{aligned}$$

Now, $G/A - S \cong G/B - T$. Since $\mathrm{supp}(A)^{\downarrow G/A} \subseteq S$, the vertices to decontract in G/A to obtain G are not in $G/A - S$. Similarly, since $\mathrm{supp}(B)^{\downarrow G/B} \subseteq T$, the vertices to decontract in G/B to obtain G are not in $G/B - T$. Therefore, $G - S^{\uparrow G} \cong G - T^{\uparrow G}$. Now $\mathrm{supp}(A)^{\downarrow G/A}$ and $\alpha^{-1}\left(\mathrm{supp}(B)^{\downarrow G/B}\right)$ each contain at most t vertices, while $\mathrm{supp}(B)^{\downarrow G/A}$ and $\alpha^{-1}\left(\mathrm{supp}(A)^{\downarrow G/B}\right)$ each contain at most $2t$ vertices, hence $|S| \leq 5t$. It implies that $|S^{\uparrow G}| \leq 6t$, since there are t decontractions to make in G/A to obtain G, thus creating t new vertices. Hence we have shown that G does not have property P_i for some $i \leq 6t$, unless $S^{\uparrow G} = T^{\uparrow G}$. We henceforth assume that $S^{\uparrow G} = T^{\uparrow G}$.

We say that the isomorphism α ***acts pseudo-trivially*** on a subset of vertices V of G/A if for every $v \in V$, $v^{\uparrow G} = \alpha(v)^{\uparrow G}$. The proof is now split into two cases, whether α acts pseudo-trivially on $V(G/A) - S$ or not.

Suppose α does not act pseudo-trivially on $V(G/A) - S$. Therefore, there exists a vertex $v \in V(G/A) - S$ such that $v^{\uparrow G} \neq \alpha(v)^{\uparrow G}$. This implies $G/A - S - v \cong G/B - T - \alpha(v)$, and since every contracted vertex of G/A and G/B are in S and T respectively, we get that $|v^{\uparrow G}| = |\alpha(v)^{\uparrow G}| = 1$, hence $G - S^{\uparrow G} - v^{\uparrow G} \cong G - T^{\uparrow G} - \alpha(v)^{\uparrow G}$, with $S^{\uparrow G} \cup v^{\uparrow G} \neq T^{\uparrow G} \cup \alpha(v)^{\uparrow G}$, since $S^{\uparrow G} = T^{\uparrow G}$ and $v^{\uparrow G} \neq \alpha(v)^{\uparrow G}$. Therefore, G does not have property P_i for some $i \leq 6t$.

Suppose then that α acts pseudo-trivially on $V(G/A) - S$. If α also acts pseudo-trivially on S, then for any contracted vertex $v \in S$, we get $v^{\uparrow G} = \alpha(v)^{\uparrow G}$. In particular, if $v \in S$ is such that $|v^{\uparrow G}| \geq 2$, then $v^{\uparrow G} \subseteq \mathrm{supp}(A)$. Then $|\alpha(v)^{\uparrow G}| \geq 2$ as well, hence $\alpha(v)^{\uparrow G} \subseteq \mathrm{supp}(B)$. This implies that $\mathrm{supp}(A) = \mathrm{supp}(B)$, which contradicts the assumption that $\mathrm{supp}(A) \neq \mathrm{supp}(B)$. We henceforth assume that α does not act pseudo-trivially on S, hence there exists a vertex $v \in S$ such that $v^{\uparrow G} \neq \alpha(v)^{\uparrow G}$. Since every vertex in S has distinct neighbour sets in $V(G/A) - S$, the neighbours U of v in $V(G/A)$ are distinct from the neighbours $\alpha(U)$ of $\alpha(v)$ in $V(G/B) - T$. It implies that the neighbours $U^{\uparrow G}$ of $v^{\uparrow G}$ in $V(G) - S^{\uparrow G}$ are distinct from $\alpha(U)^{\uparrow G}$, the neighbours of $\alpha(v)^{\uparrow G}$ in $V(G) - S^{\uparrow G}$. However, $U^{\uparrow G} = \alpha(U)^{\uparrow G}$ since α acts pseudo-trivially on $V(G/A) - S$. This contradiction completes the proof. ∎

The next step is to show that most graphs with property CP_t also have property CP_{t-1}. We will then apply this lemma to $t = 3$.

Lemma 8.2.7. *Let $t \geq 2$. Let G be a graph on at least $5t+2$ vertices, with a minimal degree of at least $5t+1$. If G has property CP_t, then G also has property CP_{t-1}.*

Proof: Suppose G is a graph on at least $5t+2$ vertices, with a minimal degree of at least $5t+1$, and suppose G does not have property CP_{t-1}, that is, there are sets of edges $A, B \subseteq E(G)$ such that $|A| = |B| = t-1$, $\mathrm{supp}(A) \neq \mathrm{supp}(B)$, and $G/A \cong G/B$. Let α be an isomorphism from G/A to G/B. Define $S \subseteq V(G/A)$ as follows:

$$S = \mathrm{supp}(A)^{\downarrow G/A} \cup \mathrm{supp}(B)^{\downarrow G/A} \cup \alpha^{-1}\left(\mathrm{supp}(A)^{\downarrow G/B}\right) \cup \alpha^{-1}\left(\mathrm{supp}(B)^{\downarrow G/B}\right).$$

Now $|S| \leq 5t$, and since the minimal degree is at least $5t+1$, there exists adjacent vertices v, w in $V(G/A) - S$. Then, $G/A/vw \cong G/B/\alpha(v)\alpha(w)$. Also, $\mathrm{supp}(A) \cup \{v^{\uparrow G}, w^{\uparrow G}\} \neq \mathrm{supp}(B) \cup \{\alpha(v)^{\uparrow G}, \alpha(w)^{\uparrow G}\}$, since none of $v^{\uparrow G}, \alpha(v)^{\uparrow G}, w^{\uparrow G}$ or $\alpha(w)^{\uparrow G}$ is in $\mathrm{supp}(A) \cup \mathrm{supp}(B)$. We conclude that G does not have property CP_t. ∎

We now show that a subset of the graphs with properties CP_3 and CP_2 are weakly reconstructible. Again, it is not necessary to show that the class of graphs with CP_2 or CP_3 is recognizable, as it will be shown later that almost all graphs have those two properties.

Proposition 8.2.8. *Simple graphs with at least one triangle, with a minimum degree of at least 2, with properties CP_3 and CP_2, are weakly reconstructible.*

Proof: Let G be a graph with properties CP_2 and CP_3, with a minimum degree of at least 2 and with a triangle.

Let G/a and G/b be cards such that there exist edges $x \in E(G/a)$ and $y \in E(G/b)$ where $G/\{a, x\} \cong G/\{b, y\}$ and where $G/\{a, x\}$ and $G/\{b, y\}$ contains a loop. The edges a, b, x and y exist if and only if $x = y$ and a, b and x form a triangle. Since G does contain a triangle, such edges exist. Since a, b and x form a triangle, in $G/\{a, x\}$ and in $G/\{b, y\}$, there is only one contracted vertex, which we name v^* and v' respectively.

Let α be an isomorphism from $G/\{a, x\}$ to $G/\{b, x\}$. We claim that this isomorphism is unique. Proposition 7.1.1 states that unless G has more than one isolated vertices or G has a connected component isomorphic to K_2, automorphisms of G can be seen as bijections on the set of edges. Since $\delta(G) \geq 2$, we can assume that $G, G/\{a, x\}$ and $G/\{b, y\}$ do not contain any isolated vertices or connected components isomorphic to K_2. Now, if there are two distinct isomorphisms from $G/\{a, x\}$ to $G/\{b, x\}$, say α_1 and α_2, there is an edge $e \in E(G/\{a, x\})$ such that $\alpha_1(e) \neq \alpha_2(e)$, one of which, say $\alpha_1(e)$, does not correspond to e in G. This implies $G/\{a, x, e\} \cong G/\{b, x, \alpha_1(e)\}$, with $\mathrm{supp}(\{a, x, e\}) \neq \mathrm{supp}(\{b, x, \alpha_1(e)\})$. This contradicts the supposition that G

has property CP_3. We conclude that α is the unique isomorphism from $G/\{a,x\}$ to $G/\{b,x\}$.

From there, label all the vertices in G/a. This induces a labeling of the vertices in $G/\{a,x\}$. By the uniqueness of the isomorphism α, this induces a labeling of the vertices in $G/\{b,x\}$, which in turn determines a labeling of G/b. Since G is simple, and since $G/\{a,x\}$ has a loop, the isomorphism maps v^* in $G/\{a,x\}$, to v' in $G/\{b,x\}$. Let $a = v_1v_2$, $b = v_2v_3$ and $x = v_3v_2$. It remains to determine the neighbours of v_1, v_2 and v_3 in G to reconstruct G. In G/a, the vertices v_1 and v_2 have been contracted, so the neighbours of v_3 can be identified. Similarly, in G/b, the neighbours of v_1 can be identified. By elimination, one can find the neighbours of v_2 using either G/a or G/b. Therefore, G is reconstructed. ∎

Once we gather Lemmas 8.2.2, 8.2.6 and 8.2.7, we can show that almost all graphs have the properties CP_3 and CP_2, a sufficiently large minimum degree and a triangle. Then, using Proposition 8.2.8, we can show that those graphs are weakly reconstructible.

Theorem 8.2.9. *Almost all simple graphs are reconstructible.*

Proof: Let $\mathcal{A}$ be the class of graphs with at least 17 vertices, a minimum degree of at least 16, the property that any two vertices have a common neighbour, the property of Lemma 8.1.3 and property P_i for all $i \leq 18$. By Lemmas 8.1.1, 8.1.2, 8.1.3 and 8.2.2, the class $\mathcal{A}$ is dense. By Lemma 8.2.6, every graph of $\mathcal{A}$ has the property CP_3, and by Lemma 8.2.7, they also have property CP_2. Thus, by Proposition 8.2.8, the graphs of $\mathcal{A}$ are weakly reconstructible.

We now show that almost all graphs are reconstructible. Let $\mathcal{G}_n$ be the class of graphs with n vertices. Let

$$\epsilon_n = \frac{|\mathcal{G}_n - \mathcal{A}|}{|\mathcal{G}_n|}$$

Then, a proportion of at most ϵ_n of the graphs in $\mathcal{A}$ with n vertices are not reconstructible. Therefore, we have shown that a ratio of at least $1 - 2\epsilon_n$ of the graphs on n vertices are reconstructible, which tends to 1 as n approaches ∞. ∎

Theorem 8.2.9 solidifies the validity of Conjecture 1.1.2 by indicating that counterexamples to the conjecture should be somewhat rare. Note that the proof of Theorem 8.2.9 is valid for any dense class of graphs, and not just the class we presented here. Theorem 8.2.9 states, in other words, that almost all graphs are weakly reconstructible if and only if almost all graphs are reconstructible. We should mention that this equivalence does not say that a dense class of graph is weakly reconstructible if and only if it is reconstructible. Indeed, we have not shown here that the graphs in Proposition 8.2.8 are reconstructible, but only that a large subset of them are reconstructible. Moreover, this large subset of reconstructible graphs is not clearly defined since we have not shown that the graphs in Proposition 8.2.8 are recognizable. What we have shown is that for any graph G in a large subset of a dense class of graphs, the reconstructions of G are either isomorphic to G or not in the dense class. This is sufficient to conclude that almost all graphs are reconstructible.

Chapter 9

Conclusion

To conclude, we discuss some open questions related contraction reconstruction.

9.1 Minor reconstruction

In this section, we introduce another reconstruction problem, closely related to contraction and edge deletion reconstruction.

A ***minor*** of a connected graph G is a graph obtainable from G by a sequence of edge deletions and edge contractions. A ***maximal proper minor*** of G is a minor of G obtained by one edge deletion or one edge contraction. By this definition, one concludes that $\mathcal{E}(G) \cup \mathcal{C}(G)$ is the multiset of all the maximal proper minors of G. Taking $\mathcal{E}(G) \cup \mathcal{C}(G)$ as the deck of G, one can then ask a similar reconstruction question.

Definition 9.1.1. *For a graph G, the multiset of unlabeled graphs $\mathcal{M}(G) = \mathcal{E}(G) \cup \mathcal{C}(G)$ is the* ***minor deck*** *of G. Any graph $H \in \mathcal{M}(G)$ is a* ***minor card****. A* ***minor reconstruction*** *of G is a graph F such that there exists two bijections $\phi_1 : E(G) \to E(F)$ and $\phi_2 : E(G) \to E(F)$ where $G/e \cong F/\phi_1(e)$ and $G - e \cong F - \phi_2(e)$ for all $e \in E(G)$. A graph G is* ***minor reconstructible*** *if every minor reconstruction of G is isomorphic to G. A class $\mathcal{G}$ of graphs is* ***minor recognizable*** *if every minor reconstruction of any $G \in \mathcal{G}$ is also in $\mathcal{G}$. The class $\mathcal{G}$ is* ***minor weakly reconstructible*** *if for every $G \in \mathcal{G}$, all minor reconstructions of G that are in $\mathcal{G}$ are isomorphic to G. A property P of a graph is* ***minor reconstructible*** *if for any graph G with property P, every minor reconstruction of G has property P.*

Some graphs with two edges are not minor reconstructible. They are shown in Figure 9.1.

Conjecture 9.1.2. *All undirected graphs with at least three edges are reconstructible.*

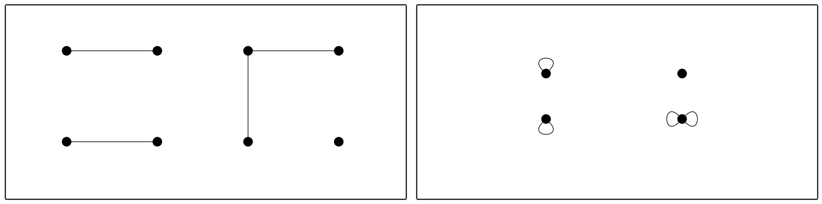

Figure 9.1: In each box are graphs that are minor reconstructions of one another.

From Figure 9.1, one may conclude that a graph is not minor reconstructible if it is both not contraction reconstructible and not edge reconstructible. The reason behind this implication is that the edge deck $\mathcal{E}(G)$ and the contraction deck $\mathcal{C}(G)$ can both be determined from the minor deck $\mathcal{M}(G)$. To prove this result, this basic proposition is required:

Proposition 9.1.3. *If G has at least one edge, then the number of vertices $v(G)$ is minor reconstructible.*

Proof: Since $v(G-e) = v(G)$ and $v(G/e) \leq v(G)$, and because G has at least one edge, then at least one card of $\mathcal{M}(G)$ corresponds to the deletion of an edge, and so $v(G) = \max\{v(H) \mid H \in \mathcal{M}(G)\}$. ∎

For a simple graph G, cards in $\mathcal{E}(G)$ have one more vertex than the cards in $\mathcal{C}(G)$. It is therefore possible to partition $\mathcal{M}(G)$ into these two sets, hence any simple graph that is edge reconstructible or contraction reconstructible is also minor reconstructible. For a multigraph, this partition can still be made.

Proposition 9.1.4. *For any graph G, $\mathcal{E}(G)$ and $\mathcal{C}(G)$ are minor reconstructible.*

Proof: Assume G has at least one edges. Since $v(G)$ is reconstructible by Proposition 9.1.3, any card with $v(G)-1$ vertices belongs to $\mathcal{C}(G)$. If none of the cards have $v(G)-1$ vertices, then all the edges of G are loops, in which case for every card $H \in \mathcal{C}(G)$, there is a card $H' \in \mathcal{E}(G)$ such that $H \cong H'$. If some edges of G are links, then the number of loops $\ell(G)$ in G is equal to $\max\{\ell(H) \mid H \in \mathcal{M}(G),\ v(H) = v(G)\}$. Now that $\ell(G)$ is known, cards with $v(G)$ vertices and $\ell(G)-1$ loops come in pairs in $\mathcal{M}(G)$, one of them corresponding to the contraction of a loop, and the other to its deletion. Both cards are isomorphic, so taking one from each pair and adding it to $\mathcal{C}(G)$ implies that both $\mathcal{C}(G)$ and $\mathcal{E}(G)$ are minor reconstructible. ∎

Corollary 9.1.5. *If G is edge reconstructible or contraction reconstructible, then G is minor reconstructible.*

Since $\mathcal{E}(G)$ is minor reconstructible, Lemma 2.2.4 applies. Denote by $m(F, G)$ the number of minors of G isomorphic to F, then a similar lemma exists.

Lemma 9.1.6. *Let F and G be graphs with $e(F) < e(G)$. Then, the number $m(F, G)$ of minors of G isomorphic to F is*

$$m(F, G) = \frac{1}{e(G) - e(F)} \sum_{H \in \mathcal{M}(G)} m(F, H).$$

Proof: Let F be a minor of G. Then F is obtained from G by a sequence of edge deletions and contractions. Let $\{e_1, ..., e_k\}$ be the set of edges that need to be deleted and $\{\epsilon_1, ..., \epsilon_{k'}\}$ be the set of edges that need to be contracted, to obtain F from G. Now F is a minor of $G - e_i$ for $i = 1, ..., k$ and is also a minor of G/ϵ_j for $j = 1, ..., k'$. F is not a minor of any other card. Note that $k + k' = e(G) - e(F)$, so the minor F is found in $e(G) - e(F)$ cards. From these we retrieve the formula stated in the lemma. ∎

9.2 Open problems in contraction reconstruction

The main contraction reconstruction conjecture may feel, as does many other reconstruction problems, out of reach for the moment. However, there are many smaller problems we think are accessible and interesting. We present some of them in this section while also highlighting our contributions.

Question 9.2.1. *Are separable graphs with one non-trivial block reconstructible?*

Separable graphs with more than one non-trivial block were shown to be reconstructible by Theorem 4.3.14. Some separable graphs with one non-trivial block were also shown to be reconstructible by Theorems 4.3.18 and 4.3.21. However, those two theorems make very specific assumptions about the structure of the non-trivial block. We feel the techniques used in Theorems 4.3.18 and 4.3.21 cannot solve Question 9.2.1, but that a better understanding of the relation between the cards of a known block and the block itself would help.

Question 9.2.2. *Are separable multi-graphs with more than one non-trivial block reconstructible?*

Recall that a block is trivial if it contains exactly two vertices. However, it might be worthwhile to not consider loops as blocks, but more as a decoration of vertices.

The reason for this is to keep control of the number of blocks in the cards. If we do allow loops as blocks, any contraction of a link with a multiplicity $m \geq 2$ creates $m - 1$ loop blocks. Going back to the proof of Theorem 4.3.14, we see that a major part of the proof relies on the observation that for any simple separable graph, the list of non-trivial blocks is not closed under taking contractions. When considering multi-graphs, this is no longer true: a block isomorphic to K_3 is not trivial, but any of its contractions yields a trivial block.

An interesting approach to this problem would be to consider the block tree graph of G. Recall that the ***block tree*** of a separable graph G is a bipartite graph $BT(G)$ with bipartition $(\mathcal{L}(G), S)$, where $\mathcal{L}(G)$ is the set of blocks of G and S is the set of separating vertices of G. A block B and a separating vertex v are adjacent if and only if $v \in V(B)$.

It can be shown that the centre of a block tree graph is a single vertex, meaning there is exactly one vertex v in the block tree graph $BT(G)$ where $\max\{d(v, u) \mid u \in V(BT(G))\}$ is minimum. That centre vertex either corresponds to a block or a cut vertex of G. Recall that a branch of a tree T centred at a vertex v is a maximal subgraph B of T containing v and exactly one of its neighbours. Each branch of $BT(G)$ corresponds to a subgraph of G. It might be possible to devise a strategy to reconstruct the list of these subgraphs of G, since a contraction can affect only one branch. Should the centre of G be a cut-vertex, simply attaching together all the branches at their root would certainly reconstruct G. However, if the centre of G is a block B, reconstructing the branches of G would not suffice. It would be necessary to identify how to attach each branch to the centre block.

Note that this general strategy could also be applied to separable multi-graphs with any number of non-trivial blocks, which leads to the following more specific problem:

Question 9.2.3. *Are multi-trees reconstructible?*

Again, a strategy similar to the one discussed for Question 9.2.2 may be applied to multi-trees. Recall that a ***multi-tree*** is a graph whose underlying simple graph is a tree. See the end of section 4.3.1 for a discussion of this problem.

Question 9.2.4. *Can the concept of sub-blocks be adequately generalized to higher edge connectivities?*

In the proof of Theorems 4.5.4 and 4.5.6, the concept of sub-blocks is essential. To adapt the proof that some graphs with $\kappa'(G) = 2$ are reconstructible to higher edge connectivities, it would be interesting to adapt the concept of sub-blocks as well.

Question 9.2.5. *Is the number of cycles of each length reconstructible?*

Some progress towards a positive answer to Question 9.2.5 has been made in Propositions 5.1.6 and 5.1.7, which shows that the number of cycles of each length less than twice the girth is reconstructible, and that the number of cycles of each length, up to the first odd length, is reconstructible.

In the context of edge deletion reconstruction, the reconstruction of the degree sequence is done in a very straightforward manner using Lemma 2.2.4 to count subgraphs isomorphic to stars. The dual operation of edge deletion being edge contraction, and the dual of a star being a cycle, we feel that the reconstruction of the number of cycles of each length is attainable. The main obstacle at the moment is that the contraction of an edge e not only reduces the length of every cycle is contains by 1, but it also destroy the cycles that contain both ends of e, but not e itself.

A positive answer to Question 9.2.5 would directly yield the reconstruction of many graph parameters, such as girth, circumference and hamiltonicity.

Question 9.2.6. *Are regular graphs reconstructible?*

Progress has been made towards the reconstruction of regular graphs in this thesis. Theorem 6.3.12 shows that distance regular graphs, which are a subset of regular graphs, are reconstructible. Also Theorem 6.2.4 shows that some cubic graphs are reconstructible. It seems unlikely that the techniques used in Theorem 6.2.4 can be applied to anything other than cubic graphs.

Question 9.2.7. *Is the automorphism group of G reconstructible?*

This seems like a hard problem. We have seen in Section 7.3 that the relation between the automorphism group of G and the automorphism groups of its cards is weak: given the automorphism group of some cards of G, nothing can be said about the automorphism group of G. However, there is no doubt that much more information is given by the deck than simply the automorphism group of the cards.

Question 9.2.8. *Given the automorphism group of every card of G, what can be said of $Aut(G)$?*

Section 7.3 falls very short of answering Question 9.2.8. It however gives the sense that this might be a hard question. It is unknown if the presence of large automorphism groups in the cards forces a large $Aut(G)$, or vice versa.

Question 9.2.9. *Is the presence of contraction pseudo-similar edges recognizable?*

Section 7.2 classifies every type of contraction pseudo-similar edges. However, it does not relate the concept of pseudo-similar edges to the contraction reconstruction problem. A positive answer to Question 9.2.9 would allow, in particular, to recognize edge transitive graphs.

Question 9.2.10. *Does there exist a graph G with $\delta(G) \geq 3$ where every edge has a pseudo-similar mate?*

Proposition 7.2.12 comes close to answering this question, by constructing an infinite sequence of graphs, each with a minimum degree of at least 3, where the ratio of edges with a pseudo-similar mate to $e(G)$ tends to 1. While it would be interesting to see what a graph where every edge has a pseudo-similar mate, a negative answer to Question 9.2.10 would also be interesting, as it would guarantee the deck of any graph G with $\delta(G) \geq 3$ contains cards $G/e_1 G/e_2, ..., G/e_k$, all pairwise isomorphic, that would correspond to an edge orbit of G.

Question 9.2.11. *Are graphs with property CP_k recognizable?*

Recall that a graph has property CP_k if $G/A \ncong G/B$ for any two distinct subsets A, B of $E(G)$, with $|A| = |B| = k$ and $\text{supp}(A) \neq \text{supp}(B)$. Theorem 8.2.9 shows that almost all graphs are reconstructible, using the claim that almost all graphs have property CP_k. While the proof of Theorem 8.2.9 ands its lemmas does not require that graphs with CP_k are recognizable, it would be interesting to determine if they are. Indeed, should graphs with CP_k be recognizable, it would, by complementarity, show that graphs without CP_k are recognizable too. Showing that the latter is reconstructible could be a great step towards the resolution of Conjecture 1.1.2.

Appendix A

Reconstructing simple graphs with few vertices

We include some Mathematica functions that can be used to determine if graphs are reconstructible. Since the number of vertices, the number of edges and degree sequence is reconstructible by Propositions 3.1.1, 3.1.3 and 6.1.1, it suffices to verify that two graphs with the same number of edges, vertices and the same degree sequence have distinct decks.

This program first partitions a list of simple graphs on a set number of vertices, according to the number of edges. It then further partitions that list according to the degree sequences. Then this program considers the deck of every graph in a partition. It compares the deck $\mathcal{C}(G_1)$ with the deck $\mathcal{C}(G_2)$ by removing from them, one by one, a card $H_1 \in \mathcal{C}(G_1)$ and a card $H_2 \in \mathcal{C}(G_2)$ if $H_1 \cong H_2$. If, at the end, the decks are empty, then G_1 is a reconstruction of G_2.

The inital list of graphs is taken from [22]. The code below verifies if connected graphs with eight vertices are reconstructible. It can be modified to verify other classes of graph by importing a different data set as "list" from [22], and by modifying, accordingly, the bounds on the number of edges i in the "For" loop.

The function "deck[g] " generates the contraction deck of a given graph g. The function "isodeck[g,h]" checks if g is a reconstruction of h. The function "selectedgenum[liste ,e]" partitions a list of graphs according to the number of edges. Similarly, the function "selectdegsequence[liste]" partitions a list of graphs according to their degree sequences.

```
<< Combinatorica`
list = Import["graph8c.g6"];
list1 = Map[FromAdjacencyMatrix , Map[Normal, Map[AdjacencyMatrix , list]]];
deck[g_] := Table[Contract[g, Edges[g][[i]]], {i, 1, Length[Edges[g]]}]
isodeck[g_, h_] := Module[{deck1, deck2, ans, temp},
  (deck1 = deck[g]; deck2 = deck[h]; ans = False;
   While[Length[deck1] == Length[deck2],
    (If[Length[deck1] == 0, (ans = True; Break[])]; temp = deck1[[1]];
     deck1 = Select[deck1, IsomorphicQ[#, temp] == False &];
     deck2 = Select[deck2, IsomorphicQ[#, temp] == False &])]; ans)]
selectedgenum [liste_, e_] := Select[liste, Length[Edges[#]] == e &]
selectdegsequence [liste_] :=
 Module[{ans, templist, tempgraph},
  (ans = {}; templist = liste;
   While[templist ≠ {}, (tempgraph = templist[[1]];
     AppendTo[ans, Select[templist,
       Sort[DegreeSequence [#]] == Sort[DegreeSequence [tempgraph]] &]];
     templist = Select[templist,
       Sort[DegreeSequence [#]] ≠ Sort[DegreeSequence [tempgraph]] &])];
   ans)]
For[i = 7, i < 27, i++, (tlist = selectedgenum [list1, i];
  degtlist = selectdegsequence [tlist];
  For[j = 1, j < Length[degtlist], j++,
   Print[
    {i, j,
     Union[Flatten[Table[isodeck[degtlist[[j, u]], degtlist[[j, v]]],
        {u, 1, Length[degtlist[[j]]] - 1},
        {v, u + 1, Length[degtlist[[j]]]}]]]}]])]
```

This program outputs triples (i, j, k), where i is the number of edges being checked, j is the index of the degree sequence being checked, and k is "{False}" if the graphs with i edges in the j^{th} partition according to the degree sequence, are reconstructible. Furthermore, k is "{}" if there is only one graph with its degree sequence, so this output also indicates that the graph in consideration is reconstructible. So far, this program has been used to verify that simple connected graphs with four to eight vertices are reconstructible.

Theorem A.0.12. *If G is a simple connected graph with $e(G) \geq 4$ and $4 \leq v(G) \leq 8$, then G is reconstructible.*

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List of Symbols

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