Branching rules for irreducible smooth representations of unramified $\mathrm{U}(1,1)$

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Abstract

Let $G = \mathrm{U}(1,1)$ denote the group of F-points of the quasi-split E/F-form of GL_2, where F is a non-archimedean local field of residual characteristic $p \neq 2$, and E is the quadratic unramified extension of F. In this thesis, we determine the branching of almost all irreducible smooth representations of G upon restriction to a fixed maximal compact subgroup $\mathcal{K}$. We prove that each such restriction decomposes as a multiplicity-free direct sum of irreducible components of distinct depth and degree, up to twisting by a quasi-character of G. Moreover, we give an explicit description of all irreducible components that occur in this decomposition in terms of irreducible representations of $\mathcal{K}$ constructed herein.

We analyze the branching rules by dividing the irreducible representations of G into three classes: depth-zero supercuspidal representations, positive-depth supercuspidal representations, and principal series representations. We provide two applications of this explicit description. First, we show that the higher-depth components arising in these decompositions exhibit a striking uniformity: up to twisting by a quasi-character of G, they coincide with the higher-depth components obtained from a fixed collection of four depth-zero irreducible supercuspidal representations. Second, we prove that the restriction of irreducible representations of G to a smaller subgroup of $\mathcal{K}$ can be described entirely in terms of the trivial representation and certain representations arising from nilpotent orbits in the Lie algebra of G, thereby establishing a new case of a recent conjecture in the literature.

Dedications

I lovingly dedicate this thesis to my dadi, Gulaba ♡, and my nani, Savitri ♡.

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Chapter 1

Introduction

A very classical problem in representation theory is to study how an irreducible representation of a group decomposes upon restriction to its subgroups. Such problems are referred to as branching problems. In the case of real reductive groups, one such problem is to study the restriction to the maximal compact subgroup. The motivation behind this was that, historically, much was known about representations of the maximal compact subgroups of real reductive groups, while comparatively little was known about representations of the group itself. Studying these restrictions therefore helped in gaining a better understanding of representations of the group.

An analogous problem also makes sense in the case of p-adic groups; however, the situation here is essentially reversed. While a great deal is known about representations of p-adic groups, very little is known about representations of their maximal compact subgroups. Thus, by studying branching problems, the hope is to gain a deeper understanding of representations of the group and, at the same time, to obtain new insight into the representation theory of its maximal compact subgroups. What makes the problem even more interesting in the p-adic case is the fact that there are several conjugacy classes of maximal compact subgroups, making it natural to study how restrictions to these different classes are related to one another.

Let F be a non-archimedean local field of residual characteristic $p \neq 2$, and E be an unramified quadratic extension of F. In this thesis, we focus on the unramified quasi-split unitary group $G := \mathrm{U}(1,1)$, viewed as the group of F-points of an E/F-form $\mathbb{G}$ of GL_2. We determine the branching rules for all irreducible supercuspidal representations of G and for its principal series representations upon restriction to a maximal compact open subgroup $\mathcal{K}$. Since G is a rank one group, Jacquet's theorem implies that every irreducible admissible representation of G is either supercuspidal, a principal series representation, or an irreducible subrepresentation of a reducible principal series. Up to twist by a character of G, exactly three principal series representations are reducible. The irreducible constituents occurring in these cases are treated separately in [Tiw25b].

Although G is closely related to $\mathrm{SL}_2(F)$, since the F-points of the derived subgroup of $\mathbb{G}$ is isomorphic to $\mathrm{SL}_2(F)$, one can at best form a ballpark expectation for the decomposition based on the $\mathrm{SL}_2(F)$ case. Writing down the explicit decomposition for G is considerably more delicate, and it is challenging to deduce it directly from the known results for $\mathrm{SL}_2(F)$. The papers of M. Nevins on branching rules for $\mathrm{SL}_2(F)$ [Nev13, Nev05] have served as a guiding framework for this work, and many of the resulting branching rules exhibit strong similarities, with the main difference arising in the depth-zero supercuspidal representations.

In the p-adic setting, to date, branching rules have been established for only a few groups, including $\mathrm{GL}_2(F)$ [Cas73, Han87], $\mathrm{PGL}_2(F)$ [Sil70, Sil77], the Weil representation of $\mathrm{Sp}_{2n}(F)$ [Pra98, MT11], principal series representations of $\mathrm{GL}_3(F)$ [CN09, CN10, OS14], and $\mathrm{SL}_2(F)$ [Nev05, Nev13], minimal representations of split simply connected group of type D_n, $(n \geq 4)$, and E_n, $(n = 6, 7, 8)$ [Sav96], often under the standing assumption that the residual characteristic $p \neq 2$.

A key component of this work is the explicit construction of certain irreducible smooth representations of our fixed maximal compact subgroup $\mathcal{K}$. Since maximal compact subgroups of reductive groups are not themselves linear algebraic groups, comparatively little is known about their representation theory. In fact, full classifications have only been achieved in a few cases: $\mathrm{PGL}_2(\mathcal{O})$ by A. Silberger [Sil70], for $p \neq 2$; $\mathrm{SL}_2(\mathcal{O})$ by J. Shalika [Sha04], again under the assumption $p \neq 2$; $\mathrm{GL}_2(\mathcal{O})$ by A. Stasinski [Sta09], valid for all p; and ramified $\mathrm{U}(1,1)(\mathcal{O})$ by L. G. Frez [GF16], also for $p \neq 2$. Earlier contributions include work of Kloosterman [Klo46a, Klo46b], Tanaka [Tan66, Tan67] and Kutzko (unpublished) on $\mathrm{SL}_2(\mathbb{Z}_p)$ for $p \neq 2$. Nobs–Wolfart also provided a description of representations of $\mathrm{SL}_2(\mathbb{Z}_p)$ for all p [Nob76, NW76]. Nagornyi provided a description for the group $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ [Nag81]. Beyond these complete results, only partial progress is available in higher rank, where families of representations of $\mathrm{GL}_N(\mathcal{O})$ have been constructed by Aubert, Onn, Prasad, Stasinski [AOPS10], Singla [Sin10], Crisp, Meir, Onn [CMO24] and Hill [Hil94, Hil95], though without full classification. We do not attempt to produce *all* irreducible representations of $\mathcal{K}$; however, we do construct a large class of such representations.

To construct irreducible representations of $\mathcal{K}$ of depth $d > 0$, we select certain special elements X in the Lie algebra $\mathfrak{g}$ of G whose centralizer $T(X)$ inside $\mathcal{K}$ is abelian, together with a subgroup $\mathcal{J}_d$ of $\mathcal{K}$ defined in (3.2.3). Each such element X determines a character Ψ_X of $\mathcal{J}_d$ depth d .

Theorem 1. *Let ζ be a character of $T(X)$ whose restriction to $T(X) \cap \mathcal{J}_d$ agrees with Ψ_X. Let $\Psi_{X,\zeta}$ denote the unique extension of Ψ_X and ζ to $T(X)\mathcal{J}_d \subset \mathcal{K}$. Then*

$$\mathcal{S}_d(X,\zeta) \;:=\; \mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \Psi_{X,\zeta}$$

is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$.

A complete proof appears later as Theorem 3.2.6 in Chapter 3.

Our principal result, stated precisely in Theorems 4.3.3, 5.2.12, 6.3.1, and Corollary 6.3.2, gives a complete description of the restriction to $\mathcal{K}$ of all irreducible representations of G, with the exception of the irreducible constituents of the three reducible principal series representations. In terms of the notation introduced above, it may be summarized as follows.

Theorem 2. *Let π be an irreducible supercuspidal representation or a principal series representation of G of depth r with central character θ. Then*

$$\mathrm{Res}_{\mathcal{K}}\pi \;\cong\; \pi_r \;\oplus\; \bigoplus_d \mathcal{S}_d(X,\zeta),$$

where, up to twisting by a character of G,

- *π_r appears only for certain supercuspidal representations and for all principal series representations, where π is of depth r and is either irreducible or a twist of $\mathbb{1}_{\mathcal{K}} \oplus \mathrm{St}_{\mathcal{K}}$; here $\mathbb{1}_{\mathcal{K}}$ and $\mathrm{St}_{\mathcal{K}}$ denote the inflations to $\mathcal{K}$ of the trivial and Steinberg representations of the corresponding unitary group $\mathcal{K}/\mathcal{K}_{0+}$ over the residue field;*
- *the sum is taken over either all even or all odd integers $d \geq r + \frac{1}{2}$ when π is a supercuspidal representations of integral depth, and over all integers $d \geq r + \frac{1}{2}$ otherwise;*
- *when π has depth zero, X is a nilpotent element and $\zeta = \theta$, and when π has positive-depth, X and ζ are twists of the datum used to construct π.*

Since the components occurring in the decomposition have distinct depths and degrees, the decomposition is multiplicity-free.

More recently, there has been growing interest in understanding the decomposition of representations of a group upon restriction to smaller neighborhoods of the identity, with the expectation that such decompositions should involve representations of a simpler nature. Some recent work in this direction includes results of M. Nevins on complex representations of $\mathrm{SL}_2(F)$ for $p \neq 2$ [Nev24], and the analysis of depth-zero supercuspidal representations of $\mathrm{SL}_2(F)$ in characteristic 2, due to Z. Karaganis and M. Nevins [KN25]. Related questions were also studied by G. Henniart and M. F. Vignéras [HV25, HV24], who considered representations of $\mathrm{SL}_2(F)$ and of inner forms of $\mathrm{GL}(n,F)$ over fields $\mathcal{R}$ of characteristic different from p. These ideas led to a set of conjectures, two of which are formulated as [HV24, Question 1.2] and [Nev24, Theorem 1.1], and which we resolve for our unitary group G.

We first answer [HV24, Question 1.2] for all smooth irreducible representations of G. This questions asks, roughly, if there exist a finite set of representations P of G such that for each irreducible representation π of G, up to twisting π by a character of G, we can express $\pi|_{K'}$ as an integral linear combination of restrictions to K' of elements of P in the Grothendiek group of smooth representations of G. To state our solution, we first observe that the higher-depth components appearing in the restriction of any irreducible representation π of G already coincide with those arising from depth-zero irreducible supercuspidal representations with the same central character (Theorems 5.3.1, 6.4.1) after potentially twisting by a quasi-character of G.

Theorem 3. *Let π be an irreducible representation of G of depth r with depth-zero central character θ. Then, for all $d > 2r$,*

$$\mathcal{S}_d(X, \zeta) \cong \mathcal{S}_d(X_d, \theta),$$

where X_d is a nilpotent element.

In fact, we show that one can fix four depth-zero irreducible supercuspidal representations such that the higher-depth components of any irreducible representation π, upon restriction to $\mathcal{K}$, coincide with the higher-depth components, upon restriction to $\mathcal{K}$, of a subset of these four fixed representations, up to twisting by a quasi-character of G (Theorem 5.3.2, 6.4.2). This implies a positive answer to their question, with $K' = G_{x,2r+}$.

The other conjecture, formulated as a theorem for $\mathrm{SL}_2(F)$ in [Nev24, Theorem 1.1], is a slight variation: instead of asking for a set P of representations of G, it asks for a set P' of representations of $\mathcal{K}$ with the property that each one is constructed from the geometry of a nilpotent orbit. We note that our unitary group G admits exactly two non-zero nilpotent G-orbits in its Lie algebra $\mathfrak{g}$, which we denote by $\mathcal{N}_1$ and $\mathcal{N}_\varpi$. For each depth-zero character θ of the center of G and $t \in \mathbb{R}_{\geq 0}$, we define the following highly reducible representations associated with these nilpotent orbits

$$\tau^t_{\mathcal{N}_1}(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{>0},\, d>t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta), \qquad \tau^t_{\mathcal{N}_\varpi}(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{\geq 0}+1,\, d>t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

Our resolution of this conjecture for G is the following decomposition (Theorems 5.3.3, 6.4.3).

Theorem 4. *Let π be an irreducible representation of G of depth r, and let θ denote its central character. Then there exists an integer $n(\pi)$ such that*

$$\mathrm{Res}_{\mathcal{K}_{2r+}}\pi \;=\; n(\pi)\,\mathbb{1} \;\oplus\; a_{\mathcal{N}_1}\,\mathrm{Res}_{\mathcal{K}_{2r+}}\big(\tau^{2r}_{\mathcal{N}_1}(\theta)\big) \;\oplus\; a_{\mathcal{N}_\varpi}\,\mathrm{Res}_{\mathcal{K}_{2r+}}\big(\tau^{2r}_{\mathcal{N}_\varpi}(\theta)\big),$$

where $a_{\mathcal{N}_1}, a_{\mathcal{N}_\varpi} \in \{0,1\}$ and the values of $n(\pi)$ are given in Table 6.1.

This thesis is organized as follows. In Chapter 2, we present the necessary background, including results on p-adic fields. We discuss the structure theory of the group G, where we give explicit descriptions of G and of several important subgroups, including $\mathcal{K}$. In Proposition 2.2.11, we classify all anisotropic tori of G up to conjugacy, which is essential for constructing positive-depth supercuspidal representations of G. Finally, we recall some classical results and definitions from the representation theory of p-adic groups, together with examples adapted to our setting.

In Chapter 3, we construct certain irreducible representations of the maximal compact subgroup $\mathcal{K}$ in Theorem 3.2.6, adapting Shalika's method [Sha04] to the setting of G. The key tool used in this construction is Clifford theory. We then classify the G- and $\mathcal{K}$-orbits of nilpotent elements in the Lie algebra of G and use these elements to construct irreducible representations of $\mathcal{K}$. Later, in §5.3 of Chapter 5, we associate a certain highly reducible representation to each nilpotent $\mathcal{K}$ orbit and use it to describe the restriction of representations to a smaller subgroup of $\mathcal{K}$.

In order to study the branching, we divide the representations into three classes: the depth-zero irreducible supercuspidal representations, the positive-depth irreducible supercuspidal representations, and the principal series representations. Together, these irreducible representations generate the category of smooth representations of G. For each class, we will see that the tools used to prove the irreducibility of the components occurring in the decomposition are quite different.

In Chapter 4, we determine the branching laws for our first class of representations, the depth-zero supercuspidal representations. We first provide an explicit description of all such representations using the construction due to Moy and Prasad [MP94, MP96] and independently due to Morris [Mor99]. Using Mackey theory, we then write down their decomposition upon restriction to $\mathcal{K}$. We prove that each component occurring in this decomposition, which we refer to as its Mackey components, is irreducible and has distinct depth and degree (Theorem 4.3.3). Consequently, the decomposition is multiplicity-free. The irreducibility of the Mackey components is established by showing that they intertwine with irreducible representations (Lemma 3.3.2) of $\mathcal{K}$ of the same degree, constructed using nilpotent elements in the Lie algebra of G. This intertwining is verified using character theory. We also note that all components in the decomposition of a given representation are irreducible and have distinct depth and degree.

In Chapter 5, we discuss the branching laws for positive-depth irreducible supercuspidal representations of G. As in the depth-zero case, we first provide an explicit description of all such representations (Theorem 5.1.3), this time using the Adler–Fintzen–Yu method [Adl98, Yu01, Fin21a]. We then restrict these representations to $\mathcal{K}$, and Mackey theory again yields several components. We prove that each of these Mackey components is irreducible and has distinct depth and degree, and thus the decomposition is again multiplicity free (Theorems 5.2.1, 5.2.12). To establish irreducibility, we follow the same strategy as in Chapter 4, showing that these components intertwine with irreducible representations of the same degree constructed in Theorem 3.2.6. This time, the intertwining is verified by observing that the data defining the $\mathcal{K}$-representations appearing in the decomposition of an irreducible representation π are simply twists of the datum used to construct π itself. This provides an explicit decomposition in terms of representations of $\mathcal{K}$. We also show that the higher-depth components occurring in the decomposition coincide with the higher-depth components appearing in the depth-zero irreducible supercuspidal representations with the same central character (Theorem 5.3.1, Theorem 5.3.2). Finally, we restrict these representations to a smaller subgroup of $\mathcal{K}$ and show that, up to a number $n(\pi)$ of copies of the trivial representation, the decomposition is expressed entirely in terms of the representations we constructed in §3.3 of Chapter 3. (Theorem 5.3.3).

In Chapter 6, we discuss the branching rules for principal series representations of G. Unlike the previous two classes of representations, where Mackey theory played a crucial role, in this case it yields only a single, highly reducible component. Therefore, we return to the definition of smooth representations and deduce certain key properties, which helps in obtaining the desired decomposition. For this class, we obtain a canonical decomposition into irreducible $\mathcal{K}$-representations without making use of the representations constructed in Theorem 3.2.6 (Theorem 6.2.6). However, the irreducible components in this decomposition arise as quotients of certain induced representations. Therefore, to make the decomposition explicit, we show that each such component is in fact isomorphic to a representation constructed in Theorem 3.2.6. It was challenging to identify these representations, but once found, the explicit decomposition is remarkably satisfying to observe (Theorem 6.3.1 and Corollary 6.3.2). As in the previous chapters, we observe the same phenomenon: the higher-depth components coincide with those appearing in the depth-zero supercuspidal representations with the same central character (Theorem 6.4.1, Theorem 6.4.2). Finally, we consider the restriction to a smaller subgroup of $\mathcal{K}$ and obtain an analogous decomposition, governed by the trivial representation and representations of $\mathcal{K}$ constructed in §5.3.2. (Theorem 6.4).

As a beginner in this area, working on this problem became my training ground. The indices and double coset representatives that at first glance seemed impossible to compute gradually became routine calculations as the project progressed. Working explicitly with objects that are infinite dimensional has been challenging, yet very satisfying. With the exception of Chapter 2, the material presented in this thesis is original. The results of this thesis may be viewed as a first step toward understanding branching problems for higher-rank unitary groups, with a natural next objective being the analysis of branching rules for $\mathrm{U}(2,1)$. There is an intermediate step as well, namely, to solve the branching problem for the ramified $\mathrm{U}(1,1)$, which we hope to accomplish by adapting the results obtained in the unramified setting.

Chapter 2

Background

In this chapter, we present the necessary background for the subsequent chapters. We begin with some basic facts and lemmas about non-archimedean local fields of residual characteristic p, which are $\mathbb{Q}_p$, finite extensions of $\mathbb{Q}_p$, and the field of Laurent series $\mathbb{F}_q((t))$ where q is a power of p. We assume that $p \neq 2$, and for convenience, we refer to all such fields as p-adic fields. In Section 2.2, we discuss aspects of the structure theory of p-adic groups, specializing to the case of the unramified quasi-split unitary group $\mathrm{U}(1,1)$. We conclude the chapter by recalling some basic definitions and results from the representation theory of p-adic groups.

2.1 Preliminaries on p-adic fields

Let F be a p-adic field of residual characteristic $p \neq 2$. Let $E = F[\sqrt{\epsilon}]$ be the quadratic unramified extension of F, where ϵ is a non-square element of $\mathcal{O}_F^\times$. We fix ϖ a uniformizer of F and normalize the valuation ν on F so that $\nu(\varpi) = 1$. The valuation ν is extended to E, and we use the same symbol ν for this extension. We define the *absolute value* (also called the analytic norm) associated with ν, denoted by $|\cdot|$, as

$$|x| := q^{-\nu(x)}, \quad x \in E,$$

where $q = |\mathcal{O}_F/\mathfrak{p}_F|$ is the cardinality of the residue field of F.

We denote the ring of integers of F by $\mathcal{O}_F$ and that of E by $\mathcal{O}_E$. Their respective maximal ideals are denoted by $\mathfrak{p}_F$ and $\mathfrak{p}_E$. The residue fields of F and E are given by

$$\mathfrak{f} \cong \mathcal{O}_F/\mathfrak{p}_F \quad \text{and} \quad \mathfrak{e} \cong \mathcal{O}_E/\mathfrak{p}_E,$$

respectively. Since the cardinality of $\mathfrak{f}$ is q, the cardinality of $\mathfrak{e}$ is q^2.

Let σ denote the nontrivial element of $\mathrm{Gal}(E/F)$. For $x \in E$, we write $\overline{x} := \sigma(x)$. The *norm* and *trace* from E to F are defined by

$$\mathrm{N}_{E/F}(x) = x\,\overline{x}, \quad \text{for all } x \in E^\times,$$

$$\mathrm{Tr}_{E/F}(x) = x + \overline{x}, \quad \text{for all } x \in E.$$

We denote by $E^1 \subset E^\times$ the subgroup consisting of elements of norm one, *i.e.*,

$$E^1 := \{x \in E^\times \mid \mathrm{N}_{E/F}(x) = 1\}.$$

By [Chi21, Proposition 2.3.8], the norm map induces a surjection

$$\mathrm{N}_{E/F} \colon \mathcal{O}_E^\times \twoheadrightarrow \mathcal{O}_F^\times, \tag{2.1.1}$$

and by [CF67, Proposition 3] this filters to a surjection

$$\mathrm{N}_{E/F} \colon 1 + \mathfrak{p}_E^d \twoheadrightarrow 1 + \mathfrak{p}_F^d \tag{2.1.2}$$

for every $d \in \mathbb{Z}_{\geq 1}$.

Proposition 2.1.1. *[LSp] For any integer $d \geq 1$, one has*

$$\left[E^1 : E^1 \cap (1 + \mathfrak{p}_E^d)\right] = (q+1)q^{d-1}.$$

Proof. By (2.1.1),(2.1.2), the norm map induces a surjection

$$\mathrm{N}_{E/F} \colon \mathcal{O}_E^\times/(1 + \mathfrak{p}_E^d) \twoheadrightarrow \mathcal{O}_F^\times/(1 + \mathfrak{p}_F^d),$$

whose kernel is $E^1/(E^1 \cap (1 + \mathfrak{p}_E^d))$. Therefore,

$$\left[E^1 : E^1 \cap (1 + \mathfrak{p}_E^d)\right] = \frac{[\mathcal{O}_E^\times : 1 + \mathfrak{p}_E^d]}{[\mathcal{O}_F^\times : 1 + \mathfrak{p}_F^d]}.$$

Since $|\mathcal{O}_E^\times/(1 + \mathfrak{p}_E)| = |\mathfrak{e}| - 1 = q^2 - 1$, and $|(1 + \mathfrak{p}_E)/(1 + \mathfrak{p}_E^d)| = |\mathfrak{p}_E/\mathfrak{p}_E^d| = q^{2(d-1)}$, we have

$$[\mathcal{O}_E^\times : 1 + \mathfrak{p}_E^d] = (q^2 - 1)q^{2(d-1)}, \quad [\mathcal{O}_F^\times : 1 + \mathfrak{p}_F^d] = (q-1)q^{d-1},$$

and hence

$$\left[E^1 : E^1 \cap (1 + \mathfrak{p}_E^d)\right] = \frac{(q^2 - 1)q^{2(d-1)}}{(q-1)q^{d-1}} = (q+1)q^{d-1}.$$

∎

Lemma 2.1.2. *If $x \neq 0$ is in the image of the norm map $N_{E/F}$, then x has even valuation.*

Proof. Suppose $x \in \mathrm{Im}(N_{E/F}(E^\times))$. Then there exists $y \in E^\times$ such that

$$x = N_{E/F}(y) = y\overline{y}.$$

Since $\nu(\overline{y}) = \nu(y)$, it follows that

$$\nu(x) = \nu(y\overline{y}) = \nu(y) + \nu(\overline{y}) = 2\nu(y),$$

which is even. ∎

We now provide a set of coset representatives for the quotient $F^\times/N_{E/F}(E^\times)$. Note that $F^\times/F^{\times 2} \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{O}_F^\times/\mathcal{O}_F^{\times 2}$; since $p \neq 2$, this has four elements. A standard choice of representatives is $\{1, \epsilon, \varpi, \epsilon\varpi\}$, where ϵ is a nonsquare unit of $\mathcal{O}_F$. By local class field theory ([CF67, Chapter VI, Theorem 2]), this quotient has index 2. For completeness, we also give a direct proof in the lemma below.

Lemma 2.1.3. *A set of coset representatives for $F^\times/N_{E/F}(E^\times)$ is given by $\{1, \varpi\}$.*

Proof. Since every non-zero element in the image of the norm map has even valuation and $\nu(\varpi) = 1$, we obtain that $\varpi \notin \mathrm{Im}(N_{E/F})$ and hence the norm map $N_{E/F} \colon E^\times \to F^\times$ is not surjective. This gives the chain of inclusions

$$F^{\times 2} \subsetneq N_{E/F}(E^\times) \subsetneq F^\times.$$

Passing to the quotient by $F^{\times 2}$, we obtain

$$\{1\} \subsetneq N_{E/F}(E^\times)/F^{\times 2} \subsetneq F^\times/F^{\times 2}.$$

Since $\epsilon \in N_{E/F}(E^\times)$, it follows that $N_{E/F}(E^\times)/F^{\times 2} = \{1, \epsilon\}$. Therefore,

$$|F^\times/N_{E/F}(E^\times)| = \frac{|F^\times/F^{\times 2}|}{|N_{E/F}(E^\times)/F^{\times 2}|} = \frac{4}{2} = 2.$$

Since $\varpi \notin N_{E/F}(E^\times)$, the set $\{1, \varpi\}$ forms a complete set of coset representatives for $F^\times/N_{E/F}(E^\times)$. ∎

We conclude this section with an observation. If $u \in \sqrt{\epsilon}\,\mathfrak{p}_F$, then

$$\nu(1 - u^2) = \min\{\nu(1), \nu(u^2)\} = 0,$$

which shows that $1 - u^2 \in \mathcal{O}_F^\times$. In fact, the following stronger statement holds. The next lemma provides a proof.

Lemma 2.1.4. *If $u \in \sqrt{\epsilon}\mathcal{O}_F$, then $(1 - u^2) \in \mathcal{O}_F^\times$.*

Proof. Let $u = \sqrt{\epsilon}x$, where $x \in \mathcal{O}_F$. Then we have $u^2 = \epsilon x^2$. If $(1 - u^2) \equiv 0 \mod \mathfrak{p}_F$, then $u^2 \equiv \epsilon x^2 \equiv 1 \mod \mathfrak{p}_F$, which is not possible as ϵ is not a square in the residue field. Therefore $(1 - u^2) \in \mathcal{O}_F^\times$. ∎

2.2 Some structure theory

We adopt the convention of using blackboard bold font to denote an algebraic group $\mathbb{G}$ defined over F, and the corresponding roman letter for its group of F-rational points, that is, $G = \mathbb{G}(F)$. For simplicity of notation, we may refer to a group T as a "maximal torus of

G" when we mean that T is the group of F-rational points of an algebraic torus $\mathbb{T} \subset \mathbb{G}$ that is defined over F. Given subsets L_i of F or E, we use the shorthand notation

$$\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \ a_i \in L_i \right\}.$$

For $g \in G$ and a subgroup $K \subset G$, we write $K^g := gKg^{-1}$ and ${}^gK := g^{-1}Kg$.

In this section, we provide an explicit description of the group $G := \mathrm{U}(1,1)$, along with its Lie algebra and certain important subgroups. We will also classify the maximal F-split tori and maximal anisotropic tori in G up to conjugacy. In addition, we determine representatives for certain double cosets that will play a significant role in subsequent chapters.

2.2.1 The group $\mathrm{U}(1,1)$

Let E be the unramified quadratic extension of F. The quasi-split unramified unitary group $G = \mathrm{U}(1,1)$ is the group of F-points of an E/F-form $\mathbb{G}$ of GL_2 defined by

$$G := \mathbb{G}(F) = \left\{ g \in \mathrm{GL}_2(E) \mid \overline{g}^{\top} \mathrm{w} g = \mathrm{w} \right\},$$

where $\mathrm{w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\overline{g}^{\top} = (\overline{g_{ij}})^{\top} = (\overline{g_{ji}})$. More explicitly, by choosing coordinates G has the form

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(E) \ \middle| \ \begin{matrix} \overline{a}d + \overline{c}b = 1, \\ \overline{a}c, \overline{b}d \in \sqrt{\epsilon}F \end{matrix} \right\}. \tag{2.2.1}$$

Note that this endows $\mathbb{G}$ with the structure of a group scheme over $\mathcal{O}_F$. The F-points of the Lie algebra of $\mathbb{G}$, denoted by $\mathfrak{g}$, is $\{X \in \mathfrak{gl}_2(E) \mid \overline{X}^{\top}\mathrm{w} + \mathrm{w}X = 0\}$ and thus has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} u + \sqrt{\epsilon}v & y\sqrt{\epsilon} \\ z\sqrt{\epsilon} & -u + \sqrt{\epsilon}v \end{pmatrix} \middle| \ u, v, y, z \in F \right\}.$$

The center of G is the subgroup Z of scalar matrices, which we identify with E^1. Let $\mathbb{G}_{\mathrm{der}}$ denote the derived subgroup of $\mathbb{G}$, and we denote its group of F-points by $G_{\mathrm{der}} = \mathrm{SU}(1,1)$. Explicitly,

$$G_{\mathrm{der}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \ \middle| \ ad - bc = 1 \right\}.$$

The F-points of the Lie algebra of $\mathbb{G}_{\mathrm{der}}$ is

$$\mathfrak{g}_{\mathrm{der}} = \left\{ \begin{pmatrix} x & y\sqrt{\epsilon} \\ z\sqrt{\epsilon} & -x \end{pmatrix} \middle| x, y, z \in F \right\}.$$

Lemma 2.2.1. *We have* $G_{\mathrm{der}} = \left\{ \begin{pmatrix} a & b\sqrt{\epsilon} \\ c\sqrt{\epsilon} & d \end{pmatrix} \ \middle| \ a, b, c, d \in F, ad - bc\epsilon = 1 \right\}.$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_{\mathrm{der}}$, with $a, b, c, d \in E$ satisfying the four conditions

$$ad - bc = 1, \quad \overline{a}d + \overline{c}b = 1, \quad \overline{a}c \in \sqrt{\epsilon}F, \quad \overline{b}d \in \sqrt{\epsilon}F.$$

If $c \in F^\times$, then the third condition implies that $a \in \sqrt{\epsilon}F$, so $a + \overline{a} = 0$. Since $\overline{c} = c$, adding the first two conditions gives $(a + \overline{a})d = 2$, a contradiction.

We may write $c = c_0 + c_1\sqrt{\epsilon}$, with $c_0, c_1 \in F$ and $c_1 \neq 0$. Suppose for the sake of contradiction that $c_0 \neq 0$. Then the third condition yields

$$a = \frac{a_1}{c_0}\sqrt{\epsilon}c$$

where we have written $a = a_0 + a_1\sqrt{\epsilon}$ with $a_0, a_1 \in F$. Adding the first two conditions yields $\overline{a}(d + \overline{d}) + \overline{c}(b - \overline{b}) = 2$. When we factor out $\overline{c}$ from the expression on the left, we obtain

$$\overline{c}\left(-\frac{a_1}{c_0}\sqrt{\epsilon}(d + \overline{d}) + (b - \overline{b})\right) = 2.$$

Since the term in parentheses lies in $\sqrt{\epsilon}F$, this contradicts our assumption that $c_0 \neq 0$.

Thus $c \in \sqrt{\epsilon}F$. If $c = 0$, then the first two relations together yield $a \in F^\times$. If $c \neq 0$, then the third condition forces $a \in F$.

Recall that $\mathrm{w} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G$. Since G_{der} is stable under conjugation by w, we have

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} = \mathrm{w}g\mathrm{w}^{-1},$$

so $b \in \sqrt{\epsilon}F$ and $d \in F$, as required. ∎

Lemma 2.2.2. *The derived subgroup G_{der} is isomorphic to $\mathrm{SL}_2(F)$.*

Proof. We define a map

$$\Xi\colon \mathrm{SL}_2(F) \longrightarrow G_{\mathrm{der}}$$

by

$$\Xi(g) = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} \sqrt{\epsilon}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } g \in \mathrm{SL}_2(F). \tag{2.2.2}$$

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F)$. Then

$$\Xi(g) = \begin{pmatrix} a & b\sqrt{\epsilon} \\ c\epsilon^{-1}\sqrt{\epsilon} & d \end{pmatrix}.$$

Since $c\epsilon^{-1}\sqrt{\epsilon} = (c/\epsilon)\sqrt{\epsilon} \in \sqrt{\epsilon}F$, and $ad - bc\epsilon^{-1}\epsilon = 1$, it follows that $\Xi(g) \in G_{\mathrm{der}}$, and hence Ξ is well-defined. It is immediate from the definition that Ξ is a group homomorphism. Furthermore, the kernel of Ξ is trivial, so Ξ is injective.

To show surjectivity, let

$$h = \begin{pmatrix} a & b\sqrt{\epsilon} \\ c\sqrt{\epsilon} & d \end{pmatrix} \in G_{\mathrm{der}},$$

with $a, b, c, d \in F$ and $ad - bc\epsilon = 1$. Then

$$g := \begin{pmatrix} a & b \\ c\epsilon & d \end{pmatrix} \in \mathrm{SL}_2(F),$$

and a direct computation shows that $\Xi(g) = h$, establishing that Ξ is surjective. Therefore Ξ is an isomorphism of groups. ∎

Lemma 2.2.3. *The quotient group G/ZG_{der} has order two and the nontrivial coset of ZG_{der} in G is represented by the matrix*

$$\xi := \begin{pmatrix} \varepsilon_E & 0 \\ 0 & \overline{\varepsilon_E}^{-1} \end{pmatrix},$$

where $\varepsilon_E \in \mathcal{O}_E^\times$ represents the nontrivial class of $\mathcal{O}_E^\times/(\mathcal{O}_E^\times)^2$.

Proof. If $g \in G$, then $\overline{g}^T \mathrm{w} g = \mathrm{w}$ and it follows that $\det(g) \in E^1$. The determinant of any element of ZG_{der} lies in $(E^1)^2$.

By Hilbert's Theorem 90 [Bou03, V.11.6 Theorem 3], the map $x \mapsto x\overline{x}^{-1}$ induces an isomorphism $\mathcal{O}_E^\times/\mathcal{O}_F^\times \to E^1$, so in particular $(E^1)^2$ is the image of the subgroup $(\mathcal{O}_E^\times)^2\mathcal{O}_F^\times/\mathcal{O}_F^\times$. Since every element of $\mathcal{O}_F^\times$ is a square of an element of $\mathcal{O}_E^\times$ (since E/F is unramified), it follows that $E^1/(E^1)^2 \cong \mathcal{O}_E^\times/\mathcal{O}_E^{\times 2}$. We conclude that $E^1/(E^1)^2$ has order 2, and that the nontrivial coset is represented by $\gamma = \varepsilon_E\overline{\varepsilon_E}^{-1} \in E^1$. Now the matrix ξ lies in G and has determinant $\gamma \in E^1 \setminus (E^1)^2$. It follows that

$$G = ZG_{\mathrm{der}} \sqcup \xi ZG_{\mathrm{der}}. \quad ∎$$

Similar to the case of $\mathrm{SL}_2(F)$, the group G also has two conjugacy classes of maximal compact subgroups [MÍ1, §3.2.2], represented by

$$\mathcal{K} = \mathbb{G}(\mathcal{O}_F) = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

and

$$\mathcal{K}^\eta = \begin{pmatrix} \mathcal{O}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathcal{O}_E \end{pmatrix} \cap G,$$

where $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right) \in \mathrm{GL}_2(E)$ is such that it normalizes our unitary group G.

Lemma 2.2.4. *Let $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{K}$. Then $x, w \in \mathcal{O}_E^\times$ or $y, z \in \mathcal{O}_E^\times$.*

Proof. Note that $\det(k) \in \mathcal{O}_E^\times$ for all $k \in \mathcal{K}$, as otherwise the inverse wouldn't be in the group. Therefore, if $k := \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{K}$, then we have

$$0 = \nu(xw - yz) \geq \min\{\nu(xw), \nu(yz)\} \geq 0.$$

Thus at least one of xw or yz must have valuation 0, and hence $x, w \in \mathcal{O}_E^\times$ or $y, z \in \mathcal{O}_E^\times$. ∎

We now describe certain double coset representatives, which will be of use in the arguments developed in the subsequent chapters.

Lemma 2.2.5. *A set of double coset representatives for $\mathcal{K}\backslash G/\mathcal{K}$, $\mathcal{K}\backslash G/\mathcal{K}^\eta$, and $\mathcal{K}^\eta\backslash G/\mathcal{K}^\eta$ is*

$$\left\{\alpha^t := \begin{pmatrix} \varpi^{-t} & 0 \\ 0 & \varpi^t \end{pmatrix} \;\middle|\; t \geq 0\right\}.$$

Proof. By the Cartan decomposition [BT72, Proposition 4.2.1], where T consists of elements of the form $\operatorname{diag}(u\varpi^t, \overline{u}^{-1}\varpi^{-t})$ with $u \in \mathcal{O}_E^\times$ and $t \in \mathbb{Z}$. For any such element,

$$\begin{pmatrix} u\varpi^t & 0 \\ 0 & \overline{u}^{-1}\varpi^{-t} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} \varpi^t & 0 \\ 0 & \varpi^{-t} \end{pmatrix} \in \mathcal{K}\,\alpha^t\,\mathcal{K},$$

so every double coset $\mathcal{K}g\mathcal{K}$ meets some α^t with $t \in \mathbb{Z}$.

Recall that $\mathrm{w} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{K}$. Then $\mathrm{w}\alpha^t\mathrm{w}^{-1} = \alpha^{-t}$, so α^t and α^{-t} lie in the same coset. Thus we may assume $t \geq 0$. Using Lemma 2.2.4, an entrywise case analysis of $\mathcal{K}\alpha^t\mathcal{K}$ shows that the minimal valuation among its entries is $-t$, and this minimum is attained by at least one entry; hence distinct t yield distinct double cosets. Hence $\{\alpha^t\}_{t\geq 0}$ is a complete set of representatives for $\mathcal{K}\backslash G/\mathcal{K}$. Since η commutes with α^t, conjugating the Cartan decomposition of G with respect to $\mathcal{K}$ by η yields $G = \bigsqcup_{t\geq 0} \mathcal{K}^\eta\alpha^t\mathcal{K}^\eta$. A similar argument gives $G = \bigsqcup_{t\geq 0} \mathcal{K}\alpha^t\mathcal{K}^\eta$. ∎

Let $\mathcal{B}$ denote the subgroup of upper triangular matrices in G intersected with $\mathcal{K}$, and let $\mathcal{B}^{\mathrm{op}}$ denote the subgroup of lower triangular matrices in G intersected with $\mathcal{K}$. Then

$$\mathcal{B}^{\mathrm{op}} = \left\{ \begin{pmatrix} x & 0 \\ z & \overline{x}^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} \overline{x}z \in \sqrt{\epsilon}\mathcal{O}_F, \\ x \in \mathcal{O}_E^\times, z \in \mathcal{O}_E \end{array} \right\},$$

and $\mathcal{B} = \mathcal{B}^{\mathrm{op}\top}$.

Lemma 2.2.6. *A set of right coset representatives for $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}$ is given by*

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} \;\middle|\; u \in \sqrt{\epsilon}\mathfrak{p}_F, v \in \sqrt{\epsilon}\mathcal{O}_F \right\}.$$

Proof. Let $k := \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{K}$. If $y = 0$, then k belongs to the identity coset $\mathcal{B}^{\mathrm{op}} I$. If $y \neq 0$ but $y \in \mathfrak{p}_E$, then Lemma 2.2.4 yields $x \in \mathcal{O}_E^\times$. Since $x^{-1} = \frac{\overline{x}}{x\overline{x}}$ and $\overline{x}z \in \sqrt{\epsilon}F$, we conclude that $x^{-1}z \in \sqrt{\epsilon}F$. We also have $\overline{x}w + \overline{z}y = 1$, implying that $w = \overline{x}^{-1} - \overline{x}^{-1}\overline{z}y = \overline{x}^{-1} + x^{-1}zy$.

Hence we can express k as

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & \overline{x}^{-1} \end{pmatrix} \begin{pmatrix} 1 & yx^{-1} \\ 0 & 1 \end{pmatrix}.$$

Observe that if $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{K}$, then $\begin{pmatrix} x & 0 \\ z & \overline{x}^{-1} \end{pmatrix} \in \mathcal{K}$, thus $\begin{pmatrix} 1 & yx^{-1} \\ 0 & 1 \end{pmatrix}$ has to be an element of $\mathcal{K}$, and therefore it follows that $x^{-1}y \in \sqrt{\epsilon}\mathfrak{p}_F$. Lastly, if $y \in \mathcal{O}_E^\times$, then we can express k as

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} y & 0 \\ w & \overline{y}^{-1} \end{pmatrix} \begin{pmatrix} xy^{-1} & 1 \\ 1 & 0 \end{pmatrix},$$

and $xy^{-1} \in \sqrt{\epsilon}\mathcal{O}_F$. Thus we have shown that each element of k either lies in the right coset space $\mathcal{B}^{\mathrm{op}} \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with $u \in \sqrt{\epsilon}\mathfrak{p}_F$ or in the right coset space $\mathcal{B}^{\mathrm{op}} \left(\begin{smallmatrix} v & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $v \in \sqrt{\epsilon}\mathcal{O}_F$.

To verify disjointness, consider $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & u' \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{K}$ with $u, u' \in \sqrt{\epsilon}\mathfrak{p}_F$. Then $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u' \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & u-u' \\ 0 & 1 \end{smallmatrix}\right)$. Hence they represent same right coset in $B^{\mathrm{op}}\backslash\mathcal{K}$ if and only if $u = u'$. Similarly, $\left(\begin{smallmatrix} v & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} v' & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $v, v' \in \sqrt{\epsilon}\mathcal{O}_F$, represent same right coset in $B^{\mathrm{op}}\backslash\mathcal{K}$ if and only if $v = v'$. Now consider $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} v & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{K}$, where $u \in \sqrt{\epsilon}\mathfrak{p}_F$, and $v \in \sqrt{\epsilon}\mathcal{O}_F$. Then we have $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} v & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} u & 1-uv \\ 1 & -v \end{smallmatrix}\right)$. Hence they will represent same right coset if and only if $u = v^{-1}$. This happens only when $u, v \in \sqrt{\epsilon}\mathcal{O}_F^\times$. This concludes the proof. ∎

Lemma 2.2.7. *A set of representatives for the right cosets in $\mathcal{B}\backslash\mathcal{K}$ is given by*

$$\left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} \;\middle|\; u \in \sqrt{\epsilon}\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\mathcal{O}_F \right\},$$

and a set of right coset representatives for $\mathcal{B}^\eta\backslash\mathcal{K}^\eta$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ u\varpi & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & v \end{pmatrix} \;\middle|\; u \in \sqrt{\epsilon}\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\mathcal{O}_F \right\}.$$

Proof. By Lemma 2.2.6, the set

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} \;\middle|\; u \in \sqrt{\epsilon}\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\mathcal{O}_F \right\}$$

is a complete set of representatives for the right cosets in $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}$. Let $\mathrm{w} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{K}$. Then conjugation by w defines an isomorphism between $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}$ and $\mathcal{B}\backslash\mathcal{K}$, since $\mathcal{B}^{\mathrm{op}} = \mathrm{w}\mathcal{B}\mathrm{w}^{-1}$. This map acts on the coset representatives by

$$\mathrm{w}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\mathrm{w}^{-1} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad \mathrm{w}\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}\mathrm{w}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix},$$

where $u \in \sqrt{\epsilon}\mathfrak{p}_F$ and $v \in \sqrt{\epsilon}\mathcal{O}_F$. Therefore, the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} \,\middle|\, u \in \sqrt{\epsilon}\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\mathcal{O}_F \right\}$$

forms a complete and disjoint set of right coset representatives for $\mathcal{B}\backslash\mathcal{K}$.

Similarly, by definition $\mathcal{K}^\eta = \eta\mathcal{K}\eta^{-1}$ and $\mathcal{B}^\eta = \eta\mathcal{B}\eta^{-1}$. Conjugation by η therefore induces an isomorphism between $\mathcal{B}\backslash\mathcal{K}$ and $\mathcal{B}^\eta\backslash\mathcal{K}^\eta$. This map acts on the coset representatives by

$$\eta \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \eta^{-1} = \begin{pmatrix} 1 & 0 \\ u\varpi & 1 \end{pmatrix}, \quad \eta \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} \eta^{-1} = \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & v \end{pmatrix},$$

where $u \in \sqrt{\epsilon}\mathfrak{p}_F$ and $v \in \sqrt{\epsilon}\mathcal{O}_F$. Therefore, the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ u\varpi & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \varpi^{-1} \\ \varpi & v \end{pmatrix} \,\middle|\, u \in \sqrt{\epsilon}\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\mathcal{O}_F \right\}$$

forms a complete and disjoint set of right coset representatives for $\mathcal{B}^\eta\backslash\mathcal{K}^\eta$. ∎

Proposition 2.2.8. *A set of double coset representatives for $\mathcal{B}\backslash\mathcal{K}/\mathcal{B}$ is*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^k & 1 \end{pmatrix} \,\middle|\, k \geq 1 \right\}.$$

Proof. By Lemma 2.2.7, a set of right coset representatives for $\mathcal{B}\backslash\mathcal{K}$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} \,\middle|\, u \in \sqrt{\epsilon}\,\mathfrak{p}_F,\ v \in \sqrt{\epsilon}\,\mathcal{O}_F \right\}.$$

It remains to determine which of these yield distinct double cosets in $\mathcal{B}\backslash\mathcal{K}/\mathcal{B}$.

First, we claim that for all $v \in \sqrt{\epsilon}\,\mathcal{O}_F$ and all $u \in \sqrt{\epsilon}\,\mathcal{O}_F^\times$ we have

$$\begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in \mathcal{B}\,\mathrm{w}\,\mathcal{B}.$$

Indeed, given $v \in \sqrt{\epsilon}\,\mathcal{O}_F$ and any $x \in \mathcal{O}_E^\times$,

$$\begin{pmatrix} 0 & 1 \\ 1 & v \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & \overline{x}^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{x} & v\,\overline{x} \\ 0 & x^{-1} \end{pmatrix} \in \mathcal{B}\,\mathrm{w}\,\mathcal{B}.$$

Next, suppose

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & \overline{x}^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & t \\ 0 & \overline{z}^{-1} \end{pmatrix}, \qquad x, z \in \mathcal{O}_E^\times,\ y, t \in \mathcal{O}_E.$$

Comparing the $(2,1)$-entries gives $u = \overline{x}^{-1}z \in \mathcal{O}_E^\times$, whence $u \in \sqrt{\epsilon}\,\mathcal{O}_F^\times$. Thus $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in \mathcal{B}\,\mathrm{w}\,\mathcal{B}$ if and only if $u \in \sqrt{\epsilon}\,\mathcal{O}_F^\times$.

Now let $u \in \sqrt{\epsilon}\,\mathfrak{p}_F \setminus \{0\}$. Write $u = \sqrt{\epsilon}\,m\,\varpi^k$ with $m \in \mathcal{O}_F^\times$ and $k = \nu(u) \geq 1$. Since the norm map $\mathcal{O}_E^\times \to \mathcal{O}_F^\times$ is surjective, choose $z \in \mathcal{O}_E^\times$ with $z\overline{z} = m$. Then

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ 0 & \overline{z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^k & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \overline{z}^{-1} \end{pmatrix},$$

so any such element is $\mathcal{B}$-conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^k & 1 \end{smallmatrix}\right)$.

Finally, we show that the double cosets represented by $\left(\begin{smallmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^k & 1 \end{smallmatrix}\right)$ are pairwise distinct as k varies. Suppose for contradiction that $k > l \geq 1$ and

$$\begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^k & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & \overline{x}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\,\varpi^l & 1 \end{pmatrix} \begin{pmatrix} z & t \\ 0 & \overline{z}^{-1} \end{pmatrix}, \qquad x, z \in \mathcal{O}_E^\times,\ y, t \in \mathcal{O}_E.$$

Comparing $(2,1)$-entries yields $\overline{x}^{-1}z = \varpi^{k-l} \in \mathfrak{p}_F$, which is impossible since $\overline{x}^{-1}z \in \mathcal{O}_E^\times$. Hence the double cosets are distinct. ∎

Finally, we conclude this section with a lemma that lets us build new double coset representatives from known ones; we will apply it in later chapters.

Lemma 2.2.9. *Let $\mathcal{G}$ be a group and let K, J be subgroups of $\mathcal{G}$ such that $J \subseteq K$. If B is a set of double coset representatives of $K\backslash\mathcal{G}/K$, and for all $g \in B$, A_g is a set of double coset representatives for $(K \cap K^{g^{-1}})\backslash K/J$, then $\{gh \mid g \in B, h \in A_g\}$ is a set of double coset representatives for $K\backslash\mathcal{G}/J$.*

Proof. Let $x \in \mathcal{G}$. Then $x = k_1 g k_2$ where $k_1, k_2 \in K$, and $g \in B$. Since $k_2 \in K$, there exist $k \in K \cap K^{g^{-1}}$, $h \in A_g$, and $j \in J$ such that $k_2 = khj$. Since $gkg^{-1} \in K$, we obtain that

$$x = k_1 gkhj = \underbrace{k_1 gkg^{-1}}_{\in K} ghj.$$

Hence gh is a double coset representative for x in $K\backslash\mathcal{G}/J$. Next we want to show that all elements of $\{gh \mid g \in B, h \in A_g\}$ represent distinct double cosets in $K\backslash\mathcal{G}/J$. Let $g_1, g_2 \in B$, $h_1 \in A_{g_1}$, and $h_2 \in A_{g_2}$. Let us assume that g_1h_1 and g_2h_2 represent the same double coset in $K\backslash G/J$, *i.e.*, $Kg_1h_1J = Kg_2h_2J$. Since $h_1, h_2 \in K$ and $J \subset K$, g_1 and g_2 represent the same double coset in $K\backslash\mathcal{G}/K$, therefore, $g_1 = g_2$. Thus we have

$$Kg_1h_1J = Kg_1h_2J \implies h_1 = \underbrace{g_1^{-1}kg_1}_{\in K^{g_1^{-1}}} h_2 j$$

for some $k \in K$ and $j \in J$. Since $h_2 j, h_1 \in K$, we have that $g_1^{-1}kg_1 \in K \cap K^{g_1^{-1}}$, implying that h_1 and h_2 represent the same double coset in $K \cap K^{g_1^{-1}}\backslash K/J$, and therefore $h_1 = h_2$ as was required. ∎

2.2.2 Moy–Prasad filtrations of G

In this subsection, we introduce the Moy–Prasad filtrations of parahoric subgroups of G and recall the structure of the building of G. Our interest lies in the explicit description of these filtrations for our unitary group $G = \mathrm{U}(1,1)$, so most of the definitions are presented in a specialized form adapted to this case. For the general theory, standard references include [KP23, MP94, MP96, BT72].

Let $\mathbb{G}_m$ denote the multiplicative group GL_1. A *torus* $\mathbb{L}$ is an algebraic group over F that is isomorphic to $(\mathbb{G}_m)^n$ over a separable closure $\overline{F}$ of F for some $n \geq 0$. It is F*-split* if the isomorphism $\mathbb{L}$ to $\mathbb{G}_m^n$ is defined over F and at the other extreme it is *anisotropic* if it contains no F-split subtori.

Suppose $\mathbb{L}$ splits over F'. The *group of characters* $X^*(\mathbb{L}_{F'})$ of $\mathbb{L}$ is the homomorphism space $\mathrm{Hom}_{F'}(\mathbb{L}(F'), F'^{\times})$, and the *group of cocharacters* $X_*(\mathbb{L}_{F'})$ is the homomorphism space $\mathrm{Hom}_{F'}(F'^{\times}, \mathbb{L}(F'))$. Moreover, there is a natural bilinear pairing

$$\langle \cdot, \cdot \rangle \colon X^*(\mathbb{L}_{F'}) \times X_*(\mathbb{L}_{F'}) \longrightarrow \mathbb{Z} \tag{2.2.3}$$

defined by the condition that, for every $\chi \in X^*(\mathbb{L}_{F'})$ and $\gamma \in X_*(\mathbb{L}_{F'})$, the integer $d = \langle \chi, \gamma \rangle$ satisfies $\chi(\gamma(t)) = t^d$ for all $t \in \mathbb{G}_m(F')$.

We now turn to our unitary group G. Note that every maximal torus of G contains the center Z of G. Let $\mathbb{T}$ be the diagonal torus of $\mathbb{G}$, whose group of F-points is

$$T := \mathbb{T}(F) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{smallmatrix}\right) \mid a \in E^{\times} \right\} \cong E^{\times}. \tag{2.2.4}$$

Note that $\mathbb{T}$ is a maximally F-split maximal F-torus of $\mathbb{G}$. Since $\mathbb{G}$ is quasi-split any other such maximal torus of $\mathbb{G}$ is a G-conjugate of $\mathbb{T}$ [Bor91, Theorem V 20.9]. The maximal F-split subtorus of $\mathbb{T}$ is $\mathbb{S}$ where

$$S := \mathbb{S}(F) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in F^{\times} \right\}.$$

A direct computation shows that the centralizer of S in G, denoted $C_G(S)$, is the same as the centralizer of T in G. In fact, $C_G(S) = C_G(T) = T$.

One of the ingredients in defining a parahoric subgroup is the family of filtration subgroups of T for each real number $r \geq 0$. Since $\mathbb{T} = \mathrm{Res}_{E/F}\mathbb{G}_m$, it is an induced torus, therefore by [KP23, Lemma 2.5.18], we have $T_0 \cong \mathcal{O}_E^{\times}$. For real numbers $r > 0$, the filtration on T is defined as follows [KP23, Definition 7.2.2]:

$$T_r = \{ t \in T_0 \mid \nu(\chi(t) - 1) \geq r \text{ for all } \chi \in X^*(\mathbb{T}_E) \}. \tag{2.2.5}$$

In general, these filtration subgroups are rather difficult to compute. However, in our case they admit a nice description given by

$$T_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \middle| \, a \in \mathcal{O}_E^{\times} \right\}, \qquad T_r = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \middle| \, a \in 1 + \mathfrak{p}_E^{\lceil r \rceil} \right\}, \qquad \text{for } r > 0. \tag{2.2.6}$$

A second ingredient required for defining parahoric subgroups are *affine apartments* and *affine root subgroups*. We begin by describing the root system of $\mathbb{G}$ associated with $\mathbb{S}$.

The root system $\Phi := \Phi(\mathbb{G}, \mathbb{S})$ associated with $\mathbb{S}$ consists of α and $-\alpha$, where for $a \in F^\times$,

$$\alpha\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = a^2, \qquad (-\alpha)\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = a^{-2}.$$

The corresponding root subgroups are

$$U_\alpha := \mathbb{U}_\alpha(F) = \left\{ \begin{pmatrix} 1 & \sqrt{\epsilon}b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in F \right\},$$
$$U_{-\alpha} := \mathbb{U}_{-\alpha}(F) = \left\{ \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}b & 1 \end{pmatrix} \,\middle|\, b \in F \right\}.$$

The elements $X_\alpha = \left(\begin{smallmatrix} 0 & \sqrt{\epsilon} \\ 0 & 0 \end{smallmatrix}\right)$ and $X_{-\alpha} = \left(\begin{smallmatrix} 0 & 0 \\ \sqrt{\epsilon}^{-1} & 0 \end{smallmatrix}\right)$ of $\mathfrak{g}$ form a Chevalley system. Let $\Phi^\vee = \{\alpha^\vee, -\alpha^\vee\}$ denote the set of coroots of $\mathbb{S}$, where for $a \in F^\times$,

$$\alpha^\vee(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad -\alpha^\vee(a) = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}. \tag{2.2.7}$$

Since $\mathbb{S} \cong \mathbb{G}_m$, it follows that $X_*(\mathbb{S}_F) \cong \mathbb{Z}$. The pairing $\langle \cdot, \cdot \rangle$ (2.2.3) extends linearly to $X_*(\mathbb{S}_F) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$.

The *affine apartment* of G associated with $\mathbb{S}$, denoted $\mathcal{A}(\mathbb{G}, \mathbb{S}, F)$, is the affine space over the real Euclidean space $X_*(\mathbb{S}_F) \otimes_{\mathbb{Z}} \mathbb{R}$. It is convenient for us to fix an origin and thus a parametrization of $\mathcal{A} := \mathcal{A}(\mathbb{G}, \mathbb{S}, F)$ with $y \in \mathbb{R}$ corresponding to $\frac{y}{2}\alpha^\vee$. Note that $\langle \alpha, \alpha^\vee \rangle = 2$ and thus we may write $\alpha(y) := \langle \alpha, y \rangle = y$.

Given the root system $\Phi = \Phi(\mathbb{G}, \mathbb{S})$, we define the set of *affine roots* by

$$\Phi^{\mathrm{aff}} = \{\alpha + n \mid \alpha \in \Phi,\ n \in \mathbb{Z}\}.$$

If $\varphi \in \Phi^{\mathrm{aff}}$, then $\varphi = \alpha + n$ for some $\alpha \in \Phi$ and $n \in \mathbb{Z}$. Each $\varphi = \alpha + n$ defines an affine function on $\mathcal{A}$ via

$$\varphi(y) := \langle \varphi, y \rangle = \langle \alpha, y \rangle + n = \alpha(y) + n, \qquad y \in \mathcal{A}.$$

For $\varphi = \alpha + n \in \Phi^{\mathrm{aff}}$, the associated *affine root subgroup* is defined by

$$U_\varphi := \mathbb{U}_\varphi(\mathcal{O}_F) = \mathbb{U}_\alpha(\mathfrak{p}_F^n).$$

Finally, for $y \in \mathcal{A}$, the corresponding *parahoric subgroup* of G is

$$G_{y,0} = \langle\, T_0, U_\varphi \;\mid\; \varphi \in \Phi^{\mathrm{aff}},\ \varphi(y) \geq 0 \,\rangle$$

where T_0 is defined in (2.2.6). If $\varphi = \alpha + n$, then for $y \in \mathcal{A}$ we have $\varphi(y) \geq 0$ if and only if $\alpha(y) + n \geq 0$ if and only if $n \geq -\alpha(y)$. In particular, with respect to our identification of $\mathcal{A}$ with $\mathbb{R}$ we have

$$G_{y,0} = \begin{pmatrix} \mathcal{O}_E & \mathfrak{p}_E^{\lceil -y \rceil} \\ \mathfrak{p}_E^{\lceil y \rceil} & \mathcal{O}_E \end{pmatrix} \cap G.$$

Note that by our choice of coordinates $G_{0,0} = \mathcal{K}$ and $\mathcal{K}^\eta$ corresponds to the parahoric subgroup $G_{1,0}$.

The parahoric subgroups admit a further filtration, called the *Moy–Prasad filtration*, given for $r > 0$ by

$$G_{y,r} = \langle T_r, U_\varphi \mid \varphi \in \Phi^{\mathrm{aff}},\ \varphi(y) \geq r \rangle.$$

For all $y \in \mathcal{A}$ and $r \geq 0$, the subgroup $G_{y,r}$ is normal in $G_{y,0}$. Moreover, for each parahoric subgroup $G_{y,0}$, the collection $\{G_{y,r}\}_{r>0}$ forms a neighborhood basis of the identity in $G_{y,0}$.

Explicitly, we have

$$G_{y,r} = \begin{pmatrix} 1+\mathfrak{p}_E^{\lceil r \rceil} & \mathfrak{p}_E^{\lceil r-y \rceil} \\ \mathfrak{p}_E^{\lceil r+y \rceil} & 1+\mathfrak{p}_E^{\lceil r \rceil} \end{pmatrix} \cap G, \tag{2.2.8}$$

and we set

$$G_{y,r+} = \bigcup_{s>r} G_{y,s}. \tag{2.2.9}$$

To understand the role of these filtrations, we now describe the structure of the corresponding quotients. The quotient $G_{y,0}/G_{y,0+}$ is the group of $\mathfrak{f}$-points of a connected reductive group $\mathbf{G}_y$ defined over the residue field $\mathfrak{f}$ of F. For $r \in \mathbb{R}_{>0}$, the quotient $G_{y,r}/G_{y,r+}$ is abelian and can be naturally identified with an $\mathfrak{f}$-vector space [MP94, MP96].

For example, the Moy–Prasad filtration subgroups of $\mathcal{K} = G_{0,0}$ are indexed by nonnegative integers and form a descending chain

$$\mathcal{K} = \mathcal{K}_0 \supsetneq \mathcal{K}_1 \supsetneq \mathcal{K}_2 \supsetneq \mathcal{K}_3 \supsetneq \cdots$$

of compact normal subgroups of $\mathcal{K}$. For each $n \in \mathbb{Z}_{\geq 0}$, these subgroups are given by

$$\mathcal{K}_n = \begin{pmatrix} 1+\mathfrak{p}_E^n & \mathfrak{p}_E^n \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n \end{pmatrix} \cap G.$$

Moreover, the collection $\{\mathcal{K}_n\}_{n\geq 0}$ forms a neighborhood basis of the identity in $\mathcal{K}$; that is, for any open subset $U \subseteq \mathcal{K}$ containing the identity, there exists n such that $\mathcal{K}_n \subseteq U$. In fact, if $\mathrm{U}(1,1)(\mathfrak{f})$ denotes the unitary group over $\mathfrak{f}$ defined by w and the unique quadratic extension of $\mathfrak{f}$, then $\mathcal{K}/\mathcal{K}_{0+} \cong \mathrm{U}(1,1)(\mathfrak{f})$.

For $y \in \mathcal{A}$ and $r \in \mathbb{R}$, we can similarly define a filtration $\mathfrak{g}_{y,r}$ of the Lie algebra $\mathfrak{g}$ and a filtration $\mathfrak{g}^*_{y,r}$ of the F-linear dual of the Lie algebra $\mathfrak{g}$ as follows. Let $\mathfrak{t}$ denote the Lie algebra of the torus T. Then we set

$$\mathfrak{t}_r = \{X \in \mathfrak{t} \mid \nu(d\chi(X)) \geq r \ \text{ for all } \chi \in X^*(\mathbb{T}_E)\},$$

where $d\chi$ denotes the derivative of χ,

$$\mathfrak{g}_{\alpha,y,r} = \varpi^{\lceil r-y \rceil}\mathcal{O}_F X_\alpha,$$

for $\alpha \in \Phi$, and

$$\mathfrak{g}_{y,r} = \mathfrak{t}_r \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_{\alpha,y,r}.$$

Thus we have

$$\mathfrak{g}_{y,r} = \begin{pmatrix} \mathfrak{p}_E^{\lceil r\rceil} & \sqrt{\epsilon}\mathfrak{p}_F^{\lceil r-y\rceil} \\ \sqrt{\epsilon}\mathfrak{p}_F^{\lceil r+y\rceil} & \mathfrak{p}_E^{\lceil r\rceil} \end{pmatrix} \cap \mathfrak{g}. \tag{2.2.10}$$

We define the filtration subspace $\mathfrak{g}^*_{y,-r}$ of the dual of the Lie algebra by

$$\mathfrak{g}^*_{y,-r} = \{\lambda \in \mathfrak{g}^* \mid \lambda(Y) \in \mathfrak{p}_E,\ \forall Y \in \mathfrak{g}_{y,s},\ s > r\}. \tag{2.2.11}$$

We can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the trace form. In particular, for $X \in \mathfrak{g}$, we define $\lambda_X \in \mathfrak{g}^*$ by the equation $\lambda_X(Y) = \mathrm{Tr}(XY)$ for all $Y \in \mathfrak{g}$. Under this identification, $\mathfrak{g}^*_{y,r} = \mathfrak{g}_{y,r}$ for all $y \in \mathcal{A}$ and all $r \in \mathbb{R}_{\geq 0}$.

There is an isomorphism, often referred to as the *Moy–Prasad isomorphism*, relating the filtration on the group to the corresponding filtration on its Lie algebra. It has the form

$$G_{x,r}/G_{x,r+} \;\cong\; \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+},$$

and is valid for all $r \in \mathbb{R}_{>0}$. More generally, for $r, s \in \mathbb{R}_{>0}$ with $2r \geq s > r$, one has an isomorphism

$$G_{x,r}/G_{x,s} \;\cong\; \mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}.$$

For our unitary group $\mathrm{U}(1,1)$, it is the isomorphism that sends a coset represented by g to the coset represented by $g - I$.

We are now ready to introduce the building of G. The Weyl group W of G is

$$W := N_G(S)/C_G(S) = N_G(S)/T \;\cong\; \mathbb{Z}/2\mathbb{Z} \;=\; \{I, \mathrm{w}\}.$$

The *affine Weyl group*, denoted by W^{aff}, is defined as

$$W^{\mathrm{aff}} := N_G(T)/T_0 \;\cong\; T/T_0 \ltimes W.$$

The group T/T_0 acts on the apartment $\mathcal{A}$ by translations. Specifically, for each $t \in T$ and $y \in \mathcal{A}$ we have

$$t \cdot y \;=\; y - 2\nu(t),$$

where $\nu(t)$ denotes the valuation of the upper-left entry of the matrix $t \in T$ [Tit79, §1.2]. Moreover the nontrivial element w acts by reflection with fixed point $y = 0$ [KP23, Chapter 3(d)].

With this action in place, the *enlarged Bruhat–Tits building* $\mathscr{B}^{\mathrm{en}}(\mathbb{G}, F)$ [BT72, 7.4.1] is defined as

$$\mathscr{B}^{\mathrm{en}}(\mathbb{G}, F) = \big(G \times \mathcal{A}(\mathbb{G}, \mathbb{S}, F)\big)\big/ \sim,$$

where the equivalence relation is given by

$$(g, x) \sim (h, y) \quad\Longleftrightarrow\quad \exists\, n \in N_G(T):\ y = n \cdot x \ \text{ and }\ g^{-1}hn \in G_{x,0}.$$

The reduced Bruhat–Tits building of $\mathbb{G}$ over F, denoted $\mathscr{B}(G)$ is the enlarged Bruhat–Tits building of $\mathbb{G}_{\text{der}}$ over F.

For our unitary group G, since the center Z of G is compact, we have

$$\mathscr{B}(G) = \mathscr{B}^{\text{en}}(\mathbb{G}, F) \; = \; \mathscr{B}(\mathbb{G}_{\text{der}}, F).$$

We now define parahoric subgroups corresponding to points in the building $\mathscr{B}(G)$. The group G acts on the building by $g[(h,x)] = [(gh,x)]$ for $g,h \in G$ and $x \in \mathcal{A}$ and for every $y \in \mathscr{B}(G)$, there exists $g \in G$ such that $g \cdot y \in \mathcal{A}$. The parahoric subgroup attached to y is then defined by

$$G_{y,0} := {}^{g}G_{g\cdot y,0},$$

and for $r > 0$, the corresponding Moy–Prasad filtration subgroups are

$$G_{y,r} := {}^{g}G_{g\cdot y,r}.$$

For $y \in \mathcal{B}(G)$ we denote by G_y its stabilizer under the action of G.

Lemma 2.2.10. *We have $\mathcal{K} = G_{0,0} = G_0$, and $\mathcal{K}^\eta = G_{1,0} = G_1$.*

Proof. Since G is quasi-split and unramified over F, and has bounded center, the stabilizer G_y of any special point $y \in \mathscr{B}(G)$ coincides with the corresponding parahoric subgroup $G_{y,0}$. This follows from the discussion preceding Remark 7.7.2 together with Lemma 7.7.10 and Proposition 7.7.11 of [KP23].

With our chosen identification of $\mathcal{A}$ with $\mathbb{R}$, the maximal compact subgroups $\mathcal{K}$ and $\mathcal{K}^\eta$ stabilize the special vertices 0 and 1, respectively. Hence $\mathcal{K} = G_0 = G_{0,0}$ and $\mathcal{K}^\eta = G_1 = G_{1,0}$, as required. ∎

2.2.3 Anisotropic tori of G

As mentioned in the introduction, to describe the branching rules for positive-depth supercuspidal representations we first provide an explicit description of these representations using the Adler–Fintzen–Yu method. One of the key ingredients required in this construction is a sequence of twisted Levi subgroups. In our case the first element of such a sequence is an anisotropic torus of G. We describe these sequences in greater detail in Chapter 5, but for the present subsection our goal is to classify all anisotropic tori of G up to conjugacy and provide filtrations on these tori.

We recall our simplified notation: by a "maximal torus of G" we mean a group T that is the group of F-rational points of an algebraic torus $\mathbb{T} \subset \mathbb{G}$ defined over F.

2.2.3.1 Conjugacy classes of anisotropic tori of G

In this section, we use $\mathbb{L}$ to refer to a maximal anisotropic torus of $\mathbb{G}$ as $\mathbb{T}$ is our maximal diagonal torus. We nonetheless set $\mathcal{T} = \mathbb{L}(F)$, whereas $T = \mathbb{T}(F)$. Each maximal anisotropic torus $\mathbb{L}$ of $\mathbb{G}$ splits over a finite extension of F. We call $\mathcal{T} = \mathbb{L}(F)$ *unramified* if its minimal splitting field over F is unramified, and *ramified* if its minimal splitting field is ramified.

Proposition 2.2.11. *The group G has four conjugacy classes of maximal anisotropic tori. They are given by*

$$\mathcal{T}_{\gamma_1,\gamma_2} = \left\{ \begin{pmatrix} a & b\gamma_1 \\ b\gamma_2 & a \end{pmatrix} \;\middle|\; a\bar{a} + b\bar{b}\,\gamma_1\gamma_2 = 1,\ \bar{a}b \in \sqrt{\epsilon}F \right\},$$

where

$$(\gamma_1,\gamma_2) \in \big\{(1,1),\ (\varpi^{-1},\varpi),\ (1,\varpi),\ (1,\epsilon^{-1}\varpi)\big\}.$$

The pairs $(1,1)$ and (ϖ^{-1},ϖ) correspond to unramified tori, while $(1,\varpi)$ and $(1,\epsilon^{-1}\varpi)$ correspond to ramified tori.

Proof. By [Bou21, Theorem 2.2] and [Bou20, Corollary 2.1.22], the maximal tori of $\mathbb{G}$ are in bijection with the maximal tori of $\mathbb{G}_{\mathrm{der}}$ via the map sending $\mathbb{L}$ to $\mathbb{L} \cap \mathbb{G}_{\mathrm{der}}$. To enumerate all maximal anisotropic tori of G (up to conjugacy), it therefore suffices to proceed as follows: for each conjugacy class of maximal anisotropic torus in G_{der}, choose a representative $\mathcal{T}'$, find a torus $\mathcal{T}$ of G containing $\mathcal{T}'$, and then determine when two such tori are conjugate. Such a torus G is automatically anisotropic, since the center of G is compact.

In our case, there is a simple isomorphism $\Xi : \mathrm{SL}_2(F) \to G_{\mathrm{der}}$ given in Lemma 2.2.2. The $\mathrm{SL}_2(F)$-conjugacy classes of maximal anisotropic tori of $\mathrm{SL}_2(F)$ are well-known. By [Nev13, §2.3], each such class is represented by a matrix of the form

$$\mathcal{T}^{\mathrm{SL}_2(F)}_{\delta_1,\delta_2} = \left\{ \begin{pmatrix} a & b\delta_1 \\ b\delta_2 & a \end{pmatrix} \;\middle|\; a,b \in F, a^2 - b^2\delta_1\delta_2 = 1 \right\} \cong F[\sqrt{\delta_1\delta_2}]^1$$

where $(\delta_1,\delta_2) \in \{(1,\epsilon),(\varpi^{-1},\epsilon\varpi),(1,\varpi),(\epsilon,\epsilon^{-1}\varpi),(1,\epsilon\varpi),(\epsilon,\varpi)\}$, understanding that if -1 is not a square, then $\mathcal{T}^{\mathrm{SL}(2,F)}_{1,\varpi} \simeq \mathcal{T}^{\mathrm{SL}_2(F)}_{\epsilon,\epsilon^{-1}\varpi}$ and $\mathcal{T}^{\mathrm{SL}_2(F)}_{1,\epsilon\varpi} \simeq \mathcal{T}^{\mathrm{SL}_2(F)}_{\epsilon,\varpi}$. Moreover, $T^{\mathrm{SL}_2(F)}_{1,\varpi} \not\cong \mathcal{T}^{\mathrm{SL}_2(F)}_{1,\epsilon\varpi}$ as they split over different field extensions. Here, $\simeq$ denotes tori representing the same conjugacy class, whereas $\cong$ denotes isomorphic tori.

The centralizer in $\mathrm{U}(1,1)$ of $\Xi(\mathcal{T}^{\mathrm{SL}_2(F)}_{\delta_1,\delta_2})$ is

$$\mathcal{T}_{\gamma_1,\gamma_2} = C_{U(1,1)}\left(\begin{pmatrix} 0 & \sqrt{\epsilon}\delta_1 \\ \sqrt{\epsilon}^{-1}\delta_2 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} t & r\delta_1 \\ r\epsilon^{-1}\delta_2 & t \end{pmatrix} \;\middle|\; t,r \in E \right\} \cap G,$$

where $\gamma_1 = \delta_1$ and $\gamma_2 = \epsilon^{-1}\delta_2$, which is already F-points of an anisotropic torus of $\mathbb{G}$. Thus we get six anisotropic tori $\mathcal{T}_{\gamma_1,\gamma_2}$ where

$$(\gamma_1,\gamma_2) = \{(1,1),(\varpi^{-1},\varpi),(1,\epsilon^{-1}\varpi),(\epsilon,\epsilon^{-2}\varpi),(1,\varpi),(\epsilon,\epsilon^{-1}\varpi)\}.$$

Let $g := \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & -\sqrt{\epsilon}^{-1} \end{pmatrix} \in G$; then we have $\mathcal{T}_{1,\varpi}^g = \mathcal{T}_{\epsilon,\epsilon^{-1}\varpi}$, and $\mathcal{T}_{1,\epsilon^{-1}\varpi}^g = \mathcal{T}_{\epsilon,\epsilon^{-2}\varpi}$. Since $\mathcal{T}_{1,\epsilon^{-1}\varpi}$ and $\mathcal{T}_{1,\varpi}$ correspond to the non-isomorphic tori $\mathcal{T}_{1,\varpi}^{\mathrm{SL}_2(F)}$ and $\mathcal{T}_{1,\epsilon\varpi}^{\mathrm{SL}_2(F)}$ respectively, they are not G-conjugates. Moreover, the unramified and ramified tori lie in distinct conjugacy classes, since their minimal splitting fields are different.

Now let us verify the nonconjugacy of the two tori

$$\mathcal{T}_{1,1} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\} \quad \text{and} \quad \mathcal{T}_{\varpi^{-1},\varpi} = \left\{ \begin{pmatrix} a & b\varpi^{-1} \\ b\varpi & a \end{pmatrix} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\}.$$

If there were an element $zh \in ZG_{\mathrm{der}}$ conjugating $\mathcal{T}_{1,1}$ to $\mathcal{T}_{\varpi^{-1},\varpi}$, then since G_{der} is normal, h would conjugate $\mathcal{T}_{1,1} \cap G_{\mathrm{der}}$ to $\mathcal{T}_{\varpi^{-1},\varpi} \cap G_{\mathrm{der}}$, a contradiction. Let $\xi = \begin{pmatrix} \epsilon_E & 0 \\ 0 & \overline{\epsilon_E}^{-1} \end{pmatrix}$ be the non-trivial coset representative of $\mathrm{U}(1,1)/ZG_{\mathrm{der}}$ as in Lemma 2.2.3. To prove that $\mathcal{T}_{1,1}$ and $\mathcal{T}_{\varpi^{-1},\varpi}$ are not $\mathrm{U}(1,1)$-conjugate, it suffices to prove that $\xi\mathcal{T}_{1,1}\xi^{-1}$ is conjugate to $\mathcal{T}_{1,1}$ via an element of G_{der}.

Note that since $\epsilon_E\overline{\epsilon_E} \in \mathcal{O}_F^\times$, Hensel's Lemma assures us that $\mathcal{O}_E^\times$ contains a square root of $\epsilon_E\overline{\epsilon_E}$. Set

$$h = \begin{pmatrix} 0 & -\sqrt{\epsilon_E\overline{\epsilon_E}} \\ \sqrt{\epsilon_E\overline{\epsilon_E}}^{-1} & 0 \end{pmatrix}.$$

We readily verify that $h \in G_{\mathrm{der}}$. Then

$$\begin{aligned} \xi\mathcal{T}_{1,1}\xi^{-1} &= \left\{ \begin{pmatrix} a & b\epsilon_E\overline{\epsilon_E} \\ b(\epsilon_E\overline{\epsilon_E})^{-1} & a \end{pmatrix} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\} \\ &= \left\{ \begin{pmatrix} a & -b\epsilon_E\overline{\epsilon_E} \\ -b(\epsilon_E\overline{\epsilon_E})^{-1} & a \end{pmatrix} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\} \\ &= h\mathcal{T}_{1,1}h^{-1}. \end{aligned}$$

Therefore, the conjugacy class of $\mathcal{T}_{1,1}$ under $\mathrm{U}(1,1)$ is equal to its conjugacy class under G_{der}, whence $\mathcal{T}_{1,1} \not\sim \mathcal{T}_{\varpi^{-1},\varpi}$. ∎

Since we have already introduced the definition of the building, it is useful to recall an important fact. Let $\mathbb{L}$ be a maximal anisotropic torus of $\mathbb{G}$ with minimal splitting field F'. Since $p \neq 2$, F' is tamely ramified and note that it also splits $\mathbb{T}$. Therefore the Bruhat–Tits building $\mathcal{B}(\mathbb{G}, F)$ can be realized as the subset of $\mathcal{B}(\mathbb{G}, F')$ fixed by $\mathrm{Gal}(F'/F)$ [Tit79, 2.6.1]. Since the maximal F-split subtorus of $\mathbb{L}$ is trivial, by [Tit79, 2.6.1] $\mathcal{A}(\mathbb{G}, \mathbb{L}, F')^{\mathrm{Gal}(F'/F)}$ is a single point, and this point coincides with $\mathcal{A}(\mathbb{G}, \mathbb{L}, F') \cap \mathscr{B}(\mathbb{G}, F)$.

Lemma 2.2.12. *Let $\mathbb{L}(F)$ be one of the four anisotropic tori in G, represented by $\mathcal{T}_{1,1}$, $\mathcal{T}_{\varpi^{-1},\varpi}$, $\mathcal{T}_{1,\varpi}$, and $\mathcal{T}_{1,\epsilon^{-1}\varpi}$, as in Proposition 2.2.11. Then the unique Galois-fixed point $y \in \mathcal{A}(\mathbb{G}, \mathbb{L}, F') \cap \mathcal{B}(\mathbb{G}, F)$ is given by*

$$y = \begin{cases} 0 & \text{if } \mathbb{L}(F) = \mathcal{T}_{1,1}, \\ 1 & \text{if } \mathbb{L}(F) = \mathcal{T}_{\varpi^{-1},\varpi}, \\ \frac{1}{2} & \text{if } \mathbb{L}(F) = \mathcal{T}_{1,\varpi} \text{ or } \mathcal{T}_{1,\epsilon^{-1}\varpi}. \end{cases}$$

Proof. The argument is adapted from [Nev13, §2.3]. We first recall that $\mathscr{B}(\mathbb{G}, F')$ can be described as

$$\mathscr{B}(\mathbb{G}, F') \;\cong\; (\mathbb{G}(F') \times \mathcal{A}(\mathbb{G}, \mathbb{T}, F'))/\!\sim,$$

where $(g, y) \sim (h, x)$ if there exists $n \in N_{\mathbb{G}(F')}(C_{\mathbb{G}(F')}(\mathbb{S}(F'))) = N_{\mathbb{G}(F')}(\mathbb{T}(F'))$ such that $n \cdot y = x$ and $g^{-1}hn \in \mathbb{G}(F')_{y,0}$. Since $\mathbb{T}$ and $\mathbb{L}$ split over F', there exists $\mathsf{h} \in \mathbb{G}(F')$ such that $\mathbb{L} = \mathbb{T}^{\mathsf{h}}$, and hence $\mathcal{A}(\mathbb{G}, \mathbb{L}, F') \;=\; \mathsf{h} \cdot \mathcal{A}(\mathbb{G}, \mathbb{T}, F') \subset \mathscr{B}(\mathbb{G}, F')$.

By the action of $\mathrm{Gal}(F'/F)$ on both $\mathbb{G}(F')$ and $\mathcal{A}(\mathbb{G}, \mathbb{T}, F')$, the Galois action on the building is given by

$$\eta([g, y]) \;=\; [\eta(g), \eta(y)], \qquad \eta \in \mathrm{Gal}(F'/F),\ g \in \mathbb{G}(F'),\ y \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F') \text{ [Pra01, §1.13]}.$$

Thus, the class $[\mathsf{h}, y]$ is fixed by $\mathrm{Gal}(F'/F)$ if and only if for all $\eta \in \mathrm{Gal}(F'/F)$, there exists $n \in N_{\mathbb{G}(F')}(\mathbb{T}(F'))$ satisfying $n \cdot y = \eta(y)$ and $\mathsf{h}^{-1}\eta(\mathsf{h}) \in \mathbb{G}(F')_{y,0}$.

The splitting field F' and the Galois group depend on the anisotropic torus $\mathbb{L}(F)$, as summarized in Table 2.1. In particular, if $\mathbb{L}(F) = \mathcal{T}_{1,1}$ or $\mathcal{T}_{\varpi^{-1},\varpi}$, then we have $F' = E$, with $\mathrm{Gal}(E/F) = \{1, \sigma\}$, and it acts on $\mathbb{G}(E)$ by $\sigma(g) = J^{\top}\overline{g}^{-1}J$. If $\mathbb{L}(F) = \mathcal{T}_{1,\varpi}$ or $\mathcal{T}_{1,\epsilon^{-1}\varpi}$, then we have $F' = E[\sqrt{\varpi}]$, with $\mathrm{Gal}(F'/F) = \{1, \sigma, \delta, \sigma\delta\}$, where we again write σ for the unique element of $\mathrm{Gal}(F'/F[\sqrt{\varpi}])$ whose restriction to E is σ. Here δ acts entry-wise on the elements of $\mathbb{G}(F[\sqrt{\epsilon}, \sqrt{\varpi}])$.

A direct computation shows that the element

$$\mathsf{h} := \begin{pmatrix} 1 & -\sqrt{\gamma_1^{-1}\gamma_2}^{\,-1} \\ \sqrt{\gamma_1^{-1}\gamma_2} & 1 \end{pmatrix}$$

satisfies $\mathbb{L} = \mathbb{T}^{\mathsf{h}}$. One verifies that for each such h and corresponding $\mathbb{L}$, and for all $\eta \in \mathrm{Gal}(F'/F)$, the elements h and $\mathsf{h}^{-1}\eta(\mathsf{h})$ lie in $\mathbb{G}(F')_{y,0}$ for the value of y stated in the lemma. Hence the unique Galois-fixed point of $\mathcal{A}(\mathbb{G}, \mathbb{L}, F')$ is $[\mathsf{h}, y] = [1, y] = y \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F')^{\mathrm{Gal}(E/F)} = \mathcal{A} \subset \mathcal{B}$. ∎

The biquadratic extension $\widetilde{E} = F(\sqrt{\epsilon}, \sqrt{\varpi})$ of F has three quadratic intermediate fields

$$F(\sqrt{\epsilon}), \qquad F(\sqrt{\varpi}), \qquad F(\sqrt{\epsilon\varpi}).$$

These are precisely the distinct quadratic extensions of F. For any quadratic subfield $F' \subset \widetilde{E}$, we write $U(1)_{\widetilde{E}/F'}$ for the group of elements of $\widetilde{E}$ of norm one relative to the extension $\widetilde{E}/F'$. Table 2.2.3.1 below provides a complete list of the conjugacy classes of anisotropic tori in G and the corresponding points in $\mathscr{B}(G)$.

2.2.3.2 Filtrations on anisotropic tori of G

In §2.2.2, we described a filtration on the maximal parahoric subgroups of G. We now use this to define a filtration on the anisotropic tori of G as follows. Let $\mathcal{T} = \mathcal{T}_{\gamma_1,\gamma_2}$ where (γ_1, γ_2)

F'	Conjugacy classes $\mathcal{L}_{\gamma_1,\gamma_2}$ of $\mathbb{T}(F)$	$\mathrm{Gal}(F'/F)$	B–basis for F'/F	Action of $\mathrm{Gal}(F'/F)$ on elements of B
E	$\mathcal{T}_{1,1}$, $\mathcal{T}_{\varpi^{-1},\varpi}$ (unramified)	$\{1,\sigma\}$	$\{1,\sqrt{\epsilon}\}$	$\sigma(\sqrt{\epsilon}) = -\sqrt{\epsilon}$
$E[\sqrt{\varpi}]$	$\mathcal{T}_{1,\varpi}$, $\mathcal{T}_{1,\epsilon^{-1}\varpi}$ (ramified)	$\{1,\sigma,\delta,\sigma\delta\}$	$\{1,\sqrt{\epsilon},\sqrt{\varpi},\sqrt{\epsilon\varpi}\}$	$\sigma(\sqrt{\varpi}) = \sqrt{\varpi}$ $\delta(\sqrt{\epsilon}) = \sqrt{\epsilon}$ $\delta(\sqrt{\varpi}) = -\sqrt{\varpi}$

Table 2.1: Different possibilities of the field F' over which an anisotropic torus $\mathbb{T}$ splits, corresponding Galois group $\mathrm{Gal}(F'/F)$, and the action of the Galois group on the basis elements of F' over F.

Isomorphism type	Anisotropic torus $\mathbb{T}(F) = \mathcal{T}_{\gamma_1,\gamma_2}$	$\mathcal{A}(G,\mathcal{T},F) = \{y\}$
$U(1)_{E/F} \times U(1)_{E/F}$ unramified torus	$\mathcal{T}_{1,1}$ $\mathcal{T}_{\varpi,\varpi^{-1}} = \mathcal{T}_{1,1}^{\eta}$	$y = 0$ $y = 1$
$U(1)_{\widetilde{E}/F[\sqrt{\varpi}]}$ ramified torus	$\mathcal{T}_{1,\varpi}$	$y = \frac{1}{2}$
$U(1)_{\widetilde{E}/F[\sqrt{\epsilon\varpi}]}$ ramified torus	$\mathcal{T}_{1,\epsilon^{-1}\varpi}$	$y = \frac{1}{2}$

Table 2.2: Representatives of conjugacy classes of anisotropic tori $\mathcal{T}_{\gamma_1,\gamma_2}$ in G.

are as per Proposition 2.2.11. By [HM08, §2.5], for $r > 0$ the filtration on $\mathcal{T}$ is simply the intersection of $\mathcal{T}$ with filtrations of $G_{y,r}$ where y is the point corresponding to $\mathcal{T}$ in the building of G. In particular, for $r > 0$ we have

$$\mathcal{T}_r = \left\{ \begin{pmatrix} a & b\gamma_1 \\ b\gamma_2 & a \end{pmatrix} \in \mathcal{T}_{\gamma_1,\gamma_2} \;\middle|\; a \in 1 + \mathfrak{p}_E^{\lceil r \rceil},\ b\gamma_1 \in \mathfrak{p}_E^{\lceil r-y \rceil} \right\}.$$

The Lie algebra $\mathfrak{t}$ of the torus $\mathcal{T}$ is a two-dimensional subalgebra of $\mathfrak{g}$, spanned by

$$\mathsf{z} = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \sqrt{\epsilon} \end{pmatrix}, \qquad Y_{\mathcal{T}} = \begin{pmatrix} 0 & \gamma_1\sqrt{\epsilon} \\ \gamma_2\sqrt{\epsilon} & 0 \end{pmatrix}.$$

For any $r \in \mathbb{R}$, by [HM08, §2.5] the corresponding filtration subring of $\mathfrak{t}$ is given by $\mathfrak{t} \cap \mathfrak{g}_{y,r}$, and therefore

$$\mathfrak{t}_r = \left\{ a\mathsf{z} + bY_{\mathcal{T}} \mid a \in \mathfrak{p}_F^{\lceil r \rceil},\ b \in \mathfrak{p}_F^{\lceil r-y \rceil} \right\}.$$

For $r \in \mathbb{R}$, we define the filtration on the F-linear dual of $\mathfrak{t}$ by

$$\mathfrak{t}_r^* = \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, Y \rangle \in \varpi \mathcal{O}_E \ \ \forall\, Y \in \mathfrak{t}_s \text{ with } s > -r\}\,.$$

We can again identify $\mathfrak{t}^*$ with $\mathfrak{t}$ using the trace form. In particular, for $\Gamma_{u,v} = u\mathsf{z} + vY_{\mathcal{T}}$ we define $\lambda_{u,v} \in \mathfrak{t}^*$ by the equation $\lambda_{u,v}(Y) = \mathrm{Tr}(\Gamma_{u,v} Y)$ for all $Y \in \mathfrak{t}$.

2.3 Representation theory

In this section, we recall some basic definitions and concepts from representation theory, and we state several classical results that will be used throughout the thesis.

2.3.1 Representations of locally compact and totally disconnected groups

Let G be a locally compact and totally disconnected group. In this subsection, we recall some basic definitions and classical results that hold in this general setting (see [Car79, §I]).

Definition 2.3.1. A *representation* (π, V) of G is a group homomorphism

$$\pi : G \longrightarrow \mathrm{End}(V),$$

where V is a complex vector space. We call (π, V) *smooth* if for every $v \in V$, the stabilizer

$$\mathrm{Stab}_G(v) = \{g \in G \mid \pi(g)v = v\}$$

is an open subgroup of G. The representation (π, V) is called *admissible* if it is smooth and for every compact subgroup $K \subseteq G$, the subspace

$$V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$$

is finite-dimensional. Finally, a one-dimensional complex representation of G is called a *quasi-character*.

The definition of smoothness has an immediate consequence for compact open subgroups. In particular, for our group G, we consider the maximal compact subgroup

$$\mathcal{K} = G_{0,0} = \mathbb{G}(\mathcal{O}_F)$$

of G. By (2.2.8), $\mathcal{K}$ admits a nice filtration

$$\mathcal{K} = \mathcal{K}_0 \supsetneq \mathcal{K}_1 \supsetneq \mathcal{K}_2 \supsetneq \cdots$$

by compact normal subgroups, and the collection $\{\mathcal{K}_n\}_{n \geq 0}$ forms a neighborhood basis of the identity in $\mathcal{K}$. This leads to the following proposition.

Proposition 2.3.2. *Let (π, V) be a smooth representation of G and let $\mathcal{K}$ be a maximal compact subgroup. Then*

$$V = \bigcup_{n \geq 0} V^{\mathcal{K}_n}.$$

Moreover, each subspace $V^{\mathcal{K}_n}$ is $\mathcal{K}$-invariant, and any irreducible $\mathcal{K}$-subrepresentation of V is a subrepresentation of $V^{\mathcal{K}_n}$ for some $n \geq 0$.

Proof. Let $v \in V$. Then by the smoothness of the representation π, the stabilizer $\mathrm{stab}_G(v)$ is open in G. Since $\mathcal{K}$ is an open subgroup of G, the intersection $\mathrm{stab}_G(v) \cap \mathcal{K}$ is open in $\mathcal{K}$. As the filtration $\{\mathcal{K}_n\}_{n \geq 0}$ is a neighborhood basis of the identity in $\mathcal{K}$, the open subgroup $\mathrm{stab}_G(v) \cap \mathcal{K}$ contains $\mathcal{K}_n$ for some n. Therefore,

$$\pi(k)v = v \quad \text{for all } k \in \mathcal{K}_n,$$

i.e., $v \in V^{\mathcal{K}_n}$. Hence,

$$V = \bigcup_{n \geq 0} V^{\mathcal{K}_n}.$$

To show that each $V^{\mathcal{K}_n}$ is $\mathcal{K}$-invariant, let $v \in V^{\mathcal{K}_n}$, $h \in \mathcal{K}$, and $k \in \mathcal{K}_n$. Since $\mathcal{K}_n \trianglelefteq \mathcal{K}$, $hk = kh'$ for some $h' \in \mathcal{K}_n$ and we have

$$\pi(h)(\pi(k)v) = \pi(hk)v = \pi(kh')v = \pi(k)(\pi(h')v) = \pi(k)v,$$

as desired. Lastly, let W be an irreducible subrepresentation of $(\pi|_{\mathcal{K}}, V)$. Since $\mathcal{K}$ is compact, W is finite-dimensional. Let $\{w_1, \ldots, w_m\}$ be a basis for W. Then for each w_i, there exists $n_i \in \mathbb{Z}_{\geq 0}$ such that $w_i \in V^{\mathcal{K}_{n_i}}$. Let $l = \max\{n_1, \ldots, n_m\}$. Then $W \subseteq \bigcup_i V^{\mathcal{K}_{n_i}} \subseteq V^{\mathcal{K}_l}$, as desired. ∎

Given a closed subgroup H of G, and a representation (ρ, W) of H, one can define a representation $\left(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W\right)$ of G, called the *induced representation*, where

$$\mathrm{Ind}_H^G W = \left\{ f : G \to W \;\middle|\; \begin{array}{l} f \text{ is locally constant, and} \\ f(hg) = \rho(h) f(g) \text{ for all } h \in H,\ g \in G \end{array} \right\},$$

and the action of G is given by

$$\left(\mathrm{Ind}_H^G(g) f\right)(x) = f(xg), \quad \text{for all } x, g \in G.$$

We say that this induction is *compact induction* if, in addition to being locally constant, f has compact support modulo H. The resulting representation of G is denoted by

$$(c\text{-}\mathrm{Ind}_H^G \rho, c\text{-}\mathrm{Ind}_H^G W).$$

Lemma 2.3.3. *Let $(\lambda, \mathbb{C})$ be a quasi-character of G, and let (ρ, V) be a smooth representation of a subgroup H of G. Then*

$$\mathrm{Ind}_H^G(\lambda \otimes \rho) \;\cong\; \lambda \otimes \mathrm{Ind}_H^G \rho.$$

Proof. For simplicity, set $\pi_1 := \mathrm{Ind}_H^G(\lambda \otimes \rho)$ and $\pi_2 := \mathrm{Ind}_H^G \rho$.

The space of π_1 is

$$V_1 := \Big\{ f\colon G \to \mathbb{C} \otimes V \;\Big|\; f(hg) = (\lambda \otimes \rho)(h) f(g), \;\; \forall h \in H,\, g \in G \Big\},$$

while the space of π_2 is

$$V_2 := \Big\{ f\colon G \to V \;\Big|\; f(hg) = \rho(h) f(g), \;\; \forall h \in H,\, g \in G \Big\}.$$

Define $\Phi\colon V_1 \to V_2$ by

$$\Phi(f)(g) = \lambda(g)^{-1} f(g).$$

We first check that $\Phi(f) \in V_2$. For $h \in H$, $g \in G$,

$$\Phi(f)(hg) = \lambda(hg)^{-1} f(hg) = \lambda(g)^{-1}\lambda(h)^{-1}\lambda(h)\rho(h)f(g) = \rho(h)(\lambda(g)^{-1}f(g)) = \rho(h)\Phi(f)(g).$$

Hence $\Phi(f) \in V_2$.

Injectivity of Φ is immediate. For surjectivity, take $f \in V_2$ and define $f'\colon G \to \mathbb{C} \otimes V$ by $f'(g) = \lambda(g)f(g)$. Then $f' \in V_1$, and $\Phi(f')(g) = \lambda(g)^{-1}f'(g) = \lambda(g)^{-1}\lambda(g)f(g) = f(g)$. Finally, we need to show that, for $g \in G$,

$$\Phi \circ \pi_1(g) = (\lambda \otimes \pi_2)(g) \circ \Phi.$$

Indeed, for $f \in V_1$ and $g \in G$ we have

$$((\Phi \circ \pi_1(g))f)(h) = (\Phi(\pi_1(g)f))(h) = \lambda(h)^{-1}(\pi_1(g)f)(h) = \lambda(h)^{-1}f(hg),$$

and

$$\begin{aligned}(((\lambda \otimes \pi_2)(g) \circ \Phi)(f))(h) &= ((\lambda(g)\pi_2(g))(\Phi(f)))(h) = \lambda(g)(\Phi(f))(hg) \\ &= \lambda(g)\lambda(hg)^{-1}f(hg) = \lambda(h)^{-1}f(hg).\end{aligned}$$

Hence, $\mathrm{Ind}_H^G(\lambda \otimes \rho) \cong \lambda \otimes \mathrm{Ind}_H^G \rho$. ∎

An entirely analogous argument yields the following result for compact induction.

Lemma 2.3.4. *Let $(\lambda, \mathbb{C})$ be a quasi-character of G and (ρ, V) be a smooth representation of subgroup H of G whose image in $G/Z(G)$ is compact open. Then*

$$c\text{-}\mathrm{Ind}_H^G(\lambda \otimes \rho) \cong \lambda \otimes c\text{-}\mathrm{Ind}_H^G \rho.$$

We now recall two important results that we will use quite frequently in the chapters to come [BH06, §2.4].

Let H be a closed subgroup of G and (ρ, W) be a smooth representation of G. Then there is a canonical H-homomorphism

$$\begin{aligned}\alpha_\rho \colon \mathrm{Ind}_H^G \rho &\to W \\ f &\mapsto f(1).\end{aligned}$$

The pair $\left(\mathrm{Ind}_H^G, \alpha\right)$ has the following fundamental property.

Theorem 2.3.5 (Frobenius Reciprocity). *Let H be a closed subgroup of G. For a smooth representation (ρ, W) of H and a smooth representation (π, V) of G, the canonical map*

$$\begin{aligned}\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) &\to \mathrm{Hom}_H(\pi, \rho) \\ \phi &\mapsto \alpha_\rho \circ \phi\end{aligned}$$

is an isomorphism that is functorial in both variables π, ρ.

Similarly, for open subgroups H of G, compact induction has its own form of Frobenius Reciprocity property. Let H be an open subgroup of G and let (ρ, W) be a smooth representation of H. Then again there is a canonical H-homomorphism

$$\begin{aligned}\alpha_\rho^c \colon W &\to c\text{-}\mathrm{Ind}_H^G \rho \\ w &\mapsto f_w\end{aligned}$$

where $f_w \in c\text{-}\mathrm{Ind}_H^G W$ is supported in H, and $f_w(h) = \rho(h)w$ for all $h \in H$.

Theorem 2.3.6. *Let H be an open subgroup of G, let (ρ, W) be a smooth representation of H and (π, V) a smooth representation of G. The canonical map*

$$\begin{aligned}\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \rho, \pi) &\to \mathrm{Hom}_H(\rho, \pi) \\ f &\mapsto f \circ \alpha_\rho^c,\end{aligned}$$

is an isomorphism which is functorial in both variables π, ρ.

Remark 2.3.7. Note that if G/H is compact then $c\text{-}\mathrm{Ind}_H^G \rho = \mathrm{Ind}_H^G \rho$. Thus, in that case, induction is both left and right adjoint to restriction.

We conclude this section with a theorem that will be used repeatedly throughout the thesis, as it describes how a compactly induced representation decomposes when restricted to a closed subgroup [Yam22, Theorem 1.1].

Theorem 2.3.8 (Mackey decomposition). *Let H and K be closed subgroups of G, and let (ρ, W) be a smooth representation of H. Suppose that at least one of H and K is open in G. Then*

$$\mathrm{Res}_K^G \ \mathrm{c\text{-}Ind}_H^G \rho \ = \bigoplus_{c \in K \backslash G/H} \mathrm{c\text{-}Ind}_{K \cap H^c}^K \mathrm{Res}_{K \cap H^c}^{H^c} \rho^c.$$

Remark 2.3.9. In our setting, we shall see that the supercuspidal representations of G arise as compact inductions from compact open subgroups of the unitary group $G = \mathrm{U}(1,1)$, and to describe the branching rules we will restrict these representations to the compact open subgroup $\mathcal{K}$ defined in §2.2.1. Since for topological groups every open subgroup is also closed, the above theorem applies in our case.

2.3.2 Representation theory of p-adic groups

Let F be a p-adic field and let $\mathbb{G}$ be a connected reductive group defined over F. We denote by $G = \mathbb{G}(F)$ the group of F-rational points, called a *p-adic group*. We now recall some standard definitions and results in this setting.

Let P be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$, and (ρ, W) a smooth representation of M. Then the *parabolic induction* $(\mathrm{Ind}_P^G\rho, \mathrm{Ind}_P^G W)$ is defined by inflating the representation (ρ, W) to a representation of P that is trivial on N, and then inducing the resulting representation from P to G.

Definition 2.3.10. An irreducible smooth representation (π, V) of G is called *supercuspidal* if for every proper parabolic subgroup $P \subsetneq G$ with Levi subgroup M and all irreducible smooth representations (σ, W) of M, the representation (π, V) is not a subrepresentation of $\mathrm{Ind}_P^G\sigma$.

In fact, a theorem of Jacquet [Jac71] promises that every irreducible smooth representation π of G can be realized as a subrepresentation of a representation parabolically induced from a supercuspidal representation σ of a Levi subgroup $M \subseteq P$, for some parabolic subgroup P of G. Therefore, supercuspidal representations are called the building blocks for all smooth representations of G.

When $\mathbb{G}$ is split over a tamely ramified extension of F, and p doesn't divide the order of the Weyl group of $\mathbb{G}$, the Adler-Fintzen-Yu method provides a well-established algorithm for constructing all supercuspidal representations of $\mathbb{G}(F)$. [Adl98, Yu01, Fin21a, Fin21b, FKS23].

The strategy for constructing supercuspidal representations is based on Mautner's Theorem, which we state below. All known constructions arise as applications of this result.

Theorem 2.3.11 (Mautner's Theorem [Mau64])**.** *Let (π, V) be a smooth representation of a subgroup K of G. Assume that K is open and compact modulo the center of G. If c-$\mathrm{Ind}_K^G\pi$ is irreducible, then it is supercuspidal (and admissible).*

We now introduce another method for obtaining a representation of a larger group from one of its quotients, namely the *inflation* of a representation.

Definition 2.3.12 (Inflation of a representation)**.** Let H be a quotient group of G, and let (ρ, W) be a representation of H. The *inflation* of ρ to G is the representation $(\widetilde{\rho}, W)$ of G defined by

$$\widetilde{\rho}(g) := \rho(\widetilde{g}), \quad g \in G,$$

where $\widetilde{g}$ denotes the image of g in H under the natural projection $G \twoheadrightarrow H$.

In our setting, the groups under consideration are infinite. However, as we saw in §2.2.2, the quotients of the Moy–Prasad filtration subgroups are finite. In particular, for the maximal compact subgroup $\mathcal{K}$ we have

$$\mathcal{K}/\mathcal{K}_{0+} \;\cong\; \mathrm{U}(1,1)(\mathfrak{f}),$$

and, more generally, $\mathcal{K}/\mathcal{K}_{r+}$ is finite for every $r > 0$. By the definition of smoothness, given any smooth representation (π, V) of $\mathcal{K}$, there exists $n \in \mathbb{N}$ such that $\mathcal{K}_n \subseteq \ker(\pi)$. Consequently, every smooth representation of $\mathcal{K}$ factors through a finite quotient $\mathcal{K}/\mathcal{K}_n$, and thus can be obtained by inflating a representation of $\mathcal{K}/\mathcal{K}_n$.

Remark 2.3.13. This observation will be fundamental in our later constructions of $\mathcal{K}$-representations. In particular, many of the representations of $\mathcal{K}$ that arise in the study of branching rules will be realized as inflations of representations of the finite quotients $\mathcal{K}/\mathcal{K}_n$.

The following lemma shows the compatibility of induction with inflation.

Lemma 2.3.14. *Let $H_2 \leq H_1 \leq K$, where $H_2 \trianglelefteq K$. Then the following diagram commutes up to isomorphism.*

$$\begin{array}{ccc} \text{Representations of } H_1/H_2 & \xrightarrow{\text{inflate}} & \text{Representations of } H_1 \\ \big\downarrow{\scriptstyle \mathrm{Ind}} & & \big\downarrow{\scriptstyle \mathrm{Ind}} \\ \text{Representations of } K/H_2 & \xrightarrow{\text{inflate}} & \text{Representations of } K. \end{array}$$

Proof. Let (ρ, W) be a representation of H_1/H_2. Its inflation to H_1 is $(\widetilde{\rho}, W)$, defined by

$$\widetilde{\rho}(h_1) := \rho(h_1 H_2), \quad h_1 \in H_1.$$

Inducing to K gives $\big(\mathrm{Ind}_{H_1}^K \widetilde{\rho}, W_{\widetilde{\rho}}\big)$, where

$$W_{\widetilde{\rho}} = \{f\colon K \to W \;\mid\; f \text{ is locally constant and } f(h_1 k) = \widetilde{\rho}(h_1) f(k),\ h_1 \in H_1,\ k \in K\},$$

with the action

$$(\mathrm{Ind}_{H_1}^K \widetilde{\rho}(g) f)(x) = f(xg), \quad g, x \in K.$$

Since $H_2 \trianglelefteq K$, each $f \in W_{\widetilde{\rho}}$ is constant on left cosets of H_2. Indeed, for $x \in K$ and $h_2 \in H_2$ we can write $x h_2 = h_2' x$ for some $h_2' \in H_2$, and hence

$$f(x h_2) = f(h_2' x) = \widetilde{\rho}(h_2') f(x) = f(x),$$

since $\widetilde{\rho}(h_2') = \rho(H_2) = \mathrm{Id}$.

On the other hand, if we first induce ρ to a representation of K/H_2 and then inflate to K, we obtain $\big(\mathrm{Ind}_{H_1/H_2}^{K/H_2} \rho, W_\rho\big)$, where

$$W_\rho = \left\{ f\colon K/H_2 \to W \;\middle|\; \begin{array}{l} f \text{ is locally constant, and} \\ f((h_1 H_2)(k H_2)) = \rho(h_1 H_2) f(k H_2), \quad h_1 \in H_1,\ k \in K \end{array} \right\},$$

with the action

$$(\widetilde{\mathrm{Ind}_{H_1/H_2}^{K/H_2}}\rho(g)f)(x) = f(xg), \quad g, x \in K/H_2.$$

Define $\Phi\colon W_\rho \to W_{\widetilde{\rho}}$ by

$$(\Phi f)(x) := f(xH_2), \quad x \in K.$$

This map is well-defined, bijective, and intertwines the K-actions. Hence

$$\mathrm{Ind}_{H_1}^{K}\widetilde{\rho} \;\cong\; \widetilde{\mathrm{Ind}_{H_1/H_2}^{K/H_2}}\rho,$$

and the diagram commutes. ∎

We end this section with the definition of the *depth* of a representation, which provides a fundamental link between representations, the Moy–Prasad filtrations, and the building of G.

Definition 2.3.15. [MP96, Theorem 3.5] Let (π, V) be a smooth representation of G. The *depth* of π is defined as the minimum real number $r \geq 0$ such that there exists a point x in the building $\mathscr{B}(G)$ with

$$V^{G_{x,r+}} \neq \{0\}.$$

Analogously, for a representation (σ, W) of $G_{x,t}$ with $t \geq 0$, we define

$$\mathrm{depth}(\sigma) := \min\{r \geq t \;\mid\; W^{G_{x,r+}} \neq \{0\}\}. \tag{2.3.1}$$

Lemma 2.3.16. *Let $t \in \mathbb{Z}_{\geq 0}$, $x \in \mathscr{B}(G)$, and let σ be a nontrivial irreducible representation of $G_{x,t}$. Then σ has depth $r \in \mathbb{Z}_{\geq 0}$ if and only if*

$$G_{x,r} \nsubseteq \ker(\sigma) \qquad \text{and} \qquad G_{x,r+} \subseteq \ker(\sigma).$$

Proof. Let (σ, W) be a nontrivial representation of $G_{x,t}$ of depth r. Then $W^{G_{x,r+}} \neq \{0\}$ by definition. This is equivalent to saying that the homomorphism space

$$\mathrm{Hom}_{G_{x,r+}}(\sigma|_{G_{x,r+}}, \mathbb{1})$$

is not equal to zero. By applying Frobenius reciprocity we obtain

$$\mathrm{Hom}_{G_{x,r+}}(\sigma|_{G_{x,r+}}, \mathbb{1}) \cong \mathrm{Hom}_{G_{x,t}}(\sigma, \mathrm{Ind}_{G_{x,r+}}^{G_{x,t}}\mathbb{1}) \neq 0.$$

Since σ is irreducible, the above homomorphism space being non-zero implies that σ is a subrepresentation of $\mathrm{Ind}_{G_{x,r+}}^{G_{x,t}}\mathbb{1}$. As $G_{x,r+} \trianglelefteq G_{x,t}$, by Theorem 2.3.8 we have

$$\mathrm{Res}_{G_{x,r+}}\mathrm{Ind}_{G_{x,r+}}^{G_{x,t}}\mathbb{1} \cong \bigoplus_{g \in G_{x,t}/G_{x,r+}} \mathbb{1},$$

i.e. the restriction of $\mathrm{Ind}_{G_{x,r+}}^{G_{x,t}} \mathbb{1}$ to $G_{x,r+}$ is $\mathbb{1}$-isotypic. Since σ is a subrepresentation of $\mathrm{Ind}_{G_{x,r+}}^{G_{x,t}} \mathbb{1}$, the restriction of σ to $G_{x,r+}$ must also be $\mathbb{1}$-isotypic and hence $G_{x,r+} \subseteq \ker(\sigma)$. Thus we have shown that if $G_{x,r+}$ fixes one non-zero vector in W, then it must fix all vectors in W. Since r was the minimal element satisfying the property that $G_{x,r+}$ fixes a vector in W, we obtain that $G_{x,r} \nsubseteq \ker(\sigma)$.

Now, for the converse direction, suppose that $G_{x,r+} \subseteq \ker(\sigma)$ while $G_{x,r} \nsubseteq \ker(\sigma)$. Then $W^{G_{x,r+}} \neq \{0\}$, and moreover $W^{G_{x,r}} = \{0\}$, since, as noted earlier, if $G_{x,r}$ fixes a non-zero, then vector it must fix all of W. Thus r is the minimal integer with the property that $W^{G_{x,r+}} \neq \{0\}$, therefore the depth of σ is r, as required. ∎

Remark 2.3.17. In particular, since $\mathcal{K} = G_{0,0}$, a nontrivial irreducible representation σ of $\mathcal{K}$ has depth $r \geq 0$ if and only if

$$\mathcal{K}_r \nsubseteq \ker(\sigma) \quad \text{and} \quad \mathcal{K}_{r+} \subseteq \ker(\sigma).$$

2.3.3 Characters and genericity

In this subsection, we discuss characters of the center Z of our unitary group G, and the notion of genericity for characters of maximal tori in G. Both these concepts will play a central role in the construction of representations in later chapters.

2.3.3.1 Characters of Z

As noted in §2.2.1, the center Z of G can be identified with E^1, the group of norm-one elements of E over F. Let $\chi(E^1)$ denote the group of continuous quasi-characters of E^1, and define

$$\chi^2(E^1) = \{\, \theta^2 \mid \theta \in \chi(E^1) \,\}.$$

Lemma 2.3.18. *The quotient group $\chi(E^1)/\chi^2(E^1)$ has index 2. In fact, there exists a unique nontrivial quadratic quasi-character δ of E^1 . It has depth zero, and every character θ of E^1 can be written as either $\theta = \mu^2$ or $\theta = \delta\mu^2$ for some quasi-character μ of E^1.*

Proof. Consider the squaring homomorphism

$$s : \chi(E^1) \longrightarrow \chi(E^1), \qquad \theta \longmapsto \theta^2.$$

Its kernel is precisely the subgroup of quadratic quasi-characters of E^1. If $\theta \in \ker(s)$, then $\theta \colon E^1 \to \{\pm 1\}$.

We claim that θ is trivial on $U := E^1 \cap (1 + \mathfrak{p}_E)$. Indeed, since $\ker(\theta)$ is an open subgroup of E^1, there exists $n \geq 1$ such that

$$E^1 \cap (1 + \mathfrak{p}_E^n) \subseteq \ker(\theta).$$

If $n = 1$, we are done. Suppose $n > 1$. Then the restriction of θ to $E^1 \cap (1 + \mathfrak{p}_E)$ factors through the quotient

$$(E^1 \cap (1 + \mathfrak{p}_E)) \big/ (E^1 \cap (1 + \mathfrak{p}_E^n)).$$

This group has order q^{n-1}, a power of q. If the restriction of θ were nontrivial, its image would have order 2, forcing the index of $\ker(\theta)$ in this quotient to be 2, which is impossible since q is odd. Thus θ is trivial on U.

Consequently, every quadratic quasi-character of E^1 factors through E^1/U. The quotient E^1/U is canonically isomorphic to the cyclic group μ_{q+1} of $(q+1)$-st roots of unity. This cyclic group has a unique nontrivial quadratic character. Let δ denote its inflation to E^1.

Then δ is nontrivial, quadratic, and (since it is trivial on U) has depth zero. Thus $\ker(s) = \{1, \delta\}$ has order 2, and so the image $\chi^2(E^1) = \mathrm{im}(s)$ has index 2 in $\chi(E^1)$. Equivalently,

$$\chi(E^1)/\chi^2(E^1) \cong \mathbb{Z}/2\mathbb{Z},$$

with nontrivial coset represented by δ. ∎

2.3.3.2 Generic characters

To construct positive-depth supercuspidal representations of G, one requires G-generic characters of positive depth of the anisotropic torus $\mathcal{T}$. In this subsection, we define and describe G-generic characters of positive depth of both the diagonal torus (2.2.4) and the anisotropic tori listed in Table 2.2.3.1.

Let $\mathbb{L}$ be a maximal torus of $\mathbb{G}$, and let $\Phi_{\text{abs}}(\mathbb{G}, \mathbb{L})$ denote the absolute root system of $\mathbb{G}$ with respect to $\mathbb{L}$. For each root $a \in \Phi_{\text{abs}}(\mathbb{G}, \mathbb{L})$, set

$$H_a = da^\vee(1) \in \mathfrak{g} \otimes_F \overline{F},$$

where $da^\vee$ denotes the derivative of the coroot $a^\vee$ associated to a, and $\overline{F}$ is a separable closure of F.

Definition 2.3.19. An element $\lambda \in \mathfrak{t}^*_{-r}$ is called *G-generic of depth $-r$* if

$$\nu(\lambda(H_a)) = -r \quad \text{for each } a \in \Phi_{\text{abs}}(\mathbb{G}, \mathbb{L}).$$

Since we have identified $\mathfrak{t}^*$ with $\mathfrak{t}$ §2.2.3.2, we may equivalently say that an element $\Gamma \in \mathfrak{t}_{-r}$ is G-generic of depth r if

$$\nu\big(\mathrm{Tr}(\Gamma H_a)\big) = -r \quad \text{for each } a \in \Phi_{\text{abs}}(\mathbb{G}, \mathbb{L}).$$

Lemma 2.3.20. *Let $T = \mathbb{L}(F)$ be a maximal F-torus of G.*

If T is the maximally F-split torus (2.2.4), then the depth r of any G-generic element of $\mathfrak{t}$ satisfies $r \in \mathbb{Z}$.

If $T = \mathcal{T}_{\gamma_1,\gamma_2}$ is an anisotropic torus from Table 2.2.3.1, then the depth r of any G-generic element in $\mathfrak{t}$ satisfies

$$r \in \mathbb{Z} \quad \text{when } \mathcal{T}_{\gamma_1,\gamma_2} \text{ is unramified,} \qquad r \in \tfrac{1}{2} + \mathbb{Z} \quad \text{when } \mathcal{T}_{\gamma_1,\gamma_2} \text{ is ramified.}$$

Proof. Note that for our unitary group $\mathbb{G}$, the absolute root system $\Phi_{\text{abs}}(\mathbb{G}, \mathbb{T})$ has a unique positive root α. In the split case we have

$$H_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Let $\Gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & -\overline{a} \end{smallmatrix}\right) \in \mathfrak{t}_{-r}$ with $a = a_0 + a_1\sqrt{\epsilon}$. Then $\nu(a_i) \geq r$ and

$$\mathrm{Tr}(\Gamma H_\alpha) = a + \overline{a} = 2a_0.$$

The genericity condition is $\nu(2a_0) = -r$, hence $\nu(a_0) = -r$. Since $a_0 \in F$ and valuations on F are integral, this forces $r \in \mathbb{Z}$.

On the other hand, if we take $\mathbb{L}(F) = \mathcal{T}_{\gamma_1,\gamma_2}$ to be one of the anisotropic tori listed in Table 2.2.3.1, then

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\mathsf{h}},$$

where $\mathsf{h} = \begin{pmatrix} \frac{1}{\sqrt{\gamma_1^{-1}\gamma_2}} & -\sqrt{\gamma_1^{-1}\gamma_2}^{\,-1} \\ & 1 \end{pmatrix}$ is the matrix conjugating $\mathbb{L}$ to $\mathbb{T}$. An element $\Gamma_{u,v} = \left(\begin{smallmatrix} u\sqrt{\epsilon} & v\gamma_1\sqrt{\epsilon} \\ v\gamma_2\sqrt{\epsilon} & u\sqrt{\epsilon} \end{smallmatrix}\right) \in \mathfrak{t}_{-r}$ is G-generic of depth $-r$ if and only if

$$\nu(\mathrm{Tr}(\Gamma_{u,v} H_\alpha)) = \nu\big(2v\sqrt{\epsilon}\,\sqrt{\gamma_1\gamma_2}\big) = -r.$$

Thus the condition $\nu(\mathrm{Tr}(\Gamma_{u,v} H_\alpha)) = -r$ is equivalent to

$$\nu(v) = -r - \nu(\sqrt{\gamma_1\gamma_2}).$$

Since $\nu(\sqrt{\gamma_1\gamma_2}) \in \{0, \frac{1}{2}\}$ and $\nu(v) \in \mathbb{Z}$, we deduce that

$$r \in \mathbb{Z} \quad \text{when } \nu(\sqrt{\gamma_1\gamma_2}) = 0, \qquad r \in \tfrac{1}{2} + \mathbb{Z} \quad \text{when } \nu(\sqrt{\gamma_1\gamma_2}) = \tfrac{1}{2}. \tag{2.3.2}$$

Since $\nu(\sqrt{\gamma_1\gamma_2}) = 0$ precisely when $\mathcal{T}_{\gamma_1,\gamma_2}$ is unramified, and $\nu(\sqrt{\gamma_1\gamma_2}) = \frac{1}{2}$ precisely when $\mathcal{T}_{\gamma_1,\gamma_2}$ is ramified, the conclusion follows. ∎

We are ready to define G-generic characters of $\mathbb{L}(F)$ of depth r. Set $s := \frac{r}{2}$ and let e be the Moy–Prasad isomorphisms of the abelian groups

$$\mathfrak{g}_{y,s+}/\mathfrak{g}_{y,r+} \to G_{y,s+}/G_{y,r+},$$

which restricts to an isomorphism of $\mathfrak{t}_{s+}/\mathfrak{t}_{r+} \to \mathcal{T}_{s+}/\mathcal{T}_{r+}$. Let ψ' be an additive character of the field F, chosen to be trivial on the maximal ideal $\mathfrak{p}_F$ and nontrivial on the ring of integers $\mathcal{O}_F$. Using ψ', we define an additive character ψ of the quadratic extension E by

$$\psi(x) \;=\; \psi'\left(\frac{x+\overline{x}}{2}\right) \quad \text{for all } x \in E.$$

Note that the ψ is a character of E that is nontrivial on $\mathcal{O}_E$ but trivial on $\mathfrak{p}_E$. We will fix this choice of ψ for the remainder of the thesis.

Definition 2.3.21. A smooth complex-valued character ϕ of $\mathbb{L}(F)$ is called *G-generic of depth r* if it is trivial on $\mathbb{L}(F)_{r+}$, non-trivial on $\mathbb{L}(F)_r$ and is realized by an element $\Gamma \in \mathfrak{t}_{-r}$ that is G-generic of depth r, *i.e.*, for every $t \in \mathbb{L}(F)_{s+}$,

$$\phi(t) = \psi(\mathrm{Tr}(\Gamma(e^{-1}(t)))).$$

Note that the image of Γ in $\mathfrak{t}_{-r}/\mathfrak{t}_{-s}$ is uniquely determined by this relation.

Remark 2.3.22. To define a G-generic character ϕ of depth r for an anisotropic torus $\mathcal{T}$, one requires a G-generic element of $\mathfrak{t}_{-r}$ of depth r. By Lemma 2.3.20, such an element can exist only when r is an integer or a half-integer. Therefore, the depths of G-generic characters are integral when $\mathcal{T}$ is unramified and half-integral when $\mathcal{T}$ is ramified.

Chapter 3

Representations of $\mathcal{K}$

Shalika constructed all irreducible representations of $\mathrm{SL}_2(\mathcal{O}_F)$ for $p \neq 2$ in his thesis [Sha04]. In this chapter, we extend the method developed by Shalika to construct certain positive-depth irreducible representations of the compact open subgroup $\mathcal{K}$. Since $\mathbb{G}_{\mathrm{der}}(\mathcal{O}_F)$ is isomorphic to $\mathrm{SL}_2(\mathcal{O}_F)$, the construction is similar, but has not been carried out in the literature. We also construct representations associated with the nilpotent orbits of elements in the Lie algebra of G, using the same approach. These representations play a crucial role in describing the branching rules in a neighborhood of the identity element.

3.1 Key concepts involved in the construction

We begin by recalling a foundational result from Clifford theory, as given in [Sha04, Lemmas 4.1.1 and 4.1.3], which serves as a key tool for extending irreducible representations from a normal subgroup to the whole group.

Theorem 3.1.1 (Clifford Theory [Sha04, Lemmas 4.1.1 and 4.1.3])**.** *Let K be a finite group, and let $N \lhd K$ be a normal subgroup.*

(a) Let ϕ be an irreducible representation of N, and define

$$N_K(\phi) = \{k \in K \mid \phi^k \cong \phi\}$$

to be its normalizer in K. If σ_i is an irreducible representation of $N_K(\phi)$ such that ϕ occurs as a subrepresentation of $\sigma_i|_N$, then the induced representation $\mathrm{Ind}_{N_K(\phi)}^K \sigma_i$ is irreducible. Moreover,

$$\mathrm{Ind}_{N_K(\phi)}^K \sigma_i \;\cong\; \mathrm{Ind}_{N_K(\phi)}^K \sigma_j \quad \textit{if and only if} \quad \sigma_i \cong \sigma_j.$$

(b) Suppose $K = AN$ with $A \leq K$ a subgroup and $N \lhd K$. Let Ψ be a one-dimensional representation of N that is K-invariant, i.e. $N_K(\Psi) = K$. Then

(a) *The irreducible representations σ of K satisfying $\sigma|_N \supset \Psi$ are in one-to-one correspondence with the irreducible representations ζ of A satisfying*

$$\zeta|_{A\cap N} \supset \Psi|_{A\cap N}$$

where $\sigma(an) = \zeta(a)\Psi(n)$ for all $a \in A$ and $n \in N$.

(b) *For each σ as in (a), the multiplicity of σ in $\mathrm{Ind}_N^K \Psi$ is equal to the degree of σ.*

Our goal is to use the above theorem to construct irreducible positive-depth representations of $\mathcal{K}$. Although $\mathcal{K}$ itself is not a finite group, by Remark 2.3.17, if an irreducible representation π of $\mathcal{K}$ has depth d, then $\mathcal{K}_{d+} \subseteq \ker(\pi)$, and therefore π factors through the finite quotient $\mathcal{K}/\mathcal{K}_{d+}$. Hence the problem reduces to the finite group setting, and irreducible positive-depth representations of $\mathcal{K}$ can be obtained by inflating irreducible representations of $\mathcal{K}/\mathcal{K}_{d+}$. Thus the theorem above will be applicable in the constructions that follow.

Since the construction proceeds by induction from a subgroup to the whole group, we recall from Lemma 2.3.14 that if $H_2 \leq H_1 \leq K$ and σ is a representation of a quotient group H_1/H_2, then inducing σ to K/H_2 and subsequently inflating to K yields the same representation as first inflating σ to H_1 and then inducing to K. We will use this identification throughout the chapter.

3.2 Positive-depth representations of $\mathcal{K}$

In this section, we construct certain positive-depth representations of $\mathcal{K}$ by using Theorem 3.1.1.

Let $d \in \mathbb{Z}_{>0}$ and let $z, u, v \in F$ such that $\nu(z) \geq -d$ and $\nu(v) > \nu(u) = -d$. Define

$$X(z) = \begin{pmatrix} z\sqrt{\epsilon} & 0 \\ 0 & z\sqrt{\epsilon} \end{pmatrix} \qquad \text{and} \qquad \widetilde{X}(u,v) = \begin{pmatrix} 0 & u\sqrt{\epsilon} \\ v\sqrt{\epsilon} & 0 \end{pmatrix}.$$

We will choose particular elements of $\mathfrak{g}_{0,-d}$ of the form

$$X = X(z) + \widetilde{X}(u,v) \in \mathfrak{g}_{0,-d}.$$

Recall that in §2.3.3.2 we fixed an additive character ψ of E that is trivial on $\mathfrak{p}_E$ but nontrivial on $\mathcal{O}_E$. We also recall that our maximal compact subgroup $\mathcal{K}$ corresponds to the parahoric subgroup $G_{0,0}$, and it has a filtration by normal subgroups indexed by the non-negative integers. By the Moy–Prasad isomorphism we have

$$\begin{aligned} \mathcal{K}_{\frac{d}{2}+}/\mathcal{K}_{d+} &\cong \mathfrak{g}_{0,\frac{d}{2}+}/\mathfrak{g}_{0,d+} \\ k + \mathcal{K}_{d+} &\mapsto (k - I) + \mathfrak{g}_{0,d+} \end{aligned}$$

Therefore, the function

$$k \mapsto \Psi_X(k) = \psi(\mathrm{Tr}(X(k-I))) \tag{3.2.1}$$

defines a character of the group $\mathcal{K}_{\frac{d}{2}+}$ of depth d. Note that z, v and u are uniquely determined by the character Ψ_X only modulo $\mathfrak{p}_F^{-\lceil \frac{d}{2} \rceil}$.

Thus, we are precisely in the setting of Theorem 3.1.1(a). We have a normal subgroup $\mathcal{K}_{\frac{d}{2}+}$ of $\mathcal{K}$ and an irreducible representation of it given by Ψ_X. To produce an irreducible representation of $\mathcal{K}$, we need an irreducible representation of the normalizer of Ψ_X in $\mathcal{K}$ whose restriction to $\mathcal{K}_{\frac{d}{2}+}$ contains Ψ_X. Therefore, we now compute the normalizer $N_{\mathcal{K}}(\Psi_X)$ of Ψ_X in $\mathcal{K}$.

Lemma 3.2.1. *Let* $\mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,-\frac{d}{2}})$ *denote the stabilizer in* $\mathcal{K}$ *of the coset* $X+\mathfrak{g}_{0,-\frac{d}{2}}$ *under the adjoint action. Then* $N_{\mathcal{K}}(\Psi_X) = \mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,-\frac{d}{2}})$.

Proof. Since Ψ is a character, the normalizer $N_{\mathcal{K}}(\Psi_X)$ equals

$$\begin{aligned} N_{\mathcal{K}}(\Psi_X) &= \left\{k \in \mathcal{K} \mid \Psi_X^k = \Psi_X\right\} \\ &= \{k \in \mathcal{K} \mid \Psi_X(k^{-1}gk) = \Psi_X(g)\ \forall g \in \mathcal{K}_{\frac{d}{2}+}\} \\ &= \{k \in \mathcal{K} \mid \psi(\mathrm{Tr}(X(k^{-1}gk-I))) = \psi(\mathrm{Tr}(X(g-I)))\ \forall g \in \mathcal{K}_{\frac{d}{2}+}\} \\ &= \{k \in \mathcal{K} \mid \psi(\mathrm{Tr}((kXk^{-1}-X)(g-I))) = 1\ \forall g \in \mathcal{K}_{\frac{d}{2}+}\}. \end{aligned}$$

Note that $g - I \in \mathfrak{g}_{0,\frac{d}{2}+}/\mathfrak{g}_{0,d+}$, therefore we may write

$$N_{\mathcal{K}}(\Psi_X) = \{k \in \mathcal{K} \mid \psi(\mathrm{Tr}((kXk^{-1}-X)Y)) = 1\ \forall Y \in \mathfrak{g}_{0,\frac{d}{2}+}\}.$$

A similar argument as in the proof of [Bou20, Proposition 3.1.12] yields

$$N_{\mathcal{K}}(\Psi_X) = \left\{k \in \mathcal{K} \,\middle|\, \mathrm{Tr}\big((kXk^{-1}-X)Y\big) \in \mathfrak{p}_F \text{ for all } Y \in \mathfrak{g}_{0,\frac{d}{2}+}\right\}.$$

For completeness, we include the argument. First note that for all $Y \in \mathfrak{g}$, $\mathrm{Tr}\big((kXk^{-1} - X)Y\big) \in F$. Assume, for contradiction, that there exists $Y' \in \mathfrak{g}_{0,\frac{d}{2}+}$ such that $b := \mathrm{Tr}\big((kXk^{-1}-X)Y'\big) \in \ker(\psi) \setminus \mathfrak{p}_F$. Choose $b' \in \mathcal{O}_F^\times$ with $b' \notin \ker(\psi)$; such a choice exists since ψ is non-trivial on $\mathcal{O}_F$. Then $b^{-1}b' \in \mathcal{O}_F$. Since $\mathfrak{g}_{0,\frac{d}{2}+}$ is an $\mathcal{O}_F$-module, we have $b^{-1}b'Y' \in \mathfrak{g}_{0,\frac{d}{2}+}$, and hence

$$\mathrm{Tr}\big((kXk^{-1}-X)(b^{-1}b'Y')\big) = b^{-1}b'\,\mathrm{Tr}\big((kXk^{-1}-X)Y'\big) = b' \notin \ker(\psi),$$

a contradiction.

Thus by (2.2.11), we have $N(\Psi_X) = \{k \in \mathcal{K} \mid kXk^{-1} - X \in \mathfrak{g}_{0,-\frac{d}{2}}\}$ which is the stabilizer of the coset $X + \mathfrak{g}_{0,-\frac{d}{2}}$ in $\mathcal{K}$ under the adjoint action, as required. ∎

Thus to compute the normalizer of Ψ_X, we need to compute $\mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,-\frac{d}{2}})$. Since conjugation by $\mathcal{K}$ preserves $\mathfrak{g}_{0,-\frac{d}{2}}$, and $\mathcal{K}$ centralizes $X(z)$, it suffices to find the stabilizer of the coset $\widetilde{X}(u,v)+\mathfrak{g}_{0,-\frac{d}{2}}$ in $\mathcal{K}$.

Note that the centralizer of X in $\mathcal{K}$ coincides with the centralizer of $\widetilde{X}(u,v)$ in $\mathcal{K}$. A direct computation yields that the centralizer $T(X)$ of X in $\mathcal{K}$ is given by

$$T(X)=\left\{\begin{pmatrix} a & b \\ bu^{-1}v & a\end{pmatrix} \,\middle|\, a,b\in\mathcal{O}_E, a\bar{a}+b\bar{b}u^{-1}v=1, \bar{a}b\in\sqrt{\epsilon}\mathcal{O}_F\right\}=\mathcal{T}_{1,u^{-1}v}\cap\mathcal{K}. \quad (3.2.2)$$

Proposition 3.2.2. *Let $X=\widetilde{X}(u,v)$ with $\mathrm{val}(u)=0$ and $\mathrm{val}(v)>0$. Let $s>0$. Let $k=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathcal{K}$. Then $k^{-1}Xk\in X+\mathfrak{g}_{0,s}$ if and only if $k=cg$ for some $c\in T(X)$ and $g\in G_{0,s}$.*

Proof. Replacing s with $\lceil s\rceil$ we may assume $s\in\mathbb{N}_{>0}$. Since $\mathfrak{g}_{0,s}$ is the intersection with $\mathfrak{g}$ of $\mathfrak{p}_E^s\mathfrak{gl}_2(\mathcal{O}_E)$, and conjugation by $k\in\mathcal{K}=G_{0,0}$ preserves $\mathfrak{g}$ and $\mathfrak{g}_{0,s}$, we can determine the elements $k\in\mathcal{K}$ that satisfy $k^{-1}Xk\in X+\mathfrak{g}_{0,s}$ by a system of congruences of matrix entries. The advantage is that kX is well-defined as an element of $\mathfrak{gl}_2(\mathcal{O}_E)$.

We note first that $T(X)G_{x,s}\subset\mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,s})$. Namely, suppose $k=cg$ with $c\in T(X)$ and $g\in G_{x,s}$. Since $c\in\mathcal{K}$, it preserves $\mathfrak{g}_{0,s}$, so we have $c\in\mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,s})$. As we can write $g=I+U$ where U is a matrix with all entries in $\mathfrak{p}_E^s$, it follows that $gX\equiv Xg \mod \mathfrak{p}_E^s$, where this indicates a congruence of matrix coefficients. Therefore $gXg^{-1}\in X+\mathfrak{g}_{0,s}$, yielding the result.

Now suppose $g\in\mathrm{Stab}_{\mathcal{K}}(X+\mathfrak{g}_{0,s})$. We write $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ with (among other conditions) $a,b,c,d\in\mathcal{O}_E$ and $ad-bc\in\mathcal{O}_E^\times$. Noting that $gXg^{-1}\equiv X$ modulo $\mathfrak{p}_E^s$ yields the equality $gX\equiv Xg$ modulo $\mathfrak{p}_E^s$. Computing the products on both sides yields the linear system

$$cu\equiv bv,\quad du\equiv au\quad\text{and}\quad av\equiv dv.$$

Since $u\in\mathcal{O}_E^\times$, this implies $a\equiv d \mod \mathfrak{p}_E^s$ and $c\equiv u^{-1}vb \mod \mathfrak{p}_E^s$. Referring back to (3.2.2), we see that k is congruent as a matrix modulo $\mathfrak{p}_E^s$ to an element of $T(X)$. In fact, we can argue by induction that there exists $k'\in T(X)$ such that $(k')^{-1}g\in G_{0,s}$, whence $g\in T(X)G_{0,s}$, as required. As this inductive argument is an involved matrix calculation, we have relegated it to an appendix A.1. ∎

We now have all the necessary tools to compute the normalizer $N(\Psi_X)$.

Proposition 3.2.3. *Let $d\in\mathbb{Z}_{>0}$ and let $z,u,v\in F$ such that $\nu(z)\geq -d$ and $\nu(v)>\nu(u)=-d$. Consider the element*

$$X=X(z)+\widetilde{X}(u,v)\in\mathfrak{g}_{0,-d}.$$

Then the normalizer of Ψ_X in $\mathcal{K}$ is $T(X)\,\mathcal{K}_{\frac{d}{2}}$.

Proof. By Lemma 3.2.1 we have $N_{\mathcal{K}}(\Psi_X) = \mathrm{Stab}_{\mathcal{K}}\left(X + \mathfrak{g}_{0,-\frac{d}{2}}\right)$, and by the discussion preceding Proposition 3.2.2 it follows that

$$\mathrm{Stab}_{\mathcal{K}}\left(X + \mathfrak{g}_{0,-\frac{d}{2}}\right) = \mathrm{Stab}_{\mathcal{K}}\left(\widetilde{X}(u,v) + \mathfrak{g}_{0,-\frac{d}{2}}\right).$$

Multiplying the coset $\widetilde{X}(u,v) + \mathfrak{g}_{0,-\frac{d}{2}}$ by ϖ^d yields $\widetilde{X}(\varpi^d u, \varpi^d v) + \mathfrak{g}_{0,\frac{d}{2}}$, and therefore

$$\mathrm{Stab}_{\mathcal{K}}\left(\widetilde{X}(u,v) + \mathfrak{g}_{0,-\frac{d}{2}}\right) = \mathrm{Stab}_{\mathcal{K}}\left(\widetilde{X}(\varpi^d u, \varpi^d v) + \mathfrak{g}_{0,\frac{d}{2}}\right).$$

Since $\nu(u) = -d$ and $\nu(v) > -d$, we have $\nu(\varpi^d u) = 0$ and $\nu(\varpi^d v) > 0$. Moreover $\frac{d}{2} > 0$, so by Proposition 3.2.2 we obtain

$$\mathrm{Stab}_{\mathcal{K}}\left(\widetilde{X}(\varpi^d u, \varpi^d v) + \mathfrak{g}_{0,\frac{d}{2}}\right) = T(\widetilde{X}(\varpi^d u, \varpi^d v))\,\mathcal{K}_{\frac{d}{2}} = T(X)\,\mathcal{K}_{\frac{d}{2}}.$$

Hence

$$\mathrm{Stab}_{\mathcal{K}}\left(\widetilde{X}(u,v) + \mathfrak{g}_{0,-\frac{d}{2}}\right) = T(X)\,\mathcal{K}_{\frac{d}{2}},$$

as required. ∎

Note that when d is odd we have $\mathcal{K}_{\frac{d}{2}+} = \mathcal{K}_{\frac{d}{2}}$. Thus in this case Ψ_X is a character of $\mathcal{K}_{\frac{d}{2}}$, and its normalizer satisfies $N(\Psi_X) = T(X)\,\mathcal{K}_{\frac{d}{2}}$. By part (b) of Theorem 3.1.1, irreducible representations σ of

$$T(X)\mathcal{K}_{\frac{d}{2}}$$

satisfying $\sigma|_{\mathcal{K}_{\frac{d}{2}}} \supset \Psi_X$ are in one-to-one correspondence with the irreducible representations ζ of $T(X)$ satisfying

$$\zeta|_{T(X)\cap\mathcal{K}_{\frac{d}{2}}} = \Psi_X|_{T(X)\cap\mathcal{K}_{\frac{d}{2}}}.$$

Since $T(X) \subset \mathcal{K}$ is abelian, we may extend $\Psi_X|_{T(X)\cap\mathcal{K}_{\frac{d}{2}}}$ to a character of $T(X)$. Let ζ denote such an extension, and let $\Psi_{X,\zeta}$ be the unique extension of these characters to $T(X)\mathcal{K}_{\frac{d}{2}}$. Then, by part (a) of Theorem 3.1.1, $\mathrm{Ind}_{T(X)\mathcal{K}_{\frac{d}{2}}}^{\mathcal{K}} \Psi_{X,\zeta}$ is an irreducible representation of $\mathcal{K}$. We record this result in the following lemma.

Lemma 3.2.4. *Let $d \in 2\mathbb{Z}_{>0}$, and let $X \in \mathfrak{g}_{0,-d}$ be as in Proposition 3.2.3. Let ζ be a character of $T(X)$ which coincides with Ψ_X on the intersection $T(X) \cap \mathcal{K}_{\frac{d}{2}}$. Write $\Psi_{X,\zeta}$ for the unique character of $T(X)\mathcal{K}_{\frac{d}{2}}$ which extends ζ and Ψ_X. Then $\mathrm{Ind}_{T(X)\mathcal{K}_{\frac{d}{2}}}^{\mathcal{K}} \Psi_{X,\zeta}$ is an irreducible representation of $\mathcal{K}$ of depth d and of degree $q^{d-1}(q^2-1)$.*

We now turn to the case where d is even. In this case, $\mathcal{K}_{\frac{d}{2}+} = \mathcal{K}_{\frac{d}{2}+1}$ is a proper normal subgroup of $\mathcal{K}_{\frac{d}{2}}$, and one verifies that Ψ_X does not extend to a character of $\mathcal{K}_{\frac{d}{2}}$. Therefore, the simplified argument used above no longer applies. Instead, we define the subgroup

$$\mathcal{J}_d = \mathcal{K}_{\frac{d}{2}} \cap G_{\frac{1}{2},\frac{d}{2}} = \left\{ \begin{pmatrix} 1+\mathfrak{p}_E^{\lceil \frac{d}{2} \rceil} & \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil} \\ \mathfrak{p}_E^{\lceil \frac{d+1}{2} \rceil} & 1+\mathfrak{p}_E^{\lceil \frac{d}{2} \rceil} \end{pmatrix} \right\} \cap G. \tag{3.2.3}$$

Then we have proper inclusions $\mathcal{K}_{\frac{d}{2}+} \subset \mathcal{J}_d \subset \mathcal{K}_{\frac{d}{2}}$ of subgroups and Ψ_X extends to a character of $\mathcal{J}_d$. We note that when d is odd, we have $\mathcal{J}_d = \mathcal{K}_{\frac{d}{2}}$.

Lemma 3.2.5. *Let $d \in 2\mathbb{Z}$. Then the normalizer of Ψ_X in $T(X)\mathcal{K}_{\frac{d}{2}}$, considered as a character of $\mathcal{J}_d$, is $T(X)\mathcal{J}_d$.*

Proof. It is straightforward to verify that $T(X)\mathcal{J}_d$ normalizes Ψ_X. To prove the reverse inclusion, we consider a set of coset representatives

$$R = \left\{ \begin{pmatrix} 1 & 0 \\ y\sqrt{\epsilon}\varpi^{\frac{d}{2}} & 1 \end{pmatrix} \,\middle|\, y \in \mathcal{O}_F^\times \right\}$$

for $\mathcal{K}_{\frac{d}{2}}/\mathcal{J}_d$. We will show that none of these representatives normalizes Ψ_X. As in the proof of Proposition 3.2.3 we may without loss of generality assume that $X = \widetilde{X}(u,v) \in \mathfrak{g}_{0,-d}$. We now show that for all $k \in R$ and $g \in \mathcal{J}_d$, $\Psi_X(g) \neq \Psi_{kXk^{-1}}(g)$.

Let $k = \begin{pmatrix} 1 & 0 \\ y\sqrt{\epsilon}\varpi^{\frac{d}{2}} & 1 \end{pmatrix} \in R$, and let $g = \begin{pmatrix} 1+a\varpi^{\frac{d}{2}} & b\varpi^{\frac{d}{2}} \\ c\varpi^{\frac{d}{2}+1} & 1-\overline{a}\varpi^{\frac{d}{2}} \end{pmatrix} \in \mathcal{J}_d$. Then

$$\Psi_X(g) = \psi(\mathrm{Tr}(X(g-I))) = \psi\left(uc\sqrt{\epsilon}\varpi^{\frac{d}{2}+1} + vb\sqrt{\epsilon}\varpi^{\frac{d}{2}} \right), \tag{3.2.4}$$

and

$$\Psi_{kXk^{-1}}(g) = \psi\left(uc\sqrt{\epsilon}\varpi^{\frac{d}{2}+1} + vb\sqrt{\epsilon}\varpi^{\frac{d}{2}} - yua\epsilon\varpi^{d} - yu\overline{a}\epsilon\varpi^{d} \right), \tag{3.2.5}$$

where we have simplified by removing all terms of valuation at least 1. Comparing (3.2.4) and (3.2.5) and writing $a = a_1 + \sqrt{\epsilon}a_2$, we find that the characters to agree if and only if

$$\psi(-2a_1yu\epsilon\varpi^d) = 1 \quad \text{for all } a_1 \in \mathcal{O}_F. \tag{3.2.6}$$

Since $y \in \mathcal{O}_F^\times$, $\epsilon \in \mathcal{O}_F^\times$, and $\nu(u) = -d$, we have $yu\epsilon\varpi^d \in \mathcal{O}_F^\times$. Thus, as a_1 ranges over $\mathcal{O}_F$, the expression $-2a_1yu\epsilon\varpi^d$ ranges over all of $\mathcal{O}_F$, and then (3.2.6) contradicts the nontriviality of ψ on $\mathcal{O}_F$. Therefore, $\Psi_X \neq \Psi_{kXk^{-1}}$, and k does not normalize Ψ_X. Hence, the normalizer of Ψ_X in $T(X)\mathcal{K}_{\frac{d}{2}}$ is $T(X)\mathcal{J}_d$. ∎

We now present the following theorem, which gives us the key representations of $\mathcal{K}$ that we will need in the sequel. It is a generalization of Shalika's results [Sha04, Theorems 4.2.1 and 4.2.5] to the context of G.

Theorem 3.2.6. *Let $X \in \mathfrak{g}_{0,-d}$ be as in Proposition 3.2.3, and let ζ be a character of $T(X)$ which coincides with Ψ_X on the intersection $T(X) \cap \mathcal{J}_d$. Write $\Psi_{X,\zeta}$ for the unique character of $T(X)\mathcal{J}_d$ which extends ζ and Ψ_X. Then*

$$\mathcal{S}_d(X,\zeta) := \mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \Psi_{X,\zeta} \tag{3.2.7}$$

is an irreducible representation of $\mathcal{K}$ of depth d and of degree $q^{d-1}(q^2-1)$.

Proof. When d is odd, we have already established in Lemma 3.2.4 that

$$S_d(X,\zeta) := \mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \Psi_{X,\zeta}$$

is an irreducible representation of $\mathcal{K}$. Now suppose that d is even. A direct computation yields that $\mathcal{J}_d$ is a normal subgroup of $T(X)\mathcal{K}_{\frac{d}{2}}$. By Lemma 3.2.5 the normalizer in $T(X)\mathcal{K}_{\frac{d}{2}}$ of the character Ψ_X of $\mathcal{J}_d$ is $T(X)\mathcal{J}_d$, and $\Psi_{X,\zeta}$ is an extension of Ψ_X to this group. Thus by Theorem 3.1.1, $\mathrm{Ind}_{T(X)\mathcal{J}_d}^{T(X)\mathcal{K}_{\frac{d}{2}}} \Psi_{X,\zeta}$ is an irreducible representation of $T(X)\mathcal{K}_{\frac{d}{2}}$ that contains Ψ_X upon restriction to $\mathcal{K}_{\frac{d}{2}+1}$. Once again using Clifford theory for $\mathcal{K}$, and $\mathrm{Ind}_{T(X)\mathcal{J}_d}^{T(X)\mathcal{K}_{\frac{d}{2}}} \Psi_{X,\zeta}$ as a representation of $T(X)\mathcal{K}_{\frac{d}{2}}$ that contains Ψ_X upon restriction to $\mathcal{K}_{\frac{d}{2}+1}$, we have that

$$\mathrm{Ind}_{T(X)\mathcal{K}_{\frac{d}{2}}}^{\mathcal{K}} \mathrm{Ind}_{T(X)\mathcal{J}_d}^{T(X)\mathcal{K}_{\frac{d}{2}}} \Psi_{X,\theta} = \mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \Psi_{X,\zeta} =: \mathcal{S}_d(X,\zeta)$$

is an irreducible representation of $\mathcal{K}$.

As Ψ_X has depth d and $\mathcal{K}_{d+} \trianglelefteq \mathcal{K}$, we have $\mathcal{K}_{d+} \subseteq \ker(\mathcal{S}_d(X,\zeta))$. Now let $k_d \in \mathcal{K}_d$ such that $\Psi_X(k_d) \neq 1$ and let $k \in \mathcal{K}$ and $f \in \mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \mathbb{C}$ such that $f(k) \neq 0$. Since $\mathcal{K}_d \trianglelefteq \mathcal{K}$, there exists $k'_d \in \mathcal{K}_d$ such that $k^{-1}k_d k = k'_d$. We then have $\mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}} \Psi_{X,\zeta}(k'_d) f(k) = f(kk'_d) = f(k_d k) = \Psi_X(k_d) f(k)$. Since $\Psi_X(k_d) \neq 1$ and $f(k) \neq 0$, we obtain that $k'_d \notin \ker(\mathcal{S}_d(X,\zeta))$, and hence the depth of $\mathcal{S}_d(X,\zeta)$ is also d.

Since $\Psi_{X,\zeta}$ is one-dimensional, the degree of $\mathcal{S}_d(X,\zeta)$ is equal to the index $[\mathcal{K}\colon T(X)\mathcal{J}_d]$. We have $\mathcal{K}_{\frac{d}{2}+} \leq T(X)\mathcal{J}_d \leq \mathcal{K}$, and so

$$[\mathcal{K}\colon T(X)\mathcal{J}_d] = \frac{[\mathcal{K}\colon \mathcal{K}_{\frac{d}{2}+}]}{[T(X)\mathcal{J}_d\colon \mathcal{J}_d][\mathcal{J}_d\colon \mathcal{K}_{\frac{d}{2}+}]}.$$

Since $T(X) \mod \mathfrak{p}_E \cong ZU$, $Z \cong E^1$, and $U \cong F$, we have $[T(X)\colon T(X) \cap \mathcal{K}_{0+}] = q(q+1)$. The indices $[T(X) \cap \mathcal{K}_{0+}\colon T(X) \cap \mathcal{J}_d], [\mathcal{J}_d\colon \mathcal{K}_{\frac{d}{2}+}], [\mathcal{K}_{0+}\colon \mathcal{K}_{\frac{d}{2}+}]$ are the same as the indices of corresponding $\mathcal{O}_F$-module in the Lie algebra. Thus we have

$$[T(X)\mathcal{J}_d\colon \mathcal{J}_d] = [T(X)\colon T(X) \cap \mathcal{K}_{0+}][T(X) \cap \mathcal{K}_{0+}\colon T(X) \cap \mathcal{J}_d] = q(q+1)q^{2\lceil \frac{d}{2} \rceil - 2},$$

$$[\mathcal{J}_d\colon \mathcal{K}_{\frac{d}{2}+}] = q^{3\left(\lceil \frac{d+1}{2} \rceil - \lceil \frac{d}{2} \rceil\right)},$$

and

$$[\mathcal{K}\colon \mathcal{K}_{\frac{d}{2}+}] = [\mathcal{K}\colon \mathcal{K}_{0+}][\mathcal{K}_{0+}\colon \mathcal{K}_{\frac{d}{2}+}] = |\mathrm{U}(1,1)(\mathfrak{f})|[\mathcal{K}_{0+}\colon \mathcal{K}_{\frac{d}{2}+}] = q(q-1)(q+1)^2 q^{4(\lceil \frac{d+1}{2}\rceil -1)}.$$

Putting everything together we obtain

$$[\mathcal{K}\colon T(X)\mathcal{J}_d] = \frac{q(q-1)(q+1)^2 q^{4(\lceil \frac{d+1}{2}\rceil -1)}}{q(q+1)q^{2\lceil \frac{d}{2}\rceil -2} q^{3\left(\lceil \frac{d+1}{2}\rceil - \lceil \frac{d}{2}\rceil\right)}} = (q^2-1)q^{\lceil \frac{d+1}{2}\rceil + \lceil \frac{d}{2}\rceil} q^{-2} = (q^2-1)q^{d-1}. \blacksquare$$

3.3 Representations associated to nilpotent orbits

Now let us focus on a special case of particular importance: when the element X is nilpotent. We will first determine a set of representatives for nilpotent orbits of $\mathcal{K}$ on $\mathfrak{g}$.

Recall that for matrix Lie algebras, an element X in $\mathfrak{g}$ nilpotent if and only if $X^n = 0$ for some positive integer n. By [Hum12, Theorem 3.2], it follows that all nilpotent elements of $\mathfrak{g}$ are G-conjugate to a strictly upper triangular matrix. For $\delta \in F$, we define a nilpotent element $X_\delta \in \mathfrak{g}$ as

$$X_\delta := \begin{pmatrix} 0 & \delta\sqrt{\epsilon} \\ 0 & 0 \end{pmatrix}. \tag{3.3.1}$$

Lemma 3.3.1. *The nilpotent G-orbits in $\mathfrak{g}$ are parametrized by the elements*

$$\{X_\delta \mid \delta \in \{0, 1, \varpi\}\}.$$

For $\delta \in \{0,1,\varpi\}$, let $\mathcal{N}_\delta$ denote the G-orbit of X_δ. Then each G-orbit $\mathcal{N}_\delta$ further decomposes as

$$\mathcal{N}_0 = \mathcal{K}\cdot X_0, \qquad \mathcal{N}_1 = \bigsqcup_{d\in 2\mathbb{Z}} \mathcal{K}\cdot X_{\varpi^{-d}}, \qquad \mathcal{N}_\varpi = \bigsqcup_{d\in 2\mathbb{Z}+1} \mathcal{K}\cdot X_{\varpi^{-d}}.$$

Proof. Let $X_{\delta_1}, X_{\delta_2} \in \mathfrak{g}$ with $\delta_1, \delta_2 \in F$, and suppose there exists

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{such that} \quad gX_{\delta_1}g^{-1} = X_{\delta_2}.$$

A direct calculation shows that if g satisfies this relation, then necessarily $c = 0$, $d = \overline{a}^{-1}$, and $a\overline{a}\,\delta_1 = \delta_2$.

Since $a \in E^\times$, we deduce that $\delta_2 \in \delta_1 \cdot N_{E/F}(E^\times)$. In particular, if $\delta_1 = 0$, then $\delta_2 = 0$, so X_0 lies in its own orbit. For $\delta_1 \neq 0$, the G-orbits of X_δ are thus parametrized by the cosets of $F^\times/N_{E/F}(E^\times)$. By Lemma 2.1.3, a set of representatives is given by $\{1, \varpi\}$, proving the first claim.

For the $\mathcal{K}$-orbits, we now require $g \in \mathcal{K}$. Then the above computation forces $a \in \mathcal{O}_E^\times$, so $a\overline{a} \in \mathcal{O}_F^\times$, and hence $a\overline{a}\,\delta_1 = \delta_2$ implies $\nu(\delta_1) = \nu(\delta_2)$. Conversely, if $\nu(\delta_1) = \nu(\delta_2)$, then

$\delta_2/\delta_1 \in \mathcal{O}_F^\times$, and since E/F is unramified, the norm map $N_{E/F} : \mathcal{O}_E^\times \to \mathcal{O}_F^\times$ is surjective, so we may choose $a \in \mathcal{O}_E^\times$ such that $a\overline{a} = \delta_2/\delta_1$, which gives $gX_{\delta_1}g^{-1} = X_{\delta_2}$ with $g \in \mathcal{K}$. This establishes the stated parametrization of $\mathcal{K}$-orbits. ∎

Let $d \in \mathbb{Z}_{>0}$, and $a \in \mathcal{O}_F^\times$. Then $X_{a\varpi^{-d}} \in \mathfrak{g}_{0,-d}$. We now apply the technique (3.2.1) of the preceding section to $X_{a\varpi^{-d}}$ to obtain a character $\Psi_{X_{a\varpi^{-d}}}$ of $\mathcal{J}_d$ given by $\Psi_{X_{a\varpi^{-d}}}(g) = \psi(\mathrm{Tr}(X_{a\varpi^{-d}}(g-I)))$. By direct computation it follows that the centralizer of $X_{a\varpi^{-d}}$ in $\mathcal{K}$ is given by $(ZU) \cap \mathcal{K}$ where

$$U = \left\{ \begin{pmatrix} 1 & \sqrt{\epsilon}b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in F \right\}.$$

Since $Z \subset \mathcal{K}$, $(ZU) \cap \mathcal{K} = Z(U \cap \mathcal{K})$. For convenience, set $\mathcal{U} := U \cap \mathcal{K}$. By Proposition 3.2.3 the normalizer of $\Psi_{X_{a\varpi^{-d}}}$ in $\mathcal{K}$ is given by $Z\mathcal{U}\mathcal{K}_{\frac{d}{2}}$. Observe that $\Psi_{X_{a\varpi^{-d}}}$ acts trivially on $Z\mathcal{U} \cap \mathcal{J}_d$.

We are interested only in the special case where the character ζ of $T(X) = Z\mathcal{U}$ is also trivial on $\mathcal{U}$, so that it is entirely determined by its value on Z. In this case, let θ be a character of Z such that $\theta\,|_{Z\cap\mathcal{J}_d} = \mathbb{1}$. We extend θ trivially across $\mathcal{U}$ to $Z\mathcal{U}$ and write $\Psi_{X_{a\varpi^{-d}},\theta}$ for the unique character of $Z\mathcal{U}\mathcal{J}_d$ that extends θ and $\Psi_{X_{a\varpi^{-d}}}$. Then from Theorem 3.2.6 we have that

$$\mathcal{S}_d(X_{a\varpi^{-d}}, \theta) := \mathrm{Ind}_{Z\mathcal{U}\mathcal{J}_d}^{\mathcal{K}} \Psi_{X_{a\varpi^{-d}},\theta} \tag{3.3.2}$$

is an irreducible representation of $\mathcal{K}$ of depth d and of degree $q^{d-1}(q^2-1)$.

Lemma 3.3.2. *Let $a \in \mathcal{O}_F^\times$. Then $\mathcal{S}_d(X_{\varpi^{-d}}, \theta) \cong \mathcal{S}_d(X_{a\varpi^{-d}}, \theta)$.*

Proof. By Theorem 3.1.1, conjugation by any $k \in \mathcal{K}$ sends $\mathcal{S}_d(X_{\varpi^{-d}}, \theta)$ to $\mathcal{S}_d(\mathrm{Ad}(k)X_{\varpi^{-d}}, \theta^k)$. Since θ is a character of the center Z, we have $\theta^k = \theta$. Moreover, $X_{\varpi^{-d}}$ and $X_{a\varpi^{-d}}$ lie in the same $\mathcal{K}$-orbit for all $a \in \mathcal{O}_F^\times$. Therefore,

$$\mathcal{S}_d(X_{\varpi^{-d}}, \theta) \cong \mathcal{S}_d(X_{a\varpi^{-d}}, \theta),$$

as claimed. ∎

Recall that for $z, u, v \in F$ such that $\nu(z) \geq -d$ and $\nu(v) > \nu(u) = -d$ and $X = X(z) + \widetilde{X}(u,v) \in \mathfrak{g}_{0,-d}$, we defined a character Ψ_X of $\mathcal{J}_d$. We conclude this chapter with a lemma that relates the characters Ψ_X and Ψ_{X_u} in a special but very informative case.

Lemma 3.3.3. *When $\nu(z), \nu(v) > -\lceil \frac{d}{2} \rceil$ and $\nu(u) = -d$, Then*

$$\Psi_X = \Psi_{X_u} \qquad \text{and} \qquad T(X)\mathcal{J}_d = T(X_u)\mathcal{J}_d = Z\mathcal{U}\mathcal{J}_d.$$

Proof. Since $X \equiv X_u$ entrywise modulo $\mathfrak{p}_E^{-\lceil d/2 \rceil}$, it follows that $\Psi_X = \Psi_{X_u}$ as characters of $\mathcal{J}_d$. Hence, they have the same normalizer in $T(X)\mathcal{K}_{\frac{d}{2}}$. By Proposition 3.2.3 and Lemma 3.2.5, we obtain

$$N_{T(X)\mathcal{K}_{\frac{d}{2}}}(\Psi_X) = T(X)\mathcal{J}_d \qquad \text{and} \qquad N_{T(X)\mathcal{K}_{\frac{d}{2}}}(\Psi_{X_u}) = T(X_u)\mathcal{J}_d = Z\mathcal{U}\mathcal{J}_d,$$

which proves the lemma. ∎

Chapter 4

Branching rules for depth-zero supercuspidal representations of G

In this chapter, we provide an explicit description of all depth-zero irreducible supercuspidal representations of G, together with their branching rules upon restriction to $\mathcal{K}$.

4.1 Construction of the representations

The depth-zero irreducible supercuspidal representations of G are induced from the cuspidal representations of $\mathrm{U}(1,1)(\mathfrak{f})$, which are well-known; see, for example, [Cam14]. Briefly: let $\mathfrak{e}$ denote the unique quadratic extension field of the residue field $\mathfrak{f}$ of $\mathcal{O}_F$, and let $N\colon \mathfrak{e}^\times \to \mathfrak{f}^\times$ be the norm map. For distinct characters α and β of $\ker(N) = \mathfrak{e}^1$, Deligne-Lusztig induction associates a representation $\sigma = \sigma(\alpha,\beta)$ of $\mathrm{U}(1,1)(\mathfrak{f})$ of degree $q-1$. This representation is cuspidal, and all cuspidal representations of $\mathrm{U}(1,1)(\mathfrak{f})$ arise in this way.

Since $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the matrices $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)$ belong to the same conjugacy class of G. Therefore, Table 4.1 (cf. [Cam14]) below summarizes the values of the character $\chi_{\sigma(\alpha,\beta)}$ corresponding to the cuspidal representation $\sigma(\alpha,\beta)$ of $\mathrm{U}(1,1)(\mathfrak{f})$.

Conjugacy class representatives	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, $x \in \mathfrak{e}^1$	$\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)$, $x \in \mathfrak{e}^1, y \neq 0$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $x \neq y$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, $y \neq 0$
$\chi_{\sigma(\alpha,\beta)}$	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$\alpha(x+y)\beta(x-y)+$ $\alpha(x-y)\beta(x+y)$

Table 4.1: Character $\chi_{\sigma(\alpha,\beta)}$ of the cuspidal representation $\sigma(\alpha,\beta)$ of $\mathrm{U}(1,1)(\mathfrak{f})$.

Let σ be a cuspidal representation of $\mathrm{U}(1,1)(\mathfrak{f}) \cong \mathcal{K}/\mathcal{K}_{0+}$. Inflate σ to a representation (also denoted by σ) of $\mathcal{K}$, and let σ^η denote the corresponding representation of $\mathcal{K}^\eta$, where $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$. By Lemma 2.2.10 and [MP96, Proposition 6.6], [Mor99] we obtain the following proposition.

Proposition 4.1.1. *Let σ be a cuspidal representation of $\mathrm{U}(1,1)(\mathfrak{f}) = \mathcal{K}/\mathcal{K}_{0+}$. Then the compact inductions*

$$c\text{-}\mathrm{Ind}_{\mathcal{K}}^{G}\sigma \qquad \text{and} \qquad c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta$$

are depth-zero irreducible supercuspidal representations of G, and every depth-zero irreducible supercuspidal representation of G arises in this way.

Remark 4.1.2. From now on, we identify a cuspidal representation σ of $\mathcal{K}/\mathcal{K}_{0+}$ with its inflation to $\mathcal{K}$.

4.2 Restriction to $\mathcal{K}$

In this section, we restrict the depth-zero irreducible supercuspidal representations of G to $\mathcal{K}$ and obtain a canonical decomposition using the Mackey decomposition. The Mackey components occurring in this decomposition may *a priori* be reducible. Therefore, we compute the degree of each component and determine the maximal depth of any irreducible subrepresentation it contains. The proof that these components are in fact irreducible is deferred to the next section, which builds on the results established here.

Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$. Then by Theorem 2.3.8 it follows that

$$\mathrm{Res}_{\mathcal{K}}^{G}\text{c-}\mathrm{Ind}_{\mathcal{K}}^{G}\sigma \cong \bigoplus_{g\in\mathcal{K}\backslash G/\mathcal{K}} \mathrm{Ind}_{\mathcal{K}\cap(\mathcal{K})^g}^{\mathcal{K}}\sigma^g \quad \text{and} \quad \mathrm{Res}_{\mathcal{K}}c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta = \bigoplus_{g\in\mathcal{K}\backslash G/\mathcal{K}^\eta} \mathrm{Ind}_{\mathcal{K}\cap\mathcal{K}^{g\eta}}^{\mathcal{K}}\sigma^{g\eta}.$$

By Lemma 2.2.5, a set of double coset representatives for $\mathcal{K}\backslash G/\mathcal{K}$ or $\mathcal{K}\backslash G/\mathcal{K}^\eta$ is given by

$$\left\{ \alpha^t := \begin{pmatrix} \varpi^{-t} & 0 \\ 0 & \varpi^t \end{pmatrix} \middle| t \geq 0 \right\}.$$

Therefore, we obtain the following decomposition:

$$\mathrm{Res}_{\mathcal{K}}c\text{-}\mathrm{Ind}_{\mathcal{K}}^{G}\sigma = \bigoplus_{t\geq 0} \mathrm{Ind}_{\mathcal{K}\cap\mathcal{K}^{\alpha^t}}^{\mathcal{K}}\sigma^{\alpha^t} \quad \text{and} \quad \mathrm{Res}_{\mathcal{K}}c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta = \bigoplus_{t\geq 0} \mathrm{Ind}_{\mathcal{K}\cap\mathcal{K}^{\alpha^t\eta}}^{\mathcal{K}}\sigma^{\alpha^t\eta}. \tag{4.2.1}$$

For each $t > 0$, the matrix $z_t = \varpi^{-t}I$ satisfies $\alpha^t = z_t\eta^{2t}$. Since z_t centralizes G, the conjugated representations σ^{α^t} and $\sigma^{\eta^{2t}}$ are equal. Therefore, we may write (4.2.1) as

$$\mathrm{Res}_{\mathcal{K}}c\text{-}\mathrm{Ind}_{\mathcal{K}}^{G}\sigma = \bigoplus_{t\geq 0} \mathrm{Ind}_{\mathcal{K}\cap\mathcal{K}^{\eta^{2t}}}^{\mathcal{K}}\sigma^{\eta^{2t}} \quad \text{and} \quad \mathrm{Res}_{\mathcal{K}}c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta = \bigoplus_{t\geq 0} \mathrm{Ind}_{\mathcal{K}\cap\mathcal{K}^{\eta^{2t+1}}}^{\mathcal{K}}\sigma^{\eta^{2t+1}}. \tag{4.2.2}$$

Lemma 4.2.1. *Let $d \in \mathbb{Z}_{>0}$. Then*

$$\mathcal{K} \cap \mathcal{K}^{\eta^d} = \mathcal{B}\mathcal{K}_d$$

where $\mathcal{B}$ denotes the subgroup of upper triangular matrices in $\mathcal{K}$.

Proof. Let $a = (a_{ij}) \in \mathcal{K} \cap \mathcal{K}^{\eta^d}$. Then

$$a \in \mathcal{K} \cap \mathcal{K}^{\eta^d} \quad \text{if and only if} \quad (\eta^d)^{-1} a \eta^d \in \mathcal{K} \text{ and } a \in \mathcal{K}.$$

Solving this condition yields $\nu(a_{21}) \geq d$. By Lemma 2.2.4, either $a_{11}, a_{22} \in \mathcal{O}_E^\times$ or $a_{12}, a_{21} \in \mathcal{O}_E^\times$. Since $a_{21} \in \mathfrak{p}_E^d$, it follows that $a_{11}, a_{22} \in \mathcal{O}_E^\times$. We claim that $a = bk$ with

$$b = \begin{pmatrix} \overline{a_{22}}^{-1} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in \mathcal{B} \quad \text{and} \quad k = \begin{pmatrix} 1 & 0 \\ a_{21}a_{22}^{-1} & 1 \end{pmatrix} \in \mathcal{K}_d.$$

Indeed, since $a \in \mathcal{K}$, we have $\overline{a_{12}} a_{22} + a_{12}\overline{a_{22}} = 0$ and $a_{11}\overline{a_{22}} + a_{21}\overline{a_{12}} = 1$, therefore

$$\overline{a_{22}}^{-1} + a_{12}a_{21}a_{22}^{-1} = \frac{1 + \overline{a_{22}} a_{12} a_{21} a_{22}^{-1}}{\overline{a_{22}}} = \frac{1 - a_{22}\overline{a_{12}} a_{21} a_{22}^{-1}}{\overline{a_{22}}} = \frac{a_{11}\overline{a_{22}}}{\overline{a_{22}}} = a_{11}.$$

Since $a_{22} \in \mathcal{O}_E^\times$ and $\overline{a_{12}} a_{22} \in \sqrt{\epsilon}\mathcal{O}_F$, we have $b \in \mathcal{B}$. As $a = bk$ with $a, b \in \mathcal{K}$, we must have $k \in \mathcal{K}$. Moreover, since $a_{21} \in \mathfrak{p}_E^d$ and $a_{22} \in \mathcal{O}_E^\times$, it follows that $a_{21}a_{22}^{-1} \in \mathfrak{p}_E^d$, and hence $k \in \mathcal{K}_d$. Therefore we have shown that $\mathcal{K} \cap \mathcal{K}^{\eta^d} \subseteq \mathcal{B}\mathcal{K}_d$.

Conversely, let $h \in \mathcal{B}\mathcal{K}_d$ with $h = bk$, where $b = (b_{ij}) \in \mathcal{B}$ and $k = (k_{ij}) \in \mathcal{K}_d$. Clearly, $h \in \mathcal{K}$. Moreover, since $b_{21} = 0$ and $k_{21} \in \mathfrak{p}_E^d$, a direct computation shows that

$$(\eta^d)^{-1} h \eta^d = (\eta^d)^{-1} b \eta^d \, (\eta^d)^{-1} k \eta^d \in \mathcal{K},$$

and hence $h \in \mathcal{K} \cap \mathcal{K}^{\eta^d}$. Therefore, $\mathcal{B}\mathcal{K}_d \subseteq \mathcal{K} \cap \mathcal{K}^{\eta^d}$. Combining both inclusions, we conclude that $\mathcal{K} \cap \mathcal{K}^{\eta^d} = \mathcal{B}\mathcal{K}_d$, as required. ∎

By the preceding lemma, for any $d > 0$, $\mathcal{K} \cap \mathcal{K}^{\eta^d} = \mathcal{B}\mathcal{K}_d$, that is, elements of $\mathcal{B}\mathcal{K}_d$ can be represented by matrices $(a_{ij}) \in \mathcal{K}$ such that $a_{21} \in \mathfrak{p}_E^d$. Consequently, (4.2.2) may be rewritten as

$$\mathrm{Res}_{\mathcal{K}} c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma = \sigma \oplus \bigoplus_{d \in 2\mathbb{Z}_{>0}} \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d} \quad \text{and} \quad \mathrm{Res}_{\mathcal{K}} c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma^\eta = \bigoplus_{d \in 2\mathbb{Z}_{\geq 0}+1} \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}. \quad (4.2.3)$$

Lemma 4.2.2. *Let $d \in \mathbb{Z}_{>0}$. Then $(\mathcal{B}\mathcal{K}_d)^{\eta^{-d}} = \mathcal{B}^{\mathrm{op}}\mathcal{K}_d$.*

Proof. Let $a = (a_{ij}) \in \mathcal{B}\mathcal{K}_d$. Then $a^{\eta^{-d}} = \begin{pmatrix} a_{11} & a_{12}\varpi^d \\ a_{21}\varpi^{-d} & a_{22} \end{pmatrix}$. By Lemma 4.2.1 we have $a_{21} \in \mathfrak{p}_E^d$ and $a_{11}, a_{22} \in \mathcal{O}_E^\times$. We claim that we can factor the matrix $a^{\eta^{-d}}$ as bk, where

$$b = \begin{pmatrix} a_{11} & 0 \\ a_{21}\varpi^{-d} & \overline{a_{11}}^{-1} \end{pmatrix} \in \mathcal{B}^{\mathrm{op}} \quad \text{and} \quad k = \begin{pmatrix} 1 & a_{11}^{-1}a_{12}\varpi^d \\ 0 & 1 \end{pmatrix} \in \mathcal{K}_d.$$

Since a is an element of $\mathcal{K}$, we have $\overline{a_{11}}a_{21} + a_{11}\overline{a_{21}} = 0$, and it follows that

$$a_{11}^{-1}a_{12}a_{21} + \overline{a_{11}}^{-1} = \frac{a_{11}^{-1}\overline{a_{11}}a_{21}a_{12} + 1}{\overline{a_{11}}} = \frac{-a_{11}^{-1}a_{11}\overline{a_{21}}a_{12} + 1}{\overline{a_{11}}} = \frac{\overline{a_{11}}a_{22}}{\overline{a_{11}}} = a_{22}.$$

Thus $a = bk$. As $a_{11} \in \mathcal{O}_E^\times$ and $\overline{a_{11}}a_{21} \in \sqrt{\epsilon}\mathcal{O}_F$, we have $b \in \mathcal{B}^{\mathrm{op}}$. Since $a = bk$ with $a, b \in \mathcal{K}$ and $a_{12} \in \mathfrak{p}_E^d$, we have $k \in \mathcal{K}_d$. Thus we have shown that $(\mathcal{B}\mathcal{K}_d)^{\eta^{-d}} \subseteq \mathcal{B}^{\mathrm{op}}\mathcal{K}_d$.

Conversely, let $a = (a_{ij}) \in \mathcal{B}^{\mathrm{op}}\mathcal{K}_d$. Then $a \in (\mathcal{B}\mathcal{K}_d)^{\eta^{-d}}$ if and only if $\eta^d a \eta^{-d} = a^{\eta^d} \in \mathcal{B}\mathcal{K}_d$. Since the $(1, 1)$-entry of any element of $\mathcal{B}^{\mathrm{op}}$ lies in $\mathcal{O}_E^\times$ and the $(1, 2)$-entry of any element of $\mathcal{K}_d$ lies in $\mathfrak{p}_E^d$, we deduce that $a_{12} \in \mathfrak{p}_E^d$. Thus we can factor a^{η^d} as $a^{\eta^d} = bk$, where

$$b = \begin{pmatrix} \overline{a_{22}}^{-1} & a_{12}\varpi^{-d} \\ 0 & a_{22} \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} 1 & 0 \\ a_{21}a_{22}^{-1}\varpi^d & 1 \end{pmatrix}.$$

As in the proof of Lemma 4.2.1, since $a \in \mathcal{K}$, we have $\overline{a_{12}}a_{22} + a_{12}\overline{a_{22}} = 0$ and $a_{11}\overline{a_{22}} + a_{21}\overline{a_{12}} = 1$, therefore

$$\overline{a_{22}}^{-1} + a_{12}a_{21}a_{22}^{-1} = \frac{1 + \overline{a_{22}}a_{12}a_{21}a_{22}^{-1}}{\overline{a_{22}}} = \frac{1 - a_{22}\overline{a_{12}}a_{21}a_{22}^{-1}}{\overline{a_{22}}} = \frac{a_{11}\overline{a_{22}}}{\overline{a_{22}}} = a_{11}.$$

Since $a \in \mathcal{K}$ and $a_{12} \in \mathfrak{p}_E^d$, it follows that $a_{22} \in \mathcal{O}_E^\times$ and $\overline{a_{22}}a_{12} \in \sqrt{\epsilon}\mathcal{O}_F$, hence $b \in \mathcal{B}$. As $a = bk$ with $a, b \in \mathcal{K}$ and $a_{12} \in \mathfrak{p}_E^d$, we have $k \in \mathcal{K}_d$. Thus we have shown that $a^{\eta^d} \in \mathcal{B}\mathcal{K}_d$, hence $a \in (\mathcal{B}\mathcal{K}_d)^{\eta^{-d}}$, and therefore the inclusion $\mathcal{B}^{\mathrm{op}}\mathcal{K}_d \subseteq (\mathcal{B}\mathcal{K}_d)^{\eta^{-d}}$ follows. Combining both the inclusions we obtain $\mathcal{B}^{\mathrm{op}}\mathcal{K}_d = (\mathcal{B}\mathcal{K}_d)^{\eta^{-d}}$. ∎

Proposition 4.2.3. *Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$. Then for any $d \geq 0$, the maximum depth of any irreducible component of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is d. Moreover, there exists at least one component that has depth d.*

Proof. When $d = 0$, the summand is simply σ, which has depth zero by hypothesis. Let $d > 0$. By Lemma 4.2.2, the action of σ^{η^d} on $\mathcal{B}\mathcal{K}_d$ is given by the action of σ on $\mathcal{B}^{\mathrm{op}}\mathcal{K}_d$, and since σ is trivial on $\mathcal{K}_d$, this is determined by $\mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma$. Explicitly, for $a = (a_{ij}) \in \mathcal{B}\mathcal{K}_d$ we have

$$\sigma^{\eta^d}(a) = \sigma\left(a^{\eta^{-d}}\right) = \sigma\left(\begin{pmatrix} a_{11} & a_{12}\varpi^d \\ a_{21}\varpi^{-d} & a_{22} \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} a_{11} & 0 \\ a_{21}\varpi^{-d} & \overline{a_{11}}^{-1} \end{pmatrix}\right).$$

Since $\mathcal{K}_{d+1} \subset \mathcal{K}_d$ and σ has depth zero, we deduce that $\mathcal{K}_{d+1} \subseteq \ker(\sigma^{\eta^d})$. As $\mathcal{K}_{d+1}$ is a normal subgroup of $\mathcal{K}$, it follows that $\mathcal{K}_{d+1} \subseteq \ker(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d})$. Therefore, the maximum possible depth of any irreducible component of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is d.

We claim that $\mathcal{K}_d \nsubseteq \ker(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d})$, and hence at least one irreducible component of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ has depth d. Suppose, for contradiction, that $\mathcal{K}_d \subseteq \ker(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d})$. Since $\mathcal{K}_d$ is a normal subgroup of $\mathcal{K}$, it would then follow that $\mathcal{K}_d \subseteq \ker(\sigma^{\eta^d})$. Therefore, to prove that $\mathcal{K}_d \nsubseteq \ker(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d})$, it suffices to show that $\mathcal{K}_d \nsubseteq \ker(\sigma^{\eta^d})$.

Let $b = \left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right) \in \mathcal{B}^{\mathrm{op}}$, where $c \in \sqrt{\epsilon}\mathcal{O}_F^\times$, and let χ_σ denote the character of σ. From the third column of Table 4.1, we see that $\chi_\sigma(b) = -1$, hence $\sigma(b) \neq \mathrm{Id}$. Now consider the element $\left(\begin{smallmatrix}1 & 0\\ c\varpi^d & 1\end{smallmatrix}\right) \in \mathcal{K}_d$. Applying σ^{η^d} to this element, we obtain

$$\sigma^{\eta^d}\left(\begin{pmatrix}1 & 0\\ c\varpi^d & 1\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right) \neq \mathrm{Id}.$$

This shows that the element $\left(\begin{smallmatrix}1 & 0\\ c\varpi^d & 1\end{smallmatrix}\right)$ is not in the kernel of σ^{η^d}, and we conclude that $\mathcal{K}_d \not\subseteq \ker(\sigma^{\eta^d})$. ∎

Proposition 4.2.4. *Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$. Then for any $d > 0$, the degree of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is $q^{d-1}(q^2-1)$.*

Proof. Since σ has degree $q-1$, the degree of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is given by $(q-1)$ times the index $[\mathcal{K} : \mathcal{B}\mathcal{K}_d]$. Noting the inclusions $\mathcal{K}_d \subseteq \mathcal{B}\mathcal{K}_d \subseteq \mathcal{K}$, and using group isomorphism theorems we have

$$[\mathcal{K} : \mathcal{B}\mathcal{K}_d] = \frac{[\mathcal{K} \colon \mathcal{K}_d]}{[\mathcal{B}\mathcal{K}_d \colon \mathcal{K}_d]} = \frac{[\mathcal{K} \colon \mathcal{K}_1][\mathcal{K}_1 \colon \mathcal{K}_d]}{[\mathcal{B} \colon (\mathcal{B} \cap \mathcal{K}_d)]} = \frac{[\mathcal{K} \colon \mathcal{K}_1][\mathcal{K}_1 \colon \mathcal{K}_d]}{[\mathcal{B} \colon (\mathcal{B} \cap \mathcal{K}_1)][(\mathcal{B} \cap \mathcal{K}_1) \colon (\mathcal{B} \cap \mathcal{K}_d)]}.$$

Since $\mathcal{K}/\mathcal{K}_1 \cong \mathrm{U}(1,1)(\mathfrak{f})$ and $\mathcal{B}/(\mathcal{B} \cap \mathcal{K}_1) \cong \mathrm{B}(\mathfrak{f})$, where $\mathrm{B}(\mathfrak{f})$ denotes the subgroup of $\mathrm{U}(1,1)(\mathfrak{f})$ consisting of upper triangular matrices, we obtain

$$[\mathcal{K} \colon \mathcal{K}_1] = |\mathrm{U}(1,1)(\mathfrak{f})| = q(q-1)(q+1)^2 \quad \text{and} \quad [\mathcal{B} \colon \mathcal{B} \cap \mathcal{K}_1] = |\mathrm{B}(\mathfrak{f})| = q(q^2-1).$$

The indices $[\mathcal{K}_1 \colon \mathcal{K}_d]$ and $[(\mathcal{B} \cap \mathcal{K}_1) \colon (\mathcal{B} \cap \mathcal{K}_d)]$ are the same as the indices of the corresponding $\mathcal{O}_F$-modules in the Lie algebra. Thus we have

$$[\mathcal{K}_1 \colon \mathcal{K}_d] = q^{4(d-1)} \text{ and } [(\mathcal{B} \cap \mathcal{K}_1) \colon (\mathcal{B} \cap \mathcal{K}_d)] = q^{3(d-1)}.$$

Thus, the degree of $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ becomes

$$(q-1) \cdot [\mathcal{K} : \mathcal{B}\mathcal{K}_d] = (q-1)\frac{q(q-1)(q+1)^2 q^{4(d-1)}}{q(q^2-1)q^{3(d-1)}} = (q^2-1)q^{d-1}.$$ ∎

The central character of σ is the map $\theta \colon \mathfrak{e}^1 \to \mathbb{C}^\times$ defined by $\sigma(zI) = \theta(z)\mathrm{Id}$ for all $z \in \mathfrak{e}^1$. From Table 4.1, we infer that $\theta(z) = \alpha(z)\beta(z)$. As $Z \subset \mathcal{K}$, the induced representations c-$\mathrm{Ind}_{\mathcal{K}}^{G}\sigma$ and c-$\mathrm{Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta$ have the same central character as σ.

Lemma 4.2.5. *Let $d > 0$. For $i \in \{1,2\}$, let σ_i be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ_i. Then $\mathrm{Res}_{\mathcal{B}\mathcal{K}_d}\sigma_1^{\eta^d} \cong \mathrm{Res}_{\mathcal{B}\mathcal{K}_d}\sigma_2^{\eta^d}$ if and only if $\theta_1 = \theta_2$. Furthermore, $\mathrm{Res}_{\mathcal{B}\mathcal{K}_d}\sigma^{\eta^d}$ is irreducible for all σ.*

Proof. By Lemma 4.2.2, the action of $\sigma_i^{\eta^d}$ on $\mathcal{B}\mathcal{K}_d$ is given by the action of σ_i on $\mathcal{B}^{\mathrm{op}}\mathcal{K}_d$, and since σ_i is trivial on $\mathcal{K}_d$, this is determined by $\mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma_i$. It thus suffices to show that

$\mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma_1 \cong \mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma_2$ if and only if $\theta_1 = \theta_2$. Since these representations factor through the finite group quotient $\mathrm{U}(1,1)(\mathfrak{f}) \cong \mathcal{K}/\mathcal{K}_{0+}$, it suffices to compare their characters. Write $\mathcal{B}^{\mathrm{op}}$ also for the image of $\mathcal{B}^{\mathrm{op}}$ in $\mathrm{U}(1,1)(\mathfrak{f})$. Using Table 4.1, the character χ_{σ_i} of $\mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma_i$ is given on elements of $\mathcal{B}^{\mathrm{op}}$ by

$$\chi_{\sigma_i}\left(\begin{pmatrix} a & 0 \\ c & \overline{a}^{-1}\end{pmatrix}\right) = \begin{cases} (q-1)\theta_i(a) & \text{if } a \in \mathfrak{e}^1,\ c = 0; \\ -\theta_i(a) & \text{if } a \in \mathfrak{e}^1,\ c \neq 0; \\ 0 & \text{otherwise.}\end{cases}$$

We now calculate the intertwining number between χ_{σ_1} and χ_{σ_2} to be

$$\begin{aligned}\mathcal{I}(\chi_{\sigma_1}, \chi_{\sigma_2}) &= \frac{1}{|\mathcal{B}^{\mathrm{op}}|}\sum_{g \in \mathcal{B}^{\mathrm{op}}} \chi_{\sigma_1}(g)\overline{\chi_{\sigma_2}(g)} \\ &= \frac{1}{q(q^2-1)}\left(\sum_{a\in\mathfrak{e}^1, c=0}\theta_1(a)\overline{\theta_2(a)}(q-1)^2 + \sum_{a \in \mathfrak{e}^1, c\neq 0}\theta_1(a)\overline{\theta_2(a)}\right) \\ &= \begin{cases}\frac{1}{q(q^2-1)}\left((q+1)(q-1)^2 + (q-1)(q+1)\right) = 1 & \text{if } \theta_1 = \theta_2; \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since $\mathcal{I}(\chi_\sigma, \chi_\sigma) = 1$ for all σ, it follows from [Ste, Corollary 4.3.15] that $\mathrm{Res}_{\mathcal{B}^{\mathrm{op}}}\sigma$ is irreducible. Moreover, as $\mathcal{I}(\chi_{\sigma_1}, \chi_{\sigma_2}) = 1$ if and only if $\theta_1 = \theta_2$, and is zero otherwise, the lemma follows from [Ste, Theorem 4.3.9]. ∎

The following proposition shows that, for each $d > 0$ and each cuspidal representation σ of $\mathcal{K}/\mathcal{K}_{0+}$, each of the Mackey components $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is independent of the choice of σ up to its central character θ.

Proposition 4.2.6. *Let $d > 0$. For $i \in \{1, 2\}$, let σ_i be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ_i. Then $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_1^{\eta^d} \cong \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_2^{\eta^d}$ if and only if $\theta_1 = \theta_2$.*

Proof. By the preceding lemma, if $\theta_1 = \theta_2$ then $\mathrm{Res}_{\mathcal{B}\mathcal{K}_d}\sigma_1^{\eta^d} \cong \mathrm{Res}_{\mathcal{B}\mathcal{K}_d}\sigma_2^{\eta^d}$, and therefore we have $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_1^{\eta^d} \cong \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_2^{\eta^d}$.

Conversely, assume that $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_1^{\eta^d} \cong \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_2^{\eta^d}$. As $Z \subseteq \mathcal{B}\mathcal{K}_d$, the induced representations $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_i^{\eta^d}$ have the same central character as σ_i. Since $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_1^{\eta^d} \cong \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma_2^{\eta^d}$, it follows that $\theta_1 = \theta_2$. ∎

Since $Z \cong E^1$, and $E^1/(E^1)^2$ has index two, there is a unique non-trivial quadratic character of E^1. Let δ denote that character. It can be shown that δ has depth-zero. We now prove a lemma that will be used in §5.3.

Lemma 4.2.7. *Every depth-zero irreducible supercuspidal representation π of G can be expressed as $\pi \cong \lambda \otimes \pi'$, where λ is a character of G and π' is a depth-zero irreducible supercuspidal representation of G whose central character is either $\mathbb{1}$ or δ.*

Proof. Let π be a depth-zero supercuspidal representation of G with central character θ. Then there exists a cuspidal representation σ of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ such that $\pi \cong c\text{-Ind}_{\mathcal{K}}^{G}\sigma$ or $\pi \cong c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta$. We can write $\theta = \delta^k\phi^2$ for some $k \in \{0,1\}$. Let $\widetilde{\sigma} = (\phi^{-1}\circ\det)\otimes\sigma$, which is again a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$. Then $c\text{-Ind}_{\mathcal{K}}^{G}\sigma \cong (\phi\circ\det)\otimes c\text{-Ind}_{\mathcal{K}}^{G}\widetilde{\sigma}$ and $c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta \cong (\phi\circ\det)\otimes c\text{-Ind}_{\mathcal{K}^\eta}^{G}\widetilde{\sigma}^\eta$, and the result follows. ∎

4.3 Irreducibility of the Mackey components

Let σ be a depth-zero irreducible cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ. In the previous section we saw that the restrictions to $\mathcal{K}$ of the depth-zero irreducible supercuspidal representations $c\text{-Ind}_{\mathcal{K}}^{G}\sigma$ and $c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta$ are given by

$$\mathrm{Res}_{\mathcal{K}}\, c\text{-Ind}_{\mathcal{K}}^{G}\sigma = \sigma \oplus \bigoplus_{d\in 2\mathbb{Z}_{>0}} \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d} \quad \text{and} \quad \mathrm{Res}_{\mathcal{K}}\, c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta = \bigoplus_{d\in 2\mathbb{Z}_{>0}+1} \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}.$$

In this section, we prove that for each $d \in \mathbb{Z}_{>0}$, each of the Mackey components $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ intertwines with the irreducible representation of the same degree constructed in §3.3, and, as a consequence, we deduce their irreducibility.

We first briefly review the construction of the irreducible representations $\mathcal{S}_d(X_{a\varpi^{-d}},\theta)$ from §3.3 where $a \in \mathcal{O}_F^\times$. For each $\delta \in F$, we denoted

$$X_\delta := \begin{pmatrix} 0 & \delta\sqrt{\epsilon} \\ 0 & 0 \end{pmatrix},$$

the nilpotent element of the Lie algebra of G. For each integer $d > 0$, we defined the compact open subgroup

$$\mathcal{J}_d := \mathcal{K}_{\frac{d}{2}} \cap G_{\frac{1}{2},\frac{d}{2}},$$

together with the character

$$\eta_d(j) := \Psi_{X_{a\varpi^{-d}}}(j) = \psi\big(\mathrm{Tr}(X_{a\varpi^{-d}}(j-I))\big), \quad j \in \mathcal{J}_d,$$

where $a \in \mathcal{O}_F^\times$. If θ is a character of the center Z such that $\theta|_{Z\cap\mathcal{J}_d} = \mathbb{1}$, we wrote $\Psi_{X_{a\varpi^{-d}},\theta}$ for the unique character of $Z\mathcal{U}\mathcal{J}_d$ extending both θ and $\Psi_{X_{a\varpi^{-d}}}$. We proved in §3.3 that the induced representation

$$\mathcal{S}_d(X_{a\varpi^{-d}},\theta) := \mathrm{Ind}_{Z\mathcal{U}\mathcal{J}_d}^{\mathcal{K}}\Psi_{X_{a\varpi^{-d}},\theta} \tag{4.3.1}$$

is irreducible, of depth d and degree $q^{d-1}(q^2-1)$. Moreover, by Lemma 3.3.2 $\mathcal{S}_d(X_{a\varpi^{-d}},\theta) \cong \mathcal{S}_d(X_{\varpi^{-d}},\theta)$ for all $a \in \mathcal{O}_F^\times$.

Theorem 4.3.1. *Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ. Then for each $d \in \mathbb{Z}_{>0}$, we have $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d} \cong \mathcal{S}_d(X_{\varpi^{-d}},\theta)$. Consequently, each $\mathcal{K}$-representation $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}}\sigma^{\eta^d}$ is irreducible.*

Proof. Since σ has depth zero, for each $d \in \mathbb{Z}_{>0}$, we have that $\theta|_{Z\cap\mathcal{J}_d} = \mathbb{1}$. Therefore, by (3.3.2)

$$S_d(X_{-\varpi^{-d}}, \theta) = \mathrm{Ind}_{Z\mathcal{U}\mathcal{J}_d}^{\mathcal{K}} \Psi_{X_{-\varpi^{-d}},\theta}$$

is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$. Since both the representations $S_d(X_{-\varpi^{-d}}, \theta)$ and $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}$ have the same degree (Proposition 4.2.4) and one of them is irreducible, it suffices to show that the space of intertwining operators between them is nonzero. Since

$$\mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}, \mathcal{S}_d(X_{-\varpi^{-d}}, \theta)) \cong \bigoplus_{g \in \mathcal{B}\mathcal{K}_d \backslash \mathcal{K}/Z\mathcal{U}\mathcal{J}_d} \mathrm{Hom}_{\mathcal{B}\mathcal{K}_d \cap (Z\mathcal{U}\mathcal{J}_d)^g}(\sigma^{\eta^d}, \Psi^g_{X_{-\varpi^{-d}},\theta}),$$

it is enough to show that

$$\mathrm{Hom}_{\mathcal{B}\mathcal{K}_d \cap (Z\mathcal{U}\mathcal{J}_d)}(\sigma^{\eta^d}, \Psi_{X_{-\varpi^{-d}},\theta}) \neq 0.$$

Since σ^{η_d} and $\Psi_{X_{-\varpi^{-d}},\theta}$ have depth d, they factor through the finite group quotient $(\mathcal{B}\mathcal{K}_d \cap Z\mathcal{U}\mathcal{J}_d)/\mathcal{K}_{d+1}$. Thus our approach is to evaluate the characters of these representations of finite groups and calculate their intertwining number $\mathcal{I}$ and show that $\mathcal{I} \neq 0$.

Computing the characters is straightforward. The character χ_{η^d} of σ^{η^d} on an element $g = (g_{ij}) \in \mathcal{B}\mathcal{K}_d$ is given by

$$\chi_{\eta^d}(g) = \mathrm{Tr}\left(\sigma^{\eta^d}\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right)\right) = \mathrm{Tr}\left(\sigma\left(\begin{pmatrix} g_{11} & 0 \\ g_{21}\varpi^{-d} & \overline{g_{11}}^{-1} \end{pmatrix}\right)\right)$$

where $g_{11} \in \mathcal{O}_E^\times$, and $g_{21} \in \mathfrak{p}_E^d$. Therefore using Table 4.1, we have

$$\chi_{\eta^d}(g) = \begin{cases} (q-1)\theta(g_{11}) & \text{if } g_{11} \in z + \mathfrak{p}_E \text{ for some } z \in E^1, \text{ and } g_{21} \in \mathfrak{p}_E^{d+1}; \\ -\theta(g_{11}) & \text{if } g_{11} \in z + \mathfrak{p}_E \text{ for some } z \in E^1, \text{ and } g_{21} \in \mathfrak{p}_E^d \setminus \mathfrak{p}_E^{d+1}; \\ 0 & \text{otherwise.} \end{cases}$$

We now provide a formula for the character $\Psi_{X_{-\varpi^{-d}},\theta}$. Let $g \in \mathcal{B}\mathcal{K}_d \cap Z\mathcal{U}\mathcal{J}_d$. Then by Lemma 4.2.1, $g_{21} \in \mathfrak{p}_E^d$, and $g_{11}^{-1}g_{12} \in \sqrt{\epsilon}\mathcal{O}_F$. Since $Z\mathcal{U} \subset \mathcal{B}$, we may factor g as $g = th$ where $t \in Z\mathcal{U}$ and $h = (h_{ij}) \in \mathcal{B}\mathcal{K}_d \cap \mathcal{J}_d$. Then there exists $z \in E^1$, such that $g_{11} \equiv z \mod \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$ and $h_{21} = z^{-1}g_{21}$. Since $g_{11} \equiv z \mod \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$ we have $z \in g_{11}(1 + g_{11}^{-1}\mathfrak{p}_E^{\lceil \frac{d}{2} \rceil})$ which implies that $z^{-1} \equiv g_{11}^{-1} \mod \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$ and therefore we have $h_{21} = z^{-1}g_{21} \equiv g_{11}^{-1}g_{21} \mod \mathfrak{p}_E^{\lceil d+1 \rceil}$. As $g_{21} \in \mathfrak{p}_E^d$, $h_{21} \equiv g_{11}^{-1}g_{21} \mod \mathfrak{p}_E^{\lceil d+1 \rceil}$ and ψ is trivial on $\mathfrak{p}_E$, it follows that

$$\Psi_{X_{-\varpi^{-d}},\theta}(g) = \theta(t)\psi(\mathrm{Tr}(X_{-\varpi^{-d}}(h - I))) = \theta(g_{11})\psi(-\varpi^{-d}h_{21}) = \theta(g_{11})\psi(-g_{11}^{-1}g_{21}\varpi^{-d}).$$

Computing the intertwining number is technically challenging, but follows easily once we arrange the sum in an appropriate way. Note that if $g_{21} \in \mathfrak{p}_E^{d+1}$, then $\psi(-g_{11}^{-1}g_{21}\varpi^{-d}) = 1$ as ψ is trivial on $\mathfrak{p}_E$, and it follows that

$$\Psi_{X_{-\varpi^{-d}},\theta}(g) = \theta(g_{11})\psi(-g_{11}^{-1}g_{21}\varpi^{-d}) = \theta(g_{11}).$$

Thus, to evaluate $\Psi_{X_{-\varpi^{-d}},\theta}$, it suffices to consider its restriction to the following two subsets of

$$N := (\mathcal{B}\mathcal{K}_d \cap Z\mathcal{U}\mathcal{J}_d)/\mathcal{K}_{d+1}.$$

$$S_1 := \{g \in N \mid g_{11} \in z + \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil} \text{ for some } z \in E^1 \text{ and } g_{21} \in \mathfrak{p}_E^{d+1}\}.$$
$$S_2 := \{g \in N \mid g_{11} \in z + \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil} \text{ for some } z \in E^1 \text{ and } g_{21} \in \mathfrak{p}_E^{d} \setminus \mathfrak{p}_E^{d+1}\}.$$

Note that $N = S_1 \sqcup S_2$. Thus we have

$$\Psi_{X_{-\varpi^{-d}},\theta}(g) = \begin{cases} \theta(g_{11}) & \text{if } g \in S_1, \\ \theta(g_{11})\psi(-g_{11}^{-1}g_{21}\varpi^{-d}) & \text{if } g \in S_2. \end{cases}$$

Thus the intertwining number $\mathcal{I}(\chi_{\eta^d}, \Psi_{X_{-\varpi^{-d}},\theta}) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{B}\mathcal{K}_d \cap (Z\mathcal{U}\mathcal{J}_d)}(\sigma^{\eta^d}, \Psi_{X_{-\varpi^{-d}},\theta}))$ is given by

$$\begin{aligned} \mathcal{I}(\chi_{\eta^d}, \Psi_{X_{-\varpi^{-d}},\theta}) &= \frac{1}{|N|}\sum_{g\in N} \chi_{\eta^d}(g)\overline{\Psi_{X_{-\varpi^{-d}},\theta}(g)} \\ &= \frac{1}{|N|}\sum_{g\in S_1}(q-1)\theta(g_{11})\overline{\theta(g_{11})} + \frac{1}{|N|}\sum_{g\in S_2}(-\theta(g_{11}))\overline{\theta(g_{11})\psi(-g_{11}^{-1}g_{21}\varpi^{-d})}. \end{aligned}$$

For convenience, set

$$M_1 := \sum_{g\in S_1}(q-1)\theta(g_{11})\overline{\theta(g_{11})}$$
$$M_2 := \sum_{g\in S_2}(-\theta(g_{11}))\overline{\theta(g_{11})\psi(-g_{11}^{-1}g_{21}\varpi^{-d})}$$

Then

$$\mathcal{I}(\chi_{\eta^d}, \Psi_{X_{-\varpi^{-d}},\theta}) = \frac{M_1 + M_2}{|N|}.$$

We now evaluate the sums M_1 and M_2. For M_1 note that $Z/(Z \cap \mathcal{K}_{d+1})$ is a subgroup of N, and is contained inside S_1, and therefore using character orthogonality relations we have

$$M_1 = (q-1)|S_1| > 0.$$

We now proceed to compute M_2. Since ψ is unitary, $\psi(-g_{11}^{-1}g_{21}\varpi^{-d}) = \overline{\psi(g_{11}^{-1}g_{21}\varpi^{-d})}$ and therefore

$$M_2 = \sum_{g\in S_2}(-\theta(g_{11}))\overline{\theta(g_{11})}\psi(g_{11}^{-1}g_{21}\varpi^{-d}).$$

Observe that for each $g_{11} \in z + \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$, the sum

$$\sum_{g_{21} \in \mathfrak{p}_E^d \setminus \mathfrak{p}_E^{d+1}} \psi(g_{11}^{-1} g_{21} \varpi^{-d} \sqrt{\epsilon}) = \sum_{y \in \mathfrak{e} \setminus \{0\}} \psi(y).$$

Since $\sum_{y \in \mathfrak{e}} \psi(y) = 0$, and $\psi(0) = 1$, we obtain that

$$\sum_{g_{21} \in \mathfrak{p}_E^d \setminus \mathfrak{p}_E^{d+1}} \psi(g_{11}^{-1} g_{21} \varpi^{-d} \sqrt{\epsilon}) = -1.$$

Therefore,

$$M_2 = \sum_{g_{11}} \theta(g_{11}) \overline{\theta(g_{11})} \geq 0.$$

Since $M_1 > 0$, and $M_2 \geq 0$, we obtain that $\mathcal{I} > 0$. Hence the representations $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}$ and $\mathcal{S}_d(X_{-\varpi^{-d}}, \theta)$ intertwine. Since $\mathcal{S}_d(X_{-\varpi^{-d}}, \theta)$ is irreducible and has the same degree as $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}$, we conclude that $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}$ is irreducible and isomorphic to $\mathcal{S}_d(X_{-\varpi^{-d}}, \theta)$, which by Lemma 3.3.2 is isomorphic to $\mathcal{S}_d(X_{\varpi^{-d}}, \theta)$, proving the theorem. ∎

Remark 4.3.2. We note that in the proof of Proposition 4.3.1, it was sufficient to show that $\mathcal{I} \neq 0$. However, by explicitly computing the sums M_1, M_2, and the cardinality of N, one obtains the precise value $\mathcal{I} = 1$.

As a corollary, we obtain our main results of this section, and our first set of branching rules.

Corollary 4.3.3 (Branching rules for depth-zero supercuspidal representations)**.** *Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ. Then the decomposition into irreducible $\mathcal{K}$-representations of the restrictions to $\mathcal{K}$ of the corresponding depth-zero supercuspidal representations of G are given by*

$$\mathrm{Res}_{\mathcal{K}} c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma \cong \sigma \oplus \bigoplus_{d \in 2\mathbb{Z}_{\geq 1}} \mathcal{S}_d(X_{\varpi^{-d}}, \theta), \quad \mathrm{Res}_{\mathcal{K}} c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma^\eta \cong \bigoplus_{d \in 2\mathbb{Z}_{\geq 0}+1} \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

We conclude by noting that the components occurring in the decomposition specified in Corollary 4.3.3 have distinct depths and degrees. Even more, the origin of the representation determines the parity of the depths of the irreducible constituents occurring in the restriction to $\mathcal{K}$, for compact induction from $\mathcal{K}$ yields only even depth components, while compact induction from $\mathcal{K}^\eta$ yields only odd depth components. We also observe that if σ_1 and σ_2 are cuspidal representations of $\mathcal{K}/\mathcal{K}_{0+}$ with the same central character, then the corresponding depth-zero supercuspidal representations of G have isomorphic restrictions to $\mathcal{K}$, *i.e.*,

$$\mathrm{Res}_{\mathcal{K}}\, c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma_1 \;\cong\; \mathrm{Res}_{\mathcal{K}}\, c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma_2 \qquad \text{and} \qquad \mathrm{Res}_{\mathcal{K}}\, c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma_1^\eta \;\cong\; \mathrm{Res}_{\mathcal{K}}\, c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma_2^\eta.$$

It is also worth comparing this behaviour with that of depth-zero irreducible supercuspidal representations of $\mathrm{SL}_2(F)$ [Nev13]. In the case of $\mathrm{SL}_2(F)$, each Mackey component $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \sigma^{\eta^d}$ decomposes further into two irreducible pieces, whereas for our unitary group each such component remains irreducible.

Chapter 5

Branching rules for positive-depth supercuspidal representations of G

In this chapter, we first construct all positive-depth irreducible supercuspidal representations of G using the construction developed by Adler and Yu [Adl98, Yu01], further refined by Fintzen [Fin21a], often referred to as the Adler–Fintzen–Yu method. This construction dates back to Corwin [Cor91] and Howe [How77]. Since $p > 2$, this method yields all irreducible supercuspidal representations of G [Fin21b, Theorem 8.1]. We also note that, under the assumption that p is sufficiently large, the exhaustiveness of this construction was established earlier in [Kim07, Theorem 19.1].

In §5.2, we restrict these representations to $\mathcal{K}$ and describe their decomposition in terms of the irreducible representations of $\mathcal{K}$ constructed in Theorem 3.2.6 of Chapter 3. We prove that the various irreducible components of each restriction have distinct depths and degrees. In particular, the decompositions are multiplicity free.

In §5.3.3, we show that the higher-depth components occurring in the decomposition of these representations, when restricted to $\mathcal{K}$, are the same as those of the depth-zero irreducible supercuspidal representations with the same central character. Moreover, we prove that one can fix four depth-zero irreducible supercuspidal representations such that, up to twisting by a character of G, the higher-depth components of any positive-depth irreducible supercuspidal representation coincide with the higher-depth components, upon restriction to $\mathcal{K}$, of a subset of these four depth-zero irreducible supercuspidal representations. This answers [HV24, Question 1.2] for all smooth irreducible representations of G.

We conclude this chapter by showing that, given an irreducible supercuspidal representation π of G of depth $r > 0$, the restriction of π to $\mathcal{K}_{2r+}$, up to twist by a quasi-character of G, decomposes into representations constructed using nilpotent elements of the Lie algebra of G together with the central character θ of the original representation. This yields a decomposition in the Grothendieck group of representations, where the components consist of copies of the trivial representation and certain fixed, highly reducible representations associated with nilpotent $\mathcal{K}$-orbits in the Lie algebra of G, thereby verifying the analogue for G of [Nev24, Theorem 1.1], which concerns $\mathrm{SL}_2(F)$.

5.1 Construction of the representations

Our goal in this section is to provide an explicit parametrization of all positive-depth irreducible supercuspidal representations of G.

Note that if λ is a positive-depth quasi-character of G and π_0 is an irreducible depth-zero supercuspidal representation of G, then $\lambda \otimes \pi_0$ is an irreducible positive-depth supercuspidal representation of G. Since we have already computed the branching rules for all depth-zero irreducible supercuspidal representations of G in Chapter 4, we now turn to the construction of those positive-depth supercuspidal representations not arising in this way.

We begin by recalling the Adler–Fintzen–Yu method for constructing positive-depth supercuspidal representations of connected reductive p-adic groups (together with the twist introduced by Fintzen, Kaletha and Spice), specializing to the case of our unitary group G. We first present the datum required for the construction and then briefly summarize the latter. There are various equivalent ways to present the datum; here we follow the formulation of [Fin21a, FKS23].

5.1.1 Datum for the construction

The datum for constructing positive-depth supercuspidal representations of a general group G is quite complex, but since our group G has rank one, it can be simplified as follows.

(D1) An anisotropic torus $\mathcal{T}$ of G;

(D2) The unique point $y \in \mathscr{B}(\mathcal{T}) \subset \mathscr{B}(G)$;

(D3) A real number $r > 0$, and we set $s := \frac{r}{2}$;

(D4) A G-generic character ϕ of $\mathcal{T}$ of depth r relative to y;

(D4') A quasi-character ϕ' of G that is either trivial or of depth $r' > r$.

For the remainder of this subsection, we briefly recall Yu's datum for constructing positive-depth supercuspidal representations as formulated in [Fin21a, §2.1], and explain why, in our setting, it reduces to the simplified form above.

(a) The general construction requires a sequence

$$G = G_1 \supseteq G_2 \supset G_3 \supset \cdots \supset G_{n+1}$$

of twisted Levi subgroups[1] of G that split over a tamely ramified extension of F, with $Z(G_{n+1})/Z(G)$ anisotropic. Since G has rank one, the only possible sequences are $G \supset \mathcal{T}$ or $G \supseteq G \supset \mathcal{T}$, where $\mathcal{T}$ is an anisotropic torus. We shall see later (5.1.3) that representations arising from the latter are simply twists (by quasi-characters of G as in (D4')) of those from the former, so it suffices to consider the sequence $G \supset \mathcal{T}$, as in (D1).

(b) In the general setting, one requires a point $y \in \mathscr{B}^{\mathrm{en}}(G_{n+1}) \subset \mathscr{B}^{\mathrm{en}}(G)$ such that the image of y in $\mathscr{B}^{\mathrm{en}}((G_{n+1})_{\mathrm{der}})$ is a vertex. Here $\mathscr{B}^{\mathrm{en}}$ denotes the enlarged Bruhat–Tits building. For our group G, the center is compact, and so $\mathscr{B}^{\mathrm{en}}(G_{\mathrm{der}})$ is the reduced building $\mathscr{B}(G)$. Moreover, since $\mathcal{T}$ is compact its building (enlarged or reduced) is a single point, which is thus a vertex of $\mathscr{B}(\mathcal{T})$, justifying (D2).

(c) The general datum requires a decreasing sequence of real numbers $r_1 > r_2 > \cdots > r_n > 0$, depending on the length of the chain of twisted Levi subgroups. In our case, the two possibilities are a single parameter $r > 0$, or a pair $r' > r > 0$, corresponding to the sequences noted in (a). So if the sequence is $G \supset \mathcal{T}$, we have a quasi-character ϕ of $\mathcal{T}$ of depth r, and if the sequence is $G \supseteq G \supset \mathcal{T}$, then in addition to ϕ, we have a quasi-character ϕ' of G of depth r'.

(d) One also requires, for each $1 \leq i \leq n$, a quasi-character ϕ_i of G_{i+1} of depth r_i, trivial on $(G_{i+1})_{y,r_i+}$. If $G_i \neq G_{i+1}$, we also require ϕ_i to be G_i-generic of depth r_i relative to y. In particular for our case, for either sequence, we need a quasi-character of $\mathcal{T}$, which we will denote by ϕ, that is G-generic of depth $r > 0$, which is the condition (D4); and for the sequence $G \supseteq G \supset \mathcal{T}$ we additionally need a character ϕ' of G of depth $r' > r$, which is the optional condition (D4').

(e) Another ingredient is an irreducible representation σ of $(G_{n+1})_{[y]}$ that is trivial on $(G_{n+1})_{y,0+}$ and is a cuspidal representation of

$$(G_{n+1})_{y,0}/(G_{n+1})_{y,0+},$$

where $[y]$ denotes the image of y in the reduced building $\mathscr{B}(G_{n+1})$. In our case, $(G_{n+1})_{[y]} = \mathcal{T}$, since $\mathscr{B}^{\mathrm{en}}(\mathcal{T}) = \mathscr{B}^{\mathrm{red}}(\mathcal{T}) = \{y\}$, and σ is a cuspidal representation of $\mathcal{T}/\mathcal{T}_{0+}$. Since this group is abelian, σ is one-dimensional, and without loss of generality we may absorb it into ϕ by replacing ϕ with $\sigma \otimes \phi$ [HM08, Theorem 6.7].

The datum in particular defines a compact open subgroup K' and a representation ρ thereof such that $c\text{-}\mathrm{Ind}_{K'}^{G}\rho$ is an irreducible supercuspidal representation as we will summarize in the next section. In our case, the inducing subgroup is simply $K' = G_{y,s}\mathcal{T}$.

Remark 5.1.1. Fintzen–Kaletha–Spice [FKS23, Definition 4.1.10] introduced an additional sign character ε of the inducing subgroup K'. However, in our case, since $G_{n+1} = \mathcal{T}$ is just a torus, ε is a sign character of $G_{y,s}\mathcal{T}$, trivial on $G_{y,s}\mathcal{T}_{0+}$. Since

$$G_{y,s}\mathcal{T}/G_{y,s}\mathcal{T}_{0+} \cong \mathcal{T}/\mathcal{T}_{0+},$$

[1]For us the symbol $\supset$ always denotes a proper subset inclusion.

ε is simply a depth-zero character of $\mathcal{T}$. As in (e), we may replace ϕ with $\varepsilon \otimes \phi$. We note that Yu's original construction did not include ε; this correction was introduced in [FKS23] to make Yu's proof work – his proof relied on a typo in a published reference. Fintzen also showed in [Fin21a] that Yu's original construction is valid even without ε, and provided a different proof of Yu's main theorem. For our purposes, the presence or absence of ε does not affect the outcome, though it is crucial in broader contexts such as the local Langlands correspondence [Kal21] and character theory [Spi18].

By a quadruple $(\mathcal{T}, y, r, \phi)$ we shall always mean one satisfying conditions (D1)–(D4). Note that we have already classified all anisotropic tori $\mathcal{T}$ of G up to conjugacy in Proposition 2.2.11, with corresponding vertices y listed in Table 2.1. We have also provided a description of G-generic characters of $\mathcal{T}$ in §2.3.3.2. Having assembled the necessary ingredients, we now proceed with the construction.

5.1.2 The construction

Given a quadruple $(\mathcal{T}, y, r, \phi)$ as in Section 5.1.1, the idea of the construction is to extend ϕ to a uniquely determined depth r representation $\rho = \rho(\mathcal{T}, y, r, \phi)$ of the compact open subgroup $G_{y,s}\mathcal{T}$, whose compact induction to G is irreducible, and hence supercuspidal. It proceeds as follows. Recall that we denoted by e the Moy–Prasad isomorphisms of abelian groups

$$\mathfrak{t}_{s+}/\mathfrak{t}_{r+} \longrightarrow \mathcal{T}_{s+}/\mathcal{T}_{r+}, \qquad \mathfrak{g}_{x,s+}/\mathfrak{g}_{x,r+} \longrightarrow G_{x,s+}/G_{x,r+}.$$

Since the character ϕ of $\mathcal{T}$ has depth r, its restriction to $\mathcal{T}_{s+}$ factors through $\mathcal{T}_{s+}/\mathcal{T}_{r+}$. As established in Section 2.3.3.2, every such character is represented by an element of $\mathfrak{t}_{-r}$; that is, there exists $\Gamma_{u,v} = \left(\begin{smallmatrix} u\sqrt{\epsilon} & v\sqrt{\epsilon} \\ v\sqrt{\epsilon} & u\sqrt{\epsilon} \end{smallmatrix}\right) \in \mathfrak{t}_{-r} \subset \mathfrak{g}$ such that

$$\phi(t) = \Psi\big(\mathrm{Tr}\left(\Gamma_{u,v}\, e^{-1}(t)\right)\big) \qquad \text{for all } t \in \mathcal{T}_{s+}. \tag{5.1.1}$$

Moreover, as explained in that section, the image of $\Gamma_{u,v}$ in $\mathfrak{t}_{-r}/\mathfrak{t}_{-s}$ is uniquely determined by this relation, and the genericity of ϕ forces $\nu(v\gamma_1) = -r - y$.

The element $\Gamma := \Gamma_{u,v}$ also defines a character Ψ_Γ of $G_{y,s+}/G_{y,r+}$ given by

$$\Psi_\Gamma(g) = \Psi(\mathrm{Tr}(\Gamma e^{-1}(g))) \quad \text{for all } g \in G_{y,s+}.$$

Since ϕ and Ψ_Γ agree on the intersection $\mathcal{T}_{s+}$ of their domains, together they define a unique character $\hat{\phi}$ of $G_{y,s+}\mathcal{T}$, given by

$$\hat{\phi}(gt) = \Psi_\Gamma(g)\, \phi(t) \qquad (g \in G_{y,s+},\ t \in \mathcal{T}). \tag{5.1.2}$$

The following lemma is a summary of the consequences of the key technical step of Yu's construction applied to our case.

Lemma 5.1.2. *If $\mathcal{T}$ is ramified, or if $\mathcal{T}$ is unramified and r is odd, then $G_{y,s+} = G_{y,s}$. In these two cases, $\rho = \hat{\phi}$ is a one-dimensional representation of $G_{y,s}\mathcal{T}$ of depth r.*

If $\mathcal{T}$ is unramified and r is even, then there exists an irreducible representation ρ of $G_{y,s}\mathcal{T}$ of dimension q such that $\rho|_{G_{y,s+}\mathcal{T}_s}$ is $\hat{\phi}$-isotypic.

In all cases $\rho|_{Z\mathcal{T}_{0+}}$ is ϕ-isotypic, and $\rho|_{G_{y,s+}}$ is Ψ_Γ-isotypic.

Proof. When $\mathcal{T}$ is ramified, Remark 2.3.22 implies that r is an half-integer, hence $G_{y,s} = G_{y,s+}$. Similarly, when $\mathcal{T}$ is unramified and r is odd, we have $G_{y,s} = G_{y,s+}$. In both these cases, we already have a description of $\rho = \hat{\phi}$ given by (5.1.2) and by the description of $\hat{\phi}$ it follows that $\rho|_{Z\mathcal{T}_{0+}}$ is ϕ-isotypic, and $\rho|_{G_{y,s+}}$ is Ψ_Γ-isotypic.

If $G_{y,s+} \neq G_{y,s}$, then we have a representation of $G_{y,s+}\mathcal{T} \subsetneq G_{y,s}\mathcal{T}$. By [Yu01, §11] $\hat{\phi}$ can be used to define a symplectic $\mathbb{F}_p$-vector space structure on $G_{y,s}/G_{y,s+}\mathcal{T}_s$ such that the conjugation action by the anisotropic torus $\mathcal{T}$ preserves the symplectic form. There is a unique representation of $G_{y,s}$, called the Heisenberg representation, whose restriction to $G_{y,s+}\mathcal{T}_s$ is $\hat{\phi}$-isotypic. Its dimension is $\sqrt{|G_{y,s}/G_{y,s+}\mathcal{T}_s|}$ (see [Fin21a, §2.5]). Since $\mathcal{T}$ acts by symplectic automorphisms on the quotient, the Heisenberg representation of $G_{y,s}$ extends to a representation of $G_{y,s}\mathcal{T}$ on the same space, called the Heisenberg–Weil representation (see [Yu01, §10]). The resulting representation remains $\hat{\phi}$-isotypic when restricted to $G_{y,s+}\mathcal{T}_s$. We denote this representation by ρ. Since $G_{y,s+} \subseteq G_{y,s+}\mathcal{T}_s \subseteq G_{y,s}\mathcal{T}_s = G_{y,s}$, using group isomorphism theorems we have

$$[G_{y,s} \colon G_{y,s+}\mathcal{T}_s] = \frac{[G_{y,s} \colon G_{y,s+}]}{[G_{y,s+}\mathcal{T}_s \colon G_{y,s+}]} = \frac{[G_{y,s} \colon G_{y,s+}]}{[\mathcal{T}_s \colon \mathcal{T}_{s+}]}.$$

Since $s > 0$, by properties of Moy–Prasad filtration subgroups, we have $G_{y,s}/G_{y,s+} \cong \mathfrak{g}_{y,s}/\mathfrak{g}_{y,s+}$ and $\mathcal{T}_s/\mathcal{T}_{s+} \cong \mathfrak{t}_s/\mathfrak{t}_{s+}$. Therefore

$$[G_{y,s} \colon G_{y,s+}\mathcal{T}_s] = \frac{[\mathfrak{g}_{y,s} \colon \mathfrak{g}_{y,s+}]}{[\mathfrak{t}_s \colon \mathfrak{t}_{s+}]} = \frac{q^4}{q^2} = q^2,$$

and hence $\dim(\rho) = q$. The remaining properties, namely that $\rho|_{Z\mathcal{T}_{0+}}$ is ϕ-isotypic and that $\rho|_{G_{y,s+}}$ is Ψ_Γ-isotypic, are established in [Nev13, Lemma 3]. ∎

We write $\rho = \rho(\mathcal{T}, y, r, \phi)$ for the representation produced by Lemma 5.1.2. Then by [Fin21a, Theorem 3.1] the compactly induced representation

$$\pi_\rho := c\text{-}\mathrm{Ind}_{G_{y,s}\mathcal{T}}^{G}\,\rho$$

is an irreducible supercuspidal representation of G of depth r.

Now suppose our datum includes a character ϕ' of G of depth $r' > r$, as in (D4'). Then the representation $\rho' = \rho(\mathcal{T}, y, r, \phi, r', \phi')$ is just $\phi' \otimes \rho$. Since ϕ' is a quasi-character of G, by applying Lemma 2.3.4 we have

$$c\text{-}\mathrm{Ind}_{G_{y,s}\mathcal{T}}^{G}(\phi' \otimes \rho) \;\cong\; \phi' \otimes \left(c\text{-}\mathrm{Ind}_{G_{y,s}\mathcal{T}}^{G}\,\rho\right), \tag{5.1.3}$$

and therefore $c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}(\phi' \otimes \rho)$ is also an irreducible supercuspidal representation of G, this time of depth r'. Hence the simplification in (D1).

Theorem 5.1.3. *[Fin21b, Theorem 8.1] All positive-depth irreducible supercuspidal representations of G have the form*

$$\lambda \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\rho$$

for some quasi-character λ of G, where $\rho = \rho(\mathcal{T}, y, r, \phi)$ for a choice of datum; or $\lambda \otimes \pi_0$, where λ is a positive-depth quasi-character of G, and π_0 is a depth-zero supercuspidal representation of G.

We denote the simplified datum $(\mathcal{T}, y, r, \phi)$ arising from (D1)–(D4) by Σ. When (D4') is also included, we denote the extended datum $(\mathcal{T}, y, r, \phi, r', \phi')$ by Σ'.

A natural question is: if we tensor ρ (or π_ρ) by a quasi-character of G of depth $r' > r$, then it is the representation arising from the datum $(\mathcal{T}, y, r, \phi, r', \phi')$, but what happens if we tensor by a quasi-character of G of depth $\leq r$? How does it arise from a datum?

Lemma 5.1.4. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$, let λ be a quasi-character of G of depth $\leq r$, and let $\widetilde{\phi}$ be a character of $\mathcal{T}$ such that $\phi = \lambda|_{\mathcal{T}} \otimes \widetilde{\phi}$. Then*

1. *$\widetilde{\phi}$ is a G-generic character of $\mathcal{T}$ of depth r; and*
2. *if $\widetilde{\rho} := \rho(\mathcal{T}, y, r, \widetilde{\phi})$, then with $s = \frac{r}{2}$ we have*

$$c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\,\widetilde{\rho} \;\cong\; \lambda \otimes \left(c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\,\rho\right).$$

Proof. We leverage the work of Hakim and Murnaghan [HM08], who showed when two data produce equivalent representations. Let ϕ_G be a quasi-character of G of depth $r_1 > r$. Then

$$\Sigma' := (\mathcal{T}, y, r, \phi, r_1, \phi_G)$$

is a well-defined input for the construction of a supercuspidal representation. Consequently, by (5.1.3),

$$\phi_G \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\,\rho$$

is an irreducible supercuspidal representation of G of depth r_1, where $\rho = \rho(\mathcal{T}, y, r, \phi)$.

To prove the lemma, consider

$$\dot{\Sigma} = (\mathcal{T}, y, r, \widetilde{\phi}, r_1, \lambda\phi_G).$$

To verify that it is a valid input, we have to prove that $\widetilde{\phi}$ is a G-generic quasi-character of $\mathcal{T}$ of depth r. To do so, we prove that $\dot{\Sigma}$ is a refactorization of Σ' in the sense of [HM08, Definition 4.19]. To verify that $\dot{\Sigma}$ is a refactorization of Σ', we must check conditions (F0), (F1), and (F2) of Definition 4.19 in [HM08], which in our setting reduce to the following two conditions:

(F0) If $\phi_G = \mathbb{1}$, then $\lambda\phi_G = \mathbb{1}$.

(F1) $\widetilde{\phi}|_{\mathcal{T}_{0+}} = (\phi\lambda^{-1})|_{\mathcal{T}_{0+}}$ and $(\lambda\phi_G)|_{G_{y,r+}} = \phi_G|_{G_{y,r+}}$.

Condition (F0) is vacuously satisfied because ϕ_G has depth $r_1 > 0$ and is therefore nontrivial. For (F1), note that $\phi = \lambda|_{\mathcal{T}} \otimes \widetilde{\phi}$, so the first equality holds immediately. Moreover, since λ has depth $\leq r$, it is trivial on $G_{y,r+}$, which implies the second equality as well.

In [HM08, Proposition 4.24], Hakim and Murnaghan prove that if $\dot{\Sigma}$ is a refactorization of Σ', then $\dot{\Sigma}$ is also a well-defined input, and the representations

$$\rho(\Sigma) := \phi_G|_{G_{y,s}\mathcal{T}} \otimes \rho \qquad \text{and} \qquad \rho(\dot{\Sigma}) := (\lambda\phi_G)|_{G_{y,s}\mathcal{T}} \otimes \widetilde{\rho}$$

are equivalent. As a consequence, we obtain an isomorphism

$$\phi_G \otimes \left(c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\, \rho\right) \;\cong\; (\lambda\phi_G) \otimes \left(c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\, \widetilde{\rho}\right),$$

of supercuspidal representations. Finally, canceling ϕ_G on both sides yields

$$c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\, \rho \;\cong\; \lambda \otimes \left(c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\, \widetilde{\rho}\right). \qquad \blacksquare$$

Remark 5.1.5. In the proof of the preceding lemma, we used a simplified form of the three conditions defining refactorization from [HM08, Definition 4.19]. In Appendix A.2, we justify why those three conditions reduce to the two that we verified.

We can now give a further simplification to our data, one that will be crucial in Section 5.3.

Proposition 5.1.6. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$, and set $s := \frac{r}{2}$. Then there exists a quasi-character λ of G such that*

$$c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\rho \;\cong\; \lambda \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\widetilde{\rho},$$

where $\widetilde{\rho} = \rho(\mathcal{T}, y, r, \phi')$ is chosen so that $c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\widetilde{\rho}$ is an irreducible supercuspidal representation of depth r with depth-zero central character.

Proof. Let θ denote the central character of $c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\rho$ and let δ denote the non-trivial quadratic character of E^1. Since $Z \subset \mathcal{T}$, θ coincides with the restriction to Z of the character ϕ. By Lemma 2.3.18, δ has depth zero and we may write $\theta = \delta^k\mu^2$ for some $k \in \{0, 1\}$ and some quasi-character μ of E^1. Consider the character $\mu \circ \det$ of G. Since $c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G}\, \rho$ has depth r, its central character θ has depth at most r. Note that $\mu \circ \det$ has same depth as that of θ. Thus $\mu \circ \det$ is a quasi-character of G of depth $\leq r$.

Define

$$\phi' := (\mu^{-1} \circ \det) \otimes \phi.$$

Then $\phi'|_Z$ is δ^k, and we may write $\phi = (\mu \circ \det) \otimes \phi'$. Applying Lemma 5.1.4 yields that ϕ' is a G-generic character of $\mathcal{T}$ of depth r, and

$$c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \rho \cong (\mu \circ \det) \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \widetilde{\rho}$$

where $\widetilde{\rho} = \rho(\mathcal{T}, y, r, \phi')$. Since $\phi'|_Z$ is δ^k, it follows that the central character of $c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \widetilde{\rho}$ is δ^k, which has depth zero for either value of k. This completes the proof. ∎

The following is a direct consequence of Theorem 5.1.3 and Proposition 5.1.6. This corollary will be used in §5.3.

Corollary 5.1.7. *Every irreducible supercuspidal representation π of G of depth $r > 0$ can be expressed as*

$$\pi \cong \lambda \otimes \pi',$$

where λ is a quasi-character of G and π' is an irreducible supercuspidal representation of G of depth r whose central character is either $\mathbb{1}$ or δ.

5.2 Restriction to $\mathcal{K}$

By Theorem 5.1.3, we know that the positive-depth supercuspidal representations of G have the form

$$\lambda \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \rho$$

for some quasi-character λ of G, or $\lambda \otimes \pi_0$, where λ is a positive-depth quasi-character of G, and π_0 is a depth-zero supercuspidal representation of G. Since the decomposition of depth-zero representations of G was already described in §4.2, and for a quasi-character λ of G

$$\mathrm{Res}_{\mathcal{K}} \left(\lambda \otimes c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \rho \right) \cong \lambda|_{\mathcal{K}} \otimes \mathrm{Res}_{\mathcal{K}} \left(c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \rho \right),$$

it suffices to describe the branching for $\pi_\rho = c\text{-Ind}_{G_{y,s}\mathcal{T}}^{G} \rho$, where $\rho = \rho(\mathcal{T}, y, r, \phi)$. In this section, we do not need to assume that the depth of the central character of π_ρ is zero.

By Mackey theory we have

$$\mathrm{Res}_{\mathcal{K}}^{G} \pi_\rho \;\cong \bigoplus_{g \in \mathcal{K}\backslash G/G_{y,s}\mathcal{T}} c\text{-Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}} \rho^g.$$

For each double coset representative $g \in \mathcal{K}\backslash G/G_{y,s}\mathcal{T}$, the depth of the Mackey component

$$c\text{-Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}} \rho^g$$

depends on $\mathcal{T}$ (via the associated point y), on g, and on r. We denote this depth by $d = d(\mathcal{T}, g, r)$.

Theorem 5.2.1. *For each double coset representative $g \in \mathcal{K}\backslash G/G_{y,s}\mathcal{T}$, the corresponding Mackey component*

$$c\text{-}\mathrm{Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}} \rho^g$$

is an irreducible representation of $\mathcal{K}$.

The proof of Theorem 5.2.1 proceeds in several steps. In §5.2.1 we give an explicit description of double coset representatives for the space $\mathcal{K}\backslash G/G_{y,s}\mathcal{T}$. Using these, we compute the degrees of the corresponding Mackey components in §5.2.2. Finally, in §5.2.3, we show that these Mackey components intertwine with the irreducible representations of the same degree constructed in Theorem 3.2.6, thereby establishing their irreducibility.

5.2.1 Double coset representatives

Our goal in this section is to explicitly compute a set of double coset representatives for the double coset space

$$\mathcal{K}\backslash G/G_{y,s}\mathcal{T}$$

for each of the anisotropic tori listed in Table 2.2.3.1, together with their corresponding point y and $s > 0$. Note that, except for $\mathcal{T}_{\varpi^{-1},\varpi}$, all such tori are contained in $\mathcal{K}$. In contrast, the torus $\mathcal{T}_{\varpi^{-1},\varpi} = \mathcal{T}_{1,1}^{\eta}$ lies inside the conjugate subgroup $\mathcal{K}^{\eta}$, where $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$. By Lemma 2.2.5 a set of representatives for $\mathcal{K}\backslash G/\mathcal{K}$ (and likewise for $\mathcal{K}^{\eta}\backslash G/\mathcal{K}^{\eta}$) is

$$\left\{ \alpha^t := \begin{pmatrix} \varpi^{-t} & 0 \\ 0 & \varpi^t \end{pmatrix} \;\middle|\; t \geq 0 \right\}.$$

When $\mathcal{T} \subset \mathcal{K}$, we also have $G_{y,s}\mathcal{T} \subset \mathcal{K}$; and for the torus corresponding to $y = 1$, we have $\mathcal{T} = \mathcal{T}_{1,1}^{\eta} \subset G_{y,s}\mathcal{T}_{1,1}^{\eta} \subset \mathcal{K}^{\eta}$. In both situations, applying Lemma 2.2.9 shows that each double coset in $\mathcal{K}\backslash G/\mathcal{T}G_{y,s}$ has a representative of the form $\alpha^t\beta$, with $t \geq 0$ and β a representative of

$$(\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})\backslash\mathcal{K}/G_{y,s}\mathcal{T} \qquad \text{or} \qquad (\mathcal{K}^{\eta} \cap (\mathcal{K}^{\eta})^{\alpha^{-t}})\backslash\mathcal{K}^{\eta}/G_{y,s}\mathcal{T},$$

respectively. Therefore, we begin by computing sets of double coset representatives for these latter double coset spaces.

Note that $\alpha^{-t} = \varpi^t I \eta^{-2t}$, therefore $\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}} = \mathcal{K} \cap \mathcal{K}^{\eta^{-2t}} = (\mathcal{K} \cap \mathcal{K}^{\eta^{2t}})^{\eta^{-2t}}$.

By Lemma 4.2.1, $(\mathcal{K} \cap \mathcal{K}^{\eta^{2t}}) = \mathcal{B}\mathcal{K}_{2t}$ and by Lemma 4.2.2 we have $(\mathcal{B}\mathcal{K}_{2t})^{\eta^{-2t}} = \mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}$. Hence

$$\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}} = \mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}. \tag{5.2.1}$$

Since $\mathcal{B}^{\mathrm{op}} \subseteq \mathcal{K} \cap \mathcal{K}^{\alpha^{-t}}$ and $\mathcal{T} \subset G_{y,s}\mathcal{T}$, each double coset is a union of smaller double cosets, namely,

$$(\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})\backslash\mathcal{K}/G_{y,s}\mathcal{T} \text{ is a union of double cosets from } \mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}/\mathcal{T}.$$

Similarly, as η and α^{-t} commute, we have $(\mathcal{B}^{\mathrm{op}})^\eta \subseteq \mathcal{K}^\eta \cap (\mathcal{K}^\eta)^{\alpha^{-t}}$ and $\mathcal{T}_{1,1}^\eta \subset (G_{0,s}\mathcal{T}_{1,1})^\eta = G_{y,s}\mathcal{T}$. Thus

$$(\mathcal{K}^\eta \cap (\mathcal{K}^\eta)^{\alpha^{-t}})\backslash \mathcal{K}^\eta / G_{y,s}\mathcal{T} \text{ is a union of double cosets from } \ (\mathcal{B}^{\mathrm{op}})^\eta \backslash \mathcal{K}^\eta / \mathcal{T}_{1,1}^\eta.$$

Proposition 5.2.2. *Let $\mathcal{T}$ be one of the tori listed in Table 2.2.3.1, with $\mathcal{T} \subset \mathcal{K}$. Then*

$$\mathcal{K} = \begin{cases} \mathcal{B}^{\mathrm{op}}\mathcal{T}, & \text{if } \mathcal{T} = \mathcal{T}_{1,1} \\ \mathcal{B}^{\mathrm{op}}\mathcal{T} \sqcup \mathcal{B}^{\mathrm{op}}\mathrm{w}\mathcal{T}, & \text{if } \mathcal{T} \text{ is ramified} \end{cases}$$

where $\mathrm{w} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. *Similarly,* $\mathcal{K}^\eta = (\mathcal{B}^{\mathrm{op}})^\eta (\mathcal{T}_{1,1})^\eta$.

Proof. Assume that $\mathcal{T}_{\gamma_1,\gamma_2} \subset \mathcal{K}$. In Lemma 2.2.6 we saw that

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} \;\middle|\; u \in \sqrt{\epsilon}\mathfrak{p}_F, v \in \sqrt{\epsilon}\mathcal{O}_F \right\} \tag{5.2.2}$$

is a set of coset representatives for the right coset space $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}$. In order to find a set of coset representatives for the double coset space $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}/\mathcal{T}_{\gamma_1,\gamma_2}$, we consider the elements listed in (5.2.2) and check which of them give rise to distinct double cosets.

We claim that for $u \in \sqrt{\epsilon}\mathfrak{p}_F$ and $v \in \sqrt{\epsilon}\mathcal{O}_F^\times$, the representatives $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$ belong to the identity double coset, and if $v \in \sqrt{\epsilon}\mathfrak{p}_F$, then $\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}$ lies in the double coset space $\mathcal{B}^{\mathrm{op}}\mathrm{w}\mathcal{T}$.

Indeed, first assume that $u \in \sqrt{\epsilon}\mathfrak{p}_F$. Then $(1 - u^2\gamma_1^{-1}\gamma_2) \in \mathcal{O}_F^\times$. Choose $a \in \mathcal{O}_E^\times$ such that $a\overline{a} = 1 - u^2\gamma_1^{-1}\gamma_2$. Note that such a choice of a is possible because the norm map $N_{E/F}$ maps $\mathcal{O}_E^\times$ surjectively onto $\mathcal{O}_F^\times$. We then have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\overline{a}^{-1}u\gamma_1^{-1}\gamma_2 & \overline{a}^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & ua^{-1}\gamma_1^{-1}\gamma_1 \\ ua^{-1}\gamma_1^{-1}\gamma_2 & a^{-1} \end{pmatrix} \in \mathcal{B}^{\mathrm{op}}\mathcal{T}.$$

Now let $v \in \sqrt{\epsilon}\mathcal{O}_F^\times$. Then $(\gamma_1^{-1}\gamma_2 - v^2) \in \mathcal{O}_F^\times$. Choose $a \in \mathcal{O}_E^\times$ such that $a\overline{a} = \gamma_1^{-1}\gamma_2 - v^2$. We then have

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\overline{a}^{-1}v\gamma_1^{-1}\gamma_2 & \overline{a}^{-1} \end{pmatrix} \begin{pmatrix} a^{-1}v & a^{-1}\gamma_1^{-1}\gamma_1 \\ a^{-1}\gamma_1^{-1}\gamma_2 & a^{-1}v \end{pmatrix} \in \mathcal{B}^{\mathrm{op}}\mathcal{T}.$$

Finally, let $v \in \sqrt{\epsilon}\mathfrak{p}_F$. Then $1 - \gamma_1\gamma_2^{-1}v^2 \in \mathcal{O}_F^\times$. Choose $a \in \mathcal{O}_E^\times$ such that $a\overline{a} = 1 - \gamma_1\gamma_2^{-1}v^2$. We then have

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\overline{a}^{-1}v\gamma_1\gamma_2^{-1} & \overline{a}^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & (a\gamma_2)^{-1}v\gamma_1 \\ (a\gamma_2)^{-1}v\gamma_2 & a^{-1} \end{pmatrix} \in \mathcal{B}^{\mathrm{op}}\mathrm{w}\mathcal{T}.$$

It remains to show that when $\mathcal{T}$ is unramified, then w belongs to the identity double coset; and when $\mathcal{T}$ is ramified, then $\mathcal{B}^{\mathrm{op}}\mathcal{T} \neq \mathcal{B}^{\mathrm{op}}\mathrm{w}\mathcal{T}$. Indeed, from direct computation one can show that $\mathrm{w} \in \mathcal{B}^{\mathrm{op}}\mathcal{T}$ if and only if there exists some $a \in \mathcal{O}_E^\times$ such that $a\bar{a} = \gamma_1^{-1}\gamma_2$. Since $N_{E/F}$ maps $\mathcal{O}_E^\times$ surjectively onto $\mathcal{O}_F^\times$, such a choice of a is possible exactly when $\gamma_1^{-1}\gamma_2 \in \mathcal{O}_F^\times$, and this happens only when $\mathcal{T}$ is unramified.

Finally, since the groups $\mathcal{B}^{\mathrm{op}}$, $\mathcal{K}$, and $\mathcal{T}_{1,1}$ are conjugated simultaneously, the map $x \mapsto \eta x \eta^{-1}$ induces a bijection between the double coset spaces $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}/\mathcal{T}_{1,1}$ and $(\mathcal{B}^{\mathrm{op}})^\eta\backslash\mathcal{K}^\eta/\mathcal{T}_{1,1}^\eta$. Therefore, $\mathcal{K}^\eta = (\mathcal{B}^{\mathrm{op}})^\eta(\mathcal{T}_{1,1})^\eta$. ∎

Lemma 5.2.3. *Let $\mathcal{T}$ be one of the tori listed in Table 2.2.3.1 with $\mathcal{T} \subset \mathcal{K}$ and let $y = \mathcal{A}(G,\mathcal{T})$. Then*

$$\mathcal{K} = \begin{cases} (\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})G_{y,s}\mathcal{T}, & \text{if } \mathcal{T} = \mathcal{T}_{1,1} \\ (\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})G_{y,s}\mathcal{T} \sqcup (\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})\mathrm{w}G_{y,s}\mathcal{T}, & \text{if } \mathcal{T} \text{ is ramified.} \end{cases}$$

Similarly, $\mathcal{K}^\eta = (\mathcal{K}^\eta \cap (\mathcal{K}^\eta)^{\alpha^{-t}})G_{y,s}\mathcal{T}$ if $\mathcal{T} = \mathcal{T}_{1,1}^\eta$.

Proof. As noted earlier $(\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})\backslash\mathcal{K}/G_{y,s}\mathcal{T}$ (resp. $(\mathcal{K}^\eta \cap (\mathcal{K}^\eta)^{\alpha^{-t}})\backslash\mathcal{K}^\eta/G_{y,s}\mathcal{T}$) is a union of double cosets from $\mathcal{B}^{\mathrm{op}}\backslash\mathcal{K}/\mathcal{T}$ $\left(\text{resp. } (\mathcal{B}^{\mathrm{op}})^\eta\backslash\mathcal{K}^\eta/\mathcal{T}_{1,1}^\eta\right)$. When $\mathcal{T}$ is unramified, by Proposition 5.2.2 we have $\mathcal{K} = \mathcal{B}^{\mathrm{op}}\mathcal{T}$ $\left(\text{resp. } \mathcal{K}^\eta = (\mathcal{B}^{\mathrm{op}})^\eta\mathcal{T}_{1,1}^\eta\right)$, from which we conclude that, when $\mathcal{T} = \mathcal{T}_{1,1}$, $\mathcal{K} = (\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})G_{y,s}\mathcal{T}$, and, when $\mathcal{T} = \mathcal{T}_{1,1}^\eta$, $\mathcal{K}^\eta = (\mathcal{K}^\eta \cap (\mathcal{K}^\eta)^{\alpha^{-t}})G_{y,s}\mathcal{T}$.

It remains to show that I and w represent distinct double cosets of $\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}}\backslash\mathcal{K}/G_{y,s}\mathcal{T}$ when $\mathcal{T}$ is ramified. Let $\mathcal{T}$ be a ramified torus with corresponding point $y = \frac{1}{2}$. Since $(\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}}) = \mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}$ (5.2.1), we have $(\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})G_{y,s}\mathcal{T} \subset \mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}G_{y,s}\mathcal{T}$. Since the $(1,1)$-entry of any element of $\mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}G_{y,s}\mathcal{T}$ will always be an element of $\mathcal{O}_E^\times$, we deduce that $\mathrm{w} \notin (\mathcal{K} \cap \mathcal{K}^{\alpha^{-t}})G_{y,s}\mathcal{T} \subset \mathcal{B}^{\mathrm{op}}\mathcal{K}_{2t}G_{y,s}\mathcal{T}$, and hence I and w must represent distinct double cosets. ∎

Theorem 5.2.4. *A set of representatives for the double coset space $\mathcal{K}\backslash G/\mathcal{T}G_{y,s}$ is given by*

$$M(\mathcal{T}) := \begin{cases} \{\alpha^t \mid t \geq 0\} & \text{if } \mathcal{T} \text{ is unramified} \\ \{I, \alpha^t, \alpha^t\mathrm{w} \mid t > 0\} & \text{if } \mathcal{T} \text{ is ramified} \end{cases}$$

where $\alpha^t = \begin{pmatrix} \varpi^{-t} & 0 \\ 0 & \varpi^t \end{pmatrix}$, *and* $\mathrm{w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

5.2.2 The degree of the Mackey components

For each $g \in M(\mathcal{T})$, we now compute the degrees of the corresponding Mackey component

$$\mathrm{Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}} \rho^g.$$

Recall that for $r > 0$ and $s = \frac{r}{2}$, the representation $\rho = \rho(\mathcal{T}, y, r, \phi)$ of $G_{y,s}\mathcal{T}$ is an irreducible representation of depth r that has degree 1 when $G_{y,s} = G_{y,s+}$, and degree q otherwise. Thus, for all $g \in M(\mathcal{T})$, the degree of the corresponding Mackey component is

$$\deg\left(\mathrm{Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}}\rho^g\right) = \deg(\rho)\,[\mathcal{K} : \mathcal{K}\cap(G_{y,s}\mathcal{T})^g].$$

Thus, the computation reduces to evaluating the index $[\mathcal{K} : \mathcal{K}\cap(G_{y,s}\mathcal{T})^g]$ for each $g \in M(\mathcal{T})$.

Proposition 5.2.5. *Let $g \in M(\mathcal{T})$. Then $\mathcal{K}\cap(G_{y,s}\mathcal{T})^g \;=\; (\mathcal{K}\cap G_{y,s}^g)(\mathcal{K}\cap\mathcal{T}^g)$.*

Proof. Since $G_{y,s} \trianglelefteq G_y$ and $\mathcal{T} \subseteq G_y$, the torus $\mathcal{T}$ normalizes $G_{y,s}$. In particular, $\mathcal{K}\cap\mathcal{T}^g$ normalizes $\mathcal{K}\cap G_{y,s}^g$. Hence the product $(\mathcal{K}\cap G_{y,s}^g)(\mathcal{K}\cap\mathcal{T}^g)$ is a subgroup of $\mathcal{K}$.

Consider first the case $g = \alpha^t$. If $t = 0$, then $g = I$. Since for all $s > 0$ we have $G_{y,s} \subseteq \mathcal{K}$, we have $(\mathcal{K}\cap G_{y,s}\mathcal{T}) = G_{y,s}(\mathcal{K}\cap\mathcal{T})$, and there is nothing to prove. Assume $t > 0$, and write $k = hu \in G_{y,s}\mathcal{T}$ with $h \in G_{y,s}$ and $u \in \mathcal{T}$. Then

$$k^{\alpha^t} = h^{\alpha^t}u^{\alpha^t} \in G_{y,s}^{\alpha^t}\mathcal{T}^{\alpha^t}.$$

Explicitly,

$$k^{\alpha^t} = \begin{pmatrix} h_{11}u_{11} + h_{12}u_{21} & (h_{11}u_{12} + h_{12}u_{22})\varpi^{-2t} \\ (h_{21}u_{11} + h_{22}u_{21})\varpi^{2t} & h_{21}u_{12} + h_{22}u_{22} \end{pmatrix}.$$

For k^{α^t} to lie in $\mathcal{K}$, the following conditions must be satisfied. Since $t > 0$, the $(2,1)$ entry lies in $\mathfrak{p}_E$; hence, by Lemma 2.2.4, the $(1,1)$ and $(2,2)$ entries must lie in $\mathcal{O}_E^\times$, and the $(1,2)$ entry must lie in $\mathcal{O}_E$.

Since $h_{21} \in \mathfrak{p}_E^{\lceil s+y\rceil} \subseteq \mathfrak{p}_E$ and $h_{22} \in 1 + \mathfrak{p}_E^{\lceil s\rceil}$, solving the condition on the $(2,2)$ entry of k^{α^t} gives $u_{22} \in \mathcal{O}_E^\times$.

Next, consider the $(1,2)$ entry of hu. Since $h_{11}, u_{22} \in \mathcal{O}_E^\times$,

$$\nu(h_{11}u_{12} + h_{12}u_{22}) \;\geq\; \min\{\nu(u_{12}), \nu(h_{12})\}.$$

If $\nu(u_{12}) \neq \nu(h_{12})$, then in order to have $k^{\alpha^t} \in \mathcal{K}$ it is necessary that $\nu(u_{12}), \nu(h_{12}) \geq 2t$. This implies that both h^{α^t} and u^{α^t} lie in $\mathcal{K}$ whenever $k^{\alpha^t} \in \mathcal{K}$. Thus we have the desired factorization in this case.

On the other hand, if $\nu(h_{12}) = \nu(u_{12}) \geq \lceil s-y\rceil$, more care is required. Since $u \in \mathcal{T}$, we have $u_{11} = u_{22}$ and $u_{21} = u_{12}\gamma_1^{-1}\gamma_2$. Moreover,

$$u_{11}\overline{u_{11}} + u_{12}\overline{u_{12}}\gamma_1^{-1}\gamma_2 = 1 \quad\Longrightarrow\quad u_{11}\overline{u_{11}} \equiv 1 \pmod{\mathfrak{p}_E^{2\lceil s-y\rceil}}.$$

Since for all $n \geq 1$ the norm map $N_{E/F}\colon 1+\mathfrak{p}_E^n \to 1+\mathfrak{p}_F^n$ is surjective when E/F is a unramified quadratic extension, there exists $c \in 1 + \mathfrak{p}_E^{2\lceil s-y\rceil}$ with $N_{E/F}(u_{11}) = N_{E/F}(c)$. As $N_{E/F}(u_{11}c^{-1}) = 1$, $u_{11}c^{-1} \in E^1$. Therefore, we can factor

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{12}\gamma_1^{-1}\gamma_2 & u_{11} \end{pmatrix} = \begin{pmatrix} c & cu_{11}^{-1}u_{12} \\ cu_{11}^{-1}u_{12}\gamma_1^{-1}\gamma_2 & c \end{pmatrix}\begin{pmatrix} u_{11}c^{-1} & 0 \\ 0 & u_{11}c^{-1} \end{pmatrix} \in G_{y,s}Z.$$

Thus $u = \dot{h}z$ for some $\dot{h} \in G_{y,s}$ and $z \in Z$. Hence $k = h\dot{h}z = \tilde{h}z$ with $\tilde{h} \in G_{y,s}$. Since $z^{\alpha^t} = z \in \mathcal{K}$, $k^{\alpha^t} \in \mathcal{K}$ if and only if $\tilde{h}^{\alpha^t} \in \mathcal{K}$. This yields the desired factorization.

The case $g = \alpha^t \mathrm{w}$ is analogous, replacing $G_{y,s}$ with $G^{\mathrm{w}}_{y,s}$ and $\lceil s-y \rceil$ with $\lceil s+y \rceil$. ∎

Let $\mathcal{T} = \mathcal{T}_{\gamma_1,\gamma_2}$, and take $\{y\} = \mathcal{A}(G,\mathcal{T})$ as in Table 2.2.3.1. Before proceeding, for each $g \in M(\mathcal{T})$ and $s > 0$, we describe explicitly the subgroups $\mathcal{K} \cap \mathcal{T}^g$ and $\mathcal{K} \cap G^g_{y,s}$.

For $g \in M(\mathcal{T})$, we define

$$\delta(g) = \begin{cases} 2t - y & \text{if } g = \alpha^t \mathrm{w} \\ 2t + y & \text{if } g = \alpha^t. \end{cases}$$

Let $k \in \mathcal{K}$. Then $k \in \mathcal{T}^g$ if and only if $g^{-1}kg \in \mathcal{T}$. A direct computation shows that

$$\mathcal{K} \cap \mathcal{T}^g = \begin{cases} \left\{ \begin{pmatrix} a & b \\ b\gamma_1^{-1}\gamma_2\varpi^{4t} & a \end{pmatrix} \middle| a, b \in \mathcal{O}_E \right\} \cap G & \text{if } g = \alpha^t, \\ \left\{ \begin{pmatrix} a & b \\ b\gamma_1\gamma_2^{-1}\varpi^{4t} & a \end{pmatrix} \middle| a, b \in \mathcal{O}_E \right\} \cap G & \text{if } g = \alpha^t \mathrm{w}. \end{cases} \tag{5.2.3}$$

Similarly, $k \in G^g_{y,s}$ if and only if $g^{-1}kg \in G_{y,s}$, and we have

$$\mathcal{K} \cap G^g_{y,s} = \mathcal{K} \cap \left\{ \begin{pmatrix} 1 + \mathfrak{p}_E^{\lceil s \rceil} & \mathfrak{p}_E^{\lceil s-\delta(g) \rceil} \\ \mathfrak{p}_E^{\lceil s+\delta(g) \rceil} & 1 + \mathfrak{p}_E^{\lceil s \rceil} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 + \mathfrak{p}_E^{\lceil s \rceil} & \mathfrak{p}_E^{M} \\ \mathfrak{p}_E^{\lceil s+\delta(g) \rceil} & 1 + \mathfrak{p}_E^{\lceil s \rceil} \end{pmatrix} \right\} \cap \mathcal{K} \tag{5.2.4}$$

where $M = \max\{0, \lceil s - \delta(g) \rceil\}$.

Lemma 5.2.6. *Let g be α^t ($t \in \mathbb{Z}_{\geq 0}$) or $\alpha^t \mathrm{w}$ ($t \in \mathbb{Z}_{>0}$), and let $\{y\} = \mathcal{A}(G,\mathcal{T})$. Then for $t > 0$ or $y \neq 0$, we have $(\mathcal{K} \cap \mathcal{T}^g) = Z(\mathcal{K} \cap \mathcal{T}^g_{0+})$.*

Proof. Since y is the point associated to $\mathcal{T}$, $\mathcal{T}_{0+} = \mathcal{T} \cap G_{y,0+}$. Explicitly, we have

$$\mathcal{T}_{0+} = \begin{cases} \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap \mathcal{T} & \text{if } y = 0, \\ \begin{pmatrix} 1 + \mathfrak{p}_E & \mathcal{O}_E \\ \mathfrak{p}_E^2 & 1 + \mathfrak{p}_E \end{pmatrix} \cap \mathcal{T} & \text{if } y = 1, \\ \begin{pmatrix} 1 + \mathfrak{p}_E & \mathcal{O}_E \\ \mathfrak{p}_E & 1 + \mathfrak{p}_E \end{pmatrix} \cap \mathcal{T} & \text{if } y = \frac{1}{2}. \end{cases}$$

The computations required for the proof of this lemma are similar to those of (5.2.3). Let $k = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$; then $k \in \mathcal{T}_{0+}^g$ if and only if $g^{-1}kg \in \mathcal{T}_{0+}$. Suppose first that $g = \alpha^t$. Then $\alpha^{-t}k\alpha^t = \left(\begin{smallmatrix} a & b\varpi^{2t} \\ c\varpi^{-2t} & d\end{smallmatrix}\right)$ and we see that

$$\mathcal{K} \cap \mathcal{T}_{0+}^{\alpha^t} = \left\{ \begin{pmatrix} a & b \\ b\gamma_1^{-1}\gamma_2\varpi^{4t} & a \end{pmatrix} \mid a \in 1+\mathfrak{p}_E, b \in \mathcal{O}_E \right\} \cap G.$$

Similarly, when $g = \alpha^t \mathrm{w}$, we have $y = \frac{1}{2}$ and $t > 0$, therefore

$$\mathcal{K} \cap \mathcal{T}_{0+}^{\alpha^t \mathrm{w}} = \left\{ \begin{pmatrix} a & b \\ b\gamma_1\gamma_2^{-1}\varpi^{4t} & a \end{pmatrix} \mid a \in 1+\mathfrak{p}_E, b \in \mathcal{O}_E \right\} \cap G.$$

Now let $h = \begin{pmatrix} a & b \\ b\gamma_1^{-1}\gamma_2\varpi^{4t} & a \end{pmatrix} \in \mathcal{K} \cap \mathcal{T}^{\alpha^t}$. Then $b\gamma_1^{-1}\gamma_2\varpi^{4t} \in \mathfrak{p}_F$, and $a\overline{a} + b\overline{b}\varpi^{4t}\gamma_1^{-1}\gamma_2 = 1$. It follows that $a\overline{a} \in 1+\mathfrak{p}_F$. Since $N_{E/F}\colon 1+\mathfrak{p}_E \to 1+\mathfrak{p}_F$ is surjective, there exists $c \in 1+\mathfrak{p}_E$ such that $N_{E/F}(a) = N_{E/F}(c)$, and we have the decomposition

$$\begin{pmatrix} a & b \\ b\gamma_1^{-1}\gamma_2\varpi^{4t} & a \end{pmatrix} = \begin{pmatrix} ac^{-1} & 0 \\ 0 & ac^{-1} \end{pmatrix} \begin{pmatrix} c & ca^{-1}b \\ ca^{-1}b\gamma_1^{-1}\gamma_2\varpi^{4t} & c \end{pmatrix} \in Z(\mathcal{K} \cap \mathcal{T}_{0+}^{\alpha^t}).$$

Similarly, when $h = \begin{pmatrix} a & b \\ b\gamma_1\gamma_2^{-1}\varpi^{4t} & a \end{pmatrix} \in \mathcal{K} \cap \mathcal{T}^{\alpha^t\mathrm{w}}$, we have $b\gamma_1\gamma_2^{-1}\varpi^{4t} \in \mathfrak{p}_F$, $a\overline{a} \in 1+\mathfrak{p}_F$, and

$$\begin{pmatrix} a & b \\ b\gamma_1\gamma_2^{-1}\varpi^{4t} & a \end{pmatrix} = \begin{pmatrix} ac^{-1} & 0 \\ 0 & ac^{-1} \end{pmatrix} \begin{pmatrix} c & ca^{-1}b \\ ca^{-1}b\gamma_1\gamma_2^{-1}\varpi^{4t} & c \end{pmatrix} \in Z(\mathcal{K} \cap \mathcal{T}_{0+}^{\alpha^t\mathrm{w}}),$$

as was required. ∎

Proposition 5.2.7. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$ and let $g \in M(\mathcal{T})$. Then*

$$\deg\left(\mathrm{Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}^{\mathcal{K}} \rho^g\right) = \begin{cases} (q-1)q^r & \text{when } t=0 \text{ and } y=0, \\ (q^2-1)q^{d-1} & \text{otherwise} \end{cases}$$

where $d = r + \delta(g)$.

Proof. We want to compute the index $[\mathcal{K} : \mathcal{K} \cap (G_{y,s}\mathcal{T})^g]$. By Proposition 5.2.5, this equals $[\mathcal{K}\colon (\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)]$. From the description of these intersections (5.2.4), $G_{y,s+2t+1}$ is contained in $(\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)$ and therefore in $\mathcal{K}$. Hence

$$[\mathcal{K}\colon (\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)] = \frac{[\mathcal{K}\colon G_{y,s+2t+1}]}{[(\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)\colon G_{y,s+2t+1}]}.$$

We also have $G_{y,s+2t+1} \subseteq (\mathcal{K} \cap G_{y,s}^g) \subseteq (\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)$, so by the third isomorphism theorem for groups we have

$$[(\mathcal{K} \cap G_{y,s}^g)(\mathcal{K} \cap \mathcal{T}^g)\colon G_{y,s+2t+1}]$$

$$=[(\mathcal{K}\cap G^g_{y,s})(\mathcal{K}\cap \mathcal{T}^g)\colon \mathcal{K}\cap G^g_{y,s}][\mathcal{K}\cap G^g_{y,s}\colon G_{y,s+2t+1}].$$

Since $(\mathcal{K}\cap \mathcal{T}^g)$ normalizes $(\mathcal{K}\cap G^g_{y,s})$, the second isomorphism theorem yields

$$\begin{aligned}[(\mathcal{K}\cap \mathcal{T}^g)(\mathcal{K}\cap G^g_{y,s})\colon \mathcal{K}\cap G^g_{y,s}] &= [(\mathcal{K}\cap \mathcal{T}^g)\colon (\mathcal{K}\cap \mathcal{T}^g)\cap(\mathcal{K}\cap G^g_{y,s})]\\ &= [(\mathcal{K}\cap \mathcal{T}^g)\colon (\mathcal{K}\cap \mathcal{T}^g\cap G^g_{y,s})].\end{aligned}$$

Hence,

$$[\mathcal{K}\colon (\mathcal{K}\cap G^g_{y,s})(\mathcal{K}\cap \mathcal{T}^g)] =\frac{[\mathcal{K}\colon G_{y,s+2t+1}]}{[(\mathcal{K}\cap \mathcal{T}^g)\colon (\mathcal{K}\cap \mathcal{T}^g\cap G^g_{y,s})][(\mathcal{K}\cap G^g_{y,s})\colon G_{y,s+2t+1}]}.$$

We next compute these indices explicitly. Since $G_{y,s+2t+1}\subseteq \mathcal{K}_{0+}\subseteq \mathcal{K}$, the third isomorphism theorem gives

$$[\mathcal{K}\colon G_{y,s+2t+1}] =[\mathcal{K}\colon \mathcal{K}_{0+}][\mathcal{K}_{0+}\colon G_{y,s+2t+1}] =\mid \mathrm{U}(1,1)(\mathfrak{f})\mid [\mathcal{K}_{0+}\colon G_{y,s+2t+1}].$$

The index $[\mathcal{K}_{0+}\colon G_{y,s+2t+1}]$ equals the index of the corresponding $\mathcal{O}_F$-modules in the Lie algebra, which is

$$[\mathcal{K}_{0+}\colon G_{y,s+2t+1}] = q^{2\lceil s+2t\rceil}q^{\lceil s+2t-y\rceil}q^{\lceil s+2t+y\rceil},$$

so

$$[\mathcal{K}\colon G_{y,s+2t+1}] = q(q-1)(q+1)^2q^{2\lceil s+2t\rceil}q^{\lceil s+2t-y\rceil}q^{\lceil s+2t+y\rceil} = (q-1)(q+1)^2q^{2\lceil s\rceil+8t+1}q^{\lceil s-y\rceil}q^{\lceil s+y\rceil}.$$

From (5.2.4), the quotient $(\mathcal{K}\cap G^g_{y,s})/(\mathcal{K}_{0+}\cap G^g_{y,s})$ is a unipotent subgroup of $\mathcal{K}/\mathcal{K}_{0+}$, and therefore is in bijection with the corresponding $\mathcal{O}_F$-module in the Lie algebra. This allows us to compute

$$\begin{aligned}[(\mathcal{K}\cap G^g_{y,s})\colon G_{y,s+2t+1}] &=q^{2(\lceil s+2t+1\rceil-\lceil s\rceil)}q^{\lceil s+2t+1-y\rceil-M}q^{\lceil s+2t+1+y\rceil-\lceil s+\delta(g)\rceil}\\ &=q^{8t+4-M-\lceil s+\delta(g)\rceil}q^{\lceil s-y\rceil}q^{\lceil s+y\rceil}\end{aligned}$$

where $M=\max\{0,\lceil s-\delta(g)\rceil\}$.

When $t=0$ and $y=0$, let $\mathbb{T}(\mathfrak{f})=\mathcal{T}/\mathcal{T}_{0+}$, which is an anisotropic torus over $\mathfrak{f}$. We then have $[(\mathcal{K}\cap \mathcal{T}^g)\colon (\mathcal{K}\cap \mathcal{T}^g\cap G^g_{y,s})] = [\mathcal{T}\colon \mathcal{T}_s] =\mid \mathbb{T}(\mathfrak{f})\mid [\mathcal{T}_1\colon \mathcal{T}_s] = (q+1)^2q^{2\lceil s\rceil-2}$. Hence

$$[\mathcal{K}\colon (\mathcal{K}\cap(\mathcal{T}G_{y,s})^g)] = \frac{(q-1)(q+1)^2q^{4\lceil s\rceil+1}}{q^4(q+1)^2q^{2\lceil s\rceil-2}} = (q-1)q^{2\lceil s\rceil-1}.$$

Since $\deg(\rho)=q$ exactly when $r=2\lceil s\rceil$, and $\deg(\rho)=1$ when $r=2\lceil s\rceil-1$, the total degree is $(q-1)q^{2\lceil s\rceil-1}\deg(\rho)=(q-1)q^r$.

In all other cases, that is, when $t>0$ or $y\neq 0$, since $\mathcal{T}\cap G_{y,s}=\mathcal{T}_s$, we have $\mathcal{T}^g\cap G^g_{y,s}=\mathcal{T}^g_s$. Therefore,

$$[(\mathcal{K}\cap\mathcal{T}^g)\colon (\mathcal{K}\cap\mathcal{T}^g\cap G^g_{y,s})] = [(\mathcal{K}\cap\mathcal{T}^g)\colon (\mathcal{K}\cap\mathcal{T}^g_s)] = [(\mathcal{K}\cap\mathcal{T}^g)\colon (\mathcal{K}\cap\mathcal{T}^g_{0+})][(\mathcal{K}\cap\mathcal{T}^g_{0+})\colon (\mathcal{K}\cap\mathcal{T}^g_s)].$$

By Lemma 5.2.6, $[(\mathcal{K}\cap\mathcal{T}^g)\colon(\mathcal{K}\cap\mathcal{T}^g_{0+})] = [Z\colon Z_{0+}] = [E^1 : (1+\mathfrak{p}_E)\cap E^1]$, which equals $q+1$ by Proposition 2.1.1. The second factor is again the index of the corresponding $\mathcal{O}_F$-module in the Lie algebra, which is $q^{\lceil s\rceil-1}q^M$. Hence

$$\deg\left(\mathrm{Ind}^{\mathcal{K}}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}\rho^g\right) = (q^2-1)q^{\lceil s\rceil+\lceil s+\delta(g)\rceil-2}\deg(\rho).$$

When r is an even integer, $\deg(\rho)=q$, and $\delta(g)$ is an integer, so the expression simplifies to $(q^2-1)q^{2\lceil s\rceil+\delta(g)-2}q = (q^2-1)q^{d-1}$. Otherwise, we have $\deg(\rho)=1$, and either r is an odd integer and y is an integer, or else r and y are half-integers. In either case, $\lceil s\rceil+\lceil s+\delta(g)\rceil = 2s+\delta(g)-1$, and we again obtain $(q^2-1)q^{d-1}$, as desired. ∎

5.2.3 Proof of the main theorem

In this section, we present the proof of Theorem 5.2.1. That is, given $\rho=\rho(\mathcal{T},y,r,\phi)$, we show that for each double coset representative $g\in\mathcal{K}\backslash G/G_{y,s}\mathcal{T}$, the Mackey component

$$\mathrm{Ind}^{\mathcal{K}}_{(G_{y,s}\mathcal{T})^g\cap\mathcal{K}}\rho^g$$

is an irreducible representation of $\mathcal{K}$. By Proposition 5.2.5, we have

$$\mathrm{Ind}^{\mathcal{K}}_{(G_{y,s}\mathcal{T})^g\cap\mathcal{K}}\rho^g = \mathrm{Ind}^{\mathcal{K}}_{(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G^g_{y,s})}\rho^g,$$

so it suffices to prove the irreducibility of the right-hand side.

As established in Theorem 5.2.4, the double coset space $\mathcal{K}\backslash G/G_{y,s}\mathcal{T}$ admits a set $M(\mathcal{T})$ of representatives consisting of elements of the form α^t with $t\in\mathbb{Z}_{\geq 0}$ and possibly $\alpha^t\mathrm{w}$ with $t\in\mathbb{Z}_{>0}$. Moreover, Proposition 5.2.7 gives the degree of the Mackey component corresponding to each $g\in M(\mathcal{T})$.

We divide the proof of this theorem into two cases: the first case, $t=y=0$, is straightforward.

Lemma 5.2.8. *Let $\rho=\rho(\mathcal{T},0,r,\phi)$ and let $t=0$. Then the Mackey component $\mathrm{Ind}^{\mathcal{K}}_{\mathcal{K}\cap(G_{0,s}\mathcal{T})}\rho = \mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho$ is an irreducible representation of $\mathcal{K}$.*

Proof. Note that $G_{0,s}\mathcal{T}\subseteq\mathcal{K}$, and therefore $\mathcal{K}\cap G_{0,s}\mathcal{T} = G_{0,s}\mathcal{T}$. Since c-$\mathrm{Ind}^G_{G_{0,s}\mathcal{T}}\rho$ is irreducible, $\mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho$ must be irreducible by transitivity of induction. But we prefer to offer a direct proof. To prove the irreducibility we need to show

$$\dim_{\mathbb{C}}\Big(\mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho,\ \mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho)\Big) = 1.$$

By Frobenius reciprocity and the Mackey decomposition theorem,

$$\mathrm{Hom}_{\mathcal{K}}\Big(\mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho,\ \mathrm{Ind}^{\mathcal{K}}_{G_{0,s}\mathcal{T}}\rho\Big) \cong \bigoplus_{h\in G_{0,s}\mathcal{T}\backslash\mathcal{K}/G_{0,s}\mathcal{T}} \mathrm{Hom}_{G_{0,s}\mathcal{T}\cap(G_{0,s}\mathcal{T})^h}(\rho^h,\rho).$$

It remains to show that if $\mathrm{Hom}_{G_{0,s}\mathcal{T}\cap(G_{0,s}\mathcal{T})^h}(\rho^h,\rho)\neq 0$, then $h\in G_{0,s}\mathcal{T}$. Indeed, if h intertwines ρ on $G_{0,s}\mathcal{T}\cap(G_{0,s}\mathcal{T})^h$, it also intertwines ρ upon restriction to $G_{0,s}\cap G_{0,s}^h$. By [Fin21a, Lemma 3.4], this implies $h\in G_{0,s}\mathcal{T}G_{0,s}$. Since $\mathcal{T}$ normalizes $G_{0,s}$, we conclude that $h\in\mathcal{T}G_{0,s}=G_{0,s}\mathcal{T}$, as required. ∎

Since we have already handled the case $t=y=0$, we shall, from this point onward, assume that $t>0$ or $y\neq 0$. Our strategy for this case is to construct an irreducible representation of the form $\mathcal{S}_d(X,\zeta)$ (Lemma 5.2.10), as in Theorem 3.2.6, having the same degree as the corresponding Mackey component, and to show that they intertwine, thereby deducing irreducibility (Proposition 5.2.11).

To compare with the Mackey component attached to g, it is enough to find X and ζ such that

$$\mathrm{Hom}_{\mathcal{K}}\Big(\mathrm{Ind}_{T(X)\mathcal{J}_d}^{\mathcal{K}}\Psi_{X,\zeta},\ \mathrm{Ind}_{(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G_{y,s}^g)}^{\mathcal{K}}\rho^g\Big)\neq 0.$$

Applying Frobenius reciprocity and the Mackey decomposition theorem, this homomorphism space decomposes as

$$\bigoplus_{h\in(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G_{y,s}^g)\backslash\mathcal{K}/T(X)\mathcal{J}_d}\mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G_{y,s}^g)\cap(T(X)\mathcal{J}_d)^h}\left(\Psi_{X,\zeta}^h,\rho^g\right).$$

The summand corresponding to $h=1$ is

$$\mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G_{y,s}^g)\cap T(X)\mathcal{J}_d}(\Psi_{X,\zeta},\rho^g)\,, \tag{5.2.5}$$

and the non-vanishing of this term is precisely the condition we will verify. This will show that $\mathcal{S}_d(X,\zeta)$ is a subrepresentation of the Mackey component, and hence the irreducibility will follow once we confirm that their degrees coincide.

Lemma 5.2.9. *Let $\rho=\rho(\mathcal{T},y,r,\phi)$ and let $g\in M(\mathcal{T})$. Then the restriction of ρ^g to the subgroup $\mathcal{K}\cap\mathcal{T}^g$ is ϕ^g-isotypic.*

Proof. By Lemma 5.2.6, we have $\mathcal{K}\cap\mathcal{T}^g=Z(\mathcal{K}\cap\mathcal{T}_{0+}^g)$. Moreover, by Lemma 5.1.2, the restriction of ρ to $Z\mathcal{T}_{0+}$ is ϕ-isotypic. Conjugating by g, it follows that the restriction of ρ^g to $\mathcal{K}\cap\mathcal{T}^g$ is ϕ^g-isotypic, as claimed. ∎

By Lemma 5.2.9, the representation ρ^g is ϕ^g-isotypic on $\mathcal{K}\cap\mathcal{T}^g$. Moreover, by Lemma 5.2.6, the restriction of ρ to $G_{y,s+}$ is Ψ_Γ-isotypic, where $\Gamma\in\mathfrak{t}_{-r}$ realizes ϕ on $\mathcal{T}_{s+}/\mathcal{T}_{r+}$. Consequently, the restriction of ρ^g to $\mathcal{K}\cap G_{y,s+}^g$ is Ψ_{Γ^g}-isotypic. Since the centralizer of Γ in G is $\mathcal{T}$, it follows that the centralizer $T(\Gamma^g)$ of Γ^g in $\mathcal{K}$ is $\mathcal{K}\cap\mathcal{T}^g$. Recalling that $T(X)$ denotes the centralizer of X in $\mathcal{K}$, these observations naturally suggest taking $X=\Gamma^g$ and $\zeta=\phi^g$. These choices will be justified in the next lemma and in Proposition 5.2.11.

Lemma 5.2.10. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$ and let $\Gamma \in \mathfrak{t}_{-r}$ be an element realizing ϕ on $\mathcal{T}_{s+}/\mathcal{T}_{r+}$. Then, for every $g \in M(\mathcal{T})$ satisfying our standing assumption that $t > 0$ or $y \neq 0$, the element Γ^g lies in $\mathfrak{g}_{0,-d}$ and has the form*

$$\begin{pmatrix} z\sqrt{\epsilon} & u\sqrt{\epsilon} \\ v\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix}$$

with $z, u, v \in F$ satisfying $\nu(z), \nu(v) > \nu(u) = -d$. Moreover, ϕ^g and Ψ_{Γ^g} agree on the intersection $T(\Gamma^g) \cap \mathcal{J}_d$.

Proof. Let $\mathcal{T} = \mathcal{T}_{\gamma_1,\gamma_2}$. Then by the genericity of ϕ we know that

$$\Gamma = \begin{pmatrix} z\sqrt{\epsilon} & u\gamma_1\sqrt{\epsilon} \\ u\gamma_2\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix} \in \mathfrak{t}_{-r},$$

with $z \in \mathfrak{p}_F^{\lceil -r \rceil}$ and $\nu(u\gamma_1) = -r - y$. Recall that elements of $M(\mathcal{T})$ are of the form α^t with $t \in \mathbb{Z}_{\geq 0}$ and possibly $\alpha^t \mathrm{w}$ with $t \in \mathbb{Z}_{>0}$. A direct computation yields

$$\Gamma^{\alpha^t} = \begin{pmatrix} z\sqrt{\epsilon} & u\gamma_1\varpi^{-2t}\sqrt{\epsilon} \\ u\gamma_2\varpi^{2t}\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix} \quad \text{and} \quad \Gamma^{\alpha^t \mathrm{w}} = \begin{pmatrix} z\sqrt{\epsilon} & u\gamma_2\varpi^{-2t}\sqrt{\epsilon} \\ u\gamma_1\varpi^{2t}\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix}.$$

Moreover,

$$\nu(u\gamma_1\varpi^{-2t}\sqrt{\epsilon}) = -r - y - 2t = -(r + \delta(\alpha^t)) = -d,$$

and

$$\nu(u\gamma_2\varpi^{-2t}\sqrt{\epsilon}) = -r - \tfrac{1}{2} + 1 - 2t = -(r + 2t - y) = -(r + \delta(g)) = -d,$$

where, in the latter case, $(\gamma_1, \gamma_2) \in \{(1, \varpi), (1, \epsilon^{-1}\varpi)\}$ and $y = \frac{1}{2}$. Since $d > r$ when $t > 0$ or $y \neq 0$, we conclude that in both cases the valuations of z, $u\gamma_2\varpi^{2t}$, $u\gamma_1\varpi^{2t}$ are strictly greater than $-d$. Thus Ψ_{Γ^g} is a character of $\mathcal{J}_d$.

It remains to show that ϕ^g and Ψ_{Γ^g} agree on $T(\Gamma^g) \cap \mathcal{J}_d$.

If $t > 0$ or $y \neq 0$, then $\delta(g) > 0$ and the subgroup $\mathcal{J}_d$ is given by

$$\mathcal{J}_d = \begin{pmatrix} 1 + \mathfrak{p}_E^{s+\frac{\delta(g)}{2}} & \mathfrak{p}_E^{s+\frac{\delta(g)}{2}} \\ \mathfrak{p}_E^{s+\frac{\delta(g)+1}{2}} & 1 + \mathfrak{p}_E^{s+\frac{\delta(g)}{2}} \end{pmatrix} \cap G.$$

This subgroup lies in $G_{y,s+}$. Since $T(\Gamma^g) = \mathcal{K} \cap \mathcal{T}^g$ and $\mathcal{J}_d \subseteq G_{y,s+}$, we have $(\mathcal{K} \cap \mathcal{T}^g) \cap \mathcal{J}_d \subseteq \mathcal{K} \cap \mathcal{T}^g_{s+}$. Since ϕ^g is realized by Γ^g on $\mathcal{K} \cap \mathcal{T}^g_{s+}$, the two characters agree on the intersection $(\mathcal{K} \cap \mathcal{T}^g) \cap \mathcal{J}_d$. ∎

By Theorem 3.2.6, there exists an irreducible representation of $\mathcal{K}$ constructed from the data in Lemma 5.2.10, given by

$$\mathcal{S}_d(\Gamma^g, \phi^g) := \mathrm{Ind}_{(\mathcal{K} \cap \mathcal{T}^g)\mathcal{J}_d}^{\mathcal{K}} \Psi_{\Gamma^g, \phi^g}, \tag{5.2.6}$$

which has depth d and degree $(q^2 - 1)q^{d-1}$.

Proposition 5.2.11. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$ and let $\Gamma \in \mathfrak{t}_{-r}$ be an element realizing ϕ on $\mathcal{T}_{s+}/\mathcal{T}_{r+}$. Then for all $g \in M(\mathcal{T})$ that satisfy our assumption $t > 0$ or $y \neq 0$, we have*

$$\mathrm{Ind}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}\rho^g \cong \mathcal{S}_d(\Gamma^g, \phi^g).$$

Proof. For $g \in M(\mathcal{T})$, our goal is to show that the representation

$$\mathcal{S}_d(\Gamma^g, \phi^g) := \mathrm{Ind}^{\mathcal{K}}_{(\mathcal{K}\cap\mathcal{T}^g)\mathcal{J}_d}\Psi_{\Gamma^g,\phi^g} \tag{5.2.7}$$

intertwines with the Mackey component

$$\text{c-}\mathrm{Ind}^{\mathcal{K}}_{\mathcal{K}\cap(G_{y,s}\mathcal{T})^g}\rho^g = \mathrm{Ind}^{\mathcal{K}}_{(\mathcal{K}\cap G^g_{y,s})(\mathcal{K}\cap\mathcal{T}^g)}\rho^g. \tag{5.2.8}$$

As discussed before Lemma 5.2.10, it suffices to prove that

$$\mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G^g_{y,s})\cap(\mathcal{K}\cap\mathcal{T}^g)\mathcal{J}_d}(\Psi_{\Gamma^g,\phi^g}, \rho^g) \tag{5.2.9}$$

is non-zero. The subgroup on which we compare the two representations is

$$((\mathcal{K}\cap\mathcal{T}^g)(\mathcal{K}\cap G^g_{y,s}))\cap\big((\mathcal{K}\cap\mathcal{T}^g)\mathcal{J}_d\big) = (\mathcal{K}\cap\mathcal{T}^g)(G^g_{y,s}\cap\mathcal{J}_d).$$

If $G_{y,s} = G_{y,s+}$, we have $G^g_{y,s}\cap\mathcal{J}_d \subset G^g_{y,s+}$. Therefore, for an element $tu \in (\mathcal{K}\cap\mathcal{T}^g)(G^g_{y,s}\cap\mathcal{J}_d)$, with $t \in \mathcal{K}\cap\mathcal{T}^g$ and $u \in G^g_{y,s}\cap\mathcal{J}_d$, we obtain

$$\rho^g(tu) = \phi^g(t)\Psi^g_\Gamma(u) = \phi^g(t)\Psi_{\Gamma^g}(u) = \Psi_{\phi^g,\Gamma^g}(tu).$$

Hence (5.2.7) and (5.2.8) intertwine. This establishes the isomorphism in the case $G_{y,s} = G_{y,s+}$, which occurs either when $\mathcal{T}$ is ramified or when $\mathcal{T}$ is unramified and r is odd.

We now consider the case when $G_{y,s} \neq G_{y,s+}$, that is, when $\mathcal{T}$ is unramified and r is even. Note that in this case the double coset representative is just α^t. The subgroup $G^{\alpha^t}_{y,s}\cap\mathcal{J}_d$ is then given by

$$G^{\alpha^t}_{y,s}\cap\mathcal{J}_d = \begin{pmatrix} 1+\mathfrak{p}_E^{\lceil\frac{d}{2}\rceil} & \mathfrak{p}_E^{\lceil\frac{d}{2}\rceil} \\ \mathfrak{p}_E^{s+2t+y} & 1+\mathfrak{p}_E^{\lceil\frac{d}{2}\rceil}\end{pmatrix}\cap G = \begin{pmatrix} 1+\mathfrak{p}_E^{s+t+\lceil\frac{y}{2}\rceil} & \mathfrak{p}_E^{s+t+\lceil\frac{y}{2}\rceil} \\ \mathfrak{p}_E^{s+2t+y} & 1+\mathfrak{p}_E^{s+t+\lceil\frac{y}{2}\rceil}\end{pmatrix}\cap G.$$

This subgroup is clearly not contained in

$$G^{\alpha^t}_{y,s+} = \begin{pmatrix} 1+\mathfrak{p}_E^{s+1} & \mathfrak{p}_E^{s+1-2t-y} \\ \mathfrak{p}_E^{s+1+2t+y} & 1+\mathfrak{p}_E^{s+1}\end{pmatrix}\cap G,$$

since the $(2,1)$-entry of $G^{\alpha^t}_{y,s+}$ lies in $\mathfrak{p}_E^{s+1+2t+y}$, whereas the $(2,1)$-entry of $G^{\alpha^t}_{y,s}\cap\mathcal{J}_d$ lies in $\mathfrak{p}_E^{s+2t+y}$, and $\mathfrak{p}_E^{s+2t+y} \not\subseteq \mathfrak{p}_E^{s+2t+y+1}$. Hence the simpler argument above does not apply. In this case, the proof is a bit convoluted and proceeds as follows.

The first point to note is that although Γ is uniquely determined by ϕ in $\mathfrak{t}_{-r}$ only modulo $\mathfrak{t}_{-s}$, the restriction of Ψ_{Γ^g} to the subgroup $G^{\alpha^t}_{y,s}\cap\mathcal{J}_d$ depends on the choice of Γ modulo $\mathfrak{t}_{-s+1}$.

In what follows, we show that for all distinct choices of Γ modulo $\mathfrak{t}_{-s+1}$, the representation $\Psi_{\Gamma^{\alpha^t},\phi^{\alpha^t}}$ intertwines with ρ^{α^t} on $(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)$.

Since ρ has depth r, $\rho^{\alpha^t}|_{(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)}$ factors through the group $G^{\alpha^t}_{y,r+}\cap\mathcal{J}_d$, which is given as follows. Set $A=\max\{r+1, s+t+\lceil\frac{y}{2}\rceil\}, B=\max\{r+1-\delta(\alpha^t), s+t+\lceil\frac{y}{2}\rceil\}$, and $C=\lceil r+1+\delta(\alpha^t)\rceil$. Then

$$G^{\alpha^t}_{y,r+}\cap\mathcal{J}_d=\left\{\begin{pmatrix}1+\mathfrak{p}_E^A & \mathfrak{p}_E^B\\ \mathfrak{p}_E^C & 1+\mathfrak{p}_E^A\end{pmatrix}\right\}\cap G.$$

Since $2(s+t+\lceil\frac{y}{2}\rceil)>r+1$ and $2(s+t+1)\geq r+1+2t+y$, we obtain that the quotient group $H=(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)/(G^{\alpha^t}_{y,r+}\cap\mathcal{J}_d)$ is abelian and finite. As H is abelian and finite, and $\deg(\rho^{\alpha^t})=q$, the representation ρ^{α^t} decomposes as a direct sum of q characters upon restriction to H. The distinct characters of H are given by Ψ_Y, where Y is an element of the dual lattice quotient $\widehat{H}$, which is given by

$$\widehat{H}:=\left\{\begin{pmatrix}x & y\sqrt{\epsilon}\\ z\sqrt{\epsilon} & -\overline{x}\end{pmatrix}\middle| x\in\mathfrak{p}_E^{-A+1}/\mathfrak{p}_E^{-\lceil\frac{d}{2}\rceil+1}, y\in\mathfrak{p}_F^{-C+1}/\mathfrak{p}_F^{-s-\delta(g)+1}, z\in\mathfrak{p}_F^{-B+1}/\mathfrak{p}_F^{-\lceil\frac{d}{2}\rceil+1}\right\}.$$

Thus we have

$$\operatorname{Res}_{G^{\alpha^t}_{y,s}\cap\mathcal{J}_d}\rho^{\alpha^t}=\bigoplus_{i\in I}\Psi_{Y_i} \tag{5.2.10}$$

for some $Y_i\in\widehat{H}$. Since ρ^{α^t} restricted to $\mathcal{K}\cap\mathcal{T}^{\alpha^t}$ is ϕ^{α^t}-isotypic, we obtain that

$$\rho^{\alpha^t}|_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\cap(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)}\cong\phi^{\alpha^t}|_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\cap(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)}\mathrm{Id}\cong\bigoplus_{i\in I}\Psi_{Y_i}|_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\cap(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)}. \tag{5.2.11}$$

Therefore for all $i\in I$ we have

$$\phi^{\alpha^t}=\Psi_{Y_i} \tag{5.2.12}$$

on $(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\cap(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)$, and it follows that

$$\operatorname{Res}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(G^{\alpha^t}_{y,s}\cap\mathcal{J}_d)}\rho^{\alpha^t}=\bigoplus_{i\in I}\Psi_{Y_i,\phi^{\alpha^t}}. \tag{5.2.13}$$

Since ρ^{α^t} is $\Psi_{\Gamma^{\alpha^t}}$-isotypic upon restriction to $G^{\alpha^t}_{y,s+}$, the characters Ψ_{Y_i} and Ψ_{Γ^g} must agree on $G^{\alpha^t}_{y,s+}\cap\mathcal{J}_d$. Next we compute the form of the elements of $\widehat{H}$ that satisfy this condition. Let $h:=\begin{pmatrix}1+a & b\\ c & 1+d\end{pmatrix}\in G^{\alpha^t}_{y,s+}\cap\mathcal{J}_d$, $\Gamma^{\alpha^t}=\begin{pmatrix}u\sqrt{\epsilon} & v\sqrt{\epsilon}\varpi^{-2t}\gamma_1\\ v\sqrt{\epsilon}\varpi^{2t}\gamma_2 & u\sqrt{\epsilon}\end{pmatrix}$ and $Y=\begin{pmatrix}x & y\sqrt{\epsilon}\\ z\sqrt{\epsilon} & -\overline{x}\end{pmatrix}\in\widehat{H}$. Then we have

$$\Psi_Y(h)=\psi(ax+cy\sqrt{\epsilon}+bz\sqrt{\epsilon}-d\overline{x})$$

and

$$\Psi_{\Gamma^{\alpha^t}}(h)=\psi(au\sqrt{\epsilon}+cv\sqrt{\epsilon}\varpi^{-2t}\gamma_1+bv\sqrt{\epsilon}\varpi^{2t}\gamma_2+du\sqrt{\epsilon}).$$

For these characters to agree, we must have, for all $a \in \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$, $b \in \mathfrak{p}_E^{\lceil \frac{d}{2} \rceil}$, and $c \in \mathfrak{p}^{s+1+\delta(\alpha^t)}$,

$$\begin{aligned} \psi\left(a(x - u\sqrt{\epsilon})\right) &= 1, \\ \psi\left(c(y\sqrt{\epsilon} - v\sqrt{\epsilon}\varpi^{-2t}\gamma_1)\right) &= 1, \\ \psi\left(b(z\sqrt{\epsilon} - v\sqrt{\epsilon}\varpi^{2t}\gamma_2)\right) &= 1. \end{aligned}$$

These equalities respectively imply that

$$\begin{aligned} &x \equiv u\sqrt{\epsilon} \pmod{\mathfrak{p}_E^{-\lceil \frac{d}{2} \rceil + 1}}, \\ &y \equiv v\varpi^{-2t}\gamma_1 \pmod{\mathfrak{p}_F^{-s-\delta(\alpha^t)}}, \\ &z \equiv v\varpi^{2t}\gamma_2 \pmod{\mathfrak{p}_F^{-\lceil \frac{d}{2} \rceil + 1}}. \end{aligned}$$

Thus we see that the elements of $\widehat{H}$ that satisfy the required conditions have the form

$$Y(\beta) := \beta \begin{pmatrix} 0 & \sqrt{\epsilon}\varpi^{-s-\delta(\alpha^t)} \\ 0 & 0 \end{pmatrix} + \Gamma^{\alpha^t}$$

for some $\beta \in \mathcal{O}_F$. Since the $(2,1)$-entry of $Y(\beta)$ is only modulo $\mathfrak{p}_E^{-\lceil \frac{d}{2} \rceil + 1}$, we may pick a more convenient coset representative as follows.

Set

$$Y'(\beta) = \begin{pmatrix} 0 & \sqrt{\epsilon}\beta\varpi^{-s-2t-y}\gamma_1^{-1}\gamma_1 \\ \sqrt{\epsilon}\beta\varpi^{-s-y+2t}\gamma_1^{-1}\gamma_2 & 0 \end{pmatrix} + \Gamma^{\alpha^t}.$$

Then we have $Y'(\beta) = \widetilde{\Gamma}_\beta^{\alpha^t}$ where

$$\widetilde{\Gamma}_\beta = \begin{pmatrix} u\sqrt{\epsilon} & (v + \beta\varpi^{-s-y}\gamma_1^{-1})\gamma_1\sqrt{\epsilon} \\ (v + \beta\varpi^{-s-y}\gamma_1^{-1})\gamma_2\sqrt{\epsilon} & u\sqrt{\epsilon} \end{pmatrix} \in \mathfrak{t}_{-r} \subset \mathfrak{g}_{y,-r},$$

and $\widetilde{\Gamma}_\beta^{\alpha^t} \in \mathfrak{g}_{2t+y,-r}$. Note that $Y(\beta)$ and $\widetilde{\Gamma}_\beta^{\alpha^t}$ represent the same coset in $\mathfrak{g}_{2t+y,-r}/\mathfrak{g}_{2t+y,-s+}$. By the argument above, for each Ψ_{Y_i} that occurs in the decomposition (5.2.10), we have $Y_i = Y(\beta) \cong \widetilde{\Gamma}_\beta^{\alpha^t}$ for some $\beta \in \mathcal{O}_F^\times$. Note that $\widetilde{\Gamma}_\beta \in \mathfrak{t}$; therefore, by (3.2.2), Γ and $\widetilde{\Gamma}_\beta$ have the same centralizer. Consequently, the centralizer of $Y_i = \widetilde{\Gamma}_\beta^g$ in $\mathcal{K}$ is $\mathcal{K} \cap \mathcal{T}^{\alpha^t}$. Since

$$(\mathcal{K} \cap \mathcal{T}^{\alpha^t}) \cap (G_{y,s}^{\alpha^t} \cap \mathcal{J}_d) = (\mathcal{K} \cap \mathcal{T}^{\alpha^t}) \cap \mathcal{J}_d,$$

and, for all $i \in I$, we have $\Psi_{Y_i} = \phi^{\alpha^t}$ on this intersection, it follows that $\Psi_{Y_i,\phi^{\alpha^t}}$ is a well-defined character of $(\mathcal{K} \cap \mathcal{T}^{\alpha^t})\mathcal{J}_d$. By Theorem 3.2.6, we obtain that

$$\operatorname{Ind}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d}^{\mathcal{K}} \Psi_{Y_i,\phi^{\alpha^t}} \tag{5.2.14}$$

is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$. Since Ψ_{Y_i} occurs in the decomposition (5.2.10), we obtain that

$$\operatorname{Hom}_{\mathcal{K}}(\operatorname{Ind}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d}^{\mathcal{K}} \Psi_{Y_i,\phi^{\alpha^t}}, \operatorname{Ind}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G_{y,s}^{\alpha^t})}^{\mathcal{K}} \rho^{\alpha^t})$$

$$\cong \bigoplus_{h\in(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})\backslash\mathcal{K}/(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d} \mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})\cap((\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d)^h}(\Psi^h_{Y_i,\phi^{\alpha^t}}, \rho^{\alpha^t}) \neq 0.$$

Since (5.2.14) is an irreducible representation of $\mathcal{K}$, and both (5.2.14) and (5.2.8) have the same degree, we obtain that (5.2.8) is an irreducible representation of $\mathcal{K}$. Note that at this point we have shown that the Mackey component (5.2.8) is irreducible. It remains to prove the isomorphism stated in the proposition, that is, we may take Γ_{α^t} in place of Y_i.

Since both representations (5.2.14) and (5.2.8) are irreducible, it follows that

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}^{\mathcal{K}}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d}\Psi_{Y_i,\phi^{\alpha^t}}, \mathrm{Ind}^{\mathcal{K}}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})}\rho^{\alpha^t})\\ &\cong \mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})\cap((\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d)}(\Psi_{Y_i,\phi^{\alpha^t}}, \rho^{\alpha^t}) \cong \mathbb{C}.\end{aligned}$$

As the restriction of ρ^{α^t} to $(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})\cap((\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d)$ is given by (5.2.13), we obtain

$$\mathrm{Hom}_{(\mathcal{K}\cap\mathcal{T}^{\alpha^t})(\mathcal{K}\cap G^{\alpha^t}_{y,s})\cap((\mathcal{K}\cap\mathcal{T}^{\alpha^t})\mathcal{J}_d)}(\Psi_{Y_i,\phi^{\alpha^t}}, \oplus_{i\in I}\Psi_{Y_i,\phi^{\alpha^t}}) \cong \mathbb{C}.$$

Thus all the characters Ψ_{Y_i} occurring in the decomposition (5.2.10) must be distinct. We conclude that $\mathrm{Res}_{G^{\alpha^t}_{y,s}\cap\mathcal{J}_d}\rho^{\alpha^t} = \oplus_{\beta\in\mathfrak{f}}\Psi_{\widetilde{\Gamma}^{\alpha^t}_\beta}$. Since $Y(0) = \Gamma^{\alpha^t}$, we see that ρ^{α^t} and $\Psi_{\Gamma^{\alpha^t}}$ intertwine on $G^{\alpha^t}_{y,s}\cap\mathcal{J}_d$ as was required. ∎

Having established both the irreducibility of these representations and their precise description, we may now state our main theorem, that gives the explicit branching rules for these positive-depth supercuspidal representations.

Theorem 5.2.12. *Let $\rho = \rho(\mathcal{T}, y, r, \phi)$, and let Γ be a G-generic element of depth $-r$ that realizes ϕ. The decomposition of the restriction of the associated supercuspidal representation of G into irreducible $\mathcal{K}$ representations is given as follows.*

$$\begin{aligned}&\mathrm{Res}^G_{\mathcal{K}}c\text{-}\mathrm{Ind}^G_{\mathcal{T}G_{0,s}}\rho \cong \mathrm{Ind}^{\mathcal{K}}_{\mathcal{T}G_{0,s}}\rho \oplus \bigoplus_{t>0}\mathcal{S}_{r+2t}(\Gamma^{\alpha^t}, \phi^{\alpha^t}),\\ &\mathrm{Res}^G_{\mathcal{K}}c\text{-}\mathrm{Ind}^G_{\mathcal{T}G_{1,s}}\rho \cong \bigoplus_{t\geq 0}\mathcal{S}_{r+2t+1}(\Gamma^{\alpha^t}, \phi^{\alpha^t}),\\ &\mathrm{Res}^G_{\mathcal{K}}c\text{-}\mathrm{Ind}^G_{\mathcal{T}G_{\frac{1}{2},s}}\rho \cong \mathcal{S}_{r+\frac{1}{2}}(\Gamma, \phi) \oplus \bigoplus_{t>0}\left(\mathcal{S}_{r+2t+\frac{1}{2}}(\Gamma^{\alpha^t}, \phi^{\alpha^t}) \oplus \mathcal{S}_{r+2t-\frac{1}{2}}(\Gamma^{\alpha^t\mathrm{w}}, \phi^{\alpha^t\mathrm{w}})\right).\end{aligned}$$

This provides an explicit and complete description of $\mathrm{Res}_{\mathcal{K}}\pi_\rho$, analogous to the one obtained in Theorem 4.3.3 for depth-zero supercuspidal representations of our unitary group G. As in the depth-zero case, all irreducible constituents appearing in the above decomposition have distinct depths. The resulting decomposition aligns with the restriction behaviour of irreducible supercuspidal representations of $\mathrm{SL}_2(F)$ [Nev13].

5.3 Decomposition near the identity

Let π_ρ be an irreducible supercuspidal representation of G of depth $r > 0$, associated with the datum $(\mathcal{T}, y, r, \phi)$, where $\rho = \rho(\mathcal{T}, y, r, \phi)$. Denote by θ the central character of π_ρ. By Corollary 5.1.7, we may assume that θ has depth zero, more precisely, $\theta = \mathbb{1}$ or the unique nontrivial quadratic character δ of E^1, at the expense of twisting our branching rules by characters of G.

In this section, we study the relationship between the higher-depth components appearing in the decomposition of π_ρ and those appearing in the decomposition of depth-zero supercuspidal representations with the same central character as π_ρ.

We also examine the restriction of π_ρ to the subgroup $\mathcal{K}_{2r+} \subseteq \mathcal{K}$ and show that its decomposition is entirely determined by the representations of the form (3.3.2), constructed from the nilpotent orbits in the Lie algebra of G together with the character θ of the center of G.

5.3.1 Connection with depth-zero representations

Recall that in Chapter 4 we constructed all depth-zero irreducible supercuspidal representations of G. Given any two distinct characters α and β of $\mathfrak{e}^1$, we obtained two irreducible depth-zero supercuspidal representations

$$c\text{-Ind}_{\mathcal{K}}^{G}\sigma \qquad \text{or} \qquad c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta \tag{5.3.1}$$

of G with central character $\alpha \otimes \beta$.

Since the centre Z can be identified with E^1, and θ has depth zero, the character θ factors through

$$E^1 / \left(E^1 \cap (1 + \mathfrak{p}_E)\right) \cong \mathfrak{e}^1.$$

Therefore, if θ is nontrivial, we may take $\alpha = \theta$ and β to be the trivial character. If θ is trivial, we may take α to be any nontrivial character of $\mathfrak{e}^1$ and set $\beta = \alpha^{-1}$. Thus, given any depth-zero character θ of the centre Z, we can always produce a depth-zero irreducible supercuspidal representation of G having central character θ.

At this stage, we have three representations: a positive-depth irreducible supercuspidal representation π_ρ, and two depth-zero irreducible supercuspidal representations

$$c\text{-Ind}_{\mathcal{K}}^{G}\sigma \qquad \text{and} \qquad c\text{-Ind}_{\mathcal{K}^\eta}^{G}\sigma^\eta, \tag{5.3.2}$$

all sharing the same central character θ.

By Theorem 4.3.3, the decomposition of a depth-zero supercuspidal representation upon restriction to $\mathcal{K}$ is multiplicity free, and each component occurring in the decomposition has distinct depth d and is isomorphic to $\mathcal{S}_d(X_{\varpi^{-d}}, \theta)$. Similarly, by Theorem 5.2.12, the corresponding decomposition in the positive-depth case is again multiplicity free, with each component of distinct depth d, and is isomorphic to $\mathcal{S}_d(\Gamma^g, \phi^g)$ for some $g \in M(\mathcal{T})$, where $\Gamma \in \mathfrak{t}_{-r}$ is an element realizing ϕ on $\mathcal{T}_{s+}/\mathcal{T}_{r+}$.

The following theorem formalizes the connection between these two decompositions.

Theorem 5.3.1. *For $d > 2r$ and all $g \in M(\mathcal{T})$,*

$$\mathcal{S}_d(\Gamma^g, \phi^g) \cong \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

Proof. Let

$$\Gamma = \begin{pmatrix} u\sqrt{\epsilon} & v\sqrt{\epsilon}\gamma_1 \\ v\sqrt{\epsilon}\gamma_2 & u\sqrt{\epsilon} \end{pmatrix} \in \mathfrak{t}_{-r}$$

be a G-generic element of depth $-r$ representing ϕ. For $g \in M(\mathcal{T})$, we have $\Gamma^g \in \mathfrak{g}_{0,-d}$ where $d = r + \delta(g)$, and we assume $d > 2r$.

If $g = \alpha^t$, then

$$\Gamma^g = \begin{pmatrix} u\sqrt{\epsilon} & v\sqrt{\epsilon}\varpi^{-2t}\gamma_1 \\ v\sqrt{\epsilon}\varpi^{2t}\gamma_2 & u\sqrt{\epsilon} \end{pmatrix},$$

and $\Gamma^g - X_{v\varpi^{-2t}\gamma_1} \in \mathfrak{g}_{0,-\lceil d/2 \rceil}$. Hence, by Lemma 3.3.3, $\Psi_{\Gamma^g} = \Psi_{X_{v\varpi^{-2t}\gamma_1}}$ on $\mathcal{J}_d$.

Similarly, if $g = \alpha^t \mathrm{w}$ with $t > 0$, then

$$\Gamma^g = \begin{pmatrix} u\sqrt{\epsilon} & v\sqrt{\epsilon}\varpi^{-2t}\gamma_2 \\ v\sqrt{\epsilon}\varpi^{2t}\gamma_1 & u\sqrt{\epsilon} \end{pmatrix}.$$

Since $t > 0$, we again have $\Gamma^g - X_{v\varpi^{-2t}\gamma_2} \in \mathfrak{g}_{0,-\lceil d/2 \rceil}$, so $\Psi_{\Gamma^g} = \Psi_{X_{v\varpi^{-2t}\gamma_2}}$ on $\mathcal{J}_d$, and $T(X)\mathcal{J}_d = T(X_{v\gamma_i\varpi^{-2t}})\mathcal{J}_d$.

Since in both cases $\phi^g|_Z = \theta$, and the central character θ has depth zero, it coincides with $\Psi_{X_{v\gamma_i\varpi^{-2t}}}$ on $Z\mathcal{U} \cap \mathcal{J}_d$, because $\Psi_{X_{v\gamma_i\varpi^{-2t}}}$ is also trivial on this intersection. Extend θ trivially to $\mathcal{U}$, and let $\Psi_{X_{v\gamma_i\varpi^{-2t}},\theta}$ denote the unique extension of θ and $\Psi_{X_{v\gamma_i\varpi^{-2t}}}$ to $T(X_{v\gamma_i\varpi^{-2t}})\mathcal{J}_d$. Then, by (3.3.2), $\mathcal{S}_d(X_{v\gamma_i\varpi^{-2t}}, \theta)$ is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$.

Since $T(X) = (\mathcal{K} \cap \mathcal{T}^g)$, by Lemma 5.2.6, we have $T(X) = Z(\mathcal{K} \cap \mathcal{T}^g_{0+})$, and a similar computation as in proof of Lemma 5.2.6 yields $T(X)^{g^{-1}} = Z(\mathcal{K}^{g^{-1}} \cap \mathcal{T}_{0+}) \subset Z\mathcal{T}_{2t}$. As $d > 2r$, we have $T(X)^{g^{-1}} \subset Z\mathcal{T}_{2t} \subset Z\mathcal{T}_{r+}$, and therefore $\phi^g|_{T(X)} = \phi|_Z = \theta$. Hence,

$$\Psi_{\Gamma^g,\phi^g} = \Psi_{X_{v\gamma_i\varpi^{-2t}},\theta}. \tag{5.3.3}$$

Since $T(X)\mathcal{J}_d = T(X_{v\gamma_i\varpi^{-2t}})\mathcal{J}_d$ and $\Psi_{\Gamma^g,\phi^g} = \Psi_{X_{v\gamma_i\varpi^{-2t}},\theta}$, we have

$$\mathcal{S}_d(\Gamma^g, \phi^g) = \mathrm{Ind}_{T(X)\mathcal{J}_d}^{K} \Psi_{\Gamma^g,\phi^g} = \mathrm{Ind}_{T(X)\mathcal{J}_d}^{K} \Psi_{X_{v\gamma_i\varpi^{-2t}},\theta} = \mathcal{S}_d(X_{v\gamma_i\varpi^{-2t}}, \theta)$$

Finally, by Lemma 3.3.2, we obtain

$$\mathcal{S}_d(\Gamma^g, \phi^g) = \mathcal{S}_d(X_{v\gamma_i\varpi^{-2t}}, \theta) \cong \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$ ∎

Thus, depending on the choice of datum $(\mathcal{T}, y, r, \phi)$, the components occurring in the decomposition of π_ρ with depth $d > 2r$ coincide with those appearing in the decomposition of a depth-zero irreducible supercuspidal representation having the same central character as π_ρ. As noted earlier in Corollary 5.1.7, we may assume that π_ρ has central character δ^k for some $k \in \{0, 1\}$, where δ denotes the nontrivial quadratic character of E^1, at the expense of twisting our branching rules by a character of G. We fix $\sigma_{\mathbb{1}}$ and σ_δ to be cuspidal representations of $\mathcal{K}/\mathcal{K}_{0+}$ with central characters $\theta = \mathbb{1}$ and $\theta = \delta$, respectively. Define

$$\tau_{\delta^k}^{\text{even}} = c\text{-Ind}_{\mathcal{K}}^G \sigma_{\delta^k}, \quad \tau_{\delta^k}^{\text{odd}} = c\text{-Ind}_{\mathcal{K}^\eta}^G \sigma_{\delta^k}^\eta, \qquad k \in \{0, 1\}.$$

These are depth-zero irreducible supercuspidal representations of G with central character δ^k where $k \in \{0, 1\}$. Let $(\pi_\rho)_d$ denote the depth-d component occurring in the decomposition of $\pi_\rho|_{\mathcal{K}}$ if it exists. Then, combined with Lemma 4.2.7, we obtain the following theorem.

Theorem 5.3.2. *Let π_ρ be an irreducible supercuspidal representations of depth $r \geq 0$. Then for all $d > 2r$,*

$$(\pi_\rho)_d \in \left\{ (\tau_{\delta^k}^{\text{even}})_d,\ (\tau_{\delta^k}^{\text{odd}})_d \mid k \in \{0, 1\} \right\},$$

up to a twist by a character of G.

The following table lists the various possibilities: the first two columns specify the condition on $(\mathcal{T}, r)$, while the third column indicates the corresponding depth-zero supercuspidal representation whose higher-depth components agree with those of π_ρ when restricted to $\mathcal{K}$.

$\mathcal{T}$	r	Depth-zero supercuspidal
$\mathcal{T}_{1,1}$	r is even	$\tau_{\delta^k}^{\text{even}}$
$\mathcal{T}_{1,1}$	r is odd	$\tau_{\delta^k}^{\text{odd}}$
$\mathcal{T}_{\varpi^{-1},\varpi}$	r is even	$\tau_{\delta^k}^{\text{odd}}$
$\mathcal{T}_{\varpi^{-1},\varpi}$	r is odd	$\tau_{\delta^k}^{\text{even}}$
$\mathcal{T}$ is ramified	$r \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$	$\tau_{\delta^k}^{\text{even}} \oplus \tau_{\delta^k}^{\text{odd}}$

Table 5.1: Correspondence between positive-depth and depth-zero supercuspidal components upon restriction to $\mathcal{K}$.

5.3.2 Representations associated with nilpotent orbits

Recall from Chapter 3 that there are three nilpotent G-orbits in $\mathfrak{g}$, and they are parametrized by

$$\{X_\delta \mid \delta \in \{0, 1, \varpi\}\},$$

where

$$X_\delta = \begin{pmatrix} 0 & \delta\sqrt{\epsilon} \\ 0 & 0 \end{pmatrix}.$$

For $\delta \in \{0, 1, \varpi\}$, we recall that the nilpotent orbits were denoted by $\mathcal{N}_\delta$, and that each G-orbit $\mathcal{N}_\delta$ decomposes into the following $\mathcal{K}$-orbits

$$\mathcal{N}_0 = \mathcal{K} \cdot X_0, \qquad \mathcal{N}_1 = \bigsqcup_{m \in 2\mathbb{Z}} \mathcal{K} \cdot X_{\varpi^{-m}}, \qquad \mathcal{N}_\varpi = \bigsqcup_{n \in 2\mathbb{Z}+1} \mathcal{K} \cdot X_{\varpi^{-n}}.$$

For each depth-zero character θ of the center of G and $t \in \mathbb{R}_{\geq 0}$ we define highly reducible representations

$$\tau^t_{\mathcal{N}_1}(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{>0},\, d > t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta), \qquad \tau^t_{\mathcal{N}_\varpi}(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{\geq 0}+1,\, d > t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

5.3.3 Restriction to $\mathcal{K}_{2r+}$

Having introduced the representations associated with nilpotent orbits, we now study the restriction of π_ρ to $\mathcal{K}_{2r+}$. Throughout this section, we assume that the central character θ of π_ρ is of depth zero.

We now state the main theorem of this section.

Theorem 5.3.3. *Let π be an irreducible supercuspidal representation of G of depth $r \geq 0$, and let θ denote its central character. Then*

$$\mathrm{Res}_{\mathcal{K}_{2r+}} \pi_\rho \;=\; n(\pi_\rho)\, \mathbb{1} \;\oplus\; a_{\mathcal{N}_1}\, \mathrm{Res}_{\mathcal{K}_{2r+}}\big(\tau^{2r}_{\mathcal{N}_1}(\theta)\big) \;\oplus\; a_{\mathcal{N}_\varpi}\, \mathrm{Res}_{\mathcal{K}_{2r+}}\big(\tau^{2r}_{\mathcal{N}_\varpi}(\theta)\big),$$

where $a_{\mathcal{N}_1}, a_{\mathcal{N}_\varpi} \in \{0, 1\}$ and values of $n(\pi_\rho)$ are listed in Table 6.1.

Proof. If π is a depth-zero irreducible supercuspidal representation, then Lemma 4.2.7 together with Corollary 4.3.3 gives the desired decomposition. Explicitly, there are two cases. If $\pi = c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma$, where σ is a cuspidal representation of $\mathrm{U}(1,1)(\mathfrak{f})$, then $n(\pi) = \dim(\sigma) = q-1$, $a_{\mathcal{N}_1} = 1$, and $a_{\mathcal{N}_\varpi} = 0$. If $\pi = c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma^\eta$, where σ is a cuspidal representation of $\mathrm{U}(1,1)(\mathfrak{f})$, then $n(\pi) = 0$, $a_{\mathcal{N}_1} = 0$, and $a_{\mathcal{N}_\varpi} = 1$.

We now turn to the positive depth irreducible supercuspidal representaitons of G. Let $\pi_\rho = c\text{-}\mathrm{Ind}_{G_{y,s}\mathcal{T}}^G \rho$ be an irreducible supercuspidal representation of G of depth $r \geq 0$, associated with the datum $\rho = \rho(\mathcal{T}, y, r, \phi)$, and let θ denote its central character. By Theorem 5.2.12, the restriction of π_ρ decomposes as a direct sum of irreducible representations of the form $\mathcal{S}_d(\Gamma^g, \phi^g)$ with $d \geq r$. Since $\mathcal{K}_{2r+}$ acts trivially on the components with $d \leq 2r$, only the terms with $d > 2r$ contribute nontrivially to the restriction to $\mathcal{K}_{2r+}$. If $d > 2r$, then by Lemma 5.3.1 we have

$$\mathcal{S}_d(\Gamma^g, \phi^g) = \mathcal{S}_d(X_{v\gamma_i\varpi^{-2t}}, \theta) \cong \mathcal{S}_d(X_{\varpi^{-d}}, \theta),$$

therefore we have the desired decomposition with $n(\pi_\rho) = \dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr)$.

As g varies, the depths $d > 2r$ occur with a fixed parity determined by $\mathcal{T}$ and r; even d assemble into $\tau_{\mathcal{N}_1}(\theta)$, and odd d into $\tau_{\mathcal{N}_\varpi}(\theta)$. Therefore,

$$a_{\mathcal{N}_1} = \begin{cases} & \text{if } \mathcal{T} = \mathcal{T}_{1,1} \text{ with } r \text{ even},\\ 1, & \text{or } \mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi} \text{ with } r \text{ odd},\\ & \text{or } \mathcal{T} \text{ is ramified},\\ 0, & \text{otherwise}, \end{cases} \qquad a_{\mathcal{N}_\varpi} = \begin{cases} & \text{if } \mathcal{T} = \mathcal{T}_{1,1} \text{ with } r \text{ odd},\\ 1, & \text{or } \mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi} \text{ with } r \text{ even},\\ & \text{or } \mathcal{T} \text{ is ramified},\\ 0, & \text{otherwise}. \end{cases}$$

∎

We now compute $n(\pi_\rho)$ explicitly and record the values in Table 5.2.

Since we have already we know the values of $a_{\mathcal{N}_1}$ and $a_{\mathcal{N}_\varpi}$ for the various choices of the datum $(\mathcal{T}, y, r, \phi)$ with associated supercuspidal representation π_ρ from the preceding proof, it remains to compute $\dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr)$ for all such choices of π_ρ.

Let $\Gamma \in \mathfrak{t}_{-r}$ be an element realizing ϕ on $\mathcal{T}_{s+}/\mathcal{T}_{r+}$. By Theorem 5.2.12, the restriction of π_ρ decomposes as a direct sum of irreducible representations of the form $\mathcal{S}_d(\Gamma^g, \phi^g)$ where $g \in M(\mathcal{T})$. Moreover, the restriction of π_ρ to the subgroup $\mathcal{K}_{2r+1}$ acts trivially on those components $\mathcal{S}_d(\Gamma^g, \phi^g)$ whose depths satisfy $d \leq 2r$. Consequently, the dimension of $\pi_\rho^{\mathcal{K}_{2r+}}$ is equal to the sum of the dimensions of those $\mathcal{S}_d(\Gamma^g, \phi^g)$ occurring in the decomposition with depth parameter d ranging from r to $2r$.

If $\mathcal{T} = \mathcal{T}_{1,1}$, then

$$\begin{aligned} \dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr) &= \dim\bigl(\mathrm{Ind}_{\mathcal{T}\mathcal{K}_s}^{\mathcal{K}} \rho\bigr) + \sum_{t=1}^{\lfloor \frac{r}{2} \rfloor} (q^2-1) q^{(r+2t)-1} \\ &= (q-1)q^r + (q^2-1)q^{r-1} \sum_{t=1}^{\lfloor \frac{r}{2} \rfloor} (q^2)^t \\ &= q^r \left(q^{2\lfloor \frac{r}{2} \rfloor + 1} - 1\right). \end{aligned}$$

Hence, when r is even we have

$$\dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr) = q^r \left(q^{r+1} - 1\right),$$

and when r is odd,

$$\dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr) = q^r \left(q^r - 1\right).$$

If $\mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi}$, then

$$\dim\bigl(\pi_\rho^{\mathcal{K}_{2r+}}\bigr) = \sum_{t=0}^{\lfloor \frac{r-1}{2} \rfloor} (q^2-1) q^{r+2t} = (q^2-1) q^r \sum_{t=0}^{\lfloor \frac{r-1}{2} \rfloor} (q^2)^t = q^r \left(q^{2\lfloor \frac{r-1}{2} \rfloor + 2} - 1\right).$$

Thus, when r is even,

$$\dim\left(\pi_\rho^{\mathcal{K}_{2r+}}\right) = q^r\left(q^r - 1\right),$$

and when r is odd,

$$\dim\left(\pi_\rho^{\mathcal{K}_{2r+}}\right) = q^r\left(q^{r+1} - 1\right).$$

Finally, when $\mathcal{T}$ is ramified,

$$\begin{aligned}\dim\left(\pi_\rho^{\mathcal{K}_{2r+}}\right) &= (q^2-1)q^{r-\frac{1}{2}} + \sum_{t=1}^{\lfloor \frac{r}{2}-\frac{1}{4}\rfloor}(q^2-1)q^{r+2t-\frac{1}{2}} + \sum_{t=1}^{\lfloor \frac{r}{2}+\frac{1}{4}\rfloor}(q^2-1)q^{r+2t-\frac{3}{2}} \\ &= (q^{2r}-q^{r-\frac{1}{2}})(q+1).\end{aligned}$$

$n(\pi_\rho)$	Choice of $\mathcal{T}$ and r
$q^r\left(q^{r+1}-1\right)$	$\mathcal{T} = \mathcal{T}_{1,1}$, and r is even; or $\mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi}$, and r is odd
$q^r\left(q^r-1\right)$	$\mathcal{T} = \mathcal{T}_{1,1}$, and r is odd; or $\mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi}$, and r is even
$(q^{2r}-q^{r-\frac{1}{2}})(q+1)$	$\mathcal{T}$ is ramified

Table 5.2: Values of $n(\pi_\rho)$ for various choices of $\rho = \rho(\mathcal{T}, y, r, \phi)$.

Remark 5.3.4. The values of $n(\pi_\rho)$ obtained for G agree with those for the group $\mathrm{SL}_2(F)$ (see [Nev24, Table 2]) when $\mathcal{T}$ is unramified. In the ramified case, however, the values for G are precisely twice those for $\mathrm{SL}_2(F)$.

Chapter 6

Branching rules for principal series representations of G

In this chapter, we describe the branching rules upon restriction to $\mathcal{K}$ for our third and final class of representations, namely, the principal series representations of G. For this class of representations, we first provide a canonical decomposition without making any use of the representations constructed in Theorem 3.2.6. Unlike the case for supercuspidal representations where Mackey theory did most of the work and every Mackey component turned out to be irreducible, here Mackey theory produces a single highly reducible piece. So, instead of stopping there, we proceed in a different manner: we bring in extra representation-theoretic tools to peel away the layers and expose the real structure hiding inside that big Mackey component.

To obtain a more explicit and satisfying description, we then express the decomposition upon restriction to $\mathcal{K}$ in terms of the irreducible representations of $\mathcal{K}$ constructed in Theorem 3.2.6. Similar to the other two classes of representations, we again obtain a multiplicity-free decomposition characterized by distinct depth and degree.

As in the case of supercuspidal representations, we again show that the higher-depth components occurring in the decomposition of principal series representations upon restriction to $\mathcal{K}$ coincide with the higher-depth components occurring upon restriction to $\mathcal{K}$ of a subset of the four fixed depth-zero irreducible supercuspidal representations of G, up to twisting by a quasi-character of G.

Similar to the case of supercuspidal representations, we conclude this chapter by restricting these principal series representations further to a smaller subgroup of $\mathcal{K}$, obtaining a decomposition in terms of representations arising from nilpotent elements of the Lie algebra of G together with the central character of the original representation.

6.1 Construction of the representations

Let B be the Borel subgroup of upper triangular matrices in G. Write $B = TU$ where

$$T = \left\{ t(a) := \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \,\middle|\, a \in E^\times \right\}$$

is a maximally F-split maximal F-torus T of G, and

$$U = \left\{ u(b) := \begin{pmatrix} 1 & \sqrt{\epsilon}b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in F \right\}$$

is the unipotent radical of B. Since the commutator of B is U, a character of B is defined by its restriction to T, and any character of T can be extended trivially to a character of B. When convenient, identify χ with a character $\chi^\dagger$ of $E^\times$ via

$$\chi \begin{pmatrix} a & \sqrt{\epsilon}b \\ 0 & \overline{a}^{-1} \end{pmatrix} = \chi^\dagger(a).$$

A *principal series representation* of G is the induced representation $\pi_\chi = \mathrm{Ind}_B^G \chi$, where χ is a character of T extended trivially to B.

Remark 6.1.1. In the literature, the principal series representation associated to χ is often instead defined as the normalized induced representation $n\text{-}\mathrm{Ind}_B^G\chi = \pi_{\delta^{\frac{1}{2}}\otimes\chi}$ where $\delta(t) = |\det(Ad(t)|_{\mathfrak{u}})|_p$ for $t \in T$, and $\mathfrak{u} = \mathrm{Lie}(U)$. Here $|\cdot|_p$ represents the p-adic norm. In our setting, given $t(a) \in T$, we have $\delta(t) = |a\overline{a}|_p$. If $t(a) \in \mathcal{K} \cap T$, then $a \in \mathcal{O}_E^\times$ and so $\delta(t) = 1$. We shall see that the branching rules of π_χ and $\pi_{\chi'}$ agree whenever the restriction of χ and χ' to $T \cap \mathcal{K}$ agree. So in particular $n\text{-}\mathrm{Ind}_B^G\chi$ and $\mathrm{Ind}_B^G\chi$ have the same branching rules.

A character χ of T is said to have *depth* r, for some real number $r \geq 0$, if

$$\chi|_{T_r} \neq \mathbb{1} \quad \text{and} \quad \chi|_{T_{r+}} = \mathbb{1}.$$

As is standard, we also say that a character which is trivial on T_0 has *depth zero*. We say that χ is *unramified* if $\chi|_{T_0} = \mathbb{1}$; otherwise, we say that χ is *ramified*.

Lemma 6.1.2. *Let χ be a character of T, and let (π_χ, V_χ) be the corresponding principal series representation of G. If there exists a nonzero G-stable subspace $W \subset V_\chi$ such that $W^K \neq \{0\}$, then χ is unramified.*

Proof. Since $W^{\mathcal{K}} \neq \{0\}$, there exists $0 \neq f \in W^{\mathcal{K}}$. Then for every $k \in \mathcal{K}$ we have $\pi_\chi(k)f = f$, and hence $f(k) = (\pi_\chi(k)f)(I) = f(I)$. Since $G = B\mathcal{K}$, for any $g \in G$ we may write $g = bk$ with $b \in B$ and $k \in \mathcal{K}$, and therefore $f(g) = f(bk) = \chi(b)f(k) = \chi(b)f(I)$. In particular, if $f \neq 0$, then necessarily $f(I) \neq 0$.

As $T_0 \subset \mathcal{K}$, for all $t \in T_0$ we have

$$f(I) = (\pi_\chi(t)f)(I) = f(t) = \chi(t)f(I).$$

Since $f(I) \neq 0$, it follows that $\chi(t) = 1$ for all $t \in T_0$. Hence χ is an unramified character of T. ∎

Lemma 6.1.3. *Let χ be a character of T, and let (π_χ, V_χ) be the corresponding principal series representation of G. If (π_χ, V_χ) is reducible, then χ is unramified.*

Proof. Suppose (π_χ, V_χ) is reducible. Then there exists a nonzero proper G-stable subspace $W \subset V_\chi$. By [Cas95, Proposition 3.3.6], we have $W^{\mathcal{K}} \neq \{0\}$. By Lemma 6.1.2, it follows that χ is unramified. ∎

Thus, if χ is a ramified character of T, then π_χ is irreducible. We denote by $\mathrm{sgn}_{E/F}$ the quadratic character of $F^\times$ defined by

$$\mathrm{sgn}_{E/F}(x) = \begin{cases} 1 & \text{if } x \in \mathrm{Im}(N_{E/F}(E^\times)), \\ -1 & \text{otherwise.} \end{cases}$$

Lemma 6.1.4. *Let χ' be the character of T defined by $\chi'(t) = (-1)^{\nu(t)}$. Suppose χ is a character of T such that $\chi|_{F^\times} = \mathrm{sgn}_{E/F}$. Then there exists a character ϕ of E^1 such that $\chi = (\phi \circ \det) \otimes \chi'$.*

Proof. Since $\chi|_{F^\times} = \mathrm{sgn}_{E/F}$ and $N_{E/F} \colon \mathcal{O}_E^\times \to \mathcal{O}_F^\times$ is surjective for an unramified quadratic extension, it follows that $\chi|_{\mathcal{O}_F^\times} = \mathbb{1}$.

By Hilbert's Theorem 90 every element of E^1 can be written as $u\overline{u}^{-1}$ for some $u \in \mathcal{O}_E^\times$. Define a character ϕ of E^1 by $\phi(u\overline{u}^{-1}) = \chi(u)$. For $u, v \in \mathcal{O}_E^\times$, if $u\overline{u}^{-1} = v\overline{v}^{-1}$, then $u/v = \overline{u/v}$, hence $u/v \in \mathcal{O}_F^\times$, and therefore $\chi(u) = \chi(v)$. Thus ϕ is well defined.

Now let $t = \begin{pmatrix} u\varpi^n & 0 \\ 0 & \overline{u}^{-1}\varpi^{-n} \end{pmatrix} \in T$. Then

$$\chi(t) = \chi^\dagger(u\varpi^n) = \chi^\dagger(u)\chi^\dagger(\varpi^n) = \chi^\dagger(u)(-1)^n = \phi(u\overline{u}^{-1})\chi'^\dagger(\varpi^n) = (\phi \circ \det)(t) \otimes \chi'(t)$$

which proves the result. ∎

Among the principal series representations induced from unramified characters of T, there are exactly three reducible ones up to twisting by a character of G, namely, the principal series representations (π_χ) corresponding to $\chi = \mathbb{1}$, δ, and $\delta^{\frac{1}{2}} \otimes \chi'$ [Key87, Theorem 3.4]. For simplicity, in this thesis we describe the branching rules at the level of the principal series representations themselves, treating each π_χ as a whole, even in the reducible cases. A detailed analysis of the branching of the irreducible constituents in these reducible cases can be found in [Tiw25b].

Recall that in Chapter 2 we defined

$$S := \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \,\middle|\, a \in F^\times \right\},$$

F-points of a maximal F-split, F-torus contained inside T. Given a character χ of T, we define the *true depth* of χ to be the depth of its restriction to S. We say that χ has *minimal depth* r if the depth of χ and its true depth both equal r.

Note that restricting χ to S corresponds to restricting $\chi^\dagger$ to $F^\times$. Since $T \cong E^\times$ and $S \cong F^\times$ whose filtrations are indexed by integers, it follows that both the depth and the true depth of characters of T are integers. Moreover, for every $r \in \mathbb{Z}_{\geq 0}$, we have $S_r \subset T_r$, and hence the depth of a character of T is always greater than or equal to its true depth.

Remark 6.1.5. It is worth noting that a character χ of T has minimal depth r if and only if it is G-generic of depth r (see §2.3.3.2).

Lemma 6.1.6. *Let $m \in \mathbb{Z}_{\geq 0}$, and let $\chi\colon T \to \mathbb{C}^\times$ be a character of T of true depth m. Then there exists a character $\phi\colon E^1 \to \mathbb{C}^\times$ such that $\phi \circ \det|_{T_{m+1}} = \chi|_{T_{m+1}}$. Moreover, if $\chi|_{S_0} = \mathbb{1}$, then $\chi|_{T_0} = \phi \circ \det|_{T_0}$ for some character ϕ of E^1.*

Proof. For each $n \in \mathbb{Z}_{\geq 0}$, consider the map $\gamma_n\colon E_n^\times/F_n^\times \to E^1 \cap E_n^\times$ given by $\gamma_n(x) = x\overline{x}^{-1}$. Then by Lang's theorem γ_n is an isomorphism. Let χ be a character of T, that is trivial on S_n. We define $\widetilde{\phi}\colon E^1 \cap E_n^\times \to \mathbb{C}^\times$ given by $\widetilde{\phi}(x) = ((\chi^\dagger) \circ \gamma_n^{-1})(x)$ as in Figure 6.1.1.

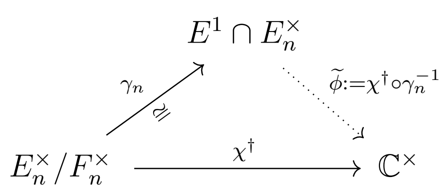

Figure 6.1.1: Commutative diagram defining the character $\widetilde{\phi}$ via γ and $\chi^\dagger$.

Since γ_n is an isomorphism and $\chi^\dagger$ is trivial on $F_n^\times$, $\widetilde{\phi}$ is well-defined. Let $x \in E_n^\times/F_n^\times$. We then have

$$\widetilde{\phi}(x\overline{x}^{-1}) = \widetilde{\phi}(\gamma_n(x)) = (\chi^\dagger \circ \gamma_n^{-1})(\gamma_n(x)) = \chi^\dagger(x). \tag{6.1.1}$$

Since E^1 is a compact abelian group and $\widetilde{\phi}$ is a character of an open subgroup of E^1, there exists an extension $\phi\colon E^1 \to \mathbb{C}^\times$ of $\widetilde{\phi}$. Since the determinant of matrices in G lie in E^1, $\phi \circ \det$ is a well-defined character of G. Note that for any element $t(x) \in T$, $\det(t(x)) = x\overline{x}^{-1}$, therefore taking $n = m+1$, we obtain $\phi \circ \det|_{T_{m+1}} = \chi|_{T_{m+1}}$, as was required. In particular, if $\chi|_{S_0} = \mathbb{1}$, then taking $n = 0$ there exists a character ϕ of E^1, such that for all $x \in \mathcal{O}_E^\times/\mathcal{O}_F^\times$,

$$\phi(x\overline{x}^{-1}) = \phi(\gamma_0(x)) = (\chi^\dagger \circ \gamma_0^{-1})(\gamma_0(x)) = \chi^\dagger(x), \tag{6.1.2}$$

and hence $\phi \circ \det|_{T_0} = \chi|_{T_0}$. ∎

Theorem 6.1.7. *Every character χ of T of true depth r is of the form $\chi = (\phi \circ \det) \otimes \chi'$, where ϕ is a character of E^1, and χ' is a character of minimal depth r.*

Proof. First suppose that χ is a character of T of true depth r such that $\chi|_{S_0} \neq \mathbb{1}$. Then $\chi|_{S_{r+1}} = \mathbb{1}$. By Lemma 6.1.6, there exists a character ϕ of E^1 such that

$$\chi|_{Tr+1} = (\phi \circ \det)|_{T_{r+1}}.$$

Equivalently, $((\phi^{-1} \circ \det) \otimes \chi)|_{T_{r+1}} = \mathbb{1}$. Since $(\phi^{-1} \circ \det)$ is trivial on S, if $\chi|_{S_0} \neq \mathbb{1}$, then

$$(\phi^{-1} \circ \det) \otimes \chi|_{S_r} = \chi|_{S_r} \neq \mathbb{1}.$$

So, in particular χ is non-trivial on T_r. Thus $((\phi^{-1} \circ \det) \otimes \chi)$ has minimal depth r. Taking $\chi' = (\phi^{-1} \circ \det) \otimes \chi$ we have

$$\chi = (\phi \circ \det) \otimes (\phi^{-1} \circ \det) \otimes \chi = (\phi \circ \det) \otimes \chi',$$

as was required. Next suppose that $\chi|_{S_0} = \mathbb{1}$. Then by Lemma 6.1.6 there exists a character $\phi \colon E^1 \to \mathbb{C}^\times$ such that $\phi \circ \det|_T = \chi$, implying that $(\phi^{-1} \circ \det) \otimes \chi = \mathbb{1}$, and we may write $\chi = (\phi \circ \det) \otimes \mathbb{1}$, and $\mathbb{1}$ has minimal depth 0. ∎

Corollary 6.1.8. *Suppose the decomposition into irreducible representations of π_χ is known for all χ of minimal depth. Then so is the decomposition of $\pi_{(\phi \circ \det) \otimes \chi}$ for every character ϕ of E^1, and thus for all principal series.*

Proof. Let χ be a character of T of minimal depth, and suppose

$$\mathrm{Res}_{\mathcal{K}} \pi_\chi = \bigoplus_{i \in I} \sigma_i,$$

where σ_i are irreducible $\mathcal{K}$-representations and I is some index set. Let ϕ be a character of E^1. By Lemma 2.3.3, multiplying an induced representation by a character of the group is equivalent to instead just multiplying the inducing character, therefore we have $\pi_{(\phi \circ \det) \otimes \chi} \cong (\phi \circ \det) \otimes \pi_\chi$. Hence, the restriction of $\pi_{(\phi \circ \det) \otimes \chi}$ to $\mathcal{K}$ decomposes as

$$\mathrm{Res}_{\mathcal{K}} \pi_\chi = \bigoplus_{i \in I} (\phi \circ \det)|_{\mathcal{K}} \otimes \sigma_i,$$

and each of these components is again irreducible. Since every character of T is of the form $(\phi \circ \det) \otimes \chi'$, where χ' has minimal depth (by Theorem 6.1.7), it follows that the decomposition of all principal series representations is determined by that of those associated to characters of minimal depth. ∎

We now introduce a normalization similar to Corollary 5.1.7 that will play a crucial role in Section 6.4.

Lemma 6.1.9. *Let χ be a character of T of minimal depth $r \geq 0$, and let $\pi_\chi = \mathrm{Ind}_B^G \chi$ be the corresponding principal series representation of G. Then there exist a character ϕ of E^1 and a character χ_0 of T such that*

$$\chi = (\phi \circ \det) \otimes \chi_0, \qquad \pi_\chi \;\cong\; (\phi \circ \det) \otimes \pi_{\chi_0},$$

and moreover the central character θ of π_{χ_0} equals δ^k for some $k \in \{0, 1\}$, where δ denotes the non-trivial quadratic character of E^1.

Proof. Let $\theta = \mathrm{Res}_Z \chi$ denote the restriction of χ to the center Z. As noted in §**??**, the center Z of G can be identified with E^1, the group of norm-one elements of E over F. The group $E^1/(E^1)^2$ is size 2. Let δ denote the non-trivial quadratic character of E^1. Then δ has depth-zero and every character θ of E^1 can be written as either $\theta = \phi^2$ or $\theta = \delta\phi^2$ for some character ϕ of E^1. Define $\chi_0 := (\phi^{-1} \circ \det) \otimes \chi$; then $\mathrm{Res}_Z \chi_0 = \delta^k$. We then have $\chi = (\phi \circ \det) \otimes \chi_0$. Applying Lemma 2.3.3, the lemma follows. ∎

6.2 A canonical decomposition upon restriction to $\mathcal{K}$

Let χ be a character of T of minimal depth r, and let (π_χ, V_χ) be the associated principal series representation. In this section, we describe the decomposition of the restriction $\pi_\chi|_{\mathcal{K}}$ into irreducible $\mathcal{K}$-representations.

Since $G = \mathcal{K}B$, applying Mackey theory we obtain

$$\mathrm{Res}_{\mathcal{K}}^G \mathrm{Ind}_B^G \chi = \mathrm{Ind}_{B\cap\mathcal{K}}^{\mathcal{K}} \chi. \tag{6.2.1}$$

Since irreducible representations of compact groups are finite dimensional, we note that V_χ is highly reducible as a representation of $\mathcal{K}$. Thus we need to decompose V_χ further into irreducible $\mathcal{K}$- representations. We also note that the branching rules of π_χ and $\pi_{\chi'}$ agree whenever the restriction of χ and χ' to $T \cap \mathcal{K}$ agree. Proposition 2.3.2 implies that the decomposition of V_χ reduces to decomposing each $V_\chi^{\mathcal{K}_n}$ for all $n \geq 0$.

Lemma 6.2.1. *If χ has depth r then so does π_χ. In particular, if χ is ramified then*

$$V_\chi^{\mathcal{K}_n} = \{0\} \quad \text{for all } n \leq r.$$

Proof. Let $n \leq r$. We want to show that $V_\chi^{\mathcal{K}_n} = \{0\}$. Assume on the contrary that there exists $f \neq 0 \in V_\chi^{\mathcal{K}_n}$. Then there exists $x \in \mathcal{K}$ such that $f(x) \neq 0$. Since χ has depth r, there exists $b \in B \cap \mathcal{K}_n$ such that $\chi(b) \neq 1$. As $\mathcal{K}_n \trianglelefteq \mathcal{K}$, the conjugate $t := x^{-1}bx \in \mathcal{K}_n$, and it acts trivially on f. This gives

$$f(x) = \pi_\chi(t) f(x) = f(xt) = f(bx) = \chi(b) f(x),$$

which is impossible since $\chi(b) \neq 1$ and $f(x) \neq 0$. Therefore, $V_\chi^{\mathcal{K}_n} = \{0\}$ when $n \leq r$. ∎

Observe that if n is greater than the depth of χ, then χ acts trivially on $B \cap \mathcal{K}_n$. Thus the restriction of χ to $B \cap \mathcal{K}$ extends trivially to a character of $(B \cap \mathcal{K})\mathcal{K}_n$. As in Chapter 2, for each $n \geq 0$ let us denote $B \cap \mathcal{K}_n$ by $\mathcal{B}_n$. Since $\mathcal{K}_0 = \mathcal{K}$ we simply denote $\mathcal{B}_0$ by $\mathcal{B}$.

Lemma 6.2.2. *Let $r \in \mathbb{Z}_{\geq 0}$, and let χ be a character of T of depth r. Then for any $n \geq r+1$, we have*

$$V_\chi^{\mathcal{K}_n} \cong \operatorname{Ind}_{\mathcal{B}\mathcal{K}_n}^{\mathcal{K}} \chi.$$

Proof. We show that the vectors spaces on which the two representation are acting are in fact the same. We have

$$\begin{aligned} W_1 &:= V_\chi^{\mathcal{K}_n} \\ &= \Big\{ f : \mathcal{K} \to \mathbb{C} \;\Big|\; f(bk) = \chi(b)f(k) \text{ and } f(kk_n) = f(k),\ \forall b \in B \cap \mathcal{K},\ \forall k_n \in \mathcal{K}_n \Big\}. \end{aligned}$$

and set

$$\begin{aligned} W_2 &:= \operatorname{Ind}_{\mathcal{B}\mathcal{K}_n}^{\mathcal{K}} \chi \\ &= \Big\{ \ell : \mathcal{K} \to \mathbb{C} \;\Big|\; \ell(bk) = \chi(b)\ell(k) \text{ and } \ell(k_n k) = \ell(k),\ \ \forall b \in \mathcal{B},\ \forall k_n \in \mathcal{K}_n\ \forall k \in \mathcal{K} \Big\}. \end{aligned}$$

Note that W_1 consists of function $f\colon \mathcal{K} \to \mathbb{C}$ that are constant on the left cosets of $\mathcal{K}_n$. Since $\mathcal{K}_n$ is a normal subgroup of $\mathcal{K}$, the left and the right cosets agree. Therefore, the spaces W_1 and W_2 are equal, and hence the corresponding representations are isomorphic. ∎

Lemma 6.2.3. *Let $r \in \mathbb{Z}_{\geq 0}$, and let χ be a character of T of depth r. Then for any $n \geq r+1$, the depth of every irreducible component of $V_\chi^{\mathcal{K}_n}$ is less than n, and the degree of $V_\chi^{\mathcal{K}_n}$ equals $q^{n-1}(q+1)$.*

Proof. Let $n \geq r+1$, by definition of $V_\chi^{\mathcal{K}_n}$, $\mathcal{K}_n$ acts trivially on the entire representation, hence on every irreducible component. Hence the maximal depth of any irreducible component of $V_\chi^{\mathcal{K}_n}$ must be strictly less that n.

The degree of $V_\chi^{\mathcal{K}_n}$ is given by the index

$$[\mathcal{K} \colon \mathcal{B}\mathcal{K}_n].$$

Since $\mathcal{K}_n \trianglelefteq \mathcal{K}$, and $\mathcal{K}_n \trianglelefteq \mathcal{B}\mathcal{K}_n$, applying group isomorphism theorems the above index can be computed as

$$[\mathcal{K} \colon \mathcal{B}\mathcal{K}_n] = \frac{[\mathcal{K} \colon \mathcal{K}_1][\mathcal{K}_1 \colon \mathcal{K}_n]}{[\mathcal{B} \colon \mathcal{B}_1][\mathcal{B}_1 \colon \mathcal{B}_n]}.$$

Since $\mathcal{K}/\mathcal{K}_1 \cong \mathrm{U}(1,1)(\mathfrak{f})$ and $\mathcal{B}/\mathcal{B}_1 \cong \mathrm{B}(\mathfrak{f})$, the index $[\mathcal{K} \colon \mathcal{K}_1] = |\mathrm{U}(1,1)(\mathfrak{f}| = q(q+1)(q-1)^2$, and the index $[\mathcal{B} \colon \mathcal{B}_1] = |\mathrm{B}(\mathfrak{f})| = q(q^2-1)$ respectively. The remaining indices $[\mathcal{K}_1 \colon \mathcal{K}_n], [\mathcal{B}_1 \colon \mathcal{B}_n]$ are same as the indices of corresponding $\mathcal{O}_F$-modules in the Lie algebra. Thus we have

$$[\mathcal{K} \colon \mathcal{B}\mathcal{K}_n] = \frac{[\mathcal{K} \colon \mathcal{K}_1][\mathcal{K}_1 \colon \mathcal{K}_n]}{[\mathcal{B} \colon \mathcal{B}_1][\mathcal{B}_1 \colon \mathcal{B}_n]} = \frac{q(q^2-1)(q+1)q^{4(n-1)}}{q(q^2-1)q^{3(n-1)}} = (q+1)q^{n-1}.$$ ∎

We now come to the major technical result of this section, which will serve as the key ingredient in establishing a canonical decomposition upon restriction to $\mathcal{K}$.

Proposition 6.2.4. *Let χ be a character of T of minimal depth r. Then for $d \geq r+1$ we have*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_d}, V_\chi^{\mathcal{K}_d})) = \begin{cases} d+1 & \text{if } r=0 \text{ and } \chi|_{S_0} = \mathbb{1}, \\ d-r & \text{otherwise.} \end{cases}$$

Proof. Let $d \geq r+1$. By Lemma 6.2.2 we have $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \chi \cong V_\chi^{\mathcal{K}_d}$. Thus

$$\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_d}, V_\chi^{\mathcal{K}_d}) \cong \mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \chi, \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \chi).$$

Using Frobenius reciprocity and Mackey theory we have

$$\mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \chi, \mathrm{Ind}_{\mathcal{B}\mathcal{K}_d}^{\mathcal{K}} \chi) = \bigoplus_{g \in \mathcal{B}\mathcal{K}_d \backslash \mathcal{K} / \mathcal{B}\mathcal{K}_d} \mathrm{Hom}_{(\mathcal{B}\mathcal{K}_d) \cap (\mathcal{B}\mathcal{K}_d)^g}(\chi, \chi^g).$$

From Proposition 2.2.8 it follows that a set of double coset representatives for $(\mathcal{B}\mathcal{K}_d)\backslash\mathcal{K}/(\mathcal{B}\mathcal{K}_d)$ is given by

$$P := \left\{ \mathrm{id} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathrm{w} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\varpi^k & 1 \end{pmatrix} \middle| \ 1 \leq k < d \right\}. \tag{6.2.2}$$

Thus, the dimension of the homomorphism space $\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_d}, V_\chi^{\mathcal{K}_d})$ is equal to the number of elements $g \in P$ such that

$$\chi(h) = \chi^g(h) \qquad \text{for all } h \in (\mathcal{B}\mathcal{K}_d) \cap (\mathcal{B}\mathcal{K}_d)^g. \tag{6.2.3}$$

Let $bk_d \in \mathcal{B}\mathcal{K}_d$, where $b = \begin{pmatrix} a & u \\ 0 & \overline{a}^{-1} \end{pmatrix} \in \mathcal{B}$, and $k_d \in \mathcal{K}_d$. Since $\mathcal{K}_d \trianglelefteq \mathcal{K}$, we have $g^{-1}k_d g \in \mathcal{K}_d$ for all $g \in P$. Therefore, for all $g \in P$, $bk_d \in (\mathcal{B}\mathcal{K}_d)^g$ if and only if $g^{-1}bg \in \mathcal{B}\mathcal{K}_d$. Moreover,

$$\chi^g(bk_d) = \chi(g^{-1}bgg^{-1}k_d g) = \chi(g^{-1}bg)\chi(g^{-1}k_d g) = \chi(g^{-1}bg).$$

Thus, the dimension of homomorphism space $\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_d}, V_\chi^{\mathcal{K}_d})$ reduces to finding the number of elements $g \in P$ such that

$$\chi(b) = \chi^g(b) \qquad \text{for all } b \in (\mathcal{B} \cap (\mathcal{B}\mathcal{K}_d)^g). \tag{6.2.4}$$

Clearly $g = \mathrm{id}$ satisfies (6.2.4). Now let $g = \mathrm{w}$. We compute

$$\mathrm{w}^{-1}b\mathrm{w} = \begin{pmatrix} \overline{a}^{-1} & 0 \\ u & a \end{pmatrix} = \begin{pmatrix} \overline{a}^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1}u & 1 \end{pmatrix}.$$

Thus $\mathrm{w}^{-1}b\mathrm{w} \in \mathcal{B}\mathcal{K}_d$, for all $a \in \mathcal{O}_E^\times$, and for all $u \in \mathfrak{p}_E^d$ such that $\overline{a}u \in \sqrt{\epsilon}F$. Since χ is trivial on $\mathcal{K}_d$, we then have

$$\chi^{\mathrm{w}}(b) = \chi(\mathrm{w}^{-1}b\mathrm{w}) = \chi^\dagger(\overline{a}^{-1}).$$

For the characters to agree we must have $\chi^\dagger(a) = \chi^\dagger(\overline{a}^{-1})$ for all $a \in \mathcal{O}_E^\times$, which implies that $\chi^\dagger(a\overline{a}) = 1$ for all $a \in \mathcal{O}_E^\times$ which happen only when $\chi^\dagger|_{\mathcal{O}_F^\times} = \mathbb{1}$ since the norm map is surjection on $\mathcal{O}_F^\times$. Thus w satisfies (6.2.4) only if $\chi|_{S_0} = \mathbb{1}$.

Lastly, let $g_k = \begin{pmatrix} 1 & 0 \\ \sqrt{\epsilon}\varpi^k & 1 \end{pmatrix}$. Then

$$g_k^{-1} b g_k = \begin{pmatrix} a + u\sqrt{\epsilon}\varpi^k & u \\ (-a + \overline{a}^{-1})\sqrt{\epsilon}\varpi^k - u\epsilon\varpi^{2k} & \overline{a}^{-1} - u\sqrt{\epsilon}\varpi^k \end{pmatrix}.$$

If $g_k^{-1} b g_k \in \mathcal{B}\mathcal{K}_d$, then there exists elements $t := \begin{pmatrix} x & y \\ 0 & \overline{x}^{-1} \end{pmatrix} \in \mathcal{B}$, and $k_d := \begin{pmatrix} 1 + c\varpi^d & e\varpi^d \\ f\varpi^d & 1 + l\varpi^d \end{pmatrix} \in \mathcal{K}_d$ with $x \in \mathcal{O}_E^\times$, $y, c, e, f, l \in \mathcal{O}_E$ such that

$$g_k^{-1} b g_k = t k_d. \tag{6.2.5}$$

Solving (6.2.5) for x yields $x = a(1 + c\varpi^d)^{-1}(1 + a^{-1}\sqrt{\epsilon}\varpi^k - a^{-1} f y \varpi^d)$. If $k > r$, then $(1 + a^{-1}\sqrt{\epsilon}\varpi^k - a^{-1} f y \varpi^d), (1 + c\varpi^d)^{-1} \in \mathfrak{p}_E^{r+1}$. Since χ has depth r, we have

$$\chi^{g_k}(b) = \chi^\dagger(x) = \chi^\dagger(a)\chi^\dagger((1 + c\varpi^d)^{-1})\chi^\dagger(1 + a^{-1}\sqrt{\epsilon}\varpi^k - a^{-1} f y \varpi^d) = \chi^\dagger(a) = \chi(b).$$

Hence for $k > r$, g_k satisfies (6.2.4) on the given intersection.

We claim that for $1 \leq k \leq r$, the element g_k does not intertwine χ. Since $k \leq r$ and χ has true depth r, we have $\chi^\dagger|_{1+\mathfrak{p}_F^k} \neq 1$. This key observation allows us to find elements $h \in \mathcal{B}\mathcal{K}_d \cap (\mathcal{B}\mathcal{K}_d)^{g_k}$ such that $\chi(h) \neq \chi^{g_k}(h)$, thereby proving the claim.

Given any $a = 1 + a_0\varpi^k \in 1 + \mathfrak{p}_F^k$, we set

$$u := \frac{a^{-1} - a}{\sqrt{\epsilon}\varpi^k} \in \sqrt{\epsilon}\mathcal{O}_F,$$

and we define

$$h := \begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix} \in \mathcal{B}.$$

Then the (2, 1) entry of $g_k^{-1} h g_k$ equals 0, and hence h lies in $\mathcal{B}\mathcal{K}_d \cap (\mathcal{B}\mathcal{K}_d)^{g_k}$. We also have

$$g_k^{-1} h g_k = \begin{pmatrix} a + u\sqrt{\epsilon}\varpi^k & u \\ 0 & a^{-1} - u\sqrt{\epsilon}\varpi^k \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 + a^{-1}u\sqrt{\epsilon}\varpi^k & a^{-1}u \\ 0 & 1 - au\sqrt{\epsilon}\varpi^k \end{pmatrix}.$$

We compute

$$\chi^{g_k}(h) = \chi(g_k^{-1} h g_k) = \chi^\dagger(a)\chi^\dagger(1 + a^{-1}u\sqrt{\epsilon}\varpi^k).$$

As a ranges over all elements of $1 + \mathfrak{p}_F^k$, the term $1 + a^{-1}u\sqrt{\epsilon}\varpi^k$ ranges over all of $1 + \mathfrak{p}_F^k$ as well. Therefore, we can choose a such that $\chi^\dagger(1 + a^{-1}u\sqrt{\epsilon}\varpi^k) \neq 1$, and hence

$$\chi(h) \neq \chi^{g_k}(h)$$

as required.

Therefore, if $\chi|_{S_0} = \mathbb{1}$, then all elements of P satisfy (6.2.4), therefore the dimension equals the cardinality of P which is $d+1$. On the other hand if $\chi|_{S_0} \neq \mathbb{1}$, then the elements of P that satisfy (6.2.4) are id, and $\{g_k \mid r < k < d\}$, which counts to $1+d-r-1 = d-r$. This proves the proposition. ∎

As noted earlier, Proposition 2.3.2 reduces the decomposition of V_χ to determining the decomposition of each $V_\chi^{\mathcal{K}_n}$ for all $n \geq 0$. Observe that if $\chi|_{S_0}$ is the trivial character, then by the preceding proposition,

$$\dim_{\mathbb{C}}\big(\mathrm{Hom}_{\mathcal{K}}\big(V_\chi^{\mathcal{K}_1},\, V_\chi^{\mathcal{K}_1}\big)\big) = 2.$$

In contrast, if χ is a character of T of true depth $r \geq 0$ such that $\chi|_{S_0} \neq \mathbb{1}$, then

$$\dim_{\mathbb{C}}\big(\mathrm{Hom}_{\mathcal{K}}\big(V_\chi^{\mathcal{K}_{r+1}},\, V_\chi^{\mathcal{K}_{r+1}}\big)\big) = 1.$$

Hence, in the latter case $V_\chi^{\mathcal{K}_{r+1}}$ is irreducible. In the former case, by Lemma 6.1.7, there exists a character $\phi\colon E^1 \to \mathbb{C}^\times$ and an unramified character χ' of T such that $\chi = (\phi \circ \det) \otimes \chi'$.

Since the branching rules of π_χ and $\pi_{\chi'}$ coincide whenever χ and χ' agree on $T \cap \mathcal{K}$, we may in this case choose the trivial character $\mathbb{1}$ as a representative and compute the branching rules for $\mathrm{Ind}_B^G \mathbb{1}$ which is given by the following lemma.

Lemma 6.2.5. *If $\chi = \mathbb{1}$ is the trivial character, then*

$$V_\chi^{\mathcal{K}_1} \cong \mathbb{1}_{\mathcal{K}} \oplus \mathrm{St}_{\mathcal{K}},$$

where $\mathbb{1}$ and St denote the inflations to $\mathcal{K}$ of the trivial and Steinberg representations $\mathbb{1}_q$ and St_q of $\mathrm{U}(1,1)(\mathfrak{f})$, respectively.

Proof. Recall that the Steinberg representation St_q of $\mathrm{U}(1,1)(\mathfrak{f})$ is defined by the property that $\mathrm{Ind}_{B(\mathfrak{f})}^{\mathrm{U}(1,1)(\mathfrak{f})} \mathbb{1} \cong \mathbb{1}_q \oplus \mathrm{St}_q$, where $\mathbb{1}_q$ denotes the trivial representation of $\mathrm{U}(1,1)(\mathfrak{f})$, and the degree of St_q is q. Let $\chi = \mathbb{1}$ be the trivial character of T. Since the induced module $\mathrm{Ind}_{\mathcal{B}\mathcal{K}_1}^{\mathcal{K}} \mathbb{1}$ is same as the induced module $\mathrm{Ind}_{\mathcal{B}/\mathcal{B}_1}^{\mathcal{K}/\mathcal{K}_1} \mathbb{1} \cong \mathrm{Ind}_{\mathrm{B}(\mathfrak{f})}^{\mathrm{U}(1,1)(\mathfrak{f})}$, we have

$$V_\chi^{\mathcal{K}_1} \cong \mathrm{Ind}_{\mathcal{B}\mathcal{K}_1}^{\mathcal{K}} \mathbb{1} \cong \mathbb{1}_{\mathcal{K}} \oplus \mathrm{St}_{\mathcal{K}},$$

where $\mathbb{1}_{\mathcal{K}}$ and $\mathrm{St}_{\mathcal{K}}$ denote the inflations to $\mathcal{K}$ of $\mathbb{1}_q$ and St_q, respectively. ∎

We now state the main theorem of this section.

Theorem 6.2.6. *Let χ be a character of T of minimal depth r, and let (π_χ, V_χ) denote the principal series representation associated to χ. Then there exists irreducible representations $\mathcal{W}_{d,\chi}$ of degree $(q^2-1)q^{d-1}$, for $d \geq r+1$ such that*

$$\mathrm{Res}_{\mathcal{K}} \pi_\chi \cong V_\chi^{\mathcal{K}_{r+1}} \oplus \bigoplus_{d \geq r+1} \mathcal{W}_{d,\chi}$$

where $V_\chi^{\mathcal{K}_1} = \mathbb{1}_q + \mathrm{St}_q$ *if* $\chi = \mathbb{1}$*, and* $V_\chi^{\mathcal{K}_{r+1}}$ *is an irreducible representation of degree* $(q+1)q^r$ *otherwise.*

Proof. If $\chi = \mathbb{1}$, then $V_\chi^{\mathcal{K}_1} = \mathbb{1}_{\mathcal{K}} + \mathrm{St}_{\mathcal{K}}$ by Lemma 6.2.5. If $\chi \neq \mathbb{1}$, then from Lemma 6.2.4 we have that $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}^{r+1}}(V_\chi^{\mathcal{K}_{r+1}}, V_\chi^{\mathcal{K}_{r+1}})) = 1$, hence $V_\chi^{\mathcal{K}_{r+1}}$ is irreducible. Since χ is one-dimensional the degree is given by the index $[\mathcal{K} \colon \mathcal{B}\mathcal{K}_{r+1}] = (q+1)q^r$ by Lemma 6.2.3.

Now let $d \geq r+1$. Since $V_\chi^{\mathcal{K}_d} \subsetneq V_\chi^{\mathcal{K}_{d+1}}$, by Maschke's theorem there exists a representation $\mathcal{W}_{d,\chi}$ of $\mathcal{K}$ such that $V_\chi^{\mathcal{K}_{d+1}} = V_\chi^{\mathcal{K}_d} \oplus \mathcal{W}_{d,\chi}$. We therefore have

$$\begin{aligned}\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_{d+1}}, V_\chi^{\mathcal{K}_{d+1}})) =& \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_d}, V_\chi^{\mathcal{K}_d})) + 2\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}^n}(V_\chi^{\mathcal{K}_d}, \mathcal{W}_{d,\chi})) \\ &+ \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{K}}(\mathcal{W}_{d,\chi}, \mathcal{W}_{d,\chi})).\end{aligned}$$

By Lemma 6.2.4 the first term is $d+1-r$, and the second term is $d-r$. Since the last term is at least one, it follows that the cross terms are zero and $\mathcal{W}_{d,\chi}$ is irreducible. Since the cross terms are zero, it follows that $\mathcal{W}_{d,\chi}$ has zero $\mathcal{K}_d$ invariants, and hence it has depth d.

We now prove by induction on d that for all $d \geq r+1$, $V_\chi^{\mathcal{K}_d} = \bigoplus_{i=r}^{d-1} \mathcal{W}_{i,\chi}$. For the base case, we denote $V_\chi^{\mathcal{K}_{r+1}}$ by $\mathcal{W}_{r,\chi}$. Let us assume that the result holds for some $d \geq r+1$, then we want to show that it holds for $d+1$. We have already shown that $V_\chi^{\mathcal{K}_{d+1}} = V_\chi^{\mathcal{K}_d} \oplus \mathcal{W}_{d,\chi}$. By the induction hypothesis we have $V_\chi^{\mathcal{K}_d} = \bigoplus_{i=r}^{d-1} \mathcal{W}_{i,\chi}$; hence the claim. Finally, we have

$$V_\chi = \bigcup_{n\geq 0} V_\chi^{\mathcal{K}_n} = \bigcup_{n \geq r+1} \left(\bigoplus_{i=r}^{n-1} \mathcal{W}_{i,\chi} \right) = \bigoplus_{d \geq r} \mathcal{W}_{d,\chi}.$$

The degree of $\mathcal{W}_{d,\chi}$ is given by $\dim_{\mathbb{C}}(V_\chi^{\mathcal{K}_{d+1}}) - \dim_{\mathbb{C}}(V_\chi^{\mathcal{K}_d}) = (q^2-1)q^{d-1}$. ∎

Remark 6.2.7. Since the representations $\mathcal{W}_{d,\chi}$ have distinct degrees for distinct d's, the resulting decomposition is multiplicity-free and therefore the decomposition in Theorem 6.2.6 is canonical.

6.3 An explicit description of $\mathcal{W}_{d,\chi}$

In the previous section we obtained a canonical decomposition of $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{K}} \chi$. For $d \geq r+1$, however, the representations $\mathcal{W}_{d,\chi}$ appear only as quotients of certain induced representations.

As in the supercuspidal case, we now show that each of these representations are isomorphic to one of the explicit representations that we constructed in Theorem 3.2.6 of Chapter 3. More precisely, we show that for each character χ of T of minimal depth $r \geq 0$, and for each integer $d \geq r+1$, there exist an element $Y_\chi \in \mathfrak{g}_{0,-d}$ of the form $\begin{pmatrix} z\sqrt{\epsilon} & u\sqrt{\epsilon} \\ v\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix}$ with

$z, u, v \in F$, $\nu(z) \geq -d$, $\nu(v) > \nu(u) = -d$, and a character ζ_χ of the centralizer $T(Y_\chi)$ of Y_χ in $\mathcal{K}$ satisfying

$$\zeta_\chi|_{T(Y_\chi)\cap\mathcal{J}_d} = \Psi_{Y_\chi}|_{T(Y_\chi)\cap\mathcal{J}_d}, \tag{6.3.1}$$

such that $\mathcal{W}_{d,\chi} \cong \mathcal{S}_d(Y_\chi, \zeta_\chi)$.

For all $d \geq r+1$, our strategy is to first identify elements Y_χ and ζ_χ such that $\mathcal{S}_d(Y_\chi, \zeta_\chi)$ is a well-defined irreducible representation in the sense of Theorem 3.2.6 and the homomorphism space

$$\mathrm{Hom}_{\mathcal{K}}(V_\chi^{\mathcal{K}_{d+1}}, \mathcal{S}_d(Y_\chi, \zeta_\chi)) \tag{6.3.2}$$

is not equal to zero, *i.e.*, $\mathcal{S}_d(Y_\chi, \zeta_\chi)$ is a subrepresentation of $V_\chi^{\mathcal{K}_{d+1}}$.

We now work toward identifying Y_χ and ζ_χ. As mentioned in the introduction, determining these objects is a challenging task, and a major portion of this section is devoted to that goal.

Applying Frobenius reciprocity and Mackey theory to (6.3.2) we obtain

$$\mathrm{Hom}_{\mathcal{K}}(\mathrm{Ind}_{\mathcal{B}\mathcal{K}_{d+1}}^{\mathcal{K}}\chi, \mathrm{Ind}_{T(Y_\chi)\mathcal{J}_d}^{\mathcal{K}}\Psi_{Y_\chi,\zeta_\chi}) \cong \bigoplus_{\alpha\in\mathcal{B}\mathcal{K}_{d+1}\backslash\mathcal{K}/T(Y_\chi)\mathcal{J}_d} \mathrm{Hom}_{\mathcal{B}\mathcal{K}_{d+1}\cap(T(Y_\chi)\mathcal{J}_d)^\alpha}(\chi, \Psi^\alpha_{Y_\chi,\zeta_\chi}).$$

Let $hk \in \mathcal{B}\mathcal{K}_{d+1} \cap (T(Y_\chi)\mathcal{J}_d)^\alpha$ where $h \in \mathcal{B}$ and $k \in \mathcal{K}_{d+1}$, and let $\alpha \in \mathcal{B}\mathcal{K}_{d+1}\backslash\mathcal{K}/T(Y_\chi)\mathcal{J}_d$. Note that χ is trivial on $\mathcal{K}_{d+1}$ by definition; therefore $\chi(hk) = \chi(h)$. Since $\mathcal{K}_{d+1} \trianglelefteq \mathcal{K}$, and Ψ_{Y_χ,ζ_χ} has depth d, we have $\alpha^{-1}k\alpha \in \mathcal{K}_{d+1}$ and $\Psi^\alpha_{Y_\chi,\zeta_\chi}(k) = \Psi^\alpha_{Y_\chi,\zeta_\chi}(\alpha^{-1}k\alpha) = 1$. Thus it suffices to show that for some choice of coset representative α, and Y_χ, ζ_χ,

$$\chi(h) = \Psi^\alpha_{Y_\chi,\zeta_\chi}(h), \qquad \text{for all } h \in \mathcal{B} \cap (T(Y_\chi)\mathcal{J}_d)^\alpha.$$

Note that if $h \in \mathcal{B} \cap (T(Y_\chi)\mathcal{J}_d)^\alpha$, then $h = \alpha t\alpha^{-1}\alpha j\alpha^{-1}$ for some $t \in T(Y_\chi)$, and $j \in \mathcal{J}_d$. We now evaluate $\Psi^\alpha_{Y_\chi,\zeta_\chi}(h) = \Psi_{Y_\chi,\zeta_\chi}(tj) = \zeta_\chi(t)\Psi_{Y_\chi}(j)$. If there exists a choice of α such that $\alpha t\alpha^{-1}$ is upper triangular, then h being upper triangular forces $\alpha j\alpha^{-1}$ to be upper triangular. For such a choice of α we would have $\chi(h) = \chi(\alpha t\alpha^{-1})\chi(\alpha j\alpha^{-1})$. Thus it suffices to find Y_χ, ζ_χ satisfying (6.3.1) such that

$$\zeta_\chi(t) = \chi(\alpha t\alpha^{-1}), \quad \text{and} \quad \Psi_{Y_\chi}(j) = \chi(\alpha j\alpha^{-1}), \tag{6.3.3}$$

for some choice of coset representative α that makes t upper triangular whenever $h = \alpha t\alpha^{-1}\alpha j\alpha^{-1} \in \mathcal{B} \cap (T(Y_\chi)\mathcal{J}_d)^\alpha$, where $j \in \mathcal{J}_d$.

We first assume that χ has depth $r > 0$. Then $\chi|_{T_{\frac{r}{2}+}/T_{r+}}$ is realized by an element $\Gamma = \begin{pmatrix} x & 0 \\ 0 & -\overline{x} \end{pmatrix} \in \mathfrak{t}_{-r}/\mathfrak{t}_{\frac{-r}{2}}$ with $x \in \mathfrak{p}_E^{-r}$, *i.e.*, for all $t \in T_{\frac{r}{2}+}/T_{r+}$ we have

$$\chi(t) = \psi(\mathrm{Tr}(\Gamma e^{-1}(t))). \tag{6.3.4}$$

where $e : \mathfrak{t}_{\frac{r}{2}+}/\mathfrak{t}_{r+} \to T_{\frac{r}{2}+}/T_{r+}$ denotes the Moy-Prasad isomorphism. In particular, for $t \in T_{\frac{r}{2}+}/T_{r+}$ we have

$$\chi(t) = \psi(\mathrm{Tr}(\Gamma(t - I))). \tag{6.3.5}$$

Let $d \geq r+1$. Recall that in Chapter 3, ζ_χ was the character of the centralizer of some element $Y_\chi \in \mathfrak{g}_{0,-d}$ having the given form. If we can find Y_χ such that it is conjugate to Γ by an element of G, then we can easily relate the characters χ and ζ_χ. Since Γ is a diagonal matrix we just need to equate the eigenvalues of $\begin{pmatrix} z\sqrt{\epsilon} & u\sqrt{\epsilon} \\ v\sqrt{\epsilon} & z\sqrt{\epsilon} \end{pmatrix}$ with that of Γ. Therefore writing $x = x_1 + x_2\sqrt{\epsilon}$ we should have

$$z\sqrt{\epsilon} + \sqrt{uv\epsilon} = x_1 + x_2\sqrt{\epsilon}, \quad z\sqrt{\epsilon} - \sqrt{uv\epsilon} = -x_1 + x_2\sqrt{\epsilon}.$$

Solving this yields, $z = x_2$, and $uv\epsilon = x_1^2$. We also want $\nu(v) > \nu(u) = -d$, and $\nu(z) \geq -d$. Therefore, one possible choice is to take $u = \epsilon^{-1}\varpi^{-d}$, and $v = x_1^2\varpi^d$. Since $d > r$, and $x_i \in \mathfrak{p}_E^{-r}$, the above choices of u and v satisfy the valuation conditions. Thus we set

$$Y_\chi := \begin{pmatrix} x_2\sqrt{\epsilon} & \epsilon^{-1}\varpi^{-d}\sqrt{\epsilon} \\ x_1^2\varpi^d\sqrt{\epsilon} & x_2\sqrt{\epsilon} \end{pmatrix} \in \mathfrak{g}_{0,-d}.$$

Let $\gamma = x_1\varpi^d\sqrt{\epsilon} \in \mathfrak{p}_E^{d-r}$. Then a matrix g_d in G that conjugates Y_χ to Γ is given by

$$g_d := \begin{pmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ \gamma & \frac{1}{2} \end{pmatrix} \in G,$$

and we have $Y_\chi = \Gamma^{g_d}$. Since the centralizer of Γ in G is T, the centralizer $T(Y_\chi)$ of Y_χ in $\mathcal{K}$ equals to the centralizer of Γ^{g_d} in $\mathcal{K}$ which is $T^{g_d} \cap \mathcal{K}$. As χ is a character of T, $\zeta_\chi := \chi^{g_d}$ gives a character of $T(Y_\chi)$. A direct computation gives that

$$T(Y_\chi) = \left\{ \begin{pmatrix} a & b \\ b\gamma^2 & a \end{pmatrix} \mid \overline{a}a + \overline{b}b\gamma^2 = 1, \overline{a}b \in \sqrt{\epsilon}F \right\} \cap \mathcal{K}.$$

We choose the coset representative $\alpha = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$. Then for any $t = \begin{pmatrix} a & b \\ b\gamma^2 & a \end{pmatrix} \in T(Y_\chi)$, we have

$$\alpha t \alpha^{-1} = \begin{pmatrix} a + b\gamma & b \\ 0 & a - b\gamma \end{pmatrix}$$

and

$$\chi(\alpha t \alpha^{-1}) = \chi^\dagger(a + b\gamma) = \chi^{g_d}(t) = \zeta_\chi(t),$$

verifying the first relation in (6.3.3).

Now let $j = \begin{pmatrix} 1 + c\varpi^{\lceil \frac{d}{2} \rceil} & y\varpi^{\lceil \frac{d}{2} \rceil} \\ z\varpi^{\lceil \frac{d+1}{2} \rceil} & 1 - \overline{c}\varpi^{\lceil \frac{d}{2} \rceil} \end{pmatrix} \in \mathcal{J}_d$, where $c, y, z \in \mathcal{O}_E$ such that $h = \alpha t \alpha^{-1} \alpha j \alpha^{-1} \in \mathcal{B} \cap (T(Y_\chi)\mathcal{J}_d)^\alpha$ where $t \in T(Y_\chi)$. Then

$$\alpha j \alpha^{-1} = \begin{pmatrix} 1 + c\varpi^{\lceil \frac{d}{2} \rceil} + y\gamma\varpi^{\lceil \frac{d}{2} \rceil} & y\varpi^{\lceil \frac{d}{2} \rceil} \\ -c\gamma\varpi^{\lceil \frac{d}{2} \rceil} + z\varpi^{\lceil \frac{d+1}{2} \rceil} - y\gamma^2\varpi^{\lceil \frac{d}{2} \rceil} - \overline{c}\gamma\varpi^{\lceil \frac{d}{2} \rceil} & -y\gamma\varpi^{\lceil \frac{d}{2} \rceil} + 1 - \overline{c}\varpi^{\lceil \frac{d}{2} \rceil} \end{pmatrix}.$$

Since h and $\alpha t \alpha^{-1}$ are upper triangular, $\alpha j \alpha^{-1}$ must be upper triangular as well. Therefore, the $(2,1)$ matrix entry of $\alpha j \alpha^{-1}$ vanishes; that is,

$$-c\gamma\varpi^{\lceil \frac{d}{2} \rceil} + z\varpi^{\lceil \frac{d+1}{2} \rceil} - y\gamma^2\varpi^{\lceil \frac{d}{2} \rceil} - \overline{c}\gamma\varpi^{\lceil \frac{d}{2} \rceil} = 0. \tag{6.3.6}$$

Multiplying the above relation by $\gamma^{-1}x_1$, and grouping the terms with c together we obtain

$$(c+\bar{c})x_1\varpi^{\lceil\frac{d}{2}\rceil} + x_1 y\gamma\varpi^{\lceil\frac{d}{2}\rceil} - \epsilon^{-1}z\sqrt{\epsilon}\varpi^{\lceil\frac{d+1}{2}\rceil - d} = 0. \tag{6.3.7}$$

Since $1 + c\varpi^{\lceil\frac{d}{2}\rceil} + y\gamma\varpi^{\lceil\frac{d}{2}\rceil} \in \mathfrak{p}_E^{\frac{d}{2}} \subseteq \mathfrak{p}_E^{\frac{r}{2}+}$, applying (6.3.5) we obtain

$$\chi(\alpha j\alpha^{-1}) = \psi(\mathrm{Tr}(\Gamma(\alpha j\alpha^{-1} - I))) = \Psi_{\Gamma^\alpha}(j). \tag{6.3.8}$$

From (6.3.7) it follows that

$$\psi(\mathrm{Tr}(\Gamma^\alpha - Y_\chi)(j-I)) = \psi((c+\bar{c})x_1\varpi^{\lceil\frac{d}{2}\rceil} + x_1y\gamma\varpi^{\lceil\frac{d}{2}\rceil} - \epsilon^{-1}z\sqrt{\epsilon}\varpi^{\lceil\frac{d+1}{2}\rceil-d}) = \psi(0) = 1.$$

Hence $\chi(\alpha j\alpha^{-1}) = \Psi_{Y_\chi}(j)$, thus verifying the second relation in (6.3.3).

It remains to show that Ψ_{Y_χ}, and ζ_χ agree on the intersection $T(Y_\chi)\cap\mathcal{J}_d$. Indeed, let $h := \begin{pmatrix} a & b \\ b\gamma^2 & a\end{pmatrix} \in T(Y_\chi)\cap\mathcal{J}_d$. Then $g_d^{-1}hg_d = \begin{pmatrix} a+b\gamma & 0 \\ 0 & a-b\gamma\end{pmatrix} \in T_{\lceil\frac{d}{2}\rceil} \subset T_{\frac{r}{2}+}$.

Therefore, applying (6.3.5) we obtain

$$\zeta_\chi(h) = \chi^{g_d}(h) = \chi(g_d^{-1}hg_d) = \psi(\mathrm{Tr}(\Gamma(g_d^{-1}hg_d - I))) = \psi(\mathrm{Tr}(g_d\Gamma g_d^{-1}(h-I))) = \Psi_{Y_\chi}(h).$$

Let Ψ_{Y_χ,ζ_χ} denote the unique extension of ζ_χ and Ψ_{Y_χ} to a character of $T(Y_\chi)\mathcal{J}_d$. Then from Theorem 3.2.6 it follows that

$$\mathcal{S}_d(Y_\chi,\zeta_\chi) = \mathrm{Ind}_{T(Y_\chi)\mathcal{J}_d}^{\mathcal{K}}\Psi_{Y_\chi,\zeta_\chi} \tag{6.3.9}$$

is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$. Thus, given a character χ of minimal depth $r\in\mathbb{Z}_{>0}$, for each $d\geq r+1$, we have constructed an irreducible representation $\mathcal{S}_d(Y_\chi,\zeta_\chi)$ which is a subrepresentation of $V_\chi^{\mathcal{K}_{d+1}}$.

We now turn our attention to the case when χ has depth-zero. In this case, there are no elements of $\mathfrak{t}$ realizing χ. Instead, for all $d\geq 1$, we take

$$Y_\chi = X_{\epsilon^{-1}\varpi^d} = \begin{pmatrix} 0 & \epsilon^{-1}\varpi^d\sqrt{\epsilon} \\ 0 & 0\end{pmatrix},$$

and set $\zeta_\chi = \theta$, where $\theta = \chi|_Z$ is the central character of π_χ. In this case, the centralizer of $X_{\epsilon^{-1}\varpi^d}$ in $\mathcal{K}$ equals $Z\mathcal{U}\mathcal{J}_d$, and $\zeta_\chi|_{Z\mathcal{U}\cap\mathcal{J}_d} = \Psi_{Y_\chi}|_{Z\mathcal{U}\cap\mathcal{J}_d} = \mathbb{1}$; thus applying (3.3.2) we obtain that

$$\mathcal{S}_d(X_{\epsilon^{-1}\varpi^d},\theta) = \mathrm{Ind}_{Z\mathcal{U}\mathcal{J}_d}^{\mathcal{K}}\Psi_{X_{\epsilon^{-1}\varpi^d},\theta} \tag{6.3.10}$$

is an irreducible representation of $\mathcal{K}$ of depth d and degree $(q^2-1)q^{d-1}$. Moreover, if $h\in\mathcal{B}\cap Z\mathcal{U}\mathcal{J}_d$, then $h = tj$, where $t\in Z\mathcal{U}$ and j is an upper triangular matrix in $\mathcal{J}_d$ on which χ is therefore trivial. Hence we have

$$\chi(h) = \chi(t)\chi(j) = \chi(t) = \zeta_\chi(t) = \Psi_{X_{\epsilon^{-1}\varpi^d},\theta}(h).$$

Hence, $S_d(X_{\epsilon^{-1}\varpi^d},\theta)$ is a subrepresentation of $V_\chi^{\mathcal{K}_d}$.

We summarize the main result of this section in the following theorem.

Theorem 6.3.1. *Let χ be a character of T of minimal depth $r \in \mathbb{Z}_{\geq 0}$. Then for each $d \geq r+1$, there exists $Y_\chi \in \mathfrak{g}_{0,-d}$, and ζ_χ a character of the centralizer of Y_χ in $\mathcal{K}$, satisfying $\zeta_\chi|_{T(Y_\chi)\cap\mathcal{J}_d} = \Psi_{Y_\chi}|_{T(Y_\chi)\cap\mathcal{J}_d}$, such that*

$$\mathcal{W}_{d,\chi} \cong \mathcal{S}_d(Y_\chi, \zeta_\chi).$$

Proof. To show the required isomorphism of $\mathcal{K}$-representations, we begin by recalling that we have already established the decomposition

$$V_\chi^{\mathcal{K}_{d+1}} = V_\chi^{\mathcal{K}_d} \oplus \mathcal{W}_{d,\chi}.$$

Since any irreducible component of $V_\chi^{\mathcal{K}_d}$ must have depth at most $d-1$, while $S_d(Y_\chi, \zeta_\chi)$ has depth exactly d, it follows that $S_d(Y_\chi, \zeta_\chi)$ cannot be isomorphic to a subrepresentation of $V_\chi^{\mathcal{K}_d}$ and must therefore be isomorphic to a subrepresentation of $\mathcal{W}_{d,\chi}$. But $\mathcal{W}_{d,\chi}$ is irreducible and has the same dimension as $S_d(Y_\chi, \zeta_\chi)$, hence we conclude that $\mathcal{W}_{d,\chi} \cong S_d(Y_\chi, \zeta_\chi)$. ∎

The following is now an immediate corollary of Lemma 6.2.5, and Theorems 6.2.6 and 6.3.1.

Corollary 6.3.2. *Let χ be a character of T of minimal depth $r \in \mathbb{Z}_{\geq 0}$. Then the restriction of the principal series representation π_χ to $\mathcal{K}$ decomposes as a direct sum of irreducible representations of $\mathcal{K}$ as follows*

$$\mathrm{Res}_{\mathcal{K}}\pi_\chi \cong \begin{cases} \mathrm{Ind}_{\mathcal{B}\mathcal{K}_{r+1}}^{\mathcal{K}}\chi \oplus \bigoplus_{d>r} \mathcal{S}_d(X_\chi, \zeta_\chi) & \text{if } \chi \text{ is ramified and } r > 0, \\ \mathrm{Ind}_{\mathcal{B}\mathcal{K}_1}^{\mathcal{K}}\chi \oplus \bigoplus_{d>0} \mathcal{S}_d(X_{\varpi^{-d}}, \chi) & \text{if } \chi \text{ is ramified and } r = 0, \\ \mathbb{1}_{\mathcal{K}} \oplus \mathrm{St}_{\mathcal{K}} \oplus \bigoplus_{d>0} \mathcal{S}_d(X_{\varpi^{-d}}, \mathbb{1}) & \text{if } \chi \text{ is unramified,} \end{cases}$$

Here,

$$Y_\chi = \Gamma^{g_d}, \quad \text{and} \quad \zeta_\chi = \chi^{g_d},$$

where $\Gamma = \begin{pmatrix} x & 0 \\ 0 & -\overline{x} \end{pmatrix} \in \mathfrak{t}_{-r}/\mathfrak{t}_{-r/2}$ *is an element that realizes χ on $T_{r/2+}/T_{r+}$, and*

$$g_d = \begin{pmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ \gamma & \frac{1}{2} \end{pmatrix} \in G, \quad \text{with } \gamma = \frac{(x+\overline{x})\varpi^d\sqrt{\epsilon}}{2}.$$

Thus we once again obtain a multiplicity free decomposition in which each constituent has distinct depth and degree, with the sole exception of the case $\chi = \mathbb{1}$ where both $\mathbb{1}_{\mathcal{K}}$ and $\mathrm{St}_{\mathcal{K}}$ have depth zero, although their degrees are different. In contrast with the supercuspidal case, where certain representations, namely those constructed from unramified anisotropic tori and depth-zero supercuspidal representations, exhibited the phenomenon that all constituents in the decomposition had depths of the same parity, no such distinction arises here.

6.4 Decomposition near the identity

In this section, we carry out an analogous decomposition to that of §5.3 of Chapter 5.

Let χ be a character of T of minimal depth $r \in \mathbb{Z}_{\geq 0}$, and let π_χ be the associated principal series representation. By Remark 6.1.9, we may assume that the central character θ of π_χ is either $\mathbb{1}$ or δ at the expense of twisting our branching rule by a character of G. Recall that δ denotes the nontrivial quadratic character of E^1.

6.4.1 Connection with depth-zero representations

As in the case of positive-depth supercuspidal representations (see Theorem 5.3.1), in this section we prove that the higher-depth components appearing in the decomposition upon restriction to $\mathcal{K}$ of a depth-zero supercuspidal representation are identical to those appearing in the corresponding decomposition of a principal series representation, provided the two representations share the same central character.

Let σ be a cuspidal representation of $\mathcal{K}/\mathcal{K}_{0+}$ with central character θ, and let $c\text{-Ind}_{\mathcal{K}}^G\sigma$ and $c\text{-Ind}_{\mathcal{K}^\eta}^G\sigma$ be the corresponding irreducible depth-zero representation.

Note that if χ is a character of minimal depth zero with central character θ, then by Corollary 6.3.2, for every $d \geq 1$, the components occurring in the decomposition of $\mathrm{Res}_{\mathcal{K}}\pi_\chi$ are of the form $\mathcal{S}_d(X_{\epsilon^{-1}\varpi^{-d}}, \theta)$, which, by Lemma 3.3.2, is isomorphic to $\mathcal{S}_d(X_{\varpi^{-d}}, \theta)$. Thus, in this case, all positive even-depth components coincide with those occurring in the decomposition of $c\text{-Ind}_{\mathcal{K}}^G\sigma$, and all odd-depth components coincide with those occurring in the decomposition of $c\text{-Ind}_{\mathcal{K}^\eta}^G\sigma$.

We now focus on the case when χ has minimal depth $r > 0$. By Corollary 6.3.2, for every $d \geq r+1$, the irreducible components occurring in the decomposition are of the form $\mathcal{S}_d(Y_\chi, \zeta_\chi)$, where

$$Y_\chi = \Gamma^{g_d} \qquad \text{and} \qquad \zeta_\chi = \chi^{g_d},$$

with $\Gamma = \begin{pmatrix} x & 0 \\ 0 & -\overline{x} \end{pmatrix} \in \mathfrak{t}_{-r}/\mathfrak{t}_{-r/2}$ an element that realizes χ on $T_{r/2+}/T_{r+}$, and

$$g_d = \begin{pmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ \gamma & \frac{1}{2} \end{pmatrix} \in G, \qquad \text{with } \gamma = \frac{(x+\overline{x})\varpi^d\sqrt{\epsilon}}{2}.$$

Theorem 6.4.1. *For $d > 2r$, we have an isomorphism*

$$\mathcal{S}_d(Y_\chi, \zeta_\chi) \cong \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

Proof. We have $Y_\chi = \Gamma^{g_d} = \begin{pmatrix} x_2\sqrt{\epsilon} & \epsilon^{-1}\varpi^{-d}\sqrt{\epsilon} \\ x_1^2\varpi^d\sqrt{\epsilon} & x_2\sqrt{\epsilon} \end{pmatrix} \in \mathfrak{g}_{0,-d}$. Consider the nilpotent element $X_{\epsilon^{-1}\varpi^{-d}}$. Since $d > 2r$, we have $-r > -\frac{d}{2}$, and therefore

$$\nu(x_2) \geq -r > -\frac{d}{2} \geq -\left\lceil \frac{d}{2} \right\rceil$$

and

$$\nu(x_1^2 \varpi^d \sqrt{\epsilon}) \geq d - 2r > 0 > -\left\lceil \frac{d}{2} \right\rceil.$$

Thus by Lemma 3.3.3, we have $\Psi_{Y_\chi} = \Psi_{X_{\epsilon^{-1}\varpi^d}}$ on $\mathcal{J}_d$ and $T(Y_\chi)\mathcal{J}_d = T(X_{\epsilon^{-1}\varpi^{-d}}) = Z\mathcal{U}\mathcal{J}_d$. Since $\Psi_{X_{\epsilon^{-1}\varpi^{-d}}}$ is trivial on the intersection $Z\mathcal{U} \cap \mathcal{J}_d$ and we assumed that θ has depth-zero, θ and $\Psi_{X_{\epsilon^{-1}\varpi^{-d}}}$ agree on $Z\mathcal{U} \cap \mathcal{J}_d$. Therefore, applying (3.3.2), we obtain that $\mathcal{S}_d(X_{\epsilon^{-1}\varpi^{-d}}, \theta)$ is an irreducible representation of $\mathcal{K}$ of depth d and degree $q^{d-1}(q^2-1)$.

We claim that

$$\mathcal{S}_d(Y_\chi, \zeta_\chi) = \mathcal{S}_d(X_{\epsilon^{-1}\varpi^{-d}}, \theta).$$

As in the case of positive-depth supercuspidal representations. It suffices to show that

$$\Psi_{Y_\chi, \zeta_\chi} = \Psi_{X_{\epsilon^{-1}}, \theta} \quad \text{on} \quad Z\mathcal{U}\mathcal{J}_d.$$

Since $\Psi_{Y_\chi} = \Psi_{X_{\epsilon^{-1}\varpi^{-d}}}$ on $\mathcal{J}_d$, we only need to show that $\zeta|_{T(Y_\chi)} = \theta|_Z$. Indeed, let $t := \left(\begin{smallmatrix} a & b \\ b\gamma^2 & a \end{smallmatrix}\right) \in T(Y_\chi)$. Then $\zeta_\chi(t) = \chi^{g_d}(t) = \chi(t^{g_d^{-1}})$.

Note that $\gamma \in \mathfrak{p}_E^{d-r}$, as $d > 2r$, we have that $\gamma \in \mathfrak{p}_E^{r+1}$. Since $a\bar{a} + b\bar{b}\gamma^2 = 1$, we obtain that $a\bar{a} \in 1 + \mathfrak{p}_F^{2(r+1)}$, as the norm map $N_{E/F} : 1 + \mathfrak{p}_E^{2(r+1)} \to 1 + \mathfrak{p}_F^{2(r+1)}$ is surjective, there exists $c \in 1 + \mathfrak{p}_E^{2(r+1)}$ such that $c\bar{c} = a\bar{a}$. Hence we may write

$$t^{g_d^{-1}} = \begin{pmatrix} a + b\gamma & 0 \\ 0 & a - b\gamma \end{pmatrix} = \begin{pmatrix} ac^{-1} & 0 \\ 0 & ac^{-1} \end{pmatrix} \begin{pmatrix} c(1 + a^{-1}b\gamma) & 0 \\ 0 & c(1 - a^{-1}b\gamma) \end{pmatrix} \in ZT_{r+1}$$

Since χ has depth r, we conclude that $\zeta|_{T(Y_\chi)} = \theta|_Z$ as required. Finally, by Lemma 3.3.2, we have

$$\mathcal{S}_d(Y_\chi, \zeta_\chi) = \mathcal{S}_d(X_{\epsilon^{-1}\varpi^{-d}}, \theta) \cong \mathcal{S}_d(X_{\varpi^{-d}}, \theta). \quad \blacksquare$$

We note that, unlike in the case of positive-depth supercuspidal representations where certain representations contributed only even depth or only odd depth components and therefore shared components with a single depth-zero representation, in the principal series case all depths occur. Consequently, each principal series representation shares components with both the depth-zero supercuspidal representations $c\text{-Ind}_{\mathcal{K}}^G \sigma$ and $c\text{-Ind}_{\mathcal{K}^\eta}^G \sigma$.

Recall that we fixed four depth-zero irreducible supercuspidal representations

$$\tau_{\delta^k}^{\text{even}} = c\text{-Ind}_{\mathcal{K}}^G \sigma_{\delta^k}, \quad \tau_{\delta^k}^{\text{odd}} = c\text{-Ind}_{\mathcal{K}^\eta}^G \sigma_{\delta^k}^\eta, \qquad k \in \{0, 1\}$$

where $\sigma_{\mathbb{1}}$ and σ_δ are cuspidal representations of $\mathcal{K}/\mathcal{K}_{0+}$ with central characters $\theta = \mathbb{1}$ and $\theta = \delta$, respectively. We denoted by $(\pi_\chi)_d$, the depth d component occuring in the decomposition of π_χ. Given any principal series representation π_χ, applying Theorem 6.4.1 and Remark 6.1.9, we obtain the following theorem

Theorem 6.4.2. *Let π_χ be a prinicipal series representation of G of depth $r \geq 0$. Then for all $d > 2r$,*

$$(\pi_\rho)_d \in \left\{ \tau_{\delta^k}^{\text{even}} \oplus \tau_{\delta^k}^{\text{odd}} \mid k \in \{0,1\} \right\},$$

up to a twist by a character of G.

6.4.2 Restriction to $\mathcal{K}_{2r+}$

Similar to §5.3.3, we now restrict π_χ to $\mathcal{K}_{2r+}$. Recall that in §5.3.2, for each $\delta \in F^\times$ we wrote

$$X_\delta := \begin{pmatrix} 0 & \delta\sqrt{\epsilon} \\ 0 & 0 \end{pmatrix},$$

and defined the two non-zero nilpotent G-orbits in the Lie algebra $\mathfrak{g}$ of G by $\mathcal{N}_1 := G \cdot X_1$ and $\mathcal{N}_\varpi := G \cdot X_\varpi$. Then, for each depth-zero character θ of the center of G and $t \in \mathbb{R}_{\geq 0}$ we defined highly reducible representations

$$\tau_{\mathcal{N}_1}^t(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{>0},\, d>t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta), \qquad \tau_{\mathcal{N}_\varpi}^t(\theta) = \bigoplus_{d \in 2\mathbb{Z}_{\geq 0}+1,\, d>t} \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

We now state the main theorem to be proved in this section.

Theorem 6.4.3. *Let χ be a character of T of minimal depth $r \in \mathbb{Z}_{\geq 0}$, and let π_χ be the associated principal series representation. Then*

$$\mathrm{Res}_{\mathcal{K}_{2r+}} \pi_{\pi_\chi} \;\cong\; q^{2r}(q+1)\mathbb{1} \;+\; \mathrm{Res}_{\mathcal{K}_{2r+}} \tau_{\mathcal{N}_1}^{2r}(\theta) \;+\; \mathrm{Res}_{\mathcal{K}_{2r+}} \tau_{\mathcal{N}_\varpi}^{2r}(\theta).$$

Proof. We first assume that χ is a character of T of minimal depth $r > 0$. By Theorem 6.3.2 the restriction of π_χ to $\mathcal{K}$ has the following decomposition

$$\mathrm{Res}_{\mathcal{K}} \pi_\chi \cong V_\chi^{\mathcal{K}_{r+1}} \oplus \bigoplus_{d \geq r+1} \mathcal{S}_d(Y_\chi, \zeta_\chi).$$

The restriction of π_χ to the subgroup $\mathcal{K}_{2r+}$ will act trivially on the components $\mathcal{S}_d(Y_\chi, \zeta_\chi)$ that have depth less than or equal to $2r$. Therefore, applying Theorem 6.4.1 we obtain

$$\mathrm{Res}_{\mathcal{K}_{2r+}} \pi_\chi \cong V_\chi^{\mathcal{K}_{2r+}} \oplus \bigoplus_{d > 2r} \mathrm{Res}_{\mathcal{K}_{2r+}} \mathcal{S}_d(X_{\varpi^{-d}}, \theta).$$

Hence, we have the desired decomposition

$$\mathrm{Res}_{\mathcal{K}_{2r+}} \pi_{\pi_\chi} \;\cong\; n(\pi_\chi)\mathbb{1} \;+\; \mathrm{Res}_{\mathcal{K}_{2r+}} \tau_{\mathcal{N}_1}^{2r}(\theta) \;+\; \mathrm{Res}_{\mathcal{K}_{2r+}} \tau_{\mathcal{N}_\varpi}^{2r}(\theta),$$

where $n(\pi_\chi) = \dim(V_\chi^{\mathcal{K}_{2r+}}) = q^{2r}(q+1)$. Now let us assume that χ has minimal depth 0. Then by Corollary 6.3.2 and Theorem 6.4.1 we have

$$\mathrm{Res}_{\mathcal{K}_1} \pi_\chi \cong V_\chi^{\mathcal{K}_1} \oplus \bigoplus_{d \geq 1} \mathrm{Res}_{\mathcal{K}_1} \mathcal{S}_d(X_{\varpi^{-d}}, \theta)$$

where $V_\chi^{\mathcal{K}_1} = \mathbb{1}_{\mathcal{K}} \oplus \mathrm{St}_{\mathcal{K}}$ if $\chi = \mathbb{1}$, and an irreducible representation of degree $(q+1)$ otherwise. Therefore we have the following decomposition of π_χ when restricted to $\mathcal{K}_1$

$$\mathrm{Res}_{\mathcal{K}_1} \pi_\chi \cong \mathrm{n}(\pi_\chi)\mathbb{1} \oplus \mathrm{Res}_{\mathcal{K}_1} \tau^0_{\mathcal{N}_1}(\theta) \oplus \mathrm{Res}_{\mathcal{K}_1} \tau^0_{\mathcal{N}_\varpi}(\theta)$$

where $n(\pi_\chi) = \dim(V_\chi^{\mathcal{K}_1}) = (q+1)$ by Lemma 6.2.3. ∎

$\pi \in \mathrm{Rep}(\mathrm{U}(1,1))$ **with central character** $\theta = \mu^2\delta^k$**, where** $\mu \in \chi(E^1)$ **and** $k \in \{0,1\}$		**depth of** π	$\tau^{\mathrm{even}}_{\delta^k}$, $\tau^{\mathrm{odd}}_{\delta^k}$	**value of** $n(\pi)$	$(a_{\mathcal{N}_1}, a_{\mathcal{N}_\varpi})$
Depth-zero irreducible supercuspidal representations	$c\text{-}\mathrm{Ind}_{\mathcal{K}}^G \sigma$	0	$\tau^{\mathrm{even}}_{\delta^k}$	$q-1$	$(1,0)$
	$c\text{-}\mathrm{Ind}_{\mathcal{K}^\eta}^G \sigma^\eta$	0	$\tau^{\mathrm{odd}}_{\delta^k}$	0	$(0,1)$
Positive-depth irreducible supercuspidal representations π_ρ	$\mathcal{T} = \mathcal{T}_{1,1}$	even	$\tau^{\mathrm{even}}_{\delta^k}$	$q^r(q^{r+1}-1)$	$(1,0)$
	$\mathcal{T} = \mathcal{T}_{1,1}$	odd	$\tau^{\mathrm{odd}}_{\delta^k}$	$q^r(q^r-1)$	$(0,1)$
	$\mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi}$	even	$\tau^{\mathrm{odd}}_{\delta^k}$	$q^r(q^r-1)$	$(0,1)$
	$\mathcal{T} = \mathcal{T}_{\varpi^{-1},\varpi}$	odd	$\tau^{\mathrm{even}}_{\delta^k}$	$q^r(q^{r+1}-1)$	$(1,0)$
	$\mathcal{T}$ ramified	$\frac{1}{2}+\mathbb{Z}$	$\tau^{\mathrm{even}}_{\delta^k} \oplus \tau^{\mathrm{odd}}_{\delta^k}$	$(q^{2r} - q^{r-\frac{1}{2}})(q+1)$	$(1,1)$
Principal series representations π_χ	$\mathcal{T} \cong E^\times$	$\mathbb{Z}_{\geq 0}$	$\tau^{\mathrm{even}}_{\delta^k} \oplus \tau^{\mathrm{odd}}_{\delta^k}$	$(q+1)q^{2r}$	$(1,1)$

Table 6.1: Summary of values of $n(\pi)$, $\tau^{\mathrm{even}}_{\delta^k}$, $\tau^{\mathrm{odd}}_{\delta^k}$, $a_{\mathcal{N}_1}$, and $a_{\mathcal{N}_\varpi}$ for the various representations of G .

Thus we see that the value of $n(\pi_\chi)$ differs from that of the irreducible supercuspidal representations of positive depth. However, it agrees with the corresponding value for the principal series representations of $\mathrm{SL}_2(F)$ [Nev24].

The remarkable similarity in the branching rules across these three classes of representations motivates us to explore this problem for higher-rank unitary groups. In particular, it would be natural to investigate whether analogous patterns persist for groups such as $\mathrm{U}(2,1)$ or, more generally, $\mathrm{U}(n,1)$ or $\mathrm{U}(n,m)$.

The results of this thesis are summarized in the papers [Tiw25a] and [Tiw25b].

Appendix A

Supplementary computations and technical details

In this appendix, we include two technical components that were deferred from the main text. In Section A.1, we provide a detailed proof that is essentially an application of Hensel's lemma, which was postponed from Proposition 3.2.2 in Chapter 3. In Section A.2, we justify Remark 5.1.5 from Chapter 5, by showing that in our setting the three refactorization conditions of [HM08, Definition 4.19] reduce to the two that were verified.

A.1 The magic of Hensel's lemma

Lemma A.1.1. *Let $s > 0$. Suppose $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K}$ satisfies $a \equiv d \mod \mathfrak{p}_E^s$ and $c \equiv u^{-1}vb \mod \mathfrak{p}_E^s$, where $u^{-1}v \in \mathcal{O}_F$ and $\mathrm{val}(u) = 0$ and $\mathrm{val}(v) > 0$. Then there exists a matrix $k' = \begin{pmatrix} a' & b' \\ b'u^{-1}v & a' \end{pmatrix} \in \mathcal{K}$ such that $a' \equiv a \mod \mathfrak{p}_E^s$ and $b' \equiv b \mod \mathfrak{p}_E^s$; in particular, $(k')^{-1}k \in G_{0,s}$.*

Proof. We wish to find $x, y \in \mathcal{O}_E$ such that setting $a' = a + x\varpi^s$ and $b' = b + y\varpi^s$ yields a matrix $k' \in \mathcal{K}$. That is, we require the following defining identities of $U(1,1)$ to hold:

$$\overline{(a + x\varpi^s)}(a + x\varpi^s) + u^{-1}v\overline{(b + y\varpi^s)}(b + y\varpi^s) = 1$$
$$\overline{(a + x\varpi^s)}u^{-1}v(b + y\varpi^s) \in \sqrt{\epsilon}F$$
$$\overline{(b + y\varpi^s)}(a + x\varpi^s) \in \sqrt{\epsilon}F$$

Since $u^{-1}v \in F$, the second equation follows from the third. We produce the solution (x, y) by induction.

We begin with depth $t \leq s$. Then choosing $x = y = 0$ is a solution modulo $\mathfrak{p}_E^t$. Suppose we have a pair (x, y) that solves this system modulo $\mathfrak{p}_E^t$. Then setting $a' = a + x\varpi^s$, $b' = b + y\varpi^s$ we know that there exists $\alpha, \gamma \in \mathcal{O}_F$ and $\beta \in \mathcal{O}_E$, such that

$$\overline{a'}a' + u^{-1}v\overline{b'}b' = 1 + \alpha\varpi^t \tag{A.1.1}$$

and

$$\overline{b'}a' = \sqrt{\epsilon}\gamma + \beta\varpi^t \tag{A.1.2}$$

We claim that we can find $x', y' \in \mathcal{O}_E$ such that $(a'', b'') = (a' + x'\varpi^t, b' + y'\varpi^t)$ satisfies

$$\overline{a''}a'' + u^{-1}v\overline{b''}b'' \in 1 + \mathfrak{p}_E^{t+1} \tag{A.1.3}$$

and

$$\overline{b''}a'' \in \sqrt{\epsilon}\gamma + \mathfrak{p}_E^{t+1}. \tag{A.1.4}$$

Write $a' = a'_0 + a'_1\sqrt{\epsilon}$, $b' = b'_0 + b'_1\sqrt{\epsilon}$ and $\beta = \beta_0 + \beta_1\sqrt{\epsilon}$; our unknowns are $x' = x'_0 + x'_1\sqrt{\epsilon}$ and $y' = y'_0 + y'_1\sqrt{\epsilon}$. We will show that x'_0, x'_1, y'_0, y'_1 are a solution to a nondegenerate linear system over the residue field $\mathfrak{f}$ of F.

Expanding the left hand side of (A.1.3) and subtracting the left hand side of (A.1.1) reveals that we need to solve only $a'\overline{x'} + \overline{a'}x' \equiv -\alpha \mod \mathfrak{p}_F$. In terms of our coefficients, we need to satisfy

$$2(a'_0x'_0 - \epsilon a'_1x'_1) \equiv -\alpha \mod \mathfrak{p}_F.$$

Similarly, expanding the left hand side of (A.1.4) and subtracting the left hand side of (A.1.2) reveals that we only require $\overline{b'}x' + a'\overline{y'} \equiv -\beta \mod \mathfrak{p}_E$. In terms of our coefficients, this is the system

$$\begin{aligned} b'_0x'_0 - b'_1\epsilon x'_1 + a'_0y'_0 - a'_1\epsilon y'_1 &\equiv -\beta_0 \mod \mathfrak{p}_F \\ -b'_1x'_0 + b'_0x'_1 + a'_1y'_0 - a'_0y'_1 &\equiv -\beta_1 \mod \mathfrak{p}_F. \end{aligned}$$

Since our original matrix k has determinant in $\mathcal{O}_E^\times$, and $a' \equiv a \mod \mathfrak{p}_E$, $b' \equiv b \mod \mathfrak{p}_E$, we infer that $(a'_0)^2 - \epsilon(a'_1)^2 \in \mathcal{O}_E^\times$, whence the linear system composed of the three preceding displayed equations is consistent over the field $\mathcal{O}_F/\mathfrak{p}_F$, and thus has a solution. By the principle of mathematical induction, this process gives as a limit an element $k' \in \mathcal{K}$ satisfying the lemma. ∎

A.2 Refactorization

In this section, we recall the definition of refactorization from [HM08, Definition 4.19] for a generic cuspidal G-datum in the case $d = 1$ and justify Remark 5.1.5. Here, a generic cuspidal

G-datum refers to a valid datum used in the construction of an irreducible supercuspidal representation of G.

We first align the notational convention of [HM08] with that of ours and state the definition in the case $d = 1$. Let

$$\Sigma' = (\mathcal{T}, y, r, \phi, r', \phi')$$

be an extended datum for the construction, where $r' > r > 0$. In the notation of [HM08], this corresponds to the sequence

$$(\vec{G}, y, \mathbb{1}, \vec{\phi}),$$

where $\vec{G} = (G^0, G^1) = (\mathcal{T}, G)$, and $\vec{\phi} = (\phi_0, \phi_1) = (\phi, \phi')$, with ϕ_0 a G-generic character of $\mathcal{T}$ of depth $r_0 = r$ and ϕ_1 a quasi-character of G of depth $r_1 = r'$. The symbol $\mathbb{1}$ denotes the trivial representation of $\mathcal{T}$.

Let

$$\dot{\Sigma} = (\mathcal{T}, y, r, \dot{\phi}, r', \dot{\phi}')$$

be such that $\dot{\phi}$ is a quasi-character of $\mathcal{T}$ and $\dot{\phi}'$ a quasi-character of G. For $i \in \{0, 1\}$, define

$$\chi_0(g) = \phi(g)\dot{\phi}(g)\phi'(g)\dot{\phi}'(g), \qquad \chi_1(g) = \phi'(g)\dot{\phi}'(g),$$

where χ_0 is a quasi-character of $\mathcal{T}$ and χ_1 a quasi-character of G. Then $\dot{\Sigma}$ is a *refactorization* of Σ if the following conditions hold.

(F0) If $\phi' = \mathbb{1}$, then $\dot{\phi}' = \mathbb{1}$.

(F1) $\dot{\phi}|_{\mathcal{T}_{0+}} = (\phi\chi_1)|_{\mathcal{T}_{0+}}$ and $\dot{\phi}'|_{G_{y,r+}} = \phi'|_{G_{y,r+}}$.

(F2) $\mathbb{1} = \mathbb{1} \otimes \big(\chi_0|_{\mathcal{T}}\big)$.

In Lemma 5.1.4, given $\rho = \rho(\mathcal{T}, y, r, \phi)$, λ be a quasi-character of G of depth $\leq r$, and $\widetilde{\phi}$ a character of $\mathcal{T}$ such that

$$\phi = \lambda|_{\mathcal{T}} \otimes \widetilde{\phi},$$

we considered extended data

$$\Sigma' = (\mathcal{T}, y, r, \lambda\widetilde{\phi}, r_1, \phi_G), \quad \text{and} \quad \dot{\Sigma} = (\mathcal{T}, y, r, \widetilde{\phi}, r_1, \lambda\phi_G).$$

where ϕ_G was a quasi-character of G of depth $r_1 > r$. In Lemma 5.1.4, we show $\dot{\Sigma}$ is a refactorization of Σ' by verifying the conditions

(F0) If $\phi_G = \mathbb{1}$, then $\lambda\phi_G = \mathbb{1}$.

(F1) $\widetilde{\phi}|_{\mathcal{T}_{0+}} = (\phi\lambda^{-1})|_{\mathcal{T}_{0+}}$ and $(\lambda\phi_G)|_{G_{y,r+}} = \phi_G|_{G_{y,r+}}$.

We now explain why these two conditions are sufficient. Evaluating χ_0 and χ_1 for these data, we obtain

$$\chi_0 = \phi\widetilde{\phi}^{-1}\phi_G(\lambda\phi_G)^{-1} = \lambda\widetilde{\phi}\widetilde{\phi}^{-1}\phi_G(\lambda\phi_G)^{-1} = \mathbb{1},$$
$$\chi_1 = \phi_G(\lambda\phi_G)^{-1} = \lambda^{-1}.$$

Since $\phi' = \phi_G$ and $\dot{\phi}' = \lambda\phi_G$, condition (F0) takes the simplified form stated in the lemma. For $i = 0$, we have $\dot{\phi} = \widetilde{\phi}$, $\phi = \lambda\widetilde{\phi}$, and $\chi_1 = \lambda^{-1}$, which gives the first equality in the simplified condition (F1). For $i = 1$, the equality $\dot{\phi}' = \lambda\phi_G$ with $\phi' = \phi_G$ gives the second part of (F1). Finally, since $\chi_0 = \mathbb{1}$, condition (F2) is automatically satisfied and can be omitted, explaining the reduction to two conditions in our argument.

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