Correspondences of von Neumann Algebras

Céline Rochon

A M.Sc. Thesis

submitted to the School of Graduate Studies and Research
in partial fulfilment of the requirements for
the Master's degree in Mathematics*

University of Ottawa
Ottawa, Ontario
Canada

1996

*The M.Sc. Program is a joint program with
Carleton University, administered by the Ottawa-Carleton
Institute of Mathematics and Statistics

National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file   Votre référence

Our file   Notre référence

ISBN 0-612-16460-8

UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

M.Sc. Thesis

Céline Rochon

Abstract

Correspondences of von Neumann Algebras

In the early 1980's, Alain Connes introduced a morphism between von Neumann algebras in order to define a property formerly defined only for groups and extending Kazhdan's property T. This morphism is called a correspondence.

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A *correspondence* from $\mathcal{M}$ to $\mathcal{N}$ is a Hilbert space $\mathcal{H}$ which is an $\mathcal{N} - \mathcal{M}$ bimodule. Equivalently, a correspondence from $\mathcal{M}$ to $\mathcal{N}$ is a unital *-representation of the algebraic tensor product $\mathcal{N} \odot \mathcal{M}^o$ which is normal when restricted to $\mathcal{N} \odot 1$ and $1 \odot \mathcal{M}^o$.

In this thesis, we show that a correspondence is indeed a morphism between von Neumann algebras and we show its link to two other morphisms between von Neumann algebras: normal involutive algebra homomorphisms and completely positive normal linear maps.

To achieve this objective, we study some important examples of von Neumann algebras and the Tomita-Takesaki theory.

University of Ottawa, June 1996.

À mes parents

Introduction

In the early 1980's, Alain Connes introduced a morphism between von Neumann algebras in order to define a property formerly defined only for groups and extending Kazhdan's property T. This morphism is called a correspondence.

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A *correspondence* from $\mathcal{M}$ to $\mathcal{N}$ is a Hilbert space $\mathcal{H}$ which is an $\mathcal{N}-\mathcal{M}$ bimodule. Equivalently, a correspondence from $\mathcal{M}$ to $\mathcal{N}$ is a unital *-representation of the algebraic tensor product $\mathcal{N} \odot \mathcal{M}^\circ$ which is normal when restricted to $\mathcal{N} \odot 1$ and $1 \odot \mathcal{M}^\circ$.

Normal involutive algebra homomorphisms and completely positive normal linear maps are well known morphisms between von Neumann algebras. Our goal will be to demonstrate that a correspondence is indeed a morphism between von Neumann algebras and to relate this new morphism to those mentioned above.

We first consider the case wherein both von Neumann algebras are commutative. Then, for an arbitrary von Neumann algebra $\mathcal{M}$, we construct, giving all details, the identity correspondence, denoted $L^2(\mathcal{M})$. We demonstrate that this correspondence is equivalent to the standard form of $\mathcal{M}$.

Subsequently, we will define the composition of correspondences, which is known as *Connes' fusion*, and show that it is associative. We will also see that the composition of correspondences is connected to the composition of *-homomorphisms.

Finally, we look at some properties of the contragredient of a correspondence.

The presentation of chapter 3 will essentially follow that of Connes in [Co$_1$]. The object of the thesis is to elaborate upon Connes' exposition and to combine in one manuscript the mathematical tools needed to understand his work.

In order to study correspondences, we define and examine in chapter 1 some important examples of von Neumann algebras. Among them, the group von Neumann algebra, abelian von Neumann algebras, tensor products of von Neumann algebras and crossed products. We will also give the general form of any normal involutive algebra homomorphism and present some properties of completely positive maps.

In chapter 2, we review the Tomita-Takesaki theory. We shall see that any von Neumann algebra $\mathcal{M}$ is isomorphic to a von Neumann algebra in standard form. To reach this result, we will introduce the notion of left Hilbert algebras and see that to any pair $\{\mathcal{M}, \phi\}$ (where $\mathcal{M}$ is a von Neumann algebra and ϕ is a faithful seminite normal weight on $\mathcal{M}$) there corresponds a von Neumann algebra associated to a left Hilbert algebra (depending on ϕ).

Furthermore, we shall see that to any pair $\{\mathcal{M}, \phi\}$, there is associated a one parameter group $\{\sigma_t^\phi\}$ of automorphisms of $\mathcal{M}$, known as the modular automorphism group. We will then present the Kubo-Martin-Schwinger (KMS) boundary condition, which characterizes the group $\{\sigma_t^\phi\}$ without making use of the GNS construction for ϕ.

We demonstrate in chapter 2 that the KMS boundary condition can be generalized. This result will be important in the construction of the identity correspondence. Finally, we present the Radon-Nikodym theorem.

Some terminology and important characteristics of unbounded operators in Hilbert spaces will be presented in the appendix. These notions will be used freely in chapter 2, where such operators occur repeatedly.

Acknowledgement

I would like to express my gratitude to my advisor, Thierry Giordano. The last two years have been very enriching. I would also like to thank my husband, Paul-Eugène, for his sense of humor and understanding.

Finally, I acknowledge the support of the Department of Mathematics, the School of Graduate Studies and Research and the Natural Sciences and Engineering Research Council.

Céline Rochon
University of Ottawa
May 1996

Contents

Chapter I - von Neumann Algebras

Chapter II - Tomita-Takesaki Theory

Chapter III - **Correspondences**

This thesis was typesetted using Plain TEX.

Chapter I

von Neumann Algebras

In this chapter, we present the elements of the theory of von Neumann algebras needed in our study of correspondences. We treat maximal abelian von Neumann algebras, tensor products of von Neumann algebras, and crossed products. Moreover, we present some important facts concerning linear forms on von Neumann algebras, projections and completely positive maps. Trace-class and Hilbert-Schmidt operators will be considered. In addition, we discuss measurable fields on Hilbert spaces.

For the sake of brevity, complete proofs will be omitted; the results will be illustrated by means of examples.

1.1 Definitions and examples

In this section, we give the algebraic definition of von Neumann algebras; the topological description will be given in section 4.

The notion of reduced von Neumann algebra and group von Neumann algebra will be introduced and some common notation will be recalled.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of continuous linear operators of $\mathcal{H}$ into $\mathcal{H}$. For every $x \in \mathcal{L}(\mathcal{H})$, there exists a unique bounded linear operator x^* on $\mathcal{H}$, called the adjoint operator, such that

$$(x\xi \mid \eta) = (\xi \mid x^* \eta) \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

Moreover, the map

$$\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \qquad x \longmapsto x^*$$

is an involution on $\mathcal{L}(\mathcal{H})$.

Proposition 1.1.1: $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ *is a Banach *-algebra with the norm defined by:*

$$\|x\| = \sup\{\|x\xi\| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}, \quad x \in \mathcal{L}(\mathcal{H}).$$

Definition 1.1.2: Let $\mathcal{M}$ be any subset of $\mathcal{L}(\mathcal{H})$. The *commutant* of $\mathcal{M}$, denoted $\mathcal{M}'$, is the set of elements of $\mathcal{L}(\mathcal{H})$ which commute with all the elements of $\mathcal{M}$. The *bicommutant* of $\mathcal{M}$, $(\mathcal{M}')'$, will be denoted $\mathcal{M}''$.

Example 1.1 a: $\mathcal{L}(\mathcal{H})' = \mathbf{C}\,1_H$.

Remarks:

i) $\mathcal{M} \subset \mathcal{M}''$;

ii) if $\mathcal{M}$ and $\mathcal{N}$ are subsets of $\mathcal{L}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{N}' \subset \mathcal{M}'$ and $\mathcal{M}'' \subset \mathcal{N}''$; in particular, $\mathcal{M}''' = (\mathcal{M}'')' \subset \mathcal{M}'$ and $\mathcal{M}' \subset (\mathcal{M}')'' = \mathcal{M}'''$. We thus have:

$$\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(5)} = \ldots \quad \text{and} \quad \mathcal{M} \subset \mathcal{M}'' = \mathcal{M}^{(4)} = \ldots$$

A subset of $\mathcal{L}(\mathcal{H})$ is said to be *self-adjoint* if it contains the adjoint of each of its elements. A self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ is called a **-subalgebra*.

Definition 1.1.3: A *von Neumann algebra* in $\mathcal{H}$ is a *-subalgebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{M} = \mathcal{M}''$.

Remark: For every von Neumann algebra $\mathcal{M}$, the identity element 1 of $\mathcal{L}(\mathcal{H})$ belongs to $\mathcal{M}$.

Example 1.1 b: The algebra $\mathcal{L}(\mathcal{H})$ is a von Neumann algebra.

Definition 1.1.4: A *factor* is a von Neumann algebra $\mathcal{M}$ whose centre $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ only contains the scalar operators.

Remark: By Example 1.1 a, $\mathcal{L}(\mathcal{H})$ is a factor.

Lemma 1.1.5: *Let $\mathcal{N} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Suppose $\mathcal{K}$ is a $\mathcal{N}$-invariant closed subspace of $\mathcal{H}$; then the orthogonal projection $e : \mathcal{H} \to \mathcal{K}$ belongs to the commutant of $\mathcal{N}$.*

Proof: Let $\xi \in \mathcal{K}$, $y \in \mathcal{N}$; then $e\xi = \xi$ and $y\xi \in \mathcal{K}$. Thus $ye\xi = y\xi = ey\xi$. Now, let $\xi \in \mathcal{K}^\perp$. Then, $e\xi = 0$; thus $ye\xi = 0$ for all $y \in \mathcal{N}$. As $\mathcal{K}^\perp$ is also $\mathcal{N}$-invariant, $y\xi \in \mathcal{K}^\perp$ for all $y \in \mathcal{N}$ and $ey\xi = 0$. We thus have the desired result.

Q.E.D.

Definition 1.1.6: Let $\mathcal{M}$ be a von Neumann algebra. The *opposite von Neumann algebra*, $\mathcal{M}^\circ$, is defined to be the set $\mathcal{M}$ equipped with the same structure as $\mathcal{M}$ except that multiplication is defined by:

$$x_1 \cdot x_2 := x_2 x_1.$$

Definition 1.1.7: Let $\mathcal{M}$ be a *-subalgebra of $\mathcal{L}(\mathcal{H})$.

i) $\mathcal{M}^+$ will denote the set of positive self-adjoint elements of $\mathcal{M}$; thus $x \in \mathcal{M}^+$ if and only if $x = y^*y$, for some $y \in \mathcal{M}$.

ii) $U(\mathcal{M})$ will denote the set of unitary operators in $\mathcal{M}$; thus $x \in U(\mathcal{M})$ if and only if $x^*x = xx^* = 1$.

Remarks:

i) Let $\mathcal{M}$ be a *-subalgebra of $\mathcal{L}(\mathcal{H})$. Every $x \in \mathcal{M}$ can be uniquely written in the form $x = x_1 + ix_2$, where x_1 and x_2 are self-adjoint elements of $\mathcal{M}$. Indeed,

$$x_1 = \frac{1}{2}(x + x^*) \in \mathcal{M} \quad \text{and} \quad x_2 = \frac{i}{2}(x^* - x) \in \mathcal{M}.$$

ii) Each element of a von Neumann algebra $\mathcal{M}$ is a finite linear combination of elements of $U(\mathcal{M})$. [K-R; 242]

iii) Each element of a von Neumann algebra $\mathcal{M}$ is a linear combination of at most four elements of $\mathcal{M}^+$. [K-R; p.247]

Definition 1.1.8:

i) Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A **-homomorphism* $\Phi : \mathcal{M} \to \mathcal{N}$ is an algebra homomorphism from $\mathcal{M}$ to $\mathcal{N}$ such that for all $x \in \mathcal{M}$, $\Phi(x^*) = \Phi(x)^*$. If in addition Φ is a bijection, it is called a **-isomorphism.*

ii) Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ and $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ be two von Neumann algebras. A *-isomorphism $\pi : \mathcal{M} \to \mathcal{N}$ is *spatial* if there exists a bijective isometric linear mapping $U : \mathcal{H} \to \mathcal{K}$ such that $UxU^* = \pi(x)$ for all $x \in \mathcal{M}$.

iii) Let $\mathcal{N}$ be a von Neumann algebra. A *-isomorphism $\phi : \mathcal{N} \to \mathcal{N}$ is called an *automorphism* of $\mathcal{N}$. The set of *-automorphisms of $\mathcal{N}$ will be denoted Aut($\mathcal{N}$). The set of *inner automorphisms* of $\mathcal{N}$, that is,

$$\{\theta \in \mathrm{Aut}(\mathcal{N}) : \text{ there exists } V \in U(\mathcal{N}) \text{ such that } \theta(x) = VxV^* \text{ for all } x \in \mathcal{N}\}$$

will be denoted Inn($\mathcal{N}$). Moreover, Out($\mathcal{N}$) = Aut($\mathcal{N}$)/Inn($\mathcal{N}$) will denote the set of *outer automorphisms* of $\mathcal{N}$.

We now describe the *reduced von Neumann algebra.*

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ a closed linear subspace of $\mathcal{H}$. Let $e = P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$ and $x \in \mathcal{L}(\mathcal{H})$. Recall that $e = e^2 = e^*$. Let us denote by x_e the restriction of ex to $\mathcal{K}$, which is an element of $\mathcal{L}(\mathcal{K})$. Note that $x_e = (xe)_e = (ex)_e = (exe)_e$.

If $\mathcal{M}$ is a subset of $\mathcal{L}(\mathcal{H})$, let $\mathcal{M}_e = \{x_e : x \in \mathcal{M}\}$. Consider the two following cases:

Case 1: $\mathcal{M}$ is a *-algebra of operators and $e \in \mathcal{M}$.

Let $\mathcal{N} = \{x \in \mathcal{M} : xe = ex = x\} = \{x \in \mathcal{M} : x(\mathcal{K}) \subset \mathcal{K} \text{ and } x(\mathcal{K}^\perp) = 0\}$.

The set $\mathcal{N}$ is a *-subalgebra of $\mathcal{M}$ and $\mathcal{N} = e\mathcal{M}e$. The map $x \rightarrow x_e$ is a *-isomorphism of $\mathcal{N}$ onto $\mathcal{M}_e$.

Remark: Suppose $\mathcal{M}$ is a von Neumann algebra and $e \in \mathcal{M}$. Then $e\mathcal{M}e$ is a *-subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed with respect to the weak operator topology. As $1 \notin e\mathcal{M}e$, then $e\mathcal{M}e$ is not a von Neumann algebra. However, $e\mathcal{M}e$ can be viewed as a von Neumann algebra on the closed subspace $e\mathcal{H}$ since its elements operate only in $e\mathcal{H}$.

Case 2: $\mathcal{M}$ is a *-algebra of operators and $e \in \mathcal{M}'$.

Every $x \in \mathcal{M}$ leaves $\mathcal{K}$ and $\mathcal{K}^\perp$ invariant. The map $x \rightarrow x_e$ is a *-homomorphism of $\mathcal{M}$ onto $\mathcal{M}_e$.

Proposition 1.1.9: *Let $\mathcal{M}$ be a von Neumann algebra and $e = P_{\mathcal{K}}$ a projection in $\mathcal{M}$. Then $\mathcal{M}_e$ and $(\mathcal{M}')_e$ are von Neumann algebras and $(\mathcal{M}')_e = (\mathcal{M}_e)'$.*

Proof: [Di; p.18]

Definition 1.1.10:

i) The algebra $\mathcal{M}'_e$ is called the *von Neumann algebra induced by $\mathcal{M}'$ in $\mathcal{K}$.*

ii) The homomorphism $x' \rightarrow x'_e$ of $\mathcal{M}'$ onto $\mathcal{M}'_e$ is called the *induction* of $\mathcal{M}'$ on $\mathcal{M}'_e$.

iii) Every von Neumann algebra of the form $\mathcal{M}_e$ is called a *reduced* von Neumann algebra of $\mathcal{M}$.

Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra.

Definition 1.1.11: Let S be a subset of $\mathcal{H}$. Then S is

i) *cyclic* for $\mathcal{M}$ if the closed subspace spanned by $\mathcal{M}S$, denoted $[\mathcal{M}S]$, equals $\mathcal{H}$;

ii) *separating* for $\mathcal{M}$ if for $x \in \mathcal{M}$, $x = 0$ if and only if $xS = \{0\}$.

Remark: A subset S of $\mathcal{H}$ is cyclic for a von Neumann algebra $\mathcal{M}$ in $\mathcal{H}$ if and only if it is separating for $\mathcal{M}'$. [K-R; p.337]

Example 1.1 c:

Let G be a countable discrete group and

$$l^2(G) = \left\{ \xi : G \rightarrow \mathbf{C} : \sum_{t \in G} |\xi(t)|^2 < \infty \right\},$$

the Hilbert space of square-summable functions on G. Consider the canonical orthonormal basis of $l^2(G)$:

$$\{\xi_t : t \in G\}, \quad \text{where} \quad \xi_t(s) = \delta_t(s) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases}$$

For all $t \in G$, let λ_t be the unitary operator corresponding to left translation by t:

$$(\lambda_t\xi)(s) = \xi(t^{-1}s), \quad \text{for all } s \in G, \xi \in l^2(G).$$

The mapping $t \longrightarrow \lambda_t$ gives a unitary representation of G in $l^2(G)$, called the *left regular representation* of G. The von Neumann algebra $\mathcal{M} = \{\lambda_t : t \in G\}''$ is called *the group von Neumann algebra* of G. We denote it $vN(G)$.

The *right regular representation* $t \longrightarrow \rho_t$ is the unitary representation of G in $l^2(G)$ defined by

$$(\rho_t\xi)(s) = \xi(st), \quad \text{for all } s \in G, \xi \in l^2(G).$$

Note that $\{\rho_t : t \in G\} \subset \mathcal{M}'$.

We shall see in the next chapter that the commutant of the left-regular representation is the von Neumann algebra generated by the right-regular representation.

We observe that $\Omega = \xi_e$ is a cyclic and separating vector for $\mathcal{M}$, where e denotes the identity in G. Indeed, $\lambda_t\Omega = \xi_t$ for all t and therefore, Ω is cyclic for $\mathcal{M}$. Moreover, $\rho_t\Omega = \xi_{-t}$ for all t, and thus, Ω is cyclic for $\mathcal{M}'$.

1.2 Maximal abelian von Neumann algebras

Definition 1.2.1: An abelian von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is *maximal* if it is not contained in any other abelian von Neumann algebra on $\mathcal{H}$.

Remarks:

i) By Zorn's Lemma, every abelian von Neumann algebra is contained in a maximal such.

ii) An abelian von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is maximal if and only if $\mathcal{M}=\mathcal{M}'$. Indeed, if $\mathcal{M}$ is abelian, then $\mathcal{M} \subset \mathcal{M}'$. Now $\mathcal{M}$ is not maximal abelian in $\mathcal{L}(\mathcal{H})$ if and only if there exists $x \in \mathcal{L}(\mathcal{H}) \setminus \mathcal{M}$ that commutes with x^* and with $\mathcal{M}$ and thus $\mathcal{M} \neq \mathcal{M}'$. Consequentiy, $\mathcal{M}$ is maximal abelian in $\mathcal{L}(\mathcal{H})$ if and only if $\mathcal{M} = \mathcal{M}'$.

Theorem 1.2.2: *If $\mathcal{M}$ is a maximal abelian von Neumann algebra on a separable Hilbert space, then $\mathcal{M}$ has a cyclic vector.*

Proof: [M; p.135]

Example 1.2 a: Let $(X, \mathcal{F}, \mu)$ be a separable σ-finite measure space. The map

$$L^\infty(X,\mu) \rightarrow \mathcal{L}(L^2(X,\mu)), \qquad f \longmapsto m_f,$$

where m_f denotes the multiplication operator: $(m_f\xi)(x) = f(x)\xi(x)$, $\xi \in L^2(X,\mu)$, is an isometric *-isomorphism of $L^\infty(X,\mu)$ into $\mathcal{L}(L^2(X,\mu))$.

Let $\mathcal{M} = \{m_f : f \in L^\infty(X, \mu)\}$; then $\mathcal{M} = \mathcal{M}'$ [M: p.117 for the case of finite μ]. Thus $\mathcal{M}$ is a maximal abelian von Neumann algebra.

Theorem 1.2.3: *Let $\mathcal{M}$ be an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then there exists a second countable compact Hausdorff space X and a positive regular Borel complex measure μ on X such that $\mathcal{M}$ is *-isomorphic to $L^\infty(X, \mu)$.*

Proof: [M; p.136]

1.3 Trace-class and Hilbert-Schmidt operators

In this section, we will see that there exists a connection between $\mathcal{L}(\mathcal{H})$ and the trace-class operators. In addition, we will introduce the Hilbert-Schmidt operators; these operators will provide us with an example of a correspondence, namely, the coarse correspondence.

Let $\mathcal{K}(\mathcal{H})$ denote the set of compact linear operators of $\mathcal{H}$ into $\mathcal{H}$. Recall that a linear operator x on $\mathcal{H}$ is *compact* if the norm-closure of $\{x\xi : \|\xi\| \le 1\}$ is compact in $\mathcal{H}$.

Let $\{\xi_i : i \in I\}$ be an orthonormal basis of $\mathcal{H}$. Let us define a trace on $\mathcal{L}(\mathcal{H})$ as follows:

$$\operatorname{Tr} x = \sum_{i\in I} (x\xi_i \mid \xi_i), \qquad x \in \mathcal{L}(\mathcal{H})^+.$$

The definition of Tr is independent of the choice of the orthonormal basis. Indeed let $\{\xi_i : i \in I\}$ and $\{\eta_j : j \in J\}$ be two orthonormal basis of $\mathcal{H}$. Define

$$P_{\eta_j} : \mathcal{H} \to \mathcal{H}, \qquad w \mapsto (w \mid \eta_j)\,\eta_j.$$

Then P_{η_j} is a projection and

$$\Big(\sum_{j\in J} P_{\eta_j}\Big)(w) = \sum_{j\in J} P_{\eta_j}(w) = \sum_{j\in J} (w \mid \eta_j)\,\eta_j = w.$$

Thus $\sum_j P_{\eta_j} = 1$. Let $x \in \mathcal{L}(\mathcal{H})^+$; then

$$\begin{aligned}
\operatorname{Tr} x &= \sum_{i\in I} (x\xi_i \mid \xi_i) = \sum_{i\in I} \Big(\sum_{j\in J} P_{\eta_j} x\xi_i \mid \xi_i\Big) = \sum_{i,j} (P_{\eta_j} x\xi_i \mid \xi_i) \\
&= \sum_{i,j} (x\xi_i \mid P_{\eta_j}\xi_i) = \sum_{i,j} (x\xi_i \mid (\xi_i \mid \eta_j)\eta_j) = \sum_{i,j} \overline{(\xi_i \mid \eta_j)}(x\xi_i \mid \eta_j) \\
&= \sum_{i,j} (\eta_j \mid \xi_i)(x\xi_i \mid \eta_j) = \sum_{i,j} (x(\eta_j \mid \xi_i)\xi_i \mid \eta_j) = \sum_{i,j} (x P_{\xi_i}\eta_j \mid \eta_j) \\
&= \sum_{j\in J} \Big(x\Big(\sum_{i\in I} P_{\xi_i}\Big)\eta_j \mid \eta_j\Big) = \sum_{j\in J} (x\eta_j \mid \eta_j).
\end{aligned}$$

This satisfies:

i) $$\mathrm{Tr}\,(x^*x) = \mathrm{Tr}\,(xx^*), \quad x \in \mathcal{L}(\mathcal{H});$$

ii) $$\mathrm{Tr}\,(uxu^*) = \mathrm{Tr}\,x, \quad x \in \mathcal{L}(\mathcal{H})^+, u \in U(\mathcal{L}(\mathcal{H});$$

iii) $$\|x\| \leq \mathrm{Tr}\,x, \quad x \in \mathcal{L}(\mathcal{H})^+;$$

iv) $$\mathrm{Tr}\,(y^*xy) \leq \|yy^*\|\mathrm{Tr}\,x, \quad x \in \mathcal{L}(\mathcal{H})^+, y \in \mathcal{L}(\mathcal{H}).$$

Relation i) is obtained as follows:

$$\begin{aligned}\mathrm{Tr}\,(xx^*) &= \sum_i (xx^*\,\xi_i \mid \xi_i) = \sum_i (x \sum_j P_{\xi_j} x^*\,\xi_i \mid \xi_i) = \sum_{i,j} (x\,P_{\xi_j} x^*\,\xi_i \mid \xi_i) \\ &= \sum_{i,j} (x\,(x^*\,\xi_i \mid \xi_j)\,\xi_j \mid \xi_i) = \sum_{i,j} (x^*\,\xi_i \mid \xi_j)(x\,\xi_j \mid \xi_i) = \sum_{i,j} (x^*\,(x\,\xi_j \mid \xi_i)\,\xi_i \mid \xi_j) \\ &= \sum_{i,j} (x^*\,P_{\xi_i}\,x\,\xi_j \mid \xi_j) = \sum_j (x^* \sum_i P_{\xi_i}\,x\,\xi_j \mid \xi_j) = \sum_j (x^*x\,\xi_j \mid \xi_j) = \mathrm{Tr}\,(x^*x).\end{aligned}$$

Note that given an orthonormal basis $\{\xi_i : i \in I\}$ of $\mathcal{H}$, the set $\{u\xi_i : i \in I\}$ is an orthonormal basis of $\mathcal{H}$ for any $u \in U(\mathcal{M})$. Hence we get relation ii). Relation iii) is immediate and relation iv) is proved using:

$$\mathrm{Tr}\,(y^*xy) = \mathrm{Tr}\,(x^{\frac{1}{2}}yy^*x^{\frac{1}{2}}) \quad \text{and} \quad x^{\frac{1}{2}}yy^*x^{\frac{1}{2}} \leq \|yy^*\|\,x$$

for any $x \in \mathcal{L}(\mathcal{H})^+$ and $y \in \mathcal{L}(\mathcal{H})$.

Definition 1.3.1:

i) $\mathcal{L}^1(\mathcal{H}) =$ span $\{x \in \mathcal{K}(\mathcal{H}) : x \geq 0, \mathrm{Tr}\,x < \infty\}$ is called the set of *trace-class operators.*

ii) $\mathcal{L}^2(\mathcal{H}) = \{x \in \mathcal{K}(\mathcal{H}) : \mathrm{Tr}\,(x^*x) < \infty\}$ is called the set of *Hilbert-Schmidt operators.*

Proposition 1.3.2:

i) *$\mathcal{L}^1(\mathcal{H})$ and $\mathcal{L}^2(\mathcal{H})$ are self-adjoint ideals in $\mathcal{L}(\mathcal{H})$.*

ii) *$\mathcal{L}^1(\mathcal{H}) = \{x \in \mathcal{L}(\mathcal{H}) : \mathrm{Tr}|x| < \infty\}$, where $|x| = (x^*x)^{\frac{1}{2}}$.*

Proof: [P; p.118]

Lemma 1.3.3:

i) *$\mathcal{L}^2(\mathcal{H})$ is a Hilbert space under the inner product*

$$(x \mid y)_{\mathrm{Tr}} = \mathrm{Tr}(y^*x), \qquad x, y \in \mathcal{L}^2(\mathcal{H}).$$

ii) *$\mathcal{L}^1(\mathcal{H})$ is a Banach algebra under the norm*

$$\|x\|_{\mathrm{Tr}} = \mathrm{Tr}\,|x|, \qquad x \in \mathcal{L}^1(\mathcal{H}).$$

Proof: [P; p.118]

Consider the following rank one operator:

$$t_{\xi,\eta}(\zeta) = (\zeta \mid \eta)\,\xi, \qquad \zeta \in \mathcal{H},$$

which is determined by $\xi, \eta \in \mathcal{H}$. Note that every self-adjoint operator x in $\mathcal{L}^1(\mathcal{H})$ can be written in the form

$$x = \sum \lambda_j t_{\xi_j,\xi_j}, \qquad \text{[P; p.107]}$$

for some orthonormal basis $\{\xi_j : j \in J\}$ and real eigenvalues λ_j with $\sum |\lambda_j| = \|x\|_{\mathrm{Tr}}$.

Moreover, for every orthonormal basis $\{\xi_j : j \in J\}$ in $\mathcal{H}$, the set $\{t_{\xi_i,\xi_j} : (i,j) \in J^2\}$ of rank one operators form an orthonormal basis for $\mathcal{L}^2(\mathcal{H})$ [P; p.121]. These operators are important in the proof of the next theorem.

Theorem 1.3.4: *The dual space of $\mathcal{K}(\mathcal{H})$ is $\mathcal{L}^1(\mathcal{H})$ and the dual space of $\mathcal{L}^1(\mathcal{H})$ is $\mathcal{L}(\mathcal{H})$.*

Proof: [P; p.120]

1.4 Locally convex topologies on $\mathcal{L}(\mathcal{H})$

We now describe different locally convex topologies on $\mathcal{L}(\mathcal{H})$; these will allow us to characterize von Neumann algebras topologically.

Theorem 1.4.1: *Suppose V is a complex vector space and Γ is a family of seminorms on V that separates the points of V. Then there is a locally convex topology on V in which, for each $x_o \in V$, the family of all sets*

$$V(x_o; p_1, ..., p_n; \epsilon) = \{x \in V : p_j(x - x_o) < \epsilon,\ j = 1, ..., n\},$$

(where $\epsilon > 0$, $p_1, ..., p_n \in \Gamma$) is a base of neighborhoods of x_o. With this topology, each of the seminorms in Γ is continuous. Moreover, every locally convex topology on V arises in this way, from a suitable family of seminorms.

Proof: [K-R; p.17]

1. Norm topology

The operator norm determines a locally convex topology on $\mathcal{L}(\mathcal{H})$ called the *norm topology*. The family of all sets

$$V(x_o; \epsilon) = \{x \in \mathcal{L}(\mathcal{H}) : \|x - x_o\| < \epsilon\},$$

where $\epsilon > 0$, forms a base of neighborhoods of $x_o \in \mathcal{L}(\mathcal{H})$.

2. Strong operator topology

The equation $p_\xi(x) = \|x\xi\|$, $\xi \in \mathcal{H}$, defines a seminorm on $\mathcal{L}(\mathcal{H})$. The collection of all these seminorms determines a locally convex topology on $\mathcal{L}(\mathcal{H})$, called the *strong operator topology*. The family of all sets

$$V(x_o;\xi_1,\ldots,\xi_n;\epsilon) = \{x \in \mathcal{L}(\mathcal{H}) : \|(x-x_o)\xi_j\| < \epsilon,\ j = 1,\ldots,n\},$$

where $\epsilon > 0$, $\xi_1,\ldots,\xi_n \in \mathcal{H}$, forms a base of neighborhoods of $x_o \in \mathcal{L}(\mathcal{H})$.

For $\xi, \eta \in \mathcal{H}$, we define a linear form $w_{\xi,\eta}$ on $\mathcal{L}(\mathcal{H})$ by:

$$w_{\xi,\eta}(x) = (x\xi \mid \eta), \quad x \in \mathcal{L}(\mathcal{H}).$$

Let $\mathcal{L}(\mathcal{H})_\sim$ be the vector subspace of the dual space $\mathcal{L}(\mathcal{H})^*$ of $\mathcal{L}(\mathcal{H})$ generated by the forms $w_{\xi,\eta}$ (for ξ, $\eta \in \mathcal{H}$), and let $\mathcal{L}(\mathcal{H})_*$ be the norm-closure of $\mathcal{L}(\mathcal{H})_\sim$ in $\mathcal{L}(\mathcal{H})^*$.

3. Weak operator topology

For $\xi, \eta \in \mathcal{H}$, the equation $p_{\xi,\eta}(\cdot) = |(\cdot\xi \mid \eta)|$ defines a seminorm on $\mathcal{L}(\mathcal{H})$. The collection of all these seminorms determines a locally convex topology on $\mathcal{L}(\mathcal{H})$, called the *weak operator topology*.

The family of all sets

$$V(x_o;p_{\xi_1,\eta_1},\ldots,p_{\xi_n,\eta_n};\epsilon) = \{x \in \mathcal{L}(\mathcal{H}) : |((x-x_o)\xi_j \mid \eta_j)| < \epsilon,\ j = 1,\ldots,n\},$$

where $\epsilon > 0$, $\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n \in \mathcal{H}$, forms a base of neighborhoods of $x_o \in \mathcal{L}(\mathcal{H})$. The weak operator topology is therefore the $\sigma(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{H})_\sim)$-topology.

4. Ultra-strong topology

Let $(\xi_1,\xi_2,\ldots)$ be a sequence of elements of $\mathcal{H}$ such that $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$. The equation

$$p_{\{\xi_i\}}(x) = \left[\sum_{j=1}^{\infty} \|x\xi_j\|^2\right]^{\frac{1}{2}},$$

defines a seminorm on $\mathcal{L}(\mathcal{H})$. The collection of all these seminorms determines a locally convex topology on $\mathcal{L}(\mathcal{H})$, called the *ultra-strong topology*.

For every sequence $\{\xi_i\}$ of elements of $\mathcal{H}$ satisfying $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$, the family of all sets

$$V(x_o;\epsilon) = \{x \in \mathcal{L}(\mathcal{H}) : \sum_{j=1}^{\infty} \|(x-x_o)\xi_j\|^2 < \epsilon^2\},$$

where $\epsilon > 0$, forms a base of neighborhoods of $x_o \in \mathcal{L}(\mathcal{H})$.

5. Ultra-weak topology

Let $(\xi_1, \xi_2, \ldots)$ and $(\eta_1, \eta_2, \ldots)$ be sequences of elements of $\mathcal{H}$ such that both

$$\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty, \qquad \sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty.$$

The function

$$x \longmapsto \left| \sum_{j=1}^{\infty} (x\xi_j \,|\, \eta_j) \right|$$

defines a seminorm on $\mathcal{L}(\mathcal{H})$. The collection of all these seminorms determines a locally convex topology on $\mathcal{L}(\mathcal{H})$, called the *ultra-weak topology*.

The ultra-weak topology is therefore the $\sigma(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H})_*)$-topology.

We will define two other important topologies in the next section, once we have discussed normal forms on a von Neumann algebra $\mathcal{M}$.

Theorem 1.4.2: *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a *-subalgebra of $\mathcal{L}(\mathcal{H})$ containing the identity operator 1. Then $\mathcal{M} = \mathcal{M}''$ if and only if $\mathcal{M}$ is a strongly closed *-subalgebra of $\mathcal{L}(\mathcal{H})$.*

Proof: [M; p.116]

Theorem 1.4.3: *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a *-subalgebra of $\mathcal{L}(\mathcal{H})$ containing the identity operator. $\mathcal{M}$ is a von Neumann algebra if and only if it is weakly closed.*

Proof: [M; p.127]

1.5 Linear forms

Let $\mathcal{M}$ be a unital *-subalgebra of $\mathcal{L}(\mathcal{H})$. A linear functional ϕ on $\mathcal{M}$ is *positive* if $\phi(x) \geq 0$ for all $x \in \mathcal{M}^+$, and *faithful* if for $x \in \mathcal{M}^+$, the condition $\phi(x) = 0$ implies $x = 0$. A positive linear functional ϕ on $\mathcal{M}$ satisfies:

$$\phi(x^*) = \overline{\phi(x)} \quad \text{and} \quad \phi(x^*x) \geq 0, \quad \text{for all } x \in \mathcal{M}.$$

Moreover, ϕ satisfies the Cauchy Schwarz's inequality:

$$|\phi(x^*y)|^2 \leq \phi(x^*x)\phi(y^*y), \quad \text{for all } x, y \in \mathcal{M}.$$

In particular, we have

$$|\phi(x)|^2 \leq \phi(1)\phi(x^*x) \leq \phi(1)^2 \|x^*x\| = \phi(1)^2 \|x\|^2.$$

We thus observe that ϕ is continuous and has norm equal to $\phi(1)$.

Definition 1.5.1: Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. A positive linear functional ϕ on $\mathcal{M}$ is *normal* if for all monotone increasing net $\{x_i\}$ in $\mathcal{M}^+$,

$$\phi(\sup_i(x_i)) = \sup_i(\phi(x_i)).$$

Theorem 1.5.2: *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and ϕ a positive linear functional on $\mathcal{M}$. The following conditions are equivalent:*

- i) *ϕ is normal;*
- ii) *ϕ is ultra-weakly continuous;*
- iii) *$\phi = \sum_{i=1}^{\infty} w_{\xi_i}$, with $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, where w_{ξ_i} is the linear form on $\mathcal{L}(\mathcal{H})$ defined by $w_{\xi_i}(x) = (x\,\xi_i \mid \xi_i)$, for all $x \in \mathcal{M}$.*

Every ultra-weakly continuous linear form on $\mathcal{M}$ is a linear combination of normal positive linear forms.

Proof: [Di; p.57]

Proposition 1.5.3: *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{M} = (\mathcal{M}_*)^*$, where $\mathcal{M}_*$ denotes the set of all normal forms on $\mathcal{M}$.*

Proof: [S-Z; p.19]

Definition 1.5.4: $\mathcal{M}_*$ is called the *predual* of $\mathcal{M}$.

Example 1.5 a:

- i) The predual of $\mathcal{L}(\mathcal{H})$ is the set of trace-class operators, $\mathcal{L}^1(\mathcal{H})$. [Thm 1.1.19]
- ii) Let (X, μ) be a σ-finite measure space. The predual of $L^\infty(X, \mu)$ is $L^1(X, \mu)$. [C; p.75]

Theorem 1.5.5 (GNS construction): *Let ϕ be a positive linear functional on a C^*-algebra A satisfying $\phi(1) = 1$. Then there exists a triple $(\mathcal{H}_\phi, \pi_\phi, \Omega_\phi)$ where*

- i) *$\mathcal{H}_\phi$ is a Hilbert space and $\pi_\phi : A \to \mathcal{L}(\mathcal{H}_\phi)$ is a *-homomorphism;*
- ii) *$\Omega_\phi \in \mathcal{H}_\phi$ and $\mathcal{H}_\phi = \overline{\pi_\phi(A)\Omega_\phi}$;*
- iii) *$\phi(x) = (\pi_\phi(x)\Omega_\phi \mid \Omega_\phi)$ for all $x \in A$.*

Such a triple is unique in the sense that if $(\mathcal{H}', \pi', \Omega')$ is another such triple, there exists a unique unitary operator $w : \mathcal{H}_\phi \to \mathcal{H}'$ such that $w\Omega_\phi = \Omega'$ and $\pi'(x) = w\pi_\phi(x)w^*$, for all $x \in A$.

Proof: [K-R; p.278]

Remark: Let $\mathcal{M}$ be a von Neumann algebra and let ϕ be a faithful normal positive linear functional on $\mathcal{M}$. Then Theorem 1.5.5 remains valid. Moreover, the image $\pi_\phi(\mathcal{M})$ is a von Neumann algebra of operators on $\mathcal{H}_\phi$; π_ϕ is norm-preserving and a ultra-weak homeomorphism of $\mathcal{M}$ onto $\pi_\phi(\mathcal{M})$. [Su; p.40]

Example 1.5 b: Let $\mathcal{M} = M_2(\mathbf{C})$ and ϕ be the trace on $M_2(\mathbf{C})$. The GNS triple is given by:

$$\mathcal{H}_\phi \cong \mathbf{C}^4, \quad \Omega_\phi = \eta_\phi(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \pi_\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \cong \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1_2,$$

where $\eta_\phi : M_2(\mathbf{C}) \rightarrow \mathcal{H}_\phi$ is the mapping defined by:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\pi_\phi(x) : \mathcal{H}_\phi \rightarrow \mathcal{H}_\phi, \qquad \pi_\phi(x)\eta_\phi(y) = \eta_\phi(xy).$$

Now, let

$$\rho_\phi : \mathcal{M} \rightarrow \mathcal{L}(\mathcal{H}_\phi), \qquad \rho_\phi(x)\eta_\phi(y) = \eta_\phi(yx^t);$$

then

$$\rho_\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \cong 1_2 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that $\rho_\phi(\mathcal{M}) = \pi_\phi(\mathcal{M})'$.

Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. In addition to the topologies defined on $\mathcal{L}(\mathcal{H})$ in section 1.4, we can consider the following topologies on $\mathcal{M}$:

The s-topology

Let ϕ be a normal form on $\mathcal{M}$. The equation $s_\phi(x) = \phi(x^*x)^{\frac{1}{2}}$ defines a seminorm on $\mathcal{M}$. The collection of all these seminorms determines a locally convex topology on $\mathcal{M}$, called the *s-topology*.

The family of all sets

$$V(x_o; \phi_1, ..., \phi_n; \epsilon) = \{x \in \mathcal{M} : \phi_j((x - x_o)^*(x - x_o))^{\frac{1}{2}} < \epsilon, \ j = 1, ..., n\},$$

where $\epsilon > 0$ and $\phi_1, ..., \phi_n$ are normal forms on $\mathcal{M}$, forms a base of neighborhoods of $x_o \in \mathcal{M}$.

The s^*-topology

Let ϕ be a normal form on $\mathcal{M}$. The equations

$$s_\phi(x) = \phi(x^*x)^{\frac{1}{2}} \quad \text{and} \quad s_\phi^*(x) = \phi(xx^*)^{\frac{1}{2}}$$

define seminorms on $\mathcal{M}$. The collection of all these seminorms determines a locally convex topology on $\mathcal{M}$, called the *s^*-topology.*

The family of all sets

$$\begin{aligned} &V(x_c;\phi_1,\dots,\phi_n;\epsilon) \\ &\quad = \{x \in \mathcal{M} : \phi_j((x-x_o)^*(x-x_o))^{\frac{1}{2}} < \epsilon,\ \phi_j((x-x_o)(x-x_o)^*)^{\frac{1}{2}} < \epsilon\ \ j = 1,\dots,n\}, \end{aligned}$$

where $\epsilon > 0$ and $\phi_1,\dots,\phi_n$ are normal forms on $\mathcal{M}$, forms a base of neighborhoods of $x_o \in \mathcal{M}$.

1.6 Projections

Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{M}$ be a self-adjoint element. Then the spectral projections of x belong to $\mathcal{M}$. Indeed, the Spectral Theorem [R; p.324] states that there exists a unique resolution of the identity $E : \mathcal{B}(\sigma(x)) \to \mathcal{M} \subset \mathcal{L}(\mathcal{H})$ defined on the Borel subsets of the spectrum of x which satisfies

$$x = \int_{\sigma(x)} \lambda \, dE(\lambda).$$

In addition, every projection $E(w)$ belongs to $\mathcal{M}'' = \mathcal{M}$. Consequently, projections abound in any von Neumann algebra.

Definition 1.6.1: Let $\mathcal{M}$ be a von Neumann algebra and p, q two projections in $\mathcal{M}$.

i) The projections p and q are said to be *equivalent*, written $p \sim q$, if there exists a partial isometry $u \in \mathcal{M}$ such that $uu^* = p$ and $u^*u = q$.

ii) We will write $p \preceq q$ whenever the projection p is equivalent to a subprojection of q; that is, there exists a projection $p_1 \in \mathcal{M}$ such that $p \sim p_1 \leq q$.

iii) A projection p in $\mathcal{M}$ is *finite* relative to $\mathcal{M}$ if the condition $p \sim q < p$ for some projection q in $\mathcal{M}$ implies $p = q$.

iv) A projection p in $\mathcal{M}$ is *infinite* relative to $\mathcal{M}$ if $p \sim q < p$ for some projection q in $\mathcal{M}$.

v) A projection p in $\mathcal{M}$ is *properly infinite* relative to $\mathcal{M}$ if p is infinite and pq is either 0 or infinite for each central projection q in $\mathcal{M}$.

Definition 1.6.2:

i) A von Neumann algebra $\mathcal{M}$ is *finite* if 1 is finite.
ii) A von Neumann algebra $\mathcal{M}$ is *properly infinite* if there is a sequence $\{p_n\}_{n=1}^{\infty}$ of pairwise orthogonal projections in $\mathcal{M}$ such that $\sum p_n = 1$ and $p_n \sim 1$ (relative to $\mathcal{M}$) for all n.

Remarks:

i) A von Neumann algebra $\mathcal{M}$ is finite if and only if for every nonzero element $x \in \mathcal{M}^+$, there exists a finite normal trace τ on $\mathcal{M}^+$ such that $\tau(x) \neq 0$.
ii) A von Neumann algebra $\mathcal{M}$ is properly infinite if and only if the only finite normal trace on $\mathcal{M}^+$ is 0.

Example 1.6 a:

i) $M_n(\mathbf{C})$ is a finite von Neumann algebra.
ii) If G is a discrete group, then $vN(G)$ is a finite von Neumann algebra.
iii) $\mathcal{L}(\mathcal{H})$ is a properly infinite von Neumann algebra.

Definition 1.6.3: A projection p in $\mathcal{M}$ is said to be *countably decomposable* relative to $\mathcal{M}$ if each orthogonal family of non-zero subprojections of p is countable. We say that $\mathcal{M}$ is countably decomposable if the identity 1 is countably decomposable relative to $\mathcal{M}$.

Definition 1.6.4: Let ϕ be a normal positive linear form on the von Neumann algebra $\mathcal{M}$. The set of all projections p in $\mathcal{M}$ such that $\phi(p) = 0$ has a greatest element [Di; p.63]; let q denote this projection. The projection $1 - q$ is called the *support* of ϕ. It will be denoted $\mathrm{s}(\phi)$.

Remarks:

i) A projection p in $\mathcal{M}$ is countably decomposable if and only if it is the support of a normal form on $\mathcal{M}$; a von Neumann algebra $\mathcal{M}$ is countably decomposable if and only if there exists a normal form ϕ on $\mathcal{M}$ such that $\mathrm{s}(\phi) = 1$ [S-Z; E.5.6].
ii) If $\mathcal{H}$ is a separable Hilbert space, $\mathcal{L}(\mathcal{H})$ and each von Neumann algebra on $\mathcal{H}$ are countably decomposable. [K-R; p.338]

Definition 1.6.5:

i) Let $x \in \mathcal{L}(\mathcal{H})$. Consider the following orthogonal projections:

$$\mathbf{n}(x) : \mathcal{H} \rightarrow \{\xi \in \mathcal{H} : x\xi = 0\}, \quad \text{the projection on the kernel of } x;$$

$$\mathbf{l}(x) : \mathcal{H} \rightarrow \overline{x\mathcal{H}}, \quad \text{the projection on the closure of the image of } x;$$

$$\mathbf{r}(x) = 1 - \mathbf{n}(x).$$

$\mathbf{l}(x)$, called the *left support* of x, is the smallest projection in $\mathcal{L}(\mathcal{H})$ such that $\mathbf{l}(x)x = x$ and $\mathbf{r}(x)$, called the *right support* of x, is the smallest projection in $\mathcal{L}(\mathcal{H})$ such that $x\mathbf{r}(x) = x$. If x is self-adjoint, then $\mathrm{s}(x) = \mathbf{l}(x) = \mathbf{r}(x)$ is called the *support* of x.

ii) Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $x \in \mathcal{M}$. The projection

$$\mathrm{z}(x) : \mathcal{H} \to [(\mathcal{M}x)\mathcal{H}]$$

is called the *central support* of x. It is the smallest projection p in $\mathcal{M} \cap \mathcal{M}'$ such that $px = x$.

Lemma 1.6.6: *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra such that $\mathcal{M}' \subset \mathcal{L}(\mathcal{H})$ is properly infinite. Then $\mathcal{M}$ has a separating vector if and only if $\mathcal{M}$ is countably decomposable.*

Proof: If $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ has a separating vector $\xi \in \mathcal{H}$, then $\mathrm{s}(w_\xi) = 1$ and thus, by the above remark, $\mathcal{M}$ is countably decomposable.

Now, suppose $\mathcal{M}$ is countably decomposable and $\mathcal{M}'$ is properly infinite. We can also suppose that there exists $\eta \in \mathcal{H}$ such that the projection $p'_\eta : \mathcal{H} \to [\mathcal{M}\eta]$ is the identity. Then the projections $p_\eta : \mathcal{H} \to [\mathcal{M}'\eta]$ and 1 in $\mathcal{M}$ are countably decomposable, properly infinite and they have the same central support. It follows that $p_\eta \sim 1$. Thus there exists $v \in \mathcal{M}$ such that $v^*v = p_\eta$ and $vv^* = 1$. Let $\xi = v\eta$. Then the projection $p_\xi : \mathcal{H} \to [\mathcal{M}'\xi]$ is the identity and thus ξ is a separating vector for $\mathcal{M}$.

Q.E.D.

1.7 Tensor products of Hilbert spaces and of von Neumann algebras

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let us denote by $\mathcal{H}_1 \odot \mathcal{H}_2$ their algebraic tensor product. A general element of $\mathcal{H}_1 \odot \mathcal{H}_2$ is of the form:

$$\sum_{i=1}^{n} \xi_{1,i} \odot \xi_{2,i} \qquad \xi_{1,i} \in \mathcal{H}_1, \quad \xi_{2,i} \in \mathcal{H}_2, \ i = 1, \dots, n.$$

In $\mathcal{H}_1 \odot \mathcal{H}_2$, if $\xi = \sum_{i=1}^{n} \xi_{1,i} \odot \xi_{2,i}$ and $\eta = \sum_{j=1}^{m} \eta_{1,j} \odot \eta_{2,j}$, then

$$(\xi \mid \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_{1,i} \mid \eta_{1,j})(\xi_{2,i} \mid \eta_{2,j})$$

defines an inner product.

Definition 1.7.1: The completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the norm induced by the above inner product is called the ***Hilbert space tensor product*** of $\mathcal{H}_1$ and $\mathcal{H}_2$. It will be denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Recall that if $\mathcal{H}$ is a Hilbert space, then the ***conjugate Hilbert space***, $\overline{\mathcal{H}}$, is the same set $\mathcal{H}$, with the algebraic structure and inner product defined by the mappings:

$$\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \longmapsto \xi + \eta \in \mathcal{H},$$
$$\mathbf{C} \times \mathcal{H} \ni (a, \eta) \longmapsto \overline{a}\xi \in \mathcal{H},$$
$$\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \longmapsto \overline{(\xi \mid \eta)} \in \mathbf{C}.$$

Then we have the following proposition:

Proposition 1.7.2: *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then for each $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, the equation*

$$x_{\xi,\eta}u = \overline{(u \mid \xi)}\,\eta = (\xi \mid u)\,\eta, \quad u \in \overline{\mathcal{H}}$$

defines a Hilbert-Schmidt operator $x_{\xi,\eta}$ from $\overline{\mathcal{H}}$ into $\mathcal{K}$. There is a unitary transformation $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{L}^2(\overline{\mathcal{H}}, \mathcal{K})$ such that

$$U(\xi \otimes \eta) = x_{\xi,\eta} \qquad \xi \in \mathcal{H},\ \eta \in \mathcal{K}.$$

Proof: [K-R; p.142]

Let $x_1 \in \mathcal{L}(\mathcal{H}_1)$ and $x_2 \in \mathcal{L}(\mathcal{H}_2)$. Let us denote by $x_1 \otimes x_2$ the operator on $\mathcal{H}_1 \odot \mathcal{H}_2$ defined by:

$$(x_1 \otimes x_2)(\xi_1 \odot \xi_2) = (x_1 \xi_1) \odot (x_2 \xi_2).$$

The operator $x_1 \otimes x_2$ extends to a bounded operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, called the tensor product of x_1 and x_2, with the following properties:

$$\begin{aligned}
(\lambda x_1 + \mu y_1) \otimes x_2 &= \lambda(x_1 \otimes x_2) + \mu(y_1 \otimes x_2);\\
x_1 \otimes (\lambda x_2 + \mu y_2) &= \lambda(x_1 \otimes x_2) + \mu(x_1 \otimes y_2);\\
(x_1 \otimes x_2)(y_1 \otimes y_2) &= x_1 y_1 \otimes x_2 y_2;\\
(x_1 \otimes x_2)^* &= x_1^* \otimes x_2^*;\\
\|x_1 \otimes x_2\| &= \|x_1\| \|x_2\|.
\end{aligned}$$

Definition 1.7.3: Let $\mathcal{M}_1 \subset \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{M}_2 \subset \mathcal{L}(\mathcal{H}_2)$ be von Neumann algebras. The von Neumann algebra on $\mathcal{H}_1 \otimes \mathcal{H}_2$ generated by $x_1 \otimes x_2$, $x_1 \in \mathcal{M}_1$, $x_2 \in \mathcal{M}_2$ is called the tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$. It will be denoted $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

Remark: Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. If $\mathcal{M}$ is properly infinite, then so is $\mathcal{M} \overline{\otimes} \mathcal{N}$. [Di; p.126]

Definition 1.7.4: Let $\mathcal{M}$ be a von Neumann algebra. The *-isomorphism

$$\mathcal{M} \to \mathcal{M} \overline{\otimes} \mathbf{C} 1_H, \qquad x \longmapsto x \otimes 1_H$$

is called an *ampliation*.

Definition 1.7.5: Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A linear mapping $\Phi : \mathcal{M} \to \mathcal{N}$ is *positive* is $\Phi(\mathcal{M}^+) \subset \mathcal{N}^+$; such a Φ is *normal* if for every monotone increasing net $\{x_i\}$ in $\mathcal{M}^+$, $\Phi(\sup(x_i)) = \sup(\Phi(x_i))$.

Theorem 1.7.6: *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and let Φ be a normal *-homomorphism of $\mathcal{M}$ into $\mathcal{N}$. Then there exists an ampliation Φ_1 of $\mathcal{M}$ into a von Neumann algebra $\mathcal{M}_1$, an induction Φ_2 of $\mathcal{M}_1$ into a von Neumann algebra $\mathcal{N}_1$, and a spatial isomorphism Φ_3 of $\mathcal{N}_1$ onto $\mathcal{N}$ such that $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$.*

Proof: [Di; p.61]

1.8 Tensor products of C*-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and consider their algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. Any $z \in \mathcal{A} \odot \mathcal{B}$ has the following form:

$$\sum_i x_i \otimes y_i, \qquad x_i \in \mathcal{A},\ y_i \in \mathcal{B}.$$

Note that $\mathcal{A} \odot \mathcal{B}$ is a *-algebra:

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2, \quad (x_1 \otimes y_1)^* = x_1^* \otimes y_1^*, \quad x_1, x_2 \in \mathcal{A},\ y_1, y_2 \in \mathcal{B}.$$

Definition 1.8.1: Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras and $\mathcal{A} \odot \mathcal{B}$ their algebraic tensor product. A norm $\|\cdot\|$ on $\mathcal{A} \odot \mathcal{B}$ is called a *cross-norm* if

$$\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|, \quad x_1 \in \mathcal{A},\ x_2 \in \mathcal{B}.$$

Definition 1.8.2: Let $\mathcal{A}$ and $\mathcal{B}$ be two C*-algebras. A seminorm p on $\mathcal{A} \odot \mathcal{B}$ is

i) a *C^*-seminorm* if $p(z^*z) = p(z)^2$, $z \in \mathcal{A} \odot \mathcal{B}$,

ii) a *C^*-norm* if p is also a norm.

Let

$$S(\mathcal{A} \odot \mathcal{B}) = \{f : \mathcal{A} \odot \mathcal{B} \to \mathbf{C} : f \text{ is linear, } f(z^*z) \geq 0\ (z \in \mathcal{A} \odot \mathcal{B}),\ f(1) = 1\}$$

be the state space of $\mathcal{A} \odot \mathcal{B}$. If we apply the GNS construction [section 1.5] to any $f \in S(\mathcal{A} \odot \mathcal{B})$ we obtain a Hilbert space $\mathcal{H}_f$, a homomorphism $\pi_f : \mathcal{A} \odot \mathcal{B} \to \mathcal{L}(\mathcal{H}_f)$ and a cyclic unit vector ξ_f such that:

$$f(z) = (\pi_f(z)\xi_f \mid \xi_f), \quad z \in \mathcal{A} \odot \mathcal{B}.$$

Let $f \in S(\mathcal{A} \odot \mathcal{B})$ and define

$$\begin{aligned} p_f(z) &= \|\pi_f(z)\|, && z \in \mathcal{A} \odot \mathcal{B}, \\ p_\Gamma &= \sup\{p_f : f \in \Gamma\}, && \Gamma \subset S(\mathcal{A} \odot \mathcal{B}). \end{aligned}$$

Then p_f and p_Γ are C^*-seminorms on $\mathcal{A} \odot \mathcal{B}$.

Definition 1.8.3: We say that the set Γ is a *separating* subset of $S(\mathcal{A} \odot \mathcal{B})$ when p_Γ is a C^*-norm: the C^*-algebra obtained by completing $\mathcal{A} \odot \mathcal{B}$ with respect to p_Γ will be denoted $\mathcal{A} \otimes_\Gamma \mathcal{B}$.

Let $\mathcal{A}^* \odot \mathcal{B}^*$ denote the vector space tensor product of $\mathcal{A}^*$ and $\mathcal{B}^*$, viewed as a space of linear functions on $\mathcal{A} \odot \mathcal{B}$.

Definition 1.8.4: The *minimal* and *maximal* tensor products of $\mathcal{A}$ and $\mathcal{B}$, denoted

$$\mathcal{A} \otimes_{min} \mathcal{B} \qquad \text{and} \qquad \mathcal{A} \otimes_{max} \mathcal{B}$$

respectively, are the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to p_{min} and p_{max}, where

$$min = (\mathcal{A}^* \odot \mathcal{B}^*) \cap S(\mathcal{A} \odot \mathcal{B}) \qquad \text{and} \qquad max = S(\mathcal{A} \odot \mathcal{B}).$$

p_{min} and p_{max} are respectively the least and the greatest C^*-norms on $\mathcal{A} \odot \mathcal{B}$. They are both cross-norms.

Remark: Let ρ be a state on $\mathcal{A} \otimes_{max} \mathcal{B}$; its restriction to $\mathcal{A} \odot \mathcal{B}$ is in $S(\mathcal{A} \odot \mathcal{B})$. We can thus define a bijective mapping from $S(\mathcal{A} \otimes_{max} \mathcal{B})$ onto $S(\mathcal{A} \odot \mathcal{B})$.

Definition 1.8.5 [E-L]: Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The *binormal tensor product* $\mathcal{M} \otimes_{bin} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is the completion of $\mathcal{M} \odot \mathcal{N}$ with respect to p_{bin}, where

$$bin = bin(\mathcal{M} \odot \mathcal{N}) = \{f \in S(\mathcal{M} \odot \mathcal{N}) : (x, y) \to f(x, y) \text{ is separately normal}\}.$$

Remarks:

i) The set *bin* defined above will be referred to as the set of *binormal states.*

ii) On $\mathcal{M} \odot \mathcal{N}$, it can be shown [E-L] that

$$p_{min} \le p_{bin} \le p_{max}.$$

1.9 Completely positive maps

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. In section 3.4, we demonstrate that to any completely positive normal map $P : \mathcal{M} \to \mathcal{N}$ is associated a correspondence from $\mathcal{M}$ to $\mathcal{N}$. Let us now study some properties of these maps.

Let $\mathcal{A}$ be a C^*-algebra. Let us denote by $\mathbf{M}_n(\mathcal{A})$ the set of all $n \times n$ matrices $\mathbf{a} = [a_{ij}]$ with entries in $\mathcal{A}$. Equipped with the following operations, $\mathbf{M}_n(\mathcal{A})$ is an involutive algebra:

$$\lambda\mathbf{a} + \mu\mathbf{b} = [\lambda a_{ij} + \mu b_{ij}],$$

$$\mathbf{ab} = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right],$$

$$\mathbf{a}^* = [a_{ji}^*],$$

where $\mathbf{a} = [a_{ij}]$, $\mathbf{b} = [b_{ij}] \in \mathbf{M}_n(\mathcal{A})$ and λ, $\mu \in \mathbf{C}$.

Let $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$. Let $\mathcal{H}_n = \overbrace{\mathcal{H} \oplus \ldots \oplus \mathcal{H}}^{n \text{ copies}}$, and define $\pi : \mathrm{M}_n(\mathcal{A}) \to \mathcal{L}(\mathcal{H}_n)$ by

$$\pi(\mathrm{a})\xi = \left(\sum_j \pi_{\mathcal{A}}(a_{1j})\xi_j, \ldots, \sum_j \pi_{\mathcal{A}}(a_{nj})\xi_j\right), \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}_n.$$

$\mathrm{M}_n(\mathcal{A})$ is a C^*-algebra under the norm defined by $\|\mathrm{a}\| = \|\pi(\mathrm{a})\|$. Note that this norm does not depend on the particular space $\mathcal{H}$ on which $\mathcal{A}$ acts faithfully, as an isomorphism of C^*-algebras is an isometry.

Remark: An element $\mathrm{a} \in \mathrm{M}_n(\mathcal{A})$ is positive if $(\mathrm{a}\xi \mid \xi) \geq 0$, for all $\xi \in \mathcal{H}_n$, that is

$$\sum_{i,j} (a_{ij}\xi_j \mid \xi_i) \geq 0, \qquad \xi_1, \ldots, \xi_n \in \mathcal{H}.$$

Lemma 1.9.1:

i) *An element of* $\mathrm{M}_n(\mathcal{A})$ *is positive if and only if it is a finite sum of matrices of the form* $[a_i^* a_j]$, $a_1, \ldots, a_n \in \mathcal{A}$.

ii) *A matrix* $\mathrm{a} = [a_{ij}] \in \mathrm{M}_n(\mathcal{A})$ *is positive if and only if*

$$\sum_{i,j=1}^{n} x_i^* a_{ij} x_j \geq 0, \quad \text{for all } x_1, \ldots, x_n \in \mathcal{A}.$$

Proof: [Tak$_1$; p.193]

Definition 1.9.2: Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear mapping. Define a linear mapping $\Phi_n : \mathrm{M}_n(\mathcal{A}) \to \mathrm{M}_n(\mathcal{B})$ by

$$\Phi_n([a_{ij}]) = [\Phi(a_{ij})].$$

The map Φ is said to be *n-positive* if Φ_n is positive and *completely positive* if Φ is n-positive for all n.

Example 1.9 a:

i) Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\mathcal{B}$ is abelian, then any positive linear map $\phi : \mathcal{A} \to \mathcal{B}$ is completely positive. In particular, a positive linear functional on a C^*-algebra is completely positive.

ii) Any *-homomorphism between C^*-algebras is completely positive.

From Lemma 1.9.1. we get

Lemma 1.9.3: *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is n-positive if and only if*

$$\sum_{i,j=1}^{n} y_i^* \Phi(x_i^* x_j) y_j \geq 0, \quad \text{for all } x_1, \dots, x_n \in \mathcal{A},\ y_1, \dots, y_n \in \mathcal{B}.$$

Proof: [Tak$_1$; p.194]

Lemma 1.9.4: *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a completely positive map, then*

$$\Phi(a)^* \Phi(a) \leq \|\Phi\| \Phi(a^* a), \qquad a \in \mathcal{A}.$$

Proof: [Tak$_1$; p.199]

Definition 1.9.5: Let $\mathcal{A}$ be a C^*-algebra. $\mathbf{M}_n(\mathcal{A}^*)$ will denote the set of $n \times n$ matrices $\mathbf{f} = [f_{ij}]$ with entries $f_{ij} \in \mathcal{A}^*$.

Remark: It is possible to identify $\mathbf{M}_n(\mathcal{A}^*)$ with the dual $\mathbf{M}_n(\mathcal{A})^*$ of $\mathbf{M}_n(\mathcal{A})$. Indeed, the equation

$$\mathbf{f}(\mathbf{a}) = \sum_{i,j} f_{ij}(a_{ij})$$

gives the desired correspondence.

From Lemma 1.9.1, we obtain:

$$\mathbf{f} \geq 0 \qquad \Longleftrightarrow \qquad \sum_{i,j} f_{ij}(a_i^* a_j) \geq 0, \quad a_1, \dots, a_n \in \mathcal{A}.$$

1.10 Crossed products

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. In chapter 3, we study the link between normal *-homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ and correspondences from $\mathcal{M}$ to $\mathcal{N}$. As an example, we will consider the case where $\mathcal{M} = \mathcal{N}$ is the crossed product of a von Neumann algebra by an action of a locally compact group.

We will now see how this new von Neumann algebra is constructed.

Throughout this section, $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ will be a von Neumann algebra and G will be a locally compact group.

Let $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ be a continuous action of G on $\mathcal{M}$, that is, α is a homomorphism of G into the group of automorphisms of $\mathcal{M}$ such that for all $x \in \mathcal{M}$, the map $s \to \alpha_s(x)$ is continuous, where $\mathcal{M}$ is considered with its strong topology.

Let ds denote the left invariant Haar measure on G and let $L^2(G)$ be the Hilbert space of (equivalence classes of) square integrable functions from G into $\mathbf{C}$ with respect to the Haar measure.

Given a continuous action α of a locally compact group G on a von Neumann algebra $\mathcal{M}$, it is possible to construct a new von Neumann algebra, denoted

$$\mathcal{M} \otimes_\alpha G,$$

called the *crossed product of* $\mathcal{M}$ *by the action* α *of* G. This von Neumann algebra acts in the Hilbert space tensor product $\mathcal{H} \otimes L^2(G)$.

Let us denote by $C_C(G, \mathcal{H})$ the complex vector space of $\mathcal{H}$-valued functions on G with compact support. Denote $(\cdot \mid \cdot)$ the scalar product in $\mathcal{H}$.

For every pair $\xi, \eta \in C_C(G, \mathcal{H})$, the function $s \to (\xi(s) \mid \eta(s))$ is a continuous complex-valued function with compact support in G. Then

$$(\xi \mid \eta) = \int (\xi(s) \mid \eta(s))\, ds$$

is a scalar product on $C_C(G, \mathcal{H})$. The completion of $C_C(G, \mathcal{H})$ with respect to this scalar product will be denoted $L^2(G, \mathcal{H})$.

Proposition 1.10.1: *There is an isomorphism U of $\mathcal{H} \otimes L^2(G)$ onto $L^2(G, \mathcal{H})$ such that*

$$(U(\xi_0 \otimes f))(s) = f(s)\xi_0 \quad \text{for all } \xi_0 \in \mathcal{H},\ f \in C_C(G),$$

where $C_C(G)$ is the set of complex-valued continuous functions with compact support in G.

Proof: [VD; p.4]

For all $x \in \mathcal{M}$, define a bounded operator $\pi_\alpha(x)$ on $L^2(G, \mathcal{H})$ by

$$(\pi_\alpha(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s), \qquad \xi \in C_C(G, \mathcal{H}).$$

Proposition 1.10.2: *π_α is a faithful normal *-representation of $\mathcal{M}$ in $L^2(G, \mathcal{H})$.*

Proof: [VD; p.7]

Example 1.10 a: Let $G = \mathbf{Z}_n = \{0, 1, ..., n-1\}$. Using the appropriate normalization of the Haar measure, we get the following isomorphism:

$$L^2(G, \mathcal{H}) \xrightarrow{\cong} \underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \ldots \oplus \mathcal{H}}_{n \text{ copies}}, \qquad \xi \mapsto (\xi(0), \xi(1), ..., \xi(n-1)).$$

The matrix representation of $\pi_\alpha(x)$ is then given by:

$$\pi_\alpha(x) = \begin{pmatrix} \alpha_0(x) & 0 & \dots & 0 \\ 0 & \alpha_{-1}(x) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \alpha_{-(n-1)}(x) \end{pmatrix}.$$

Now, for every $t \in G$, define a bounded operator $\lambda(t)$ on $L^2(G, \mathcal{H})$ by

$$(\lambda(t)\xi)(s) = \xi(t^{-1}s), \qquad \xi \in C_C(G, \mathcal{H}),\ s \in G.$$

Proposition 1.10.3: *λ is a strongly continuous unitary representation of G in $L^2(G, \mathcal{H})$.*
Proof: [VD; p.9]

Lemma 1.10.4:

$$\lambda(t)\pi_\alpha(x)\lambda(t)^* = \pi_\alpha(\alpha_t(x)), \quad \textit{for all } x \in \mathcal{M},\ t \in G.$$

Proof: [VD; p.10]

Note that by the above Lemma, the linear combinations of operators of the form $\pi_\alpha(x)\lambda(t)$ (for $x \in \mathcal{M}$, $t \in G$) form a *-algebra. Indeed, for $x, y \in \mathcal{M}$ and $t, s \in G$,

$$\begin{aligned}(\pi_\alpha(x)\lambda(t))^* &= \lambda(t)^*\pi_\alpha(x^*) = \lambda(t^{-1})\pi_\alpha(x^*)\lambda(t^{-1})^*\lambda(t^{-1}) \\ &= \pi_\alpha(\alpha_{t^{-1}}(x^*))\lambda(t^{-1})\end{aligned}$$

and

$$\begin{aligned}\pi_\alpha(x)\lambda(t)\pi_\alpha(y)\lambda(s) &= \pi_\alpha(x)\lambda(t)\pi_\alpha(y)\lambda(t)^*\lambda(t)\lambda(s) \\ &= \pi_\alpha(x)\pi_\alpha(\alpha_t(y))\lambda(ts) \\ &= \pi_\alpha(x\alpha_t(y))\lambda(ts).\end{aligned}$$

Definition 1.10.5: The crossed product of $\mathcal{M}$ by the action α of G is the von Neumann algebra generated by the operators

$$\{\pi_\alpha(x),\ \lambda(s) : x \in \mathcal{M},\ s \in G\},$$

and is denoted $\mathcal{M} \otimes_\alpha G$. It is the strong closure of the *-algebra of linear combinations of products $\pi_\alpha(x)\lambda(s)$ for $x \in \mathcal{M}$, $s \in G$.

Example 1.10 b: Let $G = \mathbb{Z}_3$. As in the previous example, $L^2(G, \mathcal{H})$ is identified with $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ and

$$\pi_\alpha(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & \alpha_2(x) & 0 \\ 0 & 0 & \alpha_1(x) \end{pmatrix}.$$

as $\alpha_0(x) = x$, $\alpha_{-1}(x) = \alpha_2(x)$ and $\alpha_{-2}(x) = \alpha_1(x)$. Moreover,

$$\lambda(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Example 1.10 c: Let $\mathcal{H} = \mathbf{C}$ and $\mathcal{M} = \mathcal{L}(\mathcal{H}) \cong \mathbf{C}$. Let G be a countable discrete group and $\alpha_t = 1_{\mathcal{M}}$ for all $t \in G$. In this case, $\mathcal{H} \otimes l^2(G)$ is identified with $l^2(G)$. The unitary representation $\pi_\alpha : \mathcal{M} \to \mathcal{L}(l^2(G))$ is given by

$$(\pi_\alpha(\beta)\,\xi)(s) = \alpha_{s^{-1}}(\beta)\,\xi(s) = \beta\,\xi(s), \qquad \text{for all } \beta \in \mathbf{C},\ \xi \in l^2(G),$$

and the representation $\lambda : G \to \mathcal{L}(l^2(G))$ is given by

$$(\lambda(u)\,\xi)(s) = \xi(u^{-1}s), \qquad \xi \in l^2(G),$$

that is, $\lambda(u)$ is identified with the left regular representation λ_u of G. Hence

$$\mathcal{M} \otimes_\alpha G = \lambda(G)'' = vN(G),$$

the group von Neumann algebra of G. [Example 1.1 c]

1.11 Measurable fields of Hilbert spaces

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. In our study of correspondences from $\mathcal{M}$ to $\mathcal{N}$, we first consider the particular case where $\mathcal{M}$ and $\mathcal{N}$ are both commutative. We shall see that in such a case, the correspondence that we get is the direct integral of a measurable field of Hilbert spaces.

Definition 1.11.1:

i) Let Z be a Borel space equipped with a σ-finite Borel measure μ. A ***field of complex Hilbert spaces*** over Z is a mapping $z \mapsto \mathcal{H}(z)$, defined on Z, such that $\mathcal{H}(z)$ is a complex Hilbert space for every $z \in Z$.

ii) $V = \prod_{z \in Z} \mathcal{H}(z)$ is a complex vector space. The elements of V are called ***vector fields*** over Z. Specifically, an element $\xi \in V$ is a mapping $z \mapsto \xi(z)$ defined on Z such that $\xi(z) \in \mathcal{H}(z)$, for all $z \in Z$.

Let $z \mapsto \mathcal{H}(z)$ be a field of complex Hilbert spaces over Z and V the vector space of the vector fields.

Definition 1.11.2: The $\mathcal{H}(z)$ form a *μ-measurable field of complex Hilbert spaces* if there exists a subspace W of V with the following properties:

i) for all $\zeta \in W$, the function $z \mapsto \|\zeta(z)\|$ is μ-measurable;

ii) If $\xi \in V$ is such that for all $\zeta \in W$, the function $z \in Z \mapsto (\zeta(z) \mid \xi(z)) \in \mathbf{C}$ is μ-measurable, then $\xi \in W$.

iii) There exists a sequence $(\xi_1, \xi_2, ...)$ of elements of W such that for all $z \in Z$, the set $\{\xi_n(z) : n = 1, 2, ...\}$ is total in $\mathcal{H}(z)$.

The vector fields belonging to W are called *μ-measurable vector fields.* The family in iii) is called a *fundamental sequence of μ-measurable vector fields.*

Remark: The above definition depends only on the equivalence class of μ.

Proposition 1.11.3: *Let (Z, μ) be as above and $z \mapsto \mathcal{H}(z)$ be a μ-measurable field of Hilbert spaces over Z. The function $z \in Z \mapsto n(z) = dim(\mathcal{H}(z))$ is measurable.*

Proof: [Tak$_1$; Lemma 8.12 p.270]

Let $z \mapsto \mathcal{H}(z)$ be a μ-measurable field of complex Hilbert spaces over Z. A μ-measurable vector field ξ satisfying

$$\|\xi\| = \left\{\int_Z \|\xi(z)\|^2 \, d\mu(z)\right\}^{\frac{1}{2}} < +\infty$$

is said to be *square-integrable*. The set of square-integrable fields is a complex vector space $\mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, $(\xi(z) \mid \eta(z))$ is an integrable function of z and the sesquilinear form

$$(\xi \mid \eta) = \int_Z (\xi(z) \mid \eta(z)) \, d\mu(z)$$

makes $\mathcal{H}$ a pre-Hilbert space. Let us still denote by $\mathcal{H}$ the above pre-Hilbert space in which two vector fields will be identified if they are equal μ-almost everywhere. It is shown in [Di; prop 5i) p.169] that $\mathcal{H}$ is a Hilbert space.

Definition 1.11.4: The Hilbert space $\mathcal{H}$ is called the *direct integral* of measurable fields of Hilbert spaces and is denoted

$$\int_Z^{\oplus} \mathcal{H}(z) \, d\mu(z).$$

Example 1.11 a: Let $\mathcal{H}_0$ be a separable complex Hilbert space. The constant field corresponding to $\mathcal{H}_0$ over Z is the μ-measurable field $z \mapsto \mathcal{H}(z)$ defined by

i) $\mathcal{H}(z) = \mathcal{H}_0$ for all $z \in Z$,

ii) the μ-measurable vector fields are the μ-measurable mappings of Z into $\mathcal{H}_0$.

In this case, the square-integrable vector fields are the square-integrable mappings of Z into $\mathcal{H}_0$ and

$$\int_Z^{\oplus} \mathcal{H}(z)\, d\mu(z) = L^2_{\mathcal{H}_0}(Z,\mu) \cong L^2(Z,\mu) \otimes \mathcal{H}_0.$$

Let Z be a Borel space, μ a σ-finite measure on Z and $z \mapsto \mathcal{H}(z)$ a μ-measurable field of complex Hilbert spaces over Z. Let

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z)\, d\mu(z).$$

For every $z \in Z$, let $T(z) \in \mathcal{L}(\mathcal{H}(z))$.

Definition 1.11.5:

i) The mapping $z \mapsto T(z)$ is called a *measurable field of operators* over Z if for every measurable vector field $z \mapsto \xi(z) \in \mathcal{H}(z)$, the vector field $z \mapsto T(z)\xi(z) \in \mathcal{H}(z)$ is measurable.

ii) A measurable field $z \mapsto T(z) \in \mathcal{L}(\mathcal{H}(z))$ is *essentially bounded* if the essential supremum of the function $z \mapsto \|T(z)\|$ is finite.

Definition 1.11.6: A mapping $T \in \mathcal{L}(\mathcal{H})$ is *decomposable* if it is defined by an essentially bounded measurable field $z \mapsto T(z)$. We then write:

$$T = \int_Z^{\oplus} T(z)\, d\mu(z).$$

Let $L^\infty_{\mathbb{C}}(Z,\mu)$ be the set of essentially bounded measurable complex-valued functions on Z (in which we identify two functions that are equal almost everywhere). If Z is a locally compact space, countable at infinity, let $L_\infty(Z)$ be the set of complex-valued functions on Z which are continuous and vanish at infinity. Consider $L^\infty_{\mathbb{C}}(Z,\mu)$ and $L_\infty(Z)$ with their usual *-algebra and Banach space structures. For $f \in L^\infty_{\mathbb{C}}(Z,\mu)$ or $f \in L_\infty(Z)$, the field of operators

$$z \in Z \mapsto f(z)1 \in \mathcal{L}(\mathcal{H}(z))$$

is measurable and essentially bounded. T_f will denote the operator in $\mathcal{L}(\mathcal{H})$ associated to this field of operators:

$$T_f = \int_Z^{\oplus} f(z)1\, d\mu(z).$$

Definition 1.11.7: The operators of the form T_f, where $f \in L^\infty_{\mathbb{C}}(Z,\mu)$ (resp. $f \in L_\infty(Z)$) are said to be *diagonalisable* (resp. *continuously diagonalisable*).

Remark: The mapping $f \mapsto T_f$ is a homomorphism of the *-algebra $L^\infty_{\mathbb{C}}(Z,\mu)$ (resp. $L_\infty(Z)$) onto the *-algebra $\mathcal{D}$ (resp. $\mathcal{D}^\dagger$) of diagonalisable (resp. continuously diagonalisable) operators.

Let us now recall a result due to von Neumann:

Proposition 1.11.8: *Let Z and Z_1 be two second-countable locally compact spaces, ν a positive measure on Z, ν_1 a positive measure on Z_1 and Ψ an isomorphism of the *-algebra $L^\infty_{\mathbb{C}}(Z,\nu)$ onto the *-algebra $L^\infty_{\mathbb{C}}(Z_1,\nu_1)$. Then, there exists:*

i) *a ν-negligible set A in Z and a ν_1-negligible set A_1 in Z_1;*
ii) *a Borel isomorphism π of $Z \setminus A$ onto $Z_1 \setminus A_1$ which transforms ν into a measure equivalent to ν_1 with $(\Psi(f))(\pi(\zeta)) = f(\zeta)$ for every $f \in L^\infty_{\mathbb{C}}(Z,\nu)$ and $\zeta \in Z \setminus A$.*

Proof: [Di, p. 367]

Theorem 1.11.9: *Let $\mathcal{H}$ be a separable complex Hilbert space, and $\mathcal{D}$ an abelian von Neumann algebra in $\mathcal{H}$. Then, there exist a compact metrizable space Z, a positive measure μ on Z with support Z, a μ-measurable field $z \mapsto \mathcal{H}(z)$ of non-zero complex Hilbert spaces over Z, and an isomorphism of $\mathcal{H}$ onto $\int_Z^\oplus \mathcal{H}(z)\,d\mu(z)$ which transforms $\mathcal{D}$ into the algebra of diagonalisable operators.*

Proof: [Di; Thm 2 p.236]

Theorem 1.11.10: *Let Z be a Borel space, ν a standard positive measure on Z, $E = (\mathcal{H}(z))$ a ν-measurable field of non-zero complex Hilbert spaces over Z, $\mathcal{H} = \int^\oplus \mathcal{H}(z)\,d\nu(z)$, and $\mathcal{D}$ the algebra of diagonalisable operators in $\mathcal{H}$. Define Z_1, ν_1, $E_1 = (\mathcal{H}_1(z_1))$, $\mathcal{H}_1$ and $\mathcal{D}_1$ analogously. Let U be an isomorphism of $\mathcal{H}$ onto $\mathcal{H}_1$ tranforming $\mathcal{D}$ into $\mathcal{D}_1$. Then there exist:*

i) *a ν-negligible Borel set N in Z and a ν_1-negligible Borel set N_1 in Z_1;*
ii) *a Borel isomorphism π of $Z \setminus N$ onto $Z_1 \setminus N_1$ which transforms ν into a measure $\tilde{\nu}_1$ equivalent to ν_1;*
iii) *an isomorphism $(V(z))$ of $E \mid Z \setminus N$ onto $E_1 \mid Z_1 \setminus N_1$, which defines an isomorphism V of $\mathcal{H}$ onto $\tilde{\mathcal{H}}_1 = \int^\oplus \mathcal{H}_1(z_1)\,d\tilde{\nu}_1(z_1)$ in such a way that $U = WV$, where W is the canonical isomorphism of $\tilde{\mathcal{H}}_1$ onto $\mathcal{H}_1$.*

Proof: [Di; Thm 4 p.238]

Chapter II

Tomita-Takesaki Theory

Given a von Neumann algebra $\mathcal{M}$ and a weight ϕ on $\mathcal{M}$, Tomita and Takesaki showed, among numerous results, the existence of a one-parameter group of automorphisms of $\mathcal{M}$, the group of modular automorphisms associated with the left Hilbert algebra $\mathcal{U}_\phi$.

In this chapter, we present Tomita's theorem, the GNS construction for weights, the KMS boundary condition, the Radon-Nikodym theorem and the notion of standard form. These notions will play a significant role in the study of correspondences.

Instead of presenting detailed proofs, examples will be given to illustrate the results and complete references will be provided.

2.1 Left Hilbert algebras

We shall see that every von Neumann algebra $\mathcal{M}$ is isomorphic to the von Neumann algebra of a left Hilbert algebra. Let us now present the notion of left Hilbert algebras, which will lead to Tomita's theorem.

Definition 2.1.1: A *left Hilbert algebra* is an involutive algebra $\mathcal{U}$ over $\mathbf{C}$, where the involution in $\mathcal{U}$ is denoted by $\xi \in \mathcal{U} \mapsto \xi^\sharp \in \mathcal{U}$, equipped with an inner product and satisfying the following conditions:

i) $\eta \in \mathcal{U} \mapsto \xi\eta \in \mathcal{U}$ is a bounded linear operator, for every $\xi \in \mathcal{U}$;

ii) $(\xi\eta \mid \zeta) = (\eta \mid \xi^\sharp \zeta)$, for any $\xi, \eta, \zeta \in \mathcal{U}$;

iii) The involution $\xi \mapsto \xi^\sharp$ is a preclosed antilinear operator;

iv) The subalgebra $\mathcal{U}^2$ spanned linearly by $\xi\eta$ for $\xi, \eta \in \mathcal{U}$ is dense in $\mathcal{U}$.

Similarly, one can define a *right Hilbert algebra* $\mathcal{U}$. The involution is then denoted by $\eta \mapsto \eta^\flat$ and it satisfies $(\xi\eta \mid \zeta) = (\xi \mid \zeta\eta^\flat)$, for any $\xi, \eta, \zeta \in \mathcal{U}$.

Fix a left Hilbert algebra $\mathcal{U}$, and denote by $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathcal{U}$.

Remarks:

- Condition i) implies that to each $\xi \in \mathcal{U}$ one can associate a bounded operator $\pi_l(\xi)$ on $\mathcal{H}$ defined by
$$\pi_l(\xi)\,\eta = \xi\,\eta, \qquad \xi, \eta, \in \mathcal{U}.$$
- Condition ii) implies that
$$\pi_l(\xi^\sharp) = \pi_l(\xi)^*, \qquad \xi \in \mathcal{U}.$$
$\pi_l : \mathcal{U} \to \mathcal{L}(\mathcal{H})$ is therefore a $*$-representation of $\mathcal{U}$.
- Condition iv) implies that $\pi_l(\mathcal{U})$ is non-degenerate on $\mathcal{H}$, that is $[\pi_l(\mathcal{U})\,\mathcal{H}] = \mathcal{H}$.

Example 2.1 a: Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ a cyclic and separating vector. In the set $\mathcal{U} = \mathcal{M}\,\xi_0$, define the following operations:

$$(x\,\xi_0)(y\,\xi_0) = xy\,\xi_0, \qquad (x\,\xi_0)^\sharp = x^*\,\xi_0 \quad x, y \in \mathcal{M},$$

and take as inner product the one inherited from $\mathcal{H}$. Then $\mathcal{U}$ is a left Hilbert algebra and ξ_0 is the unit of $\mathcal{U}$. If $x, y \in \mathcal{M}$, then $\pi_l(x\,\xi_0)(y\,\xi_0) = (x\,\xi_0)(y\,\xi_o) = xy\,\xi_0 = x(y\,\xi_0)$. Thus $\pi_l(\mathcal{M}\,\xi_0) = \mathcal{M}$.

Example 2.1 b: Let G be a locally compact group, equipped with a left Haar measure dg, and let δ_G be the modular function on G. Let $K(G)$ be the linear space of all continuous complex-valued functions on G with compact support. With the following algebraic structure and inner product, $K(G)$ is a left Hilbert algebra:

$$(\xi\eta)(g) = \int_G \xi(h)\eta(h^{-1}g)\,dh,$$
$$\xi^\sharp(g) = \delta_G(g)^{-1}\overline{\xi(g^{-1})},$$
$$(\xi \mid \eta) = \int_G \xi(g)\overline{\eta(g)}\,dg.$$

If we define the involution $\flat$ in $K(G)$ as

$$\xi^\flat(g) = \overline{\xi(g^{-1})},$$

then $K(G)$ is a right Hilbert algebra.

Definition 2.1.2:

i) The von Neumann algebra generated by $\pi_l(\mathcal{U})$ is called the *left von Neumann algebra of* $\mathcal{U}$. It is denoted by $\mathcal{R}_l(\mathcal{U})$.

ii) If $\mathcal{U}$ is a right Hilbert algebra, then to each $\eta \in \mathcal{U}$, one can associate a bounded operator $\pi_r(\eta)$ on $\mathcal{H}$ defined by $\pi_r(\eta)\xi = \xi\eta$ for $\xi, \eta \in \mathcal{U}$. π_r is an anti $*$-representation of $\mathcal{U}$, and the von Neumann algebra generated by $\pi_r(\mathcal{U})$ is called the *right von Neumann algebra of* $\mathcal{U}$. It is denoted by $\mathcal{R}_r(\mathcal{U})$.

Example 2.1 c: Consider the left Hilbert algebra $\mathcal{U} = \mathcal{M}\xi_0$ introduced in Example 2.1 a. The left von Neumann algebra $\mathcal{R}_l(\mathcal{U})$ of $\mathcal{U}$ is the weak closure of $\pi_l(\mathcal{M}\xi_0)$, which is precisely $\mathcal{M}$.

Fix a left Hilbert algebra $\mathcal{U}$ and denote by $\mathcal{H}$ the Hilbert space obtained by completion of $\mathcal{U}$.

The involution $\xi \mapsto \xi^\sharp$ is a conjugate linear preclosed operator. Let S be the closure of this operator. By definition, the involution is its own inverse; therefore, $S = S^{-1}$. Let $D^\sharp$ be the domain of S and $D^\flat$ the domain of the adjoint operator S^* of S, defined by

$$(S\xi \mid \eta) = (S^*\eta \mid \xi), \qquad \xi \in D^\sharp, \eta \in D^\flat.$$

The polar decomposition of S is given by

$$S = J\Delta^{1/2}$$

where $\Delta = S^*S$ is a positive self-adjoint linear operator and J is a conjugate linear isometry. As $S = S^{-1}$, we get:

$$J\Delta J = \Delta^{-1}, \quad S = \Delta^{-1/2}J, \quad S^* = J\Delta^{-1/2} = \Delta^{1/2}J, \quad J^2 = 1, \quad J = J^* = J^{-1}.$$

Definition 2.1.3:

i) The conjugate linear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ is called the ***modular conjugation*** associated to the left Hilbert algebra $\mathcal{U}$.

ii) The positive self-adjoint linear operator Δ is called the ***modular operator*** associated to the left Hilbert algebra $\mathcal{U}$.

Definition 2.1.4: A vector $\eta \in \mathcal{H}$ is ***right bounded*** if

$$\sup\{\| \pi_l(\xi)\eta \| : \xi \in \mathcal{U}, \| \xi \| \leq 1\} = c < +\infty.$$

Let us denote by $\mathcal{B}'$ the set of all right bounded vectors.

Thus a vector $\eta \in \mathcal{H}$ is right bounded if and only if there exists $b \in \mathcal{L}(\mathcal{H})$ such that

$$b\xi = \pi_l(\xi)\eta, \qquad \xi \in \mathcal{U}.$$

This operator b is unique. Denote it by $\pi_r(\eta)$. Note that $\pi_r(\eta) \in \mathcal{R}_l(\mathcal{U})'$ and $\pi_r(\mathcal{B}')$ is a left ideal of $\mathcal{R}_l(\mathcal{U})'$. Define a product in $\mathcal{B}'$ as follows:

$$\eta_1\eta_2 = \pi_r(\eta_2)\eta_1, \qquad \eta_1, \eta_2 \in \mathcal{B}'.$$

Then, $\mathcal{B}'$ is an algebra and $\mathcal{U}' = \mathcal{B}' \cap D^\flat$ is a right Hilbert algebra under the involution $\eta^\flat = S^*\eta$ (for $\eta \in \mathcal{U}'$). Note that for $\xi, \eta, \zeta \in \mathcal{U}'$, we have:

$$\begin{aligned}(\xi \mid \zeta\eta^\flat) &= (\xi \mid \zeta S^*\eta) = (\zeta^\sharp\xi \mid S^*\eta) = (\eta \mid S(\zeta^\sharp\xi)) \\ &= (\eta \mid (\zeta^\sharp\xi)^\sharp) = (\eta \mid \xi^\sharp\zeta) = (\xi\eta \mid \zeta).\end{aligned}$$

We say that $\mathcal{U}'$ is the right Hilbert algebra associated to $\mathcal{U}$.

Theorem 2.1.5: $\mathcal{R}_l(\mathcal{U})' = \mathcal{R}_r(\mathcal{U}')$.

Starting from the right Hilbert algebra $\mathcal{U}' = \mathcal{B}' \cap D^\flat$, we can define the notion of left boundedness for a vector $\xi \in \mathcal{H}$ analogously. $\mathcal{B}$ will denote the set of all left bounded vectors of $\mathcal{H}$.

Definition 2.1.6: Let $\mathcal{U}''$ be the left Hilbert algebra $\mathcal{B} \cap D^\sharp$. If $\mathcal{U} = \mathcal{U}''$, then $\mathcal{U}$ is said to be *full*.

Example 2.1 d: As in Example 2.1 a, let $\mathcal{U} = \mathcal{M}\xi_0$. Then $\mathcal{U} = \mathcal{U}''$. Moreover, $\mathcal{U}' = \mathcal{M}'\xi_0$ and $\mathcal{R}_l(\mathcal{U})' = \mathcal{M}' = \mathcal{R}_r(\mathcal{U}')$.

Theorem 2.1.7 (Tomita): *Let $\mathcal{U}$ be a left Hilbert algebra with modular operator Δ and modular conjugation J.*

i) *If $\mathcal{R}_l(\mathcal{U})$ is the left von Neumann algebra of $\mathcal{U}$, then*

$$\begin{aligned} J\mathcal{R}_l(\mathcal{U})J &= \mathcal{R}_l(\mathcal{U})', & J\mathcal{R}_l(\mathcal{U})'J &= \mathcal{R}_l(\mathcal{U}), \\ \Delta^{it}\mathcal{R}_l(\mathcal{U})\Delta^{-it} &= \mathcal{R}_l(\mathcal{U}), & \Delta^{it}\mathcal{R}_l(\mathcal{U})'\Delta^{-it} &= \mathcal{R}_l(\mathcal{U})', \qquad t \in \mathbf{R}.\end{aligned}$$

ii) *The one parameter unitary group $\{\Delta^{it} : t \in \mathbf{R}\}$ gives rise to a one parameter automorphism group of $\mathcal{U}''$ and $\mathcal{U}'$ respectively.*

iii) *The modular conjugation J gives an anti-isomorphism of $\mathcal{U}$ onto $\mathcal{U}'$.*

iv) *If $a \in \mathcal{R}_l(\mathcal{U}) \cap \mathcal{R}_l(\mathcal{U})'$, then $JaJ = a^*$.*

Proof: [Tak$_2$; Thm 1.13 p.12]

Definition 2.1.8: The one parameter automorphism group $\{\sigma_t\}$ of $\mathcal{R}_l(\mathcal{U})$ defined by

$$\sigma_t(x) = \Delta^{it}x\Delta^{-it}, \qquad x \in \mathcal{R}_l(\mathcal{U}), t \in \mathbf{R},$$

is called the *modular automorphism group* of the left von Neumann algebra of $\mathcal{U}$.

2.2 Weights and representations

Weights are important in the study of von Neumann algebras as they are the equivalent of infinite positive measures. The interesting fact is that while it is not always possible to find a faithful semifinite normal trace on a von Neumann algebra, every such algebra has a faithful semifinite normal weight. In this section, we present the GNS construction for these "unbounded forms".

Definition 2.2.1: A *weight* on a von Neumann algebra $\mathcal{M}$ is a mapping $\phi : \mathcal{M}^+ \to [0, \infty]$ such that

$$\phi(\lambda x + y) = \lambda\phi(x) + \phi(y), \qquad x, y \in \mathcal{M}^+, \ \lambda \in [0, \infty),$$

where $0 \cdot \infty = 0$.

The weight ϕ is

i) *faithful*, if $\phi(x) = 0$ implies $x = 0$;
ii) *normal*, if $\phi(\sup x_i) = \sup \phi(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{M}^+$;
iii) *finite* if $\phi(x) < +\infty$ for all $x \in \mathcal{M}^+$. As $x^*x \leq \|x\|^2 1$, then ϕ is finite if and only if $\phi(1) < +\infty$;
iv) *a trace* if $\phi(x^*x) = \phi(xx^*)$ for all $x \in \mathcal{M}$.

It is to note that the proof of the following lemma relies on the next two inequalities: for $a, x, y \in \mathcal{M}$,

$$(x \pm y)^*(x \pm y) \leq 2(x^*x + y^*y), \qquad (ax)^*(ax) \leq \| a \|^2 x^*x.$$

Lemma 2.2.2: *For a weight ϕ on $\mathcal{M}$, define*

$$\begin{aligned}
\mathfrak{P}_\phi &= \{x \in \mathcal{M}^+ : \phi(x) < +\infty\}, \\
\mathfrak{N}_\phi &= \{x \in \mathcal{M} : \phi(x^*x) < +\infty\}, \\
\mathfrak{L}_\phi &= \{x \in \mathcal{M} : \phi(x^*x) = 0\}, \\
\mathfrak{M}_\phi &= \mathfrak{N}_\phi{}^*\mathfrak{N}_\phi = \{\sum_{i=1}^{n} y_i^* x_i : x_i, y_i \in \mathfrak{N}_\phi, i = 1, ..., n\}.
\end{aligned}$$

i) *$\mathfrak{N}_\phi$ and $\mathfrak{L}_\phi$ are left ideals of $\mathcal{M}$.*
ii) *ϕ can be naturally extended to a linear functional on $\mathfrak{M}_\phi$, that will also be denoted ϕ.*
iii) *$\mathfrak{P}_\phi = \mathfrak{M}_\phi \cap \mathcal{M}^+$ and every element of $\mathfrak{M}_\phi$ can be expressed as a linear combination of four elements of $\mathfrak{P}_\phi$.*

Proof: [Tak$_2$; Lemma 2.2 p.13]

Definition 2.2.3: A weight ϕ on $\mathcal{M}$ is *semifinite* if $\mathfrak{M}_\phi$ is $\sigma(\mathcal{M}, \mathcal{M}_*)$-dense in $\mathcal{M}$.

Remarks:

i) Let ϕ be a normal weight on a von Neumann algebra $\mathcal{M}$. There exists two projections p and q such that $\mathfrak{L}_\phi = \mathcal{M}q$ and $\overline{\mathfrak{N}_\phi} = \mathcal{M}p$ (where $\overline{\mathfrak{N}_\phi}$ denotes the $\sigma(\mathcal{M}, \mathcal{M}_*)$ closure of $\mathfrak{N}_\phi$). Consequently, $\overline{\mathfrak{M}_\phi} = p\mathcal{M}p$ and for $x \in \mathcal{M}^+$,

$$\phi(x) = \phi(pxp) = \phi((p - q)x(p - q)).$$

The projection $p - q$ is called the *support* of ϕ. It will be denoted $\mathrm{s}(\phi)$. Note that ϕ is a faithful semifinite normal weight on the reduced algebra

$$\mathcal{M}_{\mathrm{s}(\phi)} = \{x \in \mathcal{M} : \mathrm{s}(\phi)x = x\mathrm{s}(\phi) = x\}.$$

ii) Any von Neumann algebra $\mathcal{M}$ has a faithful semifinite normal weight. Indeed, if $\{\phi_i\}$ is a maximal family of normal forms on $\mathcal{M}$, whose supports are mutually orthogonal, then

$$\phi(x) = \sum_i \phi_i(x), \quad x \in \mathcal{M}^+$$

is a faithful semifinite normal weight on $\mathcal{M}^+$.

Proposition 2.2.4 (GNS construction for weights): *Let ϕ be a faithful, normal, semifinite weight on $\mathcal{M}$. Let $\mathfrak{P}_\phi$, $\mathfrak{N}_\phi$ and $\mathfrak{M}_\phi$ be the associated subspaces of $\mathcal{M}$, as defined above. Then there exists a triple $(\mathcal{H}_\phi, \pi_\phi, \eta_\phi)$, where*

i) *$\mathcal{H}_\phi$ is a Hilbert space,*

ii) *π_ϕ is a $*$-algebra homomorphism of $\mathcal{M}$ into $\mathcal{L}(\mathcal{H}_\phi)$,*

iii) *$\eta_\phi : \mathfrak{N}_\phi \to \mathcal{H}_\phi$ is a linear map which satisfies*

$$(\eta_\phi(x) \mid \eta_\phi(y)) = \phi(y^* x), \qquad \pi_\phi(z)\eta_\phi(x) = \eta_\phi(zx), \qquad x, y \in \mathfrak{N}_\phi, \quad z \in \mathcal{M}$$

and such that $\eta_\phi(\mathfrak{N}_\phi)$ is dense in $\mathcal{H}_\phi$.

The triple is unique, i.e. if $(\mathcal{H}', \pi', \eta')$ is another such triple, there exists a unique unitary operator $U : \mathcal{H}_\phi \to \mathcal{H}'$ such that $U\eta_\phi(x) = \eta'(x)$ for all $x \in \mathfrak{N}_\phi$ and $\pi'(z) = U\pi_\phi(z)U^*$ for every $z \in \mathcal{M}$. Moreover, π_ϕ is isometric and is a ultra-weak homeomorphism of $\mathcal{M}$ onto $\pi_\phi(\mathcal{M})$.

Proof: [Su; Prop. 2.4.10 p.58]

Example 2.2 a: Let $(X, \mathcal{F}, \mu)$ be a separable σ-finite measure space and $\mathcal{M} = L^\infty(X, \mathcal{F}, \mu)$. The equation $\phi(f) = \int f \, d\nu$, where ν is a σ-finite measure absolutely continuous with respect to μ, defines a normal weight ϕ on $\mathcal{M}$. Note that ϕ is a trace as $\mathcal{M}$ is abelian. We have

$$\mathfrak{N}_\phi = L^\infty(X, \mu) \cap L^2(X, \nu), \qquad \mathcal{H}_\phi = \overline{\mathfrak{N}_\phi}^{\|\cdot\|_\phi}, \qquad \pi_\phi(f) = m_f,$$

where $(m_f\xi)(s) := f(s)\xi(s)$ for all $\xi \in \mathcal{H}_\phi$. Moreover, $\eta_\phi(f) = f$ for all $f \in \mathfrak{N}_\phi$.

Remark: Let $(X, \mathcal{F}, \mu)$ be a separable σ-finite measure space and $\mathcal{M} = L^\infty(X, \mathcal{F}, \mu)$. The σ-finite measures which are absolutely continuous with respect to μ are in bijection with the semifinite normal weights on $\mathcal{M}$.

Indeed, every positive σ-finite measure ν on $(X, \mathcal{F})$ which is absolutely continuous with respect to μ, defines a normal semifinite weight ψ_ν on $\mathcal{M}$ by the equation $\psi_\nu(f) = \int f \, d\nu$, where $f \in \mathcal{M}^+$.

Conversely, if ψ is a normal semifinite weight on $\mathcal{M}$, then the equation $\nu(E) = \psi(\chi_E)$ defines a countably additive σ-finite measure ν on $(X, \mathcal{F})$. Moreover, $\psi = \psi_\nu$.

The normal semifinite weight ψ_ν on $\mathcal{M}$ is faithful when ν is equivalent to μ.

Example 2.2 b: Let $\mathcal{M} = \mathrm{M}_2(\mathbf{C})$, equipped with the following linear form:

$$\phi(\cdot) = \tau(h\cdot), \quad \text{where} \quad h = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 > 0.$$

As $\lambda_1, \lambda_2 > 0$, then ϕ is faithful. The GNS triple is given by: $\mathcal{H}_\phi \cong \mathbf{C}^4$, the mapping $\eta_\phi : \mathrm{M}_2(\mathbf{C}) \to \mathcal{H}_\phi$, defined by:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{\lambda_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv f_1, \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \sqrt{\lambda_2} \\ 0 \\ 0 \end{pmatrix} \equiv f_2,$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ \sqrt{\lambda_1} \\ 0 \end{pmatrix} \equiv f_3, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\lambda_2} \end{pmatrix} \equiv f_4,$$

and the representation $\pi_\phi : \mathrm{M}_2(\mathbf{C}) \to \mathcal{L}(\mathbf{C}^4)$ defined by

$$\pi_\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

with respect to the basis $\{f_1, f_2, f_3, f_4\}$. Consider the mapping $S : \mathcal{H}_\phi \to \mathcal{H}_\phi^{o}$ defined by:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto x^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}, \qquad S(\lambda x) = \overline{\lambda} S(x), \qquad \text{for all } x \in \mathcal{H}_\phi,\ \lambda \in \mathbf{C}.$$

The matrix representations of S and S^* are

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda_1}{\lambda_2} & 0 \\ 0 & \frac{\lambda_2}{\lambda_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have:

$$\Delta_\phi = S^* S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda_1}{\lambda_2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{\lambda_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and thus} \qquad \Delta_\phi^{it} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{\lambda_1}{\lambda_2}\right)^{it} & 0 & 0 \\ 0 & 0 & \left(\frac{\lambda_2}{\lambda_1}\right)^{it} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover

$$J_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\lambda_1}{\lambda_2}} & 0 \\ 0 & \sqrt{\frac{\lambda_2}{\lambda_1}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, we obtain

$$\sigma_t^\phi(e_{kl}) = \pi_\phi^{-1}(\Delta_\phi^{it}\pi_\phi(e_{kl})\Delta_\phi^{-it}) = \left(\frac{\lambda_k}{\lambda_l}\right)^{it} e_{kl}.$$

Example 2.2 c: Let $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and $\phi(x) = tr(\rho x)$ where ρ is a positive trace-class operator [section 1.3], with canonical decomposition given by:

$$\rho = \sum \alpha_n t_{\xi_n, \xi_n},$$

where $\alpha_n > 0$, $\sum \alpha_n < \infty$ and $\{\xi_n\}$ is an orthonormal basis for $\mathcal{H}$, and

$$tr(\rho x) = \sum \alpha_n \, (x\xi_n \mid \xi_n).$$

The GNS triple is given by:

$$\mathcal{H}_\phi = l^2(\mathbf{N}, \mathcal{H}) = \{\bar{\xi} : \mathbf{N} \to \mathcal{H} : \sum_n \|\bar{\xi}(n)\|^2 < \infty\},$$

$$(\pi_\phi(x)\bar{\xi})(n) = x\bar{\xi}(n), \qquad \Omega_\phi(n) = \alpha_n^{\frac{1}{2}}\xi_n.$$

An orthonormal basis for $\mathcal{H}_\phi$ is given by

$$\{\bar{\xi}_n^{(m)} : n, m \in \mathbf{N}\}, \text{ where } \bar{\xi}_n^{(m)}(k) = \delta_m(k)\xi_n.$$

For $n, m \in \mathbf{N}$, define

$$x_n^{(m)} = \alpha_m^{-\frac{1}{2}} t_{\xi_n, \xi_m};$$

then,

$$\pi_\phi(x_n^{(m)})\Omega_\phi = \bar{\xi}_n^{(m)}, \qquad \pi_\phi((x_n^{(m)})^*)\Omega_\phi = \left(\frac{\alpha_n}{\alpha_m}\right)^{\frac{1}{2}} \bar{\xi}_m^{(n)}.$$

Hence

$$S\bar{\xi}_n^{(m)} = \left(\frac{\alpha_n}{\alpha_m}\right)^{\frac{1}{2}} \bar{\xi}_m^{(n)}, \qquad S^*\bar{\xi}_n^{(m)} = \left(\frac{\alpha_m}{\alpha_n}\right)^{\frac{1}{2}} \bar{\xi}_m^{(n)}.$$

As $\Delta = S^* S$ we get:

$$(\Delta\tilde{\xi})(k) = (\alpha_k)^{-1}\rho\tilde{\xi}(k)$$

for all $\tilde{\xi}$ in the linear span of $\{\tilde{\xi}_n^{(m)} : n, m \in \mathbf{N}\}$. Finally, from the equality $S = J\Delta^{\frac{1}{2}}$ we get

$$J\tilde{\xi}_n^{(m)} = \tilde{\xi}_m^{(n)}, \quad \text{for all } n, m \in \mathbf{N}$$

and

$$\sigma_t^{\varphi}(x) = \rho^{it}\, x\, \rho^{-it}, \quad \text{for all } x \in \mathcal{M},\ t \in \mathbf{R}.$$

Example 2.2 d: Let G be a countable discrete group. Let $\mathcal{M} = vN(G)$, the group von Neumann algebra of G, as described in Example 1.1 c. As seen previously, $\Omega = \xi_e$ is a cyclic and separating vector for $\mathcal{M}$, where e denotes the identity in G. Hence we can view $\mathcal{M}\Omega$ as a left Hilbert algebra [Ex. 2.1 a]. We have:

$$(S\xi)(t) = \overline{\xi(t^{-1})}, \text{ for all } t \in G, \xi \in l^2(G), \quad J = S^* = S \quad \text{and} \quad \Delta = 1.$$

Moreover, $J\lambda_t J = \rho_t$; therefore, $\mathcal{M}' = J\mathcal{M}J = \{\rho_t : t \in G\}''$, that is, the commutant of the left-regular representation is the von Neumann algebra generated by the right-regular representation.

2.3 Left Hilbert algebras and weights

In this section, the notation introduced in the preceding sections will be used.

Theorem 2.3.1:

i) *If $\mathcal{U}$ is a full left Hilbert algebra with $\mathcal{M} = \mathcal{R}_l(\mathcal{U})$, the function ϕ on $\mathcal{M}^+$ defined by*

$$\phi(x) = \begin{cases} \|\xi\|^2 & \text{if } x^{1/2} = \pi_l(\xi),\ \xi \in \mathcal{U}, \\ +\infty & \text{otherwise,} \end{cases}$$

is a faithful semifinite normal weight on $\mathcal{M}$ such that

$$\mathfrak{N}_\phi = \pi_l(\mathcal{B}) \subset \mathcal{M}, \qquad \mathfrak{M}_\phi = \mathfrak{N}_\phi^* \mathfrak{N}_\phi = \{\sum_{i=1}^{k} x_i^* y_i : x_i, y_i \in \mathfrak{N}_\phi, i = 1, \dots, k\},$$

$$\phi(\pi_l(\eta)^* \pi_l(\xi)) = (\xi \mid \eta), \qquad \xi, \eta \in \mathcal{B}.$$

Note that $\pi_l(\mathcal{U}) = \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$.

ii) *Conversely, if ϕ is a faithful semifinite normal weight on a von Neumann algebra $\mathcal{M}$, then the subspace*

$$\mathcal{U}_\phi = \eta_\phi(\mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*)$$

of $\mathcal{H}_\phi$ becomes a full Hilbert algebra with respect to the following algebraic structure:

$$\eta_\phi(x)\eta_\phi(y) = \eta_\phi(xy), \qquad \eta_\phi(x)^\sharp = \eta_\phi(x^*), \qquad x, y \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*.$$

Moreover, the left von Neumann algebra $\mathcal{R}_l(\mathfrak{U}_\phi)$ is the image $\pi_\phi(\mathcal{M})$ of $\mathcal{M}$ under the GNS representation, and the weight corresponding to $\mathfrak{U}_\phi$ in i) is given by $\phi \circ \pi_\phi^{-1}$.

Proof: [Tak$_2$; Thm 3.1 p.15]

Consequently, any von Neumann algebra is *-isomorphic to a von Neumann algebra of the form $\mathcal{R}_l(\mathfrak{U})$, where $\mathfrak{U}$ is a left Hilbert algebra.

2.4 Weights and modular automorphism groups

Let ϕ be a faithful semifinite normal weight on a von Neumann algebra $\mathcal{M}$ and $\mathcal{R}_l(\mathfrak{U})$ the left von Neumann algebra of $\mathfrak{U}$ given by Theorem 2.3.1. Let Δ and J be the modular operator and the modular conjugation associated to $\mathfrak{U}$, and $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathfrak{U}$.

The following lemma is an important tool in the proof of the next theorem:

Lemma 2.4.1: *If $\xi, \eta \in D(\Delta^{1/2})(= D^\sharp)$, the function $f(z) = \langle \Delta^z \xi, \eta \rangle$ is defined, bounded and continuous on the strip $\overline{\mathbf{D}} = \{z \in \mathbf{C} : 0 \leq Re(z) \leq \frac{1}{2}\}$, and is analytic on the interior $\mathbf{D}$ of $\overline{\mathbf{D}}$.*

Proof: Note that by polarization, it is enough to prove the result for $\xi = \eta$. Let E be the spectral resolution of the operator Δ. Then, using the spectral theory for unbounded operators, we get

$$\int_{[0,\infty]} d\langle E_\lambda \xi, \xi \rangle = \|\xi\|^2 < \infty \quad \text{and} \quad \int_{[0,\infty]} \lambda \, d\langle E_\lambda \xi, \xi \rangle = \|\Delta^{\frac{1}{2}} \xi\|^2 < \infty.$$

For $z = x + iy \in \overline{\mathbf{D}}$ and $\lambda > 0$, we have:

$$|\lambda^z|^2 = \lambda^{2x} \leq \max(1, \lambda) < 1 + \lambda$$

as $0 \leq x \leq \frac{1}{2}$. Thus

$$\begin{aligned}\int_{[0,\infty]} |\lambda^z|^2 \, d\langle E_\lambda \xi, \xi \rangle &\leq \int_{[0,\infty]} (1 + \lambda) \, d\langle E_\lambda \xi, \xi \rangle \\ &\leq \|\xi\|^2 + \|\Delta^{\frac{1}{2}} \xi\|^2 < \infty.\end{aligned}$$

We observe that $\xi \in D(\Delta^z)$ and $\|\Delta^z \xi\|$ is bounded for $z \in \overline{\mathbf{D}}$. Hence the function $f(z) = \langle \Delta^z \xi, \xi \rangle$ is defined and bounded on $\overline{\mathbf{D}}$ and

$$f(z) = \int_{[0,\infty]} \lambda^z \, d\langle E_\lambda \xi, \xi \rangle .$$

Define a complex-valued function f_n (for $n \in \mathbf{N}$) on $\mathbf{C}$ by

$$f_n(z) = \int_{[\frac{1}{n},n]} \lambda^z \, d\langle E_\lambda \xi, \xi \rangle .$$

Note that the power series expansion of $\lambda^z = \exp(z\mathrm{Log}\lambda)$ in z converges uniformly for $\lambda \in [\frac{1}{n}, n]$. Hence for all n, the function f_n has a power series expansion and thus is entire. In addition, $|\lambda^z| = \lambda^x \leq 1 + \lambda$ for $z = x + iy \in \overline{\mathbf{D}}$ and

$$\begin{aligned} |f(z) - f_n(z)| &= \left| \int_{[0,\frac{1}{n}]\cup[n,\infty]} \lambda^z \, d\langle E_\lambda \xi, \xi \rangle \right| \\ &\leq \int_{[0,\frac{1}{n}]\cup[n,\infty]} (1+\lambda) \, d\langle E_\lambda \xi, \xi \rangle . \end{aligned}$$

The last term of the inequality is independent of z and converges to zero as $n \to \infty$. Hence $f_n(z) \to f(z)$ uniformly on $\overline{\mathbf{D}}$. Finally, as each f_n is analytic on $\mathbf{C}$, it follows that f is continuous on $\overline{\mathbf{D}}$ and analytic on $\mathbf{D}$.

Q.E.D.

Theorem 2.4.2: *With the notation introduced in the beginning of the section, let*

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it}, \qquad t \in \mathbf{R}, x \in \mathcal{M}$$

be the modular automorphism group of the von Neumann algebra $\mathcal{M}$ *[Theorem 2.1.7]. Then*

i) $\phi \circ \sigma_t = \phi$, $t \in \mathbf{R}$.

ii) *For every pair* $x, y \in \mathfrak{N}_\phi \cap \mathfrak{N}_\phi^*$, *there exists a bounded continuous function*

$$F = F_{x,y} : \overline{\mathbf{D}} = \{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq 1\} \to \mathbf{C}$$

holomorphic on the interior $\mathbf{D}$ *of* $\overline{\mathbf{D}}$ *and satisfying*

$$F(it) = \phi(x\sigma_t(y)), \qquad F(1+it) = \phi(\sigma_t(y)x), \qquad t \in \mathbf{R}.$$

The one parameter automorphism group $\{\sigma_t\}$ *is uniquely determined by* ϕ *subject to conditions* i) *and* ii).

Proof: [Tak$_2$; Thm 4.1 p.16] / [S-Z; 10.17 p.288]

Definiton 2.4.3: The automorphism group $\{\sigma_t\}$ determined by conditons i) and ii) in the above theorem is called the ***modular automorphism group associated with*** ϕ. It will be denoted by $\{\sigma_t^\phi\}$.

Conditions i) and ii) together are called the *modular condition* for ϕ and $\{\sigma_t\}$. We also say that the weight ϕ satisfies the *KMS boundary condition* (Kubo-Martin-Schwinger) with respect to $\{\sigma_t\}$

2.5 Generalized KMS boundary condition

To define the identity correspondence, we will need a generalized version of the KMS boundary condition. Before presenting this new version of the KMS condition, let us first recall the Three line theorem and Hartogs' theorem.

Theorem 2.5.1 [Three line theorem]: *Let F be a bounded analytic function of the complex variables $z_1, \dots, z_n$, where $z_j = x_j + iy_j$, which is defined for $x = (x_1, \dots, x_n)$ lying in a convex set $D \subset \mathbf{R}^n$ and $y = (y_1, \dots, y_n)$ unrestricted. Suppose F has values in a complex Banach space B, and let M be defined on D by*

$$M(x) = \sup\{|F(x+iy)| : -\infty < y_j < \infty,\ j = 1, \dots, n\}.$$

Then $Log\, M(x)$ is a convex function of the vector variable x in D.

Proof: [D-S; p.521]

Let D be an open set in $\mathbf{C}^n$ and $w_1, \dots, w_n \in \mathbf{C}$. Let

$$D_{j,w} = \{z \in \mathbf{C} : (w_1, \dots, w_{j-1}, z, w_{j+1}, \dots, w_n) \in D\}.$$

For any function f on D, let us denote by $f_{j,w}$ the function on $D_{j,w}$ defined by

$$f_{j,w}(z) = f(w_1, \dots, w_{j-i}, z, w_{j+1}, \dots, w_n).$$

Theorem 2.5.2 [Hartogs]: *Let f be a function defined on D such that for any j, $1 \leq j \leq n$ and $w_1, \dots, w_n \in \mathbf{C}$, the function $f_{j,w}$ is analytic on $D_{j,w}$. Then f is analytic on D.*

Proof: [N; p.43]

Let $\mathcal{M}$ be a von Neumann algebra and ϕ a faithful normal positive linear functional on $\mathcal{M}$. As seen in sections 1.5 and 2.1, associated to $\{\mathcal{M}, \phi\}$ are $\{\pi_\phi, \mathcal{H}_\phi, \eta_\phi, \Omega_\phi, \Delta_\phi, J_\phi, \mathcal{U}_\phi\}$. Recall that the vector $\Omega_\phi \in \mathcal{H}_\phi$, which is cyclic and separating for $\pi_\phi(\mathcal{M})$, satisfies

$$\phi(x) = (x\,\Omega_\phi \mid \Omega_\phi) \qquad \text{for all } x \in \mathcal{M}.$$

(Note that we have identified $\mathcal{M}$ and $\pi_\phi(\mathcal{M})$). Moreover, $\Delta_\phi\, \Omega_\phi = \Omega_\phi$. Indeed, as

$$S_\phi : \overline{\mathcal{M}\Omega_\phi} \rightarrow \overline{\mathcal{M}\Omega_\phi} \qquad x\,\Omega_\phi \mapsto x^*\,\Omega_\phi$$

and

$$S_\phi^* : \overline{\mathcal{M}'\Omega_\phi} \to \overline{\mathcal{M}'\Omega_\phi} \qquad x\,\Omega_\phi \mapsto x^*\,\Omega_\phi.$$

we get $\Delta_\phi\,\Omega_\phi = S_\phi^* S_\phi \Omega_\phi = \Omega_\phi$. Hence $\Delta_\phi^{it}\,\Omega_\phi = \Omega_\phi$ for all $t \in \mathbf{R}$.
Define

$$D_n = \{(x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1} : x_i > 0 \text{ and } \sum x_i < 1\}$$

$$C_n = \{(z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1} : (Re(z_1), \ldots, Re(z_{n-1})) \in D_n\}.$$

We are now ready to state the generalized version of the KMS condition:

Proposition 2.5.3: *Let $a_1, \ldots, a_n \in \mathcal{M}$. There exists a function $F : \overline{C_n} \to \mathbf{C}$ continuous and bounded in $\overline{C_n}$ and analytic in C_n such that*

$$F(it_1, it_2, \ldots, it_{n-1}) = \phi(a_1 \sigma_{t_1}^\phi(a_2) \ldots \sigma_{t_1+\ldots+t_{n-1}}^\phi(a_n)).$$

Remark: If $n = 2$, then $D_2 = (0,1)$ and $C_2 = \{z \in \mathbf{C} : 0 < Re(z) < 1\}$. Consequently, the above proposition corresponds to Theorem 2.4.2.

Let us now consider the case $n = 3$; then

$$D_3 = \{(x_1, x_2) \in \mathbf{R}^2 : x_i > 0 \text{ and } x_1 + x_2 < 1\}$$

$$C_3 = \{(z_1, z_2) \in \mathbf{C}^2 : Re(z_i) > 0 \text{ and } Re(z_1) + Re(z_2) < 1\}.$$

Consider the function

$$f : B \subset \overline{C_3} \to \mathbf{C} \qquad (z, w) \mapsto (\Delta_\phi^z\, a_2\, \Delta_\phi^w\, a_3\, \Omega_\phi \mid a_1^*\, \Omega_\phi)$$

where $a_i \in \mathcal{M}$ for all i and $B = \{(z, w) \in \overline{C_3} : 0 \leq Re(z), Re(w) \leq \frac{1}{2}\}$.

i) Let us fix $s \in \mathbf{R}$. Then

$$z \mapsto (\Delta_\phi^z\, a_2\, \Delta_\phi^{is}\, a_3\, \Omega_\phi \mid a_1^*\, \Omega_\phi)$$

is continuous and bounded in

$$\{z \in \mathbf{C} : 0 \leq Re(z) \leq \tfrac{1}{2}\},$$

and analytic in $\{z \in \mathbf{C} : 0 < Re(z) < \frac{1}{2}\}$. Indeed,

$$a_2\, \Delta_\phi^{is}\, a_3\, \Omega_\phi = a_2\, \Delta_\phi^{is}\, a_3\, \Delta_\phi^{-is}\, \Delta_\phi^{is}\, \Omega_\phi = a_2\, \sigma_s^\phi(a_3)\, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$$

and $a_1^*\, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$. Hence, by Lemma 2.4.1, we have the desired result. Note that for $z = it$, we have
(*)

$$(\Delta_\phi^{it}\, a_2\, \Delta_\phi^{is}\, a_3\, \Omega_\phi \mid a_1^*\, \Omega_\phi) = (\sigma_t^\phi(a_2)\, \sigma_{t+s}^\phi(a_3)\, \Omega_\phi \mid a_1^*\, \Omega_\phi) = \phi(a_1\, \sigma_t^\phi(a_2)\, \sigma_{t+s}^\phi(a_3)).$$

For $s \in \mathbf{R}$ fixed, the map

$$w \longmapsto (\Delta_\phi^w \, a_3 \, \Omega_\phi \mid a_2^* \, \Delta_\phi^{-is} \, a_1^* \, \Omega_\phi)$$

is continuous and bounded in

$$\{w \in \mathbf{C} : 0 \leq Re(w) \leq \tfrac{1}{2}\},$$

and analytic in $\{w \in \mathbf{C} : 0 < Re(w) < \frac{1}{2}\}$. Indeed,

$$a_2^* \, \Delta_\phi^{-is} \, a_1^* \, \Omega_\phi = a_2^* \, \Delta_\phi^{-is} \, a_1^* \, \Delta_\phi^{is} \, \Delta_\phi^{-is} \, \Omega_\phi = a_2^* \, \sigma_{-s}^\phi(a_1^*) \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$$

and $a_3 \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$. Hence, by Lemma 2.4.1, we have the desired result.

ii) Now, for $z \in \mathbf{C}$ fixed, with $0 \leq Re(z) \leq \frac{1}{2}$, the map

$$w \longmapsto (\Delta_\phi^w \, a_3 \, \Omega_\phi \mid a_2^* \Delta_\phi^{\bar{z}} \, a_1^* \, \Omega_\phi)$$

is continuous and bounded in

$$\{w \in \mathbf{C} : 0 \leq Re(w) \leq \tfrac{1}{2}\}$$

and analytic in $\{w \in \mathbf{C} : 0 < Re(w) < \frac{1}{2}\}$. Indeed, as $a_1^* \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$, then $a_1^* \, \Omega_\phi \in D(\Delta_\phi^{\bar{z}})$, by Proposition A.1.11. Moreover, $a_3 \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$. Hence by Lemma 2.4.1, we have the desired result. For $w \in \mathbf{C}$ fixed, with $0 \leq Re(w) \leq \frac{1}{2}$, the map

$$z \longmapsto (\Delta_\phi^z \, a_2 \, \Delta_\phi^w \, a_3 \, \Omega_\phi \mid a_1^* \, \Omega_\phi)$$

is continuous and bounded in

$$\{z \in \mathbf{C} : 0 \leq Re(z) \leq \tfrac{1}{2}\}$$

and analytic in $\{z \in \mathbf{C} : 0 < Re(z) < \frac{1}{2}\}$. Indeed as $a_3 \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$ then by Proposition A.1.11, $a_3 \, \Omega_\phi \in D(\Delta_\phi^w)$. Moreover, $a_1^* \, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}})$. Thus by Lemma 2.4.1 again, we get the result.

iii) Note that for $z = \frac{1}{2} + it$ we have:

$$\begin{aligned}
(\Delta_\phi^{\frac{1}{2}+it} \, a_2 \, \Delta_\phi^w \, a_3 \, \Omega_\phi \mid a_1^* \, \Omega_\phi) &= (\Delta_\phi^{\frac{1}{2}+it} \, a_2 \, \Delta_\phi^w \, a_3 \, \Omega_\phi \mid S_\phi(a_1 \, \Omega_\phi)) \\
&= (\Delta_\phi^{\frac{1}{2}+it} \, a_2 \, \Delta_\phi^w \, a_3 \, \Omega_\phi \mid \Delta_\phi^{-\frac{1}{2}} \, J_\phi \, a_1 \, \Omega_\phi) \\
&= (\Delta_\phi^{it} \, a_2 \, \Delta_\phi^w \, a_3 \, \Omega_\phi \mid J_\phi \, a_1 \, \Omega_\phi).
\end{aligned}$$

For $(z,w) \in B_1 = \{(z,w) \in \overline{C_3} : \frac{1}{2} \leq Re(z) \leq 1,\ 0 \leq Re(w) \leq \frac{1}{2}\}$, put

$$f_1(z,w) = (a_2\, \Delta_\phi^w\, a_3\, \Omega_\phi \mid J_\phi\, \Delta_\phi^{z-\frac{1}{2}}\, a_1\, \Omega_\phi);$$

then $f_1(\frac{1}{2}+it, w) = f(\frac{1}{2}+it, w)$. For fixed w, the function

$$z \longmapsto f_1(z,w)$$

is continuous in $\{z \in \mathbf{C} : \frac{1}{2} \leq Re(z) \leq 1\}$ and analytic in $\{z \in \mathbf{C} : \frac{1}{2} < Re(z) < 1\}$. Moreover, for $(z,w) \in B_1$, the map f_1 is bounded. Indeed, for any pair $(z,w) \in B_1$, with $z = x + it$ and $w = u + is$, we have:

$$\begin{aligned} f_1(z,w) &= (a_2\, \Delta_\phi^u\, \Delta_\phi^{is}\, a_3\, \Omega_\phi \mid J_\phi\, \Delta_\phi^{x-\frac{1}{2}}\, \Delta_\phi^{it}\, a_1\, \Omega_\phi) \\ &= (a_2\, \Delta_\phi^u \sigma_s^\phi(a_3)\, \Omega_\phi \mid J_\phi\, \Delta_\phi^{x-\frac{1}{2}}\, \sigma_t^\phi(a_1)\, \Omega_\phi). \end{aligned}$$

Thus, by Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} &|f_1(z,w)|^2 \leq \|a_2\|^2 \|\Delta_\phi^u\, \sigma_s^\phi(a_3)\, \Omega_\phi\|^2 \|\Delta_\phi^{x-\frac{1}{2}}\, \sigma_t^\phi(a_1)\, \Omega_\phi\|^2 \\ &\leq \|a_2\|^2 (\|\sigma_s^\phi(a_3)\, \Omega_\phi\|^2 + \|\Delta_\phi^{\frac{1}{2}}\, \sigma_s^\phi(a_3)\, \Omega_\phi\|^2)(\|\sigma_t^\phi(a_1)\, \Omega_\phi\|^2 + \|\Delta_\phi^{\frac{1}{2}}\, \sigma_t^\phi(a_1)\, \Omega_\phi\|^2) \\ &< \infty \end{aligned}$$

since we have:

$$\sigma_s^\phi(a_3)\, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}}), \quad \sigma_t^\phi(a_1)\, \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}}), \quad 0 \leq u \leq \tfrac{1}{2} \text{ and } 0 \leq x - \tfrac{1}{2} \leq \tfrac{1}{2}.$$

[Proof of Lemma 2.4.1].

iv) Finally, note that for $w = \frac{1}{2} + it$, we have:

$$\begin{aligned} (\Delta_\phi^{\frac{1}{2}+it}\, a_3\, \Omega_\phi \mid a_2^*\, \Delta_\phi^{\bar{z}}\, a_1^*\, \Omega_\phi) &= (\Delta_\phi^{\frac{1}{2}+it}\, S_\phi(a_3^*\, \Omega_\phi) \mid a_2^*\, \Delta_\phi^{\bar{z}}\, a_1^*\, \Omega_\phi) \\ &= (\Delta_\phi^{\frac{1}{2}+it}\, \Delta_\phi^{-\frac{1}{2}}\, J_\phi\, a_3^*\, \Omega_\phi \mid a_2^*\, \Delta_\phi^{\bar{z}}\, a_1^*\, \Omega_\phi) \\ &= (\Delta_\phi^{it}\, J_\phi\, a_3^*\, \Omega_\phi \mid a_2^*\, \Delta_\phi^{\bar{z}}\, a_1^*\, \Omega_\phi). \end{aligned}$$

For $(z,w) \in B_2 = \{(z,w) \in \overline{C_3} : 0 \leq Re(z) \leq \frac{1}{2},\ \frac{1}{2} \leq Re(w) \leq 1\}$, put

$$f_2(z,w) = (J_\phi\, \Delta_\phi^{w-\frac{1}{2}}\, a_3^*\, \Omega_\phi \mid a_2^*\, \Delta_\phi^{\bar{z}}\, a_1^*\, \Omega_\phi);$$

then $f_2(z, \frac{1}{2}+it) = f(z, \frac{1}{2}+it)$. For fixed z, the function

$$w \longmapsto f_2(z,w)$$

is continuous in $\{w \in \mathbf{C} : \frac{1}{2} \leq Re(w) \leq 1\}$, analytic in $\{w \in \mathbf{C} : \frac{1}{2} < Re(w) < 1\}$. Moreover, for $(z.w) \in B_2$, the map f_2 is bounded. Indeed, for any pair $(z, w) \in B_2$ with $z = x + it$ and $w = u + is$, we have:

$$\begin{aligned} f_2(z,w) &= (J_\phi \Delta_\phi^{u-\frac{1}{2}} \Delta_\phi^{is} a_3^* \Omega_\phi \mid a_2^* \Delta_\phi^{x} \Delta_\phi^{-it} a_1^* \Omega_\phi) \\ &= (J_\phi \Delta_\phi^{u-\frac{1}{2}} \sigma_s^\phi(a_3^*) \Omega_\phi \mid a_2^* \Delta_\phi^{x} \sigma_{-t}^\phi(a_1^*) \Omega_\phi). \end{aligned}$$

Thus, by Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} &|f_2(z,w)|^2 \leq \|\Delta_\phi^{u-\frac{1}{2}} \sigma_s^\phi(a_3^*) \Omega_\phi\|^2 \|a_2^*\|^2 \|\Delta_\phi^{x} \sigma_{-t}^\phi(a_1^*) \Omega_\phi\|^2 \\ &\leq (\|\sigma_s^\phi(a_3^*) \Omega_\phi\|^2 + \|\Delta_\phi^{\frac{1}{2}} \sigma_s^\phi(a_3^*) \Omega_\phi\|^2) \|a_2^*\|^2 (\|\sigma_{-t}^\phi(a_1^*) \Omega_\phi\|^2 + \|\Delta_\phi^{\frac{1}{2}} \sigma_{-t}^\phi(a_1^*) \Omega_\phi\|^2) \\ &< \infty \end{aligned}$$

since we have

$$\sigma_s^\phi(a_3^*)\Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}}), \quad \sigma_{-t}^\phi(a_1^*) \Omega_\phi \in D(\Delta_\phi^{\frac{1}{2}}), \quad 0 \leq u - \tfrac{1}{2} \leq \tfrac{1}{2} \text{ and } 0 \leq s \leq \tfrac{1}{2}.$$

[Proof of Lemma 2.4.1].

v) Consequently, the function $F : \overline{C_3} \to \mathbf{C}$ defined by:

$$F(z,w) = \begin{cases} f(z,w), & (z,w) \in B = \{(z,w) \in \overline{C_3} : 0 \leq Re(z) \leq \frac{1}{2}, 0 \leq Re(w) \leq \frac{1}{2}\} \\ f_1(z,w), & (z,w) \in B_1 = \{(z,w) \in \overline{C_3} : \frac{1}{2} \leq Re(z) \leq 1, 0 \leq Re(w) \leq \frac{1}{2}\} \\ f_2(z,w), & (z,w) \in B_2 = \{(z,w) \in \overline{C_3} : 0 \leq Re(z) \leq \frac{1}{2}, \frac{1}{2} \leq Re(w) \leq 1\} \end{cases}$$

is bounded and continuous in $\overline{C_3}$ and analytic in C_3 by Hartog's Theorem. Moreover

$$F(it_1, it_2) = (\Delta_\phi^{it_1} a_2 \Delta_\phi^{it_2} a_3 \Omega_\phi \mid a_1^* \Omega_\phi) = \phi(a_1 \sigma_{t_1}^\phi(a_2) \sigma_{t_1+t_2}^\phi(a_3)).$$

For $n = k$, one would proceed in the same way.

2.6 The Tomita algebra

Let $\mathcal{U} \subset \mathcal{H}$ be a left Hilbert algebra. Consider the vector space

$$\mathcal{T} = \left\{ \xi \in \bigcap_{\alpha \in \mathbf{C}} D_{(\Delta^\alpha)} \;\middle|\; \begin{array}{c} \text{for all } \alpha \in \mathbf{C}, \Delta^\alpha \xi \in \mathcal{U}'' \cap \mathcal{U}', \\ D_{(\Delta^\alpha \pi_l(\xi) \Delta^{-\alpha})} = D_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha \pi_l(\xi) \Delta^{-\alpha} \subset \pi_l(\Delta^\alpha \xi), \\ D_{(\Delta^\alpha \pi_r(\xi) \Delta^{-\alpha})} = D_{(\Delta^{-\alpha})} \text{ and } \Delta^\alpha \pi_r(\xi) \Delta^{-\alpha} \subset \pi_r(\Delta^\alpha \xi). \end{array} \right\}$$

Theorem 2.6.1: *$\mathcal{T}$ is a left Hilbert subalgebra of $\mathfrak{U}''$ and*

$$\mathcal{T}' = \mathfrak{U}', \mathcal{T}'' = \mathfrak{U}''.$$

Moreover,

$$\mathcal{T} \subset D_{(\Delta^{\alpha})}, \ \Delta^{\alpha}\mathcal{T} = \mathcal{T} \text{ and } \overline{\Delta^{\alpha} \mid \mathcal{T}} = \Delta^{\alpha}, \ \alpha \in \mathbf{C};$$

$$J\mathcal{T} = \mathcal{T};$$

$$\Delta^{\alpha} J\xi = J\Delta^{-\overline{\alpha}}\xi, \quad \xi \in \mathcal{T}, \quad \alpha \in \mathbf{C};$$

$$\Delta^{\alpha}(\xi\eta) = (\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta), \quad \xi, \eta \in \mathcal{T}, \quad \alpha \in \mathbf{C};$$

$$J(\xi\eta) = J(\eta)J(\xi), \quad \xi, \eta \in \mathcal{T}.$$

Proof: [S-Z; 10.20 p.298]

$\mathcal{T}$ is called the *Tomita algebra* associated to the left Hilbert algebra $\mathfrak{U}$.

Let ϕ be a normal semifinite faithful weight on the von Neumann algebra $\mathcal{M}$ and $\mathfrak{U}_{\phi} \subset \mathcal{H}_{\phi}$ be the associated left Hilbert algebra. Denote $\mathcal{T}_{\phi}$ the associated Tomita algebra.

Proposition 2.6.2:

i) *For $\xi \in \mathcal{H}_{\phi}$ we have*

$$\xi \in \mathcal{T}_{\phi} \quad \Longleftrightarrow \quad \xi \in \bigcap_{\alpha \in \mathbf{C}} D_{(\Delta_{\phi}^{\alpha})} \text{ and } \Delta_{\phi}^{n}\xi \in \mathfrak{N}_{\phi} \subset \mathcal{H}_{\phi}, \quad \text{for all } n \in \mathbf{Z}.$$

ii) *For every $x \in \mathfrak{N}_{\phi}$, there exists a sequence $\{x_n\} \subset \mathcal{T}_{\phi}$ such that*

$$\|x_n\| \leq \|x\|, \quad x_n \xrightarrow{s} x \quad \text{and} \quad \|\eta_{\phi}(x_n) - \eta_{\phi}(x)\|_{\phi} \to 0;$$

the sequence $\{x_n\}$ can be chosen to be:

$$x_n = \sqrt{n/\pi} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t^{\phi}(x)\, dt.$$

If $x \in \mathfrak{N}_{\phi} \cap \mathfrak{N}_{\phi}^{*}$ we also have

$$x_n^{*} \xrightarrow{s} x^{*} \quad \text{and} \quad \|\eta_{\phi}(x_n^{*}) - \eta_{\phi}(x^{*})\|_{\phi} \to 0.$$

Proof: [S; 2.12] / [S-Z; 10.21]

2.7 Analytic elements

Let ϕ be a normal semifinite faithful weight on the von Neumann algebra $\mathcal{M}$, and let $a \in \mathcal{M}$.

Definition 2.7.1: The element a is called *analytic in the vertical strip*

$$\mathbf{D} = \{\alpha \in \mathbf{C} : -\epsilon_1 \leq Re(\alpha) \leq \epsilon_2\}$$

(where $\epsilon_1, \epsilon_2 \in \mathbf{R}^+$) if there exists an $\mathcal{M}$-valued function F, defined and ultra-weakly continuous on $\mathbf{D}$ and analytic [A.1.19] in the interior of $\mathbf{D}$ such that

$$F(it) = \sigma_t^\phi(a) \quad \text{for all } t \in \mathbf{R}.$$

Then, for all $\alpha \in \mathbf{C}$ with $-\epsilon_1 \leq Re(\alpha) \leq \epsilon_2$, we write

$$\sigma_{-i\alpha}^\phi(a) = F(\alpha).$$

Furthermore, for $\alpha \in \mathbf{C}$, we will write $a \in D(\sigma_\alpha^\phi)$ whenever the element a is analytic in some vertical strip containing $i\alpha$.

Remarks:

i) If $a \in D(\sigma_\alpha^\phi)$, then $a^* \in D(\sigma_{\bar{\alpha}}^\phi)$ and $\sigma_{\bar{\alpha}}^\phi(a^*) = \sigma_\alpha^\phi(a)^*$.

ii) If $a, b \in D(\sigma_\alpha^\phi)$, then $ab \in D(\sigma_\alpha^\phi)$ and $\sigma_\alpha^\phi(ab) = \sigma_\alpha^\phi(a)\sigma_\alpha^\phi(b)$.

iii) If $a \in D(\sigma_\beta^\phi)$ and $\sigma_\beta^\phi(a) \in D(\sigma_\alpha^\phi)$, then $a \in D(\sigma_{\alpha+\beta}^\phi)$ and $\sigma_{\alpha+\beta}^\phi(a) = \sigma_\alpha^\phi(\sigma_\beta^\phi(a))$.

iv) If $a \in D(\sigma_\alpha^\phi)$, then $\sigma_\alpha^\phi(a) \in D(\sigma_{-\alpha}^\phi)$ and $\sigma_{-\alpha}^\phi(\sigma_\alpha^\phi(a)) = a$.

v) From the relation $\pi_\phi(\sigma_t^\phi(a))\Delta_\phi^{it}\xi = \Delta_\phi^{it}\pi_\phi(a)\xi$, it follows, using Proposition A.1.17, that if $a \in D(\sigma_{-i\alpha}^\phi)$ and $\xi \in D(\Delta_\phi^\alpha)$, then

$$\pi_\phi(a)\xi \in D(\Delta_\phi^\alpha) \quad \text{and} \quad \Delta_\phi^\alpha\pi_\phi(a)\xi = \pi_\phi(\sigma_{-i\alpha}^\phi(a))\Delta_\phi^\alpha\xi.$$

Equivalently, using Proposition A.1.20, it follows that if $a \in D(\sigma_\alpha^\phi)$ then

$$\pi_\phi(\sigma_\alpha^\phi(a)) = \overline{\Delta_\phi^{i\alpha}\pi_\phi(a)\Delta_\phi^{-i\alpha}|D(\Delta_\phi^{-i\alpha})}.$$

It can be shown that if the element a is analytic in $C = \{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq \epsilon\}$, then the function $\alpha \mapsto \sigma_{-i\alpha}^\phi(a)$ is norm continuous and norm bounded on C.

Proposition 2.7.2: *Let ϕ be a normal faithful semifinite weight on the von Neumann algebra $\mathcal{M}$, $a \in \mathcal{M}$, and $\lambda \in (0, \infty)$. The following statements are equivalent:*

i) $\phi(ax^*xa^*) \leq \lambda^2\phi(x^*x)$ *for all* $x \in \mathcal{M}$;

ii) $x \in \mathfrak{N}_\phi$ *implies* $xa^* \in \mathfrak{N}_\phi$ *and* $\|\eta_\phi(xa^*)\|_\phi \leq \lambda\|\eta_\phi(x)\|_\phi$;

iii) $a \in D(\sigma^{\phi}_{-\frac{i}{2}})$ *and* $\|\sigma^{\phi}_{-\frac{i}{2}}(a)\| \leq \lambda$.

If $a \in D(\sigma^{\phi}_{-\frac{i}{2}})$, *then*

$$\eta_\phi(xa^*) = J_\phi \pi_\phi(\sigma^{\phi}_{-\frac{i}{2}}(a)) J_\phi \eta_\phi(x), \qquad x \in \mathfrak{N}_\phi.$$

and if moreover $\sigma^{\phi}_t(aa^*) = aa^*$ *for all* $t \in \mathbf{R}$, *then*

$$\phi(\sigma^{\phi}_{-\frac{i}{2}}(a)^* x^* x \sigma^{\phi}_{-\frac{i}{2}}(a)) = \phi(aa^* x^* x), \qquad x \in \mathfrak{N}_\phi.$$

Proof: [S; 2.14 p.31]

Definition 2.7.3: An element $a \in \mathcal{M}$ such that $a \in D(\sigma^{\phi}_{\alpha})$ for all $\alpha \in \mathbf{C}$ is called *entire analytic*. The subset of $\mathcal{M}$ of all entire analytic elements will be denoted $\mathcal{M}^{\phi}_{\infty}$.

Remark: If $a \in \mathcal{T}_\phi$, then $a \in \mathcal{M}^{\phi}_{\infty}$ and $\sigma^{\phi}_{\alpha}(a) \in \mathcal{T}_\phi$ for all $\alpha \in \mathbf{C}$, where $\mathcal{T}_\phi$ is the Tomita algebra associated to the left Hilbert algebra $\mathfrak{U}_\phi$.

Proposition 2.7.4: $\mathcal{M}^{\phi}_{\infty}$ *is a* s^**-dense* **-subalgebra of* $\mathcal{M}$.

Proof: [S-Z; 10.16 p.284]

Moreover we have:

Proposition 2.7.5: *Let* ϕ *be a normal semifinite faithful weight on the von Neumann algebra* $\mathcal{M}$. *Let* $\{x_k\}_{k \in K} \subset \mathfrak{U}_\phi$ *be a net such that*

$$x_k \xrightarrow{s^*} 1, \quad \sup_k \|x_k\| \leq 1 \quad \text{and} \quad a_k = \sqrt{1/\pi} \int_{-\infty}^{+\infty} e^{-t^2} \sigma^{\phi}_t(x_k)\, dt \quad (k \in K).$$

Then, $\{a_k\}_{k \in K} \subset \mathcal{T}_\phi \subset \mathcal{M}^{\phi}_{\infty}$ *and for every* $\alpha \in \mathbf{C}, k \in K$, *we have*

$$\sigma^{\phi}_{\alpha}(a_k) \xrightarrow{s^*} 1, \qquad \|\sigma^{\phi}_{\alpha}(a_k)\| \leq \exp((Im\,\alpha)^2).$$

Proof: [S; 2.16 p.33]

Proposition 2.7.6: *Let* ϕ *be a normal semifinite faithful weight on the von Neumann algebra* $\mathcal{M}$, $x, y \in \mathcal{M}$ *and* $\alpha \in \mathbf{C}$. *If*

$$x \in \mathfrak{N}^*_\phi \cap D(\sigma^{\phi}_{\alpha - i}),\ \sigma^{\phi}_{\alpha - i}(x) \in \mathfrak{N}_\phi \quad \text{and} \quad y \in \mathfrak{N}_\phi \cap D(\sigma^{\phi}_{\alpha}),\ \sigma^{\phi}_{\alpha}(y) \in \mathfrak{N}^*_\phi,$$

then

$$\phi(xy) = \phi(\sigma^{\phi}_{\alpha}(y)\sigma^{\phi}_{\alpha - i}(x)).$$

Proof: [S; 2.17 p.34]

2.8 The Radon-Nikodym derivative

Theorem 2.8.1 (Connes): *Let $\mathcal{M}$ be a von Neumann algebra, ϕ a faithful semifinite normal weight on $\mathcal{M}$, and ψ a normal semifinite weight on $\mathcal{M}$. There exists a unique s^*-continuous mapping $u : \mathbf{R} \to \mathcal{M}$ such that:*

i)

$$u_t u_t^* = \mathrm{s}(\psi) = u_0, \qquad u_t^* u_t = \sigma_t^\phi(\mathrm{s}(\psi)),$$
$$u_{t+t'} = u_t \sigma_t^\phi(u_{t'}),$$
$$u_{-t} = \sigma_{-t}^\phi(u_t^*),$$
$$\sigma_t^\psi(x) = u_t \sigma_t^\phi(x) u_t^*, \qquad (x \in \mathrm{s}(\psi)\mathcal{M}\mathrm{s}(\psi)).$$

ii) *For every $x = x\mathrm{s}(\psi) \in \mathfrak{N}_\psi \cap \mathfrak{N}_\phi^*$ and every $y = \mathrm{s}(\psi)y \in \mathfrak{N}_\phi \cap \mathfrak{N}_\psi^*$, there exists a function F defined, continuous and bounded on the strip $\{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq 1\}$, analytic in the interior of this strip, such that*

$$F(it) = \phi(x u_t \sigma_t^\phi(y)), \qquad F(1+it) = \psi(\sigma_t^\psi(y) u_t x)$$

for all $t \in \mathbf{R}$.

Proof: [S; 3.1 p.46]

Definition 2.8.2: The mapping $\mathbf{R} \ni t \to u_t \in \mathcal{M}$ uniquely determined above is called the *Connes Radon-Nikodym cocycle* associated with the normal semifinite weight ψ with respect to the normal semifinite faithful weight ϕ, and is denoted

$$u_t = (D\psi : D\phi)_t.$$

Remark: In the above theorem, if ψ is faithful, then $u_t \in U(\mathcal{M})$. Indeed, in that case, $\mathrm{s}(\psi) = 1$ and thus $u_t u_t^* = \mathrm{s}(\psi) = 1$. Moreover, when ϕ is faithful, $\sigma_t^\phi \in \mathrm{Aut}(\mathcal{M})$, and hence $u_t^* u_t = \sigma_t^\phi(\mathrm{s}(\psi)) = 1$.

Example 2.8 a: Let $\mathcal{M} = L^\infty(X,\mu)$, where (X,μ) is a σ-finite measure space. Let ϕ be the weight on $\mathcal{M}$ given by μ, that is,

$$\phi(f) = \int_X f(x)\, d\mu(x), \qquad f \in \mathcal{M}^+.$$

As seen in section 2.2, every faithful semifinite normal weight ψ on $\mathcal{M}$ corresponds to a σ-finite measure ν which is absolutely continuous with respect to μ. Hence

$$\psi(f) = \int_X f(x)\, d\nu(x), \qquad f \in \mathcal{M}^+.$$

Note that as $\mathcal{M}$ is abelian, then ϕ and ψ are both traces; thus $\Delta_\phi = 1 = \Delta_\psi$. Let $h(x) = (d\nu/d\mu)(x)$ for $x \in X$. Then h^{it} satisfies the conditions stated in Theorem 2.8.1 for all $t \in \mathbf{R}$. Hence by unicity of the cocycle, $h^{it} = (D\psi : D\phi)_t$ for all $t \in \mathbf{R}$.

Example 2.8 b: Let $\mathcal{M} = M_2(\mathbf{C})$ and ϕ, ψ be defined by:

$$\phi(\cdot) = \tau(h\cdot), \quad \text{where} \quad h = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 > 0$$

and

$$\psi(\cdot) = \tau(k\cdot), \quad \text{where} \quad k = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \mu_1 + \mu_2 = 1, \mu_1, \mu_2 > 0.$$

As seen in Example 2.2 b,

$$\sigma_t^{\phi}(e_{kl}) = \left(\frac{\lambda_k}{\lambda_l}\right)^{it} e_{kl} \quad \text{and} \quad \sigma_t^{\psi}(e_{kl}) = \left(\frac{\mu_k}{\mu_l}\right)^{it} e_{kl}.$$

Then,

$$(D\psi : D\phi)_t = k^{it}h^{-it} = \begin{pmatrix} \mu_1^{it}\lambda_1^{-it} & 0 \\ 0 & \mu_2^{it}\lambda_2^{-it} \end{pmatrix}.$$

Lemma 2.8.3 (Chain rule): *Let ϕ_1, ϕ_2, ϕ_3 be normal semifinite faithful weights on the von Neumann algebra $\mathcal{M}$. Then*

$$(D\phi_1 : D\phi_3)_t = (D\phi_1 : D\phi_2)_t(D\phi_2 : D\phi_3)_t.$$

Proof: [S; 3.5 p.49]

The following lemma is the tool used to prove the next proposition:

Lemma 2.8.4: *Let $\mathcal{M}$ be a von Neumann algebra, ϕ a faithful normal semifinite weight on $\mathcal{M}$ and $x \in \mathcal{M}$. The map $t \mapsto \sigma_t^{\phi}(x)$ (for $t \in \mathbf{R}$) extends to an $\mathcal{M}$-valued bounded ultra-weakly continuous map on $B_{-\frac{1}{2}} = \{z \in \mathbf{C} : Im(z) \in [-\frac{1}{2}, 0]\}$ which is analytic in the interior of $B_{-\frac{1}{2}}$ and satisfies $\|\sigma^{\phi}_{-\frac{i}{2}}(x)\| \leq 1$ if and only if*

$$\phi(xyx^*) \leq \phi(y) \quad \text{for all } y \in \mathcal{M}^+.$$

Proof: [Co$_2$]

Proposition 2.8.5: *Let ϕ and ψ be normal semifinite faithful weights on the von Neumann algebra $\mathcal{M}$. The following statements are equivalent:*

i) *$\psi \leq \phi$ (that is, $\psi(x) \leq \phi(x)$ for all $x \in \mathcal{M}^+$);*

ii) *there exists an $\mathcal{M}$-valued function u, defined and ultra-weakly continuous on the strip $B_{-\frac{1}{2}} = \{z \in \mathbf{C} : Im(z) \in [-\frac{1}{2}, 0]\}$, analytic in the interior of $B_{-\frac{1}{2}}$, such that*

$$u_t = (D\psi : D\phi)_t, \quad (t \in \mathbf{R}), \quad \|u_{-\frac{i}{2}}\| \leq 1.$$

If condition i) *or* ii) *hold, then*

$$\psi(x^*x) = \phi(u^*_{-\frac{i}{2}} x^* x u_{-\frac{i}{2}}), \quad \textit{for all } x \in \mathfrak{N}_\psi.$$

Proof: [Co$_2$] / [S; 3.13 p.56]

Lemma 2.8.6: *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and ϕ a normal positive linear functional on $\mathcal{M}$. Let $h \in \mathcal{M}$ be such that the functional $\phi(h\cdot)$ satisfies $\phi(hx^*) = \overline{\phi(hx)}$ for all $x \in \mathcal{M}$. Then $|\phi(hx)| \leq \|h\|\phi(x)$ for all $x \in \mathcal{M}^+$.*

Proof: [Di; p.69]

Definition 2.8.7: The fixed point algebra under the modular automorphism group $\{\sigma_t^\phi\}$, called the *centralizer* of ϕ, is defined by

$$\mathcal{M}^\phi = \{x \in \mathcal{M} : \sigma_t^\phi(x) = x \text{ for all } t \in \mathbf{R}\}.$$

Remark: Let ϕ be a faithful normal positive linear functional on a von Neumann algebra $\mathcal{M}$ and $h \in (\mathcal{M}^\phi)^+$. The implication i) $\Rightarrow$ ii) of the next proposition depends on the following observations:

i) By Lemma 2.8.6, $|\phi(hx)| \leq \|h\|\phi(x)$ for all $x \in \mathcal{M}^+$.

ii) $\phi \circ \sigma_t^\phi = \phi$.

Proposition 2.8.8: *Let ϕ be a faithful normal positive linear functional on $\mathcal{M}$. The following conditions on a normal positive linear functional ψ on $\mathcal{M}$ are equivalent:*

i) *$\psi(\cdot) = \phi(h\cdot)$, for some $h \in (\mathcal{M}^\phi)^+$;*

ii) *$\psi \leq c\phi$, for some $c > 0$, and $\psi \circ \sigma_t^\phi = \psi$ for all $t \in \mathbf{R}$.*

Proof: [Su; Prop. 2.6.2 p.74]

Definition 2.8.9: Let ϕ be a normal semifinite weight on the von Neumann algebra $\mathcal{M}$ and $a \in (\mathcal{M}^\phi)^+$. The map $\phi_a : \mathcal{M}^+ \to [0, \infty]$ given by

$$\phi_a(x) = \phi(a^{\frac{1}{2}} x a^{\frac{1}{2}}), \qquad x \in \mathcal{M}^+$$

defines a weight on $\mathcal{M}$.

Remarks:

i) ϕ_a is normal and as $a \in \mathcal{M}^\phi$, then $\phi_a(x) = \phi(ax) = \phi(xa)$ for all $x \in \mathfrak{M}_\phi$. Moreover $s(\phi_a) = s(a)$.

ii) Let $a, b \in (\mathcal{M}^\phi)^+$. Then $\phi_{a+b} = \phi_a + \phi_b$ and thus, $\phi_a \leq \phi_b$ if $a \leq b$.

iii) Let $a \in (\mathcal{M}^\phi)^+$ and $\{a_i\}_{i \in I} \subset \mathcal{M}^\phi$ be a net of positive elements. If $a_i \nearrow a$, then $\phi_{a_i} \nearrow \phi_a$.

Now, let ϕ be a normal semifinite weight on the von Neumann algebra $\mathcal{M}$ and A a positive self-adjoint operator affiliated to $\mathcal{M}^{\phi}$. We would like to give meaning to ϕ_A.

Consider the bounded positive operators

$$A_\epsilon = A(1 + \epsilon A)^{-1} \in \mathcal{M}^{\phi} \qquad (\epsilon > 0).$$

It can be shown that $A_\epsilon \nearrow A$ as $\epsilon \searrow 0$. [S; p.65]

Definition 2.8.10: Let ϕ be a normal semifinite weight on the von Neumann algebra $\mathcal{M}$ and A a positive self-adjoint operator affiliated to $\mathcal{M}^{\phi}$. The map $\phi_A : \mathcal{M}^+ \to [0, \infty]$ given by

$$\phi_A(x) = \sup_{\epsilon > 0} \phi_{A_\epsilon}(x) = \lim_{\epsilon \to 0} \phi_{A_\epsilon}(x) \qquad x \in \mathcal{M}^+$$

defines a normal weight on $\mathcal{M}$.

Remarks:

i) If the operator A in Definition 2.8.10 were bounded, then ϕ_A would coincide with the weight described in Definition 2.8.9.

ii) $s(\phi_A) = s(A)$.

Lemma 2.8.11: *Let ϕ be a normal semifinite weight on the von Neumann algebra $\mathcal{M}$ and A a positive self-adjoint operator affiliated to $\mathcal{M}^{\phi}$. Then*

i) $\sigma_t^{\phi_A}(x) = A^{it}\sigma_t^{\phi}(x)A^{-it}$ *for* $x \in s(A)\mathcal{M}s(A)$, $t \in \mathbb{R}$.

ii) $(D\phi_A : D\phi)_t = A^{it}$ *for* $t \in \mathbb{R}$.

Proof: [S; p.67]

2.9 The standard form of von Neumann algebras

We shall see that given a von Neumann algebra $\mathcal{M}$, the identity correspondence from $\mathcal{M}$ to $\mathcal{M}$ is in fact equivalent to the standard form of $\mathcal{M}$.

Let us now examine the properties of the standard form of a von Neumann algebra.

Definition 2.9.1: Let $\mathcal{H}$ be a Hilbert space. A subset $\mathcal{P}$ of $\mathcal{H}$ is a *convex cone with vertex origin* if:

i) $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$;

ii) $\lambda\mathcal{P} \subset \mathcal{P}$ for all $\lambda > 0$.

If $0 \in \mathcal{P}$, $\mathcal{P}$ is said to be a convex *pointed* cone with vertex origin.

Definition 2.9.2: Let $\mathcal{P}$ be a convex pointed cone with vertex origin in a Hilbert space $\mathcal{H}$.

i) The set $\mathcal{P}^0 = \{\eta \in \mathcal{H} : (\xi \mid \eta) \geq 0, \text{ for all } \xi \in \mathcal{P}\}$ is called the *dual cone*.

ii) If $\mathcal{P} = \mathcal{P}^0$, then $\mathcal{P}$ is said to be *selfdual*.

Let us fix a von Neumann algebra $\mathcal{M}$ and a normal semifinite faithful weight ϕ on $\mathcal{M}$. To the pair $\{\mathcal{M}, \phi\}$, we associate $\{\pi_\phi, \mathcal{H}_\phi, \eta_\phi, \Delta_\phi, J_\phi, \mathcal{U}_\phi\}$ [Prop. 2.2.4, Thm 2.3.1]. Let us identify $\mathcal{M}$ and $\pi_\phi(\mathcal{M})$ (and omit π_ϕ). Define

$$\mathcal{P}^\sharp_\phi = \overline{\{\eta_\phi(x^*x) : x \in \mathfrak{N}_\phi\}}, \quad \mathcal{P}^\flat_\phi = J_\phi \mathcal{P}^\sharp_\phi, \quad \mathcal{P}^\natural_\phi = \overline{(\Delta^{\frac{1}{4}}_\phi \mathcal{P}^\sharp_\phi)}.$$

Proposition 2.9.3 *With the above setting, we have:*

i) $\mathcal{P}^\sharp_\phi$ *and* $\mathcal{P}^\flat_\phi$ *are mutually dual convex cones.*

ii) $\mathcal{P}^\natural_\phi$ *is a selfdual convex cone. Every vector in* $\mathcal{H}_\phi$ *is represented as a linear combination of four elements of* $\mathcal{P}^\natural_\phi$. *If* $\xi = J_\phi\xi$, *then* ξ *is represented uniquely as the difference of orthogonal vectors of* $\mathcal{P}^\natural_\phi$.

iii) $aJ_\phi aJ_\phi \mathcal{P}^\natural_\phi \subset \mathcal{P}^\natural_\phi, \quad a \in \mathcal{M}$.

iv) *To each* $\omega \in \mathcal{M}^+_*$, *there corresponds a unique* $\xi \in \mathcal{P}^\natural_\phi$ *such that* $\omega = \omega_\xi$. *Moreover,*

$$\|\xi - \eta\|^2 \leq \|\omega_\xi - \omega_\eta\| \leq \|\xi - \eta\|\|\xi + \eta\|, \qquad \xi, \eta \in \mathcal{P}^\natural_\phi.$$

Proof: [Tak$_2$; p.30]

Example 2.9 a: In example 2.2 b, we calculated, for $\mathcal{M} = \mathbf{M}_2(\mathbf{C})$ and $\phi = \tau(h\cdot)$ (where $h = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 > 0$), the associated $\pi_\phi, \mathcal{H}_\phi, \eta_\phi, \Delta_\phi, J_\phi$. Note that $\mathfrak{N}_\phi = \mathbf{M}_2(\mathbf{C})$. Therefore, $\mathcal{U}_\phi = \eta_\phi(\mathbf{M}_2(\mathbf{C}))$, by Theorem 2.3.1.

The cones introduced above are given by:

$$\begin{aligned}
\mathcal{P}^\sharp_\phi &= \left\{\eta_\phi(x^*x) : x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbf{C})\right\} \\
&= \left\{ \begin{pmatrix} \sqrt{\lambda_1}(|a|^2 + |c|^2) \\ \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} \\ \sqrt{\lambda_2}(|b|^2 + |d|^2) \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbf{C}) \right\},
\end{aligned}$$

$$\mathcal{P}^\flat_\phi = \left\{ \begin{pmatrix} \sqrt{\lambda_1}(|a|^2 + |c|^2) \\ \sqrt{\frac{\lambda_1}{\lambda_2}}(a\overline{b} + c\overline{d}) \\ \sqrt{\frac{\lambda_2}{\lambda_1}}(\overline{a}b + \overline{c}d) \\ \sqrt{\lambda_2}(|b|^2 + |d|^2) \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbf{C}) \right\}, \quad \text{and}$$

$$\mathcal{P}^\natural_\phi = \left\{ \begin{pmatrix} \sqrt{\lambda_1}(|a|^2 + |c|^2) \\ (\frac{\lambda_1}{\lambda_2})^{\frac{1}{4}}(\overline{a}b + \overline{c}d) \\ (\frac{\lambda_2}{\lambda_1})^{\frac{1}{4}}(a\overline{b} + c\overline{d}) \\ \sqrt{\lambda_2}(|b|^2 + |d|^2) \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbf{C}) \right\}.$$

Example 2.9 b: Let $\mathcal{M} = L^\infty(\mathbf{R}, \lambda) \subset \mathcal{L}(L^2(\mathbf{R}, \lambda))$, where λ represents the Lebesgue measure.

$$\phi(f) = \int_{\mathbf{R}} f \, d\lambda, \quad f \in L^\infty(\mathbf{R}, \lambda)$$

is a faithful normal semifinite trace on $L^\infty(\mathbf{R}, \lambda)$.

$\mathfrak{N}_\phi = \{f \in L^\infty(\mathbf{R}, \lambda) : \phi(|f|^2) < \infty\} = L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$.

$\eta_\phi : L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}) \to L^2(\mathbf{R}), \quad f \mapsto f$.

$S_\phi = S_\phi^* = J_\phi : f \mapsto \overline{f}$; thus $\Delta_\phi = S^*S = 1$.

$$\mathcal{P}_\phi^\sharp = \{\eta_\phi(|f|^2) : f \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})\} = \{|f|^2 : f \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})\},$$

$$\mathcal{P}_\phi^\flat = J_\phi \mathcal{P}_\phi^\sharp = \mathcal{P}_\phi^\sharp \quad \text{and} \quad \mathcal{P}_\phi^\natural = \overline{(\Delta_\phi^{\frac{1}{4}} \mathcal{P}_\phi^\sharp)} = \mathcal{P}_\phi^\sharp.$$

Definition 2.9.4: If a von Neumann algebra $\{\mathcal{M}, \mathcal{H}\}$ admits a conjugate linear isometric involution $J : \mathcal{H} \to \mathcal{H}$ and a selfdual convex cone $\mathcal{P}$ in $\mathcal{H}$ with the following properties, then $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ is said to be a *standard form:*

i) $J\mathcal{M}J = \mathcal{M}'$,

ii) $JaJ = a^*, \quad a \in \mathcal{M} \cap \mathcal{M}' = Z(\mathcal{M})$,

iii) $J\xi = \xi, \quad \xi \in \mathcal{P}$,

iv) $aJaJ\mathcal{P} \subset \mathcal{P}, \quad a \in \mathcal{M}$.

Theorem 2.9.5: *Let $\{\mathcal{M}_i, \mathcal{H}_i, J_i, \mathcal{P}_i\}, i = 1, 2$ be standard forms. If π is an isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$, then there exists a unique unitary U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that:*

i) $\pi(x) = UxU^*, \quad x \in \mathcal{M}$,

ii) $J_2 = UJ_1U^*$,

iii) $\mathcal{P}_2 = U\mathcal{P}_1$.

Proof: [Tak$_2$; p.30] / [H; Thm 2.3]

Proposition 2.9.6: *For any left Hilbert algebra $\mathfrak{U} \subset \mathcal{H}$, the von Neumann algebra $\mathcal{R}_l(\mathfrak{U}) \subset \mathcal{L}(\mathcal{H})$ is a standard form.*

Proof: [S-Z; 10.23 p.308]

Remark: By Thm 2.3.1 and Prop. 2.9.6, any von Neumann algebra is *-isomorphic to a von Neumann algebra which is a standard form.

Lemma 2.9.7: *Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form.*

i) *Any $\phi \in \mathcal{M}_*^+$ has the form $\phi = w_\xi$ for a unique vector $\xi \in \mathcal{P}$.*

ii) *For $\xi, \eta \in \mathcal{P}$,*

$$\|\xi - \eta\|^2 \le \|w_\xi - w_\eta\| \le \|\xi - \eta\| \|\xi + \eta\|.$$

In particular, the map $\xi \to w_\xi$ is a homeomorphism of $\mathcal{P}$ onto $\mathcal{M}_*^+$.

Proof: [H; Lemma 2.10]

Proposition 2.9.8: *Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form and ξ a vector in $\mathcal{H}$. Then there exists a unique pair (u, η), where u is a partial isometry in $\mathcal{M}$ and η is a vector in $\mathcal{P}$, such that*

$$\xi = u \cdot \eta, \qquad u^* u = \mathrm{s}(\eta),$$

where $\mathrm{s}(\eta) \in \mathcal{M}$ denotes the projection onto $\overline{\mathcal{M}'\eta}$.

Proof: [H; Prop. 4.3]

Definition 2.9.9:

i) The vector η in the above proposition is denoted $|\xi|$, and the equation $\xi = u|\xi|$ is called the *polar decomposition* of ξ.

ii) Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form and $\phi \in \mathcal{M}_*^+$. Then $\phi^{\frac{1}{2}}$ will denote the unique vector in $\mathcal{P}$ such that $w_{\phi^{\frac{1}{2}}} = \phi$.

Proposition 2.9.10: *Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form and ϕ, ψ be positive normal forms on $\mathcal{M}$ such that $\phi \leq \psi$. Then there is a unique operator $a \in \mathcal{M}$ such that*

$$\phi^{\frac{1}{2}} = a\psi^{\frac{1}{2}}, \qquad a^* a \leq \mathrm{s}(\psi).$$

Proof: [H; Prop. 4.8]

Chapter III

Correspondences

In this chapter, we will only consider separable Hilbert spaces and von Neumann algebras with separable preduals.

3.1 Definitions and examples

Definition 3.1.1: Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A *correspondence* from $\mathcal{M}$ to $\mathcal{N}$ is a Hilbert space $\mathcal{H}$ which is an $\mathcal{N}$-$\mathcal{M}$ bimodule, i.e. there are two bilinear product maps,

$$b_1 : \mathcal{N} \times \mathcal{H} \to \mathcal{H}, \qquad b_1(y,\xi) = y \cdot \xi$$

$$b_2 : \mathcal{H} \times \mathcal{M} \to \mathcal{H}, \qquad b_2(\xi,x) = \xi \cdot x$$

which are separately weakly continuous and satisfy:

$$1_{\mathcal{N}} \cdot \xi = \xi \cdot 1_{\mathcal{M}} = \xi,$$

$$y_1 \cdot (y_2 \cdot \xi) = (y_1 y_2) \cdot \xi, \qquad (y_1 + y_2) \cdot \xi = y_1 \cdot \xi + y_2 \cdot \xi,$$

$$(\xi \cdot x_1) \cdot x_2 = \xi \cdot (x_1 x_2), \qquad (x_1 + x_2) \cdot \xi = x_1 \cdot \xi + x_2 \cdot \xi,$$

$$(y \cdot \xi) \cdot x = y \cdot (\xi \cdot x)$$

for all $y, y_1, y_2 \in \mathcal{N}$, $x, x_1, x_2 \in \mathcal{M}$.

Remarks:

i) The maps b_1 and b_2 define mutually commuting normal unital *-representations $\pi_{\mathcal{N}}$ of $\mathcal{N}$ and $\pi_{\mathcal{M}^o}$ of $\mathcal{M}^o$ (the opposite von Neumann algebra of $\mathcal{M}$ [section 1.1]) in $\mathcal{H}$. As a matter of notation, we will often write:

$$\pi_{\mathcal{N}}(y)\pi_{\mathcal{M}^o}(x^o)\xi = y \cdot \xi \cdot x, \quad \text{for all } \xi \in \mathcal{H},\, y \in \mathcal{N},\, x \in \mathcal{M}.$$

ii) A correspondence from $\mathcal{M}$ to $\mathcal{N}$ can be viewed as a unital *-representation $\pi_{\mathcal{N},\mathcal{M}^o}$ of the algebraic tensor product $\mathcal{N} \odot \mathcal{M}^o$ on the Hilbert space $\mathcal{H}$, which is normal when restricted to $\mathcal{N} \odot 1$ and $1 \odot \mathcal{M}^o$. As seen in section 1.8, $\mathcal{N} \otimes_{max} \mathcal{M}^o$ is the completion of $\mathcal{N} \odot \mathcal{M}^o$ with respect to the C^*-norm p_{max}. Therefore, any unital *-representation $\pi^\dagger : \mathcal{N} \otimes_{max} \mathcal{M}^o \to \mathcal{L}(\mathcal{H})$ defines a correspondence (as long as $\pi^\dagger$ is normal when restricted to $\mathcal{N} \odot 1$ and $1 \odot \mathcal{M}^o$).

Definition 3.1.2:

i) Let $\mathcal{H}$ and $\mathcal{H}'$ be two correspondences from $\mathcal{M}$ to $\mathcal{N}$. We will say that these correspondences are *equivalent*, written $\mathcal{H} \sim \mathcal{H}'$, if they are equivalent as $\mathcal{N}-\mathcal{M}$ bimodules. If we make use of remark ii), it means that the corresponding representations of $\mathcal{N} \odot \mathcal{M}^o$ are unitarily equivalent. The set of all equivalence classes of correspondences from $\mathcal{M}$ to $\mathcal{N}$ will be denoted $Corr(\mathcal{N}, \mathcal{M})$.

ii) Let $\mathcal{H}$ be a correspondence from $\mathcal{M}$ to $\mathcal{N}$. A *subcorrespondence* $\mathcal{H}_0$ of $\mathcal{H}$ is a Hilbert subspace of $\mathcal{H}$, invariant under the actions of $\mathcal{N}$ and $\mathcal{M}$ (i.e. $\mathcal{N}\mathcal{H}_0\mathcal{M} \subset \mathcal{H}_0$). A subcorrespondence is therefore a subrepresentation of $\mathcal{N} \odot \mathcal{M}^o$.

iii) A correspondence $\mathcal{H}^\dagger$ from $\mathcal{M}$ to $\mathcal{N}$ is *subequivalent* (or *contained*) in a correspondence $\mathcal{H}$, written $\mathcal{H}^\dagger \subset \mathcal{H}$, if it is equivalent to a subcorrespondence of $\mathcal{H}$.

iv) Let $\mathcal{H}$ be a correspondence from $\mathcal{M}$ to $\mathcal{N}$ and let $\mathcal{M}_1$ and $\mathcal{N}_1$ be von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{N}$ respectively. By restriction of the bimodule structure of $\mathcal{H}$, we obtain a correspondence from $\mathcal{M}_1$ to $\mathcal{N}_1$.

Before considering in details particular aspects of correspondences, let us examine a few examples.

1. Commutative case:

Let $\mathcal{M} = L^\infty(X, \mu_X)$ and $\mathcal{N} = L^\infty(Y, \mu_Y)$, where (X, μ_X) and (Y, μ_Y) are standard measure spaces. Let μ be a measure on $X \times Y$ whose projections $pr_X(\mu)$, $pr_Y(\mu)$ defined by:

$$pr_X(\mu)(A) = \mu(A \times Y),\, \forall A \in \mathcal{B}(X) \quad \text{and} \quad pr_Y(\mu)(A') = \mu(X \times A'),\, \forall A' \in \mathcal{B}(Y),$$

are absolutely continuous with respect to μ_X and μ_Y.

Consider the measure space $(X \times Y, \mu)$. As seen in section 1.11, given $(X \times Y, \mu)$ and a μ-measurable field $(x,y) \mapsto \mathcal{H}(x,y)$ of Hilbert spaces over $X \times Y$, we can construct a Hilbert space, namely the direct integral of the $\mathcal{H}(x,y)$'s. For all $(x,y) \in X \times Y$, the dimension $n(x,y)$ of $\mathcal{H}(x,y)$ is μ-measurable [Prop. 1.11.3]. The Hilbert space

$$\mathcal{H} = \int_{X \times Y}^{\oplus} \mathcal{H}(x,y)\, d\mu(x,y)$$

is in fact a correspondence for the action

$$(g\xi f)(x,y) := g(y) f(x)\, \xi(x,y), \quad \text{for all } \xi \in \mathcal{H},\ f \in \mathcal{M},\ g \in \mathcal{N},$$

where for all $(x,y) \in X \times Y$, $\xi(x,y) \in \mathcal{H}(x,y)$. Indeed, the mappings $\pi_{\mathcal{N}} : \mathcal{N} \to \mathcal{L}(\mathcal{H})$ and $\pi_{\mathcal{M}^o} : \mathcal{M}^o \to \mathcal{L}(\mathcal{H})$ given by

$$\pi_{\mathcal{N}}(g)\,\xi := g\,\xi, \qquad \pi_{\mathcal{M}^o}(f)\,\xi := f\,\xi$$

are normal *-representations that commute. $\mathcal{H}$ is therefore a $\mathcal{N} - \mathcal{M}$ correspondence.

This example is in fact the general case:

Proposition 3.1.3: *Let (X, μ_X) and (Y, μ_Y) be standard measure spaces and $\mathcal{H}$ a correspondence from $\mathcal{M} = L^\infty(X, \mu_X) \subset \mathcal{L}(\mathcal{H})$ to $\mathcal{N} = L^\infty(Y, \mu_Y) \subset \mathcal{L}(\mathcal{H})$. Then there exists a measure ν on $X \times Y$, a ν-measurable function $n : X \times Y \to \overline{\mathbb{N}}$ and a ν-measurable field of Hilbert spaces $(x,y) \mapsto \mathcal{H}(x,y)$, where $\mathcal{H}(x,y)$ is of dimension $n(x,y)$ for all $(x,y) \in X \times Y$, such that $\mathcal{H} = \int_{X \times Y}^{\oplus} \mathcal{H}(x,y)\, d\nu(x,y)$, with $\mathcal{N} - \mathcal{M}$ bimodule structure described in the above example.*

Proof: Let $\pi : L^\infty(X \times Y, \mu_X \times \mu_Y) \to \mathcal{L}(\mathcal{H})$ be the normal *-representation describing the correspondence $\mathcal{H}$ and $\mathcal{D} = \pi(L^\infty(X \times Y, \mu_X \times \mu_Y)) \subset \mathcal{L}(\mathcal{H})$. As a countably decomposable abelian von Neumann algebra possesses a separating element [Di; p.20], $\mathcal{D}'$ has a cyclic vector, say $\xi \in \mathcal{H}$. The equation

$$\nu(F) = \nu_{\xi,\xi}(F) = (\pi(F)\xi \mid \xi), \qquad \text{for } F \in L^\infty(X \times Y, \mu_X \times \mu_Y)$$

defines a finite positive measure on $X \times Y$. Hence

$$\mathcal{H} = \int_{X \times Y}^{\oplus} \mathcal{H}(x,y)\, d\nu(x,y)$$

by the existence and unicity theorems 1.11.9 and 1.11.10. By Theorem 1.11.3, the map $n(x,y) = dim(\mathcal{H}(x,y))$ is ν-measurable. Moreover, by definition, the projections

$$pr_X(\nu)(A) := \nu(A \times Y), \quad \forall A \in \mathcal{B}(X),$$
$$pr_Y(\nu)(A') := \nu(X \times A'), \quad \forall A' \in \mathcal{B}(Y),$$

are absolutely continuous with respect to μ_X and μ_Y respectively. The structure of bimodule is given by:

$$(g\xi f)(x,y) = g(y)f(x)\xi(x,y), \qquad \forall f \in \mathcal{M},\ g \in \mathcal{N},\ \xi \in \mathcal{H}.$$

Q.E.D.

Note that in general the measure ν is not absolutely continuous with respect to the measure $\mu_X \times \mu_Y$.

Connes noted that this measure represents the graph G of the correspondence and the function n represents the multiplicity of the correspondence, where

$$\begin{aligned} G &= \{(x,y) \in X \times Y : y \in \operatorname{supp}(\nu_x) \text{ for almost all } x\} \\ &= \{(x,y) \in X \times Y : x \in \operatorname{supp}(\nu_y) \text{ for almost all } y\} \end{aligned}$$

and $n : X \times Y \to \mathbf{N}$ is ν-measurable.

2. Identity correspondence:

Let us now consider a particular case of Example 1. In the above example, take $(X, \mu_X) = (Y, \mu_Y)$, μ equal to the image of $\mu_X = \mu_Y$ on the diagonal $\Delta = \{(x,x) : x \in X\}$, and $n(x,x) = 1$ for all $x \in X$. In that case, the measure μ on $X \times X$ is nothing but

$$\mu = \int_X \delta_x \, d\mu_X(x),$$

where δ_x is the point mass concentrated at x. Following the above construction, the associated $\mathcal{M} - \mathcal{M}$ correspondence $\mathcal{H}$ takes the following form:

$$\begin{aligned} \mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)\, d\mu_X(x) &= \int_X^{\oplus} \left(\int_X^{\oplus} \mathcal{H}(x,y)\, d\delta_x(y) \right) d\mu_X(x) \\ &= \int_X^{\oplus} \left(\int_X^{\oplus} \mathbf{C}_y\, d\delta_x(y) \right) d\mu_X(x) \\ &= \int_X^{\oplus} \mathbf{C}_x\, d\mu_X(x) \\ &= L^2(X, \mu_X) \qquad \text{[Example 1.11 a, with } \mathcal{H}_0 = \mathbf{C}] \end{aligned}$$

The bimodule structure on $\mathcal{H}$ coincides with the standard representation $L^2(\mathcal{M})$ of $\mathcal{M}(= L^\infty(X, \mu_X))$. Indeed, the standard representation $L^2(L^\infty(X,\mu_X))$ of $L^\infty(X,\mu_X)$ is by definition the closure of $L^\infty(X,\mu_X)$ with respect to the norm induced by the following inner product:

$$(f \mid g) = \int_X f\bar{g}\, d\mu_X, \quad \text{for all } f, g \in L^\infty(X,\mu_X) \cap L^2(X,\mu_X),$$

which is exactly $L^2(X,\mu_X)$ [Example 2.9 b].

3.2 Half densities and the identity correspondence

We now describe the identity correspondence, denoted by $L^2(\mathcal{M})$, for an arbitrary von Neumann algebra $\mathcal{M}$.

Let $\mathcal{M}$ be a countably decomposable von Neumann algebra and $\mathcal{M}_*^+$ the positive cone of the predual of $\mathcal{M}$. Any $\phi \in \mathcal{M}_*^+$ has a well defined support, denoted $s(\phi) = e \in \mathcal{M}$, and the modular automorphism group σ_t^ϕ is a one parameter group of automorphisms of the reduced von Neumann algebra $\mathcal{M}_e = \{x \in \mathcal{M} : xe = ex = x\}$.

Let ν be a faithful semifinite normal weight on $\mathcal{M}$. Using the GNS construction for weights [Proposition 2.2.4], we get a Hilbert space $\mathcal{H}_\nu = L^2(\mathcal{M}, \nu)$ (which is the completion of $\mathfrak{N}_\nu = \{x \in \mathcal{M} : \nu(x^*x) < \infty\}$ with respect to the inner product $(x \mid y)_\nu = \nu(y^*x)$), equipped with a normal *-representation

$$\pi_\nu : \mathcal{M} \to \mathcal{L}(L^2(\mathcal{M}, \nu))$$

defined as follows: let $\eta_\nu : \mathfrak{N}_\nu \to L^2(\mathcal{M}, \nu)$ be the canonical map from $\mathfrak{N}_\nu$ to $L^2(\mathcal{M}, \nu)$. Then

$$\pi_\nu(x)\eta_\nu(y) = \eta_\nu(xy), \quad \text{for all } x \in \mathcal{M}, y \in \mathfrak{N}_\nu.$$

As seen in Theorem 2.3.1, associated with ν is a full left Hilbert algebra $\mathfrak{U}_\nu$ whose left von Neumann algebra $\mathcal{R}_l(\mathfrak{U}_\nu)$ is canonically identified with $\mathcal{M}$ (in fact, $\mathcal{R}_l(\mathfrak{U}_\nu)$ is the image $\pi_\nu(\mathcal{M})$ of $\mathcal{M}$ under π_ν, defined above).

Let us denote by Δ_ν and J_ν the corresponding modular operator and modular conjugation (which is an antilinear isometry). By Theorem 2.1.7,

$$\pi_\nu(\sigma_t^\nu(x)) = \Delta_\nu^{it}\pi_\nu(x)\Delta_\nu^{-it}, \qquad t \in \mathbf{R}.$$

Note that for any $\phi \in \mathcal{M}_*^+$, we have, by Theorem 2.8.1,

$$\sigma_t^\phi(x) = (D\phi : D\nu)_t \sigma_t^\nu(x)(D\phi : D\nu)_t^* \quad \text{for all } x \in \mathcal{M}_{s(\phi)}.$$

Thus

$$\pi_\nu(\sigma_t^\phi(x)) = \pi_\nu((D\phi : D\nu)_t)\Delta_\nu^{it}\pi_\nu(x)\Delta_\nu^{-it}\pi_\nu((D\phi : D\nu)_t^*), \qquad t \in \mathbf{R}.$$

To simplify the notation, we shall no longer write π_ν and we shall write

$$\phi^{it} = (D\phi : D\nu)_t \Delta_\nu^{it}.$$

Furthermore, we shall write $\nu^{it} = \Delta_\nu^{it}$.

Remarks:

i) $\phi^{it} \in \mathcal{L}(L^2(\mathcal{M},\nu))$ for all $\phi \in \mathcal{M}_*^+$.

ii) If ϕ is a faithful semifinite normal weight on $\mathcal{M}$, then ϕ^{it} exists and is an operator on $L^2(\mathcal{M},\nu)$.

iii) $\sigma_t^\phi(x) = \phi^{it}x\phi^{-it}$ for all $x \in \mathcal{M}_{s(\phi)}$. If $\sigma_t^\phi(x)$ exists by analytic continuation, then $\sigma_z^\phi(x) = \phi^{iz}x\phi^{-iz}$ [section 2.7].

iv) Using the properties of the Connes Radon-Nikodym cocycle [Theorem 2.8.1], we get:
   i) $\phi^{it}\phi^{is} = \phi^{i(t+s)}$.
   ii) $(\phi^{it})^* = \phi^{-it}$.
   iii) $\phi^{i0} = s(\phi)$.
   iv) $\psi^{it} = (D\psi : D\phi)_t\phi^{it}$, for $\psi, \phi \in \mathcal{M}_*^+$.

The above discussion motivates the construction of the identity correspondence that now follows:

Let us consider the following set:

$$D_\alpha = \{a_1\phi_1^{z_1}a_2\phi_2^{z_2}...a_n\phi_n^{z_n}a_{n+1} : a_i \in \mathcal{M},\ \phi_i \in \mathcal{M}_*^+,\ z_i \in \mathbb{C},\ Re(z_i) > 0,\ \sum_i z_i = \alpha\}.$$

Following Connes, the elements of D_α are called α-densities. By concatenation, we get a composition law:

$$D_\alpha \times D_\beta \to D_{\alpha+\beta}, \quad (\delta_1, \delta_2) \mapsto \delta_1\delta_2.$$

On D_α, consider the equivalence relation $\sim$ generated by:

i) $\phi^{z_1+z_2} = \phi^{z_1}\phi^{z_2}$,

ii) $\phi^z = (D\phi : D\psi)_{-iz}\psi^z$, if $(D\phi : D\psi)_{-iz} \in \mathcal{M}$,

iii) $\sigma_t^\phi(x) = \phi^{it}x\phi^{-it}$, for all $x \in \mathcal{M}_{s(\phi)}$, $t \in \mathbb{R}$. This relation will also be used for imaginary values of t and elements of $\mathcal{M}_{s(\phi)}$ for which $\sigma_t^\phi(x)$ exists by analytic continuation [section 2.7].

Note that we will identify $D_\alpha/\sim$ and D_α. Let $E_\alpha = \langle D_\alpha \rangle$ denote the vector space generated by D_α.

Definition 3.2.1: For $\alpha \in \mathbb{R}$, the map

$$* : D_\alpha \to D_\alpha, \quad a_1\phi_1^{z_1}a_2\phi_2^{z_2}...a_n\phi_n^{z_n}a_{n+1} \mapsto a_{n+1}^*\phi_n^{\overline{z_n}}a_n^*...a_2^*\phi_1^{\overline{z_1}}a_1^*$$

defines an involution on E_α.

Recall that by Proposition 2.8.5, we have:

Lemma 3.2.2: *Let $\phi, \psi \in \mathcal{M}_*^+$ be faithful and suppose $\phi \le \psi$. If $Re(z) \le \frac{1}{2}$, then $(D\phi : D\psi)_{-iz} \in \mathcal{M}$.*

Remarks:

i) Let $\delta = a_1\,\phi_1^{z_1}\ldots a_n\,\phi_n^{z_n}\,a_{n+1} \in D_1$. We can always assume that $Re(z_j) \leq \frac{1}{2}$ for all j. Indeed, suppose that $Re(z_j) > \frac{1}{2}$ for some j. Then

$$\delta \sim a_1\,\phi_1^{z_1}\ldots a_j\,\phi_j^{\frac{z_j}{2}}\,\phi_j^{\frac{z_j}{2}}\,a_{j+1}\ldots a_n\,\phi_n^{z_n}\,a_{n+1}.$$

ii) It is always possible to find a faithful form ψ on $\mathcal{M}$ such that $\psi \geq \sum \phi_j$. Indeed, as $\mathcal{M}$ is countably decomposable, there exists $\phi \in \mathcal{M}_*^+$ faithful. Now let $\psi = \phi + \sum \phi_j$. Then ψ has the desired properties.

iii) It can be shown that Proposition 2.8.5 and hence Lemma 3.2.2 remain valid if only one of the weights is faithful.

Define

$$\bar{D}_\alpha = \{a_1\,\psi^{z_1}\,a_2\,\psi^{z_2}\ldots a_n\,\psi^{z_n}\,a_{n+1} \in D_\alpha : \psi \in \mathcal{M}_*^+ \text{ is faithful }\}.$$

Lemma 3.2.3: *For all $\delta \in D_1$, there exists an element $\delta_1 \in \bar{D}_1$ such that $\delta \sim \delta_1$.*

Proof: Let $\delta = a_1\,\phi_1^{z_1}\ldots a_n\,\phi_n^{z_n}\,a_{n+1} \in D_1$; by Remark i) above, we may assume that $Re(z_j) \leq \frac{1}{2}$ for all j. Let $\psi \in \mathcal{M}_*^+$ be faithful and such that $\psi \geq \sum \phi_j$. By Lemma 3.2.2 and Remark iii) above, $u_{-iz_j} = (D\phi_j : D\psi)_{-iz_j} \in \mathcal{M}$ for all j. Hence

$$\delta \sim a_1\,u_{-iz_1}\,\psi^{z_1}\ldots a_n\,u_{-iz_n}\,\psi^{z_n}\,a_{n+1} \in \bar{D}_1.$$

Q.E.D.

We now define a notion of integral or expectation value for the elements of $\bar{D}_1$.

Definition 3.2.4: For all $\delta = a_1\,\psi^{z_1}\,a_2\,\psi^{z_2}\ldots a_n\psi^{z_n}\,a_{n+1} \in \bar{D}_1$, define

$$\langle\delta\rangle = F(z_1, \ldots, z_n)$$

where F is the unique bounded holomorphic function in the tube

$$\{(z_1, \ldots, z_n) \in \mathbf{C}^n : Re(z_j) > 0 \text{ and } \sum z_j = 1\}$$

with boundary values given for $t_1, \ldots, t_{n-1} \in \mathbf{R}$ by

$$F(it_1, \ldots, it_{n-1}, 1 - i\sum_{j=1}^{n-1} t_j) = \psi(a_1\,\sigma_{t_1}^{\psi}(a_2)\,\sigma_{t_1+t_2}^{\psi}(a_3)\ldots a_{n+1}).$$

Remark: The function F defined above is a function of n variables, with $z_n = 1 - \sum_{j=1}^{n-1} z_j$. Hence, if we view F as a function of $n-1$ independent variables, it corresponds to the function F defined in section 2.5.

Lemma 3.2.5: *If* $\delta_1, \delta_2 \in \bar{D}_1$ *and* $\delta_1 \sim \delta_2$ *then* $\langle \delta_1 \rangle = \langle \delta_2 \rangle$.

Proof: According to the definition of $\sim$, it is enough to check the following special cases:

i) $\delta_1 = a_1\, \phi^{z_1+\bar{z}_1}\, a_2 \ldots a_n\, \phi^{z_n}\, a_{n+1} \in \bar{D}_1$ and $\delta_2 = a_1\, \phi^{z_1}\, \phi^{\bar{z}_1}\, a_2 \ldots a_n\, \phi^{z_n}\, a_{n+1} \in \bar{D}_1$;

ii) $\delta_1 = a_1\, \phi^{z_1} \ldots a_n\, \phi^{z_n}\, a_{n+1} \in \bar{D}_1$ and $\delta_2 = a_1\, u_{-iz_1}\, \psi^{z_1} \ldots a_n\, u_{-iz_n}\, \psi^{z_n}\, a_{n+1} \in \bar{D}_1$, where $\psi \geq \phi$ is faithful and $u_{-iz_j} = (D\phi : D\psi)_{-iz_j}$ for all j;

iii) $\delta_1 = a_1\, \phi^{z_1}\, \sigma_t^{\phi}(x)\, \phi^{z_2} \ldots \phi^{z_n}\, a_{n+1} \in \bar{D}_1$ and $\delta_2 = a_1\, \phi^{z_1}\, \phi^{it}\, x\, \phi^{-it}\, \phi^{z_2} \ldots \phi^{z_n}\, a_{n+1} \in \bar{D}_1$ where $x \in \mathcal{M}_{s(\phi)}$.

Case i)

$\langle \delta_1 \rangle = F_1(z_1 + \bar{z}_1, \ldots, z_n)$ where F_1 is the unique bounded holomorphic function in the tube

$$B_1 = \{(w_1, \ldots, w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, \bar{t}_1, \ldots, t_{n-1} \in \mathbf{R}$ by

$$F_1\big(it_1 + i\bar{t}_1, \ldots, it_{n-1}, 1 - i(\sum_{j=1}^{n-1} t_j + \bar{t}_1)\big) = \phi(a_1\, \sigma^{\phi}_{t_1+\bar{t}_1}(a_2) \ldots a_{n+1})$$

and $\langle \delta_2 \rangle = F_2(z_1, \bar{z}_1, \ldots, z_n)$ where F_2 is the unique bounded holomorphic function in the tube

$$B_2 = \{(w_1, \ldots, w_{n+i}) \in \mathbf{C}^{n+1} : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, \bar{t}_1, \ldots, t_{n-1} \in \mathbf{R}$ by

$$F_2\big(it_1, i\bar{t}_1, \ldots, it_{n-1}, 1 - i(\sum_{j=1}^{n-1} t_j + \bar{t}_1)\big) = \phi(a_1\, \sigma^{\phi}_{\bar{t}_1}(1) \sigma^{\phi}_{t_1+\bar{t}_1}(a_2) \ldots a_{n+1}).$$

Consider the function

$$f : B_2 \to B_1 \qquad (w_1, w_2, \ldots, w_{n+1}) \mapsto (w_1 + w_2, w_3, \ldots, w_{n+1}).$$

Note that F_2 and $F_1 \circ f$ are two bounded holomorphic functions in the tube B_2 which are equal on the boundary. They therefore coincide everywhere, that is, $\langle \delta_1 \rangle = \langle \delta_2 \rangle$.

Case ii)

$\langle \delta_1 \rangle = F_1(z_1, \ldots, z_n)$ where F_1 is the unique bounded holomorphic function in the tube

$$\{(w_1, \ldots, w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, \ldots, t_{n-1} \in \mathbf{R}$ by

$$F_1(it_1, \ldots, it_{n-1}, 1 - i\sum_{j=1}^{n-1} t_j) = \phi(a_1\, \sigma^{\phi}_{t_1}(a_2) \ldots a_{n+1})$$

and $\langle \delta_2 \rangle = F_2(z_1, ..., z_n)$ where F_2 is the unique bounded holomorphic function in the tube

$$\{(w_1, ..., w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, ..., t_{n-1} \in \mathbf{R}$ by

$$F_2(it_1, ..., it_{n-1}, 1 - i\sum_{j=1}^{n-1} t_j)$$
$$= \psi((u_{-i/2})^* \, a_1 \, \sigma^\phi_{t_1}(a_2)...a_{n+1} \, u_{-i/2}) = \phi(a_1 \, \sigma^\phi_{t_1}(a_2)...a_{n+1})$$

by Proposition 2.8.5. As F_1 and F_2 are two bounded holomorphic functions in the tube

$$\{(w_1, ..., w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

which are equal on the boundary, they are equal everywhere and thus $\langle \delta_1 \rangle = \langle \delta_2 \rangle$.

Case iii)

$\langle \delta_1 \rangle = F_1(z_1, ..., z_n)$ where F_1 is the unique bounded holomorphic function in the tube

$$\{(w_1, ..., w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, ..., t_{n-1} \in \mathbf{R}$ by

$$F_1(it_1, ..., it_{n-1}, 1 - i\sum_{j=1}^{n-1} t_j) = \phi(a_1 \, \sigma^\phi_{t_1}(\sigma^\phi_t(x))...a_{n+1})$$

and $\langle \delta_2 \rangle = F_2(z_1, ..., z_n)$ where F_2 is the unique bounded holomorphic function in the tube

$$\{(w_1, ..., w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

with boundary values given for $t_1, ..., t_{n-1} \in \mathbf{R}$ by

$$F_2(it_1, ..., it_{n-1}, 1 - i\sum_{j=1}^{n-1} t_j) = \phi(a_1 \, \sigma^\phi_{t_1+t}(x) \, \sigma^\phi_{t_1+t_2}(a_3)...a_{n+1})$$
$$= \phi(a_1 \, \sigma^\phi_{t_1}(\sigma^\phi_t(x)) \, \sigma^\phi_{t_1+t_2}(a_3)...a_{n+1}).$$

As F_1 and F_2 are two bounded holomorphic functions in the tube

$$\{(w_1, ..., w_n) \in \mathbf{C}^n : Re(w_j) > 0 \text{ and } \sum w_j = 1\}$$

which are equal on the boundary, they coincide everywhere and thus $\langle \delta_1 \rangle = \langle \delta_2 \rangle$.

Q.E.D.

Definition 3.2.6: For all $\delta \in D_1$, define $\langle\delta\rangle := \langle\delta_1\rangle$, where $\delta_1 \in \tilde{D}_1$ and $\delta \sim \delta_1$.

Define a map $\langle\cdot,\cdot\rangle : D_{\frac{1}{2}} \times D_{\frac{1}{2}} \to \mathbf{C}$ by $\langle\delta_1, \delta_2\rangle = \langle\delta_1\, \delta_2^*\rangle$.

Remark: If $\delta_1 \sim \delta_1'$ and $\delta_2 \sim \delta_2'$ (where $\delta_i, \delta_i' \in D_{\frac{1}{2}}$), then $\langle\delta_1, \delta_2\rangle = \langle\delta_1', \delta_2'\rangle$.

Proposition 3.2.7: *Let* $u = \sum_i \alpha_i \delta_i^1$, $v = \sum_j \beta_j \delta_j^2 \in E_{\frac{1}{2}}$. *The map*

$$\langle\cdot,\cdot\rangle : E_{\frac{1}{2}} \times E_{\frac{1}{2}} \to \mathbf{C} \qquad (u,v) \mapsto \sum_{i,j} \alpha_i \overline{\beta_j} \langle\delta_i^1 (\delta_j^2)^*\rangle$$

defines a positive sesquilinear form on $E_{\frac{1}{2}}$, *satisfying* $\langle\delta_1, \delta_2\rangle = \overline{\langle\delta_2, \delta_1\rangle}$ *for all* $\delta_1, \delta_2 \in E_{\frac{1}{2}}$.

Proof: By definition, the map $\langle\cdot,\cdot\rangle$ is a sesquilinear form. Let us show that $\langle\delta,\delta\rangle \geq 0$ for $\delta \in D_{\frac{1}{2}}$. Write $\delta = a_1\phi^{z_1}a_2 ... \phi^{z_n}a_{n+1} \in D_{\frac{1}{2}}$; then

$$\begin{aligned}
\langle\delta,\delta\rangle = \langle\delta\delta^*\rangle &= \langle a_1\phi^{z_1}a_2 ... \phi^{z_n}a_{n+1}a_{n+1}^*\phi^{\overline{z_n}}a_n^* ... a_2^*\phi^{\overline{z_1}}a_1^*\rangle \\
&= F(z_1, z_2, ..., z_n, \overline{z_n}, ..., \overline{z_1}) \\
&= \left(a_1\Delta_\phi^{z_1}a_2 ... \Delta_\phi^{z_n}a_{n+1}\Omega_\phi \mid a_1\Delta_\phi^{z_1}a_2 ... \Delta_\phi^{z_n}a_{n+1}\Omega_\phi\right) \geq 0.
\end{aligned}$$

Now, let $\delta_1, \delta_2 \in D_{\frac{1}{2}}$; we may assume that

$$\delta_1 = a_1\phi^{z_1} ... a_n\phi^{z_n}a_{n+1} \qquad \text{and} \qquad \delta_2 = b_1\phi^{w_1} ... b_m\phi^{w_m}b_{m+1}$$

for some $\phi \in \mathcal{M}_*^+$ faithful. Then

$$\begin{aligned}
\langle\delta_1, \delta_2\rangle = \langle\delta_1\delta_2^*\rangle &= \left(a_1\Delta_\phi^{z_1} ... \Delta_\phi^{z_n}a_{n+1}b_{m+1}^*\Delta_\phi^{\overline{w_m}} ... b_2^*\Delta_\phi^{\overline{w_1}}b_1^*\Omega_\phi \mid \Omega_\phi\right) \\
&= \left(b_{m+1}^*\Delta_\phi^{\overline{w_m}} ... b_2^*\Delta_\phi^{\overline{w_1}}b_1^*\Omega_\phi \mid a_{n+1}^*\Delta_\phi^{\overline{z_n}} ... a_2^*\Delta_\phi^{\overline{z_1}}a_1^*\Omega_\phi\right) \\
&= \overline{\left(a_{n+1}^*\Delta_\phi^{\overline{z_n}} ... a_2^*\Delta_\phi^{\overline{z_1}}a_1^*\Omega_\phi \mid b_{m+1}^*\Delta_\phi^{\overline{w_m}} ... b_2^*\Delta_\phi^{\overline{w_1}}b_1^*\Omega_\phi\right)} \\
&= \overline{\left(b_1\Delta_\phi^{w_1} ... \Delta_\phi^{w_m}b_{m+1}a_{n+1}^*\Delta_\phi^{\overline{z_n}} ... a_2^*\Delta_\phi^{\overline{z_1}}a_1^*\Omega_\phi \mid \Omega_\phi\right)} \\
&= \overline{\langle\delta_2\delta_1^*\rangle} = \overline{\langle\delta_2, \delta_1\rangle}.
\end{aligned}$$

Q.E.D.

Now, let

$$N = \{\delta \in E_{\frac{1}{2}} : \langle\delta,\delta\rangle = \|\delta\|^2 = 0\}.$$

Then, $\langle\cdot,\cdot\rangle$ defines an inner product on $E_{\frac{1}{2}}/N$. Let us denote by $L^2(\mathcal{M})$ the Hilbert space that we obtain by completing $E_{\frac{1}{2}}/N$ with respect to the inner product $\langle\cdot,\cdot\rangle$.

Lemma 3.2.8: *For $\delta \in \bar{D}_1$ fixed, the map*

$$T : \mathcal{M} \to \mathbf{C} \qquad x \longmapsto \langle x\,\delta\rangle\,.$$

defines a normal linear form on $\mathcal{M}$.

Proof: Note that δ can be written as the composition of two elements of $\bar{D}_{\frac{1}{2}}$. Then $\langle x\,\delta\rangle = \langle x\,\delta_1\,\delta_2\rangle = \langle x\,\delta_1, \delta_2^*\rangle$, where $\delta_1, \delta_2 \in \bar{D}_{\frac{1}{2}}$. Note that:

$$\begin{aligned} 4\,\langle x\,\delta_1, \delta_2^*\rangle = &\langle x\,(\delta_1 + \delta_2^*), (\delta_1 + \delta_2^*)\rangle - \langle x\,(\delta_1 - \delta_2^*), (\delta_1 - \delta_2^*)\rangle \\ &+ i\,\langle x\,(\delta_1 + i\delta_2^*), (\delta_1 + i\delta_2^*)\rangle - i\,\langle x\,(\delta_1 - i\delta_2^*), (\delta_1 - i\delta_2^*)\rangle\,. \end{aligned}$$

It is therefore enough to consider the case $\delta = \delta_1\,\delta_1^*$, for some $\delta_1 \in \bar{D}_{\frac{1}{2}}$. Let

$$\delta_1 = c_1\phi^{z_1}c_2 ... c_n\phi^{z_n}c_{n+1} \in \bar{D}_{\frac{1}{2}}$$

and let $(b_m)_{m\geq 1}$ be a monotone increasing net in $\mathcal{M}^+$; then

$$\begin{aligned} \langle \sup(b_m)\,\delta\rangle &= \langle \sup(b_m)\,\delta_1\delta_1^*\rangle \\ &= \left(\sup(b_m)\,c_1\Delta_\phi^{z_1}c_2 ... c_n\Delta_\phi^{z_n}c_{n+1}\Omega_\phi \mid c_1\Delta_\phi^{z_1}c_2 ... c_n\Delta_\phi^{z_n}c_{n+1}\Omega_\phi\right) \\ &= (\sup(b_m)\,\eta_\phi \mid \eta_\phi) \end{aligned}$$

where $\eta_\phi = c_1\Delta_\phi^{z_1}c_2 ... c_n\Delta_\phi^{z_n}c_{n+1}\Omega_\phi \in \mathcal{H}_\phi$. As for any $\eta \in \mathcal{H}_\phi$ the form

$$\mathcal{M} \to \mathbf{C}, \qquad y \mapsto (y\,\eta \mid \eta)$$

is normal, we have the desired result.

Q.E.D.

Lemma 3.2.9: *For any $x \in \mathcal{M}$ and $\delta \in \bar{D}_1$, $\langle x\delta\rangle = \langle \delta x\rangle$.*

Proof: We will only give the proof in the case $\delta = b_1\phi^z b_2\phi^{1-z}$. For such a δ,

$$\langle x\delta\rangle = \left\langle x b_1\phi^z b_2\phi^{1-z}\right\rangle = F_1(z)$$

where F_1 is the unique bounded holomorphic function in $\{z \in \mathbf{C} : 0 \leq Re(z) \leq 1\}$ such that

$$F_1(it) = \left\langle x b_1\phi^{it}b_2\phi^{1-it}\right\rangle = \phi(x b_1\,\sigma_t^\phi(b_2)) \qquad \text{and}$$
$$F_1(1+it) = \left\langle x b_1\phi^{1+it}b_2\phi^{-it}\right\rangle = \phi(\sigma_t^\phi(b_2)x b_1).$$

Now,

$$\langle \delta x\rangle = \left\langle b_1\phi^z b_2\phi^{1-z}x\right\rangle = F_2(z)$$

where F_2 is the unique bounded holomorphic function in $\{z \in \mathbf{C} : 0 \leq Re(z) \leq 1\}$ such that

$$F_2(it) = \langle b_1 \phi^{it} b_2 \phi^{1-it} x \rangle = \phi(x b_1 \sigma_t^{\phi}(b_2)) \qquad \text{and}$$
$$F_2(1+it) = \langle b_1 \phi^{1+it} b_2 \phi^{-it} x \rangle = \phi(\sigma_t^{\phi}(b_2) x b_1).$$

As F_1 and F_2 are two bounded holomorphic functions in $\{z \in \mathbf{C} : 0 \leq Re(z) \leq 1\}$ that coincide on the boundary, they therefore coincide everywhere.

Q.E.D.

The canonical involution J of $L^2(\mathcal{M})$, defined as follows:

$$J\delta = \delta^*, \quad \text{for all } \delta \in L^2(\mathcal{M})$$

is isometric.

Lemma 3.2.10: *$L^2(\mathcal{M})$ is a normal bimodule for the action:*

$$(x, \delta, y) \longmapsto x\,\delta\,y, \qquad \textit{for all } x, y \in \mathcal{M},\ \delta \in L^2(\mathcal{M}).$$

Proof: Let $\pi_1 : \mathcal{M} \to \mathcal{L}(L^2(\mathcal{M}))$ be defined by $\pi_1(x)\delta = x\delta$ and $\pi_2 : \mathcal{M}^o \to \mathcal{L}(L^2(\mathcal{M}))$ be defined by $\pi_2(y^o)\delta = \delta y$, for all $\delta \in L^2(\mathcal{M})$ and $x, y \in \mathcal{M}$. Let us show that π_1 and π_2 are normal: let $(x_i)_{i \geq 1}$ be a monotone increasing net in $\mathcal{M}^+$ which converges to the element $x \in \mathcal{M}^+$. For all $\delta \in L^2(\mathcal{M})$, we have:

$$\langle \pi_1(x_i)\delta, \delta \rangle = \langle x_i\delta, \delta \rangle = \langle x_i \delta \delta^* \rangle \to \langle x \delta \delta^* \rangle = \langle \pi_1(x)\delta, \delta \rangle$$

by Lemma 3.2.8. Thus $\pi_1(x_i) \to \pi_1(x)$. We also have

$$\langle \pi_2(x_i^o)\delta, \delta \rangle = \langle \delta x_i, \delta \rangle = \langle J\delta, J(\delta x_i) \rangle = \langle \delta^*, x_i^* \delta^* \rangle = \langle \delta^* \delta x_i \rangle \to \langle \delta^* \delta x \rangle = \langle \pi_2(x^o)\delta, \delta \rangle$$

by Lemma 3.2.9. Thus $\pi_2(x_i^o) \to \pi_2(x^o)$.

Q.E.D.

Remark: J exchanges the left and right actions of $\mathcal{M}$ on $L^2(\mathcal{M})$.

Let $\phi \in \mathcal{M}_*^+$ be faithful. Then $\phi^{\frac{1}{2}} \in L^2(\mathcal{M})$ is cyclic and separating for $\mathcal{M}$. Indeed, let $y \in \mathcal{M}$. If $y\phi^{\frac{1}{2}} = 0$, then

$$0 = \left\langle y\phi^{\frac{1}{2}}, y\phi^{\frac{1}{2}} \right\rangle = \phi(y^* y);$$

as ϕ is faithful, then $y = 0$. Thus $\phi^{\frac{1}{2}}$ is separating for $\mathcal{M}$. Moreover, $\mathcal{M}\phi^{\frac{1}{2}}$ is dense in $L^2(\mathcal{M})$. We can thus write:

$$L^2(\mathcal{M}) = [\mathcal{M}\phi^{\frac{1}{2}}].$$

Remark: As seen in Example 2.1 d, $\mathcal{M}\phi^{\frac{1}{2}}$ is a full left Hilbert algebra with the following operations:

$$(x\phi^{\frac{1}{2}})(y\phi^{\frac{1}{2}}) = (xy)\phi^{\frac{1}{2}}, \qquad (x\phi^{\frac{1}{2}})^{\sharp} = x^*\phi^{\frac{1}{2}} \quad \text{for all } x, y \in \mathcal{M}.$$

Moreover, $\mathcal{M}'$ corresponds to the right action of $\mathcal{M}$ on $L^2(\mathcal{M})$ [Theorem 2.1.5] and the set

$$\mathcal{P}_\phi = \overline{\{xJ_\phi xJ_\phi\phi^{\frac{1}{2}} : x \in \mathcal{M}\}} \subset L^2(\mathcal{M})$$

is a selfdual closed convex cone.

Consider the operator S_ϕ on $L^2(\mathcal{M})$ defined by $S_\phi(x\phi^{\frac{1}{2}}) = x^*\phi^{\frac{1}{2}}$. Then $S_\phi = J_\phi\Delta_\phi^{\frac{1}{2}}$ where

$$J_\phi(x\phi^{\frac{1}{2}}) = \phi^{\frac{1}{2}}x^*, \qquad \Delta_\phi^{\frac{1}{2}}(x\phi^{\frac{1}{2}}) = \phi^{\frac{1}{2}}x.$$

Thus $J_\phi = J$ for all $\phi \in \mathcal{M}_*^+$ faithful.

Remarks:

i) On $\mathcal{M}\phi^{\frac{1}{2}} \subset D(\Delta_\phi^{\frac{1}{2}})$, we have $\Delta_\phi^{\frac{1}{2}}(\cdot) = \phi^{\frac{1}{2}} \cdot \phi^{-\frac{1}{2}}$, as $\Delta_\phi^{\frac{1}{2}}(x\phi^{\frac{1}{2}}) = \phi^{\frac{1}{2}}x = \phi^{\frac{1}{2}}(x\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}$.

ii) For all $\alpha \in \mathbf{C}$ with $0 \leq Re(\alpha) \leq \frac{1}{2}$,

$$D(\Delta_\phi^\alpha) = D(\Delta_\phi^{\frac{1}{2}}) \qquad \text{and} \qquad \Delta_\phi^\alpha(x\phi^{\frac{1}{2}}) = \phi^\alpha x\phi^{\frac{1}{2}-\alpha}.$$

In particular, $\Delta_\phi^{\frac{1}{4}}(x\phi^{\frac{1}{2}}) = \phi^{\frac{1}{4}}x\phi^{\frac{1}{4}}$.

iii) $\mathcal{P}_\phi^\sharp = \overline{\mathcal{M}^+\phi^{\frac{1}{2}}}$ and $\mathcal{P}_\phi^\natural = \overline{\Delta_\phi^{\frac{1}{4}}\mathcal{P}_\phi^\sharp} = \overline{\{\phi^{\frac{1}{4}}x\phi^{\frac{1}{4}} : x \in \mathcal{M}^+\}}$.

iv) If $x \in \mathcal{T}_\phi$, then

$$\phi^{\frac{1}{4}}x^*x\phi^{\frac{1}{4}} = \phi^{\frac{1}{4}}x^*\phi^{-\frac{1}{4}}\phi^{\frac{1}{2}}\phi^{-\frac{1}{4}}x\phi^{\frac{1}{4}} = \sigma^\phi_{-\frac{i}{4}}(x^*)\,\phi^{\frac{1}{2}}\sigma^\phi_{\frac{i}{4}}(x) = \sigma^\phi_{-\frac{i}{4}}(x^*)\,\phi^{\frac{1}{2}}(\sigma^\phi_{-\frac{i}{4}}(x^*))^*.$$

Hence

$$\mathcal{P}_\phi^\natural = \overline{\{x\phi^{\frac{1}{2}}x^* : x \in \mathcal{T}_\phi\}} = \overline{\{x\phi^{\frac{1}{2}}x^* : x \in \mathcal{M}\}}.$$

Lemma 3.2.11: *Let $\mathcal{M}$ be a countably decomposable von Neumann algebra and $\phi \in \mathcal{M}_*^+$ be faithful. Then the set $Face(\phi) = \{\psi \in \mathcal{M}_*^+ : \psi \leq \alpha\phi$ for some $\alpha > 0\}$ is dense in $\mathcal{M}_*^+$.*

Proof: [H; Lemma 2.16]

Let ϕ be a normal semifinite faithful weight on $\mathcal{M}$ and ψ a normal semifinite weight on $\mathcal{M}$ such that $\psi \leq \phi$. Let $\theta = \theta(\phi, \psi)$ be the normal semifinite weight on $\mathbf{M}_2(\mathcal{M})$ defined by

$$\theta\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \phi(x_{11}) + \psi(x_{22}), \quad \text{for } \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathbf{M}_2(\mathcal{M})^+.$$

Then the support $\mathrm{s}(\theta)$ of θ is equal to $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{s}(\psi) \end{pmatrix}$. We have:

$$\mathrm{s}(\theta)\mathrm{M}_2(\mathcal{M})\mathrm{s}(\theta) = \begin{pmatrix} \mathcal{M} & \mathcal{M}\mathrm{s}(\psi) \\ \mathrm{s}(\psi)\mathcal{M} & \mathrm{s}(\psi)\mathcal{M}\mathrm{s}(\psi) \end{pmatrix}.$$

Now, let $x_o = \mathrm{s}(\psi) \otimes e_{21}$ and $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathcal{M})^+$; we have:

$$\begin{aligned} \theta(x_o \mathrm{s}(\theta) y \mathrm{s}(\theta) x_o^*) &= \theta\left(\begin{pmatrix} 0 & 0 \\ \mathrm{s}(\psi) & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{s}(\psi) \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{s}(\psi) \end{pmatrix}\begin{pmatrix} 0 & \mathrm{s}(\psi) \\ 0 & 0 \end{pmatrix}\right) \\ &= \theta\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{s}(\psi) a \mathrm{s}(\psi) \end{pmatrix} = \psi(\mathrm{s}(\psi) a \mathrm{s}(\psi)) = \psi(a) \leq \theta(y). \end{aligned}$$

Lemma 2.8.4 applied to $(\theta,\ \mathrm{s}(\theta)\mathrm{M}_2(\mathcal{M})\mathrm{s}(\theta))$ and x_o implies that the map $t \mapsto \sigma_t^\theta(x_o)$, for $t \in \mathbf{R}$, extends to an $\mathrm{s}(\theta)\mathrm{M}_2(\mathcal{M})\mathrm{s}(\theta)$-valued bounded ultra-weakly continuous map on $B_{-\frac{1}{2}} = \{z \in \mathbf{C} : Im(z) \in [-\frac{1}{2}, 0]\}$ which is analytic in the interior of $B_{-\frac{1}{2}}$ and satisfies $\|\sigma^\theta_{-\frac{i}{2}}(x_o)\| \leq 1$. Note that for $t \in \mathbf{R}$,

$$\sigma_t^\theta(x_o) = u_t \sigma_t^\phi(\mathrm{s}(\psi)) \otimes e_{21} = \mathrm{s}(\psi) u_t \otimes e_{21}$$

where $u_t = (D\psi : D\phi)_t$. Thus, $t \mapsto u_t$ extends to a $\mathrm{s}(\psi)\mathcal{M}\mathrm{s}(\psi)$-valued bounded ultra-weakly continuous map on $B_{-\frac{1}{2}}$ which is analytic in the interior of $B_{-\frac{1}{2}}$ and satisfies $\|u_{-\frac{i}{2}}\| \leq 1$. Note that by the properties of the Radon-Nikodym cocycle, we have

$$u_t = u_{t+0} = u_t \sigma_t^\phi(u_0) = u_t u_t^* u_t = \mathrm{s}(\psi) u_t.$$

Consequently, we have:

Lemma 3.2.12: *Let $\phi \in \mathcal{M}_*^+$ be faithful. Then $\{\psi^{\frac{1}{2}} : \psi \in Face(\phi)\} \subset \mathcal{P}_\phi^\natural$.*

Proof: Let $\psi \in \mathrm{Face}(\phi)$ and $u_t = (D\psi : D\phi)_t$ be the Radon-Nikodym cocycle associated to the forms ϕ and ψ. We have

$$u_{s+t} = u_s \sigma_s^\phi(u_t) = u_s \phi^{is} u_t \phi^{-is};$$

hence, $u_s \phi^{is} = u_{s+t} \phi^{is} u_t^*$. Let $t = -\frac{s}{2}$; then

$$u_s \phi^{is} = u_{\frac{s}{2}} \phi^{is} u_{-\frac{s}{2}}^* = u_{\frac{s}{2}} \phi^{is} (\sigma^\phi_{-\frac{s}{2}}(u_{\frac{s}{2}}^*))^* = u_{\frac{s}{2}} \phi^{is} \sigma^\phi_{-\frac{s}{2}}(u_{\frac{s}{2}}) = u_{\frac{s}{2}} \phi^{\frac{is}{2}} u_{\frac{s}{2}} \phi^{\frac{is}{2}}.$$

By the preceding discussion, the mapping $s \to u_s$ extends analytically from $s \in \mathbf{R}$ to $Im(s) \in [-\frac{1}{2}, 0]$. By setting $s = -\frac{i}{2}$ in the equation $\psi^{is} = u_s \phi^{is}$, we get:

$$\psi^{\frac{1}{2}} = u_{-\frac{i}{4}} \phi^{\frac{1}{4}} u_{-\frac{i}{4}} \phi^{\frac{1}{4}} = u_{-\frac{i}{4}} \phi^{\frac{1}{2}} (u_{-\frac{i}{4}})^*,$$

as $(\psi^{\frac{1}{4}})^* = \psi^{\frac{1}{4}}$. We thus have the desired result.

Q.E.D.

Proposition 3.2.13: *The map*

$$T : \mathcal{M}_*^+ \longrightarrow L^2(\mathcal{M})^+, \qquad \phi \longmapsto \phi^{\frac{1}{2}}$$

is a homeomorphism, where $L^2(\mathcal{M})^+ = \{\phi^{\frac{1}{2}} : \phi \in \mathcal{M}_*^+\}$.

Proof: As the surjectivity of T is clear, let us verify that T is indeed injective. Suppose $T(\phi) = T(\psi)$ for any $\phi, \psi \in \mathcal{M}_*^+$. Then

$$\phi(x) = \left\langle x\phi^{\frac{1}{2}}, \phi^{\frac{1}{2}} \right\rangle = \left\langle x\psi^{\frac{1}{2}}, \psi^{\frac{1}{2}} \right\rangle = \psi(x)$$

for all $x \in \mathcal{M}$. Thus $\phi = \psi$. Now, let us show that T^{-1} is continuous. Let $\phi^{\frac{1}{2}} \in L^2(\mathcal{M})^+$ and $(\phi_m^{\frac{1}{2}})_{m\geq 1} \subset L^2(\mathcal{M})^+$ such that $\phi_m^{\frac{1}{2}} \rightarrow \phi^{\frac{1}{2}}$. Then for all $x \in \mathcal{M}$,

$$\begin{aligned} \left| \left\langle x\phi_m^{\frac{1}{2}}, \phi_m^{\frac{1}{2}} \right\rangle - \left\langle x\phi^{\frac{1}{2}}, \phi^{\frac{1}{2}} \right\rangle \right| &\leq \left| \left\langle x(\phi_m^{\frac{1}{2}} - \phi^{\frac{1}{2}}), \phi_m^{\frac{1}{2}} \right\rangle \right| + \left| \left\langle x\phi^{\frac{1}{2}}, \phi_m^{\frac{1}{2}} - \phi^{\frac{1}{2}} \right\rangle \right| \\ &\leq \|x(\phi_m^{\frac{1}{2}} - \phi^{\frac{1}{2}})\|^{\frac{1}{2}} \|\phi_m^{\frac{1}{2}}\|^{\frac{1}{2}} + \|x\phi^{\frac{1}{2}}\|^{\frac{1}{2}} \|\phi_m^{\frac{1}{2}} - \phi^{\frac{1}{2}}\|^{\frac{1}{2}} \\ &\rightarrow 0. \end{aligned}$$

Let $\phi \in \mathcal{M}_*^+$ be faithful; then by Lemma 3.2.11, Face(ϕ) is dense in $\mathcal{M}_*^+$. Moreover, note that $T(\text{Face}(\phi)) \subset L^2(\mathcal{M})^+$. Let $(\psi_k)_{k\geq 1} \subset \text{Face}(\phi)$ such that $\psi_k \rightarrow \psi$ in $\mathcal{M}_*^+$. Let us show that $\psi_k^{\frac{1}{2}} \rightarrow \psi^{\frac{1}{2}}$ in $L^2(\mathcal{M})^+$. By Lemma 3.2.12, $\psi_k^{\frac{1}{2}} \in \mathcal{P}_\phi^\natural$ for all k. Suppose first that $\psi \in \text{Face}(\phi)$; then $\psi^{\frac{1}{2}} \in \mathcal{P}_\phi^\natural$ and

$$\|\psi_k^{\frac{1}{2}} - \psi^{\frac{1}{2}}\|^2 \leq \|\psi_k - \psi\| \qquad \text{[Prop. 2.9.3]}.$$

Hence $T|_{\text{Face}(\phi)}$ is continuous. Now suppose that $\psi \notin \text{Face}(\phi)$. Then $\chi = \psi + \phi \in \mathcal{M}_*^+$ and is faithful. In addition, $\psi \in \text{Face}(\chi)$ and $\psi_k \in \text{Face}(\chi)$ for all k. Thus $\psi^{\frac{1}{2}}, \psi_k^{\frac{1}{2}} \in \mathcal{P}_\chi^\natural$ [Lemma 3.2.12] and

$$\|\psi_k^{\frac{1}{2}} - \psi^{\frac{1}{2}}\|^2 \leq \|\psi_k - \psi\| \qquad \text{[Prop. 2.9.3]}.$$

Consequently, T has a continuous extension to $\mathcal{M}_*^+$.

Q.E.D.

Lemma 3.2.14: *Let* $\phi \in \mathcal{M}_*^+$ *be faithful. Then* $L^2(\mathcal{M})^+ = \mathcal{P}_\phi^\natural$.

Proof: Let $\xi \in \mathcal{P}_\phi^\natural$ and w_ξ the associated form in $\mathcal{M}_*^+$ which is defined by $w_\xi(x) = (x\xi \mid \xi)$. By Lemma 3.2.11 there exists a sequence $(\phi_k)_{k\geq 1} \subset \text{Face}(\phi)$ such that $\phi_k \rightarrow w_\xi$. Thus $\phi_k^{\frac{1}{2}} \rightarrow \xi$ in $\mathcal{P}_\phi^\natural$ [Prop. 2.9.3] and $\phi_k^{\frac{1}{2}} \rightarrow w_\xi^{\frac{1}{2}}$ [Prop. 3.2.13]. Hence $w_\xi^{\frac{1}{2}} = \xi \in L^2(\mathcal{M})^+$, and consequently, $\mathcal{P}_\phi^\natural = L^2(\mathcal{M})^+$.

Q.E.D.

$(\mathcal{M}, L^2(\mathcal{M}), L^2(\mathcal{M})^+, J)$ is a standard form [section 2.9] and by Proposition 2.9.8, any element of $L^2(\mathcal{M})$ admits a left polar decomposition:

$$\delta = u\phi^{\frac{1}{2}}$$

where $\phi \in \mathcal{M}_*^+$ and u is a partial isometry with initial support $u^*u = s(\phi)$.

Definition 3.2.15: Let $\mathcal{M}$ be a von Neumann algebra. The identity correspondence $L^2(\mathcal{M})$ between $\mathcal{M}$ and $\mathcal{M}$ is the bimodule of $\frac{1}{2}$-densities on $\mathcal{M}$.

Let ν be a faithful normal semifinite weight on $\mathcal{M}$. As seen in the beginning of this section, we can associate to ν, via the GNS construction and Tomita-Takesaki theory, $(L^2(\mathcal{M}, \nu), \pi_\nu, \eta_\nu, \mathcal{U}_\nu, J_\nu, \Delta_\nu)$.

Remark: π_ν is normal since ν is.

By Theorem 2.1.7,

$$J_\nu \pi_\nu(\mathcal{M}) J_\nu = \pi_\nu(\mathcal{M})'.$$

Now, consider the map

$$\pi_\nu^o : \mathcal{M}^o \to \mathcal{L}(L^2(\mathcal{M}, \nu))$$

defined by

$$\pi_\nu^o(x^o) = J_\nu \pi_\nu(x)^* J_\nu.$$

This map is a normal *-representation of $\mathcal{M}^o$ in $L^2(\mathcal{M}, \nu)$ and therefore $L^2(\mathcal{M}, \nu)$ is an $\mathcal{M} - \mathcal{M}$ bimodule. In addition, the Hilbert space $L^2(\mathcal{M}, \nu)$ comes equipped with a natural selfdual cone $L^2(\mathcal{M}, \nu)^+$, given by:

$$L^2(\mathcal{M}, \nu)^+ = \mathcal{P}_\nu^\natural = \overline{(\Delta_\nu^{\frac{1}{4}} \mathcal{P}_\nu^\sharp)} \qquad \text{[section 2.9]}.$$

Hence $(\mathcal{M}, L^2(\mathcal{M}, \nu), L^2(\mathcal{M}, \nu)^+, J_\nu)$ is a standard form [section 2.9].

By Lemma 2.9.7, the elements of $L^2(\mathcal{M}, \nu)^+$ are in bijection with the positive cone $\mathcal{M}_*^+$ of the predual of $\mathcal{M}$:

$$\xi \in L^2(\mathcal{M}, \nu)^+ \longmapsto \omega_{\xi,\xi} \in \mathcal{M}_*^+$$

where $\omega_{\xi,\xi}(x) = (\pi_\nu(x)\xi \mid \xi)$ for all $x \in \mathcal{M}$. It follows that there exists a unique unitary equivalence $U : L^2(\mathcal{M}, \nu) \to L^2(\mathcal{M})$ satisfying:

$$U(\xi) = (\omega_{\xi,\xi})^{\frac{1}{2}}, \qquad \text{for all } \xi \in L^2(\mathcal{M}, \nu)^+.$$

Remark: Denote $U^\dagger : L^2(\mathcal{M}, \nu)^+ \to L^2(\mathcal{M})$ the map defined as above (that is, $U^\dagger(\xi) = (\omega_{\xi,\xi})^{\frac{1}{2}}$). $U^\dagger$ can easily be extended to U as $L^2(\mathcal{M}, \nu)^+$ is defined in such a way that any element of $L^2(\mathcal{M}, \nu)$ is a linear combination of four elements of $L^2(\mathcal{M}, \nu)^+$ [Proposition 2.9.3]. Thus U is completely determined by $U^\dagger$ and it is unique.

Using the canonical map η_ν from $\mathfrak{N}_\nu$ (also denoted $Dom_{\frac{1}{2}}(\nu)$) to $L^2(\mathcal{M}, \nu)$, the isometry U allows to extend the notation $\phi^{\frac{1}{2}}$, $\phi \in \mathcal{M}_*^+$ to weights:

$$x\nu^{\frac{1}{2}} := U\eta_\nu(x), \qquad \text{for all } x \in \mathfrak{N}_\nu.$$

3.3 Correspondences and *-homomorphisms

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Let $\rho : \mathcal{M} \to \mathcal{N}$ be a normal *-homomorphism. Suppose we do not necessarily have $\rho(1) = 1$. Then, $\rho(1) = e$ is a projection. We will denote by $L^2(\rho)$ the subspace of $L^2(\mathcal{N})$:

$$\{\xi \in L^2(\mathcal{N}) : \xi e = \xi\}.$$

As e is a projection of $L^2(\mathcal{N})$, then $L^2(\rho)$ is a Hilbert space. Moreover, $L^2(\rho)$ is an $\mathcal{N} - \mathcal{M}$ bimodule with:

$$\pi_{\mathcal{N}}(y)\, \pi_{\mathcal{M}^\circ}(x^\circ)\, \xi = y\, \xi\, \rho(x), \quad \text{for all } y \in \mathcal{N},\ x \in \mathcal{M}.$$

The properties that a bimodule must satisfy are verified easily. Note that for $\xi \in L^2(\rho)$ and $x \in \mathcal{M}$, as $\xi\, \rho(x) = \xi\, \rho(x1) = \xi\, \rho(x)\, \rho(1) = \xi\, \rho(x) e$, then $\xi\, \rho(x) \in L^2(\rho)$.

Assume that both $\mathcal{M}$ and $\mathcal{N}$ have separable preduals and $\mathcal{H}$ is separable.

Proposition 3.3.1: *Suppose $\mathcal{N}$ is a properly infinite von Neumann algebra.*

i) *Every correspondence $\mathcal{H}$ between $\mathcal{M}$ and $\mathcal{N}$ is equivalent to an $L^2(\rho)$.*

ii) *If ρ_i is a normal *-homomorphism of $\mathcal{M}$ in $\mathcal{N}$ for $i = 1, 2$, then the intertwining operators $T : L^2(\rho_1) \to L^2(\rho_2)$ are the elements y of $\rho_2(1)\, \mathcal{N}\, \rho_1(1)$ such that:*

$$\rho_2(x)\, y = y\, \rho_1(x), \quad \textit{for all } x \in \mathcal{M}.$$

Proof:

i) As $\mathcal{H}$ is separable, $\mathcal{N}'$ is countably decomposable [section 1.6]; by applying Lemma 1.6.6 to $\mathcal{N}'$, it follows that $\mathcal{N}'$ has a separating vector, say ξ, which is therefore cyclic for $\mathcal{N}$ [section 1.1]. Hence, $[\mathcal{N}\xi] = \mathcal{H}$. Let $\nu \in \mathcal{N}_*^+$ be defined by $\nu(\cdot) = (\cdot\, \xi \mid \xi)$ and consider the map $U : \mathcal{N}\xi \subset \mathcal{H} \to L^2(\mathcal{N})$ defined by $y\xi \mapsto y\nu^{\frac{1}{2}}$. Note that U is a $\mathcal{N}$-module linear isometry:

$$U(y\eta) = yU(\eta), \quad \text{for all } y \in \mathcal{N},\ \eta \in \mathcal{N}\xi;$$

$$\|y\xi\|^2 = (y\xi \mid y\xi) = (y^* y \xi \mid \xi) = \nu(y^* y);$$

$$\langle U(y\xi), U(y\xi)\rangle = \left\langle y\nu^{\frac{1}{2}}, y\nu^{\frac{1}{2}}\right\rangle = \left\langle y\nu^{\frac{1}{2}}\nu^{\frac{1}{2}}y^*\right\rangle = \langle y\nu y^*\rangle = \nu(y^*y).$$

U can therefore be extended to a linear map from $\mathcal{H}$ to $L^2(\mathcal{N})$. Let

$$e : L^2(\mathcal{N}) \to U(\mathcal{H}) = [\mathcal{N}\nu^{\frac{1}{2}}]$$

be the orthogonal projection. As $U(\mathcal{H})$ is $\mathcal{N}$-invariant, e belongs to the commutant of $\mathcal{N}$ in $L^2(\mathcal{N})$ by Lemma 1.1.5. The Hilbert space $\mathcal{H}$ will therefore be identified to $U(\mathcal{H}) = L^2(\mathcal{N})\,e$. Let

$$\pi_{\mathcal{N}} : \mathcal{N} \to \mathcal{L}(L^2(\mathcal{N})\,e), \quad \pi_{\mathcal{N}}(y)\,\xi = y\,\xi, \quad \forall y \in \mathcal{N},\, \xi \in L^2(\mathcal{N})\,e,$$

$$\pi : \mathcal{N} \to \mathcal{L}(L^2(\mathcal{N})), \quad \pi(y)\,\xi = y\,\xi, \quad \forall y \in \mathcal{N},\, \xi \in L^2(\mathcal{N}).$$

Then, we have:

$$\pi_{\mathcal{N}}(\mathcal{N}) = \pi(\mathcal{N})_e = \{z \in \pi(\mathcal{N}) : ze = ez = z\} \qquad \text{and}$$

$$\pi_{\mathcal{N}}(\mathcal{N})' = (\pi(\mathcal{N})')_e = \{z' \in \pi(\mathcal{N})' : z'e = ez' = z'\} = e\pi(\mathcal{N})'e,$$

that is, $\pi_{\mathcal{N}}(\mathcal{N})'$ is the algebra of right multiplications in $L^2(\mathcal{N})\,e$ by elements of $e\mathcal{N}e$. As $\pi_{\mathcal{M}^o}$ describes the right action, we get a normal *-homomorphism $\rho : \mathcal{M} \to e\mathcal{N}e$ satisfying $\rho(1) = e$ and defined by

$$\pi_{\mathcal{M}^o}(x^o)\,\xi = \xi\rho(x), \quad \text{for all } x \in \mathcal{M},\, \xi \in L^2(\mathcal{N})\,e.$$

ii) Let us write $L^2(\rho_i) = L^2(\mathcal{N})\,e_i$, where $e_i = \rho_i(1)$ for $i = 1, 2$. Let

$$\pi^i_{\mathcal{N}} : \mathcal{N} \to \mathcal{L}(L^2(\mathcal{N})\,e_i), \quad \pi^i_{\mathcal{N}}(y)\,\xi_i = y\,\xi_i, \quad \forall y \in \mathcal{N},\ \xi_i \in L^2(\mathcal{N})\,e_i,\ i = 1, 2;$$

$$\pi^i_{\mathcal{M}^o} : \mathcal{M}^o \to \mathcal{L}(L^2(\mathcal{N})\,e_i), \quad \pi^i_{\mathcal{M}^o}(x^o)\,\xi_i = \xi_i\,\rho_i(x), \quad \forall x \in \mathcal{M},\ \xi_i \in L^2(\mathcal{N})\,e_i,\ i = 1, 2.$$

Let $V : L^2(\mathcal{N})\,e_2 \to L^2(\mathcal{N})\,e_1$ be an intertwining operator. Then

$$y\,(V\,\xi) = V(y\,\xi), \qquad \text{for all } \xi \in L^2(\mathcal{N})\,e_2,\ y \in \mathcal{N}.$$

Hence, $V \in \mathcal{N}'$. Note that

$$\mathcal{N}(L^2(\mathcal{N})\,e_i) = L^2(\mathcal{N})\,e_i,\ i = 1, 2;$$

therefore, $V \in e_2\,\mathcal{N}'\,e_1$. Moreover, V satisfies:

$$(\pi^2_{\mathcal{M}^o}(x)\,\xi)\,V = \pi^1_{\mathcal{M}^o}(x)(\xi\,V), \qquad \xi \in L^2(\mathcal{N})\,e_2,\ x \in \mathcal{M},$$

that is $(\xi\,\rho_2(x))\,V = (\xi\,V)\,\rho_1(x)$ for all $\xi \in L^2(\mathcal{N})\,e_2$. Therefore, $\rho_2(x)\,V = V\,\rho_1(x)$ for all $x \in \mathcal{M}$.

Q.E.D.

Let $\mathcal{N}$ be a von Neumann algebra and $\epsilon : \mathrm{Aut}(\mathcal{N}) \to \mathrm{Out}(\mathcal{N})$ denote the canonical surjection. We have:

Corollary 3.3.2: *Let $\theta_1, \theta_2 \in Aut(\mathcal{N})$; then $L^2(\theta_1)$ is equivalent to $L^2(\theta_2)$ if and only if $\epsilon(\theta_1) = \epsilon(\theta_2)$ in $Out(\mathcal{N})$.*

Proof: First recall that if $\theta \in \mathrm{Aut}(\mathcal{N})$, then θ is normal. For $i = 1, 2$, we have

$$L^2(\theta_i) = \{\xi \in L^2(\mathcal{N}) : \xi\theta_i(1) = \xi\} = L^2(\mathcal{N}).$$

Suppose $L^2(\theta_1) \sim L^2(\theta_2)$; by definition 3.1.2 i) and Proposition 3.3.1 ii), there exists a unitary $V : L^2(\theta_1) \to L^2(\theta_2)$ such that $V \in \theta_2(1)\mathcal{N}\theta_1(1) = \mathcal{N}$ and

$$\theta_2(x)V = V\theta_1(x), \quad \text{for all } x \in \mathcal{N}.$$

Now, suppose $\theta_1(x) = V\theta_2(x)V^*$, for some $V \in U(\mathcal{N})$. Let $\pi_i : \mathcal{N} \odot \mathcal{N}^o \to \mathcal{L}(L^2(\theta_i))$ be the *-representation corresponding to $L^2(\theta_i)$, for $i = 1, 2$ and $\Psi : L^2(\mathcal{N}) \to L^2(\mathcal{N})$ be defined by $\Psi(\xi) = \xi V^*$, for $\xi \in L^2(\mathcal{N})$. Then Ψ is a linear surjection and

$$\langle \Psi(\xi), \Psi(\xi) \rangle = \langle \xi V^*, \xi V^* \rangle = \langle \xi V^* V, \xi \rangle = \langle \xi, \xi \rangle .$$

Moreover, for all $(x, y) \in \mathcal{N} \odot \mathcal{N}^o$, $\xi \in L^2(\mathcal{N})$, we have:

$$\pi_1(x,y)\Psi(\xi) = x\Psi(\xi)\theta_1(y) = x\xi V^*\theta_1(y) = x\xi V^* V\theta_2(y)V^* = x\xi\theta_2(y)V^*$$

and

$$\Psi(\pi_2(x,y)\xi) = \Psi(x\xi\theta_2(y)) = x\xi\theta_2(y)V^*.$$

Q.E.D.

Remark: If $\mathcal{N}$ is not properly infinite, Proposition 3.3.1 does not hold. Indeed, let $\mathcal{N}$ be a finite von Neumann algebra and $\mathcal{M}$ a properly infinite von Neumann algebra. The coarse correspondence between $\mathcal{M}$ and $\mathcal{N}$ always exists [Example 3.3 a] but there exists no nonzero *-homomorphism of $\mathcal{M}$ into $\mathcal{N}$.

Lemma 3.3.3: *Let $\rho : \mathcal{M} \to \mathcal{N}$ be a nonzero normal *-homomorphism from $\mathcal{M}$ to $\mathcal{N}$. If $\mathcal{N}$ is finite, then $\mathcal{M}$ is not properly infinite.*

Proof: Let $x \in \mathcal{M}$ be such that $\rho(x^*x) \neq 0$. As $\mathcal{N}$ is finite [section 1.6], there exists a finite normal trace τ on $\mathcal{N}$ such that $\tau(\rho(x^*x)) \neq 0$. As $\tau \circ \rho$ is a finite normal trace on $\mathcal{M}$, we have the desired result.

Q.E.D.

Fortunately it is possible to circumvent the problem. Let F be the factor of type I_∞ of all bounded operators in $l^2(\mathbb{N})$. The von Neumann algebra $\tilde{\mathcal{N}} = \mathcal{N} \bar{\otimes} F$ is properly infinite [section 1.7] and the correspondences from $\mathcal{M}$ to $\mathcal{N}$ are not modified if $\mathcal{N}$ is replaced by $\tilde{\mathcal{N}}$. Indeed, consider

$$\begin{tikzcd} Corr(\mathcal{N},\mathcal{M}) \arrow[r, "\phi"] & Corr(\tilde{\mathcal{N}} = \mathcal{N}\overline{\otimes}\mathcal{L}(l^2(\mathbb{N})),\mathcal{M}) \\ \mathcal{H} \arrow[r] & \tilde{\mathcal{H}} = \mathcal{H}\otimes l^2(\mathbb{N}) \end{tikzcd}$$

$\tilde{\mathcal{H}} = \mathcal{H}\otimes l^2(\mathbb{N})$ is a correspondence from $\mathcal{M}$ to $\tilde{\mathcal{N}}$: note that

$$\tilde{\mathcal{H}} \cong l^2(\mathbb{N},\mathcal{H}) = \{(\xi_n)_{n\geq 1} : \xi_n \in \mathcal{H} \text{ and } \sum_n \|\xi_n\|^2 < \infty\}.$$

Let $\tilde{\xi} \in \tilde{\mathcal{H}}$ and $x = (x_{ij}) \in \tilde{\mathcal{N}}$; the left action is given by:

$$x\tilde{\xi} = \left(\sum_j x_{1j}\xi_j, \sum_j x_{2j}\xi_j, \ldots\right).$$

Note that $(\mathcal{N}\overline{\otimes}F)' = \mathcal{N}'\overline{\otimes}F' = \mathcal{N}'\overline{\otimes}\mathbb{C}$; the right action is given by:

$$\tilde{\xi}y = (\xi_1 y, \xi_2 y, \ldots), \qquad y \in \mathcal{M} \subset \tilde{\mathcal{N}}'.$$

Now, let $\tilde{e_{ij}} = 1\otimes e_{ij} \in \tilde{\mathcal{N}}$ be a system of matrix units in $\tilde{\mathcal{N}}$, where $(e_{ij})_{i,j\in\mathbb{N}}$ is the canonical system of matrix units in F and write $e = 1\otimes e_{11}$. Consider

$$\begin{tikzcd} Corr(\tilde{\mathcal{N}},\mathcal{M}) \arrow[r, "\psi"] & Corr(\mathcal{N} = \tilde{\mathcal{N}}_e,\mathcal{M}) \\ \tilde{\mathcal{H}} \arrow[r, mapsto] & \mathcal{H} = e\tilde{\mathcal{H}} \end{tikzcd}$$

$\mathcal{H} = e\tilde{\mathcal{H}}$ is a correspondence from $\mathcal{M}$ to $\mathcal{N}$ with left action given by:

$$\begin{aligned} e\tilde{x}e\,(e\tilde{\xi}) &= e\tilde{x}e\,(\xi_1, 0, 0, \ldots) = e\tilde{x}\,(\xi_1, 0, 0, \ldots) \\ &= e\,(\eta_1, \eta_2, \ldots) = (\eta_1, 0, 0, \ldots) \end{aligned}$$

where $\tilde{x} \in \tilde{\mathcal{N}}$, $\tilde{\xi} \in \tilde{\mathcal{H}}$. The right action is defined as on $\tilde{\mathcal{H}}$.

The important fact is that ϕ and ψ are inverses of each other, that is, the $\mathcal{N}-\mathcal{M}$ correspondence $\mathcal{H}$ is equivalent to the $\mathcal{N} = e(\mathcal{N}\overline{\otimes}F)e - \mathcal{M}$ correspondence $\mathcal{H} = e(\mathcal{H}\otimes l^2(\mathbb{N}))$. Indeed

$$u : \mathcal{H} \to e(\mathcal{H}\otimes l^2(\mathbb{N})), \quad \xi \mapsto (\xi, 0, 0, \ldots)$$

is the unitary which gives the equivalence.

Remark: In the above discussion, we have used the same notation for a correspondence and the class of that correspondence.

Example 3.3 a: Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The *coarse correspondence* is the bimodule of Hilbert-Schmidt operators from $L^2(\mathcal{M})$ to $L^2(\mathcal{N})$. For $T \in \mathcal{L}^2(L^2(\mathcal{M}), L^2(\mathcal{N}))$,

$$\pi_{\mathcal{N}}(y)\pi_{\mathcal{M}^\circ}(x^\circ)T = yTx, \quad \text{for all } y \in \mathcal{N}, x \in \mathcal{M}.$$

Note that $yTx \in \mathcal{L}^2(L^2(\mathcal{M}), L^2(\mathcal{N}))$ [K-R; p.141]. As seen in section 1.7, $\mathcal{L}^2(L^2(\mathcal{M}), L^2(\mathcal{N}))$ is isomorphic to $\overline{L^2(\mathcal{M})} \otimes L^2(\mathcal{N})$. Therefore, the coarse correspondence can be viewed as $\overline{L^2(\mathcal{M})} \otimes L^2(\mathcal{N})$ with bimodule structure given by:

$$\pi_{\mathcal{N}}(y)\pi_{\mathcal{M}^\circ}(x^\circ)(\overline{\xi} \otimes \eta) = \overline{x^*\xi} \otimes y\eta, \quad \text{for all } y \in \mathcal{N}, x \in \mathcal{M}.$$

Suppose $\mathcal{M} = L^\infty(X, \mu_X)$ and $\mathcal{N} = L^\infty(Y, \mu_Y)$, where (X, μ_X) and (Y, μ_Y) are σ-finite measure spaces; then

$$\overline{L^2(\mathcal{M})} \otimes L^2(\mathcal{N}) = \overline{L^2(X, \mu_X)} \otimes L^2(Y, \mu_Y) \cong L^2(X \times Y, \mu_X \times \mu_Y) \qquad \text{[K-R; p.143]}.$$

Hence, if we use the notation introduced in section 3.1, we get:

$$L^2(X \times Y, \mu_X \times \mu_Y) = \int_{X\times Y}^{\oplus} \mathbf{C}\, d(\mu_X \times \mu_Y);$$

in other words, the coarse correspondence is determined by the measure $\mu = \mu_X \times \mu_Y$ and the multiplicity function $n(x, y) = 1$, for all $(x, y) \in X \times Y$. The graph of the correspondence is $X \times Y$.

Example 3.3 b: Let Γ be a discrete countable group acting freely by nonsingular transformations of the measure space (X, μ_X). Suppose that the measure μ_X is Γ-invariant. Let $\mathcal{R}_\Gamma$ be the relation on X defined by:

$$x \sim y \iff \text{there exists } \gamma \in \Gamma \text{ such that } \gamma x = y.$$

Note that as Γ acts freely,

$$x \sim y \iff \text{there exists a unique } \gamma \in \Gamma \text{ such that } \gamma x = y;$$

this property also infers that $\Gamma.x \cong \Gamma$ μ_X-almost everywhere. We can thus write

$$\mathcal{R}_\Gamma = \{(x, y) \in X \times X : x \sim y\} = \{(x, \gamma x) : x \in X, \gamma \in \Gamma\}.$$

Let ν be the measure on $\mathcal{R}_\Gamma$ defined by

$$\nu(A) = \int_X \nu_x(A_x)\, d\mu_X(x), \qquad A \subset \mathcal{R}_\Gamma$$

where $A_x = \{y \in X : (x, y) \in A\} = \{\gamma x : (x, \gamma x) \in A\}$ and ν_x is the counting measure on Γ. Then

$$L^2(\mathcal{R}_\Gamma, \nu) = \{f : \mathcal{R}_\Gamma \to \mathbf{C} : f \text{ is } \nu\text{-measurable and } \int_{\mathcal{R}_\Gamma} |f(x,y)|^2 \, d\nu(x,y) < \infty\}.$$

By definition of ν, we have:

$$\int_{\mathcal{R}_\Gamma} |f(x,y)|^2 \, d\nu(x,y) = \int_X d\mu_X \left(\sum_{y \in \Gamma . x} |f(x,y)|^2 \right) = \int_X d\mu_X \left(\sum_{\gamma \in \Gamma} |f(x, \gamma x)|^2 \right)$$

Note that the application

$$L^2(X, \mu_X) \otimes l^2(\Gamma) \to L^2(\mathcal{R}_\Gamma, \nu) \qquad f \otimes g \mapsto F$$

where $F(x,y) = F(x, \gamma x) = f(x) g(\gamma)$ for all $(x,y) \in \mathcal{R}_\Gamma$ extends to an isomorphism. We will now recall the construction of the crossed product, as presented in [F-M]. Let $a : \mathcal{R}_\Gamma \to \mathbf{C}$ be such that there exists $n \in \mathrm{N}$ such that for all $x, y \in X$,

(i) $$|\{\gamma x \in X : a(x, \gamma x) \neq 0\}| + |\{\gamma y \in X : a(\gamma y, y) \neq 0\}| \leq n.$$

Define, for $\psi \in L^2(\mathcal{R}_\Gamma, \nu)$,

$$L_a(\psi)(x,z) = \sum_{y \sim x} a(x,y) \psi(y,z),$$

$\left(\text{that is, } L_a(\psi)(x, \gamma x) = \sum_{\eta \in \Gamma} a(x, \eta x) \psi(\eta x, \gamma x)\right)$ and

$$R_a(\psi)(x,z) = \sum_{y \sim x} \psi(x,y) a(y,z),$$

$\left(\text{that is, } R_a(\psi)(x, \gamma x) = \sum_{\eta \in \Gamma} \psi(x, \eta x) a(\eta x, \gamma x)\right)$. Then L_a, $R_a \in \mathcal{L}(L^2(\mathcal{R}_\Gamma, \nu))$.

Suppose $a, b : \mathcal{R}_\Gamma \to \mathbf{C}$ both satisfy (i). Define

$$ab(x,z) = \sum_{y \sim x} a(x,y) b(y,z),$$

that is, $ab(x, \gamma x) = \sum_{\eta \in \Gamma} a(x, \eta x) b(\eta x, \gamma x)$ and

$$a^*(x,y) = \overline{a(y,x)}.$$

Then the sets

$$L = \{L_a : \text{ the map } a : \mathcal{R}_\Gamma \to \mathbf{C} \text{ satisfies (i)}\}$$

and

$$R = \{R_a : \text{ the map } a : \mathcal{R}_\Gamma \to \mathbf{C} \text{ satisfies (i)}\}$$

are *-subalgebras of $\mathcal{L}(L^2(\mathcal{R}_\Gamma, \nu))$.

Definition b1: $\mathcal{M}(\mathcal{R}_\Gamma) = L''$ and $\tilde{\mathcal{M}}(\mathcal{R}_\Gamma) = R''$.

Proposition b2:

i) $\mathcal{M}(\mathcal{R}_\Gamma)' = \tilde{\mathcal{M}}(\mathcal{R}_\Gamma)$.

ii) *If* $\phi_o \in L^2(\mathcal{R}_\Gamma, \nu)$ *is defined by*

$$\phi_o(x,y) = \begin{cases} 0 & \textit{if } y \neq x \\ 1 & \textit{if } y = x \end{cases}$$

then $L_a(\phi_o)(x,z) = a(x,z)$ *and* $R_a(\phi_o)(x,z) = a(x,z)$. *Moreover,* ϕ_o *is cyclic and separating for* $\mathcal{M}(\mathcal{R}_\Gamma)$ *and* $\tilde{\mathcal{M}}(\mathcal{R}_\Gamma)$.

Remark: If $a \in L^\infty(X, \mu_X)$, we can view a as a measurable function defined on $\mathcal{R}_\Gamma$ as follows:

$$a(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ a(x) & \text{if } x = y \end{cases}$$

Then $\{L_a : a \in L^\infty(X, \mu_X)\}$ is an abelian subalgebra of $\mathcal{M}(\mathcal{R}_\Gamma)$ and $\{R_a : a \in L^\infty(X, \mu_X)\}$ is an abelian subalgebra of $\mathcal{M}(\mathcal{R}_\Gamma)'$. Moreover, for any $\psi \in L^2(\mathcal{R}_\Gamma, \nu)$, we have

$$L_a(\psi)(x,y) = a(x)\psi(x,y) \qquad \text{and} \qquad R_a(\psi)(x,y) = \psi(x,y)a(y).$$

Let us identify the subalgebras $\{L_a : a \in L^\infty(X, \mu_X)\}$ and $\{R_a : a \in L^\infty(X, \mu_X)\}$ with $L^\infty(X, \mu_X)$. Note that by Proposition b2, $L^2(\mathcal{R}_\Gamma, \nu) = [\mathcal{M}(\mathcal{R}_\Gamma)\,\phi_o]$. Thus $(\mathcal{M}, L^2(\mathcal{R}_\Gamma, \nu))$ is a standard form. Consider the restriction of the identity correspondence of $\mathcal{M}(\mathcal{R}_\Gamma)$ to $L^\infty(X, \mu_X)$. Then

$$\begin{aligned} L^2(\mathcal{M}(\mathcal{R}_\Gamma)) = L^2(\mathcal{R}_\Gamma, \nu) &= \int_X^\oplus \mathcal{H}_x \, d\mu_X(x) = \int_X^\oplus \left(\int_X^\oplus \mathcal{H}_{(x,y)} \, d\nu_x(y) \right) d\mu_X(x) \\ &= \int_{\mathcal{R}_\Gamma}^\oplus \mathcal{H}_{(x,y)} \, d\nu(x,y) \cong \int_X l^2(\Gamma) \, d\mu_X \end{aligned}$$

with actions described as follows:

$$(L_a \cdot \psi \cdot R_b)(x,y) = a(x)\psi(x,y)b(y) \quad \text{for all } a, b \in L^\infty(X, \mu_X),\ \psi \in L^2(\mathcal{R}_\Gamma, \nu).$$

The graph of the correspondence is $\mathcal{R}_\Gamma$ and the multiplicity function $n : X \times X \to \mathbf{N}$ is defined as

$$n(x,y) = \begin{cases} 1 & x \sim y \\ 0 & \text{otherwise} \end{cases}$$

Example 3.3 c: Let $\mathcal{M} = \mathcal{N} \otimes_\alpha G$ be the crossed product of a von Neumann algebra $\mathcal{N}$ by the action α of the locally compact group G [section 1.10], and $\pi : G \longrightarrow \mathcal{L}(H_\pi)$ be a unitary representation of G in a Hilbert space H_π. Then π defines a correspondence from $\mathcal{M}$ to $\mathcal{M}$ as follows:

$$\mathcal{H}_\pi = L^2(\mathcal{M}) \otimes H_\pi;$$

the right action is given by:

$$\pi_{\mathcal{M}^o}(x^o)(\xi \otimes \eta) = \xi x \otimes \eta, \quad \text{for all } \xi \in L^2(\mathcal{M}),\ \eta \in H_\pi,\ x \in \mathcal{M},$$

and the left action is given by:

$$\pi_{\mathcal{M}}(x)(\xi \otimes \eta) = x\xi \otimes \eta, \quad \text{for all } x \in \mathcal{N},\ \xi \in L^2(\mathcal{M}),\ \eta \in H_\pi$$
$$\pi_{\mathcal{M}}(g)(\xi \otimes \eta) = g\xi \otimes \pi(g)\eta, \quad \text{for all } g \in G,\ \xi \in L^2(\mathcal{M}),\ \eta \in H_\pi.$$

3.4 Coefficients of correspondences and completely positive maps

As seen in section 3.1, a correspondence from the von Neumann algebra $\mathcal{M}$ to the von Neumann algebra $\mathcal{N}$ is a representation π of the C*-algebra $\mathcal{N} \otimes_{max} \mathcal{M}^o$ which is binormal, that is, whose restrictions to both $\mathcal{M}^o$ and $\mathcal{N}$ are normal representations. The functionals

$$z \in \mathcal{N} \otimes_{max} \mathcal{M}^o \mapsto \langle \pi(z)\xi, \xi \rangle \in \mathbf{C},$$

are called the *coefficients* of such representations and correspond to the so-called *binormal states* that enter in the definition of $\mathcal{N} \otimes_{bin} \mathcal{M}^o$ [section 1.8]. In this section, we will see the link between these coefficients and the completely positive normal maps from $\mathcal{M}$ to $\mathcal{N}$.

Proposition 3.4.1 [Co$_3$]: *Let ν be a faithful semifinite normal weight on the von Neumann algebra $\mathcal{N}$ and $\mathcal{H}$ a normal left $\mathcal{N}$-module. Then*

$$D(\mathcal{H}, \nu) = \{\xi \in \mathcal{H} : \text{ there exists } c < \infty \text{ such that } \|y\xi\| \leq c\nu(y^*y)^{\frac{1}{2}} \text{ for all } y \in \mathfrak{N}_\nu\}$$

is dense in $\mathcal{H}$.

Proof: Let $U : \mathcal{N} \rightarrow \mathcal{L}(\mathcal{H})$ be the representation associated to the left $\mathcal{N}$-module $\mathcal{H}$. Note that the kernel $Ker(U)$ of U is a ultra-weakly closed two-sided ideal of $\mathcal{N}$. Therefore, there exists a central projection c in $\mathcal{N}$ such that

$$Ker(U) = \{x \in \mathcal{N} : xc = cx = x\} = N_c \qquad \text{[Di; p.46]}.$$

Hence $U(\mathcal{N}) \cong \mathcal{N}/Ker(U) := \mathcal{N}_{1-c}$. Let ν_{1-c} be the restriction of ν to $\mathcal{N}_{1-c}$ and $(\xi_i)_{i\in I}$ be a family of vectors in $\mathcal{H}$ such that

$$\nu_{1-c}(x) = \sum_{i\in I}(U(x)\,\xi_i \mid \xi_i), \quad \text{for all } x \in (\mathcal{N}_{1-c})^+.$$

For all $i \in I$ and $y \in \mathcal{N}$,

$$\begin{aligned}\|U(y)\,\xi_i\|^2 &= (U(y)\,\xi_i \mid U(y)\,\xi_i) = (U(y^*y)\,\xi_i \mid \xi_i)\\ &\leq \sum_{i\in I}(U(y^*y)\,\xi_i \mid \xi_i) = \nu(y^*y).\end{aligned}$$

Hence ξ_i is a ν-bounded vector. Let $E : \mathcal{H} \to \overline{D(\mathcal{H},\nu)}$ be the orthogonal projection of $\mathcal{H}$ on the closure of $D(\mathcal{H},\nu)$. We must show that $E = 1$. First note that $\overline{D(\mathcal{H},\nu)}$ is $U(\mathcal{N})'$-invariant. Indeed, for any $\xi \in D(\mathcal{H},\nu)$, $x' \in U(\mathcal{N})'$ and $y \in \mathfrak{N}_\nu$,

$$\begin{aligned}\|yx'\,\xi\|^2 &= (yx'\,\xi \mid yx'\,\xi) = (yx'\,\xi \mid x'y\,\xi)\\ &= ((x')^*x'y\,\xi \mid y\xi) \leq \|(x')^*x'\|\|y\,\xi\|^2\\ &\leq c\nu(y^*y), \quad \text{as } \xi \in D(\mathcal{H},\nu).\end{aligned}$$

Hence $E \in U(\mathcal{N})''$ by Lemma 1.1.5. It follows that $E = U(e)$, for some $e \in \mathcal{N}_{1-c}$ (recall that $\mathcal{N}_{1-c} \cong U(\mathcal{N})$). Suppose $e \neq 1-c$; then as ν is faithful,

$$\nu((1-c)-e) = \sum(U((1-c)-e)\,\xi_i \mid \xi_i) > 0.$$

Thus there exists i such that $U((1-c)-e)\,\xi_i \neq 0$. By construction, $E\,\xi_i = \xi_i$ (since $\xi_i \in D(\mathcal{H},\nu)$). However, this equality does not hold when $e \neq 1-c$:

$$0 \neq U((1-c)-e)\,\xi_i = U(1-c)\,\xi_i - U(e)\,\xi_i = \xi_i - U(e)\,\xi_i = \xi_i - E\,\xi_i.$$

Therefore $e = 1-c$, which implies that $E = U(1-c) = 1$. We thus have the desired result.

Q.E.D.

Definition 3.4.2: With the notation of the above proposition, a vector $\xi \in \mathcal{H}$ is called *ν-bounded* if $\xi \in D(\mathcal{H},\nu)$.

Lemma 3.4.3: *Let ν and ψ be weights on a von Neumann algebra $\mathcal{N}$ such that $\psi \leq \nu$ (i.e. $\psi(x) \leq \nu(x)$ for all $x \in \mathcal{N}^+$). Then there exists a unique element $a' \in \mathcal{N}' \subset \mathcal{L}(\mathcal{H}_\nu)$ (where $\mathcal{H}_\nu$ is the Hilbert space obtained from the GNS construction for ν) such that $0 \leq a' \leq 1$ and $\psi(y^*x) = (a'\eta_\nu(x) \mid \eta_\nu(y))$ for all $x, y \in \mathfrak{N}_\nu$.*

Proof: Let us apply the GNS construction to both ν and ψ. We thus obtain two triples, $(\mathcal{H}_\nu, \pi_\nu, \eta_\nu)$ and $(\mathcal{H}_\psi, \pi_\psi, \eta_\psi)$ such that $\eta_\nu(\mathfrak{N}_\nu)$ is dense in $\mathcal{H}_\nu \cong L^2(\mathcal{N}, \nu)$ and $\eta_\psi(\mathfrak{N}_\psi)$ is dense in $\mathcal{H}_\psi$. As $\psi \le \nu$, then $\mathfrak{N}_\nu \subset \mathfrak{N}_\psi$; by hypothesis, for all $x \in \mathfrak{N}_\nu$ we have:

$$\|\eta_\psi(x)\|_\psi^2 = \psi(x^*x) \le \nu(x^*x) = \|\eta_\nu(x)\|_\nu^2.$$

Hence there exists a unique linear operator $b' : \mathcal{H}_\nu \to \mathcal{H}_\psi$ with $\|b'\| \le 1$ such that $b'(\eta_\nu(x)) = \eta_\psi(x)$ for all $x \in \mathfrak{N}_\nu$. Note that b'^*b' is positive as

$$(b'^*b'\eta_\nu(x) \mid \eta_\nu(x)) = (\eta_\psi(x) \mid \eta_\psi(x)) = \|\eta_\psi(x)\|_\psi^2 \ge 0.$$

Therefore, $c' = (b'^*b')^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}_\nu) = \mathcal{L}(L^2(\mathcal{N}, \nu))$ and $0 \le c' \le 1$. Now, for all $x \in \mathcal{N}$ and $y_1, y_2 \in \mathfrak{N}_\nu$ we have:

$$\psi(y_2^*y_1) = (\eta_\psi(y_1) \mid \eta_\psi(y_2))_\psi = (b'\eta_\nu(y_1) \mid b'\eta_\nu(y_2))_\nu = (c'^2\eta_\nu(y_1) \mid \eta_\nu(y_2))_\nu$$

and

$$\begin{aligned}
(b'^*b'\pi_\nu(x)\eta_\nu(y_1) \mid \eta_\nu(y_2))_\nu &= (b'^*b'\eta_\nu(xy_1) \mid \eta_\nu(y_2))_\nu \\
&= (\eta_\psi(xy_1) \mid \eta_\psi(y_2))_\psi = \psi(y_2^*xy_1) = \psi((x^*y_2)^*y_1) \\
&= (b'\eta_\nu(y_1) \mid b'\eta_\nu(x^*y_2))_\nu = (b'\eta_\nu(y_1) \mid b'\pi_\nu(x^*)\eta_\nu(y_2))_\nu \\
&= (b'\eta_\nu(y_1) \mid (\pi_\nu(x)b'^*)^*\eta_\nu(y_2))_\nu \\
&= (\pi_\nu(x)b'^*b'\eta_\nu(y_1) \mid \eta_\nu(y_2))_\nu.
\end{aligned}$$

Hence $c'^2 = b'^*b' \in \pi_\nu(\mathcal{N})' \sim \mathcal{N}'$. Let $a' = b'^*b'$; as

$$\psi(y_2^*y_1) = (a'\eta_\nu(y_1) \mid \eta_\nu(y_2))_\nu, \qquad y_1, y_2 \in \mathfrak{N}_\nu,$$

is uniquely determined by ν and ψ, a' is unique.

Q.E.D.

Let ν be a faithful semifinite normal weight on $\mathcal{N} \subset \mathcal{L}(L^2(\mathcal{N}))$. Recall that the domain of ν in $\mathcal{N}$ is the set $\mathfrak{M}_\nu = \mathfrak{N}_\nu^*\mathfrak{N}_\nu$ and that every element of $\mathfrak{M}_\nu$ can be written as a linear combination of four elements of $\mathfrak{P}_\nu = \mathfrak{M}_\nu \cap \mathcal{N}^+$ [section 2.2]. Consider the map

$$I_\nu : \mathrm{Face}\,(\nu) \to \mathfrak{M}_\nu, \qquad \phi \mapsto a_\phi$$

where $\mathrm{Face}\,(\nu) = \{\phi \in \mathcal{N}_*^+ : \phi \le \lambda\nu \text{ for some } \lambda > 0\}$ and $a_\phi \in \mathcal{N}'$ is the linear operator obtained by applying Lemma 3.4.3 to ϕ. Note that we view $\mathcal{N}'$ as $\mathcal{N}$ acting on the right on $L^2(\mathcal{N})$. Symbolically, we shall write $I_\nu(\phi) = \nu^{-\frac{1}{2}}\,\phi\,\nu^{-\frac{1}{2}}$ for all $\phi \in \mathrm{Face}\,(\nu)$. We have:

Proposition 3.4.4 [E-L]: *Let ν be a faithful semifinite normal weight on $\mathcal{N}$. The equality*

$$I_\nu(\phi) = \nu^{-\frac{1}{2}}\phi\nu^{-\frac{1}{2}}, \quad \text{for all} \quad \phi \in \mathrm{Face}\,(\nu)$$

defines a completely positive linear map I_ν from the face of ν in $\mathcal{N}_*$ to the domain of ν in $\mathcal{N}$. Its inverse, $x \mapsto \nu^{\frac{1}{2}}x\nu^{\frac{1}{2}}$, for all $x \in \mathfrak{M}_\nu$, is also completely positive.

Proof: As above, let

$$I_\nu : \mathrm{Face}(\nu) \to \mathfrak{M}_\nu, \qquad \phi \mapsto \nu^{-\frac{1}{2}} \phi \nu^{-\frac{1}{2}} = a_\phi$$

where a_ϕ is the linear operator obtained by applying Lemma 3.4.3 to ϕ. Consider the map

$$I_{\nu,n} : \mathrm{M}_n(\mathrm{span}(\mathrm{Face}(\nu))) \to \mathrm{M}_n(\mathfrak{M}_\nu), \qquad (\phi_{ij})_{ij} \mapsto (I_\nu(\phi_{ij}))_{ij}.$$

We need to show that $(I_\nu(\phi_{ij}))_{ij}$ is positive, that is $((a_{\phi_{ij}})\xi \mid \xi) \geq 0$, for all $\xi \in \oplus_{i=1}^n L^2(\mathcal{N}, \nu)$ [section 1.9], where $a_{\phi_{ij}} = b_{ij}^* b_{ij}$ is the operator obtained by applying Lemma 3.4.3 to ϕ_{ij}. Write $\xi = (\eta_\nu(x_1), ..., \eta_\nu(x_n))$ where $x_i \in \mathfrak{N}_\nu$ for all i; then

$$\begin{aligned} ((a_{\phi_{ij}})\xi \mid \xi) &= \sum_{i,j} (b_{ij}^* b_{ij} \eta_\nu(x_j) \mid \eta_\nu(x_i)) \\ &= \sum_{i,j} (\eta_\phi(x_j) \mid \eta_\phi(x_i)) \\ &= \|\sum_i \eta_\phi(x_i)\|^2 \geq 0 \end{aligned}$$

and thus I_ν is completely positive. Let us now show that the inverse $\tilde{I}_\nu$ of I_ν is also completely positive. Consider the maps

$$\tilde{I}_\nu : \mathfrak{P}_\nu \subset \mathcal{N}^+ \to \mathrm{Face}(\nu) \subset \mathcal{N}_*^+, \qquad d = z^* z \mapsto \nu^{\frac{1}{2}} d \nu^{\frac{1}{2}} = \phi_d$$

where $\phi_d(x) = \left\langle x \nu^{\frac{1}{2}} z^* z \nu^{\frac{1}{2}} \right\rangle = \left\langle x \nu^{\frac{1}{2}} z^*, \nu^{\frac{1}{2}} z^* \right\rangle$ and

$$\tilde{I}_{\nu,n} : \mathrm{M}_n(\mathfrak{P}_\nu) \to \mathrm{M}_n(\mathcal{N}_*^+), \qquad (d_{ij})_{ij} = (z_{ij}^* z_{ij})_{ij} \mapsto (\phi_{z_{ij}^* z_{ij}})_{ij}.$$

Let $y = (y_1, ..., y_n) \in \oplus_{i=1}^n \mathcal{N}$. Then

$$\begin{aligned} \sum_{i,j} \phi_{z_{ij}^* z_{ij}}(y_i^* y_j) &= \sum_{i,j} \left\langle y_i^* y_j \nu^{\frac{1}{2}} z_{ij}^* z_{ij} \nu^{\frac{1}{2}} \right\rangle \\ &= \sum_{i,j} \left\langle y_j \nu^{\frac{1}{2}} z_{ij}^*, y_i \nu^{\frac{1}{2}} z_{ij}^* \right\rangle \geq 0 \end{aligned}$$

by the remark following Definition 1.9.5. Thus $\tilde{I}_\nu$ is completely positive.

Q.E.D.

Remark: Let $\mathcal{H}$ be a correspondence from $\mathcal{M}$ to $\mathcal{N}$. If $\xi \in D(\mathcal{H}, \nu)$ and $x \in \mathcal{M} \subset \mathcal{N}'$, then $\xi x \in D(\mathcal{H}, \nu)$.

Proposition 3.4.5: *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and ν a faithful semifinite normal weight on $\mathcal{N}$.*

i) *Let $\mathcal{H}$ be a correspondence from $\mathcal{M}$ to $\mathcal{N}$, and ξ a ν-bounded vector. Then there exists a unique completely positive map P from $\mathcal{M}$ to $\mathcal{N}$ such that for any $x \in \mathcal{M}$, $y \in \mathcal{N}$, one has*
$$\langle y\,\xi\,x, \xi\rangle = \left\langle y\,\nu^{\frac{1}{2}}P(x)\,\nu^{\frac{1}{2}}\right\rangle.$$

ii) *Let P be a completely positive normal map from $\mathcal{M}$ to $\mathcal{N}$ such that $\nu(P(1)) < \infty$. Then there exists a unique pair $(\mathcal{H}, \xi)$ where $\mathcal{H}$ is a correspondence from $\mathcal{M}$ to $\mathcal{N}$, $\xi \in \mathcal{H}$ and:*
   - α) *$\mathcal{N}\xi\mathcal{M}$ is dense in $\mathcal{H}$,*
   - β) *ξ is a ν-bounded vector and for any $x \in \mathcal{M}$ and $y \in \mathcal{N}$ such that $\nu(y^*y) < \infty$, one has*
$$\langle y\,\xi\,x, \xi\rangle = \left\langle y\,\nu^{\frac{1}{2}}P(x)\,\nu^{\frac{1}{2}}\right\rangle.$$

Proof:

i) For any $x \in \mathcal{M}$, define
$$\psi_x : \mathcal{N} \to \mathbf{C}, \qquad y \mapsto \langle y\,\xi x, \xi\rangle.$$

For $x \in \mathcal{M}^+ \subset (\mathcal{N}')^+$, then $\psi_x \in \mathrm{Face}(\nu)$. Indeed, if $x \in \mathcal{M}^+$, then $x = zz^*$ for some $z \in \mathcal{M}$; thus
$$\psi_x(y^*y) = \langle y^*y\,\xi zz^*, \xi\rangle = \langle y\,\xi z, y\,\xi z\rangle = \|y\,\xi z\|^2 \leq k\nu(y^*y)$$
for any $y \in \mathfrak{N}_\nu$ by the above Remark. Define
$$P : \mathcal{M} \to \mathcal{N}' \subset \mathcal{L}(L^2(\mathcal{N}, \nu)), \qquad P(x) = I_\nu(\psi_x) = \nu^{-\frac{1}{2}}\,\psi_x\,\nu^{-\frac{1}{2}}.$$

Then by Proposition 3.4.4, P is completely positive and
$$\begin{aligned}\langle z^*y\,\xi x, \xi\rangle = \psi_x(z^*y) &= (P(x)\,\eta_\nu(y) \mid \eta_\nu(z)) \quad \text{[Lemma 3.4.3]}\\ &= \left\langle y\,\nu^{\frac{1}{2}}P(x), z\,\nu^{\frac{1}{2}}\right\rangle \qquad \text{[section 3.2]}\\ &= \left\langle z^*y\,\nu^{\frac{1}{2}}P(x)\,\nu^{\frac{1}{2}}\right\rangle\end{aligned}$$
for all $y, z \in \mathfrak{N}_\nu$. Note that as ν is semifinite, $\mathfrak{M}_\nu = \mathfrak{N}_\nu^*\mathfrak{N}_\nu$ is dense in $\mathcal{N}$ [section 2.2]; thus the above equality remains valid for the elements of $\mathcal{N}$. Now, suppose $P, Q : \mathcal{M} \to \mathcal{N}$ are two completely positive maps such that
$$\left\langle y\,\nu^{\frac{1}{2}}P(x)\,\nu^{\frac{1}{2}}\right\rangle = \left\langle y\,\nu^{\frac{1}{2}}Q(x)\,\nu^{\frac{1}{2}}\right\rangle, \quad \text{for all } x \in \mathcal{M},\ y \in \mathcal{N},$$

that is

$$\left\langle y\nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}P(x)^*\right\rangle = \left\langle y\nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}Q(x)^*\right\rangle.$$

For fixed $x \in \mathcal{M}$, the equation

$$\left\langle y\nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}P(x)^*\right\rangle = \left\langle y\nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}Q(x)^*\right\rangle, \quad \text{for all } y \in \mathcal{N}$$

implies that $\nu^{\frac{1}{2}}P(x)^* = \nu^{\frac{1}{2}}Q(x)^*$. As $\nu^{\frac{1}{2}}$ is a cyclic and separating vector for $\mathcal{N}$, then $P = Q$.

ii) As $\nu(P(1)) < \infty$ and P is completely positive, $P(\mathcal{M})$ belongs to the domain $\mathfrak{M}_\nu$ of ν [section 2.2]. Indeed, let $z \in \mathcal{M}^+$; then $z = x^*x$ for some $x \in \mathcal{M}$ and we have:

$$\nu(P(z)) \le \|x\|^2\nu(P(1)) < \infty.$$

Thus $P(z) \in \mathfrak{P}_\nu$ for all $z \in \mathcal{M}^+$; since every element of $\mathcal{M}$ is a linear combination of four elements of $\mathcal{M}^+$ and every element of $\mathfrak{M}_\nu$ is a linear combination of four elements of $\mathfrak{P}_\nu$, we have the result. Hence, $\nu^{\frac{1}{2}}P(x)\nu^{\frac{1}{2}} \in \text{Face}(\nu)$ for all $x \in \mathcal{M}$ [Proposition 3.4.4]. Let

$$\phi : \mathcal{N} \odot \mathcal{M}^o \to \mathbf{C}, \qquad \phi(y \otimes x^o) = \nu^{\frac{1}{2}}P(x)\nu^{\frac{1}{2}}(y).$$

Then ϕ is normal and finite as

$$\phi(1 \otimes 1) = \nu^{\frac{1}{2}}P(1)\nu^{\frac{1}{2}}(1) = \left\langle \nu^{\frac{1}{2}}P(1)\nu^{\frac{1}{2}}\right\rangle = \nu(P(1)) < \infty$$

by hypothesis. Applying the GNS construction to ϕ, we get $(\pi_\phi, \mathcal{H}_\phi, \xi)$, where:

$$\mathcal{H}_\phi = \overline{\pi_\phi(\mathcal{N} \odot \mathcal{M}^o)\xi} = \overline{\mathcal{N}\xi\mathcal{M}},$$

and for all $x \in \mathcal{M}$, $y \in \mathfrak{N}_\nu$,

$$\begin{aligned}\phi(y \otimes x^o) &= (\pi_\phi(y \otimes x^o)\xi \mid \xi) = \langle y\xi x, \xi\rangle \\ &= \nu^{\frac{1}{2}}P(x)\nu^{\frac{1}{2}}(y) = \left\langle y\nu^{\frac{1}{2}}P(x)\nu^{\frac{1}{2}}\right\rangle.\end{aligned}$$

Finally, $\xi \in D(\mathcal{H}, \nu)$ since for all $y \in \mathfrak{N}_\nu$,

$$\begin{aligned}\|y\xi\|^2 &= \langle y^*y\xi, \xi\rangle = \left\langle y^*y\nu^{\frac{1}{2}}P(1)\nu^{\frac{1}{2}}\right\rangle \\ &= \nu^{\frac{1}{2}}P(1)\nu^{\frac{1}{2}}(y^*y) \le k\,\nu(y^*y)\end{aligned}$$

as $\nu^{\frac{1}{2}}P(1)\nu^{\frac{1}{2}} \in \text{Face}(\nu)$.

Q.E.D.

Corollary 3.4.6: *If $\mathcal{N}$ is properly infinite and $P : \mathcal{M} \to \mathcal{N}$ is a completely positive normal map, there exists a normal *-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ and a partial isometry $V \in \mathcal{N}$, $V^*V \leq \rho(1)$, $VV^* = \mathrm{s}(P(1))$ with $P(x) = P(1)^{\frac{1}{2}} V\rho(x)V^*P(1)^{\frac{1}{2}}$.*

Proof: Let ϕ be a faithful normal positive linear form on $\mathcal{N}$. Then $\phi(P(1)) < \infty$ and by Proposition 3.4.5 ii), there exists a unique pair $\{\mathcal{H}, \xi\}$ where $\mathcal{H}$ is a correspondence from $\mathcal{M}$ to $\mathcal{N}$, $\xi \in \mathcal{H}$ and for $x \in \mathcal{M}$ and $y \in \mathcal{N}$, we have:

$$\langle y\xi x, \xi\rangle = \left\langle y\,\phi^{\frac{1}{2}}\,P(x)\,\phi^{\frac{1}{2}}\right\rangle.$$

As $\mathcal{N}$ is properly infinite, it follows from Proposition 3.3.1 that $\mathcal{H} \sim L^2(\rho)$ for some normal *-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$. Let us assume that $\mathcal{H} = L^2(\rho)$. Consider the two subspaces of $L^2(\mathcal{N})$:

$$\mathcal{H}_o = \overline{\mathcal{N}\xi} \qquad \tilde{\mathcal{H}}_o = \overline{\mathcal{N}\phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}}.$$

Note that $\|y\xi\|^2 = \|y\,\phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}\|^2$ for all $y \in \mathcal{N}$. Let V be the partial isometry from $\tilde{\mathcal{H}}_o$ to $\mathcal{H}_o$ which maps $y\,\phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}$ to $y\xi$ (for all $y \in \mathcal{N}$). As $\mathcal{H}_o$ and $\tilde{\mathcal{H}}_o$ are both left $\mathcal{N}$-invariant, V belongs to $\mathcal{N}'$. Moreover, as $\mathcal{H}_o = L^2(\mathcal{N})e$ for some projection $e \in \mathcal{N}$, $e \leq \rho(1)$ and $\tilde{\mathcal{H}}_o = L^2(\mathcal{N})\mathrm{s}(P(1))$, we have

$$V^*V \leq \rho(1) \qquad \text{and} \qquad VV^* = \mathrm{s}(P(1)).$$

Finally, for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$, we have:

$$\begin{aligned}\langle y\xi\rho(x), \xi\rangle &= \left\langle y\phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}\,V\rho(x), \phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}\,V\right\rangle \\ &= \left\langle y\phi^{\frac{1}{2}}P(1)^{\frac{1}{2}}\,V\rho(x)\,V^*P(1)^{\frac{1}{2}}\,\phi^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\phi^{\frac{1}{2}}P(x)\,\phi^{\frac{1}{2}}\right\rangle.\end{aligned}$$

Hence $P(x) = P(1)^{\frac{1}{2}}\,V\rho(x)\,V^*\,P(1)^{\frac{1}{2}}$ for all $x \in \mathcal{M}$.

Q.E.D.

Definition 3.4.7: Let $\mathcal{M}, \mathcal{N}, \nu$ and $\mathcal{H}$ be as in Prop. 3.4.5 i). For any ν-bounded vectors $\xi_1, \xi_2 \in \mathcal{H}$, the unique normal map P from $\mathcal{M}$ to $\mathcal{N}$ such that

$$\langle y\xi_1 x, \xi_2\rangle = \left\langle y\nu^{\frac{1}{2}}P(x)\,\nu^{\frac{1}{2}}\right\rangle, \qquad \text{for all } x \in \mathcal{M},\ y \in \mathcal{N}$$

will be denoted $(\xi_1, \xi_2)_\nu$.

Lemma 3.4.8: *For $a \in \mathcal{M}$, we have*

$$(\xi_1 a, \xi_2)_\nu(x) = (\xi_1, \xi_2)_\nu(ax) \qquad \text{and} \qquad (\xi_1, \xi_2 a)_\nu(x) = (\xi_1, \xi_2)_\nu(xa^*), \quad \forall x \in \mathcal{M}.$$

Proof: By definition, $(\xi_1 a, \xi_2)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,(\xi_1 a)x, \xi_2\rangle = \langle y\,\xi_1(ax), \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} P(ax)\,\nu^{\frac{1}{2}}\right\rangle, \qquad \forall y \in \mathcal{N}, x \in \mathcal{M}$$

and $(\xi_1, \xi_2)_\nu$ is the unique normal map $Q : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 z, \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} Q(z)\,\nu^{\frac{1}{2}}\right\rangle, \qquad \forall y \in \mathcal{N}, z \in \mathcal{M}.$$

If we put $z = ax$ in the above equation, we obtain the desired result. Similarly, $(\xi_1, \xi_2 a)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 x, \xi_2 a\rangle = \langle y\,\xi_1(xa^*), \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} P(xa^*)\,\nu^{\frac{1}{2}}\right\rangle, \qquad \forall y \in \mathcal{N}, x \in \mathcal{M}$$

and $(\xi_1, \xi_2)_\nu$ is the unique normal map $Q : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 z, \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} Q(z)\,\nu^{\frac{1}{2}}\right\rangle, \qquad \forall y \in \mathcal{N}, z \in \mathcal{M}.$$

If we put $z = xa^*$ in the above equation, we obtain the desired result.

Q.E.D.

Lemma 3.4.9:

i) *Let $b \in \mathcal{N}$ be such that the map $t \mapsto \sigma_t^\nu(b) \in \mathcal{N}$ extends analytically from $t \in \mathbf{R}$ to $Im(t) \in [0, \frac{1}{2}]$. Then for any $\xi_1 \in D(\mathcal{H}, \nu)$, one has $b\xi_1 \in D(\mathcal{H}, \nu)$ and for all $x \in \mathcal{M}$,*

$$(b\xi_1, \xi_2)_\nu(x) = \sigma_{\frac{i}{2}}^\nu(b)(\xi_1, \xi_2)_\nu(x) \qquad \text{and} \qquad (\xi_1, b\xi_2)_\nu(x) = (\xi_1, \xi_2)_\nu(x)(\sigma_{\frac{i}{2}}^\nu(b))^*.$$

ii) *Let ν' be another weight on $\mathcal{N}$, with $\nu' \geq \lambda\nu$ for some $\lambda > 0$; then $D(\mathcal{H}, \nu) \subset D(\mathcal{H}, \nu')$, the Radon-Nikodym derivative $(D\nu : D\nu')_t \in \mathbf{R}$ extends analytically from $t \in \mathbf{R}$ to $Im(t) \in [-\frac{1}{2}, 0]$, and with $b = (D\nu : D\nu')_{-i/2}$ one has for any $\xi_1, \xi_2 \in D(\mathcal{H}, \nu)$, $x \in \mathcal{M}$,*

$$(\xi_1, \xi_2)_{\nu'}(x) = b^*(\xi_1, \xi_2)_\nu(x)b.$$

Proof:

i) Let $\xi_1 \in D(\mathcal{H}, \nu)$ and $b \in \mathcal{N}$ such that the map $t \mapsto \sigma_t^\nu(b) \in \mathcal{N}$ extends analytically from $t \in \mathbf{R}$ to $Im(t) \in [0, \frac{1}{2}]$. As noted in section 3.2, for $t \in \mathbf{C}$ with $Im(t) \in [0, \frac{1}{2}]$, we will still write $\sigma_t^\nu(b) = \nu^{it} b \nu^{-it}$. Let $y \in \mathfrak{N}_\nu$; then $yb \in \mathfrak{N}_\nu$ as

$$\nu((yb)^* yb) = \nu(b^* y^* yb) \leq k\nu(y^* y) < \infty \qquad \text{[Prop. 2.7.2]},$$

and

$$\|yb\xi_1\| \leq \bar{k}\nu(y^* y)^{\frac{1}{2}} \qquad \text{as } \xi_1 \in D(\mathcal{H}, \nu).$$

Thus $b\xi_1 \in D(\mathcal{H}, \nu)$. By definition, $(b\xi_1, \xi_2)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that for all $x \in \mathcal{M}$, $y \in \mathcal{N}$,

$$\begin{aligned}\langle y(b\xi_1)x, \xi_2\rangle = \langle (yb)\,\xi_1\, x, \xi_2\rangle &= \left\langle yb\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\,\nu^{\frac{1}{2}}\nu^{-\frac{1}{2}} b\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\,\nu^{\frac{1}{2}}\sigma^{\nu}_{\frac{i}{2}}(b) P(x)\,\nu^{\frac{1}{2}}\right\rangle.\end{aligned}$$

As $(\xi_1, \xi_2)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 x, \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}}\right\rangle,$$

we have: $\sigma^{\nu}_{\frac{i}{2}}(b)(\xi_1, \xi_2)_\nu(x) = (b\xi_1, \xi_2)_\nu(x)$ for all $x \in \mathcal{M}$. Similarly, $(\xi_1, b\xi_2)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that for all $x \in \mathcal{M}$, $y \in \mathcal{N}$,

$$\begin{aligned}\langle y\,\xi_1 x, b\xi_2\rangle = \langle (b^* y)\,\xi_1 x, \xi_2\rangle &= \left\langle b^* y\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}} b^*\right\rangle \quad \text{[section 3.2]} \\ &= \left\langle y\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}} b^* \nu^{-\frac{1}{2}}\nu^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\,\nu^{\frac{1}{2}} P(x)(\nu^{-\frac{1}{2}} b \nu^{\frac{1}{2}})^*\,\nu^{\frac{1}{2}}\right\rangle \\ &= \left\langle y\,\nu^{\frac{1}{2}} P(x)(\sigma^{\nu}_{\frac{i}{2}}(b))^*\,\nu^{\frac{1}{2}}\right\rangle.\end{aligned}$$

As $(\xi_1, \xi_2)_\nu$ is the unique normal map $P : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 x, \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} P(x)\,\nu^{\frac{1}{2}}\right\rangle,$$

we have: $(\xi_1, \xi_2)_\nu(x)(\sigma^{\nu}_{\frac{i}{2}}(b))^* = (\xi_1, b\xi_2)_\nu(x)$ for all $x \in \mathcal{M}$.

ii) We will first show that $D(\mathcal{H}, \nu) \subset D(\mathcal{H}, \nu')$. Let $\xi \in D(\mathcal{H}, \nu)$; then

$$\|y\xi\| \leq k\nu(y^* y)^{\frac{1}{2}} \leq \frac{k}{\lambda^{\frac{1}{2}}}\nu'(y^* y)^{\frac{1}{2}}$$

for all $y \in \mathfrak{N}_\nu$. Thus $\xi \in D(\mathcal{H}, \nu')$. By application of Proposition 2.8.5, the Radon-Nikodym derivative $(D\nu : D\nu')_t$ extends analytically from $t \in \mathbf{R}$ to $Im(t) \in [-\frac{1}{2}, 0]$. Note that

$$(\lambda\nu)^{it} = (D\lambda\nu : D\nu')_t(\nu')^{it} = \lambda^{it}(D\nu : D\nu')_t(\nu')^{it};$$

hence, by analytic continuation we get

$$(\lambda\nu)^{\frac{1}{2}} = \lambda^{\frac{1}{2}}(D\nu : D\nu')_{-i/2}(\nu')^{\frac{1}{2}} = \lambda^{\frac{1}{2}} b(\nu')^{\frac{1}{2}}.$$

Thus, $\nu^{\frac{1}{2}} = b(\nu')^{\frac{1}{2}}$. Now, $(\xi_1, \xi_2)_{\nu'}$ is the unique normal map $P_{\nu'} : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 x, \xi_2\rangle = \left\langle y\,(\nu')^{\frac{1}{2}} P_{\nu'}(x)\,(\nu')^{\frac{1}{2}}\right\rangle, \quad \text{for all } y \in \mathcal{N}, x \in \mathcal{M}$$

and $(\xi_1, \xi_2)_\nu$ is the unique normal map $P_\nu : \mathcal{M} \to \mathcal{N}$ such that

$$\langle y\,\xi_1 x, \xi_2\rangle = \left\langle y\,\nu^{\frac{1}{2}} P_\nu(x)\,\nu^{\frac{1}{2}}\right\rangle, \quad \text{for all } y \in \mathcal{N}, x \in \mathcal{M}.$$

Using the fact that $\nu^{\frac{1}{2}} = b(\nu')^{\frac{1}{2}}$, we get

$$\langle y\,\xi_1 x, \xi_2\rangle = \left\langle yb\,(\nu')^{\frac{1}{2}} P_\nu(x) b\,(\nu')^{\frac{1}{2}}\right\rangle = \left\langle y\,(\nu')^{\frac{1}{2}} b^* P_\nu(x) b\,(\nu')^{\frac{1}{2}}\right\rangle$$

and thus $b^* P_\nu(x) b = P_{\nu'}(x)$, for all $x \in \mathcal{M}$ as desired.

Q.E.D.

3.5 Composition of correspondences

Let $\mathcal{N}$ be a von Neumann algebra, $\mathcal{H}_1$ a left $\mathcal{N}$-module and $\mathcal{H}_2$ a right $\mathcal{N}$-module. As seen in section 3.4, given a faithful positive normal form ν on $\mathcal{N}$, the set of ν-bounded elements $D(\mathcal{H}_1, \nu)$ is dense in $\mathcal{H}_1$. Let us consider the algebraic tensor product

$$\mathcal{H}_2 \odot D(\mathcal{H}_1, \nu)$$

of $\mathcal{H}_2$ and $D(\mathcal{H}_1, \nu)$. We will now define a sesquilinear form on this algebraic tensor product which will enable us to define a new Hilbert space, namely the tensor product $\mathcal{H} = \mathcal{H}_2 \otimes_\nu \mathcal{H}_1$ of the modules $\mathcal{H}_2$ and $\mathcal{H}_1$. Note that the basic tensors generating $\mathcal{H}$ are symbolically (and conveniently) written in the following form:

$$\xi_2 \otimes \nu^{-\frac{1}{2}} \xi_1, \qquad \xi_j \in \mathcal{H}_j. \tag{1}$$

These elements are bilinear in ξ_1 and ξ_2. We will show that the tensor product is not modified when ν is replaced by a semifinite faithful normal weight. Moreover, we will show that the above basic tensors possess the following properties:

$$\xi_2 x \otimes \nu^{-\frac{1}{2}} \xi_1 = \xi_2 \otimes \nu^{-\frac{1}{2}} (\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}) \xi_1, \quad \text{for all } x \in \mathcal{N} \cap D(\sigma^\nu_{-\frac{i}{2}}), \tag{2}$$

and

$$\xi_2 \otimes (\nu')^{-\frac{1}{2}} \xi_1 = \xi_2 \otimes \nu^{-\frac{1}{2}} (\nu^{\frac{1}{2}} (\nu')^{-\frac{1}{2}}) \xi_1, \tag{3}$$

whenever one has $\nu \leq \nu'$, so that $\nu^{\frac{1}{2}}(\nu')^{-\frac{1}{2}} = (D\nu : D\nu')_{-i/2} \in \mathcal{N}$ [Prop. 2.8.5]. Note that relation (3) allows, using $\nu' = \nu_1 + \nu_2$ for $\nu_i \in \mathcal{N}_*^+$, comparison of simple tensors (1) for different faithful $\nu_j \in \mathcal{N}_*^+$.

The sesquilinear form on $\mathcal{H}_2 \odot D(\mathcal{H}_1, \nu)$ that we consider is defined in the following way:

$$\left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1, \eta_2 \otimes \nu^{-\frac{1}{2}}\eta_1 \right\rangle = \phi_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}}) \tag{4}$$

where $\phi_j \in \mathcal{N}_*$ are given by

$$\phi_1(y) = \langle y\xi_1, \eta_1\rangle \quad \text{for all } y \in \mathcal{N},$$
$$\phi_2(y) = \langle \xi_2 y, \eta_2\rangle \quad \text{for all } y \in \mathcal{N}.$$

Remark: As seen in section 3.4, for all $\phi \in \text{Face}(\nu)$, the elements of the form $\nu^{-\frac{1}{2}}\phi\nu^{-\frac{1}{2}}$ belong to $\mathcal{N}$. Thus the above notation makes sense.

Proposition 3.5.1:

i) *The equality (4) defines a positive sesquilinear form and relation (2) holds in the associated Hilbert space* $\mathcal{H}_2 \otimes_\nu \mathcal{H}_1$.

ii) *Let* $\nu, \nu' \in \mathcal{N}_*^+$ *be faithful and suppose* $\nu \leq \nu'$; *then with* $b = (D\nu : D\nu')_{-i/2}$, *an isometry* $\mathcal{H}_2 \otimes_{\nu'} \mathcal{H}_1 \xrightarrow{V} \mathcal{H}_2 \otimes_\nu \mathcal{H}_1$ *is defined by*

$$V(\xi_2 \otimes (\nu')^{-\frac{1}{2}}\xi_1) = \xi_2 \otimes \nu^{-\frac{1}{2}}b\xi_1, \quad \textit{for all } \xi_2 \in \mathcal{H}_2,\ \xi_1 \in D(\mathcal{H}_1, \nu').$$

Proof:

i) The sesquilinear form defined by equation (4) is positive as the map $I_\nu(\cdot) = \nu^{-\frac{1}{2}} \cdot \nu^{-\frac{1}{2}}$ from $\mathcal{N}_*$ to $\mathcal{N}$ is completely positive [Prop. 3.4.4]. To prove relation (2), we will show that for $x \in D(\sigma^\nu_{-\frac{i}{2}})$,

$$\left\langle \xi_2 x \otimes \nu^{-\frac{1}{2}}\xi_1, \eta_2 \otimes \nu^{-\frac{1}{2}}\eta_1 \right\rangle = \left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}(\nu^{\frac{1}{2}}x\nu^{-\frac{1}{2}})\xi_1, \eta_2 \otimes \nu^{-\frac{1}{2}}\eta_1 \right\rangle$$

for all ξ_2, $\eta_2 \in \mathcal{H}_2$ and ξ_1, $\eta_1 \in D(\mathcal{H}_1, \nu)$. Let $x \in D(\sigma^\nu_{-\frac{i}{2}})$; by definition,

$$\left\langle \xi_2 x \otimes \nu^{-\frac{1}{2}}\xi_1, \eta_2 \otimes \nu^{-\frac{1}{2}}\eta_1 \right\rangle = \bar{\phi}_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}})$$

where

$$\phi_1(y) = \langle y\xi_1, \eta_1\rangle, \qquad \bar{\phi}_2(y) = \langle (\xi_2 x)y, \eta_2\rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\bar{\phi}_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}}) = \left\langle \xi_2 x\, \nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}}, \eta_2 \right\rangle = \langle \xi_2 x I_\nu(\phi_1), \eta_2\rangle.$$

Now,

$$\left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}})\xi_1, \eta_2 \otimes \nu^{-\frac{1}{2}}\eta_1 \right\rangle = \phi_2(\nu^{-\frac{1}{2}} \bar{\phi}_1 \nu^{-\frac{1}{2}})$$

where

$$\bar{\phi}_1(y) = \left\langle y(\nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}\xi_1), \eta_1 \right\rangle, \qquad \phi_2(y) = \langle \xi_2 y, \eta_2 \rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\phi_2(\nu^{-\frac{1}{2}} \bar{\phi}_1 \nu^{-\frac{1}{2}}) = \left\langle \xi_2 \nu^{-\frac{1}{2}} \bar{\phi}_1 \nu^{-\frac{1}{2}}, \eta_2 \right\rangle = \left\langle \xi_2 I_\nu(\bar{\phi}_1), \eta_2 \right\rangle.$$

To have the desired equality, we need to show that $x I_\nu(\phi_1) = I_\nu(\bar{\phi}_1)$. Let $b = \nu^{\frac{1}{2}} x \nu^{-\frac{1}{2}}$; as $x \in D(\sigma^{\nu}_{-\frac{i}{2}})$ we have $\sigma^{\nu}_{\frac{i}{2}}(b) = x$. Note that $\bar{\phi}_1(y) = \phi_1(yb)$. Using Lemma 3.4.9, we get ·

$$I_\nu(\bar{\phi}_1) = (b\xi_1, \eta_1)_\nu(1) = \sigma^{\nu}_{\frac{i}{2}}(b)(\xi_1, \eta_1)_\nu(1) = x I_\nu(\phi_1).$$

ii) Let $h = \xi_2 \otimes (\nu')^{-\frac{1}{2}}\xi_1$, where $\xi_2 \in \mathcal{H}_2$, $\xi_1 \in D(\mathcal{H}_1, \nu')$. We need to show that $\|h\|^2 = \|Vh\|^2$.

$$\|h\|^2 = \left\langle \xi_2 \otimes (\nu')^{-\frac{1}{2}}\xi_1, \xi_2 \otimes (\nu')^{-\frac{1}{2}}\xi_1 \right\rangle = \phi_2((\nu')^{-\frac{1}{2}} \phi_1 (\nu')^{-\frac{1}{2}})$$

where

$$\phi_1(y) = \langle y\xi_1, \xi_1 \rangle, \qquad \phi_2(y) = \langle \xi_2 y, \xi_2 \rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\phi_2((\nu')^{-\frac{1}{2}} \phi_1 (\nu')^{-\frac{1}{2}}) = \langle \xi_2 I_{\nu'}(\phi_1), \xi_2 \rangle.$$

Now,

$$\|Vh\|^2 = \left\langle \xi_2 \otimes \nu^{-\frac{1}{2}} b\xi_1, \xi_2 \otimes \nu^{-\frac{1}{2}} b\xi_1 \right\rangle = \phi_2(\nu^{-\frac{1}{2}} \bar{\phi}_1 \nu^{-\frac{1}{2}})$$

where

$$\bar{\phi}_1(y) = \langle yb\xi_1, b\xi_1 \rangle, \qquad \phi_2(y) = \langle \xi_2 y, \xi_2 \rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\phi_2(\nu^{-\frac{1}{2}} \bar{\phi}_1 \nu^{-\frac{1}{2}}) = \left\langle \xi_2 I_\nu(\bar{\phi}_1), \xi_2 \right\rangle.$$

To have the desired equality, we need to show that $I_{\nu'}(\phi_1) = I_\nu(\bar{\phi}_1)$. Using Lemma 3.4.9, we get:

$$I_{\nu'}(\phi_1) = (\xi_1, \xi_1)_{\nu'}(1) = b^*(\xi_1, \xi_1)_\nu(1) b = I_\nu(\bar{\phi}_1).$$

Moreover, $\overline{V(\mathcal{H}_2 \otimes_{\nu'} \mathcal{H}_1)} = \mathcal{H}_2 \otimes_\nu \mathcal{H}_1$. Indeed, let $h = \xi_2 \otimes \nu'^{-\frac{1}{2}}\xi_1$, where $\xi_2 \in \mathcal{H}_2$ and $\xi_1 \in D(\mathcal{H}_1, \nu')$; then $V(h) = \xi_2 \otimes \nu^{-\frac{1}{2}} b\xi_1$. Note that by Proposition 3.4.1, $D(\mathcal{H}_1, \nu)$ is dense in $\mathcal{H}_1$. Therefore, we need to show that $b\xi_1 \in D(\mathcal{H}_1, \nu)$ and $bD(\mathcal{H}_1, \nu')$ is dense in $D(\mathcal{H}_1, \nu)$. Note that for $y \in \mathfrak{N}_\nu$,

$$\nu'((yb)^* yb) = \nu'(b^* y^* yb) = \nu(y^* y) \qquad \text{[Prop 2.8.5]}.$$

Thus, $yb \in \mathfrak{N}_{\nu'}$. As $\xi_1 \in D(\mathcal{H}_1, \nu')$, for any $y \in \mathfrak{N}_{\nu}$, we have

$$\|yb\xi_1\|^2 \leq k\nu'((yb)^*yb) = k\nu(y^*y) \qquad \text{[Prop. 2.8.5]}.$$

Thus $b\xi_1 \in D(\mathcal{H}_1, \nu)$. Now, let $\eta \in D(\mathcal{H}_1, \nu)$; we will show that $b^*\eta \in D(\mathcal{H}_1, \nu')$. Let $y \in \mathfrak{N}_{\nu'}$; then $yb^* \in \mathfrak{N}_{\nu}$:

$$\nu((yb^*)^*yb^*) = \nu(by^*yb^*) = \nu'(b^*(by^*yb^*)b) = \nu'(y^*y) \qquad \text{[Prop. 2.8.5]}$$

as b is unitary. Hence, for $y \in \mathfrak{N}_{\nu'}$, we have:

$$\|yb^*\eta\|^2 \leq k\nu((yb^*)^*yb^*) = k\nu(by^*yb^*) = k\nu'(b^*(by^*yb^*)b) = k\nu'(y^*y).$$

Q.E.D.

Definition 3.5.2: Let $\tilde{\nu}$ be a faithful semifinite normal weight on $\mathcal{N}$; using Proposition 3.5.1, we define $\mathcal{H}_2 \otimes_{\tilde{\nu}} \mathcal{H}_1$ as $\mathcal{H}_2 \otimes_{\phi} \mathcal{H}_1$, for any $\phi \in \mathrm{Face}(\tilde{\nu})$.

Remark: The second part of the above proposition ensures that $\mathcal{H} = \mathcal{H}_2 \otimes_{\nu} \mathcal{H}_1$ is independent of the choice of a faithful positive normal form ν on $\mathcal{N}$. From now on, we will therefore write $\mathcal{H} = \mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1$.

Theorem 3.5.3:

i) *Let $\mathcal{N}$ be a von Neumann algebra, $\mathcal{H}_1$ a normal left module and $\mathcal{H}_2$ a normal right module over $\mathcal{N}$. There is a canonical Hilbert space $\mathcal{H} = \mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1$ generated by the elements $\xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1$, where $\xi_2 \in \mathcal{H}_2$, $\nu \in \mathcal{N}_*^+$ is faithful and $\xi_1 \in D(\mathcal{H}_1, \nu)$ and satisfying the relations (2), (3) and (4).*

ii) *Let $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) be a correspondence from $\mathcal{M}_1$ to $\mathcal{N}$ (resp. from $\mathcal{N}$ to $\mathcal{M}_2$) and $\mathcal{H} = \mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1$ as in* i). *Then the following equality defines a correspondence between $\mathcal{M}_1$ and $\mathcal{M}_2$:*

$$x_2(\xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1)x_1 = (x_2\xi_2) \otimes \nu^{-\frac{1}{2}}(\xi_1 x_1)$$

for all $x_j \in \mathcal{M}_j$, $\nu \in \mathcal{N}_*^+$, $\xi_2 \in \mathcal{H}_2$, $\xi_1 \in D(\mathcal{H}_1, \nu)$.

Proof:

i) Follows from Proposition 3.5.1.

ii) We will show that the map

$$V : \mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1 \to \mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1 \qquad \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1 \mapsto (x_2\xi_2) \otimes \nu^{-\frac{1}{2}}(\xi_1 x_1)$$

is an isometry, for all $x_j \in U(\mathcal{M}_j)$, $\nu \in \mathcal{N}_*^+$, $\xi_2 \in \mathcal{H}_2$, $\xi_1 \in D(\mathcal{H}_1, \nu)$. We have:

$$\left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1, \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1 \right\rangle = \phi_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}})$$

where

$$\phi_1(y) = \langle y\xi_1, \xi_1 \rangle, \qquad \phi_2(y) = \langle \xi_2 y, \xi_2 \rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\phi_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}}) = \langle \xi_2 I_\nu(\phi_1), \xi_2\rangle .$$

Now,

$$\Big\langle (x_2\,\xi_2)\otimes \nu^{-\frac{1}{2}}(\xi_1 x_1), (x_2\,\xi_2)\otimes \nu^{-\frac{1}{2}}(\xi_1 x_1)\Big\rangle = \tilde{\phi}_2(\nu^{-\frac{1}{2}}\tilde{\phi}_1\nu^{-\frac{1}{2}})$$

where

$$\begin{aligned} \tilde{\phi}_1(y) &= \langle y\,\xi_1 x_1, \xi_1 x_1\rangle = \langle y\,\xi_1 x_1 x_1^*, \xi_1\rangle = \langle y\,\xi_1, \xi_1\rangle , \qquad && y\in \mathcal{N},\\ \tilde{\phi}_2(y) &= \langle x_2\,\xi_2 y, x_2\,\xi_2\rangle = \langle x_2^* x_2\,\xi_2 y, \xi_2\rangle = \langle \xi_2 y, \xi_2\rangle , \qquad && y\in \mathcal{N}, \end{aligned}$$

as $x_j \in U(\mathcal{M}_j)$. Thus

$$\tilde{\phi}_2(\nu^{-\frac{1}{2}}\tilde{\phi}_1\nu^{-\frac{1}{2}}) = \Big\langle \xi_2 I_\nu(\tilde{\phi}_1), \xi_2\Big\rangle = \langle \xi_2 I_\nu(\phi_1), \xi_2\rangle = \phi_2(\nu^{-\frac{1}{2}}\phi_1\nu^{-\frac{1}{2}})$$

as desired. By the remark following Definition 1.1.7, the result follows.

Q.E.D.

Definition 3.5.4: The correspondence $\mathcal{H}_2\otimes_{\mathcal{N}}\mathcal{H}_1$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ is called the *composition* of the correspondences $\mathcal{H}_2$ and $\mathcal{H}_1$. It is commonly referred to as *Connes Fusion.*

In the next proposition, we show that the coefficients of the composition are obtained by composing the coefficients of the correspondences.

Proposition 3.5.5: *Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H} = \mathcal{H}_2\otimes_{\mathcal{N}}\mathcal{H}_1$ be as above. Let ν be a faithful semifinite normal weight on $\mathcal{N}$ and let ν_2 be a faithful semifinite normal weight on $\mathcal{M}_2$. Then for any ξ_1, $\eta_1 \in D(\mathcal{H}_1,\nu)$ and ξ_2, $\eta_2 \in D(\mathcal{H}_2,\nu_2)$,*

$$(\xi_2\otimes\nu^{-\frac{1}{2}}\xi_1, \eta_2\otimes\nu^{-\frac{1}{2}}\eta_1)_{\nu_2} = (\xi_2,\eta_2)_{\nu_2}\circ(\xi_1,\eta_1)_\nu .$$

Proof: By definition, $(\xi_2\otimes\nu^{-\frac{1}{2}}\xi_1, \eta_2\otimes\nu^{-\frac{1}{2}}\eta_1)_{\nu_2}$ is the unique normal map $P:\mathcal{M}_1\to\mathcal{M}_2$ such that for all $x_2\in\mathcal{M}_2$, $x_1\in\mathcal{M}_1$,

$$\Big\langle x_2(\xi_2\otimes\nu^{-\frac{1}{2}}\xi_1)x_1, \eta_2\otimes\nu^{-\frac{1}{2}}\eta_1\Big\rangle = \Big\langle x_2\,\nu_2^{\frac{1}{2}}P(x_1)\,\nu_2^{\frac{1}{2}}\Big\rangle .$$

Note that

$$\Big\langle x_2(\xi_2\otimes\nu^{-\frac{1}{2}}\xi_1)x_1, \eta_2\otimes\nu^{-\frac{1}{2}}\eta_1\Big\rangle = \tilde{\phi}_2(\nu^{-\frac{1}{2}}\tilde{\phi}_1\nu^{-\frac{1}{2}})$$

where

$$\tilde{\phi}_1(y) = \langle y\,\xi_1 x_1, \eta_1\rangle , \qquad \tilde{\phi}_2(y) = \langle x_2\,\xi_2\,y, \eta_2\rangle , \qquad y\in\mathcal{N}.$$

Thus

$$\tilde{\phi}_2(\nu^{-\frac{1}{2}}\tilde{\phi}_1\nu^{-\frac{1}{2}}) = \tilde{\phi}_2(I_\nu(\tilde{\phi}_1)) = \Big\langle x_2\,\xi_2 I_\nu(\tilde{\phi}_1), \eta_2\Big\rangle .$$

Now, $(\xi_2, \eta_2)_{\nu_2}$ is the unique normal map $P_2 : \mathcal{N} \to \mathcal{M}_2$ such that

$$\langle x_2\, \xi_2 x, \eta_2\rangle = \left\langle x_2\, \nu_2^{\frac{1}{2}} P_2(x)\, \nu_2^{\frac{1}{2}} \right\rangle, \quad x_2 \in \mathcal{M}_2,\ x \in \mathcal{N}.$$

Note that $\langle x_2\, \xi_2 x, \eta_2\rangle = \tilde{\phi}_2(x)$. Moreover, $(\xi_1, \eta_1)_\nu$ is the unique normal map $P_1 : \mathcal{M}_1 \to \mathcal{N}$ such that

$$\langle x\, \xi_1 x_1, \eta_1\rangle = \left\langle x\, \nu^{\frac{1}{2}} P_1(x_1)\, \nu^{\frac{1}{2}} \right\rangle, \quad x_1 \in \mathcal{M}_1,\ x \in \mathcal{N}.$$

Note that $\langle x\, \xi_1 x_1, \eta_1\rangle = \tilde{\phi}_1(x)$ and $P_1(x_1) = I_\nu(\tilde{\phi}_1)$. As

$$\tilde{\phi}_2(P_1(x_1)) = \left\langle x_2\, \nu_2^{\frac{1}{2}} P_2(P_1(x_1))\, \nu_2^{\frac{1}{2}} \right\rangle$$

we have the desired result, that is, $P = P_2 \circ P_1$.

Q.E.D.

Let $\rho_1 : \mathcal{M}_1 \to \mathcal{N}$ and $\rho_2 : \mathcal{N} \to \mathcal{M}_2$ be *-homomorphisms and $L^2(\rho_i)$ the associated correspondences.

Proposition 3.5.6: *The correspondence $L^2(\rho_2) \otimes_{\mathcal{N}} L^2(\rho_1)$ is canonically equivalent to $L^2(\rho_2 \circ \rho_1)$.*

The proof of the above proposition depends on the next lemma.

Let $\mathcal{H}_2$ be any correspondence from $\mathcal{N}$ to $\mathcal{M}_2$. Consider the following $\mathcal{M}_2$-$\mathcal{M}_1$ bimodule:

$$\mathcal{H}_2\, \rho_1(1), \quad \pi(x_2 \otimes x_1^\circ)\xi = x_2\, \xi\, \rho_1(x_1), \quad \text{for all } x_j \in \mathcal{M}_j \text{ and } \xi \in \mathcal{H}_2\, \rho_1(1).$$

Then

Lemma 3.5.7: *The bimodule $\mathcal{H}_2\, \rho_1(1)$ is canonically equivalent to $\mathcal{H}_2 \otimes_{\mathcal{N}} L^2(\rho_1)$.*

Proof: Let ν be a faithful semifinite normal weight on $\mathcal{N}$. Let $\xi \in L^2(\rho_1) \subset L^2(\mathcal{N})$ be of the form $\xi = a\nu^{\frac{1}{2}}\rho_1(1)$, where $a \in \mathcal{N}_\infty^\nu$ [section 2.7]; then in particular, $a \in D(\sigma^\nu_{\frac{i}{2}})$ and thus $a^* \in D(\sigma^\nu_{-\frac{i}{2}})$. Hence for all $y \in \mathfrak{N}_\nu$,

$$\|y\xi\|^2 \leq \|\rho_1(1)\|^2 \|ya\nu^{\frac{1}{2}}\|^2 \leq \|\rho_1(1)\|^2 \nu(a^*y^*ya) \leq \|\rho_1(1)\|^2 k\nu(y^*y) \quad \text{[Prop. 2.7.2]},$$

that is, $\xi = a\nu^{\frac{1}{2}}\rho_1(1) \in D(L^2(\rho_1), \nu)$ and $\nu^{-\frac{1}{2}}\xi = \nu^{-\frac{1}{2}} a \nu^{\frac{1}{2}} \rho_1(1) = \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1) \in \mathcal{N}$. As $L^2(\mathcal{N}) = [\mathfrak{N}_\nu \nu^{\frac{1}{2}}]$, then

$$L^2(\rho_1) = L^2(\mathcal{N})\rho_1(1) = [\mathfrak{N}_\nu \nu^{\frac{1}{2}} \rho_1(1)].$$

By Proposition 2.7.4, for all $y \in \mathfrak{N}_\nu$, there exists a sequence $(a_n)_{n \geq 1} \subset \mathcal{N}_\infty^\nu$ such that $a_n \xrightarrow{s^*} y$, that is, for all $\epsilon > 0$, there exists $K \in \mathbf{N}$ such that for all $n \geq K$, then

$$\nu((y - a_n)^*(y - a_n)) < \epsilon, \qquad \nu((y - a_n)(y - a_n)^*) < \epsilon.$$

Hence, for all $n \geq K$,

$$\begin{aligned}\|y\nu^{\frac{1}{2}}\rho_1(1) - a_n\nu^{\frac{1}{2}}\rho_1(1)\|^2 &\leq \|\rho_1(1)\|^2 \left\langle (y-a_n)\nu^{\frac{1}{2}}, (y-a_n)\nu^{\frac{1}{2}} \right\rangle \\ &= \nu((y-a_n)^*(y-a_n)) < \epsilon\end{aligned}$$

and

$$\begin{aligned}\|y^*\nu^{\frac{1}{2}}\rho_1(1) - a_n^*\nu^{\frac{1}{2}}\rho_1(1)\|^2 &\leq \|\rho_1(1)\|^2 \left\langle (y-a_n)^*\nu^{\frac{1}{2}}, (y-a_n)^*\nu^{\frac{1}{2}} \right\rangle \\ &= \nu((y-a_n)(y-a_n)^*) < \epsilon.\end{aligned}$$

Note that the set $A = \{\xi_2 \otimes \nu^{-\frac{1}{2}} a\nu^{\frac{1}{2}}\rho_1(1) : a \in \mathcal{N}_\infty^\nu, \xi_2 \in \mathcal{H}_2\}$ is dense in $\mathcal{H}_2 \otimes_{\mathcal{N}} L^2(\rho_1)$. Now, for all $\xi_2 \in \mathcal{H}_2$ and $\xi_1 = a\nu^{\frac{1}{2}}\rho_1(1)$ for some $a \in \mathcal{N}_\infty^\nu$, let

$$V : A \to \mathcal{H}_2\,\rho_1(1), \qquad \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1 \mapsto \xi_2(\nu^{-\frac{1}{2}}\xi_1) = \xi_2\,\sigma^\nu_{\frac{i}{2}}(a)\rho_1(1)$$

and π and π_1 be the representations of $\mathcal{M}_2 \odot \mathcal{M}_1^{\circ}$ associated to the correspondences $\mathcal{H}_2\,\rho_1(1)$ and $\mathcal{H}_2 \otimes_{\mathcal{N}} L^2(\rho_1)$. Let $\xi = \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1$; then

$$\pi(x_2 \otimes x_1^{\circ})V\xi = \pi(x_2 \otimes x_1^{\circ})(\xi_2(\nu^{-\frac{1}{2}}\xi_1)) = x_2(\xi_2(\nu^{-\frac{1}{2}}\xi_1))\rho_1(x_1) \qquad \text{and}$$

$$\begin{aligned}V\pi_1(x_2 \otimes x_1^{\circ})\xi &= V(x_2(\xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1)\rho_1(x_1)) \\ &= V((x_2\xi_2) \otimes \nu^{-\frac{1}{2}}(\xi_1\rho_1(x_1))) = x_2\,\xi_2(\nu^{-\frac{1}{2}}\xi_1\rho_1(x_1)).\end{aligned}$$

Thus V intertwines the representations associated to the correspondences $\mathcal{H}_2\,\rho_1(1)$ and $\mathcal{H}_2 \otimes_{\mathcal{N}} L^2(\rho_1)$. Let us show that V is an isometry.

$$\left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1, \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1 \right\rangle = \phi_2(\nu^{-\frac{1}{2}}\phi_1\,\nu^{-\frac{1}{2}}) = \phi_2(I_\nu(\phi_1))$$

where

$$\begin{aligned}&\phi_1(y) = \langle y\xi_1, \xi_1 \rangle = \left\langle ya\nu^{\frac{1}{2}}\rho_1(1), a\nu^{\frac{1}{2}}\rho_1(1) \right\rangle, \qquad a \in \mathcal{N}_\infty^\nu, \\ &\phi_2(y) = \langle \xi_2\,y, \xi_2 \rangle, \qquad y \in \mathcal{N}.\end{aligned}$$

Thus

$$\phi_2(I_\nu(\phi_1)) = \langle \xi_2 I_\nu(\phi_1), \xi_2 \rangle.$$

Note that

$$\begin{aligned}\phi_1(y) &= \left\langle ya\nu^{\frac{1}{2}}\rho_1(1), a\nu^{\frac{1}{2}}\rho_1(1) \right\rangle = \left\langle y\nu^{\frac{1}{2}}(\nu^{-\frac{1}{2}}a\nu^{\frac{1}{2}})\rho_1(1), \nu^{\frac{1}{2}}(\nu^{-\frac{1}{2}}a\nu^{\frac{1}{2}})\rho_1(1) \right\rangle \\ &= \left\langle y\nu^{\frac{1}{2}}\sigma^\nu_{\frac{i}{2}}(a)\rho_1(1), \nu^{\frac{1}{2}}\sigma^\nu_{\frac{i}{2}}(a)\rho_1(1) \right\rangle = \left\langle y\nu^{\frac{1}{2}}\sigma^\nu_{\frac{i}{2}}(a)\rho_1(1)\sigma^\nu_{-\frac{i}{2}}(a^*), \nu^{\frac{1}{2}} \right\rangle.\end{aligned}$$

By definition of I_ν, we have $I_\nu(\phi_1) = \sigma^\nu_{\frac{i}{2}}(a)\,\rho_1(1)\,\sigma^\nu_{-\frac{i}{2}}(a^*)$: hence

$$\begin{aligned}\left\langle \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1, \xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1 \right\rangle &= \langle \xi_2 I_\nu(\phi_1), \xi_2\rangle \\ &= \left\langle \xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\,\rho_1(1)\,\sigma^\nu_{-\frac{i}{2}}(a^*), \xi_2 \right\rangle \\ &= \left\langle \xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1), \xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1) \right\rangle .\end{aligned}$$

Now

$$\begin{aligned}\left\langle V(\xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1), V(\xi_2 \otimes \nu^{-\frac{1}{2}}\xi_1) \right\rangle &= \left\langle \xi_2(\nu^{-\frac{1}{2}}\xi_1), \xi_2(\nu^{-\frac{1}{2}}\xi_1) \right\rangle \\ &= \left\langle \xi_2(\nu^{-\frac{1}{2}} a\, \nu^{\frac{1}{2}})\rho_1(1), \xi_2(\nu^{-\frac{1}{2}} a\, \nu^{\frac{1}{2}})\rho_1(1) \right\rangle \\ &= \left\langle \xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1), \xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1) \right\rangle .\end{aligned}$$

The isometry V extends to an isometry $V : \mathcal{H}_2 \otimes_{\mathcal{N}} L^2(\rho_1) \to \mathcal{H}_2\, \rho_1(1)$. In addition, V is surjective as $V(A) = \{\xi_2\, \sigma^\nu_{\frac{i}{2}}(a)\rho_1(1)\}$ is dense in $\mathcal{H}_2\, \rho_1(1)$. Note that for any $\xi_1 \in L^2(\rho_1)$, then $\xi_2(\nu^{-\frac{1}{2}}\xi_1) = \xi_2(\nu^{-\frac{1}{2}}\xi_1\rho_1(1))$; moreover $\nu^{-\frac{1}{2}}\xi_1 \in \mathcal{N}$ for all $\xi_1 \in D(L^2(\rho_1), \nu)$. Thus $\xi_2(\nu^{-\frac{1}{2}}\xi_1) \in \mathcal{H}_2\, \rho_1(1)$.

Q.E.D.

Proof of Prop. 3.5.6: Let us use Lemma 3.5.7, with $\mathcal{H}_2 = L^2(\rho_2)$; then

$$L^2(\rho_2)\,\rho_1(1) \sim L^2(\rho_2) \otimes_{\mathcal{N}} L^2(\rho_1).$$

As

$$\pi : \mathcal{M}_2 \odot \mathcal{M}_1^\circ \to \mathcal{L}(L^2(\rho_2)\,\rho_1(1)), \qquad \pi(x_2 \otimes x_1^\circ)\xi = x_2\,\xi\rho_2(\rho_1(x_1))$$

and

$$\bar{\pi} : \mathcal{M}_2 \odot \mathcal{M}_1^\circ \to \mathcal{L}(L^2(\rho_2 \circ \rho_1)), \qquad \bar{\pi}(x_2 \otimes x_1^\circ)\xi = x_2\,\xi(\rho_2 \circ \rho_1)(x_1),$$

then $L^2(\rho_2)\rho_1(1) \sim L^2(\rho_2 \circ \rho_1)$ and we have the desired equivalence.

Q.E.D.

Proposition 3.5.8: *The composition of correspondences is associative.*

Proof: Let $\mathcal{H}_i$ be a correspondence from $\mathcal{M}_{i+1}$ to $\mathcal{M}_i$, for $1 \le i \le 3$. As seen in Proposition 3.3.1 and the remark following Corollary 3.3.2, $\mathcal{H}_i$ is equivalent to an $L^2(\rho_i)$ (for all i), where $\rho_i : \mathcal{M}_{i+1} \to \mathcal{M}_i$ is a normal *-homomorphism. Thus, by Proposition 3.5.6,

$$L^2(\rho_1) \otimes_{\mathcal{M}_2} (L^2(\rho_2) \otimes_{\mathcal{M}_3} L^2(\rho_3)) \sim L^2(\rho_1) \otimes_{\mathcal{M}_2} L^2(\rho_2 \circ \rho_3) \sim L^2(\rho_1 \circ (\rho_2 \circ \rho_3))$$

and

$$(L^2(\rho_1) \otimes_{\mathcal{M}_2} L^2(\rho_2)) \otimes_{\mathcal{M}_3} L^2(\rho_3) \sim L^2(\rho_1 \circ \rho_2) \otimes_{\mathcal{M}_3} L^2(\rho_3) \sim L^2((\rho_1 \circ \rho_2) \circ \rho_3).$$

As the composition of *-homomorphisms is associative, we have the desired result.

Q.E.D.

Definition 3.5.9: Let $\mathcal{H}$ be a correspondence from $\mathcal{M}$ to $\mathcal{N}$; then its *contragredient* $\overline{\mathcal{H}}$ is the correspondence from $\mathcal{N}$ to $\mathcal{M}$ given by

$$x\,\overline{\xi}\,y = \overline{(y^*\,\xi\,x^*)}, \quad \text{for all } \xi \in \mathcal{H},\ x \in \mathcal{M},\ y \in \mathcal{N},$$

with $\overline{\mathcal{H}}$ the conjugate Hilbert space of $\mathcal{H}$ [section 1.7].

Remark: We have used the canonical antilinear isometry $\xi \mapsto \overline{\xi}$ from $\mathcal{H}$ to $\overline{\mathcal{H}}$.

Proposition 3.5.10: *Let $\rho : \mathcal{M} \to \mathcal{N}$ be a normal *-isomorphism; then $\overline{L^2(\rho)} = L^2(\rho^{-1})$.*

Proof: $\overline{L^2(\rho)} = \overline{L^2(\mathcal{N})}$ and $L^2(\rho^{-1}) = L^2(\mathcal{M})$ are $\mathcal{M}$-$\mathcal{N}$ correspondences, with actions described by the following representations of $\mathcal{M} \odot \mathcal{N}^o$:

$$\pi_{\nu\circ\rho^{-1}} : \mathcal{M} \odot \mathcal{N}^o \to \mathcal{L}(\overline{L^2(\mathcal{N}, \nu\circ\rho^{-1})}), \qquad \pi_{\nu\circ\rho^{-1}}(x \otimes y^o)\,\overline{\xi} = \overline{y^*\,\xi\rho(x^*)}$$

$$\pi_\nu : \mathcal{M} \odot \mathcal{N}^o \to \mathcal{L}(L^2(\mathcal{M}, \nu)), \qquad \pi_\nu(x \otimes y^o)\xi = x\xi\rho^{-1}(y)$$

where we have used the identifications

$$L^2(\mathcal{M}) \cong L^2(\mathcal{M}, \nu) \quad (\text{for some faithful } \nu \in \mathcal{M}_*^+) \quad \text{and} \quad \overline{L^2(\mathcal{N})} \cong \overline{L^2(\mathcal{N}, \nu\circ\rho^{-1})}$$

introduced in section 3.2. Let

$$V : L^2(\mathcal{M}, \nu) \to \overline{L^2(\mathcal{N}, \nu\circ\rho^{-1})}, \qquad z \in \mathcal{M} \mapsto \overline{\rho(z^*)}.$$

Consider the map

$$L^2(\mathcal{M}) \to \overline{L^2(\mathcal{N})}, \qquad x\nu^{\frac{1}{2}} \mapsto \overline{(\nu\circ\rho^{-1})^{\frac{1}{2}}\rho(x^*)}$$

for all $x \in \mathcal{M}$; via this identification, V is an isometry. Indeed:

$$(z \mid z) = \left\langle z\nu^{\frac{1}{2}}, z\nu^{\frac{1}{2}} \right\rangle = \nu(z^*z), \qquad z \in \mathcal{M}$$

and

$$(Vz \mid Vz) = \left\langle (\nu\circ\rho^{-1})^{\frac{1}{2}}\rho(z^*), (\nu\circ\rho^{-1})^{\frac{1}{2}}\rho(z^*) \right\rangle = (\nu\circ\rho^{-1})(\rho(z^*z)) = \nu(z^*z).$$

Moreover, for all $z \in \mathcal{M}$, we have:

$$V(\pi_\nu(x \otimes y^o)z) = V(xz\,\rho^{-1}(y)) = \overline{\rho((xz\,\rho^{-1}(y))^*)} = \overline{\rho(\rho^{-1}(y^*)z^*x^*)} = \overline{y^*\,\rho((xz)^*)}$$

and

$$\pi_{\nu\circ\rho^{-1}}(x \otimes y^o)Vz = \pi_{\nu\circ\rho^{-1}}(x \otimes y^o)\overline{\rho(z^*)} = \overline{y^*\rho(z^*)\rho(x^*)} = \overline{y^*\rho((xz)^*)}.$$

Note that as $\nu \in \mathcal{M}_*^+$ is faithful, then $L^2(\mathcal{M}, \nu) = \overline{\mathcal{M}}^{\|\cdot\|_\nu}$; furthermore, ρ is normal. Thus for any $\xi \in L^2(\mathcal{M}, \nu)$, the above argument applies. Consequently, we have the desired equivalence.

Q.E.D.

Theorem 3.5.11: *Let $\mathcal{H}_1$ be a correspondence from $\mathcal{M}_1$ to $\mathcal{N}$ and $\mathcal{H}_2$ a correspondence from $\mathcal{N}$ to $\mathcal{M}_2$; then one has a canonical equivalence of correspondences from $\mathcal{M}_2$ to $\mathcal{M}_1$:*

$$\overline{(\mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1)} = \overline{\mathcal{H}_1} \otimes_{\mathcal{N}} \overline{\mathcal{H}_2}.$$

Proof: Let ν be a faithful normal semifinite weight on $\mathcal{N}$. Note that in the construction of $\mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1$, we could have used expressions of the form $\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1$, with $\xi_2 \in D(\mathcal{H}_2, \nu)$ and $\xi_1 \in \mathcal{H}_1$, and defined the inner product as follows:

$$\left\langle \xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1, \eta_2 \nu^{-\frac{1}{2}} \otimes \eta_1 \right\rangle = \phi_1(\nu^{-\frac{1}{2}} \phi_2 \nu^{-\frac{1}{2}})$$

with

$$\phi_1(y) = \langle y\xi_1, \eta_1 \rangle, \qquad \phi_2(y) = \langle \xi_2 y, \eta_2 \rangle, \qquad y \in \mathcal{N}.$$

Let

$$V : \overline{\mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1} \to \overline{\mathcal{H}_1} \otimes_{\mathcal{N}} \overline{\mathcal{H}_2}, \qquad \overline{\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1} \mapsto \overline{\xi_1} \otimes \nu^{-\frac{1}{2}} \overline{\xi_2}$$

for all $\xi_j \in D(\mathcal{H}_j, \nu)$ and

$$\pi_1 : \mathcal{M}_1 \odot \mathcal{M}_2^\circ \to \mathcal{L}(\overline{\mathcal{H}_2 \otimes_{\mathcal{N}} \mathcal{H}_1}), \qquad \pi_1(x_1 \otimes x_2^\circ)\overline{\xi} = \overline{(x_2^* \xi x_1^*)}$$

$$\pi_2 : \mathcal{M}_1 \odot \mathcal{M}_2^\circ \to \mathcal{L}(\overline{\mathcal{H}_1} \otimes_{\mathcal{N}} \overline{\mathcal{H}_2}), \qquad \pi_2(x_1 \otimes x_2^\circ)(\overline{\xi_1} \otimes \nu^{-\frac{1}{2}} \overline{\xi_2}) = \overline{\xi_1 x_1^*} \otimes \nu^{-\frac{1}{2}} \overline{x_2^* \xi_2}$$

the associated representations. Then

$$\begin{aligned} V(\pi_1(x_1 \otimes x_2^\circ)\overline{(\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1)}) &= V(\overline{x_2^*(\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1) x_1^*}) \\ &= V(\overline{(x_2^* \xi_2)\nu^{-\frac{1}{2}} \otimes (\xi_1 x_1^*)}) = \overline{\xi_1 x_1^*} \otimes \nu^{-\frac{1}{2}} \overline{x_2^* \xi_2} \end{aligned}$$

and

$$\pi_2(x_1 \otimes x_2^\circ) V(\overline{(\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1)}) = \pi_2(x_1 \otimes x_2^\circ)(\overline{\xi_1} \otimes \nu^{-\frac{1}{2}} \overline{\xi_2}) = \overline{\xi_1 x_1^*} \otimes \nu^{-\frac{1}{2}} \overline{x_2^* \xi_2}.$$

Moreover, V is an isometry:

$$\left\langle \overline{\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1}, \overline{\xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1} \right\rangle = \left\langle \xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1, \xi_2 \nu^{-\frac{1}{2}} \otimes \xi_1 \right\rangle = \phi_1(\nu^{-\frac{1}{2}} \phi_2 \nu^{-\frac{1}{2}})$$

where

$$\phi_1(y) = \langle y\xi_1, \xi_1 \rangle, \qquad \phi_2(y) = \langle \xi_2 y, \xi_2 \rangle, \qquad y \in \mathcal{N}.$$

Thus

$$\phi_1(I_\nu(\phi_2)) = \langle I_\nu(\phi_2)\,\xi_1, \xi_1\rangle\,.$$

Now

$$\left\langle \overline{\xi_1} \otimes \nu^{-\frac{1}{2}}\overline{\xi_2}, \overline{\xi_1} \otimes \nu^{-\frac{1}{2}}\overline{\xi_2} \right\rangle = \bar{\phi}_1(\nu^{-\frac{1}{2}}\bar{\phi}_2\,\nu^{-\frac{1}{2}})$$

where for all $y \in \mathcal{N}$,

$$\bar{\phi}_2(y) = \langle y\,\overline{\xi_2}, \overline{\xi_2}\rangle = \langle \overline{\xi_2\,y^*}, \overline{\xi_2}\rangle = \langle \xi_2, \xi_2\,y^*\rangle = \langle \xi_2\,y, \xi_2\rangle = \phi_2(y)$$

and

$$\bar{\phi}_1(y) = \langle \overline{\xi_1}\,y, \overline{\xi_1}\rangle = \langle \overline{y^*\,\xi_1}, \overline{\xi_1}\rangle = \langle \xi_1, y^*\,\xi_1\rangle = \langle y\,\xi_1, \xi_1\rangle = \phi_1(y).$$

Thus

$$\bar{\phi}_1(I_\nu(\bar{\phi}_2)) = \langle I_\nu(\phi_2)\,\xi_1, \xi_1\rangle\,.$$

Q.E.D.

Appendix I

A.1 Unbounded linear operators in Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.

Definition A.1.1:

i) T is a *linear operator* from $\mathcal{H}$ to $\mathcal{K}$ if T is a linear mapping of a vector subspace $\mathcal{D}_T$ of $\mathcal{H}$ into the vector space $\mathcal{K}$; $\mathcal{D}_T$ is called the *domain (of definition)* of T. If $\mathcal{H} = \mathcal{K}$, then T is called a linear operator in $\mathcal{H}$.

ii) Let S, T be linear operators from $\mathcal{H}$ to $\mathcal{K}$. We say that these operators are equal if $\mathcal{D}_T = \mathcal{D}_S$ and $T\xi = S\xi$ for all $\xi \in \mathcal{D}_T = \mathcal{D}_S$. This relation is denoted $T = S$.

iii) Let S, T be linear operators from $\mathcal{H}$ to $\mathcal{K}$. We say that T is an extension of S (or S is a restriction of T), if $\mathcal{D}_T \supset \mathcal{D}_S$ and $T\xi = S\xi$ for all $\xi \in \mathcal{D}_S$. This relation is denoted $T \supset S$.

iv) Let S, T be linear operators from $\mathcal{H}$ to $\mathcal{K}$.
The multiplication by scalars $\lambda \in \mathbf{C}$, λT, is defined by:

$$\mathcal{D}_{\lambda T} = \mathcal{D}_T, \qquad (\lambda T)\xi = \lambda(T\xi), \quad \xi \in \mathcal{D}_{\lambda T};$$

- the sum $T + S$ is defined by:

$$\mathcal{D}_{T+S} = \mathcal{D}_T \cap \mathcal{D}_S, \qquad (T+S)\xi = T\xi + S\xi, \quad \xi \in \mathcal{D}_{T+S};$$

- the composition $S \circ T = ST$ is defined by:

$$\mathcal{D}_{ST} = \{\xi \in \mathcal{D}_T : T\xi \in \mathcal{D}_S\}, \qquad (ST)\xi = (S(T\xi)), \quad \xi \in \mathcal{D}_{ST};$$

- the inverse T^{-1} (if the mapping $T : \mathcal{D}_T \to \mathcal{K}$ is injective) is defined by:

$$\mathcal{D}_{T^{-1}} = T\mathcal{D}_T, \qquad T^{-1}\eta = \xi \Leftrightarrow T\xi = \eta, \quad \eta \in \mathcal{D}_T^{-1}.$$

Definition A.1.2: Let T be a linear operator from $\mathcal{H}$ into $\mathcal{K}$.

i) The *graph* G_T of T is the vector subspace of $\mathcal{H} \oplus \mathcal{K}$ defined as follows:

$$G_T = \{(\xi, T\xi) : \xi \in \mathcal{D}_T\}.$$

ii) T is *densely defined* if $\mathcal{D}_T$ is dense in $\mathcal{H}$.

iii) T is *preclosed* if it is densely defined and if the closure of G_T in $\mathcal{H} \oplus \mathcal{K}$ is the graph of a linear operator, denoted $\overline{T}$ and called the *closure* of T.

iv) T is *closed* if it is preclosed and $T = \overline{T}$.

v) T is *bounded* if $\sup\{\|T\xi\| : \xi \in \mathcal{D}_T, \|\xi\| \leq 1\} < +\infty$. Whenever this condition is not satisfied, T is said to be *unbounded* and then T is continuous at no point of its domain of definition.

vi) If T is closed, the kernel of T is a closed vector subspace of $\mathcal{H}$. The projection of $\mathcal{H}$ onto this subspace will be denoted $\mathbf{n}(T)$ and $\mathbf{r}(T) = 1 - \mathbf{n}(T)$ will be called the *right support* of T. The projection of $\mathcal{K}$ onto the closure of the vector subspace $T\mathcal{D}_T$ will be denoted $\mathbf{l}(T)$ and called the *left support* of T.

Let T be a densely defined linear operator from $\mathcal{H}$ into $\mathcal{K}$. The set

$$\mathcal{D} = \{\eta \in \mathcal{K} : \mathcal{D}_T \ni \xi \mapsto (T\xi \mid \eta) \text{ is bounded}\}$$

is a vector subspace of $\mathcal{K}$. As $\mathcal{D}_T$ is dense in $\mathcal{H}$, we conclude (from the Riesz theorem) that for all $\eta \in \mathcal{D}$, there exists a unique $\eta^* \in \mathcal{H}$ satisfying

$$(T\xi \mid \eta) = (\xi \mid \eta^*), \qquad \xi \in \mathcal{D}_T.$$

Definition A.1.3: The *adjoint* T^* of T is the linear operator from $\mathcal{K}$ into $\mathcal{H}$ defined by:

$$\mathcal{D}_{T^*} = \mathcal{D}, \qquad T^*\eta = \eta^*, \quad \eta \in \mathcal{D}_{T^*}.$$

Remarks:

i) T^* is determined by: $(T\xi \mid \eta) = (\xi \mid T^*\eta), \quad \xi \in \mathcal{D}_T, \eta \in \mathcal{D}_{T^*}$.

ii) If the operators $S, T, T+S, ST$ are densely defined and $\lambda \in \mathbf{C}$, then

$$(\lambda T)^* = \overline{\lambda} T^*, \qquad T \supset S \Rightarrow T^* \subset S^*, \qquad (T+S)^* \supset T^* + S^*, \qquad (ST)^* \supset T^* S^*,$$

and if T^{-1} exists and is densely defined, then $(T^{-1})^* = (T^*)^{-1}$.

Definition A.1.4: Let T be a linear operator in $\mathcal{H}$.

i) T is *symmetric* if it is densely defined and $T \subset T^*$. Equivalently, T is symmetric if and only if it is densely defined and $(T\xi \mid \eta) = (\xi \mid T\eta), \ \xi, \eta \in \mathcal{D}_T$.

ii) T is *positive* if it is densely defined and $(T\xi \mid \xi) \geq 0,\ \xi \in \mathcal{D}_T$.

iii) T is *self-adjoint* if it is densely defined and $T = T^*$.

Remark: If T is a self-adjoint operator in $\mathcal{H}$, then one denotes $\mathrm{s}(T) = \mathrm{r}(T) = \mathrm{l}(T)$ and the projection $\mathrm{s}(T)$ is called the *support* of T.

Lemma A.1.5: *Let A be a positive linear operator in $\mathcal{H}$. Then A is self-adjoint if and only if* $(1 + A)\mathcal{D}_A = \mathcal{H}$.

Proof: [S-Z; 9.5 p.193]

Definition A.1.6: Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and T be a linear operator in $\mathcal{H}$. If for all unitary $u' \in \mathcal{M}'$, $u'^*Tu' = T$, then T is said to be *affiliated* to $\mathcal{M}$.

Note that if T is densely defined and affiliated to $\mathcal{M}$, then T^* is affiliated to $\mathcal{M}$. Moreover, if T is preclosed and affiliated to $\mathcal{M}$, then $\overline{T}$ is affiliated to $\mathcal{M}$.

Let A be a positive self-adjoint operator in $\mathcal{H}$ and $f \in \mathcal{B}([0,\infty))$, (where $\mathcal{B}([0,\infty))$ is the *-algebra of all Borel measurable complex functions defined on $[0,\infty)$ and bounded on compact sets). We would like to give meaning to $f(A)$.

Let $a = (1 + A)^{-1}$; then a is self-adjoint and by Lemma A.1.5, $a \in \mathcal{L}(\mathcal{H})$. By the symbolic calculus for bounded self-adjoint operators [R; p.325], there exists a unique resolution of the identity E on the Borel subsets of the spectrum $\sigma(a)$ of a such that

$$a = \int_{\sigma(a)} \lambda \, dE(\lambda).$$

Denote by χ_n the characteristic function of the set $((n+1)^{-1}, \infty)$. Then

$$e_n = \chi_n(a) = \int_{(n+1)^{-1}}^{\infty} dE(\lambda).$$

Note that $e_n \in \{a\}''$, the von Neumann algebra generated by a. Moreover, $e_n\mathcal{H} \subset \mathcal{D}_A$ and $Ae_n \subset \mathcal{L}(\mathcal{H})$.

For any $f \in \mathcal{B}([0,\infty))$, $f \mid \sigma(Ae_n) \in \mathcal{B}(\sigma(Ae_n))$, where $\sigma(Ae_n)$ denotes the spectrum of Ae_n; making use of the symbolic calculus for bounded self-adjoint operators again, the operator $f(Ae_n) \in \mathcal{L}(\mathcal{H})$ is defined by

$$f(Ae_n) = \int_{\sigma(Ae_n)} f(\lambda) \, dE(\lambda)$$

where E is the resolution of the identity of the operator Ae_n.

For any $f \in \mathcal{B}([0,\infty))$, $f(A)$ will denote the linear operator in $\mathcal{H}$, defined by the following relations:

$$\mathcal{D}_{f(A)} = \{\xi \in \mathcal{H} : \text{ the sequence } \{f(Ae_n)\xi\}_n \text{ is norm convergent }\},$$

$$f(A)\xi = \lim_{n\to\infty} f(Ae_n)\xi, \quad \xi \in \mathcal{D}_{f(A)}.$$

We will now state the rules of the operational calculus for positive self adjoint operators.

Theorem A.1.7: *Let A be a positive self-adjoint linear operator in the Hilbert space $\mathcal{H}$;*

i) *for* $f_o(\lambda) = c \in \mathbf{C}$, $\lambda \in [0, +\infty)$, $f_o(A) = c$;

ii) *for* $f_o(\lambda) = \lambda$, $\lambda \in [0, +\infty)$, $f_o(A) = A$;

iii) *for all* $f \in \mathcal{B}([0, \infty))$,

$$\mathcal{D}_{f(A)} = \{\xi \in \mathcal{H} : \sup_n \|f(Ae_n)\xi\| < +\infty\},$$

the linear operator $f(A)$ *is closed and* $\overline{f(A) \mid \mathcal{S}_A} = f(A)$, *where* $\mathcal{S}_A = \bigcup_{n=1}^{\infty} e_n\mathcal{H}$;

iv) *for all* $f \in \mathcal{B}([0, \infty))$, $f(A)^* = \overline{f}(A)$;

v) *for all* $f, g \in \mathcal{B}([0, +\infty))$, *the linear operator* $f(A) + g(A)$ *is preclosed and*

$$\overline{f(A) + g(A)} = (f + g)(A);$$

vi) *for all* $f, g \in \mathcal{B}([0, +\infty))$, *the linear operator* $f(A)g(A)$ *is preclosed,*

$$\mathcal{D}_{f(A)g(A)} = \mathcal{D}_{(fg)(A)} \cap \mathcal{D}_{g(A)} \quad \text{and} \quad \overline{f(A)g(A)} = (fg)(A);$$

vii) *for any sequence* $\{f_k\}_k \subset \mathcal{B}([0, +\infty))$, *which is uniformly bounded on compact sets and pointwise convergent to* $f_o \in \mathcal{B}([0, +\infty))$,

$$f_o(A)\xi = \lim_{k \to \infty} f_k(A)\xi, \quad \xi \in \mathcal{S}_A.$$

viii) *if* $f, g \in \mathcal{B}([0, +\infty))$ *and* $|f| \le |g|$, *then*

$$\mathcal{D}_{g(A)} \subset \mathcal{D}_{f(A)}, \qquad \|f(A)\xi\| \le \|g(A)\xi\|, \quad \xi \in \mathcal{D}_{g(A)};$$

if f *is bounded, then* $f(A) \in \mathcal{L}(\mathcal{H})$ *and* $\|f(A)\| \le \sup\{|f(\lambda)| : \lambda \in [0, +\infty)\}$.

Proof: [S-Z; 9.11-9.12]

Corollary A.1.8: *Let A be a positive self-adjoint linear operator in $\mathcal{H}$;*

i) *for any real* $f \in \mathcal{B}([0, +\infty))$, $f(A)$ *is self-adjoint;*

ii) *for any positive* $f \in \mathcal{B}([0, +\infty))$, $f(A)$ *is self-adjoint and positive;*

iii) *for any characteristic function* $f \in \mathcal{B}([0, +\infty))$, $f(A) \in \mathcal{L}(\mathcal{H})$ *is a projection; moreover,* $\mathrm{s}(A) = \chi_{(0, +\infty)}(A)$;

iv) *for all* $f \in \mathcal{B}([0, +\infty))$ *such that* $|f| = 1$, $f(A) \in \mathcal{L}(\mathcal{H})$ *is unitary.*

Proof: [S-Z; 9.13]

Definition A.1.9: Let T be a linear operator in $\mathcal{H}$. T is *normal* if it is closed and $TT^* = T^*T$.

Remark: For any positive self-adjoint linear operator A in $\mathcal{H}$ and any $f \in \mathcal{B}([0, +\infty))$, the linear operator $f(A)$ is normal.

Definition A.1.10: Let $\mathcal{H}$ be a Hilbert space.

i) A function $f : D \subset \mathbf{C} \to \mathcal{H}$ is said to be *weakly analytic* in D if $\phi \circ f : D \subset \mathbf{C} \to \mathbf{C}$ is analytic for any continuous form ϕ on $\mathcal{H}$.

ii) A function $f : D \subset \mathbf{C} \to \mathcal{H}$ is said to be *analytic* in D if

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists in the norm topology of $\mathcal{H}$ for every $z \in D$.

In the set $C = \mathbf{C} \setminus \{\lambda : Re(\lambda) \leq 0 \text{ and } Im(\lambda) = 0\}$, define the function Log as follows:

$$\text{Log}\lambda = \ln|\lambda| + arg(\lambda), \quad -\pi < arg(\lambda) < \pi.$$

Let $\alpha \in \mathbf{C}, Re(\alpha) \geq 0$. Consider the mapping f_α defined by

$$\begin{aligned} f_\alpha(\lambda) = \lambda^\alpha &:= \exp(\alpha \text{Log}\lambda) \quad \text{for all } \lambda \in (0, +\infty), \\ &= 0 \qquad \text{if } \lambda = 0. \end{aligned}$$

Then $f_\alpha \in \mathcal{B}([0, +\infty))$. For any positive self-adjoint linear operator A in $\mathcal{H}$, we define the operator

$$f_\alpha(A) = A^\alpha.$$

Proposition A.1.11: *Let A be a positive self-adjoint linear operator in $\mathcal{H}$, $\xi \in \mathcal{H}$ and $\epsilon \geq 0$. The following statements are equivalent:*

i) $\xi \in \mathcal{D}_{A^\epsilon}$;

ii) $\xi \in \mathcal{D}_{A^\alpha}$ *for all $\alpha \in \mathbf{C}$, $0 \leq Re(\alpha) \leq \epsilon$, and the mapping $\alpha \mapsto A^\alpha \xi$ is continuous on $\{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq \epsilon\}$ and analytic in $\{\alpha \in \mathbf{C} : 0 < Re(\alpha) < \epsilon\}$;*

iii) *the mapping*

$$it \mapsto A^{it}\xi,$$

defined on the imaginary axis, has a continuous extension to $\{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq \epsilon\}$, which is analytic in $\{\alpha \in \mathbf{C} : 0 < Re(\alpha) < \epsilon\}$.

Proof: [S-Z; 9.15 p.205]

Definition A.1.12: Let $\mathcal{H}$ be a Hilbert space.

i) A family $\{u_t : t \in \mathbf{R}\} \subset \mathcal{L}(\mathcal{H})$ of unitary operators satisfying

$$u_0 = 1 \qquad \text{and} \qquad u_{t+s} = u_t u_s, \quad t, s \in \mathbf{R}$$

is called a *one parameter group of unitary operators*.

ii) The group $\{u_t\}$ is *so-continuous* (resp. *wo-continuous*) if the mapping

$$\mathbf{R} \ni t \mapsto u_t \in \mathcal{L}(\mathcal{H})$$

is so-continuous (resp. wo-continuous).

Let A be a positive self-adjoint linear operator in $\mathcal{H}$ such that $s(A) = 1$. With the help of A.1.7, A.1.8, and A.1.11, it can be shown that the operators A^{it}, $t \in \mathbf{R}$ are unitary and that $\{A^{it}\}$ is a so-continuous group of unitary operators.

Definition A.1.13: A mapping into $\mathcal{H}$ is *weakly continuous* (resp. *weakly analytic*; resp. *weakly entire*) if it is continuous (resp. analytic; resp. entire) for the weak topology in $\mathcal{H}$, that is, the topology induced by the family of seminorms $p_\eta(\xi) = |(\xi \mid \eta)|$, $\xi, \eta \in \mathcal{H}$.

Lemma A.1.14: *Let $\{u_t\}$ be a wo-continuous group of unitary operators in $\mathcal{H}$. Then the set*

$$\{\xi \in \mathcal{H} : \text{ the mapping } it \mapsto u_t\xi \text{ has a weakly entire extension}\}$$

is a dense vector subspace of $\mathcal{H}$.

Proof: [S-Z; 9.17]

Let $\{u_t\}$ be a wo-continuous group of unitary operators in $\mathcal{H}$. For any $\epsilon \geq 0$, denote

$$\mathcal{D}_\epsilon = \left\{ \xi \in \mathcal{H} : \begin{array}{c} \text{the mapping } it \mapsto u_t\xi \text{ has a weakly continuous extension to} \\ \mathbf{D} = \{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq \epsilon\}, \text{ weakly analytic in the interior of } \mathbf{D} \end{array} \right\}$$

For any $\xi \in \mathcal{D}_\epsilon$, we will denote by F_ξ the weakly continuous extension of the mapping $it \mapsto u_t\xi$ to the set $\mathbf{D} = \{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq \epsilon\}$, which is weakly analytic in the interior of $\mathbf{D}$.

For any $\epsilon \geq 0$, let A_ϵ be the linear operator in $\mathcal{H}$ defined by the relations:

$$\mathcal{D}_{A_\epsilon} = \mathcal{D}_\epsilon, \qquad A_\epsilon\xi = F_\xi(\epsilon), \quad \xi \in \mathcal{D}_{A_\epsilon}.$$

Lemma A.1.15: *For any $\epsilon \geq 0$, the operator A_ϵ is self-adjoint and positive; moreover*

$$A_{\epsilon_1+\epsilon_2} = A_{\epsilon_1} + A_{\epsilon_2}, \quad \text{for any } \epsilon_1, \epsilon_2 \geq 0.$$

The above self-adjoint operators are important in the proof of the following theorem.

Theorem A.1.16 (Stone representation thm): *Let $\{u_t : t \in \mathbf{R}\} \subset \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

i) *$\{u_t\}$ is a wo-continuous group of unitary operators;*

ii) *$\{u_t\}$ is a so-continuous group of unitary operators;*

iii) *there exists a positive self-adjoint linear operator* A *in* $\mathcal{H}$ *such that* $s(A) = 1$ *and*

$$u_t = A^{it}, \quad t \in \mathbf{R};$$

A is given by the equivalence

$$(\xi, \eta) \in G_A \Leftrightarrow \left[\begin{array}{l} \text{the mapping } it \mapsto u_t\xi \text{ has a weakly continuous extension to} \\ \mathbf{D} = \{\alpha \in \mathbf{C} : 0 \leq Re(\alpha) \leq 1\}, \text{ which is weakly analytic in} \\ \{\alpha \in \mathbf{C} : 0 < Re(\alpha) < 1\} \text{ and has the value } \eta \text{ at } 1 \end{array}\right]$$

Moreover, the relation $u_t = A^{it}$, $t \in \mathbf{R}$ *establishes a one-to-one correspondence between the so-continuous groups* $\{u_t\}$ *of unitary operators on* $\mathcal{H}$ *and the positive self-adjoint linear operators in* $\mathcal{H}$, *such that* $s(A) = 1$.

Proof: [S-Z; 9.20]

Remark: Let A be a positive self-adjoint operator in $\mathcal{H}$ such that $s(A) = 1$. We have seen that A^{it} is a unitary operator, and $(A^{it})^{-1} = A^{-it}$. Moreover, A^{-1} is also a positive self-adjoint operator such that $s(A) = 1$. In fact, we have: $(A^{-1})^{it} = A^{-it}$, $t \in \mathbf{R}$.

For any positive self-adjoint operator A in $\mathcal{H}$ and $\alpha \in \mathbf{C}$, $Re(\alpha) \geq 0$, we have defined the operator A^α; if $s(A) = 1$, we can define A^α for any $\alpha \in \mathbf{C}$:

$$A^\alpha = \begin{cases} A^\alpha & \text{if } Re(\alpha) \geq 0, \\ (A^{-1})^{-\alpha} & \text{if } Re(\alpha) \leq 0. \end{cases}$$

We thus obtain the following extension of Proposition A.1.11:

Proposition A.1.17: *Let* A *be a positive self-adjoint linear operator in* $\mathcal{H}$ *such that* $s(A) = 1$, $\xi \in \mathcal{H}$ *and* $\epsilon_1 \leq 0$, $\epsilon_2 \geq 0$. *The following statements are equivalent:*

i) $\xi \in \mathcal{D}_{A^{\epsilon_1}} \cap \mathcal{D}_{A^{\epsilon_2}}$;

ii) $\xi \in \mathcal{D}_{A^\alpha}$ *for all* $\alpha \in \mathbf{C}$, $\epsilon_1 \leq Re(\alpha) \leq \epsilon_2$, *and the mapping* $\alpha \mapsto A^\alpha\xi$ *is continuous on* $\mathbf{D} = \{\alpha \in \mathbf{C} : \epsilon_1 \leq Re(\alpha) \leq \epsilon_2\}$ *and analytic in* $\{\alpha \in \mathbf{C} : \epsilon_1 < Re(\alpha) < \epsilon_2\}$ *(the interior of* $\mathbf{D}$*);*

iii) *the mapping*

$$it \mapsto A^{it}\xi,$$

defined on the imaginary axis, has a continuous extension to the set $\mathbf{D}$ *defined in* ii), *which is analytic in the interior of* $\mathbf{D}$.

Proof: [S-Z; 9.21]

Proposition A.1.18: *Let* A *be a positive self-adjoint operator, such that* $s(A) = 1$, *and let* $f, g \in \mathcal{B}([0, +\infty))$ *be bounded functions. If* $f(\lambda) = g(\lambda^{-1})$, $\lambda \in (0, +\infty)$, *then* $f(A) = g(A^{-1})$.

Proof: [S-Z; 9.22]

Lemma A.1.19: *Let $\Omega \subset \mathbf{C}$ be an open set and $F : \Omega \to \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

i) *F is analytic for the norm topology;*

ii) *for any $\xi, \eta \in \mathcal{H}$, the function $\Omega \ni \alpha \mapsto (F(\alpha)\xi \mid \eta)$ is analytic.*

Proof: [S-Z; 9.24]

Proposition A.1.20: *Let A, B be positive self-adjoint operators in $\mathcal{H}$ such that $\mathrm{s}(A) = \mathrm{s}(B) = 1$, $x \in \mathcal{L}(\mathcal{H})$ and $\epsilon_1 \leq 0 \leq \epsilon_2$. The following statements are equivalent:*

i) *there exists vector subspaces $\mathcal{D}_1 \subset \mathcal{D}_{(A^{\epsilon_1}xB^{-\epsilon_1})}$, $\mathcal{D}_2 \subset \mathcal{D}_{(A^{\epsilon_2}xB^{-\epsilon_2})}$, such that*

$$\overline{B^{-\epsilon_1} \mid \mathcal{D}_1} = B^{-\epsilon_1}, \qquad \overline{B^{-\epsilon_2} \mid \mathcal{D}_2} = B^{-\epsilon_2}$$

and the operators

$$A^{\epsilon_1}xB^{-\epsilon_1} \mid \mathcal{D}_1, \qquad A^{\epsilon_2}xB^{-\epsilon_2} \mid \mathcal{D}_2$$

are bounded;

ii) *for any $\alpha \in \mathbf{C}$, $\epsilon_1 \leq Re(\alpha) \leq \epsilon_2$, we have $\mathcal{D}_{(A^{\alpha}xB^{-\alpha})} = \mathcal{D}_{B^{-\alpha}}$, the operator $A^{\alpha}xB^{-\alpha}$ is bounded and the mapping*

$$\alpha \mapsto A^{\alpha}xB^{-\alpha}$$

is so-continuous on $\mathbf{D} = \{\alpha \in \mathbf{C} : \epsilon_1 \leq Re(\alpha) \leq \epsilon_2\}$ and analytic in the interior of $\mathbf{D}$, $\{\alpha \in \mathbf{C} : \epsilon_1 < Re(\alpha) < \epsilon_2\}$;

iii) *the mapping*

$$it \mapsto A^{it}xB^{-it}, \quad t \in \mathbf{R},$$

has a wo-continuous extension to the set $\mathbf{D} = \{\alpha \in \mathbf{C} : \epsilon_1 \leq Re(\alpha) \leq \epsilon_2\}$, which is analytic in $\{\alpha \in \mathbf{C} : \epsilon_1 < Re(\alpha) < \epsilon_2\}$.

Proof: [S-Z; 9.24]

Theorem A.1.21 (Polar decomposition): *Let T be a closed linear operator from $\mathcal{H}$ into $\mathcal{K}$. Then there exists a positive self-adjoint linear operator A in $\mathcal{H}$, and a partial isometry $v : \mathcal{H} \to \mathcal{K}$, such that*

$$T = vA, \qquad v^*v = \mathrm{s}(A).$$

These conditions determine in a unique manner the operators A and v. Moreover,

$$A = |T|, \qquad v^*v = \mathrm{r}(T), \qquad vv^* = \mathrm{l}(T).$$

Proof: [S-Z; 9.29]

Bibliography

[Co$_1$] A. Connes. Noncommutative geometry. Academic Press, 1994.

[Co$_2$] A. Connes. Sur le théorème de Radon-Nikodym pour les poids normaux fidèles semi-finis. *Bulletin des Sciences Mathématiques*, 2e série, **97** (1973), 253-258.

[Co$_3$] A. Connes. On the spatial theory of von Neumann algebras. *Journal of functional analysis*, **35** (1980), 153-164.

[C] J. Conway. A course in functional analysis. Springer Verlag, 1990.

[Di] J. Dixmier. Von Neumann algebras. North Holland Mathematical Library, North Holland, 1981.

[D-S] N. Dunford and J.T. Schwartz. Linear Operators. Interscience, New York, 1958.

[E-L] E. Effros and C. Lance. Tensor products of operator algebras. *Advances in Mathematics*, **25** (1977), 1-34.

[F-M] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology and von Neumann algebras II. *Transactions of the American Mathematical Society*, Vol 234 (1977).

[H] U. Haagerup. The standard form of von Neumann algebras. *Math. Scand.* **37** (1975), 271-283.

[K-R] R.V. Kadison and J.R. Ringrose. Fundamentals of the theory of operator algebras, Vol I and II. Academic Press, 1983.

[M] G. Murphy. C^*-Algebras and Operator Theory. Academic Press, 1990.

[N] R. Narasimhan. Several complex variables. Chicago Lectures in Mathematics, The University of Chicago Press, 1971.

[P] G. Pedersen. Analysis Now. Springer Verlag, 1988.

[R] W. Rudin. Functional analysis. International Series in Pure and Applied Mathematics, McGraw-Hill Inc. 1991.

[S] S. Stratila. Modular theory in operator algebras. Editura Academiei and Abacus Press, 1981.

[S-Z] S. Stratila and L. Zsido. Lectures on von Neumann algebras. Editura Academiei and Abacus Press, 1979.

[Su] V.S. Sunder. An invitation to von Neumann algebras. Springer Verlag, 1987.

[Tak$_1$] M. Takesaki. Theory of operator algebras I. Springer Verlag, 1979.

[Tak$_2$] M. Takesaki. Structure of factors and automorphism groups. Conference Board of the Mathematical Sciences, Regional conference series in mathematics, number 51, AMS, 1983.

[VD] A. Van Daele. Continuous crossed products and type III von Neumann algebras. London Math. Soc. Lecture Note Series **31**. Cambridge University Press, 1978.